D-branes and Azumaya noncommutative geometry:
From Polchinski to Grothendieck

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Abstract

In this continuation of [L-Y3] [arXiv:0709.1515 [math.AG]], [L-L-S-Y] [arXiv:0809.2121 [math.AG]], [L-Y4] [arXiv:0901.0342 [math.AG]], [L-Y5] [arXiv:0907.0268 [math.AG]], and [L-Y6] [arXiv:0909.2291 [math.AG]], we give an overview of the posted part of the project and then take it as background to introduce Azumaya noncommutative $C^\infty$-manifold and the four aspects of morphisms therefrom to a projective complex manifold. This gives us then a description of supersymmetric D-branes of A-type in a Calabi-Yau manifold along the line of the Polchinski-Grothendieck Ansatz. The notion of Kähler differentials and their tensors for an Azumaya noncommutative space are introduced. Donaldson’s picture of Lagrangian and special Lagrangian submanifolds as selected from the zero-locus of a moment-map on a related space of maps can be merged into the setting of morphisms from Azumaya manifolds with a fundamental module. As a pedagogical toy model for illustration, we study D-branes of A-type in a Calabi-Yau torus. Simple as it is, it reveals already several features of D-branes of A-type, including their assembling/disassembling. The short-vs.-long string wrapping behavior of matrix-strings in the string-theory literature can be produced in this context as well. In addition to the previous comparison with stringy works made, the 4th theme (subtitled: “Gómez-Sharpe vs. Polchinski-Grothendieck”) of Sec. 2.4 is to be read with the work [G-S] [arXiv:hep-th/0008150], while the 2nd theme (subtitled: “Donagi-Katz-Sharpe vs. Polchinski-Grothendieck”) of Sec. 4.2 is to be read with the work [D-K-S] [arXiv:hep-th/0309270]. Sec. 4.3, though not yet ready to be subtitled “Denef vs. Polchinski-Grothendieck”, is to be read with the work [De] [arXiv:hep-th/0107152]. Some directly related string-theory remarks are added to the end of each section.

Key words: D-brane, A-type, B-type; Polchinski-Grothendieck Ansatz, Azumaya scheme, fundamental module, morphism; stack of D0-branes, representation-theoretical atlas; B-field, gerbe, twisted sheaf, D-module, Azumaya quantum scheme, flat connection; Azumaya manifold, Kähler differential, tensor product; special Lagrangian morphology, moment map; matrix string, filtered structure, amalgamation/decomposition (assembling/disassembling) of D-branes; short vs. long string wrapping.

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• A reflection on string/M/F/···? theory, branes, and dualities:

玄之又玄, 衆妙之門。

(Mystery and beyond mystery, door to all magics.)

大方無隅, 大器晚成, 大音希聲, 大象無形。

( So large that it has no bounds; so big that it takes a long time to make; so harmonious that it fits no tunes; so beautiful that it assumes no shapes.)

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a (From C.-H.L.) Before moving up, I went to see Prof. Freed, who is for sure a role model for mathematicians who pursue physics issues. There in his office he delivered to me one of the most important advices I ever received, “No matter how you’re interested in physics, you are still a mathematician and you have to find your ground in mathematics.” With the unexpected luck to come to one of the major and most active centers of algebraic geometry in the world after being exposed to Calabi-Yau spaces and toric geometry in Prof. Candelas’ weekly group meetings, I quietly decided to take algebraic geometry as my new starting point, despite a well-disposed warning passed to me as a quote of a famous mathematician to a renowned string-theorist: “If you haven’t learned algebraic geometry by the age of 28, then you’d better just forget about it.” It is the friendly, concrete, and down-to-earth way Joe Harris thinks about and teaches algebraic geometry and Mihnea Popa’s insightful, systematic, and wide-spanned series of courses and seminars at Harvard from Grothendieck’s foundations to research topics at the frontier during the years 2001–2005 that turned around my supposedly doomed fate of failure.

A miraculous coincidence happened: During these years Shiraz Minwalla was giving an equally insightful, systematic, and wide-spanned series of courses at Physics Department from quantum field theory foundation, supersymmetry, phase structures in a Wilson’s theory-space, to basic and frontier topics in string theory. His enthusiastic lectures constantly filled the classroom with heat and turned the would-be terrifying courses into intellectual enrichment. Thus, two exceptionally energetic teachers - one from the mathematics side and the other from the physics side - and one common student with his mind quietly set on a central question: What is a D-brane? I would not expect this project to finally get started in year 2007 without having met them both at the right time and right place in the first half of 2000s. With the above input from Miheea and Shiraz and the initiation of this long-delayed and almost-abandoned project, the special unexpected sequence of lucks given to me – namely an accidental stringy journey: Thurston ⇒ Alvarez & Nepomechie ⇒ Candelas & Distler ⇒ Yau – at last have a purpose and acquire their meaning as a whole.

Above all these, the daily summary of works to each other with Ling-Miao over the years has been providing me with tremendous momenta to the theme. Only an extremely lucky person is given simultaneously a demanding stringy detour/journey, an unusual opportunity, and a long-standing supporting soulmate and it takes not only effort but also great fortune so that these coincidences and accidental encounters are not just in vain.

b Lao-Tzu (600 B.C.), Tao-te Ching (The Scripture on the Way and its Virtue), excerpt from Chapter 1 and Chapter 41; English translation by Ling-Miao Chou.
0. Introduction and outline.

D-branes, defined in string theory as boundary conditions for the end-points of open strings, appeared in the theory in the second half of 1980s and have become a central object in string theory since year 1995. They reveal themselves in various faces/formats, depending on where they are looked at on the related Wilson’s theory-space – either of superstring theory, locally parameterized by \((\tau^F, g_s)\) with \(\tau^F\) being the tension of a fundamental open/closed string and \(g_s\) being the string coupling constant related to the condensation of the dilaton field, or of a 2-dimensional superconformal field theory with boundary \((d = 2 \text{ SCFT})\).

Where we are in the Wilson’s theory-space of strings and when we are in the history of D-branes.

In this review of [L-Y3] (D(1)), [L-L-S-Y] (D(2), with Si Li and Ruifang Song), [L-Y4] (D(3)), [L-Y5] (D(4)), [L-Y6] (D(5)), and some preparatory part of [L-Y7], we stand

- (where)
  - in the regime of the related Wilson’s theory-space either where the D-brane tension is small but still large enough compared to \(\tau^F\) – so that the D-branes remain easily excitable by open strings with a boundary attached to them and that they won’t bend the causal structure of the target space-time to form a surrounding event horizon to close themselves up to black branes – or where a boundary state of a \(d = 2 \text{ SCFT}\) remains having a space-time interpretation – so that “branes” are really branes; and

- (when)
  - either at the year 1995 when Polchinski [Pol2] realized that D-branes serve as a source for Ramond-Ramond fields created by excitations of closed superstrings and Witten [Wi4] gave an immediate follow-up to consider bound systems of D-branes or, exactly speaking, at the year 1988 when D-branes came into light, in Polchinski and Cai’s work [P-C], and the mass-tower of fields thereupon and the dynamics of these fields follow respectively from open-superstring spectrum and from the vanishing requirement of the conformal anomaly on the superconformal field theory on the open-string world-sheet, (see [Pol4] for related references),

and ask, based on what string-theorists taught us,
Q. What truly is a D-brane in its own right?

Our starting point is a paragraph in Polchinski’s textbook [Pol4: vol. I, Sec. 8.7, theme: The D-brane action, p. 272] concerning

- a matrix-type noncommutative enhancement of target space-time when probed by stacked D-branes;

see also [Joh: Sections 4.10, 5.5, 9.7, 16.3] and Sec. 1.1 of the current work for more discussions/review. It turns out that understanding this mysterious behavior of D-branes holds a key to realize a fundamental mathematical nature of D-branes. In the current review, we explain this particular aspect of D-branes, namely the one from

merging the above mysterious behavior of stacked D-branes and Grothendieck’s viewpoint of local contravariant equivalence of function rings and geometries.

This brings in the notion of Azumaya noncommutative schemes with a fundamental module in the algebro-geometric category or Azumaya noncommutative manifolds with a fundamental module in the differential/symplectic topological category. The correct notion of morphisms therefrom to a string target-space gives us then a re-formulation of the notion of D-branes that can reproduce several key features of D-branes in the string-theory literature, originally derived from open-string-induced quantum field theory on D-branes, cf. [L-Y3], [L-Y4], and [L-Y5]. Furthermore, in the algebro-geometric side, the moduli space/stack of such morphisms has a surprising feature of serving as a master moduli space/stack that simultaneously incorporates several different moduli spaces/stacks in commutative algebraic geometry, cf. [L-L-S-Y] and [L-Y6]. In the symplecto-geometric side, the notion rings naturally with how Donaldson looks at special Lagrangian submanifolds in a Calabi-Yau manifold as a special class of maps into the Calabi-Yau space, cf. [Don] and [Hi2]. This matches nicely with the role of D-branes as a master object in string theory. All these together give us an evidence that

the Azumaya-type noncommutative geometric structure on a D-brane world-volume, rather than on the string target-space(-time), can provide us with an alternative starting point to understanding D-branes.

As a hindsight, this is a step one could already take in year 1988, instead of nearly twenty years later. In particular, one may try to re-do everything about D-branes with this Ansatz.

The DOR triangle.

While this review and the so-far-posted part of the project discuss only D-branes, one should always keep in mind the mysteries of the other two closely related disciplines as well: (Cf. bottom of the D-brane/open-string/Ramond-Ramond-field triangle.)

Indeed, it is a pursuit of understanding open Gromov-Witten theory that turned us back to re-thinking D-branes, cf. [L-Y3: footnote 1]. Furthermore, it is known that Ramond-Ramond
fields on a string target-space(-time) are not just differential forms thereupon. Their complications are already manifest from both the F-theory interpretation of the rank-0 Ramond-Ramond field $C^{(0)}$ ([Va2] and [M-V]) and the $SL(2,\mathbb{Z})$-duality of type IIB superstring theory that exchanges the rank-2 Ramond-Ramond field $C^{(2)}$, which sources D-strings, with the $B$-field, which sources fundamental strings ([Schw]). In particular, such yet-to-be-understood fundamentals of Ramond-Romand fields are necessary to realize how and why a general collection of D-branes can live on a compact Calabi-Yau space without violating the charge conservation law of D-branes.

**String-theoretical remarks.**

As an object developed for more than twenty years since late 1980s, that have led to numerous new insights to string theory, string-theorists do have a very good reason to question that,

- *Didn’t we string-theorists already know all the fundamentals of D-branes?*

For string-theorists who do have this doubt, we suggest them to recall a most important example of a similar gap or concept-delay between physics and mathematics, namely the notion of “path-integrals in quantum field theory”. It began with Richard Feynman’s Ph.D. thesis under John A. Wheeler at Princeton, 1942, with title “The principle of least action in quantum mechanics” and has now become a central/standard language in quantum field theory. It provides even the very definition of what it means when claiming two quantum field theories are “the same”, a concept that pervades string literatures nowadays. Yet, despite the sixty-eight years that have passed, mathematicians still look at this notion with amazement and awe. During this time, there have been mathematical attempts to understand it – first in an infinite-dimensional analysis/measure-theory aspect, later in a functor-and-category aspect and in a combinatorics aspect, and more recently in a motive aspect. Each of these reveals some mathematical nature of path-integrals behind physicists’ formal rules. With this example in mind, string-theorists may be willing to re-think about the phenomenon of space-time being enhanced to matrix-valued noncommutativity when probed by stacked D-branes.

- *Is it really the space-time geometry that is enhanced or, indeed, is it the world-volume of the stacked D-branes that is enhanced first?*

While it looks plausible in some situations for a trading between noncommutativity from these two opposing aspects, to our best understanding at the moment, such trading can only be at best partial and the full moduli problems different aspects lead to are different mathematically. Indeed, the very question turns out to be related to another question:

- *Taking D-branes, say, of the lowest dimensions in a theory, as truly fundamental objects, then can they recover their signatory features by themselves without resorting to open strings?*

In this notes/review, we are not trying to tell string-theorists what they should think. Rather, we explain our thoughts here in a hopefully language-wise-friendlier way than our previous works and welcome string-theorists to ask themselves the above same questions and find their own answers. This is not an issue of rigor; we regularly mix ourselves with string-theorists and try to learn/appreciate the way they think and hence already pass that long time ago. *This is an issue of what exactly is fundamentally at work for D-branes.* In retrospect, the ansatz we propose in [L-Y3] (D(1)), cf. Sec. 1.1 of this review, could have been observed by string-theorists themselves in year 1988 as well, due to its simplicity and the fact that the computation of open string spectrum and the induced massless field contents on stacked D-branes were already ready at that time, if Grothendieck’s EGA and SGA series of works had entered the stringy community.
before then. Indeed, there have been efforts from different string-theory groups to understand better the foundation of D-branes. E.g. the work [G-Sh] of Tomás Gómez and Eric Sharpe at year 2000 (cf. Sec. 2.4, Test (4), of the current review) and see [L-Y3: Remark 2.2.5] for a sample of other stringy groups. They all influenced our thoughts on D-branes one way or another during the brewing years. In Spring 2007, after a train of discussions with Duiliu-Emanuel Diaconescu in December 2006 on open-string world-sheet instantons and a vanishing lemma of open Gromov-Witten invariants (cf. [D-F] and [L-Y2]) that turned us back to re-thinking D-branes, we realized that it is a carefully selected spirit of Grothendieck’s works on commutative algebraic geometry that we have to take, but not necessarily its full contents. It is this realization that finally enabled us to re-read stringy works on D-branes in Grothendieck’s eyes. Occasionally, we wonder if the history were rerun and the ansatz were observed by string-theorists either in 1995 or 1988, how things would have been different in the physics and the mathematics side. We would never know. All we can do is explain and push this ansatz to the extreme and let string-theorists decide for themselves. Overall, D-branes remain a very complicated and mysterious object to us. We have yet much to learn and to be amazed.

The very limited short list of stringy literatures quoted in the current work and the more special ones in the previous D(1) - D(5) of the project are among those we constantly go back to to find new insights, new understandings, and new guides. In string-theorists’ standard, a paper passing five years can be regarded as an old paper. Yet, these “very old” papers remain to us a reservoir of inspirations. They remain rich deposits of gold for string-theory-oriented mathematical minds. On the other hand, while they influenced us greatly, they by no means reflect the activity of this field. Unfamiliar readers are suggested to read the book ‘D-branes’ [Joh] by Johnson and the string textbook [B-B-Sc] by Becker, Becker, and Schwarz and the original (mainly physical) works quoted therein to gain balanced and more complete insights and feelings of the various aspects of D-branes in the first seven years to a decade, 1995–2002/1995–2006, after the second revolution of string theory in 1995.

**Convention.** Standard notations, terminology, operations, facts in (1) physics aspects of D-branes; (2) supersymmetry; (3) (commutative) algebraic geometry / stacks; (4) associative rings and algebras; (5) symplectic / calibrated geometry; (6) sheaves on manifolds can be found respectively in (1) [Po], [Joh]; (2) [W-B], [Arg], [Te]; (3) [E-H], [Ha] / [L-MB]; (4) [Pi], [Re]; (5) [McD-S] / [Ha-L], [McL]; (6) [Kas-S].

- All schemes are Noetherian over \( \mathbb{C} \).
- All associative rings and algebras are unital.
- All manifolds are smooth, closed, and orientable unless otherwise noted.
- Index whose precise value is either clear from the text or irrelevant to the discussion is occasionally omitted to a \( \bullet \) to make the expression cleaner/simpler.
- \( \mathbb{C} \) as a field in algebra vs. \( \mathbb{C} \) as the complex line in complex/symplectic geometry.
- D-branes of A-type (resp. B-type) are supported on special Lagrangian cycles (resp. holomorphic cycles). The name follows [B-B-St], [O-O-Y], and [H-I-V].
- Field \( B \) in the sense of quantum field theory vs. base scheme \( B \) in algebraic geometry vs. D-branes of B-type in string theory.
Noncommutative algebraic geometry is a very technical topic, with various nonequivalent points of view and formulations. For the current work, [Art] of Artin and [A-N-T] of Artin, Nesbitt, and Thrall on Azumaya algebras, [K-R] of Kontsevich and Rosenberg on noncommutative smooth spaces, and [LeB1], [LeB2], [LeB3] of Le Bruyn on noncommutative geometry are particularly relevant. See [L-Y3: References] for more references.

Mathematicians without quantum field theory background may still get a feel of D-branes in physicists’ perspective from [Zw] of Zwiebach; see also [B-B-Sc] of Becker, Becker, and Schwarz for an updated account of string theory up to year 2006.

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1 D-branes and the Polchinski-Grothendieck Ansatz.

We review in this section how the Polchinski-Grothendieck Ansatz for D-branes arises when a crucial open-string-induced behavior of fields on stacked D-branes is re-read from Grothendieck’s viewpoint. Readers are referred to [L-Y3: Sec. 2] for further discussions.

1.1 From stacked D-branes to the Polchinski-Grothendieck Ansatz.

From a fundamental question, we are led to an intrinsic nature of D-branes.

What is a D-brane?

A D-brane (i.e. Dirichlet membrane) is meant to be a boundary condition for open strings in whatever form it may take, depending on where we are in the related Wilson’s theory-space. A realization of D-branes that is most related to the current work is an embedding $f : X \to Y$ of a manifold $X$ into the open-string target space-time $Y$ with the end-points of open strings being required to lie in $f(X)$. This sets up a 2-dimensional Dirichlet boundary-value problem from the field theory on the world-sheet of open strings. Oscillations of open strings with end-points in $f(X)$ then create a mass-tower of various fields on $X$, whose dynamics is governed by open string theory. This is parallel to the mechanism that oscillations of closed strings create fields in space-time $Y$, whose dynamics is governed by closed string theory. Cf. Figure 1-1-1.

Let $\xi := (\xi^a)_a$ be local coordinates on $X$ and $\Phi := (\Phi^a; \Phi^\mu)_{a,\mu}$ be local coordinates on $Y$ such that the embedding $f : X \leftrightarrow Y$ is locally expressed as

$$\Phi = \Phi(\xi) = (\Phi^a(\xi); \Phi^\mu(\xi))_{a,\mu} = (\xi^a, \Phi^\mu(\xi))_{a,\mu};$$

i.e., $\Phi^a$’s (resp. $\Phi^\mu$’s) are local coordinates along (resp. transverse to) $f(X)$ in $Y$. This choice of local coordinates removes redundant degrees of freedom of the map $f$, and $\Phi^\mu = \Phi^\mu(\xi)$ can be regarded as (scalar) fields on $X$ that collectively describes the positions/shapes/fluctuations
of $X$ in $Y$ locally. Here, both $\xi^a$’s, $\Phi^a$’s, and $\Phi^{\alpha}$’s are $\mathbb{R}$-valued. The open-string-induced gauge field on $X$ is locally given by the connection 1-form $A = \sum_a A_a(\xi)d\xi^a$ of a $U(1)$-bundle on $X$.

When $r$-many such D-branes $X$ are coincident/stacked, from the associated massless spectrum of (oriented) open strings with both end-points on $f(X)$ one can draw the conclusion that

1. The gauge field $A = \sum_a A_a(\xi)d\xi^a$ on $X$ is enhanced to $u(r)$-valued.

2. Each scalar field $\Phi^\mu(\xi)$ on $X$ is also enhanced to matrix-valued.

Property (1) says that there is now a $U(r)$-bundle on $X$. But

- **Q. What is the meaning of Property (2)?**

For this, Polchinski remarks that: (Note: Polchinski’s $X^\mu$ and $n = \Phi^\mu$ and $r$.)

- [quote from [Pol4: vol. I, Sec. 8.7, p. 272]] “For the collective coordinate $X^\mu$, however, the meaning is mysterious: the collective coordinates for the embedding of $n$ D-branes in space-time are now enlarged to $n \times n$ matrices. This ‘noncommutative geometry’ has proven to play a key role in the dynamics of D-branes, and there are conjectures that it is an important hint about the nature of space-time.”

(See also a comment in [Joh: Sec. 4.10 (p. 125)].) From the mathematical/geometric perspective,

- Property (2) of D-branes, the above question, and Polchinski’s remark

can be incorporated into the following single guiding question:

- **Q. [D-brane] What is a D-brane intrinsically?**

In other words, what is the intrinsic nature/definition of D-branes so that by itself it can produce the properties of D-branes (e.g. Property (1) and Property (2) above) that are consistent with, governed by, or originally produced by open strings as well?

From Polchinski to Grothendieck.

To understand Property (2), one has two perspectives:

(A1) [coordinate tuple as point] A tuple $\langle \xi^a \rangle_a$ (resp. $\langle \Phi^a; \Phi^\mu \rangle_{a,\mu}$) represents a point on the world-volume $X$ of the D-brane (resp. on the target space-time $Y$).

(A2) [local coordinates as generating set of local functions] Each local coordinate $\xi^a$ of $X$ (resp. $\Phi^a$, $\Phi^\mu$ of $Y$) is a local function on $X$ (resp. on $Y$) and the local coordinates $\xi^a$’s (resp. $\Phi^a$’s and $\Phi^\mu$’s) together form a generating set of local functions on the world-volume $X$ of the D-brane (resp. on the target space-time $Y$).

While Aspect (A1) leads one to the anticipation of a noncommutative space from a noncommutatization of the target space-time $Y$ when probed by coincident D-branes, Aspect (A2) of Grothendieck leads one to a different – seemingly dual but not quite – conclusion: a noncommutative space from a noncommutatization of the world-volume $X$ of coincident D-branes, as follows.

Denote by $\mathbb{R}\langle \xi^a \rangle_a$ (resp. $\mathbb{R}\langle \Phi^a; \Phi^\mu \rangle_{a,\mu}$) the local function ring on the associated local coordinate chart on $X$ (resp. on $Y$). Then the embedding $f : X \to Y$, locally expressed as
\[ \Phi = \Phi(\xi) = (\Phi^a(\xi); \Phi^\mu(\xi))_{a,\mu} = (\xi^a; \Phi^\mu(\xi)), \]\n
is locally contravariantly equivalent to a ring-homomorphism\footnote{For string-theorists: I.e. pull-back of functions from the target-space \( Y \) to the domain-space \( X \) via \( f \). See Sec. 2.1 for more about Grothendieck’s philosophy for constructing ‘geometric spaces’ and ‘morphisms’ among them; cf. \([E-H]\) and \([Ha]\).}

\[ f^\sharp : \mathbb{R} \langle \Phi^a; \Phi^\mu \rangle_{a,\mu} \rightarrow \mathbb{R} \langle \xi^a \rangle_a, \text{ generated by } \Phi^a \mapsto \xi^a, \Phi^\mu \mapsto \Phi^\mu(\xi). \]

When \( r \)-many such D-branes are coincident, \( \Phi^\mu(\xi) \)’s become \( M_r(\mathbb{C}) \)-valued. Thus, \( f^\sharp \) is promoted to a new local ring-homomorphism:

\[ \hat{f}^\sharp : \mathbb{R} \langle \Phi^a; \Phi^\mu \rangle_{a,\mu} \rightarrow M_r(\mathbb{C} \langle \xi^a \rangle_a), \text{ generated by } \Phi^a \mapsto \xi^a \cdot 1, \Phi^\mu \mapsto \Phi^\mu(\xi). \]

Under Grothendieck’s contravariant local equivalence of function rings and spaces, \( \hat{f}^\sharp \) is equivalent to saying that we have now a map \( \hat{f} : X_{\text{noncommutative}} \rightarrow Y \), where \( X_{\text{noncommutative}} \) is the new domain-space, associated now to the enhanced function-ring \( M_r(\mathbb{C} \langle \xi^a \rangle_a) \). Thus, the D-brane-related noncommutativity in Polchinski’s treatise [Pol4], as recalled above, implies the following ansatz when it is re-read from the viewpoint of Grothendieck:

**Polchinski-Grothendieck Ansatz [D-brane: noncommutativity].** The world-volume of a D-brane carries a noncommutative structure locally associated to a function ring of the form \( M_r(R) \), where \( r \in \mathbb{Z}_{\geq 1} \) and \( M_r(R) \) is the \( r \times r \) matrix ring over \( R \).

