ON GROMOV’S SCALAR CURVATURE CONJECTURE

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Abstract. We prove the Gromov conjecture on the macroscopic dimension of the universal covering of a closed spin manifold with a positive scalar curvature under the following assumptions on the fundamental group.

0.1. Theorem. Suppose that a discrete group $\pi$ has the following properties:
1. The Strong Novikov Conjecture holds for $\pi$.
2. The natural map $\text{per}: \text{ko}_n(B\pi) \to KO_n(B\pi)$ is injective.
Then the Gromov Macroscopic Dimension Conjecture holds true for spin $n$-manifolds $M$ with the fundamental group $\pi_1(M) = \pi$.

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1. Introduction

In his study of manifolds with positive scalar curvature M. Gromov observed some large scale dimensional deficiency of their universal coverings: For an $n$-dimensional manifold $M$, its universal covering has to be at most $(n - 2)$-dimensional from the macroscopic point of view. For example, the product of a closed $(n - 2)$-manifold $N^{n-2}$ and the standard 2-sphere $M = N^{n-2} \times S^2$ admits a metric of positive scalar curvature.

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scalar curvature (by making the 2-sphere small). The universal covering \( \tilde{M} = \tilde{N}^{n-2} \times S^2 \) looks like an \((n-2)\)-dimensional space \( \tilde{N}^{n-2} \). Gromov predicted similar behavior for all manifolds with positive scalar curvature. He stated it in \([G1]\) as the following.

1.1. Conjecture (Gromov). For every closed Riemannian \( n \)-manifold \((M, g)\) with a positive scalar curvature there is the inequality

\[
\dim_{mc}(\tilde{M}, \tilde{g}) \leq n - 2
\]

where \((\tilde{M}, \tilde{g})\) is the universal cover of \( M \) with the pull-back metric.

Here \( \dim_{mc} \) stands for the macroscopic dimension \([G1]\). First time this conjecture was stated in the famous “filling” paper \([G2]\) in a different language. In \([GL]\) the conjecture was proved for 3-manifolds.

1.2. Definition. A map \( f : X \to K \) of a metric space is called uniformly cobounded if there is \( D > 0 \) such that \( \text{diam}(f^{-1}(y)) \leq D \).

A metric space \( X \) has the macroscopic dimension \( \dim_{mc} X \leq n \) if there is a uniformly cobounded proper continuous map \( f : X \to K^n \) to an \( n \)-dimensional polyhedron.

In \([G1]\) Gromov asked the following questions related to his conjecture which were stated in \([B1]\),\([B2]\) in the form of a conjecture:

1.3. Conjecture (C1). Let \((M^n, g)\) be a closed Riemannian \( n \)-manifold with torsion free fundamental group, and let \( \tilde{M}^n \) be the universal covering of \( M^n \) with the pull-back metric. Suppose that \( \dim_{mc} \tilde{M}^n < n \). Then

(A) If \( \dim_{mc} \tilde{M}^n \leq n \) then \( \dim_{mc} \tilde{M}^n \leq n - 2 \).

(B) If a classifying the universal covering map \( f : M^n \to B\pi \) can be deformed to an map with \( f(M^n) \subset B\pi^{(n-1)} \), then it can be deformed to a map with \( f(M^n) \subset B\pi^{(n-2)} \).

The Conjecture C1 is proven for \( n = 3 \) by D. Bolotov in \([B1]\). In \([B2]\) it was disproved for \( n > 3 \) by a counterexample. It turns out that Bolotov’s example does not admit a metric of positive scalar curvature \([B3]\) and hence it does not affect the Gromov Conjecture 1.1.

