CLOAKING PROPERTY OF A PLASMONIC STRUCTURE IN DOUBLY COMPLEMENTARY MEDIA AND THREE-SPHERE INEQUALITIES WITH PARTIAL DATA

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Abstract. We investigate cloaking property of negative-index metamaterials in the time-harmonic electromagnetic setting for the so-called doubly complementary media. These are media consisting of negative-index metamaterials in a shell (plasmonic structure) and positive-index materials in its complement for which the shell is complementary to a part of the core and a part of the exterior of the core-shell structure. We show that an arbitrary object is invisible when it is placed close to a plasmonic structure of a doubly complementary medium as long as its cross section is smaller than a threshold given by the property of the plasmonic structure. To handle the loss of the compactness and of the ellipticity of the modeling Maxwell equations with sign-changing coefficients, we first obtain Cauchy’s problems associated with two Maxwell systems using reflections. We then derive information from them, and combine it with the removing localized singularity technique to deal with the localized resonance. A central part of the analysis on the Cauchy’s problems is to establish three-sphere inequalities with partial data for general elliptic systems, which are interesting in themselves. The proof of these inequalities first relies on an appropriate change of variables, inspired by conformal maps, and is then based on Carleman’s estimates for a class of degenerate elliptic systems.

Key words: Maxwell equations, sign-changing coefficients, localized resonance, three-sphere inequalities, Carleman’s estimates, Cauchy’s problems, degenerate elliptic equations, complementary media.

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1. Introduction

Negative-index metamaterials are artificial structures whose refractive index has a negative value over some frequency range. Their existence was postulated by Veselago in 1964 [54] and confirmed experimentally by Shelby, Smith, and Schultz in 2001 [53]. Negative-index metamaterial research has been a very active topic of investigation not only because of potentially interesting applications, such as superlensing [51, 54], cloaking [28, 33, 39], biomedical imaging [20], and heat generation [6], but also because of challenges in understanding their surprising properties. From a mathematical point of view, the subtlety and the challenging in the study of negative-index metamaterials come from the sign-changing coefficients in the modeling equations, hence the ellipticity and the compactness are lost in general. Moreover, localize resonance, i.e., a phenomenon in which the field blows up in some regions and remains bounded in some others as the loss goes to 0, might occur.

In this paper, we investigate cloaking property of negative-index metamaterials for electromagnetic waves in the time-harmonic regime for the so-called doubly complementary media. This is a part of our program on understanding of properties and applications of negative-index metamaterials in the electromagnetic setting from mathematical perspectives [37, 40, 43, 46]. Doubly complementary media, introduced in [43], are media consisting of negative-index metamaterials in a shell and positive-index materials in its complement for which the shell is complementary to a part of the core and a part of the exterior of the core-shell structure (Definition 5.2). We show that an arbitrary object with small cross-section placed close to a plasmonic structure of a doubly complementary medium is cloaked (Theorem 5.1). This cloaking property is know as cloaking an object via anomalous localized resonance. We also address the necessity for having doubly complementary properties in various schemes of cloaking and superlensing using complementary media (Propositions 6.1 and 6.2). In particular, the possibility that a cloak can act like a lens and conversely is confirmed.

One of the consequences of our result on cloaking properties established in Section 5 can be described as follows (see Remark 5.2). Denote \( B_R(x) \) as the open ball in \( \mathbb{R}^d \) \((d \geq 2)\) centered at \( x \in \mathbb{R}^d \) and of radius \( R > 0 \); when \( x = 0 \), we simply denote \( B_R \). Let \( d = 3 \), \( 0 < r_1 < r_2 \), and set \( m = r_2^2 / r_1^2 \). Assume that, for \( \delta > 0 \),

\[
(\varepsilon, \mu) = \begin{cases} 
(\varepsilon \delta, \mu \delta) & \text{in } B_{r_2} \setminus B_{r_1}, \\
(\varepsilon \delta^2 I + i\delta I, \mu \delta^2 I - i\delta I) & \text{in } B_{r_2}, \\
(I, I) & \text{otherwise}.
\end{cases}
\]

Denote \( \Gamma_2 = \{ x \in \mathbb{R}^3; |x| = r_2 \text{ and } x_3 = 0 \} \) and \( \Gamma_1 = \{ x \in \mathbb{R}^3; |x| = r_1 \text{ and } x_3 = 0 \} \), and set \( O_{\gamma,j} := \{ x \in \mathbb{R}^3; \text{dist}(x, \Gamma_j) < \gamma \} \) for \( j = 1, 2 \), and \( \gamma > 0 \). Let \((\varepsilon_c, \mu_c)\) be a pair of piecewise \( C^1 \), real, symmetric, uniformly elliptic, matrix-valued functions defined in \( \mathcal{O}_\gamma := \left( O_{1,\gamma} \cup O_{2,\gamma} \right) \setminus (B_{r_2} \setminus B_{r_1}) \). Define

\[
(\varepsilon_{c,\delta}, \mu_{c,\delta}) = \begin{cases} 
(\varepsilon_c, \mu_c) & \text{in } \mathcal{O}_\gamma, \\
(\varepsilon_{\delta}, \mu_{\delta}) & \text{otherwise}.
\end{cases}
\]
Set \( r_3 = r_2^2 / r_1 \) and let \( \omega > 0 \). There exists \( \gamma_0 = \gamma_0(r_2, r_3) \) depending only on \( r_2 \), and \( r_3 \) (\( r_0 \) is independent of \( (\varepsilon_c, \mu_c) \)) such that for \( 0 < \gamma < \gamma_0 \), and for \( J \in [L^2(\mathbb{R}^3)]^3 \) with compact support in \( \mathbb{R}^3 \setminus B_{r_3} \), we have

\[
\lim_{\delta \to 0} \|(E_{c,\delta}, H_{c,\delta}) - (\widetilde{E}, \widetilde{H})\|_{L^2(B_R \setminus B_{r_3})} = 0.
\]

Here \((E_{c,\delta}, H_{c,\delta}), (\widetilde{E}, \widetilde{H})\) are respectively the unique radiating solution of the Maxwell equations

\[
\begin{aligned}
\nabla \times E_{c,\delta} &= i\omega \mu_{c,\delta} H_{c,\delta} \quad \text{in } \mathbb{R}^3, \\
\nabla \times H_{c,\delta} &= -i\omega \varepsilon_{c,\delta} E_{c,\delta} + J \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

\[
\begin{aligned}
\nabla \times \widetilde{E} &= i\omega \widetilde{H} \quad \text{in } \mathbb{R}^3, \\
\nabla \times \widetilde{H} &= -i\omega \widetilde{E} + J \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]

Physically, \( \varepsilon_\delta \) and \( \mu_\delta \) describe the permittivity and the permeability of the considered medium. \( B_{r_2} \setminus B_{r_1} \) is a (shell) plasmonic structure in which the permittivity and the permeability are negative, and \( \delta \partial I \) describes its loss, \( \omega \) is the frequency, and \( J \) is a density of charge. As a consequence of (1.3), \( \lim_{\delta \to 0} (E_{c,\delta}, H_{c,\delta}) = (\widetilde{E}, \widetilde{H}) \) in \( \mathbb{R}^3 \setminus B_{r_3} \) for all \( J \) with compact support outside \( B_{r_3} \). One therefore cannot detect the difference between \( (\varepsilon_{c,\delta}, \mu_{c,\delta}) \) and the homogeneous medium \((I, I)\), where \( I \) denotes the \( 3 \times 3 \) identity matrix, as \( \delta \to 0 \) by observation of \((E_{c,\delta}, H_{c,\delta})\) outside \( B_{r_3} \) using the excitation \( J \): cloaking is achieved for observers outside \( B_{r_3} \) in the limit as \( \delta \to 0 \).

Cloaking property of a plasmonic structure for small objects/sources near to in some superlensing settings satisfying doubly complementary property was raised in the literature about a decade ago. The possibility that a lens consisting of negative-index materials can act like a cloak and conversely was also debated in the literature, see e.g. [14,33]. This is in part due to the complexity of these phenomena and the occurrence of localized resonance which make numerical simulations and experiments difficult to check. The mathematical study for these problems was given in [39] for the acoustic setting for a subclass of complementary media. This class contains some but not all plasmonic structures which are complementary with homogeneous medium in three dimensions. This left widely open the question of whether cloaking property of plasmonic structures holds for the whole class of doubly complementary media in the electromagnetic setting. This work answers this question completely. In fact, we establish a stronger statement saying that not only small objects but also objects with small cross section near to the plasmonic structure are cloaked.

The cloaking method/property considered in this paper is related to but different from the so-called cloaking using complementary media [41] and is inspired by cloaking a source via anomalous localized resonance [43] with its roots in [33,39] (see also [50]). Mathematical works on applications and properties of negative-index metamaterials in the acoustic setting such as superlensing, cloaking using complementary media, cloaking via anomalous localized resonance for a source or for an object, and stability aspects of negative-index materials can be found in [36], [37,45], [4,26,35,42], [39], [9,13,38], respectively, and the references therein.

Our analysis is in the spirit of [39] but requires essentially new ideas and techniques. To deal with the loss of ellipticity and compactness, and the occurrence of localized resonance, we first derive Cauchy’s problems associated with two Maxwell systems from reflections originally proposed in [34] for the acoustic setting. We then apply the removing localized singularity technique. To be able to apply these techniques, the crucial and difficult point is to establish three-sphere inequalities with partial data for Maxwell equations (Theorem 2.2). To this end, we first prove three-sphere inequalities with partial data for general elliptic systems (Theorems 2.1). We then derive the corresponding ones for the Maxwell equations using their weakly coupled, second-order elliptic property. These inequalities are the core part of our analysis. They are interesting in
themselves, and can be used in other contexts, e.g. control theory [12, 31] or inverse problems [23, 55].

Outline of the paper: The rest of the paper is organized as follows. In Section 2, we state three-sphere inequalities for second-order elliptic systems and Maxwell equations with partial data. Sections 3 and 4 are devoted to the proof of these inequalities for elliptic systems and Maxwell equations, respectively. In Section 5, we state and give the proof of the main cloaking results for doubly complementary media. In Section 6, we make several comments on the construction of cloaking and superlensing devices using complementary media used in the literature and give the analysis for various contexts where a lens can act like a cloak and conversely.

2. Three-sphere inequalities with partial data

Let \( v \) be an holomorphic function defined in \( B_{R_3} \), Hadamard [18] proved the following famous three-sphere inequality:

\[
\|v\|_{L^\infty(\partial B_{R_2})} \leq \|v\|_{L^\infty(\partial B_{R_1})} \|v\|_{L^\infty(\partial B_{R_3})}^{1-\alpha}
\]

(2.1)

for all \( 0 < R_1 < R_2 < R_3 \), where

\[ \alpha = \ln \left( \frac{R_3}{R_2} \right) / \ln \left( \frac{R_3}{R_1} \right). \]

A three-sphere inequality for general second-order elliptic equations was established by Landis [29] using Carleman type estimates with its roots in [10]. His result [29, Theorem 2.1] can be stated as follows: if \( v \) is a solution to

\[
\text{div}(\mathcal{M}\nabla v) + c \cdot \nabla v + bv = 0 \quad \text{in} \quad B_{R_3},
\]

(2.2)

where \( \mathcal{M} \) is elliptic, symmetric, matrix-valued defined in \( B_{R_3} \) of class \( C^2, \ c \in [C^1(\bar{B}_{R_3})]^d, \ b \in C^1(\bar{B}_{R_3}), \) and \( b \leq 0 \), then there is a constant \( C > 0 \) such that

\[
\|v\|_{L^\infty(\partial B_{R_2})} \leq C \|v\|_{L^\infty(\partial B_{R_1})} \|v\|_{L^\infty(\partial B_{R_3})}^{1-\alpha}
\]

(2.3)

for some \( \alpha \in (0,1) \) depending only on \( R_2/R_1, R_2/R_3 \), the ellipticity constant of \( \mathcal{M} \), and the regularity constants of \( \mathcal{M} \), \( b \), and \( c \). The assumption \( b \leq 0 \) is necessary to avoid the scenario in which \( v = 0 \) on \( \partial B_{R_1} \) or on \( \partial B_{R_3} \) and \( v \neq 0 \) on \( \partial B_{R_2} \), see, e.g. [45] for comments on this point. Another proof of this inequality was obtained by Agmon [1] in which he used the logarithmic convexity. Garofalo and Lin [16, 17] established similar results for singular coefficients where the \( L^\infty \)-norm is replaced by the \( L^2 \)-norm, and \( \mathcal{M} \) is of class \( C^1, \ c \) and \( b \) are in \( L^\infty \):

\[
\|v\|_{L^2(\partial B_{R_2})} \leq C \|v\|_{L^2(\partial B_{R_1})} \|v\|_{L^2(\partial B_{R_3})}^{1-\alpha}
\]

(2.4)

using the Almgren type frequency function approach. A closely related topic is the unique continuation principle. Some seminar contributions in this context include the work of Aronszajn [5], Protter [52], Hörmander [19], Kenig, Ruiz, and Sogge [22], Jerison and Kenig [21], and Koch and Tataru [25]. Interesting surveys on these aspects can be found in [2, 30].

In the case \( b > 0 \), (2.4) holds under the smallness of \( R_3 \) (see e.g. [2, Theorem 4.1]); this condition is equivalent to the smallness of \( b \) for a fixed \( R_3 \) by a scaling argument. Three-sphere inequalities without imposing the smallness condition on \( R_3 \) were established [45, Theorem 2]. In particular, we showed that (2.3) holds with the \( \|v\|_{L^\infty(\partial B_r)} \)-norm replaced by

\[
\|v\|_{H(\partial B_r)} = \|v\|_{H^{1/2}(\partial B_r)} + \|\mathcal{M}\nabla v \cdot e_r\|_{H^{-1/2}(\partial B_r)}
\]

(2.5)
for \( r = R_1, R_2, \) or \( R_3, \) where \( H^{-1/2}(\partial B_r) \) denotes the dual space of \( H^{1/2}(\partial B_r) \) and is equipped with the corresponding norm.

In this section, we are concerned about three-sphere inequalities for second-order, elliptic systems and Maxwell equations with partial data. These inequalities have their own interests beside their applications in cloaking studied in this paper. For \( d \geq 2, \) denote

\[
\mathbb{R}_+^d = \left\{ x \in \mathbb{R}^d; x_1 > 0 \right\} \quad \text{and} \quad \mathbb{R}_0^d = \left\{ x \in \mathbb{R}^d; x_1 = 0 \right\}.
\]

Set \( Q = (-1, 1)^d \) and \( Q_+ = Q \cap \mathbb{R}_+^d \) and \( Q_0 = Q \cap \mathbb{R}_0^d. \) We first introduce

**Definition 2.1.** Let \( \Omega \) be a bounded, open subset \( \Omega \subset \mathbb{R}^d \) of class \( C^1. \) A compact subset \( \Gamma \) of \( \partial \Omega \) is called a \((d-2)\)-compact, smooth submanifold of \( \partial \Omega \) if for every \( x \in \Gamma, \) there exists a diffeomorphism \( F : Q \rightarrow U \) for some open neighborhood \( U \) of \( x \) such that

\[
F(Q_+) = U \cap \Omega, \quad F(Q_0) = U \cap \partial \Omega, \quad F(Q_0 \cap \{ x_2 = 0 \}) = \Gamma \cap U.
\]

When \( d = 3, \) a 1-compact, smooth submanifold of \( \partial \Omega \) is simply called a compact, smooth curve of \( \partial \Omega. \)

Our main result on three-sphere inequalities for second-order elliptic systems with partial data is

**Theorem 2.1.** Let \( d \geq 2, m \geq 1, \Lambda \geq 1, 0 < R_1 < R_3, \) and let \( \Gamma \) be a \((d-2)\)-compact, smooth submanifold of \( \partial B_{R_1}. \) Denote \( O_r = \left\{ x \in \mathbb{R}^d; \text{dist}(x, \Gamma) < r \right\}, \) \( D_r = B_{R_3} \setminus (\overline{B_{R_1} \cup O_r}), \) and \( \Sigma_r = \partial B_{R_1} \setminus \overline{O_r} \) for \( r > 0. \) Then, for every \( \alpha \in (0, 1), \) there exists \( r_2 \in (0, R_3 - R_1) \) depending only on \( \alpha, \Lambda, \Gamma, R_1, \) and \( R_3 \) such that for every \( r_1 \in (0, r_2) \), there exists \( r_0 \in (0, r_1) \) depending only on \( r_1, \alpha, \Lambda, \Gamma, R_1, \) and \( R_3 \) such that for \((d \times d)\) Lipschitz, uniformly elliptic, matrix-valued function \( M^\ell \) defined in \( D_{r_0} \) for \( 1 \leq \ell \leq m, \) verifying, in \( D_{r_0}, \)

\[
\Lambda^{-1} |\xi|^2 \leq \langle M^\ell(x) \xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M^\ell(x)| \leq \Lambda,
\]

for \( g \in L^2(D_{r_0}), \) and for \( V \in [H^2(D_{r_0})]^m \) satisfying, for \( 1 \leq \ell \leq m, \)

\[
|\text{div}(M^\ell \nabla V_\ell)| \leq \Lambda_1 (|\nabla V| + |V| + |g|) \quad \text{in} \ D_{r_0} \quad \text{for some} \ \Lambda_1 \geq 0,
\]

we have

\[
\|V\|_{H^1(B_{R_1 + r_2 \setminus B_{R_1 + r_1}})} \leq C \left( \|V\|_{H^1(\Sigma)} + \|g\|_{L^2(D_{r_0})} \right)^\alpha \left( \|V\|_{H^1(D_{r_0})} + \|g\|_{L^2(D_{r_0})} \right)^{1-\alpha},
\]

for some positive constant \( C \) depending only on \( \alpha, \Lambda, \Lambda_1, \Gamma, R_1, R_3, m, \) and \( d. \)

It is worth noting that the Cauchy’s problems for second-order elliptic equations are unstable, see the survey [2] for a discussion. Even so, three-sphere inequalities with partial data hold surprisingly in a very general setting considered in Theorem 2.1. Note that, for \( d \geq 3, \) the set \( B_{R_1} \cap O_r \) in Theorem 2.1 is not small in term of radius but has a small cross-section. To our knowledge, Theorem 2.1 is new even for the Laplace equation, i.e., \( m = 1, M^1 = I, \) and \( \Lambda_1 = 0. \) For \( d = 2, \) the setting for the Laplace equation is previously established in [39], which is indeed one of the main motivations of our present work.

As a consequence of Theorem 2.1, one can derive three-sphere inequalities of partial data for \( R_1 < R_2 < R_3 \) from classical three-sphere inequalities. Here is an illustration in the spirit of
Hadamard. Let \( d = 2, 3, \omega > 0, \) and \( R_1 < R_2 < R_3, \) and let \( v \in H^1(B_{R_3} \setminus B_{R_1}) \) be a solution of \( \Delta v + \omega^2 v = 0. \) We have, see e.g. [42, Lemma 4.2], with \( \alpha_0 = \ln(R_3/R_2)/\ln(R_3/R_1), \)

\[
\|v\|_{H^2(\partial B_{R_2})} \leq C \|v\|_{H^2(B_{R_3})}^{\alpha_0} \|v\|_{H^2(B_{R_2})}^{1-\alpha_0},
\]

for some positive constant \( C \) depending only on \( \omega, R_1, R_2, \) and \( R_3. \) Using Theorem 2.1, we establish the following (new) variant of (2.8):

**Corollary 2.1.** Let \( d = 2, 3, \) and \( 0 < R_1 < R_2 < R_3, \) and let \( \Gamma \) be a \((d-2)\)-compact, smooth submanifold of \( \partial B_{R_1}. \) Denote \( O_r = \{ x \in \mathbb{R}^d; \text{dist}(x, \Gamma) < r \}, \) \( D_r = B_{R_3} \setminus (B_{R_1} \cup O_r), \) and \( \Sigma_r = \partial B_{R_1} \setminus O_r \) for \( r > 0. \) Set \( \alpha_0 = \ln(R_3/R_2)/\ln(R_3/R_1). \) Then, for any \( \alpha \in (0, \alpha_0), \) there exists \( r_0 \in (0, R_2 - R_1), \) depending only on \( R_1, R_2, R_3, \) and \( \alpha \) such that, for \( \omega > 0 \) and for \( v \in H^1(D_{r_0}) \) satisfying \( \Delta v + \omega^2 v = 0 \) in \( D_{r_0}, \) we have

\[
\|V\|_{H^2(\partial B_{R_2})} \leq C \|V\|_{H^2(\Sigma_{r_0})} \|V\|_{H^2(D_{r_0})}^{1-\alpha_0},
\]

for some positive constant \( C \) depending only on \( \alpha, \omega, \Gamma, R_1, \) and \( R_3. \)

The proofs of Theorem 2.1 and Corollary 2.1, and variants of Theorem 2.1 (Theorem 3.1 and Corollary 3.1) are given in Section 3.

We next discuss the Maxwell equations. Our main result in this direction is

**Theorem 2.2.** Let \( d = 3, \Lambda \geq 1, 0 < R_1 < R_3, \) and let \( \Gamma \) be a compact, smooth curve of \( \partial B_{R_1}. \) Denote \( O_r = \{ x \in \mathbb{R}^d; \text{dist}(x, \Gamma) < r \}, \) \( D_r = B_{R_3} \setminus (B_{R_1} \cup O_r), \) and \( \Sigma_r = \partial B_{R_1} \setminus O_r \) for \( r > 0. \) Then for any \( \alpha \in (0, 1), \) there exists \( r_2 \in (0, R_3 - R_1) \) depending only on \( \Lambda, \Gamma, R_1, \) and \( R_3 \) such that for every \( r_1 \in (0, r_2), \) there exists \( r_0 \in (0, r_1) \) depending only on \( r_1, \alpha, \Lambda, \Gamma, \) and \( R_3 \) such that for \( (\varepsilon, \mu) \) a pair of \((3 \times 3)\) real, uniformly elliptic, matrix-valued functions defined in \( D_{r_0} \) of class \( C^2 \) verifying, in \( D_{r_0}, \)

\[
\Lambda^{-1} \xi^2 \leq \langle \mathcal{M}^\ell(x)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla \mathcal{M}^\ell(x)| \leq \Lambda,
\]

for \( \omega > 0, \) for \( J_e, J_m \in [L^2(D_{r_0})]^3, \) and for \( (E, H) \in [H(\text{curl}, D_{r_0})]^2 \) satisfying

\[
\begin{cases}
\nabla \times E = i\omega \mu H + J_e & \text{in } D_{r_0}, \\
\nabla \times H = -i\omega \varepsilon H + J_m & \text{in } D_{r_0},
\end{cases}
\]

we have

\[
\|(E, H)\|_{L^2(B_{R_1+r_2} \setminus B_{R_1+r_1})} \leq C \left( \|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\text{div}_\Gamma, \Sigma_{r_0})} + \|(J_e, J_m)\|_{L^2(D_{r_0})} \right)^\alpha \times \left( \|(E, H)\|_{L^2(D_{r_0})} + \|(J_e, J_m)\|_{L^2(D_{r_0})} \right)^{1-\alpha},
\]

for some positive constant \( C \) depending only on \( r_1, \alpha, \omega, \Lambda, \Gamma, R_1, R_3, \) and the upper bound of \( \|(\varepsilon, \mu)\|_{C^2(D_{r_0})}. \)

Here and in what follows, for an open, bounded subset \( \Omega \) of \( \mathbb{R}^3 \) of class \( C^1, \) one denotes, with \( \Gamma = \partial \Omega, \)

\[
H^{-1/2}(\text{div}_\Gamma, \Gamma) := \{ \phi \in [H^{-1/2}(\Gamma)]^3; \phi \cdot \nu = 0 \text{ and } \text{div}_\Gamma \phi \in H^{-1/2}(\Gamma) \},
\]

\[
\|\phi\|_{H^{-1/2}(\text{div}_\Gamma, \Gamma)} := \|\phi\|_{H^{-1/2}(\Gamma)} + \|\text{div}_\Gamma \phi\|_{H^{-1/2}(\Gamma)}.
\]
For an open subset Ω of ℝ³, the following standard notations are used:

\[ H(\text{curl}, \Omega) := \left\{ u \in [L^2(\Omega)]^3; \nabla \times u \in [L^2(\Omega)]^3 \right\}, \]

\[ \|u\|_{H(\text{curl}, \Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla \times u\|_{L^2(\Omega)}, \]

\[ H_{\text{loc}}(\text{curl}, \Omega) := \left\{ u \in [L^2_{\text{loc}}(\Omega)]^3; \nabla \times u \in [L^2_{\text{loc}}(\Omega)]^3 \right\}. \]

**Remark 2.1.** In Theorem 2.2, one requires \((\varepsilon, \mu)\) to be of class \(C^2\). Nevertheless, the constant \(r_2\) depends on Λ not on \(\|(\varepsilon, \mu)\|_{C^2(D_{\gamma_0})}\).

