A REMARK ON CONICAL KÄHLER-EINSTEIN METRICS

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Abstract. We give some non-existence results for Kähler-Einstein metrics with conical singularities along a divisor on Fano manifolds. In particular we show that the maximal possible cone angle is in general smaller than the invariant $R(M)$. We study this discrepancy from the point of view of log K-stability.

1. Introduction

Given a Fano manifold $M$ and a smooth anticanonical divisor $D \subset M$, the existence of a Kähler-Einstein metric on $M$ with conical singularities along $D$ has received considerable attention recently. Interest in such metrics goes back to at least McOwen [10] on Riemann surfaces, and Tian [18] for higher dimensions. The renewed interest has been sparked by a proposal by Donaldson [5, 3] to use such singular metrics in a continuity method for finding smooth Kähler-Einstein metrics, which has recently led to a solution of the problem of when Kähler-Einstein metrics exist on Fano manifolds [2]. There is by now a large body of work on such conical Kähler-Einstein metrics, see for instance Mazzeo-Rubinstein [9], Song-Wang [13], Li-Sun [8], and many others.

In this paper we give some simple calculations implying non-existence results. A Kähler-Einstein metric $\omega$ on $M$ with conical singularities along a divisor $D \in |-K_M|$ satisfies the equation

$$(1) \quad \text{Ric}(\omega) = \beta \omega + (1 - \beta) [D],$$

where the cone angle is $2\pi \beta$ for some $\beta \in (0, 1]$, and $[D]$ denotes the current of integration along $D$. Let us write

$$(2) \quad R(M, D) = \sup\{\beta > 0 \mid \text{there is a cone-singularity solution of (1)}\}.$$

Let $M_1$ and $M_2$ be the blowup of $\mathbb{P}^2$ in one or two points respectively.

**Theorem 1.** On $M_1$, for any smooth $D \in |-K_{M_1}|$ we have $R(M_1, D) \leq 12/15$. On $M_2$, if $D \in |-K_{M_2}|$ passes through the intersection of two $(-1)$-curves, then $R(M_2, D) \leq 7/9$.

Recall that for any Fano manifold $M$ one can define an invariant $R(M) \in (0, 1]$

$$(3) \quad R(M) = \sup\{t \mid \exists \omega \in c_1(M) \text{ such that } \text{Ric}(\omega) > t \omega\}. $$

We computed in [15] that $R(M_1) = 6/7$, and the invariant for all toric Fano manifolds has been computed by Li [17] (see also Tian [17] for earlier results bounding $R(M)$). In particular $R(M_2) = 21/25$. In [15] we proved that if $\alpha \in c_1(M)$ is a Kähler form, then the equation

$$(4) \quad \text{Ric}(\omega) = \beta \omega + (1 - \beta) \alpha$$

satisfies the equation (1) with cone angle $2\pi \beta$.
can be solved if and only if $\beta < R(M)$. In relation to conical Kähler-Einstein metrics, i.e. when replacing $\alpha$ by a current of integration along a smooth divisor, Donaldson \[3\] conjectured the following.

**Conjecture 2.** Suppose $D \in |-K_M|$ is smooth. For all $0 < \beta < R(M)$ there exists a cone-singularity solution to (1), and there is no solution for $R(M) < \beta < 1$. In other words, $R(M, D) = R(M)$ for any smooth $D \in |-K_M|$.

Since $12/15 < 6/7 = R(M_1)$, and $7/9 < 21/25 = R(M_2)$, our result gives counterexamples to this conjecture.

An important generalization of Equation (1) was studied by Song-Wang \[13\], where $D$ is allowed to be an element of the linear system $|-mK_M|$ for some $m > 0$.

In Section 3 we will give a non-existence result for conical Kähler-Einstein metrics along such $D$, complementing the results of Song-Wang to some extent.

The proof of Theorem 1 will be given in Section 2. It is based on a log K-stability calculation of Li \[6\], together with the result of Berman \[1\] which says that log K-stability is a necessary condition for the existence of a conical Kähler-Einstein metric. In Section 4 we will give a discussion of the difference between $R(M)$ and $R(M, D)$ from the point of view of algebro-geometric stability conditions.