Perhaps the most famous conjecture on manifolds of positive scalar curvature is

The Gromov-Lawson Conjecture \([GL]\): A closed spin manifold \( M^n \) admits a metric of positive scalar curvature if and only if \( f_*([M]_{KO}) = 0 \) in \( KO_n(B\pi) \) where \( f : M^n \to B\pi \) is a classifying map for the universal covering of \( M^n \).
J. Rosenberg connected the Gromov-Lawson conjecture with the Novikov conjecture. Namely, he proved [R] that $\alpha f_*([M]_{KO}) = 0$ in $KO_n(C^*(\pi))$ in the presence of positive scalar curvature where $\alpha$ is the assembly map.

1.4. Conjecture (Strong Novikov Conjecture). The analytic assembly map

$$\alpha : KO_*(B\pi) \to KO_*(C^*(\pi))$$

is a monomorphism.

Then Rosenberg and Stolz proved the Gromov-Lawson conjecture for manifolds with the fundamental group $\pi$ which satisfies the Strong Novikov conjecture and has the natural transformation map

$$per : ko_*(B\pi) \to KO_*(B\pi)$$

injective ([RS], Theorem 4.13).

The main goal of this paper is to prove the Gromov Conjecture 1.1 under the Rosenberg-Stolz conditions.

2. Connective spectra and n-connected complexes

We refer to the textbook [Ru] on the subject of spectra. We recall that for every spectrum $E$ there is a connective cover $e \to E$, i.e., the spectrum $e$ with the morphism $e \to E$ that induces the isomorphisms for $\pi_i(e) = \pi_i(E)$ for $i \geq 0$ and with $\pi_i(e) = 0$ for $i < 0$. By $KO$ we denote the spectrum for real $K$-theory, by $ko$ its connective cover, and by $per : ko \to KO$ the corresponding transformation (morphism of spectra). We will use both notations for an $E$-homology of a space $X$: old-fashioned $E_*(X)$ and modern $H_*(X; E)$. We recall that $KO_n(pt) = \mathbb{Z}$ if $n = 0$ or $n = 4 \mod 8$, $KO_n(pt) = \mathbb{Z}_2$ if $n = 1$ or $n = 2 \mod 8$, and $KO_n(pt) = 0$ for all other values of $n$. By $S$ we denote the spherical spectrum. Note that for any spectrum $E$ there is a natural morphism $S \to E$ which leads to the natural transformation of the stable homotopy to $E$-homology $\pi_*^s(X) \to H_*(X; E)$.

2.1. Proposition. Let $X$ be an $(n - 1)$ connected $(n + 1)$-dimensional CW complex. Then $X$ is homotopy equivalent to the wedge of spheres of dimensions $n$ and $n + 1$ together with the Moore spaces $M(\mathbb{Z}_m, n)$.

Proof. It is a partial case of the Minimal Cell Structure Theorem (see Proposition 4C.1 and Example 4C.2 in [Ha]).

2.2. Proposition. The natural transformation $\pi_*^s(pt) \to ko_*(pt)$ induces an isomorphism $\pi_*^s(K/K^{(n-2)}) \to ko_n(K/K^{(n-2)})$ for any CW complex $K$. 
Proof. Since $\pi^s$ and $ko$ are both connective, it suffices to show that
$$\pi^s_n(K^{(n+1)}/K^{(n-2)}) \to ko_n(K^{(n+1)}/K^{(n-2)})$$
is an isomorphism. Consider the diagram generated by exact sequences of the pair $(K^{(n+1)}/K^{(n-2)}, K^n/K^{(n-2)})$
$$\oplus \mathbb{Z} \longrightarrow \pi^s_n(K^n/K^{(n-2)}) \longrightarrow \pi^s_n(K^{(n+1)}/K^{(n-2)}) \longrightarrow 0$$
$$\oplus \mathbb{Z} \longrightarrow ko_n(K^n/K^{(n-2)}) \longrightarrow ko_n(K^{(n+1)}/K^{(n-2)}) \longrightarrow 0.$$ Since the left vertical arrow is an isomorphism and the right vertical arrow is an isomorphism of zero groups, it suffices to show That $\pi^s_n(K^n/K^{(n-2)}) \to ko_n(K^n/K^{(n-2)})$ is an isomorphism.
Note that $\pi^s_n(S^k) \to ko_n(S^k)$ is an isomorphism for $k = n, n - 1$. In view of Proposition 3.2 it suffices to show that $\pi^s_n(M(Z_m, n - 1)) \to ko_n(M(Z_m, n - 1))$ is an isomorphism for any $m$ and $n$. This follows from the Five Lemma applied to the co-fibration $S^{n-1} \to S^{n-1} \to M(Z_m, n - 1)$.

