B-strings on non-Kählerian manifolds

Camillo Imbimbo$^{1,2,a}$

$^1$ Dipartimento di Fisica, Università di Genova, Via Dodecaneso 33, 16146, Genoa, ITALY
$^2$ INFN, Sezione di Genova, Via Dodecaneso 33, 16146, Genoa, ITALY

$^acamillo.imbimbo@ge.infn.it$

Abstract

We explain how to couple topological B-models whose targets are non-Kählerian manifolds to topological gravity and to thus define corresponding topological strings. We emphasize the need to take into account the coupling to the superghost field of topological gravity in order to obtain a consistent definition of the string model. We also review the importance of the superghost for correctly interpreting the holomorphic anomaly of the string amplitudes. We perform our analysis in the BV framework in order to make it completely gauge independent.
Contents

1 Introduction 2

2 BRST transformations of the matter B-model 9

3 The coupling to topological gravity 11

4 The gauge fermion 12

5 BV formulation 14

6 Observables 16
   6.1 The differential geometry of complex structure moduli space . . . . . . . 18
   6.2 Observables associated to complex structure deformations . . . . . . . 21

7 Varying the parameters of the BV action 22
   7.1 Dependence on the target space metric . . . . . . . . . . . . . . . . . . . . 25
   7.2 Dependence on the holomorphic complex structure moduli . . . . . . . . 26
   7.3 The dependence on the anti-holomorphic complex structure moduli . . . . 28
1 Introduction

The original formulation of topological gravity [1] is deceptively simple: it is characterized by a BRST operator which acts as an exterior differential on the space of space-time metrics

\[ S_0 g_{\alpha\beta} = \psi_{\alpha\beta} \quad S_0 \psi_{\alpha\beta} = 0 \]  

(1.1)

where \( g_{\alpha\beta} \) is a space-time Riemannian metric and \( \psi_{\alpha\beta} \) the topological gravitino field.

However, if Eq. (1.1) were all, topological gravity would have no physical content, since the local BRST local cohomology of such nilpotent transformations is obviously empty. Raymond Stora [2] (together with others [3]), had the crucial insight which clarified the physical meaning of the theory. He understood that the relevant notion for topological gravity was local BRST cohomology equivariant with respect to diffeomorphisms. The definition of the equivariant BRST operator requires introducing the reparametrization ghost fields \( c^\alpha \) of ghost number +1 together with the ghost-for-ghost, or superghost field \( \gamma^\alpha \) of ghost number +2. The equivariant nilpotent BRST transformations which, according to Raymond, must replace (1.1) are then

\[
\begin{align*}
    s g_{\alpha\beta} &= -\mathcal{L}_c g_{\alpha\beta} + \psi_{\alpha\beta} \\
    s \psi_{\alpha\beta} &= -\mathcal{L}_c \psi_{\alpha\beta} + \mathcal{L}_\gamma g_{\alpha\beta} \\
    s c^\alpha &= -\frac{1}{2} \mathcal{L}_c c^\alpha + \gamma^\alpha \\
    s \gamma^\alpha &= -\mathcal{L}_c \gamma^\alpha
\end{align*}
\]  

(1.2)

\( \mathcal{L}_c \) is the Lie derivative implementing reparametrizations associated to the vector field \( c^\alpha \).

The usual local cohomology of the BRST operator \( s \) is just as trivial as that of the simple minded \( S_0 \) (1.1). However the equivariant cohomology of \( s \), which is the local cohomology of \( s \) on the space of field functionals which do not include the reparametrization ghost \( c^\alpha \), is not: this is the cohomology which characterizes the physical observables of the theory.

When working with the equivariant cohomology it is convenient to define the operator

\[ S = s + \mathcal{L}_c \]  

(1.3)

which acts as follows

\[
\begin{align*}
    S g_{\alpha\beta} &= \psi_{\alpha\beta} \\
    S \psi_{\alpha\beta} &= \mathcal{L}_\gamma g_{\alpha\beta} \\
    S \gamma^\alpha &= 0
\end{align*}
\]  

(1.4)

Nilpotency of \( s \) is equivalent to

\[ S^2 = \mathcal{L}_\gamma \]  

(1.5)
on the space of fields \( g_{\alpha\beta}, \psi_{\alpha\beta} \) and \( \gamma^\alpha \). The equivariant cohomology of \( s \) is therefore the same as the cohomology of \( S \) on \textit{reparametrization invariant functionals} of the fields \( g_{\alpha\beta}, \psi_{\alpha\beta} \) and \( \gamma^\alpha \). It is useful to decompose \( S \) as the sum of two nilpotent operators

\[
S = S_0 + G_\gamma
\]  

where

\[
\begin{align*}
S_0 g_{\alpha\beta} &= \psi_{\alpha\beta} & S_0 \psi_{\alpha\beta} &= 0 & S_0 \gamma^\alpha &= 0 \\
G_\gamma g_{\alpha\beta} &= 0 & G_\gamma \psi_{\alpha\beta} &= \mathcal{L}_\gamma g_{\alpha\beta} & G_\gamma \gamma^\alpha &= 0
\end{align*}
\]

Eq. (1.5) is equivalent to the super-algebra

\[
S_0^2 = G_\gamma^2 = 0 \quad \{S_0, G_\gamma\} = \mathcal{L}_\gamma
\]  

\[\text{(1.8)}\]

\( S_0 \) is the “naive” topological gravity BRST operator (1.1) whose local cohomology is empty. The \( G_\gamma \) part of the BRST transformations, linear in the superghost \( \gamma^\alpha \), provides the extension of \( S_0 \) to the equivariant, non-trivial, \( S \).

In two space-time dimensions one can couple topological gravity to topological matter and define in this way \textit{topological strings} [4]— much in the same way as ordinary (super)strings are defined by coupling 2-dimensional (super)gravity to (super)conformal matter field theories.

Topological matter theories are characterized by nilpotent BRST operators \( S_{\text{matter}}^0 \). Coupling the topological matter field theory to topological gravity means to extend \( S_{\text{matter}}^0 \) to the gravity sector, in accordance to (1.2), by including diffeomorphisms acting on the matter fields

\[
s = -\mathcal{L}_c + S_0 + G_\gamma
\]  

\[\text{(1.9)}\]

Nilpotency of \( s \) on the matter fields requires adding to \( S_{\text{matter}}^0 \) a piece, denoted by \( G_{\text{matter}}^\gamma \), which is linear in the superghost \( \gamma^\alpha \) and which satisfies, together with \( S_{\text{matter}}^0 \), the same super-algebra (1.8) which holds in the gravity sector.

From what we just said, it is apparent that the superghost dependent part of the equivariant BRST transformations is, from the algebraic point of view, the crucial ingredient necessary for the consistent, equivariant, coupling of topological matter to topological gravity. It is hence curious that, as a matter of fact, the superghost \( \gamma^\alpha \) rarely makes its appearance, explicitly, in the immense literature devoted to topological strings. To understand why, we need to recall the general features of the topological strings construction.
The action of topological matter quantum field theory has the form

$$\Gamma^{\text{matter}}[\Phi, g_{\alpha\beta}] = \Gamma_0[\Phi] + \int S^\text{matter}_0[\Phi, g_{\alpha\beta}]$$  \hspace{1cm} (1.10)

where $\Phi$ denotes schematically the collection of matter fields; $\Gamma_0[\Phi]$ is both $S^\text{matter}_0$ invariant and invariant under space-time diffeomorphisms without the help of a space-time metric — i.e. it is a topological term. In certain cases one can take the “classical” term $\Gamma_0[\Phi]$ to vanish: this happens when the local cohomology of the matter BRST operator is empty. The action for these theories — which are called of “cohomological” type — reduces to a pure gauge-fixing term. Semi-classical approximation is exact for cohomological theories and we will restrict, for simplicity, the following discussion to this class of topological theories.

The second term in the action (1.10) is a gauge-fixing term: the gauge fermion $\Psi[\Phi, g_{\alpha\beta}]$ is arbitrary as long as it provides non-degenerate kinetic terms for all the matter fields. To this end, it necessarily depends on a background space-time metric $g_{\alpha\beta}$. For matter topological quantum field theories $g_{\alpha\beta}$ plays the role of a gauge-fixing parameter: the physics, thanks to the nilpotency of $S^\text{matter}_0$, does not depend on the specific choice for $g_{\alpha\beta}$.

To construct the topological string model based on a given matter topological QFT one needs to extend $S^\text{matter}_0$ to the gravity sector in the equivariant way, as prescribed in (1.9). For cohomological theories, the action of the coupled system takes therefore the form

$$\Gamma^{\text{mat+t.g.}}[\Phi, g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^\alpha] = \int S[\Phi, g_{\alpha\beta}] =$$

$$= \Gamma^{\text{matter}}[\Phi, g_{\alpha\beta}] + \int \psi^{\alpha\beta} S_{\alpha\beta}[\Phi, g_{\alpha\beta}] + \int G_{\gamma} \Psi[\Phi, g_{\alpha\beta}]$$  \hspace{1cm} (1.11)

where $S_{\alpha\beta}$ is the topological super-current

$$S_{\alpha\beta} = \frac{\delta \Psi[\Phi, g_{\alpha\beta}]}{\delta g^{\alpha\beta}}$$  \hspace{1cm} (1.12)

From (1.11) we see that topological matter couples to the topological gravity multiplet not only via the super-current and the gravitino field $\psi^{\alpha\beta}$, but also by means of terms proportional to the superghost $\gamma^\alpha$.

The partition function obtained by integrating the matter fields

$$Z[g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^\alpha] = \int [d \Phi] e^{-\Gamma^{\text{mat+t.g.}}[\Phi, g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^\alpha]}$$  \hspace{1cm} (1.13)

is a functional of the topological gravity multiplet which satisfies the BRST identity

$$SZ[g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^\alpha] = 0$$  \hspace{1cm} (1.14)
This identity says that $Z[g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^\alpha]$ is an equivariant closed form on the space of space-time metrics: because of this, it can be pulled back to a closed form on the moduli space of Riemann surfaces [5]. The component of this form of the appropriate fermionic number can be integrated on the moduli space of genus $g$ surfaces: this operation defines topological string amplitudes of the corresponding genus.

