PSEUDOLOCALITY AND UNIQUENESS OF RICCI FLOW ON ALMOST EUCLIDEAN NONCOMPACT MANIFOLDS

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Abstract. In this paper, we prove a pseudolocality-type theorem for \( \mathcal{L} \)-complete noncompact Ricci flow which may not have bounded sectional curvature; with the help of it we study the uniqueness of the Ricci flow on noncompact manifolds. In particular, we prove the strong uniqueness theorem for the \( \mathcal{L} \)-complete Ricci flow on the Euclidean space. This partially answers a question proposed by B-L. Chen [5].

1. Introduction

The Ricci flow is a geometric evolution equation introduced by Hamilton [13], which deforms a Riemannian manifold by the Ricci curvature, namely,

\[
\frac{\partial}{\partial t} g_t = -2 \text{Ric}_g .
\]

On the one hand, the Ricci flow equation is nonlinear, which implies that almost certainly a Ricci flow develops singularity and does not exist for all time; see [15]. By a thorough analysis of the singularities in dimension three [19], Perelman successfully overcame the obstacles in solving the Poincaré and geometrization conjectures [19, 20, 21] using Hamilton’s program [15]. On the other hand, the Ricci flow equation is parabolic, and consequently, many problems related to parabolic partial differential equations arise in this field. For instance, the short-time existence problem (c.f. [11, 14, 23]), the long-time existence problem (c.f. [8, 15, 22, 25, 26]), and the uniqueness problem (c.f. [6, 17, 18]).

Another interesting problem in the study of parabolic equations is that of the strong uniqueness. In general, the solution to the initial value problem of a parabolic equation on an unbounded region is not unique. For instance, the 1-dimensional linear heat equation \( u_t - u_{xx} = 0 \) on \( \mathbb{R}^1 \) admits a nontrivial solution with the initial value \( u(\cdot, 0) \equiv 0 \), namely, the well-known Tychonoff example

\[
u(x, t) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \frac{d^k}{dt^k} \exp \left( -\frac{1}{t} \right).
\]

So it is natural to ask whether or not the solution to the Ricci flow equation is always unique provided that the initial data is good.

In this respect, B-L. Chen [5] proved a strong uniqueness result in dimension three, namely, starting from a 3-dimensional complete Riemannian manifold with bounded and nonnegative sectional curvature, two Ricci flows must be identical to each other for a short time. As a consequence, a Ricci flow starting from the standard 3-dimensional Euclidean...
space must always remain Euclidean. It is therefore interesting to ask whether such property is true for the Euclidean space in higher dimensions. We shall give an affirmative answer to this question with some additional assumptions.

At the end of his paper [5], Chen mentioned that the strong uniqueness theorem can be obtained from a (more general) pseudolocality theorem, which states that the Ricci flow cannot evolve an almost Euclidean region immediately into a high-curvature region. Let us recall Perelman’s well-known pseudolocality theorem [19] (c.f. [4]).

**Theorem 1.1** (Perelman’s pseudolocality theorem). For any \( \alpha > 0 \) there exist \( \delta > 0 \) and \( \epsilon_0 > 0 \) depending on \( \alpha \) and \( n \) with the following property. Let \((M^n, g_t)_{t \in [0, \epsilon^2]}\) be a complete Ricci flow on a noncompact manifold with bounded sectional curvature at each time on \((0, \epsilon^2)\), where \( \epsilon \in (0, \epsilon_0] \), such that

\[
R_{g_0} \geq -1 \quad \text{on} \quad B_{g_0}(x_0, 1),
\]

and

\[
\left( \text{Area}_{g_0}(\partial \Omega) \right)^n \geq (1 - \delta)n^\omega \left( \text{Vol}_{g_0}(\Omega) \right)^{n-1}
\]

for any regular domain \( \Omega \subset B_{g_0}(x_0, 1) \). Then we have

\[
|Rm|(x, t) \leq \frac{\alpha}{t} + \frac{1}{\epsilon^2},
\]

for any \( d_{g_0}(x_0, x) < \epsilon \) and \( t \in (0, \epsilon^2) \).

The original version of Perelman’s pseudolocality theorem [19] was proved under the assumption of the manifold being closed. In the complete and noncompact case, this result was verified by Chau, Tam and Yu [4]. Later, Tian and Wang [24] proved another version of Perelman’s pseudolocality theorem in which they showed that the conditions (1.1) and (1.2) in Theorem 1.1 can be replaced by small Ricci curvature and almost Euclidean volume ratio. Wang [27] improved both Perelman’s and Tian-Wang’s pseudolocality theorems and proved that (1.1) and (1.2) in Theorem 1.1 can be replaced by the smallness of the localized version of Perelman’s \( \mu \)-functional. One may also see [3] for another proof of the pseudolocality theorem based on Bamler’s \( \epsilon \)-regularity theorem [11].

It is to be noted that the above pseudolocality theorems require bounded sectional curvature on the whole manifold within each compact time-interval, under which condition the uniqueness of the Ricci flow is already established by Chen-Zhu [17]. Hence we cannot use the previous pseudolocality theorems to study the uniqueness problems of the Ricci flow. In this paper we are bound to find a pseudolocality theorem valid in a more general setting, or at least with weaker regularity requirements after the initial time. It turns out that the \( L \)-complete condition is a notion that fits our purpose.

**Definition 1.2.** A complete smooth Ricci flow \((M^n, g_t)_{t \in [0, T]}\) is called \( L \)-complete on \( M^n \times [0, T] \) if the following holds. For any \( s, t \in [0, T] \) with \( s < t \) and any compact set \( E \subset M \), there exists another compact set \( F \subset M \) that contains \( E \), such that if \( x, y \in E \), then there exists at least one minimal \( L \)-geodesic connecting \((x, t)\) and \((y, s)\), and all possible minimal \( L \)-geodesics connecting \((x, t)\) and \((y, s)\) are contained (spatially) in \( F \).

First of all, we will show that under certain conditions the Ricci flow is \( L \)-complete.

**Theorem 1.3.** Let \((M^n, g_t)_{t \in [0, T]}\) be a Ricci flow such that either one of the following is true.
\( R_g(x) \geq -k \) for all \((x, t) \in M \times [0, T]\), where \( k \) is a nonnegative constant, and there exist a positive constant \( c \) and a complete smooth Riemannian metric \( \bar{g} \) on \( M \), such that
\[
g_t \geq c \bar{g} \quad \text{for all} \quad t \in [0, T].
\]

(2) \( R_g(x) \geq -k \) for all \((x, t) \in M \times [0, T]\), where \( k \) is a nonnegative constant, and there exist constants \( \alpha > 0, \beta > 0, \gamma \in (0, 1) \), and a fixed point \( x_0 \in M \), such that
\[
\text{Ric}_{g_t}(x) \geq -\alpha t^{-\gamma} \ln \left( \text{dist}_{g_t}(x_0, x) + \beta \right) g_t \quad \text{for all} \quad (x, t) \in M \times [0, T].
\]

Moreover, \( T \leq T(\alpha, \gamma) \), where \( T(\alpha, \gamma) \) signifies a positive constant depending only on \( \alpha \) and \( \gamma \).

Then \((M^n, g_t)_{t \in [0, T]}\) is \( \mathcal{L} \)-complete on \([0, T]\).

On the other hand, the significance of the \( \mathcal{L} \)-complete assumption is that with it a version of Perelman’s pseudolocality theorem can be proved. Indeed, it is well-known that Perelman’s entropy and reduced geometry reflect the same geometric property of the Ricci flow from differently perspectives. While the bounded curvature condition enables the estimates for the entropy, through which the pseudolocality theorem was proved, the \( \mathcal{L} \)-complete assumption leads to a similar conclusion via the reduced geometry. In this way, we prove the following pseudolocality-type theorem for \( \mathcal{L} \)-complete Ricci flows.

