LOCAL STRONG FACTORIZATION OF BIRATIONAL MAPS

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Abstract. The strong factorization conjecture states that a proper birational map between smooth algebraic varieties over a field of characteristic zero can be factored as a sequence of smooth blowups followed by a sequence of smooth blowdowns. We prove a local version of the strong factorization conjecture for toric varieties. Combining this result with the monomialization theorem of S. D. Cutkosky, we obtain a strong factorization theorem for local rings dominated by a valuation.

0. Introduction

Let \( \phi : X_1 \rightarrow X_2 \) be a proper birational map between smooth varieties over a field of characteristic zero. A commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi_1} & X_1 \\
\downarrow & & \downarrow \phi \\
X_2 & \xrightarrow{\psi_2} & \\
\end{array}
\]

where \( \psi_1 \) and \( \psi_2 \) are sequences of blowups of smooth centers, is called a strong factorization of \( \phi \). The existence of a strong factorization is an open problem in dimension \( n = 3 \) and higher.

The local version of the strong factorization conjecture replaces the varieties by local rings dominated by a valuation on their common fraction field, and the smooth blowups by monoidal transforms along the valuation. The local strong factorization was proved by C. Christensen [2] in dimension 3 for certain valuations. A complete proof of the 3-dimensional case was given by S. D. Cutkosky in [3, 4], where he also made considerable progress towards proving the conjecture in general. We prove the local factorization conjecture in any dimension (see Section 2 for notation):

**Theorem 0.1.** Let \( R \) and \( S \) be excellent regular local rings containing a field \( k \) of characteristic zero. Assume that \( R \) and \( S \) have a common fraction field \( K \) and \( \nu \) is a valuation on \( K \). Then there exists a local ring \( T \), obtained from both \( R \) and \( S \) by sequences of monoidal transforms along \( \nu \).

The toric version of the strong factorization problem considers two nonsingular fans \( \Sigma_1 \) and \( \Sigma_2 \) with the same support and asks whether there exists a common refinement \( \Delta \)

\[
\begin{array}{ccc}
\Delta & \xleftarrow{\Sigma_1} & \xrightarrow{\Sigma_2} \\
\end{array}
\]

obtained from both \( \Sigma_1 \) and \( \Sigma_2 \) by sequences of smooth star subdivisions. Again, this is not known in dimension 3 or higher. The local toric version replaces a fan by a single

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cone and a smooth star subdivision of the fan by a smooth star subdivision of the cone together with a choice of one cone in the subdivision. We assume that the choice is given by a vector \( v \) in the cone: we choose a cone in the subdivision containing \( v \). If \( v \) has rationally independent coordinates, then it determines a unique cone in every subdivision (all cones are rational). We call such a vector \( v \) a valuation and the subdivision with a choice of a cone a subdivision along the valuation. We prove:

**Theorem 0.2.** Let \( \sigma \) and \( \tau \) be nonsingular cones, and let \( v \in \sigma \cap \tau \) be a vector with rationally independent coordinates. Then there exists a nonsingular cone \( \rho \) obtained from both \( \sigma \) and \( \tau \) by sequences of smooth star subdivisions along \( v \).

The proof of Theorem 0.2 is a generalization of the proof given by C. Christensen [2] in dimension 3. Theorem 0.1 follows directly from Theorem 0.2 and the monomialization theorem proved by S. D. Cutkosky [4].

**Remark 0.3.** One can define a more general version of local toric factorization. Consider a game between two players \( A \) and \( B \), where the player \( A \) subdivides the cone \( \tau \) or \( \sigma \) and the player \( B \) chooses one cone in the subdivision (and renames it again \( \tau \) or \( \sigma \)). Then the strong factorization conjecture states that \( A \) always has a winning strategy: after a finite number of steps either \( \tau = \sigma \) or the interiors of \( \tau \) and \( \sigma \) do not intersect. The proof of Theorem 0.2 given in Section 1 does not extend to this more general case. A positive answer to the global strong factorization conjecture for toric varieties would imply that \( A \) always has a winning strategy. Conversely, a counterexample to the local factorization problem would give a counterexample to the global strong factorization conjecture.

