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Iteratively regularized Gauss–Newton type methods for approximating quasi–solutions of irregular nonlinear operator equations in Hilbert space with an application to COVID–19 epidemic dynamics

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Abstract

We investigate a class of iteratively regularized methods for finding a quasi–solution of a noisy nonlinear irregular operator equation in Hilbert space. The iteration uses an a priori stopping rule involving the error level in input data. In assumptions that the Frechet derivative of the problem operator at the desired quasi–solution has a closed range, and that the quasi–solution fulfills the standard source condition, we establish for the obtained approximation an accuracy estimate linear with respect to the error level. The proposed iterative process is applied to the parameter identification problem for a SEIR–like model of the COVID–19 pandemic.

Keywords:
Nonlinear equation
Iterative regularization
Closed range
Accuracy estimate
Parameter identification
Epidemic dynamics

1. Statement of the problem

The aim of this paper is to investigate a class of iterative methods for solving operator equations

\[ \Phi(x) = f. \]  

(1.1)

In (1.1), \( \Phi : X \to Y \) is a nonlinear operator from \( X \) into \( Y \), \( X \) and \( Y \) are Hilbert spaces. Suppose that \( \Phi \) is Frechet differentiable, and the derivative \( \Phi' \) satisfies the Lipschitz condition

\[ \|\Phi'(x) - \Phi'(y)\|_{B(X,Y)} \leq L\|x - y\|_X \quad \forall x, y \in B_R(x^*). \]  

(1.2)

By \( x^* \) we denote a solution of (1.1), and we let \( B_R(x^*) = \{x \in X : \|x - x^*\|_X \leq R\} \). Below, \( \| \cdot \|_Z \) is the norm of a Banach space \( Z \), \( B(X,Y) \) denotes the Banach algebra of bounded linear operators from \( X \) to \( Y \). Motivated by applied inverse problems we...
assume that instead of the element \( f \) in (1.1), an approximation \( f_\delta \in Y \) is given, where
\[
\| f_\delta - f \|_Y \leq \delta.
\] (1.3)
The error level \( \delta \) is assumed to be available.

In this paper, the object of our interest are iteratively regularized Gauss–Newton type methods for equation (1.1). The following class of such methods is studied in detail, see [1,2] and references therein:
\[
x_{n+1} = \xi - \Theta(\Phi'(x_n), \alpha_n) \Phi'(x_n)[\Phi(x_n) - f_\delta - \Phi'(x_n)(x_n - \xi)].
\] (1.4)

For generalizations to the Banach space case we refer to [3]. Here \( x_0 \in B_R(x^*) \) and \( \xi \in X \) are initial guesses for \( x^* \), and \( \{\alpha_n\}_{n=0}^{\infty} \) is a sequence of regularization parameters satisfying
\[
\lim_{n \to \infty} \alpha_n = 0, \quad 0 < \alpha_{n+1} \leq \alpha_n, \quad r \triangleq \sup_{n=0,1,\ldots} \frac{\alpha_n}{\alpha_{n+1}} < \infty.
\] (1.5)

The real valued filter function \( \Theta = \Theta(\lambda, \alpha) \) in (1.4), is continuous for \( \lambda \in [0, M^2] \) and \( \alpha \in (0, \alpha_0] \), where \( \| \Phi'(x) \|_{B(Y,X)} \leq M, x \in B_R(x^*) \). As an example we point to the function \( \Theta(\lambda, \alpha) = (\lambda + \alpha)^{-1} \). Then (1.4) leads to the iteratively regularized Gauss–Newton method also known as the Levenberg–Marquardt iteration
\[
x_{n+1} = \xi - (\Phi'^*(x_n) \Phi'(x_n) + \alpha_n E_n)^{-1} \Phi'^*(x_n)[\Phi(x_n) - f_\delta - \Phi'(x_n)(x_n - \xi)].
\] (1.6)

Throughout the paper, by \( E_x \) we denote the identity operator in \( X \). Given \( A \in B(X,Y) \), we denote
\[
\mathcal{R}(A) = \{ y \in Y : y = Ax, x \in X \}, \quad \mathcal{N}(A) = \{ x \in X : Ax = 0 \}.
\]

Approximating properties of the process (1.4) are usually investigated under the assumption that \( x^* \) satisfies the source condition
\[
x^* - \xi \in \mathcal{R}\left((\Phi'^*(x^*) \Phi'(x^*))^\nu\right), \quad \nu \geq \frac{1}{2}.
\] (1.7)

For the process (1.4) under the condition (1.7), with an appropriate stopping rule \( N = N(\delta) \), we have the accuracy estimate, e.g., [1, Sect.4.2]
\[
\| x_{N(\delta)} - x^* \|_X = O(\delta^{2\nu/(2\nu+1)}).
\]

The development of numerically implementable solution methods for (1.1) dictate the need of finite–dimensional approximations of spaces and operators. A fairly general scheme of such approximation is as follows. Consider finite–dimensional approximations \( X \) and \( Y \) to spaces \( X \) and \( Y \), respectively. Also choose connecting mappings \( P \in B(X,Y) \) and \( Q \in B(Y,Y) \).

We approximate the operator \( \Phi \) by a mapping \( \mathcal{F} : X \to Y \) and consider for (1.1) the approximating equation
\[
\mathcal{F}(\tilde{x}) = Qf_\delta, \quad \tilde{x} \in X.
\] (1.8)

Various schemes for constructing numerically implementable iterative methods are applicable to the discretized equation (1.8) [4,5–7]. Particularly the method (1.4) can be applied to (1.8) for evaluating approximations of \( P x^* \), e.g., [4,5].

We remark that in (1.8), \( \mathcal{F} \) acts between finite–dimensional spaces. A characteristic feature of such operators is that the derivative \( \mathcal{F}'(\tilde{x}) \) is a closed range operator for all \( \tilde{x} \). We say that \( A \in B(X,Y) \) is a closed range operator if the range \( \mathcal{R}(A) \) is a closed subspace in \( Y \).

The problem statement in this article is motivated by two factors. Firstly, from a theoretical point of view, having in mind the closed range property of (1.8), we investigate equations (1.11) with mappings \( \Phi : X \to Y \) in arbitrary Hilbert spaces satisfying the following assumption.