**Theorem 1.4.** For any \( \alpha > 0 \) there exist \( \delta > 0 \) and \( \epsilon_0 > 0 \) depending on \( \alpha \), and \( n \) with the following property. Let \((M^n, g_t)_{t \in [0, \epsilon^2]}\) be a smooth \( \mathcal{L} \)-complete Ricci flow on a noncompact manifold, where \( \epsilon \in (0, \epsilon_0] \), such that
\[
R(g_0) \geq -1,
\]
and
\[
\left( \text{Area}_{g_0}(\partial \Omega) \right)^n \geq (1 - \delta)n^n \omega_n \left( \text{Vol}_{g_0}(\Omega) \right)^{n-1}
\]
for any regular domain \( \Omega \subset M \). Then we have
\[
|\text{Rm}|(x, t) \leq \frac{\alpha}{t} \quad \text{for all} \quad (x, t) \in M \times (0, \epsilon^2].
\]

Applying the above pseudolocality theorem, we can study the strong uniqueness of the \( \mathcal{L} \)-complete Ricci flow in higher dimensions. In particular, we show the strong uniqueness theorem for the \( \mathcal{L} \)-complete Ricci flow on Euclidean space in higher dimensions, which partially answers a question proposed by B-L. Chen [5].

**Theorem 1.5.** Let \((M^n, g_t)_{t \in [0, T]}\) be a smooth \( \mathcal{L} \)-complete Ricci flow on a noncompact manifold \( M \) such that
\[
|\text{Rm}_{g_0}| \leq C,
\]
and
\[
\left( \text{Area}_{g_0}(\partial \Omega) \right)^n \geq (1 - \delta)n^n \omega_n \left( \text{Vol}_{g_0}(\Omega) \right)^{n-1}
\]
for any regular domain \( \Omega \subset M \). Then we have
\[
|\text{Rm}_{g_t}| \leq C \quad \text{for all} \quad t \in [0, T].
\]

Consequently, the solution \( g_t, \ t \in [0, T] \), is uniquely determined by \( g_0 \). In particular, two \( \mathcal{L} \)-complete Ricci flows starting from a Riemannian manifold satisfying the assumptions of the theorem must always be isometric to each other.

Combining Theorem 1.3(1)(2) and Theorem 1.5 we obtain the following result.
Theorem 1.6. Let \((\mathbb{R}^n, g_t)_{t \in [0, T]}\) be a complete and smooth solution to the Ricci flow with \(g_0 = g_E\), where \(g_E\) is the standard Euclidean metric. Assume that either one of the following is true.

1. there exists a positive constant \(c\), such that
   \[ g_t \geq c\overline{g} \quad \text{for all} \quad t \in [0, T], \]
   where \(\overline{g}\) is a complete smooth Riemannian metric on \(\mathbb{R}^n\)
2. there exist constants \(\alpha > 0, \beta > 0, \gamma \in (0, 1)\), and a fixed point \(x_0 \in \mathbb{R}^n\), such that
   \[ \operatorname{Ric}_{g_t}(x) \geq -\alpha t^{-\gamma} \cdot \ln \left( \operatorname{dist}_{g_t}(x_0, x) + \beta \right) \overline{g}, \quad \text{for all} \quad (x, t) \in \mathbb{R}^n \times [0, T]. \]

Then we have \(g_t \equiv g_E\) for all \(t \in [0, T]\).

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2. Preliminaries

In this section, we collect some well-known results applied in the proofs of our main theorems. These results are chiefly about Perelman’s reduced distance and reduced volume; most of them can be found in [19, 23]. Let \((M^n, g(\tau))_{\tau \in [0, T]}\) be a solution to the backward Ricci flow equation
\[
\frac{\partial}{\partial \tau} g(\tau) = 2\operatorname{Ric}_{g(\tau)},
\]
where \(\tau\) stands for the backward time. In fact, if \((M^n, g_t)_{t \in [0, T]}\) is a Ricci flow, then \(g(\tau) := g_{T-\tau}\) solves the backward Ricci flow equation. In the remaining part of the paper, since we often shift between Ricci flow and backward Ricci flow, we remind the reader that if the “time variable” is a subindex, such as \(g_t\), then the notation stands for a Ricci flow; if the “time variable” is placed in a pair of parentheses, such as \(g(\tau)\), then the notation represents a backward Ricci flow.

Let \(\gamma(s) : [0, \tau] \to M\), where \(\tau \in (0, T]\), be a piecewise \(C^1\) curve, then Perelman’s \(L\)-energy is defined as
\[
L(\gamma) = \int_0^\tau \sqrt{3} \left( \operatorname{Ric}(\gamma(s)) + |\gamma'(s)|_{g(s)}^2 \right) ds.
\]
One may view \(\gamma\) as a curve in the (backward) Ricci flow space-time from \((\gamma(0), 0)\) to \((\gamma(\tau), \tau) \in M \times [0, T]\), satisfying \((\gamma(s), s) \in M \times \{s\}\) for all \(s \in [0, \tau]\). Let \(x_0 \in M\) be a fixed base point. For any \((x, \tau) \in M \times (0, T]\), we define
\[
L(x, \tau) = \inf_{\gamma} L(\gamma(s)), \quad \ell(x, \tau) = \frac{L(x, \tau)}{2 \sqrt{\tau}},
\]
where the infimum is taken over all piecewise \(C^1\) curves \(\gamma(s) : [0, \tau] \to M\) satisfying \(\gamma(0) = x_0\) and \(\gamma(\tau) = x\). A minimizer of \(L\) (should it exist) is called a minimal \(L\)-geodesic, and \(\ell(\cdot, \cdot) : M \times (0, T] \to \mathbb{R}\) is called Perelman’s reduced distance function or the \(\ell\)-function based at \((x_0, 0)\).

Remark 2.1. When considering a forward Ricci flow \((M, g_t)_{t \in [0, T]}\), then a minimal \(L\)-geodesic is said to connect \((x, t)\) and \((y, s)\), where \(s < t\), if it minimizes \((2.1)\) with respect to the backward Ricci flow \((M, g(\tau) := g_{t-\tau})_{\tau \in [0, t]}\), starts at \((x, 0)\), and ends at \((y, t-s)\).
For $v \in T_{x_0}M$, let $\gamma_v$ denote the $L$-geodesic satisfying $\lim_{s \to 0} |\nabla_{d(s)}v(s)| = v$. If $\gamma_v$ exists on $[0, \tau]$, then the $L$-exponential map $\text{Lexp}^\tau_{x_0} : T_{x_0}M \to M$ at time $\tau$ is defined as

$$\text{Lexp}^\tau_{x_0}(v) = \gamma_v(\tau).$$

Let $U(\tau) \subset T_{x_0}M \cong \mathbb{R}^n$ be the maximal domain of $\text{Lexp}^\tau_{x_0}$. By applying the basic ODE theory to the $L$-geodesic equation—a linear ODE depending only on the geometry near the $L$-geodesic—one may conclude that $U(\tau)$ is an open set and that $\text{Lexp}^\tau_{x_0}$ is a smooth map from $U(\tau)$ to $M$. The injectivity domain at time $\tau$ is defined as

$$\Omega^{T_{x_0}M}(\tau) = \{ v \in U(\tau) \mid \gamma_v|_{[0, \tau]} : [0, \tau] \to M \text{ is the unique minimal } L \text{-geodesic from } (x_0, 0) \text{ to } (\gamma_v(\tau), \tau); \gamma_v(\tau) \text{ is not conjugate to } x_0 \text{ along } \gamma_v \}. $$

It is easily seen that

$$\Omega^{T_{x_0}M}(\tau_2) \subset \Omega^{T_{x_0}M}(\tau_1) \quad \text{if} \quad \tau_1 \leq \tau_2.$$

