A NEW MATRIX LI-YAU-HAMILTON
ESTIMATE FOR KÄHLER-RICCI FLOW

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Abstract. In this paper we prove a new matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow. The form of this new Li-Yau-Hamilton estimate is obtained by the interpolation consideration originated in [Ch1]. This new inequality is shown to be connected with Perelman’s entropy formula through a family of differential equalities. In the rest of the paper, We show several applications of this new estimate and its linear version proved earlier in [CN]. These include a sharp heat kernel comparison theorem, generalizing the earlier result of Li and Tian, a manifold version of Stoll’s theorem on the characterization of ‘algebraic divisors’, and a localized monotonicity formula for analytic subvarieties.

Motivated by the connection between the heat kernel estimate and the reduced volume monotonicity of Perelman, we prove a sharp lower bound heat kernel estimate for the time-dependent heat equation, which is, in a certain sense, dual to Perelman’s monotonicity of the ‘reduced volume’. As an application of this new monotonicity formula, we show that the blow-down limit of a certain type III immortal solution is a gradient expanding soliton. In the last section we also illustrate the connection between the new Li-Yau-Hamilton estimate and the earlier Hessian comparison theorem on the ‘reduced distance’, proved in [FIN].

§0 Introduction.

In [LY], Peter Li and S.-T. Yau developed the fundamental gradient estimates for positive solution \( u(x,t) \) to the heat equation \( \left( \frac{\partial}{\partial t} - \Delta \right) u(x,t) = 0 \). On a complete Riemannian manifold \( M \) with nonnegative Ricci curvature, they also derived a sharp form of the classical Harnack inequality (cf. [Mo]) out of their gradient estimates. Later in [H2], Richard Hamilton extended the estimate of Li-Yau to the full matrix version on the Hessian of \( \log u \), under the stronger assumption that \( M \) is Ricci parallel and of nonnegative sectional curvature. More recently, in [CN], H.-D. Cao and the author observed that if \( M \) is a Kähler manifold with nonnegative bisectional curvature, one can obtain the Hamilton’s matrix version estimate on the complex Hessian of \( \log u \) without the assumption of Ricci being parallel. Following [NT1], We called our estimate in [CN] a Li-Yau-Hamilton estimate (or inequality, which is also referred as differential Harnack inequality in some literatures). For Ricci flow (Kähler-Ricci) flow there also exists the fundamental work of Hamilton...
In this paper we shall prove a nonlinear version of the Li-Yau-Hamilton estimate of [CN], for time dependent Kähler metrics evolving by Kähler-Ricci flow. The new matrix inequality asserts that if \((M, g(t))\) is a complete solution to Kähler-Ricci flow \(\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x, t) = -R_{\alpha\bar{\beta}}(x, t)\) with (bounded, in case \(M\) is not compact) nonnegative bisectional curvature, and if \(u\) is a positive solution to the forward conjugate heat equation: 
\[
\left( \frac{\partial}{\partial t} - \Delta - R \right) u(x, t) = 0,
\]
where \(R\) is the scalar curvature, then
\[
(0.1) \quad u_{\alpha\bar{\beta}} + \frac{u}{t} g_{\alpha\bar{\beta}} + u R_{\alpha\bar{\beta}} + u \alpha V_{\bar{\beta}} + u \bar{\beta} V_{\alpha} + u V_{\alpha} V_{\bar{\beta}} \geq 0
\]
for any \((1, 0)\) vector field \(V\). The form of this new matrix Li-Yau-Hamilton type estimate is a natural one after we found it through an interpolation consideration which was originated in [Ch1] by Ben Chow. We shall illustrate this interpolation consideration further in the following paragraph.

In [CH], Ben Chow and Richard Hamilton proved a linear trace Li-Yau-Hamilton inequality for the symmetric positive definite 2-tensors evolved by the (time-dependent) Lichnerowicz heat equation, coupled with the Ricci flow, whose complete solution metrics have bounded non-negative curvature operator. This result in particular generalizes the trace form of Hamilton’s fundamental matrix Li-Yau-Hamilton estimate on curvature tensors, for solutions to Ricci flow in [H1]. Later, in [Ch1], Chow discovered a very interesting interpolation phenomenon. Namely he shows a family of Li-Yau-Hamilton estimates in the case of Riemann surfaces with positive curvature such that this family connects Li-Yau’s estimate for the positive solutions to a heat equation with Chow-Hamilton’s linear trace estimate on the solutions to the Lichnerowicz heat equation, coupled with the Ricci flow, in the case of Riemann surfaces. Seeking the analogue of such interpolation in higher dimensions turns out to be fruitful. Even though the interpolation itself has not been found directly useful in geometric problems, it does play a crucial role in discovering new (useful) estimates. For example in [N4], such consideration led the author to discover a Li-Yau-Hamilton estimate for the Hermitian-Einstein flow (what proved there is more general). This estimate of [N4] is (crucial) one of the new ingredients in obtaining the (sharp) estimates on the dimension of the spaces of holomorphic functions (sections of certain line bundles) of the polynomial growth. (For more details, please see [N4], as well as [CFYZ].) The Li-Yau-Hamilton inequality proved in [N4] can be interpolated with the earlier one proved by L.-F. Tam and the author in [NT1]. This interpolation consideration further suggests that there should be a one-one correspondence, for the Li-Yau-Hamilton estimates, between the linear case and the (nonlinear) case with Ricci flow. Seeking the linear correspondence of the linear trace Li-Yau-Hamilton inequality for Kähler-Ricci flow, proved in [NT1], led the correct formulation of the Li-Yau-Hamilton’s inequality in [N4] for the linear case. On the other hand, looking for the nonlinear version of matrix Li-Yau-Hamilton estimate for the linear equation in [CN] leads us to formulate the correct form of the (nonlinear version) matrix Li-Yau-Hamilton inequality in this paper for the Kähler-Ricci flow. The proof of this result is applying the tensor maximum principle of Hamilton, which can not be completed.
without the generous help from Professor Ben Chow on a certain crucial step. We want to record our gratitude to him here.

As in the most other cases, the new estimate proved in this paper is sharp since it holds equality if and only if on expanding Kähler-Ricci solitons. Its proof also makes use of the earlier fundamental Li-Yau-Hamilton estimate of H.-D. Cao [C1] (see also [C2]). The Riemannian version of (0.1) suggests a new matrix differential inequality on curvature tensors, which is different from Hamilton’s one in [H1]. Please see section 5 for details. This new expression also vanishes identically on expanding solitons. Unfortunately, we have not been able to verify this new matrix estimates at this moment. (Please see Remark 5.2 for details.)

A little surprisingly, this new Li-Yau-Hamilton estimate for the Kähler-Ricci flow can be shown to be related to Perelman’s celebrated entropy formula for the Ricci flow [P], at least for Kähler case. Again this is done through Chow’s interpolation consideration. Namely, one in fact can obtain a family of \textit{pre-Li-Yau-Hamilton equalities} (a notion suggested to us by Tom Ilmanen) such that at one end, the trace of this \textit{pre-Li-Yau-Hamilton equality} gives the Perelman’s entropy formula after integration on the manifold, and at the other end one obtains the Li-Yau-Hamilton inequality of this paper by applying the tensor maximum principle of Hamilton. At the midpoint one can obtain both the entropy formula for the linear heat equation proved in [N3] and the Li-Yau’s estimate in [LY] for the linear heat equation. (The \textit{pre-Li-Yau-Hamilton equalities} was proved earlier by Ben Chow in [Ch2] (see also [CLN]) for the \textit{backward Ricci flow} on Riemannian manifolds soon after the proof of (0.1).) This connection between (0.1) and Perelman’s entropy formula suggests that the new Li-Yau-Hamilton estimate proved here may be of some fundamental importance. Please see Section 3 for the detailed exposition on this interpolation between Perelman’s entropy formula and the new Li-Yau-Hamilton estimate. One should also refer to the recent beautiful survey [Ev1] and the stellar notes [Ev2] by Evans for the relation between entropy and the Harnack estimates for the linear heat equation (see also [N3] for a different relation). [Ev1-2] also contain many other applications of entropy consideration in the study of PDE.

The rest of the paper is on applications of this new estimate, as well as the corresponding linear one of [CN]. The immediate consequences include the monotonicity of a quantity which is called Nash’s entropy, a Perelman type monotonicity (or Huisken type in the linear case) of the ‘reduced volume of analytic subvarieties’ and a sharp form of Harnack estimate for positive solutions to the \textit{forward conjugate heat equation}. It also can be applied to proved a sharp heat kernel comparison theorem for any subvariety in complete Kähler manifolds with nonnegative bisectional curvature. This generalizes a previous result of Peter Li and Gang Tian [LT], in which the authors proved the sharp comparison on heat kernels of algebraic manifolds, equipped with the induced Fubini-Study metric (also called Bergmann metric in [LT]) from the ambient $\mathbb{P}^m$. More precisely, we proved the following result.

\textit{Let $M$ be a complete Kähler manifold with nonnegative bisectional curvature. Let $H(x,y,t)$ be the fundamental solution of the heat equation. Let $\mathcal{V} \subset M$ be any complex subvariety of dimension $s$. Let $K_{\mathcal{V}}(x,y,t)$ be the fundamental solution of heat equation on $\mathcal{V}$. Then}

\begin{equation}
K_{\mathcal{V}}(x,y,t) \leq (\pi t)^{m-s} H(x,y,t), \text{ for any } x, y \in \mathcal{V}.
\end{equation}
The equality implies that $\mathcal{V}$ is totally geodesic.

A more involved application is the localized monotonicity formula and an elliptic ‘monotonicity principle’ for the analytic subvarieties. The later leads to a manifold (curved or nonlinear) version of Stoll’s characterization on ‘algebraic divisors’. This localization uses, substantially, the beautiful ideas from the the study of mean curvature flow of Ecker in [E1-2]. We shall give a brief sketch on these results below.

Let $\mathcal{V}$ be a subvariety of complex dimension $s$ as above. Denote by $A_{\mathcal{V},x_0}(\rho)$ the $2s$-dimensional Hausdorff measure of $\mathcal{V} \cap B_{x_0}(\rho)$. Here $B_{x_0}(\rho)$ is the ball (inside $M$) of radius $\rho$ centered at $x_0$. The elliptic ‘monotonicity principle’ states the following.

There exists $C = C(m, s)$ such that for any $\rho' \in (0, \delta(s)\rho)$

\[ \frac{A_{\mathcal{V},x_0}(\rho')}{V_{x_0}(\rho')}^{2(m-s)} \leq C(m, s) \frac{A_{\mathcal{V},x_0}(\rho)}{V_{x_0}(\rho)}^{2(m-s)} \]  

(0.3)

Here $\delta(s) = \frac{1}{\sqrt{2+4s}}$.

The following consequence of (0.3) is somewhat interesting.

Let $M^m$ be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that $M$ contains a compact subvariety $\mathcal{V}$ of complex dimension $s$. Then there exists $C = C(m, s) > 0$ such that for $\delta(s)\rho \geq \rho' \gg 1$,

\[ \frac{V_{x_0}(\rho)}{V_{x_0}(\rho')} \leq C \left( \frac{\rho}{\rho'} \right)^{2(m-s)}. \]  

(0.4)

In particular,

\[ \lim_{\rho \to \infty} \frac{V_{x_0}(\rho)}{\rho^{2(m-s)}} < \infty. \]

The result sharpens the Bishop-Gromov volume comparison theorem in the presence of compact subvarieties. When $M$ is simply-connected the result is in fact a consequence of the splitting theorem proved in [NT2, Theorem 0.4], via a completely different approach. It is not clear it can be derived out of any previously known result in the general case. For an entire analytic set $\mathcal{V}$ in $\mathbb{C}^m$ (of dimension $s$), one can define the Lelong number by

\[ \nu_\infty(\mathcal{V}) = \sup_{x_0 \in M} \limsup_{\rho \to \infty} \frac{(\pi \rho^2)^{(m-s)} A_{\mathcal{V},x_0}(\rho)}{V_{x_0}(\rho)}. \]

Stoll showed that $\mathcal{V}$ is algebraic if and only if $\nu_\infty(\mathcal{V}) < \infty$. The following result generalizes his result to analytic sets in curved manifold (we only succeeded in codimension one case).

Let $M$ be a complete Kähler manifold with nonnegative bisectional curvature. Let $\mathcal{V}$ be a analytic divisor of $M$. Then $\mathcal{V}$ is defined by a ‘polynomial function’ (holomorphic function of polynomial growth) if and only if $\nu_\infty(\mathcal{V}) < \infty$. 
In Section 4 we also obtained other results, one of which generalizes the classical transcendental Bézout estimate for codimension one analytic sets. (It has been known that the result fails for the high codimension case by the famous example of Cornalba and Shiffman [CS], even for the Euclidean case.) We found the connection between the parabolic approach, especially the Li-Yau-Hamilton estimate, and the classical Nevanlinna theory interesting and believe in that the parabolic approach should be the most nature/effective approach in extending sharp results from Euclidean spaces (linear) to the curved complex manifolds (nonlinear, in certain sense).

Finally, we discussed the relation between the Li-Yau-Hamilton inequality proved in this paper and the previous computations in [FIN] on the reduced distance and the reduced volume modelled on Ricci expanders. In particular, we prove a sharp lower bound on the heat kernel for the time dependent case, which, in a sense, is dual to Perelman’s monotonicity of the reduced volume. We also explain how one can view this result, and more importantly, Perelman’s monotonicity of the reduced volume as a nonlinear version of earlier work of Cheeger-Yau [CY] and Li-Yau [LY] on the heat kernel estimates for heat equations/Schrödinger equations. We also proved several local monotonicity formulae for Ricci flow on the forward reduced volume defined in [FIN] and Perelman’s entropy, without any curvature sign assumption. This again follows the observation of Ecker in [Ec2], where he derived a localized version of Huisken’s monotonicity formula for the mean curvature flow. As an application, we prove that the blow-down limit of a so-called type III solution to Kähler-Ricci flow with bounded nonnegative bisectional curvature must be a gradient expanding soliton. This is, in certain sense, dual to Perelman’s result on ancient solutions.

Here is how we organize the paper. In Section 1 we prove the interpolating version of the new Li-Yau-Hamilton estimate, which in particular, includes (0.1). In Section 2 we derive some monotonicity formulae and the heat kernel comparison theorem. In Section 3 we show the interpolation between the new Li-Yau-Hamilton inequality and Perelman’s entropy formula in [P]. In Section 4 we derive the localized version and prove the manifold version of Stoll’s theorem. In Section 5, we discuss the relation of the new inequality with the work of [FIN], formulate a new matrix Li-Yau-Hamilton expression on curvature tensors for Ricci flow, and show a sharp heat kernel lower bound estimate.

Acknowledgement. The special thanks go to Professor B. Chow as one can see that his contribution is crucial to a couple of results in this paper. He generously encouraged the author to publish the results alone even though it should really be a joint paper. The author would also like to thank Professors H.-D. Cao, Tom Ilmanen, Peter Li and Jiaping Wang for helpful discussions, Professors Klaus Ecker, Luen-Fai Tam, H. Wu for their interests.

§1 A new matrix Li-Yau-Hamilton inequality for Kähler-Ricci flow.

Let $M^m$ be a complete Kähler manifold of complex dimension $m$. Let $(M, g(t))$ be a solution to Kähler-Ricci flow:

\[
\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}(x, t) = -R_{\alpha\bar{\beta}}(x, t).
\]

(1.1)

Here $R_{\alpha\bar{\beta}}(x, t)$ is the Ricci tensor of the metric $g_{\alpha\bar{\beta}}(x, t)$. Let $u$ be a positive solution to
the forward conjugate heat equation:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) u(x, t) = R(x, t)u(x, t).
\]

Here $R(x, t)$ is the scalar curvature. We shall prove the following new matrix Li-Yau-Hamilton/differential Harnack inequality.

**Theorem 1.1.** Let $(M, g(t))$ be a solution to (1.1) with nonnegative bisectional curvature. In the case that $M$ is complete noncompact, assume further that the bisectional curvature is bounded. Let $u$ be a positive solution to (1.2). Then

\[
u_{\alpha \bar{\beta}} + \frac{u}{t} g_{\alpha \bar{\beta}} + uR_{\alpha \bar{\beta}} + u_\alpha V_{\bar{\beta}} + u_{\bar{\beta}} V_\alpha + uV_\alpha V_{\bar{\beta}} \geq 0
\]

for any $(1, 0)$ vector field $V$.

The assumption on $(M, g(t))$ having bounded nonnegative bisectional curvature can be replaced by $(M, g(0))$ having nonnegative bisectional curvature and $(M, g(t))$ with bounded curvature, thanks to the result of Bando [B], Mok [M2] and Shi [Sh2]. The same applies to other results of the paper. We state the result under the stronger assumption just for the simplicity.

Recall that in [CN] the authors proved that for a fixed Kähler metric $(M, g_{\alpha \bar{\beta}}(x))$ with the nonnegative bisectional curvature and the positive solution $u$ to the heat equation $\left( \frac{\partial}{\partial t} - \Delta \right) u = 0$ one has the matrix Li-Yau-Hamilton inequality:

\[
u_{\alpha \bar{\beta}} + \frac{u}{t} g_{\alpha \bar{\beta}} + u_\alpha V_{\bar{\beta}} + u_{\bar{\beta}} V_\alpha + uV_\alpha V_{\bar{\beta}} \geq 0
\]

Therefore, one can think (1.3) is the nonlinear version of (1.4). In fact we shall show that there exists a linear interpolation between these two inequalities. The similar interpolation was established by Ben Chow [Ch1] originally for the Li-Yau’s gradient estimates for the heat equation and Hamilton’s differential Harnack for the Ricci flow in the case of surfaces. We shall show such interpolation between the matrix differential inequalities (1.3) and (1.4) for any dimensions. In [N4] we have shown another family of Li-Yau-Hamilton inequalities which also serves as a higher dimensional generalization of Chow’s result.

