NOTES ON THE GABRIEL-ROITER MEASURE

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In his proof of the first Brauer-Thrall conjecture [6], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [2]. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed[4] that the formalism of Gabriel and Roiter is also useful for studying the representations of algebras having unbounded representation type.

In these notes we present a purely combinatorial definition of the Gabriel-Roiter measure and combine this with an axiomatic characterization; see also [3]. Given a finite dimensional algebra $\Lambda$, the Gabriel-Roiter measure is characterized as a universal morphism $\text{ind}\, \Lambda \rightarrow P$ of partially ordered sets. The map is defined on the isomorphism classes of finite dimensional indecomposable $\Lambda$-modules and is a suitable refinement of the length function $\text{ind}\, \Lambda \rightarrow \mathbb{N}$ which sends a module to its composition length. The axiomatic treatment is complemented by a recursive definition of the Gabriel-Roiter measure.

The second part of these notes discusses the Gabriel-Roiter measure for a fixed abelian length category. This is the original setting for Gabriel’s work. In particular, Gabriel’s main property of the measure is proved. This is used to extend the Gabriel-Roiter measure from indecomposable to arbitrary objects. Our main example is the category of finite dimensional $\Lambda$-modules over some finite dimensional algebra $\Lambda$. We report on Ringel’s work [4, 5], presenting for instance his refinement of the first Brauer-Thrall conjecture.

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1. CHAINS AND LENGTH FUNCTIONS

1.1. The Gabriel-Roiter measure. There are a number of possible approaches to define the Gabriel-Roiter measure. Fix a partially ordered set $(S, \leq)$ which is equipped with a length function $\lambda: S \rightarrow \mathbb{N}$. We start off by defining the Gabriel-Roiter measure for $S$ as a morphism $\mu: S \rightarrow P$ of partially ordered sets which refines the length function $\lambda$. Let us stress right away that the values $\mu(x)$ for $x \in S$ are not relevant. All we need to know is whether for a pair $x, y$ of elements in $S$, the relation $\mu(x) \leq \mu(y)$ holds or not. This is the essence of a measure and we make this precise in the following definition.

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1Cf. the footnote on p. 91 of [2].
Definition. Let $(S, \leq)$ be a partially ordered set. A measure $\mu$ for $S$ is a relation on $S$, written $\mu(x) \leq \mu(y)$, for a pair $x, y$ of elements in $S$, such that for all $x, y, z$ in $S$ the following holds:

(M1) $\mu(x) \leq \mu(y)$ and $\mu(y) \leq \mu(z)$ imply $\mu(x) \leq \mu(z)$.
(M2) $\mu(x) \leq \mu(y)$ or $\mu(y) \leq \mu(x)$.
(M3) $x \leq y$ implies $\mu(x) \leq \mu(y)$.

We write $\mu(x) = \mu(y)$ if both $\mu(x) \leq \mu(y)$ and $\mu(y) \leq \mu(x)$ hold.

A measure $\mu$ for $S$ gives rise to an equivalence relation on $S$ as follows: Call two elements $x$ and $y$ equivalent if $\mu(x) = \mu(y)$. The set $S/\mu$ of equivalence classes is totally ordered via $\mu$ and the canonical map $S \to S/\mu$ is a morphism of partially ordered sets.

Conversely, any morphism $\phi: S \to P$ to a totally ordered set $P$ gives rise to a measure $\mu$ for $S$ provided one defines $\mu(x) \leq \mu(y)$ if $\phi(x) \leq \phi(y)$ holds.

In this section we present three different approaches defining the Gabriel-Roiter measure as a morphism

\[ \lambda: S \to \text{Ch}(\mathbb{N}) \]

where $\text{Ch}(\mathbb{N})$ denotes the lexicographically ordered set of finite sets of natural numbers. We complement this by a recursive and an axiomatic definition. Note that all three concepts are equivalent in the sense that they yield the same measure for $S$.

1.2. The lexicographic order on finite chains. Let $(S, \leq)$ be a partially ordered set. A subset $X \subseteq S$ is a chain if $x_1 \leq x_2$ or $x_2 \leq x_1$ for each pair $x_1, x_2 \in X$. For a finite chain $X$, we denote by $\min X$ its minimal and by $\max X$ its maximal element, using the convention

\[ \max \emptyset < x < \min \emptyset \quad \text{for all} \quad x \in S. \]

We write $\text{Ch}(S)$ for the set of all finite chains in $S$ and let

\[ \text{Ch}(S, x) := \{ X \in \text{Ch}(S) \mid \max X = x \} \quad \text{for} \quad x \in S. \]

On $\text{Ch}(S)$ we consider the lexicographic order which is defined by

\[ X \leq Y \iff \min(Y \setminus X) \leq \min(X \setminus Y) \quad \text{for} \quad X, Y \in \text{Ch}(S). \]

Remark. (1) $X \subseteq Y$ implies $X \leq Y$ for $X, Y \in \text{Ch}(S)$.

(2) Suppose that $S$ is totally ordered. Then $\text{Ch}(S)$ is totally ordered. We may think of $X \in \text{Ch}(S) \subseteq \{0, 1\}^S$ as a string of 0s and 1s which is indexed by the elements in $S$.

The usual lexicographic order on such strings coincides with the lexicographic order on $\text{Ch}(S)$.

Example. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Q}$ be the set of rational numbers together with the natural ordering. Then the map

\[ \text{Ch}(\mathbb{N}) \to \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x} \]

is injective and order preserving, taking values in the interval $[0, 1]$. For instance, the subsets of $\{1, 2, 3\}$ are ordered as follows:

\[ \{\} < \{3\} < \{2\} < \{2, 3\} < \{1\} < \{1, 3\} < \{1, 2\} < \{1, 2, 3\}. \]

We need the following properties of the lexicographic order.

Lemma. Let $X, Y \in \text{Ch}(S)$ and $X^\ast := X \setminus \{\max X\}$.
Notes on the Gabriel-Roiter Measure

(1) \( X^* = \max\{X' \in \text{Ch}(S) \mid X' < X \text{ and } \max X' < \max X\} \).

(2) If \( X^* < Y \) and \( \max X \geq \max Y \), then \( X \leq Y \).

Proof. (1) Let \( X' < X \) and \( \max X' < \max X \). We show that \( X' \leq X^* \). This is clear if \( X' \subseteq X^* \). Otherwise, we have
\[
\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),
\]
and therefore \( X' \leq X^* \).