Theorem 2.2 is the new crucial ingredient in the proof of cloaking property for doubly complementary media. A consequence of Theorem 2.2 in the spirit of Hadamard is given in Corollary 4.2 in Section 4.

The proofs of Theorem 2.1 and Corollary 2.1 are given in Section 3. The proofs of Theorem 2.2 and its consequence (Corollary 4.2) are given in Section 4. The most important ingredient of these proofs is Theorem 3.1 in Section 3. Concerning the proof of Theorem 3.1, we first use an appropriate change of variables inspired by conformal maps. We then establish a new type of three-sphere inequalities for a class of degenerate second-order, elliptic inequalities in which not only the properties of the coefficients but also the way they interact with the domain considered play an important role (see also the paragraph right after Theorem 3.1).

### 3. Three-sphere inequalities for second-order elliptic inequalities

This section is on three-sphere inequalities for second-order elliptic inequalities with partial data. The key ingredient is their variant in a half plane given in Theorem 3.1. Throughout this section, for \(d \geq 2\) and \(x = (x_1, x_2, \tilde{x}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2}\), we use the polar coordinate \((\hat{r}, \theta)\) for the pair \((x_1, x_2)\), the variable \(\tilde{x}\) being irrelevant for \(d = 2\). For \(0 < \gamma_1 < \gamma_2 < 1\) and for \(R > 0\), we denote

\[ Y_{\gamma_1, \gamma_2, R} = \left\{ x \in \mathbb{R}^d; \theta \in (-\pi/2, \pi/2), \gamma_1 R < \hat{r} < \gamma_2 R, \text{ and } |\tilde{x}| < R \right\}. \tag{3.1} \]

We have

**Theorem 3.1.** Let \(d \geq 2, m \geq 1, \Lambda \geq 1, \text{ and } R_* < R < R^*\). Then, for any \(\alpha \in (0, 1)\), there exists a constant \(\gamma_2 \in (0, 1)\), depending only on \(\alpha, \Lambda, R_*, R^*, m, \text{ and } d\) such that for every \(\gamma_1 \in (0, \gamma_2)\), there exists \(\gamma_0 \in (0, \gamma_1)\) depending only on \(\alpha, \gamma_1, \Lambda, R_*, R^*, m, \text{ and } d\) such that, for real, symmetric, uniformly elliptic, Lipschitz matrix-valued functions \(\mathcal{M}^\ell\) with \(1 \leq \ell \leq m\) defined in \(D_{\gamma_0} := Y_{\gamma_0, 1, R} \) verifying, in \(D_{\gamma_0}\),

\[ \Lambda^{-1}|\xi|^2 \leq \langle \mathcal{M}^\ell(x) \xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla \mathcal{M}^\ell(x)| \leq \Lambda, \tag{3.2} \]

for \(g \in L^2(D_{\gamma_0})\), and for \(V \in [H^2(D_{\gamma_0})]^m \) satisfying, for \(1 \leq \ell \leq m\),

\[ |\text{div}(\mathcal{M}^\ell \nabla V)| \leq \Lambda_1 (|\nabla V| + |V| + |g|) \quad \text{in } D_{\gamma_0} \text{ for some } \Lambda_1 \geq 0, \]

we have, with \(\Sigma_{\gamma_0} = \partial D_{\gamma_0} \cap \{x_1 = 0\}\),

\[ \|V\|_{H^1(Y_{\gamma_1, \gamma_2, R/4})}^\alpha \leq C \left(\|V\|_{H(\Sigma_{\gamma_0})} + \|g\|_{L^2(D_{\gamma_0})}\right) \left(\|V\|_{H^1(D_{\gamma_0})} + \|g\|_{L^2(D_{\gamma_0})}\right)^{1-\alpha}, \tag{3.4} \]

for some positive constant \(C\) depending only on \(\alpha, \gamma_1, \Lambda, \Lambda_1, R_*, R^*, m, \text{ and } d\).
The proof of Theorem 3.1 is based on Carleman’s estimates. The weight used is $e^{\beta r - p}$ for which $\beta$ and $p$ are two (large) parameters as in [45] (the work of Protter’s [52] and of Fursikov and Imanuvilov’s [15] are also worth mentioning). A simple but critical step of the proof is the use of transformations related to the conformal type map $(x_1, x_2, \tilde{x}) \rightarrow (\tilde{r}^{1/n} \cos(\theta/n), \tilde{r} \sin(\theta/n), \tilde{x})$ to transform the domain $Y_{\gamma, 1, R}$ into a domain for which the first two variables are in a sector of circulars with a small angle. We then apply three-sphere inequalities for this domain to deduce the desired estimate. The advantage of this process is that three-sphere inequalities with partial data are easier to handle for this new geometry, as noted in [39]. However, new difficulties appear in establishing three-sphere inequalities in the new geometry. On one hand, the lower bound of the ellipticity of the new set of matrix-valued functions obtained from $M^\ell$ ($1 \leq \ell \leq m$) goes to 0 as $n \to \infty$ in general. On the other hand, to be able to carry on three-sphere inequalities with partial data in this domain, one requires to establish three-sphere inequalities associated with the new set of matrices in which the output (the parameter $\alpha$) is independent of $n$. To overcome this obstacle, a structure of the new set of matrix-valued functions is formulated (see e.g. Remark 3.1 and (3.49)) and new Carleman’s estimates capturing this structure are derived.

As a direct consequence of Theorem 3.1, we have the following result whose proof is omitted.

**Corollary 3.1.** Let $d \geq 2$, $m \geq 1$, $\Lambda \geq 1$, let $\Omega$ be a bounded, open subset of $\mathbb{R}^d$ of class $C^1$, and let $\Gamma$ be a $(d - 2)$-compact, smooth submanifold of $\partial \Omega$ and belong to a connected component $\Sigma$ of $\partial \Omega$. Denote $O_r = \{ x \in \mathbb{R}^d; \text{dist}(x, \Gamma) < r \}$, $D_r = \Omega \setminus O_r$, and $\Sigma_r = \Sigma \setminus O_r$ for $r > 0$. Then, for every $\alpha \in (0, 1)$, there exists $r_2 > 0$ depending only on $\alpha$, $\Lambda$, $\Gamma$, and $\Omega$ such that for $r_1 \in (0, r_2)$, there exists $r_0 \in (0, r_1)$, depending only on $r_1$, $\alpha$, $\Lambda$, $\Gamma$, and $\Omega$ such that for $(d \times d)$ Lipschitz, uniformly elliptic, matrix-valued function $M^\ell$ defined in $\Omega$ with $1 \leq \ell \leq m$, verifying, in $D_{r_0}$,

$$\Lambda^{-1} |\xi|^2 \leq \langle M^\ell(x)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M^\ell(x)| \leq \Lambda,$$
for $g \in L^2(D_{r_0})$, and for $V \in [H^2(D_{r_0})]^m$ satisfying, for $1 \leq \ell \leq m$,

$$\text{div}(M^\ell \nabla V_\ell) \leq \Lambda_1 (|\nabla V| + |V| + |g|) \text{ in } D_{r_0} \text{ for some } \Lambda_1 \geq 0,$$

we have

$$\|V\|_{H^1(\Omega_2 \setminus \Omega_1)} \leq C \left(\|V\|_{H^\alpha(\Sigma_{r_0})} + \|g\|_{L^2(D_{r_0})}\right)\left(\|V\|_{H^1(D_{r_0})} + \|g\|_{L^2(D_{r_0})}\right)^{1-\alpha},$$

for some positive constant $C$ depending only on $r_1$, $\alpha$, $\Lambda_1$, $\Gamma$, $\Omega$, $m$, and $d$.

The rest of this section is organized as follows. In Section 3.1, we establish several lemmas used in the proof of Theorem 3.1. The main step of the proof is given in Section 3.2. The complete proof of Theorem 3.1 is given in Section 3.3.

### 3.1. Preliminaries

In this section, we establish several lemmas used in the proof of Theorem 3.1. The computations are in the spirit of Carleman’s estimates in [45], nevertheless, the assumptions and conclusions are importantly formulated/revealed in a way that can be used in the context of partial data. In what follows, $\Omega$ denotes a bounded connected open subset of $\mathbb{R}^d$ with Lipschitz boundary, for $x \in \mathbb{R}^d$ ($d \geq 2$), $r$ denotes its (Euclidean) length, i.e., $r = |x|$, and $(\cdot, \cdot)$ denotes the standard Euclidean scalar product unless otherwise stated. All quantities considered in this section are real. The key results of this section are Lemma 3.4 and its consequence Lemma 3.6. We begin with

**Lemma 3.1.** Let $w \in H^2(\Omega)$ and let $M$ be a Lipschitz, symmetric, uniformly elliptic, matrix-valued function defined in $\Omega$. We have

$$\int_{\Omega} (x \cdot M \nabla w) \text{div}(M \nabla w) \geq -\int_{\Omega} \langle B \nabla w, \nabla w \rangle - \int_{\partial \Omega} Cr|w|^2|\nabla w|^2,$$

for some positive constant $C$ depending only on $d$. Here, for $x \in \Omega$,

$$\langle B(x)y, y \rangle := \langle [(M(x)y) \cdot \nabla]M(x)y, y \rangle + \frac{1}{2} \langle \text{div}(M(x)y)M(x)y, y \rangle + \frac{1}{2} \langle [(M(x)y) \cdot \nabla]M(x)y, y \rangle \quad \text{for } y \in \mathbb{R}^d.$$

**Proof.** An integration by parts gives

$$\int_{\Omega} (x \cdot M \nabla w) \text{div}(M \nabla w) = -\int_{\Omega} \nabla(x \cdot M \nabla w) \cdot M \nabla w + \int_{\partial \Omega} (x \cdot M \nabla w) M \nabla w \cdot \nu.$$

Using the symmetry of $M$, we have

$$\frac{\partial}{\partial x_i} (x \cdot M \nabla w) = \frac{\partial}{\partial x_i} \left(M_{kj} x_j \frac{\partial w}{\partial x_k}\right) = M_{kj} x_j \frac{\partial^2 w}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_i} (M(x) \cdot \nabla w)$$

and

$$-\int_{\Omega} 2x_j M_{kj} \frac{\partial^2 w}{\partial x_i \partial x_k} M^i \frac{\partial w}{\partial x_i} = -\int_{\Omega} x_j M_{kj} M^i \frac{\partial}{\partial x_i} \left(\frac{\partial w}{\partial x_k}\frac{\partial w}{\partial x_i}\right) = \int_{\Omega} \frac{\partial (x_j M_{kj} M^i)}{\partial x_k} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_i} - \int_{\partial \Omega} x_j M_{kj} M^i \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_i} \nu_k.$$

The conclusion now follows from (3.7), (3.8), and (3.9). \qed

The second lemma is...

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1In what follows, the repeated summation is used.
Lemma 3.2. Let $p \geq 1$, $\beta \in \mathbb{R}$, $w \in H^2(\mathcal{O})$, and let $M$ be a Lipschitz, symmetric, uniformly elliptic, matrix-valued function defined in $\mathcal{O}$. Set, for $x \in \mathcal{O}$,

$$T_1(x) = (2p + 4)(x \cdot Mx)^2 r^{-4} - r^{-2} \text{div} [(x \cdot Mx)Mx] \leq \text{div}(Mx)^2 \langle Mx, x \rangle$$

and

$$T_2(x) = -(p + 4)(x \cdot Mx)^2 r^{-4} + r^{-2} \text{div} [(x \cdot Mx)Mx].$$

Assume that $|x| \leq 1$ for $x \in \mathcal{O}$. We have

$$\int_{\mathcal{O}} e^{\beta r^{-p}} (Mx \cdot \nabla |w|^2) \text{div}(M\nabla e^{-\beta r^{-p}})$$

$$\geq \int_{\mathcal{O}} \left( p^2 \beta^2 r^{-2p-2} T_1 + \beta p(p + 2) r^{-p-2} T_2 \right) |w|^2 - \int_{\mathcal{O}} \langle M\nabla w, \nabla w \rangle - \int_{\partial \mathcal{O}} C \beta^2 p^2 r^{-2p-1} |M|^2 |w|^2,$$

for some positive constant $C$ depending only on $d$.

Proof. We have, for $x \in \mathcal{O}$,

$$\text{div}(M\nabla e^{-\beta r^{-p}}) = p \beta e^{-\beta r^{-p}} \left[ p \beta r^{-2p-4} - (p + 2)r^{-p-4} \right] x \cdot Mx + p \beta r^{-p-2} e^{-\beta r^{-p}} \text{div}(Mx).$$

An integration by parts gives

$$\int_{\mathcal{O}} e^{\beta r^{-p}} (Mx \cdot \nabla |w|^2) \text{div}(M\nabla e^{-\beta r^{-p}}) = P + Q.$$  

Here

$$P = P_1 + P_2 + P_3$$

with

$$\begin{cases} P_1 = - \int_{\mathcal{O}} p^2 \beta^2 |w|^2 \text{div} \left[ r^{-2p-4}(x \cdot Mx)Mx \right], \\ P_2 = \int_{\mathcal{O}} p(p + 2) \beta |w|^2 \text{div} \left[ r^{-p-4}(x \cdot Mx)Mx \right], \\ P_3 = \int_{\mathcal{O}} 2p \beta r^{-p-2} \text{div}(Mx)w \nabla w \cdot Mx, \end{cases}$$

and

$$Q = \int_{\partial \mathcal{O}} p \beta |w|^2 \left( [p \beta r^{-2p-4} - (p + 2)r^{-p-4}] x \cdot Mx \right) Mx \cdot \nu.$$  

We next estimate $P$ and $Q$. By a straightforward computation, we have

$$- \text{div} \left[ r^{-2p-4}(x \cdot Mx)Mx \right] = (2p + 4)(x \cdot Mx)^2 r^{-2p-6} - r^{-2p-4} \text{div} [(x \cdot Mx)Mx].$$

This implies

$$P_1 = \int_{\mathcal{O}} p^2 \beta^2 \left( (2p + 4)(x \cdot Mx)^2 r^{-2p-6} - r^{-2p-4} \text{div} [(x \cdot Mx)Mx] \right) |w|^2.$$  

Similarly,

$$P_2 = - \int_{\mathcal{O}} p(p + 2) \beta \left( (p + 4)(x \cdot Mx)^2 r^{-p-6} - r^{-p-4} \text{div} [(x \cdot Mx)Mx] \right) |w|^2.$$  

Using Cauchy’s inequality, for $a \in \mathbb{R}$,

$$2|a \nabla w \cdot Mx| \leq |a|^2 \langle M\nabla w, \nabla w \rangle + \langle Mx, x \rangle,$$

we have

$$P_3 \leq \int_{\mathcal{O}} p^2 \beta^2 r^{-2p-4} |\text{div}(Mx)|^2 \langle Mx, x \rangle |w|^2 + \langle M\nabla w, \nabla w \rangle.$$
Combining (3.12), (3.13), and (3.14) yields
\[ P \geq \int_{\mathcal{O}} \left( p^2 \beta^2 r^{-2p-2} T_1 - \beta p(p + 2) r^{-p-2} T_2 \right) |w|^2 - \int_{\partial \mathcal{O}} (M \nabla w, \nabla w). \]

Since
\[ |Q| \leq \int_{\partial \mathcal{O}} C \beta^2 p^2 r^{-2p-1} |M|^2 |w|^2, \]
assertion (3.10) follows. \hfill \Box

Using Lemma 3.2, one can derive

**Lemma 3.3.** Let \( w \in H^2(\mathcal{O}) \), and let \( M \) be a Lipschitz, symmetric, uniformly elliptic, matrix-valued function defined in \( \mathcal{O} \). Assume that \( |x| \leq 1 \) for \( x \in \mathcal{O} \), and for some \( \Lambda \geq 1 \),
\[ \langle Mx, x \rangle \geq \Lambda^{-1} |x|^2 \quad \text{for } x \in \mathcal{O} \]
and
\[ |M| + |\text{div}(Mx)| + |x|^{-2} |\nabla (x \cdot Mx) \cdot Mx| \leq \Lambda \quad \text{for } x \in \mathcal{O}. \]

There exist two constants \( p_\Lambda, \beta_\Lambda \geq 1 \), depending only on \( \Lambda \) and \( d \), such that, for \( p \geq p_\Lambda \) and \( |\beta| \geq \beta_\Lambda \), we have

\[ \int_{\mathcal{O}} e^{\beta r - p} (Mx \cdot \nabla |w|^2) \text{div}(M \nabla e^{-\beta r - p}) \]
\[ \geq \int_{\mathcal{O}} p^3 \beta^2 \Lambda^{-2} r^{-2p-2} |w|^2 - \int_{\partial \mathcal{O}} (M \nabla w, \nabla w) - \int_{\partial \mathcal{O}} C \beta^2 p^2 r^{-2p-1} \Lambda^2 |w|^2 \]
for some positive constant \( C \) depending only on \( d \).

**Proof.** Estimate (3.15) is a direct consequence of (3.10) for large \( \beta \) and \( p \). \hfill \Box

Using Lemmas 3.1 and 3.3, we can establish the following result.

**Lemma 3.4.** Let \( v \in H^2(\mathcal{O}) \), and let \( M \) be a Lipschitz, symmetric, uniformly elliptic, matrix-valued function defined in \( \mathcal{O} \). Assume that, for \( x \in \mathcal{O} \), the following conditions hold: \( |x| \leq 1 \),
\[ \langle Mx, x \rangle \geq \Lambda^{-1} |x|^2, \]
\[ |M| + |\text{div}(Mx)| + |x|^{-2} |\nabla (x \cdot Mx) \cdot Mx| \leq \Lambda, \]
\[ |\langle By, y \rangle| \leq \Lambda |My, y| \quad \text{for } y \in \mathbb{R}^d, \]
for some \( \Lambda \geq 1 \) where \( \langle By, y \rangle \) is defined in (3.6). There exist two constants \( p_\Lambda, \beta_\Lambda \geq 1 \), depending only on \( \Lambda \) and \( d \), such that, if \( p \geq p_\Lambda \) and \( |\beta| \geq \beta_\Lambda \) then
\[ \int_{\mathcal{O}} \frac{r^{p+2} e^{2\beta r - p}}{2p|\beta|} |\text{div}(M \nabla v)|^2 \geq \int_{\mathcal{O}} \Lambda^{-2} p^3 \beta^2 r^{-2p-2} e^{2\beta r - p} |v|^2 - C \Lambda e^{2\beta r - p} (M \nabla v, \nabla v) \]
\[ - \int_{\partial \mathcal{O}} C \Lambda^2 r e^{2\beta r - p} (|\nabla v|^2 + \beta^2 p^2 r^{-2p-2} |v|^2) \]
for some positive constant \( C \) depending only on \( d \).
Remark 3.1. It is worth noting that we do not assume that $M$ has a positive lower bound in Lemma 3.4; the term $(M\nabla v, \nabla v)$ still appears in the conclusion. We instead assume (3.16) only for $x \in \mathcal{O}$. Moreover, the constant $\Lambda$ encodes only partly the information of the Lipschitz property of $M$ through (3.16) and (3.18). Conditions (3.16)-(3.18) are satisfied for the new set of matrix-valued functions obtained from $M^\ell$ by the conformal type map $(\hat{r} \cos(\theta/n), \hat{r} \sin(\theta/n), \hat{x})$, see the proof of Proposition 3.1.

