Approximate transitivity of the ergodic action of the group of finite permutations of \(\mathbb{N}\) on \(\{0, 1\}^\mathbb{N}\)

B. MITCHELL BAKER†, THIERRY GIORDANO‡ and RADU B. MUNTEANU§¶
† Mathematics Department, U.S. Naval Academy, Chauvenet Hall, 572C Holloway Road, Annapolis, MD 21402-5002, USA
(e-mail: bmb@usna.edu)
‡ Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada K1N 6N5
(e-mail: giordano@uottawa.ca)
§ Department of Mathematics, University of Bucharest, 14 Academiei Street, 010014, Bucharest, Romania
¶ Simion Stoilow Institute of Mathematics of the Romanian Academy, 21 Calea Grivitei Street, 010702, Bucharest, Romania
(e-mail: radu-bogdan.munteanu@g.unibuc.ro)

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Abstract. In this paper we show that the natural action of the symmetric group acting on the product space \(\{0, 1\}^\mathbb{N}\) endowed with a Bernoulli measure is approximately transitive. We also extend the result to a larger class of probability measures.

1. Introduction
In 1985, Connes and Woods introduced in [CW1] the notion of an approximately transitive (AT) action, a new ergodic property, to characterize, among approximately finite-dimensional (AFD) von Neumann algebras, the Araki–Woods (or ITPFI) factors. Equivalently, using Krieger’s result from [K] (see [HO] or [S] for a detailed description), their result says that a countable, ergodic, non-singular equivalence relation on a Lebesgue space is orbit equivalent to the ergodic equivalence relation induced by a product odometer if and only if its associated flow is AT. In 1989, for a locally compact group \(G\), Connes and Woods [CW2] proved that the asymptotic boundary of a group invariant, time-dependent Markov random walk on \(G\) is an approximately transitive, amenable \(G\)-space. The converse statement was proved in two steps. First, that any amenable \(G\)-action can be realized as the asymptotic boundary of a generalized or matrix-valued random walk on \(G\) was proved in [EG1] in the discrete case and in [AEG] for \(G\) locally compact.
Then the characterization of an AT, amenable (ergodic) $G$-space as the asymptotic boundary of a random walk was given in [EG2]; a different proof, for $G$ discrete, was given in [GH].

In [CW1] Connes and Woods proved that any funny rank-one (a generalization of rank-one) transformation is AT and that any AT transformation has zero entropy. Apart from some recent results [AL, DQ], there are not many known ‘concrete’ examples of AT group actions.

Let $S_\infty = \bigcup_{k \geq 1} S_k$ be the group of finite permutations of $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $(X, \mathcal{B}, \nu)$ be the product space $\prod_{k \geq 1} \{0, 1\}$ endowed with the product $\sigma$-algebra and the product probability measure $\nu = \bigotimes_{k \geq 1} \nu_k$.

In this paper we concentrate our attention on the following well-known action of $S_\infty$ on the product space $(X, \mathcal{B}, \nu)$ that associates to each permutation $\sigma \in S_\infty$ a non-singular automorphism of $(X, \nu)$ (also denoted $\sigma$ and) defined by

$$\sigma(x_1, x_2, x_3, \ldots) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots) \quad \text{for} \quad x = (x_k)_{k \geq 1} \in X. \quad (1)$$

Before stating the main result of this paper, let us recall the definition of approximate transitivity.

**Definition 1.** [CW1] An action $\sigma$ of a Borel group $G$ on a Lebesgue measure space $(X, \nu)$ is approximately transitive if, given $n < \infty$, functions $f_1, f_2, \ldots, f_n \in L^1(X, \nu)$ and $\varepsilon > 0$, there exist a function $f \in L^1(X, \nu)$, elements $g_1, \ldots, g_m \in G$ for some $m < \infty$, and $\lambda_{j,k} \geq 0$, for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, such that

$$\left\| f_j - \sum_{k=1}^m \lambda_{j,k} \beta_{g_k}(f) \right\| \leq \varepsilon, \quad j = 1, \ldots, n,$$

where $\| \cdot \|$ represents the $L^1$ norm and $\beta_{g_k}(f)(x) = f \circ \alpha_{g_k}^{-1}(x)((d\mu \circ g_k^{-1})/d\mu)(x)$.

Recall that Bernoulli measures on $X = \prod_{k \geq 1} \{0, 1\}$ are the product measures $\nu_\lambda = \bigotimes_{k \geq 1} \nu_{\lambda,k}$, with $\nu_{\lambda,k}(0) = 1/(1 + \lambda)$ and $\nu_{\lambda,k}(1) = \lambda/(1 + \lambda)$, where $0 < \lambda \leq 1$.

Our main result, proved in §3, is the following theorem.

**Theorem 1.** For $0 < \lambda \leq 1$, the natural action of $S_\infty$ on $(X, \nu_\lambda)$ is approximately transitive.

In §4 we generalize our main result to a larger class of product probability measures and show that the corresponding associated flow of $S_\infty$ is AT.

2. **Preliminaries**

Throughout this section, $(X, \nu)$ will denote the Lebesgue space $X = \prod_{k \geq 1} \{0, 1\}$ and $\nu$ the product measure $\nu = \bigotimes_{k \geq 1} \nu_k$, with $\nu_k(0) = 1/(1 + \lambda_k)$ and $\nu_k(1) = \lambda_k/(1 + \lambda_k)$, $0 < \lambda_k \leq 1$. In this section we prove some technical results we will need in §3.