(2) The assumption \( X^* < Y \) implies by definition
\[
\min(Y \setminus X^*) < \min(X^* \setminus Y).
\]

We consider two cases. Suppose first that \( X^* \subseteq Y \). If \( X \subseteq Y \), then \( X \leq Y \). Otherwise, \( \min(Y \setminus X) < \max X = \min(X \setminus Y) \) and therefore \( X < Y \). Now suppose that \( X^* \nsubseteq Y \). We use again that \( \max X \geq \max Y \), exclude the case \( Y \subseteq X \), and obtain
\[
\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).
\]

Thus \( X \leq Y \) and the proof is complete. \( \square \)

1.3. Length Functions. Let \((S, \leq)\) be a partially ordered set. A length function on \( S \) is by definition a map
\[
\lambda: S \rightarrow \mathbb{N} = \{1, 2, 3, \ldots\}
\]
such that \( x < y \) in \( S \) implies \( \lambda(x) < \lambda(y) \). A length function \( \lambda: S \rightarrow \mathbb{N} \) induces for each \( x \in S \) a map
\[
\text{Ch}(S, x) \rightarrow \text{Ch}(\mathbb{N}, \lambda(x)), \quad X \mapsto \lambda(X),
\]
and therefore the following chain length function
\[
S \rightarrow \text{Ch}(\mathbb{N}), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) \mid X \in \text{Ch}(S, x)\}.
\]

This chain length function is by definition the Gabriel-Roiter measure for \( S \) with respect to \( \lambda \).

We continue with a list of basic properties (C0) – (C5) of \( \lambda^* \).

1.4. A Recursive Definition. The following property (C0) of the chain length function \( \lambda^*: S \rightarrow \text{Ch}(\mathbb{N}) \) can be used to define \( \lambda^* \) by induction on the length of the elements in \( S \). We take this as our second definition of the Gabriel-Roiter measure for \( S \) with respect to \( \lambda \). Note that \( \lambda^*(x) = \{\lambda(x)\} \) if \( x \) is a minimal element of \( S \).

Proposition. Let \( x \in S \).

(C0) \( \lambda^*(x) = \max_{x' < x} \lambda^*(x') \cup \{\lambda(x)\} \).

Proof. Let \( X = \lambda^*(x) \) and note that \( \max X = \lambda(x) \). The assertion follows from Lemma 1.2 because we have
\[
X \setminus \{\max X\} = \max\{X' \in \text{Ch}(\mathbb{N}) \mid X' < X \text{ and } \max X' < \max X\}.
\]

\( \square \)
1.5. Basic properties. Let $\lambda: S \to \mathbb{N}$ be a length function and $\lambda^*: S \to \text{Ch}(\mathbb{N})$ the induced chain length function. The following basic properties suggest to think of $\lambda^*$ as a refinement of $\lambda$.

Proposition. Let $x, y \in S$.

(C1) $x \leq y$ implies $\lambda^*(x) \leq \lambda^*(y)$.
(C2) $\lambda^*(x) = \lambda^*(y)$ implies $\lambda(x) = \lambda(y)$.
(C3) $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\lambda^*(x) \leq \lambda^*(y)$.

Proof. Suppose $x \leq y$ and let $X \in \text{Ch}(S, x)$. Then $Y = X \cup \{y\} \in \text{Ch}(S, y)$ and we have $\lambda(X) \leq \lambda(Y)$ since $\lambda(X) \subseteq \lambda(Y)$. Thus $\lambda^*(x) \leq \lambda^*(y)$. If $\lambda^*(x) = \lambda^*(y)$, then $\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y)$.

To prove (C3), we use (C0) and apply Lemma 1.2 with $X = \lambda^*(x)$ and $Y = \lambda^*(y)$. In fact, $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ implies $X^* < Y$, and $\lambda(x) \geq \lambda(y)$ implies $\max X \geq \max Y$. Thus $X \leq Y$.

We state some further elementary properties of the map $\lambda^*$.

Proposition. Let $x, y \in S$.

(C4) $\lambda^*(x) \leq \lambda^*(y)$ or $\lambda^*(y) \leq \lambda^*(x)$.
(C5) $\{\lambda^*(x) \mid x \in S \text{ and } \lambda(x) \leq n\}$ is finite for all $n \in \mathbb{N}$.

Proof. (C4) is clear since $\text{Ch}(\mathbb{N})$ is totally ordered. (C5) follows from the fact that $\{X \in \text{Ch}(\mathbb{N}) \mid \max X \leq n\}$ is finite for all $n \in \mathbb{N}$.

The map $\lambda^*$ induces a measure $\mu$ for $S$ in the sense of Definition 1.1.

Corollary. The chain length function $\lambda^*$ induces via

$$\mu(x) \leq \mu(y) :\iff \lambda^*(x) \leq \lambda^*(y) \text{ for } x, y \in S$$

a measure for $S$. Moreover, we have for all $x, y$ in $S$

$$\mu(x) = \mu(y) \iff \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) = \lambda(y).$$

Proof. (C1) and (C4) imply that the map $\lambda^*$ induces a measure $\mu$ for $S$. The characterization for $\mu(x) = \mu(y)$ follows from (C0).

1.6. An axiomatic definition. Let $\lambda: S \to \mathbb{N}$ be a length function. We present an axiomatic characterization of the induced chain length function $\lambda^*$. Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that $\lambda^*$ refines $\lambda$. We take this as our third definition of the Gabriel-Roiter measure for $S$ with respect to $\lambda$.

Theorem. Let $\lambda: S \to \mathbb{N}$ be a length function. Then there exists a map $\mu: S \to P$ into a partially ordered set $P$ satisfying for all $x, y \in S$ the following:

(P1) $x \leq y$ implies $\mu(x) \leq \mu(y)$.
(P2) $\mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y)$.
(P3) $\mu(x') < \mu(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\mu(x) \leq \mu(y)$.

Moreover, for any map $\mu': S \to P'$ into a partially ordered set $P'$ satisfying the above conditions, we have for all $x, y$ in $S$

$$\mu'(x) \leq \mu'(y) \iff \mu(x) \leq \mu(y) \iff \lambda^*(x) \leq \lambda^*(y).$$
Proof. We have seen in \([1,3]\) that \(\lambda^*\) satisfies (P1) – (P3). So it remains to show that for any map \(\mu: S \to P\) into a partially ordered set \(P\), the conditions (P1) – (P3) uniquely determine the relation \(\mu(x) \leq \mu(y)\) for any pair \(x, y \in S\). In fact, we claim that (P1) – (P3) imply \(\mu(x) \leq \mu(y)\) or \(\mu(y) \leq \mu(x)\). We proceed by induction on the length of the elements in \(S\). For elements of length \(n = 1\), the assertion is clear. In fact, \(\lambda(x) = 1 = \lambda(y)\) implies \(\mu(x) = \mu(y)\) by (P3). Now let \(n > 1\) and assume the assertion is true for all elements \(x \in S\) of length \(\lambda(x) < n\). We choose for each \(x \in S\) of length \(\lambda(x) \leq n\) a Gabriel-Roiter filtration, that is, a sequence

