MULTIPARAMETRIC AND COLOURED EXTENSIONS OF THE QUANTUM GROUP $GL_q(N)$ AND THE YANGIAN ALGEBRA $Y(gl_N)$ THROUGH A SYMMETRY TRANSFORMATION OF THE YANG-BAXTER EQUATION

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Inspired by Reshetikhin’s twisting procedure to obtain multiparametric extensions of a Hopf algebra, a general ‘symmetry transformation’ of the ‘particle conserving’ $R$-matrix is found such that the resulting multiparametric $R$-matrix, with a spectral parameter as well as a colour parameter, is also a solution of the Yang-Baxter equation (YBE). The corresponding transformation of the quantum YBE reveals a new relation between the associated quantized algebra and its multiparametric deformation. As applications of this general relation to some particular cases, multiparametric and coloured extensions of the quantum group $GL_q(N)$ and the Yangian algebra $Y(gl_N)$ are investigated and their explicit realizations are also discussed. Possible interesting physical applications of such extended Yangian algebras are indicated.

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1. Introduction

The Yang-Baxter equation and its solutions have attracted much attention in recent years due to their connection with diverse subjects like exactly solvable systems, quantum groups, knot theory and conformal field theory.\textsuperscript{1−8} In particular, the universal $R$-matrix associated with a quantum algebra\textsuperscript{5} plays a significant role in generating solutions of the Yang-Baxter equation (YBE)

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu) ,$$

(1.1)

where $\lambda, \mu, \nu$ are spectral parameters, elements of $R(\lambda, \mu)$-matrix are c-numbers and the standard notations, like $R_{12}(\lambda, \mu) = R(\lambda, \mu) \otimes 1$, are used. Furthermore, by making some ‘twisting transformation’ on the related quantum algebra structure\textsuperscript{9}, one can also generate multiparametric, as well as coloured, solutions of YBE.\textsuperscript{9−11} Such a twisting transformation on a Hopf algebra $U$ yields a new universal matrix $\tilde{R}$, which is related to the original $R$-matrix as

$$\tilde{R} = F^{-1} R F^{-1}$$

(1.2)

where $F \in U \otimes U$ satisfies certain conditions and can be calculated explicitly for specific cases.\textsuperscript{9}

However, it is worth noting that, in spite of the presence of many additional parameters, solutions generated through the above mentioned ‘twisting transformation’ often preserve some symmetries of the initial $R$-matrix elements. To see this through a simple example, one may consider the spectral parameter independent solution of YBE (1.1)

$$R = q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} ,$$

(1.3)

which can be obtained from the fundamental representation of the universal $R$-matrix associated with the quantum algebra $U_q(sl(N))$. Twisting transformation on this $U_q(sl(N))$ algebra leads to a multiparametric generalization of (1.3) given by

$$\tilde{R} = q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i \neq j} \phi_{ij} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} ,$$

(1.4)

where $\phi_{ij}$ are additional deformation parameters satisfying the condition $\phi_{ij} \phi_{ji} = 1$. Now, it is to be noted that both the solutions (1.3) and (1.4) obey a ‘particle conserving’ symmetry - i.e., the elements like $R_{kl}^{ij}$ and $\tilde{R}_{kl}^{ij}$ are nonvanishing only if the pair of ‘outgoing particles’, $k$ and $l$, is a permutation of the pair of ‘incoming particles’, $i$ and $j$. In the present article we like to show that such a symmetry condition can be employed in a rather active way to construct new solutions ($\tilde{R}(\lambda, \mu)$) of YBE from a given initial solution ($R(\lambda, \mu)$); this is achieved through a simple transformation which entails a modification of the $R$-matrix elements by just multiplying each of them with an appropriate factor. This procedure, discussed in Sec. 2, which we call ‘symmetry transformation of YBE’, is applicable even for the cases of spectral parameter dependent, nontriangular, initial $R$-matrix solutions.
As is well known, YBE (1.1) can be interpreted as the associativity condition for the quantum Yang-Baxter equation (QYBE) given by
\[
R(\lambda, \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda, \mu),
\]
where \(T_1(\lambda) = T(\lambda) \otimes 1\), \(T_2(\mu) = 1 \otimes T(\mu)\), and \(T(\lambda)\) is a matrix with operator valued elements. The above form of QYBE plays a central role in the context of quantum integrable lattice as well as field models, Yangian algebra and quantum groups. So it is natural to look for some transformation on both \(R\) and \(T\)-matrices, which would lead to new solutions of QYBE (1.5) from a given initial solution. In Sec. 2 we explore such a symmetry transformation of QYBE (1.5), using again the ‘particle conserving’ restriction on the corresponding \(R\)-matrix elements in a crucial way.

Next, we consider the applications of the above mentioned symmetry transformations of YBE and QYBE to some specific cases. The symmetry transformation of the spectral parameter independent initial solution (1.3) yields the multiparametric solution (1.4) as a particular case. More interesting things happen when one applies the symmetry transformation of QYBE to the initial \(R\)-matrix (1.3) and corresponding \(T\)-matrix elements. In Sec. 3 it is shown that this transformation leads to a new Hopf algebra, which contains the algebra of the standard \(GL_q(N)\) as a subalgebra. Moreover, it turns out that all the generators of the multiparametric deformation of \(GL(N)\) can be realized in terms of the generators of this new Hopf algebra in an elegant way.

Subsequently, in Sec. 4, we focus our attention on the ‘particle conserving’ rational solution of YBE given by
\[
R(\lambda, \mu) = (\lambda - \mu) \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + h \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji},
\]
where \((e_{ij})_{kl} = \delta_{ik}\delta_{jl}\). This type of \(R\)-matrix is intimately connected with the Yangian algebra, which was recently found to play an important role in the analysis of some quantum spin chains with long-ranged interactions. Applying the symmetry transformation of YBE to \(R(\lambda, \mu)\)-matrix (1.6) we are able to construct another solution of YBE, which in turn leads to a multiparametric and coloured extension of the standard Yangian algebra \(Y(gl_N)\). Furthermore, the symmetry transformation of QYBE helps us realize these deformed Yangian algebras through their standard counterpart. Finally, Sec. 5 contains the concluding remarks including hints on some possible interesting physical applications of the extended Yangian algebras.

