NOTES ON GOMPF’S INFINITE ORDER CORK

MOTOO TANGE

Abstract. For any positive integer $n$ we give a $\mathbb{Z}^n$-cork with a $\mathbb{Z}^n$-effective embedding in a 4-manifold being homeomorphic to $E(n)$. This means that a cork gives a subset $\mathbb{Z}^n$ in the differential structures on $E(n)$. Further, we describe handle decompositions of the twisted doubles (homotopy $S^4$) of Gompf’s infinite order cork and show that they are log transforms of $S^4$.

1. Introduction

1.1. Twist and cork. Let $X$ be a smooth manifold and $Y$ a codimension 0 submanifold with a smooth embedding $i : Y \hookrightarrow X$. Removing $Y$ from $X$ and regluing by a self-diffeomorphism $f : \partial Y \to \partial Y$, we obtain a new smooth manifold and denote the manifold by $X(i, Y, f)$ or simply $X(Y, f)$. We call the map $f$ a gluing map or a twist map. This operation is called a twist and denote it by $(Y, f)$. If the gluing map $f$ extends to $Y$ as a diffeomorphism, then we call $(Y, f)$ a trivial twist.

Let $Y$ be a contractible 4-manifold. We call a nontrivial twist $(Y, f)$ a cork. Then the gluing map $f$ is called a cork map. Cork plays a significant role in studying exotic 4-manifolds. ‘Exotic’ means that the manifolds are homeomorphic but non-diffeomorphic each other. In fact, the following fact is well-known.

Fact 1.1 ([8], [3]). Let $X, X'$ be two closed simply connected exotic 4-manifolds. Then there exists a cork $(C, \tau)$ such that $X' = X(C, \tau)$ and $\tau^2 = e$.

1.2. Gompf’s infinite order corks. Gompf in [5] gave an infinite exotic family using infinite order corks as below. Let $X$ be a certain 4-manifold (including a square zero torus with two vanishing cycles).

Fact 1.2 ([5]). Suppose that $K_n$ is the $2n$-twist knot. Then there exists an infinite order cork $(C, f)$ satisfying $X_{K_n} = X(C, f^n)$.

$X_K$ is Fintushel-Stern knot-surgery of $X$ by $K$. Cork order of cork is defined to be the minimal positive number whose power of the boundary
diffeomorphism extends to the whole cork $C$. For finite higher order corks, [11] and [2] are known ever.

**Definition 1.3** ($G$-effective embedding (defined in [2])). Let $G$ be a group acting on $\partial C$ effectively. If there exists an embedding $i$ of $C$ into a 4-manifold $X$ such that $X(i, C, g)$ is not diffeomorphic to $X(i, C, g')$ for any $g, g' \in G$ with $g \neq g'$, then we call the embedding $i$ a $G$-effective.

1.3. **Galaxy.** Here we give terminologies related to cork and cork twist. These terminologies make easier to understand our results. Let $X$ be a smooth 4-manifold. We call the set of exotic structures on $X$ the galaxy of $X$ and denote it by $\text{gal}(X)$.

Our interest is to understand some kind of structures on the set $\text{gal}(X)$. Any cork twist can be regarded as some relationship among subsets of $\text{gal}(X)$.

Let $C$ be a contractible 4-manifold. Let $G$ be a nontrivial subgroup in $\text{Diff}(\partial C)$. Let $(C, G)$ be a $G$-cork and $C \hookrightarrow X$ a $G$-effective embedding. Then the collection $S = \{X(C, g) | g \in G\}$ is a subset of $\text{gal}(X)$ with one to one correspondence $g \mapsto X(C, g)$. We call such a subset $S$ a $(G\text{-})$constellation and the embedding $G \xrightarrow{\sim} S \subset \text{gal}(X)$ a constellation embedding.

Fact 1.1 means that a pair of every two points in $\text{gal}(X)$ is a $\mathbb{Z}_2$-constellation. If $G \hookrightarrow \text{gal}(X)$ is a constellation with respect to a $G$-cork $(C, G)$, then any subgroup $e \neq H < G$ gives an $H$-constellation $H \hookrightarrow \text{gal}(X)$ with respect to an $H$-cork $(C, H)$. We call this constellation a subconstellation. The main theorem in [15] says that any infinite family in $\text{gal}(X)$ is not always a constellation.

1.4. **Results.** The first result (Theorem 1) gives a construction of $\mathbb{Z}^n$-cork. Furthermore we show that this cork gives a $\mathbb{Z}^n$-constellation in $\text{gal}(E(n))$ by knot-surgeries on a single fibered knot. This construction is due to $n$-fold boundary sum of Gompf’s $C$.

