Increasing Peer Pressure on any Connected Graph Leads to Consensus

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In this paper, we propose a novel generic model of opinion dynamics over a social network, in the presence of communication among the users leading to interpersonal influence i.e., peer pressure. Each individual in the social network has a distinct objective function representing a weighted sum of internal and external pressures. We prove conditions under which a connected group of users converges to a fixed opinion distribution, and under which conditions the group reaches consensus. Through simulation, we study the rate of convergence on large scale-free networks as well as the impact of user stubbornness on convergence in a simple political model.

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I. INTRODUCTION

The existence of peer pressures in social networks is well-documented and easily believed. Adoption of trends [1, 2], purchasing behaviors [3], beliefs and cultural norms [4], privacy behaviors [5], bullying [6], and health behaviors [7, 8] have all been linked to peer influence, among countless others. Ultimately, we have even seen peer influence over time aggregate into emergent phenomena like collective identity, catalyzing social movements and large-scale group actions that have marked many of the most significant moments in human history.

However, despite the seemingly straightforward intuition of peer influence and conformity, the specific nature and strength of this influence over time and across contexts remains exceedingly hard to capture. In part, this is because humans are variable and complex. Treating this variability and complexity as noise has limited the application of human behavioral models to understanding real-world phenomena.

In this paper, we present a novel generalized framework for expressing peer influence dynamics over time in a set of connected individuals, or agents. The proposed framework supports the representation of individual variability through parametrized accounting for differences in susceptibility to peer influence and pairwise relationship strengths. Modeling agents’ individual opinions and behaviors as strategies changing discretely and simultaneously, we formally describe the evolution of strategies in a social network as the composition of continuous maps. We identify points of convergence and analyze these points under various conditions.

Specifically, we show that agents’ strategies converge to the optimal solution, namely consensus, if and only if peer pressure increases without bound. If the peer pressure is bounded and individual agents fail to change their opinions in the presence of dissonance with their peers, then these agents will not converge to an efficient distribution.

Work in the field of sociology, anthropology and more recently in computer science has long studied the importance of peer pressure and social influence in affecting users’ actions and decisions. We briefly discuss here the most important results that are relevant for us in all such fields.

Our notion of persuasion and peer-pressure is related to psychology literature on belief formation and social influence. In particular, our work draws inspiration by studies on periodicity in human behavior, and social influence theories [11]. We follow Friedkin’s foundational theory that strong ties are more likely to affect users’ opinions and result in persuasion or social influence. Underpinning our model is also the notion of mimicking. Brewer and more recently Van Bareen [12, 13] suggest that mimicking is used when individuals feel out of a group and therefore will alter their behavior (to a point [13]) to be more socially accepted.

This is observable especially in large scale social networks and in online settings, where several studies have shown that users are likely to follow popular trends as they become viral [14, 18]. Berger and colleagues have studied properties of viral content from a psychological perspective, noting that users are more likely to agree or share content if it triggers highly positive or highly negative sentiments [16]. Crandall and colleagues [18] also study social influence (in online social networks), emphasizing however the importance of selection in users’ decision to follow suit. That is, the authors posit that users will adapt behavior similar not all users’ whom they are exposed, but only or primarily of those which share some inherent characteristics with them. In this work, we provide a generic framework and study both cases, where users tend to agree without posing any “constraint” to
their willingness to change their mind, and when such power of persuasion is bounded (possibly by users’ inherent selection of peers whom to draw inspiration from). Note that in Crandall’s model for the convergence of behavior, both behavior and social ties were drawn from specified probability distributions [18]. In contrast, we model opinion dynamics and do not restrict to a finite set of actions. By modeling opinions as a deterministic process on a continuous set, we derive theoretical results about the model which can be validated on real data, allowing for a prediction of system behavior without modeling its evolution.

Subsequently, Cheng [19] analyzed whether cascade effects on social networks can be predicted. Authors formulate the problem of cascade prediction based on the specific features of the content being spread, and build a learning model based on the cascade size and other features. Their data shows wide variation in cascades for the same content, but also predictability despite this variation. In this paper, we take upon a related and yet more challenging questions, related to consensus in online settings wherein users are involved. The distinction, while subtle, is of importance as it includes the notion of “Change” of opinion due to the peer effect.

