CHARACTERIZATIONS FOR INNER FUNCTIONS IN CERTAIN FUNCTION SPACES

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Abstract. For $\frac{1}{2} < p < \infty$, $0 < q < \infty$ and a certain two-sided doubling weight $\omega$, we characterize those inner functions $\Theta$ for which

$$\|\Theta\|^q_{A^p,q} = \left( \int_0^1 \left( \int_0^{2\pi} |\Theta'(re^{i\theta})|^p d\theta \right)^{q/p} \omega(r) dr \right)^{1/q} < \infty.$$ 

Then we show a modified version of this result for $p \geq q$. Moreover, two additional characterizations for inner functions whose derivative belongs to the Bergman space $A^p_p$ are given.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions in the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. For $0 < p < \infty$, the Hardy space $H^p$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty,$$

is bounded. A function $\omega : \mathbb{D} \to [0, \infty)$ is called a (radial) weight if it is integrable over $\mathbb{D}$ and $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p, q < \infty$ and a weight $\omega$, the weighted mixed norm space $A^p_q$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^q_{A^p,q} = \left( \int_0^1 M^q_p(r, f) \omega(r) dr \right)^{1/q} < \infty.$$ 

Class $D$ of two-sided doubling weights offers us a sufficient ballpark [20, 21]. This class can be defined, for instance, as follows: A weight $\omega$ belongs to $D$ if and only if there exist $C = C(\omega) \geq 1$, $\alpha = \alpha(\omega) > 0$ and $\beta = \beta(\omega) \geq \alpha$ such that

$$C^{-1} \left( \frac{1-r}{1-s} \right)^\alpha \hat{\omega}(s) \leq \hat{\omega}(r) \leq C \left( \frac{1-r}{1-s} \right)^\beta \hat{\omega}(s), \quad 0 \leq r \leq s < 1, \quad (1.1)$$

where

$$\hat{\omega}(z) = \int_{|z|}^1 \omega(s) ds, \quad z \in \mathbb{D}.$$ 

An inner function is a bounded analytic function having unimodular radial limits almost everywhere on the boundary $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ [6, 17]. An important subclass of inner functions consists of Blaschke products [5]. For a given sequence $\{z_n\} \subset \mathbb{D}$ satisfying $\sum_n (1 - |z_n|) < \infty$ and a real constant $\lambda$, the Blaschke product with zeros $\{z_n\}$ is defined by

$$B(z) = e^{i\lambda} \prod_n \left[ 1 - \frac{|z_n|}{z_n} \frac{z - z_n}{1 - z_n \bar{z}} \right], \quad z \in \mathbb{D}.$$ 

For $z_n = 0$, the interpretation $|z_n|/z_n = -1$ is used. Our first result characterizes those inner functions whose derivative belongs to the mixed norm space $A^p_q$ satisfying certain regularity.

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conditions. This characterization is taking advantage of the fact that the Frostman shift \( \Theta_a \) of an arbitrary inner function \( \Theta \), defined by
\[
\Theta_a(z) = \frac{\Theta(z) - a}{1 - \overline{a} \Theta(z)}, \quad z \in \mathbb{D},
\]
is a Blaschke product for almost every \( a \in \mathbb{D} \). More precisely, Frostman’s result states that the exceptional set \( E_{\Theta} \) where \( \Theta_a \) is not a Blaschke product has logarithmic capacity zero; see for instance [8, Chapter 2, Theorem 6.4].

Before our first main result, we define \( r_j = 1 - 2^{-j} \) for \( j \in \mathbb{N} \cup \{0\} \) and \( D_\delta = \{ z \in \mathbb{C} : |z| \leq \delta \} \) for \( 0 < \delta < 1 \). Moreover, recall that \( f \leq g \) if there exists a constant \( C = C(\cdot) > 0 \) such that \( f \leq Cg \), while \( f \geq g \) is understood in an analogous manner. If \( f \leq g \) and \( f \geq g \), then we write \( f = g \). Here the letter \( C = C(\cdot) \) is a positive constant whose value depends only on the parameters indicated in the parenthesis, and may change from one occurrence to another.

**Theorem 1.** Let \( \frac{1}{2} < p < \infty \), \( 0 < q < \infty \), \( 0 < \delta < 1 \) and \( \omega \in \mathcal{D} \). If \( \Theta \) is an inner function and either

\( a) \ \frac{1}{2} < p \leq 1 \) and \( \omega \) satisfies the right-hand inequality of (1.1) for some \( \beta < 2q - \frac{q}{p} \), or

\( b) \ 1 < p < \infty \), \( \omega \) satisfies the right-hand inequality of (1.1) for some \( \beta < q \) and the left-hand inequality for some \( \alpha > q - \frac{q}{p} \),

then
\[
\|\Theta^\omega\|_{A_{p,q}^\omega} \leq \sum_n \frac{\tilde{\omega}(r_n)}{(1 - r_n)^{q - q/p}} \int_{D_\delta} \nu_n(a)^{q/p} dA(a), \quad (1.2)
\]
where \( \{ z_n(a) \} \) is the zero-sequence of \( \Theta_a \) and \( \nu_n(a) = \# \{ j : r_n \leq |z_j(a)| < r_{n+1} \} \).

Here and hereafter, \( dA(z) \) will stand for the 2-dimensional Lebesgue measure \( dx dy \). The argument of Theorem 1 utilizes the recent results regarding the derivative of inner functions in \( A_{p,q}^\omega \) [24, 25], and certain estimates for \( \Theta \) originated to [14]. Using Theorem 1 together with a connection between the mixed norm and Besov spaces [27], we can prove a streamlined version of [9, Theorem 3.3]; see Corollary 7 in Section 4. Hence it does not come as a surprise that the proofs of Theorem 1 and [9, Theorem 3.3] have some similarities.

Modifying the argument of Theorem 1 for \( p \geq q \), we may remove the integral over \( D_\delta \) in the statement by assuming \( a \in \mathbb{D} \setminus E_{\Theta} \); see Theorem 2 below. More precisely, first we have to verify that the Lebesgue integral over \( D_\delta \) in an auxiliary result for \( p = 1 \) can be replaced by a certain integral with respect to a probability measure supported in a compact subset of \( \mathbb{D} \); see Lemma 8 in Section 4. Using this observation, we can prove the desired result for \( p = 1 \) in a similar manner as in Theorem 1. Finally the assertion follows from an application of some nesting properties for the derivative of inner functions in the mixed norm spaces. These nesting properties can be obtained by using a consequence of Theorem 1.

