Research Article

Approach to Multiattribute Decision-Making Problems Based on Neutrality Aggregation Operators of Picture Fuzzy Information

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This manuscript is aimed at developing some novel operational laws named scalar neutrality operation and neutrality addition on picture fuzzy numbers (PFNs). The main focus of this work is to involve the neutral behaviour of the experts towards the priorities of entities where it presents equal degrees to independent membership functions. Moreover, based on these operations, some novel aggregation operators are established to aggregate the different priorities of experts. Some useful relations and characteristics are examined thoroughly. Lastly, the multiattribute group decision-making algorithm in accordance with the suggested operation is illustrated and examined a case study in order to choose a suitable mining company for a mining project along with several numerical examples. The advantages, as well as the superiority of the suggested approach, are exhibited by comparing the results with a few existing methods.

1. Introduction

Zadeh’s idea of fuzzy set (FS) [1] opened a new horizon for the researchers to deal with real-life scenarios which includes uncertainty. Atanassov [2] generalized the idea of FSs and suggested the framework of intuitionistic fuzzy set (IFS) which provided us a significant tool to handle the imprecision. Atanassov and Gargov [3] and later on Atanassov [4] introduced interval valued intuitionistic fuzzy sets (IVIFSs) which used real numbers in the form of intervals to represent a membership degree (MD), a nonmembership degree (NMD), and hesitancy level. Researchers have frequently applied the concept of FSs to many real-life situations where decision-making is one of the prominent areas among them. Some very interesting fields including Xu et al. [5, 6] developed clustering algorithms, De et al. and Xiao and Ding [7, 8] presented their theories on medical diagnosis, and studies on TOPSIS and TIDIM methods were conducted by several authors [9–12]. Pamučar et al. [13] introduced the MABAC method for interval valued fuzzy rough numbers. Mu et al. [14] extended the study by working on Maclaurin symmetric mean based on interval valued Pythagorean fuzzy set (PyFS). As (3,2) FS has larger domain in terms of membership degree as compared to PyFS, so Ibrahim et al. [15] recently investigated the relation of (3,2) FS with other fuzzy sets. Zeng [16] made decision-making more effective by utilizing uncertain intelligence. Muhuiddin et al. [17, 18] recorded their extension in the field of interval valued m polar fuzzy structure. Pan and Deng [19] established a new similarity measure to discuss the clustering problems under IFS. Similarly, Yager [20, 21] adopted the fusion process to aggregate the information.
during the multiattribute decision-making (MADM) problems. Xu and Yager [22] introduced some innovative aggregation operators (AOs) under IFS environment. Wang and Liu [23] made extension to these AOs by using Einstein operations. Ghani and Isa [24] proposed some interactive AOs based on IFS. Xu et al. [25] established innovative operators by fusing Einstein operator under IFS. Wan et al. [26] have established a decision-making scheme by using IVIFS with the help of statistical tool and sample data. Recently, few authors utilized linear Diophantine fuzzy sets and presented their work [27, 28]. Some useful approaches [29–31] have been adopted to use the fuzzy information more smoothly. Furthermore, many authors dealt with MADM problems with more powerful aggregating tools [32].

It is clear from above-studies that PFSs have the ability to handle the vagueness of data efficiently. Many researchers have taken keen interest in IFS and contributed a lot (see [33, 34] for reference). Although IFSs have been utilized to solve many issues, there are so many real-life situations which cannot be represented by using IFSs. One of them is voting which needs three independent functions membership, nonmembership, and neutral membership function to represent the human opinion. To handle such situations, Cuong [35, 36] proposed the concept of picture fuzzy set (PFS) and examined some basic characteristics of PFS. Singh [37] used correlation coefficients to conduct the clustering analysis for PFS. Son [38] presented a new clustering scheme and weather forecasting technique on the basis of PFS. Thong [39] applied PFNs on medical diagnosis to study and support the health care system. Wei and Jana et al. [40–42] presented AOs based on PFNs and solved MADMPs. Due to immense utility of the PFS and SFS, the authors really paid their attention and produced valuable researches [43–46]. Recently, Liu et al. [47], Ullah et al. [48, 49], and Akram et al. [50] made valuable addition in the field of pattern recognition, decision-making, and performance evaluation of solar energy cells by using similarity measures and aggregation operators under picture fuzzy and interval valued T-spherical fuzzy environment.

The above-given studies show that different authors have made their contributions to establish different AOs under IFS, IVIFS, PFS, etc. to manage the ambiguities in the data. Yet, it has been observed that all are not unbiased solely. For instance, if a decision-maker has assigned equal degree to MD, NMD, and AD, then the comprehensive values are not equal by utilizing existing tools for aggregation. It shows that the obtained value is not neutral. In order to enhance the utility of the AOs, it is necessary to add the neutrality character to the operational laws and their respective AOs. To achieve this, we construct probability sum (PS) function and the interaction between the MD, NMD, and AD to establish some novel operational laws on PFS. Furthermore, we suggest the picture fuzzy weighted neutral averaging (PFWNA) operator and picture fuzzy ordered neutral averaging (PFOWNA) operator to aggregate the different values of the experts. Finally, we develop a scheme to treat the MAGDMPs and give a numerical example to explore the validity of the scheme.

The rest of the paper is set out as follows. In the next section, we review some basic definitions regarding PFS. In Section 3, we define some neutral operational laws and its salient features for PFS. In Section 4, we define AOs on the basis of newly proposed operational laws. An innovative algorithm is presented in Section 5 to manage the MAGDMPs. A feasibility study of the scheme and comparison study are conducted in Section 6. Finally, Section 7 gives us final notes to conclude the manuscript.

2. Preliminaries

In this section, some fundamental definitions associated with PFS are revised. In this manuscript, X denote a non-empty set and $\tilde{s}, \tilde{i}, \tilde{d}$, and $r$ represent membership degree (MD), abstinence degree (AD), nonmembership degree (NMD), and degree of refusal (RD), respectively. The concept of PFS was proposed by Cuong [36] by narrating the fuzzy data by using a MD, NMD, AD, and $r$.

**Definition 1** (see [36]). A PFS $\tilde{N}$ on X is defined as

$$\tilde{N} = \left\{ \left( x, \left( \tilde{s}(x), \tilde{i}(x), \tilde{d}(x) \right) : \tilde{s}(x) : X \to [0, 1], \tilde{i}(x) : X \to [0, 1] \text{ and } \tilde{d}(x) : X \to [0, 1] \forall x \in X, \right. \right\}$$

(1)

**Definition 2.** For two PFNs $\tilde{N}_1 = (\tilde{s}_1, \tilde{i}_1, \tilde{d}_1)$ and $\tilde{N}_2 = (\tilde{s}_2, \tilde{i}_2, \tilde{d}_2)$ and a real $\lambda > 0$, we have some dcharacteristic axioms defined as

(1) $\tilde{N}_1^C = (\tilde{d}_1, \tilde{i}_1, \tilde{s}_1)$

(2) $\tilde{N}_1 \subseteq \tilde{N}_2$ if $\tilde{s}_1 \leq \tilde{s}_2$, $\tilde{i}_1 \leq \tilde{i}_2$, and $\tilde{d}_1 \geq \tilde{d}_2$

(3) $\tilde{N}_1 \equiv \tilde{N}_2$ if $\tilde{N}_1 \subseteq \tilde{N}_2$ and $\tilde{N}_2 \subseteq \tilde{N}_1$

(4) $\tilde{N}_1 \oplus \tilde{N}_2 = (\tilde{s}_1 + \tilde{s}_2 - \tilde{s}_1 \tilde{s}_2, \tilde{i}_1 \tilde{i}_2, \tilde{d}_1 \tilde{d}_2)$

(5) $\tilde{N}_1 \otimes \tilde{N}_2 = (\tilde{s}_1 \tilde{s}_2, \tilde{i}_1 + \tilde{i}_2 - \tilde{i}_1 \tilde{i}_2, \tilde{d}_1 + \tilde{d}_2 - \tilde{d}_1 \tilde{d}_2)$