Our work can be seen to reside in the body of literature modeling belief and behavior dynamics in a networked population, particularly related to conditions under which that population moves toward consensus.

Work by DeMarzo and colleagues [20] present a model of belief dynamics in a fixed, weighted network. This work focuses on capturing beliefs rather than preferences. In this context, beliefs are entirely dependent on recent beliefs, without a notion of inherent, internal preference. That is, users can be stubborn and stick to their most recent belief but do not gravitate towards an underlying truth. This locality allows authors to analyze the evolution of beliefs with Markov machine techniques. We introduce fixed, inherent individual preferences, which allows us to model group compromise and any phenomena wherein personal values affect the outcome. Our results are similar in structure, differing in that our fixed point is dependent on individual preferences rather than initial values.

II. PROBLEM STATEMENT AND MODEL

Consider a network of agents, representing users of a social networking site or application in which each user communicates with his friends, but not necessarily the entire network. We represent the strategy of each agent, generically, his opinion or behavior related to a given question of interest, as a continuous value on a fixed interval. For example, opinions might reflect the perceived value of a piece of art to various collectors at an auction [21], or the sentiment within a population toward a political candidate or party [22]. Privacy preferences for a piece of content shared among multiple users might also be framed as opinions, where the corresponding interval represents access settings ranging from totally private to completely public [5].

Assume that each user (agent) has an initial belief representing her individual preference independent of external influence. Upon interacting with their peers in the social network, users move to align their opinions with their cohort, a process that may be viewed as interpersonal influence or peer pressure. Over time, this peer pressure to agree increases, particularly in cases where facilitating agreement is important and time-constrained, e.g., choosing a group leader or managing shared assets. This is especially true in the case when artificial agents are reflecting the views of their users, but required to converge to consensus in order to ensure a decision is executed. The urge to conform is also common in human populations [12, 13], where fitting in often involves mimicking of behaviors, with a potentially increasing pressure experienced, depending on the situation. Figure ?? gives a schematic representation of this trajectory. Individuals with different inherent preferences (left) co-evolve and, over time, come to consensus (center, right).

Given this, we assume that each user’s utility is modeled by a decomposable objective function with additive internal and external pressures, consisting of:

1. Inherent user comfort (distance between initial belief and a candidate choice)

2. Peer pressure (a time-dependent measure of distance between candidate choice and weighted average of peers’ beliefs).

Individual objective functions represent social stress, and therefore each user seeks to minimize this function at each (discrete) moment in time.

To account for the heterogeneity in individual weighting of peer influences, we include unique weights for individual pairwise links in the network, as well as a weight each individual places on their initial belief (which we call stubbornness). This formulation allows the model to be generalizable across a variety of application scenarios. It also allows us to leverage additional (historical, observed or reported) data about individual users and relationships.

Assume that the agents’ network is represented by a simple graph \( G = (V,E) \) where vertexes \( V \) are users (or agents), and edges \( E \) are the social (communications) connections between them. It is clear that disconnected sections of the graph are independent, so we assume that \( G \) is connected. For the remainder of the paper, let \( V = \{1,2,\ldots,n\} \), so \( E \) is a subset of the two-element subsets of \( V \). Assume we are given a set of non-negative vertex weights \( s_i \) and positive edge weights \( w_{ij} \) respectively for \( i,j \in V \). In addition, we assume at least one vertex weight is positive. Without loss of generality, assume the range of opinions to be the interval \([0,1]\) and each user has a private constant preference \( x_i^+ \in [0,1] \).