**Lemma 2.1.** Let $(X, \nu)$ be as above. For $0 \leq r \leq n$, let $A(n, r) = \{x \in X : \# \{1 \leq k \leq n : x_k = 1\} = r\}$ denote the union of cylinder sets on $n$ symbols with exactly $r$ 1s. Then

$$\nu(A(n, r)) < \left(\frac{\pi}{\sum_{k=1}^n \lambda_k}\right)^{1/2}, \quad 0 \leq r \leq n.$$
Proof. Let
\[ P_n(t) = \prod_{k=1}^{n} \left( \frac{1}{1 + \lambda_k} + \frac{\lambda_k}{1 + \lambda_k} e^{it} \right). \]

Then it is easy to check that
\[ v(A(n, r)) = \frac{1}{2\pi} \int_{0}^{2\pi} P_n(t)e^{-irt} \, dt. \]

As \(0 < \lambda_k \leq 1\), we have
\[ v(A(n, r)) \leq \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(t)| \, dt \]
\[ = \frac{1}{2\pi} \int_{0}^{2\pi} \prod_{k=1}^{n} \left( 1 - \frac{2\lambda_k}{1 + \lambda_k} \right) \left( 1 - \cos t \right)^{1/2} \, dt \]
\[ \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left\{ \prod_{k=1}^{n} \left( 1 - \frac{(1 - \cos t) \lambda_k}{2} \right) \right\}^{1/2} \, dt. \]

Now note that \(\sqrt{1 - x} \leq 1 - x/2\) and \(\log(1 - x) \leq -x\) for \(0 < x < 1\), and so
\[ v(A(n, r)) \leq \frac{1}{\pi} \int_{0}^{\pi} \exp \left( \frac{(\cos t - 1)}{4} \sum_{k=1}^{n} \lambda_k \right) \, dt \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} \exp \left( \frac{(\cos t - 1)}{4} \sum_{k=1}^{n} \lambda_k \right) \, dt \]
as \(\cos t - 1\) is symmetric about \(t = \pi\). Since \(-t^2/\pi^2 \geq \cos t - 1\) for \(0 \leq t \leq \pi\), we have
\[ v(A(n, r)) \leq \frac{1}{\pi} \int_{0}^{\pi} e^{-\left(\sum_{k=1}^{n} \lambda_k\right)t^2/4\pi^2} \, dt \]
\[ < \frac{1}{\pi} \int_{0}^{\infty} e^{-\left(\sum_{k=1}^{n} \lambda_k\right)t^2/4\pi^2} \, dt \]
\[ = \left( \frac{\pi}{\sum_{k=1}^{n} \lambda_k} \right)^{1/2}. \]

\[ \square \]

Lemma 2.2. Let \((X, \nu)\) be as above and \(p\) be a fixed positive integer. Then, for \(n \geq p\),
\[ \sum_{r=p}^{n} |v(A(n, r)) - v(A(n, r - p))| < 2p \left( \frac{\pi}{\sum_{k=1}^{n} \lambda_k} \right)^{1/2}. \]

Proof. Let
\[ P_n(t) = \prod_{k=1}^{n} \left( \frac{1}{1 + \lambda_k} + \frac{\lambda_k}{1 + \lambda_k} e^{ikt} \right) = \sum_{k=0}^{n} \alpha_k e^{ikt} \]
with \(\alpha_k = v(A(n, k))\). Then, for \(p \leq n\),
\[ \| (1 - e^{ipt}) P_n \|_1 = |\alpha_0| + |\alpha_1| + \cdots + |\alpha_{p-1}| \]
\[ + \sum_{r=p}^{n} |\alpha_r - \alpha_{r-p}| + |\alpha_{n-p+1}| + \cdots + |\alpha_n|, \]
where $\| \cdot \|_1$ denotes the Fourier 1-norm, that is, the sum of the absolute values of the Fourier coefficients. Thus

$$\sum_{r=p}^{n} |\nu(A(n, r)) - \nu(A(n, r-p))| = \sum_{r=p}^{n} |\alpha_r - \alpha_{r-p}|$$

\[ \leq \left\| (1 - e^{it}) P_n \right\|_1 = \left\| \sum_{m=0}^{p-1} e^{imt} (1 - e^{it}) P_n \right\|_1 \]

\[ \leq \sum_{m=0}^{p-1} \left\| e^{imt} (1 - e^{it}) P_n \right\|_1 = p \left(1 - e^{it}\right) P_n \left\|_1.\right]

Now, by [B, Lemma 3.18 and Proposition 3.19], we have

$$\left\| (1 - e^{it}) P_n \right\|_1 = 2 \max_{0 \leq r \leq n} \alpha_r = 2 \max_{0 \leq r \leq n} \nu(A(n, r)),$$

and by the above lemma, we conclude that

$$\sum_{r=p}^{n} |\nu(A(n, r)) - \nu(A(n, r-p))| < 2p \left(\frac{\pi}{\sum_{k=1}^{n} \lambda_k}\right)^{1/2}.$$

The lemma is proved. □

**Remark 2.1.** Lemmas 2.1 and 2.2 are certainly well known to probabilists.

For $\lambda \in (0, 1]$ and the measure $\nu_{\lambda} = \otimes \nu_{\lambda, k}$ (with $\nu_{\lambda, k}(0) = 1/(1 + \lambda)$, $\nu_{\lambda, k}(1) = \lambda/(1 + \lambda)$), we have

$$\nu_{\lambda}(A(n, r)) = \binom{n}{r} \frac{\lambda^r}{(1 + \lambda)^n}.$$

Hence, from Lemmas 2.1 and 2.2 we have the following corollary.

**Corollary 2.1.** Let $p$ be a positive integer and $\lambda \in (0, 1]$. Then:

1. \[
\binom{p}{r} \frac{\lambda^r}{(1 + \lambda)^p} \leq \left(\frac{\pi(1 + \lambda)}{p\lambda}\right)^{1/2} \] for every $0 \leq r \leq p$;

2. \[
\sum_{r=n}^{p} \left| \binom{p}{r} \frac{\lambda^r}{(1 + \lambda)^p} - \binom{p}{r-n} \frac{\lambda^{r-n}}{(1 + \lambda)^p} \right| \leq 2n \left(\frac{\pi(1 + \lambda)}{p\lambda}\right)^{1/2} \] for every $0 < n \leq p$.

**3. Main result**

Let $0 < \lambda \leq 1$ and $\nu_{\lambda} = \otimes_{k \geq 1} \nu_{\lambda, k}$ be a Bernoulli measure on $X = \prod_{k=1}^{\infty} \{0, 1\}$. Recall (see, for example, [AP] or [SV]) that the natural action of $S_\infty$ on $(X, \nu_{\lambda})$ is then ergodic and measure preserving. Moreover, any $S_\infty$-invariant, ergodic probability measure on $X$ is a Bernoulli measure. In this section, we prove the theorem stated in the introduction.
**Theorem 3.1.** The natural action of $S_{\infty}$ on $(X, v_\lambda)$ is approximately transitive.