Although the functional (1.14) which defines the string amplitudes does in general depend on the superghost field $\gamma^\alpha$, there are topological models for which one can choose gauge fermions $\Psi[\Phi, g_{\alpha\beta}]$ invariant under $G_\gamma$

$$G_\gamma \int \Psi[\Phi, g_{\alpha\beta}] = 0$$

(1.15)

Whenever such a choice of the gauge fermion is possible, the last term in the action (1.11) vanishes, the resulting topological string action does not depend on the $\gamma^\alpha$ superghost and the coupling to topological gravity only occurs via the super-current:

$$\Gamma^{\text{mat-t.g.}}[\Phi, g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^\alpha] = \Gamma^{\text{matter}}[\Phi, g_{\alpha\beta}] + \int \psi^{\alpha\beta} S_{\alpha\beta}[\Phi, g_{\alpha\beta}]$$

(1.16)

This is in precisely the situation mostly considered in the literature on topological strings and the reason why, in those contexts, the superghost $\gamma^\alpha$ is usually neglected and one does not bother with the equivariant paradigm. In this paper we will show that this point of view is however limited and it comes with a price. First of all it is too restrictive when analysing situations in which the $G_\gamma$-invariant choice (1.15) for the gauge fermion is not allowed. We will elaborate on a specific example when this occurs. Moreover, we will explain that even in the familiar situation in which (1.15) is possible, neglecting the superghost leads to conceptual puzzles when one attempts to understand such an important feature of topological strings as the holomorphic anomaly.

The typical context in which choice (1.15) for the gauge fermion is usually possible are the topological matter theories which are obtained by twisting supersymmetric non-linear sigma models models with extended $N = 2$ supersymmetry [6]. The spinorial supercharges of $N = 2$ supersymmetric model transform, upon twisting, into a scalar nilpotent supercharge which can be identified with the BRST matter operator $S^\text{matter}_0$ together with a vector supercharge $\hat{G}_\alpha^\alpha$. The twisting turns the $N = 2$ extended supersymmetry algebra into the topological

$\hat{G}_\alpha^\alpha$.

$^1$We denote by $\hat{G}_\alpha$ the vector supercharge of the matter theory, where $\alpha = 1, 2$ is a vector world-sheet index. This should not generate confusion with $G_\gamma$, which is the scalar BRST-operator, where the index $\gamma$ refers to the superghost $\gamma^\alpha$. 
super-algebra

$$\{ S_0, \hat{G}_\alpha \} = P_\alpha \quad S_0^2 = 0 = \{ \hat{G}_\alpha, \hat{G}_\beta \}$$  \hspace{1cm} (1.17)

where $P_\alpha$ are the space-time momentum generators. Comparing this with (1.8), one is lead to conjecture that the equivariant extension $G^\text{matter}_\gamma$ of the BRST symmetry of twisted $N = 2$ matter is obtained by promoting the global vector supersymmetry of the twisted matter model to a local symmetry

$$G^\text{matter}_\gamma = \gamma^\alpha(x) \hat{G}_\alpha$$  \hspace{1cm} (1.18)

A legitimate action for the topological model is the action of the supersymmetric model, appropriately twisted: this is invariant under both the scalar $S_0$ supersymmetry and the global vector $\hat{G}_\alpha$ supersymmetry:

$$\hat{G}_\alpha \Gamma^\text{matter}[\Phi, g_{\alpha\beta}] = S_0^\text{matter} \Gamma^\text{matter}[\Phi, g_{\alpha\beta}] = 0$$  \hspace{1cm} (1.19)

Therefore, Noether theorem ensures that, for $\gamma^\alpha(x)$ local

$$G^\text{matter}_\gamma \Gamma^\text{matter}[\Phi, g_{\alpha\beta}] = - \int D^\alpha \gamma^\beta \tilde{S}_{\alpha\beta}$$  \hspace{1cm} (1.20)

where $\tilde{S}_{\alpha\beta}$ is the super-current associated to global supercharge $G_\beta$. It turns out that, in the specific case of the twisted $N = 2$ supersymmetric non-linear sigma model, the super-current $\tilde{S}_{\alpha\beta}$ can be taken to be symmetric in the indices $\alpha$ and $\beta$ and the super-algebra (1.17) extends to an algebra of local currents

$$\{ S_0, \tilde{S}_{\alpha\beta} \} = T_{\alpha\beta}$$  \hspace{1cm} (1.21)

where $T_{\alpha\beta}$ is the stress energy tensor. Comparing this with (1.12), one identifies $\tilde{S}_{\alpha\beta}$ with $S_{\alpha\beta}$, which is therefore conserved

$$D^\beta S_{\alpha\beta} = 0$$  \hspace{1cm} (1.22)

In this situation the action one obtains by using the “naive” $S_0$, rather than the equivariant $S$,

$$S_0 \int \Psi[\Phi, g_{\alpha\beta}] = \Gamma^\text{matter}[\Phi, g_{\alpha\beta}] + \int \psi^{\alpha\beta} S_{\alpha\beta}[\Phi, g_{\alpha\beta}]$$  \hspace{1cm} (1.23)

is invariant under both $S_0$ and the local symmetry $G_\gamma$ which acts on the gravitino as $G_\gamma \psi^{\alpha\beta} = D^{(\alpha\gamma\beta)}$ (see Eq. (1.7)). In other words $\Psi[\Phi, g_{\alpha\beta}]$ can be chosen to be invariant under the local $G_\gamma$ transformations.
Two-dimensional supersymmetric non-linear sigma models enjoy extended $N = 2$ supersymmetry when the target space is a complex manifold equipped with a Kählerian metric. One way to twist the $N = 2$ two-dimensional supersymmetric non-linear sigma models leads to the $B$-model [6], which is a topological model of the cohomological type. The physics of the $B$-topological sigma model is expected to depend on the complex structure of the target manifold but not on the target space metric. In particular we will see that the $B$-model can be defined also when the metric on the complex target manifold is not Kählerian. In this case the supersymmetric sigma model does not enjoy extended supersymmetry, the corresponding topological action is not invariant under the vector supersymmetry, and topological super-current $S_{\alpha\beta}$ is not conserved. From our discussion above, one does not expect that a choice of a $G_{\gamma}$-invariant gauge fermion be possible in this situation: the coupling to the superghost must necessarily be taken into account for a consistent definition of the string amplitudes. If one neglects the superghost dependent term in the string action (1.11) the resulting partition function does not define an equivariant form in the space of metrics which can be integrated on the moduli space of Riemann surfaces to produce consistent string amplitudes. We will see that the equivariant formulation of the $B$-model coupled to topological gravity restores target space metric independence even for complex manifolds for which one cannot pick a Kähler metric.

Even when the $G_{\gamma}$-invariant choice for the gauge fermion is possible for a fixed topological matter model, one is often interested in deforming a given matter model and consider the dependence of the physics on the moduli which parametrize such deformations. In the case of the $B$-model among those deformations are the anti-holomorphic deformations of the complex structure of the target space variety. It turns out that the matter BRST operator $S_{0}^{\text{matter}}$ is independent of such anti-holomorphic deformations. Let us denote by $\partial_{\bar{a}}$ the anti-holomorphic derivative with respect to the complex moduli $(m^{a}, m^{\bar{a}})$ which parametrize complex structures of the target space variety. Assume that for a given complex structure of the target space (1.15) applies, so that one can neglect the $\gamma^{\alpha}$ dependent term in the string action

$$\Gamma^{\text{mat+t.g.}} [\Phi, g_{\alpha\beta}, \psi_{\alpha\beta}, \gamma^{\alpha}] = \int S \Psi[\Phi, g_{\alpha\beta}] = \int S_{0} \Psi[\Phi, g_{\alpha\beta}]$$

Taking the anti-holomorphic derivative $\partial_{\bar{a}}$ of the string action, one would then obtain

$$\partial_{\bar{a}} \Gamma^{\text{mat+t.g.}} = \int \partial_{\bar{a}} S \Psi[\Phi, g_{\alpha\beta}] = \int S_{0} \left( \partial_{\bar{a}} \Psi[\Phi, g_{\alpha\beta}] \right)$$

since, as stated above, $S_{0}$ is holomorphic in the complex moduli $(m^{a}, m^{\bar{a}})$. One would then be led to think that anti-holomorphic deformations of the target space complex structure
are BRST trivial and that the string amplitudes are holomorphic functions of the complex moduli \((m^a, m^\bar{a})\). Explicit computations show that this is actually not the case \cite{7}: in the formulation which neglects \(G_\gamma\), the non-holomorphicity of the string amplitudes seems therefore to signal a BRST anomaly.

This however cannot be the case: a genuine BRST anomaly, like any anomaly of local gauge symmetries, would destroy the consistency of the corresponding quantum topological string theory. Fortunately for topological strings, non-holomorphicity of the string amplitudes is in fact not associated to any anomaly of the *equivariant* BRST symmetry — which is, as explained, the relevant notion of BRST symmetry in this context. To understand this, consider the anti-holomorphic derivative of the (1.15)

\[
0 = \partial_a(G_\gamma \int \Psi[\Phi, g_{\alpha\beta}] = \int [\partial_a, G_\gamma] \Psi[\Phi, g_{\alpha\beta}] + \int G_\gamma (\partial_a \Psi[\Phi, g_{\alpha\beta}])
\]

(1.26)

It turns out that \(G_\gamma\) is *not* holomorphic in the complex moduli \((m^a, m^\bar{a})\), and, correspondingly, that the deformation of the gauge fermion \(\partial_a \Psi[\Phi, g_{\alpha\beta}]\) is not \(G_\gamma\)-invariant. Therefore the anti-holomorphic deformation in (1.25), although trivial with respect to the “naive” \(S_0\), is *not* trivial with respect to the equivariant \(S\). Hence anti-holomorphicity is perfectly consistent with BRST invariance with respect to the full, equivariant, \(S\). In reality although the equivariant \(S\) is not holomorphic, its anti-holomorphic variation is a \(S\)-commutator. This ensures that the anti-holomorphicity of the string amplitudes be captured by local contact terms which are explicitly calculable.

The focus of this paper is the relevance of the equivariant superghost of topological gravity to topological strings. It might be useful to add that more recently it has been understood that the superghost of topological gravity plays a prominent role also in the topological formulation of localization of supersymmetric quantum field theories in arbitrary dimension \cite{8}.

