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The Heyde characterization theorem on compact totally disconnected and connected Abelian groups

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Abstract

By the well-known Heyde theorem, the Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another. In the case of two independent random variables we give a complete description of compact totally disconnected Abelian groups \( X \), where an analogue of this theorem is valid. We also prove that even a weak analogue of the Heyde theorem fails on compact connected Abelian groups \( X \). Coefficients of considered linear forms are topological automorphisms of \( X \). The proofs are based on the study of solutions of a functional equation on the character group of the group \( X \) in the class of Fourier transforms of probability distributions.

Keywords
Compact totally disconnected Abelian group · Compact connected Abelian group · Topological automorphism · Haar distribution

Mathematics Subject Classification
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1 Introduction

Let \((\Omega, \mathcal{A}, P)\) be a probability space, \(X\) be a topological Abelian group, \(\mathcal{B}\) be the \(\sigma\)-algebra of Borel sets in \(X\). Consider random variables \(\xi\) and \(\eta\) defined on \((\Omega, \mathcal{A}, P)\) with values in \((X, \mathcal{B})\). The definition of the conditional distribution of the random variable \(\eta\) given \(\xi\) see e.g. in [1, Appendix, §5]. We note that the conditional distribution of \(\eta\) given \(\xi\) is symmetric if and only if the random vectors \((\xi, \eta)\) and \((\xi, -\eta)\) are identically distributed.

According to the classical Skitovich–Darmois theorem the independence of two linear forms of independent random variables is a characteristic property of the Gaussian distribution on the real line. Another important characterization result was proved by C.C. Heyde, where instead of the independence the symmetry of the conditional distribution of one linear form given another is considered ([19], see also [20, § 13.4.1]). For two independent random variables the Heyde theorem can be formulated as follows.

**Theorem A**  Let \(\xi_1\) and \(\xi_2\) be independent real random variables with distributions \(\mu_1\) and \(\mu_2\). Let \(\alpha_j, \beta_j\) be non zero real numbers such that \(\alpha_1\beta_1^{-1} + \alpha_2\beta_2^{-1} \neq 0\). If the conditional distribution of the linear form \(L_2 = \beta_1\xi_1 + \beta_2\xi_2\) given \(L_1 = \alpha_1\xi_1 + \alpha_2\xi_2\) is symmetric, then \(\mu_j\) are Gaussian distributions.

Some generalisations of the Heyde theorem, where independent random variables take values in a locally compact Abelian group \(X\), and coefficients of linear forms are topological automorphisms of \(X\), were studied in [4, 6, 8, 11, 21, 23], see also [7, Chapter VI]. We continue this research. The main result of the article is a complete description of compact totally disconnected Abelian groups for which a natural analogue of the Heyde theorem for two independent random variables holds true. It is well known that Gaussian distributions on a locally compact totally disconnected Abelian group \(X\) are
Define the distribution \( \hat{\mu} \) the characteristic function (Fourier transform) of the distribution \( \mu \in \mathbb{M} \) to a non-negative power of \( \Delta \) x in the discrete topology. We will always consider a finite Abelian group in the discrete topology. Denote by \( \mathcal{G} \) the class of Fourier transforms of probability distributions.

Let \( \{G_i : i \in I\} \) be a nonempty set of compact Abelian groups. Denote by \( \prod_i G_i \) the direct product of the groups \( G_i \), equipped with the product topology. If \( \{H_i : i \in I\} \) is a nonempty set of discrete Abelian groups, then denote by \( \prod_i H_i \) the weak direct product of the groups \( H_i \), equipped with the discrete topology. We will always consider a finite Abelian group in the discrete topology. Denote by \( \mathbb{Z}(n) = \{0, 1, \ldots, n - 1\} \) the group of residue classes modulo \( n \). Let \( p \) be a prime number. Denote by \( \Delta_p \) the group of \( p \)-adic integers. A \( p \)-group is defined to be a group whose elements have order equal to a non-negative power of \( p \). In particular, for power 0, it is possible that \( X_p = \{0\} \). Denote by \( \mathcal{P} \) the set of prime numbers.

Denote by \( \mathbb{M}^1(X) \) the convolution semigroup of probability distributions on the group \( X \). Let \( \mu \in \mathbb{M}^1(X) \). Denote by

\[
\hat{\mu}(y) = \int_X (x, y) d\mu(x), \quad y \in Y,
\]

the characteristic function (Fourier transform) of the distribution \( \mu \), and by \( \sigma(\mu) \) the support of \( \mu \). Define the distribution \( \tilde{\mu} \in \mathbb{M}^1(X) \) by the formula \( \tilde{\mu}(B) = \mu(-B) \) for any Borel subset \( B \) of \( X \). Then \( \hat{\mu}(y) = \overline{\hat{\mu}(y)} \).

Denote by \( m_K \) the Haar distribution of a compact subgroup \( K \) of the group \( X \), and by \( I(X) \) the set of shifts of Haar distributions \( m_K \) of compact subgroups \( K \) of the group \( X \). We note that the characteristic function of a distribution \( m_K \) is of the form

\[
\hat{m}_K(y) = \begin{cases} 1, & \text{if } y \in A(Y, K), \\ 0, & \text{if } y \notin A(Y, K). \end{cases}
\]

A distribution \( \gamma \in \mathbb{M}^1(X) \) is called Gaussian (Chapter IV, §6), see also [2] if its characteristic function is represented in the form

\[
\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\},
\]

where \( x \in X \), and \( \varphi(y) \) is a continuous nonnegative function on the group \( Y \) satisfying the equation

\[
\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.
\]
Denote by $E_x$ the degenerate distribution concentrated at an element $x \in X$. Observe that degenerate distributions are Gaussian. Denote by $\Gamma(X)$ the set of Gaussian distributions on $X$.

2 The Heyde theorem on compact totally disconnected Abelian groups

Let $\xi_j$, $j = 1, 2$, be independent random variables with values in a locally compact Abelian group $X$ and distributions $\mu_j$, and let $\alpha_j, \beta_j \in \text{Aut}(X)$. Assume that the conditional distribution of the linear form $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ given $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ is symmetric. Obviously, if we are interested in the description of $\mu_j$, then we can suppose without loss of generality that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$, where $\alpha \in \text{Aut}(X)$.

Denote by $G$ the subgroup of $X$ generated by all elements of $X$ of order 2. Then $\alpha(G) = G$ and $g = -g$ for all $g \in G$. Assume that $\xi_1$ and $\xi_2$ take values in $G$. Taking into account that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if the random vectors $(L_1, L_2)$ and $(L_1, -L_2)$ are identically distributed, we see that for any $\mu_j$ and any topological automorphism $\alpha$ the conditional distribution of $L_2$ given $L_1$ is symmetric.

Put $K = \text{Ker}(I + \alpha)$ and suppose that $K \neq \{0\}$. Then $\alpha(k) = -k$ for all $k \in K$. Assume that $\xi_1$ and $\xi_2$ are arbitrary independent identically distributed random variables with values in $K$. Consider the linear form $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$. Then $L_2 = \xi_1 - \xi_2$, and the characteristic functions of the random vectors $(L_1, L_2)$ and $(L_1, -L_2)$ are equal. Hence the random vectors $(L_1, L_2)$ and $(L_1, -L_2)$ are identically distributed. So, the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.

Let $X$ be a compact totally disconnected Abelian group. Our goal is to describe all groups $X$ for which shifts of the Haar distributions of compact subgroups of $X$ are characterized by the symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$. From what has been said above it follows that we should suppose that $X$ contains no elements of order 2, and condition

$$\text{Ker}(I + \alpha) = \{0\}$$

is satisfied.

The main result of this section is the following theorem.

**Theorem 2.1** Let $X$ be a compact totally disconnected Abelian group containing no elements of order 2, and let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. The symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ implies that $\mu_j \in I(X)$, $j = 1, 2$, if and only if the group $X$ is topologically isomorphic to a group of the form

$$P_{p \in P, \, p > 2} X_p,$$

where $X_p$ is a finite $p$-group.

Since shifts of the Haar distributions of compact subgroups of the group $X$ play the role of Gaussian distributions on $X$, Theorem 2.1 gives a complete description of compact totally disconnected Abelian groups where an analogue of the Heyde theorem for two independent random variables holds. We note that an analogue of the Heyde theorem for two independent random variables is valid on an arbitrary discrete Abelian group (see [12] Theorem 1).

