A singular perturbation problem

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Abstract

Consider the equation
\[-\varepsilon^2 \Delta u_\varepsilon + q(x)u_\varepsilon = f(u_\varepsilon) \text{ in } \mathbb{R}^3, \quad |u(\infty)| < \infty, \quad \varepsilon = \text{const} > 0.\]
Under what assumptions on $q(x)$ and $f(u)$ can one prove that the solution $u_\varepsilon$ exists and $\lim_{\varepsilon \to 0} u_\varepsilon = u(x)$, where $u(x)$ solves the limiting problem $q(x)u = f(u)$? These are the questions discussed in the paper.

1 Introduction

Let
\[-\varepsilon^2 \Delta u_\varepsilon + q(x)u_\varepsilon = f(u_\varepsilon) \text{ in } \mathbb{R}^3, \quad |u_\varepsilon(\infty)| < \infty, \quad \varepsilon = \text{const} > 0, \quad f \text{ is a nonlinear smooth function, } q(x) \in C(\mathbb{R}^3) \text{ is a real-valued function, } \]
\[a^2 \leq q(x), \quad a = \text{const} > 0.\]

We are interested in the following questions:
1) Under what assumptions does problem \eqref{eq:1.1} have a solution?
2) When does $u_\varepsilon$ converge to $u$ as $\varepsilon \to 0$?

Here $u$ is a solution to
\[q(x)u = f(u).\]

The following is an answer to the first question.

Theorem 1.1. Assume $q \in C(\mathbb{R}^3)$, \eqref{eq:1.2} holds, $f(0) \neq 0$, and $a$ is sufficiently large (see \eqref{eq:2.7} and \eqref{eq:2.9} below). Then equation \eqref{eq:1.1} has a solution $u_\varepsilon \neq 0$, $u_\varepsilon \in C(\mathbb{R}^3)$, for any $\varepsilon > 0$.

In Section 4 the potential $q$ is allowed to grow at infinity.

An answer to the second question is:

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Theorem 1.2. If \( f'(u) \) is a monotone, growing function, such that \( f'(u) \to \infty \) and \( \min_{u \geq u_0} f'(u) < a^2 \), where \( u_0 > 0 \) is a fixed number, then there is a solution \( u_\varepsilon \) to (1.1) such that
\[
\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x),
\] (1.4)
where \( u(x) \) solves (1.3).

Singular perturbation problems have been discussed in the literature [1], [3], [4]. In Section 2 proofs are given. In Section 3 an alternative approach is proposed. In Section 4 an extension of the results to a larger class of potentials is given.

2 Proofs

Proof of Theorem 1.1. The existence of a solution to (1.1) is proved by means of the contraction mapping principle.

Let \( g \) be the Green’s function
\[
(-\varepsilon^2 \Delta + a^2)g = \delta(x-y) \text{ in } \mathbb{R}^3, \quad g := g_a(x, y, \varepsilon) \to 0, \quad g = \frac{e^{-\frac{a}{\varepsilon}|x-y|}}{4\pi|x-y|\varepsilon^2}. \quad (2.1)
\]
Let \( p := q - a^2 \geq 0 \). Then (1.1) can be written as:
\[
u_\varepsilon(x) = -\int_{\mathbb{R}^3} gp_{\varepsilon}dy + \int_{\mathbb{R}^3} gf(u_\varepsilon)dy := T(u_\varepsilon). \quad (2.2)
\]
Let \( X = C(\mathbb{R}^3) \) be the Banach space of continuous and globally bounded functions, \( B_R := \{v : \|v\| \leq R\} \), and \( \|v\| := \sup_{x \in \mathbb{R}^3} |v(x)| \).

We choose \( R \) such that
\[
T(B_R) \subset B_R \quad (2.3)
\]
and
\[
\|T(v) - T(w)\| \leq \gamma\|v - w\|, \quad v, w \in B_R, \quad 0 < \gamma < 1. \quad (2.4)
\]
If (2.3) and (2.4) hold, then the contraction mapping principle yields a solution \( u_\varepsilon \in B_R \) to (2.2), and, therefore, to problem (1.1).

The assumption \( f(0) \neq 0 \) guarantees that \( u_\varepsilon \neq 0 \).