The work contained in this article builds on and extends results obtained years ago with my longtime collaborators, Carlo Becchi and Stefano Giusto. Those earlier results are contained in Giusto’s doctoral dissertation \cite{9}, but were never published. At that time we limited ourselves to considering B-strings that are obtainable by twisting supersymmetric sigma models with Kähler target manifolds and emphasized the importance of the superghost of topological gravity for the correct interpretation of the holomorphic anomaly. Many years later, Alessandro Tomasiello informed me that he, in collaboration with Anton Kapustin, had considered B-models with non-Kählerian target spaces and had attempted to build
topological string models based on them. Tomasiello and Kapustin were able to define the
topological matter model by making use of a target space connection built with the aid of
a hermitian but non-Kählerian metric, a construction which I review in Section 2. As they
realized, however, the non-conservation of the super-current did not allow for a consistent
definition of topological string amplitudes. This result provided me with the motivation
for returning to the unpublished work from my collaboration with Becchi and Giusto and
applying it to the non-Kähler situation in order to show that the difficulty encountered
by Kapustin and Tomasiello could be solved by taking into account the coupling of the
matter B-model to the equivariant superghost of topological gravity. As explained in the
introduction the coupling to the superghost is, to a certain extent, gauge-dependent. In
order to make my analysis and considerations completely gauge-independent, and thus more
widely applicable, I decided to extend the work in [9] to the more general BV framework
(Section 5). In the last section of the present paper, I also generalize the discussion of the
holomorphic anomaly found in [9], extending it to B-strings on non-Kähler manifolds in a
completely gauge-independent set up.

2 BRST transformations of the matter B-model

The topological matter B-model is defined on a complex variety whose complex coordinates
we will denote by \((\phi^i, \phi^\bar{i})\). The BRST nilpotent transformation rules are
\[\begin{align}
S_0 \phi^i &= 0 \\
S_0 \rho^i &= -d \phi^i \\
S_0 F^i &= -D \rho^i + \frac{1}{2} R^i_{j;k} \sigma^j \rho^j \rho^k \\
S_0 \phi^\bar{i} &= \sigma^\bar{i} \\
S_0 \sigma^\bar{i} &= 0
\end{align}\] (2.1)

In order to preserve covariance of the model under holomorphic reparametrizations of the
target space coordinates we introduced a hermitian, but not necessarily Kähler, metric \(g_{ij}\)

---

\(\rho^i = \rho^i_\alpha \, dx^\alpha\) is a 1-form of ghost number -1, \(F^i = \frac{1}{2} F^i_{\alpha\beta} \, dx^\alpha \, dx^\beta\), a 2-form of ghost number -2. The
sum of form degree and ghost number defines the total fermionic number. Both the BRST operator and the
exterior differential are odd with respect to total fermionic number.
and a connection whose non-vanishing components are $\Gamma_{jk}^i$ and $\bar{\Gamma}_{jk}^i$:

$$\Gamma_{jk}^i = \frac{1}{2} g^{ij} \left( \partial_j g_{ki} + \partial_k g_{ji} \right) \equiv g^{ij} \partial_{(j} g_{k)i}$$

$$\bar{\Gamma}_{jk}^i = \frac{1}{2} g^{ij} \left( \partial_j g_{\bar{k}i} + \partial_{\bar{k}} g_{ji} \right) \equiv g^{ij} \partial_{(j} g_{k)i}$$

(2.2)

The non-vanishing components of the curvature are

$$R_{ijk}^l = \partial_l \Gamma_{jk}^i - \partial_j \Gamma_{lk}^i + \Gamma_{jm}^i \Gamma_{kl}^m - \Gamma_{jm}^i \Gamma_{kl}^m$$

(2.3)

and their complex conjugates. If $g_{ij}$ is not Kähler, it is not covariantly constant

$$D_i g_{jk} = \frac{1}{2} \left( \partial_i g_{jk} - \partial_j g_{ik} \right) \equiv \partial_{[i} g_{j]k} \equiv C_{[ij];k}$$

(2.4)

The action is a pure gauge

$$\Gamma_0 = \int S_0 \Psi$$

(2.5)

where $\Psi$ is the gauge fermion. The traditional choice is

$$\Psi = \theta^i F_i + g_{ij} \rho^i \ast d \phi^j$$

(2.6)

where one has introduced a trivial BRST doublet

$$S_0 \theta_i = H_i, \quad S_0 H_i = 0$$

(2.7)

$\theta_i$ has ghost number +1 and plays the function of Nakanishi-Laudrup field, $H_i$ is a Lagrangian multiplier. The gauge fermion (2.6) gives non-degenerate kinetic terms for all fields:

$$\Gamma_0 = H_i F_i^i - \theta_i D \rho^i - \frac{1}{2} \theta_i R_{ij,k}^{i,j} \rho^i \rho^k + g_{ij} d\phi^i \ast d\phi^j - g_{ij} \rho^i \ast d\phi^j + \nu_{ij} d\phi^i \ast d\phi^j +$$

$$= H_i F_i^i - \theta_i D \rho^i - \frac{1}{2} \theta_i R_{ij,k}^{i,j} \rho^i \rho^k + g_{ij} d\phi^i \ast d\phi^j +$$

$$- g_{ij} \rho^i \ast D \sigma^j - \rho^i \ast d\phi^j \ast \sigma^k C_{[k\bar{i}]j} \bar{\sigma}^k \partial_{\bar{k}} g_{ij} =$$

(2.8)

In the Kähler case this action is obtained by twisting of the action of the $N = (2,2)$ supersymmetric non-linear sigma model.

---

3As I recalled in the Introduction, this specific connection has been suggested to me by A. Tomasiello.
3 The coupling to topological gravity

The coupling of the B-model to topological gravity is determined by requiring the validity of the super-algebra (1.8). For example, applying both two sides of Eq. (1.8) to $\phi^i$ one obtains

$$L_\gamma \phi^i = i_\gamma (d \phi^i) = \{ S_0, G_\gamma \} \phi^i = S_0 (G_\gamma (\phi^i)) = -i_\gamma (S_0 \phi^i) = S_0 i_\gamma (\rho^j)$$

(3.1)

where we introduced $i_\gamma$, the operation which contract a form along the superghost vector field $\gamma^\alpha$, and used the fact that

$$L_\gamma = \{ d, i_\gamma \}$$

(3.2)

on forms. One thus derives

$$G_\gamma (\phi^i) = i_\gamma (\rho^j)$$

(3.3)

Proceeding in this way one obtains the following BRST transformations

$$\hat{S} F^i \equiv S F^i + \Gamma^i_{i\gamma(\rho)j} F^j = -D \rho^j + \frac{1}{3} R_{kjl}^i i_\gamma (\rho^k) \rho^j \rho^l + \frac{1}{2} R_{kjl}^i \sigma^{ij} \rho^j \rho^l$$

$$\hat{S} \rho^i \equiv S \rho^i + \Gamma^i_{i\gamma(\rho)j} \rho^j = -d \phi^i + i_\gamma (F^i)$$

$$S \phi^i = i_\gamma (\rho^j)$$

$$S \sigma^i = \sigma^i$$

$$\hat{S} \sigma^i \equiv S \sigma^i + \Gamma^i_{i\gamma(\rho)j} \sigma^j = i_\gamma d \phi^i$$

$$\hat{S} \theta_i \equiv S \theta_i - \Gamma^j_{i\gamma(\rho)i} \theta_j = H^i$$

$$\hat{S} H_i \equiv S H_i - \Gamma^j_{i\gamma(\rho)i} H_j = i_\gamma D \theta_i - R_{kli}^j i_\gamma (\rho^k) \theta_j - \frac{1}{2} R_{kli}^j i_\gamma (\rho^k) i_\gamma (\rho^l) \theta_j$$

(3.4)

To simplify computations and notation we introduced, for all fields but the coordinate fields $(\phi^i, \phi^\bar{i})$, the BRST operator $\hat{S}$, covariant under target space holomorphic reparametrizations

$$\hat{S} \equiv S + \Gamma_{i\gamma(\rho)} \oplus \Gamma_{i\sigma}$$

(3.5)

where the connection pieces are matrices acting on (anti-)holomorphic indices in the standard way. $\hat{S}$ acts on the coordinates fields via covariant connections

$$\hat{S} = \sigma^i D_i + i_\gamma (\rho^j) D_i + \cdots$$

(3.6)

Therefore

$$\hat{S} \Psi = S \Psi$$

(3.7)
on functionals $\Psi$ which are invariant under target space holomorphic reparametrizations. The relation $S^2 = \mathcal{L}_\gamma$ translates into the following relation for $\hat{S}$:

$$
\hat{S}^2 = \{i_\gamma, D\} + \left( \frac{1}{2} R_{i_\gamma(i_\gamma)} + R_{\sigma i_\gamma}\right) \oplus \left( \frac{1}{2} R_{\sigma} + R_{\sigma i_\gamma}\right) = \\
\equiv \{i_\gamma, D\} + \frac{1}{2} R_{\chi\chi} + \frac{1}{2} \hat{R}_{\chi\chi}
$$

(3.8)

where

$$
R_{i_\gamma(i_\gamma)} \equiv R_{ij} \ i_\gamma(\rho^j) i_\gamma(\rho^i) \\
R_{\sigma i_\gamma} \equiv R_{\bar{i}i} \ \sigma^i i_\gamma(\rho^i) \\
\hat{R}_{\sigma} \equiv \hat{R}_{ij} \ \sigma^i \sigma^j \\
\hat{R}_{\sigma i_\gamma} \equiv \hat{R}_{ij} \ i_\gamma(\rho^i) \ \sigma^j \\
R_{\chi\chi} \equiv R_{i_\gamma(i_\gamma)} + R_{\sigma i_\gamma} + R_{\sigma i_\gamma}\ }
$$

(3.9)

are the matrix-valued curvature 2-forms acting on (anti-)holomorphic indices in the standard way,

$$
D = d + \Gamma_{d\phi} \oplus \bar{\Gamma}_{d\bar{\phi}}
$$

(3.10)

is the covariant derivative of ghost number 0, and

$$
\chi^I \equiv (i_\gamma(\rho^i), \sigma^i) \quad I = (i, \bar{i})
$$

(3.11)

is a world-sheet scalar of ghost number 1 which acts like the 1-differential on the target space.