To prove Theorem 2.1 we need some lemmas.
Define a continuous homomorphism 

\[ \tilde{\alpha} \quad \text{easy to verify that the adjoint homomorphism} \]

\[ \alpha \quad \text{easy to see that} \]

\[ \xi \quad \text{We denote by} \]

\[ \alpha \quad \text{Then there exist a topological automorphism} \]

\[ \alpha \quad \text{conditional distribution of the linear form} \]

\[ \alpha \quad \text{Lemma 2.5 (10, Theorem 2)} \]

\[ \alpha \quad \text{Let} \]

\[ \alpha \quad \text{Let} \]

\[ \alpha \quad \text{Let} \]

\[ \alpha \quad \text{Lemma 2.3 ([4], see also [7, Corollary 17.2 and Remark 17.5])} \]

\[ \alpha \quad \text{Lemma 2.4 ([10, Theorem 2}) \]

\[ \alpha \quad \text{Let} \]

\[ \alpha \quad \text{Lemma 2.5} \]

\[ \alpha \quad \text{Let} \]

\[ \alpha \quad \text{Lemma 2.2 ([7, Lemma 16.1])} \] Let \( X \) be a locally compact Abelian group, and let \( \alpha \) be a topological automorphism of the group \( X \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). The conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric if and only if the characteristic functions \( \hat{\mu}_j(y) \) satisfy the equation

\[ \hat{\mu}_1(u + v)\hat{\mu}_2(u + \tilde{\alpha}v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - \tilde{\alpha}v), \quad u, v \in Y. \quad (4) \]

\[ \alpha \quad \text{Lemma 2.3 ([4], see also [7, Corollary 17.2 and Remark 17.5])} \] Let \( X \) be a finite Abelian group containing no elements of order 2. Let \( \alpha \) be an automorphism of the group \( X \) satisfying condition \( (2) \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = m_K * E_{x_j} \), where \( K \) is a subgroup of \( X \), \( x_j \in X \), \( j = 1, 2 \). Moreover, \( \alpha(K) = K \).

\[ \alpha \quad \text{Lemma 2.4 ([10, Theorem 2])} \] Let \( X = \Delta_p \) be the group of \( p \)-adic integers. Then there exist a topological automorphism \( \alpha \) of the group \( X \) satisfying condition \( (2) \), and independent identically distributed random variables \( \xi_1 \) and \( \xi_2 \) with values in \( X \) and distribution \( \mu \) such that the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, while \( \mu_j \notin I(X) \), \( j = 1, 2 \).

The following lemma plays a key role in the proof of Theorem 2.1.

\[ \alpha \quad \text{Lemma 2.5} \] Let \( p \) be a prime number, and let \( p > 2 \). Consider the group

\[ \alpha \quad X = \prod_{n=1}^{\infty} \mathbb{Z}(p^{k_n}), \quad k_n \leq k_{n+1}, \quad n = 1, 2, \ldots \quad (5) \]

Then there exist a topological automorphism \( \alpha \) of the group \( X \) satisfying condition \( (2) \), and independent identically distributed random variables \( \xi_1 \) and \( \xi_2 \) with values in \( X \) and distribution \( \mu \) such that the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, while \( \mu \notin I(X) \).

**Proof** The group \( Y \) is topologically isomorphic to the weak direct product of the groups \( \mathbb{Z}(p^{k_n}) \). In order not to complicate the notation we assume that

\[ \alpha \quad Y = \prod_{n=1}^{\infty} \mathbb{Z}(p^{k_n}). \quad (6) \]

We denote by \( t = \{t_n\}_{n=1}^{\infty}, t_n \in \mathbb{Z}(p^{k_n}) \), elements of the group \( X \) and by \( s = \{s_n\}_{n=1}^{\infty}, s_n \in \mathbb{Z}(p^{k_n}) \), elements of the group \( Y \). Let \( s = \{s_n\}_{n=1}^{\infty} \in Y \). We say that \( s \in \mathbb{Z}(p^{k_i}) \) if \( s_2 = s_3 = \cdots = 0 \).

Let \( i \leq j \). By \( \alpha_{i,j} \) denote the homomorphism \( \alpha_{i,j} : \mathbb{Z}(p^{k_i}) \to \mathbb{Z}(p^{k_j}) \) of the form

\[ \alpha_{i,j}t_i = p^{k_j-k_i}t_i, \quad t_i \in \mathbb{Z}(p^{k_i}). \]

It is easy to verify that the adjoint homomorphism \( \tilde{\alpha}_{i,j} : \mathbb{Z}(p^{k_j}) \to \mathbb{Z}(p^{k_i}) \) is of the form

\[ \tilde{\alpha}_{i,j}s_j = s_j(\text{mod} \ p^{k_i}), \quad s_j \in \mathbb{Z}(p^{k_j}). \]

Define a continuous homomorphism \( \alpha : X \to X \) by the formula \( \alpha\{t_n\}_{n=1}^{\infty} = \{h_n\}_{n=1}^{\infty}, \) where

\[ h_n = \begin{cases} 
(p^{k_1} - t_1)(\text{mod} \ p^{k_1}), & \text{if } n = 1, \\
(\alpha_{n-1,n}t_{n-1} + p^{k_n} - t_n)(\text{mod} \ p^{k_n}), & \text{if } n \geq 2.
\end{cases} \quad (7) \]

It is easy to see that \( \alpha \) is a topological automorphism of the group \( X \) satisfying condition \( (2) \). We find
from (7) that the adjoint automorphism \( \bar{\alpha} : Y \to Y \) is of the form \( \bar{\alpha}\{s_n\}_{n=1}^\infty = \{u_n\}_{n=1}^\infty \), where
\[
\begin{align*}
u_n = (\bar{\alpha}_{n,n+1}s_{n+1} + p^{k_n} - s_n)(\text{mod } p^{k_n}), \quad n \geq 1.
\end{align*}
\]

It follows from (5) and (6) that \( X = \mathbb{Z}(p^{k_1}) \times G \), where \( G = \prod_{n=2}^\infty \mathbb{Z}(p^{k_n}) \) and \( Y = \mathbb{Z}(p^{k_1}) \times H \), where \( H = \prod_{n=2}^\infty \mathbb{Z}(p^{k_n}) \). Obviously, \( A(Y,G) = \mathbb{Z}(p^{k_1}) \).

Take \( 0 < a < 1 \). Consider on the group \( X \) the distribution \( \mu = am_X + (1-a)m_G \). It follows from (11) that the characteristic function \( \hat{\mu}(y) \) is of the form
\[
\hat{\mu}(y) = \begin{cases}
1, & \text{if } y = 0, \\
1 - a, & \text{if } y \in \mathbb{Z}(p^{k_1}) \setminus \{0\}, \\
0, & \text{if } y \notin \mathbb{Z}(p^{k_1}).
\end{cases}
\]

Let \( \xi_1 \) and \( \xi_2 \) be independent identically distributed random variables with values in \( X \) and distribution \( \mu \). Let us verify that the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha\xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric. By Lemma 2.2, it suffices to verify that the characteristic function \( \hat{\mu}(y) \) satisfies the equation
\[
\hat{\mu}(u+v)\hat{\mu}(u + \bar{\alpha}v) = \hat{\mu}(u-v)\hat{\mu}(u - \bar{\alpha}v), \quad u, v \in Y.
\]

Let \( u, v \in Y \). Consider 3 cases.

1. \( u, v \in \mathbb{Z}(p^{k_1}) \). It follows from (8) that \( \bar{\alpha}y = -y \) when \( y \in \mathbb{Z}(p^{k_1}) \). Hence, the restriction of equation (10) to the subgroup \( \mathbb{Z}(p^{k_1}) \) takes the form
\[
\hat{\mu}(u+v)\hat{\mu}(u - v) = \hat{\mu}(u-v)\hat{\mu}(u + v), \quad u, v \in \mathbb{Z}(p^{k_1}).
\]

It is obvious that (10) is fulfilled.

2. Either \( u \in \mathbb{Z}(p^{k_1}), v \notin \mathbb{Z}(p^{k_1}) \) or \( u \notin \mathbb{Z}(p^{k_1}), v \in \mathbb{Z}(p^{k_1}) \). Then \( u \pm v \notin \mathbb{Z}(p^{k_1}) \). It follows from (9) that \( \hat{\mu}(u \pm v) = 0 \). Then both sides of equation (10) are equal to zero.

3. \( u, v \notin \mathbb{Z}(p^{k_1}) \). Assume that the left-hand side in (10) is not equal to zero. Then (9) implies that \( u + v, u + \bar{\alpha}v \in \mathbb{Z}(p^{k_1}) \). It follows from this that
\[
(I - \bar{\alpha})v \in \mathbb{Z}(p^{k_1}).
\]

Let \( v = \{v_n\}_{n=1}^\infty, v_n \in \mathbb{Z}(p^{k_n}) \). We find from (8) and (11) that
\[
(2v_n + p^{k_n} - \bar{\alpha}_{n,n+1}v_{n+1})(\text{mod } p^{k_n}) = 0, \quad n \geq 2.
\]

It follows from \( p > 2 \) that if \( v_n \neq 0 \), then \( 2v_n(\text{mod } p^{k_n}) \neq 0, n \geq 1 \). Assume that \( v_2 \neq 0 \). Then we find from (12) that \( v_n \neq 0 \) for each \( n \geq 2 \). But this contradicts the fact that \( v_n = 0 \) for all but a finite set of indices \( n \). Hence, \( v_2 = 0 \). Reasoning by induction we prove that \( v_3 = 0 \), then \( v_4 = 0 \), and so on. Thus, we got that \( v = \{v_n\}_{n=1}^\infty \in \mathbb{Z}(p^{k_1}) \), contrary to assumption. The obtained contradiction shows that the left-hand side of (10) is equal to zero. Similarly, we prove that when \( u, v \notin \mathbb{Z}(p^{k_1}) \) the right-hand side of (10) is also equal to zero. So, both sides of equation (10) are equal to zero. 