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## 1. Local factorization for toric varieties

Let \( N \simeq \mathbb{Z}^n \) be a lattice and \( \sigma \) a rational polyhedral cone in \( N_{\mathbb{R}} = N \otimes \mathbb{R} \) generated by a finite set of vectors \( w_i \in N \)

\[
\sigma = \mathbb{R}_{\geq 0}w_1 + \ldots + \mathbb{R}_{\geq 0}w_m.
\]

We say that \( \sigma \) is nonsingular if it can be generated by a part of a basis of \( N \). A nonsingular \( m \)-dimensional cone has a unique set of minimal generators \( w_1, \ldots, w_m \in N \), and we write

\[
\sigma = \langle w_1, \ldots, w_m \rangle.
\]

We consider nonsingular cones only. When we draw a picture of a cone, we only show a cross-section. Thus, a 3-dimensional cone is drawn as a triangle.

Let \( \sigma = \langle w_1, \ldots, w_n \rangle \) be a nonsingular \( n \)-dimensional cone, and let \( v \in \sigma \) be a vector \( v = c_1w_1 + \ldots + c_nw_n \) such that \( c_1, \ldots, c_n \) are linearly independent over \( \mathbb{Q} \). If \( 1 \leq i < j \leq n \), then precisely one of the cones

\[
\langle w_i + w_j, w_1, \ldots, \hat{w}_i, \ldots, w_n \rangle, \quad \langle w_i + w_j, w_1, \ldots, \hat{w}_j, \ldots, w_n \rangle,
\]

is a nonsingular cone.
contains \( v \). The cone containing \( v \) is called a *star subdivision of \( \sigma \) at \( w_i + w_j \) along \( v \). The subdivision is again a nonsingular cone. We often denote a star subdivision of a cone \( \sigma \) again \( \sigma \), and name its generators also \( w_1, \ldots, w_n \).

Let us consider the situation of Theorem 0.2. It is easy to see that after star subdividing \( \tau \) sufficiently many times we may assume that \( v \in \tau \subset \sigma \). We say that a configuration \( v \in \tau \subset \sigma \) is factorizable if the statement of Theorem 0.2 holds. We say that \( v \in \tau \subset \sigma \) is directly factorizable if the statement of Theorem 0.2 holds with \( \rho = \tau \). The vector \( v \) is not needed for direct factorizability.

The following lemma is well-known:

**Lemma 1.1.** If the dimension \( n = 2 \), then \( v \in \tau \subset \sigma \) is directly factorizable. \( \Box \)

**Lemma 1.2** (C. Christensen [2]). Let \( n = 3 \) and consider \( v \in \tau \subset \sigma \), where

\[
\tau = \langle u_1, u_2, u_3 \rangle, \quad \sigma = \langle w_1, w_2, w_3 \rangle
\]

are nonsingular cones such that \( w_1, u_1, u_2 \) are linearly dependent. Then \( v \in \tau \subset \sigma \) is directly factorizable.

**Proof.** Let \( \pi : N_R \rightarrow N_R / \mathbb{R}w_1 \) be the quotient map. We claim that \( \pi(\tau) \subset \pi(\sigma) \) are nonsingular cones with respect to the lattice \( \pi(N) \). This is clear for the cone \( \sigma \); for \( \tau \) note that the generators \( u_1, u_2, u_3 \) of \( N \) map to generators of \( \pi(N) \). (More precisely, \( \pi(u_1) = au', \pi(u_2) = bu' \), where \( u' \in \pi(N) \) is primitive, \( \gcd(a, b) = 1 \).

Now we apply Lemma 1.1 to the configuration \( \pi(v) \in \pi(\tau) \subset \pi(\sigma) \). Then after a finite sequence of star subdivisions of \( \sigma \) at vectors lying in \( \langle w_2, w_3 \rangle \), we may assume that

\[
\langle u_3 \rangle \subset \langle w_1, w_3 \rangle, \quad \langle u_1, u_2 \rangle \subset \langle w_1, w_2 \rangle.
\]

If we express \( u_3 = \alpha w_1 + \beta w_3 \), then it follows from the nonsingularity of \( \tau \) that \( \beta = 1 \). In other words, the cone \( \tau \) lies in the subdivision \( \langle w_1 + w_3, w_1, w_2 \rangle \) of \( \sigma \). Performing a sequence of such star subdivisions, we get to the situation where \( u_3 = w_3 \).
Finally, \( \langle u_1, u_2 \rangle \subset \langle w_1, w_2 \rangle \) is strongly factorizable by Lemma 1.1, thus a sequence of star subdivisions of \( \sigma \) at vectors lying in \( \langle w_1, w_2 \rangle \) finishes the proof. \( \Box \)

By the previous lemma, to show that \( v \in \tau \subset \sigma \) is factorizable, we have to find a sequence of star subdivisions of \( \tau \) such that the condition of the lemma is satisfied. We prove this in any dimension.