**Theorem 2.** Let \( \frac{1}{2} < p < \infty \) and \( 0 < q \leq p \). Assume that \( \omega, \Theta \) and \( \nu_n(a) \) are as in Theorem 1. Then the following statements are equivalent:

(i) \( \Theta^\omega \in A_{p,q}^\omega \).

(ii) There exists a set \( E_{\Theta}^{\omega} \subset \mathbb{D} \) of logarithmic capacity zero such that
\[
\sum_n \frac{\tilde{\omega}(r_n)\nu_n(a)^{q/p}}{(1 - r_n)^{q - q/p}} < \infty \quad (1.3)
\]
for every \( a \in \mathbb{D} \setminus E_{\Theta}^{\omega} \).

(iii) There exists a \( E_{\Theta} \subset \mathbb{D} \) such that (1.3) holds.

For instance, \( \omega_1(z) = (1 - |z|)^{\alpha} \) and \( \omega_2(z) = (1 - |z|)^{\alpha} \left( \log \frac{1}{1 - |z|} \right)^\beta \) satisfy the hypotheses of \( \omega \) in Theorems 1 and 2 if \( \max \{ -1, q - \frac{q}{p} - 1 \} < \alpha < \min \{ 2q - \frac{q}{p} - 1, q - 1 \} \) and \( \beta \in \mathbb{R} \). Using this observation for \( \omega_1 \) together with [9, Lemma 3.1], it is easy to check that [27, Theorem 3.1] for \( q \leq p \) follows from Theorem 2. Note that our result contains also the case \( \frac{1}{2} < p < 1 \),
Unlike [27, Theorem 3]. Moreover, it is worth mentioning that [27, Theorem 3] was stated without any proof.

Next we turn our attention to the case of the weighted Bergman space $A^p_b$, which is the mixed norm space with $p = q$. If $\omega(z) = (1 - |z|)^\alpha$ for some $-1 < \alpha < \infty$, then the notation $A^p_b$ is used for $A^p_{\alpha}$.

Our first characterizations for inner functions $\Theta$ whose derivative belongs to $A^p_{\alpha}$ are straightforward consequences of Theorem 2. In addition, we give a generalization of the equivalence (a) $\iff$ (b) in [11, Theorem 1]. The proof of this result is based on the existence of approximating Blaschke products [3] and some estimates for $\|\Theta^\prime\|_{A^p_{\alpha}}$ [22, 23]. The argument used here is essentially different from that used in [11]. Applying the above-mentioned tools, we can also prove a characterization which utilizes the so-called Carleson curve $\Gamma_z = \Gamma_z(\Theta)$ (in the sense of W. S. Cohn) associated with $0 < \varepsilon < 1$ and an inner function $\Theta$. This characterization together with our auxiliary results generalizes [11, Theorem 7]. The construction of Carleson curves in the case the upper half-plane can be found in [8, pp. 328–330], and its unit disc analogue has been studied, for instance, in [3, 4]. Some properties of $\Gamma_z$ are recalled also in Section 6.

**Theorem 3.** Let $\frac{1}{2} < p < \infty$, $\omega \in \mathcal{D}$, $\Theta$ be an inner function and $\{z_n(a)\}$ the zero-sequence of $\Theta_a$. Moreover, assume either

(a) $\frac{1}{2} < p \leq 1$ and $\omega$ satisfies the right-hand inequality of (1.1) for some $\beta < 2p - 1$, or

(b) $1 < p < \infty$, $\omega$ satisfies the right-hand inequality of (1.1) for some $\beta < p$ and the left-hand inequality for some $\alpha > p - 1$.

Then the following statements are equivalent:

(i) $\Theta^\prime \in A^p_{\alpha}$.

(ii) There exists a set $E'_{\Theta} \subset \mathcal{D}$ of logarithmic capacity zero such that

$$\sum_{n} \frac{\hat{\omega}(z_n(a))}{(1 - |z_n(a)|)^{p-1}} < \infty$$

(1.4)

for every $a \in \mathcal{D}\setminus E'_{\Theta}$.

(iii) There exists $a \in \mathcal{D}\setminus E'_{\Theta}$ such that (1.3) holds.

(iv) There exists $0 < C < 1$ such that

$$\int_{\{z \in \mathcal{D} : |\Theta(z)| < C\}} \frac{\hat{\omega}(z)}{(1 - |z|)^{p+1}} dA(z) < \infty.$$

(v) There exists $0 < \varepsilon < 1$ such that

$$\int_{\Gamma_z} \frac{\hat{\omega}(z)}{(1 - |z|)^p} |dz| < \infty.$$

As mentioned above, Theorem 3 implies a part of [11, Theorem 1]. In addition, the essential contents of classical results [11, Theorem 6.2] and [3, Theorem 3] are consequences of Theorem 3. All of these results are contained in the following corollary.

**Corollary 4.** Let $\frac{1}{2} < p < 1$, $\Theta$ be an inner function and $\{z_n(a)\}$ the zero-sequence of $\Theta_a$. Then the following statements are equivalent:

(i) $\Theta^\prime \in H^p$.

(ii) $\Theta^\prime \in A^p_{\alpha} + \alpha + 1$ for every $-1 < \alpha < \infty$.

(iii) $\Theta^\prime \in A^p_{\alpha} + \alpha + 1$ for some $-1 < \alpha < \infty$.

(iv) There exists a set $E'_{\Theta} \subset \mathcal{D}$ of logarithmic capacity zero such that

$$\sum_{n} (1 - |z_n(a)|)^{1-p} < \infty$$

(1.5)

for every $a \in \mathcal{D}\setminus E'_{\Theta}$.

(v) There exists $a \in \mathcal{D}\setminus E'_{\Theta}$ such that (1.3) holds.
(vi) There exists $0 < C < 1$ such that
\[
\int_{\{z \in \mathbb{D} : |\Theta(z)| < C\}} \frac{dA(z)}{(1 - |z|)^{p+1}} < \infty.
\]

(vii) There exists $0 < \varepsilon < 1$ such that
\[
\int_{\Gamma_\varepsilon} \frac{|dz|}{(1 - |z|)^p} < \infty.
\]

The remainder of this note is organized as follows. Some auxiliary results are stated in Section 2. Theorems 1, 2 and 3 are proved in Sections 3, 5 and 6, respectively. Consequences of Theorem 1 are stated in Section 4 and the proof of Corollary 4 can be found in Section 7. In addition, the last section contains an example and some remarks.