Let us first set up the notations. For any $\epsilon > 0$, we consider the Kähler-Ricci flow:

\[
\frac{\partial}{\partial t} g_{\alpha \bar{\beta}}(x, t) = -\epsilon R_{\alpha \bar{\beta}}(x, t).
\]

Consider the positive solution $u$ to the parabolic equation:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) u(x, t) = \epsilon R(x, t)u(x, t).
\]

We shall call (1.6) forward conjugate heat equation. We shall prove the following general interpolation theorem.
The commutator formula gives that
\[ u_{\alpha\beta} + \frac{\partial}{\partial t}g_{\alpha\beta} + \epsilon uR_{\alpha\beta} + u_{\alpha}V_{\beta} + u_{\beta}V_{\alpha} + uV_{\alpha}V_{\beta} \geq 0 \]  
for any \((1,0)\) vector field \(V\).

**Remark 1.1.** Since Ben Chow helped with the proof at a critical step it is well-justified to attribute the above theorem as a joint result.

It is easy to see that Theorem 1.2, serving an interpolation between (1.3) and (1.4), implies Theorem 1.1 and the earlier result (1.4) for the linear heat equation. Therefore, we only need to prove Theorem 1.2. The proof consists of the following several lemmas.

**Lemma 1.1.**
\[ (\mathcal{R})_{\alpha\beta} = \Delta R_{\alpha\beta} + R_{\alpha\beta\gamma\delta}R_{\delta\gamma} - R_{\alpha\beta}R_{\gamma\delta}. \]

Here \(R_{\alpha\beta\gamma\delta}\) is the bisectional curvature.

**Proof.** The second Bianchi identity yields
\[ R_{\alpha\beta} = (R_{\gamma\gamma})_{,\alpha\beta} = R_{\alpha\gamma,\gamma\beta} \]
\[ = R_{\alpha\gamma,\beta\gamma} - R_{p\gamma\gamma\beta}R_{\alpha\bar{p}} + R_{\alpha\bar{p}\gamma\beta}R_{p\gamma} \]
\[ = R_{\alpha\beta,\gamma\gamma} - R_{\beta\gamma\gamma\beta}R_{\alpha\bar{p}} + R_{\alpha\bar{p}\gamma\beta}R_{p\gamma}. \]

The commutator formula gives that
\[ R_{\alpha\beta,\gamma\gamma} = R_{\alpha\bar{\beta},\gamma\gamma} - R_{\beta\gamma\gamma\beta}R_{\alpha\bar{p}} + R_{\alpha\bar{p}\gamma\beta}R_{p\gamma} \]
\[ = R_{\alpha\bar{\beta},\gamma\gamma} - R_{\beta\gamma\gamma\beta}R_{\alpha\bar{p}} + R_{\alpha\bar{p}\gamma\beta}R_{p\gamma}. \]

The lemma now follows from the definition \(\Delta R_{\alpha\beta} = \frac{1}{2} (R_{\alpha\bar{\beta},\gamma\gamma} + R_{\alpha\bar{\beta},\gamma\gamma})\).

**Lemma 1.2.** Let \(u(x,t)\) be a solution to (1.6). Then
\[ \left( \frac{\partial}{\partial t} - \Delta \right) u_{\alpha\beta} = R_{\alpha\beta\gamma\delta}u_{\gamma\delta} - \frac{1}{2} (R_{\alpha\bar{p}u_{\beta\bar{p}}} + R_{\beta\bar{p}u_{\alpha\bar{p}}}) + \epsilon (\mathcal{R}u)_{\alpha\beta}. \]

**Proof.** Differentiate (1.6) we have
\[ (u_t)_{\gamma\delta} = R_{\beta\alpha\gamma\delta}u_{\alpha\beta} + g^{\alpha\bar{\beta}}u_{\alpha\beta\gamma\delta} + \epsilon (\mathcal{R}u)_{\gamma\delta}. \]

By definition \(\Delta u_{\alpha\beta} = \frac{1}{2} (u_{\alpha\bar{\beta},\gamma\gamma} + u_{\alpha\beta,\gamma\gamma})\), in normal coordinates at a point. We need to calculate the difference between the partial derivative \(u_{\alpha\beta\gamma\delta}\) and the covariant derivative \(u_{\alpha\beta,\gamma\delta}\). Direct computations show that, for normal coordinates at a point

\[ u_{\gamma\delta,\alpha\beta} = u_{\gamma\delta\alpha\beta} + u_{s\bar{s}}R_{\alpha\beta\gamma\delta}. \]
Using the fact that
\begin{equation}
(1.14) \quad u_{\gamma\delta,\alpha\bar{\alpha}} = u_{\gamma\bar{\delta},\alpha\bar{\alpha}} + R_{\gamma\bar{p}}u_{p\bar{\delta}} - R_{p\bar{\delta}}u_{\gamma\bar{p}}
\end{equation}
and (1.13) we have
\begin{equation}
(1.15) \quad \Delta u_{\gamma\delta} = \frac{1}{2}(u_{\gamma\delta,\alpha\bar{\alpha}} + u_{\gamma\bar{\delta},\alpha\bar{\alpha}})
= u_{\gamma\delta,\alpha\bar{\alpha}} + \frac{1}{2}(R_{\gamma\bar{p}}u_{p\bar{\delta}} + R_{p\bar{\delta}}u_{\gamma\bar{p}}).
\end{equation}

Combining the above with (1.12), we conclude that $u_{\alpha\bar{\beta}}$ satisfies (1.11).

The direct calculations give the following lemma.

**Lemma 1.3.** Let $(M,g(t))$ be a solution to (1.5). Let $u(x, t)$ be a positive solution to (1.6). Then
\begin{equation}
(1.16) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \left(\frac{u_{\alpha}u_{\bar{\beta}}}{u}\right) = \epsilon\left(Ru\right)_{\alpha}u_{\bar{\beta}} + \epsilon\left(Ru\right)_{\bar{\beta}}u_{\alpha} - \epsilon(Ru)\frac{u_{\alpha}u_{\bar{\beta}} + u_{\alpha\bar{s}}u_{\bar{\beta}s}}{u} - 2\frac{u_{\alpha}u_{\bar{\beta}}u_{s}u_{\bar{s}}}{u} - \frac{1}{2}R_{\alpha\bar{s}u_{s}u_{\bar{\beta}}} + \frac{1}{2}R_{\bar{s}\beta u_{s}u_{\alpha}} + \frac{u_{\alpha\bar{s}}u_{s}u_{\bar{\beta}} + u_{\bar{s}\beta u_{s}u_{\alpha}} + u_{\alpha\bar{s}}u_{\bar{\beta}s}u_{s} + u_{\bar{s}\beta u_{s}u_{\alpha}}}{u^2}.
\end{equation}

\begin{equation}
(1.17) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \left(u_{\epsilon\bar{\gamma}\delta}(x, t)\right) = \epsilon Ru\frac{u_{\alpha}u_{\bar{\beta}}}{u} - \frac{u_{\alpha}u_{\bar{\beta}}}{u} - \frac{u_{\alpha}u_{\bar{\beta}}}{u} - \epsilon u_{\epsilon}R_{\alpha\bar{\beta}}.
\end{equation}

\begin{equation}
(1.18) \quad \left(\frac{\partial}{\partial t} - \Delta\right) (\epsilon uR_{\alpha\bar{\beta}}) = \epsilon^2 RuR_{\alpha\bar{\beta}} + \epsilon u(R_{\alpha\bar{\gamma}\gamma\delta}R_{\gamma\delta} - R_{\alpha\bar{p}}R_{p\bar{\delta}}) + \epsilon(R_{\alpha\bar{\gamma}\gamma\delta}R_{\gamma\delta} - R_{\alpha\bar{p}}R_{p\bar{\delta}} + \Delta R_{\alpha\bar{\beta}}).
\end{equation}

**Proof.** In the derivation of (1.16), a commutator formula has been used. In the derivation of (1.18) we have used
\begin{equation}
\frac{\partial}{\partial t} R_{\alpha\bar{\beta}} = \epsilon\left(R_{\alpha\bar{\gamma}\gamma\delta}R_{\gamma\delta} - R_{\alpha\bar{p}}R_{p\bar{\delta}} + \Delta R_{\alpha\bar{\beta}}\right).
\end{equation}

We also need the following result which is an easy consequence of Cao’s differential Harnack for the Kähler-Ricci flow [C1].
Lemma 1.4. Let \((M, g(t))\) be a complete solution to (1.5) with nonnegative bisectional curvature. In the case that \(M\) is complete noncompact we further assume that the bisectional curvature is bounded. Let \(u(x, t)\) be a positive solution to (1.6). Then

\[
\Delta R_{\alpha\beta} + R_{\alpha\beta} \gamma \delta R_{\gamma \delta} - \left( \frac{\nabla_s u}{eu} \nabla_s R_{\alpha\beta} + \frac{\nabla_s u}{eu} \nabla_s R_{\alpha\beta} \right) + R_{\alpha\beta} \gamma \delta \frac{\nabla_s u}{eu} \frac{\nabla_s u}{eu} + \frac{R_{\alpha\beta}}{\epsilon t} \geq 0.
\]

Once we have Lemma 1.1–1.4 we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let

\[
N_{\alpha\beta} = u_{\alpha\beta} + \frac{u}{t} g_{\alpha\beta} - \frac{u_{\alpha\beta} \bar{u}}{u}
\]

and

\[
\tilde{N}_{\alpha\beta} = N_{\alpha\beta} + \epsilon u R_{\alpha\beta}.
\]

By taking the minimizing vector in (1.7), one can see that Theorem 1.2 is equivalent to \(\tilde{N}_{\alpha\beta} \geq 0\). Since the maximum principle needs some growth conditions for the noncompact manifolds, we first prove the theorem for the case that \(M\) is compact.

Compact case. By Lemma 1.1–1.3, we have that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \tilde{N}_{\alpha\beta} = R_{\alpha\beta} \gamma \delta u_{\gamma \delta} - \frac{1}{2} \left( R_{\alpha\beta} u_{p\beta} + R_{p\beta} u_{\alpha\beta} \right) + \epsilon (Ru)_{\alpha\beta}
\]

\[
-\epsilon \frac{(Ru)_{\alpha\beta}}{u} - \epsilon \frac{(Ru)_{\beta\alpha}}{u} + \epsilon (Ru) \frac{u_{\alpha\beta}}{u^2} + \frac{u_{\alpha\beta} u_{\beta\gamma} u_{\gamma\delta} + u_{\alpha\beta} u_{\beta\gamma} u_{\gamma\delta}}{u}
\]

\[
+ \frac{2 u_{\alpha\beta} |u_s|^2}{u} \left( \frac{1}{2} R_{\alpha\beta} u_{\delta} + R_{\beta\delta} u_{\alpha} \right)
\]

\[
- \frac{2 u_{\alpha\beta} u_{\beta\gamma} u_{\gamma\delta} + u_{\alpha\beta} u_{\beta\gamma} u_{\gamma\delta}}{u} u + \epsilon R u R_{\alpha\beta} + \epsilon u \left( R_{\alpha\beta} \gamma \delta R_{\gamma \delta} - R_{\alpha\beta} R_{p\beta} \right)
\]

\[
- \epsilon (\nabla_s u \nabla_s R_{\alpha\beta} + \nabla_s u \nabla_s R_{\alpha\beta})
\]

\[
+ (\epsilon^2 - \epsilon) u \left( R_{\alpha\beta} \gamma \delta R_{\gamma \delta} - R_{\alpha\beta} R_{p\beta} + \Delta R_{\alpha\beta} \right)
\]

\[
= R_{\alpha\beta} \gamma \delta \tilde{N}_{\gamma \delta} - \frac{1}{2} \left( R_{\alpha\beta} \tilde{N}_{p\beta} + R_{\beta\delta} \tilde{N}_{\alpha p} \right) + \frac{1}{u} N_{\alpha\beta} N_{p\beta} - \frac{2}{u} \tilde{N}_{\alpha\beta}
\]

\[
+ \frac{1}{u} \left( u_{\alpha\beta} - \frac{u_{\alpha\beta} u_{\beta\gamma}}{u} \right) \left( u_{\beta\delta} - \frac{u_{\beta\delta} u_{\gamma\delta}}{u} \right) + \epsilon R \tilde{N}_{\alpha\beta} - u \epsilon^2 R_{\alpha\beta} R_{p\beta}
\]

\[
+ \epsilon^2 u \tilde{N}_{\alpha\beta}
\]

where

\[
\tilde{N}_{\alpha\beta} = \Delta R_{\alpha\beta} + R_{\alpha\beta} \gamma \delta R_{\gamma \delta} - \left( \frac{\nabla_s u}{eu} \nabla_s R_{\alpha\beta} + \frac{\nabla_s u}{eu} \nabla_s R_{\alpha\beta} \right) + R_{\alpha\beta} \gamma \delta \frac{\nabla_s u}{eu} \frac{\nabla_s u}{eu} + \frac{R_{\alpha\beta}}{\epsilon t}.
\]
By Lemma 1.4 and (1.20) we have that

\begin{equation}
(\partial_t - \Delta) \bar{N}_{\alpha\beta} = R_{\alpha\beta\gamma\delta} \bar{N}_{\gamma\delta} - \frac{1}{2} \left( R_{\alpha\beta\gamma\delta} + R_{\beta\alpha\gamma\delta} \right) + \left( \epsilon R - \frac{2}{t} \right) \bar{N}_{\alpha\beta}
\end{equation}

\begin{align*}
&+ \frac{1}{2u} \bar{N}_{\alpha\beta} \left( N_{\rho\delta} - \epsilon u R_{\rho\beta} \right) + \frac{1}{2u} \left( N_{\alpha\rho} - \epsilon u R_{\alpha\rho} \right) \bar{N}_{\rho\delta} \\
&+ \epsilon^2 u \bar{Y}_{\alpha\beta} + \frac{1}{u} \left( u_{\alpha\rho} - u_{\alpha\beta} \right) \left( u_{\rho\beta} - u_{\rho\beta} \right) \\
&\geq R_{\alpha\beta\gamma\delta} \bar{N}_{\gamma\delta} - \frac{1}{2} \left( R_{\alpha\beta\gamma\delta} + R_{\beta\alpha\gamma\delta} \right) + \left( \epsilon R - \frac{2}{t} \right) \bar{N}_{\alpha\beta}
\end{align*}

Using the observation that the right hand side of (1.21) satisfies the null-vector condition of the tensor maximum principle of Hamilton [H2], we have proved the case $M$ being compact.

**Noncompact case.** First recall the fundamental derivative estimate of Shi. For $g_{\alpha\beta}(x,t)$, a solution to (1.5) on $M \times [0,T]$ with bounded (in space-time) nonnegative bisectional curvature, there exist $A_k > 0$ such that

\begin{equation}
\| \nabla^k R_{\alpha\beta\gamma\delta} \|^2 \leq \frac{A_k}{t^k}
\end{equation}

on $M \times [0,T]$. For our consideration we only need to prove it for $T$ small. The estimate (1.22) is proved in [Sh1]. For the sake of simplicity, we will show the matrix differential Harnack inequality Theorem 1.2 for the case $\epsilon = 1$ under the above assumption (1.22). In fact, what needed is (1.22) for $k \leq 2$. We also need the perturbation trick from [NT1] (see also [CN]). Namely we first shift $t$ by $2\delta$, where $\delta$ is a small positive number. After the shifting we can have estimates on $u$, $|\nabla u|$ and $|u_{\alpha\beta}|$. The goal is to show that there exits a $b > 0$ such that

\begin{equation}
\int_{\delta}^{T} \int_M \exp(-b(r_0^2(x) + 1)) \left( \frac{1}{u} + \frac{|\nabla u|^2}{u} + |u_{\alpha\beta}|^2 \right) d\mu dt < \infty.
\end{equation}

Here $r_0(x)$ is the distance to $x$ from a fixed point $o \in M$ with respect to the initial metric. We need the estimate (1.23) to apply the tensor maximum principle from [N5] (see also [NT2] for the original time-independent version).

In order to get control on $u$ (or $\frac{1}{u}$) first we need the following Harnack inequality of Guenther [Gu].

**Theorem 1.3.** (Guenther) Let $(M, g(t))$ be a solution to Ricci flow satisfying (1.22). Let $u$ be a positive solution to the forward conjugate heat equation $(\partial_t - \Delta) u = R u$. Then for sufficient small $T$ (only depending on $A_k$) there exist $\alpha, B_1 > 0$ only depending on $A_k$ such that

\begin{equation}
u(x_1, t_1) \leq u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{\frac{\alpha}{2}} \exp \left( \frac{r^2(x_1, x_2, t_1)}{\alpha(t_2 - t_1)} + B_1(t_2 - t_1) \right).\end{equation}
for any \( T \geq t_2 > t_1 > 0 \). Here \( r(x_1, x_2, t_1) \) denotes the distance between \( x_1 \) and \( x_2 \) with respect to the metric at \( t = t_1 \).

The result above was proved in [Gu] through a gradient estimate of Li-Yau type on compact manifold. Since one can apply the localization techniques as in [LY, page 161] (see also [NT1, page 647], [Sh1] ) one can easily generalize the gradient estimate, thus the above Harnack estimate, proved in [Gu] to complete noncompact case. From (1.24), one can deduce that for small \( \delta \) there exists a constant \( b_2, B_2 > 0 \), where \( b_2 = b_2(A_k, \delta) \) and \( B_2 = B_2(u(o, \frac{\delta}{4}), u(o, T - \frac{\delta}{4}), A_k, \delta) \) such that

\[
\left( \frac{1}{u} + u \right)(x, t) \leq B_2 \exp(b_2(r^2_0(x) + 1))
\]

for \((x, t) \in M \times [\frac{\delta}{2}, T - \frac{\delta}{2}]\).