\[
x_1 < x_2 < \ldots < x_{\gamma(x)-1} < x_{\gamma(x)} = x
\]

in \(S\) such that \(x_1\) is minimal and \(\max_{x' < x_i} \mu(x') = \mu(x_{i-1})\) for all \(1 < i \leq \gamma(x)\). Such a filtration exists because the elements \(\mu(x')\) with \(x' < x\) are totally ordered. Now fix \(x, y \in S\) of length at most \(n\) and let \(I = \{i \geq 1 \mid \mu(x_i) = \mu(y_i)\}\). We consider \(r = \max I\) and put \(r = 0\) if \(I = \emptyset\). There are two possible cases. Suppose first that \(r = \gamma(x)\) or \(r = \gamma(y)\). If \(r = \gamma(x)\), then \(\mu(x) = \mu(x_r) = \mu(y_r) \leq \mu(y)\) by (P1). Now suppose \(\gamma(x) \neq r \neq \gamma(y)\). Then we have \(\lambda(x_{r+1}) \neq \lambda(y_{r+1})\) by (P2) and (P3). If \(\lambda(x_{r+1}) > \lambda(y_{r+1})\), then we obtain \(\mu(x_{r+1}) < \mu(y_{r+1})\), again using (P2) and (P3). Iterating this argument, we get \(\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1})\). From (P1) we get \(\mu(x) < \mu(y)\). Thus \(\mu(x) \leq \mu(y)\) or \(\mu(y) \leq \mu(x)\) and the proof is complete. \(\Box\)

2. ABELIAN LENGTH CATEGORIES

2.1. Additive categories. A category \(\mathcal{A}\) is additive if every finite family \(X_1, X_2, \ldots, X_n\) of objects has a coproduct

\[
X_1 \oplus X_2 \oplus \ldots \oplus X_n,
\]

each set \(\text{Hom}_\mathcal{A}(A, B)\) is an abelian group, and the composition maps

\[
\text{Hom}_\mathcal{A}(B, C) \times \text{Hom}_\mathcal{A}(A, B) \to \text{Hom}_\mathcal{A}(A, C)
\]

are bilinear.

2.2. Abelian categories. An additive category \(\mathcal{A}\) is abelian, if every map \(\phi: A \to B\) has a kernel and a cokernel, and if the canonical factorization

\[
\begin{array}{ccc}
\text{Ker} \phi & \xrightarrow{\phi'} & A \\
\downarrow & & \downarrow \phi \\
\text{Coker} \phi' & \xrightarrow{\bar{\phi}} & \text{Ker} \phi'' \\
& & \downarrow \phi''
\end{array}
\]

of \(\phi\) induces an isomorphism \(\bar{\phi}\).

Example. The category of modules over any associative ring is an abelian category.

2.3. Subobjects. Let \(\mathcal{A}\) be an abelian category. We say that two monomorphisms \(X_1 \to X\) and \(X_2 \to X\) are equivalent, if there exists an isomorphism \(X_1 \to X_2\) making the following diagram commutative.

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X_2 \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi'} & X
\end{array}
\]

An equivalence class of monomorphisms into \(X\) is called a subobject of \(X\). Given subobjects \(X_1 \to X\) and \(X_2 \to X\), we write \(X_1 \subseteq X_2\) if there is a morphism \(X_1 \to X_2\) making
the above diagram commutative. An object $X \neq 0$ is simple if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

2.4. Length categories. Let $\mathcal{A}$ be an abelian category. An object $X$ has finite length if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X,$$

that is, each $X_i/X_{i-1}$ is simple. In this case the length of a composition series is an invariant of $X$ by the Jordan-Hölder Theorem; it is called the length of $X$ and is denoted by $\ell(X)$. For instance, $X$ is simple if and only if $\ell(X) = 1$. Note that $X$ has finite length if and only if $X$ is both artinian (i.e. satisfies the descending chain condition on subobjects) and noetherian (i.e. satisfies the ascending chain condition on subobjects).

An abelian category is called a length category if all objects have finite length and the isomorphism classes of objects form a set.

An object $X \neq 0$ is called indecomposable if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$. A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

We denote by $\text{ind}\mathcal{A}$ the set of isomorphism classes of indecomposable objects of $\mathcal{A}$.

Example. (1) Let $\Lambda$ be a artinian ring. Then the category of finitely generated $\Lambda$-modules form a length category which we denote by $\text{mod}\Lambda$.

(2) Let $k$ be a field and $Q$ be any quiver. Then the finite dimensional $k$-linear representations of $Q$ form a length category.

3. The Gabriel-Roiter measure

Let $\mathcal{A}$ be an abelian length category. We give the definition of the Gabriel-Roiter measure for $\mathcal{A}$ which is due to Gabriel [2] and was inspired by the work of Roiter [6]. Then we discuss some specific properties, including Ringel’s results about Gabriel-Roiter inclusions [4].

3.1. The definition. Let $\mathcal{A}$ be an abelian length category. The isomorphism classes of objects of $\mathcal{A}$ are partially ordered via the subobject relation

$$X \subseteq Y :\iff \exists \text{ a monomorphism } X \rightarrow Y.$$

We consider the length function $\ell: \text{ind}\mathcal{A} \rightarrow \mathbb{N}$ which takes an object $X$ to its composition length $\ell(X)$. Then the induced chain length function $\ell^*: \text{ind}\mathcal{A} \rightarrow \text{Ch}(\mathbb{N})$ is by definition the Gabriel-Roiter measure for $\mathcal{A}$. We will only work with this definition when making explicit computations. Otherwise, we take the induced measure in the sense of Definition [1], which is characterized as follows.

Theorem. Let $\mathcal{A}$ be an abelian length category. The Gabriel-Roiter measure induces a relation on $\text{ind}\mathcal{A}$. This is the unique transitive relation on $\text{ind}\mathcal{A}$ satisfying for all objects $X,Y$ the following:

\begin{align*}
(\text{GR1}) \quad & X \subseteq Y \implies \mu(X) \leq \mu(Y). \\
(\text{GR2}) \quad & \mu(X) = \mu(Y) \implies \ell(X) = \ell(Y). \\
(\text{GR3}) \quad & \mu(X') < \mu(Y) \quad \text{for all } X' \subseteq X \quad \text{and } \ell(X) \geq \ell(Y) \implies \mu(X) \leq \mu(Y).
\end{align*}
Here we use the following convention: We write \( \mu(X) = \mu(Y) \) if \( \mu(X) \leq \mu(Y) \) and \( \mu(Y) \leq \mu(X) \) hold. Moreover, we write \( \mu(X) < \mu(Y) \) if \( \mu(X) \leq \mu(Y) \) and \( \mu(X) \neq \mu(Y) \) hold.

