ABUNDANCE FOR LARGE KODAIRA DIMENSION

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Abstract. In this note, we apply the semi-ampleness criterion in Lemma 3.1 to prove many classical results in the study of abundance conjecture. As a corollary, we prove abundance for large Kodaira dimension depending only on [BCHM10].

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1. INTRODUCTION

We work over $\mathbb{C}$ throughout. The following conjecture lies in the central position in minimal model theory.

Conjecture 1.1. Let $(X, \Delta)$ be a projective klt log pair of dimension $n \geq 2$ such that $\kappa(X, \Delta) \geq 0$, where $\kappa(X, \Delta)$ is the Kodaira dimension of $K_X + \Delta$. Suppose $K_X + \Delta$ is nef, then it is semi-ample.

We first note here that nothing in this note is essentially new. However, as we found that the semi-ampleness criterion in Lemma 3.1 is powerful and convenient, the purpose of this note is to apply it to quickly prove many classical results in the study of Conjecture 1.1 in a slightly different way, including Theorem 3.2, Theorem 3.5, and Theorem 6.3. We also derive an additive formula which precisely tells the difference between Iitaka dimension and numerical dimension, see theorem 5.1. Finally, as a corollary, we prove the following widely known result depending only on [BCHM10].

Theorem 1.2. (=Corollary 6.5) Suppose Conjecture 6.1 is known up to dimension $l \in \mathbb{Z}^+$. Let $(X, \Delta)$ be a minimal projective klt log pair of dimension $n \geq l + 1$. If $\kappa(X, \Delta) \geq n - l$, then $K_X + \Delta$ is semi-ample.

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2. Preliminaries

In this section, we list some preliminary results on very exceptional divisors. Readers may refer to [Bir12, Section 3] for more details. We also refer to [KM98, Laz04] for the basic notion in birational geometry such as klt singularities, Iitaka dimension, and Kodaira dimension.

**Definition 2.1.** ([Sho03, Definition 3.2] or [Bir12, Definition 3.1]) Let $g : W \to Z$ be a contraction morphism of projective normal varieties (i.e. $g_*\mathcal{O}_W = \mathcal{O}_Z$), $D$ an $\mathbb{R}$-divisor on $W$, and $V \subset W$ a closed subset. We say that $V$ is vertical over $Z$ if $g(V)$ is a proper subset of $Z$. We say that $D$ is very exceptional over $Z$ if $D$ is vertical over $Z$ and for any prime divisor $P$ on $Z$ there is a prime divisor $Q$ on $W$ which is not a component of $D$ but $g(Q) = P$, i.e. over the generic point of $P$ we have $\text{Supp} g^*P \nsubseteq \text{Supp} D$.

We fix some notation for this section.

Let $X$ be a projective normal variety and $L$ a line bundle on $X$. Write $I(L)$ to be the Iitaka dimension of $L$. Suppose $I(L) \geq 1$ and $R(X, L)$ is finitely generated in degree one. Let $f : W \to X$ be a resolution of $X$ and $f^*L \sim M + F$, where $M$ is globally generated and $F$ is the fixed part. Denote by $g : W \to Z$ the algebraic fiber space induced by $|kM|$ for a sufficiently divisible positive integer $k$. We note here that $\dim Z = I(L)$.

**Lemma 2.2.** Suppose $F$ is vertical over $Z$, then $F$ is very exceptional over $Z$.

*Proof.* This lemma is in fact implied by [Bir12, Lemma 3.2]. We provide a proof here for readers’ convenience. First note that there is an ample line bundle $\tilde{A}$ on $Z$ such that $kM \sim g^*\tilde{A}$. Since $R(X, L)$ is finitely generated by degree one, we know that $f^*rL \sim rM + rF$ where $rF$ is the fixed part for any positive integer $r$. Fix a positive integer $l$, we have the following

$$H^0(W, mkM + lF) \cong H^0(W, mg^*\tilde{A} + lF) \cong H^0(W, mg^*\tilde{A})$$

for $m \gg 1$. By projection formula, we see

$$H^0(Z, \mathcal{O}_Z(m\tilde{A}) \otimes g_*\mathcal{O}_W(lF)) \cong H^0(Z, \mathcal{O}_Z(m\tilde{A}))$$

for $m \gg 1$. As there is an injection $\mathcal{O}_Z \hookrightarrow g_*\mathcal{O}_W(lF)$ and one can choose $m$ sufficiently large such that $m\tilde{A}$ is very ample, we have that $g_*\mathcal{O}_W(lF) = \mathcal{O}_Z = g_*\mathcal{O}_W$.

