Asymptotic symmetries in Carrollian theories of gravity with a negative cosmological constant

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Abstract: Asymptotic symmetries of electric and magnetic Carrollian gravitational theories with a negative cosmological constant $\Lambda$ are analyzed in 3+1 space-time dimensions. In the magnetic theory, the asymptotic symmetry algebra is given by the conformal Carroll algebra in three dimensions, which is infinite-dimensional and isomorphic to the BMS$_4$ algebra. These results are in full agreement with holographic expectations, providing a new framework for the study of Carrollian holography. On the contrary, in the case of the electric theory, the presence of a negative $\Lambda$ turns out to be incompatible with a consistent set of asymptotic conditions, that can be traced back to the absence of a sensible ground state configuration. This can be improved if the Carrollian theory obtained from an electric contraction of Euclidean General Relativity is considered. In this case, asymptotic conditions can be constructed with an asymptotic symmetry algebra given by $\mathfrak{so}(1,4)$. However, it is shown that the space of spherically symmetric solutions of this theory is degenerate.

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1 Introduction

Asymptotic symmetries are of fundamental importance in the description of General Relativity. They define the physical symmetries of the theory. In the presence of a negative cosmological constant, the asymptotic symmetry algebra can be interpreted as a conformal symmetry acting in one lower dimension, which is one of the cornerstones of the AdS/CFT correspondence [1–3]. For instance, in four spacetime dimensions the asymptotic symmetry algebra with Henneaux-Teitelboim boundary conditions [4] is finite dimensional and given
by the $AdS_4 \simeq so(2,3)$ algebra, that can be regarded as a conformal algebra acting on a three-dimensional spacetime.\(^1\)

Recently, a new gravitational theory with a Carrollian structure was introduced in ref. [7] as a particular ultrarelativistic ($c \to 0$) contraction of General Relativity, known as “Magnetic Carroll gravity.” A different Carrollian contraction of Einstein gravity leads to the so-called “Electric Carroll gravity” that was introduced long ago in the context of Strong Gravity [8] or as a zero signature limit of General Relativity [9–11]. These Carrollian gravitational theories have a simpler structure than Einstein gravity (the algebra of constraints and the constraints themselves acquire a simpler form in the Carroll limit), and therefore they provide natural models to explore certain aspects of quantum gravity from a new perspective. In this sense, the study of the symmetries of these theories acquires a particular importance. For a vanishing cosmological constant, the asymptotic symmetries were recently studied in 3+1 space-time dimensions in ref. [12] for both contractions, magnetic and electric.

It is then natural to wonder if in the presence of a negative cosmological constant there exists a similar interpretation as in the case of Einstein gravity, i.e., that the asymptotic symmetries in the Carrollian gravitational theories correspond to conformal Carrollian algebras in one lower dimension. This question is particularly relevant in the context of holography.

Carroll symmetry was introduced independently by Levy-Leblond in ref. [13] and Sen Gupta in ref. [14]. Its Lie algebra was obtained from a contraction of the Poincaré algebra in the limit when the speed of light vanishes, and it has been used in diverse physical contexts. For example, in gravitation, electric Carroll gravity turns out to be well suited to describe the behavior of spacetime near space-like singularities along the lines of the Belinsky-Khalatnikov-Lifshitz (BKL) approach [15–18]. In this regime, time derivatives dominate over spatial gradients, which is precisely one of the main properties of Carrollian theories, the so-called “ultra-locality.” This theory was also used as the starting point of an alternative perturbation scheme for quantum gravity in terms of the signature parameter that mimicks the quantization of a relativistic free particle [11, 19, 20]. Other recent applications of the Carroll symmetry have been found in the study of the symmetries of plane gravitational waves [21], physics near black hole horizons [22, 23], cosmology and dark matter [24], fractons [25–27], the asymptotic structure of General Relativity near null infinity [28–30], and a Carrollian description of celestial holography [31]. Further applications of the Carroll symmetry can be found in refs. [32–57].

In the classification of the possible “kinematical groups” performed by Bacry and Levi-Leblond in ref. [58], it was shown that there exists an extension of the Carroll algebra that admits a non-vanishing cosmological constant $\Lambda$. This cosmological extension of the Carroll algebra can be obtained from an ultrarelativistic limit of the de Sitter or anti-de Sitter algebras depending on the sign of $\Lambda$. In the particular case when the cosmological

\(^1\)In the asymptotic conditions of Henneaux and Teitelboim [4], the boundary metric is fixed (Dirichlet boundary conditions) and is conformal to the three-dimensional Minkowski metric. Some generalizations that contain non-trivial fluxes through the AdS boundary were considered in refs. [5, 6] leading to infinite-dimensional field-dependent algebras.
constant is negative, the “Carroll AdS\textsubscript{4} algebra” turns out to be isomorphic to the Poincaré algebra.\footnote{In ref. [58], this algebra was called “Para-Poincaré”.}

On the other hand, a remarkable relation between the conformal Carroll algebra in three dimensions and the BMS\textsubscript{4} algebra was found in ref. [28], where it was shown that both algebras are isomorphic. The BMS\textsubscript{4} algebra is infinite-dimensional, and originally appeared in the study of gravitational radiation in asymptotically flat spacetimes at null infinity [59, 60]. It contains the Poincaré algebra, or equivalently the Carroll-AdS\textsubscript{4} algebra in our physical context, as a subalgebra.

In this article, it is shown that for magnetic Carroll gravity with a negative cosmological constant in 3+1 space-time dimensions, the asymptotic symmetry algebra is infinite-dimensional (in contrast to the case in Einstein gravity with Henneaux-Teitelboim boundary conditions) and is given by the conformal Carroll algebra in three dimensions, in agreement with the expectations coming from holography. The possibility of having an infinite dimensional algebra comes from the fact that there is a relaxation in the fall-off of the fields as compared with the asymptotic behavior inherited from General Relativity. This relaxation is possible because the algebra of the constraints possesses an abelian subalgebra in the Carroll limit. For a particular restriction of the asymptotic conditions, the asymptotic symmetry algebra can be consistently truncated to the finite dimensional “Carroll-AdS\textsubscript{4} algebra.”

On the contrary, the case of electric Carroll gravity is radically different. The presence of a negative cosmological constant turns out to be incompatible with a consistent set of asymptotic conditions. Indeed, the naive ground state configuration analogous to the AdS spacetime in General Relativity is not a solution of the constraints of the electric theory [12, 61]. As a consequence, it does not seems to be possible to construct a well-defined set of asymptotic conditions with non-trivial conserved charges in this case. However, if one considers the theory obtained from the electric Carrollian contraction of Euclidean General Relativity with $\Lambda < 0$, then it is possible to find an alternative background configuration that allows to construct a consistent set of asymptotic conditions as deviations from it. In this case, the asymptotic symmetry algebra is finite dimensional and given by the $so(1,4)$ algebra.