(6) $\lambda \tilde{N}_1 = (1 - (1 - \tilde{s}_1^\lambda)^{\tilde{i}_1}, \tilde{i}_1, \tilde{d}_1)$

(7) $\tilde{N}_1^\lambda = (\tilde{s}_1^\lambda, 1 - (1 - \tilde{s}_1^\lambda)^{\tilde{i}_1}, 1 - (1 - \tilde{d}_1^\lambda))$

**Definition 3** (see [51]). For a PFN $\tilde{N} = (\tilde{s}(x), \tilde{i}(x), \tilde{d}(x))$, the score function and accuracy function for PFNs are defined as under

$$\text{SC} (\tilde{N}) = \tilde{s}(x) - \tilde{d}(x), \text{ and } \text{SC}(\tilde{N}) \in [-1, 1]$$

(2)

$$\text{AC}(\tilde{N}) = \tilde{s}(x) + \tilde{i}(x) + \tilde{d}(x), \text{ and } \text{AC}(\tilde{N}) \in [0, 1]$$

(3)

**Definition 4.** Let $\tilde{N}_1$ and $\tilde{N}_2$ be two PFNs; SC($\tilde{N}$) is the “score function”; and AC($\tilde{N}$) is “accuracy function”; then, $\tilde{N}_1 > \tilde{N}_2$, where $\rangle$ refers to “preferred to” if either SC($\tilde{N}_1$) > SC($\tilde{N}_2$) or SC($\tilde{N}_1$) = SC($\tilde{N}_2$) and AC($\tilde{N}_1$) > AC($\tilde{N}_2$) hold.
For a collection of PFNs $\tilde{N}_j = (\tilde{\delta}_j, i_j, d_j)$, $j = 1, 2, \cdots, n$, the weighted averaging AOs are defined as follows.

**Definition 5** (see [52]). Let $\tilde{N}_j = (\tilde{\delta}_j, i_j, d_j), j = 1, 2, \cdots, n$, be a collection of PFNs and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weight vector of $\tilde{N}_j$ with $\sum_{j=1}^{n} \omega_j = 1$; the weighted averaging and ordered weighted averaging AOs are defined as

$$\text{PFWA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n) = \left(1 - \prod_{j=1}^{n} \left(1 - \tilde{\delta}_j \omega_j\right)^{\omega_j}, \prod_{j=1}^{n} \left(i_j \omega_j\right)^{\omega_j}, \prod_{j=1}^{n} \left(d_j \omega_j\right)^{\omega_j}\right).$$

**PFOWA**($\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n$)

$$= \left(1 - \prod_{j=1}^{n} \left(1 - \tilde{\delta}_{\sigma(j)} \omega_{\sigma(j)}\right)^{\omega_{\sigma(j)}}, \prod_{j=1}^{n} \left(i_{\sigma(j)} \omega_{\sigma(j)}\right)^{\omega_{\sigma(j)}}, \prod_{j=1}^{n} \left(d_{\sigma(j)} \omega_{\sigma(j)}\right)^{\omega_{\sigma(j)}}\right),$$

(4)

where $\sigma$ is a transformation map of $(1, 2, \cdots, n)$ such that $\tilde{N}_{\sigma(j-1)} \geq \tilde{N}_{\sigma(j)}$ for all $j = 2, 3, \cdots, n$.

### 3. New Operational Laws on PFNs

It is not possible with the help of score function given in Definition 3 to obtain the accurate ranking for all PFNs. For instance, $\tilde{N}_1 = (0.3, 0.2, 0.25)$ and $\tilde{N}_2 = (0.4, 0.2, 0.35)$ are two PFNs and by using score function according to Definition 3, we get $\text{SC}(\tilde{N}_1) = \text{SC}(\tilde{N}_2)$. But we are well aware that $\tilde{N}_1 \neq \tilde{N}_2$. To manage such cases under PFSs, we define a new score function for PFSs.

#### 3.1. A New Score Function

**Definition 6.** For a PFN $\tilde{N} = (\tilde{\delta}, i, d)$, an innovative score function is defined as

$$\delta(\tilde{N}) = \frac{e^{\tilde{\delta}(x) - i(x) - d(x)}}{1 + r},$$

(5)

where degree of refusal is denoted by $r$ and calculated as $r = 1 - \tilde{\delta}(x) - i(x) - d(x)$.

**Theorem 7.** Let $\tilde{N} = (\tilde{\delta}, i, d)$ be a PFN, and $\delta$ is the score function; then, $\delta$ monotonically increases for $\tilde{\delta}$ and decreases for $i$ and $d$.

Proof. To prove the required result, firstly, we partially differentiate $\delta$ with respect to $\tilde{\delta}$ which gives $\partial \delta / \partial \tilde{\delta} = ((2 + r) e^{\tilde{\delta}(x) - i(x) - d(x)})/1 + r > 0$ and then with respect to $i$ and $d$ gives $\partial \delta / \partial i = -(r) (e^{\tilde{\delta}(x) - i(x) - d(x)} - 1) \leq 0$ and $\partial \delta / \partial d = -(r) (e^{\tilde{\delta}(x) - i(x) - d(x) - d}) \leq 0$. Hence, the result is proved.

**Theorem 8.** For PFN $\tilde{N}$, the score function $\delta$ holds the following:

1. $e^{-1} \leq \delta \leq e(6)$
2. $\delta(\tilde{N}) = e$ iff $\tilde{N} = (1, 0, 0)$
3. $\delta(\tilde{N}) = e^{-1}$ iff $\tilde{N} = (0, 0, 0)$ and $\tilde{N} = (0, 0, 1)$

This result has trivial proof.

Proposed score function $\delta$ is validated with the help of comparison analysis given in Table 1. Clearly, we observe advantages of newly proposed score functions over the existing function. We justify our claim in Theorem 9.

**Theorem 9.** For two PFNs $\tilde{N}_1$ and $\tilde{N}_2$, if $\text{SC}(\tilde{N}_1) = \text{SC}(\tilde{N}_2)$ and $\text{AC}(\tilde{N}_1) > \text{AC}(\tilde{N}_2)$, then $\delta(\tilde{N}_1) > \delta(\tilde{N}_2)$. Further, if $\text{SC}(\tilde{N}_1) = \text{SC}(\tilde{N}_2)$ and $\text{AC}(\tilde{N}_1) = \text{AC}(\tilde{N}_2)$, then $\delta(\tilde{N}_1) = \delta(\tilde{N}_2)$.

Proof. Let $\tilde{N}_1 = (i_1, d_1)$ and $\tilde{N}_2 = (i_2, d_2)$ be two PFNs; then, by using Definition 4, if $\text{SC}(\tilde{N}_1) = \text{SC}(\tilde{N}_2)$ and $\text{AC}(\tilde{N}_1) > \text{AC}(\tilde{N}_2)$, we have $i_1 = d_1 = i_2 = d_2$ and $i_1 + i_2 = d_1 + d_2 = i_2 + d_2$. Therefore, we can have $\delta(\tilde{N}_1) = (e^{\tilde{\delta}(x) - i_1(x) - d_1(x)})/1 + r > (e^{\tilde{\delta}(x) - i_2(x) - d_2(x)})/1 + r \geq \delta(\tilde{N}_2)$.

In the next subsection, we intend to establish some novel operations on the basis of probability sum (PS) and interaction coefficients. Major advantage of such newly proposed operation for PFS environment can be obtained with a fair decision in the case when all the membership degrees are equal for a PFN.