We call a geometry associated to a local function-rings of matrix-type *Azumaya-type noncommutative geometry*; cf. [Art] and [L-Y6: footnote 23]. Note that when the closed-string-created \( B \)-field on the open-string target space(-time) \( Y \) is turned on, \( R \) in the ansatz can become noncommutative itself; cf. [S-W], [L-Y6], and Sec. 2.3.2 of the current review. Cf. Figure 1-1-2.

An additional statement hidden in this Ansatz that follows from mathematical naturality is that

- fields on \( X \) are local sections of sheaves \( \mathcal{F} \) of modules of the structure sheaf \( \mathcal{O}^{nc}_{\mathcal{X}} \) of \( X \) associated to the above noncommutative structure.

Furthermore, this noncommutative structure on D-branes (or D-brane world-volumes) is more fundamental than that of space-time in the sense that,

- from Grothendieck’s equivalence, the noncommutative structure of space-time, if any, can be detected by a D-brane only when the D-brane probe itself is noncommutative.

When D-branes are taken as fundamental objects as strings, we no longer want to think of their properties as derived from open strings. Rather, D-branes should have their own intrinsic nature in discard of open strings. Only that when D-branes co-exist with open strings in space-time, their nature has to be compatible/consistent with the originally-open-string-induced properties thereon. It is in this sense that we think of a D-brane world-volume as an Azumaya-type noncommutative space, following the Ansatz, on which other additional compatible structures – in particular, a Chan-Paton module – are defined.

### 1.2 String-theoretical remarks.

Now that a detailed explanation of the ansatz is re-given in Sec. 1.1, Readers from string-theory side are suggested to re-think about the theme ‘String-theoretical remarks’ in Sec. 0. Further remarks follow.
Figure 1-1-2. Two counter (seemingly dual but not quite) aspects on noncommutativity related to coincident/stacked D-branes: (1) (= (A2) in the text) noncommutativity of D-brane world-volume as its fundamental/intrinsic nature versus (2) (= (A1) in the text) noncommutativity of space-time as probed by stacked D-branes. (1) leads to the Polchinski-Grothendieck Ansatz and is more fundamental from Grothendieck’s viewpoint of contravariant equivalence of the category of local geometries and the category of function rings.
How about Lie-algebra type function-rings?

Naively, one may feel that even if a noncommutative structure does emerge on the world-volume of stacked D-branes, it should be of Lie-algebra type. There are reasons against taking Lie algebras as the function rings locally for the D-brane world-volume geometry:

- **(physical reason)**
  A completely physical reason is that indeed Lie algebra is not truly the algebra that is used in describing the action for the open-string-induced quantum field theory on the world-volume of stacked D-branes before taking trace. While it is possible that the potential term for the fields governing the deformation of D-branes in such an action is given by a combination of Lie-brackets of these fields, the kinetic term for them before taking an appropriate trace is not. Since every associative (unital) algebra is associated also with a Lie algebra by taking commutators, it is really the matrix product of these fields that is truly behind the action for stacked D-branes.

- **(mathematical reason)**
  There are also mathematical reasons: A Lie algebra is non-associative and has no identity element with respect to the Lie product. Such an algebra is difficult to do geometry due to the difficulty to introduce the notion of open sets by “localizing the algebra” (i.e. inverting some elements in the Lie algebra). Without this concept, the passage from local to global via gluing open sets becomes obstructed. Furthermore, there is no algebra-homomorphism from a commutative ring $\mathbb{R}$ to a Lie algebra $L$. (Even if one ignores the issue of the identity element, then any such algebra-homomorphism is the zero-homomorphism.) Thus, no morphisms

\[ \text{Space}(L) \rightarrow \text{Space}(\mathbb{R}) := \text{Spec} \mathbb{R} \]

can be constructed. This makes it difficult to talk about a stacked D-brane, say, in a Calabi-Yau space as a morphism from the Lie-type-noncommutative D-brane world-volume (even if such a notion exists) to the Calabi-Yau space.

Need supersymmetry or $B$-field?

The Azumaya-type noncommutative structure on the D-brane world-volume stated in the ansatz occurs whether or not there is a supersymmetry for the D-brane configuration or a $B$-field background on the target-space(-time). The latter may influence this structure but is not the cause for it; cf. [L-Y6] (D(5)). The only cause for this structure is simply open-strings\(^4\). This suggests that it is indeed a very fundamental property of D-branes. See also [L-Y4: footnote 1] (D(3)) for a related remark\(^5\).

A comparison with quantum mechanics and quantization of strings.

For string-theorists who still feel uncomfortable about this ansatz, the following comparison is worthy of a thought\(^6\).

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\(^4\)For this reason, as a hindsight, this noncommutativity-enhancement phenomenon and the ansatz should really be observed in year 1988 since every necessary ingredient was readily there; cf. Sec. 0.

\(^5\)We thank Lubos Motl for a related comment.

\(^6\)We thank Cumrun Vafa for the conceptual point addressed here made in his string-theory course, spring 2010, and a discussion.
In the study of quantum mechanics or quantization of strings, there are bosonic fields $X^\mu$ on the particle world-line or the string world-sheet that are associated to the space-time coordinates. Collectively, they describe the positions and deformations of the particle or string when time flows. Quantization of the particle or string moving in a space-time renders $X^\mu$ operator-valued. Thus, formally, in this process the space-time coordinates become operator-valued as well.

- **Q. [quantized or not quantized]** When a particle or string moving in a space-time gets quantized, does the space-time itself get quantized as well, or not?

Replacing ‘particle’ or ‘string’ by ‘coincident/stacked D-branes’ and ‘get(s) quantized’ by ‘have/has $X^\mu$ become matrix-valued’, one can ask exactly the same question for D-branes. If one answers the above question by: “Yes, the space-time is also (enforced to be) quantized”, then for the replaced question for D-branes, one is led to the currently more preferred view in the string-theory community that the space-time become matrix-ring-type-noncommutatively enhanced. **If one answers instead: “No, the space-time is not quantized; it remains classical”, then for the replaced question for D-branes, one is led exactly to the Polchinski-Grothendieck Ansatz!**

2 The algebro-geometric aspect: Azumaya noncommutative schemes with a fundamental module and morphisms therefrom.

Once one understands the fundamental Azumaya-algebra nature of the local function-rings of D-branes (and their world-volume), since the D-branes “observed” by open-strings are their image in the target-space(-time) via a map, cf. Sec. 1.1, an immediate question is then

- **Q. [morphism]** What should be a morphism from an Azumaya noncommutative space $\mathcal{X}^{A\text{z}}$ to a string target-space(-time) $\mathcal{Y}$?

In this section, we address this technical problem in the algebro-geometric category. This is where different viewpoints/formulations of a “noncommutative geometry” may lead to different answers. It turns out that one cannot hope to extend Grothendieck’s construction of commutative algebraic geometry in full to the noncommutative case and, hence, we have to make a choice, guided by what truly matters for D-branes. This is the place we start to diverge from the various existing versions of “noncommutative geometry”.

2.1 What should be the Azumaya-type noncommutative geometry and a morphism from an Azumaya-type noncommutative space?

**How to construct/understand a “geometric space”?**

We may start with a related fundamental question:

- **Q. How do we “construct/understand” a “geometric space $X$”?**

The lesson one learns from Grothendieck’s construction of modern (commutative) algebraic geometry and some later understandings is that there are (at least) four ways:

---

*This is a standing research problem for noncommutative algebraic geometers for more than three decades (or even nearly eight decades beginning with the work of Ore, year 1931, that touches upon the notion of “noncommutative localizations”). There are various technical issues for the notion of a “noncommutative scheme” either as a topological space or as a category, pursued by several groups of mathematicians. See [L-Y3: References].

11
(1) (as a ringed topological space) a point-set $X$ with a topology on it together with a sheaf - with respect to that topology - of rings (i.e. the structure sheaf $\mathcal{O}_X$) that encodes the data of local function rings of $X$, this gives rise to Grothendieck’s theory of schemes;

(2) (as a functor of points) fix a collection of “basic spaces” and see how they maps to $X$;

(3) (as a base for sheaves of modules) instead of the ringed topological space $(X, \mathcal{O}_X)$ itself, one looks at the category $\text{Mod}_X$ of modules of the structure sheaf $\mathcal{O}_X$ of $X$;

(4) (as a probe) fix a collection of “basic spaces” and see how $X$ maps to them.

It turns out that in commutative algebraic geometry, Methods (1), (2), and (3) are equivalent (cf. [E-H], [Ha], and [Ro]) while Method (4) in general is much weaker than any of Methods (1), (2), and (3).

Methods (1) and (3) can be applied to understand “$\text{Space} \, M_r(R)$” in the case $R$ is a commutative $\mathbb{C}$-algebra. Only that it gives us back $\text{Spec} \, R$ and whatever information/“geometry” hidden in $M_r(R)$ is completely lost; cf. Morita equivalence. Method (2) can be pursued for $M_r(R)$ and is related to smearing lower-dimensional D-branes in a higher-dimensional one. We should postpone this to later part of the project.

Comparing Sec. 1.1 on how we are led to the Polchinski-Grothendieck Ansatz, in which a D-brane plays the role of a domain of a map into an open-string target-space(-time), one sees that: among the four methods,

- it is the weakest Method (4) that fits best the purpose of D-branes.

Furthermore,

- it is ‘morphisms therefrom’ that play the central roles for understanding D-branes; whether one can truly build a satisfying/reasonable $\text{Space} \, M_r(R)$ as a ringed topological space is a secondary issue since one has $\text{Spec} \, R$ at least.

In other words, we don’t focus on what exactly $\text{Space} \, M_r(R)$ is. Rather, we address how $\text{Space} \, M_r(R)$ can map to other spaces, through which we get a feel/sense/manifestation of the hidden geometry of $\text{Space} \, M_r(R)$ and, hence, D-branes. This leads us thus to the next theme.

**How to construct/define a morphism without spaces?**

The discussion in the previous theme leads us to the next question:

- Q. [morphism without spaces] How to construct/define a morphism without having topological spaces as the domain and the target to begin with?

An answer to this question, which we follow, is that

- (space) take the notion of a “space” solely as an equivalence class $[S]$ of gluing systems $S$ of (associative unital) rings, where an equivalence $S \sim S'$ is defined via a notion of common refinement of the systems via localizations of the rings; denote symbolically this “space” by $\text{Space} \, [S]$;

---

8In the realm of projective algebraic geometry, Method (4) is part of Mori’s Program.

9Remark for noncommutative algebraic geometers: In this work, we do not adopt the Morita-equivalence-type attitude toward noncommutative algebra. In particular, any space $X$ with a noncommutative structure sheaf $\mathcal{O}^\text{nc}_X$ is called a ‘noncommutative space’ in our convention to emphasize this $\mathcal{O}^\text{nc}_X$. We thank Lieven Le Bruyn for a remark that provokes us a re-thought on our naming and the foundation of noncommutative algebraic geometry.

10I.e. for the purpose of D-branes, one has to take a carefully selected spirit of Grothendieck’s works on commutative algebraic geometry but not its full contents.
Thus, a key issue here goes back to the old one since Ore at year 1931: *localizations of (associative unital) rings*. This notion affects what class of morphisms \( \varphi \) one can define.

This is again a standing question for noncommutative algebraic geometers. Not to let this block our move towards D-branes, we take in this project the simplest class of localizations: *central localizations*, i.e. localization by a multiplicatively closed system of elements (including the identity 1 of the ring) in the center of the ring. Thus, as long as the very technical issue of localization is concerned, it is the same as in the realm of commutative algebraic geometry. This frees us to focus on true D-brane-related geometric themes, which already reveal very rich contents. Restricted to central localizations, the above general discussions can be polished into an encapsulated format, which gives Definition 2.2.2 in the next subsection when the target-space is a usual (commutative) scheme.

**Example 2.1.1. [D0-branes on \( \mathbb{A}^1 \)].** Here we illustrate the above discussions by considering *D0-branes of rank \( r \) on the (complex) affine line \( \mathbb{A}^1 = \text{Spec} \mathbb{C}[z] \).* Along the above line of thoughts, such a D-brane is described by a *morphism* \( \varphi : \text{Space} (M_r(\mathbb{C}), \mathbb{C}^r) \to \mathbb{A}^1 \) from the *Azumaya point* \( pt^{\mathbb{A}^1} := \text{Space} (M_r(\mathbb{C})) \) with the fundamental (left-)\( M_r(\mathbb{C}) \)-module \( E = \mathbb{C}^r \) to \( \mathbb{A}^1 \). To unravel it, it is then described equivalently but contravariantly by a \( \mathbb{C} \)-algebra homomorphism \( \varphi^*: \mathbb{C}[z] \to M_r(\mathbb{C}) \). The latter is determined by an assignment \( z \mapsto m_\varphi \in M_r(\mathbb{C}) \). The *image* \( \varphi(pt^{\mathbb{A}^1}) \) of the Azumaya point \( pt^{\mathbb{A}^1} \) under \( \varphi \) is given by the subscheme \( \text{Spec} (\mathbb{C}[z]/\text{Ker}(\varphi^*)) \) of \( \mathbb{A}^1 \). The *Chan-Paton module/bundle/sheaf* on D-branes as observed by open strings in \( \mathbb{A}^1 \) corresponds to the *push-forward* \( \varphi_* E \) of \( E \) to \( \mathbb{A}^1 \), defined by the \( \mathbb{C}[z] \)-module structure of \( E \) via \( \varphi^* \). In this way, *even without knowing what Space (\( M_r(\mathbb{C}) \)) really is, one can still do geometry.* See [L-Y3: Sec. 4.1] (D(1)). Cf. **Figure 2-1-1.**

Readers are referred to [L-Y3: Sec. 1 and Sec. 2.1] (D(1)) for more details and related references.

**2.2 The four aspects of a morphism from an Azumaya scheme with a fundamental module to a scheme.**

In this subsection, we give the four equivalent descriptions for a morphism from an Azumaya noncommutative scheme with a fundamental module to a (commutative) scheme. See [L-Y3] (D(1)) and [L-L-S-Y] (D(2)) for more discussions.

**I. The fundamental setting.**

Let \( X \) and \( Y \) be schemes over \( \mathbb{C} \). We assume that both \( X \) and \( Y \) are projective for the convenience of, e.g., addressing moduli problems. However, several basic definitions given do not require this condition. Summing up all the previous considerations gives us then the following fundamental definitions.

**Definition 2.2.2. [(commutative) surrogate].** Let \( \mathcal{O}^{\mathbb{A}^1}_X \) be a coherent sheaf of (associative unital) \( \mathcal{O}_X \)-algebras on \( X \), locally modeled on \( M_r(\mathcal{O}_U) \) for an affine open subset \( U \) of \( X \). Let \( \mathcal{O}_X \subset \mathcal{A} \subset \mathcal{O}^{\mathbb{A}^1}_X \) be a commutative \( \mathcal{O}_X \)-subalgebra of \( \mathcal{O}^{\mathbb{A}^1}_X \). Then \( X_{\mathcal{A}} := \text{Spec} \mathcal{A} \) is called a (commutative) *surrogate* of \( X^{\mathbb{A}^1} := (X, \mathcal{O}^{\mathbb{A}^1}_X) \).

---

\[\text{In this example, we hiddenly specify a trivialization of the fundamental module } E \text{ to make the discussion explicit throughout.} \]
Figure 2-1-1. Despite that \textit{Space} \( M_r(\mathbb{C}) \) may look only one-point-like, under morphisms the Azumaya “noncommutative cloud” \( M_r(\mathbb{C}) \) over \textit{Space} \( M_r(\mathbb{C}) \) can “split and condense” to various image schemes with a rich geometry. The latter image schemes can even have more than one component. The Higgsing/un-Higgsing behavior of the Chan-Paton module of D0-branes on \( Y (= \mathbb{A}^1 \text{ in Example}) \) occurs due to the fact that when a morphism \( \varphi : \textit{Space} \ M_r(\mathbb{C}) \to Y \) deforms, the corresponding push-forward \( \varphi_* E \) of the fundamental module \( E = \mathbb{C}^r \) on \textit{Space} \( M_r(\mathbb{C}) \) can also change/deform. These features generalize to morphisms from Azumaya schemes with a fundamental module to a scheme \( Y \). Despite its simplicity, this example already hints at a richness of Azumaya-type noncommutative geometry. In the figure, a module over a scheme is indicated by a dotted arrow. 

\[ D0\text{-brane of rank } r \]

\[ M_r(\mathbb{C}) \text{ NC cloud} \]

\[ \varphi_1 \]

\[ \varphi_2 \]

\[ \varphi_2' \]

\[ \varphi_3 \]

\[ \text{Spec} \ \mathbb{C} \]

\[ \mathbb{C}' \]

\[ \text{open-string target-space(-time) } Y \]
One should think of $X_A$ as a finite scheme over and dominating $X$ that is itself canonically dominated by $X^{A_k}$. An affine cover of $X_A$ corresponds to a gluing system of algebras from central localizations. Following this, the notion of morphisms from $X^{A_k}$ to $Y$, as an equivalence class of gluing systems of ring-homomorphisms with respect to covers, can be phrased as

**Definition 2.2.2. [morphism]**. A *morphism* from $X^{A_k}$ to $Y$, in notation $\varphi : X^{A_k} \to Y$, is an equivalence class of pairs

$$(\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{O}_X^{A_k}, f : X_A := \text{Spec} \mathcal{A} \to Y),$$

where

1. $\mathcal{A}$ is a commutative $\mathcal{O}_X$-subalgebra of $\mathcal{O}_X^{A_k}$;
2. $f : X_A \to Y$ is a morphism of (commutative) schemes;
3. two such pairs $(\mathcal{O}_X \subset \mathcal{A}_1 \subset \mathcal{O}_X^{A_k}, f_1 : X_{A_1} \to Y)$ and $(\mathcal{O}_X \subset \mathcal{A}_2 \subset \mathcal{O}_X^{A_k}, f_2 : X_{A_2} \to Y)$ are equivalent, in notation

$$(\mathcal{O}_X \subset \mathcal{A}_1 \subset \mathcal{O}_X^{A_k}, f_1 : X_{A_1} \to Y) \sim (\mathcal{O}_X \subset \mathcal{A}_2 \subset \mathcal{O}_X^{A_k}, f_2 : X_{A_2} \to Y),$$

if there exists a third pair $(\mathcal{O}_X \subset \mathcal{A}_3 \subset \mathcal{O}_X^{A_k}, f_3 : X_{A_3} \to Y)$ such that $\mathcal{A}_3 \subset \mathcal{A}_i$ and that the induced diagram

$\begin{array}{ccc}
X_{A_i} & \xrightarrow{f_i} & X_{A_3} \\
\downarrow & & \downarrow f_3 \\
Y
\end{array}$

commutes, for $i = 1, 2$.

To improve clearness, we denote the set of pairs associated to $\varphi$ by the bold-faced $\varphi$.

**Definition 2.2.3. [associated surrogate, canonical presentation, and image]**. Let

$$A_\varphi = \cap (\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{O}_X^{A_k}, f : X_A \to Y) \in \varphi A.$$

Then $\mathcal{O}_X \subset A_\varphi \subset \mathcal{O}_X^{A_k}$ and there exists a unique $f_\varphi : X_\varphi := \text{Spec} A_\varphi \to Y$ such that the induced diagram

$\begin{array}{ccc}
X_A & \xrightarrow{f} & X_\varphi \\
\downarrow & & \downarrow f_\varphi \\
Y
\end{array}$

commutes, for all $(\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{O}_X^{A_k}, f : X_A \to Y) \in \varphi$. We shall call the pair

$$(\mathcal{O}_X \subset A_\varphi \subset \mathcal{O}_X^{A_k}, f_\varphi : X_\varphi := \text{Spec} A_\varphi \to Y),$$

which is canonically associated to $\varphi$, the (canonical) presentation for $\varphi$. The scheme $X_\varphi$, which dominates $X$, is called the *surrogate of $X^{A_k}$ associated to $\varphi$*. We will denote the built-in morphism $X_\varphi \to C$ by $\pi_\varphi$. The subscheme $f_\varphi(X_\varphi)$ of $Y$ is called the *image* of $X^{A_k}$ under $\varphi$ and will be denoted $\text{Im} \varphi$ or $\varphi(X^{A_k})$ interchangeably.

**Remark 2.2.4. [minimal property of $X_\varphi$]**. By construction,
there exists no \( O_X \)-subalgebra \( O_X \subset A' \subset A' \) such that \( f' \) factors as the composition of morphisms \( X' \rightarrow \text{Spec} \ A' \rightarrow Y \).

We will call this feature the minimal property of the surrogate \( X' \) of \( X' \) associated to \( f' \).

**Definition 2.2.5.** [Azumaya scheme with a fundamental module, morphism]. An Azumaya scheme with a fundamental module is a triple \((X, O^A_X, \mathcal{E})\), denoted also as \((X^A, \mathcal{E})\), where \( \mathcal{E} \) is a locally free \( O_X \)-module of rank \( r \) and \( O^A_X = \text{End}_{O_X}(\mathcal{E}) \). (i.e. \( \mathcal{E} \) is equipped with a fixed (left)-\( O^A_X \)-module structure.) A morphism \( \varphi : (X^A, \mathcal{E}) \rightarrow Y \) is simply a morphism from \( X^A \) to \( Y \) as defined in Definition 2.2.2.

**Definition 2.2.6.** [Chan-Paton sheaf/module]. Given a morphism \( \varphi : (X^A, \mathcal{E}) \rightarrow Y \) with its canonical presentation \((O_X \subset A' \subset O^A_X, f' : X' \rightarrow Y)\). Then \( \mathcal{E} \) is automatically a \( O_{X' \varphi} \)-module in notation, \( O_{X' \varphi} \mathcal{E} \). Define the push-forward \( \varphi_* \mathcal{E} \) of \( \mathcal{E} \) to \( Y \) under \( \varphi \) by \( f'*(O_{X' \varphi} \mathcal{E}) \). It is a coherent \( O_Y \)-module supported on \( \text{Im} \varphi = f'((X')) \). We will call it also the Chan-Paton sheaf or module on \( \varphi(X^A) \) associated to \( \mathcal{E} \) under \( \varphi \).

**Definition 2.2.7.** [isomorphism between morphisms]. Two morphisms \( \varphi_1 : (X^A_1, \mathcal{E}_1) \rightarrow Y \) and \( \varphi_2 : (X^A_2, \mathcal{E}_2) \rightarrow Y \) from Azumaya schemes with a fundamental module to \( Y \) are said to be isomorphic if there exists an isomorphism \( h : X_1 \xrightarrow{\sim} X_2 \) with a lifting \( \hat{h} : \mathcal{E}_1 \xrightarrow{\sim} h^* \mathcal{E}_2 \) such that

\[
\begin{align*}
\hat{h} : \mathcal{A}_{\varphi_1} & \xrightarrow{\sim} h^* \mathcal{A}_{\varphi_2}, \\
\text{the following diagram commutes}
\end{align*}
\]

\[
\begin{tikzcd}
X_{\varphi_2} \arrow{r}{f_{\varphi_2}} \arrow[swap]{d}{\hat{h}} & Y \\
X_{\varphi_1} \arrow{r}{f_{\varphi_1}} & Y.
\end{tikzcd}
\]

Here, we denote the induced isomorphism \( O^A_{X_1} \xrightarrow{\sim} h^* O^A_{X_2} \) of \( O_{X_1} \)-algebras (or \( A_1 \xrightarrow{\sim} h^* A_2 \) of their respective \( O_{X_1} \)-subalgebras in question) via \( \hat{h} : \mathcal{E}_1 \xrightarrow{\sim} h^* \mathcal{E}_2 \) still by \( \hat{h} \) and \( \hat{h} : X_{\varphi_2} \xrightarrow{\sim} X_{\varphi_1} \) is the scheme-isomorphism associated to \( \hat{h} : \mathcal{A}_{\varphi_1} \xrightarrow{\sim} h^* \mathcal{A}_{\varphi_2} \).

The notion of a family of morphisms from (nonfixed) Azumaya schemes with a fundamental module to \( Y \) can also be defined accordingly. We refer readers to [L-L-S-Y: Sec. 2.1]. When this fundamental setting is translated to the following three equivalent settings in the realm of commutative algebraic geometry, it becomes standard how the family version should be formulated.

