Movement of external load over free surface of fluid in the ice channel

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Abstract. The solution of a steady three-dimensional problem for the wave disturbances induced by a local pressure distribution moving with uniform speed on open water lead between two semi-infinite floating ice sheets (ice channel) is presented. This external load simulates an air-cushion vehicle (ACV). The problem is formulated within the linear hydroelastic theory. The fluid is assumed to be inviscid and incompressible and its motion is potential. The ice sheets are treated as viscoelastic thin plates. The solution of this problem is constructed using the Fourier transforms and the Wiener-Hopf technique. The displacements of free surface and ice sheets are determined as well as strains in ice and wave resistance acting on ACV at various speeds of its movement: subcritical and supercritical ones relative to the minimum phase velocity of flexural-gravity waves (FGW) in fluid under an ice cover. Special attention is paid to the characteristics of edge waves. It is shown that for some values of load speed, ice thickness and external pressure, ice fracture near the edge is possible.

1. Introduction
The hydrodynamic aspects of ACV can be studied by assuming its action to be equivalent to that of pressure distribution acting on the free surface of water [1]. Disturbances induced by the motion of pressure distribution over an infinitely extended free surface have been thoroughly studied (see, e.g. [2]). However, in polar regions of the World Ocean, there are situations when the vehicle moves along the edge of ice sheet. Previously, we received the solution of steady three-dimensional problem for FGW generated by a local pressure distribution moving with uniform speed over semi-infinite free surface along the rectilinear edge of ice sheet [3, 4]. It is known [5, 6] that there are resonances and trapping modes at the movement of the vessel in an ice channel or in the channel with rigid walls. There are edge waves – waveguide modes propagating along the channel and exponentially decaying in the perpendicular direction. A crack in the ice cover is a limiting case of the ice channel when the channel width tends to zero [5]. When the load moves along the ice cover with a crack, there are two edge modes with different wave numbers and only at supercritical load speeds [7]. The wave with lower wave number propagates behind the load and one with higher wave number propagates in front of it.

In this paper, the approximate analytical solution of the steady problem at uniform motion of ACV in the ice channel is presented using the Fourier transforms and the Wiener-Hopf technique. The approximation is so that we take into account the structural damping only to evaluate the real roots of the dispersion relation for FGW to shift them from the real axis to the complex plane. Then the inverse Fourier transform is carried out without difficulties.
2. Mathematical formulation

The water is taken to be of constant density $\rho_0$ and uniform depth $H$. The ice sheets are modelled by two thin viscoelastic semi-infinite plates of identical thickness $h$ floating on the water surface and separated by an open water lead of width $L$. The edges of plates are free. The plate drafts are ignored. The pressure distribution $q(x, y)$ moves with constant speed $U$ along the rectilinear edges of plates. The moving Cartesian coordinate system $x, y, z$ is considered with the $x$-axis passing through the center of pressure region perpendicular to the plate edges, the $y$-axis is directed along the edge of left plate and the $z$-axis is directed vertically upwards.

The boundary-value problem for the velocity potential $\varphi(x, y, z)$ and the free surface elevation or plate deflection $w(x, y)$ can be written as

$$\Delta \varphi = 0 \quad (|x|, |y| < \infty, \quad -H \leq z \leq 0), \quad \Delta_3 \equiv \Delta_2 + \partial^2 / \partial z^2, \quad \Delta_2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2,$$

$$D \left(1 - \tau U \frac{\partial}{\partial y}\right) \Delta_2^2 w + \rho h U^2 \frac{\partial^2 w}{\partial y^2} + g \rho_0 w - \rho_0 U \frac{\partial \varphi}{\partial y} \bigg|_{z=0} = 0 \quad (x < 0, \quad x > L, \quad |y| < \infty),$$

$$gw - U \frac{\partial \varphi}{\partial y} \bigg|_{z=0} = -\frac{g(x, y)}{\rho_0} \quad (0 < x < L, \quad |y| < \infty), \quad \frac{\partial \varphi}{\partial z} \bigg|_{z=0} = -U \frac{\partial w}{\partial y} \quad \frac{\partial \varphi}{\partial z} \bigg|_{z=-H} = 0,$$

$$\left(\frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{\partial y^2}\right) w = 0, \quad \frac{\partial^2}{\partial x} \left[\frac{\partial^2}{\partial x^2} + (2 - \nu) \frac{\partial^2}{\partial y^2}\right] w = 0 \quad (x = 0-, \quad L+, \quad |y| < \infty).$$