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2. **Proof of Theorem 1**

We will use the notion of log K-stability, which was introduced in \[3\] (see also \[14\] for a related notion for asymptotically cuspidal metrics instead of conical ones). In particular we will use the calculation in Li \[6\], where this stability notion is analyzed for toric manifolds. We quickly recall his result. A toric Fano manifold $M$ can be viewed as a reflexive lattice polytope $P$ in $\mathbf{R}^n$. For instance $M_1$, the blowup of $\mathbf{P}^2$ in one point, corresponds to the convex hull of the points $(0, -1), (-1, 0), (-1, 2), (2, -1)$ in $\mathbf{R}^2$, shown in Figure 1.

The lattice points in $P$ correspond to sections of $K_M^{-1}$, giving a decomposition of $H^0(M, K_M^{-1})$ into one-dimensional weight spaces of the torus action. Let us write $\{s_1, \ldots, s_N\}$ for these sections, corresponding to lattice points $\{\alpha_1, \ldots, \alpha_N\}$. Given an anticanonical divisor $D$, we can write

$$D = \left\{ \sum_{i=1}^{N} a_i s_i = 0 \right\},$$

for some coefficients $a_i$. Define $P_D \subset P$ to be the convex hull of those weights $\alpha_i$, for which $a_i \neq 0$.

Let us choose $\lambda \in \mathbf{Z}^n$ giving the weights of a one-parameter subgroup in $(\mathbf{C}^*)^n$. Note that $P$ naturally lives in the dual of the Lie algebra of the torus, so here we are identifying this $\mathbf{R}^n$ with its dual, using the Euclidean inner product. This $\lambda$ defines a test-configuration for the pair $(M, D)$, which is simply a product configuration on $M$, but degenerates $D$. Let us write

$$W(\lambda) = \max_{p \in P_D} \langle p, \lambda \rangle,$$
and let $P_c \in P$ denote the barycenter of $P$. For any $\beta \in [0, 1]$, the Futaki invariant, denoted by $F(M, \beta D, \lambda)$ is computed in Li [6] (see also Section 4 for more details). The calculation there assumes that $D$ is generic, so that $a_i \neq 0$ for all $i$ and so $P_D = P$, but the same argument works if $P_D \neq P$. The result is

**Theorem 3** (Li [6]).

(7) \[ F(M, \beta D, \lambda) = -\left[ \beta \langle P_c, \lambda \rangle + (1 - \beta) W(\lambda) \right] \text{Vol}(P). \]

The sign convention is such that logarithmic K-stability requires

(8) \[ F(M, \beta D, \lambda) < 0. \]

In particular Berman [1] has shown that (8) is necessary for a conical KE metric to exist with angle $2\pi\beta$ along $D$.

**Proof of Theorem 4.** Let $D \subset M_1$ be a smooth anticanonical divisor. Suppose that $D$ intersects the exceptional divisor at the point $p$. We can choose a torus action on $M_1$ for which $p$ is a fixed point. The toric polytope $P$ can be chosen to be the convex hull of the points $(0, -1), (-1, 0), (-1, 2), (2, -1)$, so the center of mass is given by

(9) \[ P_c = \left( \frac{1}{12}, \frac{1}{12} \right). \]

Let us write $\{s_1, \ldots, s_N\}$ for the sections of $K_{M_1}^{-1}$ giving eigenvectors of the torus action, and let us assume that $s_N$ is the section corresponding to the weight $(-1, 0)$. We can assume that $p$ corresponds to the vertex $(-1, 0)$, meaning that the space of sections of $K_{M_1}^{-1}$ which vanish at $p$ are spanned by the sections $s_1, \ldots, s_{N-1}$. In Figure 1 we have indicated the lattice points corresponding to the sections $s_1, \ldots, s_{N-1}$.

![Figure 1. The polytope corresponding to $M_1$, with the sections vanishing at $p$ highlighted, and $P_D$ shaded.](image)

This implies that

(10) \[ D = \{ \sum_{i=1}^{N-1} a_is_i = 0 \}, \]

for some coefficients $a_i$, and in particular

(11) \[ P_D \subset \text{conv}\{(0, -1), (-1, 1), (-1, 2), (2, -1)\}, \]
where “conv” denotes convex hull. Let us choose $\lambda = (-2, -1)$, and consider the test-configuration corresponding to the one-parameter subgroup of $(\mathbb{C}^*)^2$ generated by $\lambda$. Using Theorem 3 we can compute

\begin{equation}
F(M_1, \beta D, \lambda) = -\left[\frac{-3}{12} \beta + 1 - \beta\right],
\end{equation}

and $F(M_1, \beta D, \lambda) < 0$ implies $\beta < 12/15$. Theorem 4.2 of Berman [1] implies that there is no conical metric solution of (1) for $\beta \geq 12/15$.