Proof. Set

$$w = e^{\beta r^p} v \quad \text{equivalently} \quad v = e^{-\beta r^{-p}} w.$$ 

Since $\text{div}(M\nabla (gh)) = 2\nabla h \cdot M \nabla g + h \text{div}(M\nabla g) + g \text{div}(M\nabla h)$ ($M$ is symmetric), it follows that

$$\text{div}(M\nabla v) = 2\beta pr^{-p-2}e^{-\beta r^{-p}} x \cdot M \nabla w + e^{-\beta r^{-p}} \text{div}(M\nabla w) + w \text{div}(M\nabla e^{-\beta r^{-p}}).$$

Using the inequality $(a + b + c)^2 \geq 2(a+b+c)$, we obtain

$$\frac{1}{2} \left[ \text{div}(M\nabla v) \right]^2 \geq 2|\beta| pr^{-p-2}e^{-\beta r^{-p}} (x \cdot M \nabla w) \left( e^{-\beta r^{-p}} \text{div}(M\nabla w) + w \text{div}(M\nabla e^{-\beta r^{-p}}) \right).$$

This implies

$$\int_{\mathcal{O}} \frac{r^{p+2} e^{2\beta r^{-p}}}{2|\beta|} \left[ \text{div}(M\nabla v) \right]^2 \geq \int_{\mathcal{O}} 2(x \cdot M \nabla w) \text{div}(M\nabla w) + \int_{\mathcal{O}} e^{\beta r^{-p}} (Mx \cdot \nabla |w|^2) \text{div}(M\nabla e^{-\beta r^{-p}}).$$

Applying Lemma 3.1 and using (3.18), we have

$$\int_{\mathcal{O}} 2(x \cdot M \nabla w) \text{div}(M\nabla w) \geq - \int_{\mathcal{O}} \Lambda \langle M \nabla w, \nabla w \rangle - \int_{\partial \mathcal{O}} C \Lambda^2 r |\nabla w|^2.$$

Applying Lemma 3.3, we obtain

$$\int_{\mathcal{O}} e^{\beta r^{-p}} (Mx \cdot \nabla |w|^2) \text{div}(M\nabla e^{-\beta r^{-p}}) \geq \int_{\mathcal{O}} p^3 \beta^2 \Lambda^{-2} r^{-2p-2} |w|^2 - \int_{\mathcal{O}} \langle M \nabla w, \nabla w \rangle - \int_{\partial \mathcal{O}} C \Lambda^2 \beta^2 p^2 r^{-2p-1} |w|^2.$$

Combining (3.19), (3.20), and (3.21) yields

$$\int_{\mathcal{O}} \frac{r^{p+2} e^{2\beta r^{-p}}}{2|\beta|} \left[ \text{div}(M\nabla v) \right]^2 \geq \int_{\mathcal{O}} p^3 \beta^2 \Lambda^{-2} r^{-2p-2} |w|^2 - \int_{\mathcal{O}} \langle M \nabla w, \nabla w \rangle - \int_{\partial \mathcal{O}} C \Lambda^2 r \left( |\nabla w|^2 + \beta^2 p^2 r^{-2p-2} |w|^2 \right).$$

Since $w = e^{\beta r^{-p}} v$,

$$\nabla w = e^{\beta r^{-p}} (\nabla v - p \beta r^{-p-2} v x),$$
we derive from (3.22) that, for large $p$,
\[
\int_{\mathcal{O}} \frac{r^{p+2}e^{2\beta r^p}}{2p|\beta|} [\text{div}(M\nabla v)]^2 \geq \int_{\mathcal{O}} \Lambda^{-2} p^3 \beta^2 r^{-2p-2} e^{2\beta r^p} |v|^2 - C \Lambda e^{2\beta r^p} (M\nabla v, \nabla v) - \int_{\partial \mathcal{O}} \Lambda^2 r e^{2\beta r^p} \left( |\nabla v|^2 + \beta^2 p^2 r^{-2p-2} |v|^2 \right).
\]
The conclusion follows. \hfill \Box

Another ingredient in the proof of the key result of this section, Lemma 3.6, is the following lemma.

**Lemma 3.5.** Let $p \geq 1$, $\beta \in \mathbb{R}$, and $v \in H^2(\mathcal{O})$, and let $M$ be a Lipschitz, symmetric, uniformly elliptic, matrix-valued function defined in $\mathcal{O}$. Then
\[
\int_{\mathcal{O}} e^{2\beta r^p} v \text{div}(M\nabla v) + \int_{\partial \mathcal{O}} \frac{1}{2} e^{2\beta r^p} (M\nabla v, \nabla v) \leq \int_{\mathcal{O}} C|\mathcal{M}| \beta^2 p^2 r^{-2p-2} e^{2\beta r^p} |v|^2 + \int_{\partial \mathcal{O}} C|\mathcal{M}| e^{2\beta r^p} \left( |\nabla v|^2 + |v|^2 \right)
\]
for some positive constant $C$ depending only on $d$.

**Proof.** We have
\[
\int_{\mathcal{O}} e^{2\beta r^p} v \text{div}(M\nabla v) = \int_{\mathcal{O}} M\nabla v \cdot \nabla (e^{2\beta r^p} v) - \int_{\partial \mathcal{O}} e^{2\beta r^p} v M\nabla v \cdot v.
\]
It is clear that
\[
\int_{\mathcal{O}} M\nabla v \cdot \nabla (e^{2\beta r^p} v) = \int_{\mathcal{O}} \left( e^{2\beta r^p} M\nabla v \cdot \nabla v - 2\beta pr^{-p-2} e^{2\beta r^p} v M\nabla v \cdot x \right)
\]
and
\[
\int_{\partial \mathcal{O}} e^{2\beta r^p} v M\nabla v \cdot v \leq \int_{\partial \mathcal{O}} C|\mathcal{M}| e^{2\beta r^p} \left( |\nabla v|^2 + |v|^2 \right).
\]
Combining (3.23), (3.24), and (3.25) yields
\[
\int_{\mathcal{O}} e^{2\beta r^p} v \text{div}(M\nabla v) + \int_{\partial \mathcal{O}} e^{2\beta r^p} M\nabla v \cdot \nabla v \leq \int_{\mathcal{O}} 2\beta pr^{-p-2} e^{2\beta r^p} v M\nabla v \cdot x + \int_{\partial \mathcal{O}} C|\mathcal{M}| e^{2\beta r^p} \left( |\nabla v|^2 + |v|^2 \right).
\]
Using Cauchy’s inequality, for $a, b \in \mathbb{R}$,
\[
2|abM\nabla v \cdot x| \leq \frac{1}{2} |a|^2 (M\nabla v, \nabla v) + 8|b|^2 (Mx, x),
\]
we obtain
\[
\int_{\mathcal{O}} 2\beta pr^{-p-2} e^{2\beta r^p} v M\nabla v \cdot x \leq \int_{\mathcal{O}} \frac{1}{2} e^{2\beta r^p} (M\nabla v, \nabla v) + C|\mathcal{M}| p^2 \beta^2 r^{-2p-2} e^{2\beta r^p} |v|^2.
\]
We derive from (3.26) and (3.27) that
\[
\int_{\mathcal{O}} e^{2\beta r_2} v \, \text{div}(M \nabla v) + \int_{\mathcal{O}} \frac{1}{2} e^{2\beta r_2} \langle M \nabla v, \nabla v \rangle \\
\leq \int_{\mathcal{O}} C |M| \beta^2 p^2 r_2^{-p} - 2p - e^{2\beta r_2} |v|^2 + \int_{\partial \mathcal{O}} C |M| e^{2\beta r_2} (|\nabla v|^2 + |v|^2),
\]
which is the conclusion. \hfill \Box

Combining the inequalities in Lemmas 3.4 and 3.5, we obtain

**Lemma 3.6.** Let \( \beta \in \mathbb{R}, v \in H^2(\mathcal{O}) \) and let \( M \) be a Lipschitz, symmetric, uniformly elliptic, matrix-valued function defined in \( \mathcal{O} \). Assume that \( |x| \leq 1 \) for \( x \in \mathcal{O} \), and, for some \( \Lambda \geq 1 \), the following three conditions hold in \( \mathcal{O} \):
\[
\langle M, x \rangle \geq \Lambda^{-1} |x|^2,
\]
\[
|M| + |\text{div}(M x)| + |x|^{-2} |\nabla (x \cdot M x) \cdot M x| \leq \Lambda,
\]
\[
|\langle B y, y \rangle| \leq \Lambda |\langle M y, y \rangle| \text{ for } y \in \mathbb{R}^d,
\]
where \( \langle B y, y \rangle \) is defined in (3.6). There exist two constants \( p_\Lambda, \beta_\Lambda \geq 1 \) such that if \( p \geq p_\Lambda \) and \( |\beta| \geq \beta_\Lambda \), then
\[
(3.28) \quad \int_{\mathcal{O}} e^{2\beta r_2} \left( p^3 \beta^2 r_2^{-2p-2} |v|^2 + \langle M \nabla v, \nabla v \rangle \right) \\
\leq C_\Lambda \int_{\mathcal{O}} \frac{1}{p |\beta|} r^p e^{2\beta r_2} |\text{div}(M \nabla v)|^2 + C_\Lambda \int_{\mathcal{O}} e^{2\beta r_2} \left( |\nabla v|^2 + p^2 \beta^2 r_2^{-2p-2} |v|^2 \right),
\]
for some positive constant \( C_\Lambda \) depending only on \( \Lambda \) and \( d \).

**Proof.** We have, by Lemma 3.4,
\[
(3.29) \quad \int_{\mathcal{O}} \Lambda^{-p} p^3 \beta^2 r_2^{-2p-2} e^{2\beta r_2} |v|^2 \leq \int_{\mathcal{O}} \frac{1}{2 p |\beta|} r^p e^{2\beta r_2} \left[ \text{div}(M \nabla v) \right]^2 \\
+ \int_{\mathcal{O}} C_\Lambda e^{2\beta r_2} \langle M \nabla v, \nabla v \rangle + \int_{\partial \mathcal{O}} C_\Lambda \beta^2 p^2 r_2^{-p} (|\nabla v|^2 + \beta^2 r_2^{-2p-2} |v|^2).
\]
We also have, by Lemma 3.5,
\[
(3.30) \quad \int_{\mathcal{O}} \frac{1}{2} e^{2\beta r_2} \langle M \nabla v, \nabla v \rangle \leq \int_{\mathcal{O}} C_\Lambda \beta^2 p^2 r_2^{-2p-2} e^{2\beta r_2} |v|^2 \\
+ \int_{\partial \mathcal{O}} C_\Lambda e^{2\beta r_2} (|\nabla v|^2 + |v|^2) - \int_{\mathcal{O}} e^{2\beta r_2} v \, \text{div}(M \nabla v).
\]
Combining (3.29) and (3.30) yields
\[
(3.31) \quad \int_{\mathcal{O}} \Lambda^{-p} p^3 \beta^2 r_2^{-2p-2} e^{2\beta r_2} |v|^2 + \int_{\mathcal{O}} \frac{1}{2} e^{2\beta r_2} \langle M \nabla v, \nabla v \rangle \\
\leq \int_{\mathcal{O}} \frac{1}{2 p |\beta|} r^p e^{2\beta r_2} \left[ \text{div}(M \nabla v) \right]^2 + \int_{\mathcal{O}} C_\Lambda \beta^2 p^2 r_2^{-p} e^{2\beta r_2} |v|^2 \\
+ \int_{\partial \mathcal{O}} C_\Lambda \beta^2 p^2 r_2^{-2p-2} e^{2\beta r_2} |v|^2 + C_\Lambda \int_{\mathcal{O}} e^{2\beta r_2} |v| \, \text{div}(M \nabla v)|.
Using the fact
\[ C \Lambda |v| \, \text{div}(M \nabla v) \leq p|\beta||v|^2 r^{-p-2} + \frac{C^2 \Lambda^2}{4p|\beta|} \left| \text{div}(M \nabla v) \right|^2 r^{p+2}, \]
for large \( p \),
\[ C \Lambda^2 \beta^2 p^2 r^{-2p-2} e^{2\beta r-p} |v|^2 \leq \frac{1}{4} \Lambda^{-p} \beta^2 r^{-2p-2} e^{2\beta r-p} |v|^2, \]
and, for large \( p \) and \( |\beta| \geq 1 \),
\[ p|\beta|r^{-p-2} e^{2\beta r-p} |v|^2 \leq \frac{1}{4} \Lambda^{-p} \beta^2 r^{-2p-2} e^{2\beta r-p} |v|^2, \]
we derive from (3.31) that, for large \( p \),
\[
\int_{\mathcal{O}} e^{2\beta r-p} \left( p^3 \beta^2 r^{-2p-2} |v|^2 + \langle M \nabla v, \nabla v \rangle \right) \leq \int_{\mathcal{O}} C \Lambda^{p+2} e^{2\beta r-p} \left[ \text{div}(M \nabla v) \right]^2 + \int_{\partial \mathcal{O}} C \Lambda e^{2\beta r-p} (|\nabla v|^2 + \beta^2 p^2 r^{-2p-2} |v|^2). 
\]
The proof is complete.

### 3.2. Main step of the proof Theorem 3.1

This section, which is the main step of the proof of Theorem 3.1, is devoted to the proof of the following result

**Proposition 3.1.** Let \( d \geq 2, m \geq 1, \Lambda \geq 1, \) and \( R_* < R < R^* \). Then, for any \( \alpha \in (0,1) \), there exists a constant \( \gamma_2 \in (0,1) \), depending only on \( \alpha, \Lambda, R_* , R^* , m, \) and \( d \) such that for every \( \gamma_1 \in (0, \gamma_2) \), there exists \( \gamma_0 \in (0, \gamma_1) \) depending only on \( \alpha, \gamma_1, \gamma_2, \Lambda, R_* , R^* , m, \) and \( d \) such that, for real, symmetric, uniformly elliptic, Lipschitz matrix-valued functions \( M^\ell \) with \( 1 \leq \ell \leq m \) defined in \( D_{\gamma_0} := Y_{\gamma_0,1,R} \) verifying, in \( D_{\gamma_0} \),

\[
\Lambda^{-1} |\xi|^2 \leq \langle M^\ell(x)\xi,\xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M^\ell(x)| \leq \Lambda,
\]

for \( g \in L^2(D_{\gamma_0}) \), and for \( V \in [H^2(D_{\gamma_0})]^m \) satisfying, for \( 1 \leq \ell \leq m \),

\[
|\text{div}(M^\ell \nabla V)| \leq \Lambda_1 (|\nabla V| + |V| + |g|) \quad \text{in } D_{\gamma_0} \text{ for some } \Lambda_1 \geq 0,
\]
we have, with \( \Sigma_{\gamma_0} = \partial D_{\gamma_0} \cap \{x_1 = 0\} \),

\[
\|V\|_{H^1(Y_{\gamma_1,\gamma_2,\tilde{\gamma}})} \leq C \left( \|V,\nabla V\|_{L^2(\Sigma_{\gamma_0})} + \|g\|_{L^2(\Sigma_{\gamma_0})} \right)^{\alpha} \left( \|V\|_{H^1(D_{\gamma_0})} + \|V,\nabla V\|_{L^2(\Sigma_{\gamma_0})} + \|g\|_{L^2(D_{\gamma_0})} \right)^{1-\alpha},
\]

for some positive constant \( C \) depending only on \( \alpha, \gamma_1, \Lambda, \Lambda_1, R_* , R^* , m, \) and \( d \).

**Proof.** By a scaling argument, one might assume that \( R = 1 \). For simplicity of presentation, we will assume that \( m = 1 \) and drop the corresponding indices (e.g. \( M^1 \) becomes \( M \), etc). Using a covering argument, it suffices to prove that there exists a constant \( \gamma_2 \in (0,1) \), depending only on \( \alpha, \Lambda, R_* , R^* , \) and \( d \) such that for every \( \gamma_1 \in (0, \gamma_2) \), there exist \( \gamma_0 \in (0, \gamma_1) \) and \( \tilde{\gamma}_0 \in (0, \gamma_1) \).
depending only on $\alpha, \gamma_1, \Lambda, R_\ast, R^\ast,$ and $d$ such that for all $\tilde{z}_0 \in \mathbb{R}^{d-2}$ with $|\tilde{z}_0| \leq 1/2,$ we have

$$
(3.35) \quad \int_{D_{\tilde{z}_0}} (|V|^2 + |\nabla V|^2) \leq C \left( \|(V, \nabla V)\|_{L^2(\Sigma_{\tilde{z}_0})} + \|g\|_{L^2(D_{\tilde{z}_0})} \right)^{2\alpha} \times
$$

$$
\times \left( \|V\|_{H^1(D_{\tilde{z}_0})}^2 + \|(V, \nabla V)\|^2_{L^2(\Sigma_{\tilde{z}_0})} + \|g\|^2_{L^2(D_{\tilde{z}_0})} \right)^{2(1-\alpha)},
$$

where $V \in H^2(D_{\tilde{z}_0})$ satisfies (3.33).

Our goal is now to establish (3.35). Fix $n \in \mathbb{N}$ and $n \geq 10,$ define $L_n : \mathbb{R}^d \cap \{x \in \mathbb{R}^d ; x_1 \geq 0\} \to \mathbb{R}^d$ by

$$
(3.36) \quad L_n(x_1, x_2, \tilde{x}) = (\tilde{x}^{1/n} \cos(\theta/n), \tilde{x}^{1/n} \sin(\theta/n), \tilde{x}).
$$

Recall that, for $d \geq 2$ and $x = (x_1, x_2, \tilde{x}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2},$ we use the polar coordinate $(\tilde{r}, \theta)$ for the pair $(x_1, x_2);$ the variable $\tilde{x}$ is irrelevant for $d = 2.$

Fix $\tilde{z}_0 \in \mathbb{R}^{d-2}$ with $|\tilde{z}_0| \leq 1/2.$ Denote

$$
z_0 = (0, 0, \tilde{z}_0).
$$

Let $Q$ be a rotation, i.e., $Q^T Q = I,$ and $\Lambda^{-1} \leq \lambda_1, \ldots, \lambda_d \leq \Lambda$ be such that

$$
Q^T \mathcal{M}(z_0) Q = \text{diag}(\lambda_1, \ldots, \lambda_d).
$$

Since $\mathcal{M}(z_0)$ is symmetric and uniformly elliptic, such $Q$ and $\lambda_j$ ($1 \leq j \leq d$) exist; in fact $\lambda_j$ ($1 \leq j \leq d$) are eigenvalues of $\mathcal{M}(z_0)$ and $Q$ is formulated from a corresponding orthogonal basis of eigenvectors of $\mathcal{M}(z_0)$.

Set

$$
S = \text{diag}(\lambda_1^{-1/2}, \ldots, \lambda_d^{-1/2}),
$$

Let $Q_1$ be a rotation which is chosen in such a way that $Q_1 SQ^T (\mathbb{R}^d \cap \{x_1 > 0\}) = \mathbb{R}^d \cap \{x_1 > 0\}$ and $Q_1 SQ^T e_2 = \lambda_2 e_2,$ e.g., one can choose a rotation $Q_1$ such that $Q_1(S^{-1}Q^T e_1/|S^{-1}Q^T e_1|) = e_1$ and $Q_1(SQ^T e_2/|SQ^T e_2|) = e_2$ by noting that $SQ^T (\mathbb{R}^d \cap \{x_1 > 0\}) = \{x \in \mathbb{R}^d; \langle x, S^{-1}Q^T e_1 \rangle > 0\}$ and $\langle S^{-1}Q^T e_1, SQ^T e_2 \rangle = 0.$ It follows that

the first two components of $Q_1 SQ^T (0, 0, \tilde{x})$ are 0 for $\tilde{x} \in \mathbb{R}^{d-2}.$

Define

$$
T_n = L_n \circ H \text{ where } H(x) = Hx \text{ with } H = Q_1 SQ^T.
$$

Denote

$$
(3.37) \quad Z_0 = T_n(z_0).
$$

By the choices of $Q_1$ and $L_n,$ the first two components of $Z_0$ are 0, which yields

$$
Z_0 = (0, 0, \tilde{Z}_0),
$$

for some $\tilde{Z}_0 \in \mathbb{R}^{d-2}.$ Set

$$
(3.38) \quad \tilde{Z}_0 = (1/n, 1/n + \pi/(2n^2), \tilde{Z}_0) = Z_0 + (1/n, 1/n + \pi/(2n^2), 0) \in \mathbb{R}^2 \times \mathbb{R}^{d-2},
$$

$$
(3.39) \quad T_n = \left\{ x = T_n(y); y \in \mathbb{R}_+^d, |\tilde{y}| < 1, 1/(4n) < \tilde{r}(x) < 2/n, -\pi/(2n) < \theta(x) < \pi/(2n) \right\},
$$

$$
(3.40) \quad Y_n = T_n^{-1}(T_n), \quad \Sigma_{Y_n} = \partial Y_n \cap \{x \in \mathbb{R}^d; x_1 = 0\}.
$$

---

2When $(x_1, x_2) = (0, 0),$ we define $L_n(x) = (0, 0, \tilde{x})$ as a convention.
Define, in $T_n$,

\[(3.41)\quad K_n(x) = \frac{\nabla T_n}{|\det \nabla T_n|^{1/2}} \circ T_n^{-1}(x),\]

\[(3.42)\quad A_n(x) = \mathcal{M} \circ T_n^{-1}(x), \quad \text{and} \quad M_n(x) = K_n A_n K_n^T(x).\]

For

\[(3.43)\quad \lambda \in \left(\frac{5}{4n}, \frac{3}{2n}\right),\]

set (see fig. 2)

\[(3.44)\quad \hat{T}_n = T_n - \hat{Z}_0 \quad \text{and} \quad \mathcal{O}_n = B_\lambda \cap \hat{T}_n,\]

\[(3.45)\quad \hat{M}_n(\cdot) = M_n(\cdot + \hat{Z}_0) \text{ in } \mathcal{O}_n,\]

and, for $x \in \mathcal{O}_n$ and $y \in \mathbb{R}^d$,

\[(3.46)\quad \langle \hat{B}_n(x)y, y \rangle = \langle [(\hat{M}_n y) \cdot \nabla]\hat{M}_n(x)x, y \rangle + \frac{1}{2}\langle \text{div}(\hat{M}_n x)\hat{M}_n(x)y, y \rangle + \frac{1}{2}\langle \text{div}(\hat{M}_n(x)x) \cdot \nabla\hat{M}_n(x)y, y \rangle.\]

Note that $\mathcal{O}_n$ also depends on $\lambda$; however, the dependence is not written explicitly for notational ease.

We claim that

\[(3.47)\quad \langle \hat{M}_n x, x \rangle \geq \hat{\Lambda}^{-1}|x|^2 \quad \text{in } \mathcal{O}_n,\]

\[(3.48)\quad |\text{div}(\hat{M}_n x)| + |x|^{-2}|\nabla(x \cdot \hat{M}_n x) \cdot \hat{M}_n x| \leq \hat{\Lambda} \quad \text{in } \mathcal{O}_n,\]

\[(3.49)\quad |\langle \hat{B}_n y, y \rangle| \leq \hat{\Lambda}\langle \hat{M}_n y, y \rangle \quad \text{in } \mathcal{O}_n,\]

for some $\hat{\Lambda} \geq 1$, for all $\lambda \in \left(\frac{5}{4n}, \frac{3}{2n}\right)$. Here and in what follows, $\hat{\Lambda}$ denotes a positive constant depending only on $\Lambda$ and $d$; it is thus independent of $\hat{z}_0$ and $n$. The proof of this claim is given in Step 1 below.

