WEIGHTED VECTOR-VALUED ESTIMATES FOR A
NON-STANDARD CALDERÓN-ZYGMUND OPERATOR

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ABSTRACT. In this paper, the author considers the weighted vector-valued
estimates for the operator defined by
\[ T_A f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)(A(x) - A(y) - \nabla A(y))f(y)}{|x-y|^{n+1}} dy \]
and its corresponding maximal operator \( T_A^* \), where \( \Omega \) is homogeneous of
degree zero, has vanishing moment of order one, \( A \) is a function in \( \mathbb{R}^n \) such that
\( \nabla A \in \text{BMO}(\mathbb{R}^n) \). By a pointwise estimate for \( \| \{ T_A f_k(x) \} \|_{L^q} \),
the author obtains some quantitative weighted vector-valued estimate for \( T_A \) and \( T_A^* \).

1. Introduction

In the remarkable work [21], Muckenhoupt characterized the class of weights \( w \) such that the
Hardy-Littlewood maximal operator \( M \) satisfies the weighted \( L^p \) (\( p \in (1, \infty) \)) estimate
\[ \| Mf \|_{L^{p, \infty}(\mathbb{R}^n, w)} \lesssim \| f \|_{L^p(\mathbb{R}^n, w)}. \] (1.1)
The inequality (1.1) holds if and only if \( w \) satisfies the \( A_p(\mathbb{R}^n) \) condition, that is,
\[ [w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{-1/p}(x) dx \right)^{p-1} < \infty, \]
where the supremum is taken over all cubes in \( \mathbb{R}^n \), \([w]_{A_p} \) is called the \( A_p \) constant
of \( w \). Also, Muckenhoupt proved that \( M \) is bounded on \( L^p(\mathbb{R}^n, w) \) if and only if
\( w \) satisfies the \( A_p(\mathbb{R}^n) \) condition. Since then, considerable attention has been paid
to the theory of \( A_p(\mathbb{R}^n) \) and the weighted norm inequalities with \( A_p(\mathbb{R}^n) \) weights
for main operators in Harmonic Analysis, see [10, Chapter 9] and related references therein.

However, the classical results on the weighted norm inequalities with \( A_p(\mathbb{R}^n) \) weights
did not reflect the quantitative dependence of the \( L^p(\mathbb{R}^n, w) \) operator norm
in terms of the relevant constant involving the weights. The question of the sharp
dependence of the weighted estimates in terms of the \( A_p(\mathbb{R}^n) \) constant specifically
raised by Buckley [2], who proved that if \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n) \), then
\[ \| Mf \|_{L^p(\mathbb{R}^n, w)} \lesssim_{n, p} [w]_{A_p}^1 \| f \|_{L^p(\mathbb{R}^n, w)}. \] (1.2)
Moreover, the estimate (1.2) is sharp since the exponent \( 1/(p-1) \) can not be replaced by a smaller one. Hytönen and Pérez [10] improved the estimate (1.2),

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and showed that
\[(1.3) \quad \|Mf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} \left( [w]_{A_p} [w^{-\frac{1}{p-1}}]_{A_\infty} \right)^\frac{1}{p} \|f\|_{L^p(\mathbb{R}^n, w)}.
\]
where and in the following, for a weight \(u\), \([u]_{A_\infty}\) is defined by
\[ [u]_{A_\infty} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u \chi_Q)(x) \, dx. \]

It is well known that for \(w \in A_p(\mathbb{R}^n)\), \([w^{-\frac{1}{p-1}}]_{A_\infty} \lesssim [w]_{A_p}\). Thus, (1.3) is more subtle than (1.2).

The sharp dependence of the weighted estimates of singular integral operators in terms of the \(A_p(\mathbb{R}^n)\) constant was much more complicated. Petermichl \cite{Petermichl2006}
solved this question for Hilbert transform and Riesz transform. Hytönen \cite{Hytönen2010} proved that for a Calderón-Zygmund operator \(T\) and \(w \in A_2(\mathbb{R}^n)\),
\[(1.4) \quad \|Tf\|_{L^2(\mathbb{R}^n, w)} \lesssim_n [w]_{A_2} \|f\|_{L^2(\mathbb{R}^n, w)}.
\]
This solved the so-called \(A_2\) conjecture. Combining the estimate (1.4) and the extrapolation theorem in \cite{Grafakos2008}, we know that for a Calderón-Zygmund operator \(T\), \(p \in (1, \infty)\) and \(w \in A_p(\mathbb{R}^n)\),
\[(1.5) \quad \|Tf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\mathbb{R}^n, w)}.\]

In \cite{Grafakos2010}, Lerner gave a much simpler proof of (1.4) by controlling the Calderón-Zygmund operator using sparse operators.

