We review the functional renormalization group (RG) approach to the BCS-BEC crossover for an ultracold gas of fermionic atoms. Formulated in terms of a scale-dependent effective action, the functional RG interpolates continuously between the atomic or molecular microphysics and the macroscopic physics on large length scales. We concentrate on the discussion of the phase diagram as a function of the scattering length and the temperature which is a paradigm example for the non-perturbative power of the functional RG. A systematic derivative expansion provides for both a description of the many-body physics and its expected universal features as well as an accurate account of the few-body physics and the associated BEC and BCS limits.

Keywords: functional RG, ultracold fermionic atoms, BCS-BEC crossover

1. Introduction

Many challenges in contemporary theoretical physics deal with strongly interacting quantum field theories or many-body systems. Progress often relies on the construction of exact or approximate solutions. In the absence of exact solutions, reliable and controlled approximation methods typically are the only source of information about the system and the underlying physical mechanisms. An approximation scheme may be considered as reliable and controlled if it is based on a systematic and consistent expansion scheme and shows convergence towards the exact result (which, however, is often not known). Textbook examples are, of course, provided by perturbative expansions or lattice discretizations both of which can be consistently evaluated to a given order or lattice refinement, systematically improved, and the convergence can at least be checked as a matter of practice.

In this contribution, we would like to demonstrate that the functional RG can be used to develop systematic and consistent expansion schemes for strongly interacting systems. Most importantly, it can be applied in the spacetime continuum and does not require a perturbative ordering scheme. Nevertheless, it offers a variety of tools to verify qualitative and quantitative reliability and practical convergence. As a prime example of strongly interacting many-body systems, we take the BCS-BEC crossover as an illustration for the use of the functional RG. The concrete physical system that we have in mind is an ultracold atomic Fermi gas with two accessible hyperfine spin states near a Feshbach resonance, showing a smooth crossover between Bardeen-Cooper-Schrieffer (BCS) superfluidity and Bose-Einstein condensation (BEC) of diatomic molecules (Eagles 1969; Leggett 1980).

By means of an external magnetic field $B$ the phenomenon of a Feshbach resonance allows to arbitrarily regulate the effective interaction strength of the atoms, parameterized by the s-wave scattering length $a$. We briefly discuss the example of $^{6}$Li (O’Hara et al. 2002), which is besides $^{40}$K...
realized in current experiments (Regal et al. 2004; Zwierlein et al. 2004; Kinast et al. 2004; Bourdel et al. 2004; Bartenstein et al. 2004; Partridge et al. 2005), see left panel of Fig. 1.

For magnetic fields larger than $\sim 1200$ G the scattering length $a$ is small and negative, giving rise to the many-body effect of Cooper-pairing and a BCS-type ground state below a critical temperature. The BCS ground state is superfluid described by a non-vanishing order parameter $\phi_0 = \langle \psi_1 \psi_2 \rangle$ bilinear in the fermion fields. An increase of the temperature leads to a second order phase transition to a normal fluid, $\phi_0 = 0$. Magnetic fields below $B \sim 600$ G induce a small and positive scattering length $a$ and the formation of a diatomic bound state, a dimer. The ground state is a BEC of repulsive dimers and again a phase transition from a superfluid, $\phi_0 > 0$, to a normal fluid, $\phi_0 = 0$, can be observed at a critical temperature. For magnetic fields in the regime $700$ G $\lesssim B \lesssim 1100$ G the modulus of the scattering length $|a|$ is large and diverges at the unitary point, $B_0 = 834$ G where unitarity of the scattering matrix solely determines the two-body scattering properties. At and near unitarity the fermions are in a strongly interacting regime. It connects the limits of BCS superfluidity and Bose-Einstein condensation by a continuous crossover and also shows a superfluid ground state, with $\phi_0 > 0$ (Eagles 1969; Leggett 1980).

A convenient parameterization of the crossover is given by the inverse concentration $c^{-1} = (ak_F)^{-1}$. Here the density of atoms $n = k_F^3/(3\pi^2)$ defines the Fermi momentum $k_F$ in natural units with $\hbar = k_B = 2M = 1$, where $M$ is the mass of the atoms. The inverse concentration $c^{-1}$ varies from large negative values on the BCS side to large positive values on the BEC side with a zero-crossing at the unitary point, see right panel of Fig. 1.

A description of the qualitative features of the BCS-BEC crossover has been achieved by Nozieres & Schmitt-Rink (1985) and Sa de Melo et al. (1993) within extended mean-field theories which account for the contribution of both fermionic and bosonic degrees of freedom. However, the quantitatively precise understanding of BCS-BEC crossover physics requires non-perturbative methods. The experimental realization of molecule condensates and the subsequent crossover to a BCS-like state of weakly attractive fermions (Regal et al. 2004; Zwierlein et al. 2004; Kinast et al. 2004; Bourdel et al. 2004; Bartenstein et al. 2004; Partridge et al. 2005) pave the way to future experimental precision measurements and provide a testing ground for non-perturbative methods. An understanding of the crossover on a quantitative level at and near the resonance has been developed through numerical Quantum Monte-Carlo (QMC) methods (Carlson et al. 2003; Astrakharchik et al. 2004; Bulgac et al. 2006; Burovski et al. 2006; Akkineni et al. 2007). The complete phase diagram has been accessed by functional field-theoretical techniques, as $t$-matrix approaches (Pieri & Strinati 2000; Perali et al. 2004), Dyson-Schwinger equations (Diehl & Wetterich 2006, b), 2PI
methods (Haussmann et al. 2007), and RG flow equations (Birse et al. 2005; Diehl et al. 2007a, b; Gubbels & Stoof 2008; Bartosch et al. 2009; Krippa 2009). These pictures of the whole phase diagram (Nikolic & Sachdev 2007; Pieri & Strinati 2000, Perali et al. 2004; Diehl & Wetterich 2006, b; Birse et al. 2005; Diehl et al. 2007a, b; Gubbels & Stoof 2008, Haussmann et al. 2007) do not yet reach a similar quantitative precision as anticipated for the QMC calculations.

We intend to fill this gap and discuss the limit of broad Feshbach resonances for which all thermodynamic quantities can be expressed in terms of two dimensionless parameters,

$$c = a k_F,$$

and the temperature in units of the Fermi temperature $T/T_F$ where $T_F = k_F^2$. In the broad resonance regime, macroscopic observables are universal (Diehl & Wetterich 2006, Nikolic & Sachdev 2007, Diehl et al. 2007a, Ho 2004), i.e. they are to a large extent independent of the concrete microscopic realization. Very similar to the notion of universality near 2nd-order phase transitions, universality in the present context can be traced back to the existence of a fixed point in the RG flow which is approached provided the Feshbach (Yukawa) coupling is large enough (Diehl et al. 2007b).