The second result (Theorem 2) is on the diffeomorphism type of the twisted double (homotopy $S^4$) of Gompf’s $C$ with respect to the cork twist $(C, f)$. As a result the twisted double is diffeomorphic to a log transform along a torus in $S^4$. The construction of Gompf’s infinite order corks is simple but it seems hard to distinguish the differential structures of twisted doubles.

1.5. **$\mathbb{Z}^n$-corks.** In [5] Gompf defined infinite order corks $(C, f)$ and asked in [5] whether there exists a $\mathbb{Z}^2$-cork by taking the full $T^2$ action of his corks. In [6] he partially gave a negative answer for this question. We construct a $\mathbb{Z}^n$-cork below, but it is not an answer of this question.

**Theorem 1.** For any natural number $n$ there exists a $\mathbb{Z}^n$-cork $C_n$. Furthermore, there exists a $\mathbb{Z}^n$-effective embedding $C_n \hookrightarrow X_n$. This $\mathbb{Z}^n$-effective embedding gives a $\mathbb{Z}^n$-constellation $\mathbb{Z}^n \hookrightarrow \text{gal}(E(n))$. 
Here $C_n$ is the boundary sum of $n$ copies of $C(1,1;-1)$ as defined in [5]. This construction is due to performing cork twistings at distinct two clasps as mentioned by Gompf in [5]. Here the following interesting questions arise:

**Question 1.4.** Is there a 4-manifold $X$ such that for arbitrary large $n$, $\text{gal}(X)$ includes a $\mathbb{Z}^n$-constellation?

**Question 1.5.** Does there exist a 4-manifold $X$ such that $G$-constellation in $\text{gal}(X)$ for an infinite non-abelian group?

Related topics to this question will be written in a sequel.

1.6. **The twisted double of Gompf’s** $(C, f)$. Let $C$ denote $C(r,s;m)$, which is defined in [5]. The twisted doubles $S_{r, s, m, k} := C \cup f_k (-C) = S^4(C, f^k)$ are homotopy 4-spheres, since the untwisted double is diffeomorphic to the standard $S^4$. We investigate these homotopy $S^4$’s in Section 3.

In [5] Gompf asks the following question:

**Conjecture 1.6.** Let $r, s, m$ be any integers with $r, s > 0 > m$. Let $k$ be a nonzero integer. Then $S_{r, s, m, k}$ is standard $S^4$.

We do not know whether $S_{r, s, m, k}$ is standard or not, but we prove the following fundamental results.

**Proposition 1.7.** $S_{r, s, m, k}$ is diffeomorphic to $S_{r, s, 0, k}$ if $m \equiv 0 \mod 2$ and $S_{r, s, 1, k}$ if $m \equiv 1 \mod 2$.

**Theorem 2.** $S_{r, s, m, k}$ is a $(1/s)$-log transform of $S^4$ along an embedded torus.

Note that exchanging the roles of $r$ and $s$, we also know that $S_{r, s, m, k}$ is $(-1/r)$-log transform of $S^4$.

If the embedding of the torus in $S^4$ extends to a fishtail neighborhood, then the log transform does not change the diffeomorphism type as discussed in [6]. Other similar situation is [14]. In [14] by using a fishtail neighborhood embedding in $S^4$ it is shown that the knot-surgeries are trivial.

It seems difficult to find a certain fishtail neighborhood in our examples $S_{r, s, m, k}$. Distinguishing the differential structures of $S_{r, s, m, k}$ is a challenging problem. Another example of log transforms of $S^4$ is in [13]. From the proof of Theorem 2 we get a handle decomposition of $S_{r, s, m, k}$.

**Proposition 1.8.** $S_{r, s, m, k}$ admits a handle decomposition with one 0-handle, two 1-handles, four 2-handles, two 3-handles, and one 4-handle.

Further, $S_{r, s, m, k}$ is also considered as 2-fold log transforms of $S^4$ as written in Section 3.1. Furthermore, the 0-log transform and $(0,0)$-log transform of $S^4$ with respect to the tori is homotopy $S^3 \times S^1 \# S^2 \times S^2$ and $S^3 \times S^1 \#^2 S^2 \times S^2$ respectively.