### 4 The gauge fermion

The action of the coupled model is

$$
\Gamma = \int (S_0 + G_\gamma) \Psi = \Gamma_0^{\text{matter}} + \int \psi_{\alpha\beta} \ \delta \Psi \ \delta g_{\alpha\beta} + \int G_\gamma \Psi
$$

(4.1)

As recalled in the Introduction, the first term is the “matter” action. The second term is the standard coupling of the topological gravitino to the matter super-current

$$
S_{\alpha\beta} = \frac{\delta \Psi}{\delta g_{\alpha\beta}}
$$

(4.2)
which produces the insertions in topological string amplitudes which are analogous to the $b$-zero modes insertions of bosonic string theory.

The third term in the action does not usually appear in the “naive” recipe for coupling topological matter to topological gravity. With the usual choice for the gauge fermion (2.6), this term is

$$G_\gamma \Psi = -\theta_i \frac{1}{3} R^{i}_{jkl} i_\gamma (\rho^j) \rho^k \rho^l + C^{k}_{ij} i_\gamma (\rho^i) g_{k\bar{k}} \rho^i \star d\phi^k + g_{ij} i_\gamma (F^i) \star d\phi^j$$

As a matter of fact, this does not vanish even when the target space metric is Kähler, in which case it reduces to

$$G_\gamma \Psi = g_{ij} i_\gamma (F^i) \star d\phi^j = F^i (g_{ij} i_\gamma (\star d\phi^j))$$

Recalling the matter action (2.8) one sees however that all the $\gamma^\alpha$ dependence of the string action is confined, in the Kähler case, to the auxiliary sector $(F^i, H_i)$

$$\Gamma = F^i (H_i + g_{ij} i_\gamma (\star d\phi^j)) + \cdots$$

Hence the field redefinition

$$\tilde{H}_i = H_i + g_{ij} i_\gamma (\star d\phi^j)$$

is sufficient to eliminate the term linear in the auxiliary field $F^i$ in $G_\gamma \Psi$ and the $\gamma^\alpha$-dependence altogether from the string action when the metric is Kähler. Note that redefinition (4.6) of the Lagrangian multiplier $H_i$ amount to modifying the BRST transformations of the Nakanishi-Laudrup field $\theta_i$

$$\tilde{S} \theta_i = \tilde{H}_i - g_{ij} i_\gamma (\star d\phi^j)$$

by a term which depends on the world-sheet metric, i.e. a term which is not topological. This is harmless for the topological character of the theory since this is confined to the BRST-trivial $(\theta_i, H_i)$ sector.

When the metric is not Kählerian, however, even after the redefinition (4.6), one remains with non-vanishing coupling to the superghost

$$G_\gamma \Psi = -\theta_i \frac{1}{3} R^{i}_{jkl} i_\gamma (\rho^j) \rho^k \rho^l + C^{k}_{ij} i_\gamma (\rho^i) g_{k\bar{k}} \rho^i \star d\phi^k$$

In this case the consistent coupling of topological matter to topological gravity cannot be described only by the standard interaction with the gravitino — it requires superghost dependent terms. The reason for this is that when the target space metric is not Kähler,
the corresponding supersymmetric model does not enjoy extended supersymmetry and the super-current $S_{\alpha\beta}$ is not conserved.

To check this, let us remark that in the $G_\gamma$ transformation laws of the matter sector, $\gamma^\alpha$ appears with no derivatives, so we can write\(^4\)

$$G_\gamma = \gamma^\alpha \hat{G}_\alpha \quad \text{in the matter sector} \quad (4.9)$$

where $\hat{G}_\alpha$ are the vector super-charges of the matter theory satisfying

$$\{ S^\text{matter}_0, \hat{G}_\alpha \} = P_\alpha \quad \text{in the matter sector} \quad (4.10)$$

Hence

$$\hat{G}_\alpha \Gamma_0 = -S_0 \int \hat{G}_\alpha \Psi \quad (4.11)$$

We see therefore that $G_\gamma$-invariance of $\Psi$ ensures conservation of the super-current $S_{\alpha\beta}$. Conversely, if $\Psi$ is not $\hat{G}_\alpha$-invariant, we obtain, via the Noether procedure, the (non-)conservation equation for the super-current

$$D^\alpha S_{\alpha\beta} = S_0 \hat{G}_\beta \Psi = S_0 \left( C^k_{ij} g_{kk} \rho_\alpha g^{\beta\gamma} \rho_\beta \partial_\gamma \phi^k - \frac{1}{3} \partial_i R^i_{jkL} \rho_\alpha \epsilon^{\beta\gamma} \rho_\beta \rho_\gamma \right) \quad (4.12)$$

which does not vanish in the non-Kählerian case.

5 BV formulation

The action of a topological model of the cohomological type, like the B-model, is BRST-trivial\(^5\)

$$\Gamma = \int S \Psi \quad (5.1)$$

The reason is that the local BRST cohomology of such class of models is empty — therefore no non-trivial invariant term can show up in the action.

All the parameters, or coupling constants, which appear in the gauge fermion $\Psi$ are gauge parameters, and the physics does not depend on them. The physical parameters of cohomological models are contained in the BRST operator itself. We have seen that, in the

\(^4\)We denote by $\hat{G}_\alpha$ the vector supercharges of the matter theory, where $\alpha = 1, 2$ is a vector world-sheet index. This should not generate confusion with $G_\gamma$, which is the scalar BRST-operator, where the index $\gamma$ refers to the superghost $\gamma^\alpha$.

\(^5\)In recent times, theories of this type are being called “localizable”.

14
case of the B-model, in order to define $S$ we had to specify not only a complex structure on the target space, but also — to preserve target space holomorphic reparametrization invariance — a metric on it. Therefore, in principle, the physics could depend on both.

Let us denote by $\delta$ a generic deformation of the parameters — complex structure and metric — on which $S$ depends. The variation of the action under such a deformation takes the form

$$\delta \Gamma = \int \delta(S) \Psi + S \delta(\Psi) \quad (5.2)$$

Let us denote by

$$I \equiv \delta(S) \quad (5.3)$$

the operator of ghost number +1 which is obtained by deforming $S$. Since

$$S^2 = \mathcal{L}_\gamma \quad (5.4)$$

we obtain

$$\{I, S\} = 0 \quad (5.5)$$

Hence the deformation of the action satisfies

$$S(\delta \Gamma) = \int S I \Psi = -I \int S \Psi = -I \Gamma \quad (5.6)$$

i.e. the deformation of the action is BRST-closed modulo the equations of motion generated by $I$. Deformations $I$ which are $S$-commutators

$$I = [S, L] \quad (5.7)$$

with $L$ bosonic, must be considered trivial solutions of the consistency equation (5.5). Indeed, in this case the corresponding deformation of the action

$$\delta \Gamma = S \int (L \Psi + \delta \Psi) - L \Gamma \quad (5.8)$$

is BRST-trivial modulo the equations of motion generated by $L$.

This is the general paradigm of cohomological theories: Since the local BRST cohomology is empty, cohomological theories have no standard — i.e. BRST invariant — observables. Their only local observables are contained in the local BRST cohomology modulo the equation of motions. Because of this, a gauge-independent analysis of a topological theory of the
cohomological type requires upgrading the usual BRST framework to the Batalin-Vilkovisky one.

The BV action corresponding to the BRST transformations (3.4) is

\[
\Gamma_{BV} = \gamma^i (\rho^i) \phi_i^* + (-d\phi^i + \gamma^i (F^i) - \Gamma_{jk}^i \gamma^i (\rho^j) \rho_k^*) + \\
(-D \rho^i + \frac{1}{3} R_{k,j|i}^i \gamma^i (\rho^k) \rho^j + \frac{1}{2} R_{k,j,i}^i \gamma^k \rho^j \rho^i - \Gamma_{jk}^i \gamma^i (\rho^j) F^k) F_i^* + \\
+\sigma^i \phi_i^* + \gamma^i (d\phi^i) \sigma_i^* + \\
=\psi_{\alpha\beta} g^{*\alpha\beta} + \mathcal{L}_\gamma g_{\alpha\beta} \psi_i^* \alpha \beta (5.9)
\]

We introduced the anti-fields, denoted with the asterisk, in correspondence of each of the fields of the model, but, for simplicity, we neglected the trivial \((\theta_i, H^i)\) doublet which plays no role in the gauge-independent physics of the model.

From this action one obtains the BRST transformations for the anti-fields

\[
\hat{S} F_i^* = i_\gamma (\rho_i^*) \\
\hat{S} \rho_i^* = -(D F_i^* + (R_{\sigma, i}^j + \frac{1}{2} R_{\tau,(\rho) i}^j) F_j^*) + i_\gamma (\tilde{\phi}_i^*) \\
\hat{S} \tilde{\phi}_i^* = -(D \rho_i^* - (R_{\sigma, i}^j + \frac{1}{2} R_{\tau,(\rho) i}^j) \rho_j^* + \\
- (R_{\sigma,F,i}^j + \frac{1}{2} R_{i,(\rho) F,i}^j + R_{d\phi,\rho,i}^j + R_{d\phi,\rho,i}^j \frac{1}{2} D_{\rho,R_{\sigma, \rho,i}}^j + \frac{1}{2} D_{\rho,R_{\tau,(\rho) \rho,i}}^j) F_j^* (5.10)
\]

where \(\tilde{\phi}_i^*\) is related to the anti-field \(\phi_i^*\) by the formula

\[
\tilde{\phi}_i^* = \phi_i^* + \frac{1}{2} R_{\gamma,j,i}^k \rho^j \rho_k F_i^* - \Gamma_{ij}^k F^j F_k^* - \Gamma_{ij}^k \rho^j \rho_k (5.11)
\]

6 Observables

In the BV formalism, the observables — which, for the B-model, as explained, are BRST classes on the space of fields modulo the equations of motion — map to BRST cohomology classes on the space of both fields and antifields.

Local observables satisfy BRST descent equations. In the equivariant context, these are descent equations which involve the BRST operator \(S\), the exterior differential on forms \(d\) and \(i_\gamma\) the contraction of forms along the vector field \(\gamma^\alpha\). They take the form

\[
SO^{(0)}(x) = i_\gamma (O^{(1)}(x)) \\
SO^{(1)}(x) + dO^{(0)} = i_\gamma (O^{(2)}(x)) \\
SO^{(2)}(x) + dO^{(1)} = 0 (6.1)
\]
where $O^{(0)}(x), O^{(1)}(x)$ are the “descendant” of the observable 2-form $O^{(2)}(x)$ whose integral is BRST invariant.