2. Symmetry Transformations of YBE and QYBE

In this section, our aim is to construct first a general transformation, which can be performed on a ‘particle conserving’ \(R\)-matrix such that the YBE (1.1) is still satisfied. For this purpose, we express (1.1) in an elementwise form as
\[
\sum_{k_1,k_2,k_3} R_{i_1i_2}^{k_1k_2}(\lambda, \mu) R_{k_1k_3}^{j_1j_3}(\lambda, \nu) R_{k_2k_3}^{i_2i_3}(\mu, \nu)
= \sum_{i_1,i_2,i_3} R_{i_1i_2}^{j_1j_2}(\lambda, \mu) R_{i_1i_3}^{j_1j_3}(\lambda, \nu) R_{i_2i_3}^{j_2j_3}(\mu, \nu),
\]
for all valid indices.
where all indices run from 1 to $N$. Throughout the paper, all indices run from 1 to $N$, unless otherwise stated, and the convention of summing over repeated indices is not used anywhere. Now, by exploiting the fact that for the ‘particle conserving’ case an element like $R^i_{ij}$ can take non-zero value only if $i = k, j = l$ or $i = l, j = k$, one can write down a formal expression of this element as

$$R^i_{ij}(\lambda, \mu) = f_{ij}(\lambda, \mu)\delta_{ik}\delta_{jl} + g_{ij}(\lambda, \mu)\delta_{il}\delta_{jk}, \quad (2.2)$$

where $f_{ij}(\lambda, \mu)$ and $g_{ij}(\lambda, \mu)$ are some yet undetermined functions. Substituting the above $R$-matrix element in YBE (2.1) and summing over its internal indices, we easily obtain the following set of equations:

\begin{align}
 f_{i_1i_2}(\lambda, \mu)f_{i_1i_3}(\lambda, \nu) &= f_{i_1i_3}(\lambda, \mu)f_{i_1i_2}(\lambda, \nu), \quad (2.3a) \\
 f_{i_2i_3}(\lambda, \nu)f_{i_1i_3}(\mu, \nu) &= f_{i_1i_3}(\lambda, \nu)f_{i_2i_3}(\mu, \nu), \quad (2.3b) \\
 f_{i_1i_2}(\lambda, \mu)g_{i_1i_3}(\lambda, \nu)f_{i_2i_3}(\mu, \nu) + g_{i_1i_2}(\lambda, \mu)g_{i_2i_3}(\lambda, \nu)g_{i_1i_3}(\mu, \nu) &= f_{i_1i_3}(\lambda, \nu)g_{i_1i_3}(\lambda, \mu)f_{i_2i_3}(\mu, \nu), \quad (2.3c) \\
 f_{i_1i_2}(\lambda, \mu)g_{i_1i_3}(\lambda, \nu)g_{i_2i_3}(\mu, \nu) + g_{i_1i_2}(\lambda, \mu)g_{i_2i_3}(\lambda, \nu)g_{i_1i_3}(\mu, \nu) &= g_{i_1i_3}(\lambda, \mu)f_{i_1i_2}(\lambda, \nu)g_{i_2i_3}(\mu, \nu), \quad (2.3d) \\
 g_{i_1i_2}(\lambda, \mu)f_{i_2i_3}(\lambda, \nu)g_{i_1i_3}(\mu, \nu) &= f_{i_2i_3}(\lambda, \mu)g_{i_2i_3}(\lambda, \nu)g_{i_1i_3}(\mu, \nu), \quad (2.3e)
\end{align}

Consequently, the $R$-matrix elements given by (2.2) can be treated as constituting a solution of YBE (2.1) if the functions $f_{ij}(\lambda, \mu)$ and $g_{ij}(\lambda, \mu)$ satisfy the set of equations (2.3a-e).

Next, we consider a transformation on the functions $f_{ij}(\lambda, \mu)$ and $g_{ij}(\lambda, \mu)$ as given by

$$f_{ij}(\lambda, \mu) \rightarrow \tilde{f}_{ij}(\lambda, \mu) = \phi_{ij} \frac{u_{i}^{(1)}(\lambda)u_{j}^{(2)}(\lambda)}{u_{i}^{(1)}(\mu)u_{j}^{(2)}(\mu)}f_{ij}(\lambda, \mu), \quad g_{ij}(\lambda, \mu) \rightarrow \tilde{g}_{ij}(\lambda, \mu) = \frac{u_{i}^{(1)}(\lambda)u_{j}^{(2)}(\lambda)}{u_{i}^{(1)}(\mu)u_{j}^{(2)}(\mu)}g_{ij}(\lambda, \mu), \quad (2.4)$$

where $\phi_{ii} = 1$, $\phi_{ij} = \phi_{ji}^{-1}$ are $N(N-1)/2$ independent constant parameters and $u_{i}^{(1)}(\lambda)$, $u_{i}^{(2)}(\lambda)$ are $2N$ arbitrary (regular and nonvanishing) functions of the spectral parameter $\lambda$. This transformation (2.4) leaves the whole set of relations (2.3a-e) invariant. Therefore, the transformed $\tilde{R}$-matrix given by

$$\tilde{R}^{i}_{ij}(\lambda, \mu) = \tilde{f}_{ij}(\lambda, \mu)\delta_{ik}\delta_{jl} + \tilde{g}_{ij}(\lambda, \mu)\delta_{il}\delta_{jk}, \quad (2.5)$$

would satisfy YBE (2.1), provided the initial $R$-matrix (2.2) is a valid solution. Due to this property of the transformation (2.4), or (2.5), we would naturally call it a
A special case of this symmetry transformation, corresponding to all $\phi_{ij} = 1$, has been used earlier for constructing non-additive type $R$-matrices from the additive ones.\(^{16}\)

To make the symmetry transformation more transparent, let us rewrite (2.3) in matrix form as

$$\tilde{R}(\lambda, \mu) = F^{-1}(\lambda, \mu) R(\lambda, \mu) \hat{F}^{-1}(\lambda, \mu), \quad (2.6)$$

where

$$F(\lambda, \mu) = \sum_{i,j} \sqrt{u_{ij}^{(1)}(\mu)} \frac{u_{ij}^{(2)}(\lambda)}{u_{ji}^{(1)}(\lambda)} e_{ii} \otimes e_{jj},$$

$$\hat{F}(\lambda, \mu) = \sum_{i,j} \sqrt{\phi_{ji}} \frac{u_{ij}^{(2)}(\mu)}{u_{ij}^{(1)}(\lambda)} e_{ii} \otimes e_{jj}, \quad (2.7)$$

and the elements of $R(\lambda, \mu)$-matrix are given by (2.2). It is worth noting that for a particular case corresponding to $u_{ij}^{(1)}(\lambda) = u_{ij}^{(2)}(\lambda)$, one gets $\hat{F}(\lambda, \mu) = F(\lambda, \mu)$, which reduces the symmetry transformation (2.6) to a form quite similar to the transformation (1.2) associated with the Reshetikhin twisting of a Hopf algebra. If one further restricts to the particular case $u_{ij}^{(1)}(\lambda) = u_{ij}^{(2)}(\lambda) = 1$, then, the symmetry transformation (2.6) generates the multiparametric solution (1.4) from the initial solution (1.3).

