NEWTON-OKOUNKOV BODIES OF BOTT-SAMELSON VARIETIES AND GROSSBERG-KARSHON TWISTED CUBES

MEGUMI HARADA AND JIHYEON JESSIE YANG

ABSTRACT. We describe, under certain conditions, the Newton-Okounkov body of a Bott-Samelson variety as a lattice polytope defined by an explicit list of inequalities. The valuation that we use to define the Newton-Okounkov body is different from that used previously in the literature. The polytope that arises is a special case of the Grossberg-Karshon twisted cubes studied by Grossberg and Karshon in connection to character formulae for irreducible $G$-representations and also studied previously by the authors in relation to certain toric varieties associated to Bott-Samelson varieties.

CONTENTS

Introduction 1
1. Preliminaries 3
2. A bijection between standard tableaux and lattice points in a polytope 6
3. Newton-Okounkov bodies of Bott-Samelson varieties 12
4. Examples 15
References 16

INTRODUCTION

The main result of this paper is an explicit computation of a Newton-Okounkov body associated to a Bott-Samelson variety, under certain hypotheses. To place our result in context, recall that the recent theory of Newton-Okounkov bodies, introduced independently by Kaveh and Khovanskii [9] and Lazarsfeld and Mustata [14], associates to a complex algebraic variety $X$ (equipped with some auxiliary data) a convex body of dimension $n = \dim \mathbb{C}(X)$. In some cases, this convex body (the Newton-Okounkov body, also called Okounkov body) is a rational polytope; indeed, if $X$ is a projective toric variety, then one can recover the usual moment polytope of $X$ as a Newton-Okounkov body. These Newton-Okounkov bodies have been shown to be related to many other research areas, including (but certainly not limited to) toric degenerations [1], representation theory [7], symplectic geometry [5], and Schubert calculus [10, 11]. However, relatively few explicit examples of Newton-Okounkov bodies have been computed so far, and thus it is an interesting problem to give new and concrete examples.

Motivated by the above, in this paper we study the Newton-Okounkov bodies of Bott-Samelson varieties; these varieties are well-known and studied in representation theory due to their relation to Schubert varieties and flag varieties (see e.g. [2]) and have been studied in the context of Newton-Okounkov bodies. For instance, Anderson computed a Newton-Okounkov body for an $SL(3, \mathbb{C})$ example in [1], they appear in the proof of Kaveh’s identification of Newton-Okounkov bodies as string polytopes in [7], and Kiritchenko conjectures a description of some Newton-Okounkov bodies of Bott-Samelson varieties using her divided-difference operators in [10]. Moreover, the global Newton-Okounkov body of Bott-Samelson varieties is studied by Seppänen and Schmitz in [19], where they show that it is rational polyhedral and also give an inductive description of it. Additionally, during the preparation of this manuscript we learned that Fujita has also (independently) computed the Newton-Okounkov bodies of Bott-Samelson varieties [3]. However, the valuation which we use in this paper (part of the auxiliary data necessary for the definition of a Newton-Okounkov body) is different from that associated to the “vertical flag” considered by Seppänen and Schmitz [19], the highest-term valuation used by Fujita and Kaveh [3, 7] and the geometric valuation used by Anderson and Kiritchenko in [1, 10] (cf. also Remark 3.3).

Date: April 21, 2015.

2000 Mathematics Subject Classification. Primary: 14M15; Secondary: 20G05.

Key words and phrases. Bott-Samelson variety, Newton-Okounkov bodies, path operators, generalized Demazure modules.
We now briefly recall the geometric objects of interest; for details see Section [1] Let $G$ be a complex semisimple connected and simply connected linear algebraic group and let $\{ \alpha_1, \ldots, \alpha_re \}$ denote the set of simple roots of $G$. Let $i = (i_1, \ldots, i_n) \in \{1, 2, \ldots, r\}^n$ be a word which specifies a sequence of simple roots $\{ \alpha_{i_1}, \ldots, \alpha_{i_n} \}$. We say that a word is reduced if the corresponding sequence of simple roots gives a reduced word decomposition $s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}$ of an element in the Weyl group. Also let $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ be a multiplicity list; this specifies a sequence of weights $\{ \lambda_1 := m_1 \omega_{\alpha_{i_1}}, \ldots, \lambda_n := m_n \omega_{\alpha_{i_n}} \}$ in the weight lattice of $G$. Associated to $i$ and $m$ one can define a Bott-Samelson variety $Z_i$ (cf. Definition [1]). We now sketch the main ideas in the proof of our main result (Theorem 3.4). To place the discussion in context, we show in Proposition 2.5 that, assuming $i$ is reduced, our geometric valuation $\nu^s_*$ defined on $H^0(Z_i, L_{i,m}) \setminus \{ 0 \}$ takes values in the polytope $P(i, m)$ (up to reordering coordinates). On the other hand, we show in Proposition 2.5 that, assuming that $(i, m)$ satisfies condition $(P)$, there is a bijection between the lattice points in $P(i, m)$ and the set of standard tableaux $T(i, m)$, so in particular $|P(i, m) \cap \mathbb{Z}^n| = |T(i, m)|$. Now a simple counting argument and the fact that $P(i, m)$ is a lattice polytope finishes the proof of the main theorem.

Both the polytope $P(i, m)$ and the “condition $(P')$” (defined precisely in Section [2]) mentioned in the theorem have appeared previously in the literature. Indeed, the polytope $P(i, m)$ is a special case of the Grossberg-Karshon twisted cubes which yield character formulae (possibly with sign) for irreducible $G$-representations [4]. Specifically, we showed in [6] Proposition 2.1 that if the pair $(i, m)$ satisfies condition $(P)$, then the Grossberg-Karshon twisted cube is equal to the polytope $P(i, m)$ and the Grossberg-Karshon character formula from [4] corresponding to $i$ and $m$ is a positive formula (i.e. with no negative signs). We also related the condition $(P)$ to the geometric condition that a certain torus-invariant divisor $D$ in a toric variety related to $Z_i$ is basepoint-free [6] Theorem 2.4]. For the purposes of the present manuscript, it is also significant that the polytope $P(i, m)$ is a lattice polytope (not just a rational polytope) whose vertices can be easily described as the Cartier data of the torus-invariant divisor $D$ mentioned above [6] Theorem 2.4]. Thus our theorem gives a computationally efficient description of the Newton-Okounkov body $\Delta(Z_i, L_{i,m}, \nu^s_*)$.

We now sketch the main ideas in the proof of our main result (Theorem 3.4). To place the discussion in context, it may be useful to recall that an essential step in the computation of a Newton-Okounkov body of a variety $X$ is to compute a certain semigroup $S = S(R, \nu)$ associated to the (graded) ring of sections $R = \oplus_k H^0(X, L^{\otimes k})$ for $L$ a line bundle over $X$ and a choice of valuation $\nu$. In general, this computation can be quite subtle; one of the main difficulties is that the semigroup may not even be finitely generated. (The issue of finite generation, in the context of Newton-Okounkov bodies, is studied in [14].) Even when $S$ is finitely generated, finding explicit generators is related to the problem of finding a “SAGBI basis” for $R$ with respect to the valuation $\nu$ which appears to be non-trivial in practice. In this manuscript, we are able to sidestep this subtle issue and compute $S$ directly by a simple observation which we now explain. It is a general fact that the valuations arising from flags of subvarieties $Y_n$ such as those above have one-dimensional leaves (cf. Definition [3]). It is also an elementary fact that a valuation $\nu$ with one-dimensional leaves, defined on a finite-dimensional vector space $V$, satisfies $|\nu(V \setminus \{ 0 \})| = \dim_{\mathbb{C}}(V)$ [2] Proposition 2.6]. As it happens, in our setting the vector spaces in question are precisely the generalized Demazure modules $H^0(Z_i, L_{i,m})$ mentioned above, and Lakshmibai-Littelmann-Magyar prove in [12] that $\dim_{\mathbb{C}}(H^0(Z_i, L_{i,m})) = |T(i, m)|$ where $T(i, m)$ is the set of standard tableaux associated to $i$ and $m$. Armed with this key theorem of Lakshmibai-Littelmann-Magyar, we are able to compute our semigroup $S$ and hence the Newton-Okounkov body explicitly in two steps. On the one hand, we show in Proposition 3.7 that, assuming $i$ is reduced, our geometric valuation $\nu^s_*$ defined on $H^0(Z_i, L_{i,m}) \setminus \{ 0 \}$ takes values in the polytope $P(i, m)$ (up to reordering coordinates). On the other hand, we show in Proposition 2.5 that, assuming that $(i, m)$ satisfies condition $(P)$, there is a bijection between the lattice points in $P(i, m)$ and the set of standard tableaux $T(i, m)$, so in particular $|P(i, m) \cap \mathbb{Z}^n| = |T(i, m)|$. Now a simple counting argument and the fact that $P(i, m)$ is a lattice polytope finishes the proof of the main theorem.

We now outline the contents of the manuscript. In Section [4] we establish basic terminology and notation, and also state the key result of Lakshmibai-Littelmann-Magyar (Theorem 1.8). The statement and proof of the bijection between $T(i, m)$ and the lattice points in $P(i, m)$ occupies Section [2]. In the process we introduce a separate “condition $(P')$”, stated directly in the language of paths and root operators as in [12][15][16], and prove in Proposition 2.12 that our polytope-theoretic condition $(P)$ implies condition $(P')$. It is then straightforward to see that condition $(P')$ implies

---

1 Such a basis is also called a Khovanskii basis in [5] Section 8], cf. also [8] Section 5.6].
that \(|P(i, m) \cap \mathbb{Z}^n| = |\mathcal{T}(i, m)|\). In Section 3 we recall in some detail the definition of a Newton-Okounkov body and define our geometric valuation \(\nu_G\) with respect to a certain flag of subvarieties. We then prove in Proposition 3.7 that \(\nu_G\) takes values in our polytope; as already explained, by using the bijection from Section 2 our main theorem then readily follows. Concrete examples and pictures for \(G = SL(3, \mathbb{C})\) are contained in Section 4.

We take a moment to comment on the combinatorics in Section 2. It may well be that our polytope \(P(i, m)\), our conditions \((P)\) and \((P')\), and our Proposition 2.5 are well-known or are minor variations on standard arguments in combinatorial representation theory. However, we were unable to locate exact references. We welcome comments from the experts. At any rate, as the discussion above indicates, Proposition 2.5 is only a stepping stone to our main result (Theorem 3.4). One final comment: in Section 2 we chose to explain conditions from the experts. At any rate, as the discussion above indicates, Proposition 2.5 is only a stepping stone to our main combinatorial representation theory. However, we were unable to locate exact references. We welcome comments conditions that arise from the toric-geometric considerations in [6]. Put another way, our condition \((P)\) is a geometrically motivated condition on \(i\) and \(m\) which suffices to guarantee the condition \((P')\).

Finally, we mention some directions for future work. Firstly, we hope to better understand the relation between our computations and those in [3]. Secondly, our condition \((P)\) on the pairs \((i, m)\) is rather restrictive and the corresponding Newton-Okounkov bodies are combinatorially extremely simple (they are essentially cubes, though they can sometimes degenerate). Hence it is a natural problem to ask for the relation, if any, between the Newton-Okounkov bodies computed in this paper and those for the line bundles which do not satisfy the condition \((P)\). It may be possible to analyze such a relationship using some results of Anderson [1], and we hope to take this up in a future paper. Thirdly, it would be of interest to examine the relation between our polytopes \(P(i, m)\) and the polytopes arising from Kiritchenko’s divided-difference operators, particularly in relation to her “degeneration of string spaces” in [10, Section 4].

Acknowledgements. We are grateful to Lauren DeDieu, Naoki Fujita, Dmitry Kerner, Eunjeong Lee, and Satoshi Naito for useful conversations, and to Henrik Seppänen for explaining his work to us. We especially thank Dave Anderson for help with our arguments in Section 3 and to Kiumars Kaveh for pointing out a critical error in a previous version of this manuscript. The first author was partially supported by an NSERC Discovery Grant, a Canada Research Chair (Tier 2) Award, an Ontario Ministry of Research and Innovation Early Researcher Award, an Association for Women in Mathematics Ruth Michler Award, and a Japan Society for the Promotion of Science Invitation Fellowship for Research in Japan (Fellowship ID L-13517). Both authors thank the Osaka City University Advanced Mathematics Institute for its hospitality while part of this research was conducted. The first author also thanks the Department of Mathematics at Cornell University for its hospitality during her tenure as the Ruth Michler Visiting Fellow, when portions of this manuscript were written. The second author also thanks Dong Youp Suh for his hospitality during her visit to the Department of Mathematical Sciences, KAIST.