2. Auxiliary results

We begin by stating a sufficient condition for the derivative of a Blaschke product $B$ to be in $A_{p,q}^\omega$ [24]. In addition, it is mentioned that the condition is necessary if the zero-sequence of $B$ is a finite union of separated sequences. We recall that a sequence $\{z_n\} \subset \mathbb{D}$ is called separated if
\[
\inf_{n \neq k} \left| \frac{z_n - z_k}{1 - z_n \bar{z}_k} \right| > 0.
\]

**Lemma A.** Let $\frac{1}{2} < p < \infty$, $0 < q < \infty$, $\omega \in \mathbb{D}$ and $B$ be the Blaschke product with zeros $\{z_j\}$. If either

(a) $\frac{1}{2} < p \leq 1$ and $\omega$ satisfies the right-hand inequality of (1.1) for some $\beta < 2q - \frac{4}{p}$, or

(b) $1 < p < \infty$, $\omega$ satisfies the right-hand inequality of (1.1) for some $\beta < q$ and the left-hand inequality for some $\alpha > q - \frac{4}{p}$,

then
\[
\|B'\|_{A_{p,q}^\omega} \leq \sum_n \frac{\hat{\omega}(r_n) v_n^{q/p}}{(1 - r_n)^{q - q/p}}
\]
where $v_n = \#\{j : r_n \leq |z_j| < r_{n+1}\}$. If, in addition to (a) or (b), $\{z_j\}$ is a finite union of separated sequences, then
\[
\|B'\|_{A_{p,q}^\omega} = \sum_n \frac{\hat{\omega}(r_n) v_n^{q/p}}{(1 - r_n)^{q - q/p}}.
\]

It is worth noting that (a) and (b) in Lemma A for $p > q$ can be replaced by the following conditions respectively:

(A) $\frac{1}{2} < p \leq 1$ and $\omega \in \tilde{D}_{2q - q/p}$,

(B) $1 < p < \infty$ and $\omega \in \tilde{D}_q \cap \tilde{D}_{q - q/p}$.

Here we say that a weight $\omega$ belongs to $\tilde{D}_p$ for $0 < p < \infty$ if
\[
\sup_{0 < r < 1} \frac{(1 - r)^p}{\hat{\omega}(r)} \int_0^r \frac{\omega(s)}{(1 - s)^p} \, ds < \infty,
\]
and $\omega \in \tilde{D}_p$ if
\[
\sup_{0 < r < 1} \frac{(1 - r)^p}{\hat{\omega}(r)} \int_r^1 \frac{\omega(s)}{(1 - s)^p} \, ds < \infty.
\]

This observation is relevant because conditions (a) and (b) imply (A) and (B), respectively. More precisely, if the right-hand inequality of (1.1) is satisfied for some $\beta = \beta(\omega) < p$, then $\omega \in \tilde{D}_p$. Similarly, if the left-hand inequality is satisfied for some $\alpha = \alpha(\omega) > p$, then $\omega \in \tilde{D}_p$. The validity of these implications can be checked by straightforward calculations based on integration by parts; see [24] for details. In addition, we recall that $\omega \in \tilde{D}$ if and only if
ω ∈ \( \hat{D}_p \) for some \( p \). The original definition of \( \hat{D} \) reads as follows [20]: \( \omega \in \hat{D} \) if there exists \( C = C(\omega) \geq 1 \) such that \( \hat{\omega}(r) \leq C\hat{\omega}(\frac{r}{2}) \) for \( 0 \leq r < 1 \).

The next auxiliary result shows that, for \( \omega \in \mathcal{D} \cap \hat{D}_q \) and an inner function \( \Theta \), we may use the Schwarz-Pick lemma inside the norm \( \|\Theta\|_{A^p_\omega} \) without losing any essential information [25]. In addition, we give some modified asymptotic estimates for \( \|\Theta\|_{A^p_\omega} \), which are consequences of the following fact [25]: For \( 0 < p, q < \infty \) and \( \omega \in \mathcal{D} \),

\[ \|f\|_{A^q_\omega}^q = \int_0^1 M_p^q(r,f) \frac{\hat{\omega}(r)}{1-r} \, dr \]

for any \( f \in \mathcal{H}(\mathbb{D}) \).

**Lemma B.** Let \( 0 < p, q < \infty \), \( \omega \in \mathcal{D} \cap \hat{D}_q \) and \( \Theta \) be an inner function. Then

\[ \|\Theta\|_{A^q_\omega}^q = \int_0^1 \left( \int_0^{2\pi} \left( \frac{1 - |\Theta(re^{i\theta})|}{1 - r} \right)^p \frac{d\theta}{r} \right)^{q/p} \hat{\omega}(r) \, dr \]

\[ = \int_0^1 \left( \int_0^{2\pi} \left( \frac{1 - |\Theta(re^{i\theta})|}{1 - r} \right)^p \frac{d\theta}{r} \right)^{q/p} \hat{\omega}(r) \frac{d\theta}{1-r} \]

\[ = \int_0^1 \left( \int_0^{2\pi} |\Theta'(re^{i\theta})|^p \frac{d\theta}{r} \right)^{q/p} \hat{\omega}(r) \frac{d\theta}{1-r} \]

We close this section by recalling that the counterparts of Lemmas [A] and [B] for \( A^p_\omega \) were originally proved in [22] [23]. In addition, we note that Lemma [B] and the first part of Lemma [A] for \( p = q \) are valid also if the hypothesis \( \omega \in \mathcal{D} \) is replaced by \( \omega \in \hat{D} \).

### 3. Proof of Theorem [1]

Let us begin by proving a modification of [14] Lemma 4.6.

**Lemma 5.** Let \( 0 < p < \infty \) and \( 0 < \delta < 1 \). Then there exists \( C = C(p, \delta) > 0 \) such that

\[ \int_{D_\delta} \left( \log \left| \frac{1 - \pi z}{z - a} \right| \right)^p \, dA(a) \leq C(1 - |z|)^p, \quad z \in \mathbb{D}. \]

**Proof.** If \( |z| < (1 + \delta)/2 \), then the assertion follows by observing that there exits a constant \( M = M(p) > 0 \) such that

\[ \int_{D_\delta} \left( \log \left| \frac{1 - \pi z}{z - a} \right| \right)^p \, dA(a) < M < \infty. \]

Hence we may assume \( |z| \geq (1 + \delta)/2 \). Since

\[ \left| \frac{1 - \pi z}{z - a} \right|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|z - a|^2} + 1, \]

we have

\[ \log \left| \frac{1 - \pi z}{z - a} \right| \leq \log \left( 8(1 - \delta)^{-2}(1 - |z|) + 1 \right) \leq 8(1 - \delta)^{-2}(1 - |z|), \quad a \in D_\delta. \]

Consequently, the assertion follows.