Here $D = Eh^3 / [12(1 - \nu^2)]$; $E$, $\nu$, $\rho$ are Young’s modulus, Poisson’s ratio, the density of ice sheets, respectively; $\tau = \eta / E$ is the retardation time, $\eta$ is the viscosity of ice; $g$ is the acceleration due to gravity. For wave motion the decaying conditions should be satisfied far from the pressure region.

We restrict our consideration to the pressure distribution in the form [1]

$$q(x, y) = q_0 \{\tanh[x(y + b)] - \tanh[x(y - b)]\} [H(x - x_0 + a) - H(x - x_0 - a)]/2,$$

where $q_0$ is a nominal pressure, $a$ and $b$ are, respectively, the half-beam and half-length of the pressure region whose center is located at the point $(x = x_0 > a, \quad y = 0)$, $x$ is the smoothing parameter, $H(\cdot)$ is the Heaviside function.

We are interested in the bending stresses of ice sheets. In particular, it is of practical interest to know whether the moving load can lead to stresses large enough to break the ice near the edge. The strain tensor $e(x, y)$ is given by the matrix

$$e(x, y) = -\frac{h}{2} \left[ \begin{array}{ccc} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2} \end{array} \right].$$

This tensor describes the strain field in the ice sheet. To find the maximum strain $e_m$ in the ice sheet, we need to find the largest eigenvalue of strain tensor (6) at each location. The fracture strain was determined for the Bering Sea ice as $4.4 - 8.5 \times 10^{-5}$ in [8]. In this study we use the estimate of the yield strain $e_*$ for destruction of an ice cover as $e_* = 8 \cdot 10^{-5}$ (see [9] and discussion of this value there).

The forces $F_x$ (side force) and $F_y$ (wave resistance) acting on ACV and its non-dimensional values $A_x$, $A_y$ are determined by formulas

$$\left(R_x, R_y\right) = \int_{x_0-a}^{x_0+a} \int_{-\infty}^{\infty} q(x, y) \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) dy dx, \quad (A_x, A_y) = \frac{g \rho_0}{2aq_0^2} (R_x, R_y).$$
3. Method of solution

We describe briefly the solution of problem (1-4) by the Wiener-Hopf technique. The dimensionless variables and parameters are introduced

\[ (x', y', z', a', b', x_0', L') = (x, y, z, a, b, x_0, L)/H, \ \mathcal{X}' = H\mathcal{X}, \ \epsilon = \tau U/H, \ \beta = D/(\rho_0 g H^4), \]

\[ F = U/\sqrt{g H}, \ \sigma = \rho h/\rho_0 H, \ q_0' = q_0/(\rho_0 g H), \ \phi = U H \phi(x, y, z), \ w = H W(x, y). \]

Below, the primes are omitted. We use the Fourier transforms to the variables \( x \) and \( y \) in the form

\[ \Phi(\alpha, s, z) = \int_{-\infty}^{\infty} e^{-isy} \int_{-\infty}^{\infty} \phi(x, y, z)e^{i\alpha x}dxdy. \]

From the Laplace equation (1) and the no-flux bottom condition (3), we have

\[ \Phi(\alpha, s, z) = C(\alpha, s)Z(\alpha, s, z), \ Z = \cosh[(z + 1)\sqrt{\alpha^2 + s^2}]/\cosh\sqrt{\alpha^2 + s^2}, \]