Suppose $F$ is not very exceptional over $Z$, then there is a prime divisor $P$ on $Z$ such that if $Q$ is any prime divisor on $W$ such that $g(Q) = P$, then $Q$ is a component of $\text{Supp} F$. Let $U$ be a smooth open subset of $Z$ such that $\mathcal{O}_U \subseteq \mathcal{O}_V(P|_U)$, $P|_U$ is Cartier, and each component of $g^*P|_U$ maps onto $P|_U$. Let $V = g^{-1}U$. Then $\text{Supp} g^*P \subset \text{Supp} F|_V$, and $g^*P|_U \leq lF|_V$ for some $l > 0$. However,

$$\mathcal{O}_U \not\subseteq \mathcal{O}_V(P|_U) \subset g_*\mathcal{O}_V(lF|_V) = \mathcal{O}_U,$$

a contradiction. \qed

Let $S \to Z$ be a projective morphism of varieties and $M$ an $\mathbb{R}$-Cartier divisor on $S$. We say that $M$ is nef on the very general curves of $S/Z$ if there is a countable union $\Lambda$ of proper closed subsets of $S$ such that $M.C \geq 0$ for any curve $C$ on $S$ contracted over $Z$ satisfying $C \not\subseteq \Lambda$. 
Lemma 2.3. ([Sho03, Lemma 3.22], [Pro03, Lemma 1.7], or [Bir12, Lemma 3.3]) Let \( g : W \to Z \) be a contraction of projective normal varieties. Let \( D \) be an \( \mathbb{R} \)-divisor on \( W \) written as \( D = D^+ - D^- \) with \( D^+, D^- \geq 0 \) having no common components. Assume that \( D^- \) is very exceptional over \( Z \), and that for each component \( S \) of \( D^- \), \( -D|_S \) is nef on the very general curves of \( S/Z \). Then \( D \geq 0 \).

Proof. See the proof of [Bir12, Lemma 3.3].

Lemma 2.4. Notation as in Lemma 2.2. Suppose \( F \) is very exceptional over \( Z \) and \( \dim g(F) > 0 \). Let \( Z' := H \) be a very general hyperplane section of \( Z \) and \( W' := g^*H \). Denote by \( g' : W' \to Z' \) the restriction morphism, then \( F|_{W'} \) is very exceptional over \( Z' \).

Proof. The proof is contained in the proof of [Bir12, Lemma 3.3].

3. A criterion for semi-ampleness

The following key lemma provides a criterion for semi-ampleness.

Lemma 3.1. Let \( X \) be a projective normal variety and \( L \) a nef line bundle on \( X \). Suppose \( I(L) \geq 1 \) and \( R(X,L) \) is finitely generated in degree one. Let \( f : W \to X \) be a resolution of \( X \) and \( f^*L \sim M + F \), where \( M \) is globally generated and \( F \) is the fixed part. Denote \( g : W \to Z \) to be the algebraic fiber space induced by \( |kM| \) for a sufficiently divisible positive integer \( k \). Suppose \( F \) is vertical over \( Z \), then \( F = 0 \) and \( L \) is semi-ample.