The following lemma follows directly from Lemma 13.24 in [7].

**Lemma 2.6** Let \( X \) be a compact totally disconnected Abelian group containing no elements of order 2. Then either \( X \) is topologically isomorphic to a group of the form (3) or for some prime number \( p \) there exists a compact subgroup \( K \) of \( X \) such that \( K \) is a topological direct factor of the group \( X \), and \( K \) is topologically isomorphic to either the group of \( p \)-adic integers \( \Delta_p \) or a group of the form (5).
Proof of Theorem 2.7 Necessity. Assume that a compact totally disconnected Abelian group $X$ contains no elements of order 2, and $X$ is not topologically isomorphic to a group of the form $(3)$. By Lemma 2.6 for some prime number $p$, $p > 2$, there exists a compact subgroup $K$ of the group $X$ such that $K$ is a topological direct factor of $X$, and $K$ is topologically isomorphic to either a group of $p$-adic integers $\Delta_p$ or a group of the form $(5)$. We have $X = K \times G$. Denote elements of the group $X$ by $x = (k, g)$, where $k \in K$, $g \in G$. Applying either Lemma 2.4 or Lemma 2.5 we find a topological automorphism $\alpha_K$ of the group $K$ satisfying condition (2), and independent random variables $\xi_1$ and $\xi_2$ with values in $K$ and distributions $\mu_1$ and $\mu_2$ such that the conditional distribution of the linear form $M_2 = \xi_1 + \alpha_K \xi_2$ given $M_1 = \xi_1 + \xi_2$ is symmetric, whereas $\mu_j \notin I(K)$, $j = 1, 2$. Put $\alpha(k, g) = (\alpha_K k, g)$. Since the group $X$ contains no elements of order 2, $\alpha$ is a topological automorphism of $X$ satisfying condition (2). Obviously, if we consider $\xi_j$ as independent random variables with values in the group $X$, then the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, whereas $\mu_j \notin I(X)$, $j = 1, 2$. Necessity is proved.

Sufficiency. Assume that a compact totally disconnected Abelian group $X$ is topologically isomorphic to a group of the form $(3)$, and let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. By Lemma 2.2 the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ implies that the characteristic functions $\hat{\nu}_j(y)$ satisfy equation (4). Put $\nu_j = \mu_j * \bar{\mu}_j$. Then $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$ for all $y \in Y$, and the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (4).

It follows from (4) that
\begin{equation}
Y = \prod_{p \in \mathbb{P}, p > 2} Y_p,
\end{equation}
where the group $Y_p$ is isomorphic to the character group of the group $X_p$. Put
\begin{equation}
A_m = \prod_{p \in \mathbb{P}, 2 < p \leq m} X_p, \quad B_m = \prod_{p \in \mathbb{P}, 2 < p \leq m} Y_p, \quad m = 3, 4, \ldots
\end{equation}
Then $A_m$ and $B_m$ are finite Abelian groups, $B_m$ is isomorphic to the character group of the group $A_m$, $B_m \subset B_{m+1}$ and $Y = \bigcup_{m=3}^{\infty} B_m$. Denote by $\alpha_m$ the restriction of $\alpha$ to the subgroup $A_m$. It is obvious that $\alpha_m \in \text{Aut}(A_m)$ and $\alpha_m$ satisfies the condition (2). Note that the restriction of $\bar{\alpha}$ to subgroup $B_m$ coincides with $\bar{\alpha}_m$. Consider the restriction of equation (4) for the characteristic functions $\hat{\nu}_j(y)$ to the subgroup $B_m$. We have
\begin{equation}
\hat{\nu}_1(u + v) \hat{\nu}_2(u + \bar{\alpha}_m v) = \hat{\nu}_1(u - v) \hat{\nu}_2(u - \bar{\alpha}_m v), \quad u, v \in B_m.
\end{equation}
Taking into account Lemma 2.2 and applying Lemma 2.3 to the finite group $A_m$, we obtain from (15) that $\hat{\nu}_1(y) = \hat{\nu}_2(y)$ for all $y \in B_m$, and the characteristic functions $\hat{\nu}_j(y)$ take only values 0 and 1 on $B_m$. This implies that $\hat{\nu}_1(y) = \hat{\nu}_2(y)$ for all $y \in Y$, and the characteristic functions $\hat{\nu}_j(y)$ take only values 0 and 1 on $Y$. Put $E = \{y \in Y : \hat{\nu}_j(y) = 1\}$. Then $E$ is a subgroup of $Y$. Put $K = A(Y, E)$. Since $E = A(Y, K)$, it follows from (1) that $\nu_1 = \nu_2 = mK$. This easily implies that $\mu_j = mK * E_{x_j}$, where $x_j \in X$, $j = 1, 2$. Sufficiency is proved.

In fact, we have proved somewhat more, namely $\mu_j$ are shifts of the same Haar distribution of a compact subgroup $K$ of the group $X$. Furthermore, we can assert that $\alpha(K) = K$. Indeed, put $E_m = E \cap B_m$. It follows from Lemma 2.3 that $\bar{\alpha}_m(E_m) = E_m$. Hence, $\bar{\alpha}(E) = E$, so that $\alpha(K) = K$.

Remark 2.7 Let us compare Theorem 2.1 with the Skitovich–Darmois theorem for compact totally disconnected Abelian groups, which has the following statement. It can be deduced from Theorem 1 in [13], see also [7] Theorem 13.25.
Theorem Let $X$ be a compact totally disconnected Abelian group containing no elements of order 2, and let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. The independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$ implies that $\mu_j \in I(X)$, $j = 1, 2$, if and only if the group $X$ is topologically isomorphic to a group of the form
\[ P_p (\Delta_{n_p} \times X_p), \]
where $n_p$ is a nonnegative integer, and $X_p$ is a finite $p$-group, $X_2 = \{0\}$.

We see that the class of compact totally disconnected Abelian groups, for which the Skitovich–Darmois theorem is true, is wider than the class of compact totally disconnected Abelian groups for which the Heyde theorem is valid.

At the same time, we note that both the Skitovich–Darmois theorem and the Heyde theorem are valid for discrete Abelian groups containing no elements of order 2.

We supplement Theorem [2.1] with the following statements.

Proposition 2.8 Let $X$ be a compact totally disconnected Abelian group of the form (3), let $K$ be a compact subgroup of $X$, and let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in $X$ and distribution $m_K$. Then the following assertions are equivalent:

(i) the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric;

(ii) $(I - \alpha)(K) = K$.

Proof (i) $\Rightarrow$ (ii). Note that every closed subgroup of a group of the form (3) is topologically isomorphic to a group of the form (4). By Theorem [2.1] $\alpha(K) = K$, i.e. the restriction of $\alpha$ to the subgroup $K$ is a topological automorphism of $K$. Thus, we can prove the statement assuming that $K = X$. We note that $(I - \alpha)(X) = X$ if and only if $\text{Ker}(I - \alpha) = \{0\}$. Denote by $\tilde{\alpha}_m$ the same topological automorphism as in the proof of sufficiency in Theorem [2.1]. We have $\text{Ker}(I - \tilde{\alpha}) = \{0\}$ if and only if $\text{Ker}(I - \tilde{\alpha}_m) = \{0\}$ for $m = 3, 4, \ldots$. Taking into account Lemma [2.2] the statement $\text{Ker}(I - \tilde{\alpha}_m) = \{0\}$ follows from the fact that the proposition is valid in the case when $X$ is a finite Abelian group containing no elements of order 2 ([12] Proposition 1)).

(ii) $\Rightarrow$ (i). This statement holds for an arbitrary locally compact Abelian group $X$ and a compact subgroup $K$ of $X$ ([12] Proposition 1]).

Proposition 2.9 Let $X$ be a compact totally disconnected Abelian group of the form
\[ X = P_p X_p, \]
where $X_p$ is a finite $p$-group. Put
\[ G = P_{p \in P, p \geq 2} X_p. \]

Let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \rho_j * m_K * E_{g_j}$, where $\rho_j$ are some distributions on $X_2$, $K$ is a compact subgroup of $G$, $g_j \in G$, $j = 1, 2$.

To prove Proposition [2.9] we need the following lemma which is a special case of Theorem 3 proved in [12].
Lemma 2.10 Let $X$ be a finite Abelian group, let $X_2$ be its 2-component, and let $G$ be the subgroup of $X$ generated by all elements of odd order. Let $\alpha$ be an automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \rho_j * m_K * E_{g_j}$, where $\sigma(\rho_j) \subset X_2$, $K$ is a subgroup of $G$, $g_j \in G$, $j = 1, 2$.

