Abstract In this paper we investigate a system of coupled inequalities consisting of a variational-hemivariational inequality and a quasi-hemivariational inequality on Banach spaces. The approach is topological, and a wide variety of existence results is established for both bounded and unbounded constraint sets in real reflexive Banach spaces. The main point of interest is that no linearity condition is imposed on the coupling functional, therefore making the system fully nonlinear. Applications to Contact Mechanics are provided in the last section of the paper. More precisely, we consider a contact model with (possibly) multivalued constitutive law whose variational formulation leads to a coupled system of inequalities. The weak solvability of the problem is proved via employing the theoretical results obtained in the previous section. The novelty of our approach comes from the fact that we consider two potential contact zones and the variational formulation allows us to determine simultaneously the displacement field and the Cauchy stress tensor.

Keywords Hemivariational inequalities · Nonlinear coupling functional · Bounded and unbounded constraint sets · Contact problems · Weak solution via bipotentials

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1 Introduction and preliminaries

Let $X, Y$ be two real reflexive Banach spaces and $K \subseteq X, A \subseteq Y$ be nonempty, closed and convex subsets. Assume $B: X \times Y \rightarrow \mathbb{R}$ is a nonlinear bifunction, $\phi: X \rightarrow (-\infty, \infty]$ is a proper, convex and lower semicontinuous such that $D(\phi) \cap K \neq \emptyset$, $Z_1, Z_2$ are Banach spaces and $\gamma_1: X \rightarrow Z_1, \gamma_2: Y \rightarrow Z_2$ are linear and compact operators, $J: Z_1 \rightarrow \mathbb{R}$ and $L: Z_2 \rightarrow \mathbb{R}$ are locally Lipschitz functionals, $H: Y \rightarrow (0, \infty)$ is a given function, and $F: X \rightarrow X^*$, $G: Y \rightarrow Y^*$ are prescribed nonlinear operators. In the present, we are interested in the following system of inequalities:

Find $(u, \sigma) \in (K \cap D(\phi)) \times A$ such that

$$
\begin{cases}
B(v, \sigma) - B(u, \sigma) + \phi(v) - \phi(u) + J^0(\gamma_1(u); \gamma_1(v - u)) \geq \langle F(u), v - u \rangle, \\
B(u, \mu) - B(u, \sigma) + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) \geq \langle G(\sigma), \mu - \sigma \rangle,
\end{cases}
$$

(1.1)

for all $v \in K$ and all $\mu \in A$.

The first line of problem (1.1) represents a nonlinear variational-hemivariational inequality, while the second line is a quasi-hemivariational inequality (since, in general, there does not exist a locally Lipschitz functional $T: Y \rightarrow \mathbb{R}$ such that $\partial_C T(\sigma) = H(\sigma)\partial_C J^0(\gamma_2(\sigma))$). Observe that the unknown pair $(u, \sigma)$ appears in both inequalities of (1.1) via the functional $B$ which will be called coupling functional, so, the system (1.1) is called coupled.

Very recently, Bai, Migórski & Zeng [1] established existence and uniqueness results for the following system with bilinear coupling function:

Find $(u, \sigma) \in X \times A$ such that

$$
\begin{cases}
\langle A(u), v - u \rangle + b(v - u, \sigma) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq \langle f, v - u \rangle, & \text{for all } v \in X, \\
b(u, \mu - \sigma) \leq 0, & \text{for all } \mu \in A,
\end{cases}
$$

(1.2)

where $X, Y$ are reflexive Banach spaces, $Z$ is a Banach space, $f \in X^*$, $A: X \rightarrow X^*$ is a nonlinear operator, $J: Z \rightarrow \mathbb{R}$ is locally Lipschitz, $\gamma: X \rightarrow Z$ is a linear and continuous operator and $b: X \times Y \rightarrow \mathbb{R}$ is a bilinear continuous function. In fact, system (1.2) can be a generalized and powerful mathematical model to solve various complicated engineering problems, such that thermopiezoelectric media problems with semi-permeable thermal boundary conditions, fluid mechanics problems with nonmonotone friction constitutive law, population dynamic problems, and so forth, see for instance, [7,8,10,11,12].

Note that, if we choose $B(u, \sigma) := b(u, \sigma), F(u) := f - A(u), K := X, \phi \equiv 0, H \equiv 0$ and $G \equiv 0$, then the system (1.1) reduces to

Find $(u, \sigma) \in X \times A$ such that

$$
\begin{cases}
\langle A(u), v - u \rangle + b(v - u, \sigma) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq \langle f, v - u \rangle, & \forall v \in X, \\
b(u, \mu - \sigma) \geq 0, & \forall \mu \in A,
\end{cases}
$$

(1.3)

which is very similar to (1.2), but the second inequality is reversed, therefore, even for bilinear coupling functions our results are new and supplement the existing knowledge in the literature.
We fix next basic notations and recall some concepts and facts which we need in this paper, and its details can be found in [3][5][7].

For a real Banach space $E$ we denote its norm by $\| \cdot \|_E$ and by $E^*$ its dual space. The duality pairing between $E^*$ and $E$ will be denoted by $\langle \cdot, \cdot \rangle_{E^* \times E}$. If there is no danger of confusion there subscripts will be ignore which means that we will simply write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ from time to time. Particularly, if $E$ is a Hilbert space, then $\langle \cdot, \cdot \rangle_E$ stands for the inner product of $E$.

Let $E, F$ be two real Banach spaces. If for each $u \in E$ there exists a corresponding subset $A(u) \subseteq F$, then we say $A : E \to 2^F$ is a set-valued mapping (or multifunction) from $E$ to $F$. For a set-valued mapping $A : E \to 2^F$ we define domain of $A$ to be the set

$$D(A) := \{ x \in E : A(x) \neq \emptyset \}.$$ 

The range of $A$ is the set

$$R(A) := \bigcup_{x \in E} A(x),$$

and the graph of $A$ is the set

$$\text{Gr}(A) := \{(x, y) \in E \times F : y \in A(x) \text{ and } x \in D(A) \}.$$ 

The inverse of the set-valued mapping $A : E \to 2^F$ is the set-valued mapping $A^{-1} : F \to 2^E$ defined by

$$A^{-1}(y) := \{ x \in E : y \in A(x) \}.$$ 

We point out the fact that every single-valued map $A : E \to F$ may be regarded as the set-valued map $E \ni x \mapsto \{ A(x) \} \in 2^F$. In this case we identify $A(x)$ with its unique element.

**Definition 1.1** Let $\beta : E \to \mathbb{R}$ be a given functional. A set-valued mapping $A : E \to 2^{E^*}$ is called $\beta$-relaxed dissipative if

$$\langle \zeta_y - \zeta_x, y - x \rangle \leq \beta(y - x) \text{ for all } (x, \zeta_x), (y, \zeta_y) \in \text{Gr}(A). \quad (1.4)$$

**Remark 1** If $\beta(x) := -m\|x\|^2$ for some $m > 0$, then $A$ is said to be strongly dissipative, while for $\beta \equiv 0$ the set-valued mapping $A$ is called dissipative. If $A$ is single-valued, then (1.4) reduces to

$$\langle A(y) - A(x), y - x \rangle \leq \beta(y - x) \text{ for all } x, y \in E.$$ 

Moreover, we have

- if $-A$ is $-\beta$-relaxed dissipative, then $A$ is called $\beta$-relaxed monotone, i.e.,

$$\langle \zeta_y - \zeta_x, y - x \rangle \geq \beta(y - x) \text{ for all } (x, \zeta_x), (y, \zeta_y) \in \text{Gr}(A).$$

- if $-A$ is strongly dissipative (namely, $A$ is $-\beta$-relaxed dissipative and $\beta(x) := m\|x\|^2$ for some $m > 0$), then $A$ is called strongly monotone, i.e.,

$$\langle \zeta_y - \zeta_x, y - x \rangle \geq m\|y - x\|^2 \text{ for all } (x, \zeta_x), (y, \zeta_y) \in \text{Gr}(A).$$
• if $-A$ is dissipative (i.e., $\beta \equiv 0$), then $A$ is called monotone, i.e.,

$$\langle \zeta_y - \zeta_x, y - x \rangle \geq 0 \text{ for all } (x, \zeta_x), (y, \zeta_y) \in Gr(A).$$

• if $-A$ is $-\beta$-relaxed dissipative and $\beta(x) := -m\|x\|^2$ for some $m > 0$, then $A$ is called relaxed monotone with constant $m$, i.e.,

$$\langle \zeta_y - \zeta_x, y - x \rangle \geq -m\|y - x\|^2 \text{ for all } (x, \zeta_x), (y, \zeta_y) \in Gr(A).$$

**Definition 1.2** A single-valued operator $A: E \to E^*$ is called hemicontinuous, if for any $u, v, w \in E$ one has

$$\lim_{t \to 0^+} \langle A(u + t(v - u)), w \rangle = \langle A(u), w \rangle.$$

Next, we recall two kinds of set-valued mappings that play an important role in Nonsmooth Analysis, namely, the subdifferential of a convex functional and the Clarke subdifferential of a locally Lipschitz functional.

Let $\phi: E \to (-\infty, +\infty]$ be a prescribed functional. We define its effective domain to be the set

$$D(\phi) := \{x \in E : \phi(x) < +\infty\}.$$

**Definition 1.3** A functional $\phi: E \to (-\infty, \infty]$ is said to be:

• proper, if $D(\phi) \neq \emptyset$;
• lower semicontinuous, if $\liminf_{n \to \infty} \phi(x_n) \geq \phi(x)$, whenever $x_n \to x$;
• convex, if $\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y)$, for all $x, y \in E$ and all $t \in [0, 1]$;
• Gâteaux differentiable at $x \in D(\phi)$, if there exists an element $\nabla \phi(x) \in E^*$ (called the gradient of $\phi$ at $x$) such that

$$\lim_{t \to 0^+} \frac{\phi(x + ty) - \phi(x)}{t} = \langle \nabla \phi(x), y \rangle \text{, for all } y \in E.$$

Conversely, we say that $\phi: [-\infty, \infty) \to \mathbb{R}$ is upper semicontinuous (resp. concave) if $-\phi$ is lower semicontinuous (resp. convex).

The following result characterizes the Gâteaux differentiability of convex functionals.

**Proposition 1.1** Let $\phi: E \to \mathbb{R}$ be a Gâteaux differentiable functional. The following statements are equivalent:

(i) $\phi$ is convex;
(ii) $\phi(y) - \phi(x) \geq \langle \nabla \phi(x), y - x \rangle$, for all $y \in E$;
(iii) $\langle \nabla \phi(y) - \nabla \phi(x), y - x \rangle \geq 0$, for all $x, y \in E$.

