Spin-polarized Bilayer Quantum Hall systems at total filling factor $\nu_T = 1$ have been under enormous experimental and theoretical investigations over the last decade [1]. When the interlayer separation $d$ is sufficiently large, the bilayer system decouples into two separate compressible $\nu = 1/2$ layers [2]. However, when $d$ is smaller than a critical distance $d_{c1}$, even in the absence of interlayer tunneling, the system undergoes a quantum phase transition into a novel spontaneous interlayer coherent incompressible phase which is an excitonic superfluid state in the pseudospin channel. [3–5].

Although the ESF phase and FL phase at the two extreme distances are well established, the picture of how the ESF phase evolves into the two weakly-coupled FL states as distance changes is still not well established. The experiment [6] discovered that although there are two critical distances and three phases in balanced case: $0 < d < d_{c1}$, the system is in the ESF, $d_{c1} < d < d_{c2}$, it is in the PSDW with square lattice structure, $d > d_{c2}$, it melts into two weakly coupled FL. We find that in the ESF side, the imbalance is irrelevant, but the merons carry continuously changing fractional charges. While in the PSDW side, we map the square lattice PSDW at the balanced case into a hard core bosons hopping on a square lattice at half filling with long range interactions, then adding a small imbalance in the BLQH corresponds to adding a small chemical potential in the boson model. Through this mapping, we find the imbalance drives two quantum phase transitions: the first one is from the commensurate PSDW state to an incommensurate PSDW (IC-PSDW), the second one is from IC-PSDW state to the ESF state. Both transitions are first order transitions. We compare our phase diagram with the recent experimental data [13]. We also discuss briefly the effects of disorders and compare our results with previous results achieved from microscopic calculations.

Consider a bi-layer system with $N_1$ ($N_2$) electrons in top (bottom) layer and with interlayer distance $d$ in the presence of magnetic field $\vec{B} = \nabla \times \vec{A}$:

$$H = H_0 + H_{\text{int}}$$

$$H_0 = \int d^2 \vec{x} c_\alpha^\dagger(\vec{x}) (-ih\vec{\nabla} + \frac{\vec{\nabla} \bar{A}(\vec{x})}{2m}) c_\alpha(\vec{x})$$

$$H_{\text{int}} = \frac{1}{2} \int d^2 x \int d^2 x' \delta \rho_\alpha(\vec{x}) V_{\alpha\beta}(\vec{x} - \vec{x}') \delta \rho_\beta(\vec{x}')$$

where electrons have bare mass $m$ and carry charge $-e$, $c_\alpha, \alpha = 1, 2$ are electron operators in top and bottom layers, $\delta \rho_\alpha(\vec{x}) = c_\alpha^\dagger(\vec{x}) c_\alpha(\vec{x}) - \bar{\rho}_\alpha, \alpha = 1, 2$ are normal ordered electron densities on each layer. The intralayer interactions are $V_{11} = V_{22} = e^2/\epsilon r, \epsilon$, while interlayer interaction is $V_{12} = V_{21} = e^2/\epsilon \sqrt{r^2 + d^2}$ where $\epsilon$ is the dielectric constant.

Performing a singular gauge transformation $\phi_\alpha(\vec{x}) = e^{i \int d^2 x' \arg(\vec{x} - \vec{x}') \rho(\vec{x'})} c_\alpha(\vec{x})$ where $\rho(\vec{x}) = c_1^\dagger(\vec{x}) c_1(\vec{x}) + c_2^\dagger(\vec{x}) c_2(\vec{x})$ is the total density of the bi-layer system. We can transform the Hamiltonian Eqn.1 into a Lagrangian
of the Composite Boson \( \phi_a \) coupled to a Chern-Simons gauge field \( a_\mu \) [8]. We can write the two bosons in terms of magnitude and phase \( \phi_a = \sqrt{\rho_+ + \delta \rho e^{i\theta}} \), then after absorbing the external gauge potential \( \vec{A} \) into \( \tilde{a} \), we get the Lagrangian in the Coulomb gauge:

\[
\mathcal{L} = i \delta \rho_+ [\frac{1}{2} \partial_\mu \theta_+ - a_0] + \frac{\tilde{\rho}}{2m} [\frac{1}{2} \nabla \theta_+ + \frac{1}{2} (v_1 - v_2) \nabla \theta_- - \tilde{a}]^2
\]