Our review is based on the RG studies of (Diehl et al. 2007a, b; Diehl et al. 2010; Floerchinger et al. 2009; Floerchinger et al. 2010). For RG studies using different expansion schemes, see (Birse et al. 2005; Gubbels & Stoof 2008; Bartosch et al. 2009; Krippa 2009). In the following, we first introduce the required techniques from RG flow equations for cold atoms, see Secs. 2 - 4. Further, we include the quantitative effect of particle-hole fluctuations, Sec. 4 and systematically extend the truncation scheme, accounting for changes of the Fermi surface due to fluctuation effects, Sec. 5. Additionally, we also consider an atom-dimer interaction term.

2. Microscopic model and the functional RG

Microscopically the BCS-BEC crossover can be described by an action including a two-component Grassmann field $\psi = (\psi_1, \psi_2)$, describing non-relativistic fermions in two hyperfine states and a complex scalar field $\phi$ as the bosonic degrees of freedom. In different regimes of the crossover $\phi$ can be seen as a field describing molecules, Cooper pairs or simply an auxiliary field. Explicitly, the microscopic action at the ultraviolet scale $\Lambda$ reads

$$S = \int_0^{1/T} d\tau \int d^3 x \left\{ \psi_1^\dagger (\partial_\tau - \Delta - \mu) \psi_1 + \phi^\dagger (\partial_\tau - \frac{\Delta}{2} - 2\mu + \nu_\Lambda(B)) \phi - h_\Lambda (\phi^\dagger \psi_1 \psi_2 + \text{h.c.}) \right\}. \tag{2.1}$$

In thermal equilibrium, the system is described by the Matsubara formalism. The variable $\mu$ is the chemical potential. The parameter $\nu_\Lambda(B) = \nu(B) + \delta \nu(\Lambda)$ includes the detuning from the Feshbach resonance $\nu(B) = \mu_M(B - B_0)$, with the magnetic moment of the boson field $\mu_M$, and a renormalization counter term $\delta \nu(\Lambda)$ that has to be adjusted to match the conditions from the physical vacuum. The Yukawa coupling $h_\Lambda$ is related to the width of the Feshbach resonance. Note that the bosonic field $\phi$ appears quadratically in Eq. (2.1), so the functional integral over $\phi$ can be carried out, corresponding to an inverse Hubbard-Stratonovich transformation, and our model is equivalent to a purely fermionic theory with an interaction term

$$S_{\text{int}} = \int_{p_1, p_2, p'_1, p'_2} \lambda_{\psi, \text{eff}} (p_1 + p_2) \phi_1^\dagger (p'_1) \psi_1 (p_1) \phi_2^\dagger (p'_2) \psi_2 (p_2) \delta (p_1 + p_2 - p'_1 - p'_2). \tag{2.2}$$
Here, $\mathbf{p} = (p_0, \mathbf{p})$ and the microscopic interaction between the fermions is described by the tree-level expression with a classical inverse boson propagator in the denominator

$$\lambda_{\psi, \text{eff}}(q) = -\frac{h^2}{-\omega + \frac{q^2}{2} - 2\mu + \nu_{\Lambda}(B)}, \quad (2.3)$$

where $\omega$ is the real-time frequency of the exchanged boson $\phi$. It is related to the Matsubara frequency $q_0$ via analytic continuation $\omega = -iq_0$. Further, $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2$ is the center of mass momentum of the scattering fermions $\psi_1$ and $\psi_2$ with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. The limit of broad Feshbach resonances corresponds to $h\Lambda \to \infty$. In this limit the microscopic interaction becomes pointlike, with strength $-\frac{h^2}{\nu_{\Lambda}}$.

The functional RG can be formulated as a functional differential equation for an action functional for which the microscopic model serves as an initial value. Whereas the microscopic interactions are governed by $S$ at the ultraviolet scale $\Lambda$, quantum and thermal fluctuations effectively modify the interactions at larger length scales which can be summarized in an effective action $\Gamma_k$ (e.g. for the 1PI proper vertices) valid at a momentum scale $k$. In other words, $\Gamma_k$ includes the effects of fluctuations with momenta higher than $k$ and governs the interactions with momenta near $k$. This effective average action or flowing action satisfies the Wetterich equation (Wetterich 1993), being an exact RG flow equation,

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \left[ \left( \Gamma_k^{(2)}[\Phi] + R_k \right)^{-1} \partial_k R_k \right]. \quad (2.4)$$

Here, the STr operation involves an integration over momenta and a summation over internal indices with appropriate minus signs for fermions. The collective field $\Phi$ summarizes all bosonic and fermionic degrees of freedom, and $\Gamma_k^{(2)}[\Phi]$ denotes the second functional derivative of $\Gamma_k$

$$(\Gamma_k^{(2)}[\Phi])_{ij}(p_1, p_2) = \frac{\delta}{\delta \Phi_i(-p_1)} \frac{\delta}{\delta \Phi_j(p_2)} \Gamma_k[\Phi]. \quad (2.5)$$

The long-wavelength regulator $R_k$ specifies the details of the regularization scheme. Specific examples will be discussed below. For reviews of the functional renormalization group see (Salmhofer & Honerkamp 2001; Berges et al. 2002; Gies 2006; Pawlowski 2007; Kopietz 2010). From the full effective action in the long wavelength limit $\Gamma[\Phi] = \Gamma_{k=0}[\Phi]$, all macroscopic properties of the system under consideration can be read off.

Equation (2.4) is the technical starting point of our investigations. It is a functional differential equation which, upon expansion of this functional into a suitable basis translates to a system of infinitely many coupled differential equations for the expansion coefficients, i.e. generalized running couplings. Identifying suitable expansion schemes is not a formal but a physics problem: expansions should be based on building blocks that encode the relevant degrees of freedom of the system possibly at all scales. In the present context, this emphasizes the usefulness of composite bosonic fields which are expected to be the relevant long-range degrees of freedom at low temperatures. Reducing the full effective action to a treatable selection of generalized couplings defines a truncation. Possible truncation schemes include vertex expansions, derivative expansions or other schemes to systematically classify all possible operators of a given system. The quantitative success of a given truncation scheme does not necessarily rely on the existence of a small expansion parameter like the interaction strength, but only requires that the operators neglected in a truncation do
not take a strong influence on the flow of the operators included. In practice, a truncation can be
tested in various ways, e.g., by verifying the practical convergence for increasing truncations or by
studying regulator-scheme independencies for universal quantities. In the present context, also the
comparison with well-known few-body physics turns out to provide a useful benchmark.

