SMOOTHNESS OF THE UNIFORMIZATION OF TWO-DIMENSIONAL LINEAR FOLIATION ON TORUS WITH NONSTANDARD METRIC

A.A. GLOTSYUK

§ 1 Main results and history

Denote \( T^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n \). Consider a two-dimensional parallel plane foliation on \( \mathbb{R}^n \). The standard projection \( \mathbb{R}^n \to T^n \) induces a foliation on the torus \( T^n \). This foliation will be denoted by \( F \).

Let \( g \) be arbitrary (smooth) Riemann metric on \( T^n \). It induces a complex structure on each leaf of the foliation \( F \) (this complex structure will be denoted by \( \sigma \)). The leaves are parabolic Riemann surfaces with respect to \( \sigma \): their universal coverings are conformally equivalent to complex plane. This follows from Riemann quasiconformal mapping theorem and boundedness of the dilatation of the metric \( g \) with respect to the standard Euclidean metric of \( T^n \). Thus, each leaf admits a complete \( \sigma \)-conformal flat metric. This means that there exists a real positive function \( \phi \) on the leaf such that the metric \( \phi g \) on the leaf is flat and complete. This function is unique up to multiplication by constant.

É. Ghys [1] stated the following problem: is it true that for any smooth metric \( g \) on the torus \( T^3 \) the corresponding function \( \phi \) (which defines the flat metric \( \phi g \) on the leaves) may be chosen to be continuous (smooth) on the whole torus: continuous (smooth) not only in the parameter of an individual leaf, but also in the transversal parameter (in particular, in all the intersections of the given leaf with a transversal circle)?

In the simplest case, when the leaves of the foliation \( F \) are tori, the answer to this question is positive. This follows from the classical theorem on dependence of uniformization on parameter of complex structure [2,3] (Theorem 4 in Subsection 2.4).

We prove the following Theorem giving the positive answer to Ghys’ question.

**Theorem 1.** Let \( F \) be as at the beginning of the paper, \( g \) be an analytic /\( C^\infty \)/measurable uniformly bounded together with dilatation along the leaves of \( F \)/ metric on \( T^n \). There exists an analytic /\( C^\infty \)//\( L_1 \)/ positive function \( \phi : T^n \to \mathbb{R}_+ \) such that the restriction of the metric \( \phi g \) to each leaf of the foliation \( F \) is flat (in the sense of distributions in the third case).

Theorem 1 is proved in Section 2.

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For a foliation $F$ satisfying appropriate Diophantine condition (from one of Definitions 2, 3 or 4) we prove the existence of a $C^\infty$ (analytic) Euclidean metric on the torus such that the leaves are totally geodesic and its restriction to each leaf is conformal and flat (Theorems 2 and 3). This is equivalent to the statement that the triple consisting of the torus, the foliation $F$ and the family of complex structures on its leaves is isomorphic to the analogous triple corresponding to a linear foliation on another torus with the complex structure on the leaves defined by the standard Euclidean metric. This Diophantine condition is exact.

**Definition 1.** We say that a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine, if there exist constants $C > 0$, $s > 1$ such that for any pair $m, k \in \mathbb{Z}$, $k \neq 0$, the following inequality holds:

$$|\alpha - \frac{m}{k}| > \frac{C}{|k|^{s+1}}.$$  

**Definition 2.** Let $x = (x_1, x_2, x_3)$ be coordinates in the space $\mathbb{R}^3$. Consider the foliation on $\mathbb{R}^3$ by level planes of the linear function $l(x) = a_1 x_1 + a_2 x_2 - x_3$. Let $F$ be the corresponding factorized foliation on $T^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$. We say that $F$ is Diophantine, if there exist constants $C > 0$, $s > 1$ such that for any $N = (p, m, k) \in \mathbb{Z}^3 \setminus 0$ the following inequality holds:

$$(1) \quad |p + ka_1| + |m + ka_2| > \frac{C}{|N|^s}, \quad |N| = |p| + |m| + |k|.$$  

**Example 1.** In the notations of the previous Definition let the additive subgroup in $\mathbb{R}$ generated by $a_1$ and $a_2$ contain a Diophantine number. Then the foliation $F$ is Diophantine.

Let us extend the Definition of Diophantine foliation to higher dimensions.

**Definition 3.** Let $x = (x_1, \ldots, x_n)$ be coordinates in $\mathbb{R}^n$. Consider the foliation on $\mathbb{R}^n$ by level planes of the linear $n - 2$-dimensional vector function $l(x) = (l_3, \ldots, l_n)(x), l_j(x) = a_1^j x_1 + a_2^j x_2 - x_j$. Let $F$ be the corresponding factorized foliation on $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. We say that $F$ is Diophantine, if there exist constants $C > 0$, $s > 1$ such that for any $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ the following inequality holds:

$$|m_1 + \sum_{j=3}^{n} a_1^j m_j| + |m_2 + \sum_{j=3}^{n} a_2^j m_j| > \frac{C}{|m|^s}, \quad |m| = \sum_{j=1}^{n} |m_j|.$$  

**Remark 1.** The previous Definition of Diophantine foliation coincides with Definition 2 in the case, when $n = 3$.

**Theorem 2.** Let $F$ be a Diophantine foliation (see Definitions 2 and 3), $g$ be an analytic ($C^\infty$) metric on $T^n$, $\sigma$ be the family of complex structures on the leaves of $F$ such that $g$ is $\sigma$-conformal. There exists an analytic ($C^\infty$) Euclidean metric on $T^n$ such that all the leaves are totally geodesic and its restriction to each leaf is $\sigma$-conformal. Or equivalently, there exists a discrete rank $n$ additive subgroup $G \subset \mathbb{R}^n$ and an analytic ($C^\infty$) diffeomorphism $T^n \to T_G = \mathbb{R}^n/G$ that transforms $F$ to a linear foliation and $\sigma$ to the standard complex structure induced by the standard Euclidean metric. Inversely, if a linear foliation on $T^n$ is not Diophantine, then
there exists a $C^\infty$ metric $g$ on the torus such that there is no $C^2$ Euclidean metric on $\mathbb{T}^n$ satisfying the previous statement.

Theorem 2 is proved in Section 3.

We give the following weaker Diophantine condition on $F$ necessary and sufficient to satisfy the analytic version of the first statement of Theorem 2 for arbitrary analytic metric.