3. INESSENTIAL MANIFOLDS

We recall the following definition which is due to Gromov.

3.1. Definition. An $n$-manifold $M$ is called essential if it does not admit a map $f : M \to K^{n-1}$ to an $(n-1)$-dimensional complex that induces an isomorphism of the fundamental groups. Note that always one can take $K^{n-1}$ to be the $(n-1)$-skeleton $B\pi^{(n-1)}$ of the classifying space $B\pi$ of the fundamental group $\pi = \pi_1(M)$.

If a manifold is not essential, it is called inessential.

The following is well-known to experts.

3.2. Proposition. An orientable $n$-manifold $M$ is inessential if and only if $f_*([M]) \in H_n(B\pi)$ is zero for a map $f : M \to B\pi_1(M)$ classifying the universal covering of $M$.

Proof. If $M$ admits a classifying map $f : M \to B\pi^{(n-1)}$, then clearly, $f_*([M]) = 0$.

Let $f_*([M]) = 0$ for some map $f : M \to B\pi_1(M)$ that induces an isomorphism of the fundamental groups. Let $o_n(f) \in H^n(M; \pi_{n-1}(F))$ be the primary obstruction to deform $f$ to the $(n-1)$-dimensional skeleton $B\pi^{(n-1)}$ and let $o_n(1_{B\pi}) \in H^n(B\pi; \pi_{n-1}(F))$ be the primary obstruction to retraction of $B\pi$ to the $(n-1)$-skeleton. Here $F$ denotes the homotopy fiber of the inclusion $B\pi^{(n-1)} \to B\pi$ and $\pi_{n-1}(F)$ is considered as a $\pi$-module. Since $f_*$ induce an isomorphism of the fundamental groups,
\[ f_* : H_0(M; \pi_{n-1}(F)) = \pi_{n-1}(F) \xrightarrow{\cong} H_0(B\pi; \pi_{n-1}(F)) = \pi_{n-1}(F) \] is an isomorphism. Then \( f_*([M] \cap o_n(f)) = f_*([M]) \cap o_n(1_{B\pi}) = 0. \) By the Poincare duality \( o_n(f) = 0. \)

### 3.3. Proposition
An orientable spin \( n \)-manifold \( M \) is inessential if \( f_*([M]_{ko}) \in ko_n(B\pi) \) is zero for a map \( f : M \to B\pi \) classifying the universal covering.

**Proof.** We assume that \( M \) is given a CW complex structure with one \( n \)-dimensional cell and \( f(M^{(n-1)}) \subset B\pi^{(n-1)} \). Let

\[ c^n_f : C_n(M) = \pi_n(M, M^{(n-1)}) \to \pi_{n-1}(B\pi^{(n-1)}) \]

