THE $\ell$-ADIC $K$-THEORY OF A $p$-LOCAL FIELD

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Abstract. We verify a special case of a conjecture of G. Carlsson that describes the $\ell$-adic $K$-theory of a field $F$ of characteristic prime to $\ell$ in terms of the representation theory of the absolute Galois group $G_F$. This conjecture is known to hold in two cases; in this article we examine the second case, in which the field in question is the field of Laurent series over a finite field $\mathbb{F}_p((x))$.

1. Introduction

We verify a special case of a conjecture of G. Carlsson that describes the $\ell$-adic $K$-theory of a field $F$ of characteristic prime to $\ell$ in terms of the representation theory of the absolute Galois group $G_F$. This conjecture employs a completion construction of Carlsson’s called the derived completion [Car07a], which generalizes Bousfield and Kan’s $\ell$-adic completion [BK72] to the categories of ring and module spectra. Throughout this article, we will work in the category of $S$-algebras (which we will refer to as simply “ring spectra”) from [EKMM97]. If $A$ is a commutative ring spectrum, $M$ is an $A$-module, and $\gamma: A \to B$ is a map of commutative ring spectra, then we may take the derived completion of $M$ with respect to $\gamma$; we will write $M_\wedge$ or $M_B$ to denote this completion.

We will be taking derived completions of $K$-theory spectra of categories of representations. For any field $E$ equipped with the action of a group $G$, denote by $\text{Rep}_E[G]$ the category of finite dimensional continuous $E$-linear representations of $G$ and by $\text{Rep}_E(G)$ the analogous category of $E$-semilinear representations. We define an $E$-semilinear representation of $G$ to be an $E$-vector space $V$ with a $G$-action satisfying the relation $g(a \cdot v) = g(a) \cdot g(v)$, where $g \in G$, $a \in E$, $v \in V$, $g(a)$ is the image of $a$ under the action by $g$ on $E$, and $g(v)$ is the image of $v$ under the action by $g$ on $V$. It was shown in [EM06] that the spectra $K\text{Rep}_E(G)$ are commutative $S$-algebras.

Denote by $\mathfrak{S}_\ell$ the symmetric monoidal category with $\ell$ objects, only identity morphisms whose monoid structure is determined by the additive structure of $\mathbb{F}_\ell$. All derived completions will be taken with respect to a map

$$\alpha_\ell: K\text{Rep}_E(G) \longrightarrow K\mathfrak{S}_\ell \cong H\mathfrak{S}_\ell,$$

for a prime $\ell$ different from the characteristic of $E$. This map is induced by the augmentation functor from $\text{Rep}_E(G)$ to $\mathfrak{S}_\ell$, which sends a vector space to its $E$-dimension modulo $\ell$.

Let $F$ be a field and $k$ an algebraically closed subfield of $\overline{F}$. Since $k$ is algebraically closed, it is preserved by the absolute Galois group $G_F$ and hence there is a category $\text{Rep}_k(G_F)$. Define $\mathfrak{E}_k^F: \text{Rep}_k(G_F) \longrightarrow \text{Rep}_{\overline{F}}(G_F)$ to be the extension of scalars functor, where the action by $G_F$ on $k$ and on $\overline{F}$ is the Galois action
and where the action by $G_F$ on $E_k^F(V)$ is the obvious one. Denote by $K F^\wedge_\ell$ the Bousfield–Kan $\ell$-adic completion.

**Conjecture (Carlsson).** For all primes $\ell$ different from the characteristic of $F$, the induced map on completions (which we also denote $E_k^F$)

$$E_k^F: (K \text{Rep}_k(G_F))_{\alpha_\ell} \overset{\sim}{\longrightarrow} (K \text{Rep}_E(G_F))_{\alpha_\ell} \simeq (KF)^\wedge_\ell$$

is a weak equivalence.

Note that the weak equivalence on the right is a result of two facts. First, the category $\text{Rep}_E(G)$ is equivalent to the category $\text{ Vect}(E^G)$ of finite dimensional $E^G$-vector spaces that arises whenever $G$ acts faithfully on $E$. Second, the derived completion $(KF)^\wedge_\ell$ becomes the Bousfield–Kan $\ell$-adic completion when $\ell$ is prime to the characteristic of $F$ [Car07b].

In [Car07a], Carlsson proved that this conjecture holds when $F$ is a characteristic zero field containing an algebraically closed subfield $k$, provided $G_F$ is abelian.

Fix primes $\ell \neq p$, and for the remainder of this article, let $k := \mathbb{F}_p$ and define the field $F$ to be the compositum of all $p$ and $\ell$-prime extensions of the field of Laurent series $\mathbb{F}_p((x))$. Denote by $\mathcal{F}^t$ the maximal tame extension of $\mathcal{F}$ and $G^t_F$ the tame Galois group. Our main goal is to prove the following theorem.

**Theorem 5.4.** The map

$$E_k: (K \text{Rep}_k(G^t_F))_{\alpha_\ell} \overset{\sim}{\longrightarrow} (K \text{Rep}_E(G^t_F))_{\alpha_\ell} \simeq (KF)^\wedge_\ell,$$

where $\mathcal{F}^t$ is the maximal tame extension of $\mathcal{F}$, is a weak equivalence.

For simplicity, we will write $\mathcal{E} = E_k^\mathcal{F}$. Note that in this theorem we use the tame Galois group $G^t_F$, which we denote by $\mathcal{F}$ throughout this article, and the maximal tame extension $\mathcal{F}^t$, whereas the conjecture regards the absolute Galois group and the separable closure.

This substitution is justified by the fact that if $P$ is a normal $p$-power subgroup of a profinite group $G$, and $G$ acts on a characteristic $p$ field $E$, then there is a weak equivalence

$$K \text{Rep}_E(G/P) \simeq K \text{Rep}_E(G).$$

Combining this with (2) gives the conjectured equivalence $E_k^\mathcal{F}$.

The first step in our strategy towards proving this theorem will be to compute the $K$-groups $K_* \text{Rep}_k(\mathcal{F})$, (Section 4). We plug this data in to the derived completion’s algebraic to geometric spectral sequence, which converges to $\pi_*((K \text{Rep}_k(\mathcal{F}))_{\alpha_\ell})$.

In order to do this, we use results from “Semisimple skew group rings and their modules” [LR07], which is joint work with K. Ribet. We summarize the relevant portions of this article in Section 3. Applying these results yields the following propositions about the homotopy groups of $K \text{Rep}_k(\mathcal{F})$.

**Proposition 4.4.** The endomorphism rings of the irreducible representations in $\text{Rep}_k(\mathcal{F})$ are matrix rings over the fixed field $k^\mathcal{F}$ and thus $K_*\text{Rep}_k(\mathcal{F}) \cong (K_0\text{Rep}_k(\mathcal{F})) \otimes K_*(k^\mathcal{F}).$

In the sequel, define $L = k^\mathcal{F}$.

**Proposition 4.5.** There is an isomorphism of rings, $K_0\text{Rep}_k(\mathcal{F}) \cong \mathbb{Z}[\mathbb{Q}/\ell].$
Finally, we compute the homotopy groups of $(\text{KRep}_k \langle G \rangle)_\wedge$ and find that they are isomorphic to those of $(\text{K}\mathcal{F})_\wedge$.

