Characterization of $k$-spectrally monomorphic Hermitian matrices

Kawtar Attas, Abderrahim Boussaïri, Imane Souktani

July 28, 2021

Abstract

This paper solves the following problem about Hermitian matrices related to the theory of 2-structures: Let $n$ be a positive integer and $k$ be an integer with $k \in \{3, \ldots, n-3\}$. Characterize the Hermitian matrices $A$ such that the characteristic polynomials of the $k \times k$ submatrices of $A$ are all equal. Such matrices are called $k$-spectrally monomorphic. A crucial step to obtain this characterization is proving that if a matrix $A$ is $k$-spectrally monomorphic then it is $l$-spectrally monomorphic for $l \in \{1, \ldots, \min\{k, n-k\}\}$.

Keywords: Hermitian matrices; spectral monomorphy; conference matrices.
MSC Classification: 15B57; 05C50.

1 Introduction

Let $n$ be a positive integer. An $n \times n$ matrix $A$ is $k$-spectrally monomorphic if all its $k \times k$ submatrices have the same characteristic polynomial. Spectral monomorphy is closely related to the notion of monomorphy introduced by Fraïssé [1]. We will define this notion for labeled 2-structure. Following [2], a labeled 2-structure on a set $V$, or shortly an $l2$-structure, is a map $g$ from the set $\{(x, y) : x \neq y \in V\}$ to a label set $C$. The elements of $V$ are called the vertices of $g$. With each subset $X$ of $V$, we associate the $l2$-substructure $g[X]$ of $g$, induced by $X$, defined by $g[X](x, y) := g(x, y)$ for any $x \neq y \in X$. $l2$-structures were introduced to generalize the notion of graphs, tournaments and other binary structures. Recall that an $n$-tournament $T$ is a digraph with $n$ vertices in which every pair of distinct vertices is joined by exactly one arc. If the arc joining vertices $u$ and $v$ of $T$ is directed from $u$ to $v$, then $u$ is said to dominate $v$ (symbolically $u \rightarrow v$). For more details about tournaments, we refer the reader to [3].

Let $g$ and $h$ be $l2$-structures with the same label set and whose vertex sets are, respectively, $V$ and $W$. We say that $g$ and $h$ are isomorphic if there exists a bijection $\sigma$ from $V$ onto $W$ such that $g(x, y) = h(\sigma(x), \sigma(y))$ for any $x \neq y \in V$. An $l2$-structure is $k$-monomorphic if all its substructures on $k$ vertices are isomorphic.
Some properties of $k$-spectral monomorphy

Several results about monomorphic relations were obtained by Assous [4], Frasnay [5], and Pouzet [6,7]. A basic example is the class of transitive tournaments. A tournament $T$ is transitive if, whenever $x \rightarrow y$ and $y \rightarrow z$, then $x \rightarrow z$. The smallest non-transitive tournament consists of 3 vertices $x$, $y$ and $z$ such that $x \rightarrow y \rightarrow z \rightarrow x$. Such a tournament is called a 3-cycle. It is not difficult to see that 3-monomorphic tournaments with at least 4 vertices are transitive. Moreover, it follows from a combinatorial lemma of Pouzet [6] that $k$-monomorphic $n$-tournaments, for $k = 3, \ldots, n-3$, are 3-monomorphic, and hence are transitive. For $k = n-2$, an $n$-tournament whose automorphism group acts transitively on the set of its arcs is $(n-2)$-monomorph. These tournaments are called arc-symmetric and were characterized independently by Kantor [8] and Berggren [9]. Conversely, Jean [10] proved earlier that $(n-2)$-monomorphic $n$-tournaments with at least 5 vertices are either transitive or arc-symmetric. The problem of the characterization of $(n-1)$-monomorphic $n$-tournaments proposed by Kotzig (see [11], problem 43, p. 252) remains unsolved. Some progress was made by Yucai et al. [12] and Issawi [13]. In [14], Boudabbous proposed a weak notion of monomorphy for tournaments using "isomporphy up to complementation" instead of "isomporphy". An analogous study for graphs was done by Boushabi and Boussaïri [15].