Let $p = p_{\hat{\Lambda}}$ where $p_{\hat{\Lambda}}$ is the constant in Lemma 3.6 corresponding to $\hat{\Lambda}$ and $\hat{M}_n$. Set

\[(3.50)\quad \tau_n = (1/n - 1/n^2)^n \quad \text{and} \quad s_n = (1/n + 1/n^2)^n.\]

Denote

\[(3.51)\quad R_1(n) = 1/n, \quad R_2(n) = R_1(n) + 8/n^2, \quad \text{and} \quad R_3(n) = 5/(4n),\]

and define

\[(3.52)\quad \rho(n) = \frac{R_1(n)^{-p} - R_3(n)^{-p}}{R_2(n)^{-p} - R_3(n)^{-p}}.\]

Note that

\[\lim_{n \to +\infty} \rho(n) = 1.\]

Let

\[n_0 = \min \left\{ n \in \mathbb{N}; n \geq 10 \text{ and } \rho(n) \geq (1 + \alpha)/2 \right\}.\]

Set

\[\gamma_2 = s_{n_0}/\Lambda \text{ where } s_n \text{ is defined in } (3.50).\]
Given γ₁ < γ₂, fix N ∈ ℤ with N > n₀ and Λτₙ ≤ γ₁ where τₙ is defined in (3.50). Set
\[ γ₀ = Λ₁/4N. \]

In what follows in this proof, we always assume that n₀ ≤ n ≤ N. Define
\[ Uₙ(x) = V ◦ Tₙ⁻¹(x) \text{ for } x ∈ Tₙ, \quad \hat{U}_n(·) = Uₙ(· + \hat{Z}_0) \text{ in } O_n, \]
\[ gₙ(x) = f ◦ Tₙ⁻¹(x) \text{ for } x ∈ Tₙ, \quad \hat{g}_n(·) = gₙ(· + \hat{Z}_0) \text{ in } O_n. \]

The proof of (3.35) is now divided into the following five steps:
- **Step 1**: We prove (3.47), (3.48), and (3.49).
- **Step 2**: Using (3.47), (3.48), and (3.49), we prove that there exists a constant βₜₙ ≥ 1 depending only on n₀, N, Λ, and d, such that, for β ≥ βₜₙ, it holds
  \[ \int_{O_n} e^{2βr_p} \left( \beta^2 e^{2βr_p} |\hat{U}_n|^2 + |\nabla \hat{U}_n|^2 \right) \leq C \int_{\partial O_n} \beta e^{2βr_p} (|\nabla \hat{U}_n|^2 + \beta^2 |\hat{U}_n|^2) + \int_{O_n} e^{2βr_p} |\hat{g}_n|^2, \]
  for all λ ∈ (5/(4n), 3/(2n)). Here and in what follows in this proof, C denotes a positive constant depending only on n₀, N, Λ, and d.
- **Step 3**: Set
  \[ \bar{Σ}_n = T_n(Σ_{Y_n}) \quad \text{and} \quad \hat{Σ}_n = \bar{Σ}_n - \hat{Z}_0 \]
  (see (3.40) for the definition of Σ_{Y_n}). Using Step 2, we prove, for β ≥ βₜₙ,
  \[ \int_{O_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) e^{2βr_p} \leq C β^2 e^{2βR_p} ||\hat{U}_n||_{H_1(\bar{T}_n)}^2 + C β^2 e^{2βR_p} \left( ||(\hat{U}_n, \nabla \hat{U}_n)||_{L^2(\bar{Σ}_n)}^2 + ||\hat{g}_n||_{L^2(O_n)}^2 \right), \]
  for some λ ∈ (5/(4n), 3/(2n)).
• Step 4: Using Step 3, we prove

\[
(3.57) \quad \int_{B_{R_2(n)} \cap \Omega_n} (|\tilde{U}_n|^2 + |\nabla \tilde{U}_n|^2) \leq C \left( \|\tilde{U}_n\|_{L^2(\Omega_n)}^2 + \|\tilde{g}_n\|_{L^2(\Omega_n)}^2 \right)^{2\alpha} \times \\
\times \left( \|\tilde{U}_n\|_{H^1(\Omega_n)}^2 + \|\tilde{U}_n\|_{L^2(\Omega_n)}^2 + \|\tilde{g}_n\|_{L^2(\Omega_n)}^2 \right)^{2(1-\alpha)}.
\]

• Step 5: Using Step 4, we prove

\[
(3.58) \quad \int_{\Lambda \tau \leq \xi \leq \xi_n/\Lambda} (|V|^2 + |\nabla V|^2) \leq C \left( \|V\|_{L^2(\Sigma_{\gamma\theta})}^2 + \|g\|_{L^2(D_{\gamma\theta})}^2 \right)^{2\alpha} \times \\
\times \left( \|V\|_{H^1(D_{\gamma\theta})}^2 + \|V\|_{L^2(\Sigma_{\gamma\theta})}^2 + \|g\|_{L^2(D_{\gamma\theta})}^2 \right)^{2(1-\alpha)},
\]

for some positive constant $C$ depending only on $\Lambda$ and $d$.

Assertion (3.35) is now a consequence of Step 5.

• Step 1: We have

\[
(3.59) \quad \nabla L_n(x) = \begin{pmatrix}
L_{2,2,n}(x) & 0_{2,d-2} \\
0_{d-2,2} & I_{d-2,d-2}
\end{pmatrix},
\]

where

\[
L_{2,2,n}(x) = \frac{1}{n r^{1-1/2}} \begin{pmatrix}
\cos \theta \cos(\theta/n) + \sin \theta \sin(\theta/n) & \sin \theta \cos(\theta/n) - \cos \theta \sin(\theta/n) \\
\cos \theta \sin(\theta/n) - \sin \theta \cos(\theta/n) & \sin \theta \sin(\theta/n) + \cos \theta \cos(\theta/n)
\end{pmatrix}
\]

\[
= \frac{1}{n r^{1-1/2}} \begin{pmatrix}
\cos ((n-1)\theta/n) & \sin ((n-1)\theta/n) \\
-\sin ((n-1)\theta/n) & \cos ((n-1)\theta/n)
\end{pmatrix}.
\]

Here and in what follows, $0_{i,j}$ denotes the zero $(i \times j)$-matrix and $I_{k,k}$ denotes the identity matrix of size $(k \times k)$ for $i, j, k \geq 0$.

Set

\[
(3.60) \quad \tilde{K}_n(x) = \begin{pmatrix}
K_{2,2,n}(x) & 0_{2,d-2} \\
0_{d-2,2} & n^{r-1} I_{d-2,d-2}
\end{pmatrix},
\]

where

\[
K_{2,2,n}(x) = \begin{pmatrix}
\cos ((n-1)\theta) & \sin ((n-1)\theta) \\
-\sin ((n-1)\theta) & \cos ((n-1)\theta)
\end{pmatrix}.
\]

It is clear from the formula of $\nabla L_n$ that, in $\Omega_n$,

\[
(3.61) \quad \nabla T_n \circ T_n^{-1}(x) = \nabla L_n(L_n^{-1}(x))H = \frac{1}{n r^{n-1}} \tilde{K}_n(x)H,
\]

\[
(3.62) \quad |\det \nabla T_n| \circ T_n^{-1}(x) = \frac{1}{\gamma n^{2n-2}} \text{ where } \gamma = |\det H|^{-1},
\]

and (see (3.41) for the definition of $K_n$)

\[
(3.63) \quad K_n(x) = \gamma^{1/2} \tilde{K}_nH.
\]
From (3.63), we have
\begin{equation}
|K_n| \leq \hat{\Lambda} \text{ in } T_n \quad \text{and} \quad |\nabla K_n| \leq C/n^2 \text{ in } T_n.
\end{equation}
Note that $1/n \leq \hat{\gamma}$ and $|x| \leq 3/(2n)$ for $x \in O_n$, and $|K_{2,2,n}(x)y| = |y|$ for $x \in T_n$ and $y \in \mathbb{R}^2$. We derive from (3.63) that
\[ |K_n^T(\cdot + \hat{Z}_0)x| \geq \hat{\Lambda}|x| \text{ for } x \in O_n. \]
It follows from the ellipticity of $M$, (3.42), and (3.45) that
\[ \langle \nabla M_n x, x \rangle \geq \hat{\Lambda}^{-1}|x|^2 \text{ in } O_n, \]
which is (3.47).

Since, by (3.60) and (3.61),
\begin{equation}
\nabla T_n^{-1}(x) = \left( \nabla T_n \circ T_n^{-1} \right)^{-1} \quad \text{for } x \in T_n,
\end{equation}
we deduce from (3.42) that
\begin{equation}
|\nabla A_n(x)| \leq \hat{\Lambda} \text{ in } T_n.
\end{equation}

From (3.37), (3.42), and (3.63), we obtain
\begin{equation}
M_n(Z_0) = K_n A_n K_n^T(Z_0) = \gamma \tilde{K}_n(Z_0) H M(z_0) H^T \tilde{K}_n^T(Z_0) = \gamma I
\end{equation}
(this is the point where $H$ must be carefully chosen). The fact $M_n(Z_0) = \gamma I$ plays an important role in establishing (3.48) and (3.49).

We have
\[ \text{div}(\tilde{M}_n x) = \sum_{j=1}^d \partial_{x_j} \langle \tilde{M}_n x, e_j \rangle \text{ in } O_n, \]

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, and
\[ |x + \hat{Z}_0 - Z_0| \leq 5/n \text{ for } x \in O_n, \quad \text{and} \quad |x| \leq 3/(2n) \text{ for } x \in O_n. \]

It follows from (3.64) and (3.66) that
\begin{equation}
|\text{div}(\tilde{M}_n x)| \leq \hat{\Lambda}.
\end{equation}

We have
\[ \langle M_n y, y \rangle \overset{(3.67)}{=} \gamma|y|^2 + \langle K_n(A_n - A_n(Z_0))K_n^Ty, y \rangle \text{ for } y \in \mathbb{R}^d, \]
\[ \nabla(x \cdot \tilde{M}_n x) \cdot \tilde{M}_n x = \sum_{j=1}^d \partial_{x_j}(x \cdot \tilde{M}_n x)\langle \tilde{M}_n x, e_j \rangle \text{ in } O_n, \]
\[ |x + \hat{Z}_0 - Z_0| \leq 5/n \text{ for } x \in O_n, \quad \text{and} \quad |x| \leq 3/(2n) \text{ for } x \in O_n. \]

Using these facts, we derive from (3.64) and (3.66) that
\begin{equation}
|\nabla(x \cdot \tilde{M}_n x) \cdot \tilde{M}_n x| \leq \hat{\Lambda}|x|^2 \text{ in } O_n.
\end{equation}

Combining (3.68) and (3.69) yields (3.48).

Using the same arguments, one also obtains (3.49). The proof of Step 1 is complete.
Step 2. Using (3.47), (3.48), and (3.49), we can apply Lemma 3.6 with \( v = \hat{U}_n \) and \( M = \hat{M}_n \) in \( \mathcal{O}_n \). We then obtain

\[
\int_{\mathcal{O}_n} e^{2\beta r - p} \left( p^3 \beta^2 r^{-2p-2} |\hat{U}_n|^2 + \langle \hat{M}_n \nabla \hat{U}_n, \nabla \hat{U}_n \rangle \right)
\]

\[
\leq C \int_{\mathcal{O}_n} \frac{1}{p |\beta|} r^{p+2} e^{2\beta r - p} |\text{div}(\hat{M}_n \nabla \hat{U}_n)|^2
\]

\[
+ C \int_{\partial \mathcal{O}_n} |\beta| p e^{2\beta r - p} (|\nabla \hat{U}_n|^2 + p^2 \beta^2 r^{-2p-2} |\hat{U}_n|^2),
\]

for \(|\beta| \geq \beta_\Lambda\) for some constant \( \beta_\Lambda \geq 1 \) depending only on \( \Lambda \) and \( d \) since \( \hat{\Lambda} \) depends only on \( \Lambda \) and \( d \).

We claim that

\[
|\text{div}(M_n \nabla U_n)| \leq C \left( \frac{1}{n^{3+3n-3}} |K^T_n \nabla U_n| + \frac{1}{n^{2+2n-2}} |U_n| + \frac{1}{n^{2+2n-2}} |g_n| \right) \text{ in } T_n.
\]

Indeed, set

\[
f(x) = \text{div} (\mathcal{M}(x) \nabla V(x)) \text{ for } x \in \Omega \quad \text{and} \quad F_n(x) = \frac{f \circ T_n^{-1}}{|\det \nabla T_n | \circ T_n^{-1}(x)} \text{ for } x \in T_n.
\]

Then, in \( T_n \),

\[
|F_n(x)| \leq C \left( \frac{1}{n^{3+3n-3}} |K^T_n \nabla U_n| + \frac{1}{n^{2+2n-2}} |U_n| + \frac{1}{n^{2+2n-2}} |g_n| \right) \text{ in } T_n.
\]

By a change of variables, see, e.g. [48, Lemma 6] (see also [27, Section 2.2]), we have

\[
\text{div}(M_n \nabla U_n) = F_n \text{ in } T_n.
\]

Recall that

\[
|f| \leq A_1 (|\nabla V| + |V| + |g|) \text{ in } T_n^{-1}(T_n).
\]

We have, for \( x \in T_n \),

\[
\nabla V(T_n^{-1}(x)) = \nabla T_n^T(T_n^{-1}(x)) \nabla U_n(x) = \frac{1}{\gamma^{1/2n^{3n-1}}} K^T_n(x) \nabla U_n(x).
\]

It follows from (3.72) that

\[
|F_n| \leq C \left( \frac{1}{n^{3+3n-3}} |K^T_n \nabla U_n| + \frac{1}{n^{2+2n-2}} |U_n| + \frac{1}{n^{2+2n-2}} |g_n| \right) \text{ in } T_n,
\]

which implies claim (3.71).

We have, in \( T_n \),

\[
\langle M_n y, y \rangle \geq C |K^T_n y|^2 \text{ for } y \in \mathbb{R}^d.
\]
Considering (3.70), and using (3.71) and (3.73), we deduce that, where \( \hat{r} \) is considered at the point \( x + \hat{Z}_0 \),

\[
(3.74) \quad \int_{\Omega_n} e^{2\beta r - p} \left( p^3 \beta^2 r^{-2p-2} |\hat{U}_n|^2 + |\hat{K}_n^T \nabla \hat{U}_n|^2 \right) \leq C \left\{ \int_{\Omega_n} \frac{1}{p\beta r^{p+2}} e^{2\beta r - p} \left( \frac{1}{n^4 r^{4n-4}} |\hat{U}_n|^2 + \frac{1}{n^{6p} r^{6n-6}} |\hat{K}_n^T \nabla \hat{U}_n|^2 + \frac{1}{n^{4} r^{4n-4}} |\hat{g}_n|^2 \right) \right\} + \int_{\partial \Omega_n} |\beta|^p e^{2\beta r - p} \left( |\nabla \hat{U}_n|^2 + p^2 \beta^2 r^{-2p-2} |\hat{U}_n|^2 \right).
\]

Fix \( \beta \) large, the largeness depends only on \( n_0, N, \Lambda \) and \( d \), so that for \( |\beta| \geq \beta \), \( n_0 \leq n \leq N \), \( 1/(4n) < \hat{r} < 2/n \), and \( r > 1/n \), we have

\[
p^3 \beta^2 r^{-2p-2} \geq \frac{C}{2p\beta r^{p+2}} \frac{1}{n^4 r^{4n-4}} \quad \text{and} \quad 1 \geq \frac{2C}{p\beta r^{p+2}} \frac{2}{n^{6p} r^{6n-6}},
\]

where \( C \) is the constant appearing in (3.74). We derive from the definition of \( \Omega_n \) and \( T_n \) that, for \( |\beta| \geq \beta \),

\[
\int_{\Omega_n} e^{2\beta r - p} \left( \beta^2 e^{2\beta r - p} |\hat{U}_n|^2 + |\nabla \hat{U}_n|^2 \right) \leq C \int_{\partial \Omega_n} |\beta|^p e^{2\beta r - p} \left( |\nabla \hat{U}_n|^2 + \beta^2 |\hat{U}_n|^2 \right) + C \int_{\Omega_n} e^{2\beta r - p} |\hat{g}_n|^2.
\]

The proof of Step 2 is complete.

- Step 3. In what follows, for notational ease, we denote \( R_1(n) \), \( R_2(n) \), and \( R_3(n) \) by \( R_1 \), \( R_2 \), and \( R_3 \). We have

\[
\partial \Omega_n = (\partial \Omega_n \cap \partial B_\lambda) \cup (\partial \Omega_n \setminus \partial B_\lambda),
\]

where \( |x| \geq R_3 \) for \( x \in \partial \Omega_n \cap \partial B_\lambda \) by (3.43) and the definition of \( R_3 \),

\[
|x| \geq R_1 \quad \text{for} \quad x \in (\partial \Omega \setminus \partial B_\lambda) \quad \text{since} \quad \text{dist}(z_0, T_n) \geq 1/n.
\]

We derive from (3.51) and (3.55) that, for \( \beta \geq \beta \),

\[
(3.75) \quad \int_{\Omega_n} \left( |\hat{U}_n|^2 + |\nabla \hat{U}_n|^2 \right) e^{2\beta r - p} \leq C \beta^2 e^{2\beta r - p} \left\{ \| \hat{U}_n, \nabla \hat{U}_n \|^2_{L^2(\partial \Omega_n \cap \partial B_\lambda)} + C \beta^2 e^{2\beta r - p} \left( \| \hat{U}_n, \nabla \hat{U}_n \|^2_{L^2(\partial \Omega_n \setminus \partial B_\lambda)} + \| \hat{g}_n \|^2_{L^2(\partial \Omega_n)} \right) \right\}.
\]

Since \( n \geq 10 \), we have

\[
\partial \Omega_n \cap \partial B_\lambda \subset \hat{T}_n \quad \text{for} \quad \lambda \in (5/(4n), 3/(2n)).
\]

This implies that, for some \( \lambda \in (5/(4n), 3/(2n)) \),

\[
(3.76) \quad \| \hat{U}_n, \nabla \hat{U}_n \|^2_{L^2(\partial \Omega_n \cap \partial B_\lambda)} \leq C \| \hat{U}_n \|^2_{H^1(\hat{T}_n)}.
\]

It is clear that

\[
\partial \Omega_n \setminus \partial B_\lambda \subset \hat{\Sigma}_n,
\]

which yields

\[
(3.77) \quad \| \hat{U}_n, \nabla \hat{U}_n \|^2_{L^2(\partial \Omega_n \setminus \partial B_\lambda)} \leq \| \hat{U}_n, \nabla \hat{U}_n \|^2_{L^2(\hat{\Sigma}_n)}.
\]
Combining (3.75), (3.76), and (3.77) yields, for $\beta \geq \beta_\Lambda$,
\[
\int_{\mathcal{O}_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) e^{2\beta r - \rho} \leq C \beta^2 e^{2\beta r - \rho} \int_{\mathcal{O}_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) e^{2\beta r - \rho}
\]
\[
+ C \beta^2 e^{2\beta r - \rho} \left( (\hat{U}_n, \nabla \hat{U}_n) + \|\hat{g}_n\|_{L^2(\mathcal{O}_n)} \right).
\]

The proof of Step 3 is complete.

- **Step 4.** Note that, for $\lambda \in \left(\frac{5}{4n}, \frac{3}{2n}\right)$,
  \[
  B_{R_2} \cap \hat{T}_n \subset B_{5/(4n)} \cap \hat{T}_n \subset \mathcal{O}_n.
  \]

As a consequence of (3.56), we have
\[
\int_{B_{R_2} \cap \hat{T}_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) e^{2\beta r - \rho} \leq C \beta^2 e^{2\beta (R_1 - R_2^p)} \int_{\mathcal{O}_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) e^{2\beta r - \rho}
\]
\[
+ C \beta^2 e^{2\beta (R_1 - R_2^p)} \left( (\hat{U}_n, \nabla \hat{U}_n) + \|\hat{g}_n\|_{L^2(\mathcal{O}_n)} \right).
\]

This implies
\[
(3.78) \quad \int_{B_{R_2} \cap \hat{T}_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) \leq C \beta^2 e^{2\beta (R_1 - R_2^p)} \|\hat{U}_n\|_{H^1(\hat{T}_n)}
\]
\[
+ C \beta^2 e^{2\beta (R_1 - R_2^p)} \left( (\hat{U}_n, \nabla \hat{U}_n) + \|\hat{g}_n\|_{L^2(\mathcal{O}_n)} \right).
\]

Set
\[
 a_n = \|\hat{U}_n, \nabla \hat{U}_n\|_{L^2(\hat{T}_n)}^2 + \|\hat{g}_n\|_{L^2(\mathcal{O}_n)}, \quad b_n = \|\hat{U}_n\|_{H^1(\hat{T}_n)}^2 + \|\hat{U}_n, \nabla \hat{U}_n\|_{L^2(\hat{T}_n)}^2 + \|\hat{g}_n\|_{L^2(\mathcal{O}_n)}^2,
\]
and
\[
 \hat{\beta} = \left(1 - \rho(n)\right) \ln(b_n/a_n),
\]
where $\rho(n)$ is given in (3.52). A straightforward estimate gives, with $\beta = \hat{\beta}$, that
\[
(3.79) \quad e^{2\beta (R_1 - R_2^p)} \|\hat{U}_n\|_{H^1(\hat{T}_n)}^2 + e^{2\beta (R_1 - R_2^p)} \left( (\hat{U}_n, \nabla \hat{U}_n) + \|\hat{g}_n\|_{L^2(\mathcal{O}_n)} \right)
\]
\[
\leq a_n^{2\rho(n)} b_n^{(1 - \rho(n))}.
\]

We claim that
\[
(3.80) \quad \int_{B_{R_2} \cap \hat{T}_n} (|\hat{U}_n|^2 + |\nabla \hat{U}_n|^2) \leq C a_n^{2\alpha} b_n^{2(1 - \alpha)},
\]
which is (3.57). Indeed, if $\hat{\beta} \geq \beta_\Lambda$, then take $\beta = \hat{\beta}$ in (3.78). We then obtain (3.80) using (3.79) and the fact $\rho(n) \geq (1 + \alpha)/2 > \alpha$. If $\hat{\beta} < \beta_\Lambda$, inequality (3.80) also holds for a different constant $C$ by taking $\beta = \beta_\Lambda$ in (3.78). The proof of Step 4 is complete.

- **Step 5.** Let $x \in \hat{T}_n$ be such that $\hat{r} \in (1/n_1 - 1/n_2, 1/n + 1/n_2)$ and $|x - \hat{Z}_0| \leq 1/n^2$. We claim that
\[
(3.81) \quad x - \hat{Z}_0 \in B_{R_2} \cap \hat{T}_n.
\]

Indeed, for such an $x$, we have
\[
(3.82) \quad |x_1 - 1/n| \leq 1/n^2 \quad \text{and} \quad |x_2| < (1/n + 1/n^2)\pi/(4n).
\]

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Since
\[ x - Z_0 = (x_1 - 1/n, x_2 - 1/n - \pi/n^2, \hat{x} - \hat{Z}_0), \]
it follows from (3.82) that
\[ |x - Z_0| \leq 1/n^2 + 1/n + \pi/n^2 + \pi(1/n^2 + 1/n^3)/4 + 1/n^2 \leq 1/n + 8/n^2, \]
since \( n \geq 10 \). Since \( x - Z_0 \in \mathcal{T}_n \), claim (3.81) follows.