Observe that we have the following two equations.

\[
\left( \frac{\partial}{\partial t} - \Delta \right) u^2 = R u^2 - |\nabla u|^2
\]

and

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla u|^2 = -|u_{\alpha\beta}|^2 - |u_{\alpha}|^2 + \langle \nabla(Ru), \nabla u \rangle + \langle \nabla u, \nabla(Ru) \rangle.
\]

The desired estimate (1.23) follows from (1.26), (1.27) by the argument through integration by parts in [CN, Lemma 3.1]. Once we have established the estimate (1.23) one can apply the perturbation argument as in [NT1] (see also [CN]) together with the tensor maximum principle in [NT2, Theorem 2.1] or [N5, Theorem 2.1] to conclude the proof of the matrix Li-Yau-Hamilton estimate (1.7) for the complete noncompact case. Note that Theorem 1.3 only provide the estimate for short time. But we can iterate the argument to prove the result for all time. An alternative is that once one has the upper bound estimate at some earlier time one can also make use of the heat kernel estimate in [N5] for the time dependent heat operator (and uniqueness of the positive solution) to get estimates of the positive solution for the later time. We also should remark that the matrix Li-Yau-Hamilton (1.7) gives a sharp Harnack (which is more precise, compared with Theorem 1.3). Please see Corollary 2.1 in the next section. However, we do need the rough estimate in the proof to apply the tensor maximum principle.

**Corollary 1.1.** Let \( u(x, t) \) be a positive solution to (1.6). Then

\[
\Delta \log u + \epsilon R + \frac{m}{t} \geq 0.
\]

If the equality holds for some \((x_0, t_0)\) with \( t_0 > 0 \), then \((M, g(t))\) is an expanding Kähler-Ricci soliton for the case \( \epsilon > 0 \), and \((M, g)\) is isometric to \( \mathbb{C}^n \) for the case \( \epsilon = 0 \).

**Proof.** Let

\[
Q = \Delta \log u + \epsilon R + \frac{m}{t}.
\]
and \( Q = g^{\alpha\bar{\beta}}\bar{N}_{\alpha\bar{\beta}} \). Since \( \bar{N}_{\alpha\bar{\beta}} \geq 0 \) and \( Q = g^{\alpha\bar{\beta}}\bar{N}_{\alpha\bar{\beta}} \) we have that \( \bar{N}_{\alpha\bar{\beta}}(x_0, t_0) = 0 \). By the strong maximum principle we know that \( \bar{N}_{\alpha\bar{\beta}} \equiv 0 \) for all \( t < t_0 \). \( \bar{N}_{\alpha\bar{\beta}} \equiv 0 \) is nothing but the equation in the definition of the gradient expanding soliton.

For the case \( \epsilon = 0 \), we apply the same line of argument. In this case we have \( Q \equiv 0 \) since \( \bar{N}_{\alpha\bar{\beta}} = N_{\alpha\bar{\beta}} \equiv 0 \), for \( t \leq t_0 \). Now from the equation

\[
0 = \left( \frac{\partial}{\partial t} - \Delta \right) t^2 Q = t^2 R_{\alpha\bar{\beta}}(\log u)_\alpha (\log u)_\beta + \frac{t^2}{u} \left( u_{\alpha p} - \frac{u_\alpha u_p}{u} \right) \left( u_{\bar{p}\bar{\alpha}} - \frac{u_{\bar{p}}u_{\bar{\alpha}}}{u} \right)
+ \frac{t^2}{u} N_{\alpha p} N_{p\bar{\alpha}} \geq 0
\]

we have that \( (\log u)_{\alpha\beta} \equiv 0 \) too. From the definition of \( N_{\alpha\bar{\beta}} \) we then have

\[
(\log u)_{\alpha\bar{\beta}} = \frac{1}{t} g_{\alpha\bar{\beta}}
\]

and

\[
(\log u)_{\alpha\beta} = 0
\]

from which it is easy to see that \((M, g)\) is flat and in fact isometric to \( \mathbb{C}^n \), since that the curvature (see, for example, [KM, page 117]) can be written as

\[
R_{\alpha\beta\gamma\delta} = -\frac{\partial^4 (tf)}{\partial z^\alpha \partial z^\beta \partial z^\gamma \partial z^\delta} + g^{p\bar{q}} \left( \frac{\partial^3 (tf)}{\partial z^p \partial z^\alpha \partial z^\gamma} \right) \left( \frac{\partial^3 (tf)}{\partial z^\bar{p} \partial z^\bar{\beta} \partial z^\bar{\delta}} \right)
\]

and that \( f = \log u \) (defined to be) is a convex function.

**Remark 1.2.** The case \( \epsilon = 0 \) case in the above Corollary 1.1 is just the original Li-Yau’s estimate [LY], which holds with nonnegativity of the Ricci curvature. The equality case implies the manifold is \( \mathbb{R}^n \) is implicit in the proofs of [LY] and proved explicitly in [N3] for Riemannian manifolds with nonnegative Ricci curvature. It would be interesting to see if Corollary 1.1 is true, for \( \epsilon > 0 \) case, for complete solutions to the Kähler-Ricci/Ricci flow without assumptions on the sign of the curvature.

When the manifold is compact, since we known that, by passing to its universal cover, it is products of \( \mathbb{C}^k \) with compact Hermitian symmetric spaces. Without the loss of generality we can assume that the first Chern class \( c_1(M) \) is a positive multiple of the Kähler class. Then the Kähler-Ricci flow will have singularity, say at \( t = 1 \), and the normalized flow has long time existence. The rescaling is given by \( \hat{g} = \frac{1}{1-t} g \) and the reparametrization is given by \( s = -\log(1-t) \). Therefore, Theorem 1.2 has the following equivalent form.

**Theorem 1.2’.** Let \( M \) be a compact Kähler manifold as above. Let \( g(x, t) \) be a solution to (1.1), and let \( \hat{u}(x, t) \) be a positive solution to (1.2). Assume that \( \hat{g}(x, s) \) is the solution to the normalized Kähler Ricci flow. Then

\[
(\log u)_{\alpha\bar{\beta}} + \hat{R}_{\alpha\bar{\beta}} + \frac{1}{e^s - 1} \hat{g}_{\alpha\bar{\beta}} \geq 0.
\]

Here \( \hat{R}_{\alpha\bar{\beta}} \) is the Ricci tensor of \( \hat{g} \).
§2 Monotonicity formulae.

In this section we derive some monotonicity formulae out of the matrix Li-Yau-Hamilton estimate, as well as its trace, proved in last section. In order to make the argument unified for both cases, with and without Ricci flow, we work with the interpolation version (1.7). Let \((M, g(t))\) be the solution to the Ricci flow (1.5) and let \(u(x, t)\) be the positive solution to (1.6).

The first result is the monotonicity of the partition function (also called Nash’s entropy in [FIN]) defined by

\[
\tilde{N}(g, u, t) = -\int_M u \log u \, dv - m \log(\pi t) - m.
\]

Simple computation shows that

\[
\frac{d\tilde{N}}{dt} = \int_M \left(-\Delta \log u - \epsilon R - \frac{m}{t}\right) u \, d\mu_t \leq 0.
\]

Here \(d\mu_t\) is the volume element of \(g(t)\). The following result is a direct consequence of Corollary 1.1.

**Proposition 2.1.** Let \((M, g(t))\) be a solution to (1.5) with nonnegative bisectional curvature. Let \(H(x, y, t)\) be the fundamental solution to (1.6). Then

\[
\frac{d}{dt} \tilde{N}(g, u, t) \leq 0
\]

for any positive solution \(u\) to (1.6) and

\[- \int_M H \log H \, d\mu_t - m \log(\pi t) \leq m\]

with the equality holds for some positive \(t\) if and only if the manifold is an expanding gradient soliton (isometric to \(\mathbb{C}^m\) in the case \(\epsilon = 0\)).

For the fixed Riemannian metric case the above was proved earlier in [N2] for the manifold with only nonnegative Ricci curvature. We believe that the result should hold for Ricci flow even without assumptions. But at this moment we can only prove it for Kähler-Ricci flow through the matrix Li-Yau-Hamilton inequality, which assumes the nonnegativity of the bisectional curvature. The result above gives a characterization of expanding solitons using the partition function. The similar formulation for the shrinking solitons also works for the partition functions related to Perelman’s entropy formula [P]. Namely, consider the backward Ricci flow \(\frac{\partial}{\partial \tau} g_{ij} = 2R_{ij}\) on \(M \times [0, \tau_0]\), where \(M\) is a Riemannian manifold of real dimension \(n\). Let \(u(x, \tau)\) be a solution to the backward adjoint heat equation
Similarly one can define

\[ (2.1') \tilde{\mathcal{N}}(g, u, \tau) = -\int_M u \log u \, d\mu_\tau - \frac{n}{2} \log(4\pi \tau) - \frac{n}{2}. \]

In Proposition 1.2 of [P], Perelman proved that

\[ (2.2') \frac{d\tilde{\mathcal{N}}}{d\tau} = \int_M \left( -\Delta \log u + R - \frac{n}{2\tau} \right) d\mu_\tau \leq 0. \]

The dual version of Proposition 2.1 states as follows.

**Proposition 2.1'**. Let \( H(x, y, \tau) \) be the fundamental solution to the adjoint heat equation. Then

\[ -\int_M H \log H \, d\mu_\tau - \frac{n}{2} \log(4\pi \tau) \leq \frac{n}{2} \]

with equality holds (or in (2.2')) for some positive \( \tau \) if and only if \((M, g_{ij}(\tau))\) is a gradient shrinking soliton.

**Proof.** Since we do not have Corollary 1.1 in this case we need other arguments. In fact, tracing the equality case of the proof of Proposition 1.2 of [P] we have that \( R_{ij} - \nabla_i \nabla_j \log H = \frac{\mathcal{R} - \Delta \log H}{n} g_{ij} \). On the other hand, the equality in (2.2') further implies that \( R_{ij} - \nabla_i \nabla_j \log H = \frac{1}{2\tau} g_{ij} \).

In [N2], the relation between the value of \( \tilde{\mathcal{N}}(t) \) (as well as the entropy functional \( \mathcal{W} \)) as \( t \to \infty \) and the asymptotic volume ratio at infinity (also called the ‘cone angle’) was proved for the linear heat equation. The similar relation, between the \( \kappa \)-constant in the \( \kappa \)-non-collapsing of volume defined in [P] and the large time limit of the partition function \( \tilde{\mathcal{N}}(g, u, \tau) \) (as well he entropy functional \( \mathcal{W} \)) should also be true for ancient solutions to Ricci flow.

Another application of Theorem 1.2 is an entropy monotonicity for the ancient solutions. Let \((M, g(t))\) be an ancient solution to (1.5) and let \( u \) be a positive solution to (1.6), both defined on \( M \times (-\infty, 0] \). Theorem 1.2 implies that

\[ (\log u)_{\alpha\bar{\beta}} + R_{\alpha\bar{\beta}} \geq 0. \]

From this one can easily see that

\[ \mathcal{N}(g, u, t) := -\int_M u \log u \, d\mu_t \]

is monotone non-increasing.

If \( M \) is compact, one can obtain such \( u \) by taking limit of solutions with initial data at \( t = -i \) as in [FIN] (where the immortal solution is studied). More precisely, let \( u_i(x, t) \) be a solution to (1.6) with \( u(x, -i) = \frac{1}{V(-i)} \). Letting \( i \to \infty \), one can extract a limit \( u_\infty > 0 \) (since \( \int_M u_i \equiv 1 \) and \( u_i > 0 \)) which is defined on \((-\infty, 0]\). In this case \( \int_M u_\infty \, dv = 1 \).

Applying the Jensen’s inequality one has that

\[ \mathcal{N}(g, u_\infty, t) \leq \log V(t). \]
The right hand side is another monotone non-increasing quantity along the flow. When \( M \) is complete noncompact one can obtain such a \( u \) similarly by solving \( u_i(x,t) \) with initial condition at \( t = -i \) and anchoring \( u_i(o,-1) = 1 \), where \( o \in M \) is a fixed point. In fact one can even get \( u_i \) integrable by taking it to be scalar multiple of the fundamental solution (with initial data being the delta function at \( t = -i \)).

The second application is to derive the heat kernel comparison theorem and Huisken type [Hu, E1-2] monotonicity formula for the analytic subvarieties in \( M \). We start with the case \( \epsilon = 0 \) since the results seem to be more useful at this moment.

**Theorem 2.1.** Let \( M \) be a complete Kähler manifold with nonnegative bisectional curvature. Let \( H(x,y,t) \) be the fundamental solution of the heat equation. Let \( \mathcal{V} \subset M \) be any complex subvariety of dimension \( s \). Let \( K_{\mathcal{V}}(x,y,t) \) be the fundamental solution of heat equation on \( \mathcal{V} \). Then

(i) \[
K_{\mathcal{V}}(x,y,t) \leq (\pi t)^{m-s} H(x,y,t), \text{ for any } x, y \in \mathcal{V}.
\]

If the equality holds, then \( \mathcal{V} \) is totally geodesic. Furthermore if \( \tilde{M} \) is the universal cover of \( M \) with covering map \( \pi \) and \( \tilde{\mathcal{V}} = \pi^{-1}(\mathcal{V}) \), then \( \tilde{M} = \tilde{M}_1 \times \mathbb{C}^k \) for some Kähler manifold \( \tilde{M}_1 \) which does not contain any Euclidean factors, with \( k \geq m - s \). Moreover \( \tilde{\mathcal{V}} = \tilde{M}_1 \times \mathbb{C}^l \) with \( l < k \).

(ii) \[
\frac{d}{dt} \int_{\mathcal{V}} (\pi t)^{m-s} H(x,y,t) dA_{\mathcal{V}}(y) \geq 0, \text{ for any } x \in M.
\]

Similarly, if the equality holds for some \( x \in M \) at some positive time \( t \), then \( \tilde{M} = \tilde{M}_1 \times \mathbb{C}^k \) with \( k \geq m - s \).

**Proof.** For any smooth point \( y \in \mathcal{V} \), choose a complex coordinate \((z_1, \cdots, z_m)\) such that \((z_1, \cdots, z_s)\) is the coordinates for \( \mathcal{V} \). Let \( i, j, k, \cdots \) denote the coordinate on \( \mathcal{V} \) and \( a, b, c, \cdots \) denote the coordinates in the normal directions. Then we compute

\[
\left( \Delta_{\mathcal{V}}^{(y)} - \frac{\partial}{\partial t} \right) (t^{m-s} H(x,y,t)) = \left( \Delta_M - g^{ab} \nabla_a \nabla_b - \frac{\partial}{\partial t} \right) (t^{m-s} H(x,y,t))
\]

\[
= (\pi t)^{m-s} \left( \Delta_M H - H_t - g^{ab} \nabla_a \nabla_b H - \frac{m-s}{t} H \right)
\]

\[
= (\pi t)^{m-s} \left( -\nabla_a \nabla_b H + \frac{\nabla_a H \nabla_b H}{H} - \frac{1}{t} H g_{ab} \right) g^{ab}
\]

\[- (\pi t)^{m-s} \left| \nabla H \right|^2.
\]

By Theorem 1.2 we have that

\[
\left( \Delta_{\mathcal{V}}^{(y)} - \frac{\partial}{\partial t} \right) (t^{m-s} H(x,y,t)) \leq 0.
\]
Noticing that for \( x \in V \), \( \lim_{t \to 0} (\pi t)^{m-s} H(x, y, t)|_V = \delta_x(y) \). This proves that (2.3) by the maximum principle, (2.4) by the integrating (2.5) on \( V \). (For the case \( V \) is singular, one can refer to [LT] for the justification on the validity of the integration by parts.)

In the case when the equality holds in (2.3) we have that \( (\pi t)^{m-s} H(x, y, t) \) satisfies the heat equation. Hence equality holds in (2.5) for any \( x, y, t, v \). This implies that

\[
|\nabla^\perp \log H|^2 = 0, \quad \text{for } x, y \in V, \quad t > 0,
\]

which implies that \( <\nabla r^2(x, y), \nu > = 0 \) for any \( x, y \in V \), and any normal direction \( \nu \). Here we have used the fact that \( \lim_{t \to 0} -t \log H(x, y, t) = r^2(x, y) \). See [CLY1], for example. This implies that for any smooth point \( x \in V \), any minimizing geodesic starting from \( x \) lies totally inside \( V \). More precisely, let \( \gamma_v(s) \) be a short minimizing geodesic emitting from \( x \) with \( \gamma_v'(0) = v \) such that \( v \in T_x V \). Denote by \( h \) be the distance function from \( V \). Then

\[
\frac{d}{ds} h(\gamma_v(s)) = 2Re(\nabla h, \gamma_v'(s))
\]

\[
= \frac{1}{s} 2Re(\nabla h, s\gamma_v'(s))
\]

\[
= \frac{1}{s} Re(\nabla h, \nabla r^2(x, \gamma_v(s))) = 0.
\]

Here \( \nabla h = \nabla^\alpha h \frac{\partial}{\partial z^\alpha} \) and \( \gamma_v'(s) = \frac{dz^\alpha(s)}{ds} \frac{\partial}{\partial z^\alpha} \). Therefore, \( V \) is totally geodesic. This in particular shows that there is no singular points in \( V \), which rules out the possibility that \( V \) is union of several totally geodesic submanifolds with singular intersections (in which case the heat kernel comparison (2.3) has strictly inequality). Moreover, by lifting the computation to the universal cover we can assume that \( M \) is simply-connected. Then the manifold \( M \) splits by Theorem 0.1 of [NT2], more precisely, Theorem 2.1 and Corollary 2.1 of [NT2], since \( N_{\alpha\beta} = \nabla^\alpha \nabla^\beta H - \frac{\nabla^\alpha H \nabla^\beta H}{H} + \frac{1}{t} H g_{\alpha\beta} \geq 0 \) and its null space is at least of \( m - s \) dimension, by (2.5), for any \( x, y \in V \) and \( t > 0 \). Notice that \( N_{\alpha\beta} \) does not satisfies the linear Lichnerowicz heat equation. However the inequality (1.21), satisfied by \( N_{\alpha\beta} \) (since \( \epsilon = 0 \)), is enough for the argument in the proof of Corollary 2.1 of [NT2]. Then \( M = M_1 \times M_2 \) such that the tangent space of \( M_2 \) consists of the null space of \( N_{\alpha\beta} \). The factor \( M_2 \) is isometric to \( \mathbb{C}^{m-s} \) follows from the same argument in the proof of Corollary 1.1, since \( -\nabla^\alpha \nabla^\beta H + \frac{\nabla^\alpha H \nabla^\beta H}{H} - \frac{1}{t} H g_{\alpha\beta} \equiv 0 \). (One can also use Corollary 1.3 of [N2, page 331].) More precisely, if we write a point \( x \in M \) as \( x = (x_1, x_2) \) according to the splitting, we can write the heat kernel \( H(x, y, t) = H_1(x_1, y_1, t) H_2(x_2, y_2, t) \). Then on \( M_2 \) we have that \( (\log H_2(x_2, y_2))_{\alpha\beta} + \frac{1}{t} g_{\alpha\beta} = 0 \) by the definition of the splitting. Therefore one can apply Corollary 1.1 to conclude that \( M_2 = \mathbb{C}^k \). If (2.4) holds equality for some \( x \in M \), it implies that the right hand side of (2.5) is zero. Then the argument as above also applies. Note that we may not have \( V \) totally geodesic since \( x \) may be a singular point.

