Asymptotic Critical Transmission Radii in Wireless Networks over a Convex Region

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Abstract

Critical transmission ranges (or radii) in wireless ad-hoc and sensor networks have been extensively investigated for various performance metrics such as connectivity, coverage, power assignment and energy consumption. However, the regions on which the networks are distributed are typically either squares or disks in existing works, which seriously limits the usage in real-life applications. In this article, we consider a convex region (i.e., a generalisation of squares and disks) on which wireless nodes are uniformly distributed. We have investigated two types of critical transmission radii, defined in terms of $k$–connectivity and the minimum vertex degree, respectively, and have also established their precise asymptotic distributions. These make the previous results obtained under the circumstance of squares or disks special cases of this work. More importantly, our results reveal how the region shape impacts on the critical transmission ranges: it is the length of the boundary of the (fixed-area) region that completely determines the transmission ranges. Furthermore, by isodiametric inequality, the smallest critical transmission ranges are achieved when regions are disks only.

Index Terms

Random geometry graph, Critical transmission radius, Asymptotic distribution, Convex region

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I. Introduction

Let $\chi_n$ be a uniform $n$-point process over a unit-area convex region $\Omega$, i.e., a set of $n$ independent points each of which is uniformly distributed over $\Omega$, and every pair of points whose Euclidean distance less than $r_n$ is connected with an undirected edge. So a random geometric graph $G(\chi_n, r_n)$ is obtained. Random geometric graphs are a basic model of wireless sensor and ad-hoc networks, and have drawn significant attention [1]–[13].

$k$–connectivity and the smallest vertex degree are two interesting topological properties of a random geometry graph. A graph $G$ is said to be $k$–connected if there is no set of $k−1$ vertices whose removal would disconnect the graph. Denote by $\kappa$ the connectivity of $G$, being the maximum $k$ such that $G$ is $k$–connected. The minimum vertex degree of $G$ is denoted by $\delta$. Let $\rho(\chi_n; \kappa \geq k)$ be the minimum $r_n$ such that $G(\chi_n, r_n)$ is $k$–connected and $\rho(\chi_n; \delta \geq k)$ be the minimum $r_n$ such that $G(\chi_n, r_n)$ has the smallest degree $k$, respectively.

It is known that various network performance metrics, such as coverage, capacity, throughput and energy consumption, depend on the critical transmission range (or radius) defined in terms of $\rho(\chi_n; \kappa \geq k)$ and $\rho(\chi_n; \delta \geq k)$. Indeed, critical transmission or sensing ranges have been extensively investigated, e.g., for connectivity in mobile ad-hoc networks [3], for coverage in wireless sensor networks [4] and in mobile heterogeneous camera sensor networks [5], [6], for capacity in large scale sensor networks [7], for routing in wireless ad-hoc networks [8], [9], for power assignment in wireless ad-hoc networks [10], and for energy consumption control in mobile heterogeneous wireless sensor networks [6]. Further, the effect of mobility on the critical transmission range for asymptotic connectivity in $k$–hop clustered networks has also been studied [11].

However, the regions on which the networks are distributed, considered in all the above mentioned works, are usually unit-area squares or disks. Here the area of considered regions is always assumed to be unit one. It has been indicated by Wan et al. in [12], [13], that the shape of the region has impacts on the critical transmission radii $\rho(\chi_n; \kappa \geq k)$ and $\rho(\chi_n; \delta \geq k)$, by comparing their precise asymptotic distributions on squares and disk regions. In realistic scenarios, the network nodes are in general placed in a more general region rather than a simple square or disk. Therefore, the assumption of square or disk regions limits the application of those existing works on the analysis of critical transmission ranges [3]–[9], [11]–[13].

In this article, we instead consider a more general unit-area convex region $\Omega$, on which $n$ wireless sensors are uniformly distributed. The critical transmission or sensing ranges $\rho(\chi_n; \kappa \geq k)$ and $\rho(\chi_n; \delta \geq k)$ over the unit-area convex region are investigated. In particular, we will establish the precise asymptotic
distributions of these two types of radii, which makes the previous results obtained under the circumstance of squares or disks special cases of our work. More importantly, our results will reveal how the region shape impacts on the critical transmission ranges. That is, it is only the length of the boundary of the region that determines the critical transmission ranges. Furthermore, by isodiametric inequality, the smallest asymptotic critical transmission radii are achieved only when the region $\Omega$ is a disk.

We use the following notations throughout this article. (1) Region $\Omega \subset \mathbb{R}^2$ is a unit-area convex region on a plane, and $B(x, r) \subset \mathbb{R}^2$ is a ball centered at $x$ with radius $r$. A bounded set means that this set can be included in a ball with a finite radius. (2) Notation $|A|$ is a shorthand for the 2-dimensional Lebesgue measure (i.e., area) of a measurable set $A \subset \mathbb{R}^2$. All integrals considered will be Lebesgue integrals. (3) Notation $\| \cdot \|$ represents the length of a line segment. For instance, $\|BC\|$ is the length of line segment $BC$. (4) Given any two nonnegative functions $f(n)$ and $g(n)$, if there exist two constants $c_1$ and $c_2$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for sufficiently large $n$ where $0 < c_1 < c_2$.

| Notations | Descriptions |
|-----------|--------------|
| $\Omega$ | An unit-area convex region on $\mathbb{R}^2$ |
| $\partial \Omega$ | The boundary of region $\Omega$ |
| $\chi_n$ | A uniform $n$-point process over $\Omega$ |
| $P_n$ | A homogeneous Poisson process of intensity $n$ (i.e., $n|\Omega|$) on $\Omega$ |
| $G(\chi_n, r_n)$ | A graph obtained by connecting each pair of points in $\chi_n$ if there distance is less than $r_n$ |
| $\rho(\chi_n; \kappa \geq k)$ | The minimum $r_n$ such that $G(\chi_n, r_n)$ is $k$-connected |
| $\rho(\chi_n; \delta \geq k)$ | The minimum $r_n$ such that $G(\chi_n, r_n)$ has the smallest degree $k$ |
| $\rho(P_n; \delta \geq k)$ | The minimum $r_n$ such that $G(P_n, r_n)$ has the smallest degree $k$ |
| $G(\Omega)$ | An uniform upper bound of all second-order derivatives of curve $\partial \Omega$ in local coordinates (see Remark 2) |
| $B(x, r)$ | A ball centered at $x$ with radius $r$ |
| $| \cdot |$ | Area of a measurable set on $\mathbb{R}^2$ |
| $\| \cdot \|$ | Length of a line segment |
| $\psi^k_{n,r}(x)$ | Short for $\frac{1}{k!} \left( n B(x, r) \cap \Omega \right)^k \exp\left( \frac{n}{k} B(x, r) \cap \Omega \right)$ |
| $a(r, t)$ (or shortly, $a(t)$) | Denotes the area $\{ x = (x_1, x_2) : x_1^2 + x_2^2 \leq r^2, x_1 \leq t \}$ |
| $\Pr(\cdot), \Pr[\cdot], \Pr\{\cdot\}$ | The probability of an event |

| $f(n) = \Theta(g(n))$ | $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for sufficiently large $n$ where $0 < c_1 < c_2$ |
| $f(n) = o(g(n))$ | $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ |
| $f(n) \sim g(n)$ | $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$ |
0 < c_1 < c_2 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for sufficiently large } n, \text{ then we denote } f(n) = \Theta(g(n)).

We also use notations \( f(n) = o(g(n)) \) and \( f(n) \sim g(n) \) to denote that \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \) and \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \), respectively. For convenience, the notations with descriptions are presented in Table I, some of which will be introduced later.

The remainder of this article is structured as follows. We first present our main results and related work in Section II. Preliminary results, i.e. some lemmas, are provided in Section III. The proof of the main result is divided into two parts, which are demonstrated in Sections IV and V respectively. We finally conclude this article in Section VI.

II. MAIN RESULTS AND RELATED WORK

There are quite a few studies on the precise asymptotic distribution of the critical transmission radius in terms of \( \rho(\chi_n; \delta \geq k) \) and \( \rho(\chi_n; \kappa \geq k) \). In this section, after reviewing closely related work, we shall provide our main results on the precise asymptotic distributions.

A. Related Work

When regions \( \Omega \) are squares, the precise asymptotic distribution of \( \rho(\chi_n; \delta \geq 1) \) has been derived by Dette and Henze in [14] as follows,

\[
\Pr \{ \rho(\chi_n; \delta \geq 1) \leq r_n \} \sim \exp \left( -e^{-c} \right),
\]

where \( c \) is a positive constant and

\[
r_n = \sqrt{\frac{\log n + c}{\pi n}}.
\]

When regions \( \Omega \) is a disk and \( k \)-connectivity is concerned, this \( r_n \) also satisfies the following

\[
\Pr \{ \rho(\chi_n; \kappa \geq 1) \leq r_n \} \sim \exp \left( -e^{-c} \right),
\]

which gives the precise asymptotic distribution of \( \rho(\chi_n; \kappa \geq 1) \). This was given by Gupta and Kumar in [15]. It has been proved by Penrose in [16] that this critical transmission radius for connectivity equals to the longest edge of the minimal spanning tree, and he has further provided the asymptotic distribution of the longest edge in [17]. In addition, a type of \((k,m)\)–connectivity was defined by Wang in his monograph [18], together with the precise asymptotic distribution of the critical transmission range over a square region.