Now let us consider a class of non-standard Calderón-Zygmund operators. For \(x \in \mathbb{R}^n\), we denote by \(x_j\) (1 \(\leq j \leq n\)) the \(j\)-th variable of \(x\). Let \(\Omega\) be homogeneous of degree zero, integrable on the unit sphere \(S^{n-1}\) and satisfy the vanishing condition that for all \(1 \leq j \leq n\),
\[(1.6) \quad \int_{S^{n-1}} \Omega(x') x'_j \, dx' = 0.\]

Let \(A\) be a function on \(\mathbb{R}^n\) whose derivatives of order one in \(\text{BMO}(\mathbb{R}^n)\). Define the operator \(T_A\) by
\[(1.7) T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y)(x-y)) f(y) \, dy.\]

The maximal singular integral operator associated with \(T_A\) is defined by
\[ T_A^* f(x) = \sup_{\epsilon > 0} |T_{A, \epsilon} f(x)|, \]
with
\[ T_{A, \epsilon} f(x) = \int_{|x-y| \geq \epsilon} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y)(x-y)) f(y) \, dy. \]

The operator \(T_A\) is closed related to the Calderón commutator, of interest in PDE, and was first consider by Cohen \cite{Cohen2001}. Cohen proved that if \(\Omega \in \text{Lip}_\alpha(S^{n-1})\) (\(\alpha \in (0, 1)\)), then for \(p \in (1, \infty)\), \(T_A\) is a bounded operator on \(L^p(\mathbb{R}^n)\) with bound \(C \|\nabla A\|_{\text{BMO}(\mathbb{R}^n)}\). In fact, the argument in \cite{Cohen2001} also leads to the boundedness on \(L^p(\mathbb{R}^n, w)\) \((w \in A_p(\mathbb{R}^n))\) for \(T_A\). Hofmann \cite{Hofmann2006} improved the result of Cohen and showed that \(\Omega \in \bigcup_{q \geq 1} L^q(S^{n-1})\) is a sufficient condition such that \(T_A\) is bounded on \(L^p(\mathbb{R}^n, w)\) for \(p \in (1, \infty)\). Hu and Yang \cite{Hu2010} established the endpoint estimate for \(T_A\), from which they deduced some weighted \(L^p\) estimates with general weights for \(T_A\).
The purpose of this paper is to establish refined weighted vector-valued estimates for the operators $T_A$ and $T_A^*$. To formulate our result, we first recall some definitions. Let $\Omega$ be a bounded function on $S^{n-1}$. The $L^\infty$ continuity modulus of $\Omega$ is defined by

$$\omega_\infty(t) = \sup_{|\rho| < t} |\Omega(\rho x') - \Omega(x')|,$$

where the supremum is taken over all rotations $\rho$ on the unit sphere $S^{n-1}$, and $|\rho| = \sup_{x' \in S^{n-1}} |\rho x' - x'|$. Let $p, r \in (0, \infty)$ and $w$ be a weight. As usual, for a sequence of numbers $\{a_k\}_{k=1}^\infty$, we denote $\|a_k\|_r = (\sum_k |a_k|^r)^{1/r}$. The space $L^p(I^r; \mathbb{R}^n, w)$ is defined as

$$L^p(I^r; \mathbb{R}^n, w) = \{ \{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} < \infty \}$$

where

$$\|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} \|f_k(x)\|_r^p w(x) \, dx \right)^{1/p}.$$

The space $L^{p, \infty}(I^r; \mathbb{R}^n, w)$ is defined as

$$L^{p, \infty}(I^r; \mathbb{R}^n, w) = \{ \{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)} < \infty \}$$

with

$$\|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)} = \sup_{\lambda > 0} \lambda^p w \left( \{ x \in \mathbb{R}^n : \|f_k(x)\|_r > \lambda \} \right).$$

When $w \equiv 1$, we denote $\|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} (\|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)})$ by $\|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n)} (\|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n)})$ for simplicity. Our first result can be stated as follows.

**Theorem 1.1.** Let $\Omega$ be homogeneous of degree zero, satisfy the vanishing moment (1.6), $A$ be a function in $A_p(\mathbb{R}^n)$ whose derivatives of order one in $\text{BMO}(\mathbb{R}^n)$. Suppose that the $L^\infty$ continuity modulus of $\Omega$ satisfies that

$$\int_0^1 \omega_\infty(t)(1 + |\log t|) \frac{dt}{t} < \infty,$$

then for $p, q \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|\{T_A f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)} + \|\{T_A^* f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)}$$

$$\lesssim_{n, p} \|\nabla A\|_{\text{BMO}(\mathbb{R}^n)} w \max_{A_p} \left( |\sigma|_{A_\infty} \right) \|\{f_k\}\|_{L^p(I^r; \mathbb{R}^n, w)}.$$

with $\sigma = w^{-1/p^*}$. In particular,

$$\|\{T_A f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)} + \|\{T_A^* f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)}$$

$$\lesssim_{n, p} \|\nabla A\|_{\text{BMO}(\mathbb{R}^n)} w \max_{A_p} \left( |\sigma|_{A_\infty} \right) \|\{f_k\}\|_{L^{p, \infty}(I^r; \mathbb{R}^n, w)}.$$