The user’s opinion value at time \( k \) is \( x_i^{(k)} \in [0,1] \). The set of all
such values are denoted by the vector $x^{(k)}$ while the set of constant private preferences is $x^+$. Each agent selects its next choice by minimizing an objective function representing social stress:

$$J_i(x_i^{(k)}, x^{(k-1)}, k) = s_i f_C \left( x_i^{(k)} - x_i^+ \right) + \rho^{(k)} \sum_{j=1}^n w_{ij} f_P \left( x_i^{(k)} - x_j^{(k-1)} \right)$$

Here:

$$s_i f_C(x_i^{(k)} - x_i^+)$$

is the internal stress felt by User $i$ as a result of deviations from her preferred state $x_i^+$. The quantity:

$$\rho^{(k)} \sum_{j=1}^n w_{ij} f_P \left( x_i^{(k)} - x_j^{(k-1)} \right)$$

is the social stress experienced by User $i$ as a result of deviations from her peers. In Expression [11], $\rho^{(k)}$ is the peer pressure factor, denoting how strongly others’ previous opinions influence person $i$. The more pressure there is to come to a consensus, the larger $\rho^{(k)}$ becomes, and therefore disagreement causes more stress.

We require both $f_C$ and $f_P$ to be quasi-convex so that the stress minimization has an unconstrained solution. For this paper, we assume that $f_C(z) = f_P(z) = z^2$, consistent with many economic models [23]. Under this assumption, the first order necessary conditions are sufficient for minimizing $J_i(x_i^{(k)}, x^{(k-1)}, k)$ and after differentiation we solve:

$$s_i \left( x_i^{(k)} - x_i^+ \right) + \rho^{(k)} \sum_{j=1}^n w_{ij} \left( x_i^{(k)} - x_j^{(k-1)} \right) = 0$$

Let $d_i = \sum_{j=1}^n w_{ij}$ be the weighted degree of vertex $i$, then:

$$x_i^{(k)} = \frac{s_i x_i^+ + \rho^{(k)} \sum_{j=1}^n w_{ij} x_j^{(k-1)}}{s_i + \rho^{(k)} \sum_{j=1}^n w_{ij}}$$

minimizes $J_i(x_i^{(k)}, x^{(k-1)}, k)$.

Let $A$ be the $n \times n$ weighted adjacency matrix of $G$. In addition, let $D$ be the $n \times n$ matrix with $d_i$ on the diagonal and let $S$ be the $n \times n$ matrix with $s_i$ on the diagonal. Using these terms, the above recurrences can be written as:

$$x^{(k)} = \left( S + \rho^{(k)} D \right)^{-1} \left( S x^+ + \rho^{(k)} A x^{(k-1)} \right)$$

We say that the agents converge to consensus $x^*$ if there is some $N$ so that for all $n > N$, $\|x^* - x^{(n)}\| < \epsilon$ for some small $\epsilon > 0$. This represents meaningful compromise on the issue under consideration.

III. THEORETICAL RESULTS

In this section, we consider the update rule in Equation [1] as a sequence of contraction mappings each with its own equilibrium. We then show that all these equilibria converge to a weighted average. The result rests on a variation of the contraction mapping theorem, which will be stated in the sequel.

Our analysis requires two lemmas, which are instrumental for our convergence results. First lemma is proved in Chapter 13, [24].

**Lemma 1.** If $L = D - A$ is the weighted graph Laplacian, then $L$ has an eigenvalue 0 with multiplicity 1 and a corresponding eigenvector $1$ where $1$ is the vector of all 1’s.

**Proof.** By definition, the graph Laplacian, is a positive semidefinite symmetric matrix. In addition, the only eigenvector with eigenvalue 0 is the vector of all 1s, written $1$.

Since $S$ is symmetric and $s_i \geq 0$, then $S + \rho^{(k)} L$ is positive semidefinite as well. Pick $x \in \mathbb{R}^n$ such that $x^T (S + \rho^{(k)} L) x = 0$. Then $x^T (S + \rho^{(k)} L) x = x^T S x + \rho^{(k)} x^T L x$, so since $S$ and $L$ are positive semidefinite and $\rho^{(k)} > 0$, this means that $x^T S x + x^T L x = 0$.

Since $L$ is symmetric, by the spectral theorem it has an orthonormal basis of eigenvectors $\{b_1, \ldots, b_n\}$ with associated eigenvectors $\{\lambda_1, \ldots, \lambda_n\}$. Then because $L$ is positive semidefinite $\lambda_i \geq 0$. This means that $x^T L x = \sum_{i=1}^n \lambda_i (x^T b_i)^2$. Because $x^T L x = 0$, if $\lambda_i \neq 0$, then $x^T b_i = 0$.