For a square region \( \Omega \), Penrose has further demonstrated the following result:

**Theorem 1:** ([1], [19]) Let \( k > 0, \lambda \in \mathbb{R}^+ \), \( \Omega \) be a unit-area square. If sequence \( (r_n)_{n \geq 1} \) satisfies

\[
\frac{n}{k!} \int_{\Omega} (n|B(x,r_n) \cap \Omega|)^k e^{-n|B(x,r_n) \cap \Omega|} \, dx \sim \lambda,
\]

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then as $n \to \infty$, the probabilities of the two events $\rho(\chi_n; \delta \geq k + 1) \leq r_n$ and $\rho(\chi_n; \kappa \geq k + 1) \leq r_n$ both converge to $e^{-\lambda}$.

The formula (3) is derived through Poisson approximation and de-Poissonized techniques, exploited by Penrose in the proof of this theorem. Let $\mathcal{P}_n$ be a homogeneous Poisson process of intensity $n$ (i.e., $n|\Omega|$) on $\Omega$, then $\mathcal{P}_n(\Omega) = \sum_{i=1}^m \mathcal{P}_n(A_i)$ where $\Omega = \bigcup_{i=1}^m A_i$ is a union of disjoint of $A_i \subset \Omega$ ($i = 1, 2, \cdots, m$), $\mathcal{P}_n(\Omega)$ is a Poisson random variable with intensity $n|\Omega|$, and $\mathcal{P}_n(A_i)$ ($i = 1, 2, \cdots, m$) are mutually independent Poisson random variables with intensities $n|A_i|$ ($i = 1, 2, \cdots, m$) respectively. That is, the number of the points in $A_i$, i.e., $\mathcal{P}_n(A_i)$, satisfies the following Poisson distribution:

$$\Pr\{\mathcal{P}_n(A_i) = j\} = \frac{e^{-n|A_i|}(n|A_i|)^j}{j!}, \quad j \geq 0.$$ 

Given the condition that there are exact $j$ points in any subregion $A \subset \Omega$, then these $j$ points are independently and uniformly distributed in $A$. This fact makes $\chi_n$ can be well approximated by $\mathcal{P}_n$.

Here Penrose assumed that $\Omega$ was a square. In [12] and [13], Wan et al. have indicated, without a proof, that this theorem also holds for a unit-area disk. Furthermore, they have provided the explicit forms of $r_n$ satisfying formula (3) when $\Omega$ is a square and disk, respectively. See the following two theorems.

**Theorem 2:** (12), (13) Assume that $\Omega \subset \mathbb{R}^2$ is a unit-area square. Let

$$r_n = \sqrt{\frac{\log n + (2k - 1) \log \log n + \xi}{\pi n}},$$

where $\xi$ satisfies

$$\xi = \begin{cases} 
-2 \log \left( \sqrt{e^{-c} + \frac{\pi}{4}} - \frac{\sqrt{\pi}}{2} \right), & k = 1, \\
2 \log \frac{\sqrt{\pi}}{\sqrt{e^{-c} + \frac{\pi}{4}}} + 2c, & k > 1,
\end{cases}$$

then formula (3) holds.

**Theorem 3:** (12), (13) Assume that $\Omega \subset \mathbb{R}^2$ is a unit-area disk. Let

$$r_n = \sqrt{\frac{\log n + (2k - 1) \log \log n + \xi}{\pi n}},$$

where $\xi$ satisfies

$$\xi = \begin{cases} 
-2 \log \left( \sqrt{e^{-c} + \frac{\pi^2}{16}} - \frac{\pi}{4} \right), & k = 1, \\
2 \log \frac{\pi}{\sqrt{e^{-c} + \frac{\pi^2}{16}}} + 2c, & k > 1.
\end{cases}$$

then formula (3) holds.

As long as formula (3) is satisfied, the probabilities of the two events $\rho(\chi_n; \delta \geq k + 1) \leq r_n$ and $\rho(\chi_n; \kappa \geq k + 1) \leq r_n$ both converge to $e^{-\lambda}$ as $n$ tends to infinity, followed by Theorem 1.

However, in realistic scenarios the regions on which networks are distributed, are typically more general than disks and squares. Unfortunately, to the best of our knowledge, there are no results on the topic of
the critical transmission radius, obtained under the circumstance of a more general region, e.g. a convex region.

B. Our Main Results

Whether Theorem 1 holds for a general convex region remains an open problem (for more details, please see page 176 in [1]). However, Penrose has pointed out that, without a proof in his monograph [1], Theorem 1 should also hold for the case of polyhedral domain. The contribution of this article is to extend the conclusion of Theorem 1 to a general convex region case, with additionally providing an explicit form of \( r_n \). See the following Theorem 4.

**Theorem 4:** Let \( \Omega \subset \mathbb{R}^2 \) be a unit-area convex region such that the length of the boundary \( \partial \Omega \) is \( l \), \( k \geq 0 \) be an integer and \( c > 0 \) be a constant.

(i) If \( k > 0 \), let

\[
    r_n = \sqrt{\frac{\log n + (2k - 1) \log \log n + \xi}{\pi n}},
\]

where \( \xi \) satisfies

\[
    \begin{cases} 
        \xi = -2 \log \left( \sqrt{e^{-c} + \frac{\pi^2}{64} - \frac{l}{8 \pi}} \right), & k = 1, \\
        \xi = 2 \log \left( \frac{l \sqrt{\pi}}{2^{k+1} \pi} \right) + 2c, & k > 1. 
    \end{cases}
\]

(ii) If \( k = 0 \), let

\[
    r_n = \sqrt{\frac{\log n + c}{\pi n}}.
\]

Then

\[
    \frac{n}{k!} \int_{\Omega} (n|B(x, r_n) \cap \Omega|)^k e^{-n|B(x, r_n) \cap \Omega|} dx \sim e^{-c},
\]

and therefore, the probabilities of the two events \( \rho(\chi_n; \delta \geq k + 1) \leq r_n \) and \( \rho(\chi_n; \kappa \geq k + 1) \leq r_n \) both converge to \( \exp(-e^{-c}) \) as \( n \to \infty \).

This theorem generalise the previous conclusions presented in Theorems 1, Theorem 2 and Theorem 3, and makes them special cases. In particular, in Theorem 4 if the region \( \Omega \) is a unit-area square (disk, respectively), then \( l = 4 \) \( (l = 2\sqrt{\pi}, \text{respectively}) \), and then the determined \( r_n \) coincides with the one presented in Theorem 2 (Theorem 3, respectively) with a simple calculation.

In [12], [13], Wan et al. mentioned that in Theorems 2 and 3, “\( \xi \) depends on the shape of \( \Omega \) and parameter \( k \)”. Our conclusions, i.e., Theorem 4 clearly indicates that the asymptotic radius is completely and only determined by the length \( l \) of the boundary \( \partial \Omega \) and the parameter \( k \). These findings have never been revealed before, to the best of our knowledge. In comparison, the claim of Wan et al. is rather rough because regions with different shapes may have the same perimeters. In addition, only two specific regions are examined in their work.

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Furthermore, Theorem 4 shows that smaller the perimeter is, the smaller the asymptotic critical radius is. By the following isoperimetric inequality on $\mathbb{R}^2$ [20]:
\[ l^2 \geq 4\pi S(\Omega) = 4\pi, \]
where $S(\Omega) = 1$ is the area of $\Omega$, and the equality only holds for a disk, we know that a disk has the smallest perimeter among all equal-area regions. From this, we have the following

**Corollary 1:** The smallest asymptotic critical radius is achieved only when the region $\Omega$ is a disk.

It is worth pointing out that when $k = 0$, i.e., when one minimum vertex degree or 1-connectivity is concerned, the shape of the region has no impacts on the radius. This conclusion also coincides with the results (1) and (2), given by Dette et al. and Gupta et al. respectively.

**Remark 1:** However, if energy saving for networks is taken into account, and the minimum degree is greater than two or $k-$connectivity ($k \geq 2$) is considered, the shape that the distributed network nodes formed should be like a “disk”, because the disk shape can lead to the smallest critical transmission ranges. Otherwise, for example, suppose $n$ nodes are placed uniformly in a rectangle with width $\epsilon$ and length $\frac{1}{\epsilon}$, the critical transmission range must be quite large to guarantee the network to be $k-$connected ($k \geq 2$), if $\epsilon$ is small (i.e., the perimeter $2\epsilon + \frac{2}{\epsilon}$ is large). Therefore, our theoretical results have a practical meaning as well.

The proof of Theorem 4 will be divided into two parts. The first part is to prove that $r_n$ given by (4) and (5) satisfies
\[ \frac{n}{k!} \int_\Omega (n|B(x, r_n) \cap \Omega|)^k e^{-\frac{n|B(x, r_n) \cap \Omega|}{k}} dx \sim e^{-e^{-c}}, \]
which will be demonstrated in Section IV. The second part, i.e., the probabilities of the two events $\rho(\chi_n; \delta \geq k + 1) \leq r_n$ and $\rho(\chi_n; \kappa \geq k + 1) \leq r_n$ both converge to $\exp \left( -e^{-c} \right)$ as $n \to \infty$, will be proved in Section V At first, we give some preliminary results in the following section.