**Definition 4.** In the notations of the previous Definition a foliation $F$ is said to be weakly Diophantine, if

(2) $\lim_{|m| \to \infty} (|m_1 + \sum_{j=3}^n a_1^j m_j| + |m_2 + \sum_{j=3}^n a_2^j m_j|)^{1/|m|} = 1.$

**Remark 2.** The limit in (2) is always less or equal to 1.

**Remark 3.** A Diophantine foliation is always weakly Diophantine.

**Theorem 3.** Let $F$ be a weakly Diophantine foliation (see Definition 4). Then for any analytic metric $g$ on $\mathbb{T}^n$ there exists an analytic Euclidean metric on $\mathbb{T}^n$ that satisfies the first statement of Theorem 2. Inversely, if $F$ is not weakly Diophantine, then there exists an analytic metric $g$ on $\mathbb{T}^n$ such that there is no $C^2$ Euclidean metric on $\mathbb{T}^n$ that satisfies the first statement of Theorem 2.

Theorem 3 is proved in Section 3. It implies the analytic version of Theorem 2.

Earlier a particular case of the problem on existence of continuous complete conformal flat metric stated at the beginning of the paper was studied by É.Ghys [1]. He proved the statement of Theorem 1 in the dimension 3 under the additional assumption that in the notations of Definition 2 the additive group generated by the numbers $a_1$ and $a_2$ contains either a nonzero rational, or a Diophantine number. In the same paper he constructed a counterexample to Theorem 1 that is a compact complex manifold equipped with a holomorphic foliation by Riemann surfaces such that for any metric $g$ on the manifold there is no measurable function $\phi$ such that the restriction to each leaf of the metric $\phi g$ would be flat and complete. Recently he proposed the following general

**Question.** Let a compact manifold equipped with a two-dimensional foliation admit at least one smooth metric flat along the leaves. Is it true that for any other smooth metric $g$ there exists a smooth function $\phi$ such that the restriction to the leaves of the metric $\phi g$ is flat?

In 1980 A.Haefliger [6] proved a theorem implying the statement analogous to Theorem 2: any smooth family of Riemann metrics on the leaves of a linear foliation $F$ on $\mathbb{T}^3$ is the restriction to the leaves of a metric on $\mathbb{T}^3$ for which the leaves are minimal, if the additive subgroup generated by the numbers $a_j$ (see the notations of Definition 2) contains a Diophantine number.

§2 Existence of fiberwise flat conformal metric. Proof of Theorem 1

2.1. Scheme of the proof of Theorem 1.

Theorem 1 is implied by the following more general statement.
Lemma 1. Let $F$ be at the beginning of the paper, $\sigma$ be a family of almost complex structures on the leaves of $F$ that is analytic $C^\infty$/measurable with uniformly bounded dilatation on $\mathbb{T}^n$. There exists a nowhere vanishing analytic $C^\infty/L^2$/differential 1-form $\omega$ on $\mathbb{T}^n$ such that its restriction to each leaf is $\sigma$-holomorphic and moreover there exists a quasiconformal homeomorphism of the universal covering of the leaf onto complex plane, whose derivative is the pullback of the form $\omega$ under the covering (in the sense of distributions in the measurable case).

Lemma 1 is proved in Subsections 2.2 ($C^\infty$ case) and 2.3 (analytic and measurable cases).

In the conditions of Theorem 1 let $\sigma$ be the family of complex structures on the leaves defined by the metric $g$. Let $\omega$ be the correspondent $\sigma$-holomorphic 1-form from Lemma 1. The metric $\phi g$ from Theorem 1 we are looking for is $\omega\bar{\omega}$.

The method of the proof of Lemma 1 presented below yields a proof of a version of the $C^\infty$ Ahlfors-Bers theorem [4] (Theorem 4 stated and proved in Subsection 2.4) that is simpler than the original one from [4].

2.2. Homotopy method. Proof of Lemma 1.

For simplicity we present a proof of Lemma 1 only in the case, when $n = 3$. This proof remains valid in higher dimensions with obvious changes.

Everywhere below we consider that the leaves of the foliation $F$ under consideration are dense in $\mathbb{T}^3$, or equivalently, they are not tori. In the opposite case the statement of Lemma 1 follows from the classical theorem on dependence of uniformization on parameter of complex structure [2,3] (cf. Theorem 4 in Subsection 2.4).

We use the notations of Definition 2 (the coordinates $(x_1, x_2, x_3)$ are chosen so that the $x_3$-axis is transversal to the lifting of the foliation $F$ to the space). Define the complex coordinate $z = x_1 + ix_2$ on the leaves of the foliation $F$ and its lifting (the coordinates $x_i$, $z$ and their projection pushforwards to the torus will be denoted by the same symbols $x_i$ and $z$).

In this Subsection we prove the smooth version of Lemma 1. Its analytic and measurable versions are proved analogously (the corresponding modifications needed are specified in Subsection 2.3). The almost complex structure $\sigma$ on the leaves is defined by its $\mathbb{C}$-linear form that is a complex-valued 1-form $\omega_\mu = dz + \mu(x)d\bar{z}$, $\mu \in C^\infty(\mathbb{T}^3)$, $|\mu| < 1$. For the proof of Lemma 1 we will construct a $C^\infty$ function $f : \mathbb{T}^3 \to \mathbb{C} \setminus 0$ such that the restriction to the leaves of the form $f \omega_\mu$ is closed. Then $\omega = f \omega_\mu$ is the $\sigma$-holomorphic differential we are looking for. The restriction to each leaf of the metric $\omega\bar{\omega}$ is conformal flat and complete, as does its pullback to the universal covering of the leaf. The quasiconformal homeomorphism from the last statement of Lemma 1 is the isometry of the universal covering equipped with the metric $\omega\bar{\omega}$ and the complex plane equipped with the standard Euclidean metric (this isometry is normalized to have the derivative $\omega$).

To define this isometry, we fix a geodesic $\Gamma$ in the universal covering and consider all the geodesics intersecting $\Gamma$ orthogonally (at least once). The flatness of the metric $\omega\bar{\omega}$ implies that no pair of the latters is intersected and no one of them intersects $\Gamma$ more than at one point. The isometry we are looking for maps all the geodesics under consideration (including $\Gamma$) to straight lines preserving lengths of their segments and the orthogonal intersections. This map is well-defined and is an isometry (by flatness of $\omega\bar{\omega}$) and hence transforms $\sigma$ to the standard complex structure. Its derivative is equal to $\omega$ up to multiplication by constant with unit.
module. Indeed the pushforward of the form $\omega$ under this isometry is holomorphic (since $\omega$ is $\sigma$-holomorphic) and hence, is equal to the standard coordinate differential times a holomorphic function of module 1. This function is constant by Liouville’s theorem on bounded entire functions. Thus, one can achieve that the derivative of the isometry be equal to $\omega$ by applying multiplication by appropriate complex number with unit module in the image $\mathbb{C}$.