A direct consequence of the above result is that convex and Gâteaux differentiable functions are in fact lower semicontinuous. Proposition 1.1 also suggests the following generalization of the gradient of a convex function.
Definition 1.4 Let \( \phi : E \to (-\infty, +\infty] \) be a convex function. The subdifferential of \( \phi \) at a point \( x \in D(\phi) \) is the set

\[
\partial \phi(x) := \{ \xi \in E^* : \langle \xi, y - x \rangle \leq \phi(y) - \phi(x) \text{ for all } y \in E \},
\]
and \( \partial \phi(x) := \emptyset \) if \( x \notin D(\phi) \).

It is well known that if \( \phi \) is convex and Gâteaux differentiable at a point \( x \in \text{int } D(\phi) \), then \( \partial \phi(x) \) contains exactly one element, namely, \( \partial \phi(x) = \{ \nabla \phi(x) \} \). Moreover, it is well-known that the set-valued mapping \( \partial \phi : E \to 2^{E^*} \) is monotone, i.e., \( -\partial \phi \) is dissipative.

The Fenchel conjugate of \( \phi : E \to (-\infty, +\infty] \) is the functional \( \phi^* : E^* \to (-\infty, +\infty] \)
defined by

\[
\phi^*(\xi) := \sup_{x \in E} \{ \langle \xi, x \rangle - \phi(x) \} \text{ for all } \xi \in E^*.
\]

Proposition 1.2 Let \( \phi : E \to (-\infty, +\infty] \) be a proper, convex and lower semicontinuous function. Then, the following statements are true:

(i) \( \phi^* \) is proper, convex and lower semicontinuous;
(ii) \( \phi(x) + \phi^*(\xi) \geq \langle \xi, x \rangle \), for all \( x \in E, \xi \in E^* \);
(iii) \( \xi \in \partial \phi(x) \iff x \in \partial \phi^*(\xi) \iff \phi(x) + \phi^*(\xi) = \langle \xi, x \rangle \).

Definition 1.5 A bipotential is a function \( b : E \times E^* \to (-\infty, +\infty] \) satisfying the following conditions:

(i) for any \( x \in E \), if \( D(b(x, \cdot)) \neq \varnothing \), then \( b(x, \cdot) \) is convex and lower semicontinuous, and
for any \( \xi \in E^* \), if \( D(b(\cdot, \xi)) \neq \varnothing \), then \( b(\cdot, \xi) \) is convex and lower semicontinuous;
(ii) for all \( x \in E \) and all \( \xi \in E^* \) it holds \( b(x, \xi) \geq \langle \xi, x \rangle \);
(iii) \( \xi \in \partial b(\cdot, \xi)(x) \iff x \in \partial b(x, \cdot)(\xi) \iff b(x, \xi) = \langle \xi, x \rangle \).

We recall that a function \( h : E \to \mathbb{R} \) is called locally Lipschitz if for every \( x \in E \) there exists a neighborhood \( U \) of \( x \) and a positive constant \( c_x = c_x(U) \) such that

\[
|h(z) - h(y)| \leq c_x \|z - y\| \text{ for all } y, z \in U.
\]

Definition 1.6 Let \( h : E \to \mathbb{R} \) be a locally Lipschitz function. The generalized derivative (in the sense of Clarke) of \( h \) at a point \( x \) in the direction \( y \), denoted by \( h^0(x; y) \), is defined by

\[
h^0(x; y) := \limsup_{\substack{t \to 0 \\, t \neq 0}} \frac{h(z + ty) - h(z)}{t}.
\]

Let \( h : E_1 \times \ldots \times E_n \to \mathbb{R} \) be locally Lipschitz with respect to the \( k^{th} \) variable. So, in the sequel, we denote by \( h^0_k(x_1, \ldots, x_k, \ldots, x_n; y_k) \) the partial generalized derivative of \( h \) with respect to the \( k^{th} \) variable, that is,

\[
h^0_k(x_1, \ldots, x_k, \ldots, x_n; y_k) := \limsup_{\substack{z_k \to x_k \\, t \neq 0}} \frac{h(x_1, \ldots, x_k + ty_k, \ldots, x_n) - h(x_1, \ldots, x_k, \ldots, x_n)}{t}.
\]

The next results recall some important properties of the generalized derivative and subgradient in the sense of Clarke, (for the details proof see Clarke [3]).
Proposition 1.3 Let $h: E \to \mathbb{R}$ be a locally Lipschitz function such that $c_x > 0$ is the constant near the point $x \in E$. Then, we have

(i) the function $y \mapsto h^0(x; y)$ is finite, positively homogeneous, subadditive and satisfies 
$$|h^0(x; y)| \leq c_x \|y\| \text{ for all } x, y \in E;$$

(ii) the function $(x, y) \mapsto h^0(x; y)$ is upper semicontinuous;

(iii) $(-h)^0(x; y) = h^0(x; -y)$.

Definition 1.7 The Clarke subdifferential of a locally Lipschitz function $h: E \to \mathbb{R}$ at a point $x$, denoted by $\partial_C h(x)$, is the following subset of $E^*$
$$\partial_C h(x) := \{ \xi \in E^* : \langle \xi, y \rangle \leq h^0(x; y) \text{ for all } y \in E \}.$$ 

Let $h: E_1 \times \ldots \times E_n \to \mathbb{R}$ be locally Lipschitz with respect to the $k^{th}$ variable. In what follows, we use the similar way to denote the partial Clarke subdifferential of $h$ with respect to the $k^{th}$ variable, thus,
$$\partial_C^k h(x_1, \ldots, x_n) := \{ \xi_k \in E_k^* : \langle \xi_k, x_k \rangle \leq h^0_k(x_1, \ldots, x_k, \ldots, x_n; y_k) \text{ for all } y_k \in E_k \}.$$ 

Proposition 1.4 Let $h: E \to \mathbb{R}$ be a locally Lipschitz functional. Then the following properties hold:

(i) For each $x \in E$, $\partial_C h(x)$ is a nonempty, convex, weak*-compact subset of $E^*$ and
$$\|\zeta\|_{E^*} \leq c_x, \text{ for all } \zeta \in \partial_C h(x),$$
where $c_x$ is the Lipschitz constant near $x$;

(ii) For each $y \in E$ one has $h^0(x; y) = \max_{\zeta \in \partial_C h(x)} \langle \zeta, y \rangle$;

(iii) The set-valued mapping $x \mapsto \partial_C h(x)$ is weakly*-closed;

(iv) The set-valued mapping $x \mapsto \partial_C h(x)$ is upper semicontinuous from $E$ into $E^*$ endowed with the weak*-topology.

We end this section with an alternative theorem for equilibrium problems due to Mosco [13] which will play a center role in proving the existence of solutions for the coupled system of inequalities (1.1).

Theorem 1.1 (Mosco [13]) Let $K$ be a nonempty, compact and convex subset of a topological vector space $E$ and $\varphi: E \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous functional such that $D(\varphi) \cap K \neq \emptyset$. Assume $g, h: E \times E \to \mathbb{R}$ are two functions that satisfy the following conditions:

(a) $g(x, y) \geq h(x, y)$ for all $x, y \in E$;

(b) $y \mapsto h(x, y)$ is convex for each $y \in E$;

(c) $x \mapsto g(x, y)$ is upper semicontinuous for each $x \in E$.

Then, for each $a \in \mathbb{R}$ the following alternative holds:

- either there exists $x_0 \in D(\varphi) \cap K$ such that
  $$g(x_0, y) + \varphi(y) - \varphi(x_0) \geq a \text{ for all } y \in E,$$

- or there exists $y_0 \in E$ such that $h(x_0, x_0) < a$. 

2 Existence results for bounded and unbounded constraint sets

The primary goal of this section is to establish the existence of solutions for the nonlinear coupled system \[(1.1)\]. Our approach is topological. More precisely we employ the alternative Theorem 1.1 for general equilibrium problems to obtain the existence theorems. The first set of assumptions under which we are able to establish the existence of solutions is listed below.

\((H_0)\) The functionals \(\phi, J, L, H\) and the operators \(\gamma_1, \gamma_2\) satisfy the following conditions:
1. \(\phi: X \to (-\infty, \infty]\) is a proper, convex and lower semicontinuous functional such that \(D(\phi) \cap K \neq \emptyset\);
2. \(J: Z_1 \to \mathbb{R}\) and \(L: Z_2 \to \mathbb{R}\) are locally Lipschitz functionals;
3. \(\gamma_1: X \to Z_1\) and \(\gamma_2: Y \to Z_2\) are linear and compact operators;
4. \(H: Y \to (0, m_H)\) is weakly continuous (i.e., \(H(\sigma_\alpha) \to H(\sigma)\) whenever \(\sigma_\alpha \to \sigma\) in \(Y\)) with some \(m_H > 0\).

\((H_B)\) \(B: X \times Y \to \mathbb{R}\) is convex and lower semicontinuous in \(X \times Y\).

\((H_F)\) \(F: X \to X^*\) is a nonlinear operator such that the mapping \(u \mapsto \langle F(u), v - u \rangle\) is weakly lower semicontinuous for each \(v \in X\), i.e.,
\[
\liminf_{n \to \infty} \langle F(u_n), v - u_n \rangle \geq \langle F(u), v - u \rangle,
\]
whenever \(u_n \to u\) in \(X\) as \(n \to \infty\) and \(v \in X\).

\((H_G)\) \(G: Y \to Y^*\) is a nonlinear operator such that the mapping \(\sigma \mapsto \langle G(\sigma), \mu - \sigma \rangle\) is weakly lower semicontinuous for each \(\mu \in Y\), i.e.,
\[
\liminf_{n \to \infty} \langle G(\sigma_n), \mu - \sigma_n \rangle \geq \langle G(\sigma), \mu - \sigma \rangle,
\]
whenever \(\sigma_n \to \sigma\) in \(Y\) as \(n \to \infty\) and \(\mu \in Y\).

Note that, if \(J: Z_1 \to \mathbb{R}\) is Lipschitz continuous, then \((H_0)\)(ii) is automatically fulfilled as \(c_x\) is the same for every \(x \in Z_1\). Moreover, \((H_F)\) is automatically fulfilled if \(F(u) \equiv f\) for some fixed \(f \in X^*\).