### III. Preliminary Results

**Lemma 1:** Let $\Omega$ be a bounded convex region, then there exists a positive constant $C$ such that for any sufficiently small $r$,
\[ \inf_{x \in \Omega} |B(x, r) \cap \Omega| \geq C\pi r^2. \]
**Proof:** See Appendix A. □

**Lemma 2:** Two disks centered at $A$ and $B$ respectively, have the same radii $r$, with the distance $\|AB\| = d$. The shadow area $S$ in Figure I is
\[ S = r^2 \left[ \frac{1}{2} \frac{d}{r} + \frac{19}{48} \left( \frac{d}{r} \right)^3 + \frac{153}{1280} \left( \frac{d}{r} \right)^5 - o \left( \frac{d}{r} \right)^5 \right]. \]
In particular, if \( r \geq d \geq \frac{r}{(nr^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{n}} \), then
\[
nS \geq \frac{1}{\sqrt{n}} \left( n\pi r^2 \right)^{\frac{1}{2}}.
\]

**Proof:** See Appendix A. \( \square \)

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**Figure 1.** Illustration of Lemma 2: area \( S \)

The next lemma needs the condition of smooth boundary of \( \Omega \). Throughout this article, the boundary \( \partial \Omega \) is smooth means that the function of curve \( \partial \Omega \) on \( \mathbb{R}^2 \) has second-order continuous derivatives everywhere.

Now we assume region \( \Omega \) has a smooth boundary \( \partial \Omega \). Let \( A, B \) be any two points on \( \partial \Omega \). Consider the \( X \)-axis in a coordinate system, which is parallel to line \( AB \) and tangent to it at point \( O \). Notice that the slope of the curve \( \partial \Omega \) increases from a negative value at point \( B \) to a positive value at point \( A \), and is zero at point \( O \). We also find a rectangle \( ABCD \), whose side \( CD \) is on \( X \)-axis and tangent to \( \partial \Omega \) at point \( O \). See Figure 2. The rectangle \( ABCD \) satisfies the following property:

**Figure 2.** \( \partial \Omega \) is tangent to \( CD \) at point \( O \)

**Lemma 3:** Denote \( h \) by the length of line segment \( AB \). Then \( S_{ABCD} \), the area of rectangle \( ABCD \), satisfies \( S_{ABCD} = \Theta(h^3) \) or \( S_{ABCD} = o(h^3) \), as \( h \to 0 \).

**Proof:** By the Taylor expansion of \( \partial \Omega \)’s function at tangent point \( O \), we obtain
\[
\|AD\| = \text{constant} \cdot \|OD\|^2 + o(\|OD\|^2) \tag{7}
\]
where $\| \cdot \|$ is the length of a line segment as mentioned, and constant is the second-order derivative of the function of $\partial \Omega$ at point $O$. Here the first-order derivative of $\partial \Omega$’s equation at point $O$ is zero in this coordinate system. Notice that $\|OD\| \leq \|AB\| = h$, so

$$S_{ABCD} = \|AB\| \cdot \|AD\| \leq \text{constant} \cdot h^3 + o(h^3).$$

The lemma is proved. $\square$

Remark 2: By the smoothness of the bounded and closed curve $\partial \Omega$, we may assume that all this kind of constants, i.e., all second-order derivatives of $\partial \Omega$ at all tangent points in all such local coordinate systems, have a uniform upper bound $G(\Omega) > 0$. Therefore, as long as $h$ is sufficiently small we will have that

$$\|OD\| \geq \sqrt{\frac{\|AD\|}{G(\Omega) + 1}}, \quad (8)$$

$$\|OC\| \geq \sqrt{\frac{\|BC\|}{G(\Omega) + 1}} = \sqrt{\frac{\|AD\|}{G(\Omega) + 1}}, \quad (9)$$

which leads to that if $\|AD\| > (G(\Omega) + 1)r^2$, then

$$\|OD\| > r, \quad \|OC\| > r. \quad (10)$$

This conclusion will also be used in the proof of Theorem 4.

The following lemma is first given by Wan et al. in [12]. For any $t \in [0, r]$, we define

$$a(r, t) = |\{x = (x_1, x_2) : x_1^2 + x_2^2 \leq r^2, x_1 \leq t\}|, \quad (11)$$

the area of the shaded region illustrated in Figure 3. We may shortly write it as $a(t)$, then

Lemma 4: (12) If $r = r_n = \sqrt{\frac{\log n + (2k-1) \log \log n + \xi}{\pi n}}$, then

1) If $k > 0$,

$$n \int_0^\xi \frac{(na(t))^k e^{-na(t)}}{k!} dt \sim \sqrt{\frac{\pi}{2k+1}} \frac{e^{-\xi}}{2^k k!}. \quad (12)$$
2) If $k = 0$, 
\[ n \int_{0}^{\frac{r}{2}} \frac{(na(t))^{k}e^{-na(t)}}{k!} dt = o(1). \]

This lemma and its proof can be found in [12] or [13]. For convenience, as we have mentioned, throughout this article we sometimes use $r$ to represent $r_n$. We also assume that $r$ is sufficiently small and $n$ is sufficiently large, if necessarily.

IV. PROOF OF THEOREM 4: PART ONE

In this section, we will present a complete proof for the first part of Theorem 4:

(i) If $k > 0$, let 
\[ r_n = \sqrt{\frac{\log n + (2k - 1) \log \log n + \xi}{\pi n}}, \]
where $\xi$ satisfies
\[
\begin{cases}
\xi = -2 \log \left( \sqrt{e^{-c} + \frac{\pi^2}{64} - \frac{\sqrt{\pi}}{8}} \right), & k = 1, \\
\xi = 2 \log \left( \frac{\sqrt{\pi}}{2^{k+1} \pi} \right) + 2c, & k > 1.
\end{cases}
\]

(ii) If $k = 0$, let 
\[ r_n = \sqrt{\frac{\log n + c}{\pi n}}. \]

Then 
\[
\frac{n}{k!} \int_{\Omega} (n|B(x, r_n) \cap \Omega|)^k e^{-n|B(x, r_n) \cap \Omega|} dx \sim e^{-c}.
\]

The following estimation about $r_n$ given in the theorem is straight forward, and will be often used. We list them as a remark.

**Remark 3:** $r_n$ given in Theorem 4 satisfies

1) $n\pi r^2 \to \infty$ as $n \to \infty$.

2) Let $\xi$ be the one given in Theorem 4 then
\[
\frac{n}{k!} (n\pi r^2)^k e^{-n\pi r^2} \sim \begin{cases} 
    e^{-\xi}, & k = 1, \\
    o(1), & k > 1.
\end{cases}
\]

3) If $C > 0$ is a constant, then
\[
(n\pi r^2)^{k+1} \exp(-C n\pi r^2) \sim \frac{(\log n + o(\log n))^{k+1}}{e^{C(\log n + o(\log n))}} = o(1).
\]

Throughout this article, we define 
\[
\psi_{n,r}^k (x) = \frac{(n|B(x, r) \cap \Omega|)^k e^{-n|B(x, r) \cap \Omega|}}{k!}.
\]

All left work is to prove 
\[
\frac{n}{\Omega} \int \psi_{n,r}^k (x) dx \sim e^{-c}.
\]
The pioneering work by Wan et al. in [12] and [13], i.e., the methods of coping with square and disk regions in the proofs of Theorem 2 and 3, stimulates us to generalise them into a uniform framework to deal with any general convex region. The proof of the above part of Theorem 4 is divided into two cases. The first case is to assume the boundary \( \partial \Omega \) to be smooth. In this case, region \( \Omega \) is divided into four disjoint parts, i.e., \( \Omega = \Omega(0) \cup \Omega(2) \cup \Omega(1, 2) \cup \Omega(1, 1) \). The proof in this case will thus be completed within three steps: Step 1 deals with the integrals on \( \Omega(0) \) and \( \Omega(2) \), Step 2 copes with \( \Omega(1, 2) \) while Step 3 with \( \Omega(1, 1) \). After that, we further treat the continuous boundary case through an approximation approach.

A. Case of smooth \( \partial \Omega \)

In this subsection, we assume the boundary \( \partial \Omega \) is smooth. Let

\[
\begin{align*}
\Omega(0) &= \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq r \}, \\
\Omega(2) &= \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq (G(\Omega) + 1)r^2 \}, \\
\Omega(1) &= \Omega \setminus (\Omega(0) \cup \Omega(2)).
\end{align*}
\]

Here the constant \( G(\Omega) \) is given by Remark 2, and \( \text{dist}(x, \partial \Omega) \), as we mentioned, represents the distance between point \( x \) and curve \( \partial \Omega \). Here \( r \) is considered sufficiently small (in fact, \( r \to 0 \) as \( n \to \infty \)). Therefore, \( (G(\Omega) + 1)r^2 < r \), and thus \( \Omega(0) \) and \( \Omega(2) \) are disjoint. Clearly, we have

\[
n \int_{\Omega} \psi_{n,r}^{k}(x) dx = n \left( \int_{\Omega(0)} + \int_{\Omega(2)} + \int_{\Omega(1)} \right) \psi_{n,r}^{k}(x) dx.
\]

The integrations of \( n\psi_{n,r}^{k}(x) \) on these subregions, i.e., \( \Omega(0), \Omega(2), \Omega(1) \), will be estimated respectively. The first two estimates are rather simple, see the following Proposition 1 and Proposition 2. The estimation of \( n \int_{\Omega(1)} \psi_{n,r}^{k}(x) dx \) will be demonstrated in Proposition 3 and Proposition 4.