3. Basic truncation

(a) Derivative expansion

Thermodynamics of a system can be obtained from its grand canonical partition function $Z$
or the corresponding grand canonical potential $\Omega_G = -T \ln Z$. It is related to the effective action
via $\Gamma[\Phi_{eq}] = \Omega_G / T$, where $\Phi_{eq}$ is obtained from the field equation $\delta \Gamma[\Phi] / \phi = \Phi_{eq} = 0$. Let us first present a basic version of a truncation which already captures all the qualitative features of the
BCS-BEC crossover:

$$\Gamma_k[\Phi] = \int_{\tau, x} \left\{ \psi^\dagger (\partial_\tau - \Delta - \mu) \psi + \bar{\phi}^* \left( \bar{Z}_\phi \partial_\tau - \frac{A_\phi \Delta}{2} \right) \bar{\phi} + \bar{U}(\bar{\rho}, \mu) - \bar{h}(\bar{\phi}^* \bar{\psi}_1 \psi_2 + \bar{\phi} \bar{\psi}_2 \psi_1^* ) \right\}. \quad (3.1)$$

The effective potential $\bar{U}(\bar{\rho}, \mu)$ is a function of $\bar{\rho} = \bar{\phi}^* \bar{\phi}$ and $\mu$. This truncation can be motivated
by a systematic derivative expansion and an analysis of the symmetries encoded in Ward identities
(Diehl et al. 2007a; Diehl et al. 2010). It does not yet incorporate, for instance, the effects of
particle-hole fluctuations and we will come back to this issue in Sect. 4. We define renormalized
$U$ of $\rho$ and $\mu$ by the requirements $(\partial_\mu U)(\rho_0(k), \mu_0) = 0$ for all $k$, and $-(\partial_\mu U)(\rho_0, \mu_0) = n$ at $k = 0$. More explicitly, we employ a simple expansion for $U(\rho, \mu)$ of the form

$$U(\rho, \mu) = U(\rho_0, \mu_0) - n k (\mu - \mu_0) + (m^2 + \alpha (\mu - \mu_0))(\rho - \rho_0) + \frac{1}{2} \lambda (\rho - \rho_0)^2. \quad (3.3)$$

In the symmetric or normal gas phase, we have $\rho_0 = 0$, while in the phase with spontaneous breaking
of $U(1)$ symmetry (superfluid phase), we have $\rho_0 > 0$ and $m^2 = 0$. The atom density $n = -\partial U / \partial \mu$ corresponds to $n_k$ in the limit $k \to 0$.

The running couplings in this truncation explicitly are $m^2(k)$, $\lambda(k)$, $\alpha(k)$, $n_k$, $Z_\phi(k)$ and $h(k)$. In the phase with spontaneous symmetry breaking $m^2$ is traded for $\rho_0$. In addition, we need the anomalous dimension $\eta = -k \partial_k \ln A_\phi$. At the microscopic scale $k = \Lambda$ the initial values of our
couplings are determined from Eq. (2.1). This gives $m^2(\Lambda) = \nu_\alpha(\Lambda) - 2\mu$, $\rho_0(\Lambda) = 0$, $\lambda(\Lambda) = 0$, $Z_\phi(\Lambda) = 1$, $h(\Lambda) = h_\Lambda$, $\alpha(\Lambda) = -2$ and $n_\Lambda = 3\pi^2 \mu \theta(\mu)$. Finally, our regularization scheme is specified by a regulator for space-like momenta which for the fermionic and bosonic field components reads

$$R_{k, \psi} = (\text{sign}(\bar{\rho}^2 - \mu) k^2 - (\bar{p}^2 - \mu)) \theta (k^2 - |\bar{p}^2 - \mu|), \quad R_{k, \phi} = A_\phi (k^2 - \bar{p}^2 / 2) \theta (k^2 - \bar{p}^2 / 2),$$

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respectively. For the fermions, it regularizes fluctuations around the Fermi surface, whereas bosonic fluctuations are suppressed for generic small momenta. This choice is optimized in the spirit of Litim (2000) and Pawlowski (2007).

For our choice of the regulator and with the basic approximation scheme Eq. (3.2) the flow equation for the effective potential can be computed straightforwardly:

\[ k \partial_k U = \eta \rho U' + \frac{\sqrt{2}k^5}{3\pi^2Z_B} \left( 1 - \frac{2\eta}{5} \right) s_B^{(0)} - \frac{k^4}{3\pi^2} \left( (\mu + k^2)^2 \theta(\mu + k^2) - (\mu - k^2)^2 \theta(\mu - k^2) \right) s_F^{(0)}, \tag{3.4} \]

with the threshold functions

\[ s_B^{(0)} = \left( \sqrt{\frac{k^2 + U'}{k^2 + U'} + \sqrt{\frac{k^2 + U' + 2\mu U''}{k^2 + U'}}} \right) \left( \frac{1}{2} + N_B \left[ \frac{\sqrt{k^2 + U'} \sqrt{k^2 + U' + 2\mu U''}}{Z_B} \right] \right), \tag{3.5} \]

\[ s_F^{(0)} = \frac{2}{\sqrt{k^4 + h^2 \rho}} \left( \frac{1}{2} - N_F \left[ \sqrt{k^4 + h^2 \rho} \right] \right). \tag{3.6} \]

The threshold functions exhibit a temperature dependence via the Bose and Fermi functions \( N_{B/F}[\epsilon] = (e^{\epsilon/T} - 1)^{-1} \). From the effective potential flow, we derive the flow equations for the running couplings \( m^2 \) or \( \rho_0 \) and \( \lambda \). For details we refer to (Diehl et al. 2010). Further we need flow equations for \( A_0 \) and \( Z_0 \) that are obtained by the projections

\[ \partial_t \bar{Z}_\phi = -\partial_t \frac{\partial}{\partial \eta_0} \langle \bar{P}_0 \rangle_{12} (q_0, 0) \bigg|_{q_0=0}, \quad \text{and} \quad \partial_t A_\phi = 2\partial_t \frac{\partial}{\partial \eta'} \left( \bar{P}_0 \right)_{22} (0, \bar{q}) \bigg|_{\bar{q}=0}, \tag{3.7} \]

with the momentum-dependent part of the propagator

\[ \frac{\delta^2 \Gamma_k}{\delta \phi_1(q) \delta \phi_j(q')} \bigg|_{\bar{q}_1 = \sqrt{2} \phi_1, \bar{q}_2 = 0} = (\bar{P}_0)_{ij}(q) \delta(q + q'). \tag{3.8} \]

Here the boson field is expressed in a basis of real fields \( \phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x)) \). These flow equations are derived by Diehl et al. (2010) and have a rather involved structure. Finally, we need the flow of the Yukawa coupling. In the symmetric regime the loop contribution vanishes and the flow is given by the anomalous dimension,

\[ \partial_t h = \frac{1}{2} \eta h, \quad \text{or in dimensionless units} \quad \partial_t \tilde{h}^2 = (-1 + \eta) \tilde{h}^2, \tag{3.9} \]

where \( \tilde{h}^2 = h^2/k \). In the regime of spontaneous symmetry breaking (\( \rho_0 > 0 \)) there is a loop contribution \( \sim h^3 \lambda \rho_0 \) from a mixed diagram involving both fermions and bosons. This contribution is quantitatively subleading which we have verified also numerically. For the basic approximation scheme, Eq. (3.2), we therefore dropped this contribution.

