CHAOTIC OSCILLATIONS OF LINEAR HYPERBOLIC PDE WITH VARIABLE COEFFICIENTS AND IMPLICIT BOUNDARY CONDITIONS

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ABSTRACT. In this paper, the chaotic oscillations of the initial-boundary value problem of linear hyperbolic partial differential equation (PDE) with variable coefficients are investigated, where both ends of boundary conditions are nonlinear implicit boundary conditions (IBCs). It separately considers that IBCs can be expressed by general nonlinear boundary conditions (NBCs) and cannot be expressed by explicit boundary conditions (EBCs). Finally, numerical examples verify the effectiveness of theoretical prediction.

1. Introduction. Over the past decades, there has been a great deal of interest in the research on chaos of dynamical systems. The theory of chaos in finite-dimensional dynamical systems, including both discrete maps and systems governed by ordinary differential equations, has been well-developed [8, 16, 19]. However, there is very little theory of chaos in systems governed by partial differential equations (PDEs). This is partly because the rigorous proof of the existence of their chaotic behaviors is challenging. For more results on the chaos in infinite-dimensional dynamical systems, we refer to [7, 15, 21, 22, 23, 24] and the references therein.

In recent twenty years, there have been lots of papers studying the chaotic oscillations in the systems governed by one-dimensional wave equations, see [1, 2, 3, 4, 5, 9, 10, 11, 12, 14] and the references therein. Interestingly, Chen et al. [6] studied chaotic dynamics of hyperbolic PDE with constant coefficients and van der Pol boundary condition. Li et al. [13] characterized the chaotic oscillations of hyperbolic PDE, which is factorizable but noncommutative, with van der Pol boundary condition. Recently, chaotic oscillations of second-order linear hyperbolic PDEs with implicit boundary conditions (IBCs) and general nonlinear boundary conditions (NBCs) are analysed respectively [18, 20]. Meanwhile, in paper [17], the superlinear boundary condition is proposed, which interacts with hyperbolic PDE to cause chaos. It is worth pointing out that most of the results on chaotic behaviors of the systems governed by hyperbolic PDE in the existing literature are obtained under the conditions of constant coefficients, one boundary condition being nonlinear and the other being linear. A naturally arising issue is whether there are

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chaotic oscillations in hyperbolic PDEs with variable coefficients and both ends of boundary conditions being IBCs including those that can be expressed by general NBCs and cannot in general be expressed by explicit boundary conditions (EBCs). This paper will address this issue.

Consider a one-dimensional linear second-order hyperbolic PDE of the form
\[
\left[ \frac{\partial}{\partial t} - d_1(x) \frac{\partial}{\partial x} + k_1(x) \right] \left[ \frac{\partial}{\partial t} + d_2(x) \frac{\partial}{\partial x} + k_2(x) \right] w(x, t) = 0, \quad x \in (0, 1), \quad t > 0
\]
where \(d_i(x) > 0\) and \(k_i(x), i = 1, 2\), are \(C^1\) on \([0, 1]\). For clarity, let us consider (1) with initial conditions
\[
w(x, 0) = w_0(x) \in C_1([0, 1]), \quad w_t(x, 0) = w_1(x) \in C([0, 1]), \quad x \in (0, 1),
\]
and IBCs
\[
B_1(w_1(0, t), w_x(0, t), w(0, t)) = 0, \quad t > 0, \quad B_2(w_1(1, t), w_x(1, t), w(1, t)) = 0, \quad t > 0,
\]
where \(B_i(\cdot, \cdot, \cdot): \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2\), is continuous function.

In [5], the authors proposed an effective method to characterize the chaotic oscillations of hyperbolic PDEs, where this method makes use of the exponential growth of total variation with time. The system (1)-(3) is called chaotic oscillations, if \(w_x(x, t)\) and \(w_t(x, t)\) are bounded and the lengths of curves \(\{w_x(x, t)|0 \leq x \leq 1\}\) and \(\{w_t(x, t)|0 \leq x \leq 1\}\) grow exponentially as time \(t\) increases. For more details on the chaotic oscillations of hyperbolic PDEs, we refer to [10, 13] and references therein.

Define linear operators
\[
L_1 = \frac{\partial}{\partial t} - d_1(x) \frac{\partial}{\partial x} + k_1(x),
\]
\[
L_2 = \frac{\partial}{\partial t} + d_2(x) \frac{\partial}{\partial x} + k_2(x).
\]
Then, it follows that
\[
L_1L_2w = 0,
\]
\[
L_2L_1w = [d_1(x)d_2'(x) - d'_1(x)d_2(x)]w_x + [d_1(x)k_2'(x) + d_2(x)k_1'(x)]w.
\]
In order to make sure \(L_1\) and \(L_2\) are commutative, i.e., \(L_1L_2 = L_2L_1\), one assumes that there exist constants \(\sigma > 0\) and \(\zeta \in \mathbb{R}\) such that
\[
(A_1') \quad d_1(x) - \sigma d_2(x) = 0, \quad \text{and} \quad \zeta = \zeta,
\]
\[
(A_2') \quad k_1(x) + \sigma k_2(x) = \zeta.
\]
We point out that such a kind of system is common but remains unexplored before. Main questions include how variable coefficients and IBCs are organized within a chaotic oscillation and whether chaotic oscillations in the system with the above variable coefficients and IBCs are different from those in the corresponding system with EBCs. In order to solve these questions, the key is to find a breakpoint of chaotic oscillations in hyperbolic PDE (1) with variable coefficients and some IBCs of (3).

The rest of this paper is organized as follows. Section 2 studies the chaotic oscillations of hyperbolic PDE with variable coefficients and general NBCs. Section 3 discusses the chaotic oscillations of hyperbolic PDE with variable coefficients and IBCs, which cannot in general be expressed by EBCs. Section 4 enumerates two examples to illustrate the correctness of theoretical results. One summarises our conclusions in Section 5.
2. Hyperbolic PDE with variable coefficients and general NBCs. In this section, we consider the chaotic oscillations of the system governed by hyperbolic PDE (1) with varying coefficients and IBCs (3) that can be expressed by general NBCs. The dynamics is given by system

\[
\begin{cases}
\left[ \frac{\partial}{\partial t} - d_1(x) \frac{\partial}{\partial x} + k_1(x) \right] \left[ \frac{\partial}{\partial t} + d_2(x) \frac{\partial}{\partial x} + k_2(x) \right] w(x,t) = 0, \\
w_x(0,t) + T_1(0)w(0,t) = F(w_1(0,t) + T_2w(0,t)), \quad t > 0, \\
w_x(1,t) + T_1(1)w(1,t) = G(w_1(1,t) + T_2w(1,t)), \quad t > 0, \\
w(x,0) = w_0(x) \in C^1([0,1]), \quad w_t(x,0) = w_1(x) \in C([0,1]), \quad x \in [0,1],
\end{cases}
\] (S₁)

where

\[
T_1(x) = \frac{k_3(x) - k_3(x)}{d_1(x) + d_2(x)} = \frac{k_2(x)}{d_2(x)} - \frac{\zeta}{\sigma + 1} \frac{1}{d_2(x)}, \\
T_2 = \frac{k_1(x) d_2(x) + k_2(x) d_1(x)}{d_1(x) + d_2(x)} = \frac{\zeta}{\sigma + 1}.
\] (4)