**Lemma 1.3.** Let \( n \geq 3 \) and consider a configuration \( v \in \tau \subset \sigma = \langle w_1, \ldots, w_n \rangle \). There exists a cone \( \tau' = \langle u_1, \ldots, u_n \rangle \), obtained from \( \tau \) by a sequence of smooth star subdivisions along \( v \), such that \( w_1, u_1, u_2 \) are linearly dependent.

Moreover, one can find \( \tau' \) such that \( w_1, u_1, u_2 \) satisfy the relation

\[
    w_1 = u_1 - u_2.
\]

**Proof.** The first part of the proof is again due to C. Christensen.

Let us start with the case \( n = 3 \) and prove the first half of the lemma. The algorithm for constructing \( \tau' \) is as follows. Let \( \pi : \mathbb{N}_R \to \mathbb{N}_R / \mathbb{R} w_1 \) be the projection and let the generators \( u_1, u_2, u_3 \) of \( \tau \) be ordered so that \( \pi(u_3) \in \pi(\langle u_1, u_2 \rangle) \). If \( \pi(u_3) \in \pi(u_1) \) or \( \pi(u_3) \in \pi(u_2) \), then we are done. Otherwise star subdivide \( \tau \) at \( u_1 + u_2 \) and repeat.

To see that this algorithm always terminates, let \( a_i, b_i \) be defined by:

\[
    w_1 = a_1 u_1 + a_2 u_2 + a_3 u_3,
    v = b_1 u_1 + b_2 u_2 + b_3 u_3.
\]

Here \( a_i \in \mathbb{Z} \), \( \gcd(a_1, a_2, a_3) = 1 \), and \( b_i \in \mathbb{R}, b_i > 0 \). Then the algorithm can be described as follows. Consider the matrix

\[
    \begin{bmatrix}
        a_1 & a_2 & a_3 \\
        b_1 & b_2 & b_3
    \end{bmatrix}.
\]

If some \( a_i = 0 \), then we are done. Otherwise, choose columns \( i \) and \( j \) such that \( a_i \) and \( a_j \) have the same sign and subtract the \( i \)’th column from the \( j \)’th if \( b_j > b_i \) and \( j \)’th column from the \( i \)’th if \( b_i > b_j \).
Since we always choose columns where $a_i$ and $a_j$ have the same sign, it is clear that \( \max_i |a_i| \) does not increase in this process, and it suffices to prove that either the algorithm terminates or \( \max_i |a_i| \) drops after a finite number of steps. Suppose that \( |a_3| = \max_i |a_i| \), and \( a_3 \) does not change as we run the algorithm. Then \( b_3 \) also does not change, and every time we choose columns \( i \) and 3, we subtract \( b_3 \) from \( b_i \). It is clear that columns 1 and 2 can be chosen only a finite number of times in a row, hence we choose column 3 infinitely many times. Since we cannot subtract \( b_3 \) from \( b_1 \) or \( b_2 \) infinitely many times and have a positive result, we get a contradiction. This proves the first half of the lemma for \( n = 3 \).

Next let us prove the “moreover” part for \( n = 3 \). We start with a matrix

\[
\begin{bmatrix}
  a_1 & a_2 & 0 \\
  b_1 & b_2 & b_3
\end{bmatrix}.
\]

If also \( a_2 = 0 \), then by nonsingularity of \( \tau \) we have \( a_1 = 1 \). We choose columns 1 and 2 the necessary number of times to get \( a_2 = -1 \):

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  b_1 & b_2 & b_3
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 0 & 0 \\
  b_1 - b_2 & b_2 & b_3
\end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix}
  1 & 0 & 0 \\
  b_1 & b_2 & b_3
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 0 & 0 \\
  b_1 - kb_2 & (k+1)b_2 - b_1 & b_3
\end{bmatrix}.
\]