1. Preliminaries

In this section we record basic notation in Section 1.1 recall the definitions of the central geometric objects in Section 1.2, and state a key result (Theorem 1.8) of Lakshmibai, Littelmann, and Magyar in Section 1.3.

1.1. Notation. We list here some notation and conventions to be used in the manuscript.

- \(G\) is a complex semisimple connected and simply connected algebraic group over \(\mathbb{C}\) and \(\mathfrak{g}\) denotes its Lie algebra.
- \(H\) is a Cartan subgroup of \(G\).
- \(B\) is a Borel subgroup of \(G\) with \(H \subset B \subset G\).
- \(r\) is the rank of \(G\).
- \(X\) denotes the weight lattice of \(G\) and \(X_\mathbb{R} = X \otimes \mathbb{Z} \otimes \mathbb{R}\) is its real form. The Killing form \(\mathcal{K}\) on \(X_\mathbb{R}\) is denoted by \(\langle \alpha, \beta \rangle\).
- For a weight \(\alpha \in X\), we let \(e^{\alpha}\) denote the corresponding multiplicative character \(e^{\alpha} : H \to \mathbb{C}^\times\).
- \(\{\alpha_1, \ldots, \alpha_r\}\) is the set of positive simple roots (with an ordering) with respect to the choices \(H \subset B \subset G\) and \(\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}\) are the corresponding coroots. Recall that the coroots satisfy

\[
\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}.
\]

The Killing form is naturally defined on the Lie algebra of \(G\) but its restriction to the Lie algebra \(\mathfrak{h}\) of \(H\) is positive-definition, so we may identify \(\mathfrak{h} \cong \mathfrak{h}^\vee\).
In particular, \( (\alpha, \alpha^\vee) = 2 \) for any simple root \( \alpha \).

- For a simple root \( \alpha \) let \( s_\alpha : X \to X, \lambda \mapsto \lambda - (\lambda, \alpha^\vee)\alpha \), be the associated simple reflection; these generate the Weyl group \( W \).
- \( \{\varpi_1, \ldots, \varpi_r\} \) is the set of fundamental weights satisfying \( \langle \varpi_i, \alpha^\vee_j \rangle = \delta_{i,j} \).
- For a simple root \( \alpha \), \( P_\alpha := B \cup B s_\alpha B \) is the minimal parabolic subgroup containing \( B \) associated to \( \alpha \).

1.2. Bott-Samelson varieties. In this section, we briefly recall the definition of Bott-Samelson varieties and some facts about line bundles on Bott-Samelson varieties. Further details may be found, for instance, in [4]. Note that the literature uses many different notational conventions.

With the notation in Section 1.1 in place, suppose given an arbitrary word \( i = (i_1, \ldots, i_n) \) with \( 1 \leq i_j \leq r \). This specifies an associated sequence of simple roots \( \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\} \). To simplify notation we define \( \beta_j := \alpha_{i_j} \), so the sequence above can be denoted \( \{\beta_1, \ldots, \beta_n\} \). Note that we do not assume here that the corresponding expression \( s_{\beta_1} s_{\beta_2} \cdots s_{\beta_n} \) is reduced; in particular, there may be repetitions. (However, we will add the reducedness as a hypothesis in Section 3.)

**Definition 1.1.** The Bott-Samelson variety corresponding to a word \( i = (i_1, \ldots, i_n) \in \{1, 2, \ldots, r\}^n \) is the quotient

\[
Z_i := (P_{\beta_1} \times \cdots \times P_{\beta_n})/B^n
\]

where \( \beta_j = \alpha_{i_j} \) and \( B^n \) acts on the right on \( P_{\beta_1} \times \cdots \times P_{\beta_n} \) by:

\[
(p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n) := (p_1 b_1 b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n).
\]

It is known that \( Z_i \) is a smooth projective algebraic variety of dimension \( n \). By convention, if \( n = 0 \) and \( i \) is the empty word, we set \( Z_i \) equal to a point.

We next describe certain line bundles over a Bott-Samelson variety. Suppose given a sequence \( \{\lambda_1, \ldots, \lambda_n\} \) of weights \( \lambda_j \in X \). We let \( \mathbb{C}^*_{(\lambda_1, \ldots, \lambda_n)} \) denote the one-dimensional representation of \( B^n \) defined by

\[
(b_1, \ldots, b_n)^{-1} \cdot k := e^{\lambda_1}(b_1) \cdots e^{\lambda_n}(b_n)k.
\]

(Notice that this is isomorphic to the representation \( \mathbb{C}(-\lambda_1, \ldots, -\lambda_n)\).)

**Definition 1.2.** Let \( \lambda_1, \ldots, \lambda_n \) be a sequence of weights. We define the line bundle \( L_i(\lambda_1, \ldots, \lambda_n) \) over \( Z_i \) to be

\[
L_i(\lambda_1, \ldots, \lambda_n) := (P_{\beta_1} \times \cdots \times P_{\beta_n}) \times_{B^n} \mathbb{C}^*_{(\lambda_1, \ldots, \lambda_n)}
\]

where the equivalence relation is given by

\[
((p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n), k) \sim ((p_1, \ldots, p_n), (b_1, \ldots, b_n) \cdot k)
\]

for \( (p_1, \ldots, p_n) \in P_{\beta_1} \times \cdots \times P_{\beta_n}, (b_1, \ldots, b_n) \in B^n \), and \( k \in \mathbb{C} \). The projection \( L_i(\lambda_1, \ldots, \lambda_n) \to Z_i \) to the base space is given by taking the first factor \( [(p_1, \ldots, p_n, k)] \mapsto [(p_1, \ldots, p_n)] \in Z_i \).

In what follows, we will frequently choose the weights \( \lambda_j \) to be of a special form. Specifically, suppose given a multiplicity list \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0} \). Then we may define a sequence of weights \( \{\lambda_1, \ldots, \lambda_n\} \) associated to the word \( i \) and the multiplicity list \( m \) by setting

\[
\lambda_1 := m_1 \varpi_{i_1}, \ldots, \lambda_n := m_n \varpi_{i_n}.
\]

In this special case we will use the notation

\[
L_{i,m} := L_i(m_1 \varpi_{i_1}, \ldots, m_n \varpi_{i_n}).
\]

In this manuscript we will study the space of global sections of these line bundles. Note that the Borel subgroup acts on both \( Z_i \) and \( L_{i,m} \) by left multiplication on the first coordinate; indeed, for \( b \in B \), the equation \( b \cdot [(p_1, \ldots, p_n)] := [(bp_1, p_2, \ldots, p_n)] \) defines the action on \( Z_i \) and \( b \cdot [(p_1, \ldots, p_n, k)] := [(bp_1, p_2, \ldots, p_n, k)] \) defines the action on \( L_{i,m} \). It is straightforward to check that both are well-defined. The space of global sections \( H^0(Z_i, L_{i,m}) \) is then naturally a \( B \)-module; these are called generalized Demazure modules (cf. for instance [12]).
1.3. Paths and root operators. We use the machinery of paths and root operators as in [12] (cf. also [15,16]) so in this section we briefly recall some necessary definitions and basic properties.

Let $X_R := X \otimes \mathbb{R}$ denote the real form of the weight lattice. By a path we will mean a piecewise-linear map $\pi : [0,1] \to X_R$ (up to reparametrization) with $\pi(0) = 0$. We consider the set $\Pi \cup \{O\}$ where $\Pi$ denotes the set of all paths and $O$ is a formal symbol. For a weight $\lambda \in X$, we let $\pi^\lambda$ denote the straight-line path: $\pi^\lambda(t) := t\lambda$. By the symbol $\pi_1 * \pi_2$ we mean the concatenation of two paths; more precisely, $\pi(t) = (\pi_1 * \pi_2)(t)$ is defined by

\[
\pi(t) := \begin{cases} 
\pi_1(2t) & \text{if } 0 \leq t \leq 1/2 \\
\pi_1(1) + \pi_2(2t - 1) & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]

By convention we take $\pi * O := \pi$ for any element $\pi \in \Pi \cup \{O\}$. For a simple root $\alpha$ and a path $\pi$, we define $s_\alpha(\pi)$ to be the path given by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$, i.e., the path $\pi$ is reflected by $s_\alpha$. We pay particular attention to endpoints so we give it a name: given $\pi$ we say the weight of $\pi$ is its endpoint, $\text{wt}(\pi) := \pi(1)$ (also denoted $v(\pi)$ in the literature, see [15]). The following is immediate from the definitions.

**Lemma 1.3.** Let $\pi, \pi_1, \pi_2$ be paths in $\Pi$ and $\alpha$ a simple root. Then $\text{wt}(\pi_1 * \pi_2) = \text{wt}(\pi_1) + \text{wt}(\pi_2)$ and $\text{wt}(s_\alpha(\pi) = s_\alpha(\text{wt}(\pi))$.

Fix a simple root $\alpha$. We now briefly recall the definitions of the raising operator $e_\alpha$ and lowering operator $f_\alpha$ on the set $\Pi \cup \{O\}$, for which we need some preparation of notation. Fix a path $\pi \in \Pi$. We cut $\pi$ into 3 pieces according to the behavior of the path $\pi$ under the projection with respect to $\alpha$. More precisely, define the function

$$h_\alpha : [0,1] \to \mathbb{R}, t \mapsto \langle \pi(t), \alpha^\vee \rangle$$

and let $Q$ denote the smallest integer attained by $h_\alpha$, i.e.,

$$Q := \min\{\text{image}(h_\alpha) \cap \mathbb{Z}\}.$$ 

Note that since $\pi(0) = 0$ by definition we always have $Q \leq 0$. Now let $q := \min\{t \in [0,1] : h_\alpha(t) = \langle \pi(t), \alpha^\vee \rangle = Q\}$ be the “first” time $t$ at which the minimum integer value of $h_\alpha$ is attained. Next, in the case that $Q \leq -1$ (note that if $Q = 0$ then, since $\pi(0) = 0$, the value $q$ must be 0 and the following discussion is not applicable) then we define $y$ to be the “last time before $q$” when the value $Q + 1$ is attained. More precisely, $y$ is defined by the conditions

$$h_\alpha(y) = Q + 1, \text{ and } Q < h_\alpha(t) < Q + 1 \text{ for } y < t < q.$$ 

We now define three paths $\pi_1, \pi_2, \pi_3$ in such a way that $\pi$ is by definition the concatenation $\pi = \pi_1 * \pi_2 * \pi_3$, where $\pi_1$ is the path $\pi$ “up to time $y$”, $\pi_2$ is the path $\pi$ “between $y$ and $q$”, and $\pi_3$ is the path $\pi$ “after time $q$”. More precisely, we define

$$\pi_1(t) := \pi(ty), \text{ and } \pi_2(t) := \pi(y + t(q - y)) - \pi(y), \text{ and } \pi_3(t) := \pi(q + t(1 - q)) - \pi(q).$$ 

See [15, Example, Section 1.2] for a figure illustrating an example in rank 2. Given this decomposition of $\pi$ into “pieces”, we may now define the raising (root) operator $e_\alpha$ as follows.

**Definition 1.4.** Fix a path $\pi$. If $Q = 0$, i.e. if the path $\pi$ lies entirely in the closed half-space defined by $\{h_\alpha > -1\}$, then $e_\alpha(\pi) = O$, where here $O$ is the formal symbol in $\Pi \cup \{O\}$. If $Q < 0$, then we define $e_\alpha(\pi) := \pi_1 * s_\alpha(\pi_2) * \pi_3$, i.e. we “reflect across $\alpha$” the portion of the path $\pi$ between time $y$ and time $q$. We also define $e_\alpha(O) = O$.