(b) Vacuum limit and contact to experiment

The vacuum limit allows us to make contact with experiment. We find that for \( n = T = 0 \) the crossover at finite density turns into a second-order phase transition in vacuum (Diehl & Wetterich 2007, Nikolic & Sachdev 2007) as a function of the initial value \( m^2(\Lambda) \). In order to see this, we consider the momentum-independent parts in both the fermion and the boson propagator, \( -\mu \) (the “chemical potential” for the fermions in vacuum) and \( m(k = 0)^2 \), which act as gaps for the
propagation of fermions and bosons. We find the following constraints, separating two different branches of the physical vacuum (Diehl & Wetterich 2007),

\[
\begin{align*}
& m^2(0) > 0, \quad \mu = 0 \quad \text{atom phase} \quad (a^{-1} < 0), \\
& m^2(0) = 0, \quad \mu < 0 \quad \text{molecule phase} \quad (a^{-1} > 0), \\
& m^2(0) = 0, \quad \mu = 0 \quad \text{resonance} \quad (a^{-1} = 0).
\end{align*}
\]  

(3.10)

The initial values \(m^2(\Lambda)\) and \(h\Lambda\) can be connected to the two-particle scattering in vacuum close to a Feshbach resonance. For this purpose, one follows the flow of \(m^2(k)\) and \(h(k)\) in vacuum, e.g. on the BCS \((a^{-1} < 0)\), i.e. \(\mu = T = n = 0\), and extracts the renormalized parameters \(m^2 = m^2(k = 0)\), \(h = h(k = 0)\). They have to match the physical conditions formulated in Eq. (3.10). We obtain the two relations

\[
\bar{m}^2(\Lambda) = \mu_M (B - B_0) - 2\mu + \frac{\hbar^2}{6\pi^2 \Lambda}, \quad a = -\frac{h^2(k = 0)}{8\pi m^2(k = 0)} = -\frac{\hbar^2(\Lambda)}{8\pi \mu_M (B - B_0)},
\]

(3.11)

where \(\mu_M\) is the relative magnetic moment of the molecules. These relations fix the initial conditions of our model completely and similar reasoning confirms their validity on the BEC side. Now we can express the parameters \(m^2(\Lambda)\) and \(h^2(\Lambda)\) by the experimentally accessible quantities \(B - B_0\) and \(a\). They remain valid also for non-vanishing density and temperature, as long as the UV cutoff \(\Lambda\) is much larger than \(T\) and \(\mu\).

\(c\) Many-body phase diagram

Although our flow equations describe accurately the vacuum limit and can be used to determine interesting few-body parameters they are not restricted to that limit. In fact, for nonzero temperature and density, the flow deviates from its vacuum form at scales with \(k^2 < T\) or \(k^2 < T_F\). The resulting system of ordinary coupled differential equations is then solved numerically for different chemical potentials \(\mu\) and temperatures \(T\). For temperatures sufficiently small compared to the Fermi temperature, \(T/T_F \ll 1\), we find that the effective potential \(U\) at the macroscopic scale \(k = 0\) develops a minimum at a nonzero field value \(\rho_0 > 0, \partial_\rho U(\rho_0) = 0\). The system is then in the superfluid phase. For larger temperatures we find that the minimum is at \(\rho_0 = 0\) and that the “mass parameter” \(m^2\) is positive, \(m^2 = \partial_\rho U(0) > 0\). The critical temperature \(T_c\) of this phase transition between the superfluid and the normal phase is then defined as the temperature where

\[
\rho_0 = 0, \quad \partial_\rho U(0) = 0 \quad \text{at} \quad k = 0.
\]

(3.12)

Throughout the whole crossover the transition \(\rho_0 \to 0\) is continuous as a function of \(T\) demonstrating that the phase transition is of second order. An analysis of the scaling of the correlation length confirms that the phase transition is governed by a Wilson-Fisher fixed point for the \(N = 2\) universality class throughout the crossover (Diehl et al. 2010). This reflects the fact that the symmetries are properly encoded also on the level of the truncated action.

From the flow equations together with the initial conditions we can already recover all the qualitative features of the BCS-BEC crossover, e.g. compute the phase diagram for the phase transition to superfluidity. The result for this basic approximation is displayed in the right panel of Fig. 6 by the dot-dashed line.

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(d) Fixed-point and universality

In the vacuum limit and in the regime where $k^2 \gg -\mu$ the flow of the anomalous dimension reads $\eta = h^2/(6\pi^2 k)$ (Diehl et al. 2010). Together with the dimensionless flow of the Yukawa coupling, Eq. (d), this reveals the existence of an IR attractive fixed point given by $\eta = 1$, $\tilde{h}^2 = 6\pi^2$. This fixed-point is approached rapidly if the initial value of $h^2(\Lambda)/\Lambda$ is large enough, i.e., in the broad-resonance limit. Then the memory of the microscopic value of $h^2(\Lambda)/\Lambda$ is lost at large length scales. Also all other parameters except for the mass term $m^2$ are attracted to IR fixed points, giving rise to universality. The fixed-point structure remains similar for non-vanishing density and temperature and these findings also apply in this regime and determine the critical physics of these non-relativistic quantum fields. For a given temperature, this fixed point has only one relevant direction which is related to the detuning of the resonance $B - B_0$.

4. Particle-hole fluctuations

(a) Gorkov’s correction to BCS theory

For small and negative scattering length $c^{-1} < 0, |c| \ll 1$ (BCS side), the system can be treated by the perturbative BCS theory of superfluidity (Cooper 1956; Bardeen et al. 1957). However, there is a significant decrease of the critical temperature as compared to the original BCS result due to a screening effect of particle-hole fluctuations in the medium (Gorkov & Melik-Barkhudarov 1961; Heiselberg et al. 2000). Here we will sketch the technique to include the effect of particle-hole fluctuations in our functional RG treatment as developed by Floerchinger et al. (2009).