#### 3.2. Meaning of PS and Rules of PFNs

For PFNs $\tilde{N}_1 = (i_1, d_1)$ and $\tilde{N}_2 = (i_2, d_2)$, the geometric characteristic of the (PS) is explained in Figure 1. In this figure, $\tilde{\delta}_1, \tilde{\delta}_2$ represents the MDs of $\tilde{N}_1$ and $\tilde{N}_2$ while $i_1, i_2$ and $d_1, d_2$ represent...
the ADs and NMDs, respectively. Obviously, \( \hat{s}_1 + i_1 + d_1 \) and \( \hat{s}_2 + i_2 + d_2 \) are two such events which do not depend on each other for PSF. Express \( \hat{s}_1 + i_1 + d_1 \) to be the PS of occurring at least independent events of \( \hat{s}_1 + i_1 + d_1 \) and \( \hat{s}_2 + i_2 + d_2 \); thus,

\[
\hat{s}_1 + i_1 + d_1 = \text{PS}(\hat{s}_1 + i_1 + d_1, \hat{s}_2 + i_2 + d_2).
\]

From it, we observe that the aggregated values of the PFNs \( \tilde{N}_1 \) and \( \tilde{N}_2 \) can be accomplished by the following:

1. Aggregate the MD, NMD, AD, and RD of the PFNs \( \tilde{N}_1 \) and \( \tilde{N}_2 \) by utilizing the algebraic sum operations \( T(x, y) = x + y \); as a result, we obtain \( \hat{s}_1 + \hat{s}_2, i_1 + i_2, d_1 + d_2 \) and \( r_1 + r_2 \) as it is shown with the help of different colours in Figure 2. It is absolutely apparent that their sum exceeds from 1 which is no more a PFN.

2. In order to calculate the overall aggregated values as PFN, we assign a mutual interaction coefficient \( (1 - r_1 r_2)/(\hat{s}_1 + \hat{s}_2 + i_1 + i_2 + d_1 + d_2) \) to \( \hat{s}_1 + \hat{s}_2, i_1 + i_2, \) and \( d_1 + d_2 \) which can be considered as MD, AD, and NMD. Also, their sum is not greater than 1 and corresponding RD is \( r_1 r_2 \). So

\[
\hat{s}_1 + i_1 + d_1 = \text{PS}(\hat{s}_1 + i_1 + d_1, \hat{s}_2 + i_2 + d_2).
\]

Moreover, PS function used in Equation (6) is a -conorm for PFS and properties like commutativity, associativity, monotonicity, and boundedness which are satisfied. Hence, on the basis of this erudition, we intend to establish some novel operation laws under PFS environment.

3.3. Neutral Operational Laws. This section presents novel operation laws under PFS environment.

**Definition 10.** Let \( \tilde{N}_1 = (\hat{s}_1, i_1, d_1) \) and \( \tilde{N}_2 = (\hat{s}_2, i_2, d_2) \) be two PFNs. The neutrality operation of \( \tilde{N}_1 \) and \( \tilde{N}_2 \) is given as

\[
\tilde{N}_1 \odot \tilde{N}_2 = \left(\begin{array}{c}
\text{MCS}(\tilde{N}_1, \tilde{N}_2) \\
\text{NCS}(\tilde{N}_1, \tilde{N}_2) + \text{ACS}(\tilde{N}_1, \tilde{N}_2)
\end{array}\right)\left(\begin{array}{c}
\text{MCS}(\tilde{N}_1, \tilde{N}_2) + \text{NCS}(\tilde{N}_1, \tilde{N}_2) + \text{ACS}(\tilde{N}_1, \tilde{N}_2)\\
\text{NCS}(\tilde{N}_1, \tilde{N}_2) + \text{ACS}(\tilde{N}_1, \tilde{N}_2)
\end{array}\right)
\]

where \( \text{MCS}(\tilde{N}_1, \tilde{N}_2) = \hat{s}_1 + \hat{s}_2 \), \( \text{NCS}(\tilde{N}_1, \tilde{N}_2) = d_1 + d_2 \), and \( \text{ACS}(\tilde{N}_1, \tilde{N}_2) = i_1 + i_2 \) represent the MD, NMD, and AD coefficient sum of \( \tilde{N}_1 \) and \( \tilde{N}_2 \), respectively. Also, \( \text{PS}(x, y) = 1 - (1 - x)(1 - y) \) is used to represent the PS of the FN \( \tilde{N}_1 \) and \( \tilde{N}_2 \).

For \( \tilde{N}_1 = (\hat{s}_1, i_1, d_1) \) and real \( r > 1 \), we get

\[
\text{PS}(r(\hat{s}_1 + i_1 + d_1)) = \text{PS}(\hat{s}_1 + i_1 + d_1, \text{PS}((r - 1)(\hat{s}_1 + i_1 + d_1))).
\]

Hence, we asserted following propositions.
Proposition 11. For PFN \( \tilde{N}_j = (\tilde{s}_j, i_j, d_j) \) and a real \( \tau > 0 \), we get
\[
PS(\tilde{s}_j + i_j + d_j) = 1 - r_j^\tau. \tag{10}
\]

**Proof.** We apply mathematical induction on \( \tau \).

Step 1. For \( \tau = 2 \) and by using Equation (9), we have
\[
PS(\tilde{s}_j + i_j + d_j) = PS(\tilde{s}_j + i_j + d_j, PS((\tau - 1)(\tilde{s}_j + i_j + d_j))
\]
\[
= PS(\tilde{s}_j + i_j + d_j, 1 - r_j)
\]
\[
= 1 - (1 - \tilde{s}_j - i_j - d_j)(1 - 1 + r_j)^\tau
\]
\[
= 1 - (r_j)^{\tau+1}. \tag{11}
\]

The given statement is satisfied for \( \tau = 2 \).

Step 2. Assume Equation (10) is true for \( \tau = n \), then, for \( \tau = n + 1 \),
\[
PS((n + 1)(\tilde{s}_j + i_j + d_j)) = PS((\tilde{s}_j + i_j + d_j), PS(n(\tilde{s}_j + i_j + d_j))
\]
\[
= PS((\tilde{s}_j + i_j + d_j), 1 - (r_j)^n)
\]
\[
= 1 - (1 - \tilde{s}_j - i_j - d_j)(1 - 1 + r_j)^\tau
\]
\[
= 1 - (r_j)^{\tau+1}. \tag{12}
\]

Hence, by using induction, Equation (10) is satisfied for all \( \tau \).

For a group of “\( n \)” PFNs \( \tilde{N}_j = (\tilde{s}_j, i_j, d_j) \) \( j = 1, 2, \ldots, n \), such that \( MCS(\tilde{N}_j) = \tilde{s}_j \) and \( MCS(\tilde{N}_j, \ldots, \tilde{N}_n) = MCS(\tilde{N}_1, \ldots, \tilde{N}_n) \). Thus, we have \( MCS(\tilde{N}_1, \ldots, \tilde{N}_n) = \sum_{j=1}^{n} \tilde{s}_j \).

Likewise, we have \( ACS(\tilde{N}_1, \ldots, \tilde{N}_n) = \sum_{j=1}^{n} \tilde{i}_j \) and \( ACS(\tilde{N}_1, \ldots, \tilde{N}_n) = \sum_{j=1}^{n} \tilde{i}_j \). By using this, we have the below given proposition.

**Proposition 12.** For PFN \( \tilde{N}_j = (\tilde{s}_j, i_j, d_j) \) and a real number \( \tau > 0 \), we have \( MCS(\tilde{N}_j) = \tau . MCS(\tilde{N}_j), ACS(\tilde{N}_j) = \tau . ACS(\tilde{N}_j) \).

**Proof.** If we take \( \tilde{N}_1 = \tilde{N}_2 \) in MCS, then \( MCS(2\tilde{N}_j) = MCS(\tilde{N}_1, \tilde{N}_2) = (\tilde{s}_1 + \tilde{s}_2) = 2MCS(\tilde{N}_1) \). Applying induction on \( \tau \), we can easily have that \( MCS(\tau\tilde{N}_j) = \tau . MCS(\tilde{N}_j), ACS(\tau\tilde{N}_j) = \tau . ACS(\tilde{N}_j) \).