**II. As a torsion sheaf on** \( X \times Y \). The minimal property of \( X' \) implies that the map \((\pi_{\varphi}, f_{\varphi}) : X \rightarrow X \times Y \) is indeed an embedding and \( O_{X_1 \varphi} \mathcal{E} \) can be identified as a torsion sheaf \( \mathcal{E} \) on \( X \times Y \) that is flat over \( X \) of relative length \( r \). The converse also holds:

**Lemma 2.2.8.** [Azumaya without Azumaya, morphisms without morphisms]. A morphism \( \varphi : (X^A, \mathcal{E}) \rightarrow Y \) from an Azumaya scheme \( X^A \) over \( X \) with a fundamental module \( \mathcal{E} \) of rank \( r \) is given by a coherent \( O_{X \times Y} \)-module \( \mathcal{E} \) on \((X \times Y)/X\) that is flat over \( X \) of relative length \( r \).
Proof. Let \( pr_1 : X \times Y \to X \) and \( pr_2 : X \times Y \to Y \) be the projection maps. Then \( \mathcal{E} \) is recovered by \( pr_1 \), \( \mathcal{E} \). \( \mathcal{O}_X^{Az} \) is thus also recovered. The scheme-theoretical support \( \text{Supp}(\mathcal{E}) \) gives \( X_\varphi \) with \( \pi_\varphi \) and \( f_\varphi \) recovered by the restriction of \( pr_1 \) and \( pr_2 \) respectively. As \( X_\varphi \) is already embedded in \( X \times Y \), the minimal property for the surrogate associated to a morphism is automatically satisfied. 

This equivalent translation of the notion ‘morphism’ to the realm of commutative algebraic geometry is technically important. However, to relate to D-branes directly, it is conceptually important to keep the Azumaya geometry in the fundamental settings in mind.

III. As a map to the stack \( \mathcal{M}_r^{0Az}(Y) \).

Let \( \mathcal{M}_r^{0Az}(Y) \) be the moduli stack of morphisms from an Azumaya point with a fundamental module of rank \( r \) to \( Y \). From Aspect II of morphisms, \( \mathcal{M}_r^{0Az}(Y) \) is identical to the Artin stack of 0-dimensional \( \mathcal{O}_Y \)-modules of length \( r \). Interpret Aspect II of morphism as a flat family of 0-dimensional \( \mathcal{O}_Y \)-module over \( X \), then a morphism \( \varphi : (X^{Az}, \mathcal{E}) \to Y \) defines a morphism \( \phi : X \to \mathcal{M}_r^{0Az}(Y) \); and vice versa.

IV. As a \( GL_r(\mathbb{C}) \)-equivariant map.

A morphism \( \phi : X \to \mathcal{M}_r^{0Az}(Y) \) is equivalent to a morphism \( \bar{\phi} \) of schemes in the commutative diagram

\[
\begin{array}{ccc}
\text{Isom}(\phi, \pi) & \xrightarrow{\bar{\phi}} & \text{Atlas} \\
pr_1 \downarrow & & \downarrow \pi \\
X & \xrightarrow{\phi} & \mathcal{M}_r^{0Az}(Y)
\end{array}
\]

from the Isom-functor construction. Here \( \pi : \text{Atlas} \to \mathcal{M}_r^{0Az}(Y) \) is an atlas of \( \mathcal{M}_r^{0Az}(Y) \). As the moduli stack of 0-dimensional \( \mathcal{O}_Y \)-modules of length \( r \), one can choose \( \text{Atlas} \) in the diagram to be the open subscheme

\[
\text{Quot}^0(\mathcal{O}_Y^{\oplus r}, r) := \{ \mathcal{O}_Y^{\oplus r} \to \mathcal{E} \to 0 \text{, length} \mathcal{E} = r \text{, } H^0(\mathcal{O}_Y^{\oplus r}) \to H^0(\mathcal{E}) \to 0 \}
\]

of the Quot-scheme \( \text{Quot}(\mathcal{O}_Y^{\oplus r}, r) \) of isomorphism classes of 0-dimensional quotients of \( \mathcal{O}_Y^{\oplus r} \) with length \( r \). The latter is known to be projective; thus \( \text{Quot}^0(\mathcal{O}_Y^{\oplus r}, r) \) is quasi-projective and it parameterizes morphisms from an Azumaya point with a fundamental module \( V \) of rank \( r \) together with a decoration \( C^r \to V \). The \( GL_r(\mathbb{C}) \)-action on \( H^0(\mathcal{O}_Y^{\oplus r}) \) induces a right \( GL_r(\mathbb{C}) \)-action on \( \text{Quot}^0(\mathcal{O}_Y^{\oplus r}, r) \).

With this choice of atlas for \( \mathcal{M}_r^{0Az}(Y) \), \( pr_1 : \text{Isom}(\phi, \pi) \to X \) becomes a principal \( GL_r(\mathbb{C}) \)-bundle \( pr : P \to X \) over \( X \) and \( \bar{\phi} : P \to \text{Quot}^0(\mathcal{O}_Y^{\oplus r}, r) \) is a \( GL_r(\mathbb{C}) \)-equivariant morphism.

Conversely, given a principal \( GL_r(\mathbb{C}) \)-bundle \( P \) over \( X \) and a \( GL_r(\mathbb{C}) \)-equivariant morphism \( \bar{\phi} : P \to \text{Quot}^0(\mathcal{O}_Y^{\oplus r}, r) \). The pullback of the universal sheaf to \((P \times Y)/X\) and the basic descent theory reproduce the torsion sheaf on \( X \times Y \) in Aspect II.
$Quot^{H_0}(O^{\oplus r}_Y, r)$ as the representation-theoretical atlas.

As Aspect IV and $Quot^{H_0}(O^{\oplus r}_Y, r)$ will play some role in Sec. 4, let us discuss more about it.

**Definition 2.2.9. [representation-theoretical atlas].** We shall call the quasi-projective scheme $Quot^{H_0}(O^{\oplus r}_Y, r)$ the representation-theoretical atlas of $\mathfrak{M}_A^{Az} (Y)$.

The purpose of this theme is to explain why $Quot^{H_0}(O^{\oplus r}_Y, r)$ generalizes the notion of the ‘representation scheme for an algebra’ and, hence, the above name.

**Lemma 2.2.10. [from global to local].** There exists a finite cover $\Pi_\alpha U_\alpha \to Y$ of $Y$ by affine open subsets such that any morphism $\varphi : (pt^k, V) \to Y$ with $V$ of rank $r$ must have its image $\varphi(pt^k)$ contained in some $U_\alpha$.

*Proof.* Since $Y$ is projective, we only need to prove the lemma for $Y = \mathbb{P}^n$ for some $n$. In this case, take any $k$-many distinct hyperplanes $H_\alpha$, $\alpha = 1, \ldots, k$, in general positions in $\mathbb{P}^n$ (i.e. any $n + 1$ of them has empty intersection) with $k > nr$ and let $U_\alpha = \mathbb{P}^n - H_\alpha$. The finite cover $\Pi_\alpha U_\alpha \to \mathbb{P}^n$ of $\mathbb{P}^n$ will do.

Let $\Pi_\alpha U_\alpha \to Y$ be such a cover of $Y$ with $U_\alpha = Spec R_\alpha$ and $Rep(R_\alpha, M_r(\mathbb{C}))$ be the representation-scheme that parameterizes $\mathbb{C}$-algebra-homomorphisms from $R_\alpha$ to $M_r(\mathbb{C})$. It follows from Sec. 2.1 that $Rep(R_\alpha, M_r(\mathbb{C}))$ is precisely the moduli space of morphisms from an Azumaya point with a fundamental module $V$ of rank $r$ with a decoration $\mathbb{C}^r \to V$. Consequently, $\Pi_\alpha Rep(R_\alpha, M_r(\mathbb{C})) \to Quot^{H_0}(O^{\oplus r}_Y, r)$ gives a finite cover of $Quot^{H_0}(O^{\oplus r}_Y, r)$. In this sense, $Quot^{H_0}(O^{\oplus r}_Y, r)$ generalizes the notion of ‘representation-schemes’.

When $R_\alpha$ is presented in a generator-relator form:

$$R_\alpha = \mathbb{C}[z_{\alpha,1}, \ldots z_{\alpha,k_\alpha}]/(h_{\alpha,1}, \ldots h_{\alpha,l_\alpha}),$$

where $h_{\alpha,j}$’s are polynomials in $z_{\alpha,i}$’s, the representation-scheme $Rep(R_\alpha, M_r(\mathbb{C}))$ is realized accordingly in a standard way as an affine subscheme in $\mathbb{A}^{r^2k_\alpha}$, described by the ideal generated by entries in a system of $(l_\alpha^2) + l_\alpha$ matrix polynomials associated to the commutativity among $z_{\alpha,i}$’s and the relators $h_{\alpha,j}$. From this aspect, one may think of $Quot^{H_0}(O^{\oplus r}_Y, r)$ as an enhancement of $Y$, with the gluing law between affine charts of $Y$ enhanced to a corresponding system of matrix gluing law between representation-schemes associated to the affine charts of $Y$.

It should be noted that while $Quot^{H_0}(O^{\oplus r}_Y, r)$ has such an elegant intrinsic representation-theoretical meaning, for $Y$ of dimension $\geq 3$ and $r >> 0$ the detail of $Quot^{H_0}(O^{\oplus r}_Y, r)$ is beyond any means of approaching at the moment.

### 2.3 D-branes in a $B$-field background à la Polchinski-Grothendieck Ansatz.

A $B$-field on a space-time $Y$ is a connection on a gerbe $\mathcal{Y}$ over $Y$. It can be presented as a Čech 0-cochain $(B_i)_i$ of local 2-forms $B_i$ with respect to a cover $\mathcal{U} = \{U_i\}_i$ on $Y$ such that on $U_i \cap U_j$, $B_i - B_j = d\Lambda_{ij}$ for some real 1-forms $\Lambda_{ij}$ that satisfies $\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = -\sqrt{-1}d\log \alpha_{ijk}$ on $U_i \cap U_j \cap U_k$, where $(\alpha_{ijk})_{ijk}$ is a Čech 2-cocycle of $U(1)$-valued functions on $Y$ in the algebrog-geometric language, $(\alpha_{ijk})_{ijk}$ is given by a presentation of an equivalence class $\alpha_B \in C^2_{\text{ét}}(Y, O^*_Y)$ of étale Čech 2-cocycles with values in $O^*_Y$. Through its coupling to the open-string current on an open-string world-sheet with boundary on a D-brane world-volume $X \subset Y$, a background $B$-field on $Y$ induces a twist to the gauge field $A$ on the Chan-Paton vector bundle $E$ on $X$ that renders $E$ itself a twisted vector bundle with the twist specified by $\alpha_B|_X \in C^2_{\text{ét}}(X, O^*_X)$.
e.g. [Hi2], [Ka1], and [Wi4]. Furthermore, the 2-point functions on the open-string world-sheet with boundary on $X$ indicate that the D-brane world-volume is deformed to a deformation-quantization type noncommutative geometry in a way that is governed by the $B$-field (and the space-time metric); cf. [C-H1], [C-H2], [Ch-K], [Scho], and [S-W].

We review here how these two effects from a $B$-field background on the target-space(-time) $Y$ modify the notion of Azumaya schemes with a fundamental module and of morphisms therefrom. More general discussions, details, and references are referred to [L-Y6] (D(5)).

2.3.1 Nontrivial Azumaya noncommutative schemes with a fundamental module and morphisms therefrom.

Polchinski-Grothendieck Ansatz with the étale topology adaptation.

Recall the Polchinski-Grothendieck Ansatz in Sec. 1.1. For the moment, let $X \hookrightarrow Y$ be an embedded submanifold of $Y$. In the smooth differential-geometric setting of Polchinski, the word “locally” in the ansatz means “locally in the $C^\infty$-topology”. This can be generalized to adapt the ansatz to fit various settings: “locally” in the analytic (resp. Zariski) topology for the holomorphic (resp. algebrao-geometric) setting. These are enough to study D-branes in a space(-time) without a background $B$-field. The Azumaya structure sheaf $O_X^{Az}$ that encodes the matrix-type noncommutative structure on $X$ in these cases is of the form $\text{End}_{O_X}(E)$ with $E$ the Chan-Paton module, a locally free $O_X$-module of rank $r$ on which the Azumaya $O_X$-algebra $O_X^{Az}$ acts tautologically as a simple/fundamental (left) $O_X^{Az}$-module. This leads to the case reviewed in Sec. 2.2. $O_X^{Az}$ in this case corresponds to the zero-class in the Brauer group $Br(X)$ of $X$.

On pure mathematical ground, one can further adapt the ansatz for $X$ equipped with any Grothendieck topology/site. On string-theoretic ground, as recalled at the beginning of the current subsection, when a background $B$-field on $Y$ is turned on, the Chan-Paton module $E$ on $X$ becomes twisted and is no longer an honest sheaf of $O_X$-modules on $X$. The interpretation of “locally” in the ansatz in the sense of (small) étale topology on $X$ becomes forced upon us. This corresponds to the case when the Azumaya structure sheaf $O_X^{Az}$ on $X$ represents a non-zero class in $Br(X)$. We will call the resulting $(X, O_X^{Az})$ a nontrivial Azumaya (noncommutative) scheme.

In this subsubsection, we review the most basic algebro-geometric aspect of D-branes along the line of the Polchinski-Grothendieck Ansatz but with this étale topology adaptation on the D-brane or D-brane world-volume. Readers are referred to [L-Y6: Sec. 1] (D(5)) for a highlight on gerbes and (general) Azumaya algebras over a scheme; and to, e.g., [Br], [Ch], [Câ], [Lie], and [Mi] for detailed treatment. In this review, we will confine ourselves only to the language of twisted sheaves.

Twisted sheaves à la Căldăraru.

Given an étale cover $p : U^{(0)} := \Pi_{i \in I} U_i \rightarrow X$ of $X$, we will adopt the following notations:

- $U_{ij} := U_i \times_X U_j =: U_i \cap U_j$, $U_{ijk} := U_i \times_X U_j \times_X U_k =: U_i \cap U_j \cap U_k$;

- $\cdots \quad \underbrace{U^{(2)} := U \times_X U \times_X U}_{p_{12}, p_{13}, p_{23}}$;

- $\underbrace{U^{(1)} := U \times_X U}_{p_{1}, p_{2}}$;

are the projection maps from fibered products as indicated; the restriction of these projections maps to respectively $U_{ijk}$ and $U_{ij}$ will be denoted the same;
the pull-back of an \( \mathcal{O}_{U_i} \)-module \( \mathcal{F}_i \) on \( U_i \) to \( U_{ij}, U_{ji}, U_{ijk}, \cdots \) via compositions of these projection maps will be denoted by \( \mathcal{F}_i|_{U_{ij}}, \mathcal{F}_i|_{U_{ji}}, \mathcal{F}_i|_{U_{ijk}}, \cdots \) respectively.

**Definition 2.3.1.1.** \( \alpha \)-twisted \( \mathcal{O}_X \)-module on an étale cover of \( X \). ([Ča: Definition 1.2.1].) Let \( \alpha \in \check{C}_2(X, \mathcal{O}_X^\alpha) \) be a Čech 2-cocycle in the étale topology of \( X \). An \( \alpha \)-twisted \( \mathcal{O}_X \)-module on an étale cover of \( X \) is a triple

\[
\mathcal{F} = (\{U_i\}_{i \in I}, \{\mathcal{F}_i\}_{i \in I}, \{\phi_{ij}\}_{i,j \in I})
\]

that consists of the following data

- an étale cover \( p : U := \coprod_{i \in I} U_i \to X \) of \( X \) on which \( \alpha \) can be represented as a 2-cocycle:

\[
\alpha = \{ \alpha_{ijk} : \alpha_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_X^\alpha) \text{ with } \alpha_{jkl}\alpha_{ikl}^{-1}\alpha_{ijk}^{-1} = 1 \text{ on } U_{ijkl} \text{ for all } i,j,k,l \in I \},
\]

such a cover will be called an \( \alpha \)-admissible étale cover of \( X \);

- \( \mathcal{F}_i \) is a sheaf of \( \mathcal{O}_{U_i} \)-modules on \( U_i \);

- (gluing data) \( \phi_{ij} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}} \) is an \( \mathcal{O}_{U_{ij}} \)-module isomorphism that satisfies

\[
\begin{align*}
(1) & \quad \phi_{ii} \text{ is the identity map for all } i \in I; \\
(2) & \quad \phi_{ij} = \phi_{ji}^{-1} \text{ for all } i,j \in I; \\
(3) & \quad (\text{twisted cocycle condition}) \quad \phi_{ki} \circ \phi_{jk} \circ \phi_{ij} \text{ is the multiplication by } \alpha_{ijk} \text{ on } \mathcal{F}_i|_{U_{ij}}.
\end{align*}
\]

\( \mathcal{F} \) is said to be coherent (resp. quasi-coherent, locally free) if \( \mathcal{F}_i \) is a coherent (resp. quasi-coherent, locally free) \( \mathcal{O}_{U_i} \)-module for all \( i \in I \). A homomorphism

\[
h : \mathcal{F} = (\{U_i\}_{i \in I}, \{\mathcal{F}_i\}_{i \in I}, \{\phi_{ij}\}_{i,j \in I}) \longrightarrow \mathcal{F}' = (\{U_i'\}_{i \in I}, \{\mathcal{F}_i'\}_{i \in I}, \{\phi_{ij}'\}_{i,j \in I})
\]

between \( \alpha \)-twisted \( \mathcal{O}_X \)-modules on the étale cover \( p \) of \( X \) is a collection \( \{h_i : \mathcal{F}_i \to \mathcal{F}_i'\}_{i \in I} \), where \( h_i \) is an \( \mathcal{O}_{U_i} \)-module homomorphism, such that \( \phi_{ij}' \circ h_i = h_j \circ \phi_{ij} \) for all \( i,j \in I \). In particular, \( h \) is an isomorphism if all \( h_i \) are isomorphisms. Denote by \( \text{Mod}(X, \alpha, p) \) the category of \( \alpha \)-twisted \( \mathcal{O}_X \)-modules on the étale cover \( p : U^{(0)} \to X \) of \( X \).

Given an \( \alpha \)-twisted sheaf \( \mathcal{F} \) on the étale cover \( p : U \to X \) of \( X \), let \( p' : U' \to X \) be an étale refinement of \( p : U \to X \). Then \( \alpha \) can be represented also on \( p' : U' \to X \) and \( \mathcal{F} \) on \( p \) defines an \( \alpha \)-twisted \( \mathcal{O}_X \)-module \( \mathcal{F}' \) on \( p' \) via the pull-back under the built-in étale cover \( U' \to U \) of \( U \). This defines an equivalence of categories:

\[
\text{Mod}(X, \alpha, p) \longrightarrow \text{Mod}(X, \alpha, p').
\]

([Ča: Lemma 1.2.3, Lemma 1.2.4, Remark 1.2.5].)

**Definition 2.3.1.2.** \( \alpha \)-twisted \( \mathcal{O}_X \)-module on \( X \). An \( \alpha \)-twisted \( \mathcal{O}_X \)-module on \( X \) is an equivalence class \([\mathcal{F}]\) of \( \alpha \)-twisted \( \mathcal{O}_X \)-modules \( \mathcal{F} \) on étale covers of \( X \), where the equivalence relation is generated by étale refinements and descents by étale covers of \( X \) on which \( \alpha \) can be represented. An \( \mathcal{F}' \in [\mathcal{F}] \) is called a representative of the \( \alpha \)-twisted \( \mathcal{O}_X \)-module \([\mathcal{F}]\). For simplicity of terminology, we will also call \( \mathcal{F}' \) directly an \( \alpha \)-twisted \( \mathcal{O}_X \)-module on \( X \).

Cf. [Ča: Corollary 1.2.6 and Remark 1.2.7].

Standard notions of \( \mathcal{O}_X \)-modules, in particular

- the scheme-theoretic support \( \text{Supp} \mathcal{E} \),
of an $\alpha$-twisted sheaf $\mathcal{E}$ on $X$ (or on $X/S$) are defined via a(ny) presentation of $\mathcal{E}$ on an $\alpha$-admissible étale cover $U \to X$.

Standard operations on $\mathcal{O}_X$-modules apply to twisted $\mathcal{O}_X$-modules on appropriate admissible étale covers by applying the operations component by component over the cover. These operations apply then to twisted $\mathcal{O}_X$-modules as well: They are defined on representatives of twisted sheaves in such a way that they pass to each other by pull-back and descent under étale refinements of admissible étale covers. In particular:

**Proposition 2.3.1.3. [basic operations on twisted sheaves].** ([Că: Proposition 1.2.10].)

1. Let $\mathcal{F}$ and $\mathcal{G}$ be an $\alpha$-twisted and a $\beta$-twisted $\mathcal{O}_X$-module respectively, where $\alpha, \beta \in C^2_{\text{ét}}(X, \mathcal{O}_X^\times)$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an $\alpha \beta$-twisted $\mathcal{O}_X$-module and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an $\alpha^{-1} \beta$-twisted $\mathcal{O}_X$-module. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are both $\alpha$-twisted $\mathcal{O}_X$-modules, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ descends to an (ordinary/untwisted) $\mathcal{O}_X$-module, still denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, on $X$.

2. Let $f : X \to Y$ be a morphism of schemes/$\mathbb{C}$ and $\alpha \in C^2_{\text{ét}}(Y, \mathcal{O}_Y^\times)$. Note that an $\alpha$-admissible étale cover of $Y$ pulls back to an $f^*\alpha$-admissible étale over of $X$ under $f$, through which the pull-back and push-forward of a related twisted sheaf can be defined. If $\mathcal{F}$ is an $\alpha$-twisted $\mathcal{O}_Y$-module on $Y$, then $f^*\mathcal{F}$ is an $f^*\alpha$-twisted $\mathcal{O}_X$-module on $X$. If $\mathcal{F}$ is an $f^*\alpha$-twisted $\mathcal{O}_X$-module on $X$, then $f_*\mathcal{F}$ is an $\alpha$-twisted $\mathcal{O}_Y$-module on $Y$.

**Morphisms from Azumaya schemes with a twisted fundamental module.**

**Definition 2.3.1.4. [Azumaya scheme with a fundamental module].** An *Azumaya scheme with a fundamental module* in class $\alpha$ is a tuple

$$(X^A, \mathcal{E}) := (X, \mathcal{O}_X^A = \mathcal{E}_{\text{nd}}(\mathcal{E}), \mathcal{E}),$$

where $X = (X, \mathcal{O}_X)$ is a (Noetherian) scheme (over $\mathbb{C}$), $\alpha \in C^2_{\text{ét}}(X, \mathcal{O}_X^\times)$ represents a class $[\alpha] \in \text{Br}(X) \subset H^2_{\text{ét}}(X, \mathcal{O}_X^\times)$, and $\mathcal{E}$ is a locally-free coherent $\alpha$-twisted $\mathcal{O}_X$-module on $X$. A commutative surrogate of $(X^A, \mathcal{E})$ is a scheme $X_\mathcal{A} := \text{Spec} \mathcal{A}$, where $\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{E}_{\text{nd}}(\mathcal{E})$ is an inclusion sequence of commutative $\mathcal{O}_X$-subalgebras of $\mathcal{E}_{\text{nd}}(\mathcal{E})$. Let $\pi : X_\mathcal{A} \to X$ be the built-in dominant finite morphism. Then $\mathcal{E}$ is tautologically a $\pi^*\alpha$-twisted $\mathcal{O}_{X_\mathcal{A}}$-module on $X_\mathcal{A}$, denoted by $\mathcal{O}_{X_\mathcal{A}}^\mathcal{E}$. We say that $X^\mathcal{A}$ is an *Azumaya scheme of rank* $r$ if $\mathcal{E}$ has rank $r$ and that it is a *nontrivial* (resp. *trivial*) Azumaya scheme if $[\alpha] \neq 0$ (resp. $[\alpha] = 0$).

Let $Y$ be a (commutative, Noetherian) scheme/$\mathbb{C}$ and $\alpha_B \in C^2_{\text{ét}}(Y, \mathcal{O}_Y^\times)$ be the étale Čech cocycle associated to a fixed $B$-field on $Y$.

