On the crystalline cohomology of Deligne-Lusztig varieties

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Abstract

Let $X \to Y^0$ be an abelian prime-to-$p$ Galois covering of smooth schemes over a perfect field $k$ of characteristic $p > 0$. Let $Y$ be a smooth compactification of $Y^0$ such that $Y - Y^0$ is a normal crossings divisor on $Y$. We describe a logarithmic $F$-crystal on $Y$ whose rational crystalline cohomology is the rigid cohomology of $X$, in particular provides a natural $W[F]$-lattice inside the latter; here $W$ is the Witt vector ring of $k$. If a finite group $G$ acts compatibly on $X$, $Y^0$ and $Y$ then our construction is $G$-equivariant. As an example we apply it to Deligne-Lusztig varieties. For a finite field $k$, if $G$ is a connected reductive algebraic group defined over $k$ and $L$ a $k$-rational torus satisfying a certain standard condition, we obtain a meaningful equivariant $W[F]$-lattice in the cohomology ($\ell$-adic or rigid) of the corresponding Deligne-Lusztig variety and an expression of its reduction modulo $p$ in terms of equivariant Hodge cohomology groups.

Introduction

Let $k$ be a perfect field of characteristic $p > 0$, let $W$ be its Witt vector ring and let $K = \text{Quot}(W)$. One of the specific interests in $p$-adic cohomology theories for $k$-varieties, as opposed to $\ell$-adic étale cohomology ($\ell \neq p$), lies in the hope to construct $W[F]$-lattices (i.e. $W$-lattices stable under the action of Frobenius) in the cohomology and to explicitly describe the Frobenius action on them; typically an estimate of the slopes of Frobenius on the cohomology should be given in terms of Hodge cohomology groups of proper smooth $k$-varieties. Rigid cohomology, as it stands, is a $K$-vector space valued $p$-adic cohomology theory and does not come, a priori, with natural $W[F]$-lattices whose reduction modulo $p$ one could control. If a $k$-scheme $X$ is the open complement of a smooth divisor with normal crossings on a proper smooth $k$-scheme $\overline{X}$ then its rigid cohomology $H^*_\text{rig}(X)$ can be computed as the rational logarithmic crystalline cohomology of $\overline{X}$ (with logarithmic poles along $\overline{X} - X$), and in this way one indeed gets a meaningful $W[F]$-lattice. However, for general smooth $X$ such $\overline{X}$ may not exist or may not be naturally at hand. Our first purpose here is to provide a computable $W[F]$-lattice in...

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$H^{*}_{rig}(X)$ in the case where $X$ admits an abelian prime-to-$p$ cover $f : X \rightarrow Y^0$ to a $k$-scheme $Y^0$ which is the open complement of a smooth divisor with normal crossings on a proper smooth $k$-scheme $Y$. Namely, we construct an explicit logarithmic $F$-crystal $E$ on $Y$ whose rational logarithmic crystalline cohomology $H^{*}_{crys}(E,Y/W) \otimes_K K$ is identified with $H^{*}_{rig}(X)$; thus (the image of) $H^{*}_{crys}(E,Y/W)$ is a $W[F]$-lattice in $H^{*}_{rig}(X)$. If a finite group $G$ acts compatibly on $X$, $Y$ and $Y^0$ then it also acts on $E$ and hence on $H^{*}_{crys}(E,Y/W)$.

The mere existence of a logarithmic $F$-crystal $E$ on $Y$ as above should certainly be expected for more general tamely ramified coverings $f : X \rightarrow Y^0$ than just abelian prime-to-$p$ coverings $f$. However, our point here is that if $f$ is abelian prime-to-$p$ we can explicitly describe $E$. This should be of algorithmic interest: for example in view of the scenario described below, but we can also conceive applications to point counting algorithms for $k$-varieties if $k$ is finite.

Our second purpose is to apply our construction to Deligne-Lusztig varieties. Let now $k$ be a finite field and let $\overline{k}$ denote an algebraic closure. Let $G$ be a connected reductive algebraic group over $\overline{k}$, defined over $k$, and let $L$ be a $k$-rational Levi subgroup of $G$. A construction due to Deligne and Lusztig [6] associates to any parabolic subgroup $P$ of $G$ with $L$ as Levi subgroup a smooth algebraic variety $X$ over $\overline{k}$, endowed with an action of $G(k) \times L(k)^{opp}$, commonly referred to as the corresponding Deligne-Lusztig variety (the pioneering paper [6] deals with the case where $L$ is a $k$-rational maximal torus). The $\ell$-adic étale cohomology with compact support ($\ell \neq p$), viewed as a virtual $G(k) \times L(k)^{opp}$-representation, plays a central role for the classification of all representations (in characteristic zero) of the finite group $G(k)$, by now a broad and successful branch of research. The techniques developed also proved useful for the study of the representation theory of $G(k)$ on vector spaces over fields of positive characteristic $\ell \neq p$ (see [4]).

Our initial observation here is that instead of $\ell$-adic étale cohomology with compact support one may equally work with the rigid cohomology with compact support of $X$: the resulting virtual $G(k) \times L(k)^{opp}$-representations are the same (upon identifying the respective characteristic-zero-coefficient fields) as for $\ell$-adic étale cohomology with compact support. If more specifically $L$ is a torus satisfying a certain standard condition then it is known that $X$ can be realized as a $G(k)$-equivariant Galois covering $f : X \rightarrow Y^0$ in such a way that our $L(k)$ acting on $X$ becomes the Galois group for this covering, and moreover such that $Y^0$ is the open complement of a $G(k)$-stable smooth divisor with normal crossings on a proper smooth $k$-scheme $Y$ with $G(k)$-action. As a result our construction provides equivariant $W(\overline{k})[F]$-lattices in the cohomology of $X$ (with $W(\overline{k})$ the Witt vector ring of $\overline{k}$). In particular we obtain a geometric and explicit construction of the reduction modulo $p$ (as virtual $\overline{k}[G(k)]$-modules) of the Deligne-Lusztig characters in terms of Hodge cohomology groups of equivariant vector bundles on $Y$. In fact, $X$, $Y^0$ and $Y$ are defined over $k$ and we actually get $W[F]$-lattices.

We briefly survey the content of each section. In section [11] we describe the logarithmic $F$-crystal $E$ on $Y$ mentioned at the beginning in the case where $X$ is obtained from the smooth $k$-variety $Y^0$ by adjoining a prime-to-$p$ root of a global invertible section of $\mathcal{O}_Y$ (the general case of a prime-to-$p$ abelian covering $f$ easily reduces to this one). As a logarithmic crystal, $E$
naturally decomposes into a direct sum of rank one logarithmic crystals $E(j)$, and if a finite group $G$ acts on $X$, $Y$ and $Y^0$ then the resulting action on $E$ respects this sum decomposition and is given on each $E(j)$ by suitable automorphy factors. In section 2 we analyse the overconvergent $F$-isocrystal $E^\dagger$ on $Y^0$ which is the push forward via $f$ of the constant $F$-isocrystal on $X$. We see that $E^\dagger$ is associated to $E$ by the general construction described in [17] and from this we conclude that $H^\ast_{rys}(E,Y/W) \otimes_W K = H^\ast_{rig}(X)$. Let $H^\ast_{rig}(X)_j$ be the direct summand of $H^\ast_{rig}(X)$ corresponding to the direct summand $E(j)$ of $E$. Using the general reduction-modulo-$p$ principle for crystalline cohomology from [12] we show that the reduction modulo $p$ of the virtual $K[G_{opp}]$-module \[ \sum_s (-1)^s H^s_{rig}(X)_j \] coincides with \[ \sum_s (-1)^s H^s(Y, \Omega^s \otimes \overline{E}(j)) \] where $\Omega^s \otimes \overline{E}(j)$ denotes the logarithmic de Rham complex of the reduction modulo $p$ of $E(j)$. An equivalent expression of this virtual $k[G_{opp}]$-module is in terms of Hodge cohomology groups for explicit equivariant vector bundles on $Y$. We also compare the rigid cohomology with the rigid cohomology with compact support. In section 3 we prove that for general Deligne-Lusztig varieties the $\ell$-adic étale cohomology coincides with the rigid cohomology (as virtual representations); the key argument is taken from the proof of the independence-of-$\ell$-result in [6]. In section 4 we look at Deligne-Lusztig varieties for $G = GL_{d+1}$ (some $d \geq 1$) and with $\mathbb{L}$ a maximally non-split $k$-rational torus such that $\mathbb{L}(k) = \mathbb{F}^\times_{q^{d+1}}$. Particular attention is paid to the direct summand $E_0$ of $E$ cut out by the trivial character of $\mathbb{L}(k) = \mathbb{F}^\times_{q^{d+1}}$: the reduction modulo $p$ of its cohomology was completely determined in [11].

Let us mention that in the case $G = SL_2$ much more substantial results have been obtained by Haastert and Jantzen [13]; they examined the crystalline cohomology of a smooth compactification of the curve $X$ itself, a method not available in higher dimensions. We hope that the present paper can be a starting point for a full generalization of the results from [13]. For general $G$ and tori $\mathbb{L}$ the reductions modulo $p$ of the characters of Deligne and Lusztig have been analysed in terms of Weyl modules in [14].

**Notations:** For a real number $r \in \mathbb{R}$ we define $\lfloor r \rfloor \in \mathbb{Z}$ as the integer satisfying $\lfloor r \rfloor \leq r < \lfloor r \rfloor + 1$. For the definition of the rigid cohomology of schemes of finite type over a perfect field $k$ with $\text{char}(k) > 0$ we refer to [2]. As coefficient field of the rigid cohomology of a $k$-scheme we generally use the fraction field $K$ of the ring of Witt vectors $W$ with coefficients in $k$. We denote by $\sigma$ the Frobenius endomorphism ($p$-power map) of $k$, and also the functorially induced endomorphisms of $W$ and $K$. For log crystalline cohomology we refer to [15].

For a polynomial ring $W[X_1, \ldots, X_n]$ in finitely many variables over $W$, Monsky and Washnitzer defined the weak formal completion $W[X_1, \ldots, X_n]^\dagger$ as a subalgebra of the $p$-adic completion $W[X_1, \ldots, X_n]^\wedge$ of $W[X_1, \ldots, X_n]$. While the elements of $W[X_1, \ldots, X_n]^\wedge$ are those power series converging on the closed unit polydisk, the elements of $W[X_1, \ldots, X_n]^\dagger$ are characterized as those power series satisfying a certain overconvergence condition. To a quotient algebra $A = W[X_1, \ldots, X_n]^\dagger/I$ (for an ideal $I \subset W[X_1, \ldots, X_n]^\dagger$), Meredith [16] associated a locally...
ringed space $\text{Spwf}(A)$, an affine weak formal scheme. Its underlying topological space is just that of $\text{Spec}(A \otimes_W k)$, its ring of global functions is $A$. A morphism of affine weak formal $W$-schemes $\text{Spwf}(A) \to \text{Spwf}(B)$ is smooth (resp. étale) if it is flat and if the induced morphism $\text{Spec}(A) \to \text{Spec}(B)$ is smooth (resp. étale).

For an affine weak formal scheme $\text{Spwf}(A)$ the $K$-algebra $A \otimes K$ is a $K$-dagger algebra in the sense of [10] so that we may form the affinoid $K$-dagger space $\text{Sp}(A \otimes W K)$. More globally, Meredith [16] defines weak formal $W$-schemes as locally ringed spaces which locally look like affine weak formal schemes. To a weak formal $W$-scheme $¥$ one may associate a "generic fibre" $¥_K$, a $K$-dagger space in the sense of [10]. There is a specialization map

$$sp : ¥_K \to ¥$$

in the category of ringed (Grothendieck topological) spaces. The situation is completely parallel to that of formal $W$-schemes (of finite type) and $K$-rigid spaces.

For a flat affine weak formal $W$-scheme $¥$ with special fibre the smooth affine $k$-scheme $¥$ the rigid cohomology $H^*_{\text{rig}}(¥)$ of $¥$ (as defined by Berthelot, see [1], [2]) is the same as the de Rham cohomology of the generic fibre $¥_K$ (as an affinoid $K$-dagger space) of $¥$ (see [10] for this comparison isomorphism).