**Remark 1.2.** Theorem 1.1 implies that for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|T_A f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n, p, q} \|\nabla A\|_{\text{BMO}(\mathbb{R}^n)} [w]_{A_p} \max_{A_p} \left( |\sigma|_{A_\infty} \right) \|f\|_{L^p(\mathbb{R}^n, w)}.$$

For the case $p \in (1, 2]$, this estimate is sharp in the sense that the exponent $2/p - 1$ can not be replaced by a smaller one, see Example 3.3. The quantitative bound in (1.9) is new, although we do not know if it is sharp for $p \in (2, \infty)$.

We are also interested in the weighted endpoint bounds for $T_A$ and $T_A^*$. We have that...
Theorem 1.3. Let $\Omega$ be homogeneous of degree zero, satisfy the vanishing moment (1.6), $A$ be a function in $\mathbb{R}^n$ whose derivatives of order one in $\text{BMO}(\mathbb{R}^n)$. Suppose that $\Omega$ satisfies (1.2), then for $q \in (1, \infty)$ and $w \in A_1(\mathbb{R}^n)$, 
\[
w(\{x \in \mathbb{R}^n : \|T_{\Omega} f_k\|_{L^q} > \lambda\}) + w(\{x \in \mathbb{R}^n : \|T_{\Omega} f_k\|_{L^q} > \lambda\}) \leq_n \|A\|_{\text{BMO}(\mathbb{R}^n)} [w]_{A_1} \Psi_2([w]_{A_1}) \int_{\mathbb{R}^n} \|f_k\|_{L^q} \log \left( e + \frac{\|f_k\|_{L^q}}{\lambda} \right) w(x) dx,
\] 
with $\Psi_2(t) = \log^2 (e + t)$.

In what follows, $C$ always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. Constant with subscript such as $C_1$, does not change in different occurrences. For any set $E \subset \mathbb{R}^n$, $\chi_E$ denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ and $\lambda \in (0, \infty)$, we use $\ell(Q) (\text{diam} Q)$ to denote the side length (diameter) of $Q$, and $\lambda Q$ to denote the cube with the same center as $Q$ and whose side length is $\lambda$ times that of $Q$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ denotes the ball centered at $x$ and having radius $r$. For locally integrable function $f$ and a cube $Q \subset \mathbb{R}^n$, $\langle f \rangle_Q = |Q|^{-1} \int_Q f(y) dy$.

2. Dominated by sparse operator

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of all cubes of the form 
\[2^{-k}([0, 1]^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n.
\]
Denote the standard grid by $\mathcal{D}$.

As usual, by a general dyadic grid $\mathcal{D}$, we mean a collection of cube with the following properties: (i) for any cube $Q \in \mathcal{D}$, it side length $\ell(Q)$ is of the form $2^k$ for some $k \in \mathbb{Z}$; (ii) for any cubes $Q_1, Q_2 \in \mathcal{D}$, $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$; (iii) for each $k \in \mathbb{Z}$, the cubes of side length $2^k$ form a partition of $\mathbb{R}^n$.

Let $\mathcal{D}$ be a dyadic grid and $M_{\mathcal{D}}$ be the maximal operator defined by
\[M_{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \langle |f| \rangle_Q.
\]

For $\delta > 0$, let $M_{\mathcal{D}, \delta} f(x) = \left\{ M_{\mathcal{D}}(|f|^{\delta})(x) \right\}^{1/\delta}$ and $M_{\delta} f(x) = \left\{ M(|f|^{\delta})(x) \right\}^{1/\delta}$. Associated with $\mathcal{D}$, define the sharp maximal function $M^\sharp_{\mathcal{D}}$ as 
\[M^\sharp_{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.
\]

For $\delta \in (0, 1)$, let $M^\sharp_{\mathcal{D}, \delta} f(x) = \left[ M^\sharp_{\mathcal{D}}(|f|^{\delta})(x) \right]^{1/\delta}$. Repeating the argument in [27, p. 153], we can verify that, if $\Phi$ is a increasing function on $[0, \infty)$ which satisfies the doubling condition that
\[\Phi(2t) \leq C \Phi(t), t \in [0, \infty),
\]
then
\[
\sup_{\lambda > 0} \Phi(\lambda)|\{x \in \mathbb{R}^n : h(x) > \lambda\}| \lesssim \sup_{\lambda > 0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M^\sharp_{\mathcal{D}, \delta} h(x) > \lambda\}|,
\]
provided that $\sup_{\lambda>0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M_{\mathcal{G}}(h(x) > \lambda)\}| < \infty$, and
\begin{equation}
(2.2) \quad \sup_{\lambda>0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M_{\mathcal{G}}(h(x) > \lambda)\}| \lesssim \sup_{\lambda>0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M_{\mathcal{G}}^2(h(x) > \lambda)\}|
\end{equation}
provided that $\sup_{\lambda>0} \Phi(\lambda)|\{x \in \mathbb{R}^n : M_{\mathcal{G}}(h(x) > \lambda)\}| < \infty$, see also [23].