**Definition 2.3.1.5. [morphism with $B$-field background].** Let $(X^\mathcal{A}, \mathcal{E})$ be an Azumaya scheme with a fundamental module in the class $\alpha \in C^2_{\text{ét}}(X, \mathcal{O}_X^\times)$. Then, a *morphism* from $(X^\mathcal{A}, \mathcal{E})$ to $(Y, \alpha_B)$, in notation $\varphi : (X^\mathcal{A}, \mathcal{E}) \to (Y, \alpha_B)$, is a pair

$$(\mathcal{O}_X \subset \mathcal{A}_\varphi \subset \mathcal{O}_X^\mathcal{A}, \ f_\varphi : X_\varphi := \text{Spec} \mathcal{A}_\varphi \to Y),$$

where

- $\mathcal{A}_\varphi$ is a commutative $\mathcal{O}_X$-subalgebra of $\mathcal{O}_X^\mathcal{A}$,
- $f_\varphi : X_\varphi \to Y$ is a morphism of (commutative) schemes,
that satisfies the following properties:

1. (minimal property of \( X_\varphi \)) there exists no \( \mathcal{O}_X \)-subalgebra \( \mathcal{O}_X \subset A' \subset A_\varphi \) such that \( f_\varphi \) factors as the composition of morphisms \( X_\varphi \to \text{Spec} A' \to Y \);

2. (matching of twists on \( X_\varphi \)) let \( \pi_\varphi : X_\varphi \to X \) be the built-in finite dominant morphism, then \( \pi_\varphi^* \alpha = f_\varphi^* \alpha_B \) in \( \check{C}_X^2 (X_\varphi, \mathcal{O}_{X_\varphi}^* \). 

\( X_\varphi \) is called the surrogate of \( X^A \) associated to \( \varphi \). Condition (2) implies that \( \varphi_* \mathcal{E} := f_\varphi^* (\mathcal{O}_X \mathcal{E}) \) is an \( \alpha_B \)-twisted \( \mathcal{O}_Y \)-module on \( Y \), supported on \( \text{Im}(\varphi) := \varphi (X^A) := f_\varphi (X_\varphi) \), where the last is the usual scheme-theoretic image of \( X_\varphi \) under \( f_\varphi \).

Given two morphisms \( \varphi_1 : (X^A_1, \mathcal{E}_1) \to (Y, \alpha_B) \) and \( \varphi_2 : (X^A_2, \mathcal{E}_2) \to (Y, \alpha_B) \), a morphism \( \varphi_1 \to \varphi_2 \) from \( \varphi_1 \) to \( \varphi_2 \) is a pair \( (h, \tilde{h}) \), where

- \( h : X_1 \to X_2 \) is an isomorphism of schemes with \( h^* \alpha_2 = \alpha_1 \), where \( \alpha_i \) is the underlying class of \( \mathcal{E}_i \) in \( \check{C}_X^2 (X_i, \mathcal{O}_X^*) \);
- \( \tilde{h} : \mathcal{E}_1 \sim \tilde{h}^* \mathcal{E}_2 \) be an isomorphism of twisted sheaves on \( X_1 \) that satisfies
  - \( \tilde{h} : A_{\mathcal{E}_1} \sim h^* A_{\mathcal{E}_2} \),
  - the following diagram commutes

\[
\begin{array}{ccc}
X_{\mathcal{E}_2} & \xrightarrow{f_{\mathcal{E}_2}} & Y \\
\downarrow h & & \\
X_{\mathcal{E}_1} & \xrightarrow{f_{\mathcal{E}_1}} & Y \\
\end{array}
\]

Here, we denote both of the induced isomorphisms, \( \mathcal{O}_{X_1}^* \sim \tilde{h}^* \mathcal{O}_{X_2}^* \) and \( A_{\mathcal{E}_1} \sim h^* A_{\mathcal{E}_2} \), of \( \mathcal{O}_{X_1} \)-algebras still by \( \tilde{h} \) and \( h : X_{\mathcal{E}_2} \to X_{\mathcal{E}_1} \) is the scheme-isomorphism associated to \( \tilde{h} : A_{\mathcal{E}_1} \sim h^* A_{\mathcal{E}_2} \).

This defines the category \( \text{Morphism}_{A^E} (Y, \alpha_B) \) of morphisms from Azumaya schemes with a fundamental module to \( (Y, \alpha_B) \).

**Definition 2.3.1.6. [D-brane and Chan-Paton module]**. Following the previous Definition, \( \varphi (X^A) \) is called the image D-brane on \( (Y, \alpha_B) \) and \( \varphi_* \mathcal{E} \) the Chan-Paton module/sheaf on the image D-brane. Similarly, for image D-brane world-volume if \( X \) is served as a (Wicked-rotated) D-brane world-volume.

**Others aspects of a morphism in the twisted case.**

The above theme gives Aspect I, ‘The fundamental setting’, of a morphism. Aspects II, III, IV of a morphism in Sec. 2.2 can be generalized to the twisted case as well. Moreover, there is now a new Aspect V of a morphism: namely, a description in terms of a morphism \( \varphi : (X^A, \mathcal{F}) \to \mathcal{Y}_{\alpha_B} \) from the Azumaya \( \mathcal{O}_X^* \)-gerbe with a fundamental module, associated to \( X^A, \mathcal{E} \), to the gerbe \( \mathcal{Y}_{\alpha_B} \), associated to \( (Y, \alpha_B) \). As we won’t use them for the rest of the current work, their discussions are omitted. We refer readers to [L-Y6: Sec. 2.2] (D(5)) themes: ‘Azumaya without Azumaya and morphisms without morphisms’ and ‘The description in terms of morphisms from Azumaya gerbes with a fundamental module to a target gerbe’ for a discussion of Aspects II and V of the twisted case.

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\(^{12}\)In other words, a morphism from \( (X^A, \mathcal{E}) \) to \( (Y, \alpha_B) \) is a usual morphism \( \varphi : X^A \to Y \) from the (possibly nontrivial) Azumaya scheme \( X^A \) to \( Y \) subject to the twist-matching Condition (2) so that \( \varphi_* \mathcal{E} \) remains a twisted sheaf in a way that is compatible with the \( B \)-field background on \( Y \).
2.3.2 Azumaya quantum schemes with a fundamental module and morphisms therefrom.

In this subsubsection, we review how the second effect - namely, the deformation quantization - of the background $B$-field to a smooth D-brane world-volume $X$ can be incorporated into Azumaya geometry along the line of the Polchinski-Grothendieck Ansatz. We focus on the case when the deformation quantizations that occur are modelled directly on that for phase spaces in quantum mechanics. This brings in the sheaf $\mathcal{D}$ of differential operators and $\mathcal{D}$-modules.

**Weyl algebras, the sheaf $\mathcal{D}$ of differential operators, and $\mathcal{D}$-modules.**

Let $X$ be a smooth variety over $\mathbb{C}$, $\Theta_X = \text{Der}_\mathbb{C}(\mathcal{O}_X, \mathcal{O}_X)$ be the sheaf of $\mathbb{C}$-derivations on $\mathcal{O}_X$, and $\Omega_X$ be the sheaf of Kähler differentials on $X$. We recall a few necessary objects and facts for our study. Their details are referred to [Bern], [Bj], and [B-E-G-H-K-M]:

1. the Weyl algebra
   
   $$A_n(\mathbb{C}) := \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]/[[x_i, x_j], [\partial_i, \partial_j], [\partial_i, x_j] - \delta_{ij} : 1 \leq i, j \leq n],$$

   which is the algebra of differential operators acting on $\mathbb{C}[x_1, \ldots, x_n]$ by formal differentiation; here, $\mathbb{C}[\cdot, \cdot]$ is the unital associative $\mathbb{C}$-algebra generated by elements $\cdot$ indicated, $[[\cdot, \cdot]]$ is the commutator, $\delta_{ij}$ is the Kronecker delta, and $(\cdot, \cdot)$ is the 2-sided ideal generated by $\cdot$ indicated;

2. the sheaf $\mathcal{D}_X$ of (linear algebraic) differential operators on $X$, which is the sheaf of unital associative algebras that extends $\mathcal{O}_X$ by new generators from the sheaf $\Theta_X$;

3. $\mathcal{D}_X$-modules (or directly $\mathcal{D}$-modules when $X$ is understood), which are sheaves on $X$ on which $\mathcal{D}_X$ acts from the left.

**Lemma 2.3.2.1.** [$A_n(\mathbb{C})$ simple]. $A_n(\mathbb{C})$ is a simple algebra: the only 2-sided ideal therein is the zero ideal $(0)$.

**Proposition 2.3.2.2.** [$\mathcal{O}$-coherent $\mathcal{D}$-module]. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module that is coherent as an $\mathcal{O}_X$-module. Then, $\mathcal{M}$ is $\mathcal{O}_X$-locally-free. Furthermore, in this case, the action of $\mathcal{D}_X$ on $\mathcal{M}$ defines a flat connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega_X$ on $\mathcal{M}$ by assigning $\nabla_\xi s = \xi \cdot s$ for $s \in \mathcal{M}$ and $\xi \in \Theta_X$; the converse also holds. This gives an equivalence of categories:

$$\{ \text{$\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules} \} \leftrightarrow \{ \text{coherent locally free $\mathcal{O}_X$-modules with a flat connection} \}. $$

$\mathcal{D}$ as the structure sheaf of the deformation quantization of the cotangent bundle.

From the presentation of the Weyl algebra $A_n(\mathbb{C})$, which resembles the quantization of a classical phase space with the position variable $(x_1, \ldots, x_n)$ and the dual momentum variable $(p_1, \ldots, p_n) = (\partial_1, \ldots, \partial_n)$, and the fact that $\mathcal{D}_X$ is locally modelled on the pull-back of $A_n(\mathbb{C})$ over $\mathbb{A}^n$ under an étale morphism to $\mathbb{A}^n$, the sheaf $\mathcal{D}_X$ of algebras with the built-in inclusion $\mathcal{O}_X \subseteq \mathcal{D}_X$ can be thought of as the structure sheaf of a noncommutative space from the quantization of the cotangent bundle, i.e. the total space $\Omega_X$ of the sheaf $\Omega_X$ of $X$.

---

13The word “quantization” has received various meanings in mathematics. Here, we mean solely the one associated to quantum mechanics. This particular quantization is also called deformation quantization.
Definition 2.3.2.3. [canonical deformation quantization of cotangent bundle]. We will formally denote this noncommutative space by $\text{Space} \mathcal{D}_X =: Q\Omega_X$ and call it the canonical deformation quantization of $\Omega_X$.

A special class of morphisms from or to $\text{Space} \mathcal{D}_X$ can be defined contravariantly as homomorphisms of sheaves of $\mathbb{C}$-algebras.

Example 2.3.2.4. [$A_n(\mathbb{C})$]. The noncommutative space $\text{Space} (A_n(\mathbb{C}))$ defines a deformation quantization of $\Omega_{\mathbb{A}^n}$. Recall the presentation of $A_n(\mathbb{C})$. The $\mathbb{C}$-algebra homomorphism

$$f'(k) : \mathbb{C}[y_1, \ldots, y_n] \rightarrow A_n(\mathbb{C})$$

$$y_i \mapsto x_i, \quad i = 1, \ldots, k,$$

$$y_j \mapsto \partial_j, \quad j = k + 1, \ldots, n,$$

defines a dominant morphism $f(k) : \text{Space} (A_n(\mathbb{C})) \rightarrow \mathbb{A}^n$, $k = 0, \ldots, n$. The $\mathbb{C}$-algebra automorphism $A_n(\mathbb{C}) \rightarrow A_n(\mathbb{C})$ with $x_i \mapsto \partial_i$ and $\partial_i \mapsto -x_i$ defines the Fourier transform on $\text{Space} (A_n(\mathbb{C}))$. Note that, since $A_n(\mathbb{C})$ is simple, any morphisms to $\text{Space} (A_n(\mathbb{C}))$ is dominant (i.e. the related $\mathbb{C}$-algebra homomorphism from $A_n(\mathbb{C})$ is injective).

$\alpha$-twisted $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules and enlargements of $\mathcal{O}_X^\mathbb{A}$ by $\mathcal{D}_X$.

Let $\alpha \in \mathcal{O}_X (X, \mathcal{O}_X^* )$ and $\mathcal{F} = (\{U_i\}_{i \in I}, \{\mathcal{F}_i\}_{i \in I}, \{\phi_{ij}\}_{i,j \in I})$ be an $\alpha$-twisted $\mathcal{O}_X$-module.

Definition 2.3.2.5. [connection on $\mathcal{F}$]. A connection $\nabla$ on $\mathcal{F}$ is a set $\{\nabla_i\}_{i \in I}$ where $\nabla_i : \mathcal{F}_i \rightarrow \mathcal{F}_i \otimes_{\mathcal{O}_{U_i}} \Omega U_i$ is a connection on $\mathcal{F}_i$, that satisfies $\phi_{ij} \circ (\nabla_i |_{U_i}) = (\nabla_j |_{U_i}) \circ \phi_{ij}$. $\nabla$ is said to be flat if $\nabla_i$ is flat for all $i \in I$.

Note that the existence of an $\alpha$-twisted $\mathcal{O}_X$-module with a connection imposes a condition on $\alpha$ that $\alpha$ has a presentation $(\alpha_{ijk})_{ijk}$ with $d\alpha := (d\alpha_{ijk})_{ijk} = (0)_{ijk}$; i.e. $\alpha_{ijk} \in \mathbb{C}^*$ for all $i, j, k$.

As the proof of Proposition 2.3.2.2 is local, it generalizes to $\alpha$-twisted $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules:

Proposition 2.3.2.6. [$\alpha$-twisted $\mathcal{O}_X$-coherent $\mathcal{D}$-module]. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module that is $\alpha$-twisted $\mathcal{O}_X$-coherent. Then, $\mathcal{M}$ is an $\alpha$-twisted $\mathcal{O}_X$-locally-free. Furthermore, in this case, the action of $\mathcal{D}_X$ on $\mathcal{M}$ defines a flat connection $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_X$ on $\mathcal{M}$ by assigning $\nabla_\xi s = \xi \cdot s$ for $s \in \mathcal{M}$ and $\xi \in \Omega_X$; the converse also holds. This gives an equivalence of categories:

$$\left\{ \alpha\text{-twisted }\mathcal{O}_X\text{-coherent }\mathcal{D}_X\text{-modules} \right\} \leftrightarrow \left\{ \alpha\text{-twisted coherent locally free }\mathcal{O}_X\text{-modules with a flat connection} \right\}.$$

Let $\mathcal{E}$ be an $\alpha$-twisted $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. Then the $\mathcal{D}_X$-module structure on $\mathcal{E}$ induces a natural $\mathcal{D}_X$-module structure on the (ordinary) $\mathcal{O}_X$-module $\mathcal{O}_X^{\mathbb{A}} := \text{End}_{\mathcal{O}_X} (\mathcal{E})$. We will denote both the connection on $\mathcal{E}$ and on $\mathcal{O}_X^{\mathbb{A}}$ by $\nabla$. As both $\mathcal{O}_X^{\mathbb{A}} := \text{End}_{\mathcal{O}_X} (\mathcal{E})$ and $\mathcal{D}_X$ act now on $\mathcal{E}$ and $\mathcal{D}_X$ acts also on $\mathcal{O}_X^{\mathbb{A}}$, one can define a sheaf $\mathcal{O}_X^{\mathbb{A}, \mathcal{D}}$ of unital associative algebras generated by $\mathcal{O}_X^{\mathbb{A}}$ and $\mathcal{D}_X$ as follows:

- Over a (Zariski) open subset $U$ of $X$, $\mathcal{O}_X^{\mathbb{A}, \mathcal{D}} (U)$ is the unital associative $\mathbb{C}$-algebra generated by $\mathcal{O}_X^{\mathbb{A}}(U) \cup \mathcal{D}_X(U)$ subject to the following rules:

  (1) for $\phi_1, \phi_2 \in \mathcal{O}_X^{\mathbb{A}}(U)$, $\phi_1 \cdot \phi_2 \in \mathcal{O}_X^{\mathbb{A}, \mathcal{D}}(U)$ coincides with the existing $\phi_1 \phi_2 \in \mathcal{O}_X^{\mathbb{A}}(U)$;
(2) for $\eta_1, \eta_2 \in \mathcal{D}_X(U)$, $\eta_1 \cdot \eta_2 \in \mathcal{O}^{\text{Ae}, \mathcal{D}}_X(U)$ coincides with the existing $\eta_1 \eta_2 \in \mathcal{D}_X(U)$;

(3) (Leibniz rule) for $\phi \in \mathcal{O}^{\text{Ae}, \mathcal{D}}_X(U)$ and $\xi \in \Theta_X(U) \subset \mathcal{D}_X(U)$,

$$\xi \cdot \phi = (\nabla_\xi \phi) + \phi \cdot \xi.$$ 

In notation, $\mathcal{O}^{\text{Ae}, \mathcal{D}}_X := \mathbb{C}(\mathcal{O}^{\text{Ae}}_X, \mathcal{D}_X)^{\nabla}$.

**Definition 2.3.2.7.** [Azumaya quantum scheme with fundamental module]. The noncommutative space

$$(X^{\text{Ae}, \mathcal{D}}, \mathcal{E}^\nabla) := (X, \mathcal{O}^{\text{Ae}, \mathcal{D}}_X = \mathbb{C}(\mathcal{E}\text{nd}\mathcal{O}_X(\mathcal{E}), \mathcal{D}_X)^{\nabla}, (\mathcal{E}, \nabla))$$

will be called an Azumaya quantum scheme with a fundamental module in the class $\alpha$.

Caution that $\mathcal{O}_X \subset \mathcal{O}^{\text{Ae}, \mathcal{D}}_X$ in general does not lie in the center of $\mathcal{O}^{\text{Ae}, \mathcal{D}}_X$.

**Remark 2.3.2.8.** [\(\mathcal{E}^\nabla\) as a module over \(\text{Space}(\mathcal{O}^{\text{Ae}, \mathcal{D}}_X)\)]. The full notation for $X^{\text{Ae}, \mathcal{D}}$ in Definition 5.1.7 is meant to make two things manifest:

1. There is a built-in diagram of dominant morphisms of X-spaces:

   \[
   \begin{array}{ccc}
   X^{\text{Ae}} := \text{Space} \mathcal{O}^{\text{Ae}}_X & \longrightarrow & \text{Space} \mathcal{O}^{\text{Ae}, \mathcal{D}}_X \\
   \downarrow & & \downarrow \\
   Q\Omega_X := \text{Space} \mathcal{D}_X & \longrightarrow & X
   \end{array}
   \]

   $\text{Space} \mathcal{O}^{\text{Ae}, \mathcal{D}}_X$ is the major space one should focus on. The other three spaces - $\text{Space} \mathcal{O}^{\text{Ae}}_X$, $\text{Space} \mathcal{D}_X$, and $X$ - should be treated as auxiliary spaces that are built into the construction to encode a special treatment that takes care of the issue of localizations of noncommutative rings in the current situation; cf. the next item.

2. Despite the fact that $\mathcal{O}_X$ is in general not in the center of $\mathcal{O}^{\text{Ae}, \mathcal{D}}_X$, there is a notion of localization and open sets on $\text{Space} \mathcal{O}^{\text{Ae}, \mathcal{D}}_X$ induced by those on $X$. I.e. $\text{Space} \mathcal{O}^{\text{Ae}, \mathcal{D}}_X$ has a built-in topology induced from the (Zariski) topology of $X$. Thus, one can still have the notion of gluing systems of morphisms and sheaves with respect to this topology.

In particular, $\mathcal{E}^\nabla$ is a sheaf of $\mathcal{O}^{\text{Ae}, \mathcal{D}}_X$-modules supported on the whole $\text{Space} \mathcal{O}^{\text{Ae}, \mathcal{D}}_X$ with this topology.

**Remark 2.3.2.9.** [Azumaya algebra over $\mathcal{D}_X$]. Note that $\mathcal{O}^{\text{Ae}, \mathcal{D}}_X$ can also be thought of as an Azumaya algebra over $\mathcal{D}_X$ in the sense that it is a sheaf of algebras on $X$, locally modelled on the matrix ring $M_r(D_U)$ over $D_U$ for $U$ an affine étale-open subset of $X$.

**Remark 2.3.2.10.** [partially deformation-quantized target]. From the fact that Weyl algebras are simple, it is anticipated that a morphism to a totally deformation-quantized space $Y = \Omega_W$ is a dominant morphism. In general, one may take $Y$ to be a partial deformation quantization of a space along a foliation. E.g. a deformation quantization of $\Omega_W/B$ along the fibers of a fibration $W/B$. For compact $Y$, one may consider the deformation quantization along torus fibers of a space fibered by even-dimensional tori.\(^{14}\) (Cf. Example 2.3.2.11.)

\(^{14}\)Though we do not touch this here, readers should be aware that this is discussed in numerous literatures.
Higgsing and un-Higgsing of quantum D-branes via deformations of morphisms.

We give here an example of morphisms from $X$ with the new structure to a target-space $Y$ being the total space $\Omega_W$ of the cotangent bundle $\Omega_W$ of a smooth variety $W$. It illustrates also the Higgsing/un-Higgsing behavior of D-branes in the current deformation-quantized situation.

**Example 2.3.2.11. [Higgsing/un-Higgsing of D-brane].** Let $(X^{\mathbb{A}^2 \times D}, \mathcal{E}^\nabla)$ be the affine Azumaya quantum scheme with a fundamental module associated to the ring $R := \mathbb{C}\langle M_2(\mathbb{C}[z]), \partial_z \rangle$ (with the implicit relation $(\partial_z, z) = 1$ and the identification of $\mathbb{C}[z]$ with the center of $M_2(\mathbb{C}[z])$) with the $R$-module $N := \mathbb{C}[z] \oplus \mathbb{C}[z]$, on which $M_2(\mathbb{C}[z])$ acts by multiplication and $\partial_z$ acts by formal differentiation, and $Y$ be the partially deformation-quantized space $Q_3 \Omega_{\mathbb{A}^2 / \mathbb{A}^1}$ associated to the ring $S_3 := \mathbb{C}\langle u, v, w \rangle / (z, [v, w], [u, v], [u, w] - \lambda)$, where $\lambda \in \mathbb{C}$. Note that the action of $\partial_z$ on $N$ induces an action of $\partial_z$ on $M_2(\mathbb{C}[z])$ by the entry-wise formal differentiation and the $\mathbb{A}^2 / \mathbb{A}^1$ corresponds to $\mathbb{C}[v] \hookrightarrow \mathbb{C}[v, w]$. Consider the following special class of morphisms:

$$X \xrightarrow{\varphi(A, B)} Y$$

$$R \xleftarrow{\varphi^2(A, B)} S_3 \quad \text{with the identification of } \mathbb{C}[z] \text{ with the center of } M_2(\mathbb{C}[z]), \quad A, B \in M_2(\mathbb{C}[z]),$$

subject to $[\lambda \partial_z + A, B] = 0$. (The other two constraints, $[B, z] = 0$ and $[\lambda \partial_z + A, z] = 0$, are automatic.) Let

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

where $a_i, b_j \in \mathbb{C}[z]$ and assume that $\lambda \neq 0$. Then, the associated system $\lambda \partial_z B + [A, B] = 0$ of homogeneous linear ordinary differential equations on $B$ has a solution if and only if $A$ satisfies

$$(a_1 - a_4)^2 + 4a_2a_3 = 0.$$ 

Under this condition on $A$, the system has four fundamental solutions:

$$B_1 = \begin{bmatrix} 1 + \lambda^{-2}a_2a_3z^2 \\ \lambda^{-1}a_3z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_3z^2 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} \lambda^{-1}a_2z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \\ \lambda^{-1}a_3z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_3z^2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -\lambda^{-1}a_2z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \\ 1 + \lambda^{-1}(a_1 - a_4)z - \lambda^{-2}a_2a_3z^2 \end{bmatrix} \quad \text{and} \quad B_4 = \begin{bmatrix} -\lambda^{-1}a_2z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \\ \lambda^{-1}a_2z + \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \end{bmatrix},$$

Denote this solution space by $\mathbb{C}^A_4$ with coordinates $(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4)$ and the correspondence

$$(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4) \leftrightarrow \hat{b}_1B_1 + \hat{b}_2B_2 + \hat{b}_3B_3 + \hat{b}_4B_4 =: B(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4).$$

Then,

\[\text{Also, we take the convention that } \partial_z \cdot m \text{ means the product in } \mathbb{C}\langle M_2(\mathbb{C}[z]), \partial_z \rangle \text{ and } \partial_z m \text{ means entry-wise formal differentiation of } m, \text{ for } m \in M_2(\mathbb{C}[z]).\]
the degree-0 term $B(0)$ of $B = B(b_1,b_2,b_3,b_4)$ (in $z$-powers) is given by
\[ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} , \]

- the characteristic polynomial of $B$ is identical to that of $B(0)$.