Proof of Proposition 2.9 We argue as in the proof of sufficiency in Theorem 2.1. We have $X = X_2 \times G$. Then $Y = L \times H$, where $L$ and $H$ are groups topologically isomorphic to the character groups of the groups $X_2$ and $G$ respectively. Denote by $y = (l, h)$, where $l \in L$, $h \in H$, elements of the group $Y$. Taking into account Lemma 2.2 and Theorem 2.1, we have that the restriction of $\alpha$ to the subgroup $G$ is a topological automorphism of $G$ satisfying condition (2), it is easy to verify that Proposition 2.9 will be proved if we prove that there exist functions $a_j(l)$, $l \in L$, and $b_j(h)$, $h \in H$, such that

$$\hat{\mu}_j(l, h) = a_j(l)b_j(h), \quad l \in L, \quad h \in H, \quad j = 1, 2.$$  

This implies that $a_j(l)$ are the characteristic functions of some distributions $\rho_j$ on $X_2$ and $b_j(h)$ are the characteristic functions of some shifts of a distribution $m_K$, where $K$ is a subgroup of $G$.

We have $Y = \prod_{p \in P} Y_p$, where the group $Y_p$ is isomorphic to the character group of the group $X_p$. Define the groups $A_m$ and $B_m$ by (14). Then $A_m$ and $B_m$ are finite Abelian groups, the group $L \times B_m$ is isomorphic to the character group of the group $X_2 \times A_m$, $B_m \subset B_{m+1}$ and $Y = L \times \bigcup_{m=3}^\infty B_m$. Denote by $\alpha_m$ the restriction of $\alpha$ to the subgroup $X_2 \times A_m$. It is obvious that $\alpha_m \in \text{Aut}(X_2 \times A_m)$ and $\alpha_m$ satisfies condition (2). Note that the restriction of $\bar{\alpha}$ to the subgroup $L \times B_m$ coincides with $\bar{\alpha}_m$. By Lemma 2.2, the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ implies that the characteristic functions $\hat{\mu}_j(l, h)$ satisfy equation (13). Consider the restriction of equation (13) to the subgroup $L \times B_m$. We have

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + \tilde{\alpha}_m v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - \tilde{\alpha}_m v), \quad u, v \in L \times B_m.$$  

Taking into account Lemma 2.2 and applying Lemma 2.10 to the finite group $X_2 \times A_m$, we obtain from (16) that there exist functions $a_j(l)$, $l \in L$, and $b_j(m)(h)$, $h \in B_m$ such that $\hat{\mu}_j(l, h) = a_j(l)b_j(m)(h)$ for all $(l, h) \in L \times B_m$. Moreover, $b_{j,m+1}(h) = b_{j,m}(h)$ for all $h \in B_m$. Put $b_j(h) = b_{j,m}(h)$ for all $h \in B_m$. Then $\hat{\mu}_j(l, h) = a_j(l)b_j(h)$ for all $(l, h) \in Y$, $j = 1, 2$.

3 A generalization of Theorem 2.1

The following theorem has been proved in [12] Theorem 2.1.

Theorem B Let $X = \mathbb{R}^n \times D$, where $n \geq 0$, and $D$ is a discrete Abelian group containing no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \gamma_j * m_K * E_{x_j}$, where $\gamma_j \in \Gamma(\mathbb{R}^n)$, $K$ is a finite subgroup of $D$, $x_j \in D$, $j = 1, 2$. Moreover, $\alpha(K) = K$.

In this section we shall prove a theorem which generalizes both Theorem 2.1 and Theorem B and describe a wide class of locally compact Abelian groups where an analogue of the Heyde theorem holds.

Theorem 3.1 Let

$$X = \mathbb{R}^n \times D \times G,$$  

(17)
where \( n \geq 0 \), \( D \) is a discrete Abelian group containing no elements of order 2, and \( G \) is a compact totally disconnected Abelian group of the form \((\mathbb{R}^n, \mathbb{Z}^n)\). Let \( \alpha \) be a topological automorphism of the group \( X \) satisfying condition \((\mathbb{R}^n, \mathbb{Z}^n)\). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = \gamma_j * m_K * E_{x_j} \), where \( \gamma_j \in \Gamma(\mathbb{R}^n) \), \( K \) is a compact subgroup of \( X \), \( x_j \in X \), \( j = 1, 2 \). Moreover, \( \alpha(K) = K \).

To prove Theorem 3.1 we need two lemmas. One of them is the following well-known statement (see e.g. \[7, Proposition 2.13\]).

**Lemma 3.2** Let \( X \) be a locally compact Abelian group, and let \( \mu \in M^1(X) \). Then the set \( E = \{ y \in Y : \hat{\mu}(y) = 1 \} \) is a closed subgroup of the group \( Y \), and \( \sigma(\mu) \subset A(X, E) \).

The following lemma generalizes Theorem 2.1 and plays a key role in the proof of Theorem 3.1.

**Lemma 3.3** Let \( X = \mathbb{R}^n \times G \), where \( n \geq 0 \), and \( G \) is a compact totally disconnected Abelian group of the form \((\mathbb{R}^n, \mathbb{Z}^n)\). Let \( \alpha \) be a topological automorphism of the group \( X \) satisfying condition \((\mathbb{R}^n, \mathbb{Z}^n)\). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = \gamma_j * m_K * E_{x_j} \), where \( \gamma_j \in \Gamma(\mathbb{R}^n) \), \( K \) is a compact subgroup of \( G \), \( x_j \in G \), \( j = 1, 2 \). Moreover, \( \alpha(K) = K \).

**Proof** Denote elements of the group \( X \) by \( x = (t, g) \), where \( t \in \mathbb{R}^n \), \( g \in G \). We have \( Y = \mathbb{R}^n \times H \), where \( H \) is a group topologically isomorphic to the character group of the group \( G \). Denote elements of the group \( Y \) by \( y = (s, h) \), where \( s \in \mathbb{R}^n \), \( h \in H \). Since \( \mathbb{R}^n \) is the connected component of zero of the group \( X \), we have \( \alpha(\mathbb{R}^n) = \mathbb{R}^n \). Since \( G \) is the subgroup of \( X \) consisting of all compact elements of the group \( X \), we have \( \alpha(G) = G \). This implies that the restriction of \( \alpha \) to each of the subgroups \( \mathbb{R}^n \) and \( G \) is a topological automorphism of the corresponding subgroup. Denote by \( \alpha_{\mathbb{R}^n} \) and \( \alpha_G \) these restrictions respectively. This means that \( \alpha \) can be written in the form \( \alpha(t, g) = (\alpha_{\mathbb{R}^n} t, \alpha_G g) \), \((t, g) \in X \). Note that the restrictions of \( \alpha \) to the subgroups \( \mathbb{R}^n \) and \( H \) coincide with \( \alpha_{\mathbb{R}^n} \) and \( \alpha_G \) respectively. Hence, \( \alpha \) can be written in the form \( \alpha(s, h) = (\alpha_{\mathbb{R}^n} s, \alpha_G h) \), \((s, h) \in Y \).

By Lemma 2.2 the symmetry of the conditional distribution of the linear form \( L_2 \) given \( L_1 \) implies that the characteristic functions \( \hat{\mu}_j(s, h) \) satisfy equation \((\mathbb{R}^n, \mathbb{Z}^n)\) which takes the form

\[
\hat{\mu}_1(s_1 + s_2, h_1 + h_2) = \hat{\mu}_1(s_1 - s_2, h_1 - h_2) \hat{\mu}_2(s_1 + \alpha_{\mathbb{R}^n} s_2, h_1 + \alpha_G h_2), \quad s_j, h_j \in \mathbb{R}^n, \quad j = 1, 2.
\]

(18)

It is obvious that the topological automorphism \( \alpha_{\mathbb{R}^n} \) satisfies condition \((\mathbb{R}^n, \mathbb{Z}^n)\). Substituting \( h_1 = h_2 = 0 \) in \((\mathbb{R}^n, \mathbb{Z}^n)\), taking into account Lemma 2.2 and Theorem B for the group \( X = \mathbb{R}^n \), we obtain from the resulting equation that

\[
\hat{\mu}_j(s, 0) = \exp\{i(t_j, s) - (A_j s, s)\}, \quad s \in \mathbb{R}^n, \quad j = 1, 2,
\]

(19)

where \( t_j \in \mathbb{R}^n \), \((.,.)\) is the scalar product, and \( A_j \) are symmetric positive semi-definite \( n \times n \)-matrices. Similarly, substituting \( s_1 = s_2 = 0 \) in \((\mathbb{R}^n, \mathbb{Z}^n)\), taking into account Lemma 2.2 and applying Theorem 2.1 to the group \( X = G \), we obtain from the resulting equation that

\[
\hat{\mu}_j(0, h) = \hat{m}_K(h)(g_j, h), \quad h \in H, \quad j = 1, 2,
\]

(20)

where \( K \) is a compact subgroup in \( G \), and \( g_j \in G \). Moreover, \( \alpha_G(K) = K \).
Substituting (20) into (3) and taking into account (1), we find that \(2(g_1 + \alpha_G g_2) \in K\). Since \(G\) is a group of the form (3) and \(K\) is a compact subgroup of \(G\), multiplication by 2 is a topological automorphism of \(K\). Hence,

\[
g_1 + \alpha_G g_2 \in K. \tag{21}
\]

Put \(x_1 = -\alpha_G g_2, x_2 = g_2\). Then \(x_1 + \alpha_G x_2 = 0\). It follows from (1), (20), and (21) that

\[
\hat{\mu}_j(0, h) = \hat{m}_K(h)(x_j, h), \quad h \in H, \quad j = 1, 2. \tag{22}
\]