**Lemma 2.1** Assume \(X\) and \(Y\) are real reflexive Banach spaces and \(K \subset X\) and \(\Lambda \subset Y\) are nonempty, bounded, closed and convex subsets. If \((H_0), (H_B), (H_F)\) and \((H_G)\) hold, then problem \[(1.1)\] possesses at least one solution.

**Proof** Let \(E := X \times Y\), \(K := K \times \Lambda\) and define \(\varphi: K \to (-\infty, \infty]\) by
\[
\varphi(u, \sigma) := \phi(u) + I_K(u) + I_\Lambda(\sigma),
\]
where \(I_K\) and \(I_\Lambda\) are the indicator functions of \(K\) and \(\Lambda\), respectively, i.e.,
\[
I_K(u) := \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise} \end{cases}, \text{ and } I_\Lambda(\sigma) := \begin{cases} 0, & \text{if } \sigma \in \Lambda \\ +\infty, & \text{otherwise} \end{cases}.
\]
Recall that $K$ and $\Lambda$ are nonempty, bounded, closed and convex subsets of real reflexive Banach spaces $X$ and $Y$, respectively, it follows at once that $\mathcal{K}$ is weakly compact, while $\varphi$ is proper, convex and lower semicontinuous with $D(\varphi) = (K \cap D(\varphi)) \times \Lambda$.

Let us consider the bifunction $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ by

$$g((u,\sigma),(v,\mu)) : = B(v,\mu) - B(u,\sigma) + J^0(\gamma_1(u);\gamma_1(v - u)) - \langle F(u),v - u \rangle + H(\sigma)L^0(\gamma_2(\sigma);\gamma_2(\mu - \sigma)) - \langle G(\sigma),\mu - \sigma \rangle.$$ 

It is not difficult to observe that $g((u,\sigma),(u,\sigma)) = 0$ for all $(u,\sigma) \in \mathcal{K}$. Let $t \in (0,1)$ and $(v_1,\mu_1),(v_2,\mu_2) \in \mathcal{K}$ be arbitrary and $v_t = tv_1 + (1-t)v_2$ and $\mu_t = t\mu_1 + (1-t)\mu_2$. From the convexity of $\mathcal{K} \ni (v,\mu) \mapsto B(v,\mu) \in \mathbb{R}$ and the positive homogeneity and subadditivity of $v \mapsto J^0(\gamma_1(u);\gamma_1(v))$ and $\mu \mapsto L^0(\gamma_2(\sigma);\gamma_1(\mu))$, we have

$$g((u,\sigma),(v_t,\mu_t)) = B(v_t,\mu_t) - B(u,\sigma) + J^0(\gamma_1(u);\gamma_1(v_t - u)) - \langle F(u),v_t - u \rangle + H(\sigma)L^0(\gamma_2(\sigma);\gamma_2(\mu_t - \sigma)) - \langle G(\sigma),\mu_t - \sigma \rangle \leq tB(v_1,\mu_1) + (1-t)B(v_2,\mu_2) + J^0(\gamma_1(u);tv_1(1-v_1) + (1-t)\gamma_1(v_2 - u)) + H(\sigma)L^0(\gamma_2(\sigma);t\gamma_2(\mu_1 - \sigma) + (1-t)\gamma_2(\mu_2 - \sigma)) - B(u,\sigma) - \langle F(u),tv_1(1-v_1) + (1-t)(v_2 - u) \rangle - \langle G(\sigma),t(\mu_1 - \sigma) + (1-t)(\mu_2 - \sigma) \rangle \leq t \left[ B(v_1,\mu_1) + J^0(\gamma_1(u);\gamma_1(v_1 - u)) + H(\sigma)L^0(\gamma_2(\sigma);\gamma_2(\mu_1 - \sigma)) - B(u,\sigma) - \langle F(u),v_1 - u \rangle - \langle G(\sigma),\mu_1 - \sigma \rangle \right] + (1-t) \left[ B(v_2,\mu_2) + J^0(\gamma_1(u);\gamma_1(v_2 - u)) + H(\sigma)L^0(\gamma_2(\sigma);\gamma_2(\mu_2 - \sigma)) - \langle F(u),v_2 - u \rangle - \langle G(\sigma),\mu_2 - \sigma \rangle - B(u,\sigma) \right] \leq \lim\inf_{n \to \infty} B(u_n,\sigma_n) + \lim\sup_{n \to \infty} J^0(\gamma_1(u_n);\gamma_1(v - u_n)) \leq J^0(\gamma_1(u);\gamma_1(v - u)),$$ 

where we have used the compactness of $\gamma_1$. Using hypotheses $H^1_\Phi$ and $H^2_\Gamma$, it yields

$$\lim\inf_{n \to \infty} \langle F(u_n),v - u_n \rangle \geq \langle F(u),v - u \rangle \quad \text{and} \quad \lim\inf_{n \to \infty} \langle G(\sigma_n),\mu - \sigma_n \rangle \geq \langle G(\sigma),\mu - \sigma \rangle.$$
Employing hypotheses \((H_0)\)(iii)-(iv) and the upper semicontinuity of the mapping \(Z_2 \times Z_2 \ni (\sigma, \mu) \mapsto L^0(\sigma; \mu) \in \mathbb{R}\) (see Proposition 1.4), one has

\[
\begin{align*}
\limsup_{n \to \infty} H(\sigma_n)L^0(\gamma_2(\sigma_n); \gamma_2(\mu - \sigma_n)) & \leq \limsup_{n \to \infty} |H(\sigma_n) - H(\sigma)| L^0(\gamma_2(\sigma_n); \gamma_2(\mu - \sigma_n)) \\
& + \limsup_{n \to \infty} H(\sigma)L^0(\gamma_2(\sigma_n); \gamma_2(\mu - \sigma_n)) \\
& \leq \limsup_{n \to \infty} H(\sigma)L^0(\gamma_2(\sigma_n); \gamma_2(\mu - \sigma_n)) \\
& \leq H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)),
\end{align*}
\]

where we have applied the compactness of \(\gamma_2\) and the following estimates

\[
|L^0(\gamma_2(\sigma_n); \gamma_2(\mu - \sigma_n))| = \max_{\zeta \in \partial_{\mathcal{C}} J(\gamma_2(\sigma_n))} \langle \zeta, \gamma_2(\mu - \sigma_n) \rangle = \|\zeta_n\| \|\gamma_2(\mu - \gamma_2(\sigma_n))\| \leq c_\mu \|\gamma_2(\mu - \gamma_2(\sigma_n))\|.
\]

The last inequality is obtained by using the fact that \(\gamma_2(\sigma_n) \to \gamma_2(\sigma)\) as \(n \to \infty\) and Proposition 1.3 and \(c_\sigma > 0\) is the Lipschitz constant of \(L\) near the point \(\gamma_2 \sigma \in Z_2\). Taking into account \((2.1), (2.4)\), we infer that

\[
\limsup_{n \to \infty} g((u_n, \sigma_n), (v, \mu)) \leq g((u, \sigma), (v, \mu)),
\]

namely, for each \((v, \mu) \in K\) the mapping \(K \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R}\) is weakly upper semicontinuous.

Therefore, all conditions of Theorem 1.1 are verified. So, we are now in position to apply this theorem with \(a := 0, h := g\) and \(E\) endowed with the weak topology to conclude that there exists an element \((u_0, \sigma_0) \in K \cap D(\varphi)\) such that

\[
g((u_0, \sigma_0), (v, \mu)) + \varphi(v, \mu) - \varphi(u_0, \sigma_0) \geq 0 \text{ for all } (v, \mu) \in E.
\]

Choosing \(\mu := \sigma_0\) in \((2.5)\), we get that

\[
B(v, \sigma_0) - B(u_0, \sigma_0) + J^0(\gamma_1(u_0); \gamma_1(v - u_0)) - \langle F(u_0), v - u_0 \rangle + \phi(v) - \phi(u_0) \geq 0
\]

for all \(v \in K\). Whereas, if we take \(v := u_0\) in \((2.5)\), then

\[
B(u_0, \mu) - B(u_0, \sigma_0) + H(\sigma_0)L^0(\gamma_2(\sigma_0); \gamma_2(\mu - \sigma_0)) - \langle G(\sigma_0), \mu - \sigma_0 \rangle \geq 0
\]

for all \(\mu \in \Lambda\). This shows that \((u_0, \sigma_0)\) is a solution of problem \((1.1)\). \(\square\)

In hypotheses \((H^-)\) and \((H^G)\), we require that \(u \mapsto \langle F(u), v - u \rangle\) and \(\sigma \mapsto \langle G(\sigma), \mu - \sigma \rangle\) are both weakly lower semicontinuous. However, this assumption strictly limits the application of our result, Lemma 2.1. In order to extend the scope of applications to problem \((1.1)\), we, further, make necessary changes to the assumptions on \(F\) and \(G\), respectively. Under these changes, we obtain a wide variety of existence results to problem \((1.1)\).

To this end, we impose the following assumptions to functions \(F\) and \(G\).
The conclusion of Lemma 2.1 still holds if we replace (H^j_k) by (H^j_k) or (H^j_k) by (H^j_k), with j, k ∈ {1, 2, 3}.