Let $\eta \in (0, 1)$ and $\mathcal{S}$ be a family of cubes. We say that $\mathcal{S}$ is $\eta$-sparse, if for each fixed $Q \in \mathcal{S}$, there exists a measurable subset $E_Q \subset Q$, such that $|E_Q| \geq \eta|Q|$ and $\{E_Q\}$ are pairwise disjoint. Associated with the sparse family $\mathcal{S}$ and constants $\beta \in [0, \infty)$, we define the sparse operator $A_{\mathcal{S}, L(\log L)^\beta}$ by
\[
A_{\mathcal{S}, L(\log L)^\beta} f(x) = \sum_{Q \in \mathcal{S}} \|f\|_{L(\log L)^\beta, Q} \chi_Q(x),
\]
here and in the following, for $\beta \in [0, \infty)$,
\[
\|f\|_{L(\log L)^\beta, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|f(y)|}{\lambda} \log^\beta \left( 1 + \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.
\]
We denote $A_{\mathcal{S}, L(\log L)^\beta}$ by $A_{\mathcal{S}, L \log L}$ for simplicity.

**Lemma 2.1.** Let $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$ and $\sigma = w^{-1/(p-1)}$. Let $\mathcal{S}$ be a sparse family. Then
\begin{equation}
(2.3) \quad \|A_{\mathcal{S}, L(\log L)^\beta} f\|_{L^p(\mathbb{R}^n, w)} \lesssim [w]_{A_p} \left( [w]_{A_\infty}^{1/\sigma} + [\sigma]_{A_\infty}^{1/\sigma} \right) \|f\|_{L^p(\mathbb{R}^n, w)}.
\end{equation}

For the proof of Lemma 2.1, see [4].

As in [18], for a sublinear operator $T$, we define the associated grand maximal operator $\mathcal{M}_T$ by
\[
\mathcal{M}_T f(x) = \sup_{Q \ni x} \sup_{\xi \in \partial Q} |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

**Lemma 2.2.** Let $q \in (1, \infty)$ and $Q_0 \subset \mathbb{R}^n$. Let $T$ be a sublinear operator. Suppose that $T$ is bounded on $L^q(\mathbb{R}^n)$. Then for a. e. $x \in Q_0$,
\[
\|\{T(f\chi_{3Q_0})(x)\}\|_{q^*} \leq C \|\{f(x)\}\|_{q^*} + \|\{\mathcal{M}_T(f\chi_{3Q_0})(x)\}\|_{q^*}.
\]

*Proof.* We employ the argument in [18]. Let $x \in \text{int}Q_0$ be a point of approximation continuity of $\|\{T_A(f\chi_{3Q_0})\}\|_{q^*}$. For $r > 0$, the set
\[
E_r(x) = \{y \in B(x, r) : \|\{T(f\chi_{3Q_0})(x)\}\|_{q^*} - \|\{T(f\chi_{3Q_0})(y)\}\|_{q^*} < \epsilon\}
\]
satisfies that $\lim_{r \to 0} \frac{|E_r(x)|}{|B(x, r)|} = 1$. Denote by $Q(x, r)$ the smallest cube centered at $x$ and containing $B(x, r)$. Let $r > 0$ small enough such that $Q(x, r) \subset Q_0$. Then for $y \in E_r(x)$,
\[
\|\{T(f\chi_{3Q_0})(x)\}\|_{q^*} < \|\{T(f\chi_{3Q_0})(y)\}\|_{q^*} + \epsilon
\]
\[
\leq \|\{T(f\chi_{3Q(x,r)})(y)\}\|_{q^*} + \|\{\mathcal{M}_T(f\chi_{3Q_0})(x)\}\|_{q^*} + \epsilon.
\]
The boundedness on $L^q(\mathbb{R}^n)$ of $T$ tells us that

$$
\| \{T(f_k \chi_{3Q_0})(x)\} \|_{L^q} \leq \left( \frac{1}{|E_s(x)|} \int_{E_s(x)} \| \{T(f_k \chi_{3Q(x,r)}(y))\} \|_{L^q}^\beta \, dy \right)^\frac{1}{\beta} + \| \{ \mathcal{M}(f_k \chi_{3Q_0})(x) \} \|_{L^q} + \epsilon \\
\leq C \left( \frac{1}{|3Q(x,r)|} \int_{3Q(x,r)} \| f_k(z) \|_{L^q}^\beta \, dz \right)^\frac{1}{\beta} + \| \{ \mathcal{M}(f_k \chi_{3Q_0})(x) \} \|_{L^q} + \epsilon.
$$

Letting $r \to 0$ then leads to the desired conclusion.

