The Quantum Wasserstein Distance of Order 1

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Quantum spin systems

• Finite set of spins $\Lambda$ endowed with distance
• Associate to each spin $x \in \Lambda$ the local Hilbert space $\mathcal{H}_x = \mathbb{C}^d$
• Hamiltonian with finite-range interactions
• Gibbs state
$$\omega = \frac{e^{-\beta H}}{\text{Tr} \, e^{-\beta H}}$$
• If correlations decay sufficiently fast, $\omega$ satisfies TCI

$$\frac{1}{n} \| \rho - \omega \|_{W_1} \leq \sqrt{\frac{O(1)}{n}} \cdot S(\rho \| \omega)$$

• Holds at high enough temperature for any finite-range commuting Hamiltonian [see also Onorati, Rouzé, França, Watson, arXiv:2301.12946]
Equivalence of ensembles

- Canonical ensemble: Gibbs states
- Microcanonical ensemble: uniform convex combination of all states in energy shell
- Assume that Gibbs state satisfies TCI

\[ \frac{1}{n} \| \rho - \omega \|_{W_1} \leq \sqrt{\frac{c}{n} S(\rho \| \omega)} \]

- Then, any \( \rho \) with same average energy as \( \omega \) and approximately same entropy as \( \omega \) is close to \( \omega \)

\[ \text{Tr} [\rho H] = \text{Tr} [\omega H] \implies \frac{1}{n} \| \rho - \omega \|_{W_1} \leq \sqrt{\frac{c}{n} (S(\omega) - S(\rho))} \]

- Ok if fraction of states in shell is \( e^{-o(n)} \)
Quantum spin systems on $\mathbb{Z}^D$

- Associate to each $x \in \mathbb{Z}^d$ local Hilbert space $\mathcal{H}_x = \mathbb{C}^d$
- Associate to each $\Lambda \subset \subset \mathbb{Z}^D$ the Hilbert space
  $$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$$
- Algebra of operators acting on $\Lambda$: $\mathcal{U}_\Lambda$
- Local algebra
  $$\mathcal{U}_{\mathbb{Z}^D} = \bigcup_{\Lambda \subset \subset \mathbb{Z}^D} \mathcal{U}_\Lambda$$
- Quantum state: Positive unital linear functional on $\mathcal{U}_{\mathbb{Z}^D}$
- We consider translation-invariant states
- Marginal states $\rho_\Lambda \in \mathcal{S}_\Lambda : \text{Tr}_{\mathcal{H}_\Lambda} [\rho_\Lambda A] = \rho(A)$ $\forall A \in \mathcal{U}_\Lambda$
Interactions

- Interaction: collection of observables \( \{ h_\Lambda \in \mathcal{O}_\Lambda \}_{\Lambda \subseteq \mathbb{Z}^D} \)
- Hamiltonian of region \( \Lambda \):
  \[
  H^h_\Lambda = \sum_{X \subseteq \Lambda} h_X
  \]

- We consider translation-invariant interactions with finite local norm

  \[
  \| h \|_r = \sum_{0 \in \Lambda \subseteq \mathbb{Z}^D} e^{r(|\Lambda|^{-1})} \| h_\Lambda \|_\infty < \infty \quad r > 0
  \]

- Specific energy of TI state

  \[
  E_h(\rho) = \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{\rho(H^h_\Lambda)}{|\Lambda|}
  \]
Gibbs states

- Specific entropy of TI state
  \[ s(\rho) = \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{S(\rho_\Lambda)}{|\Lambda|} \]

- Equilibrium states of TI interaction: Maximizers of
  \[ s(\rho) - E_h(\rho) \]

- Always exist but in general are not unique

- Satisfy KMS condition

- Local Gibbs states (NOT equal to marginals of equilibrium states)
  \[ \omega^h_\Lambda = \frac{e^{-H^h_\Lambda}}{\text{Tr} e^{-H^h_\Lambda}} \in S_\Lambda \]

- \( \rho \) is equilibrium state iff
  \[ \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{S(\rho_\Lambda \| \omega^h_\Lambda)}{|\Lambda|} = 0 \]
The specific quantum $W_1$ distance

- Specific $W_1$ distance for TI quantum states
  \[
  w_1(\rho, \sigma) = \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{\|\rho_\Lambda - \sigma_\Lambda\|_{W_1}}{|\Lambda|}
  \]

- Lipschitz constant for TI quantum interactions
  \[
  \|h\|_L = \partial_0 \sum_{0 \in \Lambda \subset \subset \mathbb{Z}^D} h_\Lambda
  \]

- Duality
  \[
  w_1(\rho, \sigma) = \sup_{\|h\|_L \leq 1} (E_h(\rho) - E_h(\sigma))
  \]

- Continuity of the specific entropy
  \[
  |s(\rho) - s(\sigma)| \leq h_2(w_1(\rho, \sigma)) + w_1(\rho, \sigma) \ln(d^2 - 1)
  \]
$w_1$-Gibbs states

- TI state $\rho$ is $w_1$-Gibbs state of TI interaction $h$ if
  \[
  \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{\|\rho_\Lambda - \omega_\Lambda^h\|_{W_1}}{|\Lambda|} = 0
  \]

- If it exists, $w_1$-Gibbs state is unique and is equilibrium state!