Descent equations can be nicely written in terms of the nilpotent coboundary operator $\delta$

$$\delta \equiv S + d - i_{\gamma} \quad (6.2)$$

acting on polyforms, which are sum of forms of different degrees. Indeed (6.1) can be recast as

$$\delta O(x) = 0 \quad O(x) = O^{(0)}(x) + O^{(1)}(x) + O^{(2)}(x) \quad (6.3)$$

Since our observables will necessarily contain the anti-fields $\phi^*_i, \rho^*_i, F^*_i$ which make up a polyform with values in the holomorphic cotangent of the target space, it will be useful to derive the analogs of (6.1) for polyforms with values in the holomorphic tangent:

$$O^i(x) = O^{i\ (0)}(x) + O^{i\ (1)}(x) + O^{i\ (2)}(x) \quad (6.4)$$

Starting from

$$\hat{S} O^{i\ (0)}(x) = i_{\gamma}(O^{i\ (1)}(x)) \quad (6.5)$$

and using the algebra (3.8) for the covariant BRST operator $\hat{S}$ one derives:

$$\hat{S} O^{i\ (0)} = i_{\gamma}(O^{i\ (1)})$$

$$\hat{S} O^{i\ (1)} + D O^{i\ (0)} + \frac{1}{2} \mathcal{R}^i_j O^{j\ (0)} = i_{\gamma}(O^{i\ (2)})$$

$$\hat{S} O^{i\ (2)} + D O^{i\ (1)} + \frac{1}{2} \mathcal{R}^i_j O^{j\ (1)} + \left[ \frac{1}{2} R_{F_{i\gamma}(\rho)} + R_{F_{\sigma}} + \frac{1}{2} R_{d\phi} + R_{d\bar{\phi}} + \frac{1}{2} D_{\rho} R_{\rho\sigma} + \frac{1}{6} D_{\rho} R_{\rho\iota}(\rho) \right]_{j}^{i} O^{j\ (0)} = 0 \quad (6.6)$$

where we introduced the matrix-valued two-form

$$\mathcal{R}^i_j \equiv (R_{\rho\iota}(\rho))_{j}^{i} + 2 (R_{\rho\sigma})_{j}^{i} \quad (6.7)$$

Comparing (6.6) with the BRST transformation rules of the anti-fields, we see that $F^*_i, \rho^*_i$ and $\tilde{\phi}^*_i$\footnote{Note that it is $\tilde{\phi}^*_i$ and not the anti-field $\phi^*_i$ which satisfies the descent together with $F^*_i$ and $\rho^*_i$.} satisfy descent equations which are the analogs of (6.6) for operators which are valued in the holomorphic cotangent bundle.
6.1 The differential geometry of complex structure moduli space

The dependence of the action on the complex structure is parametrized by Beltrami differentials $\mu^i_j$

$$d\phi^i = \Lambda^i_j \left( d\phi^i_0 + \mu^i_j d\phi^j_0 \right)$$ (6.8)

where $(\phi^i_0, \phi^j_0)$ is a fixed system of complex coordinates, and $\Lambda^i_j$ are the integrating factors, which are (non-local) functionals of the Beltrami differentials $\mu^i_j$. Derivative with respect to the complex structure moduli is performed keeping $(\phi^i_0, \phi^j_0)$ fixed. Let $\partial_a$ denote the holomorphic derivative with respect to the complex structure moduli coordinates $\{m^a\}$.

The Beltrami differentials $\mu^i_j$ are holomorphic functions of the moduli coordinates $m^a$

$$\partial_a \ d\phi^i = \partial_a \Lambda^i_j \left( (\Lambda^{-1})^i_k d\phi^k + \Lambda^i_j \partial_a \mu^i_j d\phi^j \equiv A^i_a \ d\phi^i + \mu^i_a d\phi^j \right)$$ (6.9)

where

$$A^i_a \ j = \partial_j \left[ \partial_a \phi^i(\phi_0, m) \right]_{\phi_0=\phi_0(\phi, m)} \quad \mu^i_a \ j = \partial_j \left[ \partial_a \phi^i(\phi_0, m) \right]_{\phi_0=\phi_0(\phi, m)}$$ (6.10)

$A^i_a \ j$ transforms as a connection under $m^a$-dependent holomorphic reparametrizations of the target space coordinates $\phi^i$ while

$$\mu^i_a \equiv \mu^i_a \ d\phi^j$$ (6.11)

are (0,1)-forms with values in the holomorphic tangent which are closed under the Dolbeault exterior differential in the $(\phi^i, \phi^j)$ complex structure

$$\bar{\partial} \mu^i_a = 0$$ (6.12)

Moreover

$$\partial_j A^i_a \ k = 0 \quad \partial_j A^i_k = \partial_k \mu^i_a$$ (6.13)

These equations are equivalent to the Kodaira-Spencer equation for the Beltrami differential in the fixed system of complex coordinates $(\phi^i_0, \phi^j_0)$:

$$\bar{\partial} \mu^i - \mu^i \partial_j \mu^j = 0 \quad \mu^i \equiv \mu^i_j d\phi^j_0$$ (6.14)

Eq. (6.9) leads to the definition of a covariant holomorphic derivative with respect to the complex structure moduli

$$D_a \ d\phi^i \equiv \partial_a \ d\phi^i - A^i_a \ j d\phi^j = \mu^i_a$$ (6.15)
The relations dual to (6.15) which capture the moduli dependence of the holomorphic derivatives are

\[
\begin{align*}
[\partial_a, \partial_i] & = -\mu^i_{a\bar{i}} \partial_i \\
[\partial_a, \partial_{\bar{i}}] & = -A^j_{a\bar{i}} \partial_j
\end{align*}
\]  

(6.16)

For example, on a holomorphic vector \( V^i \) one has

\[
\begin{align*}
\{D_a, \partial_i\} V^j & = \partial_i A^j_{a\bar{k}} V^k \\
\{D_a, \partial_{\bar{i}}\} V^j & = -\mu^i_{a\bar{i}} \partial_i V^j + \partial_k \mu^i_{a\bar{i}} V^k
\end{align*}
\]

(6.17)

We will also be interested in evaluating the moduli dependence of target space covariant derivatives built with a connection \( \Gamma^i_{jk} \) like (2.2).

\[
\begin{align*}
\{D_a, D_k\} & = (H_{ak}) \\
\{D_a, D_{\bar{k}}\} & = -\mu^k_{a\bar{k}} D_k + (H_{a\bar{k}})
\end{align*}
\]

(6.18)

where \((H_{ak})\) and \((H_{a\bar{k}})\) are matrices acting on the (anti-)holomorphic tangent indices. For example when acting on holomorphic vectors one finds that

\[
(H_{ak})^i_j = D_j \mu^i_{ak}
\]

(6.19)

Moreover, it turns out that

\[
(H_{ak})^i_j = \partial_k (A_a)^i_j + D_a \Gamma^i_{k\bar{j}} = \delta_a \Gamma^i_{jk} = D_a g^{ij} \partial(k g)j + g^{ij} \partial(k D_a g)j
\]

(6.20)

where \(\delta_a \Gamma^i_{jk}\) is the variation of the connection \(\Gamma^i_{jk}\) induced by the variation of the metric \(g_{ij}\)

\[
\delta_a g_{ij} = D_a g_{ij}
\]

(6.21)

which can accompany a complex structure deformation. This variation is independent of the variation of the complex structure and it is arbitrary. We can recast (6.18), when acting on holomorphic vector fields, as

\[
\{D_a, D\} = \delta_a \Gamma^i_{d\phi} + d\phi^k (D_j \mu^i_{ak})
\]

(6.22)

where \(D\) is the covariant exterior differential

\[
DV^k = dV^k + d\phi^i \Gamma^{k}_{ij} V^j
\]

(6.23)

We can analogously compute the dependence of the curvature built with \(\Gamma^i_{jk}\) on the complex structure moduli:

\[
\begin{align*}
D_a R^i_{ij,k} & = \delta_a R^i_{ij,k} - \frac{1}{2} \{D_k, D_j\} \mu^i_{a\bar{i}} + \frac{1}{2} (R^i_{k\bar{l},j} + R^i_{j\bar{l},k}) \mu^i_{a\bar{k}} \\
D_a R^i_{jk;l} & = \delta_a R^i_{jk,l} = D_j \delta_a \Gamma^i_{k\bar{l}}
\end{align*}
\]

(6.24)
Again here we denoted with $\delta_a R^i_{ij;k}$ and $\delta_a R^i_{jk;l}$ the variation of the curvatures induced by the variation (6.21) of the metric $g_{ij}$.

We will be also interested in taking the anti-holomorphic derivatives of the coordinate fields with respect to anti-holomorphic moduli $m^a$, which we will denote by $\partial_{\bar{a}}$. The anti-holomorphic Beltrami differentials are defined in a way analogous to (6.8)

$$d\phi^\bar{i} = \bar{\Lambda}^\bar{i}_j (d\phi^\bar{j}_0 + \bar{\mu}^\bar{i}_j d\phi^\bar{j})$$  \hfill (6.25)

The anti-holomorphic derivative $\partial_{\bar{a}}$ acts in a way analogous to (6.9):

$$\partial_{\bar{a}} d\phi^\bar{i} = \partial_{\bar{a}} \Lambda^\bar{i}_j (\Lambda^{-1})^\bar{j}_k d\phi^\bar{k} + \Lambda^\bar{i}_j \partial_{\bar{a}} \mu^\bar{j}_j d\phi^\bar{j} \equiv A^\bar{i}_{\bar{a}j} d\phi^\bar{j} + \bar{\mu}^\bar{i}_{\bar{a}j} d\phi^\bar{j}$$  \hfill (6.26)

where

$$\bar{\mu}^\bar{i}_{\bar{a}j} \equiv \bar{\mu}^\bar{i}_{\bar{a}j} d\phi^\bar{j}$$  \hfill (6.27)

are (1,0)-forms with values in the anti-holomorphic tangent which are \emph{closed} under the anti-Dolbeault exterior differential in the $(\phi^i, \phi^\bar{j})$ complex structure

$$\partial \bar{\mu}^\bar{i}_{\bar{a}j} = 0 \hfill (6.28)$$

The covariant derivative is defined in the same way as (6.15)

$$D_{\bar{a}} d\phi^\bar{i} \equiv \partial_{\bar{a}} d\phi^\bar{i} - A^\bar{i}_{\bar{a}j} d\phi^\bar{j} = \mu^\bar{i}_{\bar{a}}$$  \hfill (6.29)

and it satisfies

$$[D_{\bar{a}}, \partial_j] V^\bar{j} = -\bar{\mu}^\bar{i}_{\bar{a}i} \partial_i V^\bar{j} + \partial_k \bar{\mu}^\bar{i}_{\bar{a}j} V^k$$  \hfill (6.30)