**Proposition 1:** The integration of \( \psi_{n,r}^{k}(x) \) defined by [13] on subregion \( \Omega(0) \) satisfies that

\[
n \int_{\Omega(0)} \psi_{n,r}^{k}(x) dx \sim \begin{cases} e^{-\xi}, & k = 1, \\ o(1), & k > 1. \end{cases}
\]


Proof: \( \forall x \in \Omega(0), \ |B(x, r) \cap \Omega| = \pi r^2 \). Notice that \( |\Omega(0)| \sim (1 - lr) \) where \( l \) is the length of \( \partial \Omega \). So, by Remark 3

\[
n \int_{\Omega(0)} \psi^k_{n, r}(x) dx = \frac{n}{k!} (n \pi r^2)^k e^{-n \pi r^2 |\Omega(0)|}
\]

\[
\sim \frac{n}{k!} (n \pi r^2)^k e^{-n \pi r^2}
\]

\[
\sim \begin{cases} 
  e^{-\xi}, & k = 1, \\
  o(1), & k > 1.
\end{cases}
\]

\)

Proposition 2: The integration of \( \psi^k_{n, r}(x) \) defined by (13) on subregion \( \Omega(2) \) satisfies that

\[
n \int_{\Omega(2)} \psi^k_{n, r}(x) dx = o(1).
\]

Proof: By Lemma 1 there exists \( C > 0 \) such that for any \( x \in \Omega \), we have \( |B(x, r) \cap \Omega| \geq C \pi r^2 \). Notice that

\[
|\Omega(2)| \leq l \times (G(\Omega) + 1)r^2 = \Theta(1)r^2,
\]

where \( l \) is the length of the boundary \( \partial \Omega \). Then by Remark 3

\[
n \int_{\Omega(2)} \psi^k_{n, r}(x) dx \leq \frac{n}{k!} (n \pi r^2)^k e^{-Cn \pi r^2 |\Omega(2)|}
\]

\[
= \Theta(1)(n \pi r^2)^k e^{-Cn \pi r^2}
\]

\[
= o(1).
\]

Now we consider the integration of \( n \psi^k_{n, r}(x) \) on subregion \( \Omega(1) \). If \( x \in \Omega(1), \) then

\[
(G(\Omega) + 1)r^2 < \text{dist}(x, \partial \Omega) < r,
\]

and thus the circle centred at \( x \) with radius \( r \) intersects \( \partial \Omega \) at two points, namely \( E \) and \( F \) (see Figure 4 or Figure 5). The distance from \( x \) to chord \( EF \) of the circle is defined by \( t(x, r) \) or shortly \( t(x) \). Furthermore, we set

\[
\Omega(1, 1) = \{ x \in \Omega(1) : t(x) \leq \frac{r}{2} \},
\]

\[
\Omega(1, 2) = \Omega(1) \setminus \Omega(1, 1),
\]

then

\[
n \int_{\Omega(1)} \psi^k_{n, r}(x) dx = n \left( \int_{\Omega(1, 1)} + \int_{\Omega(1, 2)} \right) \psi^k_{n, r}(x) dx.
\]
In following, we will specify $n \int_{\Omega(2)} \psi_{n,r}^k(x) dx$ in Proposition 3, and then determine $n \int_{\Omega(1)} \psi_{n,r}^k(x) dx$ in Proposition 4. But at first, we will establish a lower and upper bound for $|B(x,r) \cap \Omega|$ where $x \in \Omega(1)$, which will be used in the proof of these two propositions.

For point $x \in \Omega(1)$, let $W \in \partial \Omega$ be the nearest point to $x$, i.e.,

$$\|xW\| = \text{dist}(x, \partial \Omega).$$

Let line $CD$ be tangent to $\partial \Omega$ at point $W$, and line $AB$, which parallels $CD$ and passes through point $x$, intersect $\partial \Omega$ at point $A$ and $B$. Here points $C$ and $D$ are chosen to make $ABCD$ be a rectangle. See
Figure 4. By formula (15),
\[ \| AD \| = \| xW \| > (G(\Omega) + 1)r^2, \]
then by formula (10) in Remark 2 we have that
\[ \| xA \| = \| DW \| > r. \]

The distance between point \( x \) and a point \( y \in \partial \Omega \) is a continuous function of \( y \), which is due to the smooth boundary \( \partial \Omega \). Notice that \( \| xW \| < r \) and \( \| xA \| > r \), so there exists a point \( E \) in the segment of \( \partial \Omega \) from \( W \) to \( A \) (in an anticlockwise order), such that the distance between \( x \) and \( E \) is just \( r \), i.e., \( \| xE \| = r \). Similarly, we have another point \( F \in \partial \Omega \) between \( W \) and \( B \) such that \( \| xF \| = r \). That is to say, the circle centred at \( x \) with radius \( r \) intersects \( \partial \Omega \) at points \( E \) and \( F \). So the distance from \( x \) to chord \( EF \), i.e., \( t(x) \), is positive. Thus, \( a(t(x)) \geq \frac{1}{2}\pi r^2 \), and we therefore have a lower bound for \( |B(x, r) \cap \Omega| \):
\[ |B(x, r) \cap \Omega| \geq a(t(x)) \geq \frac{1}{2}\pi r^2. \]

Now we give an upper bound for \( |B(x, r) \cap \Omega| \). Let line \( GH \) be parallel to line \( EF \) and be tangent to \( \partial \Omega \). In particular, two points \( G \) and \( H \) are chosen to make \( EFGH \) be a rectangle. See Figure 5. The area of this rectangle \( S_{EFGH} \leq \Theta(\| EF \|^3) \), according to Lemma 3. Notice that \( \| EF \| \) is less than \( 2r \), the diameter of the circle, so we have
\[ |B(x, r) \cap \Omega| \leq a(t(x)) + S_{EFGH} \leq a(t(x)) + \Theta(\| EF \|^3) \leq a(t(x)) + \Theta(r^3). \]

With \( a(t(x)) \geq \frac{\pi r^2}{2} \) that has been derived above, we obtain
\[ |B(x, r) \cap \Omega| \leq a(t(x)) \left( 1 + \frac{\Theta(r^3)}{a(t(x))} \right) \leq a(t(x))(1 + o(1)). \]

According to these lower and upper bounds for \( |B(x, r) \cap \Omega| \), we have the following estimates:
\[ (n|B(x, r) \cap \Omega|)^k e^{-n|B(x, r) \cap \Omega|} \leq (1 + o(1))^k (na(t(x)))^k e^{-na(t(x))}, \quad (16) \]
and
\[
(n|B(x,r) \cap \Omega|)^k e^{-n|B(x,r) \cap \Omega|} \\
\geq (na(t(x)))^k e^{-n(a(t(x)) + \Theta(r^3))} \\
\geq e^{-n\Theta(r^3)}(na(t(x)))^k e^{-na(t(x))}.
\]

Consider the integration on $\Omega(1,2)$. Noticing that $|\Omega(1,2)| \leq lr$ and $\frac{1}{2}\pi r^2 \leq a(t(x)) \leq \pi r^2$, and by formula (16), we have
\[
n \int_{\Omega(1,2)} \psi_{n,r}^k(x) dx = \frac{n}{k!} \int_{\Omega(1,2)} (n|B(x,r_n) \cap \Omega|)^k e^{-n|B(x,r_n) \cap \Omega|} dx \\
\leq \frac{n}{k!} \left((1 + o(1)) (na(t(x)))^k e^{-na(t(x))}\right) |\Omega(1,2)| \\
\leq \frac{n}{k!} (n\pi r^2)^k e^{-\frac{1}{2}n\pi r^2} |\Omega(1,2)| \\
\leq \Theta(1) \frac{1}{r} e^{-\frac{1}{2}n r^2} \left((n\pi r^2)^k + 1 e^{-\frac{z-1}{2}n\pi r^2}\right) \\
= o(1).
\]

That is, we have proved a proposition:

**Proposition 3**: The integration of $\psi_{n,r}^k(x)$ defined by (13) on $\Omega(1,2)$ satisfies that
\[n \int_{\Omega(1,2)} \psi_{n,r}^k(x) dx = o(1).\]

![Figure 6. $\Omega$ is divided into four parts](image)

In the following, we will determine $n \int_{\Omega(1,1)} \psi_{n,r}^k(x) dx$. By the estimates (16) and (17), we have
\[
\frac{n}{k!} \int_{\Omega(1,1)} (n|B(x,r_n) \cap \Omega|)^k e^{-n|B(x,r_n) \cap \Omega|} dx \\
\sim \frac{n}{k!} \int_{\Omega(1,1)} (na(t(x)))^k e^{-na(t(x))} dx.
\]
We define
\[ L_1 = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) = (G(\Omega) + 1)r^2 \}, \]
\[ L_2 = \{ x \in \Omega(1) \mid t(x) = \frac{r}{2} \}, \]
\[ L_3 = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) = r \}. \]

Clearly, the subregion \( \Omega(1,1) \) is the strip with the boundaries \( L_1 \) and \( L_2 \), which is illustrated by Figure 6. If \( x \in L_1 \), then \( t(x) \geq 0 \) by a similar argument as in the previous step. Clearly, \( t(x) \leq (G(\Omega) + 1)r^2 \). Therefore, \( 0 \leq t(x) \leq (G(\Omega) + 1)r^2 \) for any \( x \in L_1 \). As \( r \) tends to zero, both the curves \( L_1 \) and \( L_2 \) approximate the boundary \( \partial \Omega \). So there exists \( \epsilon_r \) such that the length of \( L_1 \) and \( L_2 \), denoted by \( l_1 \) and \( l_2 \) respectively, satisfies:
\[ l - \epsilon_r \leq l_1, l_2 \leq l, \]
where \( \epsilon_r \to 0 \) as \( r \to 0 \). In addition, it is clear that \( t(x) \) is increasing along the directed line segment started from a point in \( L_1 \) to a point in \( L_2 \). See the line segment \( PQ \) in Figure 6.