Inequality (3.58) now follows from (3.57) and (3.81) noting that \( V = U_n \circ T^{-1}_n \) and \( L_n(\{x \in \mathbb{R}^d; \hat{r} = 0\}) = \{x \in \mathbb{R}^d; \hat{r} = 0\} \). The proof of Step 5 is complete. \( \square \)

3.3. Proof of Theorem 3.1. Extend \( \mathcal{M}^\ell \) evenly for \( x_1 < 0 \) and denote
\[ \breve{Y}_{\gamma_1,\gamma_2,R} = \{x \in \mathbb{R}^d; -3\pi/4 \leq \theta \leq 3\pi/4, \gamma_1 R < \hat{r} < \gamma_2 R, \text{ and } |\hat{x}| < R \}. \]
Note that \( L_{3/2} \) is a diffeomorphism from \( \breve{Y}_{\gamma_1,\gamma_2,R} \) onto \( Y_{\gamma_1,\gamma_2,R} \), where \( L_{3/2} \) is given by (3.36) with \( n = 3/2 \). By Proposition 3.1, there exists \( \gamma_2 > 0 \) such that for \( \gamma_1 \in (0, \gamma_2) \), there exists \( \gamma_0 \in (0, \gamma_1/2) \) such that, with \( D_{\gamma_0} = \breve{Y}_{2\gamma_0,1,R/2} \), for \( h \in L^2(D_{\gamma_0}) \), \( W \in [H^2(D_{\gamma_0})]^m \) satisfying
\[ |\text{div}(\mathcal{M}^\ell \nabla W)| \leq \Lambda_1(|W| + |\nabla W| + |h|) \text{ in } D_{\gamma_0}, \]
then, with \( \breve{\Sigma}_{\gamma_0} = \partial D_{\gamma_0} \cap \{\theta = \pm 3\pi/4\} \), it holds
\[ (3.83) \quad \|W\|_{H^1(\breve{Y}_{\gamma_1,\gamma_2,R})} \leq C \left( \|(W, \nabla W)\|_{L^2(\breve{\Sigma}_{\gamma_0})} + \|h\|_{L^2(\breve{\Sigma}_{\gamma_0})} \right)^\alpha \times \]
\[ \times \left( \|V\|_{H^1(\breve{D}_{\gamma_0})} + \|(V, \nabla V)\|_{L^2(\breve{\Sigma}_{\gamma_0})} + \|h\|_{L^2(\breve{\Sigma}_{\gamma_0})} \right)^{1-\alpha}. \]
Set
\[ \hat{Y}_{\gamma_1,\gamma_2,R} = \{x \in \mathbb{R}^d; \gamma_1 R < \hat{r} < \gamma_2 R, \text{ and } |\hat{x}| < R \}, \]
and fix \( \varphi \in C^1_c(\hat{Y}_{\gamma_0,1,R}) \) such that \( \varphi = 1 \) for \( x \in \hat{Y}_{2\gamma_0,1,R/2} \). Let \( U_\ell \in H^1(\hat{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\}) \) be such that
\[ \text{div}(\mathcal{M}^\ell \nabla U_\ell) = 0 \text{ in } \hat{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\}, \quad U_\ell = 0 \text{ on } \partial \hat{Y}_{\gamma_0,1,R}, \]
and, on \( \hat{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\}, \]
\[ [U_\ell] = \varphi V_\ell \text{ and } [\mathcal{M}^\ell \nabla U_\ell \cdot \nu] = \varphi \mathcal{M}^\ell \nabla V_\ell \cdot \nu. \]
We have
\[ (3.84) \quad \|U\|_{H^1(\hat{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\})} \leq C\|V\|_{H(\Sigma_{\gamma_0})}, \]
and, by the regularity theory of elliptic equations,
\[ (3.85) \quad \|U, \nabla U\|_{L^2(\hat{Y}_{\gamma_0,1,R} \cap \{\theta = \pm 3\pi/4\})} \leq C\|V\|_{H(\Sigma_{\gamma_0})}. \]
Set, in \( \breve{D}_{\gamma_0} \),
\[ (3.86) \quad W_\ell = U_\ell \mathbb{1}_{x_1 > 0} - V_\ell \quad \text{and} \quad h = (|g| + |\nabla U| + |U|) \mathbb{1}_{x_1 > 0}. \]
Applying (3.83) with \( W \) and \( h \) given by (3.86), noting that
\[ \|h\|_{L^2(\breve{D}_{\gamma_0})} \leq C \left( \|g\|_{L^2(\breve{D}_{\gamma_0} \cap \{x_1 > 0\})} + \|U\|_{H^1(\breve{D}_{\gamma_0} \cap \{x_1 > 0\})} \right), \]
and using (3.84) and (3.85), we obtain
\begin{equation}
\|W\|_{H^1(\tilde{\Omega})} \\
\leq C \left( \|V\|_{H^1(\Sigma)} + \|g\|_{L^2(D_0)} \right) \alpha \left( \|V\|_{H^1(\Sigma)} + \|V\|_{H^1(\Sigma)} + \|g\|_{L^2(D_0)} \right)^{1-\alpha}.
\end{equation}

The conclusion now follows from (3.84) and (3.87).

3.4. Proofs of Theorem 2.1 and Corollary 2.1. We begin this section with a variant of Theorem 3.1

**Proposition 3.2.** Let \( d \geq 2, m \geq 1, \Lambda \geq 1, \) and \( R > R_* > 0. \) Then, for any \( \alpha \in (0,1), \) there exists a constant \( r \in (0,R_*), \) depending only on \( \alpha, \Lambda, R_*, m, \) and \( d \) such that for real, symmetric, uniformly elliptic, Lipschitz matrix-valued functions \( \mathcal{M}^\ell \) with \( 1 \leq \ell \leq m \) defined in \( \Omega := B_R \cap \{x_1 > 0\} \) verifying, in \( \Omega, \)

\[ \Lambda^{-1} |\xi|^2 \leq \langle \mathcal{M}^\ell(x) \xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla \mathcal{M}^\ell(x)| \leq \Lambda, \]

for \( g \in L^2(\Omega), \) and for \( V \in [H^1(\Omega)]^m \) satisfying, for \( 1 \leq \ell \leq m, \)

\begin{equation}
|\text{div}(\mathcal{M}^\ell \nabla V)| \leq \Lambda_1 (|\nabla V| + |V| + |g|) \quad \text{in} \quad \Omega \quad \text{for some} \quad \Lambda_1 \geq 0,
\end{equation}

we have, with \( \Sigma = \partial \Omega \cap \{x_1 = 0\}, \)

\[ \|V\|_{H^1(B_{r/\alpha} \cap \Omega)} \leq C \left( \|V\|_{H^1(\Sigma)} + \|g\|_{L^2(\Omega)} \right)^{\alpha \left( \|V\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)} \right)^{1-\alpha}}, \]

for some positive constant \( C \) depending only on \( \alpha, \Lambda, \Lambda_1, R_*, m, \) and \( d. \)

**Proof.** The proof of Proposition 3.2 is in the same spirit as the one of Theorem 3.1 but much simpler; one does not need to make any change of variables in the spirit of conformal maps as in the proof of Proposition 3.1 and control the corresponding process. For the convenience of the reader, we sketch here the proof. For simple presentation, we will assume that \( m = 1 \) and ignore the corresponding indices and assume that \( R_* < 1. \) Let \( p_\Lambda \) and \( \beta_\Lambda \) be the constant in Lemma 3.6 and denote \( p = p_\Lambda. \) Set

\[ R_1 = 1/n, \quad R_2 = 1/n + 1/n^2, \quad R_3 = R_*/2, \]

and

\[ \rho = \frac{R_2^{-p} + R_3^{-p}}{R_1^{-p} + R_3^{-p}}. \]

Fix \( n \) be such that \( \rho > (1 + \alpha)/2. \) Set \( x_0 = (1/n, 0, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2}, \) and

\[ \tilde{V} = V(\cdot - x_0), \quad \hat{g} = g(\cdot - x_0), \quad \tilde{M} = \mathcal{M}(\cdot - x_0). \]

Applying Lemma 3.6 with \( \mathcal{O} = B_{R_*/2} \cap \Omega, \) we have

\begin{equation}
\int_{\mathcal{O}} e^{2\beta r^{-p}} \left( p^3 \beta^2 r^{-2p-2} |\tilde{V}|^2 + \langle \tilde{M} \nabla \tilde{V}, \nabla \tilde{V} \rangle \right)
\leq C \int_{\partial \mathcal{O}} \frac{1}{p^2} e^{2\beta r^{-p}} |\text{div}(\tilde{M} \nabla \tilde{V})|^2 + C \int_{\partial \mathcal{O}} e^{2\beta r^{-p}} (|\nabla \tilde{V}|^2 + p^2 \beta^2 r^{-2p-2} |\tilde{V}|^2). \end{equation}
Using the ellipticity of \( \mathcal{M} \) and (3.88), one obtains, for \( \beta \) sufficiently large,

\[
\int_{\Omega \cap B_{R_2}} e^{2\beta r - p} \left( |\hat{V}|^2 + |\nabla \hat{V}|^2 \right) \leq C e^{2\beta r_{1} - p} \beta^2 \left( \int_{\mathcal{O}} e^{2\beta r - p} |\hat{g}|^2 + \int_{\partial \mathcal{O} \cap \{x_1 = 1/n\}} |\nabla \hat{V}|^2 + |\hat{V}|^2 \right) + C e^{2\beta r_{3} - p} \beta^2 \int_{\partial \mathcal{O} \cap \{x_1 = 1/n\}} |\nabla \hat{V}|^2 + |\hat{V}|^2.
\]

As in Step 4 in the proof of Proposition 3.1, one reaches

\[
\|\hat{V}\|_{H^1(\Omega \cap B_{R_2})} \leq C \left( \|(V, \nabla V)\|_{L^2(\Omega \cap \{x_1 = 1/n\})} + \|\hat{g}\|_{L^2(\mathcal{O})} \right)^\alpha \times \left( \|\hat{V}\|_{H^1(\mathcal{O})} + \|\hat{V}, \nabla \hat{V}\|_{L^2(\Omega \cap \{x_1 = 1/n\})} + \|\hat{g}\|_{L^2(\mathcal{O})} \right)^{1-\alpha}.
\]

This implies, with \( r = 1/n^2 \),

\[
\|V\|_{H^1(\Omega \cap B_r)} \leq C \left( \|(V, \nabla V)\|_{L^2(\Omega \cap \{x_1 = 0\})} + \|g\|_{L^2(\Omega \cap B_{R_3})} \right)^\alpha \times \left( \|V\|_{H^1(\Omega \cap B_{R_3})} + \|(V, \nabla V)\|_{L^2(\Omega \cap \{x_1 = 0\})} + \|g\|_{L^2(\Omega \cap B_{R_3})} \right)^{1-\alpha}.
\]

We now can use the arguments as in the proof of Theorem 3.1 to derive the desired conclusion. \( \square \)

We are ready to give

**Proof of Theorem 2.1.** By Corollary 3.1, there exists \( \gamma_2 > 0 \) depending only on \( \alpha, \Gamma, \Lambda, R_1 \) and \( R_3 \) such that for every \( \gamma_1 \in (0, \gamma_2) \) there exists \( \gamma_0 \in (0, \gamma_1) \), depending only on \( \gamma_1, \alpha, \Gamma, \Lambda, R_1 \) and \( R_3 \), such that

\[
\|V\|_{H^1((O_{\gamma_2} \setminus O_{\gamma_1}) \setminus B_{R_1})} \leq C_{\gamma_1} \left( \|V\|_{H^1(\Sigma_{\gamma_0})} + \|g\|_{L^2(D_{\gamma_0})} \right)^\alpha \left( \|V\|_{H^1(D_{\gamma_0})} + \|g\|_{L^2(D_{\gamma_0})} \right)^{1-\alpha},
\]

for some positive constant \( C_{\gamma_1} \) depending only on \( \gamma_1, \alpha, \Gamma, \Lambda, \Lambda_1, R_1 \) and \( R_3 \).

Fix such a \( \gamma_2 \). By Proposition 3.2, for \( x \in \partial B_{R_1} \setminus O_{\gamma_2/3} \), there exists \( \rho(x) \in (0, \gamma_2/12) \) such that

\[
\|V\|_{H^1(B_{\rho(x)}(x))} \leq C \left( \|V\|_{H^1(D_{\gamma_2/4})} + \|g\|_{L^2(D_{\gamma_2/4})} \right)^\alpha \left( \|V\|_{H^1(D_{\gamma_2/4})} + \|g\|_{L^2(D_{\gamma_2/4})} \right)^{1-\alpha}.
\]

One can also choose \( \rho(x) \) such that it depends only on \( \alpha, \Lambda, \gamma_2, \Gamma, d, \) and \( m \). This will be assumed from now on and we will simply denote it by \( \rho \) for notational ease. Since

\[
\partial B_{R_1} \setminus O_{\gamma_2/2} \subseteq \bigcup_{x \in \partial B_{R_1} \setminus O_{\gamma_2/3}} B_{\rho/2}(x),
\]

it follows that there exists a finite set \( \{x_i \in \partial B_{R_1} \setminus O_{\gamma_2/3}; i \in I\} \) such that

\[
\partial B_{R_1} \setminus O_{\gamma_2/2} \subseteq \bigcup_{i \in I} B_{\rho/2}(x_i).
\]

Then

\[
B_{R_1 + \rho/2} \setminus B_{R_1} \subseteq \bigcup_{i \in I} B_{\rho}(x_i).
\]
We derive from (3.91) and (3.92) that

\[(3.93) \quad \|V\|_{H^1(B_{r_1+\rho/2}\setminus B_{r_1})} \leq C \left( \|V\|_{H^1(S/4)} + \|g\|_{L^2(D/4)} \right)^\alpha \left( \|V\|_{H^1(D/4)} + \|g\|_{L^2(D/4)} \right)^{1-\alpha}.
\]

Set

\[r_2 = \min\{\rho/2, \gamma_2/4\},\]

For \(r_1 \in (0, r_2)\), let \(r_0 = r_0\) where \(r_0\) is the constant corresponding to \(r_1 = r_1\) in (3.90). Note that \(r_1 < \gamma_2/24\) since \(\rho \in (0, \gamma_2/12)\). Combining (3.90) and (3.93) yields

\[\|V\|_{H^1(B_{r_1+r_2}\setminus B_{r_1})} \leq C \left( \|V\|_{H^1(S/4)} + \|g\|_{L^2(D/4)} \right)^\alpha \left( \|V\|_{H^1(D/4)} + \|g\|_{L^2(D/4)} \right)^{1-\alpha}.
\]

Here we used the fact \(B_{r_1+r_2} \setminus B_{r_1} \subset \left((B_{r_1+\rho/2} \setminus B_{r_1}) \setminus O_{\gamma_2/2}\right) \cup \left(O_{\gamma_2} \setminus O_{\gamma_1}\right)\). The proof is complete.

We next give the

**Proof of Corollary 2.1.** Fix \(s \in (0, 1)\) and \(\hat{R}_1 \in (R_1, R_2)\) be such that

\[(3.94) \quad s \ln(R_3/R_2)/\ln(R_3/\hat{R}_1) > \alpha.
\]

Such \(s\) and \(\hat{R}_1\) exist since

\[\alpha < \alpha_0 = \ln(R_3/R_2)/\ln(R_3/R_1)
\]

(e.g. one can take \(\hat{R}_1\) close to \(R_1\) and \(s\) close to 1).

By Theorem 2.1, there exist \(r_s \in (R_1, \hat{R}_1)\) and \(r_0 \in (0, r_s - R_1)\) such that if \(\Delta v + \omega^2 v = 0\) in \(D_{r_0}\), then

\[(3.95) \quad \|V\|_{H^1(\partial B_{r_s})} \leq C \|V\|_{H^1(\Sigma_{r_0})} \|V\|_{H^1(D_{r_0})}^{1-s}.
\]

On the other hand, we have, by (2.8),

\[(3.96) \quad \|V\|_{H^1(\partial B_{r_2})} \leq C \|V\|_{H^1(\Sigma_{r_0})} \|V\|_{H^1(D_{r_0})}^{1-\beta}.
\]

with \(\beta = \ln(R_3/R_2)/\ln(R_3/r_s)\). Combining (3.95) and (3.96) yields

\[\|V\|_{H^1(\partial B_{r_2})} \leq C \|V\|_{H^1(\Sigma_{r_0})} \|V\|_{H^1(D_{r_0})}^{1-s} \leq C \|V\|_{H^1(D_{r_0})}^{1-\beta}.
\]

In the last inequality, we used the fact \(\|V\|_{H^1(\partial B_{r_2})} \leq C \|V\|_{H^1(D_{r_0})}\) by the trace theory. The conclusion now follows since \(\beta s > \alpha\) by (3.94) and \(\|V\|_{H^1(\Sigma_{r_0})} \leq C \|V\|_{H^1(D_{r_0})}\) by the trace theory.

\qed
4. Three-sphere inequalities for Maxwell equations

In this section, we establish three-sphere inequalities for Maxwell equations. As usual, see e.g. in [32, 43, 49] and the references therein, we also derive Theorem 2.2 from three-sphere inequalities for second-order elliptic equations with partial data. In order to be able to apply the results established in Section 3, we will use the fact that Maxwell equations can be reduced to weakly coupled second order elliptic equations. More precisely, let $\Omega$ be an open subset of $\mathbb{R}^3$. If $(E, H) \in [H^1(\Omega)]^6$ satisfies
\[
\begin{cases}
\nabla \times E = i\omega \mu H & \text{in } \Omega, \\
\nabla \times H = -i\omega \varepsilon E & \text{in } \Omega,
\end{cases}
\]
then, for $1 \leq a \leq 3$,
\begin{align}
\text{(4.1)} & \quad \operatorname{div}(\mu \nabla \mathcal{H}_a) + \operatorname{div}(\partial_a \mu \mathcal{H} - ik\mu \varepsilon^a \varepsilon \mathcal{E}) = 0 \text{ in } \Omega, \\
\text{(4.2)} & \quad \operatorname{div}(\varepsilon \nabla \mathcal{E}_a) + \operatorname{div}(\partial_a \varepsilon \mathcal{E} + ik\varepsilon \varepsilon^a \mu \mathcal{H}) = 0 \text{ in } \Omega.
\end{align}
Here $\mathcal{E}_a$ and $\mathcal{H}_a$ denote the $a$ component of $\mathcal{E}$ and $\mathcal{H}$, respectively, and the $bc$ component $\varepsilon_{bc}^a$ ($1 \leq b, c \leq 3$) of $\varepsilon^a$ ($1 \leq a \leq 3$) denotes the usual Levi Civita permutation, i.e.,
\[
\varepsilon_{bc}^a = \begin{cases} 
\operatorname{sign} (abc) & \text{if } abc \text{ is a permutation,} \\
0 & \text{otherwise.}
\end{cases}
\]

We now present a variant of Theorem 3.1 for the Maxwell equations.

**Theorem 4.1.** Let $d = 3$, $\Lambda \geq 1$, and $0 < R_* < R < R^*$. Then, for any $\alpha \in (0, 1)$, there exists a positive constant $\gamma_2 \in (0, 1)$, depending only on $\Lambda$ and $R$ such that for every $\gamma_1 \in (0, \gamma_2)$, there exists $\gamma_0 \in (0, \gamma_1)$ depending only on $\gamma_1$, $\alpha$, and $\Lambda$, such that, for a pair of symmetric, uniformly elliptic, matrix-valued functions $(\varepsilon, \mu)$ of class $C^2$ defined in $D_{\gamma_0} := Y_{\gamma_0,1,R}$ verifying, in $D_{\gamma_0}$, with $M = \varepsilon$ and $M = \mu,$
\[
\Lambda^{-1} |\xi|^2 \leq \langle M(x)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M(x)| \leq \Lambda,
\]
for $\omega > 0$, for $J_\varepsilon, J_m \in [L^2(D_{\gamma_0})]^3$, and for $(E, H) \in [H^1(D_{\gamma_0})]^2$ satisfying
\[
\begin{cases}
\nabla \times E = i\omega \mu H + J_\varepsilon & \text{in } D_{\gamma_0}, \\
\nabla \times H = -i\omega \varepsilon E + J_m & \text{in } D_{\gamma_0},
\end{cases}
\]
we have, with $\Sigma_{\gamma_0} = \partial D_{\gamma_0} \cap \{x_1 = 0\}$,
\[
\|E, H\|_{L^2(Y_{\gamma_1, \gamma_2})} \leq C \left( \|E \times \nu, H \times \nu\|_{H^{-1/2}(\Sigma_{\gamma_0})} + \|(J_\varepsilon, J_m)\|_{L^2(D_{\gamma_0})} \right)\alpha \times \left( \|E, H\|_{L^2(D_{\gamma_0})} + \|(J_\varepsilon, J_m)\|_{L^2(D_{\gamma_0})} \right)^{1-\alpha},
\]
for some positive constant $C$ depending only on $\alpha, \gamma_1, \omega, R_*, \Lambda$, and the upper bound of $\|E, H\|_{C^2(D_{\gamma_0})}$.

**Remark 4.1.** The constant $\gamma_2$ depends on $\Lambda$ but is independent of the upper bound of $\|E, H\|_{C^2(D_{\gamma_0})}$.

The proof of Theorem 4.1 is given in Section 4.2 below and is the key part of the proof of Theorem 2.2. As a consequence of Theorem 4.1, we obtain the following variant of Corollary 3.1 for the Maxwell equations whose proof is omitted.
Corollary 4.1. Let $d = 3$, $\Lambda \geq 1$, let $\Omega$ be an open subset of $\mathbb{R}^d$ of class $C^1$, and let $\Gamma$ be a compact smooth curve of $\partial \Omega$ which belongs to a connected component $\Sigma$ of $\partial \Omega$. Denote $O_r = \{ x \in \mathbb{R}^d; \text{dist}(x, \Gamma) < r \}$, $D_r = \Omega \setminus O_r$, and $\Sigma_r = \Sigma \setminus O_r$ for $r > 0$. For any $\alpha \in (0, \alpha_0)$, there exists $r_2 > 0$ depending only on $\alpha$, $\Gamma$, and $\Omega$ such that for every $r_1 \in (0, r_2)$, there exists $r_0 \in (0, r_1)$, depending only on $r_1$, $\alpha$, $\Gamma$, and $\Omega$, such that for a pair $(\varepsilon, \mu)$ of symmetric, uniformly elliptic, Lipschitz-valued functions defined in $D_{r_0}$ verifying, with $M = \varepsilon$ and $M = \mu$,

\begin{align}
\Lambda^{-1}|\xi|^2 \leq \langle M(x)\xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M(x)| \leq \Lambda,
\end{align}

for $\omega > 0$, for $J_e, J_m \in [L^2(D_{r_0})]^3$, and for $(E, H) \in [H(\text{curl}, D_{r_0})]^2$ satisfying

\begin{align}
\begin{cases}
\nabla \times E = i\omega \mu H + J_e & \text{in } D_{r_0}, \\
\nabla \times H = -i\omega \varepsilon H + J_m & \text{in } D_{r_0},
\end{cases}
\end{align}

we have

\begin{align}
\| (E, H) \|_{L^2(O_{r_2} \setminus O_{r_1})} \leq C \left( \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\text{div}, \Sigma_r)} + \| (J_e, J_m) \|_{L^2(D_{r_0})} \right)^\alpha \\
\times \left( \| (E, H) \|_{L^2(D_{r_0})} + \| (J_e, J_m) \|_{L^2(D_{r_0})} \right)^{1-\alpha},
\end{align}

for some positive constant $C$ depending only on $\alpha$, $\omega$, $\Lambda$, $\Gamma$, $\Omega$, and the upper bound of $\| (\varepsilon, \mu) \|_{C^2(D_{r_0})}$.

Applying Theorem 2.2, one can derive various three-sphere inequalities with partial data for $R_1 < R_2 < R_3$. Here is an example in the spirit of Hadamard.