**Proof.** The relation \( \mu(X) = \mu(Y) \) defines an equivalence relation on \( \text{ind} \, \mathcal{A} \) and we denote by \( \text{ind} \, \mathcal{A}/\mu \) the set of equivalence classes. This set is partially ordered via \( \mu \). The canonical map \( \text{ind} \, \mathcal{A} \rightarrow \text{ind} \, \mathcal{A}/\mu \) is a morphism of partially ordered sets satisfying the conditions (P1) – (P3) from Theorem [1.6]. Suppose we have another transitive relation, written \( \mu'(X) \leq \mu'(Y) \) for \( X, Y \) in \( \text{ind} \, \mathcal{A} \), and satisfying (GR1) – (GR3). We obtain a second morphism \( \text{ind} \, \mathcal{A} \rightarrow \text{ind} \, \mathcal{A}/\mu' \) of partially ordered sets satisfying the conditions (P1) – (P3), and we deduce from Theorem [1.6] that for all \( X, Y \)

\[
\mu'(X) \leq \mu'(Y) \quad \iff \quad \mu(X) \leq \mu(Y).
\]

□

**Example.** (1) Let \( X \in \mathcal{A} \) be uniserial, that is, \( X \) has a unique composition series. Then \( \ell^*(X) = \{1, 2, \ldots, \ell(X)\} \).

(2) Let \( X \in \mathcal{A} \) be an indecomposable object of length at most three. Then

\[
\ell^*(X) = \begin{cases} 
\{1\} & \text{if} \; \ell(X) = 1, \\
\{1, 2\} & \text{if} \; \ell(X) = 2, \\
\{1, 2, 3\} & \text{if} \; \ell(X) = 3 \text{ and } \ell(\text{soc} \, X) = 1, \\
\{1, 3\} & \text{if} \; \ell(X) = 3 \text{ and } \ell(\text{soc} \, X) \neq 1.
\end{cases}
\]

Here, \( \text{soc} \, X \) denotes the socle of \( X \), that is, the sum of all simple subobjects.

(3) Let \( k \) be a field and consider the category \( \mathcal{A} \) of \( k \)-linear representations of the following quiver.

\[
1 \leftarrow 2 \rightarrow 3
\]

An indecomposable representation \( V_1 \leftarrow V_2 \rightarrow V_3 \) is determined by its dimension vector \((d_1, d_2, d_3)\), where \( d_i = \dim_k V_i \). The following Hasse diagram displays the partial order on \( \text{ind} \, \mathcal{A} \), where the layer indicates the length of each object.

```
   3
  / \  \
 /   \ \
/     \ 
1     2
```

\[
\ell^*(010) = \ell^*(100) = \ell^*(001) = \{1\} < \ell^*(111) = \{1, 3\} < \ell^*(110) = \ell^*(011) = \{1, 2\}
\]

3.2. **Basic properties.** Recall from [1.5] that we have established the following property of the Gabriel-Roiter measure.

(\text{GR4}) \( \mu(X) \leq \mu(Y) \) or \( \mu(Y) \leq \mu(X) \) for \( X, Y \) in \( \text{ind} \, \mathcal{A} \).

(\text{GR5}) \( \{\mu(X) \mid X \in \text{ind} \, \mathcal{A} \text{ and } \ell(X) \leq n\} \) is finite for all \( n \in \mathbb{N} \).

Next we discuss further properties of the Gabriel-Roiter measure which depend on the fact that \( \mathcal{A} \) is a length category.
3.3. Gabriel-Roiter filtrations. Let $X, Y \in \text{ind}\mathcal{A}$. We say that $X$ is a Gabriel-Roiter predecessor of $Y$ if $X \subseteq Y$ and $\mu(X) = \max_{Y' \subseteq Y} \mu(Y')$. Note that each object $Y \in \text{ind}\mathcal{A}$ which is not simple admits a Gabriel-Roiter predecessor, by (GR4) and (GR5). A Gabriel-Roiter predecessor $X$ of $Y$ is usually not unique, but the value $\mu(X)$ is determined by $\mu(Y)$.

A sequence $X_1 \subset X_2 \subset \ldots \subset X_{n-1} \subset X_n = X$ in $\text{ind}\mathcal{A}$ is called a Gabriel-Roiter filtration of $X$ if $X_1$ is simple and $X_{i-1}$ is a Gabriel-Roiter predecessor of $X_i$ for all $1 < i \leq n$. Clearly, each $X$ admits such a filtration and the values $\mu(X_i)$ are uniquely determined by $X$.

**Proposition.** Let $X, Y \in \text{ind}\mathcal{A}$.

(Gr6) $X \in \text{ind}\mathcal{A}$ is simple if and only if $\mu(X) \leq \mu(Y)$ for all $Y \in \text{ind}\mathcal{A}$.

(Gr7) Suppose that $\mu(X) < \mu(Y)$, Then there are $Y' \subseteq Y'' \subseteq Y$ in $\text{ind}\mathcal{A}$ such that $Y'$ is a Gabriel-Roiter predecessor of $Y''$ with $\mu(Y') \leq \mu(Y) < \mu(Y'')$ and $\ell(Y') \leq \ell(Y)$.

**Proof.** For (Gr6), one uses that each indecomposable object has a simple subobject. To prove (Gr7), fix a Gabriel-Roiter filtration $Y_1 \subset Y_2 \subset \ldots \subset Y_n = Y$ of $Y$. We have $\mu(Y_1) \leq \mu(X)$ because $Y_1$ is simple. Using (GR4), there exists some $i$ such that $\mu(Y_i) \leq \mu(X) < \mu(Y_{i+1})$. Now put $Y'' = Y_i$ and $Y' = Y_{i+1}$. Comparing the filtration of $Y$ with a Gabriel-Roiter filtration of $X$ (as in the proof of Theorem [1,6]), we find that $\ell(Y') \leq \ell(X)$. \qed

**Example.** Let $X \in \mathcal{A}$ be uniserial. Then the composition series is a Gabriel-Roiter filtration of $X$.

3.4. The main property. The following main property of the Gabriel-Roiter measure is crucial for the whole theory.

**Proposition** (Gabriel). Let $X, Y_1, \ldots, Y_r \in \text{ind}\mathcal{A}$.

(Gr8) Suppose that $X \subseteq Y = \oplus_{i=1}^r Y_i$. Then $\mu(X) \leq \max \mu(Y_i)$ and $X$ is a direct summand of $Y$ if $\mu(X) = \max \mu(Y_i)$.