Let $g$ be a complex $l_2$-structure with $n$ vertices, that is, its label set is the complex field. With respect to an ordering $x_1, \ldots, x_n$ of the vertex set, we can identify $g$ to the $n \times n$ zero diagonal matrix $M = [m_{ij}]$ in which $m_{ij} = g(x_i, x_j)$ if $i \neq j$. This defines a one-to-one correspondence between $n \times n$ zero-diagonal complex matrices, and $l_2$-structures with vertex set $\{x_1, \ldots, x_n\}$. Note that a complex $l_2$-structure is $k$-monomorphic if all the $k \times k$ submatrices of its corresponding matrix are permutationally similar. Hence, $k$-spectral monomorphy for zero-diagonal complex matrices is a weakening of $k$-monomorphy for $l_2$-structures.

In this paper, we address the following problem.

Problem 1.1. Characterize $k$-spectrally monomorphic $n \times n$ matrices, where $n$ is a positive integer and $k \in \{3, \ldots, n-1\}$.

We restrict ourselves to Hermitian matrices. The class of $l_2$-structures corresponding to Hermitian matrices encompass many well-known classes like graphs, tournaments, and digraphs.

2 Some properties of $k$-spectral monomorphy

We will give some operations that preserve hermiticity and $k$-spectral monomorphy. Let $H$ be an $n \times n$ Hermitian matrix. The characteristic polynomial of $H$ is $\phi_H(x) = \det(xI-H)$. For a subset $\alpha$ of $\{1, \ldots, n\}$, we denote by $H[\alpha]$ the principal submatrix of $H$ whose rows and columns are indexed by $\alpha$.

- Let $a$ be a real number. Then
  $$\phi_{(H-aI)[\alpha]}(x) = \phi_{H[\alpha]}(x+a)$$ (1)

- Let $P = (p_{ij})$ be an $n \times n$ permutation matrix and let $\sigma$ be the corresponding permutation, that is $p_{ij} = 1$ if and only if $\sigma(i) = j$. For every subset $\alpha$ of $\{1, \ldots, n\}$,
we have \( PHP^t[\alpha] = H[\sigma^{-1}(\alpha)] \), and hence
\[
\phi_{PHP^t[\alpha]}(x) = \phi_{H[\sigma^{-1}(\alpha)]}(x)
\] (2)

- Let \( D \) be a diagonal matrix whose diagonal entries have modulus 1. Then, for every subset \( \alpha \) of \( \{1, \ldots, n\} \), we have
\[
\phi_{DH^\alpha HD[\alpha]}(x) = \phi_{H[\alpha]}(x)
\] (3)

- Let \( b \) be a non-zero real number. Then
\[
\phi_{bH[\alpha]}(x) = b^{[\alpha]} \phi_{H[\alpha]}(\frac{x}{b})
\] (4)

Let \( \Gamma \) be the subgroup of the general linear group, generated by the unitary diagonal matrices and the permutation matrices. Two Hermitian matrices \( H_1 \) and \( H_2 \) are \( \Gamma \)-equivalent if \( H_2 = a(SH_1S^*) + bI \) for some real numbers \( a \) and \( b \), and \( S \) in \( \Gamma \). This defines an equivalence relation between \( n \times n \) Hermitian matrices. It follows from the above equalities that this relation preserves \( k \)-spectral monomorphy.

Problem 1.1 is trivial for \( k = 1 \) and for \( k = 2 \). Indeed, if \( H \) is a \( n \times n \) Hermitian matrix with \( n \geq 3 \), then

1) \( H \) is 1-spectrally monomorphic if and only if all its diagonal entries are equal.

2) \( H \) is 2-spectrally monomorphic if and only if it is 1-spectrally monomorphic and all of its off-diagonal entries have the same modulus.

We say that a matrix \( A = (a_{ij}) \) is normalized if \( a_{i1} = a_{1i} = 1 \) for \( i \neq 1 \).

**Remark 2.1.** Let \( H \) be an \( n \times n \) Hermitian matrix with a non-zero off-diagonal entry. If \( H \) is 2-spectrally monomorphic, then \( H \) is \( \Gamma \)-equivalent to a normalized zero-diagonal Hermitian matrix.

A fundamental property of \( k \)-spectral monomorphy is given in the following proposition.

**Proposition 2.2.** If \( A \) is a \( k \)-spectrally monomorphic complex matrix, then \( A \) is \( l \)-spectrally monomorphic for each \( l \in \{1, \ldots, \min(k, n-k)\} \).

To prove this proposition, we will apply the following result which is a consequence of [6, Lemma II-2.2].