So far, we have been looking at the solutions of YBE (1.1) which depend on single-component spectral parameters. We can also treat similarly the bicomponent spectral parameter dependent YBE defined by

$$R_{12}(\lambda; \alpha; \mu; \beta) R_{13}(\lambda; \alpha; \nu; \gamma) R_{23}(\mu; \beta; \nu; \gamma) = R_{23}(\mu; \beta; \nu; \gamma) R_{13}(\lambda; \alpha; \nu; \gamma) R_{12}(\lambda; \alpha; \mu; \beta), \quad (2.8)$$

where the first components $(\lambda, \mu, \nu)$ correspond to the usual spectral parameter $(\lambda)$ and the second components $(\alpha, \beta, \gamma)$ refer to the colour parameter $(\alpha)$. It is possible to generalize the symmetry transformation (2.6), or (2.3), for such bicomponent spectral parameter dependent case and construct a solution of YBE (2.8) as

$$R(\lambda, \alpha; \mu, \beta) = F^{-1}(\alpha, \beta) R(\lambda, \mu) \hat{F}^{-1}(\alpha, \beta), \quad (2.9a)$$

$$R^{kl}_{ij}(\lambda, \alpha; \mu, \beta) = f_{ij}(\lambda, \alpha; \mu, \beta) \delta_{ik} \delta_{jl} + g_{ij}(\lambda, \alpha; \mu, \beta) \delta_{il} \delta_{jk}, \quad (2.9b)$$

where $f_{ij}(\lambda, \alpha; \mu, \beta) = \phi_{ij} \frac{u_{ij}^{(1)}(\alpha)}{u_{ij}^{(2)}(\beta)} f_{ij}(\lambda, \mu)$, $g_{ij}(\lambda, \alpha; \mu, \beta) = \frac{u_{ij}^{(1)}(\alpha)}{u_{ij}^{(2)}(\beta)} g_{ij}(\lambda, \mu)$. Note that at the limit $\alpha = \lambda, \beta = \mu$, along with the choice of notation $\tilde{R}(\lambda; \lambda; \mu; \mu) = \tilde{R}(\lambda, \mu)$, equations (2.9a,b) reduce to the transformations (2.6), (2.5) associated with the standard YBE with a single spectral parameter.

Next we want to address the more interesting issue concerning the symmetry transformation associated with QYBE (1.4), which reads, in elementwise form,

$$\sum_{m,n} R_{ij}^{mn}(\lambda, \mu) T_{mk}(\lambda) T_{nl}(\mu) = \sum_{p,q} T_{jq}(\mu) T_{ip}(\lambda) R_{pq}^{kl}(\lambda, \mu). \quad (2.10)$$
Substituting the ‘particle conserving’ $R$-matrix (2.2) in the above equation and summing over its internal indices, we get a set of algebraic relations involving the elements of $T(\lambda)$:

$$U_1 + U_2 - U_3 - U_4 = 0,$$

where

\begin{align*}
U_1 &= f_{ij}(\lambda, \mu)T_{ik}(\lambda)T_{jl}(\mu), & U_2 &= g_{ij}(\lambda, \mu)T_{jk}(\lambda)T_{il}(\mu), \\
U_3 &= T_{jl}(\mu)T_{ik}(\lambda)f_{kl}(\lambda, \mu), & U_4 &= T_{jk}(\mu)T_{il}(\lambda)g_{kl}(\lambda, \mu).
\end{align*}

(2.12)

Subsequently, in analogy with the bicomponent spectral parameter dependent YBE (2.8), we write down the bicomponent spectral parameter dependent QYBE in elementwise form as

$$
\sum_{m,n} R_{ij}^{mn}(\lambda, \alpha; \mu, \beta)T_{mk}(\lambda, \alpha)T_{nl}(\mu, \beta) = \sum_{p,q} T_{jp}(\mu, \beta)T_{ip}(\lambda, \alpha)R_{pq}^{kl}(\lambda, \alpha; \mu, \beta),
$$

(2.13)

and assume further that the corresponding $R(\lambda, \alpha; \mu, \beta)$-matrix is related to the $R(\lambda, \mu)$-matrix (2.2) through the transformation (2.9). If one substitutes such $R(\lambda, \alpha; \mu, \beta)$-matrix in (2.13), that leads to another set of algebraic relations given by

$$\tilde{U}_1 + \tilde{U}_2 - \tilde{U}_3 - \tilde{U}_4 = 0,$$

(2.14)

where the elements $\tilde{U}_r$ ($r \in [1, 4]$) can be obtained from the elements $U_r$ in (2.12) through the substitution: $T_{ij}(\lambda) \rightarrow T_{ij}(\lambda, \alpha)$, $T_{ij}(\mu) \rightarrow T_{ij}(\mu, \beta)$, $f_{ij}(\lambda, \mu) \rightarrow f_{ij}(\lambda, \alpha; \mu, \beta)$ and $g_{ij}(\lambda, \mu) \rightarrow g_{ij}(\lambda, \alpha; \mu, \beta)$. It is worth noting that the algebraic relations (2.14) can be considered as a multiparametric as well as colour parameter dependent deformation of the algebra (2.11). Particular cases of these two algebras (2.11) and (2.14), related to specific choices of functions $f_{ij}(\lambda, \mu)$ and $g_{ij}(\lambda, \mu)$ satisfying (2.3), will be discussed in detail in the following sections.

Let us now look for a transformation, similar to the transformation (2.9), through which the operator valued elements $T_{ij}(\lambda, \alpha)$ appearing in the algebra (2.14) can be expressed through the elements $T_{ij}(\lambda)$ occurring in algebra (2.11) so that the solutions of the bicomponent spectral parameter dependent QYBE (2.13) can be generated from a class of given initial solutions of QYBE (2.10) related to single-component spectral parameter: such a transformation of $T$-matrix elements, coupled with the previously derived transformation (2.9) for $R$-matrix elements, would constitute a symmetry transformation of QYBE.

It may be observed that the transformation (2.9) contains the deformation parameters $\phi_{ij}$ and colour parameter dependent functions $u_i^{(1)}(\alpha)$, $u_i^{(2)}(\alpha)$, which are not present in the initial $R$-matrix elements (2.2). So, to find out the analogous transformation for $T$-matrix elements, it is reasonable to introduce some extra generators, which do not occur in the original algebra (2.11). Let $\tau_i$ ($i \in [1, N]$) be $N$ colour parameter independent generators and $G(\alpha)$ be a colour parameter dependent
the validity of this ansatz, we substitute the above expression of $T_{\tau}$ where $S$ conditions (2.19). Now, it follows from the condition (2.14) through the generators of algebra (2.15) as given by algebra (2.11) as a subalgebra. Now, we propose a realization of deformed algebra (2.14) through the generators of algebra (2.15) as given by

$$T_{ij}(\lambda, \alpha) = r_{ij}(\alpha)\tau_{ij}G^2(\alpha)T_{ij}(\lambda) ,$$

(2.16)

where $r_{ij}(\alpha)$ are also some still undetermined $c$-number valued functions. To check the validity of this ansatz, we substitute the above expression of $T_{ij}(\lambda, \alpha)$ in the deformed algebra (2.14). Then, using the commutation relations (2.15b,c,d) we can shift the extra generators ($\tau_i, G(\alpha)$) to one side of each term in (2.14) and arrive at the relation

$$\tau_i\tau_j\tau_kG^2(\alpha)G^2(\beta) [S_1U_1 + S_2U_2 - S_3U_3 - S_4U_4] = 0 ,$$