and \( F, G \in C^1(R) \cap C^2(R \setminus \{0\}) \). Assume that \( F \) and \( G \) are odd functions satisfying the following assumptions:

\begin{itemize}
  \item [(F₁)] \( F(0) = 0 \), \( F'\) is odd,
  \item [(G₁)] \( G(0) = 0 \), \( G'\) is odd,
  \item [(F₂)] \( F'(y) < \frac{1}{d_1^2} \), \( y \in R \), \( G \) is an odd function satisfying \( G' \leq 0 \), \( y \in R \),
  \item [(G₂)] \( G'(y) < \frac{1}{d_1^2} \), \( y \in R \), \( G \) is an odd function satisfying \( G' = \frac{1}{d_1^2} \), \( y \in R \),
  \item [(F₃)] \( \lim_{y \to -\infty} \frac{F(y)}{y} = +\infty \), \( \lim_{y \to +\infty} \frac{F(y)}{y} = -\infty \),
\end{itemize}

here, \( d_i^0 = d_i(0) \) and \( d_i^1 = d_i(1) \), \( i = 1, 2 \).

Define

\[
z(x,t) = e^{\lambda(x,t)}w(x,t),
\] (5)

where \( w \) satisfies (S₁) and

\[
\lambda(x,t) = \int_0^x T_1(s)ds + T_2 t.
\] (6)

Then, (S₁) can be transformed to

\[
\begin{cases}
\left[ \frac{\partial}{\partial t} - d_1(x) \frac{\partial}{\partial x} \right] \left[ \frac{\partial}{\partial t} + d_2(x) \frac{\partial}{\partial x} \right] z(x,t) = 0, \quad x \in (0,1), \quad t > 0, \\
e^{-\lambda(0,t)}z_x(0,t) = F(e^{-\lambda(0,t)}z_1(0,t)), \quad t > 0, \\
e^{-\lambda(1,t)}z_x(1,t) = G(e^{-\lambda(1,t)}z_1(1,t)), \quad t > 0, \\
z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), \quad x \in [0,1],
\end{cases}
\] (P₁)

To treat the system (P₁), define

\[
u(x,t) = d_2(x)z_x(x,t) + z_t(x,t), \quad v(x,t) = d_1(x)z_x(x,t) - z_t(x,t). \] (7)

The boundary condition of the left end \( x = 0 \) deduces a implicit equation, i.e.,

\[
d_2^0 F \left( e^{-\lambda(0,t)}d_2 v(0,t) - d_1^0 u(0,t) \right) d_1^0 + d_2^0 \\
\quad e^{-\lambda(0,t)}d_1 v(0,t) - d_1^0 u(0,t) + e^{-\lambda(0,t)}u(0,t) = 0,
\]

which determines the reflection relation as follow

\[
e^{-\lambda(0,t)}v(0,t) = \frac{1}{d_2^0} \left[ d_1^0 + d_2^0 \right] f(e^{-\lambda(0,t)}u(0,t)) + d_1^0 e^{-\lambda(0,t)}u(0,t) \equiv -\phi(e^{-\lambda(0,t)}u(0,t)),
\] (8)
where \( \phi(x) = -\frac{1}{d_1^2}[(d_1^2 + d_2^2)f(x) + d_2^2x] \) and \( y = f(x) \) is the unique real solution to the equation
\[
d_2^2F(y) + y + x = 0.
\]
In addition, the implicit equation at \( x = 1 \),
\[
d_1^2G \left( e^{-\lambda(1,t)} \frac{d_1^2u(1,t) - d_2^2v(1,t)}{d_1^2 + d_2^2} \right) - e^{-\lambda(1,t)} \frac{d_1^2u(1,t) - d_2^2v(1,t)}{d_1^2 + d_2^2} = e^{-\lambda(1,t)}v(1,t) = 0,
\]
which determines the other reflection relation as follow
\[
e^{-\lambda(1,t)}u(1,t) = \frac{1}{d_1^2} \left[ (d_1^2 + d_2^2)g(e^{-\lambda(1,t)}v(1,t)) + d_2^2e^{-\lambda(1,t)}v(1,t) \right] = -\psi(e^{-\lambda(1,t)}v(1,t)), \quad t > 0.
\]
where \( \psi(x) = -\frac{1}{d_1^2}[g(x) + x] \) and \( y = g(x) \) is the unique real solution to the equation
\[
d_1^2G(y) - y - x = 0.
\]
The initial conditions of \( u \) and \( v \) are
\[
u_0(x) = d_2(x)z_0'(x) + z_1(x) \quad \text{and} \quad v_0(x) = d_1(x)z_0'(x) - z_1(x),
\]
respectively. Furthermore, assume that \( u_0 \) and \( v_0 \) satisfy the compatibility conditions
\[
v_0(0) = -\phi(u_0(0)), \quad e^{-\lambda(1,0)}u_0(1) = -\psi(e^{-\lambda(1,0)}v_0(1)),
\]
which ensure that system \( (P_1) \) exists a \( C^1 \)-continuous solution \( z(x,t) \).

Let
\[
l_1(x) = \int_0^x \frac{ds}{d_1(s)}, \quad l_2(x) = \int_0^x \frac{ds}{d_2(s)}
\]
and
\[
l_1 = l_1(1), \quad l_2 = l_2(1).
\]

**Lemma 2.1.** For system \( (P_1) \) and \( u, v \) defined by (7), one has
\[
u(x,t) = c_1, \quad \text{along characteristics} \quad t + l_1(x) = c_1',
\]
\[
v(x,t) = c_2, \quad \text{along characteristics} \quad t - l_2(x) = c_2.
\]

**Proof.** It is easy to see that
\[
\begin{bmatrix}
\frac{\partial}{\partial t} - d_1(x) \frac{\partial}{\partial x} \\
\frac{\partial}{\partial t} + d_2(x) \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u(x,t) \\ v(x,t)
\end{bmatrix} = 0,
\]
which imply that (2.1) holds.