Let us check (2.3). If \( \|v\| \leq R \), then
\[
\|T(v)\| \leq \|v\| \|p\| \int_{\mathbb{R}^3} g(x, y)dy + \frac{M(R)}{a^2} \leq \|p\|R + \frac{M(R)}{a^2} \leq R, \quad (2.5)
\]
where \( M(R) := \max_{|u| \leq R} |f(u)| \). Here we have used the following estimate:
\[
\int_{\mathbb{R}^3} g(x, y)dy = \int_{\mathbb{R}^3} \frac{e^{-\frac{a}{\varepsilon}|x-y|}}{4\pi|x-y|\varepsilon^2} dy = \frac{1}{a^2}. \quad (2.6)
\]
If $\|p\| < \infty$ and $a$ is such that
\[
\frac{\|p\| \cdot R + M(R)}{a^2} \leq R, \tag{2.7}
\]
then (2.3) holds.

Let us check (2.4). Assume that $v, w \in B_R$, $v - w := z$. Then
\[
\|T(v) - T(w)\| \leq \frac{\|p\|}{a^2} \|z\| + \frac{M_1(R)}{a^2} \|z\|, \tag{2.8}
\]
where $M_1(R) = \max_{|w| \leq R} \frac{|f'(u + sw)|}{0 \leq s \leq 1}$. If
\[
\frac{\|p\| + M_1(R)}{a^2} \leq \gamma < 1, \tag{2.9}
\]
then (2.4) holds. By the contraction mapping principle, (2.7) and (2.9) imply the existence and uniqueness of the solution $u_\varepsilon(x)$ to (1.1) in $B_R$ for any $\varepsilon > 0$.

Theorem 1.1 is proved. \hfill \Box

Proof of Theorem 1.2. In the proof of Theorem 1.1 one can choose $R$ and $\gamma$ independent of $\varepsilon > 0$. Let us denote by $T_\varepsilon$ the operator defined in (2.2). Then (see Remark 2.2) one has
\[
\lim_{\varepsilon \to 0} \|T_\varepsilon(v) - T_0(v)\| = 0, \tag{2.10}
\]
where
\[
T_0(v) = -pv + f(v). \tag{2.11}
\]
It is known \cite{2} and easy to prove (see Remark 2.3) that if (2.10) holds for every $v \in X$, and $\gamma$ in (2.4) does not depend on $\varepsilon$, then (1.4) holds, where $u$ solves the limiting equation (2.2):
\[
u = T_0(u) = -pu + f(u). \tag{2.12}
\]
Equation (2.12) is equivalent to (1.3). Theorem 1.2 is proved. \hfill \Box

Remark 2.1. Conditions of Theorem 1.1 and Theorem 1.2 are satisfied if, for example, $q(x) = a^2 + 1 + \sin(\omega x)$, where $\omega = \text{const} > 0$, $f(u) = (u + 1)^m$, $m > 1$, or $f(u) = e^u$.

Remark 2.2. Note that in the distribution sense
\[
g_{\varepsilon}(x, y, \varepsilon) \to \frac{1}{a^2} \delta(x - y), \quad \varepsilon \to 0. \tag{2.13}
\]

Remark 2.3. Let $u = T_\varepsilon(u)$, $v = T_{\varepsilon_0}(v) := T_0(v)$, and $T_\varepsilon(w) \to T_0(w)$ for all $w \in X$, $\|T_\varepsilon(v) - T_\varepsilon(w)\| \leq \gamma \|v - w\|$, $0 < \gamma < 1$, $\gamma$ does not depend on $\varepsilon$, $u_{n+1} = T_\varepsilon(u_n)$, $u_0 = v$. Then $u_1 = T_\varepsilon v$, and $\|u_n - v\| \leq \frac{1}{\gamma} \|u_1 - v\|$. Taking $n \to \infty$, one gets $\|u - v\| \leq \frac{1}{1-\gamma} \|T_\varepsilon(v) - T_0(v)\| \to 0$ as $\varepsilon \to \varepsilon_0$. 

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3 A different approach

Let us outline a different approach to problem (1.1). Set \( x = \xi + \varepsilon y \). Then

\[-\Delta y w_\varepsilon + a^2 w_\varepsilon + p(\varepsilon y + \xi)w_\varepsilon = f(w_\varepsilon), \quad |w_\varepsilon(\infty)| < \infty, \quad (3.1)\]

\( w_\varepsilon := u_\varepsilon(\varepsilon y + \xi), \quad p := q(\varepsilon y + \xi) - a^2 \geq 0. \) Thus

\[ w_\varepsilon = -\int_{\mathbb{R}^3} G(x, y)p(\varepsilon y + \xi)w_\varepsilon \, dy + \int_{\mathbb{R}^3} G(x, y)f(w_\varepsilon) \, dy, \quad (3.2)\]

where

\[ (-\Delta + a^2)G = \delta(x - y) \text{ in } \mathbb{R}^3, \quad G = \frac{e^{-a|x-y|}}{4\pi|x-y|}, \quad a > 0. \tag{3.3} \]