**Question 1.9.** Are $S^4[0]$ and $S^4[0,0]$ exotic $(S^3 \times S^1) \# (S^2 \times S^2)$ or $\#^2(S^3 \times S^1) \#^2(S^2 \times S^2)$?
A similar construction is a Scharlemann manifold in [9], which is a surgery of $\Sigma \times S^1$ for a rational homology sphere $\Sigma$. The surgeries are done along normally generating loops of $\pi_1(\Sigma)$ in $\Sigma \times S^1$. In the case where $\Sigma$ is a Dehn surgery of a knot and the loop is the meridian of the knot, the manifold is equivalent to a knot-surgery of the double of the fishtail neighborhood. For example see [14]. The general Scharlemann manifold gives a homotopy

$$\#(S^3 \times S^1)\#^4(S^2 \times S^2)\#^m(CP^2 \# CP^2).$$

The case where $\Sigma = \Sigma(2,3,5) = S^3_{-1}$ (left handed $3_1$) and the loop is the meridian of the trefoil is the original one in [9]. The author in [14] proved that some Scharlemann manifolds are standard. Here we give another question.

**Question 1.10.** Are $S^4[0]$ and $S^4[0,0]$ diffeomorphic to a Scharlemann manifold and a connected-sum of those manifolds respectively?

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2. **Gompf’s infinite order cork $C$.**

2.1. **Knot-surgery.** Before the introduction of Gompf’s infinite order cork, we review the definition of Fintushel-Stern knot-surgery. Let $K$ be a knot in $S^3$. Let $X$ be a 4-manifold with a square zero embedded torus $T$. Then the operation

$$X_K = [X - \nu(T)] \cup [(S^3 - \nu(K)) \times S^1]$$

is called a (Fintushel-Stern) knot-surgery along $K$. The gluing map is defined in [4]. The notation $\nu(\cdot)$ stands for the open neighborhood of a submanifold. Then the Seiberg-Witten invariant formula of knot-surgery is the following:

$$SW_{X_K} = SW_X \Delta_K(t),$$

where $\Delta_K(t)$ is the Alexander polynomial of $K$.

We remark Sunukjian’s work in [10]. The paper says that Fintushel-Stern knot-surgeries can be distinguished by the Alexander polynomial. In other words, two surgeries of an embedded torus with different Alexander polynomials give different smooth structures. To distinguish the smooth structures of knot-surgeries, we have only to consider the Alexander polynomial of the knots instead of Seiberg-Witten invariant.
2.2. A diagram of $C$. In this section we describe the diagram of Gompf’s infinite order cork $C$. In [7] the diagram is described by the different method. The manifold $C = C(r, s; m)$ is diffeomorphic to $(I \times P) \cup h_0$, where $P$ is the complement of $(r, s)$-double twist knot $\kappa(r, -s)$ as in Figure 1 in [5].

**Lemma 2.1.** The diagram of $C$ is **Figure 2** and the cork map $f$ is **Figure 3**.

The incompressible torus can be realized as an indicated torus in **Figure 2**. Note that in the diagram in **Figure 2** you apparently cannot find the embedded torus, because the torus meets the dotted 1-handles 4 times. However, by inserting 2 pairs of canceling $2/3$-handles, we can avoid the intersections. As a result we find our required embedded torus.

**Proof.** Let $\Sigma$ be a punctured torus. $\Sigma \times S^1$ is diffeomorphic to the $0$-surgered solid torus along the Bing double as in the left of **Figure 1**.

Thus by the fundamental method of handle calculus, the picture of the cylinder $I \times \Sigma \times S^1$ is the right diagram. Attaching the $-r$-framed 2-handle ($-s$-framed 2-handle on another side respectively) and removing the union of the core and the attaching sphere cross interval $I$, we have the other handle attachment and removal in **Figure 2**. The two 0-framed 2-handles in **Figure 2** are canceled out with the 3-handles when removing the cores. The similar situation appears in [13].

Hence, the handle diagram of $C$ is the picture in **Figure 2**. Reducing the diagram, we get a ribbon 1-handle and $m$-framed 2-handle along the meridian of the ribbon knot. □

The map $f$ is defined to be the right handed Dehn twist cross the identity on

$$(I \times \partial \Sigma) \times S^1.$$ 

This cork $C$ produces a knot-surgery $X_K$ for a 4-manifold $X$ for a square zero torus $T$. The torus satisfies the following condition.

We assume $r = s = -m = 1$. Let $V$ be the neighborhood of the Kodaira’s singular fibration with type III, which $V$ also appears as the same role in

**Figure 1.** $\Sigma \times S^1$ and $I \times \Sigma \times S^1$. 

2 3-handles
Let $V$ be embedded in $X$. $T$ is embedded as a general fiber of $V \subset X$. Then there exists an embedding $C \subset V \subset X$ such that

$$X_K = X(C, f^k),$$

where $K = \kappa(k, -1)$ is the $2k$-twist knot.