**Definition 13.** For PFN \( \tilde{N}_j = (\tilde{s}_j, i_j, d_j) \) and a real \( \tau \geq 0 \), the “scalar neutrality operation” is specified as
\[
\tau.\tilde{N}_j = \begin{cases} 
\text{MCS}(\tau.\tilde{N}_j) & \text{MCS}(\tau.\tilde{N}_j) + \text{NC}(\tau.\tilde{N}_j) + \text{ACS}(\tau.\tilde{N}_j) \\
\text{NC}(\tau.\tilde{N}_j) & \text{MCS}(\tau.\tilde{N}_j) + \text{NC}(\tau.\tilde{N}_j) + \text{ACS}(\tau.\tilde{N}_j) \\
\text{ACS}(\tau.\tilde{N}_j) & \text{MCS}(\tau.\tilde{N}_j) + \text{NC}(\tau.\tilde{N}_j) + \text{ACS}(\tau.\tilde{N}_j) \\
\end{cases}
\]
\[
\times \begin{pmatrix} 
\text{PS}(\tilde{s}_j + i_j + d_j) \\
\text{PS}(\tilde{s}_j + i_j + d_j) \\
\text{PS}(\tilde{s}_j + i_j + d_j) \\
\end{pmatrix} \tag{13}
\]

where \( (\tau.\tilde{N}_j) = \tau.\tilde{s}_j, \text{NC}(\tau.\tilde{N}_j) = \tau.d_j, \) and \( \text{ACS}(\tau.\tilde{N}_j) = \tau.i_j \).

From Equation (8) and Equation (13), we note that decision-makers have neutral behaviour to MD, NMD, and AD which is shown in following proposition.

**Proposition 14.** For PFNs \( \tilde{N}_1 = (\tilde{s}_1, i_1, d_1) \) and \( \tilde{N}_2 = (\tilde{s}_2, i_2, d_2) \), if \( \tilde{s}_1 = i_1 = d_1 \) and \( \tilde{s}_2 = i_2 = d_2 \), then \( \tilde{s}_1 + \tilde{i}_1 + \tilde{d}_1 = \tilde{s}_2 + \tilde{i}_2 + \tilde{d}_2 \).

**Remark 15.** This fact is observed from Definition 2 that when \( \tilde{s}_1 = i_1 = d_1 \) and \( \tilde{s}_2 = i_2 = d_2 \), we can obtain \( \tilde{s}_1 + \tilde{i}_1 + \tilde{d}_1 = \tilde{s}_2 + \tilde{i}_2 + \tilde{d}_2 \). Therefore, it shows the neutrality of newly proposed operations.

Now, by utilizing the definition of MCS, NCS, ACS, and PS function, we can write Equation (8) as
\[
\tilde{N}_j \oplus \tilde{N}_k = \begin{pmatrix} 
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j \\
\end{pmatrix} \begin{pmatrix} 
1 - r_j r_k \\
1 - r_j r_k \\
1 - r_j r_k \\
\end{pmatrix} = \begin{pmatrix} 
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\end{pmatrix}. \tag{14}
\]

Subsequently, scalar neutrality operation is introduced by using Definition 13. Equation (8) gives
\[
\tilde{N}_j \oplus \tilde{N}_k = \tilde{N}_j + \tilde{N}_k \implies \begin{pmatrix} 
\text{MCS}(\tilde{N}_j, \tilde{N}_k) \\
\text{MCS}(\tilde{N}_j, \tilde{N}_k) + \text{NC}(\tilde{N}_j, \tilde{N}_k) + \text{ACS}(\tilde{N}_j, \tilde{N}_k) \\
\text{MCS}(\tilde{N}_j, \tilde{N}_k) + \text{NC}(\tilde{N}_j, \tilde{N}_k) + \text{ACS}(\tilde{N}_j, \tilde{N}_k) \\
\end{pmatrix} \begin{pmatrix} 
\text{PS}(\tilde{s}_j + \tilde{i}_j + \tilde{d}_j) \\
\text{PS}(\tilde{s}_j + \tilde{i}_j + \tilde{d}_j) \\
\text{PS}(\tilde{s}_j + \tilde{i}_j + \tilde{d}_j) \\
\end{pmatrix} = \begin{pmatrix} 
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\end{pmatrix}. \tag{15}
\]

Similarly,
\[
\tilde{N}_j \oplus \tilde{N}_k = \begin{pmatrix} 
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\tilde{s}_j + \tilde{i}_j + \tilde{d}_j (1 - r_j r_k) \\
\end{pmatrix}. \tag{16}
\]
In the same way, for any real positive \( \tau \), we have
\[
\tau N_1 = \left( \frac{d_1}{s_1 + i_1 + d_1} (1 - (r_1)^\tau), \frac{i_1}{s_1 + i_1 + d_1} (1 - (r_1)^\tau), \frac{d_1}{s_1 + i_1 + d_1} (1 - (r_1)^\tau) \right).
\]
(17)

**Proposition 16.** The MCS, NCS, and ACS for two PFNs \( \tilde{N}_1 \) and \( \tilde{N}_2 \) satisfy the following properties and \( \tau, \tau_1, \tau_2 \) are positive real numbers; then,

1. MCS(\( \tau \tilde{N}_1, \tau \tilde{N}_2 \)) = MCS(\( \tau \tilde{N}_1, \tau \tilde{N}_2 \))
2. NCS(\( \tau \tilde{N}_1, \tau \tilde{N}_2 \)) = NCS(\( \tau \tilde{N}_1, \tau \tilde{N}_2 \))
3. ACS(\( \tau \tilde{N}_1, \tau \tilde{N}_2 \)) = ACS(\( \tau \tilde{N}_1, \tau \tilde{N}_2 \))

**Proof.** For PFNs \( \tilde{N}_1 = (s_1, i_1, d_1) \) and \( \tilde{N}_2 = (s_2, i_2, d_2) \), we have
\[
\text{MCS}(\tilde{N}_1, \tilde{N}_2) = \text{MCS}(\tilde{N}_1, \tilde{N}_2)
\]
\[
\text{NCS}(\tilde{N}_1, \tilde{N}_2) = \text{NCS}(\tilde{N}_1, \tilde{N}_2)
\]
\[
\text{ACS}(\tilde{N}_1, \tilde{N}_2) = \text{ACS}(\tilde{N}_1, \tilde{N}_2)
\]

3.4. Characteristics of the Proposed Operations. The next subsection discusses some remarkable properties of the suggested operational laws as under.

**Theorem 17.** If \( \tilde{N}_1 = (s_1, i_1, d_1) \) and \( \tilde{N}_2 = (s_2, i_2, d_2) \) are two distinct PFNs, then operations of \( \tilde{N}_1 \) and \( \tilde{N}_2 \), \( \tilde{N}_1 \cap \tilde{N}_2 \), and \( \tau \tilde{N}_1 \) are also PFNs for \( \tau > 0 \).

**Proof.** For PFNs \( \tilde{N}_1 = (s_1, i_1, d_1) \) and \( \tilde{N}_2 = (s_2, i_2, d_2) \), we have \( s_1, i_1, d_1, s_2, i_2, d_2 \in [0, 1] \), \( r_1 = 1 - s_1 - i_1 - d_1 \in [0, 1] \) and \( r_2 = 1 - s_2 - i_2 - d_2 \in [0, 1] \). Let \( \tilde{N}_1 \cap \tilde{N}_2 = (\tilde{s}, \tilde{i}, \tilde{d}, \tilde{d}_N \cap \tilde{N}_2) \). To obtain the required result \( \tilde{N}_1 \cap \tilde{N}_2 \in \text{PFN} \), we will just show that \( \tilde{s}_N \cap \tilde{N}_2, \tilde{i}_N \cap \tilde{N}_2, \tilde{d}_N \cap \tilde{N}_2 \) satisfy the given properties and \( \tau, \tau_1, \tau_2 \) are positive real numbers; then,

**Theorem 18.** Let \( \tilde{N}_1 = (s_1, i_1, d_1) \) and \( \tilde{N}_2 = (s_2, i_2, d_2) \) be two PFN and \( \tau, \tau_1, \tau_2 \geq 0 \) be a real number; then,

1. \( \tilde{N}_1 \cap \tilde{N}_2 = \tilde{N}_2 \cap \tilde{N}_1 \)
2. \( \tau (\tilde{N}_1 \cap \tilde{N}_2) = (\tau \tilde{N}_1 \cap \tilde{N}_2) \)
3. \( \tau \tilde{N}_1 \cap \tau \tilde{N}_2 = (\tau \tilde{N}_1 \cap \tau \tilde{N}_2) \)

**Proof.**

(1) Easily can be followed from Equation (14).