We are now ready to give our main result in this section.

**Theorem 2.3.** Let $q \in (1, \infty)$, $\beta \in [0, \infty)$, $T$ be a sublinear operator and $\mathcal{M}$ the corresponding grand maximal operator. Suppose that $T$ is bounded on $L^q(\mathbb{R}^n)$, and for some constants $C_1 > 0$ and any $\lambda > 0$,

$$
\left| \left\{ y \in \mathbb{R}^n : \| \{ \mathcal{M}f_k(y) \} \|_{L^q} > \lambda \right\} \right| \leq C_1 \int_{\mathbb{R}^n} \frac{\| f_k(y) \|_{L^q}^\beta}{\lambda} \log^\beta \left( 1 + \frac{\| f_k(y) \|_{L^q}}{\lambda} \right) \, dy.
$$

Then for $N \in \mathbb{N}$ and bounded functions $f_1, \ldots, f_N$ with compact supports, there exists a $\frac{1}{10\beta}$-sparse family $S$ such that for a.e. $x \in \mathbb{R}^n$,

$$
\| \{Tf_k(x)\} \|_{L^q} \lesssim \mathcal{A}_S \| f_k \|_{L(\log L)^{\beta}} \left( \| f_k \|_{L^q} \right)(x).
$$

**Proof.** We employ the ideas in [18]. We claim that for each cube $Q_0 \subset \mathbb{R}^n$, there exist pairwise disjoint cubes $\{ P_j \} \subset \mathcal{D}(Q_0)$, such that $\sum_j |P_j| \leq \frac{1}{4}|Q_0|$, and for a.e. $x \in Q_0$,

$$
\| \{T(f_k \chi_{3Q_0})(x)\} \|_{L^q} \chi_{Q_0}(x) \leq C \| f_k \|_{L(\log L)^{\beta}} \left( \| f_k \|_{L^q} \right) \chi_{Q_0}(x) + \sum_j \| \{T(f_k \chi_{3P_j})(x)\} \|_{L^q} \chi_{P_j}(x).
$$

Let $C_2 \in (1, \infty)$ be a constant which will be chosen later. It follows from (2.4) that

$$
\left| \left\{ x \in Q_0 : \| \{ \mathcal{M}(f_k \chi_{3Q_0})(x) \} \|_{L^q} > C_2 \| f_k \|_{L(\log L)^{\beta}} \right\} \right| \leq C_1 \int_{3Q_0} \frac{\| f_k(y) \|_{L^q}^\beta}{C_2 \| f_k \|_{L(\log L)^{\beta}} \| f_k \|_{L(\log L)^{\beta}} \chi_{3Q_0}} \log^\beta \left( 1 + \frac{\| f_k(y) \|_{L^q}}{C_2 \| f_k \|_{L(\log L)^{\beta}} \chi_{3Q_0}} \right) \, dy
$$

$$
\leq C_1 \frac{C_2}{C_2^\beta} \int_{3Q_0} \frac{\| f_k(y) \|_{L^q}^\beta}{C_2 \| f_k \|_{L(\log L)^{\beta}} \chi_{3Q_0}} \log^\beta \left( 1 + \frac{\| f_k(y) \|_{L^q}}{C_2 \| f_k \|_{L(\log L)^{\beta}} \chi_{3Q_0}} \right) \, dy
$$

$$
\leq 3^\beta \frac{C_1}{C_2^\beta} |Q_0|.
$$

Let

$$
E = \left\{ y \in Q_0 : \| f_k(y) \|_{L^q} > C_2 \| f_k \|_{L(\log L)^{\beta}} \chi_{3Q_0} \right\}
$$

$$
\cup \left\{ y \in Q_0 : \| \mathcal{M}(f_k \chi_{3Q_0})(y) \|_{L^q} > C_2 \| f_k \|_{L(\log L)^{\beta}} \chi_{3Q_0} \right\}.
$$