Correspondingly we also define

$$\Omega(\tau) = \{ q \in M \mid \text{There is a unique minimal } L \text{- geodesic } \gamma : [0, \tau] \to M \text{ with } \gamma(0) = x_0, \gamma(\tau) = q; q \text{ is not conjugate to } x_0 \text{ along } \gamma \}.$$ 

 Obviously,

$$\Omega(\tau) = \text{Lexp}^\tau_{x_0}(\Omega^{T_{x_0}M}(\tau)),$$

and Perelman’s reduced volume based at $(x_0, 0)$ is defined as

$$V(\tau) = \int_{\Omega^{T_{x_0}M}(\tau)} (4\pi)^{-\frac{n}{2}} e^{-\langle \ell, \cdot \rangle} d\mu(\tau).$$

The $L$-cut-locus is defined as

$$C(\tau) = M \setminus \Omega(\tau).$$

Let $J_i^\nu(\tau)$, $i = 1, \cdots, n$, be $L$-Jacobi fields along $\gamma(\tau)$ with $J_i^\nu(0) = 0, (\nabla_{\nu} J_i^\nu)(0) = E_i^0$, where $\{E_i^0\}_{i=1}^n$ is an orthonormal basis on $T_{x_0}M$ with respect to $g(0)$. Then $D(L\text{exp}^\nu)|_{\nu}(E_i^0) = J_i^\nu(\tau)$, and the reduced volume can also be computed as

$$V(\tau) = \int_{\Omega^{T_{x_0}M}(\tau)} (4\pi)^{-\frac{n}{2}} e^{-h(\nu(\tau), \nu)} L J_i(\tau) d\mu(0)(V),$$

where $L J_i(\tau) = \sqrt{\text{det} \langle J_i^\nu(\tau), J_j^\nu(\tau) \rangle}$ and $d\mu(0)$ is the standard Euclidean volume form on $(T_{x_0}M, g(x_0, 0))$.

To end this section, we recall the famous logarithmic Sobolev inequality due to Gross [12]. This inequality is important in the proofs of our main theorems, since it is strongly related to Perelman’s monotonicity formulas. The particular form of the following theorem can be found in [10] Theorem 22.5.

**Theorem 2.2** (Logarithmic Sobolev inequality). Let $(M^n, g)$ be a complete Riemannian manifold which satisfies

$$\left(\text{Area}_g(\partial \Omega)\right)^n \geq n \left(\text{Vol}_g(\Omega)\right)^{n-1}$$

for any regular compact domain $\Omega \subset M^n$. Then for any $W^{1,2}$ function $\varphi$ on $M^n$, we have

$$\int_{M^n} \left(2|\nabla \varphi|^2 - \varphi^2 \log \varphi^2\right) d\mu + \log \left(\int_{M^n} \varphi^2 d\mu\right) \int_{M^n} \varphi^2 d\mu \geq \left(\frac{n}{2} \log (2\pi) + n + \log \left(\frac{\text{Vol}_g(\Omega)}{c_n}\right)\right) \int_{M^n} \varphi^2 d\mu,$$

where $c_n = n^n \omega_n$ is the isoperimetric constant of Euclidean space.
3. Perelman’s $\mathcal{L}$-geometry on $\mathcal{L}$-complete Ricci flow

In this section, we shall first show that Perelman’s theory of $\mathcal{L}$-geometry can be extended to the case of $\mathcal{L}$-complete Ricci flows, and then show that the Ricci flow is $\mathcal{L}$-complete under the assumptions made in Theorem 1.3.

3.1. $\mathcal{L}$-complete Ricci flow. In Perelman’s study [19] of the $\mathcal{L}$-geometry, a general (although implicit) assumption is bounded sectional curvature. However, Ye [28] studied the properties of the $\ell$-function and the reduced volume assuming only a lower bound of the Ricci curvature. With a little more observation, Perelman’s theory of $\mathcal{L}$-geometry can be extended to $\mathcal{L}$-complete Ricci flows. We will leave it to the reader to check that most proofs in [28] are valid, and we summarize some important results below.

**Theorem 3.1** ([19], see also [28] Proposition 2.14, Lemma 2.22, Theorem 2.23). Let $(M^n, g(\tau))_{\tau \in [0,T]}$ be a smooth $\mathcal{L}$-complete backward Ricci flow. Let $\ell$ be the reduced distance function based at $(x_0,0)$. Then $\ell$ is locally Lipschitz on $M \times (0,T]$ and the following equation and inequalities hold almost everywhere in the smooth sense on $M \times (0,T]$.

\begin{align}
2 \frac{\partial \ell}{\partial \tau} + |\nabla \ell|^2 - R + \frac{\ell}{\tau} &= 0, \tag{3.1} \\
\frac{\partial}{\partial \tau} \ell - \Delta \ell + |\nabla \ell|^2 - R + \frac{n}{2\tau} &\geq 0, \tag{3.2} \\
2\Delta \ell - |\nabla \ell|^2 + R + \frac{\ell - n}{\tau} &\leq 0. \tag{3.3}
\end{align}

Furthermore, (3.2) and (3.3) both hold on $M \times (0,T]$ in the sense of distribution. That is to say, for any $0 < \tau_1 < \tau_2 \leq T$ and for any nonnegative Lipschitz function $\phi$ compactly supported on $M \times [\tau_1, \tau_2]$, it holds that

\begin{equation}
\int_{\tau_1}^{\tau_2} \int_M \left( -2\nabla \ell \cdot \nabla \phi + \left( \frac{\partial}{\partial \tau} \ell - |\nabla \ell|^2 + R + \frac{\ell - n}{\tau} \right) \phi \right) dg(\tau) d\tau \geq 0, \tag{3.4}
\end{equation}

and, for any $\tau \in (0,T]$ and any nonnegative Lipschitz function $\phi$ compactly supported on $M$, it holds that

\begin{equation}
\int_M \left( -2\nabla \ell \cdot \nabla \phi + \left( -|\nabla \ell|^2 + R + \frac{\ell - n}{\tau} \right) \phi \right) dg(\tau) \leq 0. \tag{3.5}
\end{equation}

Moreover, the integrand in (3.5) is pointwise monotonically non-increasing, namely,

\begin{equation}
\frac{d}{d\tau} \left( (4\pi \tau)^{-\frac{n}{2}} e^{-\frac{\ell}{2\tau}} \mathcal{L} J_{\ell}(\tau) \right) \leq 0, \tag{3.6}
\end{equation}

for any $V \in \Omega^T \omega^M(\tau)$.

**Proof.** By virtue of our definition of the $\mathcal{L}$-complete Ricci flow (Definition 1.2), when analyzing the local properties of the $\mathcal{L}$-geodesics or the reduced distance, only local geometry is involved. Thus the proofs in [28] can be wholly adopted in our case.

We emphasize that although according to Ye’s formulation, (3.5) and (3.6) hold only on $M \times (0,T)$, yet this restriction stems from the application of Shi’s estimates when obtaining the local curvature derivative bounds. However, in our assumption, the backward Ricci flow is assumed to be smooth on $M \times [0,T]$, namely, each curvature derivative is bounded locally. Thus (3.5) and (3.6) are also valid on $M \times (0,T]$. \hfill \Box

**Theorem 3.2** ([19], see also [28] Theorem 4.3, Theorem 4.5). Let $(M^n, g(\tau))_{\tau \in [0,T]}$ be a $\mathcal{L}$-complete backward Ricci flow. Let $V(\tau)$ be the reduced volume based at $(x_0,0)$. Under the same assumptions as in the above theorem, we have
Thus, we compute
\(3.2.1.\)
flow \((M, L)\) Letting idea of proving the number \(L\), such that for any \(x, \in M\) and fix \(y\) is non-increasing in \(r\);
(3) if \(V(r) = 1\) for some \(r \in (0, T)\), then \(M^n = \mathbb{R}^n\) and \(g(s) \equiv g_E\) for all \(s \in [0, r]\), where \(g_E\) is the standard Euclidean metric.