**Lemma 2.3.** Let \( V \) be a set of size \( n \). Let \( p \) and \( r \) be two arbitrary integers satisfying \( n \geq p + r \) and let \( f \) be a map from \( \binom{V}{p} \) to the complex field \( \mathbb{C} \). If \( \sum_{P \subseteq B} f(P) \) is independent of \( B \), where \( B \in \binom{V}{p+r} \), then for any subset \( X \) of \( V \) such that \( |X| \leq n - (p + r) \), the number \( \sum_{P \supseteq X} f(P) \) depends only on the cardinality of \( X \). Moreover, if \( n \geq 2p + r \), then \( f \) is a constant map.

We need also the following lemma (see for example [16, p.294]).
Lemma 2.4. Let $A$ be an $n \times n$ matrix with characteristic polynomial $\phi(x) := x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$. Then

$$a_p = (-1)^p \sum_{\alpha \in \binom{[1,...,n]}{p}} \det A[\alpha]$$

for $p = 1, \ldots, n$.

Proof of Proposition 2.2. By Lemma 2.4, it suffices to prove that for every pair of subsets $\alpha$ and $\beta$ of $\{1, \ldots, n\}$ such that $|\alpha| = |\beta| \leq \min(k, n-k)$, we have $\det A[\alpha] = \det A[\beta]$. Let $p \leq \min(k, n-k)$ and let $r := k-p$. We will apply Lemma 2.3 to the map $f(\theta) := \det A[\theta]$, where $\theta \in \binom{[1,...,n]}{p}$. For this, let $\gamma_1, \gamma_2 \subseteq \{1, \ldots, n\}$ such that $|\gamma_1| = |\gamma_2| = p+r = k$. Since $A$ is $k$-spectrally monomorphic, $A[\gamma_1]$ and $A[\gamma_2]$ have the same characteristic polynomial $x^k + a_1x^{k-1} + \cdots + a_{r-k+r} + \cdots + a_k$. From Lemma 2.4 we have

$$a_p = (-1)^p \sum_{\theta \in \binom{\gamma_1}{p}} \det A[\theta] = (-1)^p \sum_{\theta \in \binom{\gamma_2}{p}} \det A[\theta].$$

Or, equivalently,

$$\sum_{\theta \in \binom{\gamma_1}{p}} f(\theta) = \sum_{\theta \in \binom{\gamma_2}{p}} f(\theta)$$

Moreover, $n \geq 2p + r$ because $p \leq \min(k, n-k)$. Thus, by Lemma 2.3, $f$ is a constant map. \qed

The next corollary is a particular case of Proposition 2.2.

Corollary 2.5. Let $A$ be a $k$-spectrally monomorphic $n \times n$ complex matrix. If $n \geq 2k-1$, then the following equivalent assertions hold

1) For every pair of subsets $\alpha$ and $\beta$ of $\{1, \ldots, n\}$ with $|\alpha| = |\beta| \leq k$, we have $\det A[\alpha] = \det A[\beta]$.

2) The matrix $A$ is $l$-spectrally monomorphic for $l = 1, \ldots, k$.

3 $k$-spectrally monomorphic Hermitian matrices for $k \in \{3, \ldots, n-4\}$

Let $c$ be a complex number. The Hermitian matrix

$$H_n(c) = \begin{pmatrix}
0 & c & \cdots & \cdots & c \\
c & 0 & c & \cdots & c \\
\vdots & c & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & c \\
c & \cdots & \cdots & c & 0
\end{pmatrix}$$

is $k$-spectrally monomorphic for every $k \in \{1, \ldots, n\}$. 
Let $S$ be a skew-symmetric matrix whose off-diagonal entries are from the set $\{-1, 1\}$. The Hermitian matrix $iS$ is $k$-spectrally monomorphic for $k \in \{1, 2, 3\}$.

Let $n \geq 4$ and let $Q = (q_{ij})$ be a 3-spectrally monomorphic $n \times n$ zero-diagonal Hermitian matrix. Assume that there exists a non-real complex number $c$ of modulus 1 such that for every $i \neq j \in \{1, \ldots, n\}$, $q_{ij} \in \{c, \overline{c}\}$ and $q_{ij} = c$ for $j \in \{2, \ldots, n\}$.