be the primary obstruction cocycle for extending \( f|_{M^{(n-1)}} \) to the \( n \)-cell. In view of the \( \pi \)-isomorphism \( \pi_n(B\pi, B\pi^{(n-1)}) = \pi_{n-1}(B\pi^{(n-1)}) \) we may assume that \( c^n_f : \pi_n(M, M^{(n-1)}) \to \pi_n(B\pi, B\pi^{(n-1)}) \) is the induced by \( f \) homomorphism of the homotopy groups. The class \( o^n_f = [c^n_f] \) of \( c^n_f \) lives in the cohomology group \( H^n(M; \pi_n(B\pi, B\pi^{(n-1)}) \) with coefficients in a \( \pi \)-module. By the Poincare duality \( H^n(M; \pi_n(B\pi, B\pi^{(n-1)})) = H_0(M; \pi_0(B\pi, B\pi^{(n-1)})) \) is equal to the group \( \pi_n(B\pi/B\pi^{(n-1)}) \). The later group is the group of \( \pi \)-invariants of \( \pi_n(B\pi, B\pi^{(n-1)}) \) which is equal to \( \pi_n(B\pi/B\pi^{(n-1)}) \). Then \( c^n_f \) in \( \pi_n(B\pi/B\pi^{(n-1)}) \) coincides with \( \bar{f}_*(1) \) where \( \bar{f} : \hat{M}/M^{(n-1)} = S^n \to B\pi/B\pi^{(n-1)} \) is the induced map.

Assume that the obstruction \( o^n_f = [c^n_f] \neq 0. \) Then we claim that \( \bar{f}_* \) induces a nontrivial homomorphism for \( ko \) in dimension \( n \). In view of connectivity of \( ko \) it suffices to show this for the map \( \bar{f} : S^n \to B\pi^{(n+1)}/B\pi^{(n-1)}. \) By Proposition \( 2.1 \) \( B\pi^{(n+1)}/B\pi^{(n-1)} = (\vee S^n) \vee (\vee M(Z_m, n)) \vee (\vee S^{n+1}). \) Thus, it suffices to show that a non-null homotopic maps \( S^n \to S^n \) and \( S^n \to M(Z_m, n) \) induce nontrivial homomorphisms for \( ko_n \). The first is obvious, the second follows from the homotopy excision and the Five Lemma applied to the following diagram.

\[
\begin{array}{cccccc}
\pi_n(S^n) & \longrightarrow & \pi_n(S^n) & \longrightarrow & \pi_n(M(Z_m, n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
k_n(S^n) & \longrightarrow & k_n(S^n) & \longrightarrow & k_n(M(Z_m, n)) & \longrightarrow & 0.
\end{array}
\]

By the definition of the fundamental class the image of \([M]_{ko}\) is a generator in \( k_n(M, M^{(n-1)}) = k_n(M/M^{(n-1)}) = k_n(S^n) = \mathbb{Z}. \) Then
the following commutative diagram leads to the contradiction

\[
\begin{array}{ccc}
k\sigma_n(M) & \xrightarrow{f^*} & k\sigma_n(B\pi) \\
q_* & & p_* \\
k\sigma_n(M/M^{(n-1)}) & \xrightarrow{f^*} & k\sigma_n(B\pi/B\pi^{(n-1)})
\end{array}
\]

\[\square\]

There are many ways to detect essentiality of manifolds. One of them deals with the Lusternik-Schnirelmann category of \(X\), \(\text{cat}_{\text{LS}} X\), which is the minimal \(m\) such that \(X\) admits an open cover \(U_0, \ldots, U_m\) contractible in \(X\).

3.4. **Theorem.** A closed \(n\)-manifold is essential if and only if its Lusternik-Schnirelmann category equals \(n\).

We refer to [CLOT] for the proof and more facts about the Lusternik-Schnirelmann category. Note that \(\text{cat}_{\text{LS}} X\) is estimated from below by the cup-length of \(X\) possible with twisted coefficient and its estimated from above by the dimension of \(X\). The definition of the Lusternik-Schnirelmann category can be reformulated in terms of existence of a section of some universal fibration (called Ganea’s fibration). The characteristic class arising from the universal Ganea fibration over the classifying space \(B\pi\) is called the Berstein-Svartz class \(\beta_\pi \in H^1(\pi; I(\pi))\) of \(\pi\) where \(I(\pi)\) the augmentation ideal of the group ring \(\mathbb{Z}(\pi)\) (see [Ber], [Sv], [CLOT]. Formally, \(\beta_\pi\) is the image of the generator under connecting homomorphism \(H^0(\pi; \mathbb{Z}) \to H^1(\pi; I(\pi))\) in the long exact sequence generated by the short exact sequence of coefficients

\[0 \to I(\pi) \to \mathbb{Z}(\pi) \to \mathbb{Z} \to 0.\]