In an RG setting, the features of BCS theory can be described in a purely fermionic language with the fermion interaction vertex $\lambda$ as the only scale-dependent object. In general, the interaction vertex is momentum dependent, $\lambda(p_1', p_1, p_2', p_2)$, and its flow has two contributions which are depicted in Fig. 2 including the external momentum labels. For $k \to 0$, $\mu \to 0$, $T \to 0$ and $n \to 0$ this coupling is related to the scattering length, $a = \frac{1}{8\pi}\lambda(p_i = 0)$.

In the BCS approximation only the first diagram in Fig. 2 the particle-particle (pp) loop, is kept and the momentum dependence of the four fermion coupling is neglected, by replacing $\lambda(p_1', p_1, p_2', p_2)$ by the pointlike coupling evaluated at zero momentum. For $\mu > 0$, its effect increases as the temperature $T$ is lowered. For small temperatures $T \leq T_{c,BCS}$ the logarithmic divergence leads to the appearance of pairing, as $\lambda \to \infty$. In terms of the scattering length $a$, Fermi momentum $k_F$ and Fermi temperature $T_F$, the critical temperature is found to be

$$T_{c,BCS} \approx 0.61 \ T_F \ e^{\pi/(2ak_F)}.$$  

(4.1)

Figure 2: Running of the momentum-dependent vertex $\lambda$. Here $\partial_k$ indicates scale-derivatives with respect to the regulator in the propagators but does not act on the vertices.
At zero temperature the expression for the second diagram in Fig. 2, the particle-hole (ph) loop, vanishes if it is evaluated for vanishing external momenta, as both poles of the frequency integration are always either in the upper or lower half of the complex plane. The dominant part of the scattering in a fermion gas occurs, however, for momenta on the Fermi surface rather than for zero momentum. For non-zero momenta of the external particles the particle-hole loop makes an important contribution. Setting the external frequencies to zero, we find that the inverse propagators in the particle-hole loop are

\[ P_\psi(q) = iq_0 + (\vec{q} - \vec{p}_1)^2 - \mu, \quad \text{and} \quad P_\psi(q) = iq_0 + (\vec{q} - \vec{p}_2')^2 - \mu. \quad (4.2) \]

Depending on the value of the momenta \( \vec{p}_1 \) and \( \vec{p}_2' \), there are now values of the loop momentum \( \vec{q} \) for which the poles of the frequency integration are in different half-planes so that there is a nonzero contribution even for \( T = 0 \).

To include the effect of particle-hole fluctuations one could take the full momentum-dependence of the vertex \( \lambda_\psi \) into account. However, the resulting integro-differential equations represent a substantial numerical challenge. As a simple and efficient approximation, one therefore restricts the flow to the running of a single coupling \( \lambda_\psi \) by choosing an appropriate momentum projection. In the purely fermionic description, this flow equation has a simple structure and the solution for \( \lambda_\psi^{-1} \) can be written as

\[ (\lambda_\psi(k = 0))^{-1} = (\lambda_\psi(k = \Lambda))^{-1} + \text{pp-loop} + \text{ph-loop}. \quad (4.3) \]

Since the ph-loop depends only weakly on the temperature, one can evaluate it at \( T = 0 \) and add it to the initial value \( \lambda_\psi(k = \Lambda)^{-1} \). As \( T_c \) depends exponentially on the "effective microscopic coupling" \( (\lambda_\psi^{\text{eff}}(k = \Lambda))^{-1} = (\lambda_\psi(k = \Lambda)^{-1} + \text{ph-loop}) \), any shift in \( (\lambda_\psi^{\text{eff}})^{-1} \) results in a multiplicative factor for \( T_c \). The numerical value of the ph-loop and therefore of the correction factor for \( T_c/\Delta \) depends on the precise projection description. The averaging prescription used by Gorkov & Melikh-Barkhudarov (1961) leads to

\[ T_c = \frac{1}{(4e)^{1/3}} T_{c,\text{BCS}} \approx \frac{1}{2.2} T_{c,\text{BCS}} \quad (4.4) \]

and similar for the gap \( \Delta \) at zero temperature.

(b) Scale-dependent bosonization

In Sect. 2 we describe an effective four-fermion interaction by the exchange of a boson. In this picture the phase transition to the superfluid phase is indicated by the vanishing of the bosonic "mass term" \( m^2 = 0 \). Negative \( m^2 \) leads to the spontaneous breaking of U(1) symmetry, since the minimum of the effective potential occurs for a nonvanishing superfluid density \( \rho_0 > 0 \). For \( m^2 \geq 0 \) we can solve the field equation for the boson \( \phi \) as a functional of \( \psi \) and insert the solution into the effective action. This leads to an effective four-fermion vertex describing the scattering

\[ \psi_1(p_1)\psi_2(p_2) \rightarrow \psi_1(p_1')\psi_2(p_2') \]

\[ \lambda_\psi^{\text{eff}} = \frac{-\hbar^2}{i(p_1 + p_2) + \frac{1}{2}(\vec{p}_1 + \vec{p}_2)^2 + m^2}. \quad (4.5) \]

To investigate the breaking of U(1) symmetry and the onset of superfluidity, we first consider the flow of the bosonic propagator, which is mainly driven by the fermionic loop diagram. For the
effective four-fermion interaction this accounts for the particle-particle loop (see left panel, r.h.s. of Fig. 3). In the BCS limit of a large microscopic $m^2$, the running of $m^2$ for $k \to 0$ reproduces the BCS result (Cooper 1956; Bardeen et al. 1957).

The particle-hole fluctuations are not accounted for by the renormalization of the boson propagator. Indeed, we have neglected so far that a four-fermion interaction term $\lambda \psi$ in the effective action is generated by the flow. This holds even if the microscopic pointlike interaction is absorbed by a Hubbard-Stratonovich transformation into an effective boson exchange such that $\lambda \psi(A) = 0$. The strength of the total interaction between fermions

$$
\lambda_{\psi,\text{eff}} = \frac{-\hbar^2}{i(p_1 + p_2)_0 + \frac{1}{2} \bar{p}_1 \bar{p}_2 + m^2} + \lambda \psi
$$

(4.6)

adds $\lambda \psi$ to the piece generated by boson exchange. In the partially bosonized formulation, the flow of $\lambda \psi$ is generated by the box diagrams depicted in the right panel of Fig. 3. A direct connection to the particle-hole diagrams of Fig. 2 can be established on the BCS side and in the microscopic regime: There the boson gap $m^2$ is large. In this case, the effective fermion interaction in Eq. (4.6) becomes momentum independent, diagrammatically corresponding to a contracted bosonic propagator. The box diagram in Fig. 3 is then equivalent to the bare fermion interaction vertex.