For the manifold $M_2$ we can argue similarly. We have drawn the corresponding polytope $P$ in Figure 2. We can assume that we chose our torus action in such a way, that the anticanonical divisor $D$ meets two exceptional divisors at the point corresponding to the vertex $p$. It follows that $D$ is given as the zero set of a linear combination of the sections corresponding to the lattice points $(-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0)$ and $(1, 1)$. The barycenter of $P$ is

\begin{equation}
P_c = \left(\frac{2}{21}, \frac{2}{21}\right).
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The polytope corresponding to $M_2$, with the sections vanishing at $p$ highlighted.}
\end{figure}

Let us once again choose $\lambda = (-2, -1)$, and compute

\begin{equation}
F(M_2, \beta D, \lambda) = -\left[\frac{-6}{21} \beta + (1 - \beta)\right],
\end{equation}

We find that $F(M_2, \beta D, \lambda) < 0$ implies $\beta < 7/9$. Once again, Berman’s theorem [1] implies that there is no conical metric solution of (1) for $\beta \geq 7/9$.

**Remark 4.** Many other similar examples can be given. In general if $P_c$ is the barycenter of the moment polytope $P$, let $Q$ be the intersection of the ray from $P_c$ through the origin $O$, with the boundary of $P$. It is shown by Li [7] that

\begin{equation}
R(M) = \frac{|OQ|}{|P_cQ|}.
\end{equation}

Using the formula in Theorem 3 it is easy to see that we will get

\begin{equation}
F(M, R(M)D, \lambda) < 0
\end{equation}

for a suitable $\lambda$ whenever $P_D$ does not contain the point $Q$, as shown in Figure 1.
3. Pluri-anticanonical divisors

Instead of letting $D$ be an anticanonical divisor, we can allow $D$ to be a smooth divisor in the linear system $|−mK_M|$ for some $m > 1$. In this case Song-Wang have shown that for any $β \in (0, R(M))$ there exists an $m > 0$ and a smooth divisor $D \in |−mK_M|$ so that there is a conical Kähler-Einstein metric $ω$ satisfying the equation

$$\text{Ric}(ω) = βω + \frac{1 − β}{m}[D].$$

We give a related result in the converse direction.

**Theorem 5.** On the manifold $M_1$, for any $m > 0$ there is a smooth divisor $D \in |−mK_{M_1}|$ such that a cone-singularity solution of (17) must satisfy

$$β < \frac{12m}{14m + 1} < R(M_1).$$

Similarly on $M_2$ there is a smooth divisor $D \in |−mK_{M_2}|$ such that a solution of (17) must satisfy

$$β < \frac{21m}{25m + 2} < R(M_2).$$

**Proof.** We can use the same toric calculation as in the proof of Theorem 1, using the polytope in Figure 1. The only difference is that sections of $K_{M_1} − mK_{M_1}$ correspond to lattice points in $P \cap \frac{1}{m}\mathbb{Z}^2$.

Let us write $s_0, \ldots, s_N$ for the corresponding sections, ordered in such a way that $s_0, \ldots, s_{m−1}$ correspond to the lattice points along the edge joining $(-1,0)$ and $(0,−1)$, except for the point $(0,−1)$. In other words these are the $m$ sections corresponding to the lattice points

$$(-1,0), \left(\frac{m−1}{m}, \frac{1}{m}\right), \left(\frac{m−2}{m}, \frac{2}{m}\right), \ldots, \left(\frac{1}{m}, \frac{m−1}{m}\right)$$

We will take $D$ to be of the form

$$D = \{\sum_{i=m}^{N} a_is_i = 0\},$$

for generic choice of $a_i$. This will be a smooth section by Bertini’s theorem, since the base locus of the corresponding linear system consists of only the point $p$, and we can check directly that the general element is smooth at $p$. In fact to be smooth at $p$ we only need the coefficient corresponding to the lattice point $(-1,\frac{1}{m})$ to be non-zero. The divisor $D$ will meet the exceptional divisor with multiplicity $m$ at the point $p$.