Consider new independent random variables \(\eta_j = \xi_j - x_j\), and denote by \(\lambda_j\) the distribution of the random variable \(\eta_j, j = 1, 2\). Since

\[
\lambda_j = \mu_j * E_{-x_j}, \quad j = 1, 2, \tag{23}
\]

and \(x_1 + \alpha x_2 = 0\), the characteristic functions \(\hat{\lambda}_j(y)\) also satisfy equation (1). By Lemma 2.22, the conditional distribution of the linear form \(N_2 = \eta_1 + \alpha \eta_2\) given \(N_1 = \eta_1 + \eta_2\) is symmetric. It follows from (1), (22), and (23) that \(\hat{\lambda}_j(0, h) = 1\) for all \(h \in A(H, K)\). By Lemma 3.2, this implies that \(\sigma(\lambda_j) \subset A(X, A(H, K)) = \mathbb{R}^n \times K\). Since \(\alpha(\mathbb{R}^n \times K) = \mathbb{R}^n \times K\), we can assume that the independent random variables \(\eta_j\) take values in the group \(X = \mathbb{R}^n \times K\). Then \(Y = \mathbb{R}^n \times L\), where \(L\) is a group topologically isomorphic to the character group of the group \(K\). Denote elements of the group \(Y\) by \(y = (s, l)\), where \(s \in \mathbb{R}^n, l \in L\). In so doing,

\[
\hat{\lambda}_j(0, l) = \begin{cases} 1, & \text{if } l = 0, \\ 0, & \text{if } l \neq 0, \end{cases} \quad j = 1, 2. \tag{24}
\]

Denote by \(\alpha_K\) the restriction of \(\alpha\) to the subgroup \(K\) and consider equation (1) for the characteristic functions \(\hat{\lambda}_j(s, l)\) on the group \(Y = \mathbb{R}^n \times L\). We have

\[
\hat{\lambda}_1(s_1 + s_2, l_1 + l_2) = \hat{\lambda}_2(s_1 + \tilde{\alpha}_{\mathbb{R}^n} s_2, l_1 + \tilde{\alpha}_K l_2)
\]

\[
= \hat{\lambda}_1(s_1 - s_2, l_1 - l_2) \hat{\lambda}_2((s_1 - \tilde{\alpha}_{\mathbb{R}^n} s_2, l_1 - \tilde{\alpha}_K l_2), \quad s_j \in \mathbb{R}^n, \quad l_j \in L. \tag{25}
\]

Substitute \(s_1 = s_2 = s, l_1 = l, l_2 = -l\) in (3). We obtain

\[
\hat{\lambda}_1(2s, 0) \hat{\lambda}_2((I + \tilde{\alpha}_{\mathbb{R}^n}) s, (I - \tilde{\alpha}_K) l) = \hat{\lambda}_1(0, 2l) \hat{\lambda}_2((I - \tilde{\alpha}_{\mathbb{R}^n}) s, (I + \tilde{\alpha}_K) l), \quad s \in \mathbb{R}^n, \quad l \in L. \tag{26}
\]

Assume that \(l \neq 0\). Then \(2l \neq 0\), and (24) implies that \(\hat{\lambda}_1(0, 2l) = 0\). Hence, it follows from (20) that \(\hat{\lambda}_1(2s, 0) \hat{\lambda}_2((I + \tilde{\alpha}_{\mathbb{R}^n}) s, (I - \tilde{\alpha}_K) l) = 0\). Taking into account (19), this implies that

\[
\hat{\lambda}_2((I + \tilde{\alpha}_{\mathbb{R}^n}) s, (I - \tilde{\alpha}_K) l) = 0.
\]

Since \(\alpha_{\mathbb{R}^n}\) satisfies condition (2) and \(\alpha(\mathbb{R}^n) = \mathbb{R}^n\), we have \(I + \alpha_{\mathbb{R}^n} \in \text{Aut}(\mathbb{R}^n)\). Hence, \(I + \tilde{\alpha}_{\mathbb{R}^n} \in \text{Aut}(\mathbb{R}^n)\) and

\[
\hat{\lambda}_2(s, (I - \tilde{\alpha}_K) l) = 0, \quad s \in \mathbb{R}^n, \quad l \in L, \quad l \neq 0. \tag{27}
\]

In particular,

\[
\hat{\lambda}_2(0, (I - \tilde{\alpha}_K) l) = 0, \quad l \in L, \quad l \neq 0. \tag{28}
\]

It follows from (24) and (28) that

\[
\text{Ker}(I - \tilde{\alpha}_K) = \{0\}. \tag{29}
\]

Since \(K\) is a compact subgroup of the group \(G\), the group \(K\) is topologically isomorphic to a compact totally disconnected Abelian group of the form (3). This implies that the group \(L\) is topologically
isomorphic to a group of the form \((13)\). Then, it follows from \((29)\) that \(I - \tilde{\alpha}_K \in \text{Aut}(L)\). Hence, we find from \((27)\) that
\[
\hat{\lambda}_2(s, l) = 0, \quad s \in \mathbb{R}^n, \quad l \in L, \quad l \neq 0.
\]  
(30)

Similarly, we obtain
\[
\hat{\lambda}_1(s, l) = 0, \quad s \in \mathbb{R}^n, \quad l \in L, \quad l \neq 0.
\]  
(31)

It follows from \((19)\), \((30)\), and \((31)\) that
\[
\hat{\lambda}_j(s, l) = \begin{cases} 
\exp\{i(t_j, s) - (A_j s, s)\}, & \text{if } l = 0, \\
0, & \text{if } l \neq 0, \quad j = 1, 2.
\end{cases}
\]  
(32)

Taking into account \((32)\), and returning from the distributions \(\lambda_j\) to the distributions \(\mu_j\), we prove the statement of the lemma. ■

**Proof of Theorem 3.1** We have \(Y = \mathbb{R}^n \times L \times H\), where \(L\) and \(H\) are groups topologically isomorphic to the character groups of the groups \(D\) and \(G\) respectively. Let
\[
G = \bigoplus_{p \in \mathbb{P}, \, p > 2} G_p,
\]
where \(G_p\) is a finite \(p\)-group. Then
\[
H = \bigoplus_{p \in \mathbb{P}, \, p > 2} H_p,
\]
where \(H_p\) is a group topologically isomorphic to the character group of the group \(G_p\). Put
\[
A_m = \bigoplus_{p \in \mathbb{P}, \, 2 < p \leq m} G_p, \quad B_m = \bigoplus_{p \in \mathbb{P}, \, 2 < p \leq m} H_p, \quad C_m = \bigoplus_{p \in \mathbb{P}, \, p > m} G_p, \quad m = 3, 4, \ldots
\]
Since \(\mathbb{R}^n\) is the connected component of zero of the group \(X\), we have \(\alpha(\mathbb{R}^n) = \mathbb{R}^n\). It follows from the definition of the product topology that \(\alpha(C_m) \subset G\) for some \(m\). Taking into account that \(C_m\) is the only subgroup of \(G\) which is topologically isomorphic to \(C_m\), we have \(\alpha(C_m) = C_m\), and hence \(\alpha(\mathbb{R}^n \times C_m) = \mathbb{R}^n \times C_m\).

Represent the group \(X\) in the form \(X = \mathbb{R}^n \times D \times A_m \times C_m\). Consider the factor-group \(X/(\mathbb{R}^n \times C_m)\). Denote by \([x]\) its elements. Note that the factor-group \(X/(\mathbb{R}^n \times C_m)\) is topologically isomorphic to the group \(D \times A_m\). Since \(\alpha(\mathbb{R}^n \times C_m) = \mathbb{R}^n \times C_m\), the topological automorphism \(\alpha\) induces a topological automorphism \([\alpha]\) on the factor-group \(X/(\mathbb{R}^n \times C_m)\) by the formula \([\alpha][x] = [\alpha x]\). It is easy to see that any continuous monomorphism of the group \(\mathbb{R}^n \times C_m\) to itself is a topological automorphism. This easily implies that
\[
\ker(I + [\alpha]) = \{0\}.
\]  
(33)

The character group of the factor-group \(X/(\mathbb{R}^n \times C_m)\) is topologically isomorphic to the annihilator \(A(Y, \mathbb{R}^n \times C_m) = L \times B_m\). Since \(\alpha(\mathbb{R}^n \times C_m) = \mathbb{R}^n \times C_m\), we have \(\alpha(L \times B_m) = (L \times B_m)\). Note also that the restriction of \(\alpha\) to the subgroup \(L \times B_m\) is the adjoint automorphism to \([\alpha]\).