For \( x \in \mathbb{R} \) and a weight \( \omega \), we set \( \omega_x(z) = \omega(z)(1 - |z|)^x \) for all \( z \in \mathbb{D} \). If \( 0 < x < \infty \) and \( \omega \in \mathcal{D} \), then

\[ \tilde{\omega}_x(z) = \tilde{\omega}(z)(1 - |z|)^x, \quad z \in \mathbb{D}; \quad (3.1) \]

see the proof of [23 Corollary 7]. It follows that

\[ \left( \frac{1 - r}{1 - s} \right)^{\alpha + x} \tilde{\omega}_x(s) \leq \tilde{\omega}_x(r) \leq \left( \frac{1 - r}{1 - s} \right)^{\beta + x} \tilde{\omega}_x(s), \quad 0 \leq r \leq s < 1, \quad (3.2) \]
where \( \alpha \) and \( \beta \) are from [14]. With these preparations we are ready to prove Theorem 1.

**Proof of Theorem 1.** Since

\[
\theta'(z) = \theta(z) \frac{1 - |a|^2}{(1 - \overline{a} \theta(z))^2};
\]

we obtain \(|\theta'(z)| \approx |\theta'(a)|| for \( z \in \mathbb{D} \) and \( a \in D_\delta \). Hence Lemma 13 yields

\[
\|\theta'\|_{A^p_{\alpha,\delta}}^q = \int_{D_\delta} |\theta'(a)|^q dA(a) \lesssim \sum_n \frac{1}{(1 - r_n)^{q - q/p}} \int_{D_\delta} v_n(a)^{q/p} dA(a);
\]

and consequently, the upper bound for \( \|\theta'\|_{A^p_{\alpha,\delta}}^q \) is proved.

Let \( 1 \leq p < \infty \) and \( 0 < q < \infty \). Since

\[
\log \frac{1}{|\theta_n(z)|} \geq \sum_n \frac{(1 - |z|^2)(1 - |x_n(a)|^2)}{1 - \overline{a} x_n(a) z^2}, \quad z \in \mathbb{D},
\]

by [8] Chapter 7, Lemma 1.2, the super-additivity of \( x^p \) for \( 0 < x \leq 1 \) and the Forelli-Rudin estimate [13] Theorem 1.7 gives

\[
v_n(a)(1 - r_n) \lesssim \int_0^{2\pi} \left( \log \frac{1}{|\theta_n(r \exp i\theta)|} \right)^p d\theta, \quad r_n \leq r < r_{n+1},
\]

(3.4) as observed in [14] Corollary 4.5. Using (3.4) together with the hypothesis \( \omega \in \hat{D} \), we obtain

\[
S := \sum_n \frac{1}{(1 - r_n)^{q - q/p}} \int_{D_\delta} v_n(a)^{q/p} dA(a) = \sum_n \int_{r_n}^{r_{n+1}} \frac{1}{(1 - r)^{q - q/p}} \int_{D_\delta} v_n(a)^{q/p} (1 - r_n)^{q/p} dA(a) dr
\]

(3.5)

If \( p < q \), then (3.5), Minkowski’s inequality [12] Theorem 202 and Lemma 5 for \( z = \Theta(r \exp i\theta) \) yield

\[
S \lesssim \sum_n \int_{r_n}^{r_{n+1}} \frac{1}{(1 - r)^{q - q/p}} \left( \int_0^{2\pi} \left( \log \frac{1}{|\Theta_n(r \exp i\theta)|} \right)^q dA(a) \right)^{p/q} d\theta \right)^{q/p} dr
\]

\[
\lesssim \int_0^1 \left( \int_0^{2\pi} \left( \frac{1}{1 - r} \right)^p \frac{1}{1 - r} \right)^{q/p} \frac{1}{1 - r} dr.
\]

For \( p \geq q \), using (3.5), Hölder’s inequality and Lemma 5 to obtain

\[
S \lesssim \sum_n \int_{r_n}^{r_{n+1}} \frac{1}{(1 - r)^{q - q/p}} \left( \int_0^{2\pi} \left( \log \frac{1}{|\Theta_n(r \exp i\theta)|} \right)^p dA(a) \right)^{q/p} d\theta \right)^{q/p} dr
\]

\[
\lesssim \int_0^1 \left( \int_0^{2\pi} \left( \frac{1}{1 - r} \right)^p \frac{1}{1 - r} \right)^{q/p} \frac{1}{1 - r} dr.
\]

Finally the lower bound of \( \|\theta'\|_{A^p_{\alpha,\delta}}^q \) for \( 1 \leq p < \infty \) follows from these inequalities and Lemma 13. Thus we have shown (1.2) when \( p \geq 1 \).

Let \( \frac{1}{2} < p < 1 \) and \( 0 < q < \infty \). Put \( x = q/p - q \) and assume, by the hypotheses \( \omega \in \mathbb{D} \) and (a), \( \alpha + x > 0 \) and \( \beta + x < q/p \). Finally asymptotic equation (3.1), (1.2) with \( p \) and \( q \) being replaced by \( 1 \) and \( q/p \), respectively, and the Schwarz-Pick lemma yield

\[
S \approx \sum_n \int_{D_\delta} v_n(a)^{q/p} dA(a) \approx \|\theta'\|_{A^q_{\alpha,\delta}} \lesssim \|\theta'\|_{A^q_{\alpha,\delta}}^{q/p} \approx \|\theta'\|_{A^q_{\alpha,\delta}}^q.
\]

This completes the proof. \[\square\]
Note that the proof of the lower bound
\[\|\Theta\|^q_{A^p,q} \geq \sum_n \frac{\tilde{\omega}(r_n)}{(1-r_n)^{q-p/2}} \int_{D_S} v_n(a)^{q/p} dA(a)\]
relies on Lemma [3] not Lemma [A]. In particular, this means that for the lower bound it suffices to assume only the hypotheses of Lemma [3].

4. Consequences of Theorem [1]

The first consequence of Theorem [1] asserts that the derivative of an inner function \(\Theta\) belongs to \(A^p,q\) if and only if \(\Theta' \in A^{q+x/p,q+x}_p\) for every some \(0 < x < \infty\). Note that this result was originally proved in [24]. The argument there relies on the existence of approximating Blaschke products [4], unlike the proof here.