The integration on the strip \( \Omega(1,1) \) can be bounded as follows:
\[
(l - \epsilon_r) \frac{n^k}{k!} \int_{(G+1)r^2}^{\frac{r}{2}} (na(t))^k e^{-na(t)} dt \\
\leq \frac{n^k}{k!} \int_{\Omega(1,1)} (na(t))^k e^{-na(t)} dx \\
\leq l \frac{n^k}{k!} \int_{0}^{\frac{r}{2}} (na(t))^k e^{-na(t)} dt.
\]

According to Lemma 3, we obtain that
\[
l \frac{n^k}{k!} \int_{0}^{\frac{r}{2}} (na(t))^k e^{-na(t)} dt = \frac{l \sqrt{\pi}}{2^{k+1} k!} e^{-\frac{r^2}{4}}.
\]

Therefore,
\[
0 \leq \epsilon_r \int_{(G+1)r^2}^{\frac{r}{2}} \frac{n^k}{k!} (na(t))^k e^{-na(t)} dt \\
\leq \epsilon_r \int_{0}^{\frac{r}{2}} \frac{n^k}{k!} (na(t))^k e^{-na(t)} dt = o(1).
\]

Furthermore, notice that \( \pi r^2/2 \leq a(t) \leq (\pi + \delta)r^2/2 \) for any \( t \in [0,(G(\Omega) + 1)r^2] \), where \( \delta > 0 \) is a small real number, we have that
\[
l \int_{0}^{(G+1)r^2} \frac{n^k}{k!} (na(t))^k e^{-na(t)} dt = o(1).
\]
So,

\[(l - \epsilon_r) \frac{n}{k!} \int_{(G+1)^{r^2}}^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt \]

\[= \frac{n}{k!} \left( l \int_0^{\frac{\pi}{2}} - l \int_{(G+1)^{r^2}}^{\frac{\pi}{2}} - \epsilon_r \int_{(G+1)^{r^2}}^{\frac{\pi}{2}} \right) (na(t))^k e^{-na(t)} dt \]

\[= \frac{l}{k!} \int_0^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt + o(1). \]

Then by formula (18),

\[\frac{l}{k!} \int_0^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt + o(1) \]

\[= (l - \epsilon_r) \frac{n}{k!} \int_{(G+1)^{r^2}}^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt \]

\[\leq \frac{n}{k!} \int_{\Omega(1,1)} (na(t(x)))^k e^{-na(t(x))} dx \]

\[\leq l \frac{n}{k!} \int_0^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt, \]

which implies that

\[\int_{\Omega(1,1)} (na(t(x)))^k e^{-na(t(x))} dx \]

\[\sim l \frac{n}{k!} \int_0^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt. \]

So, by Lemma 4 we have

\[\frac{n}{k!} \int_{\Omega(1,1)} (n|B(x, r_n) \cap \Omega|^k e^{-n|B(x, r_n) \cap \Omega|} dx \]

\[\sim \frac{n}{k!} \int_{\Omega(1,1)} (na(t(x)))^k e^{-na(t(x))} dx \]

\[\sim l \frac{n}{k!} \int_0^{\frac{\pi}{2}} (na(t))^k e^{-na(t)} dt \]

\[= \frac{l\sqrt{\pi}}{2k+1} \frac{e^{-\frac{\xi}{2}}}{k!}. \]

Therefore, we have proved the following conclusion:

**Proposition 4:** The integration of \(\psi_{n,r}^k(x)\) defined by (13) on \(\Omega(1,1)\) satisfies that

\[n \int_{\Omega(1,1)} \psi_{n,r}^k(x) dx = \frac{l\sqrt{\pi}}{2k+1} \frac{e^{-\frac{\xi}{2}}}{k!}. \]

Relying on the above four propositions, and noticing that \(\xi\) satisfies

\[
\begin{cases}
  e^{-\xi} + \frac{l\sqrt{\pi}}{2k+1} \frac{e^{-\frac{\xi}{2}}}{k!} = e^{-c}, & k = 1, \\
  \frac{l\sqrt{\pi}}{2k+1} \frac{e^{-\frac{\xi}{2}}}{k!} = e^{-c}, & k > 1,
\end{cases}
\]
we finally complete the proof by the following calculation:

\[
\frac{n}{k!} \int_{\Omega} (n|B(x, r_n) \cap \Omega|^k e^{-n|B(x,r_n) \cap \Omega|} dx
\]

\[
= \left\{ \int_{\Omega(0)} + \int_{\Omega(2)} + \int_{\Omega(1,2)} + \int_{\Omega(1,1)} \right\} n \psi_{n,r}^k(x) dx
\]

\[
\sim e^{-c}.
\]

When \(k = 0\), notice that \(n \int_0^\infty \frac{(na(t))^k e^{-na(t)}}{k!} dt = o(1)\), repeat the above process of the proof, we know the conclusion also holds for \(k = 0\). The proof of the first part of Theorem 4 in the case of smooth \(\partial \Omega\) is now completed.

B. Case of continuous \(\partial \Omega\)

For a general unit-area convex region \(\Omega\) with a continuous rather than smooth boundary \(\partial \Omega\), we may use a family of convex regions \(\{\Omega_n\}_{n=1}^\infty \subset \Omega\) to approximate \(\Omega\), where the boundary \(\partial \Omega_n\) of each \(\Omega_n\) is smooth. We set the width of the gap between \(\partial \Omega\) and \(\partial \Omega_n\) to be less than \(r_n^2\), see Figure 7. That is, \(\sup_{x \in \Omega_n} \text{dist}(x, \partial \Omega) < r_n^2\) for any \(n\). Clearly, we have the following estimation for \(S(\Omega_n)\), the area of \(\Omega_n\):

\[
1 - lr_n^2 \leq S(\Omega_n) \leq 1.
\]

![Figure 7. Width of the Gap between \(\partial \Omega\) and \(\partial \Omega_n\) is less than \(r_n^2\)]

Because \(\partial \Omega_n\) is smooth, follow the method presented in the previous subsection, we can also similarly obtain that

\[
n \int_{\Omega_n} \psi_{n,r}^k(x) dx \sim e^{-c}.
\]

Since the area of \(\Omega \setminus \Omega_n\) is no more than \(lr_n^2\), then by the proof of Proposition 2, we have that

\[
n \int_{\Omega \setminus \Omega_n} \psi_{n,r}^k(x) dx = o(1).
\]
Therefore, we have
\[ n \int_{\Omega} \psi_{n,r}^k(x) \, dx = n \left( \int_{\Omega_n} + \int_{\Omega \setminus \Omega_n} \right) \psi_{n,r}^k(x) \, dx \sim e^{-c}. \]

This finally completes the proof of the first part of Theorem 4.

V. PROOF OF THEOREM 4: PART TWO

This section will present a proof for the second part of Theorem 4. That is, for \( r_n \) given by (4) and (5) (see Theorem 4), which satisfies
\[ \frac{n}{k!} \int_{\Omega} (n|B(x, r_n) \cap \Omega|) \, e^{-n|B(x, r_n) \cap \Omega|} \, dx \sim e^{-c} \]
as we have proved in the previous section, then the probabilities of the two events \( \rho(\chi_n; \delta \geq k + 1) \leq r_n \) and \( \rho(\chi_n; \kappa \geq k + 1) \leq r_n \) both converge to \( \exp \left( -e^{-c} \right) \) as \( n \to \infty \).

Clearly, it is sufficient to prove the following two conclusions:

**Conclusion 1:** Under the assumptions of Theorem 4
\[ \lim_{n \to \infty} \Pr \{ \rho(\chi_n; \delta \geq k + 1) \leq r_n \} = e^{-e^{-c}}. \quad (19) \]

**Conclusion 2:** Under the assumptions of Theorem 4
\[ \lim_{n \to \infty} \Pr \{ \rho(\chi_n; \delta \geq k + 1) = \rho(\chi_n; \kappa \geq k + 1) \} = 1. \quad (20) \]

We will follow Penrose’s approach to prove these two conclusions. This section mainly presents a proof for Conclusion 1, while we sketch the proof of Conclusion 2 in Appendix C. The approach is instead to prove a Poissonized version of Conclusion 1:
\[ \lim_{n \to \infty} \Pr \{ \rho(\mathcal{P}_n; \delta \geq k + 1) \leq r_n \} = \exp \left( -e^{-c} \right), \quad (21) \]
where \( \mathcal{P}_n \) means that the \( n \) points distributed over unit-area convex region \( \Omega \) are Poisson point process, which has been introduced in Section II. Because \( \chi_n \) can be well approximated by \( \mathcal{P}_n \), it is naturally to see the original (19) hold by a de-Poissonized technique. The de-Poissonized technique is standard and thus omitted here, please see [19] for details. In the following, we only demonstrate the Poissonized version (21) holds for a general convex region \( \Omega \).