(8)

where \( C(\alpha, s) \) is unknown function. We introduce the functions \( G_+ (\alpha, s), \ G_1 (\alpha, s) \) in the following manner:

\[ G_- = \int_{-\infty}^{\infty} e^{-isy} \int_{-\infty}^{0} \left[ \left( \beta(1 - \epsilon) \frac{\partial}{\partial y} \right) \Delta_2^2 + 1 + \sigma F^2 \frac{\partial^2}{\partial y^2} \right] \frac{\partial \phi}{\partial z} + F^2 \frac{\partial^2 \phi}{\partial y^2} \mid_{z=0} e^{i\alpha x}dxdy, \]

(9)

\[ G_1 = \int_{-\infty}^{\infty} e^{-isy} \int_{0}^{L} \left[ \left( \beta(1 - \epsilon) \frac{\partial}{\partial y} \right) \Delta_2^2 + 1 + \sigma F^2 \frac{\partial^2}{\partial y^2} \right] \frac{\partial \phi}{\partial z} + F^2 \frac{\partial^2 \phi}{\partial y^2} \mid_{z=0} e^{i\alpha x}dxdy, \]

(10)

\[ G_+ = \int_{-\infty}^{\infty} e^{-isy} \int_{L}^{\infty} \left[ \left( \beta(1 - \epsilon) \frac{\partial}{\partial y} \right) \Delta_2^2 + 1 + \sigma F^2 \frac{\partial^2}{\partial y^2} \right] \frac{\partial \phi}{\partial z} + F^2 \frac{\partial^2 \phi}{\partial y^2} \mid_{z=0} e^{i\alpha(x-L)}dxdy. \]

(11)

Similar to these functions, we also introduce functions \( D_+ (\alpha, s), \ D_1 (\alpha, s) \) assuming \( \beta = \sigma = 0 \) in (9-11). Using (8), we can write

\[ D_- (\alpha, s) + D_1 (\alpha, s) + e^{i\alpha L} D_+ (\alpha, s) = C(\alpha, s) K_1 (\alpha, s), \]

(12)

\[ G_- (\alpha, s) + G_1 (\alpha, s) + e^{i\alpha L} G_+ (\alpha, s) = C(\alpha, s) K_2 (\alpha, s), \]

(13)

where \( K_1 (\alpha, s) \) and \( K_2 (\alpha, s) \) are the dispersion functions for the surface waves and FGW in a moving coordinate system:

\[ K_1 (\alpha, s) = \sqrt{\alpha^2 + s^2} \tanh \sqrt{\alpha^2 + s^2} - F^2 s^2, \]

\[ K_2 (\alpha, s) = [\beta(1 - i\epsilon s)(\alpha^2 + s^2)^2 + 1 - \sigma F^2 s^2] \sqrt{\alpha^2 + s^2} \tanh \sqrt{\alpha^2 + s^2} - F^2 s^2. \]

From boundary conditions (2,3), we have

\[ G_- (\alpha, s) = 0, \ G_+ (\alpha, s) = 0, \ D_1 (\alpha, s) = i\sigma Q(\alpha, s), \]

(14)

where \( Q(\alpha, s) \) is the Fourier transform of the function \( q(x, y) \) in (5)

\[ Q(\alpha, s) = \int_{x_0-a}^{x_0+a} \int_{-\infty}^{\infty} q(x, y)e^{i(\alpha x - sy)}dydx = \frac{2\pi \rho_0 e^{i\alpha x_0} \sin(\alpha a) \sin(sb)}{\alpha \pi \sinh[\pi s/(2\epsilon)]}. \]

Using (8-14), we have

\[ G_1 (\alpha, s) = C(\alpha, s) K_2 (\alpha, s), \ D_- (\alpha, s) + i\sigma Q(\alpha, s) + e^{i\alpha L} D_+ (\alpha, s) = C(\alpha, s) K_1 (\alpha, s). \]
Excluding the function \( C(\alpha, s) \) from these equations, we obtain
\[
D_-(\alpha, s) + i s Q(\alpha, s) + e^{i\alpha L} D_+(\alpha, s) = G_1(\alpha, s) K(\alpha, s), \quad K(\alpha, s) = K_1(\alpha, s) / K_2(\alpha, s).
\]