Proof. By Lemma 2.2 \( F \) is very exceptional. If \( \dim g(F) = 0 \), then \( F|_S \sim_q (f^*L - g^*A)|_S \) is nef on the very general curves of \( S \), where \( S \) is any component of \( F \). By Lemma 2.3 we have \( F \leq 0 \). Thus \( F = 0 \) and \( f^*L \sim_q g^*A \), which implies that \( L \) is semi-ample. If \( \dim g(F) = l > 0 \), we choose \( l \) very general hyperplane sections on \( Z \), denoted by \( H_1, \ldots, H_l \), and consider the induced morphism

\[ g' : W' := g^*H_1 \cap \ldots \cap g^*H_l \to Z' := H_1 \cap \ldots \cap H_l. \]

Suppose \( F > 0 \), as \( H_i, i = 1, \ldots, l \) are very general, we have \( F|_{W'} > 0 \). By Lemma 2.4 the divisor \( F|_{W'} \) is very exceptional over \( Z' \). Note that we have the following \( \mathbb{Q} \)-linear equivalence

\[ f^*L|_{W'} \sim_q g'^*(A|_{Z'}) + F|_{W'}. \]

Since \( F|_{W'} \) is very exceptional and \( \dim g'(F|_{W'}) = 0 \), \( (F|_{W'})|_S \sim_q (f^*L|_{W'} - g'^*(A|_{Z'}))|_S \) is nef on the very general curves of \( S \), where \( S \) is any component of \( F|_{W'} \). By Lemma 2.3 \( F|_{W'} \leq 0 \), which is a contradiction. Therefore \( F = 0 \) and \( f^*L \sim_q g^*A \), which implies that \( L \) is semi-ample.

As a corollary, we have the following well-known semi-ample result for large Iitaka dimension.

Theorem 3.2. Let \( X \) be a projective normal variety of dimension \( n \geq 2 \) and \( L \) a nef line bundle on \( X \). Suppose \( I(L) \geq n - 1 \) and \( R(X,L) \) is finitely generated, then \( L \) is semi-ample.

Proof. Replace \( L \) by a sufficiently large multiple, we may assume that \( R(X,L) \) is finitely generated by degree one. Choose a resolution \( f : W \to X \) such that \( f^*L \sim M + F \), where \( M \) is globally generated and \( F \) is the fixed part. Let \( g : W \to Z \) be the algebraic fiber space induced by \( |kM| \) for a sufficiently large positive integer \( k \), then we have a natural rational
map \( \phi : X \to Z \) where \( Z = \text{Proj} R(X, L) \), and there is an ample \( \mathbb{Q} \)-line bundle \( A \) on \( Z \) such that
\[
f^*L \sim_\mathbb{Q} g^* A + F.
\]
By Lemma \[3.1\] it suffices to show that \( F \) is vertical over \( Z \). Suppose \( I(L) = n \), then \( \dim Z = n \) and \( F \) is clearly vertical over \( Z \). Suppose \( I(L) = n - 1 \), then \( \dim Z = n - 1 \). If \( F \) is not vertical over \( Z \), then \( F \) is clearly relatively big over \( Z \). Thus there exist a relatively ample/Z divisor \( D \) and an effective \( \mathbb{Q} \)-divisor \( E \) on \( W \) such that \( F \sim_\mathbb{Q} D + E + g^* B \), where \( B \) is a \( \mathbb{Q} \)-divisor on \( Z \) which is not necessarily effective. Choose a sufficiently small rational number \( 0 < \epsilon \ll 1 \) such that
\[
(1) \quad \frac{1}{2} g^* A + \epsilon D \text{ is ample on } W;
\]
\[
(2) \quad \frac{1}{2} A + \epsilon B \text{ is effective on } Z.
\]
Therefore, we have the following expression:
\[
f^*L = M + F \sim_\mathbb{Q} \frac{1}{2} g^* A + \epsilon D + g^* \left( \frac{1}{2} A + \epsilon B \right) + \epsilon E + (1 - \epsilon) F.
\]
This expression indicates that \( f^*L \) is big, which is a contradiction to the assumption that \( I(L) = n - 1 \). The contradiction implies that \( F \) is vertical over \( Z \). The proof is finished. \( \square \)

**Corollary 3.3.** Let \( X \) be a projective normal surface and \( L \) a nef line bundle on \( X \). Suppose \( I(L) \geq 1 \) and \( R(X, L) \) is finitely generated, then \( L \) is semi-ample.

