ALTERNATIVE CRITERIA FOR ADMISSIBILITY AND STABILIZATION OF SINGULAR FRACTIONAL ORDER SYSTEMS

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Abstract. This paper discusses admissibility problem of singular fractional order systems with order $1 < \alpha < 2$. The alternative necessary and sufficient admissibility conditions are proposed, in which include linear matrix inequalities (LMIs) with equality constraints and LMIs without equality constraints. Moreover, these criteria are brand-new and different from the existing results. The state feedback control to stabilize singular fractional order systems is derived. Two numerical examples are presented to shown the effectiveness of our results.

1. Introduction. With the development of control technology, fractional order systems (FOSs) have attracted increasingly attention. A lot of practical issues have been studied via fractional order systems, such as wavelet transform [13], viscoelastic systems [18] and solid mechanics [19]. Due to the great efforts of researchers, a large number of research results have been published in the analysis of fractional order systems, especially on stability and stabilization issues for fractional order systems. [17, 20, 10, 1, 30].

Singular systems are also called descriptor systems or generalized systems which are dynamic with a wider form than normal systems [2]. Since the admissibility of singular systems include regularity, impulsiveness-free and stability, the study of singular systems is more complex than that of normal systems. Recently, more and more scholars have paid attention on the admissibility of singular fractional order systems and many challenging and unsolved theoretical problems have been considered [8, 25, 21]. For example, under the assumption of regularity, Yu et al. [27] study admissibility conditions for $0 < \alpha < 1$ case. Similarly, under regular conditions, Ibrahima et al. [15] consider the admissibility of fractional order systems with order $1 < \alpha < 2$, but the obtained LMI result is only a sufficient condition, which is conservative to some extent. Then, the condition is adopted to design suitable feedback controllers to stabilize the closed-loop systems by [3]. Afterwards, Saliha et al. propose a new admissibility criterion, which was not established under the assumption of regularity, but introduced complex variables into the LMI

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formulas [12]. And based on the above results, in [11] Saliha et al. give a new admissibility condition for singular linear continuous time fractional order systems with order $1 < \alpha < 2$. Similarly an LMI criterion is presented from the region stability theory [23]. In [29], Zhang et al. give some new admissibility and robust stabilization conditions of continuous linear singular fractional order systems with the fractional order $0 < \alpha < 1$ which introduce real variables into LMI without conservativeness. On the other hand, by using the normalization method, Ibrahima et al. [14] give sufficient conditions for the robust stabilization of uncertain singular fractional order systems with order $0 < \alpha < 1$ and $1 \leq \alpha < 2$, respectively. And other results about control systems, fractional order systems and singular fractional order systems are also be used widely [7, 22, 6, 5, 24, 28, 16, 4, 26].

Motivated by the research above, the main contributions are summarized as follows:

(i) The alternative necessary and sufficient conditions of admissibility for singular fractional order systems with $1 < \alpha < 2$ are proposed. These criteria are new and different from existing results.

(ii) Generally speaking, the existing criteria of singular fractional order systems include equality constraints and involve non-strict LMIs, but our criteria are in terms of strict LMIs, which are not only of theoretical significance, but also are convenient in the process of practical numerical solution.

(iii) In terms of the linear matrix inequality approach, the state feedback controller is designed such that the closed-system is admissible.

(iv) In the case when $Y = 0$ in Theorem 1 of Formula (10), Theorem 1 coincides with the Lemma 2. Thus, Theorem 1 in this paper can be regarded as an extension of the Lemma 2 for continuous singular systems.

The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, the main results are derived and in Section 4, numerical examples that illustrate the effectiveness of the proposed results are drawn. Finally, a conclusion is given.

**Notations.** $X^T$ represents the transpose of the matrix $X$, $\text{spec}(X)$ is the spectrum of all eigenvalues of $X$, $\text{sym}(X)$ denotes the expression $X + X^T$ and $\text{asym}(X)$ stands for $X - X^T$, $\otimes$ represents the Kronecker product. For convenience, let $a = \sin(\alpha \frac{\pi}{2})$, $b = \cos(\alpha \frac{\pi}{2})$ in the sequel. $\text{arg}(x)$ represents the argument of complex $x$. $\text{rank}(\cdot)$ represents the rank of a matrix.