Even if $C$ does not satisfy $r = s = -m = 1$, we can construct the twist knot surgery on an embedded torus under certain condition as mentioned in [5].

Due to [5], for any integer $k$ the $k$-th composition $f^k$ cannot extend to the inside $C$ as any diffeomorphism.

2.3. 2-bridge knots $K_{m,n}$. In the next section we prove Theorem [1]. First we prepare a 2-bridge knot $K_{m,n}$ as in Figure 5 for integers $m, n$. The knot
Figure 4. The image of $C(r, s; m)$ by $f^3$.

Figure 5. $K_{m,n}$.

$K_{m,n}$ is classified as follows:

\[
K_{m,n} = \begin{cases} 
T_{2,2m-1} & n = 0 \\
\text{unknot} & (m, n) = (-1, -1) \\
T_{2,-3} & (m, n) = (0, 1) \\
\text{non-torus 2-bridge knot} & \text{otherwise}
\end{cases}
\]

$K_{-1,n} \approx K_{0,n+1}$

Here $T_{p,q}$ is the right handed $(|p|, |q|)$-torus knot if $pq > 0$ and is the left handed $(|p|, |q|)$-torus knot if $pq < 0$.

Let $\Delta_{m,n}$ denote the Alexander polynomial $\Delta_{K_{m,n}}$. The Alexander polynomial is computed as follows: If $m \geq 1$, then

\[
\Delta_{m,n}(t) = n(t^{m+1} + t^{-m-1}) - 3n(t^m + t^{-m}) + (4n + 1) \sum_{i=-m+1}^{m-1} (-1)^{i-m+1} t^i
\]

and if $m = 0$, then

\[
\Delta_{0,n}(t) = n(t + t^{-1}) - (2n - 1).
\]

This formula can be easily proven by using the skein relation.

The following lemma holds:

\[
\Delta_{m,n}(t) = n(t + t^{-1}) - (2n - 1).
\]
Lemma 2.2. If \( m \geq 1 \), then the polynomials \( \Delta_{m,n}(t) \) are distinct each other in \( \mathbb{Z}[t, t^{-1}]/\pm t^{\pm 1} \). In particular, as unoriented knots have \( K_{m,n} \not\approx K_{m',n'} \) for \( (m,n) \neq (m',n') \) with \( m,m' \geq 1 \).

Proof. Suppose that \( \Delta_{m,n} = \Delta_{m',n'} \). If \( n \neq 0 \) and \( n' \neq 0 \) hold, then, comparing the top degrees we have \( n = n' \) and \( m = m' \). If either \( n \) or \( n' \) is 0 and the other is not 0, then the two polynomials do not agree clearly. Because, if \( n' \neq 0 \) and \( n = 0 \) hold, \( \Delta_{m,n}(t) = \Delta_{T_2,2m-1}(t) \). Comparing the top degrees, we have \( n' = \pm 1 \). However the absolute value of the second top degree of \( \Delta_{m',n'}(t) \) must be 3. Thus, this case does not hold the equality. If \( n = n' = 0 \), then, comparing the top degrees we have \( m = m' \). \( \square \)

2.4. Proof of Theorem 1. We remark that the natural number \( k \) in this proof corresponds to \( n \) in the statement, first of all.

Let \( K(n_1, \cdots, n_k) \) be \( K_{1,n_1} \# \cdots \# K_{k,n_k} \). Hence \( K(0, \cdots, 0) = \#_{i=1}^k T_{2,2i-1} \) holds. We embed \( k \) copies of \( \Sigma \) in the exterior \( E \) of \( K(0, \cdots, 0) \) in the disjoint way. For the case of 2 copies see FIGURE 6. Thus, disjoint \( k \)

![Figure 6. A disjoint embedding of two copies of \( \Sigma \) in \( E_2 \).](image)

copies of \( I \times \Sigma \times S^1 \) in \( E \times S^1 \) are also embedded. By attaching \( 2k \) \((-1\text{-framed}) \) 2-handles on the meridians of \( E \times S^1 \) and \( k \) \((-1\text{-framed}) \) 2-handles on the \( S^1 \)-direction, we can embed \( k \) copies of \( C = C(1,1;-1) \) in \( X_k := E(k) \#_{i=1}^k T_{2,2i-1} \), because it has \( 12k \) vanishing cycles.