**Proof.** According to the definition of the reduced volume \((2.5)\), this theorem is a consequence of \((2.3)\) and \((3.6)\). The details can be found in \([28]\). \(\square\)

### 3.2. Sufficient conditions of \(L\)-completeness.

In this subsection, we shall prove Theorem \([1,3]\). To begin with, we prove the following straightforward auxiliary lemma.

**Lemma 3.3** (bounded \(L\)-energy in compact set). Let \((M, g(\tau))_{\tau \in [0, T]}\) be a complete backward Ricci flow. For any \(\tau \in (0, T)\), and any compact set \(E \subset M\), there exists a positive number \(L\), such that for any \(x_0, x \in E\), there is a piecewise \(C^1\) curve \(\gamma : [0, \tau]\) with \(\gamma(0) = x_0\) and \(\gamma(\tau) = x\), satisfying
\[
L(\gamma) \leq L.
\]

**Proof.** Since \(E\) is compact and the backward Ricci flow is complete, we can find a \(y_0 \in M\) and fix \(A > 0\) and \(K > 0\), such that
\[
\text{For any } x_0, x \in E, \text{ let } \gamma : [0, \tau] \to M \text{ a normalized shortest } g(0)-\text{geodesic with } \gamma(0) = x_0 \text{ and } \gamma(\tau) = x. \text{ Then it is obvious that}
\]
\[
\gamma(\tau) \in B_{g(0)}(y_0, 3A) \quad \text{and} \quad |\gamma'(\tau)|_{g(0)} = \frac{\text{dist}_{g(0)}(x_0, x)}{\tau} \leq \frac{2A}{\tau}, \quad \text{for all } \tau \in [0, \tau].
\]
Thus, we compute
\[
L(\gamma) = \int_0^\tau \sqrt{\text{det} g(\tau)} \left( R_{g(\tau)}(\gamma(\tau)) + |\gamma'(\tau)|_{g(\tau)}^2 \right) d\tau
\]
\[
\leq \frac{2}{3} \sqrt{nK\tau^2} + e^{K\tau} \int_0^\tau \sqrt{\text{det} g(\tau)} |\gamma'(\tau)|_{g(\tau)}^2 d\tau
\]
\[
\leq \frac{2}{3} \sqrt{nK\tau^2} + \frac{2A^2}{3 \sqrt{\tau}}.
\]
Letting \(L\) be the constant on the last line above finishes the proof of the lemma. \(\square\)

#### 3.2.1. Lower uniform equivalence.

We prove Theorem \([1,3]\). Consider a backward Ricci flow \((M, g(\tau))_{\tau \in [0, T]}\) satisfying
\[
R_{g(\tau)}(x) \geq -k \quad \text{for all } (x, \tau) \in M \times [0, T],
\]
\[
g(\tau) \geq c\bar{g} \quad \text{for all } \tau \in [0, T],
\]
where \(k \geq 0\), \(c > 0\), and \(\bar{g}\) is a complete smooth Riemannian metric \(\bar{g}\) on \(M\). The central idea of proving the \(L\)-completeness is to show that any space-time curve with bounded \(L\)-energy lies in a space-time compact set; this idea is also used in the following part of the subsection.

**Proposition 3.4.** Under the assumptions \((3.7)\) and \((3.8)\), the following holds. For any \((x_0, 0)\) and \((x, \tau) \in M \times [0, T]\), there exists a minimal \(L\)-geodesic \(\gamma : [0, \tau] \to M\) satisfying \(\gamma(0) = x_0\) and \(\gamma(\tau) = x\). Furthermore, the backward Ricci flow is \(L\)-complete on \(M \times [0, T]\).
Proof. Let us consider any piecewise $C^1$ curve $\gamma : [0, \tau] \to M$ satisfying $\gamma(0) = x_0$ and $L(\gamma) \leq L$. Then, by (3.7) and (3.8), we may estimate
\[
L \geq L(\gamma) = \int_0^\tau \sqrt{\left(R_{g(\gamma)}(\gamma(s)) + |\gamma'(s)|^2_{g(\gamma)}\right)}
\geq -\frac{2k}{3} \tau + \int_0^\tau \sqrt{3}|\gamma'(s)|^2_{g(\gamma)} \, ds
\geq -\frac{2k}{3} \tau + c \int_0^\tau \sqrt{3}|\gamma'(s)|^2 \, ds
= -\frac{2k}{3} \tau + \frac{c}{2} \int_0^\tau |\zeta'(\sigma)|^2 \, d\sigma,
\]
where we have applied the change of variable $\zeta(\sigma) := \gamma(\sigma^2)$. Then, by the Cauchy-Schwarz inequality, we have
\[
2e^{-1}L \geq -\frac{4k}{3c} \tau^2 + \int_0^\tau |\zeta'(\sigma)|^2 \, d\sigma
\geq -\frac{4k}{3c} \tau^2 + \frac{(\text{dist}_{g(x_0)}(x_0, \gamma(s)))^2}{\sqrt{T}},
\]
for all $s \in [0, \tau]$. It follows that for any $L > 0$, there exists $A = A(L, c, T)$, such that if $\gamma : [0, \tau] \to M$ satisfies $L(\gamma) \leq L$, then
\[
\gamma(s) \in B_{\tilde{g}}(x_0, A) \quad \text{for all} \quad s \in [0, \tau].
\]
In particular, $A$ is independent of the choice of $x_0$

For any $x_0$ and $x$, let us fix an arbitrary piecewise $C^1$ curve $\beta : [0, \tau] \to M$ satisfying $\beta(0) = x_0$ and $\beta(\tau) = x$. Let $L = L(\beta)$. Then we have that every minimizing sequence $\{\gamma_i\}_{i=1}^\infty$ of the $L$-energy with $L(\gamma_i) \leq L$ from $(x_0, 0)$ to $(x, \tau)$ is contained in $B_{\tilde{g}(0)}(x_0, A) \times [0, \tau]$, which is a compact set in space-time. It follows that there exists at least one minimal $L$-geodesic from $(x_0, 0)$ to $(x, \tau)$.

Finally, we show that the backward Ricci flow is $L$-complete. Let $E \in M$ be an arbitrary compact set and fix a $\tau \in (0, T)$. By Lemma 3.2 there is a constant $L(E, \tau) > 0$, such that for any $x, y \in E$, there is a piecewise $C^1$-curve $\gamma : [0, \tau] \to M$ with $\gamma(0) = x$, $\gamma(\tau) = y$, and $L(\gamma)$ not exceeding $L(E, \tau)$. The argument above shows that there exists at least one minimal $L$-geodesic connecting $(x, 0)$ and $(y, \tau)$, and all such minimal $L$-geodesics are contained in a compact set.

\[\square\]

3.2.2. Logarithmic lower bound for the Ricci curvature. Let us move on to the proof of Theorem 1.3(2). We assume that the backward Ricci flow in question has a logarithmic lower bound for the Ricci curvature. In particular, we let $k > 0, \alpha > 0, \beta > 0, \gamma \in (0, 1)$ be constants and $x_0$ be a fixed point, such that
\begin{align}
R_{g(\gamma)}(x) &\geq -k \quad \text{for all} \quad (x, \tau) \in M \times [0, T] \\
\text{and} \\
\text{Ric}_{g(\gamma)}(x) &\geq -\alpha(T - \tau)^{-\gamma} \cdot \text{ln} \left(\text{dist}_{g(\gamma)}(x_0, x) + \beta\right) g(\tau) \quad \text{for all} \quad (x, \tau) \in M \times (0, T),
\end{align}
As in the previous subsection, we show the existence of a minimal $L$-geodesic from $(x_0, 0)$ to $(x, \tau) \in M \times (0, T)$, provided that $T$ is small enough.
Lemma 3.5 (Distance distortion). For any $\tau_1$ and $\tau_2 \in [0, T]$ with $\tau_1 \leq \tau_2$ and for any $x \in M$, we have

$$\operatorname{dist}_{g(\tau_1)}(x_0, x) + \beta \leq \left( \operatorname{dist}_{g(\tau_2)}(x_0, x) + \beta \right)^{\exp \left( \frac{r}{r + \gamma} \right)}.$$