Let $¥$ be a smooth formal $W$-scheme. We say that a closed formal subscheme $¹$ of $¥$ is a divisor with normal crossings relative to $\text{Spf}(W)$ if étale locally on $¥$ the embedding of formal $W$-schemes $¹ \to ¥$ takes the form

$$\text{Spf}(W[X_1, \ldots, X_n]/(X_1 \cdots X_r)) \to \text{Spf}(W[X_1, \ldots, X_n])$$

for some $1 \leq r \leq n$. We call $¹$ a prime divisor if we can choose $r = 1$. Given a finite sum $G = \sum_V b_V V$ with $b_V \in \mathbb{Z}$ and with prime divisors $V$ on $¥$ as above, we define the invertible $O_¥$-module

$$O_¥(G) = \bigotimes_V \mathfrak{J}_V^{-b_V}$$

where $\mathfrak{J}_V \subset O_¥$ is the ideal sheaf of $V$ in $¥$, an invertible $O_¥$-module.

1 Equivariant $¥$-Crystals

Let $k$, $W$ and $K$ be as in the introduction. Let $Y$ denote a smooth proper $k$-scheme, $Y^0 \subset Y$ an open dense subscheme such that $D = Y - Y^0$ is a normal crossings divisor on $Y$ and $f : X \to Y^0$ a finite étale morphism of $k$-schemes. We suppose that a finite group $G$ acts compatibly (from the left) on $Y$, $Y^0$ and $X$ and that $f$ has the following form. There is a $t \in \mathbb{N}$ with $(p, t) = 1$ and a unit $\Pi \in \Gamma(Y^0, O_{Y^0})$ such that for all $g \in G$ there is an automorphy factor $\gamma_g \in \Gamma(Y, \mathcal{L}_Y(D))$ with $\Pi/g(\Pi) = \gamma_t^g$. We require that via $f$ we may identify

$$X = \text{Spec}(O_{Y^0}[\Xi]/(1 - \Xi^t \Pi))$$
in such a way that the lifting of the action of $G$ from $Y^0$ to $X$ is given by $g(\Xi) = \gamma_g\Xi$ for $g \in G$. (For clarification: $\text{Spec}$ signifies relative $\text{Spec}$; we do not require that $Y^0$ or $X$ be affine.)

We endow $Y$ and hence its open subschemes with the log structure associated to the normal crossings divisor $D$, and we endow $\text{Spec}(k)$ with the trivial log structure; thus $Y \to \text{Spec}(k)$ is log smooth. In this section we describe a $G$-equivariant logarithmic $F$-crystal $E$ on $Y$ which in the next section will be used to compute the rigid cohomology $H^*_\text{rig}(X)$ of $X$ (with its $G$-action).

First some more notations in the characteristic-$p$-situation. We denote by $V$ the set of irreducible components of $D$ (so its elements are prime divisors on $Y$). Let $\text{div}(\Pi)$ denote the pole-zero divisor of $\Pi$ on $Y$: by definition this is the minimal divisor (in the usual partial ordering on the set of divisors on $Y$) with $\Pi \in L_Y(\text{div}(\Pi))$. For $V \in V$ we let $\mu_V(\Pi) \in \mathbb{Z}$ denote the multiplicity of $V$ in $\text{div}(\Pi)$, i.e. $\text{div}(\Pi) = \sum_{V \in V} \mu_V(\Pi)V$. For $0 \leq j \leq t - 1$ let

$$b_{V,j} = \left\lfloor j\mu_V(\Pi) \right\rfloor$$

and define the divisor $D(j)$ on $Y$ as

$$D(j) = \sum_{V \in V} b_{V,j}V.$$ 

For $0 \leq j \leq t - 1$ we define $0 \leq \nu(j) \leq t - 1$ and $\mu(j) \in \mathbb{Z}_{\geq 0}$ by requiring

$$pj = \nu(j) + \mu(j)t. \tag{1}$$

Note that for any $m \in \mathbb{Z}$ we have

$$p\left\lfloor jmt^{-1} \right\rfloor - \mu(j)m \leq \left\lfloor \nu(j)mt^{-1} \right\rfloor.$$

Applied to the numbers $m = \mu_V(\Pi)$ we therefore get

$$pD(j) - \mu(j)\text{div}(\Pi) \leq \text{D}(\nu(j)) \tag{2}$$

(in the usual partial ordering on the set of divisors on $Y$).

Now we begin to look at liftings to characteristic 0.

**Definition:** A local lifting datum is a set of data $(U, \mathfrak{U}, D_U, \Pi, \Phi)$ as follows. $U$ is an open subscheme of $Y$ and $\mathfrak{U}$ is a lifting of $U$ to a smooth formal $W$-scheme. $D_U$ is a normal crossings divisor (relative to $\text{Spf}(W)$) on $\mathfrak{U}$ which lifts the normal crossings divisor $D \cap U$ on $U$. We have $\Pi \in \Gamma(\mathfrak{U}, L_{\mathfrak{U}}(nD_{\mathfrak{U}}))$ for some (unimportant) $n \in \mathbb{Z}$ and $\Pi$ lifts $\Pi|_U$. Finally, $\Phi : \mathfrak{U} \to \mathfrak{U}$ is an endomorphism lifting the (absolute) Frobenius endomorphism of $U$, respecting $D_U$ and such that $\Phi^* : \mathcal{O}_\mathfrak{U} \to \mathcal{O}_\mathfrak{U}$ restricts to $\sigma$ on the subring $W$ of $\mathcal{O}_\mathfrak{U}$.

**Lemma 1.1.** $Y$ can be covered by local lifting data.
PROOF: We may cover $Y$ by open affine subschemes $U = \text{Spec}(A)$ which admit charts as follows. There exists an étale morphism of $k$-schemes

$$\lambda : U = \text{Spec}(A) \rightarrow \text{Spec}(k[X_1, \ldots, X_n])$$

such that $D \cap U = \text{Spec}(A/\lambda^*(X_1) \cdots \lambda^*(X_r))$ for some $1 \leq i \leq r$. We may choose a $W$-algebra $\tilde{A}$ lifting $A$ together with an étale morphism of formal $\text{Spf}(W)$-schemes

$$\tilde{\lambda} : \text{Spf}(\tilde{A}) \rightarrow \text{Spf}(W[X_1, \ldots, X_n])$$

lifting $\lambda$. Put $U = \text{Spf}(\tilde{A})$. Choose elements $\lambda_1, \ldots, \lambda_r \in \tilde{A}$ which lift $\lambda^*(X_1), \ldots, \lambda^*(X_r)$. Put

$$D_U = \text{Spf}(A/\tilde{\lambda}(X_1) \cdots \tilde{\lambda}(X_r)),$$

Now $\Pi$ is a regular function on $Y^0$, hence $\Pi \in \Gamma(Y, \mathcal{L}(nD))$ for some $n \in \mathbb{Z}$. Since

$$\Gamma(U, \mathcal{L}(nD_U)) \rightarrow \Gamma(U, \mathcal{L}(n(D \cap U)))$$

is surjective we may lift $\Pi|_U \in \Gamma(U, \mathcal{L}(n(D \cap U)))$ to some $\tilde{\Pi} \in \Gamma(U, \mathcal{L}(nD_U))$. Finally we define the lifting $\Phi : \text{Spf}(W[X_1, \ldots, X_n]) \rightarrow \text{Spf}(W[X_1, \ldots, X_n])$ of the Frobenius endomorphism of $\text{Spec}([X_1, \ldots, X_n])$ by $\Phi|_W = \sigma$ and $\Phi(X_i) = X_i^p$. Since $\tilde{\lambda}$ is étale we get a lifting of Frobenius $\tilde{\Phi} : \tilde{U} \rightarrow \tilde{U}$ as desired.

To define the searched for $F$-crystal $E$ on $Y$, we first define its restriction $E|_U$ to $U$, for any open $U \subset Y$ which admits a local lifting datum $(U, \mathfrak{U}, D_U, \tilde{\Pi}, \Phi)$: this we can do by giving a locally free $\mathfrak{U}$-module with logarithmic connection and with a Frobenius endomorphism.

Let $(U, \mathfrak{U}, D_U, \tilde{\Pi}, \Phi)$ be a local lifting datum. Clearly $\mathfrak{U} \rightarrow \text{Spf}(W)$ is a log smooth lifting of $U \rightarrow \text{Spec}(k)$. We write $\Omega^\bullet_{\mathfrak{U}}$ for the logarithmic de Rham complex on $\mathfrak{U}$ with logarithmic poles along $D_U$. For $V \in \mathcal{V}$ we define a closed subscheme $V_{\mathfrak{U}}$ of $\mathfrak{U}$ as follows. If $V \cup U$ is empty we declare $V_{\mathfrak{U}}$ to be empty. Otherwise the prime divisor $V \cap U$ on $U$ lifts to a uniquely determined $W$-flat closed subscheme $V_{\mathfrak{U}}$ of $D_U$ (thus $V_{\mathfrak{U}}$ is a prime divisor on $\mathfrak{U}$ relative to $\text{Spf}(W)$). If we are given a divisor on $Y$ of the form $G = \sum_{V \in \mathcal{V}} b_V V$ with $b_V \in \mathbb{Z}$ then $G_{\mathfrak{U}} = \sum_{V \in \mathcal{V}} b_V V_{\mathfrak{U}}$ defines a lifting of the divisor $G \cap U$ on $U$. It gives rise to the line bundle $\mathcal{L}_{\mathfrak{U}}(G_{\mathfrak{U}})$ on $\mathfrak{U}(G)$ which by abuse of notation we simply denote by $\mathcal{L}_{\mathfrak{U}}(G)$. For example, we abusively write $\mathcal{L}_{\mathfrak{U}}(D(j))$ instead of $\mathcal{L}_{\mathfrak{U}}(D(j)_{\mathfrak{U}})$.

For $0 \leq j \leq t - 1$ we define the logarithmic integrable connection

$$\nabla_j : \mathcal{L}_{\mathfrak{U}}(D(j)) \rightarrow \mathcal{L}_{\mathfrak{U}}(D(j)) \otimes_{\mathcal{O}_{\mathfrak{U}}} \Omega^1_{\mathfrak{U}},$$

$$f \mapsto d(f) - j t^{-1} f d\log(\tilde{\Pi})$$

on the line bundle $\mathcal{L}_{\mathfrak{U}}(D(j))$ on $\mathfrak{U}$. In the proof of [1.2] below (which deals with a more general situation) we will see that the logarithmic integrable connection $\nabla_j$ gives rise to an isomorphism

$$\mathcal{D} \otimes \mathcal{L}_{\mathfrak{U}}(D(j)) \cong \mathcal{L}_{\mathfrak{U}}(D(j)) \otimes \mathcal{D}$$
where \(D\) denotes the structure sheaf of the \(p\)-adically completed divided power envelope of \(U\) in an exactification of the diagonal embedding \(U \to \Omega \times_W \Omega\) (see the proof of \([1, 2]\) for what this means). In view of this property we conclude (see \([15]\)) that the module with connection \((\mathcal{L}_U(D(j)), \nabla_j)\) defines a crystal \(E_U(j)\) on (the logarithmic crystalline site of) \(U\) relative to \(\text{Spf}(W)\).

Now comes Frobenius; unlike the connection \(\nabla_j\) it will jump between the \(\mathcal{L}_U(D(j))\) for various \(j\). We define the map

\[
F : \Phi^* \mathcal{L}_U(D(j)) = \mathcal{L}_U(D(j)) \otimes_{\mathcal{O}_U, \Phi^*} \mathcal{O}_U \to \mathcal{L}_U(pD(j) - \mu(j)\text{div}(\Pi)),
\]

\[
f \otimes 1 \mapsto \Phi^*(f)\tilde{\Pi}^{- \mu(j)}\left(\frac{\tilde{\Pi}^p}{\Phi^*(\Pi)}\right)^{jt^{-1}}
\]

\((f \in \mathcal{L}_U(D(j))). \) To understand this definition note that \(f \in \mathcal{L}_U(D(j))\) implies \(\Phi^*(f) \in \mathcal{L}_U(pD(j))\) and that \(\tilde{\Pi}^{- \mu(j)} \in \mathcal{L}_U(- \mu(j)\text{div}(\Pi))\), and finally that \(\tilde{\Pi}^p \equiv \Phi^*(\Pi)\) modulo \(p\), hence \(\tilde{\Pi}^p/\Phi^*(\Pi)\) is a 1-unit and its exponentiation with \(jt^{-1}\) makes sense. In view of \((2)\) we have \(\mathcal{L}_U(pD(j) - \mu(j)\text{div}(\Pi)) \subseteq \mathcal{L}_U(D(\nu(j)))\) so that we may and will consider \(F\) as a map

\[
F : \Phi^* \mathcal{L}_U(D(j)) \to \mathcal{L}_U(D(\nu(j))).
\]

This map commutes with the connections \(\nabla_j\) and \(\nabla_{\nu(j)}\). This could be verified by a straightforward computation; however, in section \(2\) we will get this fact for free by giving a new interpretation of the collection of crystals \(E_U(j)\) tensored with \(K\).