Remark 2.1. 1) In the case \( V \) is not smooth, the existence on \( K(x, y, t) \) was justified in the work of Li and Tian [LT]. They also obtained a similar upper bound estimate as (2.3) for the special case \( M = \mathbb{P}^m \) with the Fubini-Study metric using very different method. Their method produces better upper bound for the special case \( M = \mathbb{P}^m \). Our estimate here works for general Kähler manifolds with nonnegative bisectional curvature.
2) The similar heat kernel comparison was first proved in [CLY2] for minimal submanifolds in space forms by quite different method. The result in Theorem 2.1 is more general than [CLY2] in the sense that it holds for any Kähler manifolds with nonnegative sectional curvature in stead of space forms. However it is also more restrictive since it only applies to analytic subvarieties.

3) In the part (ii) of Theorem 2.1, the manifold $\tilde{M}_1$ may not be $\tilde{V}$. This could happen, for example, in the case $M = \mathbb{C}^m$ and $V$ is a union of two hyper-planes and $x$ lies on the intersection subvariety. If we further assume that equality holds for all smooth points $x \in V$, we do have the same conclusion as part (i).

4) The monotonicity (2.4) is enough for applications in [N4]. Namely, one can prove the comparison results on the dimensions of polynomial growth holomorphic function spaces obtained in [N4], using (2.4) in stead of the other Li-Yau-Hamilton inequality proved therein. In fact one can derive the results in [N4] through the following corollary, which is a special case of Theorem 2.1 (or Corollary 2.2) of [N4]. It is however enough for the applications considered in [N4].

**Corollary 2.1.** Let $f$ be a holomorphic function. Let $V = Z(f)$, the zero locus of $f$. Denote

$$w(x, t) = \int_M H(x, y, t) \Delta \log |f|^2(y) \, d\mu(y).$$

Then

$$tw(x, t) = (\pi t) \int_V H(x, y, t) \, dA_V(y).$$

Moreover

$$\frac{\partial}{\partial t} (tw(x, t)) \geq 0.$$

If the equality holds for some point $x \in M$ and some positive time $t$, then the universal cover (of $M$) $\tilde{M}$ splits at least a factor of $\mathbb{C}$.

**Proof.** Notice that $\Delta = g^{\alpha \beta} \frac{\partial}{\partial z^\alpha \partial \bar{z}^\beta}$, which differs a factor of 4 from [N4]. Let $f$ be a holomorphic function defined on $\tilde{M}$. Here we do require that $f$ does not grow too fast. For example, it will be sufficient if $f$ is of polynomial growth or is of finite order in the sense of Hadamard (see (3.1) of [N4] for precise definition). We can write

$$v(x, t) = \int_M H(x, y, t) \log |f|^2(y) \, d\mu(y)$$

as a solution to the heat equation with initial value $\log |f|^2(y)$. The requirement on $f$ is to make such representation formula of $v(x, t)$ meaningful. Then $w(x, t)$ is defined to be $\frac{\partial}{\partial t} v(x, t)$ as in [N4]. By the definition of $w(x, t)$ in (2.7) it is easy to see that $w(x, t)$ is also
a solution to the heat equation with initial data given by the measure $\Delta \log |f|^2$. It is easy to see that $w(x, t) = \frac{\partial}{\partial t} v(x, t)$. By the Poincaré-Lelong formula we know that

$$(\pi t) \int_M H(x, y, t) \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2(y)\right) \wedge \frac{\omega^{m-1}}{(m-1)!} = (\pi t) \int_V H(x, y, t) dA_V(y).$$

On the other hand, the direct calculation shows that the left hand side of the above equation is equal to

$$t \int_M H(x, y, t) \Delta \log |f|^2(y) \frac{\omega^m}{m!} = t \int_M H(x, y, t) \Delta \log |f|^2(y) d\mu(y) = tw(x, t).$$

This proves the first statement of the corollary. The monotonicity (2.9) is just a special case (ii) of Theorem 2.1. The proof above also shows that $w(x, t)$ has the same meaning as in Lemma 3.1 and Theorem 3.1, Theorem 4.1 of [N4].

The case with Ricci flow, namely $\epsilon = 1$, can be formulated similarly. In order to do so we have to explain some notations. For a time-dependent metrics deformed by the Kähler-Ricci flow equation (1.1), we call $H(x, y, t, t_0)$ a fundamental solution to the heat equation, if for any $u(x, t_0) = \varphi(x)$, where $\varphi(x)$ is a compact-supported smooth function, the solution $u(x, t)$ to the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right) u(x, t) = 0$ with initial condition $u(x, t_0) = \varphi(x)$ is given by the formula

$$u(x, t) = \int_M H(x, y, t, t_0) \varphi(y) d\mu(y).$$

It is easy to check that $H(x, y, t, t_0)$ must satisfies the forward conjugate heat equation (1.2). When the meaning is clear in the context (mostly $t_0 = 0$) we just simply write $H(x, y, t, t_0)$ as $H(x, y, t)$. Adapting the notation used in the proof of Theorem 2.1, the restriction of the Kähler-Ricci flow (1.5) on $V$ reads as $\frac{\partial}{\partial t} g_{ij}^V(x, t) = -R_{ij}^V(x, t)$. We denoted $g_{ij}^V R_{ij}^V$ by $\mathcal{R}^V$. It is easy to see that $\mathcal{R}^V$ is well-defined. Therefore the fundamental solution to the heat equation on $V$, $K_V(x, y, t)$ satisfies

$$(2.10) \quad \left(\frac{\partial}{\partial t} - \Delta\right) v(x, t) = \mathcal{R}_V(x, t)v(x, t).$$

We have the following comparison and monotonicity result.

**Theorem 2.1'**. Let $M$ be a complete Kähler manifold with bounded nonnegative bisectional curvature. Let $H(x, y, t)$ be a fundamental solution to the heat equation on $M$. Let $V$ be a complex subvariety of $M$ of dimension $s$. Let $K_V(x, y, t)$ be the fundamental solution to the heat equation (with respect to the induced metrics) on $V$. Then we have (2.3) and (2.4). Moreover, the equality (for positive $t$), in either cases, implies that the universal cover (of $M$) $\tilde{M}$ has the splitting $\tilde{M} = \tilde{M}_1 \times \mathbb{E}^k$, where $\mathbb{E}^k$ is an gradient expanding Kähler-Ricci soliton of dimension $k \geq m - s$. 
Remark 2.2. One can think (2.4) as a dual version of Perelman’s monotonicity of the reduced volume since the reduced volume of [P, Section 7] is, in a sense, a ‘weighted volume’ of $M$ (with weight being the heat kernel of a ‘potentially infinity dimensional manifold’ restricted to $M$, as explained in Section 6 of [P]), while here the monotonicity is on the ‘weighted volume’ of complex submanifolds with weight being the heat kernel of $M$ restricted to the submanifold. The reduced volume monotonicity of Perelman has important applications in the study of Ricci flow. We expect that (2.4) will have some applications in understanding the Kähler-Ricci flow and its effect on the complex geometry of analytic subvarieties.

Taking the trace of the matrix estimate in Theorem 1.2 and integrating along the space-time path as in [LY], we can have the following Harnack estimates for the positive solutions to the forward conjugate heat equation. This gives a sharp version of the previous rough estimate of Guenther in [Gu].

Corollary 2.1. Let $(M, g(t))$ be a solution to Ricci flow (1.1) and $u(x, t)$ be a positive solution to (1.2). Then

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} + \frac{m}{t} \geq 0$$

and for any $t_2 > t_1$,

$$u(x_2, t_2) t_2^m \geq u(x_1, t_1) t_1^m \exp\left(-\inf_\gamma \int_{t_1}^{t_2} |\gamma'(t)|^2 dt\right) \tag{2.11}$$

Here $\gamma(t)$ is a path with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.

Note that we do not have the factor 4 due to our choice of $\Delta$ and the gradient $|\nabla f|^2 = g^{\alpha\bar{\beta}} f_\alpha f_{\bar{\beta}}$. Also

$$|\gamma'(t)|^2 = g_{\alpha\bar{\beta}} \frac{dz^\alpha}{dt} \frac{dz^{\bar{\beta}}}{dt}.$$ 

We list this consequence here since it implies the monotonicity of $t^m u(x, t)$.

§3 Interpolation between Perelman’s entropy formula and the new Li-Yau-Hamilton inequality.

The purpose of this section is two-folded. First we give a different proof of Theorem 1.2. The second purpose is to show that the by-product of this different proof also implies Perelman’s monotonicity of entropy, as well as the energy. The computation in this section has its real version. See [Ch2] and the forth-coming book [CLN]. The main computation is summarized in equation (3.7) below, which we call a pre-Li-Yau-Hamilton equality. The equation (3.7) can also be viewed as a matrix version of Perelman’s entropy monotonicity formula. In a sense, one can think that the matrix Li-Yau-Hamilton inequality proved in Section 1 is dual to Perelman’s entropy formula in [P, Section 3].

Consider the Kähler-Ricci flow:

$$\frac{\partial}{\partial \tau} g_{\alpha\bar{\beta}} = \epsilon R_{\alpha\bar{\beta}} \tag{3.1}$$
where $\epsilon$ is a parameter and the *conjugate heat equation*:

\[
(3.2) \quad \left( \frac{\partial}{\partial \tau} - \Delta + \epsilon R \right) u(x, \tau) = 0.
\]

When $\epsilon < 0$, (3.1) is a forward Ricci flow equation and (3.2) become *forward conjugate heat equation*. The equations look different from those in Section 1 since in this section the case of $\epsilon < 0$ corresponds to the forward Ricci flow and the case of $\epsilon > 0$ corresponds to the backward Ricci flow. For example $\epsilon = 1$ is exactly the setting for Perelman’s entropy and energy monotonicity. Notice that (3.2) becomes the *backward conjugate heat equation* for $\epsilon = 1$. For the positive solution $u(x, \tau)$ we define the $(1, 1)$ tensor $Z_{\alpha\overline{\beta}}$ by

\[
Z_{\alpha\overline{\beta}} = -(\log u)_{\alpha\overline{\beta}} + \epsilon R_{\alpha\overline{\beta}}.
\]

Let $\Delta_L$ denote the Lichnerowicz Laplacian on $(1, 1)$ tensors, which is defined by

\[
\Delta_L \eta_{\alpha\overline{\beta}} = \Delta \eta_{\alpha\overline{\beta}} + R_{{\alpha\overline{\beta}\gamma\delta}} \eta_{\gamma\delta} - \frac{1}{2} (R_{\alpha\overline{\beta}} \eta_{p\overline{p}} + \eta_{\alpha\overline{p}} R_{p\overline{p}})
\]

for any Hermitian symmetric $(1, 1)$ tensor $\eta_{\alpha\overline{\beta}}$. It is known, see for example [C1], that

\[
\left( \frac{\partial}{\partial \tau} - \Delta_L \right) R_{\alpha\overline{\beta}} = -(1 + \epsilon) \Delta_L R_{\alpha\overline{\beta}}.
\]

The direct calculation as in [NT1, Lemma 2.1] shows that

**Lemma 3.1.** For any $C^2$-function $f(x, \tau)$

\[
(3.3) \quad \left( \frac{\partial}{\partial \tau} - \Delta_L \right) f_{\alpha\overline{\beta}} = \left[ \left( \frac{\partial}{\partial \tau} - \Delta \right) f \right]_{\alpha\overline{\beta}}.
\]

**Remark 3.1.** The similar result as the above lemma hold for Ricci flow on Riemannian manifolds. Please see [CLN] for details.

Now with the help of Lemma 3.1 and 1.1 we can calculate $\left( \frac{\partial}{\partial \tau} - \Delta_L \right) Z_{\alpha\overline{\beta}}$ as follows using the equation $\left( \frac{\partial}{\partial \tau} - \Delta_L \right) (\log u) = -\epsilon R + |\nabla \log u|^2$.

\[
(3.4) \quad \left( \frac{\partial}{\partial \tau} - \Delta_L \right) Z_{\alpha\overline{\beta}} = (\epsilon R - |\nabla \log u|^2)_{\alpha\overline{\beta}} - \epsilon (1 + \epsilon) \Delta_L R_{\alpha\overline{\beta}}
\]

\[
= \epsilon (R)_{\alpha\overline{\beta}} - \left( g^{\gamma\overline{\delta}} (\log u)_{\gamma} (\log u)_{\overline{\delta}} \right)_{\alpha\overline{\beta}} - \epsilon (1 + \epsilon) \Delta_L R_{\alpha\overline{\beta}}
\]

\[
= -R_{\alpha\overline{\beta}\gamma\delta}(\log u)_{\gamma}(\log u)_{\overline{\delta}} - (\log u)_{\alpha\gamma}(\log u)_{\overline{\gamma}\delta} - (\log u)_{\alpha\overline{\gamma}}(\log u)_{\gamma\delta} - \epsilon^2 \Delta L R_{\alpha\overline{\beta}}.
\]
Hence

\[(3.5)\]
\[
\left( \frac{\partial}{\partial \tau} - \Delta_L \right) Z_{\alpha\beta} = -\epsilon^2 \left( \Delta R_{\alpha\beta} + R_{\alpha\beta\gamma\delta} R_{\gamma\delta} + \nabla_\gamma R_{\alpha\beta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) + \nabla_\gamma R_{\alpha\beta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) \\
+ R_{\alpha\beta\gamma\delta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) \left( \frac{1}{\epsilon} \nabla_\delta \log u \right) \\
+ \epsilon^2 R_{\alpha\gamma} R_{\gamma\beta} - (\log u)_{\alpha\gamma} (\log u)_{\gamma\beta} - (\log u)_{\alpha\gamma} (\log u)_{\gamma\beta} \\
+ \nabla_\gamma (Z_{\alpha\beta}) \nabla_\gamma \log u + \nabla_\gamma (Z_{\alpha\beta}) \nabla_\gamma \log u.
\]

Regrouping terms yields

\[(3.6)\]
\[
\left( \frac{\partial}{\partial \tau} - \Delta_L \right) Z_{\alpha\beta} = -\epsilon^2 \left( \Delta R_{\alpha\beta} + R_{\alpha\beta\gamma\delta} R_{\gamma\delta} + \nabla_\gamma R_{\alpha\beta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) + \nabla_\gamma R_{\alpha\beta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) \\
+ R_{\alpha\beta\gamma\delta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) \left( \frac{1}{\epsilon} \nabla_\delta \log u \right) \\
- (\log u)_{\alpha\gamma} (\log u)_{\gamma\beta} + \nabla_\gamma (Z_{\alpha\beta}) \nabla_\gamma \log u + \nabla_\gamma (Z_{\alpha\beta}) \nabla_\gamma \log u \\
+ \frac{1}{2} Z_{\alpha\gamma} \left( \epsilon R_{\gamma\beta} + (\log u)_{\gamma\beta} \right) + \frac{1}{2} \left( \epsilon R_{\alpha\gamma} + (\log u)_{\alpha\gamma} \right) Z_{\gamma\beta}.
\]

Let

\[\tilde{Z}_{\alpha\beta} = Z_{\alpha\beta} - \frac{1}{\tau} g_{\alpha\beta}\]

and

\[Y_{\alpha\beta} = \Delta R_{\alpha\beta} + R_{\alpha\beta\gamma\delta} R_{\gamma\delta} + \nabla_\gamma R_{\alpha\beta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) + \nabla_\gamma R_{\alpha\beta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) \\
+ R_{\alpha\beta\gamma\delta} \left( \frac{1}{\epsilon} \nabla_\gamma \log u \right) \left( \frac{1}{\epsilon} \nabla_\delta \log u \right).
\]

Notice that \(\tilde{Y}_{\alpha\beta}\), defined after (1.20) in Section 1, is related to \(Y_{\alpha\beta}\) above through the equation

\[\tilde{Y}_{\alpha\beta} = Y_{\alpha\beta} - \frac{R_{\alpha\beta}}{\epsilon \tau}\]

(remember that \(-\epsilon\) here corresponding to \(\epsilon\) in Section 1). From (3.6), we can derive the equation for \(\tilde{Z}_{\alpha\beta}\) as follows.