The main property of \(\beta_\pi\) is universality: Every cohomology class \(\alpha \in H^k(\pi; L)\) is the image of \((\beta_\pi)^k\) under a suitable coefficients homomorphism \(I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to L\). We refer to [DR] (see also [Sv]) for more details.

3.5. **Lemma.** Let \(M\) be a closed inessential \(n\)-manifold, \(n \geq 4\), supplied with a CW complex structure and let \(\pi = \pi_1(M)\). Then \(M\) admits a classifying map \(f : M \to B\pi\) of the universal covering such that \(f(M) \subset B\pi^{(n-1)}\) and \(f(M^{(n-1)}) \subset B\pi^{(n-2)}\).

**Proof.** Let \(f : M \to B\pi\) be the classifying map for the universal covering of \(M\). We may assume that \(f|_{M^{(2)}, M^{(2)}} \to B\pi^{(2)}\) is the identity map. First we show that \(f^*(\beta^{n-1}) = 0\) where \(\beta = \beta_\pi\) is the Berstein-Svarz class of \(\pi\). Assume that \(f^*(\beta^{n-1}) \neq 0\). Then \(a = [M] \cap f^*(\beta^{n-1}) \neq 0\)
by the Poincare Duality Theorem \cite{B}. There is \( u \in H^1(X; \mathcal{A}) \) such that \( a \cap u \neq 0 \) for some local system \( \mathcal{A} \) (see Proposition 2.3 \cite{DKR}). Then \( f^*(\beta)^{n-1} \cup u \neq 0 \). Thus the twisted cup-length of \( M \) is at least \( n \) and hence \( \text{cat}_{\text{LS}} M = n \). It contradicts to the Theorem 3.4.

Let \( g : M \to B\pi^{(n-1)} \) be a cellular classifying map. Consider the obstruction \( o^{n-1} \in H^{n-1}(M; \pi_{n-2}(B\pi^{(n-2)})) \) for extension of \( g|_{M^{(n-2)}} \) to the \((n-1)\)-skeleton \( M^{(n-1)} \). Here \( \pi_{n-2}(B\pi^{(n-2)}) \) is considered as a \( \pi \)-module. Note that \( o^{n-1} = g^*(o_B^{n-1}) \) where \( o_B^{n-1} \) is the obstruction for retraction of \( B\pi^{(n-1)} \) to \( B\pi^{(n-2)} \). In view of the universality of the Berstein-Svarz class there is a morphisms of \( \pi \)-modules \( I(\pi)^{n-1} \to \pi_{n-2}(B\pi^{(n-2)}) \) such that \( o_B^{n-1} \) is the image of \( \beta^{n-1} \) under the induced cohomology homomorphism. The square diagram induced by \( g \) and the fact \( g^*(\beta^n_{o}) = 0 \) imply that \( o^{n-1} = 0 \). Therefore there is a map \( f' : M^{(n-1)} \to B\pi^{(n-2)} \) that coincides with \( g \) on \( M^{(n-3)} \). Clearly, for \( n \geq 5 \), this map induces an isomorphism of the fundamental groups. It is still the case for \( n = 4 \), since \( g \) is the identity on the 1-skeleton and the 2-skeletons of \( M \) and \( B\pi \) are taken to be the same.