**Definition 3.4.** (Kaw85) Let \( X \) be a projective normal variety of dimension \( n \) and \( L \) a nef line bundle on \( X \). The numerical dimension \( \nu(L) \) is defined as follows:
\[
\nu(L) := \max \{ e \in \mathbb{N} | L^e.V \neq 0 \text{ for some subvariety } V \text{ of dimension } e \}.
\]

As another application of Lemma \[3.1\] we also prove the following well-known result for all positive Iitaka dimension (cf. MR97 Corollary 1).

**Theorem 3.5.** Let \( X \) be a projective normal variety of dimension \( n \geq 2 \) and \( L \) a nef line bundle on \( X \). Suppose \( I(L) \geq 1 \) and \( R(X, L) \) is finitely generated. If \( I(L) = \nu(L) \), then \( L \) is semi-ample.

**Proof.** By Theorem \[3.2\] when \( I(L) \geq n - 1 \), we even do not need the condition \( I(L) = \nu(L) \). Thus it suffices to assume \( 1 \leq I(L) \leq n - 2 \). We divide the proof into several steps.

**Step 0.** We use the same notation as in the proof of Theorem \[3.2\]. Replace \( L \) by a sufficiently large multiple, we may assume that \( R(X, L) \) is finitely generated by degree one. Choose a resolution \( f : W \to X \) such that \( f^* L \sim M + F \), where \( M \) is globally generated and \( F \) is the fixed part. Let \( g : W \to Z \) be the algebraic fiber space induced by \( |kM| \) for a sufficiently large positive integer \( k \), then we have a natural rational map \( \phi : X \to Z \) where \( Z = \text{Proj} R(X, L) \), and there is an ample \( \mathbb{Q} \)-line bundle \( A \) on \( Z \) such that
\[
f^* L \sim_\mathbb{Q} g^* A + F.
\]
By Lemma \[3.1\] it suffices to show that \( F \) is vertical over \( Z \). Denote by \( i := n - I(L) - 1 \geq 1 \) and suppose \( F \) is not vertical over \( Z \), we aim to derive a contradiction.

**Step 1.** Let \( D_1 \) be a relatively ample line bundle on \( W \), and choose a sufficiently small rational number \( 0 < \epsilon \ll 1 \) such that \( g^* A + \epsilon D_1 \) is ample on \( W \). Take a sufficiently divisible \( m_1 \in \mathbb{Z}^+ \) such that \( |m_1 (g^* A + \epsilon D_1)| \) is a very ample linear system on \( W \). We choose a general member \( H_1 \in |m_1 (g^* A + \epsilon D_1)| \), and it is clear that \( F|_{H_1} \) still dominates \( Z \). Note that we
have the decomposition $f^*L|_{H_1} \sim M|_{H_1} + F|_{H_1}$. Replace $g : W \to Z$ with $g_1 : H_1 \to Z$ (which is the composition of $H_1 \to W$ and $W \to Z$), and $f^*L$ (resp. $M, F$) with $f^*L|_{H_i}$ (resp. $M|_{H_i}, F|_{H_i}$).

**Step 2.** If $\dim F|_{H_1} > \dim Z$ (i.e. $i \geq 2$), we repeat the process as in step 1. Let $D_2$ be a relatively ample line bundle on $H_1$, and choose a sufficiently small rational number $0 < \epsilon < 1$ such that $g_1^*A + \epsilon D_2$ is ample on $H_1$. Take a sufficiently divisible $m_2 \in \mathbb{Z}^+$ such that $|m_2(g_1^*A + \epsilon D_2)|$ is a very ample linear system on $H_1$. We choose a general member $H_2 \in |m_2(g_1^*A + \epsilon D_2)|$, and it is clear that $F|_{H_2}$ still dominates $Z$. Note that we have the decomposition $f^*L|_{H_2} \sim M|_{H_2} + F|_{H_2}$. Replace $g_1 : H_1 \to Z$ with $g_2 : H_2 \to Z$ (which is the composition of $H_2 \mapsto H_1$ and $H_1 \to Z$), and $f^*L|_{H_1}$ (resp. $M|_{H_1}, F|_{H_1}$) with $f^*L|_{H_2}$ (resp. $M|_{H_2}, F|_{H_2}$).