Let us introduce notation we will need below. For $n \geq 1$, let $X^n = \prod_{i=1}^{n} \{0, 1\}$. For $x = (x_1, x_2, \ldots, x_n) \in X^n$, let

$$C(x) = \{y \in X : y_i = x_i, i = 1, 2, \ldots, n\}$$

denote the (elementary) cylinder set of length $n$ whose first $n$ symbols are given by $x$. Let

$$S_n^+ = \left\{ \sum_{x \in X^n} \alpha_x \chi_{C(x)} : \alpha_x \in \mathbb{R}_+ \right\}.$$

Note that for $m \geq n$, every cylinder set of length $n$ decomposes into a disjoint union of cylinders of length $m$, and so $S_n^+ \subseteq S_m^+$.

The proof of Theorem 3.1 will follow from the following two technical lemmas. The first is well known and we omit its proof.

**Lemma 3.1.** Let $f \in L^1_+(X, \nu)$ be a non-negative real-valued function. Then, for every $\varepsilon > 0$, there exist $n \geq 1$ and $g \in S_n^+$ such that $\|f - g\| \leq \varepsilon$.

**Lemma 3.2.** For each $n \geq 1$ and $0 < \varepsilon \leq 1$, there exist a positive integer $p$, a function $f \in S_p^+$, a finite subset $S \subset S_{\infty}$ and non-negative constants $a_{x, \sigma}$, for $x \in X^n$ and $\sigma \in S$, such that, for all $x \in X^n$,

$$\left\| \chi_{C(x)} - \sum_{\sigma \in S} a_{x, \sigma} \chi_{S}(f) \right\| \leq \varepsilon \|\chi_{C(x)}\|.$$

**Proof.** For $x = (x_1, \ldots, x_n) \in X^n$, define $\#(x)$ to be the number of 1s in the $n$-tuple. Let $0 < \varepsilon \leq 1$ be given and choose a positive integer $p$ such that

$$p \geq \max \left\{2n, \frac{\pi(1 + \lambda)}{\lambda} \cdot \frac{18n^2}{\varepsilon^2} \right\}.$$

For $i \in \{0, 1\}$, to simplify the notation, we will write $u_i$ for a string of $l$ 1s. For $n \leq j \leq p - n$, let $C(p, j)$ denote the cylinder set $C(1, 0_{p-j})$ and define $f \in S_p^+$ by

$$f = \sum_{j=n}^{p-n} \frac{(p-n)}{(p-n)!} \chi_{C(p, j)}.$$

For each $x \in X^{k_n}$, let $x^{-1}(0)$ denote the set $\{j \in \{1, 2, \ldots, k_n\} : x_j = 0\}$ and consider the permutation $\sigma_x$ given by

$$\sigma_x = \prod_{j \in x^{-1}(0)} (j, p-n+j)$$

where $(a, b)$ denotes the transposition between $a$ and $b$.

By construction, for $n \leq j \leq p-n$, we have

$$\sigma_x(C(p, j)) = C(x, 1_{j-n}, 0_{p-n-j}, \bar{x})$$

where if $x = (x_1, x_2, \ldots, x_n)$, then $\bar{x} = (1-x_1, 1-x_2, \ldots, 1-x_n)$ (the subtraction is computed modulo 2).
Recall that, for $0 \leq j \leq p$, $A(p, j)$ denotes the union of all cylinders on $p$ symbols with exactly $j$ 1s. If

$$S_{n, p} = \{ \tau \in S_\infty : \tau(i) = i, 1 \leq i \leq n \text{ or } i \geq p + 1 \},$$

then from (2) we get, for $n \leq j \leq p - n$,

$$\bigcup_{\tau \in S_{n, p}} \tau \sigma_x(C(p, j)) = C(x) \cap A(p, j).$$

Moreover, the stabilizer in $S_{n, p}$ of $(x, 1_{j-n}, 0_{p-n-j}, \bar{x})$ is isomorphic to the product of the symmetric groups $S_{j-k}$ and $S_{p-n-j+k}$, where $k = \#(x)$. Then

$$\sum_{\tau \in S_{n, p}} \chi_{\tau \sigma_x(C(p, j))} = (j-k)!(p-n-j+k)! \chi_{C(x) \cap A(p, j)} = \frac{(p-n)!}{(j-k)!} \chi_{C(x) \cap A(p, j)}.$$

By definition (1) of $f$, we then have

$$\sum_{\tau \in S_{n, p}} \beta_{\tau \sigma_x}(f) = \sum_{j=n}^{p-n} \frac{(p-n)}{(j-k)} \chi_{C(x) \cap A(p, j)}. \quad (3)$$

Note that, for $k \leq j \leq p - (n - k)$,

$$\| \chi_{C(x) \cap A(p, j)} \| = \| \chi_{C(x)} \| \frac{(p-n) j-k}{(1 + \lambda)^{p-n}}. \quad (4)$$

Let $B(p, n) = \bigcup_{j=n}^{p-n} A(p, j)$. By Corollary 2.1, we have

$$v_\lambda(B(p, n)^c \cap C(x)) = v_\lambda \left( \bigcup_{j=k}^{n-1} (C(x) \cap A(p, j)) \right) + v_\lambda \left( \bigcup_{j=n+1}^{p-n-k} (C(x) \cap A(p, j)) \right)$$

$$= \left[ \sum_{j=k}^{n-1} \binom{p-n}{j} \frac{\lambda^{j-k}}{(1 + \lambda)^{p-n}} + \sum_{j=n+1}^{p-n-k} \binom{p-n-k}{j} \frac{\lambda^{j-k}}{(1 + \lambda)^{p-n}} \right] \| \chi_{C(x)} \|$$

$$\leq n \left( \frac{(1 + \lambda)}{(p-n) \lambda} \right)^{1/2} \| \chi_{C(x)} \|. \quad (5)$$