**2. Preliminaries.** In this section, we give some preliminaries results on normal fractional order systems and singular fractional order systems. The fractional order derivative is divided by two main classes: Riemann-Liouville derivative and Caputo derivative. Caputo approach allows using initial values of classical integer order derivatives with clear physical interpretations. Thus, Caputo definition is used in this paper [31].

**Definition 1** [17]. Caputo derivative is defined as

$$D^\alpha x(t) = \frac{1}{\Gamma(k - \alpha)} \int_0^t (t - \tau)^{k - \alpha - 1} x^{(k)}(\tau)d\tau,$$

where $k$ is an integer satisfying $k - 1 < \alpha < k$, $\Gamma(\cdot)$ is the Euler Gamma function.

Now we consider a continuous linear singular fractional order system described by

$$ED^\alpha x(t) = Ax(t) + Bu(t),$$

(1)
where $\alpha$ is the time fractional derivative order. $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input. The matrix $E \in \mathbb{R}^{n \times n}$ is singular, we assume that $\text{rank}(E) = r < n$.

For the unforced linear singular fractional order system

$$ED^\alpha x(t) = Ax(t),$$

we denote (2) with the pair $(E, A)$. If matrix $E(= I)$ becomes nonsingular system (2) reduces normal fractional order system

$$D^\alpha x(t) = Ax(t).$$

**Definition 2** [27]. The pair $(E, A)$ is said to be admissible if

(a) $\det(s^\alpha E - A)$ is not identically zero.
(b) $\deg(\det(sE - A)) = \text{rank}(E)$.
(c) all the eigenvalues of pair $(E, A)$ have negative real parts.

Now we give a lemma for normal fractional order system (3).

**Lemma 1** [28]. System (3) is asymptotically stable if and only if there exist two matrices $X, Y \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0,$$

and

$$\begin{bmatrix} \text{sym}(A(aX + bY)) & \text{asym}(A(aY - bX)) \\ -\text{asym}(A(aY - bX)) & \text{sym}(A(aX + bY)) \end{bmatrix} < 0.$$

**Lemma 2** [27]. System (2) is regular if and only if there exist two nonsingular matrices $M$ and $N$ such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & J_{n-r} \end{bmatrix}, \quad MAN = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

where $J_{n-r}$ is a nilpotent matrix.

Based on Lemma 2, System (2) can be transformed into

$$D^\alpha x_1(t) = \bar{A}_1 x_1(t),$$

and

$$J_{n-r} D^\alpha x_2(t) = x_2(t).$$

According to (6) and (7), it is easy to obtain the following lemma.

**Lemma 3** [2]. Suppose system (2) is regular, and two nonsingular matrices $M$ and $N$ are found such that (5) holds, then we have: System (2) is admissible if and only if $J_{n-r} = 0$ and $|\arg(\text{spec}(\bar{A}_1))| > \frac{\alpha \pi}{2}$ hold.

When the regularity of the pair $(E, A)$ in system (2) is not known. It is always possible to choose two nonsingular matrices $M$ and $N$ such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

And then, we have the following lemmas:

**Lemma 4** [29]. (a) System (2) is impulse-free if and only if $A_4$ is nonsingular.
(b) System (2) is admissible if and only if $A_4$ is nonsingular and

$$|\arg(\text{spec}(A_1 - A_2 A_4^{-1} A_3))| > \frac{\alpha \pi}{2}.$$
Lemma 5 [25]. Let
\[ \hat{N} = \begin{bmatrix} P & X \\ Y & Z \end{bmatrix}, \]
where \( P, X, Y \) and \( Z \) are any real given matrices with appropriate dimensions such that \( \hat{N} + \hat{N}^T < 0 \). Then, \( Z \) is nonsingular and
\[ P + P^T - XZ^{-1}Y - Y^T Z^{-T} X^T < 0. \]

3. Main results. In this section, we will give some alternative necessary and sufficient conditions for admissibility and stabilization of the system are obtained.