**Lemma 3.2.** \((Chow-Ni)\)

\[(3.7)\]
\[
\left( \frac{\partial}{\partial \tau} - \Delta_L \right) \tilde{Z}_{\alpha\beta} = \left( \frac{\partial}{\partial \tau} - \Delta_L \right) Z_{\alpha\beta} + \frac{1}{\tau^2} g_{\alpha\beta} - \frac{1}{\tau} \epsilon R_{\alpha\beta} \\
- \left( \epsilon^2 Y_{\alpha\beta} - \frac{\epsilon}{\tau} R_{\alpha\beta} \right) - (\log u)_{\alpha\gamma} (\log u)_{\gamma\beta} \\
+ \nabla_\gamma (\tilde{Z}_{\alpha\beta}) \nabla_\gamma \log u + \nabla_\gamma (\tilde{Z}_{\alpha\beta}) \nabla_\gamma \log u \\
+ \frac{1}{2} \tilde{Z}_{\alpha\gamma} \left( \epsilon R_{\gamma\beta} + (\log u)_{\gamma\beta} \right) + \frac{1}{2} \left( \epsilon R_{\alpha\gamma} + (\log u)_{\alpha\gamma} \right) \tilde{Z}_{\gamma\beta} - \frac{1}{\tau} \tilde{Z}_{\alpha\beta}.
\]
Notice that $\bar{Z}_{\alpha\bar{\beta}} = \frac{1}{u} \bar{N}_{\alpha\bar{\beta}}$. With some labor one can check that (1.21) and (3.7) are equivalent. Namely one can derive one from the other, keeping in mind that $-\epsilon$ here corresponds $\epsilon$ in Section 1. One can also write (3.7) as

$$(3.7') \quad \left( \frac{\partial}{\partial \tau} - \Delta_L \right) \bar{Z}_{\alpha\bar{\beta}} = \left( \frac{\partial}{\partial \tau} - \Delta_L \right) Z_{\alpha\bar{\beta}} + \frac{1}{\tau^2} g_{\alpha\bar{\beta}} - \frac{1}{\tau} \epsilon R_{\alpha\bar{\beta}}$$

$$= - \left( \epsilon^2 Y_{\alpha\bar{\beta}} - \frac{\epsilon}{\tau} R_{\alpha\bar{\beta}} \right) - (\log u)_{\alpha\gamma} (\log u)_{\bar{\gamma}\bar{\beta}} + \nabla_\gamma (\bar{Z}_{\alpha\bar{\beta}}) \nabla_\bar{\gamma} \log u + \nabla_\bar{\gamma} (\bar{Z}_{\alpha\bar{\beta}}) \nabla_\gamma \log u$$

$$+ \frac{1}{2} \bar{Z}_{\alpha\bar{\gamma}} \left( \epsilon R_{\gamma\bar{\beta}} + (\log u)_{\gamma\bar{\beta}} - \frac{1}{\tau} g_{\gamma\bar{\beta}} \right)$$

$$+ \frac{1}{2} \left( \epsilon R_{\alpha\bar{\gamma}} + (\log u)_{\alpha\bar{\gamma}} - \frac{1}{\tau} g_{\alpha\bar{\gamma}} \right) \bar{Z}_{\bar{\gamma}\bar{\beta}}.$$ 

For $\epsilon < 0$, applying Lemma 1.4, the fundamental result of H.-D. Cao, we know that $\epsilon^2 Y_{\alpha\bar{\beta}} - \frac{\epsilon}{\tau^2} R_{\alpha\bar{\beta}} \geq 0$ under the assumption that $M$ is a complete Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Hence the tensor maximum principle and (3.7) implies that $\bar{Z}_{\alpha\bar{\beta}} \leq 0$, which is equivalent to the statement of Theorem 1.2. Namely, (3.7) does lead to another proof of Theorem 2.1.

Even though the computation (3.7) and (1.21) are essentially equivalent, (3.7) has the advantage that when $\epsilon = 1$ it also implies Perelman’s energy/entropy monotonicity formulae. The following is a more detailed computation of this claim. Let $f = -\log u$, $Z = g_{\alpha\bar{\beta}} Z_{\alpha\bar{\beta}}$. The tracing of (3.6) gives

$$(3.8) \quad \left( \frac{\partial}{\partial \tau} - \Delta \right) Z = -\epsilon R_{\alpha\beta} Z_{\alpha\bar{\beta}} - \epsilon^2 g_{\alpha\bar{\beta}} Y_{\alpha\bar{\beta}} - (f)_{\alpha\gamma} (f)_{\bar{\gamma}\bar{\alpha}} - \nabla_\gamma Z \nabla_\bar{\gamma} f - \nabla_\bar{\gamma} Z \nabla_\gamma f$$

$$+ Z_{\alpha\bar{\beta}} (\epsilon R_{\beta\bar{\alpha}} - f_{\beta\bar{\alpha}})$$

and

$$g_{\alpha\bar{\beta}} Y_{\alpha\bar{\beta}} = \Delta R + R_{\alpha\beta} R_{\alpha\bar{\beta}} - \nabla_\gamma R \left( \frac{1}{\epsilon} \nabla_\bar{\gamma} f \right) - \left( \frac{1}{\epsilon} \nabla_\gamma f \right) \nabla_\bar{\gamma} R + R_{\alpha\bar{\beta}} \left( \frac{1}{\epsilon} f_{\bar{\alpha}} \right) \left( \frac{1}{\epsilon} f_{\beta} \right).$$

The following observation of Ben Chow is also useful.

**Lemma 3.3. (Chow)** In the case $\epsilon = 1$, we have that

$$(3.9) \quad \int_M \left( g_{\alpha\bar{\beta}} Y_{\alpha\bar{\beta}} \right) u \, d\mu = \int_M \left( R_{\alpha\bar{\beta}} (R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}}) \right) u \, d\mu.$$ 

**Proof.** Follows from the integration by parts and the second Bianchi identity $R_{\gamma} = R_{\gamma\bar{\alpha}\alpha}.$
**Remark 3.2.** Please refer to [Ch2] and [CLN] for the Riemannian version of the above identity.

Recall the definition of energy $\mathcal{F}$.

$$
\mathcal{F}(g, u, \tau) = \int_M \left( \frac{|\nabla u|^2}{u} + R u \right) d\mu.
$$

Then (3.8) (with $\epsilon = 1$) and Lemma 3.3 implies the following result.

**Proposition 3.1.** (Perelman)

(3.10)

$$
\frac{d}{d\tau} \mathcal{F}(g, u, \tau) = -\int_M (|R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}}|^2 + |f_{\alpha\bar{\beta}}|^2) u \, d\mu.
$$

Note that this is nothing but the energy monotonicity formula of Perelman in Section 1 of [P]. We show that it follows from (3.8).

**Proof of Proposition 3.1.** Let $\epsilon = 1$ in (3.8) we have that

(3.8')

$$
\left( \frac{\partial}{\partial \tau} - \Delta \right) Z = -g^{\alpha\bar{\beta}} Y_{\alpha\bar{\beta}} - (f)_{\alpha\gamma}(f)_{\bar{\gamma}\bar{\alpha}} - \nabla_{\gamma} Z \nabla_{\bar{\gamma}} f - \nabla_{\bar{\gamma}} Z \nabla_{\gamma} f - Z_{\alpha\bar{\beta}} f_{\beta\bar{\alpha}}.
$$

Now the result follows from direct computation on $\frac{d}{d\tau} \mathcal{F}(g, u, \tau) = \frac{d}{d\tau} \int_M Z \, d\mu$, by applying Lemma 3.3.

Similarly, if we trace (3.7) and denote $\tilde{Z} = g^{\alpha\bar{\beta}} \tilde{Z}_{\alpha\bar{\beta}}$, we have that

(3.11)

$$
\left( \frac{\partial}{\partial \tau} - \Delta \right) \tilde{Z} = -\epsilon R_{\alpha\bar{\beta}} \tilde{Z}_{\alpha\bar{\beta}} - \epsilon^2 g^{\alpha\bar{\beta}} Y_{\alpha\bar{\beta}} + \epsilon \mathcal{R} - (f)_{\alpha\gamma}(f)_{\bar{\gamma}\bar{\alpha}} - \nabla_{\gamma} \tilde{Z} \nabla_{\bar{\gamma}} f - \nabla_{\bar{\gamma}} \tilde{Z} \nabla_{\gamma} f + 
\tilde{Z}_{\alpha\bar{\beta}} (\epsilon R_{\beta\bar{\alpha}} - f_{\beta\bar{\alpha}}) - \frac{1}{\tau} \tilde{Z}.
$$

For $\epsilon = 1$, integration by parts as before gives

(3.12)

$$
\frac{d}{d\tau} \int_M \tilde{Z} u \, d\mu_\tau = -\int_M \left( |\tilde{Z}_{\alpha\bar{\beta}}|^2 + |f_{\alpha\bar{\beta}}|^2 \right) u \, d\mu_\tau - \frac{2}{\tau} \int_M \tilde{Z} u \, d\mu_\tau.
$$

The above equation is equivalent to Perelman’s entropy monotonicity formula due to the following consideration. Let

$$
\tilde{\mathcal{N}}(g, u, \tau) := -\int_M u \log u \, d\mu_\tau - m \log(\pi \tau) - m.
$$

Then Perelman’s entropy

$$
\mathcal{W}(g, u, \tau) = \int_M \left[ \tau(2\Delta \tilde{f} - |
\nabla \tilde{f} |^2 + \mathcal{R} + \tilde{f} - 2m \right) u \, d\mu_\tau,
$$
where $\bar{f} = -\log u - m \log(\pi \tau)$, can be expressed as

$$W(g, u, \tau) = \frac{d}{d\tau} (\tau \tilde{N}) = \tau \int_M \tilde{Z} u \, d\mu_\tau + \tilde{N}. \quad (3.13)$$

Therefore

$$\frac{d}{d\tau} W = \tau \frac{d}{d\tau} \int_M \tilde{Z} u \, d\mu_\tau + 2 \int_M \tilde{Z} u \, d\mu_\tau$$

$$= -\tau \int_M \left( |\tilde{Z}_{\alpha\beta}|^2 + |\bar{f}_{\alpha\beta}|^2 \right) u \, d\mu_\tau \quad (3.14)$$

which is nothing but the entropy formula of Perelman in [P, Section 3].

As pointed out in [P], there exists a statistical mechanics analogy of Perelman’s entropy. If we identify the quantities above with the notation of [Ev2, Chapter I and VII], $\tau$ is the temperature; $-\tilde{N}$ defined above is the log of the distribution function in [Ev2]; $-W$ is the entropy $S$ in [Ev2]; $\tau \tilde{N}$ is the free energy $F$; $\frac{\partial \tilde{N}}{\partial (1/\tau)}$ is the energy $E$ in [Ev2] (which is nonnegative by Proposition 1.2 of [P]) and the first equation of (3.13) is just the well-known equation $S = -\frac{\partial F}{\partial \tau}$ from thermodynamics. The negation of the right hand side of (3.14), measuring the deviation from an shrinking soliton, is called the heat capacity in [Ev2]. The entropy formula (3.14) implies the concavity of $S$ in $E$, one of the defining properties for the entropy. This analogy also holds for the solution to the heat equation with respect to a fixed metric Riemannian metric with nonnegative Ricci curvature [N3].

**Remark 3.3.** For the case of $\epsilon = 0$, the similar computation as above gives the monotonicity formula in [N3]. The strange thing is that one can not get nice monotonicity formula in the case $\epsilon \neq -1$ or 0. Namely, the interpolation formula (3.6)/(3.7) does not give nice interpolation for energy/entropy after integration on $M$. One can view (3.6)/(3.7) as a matrix version of the energy/entropy monotonicity formula for $\epsilon > 0$. This partially answers one of the questions raised in the end of [N3] (still not satisfactory though). On the other hand, when $\epsilon < 0$, one can not get entropy monotonicity out of (3.7). Instead we have a pointwise Li-Yau-Hamilton inequality. However, for $\epsilon = 0$, both entropy and the differential Harnack follows from (3.6) (See [N3]). The above discussion indicates that our new matrix Li-Yau-Hamilton inequality is dual to the entropy monotonicity of Perelman in some sense and there may perhaps be certain profound duality behind the scene.

Another puzzling point is that so far we have not been able to verify a matrix Li-Yau-Hamilton inequality analogue to Theorem 1.2 for the Ricci flow on Riemannian manifolds, even though the above computation (3.7) holds for $\epsilon > 0$ for the Ricci flow on Riemannian manifolds (which was done in [Ch2]). In short, the interpolation between positive and negative $\epsilon$ by now only works in Kähler category. The validity of the Riemannian case is pending on the verification of a new matrix Li-Yau-Hamilton estimate similar to Hamilton’s famous work [H1]. Please see Remark 5.2 for further details.

§4 A local monotonicity formula and its applications.

This section is inspired by the work of Ecker in [E1-2]. Let $M$ be a complete Kähler manifold with nonnegative bisectional curvature (unless specified otherwise). In this section
we study the localization of the previous established monotonicity (in Section 2) for a fixed Kähler metric. (We leave the Kähler-Ricci flow case to a later discussion.) In [E1], the localized monotonicity formula is proved for mean curvature flow in Euclidean spaces. Since we are dealing with curved spaces here, we need some extra ingredients, which includes Theorem 2.1 in Section 2, the complex Hessian comparison theorem on distance functions proved in [LW] and [CN] recently and the well-known heat kernel estimates of Li-Yau on complete Riemannian manifolds with nonnegative Ricci curvature, which states that

$$\frac{C^{-1}(n)}{V_x(\sqrt{t})} \exp\left(-\frac{r^2(x)}{3t}\right) \leq H(x, y, t) \leq \frac{C(n)}{V_x(\sqrt{t})} \exp\left(-\frac{r^2(x)}{5t}\right)$$

for some $C(n) > 0$, where $n$ is the real dimension of the manifold considered. As applications we prove an elliptic ‘monotonicity principle’ for complex subvarieties in $M$. It can then be applied to prove a manifold version of Stoll’s theorem.

Before we prove a localized version of Theorem 2.1, we need to introduce some functions (notations). Let $V$ be an analytic subvariety (of $M$) of complex dimension $s$. For the simplicity of the notation, Let $H(x, y, \tau)$ be the fundamental solution to the heat equation $\left(\frac{\partial}{\partial \tau} - \Delta\right) u(x, \tau) = 0$. When $x, y \in V$, we denote $(\pi \tau)^{m-s}H(x, y, \tau)$ by $H_V(x, y, \tau)$. For any fixed $(x_0, t_0)$ with $x_0 \in V$, we denote $H_V(x_0, y, t_0)$ by $\hat{H}_{(x_0, t_0), V}(y, t)$. For any $\rho > 0$, we also introduce a cut-off function

$$\varphi_{(x_0, t_0), \rho}(y, t) = \left(1 - \frac{r^2(x_0)}{\rho^2} + s(t - t_0)\right)_+$$

where $f_+(x) = \max(f, 0)$ for any function $f$, $r_{y_0}(x)$ is the distance function (of $M$) from $x_0$ to $y$. It is easy to see that $\varphi_{(x_0, t_0), \rho}$ is supported in $B_{x_0}(\sqrt{\rho^2 - s(t - t_0)})$.

The following simple lemma is useful.

**Lemma 4.1.** On $V$,

$$\left(\frac{\partial}{\partial t} - \Delta_V\right) \varphi_{(x_0, t_0), \rho}(y, t) \leq 0.$$  

Here $\Delta_V$ denotes the Laplacian operator with respect to the induced Kähler metric on $V$ (strictly speaking only regular part of $V$).

**Proof.** For any $y \in V$, choose a complex coordinate $(z_1, \cdots, z_m)$ as in the proof of Theorem 2.1. Namely $z_\alpha = 0$ on $V$ for any $\alpha > s$. We also use the index convention as in the proof of Theorem 2.1. Namely, $1 \leq i, j, k, \cdots \leq s$ and $s + 1 \leq a, b, c, \cdots \leq m$. Direct computation shows that

$$\left(\frac{\partial}{\partial t} - \Delta_V\right) \left(s(t - t_0) + r^2_{x_0}(y)\right) = s - g^{i\bar{j}} \left(r^2_{ij}\right) \geq s - g^{i\bar{j}} g_{ij} \geq 0.$$

Here we have used the Hessian comparison theorem on the distance functions proved in [LW] (see also [CN, Corollary 1.1]).

The following is a localized version of part (ii) of Theorem 2.1.
Proposition 4.1. Let

\[ E_{\mathcal{V},x_0,t_0,t_1}(t) = \int_{\mathcal{V}} \varphi(x_0,t_1),\rho(y,t) \hat{H}(x_0,t_0),\mathcal{V}(y,t) \, dA_{\mathcal{V}}. \]  

When in the right context we also it briefly denote by \( E_{\mathcal{V}} \). Then

\[ \frac{d}{dt}E_{\mathcal{V}}(t) \leq -\int_{\mathcal{V}} |\nabla \log \hat{H}(x_0,t_0),\mathcal{V}|^2 \varphi(x_0,t_1),\rho \hat{H}(x_0,t_0),\mathcal{V} \, dA_{\mathcal{V}}. \]

Proof. The computation (2.5) implies that

\[ \left( \frac{\partial}{\partial t} + \Delta_{\mathcal{V}} \right) \hat{H}(x_0,t_0),\mathcal{V} \leq -|\nabla \log \hat{H}(x_0,t_0),\mathcal{V}|^2 \hat{H}(x_0,t_0),\mathcal{V}. \]

By Lemma 4.1 we have that

\[
\begin{align*}
\frac{d}{dt}E_{\mathcal{V}}(t) &= \int_{\mathcal{V}} \left( \frac{\partial}{\partial t} \varphi(x_0,t_1),\rho \right) \hat{H}(x_0,t_0),\mathcal{V} + \varphi(x_0,t_1),\rho \left( \frac{\partial}{\partial t} \hat{H}(x_0,t_0),\mathcal{V} \right) \, dA_{\mathcal{V}} \\
&= \int_{\mathcal{V}} \left( \left( \frac{\partial}{\partial t} - \Delta_{\mathcal{V}} \right) \varphi(x_0,t_1),\rho \right) \hat{H}(x_0,t_0),\mathcal{V} + \varphi(x_0,t_1),\rho \left( \left( \frac{\partial}{\partial t} + \Delta_{\mathcal{V}} \right) \hat{H}(x_0,t_0),\mathcal{V} \right) \\
&\quad + \int_{\mathcal{V}} \left( \Delta_{\mathcal{V}} \varphi(x_0,t_1),\rho \right) \hat{H}(x_0,t_0),\mathcal{V} - \varphi(x_0,t_1),\rho \left( \Delta_{\mathcal{V}} \hat{H}(x_0,t_0),\mathcal{V} \right) \, dA_{\mathcal{V}} \\
&\leq -\int_{\mathcal{V}} |\nabla \log \hat{H}(x_0,t_0),\mathcal{V}|^2 \varphi(x_0,t_1),\rho \hat{H}(x_0,t_0),\mathcal{V} \, dA_{\mathcal{V}}.
\end{align*}
\]

Here we have used the observation that

\[
\int_{\partial(\mathcal{V} \cap B_{x_0}(\sqrt{\rho^2 + s(t_1 - t)}) \cup \mathcal{V}(\mathcal{V} \varphi(x_0,t_1),\rho),\mathcal{V})} \hat{H}(x_0,t_0),\mathcal{V}(\nabla_{\mathcal{V}} \varphi(x_0,t_1),\rho,\nu) \, dS \leq 0
\]

where \( \nu \) is the unit out-normal of \( \partial(\mathcal{V} \cap B_{x_0}(\sqrt{\rho^2 + s(t_1 - t)}) \cup \mathcal{V}(\mathcal{V} \varphi(x_0,t_1),\rho),\mathcal{V}) \) in \( \mathcal{V} \) and \( dS \) is the area integral of \( \partial(\mathcal{V} \cap B_{x_0}(\sqrt{\rho^2 + s(t_1 - t)}) \cup \mathcal{V}(\mathcal{V} \varphi(x_0,t_1),\rho),\mathcal{V}) \).