It follows that $x$ is the eigenvector of $L$ with eigenvalue 0; that is $x = c \cdot 1$ for some constant $c$, and therefore $x^T S x = c^2 \sum_{i=1}^n s_i$. Since $s_i \geq 0$ and not all $s_i$ are zero, this means that $c = 0$, so $x = 0$. This means that $S + \rho^{(k)} L$ is positive definite, and therefore invertible.

Define:

$$F_k(x) = \left( S + \rho^{(k)} D \right)^{-1} \left( S x^+ + \rho^{(k)} A x \right)$$

and let:

$$G_k = F_k \circ F_{k-1} \circ \cdots \circ F_1$$

Then $x^{(k)} = F_k(x^{(k-1)})$ and $x^{(k)} = G_k(x^{(0)})$. That is, iterating these $F_k$ captures the evolution of $x^{(k)}$. We show that for each $k$, $F_k$ is a contraction and therefore has a fixed point by the Banach Fixed Point Theorem [23].

**Theorem 3.** For all $k$, $F_k$ is a contraction map with fixed point given by $x^{(k)} = (S + \rho^{(k)} L)^{-1} S x^+$. 

**Proof.** Let $B$ be the $(n+1) \times (n+1)$ matrix given by adding a row and column to $\rho^{(k)} (S + \rho^{(k)} D)^{-1} A$ as follows:

$$B = \begin{bmatrix}
\rho^{(k)} (S + \rho^{(k)} D)^{-1} A & (S + \rho^{(k)} D)^{-1} A \\
0 & 1
\end{bmatrix}$$
The rows of $B$ sum to 1. To see this, replace $x^+$ and $s_{i1}$ in Equation 4 with 1. Thus $B$ is a stochastic matrix for a Markov process with a single absorbing state. Since $G$ is connected and not all $s_i$ are equal to 0, a transition exists from each state to the steady state; thus from any starting state, convergence to the steady state is guaranteed. This means that $\lim_{i<\infty}(\rho^{(k)}(S + \rho^{(k)}D)^{-1}A)^i = 0$, so $A$ is a convergent matrix. Equivalently, if $\|\cdot\|$ denotes the matrix operator norm, then, $\|\rho^{(k)}(S + \rho^{(k)}D)^{-1}A\| < 1$. Therefore for any $x, y \in [0, 1]^n$:

$$\|F_k(x) - F_k(y)\| = \|(S + \rho^{(k)}D)^{-1}\rho^{(k)}(x - y)\| \leq \|(S + \rho^{(k)}D)^{-1}\rho^{(k)}A\|\|x - y\|$$

That is, $F_k$ is a contraction map on a compact set, so by the Banach fixed-point theorem, it has a unique fixed point $x^{(k)}$.

Let $\mathbf{x}^{(k)}$ be that fixed point. Then $x^{(k)} = F_k(x^{(k)})$.

Rearranging the terms yields,

$$(S + \rho^{(k)}D)x^{(k)} - \rho^{(k)}Ax = (S + \rho^{(k)}L)x^{(k)} = Sx^+.$$

Therefore:

$$x^{(k)} = (S + \rho^{(k)}L)^{-1}Sx^+.$$

This completes the proof.

The following lemma will allow us to consider the matrices $(S + \rho^{(k)}L)^{-1}$ for $k \in \{1, 2, \ldots\}$ in $GL_n(\mathbb{R})$ (the Lie group of invertible $n \times n$ real matrices) as perturbations. This enables effective approximations of asymptotic behaviors.

**Lemma 4.** Let $\{b_1, \ldots, b_n\}$ an orthonormal basis of $\mathbb{R}^n$. Also let $M : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible symmetric linear transformation (invertible square matrix) and $\{u_1, \ldots, u_n\}$ be a set of unit vectors such that for a small constant $\delta$, $M^{-1}b_1 = \lambda b_1 + O(\delta)u_1$ and $M^{-1}b_j = O(\delta)u_j$ for $j \neq 1$.