The plan of the paper is the following. In section 2, the asymptotic symmetries in magnetic Carroll gravity with $\Lambda < 0$ are studied. After a brief review of the formulation of the theory, the ground state solution obtained from a Carroll contraction of Anti-de Sitter spacetime is introduced. This solution plays a fundamental role in the construction of the asymptotic conditions, because it is used as a background configuration. The proposed fall-off for the fields is then exhibited, and the charges and asymptotic symmetries are obtained. They are spanned by the three-dimensional conformal Carroll algebra, which according to ref. [28] is isomorphic to BMS\textsubscript{4}. Possible consistent truncations of the asymptotic symmetries are also discussed. Finally, a solution of the magnetic theory that resembles the Schwarzschild-AdS configuration is introduced and their charges are computed.

In section 3, the asymptotic structure of electric Carroll gravity with a negative cosmological constant is analyzed. In particular, in subsection 3.1 it is shown that the cons-

\[\]
configuration obtained from a Carroll contraction of AdS spacetime is not a solution of this theory. The subsequent problems for constructing consistent asymptotic conditions are then discussed. In subsection 3.2 this situation is improved by considering the Carrollian theory obtained from the electric contraction of Euclidean General Relativity with $\Lambda < 0$. Asymptotic conditions are then constructed, where the symmetry algebra is canonically realized and spanned by $so(1, 4)$ generators. The space of spherically symmetric solution of this theory is also explored, showing that is degenerate.

Finally, section 4 is devoted to the conclusions and some final remarks.

2 Asymptotic symmetries in magnetic Carroll gravity with a negative cosmological constant

2.1 Hamiltonian formulation of magnetic Carroll gravity with a non-vanishing cosmological constant

Magnetic Carroll gravity was recently introduced by Henneaux and Salgado-Rebolledo in ref. [7]. It is obtained from a “magnetic Carrollian contraction” of General Relativity, and it was originally formulated in Hamiltonian form. In the presence of a cosmological constant $\Lambda$, the Hamiltonian action reads

$$I = \int dt d^3x \left( \pi^{ij} \dot{g}_{ij} - N \mathcal{H}^M - N^i \mathcal{H}^M_i \right),$$

where the constraint $\mathcal{H}^M$ and $\mathcal{H}^M_i$ are given by

$$\mathcal{H}^M = -\sqrt{g} (R - 2\Lambda), \quad \mathcal{H}^M_i = -2\pi^{ij} \partial_j.$$  \hspace{1cm} (2.1)

Here, the canonical variables are characterized by an Euclidean three-dimensional metric $g_{ij}$, and their corresponding conjugate canonical momenta $\pi^{ij}$ ($i, j = 1, 2, 3$). They obey the following equal time Poisson brackets

$$\left\{ g_{ij} (x) , \pi^{kl} (x') \right\} = \frac{1}{2} \left( \delta^{kl}_{ij} + \delta^{kl}_{ij} \right) \delta (x, x').$$

The Lagrange multipliers are given by the lapse and shift functions $N$, $N^i$, respectively. The symbol $|$ denotes covariant differentiation with respect to the metric $g_{ij}$, where $R$ is its Ricci scalar.

The time evolution of the canonical variables is described by Hamilton’s equations, which are directly obtained from the action principle (2.1). They read

$$\dot{g}_{ij} = N_{[ij]} + N_{j[i}, \hspace{1cm} (2.2)$$

$$\dot{\pi}^{ij} = -N\sqrt{g} \left( R^{ij} - \frac{1}{2} g^{ij} R + \Lambda g^{ij} \right) + \sqrt{g} \left( N^{[ij]} - \partial_k N^{ij} \right)_k + \left( N^{ij} \partial_k \pi^{ij} \right)_k - N_{ij} \pi^{kj} - N_k \pi^{kj}.$$  \hspace{1cm} (2.3)

$^{3}$Recently, a covariant formulation of magnetic Carroll gravity was provided in ref. [61]. The covariant action was obtained from an appropriate truncation of the next-to-leading order term in a small speed of light expansion of General Relativity.
The Carrollian structure of this theory is characterized by the algebra of the first class constraints

\[
\{ \mathcal{H}^M(x), \mathcal{H}^M(x') \} = 0, \quad (2.5)
\]

\[
\{ \mathcal{H}^M(x), \mathcal{H}^M_i(x') \} = \mathcal{H}^M(x) \partial_i \delta(x, x'), \quad (2.6)
\]

\[
\{ \mathcal{H}^M_i(x), \mathcal{H}^M_j(x') \} = \mathcal{H}^M_i(x') \partial_j \delta(x, x') + \mathcal{H}^M_j(x) \partial_i \delta(x, x'), \quad (2.7)
\]

where the generators \( \mathcal{H}^M(x) \) define an abelian subalgebra, in contrast to the case of General Relativity. This particular property is due to the fact that the Hamiltonian constraint \( \mathcal{H}^M \) in (2.2) does not depend on the canonical momenta.

The constraints \( \mathcal{H}^M_i \) generate coordinate transformations on the hypersurfaces defined at slices of constant time, while \( \mathcal{H}^M \) generates normal surface deformations. The smeared form of the generators of arbitrary surface deformations then takes the form

\[
G \left[ \xi, \xi^i \right] = \int d^3x \left( \xi \mathcal{H}^M + \xi^i \mathcal{H}^M_i \right) + Q_M, \quad (2.8)
\]

where \( \xi \) and \( \xi^i \) correspond to the parameters associated with normal and tangential deformations, respectively.

Following the Regge-Teitelboim approach [62], \( Q_M \) is a surface term that must be added to ensure that the generators (2.8) possess well-defined functional derivatives. This means that all the boundary terms that are obtained from the variation of the bulk part of the generator must be cancelled with the variation of the surface term \( Q_M \). Therefore, if the generator of an improper (large) gauge transformation is evaluated on a given solution, the bulk part vanishes because is linear in the constraints, and only the boundary term remains, defining the corresponding conserved charge associated with this symmetry [63].