- TI interaction $h$ satisfies TCI with constant $c$ if for any TI state $\rho$
  \[
  \limsup_{\Lambda \uparrow \mathbb{Z}^D} \frac{\|\rho_\Lambda - \omega_\Lambda^h\|_{W_1}^2}{|\Lambda|^2} \leq \frac{c}{2} \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{S(\rho_\Lambda\|\omega_\Lambda^h)}{|\Lambda|}
  \]

- In this case, $h$ has unique equilibrium state which is also $w_1$-Gibbs state

- TCI satisfied above critical temperature by any finite-range commuting interaction
Shallow quantum circuits

- Expand $W_1$ distance by at most twice the size of the largest light-cone of a qudit

$$\| U \rho U^\dagger - U \sigma U^\dagger \|_{W_1} \leq 2B(U) \| \rho - \sigma \|_{W_1}$$
Quadratic concentration for product states

• $\omega$ product state

\[
\text{Var}_\omega H \leq n \| H \|_L^2
\]

• $\rho$ output of quantum circuit with blow-up $B$

\[
\text{Var}_\rho H \leq 4n B^2 \| H \|_L^2
\]

• See [Anshu, Metger, arXiv:2209.02715] for Gaussian concentration of observables diagonal in computational basis
Combinatorial optimization

- Goal: find bit string that maximizes cost function $C$
- Local cost: sum of functions each depending on $O(1)$ bits
- Efficient classical algorithms usually achieve
  \[
  C = a C_{\text{max}} \quad 0 < a \leq 1
  \]
- **Example**: maximum cut problem, i.e., find the bipartition of a graph that maximizes the # of edges connecting the two parts
- Associate one bit to each vertex, set to 1 bits in second half of bipartition
- NP complete!
Variational quantum algorithms

- Associate one qubit to each bit, quantum Hamiltonian to cost function

\[ H = \sum_{x \in \{0,1\}^n} C(x) |x\rangle \langle x| \]

- Train parametric quantum circuit to generate high-energy states

- **Example:** Quantum Approximate Optimization Algorithm (QAOA)

- Alternate time evolution with \( H \) and mixing Hamiltonian

\[
\left( \prod_{k=1}^{P} e^{-i \gamma_k \sum_{i=1}^{n} X_i} e^{-i \beta_k H} \right) |+\rangle \otimes n
\]
Limitations of QAOA for MaxCut

• Toy model: $D$-regular bipartite graph ($\text{maxcut} = n \frac{D}{2}$)

• Technical assumption:

$$C(x) \geq \left( \frac{D}{2} - \sqrt{D-1} \right) \min \{|x|, n - |x|\} \quad \forall x \in \{0, 1\}^n$$

• Satisfied by Ramanujan expander graphs with $D \geq 3$ and for large $n$ by random $D$-regular graphs with high probability

• Observation [Bravyi et al., PRL 125, 260505 (2020)]: QAOA circuit commutes with $X^\otimes n$

• Probability distribution of output measurement symmetric wrt flipping all bits and cannot be concentrated on single string
Limitations of QAOA for MaxCut

• Result: if

\[ \text{Tr} [\rho H] \geq C_{\text{max}} \left( \frac{5}{6} + \frac{\sqrt{D - 1}}{3D} \right) \]

then the quadratic concentration inequality implies

\[ P \geq \frac{1}{2 \log (D + 1)} \log \frac{n}{576} = \Omega(\log n) \]

• Holds for any circuit and initial state commuting with \( X^{\otimes n} \)

• Improves Bravyi et al.

\[ P \geq \frac{1}{3(D + 1)} \log_2 \frac{n}{4096} \]
Further applications

- Quantum Wasserstein Generative Adversarial Networks
  [Kiani, GdP, Marvian, Liu, Lloyd, *Quantum Sci Technol* 7, 045002 (2022)]
- Design of quantum error correcting codes
  [Zoratti, GdP, Kiani, Nguyen, Marvian, Lloyd, Giovannetti, *Phys. Rev. A* 108, 022611 (2023)]
- Efficient learning of quantum states
  [Rouzé, França, arXiv:2107.03333]
  [Onorati, Rouzé, França, Watson, arXiv:2301.12946]
  [GdP, Klein, Pastorello, arXiv:2309.08426]
- Quantum rate-distortion theory