(2) For PFNs \( \tilde{N}_1 \) and \( \tilde{N}_2 \), we get

\[
\tau (\tilde{N}_1 \cap \tilde{N}_2) = (\tau \tilde{N}_1 \cap \tau \tilde{N}_2)
\]

(18)
For $r_1, r_2 > 0$,

$$
\begin{align*}
\tau_1 \tilde{N}_i &= \left( \frac{i_1}{i_1 + i_2 + d_1} (1 - (r_1)^{i_1}) \right), \\
\tau_2 \tilde{N}_i &= \left( \frac{i_2}{i_1 + i_2 + d_1} (1 - (r_2)^{i_2}) \right).
\end{align*}
$$

(19)

Thus, by using Equation (8), we get

$$
\tau_1 \tilde{N}_i \oplus \tau_2 \tilde{N}_i = \left( \frac{\text{MCS}(\tau_1 \tilde{N}_i, \tau_2 \tilde{N}_i)}{\text{MCS}(\tau_1 \tilde{N}_i, \tau_2 \tilde{N}_i) + \text{NCS}(\tau_1 \tilde{N}_i, \tau_2 \tilde{N}_i) + \text{ACS}(\tau_1 \tilde{N}_i, \tau_2 \tilde{N}_i) + \text{PS}(\text{PS}(\tau_1 \tilde{N}_i, \tau_2 \tilde{N}_i)))}, \text{PS}(\text{PS}(\tau_1 \tilde{N}_i, \tau_2 \tilde{N}_i))) \right), (20)
$$

4. Neutral Aggregation Operators for PFNs

In the next section, the PFWN and PFOWNA operators of PFNs are discussed, including their related characteristics. To get this, we assume $\Omega$ to be the group of PFNs.

Definition 19. Let $\tilde{N}_i$ be a collection of “n” PFNs. The PFOWNA of $(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n)$ is given by

$$
\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = \otimes_{i=1}^{n} \omega_i \tilde{N}_i,
$$

(21)

where $\omega_i > 0$ is the weighting vector of $\tilde{N}_i$, and $\sum_{i=1}^{n} \omega_i = 1$.

Theorem 20. The aggregated result of PFNs with the help of Definition 19 is also a PFN and given by

$$
\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = \left( \frac{\sum_{i=1}^{n} \omega_i \tilde{N}_i}{\sum_{i=1}^{n} \omega_i (\tilde{N}_i)}, \left( 1 - \prod_{i=1}^{n} (r_i)^{\omega_i} \right) \right),
$$

(22)

Proof. For “n” PFNs $\tilde{N}_i$ and real $\omega_i > 0$, by utilizing Theorem 7, the first result is trivially satisfied. For the existence of

$$
\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k) = \left( \frac{\sum_{i=1}^{k} \omega_i \tilde{N}_i}{\sum_{i=1}^{k} \omega_i (\tilde{N}_i)}, \left( 1 - \prod_{i=1}^{k} (r_i)^{\omega_i} \right) \right),
$$

(23)
Now, for \( n = k + 1 \), we get

\[
\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_{k+1}) = \text{TSFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k) \otimes (\omega_{k+1} \tilde{N}_{k+1})
\]

\[
\text{MCS}(\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, \omega_{k+1} \tilde{N}_{k+1})) = \text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, \omega_{k+1} \tilde{N}_{k+1})
\]

\[
= \text{AC}(\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, \omega_{k+1} \tilde{N}_{k+1})) + \text{AC}(\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, \omega_{k+1} \tilde{N}_{k+1}))
\]

\[
= \text{NCS}(\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, \omega_{k+1} \tilde{N}_{k+1})) + \text{AC}(\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, \omega_{k+1} \tilde{N}_{k+1}))
\]

\[
= \frac{\mathbf{A}}{\mathbf{B}^*}
\]

(25)

Thus,

\[
\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_{k+1}) = \frac{\sum_{i=1}^{k+1} \omega_i \delta_i}{\sum_{i=1}^{k+1} \omega_i} \cdot \left( 1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i} \right)
\]

\[
= \frac{\sum_{i=1}^{k+1} \omega_i \delta_i}{\sum_{i=1}^{k+1} \omega_i} \cdot \left( 1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i} \right)
\]

(29)

Equation (22) holds for \( n = k + 1 \). Therefore, by applying induction, Equation (22) is true for all \( n \).

**Example 1.** Let us consider four PFNs \( \tilde{N}_1 = (0.4, 0.3, 0.1) \), \( \tilde{N}_2 = (0.5, 0.2, 0.2) \), \( \tilde{N}_3 = (0.1, 0.4, 0.2) \), and \( \tilde{N}_4 = (0.3, 0.3, 0.2) \); the weight vector associated with them is \( \omega = (0.2, 0.4, 0.3, 0.1) \); we use here PFWNA to calculate the four PFNs, since

\[
\text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4)
\]

\[
= \frac{\sum_{i=1}^{4} \omega_i \delta_i}{\sum_{i=1}^{4} \omega_i} \cdot \left( 1 - \prod_{i=1}^{4} (r_i)^{\omega_i} \right)
\]

(30)
Further,
\[
\sum_{i=1}^{4} \omega_i \beta_i \left( 1 - \prod_{j=1}^{k+1} (r_j)^{\omega_i} \right) = 0.3479,
\]
\[
\sum_{i=1}^{4} \omega_i \beta_i \left( 1 - \prod_{j=1}^{k+1} (r_j)^{\omega_i} \right) = 0.2353,
\]
\[
\sum_{i=1}^{4} \omega_i \beta_i \left( 1 - \prod_{j=1}^{k+1} (r_j)^{\omega_i} \right) = 0.2456.
\]
(31)

Finally, we get
\[
\text{PFWNA} (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4) = (0.3479, 0.2553, 0.2456),
\]
which is also a PFN.

**Theorem 21** (Idempotency). Let \( \tilde{N}_i = (\tilde{s}_i, i, \tilde{d}_i) (i=1, 2, 3 \ldots n) \) be a set of PFNs. If \( \tilde{N}_i = (\tilde{s}_0, i_0, \tilde{d}_0) \) for all \( i \), then \( \text{PFWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = (\tilde{s}_0, i_0, \tilde{d}_0) \).