**Lemma 3.1.** Consider the tournament $T$ with vertex set $\{1, \ldots, n\}$ such that $i$ dominates $j$ if and only if $q_{ij} = c$.

i) If $T$ is transitive, then there exists a permutation matrix $P$ such that $H_n(c) = PQP^t$.

ii) If $T$ is not transitive, then $c \in \{i, -i\}$, or equivalently $Q = iS$ for some normalized skew-symmetric matrix $S$ whose off-diagonal entries are from the set $\{-1, 1\}$. Moreover, $Q$ is not 4-spectrally monomorphic.

**Proof.** Firstly, assume that $T$ is transitive. There exists a permutation $\sigma$ such that if $i < j$, then $\sigma(i) \rightarrow \sigma(j)$. It is easy to see that $H_n(c) = PQP^t$, where $P$ is the permutation matrix corresponding to $\sigma$.

To prove ii), assume that $T$ contains a 3-cycle $j \rightarrow k \rightarrow l \rightarrow j$. Then
\[
\det Q[j, k, l] = c^3 + \overline{c}^3
\]
Since $q_{is} = c$ for $s \in \{2, \ldots, n\}$, $1 \notin \{j, k, l\}$. Moreover
\[
\det Q[1, j, k] = c + \overline{c}
\]
As $Q$ is 3-spectrally monomorphic, $c^3 + \overline{c}^3 = c + \overline{c}$. This implies that $c \in \{i, -i\}$ because $c\overline{c} = 1$ and $c$ is not real. Now, we will prove that $Q$ is not 4-spectrally monomorphic. We have $\det Q[1, j, k, l] = 9$. However, for $m \notin \{1, j, k, l\}$, it is not difficult to check that if $m \rightarrow k$, then $\det Q[1, j, k, m] = 1$ and if $k \rightarrow m$, then $\det Q[1, k, l, m] = 1$. 

For $k \in \{3, 4\}$, we have the following characterization.

**Theorem 3.2.** Let $H$ be an $n \times n$ Hermitian matrix. Then the following hold

i) If $n \geq 5$, then $H$ is 3-spectrally monomorphic if and only if $H$ is $\Gamma$-equivalent to $H_n(c)$ where $c$ is a complex number, or $\Gamma$-equivalent to $iS$ for some skew-symmetric matrix $S$ whose off-diagonal entries are from the set $\{-1, 1\}$;

ii) If $n \geq 7$, then $H$ is 4-spectrally monomorphic if and only if $H$ is $\Gamma$-equivalent to $H_n(c)$ where $c$ is a complex number.

To prove this theorem, we start with the study of the possible entries of normalized 3-spectrally monomorphic Hermitian matrices.

**Lemma 3.3.** Let $H = (h_{ij})$ be an $n \times n$ zero-diagonal Hermitian normalized matrix with $n \geq 5$. If $H$ is 3-spectrally monomorphic, then there exists a complex number $c$ of modulus 1 such that for every $i \neq j \in \{1, \ldots, n\}$, we have $h_{ij} \in \{c, \overline{c}\}$. 

Remark 3.6. Let \( H \) be a complex number. Then, by Corollary 3.4, let \( H \) be a complex number of modulus 1 such that for every \( i \neq j \in \{2, \ldots, n\} \) we have \( h_{ij} = \bar{c} \). Hence \( h_{ij} \in \{c, \bar{c}\} \).

As a consequence, we obtain the following.

\[ \text{Corollary 3.4.} \quad \text{Let } H \text{ be a } 3\text{-spectrally monomorphic } n \times n \text{ Hermitian matrix with } n \geq 5. \text{ Then } H \text{ is a real scalar matrix or } H \text{ is } \Gamma\text{-equivalent to a Hermitian zero-diagonal matrix } Q = (q_{ij}) \text{ such that } q_{ij} \in \{c, \bar{c}\} \text{ for } i \neq j \text{ and } q_{1j} = c \text{ for } j \in \{2, \ldots, n\}, \text{ where } c \text{ is a complex number of modulus 1.} \]

\[ \text{Proof.} \quad \text{By the second assertion of Corollary 2.5, } H \text{ is } \Gamma\text{-equivalent to a real scalar matrix or } H \text{ is } \Gamma\text{-equivalent to a Hermitian matrix with zero-diagonal entries. Moreover, } H \text{ is } \Gamma\text{-equivalent to a Hermitian matrix } \tilde{H} \text{ with zero-diagonal entries. As the } \Gamma\text{-equivalence preserves the spectral monomorphy, the matrix } \tilde{H} \text{ is } 3\text{-spectrally monomorphic. Thus, by Lemma 3.3 there exists a complex number } c \text{ of modulus 1 such that for every } i \neq j \in \{2, \ldots, n\}, \text{ we have } \tilde{h}_{ij} \in \{c, \bar{c}\}. \text{ To conclude, it suffices to choose } Q = \overline{DHD}, \text{ where } D = \text{diag}(1, c, \ldots, c). \]