For given \( n, x, y \in \Omega \), let
\[ v_x = |B(x, r) \cap \Omega|, \]
\[ v_y = |B(y, r) \cap \Omega|, \]
\[ v_{x,y} = |B(x, r) \cap B(y, r) \cap \Omega|, \]
\[ v_x \setminus y = v_x - v_{x,y}, \quad v_y \setminus x = v_y - v_{x,y}. \]
Define $I_i = I_i(n)(i = 1, 2, 3)$ as follows:

$$I_1 = n^2 \int_{\Omega} dx \int_{\Omega \cap B(x,3r)} dy \psi^k_{n,r}(y) \psi^k_{n,r}(x),$$

$$I_2 = n^2 \int_{\Omega} dx \int_{\Omega \cap B(x,r)} dy \Pr[Z_1 + Z_2 = Z_1 + Z_3 = k - 1],$$

$$I_3 = n^2 \int_{\Omega} dx \int_{\Omega \cap (B(x,3r) \setminus B(x,r))} dy \Pr[Z_1 + Z_2 = Z_1 + Z_3 = k],$$

where $Z_1, Z_2, Z_3$ are independent Poisson variables with means $nv_{x,y}, nv_{x,y}, nv_{y,x}$ respectively. As Penrose has pointed out in [19], by an argument similar to that of Section 7 of [16], it suffices to prove that $I_1, I_2, I_3 \to 0$ as $n \to \infty$.

However, some critical estimations in Penrose’s proof for $I_1, I_2, I_3 \to 0(n \to \infty)$ are only valid for a regular region square (see [1], [19]). Our work, i.e., Theorem 4 or Conclusion 1, extends these results to the case of a general domain, i.e., a general convex region. For this purpose, we will first establish a proposition, an estimation in the case of general convex region $\Omega$.

**Proposition 5:** Under the assumptions of Theorem 4, $Z_3$ is a Poisson variable with mean $nv_{y,x}$ defined above, then

$$\frac{1}{n \pi r^2} \int_{\Omega} n \psi^k_{n,r}(x) dx \int_{\Omega \cap B(x,r)} n \Pr(Z_3 = k - 1) dy = o(1).$$

**Proof:** The proof is presented in Appendix B.

For the remainder proof of Conclusion 1, as we mentioned, it is sufficient to prove $I_1, I_2, I_3 \to 0$, as $n \to 0$. This will be completed in the following three claims.

**Claim 1:** $I_1 \to 0$ as $n \to \infty$.

**Proof:** By Lemma 1, there exists a constant $C > 0$ such that $C \pi r^2 \leq |B(x,r) \cap \Omega| \leq \pi r^2$, so

$$\frac{(Cn\pi r^2)^k \exp(-n\pi r^2)}{k!} \leq \psi^k_{n,r} \leq \frac{(n\pi r^2)^k \exp(-Cn\pi r^2)}{k!},$$
and thus,

\[ I_1 = n^2 \int_{\Omega} dx \int_{\Omega \cap B(x, 3r)} dy \psi_{n,r}^k(y) \psi_{n,r}^k(x) \]
\[ = \int_{\Omega} n \psi_{n,r}^k(x) dx \int_{B(x, 3r) \cap \Omega} n \psi_{n,r}^k(y) dy \]
\[ \leq \int_{\Omega} n \psi_{n,r}^k(x) dx \int_{B(x, 3r) \cap \Omega} \frac{1}{k!} n (n \pi r^2)^k \exp(-C n \pi r^2) \]
\[ \leq \frac{n}{k!} |B(x, 3r)| (n \pi r^2)^k \exp(-C n \pi r^2) \int_{\Omega} n \psi_{n,r}^k(x) dx \]
\[ = \frac{9}{k!} (n \pi r^2)^{k+1} \exp(-C n \pi r^2) \int_{\Omega} n \psi_{n,r}^k(x) dx \]
\[ \sim e^{-c} \frac{9}{k!} (n \pi r^2)^{k+1} \exp(-C n \pi r^2) \]
\[ \rightarrow 0, n \rightarrow \infty. \]

The last one holds due to Remark 3, i.e., \( r_n \) given by (4) and (5) in Theorem 3 satisfying

\[ (n \pi r^2)^{k+1} \exp(-C n \pi r^2) \sim \frac{(\log n + o(\log n))^{k+1}}{e^{C(\log n + o(\log n))}} = o(1). \]

\( \square \)

**Claim 2:** \( I_2 \to 0 \), as \( n \to \infty. \)

**Proof:** It is obvious that \( Z_1 + Z_2 \) and \( Z_3 \) are Possion variables with means \( nv_x \) and \( nv_{y|x} \) respectively.

Notice that

\[ \Pr[Z_1 + Z_3 = k - 1 | Z_1 + Z_2 = k - 1] \]
\[ = \sum_{j=0}^{k-1} \Pr[Z_1 + Z_3 = k - 1; Z_1 = j | Z_1 + Z_2 = k - 1] \]
\[ \leq \sum_{j=0}^{k-1} \Pr[Z_3 = k - 1 - j | Z_1 + Z_2 = k - 1] \]
\[ = \sum_{j=0}^{k-1} \Pr[Z_3 = k - 1 - j]. \]

For any \( x \in \Omega \),

\[ \psi_{n,r}^k(x) = \frac{n |B(x, r) \cap \Omega|}{k} \psi_{n,r}^{k-1}(x), \]

then by Lemma 1 there exists a constant \( C > 0 \) such that \( C \pi r^2 \leq |B(x, r) \cap \Omega| \), and thus

\[ \psi_{n,r}^{k-1}(x) \leq \frac{k}{C n \pi r^2} \psi_{n,r}^k(x). \]
Therefore, we have

\[
I_2 = n^2 \int_{\Omega} dx \int_{\Omega \cap B(x,r)} dy \Pr[Z_1 + Z_2 = Z_1 + Z_3 = k - 1]
\]

\[
= n^2 \int_{\Omega} dx \int_{\Omega \cap B(x,r)} dy \Pr[Z_1 + Z_2 = k - 1]
\]

\[
\Pr[Z_1 + Z_3 = k - 1 | Z_1 + Z_2 = k - 1]
\]

\[
= n \int_{\Omega \cap B(x,r)} \Pr[Z_1 + Z_2 = k - 1] dx
\]

\[
\int_{\Omega \cap B(x,r)} n \Pr[Z_1 + Z_3 = k - 1 | Z_1 + Z_2 = k - 1] dy
\]

\[
\leq n \int_{\Omega} \Pr[Z_1 + Z_2 = k - 1] dx
\]

\[
\left\{ \sum_{j=0}^{k-1} \int_{\Omega \cap B(x,r)} n \Pr[Z_3 = k - 1 - j] dy \right\}
\]

\[
= \sum_{j=0}^{k-1} \int_{\Omega} n \psi_{n,r}^{k-1}(x) dx \int_{\Omega \cap B(x,r)} n \Pr[Z_3 = k - 1 - j] dy
\]

\[
\leq \frac{C}{k} \sum_{j=0}^{k-1} 1 \int_{\Omega} n \psi_{n,r}^{k}(x) dx \int_{\Omega \cap B(x,r)} n \Pr[Z_3 = k - 1 - j] dy
\]

\[
= o(1).
\]

The last equation holds due to Proposition 5 and the proved conclusion $\int_{\Omega} n \psi_{n,r}^{k}(x) dx \sim e^{-c}$. \(\square\)

Claim 3: $I_3 \to 0$, as $n \to \infty$.

Proof: The proof is similar to that of Claim 2. \(\square\)

Therefore, by these three claims, we know the Poissonized version holds, which leads to Conclusion 1 by a de-Poissonized technique as we have mentioned:

\[
\lim_{n \to \infty} \Pr\{\rho(\chi_n; \delta \geq k+1) \leq r_n\} = e^{-e^{-c}}.
\]

The proof of Conclusion 2 is presented in Appendix C. As long as these two conclusion are proved, the second part of Theorem 4 is consequently proved. Therefore, the proof of Theorem 4 is completed.

VI. CONCLUSIONS AND FUTURE WORK

In this article, we have studied a more general unit area convex region on which wireless sensor and ad-hoc networks are distributed. We have investigated two types of critical transmission radii in the networks, i.e., $\rho(\chi_n; \kappa \geq k)$ and $\rho(\chi_n; \delta \geq k)$, defined in terms of $k$-connectivity and the minimum
vertex degree respectively. We have also established the precise asymptotic distributions of these two types of radii, which makes the previous results obtained under the circumstance of squares or disks special cases of this work. More importantly, our results have clearly indicated that only the length of the boundary of regions completely determines the critical transmission radius. Further, by isodiametric inequality, the smallest asymptotic critical radius is achieved only when the region is a disk.

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APPENDIX A

PROOFS OF LEMMA 1 AND 2

Lemma 1: Let $\Omega$ be a bounded convex region, then there exists a positive constant $C$ such that for any sufficiently small $r$,

$$\inf_{x \in \Omega} |B(x, r) \cap \Omega| \geq C \pi r^2.$$  

Proof: Define

$$f(x, r) = \frac{|B(x, r) \cap \Omega|}{\pi r^2},$$

then $f(x, r)$ is a continuous function of $x$ and $r$ on $\overline{\Omega} \times (0, 1]$, where $\overline{\Omega}$ is the closure of $\Omega$. We will show that there exists a constant $C$ such that $f(x, r) > C > 0$ on $\overline{\Omega} \times (0, 1]$. Otherwise, we assume there exists a sequence $\{(x_n, r_n)\}_{n=1}^\infty$ such that $f(x_n, r_n) \to 0$ as $n$ tends to infinity. Because $\overline{\Omega}$ is a bounded and closed region, there exists a subsequence of $\{x_n\}_{n=1}^\infty$, namely $\{x_{n_k}\}_{k=1}^\infty$, converging to a limit, denoted by $x_0 \in \overline{\Omega}$, as $k$ tends to infinity. Therefore,

$$\lim_{k \to \infty} f(x_0, r_{n_k}) = \lim_{k \to \infty} f(x_{n_k}, r_{n_k}) = 0.$$  

(1) If $x_0 \in \Omega^o$, i.e., $x_0$ is an interior point in $\Omega$, then there exists a $\delta > 0$ such that $B(x_0, \delta) \subset \Omega^o \subset \Omega$. So for any $r < \delta$, we always have $f(x_0, r) = 1$, which contradicts to $\lim_{k \to \infty} f(x_0, r_{n_k}) = 0$.