**Proof.** The proof only uses the properties (GR1) – (GR3) of $\mu$. Fix a monomorphism $\phi: X \to Y$. We proceed by induction on $n = \ell(X) + \ell(Y)$. If $n = 2$, then $\phi$ is an isomorphism and the assertion is clear. Now suppose $n > 2$. We can assume that for each $i$ the $i$th component $\phi_i: X \to Y_i$ of $\phi$ is an epimorphism. Otherwise choose for each $i$ a decomposition $Y'_i = \oplus_{j} Y_{ij}$ of the image of $\phi_i$ into indecomposables. Then we use (GR1) and have $\mu(X) \leq \max \mu(Y_{ij}) \leq \max \mu(Y_i)$ because $\ell(X') + \ell(Y') < n$ and $Y_{ij} \subseteq Y_i$ for all $j$. Now suppose that each $\phi_i$ is an epimorphism. Thus $\ell(X) \geq \ell(Y_i)$ for all $i$. Let $X' \subset X$ be a proper indecomposable subobject. Then $\mu(X') \leq \max \mu(Y_i)$ because $\ell(X') + \ell(Y') < n$, and $X'$ is a direct summand if $\mu(X') = \max \mu(Y_i)$. We can exclude the case that $\mu(X') = \max \mu(Y_i)$ because then $X'$ is a proper direct summand of $X$, which is impossible. Now we apply (GR3) and obtain $\mu(X) \leq \max \mu(Y_i)$. Finally, suppose that $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$ for some $k$. We claim that we can choose $k$ such that $\phi_k$ is an epimorphism. Otherwise, replace all $Y_i$ with $\mu(X) = \mu(Y_i)$ by the image $Y_i' = \oplus_{j} Y_{ij}$ of $\phi_k$ as before. We obtain $\mu(X) \leq \max \mu(Y_{ij}) < \mu(Y_k)$ since $Y_{kj} \subset Y_k$ for all $j$, using (GR1) and (GR2). This is a contradiction. Thus $\phi_k$ is an epimorphism.
and in fact an isomorphism because $\ell(X) = \ell(Y_k)$ by (GR2). In particular, $X$ is a direct summand of $\oplus Y_k$. This completes the proof.

**Corollary.** Let $X, Y \in \text{ind}\, A$ and suppose that $X \subseteq Y$ with $\mu(X) = \max_{Y' \subseteq Y} \mu(Y')$. If $X \subseteq U \subseteq Y$ in $A$, then $X$ is a direct summand of $U$.

**Proof.** Let $U = \oplus U_i$ be a decomposition into indecomposables. Now apply (GR8). We obtain $\mu(X) \leq \max \mu(U_i) < \mu(Y)$ and our assumption on $X \subset Y$ implies that $X$ is a direct summand of $U$.

**Example.** (1) Let $Y \in \text{ind}\, A$ and suppose that $\mu(X) \leq \mu(Y)$ for all $X \in \text{ind}\, A$. Then $Y$ is an injective object, because every monomorphism $Y \to Z$ splits by (GR8).

(2) Suppose that $A$ has a cogenerator $Q$, that is, each object in $A$ admits a monomorphism into a direct sum of copies of $Q$. Let $Q = \oplus Q_i$ be a decomposition into indecomposable objects. Then $\mu(X) \leq \max \mu(Q_i)$ for all $X \in \text{ind}\, A$.

The Gabriel-Roiter measure $\ell^*: \text{ind}\, A \to \text{Ch}(\mathbb{N})$ for $A$ can be extended to a measure defined for all objects in $A$, not only the indecomposable ones. Let $X = \oplus X_i$ be an object written as a direct sum of indecomposable objects. Then we define $\ell^*(X) = \max \ell^*(X_i)$.

**Corollary.** The relation $\mu(X) \leq \mu(Y) \iff \ell^*(X) \leq \ell^*(Y)$ for $X, Y \in A$ induces a measure for the set of isomorphism classes of $A$.

**Proof.** We need to verify (M1) – (M3) from Definition [1.1]. The first two conditions are automatic and the third is an immediate consequence of (GR8).

3.5. **Gabriel-Roiter inclusions.** Let $X, Y \in \text{ind}\, A$. An inclusion $X \subseteq Y$ is called **Gabriel-Roiter inclusion** if $\mu(X) = \max_{Y' \subseteq Y} \mu(Y')$. Thus we have a Gabriel-Roiter inclusion $X \subseteq Y$ if and only if $X$ is a Gabriel-Roiter predecessor of $Y$.

**Proposition (Ringel).** Let $X, Y \in \text{ind}\, A$ and suppose that $X \subseteq Y$ is a Gabriel-Roiter inclusion. Then $Y/X$ is an indecomposable object.

**Proof.** Let $Z = Y/X$ and assume that $Z = Z' \oplus Z''$ with $Z'' \neq 0$. We obtain the following commutative diagram with exact rows and columns.

```
0 \to X \to Y' \to Z' \to 0
|     |     |     |     |
0 \to X \to Y \to Z \to 0
|     |     |     |     |
0 \to Z'' \to Z'' \to 0
```

We have $X \subseteq Y' \subset Y$ and therefore the monomorphism $X \to Y'$ splits by Corollary 3.4.

Thus the inclusion $Z' \to Z$ factors through $Y \to Z$ via a split monomorphism $Z' \to Y$. We conclude that $Z' = 0$ since $Y$ is indecomposable.

**Remark.** The argument is borrowed from Auslander and Reiten. They show that the cokernel of an irreducible monomorphism between indecomposable objects is indecomposable.

**Corollary.** Let $Y$ be an indecomposable object in $\mathcal{A}$ which is not simple. Then there exists a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ such that $X$ and $Z$ are indecomposable.

**Proof.** Take $X \subset Y$ with $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. \hfill \Box

4. **Finiteness results**

In this section, Ringel’s refinement of the first Brauer-Thrall conjecture is presented [4]. More precisely, we prove a structural result about the partial order of the values of the Gabriel-Roiter measure.

4.1. **Covariant finiteness.** A subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *covariantly finite* if every object $X \in \mathcal{A}$ admits a left $\mathcal{C}$-approximation, that is, a map $X \to Y$ with $Y \in \mathcal{C}$ such that the induced map $\text{Hom}(Y, \mathcal{C}) \to \text{Hom}(X, \mathcal{C})$ is surjective for all $C \in \mathcal{C}$. We have also the dual notion: a subcategory $\mathcal{C}$ is *contravariantly finite* if every object in $\mathcal{A}$ admits a right $\mathcal{C}$-approximation.

**Lemma.** Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$ which is closed under taking direct sums and subobjects. Then $\mathcal{C}$ is a covariantly finite subcategory of $\mathcal{A}$.

**Proof.** Fix $X \in \mathcal{A}$. Let $X' \subseteq X$ be minimal among the kernels of all maps $X \to Y$ with $Y \in \mathcal{C}$. Then the canonical map $X \to X/X'$ is a left $\mathcal{C}$-approximation. \hfill \Box

**Remark.** The proof shows that the inclusion functor $\mathcal{C} \to \mathcal{A}$ admits a left adjoint $F : \mathcal{A} \to \mathcal{C}$ which takes $X \in \mathcal{A}$ to $X/X'$. Note that the adjunction map $X \to FX$ is a left $\mathcal{C}$-approximation.