We show that there is an extension \( f : M \to B\pi^{(n-1)} \). It suffices to show that the inclusion homomorphism \( \pi_{n-1}(B\pi^{(n-2)}) \to \pi_{n-1}(B\pi^{(n-1)}) \) is trivial. This homomorphism coincides with the homomorphism

\[ \pi_{n-1}(E\pi^{(n-2)}) \to \pi_{n-1}(E\pi^{(n-1)}). \]

Since \( E\pi^{(n-2)} \) is contractible in \( E\pi \), by the Cellular Approximation Theorem it is contractible in \( E\pi^{(n-1)} \). This implies that the inclusion homomorphism is zero.

\[ \square \]

4. The Main Theorem

4.1. Lemma. Suppose that a classifying map \( f : M \to B\pi \) of a closed spin \( n \)-manifold, \( n > 3 \), takes the \( k_o \) fundamental class to 0, \( f_*([M]_{k_o}) = 0 \). Then \( f \) is homotopic to a map \( g : M \to B\pi^{(n-2)} \).

Proof. In view of Proposition 3.3 we may assume that \( f(M) \subset B\pi^{(n-1)} \).

In view of Lemma 3.5 we may additionally assume that \( f(M^{(n-1)}) \subset B\pi^{(n-2)} \). Also we assume that \( M \) has one \( n \)-dimensional cell. As in the proof of Proposition 3.3 we can say that the primary obstruction for moving \( f \) into the \((n-2)\)-skeleton is defined by the cocycle \( c_f : \pi_n(M, M^{(n-1)}) \to \pi_n(B\pi, B\pi^{(n-2)}) \) which defines the cohomology class \( o_f = [c_f] \) that lives in the group of coinvariants \( \pi_n(B\pi, B\pi^{(n-2)})_\pi = \pi_n(B\pi/B\pi^{(n-2)}) \) and is represented by \( \tilde{f}_*(1) \) for the homomorphism \( \tilde{f}_* : \pi_n(M/M^{(n-1)}) = \mathbb{Z} \to \pi_n(B\pi/B\pi^{(n-2)}) \) induced by the map of quotient spaces \( \tilde{f} : M/M^{(n-1)} \to B\pi/B\pi^{(n-2)} \).
We assume that the obstruction \([c_f]\) is nonzero. Show that \(\bar{f}_* : ko_n(S^n) \to ko_n(B\pi/B\pi^{n-2})\) is nontrivial to obtain a contradiction as in the proof of Proposition 3.3. Thus, \(\bar{f}_*(1)\) defines a nontrivial element of \(\pi_n(B\pi/B\pi^{(n-2)})\). The restriction \(n > 3\) implies that \(\bar{f}_*(1)\) survives in the stable homotopy group. In view of Proposition 2.2, the element \(\bar{f}_*(1)\) survives in the composition

\[
\pi_n(B\pi/B\pi^{(n-2)}) \to \pi^s_n(B\pi/B\pi^{(n-2)}) \to ko_n(B\pi/B\pi^{(n-2)}).
\]

The commutative diagram

\[
\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{\bar{f}_*} & \pi_n(B\pi/B\pi^{(n-2)}) \\
\cong & & \cong \\
\pi^s_n(S^n) & \xrightarrow{f_*} & \pi^s_n(B\pi/B\pi^{(n-2)}) \\
\cong & & \cong \\
ko_n(S^n) & \xrightarrow{\bar{f}_*} & ko_n(B\pi/B\pi^{(n-2)})
\end{array}
\]

implies that \(f_*(1) \neq 0\) for \(ko_n\). Contradiction.

The Strong Novikov Conjecture is connected to the Gromov Conjecture by means of the following theorem which is due to J. Rosenberg.

4.2. **Theorem** ([R]). Suppose \(M^n\) is a spin manifold with a fundamental group \(\pi\). Let \(f\) be classifying map \(f : M^n \to B\pi\). If \(M^n\) is a positive scalar curvature manifold then \(\alpha f_*([M^n]_{KO}) = 0\) where \(\alpha : KO_*(B\pi) \to KO_*(C_r^*(\pi))\) is the analytic assembly map.