From (3), (4), Corollary 2.1 and as $k \leq n$,

$$\| \chi_{C(x) \cap B(n, p)} - \lambda^k \sum_{\tau \in S_{n, p}} \beta_{\tau \sigma_x}(f) \| \leq \| \sum_{j=n}^{p-n} \left( 1 - \lambda^k \frac{(p-n)}{(j-k)} \right) \chi_{C(x) \cap A(n, j)} \|$$

$$\leq \| \chi_{C(x)} \| \sum_{j=n}^{p-n} \left| \frac{(p-n)}{j-k} \frac{\lambda^{j-k}}{(1 + \lambda)^{p-n}} - \frac{(p-n)}{j} \frac{\lambda^j}{(1 + \lambda)^{p-n}} \right|$$

$$\leq \| \chi_{C(x)} \| \sum_{j=k}^{p-n} \left| \frac{(p-n)}{j-k} \frac{\lambda^{j-k}}{(1 + \lambda)^{p-n}} - \frac{(p-n)}{j} \frac{\lambda^j}{(1 + \lambda)^{p-n}} \right|$$

$$\leq 2k \| \chi_{C(x)} \| \left( \frac{(1 + \lambda)}{(p-n) \lambda} \right)^{1/2} \leq 2n \| \chi_{C(x)} \| \left( \frac{(1 + \lambda)}{(p-n) \lambda} \right)^{1/2}.$$
By (5) and the choice of $p$, we get
\[
\|\chi_{C(x)} - \lambda^k \sum_{\tau \in S_{n,p}} \beta_{\tau \sigma_x} (f)\| \\
\leq \|\chi_{C(x) \cap B(n, p)^c}\| + \|\chi_{C(x) \cap B(n, p)} - \lambda^k \sum_{\tau \in S_{n,p}} \beta_{\tau \sigma_x} (f)\| \\
\leq \varepsilon \|\chi_{C(x)}\|.
\]

The proof of the lemma is completed by letting $S$ be the (disjoint) union of $S_{n, p} \sigma_x$, $x \in X^n$ and
\[
a_{x, \sigma} = \begin{cases} 
\lambda^{\#(x)} & \text{if } \sigma \in S_{n, p} \sigma_x, \\
0 & \text{if not.}
\end{cases}
\]

**Proof of Theorem 3.1.** Consider an arbitrary finite collection $f_1, f_2, \ldots, f_m$ of functions from $L^1_+(X, \nu)$ and $\varepsilon > 0$. Choose $\eta > 0$ such that $(1 + \|f_j\|) \cdot \eta \leq \varepsilon$ for all $j$. By Lemma 3.1, there exist $n \geq 1$ and non-negative coefficients $\lambda_{j,x}$ such that
\[
\|f_j\| = \sum_{x \in X^n} \lambda_{j,x} \|\chi_{C(x)}\| \quad \text{and} \quad \|f_j - \sum_{x \in X^n} \lambda_{j,x} \chi_{C(x)}\| \leq \eta, \quad j = 1, 2, \ldots, m.
\]

By Lemma 3.2 there exist $f \in L^1_+(X, \nu)$, a finite set $S$ of elements of $S_\infty$ and reals $a_{x, \sigma} \geq 0$, $x \in X^n$, $\sigma \in S$, which satisfy
\[
\|\chi_{C(x)} - \sum_{\sigma \in S} a_{x, \sigma} \beta_{\sigma} (f)\| \leq \eta \|\chi_{C(x)}\| \quad \text{for all } x \in X^n.
\]

Then, for $j = 1, 2, \ldots, n$, we obtain
\[
\|f_j - \sum_{x \in X^n} \sum_{\sigma \in S} \lambda_{j,x} a_{x, \sigma} \beta_{\sigma} (f)\| \\
\leq \|f_j - \sum_{x \in X^n} \lambda_{j,x} \chi_{C(x)}\| + \sum_{x \in X^n} \lambda_{j,x} \|\chi_{C(x)} - \sum_{\sigma \in S} a_{x, \sigma} \beta_{\sigma} (f)\| \\
\leq \eta \cdot \left(1 + \sum_{x \in X^n} \lambda_{j,x} \|\chi_{C(x)}\|\right) = \eta (1 + \|f_j\|) \leq \varepsilon.
\]

Hence the action of $S_\infty$ on $(X, \nu)$ is AT. \[\square\]

4. **Generalizations**

In this section we generalize Theorem 3.1 for a larger class of probability measures on $X = \prod_{k \geq 1} \{0, 1\}$, and show that the associated flow of $S_\infty$ is AT for this class of measures. Let $(L_n)_{n \geq 1}$ be a sequence of positive integers and $(\lambda_n)_{n \geq 1}$ be a sequence of real numbers in $(0, 1]$. Then $\nu = \nu(L_n, \lambda_n)$ will denote the product measure on $X = \prod_{k \geq 1} \{0, 1\} = \prod_{n \geq 1} \prod_{1}^{L_n} \{0, 1\}$, given by
\[
\nu(L_n, \lambda_n) = \bigotimes_{n \geq 1} \nu_{\lambda_n}^{\otimes L_n}.
\]
where \( v_{\lambda_n}(0) = 1/(1 + \lambda_n) \) and \( v_{\lambda_n}(1) = \lambda_n/(1 + \lambda_n) \). In this section we will assume that
\[
\sup\{L_n\lambda_n : n \geq 1\} = \infty. \tag{6}
\]

Recall (see [SV] or [AP]) that \( \nu(L_n, \lambda_n) \) is an \( S_{\infty} \)-ergodic, non-atomic measure on \( X \) if and only if \( \sum_{n \geq 1} L_n\lambda_n = \infty \).

The following lemma and its proof are a generalization of Lemma 3.2. We will use the following notation. Set \( k_0 = 0 \) and, for \( n \geq 1 \),
\[
k_n = \sum_{k=1}^{n} L_k.
\]
Recalling that, for \( n \geq 1 \),
\[
X^n = \prod_{k=1}^{n} \{0, 1\},
\]
we set for \( x \in X^n \) and \( z \in X^{L_i} \), we set for \( n \geq 1 \):
\[
C(x) = \{y \in X : y_j = x_j \text{ for } 1 \leq j \leq k_n\},
\]
\[
C(z(L_i)) = \{y \in X : y_{k_{i-1}+i} = z_i \text{ for } 1 \leq i \leq L_i\}
\]
and, for \( m > n \),
\[
C(x, z(L_m)) = C(x) \cap C(z(L_m)).
\]

**Lemma 4.1.** Let \((X, \nu)\) be as above. Then, for each \( n \geq 1 \) and \( \varepsilon > 0 \), there exist a function \( f \in L^1_+(X, \nu) \), a finite subset \( S \subset S_{\infty} \) and non-negative constants \( a_{x, \sigma} \), for \( x \in X^n \) and \( \sigma \in S \), such that
\[
\left\| \chi_{C(x)} - \sum_{\sigma \in S} a_{x, \sigma} \beta_{\sigma}(f) \right\| \leq \varepsilon \left\| \chi_{C(x)} \right\|
\]
for all \( x \in X^n \).