**Proof.** For “n” PFNs \( \tilde{N}_i = (\tilde{s}_i, i, \tilde{d}_i) \) and \( \tilde{N}_0 = (\tilde{s}_0, i_0, \tilde{d}_0) \) such that \( \tilde{N}_i = \tilde{N}_0 \), we have \( \tilde{s}_i = \tilde{s}_0, i = i_0 \) and \( \tilde{d}_i = \tilde{d}_0 \) for all \( i \). Then, by Equation (22) and \( \omega_i > 0 \) with \( \sum_{i=1}^{n} \omega_i = 1 \), we get
\[
\text{TSFWNA} (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) =
\left( \begin{array}{c}
\frac{\tilde{s}_0}{\tilde{s}_0 + i_0 + \tilde{d}_0} \left( 1 - \prod_{j=1}^{k+1} (r_j)^{\omega_i} \right),
\frac{i_0}{\tilde{s}_0 + i_0 + \tilde{d}_0} \left( 1 - \prod_{j=1}^{k+1} (r_j)^{\omega_i} \right),
\frac{\tilde{d}_0}{\tilde{s}_0 + i_0 + \tilde{d}_0} \left( 1 - \prod_{j=1}^{k+1} (r_j)^{\omega_i} \right),
\end{array} \right),
\]
(33)

Thus, we have
\[
\min \{ \tilde{s}_i + i_0 + \tilde{d}_i \} \leq 1 - \prod_{i=1}^{n} (1 - \tilde{s}_i - i_0 - \tilde{d}_i)^{\omega_i} \leq \max \{ \tilde{s}_i + i_0 + \tilde{d}_i \}.
\]
(35)

By using Theorem 20,
\[
\tilde{s}_p = \frac{\sum_{i=1}^{k} \omega_i \tilde{s}_i}{\sum_{i=1}^{k} \omega_i (\tilde{s}_i + i_0 + \tilde{d}_i)} \left( 1 - \prod_{j=1}^{k} (r_j)^{\omega_i} \right),
\]
\[
i_p = \frac{\sum_{i=1}^{k} \omega_i i}{\sum_{i=1}^{k} \omega_i (\tilde{s}_i + i_0 + \tilde{d}_i)} \left( 1 - \prod_{j=1}^{k} (r_j)^{\omega_i} \right),
\]
\[
\tilde{d}_p = \frac{\sum_{i=1}^{k} \omega_i \tilde{d}_i}{\sum_{i=1}^{k} \omega_i (\tilde{s}_i + i_0 + \tilde{d}_i)} \left( 1 - \prod_{j=1}^{k} (r_j)^{\omega_i} \right).
\]
(36)

Therefore,
\[
\tilde{s}_p + i_p + \tilde{d}_p = 1 - \prod_{i=1}^{n} (1 - \tilde{s}_i - i_0 - \tilde{d}_i)^{\omega_i}.
\]
(37)

**Theorem 22.** Consider a collection of “n” PFNs \( \tilde{N}_i \); we have
\[
(1) \min \{ \tilde{s}_i + i_0 + \tilde{d}_i \} \leq \tilde{s}_p + i_p + \tilde{d}_p \leq \max \{ \tilde{s}_i + i_0 + \tilde{d}_i \}
\]
\[
(2) \min \{ \tilde{s}_i + i_0 + \tilde{d}_i \}, \max \{ \tilde{s}_i + i_0 + \tilde{d}_i \} \leq \tilde{s}_p \leq \max \{ \tilde{s}_i + i_0 + \tilde{d}_i \}, \max \{ \tilde{s}_i + i_0 + \tilde{d}_i \}.
\]
(36)

Since, \( \tilde{s}_i \) is the minimum of \( \{ \tilde{s}_i \} \), so by expression \( \tilde{s}_p \), we have
\[ \sum_{i=1}^{k} \omega_i (\min \{ \hat{s}_i \} ) \leq \sum_{i=1}^{\min \{ \hat{s}_i + d_i \}} (1 - \min \{ \hat{s}_i + d_i \}) \omega_i. \]

Further,
\[ \sum_{i=1}^{\max \{ \hat{s}_i \} } (1 - \max \{ \hat{s}_i + d_i \}) \omega_i. \]

(3) Similar to part (2)

Theorem 23 (Monotonicity). Let \( \hat{N}_i = (\hat{s}_{N_i}, \hat{i}_{N_i}, \hat{q}_{N_i}) \) and \( M_i = (\hat{s}_{M_i}, \hat{i}_{M_i}, \hat{d}_{M_i}) \) be collection of "n" PFNs, Then,

1. If \( \hat{s}_{N_i} + \hat{i}_{N_i} + \hat{d}_{N_i} = \hat{s}_{M_i} + \hat{i}_{M_i} + \hat{d}_{M_i}, \) then we get

\[ \hat{s}_{N_i} + \hat{i}_{N_i} + \hat{d}_{N_i} \leq \hat{s}_{M_i} + \hat{i}_{M_i} + \hat{d}_{M_i} \]

2. If \( \hat{s}_{N_i} + \hat{i}_{N_i} + \hat{d}_{N_i} = \hat{s}_{M_i} + \hat{i}_{M_i} + \hat{d}_{M_i}, \) and \( \hat{s}_{N_i} \leq \hat{s}_{M_i}, \) then we get

\[ \hat{s}_{N_i} \leq \hat{s}_{M_i}. \]

Proof. For "n" PFNs \( \hat{N}_1, \hat{N}_2, \ldots, \hat{N}_n \) and \( M_1, M_2, \ldots, M_n \) and by using Theorem 20, we get PFNA \( (N_1, N_2, \ldots, N_n) = (\hat{s}_{N_1}, \hat{i}_{N_1}, \hat{d}_{N_1}), \) and PFNA \( (M_1, M_2, \ldots, M_n) = (\hat{s}_{M_1}, \hat{i}_{M_1}, \hat{d}_{M_1}), \) where

\[ d_{P_{N_i}} = \frac{\sum_{i=1}^{k} \omega_i d_{p_{N_i}}}{\sum_{i=1}^{k} \omega_i (\hat{s}_{N_i} + \hat{i}_{N_i} + \hat{d}_{N_i})} \left[ 1 - \sum_{i=1}^{n} \left( 1 - \hat{s}_{N_i} - \hat{i}_{N_i} - \hat{d}_{N_i} \right) \omega_i \right], \]

\[ d_{P_{M_i}} = \frac{\sum_{i=1}^{k} \omega_i d_{p_{M_i}}}{\sum_{i=1}^{k} \omega_i (\hat{s}_{M_i} + \hat{i}_{M_i} + \hat{d}_{M_i})} \left[ 1 - \sum_{i=1}^{n} \left( 1 - \hat{s}_{M_i} - \hat{i}_{M_i} - \hat{d}_{M_i} \right) \omega_i \right]. \]
(3) From part (2), we obtain $\bar{\delta}_{j_1} \leq \bar{\delta}_{j_2}$, $\bar{f}_{j_1} \geq \bar{f}_{j_2}$, $d_{j_1} \geq d_{j_2}$. Hence, by utilizing the score function of Equation (2), we get $\bar{\delta}_{j_1} - \bar{f}_{j_1} - d_{j_1} \leq \bar{\delta}_{j_2} - \bar{f}_{j_2} - d_{j_2}$. So, by an order relation, we get PFOWNA($\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n$) $\leq$ PFOWNA($N_1, N_2, \ldots, N_n$).

**Definition 24.** Let $\tilde{N}_i = (\bar{\delta}_i, \bar{f}_i, d_i)$ ($i = 1, 2, 3, \ldots, n$) be a set of PFNs. The PFOWNA operator of $\tilde{N}_i$ is defined by

$$\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = \langle \sum_{i=1}^{n} \omega_i \tilde{N}_{\sigma(i)} \rangle,$$

where $\sigma$ is the permutation map of $1, 2, 3, \ldots, n$ such that $\tilde{N}_{\sigma(i-1)} \geq \tilde{N}_{\sigma(i)}$ and $\tilde{N}_i \in \Omega$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector associated with PFOWNA operator, satisfying $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

**Theorem 25.** The aggregated value by using Definition 24 for “n” PFNs $\tilde{N}_i = (\bar{\delta}_i, \bar{f}_i, d_i)$ is also a PFN, where

$$\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = \left( \sum_{i=1}^{n} \omega_i \bar{\delta}_i \right) \cdot \left( 1 - \prod_{i=1}^{n} (r_{\sigma(i)} \omega_i) \right),$$

$$\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = \left( \sum_{i=1}^{n} \omega_i \bar{f}_i \right) \cdot \left( 1 - \prod_{i=1}^{n} (r_{\sigma(i)} \omega_i) \right),$$

$$\text{PFOWNA}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_n) = \left( \sum_{i=1}^{n} \omega_i d_i \right) \cdot \left( 1 - \prod_{i=1}^{n} (r_{\sigma(i)} \omega_i) \right).$$

**The proof of this Theorem 25 is omitted because it is similar to Theorem 20.**

**Remark 26.** The PFOWNA operator satisfies all the properties which are satisfied by the PFOWA operator mentioned in Theorems 20, 21, 22, and 23. Also, their proofs are omitted due to similarity.