**Proof** As in the proof of Lemma 2.1 let $E := X \times Y$, $K := K \times \Lambda$ and $\varphi: K \to (-\infty, \infty]$ be defined by

$$\varphi(u, \sigma) := \phi(u) + I_K(u) + I_\Lambda(\sigma).$$

**Case 1. Assume (H^2_k) and (H^3_k) hold.**

Let us define the functions $g, h: K \times K \to \mathbb{R}$ by

$$g((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(v), v - u \rangle
+ H(\mu) L^0(\gamma_2(\mu); \gamma_2(\mu - \sigma)) - \langle G(\mu), \mu - \sigma \rangle$$

and

$$h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle
- \beta_F(v - u) + H(\sigma) L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle - \beta_S(\mu - \sigma).$$

Note that $h((u, \sigma), (v, \mu)) = -\beta_F(0) - \beta_S(0) \geq 0$ for all $(u, \sigma) \in K$. Arguing as in the proof of Lemma 2.1 it is readily seen that $K \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R}$ is weakly upper semicontinuous and $K \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R}$ is convex. Moreover, hypothesis (H^3_k) and Proposition 1.3 point out that there exists $\xi_\sigma \in \partial CL(\gamma_2(\sigma))$ and $\xi_\mu \in \partial CL(\gamma_2(\mu))$ such that

$$L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) = \langle \xi_\sigma, \gamma_2(\mu - \sigma) \rangle_{Z_2^* \times Z_2} = \langle \gamma_2^*(\xi_\sigma), \mu - \sigma \rangle_{Y^* \times Y},$$

and

$$L^0(\gamma_2(\mu); \gamma_2(\mu - \sigma)) = \langle \xi_\mu, \gamma_2(\mu - \sigma) \rangle_{Z_2^* \times Z_2} = \langle \gamma_2^*(\xi_\mu), \mu - \sigma \rangle_{Y^* \times Y}.$$
Thus,
\[
g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) = \beta_F(v - u) - \langle F(v) - F(u), v - u \rangle + \beta_S(\mu - \sigma) - \langle G(\mu) - H(\mu)\gamma_2^*(\xi_\mu) - G(\sigma) + H(\sigma)\gamma_2^*(\xi_\sigma), \mu - \sigma \rangle \geq 0.
\]

We apply again Theorem 1.1 with \(a := 0\) to get the existence of \((u_0, \sigma_0) \in K \cap D(\varphi)\) such that
\[
g((u_0, \sigma_0), (w, \lambda)) + \varphi(w, \lambda) - \varphi(u_0, \sigma_0) \geq 0 \quad \text{for all } (w, \lambda) \in E.
\]
Let \((v, \mu) \in K\) be fixed. Then for any \(t \in (0, 1)\), by inserting \((w, \lambda) := (u_0 + t(v - u_0), \sigma_0)\) in (2.6) we get
\[
0 \leq B(u_0 + t(v - u_0), \sigma_0) - B(u_0, \sigma_0) + J^0(\gamma_1(u_0); t(\gamma_1(v - u_0))) - \langle F(u_0 + t(v - u_0)), t(v - u_0) \rangle + \phi(u_0 + t(v - u_0)) - \phi(u_0)
\]
\[
\leq t \bigg[ B(v, \sigma_0) - B(u_0, \sigma_0) + J^0(\gamma_1(u_0); \gamma_1(v - u_0)) - \langle F(u_0 + t(v - u_0)), v - u_0 \rangle + \phi(v - \phi(u_0)) \bigg],
\]
where we have utilized the convexity of \(B\) and \(\phi\) as well as the positivity homogeneity and subadditivity of \(v \mapsto J^0(w, v)\). Dividing by \(t > 0\) and then letting \(t \to 0^+\), we apply the inequality of (1.1) to get the first inequality of (1.1). In order to get the second inequality of (1.1), we fix \(t \in (0, 1)\) and choose \((w, \lambda) := (u_0, \sigma_0 + t(\mu - \sigma_0))\) in (2.6) to get
\[
0 \leq B(u_0, \sigma_0 + t(\mu - \sigma_0)) - B(u_0, \sigma_0) - \langle G(\sigma_0 + t(\mu - \sigma_0)), t(\mu - \sigma_0) \rangle + H(\sigma_0 + t(\mu - \sigma_0))L^0(\gamma_2(\sigma_0 + t(\mu - \sigma_0)); t\gamma_2(\mu - \sigma_0))
\]
\[
\leq t \bigg[ B(v, \sigma_0) - B(u_0, \sigma_0) - \langle G(\sigma_0 + t(\mu - \sigma_0)), \mu - \sigma_0 \rangle + H(\sigma_0 + t(\mu - \sigma_0))L^0(\gamma_2(\sigma_0 + t(\mu - \sigma_0)); \gamma_2(\mu - \sigma_0)) \bigg].
\]
Dividing again by \(t > 0\), then taking the lim sup as \(t \to 0^+\) we get that \((u_0, \sigma_0)\) also satisfies the second inequality of (1.1).

**Case 2. Assume that \((H^2_1, H_2^3)\) hold.**

We define the functions \(g, h : K \times K \to \mathbb{R}\) by
\[
g((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(v), v - u \rangle + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\mu), \mu - \sigma \rangle
\]
and
\[
h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle - \beta_F(v - u) + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle - \beta_S(\mu - \sigma).
\]
It is obvious that \(h((u, \sigma), (u, \sigma)) = -\beta_F(0) - \beta_S(0) \geq 0\) for all \((u, \sigma) \in K\). Moreover, the mapping \(K \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R}\) is weakly upper semicontinuous and
\[ K \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R} \text{ is convex. Additionally, we apply hypotheses } (H^2_0) \text{ and } (H^2_G) \text{ to get} \]

\[
g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) = \beta_F(v - u) - \langle F(v) - F(u), v - u \rangle + \beta_G(\mu - \sigma) - \langle G(\mu) - G(\sigma), \mu - \sigma \rangle \geq 0.
\]

This allows us to employ Theorem 1.1 with \( a := 0 \) to find \((u_0, \sigma_0) \in K \cap D(\varphi) \) such that (2.6) holds. Let \((v, \mu) \in K \) be fixed. Inserting \((w, \lambda) := (u_0 + t(v - u_0), \sigma_0)\) with \( t \in (0, 1) \) in (2.6) we get

\[
0 \leq t \left[ B(v, \sigma_0) - B(u_0, \sigma_0) + J^0(\gamma_1(u_0); \gamma_1(v - u_0)) - \langle F(u_0 + t(v - u_0), v - u_0 \rangle + \phi(v) - \phi(u_0) \right].
\]

Hence, from the hemicontinuity of \( F \), we obtain the first inequality of (1.1). Let \( t \in (0, 1) \) be arbitrary and put \((w, \lambda) := (u_0, \sigma_0 + t(\mu - \sigma_0))\) in (2.6). We have

\[
0 \leq B(u_0, \sigma_0 + t(\mu - \sigma_0)) - B(u_0, \sigma_0) - \langle G(\sigma_0 + t(\mu - \sigma_0)), t(\mu - \sigma_0) \rangle + H(\sigma_0) L^0(\gamma_2(\sigma_0); t(\gamma_2(\mu - \sigma_0))) \leq t \left[ B(u, \mu) - B(u_0, \sigma_0) - \langle G(\sigma + t(\mu - \sigma_0)), \mu - \sigma_0 \rangle + H(\sigma_0) L^0(\gamma_2(\sigma_0); \gamma_2(\mu - \sigma_0)) \right].
\]

Invoking the hemicontinuity of \( G \), we obtain the second inequality of (1.1).

**Case 3.** Suppose that \((H^0_0)\) and \((H^2_G)\) are satisfied.

Let \( g, h : K \times K \rightarrow \mathbb{R} \) be defined by

\[
g((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(v); \gamma_1(v - u)) - \langle F(v), v - u \rangle + H(\sigma) L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\mu), \mu - \sigma \rangle
\]

and

\[
h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle - \beta_T(v - u) + H(\sigma) L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\mu) - G(\sigma), \mu - \sigma \rangle - \beta_G(\mu - \sigma).
\]

It is obvious that \( h((u, \sigma), (u, \sigma)) = -\beta_T(0) - \beta_S(0) \geq 0 \) for all \((u, \sigma) \in K\). Again, \( K \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R} \) is weakly upper semicontinuous and \( K \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R} \) is convex. Additionally, we use hypotheses \((H^0_F)\) and \((H^2_G)\) to get

\[
g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) = \beta_T(v - u) - \langle F(v) - \gamma^*_1 \xi_u - F(u) + \gamma^*_1 \xi_v, v - u \rangle + \beta_G(\mu - \sigma) - \langle G(\mu) - G(\sigma), \mu - \sigma \rangle \geq 0,
\]

where \( \xi_u \in \partial_C J(\gamma_1 u) \) and \( \xi_v \in \partial J_C(\gamma_1 v) \) are such that

\[
J^0(\gamma_1(u); \gamma_1(v - u)) = \langle \xi_u, \gamma_1(v - u) \rangle_{Z^*_1 \times Z_1} = \langle \gamma^*_1(\xi_u), u - v \rangle_{X^* \times X},
\]

and

\[
J^0(\gamma_1(v); \gamma_1(v - u)) = \langle \xi_v, \gamma_1(v - u) \rangle_{Z^*_1 \times Z_1} = \langle \gamma^*_1(\xi_v), u - v \rangle_{X^* \times X}.
\]
We are now in a position to apply Theorem 1.1 with \( a := 0 \) to deduce that there exists an element \((u_0, \sigma_0) \in \mathcal{K} \cap \mathcal{D}(\varphi)\) such that (2.6) holds. Let \((v, \mu) \in \mathcal{K}\) be fixed. Then for any \(t \in (0, 1)\), we insert \((w, \lambda) := (u_0 + t(v - u_0), \sigma_0)\) in (2.6) to get
\[
0 \leq t \left[ B(v, \sigma_0) - B(u_0, \sigma_0) + J^0(\gamma_1(u_0 + t(v - u_0)); \gamma_1(v - u_0)) - \langle F(u_0 + t(v - u_0)), v - u_0 \rangle + \phi(v) - \phi(u_0) \right].
\]

Hence, from the hemicontinuity of \(F\) and upper semicontinuity of \(Z_1 \ni u \mapsto J^0(u; v)\), we obtain the first inequality of (1.1). For \(t \in (0, 1)\) plugging \((w, \lambda) := (u_0, \sigma_0 + t(\mu - \sigma_0))\) in (2.6) we have
\[
0 \leq t \left[ B(u_0, \mu) - B(u_0, \sigma_0) - \langle G(\sigma_0 + t(\mu - \sigma_0)), \mu - \sigma_0 \rangle + H(\sigma_0)L^0(\gamma_2(\sigma_0); \gamma_2(\mu - \sigma_0)) \right].
\]

Invoking the hemicontinuity of \(G\) the second inequality of (1.1) is obtained.