**Assumption 1.** The derivative \( \Phi'(x^*) \neq 0 \) is a closed range operator.

Assumption 1 is satisfied if one of the spaces \( X, Y \) is finite–dimensional, as in (1.8). It is also fulfilled in the infinite–dimensional situation when \( \Phi'(x^*) \) is a Fredholm operator. In [8,9] within Assumption 1, for (1.4) with different stopping rules \( N = N(\delta), N_\delta = N(\delta, f_\delta) \) under the source condition
\[
x^* - \xi = \Phi'^*(x^*)w, \quad w \in Y,
\] (1.9)
we have established the estimate
\[
\| x_{N(\delta)} - x^* \|_X = O(\delta).
\] (1.10)

Secondly, we remark that the approximating equation (1.8) may have no solutions, even if the original equation (1.1) is solvable. Therefore, it is natural to raise the question of approximating a quasi–solution of equation (1.8) rather than a solution. By a quasi–solution for (1.8) we mean a minimizer of the residual functional \( \| \mathcal{F}(\tilde{x}) - Qf_\delta \|_Y^2 \) over \( X \). Generalizing this problem statement, below we consider for the original equation (1.1) the problem of finding \( x^* \), such that
\[
\| \Phi(x^*) - f \|_Y^2 = \min \{ \| \Phi(x) - f \|_Y^2 : x \in B_R(x^*) \}.
\] (1.11)
A point satisfying (1.11) is called a quasi–solution of equation (1.1). Throughout the paper, the existence of a quasi–solution \( x^\ast \) is assumed. In the particular case \( \Phi(x^\ast) = f \), the quasi–solution is a solution in the usual sense. Let \( x^\ast \) be the desired quasi–solution to (1.1). In this paper our aim is to investigate the following modification of iteration (1.4):

\[
x_{n+1} = \xi - \Theta((\Phi^n(x_n)\Phi'(x_n)x_n^2, \alpha_n)(\Phi^n(x_n)x_n^2)\Phi'(x_n)(x_n - f_n) - \Phi'(x_n)(x_n - \xi)).
\] (1.12)

We establish the convergence of the method (1.12) and its projected version (2.31) to a quasi–solution \( x^\ast \) with the same accuracy estimate (1.10) that was previously obtained for approximating a classical solution by the iteration (1.4). Our main assumptions concerning (1.1) and \( x^\ast \) are Assumption 1 and the source condition (1.9). Throughout the paper we also assume that Lipschitz condition (1.2) holds with \( x^\ast \) being the desired quasi–solution.

The results of the article complement the studies of [1–3] related to iteratively regularized methods for approximating usual solutions of equations (1.1). The transition to a much more general problem of approximating a quasi–solution implemented in the work, see (1.11), was possible, firstly by modifying the iterative process used, cf. (1.4) and (1.12), and secondly due to the introduction of additional Assumption 1. However, we note that results of [2,3] also rely on various additional structural conditions on the nonlinearity of the operator \( \Phi \), e.g. the range invariance condition and the tangential cone condition [2, p.135]. We also note that for methods of regularization of linear ill-posed problems under Condition 1, estimates of the form (1.10) were obtained back in the 1980s, see [10] and Chapter 6 in [11].

The projected version (2.33) of the iterative process (1.12) with \( \Theta(\lambda, \alpha) = (\lambda + \alpha)^{-1} \) is used in the study of a SEIR–like epidemic model (3.1), as applied to the COVID–19 pandemic. The model includes a number of parameters that characterize the infectivity, lethality, and other properties of an infectious agent. To solve the system and predict the epidemic dynamics, one needs to know all its coefficients with high enough accuracy. At the same time, not all of them are known a priori. Therefore, the restoration of the unknown model coefficients from available statistical data is very relevant. We show that the proposed iterative methods can be used to solve this problem.

The article is organized as follows. Section 2 is devoted to substantiating the convergence of iterations (1.12) and (2.31). When using the a priori stopping criterion (2.7), an accuracy estimate of the obtained approximation is established, see Theorems 1, 3. Section 3 describes the investigated model of the COVID–19 epidemic dynamics. In Section 4 we present results of numerical experiments with the model. Some concluding remarks are given in Section 5.

2. Iterative process and accuracy estimate

In this section we investigate the process (1.12) and its projected version for approximating the quasi–solution \( x^\ast \) of equation (1.1). First we will give auxiliary results and introduce necessary notations. By (1.2) it follows that, e.g. [1],

\[
\Phi(x) - \Phi(y) = \Phi'(x)(x - y) + H(x, y), \quad x, y \in B_\delta(x^\ast).
\]

\[
\|H(x, y)\|_Y \leq \frac{1}{2}\|x - y\|^2_X.
\] (2.1)

Any solution \( x^\ast \) to the variational problem (1.11) fulfills the necessary minimum condition \( f'(x^\ast) = 0_X \), where \( f' \) is the Frechet gradient of the functional \( f(x) = \|\Phi(x) - f\|^2_Y \). Direct calculation proves that \( f'(x) = 2\Phi^n(x)(\Phi(x) - f) \). Hence,

\[
\Phi^n(x^\ast)(\Phi(x^\ast) - f) = 0_X.
\] (2.2)

By \( \sigma(B) \) we denote the spectrum of an operator \( B \in B(X, X) \). We recall the following known results on the closed range operators in Hilbert spaces.

**Lemma 1.** ([11]) Let \( A \in B(X, Y) \) be a closed range operator, and

\[
\mu = \inf\{\|Ax\|_Y : x \in X, x \perp \mathcal{N}(A), \|x\|_X = 1\}.
\]

Then \( \mu > 0 \), and \( \sigma(A^*A) \subset (0) \cup \{\mu^2, \|A\|^2_{B(X, Y)}\} \). Assume that

\[
\tilde{A} \in B(X, Y), \quad \|\tilde{A} - A\|_{B(X, Y)} \leq \eta, \quad 0 < \eta < \mu/2.
\]

Then

\[
\sigma(\tilde{A}^*\tilde{A}) \subset (0, \eta^2] \cup [(\mu - \eta)^2, \|\tilde{A}\|^2_{B(X, Y)}].
\]

**Lemma 2.** ([11]) Let conditions of Lemma 1 be fulfilled. Let \( P \) be the orthoprojector from \( X \) onto \( \mathcal{R}(A^*) \), and \( \tilde{P} \) be the orthoprojector onto the invariant subspace of \( A^*\tilde{A} \) corresponding to the part of \( \sigma(A^*A) \) in \( [(\mu - \eta)^2, \|\tilde{A}\|^2_{B(X, Y)}] \). Then

\[
\|(E_X - \tilde{P})P\|_{B(X, X)} \leq \frac{\eta}{\mu - \eta}.
\]

Below we need a parameter \( l > 0 \). Further restrictions on this parameter will be introduced as needed. We denote

\[
\eta(x) = l\|x - x^\ast\|_X, \quad D = (lI)^2,
\]

\[
\mu_\ast = \inf\{\|\Phi'(x^\ast)x\|_Y : x \in X, x \perp \mathcal{N}(\Phi'(x^\ast)), \|x\|_X = 1\}, \quad \nu_\ast = (1 - q)\mu_\ast.
\]
By Lemma 1, \( \mu_+, v_+ > 0 \).