**Proposition 4.1.** The homotopy groups of $(\text{Rep}_k \langle G \rangle)_\wedge$ are

$$\pi_n((\text{KRep}_k \langle G \rangle)_\wedge) \cong \pi_n((\text{K}F_p)_\wedge) \oplus \pi_n\Sigma((\text{K}\mathcal{F})_\wedge)$$

In Section 5, we show that the map $E$ induces the isomorphism of homotopy groups by factoring $E$ through the intermediate category

$$\text{colim}_{\mathcal{O}_i \in \text{OpNm}(\mathcal{G})} \text{Rep}_{\mathcal{O}_i} \langle \mathcal{G}/\mathcal{G}_i \rangle,$$

where $\text{OpNm}(\mathcal{G})$ is defined to be the opposite category of the category of open normal subgroups of $\mathcal{G}$, $\mathcal{O}_i$ is the ring of integers in the fixed field $\mathcal{F}_{\mathcal{G}_i}$, and $\text{Rep}_{\mathcal{O}_i} \langle \mathcal{G}/\mathcal{G}_i \rangle$ is the category of finitely generated $\mathcal{O}_i$-modules equipped with the continuous semi-linear action by $\mathcal{G}/\mathcal{G}_i$ on $\mathcal{F}_{\mathcal{G}_i}$.

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## 2. Completions

### 2.1. The Bousfield-Kan Completion.**

We briefly review the Bousfield-Kan completion [BK72]. Let $\mathcal{C}$ be a category.

**Definition 2.1.** A monad (or triple) on $\mathcal{C}$ is a functor $T: \mathcal{C} \to \mathcal{C}$ together with natural transformations $\eta: \text{id} \to T$ and $\mu: T \circ T \to T$ such that for any object $X \in \mathcal{C}$ the diagrams below commute.

\[
\begin{array}{ccc}
TX & \xrightarrow{T\eta(X)} & T \circ TX \\
\downarrow & & \downarrow T\eta(X) \\
TX & \xrightarrow{T\mu(X)} & T^2(X)
\end{array}
\]

\[
\begin{array}{ccc}
TX & \xrightarrow{T\eta(X)} & T \circ TX \\
\downarrow & & \downarrow T\eta(X) \\
TX & \xrightarrow{T\mu(X)} & T^2(X)
\end{array}
\]

Now let $\mathcal{C}$ be the category of simplicial sets and $R$ a ring. Define $F_R$ to be the composition of the forgetful functor with the free $R$-module functor applied levelwise. Then $F_R$ (together with the obvious natural transformations) is a monad on $\mathcal{C}$.
**Definition 2.3** (Bousfield-Kan). Let $X_{\bullet}$ be a simplicial set. Then the $R$-completion of $X_{\bullet}$, denoted $R_{\infty}X$, is defined to be the cosimplicial resolution of $X_{\bullet}$ relative to $F_R$. The $\ell$-adic completion of $X_{\bullet}$, denoted $X_{\ell}^\wedge$, is the $R$-completion for $R = \mathbb{F}_\ell$.

This construction can be extended to the category of spectra.

**Definition 2.4.** Suppose that $X = \{X_i\}$ is a spectrum. Then the $R$-completion of $X$, denoted $R_{\infty}X$, is defined to be the “spectrification” of the prespectrum $\{R_{\infty}(X_i)\}$. As before, the $\mathbb{F}_\ell$-completion is called the $\ell$-adic completion and is denoted $X_{\ell}^\wedge$.

The following propositions are about the homotopy groups of $\ell$-adically completed spectra.

**Proposition 2.5.** Let $X$ be a spectrum whose homotopy groups are all finitely generated. Then

$$\pi_n(X_{\ell}^\wedge) \cong \mathbb{Z}_\ell \otimes \pi_nX.$$  

*Proof.* This follows from Example 5.2 in [BK72]. \hfill \Box

**Proposition 2.6.** Let $F$ be a finite field of characteristic $p$ and $L \supseteq F$ an extension of degree prime to $\ell$. The inclusion $F \hookrightarrow L$ induces a weak equivalence,

$$\hat{(KF)}_{\ell} \cong \hat{(KL)}_{\ell}.$$  

*Proof.* Let $L$ be a finite extension of $F$ of degree prime to $\ell$. The composition of extension and restriction of scalars, $KF \to KL \to KF$ induces multiplication by $[L:F]$ on $\pi_*(KF)$. Since $[L:F]$ is prime to $\ell$, the $\ell$-primary components of $\pi_*KF$ are direct summands of the homotopy groups $\pi_*KL$.

Because $\pi_*KF$ and $\pi_*KL$ are both cyclic, they must be isomorphic. \hfill \Box

**Proposition 2.7.** The $\ell$-adic $K$-groups of $F$ are

$$\pi_*(K\mathcal{F})_{\ell}^\wedge \cong \pi_*(K\mathbb{F}_p)_{\ell}^\wedge \oplus \pi_*\Sigma(K\mathbb{F}_p)_{\ell}^\wedge.$$  

*Proof.* This is proven in Section 4.2 of [Car07b]. \hfill \Box

2.2. **The Derived Completion.** To define the derived completion, we observe that when $\gamma: A \to B$ is a map of commutative ring spectra, then the functor $(B \wedge_A -)$ together with the natural transformations

$$\eta: \text{id} \to (B \wedge_A -),$$

defined by $\eta = f \wedge \text{id}$ and

$$\mu: (B \wedge_A B \wedge_A -) \to (B \wedge_A -),$$

defined by multiplication on $B$, is a monad on the category Mod$(A)$ of $A$-modules. When $A$ is the sphere spectrum and $B$ is the Eilenberg-MacLane spectrum of a ring $R$, the resulting monad $(H(R) \wedge_S -)$ is equivalent in the category of spectra to the monad $F_R$. Thus for an $S$-module $M$ there is a weak equivalence of $S$-modules, $M_{HR}^\wedge \cong R_{\infty}M$. 


Denote by $\hat{B}$ the functorial fibrant replacement for $B$ in the category of commutative $A$-algebras, and by $\mathcal{R}^\bullet ((\hat{B} \wedge_A -), M)$ the functorial fibrant replacement for $\mathcal{R}^\bullet ((\hat{B} \wedge_A -), M)$ in the category of cosimplicial $A$-modules.

**Definition 2.8.** The derived completion $M_\gamma$ of $M$ with respect to the map $\gamma: A \to B$ is the total space, 

$$M_\gamma := \text{Tot}(\mathcal{R}^\bullet(\hat{B} \wedge_A -, M)).$$

The following propositions and theorems from [Car07a] are the most relevant for our purposes. We refer the reader to [Car07a] for more details.

**Proposition 2.9.** Let $A \to B$ be a map of commutative ring spectra and $M$ an $A$-module. Then the following statements all hold.

1. The construction $M \to M_B^\wedge$ is functorial for homomorphisms of $A$-modules. The map on $B$-completions induced by a homomorphism $f: M \to N$ of $A$-modules will be denoted by $f_B^\wedge$.
2. Let $f: M \to N$ be a homomorphism of $A$-modules, where $A$ is a commutative ring spectrum, and suppose that $id_B \wedge_A f$ is a weak equivalence of $B$-module spectra. Then the map on completions $f_B^\wedge$ is also a weak equivalence of spectra.
3. Let $M'' \to M \to M'$ be a cofibration sequence of $A$-module spectra. Then the sequence $(M'')_B^\wedge \to (M)_B^\wedge \to (M')_B^\wedge$ is a cofibration sequence up to homotopy.
4. Let $A'' \to A \to B$ be a diagram of commutative ring spectra, and let $M$ denote an $A$-module spectrum. Denote by $M_{A''}$ the module $M$ regarded as an $A''$-module. Suppose that the natural map $B \wedge_{A''}(M_{A''}) \to B \wedge_A M$ is a weak equivalence of spectra. Then the natural map $(M_{A''})_B \to M_B^\wedge$ is also an equivalence of spectra.
5. There is a natural map $\eta: M \to M_B^\wedge$.
6. Suppose $M$ is an $A$-module spectrum for which the $A$-module structure admits a $B$-module structure extending the given $A$-module structure. Then the natural map $\eta: M \to M_B^\wedge$ is an equivalence of spectra.