We denote the 2s-dimensional Hausdorff measure of set \( \mathcal{V} \cap B_{x_0}(\rho) \) by \( A_{\mathcal{V},x_0}(\rho) \). As a consequence we have the following elliptic ‘monotonicity principle’ (can be viewed a Bishop-Lelong lemma on manifolds).

Corollary 4.1. (Monotonicity principle) Let \( \delta(s) = \frac{1}{\sqrt{2+s}} \). There exists \( C = C(m,s) \) such that for any \( \rho' \in (0,\delta(s)\rho) \)

\[ \frac{A_{\mathcal{V},x_0}(\rho')(\rho')^{2(m-s)}}{V_{x_0}(\rho')} \leq C(m,s) \frac{A_{\mathcal{V},x_0}(\rho)\rho^{2(m-s)}}{V_{x_0}(\rho)}. \]
Proof. The proof follows essentially the argument of Proposition 3.5 in [E2]. Applying Proposition 4.1 with $t_1 = t_0 - \delta^2(s)\rho^2$ and $t_0$ replaced by $t_0 + \rho'^2$. Using the heat kernel upper bound of Li-Yau we have that

\begin{equation}
H_{(x_0, t_0 + \rho'^2), \mathcal{V}}(y, t_0 - \delta^2(s)\rho^2) \leq \left( \pi \delta^2(s)\rho^2 + \rho'^2 \right)^{m-s} \frac{C(m)}{V_{x_0}(\sqrt{\rho^2 + \delta^2(s)\rho^2})}
\end{equation}

\begin{equation}
\leq C(m, s) \frac{\pi \rho^2}{V_{x_0}(\delta(s)\rho)}^{m-s}
\end{equation}

\begin{equation}
\leq C(m, s) \frac{\rho^2}{V_{x_0}(\rho)}^{m-s}.
\end{equation}

Here we have used the Bishop volume comparison in the last inequality. Notice that $\varphi_{(x_0, t_1), \rho}(x, t_0 - \delta^2(s)\rho^2) \leq 1$ and supported inside $B_{x_0}(\rho)$. Hence (4.7) implies that

\begin{equation}
E_{\mathcal{V}}(t_0 - \delta^2(s)\rho^2) \leq C(m, s) \frac{A_{\mathcal{V}, x_0}(\rho)\rho^{2(m-s)}}{V_{x_0}(\rho)}.
\end{equation}

By Proposition 4.1 we know that

\begin{equation}
E_{\mathcal{V}}(t_0 - \delta^2(s)\rho^2) \geq E_{\mathcal{V}}(t_0 - \rho'^2).
\end{equation}

On the other hand, by Li-Yau’s heat kernel lower bound we also have that, for all $y \in B_{x_0}(\rho')$,

\begin{equation}
H_{(x_0, t_0 + \rho'^2), \mathcal{V}}(y, t_0 - \rho'^2) \geq \left( \pi (2\rho'^2) \right)^{m-s} \frac{C(m)}{V_{x_0}(\sqrt{2}\rho')} \exp \left( - \frac{r_{x_0}(y)}{6\rho'^2} \right)
\end{equation}

\begin{equation}
\geq C(m, s) \frac{\pi \rho'^2}{V_{x_0}(\rho')}^{m-s}.
\end{equation}

Again we have used the Bishop volume comparison theorem. Notice that for $y \in B_{x_0}(\rho')$

\begin{equation}
\varphi_{(x_0, t_1), \rho}(y, t_0 - \rho'^2) \geq 1 - \frac{\rho'^2 + s(\delta^2(s)\rho^2 + \rho'^2)}{\rho^2}
\end{equation}

\begin{equation}
\geq 1 - \delta^2(s)(1 + 2s)
\end{equation}

\begin{equation}
= \frac{1}{2}.
\end{equation}

Combining (4.10), (4.11) we have that

\begin{equation}
E_{\mathcal{V}}(t_0 - \rho'^2) \geq C(m, s) \frac{\rho'^2(m-s)}{V_{x_0}(\rho')} \frac{A_{\mathcal{V}, x_0}(\rho')}{V_{x_0}(\rho')}.
\end{equation}

Combining (4.8), (4.9) and (4.12) we complete the proof.
Remark 4.1. In Proposition 3.1.1 of [M1], a (considerably weaker) comparison on the relative volumes in the similar spirit of Corollary 4.1 was established for the zero divisors of holomorphic functions of polynomial growth, under further assumptions on \( M \) being of maximum volume growth and of quadratic curvature decay, using very different method.

The following consequence of Corollary 4.1 is somewhat surprising. The result sharpen the Bishop-Gromov volume comparison theorem in the presence of compact subvarieties.

Corollary 4.2. Let \( M^m \) be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that \( M \) contains a compact subvariety \( V \) of complex dimension \( s \). Then there exists \( C = C(m, s) > 0 \) such that for \( \delta(s) \rho \geq \rho' \gg 1 \),

\[
\frac{V_{x_0}(\rho)}{V_{x_0}(\rho')} \leq C \left( \frac{\rho}{\rho'} \right)^{2(m-s)}.
\]

In particular,

\[
\lim_{\rho \to \infty} \frac{V_{x_0}(\rho)}{\rho^{2(m-s)}} < \infty.
\]

An application of the above ‘monotonicity principle’ we can have another proof of Theorem 3.1 of [N4]. In fact the new proof gives a characterization of divisors defined by holomorphic functions of polynomial growth (also called ‘polynomial functions’ according to the notation in [W2]).

Theorem 4.1. Let \( M \) be a complete Kähler manifold with nonnegative bisectional curvature. Let \( V \) be a analytic divisor of \( M \). Define the Lelong number (elliptic) at infinity of \( V \) by

\[
\nu_\infty(V) = \sup_{x_0 \in M} \limsup_{\rho \to \infty} \frac{\pi \rho^2 A_{V,x_0}(\rho)}{V_{x_0}(\rho)}.
\]

Assume further that \( H^1(M, \mathcal{O}^*) = 0 \). Then \( V \) is defined by a polynomial function if and only if \( \nu_\infty(V) < \infty \). Moreover, if \( V = Z(f) \) for some \( f \in P_d(M) \) (the space of holomorphic functions of polynomial growth with degree at most \( d \)) then there exists a \( C(m) \) such that for any \( x_0 \in M \), the Lelong number \( \nu(x_0, V) \) at \( x_0 \) is bounded by

\[
\nu(x_0, V) \leq C(m) \nu_\infty(V)
\]

and

\[
\nu_\infty(V) \leq C(m) d.
\]

Proof. Notice that (4.14) follows from Corollary 4.1 directly. First assume that \( V \) is the zero divisor of a polynomial function \( f \) (\( \in P_d(M) \) for some \( d \)). We shall show that \( \nu_\infty(V) < \infty \) (in fact (4.15)). We follow the notation in Section 3 of [N4] (also Section 2 of this paper).
Let \( v(x,t) = \int_M H(x,y,t) \Delta \log |f|^2 \, dv_y \) and \( w(x,t) = \frac{\partial}{\partial t} v(x,t) \). By Corollary 2.1 we know that
\[
tw(x,t) = (\pi t) \int_V H(x,y,t) \, dA_v.
\]
By Li-Yau’s lower bound estimate on heat kernel we have that
\[
tw(x,t) \geq C(m) \frac{\pi t A_{V,x}(\sqrt{t})}{V_x(\sqrt{t})}.
\]
Then (4.15) follows from (3.13) of [N4].

Now we assume that \( \nu_\infty(V) < \infty \) we prove that \( V \) is the divisor of a polynomial function. First, by the solution to Cousin problem II (directly from the vanishing of the cohomology \( \mathcal{H}^1(M, \mathcal{O}^*) \)) we know that there exists a holomorphic function \( f \) such that \( Z(f) = V \). First we apply the ‘moment type estimate’ from [N1] to estimate \( tw(x,t) \) from above. Note that \( w(x,t) = \int_Y H(x,y,t) \, dA_v(y) \) is well-defined due to the assumption that \( \nu_\infty(V) < \infty \). By Corollary 2.1 we know that
\[
w(x,t) = \int_M H(x,y,t) \Delta \log |f|^2 \, d\mu(y).
\]
Applying the ‘moment type estimate’, Theorem 3.1 of [N1] we have that
\[
(4.16) \quad tw(x,t) \leq C(m) \nu_\infty(V).
\]
Now we define \( v(x,t) = \int_0^t w(x,\tau) \, d\tau + \log |f|^2(x) \). Then (4.16) implies that \( v(x,t) \leq C(m) \nu_\infty(V) \log(t+1) + \log |f|^2(x) \) for \( t \gg 1 \). One can also check that \( v(x,t) \) is a solution to the heat equation \( (\frac{\partial}{\partial t} - \Delta) v(x,t) = 0 \) with \( v(x,0) = \log |f|^2(x) \). Now we apply the ‘moment type estimate’ of [N1] again we have that
\[
\int_{B_x(r)} \log |f|^2(y) \, d\mu(y) \leq C(m) \nu_\infty(V) \log(r+1).
\]
Now applying the mean-value inequality of Li-Schoen [LS] we conclude that \( f \) is of polynomial growth.

**Remark 4.2.** In [St] (see also [Ru] for the codimension one case), Stoll proved that an analytic divisor \( V \) in \( M = \mathbb{C}^m \) is algebraic if and only if \( \nu_\infty(V) < \infty \). The result was later generalized to the case \( M \) being an affine algebraic variety in [GK] by Griffiths and King. In fact, in [St] the result was proved for any analytic sets of \( \mathbb{C}^m \). It is desirable to generalize our result to the high codimension case.

The assumption on vanishing of the cohomology in the above theorem is satisfied, for example, when \( M \) is Stein (cf. Theorem 5.5.2 of [Hö]).

In [N4], the author developed a parabolic approach to compare the vanishing order of a holomorphic function, at any fixed point, with the growth order at infinity. The method is sharp and effective. It turns out that the parabolic method there is also related to the...
We called that a function $s : \mathbb{R}_+ \to \mathbb{R}_+$ has finite order if

$$\text{Ord}(s) := \limsup_{r \to \infty} \frac{\log s(r)}{\log r} < \infty.$$ 

For a $f \in \mathcal{O}(M)$, the space of holomorphic functions, we define the order of $f$ in sense of Hadamard by

$$\text{Ord}_H(f) = \text{Ord}(\log(A(r)))$$

where $A(r) = \sup_{x \in B_o(r)} |f|(x)$ with $o \in M$ being a fixed point. Following [Gr, St] for any analytic subvariety of complex dimension $s$ we define

$$n_V(x_0, r) = \frac{A_{x_0}(r)(\pi r^2)^{m-s}}{V_{x_0}(r)}$$

and

$$N_V(x_0, r) = \int_0^r (n_V(x_0, \tau) - n_V(x_0, 0)) \frac{d\tau}{\tau} + n_V(x_0, 0) \log r.$$

$N_V(x_0, r)$ is called the counting function since for $m = 1$ and $V$ being the zeros of a holomorphic function, it simply counts the number of zeros in $B(x_0, r)$. If $x_0$ does not lie in $V$ one has $N_V(x_0, r) = \int_0^r n_V(x_0, \tau) \frac{d\tau}{\tau}$. One can view the estimate (4.15) as bounding the counting function by the growth order, a Nevanlinna type inequality. The following result is a further generalization of (4.15), whose correspondence in the Euclidean space is also known as the transcendental Bézout estimate.

**Corollary 4.3.** Let $M$ be a complete noncompact Kähler manifold with non-negative Ricci curvature. Let $f \in \mathcal{O}(M)$ be a holomorphic function of finite order. Let $Z(f)$ be the zero divisor of $f$. Then

$$\text{Ord}(N_Z(x_0, r)) \leq \text{Ord}_H(f).$$

**Proof.** The proof follows the same line of argument as the proof of Theorem 4.1. We leave it to the interested readers.

We found the connection between the parabolic equations (as well as the related differential Harnack inequality) and the Nevanlinna theory quite interesting.

§5 Heat kernel and the reduced volumes.

In this section we consider the fundamental solution to the time-dependent heat equation with metrics deformed by Kähler-Ricci/Ricci flow (1.1). As we explained in Section 2 that the fundamental solution $H(x, y, t, t_0)$ satisfies (1.2). We shall prove a sharp lower bound on $H(x, y, t, t_0)$. In [N5] we derived an estimate for the fundamental solution satisfying the time-dependent heat equation itself (instead of (1.2)). The estimate there is only valid for short time interval. In [Gu], a rough lower bound was obtained through the earlier
mentioned Harnack inequality, Theorem 1.3. Notice that the result in [Gu] is not sharp and the result in [N5] is only sharp in the exponents. In this section we show a sharp lower bound on the heat kernel in the case that $M$ is either a complete Riemannian bounded nonnegative curvature operator or a complete Kähler manifold with bounded nonnegative bisectional curvature. For the sake of simplicity we only state the result for the Kähler-Ricci flow and leave the Riemannian analogue to the interested readers. Before we state our result we need to recall some notations and computations from [FIN] (which follows closely the computation in [P]). For the simplicity we assume that $t_0 = 0$.

Let $g(t)$ solve the Kähler-Ricci flow on $M^m \times [0, T]$ ($m = \dim_{\mathbb{C}}(M)$ and $n = 2m$). Fix $x_0$ and let $\gamma$ be a path $(x(\eta), \eta)$ joining $(x_0, 0)$ to $(y, t)$. Following [P] and [LY, FIN] we define

\begin{equation}
L_+ (\gamma) = \int_0^t \sqrt{\eta} \left( \mathcal{R} + 4|\gamma'(\eta)|^2 \right) \ d\eta.
\end{equation}

Let $X = \gamma'(t) = \frac{dz^\alpha(t)}{dt} \frac{\partial}{\partial z^\alpha}$ and let $Y$ be a variational vector field along $\gamma$. Here $|\gamma'(t)|^2 = g_{\alpha\beta} \frac{dz^\alpha(t)}{dt} \frac{dz^\beta(t)}{dt}$. Using $L_+$ as energy we can define the $L_+$-geodesics and we denote $L_+(y, t)$ to be the length of a shortest geodesics jointing $(x_0, 0)$ to $(y, t)$. We also define

\begin{equation}
\ell_+(y, t) := \frac{1}{2\sqrt{t}} L_+(y, t).
\end{equation}

Following the first and second variation calculation of [P] (see also [FIN]) we compute that

\begin{equation}
|\nabla \ell_+|^2 = -\mathcal{R} + \frac{\ell_+}{t} + \frac{K}{t^{3/2}},
\end{equation}

\begin{equation}
\frac{\partial \ell_+}{\partial t} = \mathcal{R} - \frac{K}{2t^{3/2}} - \frac{\ell_+}{t},
\end{equation}

\begin{equation}
\Delta \ell_+ \leq \mathcal{R} + \frac{n}{2t} - \frac{K}{2t^{3/2}}.
\end{equation}

Here

\begin{equation}
K := \int_0^t \eta^{3/2} H(X) \ d\eta,
\end{equation}

where $H(X) := \partial \mathcal{R}/\partial t + 2\langle \nabla \mathcal{R}, X \rangle + 2\langle X, \nabla \mathcal{R} \rangle + 4\text{Ric}(X, X) + \mathcal{R}/t$ is exactly the traced Li-Yau-Hamilton differential Harnack expression in [C1] applying to the $(1, 0)$ vector field $2X$.

**Theorem 5.1.** Let $(M^m, g(t))$ be a complete solution to Kähler-Ricci (Ricci) flow with bounded nonnegative bisectional curvature (curvature operator). Let $H(x_0, y, t)$ be the fundamental solution to the time dependent heat equation (satisfying (1.2) and $H(x_0, y, 0) = \delta_{x_0}(y)$). Then

\begin{equation}
\tilde{u}(x, t) := \frac{1}{(\pi t)^m} \exp \left( -\ell_+(x, t) \right)
\end{equation}
satisfies
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta - \mathcal{R} \right) \tilde{u}(x,t) \leq 0.
\end{equation}
In particular,
\begin{equation}
\tilde{u}(x,t) \leq H(x_0, y, t)
\end{equation}
and
\begin{align*}
\tilde{\theta}^{(x_0, 0)}(t) := \int_M \tilde{u}(x,t) \, d\mu_t
\end{align*}
is monotone decreasing. Moreover, the equality in (5.5), or (5.6) implies that $M$ is a gradient expanding soliton.

**Proof.** First (5.2)–(5.4) implies that
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta - \mathcal{R} \right) \left( \frac{1}{(\pi t)^m} \exp(-\ell(y,t)) \right) = -\frac{K}{t^\frac{3}{2}} \left( \frac{1}{(\pi t)^m} \exp(-\ell(y,t)) \right) \leq 0.
\end{equation}
Here we have used fact that $K \geq 0$ under the assumption that $M$ has bounded non-negative bisectional curvature. Also if the equality holds it implies that $K \equiv 0$. This further implies that $M$ is an expanding soliton from the computation in [FIN]. In order to prove (5.6) one just need to apply the maximum principle (cf. [NT2, N5]) and notice that
\begin{equation}
\lim_{t \to 0} \frac{1}{(\pi t)^m} \exp(-\ell(y,t)) = \delta_{x_0}(y).
\end{equation}
The equality case follows from the consideration of the equality in [FIN].