Corollary 4.2. Let $d = 3$, and $R_1 < R_2 < R_3$, and let $\Gamma$ be a a compact smooth curve of $\partial B_{R_1}$. Denote $O_r = \{ x \in \mathbb{R}^d; \text{dist}(x, \Gamma) < r \}$, $D_r = B_{R_3} \setminus (B_{R_1} \cup O_r)$, and $\Sigma_r = \partial B_{R_3} \setminus O_r$ for $r > 0$. Set $\alpha_0 = \ln(R_3/R_2)/\ln(R_3/R_1)$. Then, for any $\alpha \in (0, \alpha_0)$, there exists $r_0 \in (0, R_2 - R_1)$, depending only on $R_1, R_2, R_3, \Gamma$, and $\alpha$ such that for $\omega > 0$ and for $(E, H) \in [H(\text{curl}, D_{r_0})]^2$ satisfying

\begin{align}
\begin{cases}
\nabla \times E = i\omega H & \text{in } D_{r_0}, \\
\nabla \times H = -i\omega H & \text{in } D_{r_0},
\end{cases}
\end{align}

we have

\begin{align}
\| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\text{div}, \partial B_{R_2})} \leq C \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\text{div}, \Sigma_{r_0})}^{\alpha} \| (E, H) \|_{L^2(D_{r_0})}^{1-\alpha},
\end{align}

for some positive constant $C$ depending on $\alpha$, $\omega$, $\Gamma$, $R_1$, $R_2$, and $R_3$.

The rest of this section is organized as follows. We first present two lemmas used in the proofs of Theorem 4.1, Theorem 2.2, and Corollary 4.2. The proofs of Theorem 4.1, Theorem 2.2, and Corollary 4.2 are then given in Section 4.2, Section 4.3, and Section 4.4, respectively.

4.1. Two useful lemmas. We begin with

Lemma 4.1. Let $\Lambda \geq 1$, $D \subseteq \Omega \subset \mathbb{R}^3$ be two connected, open, bounded subsets of $\mathbb{R}^3$, and let $(\varepsilon, \mu)$ be a pair of real, symmetric, Lipschitz, uniformly elliptic matrix-valued functions defined in $\Omega$ such that, with $M = \varepsilon$ and $M = \mu$, in $\Omega$,

\begin{align}
\Lambda^{-1}|\xi|^2 \leq \langle M(x)\xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M(x)| \leq \Lambda.
\end{align}
Let \( J_e, J_m \in [L^2(\Omega)]^3 \), and let \( f, g \in H^{-1/2}(\text{div}_\Gamma, \partial D) \). There exists a unique solution \( (E, H) \in [H(\text{curl}, \Omega \setminus \partial D)]^2 \) of the system
\[
\begin{aligned}
\nabla \times E &= i\omega \mu H + J_e & \text{in } \Omega, \\
\nabla \times H &= -i\omega \varepsilon E + J_m & \text{in } \Omega, \\
[E \times \nu] &= f, & [H \times \nu] &= g & \text{on } \partial D, \\
(H \times \nu) \times \nu - E \times \nu &= 0 & & \text{on } \partial \Omega,
\end{aligned}
\]

Moreover,
\[
\| (E, H) \|_{L^2(\Omega)} \leq C \left( \| (f, g) \|_{H^{-1/2}(\text{div}_\Gamma, \partial D)} + \| (J_e, J_m) \|_{L^2(\Omega)} \right),
\]
for some positive constant \( C \) depending only on \( \Lambda, \Omega, \) and \( D \).

**Remark 4.2.** The novelty of Lemma 4.1 lies in the fact that the constant \( C \) does not depend on \( (\varepsilon, \mu) \) but on \( \Lambda \). Lemma 4.1 is well-known if the constant \( C \) depends on \( \varepsilon \) and \( \mu \).

**Proof.** By the trace theory, see e.g. [3], there exist \( E_f, H_g \in H(\text{curl}, D) \) such that
\[
E_f \times \nu = f \text{ on } \partial D, \quad H_g \times \nu = g \text{ on } \partial D,
\]
\[
\| E_f \|_{H(\text{curl}, D)} \leq C \| f \|_{H^{-1/2}(\text{div}_\Gamma, \partial D)}, \quad \text{and} \quad \| H_g \|_{H(\text{curl}, D)} \leq C \| g \|_{H^{-1/2}(\text{div}_\Gamma, \partial D)}.
\]
By considering the pair \( (E - E_f 1_D, H - H_g 1_D) \), one may assume that \( f = g = 0 \). This fact is assumed from later on.

We now establish the existence and the uniqueness of \( (E, H) \). An integration by parts gives
\[
\int_{\Omega} \mu^{-1} \nabla \times E, \nabla \times E \, d\Omega = \int_{\partial \Omega} i\omega |E \times \nu|^2 + \int_{\Omega} \omega^2 \varepsilon E, E \, d\Omega + \int_{\Omega} i\omega J_m, E \, d\Omega + \langle \mu^{-1} J_e, \nabla \times E \rangle.
\]
This implies that \( E \times \nu = 0 \) on \( \partial \Omega \) if \( J_m = J_e = 0 \) in \( \Omega \), and this in turn yields \( H \times \nu = 0 \) on \( \partial \Omega \).
Thus \( E = H = 0 \) in \( \Omega \) if \( J_m = J_e = 0 \) in \( \Omega \) by the unique continuation principle, see [7, 49] (see also [52]). Hence the uniqueness of \( (E, H) \) holds. The existence of \( (E, H) \) can then be derived from the limiting absorption principle using the standard compactness for the Maxwell system and the uniqueness of the system, which is just proved. The details are omitted.

We next establish (4.8) by contradiction. Assume that there exist sequences \( (\varepsilon^{(n)}), (\mu^{(n)}), (J_e^{(n)}), (J_m^{(n)}) \subset [L^2(\Omega)]^3 \), and \( ((E^{(n)}, H^{(n)})) \subset [H(\text{curl}, \Omega)]^2 \) such that (4.7) holds for \( (\varepsilon^{(n)}, \mu^{(n)}), \)
\[
\begin{aligned}
\nabla \times H^{(n)} &= i\omega \varepsilon^{(n)} E^{(n)} + J_m^{(n)} & \text{in } \Omega, \\
\nabla \times E^{(n)} &= -i\omega \mu^{(n)} H^{(n)} + J_e^{(n)} & \text{in } \Omega, \\
(H^{(n)} \times \nu) \times \nu - E^{(n)} \times \nu &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
\[
\| (J_e^{(n)}, J_m^{(n)}) \|_{L^2(\Omega)} \leq \| (E^{(n)}, H^{(n)}) \|_{L^2(\Omega)} = 1.
\]
Using (4.9), we have
\[
\| E^{(n)} \times \nu \|_{L^2(\Omega)} \leq C.
\]
Without loss of generality, one may assume that \( (E^{(n)} \times \nu) \) converges in \( H^{-1/2}(\partial \Omega) \). By Ascoli’s theorem, one may also assume that \( (\varepsilon^{(n)}, \mu^{(n)}) \to (\varepsilon, \mu) \) in \( L^\infty(\Omega) \) for some \( (\varepsilon, \mu) \in W^{1,\infty}(\Omega) \). We derive that \( E^{(n)} \) is bounded in \( H(\text{curl}, \Omega) \) and
\[
(\text{div}(\varepsilon E^{(n)})) \text{ converges in } [H^{-1}(\Omega)]^3.
\]
Applying [40, Lemma 1], one may assume that \((E^{(n)})\) converges in \([L^2(\Omega)]^3\). Similarly, one may assume that \((H^{(n)})\) converges in \([L^2(\Omega)]^3\).

Let \((E, H)\) be the limit of \((E^{(n)}, H^{(n)})\) in \([L^2(\Omega)]^6\). Then \((E, H) \in [H(\text{curl}, \Omega)]^2\) and

\[
\begin{align*}
\nabla \times H &= i\omega \varepsilon E & \text{in } \Omega, \\
\nabla \times E &= -i\omega \mu H & \text{in } \Omega, \\
(H \times \nu) \times \nu - E \times \nu &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

It follows that \((E, H) = (0, 0)\) in \(\Omega\) by the uniqueness. This contradicts the fact \(\|(E, H)\|_{L^2(\Omega)} = \lim_{n \to +\infty} \|(E_n, H_n)\|_{L^2(\Omega)} = 1\). Therefore, (4.8) holds. The proof is complete. \(\square\)

The second lemma, whose proof is given in the appendix, is

**Lemma 4.2.** Let \(\omega > 0\), \(0 < R_\ast < R_1 < R_2 < R_3 < R^*\), and let \((E, H) \in [H(\text{curl}, B_{R_3} \setminus B_{R_1})]^2\) be a solution of the system

\[
\begin{align*}
\nabla \times E &= i\omega H & \text{in } B_{R_3} \setminus B_{R_1}, \\
\nabla \times H &= -i\omega H & \text{in } B_{R_3} \setminus B_{R_1}.
\end{align*}
\]

Then

\[
\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\text{div}, \partial B_{R_2})} 
\leq C\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\text{div}, \partial B_{R_1})} \|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\text{div}, \partial B_{R_3})}^{1-\alpha}
\]

with \(\alpha = \ln(R_3/R_2)/\ln(R_3/R_1)\) for some positive constant \(C\) depending only on \(R_\ast, R^*, \) and \(\omega\).

4.2. **Proof of Theorem 4.1.** The proof of Theorem 4.1, which involves Lemma 4.1, is in the same spirit as the one of Theorem 3.1. Extend \(\varepsilon\) and \(\mu\) evenly for \(x_1 < 0\) and still denote the extensions by \(\varepsilon\) and \(\mu\). Set

\[
\tilde{Y}_{\gamma_1, \gamma_2, R} = \left\{ x \in \mathbb{R}^d; \theta \in (-3\pi/4, 3\pi/4), \gamma_1 R < \hat{r} < \gamma_2 R, \text{ and } |\tilde{x}| < R \right\}.
\]

Note that \(L_{3/2}\) is a diffeomorphism from \(\tilde{Y}_{\gamma_1, \gamma_2, R}\) onto \(Y_{\gamma_1, \gamma_2, R}\), where \(L_{3/2}\) is given by (3.36) with \(n = 3/2\). By Proposition 3.1, and (4.1) and (4.2), there exists \(\gamma_2 > 0\) depending only on \(\alpha, \Lambda, \Gamma, R_1, \) and \(R_3\), such that for every \(\gamma_1 \in (0, \gamma_2)\), there exists \(\gamma_0 \in (0, \gamma_1/2)\), depending only on \(\gamma_1, \alpha, \Lambda, \Gamma, R_1, \) and \(R_3\), such that, with \(D_{\gamma_0} = \tilde{Y}_{2\gamma_0,1,R/2}\), for \(h \in L^2(\Sigma_{\gamma_0})\), for \((\tilde{E}, \tilde{H}) \in [H^1(\tilde{D}_{\gamma_0})]^6\) satisfying

\[
\begin{align*}
\nabla \times \tilde{E} &= i\omega \mu \tilde{H} & \text{in } \tilde{D}_{\gamma_0}, \\
\nabla \times \tilde{H} &= -i\omega \varepsilon \tilde{E} & \text{in } \tilde{D}_{\gamma_0},
\end{align*}
\]

we have, with \(\Sigma_{\gamma_0} = \partial \tilde{D}_{\gamma_0} \cap \{\theta = \pm 3\pi/4\}\),

\[
(4.10) \quad \|((\tilde{E}, \tilde{H})\|_{H^1(\tilde{Y}_{\gamma_1, \gamma_2, R})} 
\leq C\|(\tilde{E}, \tilde{H}, \nabla \tilde{E}, \nabla \tilde{H})\|_{L^2(\Sigma_{\gamma_0})} \left(\|((\tilde{E}, \tilde{H})\|_{H^1(\tilde{D}_{\gamma_0})} + \|((\tilde{E}, \tilde{H}, \nabla \tilde{E}, \nabla \tilde{H})\|_{L^2(\Sigma_{\gamma_0})}\right)^{1-\alpha}.
\]

Set

\[
\tilde{Y}_{\gamma_1, \gamma_2, R} = \left\{ x \in \mathbb{R}^d; \gamma_1 R < \hat{r} < \gamma_2 R, \text{ and } |\tilde{x}| < R \right\},
\]
and let \((\tilde{E}, \tilde{H}) \in [H(\text{curl}, \dot{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\})]^2\) be such that

\[
\begin{align*}
\nabla \times \tilde{E} &= i\omega \mu H + J_e \mathbb{1}_{x_1 > 0} & \text{in } \dot{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\}, \\
\nabla \times \tilde{H} &= -i\omega \varepsilon \tilde{E} + J_m \mathbb{1}_{x_1 > 0} & \text{in } \dot{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\}, \\
[\tilde{E} \times \nu] &= \varphi H \times \nu, & \text{on } \dot{Y}_{\gamma_0,1,R} \setminus \{x_1 = 0\}, \\
[\tilde{H} \times \nu] &= \varphi H \times \nu & \text{on } \dot{Y}_{\gamma_0,1,R},
\end{align*}
\]

where \(\varphi \in C^1_c(\dot{Y}_{\gamma_0,1,R})\) is fixed such that \(\varphi = 1\) for \(x \in \dot{Y}_{2\gamma_0,1,R/2} + B_{\gamma_0/4}\). By Lemma 4.1, we have

\[
\|\tilde{E}, \tilde{H}\|_{L^2(\dot{Y}_{\gamma_0,1,R})} \leq C \left( \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\text{div}, \Sigma_{\gamma_0})} + \| (J_e, J_m) \|_{L^2(D_{\gamma_0})} \right). \tag{4.11}
\]

This in turn implies, by the regularity theory of elliptic equations, and (4.1) and (4.2), that

\[
\|\tilde{E}, \tilde{H}, \nabla \tilde{E}, \nabla \tilde{H}\|_{L^2(\dot{Y}_{\gamma_0,1,R} \cap \{\theta = \pm 3\pi/4\})} \leq C \left( \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\text{div}, \Sigma_{\gamma_0})} + \| (J_e, J_m) \|_{L^2(D_{\gamma_0})} \right). \tag{4.12}
\]

Set

\[
\tilde{E}, \tilde{H} = (E, H) \mathbb{1}_{x_1 > 0} - (\tilde{E}, \tilde{H}) \quad \text{in } \dot{Y}_{2\gamma_0,1,R/2} + B_{\gamma_0/4}.
\]

Applying (4.10) to \((\tilde{E}, \tilde{H})\) given in (4.13), noting that, by the regularity theory of elliptic equations,

\[
\|\tilde{E}, \tilde{H}\|_{H^{1}(\tilde{D}_{\gamma_0})} \leq \|\tilde{E}, \tilde{H}\|_{L^2(\tilde{D}_{\gamma_0/2} + B_{\gamma_0/4})}
\]

and \(\tilde{D}_{\gamma_0} + B_{\gamma_0/4} \subset \dot{Y}_{2\gamma_0,1,R/2} + B_{\gamma_0/4}\), and using (4.11) and (4.12), one reaches the conclusion. \(\square\)

4.3. **Proof of Theorem 2.2.** The proof of Theorem 2.2 is similar to the one of Theorem 2.1. However, instead of using Theorem 4.1 and Proposition 3.2, one applies Theorem 4.1 and Proposition 4.1 below. The details are left to the reader. \(\square\)

In the proof of Theorem 2.2, we also use the following variant of Theorem 4.1:

**Proposition 4.1.** Let \(R > R_* > 0\) and \(\Lambda \geq 1\), and set \(\Omega = B_R \cap \{x_1 > 0\}\). For any \(0 \leq \alpha < 1\), there exists a positive constant \(r \in (0, R)\), depending only on \(\Lambda\) and \(R_*\) such that for a pair \((\varepsilon, \mu)\) of symmetric, uniformly elliptic, Lipschitz matrix-valued functions defined in \(\Omega\) verifying, for \(M = \varepsilon\) and \(M = \mu\),

\[
\Lambda^{-1} |\xi|^2 \leq \langle M(x) \xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \quad \text{and} \quad |\nabla M(x)||x| \leq \Lambda,
\]

for \(\omega > 0\), and for \((E, H) \in [H(\text{curl}, \Omega)]^2\) satisfying

\[
\begin{align*}
\nabla \times E &= i\omega \mu H + J_e & \text{in } \Omega, \\
\nabla \times H &= -i\omega \varepsilon E + J_m & \text{in } \Omega,
\end{align*}
\]

we have, with \(\Sigma = \partial \Omega \cap \{x_1 = 0\}\),

\[
\|E, H\|_{L^2(B, r; \Omega)} \leq C \left( \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\text{div}, \Sigma)} + \| (J_e, J_m) \|_{L^2(\Omega)} \right) \alpha \times \\
\quad \quad \times \left( \|E, H\|_{L^2(\Omega)} + \| (J_e, J_m) \|_{L^2(\Omega)} \right)^{1-\alpha},
\]

\(^3\text{Given two subsets } A \text{ and } B \text{ of } \mathbb{R}^d, \text{ one denotes } A + B = \{x + y; x \in A, y \in B\}.\)
for some positive constant $C$ depending only on $\alpha$, $\omega$, $\Lambda$, $R_*$, and the upper bound of $\|(\varepsilon, \mu)\|_{C^2(\Omega)}$.

The proof of Proposition 4.1 is in the same spirit as the one of Proposition 3.2 (see also the proof of of Theorem 4.1) and is omitted.

4.4. Proof of Corollary 4.2. Let $\hat{R}_1 \in (R_1, R_3)$ and $s \in (0, 1)$ be such that

$$\alpha < \beta s < \alpha_0,$$

where $\beta = \ln(R_3/R_2)/\ln(R_3/\hat{R}_1)$. By Theorem 2.2, there exist $r_* \in (R_1, \hat{R}_1)$ and $r_0 \in (0, r_* - R_1)$ such that

$$\| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\clos{\Gamma}, \partial B_{R_1})} \leq C \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\clos{\Gamma}, \Sigma_{r_0})}^{\beta s} \| (E, H) \|_{L^2(D_{r_0})}^{1-s}.$$  

By Lemma 4.2, we have

$$\| (E, \nu) \|_{H^{-1/2}(\clos{\Gamma}, \partial B_{R_3})} \leq C \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\clos{\Gamma}, \partial B_{R_1})}^{\beta} \| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\clos{\Gamma}, \partial B_{R_3})}^{1-\beta}.$$  

Combining (4.14) and (4.15) yields

$$\| (E \times \nu, H \times \nu) \|_{H^{-1/2}(\clos{\Gamma}, \partial B_{R_2})} \leq C \| V \|_{H^{-1/2}(\clos{\Gamma}, \Sigma_{r_0})}^{\beta s} \| (E, H) \|_{L^2(D_{r_0})}^{1-\beta s}.$$

The conclusion follows since $\beta s > \alpha$. \qed

5. Cloaking property of plasmonic structures in doubly complementary media

This section is devoted to the cloaking property of plasmonic structures in doubly complementary media. This cloaking phenomenon is also known as cloaking an object via anomalous localized resonance. Let $\omega > 0$, and let $\Omega_1 \Subset \Omega_2 \Subset \mathbb{R}^3$ be smooth, bounded, simply connected, open subsets of $\mathbb{R}^3$. Let $\varepsilon^+, \mu^+$ be defined in $\mathbb{R}^3 \setminus (\Omega_2 \setminus \Omega_1)$ and $\varepsilon^-, \mu^-$ be defined in $\Omega_2 \setminus \Omega_1$ such that $\varepsilon^+, \mu^+ \geq \varepsilon^-, \mu^-$ are real, symmetric, uniformly elliptic, matrix-valued functions in their domains of definition. Set, for $\delta \geq 0$,

$$\varepsilon^+ \! + i \delta I, \mu^+ \! + i \delta I \quad \text{in} \quad \Omega_2 \setminus \Omega_1,$$

$$\varepsilon^-, \mu^- \quad \text{in} \quad \mathbb{R}^3 \setminus (\Omega_2 \setminus \Omega_1).$$

As usual, we assume that for some $R_0 > 0$, $\Omega_2 \subset R_{B_0}$, $(\varepsilon^+, \mu^+) = (I, I)$ in $\mathbb{R}^3 \setminus B_{R_0}$. Here and in what follows, all matrix-valued functions are assumed to be piecewise $C^1$ in their domain of definition. Given $\delta > 0$ and $J \in [L^2(\mathbb{R}^3)]^3$ with compact support, let $(E_\delta, H_\delta) \in [H_{\text{loc}}(\text{curl,} \mathbb{R}^d)]^2$ be the unique radiating solution of the Maxwell equations

$$\begin{cases}
\nabla \times E_\delta = i \omega \mu_\delta H_\delta & \text{in } \mathbb{R}^3, \\
\n\nabla \times H_\delta = -i \omega \varepsilon_\delta E_\delta + J & \text{in } \mathbb{R}^3.
\end{cases}$$

Physically, $\varepsilon_\delta$ and $\mu_\delta$ describe the permittivity and the permeability of the considered medium, $\Omega_2 \setminus \Omega_1$ is a (shell) plasmonic structure in which the permittivity and the permeability are negative and $i \delta I$ describes its loss, $\omega$ is the frequency, $J$ is a density of charge, and $(E_\delta, H_\delta)$ is the electromagnetic field generated by $J$ in the medium $(\varepsilon_\delta, \mu_\delta)$. We assume here that the loss is $i \delta I$ for the simplicity of notation; any quantity of the form $i \delta M$, where $M$ is a real, symmetric, uniformly elliptic, matrix-valued function defined in $\Omega_2 \setminus \Omega_1$, is admissible.

\footnote{In this paper, the notation $D \Subset \Omega$ means $\hat{D} \subset \Omega$ for two subsets $D$ and $\Omega$ of $\mathbb{R}^d (d \geq 2)$.}
Recall that a solution \((E, H) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus B_R)]^2\), for some \(R > 0\), of the Maxwell equations

\[
\begin{cases}
\nabla \times E = i\omega H & \text{in } \mathbb{R}^3 \setminus B_R, \\
\nabla \times H = -i\omega E & \text{in } \mathbb{R}^3 \setminus B_R,
\end{cases}
\]

is called radiating if it satisfies one of the (Silver-Müller) radiation conditions

\[ H \times x - |x|E = O(1/|x|) \quad \text{or} \quad E \times x + |x|H = O(1/|x|) \quad \text{as } |x| \to +\infty. \]

For a matrix-valued function \(A\) defined in \(\Omega\), for a bi-Lipschitz homeomorphism \(T : \Omega \to \Omega'\), and for a vector field \(j\) defined in \(\Omega\), the following standard notations are used, for \(y \in \Omega'\):

\[
T_*A(y) = \frac{\nabla T(x)A(x)\nabla T(x)}{\det \nabla T(x)} \quad \text{and} \quad T_*j(y) = \frac{j(x)}{\det \nabla T(x)},
\]

with \(x = T^{-1}(y)\).