3.1. Admissibility of singular fractional order systems. We state our main result as follows.

**Theorem 1.** System (2) is admissible if and only if there exist matrices \( X, Y \in \mathbb{R}^{n \times n} \), such that
\[ \begin{bmatrix} EX & EY \\ -EY & EX \end{bmatrix} = \begin{bmatrix} X^T E^T & -Y^T E^T \\ Y^T E^T & X^T E^T \end{bmatrix} \geq 0, \] (9)
\[ \begin{bmatrix} \text{sym}(A(ax + by)) & \text{asym}(A(ax - bx)) \\ -\text{asym}(A(ax - bx)) & \text{sym}(A(ax + by)) \end{bmatrix} < 0. \] (10)

**Proof.** (Sufficiency) For system (2), we choose two arbitrary nonsingular matrices \( M \) and \( N \) satisfying (8). Let
\[ N^{-1}XM^T = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad N^{-1}YM^T = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}. \] (11)

Pre- and post-multiplying (9) by \[ \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \] and its transpose, respectively, we can deduce that
\[ \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} EX & EY \\ -EY & EX \end{bmatrix} \begin{bmatrix} M^T \\ 0 \end{bmatrix} = \begin{bmatrix} M \end{bmatrix}. \]

Then it follows that
\[ \begin{bmatrix} \text{MENN}^{-1}XM^T & \text{MENN}^{-1}YM^T \\ -\text{MENN}^{-1}YM^T & \text{MENN}^{-1}XM^T \end{bmatrix} = \begin{bmatrix} MX^T N^{-T} N^T E^T M^T & -MY^T N^{-T} N^T E^T M^T \\ MY^T N^{-T} N^T E^T M^T & MX^T N^{-T} N^T E^T M^T \end{bmatrix}. \]

By using (11), it is easy to check that
\[ \begin{bmatrix} X_1 & X_2 & Y_1 & Y_2 \\ 0 & 0 & 0 & 0 \\ -Y_1 & -Y_2 & X_1 & X_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1^T & 0 & -Y_1^T & 0 \\ X_2^T & 0 & -Y_2^T & 0 \\ Y_1^T & 0 & X_1^T & 0 \\ Y_2^T & 0 & X_2^T & 0 \end{bmatrix} \geq 0. \]