We take each point in the complements of the incompressible tori in \( \partial C \cup \partial C \) and connect the two components by a 1-handle attached on the neighborhoods of the two points in \( X_k \), which the 1-handle is disjoint from \( k \) \( C \)’s. Embedding such \( k - 1 \) 1-handles, we construct \( \natural^k C \hookrightarrow X_k \).

Let \( f_i \) be a diffeomorphism on \( \partial(\natural^k C) \) which acts as Gompf’s \( f \) on the \( i \)-th component of \( \#^k \partial C \) and acts as the identity on the other component of \( \#^k \partial C \). Since those points are taken in the complement of the incompressible tori, the two maps \( f_i \) and \( f_j \) are commutative. The twist \( (\natural^k C, f_1^{n_1} \cdots f_k^{n_k}) \) of \( X_k \) produces \( E(k)K(n_1, \cdots, n_k) \).

Then, the computation of the Seiberg-Witten invariants is as follows:

\[
SW_{E(k)K(n_1, \cdots, n_k)} = SW_{E(k)} \prod_{i=1}^k \Delta_{i,n_i}
\]
Comparing the degrees of the two results, the two Seiberg-Witten invariants do not agree, unless \( n_i = 0 \) for any \( i \). Since \( X_k \) and \( E(k)(n_1, \ldots, n_k) \) are exotic when \( (n_1, \ldots, n_k) \neq (0, \ldots, 0) \). \( (\mathbb{Z}^k C, f_1^{n_1} \cdots f_k^{n_k}) \) gives an exotic \( E(k) \). Thus, \( (\mathbb{Z}^k C, f_1^{n_1} \cdots f_k^{n_k}) \) is a cork. This means that \( (\mathbb{Z}^k C, \{f_1^{n_1} \cdots f_k^{n_k}|n_j \in \mathbb{Z}\}) \) is a \( \mathbb{Z}^k \)-cork.

To prove this embedding is \( \mathbb{Z}^k \)-embedding, we have only to show that if \( \prod_{i=1}^{k} \Delta_{i,n_i}(t) = \prod_{i=1}^{k} \Delta_{i,n_i'}(t) \), then \( (n_1, \ldots, n_k) = (n_1', \ldots, n_k') \).

**Claim 2.3.** Let \( k, p \) be integers with \( k > 0 \) and \( p \geq 0 \) and \( n_i, n_i'(i = 1, \ldots, k) \) integers. If we have

\[
\prod_{i=1}^{k-p} \Delta_{i,n_{p+i}} = \prod_{i=1}^{k-p} \Delta_{i,n_{p+i}'}
\]

then we have \( n_{p+1} = n_{p+1}' \).

If \( \Delta_{p,q} \) were irreducible, then this claim would be easy. However, since some 2-bridge knots are ribbon, such Alexander polynomials are not always irreducible.

**Proof.** By the induction of the number \( k \) in (2) we prove this claim.

Let \( \sigma_i \) and \( \sigma_i' \) be the \( i \)-th elementary symmetric polynomials in \( n_{p+1}, \ldots, n_k \) and \( n_{p+1}', \ldots, n_k' \) respectively. For example, we see

\[
\sigma_i = \sum_{\{t_1, \ldots, t_i\} \subset\{p+1, \ldots, k\}, \#\{t_1, \ldots, t_i\} = i} n_{t_1} \cdots n_{t_i}.
\]

Let \( d \) be the degree of (2). Comparing the degree \( d \) of (2), we obtain

\[
\sigma_{k-p} = \sigma_{k-p}'
\]

Let \( j_0 \) be an integer with \( 1 \leq j_0 \leq p + 1 \). Comparing the coefficients of the degree \( d - 2j \) of (2) with \( 0 \leq j \leq j_0 \) we have \( \sigma_{k-p-j} = \sigma_{k-p-j}' \).

Further, the coefficients with degree \( d - 2p - 3 \) of the left hand side of (2) is

\[
S + (-3n_{p+1}) \prod_{j=p+2}^{k-p} n_j + \sum_{j=p+2}^{k-p} (-4n_j - 1) \prod_{\ell=p+2 \atop \ell \neq j}^{k} n_\ell
\]

\[
= S_0 - \sum_{j=p+2}^{k} \prod_{\ell=p+2 \atop \ell \neq j}^{k} n_\ell = S_0 - \sigma_{k-p-1} + n_{p+2} \cdots n_k,
\]

where \( S, S_0 \) are polynomials generated by \( \sigma_{k-p}, \sigma_{k-p-1}, \ldots, \sigma_{k-p-j} \).