We next recall the definition of complementary media and doubly complementary media [40,43]. We begin with

**Definition 5.1** (Complementary media). Let \(\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \subset \mathbb{R}^3\) be smooth, bounded, simply connected, open subsets of \(\mathbb{R}^3\). The medium in \(\Omega_2 \setminus \Omega_1\) characterized by a pair of two symmetric matrix-valued functions \((\varepsilon_1, \mu_1)\) and the medium in \(\Omega_3 \setminus \Omega_2\) characterized by a pair of two symmetric, uniformly elliptic, matrix-valued functions \((\varepsilon_2, \mu_2)\) are said to be complementary if there exists a diffeomorphism \(F : \Omega_2 \setminus \Omega_1 \to \Omega_3 \setminus \Omega_2\) such that \(F \in C^1(\Omega_2 \setminus \Omega_1)\),

\[
(F_\ast \varepsilon_1, F_\ast \mu_1) = (\varepsilon_2, \mu_2) \quad \text{for } x \in \Omega_3 \setminus \Omega_2,
\]

\[
F(x) = x \quad \text{on } \partial \Omega_2,
\]

and the following two conditions hold: 1) There exists an diffeomorphism extension of \(F\), which is still denoted by \(F\), from \(\Omega_2 \setminus \{x_1\} \to \mathbb{R}^3 \setminus \Omega_2\) for some \(x_1 \in \Omega_1\); and 2) there exists a diffeomorphism \(G : \mathbb{R}^3 \setminus \Omega_3 \to \mathbb{R}^3 \setminus \{x_1\}\) such that \(G \in C^1(\mathbb{R}^3 \setminus \Omega_3)\), \(G(x) = x\) on \(\partial \Omega_3\), and \(G \circ F : \Omega_1 \to \Omega_3\) is a diffeomorphism if one sets \(G \circ F(x_1) = x_1\).

**Definition 5.2.** The medium \((\varepsilon_0, \mu_0)\) given in (5.1) with \(\delta = 0\) is said to be doubly complementary if for some \(\Omega_2 \Subset \Omega_3\), \((\varepsilon^+, \mu^+)\) in \(\Omega_3 \setminus \Omega_2\) and \((\varepsilon^-, \mu^-)\) in \(\Omega_2 \setminus \Omega_1\) are complementary, and

\[
(G_\ast F_\ast \varepsilon^+, G_\ast F_\ast \mu^+) = (\varepsilon^+, \mu^+) \quad \text{in } \Omega_3 \setminus \Omega_2
\]

for some \(F\) and \(G\) from Definition 5.1.

We now address the point that makes the doubly complementary media special. Let \((\varepsilon_\delta, \mu_\delta)\) be defined by (5.1) such that \((\varepsilon_0, \mu_0)\) is doubly complementary. Assume that \((E_\delta, H_\delta)\) is a solution of (5.2) with \(J = 0\) in \(\Omega_3\). Set, for \(x' \in \mathbb{R}^3 \setminus \Omega_2\),

\[
E_{1,\delta}(x') = \nabla F^{-T}(x)E_\delta(x) \quad \text{and} \quad H_{1,\delta}(x') = \nabla F^{-T}(x)H_\delta(x) \quad \text{with } x = F^{-1}(x'),
\]

and, for \(y' \in \Omega_3\),

\[
E_{2,\delta}(y') = \nabla G^{-T}(y)E_{1,\delta}(y) \quad \text{and} \quad E_{2,\delta}(y') = \nabla G^{-T}(y)H_{1,\delta}(y) \quad \text{with } y = G^{-1}(y').
\]

Here \(F, G, \) and \(\Omega_3\) are from the definition of doubly complementary media. By a change of variables, see e.g. [40, Lemma 7], up to a (small) perturbation, one can check that \((E_{1,\delta}, H_{1,\delta})\) and \((E_{2,\delta}, H_{2,\delta})\) satisfy the same Maxwell equations in \(\Omega_3 \setminus \Omega_2\) as the one of \((E_\delta, H_\delta)\). It is clear that

\[
E_{1,\delta} \times \nu - E_\delta \times \nu = H_{1,\delta} \times \nu - H_\delta \times \nu = 0 \quad \text{on } \partial \Omega_2
\]
and
\[ E_{2,\delta} \times \nu - E_{1,\delta} \times \nu = H_{2,\delta} \times \nu - H_{1,\delta} \times \nu = 0 \text{ on } \partial \Omega_3. \]
Here on a boundary of a bounded subset of \( \mathbb{R}^3 \), \( \nu \) denotes its normal unit vector directed to the exterior. One hence has two Cauchy’s problems with the same equations, one for \((E_{\delta}, H_{\delta})\) and \((E_{1,\delta}, H_{1,\delta})\) in \( \Omega_3 \setminus \Omega_2 \) with the boundary data given on \( \partial \Omega_2 \), and one for \((E_{1,\delta}, H_{1,\delta})\) and \((E_{2,\delta}, H_{2,\delta})\) in \( \Omega_3 \setminus \Omega_2 \) with the boundary data given on \( \partial \Omega_3 \). This is the essential property of \((\varepsilon_{\delta}, \mu_{\delta})\) for the cloaking purpose and the root of the definition of doubly complementary media.

We list here some examples of doubly complementary media for which the formulas of \((\varepsilon_{0}, \mu_{0})\) are explicit; a general way to obtain doubly complementary media is presented in [43]. Fix \( p > 1 \) and \( r_2 > r_1 > 0 \), and define \( r_3 = r_2^p/r_1^{p-1} \) and \( m = r_2^p/r_1^p \). Set, with the standard notations of polar coordinates,
\begin{equation}
M = -\frac{r_2^2}{r^p} \left[ \frac{1}{p-1} e_r \otimes e_r + (p-1) \left( e_\theta \otimes e_\theta + e_\phi \otimes e_\phi \right) \right] \text{ in } B_{r_2} \setminus B_{r_1},
\end{equation}
and define, for \( \delta \geq 0 \),
\begin{equation}
(\varepsilon_{\delta}, \mu_{\delta}) = \begin{cases}
(M + i\delta I, M + i\delta I) & \text{in } B_{r_2} \setminus B_{r_1}, \\
(mI, mI) & \text{in } B_{r_1}, \\
(I, I) & \text{otherwise.}
\end{cases}
\end{equation}
One can check that \((\varepsilon_{0}, \mu_{0})\) is doubly complementary with \( \Omega_j = B_r \) for \( j = 1, 2, 3 \), \( F(x) = r_2^p x/|x|^p \) and \( \mathcal{G} = r_2^3 x/|x|^q \) with \( q = p/(p-1) \). In the case \( p = 2 \), it is easy to see that \( M = -r_2^2 I/|x|^2 \) in \( B_{r_2} \setminus B_{r_1} \).

For a doubly complementary medium \((\varepsilon_{0}, \mu_{0})\) and \( J \in [L^2(\mathbb{R}^3)]^3 \) with \( \text{supp } J \cap \Omega_2 = \emptyset \), set
\begin{equation}
(\bar{\varepsilon}, \bar{\mu}) := \begin{cases}
(\varepsilon^+, \mu^+) & \text{in } \mathbb{R}^3 \setminus \Omega_3, \\
(\mathcal{G}_sF_+\bar{\varepsilon}^+, \mathcal{G}_sF_+\mu^+) & \text{in } \Omega_3,
\end{cases}
\end{equation}
and let \((\bar{E}, \bar{H}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2 \) be the unique radiating solution of
\begin{equation}
\begin{cases}
\nabla \times \bar{E} = i\omega \bar{\varepsilon} \bar{H}, & \text{in } \mathbb{R}^3, \\
\nabla \times \bar{H} = -i\omega \bar{\mu} \bar{E} + J, & \text{in } \mathbb{R}^3.
\end{cases}
\end{equation}
Note that if \((\varepsilon_{0}, \mu_{0})\) is doubly complementary, then \( \bar{\varepsilon} \) and \( \bar{\mu} \) are uniformly elliptic in \( \mathbb{R}^3 \) since \( \text{det } \mathcal{F} < 0 \) and \( \text{det } \mathcal{G} < 0 \).

The following result provides an interesting property of doubly complementary media [43]:

**Proposition 5.1.** Let \( 0 < \delta < 1 \), \( R_0 > 0 \), \( J \in [L^2(\mathbb{R}^3)]^3 \) with \( \text{supp } J \subset B_{R_0} \setminus \Omega_2 \), and let \((E_\delta, H_\delta) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2 \) be the unique radiating solution of (5.2). Assume that \((\varepsilon_{0}, \mu_{0})\) is doubly complementary. Then, for \( R > R_0 \),
\begin{equation}
\|(E_\delta, H_\delta)\|_{L^2(B_R \setminus \Omega_3)} \leq C_R \|J\|_{L^2(\mathbb{R}^3)}
\end{equation}
for some positive constant \( C_R \) that depends on \( R \) but is independent of \( J \) and \( \delta \). Moreover,
\begin{equation}
(E_\delta, H_\delta) \text{ converges to } (\bar{E}, \bar{H}) \text{ in } [L^2_{\text{loc}}(\mathbb{R}^3 \setminus \Omega_3)]^6 \text{ as } \delta \to 0,
\end{equation}
where \((\bar{E}, \bar{H}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2 \) is the unique radiating solution of (5.11). Assume in addition that \( \text{supp } J \cap \Omega_3 = \emptyset \). We have, for \( R > R_0 \),
\begin{equation}
\|(E_\delta, H_\delta) - (\bar{E}, \bar{H})\|_{L^2(B_R \setminus \Omega_3)} \leq C_R \delta \|J\|_{L^2(\mathbb{R}^3)}.
\end{equation}
Here $C_R$ denotes a positive constant that depends on $R$ but is independent of $J$ and $\delta$.

**Remark 5.1.** In Proposition 5.1, we assume implicitly that $\Omega_2 \subset B_{R_0}$ for (5.12) and (5.13) and $\Omega_3 \subset B_{R_0}$ for (5.14) since otherwise, there is no information.

We now present our main result on the cloaking property of doubly complementary media.

**Theorem 5.1.** Assume that $(\varepsilon_0, \mu_0)$ is doubly complementary and of class $C^2$ in $\Omega_3 \setminus \Omega_2$. Let $\Gamma_1$ be a compact smooth curve on $\partial \Omega_1$ and $\Gamma_2$ be a compact smooth curve on $\partial \Omega_2$. Set, for $\gamma > 0$,

$$O_{j, \gamma} = \left\{ x \in \mathbb{R}^3; \text{dist}(x, \Gamma_j) < \gamma \right\} \text{ for } j = 1, 2.$$

For $\gamma > 0$, let $(\varepsilon_c, \mu_c)$ be a pair of symmetric, uniformly, matrix-valued functions defined in $D_\gamma := (O_{1, \gamma} \cup O_{2, \gamma}) \setminus (\Omega_2 \setminus \Omega_1)$ and define

$$(\varepsilon_c, \mu_c, \gamma) = \begin{cases} 
(\varepsilon_c, \mu_c) & \text{in } D_\gamma, \\
(\varepsilon_0, \mu_0) & \text{otherwise}.
\end{cases}$$

For all $0 < \alpha < 1$, there exists $\gamma_0$ depending only on $\alpha$, $\Gamma_1$, $\Gamma_2$, $\Omega_j$ for $j = 1, 2, 3$, and $(\varepsilon_0, \mu_0)$ such that for $\gamma \leq \gamma_0$, and for $J \in [L^2(\mathbb{R}^3)]^3$ with $\text{supp } J \subset B_{R_0} \setminus \Omega_3$ for some $R_0 > 0$, we have, for $0 < \delta < 1$,

$$\|(E_{c, \delta}, H_{c, \delta}) - (\vec{E}, \vec{H})\|_{L^2(\mathbb{R}^3 \setminus \Omega_3)} \leq C_R \delta^\alpha \|J\|_{L^2},$$

for some $C_R$ depending only on $\alpha$, $\Gamma_1$, $\Gamma_2$, $\Omega_j$ for $j = 1, 2, 3$, $(\varepsilon_0, \mu_0)$, $\omega$, $R_0$, and $R$. Here $(\vec{E}, \vec{H}), (E_{c, \delta}, H_{c, \delta}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ are respectively the unique radiating solutions of (5.11) and of the following system

$$\begin{align*}
\nabla \times E_{c, \delta} &= i\omega \mu_c H_{c, \delta} & \text{in } \mathbb{R}^3, \\
\nabla \times H_{c, \delta} &= -i\omega \varepsilon_c E_{c, \delta} + J & \text{in } \mathbb{R}^3.
\end{align*}$$

As a consequence of Theorem 5.1, $\lim_{\delta \to 0} (E_{c, \delta}, H_{c, \delta}) = (\vec{E}, \vec{H})$ in $\mathbb{R}^3 \setminus B_{r_2}$ for all $J$ with compact support outside $\Omega_3$ if $\gamma$ is sufficiently small. One, therefore, cannot detect the difference between two media $(\varepsilon_{c, \delta}, \mu_{c, \delta})$ and $(I, I)$ as $\delta \to 0$ by the observation of $(E_{c, \delta}, H_{c, \delta})$ outside $\Omega_3$ using the excitation $J$: cloaking is achieved for observers outside $\Omega_3$ in the limit as $\delta \to 0$. It is worth noting that the constant $\alpha$ and $C_R$ in (5.15) does not depend on the ellipticity and the Lipschitz constant of $(\varepsilon_c, \mu_c)$.

**Remark 5.2.** Given $0 < r_1 < r_2$, set $r_3 = r_2^2/r_1$, $F(x) = r_2^2 x/|x|^2$, $G(x) = r_3^2 x/|x|^2$, $\Omega_j = B_{r_j}$ for $j = 1, 2, 3$, and

$$\Gamma_j = \left\{ x \in \mathbb{R}^3; \text{dist}(x, \Gamma_j) < \gamma \right\} \text{ for } j = 1, 2.$$

Consider $(\varepsilon_0, \mu_0)$ given by (5.9) where $M$ is from (5.8) with $p = 2$. Take $\alpha = 2/3$. Applying Theorem 5.1, one derives the statement on cloaking associated with doubly complementary media mentioned in the introduction.

**Remark 5.3.** It would be very interesting to understand the cloaking property considered in this paper in the time domain for dispersive materials, whose material constants are frequency dependent., see e.g. [47] for a discussion on these materials and their basis properties in the time domain.
Proof of Theorem 5.1. Set
\begin{equation}
\text{Data}(J, \delta) = \frac{1}{\delta} \left| \Im \int_{\mathbb{R}^3} i\omega J \mathcal{E}_{c, \delta} + \Im \mathcal{I}(H_{c, \delta}) \right| + \|J\|_{L^2(\mathbb{R}^3)}^2,
\end{equation}
where \( \Im \) denotes the imaginary part, and
\begin{equation}
\mathcal{I}(H_{c, \delta}) = \lim_{R \to +\infty} \int_{\partial BR} i\omega |H_{c, \delta}|^2.
\end{equation}
By Lemma 5.1 below, we have, for \( \gamma < \gamma_0 \) where \( \gamma_0 \) is positive constant determined later,
\begin{equation}
\|(E_{c, \delta}, H_{c, \delta})\|_{L^2(B_R \setminus \Omega_0)} \leq C R \text{Data}(J, \delta)^{1/2}.
\end{equation}
The starting point of the proof is the use of reflections \( \mathcal{F} \) and \( \mathcal{G} \) from the definition of doubly complementary media. Set
\begin{equation}
(E_{c, 1, \delta}, H_{c, 1, \delta}) = (\mathcal{F} * E_{c, \delta}, \mathcal{F} * H_{c, \delta}) \text{ in } \mathbb{R}^3 \setminus \Omega_2
\end{equation}
and
\begin{equation}
(E_{c, 2, \delta}, H_{c, 2, \delta}) = (\mathcal{G} * E_{c, 1, \delta}, \mathcal{G} * H_{c, 1, \delta}) \text{ in } \Omega_3.
\end{equation}
Set
\[
\Gamma_3 = F(\Gamma_1) \quad \text{and} \quad O_{3, \gamma} = \left\{ x \in \mathbb{R}^3; \text{dist}(x, \Gamma_3) < \gamma \right\}.
\]
It is clear that \( \mathcal{G} \circ \mathcal{F}(O_{1, \gamma} \cap \Omega_1) \subset \Omega_3 \cap O_{3, \lambda} \) for some positive constant \( \lambda \). For notational ease, we will assume that \( \lambda = 1 \). Then, by a change of variables, see e.g. [40, Lemma 7],
\begin{equation}
\begin{cases}
\nabla \times E_{c, 1, \delta} = i\omega \mu^+ H_{c, 1, \delta} - \delta \mathcal{F} * IH_{c, 1, \delta} & \text{in } \Omega_3 \setminus (\bar{\Omega}_2 \cup \bar{O}_{2, \gamma}), \\
\nabla \times H_{c, 1, \delta} = -i\omega \varepsilon^+ E_{c, 1, \delta} + \delta \mathcal{F} * IE_{c, 1, \delta} & \text{in } \Omega_3 \setminus (\bar{\Omega}_2 \cup \bar{O}_{2, \gamma}),
\end{cases}
\end{equation}
\begin{equation}
\begin{cases}
\nabla \times E_{c, 2, \delta} = i\omega \tilde{\mu} H_{c, 2, \delta} & \text{in } \bar{O}_{3, \gamma}, \\
\nabla \times H_{c, 2, \delta} = -i\omega \tilde{\varepsilon} H_{c, 2, \delta} & \text{in } \bar{O}_{3, \gamma},
\end{cases}
\end{equation}
\begin{equation}
E_{c, 1, \delta} \times \nu - E_{c, \delta} \times \nu = H_{c, 1, \delta} \times \nu - H_{c, \delta} \times \nu = 0 \text{ on } \partial \Omega_2,
\end{equation}
\begin{equation}
E_{c, 2, \delta} \times \nu - E_{c, \delta} \times \nu = H_{c, 2, \delta} \times \nu - H_{c, \delta} \times \nu = 0 \text{ on } \partial \Omega_3.
\end{equation}
Set \( \beta = (\alpha + 2)/3 \). Since \( (\varepsilon, \mu) = (\varepsilon^+, \mu^+) \) in \( \Omega_3 \setminus \Omega_2 \) thanks to the property of doubly complementary media, by applying Corollary 4.2 to \( (E_{c, 1, \delta} - E_{c, \delta}, H_{c, 1, \delta} - H_{c, \delta}) \) in \( \Omega_3 \setminus \Omega_2 \) with \( \Sigma = \partial \Omega_2 \) and to \( (E_{c, 2, \delta} - E_{c, 1, \delta}, H_{c, 2, \delta} - H_{c, 1, \delta}) \) in \( \Omega_3 \setminus \Omega_2 \) with \( \Sigma = \partial \Omega_3 \), there exist \( 0 < \gamma_1 < \gamma_2 \leq \gamma_2/2 \), depending only on \( \alpha \), \( (\varepsilon_0, \mu_0) \), \( \Omega_1 \), \( \Omega_2 \), \( \Gamma_1 \), \( \Gamma_2 \), and \( \omega \) such that for \( \gamma \in (\gamma_1, \gamma_2) \) with \( \gamma_1 = \gamma_2/2 \), we have, with \( O_{2, \gamma} = \Omega_2 \cup O_{2, \gamma} \),
\begin{equation}
\|(E_{c, 1, \delta} \times \nu, H_{c, 1, \delta} \times \nu) - (E_{c, \delta} \times \nu, H_{c, \delta} \times \nu)\|_{H^{-1/2}(\text{div}_\Gamma, \partial O_{2, \gamma})} \leq C \delta^\beta \text{Data}(J, \delta)^{1/2}
\end{equation}
and, with \( O_{3, \gamma} = \Omega_3 \setminus O_{3, \gamma} \),
\begin{equation}
\|(E_{c, 2, \delta} \times \nu, H_{c, 2, \delta} \times \nu) - (E_{c, 1, \delta} \times \nu, H_{c, 1, \delta} \times \nu)\|_{H^{-1/2}(\text{div}_\gamma, \partial O_{3, \gamma})} \leq C \delta^\beta \text{Data}(J, \delta)^{1/2}.
\end{equation}
Fix \( \gamma = (\gamma_1 + \gamma_2)/2 \), and set
\[
D = (\Omega_3 \setminus \Omega_2) \setminus (O_{3, \gamma} \cup O_{2, \gamma}) = (\Omega_3 \setminus \Omega_2) \setminus (O_{3, \gamma} \cup O_{2, \gamma}).
\]
A simple but important part of the proof is the introduction of \((\tilde{E}_\delta, \tilde{H}_\delta) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \partial D)]^2\), known as the removing localized singularity, as follows:

\[
(\tilde{E}_\delta, \tilde{H}_\delta) = \begin{cases} 
(E_{c,\delta}, H_{c,\delta}) - \left[ (E_{c,1,\delta}, H_{c,1,\delta}) \right] & \text{in } D, \\
(E_{c,2,\delta}, H_{c,2,\delta}) & \text{in } \Omega_2 \cup O_{2,\gamma}, \\
(E_{c,\delta}, H_{c,\delta}) & \text{otherwise.}
\end{cases}
\]

It follows from (5.21), (5.22), the definition of (\(\tilde{\varepsilon}, \tilde{\mu}\)) in (5.10), and the property of doubly complementary media that \((\tilde{E}_\delta, \tilde{H}_\delta) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2\) is a radiating solution of the system

\[
\begin{align*}
\nabla \times \tilde{E}_\delta &= i \omega \tilde{\mu} \tilde{H}_\delta + \delta \omega \mathbb{1}_D \mathcal{F} \times I H_{c,1,\delta} \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\
\nabla \times \tilde{H}_\delta &= -i \omega \tilde{\varepsilon} \tilde{E}_\delta - \delta \omega \mathbb{1}_D \mathcal{F} \times I E_{c,1,\delta} + J \quad \text{in } \mathbb{R}^3 \setminus \partial D.
\end{align*}
\]

We derive from the definition of \((\tilde{E}, \tilde{H})\) in (5.11) that \((\tilde{E}_\delta - \tilde{E}, \tilde{H}_\delta - \tilde{H}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \partial D)]^2\) is radiating and satisfies

\[
\begin{align*}
\nabla \times (\tilde{E}_\delta - \tilde{E}) &= i \omega \tilde{\mu} (\tilde{H}_\delta - \tilde{H}) + \delta \omega \mathbb{1}_D \mathcal{F} \times I H_{c,1,\delta} \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\
\nabla \times (\tilde{H}_\delta - \tilde{H}) &= -i \omega \tilde{\varepsilon} (\tilde{E}_\delta - \tilde{E}) - \delta \omega \mathbb{1}_D \mathcal{F} \times I E_{c,1,\delta} \quad \text{in } \mathbb{R}^3 \setminus \partial D.
\end{align*}
\]

Since \(\tilde{\varepsilon}\) and \(\tilde{\mu}\) are uniformly elliptic, we deduce from (5.25) and (5.26) that

\[
(5.28) \quad \| (\tilde{E}_\delta - \tilde{E}, \tilde{H}_\delta - \tilde{H}) \|_{L^2(B_R)} \leq C_R \delta^\beta \text{Data}(J, \delta)^{1/2}.
\]

Since \(\beta > 1/2\) and \(\text{Data}(J, \delta) \leq C \delta^{-1} \| J \|_{L^2(\mathbb{R}^3)}^2\), it follows that

\[
(5.29) \quad \| (E_{c,\delta}, H_{c,\delta}) \|_{L^2(B_{R_0})} \leq C \| J \|_{L^2(\mathbb{R}^3)},
\]

and

\[
(5.30) \quad \| (\tilde{E}_\delta - \tilde{E}, \tilde{H}_\delta - \tilde{H}) \|_{L^2(B_R)} \leq C_R \delta^{\beta - 1/2} \| J \|_{L^2(\mathbb{R}^3)}.
\]

Integrating by parts for \((\tilde{E}, \tilde{H})\) in \(B_R\), letting \(R \to +\infty\), and using the radiation condition give

\[
\Im \int_{\mathbb{R}^3} i \omega J \overline{\tilde{E}} + \Im I(\tilde{H}) = 0.
\]

It follows that

\[
\left| \Im \int_{\mathbb{R}^3} i \omega J E_{c,\delta} + \Im I(H_{c,\delta}) \right| = \left| \Im \int_{\mathbb{R}^3} i \omega J E_{c,\delta} + \Im I(H_{c,\delta}) - \Im \int_{\mathbb{R}^3} i \omega J \tilde{E} - \Im I(\tilde{H}) \right| \leq C \| J \|_{L^2} \| (E_{c,\delta} - \tilde{E}) \|_{L^2(B_{R_0})}.
\]

This implies, by (5.16) and (5.30),

\[
\text{Data}(J, \delta) \leq C \delta^{\beta - 1/2 - 1} \| J \|_{L^2(\mathbb{R}^3)}^2.
\]

Repeating this process, one reaches (see [44, Proof of Theorem 3.1] for related arguments), for \(\ell \geq 1\),

\[
\text{Data}(J, \delta) \leq C \delta^{\beta - (1/2 + 1/2 + \cdots + 1/2^{\ell - 1})} \| J \|_{L^2}^2
\]

and

\[
\| (\tilde{E}_\delta - \tilde{E}, \tilde{H}_\delta - \tilde{H}) \|_{L^2(B_R)} \leq C \delta^{\beta - (1/2 + 1/2 + \cdots + 1/2^{\ell + 1})} \| J \|_{L^2}.
\]

The conclusion follows by taking \(\ell\) sufficiently large. \(\square\)
Remark 5.4. The removing localized singularity technique is inspired by the idea of renormalizing energy in the theory of the Ginzburg-Landau equation [8]. The gluing argument was first suggested in [34].