Then $|E| \leq \frac{1}{2^\beta+2} |Q_0|$ if we choose $C_2$ large enough.
Now on cube $Q_0$, we apply the Calderón-Zygmund decomposition to $\chi_E$ at level $\frac{1}{2^{n+1}}$, we then obtain pairwise disjoint cubes $\{P_j\} \subset \mathcal{D}(Q_0)$, such that
\[
\frac{1}{2^{n+1}} |P_j| \leq |P_j \cap E| \leq \frac{1}{2} |P_j|
\]
and $|E \setminus \bigcup_j P_j| = 0$. Observe that $\sum_j |P_j| \leq \frac{1}{2} |Q_0|$ and $P_j \cap E^c \neq \emptyset$. Therefore,
\[
\text{ess sup}_{x \in P_j} \| T(f_k \chi_{3Q_0 \setminus 3P_j}) \|_{L^4} \leq C_2 \| \{f_k\} \|_{L(\log L)^{\beta}; 3Q_0}.
\]
By Lemma 2.2, we also have that for $a.$ e. $x \in Q_0 \setminus \bigcup_j P_j$,
\[
\| T(f_k \chi_{3Q_0})(x) \|_{L^4} \leq C_2 \| \{f_k\} \|_{L(\log L)^{\beta}; 3Q_0}.
\]
Observe that
\[
\| T(f_k \chi_{3Q_0})(x) \|_{L^4} \chi_{Q_0}(x) \leq \| T(f_k \chi_{3Q_0})(x) \|_{L^4} \chi_{Q_0 \setminus \bigcup_j P_j}(x)
\]
\[
+ \sum_j \| T(f_k \chi_{3Q_0 \setminus 3P_j})(x) \|_{L^4} \chi_{P_j}(x) + \sum_j \| T(f_k \chi_{3P_j})(x) \|_{L^4} \chi_{P_j}(x)
\]
\[
\leq 2C_2 \| \{f_k\} \|_{L(\log L)^{\beta}; 3Q_0} + \sum_j \| T_A(f_k \chi_{3P_j})(x) \|_{L^4} \chi_{P_j}(x).
\]
The inequality (2.6) now follows directly.

We can now conclude the proof of Theorem 2.3. As it was proved in [13], the last estimate shows that there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$, such that for $a.$ e. $x \in Q_0$,
\[
\| T(f_k \chi_{3Q_0})(x) \|_{L^4} \chi_{Q_0}(x) \leq \sum_{Q \in \mathcal{F}} \| \{f_k\} \|_{L(\log L)^{\beta}, 3Q} \chi_Q(x).
\]
Recalling that $\{f_k\}_{1 \leq k \leq N}$ are functions in $L^1(\mathbb{R}^n)$ with compact supports, we can take now a partition of $\mathbb{R}^n$ by cubes $Q_j$ such that $\bigcup_{j=1}^{N} \text{supp } f_k \subset 3Q_j$ for each $j$ and obtain a $\frac{1}{2}$-sparse family $\mathcal{F}_j \subset \mathcal{D}(Q_j)$ such that for $a.$ e. $x \in Q_j$,
\[
\| T(f_k \chi_{3Q_j})(x) \|_{L^4} \chi_{Q_j}(x) \leq \sum_{Q \in \mathcal{F}_j} \| \{f_k\} \|_{L(\log L)^{\beta}, 3Q} \chi_Q(x).
\]
Setting $\mathcal{S} = \{3Q : Q \in \bigcup_j \mathcal{F}_j\}$, we see that (2.3) holds for $\mathcal{S}$ and $a.$ e. $x \in \mathbb{R}^n$. □

3. PROOF OF THEOREM 1.1

To prove our theorem 1.1 we will employ some lemmas.

Lemma 3.1. Let $A$ be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some $q \in (n, \infty]$. Then
\[
|A(x) - A(y)| \lesssim |x - y| \left( \frac{1}{|I_x^y|} \int_{I_x^y} |\nabla A(y)|^q dy \right)^{\frac{1}{q}},
\]
where $I_x^y$ is the cube centered at $x$ and having side length $2|x - y|$.

For the proof of Lemma 3.1 see [6].

For a fixed $\beta \in (0, \infty)$, let $M_{L(\log L)^{\beta}}$ be the maximal operator defined by
\[
M_{L(\log L)^{\beta}} f(x) = \sup_{Q \ni x} \| f \|_{L(\log L)^{\beta}, Q},
\]
where the supremum is take over all cubes containing $x$. It is well known (see [23]) that for any $\lambda > 0$,

$$
(3.1) \quad |\{ x \in \mathbb{R}^n : M_{L(\log L)^\delta} f(x) > \lambda \}| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^\delta \left( 1 + \frac{|f(x)|}{\lambda} \right) dx.
$$

**Lemma 3.2.** Let $l \in \mathbb{N}$ and $q \in (1, \infty)$. Then the maximal operator $M_{L(\log L)^l}$ satisfies that

$$
(3.2) \quad \left| \left\{ x \in \mathbb{R}^n : \| M_{L(\log L)^l} f_k(x) \|_{L^q} > \lambda \} \right| \lesssim \int_{\mathbb{R}^n} \frac{\| f_k(x) \|_{L^q}}{\lambda} \log^l \left( 1 + \frac{\| f_k(x) \|_{L^q}}{\lambda} \right) dx.
$$