(2.17)

where

$$S_1 = \phi_{ij} \frac{u_j^{(1)}(\alpha)u_j^{(2)}(\alpha)r_{ik}(\alpha)r_{jl}(\beta)}{u_i^{(1)}(\beta)u_i^{(2)}(\beta)c_{ik}^j(\lambda)d^2_{ik}(\lambda, \beta)c_{jl}^i(\lambda)} ,$$

$$S_2 = \frac{u_i^{(1)}(\alpha)u_j^{(2)}(\alpha)r_{ik}(\alpha)r_{jl}(\beta)}{u_i^{(1)}(\beta)u_j^{(2)}(\beta)c_{ik}^j(\lambda)d^2_{ik}(\lambda, \beta)c_{jl}^i(\lambda)} ,$$

$$S_3 = \phi_{kl} \frac{u_i^{(1)}(\alpha)u_j^{(2)}(\alpha)r_{ik}(\alpha)r_{jl}(\beta)}{u_k^{(1)}(\beta)u_k^{(2)}(\beta)c_{ik}^j(\mu)d^2_{ik}(\mu, \alpha)c_{jl}^i(\mu)} ,$$

$$S_4 = \frac{u_i^{(1)}(\alpha)u_i^{(2)}(\alpha)r_{ik}(\beta)r_{il}(\alpha)}{u_i^{(1)}(\beta)u_k^{(2)}(\beta)c_{ik}^j(\mu)d^2_{ik}(\mu, \alpha)c_{jl}^i(\mu)} .$$

(2.18)

Comparing (2.17) and (2.11), it is evident that (2.17) would be automatically satisfied if one sets

$$S_1 = S_2 = S_3 = S_4 .$$

(2.19)

So, the expression (2.16) can be treated as a realization of the deformed algebra (2.14), if the yet undetermined functions $c_{jk}^i(\lambda), d_{jk}(\lambda, \alpha)$ and $r_{ij}(\alpha)$ satisfy the conditions (2.19). Now, it follows from the condition $S_1 = S_3$ that the functions $c_{jk}^i(\lambda), d_{jk}(\lambda, \alpha)$ can in fact be chosen to be independent of the spectral parameter $\lambda$ and may be given by

$$c_{jk}^i(\lambda) = c_{jk}^i = \sqrt{\frac{\phi_{ik}}{\phi_{ij}}} ,$$

(2.19)
Taking the above form of $c_{jk}$, $d_{jk}(\alpha)$ and using further the condition $S_1 = S_2 = S_4$, we obtain

$$r_{ij}(\alpha) = \frac{1}{\sqrt{\phi_{ij}}} u_{i}^{(1)}(\alpha) u_{j}^{(1)}(\alpha).$$

(2.21)

Then, the realization of the elements $T_{ij}(\lambda, \alpha)$ proposed in (2.16) takes the form

$$T_{ij}(\lambda, \alpha) = \frac{1}{\sqrt{\phi_{ij}}} \tau_{i} \tau_{j} G^{2}(\alpha) T_{ij}(\lambda).$$

(2.22)

It is thus found that there indeed exists a transformation (2.22) through which the operator valued elements $T_{ij}(\lambda, \alpha)$ appearing in the deformed algebra (2.14) can be expressed through the elements $T_{ij}(\lambda)$ of the original algebra (2.11). However, to achieve this, we have introduced the extra generators $\tau_{i}$, $G(\alpha)$ which satisfy the commutation relations

$$[\tau_{i}, \tau_{j}] = [\tau_{i}, G(\alpha)] = [G(\alpha), G(\beta)] = 0,$$

(2.23a)

$$\tau_{i} T_{jk}(\lambda) = \frac{\phi_{ik}}{\phi_{ij}} T_{jk}(\lambda) \tau_{i},$$

(2.23b)

$$G(\alpha) T_{jk}(\lambda) = \frac{u_{k}^{(1)}(\alpha) u_{k}^{(2)}(\alpha)}{u_{j}^{(1)}(\alpha) u_{j}^{(2)}(\alpha)} T_{jk}(\lambda) G(\alpha).$$

(2.23c)

It may be observed further that the transformation (2.22) can be written in an elegant matrix form as

$$T(\lambda, \alpha) = \mathcal{M}(\alpha) T(\lambda) \hat{\mathcal{M}}(\alpha)$$

(2.24)

where $\mathcal{M}(\alpha)$ and $\hat{\mathcal{M}}(\alpha)$ are diagonal matrices with operator valued elements

$$\mathcal{M}_{ij}(\alpha) = \left( \frac{u_{i}^{(1)}(\alpha)}{u_{i}^{(2)}(\alpha)} \right)^{1/2} \tau_{i} G(\alpha) \delta_{ij},$$

$$\hat{\mathcal{M}}_{ij}(\alpha) = \left( \frac{u_{i}^{(2)}(\alpha)}{u_{i}^{(1)}(\alpha)} \right)^{1/2} \tau_{i} G(\alpha) \delta_{ij}.$$  

(2.25)

For the special case corresponding to $u_{i}^{(1)}(\alpha) = u_{i}^{(2)}(\alpha)$, one evidently gets $\mathcal{M}(\alpha) = \hat{\mathcal{M}}(\alpha)$. This reduces the symmetry transformation of $T$-matrix (2.24) to a form analogous to the transformation of the $R$-matrix (1.2) associated with Reshetikhin’s twisted Hopf algebra.

Since, by using the expressions (2.9) and (2.22) simultaneously, one can construct the solutions of the bicomponent spectral parameter dependent QYBE (2.13) from
Moreover, at the particular limit $\alpha = \lambda$, $\beta = \mu$, the QYBE (2.13) tends to its single spectral parameter counterpart (2.10). So this $\alpha = \lambda$, $\beta = \mu$, limit of the symmetry transformation ((2.9), (2.22)) can be used to obtain a more general solution of the standard QYBE (2.10) itself, from a given initial solution. In another special case corresponding to the choice $u_i^{(1)}(\alpha) = u_i^{(2)}(\alpha) = 1$ for all $i$, the transformations (2.9) and (2.22) become independent of colour parameter and can also be used to construct new solutions of the standard QYBE. This type of symmetry transformation will be used shortly, in the next section, for establishing a link between the single-parametric and multiparametric deformations of $GL(N)$.

3. A New Realization of Multiparametric Deformation of $GL(N)$

As is well known, the algebra of $GL_q(N)$ is generated by $N^2$ elements $T_{ij}$ satisfying the commutation relations

$$
T_{ij}T_{ik} = q^{-1}T_{ik}T_{ij}, \quad T_{ik}T_{lj} = q^{-1}T_{lj}T_{ik}, \quad [T_{ij}, T_{lk}] = (q^{-1} - q)T_{lj}T_{ik},
$$

where $i < l$, $j < k$. This algebra is obtained by substituting the ‘particle conserving’ $R$-matrix (1.3) in the spectral parameterless limit of QYBE (1.5). One can construct a multiparametric generalization of $GL_q(N)$, by substituting the $R$-matrix (1.4) in QYBE (1.3) : with $i < l$, $j < k$,

$$
\tilde{T}_{ij}\tilde{T}_{ik} = q^{-1}\phi_{jk}\tilde{T}_{ik}\tilde{T}_{ij}, \quad \tilde{T}_{ik}\tilde{T}_{lj} = q^{-1}\phi_{il}\tilde{T}_{lk}\tilde{T}_{ik},
$$

$$
\phi_{ij}\tilde{T}_{ik}\tilde{T}_{ij} = \phi_{jk}\tilde{T}_{lj}\tilde{T}_{ik}, \quad \phi_{il}\tilde{T}_{ij}\tilde{T}_{lk} - \phi_{jk}\tilde{T}_{lj}\tilde{T}_{ik} = (q^{-1} - q)\tilde{T}_{lj}\tilde{T}_{ik},
$$

where $\phi_{ij}$ are $N(N-1)/2$ additional deformation parameters. Evidently, at the limit $\phi_{ij} = 1$ for all $i, j$, the multiparametric deformed algebra (3.2) reduces to its single parameter dependent version (3.1).