**Remark 1.** We now describe a possible way of practical evaluation of the parameter \( \mu_+ \) that is included in conditions of the following theorems, see (2.6) and (2.9). Assume that there is an a–priori approximation \( x_0 \) to the required quasi–solution \( x^* \) with known error level \( \varepsilon \), i.e., \( \| x_0 - x^* \|_X \leq \varepsilon \). Assumptions of this kind are typical for the studies of iterative methods such as (1.12) or the classical Newton method. By (1.2), we have then \( \| \Phi'(x_0) - \Phi'(x^*) \|_{(X,Y)} \leq L \varepsilon \). Assume also that \( \Phi'(x_0) \) is a closed range operator. This assumption is satisfied for \( \Phi \) from the epidemiological model we consider in Sections 3–4.

Define \( \mu \) as in Lemma 1 for \( A = \Phi'(x_0) \). Then for the unknown parameter \( \mu_+ \) the estimate holds: \( |\mu_+ - \mu| \leq L \varepsilon [11, \text{Sect.6.1}] \).

Below we also need the following result.

**Lemma 3.** Assume that
\[
\| x - x^* \|_X \leq \min \left\{ \frac{q\mu_+}{L}, R \right\}, \quad q \in (0, 1/2).
\]

Then
\[
\sigma(\Phi'(x)\Phi'(x)) \subset [0, \eta^2(x)] \cup [v_0^2, M^2].
\]

**Proof.** For all \( x \in B_k(x^*) \), by (1.2) and (2.3) we have
\[
\| \Phi'(x) - \Phi'(x^*) \|_{(X,Y)} \leq \eta(x), \quad \mu_+ - \eta(x) \geq v_+.
\]

Using Lemma 1, with \( \tilde{A} = \Phi'(x) \) and \( A = \Phi'(x^*) \), from (2.5) we obtain (2.4). The lemma is proved. \( \Box \)

The following assumptions specify in more detail the class of filter functions in (1.12).

**Assumption 2.** There exists \( D_1 \) such that for all \( \alpha \in (0, \alpha_0] \),
\[
\max \{ |\Theta(\lambda^2, \alpha) \lambda| : \lambda \in [0, D\alpha^2] \cup [v_0^2, M^2] \} \leq D_1.
\]

**Assumption 3.** There exist \( D_2 \) and \( D_3 \) such that for all \( \alpha \in (0, \alpha_0] \),
\[
\max \{ |1 - \Theta(\lambda, \alpha) \lambda| \} \leq D_2, \quad \max \{ |1 - \Theta(\lambda, \alpha) \lambda| \} \leq D_3\alpha^p.
\]

**Remark 2.** Assumptions 2 and 3 are fulfilled for the function \( \Theta(\lambda, \alpha) = (\lambda + \alpha)^{-1} \) with \( p = 1 \). This filter function is associated with iteration (1.6). Further examples can be found in [1, Sect.2.1]. We only point to functions
\[
\Theta(\lambda, \alpha) = \frac{1}{\lambda} \left[ 1 - \left( \frac{\alpha}{\lambda + \alpha} \right)^N \right], \quad \Theta(\lambda, \alpha) = \begin{cases} \lambda^{-1}[1 - (1 - \gamma)\lambda^{1/\omega}], & \lambda \neq 0, \\ \omega^{-1}, & \lambda = 0, \end{cases}
\]

with \( N \in \mathbb{N}, \gamma \in (0, 2/M^4) \), which lead to the iteratively regularized processes on the basis of iterated Tikhonov’s method and gradient method, respectively, [1, Sect.4.2]. Assumptions 2 and 3 are fulfilled for these functions with \( p = N \) and with any \( p \in \mathbb{N} \), respectively.

In original iterative regularization methods (1.4), the most commonly used generating function \( \Theta \) is \( \Theta(\lambda, \alpha) = (\lambda + \alpha)^{-1} \) with \( p = 1 \). In Sections 3 and 4, we apply the iteration (2.31) with this \( \Theta \) to parameter reconstruction problem in an epidemiological model. We remark that the accuracy order of the estimate (2.26), established below for the methods (1.12), (2.31), does not depend on \( p \).

Extensive practice of using iteratively regularized methods of type (1.4) shows that these methods do not converge, as \( n \to \infty \). Therefore the quasi–solution \( x^* \) should be approximated by a point \( x_N \) generated by a suitable stopping rule \( N = N(\delta) \) or \( N = N(\delta, f) \). In this paper, for stopping iterations we use the following a priori rule. Let \( \kappa > 0 \) and
\[
\delta \leq \kappa \alpha_0^p.
\]

We put
\[
N(\delta) = \max \{ n \in \mathbb{N} : \delta \leq \kappa \alpha_0^p \}.
\]

By \( \chi_{[a,b]} \) we denote the indicator function of the segment \( [a, b] \subset \mathbb{R} \),
\[
\chi_{[a,b]}(\lambda) = \begin{cases} 1, & \lambda \in [a, b], \\ 0, & \lambda \in \mathbb{R} \setminus [a, b]. \end{cases}
\]

First we prove the following auxiliary theorem.

**Theorem 1.** Let Assumptions 1–3 and conditions (1.2), (1.3), (1.5), and (1.9) be fulfilled, and \( \{x_n\} \) be generated by (1.12). Assume that for some \( l > 0, p \geq 1 \) and \( n \geq 0 \),
\[
0 < \alpha_0 \leq \left( \frac{q\mu_+}{L} \right)^{1/p}, \quad l\alpha_0^p \leq R.
\]
\[ \|x_n - x^*\|_X \leq l\alpha_n^p. \] \hspace{1cm} (2.10)

Then
\[ \|x_{n+1} - x^*\|_X \leq D_1 \left( \frac{1}{2} L \|x_n - x^*\|_X^2 + \delta \right) + \]
\[ + D_4 \|\xi - x^*\|_X \alpha_n^p + LD_1 \|\Phi(x^*) - f\|_Y \|x_n - x^*\|_X, \] \hspace{1cm} (2.11)

where \( D_4 = LD_2/\upsilon_+ D_3. \)

**Proof.** By (1.5) and (2.9), \( \|x_n - x^*\|_X \leq l\alpha_n^p < l\alpha_0^p \leq R, \) therefore \( x_n \in B_R(x^*). \) Hence according to (2.1), \( \Phi(x_n) = \Phi(x^*) + \Phi'(x_n)(x_n - x^*) + H(x_n, x^*), \) and from (1.12) we get
\[
x_{n+1} - x^* = (E_X - \Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]^2)(\xi - x^*) -
\]
\[-\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\Phi''(x_n).\]
\[
H(x_n, x^*) + (\Phi(x^*) - f) + (f - f_0). \]

We recall that \( \|\Phi'(x)\|_{\mathcal{B}((X, Y))} \leq M \) for all \( x \in B_R(x^*). \) Consequently,
\[
\|x_{n+1} - x^*\|_X \leq
\]
\[
\leq M \|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\|_{\mathcal{B}((X, Y))} \left( \frac{1}{2} L \|x_n - x^*\|_X^2 + \delta \right) +
\]
\[ + \|E_X - \Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]^2\)(\xi - x^*)\|_X +
\]
\[ + \|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\Phi''(x_n)\Phi'(x_n)\|_X \|\Phi(x^*) - f\|_Y \|x_n - x^*\|_X. \] \hspace{1cm} (2.12)