To construct a function $f$ as in the previous item, we use the homotopy method. Namely, we include the complex structure $\sigma$ on the leaves into the one-parametric family of complex structures (denoted by $\sigma_\nu$) defined by their $\mathbb{C}$-linear 1-forms $\omega_\nu = dz + \nu(x,t)d\bar{z}$, $\nu(x,t) = t\mu(x)$, $t \in [0, 1]$. The complex structure on the leaves that corresponds to the parameter value $t = 0$ is the standard one, the given structure $\sigma = \sigma_\mu$ corresponds to the value $t = 1$. We will find a $C^\infty$ family $f(x,t): \mathbb{T}^3 \times [0, 1] \to \mathbb{C} \setminus 0$ of complex-valued nowhere vanishing $C^\infty$ functions on $\mathbb{T}^3$ depending on the same parameter $t$, $f(x,0) \equiv 1$, such that the differential forms $f\omega_\nu$ are closed on the leaves. Then the function $f(x,1)$ is a one we are looking for.

We will find a $C^\infty$ family $f(x,t)$ of complex-valued nonidentically-vanishing $C^\infty$ functions on $\mathbb{T}^3$ such that the forms $f\omega_\nu$ are closed on the leaves. Then it follows that the functions $f(x,t)$ vanish nowhere. Let us prove this inequality by contradiction. Suppose the contrary. Then the set of the parameter values corresponding to the functions $f(x,t)$ on the torus having zeroes is nonempty (denote this set by $M$). Its complement $[0, 1] \setminus M$ is open by definition. Let us show that the set $M$ is open as well. This will imply that the parameter segment is a union of two disjoint open sets, which will bring us to contradiction. For any parameter value the restriction to each leaf of the form $f\omega_\nu$ is a $\sigma_\nu$-holomorphic differential that does not vanish identically (by the assumptions that the leaves are dense and for any fixed $t$ $f(x,t) \neq 0$ on $\mathbb{T}^3$). On the other hand, each point of the torus has a neighborhood in the corresponding leaf that admits a continuous family of $\sigma_\nu$-holomorphic univalent coordinates (depending on the same parameter $t$) by the local $C^\infty$ version of the Ahlfors-Bers theorem [4] (more precisely, by Theorem 4 (or by its weaker version Proposition 2) both stated and proved in Subsection 2.4). This is the only place in the proof of the $C^\infty$ version of Lemma 1 we use the Ahlfors-Bers theorem. The statement that $M$ is open follows from the existence of continuous family of local $\sigma_\nu$-holomorphic charts (from Proposition 2), continuity of the family $f\omega_\nu$ and openness of the set of nonidentically-vanishing holomorphic differentials having zeroes. This proves the inequality $f(x,t) \neq 0$.

Let us write down explicitly the fiberwise closeness condition on the form $f\omega_\nu$. To do this, let us introduce the following

**Definition 5.** Let $F$ be a linear foliation on $\mathbb{T}^3$. In the notations of Definition 2 let $z = x_1 + ix_2$. Let us equip each leaf of the foliation $F$ with the local coordinate $z$. Define $D_z$ ($D_{\bar{z}}$) to be the differential operator in the space of complex-valued smooth functions on $\mathbb{T}^3$ acting as follows: first restrict a function to a leaf and then apply the operator $\frac{\partial}{\partial z}$ (respectively, $\frac{\partial}{\partial \bar{z}}$) to its restriction to the leaf.

**Remark 4.** In the conditions of Definition 5

\[
D_z = \frac{\partial}{\partial z} + \frac{a_1 - ia_2}{2} \frac{\partial}{\partial x_3}, \quad D_{\bar{z}} = \frac{\partial}{\partial \bar{z}} + \frac{a_1 + ia_2}{2} \frac{\partial}{\partial x_3}
\]

(here $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ are the operators acting on functions on $\mathbb{T}^3$ in the coordinates $(z,x_3)$, not on their restrictions to the leaves).
Remark 5. In the conditions of Definition 5 let \( f, \nu : \mathbb{T}^3 \to \mathbb{C} \) be \( C^\infty \) functions, \( \omega_\nu = dz + \nu d\bar{z} \). The restriction to the leaves of the foliation \( F \) of the differential form \( f \omega_\nu \) is closed, iff
\[
D_\bar{z}f = D_\bar{z}(\nu f).
\]

Thus, Lemma 1 is implied by the previous discussion and the following

**Lemma 2.** Let \( F \) be a linear foliation on \( \mathbb{T}^3 \) with dense leaves, \( \nu(x,t) : \mathbb{T}^3 \times [0,1] \to \mathbb{C} \) be a \( C^\infty \) family of \( C^\infty \) functions on \( \mathbb{T}^3 \) depending on the parameter \( t \in [0,1] \), \( \nu(x,0) \equiv 0, |\nu| < 1 \). There exists a \( C^\infty \) family \( f(x,t) \) of complex-valued \( C^\infty \) functions on \( \mathbb{T}^3 \) depending on the same parameter \( t \), \( f(x,0) \equiv 1 \), that do not vanish identically and satisfy (4).

**Proof.** For a family \( u(x,t) \) of functions on the torus denote \( \dot{u} = \frac{\partial u}{\partial t} \). The differentiation of equation (4) in \( t \) yields
\[
(D_\bar{z} \dot{f} - (D_\bar{z} \circ \nu) \dot{f}) = (D_\bar{z} \circ \dot{\nu}) f,
\]
where \( D_\bar{z} \circ \nu (D_\bar{z} \circ \dot{\nu}) \) is the composition of the operator of the multiplication by the function \( \nu \) (respectively, \( \dot{\nu} \)) and the operator \( D_\bar{z} \). Any solution \( f \) of equation (5) with the initial condition \( f(x,0) \equiv 1 \) that does not vanish identically in the torus for no value of \( t \) is a one we are looking for.

The proof of Lemma 2 is based on the following properties of the operators \( D_\bar{z} \) and \( D_\bar{z} \).

**Remark 6.** In the conditions of Definition 5 the operators \( D_\bar{z} \) and \( D_\bar{z} \) have the common eigenfunctions \( e^{i(N,x)} \), \( N = (p,m,k) \in \mathbb{Z}^3 \) (which form an orthogonal base in the space \( L_2(\mathbb{T}^3) \)). The corresponding eigenvalues are equal to
\[
\lambda_N = \frac{i}{2}(p+ka_1-i(m+ka_2)) \quad \text{and} \quad \lambda'_N = -\lambda_N
\]
respectively. This follows from (3). In particular, \(|\lambda_N| = |\lambda'_N|\) for any \( N \). The leaves of the foliation \( F \) are dense, iff \( \lambda_N \neq 0 \) for all \( N \neq 0 \).