Now, we are able to prove Theorem 3.2.

\[ \text{Proof of Theorem 3.2:} \quad \text{i) If } H \text{ is a real scalar matrix, then } H \text{ is } \Gamma\text{-equivalent to } H_n(0). \text{ Assume that } H \text{ is not a real scalar matrix, and consider the matrix } Q, \text{ } \Gamma\text{-equivalent to } H, \text{ as described in Corollary 3.4. If } c \text{ is a real number, then } c \in \{1, -1\}, \text{ and hence } Q = \pm H_n(1). \text{ If } c \text{ is not a real number, then the result is obtained by applying Lemma 3.1.} \]

\[ \text{ii) By Corollary 2.5, } H \text{ is } 3\text{-spectrally monomorphic. Using assertion i) above, we can assume that } H \text{ is } \Gamma\text{-equivalent to } iS \text{ for some normalized skew-symmetric matrix } S \text{ whose off-diagonal entries are from the set } \{-1, 1\}. \text{ By Lemma 3.1 the matrix } iS \text{ is permutationally similar to } H_n(i). \text{ Hence } H \text{ is } \Gamma\text{-equivalent to } H_n(i). \]

The following corollary is a direct consequence of Proposition 2.2 and assertion ii) of Theorem 3.2.

\[ \text{Corollary 3.5.} \quad \text{Let } H \text{ be an } n \times n \text{ Hermitian matrix. If } n \geq 8 \text{ and } 4 \leq k \leq n - 4, \text{ then } H \text{ is } k\text{-spectrally monomorphic if and only if } H \text{ is } \Gamma\text{-equivalent to } H_n(c) \text{ where } c \text{ is a complex number.} \]

\[ \text{Remark 3.6.} \quad \text{The classes of } k\text{-spectrally monomorphic Hermitian matrices for } k = 3 \text{ and } n = 4, \text{ as well as } k = 4 \text{ and } n = 5, 6, \text{ are not easy to describe. More generally, the characterization of } k\text{-spectrally monomorphic } n \times n \text{ Hermitian matrices for } k = n - 1, n - 2 \text{ seems difficult.} \]
4 Characterization of \((n - 3)\)-spectrally monomorphic Hermitian matrices

As we have seen above, the Hermitian matrices \(H_n(c)\) are \((n - 3)\)-spectrally monomorphic. Another example is obtained from skew-symmetric conference matrices. Recall that a conference matrix is an \(n \times n\) matrix \(C\) with 0 on the diagonal and 1 and \(-1\) off the diagonal, such that \(C^tC = (n - 1)I_n\). Let \(S\) be a skew-symmetric conference matrix. Assertion iv) of the next proposition shows that the Hermitian matrix \(iS\) is \((n - 3)\)-spectrally monomorphic.

**Proposition 4.1.** Let \(S\) be a skew-symmetric conference matrix of order \(4t + 4\). Then

i) The characteristic polynomial of \(iS\) is \((x^2 - 4t - 3)^{2t+2}\).

ii) The characteristic polynomial of the matrix obtained from \(iS\) by deleting one row and the corresponding column is \(x(x^2 - 4t - 3)^{2t+1}\).

iii) The characteristic polynomial of the matrix obtained from \(iS\) by deleting two rows and the corresponding columns is \((x^2 - 1)(x^2 - 4t - 3)^{2t}\).

iv) The characteristic polynomial of the matrix obtained from \(iS\) by deleting three rows and the corresponding columns is \(x(x^2 - 3)(x^2 - 4t - 3)^{2t-1}\).

The proof of this proposition is contained implicitly in [17]. It is based on the interlacing theorem due to Cauchy [18].

The characterization of \((n - 3)\)-spectrally monomorphic Hermitian matrices is given by the following theorem.