For the third term in the right part of (2.12), using (2.2) we obtain
\[
\|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\|_X =
\]
\[
= \|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)](\Phi''(x_n) - \Phi''(x^*))\Phi'(x^*)\|_X \leq
\]
\[ \leq L \|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\|_{\mathcal{B}(X, X)} \|\Phi(x^*) - f\|_Y \|x_n - x^*\|_X. \] \hspace{1cm} (2.13)

From (2.12) and (2.13) it follows that
\[
\|x_{n+1} - x^*\|_X \leq
\]
\[
\leq \|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\|_{\mathcal{B}(X, X)} \times
\]
\[ \cdot \left\{ M \left( \frac{1}{2} L \|x_n - x^*\|_X^2 + \delta \right) + L \|\Phi(x^*) - f\|_Y \|x_n - x^*\|_X \right\} +
\]
\[ + \|E_X - \Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]^2\)(\xi - x^*)\|_X. \] \hspace{1cm} (2.14)

We observe that by (2.9),
\[
\eta(x_n) = L \|x_n - x^*\|_X \leq Ll\alpha_n^p \leq Ll\alpha_0^p \leq q\mu., \] \hspace{1cm} (2.15)

Lemma 3 and (2.15) yield
\[
\sigma(\Phi''(x_n)\Phi'(x_n)) \subset [0, D\alpha_n^{2p}] \cup [v^2, M^2]. \] \hspace{1cm} (2.16)

For the operator norm in the first term of (2.14) we have representation
\[
\|\Theta([\Phi''(x_n)\Phi'(x_n)]^2, \alpha_n)[\Phi''(x_n)\Phi'(x_n)]\|_{\mathcal{B}(X, X)} =
\]
\[ = \max\{\|\Theta(\lambda^2, \alpha_n)\lambda\| : \lambda \in \sigma(\Phi''(x_n)\Phi'(x_n))\}. \]
Now from (2.16) and (2.6) it follows that

$$
\| \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \|_{B(\mathcal{H})} \leq \\
\leq \max[|\Theta(\lambda^2, \alpha_n)\lambda| : \lambda \in [0, D\alpha_n^2] \cup [v_n^2, M^2]] \leq D_1.
$$

(2.17)

Consider the second summand in the right part of (2.14). We denote

$$
P_n = \mathcal{X}_{[v_n^2, M^2]}(\Phi^\ast(x_n) \Phi'(x_n)).
$$

By (1.9) we conclude that $\xi - x^* \in \mathcal{R}(\Phi^\ast(x^*))$, therefore $\xi - x^* = P_n(\xi - x^*)$. Consequently $\xi - x^* = P_n(\xi - x^*) + (E_X - P_n)P_n(\xi - x^*)$, and

$$
\{ E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2 \}(\xi - x^*) = \\
= (E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2)P_n(\xi - x^*) + \\
+ (E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2) (E_X - P_n)P_n(\xi - x^*).
$$

(2.18)

We observe that

$$
\| (E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2)P_n(\xi - x^*) \|_X \leq \\
= \| \xi - x^* \|_X \max_{\lambda \in [v_n^2, M^2]} \{ 1 - \Theta(\lambda^2, \alpha_n)\lambda^2 \} \mathcal{X}_{[v_n^2, M^2]}(\lambda) = \\
= \| \xi - x^* \|_X \max_{\lambda \in [v_n^2, M^2]} \{ 1 - \Theta(\lambda, \alpha_n)\lambda \}.
$$

(2.19)

Using (2.19) and Assumption 3, we get

$$
\| (E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2)P_n(\xi - x^*) \|_X \leq \\
\leq D_3 \| \xi - x^* \|_X \lambda \alpha_n^p.
$$

(2.20)

Turning to the second summand in the right part of (2.18) we apply Lemma 1 to projectors $P = P_\ast$ and $\tilde{P} = P_n$. With the use of (2.15) we obtain

$$
\| (E_X - P_n)P_\ast \|_{B(\mathcal{H})} \leq \frac{\eta(x_n)}{\mu - \eta(x_n)} \leq \frac{L\alpha_n^p}{v_n} = C\alpha_n^p,
$$

(2.21)

where $C = L/v_n$. According to Assumption 3,

$$
\| E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2 \|_{B(\mathcal{H})} \leq D_2.
$$

Using (2.21) we deduce

$$
\| (E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2) (E_X - P_n)P_\ast(\xi - x^*) \|_X \leq \\
\leq \| E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2 \|_{B(\mathcal{H})} \\
\times \| (E_X - P_n)P_\ast(\xi - x^*) \|_X \leq CD_2 \| \xi - x^* \|_X \lambda \alpha_n^p.
$$

(2.22)

From (2.18), (2.20), and (2.22) we get

$$
\| (E_X - \Theta(\{ \Phi^\ast(x_n) \Phi'(x_n) \}^2, \alpha_n) \| \Phi^\ast(x_n) \Phi'(x_n) \|^2) (\xi - x^*) \|_X \leq \\
\leq D_4 \| \xi - x^* \|_X \lambda \alpha_n^p, \quad D_4 = CD_2 + D_3.
$$

(2.23)

Combining (2.14), (2.17), and (2.23) we arrive at the desired estimate (2.11). The theorem is proved. \(\square\)

Now we are ready to prove the main result on the accuracy estimate of iteratively regularized method (1.12) with the stopping rule (2.8).

**Theorem 2.** Let Assumptions 1–3 and conditions (1.2), (1.3), (1.5), (1.9), (2.7), (2.9), and (2.10) be fulfilled. Additionally assume that

$$
\| x_0 - x^* \|_X \leq I\alpha_0^p.
$$

(2.24)
\[ r^p \left\{ \frac{1}{2} D_1 l l_0^p + (D_1 \kappa + D_4 \| x^* \|_X) / l + LD_1 \| \Phi(x) - f \|_Y \right\} \leq 1. \]  

(2.25)

Then for the process (1.12) with the stopping rule (2.8), the estimate holds:

\[ \| x_{n(\delta)} - x^* \|_X < \kappa^{-1} \delta. \]  

(2.26)

**Proof.** In assumptions of this theorem, Theorem 1 holds true. We prove by induction the estimate

\[ \| x_n - x^* \|_X \leq \kappa \alpha_n^p, \quad n = 0, 1, \ldots, N(\delta). \]  

(2.27)

From (2.9) and (2.24) we get

\[ \| x_0 - x^* \|_X \leq \kappa \alpha_0^p \leq R. \]