\[
+ \frac{1}{2} \delta \rho_+ V_+ (\vec{q}) \delta \rho_+ - \frac{i}{2 \pi} a_0 (\nabla \times \tilde{a})
\]

\[
+ \frac{q \delta \rho_- \partial_\mu \theta_+ + \tilde{\rho} f (\frac{1}{2} \nabla \theta_-)^2}{2} + \frac{1}{2} \delta \rho_- V_- (\vec{q}) \delta \rho_- - h_\perp \delta \rho_-
\]

where \( \delta \rho_+ = \delta \rho_1 + \delta \rho_2, \delta \rho_- = \rho_1 - \rho_2, \theta_\perp = \theta_1 \pm \theta_2 \), they satisfy commutation relations \( [\delta \rho_\alpha (\vec{x}), \delta \beta (\vec{x}')] = 2i \hbar \delta_\alpha \beta \delta (\vec{x} - \vec{x}') \), \( \alpha, \beta = \pm \). \( \tilde{\rho} = \rho_1 + \rho_2 \) and \( f = 4 \nu_1 \nu_2 \) which is equal to 1 at the balanced case, \( V_\pm = V_{1 \pm} \pm V_{2 \pm} \), \( h_\perp = V, \rho_- = V_\perp (\rho_1 - \rho_2) \) plays a role of a Zeeman field.

(1) Off-diagonal algebraic order and Spin-wave excitation: Neglecting vortex excitations, dropping a linear derivative term in \( \theta_- \) which is irrelevant in the ESF state and expanding the second term in Eqn.2 which includes the coupling between the spin sector and the charge sector, we can get the complete forms of the three propagators [8]: \( < \theta_+ \theta_+ >, < \theta_- \theta_- >, < \theta_+ \theta_- > \).

Performing the frequency integral of the \( \perp \) propagator carefully, we find the leading term of the equal time correlator of \( \theta_+ \) stays the same as the balanced case \( < \theta_+ (\vec{q}) \theta_+ (\vec{q}) > = 2 \times \frac{2m}{q^2} + O (\frac{1}{q^4}) \) which leads to the same off-diagonal algebraic order exponent 2 as in the balanced case. In the \( q, \omega \rightarrow 0 \) limit, we can extract the leading terms in the \( \theta_- \theta_- \) propagator \( < \theta_- \theta_- > = \frac{4 V_\perp (\vec{q})}{q^2} \nu \)

where \( v \) is the spin wave velocity in the balanced case and we can identify the spin wave velocity in the imbalanced case \( v_{\perp m}^2 = f v^2 = 4 \nu_1 \nu_2 v^2 = 4 \nu_1 (1 - \nu_1) v^2 \) which shows that the spin-wave velocity attains its maximum at the balanced case and decreases parabolically as im-balance increases.

As analyzed in detail in [8], it is hard to incorporate the Lowest Landau Level (LLL) projection in the CB approach, so the spin wave velocity can only be taken as a phenomenological parameter to be fitted to experiments. But as shown explicitly in the next subsection, the CB theory can be used to classify topological excitations correctly and systematically.

(2) Topological excitations: There are following 4 kinds of topological excitations: \( \Delta \theta_1 = \pm 2\pi, \Delta \theta_2 = 0 \) or \( \Delta \theta_1 = 0, \Delta \theta_2 = \pm 2\pi \). Namely \( (m_1, m_2) = (\pm 1, 0) \) or \( (m_1, m_2) = (0, \pm 1) \) where \( m_1 \) and \( m_2 \) are the winding numbers in \( \theta_1 \) and \( \theta_2 \) respectively. The fractional charges can be determined from the constraint \( \nabla \times \tilde{a} = 2\pi \delta \rho \) and the finiteness of the energy in the charge sector:

\[
q = \frac{1}{2\pi} \oint \tilde{a} \cdot d\tilde{l} = \frac{1}{2\pi} \times \frac{1}{2} \oint [\nabla \theta_+ + (\nu_1 - \nu_2) \nabla \theta_-] \cdot d\tilde{l}
= \frac{1}{2} m_+ + (\nu_1 - \nu_2) m_- \quad (3)
\]

where \( m_+ = m_1 + m_2 \).