From \((3)\) we get morphisms of crystals

\[
F : \Phi^* E_U(j) \to E_U(\nu(j))
\]

for \(0 \leq j \leq t - 1\). We define the crystal

\[
E_U = \bigoplus_{0 \leq j \leq t - 1} E_U(j).
\]

and endow \(E_U\) with a Frobenius structure by taking \(F : \Phi^* E_U \to E_U\) as the sum over the maps \((4)\) (each of them composed with the respective projection \(\Phi^* E_U \to \Phi^* E_U(j)\) and inclusion \(E_U(\nu(j)) \to E_U\)). We have defined an \(F\)-crystal \(E_U\) on \(U\).

Next we wish to glue these locally defined \(F\)-crystals \(E_U\) to obtain a crystal \(E\) on \(Y\). To do this we need to check the independence of our local constructions of the chosen local lifting data. Repeating the above constructions with respect to another local lifting datum \((U', \Omega', D'_U, \tilde{\Pi}', \Phi')\) we get an \(F\)-crystal \(E_{U'}\) on \(U'\). Let \(g \in G\) and let \(g^* E_{U'}\) be the pull back \(F\)-crystal on \(g^{-1}U'\).

**Lemma 1.2.** There is a canonical (depending on the local lifting data) isomorphism of \(F\)-crystals on \(U \cap g^{-1}U'\),

\[
\beta_g : E_{U | U \cap g^{-1}U'} \cong E_{U' | g^{-1}U'}. \]

Given a third local lifting datum over \(U'' \subseteq Y\) and another element \(h \in G\), we have \(\beta_{gh} = h^*(\beta_g) \circ (\beta_h)\) on \(U \cap g^{-1}U' \cap (gh)^{-1}U''\).
PROOF: Chopping of closed subschemes we may assume that \( g \) induces an isomorphism \( U \to U' \). We blow up \( \mathfrak{U} \times_W \mathfrak{U}' \) along the closed formal subschemes \( V_{\mathfrak{U}} \times gV_{\mathfrak{U}'} \) for \( V \in \mathcal{V} \), we then remove the strict transforms of all \( \mathfrak{U} \times gV_{\mathfrak{U}'} \) and of all \( V_{\mathfrak{U}} \times \mathfrak{U}' \) (for all \( V \)) and call the result \( \mathfrak{W} \).

By construction, the embedding \((1, g) : U \to \mathfrak{U} \times_W \mathfrak{U}' \) factors as canonically as

\[
U \xrightarrow{\iota} \mathfrak{W} \xrightarrow{\rho} \mathfrak{U} \times_W \mathfrak{U}'.
\]

So far this is the standard construction in logarithmic crystalline cohomology: in the terminology of logarithmic geometry, \( \iota \) is an exact closed embedding and \( \rho \) is log étale so that \((\square)\) may be called an exactification of the closed embedding of formal log schemes \((1, g) : U \to \mathfrak{U} \times_W \mathfrak{U}'\).

In our particular situation this means that the map \( u_1 : \mathfrak{W} \to \mathfrak{U} \) (resp. \( u_2 : \mathfrak{W} \to \mathfrak{U}' \)), the composite of \( \rho \) with the projection to the first (resp. the second) factor of \( \mathfrak{U} \times_W \mathfrak{U}' \), is smooth (in the classical sense), and moreover that the pull back of \( D_{\mathfrak{U}} \) via \( u_1 \) is the same as the pull back of \( D_{\mathfrak{U}'} \) via \( u_2 \): a relative normal crossings divisor on \( \mathfrak{W} \) which we denote by \( D_{\mathfrak{W}} \).

We view \( U \) as a closed subscheme of \( \mathfrak{W} \) via \( \iota \). Let \( \mathcal{D} \) denote the structure sheaf of the \( p \)-adically completed divided power envelope of \( U \) in \( \mathfrak{W} \); it is supported on \( U \). On sufficiently small open affine pieces \( T \) of \( \mathfrak{W} \) the ring \( \mathcal{D}(T) \) is via \( u_1 \) (resp. \( u_2 \)) a relative \( p \)-adically completed divided power polynomial ring over \( \mathcal{O}_{\mathfrak{U}} \) (resp. over \( \mathcal{O}_{\mathfrak{U}'} \)): this is because \( u_1 \) (resp. \( u_2 \)) is smooth and admits locally a section supported on \( U \). We claim that

\[
\alpha_j = \frac{u_1^{-1}(\widetilde{\Pi})jt^{-1}}{u_2^{-1}(\widetilde{\Pi}')jt^{-1}}
\]

is a global section and a generator (as a \( \mathcal{D} \)-module sheaf) of

\[
\mathcal{L}_{\mathfrak{W}}(j(u_1^{-1}D(j) - u_2^{-1}D(j))) \otimes_{\mathcal{O}_{\mathfrak{W}}} \mathcal{D}.
\]

In particular, \((\square)\) is a free \( \mathcal{D} \)-module of rank one. [Here and below, to define the line bundle \( \mathcal{L}_{\mathfrak{W}}(j(u_1^{-1}D(j) - u_2^{-1}D(j))) \) on \( \mathfrak{W} \) we use similar notational conventions as we did to define line bundles on \( \mathfrak{U} \); namely, we write the divisor \( j(u_1^{-1}D(j) - u_2^{-1}D(j)) \) on \( \mathfrak{W} \otimes k \) as \( \sum_{V \in \mathcal{V}} b_V V_{\mathfrak{W} \otimes k} \) with \( b_V \in \mathbb{Z} \) and prime divisors \( V_{\mathfrak{W} \otimes k} \) on \( \mathfrak{W} \otimes k \). By construction, the \( V_{\mathfrak{W} \otimes k} \) lift to \( \mathcal{W} \)-flat formal subschemes \( V_{\mathcal{W}} \) of \( D_{\mathcal{W}} \) (which are thus prime divisors relative to \( \text{Spf}(W) \)), and we define the relative normal crossings divisor \( j(u_1^{-1}D(j) - u_2^{-1}D(j))_{\mathcal{W}} \) on \( \mathfrak{W} \) as \( \sum_{V \in \mathcal{V}} b_V V_{\mathcal{W}} \), which we then use to define \( \mathcal{L}_{\mathfrak{W}}(j(u_1^{-1}D(j) - u_2^{-1}D(j))) \).

We know \( \Pi/g(\Pi) = \gamma_g \) for some \( \gamma_g \in \Gamma(Y, \mathcal{L}_Y(D)) \). Viewing \( \gamma_g \) as an element of \( \Gamma(U, \mathcal{O}_U(D \cap U)) \) we may lift it to a section \( \widetilde{\gamma}_g \in \Gamma(U, \mathcal{O}_U(D_{\mathfrak{U}})) \). Letting \( g(\widetilde{\Pi}) = \widetilde{\Pi}^{-1} \gamma_g \) we have

\[
\alpha_j = \frac{u_1^{-1}(\widetilde{\Pi})jt^{-1}}{u_1^{-1}(g(\widetilde{\Pi}))jt^{-1}} \frac{u_1^{-1}(g(\widetilde{\Pi}))jt^{-1}}{u_2^{-1}(\widetilde{\Pi}')jt^{-1}}.
\]

The first factor in \((\square)\) is simply \( u_1^{-1}(\widetilde{\gamma}_g)^j \) and this is a generator of

\[
\mathcal{L}_{\mathfrak{W}}(j(u_1^{-1}D(j) - u_2^{-1}D(j))) = \mathcal{L}_{\mathcal{W}}(j(u_1^{-1}(D(j) - g^{-1}D(j))))
\]
(note that $u_2^{-1}(V) = u_1^{-1}(g^{-1}V)$ for any divisor $V$ on $Y$). For the second factor in (8) we have
\[
\frac{u_1^{-1}(g(\mathcal{W}))}{u_2^{-1}(\mathcal{W})} = (1 + \frac{u_1^{-1}(g(\mathcal{W})) - u_2^{-1}(\mathcal{W})}{u_2^{-1}(\mathcal{W})})^{jt-1} = \sum_{\nu=0}^{\infty} \frac{jt^{-1}(jt^{-1} - 1) \cdots (jt^{-1} - \nu + 1)}{\nu!} \left( \frac{u_1^{-1}(g(\mathcal{W})) - u_2^{-1}(\mathcal{W})}{u_2^{-1}(\mathcal{W})} \right)^{\nu}.
\]
Now $\frac{u_1^{-1}(g(\mathcal{W})) - u_2^{-1}(\mathcal{W})}{u_2^{-1}(\mathcal{W})}$ is a global section of the ideal of the embedding $\iota : U \to \mathfrak{M}$, hence the infinite sum is a global section and a generator of $\mathcal{D}$. (In fact the second factor in (8) is an element of the structure sheaf of the completion of $\mathfrak{M}$ along $U$ since all $\frac{jt^{-1}(jt^{-1} - 1) \cdots (jt^{-1} - \nu + 1)}{\nu!}$ lie in $\mathbb{Z}_p$.) Our claim is established.

We have the logarithmic connection $u_1^*\nabla_j$ on
\[
\mathcal{L}_\mathfrak{M}(u_1^{-1}D(j)) = u_1^*\mathcal{L}_U(D(j))
\]
and the logarithmic connection $u_2^*\nabla_j$ on
\[
\mathcal{L}_\mathfrak{M}(u_2^{-1}D(j)) = u_2^*\mathcal{L}_U(D(j))
\]
(with logarithmic poles along $D_\mathfrak{M}$) and $u_1^*\nabla_j$ (resp. $u_2^*\nabla_j$) is characterized as follows. Outside $D_\mathfrak{M}$ we identify as usual $\mathcal{L}_\mathfrak{M}(u_1^{-1}D(j))$ (resp. $\mathcal{L}_\mathfrak{M}(u_2^{-1}D(j))$) with $\mathcal{O}_\mathfrak{M}$ and under this identification, $u_1^*\nabla_j$ (resp. $u_2^*\nabla_j$) acts on $f \in \mathcal{O}_\mathfrak{M}$ as
\[
u_j(f) = d(f) - jt^{-1}f d\log(u_1^{-1}(\mathcal{W})),
\]
\[
u_j(f) = d(f) - jt^{-1}f d\log(u_2^{-1}(\mathcal{W})).
\]

The connection $u_1^*\nabla_j$ (resp. the connection $u_2^*\nabla_j$) gives rise to a connection $u_1^*\nabla_j$ (resp. $u_2^*\nabla_j$) on $\mathcal{L}_\mathfrak{M}(u_1^{-1}D(j)) \otimes \mathcal{O}_\mathfrak{M} \mathcal{D}$ (resp. on $\mathcal{L}_\mathfrak{M}(u_2^{-1}D(j)) \otimes \mathcal{O}_\mathfrak{M} \mathcal{D}$). Since $\iota : U \to \mathfrak{M}$ is an exact closed embedding into a log smooth formal $W$-scheme, these data are equivalent with crystals on $U$; namely, the crystal $E_U(j)$ is equivalent to the one associated with $(\mathcal{L}_\mathfrak{M}(u_1^{-1}D(j)) \otimes \mathcal{O}_\mathfrak{M} \mathcal{D}, u_1^*\nabla_j)$, and the crystal $g^*E_U(j)$ is equivalent to the one associated with $(\mathcal{L}_\mathfrak{M}(u_2^{-1}D(j)) \otimes \mathcal{O}_\mathfrak{M} \mathcal{D}, u_2^*\nabla_j)$.

