Effect of time delay on the onset of synchronization of the stochastic Kuramoto model

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Abstract. We consider the Kuramoto model of globally coupled phase oscillators with time-delayed interactions, that is subject to Ornstein–Uhlenbeck (Gaussian) colored or non-Gaussian colored noise. We investigate numerically the interplay between the influences of the finite correlation time of noise $\tau$ and the time delay $\tau_d$ on the onset of the synchronization process. The cases for identical and nonidentical oscillators have both been considered. Among the obtained results for identical oscillators is a large increase of the synchronization threshold as a function of time delay for the colored non-Gaussian noise compared to the case of the colored Gaussian noise at low noise correlation time $\tau$. However, the difference reduces remarkably for large noise correlation times. For the case of nonidentical oscillators, the incoherent state may become unstable around the maximum value of the threshold (as a function of time delay) even at lower coupling strength values in the presence of colored noise as compared to the noiseless case. We have studied the dependence of the critical value of the coupling strength (the threshold of synchronization) on given parameters of the stochastic Kuramoto model in great detail and presented results for possible cases of colored Gaussian and non-Gaussian noises.

Keywords: phase diagrams (theory), phase transformations (theory), stochastic processes (theory)
1. Introduction

Since the pioneering works on coupled phase oscillators by Winfree [1] and Kuramoto [2], synchronization in nonlinear systems has been systematically studied and has attracted much attention. Recent reviews on the developments can be found in [3]–[6]. Systems where phenomena of synchronization have been observed include biological clocks [1], chemical oscillators [2], coupled map lattices [7]–[9], coupled random frequency oscillators [10], cardiorespiratory coupled system [11], etc.

The model first introduced by Kuramoto [2] is one of the basic models that describes the synchronization process when initially independent oscillators begin to move coherently. It was thoroughly studied and successfully applied in several systems which were modeled by an ensemble of coupled phase oscillators [6]. Another model of interest is the Kuramoto model with a time delay [12]–[15]. The model shows a number of interesting phenomena including, e.g., the effect of bistability as discovered in [14] where the Kuramoto model with time-delayed interactions was also considered to be subject to a white noise. Note that both time-delayed interactions and noise play very important roles in Nature (see, e.g., [16]).

In this paper we consider the Kuramoto model of globally coupled (all-to-all) phase oscillators with a time delay and subject to the Ornstein–Uhlenbeck (OU) colored noise [17], that is a Gaussian process with a finite correlation time, and a non-Gaussian colored noise [18]–[21]. The focus in our study is on the interplay between the influence of noise and the time delay on the synchronization process. Our previous study on the stochastic Kuramoto model [21] showed that the influence of the OU noise qualitatively differs from the case of white noise, as the former allows for full synchronization despite the fact that the system is subject to a noise, see e.g. figure 3 in [21].

Generally speaking, the time delay introduces de-phasing among the oscillators. This de-phasing is to interfere with the intrinsic correlations caused by the finite correlation time of the noise. As a result, we have found that the de-phasing plays an important role in the dynamics of the system when the intrinsic correlations are small. We also investigate the effect of the time delay on the onset of synchronization for different noise strengths for both OU and non-Gaussian noises. But before doing that we show that the effect of time delay for the noiseless Kuramoto model is qualitatively equivalent to
the effect of frequency fluctuations of the phase oscillators. Meanwhile we demonstrate 
that for the stochastic Kuramoto model the critical coupling grows nonlinearly with the 
increase of the time delay that is due to the additional de-phasing caused by the time 
delay. We also investigate how the transition from the Gaussian to a non-Gaussian noise 
changes the dynamical properties of the system.