The lowering (root) operator $f_\alpha$ may be defined similarly. This time, let $p$ denote the maximal real number in $[0,1]$ such that $h_\alpha(p) = Q$, i.e., it is the “last” time $t$ at which the minimal value $Q$ is attained. Then let $P$ denote the integral part of $h_\alpha(1) - Q$. If $P \geq 1$, then let $x$ denote the first time after $p$ that $h_\alpha$ achieves the value $Q + 1$; more precisely, let $x$ be the unique element in $(p,1]$ satisfying

$$h_\alpha(x) = Q + 1 \text{ and } Q < h_\alpha(t) < Q + 1 \text{ for } p < t < x.$$ 

Once again we may decompose the path $\pi$ into 3 components, $\pi = \pi_1 * \pi_2 * \pi_3$ by the equations

\[
\pi_1(t) := \pi(tp) \text{ and } \pi_2(t) := \pi(p + t(x - p)) - \pi(p) \text{ and } \pi_3(t) := \pi(x + t(1 - x)) - \pi(x).
\]

Given this decomposition, we define the lowering (root) operator $f_\alpha$ as follows.

**Definition 1.5.** Fix a path $\pi$ as above. If $P \geq 1$, then we define $f_\alpha(\pi) := \pi_1 * s_\alpha(\pi_2) * \pi_3$, so we “reflect across $\alpha$” the portion of the path $\pi$ between time $p$ and $x$. If $P = 0$, then $f_\alpha(\pi) = O$. Finally, we define $f_\alpha(O) = O$.

The following basic properties of the root operators are recorded in [15, Section 1.4].
Lemma 1.6. Let \( \pi \in \Pi \) be a path.

1. If \( e_\alpha(\pi) \neq O \), then \( wt(e_\alpha(\pi)) = wt(\pi) + \alpha \), and if \( f_\alpha(\pi) \neq O \), then \( wt(f_\alpha(\pi)) = wt(\pi) - \alpha \).
2. If \( e_\alpha(\pi) \neq O \), then \( f_\alpha(e_\alpha(\pi)) = \pi \). If \( f_\alpha(\pi) \neq O \), then \( e_\alpha(f_\alpha(\pi)) = \pi \).
3. \( e_\alpha^n(\pi) = O \) if and only if \( n > -Q \), and \( f_\alpha^n(\pi) = O \) if and only if \( n > P \).

We now recall a result (Theorem 1.8 below) of Lakshmibai, Littelmann, and Magyar which is crucial to our arguments in the remainder of this paper. Specifically, Theorem 1.8 gives a bijective correspondence between a certain set \( \mathcal{J}(i, m) \) of standard tableaux, defined below using paths and the root operators, and a basis of the vector space \( H^0(Z_i, L_{i,m}) \) of global sections of \( L_{i,m} \) over \( Z_i \). Our main result in Section 2 is that - under certain conditions on the word \( i \) and the multiplicity list \( m \) - there exists, in turn, a bijection between \( \mathcal{J}(i, m) \) and the set of integer lattice points in a certain polytope. This then allows us to compute Newton-Okounkov bodies associated to \( Z_i \) and \( L_{i,m} \) in Section 3.

We now recall the definition of standard tableaux. Suppose given a word \( i \) and multiplicity list \( m \) as above. Let \( \{ \beta_1 = \alpha_{i_1}, \ldots, \beta_n = \alpha_{i_n} \} \) be the sequence of simple roots associated to \( i \) and set \( \lambda_j := m_j \beta_j \) for \( 1 \leq j \leq n \). The following is from [12] Section 1.2.

Definition 1.7. A path \( \pi \in \Pi \) is called a (constructable) standard tableau of shape \( \lambda = (\lambda_1, \ldots, \lambda_n) \) if there exist integers \( \ell_1, \ldots, \ell_n \in \mathbb{Z}_{\geq 0} \) such that

\[
\pi = e_{\beta_1}^{\ell_1} f_{\beta_2}^{\ell_2} e_{\beta_3}^{\ell_3} f_{\beta_4}^{\ell_4} \cdots
\]

where the \( e_{\beta_j} \) are the lowering operators defined above. Given a word \( i = (i_1, \ldots, i_n) \) and multiplicity list \( m = (m_1, \ldots, m_n) \), we denote by \( \mathcal{J}(i, m) \) the set of standard tableau of shape \( (\lambda_1 = m_1 \beta_1, \ldots, \lambda_n = m_n \beta_n) \).

It turns out that there are only finitely many standard tableaux of a given shape \( \lambda \) associated to a given pair \((i, m)\). In fact, Lakshmibai, Littelmann, and Magyar prove [12, Theorems 4 and 6] the following.

Theorem 1.8. Let \( i = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n \) be a word and \( m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \) be a multiplicity list. Let \( \{\beta_1 = \alpha_{i_1}, \ldots, \beta_n = \alpha_{i_n}\} \) be the sequence of simple roots associated to \( i \) and set \( \lambda_j := m_j \beta_j \) for \( 1 \leq j \leq n \). Then

\[
|\mathcal{J}(i, m)| = \dim \mathbb{C} H^0(Z_i, L_{i,m}).
\]

2. A bijection between standard tableaux and lattice points in a polytope

The main result of this section (Proposition 2.3) is that, under a certain assumption on the word \( i \) and the multiplicity list \( m \), there is a bijection between the set of integer lattice points within a certain lattice polytope \( P(i, m) \) and the set of standard tableaux \( \mathcal{J}(i, m) \). Together with Theorem 1.8 this then implies that the cardinality of \( P(i, m) \cap \mathbb{Z}^n \) is equal to the dimension of the space \( H^0(Z_i, L_{i,m}) \) of sections of the line bundle \( L_{i,m} \) over the Bott-Samelson variety \( Z_i \). This then allows us to compute Newton-Okounkov bodies in the next section. The necessary hypothesis on \( i \) and \( m \), which we call “condition (P)”, also appeared in our previous work [6] connecting the polytopes \( P(i, m) \) with representation theory and toric geometry (cf. Remark 2.2 below).

We begin with the definition of the polytope \( P(i, m) \) by an explicit set of inequalities.

Definition 2.1. Let \( i = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n \) be a word and \( m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \) be a multiplicity list. Then the polytope \( P(i, m) \) is defined to be the set of all real points \((x_1, \ldots, x_n) \in \mathbb{R}^n\) satisfying the following inequalities:

\[
\begin{align*}
0 & \leq x_n \leq A_n := m_n, \\
0 & \leq x_{n-1} \leq A_{n-1}(x_n) := m_{n-1} \omega_{\beta_{n-1}} + m_n \omega_{\beta_n} - x_n \beta_n, \\
0 & \leq x_{n-2} \leq A_{n-2}(x_{n-1}, x_n) := m_{n-2} \omega_{\beta_{n-2}} + m_{n-1} \omega_{\beta_{n-1}} + m_n \omega_{\beta_n} - x_{n-1} \beta_{n-1} - x_n \beta_n, \\
& \quad \vdots \\
0 & \leq x_1 \leq A_1(x_2, \ldots, x_n) := m_1 \omega_{\beta_1} + m_2 \omega_{\beta_2} + \cdots + m_n \omega_{\beta_n} - x_2 \beta_2 - \cdots - x_n \beta_n.
\end{align*}
\]

Remark 2.2. • The polytopes \( P(i, m) \) have appeared previously in the literature and has connections to toric geometry and representation theory. Specifically, under a hypothesis on \( i \) and \( m \) which we call “condition (P)” (see Definition 2.3), we show in [6] that \( P(i, m) \) is exactly a so-called Grossberg-Karshon twisted cube. These twisted cubes were introduced in [2] in connection with Bott towers and character formulae for irreducible \( G \)-representations. Our proof of this fact in [6] used a certain torus-invariant divisor in a toric variety associated to Bott-Samelson varieties studied by Pasquier [17].
The functions $A_k(x_{k+1}, \ldots, x_n)$ appearing in Definition 2.1 also have a natural interpretation in terms of paths, as we shall see in Lemma 2.8 below; this is useful in our proof of Proposition 2.5.

In the statement of our main proposition of this section, we need the following technical hypothesis on the word and the multiplicity list. As noted above, the same condition appeared in our previous work [6] which related the polytope $P(i, m)$ to toric geometry and representation theory.

Definition 2.3. Let $i = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ be a word and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ be a multiplicity list. We say that the pair $(i, m)$ satisfies condition (P) if

(P-n) $m_n \geq 0$

and for every integer $k$ with $1 \leq k \leq n-1$, the following statement, which we refer to as condition (P-k), holds:

(P-k) if $(x_{k+1}, \ldots, x_n)$ satisfies

\[
0 \leq x_n \leq A_n, \\
0 \leq x_{n-1} \leq A_{n-1}(x_n) \\
\quad \vdots \\
0 \leq x_{k+1} \leq A_{k+1}(x_{k+2}, \ldots, x_n),
\]

then

\[A_k(x_{k+1}, \ldots, x_n) \geq 0.\]

In particular, condition (P) holds if and only if the conditions (P-1) through (P-n) all hold.

Remark 2.4. The condition (P) is rather restrictive. On the other hand, for a given word $i$, it is not difficult to explicitly construct (either directly from the definition, or by using the other equivalent characterizations of condition (P) in [6, Proposition 2.1]) many choices of $m$ such that $(i, m)$ satisfies condition (P).

We may now state the main result of this section.

Proposition 2.5. If $(i, m)$ satisfies condition (P), then there exists a bijection between the set of integer lattice points in the polytope $P(i, m)$ and the set of standard tableaux $T(i, m)$, therefore,

\[|P(i, m) \cap \mathbb{Z}^n| = |T(i, m)|.\]

To prove Proposition 2.5 we need some preliminaries. Let $i, m$ be as above. For any $k$ with $1 \leq k \leq n$, we define the notation

\[\iota[k] := (i_k, i_{k+1}, \ldots, i_n) \quad m[k] := (m_k, m_{k+1}, \ldots, m_n)\]

so $i[k]$ and $m[k]$ are obtained from $i$ and $m$ by deleting the left-most $k-1$ coordinates. The following lemma is immediate from the inductive nature of the definitions of the polytopes $P(i, m)$ and of the condition (P).

Lemma 2.6. Let $i = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ be a word and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ be a multiplicity list.

1. Suppose $(x_1, \ldots, x_n) \in P(i, m)$. For any $k$ with $1 \leq k \leq n-1$, we have

\[(x_{k+1}, \ldots, x_n) \in P(\iota[k], m[k+1]).\]

2. If $(i, m)$ satisfies condition (P), then for any $k$ with $1 \leq k \leq n-1$ and any $(x_{k+1}, \ldots, x_n) \in P(\iota[k+1], m[k+1])$, the vector $(0, \ldots, 0, x_{k+1}, \ldots, x_n)$ lies in $P(i, m)$, where $(0, \ldots, 0, x_{k+1}, \ldots, x_n)$ is the vector obtained by adding $k$ zeroes to the right.

3. If $(i, m)$ satisfies condition (P), then for any $k$ with $1 \leq k \leq n$, the pair $(i[k], m[k])$ also satisfies condition (P).

To prove Proposition 2.5 the plan is to first explicitly construct a map from $P(i, m) \cap \mathbb{Z}^n$ to $T(i, m)$ and then prove that it is a bijection. Actually it will be convenient to define a sequence of maps from $\varphi_k : \mathbb{Z}_{\geq 0}^n \rightarrow \Pi \cap \{O\}$; the map $\varphi := \varphi_1$ will be the desired bijection between $P(i, m) \cap \mathbb{Z}^n$ with $T(i, m)$.

Definition 2.7. Let $i = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ be a word and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ be a multiplicity list. Let $k$ be an integer with $1 \leq k \leq n$. We define a map $\varphi_k : \mathbb{Z}_{\geq 0}^n \rightarrow \Pi \cup \{O\}$ associated to $i$ and $m$ by

\[\varphi_k(x_1, \ldots, x_n) := f_{\beta_k}^x(m^\lambda_k \cdots f_{\beta_{k+1}}^x(p^\lambda_{k+1} \cdots f_{\beta_n}^x(p^\lambda_n)) \cdots)\]

where $\lambda_k := m_k \beta_k$ for $1 \leq k \leq n$. (Although the map $\varphi_k$ depends on $i$ and $m$, for simplicity we omit it from the notation.)
From the definition it is immediate that the \( \varphi_k \) are related to one another by the equation
\[
\varphi_k(x_k, \ldots, x_n) = f_{\beta_k}^k(\pi^k \star \varphi_{k+1}(x_{k+1}, \ldots, x_n))
\]
for \( 1 \leq k < n \). It will be also useful to introduce the notation
\[
\tau_k(x_{k+1}, \ldots, x_n) := \pi^k \star \varphi_{k+1}(x_{k+1}, \ldots, x_n)
\]
for \( 1 \leq k < n \) and we set \( \tau_n := \pi^0 \), from which it immediately follows that
\[
\varphi_k(x_k, \ldots, x_n) = f_{\beta_k}^k(\tau_k(x_{k+1}, \ldots, x_n)).
\]

With this notation in place we can interpret the functions \( A_k \) appearing in the definition of \( P(i, m) \) naturally in terms of paths. Recall that the endpoint \( \pi(1) \) of a path \( \pi \in \Pi \) is called its weight and we denote it by \( \text{wt}(\pi) := \pi(1) \).