We define the isomorphism of modules with connection
\[
\alpha_j : (\mathcal{L}_\mathfrak{M}(u_2^{-1}D(j)) \otimes \mathcal{O}_\mathfrak{M} \mathcal{D}, u_2^*\nabla_j) \to (\mathcal{L}_\mathfrak{M}(u_1^{-1}D(j)) \otimes \mathcal{O}_\mathfrak{M} \mathcal{D}, u_1^*\nabla_j)
\]
as the one induced by multiplication with the function $\alpha_j$ defined above. It induces an isomorphism of crystals $g^*E_U(j) \cong E_U(j)$. Taking the sum over all $j$ gives the isomorphism of crystals $g^*E_U \cong E_U$; it respects the Frobenius structures, so this is in fact an isomorphism of $F$-crystals. To check the cocycle condition one has to work on the divided power envelope of (an exactification of) $(1, g, gh) : U \to \Omega \times \Omega' \times \Omega''$ (note that $(1, g) : Y \to Y \times Y$ has the same image as $(h, gh) : Y \to Y \times Y$; this is the axiom for $G$ acting (from the left) on $Y$) and there it boils down to the obvious multiplicativity of the involved automorphy factors $\alpha_j$ (one of them being the product of the other two).
Remark: In fact all compatibilities (commutation and cocycle conditions between Frobenii, connections and the $G$-action, i.e. the $\alpha_j$) in the above proof, partly left out there, follow from our discussion in section 2 below where we see that the $E_U$ sit inside a $G$-equivariant overconvergent $F$-isocrystal $E^\dagger$ on $Y^0$ for which a priori all the compatibilities hold; from this point of view the purpose of the present section is to observe that the Frobenius, connection and $G$-action of $E^\dagger$ respect the integral structures $E_U$.

Using 1.2 with $g = h = 1$ we see that we can glue our $F$-crystals $E_U$ associated with local lifting data to a global crystal $E$ on $Y$ relative to Spf$(W)$. By construction, as a crystal it is naturally decomposed as

$$E = \bigoplus_{0 \leq j \leq t-1} E(j)$$

such that $E_U(j)$ is the restriction of $E(j)$ to $U$. Moreover 1.2 for arbitrary $g \in G$ provides an isomorphism of $F$-crystals $g^*E \cong E$, respecting the crystal decomposition (9). The cocycle condition in 1.2 ensures that these isomorphisms give $E$ the structure of a $G$-equivariant $F$-crystal on $Y$. In particular, the crystalline cohomology $H^{\ast \text{crys}}(Y/W, E)$ becomes a right representation of $G$.

Let $\Omega^\bullet_Y$ denote the logarithmic de Rham complex on $Y$. Let $E$ resp. $E(j)$ denote the reduction modulo $p$ of the $F$-crystal $E$, resp. the crystal $E(j)$. Thus $E(j)$ is equivalent with the logarithmic connection $\nabla^j : \mathcal{L}_Y(D(j)) \to \mathcal{L}_Y(D(j)) \otimes_{O_Y} \Omega^1_Y$. The action of $G$ on each $E(j)$ and hence on $E$ is then easily described follows. For $g \in G$ we need to give an isomorphism $g^*E(j) \cong E(j)$. It is the one corresponding to the isomorphism of line bundles with connection $\mathcal{L}_Y(g^{-1}D(j)) \cong \mathcal{L}_Y(D(j))$ given by multiplication with $\gamma_g^j$.\n
Remarks: (1) If $s \in \mathbb{N}$ is such that $t$ divides $p^s - 1$ then $F^s$ respects the direct sum decomposition (9) so that each $E(j)$ is an $F^s$-crystal.

(2) Suppose $m \in \mathbb{N}$ divides $t$ and there is a $\Lambda \in \Gamma(Y^0, \mathcal{O}_{Y^0})$ with $\Lambda^m = \Pi$. For $0 \leq j \leq t - 1 - tm^{-1}$ multiplication with local liftings $\tilde{\Lambda}$ of $\Lambda$ (such that $\Lambda^m = \tilde{\Pi}$ is a lifting of $\Pi$ as part of a local lifting datum) induces an isomorphism of crystals

$$E(j) \cong E(j + tm^{-1}).$$

We conclude this section by computing the residues of the logarithmic connections $\nabla^j$. Let $V \in \mathcal{V}$ and let $(U, \mathcal{U}, D_U, \tilde{\Pi}, \Phi)$ be a local lifting datum such that $V_U$ is non empty. Let $\text{Res}_V(\nabla^j)$ denote the residue along $V_U$ of the logarithmic connection $\nabla^j$ on $\mathcal{L}_U(D(j))$.

Lemma 1.3.

$$\text{Res}_V(\nabla^j) = jt^{-1}\mu_V(\Pi) - b_{V,j} = jt^{-1}\mu_V(\Pi) - \lfloor jt^{-1}\mu_V(\Pi) \rfloor.$$
Proof: Locally on \( \mathcal{U} \) we find an étale morphism of formal \( W \)-schemes

\[
\tilde{\lambda} : \mathcal{U} \longrightarrow \text{Spf}(W[X_1, \ldots, X_n])
\]
such that \( g = \tilde{\lambda}^*(X_1) \in \mathcal{O}_\mathcal{U} \) is a local equation for \( V_\mathcal{U} \) in \( \mathcal{U} \). Consider the composition

\[
\mathcal{L}_\mathcal{U}(D(j)) \otimes_{\mathcal{O}_\mathcal{V}_\mathcal{U}} \mathcal{O}_\mathcal{V}_\mathcal{U} \rightarrow \mathcal{L}_\mathcal{U}(D(j)) \otimes_{\mathcal{O}_\mathcal{U}} \mathcal{O}_{\mathcal{V}_\mathcal{U}} \xrightarrow{\beta} \mathcal{L}_\mathcal{U}(D(j)) \otimes_{\mathcal{O}_\mathcal{U}} \mathcal{O}_{\mathcal{V}_\mathcal{U}}
\]

where the first map is the one induced by \( \nabla_j \) and the second map \( \beta \) is induced by the map \( \Omega^1_{\mathcal{U}} \rightarrow \mathcal{O}_{\mathcal{V}_\mathcal{U}} \) which sends the class of \( f \text{dlog}(g) \) with \( f \in \mathcal{O}_\mathcal{U} \) to the image of \( f \) in \( \mathcal{O}_{\mathcal{V}_\mathcal{U}} \) (and vanishes on all forms which are regular along \( V_\mathcal{U} \)). This composition is given by multiplication with \( \text{Res}_{V_\mathcal{U}}(\nabla_j) \) — by definition of \( \text{Res}_{V_\mathcal{U}}(\nabla_j) \). To compute \( \text{Res}_{V_\mathcal{U}}(\nabla_j) \) it is enough to evaluate the map on an arbitrary non zero element of \( \mathcal{L}_\mathcal{U}(D(j)) \otimes_{\mathcal{O}_\mathcal{U}} \mathcal{O}_{\mathcal{V}_\mathcal{U}} \). We evaluate it on the element \( g - b_{V,j} \in \mathcal{L}_\mathcal{U}(D(j)) \otimes_{\mathcal{O}_\mathcal{U}} \mathcal{O}_{\mathcal{V}_\mathcal{U}} \). We may write \( \tilde{\Pi} = g - \mu_V(\Pi) h \) with a function \( h \) regular and non-zero along \( V_\mathcal{U} \). Then

\[
\nabla_j(g^{-b_{V,j}}) = d(g^{-b_{V,j}}) - j t^{-1} g^{-b_{V,j}} \text{dlog}(g^{-\mu_V(\Pi)} h)
\]

\[
= g^{-b_{V,j}} \text{dlog}(g^{-b_{V,j}}) - j t^{-1} g^{-b_{V,j}} \text{dlog}(g^{-\mu_V(\Pi)} h)
\]

\[
= -b_{V,j} g^{-b_{V,j}} \text{dlog}(g) + j t^{-1} \mu_V(\Pi) g^{-b_{V,j}} \text{dlog}(g) - j t^{-1} g^{-b_{V,j}} \text{dlog}(h)
\]

and the claim follows since \( \beta \) vanishes on the term \( j t^{-1} g^{-b_{V,j}} \text{dlog}(h) \). \( \square \)

2 Relative rigid cohomology of tame abelian coverings

We compute the relative rigid cohomology \( E^\dagger \) of the finite étale morphism \( f : X \rightarrow Y^0 \); thus \( E^\dagger \) will be an overconvergent \( F \)-isocrystal on \( Y^0 \), endowed with an action by \( G \). Since this is a local datum we suppose that \( Y^0 \) is affine. Then \( Y^0 \) lifts to a smooth weak formal affine \( W \)-scheme \( \mathcal{U} = \text{Spwf}(A) \), see [8]. We choose a lifting \( \tilde{\Pi} \in A \) of \( \Pi \) and let

\[
B = A[\Xi]/(1 - \Xi^t \tilde{\Pi}).
\]

As an \( A \)-module \( B \) decomposes as

\[
(11)
B = \bigoplus_{j=0}^{t-1} \Xi^j A.
\]

\( \mathcal{X} = \text{Spwf}(B) \) is a smooth weak formal \( W \)-scheme lifting \( X \) and the map \( A \rightarrow B \) defines a lifting \( \tilde{f} : \mathcal{X} \rightarrow \mathcal{U} \) of \( f \).

(1) First let us compute \( E^\dagger \) as an overconvergent isocrystal on \( Y^0 \). Let \( (\Omega^\bullet_A, d) \) resp. \( (\Omega^\bullet_B, d) \) denote the de Rham complex of \( A \) resp. \( B \) relative to \( W \). We compute the Gauss-Manin connection

\[
(12) \quad \nabla : B \rightarrow B \otimes_A \Omega^1_A.
\]
In $\Omega^t_B$ we have $0 = d(1 - \Xi^t\Pi) = d(\Xi^t\Pi)$, hence $d\Xi = -t^{-1}\Xi d\log(\Pi)$. It follows that
\[
\nabla(\Xi^j) = -jt^{-1}\Xi^j d\log(\Pi)
\]
for any $0 \leq j \leq t - 1$. In particular we see that $\nabla$ respects the direct sum decomposition (II).

We endow the $O_{U_0}$-module
\[
E = (f_0)_*O_{X_0} = \bigoplus_{j=0}^{t-1} O_{U_{t_j}} \Xi^j
\]
with an (integrable overconvergent) connection $\nabla$ as follows: it respects the indicated sum decomposition, and on the $j$-summand it is given via the isomorphism of $O_{U_0}$-modules
\[
O_{U_0} \cong \Xi^j O_{U_0}, \quad 1 \mapsto \Xi^j
\]
(for $0 \leq j \leq t - 1$) as the connection
\[
\nabla_j : O_{U_0} \to \Omega^1_{U_0},
\]
\[
f \mapsto d(f) - jt^{-1}f d\log(\Pi).
\]
This defines the overconvergent isocrystal $E^\dagger$ on $Y^0$ which decomposes accordingly as
\[
E^\dagger = \bigoplus_{j=0}^{t-1} E(j)^\dagger.
\]