**Theorem 2.10.** Suppose that we have a diagram $A \to B \xrightarrow{\beta} B'$ of commutative ring spectra and a left $A$-module spectrum $M$. Suppose further that $A$, $B$, and $M$ are all $(-1)$-connected, that the homomorphism $\pi_0(\beta)$ is an isomorphism, and that the natural homomorphisms $\pi_0(A) \to \pi_0(B)$ and $\pi_0(A) \to \pi_0(B')$ are surjections. Then the natural homomorphism $M_B^\wedge \to M_B'^\wedge$ is an equivalence of $A$-module spectra.

**Theorem 2.11.** (Algebraic to geometric spectral sequence). Let $A \to B$ be a map of commutative ring spectra, with $\pi_0 A \to \pi_0 B$ surjective, and let $M$ be an $A$-module spectrum. The ring $\pi_0 A$ is commutative, the ring $\pi_0 B$ is a commutative $\pi_0 A$-algebra, and for each $i$, the abelian group $\pi_i M$ has a $\pi_0 A$-module structure. We may therefore construct the derived completion $(\pi_i M)_B^\wedge$ for each $i$. There is a second quadrant spectral sequence with $E_1^{s,t} = \pi_{s+2t}( (\pi_i M)_B^\wedge )$, converging to $\pi_{t+s}(M_B^\wedge)$.

The following example from [Car07a] will play an important role in our computation of the homotopy groups $\pi_* \left( (K\text{Rep}_K(\mathbb{S}))_{\alpha_{\ell}}^\wedge \right)$. 

Example 2.12. Denote by $R[\mathbb{Z}_\ell]$ the ring of isomorphism classes of complex representations of the group of $\ell$-adic integers. Then $R[\mathbb{Z}_\ell]$ can be expressed as a colimit

$$R[\mathbb{Z}_\ell] = \operatorname{colim}_n R[\mathbb{Z}/\ell^n].$$

For each finite representation ring, there is an isomorphism,

$$R[\mathbb{Z}/\ell^n] \cong \mathbb{Z}[\mu_{\ell^n}],$$

and thus $R[\mathbb{Z}_\ell] \cong \mathbb{Z}[\mu_{\ell\infty}]$, where $\mu_{\ell\infty}$ is the group of all $\ell$-power roots of unity.

Define $\alpha_\ell$ as in (1). Then the homotopy groups of the derived completion of the Eilenberg-MacLane spectrum $(HR[\mathbb{Z}_\ell])_{\alpha_\ell}$ relative to $\alpha_\ell$ are

$$\pi_*((HR[\mathbb{Z}_\ell])_{\alpha_\ell}) \cong \begin{cases} 
\mathbb{Z}_\ell & \text{for } i = 0 \text{ or } 1, \\
0 & \text{otherwise}.
\end{cases}$$

3. Semisimple skew group rings

In this section we summarize the relevant results of “Semisimple skew group rings and their modules” [LR07], which presents a means of describing the objects, endomorphisms, and tensor products in the category of finitely generated modules over a skew group ring. We will compute the ring $K_0$ of isomorphism classes of finitely generated modules over a skew group ring under the operations $\oplus$ and $\otimes_E$.

Suppose that $E$ is a field equipped with the action of a finite group $G$. Then we define the skew group ring $E\langle G \rangle$ to be the $E$-vector space with basis set $G$ endowed with the multiplication structure determined by $(\alpha g)(\beta h) = \alpha \beta gh$, for $\alpha$ and $\beta \in E$, for $g$ and $h \in G$, and where $\beta g$ is the image of $\beta$ under $g$. If $G$ acts trivially on $E$, then $E\langle G \rangle$ is the ordinary group ring $E[G]$. In our motivating example, the action by $G$ on $E$ is not trivial; however it is typically not faithful either, and hence the elements of $G$ that act trivially on $E$ form a proper normal subgroup $N$.

Throughout this section, we will assume that $N$ is abelian and therefore that the action by $G$ on $N$ factors through the quotient $G/N$, which we will denote by $\Delta$. The assumption that $N$ is abelian also implies that there is a cohomology group $H^2(G/N, N)$. The group $G$ is an extension of $G/N$ by $N$ and thus determines an equivalence class in $R^2(G/N, N)$.

We will also assume that the order of $G$ is invertible in $E$ and thus, by a generalization of Maschke’s Theorem, that $E\langle G \rangle$ is semisimple. Every $E\langle G \rangle$ module can therefore be broken down as a direct sum of simple modules. Every simple module is isomorphic to a simple module over a single factor in the Artin-Wedderburn decomposition of $E\langle G \rangle$ as a product of matrix rings

$$E\langle G \rangle = \prod_{i=1}^s M(d_i, D_i),$$

where $s$ and $d_i$ are positive integers and $D_i$ is a division ring (which will contain the fixed field $F := E^G$). This decomposition is unique; therefore, describing the objects and endomorphisms of $E\langle G \rangle$-modules is equivalent to finding the division rings $D_i$ and the corresponding dimensions $d_i$. Each $D_i$ is a central simple algebra over its center $Z_i$, so $D_i$ corresponds to an element of $\operatorname{Br}(Z_i)$.

The product $Z = \prod Z_i$ is the center of $E\langle G \rangle$. 
Proposition 3.1. The map
\[ \mu : E \otimes_F Z \longrightarrow E[N] \]
defined by
\[ e \otimes z \mapsto ez, \]
where \( e \in E \) and \( z \in Z \), is an \( E \)-algebra isomorphism.

Now define \( \text{Br}(Z) := \prod \text{Br}(Z_i) \). By a theorem of Auslander and Goldman [AG60] we can identify \( \text{Br}(Z) \) with the cohomology group \( H^2(G_F, (\mathcal{F} \otimes Z)^*) \). The skew group ring \( E\langle G \rangle \) thus corresponds to a 2-cocycle in \( H^2(G_F, (\mathcal{F} \otimes Z)^*) \). There is a natural map obtained by composing the maps
\[ H^2(G/N, N) \xrightarrow{\text{inf.}} H^2(G_F, N) \]
\[ \xrightarrow{\text{inc.}} H^2(G_F, F[N]^*) \]
\[ \xrightarrow{\mu} H^2(G_F, (\mathcal{F} \otimes Z)^*) \]
\[ \xrightarrow{\cong} \text{Br}(Z), \]
where the map “inf.” is inflation and “inc.” is inclusion. Call this composite \( \Phi \).

Theorem 3.2. The map \( \Phi \) sends \([G] \in H^2(G/N, N)\) to \([E\langle G \rangle] \in \text{Br}(Z)\).

Corollary 3.3. If \( \text{Br}(F) = 1 \), there is an isomorphism of rings,
\[ E(G) \cong M([E : F], Z). \]

Proposition 3.4. As a \( Z \)-module, \( E\langle G \rangle \) is free of rank \([E : F]^2\).

This describes \( E(G) \) up to isomorphism.

To compute the multiplicative structure of \( K_0(E(G)) \), we first observe that the set \( \text{Hom}_F(Z, \mathcal{F}) \) of \( F \)-algebra homomorphisms is endowed with a natural action by the absolute Galois group \( G_F \) which is defined by
\[ (g \ast \phi)(z) := g(\phi(z)) \]
for \( g \in G_F \) and \( \phi \in \text{Hom}_F(Z, \mathcal{F}) \). This action partitions \( \text{Hom}_F(Z, \mathcal{F}) \) into a disjoint union of orbits \( \text{Hom}_F(Z_i, \mathcal{F}) = \bigsqcup \text{Hom}(Z_i, \mathcal{F}) \).