Thus \( n_{p+2} \cdots n_k = n_{p+2}' \cdots n_k' \) holds. By using \( \sigma_{k-p} = \sigma_{k-p}' \), we have \( n_{p+1} = n_{p+1}' \). \( \square \)
We go back to the proof of Theorem 1. Suppose that \( \prod_{i=1}^{k} \Delta_{i,n_i} = \prod_{i=1}^{k} \Delta_{i,n'_i} \). Then, by using Claim 2.3 in the case of \( p = 0 \) we have \( n_1 = n'_1 \).

By dividing \( \Delta_{1,n_1} \) from both sides of this equality, we have \( \prod_{i=2}^{k} \Delta_{i,n_i} = \prod_{i=2}^{k} \Delta_{i,n'_i} \). Iterating this process by using Claim 2.3, we have \( n_2 = n'_2, \cdots, n_k = n'_k \).

Thus \( \prod_{i=1}^{k} \Delta_{i,n_i} = \prod_{i=1}^{k} \Delta_{i,n'_i} \) implies \( (n_1, \cdots, n_k) = (n'_1, \cdots, n'_k) \). Therefore we show that this embedding

\[ \natural k C := C_k \hookrightarrow X_k = E_k \}_{i=1}^{k} T_{2,2i-1} \]

is \( \mathbb{Z}^k \)-effective. \( \square \)

This proof focuses on the case of \( C = C(1,1; -1) \). However, Gompf’s method in Remarks (a) in [5] would change our examples to \( C = C(r, s; m) \) with \( r, s > 0 > m \).

3. The twisted double \( S_{r,s,m,k} \) of \( C \).

3.1. The diagrams of twisted doubles. Let \( S_{r,s,m,k} \) denote the homotopy \( S^4 \) defined in Section 1.6. We prove the following proposition first of all.

**Proposition 3.1.** \( S_{r,s,m,k} \) has a handle diagram as in Figure 8. In the case of \( k = 1 \), the diagram is Figure 7.

**Proof.** The images by \( f \) of meridians of 4 2-handles in Figure 2 are the link \( \alpha, \beta, \gamma \) and \( \delta \) in Figure 4. This diagram is the \( k = 1 \) case. In the general \( k \) case, the images of the meridians of the 2-handles by \( f^k \) are \( \alpha, \beta, \gamma \) and \( \delta \) in Figure 8. Other 0-framed 2-handles \( \alpha', \beta', \gamma', \delta' \) are additional 2,3-canceling pairs to introduce \( f^k \). Thus the first diagram in Figure 8 describes \( S_{r,s,m,k} \). Since the components \( \alpha', \beta', \gamma', \delta' \) in the boundary of 2-handlebody of \( S_{r,s,m,k} \) are 0-framed unlink, we cancel those handles together with 4 3-handles. After the canceling, \( \beta, \gamma \) and meridian of the \( m \)-framed 2-handle is the 0-framed unlink on the handle decomposition of the 2-handlebody. Thus we cancel those together with 3 3-handles.

Then we obtain the second picture as in Figure 8.

**Proof of Proposition 1.8** The union of 2-handles \( \alpha, \delta \), two 3-handles and a 4-handle is diffeomorphic to \( C \). Hence \( S_{r,s,m,k} = C \cup_{f^k} (-C) \) admits two 1-handles and four 2-handles.

In particular, in Figure 8, the four 2-handles are \( \alpha, \delta \), \( s \)-framed 2-handle and \( (-\tau) \)-framed 2-handle.

If one proves that \( S_{r,s,m,k} \) is standard, the first diagram in Figure 8 would be useful as an auxiliary information of the handle decomposition.

3.2. Proofs of Proposition 1.7. Gluck twist \( (Y \leadsto Y(S^2 \times D^2, \tau)) \) is a twist along an embedded \( S^2 \) with the trivial neighborhood, where \( \tau \) is a gluing map with the non-trivial homotopy class in the diffeomorphisms on \( S^2 \times S^1 \). Note that the square of twist \( (S^2 \times D^2, \tau^2) \) is the trivial twist. In fact \( \tau^2 \) is isotopic to the identity.
In the diagram of $S_{r,s,m,k}$ the union of the $m$-framed 2-handle and a 4-handle consists of $S^2 \times D^2$. The difference between $S_{r,s,m,k}$ and $S_{r,s,m-1,k}$ is the Gluck twist with respect to this embedded $S^2$. For example see [1]. Since the square $(S^2 \times D^2, \tau^2)$ is trivial twist, we obtain a diffeomorphism $S_{r,s,m,k} \cong S_{r,s,m-2,k}$. This diffeomorphism can be also verified by the calculus in Figure 9.