Then if $\|v\| = 1$, and $s \in \mathbb{R}$, then unless $(M + svv^T)$ is not invertible, there exists a set of unit vectors $\{u'_1, \ldots, u'_n\}$ such that $(M + svv^T)^{-1}b_1 = \frac{\lambda}{1 + s\lambda(v^Tv)^{-1}b_1 + O(\delta)u'_1}$ and $(M + svv^T)^{-1}b_j = O(\delta)u'_j$ for $j \neq 1$.

Before proceeding to the proof of this result, based on the Sherman-Morrison formula, we note that we will establish an instance of the necessary conditions of this lemma in Theorem 5. Thus the lemma is not vacuous.

**Proof of Lemma 4** Since $\{b_1, \ldots, b_n\}$ is an orthonormal basis, $v = \sum_{i=1}^n a_i b_i$, where $a_i = v_i^T b_i$. This means that $M^{-1}v = \sum_{i=1}^n a_i M^{-1}(b_i) = \lambda_1 b_1 + O(\delta)\sum_{i=1}^n a_i u_i$. By Cauchy-Schwarz, $|a_i| \leq \|v\| \|b_i\| = 1$, so by the triangle inequality, $\|\sum_{i=1}^n a_i u_i\| \leq n$. Then letting $u = \sum_{i=1}^n a_i u_i$, we have that $M^{-1}v = \lambda_1 b_1 + O(\delta)u$, where $\|u\| \leq 1$.

By the Sherman-Morrison formula,

$$(M + svv^T)^{-1} = M^{-1} - \frac{svM^{-1}vv^T}{1 + sv^Tv}.$$

Using this, and choosing each $u'_j$ to be an appropriate rescaling of the $O(\delta)$ terms yields:

$$(M + svv^T)^{-1}(b_j) = M^{-1}b_j - \frac{svM^{-1}vv^TM^{-1}b_j}{1 + sv^Tv} = \lambda b_j + O(\delta)u_j - \frac{sv^T(\lambda a_1 b_1 + O(\delta)u_1)(\lambda a_1 b_1 + O(\delta)u_1)}{1 + s\lambda^2 + O(\delta)} = \lambda b_j + O(\delta)u_j = \frac{\lambda}{1 + s\lambda(v^Tv)^{-1}b_1 + O(\delta)u'_1} = O(\delta)u'_j.$$

This completes the proof.

The results in the previous section give insight into the motion of fixed points as $\rho^{(k)}$, i.e. peer-pressure, increases. We now show that these points converge to the average of the agents’ initial preferences, weighted by the stubbornness of each agent.

The following theorem states that under increasing, unbounded peer pressure, the limit of individual strategies converges to the average value of weighted inherent preferences.

**Theorem 5.** If $\lim_{k \to \infty} \rho^{(k)} = \infty$, then:

$$\lim_{k \to \infty} x^{(k)} = \frac{\sum_{i=1}^n s_i x_i^+}{\sum_{i=1}^n s_i} \cdot 1.$$

**Proof.** Since $G$ is a graph, the Laplacian $L$ is a positive semidefinite symmetric matrix, and therefore has an orthonormal basis of eigenvectors $\{b_1, \ldots, b_n\}$ with real eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Since $G$ is connected, only a single eigenvalue $\lambda_1 = 0$ and the associated unit eigenvector is $b_1 = \frac{1}{\sqrt{n}}1$.

Since every vector is an eigenvector of the identity matrix $I, \{b_1, \ldots, b_n\}$ are orthonormal bases of eigenvectors for $I + \rho^{(k)}L$ with eigenvalues $\{1, 1 + \rho^{(k)}\lambda_2, \ldots, 1 + \rho^{(k)}\lambda_n\}$. But then $(I + \rho^{(k)}L)^{-1}$ has the same basis of eigenvectors, with eigenvalues $\{\frac{1}{1 + \rho^{(k)}\lambda_2}, \ldots, \frac{1}{1 + \rho^{(k)}\lambda_n}\}$.