Lemma 2.8. Let \( (x_1, \ldots, x_n) \in \mathbb{Z}_{\geq 0} \) and let \( k \) be an integer; \( 0 \leq k \leq n-1 \). If \( \varphi_{k+1}(x_{k+1}, \ldots, x_n) \notin \mathcal{O} \) then
\[
\text{wt}(\varphi_{k+1}(x_{k+1}, \ldots, x_n)) = m_{k+1}\omega_{\beta_{k+1}} + \cdots + m_n\omega_{\beta_n} - x_{k+1}\beta_{k+1} - \cdots - x_n\beta_n.
\]
Moreover, if in addition \( k \geq 1 \) then \( \tau_k(x_{k+1}, \ldots, x_n) \notin \mathcal{O} \) and
\[
\text{wt}(\tau_k(x_{k+1}, \ldots, x_n)) = m_k\omega_{\beta_k} + m_{k+1}\omega_{\beta_{k+1}} + \cdots + m_n\omega_{\beta_n} - x_{k+1}\beta_{k+1} - \cdots - x_n\beta_n
\]
so in particular
\[
A_k(x_{k+1}, \ldots, x_n) = (\text{wt}(\tau_k(x_{k+1}, \ldots, x_n)), \beta_k^\vee).
\]
Proof. Under the hypothesis that \( \varphi_{k+1}(x_{k+1}, \ldots, x_n) \) is an honest path (i.e. it is not \( \mathcal{O} \)), the first statement of the lemma is immediate from the definition of \( \varphi_k \), Lemma 1.8 and Lemma 1.6(1). The other statements of the lemma are then straightforward from the definitions. \( \square \)

In words, the equation (2.4) says that the functions \( A_k \) measure the pairing of the endpoint of \( \tau_k(x_{k+1}, \ldots, x_n) \) against the coroot \( \beta_k^\vee \) (assuming \( \tau_k(x_{k+1}, \ldots, x_n) \) is an honest path).

Now we show that when \( \varphi_k \) is restricted to the subset \( P(i[k], m[k]) \cap \mathbb{Z}^{n-k+1} \), the output is an honest path in \( \Pi \) (i.e. it is not the formal symbol \( \mathcal{O} \)). From the definition of standard tableaux it immediately follows that the output is also in fact an element in \( \mathcal{T}(i[k], m[k]) \).

Lemma 2.9. Let \( k \) be an integer with \( 1 \leq k \leq n \). The map \( \varphi_k \) restricts to a map
\[
\varphi_k : P(i[k], m[k]) \cap \mathbb{Z}^{n-k+1} \to \mathcal{T}(i, m).
\]
Proof. We first show that the output of the maps \( \varphi_k \) are honest paths (i.e. \( \notin \mathcal{O} \)). We argue by induction, and since the definition of the \( \varphi_k \) is a composition of operators starting with \( f_{\beta_n} \) (not \( f_{\beta_1} \)) the base case is \( k = n \). From the definition of \( P(i, m) \) we know that \( x_n \leq m_n = (\pi^0(1), \beta_n^\vee) \), so it suffices to prove that for such \( x_n \), we have \( f_{\beta_n}^n(\pi^0 = \pi^m \omega_{\beta_n}) \neq \mathcal{O} \). Since \( \pi^0 \) is a straight-line path from 0 to \( \lambda_n = m_n\omega_{\beta_n} \), the constants \( Q \) and \( P \) in the definition of \( f_{\beta_n} \) (applied to \( \pi^0 \)) are 0 and \( m_0(1) - Q = m_n\omega_{\beta_n}, \beta_n^\vee \)). Thus by Lemma 1.6, we may conclude \( \varphi_n(x_n) := f_{\beta_n}^n(\pi^0) \neq \mathcal{O} \), which completes the base case. Now suppose that \( 1 \leq k < n \) and \( \varphi_{k+1}(x_{k+1}, \ldots, x_n) \neq \mathcal{O} \), which in turn implies \( \tau_k(x_{k+1}, \ldots, x_n) \neq \mathcal{O} \) since concatenation of paths always results in a path. We must show that \( \varphi_k(x_k, \ldots, x_n) = f_{\beta_k}^k(\tau_k) \neq \mathcal{O} \). Since \( \tau_k \) is a path starting at the origin 0, the constants \( Q \) and \( P \) in the definition of \( f_{\beta_k} \) (applied to \( \tau_k(x_{k+1}, \ldots, x_n) \)) are \( \leq 0 \) and \( \geq (\text{wt}(\tau_k(x_{k+1}, \ldots, x_n)), \beta_k^\vee) \) respectively. In particular, again by Lemma 1.6 it suffices to show that \( x_k \leq (\text{wt}(\tau_k(x_{k+1}, \ldots, x_n)), \beta_k^\vee) \). Since \( \tau_k(x_{k+1}, \ldots, x_n) \notin \mathcal{O} \) and \( (x_k, \ldots, x_n) \in P(i[k], m[k]) \), the result then holds by definition of \( P(i[k], m[k]) \) and the interpretation of the \( A_k \) given in Lemma 2.8. It remains to check that the paths \( \varphi_k(x_{k+1}, \ldots, x_n) \in \Pi \) are standard tableaux, but this follows directly from Definition 1.7. \( \square \)

From the above discussion we have a well-defined map
\[
\varphi := \varphi_1 : P(i, m) \cap \mathbb{Z}^n \to \mathcal{T}(i, m).
\] We need to prove that \( \varphi \) is a bijection. For this it is useful to introduce another condition on \( (i, m) \) which we call condition (P'); it is formulated in terms of the paths \( \tau_k \) and the raising operators \( e_{\beta_k} \).

Definition 2.10. Let \( i = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n \) be a word and \( m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \) be a multiplicity list. We say that the pair \( (i, m) \) satisfies condition (P') if for all \( (x_1, \ldots, x_n) \in P(i, m) \cap \mathbb{Z}_{\geq 0}^n \) and all \( k \) with \( 1 \leq k \leq n \), we have \( e_{\beta_k}(\tau_k(x_{k+1}, \ldots, x_n)) = \mathcal{O} \).
It may be conceptually helpful to note that, from our interpretation of the functions $A_{k}$ in Lemma 2.8 and the definitions of $P(i, m)$ and $\tau_{k}$, we may think of condition (P) as saying that the endpoints of certain paths $\tau_{k}$ are always contained in the affine half-space defined by $\{ (\beta_{k}, \beta_{k}) \geq 0 \}$ (i.e. the half-space pairing non-negatively against the coroot $\beta_{k}^{\vee}$). Moreover, from Lemma 1.6(3) we see that in order to show $e_{\beta_{k}}(\tau_{k}) = O$ for a given path $\tau_{k}$, it suffices to show that the entire path $\tau_{k}$ lies in the same affine half-space. Thus, roughly speaking, condition (P) is about endpoints, whereas condition (P') is about the entire path.

Remark 2.11. From the above discussion it may seem that condition (P’) is stronger than condition (P). This is not the case. For instance, for $G = SL(3, \mathbb{C})$, the pair $i = (1, 2, 1)$ and $m = (0, 1, 1)$ is an example where $(i, m)$ satisfies condition (P’) but it does not satisfy condition (P); for an illustration of the corresponding polytope $P(i, m)$ see Example 4.2. The subtilty in the definitions of the conditions (P) and (P’).

Proposition 2.12. Let $i = (i_{1}, \ldots, i_{n}) \in \{1, \ldots, r\}^{n}$ be a word and $m = (m_{1}, \ldots, m_{n}) \in \mathbb{Z}_{\geq 0}^{n}$ be a multiplicity list. If the pair $(i, m)$ satisfies condition (P) then $(i, m)$ satisfies condition (P’).

Since condition (P’) is phrased in terms of the $e_{\beta_{k}}$ and because the raising and lowering operators acts as inverses (provided the composition makes sense) as in Lemma 1.6(2), once we know Proposition 2.12 it is straightforward to show that $\varphi$ is a bijection. Indeed, we suspect that the argument given below is standard for the experts, but we include it for completeness.

Proof of Proposition 2.12 (assuming Proposition 2.7). By Proposition 2.12 we may assume that condition (P”) holds. First we prove by induction that $\varphi_{k}$ is injective for each $k$, starting with the base case $k = n$. Suppose

\begin{equation}
\varphi_{n}(x_{n}) = f_{\beta_{n}}^{x_{n}}(\pi_{\lambda_{n}}) = f_{\beta_{n}}^{y_{n}}(\pi_{\lambda_{n}}) = \varphi_{n}(y_{n})
\end{equation}

and also suppose for a contradiction that $x_{n} < y_{n}$. Applying $e_{\beta_{n}}^{x_{n}+1}$ to the LHS of (2.6) yields $e_{\beta_{n}}(\pi_{\lambda_{n}})$ since by Lemma 1.6(2) we know $e_{\beta_{n}}$ is inverse to $f_{\beta_{n}}$ whenever the image of $f_{\beta_{n}}$ is $\neq O$. By condition (P”), $e_{\beta_{n}}(\pi_{\lambda_{n}}) = e_{\beta_{n}}(\tau_{n}) = O$. On the other hand, applying $e_{\beta_{n}}^{x_{n}+1}$ to the RHS of (2.6) yields $f_{\beta_{n}}^{y_{n}-x_{n}+1}(\pi_{\lambda_{n}})$ which is $\neq O$ since $y_{n} - x_{n} - 1 \geq 0$ by assumption. This contradicts (2.6) and so $x_{n} = y_{n}$ and we conclude $\varphi_{n}$ is injective. This completes the base case. Now suppose by induction that $\varphi_{k+1}$ is injective; we need to show $\varphi_{k}$ is injective. Assume

\[ \varphi_{k}(x_{k}, \ldots, x_{n}) = f_{\beta_{k}}^{x_{k}}(\tau_{k}(x_{k+1}, \ldots, x_{n})) = f_{\beta_{k}}^{y_{k}}(\tau_{k}(y_{k+1}, \ldots, y_{n})) = \varphi_{k}(y_{k}, \ldots, y_{n}). \]

and suppose also that $x_{k} < y_{k}$. The same argument as above, namely applying $e_{\beta_{k}}^{x_{k}+1}$ to both sides, yields a contradiction due to the condition (P”). Thus $x_{k} = y_{k}$. Applying $e_{\beta_{k}}^{x_{k}}$ to both sides of the equation above we obtain $\tau_{k}(x_{k+1}, \ldots, x_{n}) = \tau_{k}(y_{k+1}, \ldots, y_{n})$. Concatenation by $\pi_{k}$ is evidently injective, so $\varphi_{k+1}(x_{k+1}, \ldots, x_{n}) = \varphi_{k+1}(y_{k+1}, \ldots, y_{n})$, but then by the inductive assumption we have $(x_{k+1}, \ldots, x_{n}) = (y_{k+1}, \ldots, y_{n})$. This proves $(x_{k}, \ldots, x_{n}) = (y_{k}, \ldots, y_{n})$ and hence that $\varphi_{k}$ is injective as desired.

Now we claim $\varphi_{k}$ is surjective for each $k$. We argue by induction on the size of $n$. First consider the base case $n = 1$, so $w = (\beta_{1} = \beta_{1})$, $m = (m_{1} = m_{1})$, and $P(w, m) = \{0, [m]\}$. By definition, a standard tableau of shape $\lambda = m_{1}\omega_{\beta}$ is of the form $f_{\beta_{1}}^{x_{1}}(\pi_{\lambda})$ for some $\xi \in \mathbb{Z}_{\geq 0}$. Since $\pi_{\lambda}$ is a straight-line path from $0$ to $m_{1}\beta$, the constants $Q$ and $P$ in the definition of $f_{\beta_{1}}$ applied to $\pi_{\lambda}$ are $0$ and $m$ respectively. Then for $x$ a non-negative integer we know by Lemma 1.6(3) that $f_{\beta_{1}}^{x}(\pi_{\lambda}) \neq O$ if and only if $x \leq m$. Since $P(i, m) = \{0, m\}$ in this case, we conclude that $\varphi_{1}$ is surjective if $n = 1$, as desired.