### 5. Proposed MAGDM Approach Based on Aggregation Operators

This section introduces an inventive scheme to work out the MAGDM problem under the PFS conditions. Consider that a collection of alternatives $F = \{F_1, F_2, \ldots, F_m\}$ and objective is to choose the best possible one under the evaluation of a set of attributes $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n\}$ with weight vector $\omega > 0$ such that $\sum_{j=1}^{n} \omega_j = 1$. The set $\mathcal{H}$ containing attributes is split into two mutually exclusive sets, that is, cost type and benefit type attributes, respectively ($F_1$) and ($F_2$). To rank the given alternatives, in this event, a team of experts

$$\mathcal{E} = \{E^{(1)}, E^{(2)}, \ldots, E^{(l)}\}$$

is selected with weight vector $\omega_k > 0$ such that $\sum_{k=1}^{l} \omega_k = 1$, with a task to propose the best possible solution. For the sake of smooth assessment, decision-makers will utilize PFS environment as $\delta_{ij}^k = (\bar{\delta}_{ij}, \bar{f}_{ij}, d_{ij})$ with $0 \leq \bar{\delta}_{ij}, \bar{f}_{ij}, d_{ij} \leq 1$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, l$. Henceforth, a PF decision matrix can be considered as

$$\mathfrak{Y}^k = \begin{pmatrix}
\delta_{11}^k & \delta_{12}^k & \cdots & \delta_{1n}^k \\
\delta_{21}^k & \delta_{22}^k & \cdots & \delta_{2n}^k \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{m1}^k & \delta_{m2}^k & \cdots & \delta_{mn}^k
\end{pmatrix}.$$

Now, the detailed strategy of new technique is presented below:

**Step 1.** A decision matrix $\mathfrak{Y}$ is formulated under PFS environment corresponding to each expert $E$.

**Step 2.** By using a weight vector and suitable AOs, that is, PFOWNA or PFOWNA aggregate all the decision-makers' rankings $\delta_{ij}^k$, $k = 1, 2, \ldots, l$ into $\delta_{ij} = (\bar{\delta}_{ij}, \bar{f}_{ij}, d_{ij})$.

**Step 3.** As there are two types of attribute indicators, namely, the cost ($F_1$) and the benefit ($F_2$), so we normalize $\delta_{ij}$ into $r_{ij}$ if it is essential.

$$r_{ij} = \begin{cases} 
\delta_{ij}, & \text{for } F_1 \text{ criteria}, \\
\bar{\delta}_{ij}, & \text{for } F_2 \text{ criteria}.
\end{cases}$$

**Step 4.** In the case if weights of the attributes are already given, then use them as they are. Contrarily, if the detail about the attributes is partially known, which is signified in set $\mathcal{G}$, then an optimization model is formulated to calculate an unknown weight vector for each attribute.

$$\max f = \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_{ij} \delta_{ij}$$

s.t. $\sum_{j=1}^{n} \omega_j = 1$, $\omega_j \geq 0$, $\omega \in \mathcal{G}$,}

where $\delta_{ij}$ is the score of each alternative, $F_j$; the above model gives $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$.

**Step 5.** Using weights and utilizing suggested PF averaging AOs to compute the comprehensive values $r_{ij}$ ($i = 1, 2, \ldots, m$).
Table 2: The decision matrix from experts.

| Experts | Alternatives | \( \mathcal{H}_1 \) | \( \mathcal{H}_2 \) | \( \mathcal{H}_3 \) | \( \mathcal{H}_4 \) |
|---------|--------------|----------------|----------------|----------------|----------------|
| \( \tilde{E}^{(1)} \) | \( F_1 \) | (0.5, 0.19, 0.3) | (0.4, 0.25, 0.3) | (0.2, 0.4, 0.3) | (0.3, 0.3, 0.3) |
|       | \( F_2 \) | (0.4, 0.39, 0.2) | (0.45, 0.24, 0.3) | (0.37, 0.5, 0.1) | (0.27, 0.32, 0.4) |
|       | \( F_3 \) | (0.5, 0.19, 0.3) | (0.34, 0.49, 0.1) | (0.29, 0.4, 0.2) | (0.4, 0.35, 0.21) |
|       | \( F_4 \) | (0.21, 0.31, 0.33) | (0.51, 0.27, 0.2) | (0.32, 0.27, 0.4) | (0.42, 0.25, 0.3) |
| \( \tilde{E}^{(2)} \) | \( F_1 \) | (0.5, 0.19, 0.3) | (0.5, 0.22, 0.2) | (0.3, 0.23, 0.2) | (0.28, 0.28, 0.28) |
|       | \( F_2 \) | (0.34, 0.42, 0.2) | (0.24, 0.3, 0.39) | (0.25, 0.3, 0.39) | (0.41, 0.28, 0.3) |
|       | \( F_3 \) | (0.35, 0.35, 0.25) | (0.44, 0.35, 0.18) | (0.35, 0.34, 0.3) | (0.5, 0.29, 0.2) |
|       | \( F_4 \) | (0.34, 0.42, 0.2) | (0.39, 0.3, 0.3) | (0.3, 0.35, 0.29) | (0.4, 0.37, 0.2) |
| \( \tilde{E}^{(3)} \) | \( F_1 \) | (0.5, 0.38, 0.1) | (0.4, 0.28, 0.3) | (0.2, 0.39, 0.4) | (0.33, 0.33, 0.33) |
|       | \( F_2 \) | (0.37, 0.31, 0.3) | (0.29, 0.33, 0.32) | (0.32, 0.25, 0.4) | (0.4, 0.55, 0.04) |
|       | \( F_3 \) | (0.4, 0.3, 0.29) | (0.54, 0.22, 0.2) | (0.34, 0.34, 0.3) | (0.31, 0.38, 0.3) |
|       | \( F_4 \) | (0.51, 0.23, 0.2) | (0.39, 0.32, 0.27) | (0.32, 0.42, 0.21) | (0.34, 0.31, 0.3) |

Table 3: Aggregated values of the experts by the PFVNA operator.

| Alternatives | \( \mathcal{H}_1 \) | \( \mathcal{H}_2 \) | \( \mathcal{H}_3 \) | \( \mathcal{H}_4 \) |
|--------------|----------------|----------------|----------------|----------------|
| \( F_1 \) | (0.5004, 0.2548, 0.2322) | (0.4350, 0.2376, 0.2781) | (0.2313, 0.3436, 0.3945) | (0.3160, 0.3160, 0.3160) |
| \( F_2 \) | (0.3748, 0.3721, 0.2347) | (0.3432, 0.2904, 0.3353) | (0.3212, 0.3617, 0.2809) | (0.3543, 0.3899, 0.2385) |
| \( F_3 \) | (0.4278, 0.2721, 0.2846) | (0.4363, 0.3615, 0.1561) | (0.3301, 0.1371, 0.2665) | (0.3982, 0.3456, 0.2390) |
| \( F_4 \) | (0.3520, 0.3169, 0.2542) | (0.44371, 0.2953, 0.2510) | (0.3176, 0.2458, 0.3086) | (0.3878, 0.3031, 0.2733) |

Step 6. Equation (49) is used to calculate the score values \( r_i = (\hat{s}_i, \hat{t}_i, \hat{d}_i), i = 1, 2, \ldots, m \).

\[
\hat{s}(r_i) = \frac{e^{\hat{s}_i - \hat{t}_i - \hat{d}_i}}{2 - \hat{s}_i - \hat{t}_i - \hat{d}_i}.
\]  

(49)

Step 7. By using Definition 4, the alternatives \( F_i (i = 1, 2, \ldots, m) \) are ranked.