**Corollary 6.** Let \(\frac{1}{2} < p < \infty\) and \(0 < q, x < \infty\). Assume that \(\omega\) and \(\Theta\) are as in Theorem [1]. Then
\[\|\Theta'\|^q_{A^p,q} \simeq \|\Theta'\|^q_{A^{q+x/p,q+x}_p}.
\]

**Proof.** Let \(p' = p + xp/q\) and \(q' = q + x\). Then, by [3,1], we have
\[\sum_n \frac{\tilde{\omega}(r_n)}{(1-r_n)^{q-p/2}} \int_{D_S} v_n(a)^{q'/p'} dA(a) = \sum_n \frac{\tilde{\omega}(r_n)}{(1-r_n)^{q'-q/p'}} \int_{D_S} v_n(a)^{q'/p'} dA(a),\]
where \(v_n(a)\) is as in Theorem [1]. Hence the assertion follows from Theorem [1] by showing that one of the following conditions holds:

(i) \(\frac{1}{2} < p' \leq 1\) and \(\omega_\alpha\) satisfies the right-hand inequality of [3.2] for some \(\beta + x < 2q' - \frac{q'}{p'}\) and the left-hand inequality for some \(\alpha + x > 0\),

(ii) \(1 < p' < \infty\), \(\omega_\alpha\) satisfies the right-hand inequality of [3.2] for some \(\beta + x < q'\) and the left-hand inequality for some \(\alpha + x > q' - \frac{q'}{p'}\).

Since the validity of (i) or (ii) can be checked by straightforward calculations, the proof is complete.

Next we turn our attention to the Besov space. For \(0 < \alpha < \infty\) and an analytic function \(f(z) = \sum_n a_n z^n\), the fractional derivative of order \(\alpha\) is defined by
\[D^\alpha f(z) = \sum_n (n+1)^\alpha a_n z^n, \quad z \in \mathbb{D}.
\]
Note that \(M_p(r, f^{(n)})\) and \(M_p(r, D^\alpha f)\) are comparable for any \(f \in \mathcal{H}(\mathbb{D})\) and \(n \in \mathbb{N}\) [7]. For \(0 < p, q < \infty\) and \(0 \leq \alpha < \infty\), the Besov space \(B^{p,q}_\alpha\) consists of those \(f \in \mathcal{H}(\mathbb{D})\) such that
\[\|f\|^q_{B^{p,q}_\alpha} = \int_0^1 M^q_p(r, D^{1+\alpha} f)(1-r)^{q-1} dr < \infty.
\]

**Corollary 7.** Let \(\Theta\) be an inner function, \(0 < \delta < 1\) and \(0 < p, q, \alpha < \infty\) be such that \(\max\{0, \frac{1}{p} - 1\} < \alpha < \frac{1}{p}\). Then \(\Theta \in B^{p,q}_\alpha\) if and only if
\[\sum_n (1-r_n)^{q/p - q\alpha} \int_{D_S} v_n(a)^{q/p} dA(a) < \infty, \quad \text{(4.1)}\]
where \(v_n(a)\) as in Theorem [7]

Set \(K_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq 1 - \delta\}\) for \(0 < \delta < \frac{1}{2}\), and recall that [9, Theorem 3.3] is a corresponding result where (4.1) is replaced by the condition
\[\int_{K_\delta} \left(\sum_n (1-r_n)^{q/p - q\alpha} v_n(a)^{q/p}\right) \frac{1}{|a|^{q/p}} dA(a) < \infty.
\]
One could say that Corollary 7 is a streamlined version of [9, Theorem 3.3], or a natural extension of the main result of [15]. Before the proof we underline that our argument for $\alpha \geq 1$ takes advantage of the original result.

**Proof of Corollary 7.** Let $0 < \alpha < 1$. Then Theorem 1 together with [7, Theorem 6] yields

$$\left\| \Theta \right\|_{B^{p,q}_\alpha}^q \leq \int_0^1 M_p^q(r, D^1 \Theta)(1 - r)^{(1 - \alpha)q - 1} \, dr = \left\| \Theta \right\|_{A^{p,q}_{(1-\alpha)q-1}}^q \leq \sum_n (1 - r_n)^{q/p - \alpha q} \int_{D_\delta} v_n(a)^{q/p} \, dA(a).$$

Note that for the last asymptotic equation it suffices to check that $\omega(z) = (1 - |z|)^{(1 - \alpha)q - 1}$ satisfies the hypotheses of Theorem 1. This gives the assertion for $0 < \alpha < 1$.

Let $1 < \alpha < \infty$ and $\alpha < t < \infty$. By [9, Lemma 3.4], we know $B^{pl,q}_{\alpha/t} \subset B^{p,q}_{\alpha/t}$. Moreover, [9, Corollary 3.6] for $t' = 1/t$ implies $\Theta \in B^{p,q}_{\alpha/t}$ if $\Theta \in B^{pl,q}_{\alpha/t}$ for $\alpha > \frac{1}{p} - 1$. Applying these facts together with the previous case, it is easy to verify the assertion for $\alpha > 1$. This completes the proof.

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5. PROOF OF THEOREM 2

Let us begin by stating an auxiliary result, which can be proved in a similar manner as Lemma 5.

**Lemma 8.** Let $0 < \delta < 1$ and $\sigma$ be a probability measure supported in $D_\delta$ and satisfying

$$\sup_{z \in D_\delta} \int_{D_\delta} \log \frac{1 - \pi_z}{z - a} \, d\sigma(a) = M < \infty. \quad (5.1)$$

Then there exists $C = C(\delta, M) > 0$ such that

$$\int_{D_\delta} \log \frac{1 - \pi_z}{z - a} \, d\sigma(a) \leq C(1 - |z|), \quad z \in D.$$

Before the proof of Theorem 2, we recall that a compact set $K \subset \mathbb{D}$ has a positive logarithmic (inner) capacity if there exits a non-zero probability measure $\sigma$ supported in $K$ and satisfying (5.1). For details, see Section 12 as well as Section 2 in [24, Chapter III].

**Proof of Theorem 2.** If (1.3) holds for some $a \in \mathbb{D} \setminus E_\Theta$, then $\Theta' \in A^{p,q}_\alpha$ by Lemma A and (3.3). Consequently, condition (iii) implies (i). Moreover, since $(\mathbb{D} \setminus E_\Theta) \cap (\mathbb{D} \setminus E_\Theta) \neq \emptyset$, the implication (ii) $\Rightarrow$ (iii) is clear. Hence it suffices to show (i) $\Rightarrow$ (ii).