**Step 3.** If $\dim F|_{H_2} > \dim Z$, we repeat the above process. After $i$ steps, we have the morphism $g_i : H_i \to Z$ and the decomposition

$$f^*L|_{H_i} \sim M|_{H_i} + F|_{H_i} \sim g_i^*A + F|_{H_i}.$$ 

Since $F|_{H_i}$ dominates $Z$ and $\dim F|_{H_i} = \dim Z$, we see that $F|_{H_i}$ is relatively big with respect to $g_i$. Thus there exist a relatively ample $/Z$ divisor $D$ and an effective $\mathbb{Q}$-divisor $E$ on $H_i$ such that $F|_{H_i} \sim_{\mathbb{Q}} D + E + g_i^*B$, where $B$ is a $\mathbb{Q}$-divisor on $Z$ which is not necessarily effective. Choose a sufficiently small rational number $0 < \epsilon' < 1$ such that

1. $\frac{1}{2}g_i^*A + \epsilon' D$ is ample on $H_i$;
2. $\frac{1}{2}A + \epsilon' B$ is effective on $Z$.

Therefore, we have the following expression:

$$f^*L|_{H_i} = M|_{H_i} + F|_{H_i} \sim_{\mathbb{Q}} \frac{1}{2}g_i^*A + \epsilon' D + g_i^*(\frac{1}{2}A + \epsilon' B) + \epsilon' E + (1 - \epsilon')F|_{H_i}.$$ 

This expression indicates that $f^*L|_{H_i}$ is big on $H_i$. As $\dim H_i = \dim Z + 1 = I(L) + 1$, we see that $\nu(f^*L) \geq I(L) + 1$, which implies that $\nu(L) \geq I(L) + 1$. Contradiction. 

## 4. Numerical dimension in family

In this section, we fix $g : W \to Z$ to be a projective contraction morphism (i.e. $g_*O_W = O_Z$) of quasi-projective normal varieties with $\dim Z \geq 1$, and $F$ a $\mathbb{Q}$-line bundle on $W$ such that $F_z$ is nef on $W_z$ for any closed point $z \in Z$, where $W_z$ is the fiber over $z \in Z$ and $F_z := F|_{W_z}$.

**Lemma 4.1.** Suppose $g : W \to Z$ is of relative dimension $d$, then there exists an open subset $U \subset Z$ such that the function $z \mapsto \nu(F_z)$ defined on $U$ is a constant function.

**Proof.** We choose $U$ to be the open subset of $Z$ such that $U$ is smooth and the morphism $W \times_Z U \to U$ is flat. Let $z_1, z_2 \in U$ be two different closed points, we show that $\nu(F_{z_1}) = \nu(F_{z_2})$. Suppose $\nu(F_{z_1}) = e \in \mathbb{N}$, then for any relatively ample line bundle $H$ on $W$, we have that $H^{d-e}.F^e.W_{z_1} \neq 0$. By [Ful84, Proposition 10.2], the following holds:

$$H^{d-e}.F^e.W_{z_1} = H^{d-e}.F^e.W_{z_2} \neq 0.$$ 

This implies that $\nu(F_{z_2}) \geq \nu(F_{z_1})$. By symmetry, $\nu(F_{z_1}) \geq \nu(F_{z_2})$, concluded. 

□
5. Additive formula

In this section, we fix $X$ to be a projective normal variety of dimension $n$ and $L$ a nef line bundle on $X$ with $I(L) \geq 1$. We always assume that the graded ring $R(X, L)$ is finitely generated by degree one. Denote by $f : W \to X$ the resolution such that $f^*L \sim M + F$, where $M$ is globally generated and $F$ is the fixed part. Let $g : W \to Z$ be the algebraic fiber space induced by $|kM|$ for $k \gg 1$. We will derive an additive formula which precisely reflects the difference between $I(L)$ and $\nu(L)$. We note here that $F_z$ is nef on $W_z$ for general closed point $z \in Z$, since $M \sim Q g^*A$ for some ample $Q$-line bundle $A$ on $Z$. Combine the following additive formula and Theorem 3.5 one clearly sees that $L$ being semi-ample is equivalent to $\nu(F_z) = 0$ for general $z \in Z$.