We can classify all the possible topological excitations in terms of \( (q, m_-) \) in the following table:

| \( m_+ \) | \( m_- \) | \( q \) | \( \nu_1 \) | \( \nu_2 \) |
|---|---|---|---|---|
| 1 | -1 | 1 | -1 | -1 |
| 1 | -1 | 1 | -1 | 1 |

Table 1: The fractional charge in im-balanced case

(2) The finiteness of the energy in the spin sector dictates that any finite energy excitation must have \( m_- = 0 \), so the merons listed in table 1 are confined into the following two possible pairs at low temperature. (1) Charge neutral pairs: \( (\pm \nu_1, \pm 1) \) or \( (\pm \nu_2, \mp 1) \). The NGM will turn into these charge neutral pairs at large wavevectors. The pairs behave as bosons. (2) Charge 1 pair \( (\nu_1, 1) + (\nu_2, -1) \) or charge \(-1\) pair \(- (\nu_1, -1) + (\nu_2, 1) \). The pairs behave as fermions. They maybe the lowest charged excitations in BLQH and the main dissipation sources for the charge transports.

(3) Im-balance driven quantum phase transitions in the PSDW side: As said in (1), the imbalance is irrelevant in the ESF side, but we expect it is important in the PSDW side. Starting from the ESF side, it would be useful to calculate how the mageto-roton minimum depends on the imbalance \([12,8]\), unfortunately this is beyond the scope of the CB theory. Here we take a different strategy: starting from the PSDW side and studying how a small imbalance affects the PSDW. If the imbalance is sufficiently small, we expect the C-PSDW at the balanced case in Fig.2a is a very good reference state. Because it is a square lattice with the "up" pseudo-spins taking sublattice \( A \) and the "down" pseudo-spins taking sublattice \( B \), it is reasonable to start from a lattice model from the PSDW side. If we think the PSDW as a charge density wave (CDW) of bosons at half filling on a square lattice, then we can view the ESF to the PSDW as a superfluid to CDW transition in a boson Hubbard model of hard core bosons near half filling hopping on square lattice of bosons:

\[
H = -t \sum_{<ij>} (b_i^\dagger b_j + h.c.) - \mu \sum_i n_i + V_1 \sum (n_i - 1/2) (n_j - 1/2) + V_2 \sum n_k n_l + \cdots \quad (4)
\]

where \( S^+ = b_i^\dagger c_i^2, S^- = b_i b_j, S^z = \frac{1}{2} (c_i^\dagger c_i - c_i c_i^\dagger) = b_i b_j - 1/2 \) are the pseudo-spin density and boson operators. At the total filling factor \( \nu = 1 \), we can impose the local constraint \( c_i^\dagger c_i + c_i c_i^\dagger = 1 \). \( n_i = b_i^\dagger b_i \) is the boson density, \( t \) is the nearest neighbor hopping amplitude, \( V_1, V_2 \) are the nearest and next nearest neighbor repulsive interactions between the bosons. The \( \cdots \) may include further neighbor interactions. Because of
the long-rang Coulomb interaction in Eqn.2, it is important to keep all the long-range interactions in the lattice model Eqn.4. If the chemical potential $\mu = 0$, the bosons are at the half filling $< n_b > = 1/2$ which corresponds to the balanced case $\nu = 1/2$. The particle-hole symmetry of Eqn.4 corresponds to the $Z_2$ exchange symmetry of the BLQH. If the chemical potential $\mu \neq 0$, the bosons are slightly away from the half filling which corresponds to the slightly imbalanced case.

The boson Hubbard model Eqn. 4 in square lattice at generic commensurate filling factors $f = p/q$ ( $p,q$ are relative prime numbers ) were systematically studied in [10] by performing the charge-vortex duality transformation. Recently, we applied the dual approach to study reentrant supersolids and quantum phase transitions from solids to the reentrant supersolids on extended boson Hubbard models Eqn.4 at in-commensurate filling factors in bipartite lattices such as square and honeycomb lattices. In the following, we apply the results achieved in [11] to the present problem on square lattice at and slightly away from half filling. At $q = 2$, there are two dual vortex fields $\psi_a$ and $\psi_b$. Moving slightly away from half filling $f = 1/2$ corresponds to adding a small mean dual magnetic field $H \sim \delta f = f - 1/2$ in the dual action. The most general action invariant under all the MSG transformation laws up to quartic terms is [10,11]:

$$L = \sum_{a=a/b} (|\partial_\mu - iA_\mu|\psi_a|^2 + r'|\psi_a|^2 + \frac{1}{4}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - 2\pi\delta f\partial_\mu\nu\lambda)^2 + \gamma_0(|\psi_a|^2 + |\psi_b|^2)^2 - \gamma_1(|\psi_a|^2 - |\psi_b|^2)^2 + \cdots$$

where $A_\mu$ is a non-compact $U(1)$ gauge field. Up to the quartic level, Eqn.5 is the same in square lattice and in honeycomb lattice. If $r > 0$, the system is in the superfluid state $< \psi_1 > = 0$ for every $l = a/b$. If $r < 0$, the system is in the insulating state $< \psi_1 > \neq 0$ for at least one $l$. We assume $r < 0$ in Eqn.5, so the system is in the insulating state. $\gamma_1 > 0$ ($\gamma_1 < 0$) corresponds to the Ising (or Easy-plane) limit. The insulating state takes the CDW state (or valence bond solid (VBS) state).

In the balanced case $\delta f = 0$, the SF to the VBS transition in the easy plane limit was argued to be 2nd order through a novel deconfined quantum critical point [10]. However, the boson Hubbard model Eqn.4 on the PSDW side corresponds to the Ising limit in the dual model Eqn.5, therefore $\gamma_1 > 0$. The SF to the CDW transition in the Ising limit is first order. This is consistent with the first order ESF to PSDW transition driven by the collapsing of magnetoroton minimum studied in [9].

In the CDW order side, the mean field solution is $\psi_a = 1$, $\psi_b = 0$ or vice versa. In the slightly imbalance case $\delta f \neq 0$, setting $\psi_b = 0$ in Eqn.5 leads to:

$$L = (|\partial_\mu - iA_\mu|\psi_a|^2 + r'|\psi_a|^2 + \gamma_0|\psi_a|^4 + \cdots + \frac{1}{4}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - 2\pi\delta f\partial_\mu\nu\lambda)^2$$

$$\text{where } \rho_A = \psi_a^\dagger \psi_a \text{ should be interpreted as the vacancy number, while the vortices in its phase winding are interpreted as boson number. Of course, a negative imbalance can simply achieved by a particle hole transformation } \psi_a \rightarrow \psi_a^\dagger, \delta f \rightarrow -\delta f \text{ in Eqn.6.}$$

Eqn.6 has the structure identical to the conventional $q = 1$ component Ginzburg-Landau model for a type II "superconductor" in a "magnetic" field. It was well known that as the magnetic field increases, there are two first order phase transitions: $H < H_{c1}$, the system is in the Messiner phase, $H_{c1} < H < H_{c2}$, it is in the vortex lattice phase, $H > H_{c2}$ it is in the normal phase. In the present boson problem with the nearest neighbor interaction $V_1 > 0$ and further neighbor interactions in Eqn.4 which stabilizes the CDW state at $f = 1/2$ (Fig.2a) and IC-CDW state at $f = 1/2 + \delta f$ (Fig.2b), this corresponds to C-CDW to IC-CDW to superfluid transition shown in Fig.1a. Transferring back to the original BLQH problem, the small imbalance will first drive the C-PSDW to the in-commensurate pseudo-spin density wave (IC-PSDW), then drive a 1st order transition from the IC-PSDW to the ESF shown in Fig.1b.

As shown in Fig.1b, the bias voltage increases, the imbalance will first introduce interstitials in the top layer and vacancies in the bottom layer, namely, turn the C-PSDW into the IC-PSDW whose charge distributions are shown in Fig.2a and 2b respectively, then the whole IC-PSDW melts into the ESF through a 1st order transition.
Disorders may smear all the 1st order transitions in Fig.1b into 2nd order transitions. The disorders may also transfer the long range lattice orders of the C-PSDW and IC-PSDW into short range ones. The fact make the observation of the commensurate and incommensurate lattice structures by light scattering experiment [14] difficult.

The dashed line in Fig.1b was investigated in a recent experiment [13]. But the first phase transition in Fig.1b was not paid attention in the experiment where the phase diagram was drawn against fixed charge imbalance $h_z$ instead of fixed bias voltage $V$. So the C-PSDW phase was crushed into the horizontal axis. A simple mean field argument leads to the linear scaling of the second transition line $V \sim d - d_{c1}$. A parabolic behavior $h_z^2 \sim d - d_{c1}$ was found for the shape of the second transition at very small imbalances. We expect the disorders may transform the linear behavior to the parabolic one. In the presence of disorders, all the properties of the C-PSDW and IC-PSDW are consistent with the experimental observations in [13] on the intermediate phase at small imbalances.