Therefore
\[ X_2 = Y_2 = 0, X_1 = X_1^T \geq 0, Y_1 = -Y_1^T. \]
Considering (10) and using the expressions in (11), we have
\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Psi_{11} & \Psi_{12} \\
\Phi_{12} & \Phi_{22} & -\Psi_{12} & \Psi_{22} \\
-\Psi_{11} & -\Psi_{12} & \Phi_{11} & \Phi_{12} \\
\Psi_{12} & -\Psi_{22} & \Phi_{12} & \Phi_{22}
\end{bmatrix} < 0,
\]
(12)
where
\[
\Phi_{11} = \text{sym}(A_1(aX_1 + bY_1)) + \text{sym}(A_2(aX_3 + bY_3)),
\]
\[
\Phi_{12} = A_2(aX_4 + bY_4) + (aX_1 + bY_1)^T A_2^T + (aX_3 + bY_3)^T A_4^T,
\]
\[
\Psi_{11} = \text{sym}(A_4(aX_4 + bY_4)),
\]
\[
\Psi_{12} = A_2(aY_4 - bX_4) - (aY_1 - bX_1)^T A_2^T - (aY_3 - bX_3)^T A_4^T,
\]
\[
\Psi_{22} = \text{sym}(A_4(aY_4 - bX_4)).
\]
From (12), we can get
\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} \\
\Phi_{12}^T & \Phi_{22}
\end{bmatrix} < 0.
\]
(13)
Moreover, the 2-2 block in (13) gives
\[
\text{sym}(A_4(aX_4 + bY_4)) < 0.
\]
Therefore, \(A_4(aX_4 + bY_4)\) is nonsingular, which implies \(A_4\) is nonsingular too. Hence by applying Lemma 4, we can obtain that the pair \((E, A)\) is regular and impulsive-free. Defining
\[
\hat{N} = \begin{bmatrix}
\hat{N}_{11} & \hat{N}_{12} & \hat{N}_{13} & \hat{N}_{14} \\
\hat{N}_{21} & \hat{N}_{22} & \hat{N}_{23} & \hat{N}_{24} \\
-\hat{N}_{13} & -\hat{N}_{14} & \hat{N}_{11} & \hat{N}_{12} \\
-\hat{N}_{23} & -\hat{N}_{24} & \hat{N}_{21} & \hat{N}_{22}
\end{bmatrix},
\]
where
\[
\hat{N}_{11} = A_1(aX_1 + bY_1) + (aX_3 + bY_3)^T A_2^T, \quad \hat{N}_{12} = A_2(aX_4 + bY_4),
\]
\[
\hat{N}_{13} = A_1(aY_1 - bX_1) - (aY_3 - bX_3)^T A_2^T, \quad \hat{N}_{14} = A_2(aY_4 - bX_4),
\]
\[
\hat{N}_{21} = A_3(aX_1 + bY_1) + A_4(aX_3 + bY_3), \quad \hat{N}_{22} = A_4(aX_4 + bY_4),
\]
\[
\hat{N}_{23} = A_3(aY_1 - bX_1) - A_4(aY_3 - bX_3), \quad \hat{N}_{24} = A_4(aY_4 - bX_4).
\]
And considering (12), it is easy to check that
\[
\hat{N} + \hat{N}^T = \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Psi_{11} & \Psi_{12} \\
\Phi_{12} & \Phi_{22} & -\Psi_{12} & \Psi_{22} \\
-\Psi_{11} & -\Psi_{12} & \Phi_{11} & \Phi_{12} \\
\Psi_{12} & -\Psi_{22} & \Phi_{12} & \Phi_{22}
\end{bmatrix} < 0.
\]
(14)
According to the contractual transformation of matrices, we can change (14) into
\[
\hat{N} + \hat{N}^T = \begin{bmatrix}
\Phi_{11} & \Psi_{11} & \Phi_{12} & \Phi_{12} \\
-\Psi_{11} & \Phi_{11} & -\Psi_{12} & \Phi_{12} \\
\Phi_{12} & -\Psi_{12} & \Phi_{12} & \Phi_{22} \\
\Psi_{12} & \Phi_{12} & -\Psi_{22} & \Phi_{22}
\end{bmatrix} < 0.
\]
where
\[
\tilde{N} = \begin{bmatrix}
\tilde{N}_{11} & \tilde{N}_{13} & \tilde{N}_{12} & \tilde{N}_{14} \\
-\tilde{N}_{13} & \tilde{N}_{11} & -\tilde{N}_{14} & \tilde{N}_{12} \\
\tilde{N}_{21} & \tilde{N}_{23} & \tilde{N}_{22} & \tilde{N}_{24} \\
-\tilde{N}_{23} & -\tilde{N}_{21} & -\tilde{N}_{24} & \tilde{N}_{22}
\end{bmatrix},
\]

Therefore, from Lemma 5, it follows that
\[
\begin{bmatrix}
\text{sym}((A_1 - A_2 A_3^{-1} A_4)(aX_1 + bY_1)) & \text{asym}((A_1 - A_2 A_3^{-1} A_4)(aY_1 - bX_1)) \\
-\text{asym}((A_1 - A_2 A_3^{-1} A_4)(aY_1 - bX_1)) & \text{sym}((A_1 - A_2 A_3^{-1} A_4)(aX_1 + bY_1))
\end{bmatrix} < 0.
\]

According to Lemma 1, we have that the pair \((E, A)\) is stable. This together with the regularity and impulsiveness-free of the pair \((E, A)\) gives that system (2) is admissible.

(Necessity) Assume that System (2) is admissible. According to Lemma 3, for two arbitrary chosen nonsingular matrices \(M_1\) and \(N_1\) satisfying (4) and
\[
|\arg(\text{spec}(\tilde{A}_1))| > \frac{\alpha \pi}{2}.
\]

Noting the above inequality and applying Lemma 1, it can be seen that there exist matrices \(X_1, Y_1\) such that (4) holds and
\[
\begin{bmatrix}
\text{sym}(\tilde{A}_1(aX_1 + bY_1)) & \text{asym}(\tilde{A}_1(aY_1 - bX_1)) \\
-\text{asym}(\tilde{A}_1(aY_1 - bX_1)) & \text{sym}(\tilde{A}_1(aX_1 + bY_1))
\end{bmatrix} < 0. \tag{15}
\]

Let
\[
X = N_1 \begin{bmatrix} X_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix} M_1^{-T}, \quad Y = N_1 \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix} M_1^{-T}.
\]