Various aspects of the algebra of $GL_q(N)$ (3.1) have been studied quite extensively in the literature. In particular, we are interested in the realization of the generating elements $T_{ij}$ of $GL_q(N)$, for $|q| = 1$, through recasting it in the Heisenberg-Weyl form ($A_iB_j = \omega_{ij}B_jA_i$, $|\omega_{ij}| = 1$) and subsequently using mutually commuting pairs of canonically conjugate quantum mechanical operators (boson realization), or, finite dimensional matrices. The multiparametric extensions of $GL_q(N)$ are, in general, more difficult to analyze and the corresponding realizations have been investigated only for some special cases like $N = 2$. So, if one can express the generators of the multiparametric algebra (3.2) in terms of the generators of another algebra structurally similar to the algebra of $GL_q(N)$ (3.1), then, such expression should be much helpful for constructing the realizations of the multiparametric deformation of $GL(N)$. To this end, we use the symmetry transformation (2.22).

As has been already noted in Sec. II, the ‘particle conserving’ $R$-matrices (1.3) and (1.4) are related through a particular limit of symmetry transformation (2.9) corresponding to the choice $u_i^{(1)}(\alpha) = u_i^{(2)}(\alpha) = 1$, for all $i$. Consequently, the pair
of algebras (3.1) and (3.2) can be considered as a special case of the pair (2.11) and (2.14), which are connected through the general symmetry transformation (2.22). More explicitly, in the present context, the algebra

\[
\tau_i, \tau_j = 0 , \quad \tau_m T_{kl} = \sqrt{\frac{\phi_{ml}}{\phi_{mk}}} T_{kl} \tau_m , \quad (3.3a)
\]

\[
T_{ij} T_{ik} = q^{-1} T_{ik} T_{ij} , \quad T_{ik} T_{ij} = T_{ij} T_{ik} , \quad [T_{ij}, T_{lk}] = (q^{-1} - q) T_{lj} T_{ik} , \quad (3.3b)
\]

\[
T_{ik} T_{lj} = T_{lj} T_{ik} , \quad [T_{ij}, T_{lk}] = (q^{-1} - 1) T_{lj} T_{ik} , \quad (3.3c)
\]

where \( \tau_i \) are \( N \) extra generators, and \( i < l, j < k \), contains \( GL_q(N) \) as a subalgebra, and is a particular case of the algebra (2.15) corresponding to the choice \( \alpha_i^{(1)} = 1 \) and \( G(\alpha) = 1 \). Consequently, from (2.22), or (2.24), we obtain the realization

\[
\tilde{T}_{ij} = \frac{1}{\sqrt{\phi_{ij}}} \tau_i \tau_j T_{ij} , \quad \text{or} \quad \tilde{T} = \mathcal{M} T \mathcal{M} , \quad (3.4)
\]

where \( \mathcal{M} \) is the diagonal \( N \times N \) matrix with operator valued elements given by \( \mathcal{M}_{ij} = \tau_i \delta_{ij} \). By using the algebra (3.3), one can also verify directly the validity of above realization (3.4). Thus, we find that it is indeed possible to express the generators of the multiparametric deformation of \( GL(N) \) (3.2) through the generators of \( GL_q(N) \) (3.1) and \( N \) additional generators \( \tau_i \) satisfying the commutation relations (3.3a).

Though the above observations are true in general, sometimes one might be able to obtain a realization of the multiparametric deformation of \( GL(N) \), using less than \( N \) additional generators \( \tau_i \), besides the generators of \( GL_q(N) \), provided the corresponding parameters \( \phi_{ij} \) satisfy certain constraints. To illustrate this point, let us assume that all \( \phi_{ij} \) can be written in the form: \( \phi_{ij} = r_i r_j \), where \( r_i \)'s are \( N \) independent parameters. Then, the structure constant \( \sqrt{\frac{\phi_{ml}}{\phi_{mk}}} \), occurring in the commutation relation (3.3a), would be independent of the index \( m \). Consequently, all generators \( \tau_i \) occurring in the algebra (3.3), as well as in the realization (3.4), can be replaced effectively by a single generator \( \tau \) satisfying the commutation relations \( \tau T_{kl} = \sqrt{\phi_{lk}} T_{kl} \tau \). Thus, in this case we can have a realization of the multiparametric deformed \( GL(N) \) algebra (3.2) by augmenting the \( GL_q(N) \) algebra (3.1) with just one additional generator \( \tau \). A realization of this type has been considered earlier for the special case \( N = 2 \).

Next, one may ask the interesting question whether the extended algebra (3.3) itself corresponds to a quantum group. To answer this question, we take a \( 2N \times 2N \) dimensional \( T' \)-matrix and a \( 4N^2 \times 4N^2 \) dimensional \( R' \)-matrix, whose nonvanishing elements are given by

\[
T'_{ij} = T_{ij} , \quad T'_{i', j'} = \delta_{ij} \tau_i , \quad (3.5a)
\]

\[
(R')_{ij}^{i'j'} = (R')_{i', j'}^{i, j} = 1 + (q - 1) \delta_{ij} , \quad (R')_{ik} = q - q^{-1} , \quad (3.5b)
\]
where \( i' = i + N, \) \( j' = j + N, \) \( 1 \leq i, j \leq N \) and \( 1 \leq k < l \leq 2N \). Then, it is found that the algebra (3.3) is generated by substituting these \( R' \) and \( T' \) matrices in the spectral-parameterless limit of QYBE (1.3). So, the extended algebra (3.3) indeed defines a quantum group, for which the coproduct and other Hopf algebra operations are readily obtained. From the block diagonal structure of \( T' \)-matrix (3.5a) it is evident that this new quantum group is a deformation of the group of \( 2N \times 2N \) matrices having only \( \{ M_{ij}, M_{N+k,N+k} | i, j, k = 1, 2, \ldots, N \} \) as nonzero elements, or in other words it is a deformation of the subgroup of \( GL(2N) \) isomorphic to \( GL(N) \otimes GL(1) \otimes \cdots \otimes GL(1) \).