Let \(\eta_1\) and \(\eta_2\) be independent random variables with values in the factor-group \(X/(\mathbb{R}^n \times C_m)\) such that the characteristic functions of their distributions coincide with the restrictions to the subgroup \(L \times B_m\) of the characteristic functions \(\hat{\mu}_j(y)\). By Lemma 2.2 the conditional distribution of the linear form \(M_2 = \eta_1 + [\alpha]\eta_2\) given \(M_1 = \eta_1 + \eta_2\) is symmetric. Obviously, the factor-group \(X/(\mathbb{R}^n \times C_m)\) is discrete and contains no elements of order 2. Hence, taking into account \((33)\) and applying Theorem B to the discrete group \(X/(\mathbb{R}^n \times C_m)\), we obtain that the restrictions of the characteristic functions \(\hat{\mu}_j(y)\) to the subgroup \(L \times B_m\) are of the form
\[
\hat{\mu}_j(y) = \begin{cases} 
\langle g_j, y \rangle, & \text{if } y \in E, \\
0, & \text{if } y \notin E, \quad j = 1, 2,
\end{cases}
\]
where \( g_j \in D \times A_m, E \) is an open subgroup in \( L \times B_m \), which is the annihilator of a finite subgroup in the group \( X/(\mathbb{R}^n \times C_m) \). Moreover,

\[
\hat{\alpha}(E) = E. \quad (34)
\]

Using the same reasoning as in the proof of Lemma 3.3 we can replace the independent random variables \( \xi_j \) by \( \eta_j = \xi_j - x_j \) with distributions \( \lambda_j \) in such a way that the conditional distribution of the linear form \( N_2 = \eta_1 + \alpha \eta_2 \) given \( N_1 = \eta_1 + \eta_2 \) is symmetric, and the restrictions of the characteristic functions \( \hat{\lambda}_j(y) \) to the subgroup \( L \times B_m \) are of the form

\[
\hat{\lambda}_j(y) = \begin{cases} 
1, & \text{if } y \in E, \\
0, & \text{if } y \notin E, \quad j = 1, 2.
\end{cases} \quad (35)
\]

By Lemma 3.2 (35) implies that \( \sigma(\lambda_j) \subset A(X, E) \). It is easy to see that \( A(X, E) = \mathbb{R}^n \times F \), where \( F \) is a compact totally disconnected group of the form \( (3) \). It follows from (34) that \( \alpha(\mathbb{R}^n \times F) = \mathbb{R}^n \times F \). Applying Lemma 3.3 to the group \( \mathbb{R}^n \times F \), we get the required representation for the distributions \( \lambda_j \), and hence, for the distributions \( \mu_j \) too.

An analogue of the Heyde theorem is valid not only for locally compact Abelian groups of the form \( (17) \). The following statement holds true (compare with [16, §3]).

**Proposition 3.4** There exists a locally compact Abelian group \( X \) containing no elements of order 2 such that \( X \) is not topologically isomorphic to any group of the form \( (17) \), and the following statement holds. If \( \alpha \) is a topological automorphism of the group \( X \) satisfying condition (2), and \( \xi_1 \) and \( \xi_2 \) are independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \), then the symmetry of the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) implies that \( \mu_j = m_K * E_{x_j} \), where \( K \) is a compact subgroup of \( X \), \( x_j \in X, j = 1, 2 \). Moreover, \( \alpha(K) = K \).

**Proof** Denote by \( p_n \) the \( n \)th prime number. Put \( T = \prod_{n=2}^{\infty} \mathbb{Z}(p_n^2) \) and consider \( T \) only as "algebraic" group. We do not consider \( T \) as a topological group equipped with the product topology.

Consider a subgroup \( K_n = \{0, p_n, 2p_n, \ldots, (p_n-1)p_n\} \) of \( \mathbb{Z}(p_n^2) \). It is obvious that \( K_n \) is isomorphic to \( \mathbb{Z}(p_n) \). Denote by \( Y \) a subgroup of \( T \) consisting of all elements \( y = (y_2, \ldots, y_n, \ldots) \), \( y_n \in \mathbb{Z}(p_n^2) \), such that \( y_n \notin K_n \) only for a finite number indexes \( n \). Put

\[
F = \prod_{n=2}^{\infty} K_n.
\]

Then \( F \) is a subgroup of \( Y \). Consider the group \( F \) in the product topology. Obviously, \( F \) is a compact Abelian group. Define a topology on \( Y \) as follows. A subset \( U \) in \( Y \) is called open if for each \( y \in U \) there is a neighborhood of zero \( V_y \) in \( F \) such that \( y + V_y \subset U \). Then \( Y \) is a second countable locally compact Abelian group, and \( F \) is an open subgroup in \( Y \). It is obvious that \( Y \) is a non-discrete, non-compact and totally disconnected group. Denote by \( X \) the character group of the group \( Y \).

Since multiplication by 2 is a topological isomorphism of the group \( Y \), the group \( X \) contains no elements of order 2. To see that the group \( X \) is not topologically isomorphic to a group of the form \( (17) \), suppose the contrary. It is obvious that then in \( (17) \) \( n = 0 \), and then the group \( Y \) can be represented in the form \( Y = L \times H \), where the group \( L \) is compact, and the group \( H \) is discrete. It is easy to see that there exists \( m \), such that \( \prod_{n=m}^{\infty} K_n \subset L \). This implies that the group \( H \) contains only a finite number different of subgroups \( K_n \), and hence, \( H \) is a finite group. It follows from this that \( Y \) is a compact group. The obtained contradiction shows that the group \( X \) is not topologically isomorphic to a group of the form \( (17) \).

We shall verify that

\[
(I + \hat{\alpha})(F) = F. \quad (36)
\]
Note that \( \alpha \) satisfies condition (2) if and only if
\[
(I + \tilde{\alpha})(Y) = Y. 
\]  
(37)

Put \( B = \bigoplus_{n=2}^{\infty} \mathbb{Z}(p_n^2) \), and consider \( B \) as a subgroup of the group \( Y \) in the topology induced on \( B \) by the topology of \( Y \). Denote by \( \tilde{\alpha}_B \) the restriction of \( \tilde{\alpha} \) to the subgroup \( B \). Assume that \( \text{Ker}(I + \tilde{\alpha}_B) \neq \{0\} \). This implies that \((I + \tilde{\alpha}_B)(B)\) is a proper subgroup of \( B \). It follows from this that \((I + \tilde{\alpha})(B)\) is contained in a proper closed subgroup of the group \( Y \). Since \( B \) is a dense subgroup in \( Y \), this implies that \((I + \tilde{\alpha})(Y)\) is also contained in a proper closed subgroup of the group \( Y \), contrary to (37). Hence, \( \text{Ker}(I + \tilde{\alpha}_B) = \{0\} \), and then \( I + \tilde{\alpha}_B \) is an automorphism of the group \( B \). Put \( C = \bigoplus_{n=2}^{\infty} K_n \), and consider \( C \) as a subgroup of the group \( F \) in the topology induced on \( C \) by the topology of \( Y \). Then \( C \) is a dense subgroup in \( F \). Since \( I + \tilde{\alpha}_B \) is an automorphism of the group \( B \), we have
\[
(I + \tilde{\alpha})(C) = C. 
\]  
(38)

Take \( y_0 \in F \). The subgroup \( C \) is dense in \( F \). Hence, there exists a sequence \( y_j \in C \) such that \( y_j \to y_0 \).

It follows from (38) that \( y_j = (I + \tilde{\alpha})z_j \), where \( z_j \in C \). Since \( F \) is a compact group, there exist a sequence \( \{j_k\} \) and an element \( z \in F \) such that \( z_{j_k} \to z \). Hence \( y_0 = (I + \tilde{\alpha})z \). Thus, (36) is proved.

By Lemma 2.2, the characteristic functions \( \hat{\mu}_j(y) \) satisfy equation (4). Put \( \nu_j = \mu_j * \hat{\mu}_j \). Then \( \hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0 \) for all \( y \in Y \), and the characteristic functions \( \hat{\nu}_j(y) \) also satisfy equation (4).

It is easy to see that Proposition 3.4 will be proved if we prove that \( \nu_1 = \nu_2 = m_K \), where \( K \) is a compact subgroup of the group \( X \), and \( \alpha(K) = K \). Therefore we can solve equation (4), assuming that \( \hat{\mu}_j(y) \geq 0 \) for all \( y \in Y \). Denote by \( \tilde{\alpha}_F \) the restriction of \( \tilde{\alpha} \) to the subgroup \( F \). It is obvious that \( \tilde{\alpha}_F \in \text{Aut}(F) \). Consider the restriction of equation (4) to the subgroup \( F \). We have
\[
\hat{\mu}_1(u + v)\hat{\mu}_2(u + \tilde{\alpha}_F v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - \tilde{\alpha}_F v), \quad u, v \in F. 
\]  
(39)

The group \( F \) is topologically isomorphic to the character group of the discrete group \( \bigoplus_{n=2}^{\infty} K_n \), and \( \tilde{\alpha}_F \) is the adjoint automorphism to an automorphism \( \tilde{\alpha} \) of the group \( \bigoplus_{n=2}^{\infty} K_n \). Note that (36) implies that \( \tilde{\alpha} \) satisfies condition (2). Taking into account Lemma 2.2 by applying Theorem B to the discrete group \( \bigoplus_{n=2}^{\infty} K_n \), we conclude from (39) that the restrictions of the characteristic functions \( \hat{\mu}_j(y) \) to the subgroup \( F \) are of the form
\[
\hat{\mu}_j(y) = \begin{cases} 
1, & \text{if } y \in E, \\
0, & \text{if } y \notin E, 
\end{cases} \quad j = 1, 2, 
\]
where \( E \) is an open subgroup of \( F \). In so doing,
\[
\tilde{\alpha}_F(E) = E. 
\]  
(40)

Put \( M = A(X, E) \). It follows from (10) that \( \alpha(M) = M \). By Lemma 3.2 \( \sigma(\mu_j) \subset M, \ j = 1, 2 \). Thus we reduced the proof of Proposition 3.4 to the case when the random variables \( \xi_j \) take values in the group \( M \). The character group of the group \( M \) is topologically isomorphic to the factor-group \( Y/E \). Since \( E \) is an open subgroup of \( F \), there is some natural number \( m \) such that the group \( Y/E \) is topologically isomorphic to a group of the form
\[
\bigoplus_{n=2}^{m} \mathbb{Z}(p_n^l) \times \bigoplus_{n=m+1}^{\infty} \mathbb{Z}(p_n), 
\]
where \( l_n \in \{0, 1, 2\} \). Hence, the group \( M \) is topologically isomorphic to a group of the form (3). The proposition follows now from Theorem 2.1.  

\[\blacksquare\]
4 The Heyde theorem fails on compact connected Abelian groups

We will prove in this section that there does not exist a compact connected Abelian group on which even a weak analogue of the Heyde theorem holds (see below Theorem 4.1).