2. Stochastic Kuramoto model with time-delayed interactions

Let us consider the stochastic Kuramoto model with time-delayed interactions. This 
model describes \( N \) coupled phase oscillators with dynamics governed by the following 
equations

\[
\frac{d\theta_i(t)}{dt} = \omega_i + \frac{\epsilon}{N} \sum_{i< j}^{N} \sin[\theta_j(t - \tau_d) - \theta_i(t)] + \eta_i(t),
\]

(1)

where \( \theta_i \) and \( \omega_i \) are, respectively, the phase and the frequency of the \( i \)th oscillators 
(\( i = 1, \ldots, N \)), \( \epsilon \) is the coupling constant, and \( \tau_d \) is the time delay. The independent 
noise processes \( \eta_i(t) \) are governed by

\[
\frac{d\eta_i(t)}{dt} = -\frac{1}{\tau} \frac{d}{d\eta_i} U_p(\eta_i) + \sqrt{\frac{D}{\tau}} \xi_i(t).
\]

(2)

The potential function is

\[
U_p(\eta_i) = \frac{D}{\tau(p-1)} \ln[1 + \alpha(p-1)\eta_i^2/2]
\]

with \( \alpha = \tau/D \). \( \xi_i(t) \) is the Gaussian white noise process defined via

\[
\langle \xi_i(t)\xi_j(t') \rangle = 2\delta_{ij}\delta(t-t')
\]

and \( \langle \xi_i(t) \rangle = 0 \). \( D \) and \( \tau \) measure the intensity and the correlation time of the noise 
process. Shortly we will discuss them in more detail. In the noise term in equation (1) 
it is apparent that in the present problem we assume the homogeneous diffusion of phase 
oscillators.

The form of the noise \( \eta_i \) allows us to control the deviation from the Gaussian behavior 
by changing a single parameter \( p \). For \( p = 1 \), equation (2) becomes

\[
\frac{d\eta_i}{dt} = -\eta_i + \frac{\sqrt{D}}{\tau} \xi_i(t)
\]

(3)

which is a well-known time evolution equation for the OU noise process [17] for which the 
auto-correlation function is given by

\[
\langle \eta_i(t)\eta_j(t') \rangle = \frac{D}{\tau} \exp\left( -\frac{|t-t'|}{\tau} \right).
\]

(4)

Thus \( D \) and \( \tau \) are the noise strength and correlation time of the OU noise. The factor 
\([1 + \alpha(p-1)\eta_i^2/2]\) in the first term of equation (2) leads to the production of colored 
non-Gaussian noise whose effective noise strength and correlation time are different from 
\( D \) and \( \tau \). However, to make the present paper self-consistent we would like to mention 
here salient features of the non-Gaussian noise only [18].
The stationary probability distribution of the noise process is given by

\[ P(\eta_i) = \frac{1}{Z_{ip}} \left[ 1 + \alpha(p-1)\frac{\eta_i^2}{2} \right]^{-1/(p-1)}, \tag{5} \]

where \( Z_{ip} \) is the normalization constant which equals

\[ Z_{ip} = \int_{-\infty}^{\infty} d\eta_i \left[ 1 + \alpha(p-1)\frac{\eta_i^2}{2} \right]^{-1/(p-1)} = \sqrt{\frac{\pi}{\alpha(p-1)}} \frac{\Gamma_1(1/(p-1) - 1/2)}{\Gamma_1(1/(p-1))}, \tag{6} \]

where \( \Gamma_1 \) is the Gamma function. This distribution can be normalized only for \( p < 3 \). Since \( P(\eta_i) \) is an even function of \( \eta_i \), the first moment, \( \langle \eta_i \rangle \), is always zero, and the second moment (variance) is given by

\[ \langle \eta_i^2 \rangle = \frac{2D}{\tau(5-3p)}, \tag{7} \]

which is finite only for \( p < 5/3 \). Furthermore, for \( p < 1 \), the distribution has a cut-off and it is only defined for \( |\eta_i| < \eta_{ic} \equiv \sqrt{2D/(\tau(1-p))} \).