**Case 4.** Let \((H^3_F)\) and \((H^3_G)\) be fulfilled.

Consider the functions \(g, h: \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}\) defined by
\[
g((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(v); \gamma_1(v - u)) - \langle F(v), v - u \rangle + H(\mu)L^0(\gamma_2(\mu); \gamma_2(\mu - \sigma)) - \langle G(\mu), \mu - \sigma \rangle
\]
and
\[
h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle - \beta_T(v - u) + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle - \beta_S(\mu - \sigma).
\]

Then, we have \(h((u, \sigma), (u, \sigma)) = -\beta_T(0) - \beta_S(0) \geq 0\) for all \((u, \sigma) \in \mathcal{K}\), the mapping \(\mathcal{K} \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R}\) is weakly upper semicontinuous, while the mapping \(\mathcal{K} \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R}\) is convex. Let \(\xi_\sigma \in \partial_C L(\gamma_2(\sigma))\) and \(\xi_\mu \in \partial_C L(\gamma_2(\mu))\) be such that
\[
L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) = \langle \xi_\sigma, \gamma_2(\mu - \sigma) \rangle_{Z^*_2 \times Z_2} = \langle \gamma_2^*(\xi_\sigma), \mu - \sigma \rangle_{Y^\ast \times Y},
\]
and
\[
L^0(\gamma_2(\mu); \gamma_2(\mu - \sigma)) = \langle \xi_\mu, \gamma_2(\mu - \sigma) \rangle_{Z^*_2 \times Z_2} = \langle \gamma_2^*(\xi_\mu), \mu - \sigma \rangle_{Y^\ast \times Y}.
\]

We apply hypotheses \((H^3_F)\) and \((H^3_G)\) to get
\[
g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) = \beta_T(v - u) - \langle F(v), v - u \rangle - \gamma_1^*\xi_u - F(u) + \gamma_1^*\xi_v, v - u \rangle + \beta_S(\mu - \sigma) - \langle G(\mu), \mu - \sigma \rangle - H(\sigma)L^0(\gamma_2(\xi_\sigma); \gamma_2(\xi_\mu), \mu - \sigma) \geq 0,
\]
where \(\xi_u \in \partial_C J(\gamma_1 u)\) and \(\xi_v \in \partial J_C(\gamma_1 v)\) are such that
\[
J^0(\gamma_1(u); \gamma_1(v - u)) = \langle \xi_u, \gamma_1(v - u) \rangle_{Z^*_1 \times Z_1} = \langle \gamma_1^*(\xi_u), u - v \rangle_{X^\ast \times X},
\]
and
\[
J^0(\gamma_1(v); \gamma_1(v - u)) = \langle \xi_v, \gamma_1(v - u) \rangle_{Z^*_1 \times Z_1} = \langle \gamma_1^*(\xi_v), u - v \rangle_{X^\ast \times X}.
\]
We apply Theorem 1.1 with $a := 0$ to find an element $(u_0, \sigma_0) \in K \cap D(\varphi)$ such that (2.6) is fulfilled. Let $(v, \mu) \in K$ be fixed. Then for any $t \in (0, 1)$, we insert $(w, \lambda) := (u_0 + t(v - u_0), \sigma_0)$ in (2.6) to get

$$\begin{align*}
  0 \leq t \left[ B(v, \sigma_0) - B(u_0, \sigma_0) + J^0(\gamma_1(u_0 + t(v - u_0)); \gamma_1(v - u_0)) \right. \\
  - \left. \langle F(u_0 + t(v - u_0)), v - u_0 \rangle + \phi(v) - \phi(u_0) \right].
\end{align*}$$

Hence, from the hemicontinuity of $F$ and upper semicontinuity of $Z_1 \ni u \mapsto J^0(u; v)$, we obtain the first inequality of (1.1). Let $t \in (0, 1)$ be arbitrary and put $(w, \lambda) := (u_0, \sigma_0 + t(\mu - \sigma_0))$ in (2.6). We have

$$\begin{align*}
  0 \leq t \left[ B(u_0, \mu) - B(u_0, \sigma_0) - \langle G(\sigma_0 + t(\mu - \sigma_0)), \mu - \sigma_0 \rangle \\
  + H(\sigma_0 + t(\mu - \sigma_0))L^0(\gamma_2(\sigma_0 + t(\mu - \sigma_0)); \gamma_2(\mu - \sigma_0)) \right].
\end{align*}$$

Invoking the hemicontinuity of $G$ and upper semicontinuity of $Y \ni \sigma \mapsto H(\sigma)L^0(\gamma_2(\mu; \mu)$, we get the second inequality of (1.1).

**Case 5.** Let $(H_B^1)$ and $(H_G^2)$ be fulfilled.

Let us introduce the functions $g, h : K \times K \to \mathbb{R}$ given by

$$\begin{align*}
  g((u, \sigma), (v, \mu)) := & B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v) - \gamma_1(u)) - \langle F(u), v - u \rangle \\
  & + H(\sigma)L^0(\gamma_2(\gamma_2(\mu) - \mu(\mu - \sigma)) - \langle G(\mu), \mu - \sigma \rangle
\end{align*}$$

and

$$\begin{align*}
  h((u, \sigma), (v, \mu)) := & B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v) - \gamma_1(u)) - \langle F(u), v - u \rangle \\
  & + H(\sigma)L^0(\gamma_2(\mu) - \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle - \beta_G(\mu - \sigma)
\end{align*}$$

for all $(u, \sigma), (v, \mu) \in K$.

Then, we have that $h((u, \sigma), (u, \sigma)) = -\beta_G(0) \geq 0$ for all $(u, \sigma) \in K$, the mapping $K \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R}$ is weakly upper semicontinuous, while the mapping $K \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R}$ is convex. It follows from hypothesis $(H_G^2)$ that

$$g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) = \beta_G(\mu - \sigma) - \langle G(\mu) - G(\sigma), \mu - \sigma \rangle \geq 0.$$

We apply Theorem 1.1 with $a := 0$ to find an element $(u_0, \sigma_0) \in K \cap D(\varphi)$ such that (2.6) is fulfilled. Let $(v, \mu) \in K$ be fixed. Then for any $t \in (0, 1)$, we insert $(w, \lambda) := (v_0, \sigma_0 + t(\mu - \sigma_0))$ in (2.6), respectively, and pass to the upper limit as $t \to 0^+$ for the second resulting inequality to get the (1.1).

**Case 6.** Let $(H_B^1)$ and $(H_G^2)$ be fulfilled.

Let us introduce the functions $g, h : K \times K \to \mathbb{R}$ given by

$$\begin{align*}
  g((u, \sigma), (v, \mu)) := & B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle \\
  & + H(\mu)L^0(\gamma_2(\mu) - \gamma_2(\mu - \sigma)) - \langle G(\mu), \mu - \sigma \rangle
\end{align*}$$
and
\[ h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle - \beta_S(\mu - \sigma). \]
for all \((u, \sigma), (v, \mu) \in \mathcal{K}\).

Then, we have that \( h((u, \sigma), (u, \sigma)) = -\beta_S(0) \geq 0 \) for all \((u, \sigma) \in \mathcal{K}\), the mapping \( \mathcal{K} \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R} \) is weakly upper semicontinuous, while the mapping \( \mathcal{K} \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R} \) is convex. It follows from hypothesis (H2_3) that \( g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) \geq 0 \).

We apply Theorem 1.1 with \( a := 0 \) to find an element \((u_0, \sigma_0) \in \mathcal{K} \cap D(\varphi)\) such that (2.6) is fulfilled. Let \((v, \mu) \in \mathcal{K}\) be fixed. Then for any \( t \in (0, 1)\), we insert \((w, \lambda) := (v, \sigma_0)\) and \((w, \lambda) := (u_0, \sigma_0 + t(\mu - \sigma_0))\) in (2.6), respectively, and pass to the upper limit as \( t \to 0^+\) for the second resulting inequality to get the (1.1).

**Case 7.** Let (H2_2) and (H1_2) be fulfilled.

Let us introduce the functions \( g, h : \mathcal{K} \times \mathcal{K} \to \mathbb{R} \) given by
\[ g((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(v), v - u \rangle + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle \]
and
\[ h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle - \beta_F(v - u) + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle. \]
for all \((u, \sigma), (v, \mu) \in \mathcal{K}\).

Then, we have that \( h((u, \sigma), (u, \sigma)) = -\beta_F(0) \geq 0 \) for all \((u, \sigma) \in \mathcal{K}\), the mapping \( \mathcal{K} \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R} \) is weakly upper semicontinuous, while the mapping \( \mathcal{K} \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R} \) is convex. It follows from hypothesis (H2_2) that \( g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) \geq 0 \).

We apply Theorem 1.1 with \( a := 0 \) to find an element \((u_0, \sigma_0) \in \mathcal{K} \cap D(\varphi)\) such that (2.6) is fulfilled. Let \((v, \mu) \in \mathcal{K}\) be fixed. Then for any \( t \in (0, 1)\), we insert \((w, \lambda) := (u_0, \sigma_0)\) and \((w, \lambda) := (u_0, \mu)\) in (2.6), respectively, and pass to the upper limit as \( t \to 0^+\) for the first resulting inequality to get the (1.1).

**Case 8.** Let (H2_2) and (H1_3) be fulfilled.

Consider the functions \( g, h : \mathcal{K} \times \mathcal{K} \to \mathbb{R} \) given by
\[ g((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(v); \gamma_1(v - u)) - \langle F(v), v - u \rangle + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle \]
and
\[ h((u, \sigma), (v, \mu)) := B(v, \mu) - B(u, \sigma) + J^0(\gamma_1(u); \gamma_1(v - u)) - \langle F(u), v - u \rangle - \beta_T(v - u) + H(\sigma)L^0(\gamma_2(\sigma); \gamma_2(\mu - \sigma)) - \langle G(\sigma), \mu - \sigma \rangle. \]
for all $(u, \sigma), (v, \mu) \in K$.

Then, we have that $h((u, \sigma), (u, \sigma)) = -\beta_T(0) \geq 0$ for all $(u, \sigma) \in K$, the mapping $K \ni (u, \sigma) \mapsto g((u, \sigma), (v, \mu)) \in \mathbb{R}$ is weakly upper semicontinuous, while the mapping $K \ni (v, \mu) \mapsto h((u, \sigma), (v, \mu)) \in \mathbb{R}$ is convex. It follows from hypothesis $(H_3^k)$ that $g((u, \sigma), (v, \mu)) - h((u, \sigma), (v, \mu)) \geq 0$.

We apply Theorem 2.1 with $a := 0$ to find an element $(u_0, \sigma_0) \in K \cap D(\varphi)$ such that $(2.6)$ is fulfilled. Let $(v, \mu) \in K$ be fixed. Then for any $t \in (0, 1)$, we insert $(w, \lambda) := (tv + (1 - t)u_0, \sigma_0)$ and $(w, \lambda) := (u_0, \mu)$ in (2.6), respectively, and pass to the upper limit as $t \to 0^+$ for the first resulting inequality to get the $(1.1)$.