(II) Next let us look at the Frobenius structure on $E^\dagger$. Choose an endomorphism $\Phi^*$ of the $W$-algebra $A$ which restricts to $\sigma$ on the subring $W$ and which lifts the Frobenius endomorphism of $A \otimes k$. We extend $\Phi^*$ further to an endomorphism $\Phi^*$ of $B$ lifting the $p$-power Frobenius endomorphism of $B$ by prescribing
\[
\Phi^*(\Xi^j) = \Xi^{\nu(j)}\Pi^{-\mu(j)}(\frac{\Pi^p}{\Phi^*(\Pi)})^{jt^{-1}}
\]
for $0 \leq j \leq t - 1$, with the numbers $0 \leq \nu(j) \leq t - 1$ and $\mu(j) \in \mathbb{Z}_{\geq 0}$ defined through equation (I). It follows that the Frobenius structure
\[
F : \Phi^* E^\dagger \to E^\dagger
\]
on $E^\dagger$ is given as the sum of maps
\[
F_j : \Phi^* E(j)^\dagger \to E(\nu(j))^\dagger
\]
where $F_j$ is given by the morphism of modules with connection
\[
(\bigoplus_{j=0}^{t-1} \bigoplus_{j=0}^{t-1} (\bigoplus_{j=0}^{t-1} O_{U_0}, \nabla(j)))
\]
\[
f \otimes 1 \mapsto \Phi^*(f)\Pi^{-\mu(j)}(\frac{\Pi^p}{\Phi^*(\Pi)})^{jt^{-1}}.
\]
Finally we make explicit the $G$-action on $E^\dagger$. Fix $g \in G$. Let

$$Z = X \times X, \quad \mathfrak{M} = \mathfrak{U} \times \mathfrak{U}$$

(products in the category of weak formal schemes), and let $Z_K$ (resp. $\mathfrak{M}_K$) denote the generic fibre (as a dagger space) of $Z$ (resp. of $\mathfrak{M}$). We view $X$ as a closed subscheme of $Z$, of $\mathfrak{U} \times X$ and of $X \times \mathfrak{U}$ via the embeddings

$$X \overset{(1,g)}{\to} X \times X = Z,$$
$$X \overset{(f \circ 1, g)}{\to} \mathfrak{U} \times X,$$
$$X \overset{(1, f \circ g)}{\to} X \times \mathfrak{U},$$
and we view $Y^0$ as a closed subscheme of $\mathfrak{M}$ via the embedding

$$Y^0 \overset{(1,g)}{\to} \mathfrak{U} \times \mathfrak{U} = \mathfrak{M}.$$ 

We denote by $]X[3$ the preimage of $X$ under the specialization map $sp : Z_K \to Z$; thus $]X[3$ is an admissible open dagger subspace of $Z_K$. Similarly we define the admissible open dagger subspace $]Y^0[\mathfrak{M}$ (resp. $]X[U \times \mathfrak{X}$, resp. $]X[\mathfrak{X} \times \mathfrak{U}$) of $\mathfrak{M}_K$ (resp. of $(\mathfrak{U} \times X)_K$, resp. of $(\mathfrak{X} \times \mathfrak{U})_K$). Denote by

$$u_i : ]Y^0[\mathfrak{M} \to \mathfrak{M}_K \to \mathfrak{U}_K$$

the projection to the $i$-th component ($i = 1, 2$). Since $\mathfrak{U}$ and $\mathfrak{M}$ are smooth weak formal schemes, the structure sheaf of the formal completion of $\mathfrak{M}$ along $Y^0$ is (locally) a relative formal power series ring over the structure sheaf of the $p$-adic completion of $\mathfrak{U}$. This implies (see [1]) that the morphism which $u_i$ induces on the associated rigid spaces is a fibration in relative open polydisks; by the principles of [10] the same is true for the morphism of dagger spaces $u_i$ itself. Moreover, with $\mathfrak{X} \to \mathfrak{U}$ also the projections $\mathfrak{X} \times \mathfrak{U} \to \mathfrak{U} \times \mathfrak{U}$ and $\mathfrak{U} \times \mathfrak{X} \to \mathfrak{U} \times \mathfrak{U}$ are finite étale. Therefore they induce isomorphisms between the respective formal completions along $X$ and this implies — again first for the associated rigid spaces, then by the principles of [10] for the dagger spaces themselves — that the projection maps

$$]X[\mathfrak{U} \times \mathfrak{X} \leftrightarrow ]X[3 \leftrightarrow ]X[\mathfrak{X} \times \mathfrak{U}$$

are isomorphisms. Now $]X[3$ is finite étale over $]Y^0[\mathfrak{M}$, hence its relative de Rham cohomology provides a module with (integrable overconvergent) connection $(\mathcal{E}', \nabla')$ on $]Y^0[\mathfrak{M}$, hence an overconvergent $F$-isocrystal $(E')^\dagger$ on $Y^0$. On the other hand the module with connection $u_1^\ast(\mathcal{E}, \nabla)$ (resp. $u_2^\ast(\mathcal{E}, \nabla)$) on $]Y^0[\mathfrak{M}$ corresponds to the previously defined $F$-isocrystal $E^\dagger$ (resp. $g^\ast E^\dagger$) on $Y^0$ by general principles of rigid cohomology (see [1]). By construction this is also the relative de Rham cohomology of $]X[\mathfrak{U} \times \mathfrak{X} \to ]Y^0[\mathfrak{M}$ (resp. of $]X[\mathfrak{X} \times \mathfrak{U} \to ]Y^0[\mathfrak{M}$). Since the maps (14) are isomorphisms the canonical maps

$$u_i^{-1} : u_i^\ast(\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$$
(i = 1, 2) are therefore isomorphisms. Thus we may define the isomorphism of $F$-isocrystals
\begin{equation}
  g : g^*E^\dagger \to E^\dagger
\end{equation}
as the one corresponding to the composite $(u_1^{-1})^{-1} \circ u_2^{-1}$.

This describes the $G$-action on $E^\dagger$. However, we also wish to trace back the decomposition (13) in this description. We keep our fixed $g \in G$ and the above notations. In view of the isomorphisms (14) we can describe the $O_{Y^0}|_{m}$-module with connection $(E', \nabla')$ as the pull back of $(E, \nabla)$ via $u_i$ for either $i = 1$ or $i = 2$, hence both
\[
  \{u_i^{-1}(\Xi)^j\}_{0 \leq j \leq t-1} \quad (i = 1, 2)
\]
are $O_{Y^0}|_{m}$-bases. We claim that the transformation matrix between these bases is diagonal: namely, we claim that for all $j$,
\[
  \alpha_j = \frac{u_2^{-1}(\Xi)^j}{u_1^{-1}(\Xi)^j},
\]
a priori an element of the fraction field of $O_{3_K}(3_K)$, is a global section of $O_{Y^0}|_{m}$. Let $I$ denote the ideal in $O_3$ which defines the diagonal embedding $(1, 1) : \mathfrak{X} \to \mathfrak{Z}$. Then $\alpha_j$ is characterized by the two properties
\[
  1 - \alpha_j \in I, \\
  \alpha_j^t = \frac{u_1^{-1}(\Pi)^j}{u_2^{-1}(\Pi)^j}.
\]
Hence necessarily
\[
  \alpha_j = \left(\frac{u_1^{-1}(\Pi)}{u_2^{-1}(\Pi)}\right)^jt^{-1}.
\]
Therefore it follows from our discussion in the proof of 1.2 that $\alpha_j$ is indeed a global section of $O_{Y^0}|_{m}$. By symmetry it is clear that it is even a unit. As announced it follows that the two decompositions
\[
  E' = \bigoplus_{j=0}^{t-1} O_{Y^0}|_{m}u_1^{-1}(\Xi)^j = \bigoplus_{j=0}^{t-1} O_{Y^0}|_{m}u_2^{-1}(\Xi)^j
\]
are in fact the same and respect the connection $\nabla'$ on $E'$. Moreover, the map (15) respects this decomposition, and its effect on the $j$-th summand can be described as follows.

Via the isomorphism
\[
  O_{u_K} \to O_{u_K} \Xi^j, \quad 1 \mapsto \Xi^j
\]
the connection which $\nabla$ induces on $O_{u_K} \Xi^j$ becomes the connection $\nabla_j$ on $O_{u_K}$ given by
\[
  f \mapsto d(f) - jt^{-1}f d\log(\Pi).
\]
Multiplication with $\alpha_j$ defines an isomorphism of $O_{Y^0}|_{m}$-modules with connection
\[
  (O_{Y^0}|_{m}, u_2^*\nabla_j) \to (O_{Y^0}|_{m}, u_1^*\nabla_j).
\]
Identifying $E(j)^\dagger$ (resp. $g^* E(j)^\dagger$) with the overconvergent isocrystal associated with $(\mathcal{O}_Y^\op, u_j^0 \nabla_j)$ (resp. associated with $(\mathcal{O}_Y^\op, u_j^2 \nabla_j)$) we therefore obtain an isomorphism of overconvergent isocrystals on $Y^0$, 
\[
g : g^* E(j)^\dagger \rightarrow E(j)^\dagger.
\]

**Definition:** We have $H^*_\rig(X) = H^*_\rig(Y^0, E^\dagger)$. For $0 \leq j \leq t - 1$ we now define the subspace $H^*_\rig(X)_j$ of $H^*_\rig(X)$ as the one corresponding to the subspace $H^*_\rig(Y^0, E^\dagger(j))$ of $H^*_\rig(Y^0, E^\dagger)$.

**Remark:** If $K^\times$ contains the cyclic group $T$ of all $t$-th roots of unity then we may view $f : X \rightarrow Y^0$ as the $T$-covering for which the action of $T$ on $X$ is given by $h, \Xi = h\Xi$ for all $h \in T$. Then $H^*_\rig(X)_j$ is the subspace of $H^*_\rig(X)$ on which $T$ acts through the character $\theta_j : T \rightarrow K^\times, h \mapsto h^j$.

**Theorem 2.1.** We have a canonical $G$-equivariant and Frobenius equivariant isomorphism
\[
(16) \quad H^*_\rig(X) = H^*_\rig(Y^0, E^\dagger) \cong H^*_\crys(Y/W, E) \otimes_W K.
\]

For $0 \leq j \leq t - 1$ it restricts to a $G$-equivariant isomorphism
\[
H^*_\rig(X)_j \cong H^*_\crys(Y/W, E(j)) \otimes_W K.
\]

**Proof:** By the construction described in [17] we can associate to the $F$-crystal $E$ on $Y$ (which is weakly non-degenerate in the terminology of [17]: it restricts to a non-degenerate $F$-crystal on the open subscheme $Y^0$ of $Y$ where the log structure is trivial) an overconvergent $F$-isocrystal on $Y^0$: but this is precisely our overconvergent $F$-isocrystal $E^\dagger$, as follows from the explicit descriptions of $E$ and $E^\dagger$ given above. These descriptions also show the coincidence of the $G$-actions. To get the isomorphism (16) we proceed as in [17] 4.2, 4.4 or [19] Corollary 2.3.9, Theorem 3.1.1, Theorem 2.4.4: the results in [19] are formulated only for (truly) non-degenerate $F$-crystals, but as remarked in [19] 2.4.14 they carry over to weakly non-degenerate $F$-crystals whose residues along the compactifying normal crossings divisor have no positive integers as eigenvalues; that this condition is met by our $E$ was checked in [13] (the other condition in [19] 2.4.14 — that the exponents of the monodromy be non-Liouville numbers — is guaranteed by the Frobenius structure, as remarked in [17] 4.2). That the isomorphism (16) respects the $j$-parts is clear. \(\square\)

**Theorem 2.2.** For $0 \leq j \leq t - 1$ the following three virtual $k[G^\opp]$-modules are the same:
(i) the reduction modulo $p$ of the virtual $K[G^\opp]$-module $\sum_s (-1)^s H^s_\rig(X)_j$
(ii) $\sum_s (-1)^s H^s(Y, (\Omega^\bullet_Y \otimes L_Y(D(j)), \nabla_j))$
(iii) $\sum_{s,m} (-1)^{s+m} H^s(Y, \Omega^m_Y \otimes L_Y(D(j)))$.