However, it is difficult to determine the auto-correlation function (ACF) of the non-Gaussian noise \( \eta_i \) exactly. To have an idea about this we have calculated it numerically and presented the result in figure 1. The two-time correlation function for the non-Gaussian noise (solid curve) is fitted well by the bi-exponentially decaying function \( \langle \eta_i(t)\eta_i(0) \rangle = ae^{-t/37} + be^{-t} \) having two correlation times 37 and 1, respectively, for \( \tau = 1.0 \), where \( a \) and \( b \) are constants related to the noise intensity. In the same figure we have plotted the numerically calculated ACF for Gaussian colored noise. It exactly mimics the function given in equation (4).

Figure 1 clearly shows that the effective noise strength and correlation time for non-Gaussian noise are greater compared to those of colored Gaussian noise. The correlation time of non-Gaussian noise \( \tau_p \) at the stationary regime

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of the process $\eta_i(t)$ diverges near $p = 5/3$ and it can be approximated \cite{18} over the range $1 \leq p < 5/3$ as

$$\tau_p \simeq 2\tau/(5 - 3p).$$  \hfill (8)

This equation is qualitatively consistent with figure 1. However, for this approximate correlation time, equation (7) becomes

$$\langle \eta_{ip}^2 \rangle \simeq \frac{4D}{\tau_p(5 - 3p)^2}.$$  \hfill (9)

Equations (7)–(9) imply that when $p \to 1$, we recover the limit that $\eta_i$ is a Gaussian colored noise since at this limit equations (7) and (9) correspond to the variance of Colored Gaussian noise as given in equation (4) and $\tau_p$ in equation (8) becomes $\tau$ which is the correlation time of the Gaussian noise. Another check in this context can be obtained in the following way. In this limit the term in the square bracket of equation (5) can be written as

$$1 + \alpha(p - 1)\eta_i^2/2 = \exp(\alpha(p - 1)\eta_i^2/2).$$  \hfill (10)

Then equation (5) becomes

$$P(\eta) = \frac{1}{Z_1}\exp(-\alpha\eta_i^2/2),$$  \hfill (11)

with $Z_1 = \sqrt{\pi/\alpha}$, which is a Gaussian distribution function. However, equation (7) shows that for a given external noise strength $D$ and the noise correlation time $\tau$, the variance of the non-Gaussian noise is higher than that of the Gaussian noise for $p > 1$, i.e. $\langle \eta_{ip}^2 \rangle > \langle \eta_{i1}^2 \rangle$. Similarly, equation (8) implies that $\tau_p > \tau$ for $p > 1$. These are consistent with the message of figure 1. Before leaving this issue we would like to emphasize that in the present study we have considered a continuous distribution of the non-Gaussian noise which is more relevant to natural systems than to two-state or discrete distributions as mostly used in the literature to study the noise driven dynamical systems \cite{22}.

The quantity of interest in the present study is

$$Z = \Gamma e^{i\Theta} = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i}$$  \hfill (12)

which is the order parameter that measures the extent of synchronization in the system of $N$ phase oscillators. Its absolute value $\Gamma$ determines the degree of synchronization. It can be seen that in the case of all the oscillators having the same phase the quantity equals one ($\Gamma = 1$) that corresponds to full synchronization. The degree of synchronization is equal to zero ($\Gamma = 0$) when all the oscillators are independent and have different phases. $\Theta$ defines an average phase of the oscillators. For the initial distribution of frequencies we choose the Lorentzian distribution

$$g(w) = \frac{1}{\pi} \frac{\lambda}{(\omega - \omega_0)^2 + \lambda^2}.$$  \hfill (13)

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3. Results and discussion

We have computed the order parameter $\Gamma$ as well as the critical coupling strength numerically since it is very difficult to study the problem analytically due to the colored noise and nonlinearity in terms of $\eta_i$ in its time evolution equation. Using Heun’s method (a stochastic version of the Euler method which reduces to the second order Runge–Kutta method in the absence of noise) we have solved the equations (1) and (2) simultaneously [23]. Based on the above method, we have studied the time evolution of $N = 5000$ coupled phase oscillators. In [21], we have shown that such an $N$ value is large enough so that the calculated quantities can represent those for systems in the thermodynamic limit.