Assume $\Theta' \in A^{p,q}_\alpha$ for some $p \geq 1$, let $0 < \delta < 1$ and $\sigma$ be a probability measure supported in $D_\delta$ and satisfying (5.1). For condition (ii) it suffices to prove

$$I := \int_{D_\delta} \sum_n \frac{\tilde{\omega}(r_n) v_n(a)^{q/p}}{(1 - r_n)^{q - q/p}} \, d\sigma(a) < \infty \quad (5.2)$$

because then

$$\sigma \left( \left\{ a \in D_\delta : \sum_n \frac{\tilde{\omega}(r_n) v_n(a)^{q/p}}{(1 - r_n)^{q - q/p}} = \infty \right\} \right) = 0.$$
Hence estimate (5.2) is satisfied for \( p \), uniformly separated zeros where \( x \) and \( \alpha \) are approximating Blaschke product of \( \Theta \). Using the existence of approximating Blaschke product for \( q \), we have to first prove a modification of Corollary 6 for \( p \). By [4, Theorem 2.1], for any inner function \( \Theta \), there exists a Blaschke product \( B_\Theta \) for \( \omega \) such that \( 1 - |\Theta(z)| = 1 - |B_\Theta(z)| \) for all \( z \) in \( D \). \( B_\Theta \) is called an approximating Blaschke product of \( \Theta \). Using the existence of approximating Blaschke product for \( q \), we obtain

\[
I = \sum_n \int_{r_n}^{r_{n+1}} \frac{\hat{\omega}(r)}{(1-r)^{q+1}} \int_{D_r} v_n(a)^{q/p} (1-r_n)^{q/p} d\sigma(a) dr
\]

\[
\leq \sum_n \int_{r_n}^{r_{n+1}} \frac{\hat{\omega}(r)}{(1-r)^{q+1}} \left( \int_{D_r} \log \left( \frac{1}{|\Theta_a(re^{i\theta})|} \right) d\theta \right)^{q/p} d\sigma(a) dr
\]

\[
\leq \int_0^1 \frac{\hat{\omega}(r)}{(1-r)^{q+1}} \left( \int_0^{2\pi} \log \left( \frac{1}{|\Theta_a(re^{i\theta})|} \right) d\theta \right)^{q/p} dr
\]

\[
\leq \int_0^1 \left( \int_0^{2\pi} \frac{1 - |\Theta(re^{i\theta})|}{1-r} d\theta \right)^{q/p} \frac{\hat{\omega}(r)}{(1-r)^{q+1-q/p}} dr
\]

\[
\leq \int_0^1 \left( \int_0^{2\pi} \frac{1 - |\Theta(re^{i\theta})|}{1-r} d\theta \right)^{q/p} \frac{\hat{\omega}(r)}{(1-r)^{q+1-q/p}} dr
\]

where \( x = q/p - q \).

Next we verify some properties for \( \omega_x \). By the second part of hypothesis (b) and its consequence \( \omega \in D_{-x} \), we find \( \alpha = \alpha(\omega) > -x \) such that

\[
\frac{\hat{\omega}_x(s)}{(1-s)^{\alpha+1}} \leq \frac{\hat{\omega}(s)}{(1-s)^{\alpha}} \leq \frac{\hat{\omega}_x(s)}{(1-s)^{\alpha+1}}, \quad 0 \leq r \leq s < 1,
\]

and

\[
\hat{\omega}(t)(1-t)^x \leq 2^{\alpha} \hat{\omega} \left( \frac{1}{2} \right) (1-t)^{\alpha+x} \to 0^+, \quad t \to 1^-
\]

Consequently, an integration by parts together with hypothesis (b) gives

\[
\left( \frac{1-r}{1-s} \right)^{\alpha+1} \hat{\omega}_x(r) \leq \hat{\omega}_x(r)(1-r)^x \leq \frac{1}{(1-t)^{\alpha}} (1-t)^{\alpha+x-1} \int_r^1 \hat{\omega}(t)(1-t)^{\alpha+x} dt
\]

\[
\leq \hat{\omega}(r)(1-r)^x \leq \left( \frac{1-r}{1-s} \right)^{\beta+1} \hat{\omega}_x(s), \quad 0 \leq r \leq s < 1.
\]

for some \( \alpha = \alpha(\omega) > -x \) and \( \beta = \beta(\omega) < q \).

Finally [5.3], Lemma [5.4] and Corollary [5.5] for \( p' = 1, q' = q/p \) and \( x = -x \) yield

\[
I \leq \|\Theta\|_{A_{p',q'}}^p \to \|\Theta\|_{A_{p,q}}^q < \infty.
\]

Hence estimate [5.2] is satisfied for \( p \geq 1 \). Since the case \( \frac{1}{2} < p < 1 \) can be verified by imitating the end part of the proof of Theorem [1], condition (i) implies (ii). This completes the proof.

It is worth noting that we can slightly weaken the hypotheses for \( \omega \) in Theorem [2]. Condition (a) can be replaced by the hypothesis \( \omega \in D_{2q'-q/p} \), and the first part of (b) by \( \omega \in D_q \). This is due to the alternative version of Lemma [2.1] for \( q \leq p \), mentioned in Section [2]. More precisely, we have to first prove a modification of Corollary [2.2] for \( q \leq p \), and then apply this result in the proof.

6. Proof of Theorem [3]

Recall that a sequence \( \{z_n\} \subset D \) is said to be uniformly separated if

\[
\inf_{n \in \mathbb{N}} \prod_{k \neq n} \frac{|z_k - z_n|}{1 - \overline{z}_k z_n} > 0.
\]

By [4, Theorem 2.1], for any inner function \( \Theta \), there exists a Blaschke product \( B_\Theta \) with uniformly separated zeros \( \{z_n\} \) such that \( 1 - |\Theta(z)| = 1 - |B_\Theta(z)| \) for all \( z \) in \( D \). \( B_\Theta \) is called an approximating Blaschke product of \( \Theta \). Using the existence of approximating Blaschke product for \( q \)}
products together with our auxiliary results, we prove the following proposition which implies the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 3.

**Proposition 9.** Let $\frac{1}{2} < p < \infty$, $\omega \in \mathcal{D}$ and $\Theta$ be an inner function. Moreover, assume either $\frac{1}{2} < p \leq 1$ and $\omega \in \mathcal{D}_{2p-1}$, or $1 < p < \infty$ and $\omega \in \mathcal{D}_p \cap \mathcal{D}_{p-1}$. Then there exists $C = C(\Theta) \in (0, 1)$ such that
\[
\|\Theta\|_{A^p_\omega}^p = I_C := \int_{\{z \in \mathbb{D} : |\Theta(z)| < C\}} \frac{\tilde{\omega}(z)}{(1 - |z|^{p+1})} dA(z),
\]
where the comparison constants may depend on $p$, $\omega$, $\Theta$ and $C$.