It is known that the dispersion relation for the free surface waves \( K_1(\gamma) \equiv \gamma \tanh \gamma - F^2 s^2 = 0 \) has two real roots \( \pm \gamma_0 \) and the countable set of imaginary roots \( \pm \gamma_j, \ j = 1, 2, \ldots \). At \( \epsilon = 0 \), the dispersion relation for FGW \( K_2(\mu) \equiv (\beta \mu^4 + 1 - \sigma F^2 s^2) \mu \tanh \mu - F^2 s^2 = 0 \) has two real roots \( \pm \mu_0 \), four complex roots \( \pm \mu_{-1}, \pm \mu_{-2}, \mu_2 = -\mu_{-1} \) (the bar denotes complex conjugation), and the countable set of imaginary roots \( \pm \mu_j, \ j = 1, 2, \ldots \). At \( \epsilon \neq 0 \), the values of roots are shifted from the real and imaginary axes. It is difficult to find them numerically since they can be close with complex roots \( \pm \mu_{-1}, \pm \mu_{-2} \). Therefore, we will take structural damping into account approximately, only for the root \( \mu_0 \), in order to shift the real root into the complex domain. Then the roots of the dispersion relations \( K_\alpha(\alpha, s) = 0 \) are \( \chi_j(s) = \sqrt{\gamma_j^2(s) - s^2} \) ( \( n = 1 \)) and \( \alpha_j(s) = \sqrt{\mu_j^2(s) - s^2} \) ( \( n = 2 \)). We will take these values in the upper half-plane or on the positive part of the real axis.

In accordance with the Wiener-Hopf technique, we factorize the function \( K(\alpha, s) \):
\[
K(\alpha, s) = K_-(\alpha, s) K_+(\alpha, s), \quad K_\pm(\alpha, s) = \frac{\mu \mp 1 \mu_2}{(\alpha \pm \alpha_{-1})(\alpha \pm \alpha_{-2})} N_\pm(\alpha, s), \quad N_\pm = \prod_{j=0}^{\infty} \frac{\mu_j(\alpha \pm \chi_j)}{\gamma_j(\alpha \pm \alpha_j)},
\]
where the functions \( K_\pm \) are analytical in the upper/lower parts of the complex plane \( \alpha \), respectively.

After some algebra we obtain the equation
\[
P(\alpha) \left[ \frac{D_+(\alpha, s)}{N_+(\alpha, s)} + V_+(\alpha, s) + q_1(s) \Omega_+(\alpha, s) \right] =
G_1(\alpha, s) K_-(\alpha, s) e^{-i\alpha L} - P(\alpha) [V_-(\alpha, s) + q_1(s) \Omega_-(\alpha, s)],
\]
where
\[
q_1(s) = \frac{\pi s q_0 \sin(sb)}{\sinh[\pi s/(2\alpha)]}, \quad \psi(\alpha) = \frac{e^{i\alpha(x_0+a)} - e^{i\alpha(x_0-a)}}{\alpha}, \quad P(\alpha) = \frac{(\alpha + \alpha_{-1})(\alpha + \alpha_{-2})}{\mu \mp 1 \mu_2},
\]
\[
V_\pm(\alpha, s) = \pm \frac{1}{2i\pi} \int_{-\infty+i\lambda}^{\infty+i\lambda} \frac{D_\pm(\zeta, s) e^{-i\zeta L} d\zeta}{(\zeta - \alpha) N_\pm(\zeta, s)}, \quad \Omega_\pm(\alpha, s) = \pm \frac{1}{2i\pi} \int_{-\infty+i\lambda}^{\infty+i\lambda} \frac{\psi(\zeta) e^{-i\zeta L} d\zeta}{(\zeta - \alpha) N_\pm(\zeta, s)}.
\]
The functions on the left-hand and right-hand sides of equation (15) are analytical in the lower and upper parts of the complex plane \( \alpha \), respectively. Then we have analytical function over the entire complex plane \( \alpha \). By Liouville’s theorem, this function is a polynomial. The degree of the polynomial is determined by the behavior of this function as \( |\alpha| \to \infty \) and is equal to one. Consequently, we can write
\[
\frac{D_+(\alpha, s)}{N_+(\alpha, s)} + V_+(\alpha, s) + q_1(s) \Omega_+(\alpha, s) = q_1(s) \left[ \frac{c_1(s) + c_2(s) \alpha}{P(\alpha)} \right],
\]
where \( c_1(s) \) and \( c_2(s) \) are constants.