**Theorem 4.2.** Let \(H\) be an \(n \times n\) Hermitian matrix. If \(n \geq 7\), then \(H\) is \((n - 3)\)-spectrally monomorphic if and only if \(H\) is \(\Gamma\)-equivalent to \(H_n(c)\) where \(c\) is a complex number or to \(iS\) where \(S\) is a skew-symmetric conference matrix.

Before proving this theorem, we recall some properties of skew-symmetric conference matrices and their relationship with doubly regular tournaments.

Skew-symmetric conference matrices are related to skew Hadamard matrices. A Hadamard matrix \(H\) is a square matrix of order \(n\) whose entries are from \{-1, 1\} and whose rows are mutually orthogonal, or equivalently, \(HH^t = H^tH = nI_n\). The order of a Hadamard matrix is necessarily 1, 2 or a multiple of 4. It is conjectured [19] that Hadamard matrices of order \(n\) always exist when \(n\) is divisible by 4. A Hadamard matrix \(H\) of order \(n\) is called skew if \(H + H^t = 2I_n\). It is easy to see that \(H\) is a skew Hadamard matrix if and only if \(H - I_n\) is a skew-symmetric conference matrix. Reid and Brown [20] gave a construction of skew Hadamard matrices from doubly regular tournaments. Recall that a tournament \(T\) of order \(n\) is doubly regular if there exists \(t > 0\), such that every pair of vertices is dominated by exactly \(t\) vertices. We have necessarily that \(n = 4t + 3\). Let \(\widehat{T}\) be the tournament obtained from \(T\) by adding a new vertex which dominates every vertex of \(T\). If \(A\) is the adjacency matrix of \(\widehat{T}\), then \(A - A^t + I_{4t+4}\) is a skew Hadamard matrix and hence \(A - A^t\) is a skew-symmetric conference matrix. Conversely, let \(H\) be a normalized skew Hadamard matrix of order \(4t + 4\) and let \(K\) be the matrix obtained
from $H$ by removing the first row and the corresponding column. We denote by $J_{4t+3}$ the all-ones matrix. Reid and Brown [20] showed that the tournament with adjacency matrix 
\[
\frac{1}{2}(K + J_{4t+3} - 2I_{4t+3})
\]
is doubly regular.

Let $T$ be a tournament and let $i, j$ be two vertices of $T$. We denote by $C_3(i, j)$ (resp. $O_3(i, j)$), the number of 3-cycles (resp. the number of transitive 3-tournaments) of $T$ containing $i$ and $j$. The tournament $T$ is homogeneous if there exists an integer $k > 0$ such that $C_3(i, j) = k$ for every vertices $i, j$ of $T$. Kotzig [21] proved that such a tournament contains exactly $4k - 1$ vertices. Moreover, Reid and Brown [20] established that it is doubly regular.

**Proof of Theorem 4.2** It suffices to prove the direct implication. By Proposition 2.2, $H$ is 3-spectrally monomorphic. Using the first assertion of Theorem 3.2, we can assume that $H$ is $\Gamma$-equivalent to $iS$ where $S = (s_{ij})$ is a skew-symmetric matrix. Without loss of generality, the matrix $S$ can be chosen to be normalized. Consider the tournament $T$ with vertex set \( \{2, \ldots, n\} \) such that $i$ dominates $j$ if $s_{ij} = 1$. We can assume that $T$ is not transitive, because otherwise $iS$ is $\Gamma$-equivalent to $H_n(i)$. We have to prove that $S$ is a skew-symmetric conference matrix, or equivalently, $T$ is homogeneous.

Consider two arbitrary vertices $i, j$ of $T$. It is easy to see that for every $k \in \{2, \ldots, n\} \setminus \{i, j\}$, \( \det S[i, i, j, k] = 9 \) if $i, j$ and $k$ form a 3-cycle of $T$ and \( \det S[i, i, j, k] = 1 \) otherwise. It follows that

\[
\sum_{|\alpha| = 4}^{\{1, i, j\} \subseteq \alpha} \det S[\alpha] = 9 \cdot C_3(i, j) + O_3(i, j)
\]

Since $C_3(i, j) + O_3(i, j) = n - 3$, we have

\[
8 \cdot C_3(i, j) = \sum_{|\alpha| = 4}^{\{1, i, j\} \subseteq \alpha} \det S[\alpha] - (n - 3)
\]