It follows that the image $\text{Im} \varphi(A,B)$ of $\varphi(A,B)$ is a (complex-)codimension-1 sub-quantum scheme in $Y$ whose associated ideal in $S_\lambda$ contains the ideal
\[ (v^2 - trB(0) v + detB(0)) . \]

Let $\mu_-$ and $\mu_+$ be the eigen-values of $B(0)$.

**Case (a) : $\nu_- \neq \nu_+$.** In this case, the above ideal $((v - \nu_-)(v - \nu_+))$ coincides with $\text{Ker} \varphi(A,B)$ and, hence, describes precisely $\text{Im} \varphi(A,B) \subset Y$. Since $\varphi(A,B)(v) = B$, let $N_- := \text{Ker}(B - \nu_-) \subset N$. This is a rank-1 $\mathbb{C}[z]$-submodule of $\mathbb{C}[z] \oplus \mathbb{C}[z]$ that is invariant also under $\varphi(A,B)(S_\lambda)$. This gives $N_-$ a $S_\lambda/(v - \nu_-)$-module structure that has rank-1 as $\mathbb{C}[w]$-module. Similarly, $N_+ := \text{Ker}(B - \nu_+) \subset N$ is invariant under $\varphi(A,B)(S_\lambda)$ and has a $\varphi(A,B)$-induced $S_\lambda/(v - \nu_+)$-module structure that is of rank-1 as $\mathbb{C}[w]$-module. Let
\[ Z := \text{Im} \varphi(A,B) = \text{Space}(S_\lambda/(v - \nu_-)(v - \nu_+)) \]
\[ = \text{Space}(S_\lambda/(v - \nu_-)) \cup N_+ \implies Z_- \cup N_+ \]
be the two connected components of the quantum subscheme $\text{Im} \varphi(A,B) \subset Y$ and denote the $\mathcal{O}_{Z_-}$-modules associated to $N_-$ and $N_+$ by $(\delta_{\lambda} N_-)\sim$ and $(\delta_{\lambda} N_+)\sim$ respectively. Then
\[ \varphi(A,B) \circ \mathcal{E} = (\delta_{\lambda} N_-)\sim \oplus (\delta_{\lambda} N_+)\sim \] with $(\delta_{\lambda} N_-)\sim$ supported on $Z_-$ and $(\delta_{\lambda} N_+)\sim$ on $Z_+$.

**Case (b) : $\nu_- = \nu_+ = \nu$.** In this case, $\text{Ker} \varphi(A,B)$ can be either $(v - \nu)$ or $(v - \nu)^2$ and both situations happen.

- When $\text{Ker} \varphi(A,B) = (v - \nu)$, $N = \mathbb{C}[z] \oplus \mathbb{C}[z]$ has a $\varphi(A,B)$-induced $S_\lambda/(v - \nu)$-module structure and $\varphi(A,B) \circ \mathcal{E}$ has support $\text{Im} \varphi(A,B) = \text{Space}(S_\lambda/(v - \nu)) \subset Y$.

- When $\text{Ker} \varphi(A,B) = ((v - \nu)^2)$, $N = \mathbb{C}[z] \oplus \mathbb{C}[z]$ has a $\varphi(A,B)$-induced $S_\lambda/(v - \nu)^2$-module structure and $\varphi(A,B) \circ \mathcal{E}$ has support $Z := \text{Im} \varphi(A,B) = \text{Space}(S_\lambda/(v - \nu)^2)) \subset Y$. It contains an $\mathcal{O}_Z$-submodule $(\delta_{\lambda} N_0)\sim$, associated to $N_0 := \text{Ker}(v - \nu) \subset N$, that is supported on $Z_0 := \text{Space}(S_\lambda/(v - \nu)) \subset Z$. In other words, in the current situation, $\varphi(A,B) \circ \mathcal{E}$ not only is of rank-2 as a $\mathbb{C}[w]$-module but also has a built-in $\varphi(A,B)$-induced filtration $(\delta_{\lambda} N_0)\sim \subset \varphi(A,B) \circ \mathcal{E}$.

Thus, by varying $(A,B)$ in the solution space of $\lambda \partial_z B + [A,B] = 0$ so that the eigen-values of $B(0)$ change from being distinct to being identical and vice versa, one realizes the Higgsing and un-Higgsing phenomena of D-branes in superstring theory for the current situation as deformations of morphisms from Azumaya quantum schemes to the open-string quantum target-space $Y$:

![Diagram](deformations-of-morphisms-phi-from-azumaya-deformation-quantized-schemes-with-a-fundamental-module-to-a-deformation-quantized-target-y.png)

This concludes the example. See also **Figure 2-1-1**.
2.4 Tests of the Polchinski-Grothendieck Ansatz for D-branes.

If the Polchinski-Grothendieck Ansatz is truly fundamental for D-branes and the notion of morphisms formulated above does capture D-branes, then we should be able to see what string-theorists see in quantum-field-theory language solely by our formulation. In this subsection, we collect six basic tests in this regard on string-theory works, 1995–2008, from our first group of examples. This group is guided by the following question:

- Q. [QFT vs. maps] Can we reconstruct the geometric object that arises in a quantum-field-theoretical study of D-branes through morphisms from Azumaya noncommutative spaces?

This subsection is not to be read alone. Rather, we recommend readers to go through the quoted string-theory works on each theme first and then compare.

(1) Bershadsky-Sadov-Vafa: Classical and quantum moduli space of D0-branes.
(Bershadsky-Sadov-Vafa vs. Polchinski-Grothendieck; [B-V-S1], [B-V-S2], and [Va1], 1995.)

The moduli stack $\mathcal{M}^{\text{Az}}_0(Y)$ of morphisms from Azumaya point with a fundamental module to a smooth variety $Y$ of complex dimension 2 contains various substacks with different coarse moduli space. One choice of such gives rise to the symmetric product $S^k(Y)$ of $Y$ while another choice gives rise to the Hilbert scheme $Y^{[k]}$ of points on $Y$. The former play the role of the classical moduli and the latter quantum moduli space of D0-branes studied in [Va1] and in [B-V-S1], [B-V-S2].

See [L-Y3: Sec. 4.4] (D(1)), theme: ‘A comparison with the moduli problem of gas of D0-branes in [Vafa1] of Vafa’ for more discussions.

(2) Douglas-Moore and Johnson-Myers: D-brane probe to an ADE surface singularity.
(Douglas-Moore/Johnson-Myers vs. Polchinski-Grothendieck; [Do-M], 1996, and [J-M], 1996.)

Here, we are compared with the setting of Douglas-Moore [Do-M]. The notion of ‘morphisms from an Azumaya scheme with a fundamental module’ can be formulated as well when the target $Y$ is a stack. In the current case, $Y$ is the orbifold associated to an ADE surface singularity. It is a smooth Deligne-Mumford stack. Again, the stack $\mathcal{M}^{\text{Az}}_0(Y)$ of morphisms from Azumaya points with a fundamental module to the orbifold $Y$ contains various substacks with different coarse moduli space. An appropriate choice of such gives rise to the resolution of ADE surface singularity.

See [L-Y4] (D(3)) for a brief highlight of [Do-M], details of the Azumaya geometry involved, and more references.

(3) Klebanov-Strassler-Witten: D-brane probe to a conifold.
(Klebanov-Strassler-Witten vs. Polchinski-Grothendieck; [Kl-W], 1998, and [Kl-S], 2000.)

Here, the problem is related to the moduli stack $\mathcal{M}^{\text{Az}}_0(Y)$ of morphisms from Azumaya points with a fundamental module to a local conifold $Y$, a singularity Calabi-Yau 3-fold, whose complex structure is given by $Y = \text{Spec}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4))$. Again, different resolutions of the conifold singularity of $Y$ can be obtained by choices of substacks from $\mathcal{M}^{\text{Az}}_0(Y)$, as in Tests (1) and (2). Such a resolution corresponds to a low-energy effective geometry “observed” by a stacked D-brane probe to $Y$ when there are no fractional/trapped brane sitting at the singularity $0$ of $Y$. 


New phenomenon arises when there are fractional/trapped D-branes sitting at 0. Instead of resolutions of the conifold singularity of $Y$, a low-energy effective geometry “observed” by a D-brane probe is a complex deformation of $Y$ with topology $T^*S^3$ (the cotangent bundle of 3-sphere). From the Azumaya geometry point of view, two things happen:

- Taking both the (stacked-or-not) D-brane probe and the trapped brane(s) into account, the Azumaya geometry on the D-brane world-volume remains.
- A noncommutative-geometric enhancement of $Y$ occurs via morphisms

$$\Xi = \text{Space } R_{\Xi}$$

$$Y \ar_{\pi^{\Xi}} \ar^A \rightarrow A^4$$

Here, $A^4 = \text{Spec } (\mathbb{C}[z_1, z_2, z_3, z_4])$,

$$R_{\Xi} = \mathbb{C} \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle / (\langle \xi_1 \xi_3, \xi_2 \xi_4 \rangle, \langle \xi_1 \xi_3, \xi_1 \xi_4 \rangle, \langle \xi_1 \xi_3, \xi_2 \xi_3 \rangle, \langle \xi_2 \xi_4, \xi_2 \xi_3 \rangle, \langle \xi_1 \xi_4, \xi_2 \xi_3 \rangle)$$

with $\mathbb{C} \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle$ being the associative (unital) $\mathbb{C}$-algebra generated by $\xi_1, \xi_2, \xi_3, \xi_4$ and $[\bullet, \cdot]$ being the commutator, $Y \hookrightarrow A^4$ via the definition of $Y$ above, and $\pi^{\Xi}$ is specified by the $\mathbb{C}$-algebra homomorphism

$$\pi^{\Xi}_*: \mathbb{C}[z_1, z_2, z_3, z_4] \rightarrow R_{\Xi}$$

$$z_1 \mapsto \xi_1 \xi_3$$

$$z_2 \mapsto \xi_2 \xi_4$$

$$z_3 \mapsto \xi_1 \xi_4$$

$$z_4 \mapsto \xi_2 \xi_3$$

One is thus promoted to studying the stack $\mathcal{M}^{\text{Az}}_\bullet(Space R_{\Xi})$, of morphisms from Azumaya points with a fundamental module to $Space R_{\Xi}$, following Sec. 2.1.

To proceed, we need the following notion:

**Definition 2.4.3.1. [Superficially infinitesimal deformation].** Given associative (unital) rings, $R = \langle r_1, \ldots, r_m \rangle / \sim$ and $S$, that are finitely-presentable and a ring-homomorphism $h : R \rightarrow S$. A **superficially infinitesimal deformation of $h$ with respect to the generators $\{r_1, \ldots, r_m\}$ of $R$** is a ring-homomorphism $h_\varepsilon : R \rightarrow S$ such that $h_\varepsilon(r_i) = h(r_i) + \varepsilon_i$ with $\varepsilon^2_i = 0$, for $i = 1, \ldots, m$.

When $S$ is commutative, a superficially infinitesimal deformation of $h_\varepsilon : R \rightarrow S$ is an infinitesimal deformation of $h$ in the sense that $h_\varepsilon(r) = h(r) + \varepsilon_r$ with $(\varepsilon_r)^2 = 0$, for all $r \in R$. This is no longer true for general noncommutative $S$. The $S$ plays the role of the Azumaya algebra $M_\bullet(\mathbb{C})$ in our current test. It turns out that a morphism $\varphi : pt^A \rightarrow Space R_{\Xi}$ that projects by $\pi^{\Xi}$ to the conifold singularity $0 \in Y$ can have superficially infinitesimal deformations $\varphi'$ such that the image $(\pi^{\Xi} \circ \varphi')(pt^A)$ contains not only 0 but also points in $A^4 - Y$. Indeed there are abundant such superficially infinitesimal deformations. Thus, beginning with a substack $\mathcal{Y}$ of $\mathcal{M}^{\text{Az}}_\bullet(Space R_{\Xi})$, that projects onto $Y$ via $\varphi \mapsto Im(\pi^{\Xi} \circ \varphi)$, one could use a 1-parameter family of superficially infinitesimal deformations of $\varphi \in \mathcal{Y}$ to drive $\mathcal{Y}$ to a new substack $\mathcal{Y}'$ that projects to $0 \cup Y' \subset A^4$, where $Y'$ is smooth (i.e. a deformed conifold). It is in this way that a deformed conifold $Y'$ is detected by the D-brane probe via the Azumaya structure on the common world-volume of the probe and the trapped brane(s). Cf. Figure 2-4-1.

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Figure 2-4-1. A generic superficially infinitesimal deformation $\tilde{\varphi}$ of $\varphi$ has a noncommutative image $\simeq \text{Space} \ M_2(\mathbb{C})$. It then descends to $\mathbb{A}^4_{[\xi_1, \xi_2, \xi_3, \xi_4]}:=\text{Spec} (\mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4])$ and becomes a pair of $\mathbb{C}$-points on $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$. One of the points is the conifold singularity $0=V(z_1, z_2, z_3, z_4) \in Y$ and the other is the point $p'=V(z_1-a_1b_1-\delta_1\eta_1, z_2-a_2b_2-\delta_2\eta_2, z_3-a_1b_2-\delta_1\eta_2, z_4-a_2b_1-\delta_2\eta_1)$ off $Y$ (generically). Through such deformations, any $\mathbb{C}$-point on $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ can be reached. Thus, one can realizes a deformation $Y'$ of $Y$ in $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ by a subvariety in $\text{Rep} (R_{\Xi}, M_2(\mathbb{C}))$. This is the Azumaya-geometry origin of the phenomenon in Klebanov-Strassler [Kl-S] that a trapped D-brane sitting on the conifold singularity may give rise to a deformation of the moduli space of SQFT on the D3-brane probe, turning a conifold to a deformed conifold. Our D0-brane here corresponds to the internal part of the effective-space-time-filling D3-brane world-volume of [Kl-S]. In this figure, $\mathbb{A}^4_{[\xi_1, \xi_2, \xi_3, \xi_4]}:=\text{Spec} (\mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4]) = \text{Spec} (R_{\Xi}/[R_{\Xi}, R_{\Xi}])$ is the maximal commutative subspace of $\text{Space} \ R_{\Xi}$ and $(\delta_1, \delta_2, \eta_1, \eta_2)$ parameterizes the superficially infinitesimal deformations of $\varphi$ in the current situation.
See [L-Y5] (D(4)) for a brief highlight of [Kl-W] and [Kl-S], details of the Azumaya geometry involved, and more references.

(4) Gómez-Sharpe: Information-preserving geometry, schemes, and D-branes.

(Gómez-Sharpe vs. Polchinski-Grothendieck; [G-Sh], 2000.)

Among the various groups who studied the foundation of D-branes, this is a work that is very close to us in spirit. There, Gómez and Sharpe began with the quest: [G-Sh: Sec. 1]

“As is well-known, on $N$ coincident D-branes, $U(1)$ gauge symmetries are enhanced to $U(N)$ gauge symmetries, and scalars that formerly described normal motions of the branes become $U(N)$ adjoints. People have often asked what the deep reason for this behavior is – what does this tell us about the geometry seen by D-branes?”,

like us. They observed by comparing colliding D-branes with colliding torsion sheaves in algebraic geometry that it is very probable that

*coincident D-branes should carry some fuzzy structure – perhaps a nonreduced scheme structure*

though the latter may carry more information than D-branes do physically. Further study on such nilpotent structure was done in [D-K-S]; cf. Sec. 4.2: theme ‘The generically filtered structure on the Chan-Patan bundle over a special Lagrangian cycle on a Calabi-Yau torus’ of the current review.

From our perspective,

*the (commutative) scheme/nilpotent structure Gómez and Sharpe proposed/observed on a stacked D-brane is the manifestation/residual of the Azumaya (noncommutative) structure on an Azumaya space with a fundamental module when the latter forces itself into a commutative space/scheme via a morphism.*

This connects our work to [G-Sh].

(5) Sharpe: $B$-field, gerbes, and D-brane bundles.

(Sharpe vs. Polchinski-Grothendieck; [Sh2], 2001.)

Recall that a $B$-field on the target space(-time) $Y$ specifies a gerbe $\mathcal{Y}_B$ over $Y$ associated to an $\alpha_B \in \tilde{C}_\text{et}^2(Y, O_Y^*)$ determined by the $B$-field. A morphism $\varphi: (X^\text{Az}, \mathcal{E}) \to (Y, \alpha_B)$ from a general Azumaya scheme with a twisted fundamental module to $(Y, \alpha_B)$ can be lifted to a morphism $\tilde{\varphi}: (X^\text{Az}, \mathcal{F}) \to \mathcal{Y}_B$ from an Azumaya $O_X^\Lambda$-gerbe with a fundamental module to the gerbe $\mathcal{Y}_B$. In this way, our setting is linked to Sharpe’s picture of gerbes and D-brane bundles in a $B$-field background.

See [L-Y6: Sec. 2.2] (D(5)) theme: ‘The description in term of morphisms from Azumaya gerbes with a fundamental module to a target gerbe’ for details of the construction.

(6) Dijkgraaf-Hollands-Sulkowski-Vafa: Quantum spectral curves.

(Dijkgraaf-Hollands-Sulkowski-Vafa vs. Polchinski-Grothendieck; [D-H-S-V], 2007, and [D-H-S], 2008.)
Here we focus on a particular theme in these works: the notion of quantum spectral curves from the viewpoint of D-branes. Let $C$ be a smooth curve, $\mathcal{L}$ an invertible sheaf on $C$, $\mathcal{E}$ a coherent locally-free $\mathcal{O}_C$-module, and $\mathcal{L} = \text{Spec} \left( \text{Sym}^* (\mathcal{L}') \right)$ be the total space of $\mathcal{L}$. Here, $\mathcal{L}'$ is the dual $\mathcal{O}_C$-module of $\mathcal{L}$. Then one has the following canonical one-to-one correspondence:

\[
\left\{ \begin{array}{l}
\mathcal{O}_C\text{-module homomorphisms} \\
\phi: \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
morphisms \varphi: (C^{\mathcal{A}_1}, \mathcal{E}) \to \mathcal{L} \\
as spaces over $C$
\end{array} \right\}
\]

induced by the canonical isomorphisms

\[
\text{Hom}_{\mathcal{O}_C}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L}) \simeq \Gamma(\mathcal{E}' \otimes \mathcal{E} \otimes \mathcal{L}) \simeq \text{Hom}_{\mathcal{O}_C}(\mathcal{L}', \text{End}_{\mathcal{O}_C}(\mathcal{E})).
\]

Let $\Sigma(\mathcal{E}, \phi) \subset \mathcal{L}$ be the (classical) spectral curve associated to the Higgs/spectral pair $(\mathcal{E}, \phi)$; cf. e.g. [B-N-S], [Hi1], and [Ox]. Then, for $\varphi$ corresponding to $\phi$, $\text{Im} \varphi \subset \Sigma(\mathcal{E}, \phi)$. Furthermore, if $\Sigma(\mathcal{E}, \phi)$ is smooth, then $\text{Im} \varphi = \Sigma(\mathcal{E}, \phi)$. This gives a morphism-from-Azumaya-space interpretation of spectral curves.

To address the notion of ‘quantum spectral curve’, let $\mathcal{L}$ be the sheaf $\Omega_C$ of differentials on $C$. Then the total space $\Omega_C \otimes \mathcal{L}$ admits a canonical $\mathbb{A}^1$-family $Q_{\mathbb{A}^1}(\mathcal{E})$ of deformation quantizations with the central fiber $Q_0 \Omega_C = \Omega_C$. Let $(\mathcal{E}, \phi: \mathcal{E} \to \mathcal{E} \otimes \Omega_C)$ be a spectral pair and $\varphi : (C^{\mathcal{A}_1}, \mathcal{E}) \to \Omega_C$ be the corresponding morphism. Denote the fiber of $Q_{\mathbb{A}^1}(\mathcal{E})$ over $\lambda \in \mathbb{A}^1$ by $Q_\lambda \Omega_C$. Then, due to the fact that the Weyl algebras are simple algebras, the spectral curve $\Sigma(\mathcal{E}, \phi)$ in $\Omega_C$ in general may not have a direct deformation quantization into $Q_\lambda \Omega_C$ by the ideal sheaf of $\Sigma(\mathcal{E}, \phi)$ in $\mathcal{O}_C$, since this will only give $Q_0 \Omega_C$, which corresponds to the empty subspace of $Q_\lambda \Omega_C$. However, one can still construct an $\mathbb{A}^1$-family $(Q_{\mathbb{A}^1} (C^{\mathcal{A}_1}), Q_{\mathbb{A}^1}(\mathcal{E}))$ of Azumaya quantum curves with a fundamental module out of $(C^{\mathcal{A}_1}, \mathcal{E})$ and a morphism $\varphi_{\mathbb{A}^1} : (Q_{\mathbb{A}^1} (C^{\mathcal{A}_1}), Q_{\mathbb{A}^1}(\mathcal{E})) \to Q_1 \Omega_C$, as spaces over $\mathbb{A}^1$, using the notion of ‘$\lambda$-connections’ and ‘$\lambda$-connection deformations of $\phi$’, such that

\[
\varphi := \varphi_{\mathbb{A}^1}|_{\lambda=0} \quad \text{is the composition} \quad (Q_0 C^{\mathcal{A}_1}, Q_0 \mathcal{E}) \to (C^{\mathcal{A}_1}, \mathcal{E}) \xrightarrow{\varphi_{\mathbb{A}^1}} \Omega_C, \quad \text{where}
\]

\[
(Q_0 C^{\mathcal{A}_1}, Q_0 \mathcal{E}) \to (C^{\mathcal{A}_1}, \mathcal{E}) \quad \text{is a built-in dominant morphism from the construction};
\]

\[
\varphi_\lambda := \varphi_{\mathbb{A}^1}|_{\lambda} : (Q_\lambda C^{\mathcal{A}_1}, Q_\lambda \mathcal{E}) \to Q_\lambda \Omega_C, \quad \text{for } \lambda \in \mathbb{A}^1 - \{0\}, \quad \text{is a morphism of}
\]

Azumaya quantum curves with a fundamental module to the deformation-quantized noncommutative space $Q_\lambda \Omega_C$.

In other words, we replace the notion of ‘quantum spectral curves’ by ‘quantum deformation $\varphi_\lambda$ of the morphism $\varphi$’. In this way, both notions of classical and quantum spectral curves are covered in the notion of morphisms from Azumaya spaces.

See [L-Y6: Sec. 5.2] (D(5)) for more general discussions, details, and more references.

### 2.5 Remarks on general Azumaya-type noncommutative schemes.

From the pure mathematical/geometric point of view, it should be clear that the notion of (trivial or nontrivial) ‘Azumaya noncommutative schemes with a fundamental module’ alone is not a final/complete picture. Beginning with such a space $(X^{\mathcal{A}_1}, \mathcal{E})$, let $X_\mathcal{A}$ be a surrogate of $X^{\mathcal{A}_1}$, the category $\mathcal{C}$ that contains all Azumaya noncommutative schemes with a fundamental module should contain also $(X_\mathcal{A}, \mathcal{E})$, where $\mathcal{O}_X^{\mathcal{A}_1} = \text{End}_X (\mathcal{A}_X \mathcal{E})$. From this one starts to extend the set of objects of $\mathcal{C}$ to include sheaves of orders with a generically fundamental module, ... Furthermore, from the naturality of operations on the category of sheaves of modules ([II] and [Kas-S]) and the later development of D-branes since 1999 ([Sh1] and [Dou6]), one expects that one finally has to consider everything in the derived(-category) sense.
String-theoretical remarks on Sec. 2.

(1) [Matrix gauged linear sigma model]

**Conjecture [matrix gauged linear sigma model].** For each gauged linear sigma model in the sense of [Wi1], there exists a canonically constructed matrix/Azumaya gauged linear sigma model so that the moduli space of vacua of the latter is the stack of D0-branes, in the sense of Polchinski-Grothendieck Ansatz, on the moduli space of vacua of the former.