Let \( n \) be a natural number. Denote by \( f_n : X \to X \) an endomorphism of the group \( X \), defined by the formula \( f_n x = nx \) for all \( x \in X \). Put \( f_n(X) = X^{(n)} \). Denote by \( \mathbb{Q} \) the additive group of rational numbers considering in the discrete topology. Let \( a = (a_0, a_1, \ldots, a_n, \ldots) \) be an arbitrary infinite sequence of integers, where each of \( a_n \) is greater than 1. Denote by \( \Sigma_a \) the corresponding \( \alpha \)-adic solenoid ([17, (10.12)]). The group \( \Sigma_a \) is compact, connected and has dimension 1 ([17, (10.13), (24.28)]). Denote by \( H_a \) a subgroup of \( \mathbb{Q} \) of the form

\[
H_a = \left\{ \frac{m}{a_0a_1 \cdots a_n} : n = 0, 1, \ldots ; m = 0, \pm 2, \ldots \right\}.
\]

The character group of the group \( \Sigma_a \) is topologically isomorphic to the group \( H_a \) ([17, (10.12)]). Note that if \( a = (2, 3, 4, \ldots) \), then \( H_a = \mathbb{Q} \).

The main result of this section is the following statement.

**Theorem 4.1** Let \( X \) be a compact connected Abelian group containing no elements of order 2. Then there exist a topological automorphism \( \alpha \) of the group \( X \) satisfying condition (2) and independent random variables \( \xi_1 \) and \( \xi_2 \) with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \) such that the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, whereas \( \mu_j \notin \Gamma(X) * I(X) \), \( j = 1, 2 \).

We note that if \( \mu \in \Gamma(X) * I(X) \), then \( \mu \) is invariant with respect to a compact subgroup \( K \) of the group \( X \) and, under the natural homomorphism \( X \to X/K \), induces a Gaussian distribution on the factor-group \( X/K \). Thus, Theorem 4.1 shows that even a weak analogue of the Heyde theorem is not valid for compact connected Abelian groups.

**Proof of Theorem 4.1** Since \( X \) is a compact connected Abelian group, \( Y \) is a discrete torsion free Abelian group. Two cases are possible: either the group \( X \) contains elements of finite order or \( X \) is a torsion-free group.

1. Assume that the group \( X \) contains elements of finite order. Denote by \( p \) the minimum of orders of nonzero elements of \( X \). Then \( p \) is a prime and taking into account that \( X \) contains no elements of order 2, \( p \geq 3 \). Since \( X \) is a connected Abelian group, we have \( X^{(h)} = X \) for all natural \( n \). This implies that if \( \text{Ker} f_n = \{0\} \), then \( f_n \in \text{Aut}(X) \). Set \( \alpha = -f_{p-1} \). Then \( \alpha \in \text{Aut}(X) \). Since \( I + \alpha = -f_{p-2} \), we have \( I + \alpha \in \text{Aut}(X) \), so that \( \alpha \) satisfies condition (2). It follows from \( Y^{(p)} = A(Y, \text{Ker} f_p) \) that \( Y^{(p)} \) is a proper subgroup in the group \( Y \). Take \( y_0 \in Y \) such that \( y_0 \notin Y^{(p)} \). Since \( p \geq 3 \), the numbers 2 and \( p \) are mutually prime. Therefore there exist integers \( m \) and \( n \) such that \( 2m + pn = 1 \). It follows from this that \( y_0 = 2my_0 + pmy_0 \). This implies that \( 2y_0 \notin Y^{(p)} \).

Consider on the group \( X \) the function

\[
\rho(x) = 1 + \text{Re}(x, y_0), \quad x \in X.
\]

Then \( \rho(x) \geq 0 \) for all \( x \in X \), and

\[
\int_X \rho(x) dm_X(x) = 1.
\]

Denote by \( \mu \) the distribution on \( X \) with the density \( \rho(x) \) with respect to \( m_X \). It is easy to see that
the characteristic function \( \hat{\mu}(y) \) is of the form
\[
\hat{\mu}(y) = \begin{cases} 
1, & \text{if } y = 0, \\
\frac{1}{2}, & \text{if } y = \pm y_0, \\
0, & \text{if } y \not\in \{0, \pm y_0\}.
\end{cases}
\] (41)

It is obvious that \( \mu \not\in \Gamma(X) \ast I(X) \).

Let \( \xi_1 \) and \( \xi_2 \) be independent identically distributed random variables with values in the group \( X \) and distribution \( \mu \). Let us verify that the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric. By Lemma 2.2, it suffices to verify that the characteristic function \( \hat{\mu}(y) \) satisfies equation (10). Obviously, equation (10) holds true if \( v = 0 \). So, assume that \( v \neq 0 \). Then the left-hand side of equation (10) vanishes. Otherwise, taking into account (41), we get \( u + v \in \{0, \pm y_0\} \) and \( u + \alpha v \in \{0, \pm y_0\} \). This implies that \((I - \tilde{\alpha})v = f_p u \in \{0, \pm y_0, \pm 2y_0\}\), but that’s impossible because it contradicts \( y_0, 2y_0 \not\in Y^{(p)} \). Reasoning similarly we obtain that if \( v \neq 0 \), then the right-hand side of equation (10) vanishes too. Thus, in the case when the group \( X \) contains elements of finite order, the theorem is proved.

2. Assume that \( X \) is a torsion-free group. Taking into account the structure theorem for compact Abelian torsion-free group ([17 (25.8)]) and the fact that \( X \) is connected, we get that for some \( n \) the group \( X \) is topologically isomorphic to a group of the form \((\Sigma_n)^n\), where \( a = (2, 3, 4, \ldots) \). Obviously, arguing as in the proof of necessity in Theorem 2.1 it suffices to prove the theorem for the group \( X = \Sigma_a \), where \( a = (2, 3, 4, \ldots) \). Then the group \( Y \) is topologically isomorphic to the group \( \mathbb{Q} \). In order not to complicate the notation we assume that \( Y = \mathbb{Q} \).

Let \( H \) be a subgroup of the group \( Y \) of the form
\[
H = \left\{ \frac{m}{5^n} : n = 0, 1, \ldots; m = 0, \pm 1, \pm 2, \ldots \right\}.
\]

Consider on the group \( H \) the function
\[
g(h) = \begin{cases} 
1, & \text{if } h \in H^{(2)}, \\
c, & \text{if } h \not\in H^{(2)},
\end{cases}
\] (42)

where \(-1 < c < 1, c \neq 0\). It is easy to verify that \( g(h) \) is a positive definite function. Consider on the group \( Y \) the function
\[
f(y) = \begin{cases} 
g(y), & \text{if } y \in H, \\
0, & \text{if } y \not\in H.
\end{cases}
\] (43)

Then \( f(y) \) is also a positive definite function ([18 (32.43)]). By the Bochner theorem, there exists a distribution \( \mu \in M^1(X) \) such that \( \hat{\mu}(y) = f(y) \). It is obvious that \( \mu \not\in \Gamma(X) \ast I(X) \).

Put \( \alpha = -f_2^{-2} \). Then \( \alpha \in \text{Aut}(X) \), and we have \( I + \alpha = f_3f_2^{-2} \). Obviously, \( I + \alpha \in \text{Aut}(X) \), so that \( \alpha \) satisfies condition (2).