The anti-holomorphic moduli dependence of target space covariant derivatives built with a connection $\Gamma^i_{jk}$ like (2.2) are captured by

$$[D_{\bar{a}}, D_k] = (\mathcal{H}_{\bar{a}k}) \quad [D_{\bar{a}}, D_k] = -\bar{\mu}^\bar{i}_{\bar{a}k} D_k + (\mathcal{H}_{\bar{a}k})$$  \hfill (6.31)

where $(\mathcal{H}_{\bar{a}k})$ and $(\mathcal{H}_{\bar{a}k})$ are matrices: when acting on the anti-holomorphic indices they are given by the complex conjugate of (6.19)

$$\mathcal{(H)}_{\bar{a}k}^j = D_j \mu^\bar{i}_{\bar{a}k} \quad \mathcal{(H)}_{\bar{a}k}^j = \delta_{\bar{a}} \Gamma^\bar{i}_{\bar{j}k}$$  \hfill (6.32)

where $\delta_a \Gamma^i_{jk}$ is the variation of the connection $\Gamma^i_{jk}$ induced by the (arbitrary) variation of the metric $g_{ij}$

$$\delta_{\bar{a}} g_{ij} = D_{\bar{a}} g_{ij}$$  \hfill (6.33)
which can accompany a complex structure anti-holomorphic deformation. When acting on holomorphic target space indices, the matrices \( (\mathcal{H}_{\bar{a}k}) \) and \((\mathcal{H}_{a\bar{k}})\) write instead

\[
(\mathcal{H}_{\bar{a}k})_j^i = 0 \\
(\mathcal{H}_{a\bar{k}})_j^i = D_a \Gamma_{kj}^i = \delta_a \Gamma_{kj}^i + D^i \bar{\mu}_{a\bar{k}j}
\]  

(6.34)

where

\[
\bar{\mu}_{a\bar{i}j} \equiv \frac{1}{2} (\mu_{a\bar{i}j}^i g_{ji} + \mu_{a\bar{i}j}^i g_{ij})
\]

(6.35)

is the anti-Beltrami differential with lower symmetrized holomorphic indices. Note that in the Kähler case \( \bar{\mu}_{a\bar{i}j} = \bar{\mu}_{a\bar{i}j}^i g_{ji} \) is automatically symmetric, but this is not so for the non-Kählerian metric we are considering. In conclusion the following commutation relation holds on holomorphic vectors

\[
[D_{\bar{a}}, D] V^i = \delta_{\bar{a}} \Gamma_{d\phi j}^i V^j + D^i \bar{\mu}_{a\bar{k}j} d\phi^k V^j
\]

(6.36)

We can analogously compute the dependence of the curvature on the anti-holomorphic moduli

\[
D_{\bar{a}} R_{j\bar{k};l}^i = \delta_{\bar{a}} R_{j\bar{j};k}^i + D_{\bar{k}} D^i (\bar{\mu}_{a\bar{j}k})
\]

\[
D_{\bar{a}} R_{j\bar{k};l}^i = \delta_{\bar{a}} R_{j\bar{k};k}^i - \mu_{a[j}^m R_{i\bar{m}k];l} + D_{[j} D^i (\bar{\mu}_{a\bar{k}l])}
\]

(6.37)

The tensor

\[
\mathcal{R}_{j\bar{k};l}^i \equiv -\mu_{a[j}^m R_{i\bar{m}k];l} + D_{[j} D^i (\bar{\mu}_{a\bar{k}l])}
\]

(6.38)

vanishes when the metric is Kähler, thanks to the following relations which hold in this case,

\[
g_{\bar{i}j} R_{j\bar{k};l}^m = g_{\bar{i}j} R_{j\bar{k};l}^i \quad \bar{\mu}_{a\bar{i}j} = g_{\bar{i}j} \mu_{a\bar{i}j} \quad D_j g_{\bar{i}j} = 0
\]

(6.39)

Finally, the anti-fields \( \phi_{\bar{i}}^* \) and \( \phi_{\bar{i}}^* \) transform as the holomorphic derivatives \( \partial_{\bar{i}} \) and \( \partial_i \) and therefore

\[
D_{\bar{a}} \phi_{\bar{i}}^* = 0 \\
D_{\bar{a}} \phi_{\bar{i}}^* = -\mu_{a\bar{i}j}^i \phi_{\bar{i}}^* \\
D_{\bar{a}} \phi_{\bar{i}}^* = -\mu_{a\bar{i}j}^i \phi_{\bar{i}}^* \\
D_{\bar{a}} \phi_{\bar{i}}^* = 0
\]

(6.40)

### 6.2 Observables associated to complex structure deformations

To any Beltrami differential \( \mu_{a\bar{j}}^i \) we can associate the 0-form of ghost number +1 with values in the holomorphic tangent

\[
M_{\bar{a}}^{i(0)} = \mu_{a\bar{j}}^i \sigma^j
\]

(6.41)
Thanks to the Beltrami equation (6.12) this satisfies
\[ \hat{S}(\mu_{a,j} \sigma^j) = i_\gamma (\mu_{a,j} \sigma^j + D_k \mu_{a,j} \rho^k \sigma^j) \] (6.42)
We can therefore construct the corresponding one-form \( M^{(1)}_a \) and two-form \( M^{(2)}_a \) operators which satisfy the descent equations (6.6). It turns out that
\[
M^{(1)}_a = \mu_{a,j} \sigma^j + D_k \mu_{a,j} \rho^k \sigma^j
\]
\[
M^{(2)}_a = D_k \mu_{a,j} \rho^k \sigma^j + D_k \mu_{a,j} \rho^j \rho^k \sigma^j
\] (6.43)
Then
\[ O_a = (M^{(0)}_a + M^{(1)}_a + M^{(2)}_a) \left( \tilde{\phi}^*_i + \rho^*_i + F^*_i \right) \] (6.44)
is a \( \delta \)-cocycle and
\[
O^{(0)}_a = \mu_{a,j} \sigma^j F^*_i
\]
\[
O^{(1)}_a = \mu_{a,j} \sigma^j \rho^*_i + (\mu_{a,j} \sigma^j + D_k \mu_{a,j} \rho^k \sigma^j) F^*_i
\]
\[
O^{(2)}_a = \mu_{a,j} \sigma^j \tilde{\phi}^*_i + (\mu_{a,j} \sigma^j + D_k \mu_{a,j} \rho^k \sigma^j) \rho^*_i +
+ (D_k \mu_{a,j} \rho^k \sigma^j + D_k \mu_{a,j} \rho^j \rho^k \sigma^j + \frac{1}{2} D_j D_k \mu_{a,j} \rho^j \rho^k \sigma^j) F^*_i =
= \mu_{a,j} \sigma^j \tilde{\phi}^*_i + (\mu_{a,j} \sigma^j + \partial_k \mu_{a,j} \rho^k \sigma^j) \rho^*_i +
+ (D_k \mu_{a,j} \rho^k \sigma^j + \partial_k \mu_{a,j} \rho^j \rho^k \sigma^j + \frac{1}{2} D_j D_k \mu_{a,j} \rho^j \rho^k \sigma^j +
+ \frac{1}{2} R^k_{ij,l} \rho^j \rho^l \mu_{a,j} \sigma^j) F^*_i
\] (6.45)
are observables of total fermionic number +2 satisfying the equivariant descent equations (6.1).
This cocycle is \( \gamma \)-independent: therefore \( O_a \) is also an observable of the matter theory. This is possible since the 2-form observable \( O^{(2)}_a \) is, in fact, \( G_\gamma \)-invariant
\[ G_\gamma O^{(2)}_a = 0 \] (6.46)
Observables which satisfy this conditions are called “chiral”, in the \( N = 2 \) supersymmetric language.

7 Varying the parameters of the BV action

The observables described in the previous section are associated to Beltrami differentials: one therefore expects that an integrated 2-form \( O^{(2)}_a \) is related to the holomorphic derivative
of the BV action with respect to the complex structure moduli $m^a$. Since however the coordinates fields do depend implicitly on the complex structure, this expectation is not completely realized: it turns out that the integrated observable $O_a^{(2)}$ is not just given by the derivative of the BV action with respect to $m^a$ but it must be supplemented with extra terms which make it invariant. To explain this let us start from the BV master equation

$$
\sum_A \int S\phi^A S\phi^*_A(-1)^{A+1} = 0 \quad (7.1)
$$

where we denoted by $\phi^A$ and $\phi^*_A$ the fields and anti-fields of the model. Let us now consider the variation of the master equation under a generic deformation $\delta$ of the parameters — which could be either the complex structure or metric — on which $S$ depends:

$$
0 = \sum_A \int \left( \delta(S\phi^A) S\phi^*_A(-1)^{A+1} + S\phi^A(\delta(S\phi^*_A(-1)^{A+1})) \right) = \sum_A \int \left( I\phi^A + \frac{\partial \Gamma_{BV}}{\partial \phi^A} I\phi^*_A \right) + \sum_A \int \left( S\delta\phi^A S\phi^*_A(-1)^{A+1} + S\phi^A(S\delta\phi^*_A(-1)^{A+1}) \right) 
$$

$$
= \sum_A \int \left( I\phi^A \frac{\partial \Gamma_{BV}}{\partial \phi^A} + \frac{\partial \Gamma_{BV}}{\partial \phi^*_A} I\phi^*_A \right) + \sum_A \int \left( S\delta\phi^A \frac{\partial \Gamma_{BV}}{\partial \phi^A} + \frac{\partial \Gamma_{BV}}{\partial \phi^*_A} (S\delta\phi^*_A) \right) = I\Gamma_{BV} + \sum_A \int \left( S\delta\phi^A \frac{\partial \Gamma_{BV}}{\partial \phi^A} + \frac{\partial \Gamma_{BV}}{\partial \phi^*_A} (S\delta\phi^*_A) \right) = 0 \quad (7.2)
$$

where we introduced, as in (5.3), the operator

$$
I = \delta(S) \quad (7.3)
$$

of ghost number +1 which is the deformation of the BRST operator and which anti-commutes with $S$. We also accounted for an implicit dependence of fields and anti-fields on the parameter which are being varied. This is the case, of course, when we vary the complex structure in the B-model since the coordinates fields depend on the complex structure in the way that has been computed in Section 6.1.