(2) [Evidence of Polchinski-Grothendieck Ansatz]

Each of the six tests reviewed/presented compactly in Sec. 2.4 has their own distinct feature. Passing one does not imply passing another. Thus, all six tests together give us a first evidence of the Polchinski-Grothendieck Ansatz as a foundational feature of D-branes.

(3) [Too much information?]

Gómez and Sharpe pointed out in [G-Sh] that the scheme structure on D-branes may carry more information than D-branes do physically. This rings also with the mathematical fact that not all morphisms are good, e.g., in the sense of the existence of a perfect obstruction theory on the moduli stack of morphisms of a fixed combinatorial type. So having too much information – i.e. the necessity to single out “good” morphisms in our setting – is a question one should definitely address – if not for stringy reasons, then for mathematical reasons.

3 The differential/symplectic topological aspect: Azumaya noncommutative $C^\infty$-manifolds with a fundamental module and morphisms therefrom.

Having reviewed Azumaya (noncommutative) geometry in the algebro-geometric setting, we now take it as background to introduce Azumaya noncommutative $C^\infty$-manifold with a fundamental module and smooth morphisms/maps therefrom to a complex projective manifold $Y$. This Azumaya geometry in the differential/symplectic topological category will be our prototypical picture of D-branes of A-type in superstring theory along the line of the Polchinski-Grothendieck Ansatz. For simplicity, we assume that there is no $B$-field on $Y$ in Sec. 3 and Sec. 4. In particular, any Azumaya structure is assumed to be (Zariski-/analytic-/$C^\infty$-topology-)locally trivializable as $M_r(R)$ for $R$ the function/coordinate ring of a local chart.

3.1 Azumaya noncommutative $C^\infty$-manifolds with a fundamental module and morphisms therefrom.

Aspect II and Aspect IV presentations of morphisms from Azumaya $C^\infty$-manifolds are given. The notion of 1-forms and of their tensor products on an Azumaya $C^\infty$-manifold are introduced.

Maps from a real manifold(-stratified space) to a complex manifold/variety/stack.

Let $X$ be a real smooth manifold.

**Convention 3.1.1. [C∞ structure sheaf].** To uniform the notation with algebro-geometry, we will denote the sheaf $C^\infty_X$ of smooth functions on $X$ interchangeably by $O^\infty_X$ or, simply, $O_X$. 

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Let $Y$ be a complex manifold. Again, we will assume that $Y$ is projective though some constructions below do not require this. Then a (smooth) map $f : X \to Y$ is specified by the pullback of functions $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X^{\infty} \otimes_R \mathbb{C}$. Identify $Y$ with a smooth variety over $\mathbb{C}$, we shall think of $f$ as specifying a (real) $X$-family of $\mathbb{C}$-points on $Y$.

This interpretation applies also for $Y$ a singular complex variety. Treating $X$ as a gluing system of local charts, the same picture applies when $Y$ is an Artin stack over $\mathbb{C}$.

The same applies also when $X$ is a topological space that is stratified by real manifolds. In this case, the smooth structure sheaf $\mathcal{O}_X^{\infty} = : \mathcal{O}_X$ of $X$ is defined to be the sheaf of continuous functions on $X$ that is smooth in the interior of all its manifold strata.

**Morphisms from Azumaya $C^\infty$-manifolds with a fundamental module.**

The theory of sheaves on real manifolds was already developed long ago, e.g. [Kas-S]. It can be used to parallelly construct Azumaya geometry in the $C^\infty$-category as is done in the algebro-geometric category.

**Definition 3.1.2. [Azumaya $C^\infty$-manifold].** Let $X$ be a real (smooth) manifold with structure sheaf $\mathcal{O}_X^{\infty}$. An Azumaya $C^\infty$-manifold with a fundamental module is a triple

$$(X , \mathcal{O}_X^{\infty, A_e} := \text{End}_{\mathcal{O}_X^{\infty} \otimes_R \mathbb{C}}(\mathcal{E}) , \mathcal{E}) =: (X^A_e, \mathcal{E})$$

where $\mathcal{E}$ is the sheaf of (smooth) sections of a (smooth) complex vector bundle $E$ over $X$. The sheaf $\mathcal{O}_X^{\infty, A_e}$ of $\mathcal{O}_X^{\infty} \otimes_R \mathbb{C}$-algebras is called the Azumaya (noncommutative) structure sheaf of $X^A_e$. It contains $\mathcal{O}_X^{\infty} \otimes_R \mathbb{C}$ as its sheaf of centers.

With the interpretation of a map from a real-manifold-stratified topological space to a complex manifold/variety/stack in the previous theme, all four aspects of a morphism from an Azumaya scheme with a fundamental module to $Y$ are expected to be adoptable to the $C^\infty$-category to give four equivalent aspects of a morphism $\varphi : (X, \mathcal{O}_X^{\infty, A_e}, \mathcal{E}) \to Y$. However, for Aspect I, to develop a theory in its own right to characterize an $\mathcal{O}_X^{\infty} \otimes_R \mathbb{C}$-subalgebra $\mathcal{O}_X^{\infty} \otimes_R \mathbb{C} \subset A \subset \mathcal{O}_X^{\infty, A_e}$ of $\mathcal{O}_X^{\infty, A_e}$ such that $A_{\text{red}}$ is the complexified structure sheaf $\mathcal{O}_X^{\infty} \otimes_R \mathbb{C}$ of a real-manifold-stratified space $X^\circ$ is a very technical language issue. An easier path to take is treat Aspects II and IV with slightly higher weight. Once either is taken as the starting point, the remaining Aspects I and III become a matter of translation. In the following two definitions, we actually mix Aspects II and IV slightly to take care of the notion of ‘flat over $X$’ and of ‘piecewise smooth surrogate’:

**Definition 3.1.3. [morphism: Aspect II].** An Aspect II presentation of a morphism from an Azumaya $C^\infty$-manifold with a fundamental module of rank $r$ to $Y$ is the following data:

1. A torsion sheaf $\widetilde{\mathcal{E}}$ of $\mathcal{O}_X \otimes_R \mathcal{O}_Y$-modules on $X \times Y$ that satisfies:
   1. (1) on each $\{p\} \times Y$, $\widetilde{\mathcal{E}}|_{\{p\} \times Y}$ is a 0-dimensional $\mathcal{O}_Y$-module of length $r$;
   2. (2) $(\text{Supp} \widetilde{\mathcal{E}})_{\text{red}} \subset X \times Y$, with the induced topology, is stratified by smooth manifolds.\(^\text{10}\)

\(^{10}\)Here a subtle issue comes in: In algebraic geometry, the support of a sheaf $\mathcal{F}$ on a scheme $Z$ is the subscheme defined by the ideal sheaf $\text{Ker}(\mathcal{O}_Z \to \text{End}_{\mathcal{O}_Z}(\mathcal{F}))$ of $\mathcal{O}_Z$. This is the most natural notion of ‘support of a sheaf’ as it encodes some fuzzy structure related to sections of $\mathcal{F}$. Here, we are in $C^\infty$-category. Naively, $\text{Supp} \widetilde{\mathcal{E}}$ would be just the set of points, with the induced subset-topology, on $X \times Y$ such that the stalk of $\mathcal{E}$ at which is non-zero. However, along the $Y$-direction, it still makes sense to talk about scheme-type structure. Indeed, one wishes to define $\text{Supp} \widetilde{\mathcal{E}}$ as close to scheme-theoretical support as possible to reflect D-branes. In the current work, we do not finalize the resolution of this issue in the $C^\infty$/symplectic category. Rather, we use Aspect IV to encode this not-yet-defined structure as much as possible. Here, the redundant notation $(\text{Supp} \widetilde{\mathcal{E}})^{\text{red}}$ for the point-set support of $\widetilde{\mathcal{E}}$ on $X \times Y$ is meant to keep this subtle point in mind. See Sec. 2.4 Test (4) with [G-Sh] and Remark 4.2.5.
(3) for any $p \in X$, there exists a neighborhood $U$ of $p$ such that there is a continuous map $f_U : U \to (\text{Quot}^H_0(\mathcal{O}_{Y}^{\oplus r}, r))_{\text{red}}$ with $f_U(p) \simeq \mathcal{E}^\sim |_{\{p\} \times Y}$.

Here we treat $(\text{Quot}^H_0(\mathcal{O}_{Y}^{\oplus r}, r))_{\text{red}}$ as a singular complex space with the analytic topology.

**Remark 3.1.4.** [from Aspect II to Aspect I]. Given an Aspect II presentation $\mathcal{E}$ on $X \times Y$ of a morphism, let $p_{1, X} : X \times Y \to X$ and $p_{2, Y} : X \times Y \to Y$ be the projection maps. Then, $\mathcal{E}$ on $X$ is recovered by $p_{1, X}^* \mathcal{E}$. The composition $\mathcal{O}_Y \xrightarrow{p_{2, Y}^*} \mathcal{O}_X \boxtimes_\mathbb{R} \mathcal{O}_Y \to \mathcal{E} \text{End}_{\mathcal{O}_X \boxtimes_\mathbb{R} \mathcal{O}_Y}(\mathcal{E})$ induces a map between sheaves of rings $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{E} \text{End}_{\mathcal{O}_X \boxtimes_\mathbb{R} \mathcal{O}_Y}(\mathcal{E}) = \mathcal{O}_X^\infty, A^e$. This recovers $\varphi : (X^e, \mathcal{E}) \to Y$.

**Definition 3.1.5.** [morphism: Aspect IV]. An Aspect IV presentation of a morphism from an Azumaya $C^\infty$-manifold with a fundamental module of rank $r$ to $Y$ is the following data:

- A $GL_r(\mathbb{C})$-equivariant continuous map $f_{P_X} : P_X \to (\text{Quot}^H_0(\mathcal{O}^{\oplus r}_X, r))_{\text{red}}$, where $P_X$ is a smooth principal $GL_r(\mathbb{C})$-bundle over $X$, that satisfies:

  1. the point-set support $(\text{Supp} \mathcal{E})_{\text{red}}$, with the subset topology, of the sheaf $\mathcal{E}$ on $X \times Y$ associated to $f_{P_X}$ is stratified by smooth manifolds.

**Remark 3.1.6.** [from Aspect IV to Aspect I]. Note that the $\mathcal{E}$ on $X \times Y$ associated to $f_{P_X}$ automatically satisfies Conditions (1) and (3) in Definition 3.1.3. Thus, it gives an Aspect II presentation of a morphism, which can recover Aspect I, ‘The fundamental setting’, of a morphism by Remark 3.1.4.

**Kähler differentials on an Azumaya noncommutative scheme $C^\infty$-manifold.**

Before leaving this subsection, we introduce the basic notion of Kähler differentials (i.e. 1-forms) and their tensor products on an Azumaya noncommutative space. This is a notion we can still bypass in this review/work (but see Remark 3.2.2); however, they become important in [L-Y7]. Such notion for an associative unital algebra appeared earlier in, e.g., [C-Q] and [K-R].

**Definition 3.1.7.** [C-linear derivation]. Let $S$ be an associative (unital) $\mathbb{C}$-algebra and $M$ be a (two-sided) $S$-module. A map $d : S \to M$ (as abelian groups) is called a C-linear derivation if it is a homomorphism of $\mathbb{C}$-modules that satisfies the Leibniz rule

$$d(fg) = (df)g + f dg \quad \text{for } f, g \in S.$$

**Definition 3.1.8.** [Kähler differential]. Let $M_r(R)$ be the $r \times r$ matrix ring over a commutative $\mathbb{C}$-algebra $R$. Denote by $\Omega_{M_r(R)}$ the module of (Kähler) differentials of $M_r(R)$ over $\mathbb{C}$ the (two-sided) $M_r(R)$-module generated by the set $\{dm : m \in M_r(R)\}$ subject to the relations

\[
\begin{align*}
\text{(C-linearity)} & \quad d(am + bm') = a\, dm + b\, dm' \quad \text{for } a, b \in \mathbb{C} \text{ and } m, m' \in M_r(R), \\
\text{(Leibniz rule)} & \quad d(mm') = (d)m' + m\, dm', \\
\text{(commutativity pass-over)} & \quad m(dm') = (dm')m \quad \text{for } m, m' \in M_r(R) \\
\end{align*}
\]

that commutes: $mm' = m'm$.

By construction, it is equipped with a built-in $\mathbb{C}$-linear derivation

$$d : M_r(R) \to \Omega_{M_r(R)} \text{ defined by } m \mapsto dm.$$
Definition 3.1.9. [tensor product]. The $s$-fold tensor product $\otimes^s_{M_r(R)} \Omega_{M_r(R)}$ of $\Omega_{M_r(R)}$ over $M_r(R)$ for $s \in \mathbb{Z}_{\geq 0}$ is the bi-$M_r(R)$-module with generators $dm_1 \otimes \cdots \otimes dm_s$, $m_j \in M_r(R)$, and relators

$$dm_1 \otimes \cdots \otimes (dm_i)m \otimes dm_{i+1} \otimes \cdots \otimes dm_s = dm_1 \otimes \cdots \otimes dm_i \otimes m dm_{i+1} \otimes \cdots \otimes dm_s,$$

$i = 1, \ldots, s - 1$, for all $m, m_j \in M_r(R)$.

For $r = 1$, the above reduces to the usual notion of Kähler differentials and their tensor products in commutative ring theory; e.g., $[Ei]$ and $[Ma]$.

As the above construction commutes with central localizations of $M_r(R)$, one has the following sheaves via central localizations and gluings:

Lemma/Definition 3.1.10. [sheaf of Kähler differential]. Given an Azumaya scheme $X^A = (X, \mathcal{O}_X^A = \text{End}_{\mathcal{O}_X}\mathcal{E})$, one obtains a sheaf $\Omega_{X^A}$ on $X^A$ via gluing $\Omega_{M_r(R)}$’s from affine open sets $U = \text{Spec } R$ of $X$. $\Omega_{X^A}$ is called the sheaf of Kähler differentials on $X^A$. Similarly, one has its tensor products $\otimes^s_{\mathcal{O}_X^A(X^A)} \Omega_{X^A} =: \Omega_{X^A}^{\otimes s}$ over $\mathcal{O}_X^A$ for $s \in \mathbb{Z}_{\geq 0}$.

Similar construction applies to Azumaya $C^\infty$-manifolds, in which case we will call the resulting Kähler differentials also 1-forms.

In terms of this, the commutativity-pass-over property of Kähler differentials/1-forms implies the following induced map defined in exactly the same way as in the commutative case:

Lemma 3.1.11. Given a morphism $\varphi : X^A \to Y$ either from an Azumaya scheme to a (commutative) scheme or from a $C^\infty$ Azumaya manifold to a complex manifold, then there is an induced map $\varphi^* : \Omega_Y \to \Omega_{X^A}$ as $\mathcal{O}_Y$-modules, defined locally by $df \mapsto d(\varphi^*(f))$. Here, $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_{X^A}^A$ is the defining pull-back-of-functions underlying $\varphi$. Similarly, for the existence of $\varphi^{\otimes s} : \Omega_{Y^{\otimes s}} \to \Omega_{X^A}^{\otimes s}$.

3.2 Lagrangian morphisms and special Lagrangian morphisms: Donaldson and Polchinski-Grothendieck.

Definition 3.2.1. [Lagrangian/special Lagrangian morphism]. Let $X$ be a smooth manifold of (real) dimension $n$ and $(Y, \omega)$ be a complex manifold of (complex) dimension $n$ from a projective variety $\mathbb{C}$ with a Kähler form $\omega$. A morphism $\varphi : (X^A, \mathcal{E}) \to Y$ is said to be a Lagrangian morphism if in its Aspect II presentation, say, by a torsion sheaf $\mathcal{E}$ of $\mathcal{O}_X \otimes \mathcal{O}_Y$-modules on $X \times Y$, the following conditions are satisfied:

1. (generically immersion) the restriction of the projection map $\text{pr}_2 : X \times Y \to Y$ to $(\text{Supp } \mathcal{E})_{\text{red}}$ is an immersion on a dense open subset;

2. (Lagrangian condition) the restriction of $\text{pr}_2^* \omega$ to $(\text{Supp } \mathcal{E})_{\text{red}}$ vanishes.

If furthermore $(Y, \omega)$ is a Calabi-Yau manifold with a calibration given by $\text{Re } \Omega$ of a holomorphic $n$-form $\Omega$, then $\varphi : (X^A, \mathcal{E}) \to Y$ is said to be a special Lagrangian morphism if it is a Lagrangian morphism and

3. (calibration condition) $\text{pr}_2^* \text{Re } \Omega = \text{pr}_2^* \text{vol}_n$ holds on an open dense subset of $(\text{Supp } \mathcal{E})_{\text{red}}$.

Here, $\text{vol}_n$ is the real volume-$n$-form associated to the Kähler metric of $Y$ from $\omega$.

Remark 3.2.2. [intrinsic Lagrangian/calibration condition]. In terms of 1-forms on $X^A$, the Lagrangian and the calibration condition can be expressed more intrinsically/compactly as

$$\varphi^* \omega = 0 \quad \text{and} \quad \varphi^* \text{Re } \Omega = \varphi^* \text{vol}_n.$$
To reflect the behavior of D-branes of A-type correctly, we need a notion of “multiplicity” of a Lagrangian cycle that takes into account not just the cycle alone but also the bundle/sheaf it carries. The following version of such is what we will use.

Definition 3.2.3. [ϕ∗(X) and ϕ∗([X, E]) for Lagrangian ϕ]. Let X be oriented with the associated fundamental cycle [X], ϕ : (X£, E) → Y a morphism, E on X × Y its presentation, and pr1 : X × Y → X, pr2 : X × Y → Y the projection maps. Then the orientation on X induces an orientation on the n-dimensional strata of (Supp E)red. This defines a fundamental cycle [(Supp E)red] of (Supp E)red. Define ϕ∗[X] to be the (real) n-cycle pr2∗((Supp E)red) on Y.

If furthermore ϕ is a Lagrangian morphism, then let

\[ [(\text{Supp} \tilde{E})_{\text{red}}] = \sum_{\alpha} \Delta_\alpha \]

be a fine enough triangulation of (Supp E)red such that the following hold:

1. \( \sum_\alpha \Delta_\alpha \) refines the manifold-stratification of (Supp E)red;
2. for each n-simplex \( \Delta_\alpha \) as a subset of \( X \times Y \), \( pr_1 : \Delta_\alpha \to X \) is an embedding;
3. for each n-simplex \( \Delta_\alpha \) as a subset of \( X \times Y \), define \( l(p), p \in \Delta_\alpha \), to be the length of the stalk \( (\tilde{E}|_{\{pr_1(p)\} \times Y})_{\{pr_1(p)\}} \) on \( \{pr_1(p)\} \times Y \); then the function \( l(\bullet) \) is constant in the interior of \( \Delta_\alpha \); denote this constant by \( l_\alpha \).

Define

\[ \varphi_*([X, E]) := pr_{2*} \left( \sum_\alpha l_\alpha \Delta_\alpha \right). \]

By taking a common refinement of triangulations of (Supp E)red, it is clear that \( \varphi_*([X, E]) \) is well-defined.

Notation 3.2.4. [Chan-Paton-sheaf-adjusted multiplicity of Lagrangian cycle]. With the notation from above, we will denote a Lagrangian cycle of the form \( \varphi_*([X, E]) \) also as \( [[\varphi_*[X], \varphi_*E]] \) to emphasize this adjustment of multiplicities along the Lagrangian cycle \( \varphi_*[X] \) due to the Chan-Paton bundle/sheaf \( \varphi_*E \) over it.

Recall Donaldson’s description [Don] of Lagrangian submanifolds/cycles and special Lagrangian submanifolds/cycles \( L \) in a Calabi-Yau manifold \( Y \) as the image of a special class of maps \( f \) from a smooth manifold \( S \), equipped with a volume-form \( \sigma \), to \( Y \), selected by a moment map associated to the (right) action of the group \( \text{Diff}(S, \sigma) \) of volume-preserving diffeomorphisms of \( (S, \sigma) \) on the space \( \text{Map}(S, Y) \) of smooth maps from \( S \) to \( Y \) by precomposition of maps; see also [Hi2]. The aspect of treating Lagrangian or special Lagrangian \( L \subset Y \) as the image \( f(S) \) of \( f : S \to Y \) and the fact that such \( L \) are candidates for supersymmetric D-branes naturally make one wonder:

· Q. [Donaldson + Polchinski-Grothendieck]

Can Donaldson’s aspect of Lagrangian and special Lagrangian submanifolds/cycles and Polchinski-Grothendieck’s aspect of D-branes merge?

\[ \text{C.-H.L. would like to thank Katrin Wehrheim for a discussion on coincident Lagrangian submanifolds and their multiplicities, spring 2008.} \]
In this subsection, we discuss a special class of morphisms $\varphi$ from Azumaya spaces $(X^c, E)$ with a fundamental module over a fixed $X$ to a Calabi-Yau manifold $Y$ when the answer is yes. The symplectic construction needed in the discussion follows [Don]. For simplicity of presentation, we identify a vector bundle $V$ with its sheaf $\mathcal{V}$ of local sections and denote both by the sheaf notation $\mathcal{V}$.

Let $(X, \sigma_X)$ be a smooth manifold of real dimension $n$ with a volume-form $\sigma_X$, $S$ be a smooth manifold of real dimension $n$, $\mathcal{V}$ be a (smooth) complex vector bundle on $S$ of $(\mathbb{C})$-rank $r_0$, and $(Y, \omega, \Omega)$ be a Calabi-Yau manifold of complex dimension $n$ with a Kähler form $\omega$ and a holomorphic $n$-form $\Omega$. Let $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y$ be the projection maps. Then a smooth maps $(c, g) : S \to X \times Y$, where $c : S \to X$ is a finite cover of $X$, and $g : S \to Y$, defines a torsion sheaf $\mathcal{E}_{(c,g)} := (c,g)_*\mathcal{V}$ on $X \times Y$. Its pushforward $\mathcal{E}_{(c,g)} := pr_{1*}\mathcal{E}_{(c,g)} = c_*\mathcal{V}$ is a complex vector bundle on $X$ of rank $r = dr_0$, where $d$ is the degree of $c$. It follows from Sec. 2.2 that $(c, g)$ induces a morphism

$$\varphi_{(c,g)} : (X^c, \mathcal{E}_{(c,g)}) \to Y$$

with surrogates $X_{\varphi_{(c,g)}} = (c, g)(S)$ and maps $\pi_{\varphi} : X_{\varphi_{(c,g)}} \to X$ and $f_{\varphi_{(c,g)}} : X_{\varphi_{(c,g)}} \to Y$ induced by $pr_1$ and $pr_2$ respectively; denote the built-in map $S \to X_{\varphi_{(c,g)}}$ by $h_{(c,g)}$:

Let $\text{Map}^{\text{cover/}X}(S, X \times Y)$ be the space of smooth maps $S \to X \times Y$ such that its first component $S \to X$ is a cover of $X$. This is an infinite-dimensional smooth manifold locally modelled on $C^\infty(S, c^\ast T_sX \oplus g^\ast T_sY)$ at a point $[(c, g)]$. The correspondence

$$(c, g) \mapsto \left(\varphi_{(c,g)} : (X^c_{(c,g)}, \mathcal{E}_{(c,g)}) \to Y\right)$$

defines then a map

$$\text{Map}^{\text{cover/}X}(S, X \times Y) \to \text{Map}^{\mathcal{A}/}(X, Y)$$

from $\text{Map}^{\text{cover/}X}(S, X \times Y)$ into the space $\text{Map}^{\mathcal{A}/}(X, Y)$ of morphisms from Azumaya noncommutative manifolds with a fundamental module, supported on the fixed $X$, to $Y$.

Let $\sigma_c = c^\ast \sigma_X$ be a volume-form on $S$ by lifting $\sigma_X$ via $c$. Then, the $\text{Diff}(S, \sigma_c)$-action on $\text{Map}(S, Y)$ in [Don], recalled in the beginning of this subsection, induces a $\text{Diff}(S, \sigma_c)$-action on $\text{Map}^{\text{cover/}X}(S, X \times Y)$ by acting on the second component $g$ of $(c, g)$.