**Proof:** Let us describe the main result from [12]. Let $Y$ be a proper and smooth $k$-scheme and suppose that the finite group $G$ acts (from the right) on $Y$. Let $E$ be a locally free, finitely generated crystal of $\mathcal{O}_{Y/W}$-modules endowed with an action by $G$ (covering the action of $G$ on
For $s \in \mathbb{Z}$ let $H^s_{\text{crys}}(Y/W, E)$ denote the $s$-th crystalline cohomology group (relative to $\text{Spf}(W)$) of the crystal $E$, a finitely generated $W$-module which is zero if $s \notin [0, 2 \dim(Y)]$. On the other hand, the reduction modulo $p$ of the crystal $E$ is equivalent with a locally free $\mathcal{O}_Y$-module $E_k$ with connection $E_k \to E_k \otimes_{\mathcal{O}_Y} \Omega^1_Y$; here $\Omega^1_Y$ denotes the $\mathcal{O}_Y$-module of differentials of $Y/k$. Let $\Omega^*_Y \otimes E_k$ denote the corresponding de Rham complex. The cohomology group $H^*(Y, \Omega^*_Y \otimes E_k)$ is a finite dimensional $k$-vector space which is zero if $s \notin [0, 2 \dim(Y)]$. The $G$-action on $E$ provides each $H^s_{\text{crys}}(Y/W, E)$, each $H^s(Y, \Omega^*_Y \otimes E_k)$ and each $H^s(Y, \Omega^1_Y \otimes E_k)$ with a $G$-action. By definition, the reduction modulo $p$ of the $K[G]$-module $H^s_{\text{crys}}(Y/W, E) \otimes_k K$ is the $k[G]$-module obtained by reducing modulo $p$ the $G$-stable $W$-lattice $H^s_{\text{crys}}(Y/W, E)/\text{(torsion)}$ in $H^s_{\text{crys}}(Y/W, E) \otimes_k K$. Then: for any $j$, the following three virtual $k[G]$-modules are the same:

(i) the reduction modulo $p$ of the virtual $K[G]$-module $\sum (-1)^s H^s(Y/W, E) \otimes_k K$
(ii) $\sum (-1)^s H^s(Y, \Omega^*_Y \otimes E_k)$
(iii) $\sum (-1)^s H^s(Y, \Omega^1_Y \otimes E_k)$.

This result is stated for crystals in the ordinary sense, but the transposition to logarithmic crystals is immediate. Therefore, combined with Theorem 2.1, it implies Theorem 2.2. (Of course, (ii)=(iii) is immediately clear anyway). □

For a collection $H^* = (H^i)_{i \in \mathbb{Z}}$ of vector spaces indexed by $\mathbb{Z}$ we write
\[
\chi(H^*) = \sum (-1)^i \dim(H^i).
\]

**Theorem 2.3.** (a) $\chi(H_{\text{rig}}^*(X)_j)$ is independent of $0 \leq j \leq t - 1$.
(b) If $0 \leq j \leq t - 1$ is such that $j \mu_V(\Pi)$ is not divisible by $t$ for all $V \in \mathcal{V}$ then
\[
H^*_{\text{crys}}(Y/W, E(j)) \otimes_k K = H^*_{\text{crys,c}}(Y/W, E(j)) \otimes_k K,
\]
\[
H^*_{\text{rig}}(Y^0, E^0_j) = H^*_{\text{rig,c}}(Y^0, E^1_j).
\]
(c) Suppose $Y^0$ is affine and of pure dimension $d$. For all $0 \leq j \leq t - 1$ and all $m > d$ we have $H^m_{\text{rig}}(X)_j = 0$. For $0 \leq j \leq t - 1$ as in (b) also $H^m_{\text{rig}}(X)_j = 0$ for all $m < d$.

**Proof:** Clearly (c) follows from (b) by Poincaré duality. Consider the divisor $D^-(j) = D(j) - \sum_{V \in \mathcal{V}} V$ on $Y$. Just as we defined the crystal $E(j)$ departing from the divisor $D(j)$ we may now define the crystal $E^-(j)$ departing from the divisor $D^-(j)$ but using the same rule for the integrable connections $\nabla_j$. Thus if $(U, \mathcal{U}, D, \Pi, \Phi)$ is a local lifting datum then $E(j)|_U$ is given by the connection $\nabla_j : \mathcal{L}_U(D(j)) \to \mathcal{L}_U(D(j)) \otimes \Omega^1_U$ while $E^-(j)|_U$ is given by the connection $\nabla_j : \mathcal{L}_U(D^-(j)) \to \mathcal{L}_U(D^-(j)) \otimes \Omega^1_U$. The crystalline cohomology with compact support $H^*_{\text{crys,c}}(Y/W, E(j))$ of $E(j)$ is just the crystalline cohomology of the subcrystal $E^-(j)$ of $E(j)$. Hence to prove (b) for crystalline cohomology we need to show that the natural map
\[
H^*_{\text{crys}}(Y/W, E^-(j)) \otimes_k K \to H^*_{\text{crys}}(Y/W, E(j)) \otimes_k K
\]
is an isomorphism. To see this it is enough to show that for sufficiently small open $U \subset Y$ the map $H^*_{\text{crys}}(U/W, E^-(j)) \otimes_k K \to H^*_{\text{crys}}(U/W, E(j)) \otimes_k K$ is an isomorphism. Thus we may
work with a local lifting datum \((U, \mathfrak{U}, D_U, \mathring{\Pi}, \Phi)\) and need to show that the map

\[
H^*(U, (\Omega^*_U \otimes_{\mathcal{O}_U} \mathcal{L}_U(D^{-}(j))) \otimes_W K, \nabla_j) \rightarrow H^*(U, (\Omega^*_U \otimes_{\mathcal{O}_U} \mathcal{L}_U(D(j))) \otimes_W K, \nabla_j)
\]

is an isomorphism. Our hypothesis on \(j\) together with [13] implies that the residue of \((\mathcal{L}_U(D(j)), \nabla_j)\) along \(V_U\) for each \(V \in \mathcal{V}\) is non-zero. This means that the quotient complex

\[
\frac{(\Omega^*_U \otimes_{\mathcal{O}_U} \mathcal{L}_U(D(j))) \otimes_W K, \nabla_j}{(\Omega^*_U \otimes_{\mathcal{O}_U} \mathcal{L}_U(D^{-}(j))) \otimes_W K, \nabla_j)
\]

is acyclic, hence (b) for crystalline cohomology. But then we also get (b) for rigid cohomology from [2] and Poincaré duality in crystalline and rigid cohomology. Now we prove (a). By [3] we know that \(\mathbb{R}\Gamma_{\text{cr}}(Y/W, E(j))\) is represented by a bounded complex of finitely generated free \(W\)-modules. Moreover we know from [3] that

\[
\mathbb{R}\Gamma_{\text{cr}}(Y/W, E(j)) \otimes^\mathbb{L}_W k = \mathbb{R}\Gamma_{\text{cr}}(Y/k, E(k) \otimes_W k).
\]

Hence

\[
\chi(H^*_\text{rig}(X)_j) = \chi(H^*_\text{cr}(Y/W, E(j)) \otimes_W K) = \chi(H^*_\text{cr}(Y/k, E(j) \otimes_W k)) = \chi(H^*(Y, \Omega^*_Y \otimes \mathcal{L}_Y(D(j)), \nabla_j)) = \chi(H^*(Y, \Omega^*_Y \otimes \mathcal{L}_Y(D(j))))
\]

where the last term is to be understood as \(\sum_{s,m} (-1)^{s+t} \dim_k (H^s(Y, \Omega^m_Y \otimes \mathcal{L}_Y(D(j))))\). In the same sense we understand \(\chi\) in the following (17). For any divisor \(Q = \sum_{V \in \mathcal{V}} b_V V\) and any \(V \in \mathcal{V}\) we claim

\[
\chi(H^*(Y, \Omega^*_Y \otimes \mathcal{L}_Y(Q) \otimes_{\mathcal{O}_Y} \mathcal{O}_V)) = 0.
\]

Choose a non empty open subset \(U\) of \(Y\) and an element \(y \in \mathcal{O}_Y(U)\) which is an equation for \(V \cap U\) in \(U\). Then we have for each \(s\) an exact sequence of sheaves on \(Y\)

\[
0 \rightarrow \mathcal{L}_Y(Q) \otimes_{\mathcal{O}_Y} \Omega^s_{Y} \otimes_{\mathcal{O}_V} \mathcal{O}_{\mathcal{V}} \xrightarrow{\text{log}(y)} \mathcal{L}_Y(Q) \otimes_{\mathcal{O}_Y} \Omega^s_{Y} \otimes_{\mathcal{O}_V} \mathcal{O}_{\mathcal{V}} \rightarrow \mathcal{L}_Y(Q) \otimes_{\mathcal{O}_V} \mathcal{O}_V \rightarrow 0
\]

where \(pr\) denotes the natural projection map. The vanishing (17) follows. Hence, if we let \(Q' = Q - V\) the exact sequence

\[
0 \rightarrow \mathcal{L}_Y(Q') \rightarrow \mathcal{L}_Y(Q) \rightarrow \mathcal{L}_Y(Q) \otimes_{\mathcal{O}_V} \rightarrow 0
\]

shows that the number \(\chi(H^*(Y, \Omega^*_Y \otimes \mathcal{L}_Y(Q)))\) is independent of the coefficients \(b_V\). Statement (a) follows. \(\square\)

Remarks: (1) Let \(Y\) and \(Y^0\) be as above and consider now the more general finite étale coverings \(f: X \rightarrow Y^0\) of the following form. There is a finite index set \(I\) and for each \(i \in I\) an element \(t_i \in \mathbb{N}\) with \((p, t_i) = 1\) and a unit \(\Pi_i \in \Gamma(Y^0, \mathcal{O}_{Y^0})\) such that

\[
X = \text{Spec} (\mathcal{O}_{Y^0}[\Xi_i]_{i \in I}/((1 - \Xi^t_i \Pi_i)_{i \in I})).
\]
Suppose again that a finite group $G$ acts compatibly on $Y$, $Y^0$ and $X$. Then all our constructions extend straightforwardly to this more general situation: the relative rigid cohomology of $f$ extends to an explicitly described $G$-equivariant logarithmic $F$-crystal $E$ on $Y^0$. As a $G$-equivariant crystal it decomposes as

$$E = \bigoplus_{i \in I} \bigoplus_{0 \leq j_i \leq t_i - 1} E((j_i)_{i \in I}),$$

such that each $E((j_i)_{i \in I})$ is of rank one. Theorems 2.1, 2.2 and 2.3 have obvious analogs in this situation.

(2) Now assume in addition that $Y^0$ is quasiaffine: this implies that every coherent $\mathcal{O}_{Y^0}$-module is generated by its global sections (see e.g. [4] A2.10). Consider a finite étale Galois covering $f : X \to Y^0$ with abelian Galois group $T$ of order $t$ prime to $p$ and assume that $k$ contains all exp($T$)-th roots of unity. We claim that $f$ has the form just described. To see this we use the action of $T$ to decompose the $\mathcal{O}_{Y^0}(Y^0)$-module $\mathcal{O}_X(X)$ into $t$ free direct summands of rank one. Explicitly, let $T = \prod_{i \in I} T_i$ be a decomposition into cyclic groups. For each $i \in I$ choose a character $\theta(i) : T_i \to k^\times$ such that any other character $T_i \to k^\times$ is a power of $\theta(i)$. Let $\Xi_i$ denote a generator of the $\mathcal{O}_{Y^0}(Y^0)$-submodule of $\mathcal{O}_X(X)$ on which $T_i$ acts through $\theta(i)$ and on which all $T_s$ for $s \neq i$ act trivially. Then $\Xi_i$ is a unit in $\mathcal{O}_{Y^0}(Y^0)$; if we let $\Pi_i = \Xi_i^{-t_i}$ we are precisely in the situation considered above.

(3) This leads to the following natural question. Suppose $X \to Y^0$ is a finite étale Galois covering with (possibly non-abelian) Galois group $T$ of order prime to $p$. Does the relative rigid cohomology of $f$ extend to a logarithmic $F$-crystal $E$ on $Y^0$? Does it so in a $G$-equivariant manner if a finite group $G$ acts compatibly on $Y$, $Y^0$ and $X$? We hope to return to this question in the future.

3 The Deligne-Lusztig functor

For the remainder of this paper we adopt the following notations. $k$ is a finite field with $q$ elements, $\overline{k}$ an algebraic closure and $W$, resp. $W(\overline{k})$, the ring of Witt vectors with coefficients in $k$, resp. in $\overline{k}$. We let $K = \text{Quot}(W)$, $K(\overline{k}) = \text{Quot}(W(\overline{k}))$ and let $\overline{K}$ denote an algebraic closure of $K$ and $K(\overline{k})$.