For a given frequency distribution of the phase oscillators and environment of specific characteristic there exists a threshold value of the coupling strength ($\epsilon_c$) above which the coherent (synchronized) state is the stable state. If $\epsilon$ is smaller than $\epsilon_c$, then the incoherent state is the stable one. $\epsilon_c$ is called critical coupling strength. To calculate the value of $\epsilon_c$, we first obtain the dependence of the ensemble average of $\Gamma$ ($\langle \Gamma \rangle$) on $\epsilon$. From this we determine numerically the derivative of $\langle \Gamma \rangle$ with respect to $\epsilon$ at stationary state. The $\epsilon$ corresponding to the maximum value of the derivative is identified as the critical coupling strength. Our numerical study satisfies all the known limiting results, i.e., (i) $\epsilon_c = 2\lambda$ in the absence of noise ($D = 0$) and $\tau_d = 0$, (ii) $\epsilon_c = 2(\lambda + D)$ for white noise and $\tau_d = 0$ (results are not shown here). We would also like to mention here that we chose stationary time to be a linear function of time delay ($\tau_d$) with the proportionality constant 100 and integration step length $h = 0.01$.

We start our numerical study to determine how stability of both coherent and incoherent states depends on the time delay. Stability of these states of white noise driven coupled oscillators in the Kuramoto model with a time delay has been studied by Young and Strogatz [14]. They have considered two cases. In the first case, all the oscillators are identical. In the second case, they have chosen Lorentzian distribution, equation (13), of frequency of the coupled oscillators. In the present paper, we extend this study to colored noise driven coupled oscillators. The noise may be OU Gaussian or non-Gaussian in characteristics. However, the critical coupling strength as mentioned above indicates that up to that value of coupling strength, the incoherent state can survive at long times. This range corresponds to the stability zone of the incoherent state of the coupled phase oscillators. Thus critical coupling strength is a measure to imply how stability of incoherent or coherent states of the coupled oscillators depends on time delay and other parameters of the system. To demonstrate the dependence of the critical coupling strength $\epsilon_c$ on the time delay $\tau_d$ for various noise properties, we have reproduced figure 2 of [14] for identical oscillators in the absence of noise (i.e. $D = 0$) and presented it by the solid curve in figure 2. In the presence of noise the phase oscillators become nonidentical and the time delay is effective in the dynamics even at its low value compared to the case where noise is absent. Since the variance of the non-Gaussian noise is higher than the Gaussian one, the shift of the damped oscillating curve towards the larger critical coupling strength from the noiseless case is much larger for the former than the latter at low noise correlation time, $\tau \approx 0.1$. Their difference reduces remarkably at large noise correlation time, $\tau \approx 2$. The reason may be as follows. The effective noise correlation time of the colored non-Gaussian noise is much higher than that of the

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Gaussian one at large $\tau$. Increase of the noise correlation time leads to the development of a better phase relationship among the oscillators reducing the phase diffusion. Thus at large $\tau$ the difference in critical coupling strength is small for a given time delay. It is apparent in figure 2 that the interplay of noise correlation time and the time delay plays some constructive role in having a synchronized state, particularly at large $\tau$ and around the first maximum of the damped oscillation curve. It also slightly shifts the position of the maximum towards the left and reduces the oscillation amplitude at large time delay values. Thus colored noise plays a role beyond the induction of diffusion behavior.

Before going to the next case we would like to demonstrate the variation of critical coupling strength as a function of frequency of the coupled identical oscillators in the presence of time delay. In figure 3 we have presented this. It exhibits that the variation is periodic as a result of interplay of the time delay and the frequency of the oscillator. However, again it shows how noise correlation can reduce the stability of the incoherent state. Because of higher effective noise strength of the non-Gaussian noise, the critical coupling strength for the case $p = 1.5$ and $\tau = 0.5$ is comparable to white Gaussian noise.