**Lemma 3.2.5.** [$h_{(c,g)}$ and $\pi_{\varphi_{(c,g)}}$: generically covers]. There exists an open dense subset $U$ of $X$ such that:

1. $\pi_{\varphi_{(c,g)}}^{-1}(U)$ is open dense in $X_{\varphi_{(c,g)}}$ and $c^{-1}(U) = h_{(c,g)}^{-1}(\pi_{\varphi_{(c,g)}}^{-1}(U))$ is open dense in $S$;

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(2) the restrictions $h_{(c,g)} : c^{-1}(U) \to \pi_{\varphi_{(c,g)}}^{-1}(U)$ and $\pi_{\varphi_{(c,g)}} : \pi_{\varphi_{(c,g)}}^{-1}(U) \to U$ are covering maps.

This follows from the fact that $(c,g)$ and hence $h_{(c,g)}$ and $\pi_{\varphi_{(c,g)}}$ are all smooth maps and that $c = \pi_{\varphi_{(c,g)}} \circ h_{(c,g)}$ is a covering map. Recall the volume-form $\sigma_X \varphi_{(c,g)}$ on the possibly singular $X_{\varphi_{(c,g)}}$ from lifting $\sigma_X$ via $\pi_{\varphi_{(c,g)}}$. The above lemma implies immediately:

**Corollary 3.2.6.** \([\varphi_{(c,g)} \text{ and } g: \text{same Lagrangian/special Lagrangian property}].\)

1. $f_{\varphi_{(c,g)}}^* \omega = 0$ if and only if $g^* \omega = 0$.
2. $f_{\varphi_{(c,g)}}^* Re \Omega = \sigma_X \varphi_{(c,g)}$ (resp. $f_{\varphi_{(c,g)}}^* Im \Omega = 0$) if and only if $g^* Re \Omega = \sigma_c$ (resp. $g^* Im \Omega = 0$).

In particular, $\varphi_{(c,g)}$ is a Lagrangian morphism if and only if $g$ is a Lagrangian morphism; and $\varphi_{(c,g)}$ is a special Lagrangian morphism if and only if $g$ is a special Lagrangian morphism.

Thus, Donaldson’s picture of Lagrangian/special Lagrangian submanifolds in a Calabi-Yau space and Polchinski-Grothendieck’s picture of supersymmetric D-branes of A-type in a Calabi-Yau space are tied together for a special class of such submanifolds/branes. The construction of Donaldson in [Don: Sec. 1.1 and Sec. 3.1] can now be applied to this special submoduli space of morphisms from Azumaya manifolds with the fixed base $X$ to the Calabi-Yau manifold $Y$ to characterize Lagrangian morphisms, with special Lagrangian morphisms in this special class selected further by the calibration condition.

To proceed, let $\text{Cover}(S, X)$ be the space of covering maps $S \to X$, $\text{Cover}(S, X) \times S \to X$ the universal covering map, and

$$\text{Map}^{cover/X}(S, X \times Y) = \text{Cover}(S, X) \times \text{Map}(S, Y) \longrightarrow \text{Cover}(S, X)$$

the forgetful map. A connected component of $\text{Map}^{cover/X}(S, X \times Y)$ is a product of that of $\text{Cover}(S, X)$ and that of $\text{Map}(S, Y)$. For each connected component of $\text{Map}^{cover/X}(S, X \times Y)$ with a base-section \([([C_0; g_0] : \text{Cover}(S, X) \times S \to X \times Y)] \text{ over } \text{Cover}(S, X), \text{ where } g_0 : \text{Cover}(S, X) \times S \to Y\) is induced from a map, also denoted by $g_0 \in \text{Map}(S, Y)$, via the composition $\text{Cover}(S, X) \times S \to S \overset{g_0}{\to} Y$, fix a base-reference relative 2-form $\nu$ in the relative de Rham cohomology class $g_0^*([\omega]) \in H^2((\text{Cover}(S, X) \times S)/\text{Cover}(S, X); \mathbb{R})$. Let

$$\text{Diff}((\text{Cover}(S, X) \times S, C_0^* \sigma_X)/\text{Cover}(S, X))$$

be the relative group of relative-volume-preserving $S$-bundle-diffeomorphisms of $(\text{Cover}(S, X) \times S, C_0^* \sigma_X)$ over $\text{Cover}(S, X)$ and

$$\text{Calabi} : \text{Diff}((\text{Cover}(S, X) \times S, C_0^* \sigma_X)/\text{Cover}(S, X)) \longrightarrow H^{n-1}(S; \mathbb{R})/H^{n-1}(S; \mathbb{Z})$$

be a relative Calabi-homomorphism. Then $G_0 := \text{Ker} (\text{Calabi})$ is a relative Lie group over $\text{Cover}(S, X)$ whose relative Lie algebra $\mathfrak{g}_0$ can be identified with the relative exact $(n - 1)$-forms on $(\text{Cover}(S, X) \times S)/\text{Cover}(S, X)$.

For $g : S \to Y$ in the same connected component as $g_0$, $g^*([\omega]) = g_0^*([\omega])$; thus, one can choose an $a \in \Omega^1(S)$ so that $g^* \omega - \nu = da$. Fix a such $a$ for $g$. Then for any $c \in \text{Cover}(S, X)$ and any vector field $\xi$ on $S$, one can define a pairing

$$\langle a, \xi \rangle := \int_S a(\xi) \sigma_c .$$

**Proposition 3.2.7.** \([\text{moment map}]. \) ([Don: Sec. 1.1].) For a vector field $\xi$ on $S$ associated to an element in $\mathfrak{g}_0$ over $c \in \text{Cover}(S, X)$, the pairing $\langle a, \xi \rangle$ depends only on $\xi$ (and $(c,g))$, not on the choice of $a$. The relative linear functional on $\mathfrak{g}_0/\text{Cover}(S, X)$ defined by $\xi \mapsto \langle a, \xi \rangle$ gives a map

$$\mu : \text{Map}^{cover/X}(S, X \times Y) \longrightarrow \mathfrak{g}_0^* \quad (\text{as spaces over } \text{Cover}(S, X))$$

which is a relative moment map for the action of $\mathfrak{g}_0$ on $\text{Map}^{cover/X}(S, X \times Y)$ over $\text{Cover}(S, X)$.

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The locus $U_{\text{Lag}} \subset \text{Map}^{\text{cover}}/X(S, X \times Y)$ of $(c, g)$ whose associated $\varphi_{(c,g)}$ is a Lagrangian morphism lies in the zero-locus $\mu^{-1}(0)$ of the relative moment map $\mu$. Inside $U_{\text{Lag}}$ resides the locus $U_{s\text{Lag}}$ of $(c, g)$ whose associated $\varphi_{(c,g)}$ is a special Lagrangian morphism.

**String-theoretical remarks on Sec. 3.**

(1) [D-branes of A-type: cycles vs. maps]

Due to the special Lagrangian submanifold nature of D-branes of A-type, they are usually thought of as a sub-object in a Calabi-Yau manifold. Donaldson’s picture changes that. Thinking of them as maps into a target-space(-time) from an Azumaya manifold with a fundamental module makes it very direct to see how they behave under deformations or collidings-into-one. It also gives us an anticipation of what structure should be there on these branes in its own right; cf. [D-K-S] and Sec. 4.2, theme: ‘The generically filtered structure on the Chan-Paton bundle over a special Lagrangian cycle on a Calabi-Yau torus’ of the current work.

4 D-branes of A-type on a Calabi-Yau torus and their transitions.

So far in this project we have illustrated featural behaviors of D-branes of B-type as they fit well in the realm of algebraic geometry. In this section\(^\text{18}\), we give an example of such behaviors for nonsupersymmetric D-branes and D-branes of A-type. This example is only a toy model but has several pedagogical meanings. Its simplicity allows one to see/learn things about D-branes in this category without being blocked/complicated by mathematical technicality before one studies the same issue at the level of, e.g., [Lee] and [Joy2].

Let $C$ be the complex plane with coordinate $z$ and $C = C_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be the complex torus of modulus $\tau \in \mathbb{C}$, defined up to the $SL(2, \mathbb{Z})$ transformation $\tau \mapsto (a\tau + b)/(c\tau + d)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, with the flat metric specified by the Kähler form $\omega = dz \wedge d\overline{z}$. With this metric, a covariantly constant holomorphic 1-form on $C$ whose real part defines a calibration on $(C, \omega)$ such that there exists a calibrated cycle on $(C, \omega)$ is given by $\Omega_\theta = e^{i\theta}dz$ for $\theta \in A_\tau \subset S^1$, parameterized by $\{\text{Arg}(z) : z \in \mathbb{Z} + \mathbb{Z}\tau\} \subset [0, 2\pi)$. The volume-minimizing property of a calibrated cycle implies that such a cycle is lifted to a collection of parallel straight lines in $\mathbb{C}$ (with the standard Kähler form also denoted by $dz \wedge d\overline{z}$). In particular, all the calibrated 1-cycles on $(C, \omega, \Omega_\theta)$ are submanifolds, with their connected components differing by isometric translations on $(C, \omega)$. For the current case, any 1-cycle on $(C, \omega)$ is a Lagrangian cycle while:

**Convention 4.0.8.** [special Lagrangian cycle/morphism on/to $(C, \omega)$]. A 1-cycle $L$ on $(C, \omega)$ is called a special Lagrangian cycle if it is a calibrated 1-cycle with respect to $\Omega_\theta$ for some $\theta \in S^1$. Similarly, for the notion of a special Lagrangian morphism $\varphi$ to $(C, \omega)$.

Naively, one may wonder that D-branes of A-type in the current situation is too trivial to even be considered as an example. However, it should be remembered that D-branes are not just the underlying supporting cycles. Among other things, they are equipped with a Chan-Paton bundle/sheaf/module which even in this example can has less trivial structures. To easily see such latter structure, we employ Aspect IV of morphisms in Sec. 2.2 for the discussion.

\(^\text{18}\)In this section, a 1-cycle means a cycle of $\mathbb{R}$-dimension 1, while the rank of a complex vector bundle over a real manifold is by definition the $\mathbb{C}$-rank.
4.1 The stack $\mathcal{M}_r^{0\text{Az}}(C)$.

Some details of the stack $\mathcal{M}_r^{0\text{Az}}(C)$ are required to understand morphisms to $C$ from Aspect IV.

The stack $\mathcal{M}_r^{0\text{Az}}(C)$ and its representation-theoretical atlas $\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r)$\(^{19}\)

Recall the representation-theoretical atlas

$$\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r) := \{ \mathcal{O}_C^{\text{br}} \rightarrow \mathcal{E} \rightarrow 0, \text{ length } \mathcal{E} = r, H^0(\mathcal{O}_C^{\text{br}}) \rightarrow H^0(\mathcal{E}) \rightarrow 0 \}$$

of the stack $\mathcal{M}_r^{0\text{Az}}(C)$ of morphisms from a (non-fixed) Azumaya point with a fundamental module $\simeq \mathbb{C}^r$ to $C$.

**Lemma 4.1.1.** [$\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r)$ smooth]. $\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r)$ is smooth of (complex) dimension $r^2$ for $C$ a smooth complex curve.

**Proof.** One only needs to prove the statement around a point $[\mathcal{O}_C^{\text{br}} \rightarrow \mathcal{E} \rightarrow 0] \in \text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r)$ with $\mathcal{E}$ supported at a point $p \in C$. In this case, since there exists a branched-covering map $C \rightarrow \mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$ that takes $p$ to $0 \in \mathbb{C}$, one reduces the problem further to the case $C = \mathbb{C}$. Since $\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r) \simeq \mathbb{C}^{r^2}$ (cf. Example 2.1.1), the lemma follows. \( \square \)

**Lemma 4.1.2.** [$\mathcal{M}_r^{0\text{Az}}(C)$ smooth]. $\mathcal{M}_r^{0\text{Az}}(C)$ is smooth of uniform stacky-dimension 0.

**Proof.** The smoothness of $\mathcal{M}_r^{0\text{Az}}(C)$ follows from Lemma 4.1.1 that it has a smooth atlas, and [Schl]. For the stacky-dimension, note that the first projection map of the fibered-product

$$\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r) \times_{\mathcal{M}_r^{0\text{Az}}(C)} \text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r) \xrightarrow{pr_1} \text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r)$$

has uniform relative-dimension $r^2$. Since $\text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r)$ is smooth of dimension $r^2$, the stacky-dimension of $\mathcal{M}_r^{0\text{Az}}(C)$ is also uniform, whose value is given by

$$\dim \mathcal{M}_r^{0\text{Az}}(C) = \dim \text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r) - \text{rel. dim } pr_1 = 0.$$  \( \square \)

It follows from the proof of Lemma 4.1.1 that the generalized Hilbert-Chow morphism

$$\rho : \text{Quot}^H_0(\mathcal{O}_C^{\text{br}}, r) \rightarrow C^{(r)} \quad [\mathcal{O}_C^{\text{br}} \rightarrow \mathcal{E} \rightarrow 0] \mapsto \sum_{p \in C} \text{length } (\mathcal{E}_p)[p],$$

where $C^{(r)}$ is the $r$-th symmetric product of $C$ and $\mathcal{E}_p$ is the stalk of $\mathcal{E}$ at $p$, is locally modelled on the morphism

$$\rho' : \mathbb{C}^{r^2} \rightarrow \mathbb{C}^r \quad m \mapsto \det(\lambda - m),$$

where $\mathbb{C}^{r^2}$ is the space of $r \times r$-matrices over $\mathbb{C}$ and $\mathbb{C}^r$ is the space of monic polynomials of degree $r$ in $\lambda$. Let $GL_r(\mathbb{C})$ acts on $C^{(r)}$ on the right trivially by the identity map, then $\rho$ is

\(^{19}\)In this theme, we adopt notations from complex/symplectic geometry. However, one can treat the whole theme in the realm of algebraic geometry over $\mathbb{C}$ and then take its valid analytic counterpart.
For two orbits \( O \) on \((P, r)\) where \( GL_r(C) \)-invariant stratification of \( \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r) \).

For \( r \geq 2 \), the fibration \( \rho \) has no sections. However, through the canonical isomorphism of \( C^{(r)} \) with the Hilbert-scheme \( C^{[r]} \) of 0-dimensional subschemes of \( C \) of length \( r \) and the universal subscheme on \((C^{[r]} \times C)/C^{[r]}\), one obtains a map

\[
\sigma_r =: \sigma_{(r)} : C^{[r]} = C^{(r)} \rightarrow \mathbb{M}^{\mathbb{P}^r}_{C^{[r]}}(C)
\]

and, hence, a \( GL_r(C) \)-equivariant morphism over \( C^{[r]} = C^{(r)} \) via the Isom-functor construction:

\[
\tilde{\sigma}_r =: \tilde{\sigma}_{(r)} : P_{C^{[r]}} =: P_{C^{(r)}} \rightarrow \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r).
\]

Here, \( P_{C^{[r]}} \) (resp. \( P_{C^{(r)}} \)) is a principal \( GL_r(C) \)-bundle over \( C^{[r]} \) (resp. \( C^{(r)} \)).

Another natural map into \( \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r) \) that is related to \( C^{(r)} \) more directly can be constructed as follows. Regard \( C \) as the moduli space of 0-dimensional subschemes of itself of length 1 and consider the universal subscheme \( O_{C \times C} \rightarrow O_{\Delta_C} \rightarrow 0 \) on \((C \times C)/C\). Here, all the products are over \( C \), \( \Delta_C \) is the diagonal of \( C \times C \), and \((C \times C)/C\) corresponds to the first projection map \( pr_1 : C \times C \rightarrow C \). Let \( pr_i : C^{x_r} \rightarrow C \), \( 1 \leq i \leq r \), be the projection map to the \( i \)-th component. Then, the direct sum of the exact complexes

\[
\bigoplus_{i=1}^r (pr_i^*O_{C \times C} \rightarrow pr_i^*O_{\Delta_C} \rightarrow 0)
\]

on \((C^{x_r} \times C)/C^{x_r}\) defines a quotient

\[
O^{\mathbb{P}^r}_{C^{x_r} \times C} \rightarrow \bigoplus_{i=1}^r pr_i^*O_{\Delta_C} \rightarrow 0
\]

on \((C^{x_r} \times C)/C^{x_r}\). This realizes \( C^{x_r} \) as the moduli space of the quotients \( \bigoplus_{i=1}^r (O_C \rightarrow O_{p_i} \rightarrow 0) \), where \( p_i \in C \), and it defines a map over \( C^{(r)} \):

\[
\hat{\sigma} : C^{x_r} \rightarrow \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r).
\]

One can view \( \hat{\sigma} \) as a canonical multi-section of \( \rho : \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r) \rightarrow C^{(r)} \). As such, the set-value \( \hat{\sigma}^{-1}(p) \) of \( \hat{\sigma} \) at a point \( p = [(p_1, \ldots, p_r)] \in C^{(r)} \) lies in a single \( GL_r(C) \)-orbit.

**Orbit-closure inclusion relations.**

The orbit-closure inclusion relations of the \( GL_r(C) \)-orbits in \( \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r) \) can be characterized in terms of Jordan forms, as follows.

**Definition 4.1.3. [support-length data].** Let \([O^{\mathbb{P}^r}_C \rightarrow \tilde{E} \rightarrow 0] \in \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r)\), define the support-length data of \([O^{\mathbb{P}^r}_C \rightarrow \tilde{E} \rightarrow 0]\) to be \( l := \{ (p, r_p) : p \in C, r_p = \text{length}(\tilde{E}_p) > 0 \} \). This is a finite set with \( \sum_p r_p = r \) and is invariant under the \( GL_r(C) \)-action on \( \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r) \).

For two orbits \( O_1 \) and \( O_2 \) in \( \text{Quot}^H_{\tilde{O}^{\mathbb{P}^r}}(O^{\mathbb{P}^r}_C, r) \) to have non-empty \( \overline{O_1 \cap O_2} \) or \( O_1 \cap \overline{O_2} \), it is necessary that \( O_1 \) and \( O_2 \) are mapped under \( \rho \) to the same point on \( C^{(r)} \). The latter condition holds if and only if their associated support-length data are identical. Observe also that if \( O_1 \cap \overline{O_2} \neq \emptyset \), then \( O_1 \subset \overline{O_2} \).

**Definition 4.1.4. [tamed representative for orbit].** Let \( O \) be an orbit with support-length data \( l = \{(p_i, r_{p_i}) : i = 1, \ldots, k\} \). Then \([O^{\mathbb{P}^r}_C \rightarrow \tilde{E} \rightarrow 0] \in O \) is called a tamed representative of \( O \) if it is a direct sum \( \bigoplus_{i=1}^k (O^{\mathbb{P}^r}_{C_{p_i}} \rightarrow \tilde{E}_i \rightarrow 0) \), where \((\text{Supp}(\tilde{E}_i))_{\text{red}} = p_i \).
Note that for an orbit \( O \) with support-length data \( l = \{(p_i, r_{p_i}) : i = 1, \ldots, k\} \), \( \prod_{i=1}^{k} \text{GL}_{r_{p_i}}(\mathbb{C}) \) acts transitively on the set of tamed representatives of \( O \).

**Lemma 4.1.5.** [big orbit relation from small orbit relation]. Let \( O_1 \) and \( O_2 \) be two \( \text{GL}_r(\mathbb{C}) \)-orbits with identical support-length data \( l = \{(p_i, r_{p_i})\}_i \) and \( \mathcal{O}_C^r \to \tilde{E}_1 \to 0 \), \( \mathcal{O}_C^r \to \tilde{E}_2 \to 0 \) are tamed representatives for \( O_1 \) and \( O_2 \) respectively. Then \( O_1 \subset \overline{O_2} \) if and only if the corresponding \( \prod_{i} \text{GL}_{r_{p_i}}(\mathbb{C}) \)-orbits satisfy the same relation

\[
(\prod_{i} \text{GL}_{r_{p_i}}(\mathbb{C})) \cdot [\mathcal{O}_C^r \to \tilde{E}_1 \to 0] \subset \overline{\prod_{i} \text{GL}_{r_{p_i}}(\mathbb{C})} \cdot [\mathcal{O}_C^r \to \tilde{E}_2 \to 0].
\]

**Proof.** We need to show the ‘only-if’ part. Assume that \( O_1 \subset \overline{O_2} \), then there exists a path \( \gamma : [0, \infty) \to \text{GL}_r(\mathbb{C}) \) with \( \gamma(0) = \text{Id} \) such that \( \lim_{t \to \infty} \gamma(t) \cdot \mathcal{O}_C^r \to \tilde{E}_1 \to 0 \).

Since \( H^0(\mathcal{O}_C^r) \to H^0(\tilde{E}_1) \to 0 \), \( H^0(\mathcal{O}_C^r) \to H^0(\tilde{E}_2) \to 0 \), and both representatives are tamed representatives for orbits with the same support-length data, through the common decomposition \( H^0(\mathcal{O}_C^r) = \mathbb{C}^r = \oplus_{i} \mathbb{C}^{r_{p_i}} \), there exists a path \( \gamma' : [0, \infty) \to \prod_{i} \text{GL}_{r_{p_i}}(\mathbb{C}) \) with \( \gamma(0) = \text{Id} \) such that \( \gamma \) and \( \gamma' \) are asymptotically the same in the sense that \( \lim_{t \to \infty} (\gamma(t) - \gamma'(t)) = 0 \) in the matrix-representation \( \text{GL}_r(\mathbb{C}) \to M_n(\mathbb{C}) \) of \( \text{GL}_r(\mathbb{C}) \) with respect to the above decomposition of \( H^0(\mathcal{O}_C^r) \). This implies that \( \lim_{t \to \infty} \gamma'(t) \cdot [\mathcal{O}_C^r \to \tilde{E}_2 \to 0] = [\mathcal{O}_C^r \to \tilde{E}_1 \to 0] \). The lemma follows.

Let \( J_2^{(\lambda)} \in M_j(\mathbb{C}) \) be the matrix

\[
\begin{bmatrix}
\lambda & & 0 \\
1 & \lambda & \\
& \ddots & \ddots \\
0 & & 1 & \lambda
\end{bmatrix}_{j \times j}
\]

A Jordan form \( J \) in \( M_n(\mathbb{C}) \) is a matrix of the following form

\[
\begin{bmatrix}
A_1 & & 0 \\
& \ddots & \\
0 & & A_k
\end{bmatrix}
\]

with each \( A_i \in M_{n_i}(\mathbb{C}) \) of the form

\[
\begin{bmatrix}
J_{d_{i_1}}^{(\lambda_1)} & & \\
& \ddots & \\
& & J_{d_{i_k}}^{(\lambda_k)}
\end{bmatrix}
\]

Here, omitted entries are all zero, \( n_1 \geq \cdots \geq n_k > 0 \), and \( d_{i_1} \geq \cdots \geq d_{i_k} > 0 \).

Let \( O \) be a \( \text{GL}_r(\mathbb{C}) \)-orbit in \( \text{Quot} H^0(\mathcal{O}_C^r, r) \) of support-length data \( l = \{(p_i, r_{p_i}) : i = 1, \ldots, k\} \) and \( \mathcal{O}_C^r \to \tilde{E} \to 0 \) be a tamed representative of \( O \). Then it follows from the proof of Lemma 4.1.1, Example 2.1.1, and a shifting of \( z \) by \( z - c \) for an appropriate \( c \in \mathbb{C} \) in that Example that each stalk \( \tilde{E}_{p_i} \) of \( \tilde{E} \) can be represented by a matrix \( m_i \in M_{r_{p_i}}(\mathbb{C}) \) with characteristic polynomial \( \det(\lambda - m_i) \) equal to \( \lambda^{r_{p_i}} \).

**Definition 4.1.6.** [Jordan-form data]. With the above notation, the tuple \( J_O := (J_{p_i})_{i=1}^{k} \), where \( J_{p_i} \) is the Jordan form of \( m_i \), is called the Jordan-form data associated to \( O \).

**Definition 4.1.7.** [partial order on the set of Jordan-form data]. Given two Jordan-form data \( J_1 = (J_{p_i})_{i=1}^{k_1} \) and \( J_2 = (J_{q_j})_{j=1}^{k_2} \) from the above construction, we say that \( J_1 \prec J_2 \) if the following two conditions are satisfied:

1. The underlying support-length data are identical: \( l_1 = l_2 \). I.e., \( k_1 = k_2 \) and, up to a relabelling, \( p_i = q_i \) with \( r_{p_i} = r_{q_i} \).
2. \( \text{rank}(J_{p_i}) \leq \text{rank}(J_{q_j}) \) for all \( j \in \mathbb{N} \).
The above discussion reduces the problem of a characterization of the orbit-closure inclusion relations to the case of [Mo-T], [Ge], and [Dj]:

**Proposition 4.1.8.** [orbit-closure inclusion relation]. ([Mo-T], [Ge], and [Dj].) Let $O_1$ and $O_2$ be two $GL_r(\mathbb{C})$-orbits in $\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r)$. Then, $O_1 \subset \overline{O_2}$ if and only of $J_{O_1} \prec J_{O_2}$.