**Corollary 1.** In the conditions of Definition 5 there exists a unitary operator \( U : L_2(\mathbb{T}^3) \to L_2(\mathbb{T}^3) \) preserving averages and the spaces of \( C^\infty \) (analytic) functions such that \( "U = D_{\bar{z}}^{-1} \circ D_\bar{z} " \) (more precisely, \( U \circ D_{\bar{z}} = D_{\bar{z}} \circ U = D_{\bar{z}} \)). This operator is unique, iff the leaves of the foliation \( F \) are dense. The operator \( U \) commutes with partial differentiations and extends up to a unitary operator to any Hilbert Sobolev space of functions on \( \mathbb{T}^3 \).

The operator \( U \) from Corollary 1 is defined to have the eigenfunctions \( e^{i(N,x)} \); the corresponding eigenvalue is equal to \( \frac{\lambda_N}{\lambda_N'} = -\frac{\lambda_N}{\lambda_N'} \) (see (6)) if \( \lambda_N \neq 0 \) and to 1 otherwise, i.e., if \( D_{\bar{z}} e^{i(N,x)} = 0 \) (in this case it could be taken to be arbitrary number with unit module, if \( N \neq 0 \)).

Let us write down equation (5) in terms of the new operator \( U \). Applying the "operator" \( D_{\bar{z}}^{-1} \) to (5) and substituting \( U = D_{\bar{z}}^{-1} \circ D_\bar{z} \) yields
\[
(Id - U \circ \nu) \dot{f} = (U \circ \dot{\nu}) f.
\]
This equation implies (5). The operator \( Id - U \circ \nu \) in the left-hand side of this equation is invertible in \( L_2(\mathbb{T}^3) \) for any \( t \) and the norm of the inverse operator is bounded uniformly in \( t \), since \( U \) is unitary and the module \( |\nu| \) is less than 1 and bounded away from 1. As it is shown below (in Proposition 1), so is it in all the Hilbert Sobolev spaces \( H^j(\mathbb{T}^3) \). Thus, the last equation can be rewritten as

\[
(7) \quad \dot{f} = (Id - U \circ \nu)^{-1}(U \circ \dot{\nu})f,
\]

which is an ordinary differential equation in \( f \in L_2(\mathbb{T}^3) \) with the right-hand side having uniformly bounded derivative in \( f \), and so it in \( f \) belonging to arbitrary space \( H^j(\mathbb{T}^3) \) (with respect to the Sobolev scalar product). Therefore, equation (7) written in arbitrary Hilbert Sobolev space has a unique solution with a given initial condition, in particular, with \( f(x,0) \equiv 1 \) (the theorem on existence and uniqueness of solution of ordinary differential equation in Banach space with the right-hand side having uniformly bounded derivative [5]). This solution does not vanish identically in \( \mathbb{T}^3 \) and belongs to all the spaces \( H^j(\mathbb{T}^3) \) for each value of \( t \). Therefore it is \( C^\infty(\mathbb{T}^3) \) for any \( t \) by Sobolev embedding theorem (see [5], p.411). Thus, Lemma 2 is implied by the following

**Proposition 1.** Let \( x = (x_1, x_2, x_3) \) be affine coordinates in \( \mathbb{R}^3 \), \( \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3 \). Let \( s \geq 0, s \in \mathbb{Z}, U \) be a linear operator in the space of \( C^\infty \) functions on \( \mathbb{T}^3 \) that commutes with the operators \( \partial_{x_1}, \partial_{x_2}, \partial_{x_3} \), \( i = 1, 2, 3 \), and extends to any Sobolev space \( H^j = H^j(\mathbb{T}^3), 0 \leq j \leq s \), up to a unitary operator. Let \( 0 < \delta < 1, \nu \in C^s(\mathbb{T}^3) \) be a complex-valued function, \( |\nu| \leq \delta \). The operator \( Id - U \circ \nu \) is invertible and the inverse operator is bounded in all these spaces \( H^j \). For any \( 0 < \delta < 1, j \leq s \), there exists a constant \( C > 0 \) (depending only on \( \delta \) and \( s \)) such that for any complex-valued function \( \nu \in C^s(\mathbb{T}^3) \) with \( |\nu| \leq \delta \)

\[
||(Id - U \circ \nu)^{-1}||_{H^j} \leq C(1 + \max\{|\partial^k\nu|_{\partial x_{i_1}, \ldots, \partial x_{i_k}}|j| \mid k \leq j\}).
\]

**Proof.** Let us prove Proposition 1 for \( s = 1 \). For higher \( s \) its proof is analogous.

By definition,

\[
(8) \quad ||U \circ \nu||_{L_2} \leq \delta < 1.
\]

Hence, the operator \( Id - U \circ \nu \) is invertible in \( L_2 = H^0 \) and

\[
(9) \quad (Id - U \circ \nu)^{-1} = Id + \sum_{k=1}^{\infty} (U \circ \nu)^k:
\]

the sum of the \( L_2 \) operator norms of the sum entries in the right-hand side of (9) is finite by (8). Let us show that the operator in the right-hand side of (9) is well-defined and bounded in \( H^1 \). To do this, it suffices to show that the sum of the operator \( H^1 \)-norms of the same entries is finite.

Let \( f \in H^1 \). Let us estimate the \( H^1 \)- norm of the images \( (U \circ \nu)^k f \). We show that for any \( k \in \mathbb{N} \)

\[
(10) \quad ||\partial_{x_r}((U \circ \nu)^k f)||_{L_2} < ck\delta^{k-1}||f||_{H^1}, \quad c = \delta + \max \left| \frac{\partial \nu}{\partial x_r} \right|, \quad r = 1, 2, 3.
\]
This will imply the finiteness of the operator $H^1$-norm of the sum in the right-hand side of (9) and Proposition 1 (with $C = 4 \sum_{k \in \mathbb{N}} k \delta^{k-1}$).

Let us prove (10), e.g., for $r = 1$. The correspondent derivative in the left-hand side of (10) is equal to

$$(11) \quad (U \circ \nu)^k \frac{\partial f}{\partial x_1} + \sum_{i=1}^{k} (U \circ \nu)^{k-i} \circ (U \circ \frac{\partial \nu}{\partial x_1}) \circ (U \circ \nu)^{i-1} f$$

(since $U$ commutes with the partial differentiation by the condition of Proposition 1). The $L_2$-norm of the first term in (11) is not greater than $\delta^k ||f||_{H^1}$ by (8). That of each entry of the sum in (11) is not greater than $\delta^{k-1} \max |\frac{\partial \nu}{\partial x_1}| ||f||_{L_2}$ by (8). This proves (10). Proposition 1 is proved. Lemma 2 is proved, as are the $C^\infty$ versions of Lemma 1 and Theorem 1.