This completes the proof of the lemma. □

We end this section with an existence result when at least one of the constraint sets is unbounded. To this end, we need a coercivity condition. Here, and hereafter, we consider the space $X \times Y$ is endowed with the norm $\|(u, \sigma)\| := \sqrt{\|u\|_X^2 + \|\sigma\|_Y^2}$. We consider the following coercivity condition:

$(C)$ There exist $\bar{w}_0 \in K \cap D(\phi)$ and $\bar{r}_0 \in \Lambda$ such that

$$\frac{2B(u, \sigma) - B(\bar{w}_0, \sigma) - B(u, \bar{r}_0) + \langle F(u), \bar{w}_0 - u \rangle + \langle G(\sigma), \bar{r}_0 - \sigma \rangle}{\|(u, \sigma)\|} \to \infty, \text{ as } \|(u, \sigma)\| \to \infty.$$

We also consider $(H'_0)$ to be set of assumptions obtained from $(H_0)$ by replacing (ii) with the slightly stronger condition:

$(i'i')$ $J : Z_1 \to \mathbb{R}$ and $L : Z_2 \to \mathbb{R}$ are locally Lipschitz functionals and there exist $\alpha_J > 0$ and $\alpha_L > 0$ such that

$$\sup_{x \in Z_1} c_x \leq \alpha_J \text{ and } \sup_{y \in Z_2} c_y \leq \alpha_L,$$

where $c_x > 0$ (resp. $c_y > 0$) denotes the Lipschitz constant of $J$ (resp. $L$) near the point $x \in Z_1$ (resp. $y \in Z_2$).

**Theorem 2.1** Let $X, Y$ be real reflexive Banach spaces. Assume that $K \subseteq X$, $\Lambda \subseteq Y$ are nonempty closed convex sets such that either $K \times \Lambda$ is bounded or $(C)$ is satisfied. If, in addition, $(H'_0), (H_B), (H_F^j), (H_G^k)$ with $j, k \in \{1, 2, 3\}$, then problem $(1.1)$ possesses at least one solution in $K \times \Lambda$.

**Proof** If $K \times \Lambda$ is bounded, then the desired conclusion can be obtained directly by using Lemmas 2.3 and 2.2.

So, we assume now that $K \times \Lambda$ is unbounded. Let $r_0 > 0$ be sufficiently large such that $K_r := K \cap B_X(\bar{w}_0, r)$ and $\Lambda_r := \Lambda \cap B_Y(\bar{r}_0, r)$ are both nonempty for all $r \geq r_0$. Let $r \geq r_0$ be arbitrary fixed. Next, we consider the following intermediate problem:

Find $(u, \sigma) \in (K_r \cap D(\phi)) \times \Lambda_r$ such that

$$\begin{cases}
B(v, \sigma) - B(u, \sigma) + \phi(v) - \phi(u) + J^0(\gamma_1 u; \gamma_1(v - u)) \geq \langle F(u), v - u \rangle, \\
B(u, \mu) - B(u, \sigma) + H(\sigma)L^0(\gamma_2 \sigma; \gamma_2(\mu - \sigma)) \geq \langle G(\sigma), \mu - \sigma \rangle,
\end{cases}$$

(2.7)

for all $v \in K_r$ and all $\mu \in \Lambda_r$. 

In virtue of Lemmas \textit{2.3} and \textit{2.2} we can see that problem \textit{2.7} has at least one solution, say \((u_r, \sigma_r) \in K_r \times A_r\).

We claim that there exists a constant \(r_1 \geq r_0\) such that each solution of problem \textit{2.7} with \(r = r_1\) satisfies the following inequality
\[
\max\{\|u_r - \bar{w}_0\|_X, \|\sigma_r - \bar{\sigma}_0\|_Y\} < r_1.
\] (\textit{2.8})

Arguing by contradiction, we assume that for each \(r \geq r_0\) it holds
\[
\max\{\|u_r - \bar{w}_0\|_X, \|\sigma_r - \bar{\sigma}_0\|_Y\} = r.
\]

Then, plugging \((\nu, \mu) := (\bar{w}_0, \bar{\sigma}_0)\) into \textit{2.7} we find that
\[
\begin{align*}
\begin{cases}
B(\bar{w}_0, \sigma_r) - B(u_r, \sigma_r) + \phi(\bar{w}_0) - \phi(u_r) + J^0(\gamma_1(u_r); \gamma_1(\bar{w}_0 - u_r)) \geq \langle F(u_r), \bar{w}_0 - u_r \rangle, \\
B(u_r, \bar{\sigma}_0) - B(u_r, \sigma_r) + H(\sigma_r)L^0(\gamma_2(\sigma_r); \gamma_2(\bar{\sigma}_0 - \sigma_r)) \geq \langle G(\sigma_r), \bar{\sigma}_0 - \sigma_r \rangle.
\end{cases}
\end{align*}
\]

Summing up the last two inequalities and we get
\[
\begin{align*}
\phi(\bar{w}_0) - \phi(u_r) - \langle \zeta_u, \gamma_1(\bar{w}_0 - u_r) \rangle - H(\sigma_r)\langle \zeta_{\sigma}, \gamma_2(\bar{\sigma}_0 - \sigma_r) \rangle & \geq 2B(u_r, \sigma_r) - B(u_r, \bar{\sigma}_0) \\
& - B(\bar{w}_0, \sigma_r) + \langle F(u_r), \bar{w}_0 - u_r \rangle - \langle G(\sigma_r), \bar{\sigma}_0 - \sigma_r \rangle,
\end{align*}
\] (\textit{2.9})

where \(\zeta_u \in \partial C J(\gamma_1(u_r))\) and \(\zeta_{\sigma} \in \partial C L(\gamma_2(\sigma_r))\) are such that
\[
\begin{align*}
\langle \zeta_u, \gamma_1(\bar{w}_0 - u_r) \rangle z_1^1 \times z_1^2 & = \max_{\zeta \in \partial C J(\gamma_1(u_r))} \langle \zeta, \gamma_1(\bar{w}_0 - u_r) \rangle z_1^1 \times z_1^2, \\
\langle \zeta_{\sigma}, \gamma_2(\bar{\sigma}_0 - \sigma_r) \rangle z_2^1 \times z_2^2 & = \max_{\zeta \in \partial C L(\gamma_2(\sigma_r))} \langle \zeta, \gamma_2(\bar{\sigma}_0 - \sigma_r) \rangle z_2^1 \times z_2^2.
\end{align*}
\]

Since any convex and l.s.c. functional is bounded below (see, e.g., \textit{2.} Proposition 1.10) there exist constants \(\alpha_\phi, \beta_\phi \geq 0\) such that
\[
\phi(u) \geq -\alpha_\phi \|u\|_X - \beta_\phi, \ \forall u \in X.
\] (\textit{2.10})

Inserting \textit{2.10} into \textit{2.9}, we have
\[
2B(u_r, \sigma_r) - B(u_r, \bar{\sigma}_0) - B(\bar{w}_0, \sigma_r) \langle F(u_r), \bar{w}_0 - u_r \rangle - \langle G(\sigma_r), \bar{\sigma}_0 - \sigma_r \rangle \leq \phi(\bar{w}_0) + \alpha_\phi \|u\|_X + \beta_\phi + \|\langle \zeta_u, \gamma_1(\bar{w}_0 - u_r) \rangle \| + H(\sigma_r)\|\langle \zeta_{\sigma}, \gamma_2(\bar{\sigma}_0 - \sigma_r) \rangle \|.
\] (\textit{2.11})

Dividing \textit{2.11} by \(\sqrt{\|u_r\|_X^2 + \|\sigma_r\|_Y^2}\) and letting \(r \to \infty\), it gives
\[
\begin{align*}
\infty = \lim_{r \to \infty} \frac{2B(u_r, \sigma_r) - B(u_r, \bar{\sigma}_0) - B(\bar{w}_0, \sigma_r) + \langle F(u_r), \bar{w}_0 - u_r \rangle + \langle G(\sigma_r), \bar{\sigma}_0 - \sigma_r \rangle}{\| (u_r, \sigma_r) \|} \\
& \leq \lim_{r \to \infty} \frac{\phi(\bar{w}_0) + \alpha_\phi \|u\|_X + \beta_\phi + \|\langle \zeta_u, \gamma_1(\bar{w}_0 - u_r) \rangle \| + H(\sigma_r)\|\langle \zeta_{\sigma}, \gamma_2(\bar{\sigma}_0 - \sigma_r) \rangle \|}{\| (u_r, \sigma_r) \|} \\
& \leq \alpha_\phi + \alpha_J \|\gamma_1\| + \alpha_L m_H \|\gamma_2\|,
\end{align*}
\]
which is obviously a contradiction. Therefore, we conclude that there exists a constant \(r_1 > 0\) such that inequality \textit{2.8} is satisfied.

Let \((u_{r_1}, \sigma_{r_1}) \in K_{r_1} \times A_{r_1}\) be a solution to problem \textit{2.7} with \(r = r_1\). Furthermore, we are going to show that \((u_{r_1}, \sigma_{r_1})\) solves problem \textit{1.1} as well. Let \((v, \mu) \in K \times A\) be arbitrarily
fixed. It follows from (2.8) (i.e., \((u_{r_1}, \sigma_{r_1}) \in B_X(\bar{\omega}_0, r_1) \times B_Y(\bar{\tau}_0, r_1)\)) that there exists a constant \(t \in (0, 1)\) such that
\[
(w, \lambda) := (u_{r_1}, \sigma_{r_1}) + t(v - u_{r_1}, \mu - \sigma_{r_1}) \in K_{r_1} \times A_{r_1}.
\]
Inserting \((u, \sigma) = (u_{r_1}, \sigma_{r_1})\) and \((v, \mu) = (w, \lambda)\) into the first inequality of (2.7), we have
\[
t(F(u_{r_1}, v - u_{r_1})) \leq B(w, \sigma_{r_1}) - B((u_{r_1}, \sigma_{r_1})) + \phi(w) - \phi(u_{r_1}) + J^0(\gamma_1(u_{r_1}); \gamma_1(w - u_{r_1}))
\leq t \left[ B(v, \sigma_{r_1}) - B(u_{r_1}, \sigma_{r_1}) + \phi(v) - \phi(u_{r_1}) + J^0(\gamma_1(u_{r_1}); \gamma_1(v - u_{r_1})) \right].
\]
Dividing by \(t > 0\) we infer directly that \((u_{r_1}, \sigma_{r_1})\) satisfies the first inequality of \((1.1)\). Likewise, we also can prove that \((u_{r_1}, \sigma_{r_1})\) satisfies the second inequality of \((1.1)\). This means that \((u_{r_1}, \sigma_{r_1}) \in K_{r_1} \times A_{r_1}\) is a solution to problem \((1.1)\), hence the proof is now complete. \(\square\)