Now, pre- and post-multiplying (10) by \(\begin{bmatrix} M_1 & 0 \\ 0 & M_1 \end{bmatrix}\) and its transpose, respectively, we can deduce that
\[
\begin{bmatrix}
\text{sym}(\tilde{A}_1(aX_1 + bY_1)) & 0 & -2I_{n-r} \\
0 & \text{sym}(\tilde{A}_1(aY_1 - bX_1)) & 0 \\
-\text{asym}(\tilde{A}_1(aY_1 - bX_1)) & 0 & \text{sym}(\tilde{A}_1(aX_1 + bY_1)) \\
0 & 0 & -2I_{n-r}
\end{bmatrix} < 0. \tag{16}
\]

And then, we can deduce that (15) holds. Therefore both the matrices \(X, Y\) satisfy (9) and (10). This completes the proof. \(\square\)

**Remark 1.** In the case when \(Y = 0\), then, Theorem 1 coincides with Theorem 1 in [17]. Therefore, Theorem 1 in this paper can be regarded as an extension of Theorem 1 in [17] for continuous singular fractional systems.

**Corollary 1.** System (2) is admissible if and only if there exist matrices \(X, Y \in \mathbb{R}^{n \times n}\) such that
\[
\begin{bmatrix} EX & EY \\ -EY & EX \end{bmatrix} = \begin{bmatrix} X^T E^T & -Y^T E^T \\ Y^T E^T & X^T E^T \end{bmatrix} \succeq 0, \tag{17}
\]

\[
\begin{bmatrix}
\text{sym}(A(aX + bY)) & \text{asym}(A(aY - bX)) \\
-\text{asym}(A(aY - bX)) & \text{sym}(A(aX + bY))
\end{bmatrix} < 0.
\]

**Proof.** The proof is similar to Theorem 1. Thus, it is omitted here. \(\square\)

**Corollary 2.** System (2) is admissible if and only if there exist matrices \(X_1, X_2 \in \mathbb{R}^{r \times r}, X_3 \in \mathbb{R}^{(n-r) \times m}, X_4 \in \mathbb{R}^{(n-r) \times (n-r)}\) such that
\[
\begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{bmatrix} > 0,
\]
\[
\begin{bmatrix}
\text{sym}(aMANX + bMANY) & \text{asym}(aMANY - bMANX) \\
\text{asym}(aMANY - bMANX) & \text{sym}(aMANX + bMANY)
\end{bmatrix} < 0.
\]

where
\[
X = \begin{bmatrix}
X_1 & 0 \\
X_3 & X_4
\end{bmatrix}, 
Y = \begin{bmatrix}
X_2 & 0 \\
0 & 0
\end{bmatrix}
\]

\(r = \text{rank}(E), M, N \in \mathbb{R}^{n \times n}\) are two arbitrary chosen nonsingular matrices satisfying (8).

**Proof.** The method of proof is similar to Theorem 1. Thus, it is omitted here. \(\square\)

**Remark 2.** Both Theorem 1 and Corollary 1 give the necessary and sufficient conditions for the pair \((E, A)\) to be admissible in terms of LMIs. However, it is noted that the conditions in (9) and (17) are non-strict LMIs and involve quality constraints which may result in numerical problems when checking such non-strict LMIs.

Considering the remark, we give a strict LMI condition for admissibility in the following theorem according to the result of Theorem 1.

**Theorem 2.** System (2) is admissible if and only if there exist matrices \(X, Y \in \mathbb{R}^{n \times n}\), \(Q \in \mathbb{R}^{(n-r) \times n}\) such that (4) holds and
\[
\begin{bmatrix}
\text{sym}(A(aXE^T + bYE^T + aSQ)) & \text{asym}(A(aYE^T - bXE^T - bSQ)) \\
\text{asym}(A(aYE^T - bXE^T - bSQ)) & \text{sym}(A(aXE^T + bYE^T + aSQ))
\end{bmatrix} < 0. \tag{18}
\]

where \(S \in \mathbb{R}^{n \times (n-r)}\) is any matrix with full column rank and satisfies \(ES = 0\).