Now assume by induction that each $\varphi_{k}$ is surjective (hence bijective) for words of length $< n$. From Lemma 1.6(3) we know that $(i[k], m[k])$ satisfies condition (P) (and hence condition (P’)). By the inductive assumption we may therefore assume that $\varphi_{k+1} : P(i[k], m[k]) \cap \mathbb{Z}^{n-k+1} \to T(i[k], m[k])$ is a bijection for $k > 1$ and we wish to show $\varphi = \varphi_{1}$ is surjective. By definition of the standard tableaux, any element in $T(i, m)$ is of the form $f_{\beta_{1}}^{x_{1}}(\pi_{\lambda_{1}} \ast \tau')$ for some $\tau' \in T[w_{2}, m_{2}]$ and some $\ell_{1} \in \mathbb{Z}_{\geq 0}$. By the inductive assumption, we know that there exists some $(x_{2}, \ldots, x_{n}) \in P(i[2], m[2])$ such that $\tau' = \varphi_{2}(x_{2}, \ldots, x_{n})$. From the definition of $P(i, m)$, in order to prove the surjectivity it would suffice to show that

\[ f_{\beta_{1}}^{x_{1}}(m_{1}\omega_{\beta_{1}} \ast \varphi_{2}(x_{2}, \ldots, x_{n})) = f_{\beta_{1}}(\tau_{1}(x_{2}, \ldots, x_{n})) = O \Rightarrow \ell_{1} \leq A_{1}(x_{2}, \ldots, x_{n}). \]

From Lemma 1.6(3) we know $f_{\beta_{1}}^{x_{1}}(\tau_{1}) \neq O$ if and only if $\ell_{1} \leq P$ where $P$ is defined to be the integral part of $(\langle \tau_{1}(x_{2}, \ldots, x_{n}), \beta_{1}^{\vee} \rangle - Q \cap Q = \min_{t \in [0, 1]} (\tau_{1}(x_{2}, \ldots, x_{n})(t), \beta_{1}^{\vee})$. Since $\tau_{1}(x_{2}, \ldots, x_{n}) \neq O$ by assumption, we know from (2.4) that $A_{1}(x_{2}, \ldots, x_{n}) = \langle \tau_{1}(x_{2}, \ldots, x_{n}), \beta_{1}^{\vee} \rangle$ and it is evident from the definition of $A_{1}$ that for $(x_{2}, \ldots, x_{n}) \in \mathbb{Z}^{n-1}$,
the value $A_1(x_2, \ldots, x_n)$ is integral. Hence it suffices to show that $Q = 0$, and again from Lemma 1.6 this is equivalent to showing that $e_{\beta_1} (\tau_k(x_2, \ldots, x_n)) = O$. Note that the vector $(0, x_2, \ldots, x_n)$ lies in $P(i, m)$ by Lemma 2.6.2. By applying the statement of condition $(P')$ to $(0, x_2, \ldots, x_n)$ and $k = 1$ we obtain that $e_{\beta_1} (\tau_k(x_2, \ldots, x_n)) = O$ as desired. This completes the proof.

It remains to justify Proposition 2.12. The following simple lemma will be helpful.

**Lemma 2.13.** Let $\pi \in \Pi$ be a piecewise linear path in $X_R$.

1. Let $\pi^\lambda$ be a linear path for some $\lambda \in X_R$. Then for any $t \in [0, 1]$, there exist non-negative real constants $a, c \geq 0$ and $s \in [0, 1]$ such that \((\pi^\lambda \star \pi)(t) = a\lambda + c \pi(t)\).

2. Let $\beta$ be a simple root. Let $x$ be a positive integer and assume that $f_{\beta}^x(\pi) \neq O$. Then for any $t \in [0, 1]$, there exists $b \in R$ with $0 \leq b \leq x$ such that $f_{\beta}^x(\pi)(t) = \pi(t) + b(-\beta)$ where $0 \leq b \leq x$.

3. Let $\pi \in \Pi$ be a path in $X_R$. Let $\{\beta_1, \ldots, \beta_j\}$ be any sequence of simple roots and $n_1, \ldots, n_j \in \mathbb{Z}_{\geq 0}$ any sequence of non-negative integers. Then any point along the path $f_{\beta_1}^{n_1}(\pi^{\beta_1}, \ldots, f_{\beta_j}^{n_j}(\pi^{\beta_j}) \star \pi)$ can be expressed as a linear combination

$$\sum_{\ell=1}^j a_{\ell t} = 0 \pi^x + \sum_{\ell=1}^j b_{\ell}(\pi^x(\pi^\ell))$$

for some $a_{\ell t}, b_{\ell} \geq 0$ non-negative real constants and some $s \in [0, 1]$.

4. Let $1 = (1, \ldots, 1) \in \{1, 2, \ldots, r\}^n$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ be a word and multiplicity list and let $k \in [1, m]$ be an integer with $1 \leq k \leq n$. Let $\varphi_k$ denote the map associated to $1, m$ as in Definition 2.7. Then any point along the path $\varphi_k(x_k, \ldots, x_n)$ can be expressed as a linear combination

$$\sum_{\ell=k}^n a_{\ell m} = 0 \pi^x + \sum_{\ell=k}^n b_{\ell}(\pi^x(\pi^\ell))$$

where $a_{\ell t}, b_{\ell} \geq 0$.

**Proof.** First we prove (1). From the definition of paths and the definition of a straight-line path $\pi^\lambda$ it follows that for $t \in [0, \frac{1}{2}]$ we may take $a = 2t$ and $c = 0$, since $(\pi^\lambda \star \pi)(t) = \pi^\lambda(2t) = 2t\lambda$ in this case. On the other hand, if $t \in [\frac{1}{2}, 1]$ then we may take $a = 1$, $c = 1$ and $s = 2t - 1$, since by (1.5) we have $(\pi^\lambda \star \pi)(t) = \pi^\lambda(1) + \pi(2t - 1)$. This proves the claim.

Next we prove (2). Recall that the reflection operator $s_\beta$ acts by $s_\beta(\alpha) := \alpha - \langle \alpha, \beta^\vee \rangle \beta$, so for any path $\pi$ and for any $t \in [0, 1]$ we have $s_\beta(\pi)(t) := s_\beta(\pi(t)) = \pi(t) - \langle \pi(t), \beta^\vee \rangle \beta = \pi(t) + (\pi(t), \beta^\vee)(-\beta)$ and in particular, $s_\beta(\pi)(t)$ is a linear combination of $\pi(t)$ and $-\beta$. Additionally, from Definition 1.5 we know that $f_{\beta}^x(\pi) := \pi_1 \star s_{\beta}(\pi) \star \pi_3$ where $\pi_1$ and $\pi_2$ are defined in (1.6) and from the discussion preceding Definition 1.5 which defines $\pi$ and $x$ it follows that $\langle \pi_2(t), \beta^\vee \rangle \in [0, 1]$ for all $t$. To prove the claim we begin with the base case $x = 1$. Consider each of the 3 components of $f_{\beta}^x(\pi)$ in turn. For the first portion of the path (corresponding to $\pi_1$), the operator $f_{\beta}$ does not alter the path at all, so for such $t$ we have $f_{\beta}(\pi)(t) = \pi(t)$ and the claim of the lemma holds with $b = 0$. For $t$ in the second portion of the path, we have $\pi(t) = \pi_1(p) + \pi_2(t')$ (here $t'$ is determined by $t$ by some reparametrization coming from the concatenation operation) and $f_{\beta}(\pi)(t) = \pi_1(p) + s_\beta(\pi_2(t')) \pi_1(p) + \pi_2(t') + \langle \pi_2(t'), \beta^\vee \rangle \beta = \pi(t) + \langle \pi_2(t'), \beta^\vee \rangle \beta$. As we have already seen, $\langle \pi_2(t'), \beta^\vee \rangle \in [0, 1]$, so choosing $b = \langle \pi_2(t'), \beta^\vee \rangle$ does the job. Finally, again from the discussion preceding the definitions of $\pi_1, \pi_2$ and $\pi_3$ it follows that $\langle \pi_2(1), \beta^\vee \rangle = 1$ so for the last (third) portion of the path we have that $f_{\beta}(\pi)(t) = (\pi(t) - \beta) + \pi_3(t'') = (\pi(t) + \pi_3(t'') - \beta = \pi(t) - \beta$ where again $t''$ is determined by $t$ by a reparametrization. By choosing $b = 1$ we see that the claim holds in this case also.

Applying the same argument $x$ times yields the result.

The statements (3) and (4) follow straightforwardly by applying (1) and (2) repeatedly. □

The following elementary observation is also conceptually useful. For two simple positive roots $\alpha, \beta$ we say that $\alpha$ and $\beta$ are adjacent if they are distinct and they correspond to two adjacent nodes in the corresponding Dynkin diagram. (From properties of the Cartan matrix, $\alpha$ and $\beta$ are adjacent precisely when the value of the pairing $\langle \alpha, \beta^\vee \rangle$ is strictly negative.) Then it is immediate that $A_k(x_{k+1}, \ldots, x_n)$ can be interpreted as

$$A_k(x_{k+1}, \ldots, x_n) = m_k + \left( \sum_{j > k \atop \beta_j = \beta_k} (m_j - 2x_j) \right) - \left( \sum_{j > k \atop \beta_j \text{ adjacent to } \beta_k} x_j \langle \beta_j, \beta^\vee_k \rangle \right).$$
Proof of Proposition \ref{prop:lower_bound} We begin by noting that the path \( \tau_n \) is by definition \( \pi^{\lambda_n} \) where \( \lambda_n := m_n \beta_n \). Thus \( Q = 0 \) in this case and by Lemma \ref{lem:lower_bound} we conclude \( e_{\beta_n}(\tau_n) = O \). So it remains to check the cases \( k < n \). As in the discussion above, by Lemma \ref{lem:lower_bound} and by the definition of the raising operators, in order to prove the claim it suffices to prove that for any \( (x_1, \ldots, x_n) \in P(i, m) \) and any \( k \) with \( 1 \leq k \leq n-1 \), we have
\begin{equation}
\min_{t \in [0,1]} \{ \langle \tau_k(x_{k+1}, \ldots, x_n)(t), \beta_k \rangle \} \geq 0
\end{equation}
which is equivalent to
\begin{equation}
\min_{t \in [0,1]} \{ \langle \varphi_{k+1}(x_{k+1}, \ldots, x_n)(t), \beta_k \rangle \} \geq -m_k
\end{equation}
by definition of the \( \tau_k \) and \( \varphi_k \).

We use induction on the size of \( n \). We already proved the case \( n = 1 \) above so the base case is \( n = 2 \) and \( k = 1 \). Let \( i = (i_1, i_2) \) with associated sequence of simple roots (\( \beta_1, \beta_2 \)) and \( m = (m_1, m_2) \). Let \( (x_1, x_2) \in P(i, m) \). Then we have \( 0 \leq x_2 \leq m_2 \) so an explicit computation shows \( \varphi_2(x_2) = f_{\beta_2}^x (\pi^{m_2 \beta_2}) = \pi^{x_2 (\beta_2)} \star (m_2 - x_2) \beta_2 \). Hence we wish to show that
\begin{equation}
\min_{t \in [0,1]} \{ \langle \varphi_2(x_2), \pi^{m_2 \beta_2} \rangle \} \geq -m_1.
\end{equation}
First consider the case \( \beta_1 \neq \beta_2 \). Since \( \langle \omega_{\beta_2}, \beta_1' \rangle = 0 \) and \( \langle \beta_2, \beta_1' \rangle \leq 0 \) for any two distinct simple roots, and \( x_2 \geq 0 \) by assumption, we can see that \( \langle \pi^{x_2 (\beta_2)} \star (m_2 - x_2) \beta_2, \beta_1' \rangle \geq 0 \) for all \( t \). In particular the minimum value taken over all \( t \) is 0, which is greater than or equal to \(-m_1\) as desired (since \( m_1 \geq 0 \) by assumption). Next consider the case \( \beta_1 = \beta_2 \). In this case, the inequalities defining \( P(i, m) \) are
\[ 0 \leq x_2 \leq m_2 \text{ and } 0 \leq x_1 \leq m_1 \omega_{\beta_1} + m_2 \omega_{\beta_2} - x_2 \beta_2, \beta_1' = m_1 + m_2 - 2x_2. \]
From the condition (P), for any choice of \( x_2 \) with \( 0 \leq x_2 \leq m_2 \) we must have \( A_1(x_2) = m_1 + m_2 - 2x_2 \geq 0 \). In particular, for \( x_2 = m_2 \) we must have \( m_1 > m_2 \geq 0 \), from which it follows \( m_1 \geq m_2 \). Next notice that, since the vector \((m_2 - x_2) \omega_{\beta_2}, \beta_1' \) pairs non-negatively with \( \beta_1' = \beta_2' \), the minimum value of the function\[ t \mapsto \langle \pi^{x_2 (\beta_2)} \star (m_2 - x_2) \omega_{\beta_2}, \beta_1' \rangle \]
occurs at the endpoint of \( \pi^{x_2 (\beta_2)} \) where the value is \(-x_2\). From the assumptions we know \( x_2 \leq m_2 \), so \(-x_2 \geq -m_2 \). Also from the above we have seen that \( m_1 \geq m_2 \), so \(-m_2 \geq -m_1 \) and finally we obtain \(-x_2 \geq -m_1 \). This completes the base case.