From the form of the constraints in eq. (2.2) one finds

\[
\delta Q_M = \oint d^2s_l \left[ G^{ijkl} \left( \xi \delta g_{ijk} - \xi^i \delta g_{ij} \right) + 2 \xi_j \delta \pi^{kl} + \left( 2 \xi^k \pi^{jl} - \xi^l \pi^{jk} \right) \delta g_{jk} \right]. \quad (2.9)
\]

Here \( G^{ijkl} \) denotes the inverse of the de Witt supermetric, given by

\[
G^{ijkl} = \frac{1}{2} \sqrt{g} \left( g^{ik} g^{jl} + g^{il} g^{jk} - 2 g^{ij} g^{kl} \right).
\]

The transformation laws of the canonical variables are obtained by acting with \( G \left[ \xi, \xi^i \right] \) on them

\[
\delta g_{ij} = \left\{ g_{ij}, G \left[ \xi, \xi^k \right] \right\}, \quad \delta \pi^{ij} = \left\{ \pi^{ij}, G \left[ \xi, \xi^k \right] \right\}.
\]

Explicitly, they read

\[
\delta g_{ij} = \xi_{ij} + \xi_{ji}, \quad (2.10)
\]

\[
\delta \pi^{ij} = -\xi \sqrt{g} \left( R^{ij} - \frac{1}{2} g^{ij} R + \Lambda g^{ij} \right) + \sqrt{g} \left( \xi^{[ij]} - g^{ij} \xi^k \right) + \left( \xi^{k \pi^{ij}} \right)_{|k} - \xi^i \pi^{kj} - \xi^j \pi^{ki}. \quad (2.11)
\]
2.2 Magnetic Carrollian ground state

In order to construct a consistent set of asymptotic conditions, generically one must consider deviations of the fields with respect to a certain background configuration in the asymptotic expansion. For instance, in the case of General Relativity with a negative cosmological constant, the corresponding background is Anti-de Sitter spacetime, with isometries that are spanned by the AdS\(_4\) algebra.

For magnetic Carroll gravity, the natural background is given by the solution obtained from a Carroll contraction of Anti-de Sitter spacetime. It is given by

\[
\bar{g}_{ij}dx^i dx^j = \frac{dr^2}{r^2 (r^2 + 1)} + r^2 \gamma_{AB} dx^A dx^B, \quad \bar{\pi}^{ij} = 0, \quad \bar{N} = \sqrt{\frac{r^2}{l^2} + 1}, \quad \bar{N}^i = 0.
\]

Here \(\Lambda = -3/l^2\), and \(\gamma_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2\), is the metric of the round 2-sphere. Its determinant is denoted by \(\gamma\). In what follows the indices \(A, B = 1, 2\) are lowered and raised with this metric.

Note that the constraints (2.2) are automatically fulfilled because the spatial metric is of constant curvature, and \(\bar{\pi}^{ij}\) vanishes. In ref. [64], this solution was obtained from the quotient of the Poincaré group by the subgroup formed by spatial rotations and translations. It defines an homogeneous space with symmetries given by the Carroll-AdS\(_4\) algebra.

To determine the Carrollian isometries of the solution in the context of this particular theory, one can use the transformation laws of the canonical variables in eqs. (2.10) and (2.11). Requiring that \(\delta \bar{g}_{ij} = \delta \bar{\pi}^{ij} = 0\), one finds that the following equations must be obeyed by the parameters \(\xi\) and \(\xi^i\)

\[
\xi_{ij} + \xi_{j|i} = 0, \quad \xi_{i|j} - \bar{g}_{ij} \xi_{j|k} + \frac{2}{l^2} \bar{g}_{ij} = 0.
\]

The explicit solution is given by

\[
\xi = r \left( \vec{\beta} \cdot \hat{r} \right) + T \sqrt{\frac{r^2}{l^2} + 1}, \quad \xi' = (\vec{\alpha} \cdot \hat{r}) \sqrt{\frac{r^2}{l^2} + 1},
\]

\[
\xi^A = \frac{\epsilon^{AB}}{\sqrt{\gamma}} \partial_B (\vec{\omega} \cdot \hat{r}) + \frac{\partial^A (\vec{\alpha} \cdot \hat{r})}{r} \sqrt{\frac{r^2}{l^2} + 1}.
\]

Here, the constant \(T\) is the parameter of time translations, and \(\vec{\alpha}, \vec{\beta}, \vec{\omega}\) are three-vectors associated with spatial translations, Carrollian boosts and spatial rotations, respectively. The vector \(\hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) denotes the unit normal to the two-sphere.

The algebra of Carrollian isometries can be obtained from the composition law of the parameters derived from the “Carrollian surface deformation algebra” in eqs. (2.5)–(2.7)

\[
\xi_3^+ = \xi_1^i \partial_i \xi_2^+ - \xi_2^i \partial_i \xi_1^+, \quad \xi_3^i = \xi_1^j \partial_j \xi_2^i - \xi_2^j \partial_j \xi_1^i.
\]
It is given by
\begin{align*}
\beta^K_3 &= \frac{1}{l^2} \left( T_2 \alpha^K_1 - T_1 \alpha^K_2 \right) + \epsilon_{IJK} \left( \beta^I_2 \omega^J_1 - \beta^J_2 \omega^I_1 \right), \\
\alpha^K_3 &= -\epsilon_{IJK} \left( \alpha^I_1 \omega^J_2 - \alpha^J_1 \omega^I_2 \right), \\
\omega^K_3 &= \frac{1}{l^2} \epsilon_{IJK} \alpha^I_1 \alpha^J_2 - \epsilon_{IJK} \omega^I_1 \omega^J_2,
\end{align*}
where \( I, J, K = 1, 2, 3 \) label the indices associated with the three-vectors.

This composition law defines the Carroll-AdS_4 algebra, which possesses the following non-vanishing commutators
\begin{align*}
\{ J_I, J_J \} &= -\epsilon_{IJK} J_K, \\
\{ P_I, J_J \} &= -\epsilon_{IJK} P_K, \\
\{ K_I, J_J \} &= -\epsilon_{IJK} K_K, \\
\{ P_I, K_J \} &= \delta_{IJ} E, \\
\{ P_I, P_J \} &= \frac{1}{l^2} \epsilon_{IJK} J_K, \\
\{ P_I, E \} &= \frac{1}{l^2} K_I.
\end{align*}
(2.15)

Here \( \vec{J}, \vec{K}, \vec{P} \) are the generators associated with spatial rotations, Carrollian boosts and spatial translations, respectively. The generator associated with time translations, i.e., the Carrollian energy, is denoted by \( E \).

Note that the equivalence with the Poincaré algebra becomes explicit if we change \( P_I \to \frac{1}{l} K_I \) and \( K_I \to l P_I \). When \( l \to \infty \) one recovers the original Carroll algebra of Levy-Leblond [13].