3.3. **Proof of Theorem 2**. We can find $T^2 \times D^2$ in the diagrams in Figure 7, in general Figure 8 of $S_{r,s,m,1}$ and $S_{r,s,m,k}$ respectively. We obtain the sub-handlebody as in Figure 11 which presents $T^2 \times D^2$. 

\[ \text{Figure 7. Twisted double } S_{r,s,m,1}. \]
Figure 8. Twisted double $S_{r,s,m,k}$ and a diagram after canceling.

Figure 9. The diffeomorphism $S_{r,s,m,k} \cong S_{r,s,m-2,k}$. 
One time \((1/1)\)-log transform corresponds to the change of diagram which is given in Figure 12. In general \((1/s)\)-log transform is the \(s\)-times iteration of this process. Hence, \(S_{r,s,m,k}\) is the \((1/s)\)-log transform of \(S_{r,0,m,k}\). Since the knot \(\kappa(r,0)\) is isotopic to the unknot, \(C(r,0;m)\) is the standard 4-ball. Thus \(S_{r,0,m,k}\) is diffeomorphic to \(S^4\).

Therefore, \(S_{r,s,m,k}\) is a \((1/s)\)-log transform along a torus embedded in \(S^4\). By exchanging the roles of \(r\) and \(s\), we give \(S_{r,s,m,k}\) is a \((-1/r)\)-log transform along another torus in \(S^4\). □

In the diagram in Figure 10 the embedding of the two tori used in the proof of Theorem 2 is described.

![Figure 10. Disjoint embedded two \(T^2\)'s in \(S^4\).](image1.png)

![Figure 11. \(T^2 \times D^2 \cup (1\text{-handle} \cup 2\text{-handle})\) in \(S_{r,s,m,1}\).](image2.png)

### 3.4. The log transform along several tori.

Here we give a notation of the log transform along linking tori with the square zero. Let \(e_i : T^2_i \hookrightarrow X\) be a disjoint embedded tori each other that the squares are all zero. Let \(c_i\) be a curve presenting a primitive element in \(H_1(T^2_i)\). Let \(p_i, q_i\) be several pairs of coprime integers. Let \(\tilde{e}_i\) be an embedding of the tubular neighborhood of \(T^2\) with respect to \(e_i\). Suppose that a gluing diffeomorphism \(g_{c_i,p_i,q_i} : T^2 \times \partial D^2 \rightarrow T^2_i \times \partial D^2\) a diffeomorphism satisfying

\[
\partial D^2 \mapsto p \cdot \partial D^2 + q \cdot c.
\]
Then the diffeomorphism type \( X(\{\tilde{e}_i\}, \{\nu(T^2_i)\}, \{g_{c_i, p_i, q_i}\}) \) depends only on \( e_i, c_i, p_i \) and \( q_i \). In fact, the image of \( \partial D^2 \) by the gluing map is the attaching sphere of the unique 2-handle in \( T^2 \times D^2 \) and the remaining handles of \( T^2 \times D^2 \) are 3-handles and 4-handles. It is well-known that the way to attach the remaining handles is unique.

We call the twist \( X(\{\tilde{e}_i\}, \{\nu(T^2_i)\}, \{g_{c_i, p_i, q_i}\}) \{ (p_i/q_i) \} \)-log transform along \( \{e_i\} \) with direction \( \{c_i\} \) and denote the resulting manifold by \( X[\{e_i\}, \{c_i\}, \{p_i/q_i\}] \).

When embeddings \( \{e_i\} \) and curves \( \{c_i\} \) are clear in the context, we omit these items.

Theorem 2 says that the twisted double \( S_{r,s,m,k} \) is obtained by two log transforms along two embedded disjoint tori in \( S^4 \) as in Figure 10. Namely, we have

\[
S_{r,s,m,k} = S^4[[e_{m,k,1}, e_{m,k,2}], \{c_1, c_2\}, \{-1/r, 1/s\}] = S^4[-1/r, 1/s].
\]

The two torus embeddings \( e_{m,k,i} : T^2_i \hookrightarrow S^4 \) (\( i = 1, 2 \)) are embedded in such a way that each torus is embedded in each component \( T^2 \times D^2 \) in \( \natural T^2 \times D^2 \). The \( T^2 \times D^2 \times D^2 \) exterior in \( S^4 \) is described in Figure 13.

### 3.5. A remark for the curves \( c_1 \) and \( c_2 \)

As mentioned in Section 1.6, if either of curves \( c_1 \) or \( c_2 \) in the boundary has an embedded slice disk in the exterior with framing \(-1\). However, it is difficult to find such a disk in terms of the following observation.