Similar to equation (15), we can get the equation
\[
P(-\alpha) \left[ \frac{D_-(\alpha, s)}{N_-(\alpha, s)} + R_-(\alpha, s) + q_1(s) \Psi_-(\alpha, s) \right] =
G_1(\alpha, s) K_+(\alpha, s) - P(-\alpha) [R_+(\alpha, s) + q_1(s) \Psi_+(\alpha, s)],
\]
where \( R_\pm(\alpha, s) \) are constants.
where
\[ R_{\pm}(\alpha, s) = \pm \frac{1}{2i\pi} \int_{-\infty}^{\infty+i\lambda} \frac{D_{\pm}(\zeta, s)e^{i\zeta L}d\zeta}{(\zeta - \alpha)N_{\pm}(\zeta, s)}, \quad \Psi_{\pm}(\alpha, s) = \pm \frac{1}{2i\pi} \int_{-\infty}^{\infty+i\lambda} \frac{\psi(\zeta)d\zeta}{(\zeta - \alpha)N_{\pm}(\zeta, s)}. \]

Reasoning as above, we can write
\[ \frac{D_{-}(\alpha, s)}{N_{-}(\alpha, s)} + R_{-}(\alpha, s) + q_{1}(s)\Psi_{-}(\alpha, s) = q_{1}(s)\left[ \frac{d_{1}(s) + d_{2}(s)s}{P(-\alpha)} \right]. \]

Equations (16,17) compose the system of two integral equations. Integrals are evaluated by the calculus of residues. Unknown functions \( c_{1}(s), \ c_{2}(s) \) in (16) and \( d_{1}(s), \ d_{2}(s) \) in (17) are defined from the edge conditions (4). We introduce new variables
\[ \xi_{j}(s) = D_{+}(\chi_{j}, s)/[q_{1}(s)N_{+}(\chi_{j}, s)], \quad \zeta_{j}(s) = D_{-}(\chi_{j}, s)/[q_{1}(s)N_{-}(\chi_{j}, s)], \]

and as a result we obtain the infinite system of linear algebraic equations for determination of coefficients \( \xi_{j}, \ zeta_{j} \ (j = 0, 1, \ldots); \ c_{k}, \ d_{k} \ (k = 1, 2) \), which is solved by the truncation method. If the vehicle moves along the central line of the channel, then \( \xi_{j} = \zeta_{j} \) and the system of equations becomes simpler.

For some values of the parameter \( s \), the corresponding homogeneous system of these equations has nontrivial solutions, which are explained by the existence of edge modes in this problem [5, 6]. In the case of ice channel, there are many edge modes and they exist both at subcritical and supercritical load speeds. The analysis of the wave numbers of symmetric edge waves showed that at subcritical load speeds, the wave numbers of edge modes are almost independent of the plate thickness. At supercritical speeds, with an increase in the thickness of plates, the lower wave numbers of the edge modes become more densely arranged, while the higher ones practically do not depend on the thickness of plates.

The vertical displacements of ice sheets and free surface \( W(x, y) \) are determined by performing the inverse Fourier transform.