**Remark 5.1.** In [CY], Cheeger and Yau proved that for complete manifolds with nonnegative Ricci curvature, the heat kernel $H(x,y,t)$ has the lower bound estimate:
\begin{equation}
\bar{H}(d(x,y),t) := \frac{1}{(4\pi t)^\frac{n}{2}} \exp\left( -\frac{d^2(x,y)}{4t} \right) \leq H(x,y,t)
\end{equation}
by showing that the transplant of the Euclidean heat kernel $\bar{H}(d(x,y),t)$ is a sub-solution to the heat equation. Recall that in [P], Perelman first discovered that there is a similar result for the backward Ricci flow, even without any curvature sign assumptions (this is the astonishing part of Perelman’s work). Namely he proved, for $\frac{\partial}{\partial \tau} g_{ij} = 2R_{ij}$ and the reduced distance $\ell(y,\tau)$, defined formally by the same expression as $\ell(y,t)$, that
\begin{equation}
\left( \frac{\partial}{\partial \tau} - \Delta + \mathcal{R} \right) \bar{u}(y,\tau) \leq 0
\end{equation}
where
\begin{equation}
\bar{u}(y,\tau) = \frac{1}{(4\pi \tau)^\frac{n}{2}} \exp(-\ell(y,\tau)).
\end{equation}
In particular, Perelman further showed (in Corollary 9.5 of [P]) that $\bar{u}$ gives a lower bound for the heat kernel of the **backward heat equation** $(\frac{\partial}{\partial \tau} - \Delta - \mathcal{R} \bar{u} = 0)$. (Notice that the heat kernel $H(x,y,\tau)$ for the backward heat equation satisfies the Schrödinger equation $(\frac{\partial}{\partial \tau} - \Delta + \mathcal{R}) H(x,y,\tau) = 0$. One should also refer to [LY, Theorem 4.3] for a precedence of Perelman’s second variation computation in [P, Section 7].) Thus both Perelman’s monotonicity of the reduce volume and Theorem 5.1 above can be viewed as nonlinear analogue of the earlier work of Cheeger-Yau in [CY] and Li-Yau in [LY].

Tracing the computation of [FIN] we also have the following estimates for $\tilde{u}(x,t)$. 
Proposition 5.1. Assume that \((M, g(t))\) be a Kähler-Ricci flow with bounded nonnegative bisectional curvature on \(M \times [0, T)\). Let \(\tilde{u}(x, t)\) be as above. Then

\[
\log(\tilde{u})_{\alpha \bar{\beta}} + R_{\alpha \bar{\beta}} + \frac{1}{t}g_{\alpha \bar{\beta}} \geq 0.
\]

The equality holds if and only if \((M, g(t))\) is an expanding Kähler-Ricci soliton.

Notice that (5.7) is equivalent to the estimate (1.3) on \(u(x, t)\) a positive solution to the forward conjugate heat equation. On the other hand \(\tilde{u}(x, t)\) is only a sub-solution to the forward conjugate heat equation.

Remark 5.2. In [FIN, Corollary 2.1] we showed that under the assumption of bounded nonnegative curvature operator there exists a same estimate as (5.8) (without the bars). This suggests that one may have the similar estimate as Theorem 1.2 for the Riemannian case on positive solution \(u\) of the forward conjugate heat equation. Indeed, this can be shown if one proves, under the assumption of \(M\) having bounded nonnegative curvature operator, that

\[
Y_{ij} := \nabla_i \nabla_j \mathcal{R} - 2 \nabla_k R_{ij} \nabla_k \log u + 2 R_{ikjl} \nabla_k \log u \nabla_l \log u + R_{ik} R_{jk} + \frac{1}{t} R_{ij} \geq 0.
\]

This claim is based on the following computation which is essentially due to Chow. Note that \(Y_{ij} \equiv 0\) on gradient expanding solitons and its trace is the same as the trace of Hamilton’s matrix Li-Yau-Hamilton expression in [H1].

Lemma 5.1. Let

\[
\tilde{Z}_{ij} = R_{ij} + (\log u)_{ij} + \frac{1}{2t}g_{ij}.
\]

Then

\[
\left( \frac{\partial}{\partial t} - \Delta_L \right) \tilde{Z}_{ij} = Y_{ij} + 2 \nabla_k \tilde{Z}_{ij} \nabla_k \log u + \tilde{Z}_{ik} \left( \log(u)_{jk} - R_{jk} - \frac{1}{2t} g_{jk} \right) + \left( \log(u)_{ik} - R_{ik} - \frac{1}{2t} g_{ik} \right) \tilde{Z}_{jk}
\]

where \(\Delta_L \eta_{ij} = \Delta \eta_{ij} + 2 R_{ikjl} \eta_{kl} - R_{ik} \eta_{kj} - R_{ik} \eta_{ik}\) is the Lichnerowicz operator acting on the symmetric 2-tensor \(\eta_{ij}\).

The above computation was first carried out in [Ch2] for the backward Ricci flow, with \(Y_{ij}\) is replaced by Hamilton’s matrix Harnack expression for shrinkers. Notice also that the matrix Harnack expression \(Y_{ij}\) and Hamilton’s expression for Ricci expanders are the same if the manifold is Kähler.

One can obtain some upper bound on the heat kernel \(H(x, y, t, 0)\) using the Harnack inequality proved in Corollary 2.1 and the fact that \(\int_M h(x, y, t, 0) d\mu_t = 1\). But the result is not as satisfactory as for the fixed metric case of Li-Yau. Hence we shall leave this to a later investigation. On the other hand, the localization technique of Ecker can be
applied to the reduced volume for the Ricci expanders (defined in [FIN]) to obtain the monotonicity of a localized reduced volume without assuming bisectional curvature (nor curvature operator) being nonnegative. Recall that in [FIN], the authors proved that, if \( M \) is a closed manifold, the forward reduced volume

\[
\theta_+^{(x_0,0)}(t) := \int_M \hat{u}(x,t) \, d\mu_t
\]

is monotone non-increasing, where \( \hat{u}(x,t) = e^{\ell_+^{(x,t)}(\pi t)/m} \). Here we define \( \hat{u} \) for the Kähler case to be coherent with our previous discussions. This monotonicity has severe restriction since the reduced volume \( \theta_+^{(x_0,0)}(t) \) is only meaningful when \( M \) is compact. We shall show the monotonicity of a localized reduced volume, which is well-defined on complete manifolds, to compensate such restriction. In order to put the result in its general form we denote by \( l_+^{(x_0,t_0)} \) the forward reduced distance with respect to \( (x_0,t_0) \) if one replaces the reference point \( (x_0,0) \) by \( (x_0,t_0) \). Correspondingly we denote by \( \hat{u}^{(x_0,t_0)} \) and \( \theta_+^{(x_0,t_0)} \), the reduced volume function and the reduced volume with respect to \( (x_0,t_0) \), respectively. Let

\[
\bar{L}_+^{(x_1,t_1)} = (t-t_1)\ell_+^{(x_1,t_1)}
\]

and

\[
\phi_{t_2,\rho}^{(x_1,t_1)}(x,t) = \left( 1 - \frac{\bar{L}_+^{(x_1,t_1)}(x,t) + m(t-t_2)}{\rho^2} \right)_+
\]

Then (5.2)–(5.4) imply that

\[
\left( \frac{\partial}{\partial t} + \Delta - \mathcal{R} \right) \hat{u}^{(x_0,t_0)}(x,t) \leq 0
\]

and

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \phi_{t_2,\rho}^{(x_1,t_1)}(x,t) \leq 0.
\]

Notice that if \( \mathcal{R} \) is uniformly bounded from below, \( \phi_{t_2,\rho}^{(x_1,t_1)}(x,t) \) is compactly supported. We then have the following localized monotonicity of the reduced volume.

**Proposition 5.2.**

\[
\frac{d}{dt} \theta_+^{(x_0,t_0)}(t) \leq 0
\]

where

\[
\theta_+^{(x_0,t_0)}(t) = \int_M \phi_{t_2,\rho}^{(x_1,t_1)}(x,t) \hat{u}^{(x_0,t_0)}(x,t) \, d\mu_t.
\]
It turns out that one can also construct a subsolution to the heat equation with compact support, similar to \( \varphi(x_1, t_1)(x, t) \) defined above. This is through Perelman’s reduced distance \( \ell(x_1, t_1)(x, \tau) \), with \( \tau = t_1 - t \). \( \ell(x_1, t_1)(x, t_1 - t) \) is only defined for \( t \leq t_1 \). Please refer to [P, Section 7] for the detailed discussions.) Here we use \((x_1, t_1)\) to specify the reference space-time point with respect to which the reduced distance is defined. Recall that from [P, Section 7], \( \bar{L}(x_1, t_1)(x, t_1) := (t_1 - t) \ell(x_1, t_1)(x, t_1 - t) \) (again we do not have the factor 4 since we are in the Kähler setting) satisfies the differential inequality:

\[
\left( -\frac{\partial}{\partial t} + \Delta \right) \bar{L}(x_1, t_1)(x, t) \leq m.
\]

Then in the case \( \mathcal{R} \) is bounded from below, we may define a compact supported function

\[
\psi_{t_2, \rho}(x, t) := \left( 1 - \frac{\bar{L}(x_1, t_1)(x, t) + m(t - t_2)}{\rho^2} \right)_+.\]

It is easy to see that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \psi_{t_2, \rho}(x, t) \leq 0. \tag{5.15}
\]

This gives another localization on the forwarded reduced volume.

**Proposition 5.3.**

\[
\frac{d}{dt} \theta^{(x_0, t_0)}_{+, \varphi}(t) \leq 0 \tag{5.16}
\]

where

\[
\theta^{(x_0, t_0)}_{+, \varphi}(t) = \int_M \psi_{t_2, \rho}^{(x_1, t_1)}(x, t) u^{(x_0, t_0)}(x, t) d\mu. \tag{5.17}
\]

The similar idea can also be applied to Perelman’s entropy. Let \((x_0, t_0)\) be a fixed space-time point. Let \( \tau = t_0 - t \) and \( u(x, \tau) \) be the fundamental solution of the backward conjugate heat equation \( \frac{\partial}{\partial \tau} - \Delta + \mathcal{R} \) centered at \((x_0, t_0)\). Write \( u = \frac{e^{-f}}{(\pi \tau)^{m}} \) and define \( v = (\tau (2\Delta f - |\nabla f|^2 + \mathcal{R}) + f - 2m) u \). Then it was proved in [P] that

\[
\left( \frac{\partial}{\partial \tau} - \Delta + \mathcal{R} \right) v = -\tau \left( |R_{\alpha\beta} + \nabla_\alpha \nabla_\beta f - \frac{1}{\tau} g_{\alpha\beta}|^2 + |\nabla_\alpha \nabla_\beta f|^2 \right) u \tag{5.18}
\]

and \( v \leq 0 \). Applying (5.12) or (5.15) we can have the following local monotonicity formulae.
Proposition 5.4.

\begin{equation}
\frac{d}{dt} \left( \int_M -v \varphi_{t_2, \rho}^{(x_1, t_1)} \, d\mu_t \right) \leq - \int_M \tau \left( |R_{\alpha \beta} + \nabla_{\alpha} \nabla_{\beta} f - \frac{1}{\tau} g_{\alpha \beta}|^2 + |\nabla_{\alpha} \nabla_{\beta} f|^2 \right) u \varphi_{t_2, \rho}^{(x_1, t_1)} \, d\mu_t
\end{equation}
and

\begin{equation}
\frac{d}{dt} \left( \int_M -\psi_{t_2, \rho}^{(x_1, t_1)} \, d\mu_t \right) \leq - \int_M \tau \left( |R_{\alpha \beta} + \nabla_{\alpha} \nabla_{\beta} f - \frac{1}{\tau} g_{\alpha \beta}|^2 + |\nabla_{\alpha} \nabla_{\beta} f|^2 \right) u \psi_{t_2, \rho}^{(x_1, t_1)} \, d\mu_t.
\end{equation}

Proof. The proof is just direct computations and integration by parts, using (5.12), (5.15) and (5.18).

There is no difference for the Riemannian cases. The formulae are exactly the same except some factors caused by the definition of the operators.

We conclude this section by an application of Theorem 5.1 to the study of the large time behavior of Kähler-Ricci flow. Let \((M^m, g_0)\) be a complete Kähler manifold with bounded nonnegative holomorphic bisectional curvature. If we further assume that \((M, g_0)\) has maximum volume growth, namely for any \(x\), the volume of the ball \(B(x, r)\), \(V_x(r)\) is bounded from below by \(\delta r^{2m}\) for some positive \(\delta\), in [N6] we proved that the Kähler-Ricci flow (1.1) has long time solution with the initial data \(g(x, 0) = g_0(x)\). Moreover, there exists a constant \(A = A(M) > 0\) such that \(t \gg 1\),

\begin{equation}
tR(x, t) \leq A.
\end{equation}

Applying Theorem 5.1 to this situation we can show the following result.

Corollary 5.1. Let \((M, g_0)\) be a complete Kähler manifold as above. Let \(g(x, t)\) be a long time solution to Kähler-Ricci flow as above. For any \((x_j, t_j)\) with \(t_j \to \infty\), define \(g_j(t) = \frac{1}{t_j} g(t_j, t)\). Then the pointed sequence \((M, x_j, g_j(x, t))\) sub-sequentially converges to a gradient expanding Kähler-Ricci soliton \((M_\infty, x_\infty, g_\infty(t))\).

Proof. For any \(t\), denote by \(B_t(x, r)\) the ball of radius \(r\) with respect to \(g(t)\), and by \(V_t(x, r)\) the volume of this ball (with respect to \(g(t)\)). By Theorem 2.2 of [NT3], we know that \(V_t(x, r) \geq \delta r^{2m}\). Together with (5.21) we have the injectivity radius bounded from below uniformly for \(g_j(t)\). Therefore, by Hamilton’s compactness theorem, \((M, x_j, g_j(t))\) sub-sequentially converges to \((M_\infty, x_\infty, g_\infty(t))\), a solution to Kähler-Ricci flow defined on \(M_\infty \times (0, \infty)\). The only thing we need to prove is that \((M_\infty, g_\infty(t))\) is an expanding soliton.

It is easy to see that \((M_\infty, g_\infty)\) also has bounded nonnegative bisectional curvature and the maximum volume growth. By Corollary 1 of [N6], we know that \(M_\infty\) is topologically \(\mathbb{R}^{2m}\). The fact that it is an expanding soliton follows from the monotonicity consideration. Let us adapt the notations from Theorem 5.1. Let \((x_0, 0)\) be a fixed reference point, with respect to which we have the forward reduced distance function \(\ell_+(x, t)\), and the second forward reduced volume function \(\tilde{u}(x, t)\) and the second forward reduced volume \(\tilde{\theta}_+(t)\). By the proof of Theorem 5.1 we have that

\begin{equation}
\frac{d}{dt} \tilde{\theta}_+(t) = - \int_M \frac{K}{t^{3/2}} \tilde{u} \, d\mu_t, \, dt
\end{equation}
from which we can conclude that
\[
\tilde{\theta}_+(t_j) - \tilde{\theta}_+(2t_i) = \int_{t_j}^{2t_j} \frac{1}{t^{3/2}} \int_M K \tilde{u} d\mu_t.
\]

Hence
\[
\tilde{\theta}_+(2t_j) - \tilde{\theta}_+(t_j) \geq \int_{t_j}^{2t_j} \frac{1}{t^{3/2}} \int_M K \tilde{u} d\mu_t dt.
\]

Taking limit we have that for the limit metric

\[
\int_1^2 \frac{1}{t^{3/2}} \int_{M_\infty} K \tilde{u} d\mu_t dt = 0.
\]

The conclusion now follows from [C2] or Theorem 4.1 of [N2] since \( K = 0 \) implies that the linear trace LYH quantity \( H \) achieves its minimum (zero) somewhere.

**Remark 5.3.** Corollary 5.1 simply says that the blow-down limit of a Type III solution is an expanding soliton. The result is in some sense dual to Proposition 11.2 of [P]. It is also similar to the situation studied in [Hu] for the mean curvature flow. The maximum volume growth assumption can be weaken to a certain \( \kappa \)-noncollapsing condition in terms of the lower bound of \( \tilde{\theta}_+(t) \).

It has been proved in [CT1] (see also [CT2] for further more recent progresses) that a gradient expanding Kähler soliton must be biholomorphic to \( \mathbb{C}^m \).

One can have a similar result for the Riemannian case, replacing the nonnegativity of the bisectional curvature by the nonnegativity of the curvature operator.

§Appendix: A parabolic relative volume comparison theorem.