**Proof.** For any $\tau \in [0, T]$, let us define $r(\tau) := \operatorname{dist}_{g(\tau)}(x_0, x)$ and let $\gamma : [0, r(\tau)] \to M$ be a $g(\tau)$-geodesic connecting $x_0$ and $x$ with unit speed. We may apply (3.10) to compute

$$\frac{dr}{d\tau} = \int_0^\tau \operatorname{Ric}_{g(\tau)}(\gamma'(s), \gamma'(s))ds$$

$$\geq -\alpha(T - \tau)^{-\gamma} \cdot r \ln(r + \beta)$$

$$\geq -\alpha(T - \tau)^{-\gamma} \cdot (r + \beta) \ln(r + \beta),$$

and equivalently

$$\frac{d}{d\tau} \ln(r + \beta) \geq -\alpha(T - \tau)^{-\gamma}.$$

Integrating the above inequality from $\tau_1$ to $\tau_2$, we obtain the lemma. \(\square\)

Next, we prove that a minimizing sequence of the $\mathcal{L}$-energy is always contained in a compact set in space-time.

**Lemma 3.6.** There exists a positive number $\overline{T} = \overline{T}(\alpha, \gamma) > 0$ with the following property. If $T < \overline{T}$, then for any $r > 0$ and $L > 0$, there is a positive number $A = A(r, L, T, \alpha, \beta, \gamma)$, such that the following holds. For any $(x, \tau) \in M \times (0, T]$ with

$$\operatorname{dist}_{g(\tau)}(x_0, x) \leq r,$$

if $\gamma : [0, \tau] \to M$ is a piecewise $C^1$ curve satisfying $\gamma(0) = x_0$, $\gamma(\tau) = x$, and

$$\mathcal{L}(\gamma) \leq L$$

Then we have

$$\gamma(s) \in \overline{B}_{g(\tau)}(x_0, A) \quad \text{for all} \quad s \in [0, \tau].$$

**Proof.** We shall argue by contradiction. Let $A \gg r$ be a large number to be fixed. Define

$$\overline{\tau} := \inf \left\{ \tau' \mid \gamma(s) \in B_{g(\tau)}(x_0, A) \text{ for all } s \in [\tau', \tau] \right\} \in [0, \tau).$$

We assume that $\overline{\tau} > 0$ and show that there is a contradiction when $A$ is large enough. By this contradictory assumption, we have

(3.11) \quad $$\operatorname{dist}_{g(\overline{\tau})}(x_0, \gamma(\overline{\tau})) = A,$$

$$\operatorname{dist}_{g(\tau)}(x_0, \gamma(s)) < A \quad \text{for all} \quad s \in [\overline{\tau}, \tau].$$

By Lemma 3.5, we have

$$\operatorname{dist}_{g(\tau)}(x_0, \gamma(\tau_2)) \leq \left( \operatorname{dist}_{g(\tau)}(x_0, \gamma(\tau_2)) + \beta \right)^{\exp \left( \frac{r}{r + \gamma} \right)}$$

for all $0 \leq \tau_1 \leq \tau_2 \leq \tau$, and consequently

(3.12) \quad $$\operatorname{dist}_{g(\tau)}(x_0, \gamma(\tau_2)) \leq (A + \beta)^{\exp \left( \frac{r}{r + \gamma} \right)}$$

for all $\tau_1 \leq \tau_2 \leq \tau$

(3.13) \quad $$\operatorname{dist}_{g(\tau)}(x_0, x) \leq (r + \beta)^{\exp \left( \frac{r}{r + \gamma} \right)}.$$
By (3.10) and (3.12), we may estimate
\[
\frac{d}{d\tau} |\gamma'(\tau)|^2_{g(\tau)} = 2\text{Ric}_{g(\tau)}(\gamma'(\tau), \gamma'(\tau)) \\
\geq - 2\alpha(T - \tau_1)^{-\gamma} \cdot \ln\left( (A + \beta)\exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) + \beta \right) |\gamma'(\tau)|^2_{g(\tau)},
\]
for all \( \bar{\tau} \leq \tau_1 \leq \tau_2 \leq \tau \). Integrating over \( \tau_1 \), we have
\[
|\gamma'(\tau_2)|^2_{g(\tau_2)} \geq \left( (A + \beta)\exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) + \beta \right) |\gamma'(\tau_1)|^2_{g(\tau_1)} \\
\geq (2A)^{-\frac{\tau_2 - \tau_1}{\tau - \tau_1}} |\gamma'(\tau_2)|^2_{g(\tau_2)} \quad \text{for all} \quad \bar{\tau} \leq \tau_2 \leq \tau,
\]
if \( A \geq A_0(\alpha, \beta, \gamma, T) \) for some large positive constant \( A_0(\alpha, \beta, \gamma, T) \). By the definition of the \( \mathcal{L} \)-energy and the assumption (3.7), we have
\[
L(\gamma) \geq -\frac{2k}{3} \tau^2 + \int_{\bar{\tau}}^{\tau} \sqrt{\gamma(s)} |\gamma'(s)|^2_{g(\bar{\tau})} ds \geq -\frac{2k}{3} \tau^2 + (2A)^{-\frac{\tau_2 - \tau_1}{\tau - \tau_1}} \exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) \int_{\bar{\tau}}^{\tau} \sqrt{\gamma(s)} |\gamma'(s)|^2_{g(\bar{\tau})} ds.
\]
Letting \( \zeta(\tau) := \gamma(\tau^2) \), we have
\[
L \geq -\frac{2k}{3} \tau^2 + (2A)^{-\frac{\tau_2 - \tau_1}{\tau - \tau_1}} \exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) \int_{\bar{\tau}}^{\tau} \sqrt{\gamma(s)} |\gamma'(s)|^2_{g(\bar{\tau})} ds \\
\geq -\frac{2k}{3} \tau^2 + (2A)^{-\frac{\tau_2 - \tau_1}{\tau - \tau_1}} \exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) \left( \text{dist}_{g(\bar{\tau})}(x, \gamma(\bar{\tau})) \right)^2 \\
\geq -\frac{2k}{3} \tau^2 + (2A)^{-\frac{\tau_2 - \tau_1}{\tau - \tau_1}} \exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) \frac{(A - (r + \beta)\exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right))^2}{2 \tau T},
\]
where we have applied (3.11) and (3.13). Finally, if we take \( T < \bar{T}(\alpha, \gamma) \) such that \( \frac{2k}{3} \tau^2 + (2A)^{-\frac{\tau_2 - \tau_1}{\tau - \tau_1}} \exp\left( \frac{\tau_2 - \tau_1}{\tau - \tau_1} \right) \leq 1 \), then we have
\[
L \geq \frac{(A - (r + \beta)\exp(\tau_1/T))^2}{4A \sqrt{T}} - \frac{2k}{3} \tau^2,
\]
and this obviously is a contradiction if \( A > A(\tau, L, T, \alpha, \beta, \gamma) \). \( \square \)

**Proposition 3.7** (Existence of minimal \( \mathcal{L} \)-geodesic). If \( T < \bar{T} \), where \( \bar{T} \) is the constant in the statement of Lemma 3.6, then for any \( (x, \tau) \in M \times (0, \bar{T}) \), there is a minimal \( \mathcal{L} \)-geodesic from \( (x_0, 0) \) to \( (x, \tau) \). In particular, the backward Ricci flow is \( \mathcal{L} \)-complete on \( [0, \bar{T}] \).