When the distance of the two layers is further increased to larger than a second critical distance $d_{c2}$, then all the signature of the interlayer coherent state are lost, the two layers are decoupled into two separate $\nu = 1/2$ CF Fermi liquid state (Fig.1b). We expect that there is a level crossing and associated first order transition at $d_{c2}$. When $d > d_{c2}$, increasing the bias voltage may not transform the two decoupled FL state back into the ESF.

(4) Discussions: By projecting into the LLL and then performing the HF approximation (LLL+HF approach), the authors in [12] found that the transition at $h_z = 0$, $d = d_{c1}$ is an instabiliy through a 1st order transition to a pseudospin density wave state driven by the gap closing of magneto-roton minimum at a finite wavevector. Starting from the ESF side, their numerical results indicated that the imbalance increases the spin stiffness and the critical distance $d_{c1}$. It is not known if the LLL+HF calculations are accurate in describing the transition from the ESF to the pseudo-spin density wave state. Our effective theory circumvents the difficulty associated with the still not precisely known wavefunction at any finite $d$ [15,16]. We study the effects of small imbalance from the PSDW side and map its effect as a chemical potential of hard core bosons with long range interactions hopping on a square lattice near half filling, namely, mapping Fig.1a to Fig.1b. Our upper phase boundary in Fig.1b is consistent with that achieved in [12] from the ESF side by the LLL+HF calculation. We also worked out the lattice structures of the C-PSDW and IC-PSDW and the whole physical picture along the dashed line in Fig.1b. It is not known how to apply the LLL+HF theory in [12] to study the PSDW side. The effective theory presented in this paper is complementary to and goes well beyond the previous LLL+HF calculations.

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[1] For reviews of bilayer quantum Hall systems, see S. M. Girvin and A. H. Macdonald, in Perspectives in Quantum Hall Effects, edited by S. Das Sarma and Aron Pinczuk (Wiley, New York, 1997).
[2] J. P. Eisenstein, L. N. Pfeiffer and K. W. West, Phys. Rev. Lett. 69, 3804 (1992); Song He, P. M. Platzman and B. I. Halperin, Phys. Rev. Lett. 71, 777 (1993).
[3] I. B. Spielman et al, Phys. Rev. Lett. 84, 5808 (2000). ibid. 87, 036803 (2001).
[4] M. Kellogg, et al, Phys. Rev. Lett. 88, 126804 (2002).
[5] M. Kellogg, et al, cond-mat/0401521. For a review, see J.P. Eisenstein and A.H. MacDonald, cond-mat/0404113.
[6] M. Kellogg, et al, Phys. Rev. Lett. 90, 246801 (2003).
[7] L. Brey, Phys. Rev. Lett. 65, 903 (1990); R. Cote, L. Brey and A. H. MacDonald, Phys. Rev. B 46, 10239 (1992); X. M. Chen and J. J. Quinn, Phys. Rev. Lett. 67, 895 (1991), Phys. Rev. B 45, 11054 (1992), Phys. Rev. B 47, 3990 (1993); L. Zheng and H. A. Fertig, Phys. Rev. B 52, 12282 (1995) S. Narasimhan and Tin-Lun Ho, Phys. Rev. B 52, 12291 (1995)
[8] Jinwu Ye, cond-mat/0310512.
[9] Jinwu Ye, cond-mat/0606369.
[10] L. Balents, L. Bartosch, A. Burkov, S. Sachdev, K. Sengupta, Phy. Rev. B 71, 144508 (2005).
[11] Jinwu Ye, cond-mat/0503113.
[12] Y. N. Joglekar and A. H. Macdonald, Phys. Rev. B 65, 235319 (2002).
[13] I.B. Spielman, et al, cond-mat/0406067.
[14] For light scattering experiments, see Aron Pinczuk, Chap. 8 in the book in [1].
[15] Gun Sang Jeon and Jinwu Ye, Phys. Rev. B 71, 035348 (2005).
[16] Longhua Jiang and Jinwu Ye, cond-mat/0607715, to be published in Phys. Rev. B.