Let $M$ be any set of values $\mu(X)$. Then we define the subcategory $\mathcal{A}(M) := \{ X \in \mathcal{A} \mid \mu(X) \in M \}$.

**Proposition** (Ringel). Let $M$ be a set of values $\mu(X)$ which is closed under predecessors, that is, $\mu(X_1) \leq \mu(X_2)$ and $\mu(X_2) \in M$ implies $\mu(X_1) \in M$. Then $\mathcal{A}(M)$ is a covariantly finite subcategory of $\mathcal{A}$.

**Proof.** The subcategory $\mathcal{A}(M)$ is closed under taking subobjects by (GR8). \hfill \Box

4.2. **Almost split morphisms.** A map $\phi : X \to Y$ in $\mathcal{A}$ is called *left almost split* if $\phi$ is not a split monomorphism and every map $X \to Y'$ in $\mathcal{A}$ which is not a split monomorphism factors through $\phi$. Dually, a map $\psi : Y \to Z$ is called *right almost split* if $\psi$ is not a split epimorphism and every map $Y' \to Z$ which is not a split epimorphism factors through $\psi$. For example, if $\mathcal{A} = \text{mod} \Lambda$ for some artin algebra $\Lambda$, then every indecomposable object $X \in \mathcal{A}$ admits a left almost split map starting at $X$ and a right almost split map ending at $X$; see [1] Cor. V.1.17.
4.3. Immediate successors. Let $X \in \text{ind } A$. An immediate successor of $\mu(X)$ is by definition a minimal element in
\[ \{ \mu(Y) \mid Y \in \text{ind } A \text{ and } \mu(X) < \mu(Y) \}. \]

**Lemma.** Let $X, Y \in \text{ind } A$ and suppose that $X$ is a Gabriel-Roiter predecessor of $Y$. If $X \to X$ is a left almost split map in $A$, then $Y$ is a factor object of $X$.

**Proof.** The monomorphism $X \to Y$ factors through $X \to X$ via a map $\phi: X \to Y$. Let $U$ be the image of $\phi$. Applying Corollary 3.4 we find that $U = Y$. \hfill \Box

**Proposition.** Let $X \in \text{ind } A$ and suppose there exists $n_X \in \mathbb{N}$ such that each $V \in \text{ind } A$ with $\mu(V) \leq \mu(X)$ and $\ell(V) \leq \ell(X)$ admits a left almost split map $V \to \bar{V}$ with $\ell(\bar{V}) \leq n_X$. Then there exists an immediate successor of $\mu(X)$ provided that $\mu(X)$ is not maximal.

**Proof.** Let $\mu(X) < \mu(Y)$. We apply (GR7) and find $Y' \subseteq Y$ in $\text{ind } A$ such that $Y'$ is a Gabriel-Roiter predecessor of $Y$ with $\mu(Y') \leq \mu(X) < \mu(Y'' \leq \mu(Y)$ and $\ell(Y') \leq \ell(X)$. The preceding lemma implies $\ell(Y'') \leq n_X$, and (GR5) implies that the number of values $\mu(Y'')$ is finite. Thus there exists a minimal element among those $\mu(Y'')$. \hfill \Box

**Corollary** (Ringel). Let $\Lambda$ be an artin algebra and $X \in \text{ind } \Lambda$. Then there exists an immediate successor of $\mu(X)$ provided that $\mu(X)$ is not maximal.

**Proof.** Use that there exists $n_{\Lambda} \in \mathbb{N}$ having the following property: for each indecomposable $V \in \text{mod } \Lambda$, there exists a left almost split map $V \to \bar{V}$ satisfying $\ell(\bar{V}) \leq n_{\Lambda}\ell(V)$. In fact, one takes $n_{\Lambda} = pq$, where $p$ denotes the maximal length of an indecomposable projective $\Lambda$-module and $q$ denotes the maximal length of an indecomposable injective $\Lambda$-module; see [1] Prop. V.6.6]. \hfill \Box

4.4. A finiteness criterion. We present a criterion for a subcategory $C$ of $A$ such that the number of indecomposable objects in $C$ is finite. This is based on the following classical lemma.

**Lemma** (Harada-Sai). Let $n \in \mathbb{N}$. A composition $X_1 \to X_2 \to \ldots \to X_{2^n}$ of non-invertible maps between indecomposable objects of length at most $n$ is zero.

**Proof.** See [1] Cor. VI.1.3]. \hfill \Box

**Proposition.** Let $\mathcal{A}$ be a length category with left almost split maps and only finitely many isomorphism classes of simple objects. Suppose that $\mathcal{C}$ is a subcategory such that

1. $\mathcal{C}$ is covariantly finite, and
2. there exists $n \in \mathbb{N}$ such that $\ell(X) \leq n$ for all indecomposable $X \in \mathcal{C}$.

Then there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$.

**Proof.** We claim that we can construct all indecomposable objects $X \in \mathcal{C}$ in at most $2^n$ steps from the finitely many simple objects in $\mathcal{A}$ as follows. Choose a non-zero map $S \to X$ from a simple object $S$ and factor this map through the left $\mathcal{C}$-approximation $S \to S'$. Take an indecomposable direct summand $X_0$ of $S'$ such that the component $S \to X_0 \to X$ of the composition $S \to S' \to X$ is non-zero. Stop if $X_0 \to X$ is an isomorphism. Otherwise take a left almost split map $X_0 \to Y_0$ and a left $\mathcal{C}$-approximation $Y_0 \to Z_0$. The map $X_0 \to X$ factors through the composition $X_0 \to Y_0 \to Z_0$ and we choose an
indecomposable direct summand $X_1$ of $Z_0$ such that the component $X_0 \to Y_0 \to X_1 \to X$ is non-zero. Again, we stop if $X_1 \to X$ is an isomorphism. Otherwise, we continue as before and obtain in step $r$ a sequence of non-invertible maps

$$X_0 \to X_1 \to X_2 \to \ldots \to X_r$$

such that the composition is non-zero. The Harada-Sai lemma implies that $r < 2^n$ because $\ell(X_i) \leq n$ for all $i$ by our assumption. Thus $X$ is isomorphic to $X_i$ for some $i < 2^n$, and we obtain $X$ in at most $2^n$ steps, having in each step only finitely many choices by taking an indecomposable direct summand. We conclude that $\mathcal{C}$ has only a finite number of indecomposable objects.

\[\square\]

**Remark.** This classical argument provides a quick proof of the first Brauer-Thrall conjecture; it is due to Auslander and Yamagata.