**Proof.** We only consider the case $l = 1$. The case $l \geq 2$ can be proved in the same way. As it was pointed out in [3] (see also [22]) that

$$
(3.3) \quad M_{L(\log L)^1} f(x) \approx M^2 f(x),
$$

with $M^2$ the operator $M$ iterated twice. Thus, it suffices to show the operator $M^1$ satisfies (3.2). On the other hand, by the well known one-third trick (see [15, Lemma 2.5]), we only need to prove that, for each dyadic grid $\mathcal{D}$, the inequality

$$
(3.4) \quad \left| \left\{ x \in \mathbb{R}^n : \| M_\mathcal{D} (M_\mathcal{D} f_k)(x) \|_{L^q} > 1 \right\} \right| \lesssim \int_{\mathbb{R}^n} \| f_k(x) \|_{L^q} \log (1 + \| f_k(x) \|_{L^q}) dx.
$$

holds when $\{ f_k \}$ is finite. As in the proof of Lemma 8.1 in [7], we can very that for each cube $Q \in \mathcal{D}$, $\delta \in (0, \frac{1}{2})$ and $\lambda \in (0, 1),

$$
(3.5) \quad \inf_{c \in \mathbb{C}} \left( \frac{1}{|Q|} \int_Q \left\| \{ M_\mathcal{D} f_k(y) \} \right\|_{L^q} - c \right)^d dy \lesssim \left( \frac{1}{|Q|} \int_Q \left\| \{ M_\mathcal{D} (f_k \chi_Q)(y) \} \right\|_{L^q}^d dy \right)^\frac{1}{d} \lesssim \langle \| f_k \chi_Q \|_{L^q} \rangle_Q,
$$

where in the last inequality, we invoked the fact that $M_\mathcal{D}$ is bounded from $L^1(I^q, \mathbb{R}^n)$ to $L^{1, \infty}(I^q, \mathbb{R}^n)$. This, in turn, implies that

$$
(3.6) \quad M_{\mathcal{D}, \delta}^d (\| \{ M_\mathcal{D} f_k \} \|_{L^q})(x) \lesssim M_{\mathcal{D}} (\| \{ f_k \} \|_{L^q})(x).
$$

Again by the argument used in the proof of Lemma 8.1 in [7], we can verify that for each cube $Q \in \mathcal{D}$,

$$
\inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q \left\| \{ M_\mathcal{D} f_k(y) \} \right\|_{L^q} - c \left| dy \right| \lesssim \frac{1}{|Q|} \int_Q \left\| \{ M(f_k \chi_Q) \} \right\|_{L^q} \left| dy \right|.
$$

Therefore,

$$
(3.7) \quad M_{\mathcal{D}}^d (\| \{ M_\mathcal{D} f_k \} \|_{L^q})(x) \lesssim \sup_{Q \ni x} \langle (\| M_\mathcal{D} (f_k \chi_Q) \|_{L^q})_Q \rangle.
$$

Now we claim that for each cube $Q,

$$
(3.8) \quad \langle \| M(f_k \chi_Q) \|_{L^q} \rangle_Q \lesssim \| \{ f_k \} \|_{L^q} \|_{L(\log L)^1, Q}.
$$
Let $\Phi(t) = t \log^{-1}(e + t^{-1})$. If we can prove (3.8), it then follows from (2.1), (3.6), (2.2), (3.7) and (3.8) that

$$\left| \left\{ x \in \mathbb{R}^n : \| M_{\mathcal{G}}(M_{\mathcal{D}} f_k)(x) \|_{\mathcal{I}_b} > 1 \right\} \right| \lesssim \sup_{\lambda > 0} \Phi(\lambda) \left| \left\{ x \in \mathbb{R}^n : M_{\mathcal{D}_{\mathcal{G}}}^L \left( \left\{ \| M_{\mathcal{D}} f_k \|_{\mathcal{I}} \right\} (x) > \lambda \right) \right\} \right| \lesssim \sup_{\lambda > 0} \Phi(\lambda) \left| \left\{ x \in \mathbb{R}^n : M_{\mathcal{D}_{\mathcal{G}}}^L \left( \left\{ \| M_{\mathcal{D}} f_k \|_{\mathcal{I}_b} \right\} (x) > \lambda \right) \right\} \right| \lesssim \sup_{\lambda > 0} \Phi(\lambda) \left| \left\{ x \in \mathbb{R}^n : M_{\mathcal{D}_{\mathcal{G}}}^L \log \left( \left\{ \| f_k \|_{\mathcal{I}_b} \right\} (x) \right) > \lambda \right) \right| \lesssim \int_{\mathbb{R}^n} \| f_k(x) \|_{\mathcal{I}_b} \log \left( 1 + \| f_k(x) \|_{\mathcal{I}_b} \right) dx,$$

which gives (3.4).