**Proof.** Let \( n \geq 1 \) and \( 0 < \varepsilon < 1 \). By (6), we can choose an integer \( m \geq 1 \) such that
\[
3k_n \left( \frac{\pi(1 + \lambda_m)}{L_{m}\lambda_m} \right)^{1/2} \leq \varepsilon. \tag{7}
\]

Notice that \( k_n < L_m \). For \( 0 \leq j \leq L_m - k_n \) and \( \tilde{z}_j = (1_j, 0_{L_m-j}) \in X^{L_m} \), set
\[
C(n, m, j) = C(1_{k_n}) \cap C(\tilde{z}_j(L_m))
\]
and
\[
C(n, m) = \bigcup_{j=0}^{L_m-k_n} C(n, m, j).
\]

We then define the function \( f \in L^1_+(X, \nu) \) by
\[
f = \sum_{j=0}^{L_m-k_n} \frac{L_m}{L_m!} \chi_{C(n, m, j)}, \tag{8}
\]
whose support is \( C(n, m) \). For an arbitrary \( x \in X^n \), let \( x^{-1}(0) \) denote the set \( \{j \in [1, 2, \ldots, k_n] : x_j = 0\} \) and consider the permutation \( \sigma_x \) given by
\[
\sigma_x = \prod_{j \in x^{-1}(0)} (j, k_m - k_n + j)
\]
where \((a, b)\) denotes the transposition between \( a \) and \( b \). For all \( 0 \leq j \leq L_m - k_n \), set
\[
z_j = (1_j, 0_{L_m-k_n-j}, \tilde{x}),
\]
where \( \bar{x} = (1 - x_1, 1 - x_2, \ldots, 1 - x_{k_n}) \in X^{k_n} \) (the subtraction is computed modulo 2). Then, by construction,

\[
\sigma_x(C(n, m, j)) = C(x, z_j(L_m))
\]

and, for every \( y \in C(x, z_j(L_m)) \),

\[
\frac{dv \circ \sigma_x^{-1}(y)}{dv} = \prod_{k \geq 1} \frac{v_k(\sigma_x^{-1}(y)_k)}{v_k(y_k)}
\]

\[
= \prod_{j \in x^{-1}(0)} \frac{v_j(y_{km-kn+j})}{v_j(y_j)} \frac{v_{km-kn+j}(y_j)}{v_{km-kn+j}(y_{km-kn+j})}
\]

\[
= \prod_{j \in x^{-1}(0)} \frac{v_j(1)}{v_j(0)} \frac{v_{km-kn+j}(0)}{v_{km-kn+j}(1)} = \prod_{j \in x^{-1}(0)} v_j(1) \lambda_m^{-1}
\]

and therefore is constant on \( \sigma_x(C(n, m)) \); let \( D_x \) be this constant. By definition of \( f \), we have

\[
\beta_{\sigma_x}(f) = \sum_{j=0}^{L_m-k_n \frac{L_m}{L_m+1}} \chi_{C(x, z_j(L_m))} D_x.
\]

Let \( S(L_m) \) be the subgroup of \( S_\infty \) of permutations \( \sigma \) such that \( \sigma(i) = i \), for \( i \notin \{k_{m-1} + 1, \ldots, k_m\} \). For \( 0 \leq l \leq L_m \), set

\[
A(x, m, l) = C(x) \bigcap \{ y \in X : \text{card}\{i : k_{m-1} + 1 \leq i \leq k_m, y_i = 1\} = l \}.
\]

Then it is easy to observe that, for \( 0 \leq j \leq L_m - k_n \)

\[
\bigcup_{\tau \in S(L_m)} \tau(C(x, z_j(L_m))) = A(x, m, k_n - k + j),
\]

where \( k = \sum_{i=1}^{k_n} x_i = \text{card}\{i : 1 \leq i \leq k_n, x_i = 1\} \). As the stabilizer in \( S(L_m) \) of \( z_j \) is isomorphic to \( S_{j+k_n-k} \times S_{L_m-k_n-j+k} \), we have, for \( 0 \leq j \leq L_m - k_n \),

\[
\sum_{\tau \in S(L_m)} \chi_{\tau(C(x, z_j(L_m)))} = (k_n - k + j)!(L_m - k_n - j + k)! \chi_{A(x, m, k_n-k+j)}
\]

\[
= \frac{L_m!}{(k_n-k+j)!} \chi_{A(x, m, k_n-k+j)}.
\]

Any \( \tau \in S(L_m) \) being \( v \)-measure preserving, we get

\[
\sum_{\tau \in S(L_m)} \beta_{\tau \sigma_x}(f) = \sum_{j=0}^{L_m-k_n} D_x \frac{L_m}{(k_n-k+j)} \chi_{A(x, m, k_n-k+j)}.
\]

Note that, for \( 0 \leq j \leq L_m \),

\[
\| \chi_{A(x, m, j)} \| = \| \chi_{C(x)} \| \left( \frac{L_m}{j} \right) \frac{\lambda_m^j}{(1 + \lambda_m) L_m}.
\]
Let $B(x, m) = \bigcup_{j=k_n^0 - k}^{L_m^0} A(x, m, j) = \bigcup_{j=0}^{L_m^0-k} A(x, m, k_n - k + j)$. From (11) and (12), we easily get

$$\|\chi C(x) \cap B(x, m) - \frac{\lambda_m^k}{D_x} \sum_{\tau \in S(L_m)} \beta_{\tau \sigma_x}(f)\| \leq \sum_{j=0}^{L_m^0-k} \left| \frac{\lambda_m^{k_n-k+j}}{(L_m^0+k_n-j)(1+\lambda_m)_{L_m^0} - (k_n^0+j)(1+\lambda_m)_{L_m^0}} \right| \|\chi C(x)\|.$$  

Then, by Corollary 2.1,

$$\|\chi C(x) \cap B(x, m) - \frac{\lambda_m^k}{D_x} \sum_{\tau \in S(L_m)} \beta_{\tau \sigma_x}(f)\| \leq 2k_n \left( \frac{\pi(1+\lambda_m)}{L_m^0 \lambda_m^0} \right)^{1/2} \|\chi C(x)\|. \quad (13)$$

and

$$\|\chi C(x) \cap B(x, m)^c\| = \sum_{j=0}^{k_n^0-k-1} \|\chi A(x, m, j)\| + \sum_{j=L_m^0-k+1}^{L_m^0} \|\chi A(x, m, j)\| \leq k_n \left( \frac{\pi(1+\lambda_m)}{L_m^0 \lambda_m^0} \right)^{1/2} \|\chi C(x)\|. \quad (14)$$