As $\rho^{(k)} \to \infty, \frac{1}{1 + \rho^{(k)}\lambda_j} \to 0$ for each $j \neq 1$. In particular, for any $\delta > 0$, for sufficiently large $\rho^{(k)}, I + \rho^{(k)}L$ satisfies the conditions of Lemma 4 with $\lambda = 1$. 


Let \( I + \rho^{(k)}L = M_0 \). Then, for each \( l \) up to \( n \), let \( M_l = (M_{l-1} + (s_l - 1)e_l e_l^T) \) where \( e_l \) is the \( l \)th vector of the standard basis. Since \( e_l e_l^T \) is the zero matrix with a one in the \( l \)th place on the diagonal, \( \sum_{l=1}^n (s_l - 1)e_l e_l^T = S - I \) and therefore \( M_n = (I + \rho^{(k)}L) + \sum_{i=1}^n (s_i - 1)e_i e_i^T = S + \rho^{(k)}L \).

By iterating Lemma 4 with \( s = s_l - 1 \) and \( v = e_l \), we have that for each \( l \) there is a \( \lambda_l \) such that \( M_l^{-1}b_1 = \lambda_l b_1 + O(\delta)u_i^{(l)} \) and \( M_l^{-1}b_1 = O(\delta)u_i^{(l)} \) for \( j \neq 1 \).

Since \( e_l^T b_1 = \frac{1}{\sqrt{n}} \), Lemma 4 gives the recurrence:

\[
\lambda_l \frac{1}{\sqrt{n}} - \frac{\lambda_{l-1}}{1 + \lambda_{l-1} - \frac{n}{n-1}}
\]

Solving this recurrence with \( \lambda_0 = 1 \) yields:

\[
M_l^{-1}b_1 = \frac{n}{n + \sum_{k=1}^l (s_k - 1)} b_1 + O(\delta)u_i^{(l)}
\]

Since \( \sum_{k=1}^n (s_k - 1) = tr(S) - n \), it is clear that

\[
M_n^{-1}b_1 = \frac{n}{tr(S)} b_1 + O(\delta)u_i^{(n)}
\]

Therefore, for \( u = \sum_{i=1}^n b_i^T Sx^+ u_i^{(n)} \):

\[
\overline{x}^{(k)} = \frac{n}{tr(S)} b_1^T Sx^+ b_1 + O(\delta) \sum_{i=1}^n b_i^T Sx^+ u_i^{(n)}
\]

\[
= \frac{1}{tr(S)} Sx^+ + O(\delta)u
\]

\[
= \frac{1}{\sum_{i=1}^n s_i x_i^+} Sx^+ + O(\delta)u
\]

Since \( \delta \to 0 \) as \( \rho^{(k)} \to \infty \), this means that if \( \lim_{k \to \infty} \rho^{(k)} = \infty \), then:

\[
\lim_{k \to \infty} \overline{x}^{(k)} = \frac{1}{\sum_{i=1}^n s_i x_i^+} 1.
\]

This completes the proof.

Since peer pressure increases in each step, no single \( F_k \) is sufficient to model the process of convergence. We need to show that this is the attractive fixed point of the entire process. From Theorem 2 of [20] and the fact that \( F_k \) are contractions whose fixed points converge, we have:

**Theorem 6.** Let \( G_k = F_k \circ G_{k-1} = F_k \circ F_{k-1} \circ \cdots \circ F_1 \) for each \( k \geq 0 \). Then \( G = \lim_{k \to \infty} G_k \) is a constant function with value \( \lim_{k \to \infty} x^{(k)} \), and the convergence is uniform.

This means that in the case of increasing and unbounded peer pressure, all the agents’ opinions always converge to consensus. In addition, the value of this consensus is the average of their preferences weighted by their stubbornness. This is irrespective of the weighting of the edges in the network, so long as the network is connected.

In the case of increasing but bounded peer pressure, we have:

\[
\lim_{k \to \infty} \rho^{(k)} \leq \rho^*
\]

Further, this limit always exists by monotone convergence. Intuitively, this means the influence of others is limited, and that personal preferences will always slightly skew the opinions of others. Again, this is consistent with social influence theories on bounded peer pressure and tradeoffs with comfort level [13].