### 4.5. The initial segment.

**Theorem** (Ringel). Let $\mathcal{A}$ be a length category such that $\text{ind} \mathcal{A}$ is infinite. Suppose also that $\mathcal{A}$ has only finitely many isomorphism classes of simple objects and that every indecomposable object admits a left almost split map. Then there exist infinitely many values $\mu(X_1) < \mu(X_2) < \mu(X_3) < \ldots$ of the Gabriel-Roiter measure for $\mathcal{A}$ having the following properties.

1. If $\mu(X) \neq \mu(X_i)$ for all $i$, then $\mu(X_i) < \mu(X)$ for all $i$.
2. The set $\{X \in \text{ind} \mathcal{A} \mid \mu(X) = \mu(X_i)\}$ is finite for all $i$.

**Proof.** We construct the values $\mu(X_i)$ by induction as follows. Take for $X_1$ any simple object. Observe that $\mu(X_1)$ is minimal among all $\mu(X)$ by (GR6) and that only finitely many $X \in \text{ind} \mathcal{A}$ satisfy $\mu(X) = \mu(X_1)$ because $\mathcal{A}$ has only finitely many simple objects. Now suppose that $\mu(X_1) < \ldots < \mu(X_n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leq i \leq n$. We can apply Proposition 4.3 and find an immediate successor $\mu(X_{n+1})$ of $\mu(X_n)$. It remains to show that the set $\{X \in \text{ind} \mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite. To this end consider $M = \{\mu(X_1), \ldots, \mu(X_{n+1})\}$. We know from Proposition 4.4 that $\mathcal{A}(M)$ is a covariantly finite subcategory. Clearly, $\ell(X)$ is bounded by $\text{max}(\ell(X_1), \ldots, \ell(X_{n+1}))$ for all indecomposable $X \in \mathcal{A}(M)$ by (GR2). We conclude from Proposition 4.4 that the number of indecomposables in $\mathcal{A}(M)$ is finite. Thus $\{X \in \text{ind} \mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite and the proof is complete.

\[\square\]

**Corollary** (Brauer-Thrall I). Let $\mathcal{A}$ be a length category satisfying the above conditions. Then for every $n \in \mathbb{N}$ there exists an indecomposable object $X \in \mathcal{A}$ with $\ell(X) > n$.

**Proof.** Use that for fixed $n \in \mathbb{N}$, there are only finitely many values $\mu(X)$ with $\ell(X) \leq n$, by (GR5).

\[\square\]

### 4.6. The terminal segment.

**Theorem** (Ringel). Let $\mathcal{A}$ be a length category such that $\text{ind} \mathcal{A}$ is infinite. Suppose also that $\mathcal{A}$ has a cogenerator (i.e. an object $Q$ such that each object in $\mathcal{A}$ admits a monomorphism into a direct sum of copies of $Q$) and that every indecomposable object admits a right almost split map. Then there exist infinitely many values $\mu(X^1) > \mu(X^2) > \mu(X^3) > \ldots$ of the Gabriel-Roiter measure for $\mathcal{A}$ having the following properties.

1. If $\mu(X) \neq \mu(X^i)$ for all $i$, then $\mu(X^i) > \mu(X)$ for all $i$.
2. The set $\{X \in \text{ind} \mathcal{A} \mid \mu(X) = \mu(X^i)\}$ is finite for all $i$.

**Remark.**
The proof is based on the following lemma.

Lemma (Auslander-Smalø). Let $A$ be a length category and let $X \in A$. Denote by $A_X$ the subcategory formed by all objects in $A$ having no indecomposable direct summand which is isomorphic to a direct summand of $X$. If every indecomposable direct summand of $X$ admits a right almost split map, then $A_X$ is contravariantly finite.

Proof. Let $X = \bigoplus_{i=1}^{n} X_{i_0}$ be a decomposition into indecomposables. It is sufficient to construct a right $A_X$-approximation for each indecomposable object $Z \in A$. We take the identity map if $Z \in A_X$. Otherwise, $Z$ is isomorphic to $X_{i_0}$ for some $i_0$ and we proceed as follows. Let $\phi_{i_0}: X_{i_0} \to X_{i_0}$ be a right almost split map and choose a decomposition

$$X_{i_0} = Y_{i_0} \oplus (\oplus_{i} X_{i_0 i_1})$$

such that $Y_{i_0} \in A_X$ and $i_0 i_1 \in \{1, \ldots, r\}$ for all $i$. Note that each map $V \to X_{i_0}$ with $V \in A_X$ factors through $\phi_{i_0}$. Also, each component $X_{i_0 i_1} \to X_{i_0}$ of $\phi_{i_0}$ is non-invertible. Now compose $\phi_{i_0}$ with $\id Y_{i_0} \oplus (\oplus_{i} \phi_{i_0 i_1})$ to obtain a map

$$Y_{i_0} \oplus (\oplus_{i} (Y_{i_0 i_1} \oplus (\oplus_{i} X_{i_0 i_1 i_2}))) \to Y_{i_0} \oplus (\oplus_{i} X_{i_0 i_1}) \to X_{i_0}.$$ 

Again, each map $V \to X_{i_0}$ with $V \in A_X$ factors through this new map, and each component $X_{i_0 i_1 i_2} \to X_{i_0 i_1}$ is non-invertible. We continue this procedure, compose this map with

$$\id Y_{i_0} \oplus (\oplus_{i} (\id Y_{i_0 i_1} \oplus (\oplus_{i} \phi_{i_0 i_1 i_2}))),$$

and so on. Now let $n = 2^m$ where $m = \max\{\ell(X_1), \ldots, \ell(X_r)\}$. Then the Harada-Sai lemma implies that any composition

$$X_{i_0 i_1 \cdots i_n} \to X_{i_0 i_1 \cdots i_{n-1}} \to \cdots \to X_{i_0 i_1} \to X_{i_0}$$

is zero. Thus the induced map

$$\bigoplus_{j=0}^{n} (\oplus_{i_1, i_2, \ldots, i_j} Y_{i_0 i_1 \cdots i_j}) \to X_{i_0}$$

is a right $A_X$-approximation of $X_{i_0}$. \hfill $\square$

Proof of the theorem. We construct the values $\mu(X^i)$ by induction as follows. Let $n \geq 0$ and suppose that $\mu(X^1) > \ldots > \mu(X^n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leq i \leq n$. Denote by $P$ the direct sum of all $X \in \ind A$ with $\mu(X) > \mu(X^n)$, and let $P = 0$ if $n = 0$. Choose a right $A_P$-approximation $P' \to Q$ and take for $X^{n+1}$ any indecomposable direct summand $P'$ such that $\mu(X)$ is maximal. Observe that every indecomposable object $X \in A_P$ is cogenerated by $Q$ and therefore by $P'$. Thus (GR8) implies that $\mu(X)$ is bounded by $\mu(X^{n+1})$. Moreover, if $\mu(X) = \mu(X^{n+1})$, then $X$ is isomorphic to a direct summand of $P'$. Thus $\{X \in \ind A \mid \mu(X) = \mu(X^{n+1})\}$ is finite and the proof is complete. \hfill $\square$

Let $A$ be an artin algebra of infinite representation type. Then $A = \text{mod} A$ satisfies the assumptions of Theorems 1.5 and 1.6. Let us summarize the structure of the partial order on the values of the Gabriel-Roiter measure as follows. We have

$$\ind A/\mu := \{\mu(X) \mid X \in \ind A\} = S_{\text{init}} \sqcup S_{\text{cent}} \sqcup S_{\text{term}} \cong \mathbb{N} \sqcup S_{\text{cent}} \sqcup \mathbb{N}^{\text{op}},$$

where the notation $S = S_1 \sqcup S_2$ for a poset $S$ means $S = S_1 \cup S_2$ and $x_1 < x_2$ for all $x_1 \in S_1, x_2 \in S_2$. 