One can also easily construct the noncommutative planes associated with this non-semisimple quantum group. For this purpose, let us consider a set of \( 2N \) coordinates \( x_i, y_i \) \( (i \in [1, N]) \) undergoing the transformation

\[
x_i' = \sum_{j=1}^{N} T_{ij} x_j, \quad y_i' = \tau_i y_i. \tag{3.6}
\]

Using the algebra (3.3), it can be readily checked that the following two sets of bilinear relations remain invariant under the transformation (3.6):

\[
x_i x_j = q^{-1} x_j x_i, \quad x_k y_l = \sqrt{\phi_{lk}} y_l x_k, \quad [y_k, y_l] = 0, \tag{3.7a}
\]

\[
x_i^2 = 0, \quad x_i x_j = -q x_j x_i, \quad x_k y_l = \sqrt{\phi_{lk}} y_l x_k, \quad [y_k, y_l] = 0, \tag{3.7b}
\]

where \( i < j \). Therefore, the relations (3.7a) and (3.7b) represent the two noncommutative planes corresponding to the quantum group (3.3). Let us now define \( X_i = x_i y_i, \quad i = 1, 2, \ldots, N \). Then, it is found that \( X_i \) obey the commutation relations

\[
X_i X_j = q^{-1} \phi_{ji} X_j X_i, \tag{3.8a}
\]

or

\[
X_i^2 = 0, \quad X_i X_j = -q \phi_{ji} X_j X_i, \tag{3.8b}
\]

depending on whether \( (x_i, y_i) \) satisfy (3.7a) or (3.7b) respectively. It is interesting to observe that (3.8a) and (3.8b) coincide with the commutation relations of the Manin \( q \)-plane and its exterior plane associated with the multiparametric deformation of \( GL(N) \).\(^{18}\)

Let us note that the extended algebra (3.3) can be recast in the Heisenberg-Weyl form in a very simple way. Suppose \( T_{ij} \) are the basis elements, through which the \( GL_q(N) \) algebra (3.1) can be expressed in the Heisenberg-Weyl form for unimodular values of the parameter \( q \); explicit construction of such \( T_{ij} \) is known.\(^{19,20}\) Then, it can be verified directly that the extended algebra (3.3) will also take the Heisenberg-Weyl form if \( T_{ij}, \tau_k \) are chosen as the basis elements. Hence, the multiparametric deformation of \( GL(N) \) can also be realized in terms of mutually commuting pairs of canonically conjugate quantum mechanical operators and finite dimensional matrices using the known methods.\(^{21,22}\)

4. Multiparametric and Coloured Extensions of \( Y(gl_N) \)
Here our aim is to study various types of deformations of the Yangian algebra $Y(gl_N)$ and their interrelations, using the symmetry transformations of YBE and QYBE. The standard $Y(gl_N)$, with the defining relations

$$(\lambda - \mu) [T_{ij}(\lambda), T_{kl}(\mu)] = h \left\{ T_{kj}(\mu)T_{il}(\lambda) - T_{kj}(\lambda)T_{il}(\mu) \right\}, \quad (4.1)$$

results from the substitution of the rational $R(\lambda)$-matrix (1.6) in QYBE (2.10). If one assumes the usual analyticity property of $T(\lambda)$ and the asymptotic condition $T(\lambda) \to 1$ at $\lambda \to \infty$, then the operator valued elements $T_{ij}(\lambda)$ can be expanded in powers of $\lambda$ as

$$T_{ij}(\lambda) = \delta_{ij} + h \sum_{n=0}^{\infty} \frac{t_{ij}^n}{\lambda^{n+1}}. \quad (4.2)$$

Substituting the above expansion in (4.1) and comparing the coefficients of equal powers in spectral parameters on both sides, one can express the $Y(gl_N)$ algebra through the modes $t_{ij}^n$ as

$$[t_{ij}^n, t_{kl}^m] = \delta_{il}t_{kj}^m - \delta_{kj}t_{il}^m, \quad (4.3a)$$

$$[t_{ij}^{n+1}, t_{kl}^m] - [t_{ij}^n, t_{kl}^{m+1}] = h(t_{kj}^{m+l}t_{il}^n - t_{kj}^{n+l}t_{il}^m). \quad (4.3b)$$

With the help of induction procedure, it is seen that the algebra (4.3a,b) can also be presented equivalently as a single relation

$$[t_{ij}^n, t_{kl}^m] = \delta_{il}t_{kj}^{m+n} - \delta_{kj}t_{il}^{n+m} + h \sum_{p=0}^{n-1} (t_{kj}^{m+p}t_{il}^{n-1-p} - t_{kj}^{n-1-p}t_{il}^{m+p}). \quad (4.3c)$$

It may be noted that at the limit $h \to 0$, (4.3c) reduces to a subalgebra of $gl(N)$ Kac-Moody algebra containing its non-negative modes. Consequently, this Yangian algebra might be considered as some nonlinear deformation of the $gl(N)$ Kac-Moody algebra through the parameter $h$. Casimir operators for $Y(gl_N)$ may be obtained by constructing the corresponding quantum determinant. Further, it turns out that all the higher level generators of $Y(gl_N)$ can be realized consistently through only the 0th and 1st level generators of $Y(sl_N)$, provided a few Serre relations are satisfied.\(^5,^{15}\)

Now, for constructing a multiparametric and coloured extension of $Y(gl_N)$, we apply the symmetry transformation of YBE (2.9) to the particle conserving rational $R$-matrix (1.6). This leads to a solution of the bicomponent spectral parameter dependent YBE (2.8), given by

$$R(\lambda, \alpha; \mu, \beta) = (\lambda - \mu) \sum_{i,j=1}^{N} u_{ij}(\alpha, \beta) e_{ii} \otimes e_{jj} + h \sum_{i,j=1}^{N} v_{ij}(\alpha, \beta) e_{ij} \otimes e_{ji}, \quad (4.4)$$
where \( u_{ij}(\alpha, \beta) = \frac{u_i^{(1)}(\alpha)u_j^{(2)}(\alpha)}{u_i^{(1)}(\beta)u_j^{(2)}(\beta)} \) and \( v_{ij}(\alpha, \beta) = \frac{u_i^{(1)}(\alpha)u_j^{(2)}(\alpha)}{u_i^{(1)}(\beta)u_j^{(2)}(\beta)} \). Substituting this into (4.5) and equating the terms on both sides with the single relation

\[
R(\lambda, \alpha; \mu, \beta) \text{ in QYBE (2.13), we get the desired generalization of } Y(gl_N) \text{ (4.1)}:
\]

\[
(\lambda - \mu) \{ u_{ik}(\alpha, \beta)T_{ij}(\lambda, \alpha)T_{kl}(\mu, \beta) - u_{jl}(\alpha, \beta)T_{kl}(\mu, \beta)T_{ij}(\lambda, \alpha) \}
\]

\[
= h \{ v_{ij}(\alpha, \beta)T_{kj}(\mu, \beta)T_{il}(\lambda, \alpha) - v_{ik}(\alpha, \beta)T_{kj}(\lambda, \alpha)T_{il}(\mu, \beta) \} . \quad (4.5)
\]

It is evident that at the particular limit \( u_i^{(1)}(\alpha) = u_i^{(2)}(\alpha) = 1 \), \( \phi_{ij} = 1 \), the above algebra (4.5) tends to its standard counterpart (4.1). However, at another limit corresponding to the choice \( u_i^{(1)}(\alpha) = u_i^{(2)}(\alpha) = 1 \), but \( \phi_{ij} \neq 1 \), the algebra (4.3) becomes colourless and reduces to

\[
(\lambda - \mu) \{ \phi_{ik}T_{ij}(\lambda)T_{kl}(\mu) - \phi_{jl}T_{kl}(\mu)T_{ij}(\lambda) \}
\]

\[
= h \{ T_{kj}(\mu)T_{il}(\lambda) - T_{kj}(\lambda)T_{il}(\mu) \} . \quad (4.6)
\]

This type of multiparametric extension of the Yangian algebra \( Y(gl_N) \) (4.1) and related Hopf algebra properties have been explored earlier.\(^{25}\)