**Theorem 7.** Suppose \( \rho^{(k)} \) is increasing and bounded and that:

\[
\lim_{k \to \infty} \rho^{(k)} = \rho^*,
\]

then

\[
\lim_{k \to \infty} x^{(k)} = (S + \rho^*L)^{-1}Sx^+.
\]

**Proof.** Since \( \rho^{(k)} \) is increasing and bounded, it converges to a finite number \( \rho^* \) by monotone convergence. From Lemma 2 \( (S + \rho^*L) \) is defined and invertible for all \( k \). Since matrix inversion is continuous in \( GL_0(\mathbb{R}) \), by Theorem 6

\[
\lim_{k \to \infty} x^{(k)} = \lim_{k \to \infty} \overline{x}^{(k)}
\]

\[
= \lim_{k \to \infty} (S + \rho^{(k)}L)^{-1}Sx^+
\]

\[
= (S + \lim_{k \to \infty} \rho^{(k)}L)^{-1}Sx^+
\]

\[
= (S + \rho^*L)^{-1}Sx^+
\]

The above theorem tells us that if peer pressure is increasing and bounded, then the agents’ opinions converge to a fixed distribution, which may not be a consensus, but is easily computable from the initial preferences. In this case, the shape of the network is important for determining the limit distribution, as the edge weights factor into the Laplacian. Now that we have proven that convergent points exist, we can now analyze their efficiency. Define the utility of these convergent points to be the sum of the stress of the agents when the state \( x \) is constant. Formally:

\[
U(x) = \sum_i \lim_{k \to \infty} J_i(x_i, x, k)
\]

\[
= \sum_{i=1}^n s_i (x_i - x_i^+)^2 + 2 \left( \sum_{i \in E} (x_i - x_j)^2 \right) \lim_{k \to \infty} \rho^k
\]

\[
= (x - x^+)^T S(x - x^+) + 2 \lim_{k \to \infty} \rho^k x^T L x
\]

\[
= \lim_{k \to \infty} x^T (S + 2\rho^k L) x - 2x^T S x^+ + (x^+)^T S x^+
\]
Using the formulation of utility shown above, the cost of anarchy is then the ratio of the utility of the convergent point from Theorems 5 and 7 to the utility (stress) minimizing solution. The following lemma is immediately clear from by construction of $U$ and $J_i$:

**Lemma 8.** The global utility function $U(x)$ is convex. □

Leveraging the above Lemma on the utility function, we are now ready to present our main optimality results on the convergence point conditions.

**Theorem 9.** The convergent point $\lim_{k \to \infty} x^{(k)}$ minimizes utility if and only if $\lim_{k \to \infty} \rho^k = \infty$.

**Proof.** By Lemma 3, $S + 2\rho^k L$ is invertible when $\rho^k > 0$. This means that by first order conditions of optimality, $x = \lim_{k \to \infty} (S + 2\rho^k L)^{-1} S x^+$ minimizes $U(x)$, since $U(x)$ is convex.

If $\lim_{k \to \infty} \rho^k < \infty$, then $\lim_{k \to \infty} x^{(k)}$ is not optimal. However if $\lim_{k \to \infty} \rho^k = \infty$, by Theorem 5 $\sum_{i=1}^n \frac{1}{\sum_{i=1}^n s_i x_i^+} = 1$ minimizes $U(x)$, and thus $\lim_{k \to \infty} x^{(k)}$ is optimal. □

**Corollary 10.** The cost of anarchy is $1$ if and only if $\lim_{k \to \infty} \rho^k = \infty$.

This means that the agents’ opinions converge to the optimal solution if and only if the peer pressure increases without bound. If the peer pressure is bounded, then the agents will not converge to an efficient distribution.

### IV. EXPERIMENTAL RESULTS

To validate our models, we used data from the well-known Social Evolution experiment [27]. The experiment tracked the everyday life of approximately 80 students in an undergraduate dormitory over 6 months using mobile phones and surveys, to mine spatio-temporal behavioral patterns and the co-evolution of individual behaviors and social network structure.