**Theorem 5.1.** Notation as above, we have the additive formula $\nu(L) = I(L) + \nu(F_z)$, where $z \in Z$ is a general closed point on $Z$.

**Proof.** By Theorem 3.2 it suffices to assume $1 \leq I(L) \leq n - 2$. By Lemma 4.1 $\nu(F_z)$ is constant for general closed point $z \in Z$. Denote by $i := n - I(L) - 1$ and $e := \nu(F_z)$ for general closed point on $Z$. Let $D$ be a relatively ample line bundle on $W$ and $A$ an ample $Q$-line bundle on $Z$ such that $M \sim Q g^*A$, we choose a sufficiently small rational number $0 < \epsilon \ll 1$ such that $g^*A + \epsilon D$ is ample on $W$. Consider the linear system $|m(g^*A + \epsilon D)|$ for $m$ sufficiently large and divisible positive number $m$. Choose $i + 1 - e$ general elements in the linear system, denoted by $H_j, j = 1, ..., i + 1 - e$, then by [Ful84] Proposition 10.2 we see that the intersection number $H_1 ... H_{i+1-e}.F^e.W_z$ is a non-zero constant for general closed point $z \in Z$. This means that $F_z|_{H_1 \cap ... \cap H_{i+1-e}}.F^e.W_z$ is big on $W_z|_{H_1 \cap ... \cap H_{i+1-e}}$ for general $z \in Z$, thus $F|_{H_1 \cap ... \cap H_{i+1-e}}$ is relatively big with respect to the morphism $H_1 \cap ... \cap H_{i+1-e} \to Z$ (which is the composition of $H_1 \cap ... \cap H_{i+1-e} \to W$ and $W \to Z$). Since

$$f^*L|_{H_1 \cap ... \cap H_{i+1-e}} \sim f^*A|_{H_1 \cap ... \cap H_{i+1-e}} + F|_{H_1 \cap ... \cap H_{i+1-e}},$$

by the same explanation as in the proof of Theorem 3.2, the line bundle $f^*L|_{H_1 \cap ... \cap H_{i+1-e}}$ is big and nef on $H_1 \cap ... \cap H_{i+1-e}$. Note that

$$\dim H_1 \cap ... \cap H_{i+1-e} = n - (i + 1 - e) = I(L) + e,$$

thus $\nu(L) = \nu(f^*L) \geq I(L) + \nu(F_z)$ for general $z \in Z$. The other direction follows from the next lemma. \hfill $\square$

**Lemma 5.2.** ([Nak04] Proposition V.2.7) Let $Y \to T$ be a projective contraction morphism from a smooth projective variety such that $\dim T \geq 1$. Suppose $D$ is a pseudo-effective line bundle on $Y$, then

$$\nu(D) \leq \dim T + \nu(D|_{Y_t}),$$

where $Y_t$ is the fiber over a general closed point $t \in T$.

**Proof.** We only note here that the proof of [Nak04] Proposition V.2.7 uses another definition of numerical dimension which defines for pseudo-effective line bundles, i.e.

$$\nu(D) = \max \{e \in \mathbb{N} | \lim_{m \to \infty} \frac{\dim H^0(Y, mD + D')}{m^e} > 0 \text{ for some ample } D' \text{ on } Y\}.$$

This definition coincides with Definition 3.4 (cf. [Nak04] Proposition V.2.7(6))] when $D$ is nef and the rest follows from the proof of [Nak04] Proposition V.2.7(9)]. \hfill $\square$
6. REDUCTION TO KODAIRA DIMENSION ZERO

It is well-known that the following conjecture implies Conjecture 1.1 (cf. [Kaw85, Theorem 7.3]). In this section, we will prove this reduction by applying Lemma 3.1.

**Conjecture 6.1.** Let \((X, \Delta)\) be a projective klt log pair of dimension \(n \geq 2\) such that \(\kappa(X, \Delta) = 0\). Then we have \(\nu(K_X + \Delta) = 0\).

**Remark 6.2.** We do not assume \((K_X + \Delta)\) to be minimal (i.e. \(K_X + \Delta\) is nef) in the above conjecture. The numerical dimension \(\nu(K_X + \Delta)\) of \(K_X + \Delta\) is defined as in the proof of Lemma 5.2.