2.3. Measurable and analytic cases of Lemma 1.

In the case, when the metric $g$ is measurable, the proof of Lemma 1 remains valid with obvious changes (e.g. all the differential equations are understood in the sense of distributions) except that of its last statement on existence of quasiconformal homeomorphism. Let us prove this statement.

Let $\omega_\mu = dz + \mu d\bar{z}$ be the $1$-form $\mathbb{C}$-linear with respect to $\sigma$, $\nu = t\mu$, $t \in [0, 1]$. Denote by $f(x, t)$ the solution of ordinary differential equation (7) in $L_2(\mathbb{T}^3)$ with $f(x, 0) \equiv 1$ and put $f = f(x) = f(x, 1)$. The form $f \omega_\mu$ is a one we are looking for. Indeed it is closed on the leaves (in the sense of distributions on the torus) by construction. Let us show that its lifting to the universal covering of a generic leaf (with respect to the Lebesgue transversal measure) is the derivative of a quasiconformal homeomorphism that maps the universal covering onto complex plane. (The universal covering of a leaf $L$ will be denoted by $\tilde{L}$, the lifting to $\tilde{L}$ of a function $f$ (a form $\omega_\mu$) on $L$ will be denoted by the same symbol $f$ ($\omega_\mu$)). To do this, let us approximate the function $\mu$ by functions $\mu_k \in C^\infty(\mathbb{T}^3)$ with moduli less than 1 and uniformly bounded away from 1: $|\mu_k| < \delta < 1$, $\mu_k \to \mu$ almost everywhere, as $k \to \infty$. Let $f_k(x, t)$ be the solution of (7) with $\nu = t\mu_k$, $t \in [0, 1]$, $f_k(x, 0) \equiv 1$. Put $f_k(x) = f_k(x, 1)$. Then $f_k \to f$ in $L_2(\mathbb{T}^3)$; the solution of (7) in $L_2(\mathbb{T}^3)$ with unit initial condition depends continuously on the functional parameter $\nu$ by theorem on dependence of solution of ordinary differential equation in Banach space on the parameter $[5]$. Let us fix a generic leaf $L$ so that the set $f \neq 0$ has a positive measure in $\tilde{L}$ and $f_k \to f$ in $L_2$ in compact subsets of $\tilde{L}$. Let us fix a point $y_0 \in \tilde{L}$. Let $\psi_k : \tilde{L} \to \mathbb{C}$ be the diffeomorphism with the derivative $f_k \omega_{\mu_k}$ such that $\psi_k(y_0) = 0$ (the diffeomorphism satisfying these conditions is unique and quasiconformal by the smooth version of Lemma 1 proved before). By construction, the derivatives $f_k \omega_{\mu_k}$ of the diffeomorphisms $\psi_k$ converge to $f \omega_\mu$ in the sense of distributions on $\tilde{L}$. Let us show that the sequence $\psi_k$ converges uniformly in compact sets to a quasiconformal homeomorphism. This homeomorphism will be a one we are looking for: its derivative will be equal to $f \omega_\mu$, as the limit of the derivatives $f_k \omega_{\mu_k}$. To do this, let us fix another point $y_1 \in \tilde{L}$, $y_1 \neq y_0$. Consider the sequence $b_k = (\psi_k(y_1))^{-1}$ and denote $\tilde{\psi}_k = b_k \psi_k$. The maps $\tilde{\psi}_k$ are quasiconformal diffeomorphisms; each of them transforms the complex structure on $\tilde{L}$ defined by the form $\omega_{\mu_k}$ to the standard one. By definition, they map the points $y_0$ and $y_1$ to 0 and 1 respectively. They converge uniformly in
compact subsets to a quasiconformal homeomorphism that transforms the complex structure defined by the form $\omega_\mu$ to the standard one (by theorem on continuous dependence of the uniformizing quasiconformal homeomorphism on the parameter of complex structure [2,3,4]). Therefore their derivatives also converge (in the sense of distributions) to a nonzero limit. Hence the sequence $b_k$ also converges to a nonzero limit, since the contrary would contradict the convergence of the derivatives $f_k\omega_{\mu_k}$ of the diffeomorphisms $\psi_k$. Therefore the initial sequence $\psi_k$ also converges uniformly in compact sets to a quasiconformal homeomorphism with the derivative $f\omega_\mu$. This proves the measurable version of Lemma 1.

The analytic version of Lemma 1 is proved analogously to its $C^\infty$ version with the following change. We consider equation (7) in the space of analytic functions on a fixed closed ”annulus” $\{x \in \mathbb{C}^3, |\text{Im} x| \leq r\}/2\pi \mathbb{Z}$ containing $\mathbb{T}^3$. This space is equipped with the scalar product

$$
(f,g) = \sum_N f_N \bar{g}_N e^{i|N|r}, \ N \in \mathbb{Z}^3, \ f(x) = \sum_N f_N e^{i(N,x)}, \ g(x) = \sum_N g_N e^{i(N,x)}
$$

(instead of the Sobolev scalar product considered in the smooth case). Lemma 1 is proved. The proof of Theorem 1 is completed.

### 2.4. A proof of the $C^\infty$ Ahlfors-Bers theorem.

Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi \mathbb{Z}^2$, $z$ be a complex coordinate in $\mathbb{R}^2$. Recall the following notation: for a complex-valued function $\nu$ on $\mathbb{T}^2(\mathbb{R}^2)$ with $|\nu| < 1$ by $\sigma_\nu$ we denote the almost complex structure on $\mathbb{T}^2(\mathbb{R}^2)$ with the $\mathbb{C}$-linear differential $\omega_\nu = dz + \nu d\bar{z}$.

In this Subsection we give a proof of the following version of the Ahlfors-Bers theorem [4].

**Theorem 4** [2,3]. Let $\nu(z,t) : \mathbb{T}^2 \times [0,1] \rightarrow \mathbb{C}$ be a $C^\infty$ family of $C^\infty$ functions on $\mathbb{T}^2$ depending on the parameter $t \in [0,1], |\nu| < 1; \omega_\nu = dz + \nu d\bar{z}$. There exists a $C^\infty$ family $f(z,t) : \mathbb{T}^2 \times [0,1] \rightarrow \mathbb{C} \setminus 0$ of complex-valued nonvanishing $C^\infty$ functions on $\mathbb{T}^2$ such that for any fixed $t \in [0,1]$ the form $f\omega_\nu$ is closed and its standard projection pullback to $\mathbb{R}^2$ is the differential of a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{C}$.