In contrast to the particle-particle fluctuations (leading to SSB for decreasing $T$), the particle-hole fluctuations lead only to quantitative corrections and depend only weakly on temperature. This can be checked explicitly in the pointlike approximation, and holds not only in the BCS regime where $T/\mu \ll 1$, but also for moderate $T/\mu$ as realized at the critical temperature in the unitary regime. We therefore evaluate the box diagrams in Fig. 2 for zero temperature. We emphasize that a temperature dependence, resulting from the couplings parameterizing the boson propagator, is implicitly taken into account. For the external momenta we use an averaging on the Fermi surface similar to the one of Gorkov & Melik-Barkhudarov (1961). For details see Floerchinger et al. (2009).

After these preliminaries, we can now incorporate the effect of particle-hole fluctuations in the RG flow. In principle one could simply take $\lambda \psi$ as an additional coupling into account. However, it is much more elegant to use a scale-dependent Hubbard-Stratonovich transformation (Gies & Wetterich 2001; Pawlowski 2007; Floerchinger & Wetterich 2009) which absorbs $\lambda \psi$ into the Yukawa-type interaction with the bosons at every scale $k$. By construction, there is then no self interaction between the fermionic quasiparticles. The general procedure of “partial bosonization” is discussed in detail in Floerchinger et al. (2009). A slightly modified scheme based on the exact flow equation derived in (Floerchinger & Wetterich 2009) has been used in (Floerchinger et al. 2010). In that formulation one finds for the renormalized coupling $m^2$ in the symmetric regime an additional

\[
\frac{\partial}{\partial t} \lambda \psi = \tilde{\partial}_h
\]

Figure 3: Left panel: Flow of the boson propagator. Right panel: Box diagram for the flow of the four-fermion interaction.
term reflecting the absorption of $\lambda_\psi$ into the fermionic interaction induced by boson exchange,

$$\partial_t m^2 = \partial_t m^2 \big|_{\text{HS}} + \frac{\bar{m}^4}{\hbar^2} \partial_t \lambda_\psi \big|_{\text{HS}}. \quad (4.7)$$

Here $\partial_t m^2 \big|_{\text{HS}}$ and $\partial_t \lambda_\psi \big|_{\text{HS}}$ denote the flow equations when the Hubbard-Stratonovich transformation is kept fixed. Since $\lambda_\psi$ remains now zero during the flow, the effective four-fermion interaction $\lambda_\psi,_{\text{eff}}$ is purely given by the boson exchange. However, the contribution of the particle-hole exchange diagrams is incorporated via the second term in Eq. (4.7). The flow equation of all other couplings are the same as with fixed Hubbard-Stratonovich transformation. In the regime with spontaneous symmetry breaking we use

$$\partial_t h = \partial_t h \big|_{\text{HS}} + \frac{\lambda \rho_0}{\hbar} \partial_t \lambda_\psi \big|_{\text{HS}},$$

$$\partial_t \rho_0 = \partial_t \rho_0 \big|_{\text{HS}} - 2 \frac{\lambda \rho_0^2}{\hbar^2} \partial_t \lambda_\psi \big|_{\text{HS}},$$

$$\partial_t \lambda = \partial_t \lambda \big|_{\text{HS}} + 2 \frac{\lambda^2 \rho_0}{\hbar^2} \partial_t \lambda_\psi \big|_{\text{HS}}. \quad (4.8)$$

We emphasize that our non-perturbative flow equations go beyond the treatment by Gorkov & Melik-Barkhudarov (1961) which includes the particle-hole diagrams only in a perturbative way. Furthermore, the inner bosonic lines $\hbar^2/P_\phi(q)$ in the box diagrams include the center-of-mass momentum dependence of the four-fermion vertex. This is neglected in Gorkov’s pointlike treatment, and thus represents a further improvement of the classic calculation. Actually, this momentum dependence becomes substantial away from the BCS regime where the physics of the bosonic bound state sets in. The continuous description of dynamically transmuting degrees of freedom is a particular strength of an RG description, as exemplified also in the context of QCD (Gies & Wetterich 2004, Braun 2009).

5. Running Fermion sector

In this section we aim at a systematic extension of the truncation scheme and consider a running fermion sector. Similar parameterizations of the fermionic self-energy have been studied in Gubbels & Stoof (2008), Bartosch et al. (2009) and Strack et al. (2008). Further, we include an atom-dimer interaction term. This section is based on the work by Floerchinger et al. (2010).

(a) Completion of the truncation

In addition to the running couplings that have occurred so far in Secs. 3, now we want to take into account $k$-dependent parameters $\bar{m}_\psi^2$ and $Z_\psi$, in order to parametrize fluctuation effects on the self-energy of the fermionic quasiparticles. The extension of the truncation explicitly reads

$$\Gamma_k = \int_0^{1/T} d\tau \int d^3 x \left\{ \bar{\psi}^\dagger \left( \bar{Z}_\psi (\partial_\tau - \Delta) \bar{\psi} + \bar{m}_\psi^2 \bar{\psi}^\dagger \bar{\psi} + \bar{\phi}^* (Z_\phi \partial_\tau - \frac{1}{2} A_\phi \Delta) \right) \bar{\phi} \right. \right.$$

$$\left. + \bar{U}(\bar{\rho}, \mu) - \bar{h} (\bar{\phi}^* \bar{\psi}_1 \bar{\psi}_2 + \bar{\phi} \bar{\psi}_2^* \bar{\psi}_1^*) + \bar{\lambda}_{\phi \psi} \bar{\phi}^* \bar{\phi} \bar{\psi}^\dagger \bar{\psi} \right\}. \quad (5.1)$$

The additional inclusion of the atom-dimer coupling $\bar{\lambda}_{\phi \psi}$ closes the truncation on the level of interaction terms quartic in the fields and describes three-body scattering (Diehl et al. 2007c). It
leads to quantitative modifications for the many-body problem. In the regime of spontaneously broken symmetry ($\rho_0 > 0$) the atom dimer coupling leads to a modification of the Fermi surface, in addition to the gap $\sqrt{\hbar^2 \rho_0}$.