Recall that Cheeger and Yau proved that on a complete Riemannian manifold with nonnegative Ricci curvature, the heat kernel \( H(x, y, \tau) \) (the fundamental solution of the operator \( \left( \frac{\partial}{\partial \tau} - \Delta \right) \)) has the lower estimate

\[
(A1) \quad H(x, y, \tau) \geq \frac{1}{(4\pi \tau)^{n/2}} \exp \left( -\frac{r^2(x, y)}{4\tau} \right)
\]

where \( r(x, y) \) is the distant function on the manifold. This fact can be derived out of the maximum principle and the differential inequality

\[
\left( \frac{\partial}{\partial \tau} - \Delta \right) \left( \frac{1}{(4\pi \tau)^{n/2}} \exp \left( -\frac{r^2(x, y)}{4\tau} \right) \right) \leq 0.
\]

Integrating on the manifold \( M \), this differential inequality also implies the monotonicity (monotone non-increasing) of the integral

\[
(A2) \quad \tilde{V}(x_0, \tau) := \int_M \frac{1}{(4\pi \tau)^{n/2}} \exp \left( -\frac{r^2(x_0, y)}{4\tau} \right)
\]
In [P], Perelman discovered a striking analogue of this comparison result for the Ricci flow geometry. More precisely, he introduced a length function, called the reduced distance,

\[ \ell_{x_0,g(\tau)}(y,\bar{\tau}) := \inf_{\gamma} \frac{1}{2\sqrt{\tau}} \int_0^{\bar{\tau}} \sqrt{\tau} |\gamma'(\tau)|^2 d\tau \]

for all \( \gamma(\tau) \) with \( \gamma(0) = x_0, \gamma(\bar{\tau}) = y \) (in the right context, we often omit the subscript \( x_0, g(\tau) \)), and a functional, called the reduced volume,

\[ (A2') \quad \tilde{V}_{g(\tau)}(x_0, \tau) := \int_M \frac{1}{(4\pi \tau)^{\frac{n}{2}}} \exp\left(-\ell(y,\tau)\right) \]

with respect to a solution \( g_{ij}(x,\tau) \) to the backward Ricci flow

\[ \frac{\partial}{\partial \tau} g_{ij} = 2R_{ij} \]

on \( M \times [0,a] \). Perelman proved further that \( \tilde{V}_{g(\tau)}(x_0, \tau) \) is monotone non-increasing in \( \tau \). In fact, he showed this monotonicity result via a space-time relative comparison result and used it to give a more flexible proof of the \( \kappa \)-noncollapsing on the solution to Ricci flow, defined on \( M \times [0,T] \), for any given finite time \( T \) with given initial data.

In [P], Perelman also discovered a more rigid monotone quantity, the entropy functional. These two functionals can be related through a differential inequality of Li-Yau-Hamilton type. One can refer to [CLN] for an exposition on this relation. The entropy is stronger but less flexible. The entropy monotonicity has its analogue for the positive solutions to linear heat equation on a fixed Riemannian manifold with non-negative Ricci curvature. This was derived in [N3], where the author also observed that the entropy defined on a complete Riemannian manifold \( M \) with non-negative Ricci curvature is related to the volume growth of the manifold. More precisely, let

\[ (A3) \quad \mathcal{W}(f, \tau) := \int_M (\tau|\nabla f|^2 + f - n) \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} d\mu \]

for any \( \tau > 0 \) and \( C^1 \) function \( f \) with \( \int_M \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} d\mu = 1 \). If \( u(x, \tau) = \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} \) is a solution to the heat equation, it was proved in [N3] that \( \mathcal{W}(f, \tau) \) is monotone non-increasing in \( \tau \). Moreover, it was shown that

\[ \lim_{\tau \to \infty} \mathcal{W}(f, \tau) = \log \left( \lim_{\tau \to \infty} \tilde{V}_{x_0}(\tau) \right) \]

if \( u(x, \tau) \) is a fundamental solution originated at \( x_0 \). In fact,

\[ \lim_{\tau \to \infty} \tilde{V}_{x_0}(\tau) = \lim_{\tau \to \infty} \frac{V_{x_0}(r)}{\omega_n r^n} \]
where $V_{x_0}(r)$ is the volume of ball of radius $r$, which is independent of $x_0$. (The limit on the right hand side above is also called the cone angle at infinity).

Motivated by this close connection between the Ricci flow geometry for a family of metrics and the Riemannian geometry of a fixed Riemannian metric, it is nature to seek a localized version of the above mentioned Cheeger-Yau’s result on the monotonicity of the reduced volume $\tilde{V}_{x_0}(\tau)$. In fact, a local version of the heat kernel comparison was also carried out in the paper of [CY] from a PDE point of view. However, Perelman’s localization in the case of Ricci flow geometry is along the line of comparison geometry, and very much different from Cheeger-Yau’s localization consideration (by considering the Dirichlet or Neumann boundary value problem for the heat equation). This leads us to formulate a new relative (local) volume comparison theorem in this section, which is the linear analogue of Perelman’s formulation for Ricci flow. It can be viewed a parabolic version of the (classical) relative volume comparison theorem in the standard Riemannian geometry. The interested readers may want to compare these two results in more details. The relation between the classical relative volume comparison theorem and the new one here is very similar to the one between the monotonicity formula for the minimal submanifolds in $\mathbb{R}^n$ and Huisken’s monotonicity for mean curvature flow in $\mathbb{R}^n$ (as well as Ecker’s localized version).

In order to state our result, let us first fix some notations. It is our hope that the exposition below is detailed enough to be helpful in understanding [P] better. Fix a point $x_0 \in M$. Let $\gamma(\tau) (0 \leq \tau \leq \bar{\tau})$ be a curve parameterized by the time variable $\tau$ with $\gamma(0) = x_0$. Here we image that we have a time function $\tau$, with which some parabolic equation is associated. Define the $L$-length by

\begin{equation}
L(\gamma)(\bar{\tau}) = \int_0^{\bar{\tau}} \sqrt{\tau} |\gamma'(\tau)|^2 \, d\tau.
\end{equation}

We can define the $L$-geodesic to be the curve which is the critical point of $L(\gamma)$. The simple computation shows that the first variation of $L$ is given by

\begin{equation}
\delta L(\gamma) = 2\sqrt{\tau} \langle Y, X \rangle(\bar{\tau}) - 2 \int_0^{\bar{\tau}} \sqrt{\tau} \left( \langle \nabla_X X + \frac{1}{2\tau} X, Y \rangle \right) \, d\tau,
\end{equation}

where $Y$ is the variational vector field, from which one can write down the $L$-geodesic equation. It is an easy matter to see that $\gamma$ is a $L$-geodesic if and only if $\gamma(\sigma)$ with $\sigma = 2\sqrt{\tau}$ is a geodesic. In another word, a $L$-geodesic is a geodesic after certain re-parametrization. Here we insist all curves are parameterized by the ‘time’-variable $\tau$. One can check that for any $v \in T_{x_0} M$ there exists a $L$-geodesic $\gamma(\tau)$ such that $\frac{d}{d\sigma} (\gamma(\sigma))|_{\sigma=0} = v$. Notice that the variable $\sigma$ scales in the same manner as the distance function on $M$. So it is more convenient to work with $\sigma$.

We then define the $L$-exponential map by

\begin{equation}
L \exp_v(\bar{\sigma}) := \gamma_v(\bar{\sigma})
\end{equation}

if $\gamma_v(\sigma)$ is a $L$-geodesic satisfying that

\[ \lim_{\sigma \to 0} \frac{d}{d\sigma} (\gamma_v(\sigma)) = v. \]
It is also illuminating to go one dimensional higher by considering the manifold \( \tilde{M} = M \times [0, 2\sqrt{T}] \) and the space-time exponential map \( \tilde{\exp}(\tilde{v}^a) = (\mathcal{L}\exp_{\tilde{v}^a}(a), a) \), where \( \tilde{v}^a = (v^a, a) \). Denote \( \frac{v^a}{a} \) simply by \( v^1 \) and \((v^1, 1)\) by \( \tilde{v}^1 \). Also let \( \tilde{\gamma}_{\tilde{v}^a}(\eta) = \exp(\eta \tilde{v}^a) \). It is easy to see that

\[
\tilde{\gamma}_{\tilde{v}^a}(\eta) = \tilde{\gamma}_{\tilde{v}^1}(\eta a) .
\]

This shows that

\[
d\tilde{\exp}|_{(0,0)} = \text{identity}.
\]

Computing the second variation of \( \mathcal{L}(\gamma) \) gives that

\[
\delta^2 \mathcal{L}(\gamma) = 2\sqrt{T}(\nabla_Y Y, X) + \int_0^T 2\sqrt{T}(|\nabla_X Y|^2 - R(X, Y, X, Y)) \, d\tau
\]

where \( Y \) is a given variational vector field. Let \((y, \tilde{\sigma}) = (\mathcal{L}\exp_y(\tilde{\sigma}), \tilde{\sigma})\). Consider the variation which is generated by \((w, 0)\). Namely consider the family

\[
U(s, \sigma) = (\mathcal{L}\exp_{v+sw}(\sigma), \sigma) = \tilde{\exp}((\sigma(v + sw), \sigma)).
\]

Direct calculation shows that

\[
\frac{DU}{\partial \sigma}(0, 0) = (v, 1), \quad \frac{DU}{\partial s}(0, 0) = 0
\]

and that the Jacobi field \( \tilde{J}_w(\sigma) = (J_w(\sigma), 0) \) is given by

\[
\frac{DU}{\partial s}|_{s=0}(\sigma) = d\tilde{\exp}((\sigma w, 0))
\]

with the initial velocity

\[
\nabla_{\frac{\partial}{\partial \sigma}} \left( \frac{DU}{\partial s}|_{s=0} \right)(0) = (w, 0).
\]

(One can define the Jacobi operator to be the linear second order operator associated with the quadratic form in the right-hand side of (A6). One call a vector field along \( \gamma \) a \( L \)-Jacobi field if it satisfies the Jacobi equation. It is easy to show that the variational vector field of a family of \( L \)-geodesics satisfies the Jacobi equation as in the standard Riemannian geometry. In fact, the \( L \)-Jacobi field turns out to be just the regular Jacobi-field after re-parametrization.) This shows that

\[
(dL \exp)_w(\sigma) = J_w(\sigma)
\]

and

\[
d\tilde{\exp}((\sigma w, 0)) = \tilde{J}_w(\sigma).
\]

Now we can conclude that \((y, \sigma)\) is a regular value of the map \( \tilde{\exp} \) (and \( y \) is a regular value of \( \mathcal{L}\exp_{\cdot}(\sigma) \)) if and only if that any Jacobi field \( \tilde{J} \) with initial condition as above does
not vanish at \( \sigma \). We can introduce the concept of conjugate point (with respect to \( x_0 \)) similarly as in the classical case. We can define the set

\[ D(\bar{\sigma}) \subseteq T_{x_0}M \]

to be the collection of vectors \( v \) such that \( (L \exp_v(\sigma), \sigma) \) is a \( L \)-geodesic along which there is no conjugate point up to \( \bar{\sigma} \). Similarly we can define the set

\[ C(\bar{\sigma}) \subseteq T_{x_0}M \]

to be the collection of vectors \( v \) such that \( (L_v(\sigma), \sigma) \) is a minimizing \( L \)-geodesic up to \( \bar{\sigma} \). One can see easily that \( D(\sigma) \) and \( C(\sigma) \) decreases (as sets) as \( \sigma \) increases. For any measurable subset \( A \subseteq T_{x_0}M \) we can define

\[ D_A(\sigma) = A \cap D(\sigma) \quad \text{and} \quad C_A(\sigma) = A \cap C(\sigma). \]

Following [P] we also introduce the \( \ell \)-'distance' function.

\[ \ell_{x_0}(y, \bar{\tau}) = \frac{1}{2\sqrt{\bar{\tau}}}L_{x_0}(y, \bar{\tau}), \quad \text{where} \quad L_{x_0}(y, \bar{\tau}) = \inf_{\gamma} L(\gamma). \]

Here our \( \ell \) is defined for a fixed background metric. We also omit the subscript \( x_0 \) in the context where the meaning is clear. The very same consideration as in [P], as well as in the standard Riemannian geometry, shows that

\[
\begin{align*}
(A9) \quad |\nabla \ell|^2 &= \frac{1}{\bar{\tau}} \ell \\
(A10) \quad \ell_{\tau} &= -\frac{1}{\bar{\tau}} \ell \\
\text{and} \\
(A11) \quad \Delta \ell &\leq \frac{n}{2\bar{\tau}} - \frac{1}{\bar{\tau}^{\frac{3}{2}}} \int_0^{\bar{\tau}} \bar{\tau}^{\frac{3}{2}} \text{Ric}(X, X) \, d\tau
\end{align*}
\]

where \( X = \gamma'(\tau) \) with \( \gamma(\tau), 0 \leq \tau \leq \bar{\tau} \) being the minimizing \( L \)-geodesic joining \( x_0 \) to \( y \). Putting (A9)–(A11) together, one obtains a new proof of the result of Cheeger-Yau, which asserts that if \( M \) has nonnegative Ricci curvature, then

\[
(A12) \quad \left( \frac{\partial}{\partial \tau} - \Delta \right) \left( \frac{e^{-\ell(y, \tau)}}{(4\pi \tau)^{\frac{n}{2}}} \right) \leq 0.
\]

Namely \( \frac{e^{-\ell(y, \tau)}}{(4\pi \tau)^{\frac{n}{2}}} \) is a sub-solution of the heat equation. In particular,

\[
\frac{d}{d\tau} \int_M e^{-\ell(y, \tau)} (4\pi \tau)^{-\frac{n}{2}} \, d\mu \leq 0.
\]
Using the above geometric consideration, one can think the above result of Cheeger-Yau as a parabolic volume comparison with the respect to the positive measure \( \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{n/2}} d\mu \). Recall that the well-known Bishop-Gromov volume comparison states that if \( M \) has nonnegative Ricci curvature

\[
\frac{d}{dr} \left( \frac{1}{r^n} \int_{S_{x_0}(r)} dA \right) \leq 0
\]

where \( S_{x_0}(r) \) denotes the boundary of the geodesic ball centered at \( x_0 \) with radius \( r \), \( dA \) is the induced area measure. The by-now standard relative volume comparison can be formulated in the similar way as above. Let \( A \) be a measurable subset of \( S^{n-1} \subset T_{x_0}M \) one can define \( C_A(r) \) to be the collection of vectors \( rv \) with \( v \in A \) such that the geodesic \( \exp_{x_0}(sv) \) is minimizing for \( s \leq r \), where \( \exp_{x_0}(. ) \) is the (classical) exponential map. Then the classical relative volume comparison theorem (cf. [Gr]) asserts that if \( M \) has nonnegative Ricci curvature

\[
\frac{d}{dr} \left( \frac{1}{r^n} \int_{\exp(C_A(r))} dA \right) \leq 0.
\]

The following is a parabolic version of such relative volume comparison theorem parallel to Perelman’s work on Ricci flow geometry.

**Theorem A1.** Assume that \( M \) has nonnegative Ricci curvature. Then

\[
(A13) \quad \frac{d}{d\tau} \int_{\mathcal{L}\exp C_A(\tau)} \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{n/2}} d\mu \leq 0.
\]

**Proof.** The proof follows the similar argument as in [P]. Using the notation in the above discussion, first we observe that

\[
\int_{\mathcal{L}\exp C_A(\tau)} \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{n/2}} d\mu = \int_{C_A(\tau)} \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{n/2}} J(\tau) d\mu_0,
\]

where \( J(\tau) \) is the Jacobian of \( \mathcal{L}\exp(\cdot)(\tau) \). Since \( C_A(\tau) \) is decreasing in \( \tau \), it suffices to show that

\[
(A14) \quad \frac{d}{d\tau} \left( \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{n/2}} J(\tau) \right) \leq 0.
\]

This follows from (A9)–(A10), the fact that

\[
\frac{d}{d\tau} e^{-\ell} = \left( -\frac{\ell}{\tau} + \langle \nabla \ell, X \rangle(\tau) \right) e^{-\ell} = 0
\]

(since \( \nabla \ell(\dot{\tau}) = X \)), and the claim that

\[
(A15) \quad \frac{d}{d\tau} \log J(\tau) \leq \frac{n}{2\tau} - \frac{1}{\tau} \int_0^\tau \tau^2 \text{Ric}(X,X) d\tau.
\]
The estimate (A15) follows exactly the same proof as in [P], via the second variation formula (A6) and its consequence on the Hessian comparison:

$$\text{Hess}(L)(\bar{Y}, \bar{Y}) \leq \int_0^\tau 2\sqrt{\tau} (|\nabla_X Y|^2 - R(X, Y, X, Y)) \, d\tau$$

where $Y(\tau)$ is a vector field along the minimizing $\mathcal{L}$-geodesic $\gamma(\tau)$ joining $x_0$ to $y$ with $\gamma(\bar{\tau}) = y$, satisfying $Y(\bar{\tau}) = \bar{Y}$ and

$$\nabla_X Y - \frac{1}{2\tau} Y = 0.$$

Better comparison between Theorem A1 and the classical relative volume comparison can be seen by comparing $\tilde{M}$ with $M$, $M \times \{a\}$ with $S_{x_0}(a)$. Notice that in [P], one does need such a result (not just the global version on the monotonicity of $\tilde{V}_g(\gamma(x_0, \tau))$) to prove the non-collapsing result on finite time solution to Ricci flow. This and the potential application to the study of Riemannian geometry of our linear version, justify the spelling out of this result in spite of its simplicity.

**Remark.** One can formulate the similar parabolic version of the volume comparison result for the case $\text{Ric}(M) \geq -(n - 1)K$. We leave that to the interested readers.

In [N3], we showed that the lower bound on the entropy $\inf_{0 \leq \tau \leq T} \mu(\tau)$ (please see [N3] for definition) implies the non-collapsing of volume of ball of radius $r$ for $r^2 \leq T$. This corresponds (but is considerably easier than) the $\kappa$-non-collapsing result of Perelman on the solutions to Ricci flow, via the monotonicity of entropy functional. Using Theorem A1, one can have the following consequence, which is just the linear version of the second proof of Perelman on his celebrated $\kappa$-non-collapsing result.

**Lemma A1.** Let $M$ be a complete Riemannian manifold with nonnegative Ricci curvature. Assume that $B(x_0, r)$ is $\kappa$-collapsed in the sense that

(A16) \[ V(x_0, r) \leq \kappa r^n. \]

Then there exists $C = C(n) > 0$

(A17) \[ \tilde{V}(\kappa^{\frac{n}{2}} r^2) \leq C \left( \sqrt{\kappa} + \exp \left( -\frac{1}{8\kappa \frac{1}{n}} \right) \right). \]

Namely, the smallness of the relative volume $\frac{V(x_0, r)}{r^n}$ is equivalent to the smallness of the ‘reduced volume’ $\tilde{V}(r^2)$.

Notice that the lemma can also be proved by direct computations and the Bishop volume comparison theorem, without using Theorem A1.
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