Corollary 3.5. The map \( \mu \) induces a bijection of sets
\[ \mu_* : \text{Hom}(Z, \mathcal{F}) \xrightarrow{\cong} \text{Hom}(N, \mathcal{F}^*) \]
that is \( G_F \) equivariant if the action by \( G_F \) on \( \text{Hom}(N, \mathcal{F}^*) \) is defined in the natural way,
\[ (g \ast \phi)(n) := g(\phi(\mathcal{F}^{-1} n)), \]
where \( g \in G_F \), \( \phi \in \text{Hom}(N, \mathcal{F}^*) \), and \( n \in N \); the element \( \mathcal{F} \) is the image of \( g \) in the quotient \( G/N \cong \text{Gal}(E/F) \) of \( G_F \) and \( \mathcal{F}^{-1} n \) is the image of \( n \) under conjugation by \( \mathcal{F} \).
Under $\mu_*$, the orbits $\text{Hom}(Z, F)$ get sent to orbits of $\text{Hom}(N, F^\ast)$ which we call $O_i$; each of the $O_i$ therefore corresponds to a factor $M(d_i, D_i)$ of $E(G)$. Note that the field $Z_i$ is isomorphic to a fixed field of $F$ under the subgroup $\text{Stab}_{G_F}(\chi)$ of elements in $G_F$ that fix any character $\chi$ in $O_i$. The degree of $Z_i$ over $F$ is equal to the number of elements in $O_i$.

Now fix a common separable closure $F$ of $F$ and all separable extensions of $F$, and write $N^\lor := \text{Hom}(N, F^\ast)$. We will associate to every $E(G)$-module $W$ an element of the group ring $Z[N^\lor]$ as follows.

We start by restricting $W$ to an $E$-linear representation of $N$. Since $N$ is abelian, we can write $F \otimes_E W$ as a sum of 1-dimensional representations

$$F \otimes_E W \cong \bigoplus_{\chi \in N^\lor} U^\ast \chi,$$

where $m_\chi \in Z_{\geq 0}$. This decomposition is unique and suggests a map $\rho$ defined by

$$\rho(W) := \sum m_\chi \chi$$

from $K_0 E(G)$ to the group ring $Z[N^\lor]$.

**Proposition 3.6.** For every $E(G)$-module $W$, the image $\rho(W)$ is stable under the action by $G_F$ defined in equation (3).

**Proposition 3.7.** For a simple module $V_i$, the set of elements that appear with a nonzero coefficient in $\rho(V_i)$ is exactly the orbit $O_i$ of $N^\lor$, defined in Corollary 3.5. The map $\rho$ is therefore injective.

If $V_i$ is a simple module over the $i$th factor $M(d_i, D_i)$ of $E(G)$, then the center $Z_j$ of a different factor, where $1 \leq j \leq s$ and $j \neq i$, acts trivially on $V_i$. The elements of $Z_i$, on the other hand, act through the characters in $\text{Hom}(Z_i, F)$ by

$$z \cdot v = \chi(z)v,$$

where $z \in Z_i$, $v \in V$, and $\chi \in \text{Hom}(Z_i, F)$. By Corollary 3.5, these characters correspond exactly to those in $O_i$.

It follows immediately from Propositions 3.6 and 3.7 that for a simple module $V_i$, there is some positive integer $m_i$, which we call the “multiplicity” of $V_i$, such that

$$\rho(V_i) = m_i \sum_{\chi \in O_i} \chi.$$

**Corollary 3.8.** If $\text{Br}(F) = 1$, then $m_i = 1$ for all irreducible $E(G)$-modules $V_i$.

**Theorem 3.9.** The map $\rho$ is multiplicative with respect to $\otimes_E$, and hence an $E(G)$-module $W$ is irreducible exactly when its image in $Z[N^\lor]$ is

$$\rho(W) = m_i \sum_{\chi \in O_i} \chi,$$

for some $i$. If $W'$ is another module over $E(G)$, then

$$W \otimes_E W' \cong \bigoplus_{r_i} V_i^{r_i},$$

where $r_i \in Z_{\geq 0}$ and $V$ is a set containing one representative $V_i$ for each isomorphism class of simple modules, exactly when $\rho(W) \cdot \rho(W') = \sum_{r_i} \rho(V_i^{r_i})$. 
Example 3.10. Suppose that the Brauer group $\text{Br}(F)$ is trivial and therefore that $\text{Hom}(N, F^\times)$ breaks into singleton orbits under the action by $G_F$. Then the map $\rho$ induces an isomorphism of rings
$$\rho : K_0(E(G)) \cong \mathbb{Z}[N^\vee].$$

Example 3.11. We will show in Proposition 4.1 that $\mathcal{G} \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ where the action is the Frobenius action. Let $\mathcal{G}_i = \mathbb{Z}/\ell^i \mathbb{Z} \times \mathbb{Z}/\ell^i \mathbb{Z}$ be the quotient of the group $\mathcal{G}$ by the subgroup $\mathcal{G}_i = \mathbb{Z}/\ell^i \mathbb{Z} \times \mathbb{Z}/\ell^i \mathbb{Z}$, where $i > i'$ are positive integers. The integer $i'$ is defined to be the degree of the Galois extension $L(\zeta)/L$, where $\zeta$ is a primitive $\ell'$th root of unity. The group $\mathcal{G}_i$ acts on $(\mathbb{F}_p)[\ell^i \mathbb{Z} \ell]$ (which we denote $k_i$) in the obvious way, and the fixed field of this action is $L$.

Let $N_i := \mathbb{Z}/\ell^i \mathbb{Z}$ be the subgroup of elements of $\mathcal{G}_i$ that act trivially on $k_i$ and define $\Delta_i$ to be the quotient $\mathbb{Z}/\ell^i \mathbb{Z} \cong \mathcal{G}_i/N_i$. Let $\eta$ be a generator of $N_i$. The quotient group $\Delta_i$ (which is also a subgroup of $\mathcal{G}_i$) is generated by the Frobenius automorphism $\phi$ and sends an automorphism $\eta^r \in N_i$, where $r$ is an integer between $0$ and $\ell^i - 1$, to its $p$th power; that is, $\phi \eta^r \phi^{-1} = \eta^{pr}$. The action by the absolute Galois group $G_L$ on $N_i$ divides $N_i$ into singleton orbits. By Example 3.10,
$$K_0\left(k_i(\mathbb{Z}/\ell^i \mathbb{Z} \times \mathbb{Z}/\ell^i \mathbb{Z})\right) \cong \mathbb{Z}[\mu_i].$$

Now we consider the case where $G$ is a profinite group. Define $\text{OpNm}(G)$ to be the category whose objects are the open normal subgroups of $G$. For every pair $G''$ and $G'$ of such subgroups, the morphism set $\text{Mor}(G'', G')$ has exactly one element if $G'$ is a subgroup of $G''$ and is empty otherwise. Let $\text{OpNm}(G)^{\text{op}}$ be the opposite category. The functor
$$\text{Rep}_E(G/-) : \text{OpNm}(G)^{\text{op}} \rightarrow \text{ExactCategs}$$
takes an open normal subgroup $G' \subseteq G$ to the exact category $\text{Rep}_{E'}(G/G')$, where $E' = E^{G'}$.

Proposition 3.12. The category $\text{Rep}_E(G)$ is a colimit
$$\text{Rep}_E(G) \cong \text{colim}_{G' \in \text{OpNm}(G)^{\text{op}}} \text{Rep}_{E'}(G/G').$$

Each constituent $\text{Rep}_{E'}(G/G')$ of the colimit may be identified with a subcategory of $\text{Rep}_E(G)$ under the functor
$$\varphi' : \text{Rep}_{E'}(G/G') \rightarrow \text{Rep}_E(G),$$
which is defined by
$$\varphi'(W) = E \otimes_{E'} W.$$ The action by an element $g \in G$ on an element $e \otimes w \in E \otimes W$ is defined by
$$g(e \otimes w) = e \otimes \varphi(g(w)),$$ where $\varphi$ is the image of $e$ under the action by $g$, the element $\varphi(g(w))$ is the image of $g$ under the quotient map $G \rightarrow G/G'$, and $\varphi(w)$ is the image of $w$ under the action by $\varphi$.