If both \( a_1 \) and \( a_2 \) are nonzero then they must have different signs. Hence, if \( \max_i |a_i| = 1 \) then we are done. Otherwise, since \( \gcd(a_1, a_2) = 1 \), we may assume that \( |a_1| > |a_2| \). We perform star subdivisions of \( \tau \) by choosing columns 2 and 3 the necessary number of times to get to the matrix

\[
\begin{bmatrix}
  a_1 & a_2 & -a_2 \\
  b_1 & b_2 - kb_3 & (k+1)b_3 - b_2
\end{bmatrix}.
\]

After this, we run the algorithm as before. For instance, since \( a_1 \) and \( -a_2 \) have the same sign, at the next step we choose columns 1 and 3. If we subtract the third column from the first, then \( \max_i |a_i| \) drops and we are done by induction; otherwise, we subtract the first column from the third. As before, if \( \max_i |a_i| \) does not decrease, then we are subtracting \( b_1 \) from \( b_2 \) or \( b_3 \) infinitely many times, and this gives a contradiction.

For \( n > 3 \) we have a matrix

\[
\begin{bmatrix}
  a_1 & \cdots & a_n \\
  b_1 & \cdots & b_n
\end{bmatrix}.
\]

We can apply the \( n = 3 \) case to the last three columns and achieve \( a_n = 0 \); then apply the same algorithm to columns \( n-3, n-2, n-1 \) to get \( a_{n-1} = 0 \), and so on, until all but two of the \( a_i \) are nonzero. To prove the second half of the lemma, we apply the \( n = 3 \) case to three columns, including the ones with \( a_i \neq 0 \).

**Proof of Theorem 0.2.** We may assume that \( \tau = \langle u_1, \ldots, u_n \rangle \subset \sigma = \langle w_1, \ldots, w_n \rangle \), and using Lemma 1.3 we may also assume that \( w_1, u_1, u_2 \) satisfy the relation

\[
w_1 = u_1 - u_2.
\]

Let \( \pi : N_R \to N_R/\mathbb{R}w_1 \) be the projection. Then \( \pi(\tau) \subset \pi(\sigma) \) are both nonsingular with respect to the lattice \( \pi(N) \). The relation \( \langle \diamond \rangle \) implies that \( \pi(u_1) = \pi(u_2) \) is a minimal
generator of \( \pi(\tau) \). In particular, \( \pi \) restricts to isomorphisms of cones and lattices:

\[
\pi : \langle u_1, u_3, \ldots, u_n \rangle \xrightarrow{\approx} \pi(\tau), \quad \pi : \bigoplus_{i \neq 2} \mathbb{Z}u_i \xrightarrow{\approx} \pi(N).
\]

\[
\pi : \langle u_2, u_3, \ldots, u_n \rangle \xrightarrow{\approx} \pi(\tau), \quad \pi : \bigoplus_{i \neq 1} \mathbb{Z}u_i \xrightarrow{\approx} \pi(N).
\]

By induction on the dimension \( n \), we have a factorization of \( \pi(v) \in \pi(\tau) \subset \pi(\sigma) \). Unlike the case \( n = 3 \), we may also have to subdivide \( \pi(\tau) \). Consider a star subdivision of \( \pi(\tau) \) at \( z = \pi(u_1) + \pi(u_j), j \geq 3 \), and define \( z' \) and \( z'' \) by:

\[
\pi^{-1}(z) \cap \langle u_2, u_3, \ldots, u_n \rangle = \{z'\}, \quad \pi^{-1}(z) \cap \langle u_1, u_3, \ldots, u_n \rangle = \{z''\}.
\]

Now star subdividing \( \tau \) first at \( z' \) and then at \( z'' \) along \( v \), the resulting cone again satisfies the relation (\( \diamond \)) (after possibly reordering the generators), and its image under \( \pi \) is the star subdivision of \( \pi(\tau) \) at \( z \) along \( \pi(v) \). In other words, every star subdivision of \( \pi(\tau) \) can be lifted to a subdivision of \( \tau \). Thus after a finite sequence of star subdivisions of \( \tau \) we may assume that \( \pi(v) \in \pi(\tau) \subset \pi(\sigma) \) is directly factorizable.