Assume that (2.27) holds for a fixed number \( 0 \leq n \leq N(\delta) - 1 \). Then \( \| x_n - x^* \|_X \leq \kappa \alpha_n^p \leq \kappa \alpha_{n+1}^p \leq R \). From (1.5) and (2.11) it follows that

\[ \| x_{n+1} - x^* \|_X \leq \frac{1}{2} D_1 l l_0^p + (D_1 \kappa + D_4 \| x^* \|_X) \alpha_n^p + LD_1 \| \Phi(x^*) - f \|_Y \alpha_n^p \leq r^p \left\{ \frac{1}{2} D_1 l l_0^p + (D_1 \kappa + D_4 \| x^* \|_X) / l + LD_1 \| \Phi(x^*) - f \|_Y \right\} \alpha_{n+1}^p. \]  

(2.28)

In view of (2.25), estimate (2.28) yields \( \| x_{n+1} - x^* \|_X \leq \kappa \alpha_{n+1}^p \). Now (2.27) follows by the induction principle. According to (2.8), \( \alpha_{n(\delta)}^p < \delta / \kappa \). Letting \( n = N(\delta) \) in (2.27), we obtain estimate (2.26). This proves the theorem. \( \square \)

Let us comment on the condition (2.25). It is not difficult to check that the following conditions are sufficient for (2.25):

\[ r^p l D_1 \| \Phi(x^*) - f \|_Y \leq 1 - \varepsilon, \quad \varepsilon \in (0, 1); \]  

(2.29)

\[ r^p D_1 l l_0^p \leq \varepsilon, \quad r^p (D_1 \kappa + D_4 \| x^* \|_X) \leq \frac{1}{2} l \varepsilon. \]  

(2.30)

Condition (2.29) is of a qualitative nature and imposes an upper bound on the residual value \( \| \Phi(x^*) - f \|_Y \), which is reached at the quasi-solution of equation (1.1). For a fixed \( \varepsilon \), the first inequality in (2.30) can be provided by selecting a sufficiently small \( l > 0 \). With a fixed \( \alpha_0 \), condition (2.9) also assumes that the value \( l \) is not large. The second condition in (2.30) is provided by choosing a sufficiently small \( \kappa > 0 \) in (2.7). In turn, by (2.7) this limits from above \( \delta \), the error level in input data. In addition, the second condition in (2.30) means that the value \( \| x^* - x \|_X \) must be small enough. This requirement is similar to the smallness condition of the initial residual \( \| x_0 - x^* \|_X \) that follows from (2.24). The proximity of an initial guess to the solution is also typical for classical iterative methods for solving regular nonlinear equations such as the Gauss–Newton method, gradient methods, e.g., [12,13].

For applied ill-posed problems, the situation is quite typical when a priori information of the form \( x^* \in Q \subset X \) is available about the desired (quasi)solution, where \( Q \) is a given closed convex set. Denote by \( P_Q \) the metric projection from \( X \) onto \( Q \).

\[ P_Q(x) \in Q, \quad \| P_Q(x) - x \|_X = \min \{ \| y - x \|_X : y \in Q \}. \]

Let us modify the process (1.12) by projection in the following way:

\[ x_{n+1} = P_Q(\bar{x}_{n+1}), \quad \bar{x}_{n+1} = x - \Theta([\Phi^*(x_n)] \Phi^*(x_n^0) \Phi(x_n) - f_\delta - \Phi^*(x_n) x - \xi_n). \]  

(2.31)

We obviously have \( x_n \in Q \) for all \( n \geq 1 \). Since \( \| P_Q(x) - P_Q(y) \|_X \leq \| x - y \|_X \) for all \( x, y \in X \), (2.31) yields

\[ \| x_{n+1} - x^* \|_X = \| P_Q(\bar{x}_{n+1}) - P_Q(x^*) \|_X \leq \| \bar{x}_{n+1} - x^* \|_X \leq \| E_x - \Theta([\Phi^*(x_n)] \Phi^*(x_n^0) \Phi(x_n) - f_\delta - \Phi^*(x_n) x - \xi_n) - \Theta([\Phi^*(x_n)] \Phi^*(x_n^0) \Phi(x_n) - f_\delta - \Phi^*(x_n) x - \xi_n) \|_X \]

\[ \leq \| (H(x_n, x^*) + (\Phi(x^*) - f) + (f - f_\delta)) \|_X \leq \| \} H(x_n, x^*) \|_X \] < 7
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\[ \leq M \frac{\Theta(\alpha_n) \Phi'(x_n) \Phi'(x_n)}{\varphi(x_n) \Phi'(x_n)} \leq \frac{1}{2} L \| x_n - x^* \|_2^2 + \delta \]

+ \left\| \Theta(\alpha_n) \Phi'(x_n) \right\|^2 + \| \Phi'(x_n) \|^2 \left( \xi - x^* \right) \|_X + \left\| \Theta(\alpha_n) \Phi'(x_n) \right\| \Phi'(x_n) \left( \Phi(x^*) - f \right) \|_X. \]

Arguing as above, we get the following result.

**Theorem 3.** In assumptions of Theorem 1, for the process (2.31) with the stopping rule (2.8), the estimate (2.26) holds.

If \( \Theta(\lambda, \alpha) = (\lambda + \alpha)^{-1} \), then (2.31) leads to the iteration

\[ x_{n+1} = \mathcal{D}_Q(\Xi_{n+1}), \quad \Xi_{n+1} = \xi - \left( \| \Phi'(x_n) \|^2 + \alpha_n \right)^{-1}. \]

\[ -\| \Phi'(x_n) \|^2 \Phi'(x_n) [ \Phi(x_n) - f_\beta - \Phi'(x_n) (x_n - \hat{\xi}) ] \]  

(2.32)

We now turn to an application of iterations (2.32).

### 3. Identification of coefficients in a SEIR-like model for the COVID-19 epidemic

Below we apply the iterative method (2.32) to recovery of unknown coefficients in a SEIR-type model of the COVID–19 epidemic in several regions. The model is described by the system of ODEs [14]:

\[
\begin{align*}
\dot{e}(t) &= (1 - e(t) - i(t) - r(t) - p(t))(\alpha(t) e(t) + \alpha(t) i(t)) - (\lambda + \rho) e(t) \\
i(t) &= \lambda e(t) - (\beta + \mu) i(t) \\
r(t) &= \beta i(t) + \rho e(t) \\
p(t) &= \mu i(t)
\end{align*}
\]

(3.1)

Here, \( e(t) \) is exposed population fraction, i.e., the number of individuals exposed to the virus but without having symptoms, \( i(t) \) is the infected population fraction (with symptoms), \( r(t) \) is the recovered population fraction, and \( p(t) \) is the number of individuals that pass away due to the disease, till the moment \( t \). All these functions are understood as relative values with respect to the total population size in the region under investigation. As in [14], we suppose that \( i(t) \) is the relative number of infected falling into the statistics, and \( e(t) \) is the relative number of infected people that do not fall into it. It is clear that the recovery statistics do not include all those who recovered, but only those who fell into the statistics while being infected, their relative number with respect to the population size we denote by \( r(t) \). Therefore in addition to the third equation in (3.1) we can write \( r(t) = \beta i(t) \). We consider the epidemic dynamics for \( t \in [t_0, T] \). In examples below, \( t_0 \) is ten days after the deaths started in the considered region, \( T = 99 \) days after \( t_0 \).