**Proof.** Without loss of generality we consider that $\nu(z,0) \equiv 0$. The proof of Theorem 4 presented below is a modification of that of Lemma 1. As in Lemma 2, there exists a $C^\infty$ family $f(z,t)$ of complex-valued $C^\infty$ functions on $\mathbb{T}^2$ such that $f(z,0) \equiv 1$, the forms $f\omega_\nu$ are closed and for any $t \in [0,1] f(z,t) \neq 0$. It is the unique solution of the ordinary differential equation (7) in $L_2(\mathbb{T}^3)$ with the initial condition $f(z,0) \equiv 1$, where the operator $U = (\partial_{\bar{z}})^{-1} \circ \partial_z$ is defined to have the eigenbase $e^{i(n_1,\text{Re} n_2 + n_2 \text{Im} z)}$, $n_1, n_2 \in \mathbb{Z}$, with the eigenvalues equal to $\frac{n_1 - in_2}{n_1 + in_2}$ for $(n_1, n_2) \neq 0$ and the unit eigenvalue at the constant function. (This base is formed by the common eigenfunctions of the operators $\partial_{\bar{z}}$ and $\partial_z$.) The functions $f(z,t)$ are $C^\infty$, since for any $s \in \mathbb{N}$ and $t \in [0,1]$ $f(z,t) \in H^s(\mathbb{T}^2)$ and by Sobolev embedding theorem ([5], p.411). The inclusions $f \in H^s$ follow from boundedness in any Hilbert Sobolev norm of the operator in the right-hand side of (7) (cf. the discussion following (7) and the inequality of Proposition 1, which remains valid for the new operator $U$). Equation (7) implies that the forms $f\omega_\nu$ are closed, as in the proof of Lemma 2.

Let us prove that the family $f$ is a one we are looking for. As at the beginning of Subsection 2.2, it suffices to show that $f(z,t)$ vanishes nowhere. The proof of this
inequality repeats the one from the proof of Lemma 1 and is done by contradiction.
Suppose the contrary, i.e., the set of parameter values corresponding to the functions having zeroes (denote it by $M$) is nonempty. Its complement is open. It suffices to show that $M$ is open: this will imply that the parameter segment is a union of two disjoint nonempty open sets, which brings us to contradiction. Zeroes of a function $f(z, t)$ are exactly zeroes of the 1-form $f\omega\nu$, which is $\sigma\nu$-holomorphic and does not vanish identically. The openness of the set $M$ is implied by the continuity of the family $f\omega\nu$ of 1-forms, the openness of the set of nonidentically-vanishing holomorphic differentials having zeroes and the following local version of the Ahlfors-Bers theorem [4]:

**Proposition 2** [4]. Let $\nu(z, t)$ be a $C^\infty$ family of complex-valued $C^\infty$ functions in unit disc (equipped with the coordinate $z$) depending on the parameter $t \in [0, 1]$, $|\nu'| < 1$. For any $t_0 \in [0, 1]$ there exist a smaller disc $V$ centered at 0 and a $C^\infty$ family of $C^\infty$ $\sigma\nu$-holomorphic univalent coordinates in $V$ defined for all $t$ close enough to $t_0$.

Proposition 2 will be proved below, more precisely, it will be reduced to the following statement.

**Proposition 3.** For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any complex-valued function $\mu \in C^3(\mathbb{T}^2)$ with $||\mu||_{C^3} < \delta$ there exists a complex-valued function $f \in C^1(\mathbb{T}^2)$ with $||f - 1||_{C^1} < \varepsilon$ such that the form $f\omega\mu = f(dz + \mu dz)$ is closed (i.e., if $\varepsilon < 1$, then locally it is the differential of a $\sigma\mu$-holomorphic univalent function). The function $f$ is unique up to multiplication by constant and can be normalized to depend continuously on $\mu \in C^3(\mathbb{T}^2)$ as a $C^1$-valued functional. If $\mu$ is $C^\infty$, then so is $f$.

**Proof.** Firstly let us prove Proposition 3 with the change of the $C^1$- norm in its statement to the $H^3$- norm. To do this, let us include $\mu$ into the family $\nu(z, t) = t\mu(z)$ of functions depending on the parameter $t \in [0, 1]$. As in the proof of Lemma 2, there exists a unique solution $f(z, t)$ of the ordinary differential equation (7) in $H^3(\mathbb{T}^2)$ (with the new operator $U$ defined at the beginning of the Subsection) such that $f(z, 0) \equiv 1$. The function $f(z) = f(z, 1)$ is the one we are looking for. Indeed, the form $f\omega\mu$ is closed, $f$ is $C^\infty$ if so is $\mu$ (as in the proof of Lemma 2). The function $f(z)$ depends continuously on the functional parameter $\mu \in C^3(\mathbb{T}^2)$ as an $H^3$-valued functional (by the inequality of Proposition 1 (which remains valid for the new operator $U$) and the theorem on continuous dependence of solution of bounded ordinary differential equation in Banach space on the parameter [5]). Therefore $f(z)$ is $H^3$-close to 1, whenever the norm $||\mu||_{C^3}$ is small enough. This together with continuity of the canonical embedding $H^3 \to C^1$ (Sobolev embedding theorem, [5], p.411) proves Proposition 3.

Let us prove Proposition 2. Without loss of generality we consider that $\nu(0, t_0) = 0$ (one can achieve this by applying appropriate $\mathbb{R}$-linear change of the coordinate $z$). Let us choose arbitrary $\delta' > 0$. Without loss of generality we assume that all the derivatives of the function $\nu(z, t_0)$ in $z$ up to the order 3 have moduli less than $\delta'$ in the disc $|z| \leq \frac{1}{2}$ (one can achieve this by applying appropriate homothety in $z$). Let $\delta, \varepsilon < 1$ be as in Proposition 3. In the case, when $\delta'$ is chosen to be small enough, the restriction to the disc $|z| \leq \frac{1}{3}$ of the function $\nu$ extends to $\mathbb{C} = \mathbb{R}^2$ up to a double $2\pi$-periodic function whose pushforward to $\mathbb{T}^2$ has $C^3$-norm less
than $\delta$. This together with Proposition 3 implies existence of local family of $\sigma_\nu$-holomorphic coordinates we are looking for. This proves Proposition 2 and Theorem 4.