**Proof.** We fix an arbitrary \( (x, \tau) \in M \times (0, \bar{T}) \) and a piecewise \( C^1 \) curve \( \gamma : [0, \tau] \to M \) with \( \gamma(0) = x_0 \) and \( \gamma(\tau) = x \). Let
\[
r := \text{dist}_{g(\tau)}(x_0, x),
\]
and let \( A = A(\tau, L, T, \alpha, \beta, \gamma) \) be the positive constant given by Lemma 3.6. Then we have that, by Lemma 3.6, any minimizing sequence \( \{\gamma_i\}_{i=1}^\infty \) of the \( \mathcal{L} \)-energy with \( \mathcal{L}(\gamma_i) \leq L \) from \( (x_0, 0) \) to \( (x, \tau) \) lies in the space-time compact set
\[
\bigcup_{s \in [0, \tau]} \overline{B} g(\tau)(x_0, A) \times \{s\},
\]
and the existence of minimal \( \mathcal{L} \)-geodesic follows from the standard theory of variation.
Finally, we need to show that \((M, g(\tau))_{\tau \in [0, T]}\) is \(\mathcal{L}\)-complete, where \(T \leq \bar{T}\) satisfies the condition in the previous Lemma. For \(0 \leq \tau_1 \leq \tau_2 \leq T\), it is obvious that \((M, g(\tau + \tau_1))_{\tau \in [0, \tau_2 - \tau_1]}\) still satisfies (4.9) and (4.10). Thus we may take \(\tau_1 = 0\) and \(\tau_2 = \bar{\tau} \in [0, T]\). Let \(E\) be an arbitrary closed set in \(M\). Since each time slice of the backward Ricci flow is complete, we may, without loss of generality, assume that \(E = \overline{B}_{g(\tau)}(x_0, r)\). Applying the standard distance distortion estimate, while using the fact that \(|\text{Ric}|\) is uniformly bounded in \(\overline{B}_{g(\bar{T})}(x_0, r) \times [0, \bar{\tau}]\), we have
\[
R_0 := \sup_{\tau \in [0, \bar{\tau}]} \sup_{y \in \overline{B}_{g(\bar{T})}(x_0, r)} \text{dist}_{g(\tau)}(x_0, y) < \infty.
\]
It clearly follows from the triangle inequality that, for any \(y_0 \in \overline{B}_{g(\bar{T})}(x_0, r)\),
\[
(3.15)
\]
\[
\text{Ric}_{g(\tau)}(x) \geq -\alpha(T - \tau)^{-\gamma} \cdot \ln \left( \text{dist}_{g(\tau)}(y_0, x) + \beta + R_0 \right) g(\tau) \quad \text{for all} \quad (x, \tau) \in M \times [0, T].
\]
Now we may argue as in the previous lemma with (3.10) replaced by (3.15), \(x_0\) replaced by an arbitrary \(x \in \overline{B}_{g(\bar{T})}(x_0, r)\), and \(\beta\) replaced by \(\beta + R_0\) (note that \(\bar{T}(\alpha, \gamma)\) is independent of \(\beta\)). There exists \(A = A(2\bar{\tau}L(E, \bar{\tau}), T, \alpha, \beta, R_0, \gamma) > 0\), where \(L(E, \bar{\tau})\) is obtained by applying Lemma [3.3] to \(E = \overline{B}_{g(\bar{T})}(x_0, r)\) and \(\bar{\tau} \in [0, T]\), such that for \(x, y \in \overline{B}_{g(\bar{T})}(x_0, r)\), any minimal \(\mathcal{L}\)-geodesic connecting \((x, 0)\) and \((y, \bar{\tau})\) lies in the space-time compact set
\[
\bigcup_{\tau \in [0, \bar{\tau}]} \overline{B}_{g(\bar{T})}(x, A) \times \{s\} \subset \bigcup_{\tau \in [0, \bar{\tau}]} \overline{B}_{g(\bar{T})}(x_0, A + R_0) \times \{s\},
\]
which is clearly contained in the cross product of a spatial compact set \(F\) and the time interval \(\{0, \bar{\tau}\}\).

4. **Perelman’s pseudolocality under the \(\mathcal{L}\)-complete assumption**

In this section, we prove Theorem 1.4. Let \((M, g(\tau))_{\tau \in [0, T]}\) be the \(\mathcal{L}\)-complete Ricci flow in the statement of Theorem 1.4 satisfying
\[
(4.1) \quad R(g_0) \geq -1,
\]
\[
(4.2) \quad \left( \text{Area}_{g_0}(\partial \Omega) \right)^n \geq (1 - \delta)n^n \omega_n \left( \text{Vol}_{g_0}(\Omega) \right)^{n-1} \quad \text{for all} \quad \Omega \subset M.
\]
When formulating an argument of the reduced distance, we shall always consider the backward Ricci flow \(g(\tau) := g_{T - \tau}\) instead.

Let \(\ell : M \times [0, T] \to \mathbb{R}\) be the reduced distance based at some point on the \(\tau = 0\) (i.e. \(t = T\) slice) \(\mathcal{V}(\tau)\) the corresponding reduced volume. First of all, we shall prove some preparatory lemmas focusing on the “final” time-slice of the \(\ell\)-function
\[
(4.3) \quad \ell := \ell(\cdot, T) : M^n \to \mathbb{R}.
\]

**Lemma 4.1.** Under the \(\mathcal{L}\)-complete assumption and (4.1), we have
\[
(4.4) \quad \int_{M^n} |\nabla \ell|^2 (4\pi T)^{\frac{n}{2}} \, e^{-\ell} \, d\mu \leq \frac{2n}{T} + 2 + 2T^2 < \infty.
\]
Consequently, \((4\pi T)^{-n/4} e^{-\ell/2}\) is a \(W^{1,2}\) function on \(M^n\).
Lemma 4.2. Under the \( L \)-complete assumption and \((4.1)(4.2)\), we have

\[ \mathcal{V}(T) \geq e^{-T(1-\delta)}. \]
Proof. Rewriting (4.5) using the same cut-off function as defined in (4.6) and the fact that $R(g_0) \geq -1$, we have

$$
\int_{M'} \left( |\nabla \ell|^2 + \frac{\ell - n}{T} \right) \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu
\leq 4 \int_{M'} (|\nabla \ell|^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu) - \int_{M'} R \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu
\leq 4 \left( \int_{M'} |\nabla \ell|^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu \right)^{\frac{1}{2}} \left( \int_{M'} |\nabla \varphi_A|^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu \right)^{\frac{1}{2}} + \int_{M'} \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu
\leq 4 \left( \frac{2n}{T} + 2 + 2T^{\frac{1}{2}} \right) \left( \frac{4}{A^2} \mathcal{V}(T) \right)^{\frac{1}{2}} + \int_{M'} \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu,
$$

where we have applied Lemma 4.1. Taking $A \to \infty$, we have

$$
(4.7) \quad \int_{M'} \left( |\nabla \ell|^2 + \ell - n \right) (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu \leq T \int_{M'} (4\pi T)^{-\frac{n}{2}} e^{-\ell} d\mu.
$$

Rescaling $g(\tau)$ as

$$\tilde{g}(\tau) = (2T)^{-1} g(2T\tau),$$

and denote by $\tilde{\ell}$ the reduced length with respect to $\tilde{g}(\tau)$. Then (4.7) becomes

$$
(4.8) \quad \int_{M'} \left( \frac{1}{2} |\nabla \tilde{\ell}|^2 + \tilde{\ell}^{\frac{1}{2}} - n \right) (2\pi)^{-\frac{n}{2}} e^{-\tilde{\ell}^{\frac{1}{2}}} d\tilde{\mu} \leq T \int_{M'} (2\pi)^{-\frac{n}{2}} e^{-\tilde{\ell}^{\frac{1}{2}}} d\tilde{\mu}.
$$