For the second case of nonidentical phase oscillators, we have done calculations similar to those for figure 2 for identical oscillators, and the results are presented in figure 4. Again, the solid curve in figure 4 is a reproduction of figure 4 of [14]. All the features of figure 2 in the presence of colored noise have appeared in figure 4. Here one important point to be noted is that around the maximum, the incoherent state is unstable even at lower coupling strength in the presence of colored noise compared to the case without noise. Thus colored noise can effect the nonlinear coupling among the phase oscillators. It is quite similar to the modification of the dynamics by the colored noise in the presence of a nonlinear potential [24, 25].

In the next step we demonstrate how the critical coupling strength varies as a function of noise parameters. First, we consider the dependence of $\epsilon_c$ on the noise strength ($D$). To identify the signature of the time delay in this context we have determined a ratio

Figure 2. Plot of the critical value of coupling strength $\epsilon_c$ versus the time delay $\tau_d$ for identical oscillators for $\omega_0 = \pi/2$. 

\[
\begin{align*}
\epsilon_c & = 2 \\
\tau_d & = 4 \\
D & = 0.0 \\
D & = 0.25, \epsilon = 0.1, p = 1.0 \\
D & = 0.25, \epsilon = 0.1, p = 1.5 \\
D & = 0.25, \epsilon = 2.0, p = 1.0 \\
D & = 0.25, \epsilon = 2.0, p = 1.5 \\
\end{align*}
\]
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Figure 3. Plot of the critical value of coupling strengths ($\epsilon_c$) versus the frequency of identical oscillators $\omega_0$ for $D = 1.0$ and $\tau_d = 2.0$.

Figure 4. Plot of the critical value of coupling strength $\epsilon_c$ versus the time delay $\tau_d$ for nonidentical oscillators for the parameter set $\omega_0 = 3.0$ and $\lambda = 0.1$.

of critical coupling strengths $\epsilon_c(\tau_d = 0)/\epsilon_c(\tau_{dm})$ for a given noise intensity, where $\tau_{dm}$ is the location of the first maximum in the $\tau_d$ axis. For identical oscillators we have chosen $\tau_{dm} = 2$ from figure 2 and for nonidentical oscillators we have considered $\tau_{dm} = 1$ from figure 4. It is not difficult to anticipate that this choice will incorporate the maximum effect of time delay. However, the ratio had been calculated for different values of the noise strength for the white noise and is presented in figure 5. At first it rapidly increases both for identical (dotted curve) and nonidentical (solid curve) oscillators then the growth rate slows down. The initial growth rate is higher for identical oscillators compared to
nonidentical oscillators because at low noise strength $\epsilon_c(\tau_d = 0)$ is close to zero for the former and it grows at a faster rate. If the noise strength is appreciably large then the ratio is higher for identical oscillators than for the nonidentical case even if $\tau_{dm}$ is greater for the former than the latter. Thus time delay is more effective in the dynamics for nonidentical oscillators compared to the case of identical oscillators. Similar features are also observed for the colored noise and are demonstrated in insets (a) and (b) for the Gaussian and non-Gaussian noises. At large noise intensities the ratio of critical coupling strengths is smaller for the colored Gaussian noise compared to other noises for the case of nonidentical oscillators. Thus on switching from the white Gaussian to colored Gaussian or non-Gaussian characteristics of noise the time delay begins to play an important role in the dynamics.

The variation of the ratio of critical coupling strengths with the non-Gaussian parameter $p$ is demonstrated in figure 6. It shows that the ratio increases as a function of $p$ for the cases of both identical and nonidentical oscillators. The rate of growth is higher for the former than the latter case. The ratio is lower for the identical oscillators compared to the case of nonidentical oscillators at low $p$. However at large $p$ the situation turns around, it becomes inverse. These aspects can be understood if one keeps in mind that the effective noise strength increases as $p$ grows. Therefore figure 5 is essentially similar to figure 6 and it can be explained in the same way as we have explained the results presented in figure 6.