Therefore, by (7), (13) and (14), we get

$$\left\|\chi C(x) - \frac{\lambda_m^k}{D_x} \sum_{\tau \in S(L_m)} \beta_{\tau \sigma_x}(f)\right\| \leq 3k_n \left( \frac{\pi(1+\lambda_m)}{L_m^0 \lambda_m^0} \right)^{1/2} \|\chi C(x)\| \leq \varepsilon\|\chi C(x)\|. \quad \square$$

We let $S$ be the (disjoint) union of $S(L_m)\sigma_x$, $x \in X^\sigma$ and we define

$$a_{x, \sigma} = \begin{cases} \frac{\lambda_m^k}{D_x} & \text{if } \sigma = \tau \sigma_x, \tau \in S(L_m), \\ 0 & \text{otherwise}. \end{cases}$$

By Lemmas 4.1 and 3.1, we then get the following theorem.

**THEOREM 4.1.** Let $\nu = \nu(L_n, \lambda_n)$ be a product probability measure on $X$ satisfying

$$\sup\{L_n \lambda_n : n \geq 1\} = \infty.$$  

Then the natural action of $S_\infty$ on $(X, \nu)$ is AT.

**Remark 4.1.** Keeping the notation of Lemma 4.1 and its proof, note that, for any $\sigma \in S \subset S_\infty$, the Radon-Nikodym derivative $d\nu \circ \sigma^{-1}/d\nu$ is constant on $\sigma(C(n, m))$, where $C(n, m)$ is the support of $f$. Indeed, $S$ is the disjoint union of $S(L_m)\sigma_x$, for $x \in X^\sigma$. Moreover, any $\tau \in S(m)$ is $\nu$-measure preserving and

$$\frac{d\nu \circ \sigma^{-1}_x}{d\nu}(y) = D_x = \prod_{j \in x^{-1}(0)} \frac{\nu_j(1)}{\nu_j(0)} \lambda_m^{-1} \quad \text{for } y \in \sigma_x(C(n, m)).$$
Recall that the associated flow of a non-singular action of a countable discrete group $G$ on a Lebesgue space $(X, \nu)$ is the Mackey range of the Radon–Nikodym cocycle of the $G$-action on $(X, \nu)$. More precisely, on $(X \times \mathbb{R}, \nu \otimes e^u du)$, let us denote by $\gamma$ the action of $G$ and by $\rho$ the action of $\mathbb{R}$, given by

$$\gamma_g(x, t) = \left( gx, t - \log \frac{d\mu \circ g}{d\mu}(x) \right)$$

and

$$\rho_s(x, t) = (x, t + s).$$

As $\rho$ commutes with the $G$-action $\gamma$, it induces an $\mathbb{R}$-action on the ergodic decomposition of the (infinite measure preserving) action $\gamma$, which is the associated flow of the action of $G$ on $(X, \nu)$. Let

$$\tilde{\gamma}_{g, s}(x, t) = \left( gx, t + s - \log \frac{d\mu \circ g}{d\mu}(x) \right)$$

be the skew product action $\tilde{\gamma}$ of $G \times \mathbb{R}$ on $(X \times \mathbb{R}, \nu \otimes \mu)$, where $d\mu = e^u du$.

As by [CW1, Remark 2.4], any factor action of an AT action is AT, to prove that the associated flow of a $G$-action on $(X, \nu)$ is AT it is sufficient to show that the above skew product action $\tilde{\gamma}$ is AT.

We introduce in Definition 4.2 a strong version of AT, which we denote by $\tilde{\gamma}T$, and show in Proposition 3.5 that if a non-singular $G$-action is $\tilde{\gamma}T$, its associated skew product $(G \times \mathbb{R})$-action is AT.

**Definition 4.2.** Let $(X, \nu, G)$ be a non-singular action of a countable discrete group $G$ on a Lebesgue space $(X, \nu)$. Then the action is $\tilde{\gamma}T$ if, given $n < \infty$, functions $f_1, f_2, \ldots, f_n \in L^1_+(X, \nu)$ and $\varepsilon > 0$, there exist a function $f \in L^1_+(X, \nu)$, elements $g_1, \ldots, g_m \in G, s_1, \ldots, s_m > 0$ for some $m < \infty$, and $\lambda_{j,k} \geq 0$, for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, such that

$$\left\| f_j - \sum_{k=1}^m \lambda_{j,k} g_k(f) \right\| \leq \varepsilon,$$

and

$$\frac{dv \circ g_k^{-1}}{dv}(x) = s_k, \quad \nu\text{-almost every } x \in g_k(Supp(f)), \quad k = 1, \ldots, m,$$

where $Supp(f)$ denotes the support of $f$.

**Proposition 4.1.** Let $(X, \nu, G)$ be a non-singular $\tilde{\gamma}T$ action of a countable group $G$ on a Lebesgue space $(X, \nu)$. Let $\mu$ denote the measure on $\mathbb{R}$ given by $d\mu = e^u du$. Then the skew product action $\tilde{\gamma}$ of $G \times \mathbb{R}$ on $(X \times \mathbb{R}, \nu \times \mu)$ is AT.

**Proof.** Let $0 < \varepsilon < 1$ and consider $n$ non-negative functions $\tilde{f}_j \in L^1_+(X \times \mathbb{R}, \nu \otimes \mu), j = 1, 2, \ldots, n$. By standard approximation arguments, we can assume that $\tilde{f}_j = f_j \otimes f_j$, where $f_j \in L^1_+(X, \nu)$ and $f_j' \in L^1_+(\mathbb{R}, \mu)$. By assumption, there exist $f \in L^1_+(X, \nu), g_1, \ldots, g_m \in G, s_1, \ldots, s_m \in \mathbb{R}^e_+$ for some $m < \infty$, and $\lambda_{j,k} \in \mathbb{R}_+$, for $k = 1, \ldots, m$ and $j = 1, \ldots, n$, such that

$$\left\| f_j - \sum_{k=1}^m \lambda_{j,k} g_k(f) \right\| \leq \varepsilon \| f_j \|,$$