Corollary 3.13. There is an isomorphism of rings,
$$K_0\text{Rep}_E(G) \cong \text{colim}_{G' \in \text{OpNm}(G)^{\text{op}}} K_0 E'(G/G').$$
4. The homotopy groups of \((K\text{Rep}_k(\mathcal{G}))^\wedge_{\alpha_t}\)

In this section we compute the homotopy groups of \(K\text{Rep}_k(\mathcal{G})\), which we will feed in to the algebraic to geometric spectral sequence to obtain the homotopy groups of the derived completion \((K\text{Rep}_k(\mathcal{G}))^\wedge_{\alpha_t}\).

Define \(\mathbb{N}\) to be the filtered category whose objects are the natural numbers and whose morphism sets \(\text{Mor}_\mathbb{N}(m,n)\) consist of one element when \(n\) divides \(m\) and are empty otherwise. For any ring \(R\) and any positive integer \(r\), we will denote by \(R[[x^{(1/r)}, x^{(1/s)}]]\) the colimit,

\[
R[[x^{(1/r)}, x^{(1/s)}]] := \colim_{i \in \mathbb{N} \text{ s.t. } r \mid i} R[[x^{1/i}]].
\]

If \(R\) is already a colimit of the form

\[
R = \colim_{i \in J} R'[[x^{1/i}]]
\]

for some ring \(R'\) and some subset \(J \subseteq \mathbb{N}\), then we denote by \(R[x^{(1/r)}]\) the colimit

\[
R[x^{(1/r)}] := \colim_{i \in J \text{ s.t. } r \mid i} R'[x^{1/i}].
\]

We define \(F((x^{(1/r)}))\) and \(F(x^{(1/r)})\) in the analogous way.

We start by describing the tame Galois group \(\mathcal{G}\). It is easy to see that there is a normal subgroup of \(\mathcal{G}\), which we denote by \(N\), that is isomorphic to \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{L}) \cong \mathbb{Z}_\ell\). The quotient by this normal subgroup, which permutes the roots of \(x\), will be denoted \(\Delta\).

**Proposition 4.1.** The tame Galois group of \(\mathcal{F}\) is the semidirect product \(\mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell\).

**Proof.** The quotient \(\Delta\) sits inside \(\mathcal{G}\) as the Galois group \(\Delta \cong \text{Gal}(\mathcal{F}/\mathcal{F}((x^{(1/f)})))\).

The subgroups \(N\) and \(\Delta\) intersect trivially and their product fixes exactly \(\mathcal{F}\). Therefore \(\mathcal{G}\) is the product of subgroups \(N \cdot \Delta\).

\(\square\)

**Remark 4.2.** The generator of \(\Delta\) (the Frobenius automorphism) acts on \(N\) by raising elements to the \(\ell\)-th power.

**Remark 4.3.** The subgroup \(\mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell\) is not necessarily normal in \(G_{\mathcal{F}_p(x)}\). This is because for all primes \(q\) different from \(\ell\) and \(p\), \(\mathcal{F}\) contains all \(q\)th roots of \(x\) but not necessarily all such roots of \(1\). For example, if \(q \mid p^\ell - 1\), then the order of \(p\) modulo \(q\) will be \(\ell\) and hence the \(q\)th roots of unity can only be adjoined by making a degree \(\ell\) extension. The extension \(\mathcal{F}/\mathcal{F}_p(x)\) is therefore not necessarily Galois.

Define \(\mathcal{G}_i\), \(\mathcal{G}'_i\), \(N_i\), and \(k_i\) as in Example 3.11. By Proposition 3.12 there is an equivalence of categories

\[
\text{Rep}_k(\mathcal{G}) \cong \colim_{\mathcal{G}'_i \in \text{OpNm}(\mathcal{G})} \text{Rep}_{k_i}(\mathcal{G}/\mathcal{G}'_i).
\]

**Proposition 4.4.** The ring of endomorphisms of every irreducible object in \(\text{Rep}_k(\mathcal{G})\) is \(\mathcal{L}\). The \(K\)-groups of \(\text{Rep}_k(\mathcal{G})\) are therefore

\[
K_*\text{Rep}_k(\mathcal{G}) \cong \left(K_0\text{Rep}_k(\mathcal{G})\right) \otimes K_*\mathcal{L}.
\]

**Proof.** The orbits of \(\text{Hom}(N_i, k^*)\) under the action by the absolute Galois group \(G_{\mathcal{L}}\) are all singleton orbits, and thus by Proposition 3.7, there is a one-to-one correspondence between characters of \(N_i\) and irreducible representations of \(\mathcal{G}'_i\). An
irreducible representation \( V \) that corresponds to the character \( \chi \in \text{Hom}(N_i, k_i^*) \) is a \( k_i \)-vector space on which \( N_i \) acts by
\[
n \cdot v = \chi(n)v,
\]
for \( n \in N_i \) and \( v \in V \). Since \( G_i \) is a semidirect product of \( N_i \) and \( \Delta_i \), every \( g \in G'_i \) can be written uniquely as a product \( g = nd \), for \( n \in N_i \) and \( d \in \Delta_i \). The action by \( g \) on \( ev \in V \) is determined by
\[
g(ev) = d \cdot \chi(n) \cdot d(v).
\]
The endomorphisms of \( V \) as a \( k_i \)-vector space are just \( k_i \); the endomorphisms of \( V \) that commute with \( G_i \) are exactly those that are fixed by the action of \( \Delta_i \), namely, the elements of \( L \).

Now since every object in \( \text{Rep}_k(G) \) has a canonical decomposition as a direct sum of irreducible representations, the result follows.

**Proposition 4.5.** The ring \( K_0\text{Rep}_k(G) \) of isomorphism classes of \( k \)-semilinear representations of \( G \) is
\[
K_0\text{Rep}_k(G) \cong \mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell].
\]

**Proof.** This follows immediately from Example 3.11 and Proposition 3.12.

Now that we have computed the homotopy groups of \( K\text{Rep}_k(G) \), we will use them to compute, in Proposition 4.9, the homotopy groups of the derived completion \( (K\text{Rep}_k(G))^\wedge_{\alpha} \). We will use the following well-known lemma which describes the homotopy groups of the \( \ell \)-adic completion \( (KL)^\wedge_{\ell} \).

**Lemma 4.6.** Let \( X \) be a spectrum with homotopy groups \( M_n := \pi_n X \), and let \( U_n \subseteq M_n \) be a uniquely \( \ell \)-divisible part of \( M_n \). Then if \( M_n/U_n \) is finitely generated, the map \( \beta: X \to X^\wedge_{\ell} \) induces a weak equivalence
\[
\pi_n(X^\wedge_{\ell}) \cong M_n/U_n \otimes \mathbb{Z}_\ell.
\]

The next two technical lemmas simplify our computations of the homotopy groups of derived completions.

**Lemma 4.7.** Let \( \alpha: \mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell] \to \mathbb{F}_\ell \) be the mod-\( \ell \) augmentation map and let \( X \) be an \( H(\mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell]) \)-module. Let \( M_n := \pi_n(X) \), and define \( U_n \subseteq M_n \) to be the uniquely \( \ell \)-divisible part of \( M_n \). Then if \( M_n/U_n \) is finitely generated, the map \( \beta: X \to X^\wedge_{\ell} \) induces a weak equivalence
\[
(\mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell] \otimes \pi_* X)^\wedge_{\alpha} \xrightarrow{\cong} (\mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell] \otimes \pi_* (X^\wedge_{\ell}))^\wedge_{\alpha}
\]
on derived completions.