**Proof.** (Sufficiency) It is easy to change (18) to the following form:
\[
\begin{bmatrix}
\text{sym}(aA(XE^T + SQ) + bAY^T) & \text{asym}(b(EX + QS^T)A^T + aEYA^T) \\
\text{asym}(b(EX + QS^T)A^T + aEYA^T) & \text{sym}(aA(XE^T + SQ) + bAY^T)
\end{bmatrix} < 0.
\]

Assume that there exist matrix \(X, Y\) and \(Q\) such that (4) and (18) hold. Let
\[
\bar{X} = XE^T + SQ, \bar{Y} = YE^T.
\]

Then thanks to (4) and (18), it is easy to verify \(\bar{X}\) and \(\bar{Y}\) satisfy (9) and (10). Therefore by Theorem 1, it follows that system (2) is admissible.

(Necessity) Suppose system (2) is admissible. According to Lemma 3, from the full rank decomposition of descriptor matrix \(E\), for the two arbitrary chosen nonsingular matrices \(M_1\) and \(N_1\) satisfying (5) and
\[
|\text{arg} (\text{spec}(\bar{A}))| > \alpha \frac{\pi}{2}.
\]

Considering the definition of matrix \(S\), we can write
\[
S = N_1 \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} T,
\]

where \(T\) is any nonsingular matrix. Noting the above equality and applying Lemma 1, it can be seen that there exist matrices \(X_1, Y_1\) such that (4) and (15) hold. Let
\[
X = N_1 \begin{bmatrix} X_1 \\ 0 \\ I_{n-r} \end{bmatrix} N_1^{-T}, 
Y = N_1 \begin{bmatrix} Y_1 \\ 0 \\ 0 \end{bmatrix} N_1^{-T},
\]

\[
Q = T^{-1} \begin{bmatrix} 0 & -I_{n-r} \end{bmatrix} M_1^{-T}.
\]
Then, pre- and post-multiplying (10) by 
\[
\begin{bmatrix}
M_1 & 0 \\
0 & M_1
\end{bmatrix}
\] and 
\[
\begin{bmatrix}
M_1^T & 0 \\
0 & M_1^T
\end{bmatrix}
\], respectively, we can deduce that (16) holds. And then, we can deduce that (15) holds. Therefore, the matrices \(X, Y\) satisfy (4) and (18). This completes the proof.

3.2. Stabilization of the closed-loop singular fractional order systems. For system (1), we consider the following state feedback controller:
\[
u(t) = Kx(t), \quad K \in \mathbb{R}^{m \times n}.
\]
Applying this controller to system (1), we obtain the closed-loop system as follows:
\[
ED^\alpha X(t) = (A + BK)x(t).
\]
Then we have the following stabilization result. Considering the closed-loop linear singular fractional order system (20) and involving in Theorem 2, we have
\[
\begin{bmatrix}
\text{sym}((A + BK)\begin{pmatrix}aXE^T + bYE^T + aSQ \end{pmatrix}) & \text{asym}((A + BK)\begin{pmatrix}aYE^T - bXE^T - bSQ \end{pmatrix}) \\
-\text{asym}((A + BK)\begin{pmatrix}aYE^T - bXE^T - bSQ \end{pmatrix}) & \text{sym}((A + BK)\begin{pmatrix}aXE^T + bYE^T + aSQ \end{pmatrix})
\end{bmatrix} < 0.
\]
If denote \(\hat{Z} = \hat{K}\hat{H}\), we have the following theorem, where
\[
\hat{H} = \begin{bmatrix}
H_1 & H_2 \\
-H_2 & H_1
\end{bmatrix} = \begin{bmatrix}
aXE^T + bYE^T + aSQ & aYE^T - bXE^T - bSQ \\
-aYE^T + bXE^T + bSQ & aXE^T + bYE^T + aSQ
\end{bmatrix},
\]
\[
\hat{K} = \begin{bmatrix}
K & 0 \\
0 & K
\end{bmatrix}.
\]

**Theorem 3.** For singular fractional order system (1), there exists a state feedback controller (19) such that the closed-loop system (20) is admissible if and only if there exist matrices \(X, Y \in \mathbb{R}^{n \times n}, Q \in \mathbb{R}^{(n-r) \times n}\) and \(\hat{Z} \in \mathbb{R}^{2m \times 2n}\) such that (4) holds and
\[
\hat{A}\hat{H} + \hat{B}\hat{Z} + \hat{H}^T \hat{A}^T + \hat{Z}^T \hat{B}^T < 0,
\]
where
\[
\hat{A} = \begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
B & 0 \\
0 & B
\end{bmatrix}.
\]
Then a stabilizing state feedback controller gain matrix can be chosen as
\[
\hat{K} = \hat{Z}\hat{H}^{-1}.
\]

4. Numerical examples. The availability of the results proposed in this paper is verified as following by using two examples.