To use the model (3.1) for forecasting the epidemic dynamics, one needs to know the functional and scalar coefficients \( \alpha(t), \beta(t), \lambda, \rho, \beta, \mu \) as well as the initial values \( e(t_0), i(t_0), r(t_0), p(t_0) \). We start the discussion of model (3.1) with determination of coefficients \( \mu \) and \( \beta \). From (3.1) it follows that \( \mu = \frac{\ddot{p}(t) \dot{i}(t)}{\dot{i}(t)} \) and \( \beta = \frac{\ddot{r}(t) \dot{i}(t)}{\dot{i}(t)} \). However functions \( \ddot{p}(t) \dot{i}(t) \) and \( \dot{r}(t) \dot{i}(t) \), if constructed according to official statistics, usually turn out to be very sharply changing, which cannot be explained by objective properties of the virus activity over time. To overcome this difficulty, we assume that instead of true values of \( p(t), i(t), r(t) \), the functions \( \ddot{p}(t) \approx p(t), \dot{i}(t) \approx i(t - \delta), \dot{r}(t) \approx r(t - \tau) \) are available from the statistics. Here, \( \delta \) is a time delay between onset of symptoms and including the patients in the statistics of infected individuals. Similarly, \( \tau \) denotes a time delay between recovery (or disappearance of contagion) and including the patients in the statistics of recovered ones. Then we have

\[
\mu = \frac{\ddot{p}(t) \dot{i}(t)}{\dot{i}(t)} \approx \frac{\lambda \ddot{p}(t) \dot{i}(t)}{\dot{i}(t)} = \frac{\lambda \ddot{r}(t + \tau)}{\dot{r}(t + \tau)} \approx \frac{\lambda \ddot{p}(t) \dot{r}(t + \tau)}{\dot{r}(t + \tau)}.
\]

(3.2)

Here, \( \Delta \dot{f}(t) \) stands for a suitable formula for numerical differentiation of \( f(t) \). We choose the differentiation scheme

\[
\Delta \dot{f}(t) = \frac{1}{\Delta t} \left( \frac{1}{12} f(t - 2 \Delta t) - \frac{2}{3} f(t - \Delta t) + \frac{2}{3} f(t + \Delta t) - \frac{1}{12} f(t + 2 \Delta t) \right)
\]

with \( \Delta t = 5 \) (days). In numerical experiments, we plot functions \( \ddot{p}(t) \) and \( \ddot{r}(t) \) defined in (3.2) and pick the values \( \tau_1 \) and \( \tau_2 \) so that values of \( \ddot{p}(t) \) and \( \ddot{r}(t) \) fluctuate around some constants during a time interval of several weeks. Then we choose the obtained constants as \( \mu \) and \( \beta \) in (3.1), respectively.

The following formulae provide initial conditions for system (3.1):

\[
\begin{align*}
p(t_0) &= \ddot{p}(t_0), \quad i(t_0) = \frac{\ddot{i}(t_0)}{\mu} \approx \frac{\Delta \ddot{p}(t_0)}{\mu}; \\
e(t_0) &= \frac{\ddot{i}(t_0) + (\beta + \mu) \dot{i}(t_0)}{x} \approx \frac{\Delta \ddot{p}(t_0)}{x}, \\
\omega &= \frac{\ddot{i}(t_0) + (\beta + \mu) i(t_0)}{x} \approx \frac{\Delta \ddot{p}(t_0)}{x}.
\end{align*}
\]

(3.3)
In (3.3), we use the notation
\[ \Delta_2f(t) = \frac{\Delta f(t + \Delta t) - \Delta f(t)}{\Delta t}. \]

Let \( t_1, \ldots, t_{K-1} \) be reference moments of adding, removing or changing quarantine restrictions in a considered region, \( t_0 < t_1 < \cdots < t_{K-1} < t_K = T \). We assume that the functions \( \alpha(t), \) \( \alpha(t), \) which describe the infectiousness of exposed and infected individuals respectively, are constants on each interval \((t_{k-1}, t_k), 1 \leq k \leq K, \) namely,
\[ \alpha_e(t) = \alpha_{ek}, \quad \alpha_i(t) = \alpha_{ik}, \quad t \in (t_{k-1}, t_k). \] (3.4)

Consider the operator equation \( \Phi(x) = p, \) where \( x = (\alpha_{e1}, \ldots, \alpha_{ek}, \alpha_{i1}, \ldots, \alpha_{ik}, \kappa, \rho, r_0) \) is the vector of unknown coefficients, and \( r_0 = r(0). \) Here, \( \Phi(x) = \Phi(x)(t) \) is the function \( p(t) \) received from the system (3.1) with coefficients and initial conditions determined by (3.2) – (3.4) and the vector \( x. \) We have \( \Phi : \mathbb{R}^{2K+3} \rightarrow L_2(t_0, T). \) For determination of the vector \( x \) we use the method (2.32), (2.8) with \( f_3 = \tilde{p} \) and \( Q = \{x \in \mathbb{R}^{2K+3} | r_0 \geq 0\}. \) The Frechet derivative \( \Phi'(x) \) is the vector function
\[ \left( \frac{\partial p(t)}{\partial \alpha_{e1}}, \ldots, \frac{\partial p(t)}{\partial \alpha_{ek}}, \frac{\partial p(t)}{\partial \alpha_{i1}}, \ldots, \frac{\partial p(t)}{\partial \alpha_{ik}}, \frac{\partial p(t)}{\partial \kappa}, \frac{\partial p(t)}{\partial \rho}, \frac{\partial p(t)}{\partial r_0} \right). \]

These partial derivatives are evaluated in the usual way, e.g., [15, Ch.5]. The existence of the second derivative \( \Phi''(x) \) follows from Theorem 4.1 in [15, Ch.5]. Therefore, the condition (1.2) is fulfilled. Obviously, the Frechet derivative \( \Phi'(x) \) at any point \( x \in \mathbb{R}^{2K+3} \) is a closed range operator, as required in Assumption 1. After each iteration of (2.32), one can solve the system (3.1), (3.3) with the received coefficients and compare the component \( p(t) \) of the solution with the function \( \tilde{p}(t) \) available from statistical observations. If these functions are close to each other on the whole segment \([0, T]\), then one can stop iterations and use the received coefficients for forecasting of future epidemic dynamics. In examples below, instead of \( p(t) \) we plot absolute values \( \tilde{p}(t) = Mp(t), \) where \( M \) is the total population size in the region.