§3. DIOPHANTINE FOLIATIONS. PROOFS OF THEOREMS 2 AND 3

We present only the proof of the three-dimensional $C_\infty$ version of Theorem 2. This proof remains valid in higher dimensions with obvious changes. The proof of Theorem 3 (which implies the analytic version of Theorem 2) is analogous with the modifications similar to those needed in the proof of the analytic version of Lemma 1 (see the end of Subsection 2.3).

Let $F$ be a Diophantine foliation, $l(x) = a_1x_1 + a_2x_2 - x_3$ be the corresponding linear function (see the notations of Definition 2), $z = x_1 + x_2$. Let $\sigma$ be a $C_\infty(\mathbb{T}^3)$ family of almost complex structures on the leaves of $F$ (e.g., defined by a smooth metric on $\mathbb{T}^3$). Let $\omega$ be the correspondent differential form from Lemma 1. It is uniquely defined modulo $dl$ up to multiplication by constant, since the leaves of the foliation $F$ are dense ($F$ is Diophantine). Everyone of the two first equivalent statements in Theorem 2 is equivalent to the possibility to choose the form $\omega$ to be closed not only on the leaves, but on the whole torus. Indeed the second one of these statements in Theorem 2 implies the existence of a differential 1-form closed on $\mathbb{T}^3$ and holomorphic on the leaves. Inversely, for a given closed fiberwise holomorphic 1-form $\omega$ the Euclidean metric $\omega\omega + dld\bar{l}$ on $\mathbb{T}^3$ satisfies the first statement of Theorem 2. Let $\omega = f(x)(dz + \mu d\bar{z})$ be a fixed $C_\infty$ differential form on $\mathbb{T}^3$ $\sigma$-holomorphic on the leaves. Let us show that there exists a $C_\infty$ function $h : \mathbb{T}^3 \to \mathbb{C}$ such that the form

\begin{equation}
(12)\ f(dz + \mu d\bar{z}) - hdl
\end{equation}

is closed on $\mathbb{T}^3$. This will prove the $C_\infty$ version of Theorem 2. To do this, we use the following equivalent reformulation of the condition of closeness of the form (12).

Remark 7. In the conditions of Definition 5 let $f$, $\mu$, $h$ be functions on $\mathbb{T}^3$ such that the form (12) is closed on the leaves of $F$. Then (12) is closed on $\mathbb{T}^3$, iff $f$, $\mu$ and $h$ satisfy the following system of differential equations:

\begin{equation}
(13)\ \begin{cases}
\frac{\partial f}{\partial x_3} = D_z h \\
\frac{\partial (\mu f)}{\partial x_3} = D_{\bar{z}} h.
\end{cases}
\end{equation}

Indeed, let us write down the closeness condition on the form (12) in the coordinates $(x_1, x_2, l)$. In the new coordinates $\frac{\partial}{\partial z} = D_z$, $\frac{\partial}{\partial \bar{z}} = D_{\bar{z}}$, the operator $\frac{\partial}{\partial l}$ coincides with the operator $-\frac{\partial}{\partial x_3}$ correspondent to the coordinates $(x_1, x_2, x_3)$. By definition, (12) is closed on the leaves (or equivalently, $f$ and $\nu = \mu$ satisfy (4)). Under this assumption, the condition that (12) is closed on $\mathbb{T}^3$ is equivalent to the system of differential equations $\frac{\partial f}{\partial x_3} = -\frac{\partial h}{\partial z}$, $\frac{\partial (\mu f)}{\partial x_3} = -\frac{\partial h}{\partial \bar{z}}$. Rewriting these equations in the coordinates $(x_1, x_2, x_3)$ yields (13).

Let us show the existence of a $C_\infty$ solution $h$ to (13). This together with the previous Remark will prove the first statement of the $C_\infty$ version of Theorem 2. To do this, we will use the following characterization of $C_\infty$ functions on the torus.
Remark 8. A function \( h \in L_1(\mathbb{T}^3) \) with the Fourier series \( \sum_N h_N e^{i(N,x)} \) is \( C^\infty \), iff
\[
\sum_N |N|^s |h_N| < \infty \text{ for any } s \in \mathbb{N}.
\]

We use the following equivalent reformulation of the Diophantiness condition.

Remark 9. Let \( F \) be a linear foliation on \( \mathbb{T}^3 \), \( D_z \) be the corresponding differential operator from Definition 5, \( \lambda_N \) be its eigenvalues from (6). The foliation \( F \) is Diophantine, iff there exist \( c > 0, s > 1 \) such that for any \( N \in \mathbb{Z}^3 \setminus 0 \)
\[
|\lambda_N^{-1}| \leq c|N|^s.
\]

Define the "inverse" operator \( D_z^{-1} \) in the space of infinitely-smooth functions on \( \mathbb{T}^3 \) with zero average (this space is contained in the Hilbert subspace in \( L_2 \) generated by the eigenfunctions \( e^{i(N,x)} \) of the operator \( D_z \) with \( N \neq 0 \)) to have the same eigenfunctions \( e^{i(N,x)} \) with the eigenvalues inverse to those of \( D_z \). This operator is well-defined at least on the eigenfunctions, since the corresponding eigenvalues \( \lambda_N \) of the operator \( D_z \) do not vanish (by (15)). It is well-defined on the space of infinitely-smooth functions on \( \mathbb{T}^3 \) with zero average by Remark 8 and (15) and has zero kernel. The function \( h = D_z^{-1}(\frac{\partial f}{\partial x_3}) \) is the one we are looking for. Indeed it is a \( C^\infty \) function satisfying the first equation in (13). It satisfies the second equation as well. Indeed, applying the operator \( D_z \) to the second equation and substituting \( h = D_z^{-1}(\frac{\partial f}{\partial x_3}) \) yields (4) with \( \nu = \mu \), which is satisfied by the assumption that the form \( f \omega_\mu \) is closed on the leaves. Therefore, the \( D_z \) images of both parts of the second equation coincide. Hence, these parts coincide themselves: they have zero average, and hence they are obtained from their (coinciding) \( D_z \) images by applying the operator \( D_z^{-1} \). This proves the first statement of the \( C^\infty \) version of Theorem 2.

Now let us prove the last statement of Theorem 2. Let \( F \) be a linear nondiophantine foliation on \( \mathbb{T}^3 \). Let us prove the existence of a \( C^\infty(\mathbb{T}^3) \) family of almost complex structures on the leaves of the foliation \( F \) such that there is no \( C^2 \) Euclidean metric on \( \mathbb{T}^3 \) satisfying the first statement of Theorem 2 (this is equivalent to the last statement of Theorem 2). In the case, when the leaves are not dense (i.e., they are tori), for a generic \( C^\infty(\mathbb{T}^3) \) family of complex structures the leaves do not have the same conformal type. This is the family we are looking for. Indeed there is no diffeomorphism satisfying the second one of the equivalent statements in Theorem 2, since otherwise all the leaves would be conformally equivalent to each other.