**Proof.** By Lemma 4.6,
\[
\pi_n(X^\wedge_{\ell}) \cong M_n/U_n \otimes \mathbb{Z}_\ell.
\]

There is an isomorphism
\[
\mathbb{F}_\ell \otimes M_n \cong \mathbb{F}_\ell \otimes M_n/U_n \otimes \mathbb{Z}_\ell.
\]
and hence, by a theorem of Elmendorf-Kriz-Mandell-May [EKMM97], \( \beta \) induces weak equivalences between the \( n \)-simplices of the derived completions.
Lemma 4.8. Let $R \to S$ be a map of commutative rings and $M$ an $R$-module with no $\mathbb{Z}$-torsion. Then if $A$ is a finitely generated abelian group,

$$\pi_* \left( (M \otimes A)_S^\wedge \right) \cong \pi_*(M_S^\wedge) \otimes A.$$

Proof. Let $0 \to \mathbb{Z}^n \xrightarrow{i} \mathbb{Z}^m \to A \to 0$ be a free resolution of $A$. The result now follows immediately from the fact that the derived completion and $\pi_*$ preserve fiber sequences. \qed

Proposition 4.9. The algebraic to geometric spectral sequence for $(K\text{Rep}_k \langle G \rangle)^\wedge_{\alpha_{\ell}}$ collapses at the $E_1$-term and thus $(K\text{Rep}_k \langle G \rangle)^\wedge_{\alpha_{\ell}}$ has homotopy groups

$$\pi_i \left( (K\text{Rep}_k \langle G \rangle)^\wedge_{\alpha_{\ell}} \right) \cong \pi_i \left( (K\mathbb{F}_p)^\wedge_{\ell} \right) \oplus \pi_{i-1} \left( (K\mathbb{F}_p)^\wedge_{\ell} \right).$$

Proof. The $E_1$-term of the algebraic to geometric spectral sequence is

$$E^{s,t}_1 = \pi_{s+2t} \left( \left( \mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell] \otimes \pi_{-t} K\mathbb{L} \right)^\wedge_{\ell} \right).$$

By Lemmas 4.6 and 4.8 along with Example 2.12, this becomes

$$E^{s,t}_1 = \begin{cases} \pi_{-t} \left( (K\mathbb{L})^\wedge_{\ell} \right) & \text{when } s + 2t = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

Our $E_1$ page is therefore as in the diagram below.
By Proposition 2.6, the diagonal lines each correspond to a copy of $\pi_* \left( (KF_p)_\ell^\wedge \right)$. All of the differentials vanish, and hence the spectral sequence collapses. Therefore the homotopy groups of the derived completion are

$$\pi_i \left( (K\text{Rep}_k \langle S \rangle)_{\alpha_\ell}^\wedge \right) \cong \pi_i \left( (KF_p)_\ell^\wedge \right) \oplus \pi_{i-1} \left( (KF_p)_\ell^\wedge \right).$$

\[ \square \]

5. The map $E$

In the proof of Theorem 5.4, we will follow a strategy similar to the one found in Section 4 of [Car07b]. Define $k_i$, $G_i$ and $S_i$ as in example 3.11, let $F_i := \langle \mathcal{F}_i \rangle^{S_i}$ and let $O_i$ be the ring of integers in $F_i$. Denote by $\text{Rep}_{O_i}(S_i)$ the category of finitely generated $O_i$-modules equipped with the continuous semilinear action by $S_i$ that is inherited from the natural action by $S_i$ on $F_i$.

The categories $\text{colim} \text{Rep}_{O_i}(S_i)$ and $\text{colim} \text{Rep}_{F_i}(S_i)$ are closed under tensor products with objects in $\text{Rep}_k(S)$, and thus their $K$-theory spectra are both $K\text{Rep}_k(S)$-modules. We may therefore take their derived completions across the augmentation map $\alpha_\ell : K\text{Rep}_k(S) \to K\tilde{\mathcal{F}}_\ell$ (equation 1). As in [Car07b], we will show that the intermediate category $\text{colim} \text{Rep}_{O_i}(S_i)$ satisfies the two conditions below. These conditions will be proven in Propositions 5.2 and 5.3, respectively.

1. Extension of scalars induces a weak equivalence of completed $K$-theory spectra,

$$\left( K\text{Rep}_k(S) \right)_{\alpha_\ell}^\wedge \xrightarrow{\sim} \left( \text{colim} K\text{Rep}_{O_i}(S_i) \right)_{\alpha_\ell}^\wedge.$$

2. Extension of scalars induces a weak equivalence of completed $K$-theory spectra,

$$\left( \text{colim} K\text{Rep}_{O_i}(S_i) \right)_{\alpha_\ell}^\wedge \xrightarrow{\sim} \left( K\text{Rep}_{F_i}(S) \right)_{\alpha_\ell}^\wedge \cong \left( K\tilde{\mathcal{F}}_\ell \right)_{\alpha_\ell}^\wedge.$$

We start by proving a fact about $K$-theory spectra of rings of the form $R = E[[x]]/\langle G \rangle$ where $E$ is a field, $G$ is a finite group, and $r$ is a positive integer. Denote by $\text{Mod} (R)$ the category of finitely generated projective $R$-modules, and by $\text{NoTor} (R) \subseteq \text{Mod} (R)$ the full subcategory of finitely generated torsion free projective $R$-modules.

**Lemma 5.1.** The inclusion functor $\text{NoTor} (R) \hookrightarrow \text{Mod} (R)$ induces a weak equivalence of $K$-theory spectra.

**Proof.** Let $M$ be a projective $R$-module on $n$ generators, where $n$ is a positive integer. Then we can construct a surjective map $R^n \twoheadrightarrow M$. The kernel of this map is torsion free, and thus we have a length 2 resolution of $M$ by torsion free modules. The result now follows by “Reduction by Resolution,” Theorem 3.3 of [Qui73]. \[ \square \]

**Proposition 5.2.** Extension of scalars induces a weak equivalence,

$$\left( \text{colim} K\text{Rep}_{k_i}(S_i) \right)_{\alpha_\ell}^\wedge \xrightarrow{\sim} \left( \text{colim} K\text{Rep}_{O_i}(S_i) \right)_{\alpha_\ell}^\wedge.$$

**Proof.** First we show that the map $t_i : \text{Rep}_{k_i}(S_i) \xrightarrow{G_i(S_i) \otimes k_i(S_i)} \text{Rep}_{O_i}(S_i)$ induces an isomorphism on $\pi_0$. 

---

---
The categories $\text{Rep}_k\langle S_i \rangle$ and $\text{Rep}_\mathcal{O}\langle S_i \rangle$ can be identified with the categories $\text{Mod}(k_i\langle S_i \rangle)$ and $\text{Mod}(\mathcal{O}_i\langle S_i \rangle)$, respectively, since the group $S_i$ is finite. By Lemma 5.1, it suffices to show that $\tilde{t}_i$ induces an isomorphism

$$\tilde{t}_i: \pi_0 K\text{NoTor}(k_i\langle S_i \rangle) \xrightarrow{\cong} \pi_0 K\text{NoTor}(\mathcal{O}_i\langle S_i \rangle).$$

Clearly, $\tilde{t}_i$ is an injection on the set of isomorphism classes of objects. To see that it is a surjection, we start with a module $M$ over the ring $\mathcal{O}_i\langle S_i \rangle$. Let $I$ be the kernel of the quotient map $\mathcal{O}_i\langle S_i \rangle \to k_i\langle S_i \rangle$; $I$ is generated by the fractional powers of $x$. The module $IM$ is closed under multiplication by $\mathcal{O}_i\langle S_i \rangle$, and hence the quotient map $M \xrightarrow{\sim} M/IM$ is a map of $\mathcal{O}_i\langle S_i \rangle$-modules. Both $M$ and $M/IM$ are, by restriction, modules over $k_i\langle S_i \rangle$ as well, and since $k_i\langle S_i \rangle$ is semisimple, the map $\gamma$ splits. Therefore we have a section $s: M/IM \to M$. This map can be extended to a map of $\mathcal{O}_i\langle S_i \rangle$-modules,

$$S: \mathcal{O}_i\langle S_i \rangle \otimes_{k_i\langle S_i \rangle} M/IM \to M.$$  

Now as a consequence of Nakayama’s Lemma, a homomorphism $f: X \to Y$ of free modules over a complete local ring with maximal ideal $M$ is an isomorphism exactly when the induced map on the quotient is.