### 2.3 Asymptotic conditions and symmetry algebra

#### 2.3.1 Fall-off of the fields

A key step in the process of finding the asymptotic symmetries is to specify the behavior of the fields, in this case the canonical variables \( g_{ij}, \pi^{ij} \), in the large \( r \) expansion. The fall-off of the fields must guarantee that the symplectic term in the action (2.1) is finite and that the charges are finite and integrable in the functional sense.\(^4\) For this purpose, we will consider deviations with respect to the background configuration defined in eq. (2.12). The proposed asymptotic conditions are given by
\begin{align*}
g_{rr} &= \frac{l^2}{r^2} - \frac{l^4}{r^4} + \frac{f_{rr}}{r^5} + O \left( r^{-6} \right), \\
g_{rA} &= \frac{f_{rA}}{r^4} + O \left( r^{-5} \right), \\
g_{AB} &= r^2 \gamma_{AB} + h_{AB} + \frac{f_{AB}}{r} + O \left( r^{-2} \right), \\
\pi^{rr} &= \frac{p^{rr}}{r} + O \left( r^{-2} \right), \\
\pi^{rA} &= -\frac{D_A \hat{k}^{(2)}}{r} + \frac{p^{rA}}{r^2} + O \left( r^{-3} \right), \\
\pi^{AB} &= \frac{k^{(2)}}{r^2} + \frac{k^{(4)}}{r^4} + \frac{p^{AB}}{r^3} + O \left( r^{-6} \right).
\end{align*}
(2.17)\(\text{--}\) (2.22)

Here, \( D_A \) is the covariant derivative associated with the metric \( \gamma_{AB} \). For a tensor \( X^{AB} \), its trace is denoted by \( \tilde{X} := X^{AB} \gamma_{AB} \) and its traceless part by \( \hat{X}^{AB} := X^{AB} - \frac{1}{2} \gamma^{AB} \hat{X} \).

\(^4\)We are assuming that there are no fluxes across the boundary.
It is worth pointing out that the terms with \( h^{AB} \), \( \tilde{k}^{AB}_{(2)} \) and \( k^{AB}_{(4)} \) are not present in the fall-off of the fields inherited from General Relativity. Indeed, this “relaxation” in the asymptotic behavior of the fields is of fundamental importance for the existence of an infinite-dimensional asymptotic symmetry algebra, and has its origin in the abelian subalgebra (2.5) of the Carrollian surface deformation algebra.

The asymptotic conditions (2.17)–(2.22) are preserved by parameters of the form

\[
\xi = \frac{r}{l} T(\theta, \phi) + \frac{l(\Delta + 2)}{4r} T(\theta, \phi) + \ldots,
\]

\[
\xi' = -\frac{r}{2} D_A Y^A(\theta, \phi) - \frac{l^2}{4} D_A Y^A(\theta, \phi) \frac{1}{r} + \ldots,
\]

\[
\xi^A = Y^A(\theta, \phi) - \frac{l^2}{4r^2} D^A D_B Y^B(\theta, \phi) + \ldots.
\]

Here, \( \Delta = D^A D_A \), and \( T(\theta, \phi) \) is an arbitrary scalar function on the sphere. The vector \( Y^A \) is constrained to obey the two-dimensional conformal Killing equation

\[
D_A Y_B + D_B Y_A - \gamma_{AB} D_C Y^C = 0.
\]

### 2.3.2 Conserved charges and asymptotic symmetry algebra

The canonical generators of the asymptotic symmetries parametrized by \( T \) and \( Y^A \) can be directly obtained by evaluating the surface term \( \delta Q_M \) in eq. (2.9). If the parameters are assumed to have a vanishing functional variation, the charge can be readily integrated and yields

\[
Q_M = \oint d^2x \sqrt{\gamma} \left( T \mathcal{P} + Y^A \mathcal{J}_A \right),
\]

where

\[
\mathcal{P} := \frac{1}{l^2} \left( 3 \dot{f} + \frac{2}{l^2} \dot{f}_{rr} \right), \quad \mathcal{J}_A := 2 \gamma^{-\frac{1}{2}} \gamma_{AB} \dot{f}^B.
\]

If \( \lambda = (T, Y^A) \) denotes the parameters, the algebra can be obtained from the identity \( \delta_\lambda Q = \{ Q | \lambda_1 \}, Q | \lambda_2 \} = Q | \lambda_3 \). Using the transformation laws for the fields \( \mathcal{P} \) and \( \mathcal{J}_A \) given by

\[
\delta \mathcal{P} = Y^A \partial_A \mathcal{P} + \frac{3}{2} \mathcal{P} D_A Y^A,
\]

\[
\delta \mathcal{J}_A = \frac{1}{2} T \partial_A \mathcal{P} + \frac{3}{2} \mathcal{P} \partial_A T + Y^B D_B \mathcal{J}_A + \mathcal{J}_B D_A Y^B + \mathcal{J}_A D_B Y^B + \frac{3}{l^2} D^B \left( T \dot{f}_{AB} \right),
\]

it can be shown that the algebra closes according to

\[
T_3 = Y^1_1 \partial_A T_2 - Y^1_2 \partial_A T_1 + \frac{1}{2} \left( T_1 D_A Y^2_A - T_2 D_A Y^1_A \right),
\]

\[
Y^A_3 = Y^C_1 \partial_C Y^A_2 - Y^C_2 \partial_C Y^A_1.
\]

This is precisely the composition law of the BMS\(_4\) algebra (see e.g. [65]). Therefore, using the results of [28], it can be concluded that the asymptotic symmetry algebra of magnetic Carroll gravity with \( \Lambda \leq 0 \) is given by the infinite-dimensional conformal Carroll algebra in three-dimensions, in full agreement with the expectations coming from holography.
Note that the canonical realization of the algebra does not contain central charges, and also admits “superrotations” in the sense of ref. [65] if the conformal Killing vectors $Y^A$ are locally-defined. Alternatively, the composition laws (2.28) and (2.29) can also be obtained using the asymptotic form of the parameters (2.23)–(2.25) in eq. (2.14).

If the parameters $T, Y^A$ are defined globally, they can be expanded in spherical harmonics according to

$$Y^A = \partial^A \left( \frac{\bar{\alpha}}{l} \cdot \hat{r} \right) + \frac{\epsilon^{AB}}{\sqrt{\gamma}} \partial_B (\bar{\omega} \cdot \hat{r}) , \quad T = T_0 + l \bar{\beta} \cdot \hat{r} + \sum_{\ell = 2}^{\infty} \sum_{m = -\ell}^{\ell} T_{\ell,m} Y_{\ell,m} .$$

Analogously, the fields $\mathcal{P}$ and $\mathcal{J}_A$ can be expanded as follows

$$\mathcal{J}_A = \frac{3l}{8\pi} \partial_A \left( \bar{p} \cdot \hat{r} \right) + \frac{3}{8\pi \sqrt{\epsilon_{AB}} \gamma} \partial^B (\bar{\hat{\bar{\omega}}} \cdot \hat{r}) , \quad \mathcal{P} = \frac{E}{4\pi} + \frac{3}{4\pi l} \hat{K} \cdot \hat{r} + \sum_{\ell = 2}^{\infty} \sum_{m = -\ell}^{\ell} P_{\ell,m} Y_{\ell,m} .$$

Therefore, the charge (2.26) can be written as

$$Q_M = T_0 E + \bar{\alpha} \cdot \bar{\bar{P}} + \bar{\beta} \cdot \bar{\bar{K}} + \bar{\omega} \cdot \bar{\bar{J}} + \sum_{\ell = 2}^{\infty} \sum_{m = -\ell}^{\ell} T_{\ell,m} P_{\ell,m} .$$

The subset of generators $E, \bar{P}, \bar{K}$ and $\bar{J}$ defines a Carroll-AdS$^4$ subalgebra, with Poisson brackets that close according to eqs. (2.15) and (2.16). The additional infinite number of generators $P_{\ell,m}$ define the supertranslations.