Next, we like to investigate whether, in analogy with (4.3a,b,c), the coloured and multiparametric Yangian algebra (4.3) can be written in terms of the modes of its generators. To this end, we expand the element \( T_{ij}(\lambda, \alpha) \) in powers of the spectral parameter \( \lambda \) as

\[
T_{ij}(\lambda, \alpha) = \tau_{ii}(\alpha)\delta_{ij} + h \sum_{n=0}^{\infty} \frac{t^{(1)}_{ij}(\alpha)}{\lambda^{n+1}} , \quad (4.7)
\]

where the mode operators depend on the continuous colour parameter \( \alpha \). Substituting this mode expansion in (4.3) and equating the terms on both sides with the same powers of \( \lambda \) and \( \mu \), we arrive at the following relations:

\[
[\tau_{ii}(\alpha), \tau_{jj}(\beta)] = 0 , \quad (4.8a)
\]

\[
\tau_{jj}(\alpha)t^{ii}_{m}(\beta) = \frac{u_{ji}(\alpha, \beta)}{u_{ji}(\beta, \alpha)}t^{ii}_{m}(\beta)\tau_{jj}(\alpha) , \quad (4.8b)
\]

\[
u_{ik}(\alpha, \beta)t^{ij}_{0}(\alpha)t^{kl}_{m}(\beta) - u_{jl}(\alpha, \beta)t^{kl}_{m}(\beta)t^{ij}_{0}(\alpha)
\]

\[
= \delta_\alpha v_{ij}(\alpha, \beta)t^{kl}_{m}(\beta)\tau_{ii}(\alpha) - \delta_\beta v_{ik}(\alpha, \beta)\tau_{kk}(\alpha)t^{kl}_{m}(\beta) , \quad (4.8c)
\]

\[
\{ u_{ik}(\alpha, \beta)t^{ij}_{n+1}(\alpha)t^{kl}_{m}(\beta) - u_{jl}(\alpha, \beta)t^{kl}_{m}(\beta)t^{ij}_{n+1}(\alpha) \}
\]

\[
- \{ u_{ik}(\alpha, \beta)t^{ij}_{n+1}(\alpha)t^{kl}_{m+1}(\beta) - u_{jl}(\alpha, \beta)t^{kl}_{m+1}(\beta)t^{ij}_{n}(\alpha) \}
\]

\[
= h \{ v_{ij}(\alpha, \beta)t^{kl}_{m}(\beta)t^{ij}_{n}(\alpha) - v_{ik}(\alpha, \beta)t^{kl}_{m}(\beta)t^{ij}_{n}(\alpha) \} . \quad (4.8d)
\]

Using the induction procedure, the last two equations (4.8c,d) can be combined into the single relation

\[
\{ u_{ik}(\alpha, \beta)t^{ij}_{n}(\alpha)t^{kl}_{m}(\beta) - u_{jl}(\alpha, \beta)t^{kl}_{m}(\beta)t^{ij}_{n}(\alpha) \}
\]

\[
= \{ \delta_\alpha v_{ij}(\alpha, \beta)t^{kl}_{m+1}(\beta)\tau_{ii}(\alpha) - \delta_\beta v_{ik}(\alpha, \beta)\tau_{kk}(\alpha)t^{kl}_{m+1}(\beta) \} .
\]
\[ +h \sum_{p=0}^{n-1} \left\{ v_{ij}(\alpha, \beta)t_{m+p}^{kj}(\beta)t_{n-1-p}^{il}(\alpha) \right. \\
- v_{ik}(\alpha, \beta)t_{n-1-p}^{kj}(\alpha)t_{m+p}^{il}(\beta) \left. \right\} . \] (4.8e)

Thus the mode operators associated with the coloured and multiparametric Yangian algebra (4.5) satisfy the relations (4.8). It is obvious from (4.8a) and (4.8b) that in the limit \( u_i^{(1)}(\alpha) = u_i^{(2)}(\alpha) = 1, \phi_{ij} = 1, \tau_{jj}(\alpha) \) can be taken to be the unit operator for any \( j \) so that in this limit the relations (4.8c), (4.8d) and (4.8e) tend to their standard counterparts (4.3a), (4.3b) and (4.3c) respectively.

It should be of physical relevance to construct the representations of the coloured and multiparametric Yangian algebra since the representation theory of the standard Yangian algebra has been found to play an important role in the analysis of the degeneracies of wavefunctions of some quantum spin chains with long-ranged interactions. One may also ask whether the new mode algebra (4.8) can be realized through some 0th and 1st level generators satisfying the Serre relations. To study these matters, we first like to see the connection between the deformed Yangian algebra (4.8) and its standard counterpart (4.3) through the symmetry transformation of QYBE (2.22). To this end, let us extend the standard \( Y(gl_N) \) algebra (4.3) by augmenting it with extra generators, \( N \) colour parameter independent \( \tau_i \) and a colour parameter dependent \( G(\alpha) \), such that

\[ [\tau_i, \tau_j] = [\tau_i, G(\alpha)] = [G(\alpha), G(\beta)] = 0 , \] (4.9a)

\[ \tau_i t^{jk}_n = \sqrt{\frac{\phi_{ik}}{\phi_{ij}}} t^{jk}_n \tau_i , \] (4.9b)

\[ G(\alpha) t^{jk}_n = \sqrt{\frac{u_k^{(1)}(\alpha)u_k^{(2)}(\alpha)}{u_j^{(1)}(\alpha)u_j^{(2)}(\alpha)}} t^{jk}_n G(\alpha) , \] (4.9c)

\[ [t^{ij}_n, t^{kl}_m] = \delta_{il} t^{kj}_{n+m} - \delta_{kj} t^{il}_{n+m} \\
+ h \sum_{p=0}^{n-1} \left( t^{kj}_{m+p} t^{il}_{n-1-p} - t^{kj}_{n-1-p} t^{il}_{m+p} \right) . \] (4.9d)

Note that the above algebra is a special case of (2.15) and the relations (4.9b) and (4.9c) are obtained from (2.23b) and (2.23c), respectively, by inserting in them the mode expansion (4.2). Subsequently, using the general symmetry transformation (2.22) and the mode expansions (4.2) and (4.7), we find that the coloured and multiparametric Yangian algebra (4.8) can be realized through the generators of the extended algebra (4.9) as

\[ \tau_{ii}(\alpha) = (\tau_i G(\alpha))^2 , \quad t_{n}^{ij}(\alpha) = \frac{1}{\sqrt{\phi_{ij}}} \frac{u_i^{(1)}(\alpha)}{u_j^{(1)}(\alpha)} \tau_i \tau_j G^2(\alpha) t_{n}^{ij} . \] (4.10)