Define
\[
\Phi(\cdot) = \phi(e^{-p\cdot}), \quad \Psi(\cdot) = \psi(e^{-q\cdot}),
\]
where
\[
p = \int_0^1 \frac{k_1(s)}{d_1(s)} ds, \quad q = \int_0^1 \frac{k_2(s)}{d_2(s)} ds.
\]
Let \( \tau \in [0, l_1 + l_2] \), \( n \) be a nonnegative integer and \( t = \tau + n(l_1 + l_2) \). Applying method of characteristics as described in [1] and the two reflections (8) and (9), \( u(1,t) \) and \( v(0,t) \) can be solved as follows
\[
e^{-\lambda(1,t)}u(1,t) = (\Psi \circ \Phi)^n(e^{-\lambda(1,\tau)}u(1,\tau)),
\]
\[
e^{-\lambda(0,t)}v(0,t) = (\Phi \circ \Psi)^n(e^{-\lambda(0,\tau)}v(0,\tau)),
\]
where
\[ \Psi \circ \Phi(x) = \psi(e^{-x} \Phi(e^{-x} x)), \quad \Phi \circ \Psi(x) = \phi(e^{-x} \psi(e^{-x} x)). \]

Note that
\[ u(x, t) = u(1, t + l_1(x) - l_1), \quad v(x, t) = v(0, t - l_2(x)). \]

Then, one has
\[
\begin{align*}
& e^{-\lambda(t, x)} u(x, t) = e^{-\int_0^t \frac{h(x)}{\tau} ds} (\Psi \circ \Phi)(e^{-\lambda(1, \tau + l_1(x) - l_1)} u(x, \tau)), \\
& e^{-\lambda(t, x)} v(x, t) = e^{-\int_0^t \frac{h(x)}{\tau} ds} (\Phi \circ \Psi)(e^{-\lambda(0, \tau - l_2(x))} v(x, \tau)),
\end{align*}
\]

(10)

where
\[
\begin{align*}
e^{-\lambda(1, \tau + l_1(x) - l_1)} u(x, \tau) &=\begin{cases}
e^{-\xi(1, \tau + l_1(x) - l_1)} u_0(h_1(x, \tau)), & 0 \leq \tau \leq l_1 - l_1(x), \\
-\Psi(e^{-\xi(1, \tau + l_1(x) - l_1)} u_0(h_2(x, \tau))), & l_1 - l_1(x) < \tau \leq l_1 + l_2 - l_1(x), \\
\Psi \circ \Phi(e^{-\xi(0, \tau + l_1(x) - l_1 - l_2)} u_0(h_3(x, \tau))), & l_1 + l_2 - l_1(x) < \tau \leq l_1 + l_2,
\end{cases} \\
e^{-\lambda(0, \tau - l_2(x))} v(x, \tau) &=\begin{cases}
e^{-\xi(0, \tau - l_2(x))} v_0(h_4(x, \tau)), & 0 \leq \tau \leq l_2(x), \\
-\Phi(e^{-\xi(0, \tau - l_2(x))} u_0(h_5(x, \tau))), & l_2(x) < \tau \leq l_1 + l_2(x), \\
\Phi \circ \Psi(e^{-\xi(0, \tau - l_2(x) - l_1)} u_0(h_6(x, \tau))), & l_1 + l_2(x) < \tau \leq l_1 + l_2,
\end{cases}
\end{align*}
\]

and \( h_i(x, \tau), i = 1, \ldots, 6, \) satisfy
\[
\begin{align*}
\int_x^{h_1(x, \tau)} \frac{d\tau}{d_1(s)} &= \tau, & \int_x^{h_2(x, \tau)} \frac{d\tau}{d_2(s)} &= \tau + l_1(x) - l_1, \\
\int_x^{h_3(x, \tau)} \frac{d\tau}{d_1(s)} &= \tau - l_1 - l_2, & \int_x^{h_4(x, \tau)} \frac{d\tau}{d_2(s)} &= \tau, \\
\int_0^{h_5(x, \tau)} \frac{d\tau}{d_1(s)} &= \tau - l_2(x), & \int_x^{h_6(x, \tau)} \frac{d\tau}{d_2(s)} &= \tau - l_1 - l_2.
\end{align*}
\]

(11)

Therefore, one can see that the dynamical behavior of the solution \( u \) or \( v \) is determined by the iterates of the maps \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \).

Now, one analyzes the properties of \( \Phi \) and \( \Psi \):
\[
\begin{align*}
\Phi(x) &= \phi(e^{-x} x) = -\frac{1}{\tau^2} [(d_0^1 + d_0^2) f(e^{-x} x) + d_0^1 e^{-x} x], \\
\Psi(x) &= \psi(e^{-x} x) = -\frac{1}{\tau^2} [(d_1^1 + d_1^2) g(e^{-x} x) + d_1^1 e^{-x} x],
\end{align*}
\]

where \( y = f(e^{-x} x) \) and \( y = g(e^{-x} x) \) satisfy
\[
d_0^2 F(y) + y + e^{-x} x = 0 \quad \text{and} \quad d_1^1 G(y) - y - e^{-x} x = 0,
\]
respectively. Since \( F \) and \( G \) are odd, it follows that \( \Phi \) and \( \Psi \) are odd.

One recalls the definitions of fixed point and critical point for an map \( h \).

**Definition 2.2.** The point \( x^* \) is called a fixed point of the map \( h \) if it satisfies \( x^* = h(x^*) \); the point \( x^* \) is called a critical point of the map \( h \) if it satisfies \( h'(x^*) = 0 \) or \( h'(x^*) = \infty \).

In the following two lemmas, one considers the fixed points, roots, critical points and extreme values of \( \Phi \) and \( \Psi \), respectively, where fixed points, roots and critical points are chosen closest to the origin.
Lemma 2.3. Let \((F_1)-(F_4)\) be satisfied. Suppose that \(e^{-p} \leq \frac{d_1^2}{d_4^1}\). Then the function \(\Phi(x)\) has the following properties:

(i) \(\Phi(0) = 0\);

(ii) \(\Phi(x) = 0\) has nonzero real roots \(-x_1^*\) and \(x_1^*\),
\[x_1^* = -e^p[d_2^pF(y_1^*) + y_1^*] = -\frac{d_1^0 + d_2^p}{d_1^0}e^p y_1^*,\]
where \(y_1^* = f(e^{-p}x_1^*)\) and \(d_1^0F(y_1^*) - y_1^* = 0\);

(iii) \(\Phi(x)\) has nonzero fixed points \(-x_1^*\) and \(x_1^*\),
\[x_1^* = -e^p[d_2^pF(y_1^*) + y_1^*] = -\frac{d_1^0 + d_2^p}{d_1^0}e^p + d_2^p y_1^*,\]
where \(y_1^* = f(e^{-p}x_1^*)\) and \(F(y_1^*) + \frac{1-e^{-p}}{e^p + d_2^p} y_1^* = 0\);

(iv) \(-\Phi(x)\) has no fixed point on \(\mathbb{R} \setminus \{0\}\);

(v) \(\Phi(x)\) has a local maximum value \(M_1\),
\[M_1 = \Phi(x_1^*) = d_1^0F(y_1^*) - y_1^*,\]
where \(x_1^*\) is critical point with
\[x_1^* = -e^p[d_2^pF(y_1^*) + y_1^*],\]
and \(y_1^* = f(e^{-p}x_1^*)\), \(d_1^0F(y_1^*) - 1 = 0\).

Proof. The proofs of (i)-(iii) and (v) are straightforward.