In this analysis, the canonical generators of the BMS$_4$ algebra are well-defined, in contrast with the case of asymptotically flat spacetimes at null infinity. In that context, there appear non-integrable terms in the charges associated with the flux of gravitational radiation across future/past null infinity [66] (see also [67]).

### 2.3.3 Consistent truncations of the asymptotic conditions

Some consistent truncations of the asymptotic conditions (2.17)–(2.22) are possible. From the transformation law of $\bar{\hat{\bar{h}}}_{AB}$ written in stereographic coordinates $z = e^{i\phi} \cot (\theta/2)$

$$\delta h_{zz} = Y^z \partial_z h_{zz} + Y_z \partial_z h_{zz} + 2 (\partial_z Y^z) h_{zz} - \frac{1}{2} l^2 \partial_z^2 Y^z ,$$

it is clear that if the condition $\bar{\hat{\hat{h}}}_{AB} = 0$ is imposed, then $\partial_z^2 Y^z = \partial_z^2 Y^{\bar{z}} = 0$, restrictions that eliminate the possibility of having superrotations.

From the transformation law of $\bar{\hat{\hat{k}}}_{(2)}^{AB}$

$$\delta \hat{k}_{(2)}^{AB} = -\frac{\sqrt{\gamma} T}{l^2} \hat{k}_{(2)}^{AB} + \sqrt{\gamma} \left( D^A D^B - \frac{1}{2} \gamma^{AB} \Delta \right) T + \frac{1}{2} \left( D_C Y^C \right) \hat{k}_{(2)}^{AB} + \mathcal{L}_Y \hat{k}_{(2)}^{AB} ,$$

if we set

$$\tilde{h}_{AB} = 0, \quad \tilde{k}_{(2)}^{AB} = 0 , \quad (2.31)$$

the following condition on the parameter $T (\theta, \phi)$ is obtained

$$\left( D^A D^B - \frac{1}{2} \gamma^{AB} \Delta \right) T = 0 .$$
This condition annihilates the modes with \( \ell \geq 2 \) in the spherical harmonics expansion. Thus,

\[
T = T_0 + l \vec{\beta} \cdot \hat{r}.
\]

Consequently, the modes of \( \mathcal{P}(\theta, \phi) \) that contribute to the charges are only those with \( \ell = 0 \) and \( \ell = 1 \), i.e., \( E \) and \( \vec{K} \). Therefore, the conditions (2.31) truncate the infinite-dimensional BMS_4 algebra to the finite-dimensional Carroll-AdS_4 algebra. With these restrictions (together with \( \vec{k}_{AB}^{(4)} = 0 \), that does not play any fundamental role), the fall-off in eqs. (2.17)–(2.22) reduces to a truncated set of asymptotic conditions with the same fall-off as the one used in General Relativity with a negative cosmological constant [4].

On the other hand, it is a well known fact that there is no a unique embedding of the Poincaré/Carroll-AdS_4 algebra within the BMS_4 algebra [60]. However, the truncation of the asymptotic conditions characterized by (2.31), selects a “preferred” Poincaré/Carroll-AdS_4 algebra that corresponds to the algebra that is inherited from General Relativity in the Carroll limit.

### 2.4 Carrollian Schwarzschild-AdS solution

Let us consider the solution of magnetic Carroll gravity with a negative cosmological constant whose spatial metric coincides with that of the Schwarzschild-AdS solution in General Relativity

\[
g_{ij} dx^i dx^j = \frac{dr^2}{(r^2 + 1 - \frac{M}{8\pi r})} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad \pi^{ij} = 0. \tag{2.32}
\]

The lapse and shift functions are given by

\[
N = \sqrt{\frac{r^2}{r^2 + 1} \frac{M}{8\pi r}}, \quad N^i = 0. \tag{2.33}
\]

This solution was discussed using isotropic coordinates in a covariant formulation of the magnetic theory in ref. [61].

From the asymptotic expansion of the spatial metric in eq. (2.32)

\[
g_{rr} = \frac{l^2}{r^2} - \frac{l^4}{r^4} + \frac{M l^4}{8\pi r^5} + \mathcal{O} \left( r^{-6} \right),
\]

\[
g_{rA} = 0,
\]

\[
g_{AB} = r^2 \gamma_{AB},
\]

it is clear that fits within the asymptotic conditions (2.17)–(2.22) with

\[
f_{rr} = \frac{l^4}{8\pi} M.
\]

Therefore, from (2.27) one finds

\[
\mathcal{P} = \frac{1}{4\pi} M.
\]
For this solution, $\mathcal{P}$ contains only the zero mode in the expansion in spherical harmonics. Consequently, according to (2.27), the only non-trivial charge is the Carroll energy, given by

$$E = M.$$  

Note that, as in the case with $\Lambda = 0$ discussed in [12], the canonical variables, as well as the lapse and the shift in eqs. (2.32), (2.33) exactly coincide with those of the Schwarzschild-AdS solution in General Relativity. In spite of the fact that in the Carrollian theory there is no notion of a four-dimensional Riemannian metric that can be reconstructed from them, in ref. [68] it is shown that the solution in (2.32) and (2.33) can be interpreted as a “Carrollian black hole” when appropriate regularity conditions are imposed on the “thermal” Carrollian geometry.

3 Asymptotic symmetries in electric Carroll gravity with a negative cosmological constant

The gravitational theory obtained from the electric Carroll contraction of General Relativity was originally introduced in the 70’s as a strong coupling limit of Einstein gravity [8], or alternatively as a “zero signature limit” of it [9–11]. In this limit, the Hamiltonian constraint does not depend on derivatives of the spatial metric, acquiring only a quadratic dependence on the momenta. This particular form of the Hamiltonian resembles that of a massive relativistic free particle, where the role of the mass is played by the cosmological constant. This similarity was used in ref. [11] as a starting point of an alternative perturbative approach to quantum gravity. In ref. [10], an action for electric Carroll gravity that is manifestly invariant under Carrollian changes of coordinates, was introduced. By virtue of their ultra-local properties (neighboring points are causally disconnected), this theory turns out to be useful to describe the properties of spacetime near space-like singularities, along the lines of the Belinsky-Khalatnikov-Lifshitz approach [15–18].