The remaining proof is the same as in the 3-dimensional case. Star subdividing \( \sigma \) at vectors lying in the face \( \langle w_2, \ldots, w_n \rangle \), we may assume that

\[
u_i \in \langle w_1, w_i \rangle, \quad i = 3, \ldots, n, \quad \langle u_1, u_2 \rangle \subset \langle w_1, w_2 \rangle.
\]

If \( u_i = \alpha_i w_1 + \beta_i w_i \), then \( \beta_i = 1 \) for \( i \geq 3 \), hence after star subdividing \( \sigma \) at vectors lying in the face \( \langle w_1, w_i \rangle \), we may assume that \( u_i = w_i \) for \( i \geq 3 \). Now \( \langle u_1, u_2 \rangle \subset \langle w_1, w_2 \rangle \) are nonsingular cones, hence directly factorizable by Lemma 1.1. A sequence of star subdivisions of \( \sigma \) at vectors lying in the face \( \langle w_1, w_2 \rangle \) finishes the proof. \( \square \)

1.1. The case of \( v \) with rationally dependent coordinates. J. Wlodarczyk has noted that it makes sense to consider the local toric factorization problem also for a vector \( v \) with rationally dependent coordinates, and this problem can be reduced to the rationally independent case. We bring here an argument for such a reduction. Similar reduction appears in S. D. Cutkosky's proof of the monomialization theorem [1].

Consider a nonsingular \( n \)-dimensional cone \( \sigma \) and a vector \( v \in \sigma \), with possibly rationally dependent coordinates. A star subdivision of \( \sigma \) along \( v \) is a star subdivision of \( \sigma \) and a choice of an \( n \)-dimensional cone in the subdivision containing \( v \) (i.e., in case there are more than one such cone, we are free to choose any one of them). The factorization problem then is: Given two \( n \)-dimensional nonsingular cones \( \sigma, \tau \) and a vector \( v \in \sigma \cap \tau \), there
exists a nonsingular cone $\rho$ obtained from both $\tau$ and $\sigma$ by sequences of star subdivisions along $v$. It is clear that the factorization problem has a solution only if the interiors of $\sigma$ and $\tau$ intersect nontrivially. We assume that this is the case initially and after every subdivision we choose a cone containing $v$ such that this condition again holds.

An extreme case of the factorization problem is when $v = 0$. Then a factorization along any vector $v' \in \sigma \cap \tau$ (for instance, $v'$ with rationally independent coordinates) is also a factorization along $v$. If $v \neq 0$, we reduce the factorization problem to the case of $v$ with rationally independent coordinates as follows.

The first reduction step is to star subdivide both $\tau$ and $\sigma$ along $v$ to get to the situation where $\tau = \langle u_1, \ldots, u_n \rangle$, $\sigma = \langle w_1, \ldots, w_n \rangle$ and

$$v \in \langle u_1, \ldots, u_m \rangle \cap \langle w_1, \ldots, w_m \rangle,$$

such that the coordinates of $v$ with respect to $u_1, \ldots, u_m$ (hence also with respect to $w_1, \ldots, w_m$) are rationally independent. For this write $v = b_1 u_1 + \ldots + b_n u_n$ and consider the vector $(b_1, \ldots, b_n)$ with nonnegative entries. It is a simple exercise to show that after a finite sequence of column operations where one subtracts $b_i$ from $b_j$ for $b_j \geq b_i$, we get to the vector (after reordering the components) $(b'_1, \ldots, b'_m, 0 \ldots, 0)$, such that $b'_1, \ldots, b'_m$ are linearly independent over $\mathbb{Q}$. After a similar sequence of star subdivisions of $\sigma$, we get $v = c_1 w_1 + \ldots + c_m w_m$. Note that $\text{Span}(u_1, \ldots, u_m) = \text{Span}(w_1, \ldots, w_m)$ is the smallest subspace of $N_R$ spanned by rational vectors and containing $v$.

The next step is to use the rationally independent case and factor $v \in \langle u_1, \ldots, u_m \rangle \cap \langle w_1, \ldots, w_m \rangle$. Thus after a finite sequence of star subdivisions of $\sigma$ and $\tau$ we may assume that the two cones have a common face $\langle u_1, \ldots, u_m \rangle = \langle w_1, \ldots, w_m \rangle$ containing $v$. After additional subdivisions of $\tau$ we may also assume that $\tau \subset \sigma$.

The final step is to consider the projection $\pi : N_R \to N_R / R w_1$, and proceed by induction on dimension the same way as in the proof of Theorem \textbf{0.2}.

\section*{2. Factorization for local rings.}

We recall in this section the monomialization theorem of S. D. Cutkosky, and then prove Theorem \textbf{0.1}.