The following lemma was used in the proof of Theorem 5.1.

Lemma 5.1. Under the assumption of Theorem 5.1, we have, for $\gamma < \hat{\gamma}$,

$$
\| (E_{c,\delta}, H_{c,\delta}) \|_{L^2(B_R \setminus D_\gamma)} \leq C_R \text{Data}(J, \delta),
$$

where $\text{Data}(J, \delta)$ is defined by (5.16). Here $C_R$ denotes a positive constant depending only on $\hat{\gamma}$, $R$, $R_0$, $\varepsilon$, $\mu$, $\Gamma_1$, $\Gamma_2$, $\Omega_1$, and $\Omega_2$.

Proof. The proof of this lemma is quite simple as follows. We have, in $\mathbb{R}^3$,

$$
\nabla \times (\mu_{c,\delta}^{-1} \nabla \times E_{c,\delta}) - \omega^2 \varepsilon_{c,\delta} E_{c,\delta} = i \omega J.
$$

Multiplying the equation by $\hat{\varepsilon}_{c,\delta}$, integrating in $B_R$, and using the fact $\text{supp } J \subset B_{R_0} \setminus \Omega_2$, we obtain, for $R > R_0$,

$$
\int_{B_R} \langle \mu_{\delta}^{-1} \nabla \times E_{c,\delta}, \nabla \times E_{c,\delta} \rangle + \int_{\partial B_R} \langle i \omega H_{c,\delta}, E_{c,\delta} \times \nu \rangle - \omega^2 \int_{B_R} \langle \varepsilon_{\delta} E_{c,\delta}, E_{c,\delta} \rangle = \int_{B_R} \langle i \omega J, E_{c,\delta} \rangle.
$$

Letting $R \to +\infty$, using the radiation condition, and considering the imaginary part, we get

$$
\| (E_{c,\delta}, H_{c,\delta}) \|_{L^2(\Omega_2 \setminus \Omega_1)}^2 \leq C \text{Data}(J, \delta).
$$

It follows from the trace theory that

$$
\| E_{c,\delta} \times \nu \|_{H^{-1/2}(\partial \Omega_1 \cup \partial \Omega_2)} + \| H_{c,\delta} \times \nu \|_{H^{-1/2}(\partial \Omega_1 \cup \partial \Omega_2)} \leq C \text{Data}(J, \delta).
$$

A compactness argument involving the unique continuation principle gives, (see e.g. the proof of [40, Lemma 3] for similar arguments), one has

$$
\| (E_{c,\delta}, H_{c,\delta}) \|_{L^2(B_R \setminus D_\gamma)}^2 \leq C_R \text{Data}(J, \delta).
$$

The proof is complete. \hfill \Box

Remark 5.5. In the proof Lemma 5.1, the complementary property of $(\varepsilon_0, \mu_0)$ is not required.

6. ON SUPERLENSEING AND CLOAKING USING COMPLEMENTARY MEDIA

In this section, we discuss the lensing and cloaking designs using complementary media given in [40, 41]. We show on one hand that it is necessary to impose additional conditions on the schemes proposed in some physics works. On the other hand, we discuss various contexts where a lens can act like a cloak and conversely.

6.1. Superlensing using complementary media. In this section, we analyse the lensing construction given in [40] motivated from [36, 51, 54]. To magnify $m$-times the region $B_{r_0}$ of material parameters $(\varepsilon_O, \mu_O)$ (a pair of uniformly elliptic symmetric matrix-valued functions), for some $r_0 > 0$ and $m > 1$, we proposed to use two layers. One layer makes use of complementary media concept

$$
(F_s^{-1} I, F_s^{-1} I) \text{ in } B_{r_2} \setminus B_{r_1},
$$

and the other layer is given by

$$
(mI, mI) \text{ in } B_{r_1} \setminus B_{r_0}.
$$

Here $F$ is the Kelvin transform with respect to $\partial B_{r_2}$, and $r_1$ and $r_2$ are required to satisfy $r_1 \geq m^{1/2} r_0$ and $r_2 = m r_0$. The construction in [40] is for the case where $r_1 = m^{1/2} r_0$, nevertheless,
the case $r_1 \geq m^{1/2}r_0$ is its direct consequence. Other choices for the first and the second layers are possible via the concept of complementary media and were analyzed there.

Set

\[
(\varepsilon_\delta, \mu_\delta) = \begin{cases} 
(I, I) & \text{in } \mathbb{R}^3 \setminus B_{mr_0}, \\
(mI, mI) & \text{in } B_{r_1} \setminus B_{r_0}, \\
(F_*^{-1}I, F_*^{-1}I) & \text{in } B_{r_2} \setminus B_{r_1}, \\
(\varepsilon_0, \mu_0) & \text{in } B_{r_0},
\end{cases}
\]

and

\[
(\tilde{\varepsilon}, \tilde{\mu}) = \begin{cases} 
(I, I) & \text{in } \mathbb{R}^3 \setminus B_{mr_0}, \\
(m^{-1}\varepsilon_0(x/m), m^{-1}\mu_0(x/m)) & \text{otherwise}.
\end{cases}
\]

Assume that with $M = \varepsilon_0$ or $\mu_0$,

\begin{equation}
\Lambda^{-1}|\xi|^2 \leq \langle M(x)\xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \text{ for some } \Lambda \geq 1.
\end{equation}

Given $J \in [L^2(\mathbb{R}^3)]^3$ with compact support outside $B_{r_3}$ with $r_3 = r_2/r_1$, let $(E_\delta, H_\delta), (\hat{E}, \hat{H}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ be, respectively, the unique radiating solution to

\begin{equation}
\begin{cases} 
\nabla \times E_\delta = i\omega \mu_\delta H_\delta & \text{in } \mathbb{R}^3, \\
\nabla \times H_\delta = -i\omega \varepsilon_\delta E_\delta + J & \text{in } \mathbb{R}^3,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases} 
\nabla \times \hat{E} = i\omega \hat{\mu}\hat{H} & \text{in } \mathbb{R}^3, \\
\nabla \times \hat{H} = -i\omega \hat{\varepsilon}\hat{E} + J & \text{in } \mathbb{R}^3.
\end{cases}
\end{equation}

We showed, as $\delta \to 0$, that [40, Theorem 1]

\[(E_\delta, H_\delta) \to (\hat{E}, \hat{H}) \text{ in } \mathbb{R}^3 \setminus B_{r_3}.
\]

For an observer outside $B_{r_3}$, measuring $(E_\delta, H_\delta)$ using the excitation $J$ gives the same results as measuring $(\hat{E}, \hat{H})$ using the same excitation. The object in $B_{r_0}$ is magnified $m$-times for such an observer.

The second layer can be chosen thinner using the technique in [36]; nevertheless, the second layer is necessary. This is derived from the following consequence of Theorem 5.1:

**Proposition 6.1.** Assume that $(\varepsilon_0, \mu_0)$ is doubly complementary with $\mathcal{F}$ and $\mathcal{G}$ being the Kelvin transforms with respect to $\partial B_{r_2}$ and $\partial B_{r_3}$ with $r_3 = r_2^2/r_1$. Let $\Gamma_1$ be a compact smooth curve on $\partial B_{r_1}$. Set, for $\gamma > 0$,

\[O_{1,\gamma} := \{x \in \mathbb{R}^3; \text{dist}(x, \Gamma_1) < \gamma\}.
\]

For $\gamma > 0$, let $(\varepsilon_c, \mu_c)$ be a pair of symmetric matrix-valued functions defined in $D_\gamma := O_{1,\gamma} \cap B_{r_1}$ and define

\[(\varepsilon_{c,\delta}, \mu_{c,\delta}) = \begin{cases} 
(\varepsilon_c, \mu_c) & \text{in } D_\gamma, \\
(\varepsilon_\delta, \mu_\delta) & \text{otherwise}.
\end{cases}
\]

Let $0 < \delta < 1$, $J \in [L^2(\mathbb{R}^3)]^3$ with supp $J \subset B_{r_0} \setminus B_{r_3}$, and let $(E_{c,\delta}, H_{c,\delta}) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ be the unique radiating solution of the Maxwell equations

\begin{equation}
\begin{cases} 
\nabla \times E_{c,\delta} = i\omega \mu_{c,\delta} H_{c,\delta} & \text{in } \mathbb{R}^3, \\
\nabla \times H_{c,\delta} = -i\omega \varepsilon_{c,\delta} E_{c,\delta} + J & \text{in } \mathbb{R}^3.
\end{cases}
\end{equation}
For all $0 < \alpha < 1$, there exists $\gamma_0 > 0$ depending only on $\alpha$, $\Gamma_1$, $r_1$, and $r_2$ such that for $\gamma \in (0, \gamma_0)$, we have
\[ \| (E_{\gamma}, H_{\gamma}) - (\hat{E}, \hat{H}) \|_{L^2(B_R \setminus \Omega_3)} \leq C_R \delta \| J \|_{L^2}, \]
for some positive constant $C_R$ depending only on $\alpha$, $\Gamma_1$, $r_1$, and $r_2$, $\Lambda$, $R_0$, and $R$.

As a consequence of Proposition 6.2, an object inside $B_{r_1}$ located near the layer $B_{r_2} \setminus B_{r_1}$ is cloaked; the second layer in the lensing construction is hence necessary to achieve superlensing.

6.2. Cloaking using complementary media. In this section, we analyse the construction of the cloaking device in [41] motivated from [28, 37]. Assume that the cloaked region is the annulus $B_{r_2} \setminus B_{r_1}$ in $\mathbb{R}^3$ for some $r_2 > 0$ in which the medium is characterized by $(\varepsilon_O, \mu_O)$ (a pair of uniformly elliptic symmetric matrix-valued functions). The cloaking device proposed in [41] then contains two parts. The first one, in $B_{r_2} \setminus B_{r_1}$, makes use of complementary media to cancel the effect of the cloaked region and the second one, in $B_{r_1}$, is to fill the space which “disappears” from the cancellation by the homogeneous medium. Concerning the first part, instead of (6.4)
\[ (\varepsilon_O, \mu_O) = \left\{ \begin{array}{ll} (\varepsilon_O, \mu_O) & \text{in } B_{r_2} \setminus B_{r_1}, \\
(I, I) & \text{in } B_{r_3} \setminus B_{r_2}. \end{array} \right. \]
The complementary medium in $B_{r_2} \setminus B_{r_1}$ is then given by
\[ (\mathcal{F}^{-1}_* \varepsilon_O, \mathcal{F}^{-1}_* \mu_O), \]
where $\mathcal{F}$ is the Kelvin transform with respect to $\partial B_{r_2}$. Concerning the second part, the medium in $B_{r_1}$ with $r_1 = r_2^3/r_3$ is given by, with $m = r_2^3/r_3 = r_2^2/r_3^2$,
\[ (mI, mI). \]
Set
\[ (\varepsilon_\delta, \mu_\delta) = \left\{ \begin{array}{ll} (\varepsilon_O, \mu_O) & \text{in } B_{r_3} \setminus B_{r_2}, \\
(F^{-1}_* \varepsilon_O + i\delta I, F^{-1}_* \mu_O + i\delta I) & \text{in } B_{r_2} \setminus B_{r_1}, \\
(mI, mI) & \text{in } B_{r_1}, \\
(I, I) & \text{in } \mathbb{R}^3 \setminus B_{r_3}. \end{array} \right. \]
Given $J \in [L^2(\mathbb{R}^3)]^3$ with compact support outside $B_{r_3}$, let $(E_\delta, H_\delta), (E, H) \in [H_{loc}(\text{curl}, \mathbb{R}^3)]^2$ be respectively the unique outgoing solutions to the Maxwell systems
\[ \begin{cases} \nabla \times E_\delta = i\omega \mu_\delta H_\delta & \text{in } \mathbb{R}^3, \\ \nabla \times H_\delta = -i\omega \varepsilon_\delta E_\delta + J & \text{in } \mathbb{R}^3, \end{cases} \]
and
\[ \begin{cases} \nabla \times E = i\omega H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega E + J & \text{in } \mathbb{R}^3. \end{cases} \]
Assume that
\[ (\varepsilon_O, \mu_O) \) is $C^2$
and $r_3/r_2$ is large enough. We have [41, Theorem 1.1]
\[ (E_\delta, H_\delta) \to (E, H) \text{ in } \mathbb{R}^3 \setminus B_{r_3} \text{ as } \delta \to 0. \]
The object in $B_{2r_2} \setminus B_{r_2}$ is cloaked.

We next show that the largeness condition on $r_3/r_2$ is necessary. More precisely, we have

**Proposition 6.2.** Let $\Gamma_3$ be a compact smooth curve on $\partial B_{r_3}$. Set, for $\gamma > 0$,

$$O_{3,\gamma} := \{x \in \mathbb{R}^3; \text{dist}(x, \Gamma_3) < \gamma\}.$$

Define, in $B_{r_3} \setminus B_{r_2}$,

$$(6.11) \quad \left(\varepsilon_O, \mu_O\right) = \begin{cases} \left(\varepsilon_O, \mu_O\right) & \text{in } (B_{r_3} \setminus B_{r_2}) \cap O_{3,\gamma}, \\ (I, I) & \text{in } (B_{r_3} \setminus B_{r_2}) \setminus O_{3,\gamma}. \end{cases}$$

Let $J \in [L^2(\mathbb{R}^3)]^3$ with compact support in $B_{R_0} \setminus B_{r_3}$ for some $R_0 > r_3$ and let $(E_\delta, H_\delta)$ be the unique radiating solution of (6.8) in which $(\varepsilon_\delta, \mu_\delta)$ is given in (6.7) with $(\varepsilon_O, \mu_O)$ defined in (6.11). Assume that with $M = \varepsilon_O$ or $\mu_O$,

$$(6.12) \quad \Lambda^{-1} |\xi|^2 \leq \langle M(x) \xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d$$

for some positive constant $\Lambda \geq 1$.

For all $0 < \alpha < 1$, there exists $\gamma_0$ depending only on $\alpha$, $r_1$, $r_2$, and $\Gamma_3$ such that for $\gamma \leq \gamma_0$, we have

$$\| (E_\delta, H_\delta) - (\hat{E}, \hat{H}) \|_{L^2(B_{2r_2} \setminus B_{r_2})} \leq C_{R, \alpha} \| J \|_{L^2},$$

for some positive constant $C_R$ depending only on $\alpha$, $\Lambda$, $(\varepsilon_0, \mu_0)$, $r_2$, $r_3$, $\Gamma_3$, $R_0$, and $R$. Here $(\hat{E}, \hat{H})$ is the unique radiating solution of the equation

$$\begin{cases} \nabla \times \hat{E} = i \omega \hat{\mu} \hat{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{H} = -i \omega \hat{\varepsilon} \hat{E} + J & \text{in } \mathbb{R}^3, \end{cases}$$

where $(\hat{\varepsilon}, \hat{\mu}) = \begin{cases} (\varepsilon_O, \mu_O) & \text{in } B_{r_3} \cap O_{3,\gamma}, \\ (I, I) & \text{otherwise}. \end{cases}$

As a consequence of Proposition 6.2, the object $(\varepsilon_c, \mu_c)$ in $B(x_3, r_0) \cap B_{r_3}$ does not disappear: cloaking is not achieved.

**Proof.** The proof of Proposition 6.2 is almost the same as the one of Theorem 5.1. Using the notations in the proof of Theorem 5.1 with $(E_{c,\delta}, H_{c,\delta}) = (E_\delta, H_\delta)$, $(\varepsilon_{c,\delta}, \mu_{c,\delta}) = (\varepsilon_\delta, \mu_\delta)$, $\Omega_j = B_{r_3}$ for $j = 1, 2, 3$, and the convention $\Gamma_1 = \emptyset$ and $\Omega_{1,\gamma} = \emptyset$, one just needs to observe that $(\hat{E}_\delta - \hat{E}, \hat{H}_\delta - \hat{H}) \in [H_\text{loc}(\text{curl, } \mathbb{R}^3)]^2$ is radiating and satisfies

$$\begin{cases} \nabla \times (\hat{E}_\delta - \hat{E}) = i \omega \hat{\mu} (\hat{H}_\delta - \hat{H}) - \delta \omega \mathbb{1}_{B_{r_3} \setminus B_{r_2}} \mathbb{F} \mathbb{I} H_{c,1,\delta} & \text{in } \mathbb{R}^3 \setminus (\partial O_{3,\gamma_1} \cap B_{r_3}), \\ \nabla \times (\hat{H}_\delta - \hat{H}) = -i \omega \hat{\varepsilon} (\hat{E}_\delta - \hat{E}) + \delta \omega \mathbb{1}_{B_{r_3} \setminus B_{r_2}} \mathbb{F} \mathbb{I} E_{c,1,\delta} & \text{in } \mathbb{R}^3 \setminus (\partial O_{3,\gamma_1} \cap B_{r_3}). \end{cases}$$

The conclusion then follows as in the proof of Theorem 5.1. 

**APPENDIX A. PROOF OF LEMMA 4.2**

Before giving the proof of Lemma 4.2, we recall some properties of the spherical Bessel and Neumann functions and the Bessel and Neumann functions of large order. We first introduce, for $n \geq 1$,

$$\begin{aligned} (A1) \quad \hat{j}_n(t) &= 1 \cdot 3 \cdots (2n + 1) j_n(t) \quad \text{and} \quad \hat{y}_n = -\frac{y_n(t)}{1 \cdot 3 \cdots (2n - 1)}, \\
(A2) \quad \hat{j}_n(r) &= r^n \left[1 + O(1/n)\right] \quad \hat{y}_n(r) = r^{-n-1} \left[1 + O(1/n)\right]. \end{aligned}$$

where $j_n$ and $y_n$ are the spherical Bessel and Neumann functions. Then, see, e.g. [11, (2.37) and (2.38)], as $n \to +\infty$,
One also has, see, e.g. [11, (2.36) and (3.56)],

(A3) \[ j_n(r)g'_n(r) - j'_n(r)g_n(r) = \frac{1}{r^2}. \]

In what follows, for \(-n \leq m \leq n, n \in \mathbb{N}\), denote \(Y^m_n\) the spherical harmonic function of order \(n\) and degree \(m\) and set

\[ U^m_n(\hat{x}) := \nabla_{\partial B_1}Y^m_n(\hat{x}) \quad \text{and} \quad V^m_n(\hat{x}) := \hat{x} \times U^m_n(\hat{x}) \quad \text{for} \quad \hat{x} \in \partial B_1. \]

We recall that \(Y^m_n(\hat{x}), U^m_n(\hat{x}), \) and \(V^m_n(\hat{x})\) for \(-n \leq m \leq n, n \in \mathbb{N}\) form an orthonormal basis of \([L^2(\partial B_1)]^3\).

**Proof of Lemma 4.2.** Without loss of generality, one may assume that \(\omega = 1\). One then can represent \(E, H\) in \(B_{R_3} \setminus B_{R_1}\) as follows, see, e.g. [24], with \(r = |x|\) and \(\hat{x} = x/|x|\),

\[
E(x) = \sum_{n=1}^{\infty} \sum_{|m| \leq n} \sqrt{n(n+1)} \frac{\alpha^m_{1,n} \hat{j}_n(r) + \alpha^m_{2,n} \hat{g}_n(r)}{r} Y^m_n(\hat{x}) \hat{x} \\
+ \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left( \frac{r[\alpha^m_{1,n} \hat{j}_n(r) + \alpha^m_{2,n} \hat{g}_n(r)]}{r} \right)' U^m_n(\hat{x}) \\
+ \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \beta^m_{1,n} \hat{j}_n(r) + \beta^m_{2,n} \hat{g}_n(r) \right] V^m_n(\hat{x})
\]

and

\[
H(x) = i \sum_{n=1}^{\infty} \sum_{|m| \leq n} \sqrt{n(n+1)} \frac{\beta^m_{1,n} \hat{j}_n(r) + \beta^m_{2,n} \hat{g}_n(r)}{r} Y^m_n(\hat{x}) \hat{x} \\
+ i \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left( \frac{r[\beta^m_{1,n} \hat{j}_n(r) + \beta^m_{2,n} \hat{g}_n(r)]}{r} \right)' U^m_n(\hat{x}) \\
+ i \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \alpha^m_{1,n} \hat{j}_n(r) + \alpha^m_{2,n} \hat{g}_n(r) \right] V^m_n(\hat{x}).
\]

One can then check that

\[
\|(E \times \nu, H \times \nu)\|_{H^{-1/2}({\text{div}} B_{r})} \lesssim \sum_{n=1}^{\infty} \sum_{|m| \leq n} \sum_{j=1}^{2} n^3 (|\alpha^m_{j,n}|^2 + |\beta^m_{j,n}|^2) r^{2n}.
\]

The conclusion now follows from the interpolation. \(\square\)

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