Recently, an analysis of the asymptotic structure of electric Carroll gravity was performed in the case of vanishing cosmological constant [12], where it was shown that, from the point of view of the asymptotic symmetries, this theory has some unusual properties. In particular, the asymptotic symmetry algebra does not contain a generator associated with time translations at the boundary, i.e., there is no a notion of energy in this theory. This effect was also observed in ref. [61] using covariant methods.

When a negative cosmological constant is present, the situation is even more dramatic. As it was pointed out in [12], the background configuration (2.12) is no longer a solution of the electric theory, and as a consequence it does not seems to be possible to construct a consistent set of asymptotic conditions with non-trivial charges. The incompatibility of the electric contraction with a negative cosmological constant was also discussed in [61] using scaling arguments.

---

5When Regge-Teitelboim [69] and “restricted Henneaux-Troessaert parity conditions” [70] are used, the asymptotic symmetry algebra of the electric theory with $\Lambda = 0$ does not contain energy and boost generators [12]. The same conclusion is obtained when “extended Henneaux-Troessaert parity conditions” [71, 72] are considered [73].
Here, it is shown that some of these problems can be circumvented if asymptotic conditions are constructed for the theory that is obtained from the electric contraction of Euclidean General Relativity with \( \Lambda < 0 \).

### 3.1 Electric theory obtained from a Carrollian contraction of General Relativity with Lorentzian signature

The action principle of the theory obtained from an electric contraction of (Lorentzian) General Relativity with a non-vanishing cosmological constant is given by

\[
I^E_{\text{Lor}} = \int dt d^3x \left( \pi^{ij} \dot{g}_{ij} - N \mathcal{H}^E_{\text{Lor}} - N^i \mathcal{H}^E_i \right),
\]

where

\[
\mathcal{H}^E_{\text{Lor}} = \frac{1}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) + 2\sqrt{g} \Lambda, \quad \mathcal{H}^E_i = -2\pi_{ij}^j.
\]

The Hamiltonian constraint \( \mathcal{H}^E_{\text{Lor}} \) does not depend on the derivatives of the spatial metric. Therefore, as it was shown in ref. [9], it obeys the following Poisson bracket

\[
\{ \mathcal{H}^E_{\text{Lor}} (x), \mathcal{H}^E_{\text{Lor}} (x') \} = 0.
\]

This abelian subalgebra reflects the Carrollian structure of the theory [7]. The Poisson brackets involving \( \mathcal{H}^E_i \) are completely determined by the invariance of the theory under spatial reparametrizations, and coincide with those in eqs. (2.6) and (2.7).

#### 3.1.1 Problem with the ground state and the lack of a consistent set of asymptotic conditions

From the form of the Hamiltonian constraint \( \mathcal{H}^E_{\text{Lor}} \) in eq. (3.2), it is clear that the solution obtained from a naive Carrollian limit of AdS spacetime given in eq. (2.12) is not a solution of this theory. The Hamiltonian constraint does not admit solutions with vanishing momenta, which is a direct consequence of the presence of the cosmological constant. Indeed, in the case when \( \Lambda = 0 \) studied in [12], the direct Carrollian contraction of Minkowski spacetime is a solution of both Carrollian gravitational theories, electric and magnetic.

One could expect that the background that is used to construct asymptotic symmetries possesses some isometries, for example invariance under rotations (that are inherited to the asymptotic symmetry algebra). However, as it is shown in appendix A, there are no spherically symmetric solutions of this theory where the spatial metric possesses an Euclidean signature, which is one of the main assumptions in the Hamiltonian formulation described here.

The absence of a sensible background configuration has relevant consequences in the possible construction of a consistent set of asymptotic conditions (usually they are defined as deviations from the corresponding background). Indeed, if one attempts to construct asymptotic conditions with finite non-trivial charges and such that the leading term of the angular components of \( \dot{g}_{ij} \) goes like the metric of a 2-sphere of radius \( r \), i.e., \( g_{AB} = r^2\gamma_{AB} + \ldots \), the Hamiltonian constraint, together with the preservation of the fall-off, eliminate all the possible terms appearing in the charges, or fix them as complex numbers.
This incompatibility with a negative $\Lambda$ was also observed in ref. [61] where the electric Carroll theory was considered as the leading order in a Carrollian expansion of General Relativity.

In the next section, it is shown that some of these problems can be solved if one considers the gravitational Carrollian theory obtained from the electrical contraction of General Relativity with Euclidean signature.

### 3.2 Electric theory obtained from a Carrollian contraction of General Relativity with Euclidean signature

The action principle of the electric Carrollian contraction of Euclidean General Relativity is given by

$$I^E_{\text{Euc}} = \int dt d^3x \left( \pi^{ij} \dot{g}_{ij} - N \mathcal{H}^E_{\text{Euc}} - N^i \mathcal{H}^E_i \right), \quad (3.3)$$

where

$$H^E_{\text{Euc}} = -\frac{1}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) + 2\sqrt{g} \Lambda, \quad H^E_i = -2\pi^j_{ij}. \quad (3.4)$$

In this case, there is a sign difference in the first term of the Hamiltonian constraint, as compared with the one derived from the Lorentzian theory in eq. (3.2). This is due the fact that in the Euclidean continuation in General Relativity one must Wick rotate the canonical momenta $\pi^{ij} \rightarrow -i\pi^{ij}$. The algebra of the constraints is insensitive to this change of sign, and consequently it closes according to the algebra in eqs. (2.5)–(2.7).

The transformation laws of the fields are given by

$$\delta g_{ij} = -\frac{2\xi}{\sqrt{g}} \left( \pi_{ij} - \frac{1}{2} g_{ij} \pi \right) + \xi_{ij} + \xi_{ji}, \quad (3.5)$$

$$\delta \pi^{ij} = -\frac{\xi}{2\sqrt{g}} g^{ij} \left( \pi^{kl} \pi_{kl} - \frac{1}{2} \pi^2 \right) + \frac{2\xi}{\sqrt{g}} \left( \pi^i_i \pi^{jl} - \frac{1}{2} \pi^{ij} \pi \right) - \xi \Lambda \sqrt{g} g^{ij} + \left( \xi^k \pi_{kj} \right)_{ik} - \xi_{ijk} \pi^{kj} - \xi_{ijk} \pi^{ki}. \quad (3.6)$$

The variation of the charge then becomes

$$\delta Q_E = \oint d^2s [2\xi_k \delta \pi^{kl} + \left( 2\xi^k \pi^{jl} - \xi_{ij} \pi^{jk} \right) \delta g_{jk}]. \quad (3.7)$$

Note that only the boundary term associated with the momentum constraint, $\mathcal{H}^E_i$, contributes to it.