In particular, over each $p = [(p_1, \ldots, p_r)] \in C(r)$, there are a unique maximal orbit, given by the image $\overline{\sigma}^{-1}(p)$, and a unique minimal orbit, given by the orbit that contains $\overline{\sigma}^{-1}(p)$.

### 4.2 Special Lagrangian cycles with a bundle/sheaf on a Calabi-Yau torus à la Polchinski-Grothendieck Ansatz.

With the description of the stack $\mathcal{M}^\text{A_{lf}}_r(C)$, its representation-theoretical atlas $\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r)$, and the orbit-closure inclusion relations of the $GL_r(\mathbb{C})$-orbits therein in Sec. 4.1, we are now ready to see, by studying Lagrangian/special Lagrangian morphisms to $(C, \omega)$ from Aspect IV in Sec. 2.2, that, indeed, even D-branes of A-type on a flat torus can have less visible structures.

**Lagrangian/special Lagrangian morphisms to $(C, \omega)$ as $GL_r(\mathbb{C})$-equivariant maps.**

Any principal $GL_r(\mathbb{C})$-bundle $P$ over $S^1$ is trivial. Fix a trivialization $P \simeq S^1 \times GL_r(\mathbb{C})$ with the identity section $S^1 \rightarrow S^1 \times \{e\}$. Here, $e$ denotes the identity of $GL_r(\mathbb{C})$. Whenever needed, we will identify the base $S^1$ with $S^1 \times \{e\} \subset P$. It follows that a morphism $\varphi : (S^1, \mathcal{E}) \rightarrow C$ is specified by a smooth map $f : S^1 \rightarrow \text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r)$. Denote the universal quotient sheaf on $(\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r) \times C)/\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r)$ by $\mathcal{O}_C^{\oplus r}$ $\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r) \times C \rightarrow \mathcal{Q} \rightarrow 0$. Then, the $S^1$-family of $0$-dimensional $\mathcal{O}_C$-modules on $C$ defines a map $S^1 \rightarrow \mathcal{M}^\text{A_{lf}}_r(C)$, Aspect III of $\varphi$. Regarding the $S^1$-family as moving along $S^1$ in $(S^1 \times C)/S^1$ gives Aspect II of $\varphi$. The further projection/push-forward to $S^1$ and to $C$ recover $\varphi$.

To incorporate the flat geometry and the calibrations on $(C, \omega)$ into the construction, it is convenient to treat $f$ as a lifting of a $\underline{f} : S^1 \rightarrow C(r)$:

$$
\begin{array}{ccc}
S^1 & \xrightarrow{f} & \text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r) \\
\downarrow & & \downarrow \rho \\
\underline{f} & \longrightarrow & C(r)
\end{array}
$$

The following lemma is then immediate.

**Lemma 4.2.1.** [Lagrangian/special Lagrangian morphism]. Let $\pi : C^{\times r} \rightarrow C(r)$ be the quotient map. Then, $\underline{f}$ defines a Lagrangian morphism $\varphi$ if and only if $\underline{f}$ is an immersion and the projection of the $1$-complex $\pi^{-1}(\underline{f}(S^1))$ to each factor of $C^{\times r}$ is also an immersion on the edges of the $1$-complex; $\underline{f}$ defines a special Lagrangian morphism $\varphi$ if and only if $\underline{f}$ is an immersion and there exists an $\Omega_0$ such that the projection of the edges of the $1$-complex $\pi^{-1}(\underline{f}(S^1))$ to each factor of $C^{\times r}$ are calibrated with respect to $\Omega_0$.

The following lemma follows from either the canonical identification $C^{(r)} = C^{[r]}$ or the fact that $\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r)$ is smooth and all fibers of $\rho$ are path-connected:

**Lemma 4.2.2.** [lifting]. Any $\underline{f} : S^1 \rightarrow C(r)$ lifts to an $f : S^1 \rightarrow \text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r}, r)$ such that $\underline{f} = \rho \circ f$. 

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Example 4.2.3. [Lagrangian morphism]. Any generic \( f : S^1 \to Quot^{H_0}(O^\oplus_C, r) \) defines a Lagrangian morphism \( \varphi : (S^1, \mathcal{A}, \mathcal{E}) \to (C, \omega) \).

Example 4.2.4. [special Lagrangian morphism]. Denote the domain \( S^1 \) by \( X \) and equip all \( S^1 \) in the discussion with an orientation via \( S^1 \simeq \mathbb{R}^1/(2\pi \mathbb{Z}) \). The Donaldson/Polchinski-Grothendieck picture and Aspects II and IV of morphisms can be combined to construct a special Lagrangian morphism \( \varphi \), as follows.

Let \( L \) be a special Lagrangian 1-cycle on \( C \). For simplicity of presentation, we assume that \( L \) is connected. Let \( S = \Pi_i S^1 \) and \( m = (m_i)_i, d = (d_i)_i \) with \( m_i, d_i \in \mathbb{Z} \) be multiplicity vectors. Let \( g : S \to L \) (resp. \( c : S \to X \)) be a covering map with degree specified by \( m \) (resp. \( d \)). This defines a map \( (e, g) : S \to X \times C \) with degree \( r = |d| = \sum d_i \) over \( X \) and, hence, a map \( \underline{f} : X \to C^{(r)} \). Any lifting \( f \) of \( f \) defines then a special Lagrangian morphism \( \varphi : (S^1, \mathcal{A}, \mathcal{E}) \to C \), with \( \mathcal{E} \) of complex rank \( r \), whose image is supported on \( L \). For example, the trivial complex line bundle on \( S \) in the Donaldson/Polchinski-Grothendieck picture induces a lifting of \( f \) while the identification \( C^{(r)} = C^{(r)} \) induces another. They give rise to different special Lagrangian morphisms. (See next theme for further explanation.)

When \( L = \Pi_j L_j \) has more than one connected component, the above separate construction for each \( L_j \) can be combined/merged to one construction to give a morphism \( \varphi \) with image support on \( L \).

The generically filtered structure on the Chan-Paton bundle over a special Lagrangian cycle on a Calabi-Yau torus\(^{20}\)

To single out and manifest better the current theme, consider the class of morphisms \( \varphi : (S^1, \mathcal{A}, \mathcal{E}) \to C \) such that both \( \pi_\varphi : S^1_\varphi \to S^1 \) and \( S^1_\varphi \to C \) are embeddings. The corresponding \( f : S^1 \to Quot^{H_0}(O^\oplus_C, r) \) has the image of \( \underline{f} : S^1 \to C^{(r)} \) contained in the lowest subdiagonal \( C \subset C^{(r)} \), consisting of points of the form \( [p, \ldots, p], p \in C \). Recall from Sec. 4.1 that the subset \( \rho^{-1}([p, \ldots, p]) \) in \( Quot^{H_0}(O^\oplus_C, r) \) consists of a partial-ordered collection of \( GL_r(C) \)-orbits. A point in \( \rho^{-1}([p, \ldots, p]) \) represents a punctual subscheme \( Z \) of \( C \) with \( Z_{red} = p \) together with an \( O_Z \)-module \( \mathcal{F} \) with a decoration \( H^0(O^\oplus_C) = C^r \to H^0(\mathcal{F}) \to 0 \). As a 0-dimensional scheme, \( Z \simeq Spec(C[z]/(z^{r'})) \) for some \( r' \leq r \). In terms of this expression, the \( z \)-action on \( \mathcal{F} \) induces a filtration \( 0 \subset z^{r'-1}\mathcal{F} \subset \ldots \subset z^2\mathcal{F} \subset z\mathcal{F} \subset \mathcal{F} \). It follows that the collection \( \{f^{-1}(O) : O \text{ is a } GL_r(C) \text{-orbit}\} \) gives a finite decomposition of \( S^1 \) into a disjoint cyclic union \( I_1 \cup \{p_{12}\} \cup I_2 \cup \{p_{23}\} \cup \ldots \cup I_k \cup \{p_{k1}\} \) with \( I_i \) an open interval in \( S^1 \) and \( p_{i,i+1} = \overline{I_i} \cap \overline{I_{i+1}} \), \( i = 1, \ldots, k \). (By convention, \( k + 1 \equiv 1 \).) For each \( I_i = f^{-1}(O_i) \) (resp. \( p_{i,i+1} = f^{-1}(O_{i,i+1}) \)), \( (\varphi|_{I_i})*(\mathcal{E}|_{I_i}) \) (resp. \( (\varphi|_{p_{i,i+1}})*(\mathcal{E}|_{p_{i,i+1}}) \)) is thus endowed with a filtration specified by \( O_i \) (resp. \( O_{i,i+1} \)). Since \( O_{i,i+1} \subset \overline{O_i} \cap \overline{O_{i+1}} \), the filtration associated to \( O_i \) and that associated to \( O_{i+1} \) can be regarded as refinements of the filtration associated to \( O_{i,i+1} \). Since \( \rho^{-1}([p, \ldots, p]) \) contains a unique maximal orbit, associated to the case \( Z \simeq Spec(C[z]/(z^{r'})) \), and a unique minimal orbit, associated to the case \( Z \simeq Spec C \). A generic \( \varphi \) in the current class of morphisms under discussion has \( \varphi_*\mathcal{E} \) filtered by a complete flag of subbundles. In contrast, the case that \( \varphi_*\mathcal{E} \) is not filtered is a most non-generic situation.

Remark 4.2.5. [hidden scheme structure in symplectic geometry]. While the above filtration is very natural from scheme-theoretical aspect, in symplectic/calibrated geometry one does not usually think of having a scheme structure on/along the special Lagrangian cycle \( L \). Since the

\(^{20}\)A hidden mild subtitle to this theme is: Donagi-Katz-Sharpe vs. Polchinski-Grothendieck. Readers are highly recommended to read the theme alongside with the work [D-K-S] of Donagi, Katz, and Sharpe from open-string-states point of view for the nilpotent/filtered structure on the Chan-Paton sheaf addressed here.
filtration is on $\varphi_\ast \mathcal{E}$, one can still have the filtration structure without resorting to its hidden scheme-like source. Symplectic geometers may think of such a filtration on bundles/sheaves over Lagrangian/special Lagrangian cycles as a (weak) datum to encode details of how the latter merge or split under deformations.

4.3 Amalgamation/decomposition (or assembling/disassembling) of special Lagrangian cycles with a bundle/sheaf.

In this subsection\(^{21}\) we discuss the amalgamation/decomposition (or assembling/disassembling) of special Lagrangian cycles with a bundle/sheaf on a Calabi-Yau torus through deformations of morphisms from an $(S^{1,A_z}, \mathcal{E})$.

For convenience, fix a basis $H_1(C; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with intersection form $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. With respect to this basis, the algebraic intersection number $\gamma_{(p_1,q_1)} \cdot \gamma_{(p_2,q_2)}$ of a $(p_1,q_1)$-curve with a $(p_2,q_2)$-curve $\gamma_{(p_2,q_2)}$ is given by $\left\| \begin{array}{c} p_1 \\ p_2 \\ q_1 \\ q_2 \end{array} \right\| = p_1 q_2 - p_2 q_1$ and a special Lagrangian cycle, oriented as a 1-cycle, in $(p,q)$-class on $(C, \omega)$ is described by a straight/geodesic $(p,q)$-curve, which can have several connected components with multiplicities if $p$ and $q$ are not co-prime. The purpose of this subsection is to explain the following proposition:

**Proposition 4.3.1. [amalgamation/decomposition of D-branes of A-type].** Let

$$(L_i, V_i) = (\varphi_\ast [S^1], \varphi_\ast \mathcal{E}_i), \quad i = 1, \ldots, k,$$

for some special Lagrangian morphisms $\varphi_i : (S^{1,A_z}, \mathcal{E}_i) \to C$, $i = 1, \ldots, k$, respectively. Let $[(L_i, V_i)] = (p_i, q_i) \in H_1(C; \mathbb{Z})$. Then there exists a special Lagrangian morphism $\varphi : (S^{1,A_z}, \mathcal{E}) \to C$ with image class $[(\varphi_\ast [S^1], \varphi_\ast \mathcal{E})]$ in $\Sigma_{i=1}^k p_i \cdot \Sigma_{i=1}^k q_i \in H_1(C; \mathbb{Z})$ such that $\varphi$ can be deformed into a Lagrangian morphism $\varphi' : (S^{1,A_z}, \mathcal{E}) \to C$ with image cycle $\varphi'[S^1] = \sum_{i=1}^k L_i$ and push-forward $\varphi'_\ast \mathcal{E} = \oplus_{i=1}^k V_i$.

This a consequence of the following two lemmas:

**Lemma 4.3.2. [amalgamation of morphisms].** Let $\varphi_i : (S^{1,A_z}, \mathcal{E}_i) \to C$, $i = 1, 2$, be two morphisms. Then there exists a morphism $\varphi_3 : (S^{1,A_z}, \mathcal{E}_1 \oplus \mathcal{E}_2) \to C$ such that

- $\text{Im} \varphi_3 = \text{Im} \varphi_1 + \text{Im} \varphi_2$ as cycles on $C$;
- $\varphi_\ast \mathcal{E}_3 = \varphi_\ast \mathcal{E}_1 \oplus \varphi_\ast \mathcal{E}_2$ as torsion sheaves of $\mathcal{O}_C$-modules on $C$.

**Proof.** This follows from the following canonical morphism induced by taking the direct sum of two complexes:

$$\text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r_1}, r_1) \times \text{Quot}^{H_0}(\mathcal{O}_C^{\oplus r_2}, r_2) \quad \longrightarrow \quad \text{Quot}^{H_0}(\mathcal{O}_C^{\oplus (r_1+r_2)}, r_1 + r_2).$$

$$(\mathcal{O}_C^{\oplus r_1} \to \overline{E}_1 \to 0, \mathcal{O}_C^{\oplus r_2} \to \overline{E}_2 \to 0) \quad \longmapsto \quad [\mathcal{O}_C^{\oplus (r_1+r_2)} \to \overline{E}_1 \oplus \overline{E}_2 \to 0].$$



**Lemma 4.3.3. [special Lagrangian representative in deformation class].** Any morphism $\varphi : (S^{1,A_z}, \mathcal{E}) \to C$ can be deformed into a special Lagrangian morphism $\varphi_{sL} : (S^{1,A_z}, \mathcal{E}) \to C$\(^{22}\)

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\(^{21}\) Though this subsection, as it is, is not yet ready to be subtitled “Denef vs. Polchinski-Grothendieck”, it is written with the work [De] of Denef in mind. One cannot help but notice the similarity of the assembling/disassembling behavior of D-branes of A-type discussed there via split attractor flows and here via deformations of morphisms from Azumaya noncommutative spaces with a fundamental module. Readers are highly recommended to read ibidem alongside with the current subsection.

\(^{22}\) For the uniformization of the statement, here we allow $\varphi_{sL}$ to have $\text{Im} \varphi_{sL}$ supported on a (possibly empty) special Lagrangian cycle plus possibly 0-cycles on $C$. 

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Proof. Suppose that $E$ is of (complex) rank $r$. Let $f : S^1 \to Quot^{H^0}(\mathcal{O}_C^{\otimes r}, r)$ be the map associated to $\varphi$. Recall $\rho : Quot^{H^0}(\mathcal{O}_C^{\otimes r}, r) \to C^{(r)}$. For the convenience of presentation, let $X = S^1$ and endow the smooth manifold $C^{(r)}$ with the natural orbifold structure from $C^{(r)} = C^{x_{r}/\text{Sym}(r)}$, where the permutation group $\text{Sym}(r)$ acts on $C^{(r)}$ by permuting the $r$-many components in the product. The union of all subdiagonals in $C^{(r)}$ corresponds to the set of orbifold-points on $C^{(r)}$. Homotope $f$ so that the corresponding $\mathbf{f} : S^1 \to C^{(r)}$ has image containing no orbifold-points of $C^{(r)}$. Denote the new map/morphism still by $f$ and $\varphi$. Then the surrogate $X_\varphi$ of $\varphi$ is a smooth curve (with possibly several connected components) that covers $X$ with degree $r$. Let $H_1(X \times C; \mathbb{Z}) \simeq \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z})$ be induced from the product and $H_1(C; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ we fixed at the beginning. Then $X_\varphi$ is a smooth curve representing a class $(r; p, q) \in H_1(X \times C; \mathbb{Z})$.

Let $L_{(r,p,q)}$ be another smooth curve representative of $(r; p, q) \in H_1(C)$. Consider the 4-manifold $M := [0, 1] \times (X \times C)$ with boundary $\{1\} \times (X \times C) - \{0\} \times (X \times C)$ and smooth curves $X_\varphi \subset \{0\} \times (X \times C)$ and $L_{(r,p,q)} \subset \{1\} \times (X \times C)$. Then from the long-exact sequence of homologies (with $\mathbb{Z}$-coefficient)

$$
\cdots \longrightarrow H_2(\partial M) \xrightarrow{\alpha_2} H_2(M) \xrightarrow{\beta_2} H_2(M, \partial M) \xrightarrow{\delta_2} H_1(\partial M) \xrightarrow{\alpha_1} H_1(M) \longrightarrow \cdots
$$

with $\alpha_1$ and $\alpha_2$ surjective and, hence, $\beta_2$ a zero-map and $\delta_2$ injective. If follows that

$$H_2(M, \partial M) \simeq \text{Ker} \alpha_1 \simeq \{(\gamma, -\gamma) : \gamma \in H_1(X \times C)\} \subset H_1((X \times C)\Pi(X \times C)) .$$

Consequently, $(-X_\varphi, L_{(r,p,q)})$ bounds a 2-chain $\Sigma$ in $(M, \partial M)$ with $\partial \Sigma = L_{(r,p,q)} - X_\varphi$. Furthermore, one can choose $\Sigma$ to be an embedded smooth, orientable surface with boundary.\footnote{This follows from the proof of a classical theorem in smooth 4-manifold topology which, in our case, says that for $M$ a smooth orientable 4-manifold with boundary, any class in $H_2(M, \partial M; \mathbb{Z})$ can be represented by a smooth embedded orientable surface with boundary. See [Ki: II.1, Theorem 1.1 and remark] and [G-St: Chap. 1, Proposition 1.2.3 and Remark 1.2.4; Chap. 4, Exercise 4.5.12(b)]. C.-H.L. would like to thank Robert Gompf for teaching him non-gauge-theory-type 4-manifold theory around 1998.}

Since $\pi_2(X \times C) = 0$, with a further deformation of the embedding of $\Sigma$ in $[0, 1] \times (X \times C)$ if necessary, one can assume that the map: $\pi : \Sigma \to [0, 1]$ from the restriction of projection map is a Morse function on $\Sigma$ with the index of any critical point of $\pi$, if exists, equal to 1 only. It follows that $\Sigma$ defines a homotopy $F : [0, 1] \times S^1 \to Quot^{H^0}(\mathcal{O}_C^{\otimes r}, r)$ such that

1. $F|_{\{0\} \times S^1} = f$,  
2. the image $\text{Im} F$ of $F = \rho \circ F$ in $C^{(r)}$ contains no other orbifold-points except possibly finitely many orbifold-points with the structure group $\mathbb{Z}/2$,  
3. $f_1 := F|_{\{1\} \times S^1}$ defines a special Lagrangian morphism $\varphi_{sL}$.

This proves the lemma. Cf. FIGURE 4-3-1.

\[\square\]

**Example 4.3.4. [brane-anti-brane cancellation].** In particular, the situation of amalgamating special Lagrangian morphisms $\varphi_1$ and $\varphi_2$ with image class $(p, q), (-p, -q) \in H_1(C)$ corresponds to a brane-anti-brane cancellation. Cf. FIGURE 4-3-2.

**Remark 4.3.5. [varying the moduli of the Calabi-Yau torus].** It should be noted that the proof of Lemma 4.3.3 is essentially topological, depending only on the homology class of the special Lagrangian cycles. Consequently, the mechanism of amalgamation/decomposition of special Lagrangian cycles with a bundle/sheaf in a Calabi-Yau torus through deformations of morphisms from an $(S^1, \mathbb{A}, E)$ as discussed works completely the same way even if the modulus of the Calabi-Yau torus varies in this process of assembling/disassembling of branes thereupon.
Figure 4-3-1. The basic local move/deformation-of-morphism that corresponds to crossing an index-1 critical point of $\pi : \Sigma \rightarrow [0,1]$. This has an effect of turning a short-string wrapping to a longer-string wrapping or a long string wrapping to a shorter string wrapping. Here, Aspect II of a morphism is used.
Figure 4.3.2. The brane-anti-brane cancellation procedure corresponds to a deformation of a morphism to one with 0-dimensional image. In this figure, each opposite pair of faces of a parallelepiped are identified. Here, Aspect II of a morphism is used.
String-theoretical remarks for Sec. 4.

(1) [The leftover/residual of brane-anti-brane cancellation]

In morphism-from-Azumaya-space picture of the brane-anti-brane cancellation, all the process is simply a deformation of a nonconstant-type morphism to a constant-type morphism. As such, after the brane cancellation process, there is no more local Ramond-Ramond charge of the original branes and yet there is still something left over, namely the 0-dimensional image-cycle/push-forward of the Azumaya space with a fundamental module under the final constant-type morphism. This could represent an energy-lump that remains from the brane-anti-brane cancellation.

(2) [Short vs. long string wrapping]

The short vs. long string wrapping behavior of matrix-strings in the string-theory literature (e.g., [D-V-V], [Joh: Sec. 16.3.3]; also [Ma-S]) can be produced in this context by the same manner via morphisms from $S^1, Az$ and their deformations as well. Cf. Figure 4-3-3.

(3) [Azumaya geometry and tensionless string]

In type IIA superstring model of the superstring theory, an open D2-brane can have its boundary attached to NS5-branes. When a pair of parallel NS5-branes become coincident, the open D2-brane sandwiched between them becomes degenerate and 1-dimensional: tensionless string\(^{24}\). As it comes from the boundary of D2-branes, one anticipates that the Azumaya noncommutative structure on the original D2-brane passes to the tensionless string in the NS5-brane. Morphisms from an Azumaya $S^1$ to the target NS5-brane become the most basic fields on the tensionless string. In this way, a tensionless string is linked with a matrix/Azumaya string in a natural way. We thus leave this section and, hence, this review/work with the following guiding question:

- Q. [Azumaya geometry and tensionless string] How do Azumaya geometry and tensionless string theory relate? Will it shed some light to or provide a mathematical language/foundation for the theory of tensionless strings?

Behind the writing of this unexpected review/work, it is a wish to transfer the following sense to readers:

- [unity in geometry vs. unity in string theory]

This would be highly surprising/un-anticipated/unthinkable on the mathematics side if not because of the Polchinski-Grothendieck Ansatz, which realizes morphisms from Azumaya-type manifolds/schemes/stacks with a fundamental module as the lowest level presentation of D-branes, and superstring theory dictates the master nature of such an object. We hope this brief review helps give both mathematicians and string-theorists working on D-branes a sense of richness hidden in the Azumaya-type noncommutative geometry or, even better, a new perspective of what they have been or are doing. There are many themes along the line of the Polchinski-Grothendieck Ansatz yet to be studied/generalized.

\(^{24}\)We thank Frederik Denef for a discussion on smeared D-branes and degenerate D-branes.
Figure 4-3-3. Under deformations of morphisms from an Azumaya string, the surrogate associated to the morphisms can change from a collection of short strings to a single long string. Here, Aspect II of a morphism is used and a short-to-long string-wrapping transition corresponding to a merging \((1; 1, 0) + (1; 1, 0) + (2; 1, 0) \rightarrow (4; 3, 0)\) in \(H_1(S^1 \times C; \mathbb{C})\) of the surrogates associated to morphisms involved is illustrated. The multiplicity of an image cycle in \(C\) in terms of the associated primitive one is indicated by the number of arrowhead.
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