**Proof.** For any $0 < C < 1$, Lemma 3 yields
\[
\|\Theta\|_{A^p_\omega}^p \geq \int_{\mathbb{D}} \left( \frac{1 - |\Theta(z)|}{1 - |z|} \right)^p \tilde{\omega}(z) \frac{dA(z)}{1 - |z|} \geq (1 - C)^p I_C.
\]
Hence the lower bound for $\|\Theta\|_{A^p_\omega}^p$ is proved.

Let $B_\Theta$ be an approximating Blaschke product of $\Theta$ with zeros $\{z_n\}$. Since $\{z_n\}$ is (uniformly) separated, we find $0 < \delta < 1$ such that discs $\Delta(z_n) = \{z : |z_n - z| < \delta (1 - |z_n|)\}$ are pairwise disjoint. Hence, using Lemma B and [23, Theorem 1] together with the hypotheses for $\omega$, we obtain
\[
\|\Theta\|_{A^p_\omega}^p \geq \sum_n \frac{\tilde{\omega}(z_n)}{(1 - |z_n|)^{p+1}} \frac{dA(z_n)}{1 - |z_n|} = \sum_n \frac{\tilde{\omega}(z_n)}{(1 - |z_n|)^{p+1}} \frac{dA(z)}{1 - |z|} \geq B_{\Theta}'^p_{A^p_\omega} = \sum_n \frac{\omega(z_n)}{(1 - |z_n|)^{p+1}} \frac{dA(z_n)}{1 - |z_n|}.
\]
Consequently, it suffices to find constants $C$ and $D$ such that $0 < C, D < 1$ and
\[
\bigcup_n \Delta(z_n) \subset \{z \in \mathbb{D} : |B_\Theta(z)| < D\} \subset \{z \in \mathbb{D} : |\Theta(z)| < C\}.
\]
Since
\[
|B_\Theta(z)| \leq \frac{|z_n - z|}{|1 - \overline{z}z_n|} \leq \frac{|z_n - z|}{1 - |z_n|} < \delta, \quad z \in \Delta(z_n),
\]
the first inclusion is valid for $D = \delta$. If $|B_\Theta(z)| < D$ for some $0 < D < 1$, then we find $M = M(\Theta) < \frac{1}{1 - \delta}$ such that
\[
|\Theta(z)| \leq 1 - M(1 - |B_\Theta(z)|) < 1 - M(1 - D), \quad z \in \mathbb{D}.
\]
Thus the second inclusion is proved and the assertion follows. \qed

Recall that the Carleson curve $\Gamma_\varepsilon \subset \overline{\mathbb{D}}$ associated with $0 < \varepsilon < 1$ and an inner function $\Theta$ has the following properties [34, 43, 53]:

1. There exists $\varepsilon_0 = \varepsilon_0(\varepsilon) \in (0, \varepsilon)$ such that $\varepsilon_0 < |\Theta(z)| < \varepsilon$ for $z \in \Gamma_\varepsilon \cap \mathbb{D}$.
2. $\Gamma_\varepsilon \cap \mathbb{D}$ is a countable union of arcs $I_n$ with pairwise disjoint interiors such that
   - each $I_n$ is either a part of a circle $|z| = r < 1$ or a radial segment in $\mathbb{D}$;
   - the end points $a_n$ and $b_n$ of $I_n$ satisfy
     \[
     \delta_1 = \left| \frac{a_n - b_n}{1 - \overline{a_n}b_n} \right| \leq \delta_2
     \]
     for all $n$ and some fixed $\delta_1, \delta_2 \in (0, 1)$.
3. If $\{z_n\}$ is the sequence of the middle points of $I_n$, then the Blaschke product $B_\Theta$ with zeros $\{z_n\}$ is an approximating Blaschke product of $\Theta$. 

Now we are ready to prove Theorem 3.

**Proof of Theorem 3.** By Proposition 9, the equivalence \( (i) \iff (iv) \) is valid. Assume without loss generality that \( \{z_n(a)\} \) is ordered by increasing moduli, and enumerate it such that, for all \( k, r_j \leq |z_{jk}(a)| < r_{j+1}, j = 0, 1, \ldots \), and \( \{z_{jk}(a)\} \) is ordered by increasing moduli with \( k \).

Then the hypothesis \( \omega \in \mathcal{D} \) yields

\[
\sum_j \frac{\hat{\omega}(r_j) v_j(a)}{(1 - r_j)^{p-1}} = \sum_j \sum_k \frac{\hat{\omega}(z_{jk}(a))}{(1 - |z_{jk}(a)|)^{p-1}} = \sum_n \frac{\hat{\omega}(z_n(a))}{(1 - |z_n(a)|)^{p-1}}, \quad a \in \mathbb{D},
\]

where \( v_j(a) \) is as in Theorem 1. Consequently, \( (i) \iff (ii) \iff (iii) \) by Theorem 2. Hence it suffices to prove \( (i) \iff (v) \).

By the hypotheses of \( \omega \), we know that \( \hat{\omega}(r)/(1 - r)^p \) is essentially increasing with \( r \). Using this fact and condition (2) of \( \Gamma_\varepsilon \), we obtain

\[
\int_{\Gamma_\varepsilon} \frac{\hat{\omega}(z)}{(1 - |z|)^p} |dz| = \sum_n \int_{I_n} \frac{\hat{\omega}(z)}{(1 - |z|)^p} |dz| \leq \sum_n |I_n| \frac{\hat{\omega}(\xi_n)}{(1 - |\xi_n|)^p},
\]

where \( \xi_n \) is the supremum of \( I_n \) in the sense of absolute value. Since, for all \( n \),

\[
|I_n| = |a_n - b_n| = 1 - |z_n| \quad \text{and} \quad \hat{\omega}(\xi_n) = \hat{\omega}(z_n)
\]

by condition (2) and the hypothesis \( \omega \in \mathcal{D} \), we obtain

\[
\int_{\Gamma_\varepsilon} \frac{\hat{\omega}(z)}{(1 - |z|)^p} |dz| \leq \sum_n \frac{\hat{\omega}(z_n)}{(1 - |z_n|)^{p-1}}.
\]

In a similar manner, one can also verify the asymptotic equation \( \asymp \). Consequently, condition (3) together with Lemmas A and B yields

\[
\int_{\Gamma_\varepsilon} \frac{\hat{\omega}(z)}{(1 - |z|)^p} |dz| = \sum_n \frac{\hat{\omega}(z_n)}{(1 - |z_n|)^{p-1}} \asymp \|B'\|_A^p \asymp \|\Theta'\|_{A^p}^p.
\]

This means that \( (i) \iff (v) \) and the proof is complete.