#### 3.2.1 Alternative ground state

For the electric theory obtained from a Carrollian contraction of Euclidean General Relativity with $\Lambda < 0$, it is possible to find an alternative ground state configuration with a spatial metric of constant curvature that solves the constraints (3.4). The canonical variables of this solution are given by

$$\tilde{g}_{ij} dx^i dx^j = \frac{dr^2}{(r^2 + 1)} + r^2 \gamma_{AB} dx^A dx^B, \quad \tilde{\pi}^{ij} = \frac{2}{l} \sqrt{\tilde{g}} g^{ij}, \quad (3.8)$$
with the following lapse and shift

\[ \tilde{N} = \sqrt{\frac{r^2}{l^2} + 1}, \quad \tilde{N}^r = -\frac{r}{l} \sqrt{\frac{r^2}{l^2} + 1}. \]

In contrast with the ground state of the magnetic theory in eq. (2.12), this configuration has non-trivial momenta and a non-vanishing shift.

The Carrollian Killing vectors can be obtained from the transformations laws of the canonical variables in eqs. (3.5) and (3.6), requiring that \( \delta \tilde{g}_{ij} = \delta \tilde{\pi}^{ij} = 0 \). Thus, they are restricted to obey the following equations

\[ \xi_{ij} + \xi_{ji} - \frac{2}{3} \tilde{g}_{ij} \xi^k_k = 0, \quad (3.9) \]

\[ \xi = -\frac{1}{3} \xi^k_k, \quad (3.10) \]

Note that eq. (3.9) is the three-dimensional conformal Killing equation defined on a space with Euclidean signature. Because the composition law (2.14) of \( \xi^i \) coincides with the Lie bracket, the algebra of isometries span the conformal group of the Euclidean three-dimensional space \( SO(1,4) \). As it is shown in the next section, this property also extends to the asymptotic symmetries that are defined in terms of deviations with respect to this background.

### 3.2.2 Asymptotic conditions

A consistent set of asymptotic conditions can be constructed by considering deviations with respect to the background configuration (3.8). The proposed fall-off of the fields is given by

\[ g_{rr} = \frac{l^2}{r^2} \frac{l^4}{r^4} + \frac{f_{rr}}{r^5} + \mathcal{O} \left( r^{-6} \right), \quad (3.11) \]

\[ g_{rA} = \frac{f_{rA}}{r^4} + \mathcal{O} \left( r^{-5} \right), \quad (3.12) \]

\[ g_{AB} = r^2\gamma_{AB} + \frac{\tilde{f}_{AB}}{r} + \mathcal{O} \left( r^{-2} \right), \quad (3.13) \]

\[ \pi^{rr} = \frac{2\sqrt{\gamma}}{l^2} r^3 + \sqrt{\gamma} r + \frac{1}{l^2} f_{rr} \left( \tilde{f} + \frac{1}{l^2} f_{rr} \right) + \frac{p_{rr}}{r} + \mathcal{O} \left( r^{-2} \right), \quad (3.14) \]

\[ \pi^{rA} = \frac{\tilde{p}^{rA}}{r^2} + \mathcal{O} \left( r^{-3} \right), \quad (3.15) \]

\[ \pi^{AB} = \frac{2\sqrt{\gamma}}{r} \gamma^{AB} r^2 - \frac{\sqrt{\gamma} \gamma^{AB} r^2}{r^3} - \frac{\sqrt{\gamma} \tilde{f}^{AB}}{2 r^4} + \frac{\tilde{p}^{AB}}{r^5} + \mathcal{O} \left( r^{-6} \right). \quad (3.16) \]

The asymptotic behavior of the canonical variables is preserved by the following parameters

\[ \xi = r T + \frac{l^2}{4} (\Delta + 2) T \frac{1}{r} + \ldots, \quad (3.17) \]

\[ \xi' = -\frac{r^2 T}{l} - \frac{r D_A Y^A}{2} + \frac{l}{4} (\Delta - 2) T + \ldots, \quad (3.18) \]

\[ \xi^A = Y^A - \left( D^A T \right) \frac{1}{r} + \ldots. \quad (3.19) \]
Here, the parameters $T$ and $Y^A$ must to obey the following conditions coming from the preservation of (3.11)–(3.16)

\[
D_A D_B T - \frac{1}{2} \gamma_{AB} \Delta T = 0, \quad (3.20)
\]

\[
D_A Y_B + D_B Y_A - \gamma_{AB} D_C Y^C = 0. \quad (3.21)
\]

The conserved charges can be obtained using the asymptotic conditions (3.11)–(3.19) in eq. (3.7)

\[
Q = \oint d^2 x \left( - \frac{4 \sqrt{\gamma}}{l^3} T f_{rr} + 2 Y^A \gamma_{AB} p^{rB} \right), \quad (3.22)
\]

where it was assumed that the parameters $T$ and $Y^A$ have a vanishing functional variation. Note that, in spite of the fact that according to eq. (3.7) the parameter $\xi$ describing normal deformations to the $t = \text{const.}$ hypersurface does not appear in $\delta Q$, the parameter $T$ in the leading order of $\xi$ enters in the expression for the charge through $\xi_r$. There is no analogue of this situation when the cosmological constant vanishes [12].

The explicit solution of eqs. (3.20) and (3.21) is

\[
T = \alpha^{40} + \alpha^{4I} \hat{r}_I, \quad Y^A = \partial^A \left( \alpha^{0I} \hat{r}_I \right) + \frac{\epsilon^{AB}}{\sqrt{\gamma}} \partial_B \left( \frac{1}{2} \epsilon_{IJK} \alpha^{JK} \hat{r}_I \right),
\]

where $\alpha^{40}, \alpha^{4I}, \alpha^{0I}$ and $\alpha^{JK}$ are constants in the angles. Therefore, the charge $Q$ in (3.22) can be written as

\[
Q = \frac{1}{2} \alpha^{mn} J_{mn},
\]

where $m, n = \{0, I, 4\}$, with $I = 1, 2, 3$ and $\alpha_{mn} = -\alpha_{nm}$, $J_{mn} = -J_{nm}$.