Letting

$$\varphi(x) := (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} |\varphi(x)|} \in W^{1,2}(M),$$

we can rewrite (4.8) as

$$
\int_{M'} \left( 2|\nabla \varphi|^2 - \varphi^2 \log \varphi \right) d\tilde{\mu}
\leq \left( n + \frac{n}{2} \log(2\pi) \right) \int_{M'} \varphi^2 d\tilde{\mu} + T \int_{M'} \varphi^2 d\tilde{\mu}
\leq \int_{M'} \left( 2|\nabla \varphi|^2 - \varphi^2 \log \varphi \right) d\tilde{\mu} + \left( \log \left( \int_{M'} \varphi^2 d\tilde{\mu} \right) - \log(1 - \delta) + T \right) \int_{M'} \varphi^2 d\tilde{\mu},
$$

where in the second inequality we have applied Theorem 2.2. Consequently, we have

$$
\int_{M'} \varphi^2 d\tilde{\mu} \geq e^{-T} (1 - \delta).
$$

Now we proceed to prove Theorem 1.4 by contradiction. Assume that there exist $\alpha$ and sequences of positive numbers $\delta_i \to 0^+$, $\epsilon_i \to 0^+$, and smooth $L$-complete pointed Ricci flows $\{M_i, g_{i, \epsilon}\}_{\epsilon \in [0, \epsilon_i^2]}$, $i \in \mathbb{N}$, satisfying (4.1), (4.2). Furthermore, there exist “bad” points in space-time $(x_i, t_i) \in M_i \times (0, \epsilon_i^2]$ satisfying

$$
|\text{Rm}_{g_i}|(x_i, t_i) > \frac{\alpha}{t_i^4}
$$

for all $i \in \mathbb{N}$.
We can apply a point-picking method as in Perelman’s proof of pseudolocality theorem. The details can be found in [10]. Arguing in the same way as [10] Lemma 21.12 and [10] Chapter 22 §1, we may adjust \((x_i, t_i)\) such that they satisfy not only (4.9), but also
\[
(4.10) \quad |Rm_{\tilde{g}_i}|(x, t) \leq 4Q_i \quad \text{for all} \quad (x, t) \in B_{\tilde{g}_{i0}} \left( x_i, A_i 1/2 \right) \times \left[ t_i - \frac{\alpha}{2} Q_i^{-1}, t_i \right]
\]
where \(Q_i := |Rm_{\tilde{g}_i}|(x_i, t_i) > \frac{\alpha}{\bar{g}}\) and \(A_i \to \infty\).

**Lemma 4.3.** For each \(i\), there exists a time
\[
\tilde{t}_i \in \left[ t_i - \frac{\alpha}{2} Q_i^{-1}, t_i \right]
\]
such that
\[
\mathcal{V}_i(t_i - \tilde{t}_i) \leq 1 - \beta_i
\]
where \(\mathcal{V}_i\) is the reduced volume of \(g_i\) based at \((x_i, t_i)\) and \(\beta_i \in (0, 1)\) is a constant independent of \(i\).

**Proof.** We apply the following rescaling and time-shifting to the Ricci flows
\[
(4.11) \quad g_i \to \tilde{g}_i := Q_i g_i, t + Q_i^{-1},
\]
and proceed to show that for each \(i \in \mathbb{N}\), there is a \(\tilde{t}_i \in [-\frac{4}{\alpha}, 0)\) with
\[
\mathcal{V}_i(-\tilde{t}_i) \leq 1 - \beta_i,
\]
where \(\mathcal{V}_i\) is the reduced volume of \(\tilde{g}_i\) based at \((x_i, 0)\). (4.10) now becomes
\[
(4.12) \quad |Rm_{\tilde{g}_i}|(x, t) \leq 4 \quad \text{for all} \quad (x, t) \in B_{\tilde{g}_{i0}}(x_i, A_i) \times \left[ -\frac{4}{\alpha}, 0 \right].
\]

We argue by contradiction. Assume that, by passing to a subsequence,
\[
(4.13) \quad \lim_{i \to \infty} \inf_{\tau \in (0, \frac{4}{\alpha})} \mathcal{V}_i(\tau) = 1.
\]

**Claim.** There are positive constants \(c(\alpha)\) and \(C(\alpha)\) with the following property. Fix any large \(i \in \mathbb{N}\) and any \(\bar{\tau} \in (0, \frac{4}{\alpha})\). Let \(\gamma_{\bar{\tau}} : [0, \bar{\tau}] \to M\) be a minimal \(L\)-geodesic with respect to \(\tilde{g}_i(\tau) = \tilde{g}_i, t - \tau\) starting from \((x_i, 0)\) with \(\lim_{\tau \to 0} \sqrt{\bar{\tau}} \gamma_{\bar{\tau}}(\tau) = V \in T_{x_i}M\). For any \(A \in [10C(\alpha), A_i]\), if \(|V|_{\tilde{g}_{i0}} \leq c(\alpha)A\), then \(\gamma_{\bar{\tau}}|_{[0, \bar{\tau}] \subset B_{\tilde{g}_{i0}}(x_i, \frac{1}{2}A)}\).

**Proof of the Claim.** Writing \(X(\tau) = \gamma_{\bar{\tau}}(\tau)\), we may apply the \(L\)-geodesic equation to obtain (c.f. [16] (26.41))
\[
\frac{d}{d\tau}(\tau X_{\tilde{g}_{i0}}^2(\tau)) = -2\tau \text{Ric}_{\tilde{g}_i}(X, X) + \tau (X, \nabla R_{\tilde{g}_i}(X, \nabla R_{\tilde{g}_i}(X)) - 2R_{\tilde{g}_i}(X, X)).
\]
By (4.12) and Shi’s estimates, we have
\[
|Rm_{\tilde{g}_i}| \leq 4, \quad |\nabla R_{\tilde{g}_i}| \leq C(n, \alpha), \quad \text{for all} \quad (x, \tau) \in B_{\tilde{g}_{i0}}(x_i, \frac{1}{2}A) \times [0, \frac{4}{\alpha}].
\]
Then, as long as \(\gamma_{\bar{\tau}}|_{[0, \bar{\tau}] \subset B_{\tilde{g}_{i0}}(x_i, \frac{1}{2}A)}\), we may estimate
\[
\frac{d}{d\tau}(\tau X_{\tilde{g}_{i0}}^2(\tau)) \leq 8\tau |X_{\tilde{g}_{i0}}^2(\tau) + C\tau |X_{\tilde{g}_{i0}}^2(\tau) \leq 9\tau |X_{\tilde{g}_{i0}}^2(\tau) + C
\]
and by integrating
\[
(4.14) \quad \tau |X_{\tilde{g}_{i0}}^2(\tau) \leq e^{4\tau} |X_{\tilde{g}_{i0}}^2(0) + C.
\]
Let us fix an \(A \leq A_i\). Assume that \(\tau' \in (0, \frac{4}{\alpha})\) is the first time when \(\gamma_{\bar{\tau}}\) has reached the boundary of \(B_{\tilde{g}_{i0}}(x_i, \frac{1}{2}A)\). Thus, \(\gamma_{\bar{\tau}}|_{[0, \tau']} \subset B_{\tilde{g}_{i0}}(x_i, \frac{1}{2}A), \text{dist}_{\tilde{g}_{i0}}(\gamma_{\bar{\tau}}(0), \gamma_{\bar{\tau}}(\tau')) = \frac{1}{2}A, \) and
\[
|X(\tau)|_{\tilde{g}_{i0}}^2(\tau) \geq e^{-\alpha}|X(\tau)|_{\tilde{g}_{i0}}^2(0) \quad \text{for all} \quad \tau \in [0, \tau'].
\]
Then,
\[ \frac{1}{2}A = \text{dist}_{\hat{g}_\infty}(\gamma_V(0), \gamma_V(\tau')) \leq \int_0^{\tau'} |X(\tau)|_{\hat{g}_\infty} d\tau \leq e^{\alpha/2} \int_0^{\tau'} |X(\tau)|_{\hat{g}_\infty} d\tau \]
\[ \leq e^{\alpha/2} \left( \int_0^{\tau'} \sqrt{V}(\tau)_{\hat{g}_\infty}^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^{\tau'} \tau^{-\frac{2}{n}} d\tau \right)^{\frac{1}{2}} \]
\[ \leq C(\alpha) \left( \int_0^{\tau'} \tau^{-\frac{2}{n}} \left( e^{\alpha|V|_{\hat{g}_\infty}(0) + C} \right) d\tau \right)^{\frac{1}{2}} \]
\[ \leq C(\alpha)(|V|_{\hat{g}_\infty}(0) + 1), \]
where we have applied (4.14). If \( A \geq 10C(\alpha) \), then the above computation yields a contradiction if we assume \( |V|_{\hat{g}_\infty}(0) \leq \frac{1}{10\alpha}A \). This proves the claim.