3 Applications to Contact Mechanics

Throughout this section \(S^m\) denotes the the linear space of second order symmetric tensors on \(\mathbb{R}^m\), i.e., \(S^m = \mathbb{R}^{m \times m}\). The inner products and the corresponding norms on \(\mathbb{R}^m\) and \(S^m\) are defined by
\[
\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad |\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{1/2}, \quad \mathbf{u} := (u_i), \quad \mathbf{v} := (v_i) \in \mathbb{R}^m,
\]
and, respectively,
\[
\mathbf{\tau} \cdot \mathbf{\sigma} = \tau_{ij} \sigma_{ij}, \quad |\mathbf{\tau}| = (\mathbf{\tau} \cdot \mathbf{\tau})^{1/2}, \quad \mathbf{\tau} := (\tau_{ij}), \quad \mathbf{\sigma} := (\sigma_{ij}) \in S^m.
\]
Here and hereafter, \(m\) is a positive integer which stands for the dimension of the spatial variable, indices \(i\) and \(j\) run from 1 to \(m\) and the summation convention of the repeated indices is adopted. For a bounded open set \(\Omega \subset \mathbb{R}^m\) with sufficiently smooth boundary \(\Gamma\) (normally, we assume that \(\Gamma\) is Lipschitz continuous) we denote by \(\mathbf{v}\) the outward unit vector to \(\Gamma\) and we introduce the following function spaces which will play a key role in applications to nonsmooth mechanics problems
\[
H := L^2(\Omega; \mathbb{R}^m), \quad H^1(\Omega; \mathbb{R}^m), \quad \mathcal{H} := \{\mathbf{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega; S^m), \quad \mathcal{H}_1 := \{\mathbf{\tau} \in \mathcal{H} : \text{Div} \mathbf{\tau} \in H\},
\]
where \(\varepsilon\) and \(\text{Div}\) are the deformation operator and the divergence operator, respectively, and are defined in the following way
\[
\varepsilon(\mathbf{u}) := (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(u_{ij} + u_{ji}), \quad \text{Div} \mathbf{\tau} = (\tau_{ij,j}),
\]
accordingly. The index following a comma represents the partial derivative with respect to the corresponding component of \(x \in \Omega\), i.e., \(u_{ij} = \partial u_i / \partial x_j\). Keep in mind that the above mentioned spaces are Hilbert spaces endowed with the following corresponding inner products
\[
(\mathbf{u}, \mathbf{v})_H := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx, \quad (\mathbf{\tau}, \mathbf{\sigma})_{\mathcal{H}} := \int_{\Omega} \mathbf{\tau} \cdot \mathbf{\sigma} \, dx, \quad (\mathbf{u}, \mathbf{v})_{H_1} := (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}},
\]
\[(\tau, \sigma)_{H^1} := (\tau, \sigma)_H + (\text{Div} \tau, \text{Div} \sigma)_H.\]

We recall that the trace operator \(\gamma : H^1(\Omega; \mathbb{R}^m) \to H^{1/2}(\Gamma; \mathbb{R}^m) \subset L^2(\Gamma; \mathbb{R}^m)\) is linear and compact. Sometimes, for simplicity, we will omit to write \(\gamma\) to indicate the Sobolev trace on the boundary, i.e., writing \(v\) instead of \(\gamma v\). Also, for a given \(v \in H^{1/2}(\Gamma; \mathbb{R}^m)\) we denote by \(v_\nu\) and \(v_\tau\) the normal and the tangential components of \(v\) on the boundary, i.e., \(v_\nu := v \cdot \nu\) and \(v_\tau := v - v_\nu \nu\), respectively. Similarly, for a tensor field \(\sigma\), we define \(\sigma_\nu\) and \(\sigma_\tau\) to be the normal and the tangential components of the Cauchy vector field \(\sigma \nu\), that is \(\sigma_\nu := (\sigma \nu) \cdot \nu\) and \(\sigma_\tau := \sigma \nu - \sigma_\nu \nu\), respectively. Recall that the following Green formula holds

\[
(\sigma, \varepsilon(v))_H + (\text{Div} \sigma, v)_H = \int_{\Gamma}(\sigma \nu) \cdot v \, d\Gamma, \quad \text{for all } v \in H_1.
\]

We consider the following contact model involving a deformable body which occupies a bounded domain \(\Omega \subset \mathbb{R}^m\) with Lipschitz boundary \(\Gamma\), which is partitioned into four disjoint measurable parts \(I_1, I_2\) and \(I_3\).

(P) Find a displacement \(u : \Omega \to \mathbb{R}^m\) and a stress tensor \(\sigma : \Omega \to \mathcal{S}^m\) such that

\[
-\text{Div} \sigma = f_0 \quad \text{in } \Omega, \quad (3.2)
\]
\[
\sigma \in \partial \chi(\varepsilon(u)) \quad \text{in } \Omega, \quad (3.3)
\]
\[
u = 0 \quad \text{on } I_1, \quad (3.4)
\]
\[
\sigma \nu = f_2 \quad \text{on } I_2, \quad (3.5)
\]
\[
\sigma_\nu = 0, -\sigma_\tau \in \partial C_j(x, u_\tau) \quad \text{on } I^a_3, \quad (3.6)
\]
\[
\sigma_\tau = 0, \sigma_\nu \leq 0, u_\nu \leq 0, \sigma_\nu u_\nu = 0 \quad \text{on } I^b_3. \quad (3.7)
\]

Problem (P) describes the contact between a deformable body and a foundation. Relation (3.2) represents the equilibrium equation, showing that the body is subjected to volume forces of density \(f_0\). The behaviour of the material is described by a nonlinear constitutive law expressed as a (convex) subdifferential inclusion, namely, (3.3). On \(I_1\) the body is clamped, therefore the displacement vanishes here. Equation (3.5) shows that surface tractions of density \(f_2\) act on \(I_2\). On \(I_3\) the body may come in frictional contact with the foundation and this contact is modelled considering there are two potential contact zones: \(I^a_3\) where a nonmonotone friction law is considered; and \(I^b_3\) where the contact is frictionless and satisfies Signorini unilateral contact model. Note that we allow the case of only one contact zone, i.e., \(\text{meas}(I^a_3) = 0\) or \(\text{meas}(I^b_3) = 0\).

For examples of nonlinear constitutive laws of the form (3.3) and nonmonotone friction laws of the form (3.6) we refer the reader to [1] [2] [3] [4] [5] [6].

We assume that the constitutive function \(\chi\) and the initial data \(f_0, f_2\) satisfy the following conditions:

\((H_1)\) \(f_0 \in L^2(\Omega; \mathbb{R}^m)\) and \(f_2 \in L^2(I_2; \mathbb{R}^m)\);

\((H_2)\) \(\chi : \mathcal{S}^m \to \mathbb{R}\) is a convex and l.s.c. functional with the property that there exist \(\alpha, \beta \in (0, 1)\) such that

\[
\alpha |\mu|^2 \leq \chi(\mu) \leq \beta |\mu|^2, \quad \forall \mu \in \mathcal{S}^m;
\]
Keeping in mind the boundary conditions on \( v \) for all \( v \) for all \( H \) of Green's formula to find that

\[
|j(x, t_1) - j(x, t_2)| \leq p(x)|t_1 - t_2|, \quad \text{for a.e. } x \in \Gamma_3^a \text{ and all } t_1, t_2 \in \mathbb{R}.
\]

In order to derive the variational formulation of problem (P) assume \( u \) and \( \sigma \) are regular functions that satisfy (3.2)-(3.7). Multiplying (3.2) by \( v - u \) and integrating over \( \Omega \) we apply Green’s formula to find that

\[
(f_0, v - u)_H = -(\text{Div } \sigma, v - u)_H = (\sigma, \varepsilon(v - u))_H - \int_{\Gamma} (\sigma v) \cdot (v - u) d\Gamma,
\]

for all \( v \in X := \{ w \in H_1 : w = 0 \text{ a.e. on } \Gamma_1 \} \). It is well known that \( X \) is a closed subspace of \( H_1 \), therefore it is a Hilbert space endowed with the inner product

\[
(u, v)_X := (\varepsilon(u), \varepsilon(v))_H.
\]

Keeping in mind the boundary conditions on \( \Gamma_3 \) we define the set of admissible displacements

\[
\mathcal{K}_0 := \{ w \in X : w_\nu = 0 \text{ on } \Gamma_3^a \text{ and } w_\nu \leq 0 \text{ on } \Gamma_3^b \},
\]

which is nonempty, unbounded, closed and convex subset of \( X \). Keeping in mind that

\[
(\sigma v) \cdot (v - u) = \sigma_v(v_\nu - u_\nu) + \sigma_\tau \cdot (v_\tau - u_\tau)
\]

for all \( v \in \mathcal{K}_0 \) we have

\[
-\int_{\Gamma_1} \sigma v \cdot (v - u) d\Gamma = 0,
\]

\[
-\int_{\Gamma_2} \sigma v \cdot (v - u) d\Gamma = -\int_{\Gamma_2} f_2 \cdot (v - u) d\Gamma,
\]

\[
-\int_{\Gamma_3^a} \sigma v \cdot (v - u) d\Gamma = \int_{\Gamma_3^a} -\sigma_\tau \cdot (v_\tau - u_\tau) d\Gamma \leq \int_{\Gamma_3^a} j_{3,2}^0(x, u_\tau; v_\tau - u_\tau) d\Gamma,
\]

and

\[
-\int_{\Gamma_3^b} \sigma v \cdot (v - u) d\Gamma = -\int_{\Gamma_3^b} \sigma_v(v_\nu - u_\nu) d\Gamma = -\int_{\Gamma_3^b} \sigma_v v_\nu d\Gamma \leq 0.
\]

Thus,

\[
((\sigma, \varepsilon(v - u))_H + \int_{\Gamma_3^a} j_{3,2}^0(x, u_\tau; v_\tau - u_\tau) d\Gamma \geq (f_0, v - u)_H + \int_{\Gamma_2} f_2 \cdot (v - u) d\Gamma, \quad \forall v \in \mathcal{K}_0. \quad (3.8)
\]