The second term in the last line of the equation (7.2) above is $S$-trivial

$$
\sum_A \int \left( S\delta\phi^A S\phi^*_A(-1)^{A+1} + S\phi^A(S\delta\phi^*_A(-1)^{A+1}) \right) = S \int \left( (S\delta\phi^A) \phi^*_A + S\phi^A \delta\phi^*_A \right) \quad (7.4)
$$

Moreover

$$
0 = \delta(S\Gamma_{BV}) = I\Gamma_{BV} + S(\delta\Gamma_{BV}) \quad (7.5)
$$
Plugging both (7.4) and (7.5) into Eq. (7.2) one obtains

\[ S\left(\delta\Gamma_{BV} - \int (S\delta\phi^A)\phi^*_A - S\phi^A\delta\phi^*_A\right) = 0 \]  

(7.6)

This equation says that, when (some of) the (anti-)fields depend implicitly on the deformation parameter, the BRST invariant observable associated to deformation parameter is not simply the variation of the action, but it must be supplemented with bilinear terms containing the anti-fields. The resulting BRST invariant integrated observable only depends on the fermionic operator \( I \)

\[ \hat{O} = \delta\Gamma_{BV} - \int ((S\delta\phi^A)\phi^*_A + S\phi^A\delta\phi^*_A) = \]

\[ = \sum_A \int \delta(S\phi^A)\phi^*_A + S\phi^A\delta\phi^*_A - (S\delta\phi^A)\phi^*_A - S\phi^A\delta\phi^*_A = \]

\[ = \sum_A \int (I\phi^A)\phi^*_A \]  

(7.7)

The gauge-independent physics associated to the deformation \( \delta \) is hence captured by the operator \( I \). The operator \( I \) satisfies the consistency condition

\[ \{I, S\} = 0 \]  

(7.8)

Deformations \( I \) which are \( S \)-commutators

\[ I = [S, L] \]  

(7.9)

correspond to integrated observables \( \hat{O} \) which are trivial in the BV sense

\[ \hat{O} = \sum_A [S, L] \phi^A\phi^*_A = S \sum_A \int L\phi^A\phi^*_A \]  

(7.10)

In the BV framework, gauge-fixing is performed by choosing a gauge-fermion functional \( \Psi[\phi^A] \) and putting

\[ \phi^*_A = \frac{\delta}{\delta\phi^A} \int \Psi[\phi^A] \]  

(7.11)

From (7.7) it follows that to the BV observable \( \hat{O} \) there corresponds the gauge-fixed integrated observable

\[ \hat{O}^{g.f} = I \int \Psi(\phi^A) \]  

(7.12)
\( \hat{O}^{g,f} \) is BRST closed modulo the equations of motion generated by \( I \)

\[
S \hat{O}^{g,f} = -I \Gamma^{g,f}. \tag{7.13}
\]

When \( I \) is a \( S \) commutator, as in \( (7.9) \), the corresponding gauge-fixed observable is BRST-trivial modulo the equations of motions associated to \( L \):

\[
\hat{O}^{g,f} = S \int (L \Psi[\phi^A]) - L \Gamma^{g,f}[\phi^A] \tag{7.14}
\]

We see therefore that deformations which generate \( I \) which are \( S \)-commutators, do not necessarily decouple in physical correlators: Eq. \( (7.14) \) says that the insertion of a trivial \( I \) operator in a physical correlator gives contact terms generated by the operator \( L \). Those contact may or may not vanish according to the specific form of both \( L \) and the physical observables. In the following we will determine the operator \( L \) for different BRST trivial deformations of the B-model, to assess, in a gauge-independent way, their decoupling — or lack thereof.

### 7.1 Dependence on the target space metric

Let us denote by \( \delta g \) the variation of the target space metric \( g_{ij} \to g_{ij} + \delta g_{ij} \). In this case the (anti-)fields are left invariant by the deformation and therefore the corresponding operator insertion is just given by the variation of the BV action\(^7\):

\[
\hat{O}_{\delta g} \equiv \delta g \Gamma_{BV} = \int \left[ \delta g \langle \hat{S} F^i \rangle F^*_i + \delta g \langle \hat{S} H_i \rangle H^{*i} + \delta g \Gamma_{i,(\rho);j} \rho^j \rho^i + \delta g \Gamma_{i,(\rho);j} F^i F^*_j + \delta g \Gamma_{i,(\rho);j} \theta_i \theta^*_j + \delta g \Gamma_{i,(\rho);j} F^j F^*_i \right] =
\]

\[
= \int \left[ \left( \delta \Gamma_{i,(\rho);j} \rho^j + \frac{1}{2} \delta g \delta \Gamma_{i,(\rho);j} \rho^j \rho^i + \frac{1}{3} \left( D_{i,(\rho)} \delta g \Gamma_{i,(\rho);j} \theta_i \theta^*_j + \delta g \Gamma_{i,(\rho);j} \theta_i \theta^*_j \right) H^{*i} \right) + \left( \delta \Gamma_{i,(\rho);j} \theta^*_j \rho^j + \delta g \Gamma_{i,(\rho);j} \theta^*_j \rho^i \right) H^{*i} \right] =
\]

\[
= S \int \left( \frac{1}{2} \delta g \Gamma_{\rho \rho} F^*_i - \delta g \Gamma_{i,(\rho);j} \theta_j H^{*i} \right) \tag{7.15}
\]

This shows that a deformation of the target space metric is BRST trivial in the space of fields and anti-fields and that \( I_{\delta g} = \delta g (S) \) is a BRST commutator

\[
I_{\delta g} = [S, L_{\delta g}] \tag{7.16}
\]

\(^7\)We included in this formula for completeness also the trivial \((\theta_i, H_i)\) sector.
where the operator $L_{\delta g}$ acts non-trivially only on the fields $F^i$ and $H_i$

$$
L_{\delta g} F^i = \frac{1}{2} \delta_{\delta g} \Gamma^i_{jk} \rho^j \rho^k \\
L_{\delta g} H_i = -\delta_{\delta g} \Gamma^i_{jk} i_j (\rho^k) \theta_j
$$

(7.17)

Since $\hat{O}_{\delta g}$ is trivial in the space of both fields and anti-fields, it follows that, upon gauge fixing, the corresponding insertion is BRST trivial up to terms proportional to the equations of motions generated by $L_{\delta g}$: From (7.17) these are the equations of motion of the auxiliary fields $F^i$ and $H_i$. If one therefore considers correlators involving only 0-form observables (6.45) associated to the complex structure moduli,

$$
O_a^{(0)} = \mu^i_{a \bar{j}} \sigma^j_i F_i^*
$$

(7.18)

these contact terms vanish, since the observables do not depend on either $F^i$ or $H_i$. Correlators of such observables are therefore independent of the target space metric.

### 7.2 Dependence on the holomorphic complex structure moduli

Since the coordinate fields $\phi_i$ and anti-fields $\bar{\phi}_i^*$ depend implicitly on the complex structure moduli, the holomorphic derivative $\partial_a \Gamma_{BV}$ of the BV action differs from the integrated BRST-invariant observable $\hat{O}_a$, as specified in (7.7)

$$
\int \hat{O}_a^{(2)} = \partial_a \Gamma_{BV} - \int [(S \delta \phi^A) \phi^*_A + S \phi^A \delta \phi^*_A] = \\
= \int (I_a \phi^j \phi^*_i + I_a \rho^j \rho^*_i + I_a F^i F^*_i + I_a \phi^j \phi^*_i + I_a \sigma^j \sigma^*_i)
$$

(7.19)

where we neglected the BRST trivial doublet $(\theta_i, H_i)$ which gives an equally trivial contribution to the observable.

To compute the corresponding $I_a = \partial_a (S)$ we need to specify the implicit dependence on the holomorphic moduli of fields and anti-fields. We discussed the dependence on the complex structure moduli of the coordinate fields $(\phi^i, \phi^j)$ in Section 6.1: it is given by the Beltrami parametrization (6.8), which also determines the dependence on the complex structure moduli of connections, curvature tensors and anti-fields as shown in Eqs. (6.22), (6.24) and (6.40).

The fields other than the coordinates take values on non-compact affine field spaces with no boundaries. Since they are integrated over in the functional integral that defines quantum averages, their specific dependence on the moduli is, in fact, irrelevant for the computation
of normalized quantum correlators. It is therefore convenient to choose a dependence for the (anti-)fields other than the coordinate fields, which is explicitly covariant under target space holomorphic reparametrization, i.e.

\[ D_a \rho^i = D_a F^i = D_a \theta_i = D_a H_i = 0 \]  \hfill (7.20)

and analogously for the corresponding anti-fields. With this choice

\begin{align*}
I_a \phi^i &= -\mu^i_{a \bar{j}} \\
I_a \rho^i &= \mu^i_{a \bar{i}} d\bar{\phi}^\bar{i} - \partial_j \mu^i_{a \bar{\sigma}} \rho^j + I_{D_{a \bar{q}}} \rho^i \\
I_a F^i &= d\bar{\phi}^\bar{k} D_{\rho} \mu^i_{a \bar{k}} - \frac{1}{2} D_{\rho}^2 \mu^i_{a \bar{\sigma}} + \frac{1}{2} R^j_{\rho \lambda \bar{\mu}} \mu^i_{a \bar{\sigma}} - \partial_j \mu^i_{a \bar{\sigma}} F^j + I_{D_{a \bar{q}}} F^i \\
I_a \phi^i &= 0 \\
I_a \sigma^{\bar{i}} &= 0 \hfill (7.21)
\end{align*}

where \( I_{D_{a \bar{q}}} \) is the deformation of the BRST under a change of the target space metric

\[ g_{ij} \rightarrow g_{ij} + D_a g_{ij} \]  \hfill (7.22)

We have just shown that \( I_{D_{a \bar{q}}} \) is an \( S \)-commutator and that the associated insertion is BRST trivial. Therefore, by comparing (7.21) with (6.45), we conclude that the integrated observable associated to the deformation \( I_a \) is, up to BRST trivial terms, precisely the one which descends from the 0-form operator (7.18).