We now take $λ = (-m − 1, −m)$. Again using Theorem 5 (or rather a slight generalization which works for pluri-anticanonical divisors), we obtain

$$F(M_1, βD, λ) = −\left[\frac{−2m−1}{12}β + m(1 − β)\right].$$

The inequality $F(M_1, βD, λ) < 0$ implies

$$β < \frac{12m}{14m + 1}.$$
Note that for any \( m \), we have \( \frac{12m}{14m+1} < R(M_1) \), since \( R(M_1) = 6/7 \). It is also worth pointing out that for \( m > 1 \) the divisor \( D \) we use here is quite special, since a generic element in \( |-mK_{M_1}| \) will meet the exceptional divisor in \( m \) distinct points.

The calculation for \( M_2 \) is completely analogous, the only difference is that in that case \( P_c = \left( \frac{2}{7}, \frac{2}{7} \right) \) as in the proof of Theorem 1. The divisor \( D \) in this case will meet the \((-1)\)-curve which intersects the two exceptional divisors, with multiplicity \( m \) at the point \( p \).

\[ \square \]

4. Stability conditions

By definition \( t < R(M) \) if and only if there is a metric \( \omega \in c_1(M) \), and a smooth positive form \( \alpha \in c_1(M) \) such that

\[ \text{Ric}(\omega) = t\omega + (1-t)\alpha. \]  

We showed in [16] that the solvability of (24) for a given \( t \) is independent of the choice of \( \alpha \in c_1(M) \). The reasoning behind Conjecture 2 is the natural expectation that the same holds if we allow \( \alpha \) to be a current supported on a divisor. We have seen that this is not the case for the manifolds \( M_1 \) and \( M_2 \).

To understand the counterexamples from the point of view of algebraic geometry, we will compare log K-stability with an analogous notion of stability where the current \( [D] \) is replaced by a smooth form in \( c_1(M) \). We plan to flesh out these ideas in more detail in future work, so for now we just give a brief sketch.

A test-configuration for \( M \) is obtained by embedding \( M \hookrightarrow \mathbb{P}^{N_r} \) using the linear system \( |-rK_M| \) for some \( r > 0 \), and then acting on \( \mathbb{P}^{N_r} \) by a \( \mathbb{C}^* \)-action \( \lambda \). The flat limit

\[ M_0 = \lim_{t \to 0} \lambda(t) \cdot M \]

is invariant under the action \( \lambda \), and this can be used to define (see Donaldson [4] for details) the Futaki invariant \( \text{Fut}(M, \lambda) \). Our sign convention, in order to match with Li [6], is such that K-semistability means \( \text{Fut}(M, \lambda) \leq 0 \) for all such test-configurations.

In [5], Donaldson outlined a modification of this, which is conjecturally equivalent to the existence of Kähler-Einstein metrics on \( M \) with conical singularities along a divisor \( D \in |-mK_M| \) for some \( m > 0 \). Given a test-configuration as above, we have \( D \subset M \subset \mathbb{P}^{N_r} \), and we can take the flat limit

\[ D_0 = \lim_{t \to 0} \lambda(t) \cdot D. \]

Suppose that \( \lambda(t) = t^A \) for some \( A \in \sqrt{-1} \mathbb{S}(N_r+1) \) with integer eigenvalues. For real \( t \), the one parameter group of automorphisms \( \lambda(t) \) is induced by the gradient flow of the function

\[ H_A = \frac{A_{ij}Z^i\overline{Z}^j}{|Z|^2}, \]

where the \( Z^i \) are homogeneous coordinates on \( \mathbb{P}^{N_r+1} \). It is well known that the function

\[ f(t) = \int_{\lambda(t) \cdot D} H_A \omega_{FS}^{n-1} \]
is increasing in \( t \), where \( n \) is the dimension of \( M \), and \( \omega_{FS} \) is the Fubini-Study metric. One defines the Chow weight to be

\[
\text{Ch}(D, \lambda) = \lim_{t \to 0} f(t).
\]

The relevant modified Futaki invariant when looking for Kähler-Einstein metrics on \( M \) with conical singularities along \( D \), is

\[
\text{Fut}(M, \beta D, \lambda) = \beta \text{Fut}(M, \lambda) + \frac{1 - \beta}{m} \text{Ch}(D, \lambda).
\]

Here, as before, the parameter \( \beta \in (0, 1] \) determines the cone angle.