Since the stress tensor \( \sigma \) satisfies (3.8) it is natural to define now the set of admissible stress tensors with respect to a displacement \( w \in \mathcal{K}_0 \) to be the following subset of \( H \)

\[
\Theta(w) := \left\{ \mu \in H : (\mu, \varepsilon(v))_H + \int_{\Gamma_3^a} j_{3,2}^0(x, w_\tau; v_\tau) d\Gamma \geq (f_0, v)_H + \int_{\Gamma_2} f_2 \cdot v d\Gamma, \quad \forall v \in \mathcal{K}_0 \right\}.
\]
We define the separable bipotential \( b : S^m \times S^m \rightarrow (-\infty, +\infty] \) by

\[
b(\tau, \mu) := \chi(\tau) + \chi^*(\mu).
\]

We point out the fact that \( b \) connects the constitutive law (3.3), the function \( \chi \) and its Fenchel conjugate \( \chi^* \) as

\[
b(\varepsilon(u), \sigma) = \sigma \cdot \varepsilon(u)
\]

and

\[
b(\varepsilon(v), \mu) \geq \mu \cdot \varepsilon(u), \quad \forall v \in X, \forall \mu \in H.
\]

Note that, if (\( H_3 \)) holds, then according to [9, Lemma 1] one has

\[
(1 - \beta)|\mu|^2 \leq \chi^*(\mu) \leq \frac{1}{4\alpha}|\mu|^2, \quad \forall \mu \in S^m.
\]

Therefore, we have

\[
\chi(\mu(\cdot)) \in L^1(\Omega) \quad \text{and} \quad \chi^*(\mu(\cdot)) \in L^1(\Omega), \quad \forall \mu \in H.
\]

Thus, the coupling function \( B : X \times H \rightarrow \mathbb{R} \) defined via the bipotential \( b \)

\[
B(v, \mu) := \int_{\Omega} b(\varepsilon(v), \mu) \, dx,
\]

is well defined and the following estimates hold

\[
B(u, \sigma) = (\sigma, \varepsilon(u))_H \quad \text{and} \quad B(v, \mu) \geq (\mu, \varepsilon(v))_H, \forall v \in X, \forall \mu \in H. \tag{3.9}
\]

Moreover, there exists a positive constant \( C = C(\alpha, \beta) \) such that

\[
B(v, \mu) \geq C(\|v\|^2_{\chi} + \|\mu\|^2_{\mu}), \quad \forall v \in X, \forall \mu \in H. \tag{3.10}
\]

Choosing \( v = u + w \) in (3.8) we deduce that \( \sigma \in \Theta(u) \), therefore \( \Theta(u) \neq \emptyset \). Keeping in mind the definition of \( \Theta(u) \) and (3.9) we get

\[
B(u, \mu) + \int_{\Gamma_3} j_2^0(x, u_\tau; u_\tau) \, d\Gamma \geq (f_0, u)_H + \int_{\Gamma_2} f_2 \cdot u \, d\Gamma. \tag{3.11}
\]

On the other hand, taking \( v := 0_X \) in (3.8)

\[
-B(u, \sigma) + \int_{\Gamma_3} j_2^0(x, u_\tau; -u_\tau) \, d\Gamma \geq -(f_0, u)_H - \int_{\Gamma_2} f_2 \cdot u \, d\Gamma. \tag{3.12}
\]

Adding (3.11) and (3.12) we get

\[
B(u, \mu) - B(u, \sigma) \geq - \int_{\Gamma_3} [j_2^0(x, u_\tau; u_\tau) + j_2^0(x, u_\tau; -u_\tau)] \, d\Gamma, \quad \forall \mu \in \Theta(u). \tag{3.13}
\]

Using relations (3.8), (3.9) and (3.13) we derive the following \textit{variational formulation in terms of bipotentials} of problem (P):
It follows immediately that
\begin{equation}
\begin{aligned}
\{ B(v, \sigma) - B(u, \sigma) + \int_{\Gamma_2^0} j^0_2(x, u_r; v_r - u_r) \, d\Gamma \geq (f_0, v - u)_H + \int_{\Gamma_2} f_2 \cdot (v - u) \, d\Gamma \\
B(u, \mu) - B(u, \sigma) \geq 0,
\end{aligned}
\end{equation}
for all \((v, \mu) \in \mathcal{K}_0 \times \Theta(u)\). Noted that, if \((u, \sigma)\) is a weak solution of problem \((P)\), i.e., it solves \((3.14)\), then \((3.8)\) and \((3.13)\) are automatically fulfilled as
\[ B(v, \sigma) - B(u, \sigma) \geq (\sigma, \varepsilon(v - u))_H, \quad \forall v \in \mathcal{K}_0, \]
and
\[ 0 = j^0_2(x, u_r; 0_X) = j^0_2(x, u_r; u_r - u_r) \leq j^0_2(x, u_r; u_r) + j^0_2(x, u_r; -u_r). \]

**Theorem 3.1** Assume \((\mathcal{H}_1), (\mathcal{H}_2)\) and \((\mathcal{H}_3)\) hold. Then problem \((P)\) possesses at least one weak solution.

**Proof** Let us define the function spaces \(Y := \mathcal{H}, Z_1 := L^2(\Gamma_3^0; \mathbb{R}^m), Z_2 := L^2(\Omega; S^m)\). Moreover, assume that \(\gamma_2 : Y \to Z_2\) is the embedding operator and \(\gamma_1 : X \to Z_1\) is defined by
\[ \gamma_1(u) := (\gamma(u_r))|_{\Gamma_3^0}, \]
where \(\gamma : X \to H^{1/2}(\Gamma; \mathbb{R}^m)\) is the trace operator. Also define \(J : Z_1 \to \mathbb{R}, \phi : X \to (-\infty, \infty), H : Y \to (0, \infty), L : Z_2 \to \mathbb{R}, F : X \to X^*\) and \(G : Y \to Y^*\) by the following instructions
\[ J(y) := \int_{\Gamma_3^0} j(x, y(x)) \, d\Gamma, \quad \phi := I_{\mathcal{K}_0}, \quad H \equiv 1, \quad L \equiv 0, \quad G \equiv 0_Y, \]
and
\[ \langle F(u), v \rangle := (f_0, v)_H + \int_{\Gamma_2} f_2 \cdot v \, d\Gamma, \]
respectively.

By virtue of definition of \(B : X \times Y \to \mathbb{R}\), it is easy to check that \((\mathcal{H}_B), (\mathcal{H}_F)\) and \((\mathcal{H}_G)\) are fulfilled. Furthermore, the Aubin-Clarke Theorem (see, e.g., Clarke [3, Theorem 2.7.5]) ensures \(J\) is Lipschitz continuous of constant \(\|p\|_{L^2}\) and
\[ J^0(y; z) \leq \int_{\Gamma_3^0} j^0_2(x, y; z) \, d\Gamma, \quad \forall y, z \in Z_1. \]
It follows immediately that
\[ J^0(\gamma_1(u); \gamma_1(v - u)) \leq \int_{\Gamma_3^0} j^0_2(x, u_r; v_r - u_r) \, d\Gamma, \quad \forall u, v \in X. \]
\[(3.15)\]
Thus, \((H_0')\) holds with \(\alpha_J := \|p\|_{L^2}^2\) and \(\alpha_L := 1\). Let \(\bar{w}_0 := 0_X\) and \(\bar{\tau}_0 \in Y\) be arbitrarily fixed. Then

\[
2B(u, \sigma) - B(0_X, \sigma) - B(u, \bar{\tau}_0) + \langle F(u), -u \rangle + \langle G(\sigma), \bar{\tau}_0 - \sigma \rangle
\frac{\|u, \sigma\|}{
\frac{\sqrt{\|u\|_X^2 + \|\sigma\|_Y^2}}{\|u\|_X^2 + \|\sigma\|_Y^2}}
\]

\[
\geq \frac{C(\|u\|_X^2 + \|\sigma\|_Y^2)}{\|u\|_X^2 + \|\sigma\|_Y^2} - c_1\|u\|_X - c_2 \rightarrow \infty \quad \text{as} \quad \|u, \sigma\| \rightarrow \infty,
\]

for some suitable constants \(c_1, c_2 > 0\). Therefore, the coercivity condition \((C)\) also holds. Consequently, problem \((P)\) possesses at least one solution for any constraint sets \(K \subseteq X\) and \(\Lambda \subseteq Y\). Choosing \(K := K_0\) and \(\Lambda := Y\), Theorem 2.1 ensures the existence of a pair \((u_1, \sigma_1) \in K_0 \times Y\) such that

\[
\begin{align*}
B(v, \sigma_1) - B(u_1, \sigma_1) + J^0(\gamma_1(u_1); \gamma_1(v - u_1)) & \geq \langle F(u_1), v - u_1 \rangle, \quad \forall v \in K_0, \\
B(u_1, \mu) - B(u_1, \sigma_1) & \geq 0, \quad \forall \mu \in Y.
\end{align*}
\]

\(3.16\)

Now, applying again Theorem 2.1 with \(K := K_0\) and \(\Lambda := \Theta(u_1)\) we infer there exist \(u_2 \in K_2\) and \(\sigma_2 \in \Theta(u_1)\) such that

\[
\begin{align*}
B(v, \sigma_2) - B(u_2, \sigma_2) + J^0(\gamma_1(u_2); \gamma_1(v - u_2)) & \geq \langle F(u_2), v - u_2 \rangle, \quad \forall v \in K_0, \\
B(u_2, \mu) - B(u_2, \sigma_2) & \geq 0, \quad \forall \mu \in \Theta(u_1).
\end{align*}
\]

\(3.17\)

But, from the definition of the coupling function \(B\) we have

\[
\begin{align*}
B(v, \sigma_1) - B(u_1, \sigma_1) = B(v, \sigma_2) - B(u_1, \sigma_2), \quad \forall v \in X,
\end{align*}
\]

and

\[
\begin{align*}
B(u_2, \mu) - B(u_2, \sigma_2) = B(u_1, \mu) - B(u_1, \sigma_2), \quad \forall \mu \in Y,
\end{align*}
\]

which combined with \((3.16)-(3.17)\) shows that \((u_1, \sigma_2)\) solves the following system

\[
\begin{align*}
B(v, \sigma_2) - B(u_1, \sigma_2) + J^0(\gamma_1(u_1); \gamma_1(v - u_1)) & \geq \langle F(u_1), v - u_1 \rangle, \quad \forall v \in K_0, \\
B(u_1, \mu) - B(u_1, \sigma_2) & \geq 0, \quad \forall \mu \in \Theta(u_1).
\end{align*}
\]

\(3.18\)

It is readily seen that, due to \((3.15)\), the pair \((u_1, \sigma_2)\) also solves \((3.14)\), i.e., it is a weak solution for problem \((P)\).

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