Finally, in figure 7 we have presented the variation of the ratio of critical coupling strength as a function of noise correlation time. It shows that the ratio decreases with increase of $\tau$. Since phase diffusion reduces as the noise becomes more colored, the critical coupling strength decreases as $\tau$ grows both in the presence and the absence of time delay.
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Figure 6. Plot of the ratio of the critical value of coupling strengths $\epsilon_c(\tau_d = 0)/\epsilon_c(\tau_{dm})$ versus the non-Gaussian parameter $p$ for the parameter set $\tau = 0.5$ and $D = 0.25$. For identical oscillators $\omega_0 = \pi/2$ and for nonidentical oscillators $\omega_0 = 3.0$ and $\lambda = 0.5$. Solid and dotted curves correspond to nonidentical and identical oscillators, respectively.

Figure 7. Plot of the ratio of the critical value of coupling strengths $(\epsilon_c(\tau_d = 0)/\epsilon_c(\tau_{dm}))$ versus the noise correlation time $\tau$ for the parameter set $p = 1.0$ and $D = 1.0$. For identical oscillators $\omega_0 = \pi/2$ and for nonidentical oscillators $\omega_0 = 3.0$ and $\lambda = 0.5$. Solid and dotted curves correspond to nonidentical and identical oscillators, respectively. In the inset, $p = 1.5$.

The ratio decreases because in the absence of time delay the colored noise is more effective in the dynamics. Similarly, the rate of decrease is a little bit higher for identical oscillators than for nonidentical ones. These kinds of features are also observed for the colored non-Gaussian noise and are presented in the inset of figure 7. In both cases the ratio is higher for the identical oscillators compared to the nonidentical oscillators for a given
noise correlation time. Thus it again supports the statement that the time delay is more effective in the dynamics for the latter than for the former.

4. Conclusion

We have considered the stochastic Kuramoto model including time-delayed interaction with a delay time $\tau_d$ subject to both OU Gaussian or non-Gaussian colored noise with a correlated time $\tau$. The main focus of our study was on the interplay between the effects of finite correlation time $\tau$ and the time delay $\tau_d$ and their influence on the onset of the synchronization. Our results can be summarized as follows.

(i) The shift of the damped oscillating curve (for the critical coupling strength as a function of time delay $\tau_d$) towards the larger critical coupling strength from the noiseless case is much larger for the non-Gaussian noise than for the Gaussian one at low noise correlation time, $\tau \approx 0.1$. Their difference reduces remarkably at large noise correlation times, $\tau \approx 2$ (figure 2).

(ii) Due to the present intrinsic correlations, the colored noise plays a role beyond the induction of diffusive behavior.

(iii) The incoherent (unsynchronized) state may be unstable around the maximum even at lower coupling strength values in the presence of colored noise compared to the noiseless case (figure 4).

(iv) The ratio of critical coupling strengths $\epsilon_c(\tau_d = 0)/\epsilon_c(\tau_{dm})$ first rapidly increases as a function of noise intensity $D$ for both the identical and nonidentical oscillators and then slows down. The initial growth rate is higher for identical oscillators compared to the case of nonidentical oscillators. If the noise strength is appreciably large then the ratio is higher for identical oscillators than for the nonidentical case even if $\tau_{dm}$ is greater for the former than the latter case. At large values of noise intensity the ratio of critical coupling strength is smaller for the colored Gaussian noise than for the other noises for the case of nonidentical oscillators (figure 5).

(v) The ratio of critical coupling strengths increases as a function of the non-Gaussian parameter $p$ for both cases of identical and nonidentical oscillators. The rate of growth is higher for the former case than the latter one. The ratio is lower for identical oscillators compared to nonidentical oscillators at low $p \geq 1$. However at large values of $p$ it becomes inverse (figure 6).

(vi) The above ratio decreases with increase of $\tau$. The rate of decrease is higher for the case of identical oscillators than for the case of nonidentical oscillators (figure 7).

We anticipate that investigation of the influence of time-delayed interactions and (generally non-Gaussian) noise in complex systems would lead to important insights into their stochastic dynamics, e.g., for the process of intercellular synchronization in biology [26] (see also the systems biology models mentioned as possible applications in [21]). The time-delayed interaction also plays an important role in a molecular model of biological evolution [27].

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