We now assume by induction that the statement of the proposition holds for words and multiplicity lists of length \( \leq n - 1 \) and we must prove the statement for \( n \). As above, we already know the statement holds for \( k = n \). Next suppose \( 1 < k < n \). By Lemma \ref{lem:word_bound}, we know that \((i[k], m[k])\) satisfies condition (P) and \((x_1, \ldots, x_k)\) lies in \( P(i[k], m[k]) \).

Since \((i[k], m[k])\) have length strictly less than \( n \), by the inductive assumption we know the statement holds for such \( k \).

Thus it remains to check the case \( k = 1 \), i.e. that \( e_{\beta_1}(\tau_1(x_2, \ldots, x_n)) = O \) for \((x_1, \ldots, x_n) \in P(i, m) \). First consider the case in which the simple root \( \beta_1 \) does not appear in the word \( \beta_2, \ldots, \beta_n \). By Lemma \ref{lem:word_bound} (4), any point along the path \( \varphi_2(x_2, \ldots, x_n) \) can be written in the form \( \sum_{t=1}^n a_t \omega_{\beta_t} + \sum_{t=2}^n b_t (-\beta_t) \), where \( a_t, b_t \geq 0 \) are non-negative real constants, and all the simple roots \( \beta_t \) are distinct from \( \beta_1 \). Then for any time \( t \) we have \( \langle \varphi_2(x_2, \ldots, x_n)(t), \beta_1' \rangle = \sum_{t=1}^n a_t \omega_{\beta_t} + \sum_{t=2}^n b_t (-\beta_t) \). For \( t \) we have \( \langle \varphi_2(x_2, \ldots, x_n)(t), \beta_1' \rangle = \langle \sum_{t=2}^n b_t (-\beta_t), \beta_1' \rangle \geq 0 \) where the second equality is because \( \langle \omega_{\beta_t}, \beta_1' \rangle = 0 \) for \( \beta_t \neq \beta_1 \) and the last inequality is because \( \langle \beta_t, \beta_1' \rangle \leq 0 \) for \( \beta_t \neq \beta_1 \). Since \( m_1 \geq 0 \) by assumption, we conclude that \( \langle \varphi_2(x_2, \ldots, x_n)(t), \beta_1' \rangle \geq 0 \geq -m_1 \) for all \( t \), which yields the desired result.

Next we consider the case when \( \beta_1 \) occurs in the sequence \( \beta_2, \ldots, \beta_n \). Let \( s \) be the smallest index with \( s \geq 2 \) such that \( \beta_s = \beta_1 \), i.e., it is the first place after \( \beta_1 \) where the repetition occurs. Since the length of \( i[s] \) is \( n - 1 \), from the inductive assumption we know that \( \min_{t \in [0,1]} \{ \langle \tau_s(x_{s+1}, \ldots, x_n)(t), \beta_1' \rangle \} \geq 0 \). Note also that the path \( \tau_s \) has the property that the minimum value \( \min_{t \in [0,1]} \{ \langle \tau_s(x_{s+1}, \ldots, x_n)(t), \beta_1' \rangle \} = \min_{t \in [0,1]} \{ \langle \tau_s^t(x_1, \ldots, x_n)(t), \beta_1' \rangle \} \). As well the endpoint pairing \( \langle \omega_{\beta_s}, \beta_1' \rangle \) is both integers; this follows from its construction. Also by definition, the operator \( f_{\beta_s} \) preserves these properties; moreover, for such a path \( \tau' \) it follows from the definition of \( f_{\beta_s} \), that \( \min_{t \in [0,1]} \{ \langle f_{\beta_s}(\tau')(t), \beta_1' \rangle \} = \min_{t \in [0,1]} \{ \langle \tau'(t), \beta_1' \rangle \} = -1, \) i.e., the minimum decreases by precisely 1. From this we conclude that \( \varphi_s(x_1, \ldots, x_n) = f_{\beta_s}^x \) satisfies
\begin{equation}
\langle \varphi_s(x_1, \ldots, x_n)(t), \beta_1' \rangle = \beta_1' \geq -x_s \text{ for all } t \in [0,1].
\end{equation}

By definition \( \varphi_2(x_2, \ldots, x_n) \) is obtained from \( \varphi_s(x_1, \ldots, x_n) \) by
\[ \varphi_2(x_2, \ldots, x_n) := f_{\beta_2}^x (\pi^{m_2 \beta_2} \star \cdots \star f_{\beta_{s-1}}^x (\pi^{m_{s-1} \beta_{s-1}} \star \varphi_s(x_1, \ldots, x_n))) \]
By assumption, $\beta_1$ is distinct from all the roots $\beta_\ell$ for $2 \leq \ell \leq s - 1$. Thus $\langle \omega_{\beta_1}, \beta_1^\vee \rangle = 0$ and $\langle -\beta_1, \beta_1^\vee \rangle \geq 0$ for $2 \leq \ell \leq s - 1$ and from Lemma 2.13(3) it follows that
\[ \min_{t \in [0,1]} \{ \langle \phi_2(x_2, \ldots, x_n)(t), \beta_1^\vee \rangle \} \geq \min_{t \in [0,1]} \{ \langle \phi_2(x_s, \ldots, x_n)(t), \beta_1^\vee \rangle \}. \]

Since we know from (2.11) that the RHS above is $\geq -x_s$, it now suffices to prove that $x_s \leq m_1$. Since $(x_1, \ldots, x_n) \in P(i, m)$, we know $(y_s, x_{s+1}, \ldots, x_n) \in P(i[2], m[2])$ if $0 \leq y_s \leq A_s(x_{s+1}, \ldots, x_n)$. Also since $(i, m)$ satisfies condition (P), from Lemma 2.6(2) we know that $(y_2, \ldots, y_n) \in P(i[2], m[2])$, where $y_2 = \cdots = y_{s-1} = 0$, $y_s = A_s(x_{s+1}, \ldots, x_n)$, and $y_k = x_k$ for $k \geq s + 1$. Then from the condition (P) we conclude that
\[
A_1(y_2, \ldots, y_n) = m_1 + (m_s - 2) + \left( \sum_{k > s, \beta_k = \beta_s} (m_k - 2x_k) \right) - \left( \sum_{k > s, \beta_k \text{ adjacent to } \beta_1 = \beta_s} x_k(\beta_k, \beta_1^\vee - \beta_s^\vee) \right) \\
= m_1 + A_s(x_{s+1}, \ldots, x_n) - 2y_s - m_1 - A_s(x_{s+1}, \ldots, x_n) \geq 0.
\]
or in other words $m_1 \geq A_s(x_{s+1}, \ldots, x_n)$. But the original $x_s$ was required to satisfy the inequality $x_s \leq A_1(x_1, \ldots, x_n)$, from which it follows that $x_s \leq m_1$ as was to be shown. This completes the inductive argument and hence the proof. \hfill \square

3. NEWTON-OKOUNKOV BODIES OF BOTT-SAMUELSON VARIETIES

The main result of this manuscript is Theorem 3.4 which gives an explicit description of the Newton-Okounkov body of $(Z_i, L_i, m)$ with respect to a certain geometric valuation (to be described in detail below), provided that the word $i$ corresponds to a reduced word decomposition and the pair $(i, m)$ satisfies condition (P).

We first very briefly recall the ingredients in the definition of a Newton-Okounkov body. For details we refer the reader to [14]. We begin with the definition of a valuation (in our setting).

**Definition 3.1.** Let $A$ be a $\mathbb{C}$-algebra and $\Gamma$ a totally ordered set with order $<$. We say that a function $\nu : A \setminus \{0\} \to \Gamma$ is a valuation if

1. $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in A \setminus \{0\}$,
2. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for all $f, g \in A \setminus \{0\}$ with $f + g \neq 0$,
3. $\nu(cf) = \nu(f)$ for all $f \in A \setminus \{0\}$ and $c \in \mathbb{C}^*$.

The image of $\nu$ is clearly a semigroup and is called the value semigroup of the pair $(A, \nu)$. Moreover, if in addition the valuation has the property that

(iii) if $\nu(f) = \nu(g)$, then there exists a non-zero constant $\lambda \neq 0 \in \mathbb{C}$ such that $\nu(g - \lambda f) > \nu(g)$ or else $g - \lambda f = 0$, then we say the valuation has one-dimensional leaves.

In the construction of Newton-Okounkov bodies, we consider valuations on rings of sections of line bundles. More specifically, let $X$ be a complex-$n$-dimensional algebraic variety over $\mathbb{C}$, equipped with a line bundle $L = \mathcal{O}_X(D)$ for some (Cartier) divisor $D$. Consider the corresponding (graded) $\mathbb{C}$-algebra of sections $R = R(L) := \oplus_{k \geq 0} R_k$ where $R_k := H^0(X, L^k)$. We now describe a way to geometrically construct a valuation. (Not all valuations arise in this manner but this suffices for our purposes.) Suppose given a flag
\[ Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{ pt \} \]
of irreducible subvarieties of $X$ where $\text{codim}_X(Y_i) = \ell$ and each $Y_i$ is non-singular at the point $Y_n = \{ pt \}$. Such a flag defines a valuation $\nu_{Y_\bullet} : H^0(X, L) \setminus \{0\} \to \mathbb{Z}^n$ by an inductive procedure involving restricting sections to each subvariety and considering its order of vanishing along the next (smaller) subvariety, as follows. We will assume that all $Y_i$ are smooth (though this is not necessary, cf. [14]). Given a non-zero section $s \in H^0(X, L = \mathcal{O}_X(D))$, we define
\[ \nu_1 := \text{ord}_{Y_1}(s) \]
i.e. the order of vanishing of $s$ along $Y_1$. By choosing a local equation for $Y_1$ in $X$, we can construct a section $s_1 \in H^0(X, \mathcal{O}_X(D - \nu_1 Y_1))$ that does not vanish identically on $Y_1$. By restricting we obtain a non-zero section $s_2 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1))$, and define $\nu_2 := \text{ord}_{Y_2}(s_1)$. We define each $\nu_1$ by proceeding inductively in the same fashion. It is not difficult to see that $\nu_{Y_\bullet}$ thus defined gives a valuation with one-dimensional leaves on each $R_k$. 


Given such a valuation \( \nu \), we may then define
\[
S(R) = S(R, \nu) := \bigcup_{k>0} \{(k, \nu(\sigma)) \mid \sigma \in R_k \setminus \{0\} \} \subset \mathbb{N} \times \mathbb{Z}^n
\]
(cf. also [14] Definition 1.6, where the notation slightly differs) which can be seen to be an additive semigroup. Now define \( C(R) \subseteq \mathbb{R} \times \mathbb{R}^n \) to be the cone generated by the semigroup \( S(R) \), i.e., it is the smallest closed convex cone centered at the origin containing \( S(R) \). We can now define the central object of interest.

**Definition 3.2.** Let \( \Delta = \Delta(R, \nu) = \Delta(R, \nu) \) be the slice of the cone \( C(R) \) at \( \ell = 1 \) projected to \( \mathbb{R}^n \) via the projection to the second factor \( \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \). In other words
\[
\Delta = \text{conv} \left( \bigcup_{k>0} \left\{ \frac{x}{k} : (k, x) \in S(R) \right\} \right).
\]
The convex body \( \Delta \) is called the **Newton-Okounkov body** of \( R \) with respect to the valuation \( \nu \).