Let $G$ be a reductive algebraic group over $\overline{k}$ with Frobenius endomorphism $F$ such that $G(\overline{k})^F = G(k)$ (in particular $G$ is $k$-rational). Let $L$ be a $k$-rational Levi subgroup of $G$ and let $P$ denote a parabolic subgroup of $G$ whose Levi subgroup is $L$ but which itself is not necessarily $k$-rational. Let $P = LU$ be the Levi decomposition. Define the associated Deligne-Lusztig variety as the following subvariety of $G/U$:

$$X = \{ g.U \mid g^{-1}F(g) \in U.F(U) \}.$$

$X$ is a smooth ([4] Theorem 7.7) and quasiaffine ([4] Theorem 7.15) $\overline{k}$-variety and $G(k) \times L(k)^{opp}$ acts from the left on $X$. 

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Let \( \ell \) be any prime number different from \( p \) and fix an identification \( \mathcal{K} \cong \overline{\mathbb{Q}}_\ell \). We define the virtual \( \mathcal{K}[G(k) \times \mathbb{L}(k)^{opp}] \)-modules
\[
H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell) = \sum_i (-1)^i H_{et,c}^i(X, \overline{\mathbb{Q}}_\ell),
\]
\[
H_{rig,c}^\otimes(X/\mathcal{K}) = \sum_i (-1)^i H_{rig,c}^i(X) \otimes_{K(\mathcal{K})} \mathcal{K}
\]
and the virtual \( \mathcal{K}[\mathbb{L}(k) \times G(k)^{opp}] \)-module
\[
H_{rig}^\otimes(X/\mathcal{K}) = \sum_i (-1)^i H_{rig}^i(X) \otimes_{K(\mathcal{K})} \mathcal{K}.
\]
The virtual \( \mathcal{K}[G(k) \times \mathbb{L}(k)^{opp}] \)-module \( H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell) \) gives rise to the Deligne-Lusztig functor associated with \( L \subset G \), see [4].

**Theorem 3.1.** (a) \( H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell) = H_{rig,c}^\otimes(X/\mathcal{K}) \) as virtual \( \mathcal{K}[G(k) \times \mathbb{L}(k)^{opp}] \)-modules.

(b) \( H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell) = \text{Hom}_\mathcal{K}(H_{rig}^\otimes(X/\mathcal{K}), \mathcal{K}) \) as virtual \( \mathcal{K}[G(k) \times \mathbb{L}(k)^{opp}] \)-modules.

**Proof:** (a) Since the group algebra of a finite group over an algebraically closed field of characteristic 0 is semisimple, the elements in the Grothendieck group of its modules are uniquely determined by their characters. Therefore we need to prove that for all \((g, l) \in G(k) \times \mathbb{L}(k)^{opp}\) we have
\[
\text{Tr}((g, l)^* | H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell)) = \text{Tr}((g, l)^* | H_{rig,c}^\otimes(X/\mathcal{K}))
\]
where \( \text{Tr} \) denotes the trace. Let more generally \( \sigma: X \to X \) be an automorphism of finite order. Then we claim
\[
\text{Tr}(\sigma^* | H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell)) = \text{Tr}(\sigma^* | H_{rig,c}^\otimes(X/\mathcal{K})).
\]
(For the following argument compare with [6] Theorem 3.2). For \( n \geq 1 \) the composite \( F^n \sigma \) is the Frobenius map relative to some new way of lowering the field of definition of \( X \) from \( \overline{k} \) to a finite subfield of \( k \) (see [7] 3.3, 3.6). The Lefschetz fixed point formula for \( \ell \)-adic and for rigid cohomology (see [9] for the latter) shows
\[
\text{Tr}((F^n \sigma)^* | H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell)) = |X(\overline{k})^{F^n \sigma}| = \text{Tr}((F^n \sigma)^* | H_{rig,c}^\otimes(X/\mathcal{K})).
\]
The automorphisms \( F^* \) and \( \sigma^* \) of the cohomology commute; hence they can be reduced to a triangular form in the same basis of \( \oplus_i H_{et,c}^i(X, \overline{\mathbb{Q}}_\ell) \) (resp. of \( \oplus_i H_{rig,c}^i(X) \otimes_{K(\mathcal{K})} \mathcal{K} \)). Hence there are \( \alpha_\lambda \) and \( \beta_\lambda \) such that
\[
\text{Tr}((F^n \sigma)^* | H_{et,c}^\otimes(X, \overline{\mathbb{Q}}_\ell)) = \sum_\lambda \alpha_\lambda \lambda^n, \quad \text{Tr}((F^n \sigma)^* | H_{rig,c}^\otimes(X/\mathcal{K})) = \sum_\lambda \beta_\lambda \lambda^n
\]
(19)
for all \( n \geq 0 \), where \( \lambda \) runs through the multiplicative group of \( \mathbb{K} \cong \mathbb{Q}_p \), and where almost all \( \alpha_\lambda \) and almost all \( \beta_\lambda \) are zero. Comparing (19) and (20) for all \( n \geq 1 \) shows \( \alpha_\lambda = \beta_\lambda \) for all \( \lambda \), hence (20) for \( n = 0 \) gives our claim.

(b) This follows from (a) and Poincaré-duality in rigid cohomology since \( X \) is smooth. \( \square \)

Now consider the following subvariety of \( \mathbb{G}/\mathbb{P} \):

\[ Y^0 = \{ g.\mathbb{P} \ ; \ g^{-1}F(g) \in \mathbb{P}.F(\mathbb{P}) \} . \]

Also \( Y^0 \) is smooth (4 Theorem 7.7) and quasiaffine (4 Theorem 7.15) and \( \mathbb{G}(k) \) acts on it from the left. In fact \( Y^0 \) is the quotient of \( X \) by the action of \( \mathbb{L}(k)^{opp} \) so that \( X \to Y^0 \) is a Galois covering with group \( \mathbb{L}(k)^{opp} \) (see 4 Theorem 7.8).

The initiating paper [6] deals with the case where \( L \) is a torus and \( \mathbb{P} \) is a Borel subgroup. Consider the following more specific assumption. Let \( B_0 \subset \mathbb{G} \) be a \( k \)-rational Borel subgroup with unipotent radical \( U_0 \) and maximal torus \( T_0 \). Let \( S \) be the set of generating reflections of the Weyl group \( W(G,T_0) = N_G(T_0)/T_0 \) corresponding to \( B_0 \). Let \( \gamma : W(G,T_0) \to N_G(T_0) \) be the section which is multiplicative on pairs of elements whose lengths add (see 4 Theorem 7.11).

Let \( w \in W(G,T_0) \) be the product of some pairwise commuting elements of \( S \) and let \( b \in \mathbb{G} \) satisfy \( b^{-1}F(b) = \bar{w} \). Then we assume that \( U \) is the \( b \)-conjugate of \( U_0 \). (Many representation theoretic questions concerning the Deligne-Lusztig functor for general \( L \subset \mathbb{G} \) can be reduced to this standard assumption.) Under this assumption there is a smooth proper \( \mathbb{K} \)-scheme \( Y \) with \( \mathbb{G}(k) \)-action and an equivariant open dense immersion \( Y^0 \to Y \) such that \( Y \to Y^0 \) is a smooth divisor with normal crossings (4 Proposition 7.13).

Now we are precisely in the situation considered in remark (2) at the end of section 2 and our results from section 2 (extended as indicated) apply. In section 4 below we will consider the example where \( \mathbb{G} = \text{GL}_{d+1} \) and where \( w \) is the longest element in the Weyl group. (In this example the Galois group \( T = \mathbb{L}(k) \) is cyclic, so we are in the simplified situation considered in the body of sections 1 and 2.)

4 Deligne-Lusztig varieties for \( \text{GL}_{d+1} \)

We fix \( d \in \mathbb{N} \). Consider the affine \( k \)-scheme \( \mathbb{V} \) associated with \( (k^{d+1})^* = \text{Hom}_k(k^{d+1}, k) \). We write it as

\[ \mathbb{V} = \text{Spec}(k[\Xi_0, \ldots, \Xi_d]) \]

where we let \( \Xi_0, \ldots, \Xi_d \) correspond to the canonical basis of \( (k^{d+1})^* \). The right action of \( \text{GL}_{d+1}(k) = \text{GL}(k^{d+1}) \) on \( (k^{d+1})^* = \text{Hom}_k(k^{d+1}, k) \) defines a left action of \( \text{GL}_{d+1}(k) \) on \( \mathbb{V} \). Explicitly, \( \text{GL}_{d+1}(k) \) acts from the right on the graded ring \( k[\Xi_0, \ldots, \Xi_d] \): if \( f(\Xi_0, \ldots, \Xi_d) \in k[\Xi_0, \ldots, \Xi_d] \), then

\[ g.f(\Xi_0, \ldots, \Xi_d) = f\left( \sum_{s=0}^d a_{sd} \Xi_s, \ldots, \sum_{s=0}^d a_{sd} \Xi_s \right) \quad \text{for} \quad g^{-1} = (a_{st})_{0 \leq s, t \leq d}. \]
Thus, the subsequent representations are to be understood as the ramified right side of GL

Denote by \( \Omega \) for \( 0 \leq d \leq s \) the sequence of projective k-varieties

is defined inductively by letting \( Y_{m+1} \rightarrow Y_m \) be the blowing up of \( Y_m \) in the strict transforms (in \( Y_m \)) of all \( Z \in \mathcal{V}_0^m \). Let \( \mathcal{V} \) denote the set of all strict transforms in \( Y \) of elements of \( \mathcal{V}_0^m \) for some \( m \), a set of divisors on \( Y \). The action of GL\(_d+1\)(k) on \( Y_0 \) naturally lifts to an action (from the left) of GL\(_d+1\)(k) on \( Y \).

On \( Y_0 \) and then by pull back on \( Y \) we have the rational functions \( z_t = \Xi_t/\Xi_0 \) for \( 0 \leq t \leq d \). Denote by \( (\Omega^*, d) \) the de Rham complex on \( Y \) with logarithmic poles along the normal crossings divisor \( \sum_{V \in \mathcal{V}} V \) on \( Y \). We give its open and GL\(_d+1\)(k)-stable complement a name,

For \( 0 \leq s \leq d \) denote by \( \mathcal{P}_s \) the set of subsets of \( \{1, \ldots, d\} \) consisting of \( s \) elements. For \( 1 \leq j \leq d \) we define the rational function

on \( Y \), and if in addition \( 0 \leq s \leq d \) we define the integer

Recall the classification of irreducible representations of GL\(_d+1\)(k)\textsuperscript{opp} on k-vector spaces according to Carter and Lusztig. For convenience we drop the superscript opp in our notation; thus, the subsequent representations are to be understood as right representations. For \( 1 \leq r \leq d \) let \( r \in \text{GL}_d(k) \) denote the permutation matrix obtained by interchanging the \( (r-1) \)-st and the \( r \)-th row (or equivalently: column) of the identity matrix (recall that we start counting with 0). Then \( S = \{t_1, \ldots, t_d\} \) is a set of Coxeter generators for the Weyl group of GL\(_d+1\)(k). Let B(k) resp. U(k) denote the subgroup of upper triangular (resp. upper triangular unipotent) matrices.

**Theorem 4.1.** [3] (i) For an irreducible representation \( \rho \) of GL\(_d+1\)(k) on a k-vector space, the subspace \( \rho^U(k) \) of \( U(k) \)-invariants is one dimensional. If the action of B(k) on \( \rho^U(k) \) is given by the character \( \chi : B(k)/U(k) \rightarrow k^\times \) and if \( J = \{ t \in S ; \ t, \rho^U(k) = \rho^U(k) \} \), then the pair \( (\chi, J) \) determines \( \rho \) up to isomorphism.

(ii) Conversely, given a character \( \chi : B(k)/U(k) \rightarrow k^\times \) and a subset \( J \) of \( \{ t \in S ; \ \chi^t = \chi \} \), there exists an irreducible representation \( \Theta(\chi, J) \) of GL\(_d+1\)(k) on a k-vector space whose associated pair (as above) is \( (\chi, J) \).
In [11] we proved the following Theorems 4.2, 4.3, 4.4:

**Theorem 4.2.** Let $0 \leq s \leq d$. Then $H^t(Y, \Omega^s_Y) = 0$ for all $t > 0$, and

$$\dim_k(H^0(Y, \Omega^s_Y)) = \sum_{\tau \in \mathcal{P}_s} q^{\sum_i \tau_i}.$$ 

Moreover, $H^0(Y, \Omega^s_Y)$ is generated (as a $k$-vector space) by logarithmic differential $s$-forms.