(iv) It is apparent from \(e^{-p} = \frac{d_2^p}{d_1^0}\) that \(-\Phi(x)\) has no fixed point on \(\mathbb{R} \setminus \{0\}\). For \(e^{-p} \neq \frac{d_2^p}{d_1^0}\), solving equation \(-\Phi(x) = x\) gives
\[F(y) + \frac{e^{-p}+1}{d_2^p e^p - d_1^0} y = 0,\]  
where \(y = f(e^{-p}x)\). By \(e^{-p} < \frac{d_2^p}{d_1^0}\) in \((F_2)\), one has
\[\frac{e^{-p}+1}{d_2^p e^p - d_1^0} < -\frac{1}{d_2^p}.\]
This together with \((F_2)\) implies that equation \((12)\) has no real solution on \(\mathbb{R} \setminus \{0\}\). □

Lemma 2.4. Let \((G_1)-(G_4)\) be satisfied. Suppose that \(e^{-q} \leq \frac{d_1^1}{d_2^1}\). Then the function \(\Psi(x)\) has the following properties:

(i) \(\Psi(0) = 0\);

(ii) \(\Psi(x) = 0\) has nonzero real roots \(-x_2^*\) and \(x_2^*\), where
\[x_2^* = e^q[d_1^1G(y_2^*)] - y_2^* = -\frac{d_1^1 + d_2^1}{d_2^1} e^q y_2^*,\]
where \(y_2^* = g(e^{-q}x_2^*)\) and \(d_1^1G(y_2^*) + y_2^* = 0\);

(iii) \(\Psi(x)\) has nonzero fixed points \(-x_2^*\) and \(x_2^*\), where
\[x_2^* = e^q[d_1^1G(y_2^*)] - y_2^* = -\frac{d_1^1 + d_2^1}{d_1^1} e^q y_2^*,\]
where \(y_2^* = g(e^{-q}x_2^*)\) and \(G(y_2^*) + \frac{e^{-q}-1}{d_1^1 + d_2^1 e^{-q}} y_2^* = 0\);

(iv) \(-\Psi(x)\) has no fixed point on \(\mathbb{R} \setminus \{0\}\).
Lemma 2.5. Let \( \Phi \) and \( \Psi \) be satisfied. Then, \([-N, N]\) is a global attractor of \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \).

Proof. Write
\[
N_1 = \max\{x_1', M_1\}, \quad N_2 = \max\{x_2', M_2\}.
\]
According to \((F_6)\), \((G_6)\), Lemmas 2.3 and 2.4, one has
\[
\begin{align*}
\Phi([-N_1, N_1]) &= [-N_1, N_1] \quad \text{and} \quad |\Phi(x)| < |x|, \quad x \in \mathbb{R} \setminus [-N_1, N_1], \\
\Psi([-N_2, N_2]) &= [-N_2, N_2] \quad \text{and} \quad |\Psi(x)| < |x|, \quad x \in \mathbb{R} \setminus [-N_2, N_2].
\end{align*}
\]
Obviously, \( N = \max\{N_1, N_2\} \) and
\[
\begin{align*}
\Phi([-N, N]) &\subseteq [-N, N], \quad \Psi([-N, N]) \subseteq [-N, N], \\
|\Phi(x)| < |x|, \quad |\Psi(x)| < |x|, &\quad x \in \mathbb{R} \setminus [-N, N].
\end{align*}
\]
Then, one can see that
\[
\Phi \circ \Psi([-N, N]) \subseteq [-N, N], \quad \Psi \circ \Phi([-N, N]) \subseteq [-N, N].
\]
Furthermore, by an argument similar that in the proof of [14, Proposition 2.1], one obtains that for \( x \in \mathbb{R} \setminus [-N, N], \Phi \circ \Psi(x) < |x| \) and \( |\Psi \circ \Phi(x)| < |x| \). Therefore, it is easy to verify that \([-N, N]\) is a global attractor of \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \).

Let
\[
z_{12} = \min\{z|\Phi \circ \Psi(z) = 0, \quad z > 0\}, \quad M_{12} = \max\{\Phi \circ \Psi(z), \quad x \in [0, z_{12}]\}, \\
z_{21} = \min\{z|\Psi \circ \Phi(z) = 0, \quad z > 0\}, \quad M_{21} = \max\{\Psi \circ \Phi(z), \quad x \in [0, z_{21}]\}.
\]

Lemma 2.6. Let assumptions \((F_1)-(F_5)\) and \((G_1)-(G_5)\) be satisfied.

(i) If \( M_1 \geq x_2^c \), then \( \Phi(z_{21}) = x_2^c \) and \( M_{21} = M_2 \), where \( z_{21} \in (0, x_1^r) \).
(ii) If \( x_2^c \leq M_1 < x_1^r \), then \( z_{21} = x_1^r \) and \( M_{21} = M_2 \).
(iii) If \( M_1 < x_2^c \), then \( z_{21} = x_1^r \) and \( M_{21} = \Psi(M_1) \).

Proof. It follows from the definition of \( z_{21} \) that
\[
\Phi(z_{21}) = 0 \quad \text{or} \quad \Phi(z_{21}) = x_2^c.
\]
(i) If \( \Phi(z_{21}) = 0 \), then \( z_{21} = x_1^r \). Since \( \Phi((0, x_1^r)) = \{0, M_1\} \) and \( M_1 \geq x_2^c \), there exist at least one point \( x^* \in (0, x_1^r) \) such that \( \Phi(x^*) = x_2^c \), which implies \( \Psi \circ \Phi(x^*) = 0 \). This leads to a contradiction. Therefore, \( \Phi(z_{21}) = x_2^c \).
By $M_1 \geq x_5^r$, one has $0 < z_{21} \leq x_1^c$, which indicates that $\Phi$ is strictly monotonously increasing on $[0, z_{21}]$. Then, from

$$\Psi \circ \Phi([0, z_{21}]) = \Psi([0, x_5^r]) = [0, M_2],$$

it follows that $M_{21} = M_2$.

(ii) If $\Phi(z_{21}) = x_2^r$, then $z_{21} > x_1^c$ by the condition $M_1 < x_5^r$. From the definition of $z_{21}$, one has $\Phi(z_{21}) = 0$ and then $z_{21} = x_1^c$. According to $x_5^r \leq M_1 < x_2^r$, one has

$$\Psi \circ \Phi([0, z_{21}]) = \Psi([0, M_1]) = [0, M_2],$$

which implies $M_{21} = M_2$.

(iii) Similar to the proof of (ii), it is easy to see that $z_{21} = x_1^c$. It follows from $M_1 < x_2^r$ that $\Psi$ is strictly monotonously increasing on $[0, M_1]$. Then,

$$\Psi \circ \Phi([0, z_{21}]) = \Psi([0, M_1]) = [0, \Psi(M_1)],$$

which implies $M_{21} = \Psi(M_1)$.

\begin{lemma}
\label{lem:2.7}
Let assumptions $(F_1)$-$(F_5)$ and $(G_1)$-$(G_5)$ be satisfied.

\begin{enumerate}
\item If $M_2 \geq x_1^r$, then $\Psi(z_{12}) = x_1^r$ and $M_{12} = M_1$, where $z_{21} \in (0, x_2^r)$.
\item If $x_1^r \leq M_2 < x_1^c$, then $z_{12} = x_2^r$ and $M_{12} = M_1$.
\item If $M_2 < x_1^c$, then $z_{12} = x_2^r$ and $M_{12} = \Phi(M_2)$.
\end{enumerate}
\end{lemma}

\begin{proof}
It can be obtained by an argument similar that in the proof of Lemma 2.6.
\end{proof}

\begin{lemma}
\label{lem:2.8}
Let assumptions $(F_1)$-$(F_5)$ and $(G_1)$-$(G_5)$ be satisfied.