In numerical experiments, the equation \( \Phi(x)(t) = p(t) \) is approximated by the finite dimensional problem \( \Phi(x)(t_k) = \tilde{p}(t_k), k = 0, 1, \ldots, K, \) where \( K = 99 \) and \( t_k = T. \) In particular, \( \Phi(x_n) \) are \( 100 \times (2K + 3) \)-matrices, whose components are evaluated using a standard procedure for solving systems of ODEs.

4. Numerical experiments

In this section, we present results of numerical experiments with the model (3.1) applied to the spread of the COVID–19 epidemic in the Moscow region (Example 1) and in Italy (Example 2). The number of reference days is 2, so \( K = 3. \)

Example 1. We consider COVID–19 epidemic dynamics in the Moscow region (Russia), April 4, – July 12, 2020. The functions \( l(t), \) \( \tilde{r}(t), \) \( \tilde{p}(t) \) are taken from the official statistics.

As reference days \( t_1 \) and \( t_2 \) we choose the days of removing restrictions, June, 1 (non–food stores opened) and June, 16 (summer verandas of cafés opened). Analyzing the plots of \( \tilde{p}(t) \) and \( \tilde{p}(t), \) see (3.2), we conclude that two possibilities can occur with the time delays \( \tau_l \) and \( \tau_r. \)

**Case 1.** \( \tau_l = 2, \tau_r = 8, \) then \( \mu \approx 0.0008, \beta \approx 0.033, \) see Figs. 1, 2.

We use the method (2.32) with regularization parameters \( \alpha_n = 10^{-9} \cdot 2^{-n}. \) The initial guess is \( x_0 = \xi \) with \( \alpha_{e1} = \alpha_e(0) = 0.07, \alpha_{i1}^{(0)} = \alpha_i(0) = 0.15, \kappa(0) = 0.02, \rho(0) = 0.02, r_0 = 2.4 \cdot 10^{-6}. \) The results are shown on Figs. 3, 4. The coefficients received after 5 iterations are as follows: \( \alpha_{e1}^{(5)} = 0.080, \alpha_{i1}^{(5)} = 0.070, \alpha_{i1}^{(5)} = \alpha_i(5) = 0.15, \kappa(5) = 0.00046, \rho(5) = 0.019, r_0 = 0. \)

**Case 2.** \( \tau_l = 14, \tau_r = 8, \) then \( \mu \approx 0.00035, \beta \approx 0.015, \) see Figs. 5, 6.

Here we used the same initial approximations and parameters. The obtained results are given on Fig. 7. The coefficients received after 5 iterations are \( \alpha_{e1}^{(5)} = 0.072, \alpha_{e2}^{(5)} = \alpha_e(5) = 0.070, \alpha_{i1}^{(5)} = \alpha_i(2) = \alpha_i(3) = 0.15, \kappa(5) = 0.00070, \rho(5) = 0.022, r_0 = 0. \)

We see that the model (3.1), (3.3) with coefficients obtained in the both cases after 5 iterations of the method (2.32) describe the dynamics of deaths quite well. Recall that the parameters \( \alpha_e \) and \( \alpha_i \) characterize the contagiousness of the virus. In both cases, the iterations show that the values \( \alpha_{e2}, \alpha_{i2}, \alpha_{i3} \) of these parameters at the period after removing restrictions are not bigger than the values \( \alpha_{e1}, \alpha_{i1} \) at the period before removing restrictions. This suggests that removing restrictions did not lead to strengthening the epidemic. On the contrary, in both cases the coefficient \( \alpha_e \) decreased in June. Probably, this phenomenon is related with coming of warm season. But for a confident conclusion, more numerical experiments with different epidemiological models are required.

Example 2. COVID–19 epidemic dynamics in Italy, March 2, – June 9, 2020. We use the data compiled by the Johns Hopkins University Center for Systems Science and Engineering. We choose March, 9 (the beginning of strict isolation) and May, 18 (the end of strict isolation) as the reference points \( t_1 \) and \( t_2. \) Graphical analysis of \( \tilde{p}(t) \) and \( \tilde{p}(t), \) see Figs. 8, 9 gives the values \( \tau_l = \tau_r = 2 \) and \( \mu \approx 0.0125, \beta \approx 0.018. \) To identify remaining model parameters we use iterations (2.32) with \( \alpha_n = \)
The coefficients received after 7 iterations of the method are as follows: 
\( \alpha_e^{(7)} = 0.32, \alpha_{e2}^{(7)} = 0.30, \alpha_{e3}^{(7)} = 0.30, \alpha_{i1}^{(7)} = \alpha_{i2}^{(7)} = \alpha_{i3}^{(7)} = 0.50, \chi^{(7)} = 0.00017, \rho^{(7)} = 0.11, r_0^{(7)} = 0 \). We see that the coefficient \( \alpha_e \) decreased at the beginning of the lockdown, and this coefficient did not change considerably after removing strict isolation.

**Remark 3.** Comparing the plots of the function \( \tilde{p}(t) \) available from statistical observations with solutions \( p(t) \) to (3.1) obtained for reconstructed coefficients, we see that the stopping moments \( N = 5 \) and \( N = 7 \) used in Examples 1 and 2, provide the optimal data fitting. The same number \( N \) we get by the stopping rule (2.8) with \( \kappa = 1.2 \cdot 10^5 \) assuming that the maximal error level of the death statistics is 10 people per day. Here we take into account that the statistical data in Examples 1 and
2 is related to the initial period of the pandemic development. Then the error level of the relative death number in $L_2$-norm is $\delta \approx 5 \cdot 10^{-6}$ for Example 1 and $\delta \approx 1.7 \cdot 10^{-6}$ for Example 2.

To justify the choice $\kappa = 1.2 \cdot 10^5$, we have performed additional experiments with data from Examples 1 and 2 but with the regularization parameter sequence $\alpha_n = 10^{-6} \cdot 2^{-n}$ instead of $\alpha_n = 10^{-9} \cdot 2^{-n}$. Then by the rule (2.8) with $\kappa = 1.2 \cdot 10^5$ we obtain $N = 15$ for the both cases in Example 1 and $N = 17$ in Example 2. In Table 1 we present values of parameters obtained in this test.

We see that the values in Table 1 are mostly close to corresponding numbers listed above in Examples 1 and 2. Corresponding plots do not differ significantly from Figs. 3, 7, 10. Hence this choice of $\kappa$ is applicable to identification of the parameters of model (3.1) for two different regions and two significantly different sequences $\{\alpha_n\}$. This allows us to recommend the mentioned value for working with other regions and time intervals.
Fig. 5. Coefficient $\mu$, Case 2.

Fig. 6. Coefficient $\beta$, Case 2.