4.7. The Kronecker algebra. Let \( \Lambda = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix} \) be the Kronecker algebra over an algebraically closed field \( k \). We consider the abelian length category which is formed by all finite dimensional \( \Lambda \)-modules. A complete list of indecomposable objects is given by the preprojectives \( P_n \), the regulars \( R_n(\alpha, \beta) \), and the preinjectives \( Q_n \); see [1, Thm. VIII.7.5]. More precisely,
\[
\text{ind } \Lambda = \{ P_n \mid n \in \mathbb{N} \} \cup \{ R_n(\alpha, \beta) \mid n \in \mathbb{N}, (\alpha, \beta) \in \mathbb{P}_k^1 \} \cup \{ Q_n \mid n \in \mathbb{N} \},
\]
and we obtain the following Hasse diagram.

The set of indecomposables is ordered via the Gabriel-Roiter measure as follows:
\[
\mu(Q_1) = \mu(P_1) < \mu(P_2) < \mu(P_3) < \ldots < \mu(R_1) < \mu(R_2) < \mu(R_3) < \ldots \]
\[
\ldots < \mu(Q_4) < \mu(Q_3) < \mu(Q_2)
\]

5. The Gabriel-Roiter measure for derived categories

Let \( \mathcal{A} \) be an abelian length category. We propose a definition of the Gabriel-Roiter measure for the bounded derived category \( \mathbf{D}^b(\mathcal{A}) \). The derived Gabriel-Roiter measure extends the Gabriel-Roiter measure for the underlying abelian category \( \mathcal{A} \).

5.1. The definition. The bounded derived category \( \mathbf{D}^b(\mathcal{A}) \) of \( \mathcal{A} \) is by definition the full subcategory of the derived category \( \mathbf{D}(\mathcal{A}) \) which is formed by all complexes \( X \) such that \( H^nX = 0 \) for almost all \( n \). Note that each object of \( \mathbf{D}^b(\mathcal{A}) \) admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism. We denote by \( \text{ind} \mathbf{D}^b(\mathcal{A}) \) the set of isomorphism classes of indecomposable objects of \( \mathbf{D}^b(\mathcal{A}) \).

We consider the functor
\[
\mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{A}, \quad X \mapsto H^*X = \oplus_{n \in \mathbb{Z}} H^nX,
\]
and the isomorphism classes of objects of \( \mathbf{D}^b(\mathcal{A}) \) are partially ordered via
\[
X \preceq Y :\iff \begin{cases} \text{there exists a map } X \to Y \text{ inducing} \\ \text{a monomorphism } H^*X \to H^*Y. \end{cases}
\]
We have the length function
\[
\ell_{H^*} : \text{ind} \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbb{N}, \quad X \mapsto \ell(H^*X)
\]
and the induced chain length function \( \ell^*_H : \text{ind} D^b(A) \to \text{Ch}(\mathbb{N}) \) is by definition the *Gabriel-Roiter measure* for \( D^b(A) \).

5.2. Derived versus abelian Gabriel-Roiter measure.

**Proposition.** The Gabriel-Roiter measure for \( D^b(A) \) extends the Gabriel-Roiter measure for \( A \). More precisely, the canonical functor \( A \to D^b(A) \) sending an object of \( A \) to the corresponding complex concentrated in degree zero induces an inclusion \( \text{ind} A \to \text{ind} D^b(A) \) of partially ordered sets, which makes the following diagram commutative.

\[
\begin{array}{ccc}
\text{ind} A & \xrightarrow{\text{inc}} & \text{ind} D^b(A) \\
\downarrow{\ell^*} & & \downarrow{\ell^*_H} \\
\text{Ch}(\mathbb{N}) & & \\
\end{array}
\]

**Proof.** Use the fact that the diagram

\[
\begin{array}{ccc}
\text{ind} A & \xrightarrow{\text{inc}} & \text{ind} D^b(A) \\
\downarrow{\ell} & & \downarrow{\ell^*_H} \\
\mathbb{N} & & \\
\end{array}
\]

is commutative and that \( \text{ind} A \) is closed under predecessors in \( \text{ind} D^b(A) \). \( \square \)

5.3. An alternative definition. For an alternative definition of the Gabriel-Roiter measure for \( D^b(A) \), consider the lexicographic order on

\[
\prod_{n \in \mathbb{N}} \mathbb{N} := \{ (x_n) \in \prod_{n \in \mathbb{N}} \mathbb{N} | x_n = 0 \text{ for almost all } n \},
\]

with \((x_n) \leq (y_n) \iff \begin{cases} x_i = y_i \text{ for all } i \in \mathbb{Z}, \text{ or} \\ x_i \leq y_i \text{ for } i = \min \{ n \in \mathbb{Z} | x_n \neq y_n \}. \end{cases}\)

Take instead of \( \ell^*_H \) the length function

\[
\lambda : \text{ind} D^b(A) \to \prod_{n \in \mathbb{N}}, \quad X \mapsto (\ell(H^n X)),
\]

and instead of \( \ell^*_H \) the induced chain length function

\[
\lambda^* : \text{ind} D^b(A) \to \text{Ch}(\prod_{n \in \mathbb{N}}).\]

We illustrate the difference between both definitions by taking a hereditary length category \( A \). Recall that \( A \) is *hereditary* if \( \text{Ext}_A^2(-,-) = 0 \). Then each indecomposable object of \( D^b(A) \) is isomorphic to a complex concentrated in a single degree. Identifying objects having the same Gabriel-Roiter measure, we obtain

\[
\text{ind} D^b(A)/\ell^*_H = \text{ind} A/\ell^*,
\]

whereas

\[
\text{ind} D^b(A)/\lambda^* = \ldots \sqcup \text{ind} A/\ell^* \sqcup \text{ind} A/\ell^* \sqcup \text{ind} A/\ell^* \sqcup \ldots .
\]
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