The collected data represents approximately 80 students, over 6 months. The dataset includes proximity, location, and call logs, collected through a mobile application. Also included is Sociometric survey data for relationships, political opinions, recent smoking behavior, attitudes towards exercise and fitness, attitudes towards diet, attitudes towards academic performance, current confidence and anxiety level, and music sharing.

The derived social network graph (shown Figure 1) represents each student as a node; an edge is present between two nodes if either student noted any level of interaction during the surveys. Edge weights were derived based on the level of interaction recorded between the students in the surveys, as well as the number of surveys in which the interaction appeared.

Political opinion was modeled on a $[0, 1]$ scale, with lower numbers representing reported Democratic leanings and higher numbers representing Republican preferences. Individual scores were assigned based on reported political party, preferred candidate and likelihood of voting (prior to the election), as well as who they voted for and their approval rating of Barack Obama (after the election). Each month’s survey questions were examined individually to put together a timeline of each person’s political views. The results of the first survey were used as proxy for their inherent personal preference, prior to peer influence.

Finally, individual stubbornness/lack of susceptibility to peer pressure was approximated using reported interest in politics on the first survey administered, as well as stated likelihood of voting. These survey questions were independent of those used in determining political preferences.

Given a list of $\rho^{(k)}$, students’ preferences were simulated aligning each iteration of play to one day in the survey period. Simulated preferences were compared to surveyed preferences for each month, and the distances were summed to get a single score for each list of $\rho^{(k)}$. This function of $\rho^{(k)}$ was minimized with fminsearch in Matlab and the best-fit peer pressure values were found to be increasing, with a best fit line $\rho^{(k)} = 1.06 * k - 11.96$ and $r^2 = .9886$ (see Figure 2).

![FIG. 1: Social network of the political data set showing a high-degree of social connectedness.](Image)

![FIG. 2: Fit of estimated peer pressure showing a clear increase over time, validating the primary hypothesis of the paper that individual consensus occurs because peer pressure increases on each round.](Image)
Notice that peer pressure increases steadily. As we further discuss in Section V, this result validates both the structure of the model, as well as the underlying assumption about increasing peer pressure.

V. CONCLUSION

In this paper, we lay the groundwork for a flexible framework which we can theoretically understand for modeling consensus dynamics in a group of connected agents. This framework is able to incorporate available knowledge about the system. The theoretical results enable modeling on any undirected network with any number of agents, each of whom can be flexible or rigid in their private preferences.

Yet, the fidelity of behavioral models, in particular in the context of social networks and in the presence of peer influence, will depend on a priori knowledge about its heterogeneous agents and the nature of relationships amongst them. In our experiments, by applying this model to surveyed political and musical opinions, not only did the subjects show marked convergence towards consensus, but also showed evidence that peer pressure does appear to increase over time. This validates the structure of the model, as well as the assumption that peer pressure increases over time. This suggests that the framework and model can be used to model opinion dynamics in other settings, such as social networks. In addition, the theoretical results can be applied (such as in a shared privacy recommender) in order to efficiently arrive at consensus.

To more accurately model similar phenomena, in addition to securing more comprehensive data, outside influences could be taken into account. People’s political ideologies remain largely unchanged over short spans of time, but new legislation or press can cause fluctuations. Due to the iterative nature of this model, accounting for those external influences could decrease error significantly.

Future studies should also address the limitation that the network is undirected, so as to account for imbalanced social influence. In addition, the network is assumed to remain static during the convergence process, with connections independent of the agents’ opinions. Sufficiently different opinions could cause enough stress between agents so as to cause them to reduce or even sever the tie between them. A dynamic network model as in [28] could accommodate this kind of network update.

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Finally, it would be interesting to study corresponding control problem, in which we are given a $x^*$, the desired convergence point and we can control a subset of agents reporting values $(x_i)$, stubbornness $(s_i)$ or initial value $(x_i^0)$ to determine conditions under which opinion control is possible. This problem becomes more interesting if the other agents attempt to determine whether agent(s) is (are) intentionally attempting to manipulate the opinion value $x^*$.

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