**Theorem 6.3.** Conjecture 6.1 implies Conjecture 1.1.

**Proof.** Let \((X, \Delta)\) be a minimal projective klt pair of dimension \(n \geq 2\). If \(\kappa(X, \Delta) = 0\), then \(\nu(K_X + \Delta) = 0\) by Conjecture 6.1. This implies that \(K_X + \Delta \equiv 0\), thus \(K_X + \Delta \sim Q\) by [Amb05, Theorem 0.1]. From now on, we assume \(\kappa(X, \Delta) \geq 1\). By [BCHM10], we know that the graded ring \(R((X, r(K_X + \Delta)))\) is finitely generated, where \(r\) is a positive integer such that \(r(K_X + \Delta)\) is Cartier. Denote by \(L := r(K_X + \Delta)\), we may assume that \(r\) is sufficiently divisible such that \(R(X, L)\) is finitely generated in degree one. Denote by \(f: W \to X\) the log resolution of \((X, \Delta)\) such that \(f^*L \equiv M + F\), where \(M\) is globally generated and \(F\) is the fixed part. Let \(g: W \to Z\) be the algebraic fiber space induced by \(|kM|\) for \(k \gg 1\). Write

\[K_W + f_*^{-1}\Delta + \Gamma = f^*(K_X + \Delta) + E,\]

where \((W, f_*^{-1}\Delta + \Gamma + E)\) is simple normal crossing, \(\Gamma\) and \(E\) are non negative divisors with no common components. It is not hard to see that \(g: W \to Z\) is also the Iitaka fibration with respect to the line bundle \(f^*r(K_X + \Delta) + rE\), thus

\[r(K_W + f_*^{-1}\Delta + \Gamma)|_{W_z} = (f^*r(K_X + \Delta) + rE)|_{W_z} \sim (F + rE)|_{W_z}\]

is of Iitaka dimension zero for general fiber \(W_z\) (cf. [Laz04, Theorem 2.1.33]). By adjunction,

\[K_{W_z} + \Delta_{W_z} := (K_W + f_*^{-1}\Delta + \Gamma)|_{W_z}\]

is a klt log pair of Kodaira dimension zero, thus by Conjecture 6.1,

\[\nu(K_{W_z} + \Delta_{W_z}) = \nu((F + rE)|_{W_z}) = 0.\]

This also implies that \(\nu(F|_{W_z}) = 0\) and thus \(F\) is vertical over \(Z\). By Lemma 3.1 \(L\) is semi-ample. The proof is finished. \(\square\)

**Remark 6.4.** By the proof of Theorem 6.3 one sees that we can assume \((X, \Delta)\) to be simple normal crossing in Conjecture 6.1 since \((W_z, \Delta_{W_z})\) is simple normal crossing for general \(z \in Z\).

**Corollary 6.5.** Suppose Conjecture 6.1 is known up to dimension \(l \in \mathbb{Z}^+\). Let \((X, \Delta)\) be a minimal projective klt log pair of dimension \(n \geq l + 1\). If \(\kappa(X, \Delta) \geq n - l\), then \(K_X + \Delta\) is semi-ample.

**Proof.** We use the same notation as in the proof of Theorem 6.3. As \(\kappa(X, \Delta) \geq n - l\), we see that \(\dim W_z \leq l\). By our assumption and the proof of Theorem 6.3 \(F\) is vertical and \(K_X + \Delta\) is semi-ample. \(\square\)
The following corollary is widely known and the proof does not depend on minimal model program once we are armed by [BCHM10] and Conjecture 6.1 up to dimension 3. One can also refer to [Fil20, Theorem 1.5] for a proof which depends on more MMP techniques.

**Corollary 6.6.** Let \((X, \Delta)\) be a minimal projective klt log pair of dimension \(n \geq 4\). Suppose \(\kappa(X, \Delta) \geq n - 3\), then \(K_X + \Delta\) is semi-ample. 

**Proof.** The proof follows from the fact that Conjecture 6.1 is known up to dimension 3 (cf. [KMM94]). □

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