In the current manuscript, the geometric objects under study are the Bott-Samelson variety \( Z_i \) and the line bundle \( L_{i,m} \) over it. Following the notation above, we study the Newton-Okounkov body of \( R(L_{i,m}) = \oplus_{k \geq 0} R^0(Z_i, L_{i,m}^k) \). We begin with a description of the flag \( Y_\bullet \) of subvarieties with respect to which we will define a valuation. Given \( \ell \) with \( 1 \leq \ell \leq n \), we define a subvariety \( Y_\ell \) of \( Z_1 \) of codimension \( \ell \) by
\[
Y_\ell := \{(p_1, \ldots, p_n) : p_s = e \text{ for the last } \ell \text{ coordinates, i.e., for } n-\ell+1 \leq s \leq n\}.
\]
The subvariety \( Y_\ell \) is smooth, since it is evidently isomorphic to the Bott-Samelson variety \( Z_{(i_1, \ldots, i_{n-\ell})} \). In Kaveh’s work on Newton-Okounkov bodies and crystal bases [7], he introduces a set of coordinates, which he denotes \( (t_1, \ldots, t_n) \), near the point \( Y_0 = \{((e, e, \ldots, e))\} \). Near \( Y_\ell \), our flag \( Y_\bullet \) can be described using Kaveh’s coordinates as
\[
\{t_n = 0\} \supset \{t_n = t_{n-1} = 0\} \supset \cdots \supset \{t_n = \cdots = t_2 = 0\} \supset \{(0, 0, \ldots, 0)\}.
\]

**Remark 3.3.** In particular, with respect to Kaveh’s coordinates, our geometric valuation \( \nu_\bullet \) is the lowest-term valuation on polynomials in \( t_1, \ldots, t_n \) with respect to the lexicographic order with \( t_1 < t_2 < \cdots < t_n \). Thus our valuation is different from the valuation used by Kaveh in [7] and Fujita in [3], since they take the highest-term valuation with respect to the lexicographic order with the variables in the reverse order, \( t_1 > t_2 > \cdots > t_n \). In general, it seems to be a rather subtle problem to understand the dependence of the Newton-Okounkov body on the choice of valuation, cf. for instance the discussion in [4] Remark 2.3.

We now state the main theorem of this section, which is also the main result of this manuscript. Let \( P(i, m) \) denote the polytope of Definition 2.1. In the statement below, \( P(i, m)^{op} \) denotes the points in \( P(i, m) \) with coordinates reversed, i.e. \( P(i, m)^{op} := \{(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in P(i, m)\} \). (The reversal of the ordering on coordinates arises because, locally near \( Y_\ell = \{((e, e, \ldots, e))\} \) and in Kaveh’s coordinates, \( Y_\ell \) is given by the equations \( \{t_{n-\ell+1} = \cdots = t_n = 0\} \), i.e. the last coordinates are 0. So for example \( \nu_1(s) \) is the order of vanishing of \( s \) along \( \{t_n = 0\} \), not \( \{t_1 = 0\} \).)

**Theorem 3.4.** Let \( i = (i_1, \ldots, i_n) \in \{1, 2, \ldots, r\}^n \) be a word and \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \) be a multiplicity list. Let \( Z_i \) and \( L_{i,m} \) denote the associated Bott-Samelson variety and line bundle respectively. Suppose that \( i \) corresponds to a reduced word decomposition and that \( (i, m) \) satisfies condition \( \mathbf{(P)} \). Consider the valuation \( \nu_\bullet \) defined above and let \( S(R(L_{i,m})) \) denote the corresponding value semigroup. Then
\begin{enumerate}
    \item the degree-1 piece \( S_1 := S(R(L_{i,m})) \cap \{1\} \times \mathbb{Z}^n \) of \( S(R(L_{i,m})) \) is equal to \( P(i, m)^{op} \cap \mathbb{Z}^n \) (where we identify \( \{1\} \times \mathbb{Z}^n \) with \( \mathbb{Z}^n \) by projection to the second factor),
    \item \( S(R(L_{i,m})) \) is generated by \( S_1 \), so in particular it is finitely generated, and
    \item the Newton-Okounkov body \( \Delta = \Delta(R(L_{i,m})) \) of \( Z_i \) and \( L_{i,m} \) with respect to \( \nu_\bullet \) is equal to the polytope \( P(i, m)^{op} \).
\end{enumerate}

Before diving into the proof of Theorem 3.4, we explain the basic structure of our argument. Our first step is Proposition 3.7 where we show that the image of \( \nu_\bullet \) is always a subset of the polytope \( P(i, m)^{op} \). This is the most important step in our argument; here we need that \( i \) is reduced. Then, under the additional assumption that \( (i, m) \) satisfies condition \( \mathbf{(P)} \), the results of Section 2 allows us to quickly conclude that \( \nu_\bullet \) gives a surjection from \( S_1 \) to \( P(i, m)^{op} \cap \mathbb{Z}^n \), from which the theorem follows.

We need some preliminaries. For each \( j \) with \( 1 \leq j \leq n \) let \( C_j \) denote the curve in \( Z_i \) given by setting all but the \( j \)-th coordinate in \( [(p_1, \ldots, p_n)] \in Z_i \) equal to \( e \). Note that the curves are isomorphic to \( \mathbb{P}^1 \). The lemma below is from [4] Section 3.7.
Lemma 3.5. Let $\lambda_1, \ldots, \lambda_n$ be a sequence of weights. The degree of the restriction of the line bundle $L_i(\lambda_1, \ldots, \lambda_n)$ on $Z_i$ to the curve $C_n$ is equal to $\langle \lambda_n, b_{\nu_i} \rangle$.

In what follows we also need the following codimension-1 subvarieties (divisors) on $Z_i$. For $1 \leq j \leq n$ let $Z_{i(j)}$ denote the subvariety of $Z_i$ obtained by requiring the $j$-th coordinate of $\{(p_1, \ldots, p_n)\} \in Z_i$ to be equal to $e$. Notice that $Z_{i(n)}$ is the same as our $Y_1$ above, and is also naturally isomorphic to the smaller Bott-Samelson variety $Z_{(i_1, \ldots, i_{n-1})}$ associated to the word obtained by deleting the last entry in $i$. Also note that since $Z_{i(n)}$ is an irreducible subvariety of codimension 1, it determines a line bundle $\mathcal{O}(Z_{i(n)})$.

We will need the following lemma, which computes the restriction of certain line bundles on $Z_i$ to $Z_{i(n)}$.

Lemma 3.6. Let $\lambda_1, \ldots, \lambda_n$ be a sequence of weights. Then the restriction to $Z_{i(n)}$ of the line bundle $L_i(\lambda_1, \ldots, \lambda_n)$ is isomorphic to $L_i(\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1} + \lambda_n)$ on $Z_{(i_1, \ldots, i_{n-1})}$. Moreover, the restriction of $\mathcal{O}(Z_{i(n)})$ to $Z_{i(n)}$ is isomorphic to $L_i(\lambda_1, \ldots, \lambda_{n-1} - e, \lambda_n)$ on $Z_{(i_1, \ldots, i_{n-1})}$.

Proof. Consider the map $\varphi : L_i(n)(\lambda_1, \ldots, \lambda_{n-1} + \lambda_n) \to L_i(\lambda_1, \ldots, \lambda_n)|_{Z_{i(n)}}$ given by $[(p_1, \ldots, p_n, s, k)] \mapsto [(p_1, \ldots, p_{n-1}, e, k)]$. Then $\varphi$ gives the required isomorphism. Indeed, $\varphi$ is well-defined as can be seen by the computation

$$
[(p_1b_1, \ldots, b_{n-2}p_{n-1}b_1, e = b_{n-1}^1b_1), k] = [(p_1, p_2, \ldots, p_{n-1}, e = -\lambda_1(b_1) \cdots e_{n-1}(b_1) e_{n-1}^1(b_{n-1}k)] \\
\quad = [(p_1, p_2, \ldots, p_{n-1}, e = -\lambda_1(b_1) \cdots e_{n-1}^1(b_{n-1}k)]
$$

in $L_i(\lambda_1, \ldots, \lambda_n)$. It can be checked similarly that $\varphi$ is injective, and surjectivity is immediate from its definition.

For the second claim, recall that the restriction $\mathcal{O}(D)|_D$ is the normal bundle to $D$ (see e.g. [20] Exercise 21.2H).

Applying this to $Z_{i(n)}$, it suffices to show that the normal bundle to $Z_{i(n)}$ in $Z_i$ is isomorphic to $L_i(\lambda_1, \ldots, \lambda_{n-1} - e, \lambda_n)$ on $Z_{(i_1, \ldots, i_{n-1})}$. Now note $Z_i$ is a $P_{\beta_i}/B$-bundle over $Z_{i(n)} \cong Z_{(i_1, \ldots, i_{n-1})}$, and since $Z_{i(n)}$ is defined by setting the last coordinate equal to $e$, the normal bundle in question can be identified with $Z_{(i_1, \ldots, i_{n-1})} \times_B T_{eB}(P_{\beta_i}/B)$. The weight of the action of $B$ on the tangent space $T_{eB}(P_{\beta_i}/B)$ at the identity coset $eB$ of $P_{\beta_i}/B$ is $-\beta_i$. Thus the normal bundle is precisely $L_i(\lambda_1, \ldots, \lambda_{n-1} - e, \lambda_n)$ as desired.

The important step towards the proof of the main result is the following, which states that the image of the valuation is contained inside the polytope $P_i(m)^{op}$.

Proposition 3.7. Let $i = (i_1, \ldots, i_n) \in \{1, 2, \ldots, r\}^n$ be a word and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ a multiplicity list. Let $Z_i$ and $L_{i,m}$ be the Bott-Samelson variety and line bundle specified by $i, m$ and let $\nu_i$, $\nu_i^*$ denote the geometric valuation specified by the flag $Y_i$ given above. Assume that $i$ corresponds to a reduced word decomposition. Then

$$
\nu_i(\mathcal{O}(Z_i, L_{i,m}) \setminus \{0\}) \subseteq P_i(m)^{op} \cap \mathbb{Z}_n.
$$

Proof. Let $0 \neq s \in H^0(Z_i, L_{i,m})$ with $\nu_i(s) = (x_1, x_{n-1}, \ldots, 1)$. We wish to show that $(x_1, \ldots, x_n) \in P_i(m)$, for which it is enough to show that $x_n \leq m_n$ and $x_k \leq a_k(x_{k+1}, \ldots, x_n)$ for $1 \leq k \leq n - 1$.

We first prove that $x_n \leq m_n$. Since $m_n \geq 0$ for all $i$, by [13] Corollary 3.3 the bundle $L_{i,m}$ is globally generated and hence effective. Moreover, $i$ is reduced by assumption, so we can conclude from [13] Proposition 3.5 that

$$
L_{i,m} \cong \mathcal{O}\left(\sum_{k=1}^n a_k Z_{i(k)}\right)
$$

for some integers $a_k \geq 0$, $1 \leq k \leq n$. Also since $x_n = \nu_1(s) = \text{ord}_{Z_i(n)}(s)$ is the order of vanishing of $s$ along $Y_1 = Z_{i(n)}$, we know $\text{div}(s) = x_n Z_{i(n)} + E$ for some effective divisor $E$. Since $\text{div}(s)$ is linearly equivalent to $\sum_{k=1}^n a_k Z_{i(k)}$, we may conclude

$$
E \sim -x_n Z_{i(n)} + \sum_{k=1}^n a_k Z_{i(k)}
$$

where $\sim$ denotes linear equivalence. Considering now the corresponding Chow classes, we may compare the (intersection) product of both sides of (3.1) with the class $[C_n] \in A^*(Z_i)$. The Chow ring $A^*(Z_i)$ and the classes $[Z_{i(k)}]$ have been extensively studied and it is known (cf. [2, 13], see also [18] Proposition 2.11) that $[C_n] \cdot [Z_{i(j)}] = \delta_{jn}$. Thus we obtain that the product (RHS of (3.1)) $\cdot [C_n] = -x_n + a_n$, whereas the product (LHS of (3.1)) $\cdot [C_n] = b_n \geq 0$ since $E$ is effective. Hence $x_n \leq a_n$. Furthermore, from [18] Proposition 2.11 and from basic properties of intersection products, we may also conclude that $a_n$ is the degree of the restriction $L_{i,m}|_{C_n}$ of the line bundle $L_{i,m}$ to
the curve $C_n$ (which is isomorphic to $\mathbb{P}^1$, so $A_0(C_n) \cong \mathbb{Z}$). By Lemma 3.5, above, this degree is precisely equal to $\langle m_n w_n, \beta_n^r \rangle = m_n$. Thus $x_n \leq m_n$ as was to be shown.