6. Illustrative Examples

To validate the suggested method, we assume a MAGDM problem and examine the performance of the proposed method further; for the sake of comparison, some existing methods are compared with the suggested method.

6.1. Case Study. Mineral resources are the most vital among the category of natural resources. Minerals provide raw material to all kinds of industries which is considered as a backbone of economic growth. Mining is the process through which different minerals are extracted from the earth. Mining sector has made great contributions for human development. Especially in developing countries, it has really helped to raise the standards of health and education. Pakistan is an Asian country and full of natural resources. Balochistan is a resource rich province of Pakistan. A well-known geological survey team has reported that Balochistan has round about ninety mineral resources with sufficient layers. Most of them have not been discovered yet. Recently, there are three main projects working in Balochistan named as Reko-diq, Dudder, and Saindak. All of them are being managed by the Chinese companies. The management of Balochistan province wants to explore more mineral resources. In order to speed up the mining work, the government has invited four different companies with four required attributes for selection named as technical capability (\( \mathcal{H}_1 \)), financial status (\( \mathcal{H}_2 \)), company background (\( \mathcal{H}_3 \)), and employability (\( \mathcal{H}_4 \)). There are three panels \( \tilde{E}^{(1)}, \tilde{E}^{(2)}, \) and \( \tilde{E}^{(3)} \) of decision-makers which are appraising the four companies: \( F_1 \) ("China National Coal Group"), \( F_2 \) ("China Northern Rare Earth Group"), \( F_3 \) ("High-Tech Mining Associates"), and \( F_4 \) ("HBIS Group") with the abovementioned four attributes. Suppose that \( w = (0.39, 0.27, 0.34) \) is the weight vector of the experts and their assessment matrices \( \mathcal{R}_1, \mathcal{R}_2, \) and \( \mathcal{R}_3 \) under PFVNs are presented in Table 2. The purpose of the study is to choose the appropriate company for new mining project. The steps of the proposed MAGDM process are performed as follows:

Step 1. All the values evaluated by the experts are summarized in Table 2.

Step 2. We assume the corresponding weight vector \( w = (0.39, 0.27, 0.34) \) and PFVNA operator to aggregate the information of each alternative. The values are listed in Table 3.
Step 3. All of the given attributes are the same type, so it is not essential to normalize the information.

Step 4. The partial weight information regarding attribute importance as proposed by decision-makers is \( \mathcal{G} = \{0.2 \leq \omega_1 \leq 0.3, 0.25 \leq \omega_2 \leq 0.4, 0.12 \leq \omega_3 \leq 0.4, 0.2 \leq \omega_4 \leq 0.35, \omega_1 + \omega_2 \leq \omega_2, \omega_1 - \omega_2 \leq \omega_3\} \); an optimization model has been formulated by using Equation (48), and after some calculations, we obtained \( \omega = (0.26, 0.4, 0.14, 0.2) \).

Step 5. In accordance with the PFOWNA operator and weights \( \omega \), we obtain the overall value \( r_i \) of each alternative; values are given as \( r_1 = (0.4018, 0.2740, 0.2917), r_2 = (0.3509, 0.4021, 0.3335), r_3 = (0.4130, 0.3376, 0.2223), \) and \( r_4 = (0.3911, 0.3117, 0.2661) \).

Step 6. By using Equation (5), score values are calculated as \( \delta(r_1) = 0.8221, \delta(r_2) = 0.7452, \delta(r_3) = 0.8407, \) and \( \delta(r_4) = 0.8046 \).

Step 7. As \( \delta(r_3) > \delta(r_1) > S(r_4) > S(r_2) \), therefore the preferences are ordered as \( F_3 > F_1 > F_4 > F_2 \). So \( F_3 \) is the optimal choice.

6.2. Alteration of AOs. We can choose a different pair of proposed AOs except from the above-given analysis. Here, we examine the ranking pattern of the alternatives if experts select distinct AOs in Step 2 and Step 5, respectively. As it is quite clear that every AO has its own importance and specific features as per given situation. For instance, PFWNAO gives more weight to PFNs; on the other hand, PFOWNAO gives more importance to the position of PFNs; on the other hand, PFWONA in step 5 generates relatively low score values of alternatives. However, the overall ranking pattern and best possible alternatives remain unchanged.

6.3. Comparative Analysis

Example 2. Again, utilizing the data from Table 2, here, we will calculate the aggregation values and validate the proposed AOs with the help of different tools which already exist in the literature. It is necessary to mention here that the already available work by Garg et al. [52, 53] on neutrality operators is not sufficient enough to manage the information recorded in Table 2. Here, we apply the operators proposed by Garg [52], Wei [40], and Jana at el. [42] on the information given in Table 2 in order to conduct a comparative study among different aggregating tools. The ranking patterns obtained by using the aforementioned existing AOs are enlisted in Table 5. Results obtained using the proposed PFOWNA and PFOWNA operators have consistency. Ranking patterns are quite similar with results obtained by

| Operators used | Step 2 | Step 5 | \( F_1 \) | \( F_2 \) | \( F_3 \) | \( F_4 \) | Ranking |
|----------------|--------|--------|----------|----------|----------|----------|---------|
| PFOWNA         | PFOWNA | 0.8221 | 0.7452   | 0.8407   | 0.8046   | \( F_3 > F_4 > F_2 > F_1 \) |
| PFOWNA         | PFOWNA | 0.8135 | 0.7368   | 0.8144   | 0.7742   | \( F_3 > F_4 > F_2 > F_1 \) |
| PFOWNA         | PFOWNA | 0.3961 | 0.3332   | 0.3978   | 0.3728   | \( F_3 > F_4 > F_2 > F_1 \) |
| PFOWNA         | PFOWNA | 0.8351 | 0.7436   | 0.8506   | 0.8107   | \( F_3 > F_4 > F_2 > F_1 \) |

Figure 3: Geometrical interpretation of alteration of AOs.
Garg [52], Wei [40], and Jana at el. [42]. Moreover, neutrality towards the selection of alternatives plays an important part during the overall decision-making process so, in this way, the proposed operators provide better environment for decision-making. The validity of suggested AOs is evident from Figure 4. Throughout the analysis, there is no change in ranking patterns which shows that results obtained using novel operators agree with the results that exist in the literature. Furthermore, the stability of the results makes it clear from Figure 4 that the approach prescribed in this manuscript is much effective than the remaining methods recorded in Table 5.

6.4. Advantages of the Suggested Approach. Some salient features and advantages of the suggested approach are elaborated in the following:

(1) Suggested AOs can handle all the human aspects; in this way, decision-makers can handle various real-life situations more efficiently

(2) Degree of refusal plays an important role to choose the best alternative; during the information aggregation, by using AOs presented in this, work we can control refusal degree

(3) The main characteristic of this novel extension is the inclusion of decision-makers’ attitude

(4) The ability to handle neutral behaviour of the experts makes this work more effective as compared to other studies presented in Table 5

7. Conclusion

Aggregation operators have a pivotal role during the MADM problems. Hence, we suggest some novel AOs in this paper named as PFWNA and PFOWNA on the basis of PFSs. Many researchers have made enormous contributions for IFSs which consider only MD and NMD. It has been noticed that some real-life scenarios cannot be represented clearly by using IFSs. During this work, we represented our information under PFS environment which extends the idea of IFS. Also, a novel MAGDM scheme is proposed on the basis of newly suggested AOs for PFSs. Some numerical examples have been demonstrated to prove the effectiveness of this approach. Finally, a comparative analysis is presented which shows the supremacy and advantages of this scheme. The concept can be further extended to develop neutrality AOs for spherical and T-spherical fuzzy sets [54, 55].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.
Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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