\begin{enumerate}
\item If $M_1 \geq x_5^r$ and $M_2 < x_1^r$,

then

$$\Phi(M_2) \geq x_2^r \iff M_{12} \geq z_{12} \iff M_{21} \geq z_{21}. $$

\item If $M_2 \geq x_1^r$ and $M_1 < x_2^r$,

then

$$\Psi(M_1) \geq x_1^c \iff M_{12} \geq z_{12} \iff M_{21} \geq z_{21}. $$
\end{enumerate}
\end{lemma}

\begin{proof}
By Lemma 2.6 and Lemma 2.7, the proofs of (i) and (ii) are straightforward.
\end{proof}

\begin{remark}
If $x_2^r \leq M_1 < x_5^r$ and $x_1^c \leq M_2 < x_1^r$, then one has

$$M_{12} \leq z_{12} \iff M_{21} \leq z_{21}.$$ 

\end{remark}

\begin{lemma}
\label{lem:2.9}
Let $(F_1)$-$(F_5)$ and $(G_1)$-$(G_5)$ be satisfied. If $M_{12} \geq z_{12}$ and $M_{21} \geq z_{21}$, then $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are turbulent.
\end{lemma}

\begin{proof}
Let $N$ be given by (13). Evidently, $N \geq M_1$ and $N \geq M_2$. Moreover, by Lemma 2.5, $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are smooth maps of the interval $[-N, N]$ into itself. Note that $M_{21} \leq M_2$ and $M_{12} \leq M_1$, due to Lemma 2.6 and Lemma 2.7. Then, it follows that $M_{21} \leq N$ and $M_{12} \leq N$. Hence, it is easy to see that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are turbulent when $M_{12} \geq z_{12}$ and $M_{21} \geq z_{21}$.
\end{proof}

We are now in a position to present our main result in this section.

\begin{theorem}
\label{thm:2.10}
Let assumptions $(F_1)$-$(F_5)$ and $(G_1)$-$(G_5)$ be satisfied. Assume that one of the following two pairs of conditions hold:

Then, by Jordan decomposition theorem, one obtains that $T$ can be represented as a subtraction of two monotonically increasing functions. Therefore, the boundedness of $w$ is obtained. Hence, $w(x,t) + T_1(x)w(x,t)$ and $w_t(x,t) + T_2w(x,t)$ are chaotic oscillations, which imply that

$$
\lim_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}[w_x(\cdot, t) + T_1(x)w(\cdot, t)] > 0,
$$

and

$$
\lim_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}[w_t(\cdot, t) + T_2w(\cdot, t)] > 0.
$$

By $T_2 = \frac{\xi}{\lambda_1}$ and $\xi \geq 0$, one has $T_2 \geq 0$. Meanwhile, it follows from the second formula of (14) that

$$
w(x,t) = \int_0^t e^{-\lambda(x,s)z(x,s)}e^{-T_2(t-s)}d\xi + w(x,0)e^{-T_2t}.
$$

It is easy to verify that $w(x,t)$ given by (16) is bounded for $x \in (0,1)$, $t > 0$. Therefore, the boundedness of $w_x(x,t)$ and $w_t(x,t)$ are obtained. Note that

$$w(x,t) = w(0,t) + \int_0^x w_x(\theta,t)d\theta, \quad 0 \leq x \leq 1.
$$

Therefore,

$$w_x(x,t) + T_1w(x,t) = w_x(x,t) + T_1w(0,t) + T_1(x) \int_0^x w_x(\theta,t)d\theta, \quad 0 \leq x \leq 1.
$$

Since $\xi \geq 0$, $d_2(x)$ is monotonically decreasing and $k_2(x) \geq 0$ is monotonically increasing on $[0,1]$, it follows that $T_1(x)$ with

$$T_1(x) = \frac{k_2(x)}{d_2(x)} - \frac{\xi}{\sigma + 1 \cdot d_2(x)}
$$

can be represented as a subtraction of two monotonically increasing functions. Then, by Jordan decomposition theorem, one obtains that $T_1(\cdot)$ is a bounded variation function on $[0,1]$. Furthermore, according to the definition of total variation, one has

$$V_{[0,1]}[w_x(\cdot, t) + T_1(\cdot)w(\cdot, t)]
\leq V_{[0,1]}[w_x(\cdot, t)] + \sup_p \sum_{i=0}^{k-1} |T_1(x_{i+1}) \int_{x_i}^{x_{i+1}} w_x(\theta, t)d\theta - T_1(x_i) \int_{x_i}^{x_{i+1}} w_x(\theta, t)d\theta |
\leq V_{[0,1]}[w_x(\cdot, t)] + T_1 \sup_p \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |w_x(\theta, t)d\theta | + L \sup_p \sum_{i=0}^{k-1} |T_1(x_{i+1}) - T_1(x_i)|,
\leq V_{[0,1]}[w(x, \cdot, t)] + (\tilde{T}_1 + T_1)L,
$$

Proof. It follows from the conditions in (i) or (ii) that $M_{12} \geq z_{12}$ and $M_{21} \geq z_{21}$ by Lemma 2.8. Then, according to $\zeta \geq 0$, $k_2(x) \geq 0$ and Lemma 2.9, one obtains that $e^{-\lambda(x,t)u(x,t)}$ and $e^{-\lambda(x,t)v(x,t)}$ expressed by (10), are chaotic, which imply that $e^{-\lambda(x,t)z_2(x,t)}$ and $e^{-\lambda(x,t)z_t(x,t)}$ are chaotic.
that chaotic oscillations of hyperbolic PDE (1) with variable coefficients and the IBCs
increases. The proof is completed. Together with (15), it follows that
which implies

\[
\liminf_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}(w_x(\cdot, t)) \geq \liminf_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}(w_x(\cdot, t) + T_1 w(\cdot, t) - (\hat{T}_1 + \hat{T}_1)L > 0.
\]

Similarly, one has
\[
\liminf_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}(w_t(\cdot, t)) \geq \liminf_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}(w_t(\cdot, t) + T_2 w(\cdot, t)) > 0.
\]

Hence, the total variations of \(w_x(x, t)\) and \(w_t(x, t)\) grow exponentially as time \(t\) increases. The proof is completed. 

**Remark 2.** Let \(d_i(x)\) and \(k_i(x)\), \(i = 1, 2\), be constants and the boundary conditions be van der Pol type or superlinear type. It is easy to see that this is actually a special case of system (S1). Therefore, Theorem 2.10 is generalization of chaotic oscillations in [2, 6, 14, 17, 20].