**Theorem 4.3.** For $0 \leq s \leq d$, the $GL_{d+1}(k)$-representation on $H^0(Y, \Omega^s_Y)$ is equivalent to $\Theta(1, \{t_{s+1}, \ldots, t_d\})$; it is a generalized Steinberg representation. The subspace of $U(k)$-invariants of $H^0(Y, \Omega^s_Y)$ is generated by

$$\omega_s = (\prod_{j=1}^d \gamma_j^{m_j^s}) dz_1 \wedge \ldots \wedge dz_s.$$ 

**Theorem 4.4.** $H_{crys}^s(Y/W)$ is torsion free for any $s$, and

$$H_{crys}^s(Y/W) \otimes W k = H^s(Y, \Omega_Y^{\bullet}) = H^0(Y, \Omega^s_Y).$$

Let $L$ be a maximally non split torus in $GL_{d+1}$. Thus $L$ is in relative position $w$ to the standard torus $T_0$ of diagonal matrices where $w$ denotes the permutation matrix with entry 1 at position $(i, d-i)$ for each $0 \leq i \leq d$. It follows that $T = L(k)$ is (abstractly) isomorphic with the set of fixed points of $wF$ acting on $T_0(k)$ if $F$ denotes the standard Frobenius which raises each matrix entry to its $q$-th power. But this is the subgroup of $GL_{d+1}(k)$ consisting of the diagonal matrices $diag(t, t^q, \ldots, t^{q_d})$ for $t \in \mathbb{F}_q^{\times d+1}$. Thus we may henceforth identify $T$ with the multiplicative group of the field $\mathbb{F}_{q^{d+1}}$ with $q^{d+1}$ elements. If we let

$$\delta = \det((\Xi^q_j)_{0 \leq i,j \leq d}) \in \mathbb{Z}[\Xi_0, \ldots, \Xi_d]$$

then it is straightforwardly checked that in $k[\Xi_0, \ldots, \Xi_d]$ we have

$$\prod_{a \in k^{d+1}-\{0\}} \sum_{i=0}^d a_i \Xi_i = (-1)^{d+1} \delta^{q-1}.$$ (23)

The Deligne-Lusztig variety as defined in section 3 is the $k$-variety

$$X_k = \text{Spec}(B),$$

$$B = \mathbb{F}_{q^{d+1}}[\Xi_0, \ldots, \Xi_d]/(\delta^{q-1} - (-1)^d).$$

In fact, only for even $d$ this is the explicit formula [6] (2.2.2); however, since (for all $d$) we have an isomorphism of $\mathbb{F}_k$-algebras

$$B \cong \mathbb{F}_{q^{d+1}}[\Xi_0, \ldots, \Xi_d]/(\delta^{q-1} - (-1)^{d+1})$$

(if $\xi$ is a $(q^{d+1} - 1)$-st root of $-1$ in $\overline{k}$, send $\Xi_i$ to $\xi \Xi_i$, for $i = 0, \ldots, d$) our formula is equivalent with [6] (2.2.2) also for odd $d$. This argument also shows that for any $d$ our $X_k$ decomposes into
Let \( z_1, \ldots, z_d \) denote free variables, let \( z_0 = 1 \) and set
\[
\Pi = - \prod_{a \in k^{d+1-\{0\}}} \sum_{i=0}^d a_i z_i,
\]
\[
A = \overline{k}[z_1, \ldots, z_d][\Pi^{-1}],
\]
\[
Y_0^0 = \text{Spec}(A).
\]
\( Y_0^0 \) is the complement in \( \mathbb{P}^d_k \) of all \( k \)-rational linear hyperplanes. Sending \( z_i \) to \( \Xi_i/\Xi_0 \) we have an isomorphism (use (23))
\[
B = A[\Xi_0]/(1 - \Xi_0^{q^{d+1}-1}\Pi).
\]
In this way we may view \( A \) as a subring of \( B \); as such it is stable under the action by \( \text{GL}_{d+1}(k) \).

In fact, \( X_k \rightarrow Y_0^0 \) is a \( \text{GL}_{d+1}(k) \)-equivariant finite étale Galois covering with group \( T \) if we let \( T \) act on \( B \) as follows: on the subring \( A \) of \( B \) it acts trivially, and on the class of \( \Xi_0 \) it acts by multiplication, \( h.\Xi_0 = h\Xi_0 \) for \( h \in T \). For \( 0 \leq j \leq q^{d+1} - 2 \) we introduce the characters
\[
\theta_j : T \rightarrow K(\overline{k})^\times, \quad h \mapsto h^j
\]
(via the Teichmüller lifting).

Using the same defining equations we see that \( X_k \rightarrow Y_0^0 \) is obtained by base change from a morphism of \( k \)-schemes \( X \rightarrow Y^0 \). We have the proper smooth \( k \)-scheme \( Y \) with \( \text{GL}_{d+1}(k) \)-action and an equivariant open immersion \( Y^0 \rightarrow Y \) such that \( D = Y - Y^0 \) is a normal crossings divisor on \( Y \). Thus all the results from sections 1 and 2 are available for \( X_k \rightarrow Y_0^0 \) and for \( X \rightarrow Y^0 \) (however the \( T \)-action does not descend from \( X_k \) to \( X \)). In fact since it is known that rigid cohomology commutes with the base change \( k \rightarrow \overline{k} \) it follows that the \( K(\overline{k}) \)-vector spaces \( H_{\text{rig}}^*(X_k) \) and their \( \theta_j \)-eigenspaces \( H_{\text{rig}}^*(X_k)_{\theta_j} \) are obtained by base extension \( K \rightarrow K(\overline{k}) \) from the \( K \)-vector spaces \( H_{\text{rig}}^*(X) \) and their subspaces \( H_{\text{rig}}^*(X)_{\theta_j} \). We point out that the automorphy factors \( \gamma_g \in \Gamma(Y, \mathcal{L}_Y(D)) \) for \( g \in \text{GL}_{d+1}(k) \) considered in section 1 are in this case given by
\[
\gamma_g = \frac{g(\Xi_0)}{\Xi_0}.
\]
The condition from (23) (b) that \( 0 \leq j \leq t - 1 \) be such that \( j \mu_V(\Pi) \) is not divisible by \( t \) for all \( V \in \mathcal{V} \) becomes the condition that \( j \) be not divisible by \( \sum_{i=1}^d q^i = (q - 1)^{-1}(q^{d+1} - 1) \). The \( \theta_j \) for such \( j \) are called non singular or in general position; equivalently, \( \theta_j \) does not factor through a norm map \( T = \mathbb{F}^{\times}_{q^{d+1}} \rightarrow \mathbb{F}^{\times}_q \) for some \( s < d + 1 \).

Since \( \Pi \) is the \( (q - 1) \)-st power of an element in \( A \) the isomorphism (10) yields isomorphisms
\[
H_{\text{rig}}^*(X_k)_{\theta_j} \cong H_{\text{rig}}^*(X_k)_{\theta_j + \sum_{i=0}^d q^i}.
\]
In particular, if \( \theta_j \) is singular (i.e. not in general position) then
\[
H_{\text{rig}}^*(X_k)_{\theta_j} \cong H_{\text{rig}}^*(X_k)_{0} = H_{\text{rig}}^*(Y_0^0_{k}).
\]
Corollary 4.5. The $\theta_j$-eigenspaces for non singular $\theta_j$ in the (compactly supported) rigid and $\ell$-adic étale cohomology groups (any $\ell \neq p$) of $X_K$ are non zero only in degree $d$, and in degree $d$ they coincide, as $GL_{d+1}(k)$-representations on characteristic zero vector spaces of dimension

$$\dim_{K(K)} H^d_{\text{rig}}(X_K)_j = \sum_{s \geq 0} (-1)^{s+d} \sum_{\tau \in P_s} q^{\sum_{i \in \tau} i}$$

$$= (q-1)(q^2-1) \cdots (q^d-1).$$

Similarly, the $\theta_0$-eigenspaces in (compactly supported) rigid and $\ell$-adic étale cohomology coincide as $GL_{d+1}(k)$-representations, individually in each degree.

Proof: The coincidence of rigid and $\ell$-adic étale cohomology as virtual $GL_{d+1}(k) \times T$-representations follows from 3.1. We saw in 2.3 that the $\theta_j$-eigenspaces in (compactly supported) rigid cohomology live only in degree $d$, and in [6] the same is shown for (compactly supported) $\ell$-adic cohomology. Since everything is semisimple we get coincidence. By the proof of 2.3 (a) the vector space dimension is given for any $j$ by $\chi(H^*(Y, \Omega^*_Y))$; but this is computed in 4.2. The statement on $\theta_0$-eigenspaces follows by comparing the explicit description from 4.3 with that from [18]. □

Of particular interest for the representation theory of $GL_{d+1}(k)$ is the virtual $GL_{d+1}(k)$-representation

$$\sum_{s=0}^{d} (-1)^s H^s_{\text{rig}}(X_K)_0 = \sum_{s=0}^{d} (-1)^s H^s_{\text{rig}}(Y^0_K)$$

(it is called unipotent). In view of 4.4 we determined its reduction modulo $p$ in 4.3. By (10) (which yields (24)) the situation for the $\theta_j$-eigenspaces of other singular $\theta_j$ is similar.

To compute the reduction modulo $p$ for non singular $\theta_j$ the general results from section 2 apply. Alternatively, one might try to understand the finitely generated free $W$-module

$$H^d_j = H^d_{\text{crys}}(Y/W, E(j))/\text{torsion}.$$ 

$H^d_j$ is a $GL_{d+1}(k)$-stable $W$-lattice in $H^d_{\text{rig}}(X_K)_j = H^d_{\text{crys}}(Y/W, E(j)) \otimes_K W$.

Corollary 4.6. If $\theta_j$ is non singular then $H^d_j \otimes_W k$ is a $GL_{d+1}(k)$-equivariant subquotient of $H^d(Y, (\Omega^*_Y \otimes \mathcal{L}_Y(D(j)), \nabla_j))$ of $k$-dimension $(q-1)(q^2-1) \cdots (q^d-1)$. Viewed as an element of the Grothendieck group of $k[GL_{d+1}(k)]$-modules it coincides with

$$\sum_{s,m} (-1)^{s+m} H^s(Y, \Omega^m_Y \otimes \mathcal{L}_Y(D(j))).$$

Proof: The $k$-dimension of $H^d_j \otimes_W k$ is the $K$-dimension of $H^d_j \otimes_W K$ and this can be read off from Corollary 1.5. We have a natural surjection

$$H^d_{\text{crys}}(Y/W, E(j)) \otimes_W k \twoheadrightarrow H^d_j \otimes_W k.$$
But $H_{\text{crys}}^d(Y/W, E(j)) \otimes_W k$ is a subobject of $H_{\text{crys}}^d(Y/k, E(j) \otimes_W k) = H^d(Y, (\Omega_Y^+ \otimes \mathcal{L}_Y(D(j)), \nabla_j))$, hence the first statement. The second one follows from Theorem 2.2.

Remark: If $d = 1$ and $j$ is arbitrary (but not divisible by $q+1$), or if $d = 2$ and $1 \leq j \leq p-1$, one can show $H^s(Y, (\Omega_Y^+ \otimes \mathcal{L}_Y(D(j)), \nabla_j)) = 0$ for $s \neq d$ and by the usual devissage for crystalline cohomology relative to $W/p^n$ for all $n$ we find $H_j^d = H_{\text{crys}}^d(Y/W, E(j))$ and

$$H_j^d \otimes_W k = H^d(Y, (\Omega_Y^+ \otimes \mathcal{L}_Y(D(j)), \nabla_j)).$$

The case $d = 2$ and $j = p$ is the first one where $H_{d-1}^d(Y, (\Omega_Y^+ \otimes \mathcal{L}_Y(D(j)), \nabla_j)) \neq 0$ and hence $H_j^d \neq H_{\text{crys}}^d(Y/W, E(j))$ and $H_j^d \otimes_W k \neq H^d(Y, (\Omega_Y^+ \otimes \mathcal{L}_Y(D(j)), \nabla_j))$ in that case.

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