4.3. **Theorem.** Suppose that a discrete group \(\pi\) has the following properties:

1. The Strong Novikov Conjecture holds for \(\pi\).
2. The natural map \(\text{per} : ko_n(B\pi) \to KO_n(B\pi)\) is injective.

Then the Gromov Conjecture holds for spin \(n\)-manifolds \(M\) with the fundamental group \(\pi_1(M) = \pi\).

**Proof.** Let \(M\) be a closed spin \(n\)-manifold that admits a metric with positive scalar curvature. By Theorem 4.2 \(\alpha \circ \text{per} \circ f_*([M]_{ko}) = 0\). The conditions on \(\pi\) imply that \(f_*([M]_{ko}) = 0\) for the classifying map \(f : M \to B\pi\). Then by Lemma 4.1 \(f\) is homotopic to \(g : \tilde{M} \to B\pi^{(n-2)}\). The induced map of the universal covering spaces \(\tilde{M} \to E\pi^{(n-2)}\) produces the inequality \(\dim_{mc} \tilde{M} \leq n - 2\).

4.4. **Corollary.** The Gromov Conjecture holds for spin \(n\)-manifolds \(M\) with the fundamental group \(\pi_1(M) = \pi\) having \(\text{cd}(\pi) \leq n + 3\) and satisfying the Strong Novikov Conjecture.
Proof. We show that $\text{per}$ is an isomorphism in dimension $n$ in this case. Let $\mathbf{F} \to k_0 \to KO$ be the fibration of spectra induced by the morphism $k_0 \to KO$. Then $\pi_k(\mathbf{F}) = 0$ for $k \geq 0$ and $\pi_k(\mathbf{F}) = \pi_k(KO) = KO_k(pt) = 0$ if $k = -1, -2, -3 \mod 8$. The Atiyah-Hirzebruch $F$-homology spectral sequence for $B\pi$ implies that $H_n(B\pi; \mathbf{F}) = 0$ since all entries on the $n$-diagonal in the $E^2$-term are 0. Then the coefficient exact sequence for homology $H_n(B\pi; \mathbf{F}) \to k_0n(B\pi) \to KO_n(B\pi) \to \ldots$ implies that $\text{per} : k_0n(B\pi) \to KO_n(B\pi)$ is a monomorphism.

We note that this Corollary for $\text{cd}(\pi) \leq n - 1$ first was proven in [B3].

4.5. Corollary. The Gromov Conjecture holds for spin $n$-manifolds $M$ with the fundamental group $\pi_1(M) = \pi$ having finite $B\pi$ and with $\text{asdim} \pi \leq n + 3$.

Proof. This is a combination of the fact that the Strong Novikov conjecture holds true for such groups $\pi$ ([Ba, DF], the above Corollary, and the inequality $\text{cd}(\pi) \leq \text{asdim} \pi$ proven in [Dr].

4.6. Corollary. The Gromov conjecture holds for spin $n$-manifolds $M$ with the fundamental group $\pi_1(M)$ equal the product of free groups $F_1 \times \cdots \times F_n$. In particular, it holds for free abelian groups.

Proof. The formula for homology with coefficients in a spectrum $\mathbf{E}$:

$$H_i(X \times S^1; \mathbf{E}) \cong H_i(X; \mathbf{E}) \oplus H_{i-1}(X; \mathbf{E})$$

implies that if $k_0s(X) \to KO_s(X)$ is monomorphism, then $k_0s(X \times S^1) \to KO_s(X \times S^1)$ is a monomorphism. By induction on $m$ using the Mayer-Vietoris sequence this formula can be generalize to the following

$$H_i(X \times (\bigvee_m S^1); \mathbf{E}) \cong H_i(X; \mathbf{E}) \oplus \bigoplus_m H_{i-1}(X; \mathbf{E}).$$

Therefore,

$$k_0s(X \times (\bigvee_m S^1)) \to KO_s(X \times (\bigvee_m S^1))$$

is a monomorphism.

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