One can also verify the validity of this realization directly by using the extended algebra (4.9).
The above realization of the deformed Yangian algebra is seen to be quite similar to the previously constructed realization (3.4) of the multiparametric extension of $GL_q(N)$. However, the extra generators $\tau_i$ appearing in the algebra (3.3) seem to be independent of the original $GL_q(N)$ generators $T_{ij}$ and that is why it is difficult to give a realization of the algebra of multiparametric deformed $GL(N)$ in terms of only the $GL_q(N)$ generators. On the other hand, it turns out interestingly that one can express the extra generators $\tau_i$, $G(\alpha)$ appearing in the extended $Y(gl_N)$ algebra (4.9) through the original $Y(sl_N)$ generators, satisfying the algebra (4.3), as

$$\tau_j = \exp \left( i \sum_{l=1}^{N} A_{jl} t_0^l \right), \quad G(\alpha) = \exp \left( i \sum_{l=1}^{N} B_l(\alpha) t_0^l \right), \quad (4.11)$$

where $\exp(iA_{jl}) = \sqrt{\phi_{jl}}$ and $\exp(iB_l(\alpha)) = \sqrt{u_l^{(1)}(\alpha) u_l^{(2)}(\alpha)}$. Now, substituting the expressions for $\tau_s$ and $G(\alpha)$ given by (4.11) in (4.10), we have a realization of the coloured and multiparametric Yangian algebra (4.8) entirely in terms of the standard Yangian generators:

$$\tau_{jj}(\alpha) = \exp \left( 2i \sum_{l} [A_{jl} + B_l(\alpha)] t_0^l \right),$$

$$t_{nk}^{jk}(\alpha) = \frac{1}{\sqrt{\phi_{jk} u_k^{(1)}(\alpha)}} \exp \left( i \sum_{l=1}^{N} [A_{jl} + A_{kl} + 2B_l(\alpha)] t_0^l \right) t_{nk}^{jk}. \quad (4.12)$$

However, it should be noted that, though expressible entirely in terms of the generators of its standard counterpart, a multiparametric generalization of the Yangian algebra is endowed with a new coalgebra structure as can be seen by following the treatment of the Hopf algebraic structure of the algebra (4.3) presented in Ref. 25.

As has been already mentioned, all the higher level generators of the standard $Y(gl_N)$ (4.3) can be expressed in terms of the level-0 and level-1 generators of $Y(sl_N)$ provided they satisfy a few Serre relations. Then, using the realization (4.12), one can also express all higher level generators of the coloured and multiparametric Yangian algebra (4.8) through the level-0 and level-1 generators of $Y(sl_N)$. Moreover, it is also possible to construct the representations of the deformed Yangian algebra by using the realization (4.12) and the known representations of the standard Yangian algebra. It is thus found that, in spite of the apparently complicated nature of the coloured and multiparametric Yangian algebra (4.8), one can easily understand it through its realization in terms of the standard Yangian generators.

5. Concluding Remarks

The YBE and its solutions, $R$-matrices, are well known to play the central role in several problems associated with exactly solvable models, quantum algebras, knot theory, conformal field theory, etc. So, it is useful to search for new solutions of the YBE. In this article, inspired by Reshetikhin’s twisting procedure for obtaining the multiparametric extensions of a Hopf algebra, we have demonstrated the operation of a ‘symmetry transformation’ which, applied to a known ‘particle conserving’ $R$-matrix, leads to a new spectral parameter dependent solution of the YBE depending,
in general, on multiple $q$-parameters as well as a colour parameter. The YBE follows from the associativity condition of QYBE. Hence, naturally, we find that a certain set of transformations on both the $R$-matrix and the $T$-matrix, constituting a symmetry transformation of QYBE, leads to a multiparametric and coloured extension of the associated quantum group structure; here again, the $R$-matrix is considered to have the ‘particle conserving’ symmetry.

As an application of the symmetry transformations of YBE and QYBE, first we have analysed the construction of the multiparametric extension of $GL_q(N)$. It is found that the resulting quantum group admits a realization through the generators of $GL_q(N)$ and some mutually commuting extra generators. We hope that such a simple realization of the multiparametric extension of $GL_q(N)$ would be helpful in building up the corresponding representation theory. Next, by applying the symmetry transformation to a class of rational solutions of YBE, we have constructed a multiparametric and coloured extension of the Yangian algebra $Y(gl_N)$. It is found that this extended Yangian algebra can also be realized completely in terms of the standard Yangian generators, indicating that the extra deformation occurs essentially in the coalgebra sector of the Hopf algebra.

Finally, we like to hint at some possible interesting physical applications of the extended Yangian algebras which have been discussed here. As is well known, the conserved quantities of $su(N)$-invariant spin Calogero-Sutherland (CS) model associated with the Hamiltonian

$$H = -\frac{1}{2} \sum_{k=1}^{M} \left( \frac{\partial}{\partial x_k} \right)^2 + \frac{\pi^2}{L^2} \sum_{k<l} \frac{\beta (\beta + P_{kl})}{\sin^2 \frac{\pi}{L} (x_k - x_l)},$$

(5.1)

where $P_{kl}$ is the permutation operator which interchanges the ‘spins’ of $k$-th and $l$-th particles, yield a realization of the standard Yangian algebra $Y(gl_N)$. So it is natural to ask whether there exists some new types of spin CS models whose conserved quantities would be similarly related to the present multiparameter deformation of the Yangian algebra. Working along these lines it has been found very recently that the Hamiltonians of such spin CS models can be easily generated by substituting the following ‘anyon-like’ representation of the permutation operator $P_{kl}$ in the expression (5.1):

$$P_{kl} |\alpha_1 \cdots \alpha_k \cdots \alpha_l \cdots \alpha_M \rangle = \left\{ \phi_{\alpha_k \alpha_l} \prod_{\tau=1}^{N} \left( \frac{\phi_{\tau \alpha_l}}{\phi_{\tau \alpha_k}} \right)^{n_{\tau}} \right\} |\alpha_1 \cdots \alpha_k \cdots \alpha_l \cdots \alpha_M \rangle,$$

(5.2)

where $\alpha_i \in [1, N]$ denotes the spin degrees of freedom of the $i$-th particle, $n_{\tau}$ is the number of occurrences of the particular spin orientation $\alpha_i = \tau$ in the configuration $|\alpha_1 \cdots \alpha_k \cdots \alpha_l \cdots \alpha_M \rangle$ when the index $i$ runs from $k+1$ to $l-1$, $\phi_{\alpha \alpha} = \pm 1$ and $\phi_{\alpha \beta}$ (for $\alpha \neq \beta$) are the multiple deformation parameters, the above defined operator $P_{kl}$ not only interchanges the spins of $k$-th and $l$-th particles but also picks up a phase factor depending on the orientations of all the intermediate spins. Consequently such representations of permutation operators would lead to novel variants of the CS model which contain highly nonlocal spin dependent interactions and can be solved...
exactly\(^{29}\) by the action of a generalized antisymmetric projection operator on the eigenvectors of the Dunkl operators. Since a ‘frozen’ limit of the spin CS model \(^{29}\) corresponds to the Haldane-Shastry (HS) spin chain, it is to be expected that the HS model would also admit similar generalization respecting the multiparametric Yangian symmetry. Moreover, it may be noted that the Hamiltonian of the one-dimensional Hubbard model on an infinite chain respects\(^{30}\) the Yangian symmetry \(Y(\mathfrak{sl}_2) \oplus Y(\mathfrak{sl}_2)\) as an extension of the known \(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2\) symmetry. Thus, it will be of interest to search for a generalized form of the Hubbard model which would possess the multiparametric and coloured extension of the Yangian symmetry.

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