5. Concluding remarks

The theoretical part of this paper is an extension of previous work [8,9] on the behavior of Gauss–Newton type methods (1.4) for finding solutions of irregular nonlinear equations whose operators have closed range derivative at the solution. In the present work we prove that modified Gauss–Newton type processes (1.12) and (2.31) are applicable to stable approximation of quasi–solutions to such equations, and establish corresponding accuracy estimate, see Theorems 2,3. The obtained
estimate has the same order in $\delta$ as the previously known estimate for scheme (1.4), which relates to the case of solvable equations. The results also develop previous studies of [1–3] related to iteratively regularized methods for approximating classical solutions of equations (1.1). The transition to a more general problem of approximating a quasi–solution implemented in the work was possible, firstly by modifying the iterative process used, and secondly due to the introduction of additional Assumption 1.

The need to find quasi–solutions rather than solutions in the classical sense naturally arises when solving parameter identification problems in applied models. This is due to the approximate nature of models under consideration, which often excludes the presence of classical solutions of the associated operator equations. Typical examples are models of epidemiology, one of which is analyzed in this paper.

For a detailed overview of SEIR–like epidemic models and approaches to their analysis, see e.g., [14,16,17] and references therein. The approach to identification of coefficients of epidemiological models by reducing the problem to a nonlinear irregular equation and application of regularization methods, is fairly standard and widespread, e.g., [16,18,19]. In [18] the

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**Fig. 7.** Number of deaths in Moscow region received from the statistics (thick line) and from the model after 5 iterations (thin line), Case 2.

**Fig. 8.** Coefficient $\mu$, Italy.
approach was used to find a time-dependent coefficient in an epidemiological model similar to (3.1) describing the spread of Spanish flu in 1918 and Ebola in West Africa in 2014–2015. An alternative approach based on stochastic methods is developed in a recent paper [20].

We remark that the epidemiological model used in this paper is rather crude. To describe several waves of the epidemic, models should be developed in which all coefficients depend on time, and there is also a decline in immunity to the virus in those who have already had an infection. However, the numerical experiments presented in Section 4 prove that even this simplest model can predict the number of deaths for small time periods of about 20–30 days, not only when the increase in deaths is close to exponential, but also when it reaches a plateau (see Example 2). Note that large discrepancies between the function $i(t)$ obtained from the model with the coefficients found by method (2.32) and the incidence statistics are unavoidable in our approach. Indeed, the function $i(t)$ for a fixed $\mu$ is uniquely determined by $p(t)$ due to the fourth equation in (3.1). At the same time, our approach to the operator equation $\Phi(x) = p$ is aimed at finding coefficients that most accurately correspond to the observed number of deaths, with the possibility of predicting them. In turn, this approach
was chosen due to the greater objectivity of mortality statistics in comparison with morbidity statistics, which is why relying on the former in mathematical modeling is more preferable. Modification of the model with a time-dependent coefficient \( \mu \) can potentially help to solve this problem, and allow predicting morbidity as well.

**Supplementary materials**

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.amc.2022.127312.

**References**

[1] A.B. Bakushinsky, M.M. Kokurin, M.Y. Kokurin, Regularization algorithms for III-posed problems, Walter de Gruyter, Berlin, 2018.

[2] B. Kaltenbacher, A. Neubauer, O. Scherzer, Iterative regularization methods for nonlinear III-posed problems, Walter de Gruyter, Berlin, 2008.

[3] T. Schuster, B. Kaltenbacher, B. Hofmann, K. Kazimierski, K. regularization methods in banach spaces, Walter de Gruyter, Berlin (2012).

[4] O. Karabanova, M. Kokurin, A. Kozlov, Finite dimensional iteratively regularized Gauss-Newton type methods and application to an inverse problem of the wave tomography with incomplete data range, Inverse Probl Sci Eng 28 (2020) 637–661, doi:10.1080/17415777.2019.1628743.

[5] M.Y. Kokurin, A.I. Kozlov, Finite–dimensional iteratively regularized processes with an a posteriori stopping for solving regular nonlinear operator equations, Journal of Inverse and Ill–Posed Problems (2020) 10.1515/jiip–2020–0091.

[6] B. Kaltenbacher, J. Schicho, A multi-grid method with a priori and a posteriori level choice for the regularization of nonlinear ill–posed problems, Numerische Mathematik 93 (2002) 77–107.

[7] B. Kaltenbacher, A. Neubauer, Convergence of projected iterative regularization methods for nonlinear problems with smooth solutions, Inverse Probl 22 (2006) 1105–1119.

[8] M.Y. Kokurin, Iteratively regularized methods for irregular nonlinear operator equations with a normally solvable derivative at the solution, Comput. Math. Math. Phys. 56 (2016) 1523–1535, doi:10.1134/S0965542516090098.

[9] M.Y. Kokurin, Accuracy estimates of Gauss–Newton type iterative regularization methods for nonlinear equations with operations having normally solvable derivative at the solution, Journal of Inverse and Ill-posed Problems 24 (2016) 449–462, doi:10.1515/jiip-2016-0009.

[10] S.F. Gilyazov, V.A. Morozov, Optimal regularization of ill-posed normally solvable operator equations, USSR Computational Mathematics and Mathematical Physics 24 (6) (1984) 89–92.

[11] G.M. Vainikko, A.Y. Veretennikov, Iterative procedures in III-posed problems, Nauka,Moscow, 1986.

[12] A.B. Bakushinsky, M.Y. Kokurin, A. Smirnova, Iterative methods for III-posed problems, An Introduction, Walter de Gruyter, Berlin, 2010.

[13] P. Deuflhard, Newton Methods for Nonlinear Problems, Springer, Berlin, 2004.

[14] R. Sameni, Mathematical modeling of epidemic diseases, A case study of the COVID–19 Coronavirus (2020).

[15] P. Hartman, Ordinary Differential Equations, John Wiley, N.Y., 1964.

[16] S.I. Kabanihkin, O.I. Krivorotko, D.V. Ermolenco, V.N. Kashtanova, V.A. Latyschenko, Inverse problems in immunology and epidemiology, Eurasian J. Math. Comput. Appl. 5 (2017) 14–35, doi:10.32523/2306–3172–2017–5–2–14–35.

[17] W.E. Schiesser, Computational modeling of the COVID–19 disease, Numerical ODE Analysis with R Programming, World Scientific Publishing Co., Singapore, 2020.

[18] A. Smirnova, L. deCamp, G. Chowell, Forecasting epidemics through nonparametric estimation of time–dependent transmission rates using the SEIR model, Bull Math Biol 81 (2019) 4343–4365, doi:10.1007/s11538-017-0284-3.

[19] A. Smirnova, A. Bakushinsky, On iteratively regularized predictor–corrector algorithm for parameter identification, Inverse Probl 36 (2020) 125015.

[20] O.I. Krivorot’ko, S.I. Kabanihkin, N.Y. Zya’tKov, A.Y. Prikhod’ko, N.M. Prokhoshin, M.A. Shishlenin, Mathematical modeling and forecasting of COVID–19 in moscow and the novosibirsk region, Numer. Anal. Appl. 23 (2020) 332–348.