If we want to replace \( D \) with a smooth positive form \( \alpha \in c_1(M) \), then it is natural to define an analogous Chow weight as follows, as was also remarked on in Donaldson [5]. Let us write \( \iota : M \hookrightarrow \mathbb{P}^{N_r + 1} \) for our initial embedding, and \( \varphi_t = \lambda(t) \circ \iota \). One can then check that the function

\[
f(t) = \int_M \alpha \wedge \varphi_t^*(\omega_{FS}^{n-1})
\]

is monotonic in \( t \), and we define

\[
\text{Ch}(\alpha, \lambda) = \lim_{t \to 0} f(t).
\]

Then in analogy with (30) we define

\[
\text{Fut}(M, \beta \alpha, \lambda) = \beta \text{Fut}(M, \lambda) + (1 - \beta) \text{Ch}(\alpha, \lambda).
\]

The main point that we want to make is the following.

**Theorem 6.** Suppose that \( \alpha \in c_1(M) \) is a smooth positive form as above, and \( D \in |-K_M| \). Then we have

\[
\lim_{t \to 0} \int_M \alpha \wedge \varphi_t^*(\omega_{FS}^{n-1}) \leq \lim_{t \to 0} \int_{\lambda(t) \cdot D} H_A \omega_{FS}^{n-1}.
\]

In other words, we have

\[
\text{Fut}(M, \beta \alpha, \lambda) \leq \text{Fut}(M, \beta D, \lambda)
\]

for all \( \beta \in [0, 1] \), and all \( C^* \)-actions \( \lambda \).

**Proof.** First let us suppose that \( \alpha \) is the pullback of a Fubini-Study metric, i.e. \( \alpha = \frac{1}{k} \Phi^* \omega_{FS} \) for some embedding \( \Phi : M \to \mathbb{P}^{N_k} \) using the linear system \( |-kK_M| \).

In this case we can write \( \alpha \) as an average of the currents of integration \( \frac{1}{k}[C] \) as the divisor \( C \) varies over \( |-kK_M| \). This follows from Lemma 3.1 in Shiffman-Zelditch [12]. In fact the relevant measure \( d\mu \) on the linear system \( |-kK_M| \) is induced by the inner product on \( H^0(K_M^{-k}) \), for which the embedding \( \Phi \) is given by orthonormal sections.

This implies that

\[
\int_M \alpha \wedge \varphi_t^*(\omega_{FS}^{n-1}) = \frac{1}{k} \int_{C \in |-kK_M|} \left( \int_{\lambda(t) \cdot C} H_A \omega_{FS}^{n-1} \right) d\mu.
\]

For a fixed \( C \in |-kK_M| \), the limit

\[
\lim_{t \to 0} \int_{\lambda(t) \cdot C} H_A \omega_{FS}^{n-1}
\]
is the Chow weight $\text{Ch}(C, \lambda)$. For any integer $w$, let us write $E_w \subset | - kM |$ for the set
\begin{equation}
E_w = \{ C \in | - kM | ; \text{Ch}(C, \lambda) \geq w \}.
\end{equation}
This is a Zariski closed subset, since the weight can only jump up under specialization. In fact under an embedding of $| - kM |$ into a projective space using the Chow line bundle, $E_w$ is the intersection with a linear subspace. It follows that if we let $w_{\min}$ be the largest $w$ for which $E_w = | - kM |$, then $E_{w_{\min}+1}$ has measure zero in $| - kM |$. From the monotone convergence theorem we obtain
\begin{equation}
\lim_{t \to 0} \int_M \alpha \wedge \varphi_t^*(H_A \omega^n_{FS}^{-1}) = \frac{1}{k} \int_{C \in | - kM |} \left( \lim_{t \to 0} \int_{(\lambda(t) \cdot C)} H_A \omega^n_{FS}^{-1} \right) \, d\mu
= \frac{1}{k} \int_{C \in | - kM | \setminus E_{w_{\min}+1}} w_{\min} \, d\mu
= \frac{1}{k} w_{\min}.
\end{equation}
On the other hand, for a divisor $D \in | - M |$ we have $kD \in | - kM |$, and so $kD \in E_w$ for some $w \geq w_{\min}$. It follows that
\begin{equation}
\lim_{t \to 0} \int_{\lambda(t) \cdot D} H_A \omega^n_{FS}^{-1} = \frac{1}{k} \lim_{t \to 0} \int_{\lambda(t) \cdot kD} H_A \omega^n_{FS}^{-1} = \frac{1}{k} w \geq \frac{1}{k} w_{\min}.
\end{equation}
Comparing this with (39) we obtain the result for such $\alpha$.