**Example 1.** Consider system (2) with parameters as \(\alpha = 1.5\) and
\[
E = \begin{bmatrix}
1 & 1 & 0.5 \\
-0.5 & 1.5 & 1 \\
1 & 1 & 0.5
\end{bmatrix}, \quad A = \begin{bmatrix}
-10 & 2 & 6.5 \\
2 & -5.5 & -1.25 \\
-9 & 4.3 & 8.5
\end{bmatrix}.
\]
It is easy to verify that system (2) is stable because the finite eigenvalues are \((-12.2500 - 13.8180j, -12.2500 + 13.8180j)\}. Using MATLAB LMI Control Toolbox to solve LMIs (4) and (18) in Theorem 2, we can obtain the feasible solution as follows:
\[
X = \begin{bmatrix}
21.1682 & -14.5699 & 8.7770 \\
-14.5699 & 55.5498 & -64.4499 \\
8.7770 & -64.4499 & 85.5542
\end{bmatrix},
\]
Example 2. Consider system (1) with parameters $\alpha = 1.4$ and

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & -1.7 & -1 & 1 \\ 0 & -2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. $$

It is easy to verify that system (1) is unstable because the finite eigenvalues are $\{1, -4.2649, -1.7351\}$. This example can be easily solved by the approaches presented in Theorem 3. Using the MATLAB LMI Control Toolbox to solve for LMI feasibility problems of LMIs (8) and (21) in Theorem 3, we obtain the feasible solutions as follows:

$$X = \begin{bmatrix} 28.6245 & 0 & 0 & 0 \\ 0 & 10.1360 & 3.3497 & 0 \\ 0 & 3.3497 & 17.2062 & 0 \\ 0 & 0 & 0 & 28.6245 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3.5876 & -8.6567 \\ 0 & -3.5876 & 0 & 3.0844 \\ 0 & 8.6567 & -3.0844 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 11.4761 & -3.7767 & -12.5411 \end{bmatrix},$$

where

$$\hat{Z} = \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 37.4699 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$H_1 = aXE^T + bYE^T + SQ = \begin{bmatrix} 23.1577 & 0 & 0 & 0 \\ 0 & 8.2002 & 0.6013 & 0 \\ 0 & 4.8187 & 13.9201 & 0 \\ 0 & 4.1960 & -1.2425 & -10.1460 \end{bmatrix},$$

$$H_2 = aYE^T - bYE^T - bSQ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5.0111 & 0 \\ 0 & -5.0111 & 0 & 0 \\ 0 & 18.8372 & -6.5281 & -7.3715 \end{bmatrix},$$

$$\hat{K} = \frac{\hat{Z}H^{-1}}{} = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix},$$

where

$$K = \begin{bmatrix} 1.6180 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In one hand we have $\det(sE - A - BK) = s^3 + 4.6180s^2 + 9.4720s + 4.5732$ which means that system (20) is regular and impulse-free. In the other hand, we have $\text{spec}(E, A + BK) = \{-0.6180, -3.8439, -3.8439\}$ which means that the eigenvalues of the pair $(E, A + BK)$ lie in the stability region.

Therefore, the state responses of the closed-loop fractional order system are shown in Fig. 1 under initial state condition of $x(0) = [0.63 \quad 1.89 \quad -1.26 \quad 5.04]^T.$
5. **Conclusion.** Alternative LMI conditions of admissibility are proposed to solve the problem of stability and stabilization of singular fractional order systems with the fractional order $\alpha$ belonging to $1 < \alpha < 2$. Necessary and sufficient admissibility conditions are also derived for closed-loop systems. The results can be regarded as a natural extension of the Lemma 2 of singular systems. In the future, output feedback control technique can be developed similarly according to the approach of Theorem 3 in the paper.

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