### 3. Hyperbolic PDE with variable coefficients and IBCs.

Let us consider the chaotic oscillations of hyperbolic PDE (1) with variable coefficients and the IBCs (3) that cannot in general be expressed by EBCs. The dynamics is given by system

\[
(S_2) \begin{cases}
\left[ \frac{\partial}{\partial t} - d_1(x) \frac{\partial}{\partial x} + k_1(x) \right] \left[ \frac{\partial}{\partial t} + d_2(x) \frac{\partial}{\partial x} + k_2(x) \right] w(x, t) = 0, \\
\begin{align*}
& w_t(0, t) + T_2 w(0, t) = d_1^2(w_x(0, t) + T_1(0)w(0, t)) + e^{-\lambda(0, t)} f_1[e^{\lambda(0, t)}(w_t(0, t) + T_2 w(0, t) + d_2^2(w_x(0, t) + T_1(0)w(0, t))], \quad t > 0, \\
& w_t(0, t) + T_2 w(0, t) = -d_2^2(w_x(1, t) + T_1(1)w(1, t)) + e^{-\lambda(1, t)} f_2[e^{\lambda(1, t)}(w_t(1, t) + T_2 w(1, t) - d_1^2(w_x(1, t) + T_1(1)w(1, t))], \quad t > 0,
\end{align*}
\end{cases}
\]

where \(T_1(x), x \in [0, 1]\), and \(T_2\) are defined as (4), \(\lambda(x, t), x \in [0, 1], t > 0\) is defined as (6), \(f_i, i = 1, 2\), is odd and derivable. Assume further that \(f_i, i = 1, 2\), satisfies the following assumptions:

\[
\begin{align*}
(H_1) f_i(0) &= 0, \\
(H_2) f_i'(0) &\geq 0, \\
(H_3) \exists \gamma > 0, f_i(y) &\leq 0, \\
(H_4) \forall y > 0, f_i(y) &> -y.
\end{align*}
\]

Let \(w(x, t)\) satisfy (5), and further define two variables

\[
\tilde{u}(x, t) = z_t(x, t) + d_2(x)z_x(x, t), \quad \tilde{v}(x, t) = z_t(x, t) - d_1(x)z_x(x, t), \quad (17)
\]
From the boundary conditions of system \((S_2)\), it follows that
\[
\begin{align*}
\dot{v}(0,t) &= f_1(\dot{u}(0,t)), \quad t > 0, \\
\ddot{u}(1,t) &= f_2(\ddot{v}(1,t)), \quad t > 0.
\end{align*}
\] (18)

The initial conditions of \(u(x,t)\) and \(v(x,t)\) are
\[
\begin{align*}
\dot{u}_0(x) &= u(x,0) = e^{\lambda(x,0)}[w_1(x) + d_2(x)w_0'(x) + k_2(x)w_0(x)], \\
\dot{v}_0(x) &= v(x,0) = e^{\lambda(x,0)}[w_1(x) - d_1(x)w_0'(x) + k_1(x)w_0(x)].
\end{align*}
\] (19)

Therefore, by the method of characteristic, the solutions \(\dot{u}(x,t)\) and \(\ddot{v}(x,t)\) of system \((17)-(19)\) can be expressed explicitly as follows
\[
\dot{u}(x,t) = \begin{cases} 
(f_2 \circ f_1)^k(\dot{u}_0(h_1(x,\tau))), & 0 \leq \tau \leq l_1(x), \\
(f_1 \circ f_2)^{-1}(\dot{u}_0(h_2(x,\tau))), & l_1 - l_2 < \tau \leq l_1 + l_2 - l_1(x), \\
(f_2 \circ f_1)^k(\dot{u}_0(h_3(x,\tau))), & l_1 + l_2 - l_1(x) < \tau \leq l_1 + l_2,
\end{cases}
\]
\[
\ddot{v}(x,t) = \begin{cases} 
(f_1 \circ f_2)^k(\dot{v}_0(h_4(x,\tau))), & 0 \leq \tau \leq l_2(x), \\
(f_1 \circ f_2 \circ f_1)^k(\dot{u}_0(h_5(x,\tau))), & l_2(\tau) < \tau \leq l_1 + l_2, \\
(f_1 \circ f_2)^{-1}(\dot{u}_0(h_6(x,\tau))), & l_1 + l_2(x) < \tau \leq l_1 + l_2 + 1,
\end{cases}
\] (20) (21)

where \(0 \leq x \leq 1, t = (l_1 + l_2)k + \tau, \ k \in \mathbb{N}, \ 0 \leq \tau \leq l_1 + l_2, \) and \(h_i(x,\tau), \ i = 1,\ldots,6, \) satisfy \((11)\).

In order to prevent the discontinuity of \((20)\) and \((21)\) from propagating along the characteristic lines, one needs assumptions of the initial values \(u_0\) and \(v_0\) satisfying the compatibility conditions
\[
\dot{v}_0(0) = f_1(\dot{u}_0(0)), \quad \ddot{u}_0(1) = f_2(\ddot{v}_0(1)).
\]

It is obvious that the dynamical behavior of the solutions \(u\) and \(v\) of system \((17)-(19)\) can be completely determined by the iterates of maps \(f_1 \circ f_2\) and \(f_2 \circ f_1\). In the following lemma, one considers the roots, critical points and extreme values of \(f_1 \circ f_2\) and \(f_2 \circ f_1\), respectively, where the roots and critical points are chosen closest to the origin.

**Lemma 3.1.** Let assumptions \((H_1)-(H_4)\) hold. Then the functions \(f_1(y)\) and \(f_2(y)\) have the following properties:

(i) \(f_1(y) = 0\) nonzero positive real roots \(\bar{y}_1\), and \(f_2(y) = 0\) has nonzero positive real roots \(\bar{y}_2\);

(iv) \(-f_1(y)\) and \(-f_2(y)\) have no fixed point on \(\mathbb{R} \setminus \{0\}\); 

(v) \(f_1(y)\) and \(f_2(y)\), respectively, have local maximum values \(R_1\) and \(R_2\),
\[
R_1 = f_1(\bar{y}_1), \quad R_2 = f_2(\bar{y}_2),
\]

where \(\bar{y}_1\) and \(\bar{y}_2\) are critical points.

**Proof.** This can be verified directly. \(\square\)

In order to characterize the global attractor of \(f_1 \circ f_2\) and \(f_2 \circ f_1\), one needs additional assumption:

\((H_5)\) \(f_1(y) - y \neq 0\) for \(y \in [\bar{y}_1, +\infty)\);

Let
\[
R = \max\{R_1, R_2\}.
\]

**Lemma 3.2.** Let \((H_1)-(H_5)\) be satisfied. Then, \([-R, R]\) is a global attractor of \(f_1 \circ f_2\) and \(f_2 \circ f_1\).
Proof. According to \((H_5)\) and Lemma 3.1, one has

\[
f_i([-R_i, R_i]) = [-R_i, R_i] \quad \text{and} \quad |f_i(y)| < |y|, \quad \text{for } y \in \mathbb{R} \setminus [-R_i, R_i], \quad i = 1, 2.
\]

Therefore,

\[
f_i([-R, R]) \subseteq [-R, R], \quad |f_i(y)| < |y|, \quad \text{for } y \in \mathbb{R} \setminus [-R, R], \quad i = 1, 2.
\]

By an argument similar to that in the proof of Lemma 2.5, one obtains that \([-R, R]\) is a global attractor of \(f \circ f_2\) and \(f_2 \circ f_1\).

Let

\[
y_{12} = \min\{y | f_1 \circ f_2(y) = 0, \quad y > 0\}, \quad \mathcal{R}_{12} = \max\{f_1 \circ f_2(y), \quad y \in [0, y_{12}]\},
\]

\[
y_{21} = \min\{y | f_2 \circ f_1(y) = 0, \quad y > 0\}, \quad \mathcal{R}_{21} = \max\{f_2 \circ f_1(y), \quad y \in [0, y_{21}]\}.
\]

Lemma 3.3. Let assumptions \((H_1)-(H_5)\) be satisfied.