4. Numerical results
The following input data are used for ice sheets, water and external load: \( E = 5 \) GPa, \( \nu = 1/3 \), \( \rho = 900 \) kg/m\(^3\), \( \tau = 0.7 \) s, \( \rho_{0} = 10^{3} \) kg/m\(^3\), \( H = 100 \) m, \( q_{0} = 10^{2} \) Pa, \( a = 10 \) m, \( b = 20 \) m, \( \zeta = 5/b \). The width of ice channel is equal to \( L = 50 \) m, and ACV moves along its centerline, so that \( x_{0} = 25 \) m. The load speed \( U \) and the thickness of ice sheet \( h \) are varying in the calculations. The minimum phase velocity of FGWs for infinitely extended ice cover increases with increasing ice thickness and at \( h = 0.5, \ 1, \ 2 \) m is equal to 12.06, 15.59, 20.09 m/s, respectively.

Figure 1(a,b) shows the 3-D plots for the function \( w(x, y) \) for the motion of load at subcritical speed \( U = 10 \) m/s and \( h = 1 \) m and at supercritical speed \( U = 21 \) m/s and \( h = 2 \) m, respectively. At subcritical load speed, waves under the ice sheets are excited only near the edges and rapidly decay with distance from the edges. At supercritical velocities, FGWs are generated that propagate over large distances from the edges.

Figure 2(a,b) shows the vertical displacements of ice sheet and fluid along the ice edge at load speed \( U = 5 \) m/s and \( h = 0.5 \) and 1 m, respectively. The modulation of waves on the free surface is caused by the superposition of reflected waves from the edges of ice sheets. At subcritical load speeds, the amplitudes of the surface waves weakly depend on the thickness of ice and the amplitudes of FGWs become smaller with increasing thickness of ice. During the transition from a subcritical regime to a supercritical one, the amplitudes of FGWs increase significantly, and amplitudes of surface waves become smaller, since a significant portion of the energy of surface waves are converted to FGWs.
Figure 1. Wave patterns of the free surface (0 < x < 50 m) (blue) and the ice sheets (x < 0, x > 50 m) (grey) for U = 10 m/s, h = 1 m (a) and U = 21 m/s, h = 2 m (b).

Figure 2. Free surface elevation (blue curves) and the deflection of ice sheet (red curves) along the ice edge at x = 0, U = 5 m/s: h = 0.5 m (a) and h = 1 m (b).

The maximum strains of ice sheets are observed near the plate edges. These strains along the edge of ice sheet at x = 0 for h = 1 m and two different load speeds U = 12.5; 13 m/s are shown in figure 3(a). The strains are scaled with $e_\ast = 8 \times 10^{-5}$, where $e_\ast$ is the yield strain at which an ice sheet can break. We can see that at U = 12.5 m/s and at given value of the external load $q_0$, the ice sheets do not break whereas at speed U = 13 m/s and the same values h and $q_0$ the arising strains $e_m$ exceed the value $e_\ast$ and the destruction of ice can be possible.

The non-dimensional values of the wave resistance $A_y$ in (7) acting on moving ACV are presented in figure 3(b) depending on the load speed. The side force acting on ACV is identically zero when the vehicle moves along the center line of the channel. It can be seen that the influence of ice sheets on the wave resistance is manifested not only at near-critical speeds, but also at those subcritical speeds, where edge waves arise in front of the load. For comparison, the values of wave resistance for infinite free surface [1] and for the channel with rigid vertical walls at x = 0 and x = L [10] are shown.
Figure 3. (a) Strains $e_m/e_s$ along the ice edge at $x = 0$, $h = 1$ m: $U = 12.5$ m/s (1) and $U = 13$ m/s (2). (b) The non-dimensional values of wave resistance of ACV versus load speed.

5. Conclusion

The 3-D linear hydroelastic problem on the waves induced by a uniformly moving load over an open water channel embedded in an ice-covered ocean is solved. The influence of load speed and the thickness of ice sheets on the nature of generated surface waves and flexural-gravity waves in fluid under ice sheets, strains in ice sheets, as well as on the wave forces acting on moving load is investigated. Edge waves existing in the ice channel both at subcritical and supercritical speeds of the load motion are found. It is shown that at some values of load speed, the thickness of ice and external pressure, the destruction of ice sheet near the edge is possible.

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