(2) Otherwise, $x_0 \in \partial \Omega$. For small $r' > r'' > 0$, suppose the circle with the center $x_0$ and radius $r'$ intersects the boundary $\partial \Omega$ at $P_1$ and $Q_1$ (see Figure 8), then we obtain a sector $P_1x_0Q_1$, denoted by $S(x_0, r')$. Because $\Omega$ is a convex region, so $S(x_0, r') \subset \overline{\Omega}$ as long as $r'$ is sufficiently small. The angle of sector $S(x_0, r')$ is denoted by $\theta(x_0, r')$. Similarly, we assume the circle centering $x_0$ with radius $r''$ intersects the boundary $\partial \Omega$ at $P_2$ and $Q_2$, then we obtain a sector $P_2x_0Q_2$, denoted by $S(x_0, r'')$. The
angle of sector $S(x_0, r'')$ is denoted by $\theta(x_0, r'')$. We also know $S(x_0, r'') \subset \overline{\Omega}$. Notice that $r'' < r'$, by the convexity of $\overline{\Omega}$, it is easy to see that $\theta(x_0, r'') \geq \theta(x_0, r')$. Therefore, for any sufficiently small $r < r'$,

$$|B(x_0, r) \cap \Omega| \geq |S(x_0, r) \cap \Omega|$$

$$= |S(x_0, r)|$$

$$= \frac{\theta(x_0, r)}{2\pi} \pi r^2$$

$$\geq \frac{\theta(x_0, r')}{2\pi} \pi r^2,$$

leading to

$$f(x_0, r) = \frac{|B(x_0, r) \cap \Omega|}{\pi r^2} \geq \frac{\theta(x_0, r')}{2\pi},$$

which contradicts to $\lim_{k \to \infty} f(x_0, r_{n_k}) = 0$. The proof of this lemma is complete.

We should point out that if the boundary $\partial \Omega$ is smooth, then we can further obtain that the constant $C$ can approximate $\frac{1}{2}$ when $r$ tends to zero.

![Sectors on $\overline{\Omega}$](image)

**Figure 8. Sectors on $\overline{\Omega}$**

**Lemma 2:** Two disks centered at $A$ and $B$ respectively, have the same radii $r$, with the distance $\|AB\| = d$. The shadow area $S$ in Figure 9 is

$$S = r^2 \left[ \frac{1}{2} d + \frac{19}{48} \left( \frac{d}{r} \right)^3 + \frac{153}{1280} \left( \frac{d}{r} \right)^5 - o \left( \frac{d}{r} \right)^5 \right].$$

In particular, if $r \geq d \geq \frac{r}{(nr^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{n}}$, then

$$nS \geq \frac{1}{\sqrt{\pi}} \left( n \pi r^2 \right)^{\frac{1}{2}}.$$
Figure 9. Illustration of Lemma 2: Area $S$

Figure 10. Illustration of Lemma 2: Area $S_1$

**Proof:** It is clear that quadrilateral $ABCD$ is a diamond, with the angles $\alpha = \beta$. Let $S_1$ be the total area of these two overlap disks (see Figure 10), $S_2$ be the area of gray semi-disk (see Figure 11), and $S_3$ be the area of the acid blue region (see Figure 12). Then, it is easy to see

$$S_1 = 2\pi r^2 - 2 \left[ \pi r^2 \frac{\alpha}{2\pi} - \frac{d}{2} \sqrt{r^2 - \left( \frac{d}{2} \right)^2} \right],$$

$$S_2 = \frac{\pi r^2}{2},$$

$$S_3 = \pi r^2 - \left[ \pi r^2 \frac{\beta}{2\pi} - d \sqrt{r^2 - d^2} \right].$$
and the shadow area

\[ S = S_1 - S_2 - S_3 \]
\[ = \frac{\pi r^2}{2} + d \sqrt{r^2 - \left(\frac{d}{2}\right)^2} - d \sqrt{r^2 - d^2} + \pi r^2 \left(\frac{\beta}{2\pi} - \frac{\alpha}{\pi}\right) \]
\[ = \frac{\pi r^2}{2} + d \sqrt{r^2 - \left(\frac{d}{2}\right)^2} - d \sqrt{r^2 - d^2} + \pi r^2 \left(\frac{\beta}{2\pi} - \frac{\beta}{\pi}\right) \]
\[ = \frac{\pi r^2}{2} + d \sqrt{r^2 - \left(\frac{d}{2}\right)^2} - d \sqrt{r^2 - d^2} - \frac{r^2 \beta}{2}. \]  

(22)

According to Figure 9

\[ \cos\left(\frac{\beta}{2}\right) = \frac{d}{2r}, \]

or

\[ \frac{\beta}{2} = \arccos\left(\frac{d}{2r}\right). \]
Notice that \(\arccos(x) = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} + o(x^5)\). So,
\[
\frac{\beta}{2} = \arccos \left(\frac{d}{2r}\right)
= \frac{\pi}{2} - \frac{d}{2r} - \frac{3}{6} \left(\frac{d}{2r}\right)^3 - \frac{3}{40} \left(\frac{d}{2r}\right)^5 + o\left(\frac{d}{2r}\right)^5,
\]
which leads to
\[
\frac{r^2\beta}{2} = \frac{\pi}{2} r^2 - r^2 \left[\frac{1}{2r} + \frac{1}{48} \left(\frac{d}{r}\right)^3 + \frac{3}{1280} \left(\frac{d}{r}\right)^5 - o\left(\frac{d}{r}\right)^5\right].
\]
It is clearly,
\[
\sqrt{1 - \left(\frac{d}{2r}\right)^2} = 1 - \frac{1}{2} \left(\frac{d}{r}\right)^2 - \frac{1}{8} \left(\frac{d}{r}\right)^4 + o\left(\frac{d}{r}\right)^5,
\]
\[
\sqrt{1 - \left(\frac{d}{r}\right)^2} = 1 - \frac{1}{2} \left(\frac{d}{2r}\right)^2 - \frac{1}{8} \left(\frac{d}{2r}\right)^4 + o\left(\frac{d}{2r}\right)^5.
\]
So the shadow area \(S\) can be written as
\[
S = \frac{\pi r^2}{2} + dr\sqrt{1 - \left(\frac{d}{2r}\right)^2} - dr\sqrt{1 - \left(\frac{d}{r}\right)^2} - \frac{r^2\beta}{2}
= dr \left[\frac{3}{8} \left(\frac{d}{r}\right)^2 + \frac{15}{128} \left(\frac{d}{r}\right)^4\right]
+ r^2 \left[\frac{1}{2r} + \frac{1}{48} \left(\frac{d}{r}\right)^3 + \frac{3}{1280} \left(\frac{d}{r}\right)^5 - o\left(\frac{d}{r}\right)^5\right]
= r^2 \left[\frac{1}{2} \frac{d}{r} + \frac{19}{48} \left(\frac{d}{r}\right)^3 + \frac{153}{1280} \left(\frac{d}{r}\right)^5 - o\left(\frac{d}{r}\right)^5\right].
\]
Clearly, if \(r \geq d \geq \frac{r}{(nr^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{n}}\), we have
\[
nS \geq \frac{1}{2} nr^2 \frac{d}{r} \geq \frac{1}{\sqrt{\pi}} \left(n\pi r^2\right)^{\frac{1}{2}}.
\]

**APPENDIX B**

**PROOF OF PROPOSITION 5**

As we have denoted previously,
\[
v_y = |B(y, r) \cap \Omega|, v_{x,y} = |B(x, r) \cap B(y, r) \cap \Omega|,
\]
\[
v_{y \setminus x} = v_y - v_{x,y}.
\]
Z₃ is a Poisson variable with mean \( nv_{y|x} \), and
\[
\psi^k_{n,r}(x) = \frac{(n|B(x,r) \cap \Omega|)^k e^{-n|B(x,r) \cap \Omega|}}{k!}.
\]

Now we prove the following

**Proposition 5:** Let \( \Omega \) be an unit-area convex region and \( r \) be given by (4) and (5) in Theorem 4, then
\[
\frac{1}{n\pi r^2} \int_{\Omega} n\psi^k_{n,r}(x)dx \int_{\Omega \cap B(x,r)} nP(Z_3 = k - 1)dy = o(1).
\]

**Proof:** Let \( d_0 = \frac{r}{(n\pi r^2)^{\frac{1}{2}}} \), and \( \forall x \in \Omega \), let
\[
\Gamma_1(x) = \left\{ y \in B(x,r) \cap \Omega : \text{dist}(y, x) \leq d_0 = \frac{r}{(n\pi r^2)^{\frac{1}{2}}} \right\},
\]
\[
\Gamma_2(x) = \left\{ y \in B(x,r) \cap \Omega : \text{dist}(y, x) \geq d_0, \text{dist}(y, \partial\Omega) > (G(\Omega) + 1)r^2 \right\},
\]
\[
\Gamma_3(x) = \left\{ y \in B(x,r) \cap \Omega : \text{dist}(y, \partial\Omega) \leq (G(\Omega) + 1)r^2 \right\}.
\]