Since $M$ and $\mathcal{O}_i\langle S_i \rangle \otimes_{k_i\langle S_i \rangle} M/IM$ are free over $\mathcal{O}_i$ and the induced map on quotients

$$(\mathcal{O}_i\langle S_i \rangle \otimes_{k_i\langle S_i \rangle} M/IM)/I(\mathcal{O}_i\langle S_i \rangle \otimes_{k_i\langle S_i \rangle} M/IM) \to M/IM$$

is an isomorphism, the map $S$ must be an isomorphism of $\mathcal{O}_i$-modules. Thus every module over $\mathcal{O}_i\langle S_i \rangle$ is “extended” from $k_i\langle S_i \rangle$.

Next we show that the endomorphism rings of the irreducible modules over $\mathcal{O}_i\langle S_i \rangle$ are of the form $L[[x^{(1/p)}, x^{(1/\ell+1)}]]$ by examining the irreducible $k_i\langle S_i \rangle$-modules.

By Example 3.11 the irreducible $k_i\langle S_i \rangle$-modules are in bijective correspondence with the group $\text{Hom}(N_i, k)$, where $N_i = \mathbb{Z}/\ell^r\mathbb{Z}$. Suppose that $M_\alpha$ is an irreducible $k_i\langle S_i \rangle$ module corresponding to a character $\alpha \in \text{Hom}(N_i, k)$. Then because $S_i$ acts on $M_\alpha$ through $\alpha$, the tensor product $M'_\alpha := \mathcal{O}_i\langle S_i \rangle \otimes_{k_i\langle S_i \rangle} M_\alpha$ is isomorphic as an $\mathcal{O}_i\langle S_i \rangle$-module to $\mathcal{O}_i$. The endomorphisms of $M'_\alpha$ over $\mathcal{O}_i\langle S_i \rangle$ are exactly the elements of $M'_\alpha$ that commute with the action by $\mathcal{O}_i\langle S_i \rangle$—namely, $L[[x^{(1/p)}, x^{(1/\ell+1)}]]$.

The homotopy groups of $K\text{Rep}_{\mathcal{O}_i}\langle S_i \rangle$ are therefore

$$\pi_* K\text{Rep}_{\mathcal{O}_i}\langle S_i \rangle = K_0 \text{Rep}_{\mathcal{O}_i}\langle S_i \rangle \otimes K_* L[[x^{(1/p)}, x^{(1/\ell+1)}]]$$

and thus the homotopy groups of the colimit are

$$\pi_* (\text{colim } K\text{Rep}_{\mathcal{O}_i}\langle S_i \rangle) = \mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell] \otimes K_* L[[x^{(1/p)}]].$$

We now have the commutative diagram below. We set $\mathcal{L} := \mathbb{Z}[\mathbb{Q}_\ell/\mathbb{Z}_\ell]$ in all of the diagrams of this proof for typesetting purposes.

$$\begin{array}{ccc}
\pi_* K\text{Rep}_k\langle S \rangle & \cong & 3 \otimes \pi_* K\mathcal{L} \\
\downarrow_{t_*} & & \downarrow_{(t_*)^\wedge} \\
\pi_* \text{colim } K\text{Rep}_{\mathcal{O}_i}\langle S_i \rangle & \cong & 3 \otimes \pi_* K\mathcal{L}[[x^{(1/p)}]] \\
\downarrow_{q} & & \downarrow_{(q)^\wedge} \\
\pi_* K\text{Rep}_k\langle S \rangle & \cong & 3 \otimes K\mathcal{L} \\
\end{array}$$
The horizontal arrows are induced by the natural maps that come from the Bousfield completion, the map \( t \) is defined to be the colimit of the \( t_i \), and the map \( t_* \) is the induced map on homotopy groups. The map \( q \) is induced by the quotient map \( \mathcal{L}[[x^{1/p}]] \to \mathcal{L} \) and, by a theorem of Suslin [Sus84], induces a weak equivalence on the \( \ell \)-adic completions. The composition \( q \circ t \) is the identity and hence if \( (q)_\wedge^\wedge \) is a weak equivalence, \( (t_\wedge^\wedge) \wedge^\wedge \) must also be a weak equivalence. The maps in the top two rows of the diagram above induce the diagram of \( E_1 \)-terms for the algebraic to geometric spectral sequence below.

\[
\begin{align*}
\pi_{s+2t} \left( \left( 3 \otimes \pi_{-t} \mathcal{K} \mathcal{L} \right) \wedge^\wedge \right) & \xrightarrow{\epsilon(t)} \pi_{s+2t} \left( \left( 3 \otimes \pi_{-t} \mathcal{K} \mathcal{L}_t \right) \wedge^\wedge \right) \\
\pi_{s+2t} \left( \left( 3 \otimes \pi_{-t} \mathcal{K} \mathcal{L}[[x^{1/p}]] \right) \wedge^\wedge \right) & \xrightarrow{\epsilon(t)} \pi_{s+2t} \left( \left( 3 \otimes \pi_{-t} \mathcal{K} \mathcal{L}[[x^{1/p}]]_t \right) \wedge^\wedge \right).
\end{align*}
\]

By Lemma 4.7 the horizontal maps are isomorphisms on \( E_1 \)-terms. Therefore the map \( \epsilon(t) \) of \( E_1 \)-terms is an isomorphism. Since by Lemma 4.9, the spectral sequence collapses at \( E_1 \), the map \( t \) induces a weak equivalence,

\[
(K \text{Rep}_k(\langle \mathfrak{g} \rangle))^\wedge_{\alpha_t} \xrightarrow{\simeq} (\text{colim} \, K \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle))^\wedge_{\alpha_t}.
\]

\[\square\]

**Proposition 5.3.** *Extension of scalars induces a weak equivalence of completed K-theory spectra,*

\[
(\text{colim} \, K \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle))^\wedge_{\alpha_t} \xrightarrow{\simeq} (K \text{Rep}_{\mathcal{T}_i}(\langle \mathfrak{g} \rangle))^\wedge_{\alpha_t} \cong K F^\wedge_{\alpha_t}.
\]

*Proof.* As previously mentioned, the weak equivalence \((K \text{Rep}_{\mathcal{T}_i}(\langle \mathfrak{g} \rangle))^\wedge_{\alpha_t} \simeq (K F)^\wedge_{\alpha_t}\) is an immediate consequence of the equivalence of categories \( \text{Rep}_{\mathcal{T}_i}(\langle \mathfrak{g} \rangle) \cong \text{Mod}(F) \).

Therefore we need only show that

\[
(\text{colim} \, K \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle))^\wedge_{\alpha_t} \to (K \text{Rep}_{\mathcal{T}_i}(\langle \mathfrak{g} \rangle))^\wedge_{\alpha_t}
\]

is a weak equivalence. Since \( \mathfrak{g}_i \) is finite, the category \( \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle) \) can also be thought of as a category of finitely generated modules over the skew group ring \( \mathcal{O}_i(\langle \mathfrak{g}_i \rangle) \). In this ring, inverting all powers of \( x \) (including fractional ones) yields the ring \( \mathcal{T}_i \), and hence we can use Corollary 5.12 from [Swa68] and Quillen’s localization theorem [Qui73] to conclude that there is a homotopy fiber sequence of \( K \)-theory spectra,

\[
K \mathcal{T}_i \to K \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle) \to K \text{Rep}_{\mathcal{T}_i}(\langle \mathfrak{g}_i \rangle),
\]

where \( \mathcal{T}_i \) is the full subcategory of modules in \( \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle) \) that localize to 0. These are precisely the modules that are annihilated by some (possibly fractional) power of \( x \).