To conclude, it suffices to prove that the number

\[
\sum_{|\alpha| = 4}^{\{1, i, j\} \subseteq \alpha} \det S[\alpha]
\]
does not depend on $i$ and $j$. For this, we will use Lemma 2.3 applied to the map $f(\alpha) := \det S[\alpha]$ with $V := \{1, \ldots, n\}$, $p := 4$ and $r := n - 7$. Let $\gamma_1$ and $\gamma_2$ be two subsets of \( \{1, \ldots, n\} \) with $|\gamma_1| = |\gamma_2| = n - 3$. Since the matrix $iS$ is $(n - 3)$-spectrally monomorphic, the submatrices $S[\gamma_1]$ and $S[\gamma_2]$ have the same characteristic polynomial

\[
\phi(x) = x^{n-3} + a_1x^{n-4} + \cdots + a_p x^{n-3-p} + \cdots + a_{n-3}
\]

It follows from Lemma 2.4 that

\[
a_4 = \sum_{|\alpha| = 4}^{\alpha \subseteq \gamma_1} \det S[\alpha] = \sum_{|\alpha| = 4}^{\alpha \subseteq \gamma_2} \det S[\alpha]
\]

Hence by Lemma 2.3 for a subset $\zeta$ of \( \{1, \ldots, n\} \) with $|\zeta| = 3$, the number $\sum_{|\alpha| = 4}^{\zeta \subseteq \alpha} \det S[\alpha]$ does not depend on $\zeta$. 

\[\square\]
References

[1] Fraïssé R. Theory of relations. Amsterdam: North-Holland Publishing Co; 1986.

[2] Ehrenfeucht A, Harju T, Rozenberg G. The theory of 2-structures: A framework for decomposition and transformation of graphs. Singapore: World Scientific; 1999.

[3] Moon JW. Topics on tournaments. New York: Holt, Rhinehart and Winston; 1968.

[4] Assous R. Enchainabilité et seuil de monomorphie des tournois n-aires. Discrete Math. 1986;62(2):119-125.

[5] Frasnay C. Quelques problèmes combinatoires concernant les ordres totaux et les relations monomorphes. Ann Inst Fourier. 1965;15:415-524.

[6] Pouzet M. Application d’une propriété combinatoire des parties d’un ensemble aux groupes et aux relations. Math Z. 1976;150(2):117-134.

[7] Pouzet M. Application de la notion de relation presque-enchainable au dénombrement des restrictions finies d’une relation. Math Logic Q. 1981;27(19-21):289-332.

[8] Kantor WM. Automorphism groups of designs. Math Z. 1969;109(3):246-252.

[9] Berggren JL. An algebraic characterization of finite symmetric tournaments. Bull Aust Math Soc. 1972;6(1):53-59.

[10] Jean M. Line-symmetric tournaments. In: Recent Progress in Combinatorics. Academic Press, New York; 1969:265-271.

[11] Bondy JA, Murty USR. Graph theory with applications. New York: American Elsevier Publishing Co Inc; 1976.

[12] Yucai L, Guoxun H, Jiongsheng L. Score vectors of Kotzig tournaments. J Comb Theory Ser B. 1987;42(3):328-336.

[13] El-Issawi A. La (-1)-reconstruction des tournois (-1)-monomorphes. C R Acad Sci, Ser I Math. 1996;322(11):1015-1018.

[14] Boudabbous Y. Reconstructible and half-reconstructible tournaments: application to their groups of hemimorphisms. Math Logic Q. 1999;45(3):421-431.

[15] Boushabi B, Boussaïri A. Les graphes (-2)-monohémimorphes. C R Math Acad Sci Paris. 2012;350(15-16):731-735.

[16] Meyer CD. Matrix analysis and applied linear algebra. Philadelphia: Society for Industrial and Applied Mathematics; 2000.

[17] Greaves G, Suda S. Symmetric and skew-symmetric-matrices with large determinants. J Comb Des. 2017;25(11):507-522.

[18] Fisk S. A very short proof of Cauchy’s interlace theorem for eigenvalues of Hermitian matrices. Am Math Mon. 2005;112(2):118.
[19] Paley RE. On orthogonal matrices. J Math Phys. 1933;12(1-4):311-320.

[20] Reid KB, Brown E. Doubly regular tournaments are equivalent to skew Hadamard matrices. J Comb Theory Ser A. 1972;12(3):332-338.

[21] Kotzig A. Sur les tournois avec des 3-cycles régulièrement placés. Mat Časopis. 1969;19(2):126-134.