Here dist \((y, x)\) denotes the distance between points \( y \) and \( x \), i.e., \( \|yx\| \), and dist \((y, \partial\Omega)\) = \( \inf_{x \in \partial\Omega} \text{dist}(y, x) \), is the distance between point \( y \) and curve \( \partial\Omega \). Constant \( G(\Omega) \) is an uniform upper bound of all second-order derivatives of the function of \( \partial\Omega \) in local coordinates, as discussed in Remark 2. Clearly, as Figure 13 illustrates,
\[
B(x,r) \cap \Omega \subset \Gamma_1(x) \cup \Gamma_2(x) \cup \Gamma_3(x).
\]

Figure 13. Illustration of \( B(x,r) \cap \Omega \subset \Gamma_1(x) \cup \Gamma_2(x) \cup \Gamma_3(x) \)
Notice that $\Gamma_3(x)$ falls in a region with the width less than $(G(\Omega) + 1)r^2$ and the length less than $2\pi r$ (the perimeter of $B(x, r)$), therefore

$$|\Gamma_3(x)| \leq 2\pi(G(\Omega) + 1)r^3.$$ 

Let

$$A_i = \frac{1}{n\pi r^2} \int_{\Omega} n\psi_{n,r}^k(x) dx \int_{\Gamma_i(x)} n \Pr(Z_3 = k - 1) dy, i = 1, 2, 3.$$  

(24)

We will prove $A_i = o(1), i = 1, 2, 3,$ in the following three steps. Notice that we have proved in Section IV $\int_{\Omega} n\psi_{n,r}^k(x) dx \sim e^{-c}$.

**Step 1:** to prove $A_1 = o(1)$ where $A_1$ is defined by (24).

$$A_1 = \frac{1}{n\pi r^2} \int_{\Omega} n\psi_{n,r}^k(x) dx \int_{\Gamma_1(x)} n \Pr(Z_3 = k - 1) dy$$

$$\leq \frac{1}{n\pi r^2} \int_{\Omega} |\Gamma_1(x)| n\psi_{n,r}^k(x) dx$$

$$\leq \frac{\pi d_0^2}{\pi r^2} \int_{\Omega} n\psi_{n,r}^k(x) dx$$

$$\sim \frac{1}{(n\pi r^2)^{\frac{1}{2}}} e^{-c}$$

$$= o(1).$$

**Step 2:** to prove $A_2 = o(1)$ where $A_2$ is defined by (24). According to the definition of $\Gamma_2(x)$, for any $y \in \Gamma_2(x)$, the distance from $y$ to $\partial \Omega$ is larger than $(G(\Omega) + 1)r^2$. Then by Remark 2, $\|yA\| > r$ and $\|yB\| > r$, see Figure 14. This means that at least more than one half of $B(y, r)$ falling in $\Omega$. So there exists at least one joint point of $\partial B(y, r)$ and $\partial B(x, r)$ falls in $\Omega$. As a result,

$$v_{y\setminus x} = |B(y, r) \cap \Omega| - |B(x, r) \cap B(y, r) \cap \Omega| \geq \frac{S}{2},$$

and therefore,

$$nv_{y\setminus x} \geq \frac{1}{2} nS \geq \frac{1}{2\sqrt{\pi}} \left( n\pi r^2 \right)^{\frac{1}{2}},$$

where $S$ is given by Lemma 2.
Figure 14. Illustration of the proof for $A_2 = o(1)$

So we have

$$A_2 = \frac{1}{\pi r^2} \int_{\Omega} n \psi_{n,r}^k(x) dx \int_{\Gamma_2} n \Pr(Z_3 = k - 1) dy$$

$$\leq \frac{1}{\pi r^2} \int_{\Omega} n \psi_{n,r}^k(x) dx$$

$$\leq \frac{1}{\pi r^2} \int_{\Omega} n (n \pi r^2)^{k-1} \exp \left(-\frac{C}{2} \left(\frac{n \pi r^2}{2}\right)^{\frac{1}{2}}\right) dy$$

$$\leq \frac{\exp \left(-\frac{C}{2} \left(\frac{n \pi r^2}{2}\right)^{\frac{1}{2}}\right)}{(k-1)!}$$

$$\sim \frac{e^{-C}}{n \pi r^2} o(1)$$

$$= o(1).$$

Step 3: to prove $A_3 = o(1)$ where $A_1$ is defined by $[24]$.

$$A_3 = \frac{1}{\pi r^2} \int_{\Omega} n \psi_{n,r}^k(x) dx \int_{\Gamma_3(x)} n \Pr(Z_3 = k - 1) dy$$

$$\leq \frac{1}{\pi r^2} \int_{\Omega} n \psi_{n,r}^k(x) dx \int_{\Gamma_3(x)} n dy$$

$$= \frac{1}{\pi r^2} \int_{\Omega} |\Gamma_3(x)| n \psi_{n,r}^k(x) dx$$

$$\leq \frac{2\pi (G(\Omega) + 1)r^3}{\pi r^2} \int_{\Omega} n \psi_{n,r}^k(x) dx$$

$$\sim \Theta(r)$$

$$= o(1).$$
Finally, we have
\[
\frac{1}{n \pi r^2} \int_{\Omega} n \psi_{n,r}^k(x) dx \int_{\Omega \cap B(x,r)} n \Pr(Z_3 = k - 1) dy \
\leq A_1 + A_2 + A_3 = o(1).
\]
The proposition is therefore proved.

APPENDIX C
PROOF SKETCH OF CONCLUSION 2

Conclusion 2: Under the assumptions of Theorem 4,
\[
\lim_{n \to \infty} \Pr \{ \rho(\chi_n; \delta \geq k + 1) = \rho(\chi_n; \kappa \geq k + 1) \} = 1.
\]

Here we sketch the proof of Conclusion 2. Penrose has clearly proved this result when region \( \Omega \) is a square [19]. He cleverly constructed two events, \( E_n(K) \) and \( F_n(K) \), such that for any \( K > 0 \),
\[
\{ \rho(\chi_n; \delta \geq k + 1) \leq r_n < \rho(\chi_n; \kappa \geq k + 1) \} \subseteq E_n(K) \cup F_n(K),
\]
and
\[
\lim_{n \to \infty} \Pr[E_n(K)] = \lim_{n \to \infty} \Pr[F_n(K)].
\]
The definition of the events and the convergence results are organised in the following two propositions (see Proposition 5.1 and 5.2 in [19]). Here and in the following, we do not introduce the notations in detail. Please refer to [19].

Proposition 6: (19) Let \( E_n(K) \) be the event that there exists a set \( U \) of two or more points of \( \chi_n(U_{r_n}) \), of diameter at most \( Kr_n \) with \( \chi_n(U_{r_n} \setminus U) \leq k \). Then for all \( K > 0 \) we have \( \lim_{n \to \infty} \Pr[E_n(K)] = 0 \).

Proposition 7: (19) Let \( F_n(K) \) be the event that there are disjoint subsets \( U, V, W \) of \( \chi_n \), such that \( \text{card}(W) \leq k \), and such that \( U \) and \( V \) are connected components of the \( r_n \) graph on \( \chi_n \setminus W \), with \( \text{diam}(U) > Kr_n \) and \( \text{diam}(V) > Kr_n \). Then there exists \( K > 0 \), such that \( \lim_{n \to \infty} \Pr[F_n(K)] = 0 \).

Proposition 6 is immediate from the following three lemmas (see Lemma 5.1, 5.2 and 5.3 in [19]).

Lemma 5: (19) Let \( K > 0 \). Then with \( \psi_{n,r}^k(x) \) defined previously,
\[
\Pr[E_n(K)] \leq \sup_{x \in \Omega} \left( \frac{\Pr[E^x_n(K)]}{\psi_{n,r}^k(x)} \right).
\]

Lemma 6: (19) There exists \( K \in (0, 1] \), such that
\[
\lim_{n \to \infty} \sup_{x \in \Omega} \frac{\Pr[E^x_n(K)]}{\psi_{n,r}^k(x)} = 0.
\]

Lemma 7: (19) Suppose \( 0 < K' < K < \infty \). Then
\[
\lim_{n \to \infty} \sup_{x \in \Omega} \frac{\Pr[E^x_n(K) \setminus E^x_n(K')]}{\psi_{n,r}^k(x)} = 0.
\]
These lemmas deal with square regions. However, it can be straightly generalised to cope with the convex case, with their proofs not much been modified. That is,

$$|B(x,r) \cap \Omega| \in \psi_k^{n,r}(x) = \frac{(n |B(x,r) \cap \Omega|)^2 e^{-n |B(x,r) \cap \Omega|}}{k!}$$

can be bounded by $C \pi r^2 \leq |B(x,r) \cap \Omega| \leq \pi r^2$ where $C > 0$. Estimation $n \int_\Omega \psi_k^{n,r}(x) dx \sim e^{-c}$ (see (14)) can also hold.

As a result, Propositions 6 and 7 can be generalised to the case of convex region. Based on these two generalised propositions, a squeezing argument can lead to the following

$$\lim_{n \to \infty} \Pr \{ \rho(\chi_n; \delta \geq k + 1) = \rho(\chi_n; \kappa \geq k + 1) \} = 1.$$

Please see the details presented in [19].

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