Let $(R, m_R)$ be a regular local ring of dimension $n$ containing a field $k$ of characteristic zero, and let $\nu$ be a valuation on the fraction field of $R$, such that the valuation ring $V$ dominates $R$. Let $x_1, \ldots, x_m \in R$ be a subset of a system of regular parameters $x_1, \ldots, x_n$ of $R$. Then the homomorphism

$$R \to R' = (R[x_1, \ldots, x_m]_p),$$

for some $1 \leq i \leq m$, and $p$ a prime ideal lying over $m_R$, is called a monoidal transform of $R$. If $R'$ is again dominated by the valuation $\nu$, we say that the monoidal transform is a transform along the valuation $\nu$. Geometrically, a monoidal transform is obtained by blowing up a smooth center and localizing at a point $p$ above $m_R$ determined by the valuation. In the following, we will be interested in monoidal transforms with $m = 2$.

Let $R$ and $S$ be two excellent regular local rings of dimension $n$ containing a field $k$ of characteristic zero, both dominated by a valuation $\nu$ on their common fraction field. S. D. Cutkosky proved in \textbf{3, 4} that after a sequence of monoidal transforms of $R$ and $S$, one can express a system of regular parameters of $S$ as monomials in regular parameters.
of $R$. More precisely, if $\nu$ has rank 1 and rational rank $n$ (i.e., the value group can be embedded in $\mathbb{R}$ and it contains $n$ rationally independent elements), then after a finite sequence of monoidal transforms, we may assume that a system $y_1, \ldots, y_n$ of regular parameters of $S$ can be expressed in terms of regular parameters $x_1, \ldots, x_n$ of $R$ as:

$$y_1 = x_1^{a_{11}} x_2^{a_{12}} \cdots x_n^{a_{1n}}$$

$$\ldots$$

$$y_n = x_1^{a_{n1}} x_2^{a_{n2}} \cdots x_n^{a_{nn}},$$

where $a_{ij}$ are nonnegative integers and $\det(a_{ij}) = \pm 1$. Note that $\nu(y_1), \ldots, \nu(y_n)$ are rationally independent positive real numbers. If $\nu$ is an arbitrary valuation, then the matrix $(a_{ij})$ is block diagonal, with $y_i$ corresponding to the same block having rationally independent values $\nu(y_i)$ (\cite{1}, Theorem 4.4). In the following proof we will perform monoidal transforms with centers $(y_i, y_j)$ or $(x_i, x_j)$, with $i$ and $j$ lying in the same block, hence we may assume that $\nu$ has rank 1 and rational rank $n$.

Now let us consider the situation of Theorem \ref{thm:monoidal_transforms}. We assume that an embedding $S \subset R$ is given by monomials as above, and we have to show that after a sequence of monoidal transforms along $\nu$, we get (renaming parameters) $y_i = x_i$ for $i = 1, \ldots, n$. This follows directly from Theorem \ref{thm:monoidal_transforms} once we express the problem in terms of cones and subdivisions.

Let $\sigma = \langle w_1, \ldots, w_n \rangle \subset N = \mathbb{Z}^n$ be a nonsingular cone, and let $\tau = \langle u_1, \ldots, u_n \rangle \subset \sigma$ be the cone defined by

$$u_1 = a_{11} w_1 + a_{21} w_2 + \ldots + a_{1n} w_n$$

$$\ldots$$

$$u_n = a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{nn} w_n,$$

where $(a_{ij})$ is the matrix of exponents above. Since $\det(a_{ij}) = \pm 1$, the cone $\tau$ is nonsingular. We also let

$$v = \nu(y_1) w_1 + \nu(y_2) w_2 + \ldots + \nu(y_n) w_n$$

be a vector $v \in \tau \subset \sigma$. Now one can easily check that the monoidal transform of $R$ with center $(x_i, x_j)$ along $\nu$ corresponds to the star subdivision of $\tau$ at $u_i + u_j$ along $v$ (which in terms of the matrix $(a_{ij})$ corresponds to adding one column to another), and similarly for $S$ and $\sigma$ (in terms of $(a_{ij})$, subtract one row from another). Applying Theorem \ref{thm:monoidal_transforms} after a finite sequence of monoidal transforms of $R$ and $S$, the matrix $a_{ij}$ is the identity matrix, hence $R = S$. \hfill \Box

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