and

$$\frac{dv \circ g_k^{-1}}{dv}(x) = s_k, \quad \nu\text{-almost every } x \in g_k(Supp(f)), \quad k = 1, \ldots, m.$$
Note first that
\[ \nu(2892) = s_k, \quad \nu\text{-almost every } x \in g_k(\text{Supp}(f)), \quad k = 1, \ldots, m. \]
As the action \( \rho \) by translation on \((\mathbb{R}, \mu) \) is transitive and therefore AT, there exist \( f' \in L_+^1(\mathbb{R}, \mu) \), \( t_1, \ldots, t_p \in \mathbb{R} \) for some \( p < \infty \) and \( \lambda_{j,l}' \geq 0, \quad l = 1, \ldots, p, \quad j = 1, \ldots, n \), such that
\[ \left\| f_j' - \sum_{l=1}^p \lambda_{j,l}' \rho_l(f') \right\| \leq \epsilon \| f_j' \|, \]
where
\[ \rho_s(f'(t)) = f'(t-s) \frac{d\mu \circ \rho^{-s}}{d\mu}(t) = e^{-s} f'(t-s), \quad t \in \mathbb{R}. \]
Note first that
\[ \left\| f_j \otimes f_j' - \sum_{k=1}^m \lambda_{j,k} \beta_{g_k}(f) \otimes f_j' \right\| \leq \epsilon \| f_j \| \| f'_j \| = \epsilon \| f_j \otimes f_j' \| \]
and that
\[ \left\| \sum_{k=1}^m \lambda_{j,k} \beta_{g_k}(f) \otimes f_j' - \sum_{k=1}^m \sum_{j=1}^n \lambda_{j,k} \lambda_{j,l}' \beta_{g_k}(f) \otimes \rho_l(f') \right\| \]
\[ \leq 2 \epsilon \| f_j \| \| f_j' \| = 2 \epsilon \| f_j \otimes f_j' \|. \]
By definition of the \((G \times \mathbb{R})\)-action \( \tilde{\gamma} \) and as for \( 1 \leq k \leq m \), \((d\nu \circ g_k^{-1})/d\nu)(x) = s_k \) for \( \nu\text{-almost every } x \) with \( g_k^{-1} x \in \text{Supp}(f) \),
\[ \tilde{\gamma}_{g_k,t} \to \log s_k (f \otimes f')(x, t) = f(g_k^{-1} x) f' \left( t - \log \frac{d\nu \circ g_k^{-1}}{d\nu}(x) - t_l + \log s_k \right) e^{-t_l + \log s_k} \]
\[ = \beta_{g_k}(f)(x) \rho_l(f')(t) \quad \text{for } (\nu \otimes \mu)\text{-almost every } (x, t) \in X \times \mathbb{R}. \]
Hence,
\[ \left\| f_j \otimes f_j' - \sum_{k=1}^m \sum_{j=1}^n \lambda_{j,k} \lambda_{j,l}' \tilde{\gamma}_{g_k,t_l + \log s_k} (f \otimes f') \right\| \leq 3 \epsilon \| f_j \| \| f_j' \| = 3 \epsilon \| f_j \otimes f_j' \| \]
and therefore the skew action \( \tilde{\gamma} \) is AT.

By Lemma 4.1 and Remark 4.1 if \( \nu_n = \nu(L_n, \lambda_n) \) is a product probability measure on \( X = \prod_{k \geq 1} [0, 1] \) satisfying (6), then the natural action of \( S_\infty \) on \((X, \nu)\) is \( \tilde{\text{AT}} \).

By [CW1, Remark 2.4] and Proposition 4.1 we then get the following theorem.

**Theorem 4.3.** Let \( \nu = \nu(L_n, \lambda_n) \) be a product probability measure on \( X = \prod_{k \geq 1} [0, 1] \) such that
\[ \sup \{ L_n \lambda_n : n \geq 1 \} = \infty. \]
Then the associated flow of the natural action of \( S_\infty \) on \((X, \nu)\) is \( \tilde{\text{AT}} \).

The previous results can be generalized as follows.
Proposition 4.2. Let \( \nu = \bigotimes_{k \geq 1} \nu_k \) be a product probability measure on \( X = \prod_{k \geq 1} [0, 1] \). We assume that there exist real numbers \( \lambda_n \in (0, 1] \), \( n \geq 1 \), and mutually disjoint sets of positive integers \( J_n, n \geq 1 \), of cardinality \( L_n \) such that \( \nu_k = \nu_{\lambda_n} \), for \( k \in J_n \) and such that at least one of the \( J_n \) is infinite or

\[
\sup \{ L_n \lambda_n : n \geq 1 \} = \infty
\]

if all \( J_n \) are finite. Then the natural action of \( S_\infty \) on \((X, \nu)\) is \( \overline{AT} \), and the corresponding skew product action and the associated flow are \( AT \).

Using Kakutani’s theorem on equivalence of infinite product measures (see, for example, [HS, Theorem 22.36]) we get the following corollary.

Corollary 4.1. Let \( \nu = \bigotimes_{k \geq 1} \nu_{\lambda_k} \) be a product probability measure on \( X = \prod_{k \geq 1} [0, 1] \). If the sequence \( (\lambda_k)_{k \geq 1} \) has a non-zero limit point \( \lambda \), then the action of \( S_\infty \) on \((X, \nu)\), the skew product action \( \tilde{\nu} \) of \( S_\infty \times \mathbb{R} \) on \((X \times \mathbb{R}, \nu \otimes e^u du)\), and the associated flow are \( AT \).

Remark 4.2. In general, it is not known whether, for any product probability measure \( \nu \) on \( X = \prod_{k \geq 1} [0, 1] \), the natural action of \( S_\infty \) on \((X, \nu)\) or its associated flow is \( AT \).

5. Examples
If \( \nu \) is a Bernoulli measure on \( X \), then \((X, \nu, S_\infty)\) is a system of type \( \text{II}_1 \). We have shown in §3 that such a system is \( AT \).

In the first part of this section we apply Theorem 4.3 and Proposition 4.2 to give examples of product measures on \( X \) with respect to which the action of \( S_\infty \) is \( AT \) and of type \( \text{III}_0 \). We show that they are of type \( \text{III}_0 \) by using the invariant \( T \) of the ergodic systems \((X, \nu, S_\infty)\).

In the last example, we consider a product measure \( \nu \) on \( X = \prod_{k \geq 0} [0, 1] \) which does not satisfy the assumption of Proposition 4.2, but such that the associated flow of the ergodic system \((X, \nu, S_\infty)\) is \( AT \).