Let \( \xi_1 \) and \( \xi_2 \) be independent identically distributed random variables with values in the group \( X \) and distribution \( \mu \). Verify that the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric. This is equivalent to the fact that the conditional distribution of the linear form \( N_2 = 4\xi_1 - \xi_2 \) given \( N_1 = \xi_1 + \xi_2 \) is symmetric. By Lemma 2.2, it suffices to verify that the characteristic function \( \hat{\mu}(y) \) satisfies the equation (10) which takes the form
\[
f(u + 4v)f(u - v) = f(u - 4v)f(u + v), \quad u, v \in Y.
\] (44)

It follows from (42) and (43) that for \( u, v \in H \) the equation (44) becomes an equality. Note that if either \( u \in H, v \not\in H \) or \( v \in H, u \not\in H \), then \( u \pm v \not\in H \), and it follows from (43) that both sides of
equation (44) are equal to zero. So, equation (44) becomes also an equality. Assume that $u, v \notin H$. Then the left-hand side in (44) is equal to zero. In the opposite case (43) implies that $u + 4v, u - v \in H$. It follows from this that $5v \in H$, and hence, $v \in H$. The obtained contradiction shows that the left-hand side in (44) is equal to zero. Arguing similarly, we verify that if $u, v \notin H$, then the right-hand side in (44) is also equal to zero. ■

In connection with Theorem 4.1 it is interesting to note that the following statement is true (see [11, Theorem 3]).

Let $X$ be a locally compact Abelian group containing no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$ with non-vanishing characteristic functions. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j \in \Gamma(X)$, $j = 1, 2$.

We will complement Theorem 4.1 with two propositions. The first proposition shows that, generally speaking, we can not replace in Theorem 4.1 the statement:

(A) Then there exist a topological automorphism $\alpha$ of the group $X$ satisfying condition (2) and independent random variables $\xi_1$ and $\xi_2$ with values in $X$ and distributions $\mu_1$ and $\mu_2$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, whereas $\mu_j \notin \Gamma(X) \ast I(X)$, $j = 1, 2$.

by the statement:

(B) Then for each topological automorphism $\alpha$ of the group $X$, $\alpha \neq I$, satisfying condition (2) there exist independent random variables $\xi_1$ and $\xi_2$ with values in $X$ and distributions $\mu_1$ and $\mu_2$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, whereas $\mu_j \notin \Gamma(X) \ast I(X)$, $j = 1, 2$.

The second proposition shows that there is a compact connected Abelian group $X$ containing no elements of order 2 when we can replace in Theorem 4.1 (A) by (B). We should exclude the case when $\alpha = I$, because if a compact Abelian group $X$ contains no elements of order 2, then $Y^{(2)} = Y$, and by Lemma 2.2 the symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \xi_2$ given $L_1 = \xi_1 + \xi_2$ implies that $\xi_1$ and $\xi_2$ have degenerate distributions.

**Proposition 4.2** There exist a compact connected Abelian group $X$ containing no elements of order 2 and a topological automorphism $\alpha$ of the group $X$, $\alpha \neq I$, satisfying condition (2) such that the following statement holds: If $\xi_1$ and $\xi_2$ are independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j \in \Gamma(X) \ast I(X)$, $j = 1, 2$.

To prove Proposition 4.2 we need two lemmas.

**Lemma 4.3** ([22, Lemma 6]) Let $X$ be a locally compact Abelian group, and let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then the linear forms $M_1 = (I + \alpha) \xi_1 + 2 \alpha \xi_2$ and $M_2 = 2 \xi_1 + (I + \alpha) \xi_2$ are independent.

**Lemma 4.4** ([3], see also [7, Theorem 7.10]) Let $X$ be a locally compact Abelian group. Assume that the connected component of zero of the group $X$ contains no elements of order 2. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, then $\mu_j \in \Gamma(X) \ast I(X)$, $j = 1, 2$.

**Proof of Proposition 4.2** Let $X = \Sigma_2^a$, where $a = (2, 3, 4, \ldots)$. Since the character group of the group $X$ is topologically isomorphic to the group $\mathbb{Q}^2$, we have $f_n \in \text{Aut}(X)$ for any natural $n$. This
implies in particular, that the group \(X\) contains no elements of order 2. Denote by \(x = (a, b)\), where \(a, b \in \Sigma_{\alpha}\), elements of the group \(X\). Define the mapping \(\alpha : X \to X\) by the formula

\[
\alpha(a, b) = (-3a + 4b, 2a - 3b), \quad (a, b) \in X.
\]

Then \(\alpha\) is a topological automorphism of the group \(X\). Since \(I + \alpha \in \text{Aut}(X)\), condition (2) is fulfilled. By Lemma 4.3 the symmetry of the conditional distribution of the linear form \(L_2 = \xi_1 + \alpha \xi_2\) given \(L_1 = \xi_1 + \xi_2\) implies that the linear forms \(M_1 = (I + \alpha)\xi_1 + 2\alpha \xi_2\) and \(M_2 = 2\xi_1 + (I + \alpha)\xi_2\) are independent. Note that coefficients of the linear forms \(M_1\) and \(M_2\) are topological automorphisms of the group \(X\). Consider the new independent random variables \(\eta_1 = (I + \alpha)\xi_1\) and \(\eta_2 = 2\alpha \xi_2\). The independence of the linear forms \(M_1\) and \(M_2\) implies that the linear forms \(N_1 = \eta_1 + \eta_2\) and \(N_2 = 2(I + \alpha)^{-1}\eta_1 + f_2^{-1}\alpha^{-1}(I + \alpha)\eta_2\) are independent. Hence, the linear forms \(P_1 = \eta_1 + \eta_2\) and \(P_2 = \eta_1 + f_2^{-2}\alpha^{-1}(I + \alpha)^2\eta_2\) are also independent. It is easy to verify that \(f_2^{-2}\alpha^{-1}(I + \alpha)^2 = -I\). Hence, \(P_2 = \eta_1 - \eta_2\). Denote by \(\lambda_j\) the distribution of the random variable \(\eta_j\). Since \(X\) is a connected Abelian group containing no elements of order 2, by Lemma 4.4 \(\lambda_j \in \Gamma(X) * I(X), \ j = 1, 2\). In view of \(I + \alpha, 2\alpha \in \text{Aut}(X)\), it follows from \(\lambda_1 = (I + \alpha)\mu_1\) and \(\lambda_2 = 2\alpha \mu_2\) that \(\mu_j \in \Gamma(X) * I(X), \ j = 1, 2\). ■

**Proposition 4.5** There exists a compact connected Abelian group \(X\) containing no elements of order 2 such that for each topological automorphism \(\alpha\) of the group \(X\), \(\alpha \neq I\), satisfying condition (2), there exist independent random variables \(\xi_1\) and \(\xi_2\) with values in \(X\) and distributions \(\mu_1\) and \(\mu_2\) such that the conditional distribution of the linear form \(L_2 = \xi_1 + \alpha \xi_2\) given \(L_1 = \xi_1 + \xi_2\) is symmetric, while \(\mu_j \notin \Gamma(X) * I(X), \ j = 1, 2\).

**Proof** Let \(X = \Sigma_{\alpha}\), where \(\alpha = (2, 2, 2, \ldots)\). The group \(Y\) is topologically isomorphic to a subgroup of the group \(Q\) of the form

\[
H_a = \left\{ \frac{m}{2^n} : n = 0, 1, \ldots; m = 0, \pm 1, \pm 2, \ldots \right\}.
\]

Since \(Y^{(2)} = Y\), the group \(X\) contains no elements of order 2. It is easy to see that \(\text{Aut}(X) = \{ f_{2^k}, f_{2^k}^{-1} : k = 0, 1, 2, \ldots \}\). Obviously, there exist only two topological automorphisms \(\alpha, \alpha \neq I\), of the group \(X\) satisfying condition (2). They are \(\alpha = -f_2\) and \(\alpha = -f_2^{-1}\). Let \(\alpha = -f_2\). It is easy to see that the group \(X\) contains an element of order 2. Then, as has been proved in the case 1 of Theorem 4.1, there exist independent random variables \(\xi_1\) and \(\xi_2\) with values in \(X\) and distributions \(\mu_1\) and \(\mu_2\) such that the conditional distribution of the linear form \(L_2 = \xi_1 - 2\xi_2\) given \(L_1 = \xi_1 + \xi_2\) is symmetric, where \(\mu_j \notin \Gamma(X) * I(X), \ j = 1, 2\).

The proof of the existence of independent random variables \(\xi_1\) and \(\xi_2\) with values in \(X\) and distributions \(\mu_1\) and \(\mu_2\) such that the conditional distribution of the linear form \(L_2 = \xi_1 - f_2^{-1}\xi_2\) given \(L_1 = \xi_1 + \xi_2\) is symmetric, while \(\mu_j \notin \Gamma(X) * I(X), \ j = 1, 2\), is reduced to the previous case. ■

**Remark 4.6** The Heyde theorem is true for some compact connected Abelian groups if we additionally assume that the characteristic functions of random variables do not vanish. In particular, the following statement was proved in [13].

**Theorem** Let \(X = \Sigma_{\alpha}\) be an \(\alpha\)-adic solenoid containing no elements of order 2, and let \(\alpha\) be a topological automorphism of the group \(X\) satisfying condition (2). Let \(\xi_1\) and \(\xi_2\) be independent random variables with values in the group \(X\) and distributions \(\mu_1\) and \(\mu_2\) with nonvanishing characteristic functions. If the conditional distribution of the linear form \(L_2 = \xi_1 + \alpha \xi_2\) given \(L_1 = \xi_1 + \xi_2\) is symmetric, then \(\mu_j \in \Gamma(X), \ j = 1, 2\).

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