(i) If \(R_1 \geq \bar{y}_2\), then \(f_1(y_{21}) = \bar{y}_2\) and \(\mathcal{R}_{21} = R_2\), where \(y_{21} \in (0, \bar{y}_2]\).

(ii) If \(R_1 < \bar{y}_2\), then \(y_{21} = \bar{y}_1\) and \(\mathcal{R}_{21} = f_2(R_1)\).

(iii) If \(R_2 \geq \bar{y}_2\), then \(f_2(y_{12}) = \bar{y}_1\) and \(\mathcal{R}_{12} = R_1\), where \(y_{21} \in (0, \bar{y}_2]\).

(v) If \(R_2 < \bar{y}_2\), then \(y_{12} = \bar{y}_2\) and \(\mathcal{R}_{12} = f_1(R_2)\).

Proof. It can be obtained by an argument similar that in the proof of Lemma 2.6.

Lemma 3.4. Let assumptions \((H_1)-(H_5)\) be satisfied.

(i) If

\[
R_1 \geq \bar{y}_2 \quad \text{and} \quad R_2 < \bar{y}_1,
\]

then

\[
f_1(R_2) \geq \bar{y}_2 \iff \mathcal{R}_{12} \geq y_{12} \iff \mathcal{R}_{21} \geq y_{21}.
\]

(ii) If

\[
R_2 \geq \bar{y}_1 \quad \text{and} \quad R_1 < \bar{y}_2,
\]

then

\[
f_2(R_1) \geq \bar{y}_1 \iff \mathcal{R}_{12} \geq y_{12} \iff \mathcal{R}_{21} \geq y_{21}.
\]

Proof. By Lemma 3.3, the proofs of (i) and (ii) are straightforward.

The following lemma can be obtained by an argument similar that in the proof of Lemma 2.9. Here we omit the proof for simplicity.

Lemma 3.5. Let assumptions \((H_1)-(H_5)\) be satisfied. If \(\mathcal{R}_{12} \geq y_{12}\) and \(\mathcal{R}_{21} \geq y_{21}\), then \(f_1 \circ f_2\) and \(f_2 \circ f_1\) are turbulent.

The following theorem shows the chaotic oscillations of system \((S_2)\).

Theorem 3.6. Let assumptions \((H_1)-(H_5)\) be satisfied. Assume that one of the following two pairs of conditions hold:

(i) \(f_1(y_1) - y_2 \geq 0, \quad f_2(y_2) - y_1 < 0 \quad \text{and} \quad f_1(R_2) - \bar{y}_2 \geq 0\);

(ii) \(f_2(y_2) - y_1 \geq 0, \quad f_1(y_1) - y_2 < 0 \quad \text{and} \quad f_2(R_1) - \bar{y}_1 \geq 0\).

Moreover, if \(\zeta = 0\), \(d_2(x)\) is decreasing and \(k_2(x) \geq 0\) is increasing on \([0, 1]\), then \(w_x(x, t)\) and \(w_t(x, t)\) of system \((S_2)\) are chaotic oscillations.
Proof. By Lemma 3.4, one has \( R_{12} \geq y_{12} \) and \( R_{21} \geq y_{21} \). It follows from \( \zeta = 0 \) that \( T_2 = 0 \). Therefore, by (3), one has

\[
\begin{align*}
  w_x(x,t) &= e^{-\int_0^t T_1(s)ds} [z_x(x,t) - T_1(x)z(x,t)], \\
  w_t(x,t) &= e^{-\int_0^t T_1(s)ds} z_t(x,t).
\end{align*}
\]

(22)

Since

\[
z_x(x,t) = \frac{\tilde{u}(x,t) - \tilde{v}(x,t)}{d_1(x) + d_2(x)},
\]

one has

\[
z(x,t) = z(0,t) + \int_0^x z_x(\theta,t)d\theta
\]

for \( 0 \leq x \leq 1, t > 0 \). Based on the definition of total variation, one has

\[
V_{[0,1]}(z(\cdot,t)) = \sup_P \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{\tilde{u}(\theta,t) - \tilde{v}(\theta,t)}{d_1(\theta) + d_2(\theta)} d\theta
\]

which together with Lemma 3.5 yields the total variation \( V_{[0,1]}(z(\cdot,t)) \) is bounded, and then \( z(x,t) \) is bounded for \( 0 < x < 1 \) and \( t > 0 \). Moreover, in view of (17), \( \zeta = 0 \) and \( k_2(x) \geq 0 \), one has that \( z_x(x,t) \) and \( z_t(x,t) \) are bounded implying boundedness of \( w_x(x,t) \) and \( w_t(x,t) \) for \( 0 < x < 1 \) and \( t > 0 \).

Again using (22), one obtains

\[
V_{[0,1]}(z_x(\cdot,t)) = V_{[0,1]}(e^{\int_0^t T_1(s)ds} w_x(\cdot,t) + T_1(\cdot)z(\cdot,t))
\]

\[
\leq V_{[0,1]}(e^{\int_0^t T_1(s)ds} w_x(\cdot,t)) + V_{[0,1]}(T_1(\cdot)z(\cdot,t))
\]

\[
\leq e^{\int_0^T} V_{[0,1]}(w_x(\cdot,t)) + L_1
\]

and

\[
V_{[0,1]}(z_t(\cdot,t)) = V_{[0,1]}(e^{\int_0^t T_1(s)ds} w_t(\cdot,t))
\]

\[
\leq e^{\int_0^T} V_{[0,1]}(w_t(\cdot,t)) + L_2,
\]

where

\[
T_1 = \max_{0 \leq x \leq 1} \{ T_1(x) \},
\]

\[
L_1 = \sup_{0 \leq x \leq 1} |w_x(x,t)| V_{[0,1]}(e^{\int_0^t T_1(s)ds})
\]

\[
+ T_1 V_{[0,1]}(z(\cdot,t)) + \sup_{0 \leq x \leq 1} |z(x,t)| V_{[0,1]}(T_1(\cdot)),
\]

\[
L_2 = \sup_{0 \leq x \leq 1} |w_t(x,t)| V_{[0,1]}(e^{\int_0^t T_1(s)ds}).
\]

According to \( \zeta = 0 \), \( d_2(x) \) is decreasing and \( k_2(x) \geq 0 \) is increasing on \([0,1]\), one can see that \( V_{[0,1]}(e^{\int_0^t T_1(s)ds}) \) and \( V_{[0,1]}(T_1(\cdot)) \) are bounded. Since \( u_t(x,t) \) and \( v_t(x,t) \) are chaotic by Lemma 3.5, then \( z_x(x,t) \) and \( z_t(x,t) \) are also chaotic, which imply that there exists a sufficiently large \( T \) satisfying \( t > T \) such that

\[
\liminf_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}(w_x(\cdot,t)) \geq \liminf_{t \to \infty} \frac{1}{t} \ln [V_{[0,1]}(z_x(\cdot,t)) - L_1]
\]

\[
= \liminf_{t \to \infty} \frac{1}{t} \ln V_{[0,1]}(z_x(\cdot,t)) > 0
\]
Therefore, the total variations $V_{[0,1]}(w(\cdot, t))$ and $V_{[0,1]}(w_t(\cdot, t))$ grow exponentially as time $t$ increases.