Recall that an ergodic and non-singular action of a countable group \( G \) on a Lebesgue space \((Y, \nu)\) is of type \( \text{III} \) if there is no \( G \)-invariant measure equivalent to \( \nu \). Moreover, the ergodic \( G \)-space \((Y, \nu)\) is of type \( \text{III}_{\lambda} \), \( 0 \leq \lambda \leq 1 \), if its associated flow is the periodic flow on the interval \([0, -\log \lambda]\) for \( 0 < \lambda < 1 \), is the trivial flow on a singleton for \( \lambda = 1 \), and is non-transitive for \( \lambda = 0 \) (see, for example, [HO, Theorem 19]). Notice that the type of an ergodic measurable dynamical system depends only on its orbit equivalence class. The \( L^\infty \)-point spectrum of the flow is the invariant \( T \) of the ergodic system and is equal to Connes invariant \( T \) of the associated von Neumann factor.

Recall also that the ratio set \( r(Y, \nu, G) \) is the set of all \( \lambda \geq 0 \) such that, for all \( \varepsilon > 0 \) and all \( A \subset X \) of positive measure, there exist a measurable subset \( B \subset A \) of positive measure and \( \sigma \in S_\infty \), such that \( \sigma (B) \subset A \) and

\[
\left| \frac{d\nu \circ \sigma}{d\nu} (x) - \lambda \right| < \varepsilon \quad \text{for all } x \in B.
\]

The ratio set is Connes invariant \( S \) of the associated von Neumann algebra, and, by [T, Theorem 9.6], \( \lambda \) belongs to the kernel of the associated flow if and only if
\( e^k \in r(Y, v, G) \cap \mathbb{R}_+^{+} \). Therefore the ergodic system \((Y, v, G)\) is of type \(III_1\) if and only if its ratio set \(r(Y, v, G)\) is equal to \(\mathbb{R}_+\).

**Example 5.1.** For \(0 < \lambda < 1\), let \(\lambda_n = \lambda^{2^n}\), for \(n \geq 0\), and \((L_n)_{n \geq 0}\) be a sequence of positive integers such that

\[
\sup\{L_n \lambda_n : n \geq 0\} = \infty.
\]

Let \(\nu = \nu(L_n, \lambda_n)\) be the corresponding product measure on \(X = \prod_{k \geq 0}\{0, 1\}\), as in §4. If \(R\) denotes the tail equivalence relation on \((X, \nu)\), then the equivalence \(S\) induced by the action of \(S_{\infty}\) is a subequivalence of \(R\).

By Theorem 4.3, the system \((X, \nu, S_{\infty})\) is AT and is of type III by [SV, Theorem 1.2] (or [BP, Theorem 4.3]).

Then, by [C, Theorem 1.3.7], we have

\[
T(X, \nu, R) \supseteq \left\{ \frac{2k\pi}{2^n \log \lambda}, n \geq 0, k \in \mathbb{Z} \right\},
\]

which implies that \((X, \mu, R)\) is of type \(III_0\). As the associated flow of \((X, \nu, R)\) is a factor of the flow associated to \((X, \nu, S_{\infty})\), then \((X, \nu, S_{\infty})\) is also of type \(III_0\).

**Example 5.2.** Let \(0 < \lambda < 1\) be fixed and let \((k_n)_{n \geq 0}\) be an increasing sequence of positive integers with \(k_0 = 0\) such that

\[
\sum_{n \geq 1} (k_n - k_{n-1}) \lambda^{2^n} = \infty.
\]

Let \(X = \prod_{k \geq 0}\{0, 1\}\) and \(\nu = \bigotimes_{k \geq 0} \nu_k\) be the probability measure defined by

\[
v_{2k} (0) = \frac{1}{1 + \lambda}, \quad v_{2k} (1) = \frac{\lambda}{1 + \lambda}, \quad k \geq 0,
\]

and

\[
v_{2k+1} (0) = \frac{1}{1 + \lambda^{2^n+1}}, \quad v_{2k+1} (1) = \frac{\lambda^{2^n+1}}{1 + \lambda^{2^n+1}}, \quad k_n - 1 \leq k < k_n, n \geq 1.
\]

The dynamical system \((X, \nu, S_{\infty})\) is AT by Proposition 4.2. Since \(\sum_{n \geq 0} (k_n - k_{n-1}) \lambda^{2^n} = \infty\), it follows from [SV, Theorem 1.2] or [GM] that \((X, \nu, S_{\infty})\) is of type III.

Following [GM], we have

\[
T(X, \nu, S_{\infty}) \supseteq \left\{ \frac{2k\pi}{2^n \log \lambda}, n \geq 0, k \in \mathbb{Z} \right\}.
\]

Then \((X, \nu, S_{\infty})\) is a system of type \(III_0\).

This example shows that there exist product probability measures \(\bigotimes_{k \geq 1} \nu_k\) on \(X\) such that the corresponding sequence \((\lambda_k)_{k \geq 1}\) has a non-zero limit point and such that \((X, \nu, S_{\infty})\) is AT and of type \(III_0\).

**Example 5.3.** On the product space \(X = \prod_{k \geq 1}\{0, 1\}\) consider the product probability measure \(\nu = \bigotimes_{k \geq 1} \nu_k\), where \(\lambda_k = 1/k\) for \(k \geq 1\). By [SV, Theorem 1.2], the system \((X, \nu, S_{\infty})\) is of type III.

For any \(k, n \in \mathbb{N}\), if \(C_{k,n}\) denotes the cylinder set \(\{x \in X : x_n = 1, x_{kn+k} = 0\}\), and \(\sigma_{k,n}\) is the transposition \((n, kn + k)\), then \(((dv \circ \sigma_{k,n})/dv)|_{C_{k,n}} = 1/k \cdot n/(1 + n)\). Hence, we
easily get that
\[ \frac{1}{k} \in r(X, \nu, S_\infty) \quad \text{for all } k \geq 1. \tag{15} \]

As \((X, \nu, S_\infty)\) is of type III and \(r(X, \nu, S_\infty) \cap \mathbb{R}_+^*\) is a closed subgroup of \(\mathbb{R}_+\), we have \(r(X, \nu, S_\infty) = \mathbb{R}_+\), which shows that the dynamical system \((X, \nu, S_\infty)\) is of type III_1. Its associated flow, being trivial, is AT, but we do not know if the dynamical system \((X, \nu, S_\infty)\) is AT.

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