We define the renormalized fields $\phi = A^{1/2}_\phi \tilde{\phi}$, $\rho = A_\rho \tilde{\rho}$, $\psi = Z^{1/2}_\psi \tilde{\psi}$ and study the flow of the renormalized couplings $Z_\phi = Z_\phi / A_\phi$, $h = \hbar /(A^{1/2}_\phi Z_\psi)$, $\lambda_{\phi \psi} = \tilde{\lambda}_{\phi \psi} / (A_\phi Z_\psi)$, $m_{\psi}^2 = \tilde{m}_{\psi}^2 / Z_\psi$. As before, we expand the effective potential in monomials of $\rho$, see Eq. (3.3). We use again a purely space-like regulator which is adjusted to the running Fermi surface,

$$R_{k,\psi} = Z_\psi \left[ \text{sign}(\vec{p}^2 - r_F^2) k^2 - (\vec{p}^2 - r_F^2) \right] \theta \left( k^2 - |\vec{p}^2 - r_F^2| \right), \quad R_{k,\phi} = A_\phi [k^2 - \vec{p}^2/2] \theta \left( k^2 - \vec{p}^2/2 \right),$$

where $r_F^2 = -m_{\psi}^2 - \lambda_{\phi \psi} \rho_0$. For the fermions it regularizes fluctuations around the running Fermi surface, while for the bosons fluctuations with small momenta are suppressed.

As a first quantity we investigate the vacuum dimer-dimer scattering length $a_M$ expressed in units of the atom-atom scattering length $a$. On the BEC side of the resonance we can derive this quantity from the corresponding couplings by the equation

$$\frac{a_M}{a} = 2 \frac{\lambda}{\lambda_{\psi,\text{eff}}}, \quad \lambda_{\psi,\text{eff}} = 8\pi a = \frac{8\pi}{\sqrt{-\mu}}, \text{ for } \mu < 0 \text{ and } k = 0. \quad (5.2)$$

To explicitly compute the vacuum quantity $a_M/a$, we choose a value for $a$ on the far BEC side, where for broad resonances the identity $a = (-\mu)^{-1/2}$ holds. We evolve the flow of the couplings to the infrared and extract the value of $\lambda$, completely fixing $a_M/a$. In this truncation including the $\lambda_{\phi \psi}$ vertex, we find $a_M/a = 0.59$, which is in very good agreement with the well-known result from a direct solution of the Schrödinger equation, $a_M/a = 0.60$ (Petrov et al 2004). The accuracy of this result is somewhat surprising since no momentum dependence of $\lambda_{\phi \psi}$ has been taken into account. The latter has turned out to be important for the atom-dimer scattering Diehl et al. (2007c). On the other hand, the general leading-order effect of fermionic momentum-dependencies is captured by the wave function renormalization $Z_\psi$ the effect of which is largest in the strongly interacting regime, when $|c^{-1}| < 1$, cf. Fig. 4.
Figure 5: Left panel: Effective Fermi radius $r_F/k_F$ as a function of the crossover parameter $c^{-1}$ for vanishing temperature (solid line). We compare to the effective Fermi radius in an approximation without the contribution of the atom-dimer vertex $\lambda_{\phi\psi}$ (dotted line). Right panel: Positive branch of the dispersion relation $\omega(q)$ in units of $E_F$ for $c^{-1} = -1$ (dashed), $c^{-1} = 0$ (solid) and $c^{-1} = 1$ (short dashed).

(b) Fermi sphere and dispersion relation

The dispersion relation can be computed from the determinant of the renormalized fermionic propagator

$$G^{-1}_\psi = \begin{pmatrix} -\hbar \phi_0 \epsilon & -\omega - (q^2 + m^2_\psi + \lambda_{\phi\psi} \rho_0) \\ -\omega + (q^2 + m^2_\psi + \lambda_{\phi\psi} \rho_0) & \hbar \phi_0 \epsilon \end{pmatrix}. \quad (5.3)$$

by the equation $\det G^{-1}_\psi = 0$. Here, we have evaluated $G^{-1}_\psi$ in the regime of spontaneously broken symmetry and performed analytical continuation to real frequencies $\omega$. We see that the dispersion relation is affected by the running of the couplings $Z_\psi$, $m^2_\psi$ and $\lambda_{\phi\psi}$, which follows as $\omega = \pm \sqrt{\Delta^2 + (q^2 - r_F^2)^2}$, where $\Delta = \hbar \sqrt{\rho_0}$ is the gap and $r_F = \sqrt{-m^2_\psi - \lambda_{\phi\psi} \rho_0}$ is the effective radius of the Fermi sphere. On the far BCS side of the crossover the renormalization effects on $r_F$ are small and $r_F$ approaches its classical value $r_F \approx \sqrt{\mu} = k_F$ where $k_F = (3\pi^2 n)^{1/3}$. Close to the resonance, the Fermi sphere gets smaller. It finally vanishes on the BEC side at a point with $c^{-1} \approx 0.6$, see Fig. Here, the fermions are gapped even for $\Delta \to 0$ by a positive value of $m^2_\psi + \lambda_{\phi\psi} \rho_0$.

6. Results

For the studies in section 5, we have omitted the effect of particle-hole fluctuations for simplicity. In the following, however, all the results are given for the correspondingly extended truncation including particle-hole fluctuations.

(a) Single-particle gap at $T = 0$

As a first study including all the couplings introduced in this contribution we investigate the single particle gap at zero temperature. On the far BCS side it is possible to compare to the result by Gorkov & Melik-Bakhudarov (1961) which is given by $\Delta/E_F = (2/e)^{7/3} e^{\pi/2c}$. Our approach allows to extend to the strongly interacting regime and even to the BEC side of the crossover, see Fig. At the unitary point, $(ak_F)^{-1} = 0$, we obtain $\Delta/E_F = 0.46$. Further, we compare our result...
Table 1: Results for the single-particle gap and the chemical potential at $T=0$ and at the unitary point by various authors.

| Authors               | $\mu/E_F$ | $\Delta/E_F$ |
|-----------------------|-----------|--------------|
| Carlson et al. (2003) | 0.43      | 0.54         |
| Perali et al. (2004)  | 0.46      | 0.53         |
| Haussmann et al. (2007)| 0.36      | 0.46         |
| Bartosch et al. (2009)| 0.32      | 0.61         |
| Floerchinger et al. (2010)| 0.51  | 0.46         |

for chemical potential in units of the Fermi energy at the unitary point $\mu/E_F = 0.51$ to different (non-perturbative) methods in Tab. 1.

(b) Phase diagram

Our results for the critical temperature for the phase transition to superfluidity throughout the crossover are shown in Fig. 6. We plot the critical temperature in units of the Fermi temperature $T_c/T_F$ as a function of the scattering length measured in units of the inverse Fermi momentum, i.e. the concentration $c = a k_F$.