Everywhere below we consider that the foliation \( F \) has dense leaves. We consider that its lifting to the 3-space is transversal to the \( x_3 \)-axis and we use the notations of Definition 2. In the proof of the last statement of Theorem 2 we use the following

Remark 10. Let \( F \) be a linear foliation on \( \mathbb{T}^3 \) with dense leaves. In the notations of Definition 2 let \( z = x_1 + ix_2, \mu \in C^\infty(\mathbb{T}^3), |\mu| < 1, \omega_\mu = dz + \mu dz \). There exists a unique complex-valued function \( f \in C^1(\mathbb{T}^3) \) such that the restriction to the leaves of the form \( f \omega_\mu \) is closed, up to multiplication by constant. Indeed, there exists at least one function \( f \in C^\infty(\mathbb{T}^3) \) satisfying this condition and nowhere vanishing (this is an equivalent reformulation of the first statement of Lemma 1). Let \( \varphi \in C^1(\mathbb{T}^3) \) be another function such that the form \( \varphi \omega_\mu \) is closed on the leaves. Then the ratio
\( \tilde{f} \) is \( \sigma_\mu \)-holomorphic and uniformly bounded on each leaf. Therefore, it is constant on each leaf (the leaves are parabolic Riemann surfaces), and hence, on \( \mathbb{T}^3 \) as well (by density of the leaves).

For the proof of the last statement of Theorem 2 in the \( C^\infty \) case we show the existence of complex-valued functions \( \nu, f \in C^\infty(\mathbb{T}^3) \), \( |\nu| < 1 \), such that the form \( f \omega_\nu \) is closed on the leaves (or equivalently, \( f \) and \( \nu \) satisfy (4)) and that there is no \( C^2 \) (and even \( L_2 \)) complex-valued function \( h \) on \( \mathbb{T}^3 \) such that the correspondent form (12) with \( \mu = \nu \) is closed on \( \mathbb{T}^3 \) (or equivalently, (13) holds, cf. Remark 7). This will prove the \( C^\infty \) version of Theorem 2.

Let \( \lambda_N \) be the eigenvalues of the operator \( D_z \) (see (6)). To construct a pair \( (\nu, f) \) as in the previous paragraph, let us choose and fix a sequence of distinct multiindices \( N_j = (p_j, m_j, k_j) \in \mathbb{Z}^3 \) with \( k_j \neq 0 \) such that for any \( s \in \mathbb{N} \) there exists a number \( J \in \mathbb{N} \) such that for any \( j > J \)
\[
|\lambda_{N_j}^{-1}| > |N_j|^s.
\]
The existence of a sequence \( N_j \) satisfying the previous estimate (16) follows from nondivisibility of the foliation \( F \) (which is contrary to (15)). This sequence can be chosen so that \( k_j \neq 0 \) for all \( j \); one can achieve this by choosing a subsequence of multiindices \( N_j \neq 0 \) such that \( |\lambda_{N_j}^{-1}| > 1 \). Then \( k_j \neq 0 \), which is implied by the following inequality: let \( N = (p, m, k) \in \mathbb{Z}^3 \setminus 0 \) be such that \( |\lambda_N| < 1 \); then \( k \neq 0 \). This inequality follows from (6).

Consider the family \( f(x, t) = 1 + t \sum_{j=1}^\infty \frac{\lambda_{N_j}}{k_j} e^{i(N_j, x)} \) of functions on \( \mathbb{T}^3 \) depending on real parameter \( t \). This is a \( C^\infty \) family of \( C^\infty \) functions, since the coefficients of the Fourier series in its formula satisfy (14) (by (16)). As it is shown below, any function \( f = f(x, t) \) corresponding to small enough nonzero value of \( t \) is one we are looking for.

The statement on the existence of a function \( \nu \) satisfying (4) is proved below (Lemma 3). For any fixed nonzero value of the parameter \( t \) and any complex-valued function \( \mu = \nu \) there is no function \( h \in L_2(\mathbb{T}^3) \) such that the correspondent form (12) is closed, since there is no \( h \in L_2(\mathbb{T}^3) \) satisfying the first equation in (13) and by Remark 7. Indeed, if an \( L_2 \) function \( h \) satisfies the first equation in (13), then its Fourier coefficients with the indices \( N_j \) would be equal to \( it \), which is impossible if \( t \neq 0 \). Thus, now Theorem 2 is implied by the following

**Lemma 3.** Let \( F \) be a linear foliation on \( \mathbb{T}^3 \), \( f(x, t) \) be a \( C^\infty \) family of complex-valued \( C^\infty \) functions on \( \mathbb{T}^3 \) uniformly bounded away from zero depending on real parameter \( t \), \( f(x, 0) \equiv 1 \). There exists a \( C^\infty \) family \( \nu(x, t) \) of complex-valued \( C^\infty \) functions on \( \mathbb{T}^3 \) depending on the same parameter \( t \), \( \nu(x, 0) \equiv 0 \), and satisfying (4).

**Proof.** Lemma 3 is proved analogously to Lemma 2. Differentiating equation (4) in \( t \) yields (5). The solution \( \nu(x, t) \) of (5) with zero initial condition will be a solution of (4). Let \( U \) be the operator from Corollary 1. Applying subsequently the "operator" \( D_z^{-1} \) and the multiplication by \( f^{-1} \) to (5) and substituting \( U^{-1} = D_z^{-1} \circ D_z \) yields
\[
\dot{\nu} = U^{-1} f - \nu f.
\]
The last equation implies (5) and has a unique infinitely-smooth solution \( \nu(x, t) \) with any given \( C^\infty \) initial condition (e.g., \( \nu(x, 0) \equiv 0 \)): its right-hand side has bounded derivative in \( \nu \) in any Hilbert Sobolev norm (cf. the proof of Lemma 2). This proves Lemma 3. The proof of the \( C^\infty \) version of Theorem 2 is completed.
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INDEPENDENT UNIVERSITY OF MOSCOW, BOLSCHOI VLASIEVSKII PERELOK, 11, 121002 MOSCOW, RUSSIA

STEKLOV MATHEMATICAL INSTITUTE, MOSCOW

MOSCOW STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS

PRESENT ADDRESS: INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, 91440 BURES-SUR-YVETTE, FRANCE