By contradictory assumption (4.13) and the proof of Perelman’s no local collapsing theorem (c.f. [16, Theorem 26.2]), it is easy to see that
\[ \inf_i \text{inj}_{\hat{g}_\infty}(x_i) > 0. \]
Therefore, we may extract a subsequence from \( \{(M_i, \hat{g}_i, x_i)\}_{i \in [\frac{1}{10\alpha}A, \infty]} \) which converges to a smooth Ricci flow with bounded curvature
\[ (M_{\infty}, \hat{g}_{\infty}, x_{\infty}) \in \text{Lexp}_{\hat{g}}(\mathbb{R}^n \setminus \{|V| \leq cA\}), \]
for all \( \tau \in [0, \frac{1}{4}] \) and \( A \in [10C(\alpha), A]. \)

By the monotonicity of the L-jacobian (3.6) and the fact that
\[ \lim_{\tau \to 0} (4\pi)^{-\frac{1}{2}} e^{-\delta y(y, \tau)} \text{L} J_V(\tau) = (4\pi)^{-\frac{1}{2}} \exp \left( -\frac{|V|^2}{4} \right), \]
we have that, for each \( \tau \in (0, \frac{1}{4}] \),
\[ \int_{M \setminus B_{\hat{g}_{\infty}}(x_i, \frac{1}{4}A)} (4\pi)^{\frac{1}{2}} e^{-\delta \hat{g}(\tau)} d\hat{g}(\tau) \leq \int_{|V| \geq cA} (4\pi)^{-\frac{1}{2}} \exp \left( -\frac{|V|^2}{4} \right) dV, \]
for all \( A \in [10C(\alpha), A]. \)

It is then clear from (4.13) that
\[ V_{\infty}(\tau) = \lim_{i \to \infty} V_i(\tau) = 1 \quad \text{for all} \quad \tau \in (0, \frac{1}{4}], \]
where \( V_{\infty} \) is the reduced volume of \( \hat{g}_\infty \) based at \((x_\infty, 0)\). Thus, \( \hat{g}_\infty \) is the flat Euclidean space; this is a contradiction.

Now we can give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Applying Lemma 4.2 to the reduced volumes \( V_i \) of the Ricci flows \((M_i, g_i)\) based at \((x_i, t_i)\), since \( t_i \leq T_i^2 \), we have that
\[ V_i(t_i) \geq e^{-\delta_i} (1 - \delta_i). \]

By the monotonicity of the reduced volume, we also have
\[ V_i(\tau) \geq e^{-\delta i} (1 - \delta_i) \quad \text{for all} \quad \tau \in (0, t_i]. \]
This is clearly a contradiction to Lemma 4.3 when \( i \) is large.
Theorem 1.5 just follows from Theorem 1.4 and Theorem 3.1 in [5].

5. Uniqueness on Euclidean space

Proof of Theorem 1.6. By Theorem 1.3, the Ricci flow in question is $L$-complete (in the second case, we also assume $T \leq \overline{T}(\alpha, \gamma)$ for the moment). Then Theorem 1.5 implies that the Ricci flow is one with bounded curvature. The conclusion then follows from [6].

We remark here that in case (2), once we established the strong uniqueness assuming $T \leq \overline{T}(\alpha, \gamma)$, then the same result holds also for any $T > 0$ by splitting $[0, T]$ into smaller intervals, each with length no greater than $\overline{T}(\alpha, \gamma)$. □

References

[1] R. Bamler. Entropy and heat kernel bounds on a Ricci flow background. ArXiv preprint. arXiv:2008.07093.
[2] R. Bamler. Structure theory of non-collapsed limits of Ricci flows. ArXiv preprint. arXiv:2009.03243.
[3] Pak-Yeung Chan, Zilu Ma, Yongjia Zhang. A local Sobolev inequality on Ricci flow and its applications, https://arxiv.org/abs/2111.05517v2
[4] Chau, Albert, Tam, Luen-Fai, and Yu, Chengjie (2011). Pseudolocality for the Ricci Flow and Applications. Canadian Journal of Mathematics, 63(1), 55-85. doi:10.4153/CJM-2010-076-2
[5] Chen, Bing-Long. Strong uniqueness of the Ricci flow. Journal of Differential Geometry, 2009, 82(2):363-382.
[6] Chen, Bing-Long, and Xi-Ping Zhu. Uniqueness of the Ricci flow on complete noncompact manifolds. Journal of Differential Geometry, 2006, 74(1): 119-154.
[7] Cheng, Liang, and Yongjia Zhang. Perelman-type no breather theorem for noncompact Ricci flows. Transactions of the American Mathematical Society 374.11 (2021): 7991-8012.
[8] Chen, Xinxiong, and Bing Wang. On the conditions to extend Ricci flow (III). International Mathematics Research Notices 2013.10 (2013): 2349-2367.
[9] Chow, B., Chu, S-C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci flow: techniques and applications. Geometric Analysis and Geometry, Mathematical Surveys and Monographs, vol.163, AMS, Providence, RI, 2010.
[10] Chow, B.: Entropy and heat kernel bounds on a Ricci flow background. ArXiv preprint. arXiv:2008.07093.
[11] Kotschwar, Brett L. Backwards uniqueness for the Ricci flow. International Mathematics Research Notices 2010, 21: 4064-4097.
[12] Peng Lu, Gang Tian. Uniqueness of standard solutions in the work of Perelman. https://math.berkeley.edu/~lott/ricciflow/StanUniqWork2.pdf
[13] G.Perelman, The entropy formula for the Ricci flow and its geometric applications. http://arxiv.org/abs/math/0211159
[14] G.Perelman, Ricci flow with surgery on three-manifolds. http://arxiv.org/abs/math/0303109v1
[15] G.Perelman, Finite time extinction for the solutions to the Ricci flow on certain three-manifold. http://arxiv.org/abs/math/0307245
[16] Sesum, Natasa. Limiting behavior of Ricci flows. Massachusetts Institute of Technology, doctoral thesis.
[17] Shi, Wan-Xiong. Deforming the metric on complete Riemannian manifolds. J. Differential Geom. 1989, 30(1):223-301.
[24] G. Tian, and B. Wang, *On the structure of almost Einstein manifolds*, J. Am. Math. Soc. 28(2015), no. 4, 1169-1209.

[25] Wang, Bing. *On the conditions to extend Ricci flow*. International Mathematics Research Notices 2008.9 (2008): rnn012-rnn012.

[26] Wang, Bing. *On the conditions to extend Ricci flow (II)*. International Mathematics Research Notices 2012.14 (2012): 3192-3223.

[27] Wang, Bing. *The local entropy along Ricci flow—Part B: the pseudo-locality theorems*. arXiv preprint arXiv:2010.09981 (2020).

[28] Ye, Rugang. *On the $l$-Function and the Reduced Volume of Perelman I*. Transactions of the American Mathematical Society, 2008, 360(1):507-531.

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