One has

\[ \int_{\mathbb{R}^3} G(x, y) \, dy = \frac{1}{a^2}. \tag{3.4} \]

Using an argument similar to the one in the proofs of Theorem 1.1 and Theorem 1.2 one concludes that for any \( \varepsilon > 0 \) and any sufficiently large \( a \), problem (3.1) has a unique solution, which tends to a limit \( w = w(y, \xi) \) as \( \varepsilon \to 0 \), where \( w \) solves the problem

\[ -\Delta y w + q(\xi)w = f(w), \quad |w(\infty, \xi)| < \infty. \tag{3.5} \]

Problem (3.5) has an obvious solution \( w = w(\xi) \), which is independent of \( y \) and solves the equation

\[ q(\xi)w = f(w). \tag{3.6} \]

The solution to (3.5) is unique if \( a \) is sufficiently large. This is proved similarly to the proof of (2.9). Namely, let \( b^2 := q(\xi) \). Note that \( b \geq a \). If there are two solutions to (3.5), say \( w \) and \( v \), and if \( z := w - v \), then \( ||z|| \leq b^{-2}M_1(R)||z|| < ||z|| \), provided that \( b^{-2}M_1(R) < 1 \). Thus \( z = 0 \), and the uniqueness of the solution to (3.5) is proved.

Replacing \( \xi \) by \( x \) in (3.6), we obtain the solution found in Theorem 1.2.

4 Extension of the results to a larger class of potentials

Here a method for a study of problem (1.1) for a larger class of potentials \( q(x) \) is given. We assume that \( q(x) \geq a^2 \) and can grow to infinity as \( |x| \to \infty \). Note that in Sections 1 and 2 the potential was assumed to be a bounded function. Let \( g_\varepsilon \) be the Green’s function

\[ -\varepsilon^2 \Delta g_\varepsilon + q(x)g_\varepsilon = \delta(x - y) \text{ in } \mathbb{R}^3, \quad |g_\varepsilon(\infty, y)| < \infty. \tag{4.1} \]

As in Section 2 problem (1.1) is equivalent to

\[ u_\varepsilon = \int_{\mathbb{R}^3} g_\varepsilon f(u_\varepsilon(y)) \, dy, \tag{4.2} \]
and this equation has a unique solution in $B_R$ if $a^2$ is sufficiently large. The proof, similar to the one given in Section 2, requires the estimate

$$\int_{\mathbb{R}^3} g_\varepsilon(x, y) dy \leq \frac{1}{a^2}.$$  \hfill (4.3)

Let us prove the above inequality. Let $G_j$ be the Green’s function satisfying equation (4.1) with $q = q_j$, $j = 1, 2$. Estimate (4.3) follows from the inequality

$$G_1 \leq G_2 \quad \text{if} \quad q_1 \geq q_2.$$  \hfill (4.4)

This inequality can be derived from the maximum principle. If $q_2 = a^2$, then $G_2 = \frac{\varepsilon^{-|x-y|}}{4\pi|x-y|^2}$, and the inequality $g_\varepsilon(x, y) \leq \frac{\varepsilon^{-|x-y|}}{4\pi|x-y|^2}$ implies (4.3).

We prove below the following relation:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} g_\varepsilon(x, y) h(y) dy = \frac{h(x)}{q(x)} \quad \forall h \in \mathcal{C}^\infty(\mathbb{R}^3),$$  \hfill (4.5)

where $\mathcal{C}^\infty$ is the set of $C^\infty(\mathbb{R}^3)$ functions vanishing at infinity together with their derivatives. This formula is an analog to (2.13).

To prove (4.5), multiply (4.1) by $h(y)$, integrate over $\mathbb{R}^3$ with respect to $y$, then integrate the first term by parts, and then let $\varepsilon \to 0$. The result is (4.5).

Thus, Theorem 1.1 and Theorem 1.2 remain valid for $q(x) \geq a^2$, $a > 0$ sufficiently large, $\frac{f(u)}{u}$ monotonically growing to infinity, and the solution $u(x)$ to the limiting equation (1.3) is the limit of the solution to (4.2) as $\varepsilon \to 0$.

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