Now suppose that $\alpha \in c_1(M)$ is an arbitrary smooth positive form. From the asymptotic expansion of the Bergman kernel (see Tian \cite{Tian}, Ruan \cite{Ruan}, Zelditch \cite{Zelditch}), we know that we can approximate $\alpha$ with forms of the type $\frac{1}{k} \Phi^* \omega_{FS}$. In particular we can choose $\alpha_k \in c_1(M)$ for which our arguments above apply, and
\begin{equation}
\alpha = \alpha_k + \sqrt{-1} \partial \bar{\partial} f_k,
\end{equation}
where $|f_k| < \frac{1}{k}$. For any $t$ we have
\begin{equation}
\int_M (\alpha - \alpha_k) \wedge \varphi_t^*(H_A \omega^n_{FS}^{-1}) = \int_M f_k \varphi_t^*(\sqrt{-1} \partial \bar{\partial} H_A \wedge \omega^n_{FS}^{-1}).
\end{equation}
For some constant $A$ we have
\begin{equation}
-A \omega^n_{FS} < \sqrt{-1} \partial \bar{\partial} H_A \wedge \omega^n_{FS} < A \omega^n_{FS},
\end{equation}
so
\begin{equation}
\left| \int_M f_k \varphi_t^*(\sqrt{-1} \partial \bar{\partial} H_A \wedge \omega^n_{FS}^{-1}) \right| < \frac{A}{k} \text{Vol}(M).
\end{equation}
It follows, using also (39) that
\begin{equation}
\lim_{t \to 0} \int_M \alpha \wedge \varphi_t^*(H_A \omega^n_{FS}^{-1}) = \lim_{t \to 0} \int_M \alpha_k \wedge \varphi_t^*(H_A \omega^n_{FS}^{-1}) + O(1/k)
= \frac{1}{k} w_{\min} + O(1/k).
\end{equation}
Since $k$ was arbitrary, we get
\begin{equation}
\lim_{t \to 0} \int_M \alpha \wedge \varphi_t^*(H_A \omega^n_{FS}^{-1}) = w_{\min},
\end{equation}
and so the result follows for arbitrary smooth positive $\alpha \in c_1(M)$. \qed
Remark 7. It is clear from the proof that if $D$ is chosen to be in a special position, in particular if it passes through more non-minimal critical points of $H_A$ than a generic $D$ would, then one would expect strict inequality to hold in (35). This means that if we can find a cone-singularity solution of (24), with $\alpha = [D]$ for some divisor $D \in -K_M$, then we expect to be able to solve the same equation with any smooth positive form $\alpha \in c_1(M)$, at least if there are no holomorphic vector fields on $M$. The converse, however, need not be true if $D$ is in special position. This is exactly what happens in the examples that we have for $M_1$ and $M_2$. In particular for $M_1$, if $D$ is any smooth anticanonical divisor, then we can choose a $C^*$-action on $M_1$ for which $D$ is in special position and gives a discrepancy between $R(M_1)$ and $R(M_1, D)$.

It also follows from the proof that if we fix the $C^*$-action $\lambda$, then for a generic divisor $D$, we will have equality in (35). A special case of this can be observed in Theorem 3 where for generic $D$ we have $P_D = P$. Indeed in this case the formula matches up with the result we obtained in [15] for the case of a smooth positive $\alpha \in c_1(M)$, which was formulated in terms of the derivative of the twisted Mabuchi functional.

It is interesting to speculate on what happens with the conical Kähler-Einstein metrics on $M_1$, as $\beta \to 12/15$. Along the test-configuration that we used in the proof of Theorem 1, the divisor $D$ degenerates into a divisor $D_0$ given by the union of a conic passing through the exceptional divisor, and a line which is tangent to the conic. We expect that $M_1$ admits a cone-singularity solution of

$$\text{Ric}(\omega) = \beta \omega + (1 - \beta)[D_0]$$

in a suitable sense with $\beta = 12/15$, to which the conical KE metrics solving (11) on $M_1 \setminus D$ degenerate as $\beta \to 12/15$. Moreover, we expect that one can find Kähler-Ricci solitons (or extremal metrics) with conical singularities along $D_0$ in a suitable sense even for $\beta > 12/15$. This would be a natural extension of Donaldson’s deformation result in [3] to the case when there exist vector fields preserving the divisor. Finally, these conical Kähler-Ricci solitons (or extremal metrics) should converge to the smooth Kähler-Ricci soliton (or extremal metric), which is known to exist on $M_1$, as $\beta \to 1$.

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