We consider a cobordism \( C \) from \( \#^2 T^2 \times S^1 \) to \( \#^3 S^2 \times S^1 \) by removing the three 3-handles and one 4-handle from the exterior which is described in Figure 13. We take an annulus in \( C \) beginning from either of \( c_1 \) or \( c_2 \). Suppose that the annulus has no critical points in \( C \), i.e., the annulus is the trace by the gradient flow of the Morse function for the handle decomposition. Let \( \tilde{c} \) be the obtained knot in \( \#^3 S^2 \times S^1 \). Turning the union of 3 3-handles and 4-handle by the upside down calculus, we obtain a knot description in Figure 14. Clearly, this knot \( \tilde{c} \) has no \(-1\)-framed disk in \( \natural D^3 \times S^1 \). Because if there is such a disk, then by attaching a 2-handle on \( \tilde{c} \) with 0-framing, we must find a \((-1)\)-sphere in the attached manifold whose intersection form is \( (0) \). This is a contradiction.
This means that the easy way cannot be found a $-1$-framed embedded disk in the exterior for $c_{m,k,i}$.

3.6. The cases of $r$ or $s = \infty$. The cases where $r$ or $s$ is $\infty$ can be regarded as 0-log transforms along the tori from the equality $\mathbb{S}_{r,s,m,k} = S^4[-1/r, 1/s]$. Namely, $\mathbb{S}_{\infty,s,m,k} = S^4[0, 1/s]$ and $\mathbb{S}_{\infty,\infty,m,k} = S^4[0, 0]$. $\mathbb{S}_{\infty,\infty,m,k}$ is obtained by exchanging two dots and two 0’s in the sub-handle for $z^2T^2 \times D^2$. The diagram of $S_{\infty,\infty,m,k}$ is described in Figure 15. By computing the fundamental groups and homology groups, the manifolds $S_{\infty,0,m,k}$ and $S_{\infty,\infty,m,k}$ are homotopic to

$$(S^3 \times S^1) \# (S^2 \times S^2) \text{ and } \#^2(S^3 \times S^1) \#^2(S^2 \times S^2)$$

respectively. In [14] some homotopy $(S^3 \times S^1) \# (S^2 \times S^2)$’s are constructed according to Scharlemann’s method. Some of those are diffeomorphic to the standard manifold. What one knows the relationship between these manifolds would be an interesting problem.
Figure 15. The handle diagram of $S^4[0,0]$.

REFERENCES

[1] S. Akbulut, Scharlemann's manifold is standard, Ann. of Math., 149 (1999) 497-510.
[2] D. Auckly, H. Kim, P. Melvin, and D. Ruberman, Equivalent corks. [arXiv:1602.07650]
[3] C. L. Curtis; M. H. Freedman; W. C. Hsiang; R. Stong, A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds, Invent. Math. 123 (1996), no. 2, 343-348.
[4] R. Fintushel and R. Stern, Knots, links and 4-manifolds, Invent. Math. 134 (1998), 363-400.
[5] R. Gompf, Infinite order corks. [arXiv:1603.05090]
[6] R. Gompf, More Cappell-Shaneson spheres are standard, Algebr. Geom. Topol. 10 (2010), no. 3, 1665-1681.
[7] R. Gompf, Infinite order corks via handle diagrams. [arXiv:1607.04354]
[8] R. Matveyev, A decomposition of smooth simply-connected h-cobordant 4-manifolds, J. Diff. Geom. Vol. 44 (1996) 571-582.
[9] M. Scharlemann, Constructing strange manifolds with dodecahedral space, Duke Math. J 43(1976) 33-40.
[10] N. S. Sunukjian, A note on knot surgery. J. Knot Theory Ramifications 24 (2015), no. 9, 1520003, 5 pp.
[11] M. Tange, Finite order corks. [arXiv:1601.07589]
[12] M. Tange, Non-existence theorems on infinite order corks. [arXiv:1609.04344]
[13] M. Tange, On Nash’s 4-sphere and Property 2R, Turkish J. Math. 37 (2013), no. 2, 360-374.
[14] M. Tange, The link surgery of $S^2 \times S^2$ and Scharlemann’s manifolds, Hiroshima Math. J. 44 (2014), no. 1, 35-62.
[15] M. Tange, A plug with infinite order and some exotic 4-manifolds, Journal of Gökova Geometry Topology - Volume 9 (2015) 1-17.

Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan
E-mail address: tange@math.tsukuba.ac.jp