\( \square \)

7. **Proof of Corollary 4.** example and remarks

Let us begin with the proof of Corollary 4.

**Proof of Corollary 4.** Let \( B_{\Theta} \) be an approximation Blaschke product of \( \Theta \) with zeros \( \{z_n\} \). Using this fact together with [22, Theorem 2], [3, Theorem 1], Lemmas A and B, we obtain

\[
\|\Theta'\|_{H^p}^p \asymp \sup_{0 < r < 1} \int_{\Theta} \left( 1 - \frac{|\Theta(re^{i\theta})|}{1 - r} \right)^p d\theta \asymp \sup_{0 < r < 1} \int_{\Theta} \left( 1 - \frac{|B_{\Theta}(re^{i\theta})|}{1 - r} \right)^p d\theta
\]

for every/some \(-1 < \alpha < \infty\). Hence the equivalences \( (i) \iff (ii) \iff (iii) \) are valid. Moreover, Theorem 4 gives \( (iii) \iff (iv) \iff (v) \iff (vi) \iff (vii) \). This completes the proof.

**Example 10.** Let us consider the atomic singular inner function

\[
S(z) = \exp \left( \frac{z + 1}{z - 1} \right), \quad z \in \mathbb{D}.
\]

Let \( \{z_n(a)\} \) be the zero-sequence of the Frostman shift \( S_a \) of \( S \), assume \( a \in \mathbb{D}\setminus\{0\} \) and set \( -\pi < \arg a \leq \pi \). Solving the equation

\[
S(z) = \exp \left( \frac{z + 1}{z - 1} \right) = a,
\]
we can present zeros $z_n(a)$ in the form

$$z_n(a) = \frac{c_n + 1}{c_n - 1},$$

where $c_n = \log |a| + i(2\pi n + \arg a)$, for $n \in \mathbb{Z}$. It follows that

$$1 - |z_n(a)|^2 = \frac{|c_n - 1|^2 - |c_n + 1|^2}{|c_n - 1|^2} = \frac{-4 \Re c_n}{|c_n - 1|^2} \approx |a|^{-2}, \quad |n| \to \infty.$$ 

In particular, for $\alpha \in \mathbb{R}$,

$$\sum_n (1 - |z_n(a)|)^{\alpha} < \infty \quad \text{if and only if} \quad \alpha > \frac{1}{2}. \quad (7.1)$$

Hence, as a consequence of Theorem 2 and the nesting property $A_{\alpha_1}^p \subset A_{\alpha_2}^p$ for $-1 < \alpha_1 \leq \alpha_2 < \infty$, we obtain the following result: For $\frac{1}{2} < p < \infty$ and $-1 < \alpha < \infty$, the derivative of $S$ belongs to $A_p^\alpha$ if and only if $\alpha > p - \frac{3}{2}$. This result originates to [10]; see also [18]. However, the argument used here is essentially different from that used in these references.

By [10] Example 2], the Frostman shift $S_a$ for any $a \in \mathbb{D}\{0\}$ is a Blaschke product with uniformly separated zeros. Applying this fact together with (7.1), the above-mentioned result follows also from Lemma A.

We close this note with the following remarks, which indicate two open questions.

(I) A modification of Corollary 7 for $p \geq q$ can be obtained in a similar manner as the current version using Theorem 2 instead of Theorem 1. More precisely, the counterpart of (4.1) takes the form

$$\sum_n (1 - r_n)^{q/p - ap} v_n(a)^{q/p} < \infty, \quad (7.2)$$

where $a \in \mathbb{D}\{E_\Theta \}$. In addition, if $\Theta$ belongs $B_{\alpha}^{p,q}$ with the given restrictions, then there exists a set $E_\Theta' \subset \mathbb{D}$ of logarithmic capacity zero such that (7.2) holds for every $a \in \mathbb{D}\{E_\Theta' \}$.

Applying the above-mentioned result together with Corollary 3, one can show that, for $\frac{1}{2} < p < \infty$, the derivative of an inner function $\Theta$ belongs to $H^p$ if and only if $\Theta^\prime \in A_{p-1}^p$. Note that for $p \geq 1$ we are working with finite Blaschke products. Originally this result was stated as a part of [3] Theorem 3.10]. The existence of the corresponding result in the case $0 < p \leq \frac{1}{2}$ is an open question. However, since

$$\{f : f^\prime \in A_{p-1}^p\} \subset H^p, \quad 0 < p \leq 2,$$

by [28] Lemma 1.4, another implication is trivially valid also for $0 < p \leq \frac{1}{2}$.

(II) Corollary 2 contains several ways to characterize those inner functions $\Theta$ whose derivative belongs to $H^p$ for some $\frac{1}{2} < p < 1$. Nevertheless, it does not contain an important characterization given in [11] Theorem 1]: For $\frac{1}{2} < p < 1$ and $1 < \eta < \infty$, the derivative of an inner function $\Theta$ belongs to $H^p$ if and only if $\Theta$ is a Blaschke product whose zero-sequence $\{z_n\}$ satisfies the condition

$$\int_0^{2\pi} \left( \sum_{z_n \in \Gamma_\eta(e^{i\theta})} \frac{1}{1 - |z_n|} \right)^p d\theta < \infty,$$

where

$$\Gamma_\eta(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}| \leq \eta(1 - |z|)\}.$$

For instance using Corollary 2] we may replaced $H^p$ in the above-mentioned result by $A_{\alpha}^{p+1}$, where $-1 < \alpha < \infty$. Even so any corresponding result for general $A_{\alpha}^p$ has
not been verified, and proving such result seems to be laborious. A reason for this is the fact that the argument of [11, Theorem 1] utilizes the well-known identity

\[ \| B' \|^p_{H^p} = \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_n \frac{1 - |z_n|^2}{|z_n - e^{i\theta}|^2} \right)^p d\theta, \quad 0 < p < \infty, \]

where \( B \) is the Blaschke product with zeros \( \{z_n\} \) [2]: and we do not have a similar result for \( A^{\alpha}_p \).

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