The explicit form of the generators expressed in terms of surface integrals is

\[
J_{40} = - \oint d^2 x \left( \frac{4 \sqrt{\gamma}}{l^3} f_{rr} \right), \quad J_{4I} = - \oint d^2 x \left( \frac{4 \sqrt{\gamma}}{l^3} \hat{r}_I f_{rr} \right), \quad J_{0I} = - \oint d^2 x \left( 2 \hat{r}_I D_A p^{rA} \right), \quad J_{IJ} = \oint d^2 x \left( 2 \sqrt{\gamma} \epsilon_{IJK} \epsilon_{AB} \hat{r}_K D_A p^{rB} \right).
\]

The asymptotic symmetry algebra can then be obtained using the identity $\delta_2 Q [\lambda_1] = \{Q [\lambda_1], Q [\lambda_2]\}$, where the transformation laws of the fields appearing in the charges, $f_{rr}$ and $p^{rA}$, are given by

\[
\delta f_{rr} = \frac{3}{2} f_{rr} D_A Y^A + Y^A D_A f_{rr} + \frac{l^3}{2 \sqrt{\gamma}} T D_A p^{rA} + l^3 (D_A T) \frac{p^{rA}}{\sqrt{\gamma}},
\]

\[
\delta p^{rA} = - \frac{3 \sqrt{\gamma}}{l^3} f_{rr} D^A T - \sqrt{\gamma} T D^A f_{rr} + \frac{3 \sqrt{\gamma}}{2 l} T D_C J^{AC} + \frac{3 \sqrt{\gamma}}{2 l} (D_B T) \hat{f}_{AB} + (D_B Y^B) p^{rA} + \mathcal{L}_Y p^{rA}.
\]

The generators $J_{mn}$ then obey the following Poisson bracket algebra

\[
\{J_{mn}, J_{rs}\} = \eta_{ms} J_{nr} + \eta_{nr} J_{ms} - \eta_{mr} J_{ns} - \eta_{ns} J_{mr},
\]
where $\eta_{mn} = \text{diag}(-,+,+,+,+)$. Thus, the asymptotic symmetry algebra is given the $so(1,4)$ algebra.

Although this symmetry is not the Carroll algebra (or a cosmological extension of it), it could be interpreted as the Carrollian symmetry that acts on the homogeneous space given by the “Carrollian light-cone” [64].

It is worth noting that the asymptotic conditions (3.11)–(3.16) are not inherited from General Relativity, and therefore are specific of this particular electric Carrollian theory.

### 3.2.3 Spherically symmetric solutions

As it was shown in the previous sections, if one considers the Carrollian theory obtained from an electric contraction of Euclidean General Relativity with $\Lambda < 0$, a consistent set of asymptotic conditions can be constructed using the ground state (3.8) as a starting point. This does not seem to be possible for the Carrollian theory obtained from Einstein gravity with a Lorentzian signature.

In spite of this success, the electric Carrollian theory coming from Euclidean General Relativity still possesses some unusual properties that become explicit when spherically symmetric solutions are considered. For example, the spherically symmetric ansatz of the form

$$g_{ij} dx^i dx^j = \frac{dr^2}{f^2(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad \pi^{ij} = \frac{2}{l} \sqrt{g} g^{ij}, \quad (3.23)$$

solves the equation of motion for any arbitrary value of $f(r)$, with the following lapse and shift

$$N = f(r), \quad N^r = -\frac{r}{l} f(r).$$

Consequently, there is a huge degeneracy in the space of solutions of this theory. Similar problems were encountered in ref. [12] when $\Lambda = 0$.

If we choose $f(r)$ such that the spatial metric in (3.23) coincides with the one of the (Euclidean) Schwarzschild-AdS metric (that contains the ground state configuration (3.8) as a particular case), i.e.,

$$f(r) = \sqrt{\frac{r^2}{l^2} + 1 - \frac{M}{8\pi r}},$$

the only non-trivial charge is $J_{40} = -2lM$, which is proportional to the parameter $M$ that in General Relativity represents the “mass” of the solution.

### 4 Discussion

The asymptotic structure of Carrollian gravitational theories, obtained from electric and magnetic contractions of General Relativity with a negative cosmological constant, was analyzed. The electric theory, as in the case with $\Lambda = 0$, has some unusual features from the point of view of their asymptotic symmetries. In particular, for the electric theory obtained from a contraction of (Lorentzian) General Relativity, it does not seem to be possible to construct a consistent set of asymptotic conditions. This can be traced back to the absence of a sensible ground state configuration in this theory. The situation can
be improved if the Carrollian theory obtained from an electric contraction of Euclidean General Relativity is considered. In this case, asymptotic conditions are constructed with an asymptotic symmetry algebra given by the “Euclidean AdS\(_4\)” algebra \(so(1,4)\). This feature has its origin in the fact that the fall-off of the fields cannot be seen as coming from Einstein gravity, it is intrinsic to the “Euclidean” electric Carrollian theory. Additionally, it was shown that the space of a certain class of spherically symmetric solutions is degenerate, where some of the arbitrary functions in the ansatz are not determined by the equations of motion.

On the contrary, the magnetic Carrollian gravitational theory with \(\Lambda < 0\) has very interesting properties and a very rich structure from the perspective of their asymptotic symmetries. The asymptotic symmetry algebra is infinite-dimensional and corresponds to the three-dimensional Carrollian conformal algebra, that according to [28] is isomorphic to the BMS\(_4\) algebra. These results are in full agreement with the expectations coming from holography. In this sense, this theory could provide a new concrete framework for the study of Carrollian holography. This is particularly interesting because the structure of the constraints in the Carrollian magnetic theory is simpler than the one of General Relativity. Indeed, the algebra (2.5)–(2.7) is field independent. Furthermore, as it is shown in ref. [68], this theory also admits a solution (see eqs. (2.32) and (2.33)) that can be interpreted as a “Carrollian black hole.” It is a thermal regular configuration in the context of Carrollian geometry, that possesses a non-vanishing entropy. This could provide a fertile ground to explore the possible nature of their microstates, as well as different aspects related with quantum gravity and holography in the Carrollian limit.

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A Spherically symmetric configurations in “Lorentzian” electric Carroll gravity with a negative cosmological constant

In this appendix we examine spherically symmetric configuration in the electric Carrollian contraction of Lorentzian Einstein gravity with a negative cosmological constant.

Let us consider a spherically symmetric ansatz for the canonical variables of the following form

\[
\begin{align*}
  g_{rr} &= f(r), & g_{rA} &= 0, & g_{AB} &= r^2 \gamma_{AB}, \\
  \pi^{rr} &= \sqrt{gp}(r), & \pi^{rA} &= 0, & \pi^{AB} &= \sqrt{\gamma} \gamma^{AB} h(r),
\end{align*}
\]

From the constraints \(\mathcal{H}^E_{\text{Lor}} \approx 0\) and \(\mathcal{H}^E_i \approx 0\) in eq. (3.2) one finds

\[
\begin{align*}
  f(r) &= \frac{r}{p(r)^2} \left( C - \frac{4r^3}{\ell^2} \right), & h(r) &= \frac{1}{p(r)} \left( \frac{C}{4r} - \frac{4r^2}{\ell^2} \right),
\end{align*}
\]
where $C$ is an integration constant. Note that for those values of $r$ such that $r > \left(\frac{\ell C}{4}\right)^{\frac{1}{3}}$, the function $f(r)$ becomes negative and therefore violates the assumption that the spatial metric has Euclidean signature. In particular, this is always true in the asymptotic region where $r \to \infty$.

As a consequence, spherically symmetric configurations with an Euclidean spatial metric are not solutions of the electric Carrollian gravity with negative cosmological constant that is obtained from the ultra-relativistic limit of Lorentzian General Relativity.

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