4. Examples. This section gives two examples to illustrate the effectiveness of our theoretical results.

Example 4.1. Consider a linear telegraph equation with varying coefficients and general EBCs of the form

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} - (2 - x) \frac{\partial}{\partial x} + 0.1 \left( \frac{\partial}{\partial t} + (2 - x) \frac{\partial}{\partial x} + 0.1 \right) w(x, t) = 0, \\
w_x(0, t) = F(w_t(0, t) + 0.1 w(0, t)), \\
w_x(1, t) = G(w_t(0, t) + 0.1 w(0, t)), \\
w(0, t) = 0, \\
w_t(0, t) = 0.1 \sin(2\pi x),
\end{array} \right. 
$$

(23)

where $F$ and $G$ are given by

$$
F(x) = -\alpha_1 x + \beta_1 |x|^m \text{sgn} x, \quad 0 < \alpha_1 < \frac{1}{2}, \quad \beta_1 > 0, \quad m > 1,
$$

$$
G(x) = \alpha_2 x - \beta_2 |x|^n \text{sgn} x, \quad 0 < \alpha_2 < 1, \quad \beta_2 > 0, \quad n > 1,
$$

which satisfies the assumptions (F1)-(F5) and (G1)-(G5), respectively.

From system (23), one can see that

$$
d_1(x) = d_2(x) = 2 - x, \quad k_1(x) = k_2(x) = 0.1.
$$

Let $m = 3$ and $n = 3$, then $F$ and $G$ are the van der Pol boundary conditions. If $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ satisfy one of the following two pairs of conditions:

$$
\left\{ \begin{array}{l}
\frac{2\alpha_1 + 1}{6} \sqrt{\frac{2\alpha_1 + 1}{6\beta_1}} \geq e^q \sqrt{\frac{\alpha_2 + 1}{\beta_2}}, \\
\frac{\alpha_2 + 1}{3} \sqrt{\frac{\alpha_2 + 1}{3\beta_2}} < e^p (1 - \alpha_1) \sqrt{\frac{2\alpha_1 + 1}{6\beta_1}}, \\
\frac{\alpha_2 + 1}{3} \sqrt{\frac{\alpha_2 + 1}{3\beta_2}} \geq e^q (2 - \alpha_2) \sqrt{\frac{\alpha_2 + 1}{3\beta_2}}, \\
\frac{2\alpha_1 + 1}{6} \sqrt{\frac{2\alpha_1 + 1}{6\beta_1}} < e^p (1 - \alpha_1) \sqrt{\frac{2\alpha_1 + 1}{6\beta_1}}, \\
\alpha_2 + 1 \frac{\alpha_2 + 1}{3\beta_2} \geq e^q (2 - \alpha_2) \sqrt{\frac{\alpha_2 + 1}{3\beta_2}}, \\
\frac{\alpha_2 + 1}{3} \sqrt{\frac{\alpha_2 + 1}{3\beta_2}} \geq e^q (2 - \alpha_2) \sqrt{\frac{\alpha_2 + 1}{3\beta_2}}, \\
\frac{2\alpha_1 + 1}{6} \sqrt{\frac{2\alpha_1 + 1}{6\beta_1}} \geq e^q (2 - \alpha_2) \sqrt{\frac{\alpha_2 + 1}{3\beta_2}},
\end{array} \right.
$$

(24)

where $p = 0.1 \ln 2$ and $q = 0.1 \ln 2$.

Then by Theorem 2.10, the solutions $w_x(x, t)$ and $w_t(x, t)$ of system (23) have chaotic oscillations. Taking the parameter pairs $(\alpha_1, \beta_1) = (0.1, 1)$ and $(\alpha_2, \beta_2) = (0.8, 1)$, which satisfy (24), the spatiotemporal profiles of $w_x(x, t)$ and $w_t(x, t)$ of system (23) are shown in Figure 1. It is easy to see that $w_x(x, t)$ and $w_t(x, t)$ are rapidly oscillatory as $t$ increases.
**Example 4.2.** Consider a linear hyperbolic hyperbolic PDE with varying coefficients and IBCs of the form

\[
\begin{align*}
\frac{\partial}{\partial t} - (2 - x) \frac{\partial}{\partial x} - 0.1 &\left[ \frac{\partial}{\partial t} + (2 - x) \frac{\partial}{\partial x} + 0.1 \right] w(x, t) = 0, \quad x \in (0, 1), \quad t > 0 \\
w_t(0, t) &= f_1(w_t(0, t) + 2w_x(0, t) + 0.1w(0, t)) + 2w_x(0, t) + 0.1w(0, t), \quad t > 0, \\
w_t(1, t) &= 2^{-0.1} f_2(2^{0.1}(w_t(1, t) - w_x(1, t) - 0.1w(1, t))) \\
w(x, 0) &= 0, \quad w_t(x, 0) = 0.1 \sin(2\pi x), \quad x \in [0, 1],
\end{align*}
\]

where \( f_i, i = 1, 2 \), is given by

\[
f_i(x) = \gamma_i \sin x, \quad 0 < \gamma_i < 1.5\pi, \quad i = 1, 2,
\]

which satisfies the assumptions (H_1)-(H_5).
Figure 2. The spatiotemporal profiles of system (23) with $\gamma_1 = 1.1\pi$ and $\gamma_2 = 0.4\pi$, $x \in [0, 1]$ and $t \in [60, 64]$: (a) $w_x(x, t)$; (b) $w_t(x, t)$.

One can verify that the conditions of the Theorem 3.6 can be satisfied when $\gamma_1 \in [\pi, 1.5\pi)$ and $\gamma_2 \in (0, 0.5\pi)$. Then, one obtains that the solutions $w_x$ and $w_t$ of system (25) are chaotic. Taking the parameters $\gamma_1 = 1.1\pi$ and $\gamma_2 = 0.4\pi$, the spatiotemporal profiles of $w_x(x, t)$ and $w_t(x, t)$ are shown in Figure 2. It is easy to see that $w_x(x, t)$ and $w_t(x, t)$ are rapidly oscillatory as $t$ increases.

5. Conclusions. This paper rigorously proves the existence of chaotic oscillations in systems of linear hyperbolic PDEs with variable coefficients and both ends of boundary conditions being IBCs, where the IBCs can be expressed by general NBCs and cannot be expressed by EBCs. It should be pointed out that in spite of our systemic analysis on chaotic oscillations of second order linear hyperbolic PDEs with variable coefficients and nonlinear IBCs, there are still improvement rooms, e.g., whether chaotic oscillations in the system (1)-(3) when the two first order operators of (1) fails to commutative. It is expected that our analysis method proposed in this paper will stimulate further studies on chaotic dynamics of linear
PDEs with general IBCs, forward deep understanding of the geometric structures of chaotic attractors in infinite-dimensional linear PDEs with nonlinear IBCs.

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