The insertion of an integrated observable \( \hat{O}_a \) in a BRST invariant correlator is related to the holomorphic derivative \( \partial_a \) of the same correlator, but does not coincide with it. The basic reason for this is that the observable is BRST closed only up to terms which are proportional to the equations of motion. The consequence of this is that the insertion of \( \hat{O}_a \) is obtained by taking an appropriate covariant derivative of the correlator, whose connection is fixed by BRST invariance. To see this let us consider the holomorphic derivative \( \partial_a \) of a correlator involving another integrated observable \( \hat{O}_b \)

\[ \partial_a \langle \hat{O}_b \rangle = \langle [(\partial_a S) \int \Psi] (\int I_b \Psi + \partial_a I_b \int \Psi] \rangle = \]

\[ = \langle [(S \int \partial_a \Psi + \int I_a \Psi) (\int I_b \Psi + \partial_a(I_b)) \int \Psi + I_b \int \partial_a \Psi] \rangle = \]

\[ = \langle \int I_a \Psi \int I_b \Psi - \int \partial_a \Psi I_b \Gamma + \partial_a (I_b) \int \Psi + I_b \int \partial_a \Psi] \rangle = \]

\[ = \langle \int I_a \Psi \int I_b \Psi + I_{ab} \int \Psi \rangle \]  \hfill (7.23)
where in the last line we introduced the fermionic operator $I_{ab}$ symmetric in $a$ and $b$:

$$I_{ab} = \partial_a I_b = \partial_a \partial_b (S) = -D_a \mu_{b,j}^i \sigma^j \partial_i + \cdots \quad (7.24)$$

Here the the dots denote the action of the operator on fields other than the coordinate fields $\phi^i$ and $\bar{\phi}^\bar{i}$. We see that the correlator of two BRST invariant integrable observables writes as

$$\langle \hat{O}_a \hat{O}_b \rangle = \partial_a \langle \hat{O}_b \rangle - \langle \int I_{ab} \Psi \rangle \quad (7.25)$$

The last term is a local integrated operator which encodes the contact between the two local operators $\hat{O}_a$ and $\hat{O}_b$. We can think of the contribution of $I_{ab} \Psi$ as a renormalization counterterm which must be added to the correlator of two integrable observables to make it BRST-invariant — i.e. gauge-independent. The overall effect of this contact term is that the insertion of $\hat{O}_a$ is given by taking a covariant derivative

$$\langle \hat{O}_a \hat{O}_b \rangle = D_a (\Gamma) \langle \hat{O}_b \rangle \quad (7.26)$$

build with a connection $\Gamma_{ab}^c$ in the moduli space which is determined from $I_{ab}$. This connection, in the Kähler case, is precisely the connection compatible with the Zamolodchikov metric on the moduli space of complex structure. For a detailed derivation of these statements see S. Giusto, Ph. D. dissertation thesis [9]. The computation of this connection in the non-Kähler case is left to future work.

### 7.3 The dependence on the anti-holomorphic complex structure moduli

Let us now turn to consider the BRST-invariant operator insertion associated to the anti-holomorphic derivative of the BV action with respect to the complex structure: The anti-holomorphic derivative of the simple BV action, neglecting again for simplicity the $(\theta_i, H_i)$ sector, is

$$\int \hat{O}_a^{(2)} = \int (I_a \phi^i \phi^*_i + I_a \rho^i \rho^*_i + I_a F^i F^*_i + I_a \phi^i \phi^*_i + I_a \sigma^i \sigma^*_i) \quad (7.27)$$
where the deformation $I_{\bar{a}} = \partial_{\bar{a}}(S)$ is

$$
I_{\bar{a}}\phi^i = 0 , \\
I_{\bar{a}}\rho^i = I_{Dag} \rho^i - \frac{1}{2} D^i(\bar{\mu}_{\bar{a} jk}) i_{\gamma}(\rho^j \rho^k) \\
I_{\bar{a}}F^i = I_{Dag} F^i - D^i \bar{\mu}_{\bar{a} jk} i_{\gamma}(\rho^j) F^k - D^i \bar{\mu}_{\bar{a} k} d\phi^k \rho^j + \\
\frac{1}{2} D^i D^j(\bar{\mu}_{\bar{a} jk}) \sigma^r \rho^j \rho^k + \frac{1}{3} R_{\bar{a} jk l} i_{\gamma}(\rho^j) \rho^k \rho^l \\
I_{\bar{a}}\phi^j = -\bar{\mu}_{\bar{a} i} i_{\gamma}(\rho^i) , \\
I_{\bar{a}}\sigma^j = \bar{\mu}_{\bar{a} i} i_{\gamma}(d\phi^i) + \bar{\partial}_k \bar{\mu}_{\bar{a} i} \sigma^k i_{\gamma}(\rho^i) 
$$

(7.28)

where $R_{\bar{a} jk l}^i$ is the tensor defined in (6.38), which vanishes for Kaehler metrics, while $I_{Dag}$ is the deformation of $S$ associated to a shift of the metric

$$
g_{ij} \rightarrow g_{ij} + D_ag_{ij} 
$$

(7.29)

The anti-holomorphic deformation $I_{\bar{a}} = \partial_{\bar{a}}(S)$ turns out to be a $S$ commutator

$$
I_{\bar{a}} = [S, L_{\bar{a}} + L_{Dag}] 
$$

(7.30)

where $L_{\bar{a}}$ is a bosonic operator whose non-trivial action is

$$
L_{\bar{a}} \sigma^i = \bar{\mu}_{\bar{a} i} i_{\gamma}(\rho^i) \\
L_{\bar{a}} F^i = \frac{1}{2} D^i \bar{\mu}_{\bar{a} jk} \rho^j \rho^k 
$$

(7.31)

Therefore the integrated anti-holomorphic insertion is BRST trivial in the BV sense

$$
\int \hat{O}_{\bar{a}}^{(2)} = S \int (L_{\bar{a}} \sigma^i \sigma^*_i + L_{\bar{a}} F^i F^*_i) + \\
+ S \int \left( \frac{1}{2} \delta_{Dag} \Gamma^i_{pp} F^*_i - \delta_{Dag} \Gamma^i_{\gamma(\rho)} i_{\gamma(\rho)} \theta_j H^{*i} \right) 
$$

(7.32)

The terms in the second line of the r.h.s is a trivial term associated to a target space deformation of the metric, which, as we discussed above, does decouple when inserted in a correlator of holomorphic deformations. The terms in the first line, instead, correspond, upon gauge-fixing, to an insertion which is BRST trivial up to terms proportional to the equations of motions generated by $L_{\bar{a}}$: from (7.31), this means trivial up to terms proportional to equations of motion of $\sigma^i$. Since 0-form observables (6.45) associated to the complex structure moduli,

$$
O_{\bar{a}}^{(0)} = \mu_{\bar{a} j} \sigma^j F^*_i 
$$

(7.33)
do depend on $\sigma^i$, the anti-holomorphic integrated insertion does not decouple when inserted in a correlator containing holomorphic observables: the terms proportional to the equation of motions of $\sigma^i$ produce non-vanishing contact terms when observables like (7.33) are present in a quantum correlator. This is the root of the anti-holomorphic dependence of integrated correlators of observables $O_a^{(0)}$, i.e. of topological strings amplitudes. Let us note that the contacts produced by anti-holomorphic insertions are proportional to the antighost fields $\gamma^\alpha$: in the “matter” formulation in which $\gamma^\alpha$ is ignored, the contact would be — incorrectly — interpreted as a BRST anomaly.

**Acknowledgments**

It is a great pleasure to thank C. Becchi and S. Giusto for earlier collaboration with me on this topic: most of what I have presented in this article I understood as a result of working with them. I am also grateful to A. Tomasiello for sharing with me his work with A. Kapustin on non-Kählerian B-models. As noted, trying to solve their puzzle provided the original inspiration this paper. I am also greatly indebted to D. Rosa for patiently and frequently discussing the details of many puzzles and confusing moments with me over the course of the last few months.

Finally, I would like express my gratitude to Raymond Stora, who, during the time I had the fortune to know him, generously and graciously bestowed upon me not only his profound knowledge of both physics and mathematics, but also his enthusiasm for everything beautiful, his inexhaustible intellectual curiosity and his exquisite humanity.

The work of CI was supported in part by INFN, by Genoa University Research Projects (P.R.A.) 2014 and 2015.
References

[1] J. M. F. Labastida, M. Pernici and E. Witten, “Topological Gravity in Two-Dimensions,” Nucl. Phys. B 310, 611 (1988).

E. Witten, “On the Structure of the Topological Phase of Two-dimensional Gravity,” Nucl. Phys. B 340, 281 (1990).

[2] S. Ouvry, R. Stora and P. van Baal, “On the Algebraic Characterization of Witten’s Topological Yang-Mills Theory,” Phys. Lett. B 220 (1989) 159.

[3] L. Baulieu and I. M. Singer, “Topological Yang-Mills Symmetry,” Nucl. Phys. Proc. Suppl. 5B (1988) 12.

L. Baulieu and I. M. Singer, “The Topological Sigma Model,” Commun. Math. Phys. 125 (1989) 227.

H. Kanno, “Weyl Algebra Structure and Geometrical Meaning of the BRST transformation in Topological Quantum Field Theory,” Z. Phys. C 43 (1989) 477.

[4] D. Montano and J. Sonnenschein, “Topological Strings,” Nucl. Phys. B 313, 258 (1989).

[5] C. M. Becchi and C. Imbimbo, “Gribov horizon, contact terms and Cech-De Rham cohomology in 2-D topological gravity,” Nucl. Phys. B 462, 571 (1996) [arXiv:hep-th/9510003].

[6] E. Witten, “Topological Sigma Models,” Commun. Math. Phys. 118, 411 (1988).

E. Witten, Mirror manifolds and topological field theory, In *Yau, S.T. (ed.): Mirror symmetry I* 121-160 [arXiv:hep-th/9112056].

[7] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. B 405, 279 (1993) [arXiv:hep-th/9302103].

M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Commun. Math. Phys. 165, 311 (1994) [arXiv:hep-th/9309140].

[8] J. Bae, C. Imbimbo, S. J. Rey and D. Rosa, “New Supersymmetric Localizations from Topological Gravity,” JHEP 1603, 169 (2016) [arXiv:1510.00006].

C. Imbimbo and D. Rosa, “Topological anomalies for Seifert 3-manifolds,” JHEP 1507, 068 (2015) [arXiv:1411.6635].
[9] S. Giusto, “Topological Field Theories and Strings”, Ph. D Thesis, University of Genoa, 1999.