On the BCS side of the crossover, where $c^{-1} \ll -1$, the BCS approximation and the effect of particle-hole fluctuations yield a critical temperature (Gorkov & Melik-Barkhudarov 1961)

$$\frac{T_c}{T_F} = \frac{e^C}{\pi} \left( \frac{2}{e} \right)^{7/3} e^{\pi/(ak_F)} \approx 0.28 e^{\pi/(ak_F)}. \quad (6.1)$$

depicted by the short dashed line in the right panel of Fig. 6. Here, $C \approx 0.577$ is Euler’s constant.

On the BEC side for very large and positive $c^{-1}$, our result approaches the critical temperature of a free Bose gas where the bosons have twice the mass of the fermions $M_B = 2M$. In our units the critical temperature is then

$$\frac{T_{c,BEC}}{T_F} = \frac{2\pi}{(6\pi^2\zeta(3/2))^{2/3}} \approx 0.218. \quad (6.2)$$

In-between there is the unitarity regime, where the two-atom scattering length diverges ($c^{-1} \to 0$) and we deal with a system of strongly interacting fermions.

Our best result including particle-hole fluctuations is given by the solid line. This may be compared with a functional renormalization flow investigation ignoring particle-hole fluctuations as discussed in section 3 (dot-dashed line) (Diehl et al. 2007a). For $c \to 0_-$ the solid line of our result agrees with BCS theory including the correction by Gorkov & Melik-Barkhudarov (1961). Deviations from this perturbative regime appear only rather close to the regime of strong interactions $c^{-1} \to 0$.

For $c \to 0_+$ this value is approached in the form (Baym et al. 1999)

$$\frac{T_c - T_{c,BEC}}{T_{c,BEC}} = \kappa a_M n_M^{1/3} = \frac{\kappa}{a} \frac{a_M c}{(6\pi^2)^{1/3}}. \quad (6.3)$$

Here, $n_M = n/2$ is the density of molecules and $a_M$ is the molecular scattering length. Using our result $a_M/a = 0.59$ obtained from solving the flow equations in vacuum, the coefficient determining the shift in $T_c$ compared to the free Bose gas yields $\kappa = 1.39$, see also Diehl et al. (2007a). In (Arnold & Moore 2001; Kashurnikov et al. 2001), the result for an interacting BEC is determined as $\kappa = 1.31$ (dashed curve on BEC side of 6), see also (Blaizot et al. 2006a, b) for a functional RG study. This is in reasonable agreement with our result. As further characteristic quantities we give the maximum...
of the ratio $(T_c/T_F)_{\text{max}} \approx 0.31$ and the location of the maximum $(ak_F)^{-1} \approx 0.40$. For $c^{-1} > 0.5$, the effect of the particle-hole fluctuations vanishes. This is expected, since the chemical potential is now negative $\mu < 0$ such that the Fermi surface disappears.

In the unitary regime ($c^{-1} \approx 0$), the particle-hole fluctuations still have a quantitative effect. We can give an improved estimate for the critical temperature at the resonance ($c^{-1} = 0$) where we find $T_c/T_F = 0.248$ and a chemical potential $\mu_c/T_F = 0.55$. A comparison to other methods and our previous work is given in Tab. 2. We observe reasonable agreement with QMC results for the chemical potential $\mu_c/T_F$, however, our critical temperature $T_c/T_F$ is larger.

### 7. Discussion and Outlook

As illustrated with the example of the BCS-BEC crossover, the functional RG is capable of describing strongly-interacting many-body systems in a consistent and controllable fashion. Once the relevant degrees of freedom are identified – possibly in a scale-dependent manner – approximation schemes based on expansions of the effective action can be devised that facilitate systematically

| $\mu_c/E_F$ | $T_c/T_F$ |
|------------|------------|
| Burovski et al. (2006) (QMC) | 0.49 | 0.15 |
| Bulgac et al. (2006) (QMC) | 0.43 | < 0.15 |
| Akkineni et al. (2007) (QMC) | - | 0.245 |
| Previous FRG estimate by Floerchinger et al. (2009) | 0.68 | 0.276 |
| Floerchinger et al. (2010) (FRG) | 0.55 | 0.248 |

Table 2: Results for $T_c/T_F$ and $\mu_c/T_F$ at the unitary point by various authors.
improvable quantitative estimates of physical observables. For the BCS-BEC crossover, already a simple derivative expansion including fermionic and composite bosonic degrees of freedom exhibits all qualitative features of the phase diagram.

The inclusion of particle-hole fluctuations and higher-orders of the derivative expansion improve our numerical results in the BCS as well as in the BEC limit of the crossover in agreement with well-known other field-theoretical methods. We obtain satisfactory quantitative precision on the BCS and BEC sides of the resonance. Remarkably, the functional RG allows for a description of both, many-body as well as few-body physics within the same formalism. For instance, our result for the molecular scattering length ratio $a_M/a$ is in good agreement with the exact result (Petrov et al. 2004). This quantitative accuracy is remarkable, as we have started with a purely fermionic microscopic theory without propagating bosonic degrees of freedom or bosonic interactions.

In the strongly interacting regime where the scattering length diverges, no exact analytical treatments are available. Our results for the gap $\Delta/E_F$ and the chemical potential $\mu/E_F$ at zero temperature are in reasonable agreement with Monte-Carlo simulations. This holds also for the ratio $\mu_c/E_F$ at the critical temperature. The critical temperature $T_c/T_F$ itself is found to be larger than the Monte-Carlo result.

In future studies, our approximations might be improved mainly at two points. One is the frequency- and momentum dependence of the boson propagator. In the strongly interacting regime, this might be rather involved, developing structures beyond our current approximation. A more detailed resolution might lead to modifications in the contributions from bosonic fluctuations to various flow equations. Another point concerns structures in the fermion-fermion interaction that go beyond a diatom bound-state exchange process. Close to the unitary point, other contributions might arise, for example in form of a ferromagnetic channel. While further quantitative modifications in the unitarity regime are conceivable, the present approximation already allows for a coherent description of the BCS-BEC crossover for all values of the scattering length, temperature and density by one simple method and microscopic model. This includes the critical behavior of a second order phase transition as well as the vacuum, BEC and BCS limits.

In order to improve the comparison between the QMC simulations and the functional RG results in the strongly interacting regime, the Wetterich equation can also be evaluated in a finite volume. This may shed light on possible finite size effects in the QMC simulations, and can help to quantitatively compare finite-volume studies with infinite-volume calculations inherent to most analytical works.

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