Next, we consider $x_{n-1} = r_2(s) = \text{ord}_s(s_1)$, where $0 \neq s_1 \in H^0(Y_1 = Z_i(n))$, $L_{1,m} \otimes \mathcal{O}(-x_n Z_i(n))|_{Y_1 = Z_i(n)}$, and $s_1$ is constructed as in $s$ in the fashion described above. Note that $Z_i(n) \cong Z_i(i_1, \ldots, i_{n-1})$. Thus, repeating the same argument as given above, we may deduce that $x_{n-1}$ is at most the degree of the restriction of the line bundle $L_{1,m} \otimes \mathcal{O}(-x_n Z_i(n))|_{Y_1 = Z_i(n)}$ to the curve $C_{n-1}$.

From Lemma 3.6, we know that the restriction of $L_{1,m}$ to $Z_i(n) \cong Z_i(i_1, \ldots, i_{n-1})$ is isomorphic to the line bundle $L_{i_1, \ldots, i_{n-1}}(m_1 \varpi_1, \ldots, m_{n-2} \varpi_{n-2}, m_n \varpi_n - x_n \beta_n)$. Thus, since $s_1$ is a non-zero global section, the line bundle in (3.2) above is effective. Thus by again applying Proposition 3.5, we can write it as $\mathcal{O}(\sum a_k Z_k)$ where $a_k \geq 0$. By proceeding with the same argument as before, since the degree of (3.2) along $C_{n-1}$ is precisely

$$\langle m_{n-1} w_{n-1} + m_n w_n - x_n \beta_n, \beta_n^r \rangle = A_{n-1}(x_n)$$

we may conclude $x_{n-1} \leq A_{n-1}(x_n)$. Continuing similarly, we obtain $(x_1, \ldots, x_n) \in P(i, m)$ as desired.

Remark 3.8. Note that since a scalar multiple $rm$ is also a multiplicity list for any positive integer $r$, it immediately follows from the above proposition that $\nu_Y(H^0(Z_i, L_{1,m}^{\oplus r}) \setminus \{0\}) \subseteq P(i, rm)^{op} \cap \mathbb{Z}^n$

for any $r \in \mathbb{N}$.

To complete the argument we need to recall the following fact from [6].

Proposition 3.9. If $(i, m)$ satisfies condition $(P)$, then $P(i, m)$ is a lattice polytope.

We are finally ready to prove the main result.

Proof of Theorem 3.2. We begin with the first claim of the theorem. It is elementary that if a valuation $\nu : V \to \Gamma$ (for $V$ a finite-dimensional complex vector space and $\Gamma$ a totally ordered group) has one-dimensional leaves, then the cardinality $|\nu(V \setminus \{0\})|$ of the image of $\nu$ is equal to $\dim_{\mathbb{C}}(V)$ [9, Proposition 2.6]. Since our valuation $\nu_Y$ has one-dimensional leaves on $R_1$, we conclude $|\nu_Y(R_1 \setminus \{0\})| = \dim_{\mathbb{C}}(R_1) = \dim_{\mathbb{C}}(H^0(Z_i, L_{1,m}^\oplus))$. On the other hand, we know from Proposition 3.7 that the image of $\nu_Y$ on $R_1$ is $H^0(Z_i, L_{1,m})$ must lie in $P(i, m)^{op} \cap \mathbb{Z}^n$. Proposition 2.5 implies $\dim_{\mathbb{C}}(H^0(Z_i, L_{1,m})) = \dim_{\mathbb{C}}(H^0(Z_i, L_{1,m}^\oplus))$, so we conclude that $S_1 := S(R) \cap \{1\} \times \mathbb{Z}^n$ (which by definition is the image of $\nu_Y : R_1 \setminus \{0\} \to P(i, m)^{op} \cap \mathbb{Z}^n$) is precisely $P(i, m)^{op} \cap \mathbb{Z}^n$. Here we identify $\{1\} \times \mathbb{Z}^n$ with projection to the second factor. This proves the first statement of the theorem.

By Remark 3.8 we also conclude that $S_0$ is equal to $P(i, rm)^{op} \cap \mathbb{Z}^n$. From the definition of the polytopes $P(i, m)$ it follows that $P(i, rm) = r \cdot P(i, m)$. This justifies the second statement of the theorem. Finally, the last statement of the theorem now follows directly from Definition 3.2 and Proposition 3.9.

4. Examples

In this section, we give several concrete examples in order to illustrate the results in the manuscript.

Let $G = SL(3, \mathbb{C})$ with Borel subgroup $B$ the upper-triangular matrices and $T$ the diagonal subgroup. The rank $r$ is 2 in this case and we let $\{a_1, a_2\}$ be the usual positive simple roots corresponding to the simple transpositions $s_1 = (12)$ and $s_2 = (23)$ in the Weyl group $W = S_3$.

For all of the examples below, we consider the Bott-Samelson variety $Z_i$ where $i = (1, 2, 1)$ corresponds to the reduced word decomposition $s_1 s_2 s_1$ of the longest element $w_0$ in $W = S_3$.

Example 4.1. Let $m = (1, 1, 1)$. Then it can be easily checked that $(i = (1, 2, 1), m = (1, 1, 1))$ satisfies condition $(P)$. The figure below illustrates the polytope $P(i, m)$ which is (up to a re-ordering of coordinates) the Newton-Okaunov body of $Z_{(1,2,1)}$ with line bundle $L_{(1,2,1),(1,1,1)}$ with respect to our valuation $\nu_Y$. For visualization purposes, the vertices of the polytope are indicated by black dots, while the other lattice points are indicated by white dots.
Example 4.2. Let \( m = (2, 1, 1) \). Again it can be checked easily that \((i, m)\) satisfies condition (P). The polytope \( P(i, m) \), i.e. the Newton-Okounkov body of \( Z_i \) and \( L_{i, m} \) (again up to reordering of coordinates), is illustrated below.

![Polytope](image)

As a final example we consider a choice of multiplicity list for which the pair \((i, m)\) does not satisfy condition (P); it can be seen below that the corresponding \( P(i, m) \) is not a lattice polytope.

Example 4.3. Let \( m = (0, 1, 1) \). Then one can check easily that \((i, m)\) does not satisfy condition (P). The polytope \( P(i, m) \) is illustrated below. The vertex which is not a lattice point is indicated in red. This example was also mentioned in Remark 2.77.

![Polytope](image)

References

[1] D. Anderson. Okounkov bodies and toric degenerations. *Math. Ann.*, 356(3):1183–1202, 2013.

[2] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[3] N. Fujita. Newton-Okounkov bodies for Bott-Samelson varieties and string polytopes for generalized Demazure modules, 2015, arXiv:1503.08916.

[4] M. Grossberg and Y. Karshon. Bott towers, complete integrability, and the extended character of representations. *Duke Math. J.*, 76(1):23–58, 1994.
M. Harada and K. Kaveh. Integrable systems, toric degenerations and Okounkov bodies. *Invent. Math.*, 2015.

M. Harada and J. J. Yang. Grossberg-Karshon twisted cubes and basepoint-free divisors, 2014, arXiv:1407.4147.

K. Kaveh. Crystal bases and Newton-Okounkov bodies, 2011, arXiv:1101.1687.

K. Kaveh and A. Khovanskii. Convex bodies and algebraic equations on affine varieties, 2008, http://arxiv.org/abs/0804.4095v1.

K. Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math.* (2), 176(2):925–978, 2012.

V. Kiritchenko. Divided difference operators on polytopes, 2013, arXiv:1307.7234.

V. Kiritchenko. Geometric mitosis, 2014, arXiv:1409.6097.

V. Lakshmibai, P. Littelmann, and P. Magyar. Standard monomial theory for Bott-Samelson varieties. *Compositio Math.*, 130(3):293–318, 2002.

R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.

P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116(1-3):329–346, 1994.

P. Littelmann. Paths and root operators in representation theory. *Ann. of Math.* (2), 142(3):499–525, 1995.

B. Pasquier. Vanishing theorem for the cohomology of line bundles on Bott-Samelson varieties. *J. Algebra*, 323(10):2834–2847, 2010.

N. Perrin. Small resolutions of minuscule Schubert varieties. *Compos. Math.*, 143(5):1255–1312, 2007.

D. Schmitz and H. Seppänen. Global Okounkov bodies for Bott-Samelson varieties, 2014, arXiv:1409.1857.

R. Vakil. The rising sea: fundamentals of algebraic geometry, http://math.stanford.edu/~vakil/216blog/.

[5] M. Harada and K. Kaveh. Integrable systems, toric degenerations and Okounkov bodies. *Invent. Math.*, 2015.
[6] M. Harada and J. J. Yang. Grossberg-Karshon twisted cubes and basepoint-free divisors, 2014, arXiv:1407.4147.
[7] K. Kaveh. Crystal bases and Newton-Okounkov bodies, 2011, arXiv:1101.1687.
[8] K. Kaveh and A. Khovanskii. Convex bodies and algebraic equations on affine varieties, 2008, http://arxiv.org/abs/0804.4095v1.
[9] K. Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math.* (2), 176(2):925–978, 2012.
[10] V. Kiritchenko. Divided difference operators on polytopes, 2013, arXiv:1307.7234.
[11] V. Kiritchenko. Geometric mitosis, 2014, arXiv:1409.6097.
[12] V. Lakshmibai, P. Littelmann, and P. Magyar. Standard monomial theory for Bott-Samelson varieties. *Compositio Math.*, 130(3):293–318, 2002.
[13] R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.
[14] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116(1-3):329–346, 1994.
[15] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math.* (2), 142(3):499–525, 1995.
[16] B. Pasquier. Vanishing theorem for the cohomology of line bundles on Bott-Samelson varieties. *J. Algebra*, 323(10):2834–2847, 2010.
[17] N. Perrin. Small resolutions of minuscule Schubert varieties. *Compos. Math.*, 143(5):1255–1312, 2007.
[18] D. Schmitz and H. Seppänen. Global Okounkov bodies for Bott-Samelson varieties, 2014, arXiv:1409.1857.
[19] R. Vakil. The rising sea: fundamentals of algebraic geometry, http://math.stanford.edu/~vakil/216blog/.

[5] M. Harada and K. Kaveh. Integrable systems, toric degenerations and Okounkov bodies. *Invent. Math.*, 2015.
[6] M. Harada and J. J. Yang. Grossberg-Karshon twisted cubes and basepoint-free divisors, 2014, arXiv:1407.4147.
[7] K. Kaveh. Crystal bases and Newton-Okounkov bodies, 2011, arXiv:1101.1687.
[8] K. Kaveh and A. Khovanskii. Convex bodies and algebraic equations on affine varieties, 2008, http://arxiv.org/abs/0804.4095v1.
[9] K. Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math.* (2), 176(2):925–978, 2012.
[10] V. Kiritchenko. Divided difference operators on polytopes, 2013, arXiv:1307.7234.
[11] V. Kiritchenko. Geometric mitosis, 2014, arXiv:1409.6097.
[12] V. Lakshmibai, P. Littelmann, and P. Magyar. Standard monomial theory for Bott-Samelson varieties. *Compositio Math.*, 130(3):293–318, 2002.
[13] R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.
[14] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116(1-3):329–346, 1994.
[15] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math.* (2), 142(3):499–525, 1995.
[16] B. Pasquier. Vanishing theorem for the cohomology of line bundles on Bott-Samelson varieties. *J. Algebra*, 323(10):2834–2847, 2010.
[17] N. Perrin. Small resolutions of minuscule Schubert varieties. *Compos. Math.*, 143(5):1255–1312, 2007.
[18] D. Schmitz and H. Seppänen. Global Okounkov bodies for Bott-Samelson varieties, 2014, arXiv:1409.1857.
[19] R. Vakil. The rising sea: fundamentals of algebraic geometry, http://math.stanford.edu/~vakil/216blog/.