The category \( \text{colim} \, \mathcal{T}_i \) is closed under the tensor product with objects in \( \text{Rep}_k(\langle \mathfrak{g} \rangle) \). We may therefore take its derived completion across the map \( \alpha \) (see equation 1). Since colimits and derived completions preserve fiber sequences, we have a fiber sequence of \( K \text{Rep}_k(\langle \mathfrak{g} \rangle) \)-modules

\[
(\text{colim} \, K \mathcal{T}_i)^\wedge_{\alpha_t} \to (\text{colim} \, K \text{Rep}_{\mathcal{O}_i}(\langle \mathfrak{g}_i \rangle))^\wedge_{\alpha_t} \to (\text{colim} \, K \text{Rep}_{\mathcal{T}_i}(\langle \mathfrak{g}_i \rangle))^\wedge_{\alpha_t}.
\]

Therefore it suffices to show that \((\text{colim} \, K \mathcal{T}_i)^\wedge_{\alpha_t}\) is contractible.
Let \( \mathcal{T} := \text{colim} \mathcal{T}_i \), let \( \mathcal{R}_i := \text{Rep}_{K_i}(\mathcal{G}_i) \), and let \( \mathcal{R} := \text{colim} \mathcal{R}_i \). The derived completion \( (K\mathcal{T})^\wedge_{K_i} \) is the cosimplicial \( K\mathcal{R} \)-module whose \( n \)-simplices are obtained by \( n + 1 \) applications of the functor \((\mathbb{H}_e \wedge_{K\mathcal{R}} -)\) to \( K\mathcal{T}_i \). It suffices to show that \( \mathbb{H}_e \wedge_{K\mathcal{R}} K\mathcal{T}_i \) is isomorphic to \( \text{colim}(\mathbb{H}_e \wedge_{K\mathcal{R}_i} K\mathcal{T}_i) \), is contractible.

We will now show that for every \( i \), the \( K \)-theory spectrum \( K\mathcal{T}_i \) is weakly equivalent to \( K\mathcal{R}_i \). Let \( X \subset \mathbb{N} \) be the subcategory of positive integers whose reciprocal occurs as an exponent of \( x \) in the ring \( \mathbb{O}_i \). For every \( s \in X \) define \( Z_i(s) \subset \mathcal{T}_i \) to be the full subcategory of modules that are killed by \( x^{1/s} \). Then by devissage [Qui73], the \( K \)-theory spectrum \( KZ_i(s) \) is weak equivalent to \( K\mathcal{T}_i \). On the other hand, the category \( Z_i(s) \) is equivalent to the category \( \text{Rep}_{O_i[x^{1/s}]}(\mathcal{G}_i) \) via the forgetful functor.

Because for every positive integer \( r \), the category \( Z_i(rs) \) is a full subcategory of \( Z_i(s) \), the functor \( Z_i(-) \) takes the category \( X \) contravariantly to the category of full subcategories of \( \mathcal{G}_i \); it is therefore a cofiltered diagram. The category \( \mathcal{R}_i \) maps via \((\mathcal{O}_i[x^{1/s}] \otimes_{K_i} -)\) to \( \text{Rep}_{O_i[x^{1/s}]}(\mathcal{G}_i) \), and it is easy to see that it is the limit

\[
\mathcal{R}_i = \text{lim}_{s \in X} \text{Rep}_{O_i[x^{1/s}]}(\mathcal{G}_i).
\]

Applying the functor \( K \) yields the commutative diagram

\[
\begin{array}{cccccc}
K\mathcal{Z}_i & \longrightarrow & \cdots & \overset{\simeq}{\longrightarrow} & K\mathcal{Z}_i(sr) & \overset{\simeq}{\longrightarrow} & K\mathcal{Z}_i(s) & \overset{\simeq}{\longrightarrow} & K\mathcal{T}_i \\
\downarrow{\simeq} & & & & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
K\mathcal{R}_i & \longrightarrow & \cdots & \longrightarrow & K\text{Rep}_{O_i[x^{1/s}]}(\mathcal{G}_i) & \longrightarrow & K\text{Rep}_{O_i[x^{1/s}]}(\mathcal{G}_i)
\end{array}
\]

where \( Z_i \) is defined to be the limit \( \text{lim}_{s \in \mathbb{N}} Z_i(s) \). This gives us the weak homotopy equivalence,

\[
K\mathcal{T}_i \simeq K\mathcal{Z}_i \simeq K\mathcal{R}_i.
\]

Now, since \( \pi_* \) commutes with filtered colimits and smash products, the above equivalence yields

\[
(5) \quad \pi_* (\mathbb{H}_e \wedge_{K\mathcal{R}} K\mathcal{T}) \cong \text{colim} \pi_* (\mathbb{H}_e \wedge_{K\mathcal{R}_i} K\mathcal{Z}_i) \cong \text{colim} \pi_* \mathbb{H}_e.
\]

Since \( \pi_0 \mathbb{H}_e = 0 \) for all \( n \neq 0 \), we restrict our attention to \( \pi_0 \). Because \( Z_i \) is closed under tensor product, \( K\mathcal{Z}_i \) is actually a ring spectrum, so \( \pi_0 K\mathcal{Z}_i \) is a ring, and

\[
\pi_0 (\mathbb{H}_e \wedge_{K\mathcal{R}_i} K\mathcal{Z}_i) = \pi_0 \mathbb{H}_e \otimes_{\pi_0 K\mathcal{Z}_i} \pi_0 K\mathcal{Z}_i,
\]

where the tensor product is taken over \( \pi_0 K\mathcal{R}_i \). As a free \( \pi_0 K\mathcal{R}_i \)-module, \( \pi_0 K\mathcal{Z}_i \) is generated by a 1-dimensional \( k_i \)-vector space \( V_i \) that is annihilated by all (possibly fractional) powers of \( X \) in \( \mathcal{O}_i \). Under \( \tau^i \), the module \( V_i \) (which corresponds to 1 in \( \pi_0 \mathbb{H}_e \)) gets sent to the module

\[
V_i \otimes_{\mathcal{O}_i} \mathcal{O}_j = k_f \left[ \left[ x^{(1/\ell)}, x^{(1/p)} \right] \right] x^{1/\ell}
\]

(which corresponds to \( \ell \) in the ring \( \pi_0 \mathbb{H}_e = \mathbb{F}_e \)), and thus all of the maps in the colimit equation (5) are zero maps. Therefore \( \pi_* (\mathbb{H}_e \wedge_{K\mathcal{R}} K\mathcal{T}) \) is contractible.

The \( n \)-simplices of \( (K\mathcal{T})^\wedge_{K_i} \) are of the form \( \mathbb{H}_e \wedge_{K\mathcal{R}} \cdots \wedge_{K\mathcal{R}} \mathbb{H}_e \wedge_{K\mathcal{R}} K\mathcal{T} \) so they are all contractible as well. Since a level-wise weak equivalence of cosimplicial objects induces a weak equivalence on total spaces, \( K\mathcal{T}^\wedge_{K_i} \) is contractible, and thus

\[
(\text{colim} K\text{Rep}_{O_i}(\mathcal{G}_i))_{\alpha_i}^\wedge \overset{\simeq}{\longrightarrow} (K\text{Rep}_{\mathcal{T}}(\mathcal{G}))_{\alpha_i}^\wedge.
\]
The following theorem follows immediately from the two preceding propositions.

**Theorem 5.4.** *Extension of scalars induces a weak equivalence of completed $K$-theory spectra,*

\[
(K\text{Rep}_k(\mathcal{F}))^\wedge_{\alpha\ell} \xrightarrow{\cong} (K\text{Rep}_{\mathcal{F}'}(\mathcal{F}))^\wedge_{\alpha\ell} \cong (K\mathcal{F})^\wedge_{\alpha\ell}.
\]
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