We now obtain that

$$\left| \left\{ x \in \mathbb{R}^n : \| M h_k(x) \|_{\mathcal{I}_b} > \lambda \right\} \right| \lesssim \frac{1}{\lambda^2} \int_{\{ \| h_k(y) \|_{\mathcal{I}_b} \leq \lambda \}} \| h_k(y) \|_{\mathcal{I}_b}^2 dy + \frac{1}{\lambda} \int_{\{ \| h_k(y) \|_{\mathcal{I}_b} > \lambda \}} \| h_k(y) \|_{\mathcal{I}_b} dy.
$$

We now obtain that

$$\int_Q \| M(f_k \chi_Q)(y) \|_{\mathcal{I}_b} dy = \int_{\{ y \in Q : \| M(f_k \chi_Q)(y) \|_{\mathcal{I}_b} \leq 1 \}} \| M(f_k \chi_Q)(y) \|_{\mathcal{I}_b} dy + \int_{\{ y \in Q : \| M(f_k \chi_Q)(y) \|_{\mathcal{I}_b} > 1 \}} \| M(f_k \chi_Q)(y) \|_{\mathcal{I}_b} dy \lesssim |Q| + \int_1^\infty \int_{\{ y \in Q : \| f_k(x) \|_{\mathcal{I}_b} \leq \lambda \}} \| f_k(x) \|_{\mathcal{I}_b} dy dx \frac{d\lambda}{\lambda^2} + \int_1^\infty \int_{\{ y \in Q : \| f_k(x) \|_{\mathcal{I}_b} > \lambda \}} \| f_k(x) \|_{\mathcal{I}_b} dy dx \frac{1}{\lambda} d\lambda \lesssim |Q|.$$

This establishes (3.8) and completes the proof of Lemma 3.2.

Let $\Omega$ be homogeneous of degree zero. For each $j$ with $1 \leq j \leq n$, define the operator $T_j$ as

$$T_j f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)(x_j - y_j)}{|x-y|^{n+1}} f(y) dy. \quad (3.9)$$
Lemma 3.3. Let \( q \in (1, \infty) \). \( T_A \) be the operator defined by (1.7). Under the hypothesis of Theorem 1.1, for each \( \lambda > 0 \),

\[
\| x \in \mathbb{R}^n : \| \{ T_A f_k(x) \} \|_q > \lambda \| \leq \lambda \int_{\mathbb{R}^n} \frac{\| \{ f_k(x) \} \|_q}{\lambda} \log \left( 1 + \frac{\| f_k(x) \|_q}{\lambda} \right) \, dx.
\]

Proof. We will employ the argument from [H]. Applying the Calderón-Zygmund decomposition to \( \| f_k(x) \|_q \) at level \( \lambda \), we obtain a sequence of cubes \( \{ Q_j \}_j \) with disjoint interiors, such that

\[
\lambda < \langle \{ f_k \} \|_q \rangle_{Q_j} \leq 2^n \lambda,
\]

and \( \| f_k(x) \|_q \lesssim \lambda \) for a.e. \( x \in \mathbb{R}^n \setminus \bigcup_j Q_j \). Let

\[
g_k(x) = f_k(x) \chi_{\mathbb{R}^n \setminus \bigcup_j Q_j}(x) + \sum_j f_k(Q_j) \chi_{Q_j}(x),
\]

and

\[
b_k(x) = f_k(x) - g_k(x) = \sum_j (f_k(x) - \langle f_k \rangle_{Q_j}) \chi_{Q_j}(x) := \sum_j b_{k,j}(x).
\]

Let \( E_\lambda = \bigcup_n 4nQ_j \). By the fact that \( \| g_k \|_{L^\infty(\mathbb{R}^n)} \lesssim \lambda \) and the assumption (ii), we have that

\[
\left| \left\{ x \in \mathbb{R}^n : \| \{ T_A g_k(x) \} \|_q > \lambda/2 \right\} \right| \\
\lesssim |E_\lambda| + \left| \left\{ x \in \mathbb{R}^n \setminus E_\lambda : \| \{ T_A g_k(x) \} \|_q > \lambda/2 \right\} \right| \\
\lesssim \lambda^{-1} \| F_k \|_{L^1(\mathbb{R}^n \setminus E_\lambda)}.
\]

Thus, the proof of (3.10) can be reduced to showing that

\[
\left| \left\{ x \in \mathbb{R}^n \setminus E_\lambda : \| \{ T_A b_k(x) \} \|_q > \lambda/2 \right\} \right| \\
\lesssim \int_{\mathbb{R}^n} \frac{\| \{ f_k(x) \} \|_q}{\lambda} \log \left( 1 + \frac{\| f_k(x) \|_q}{\lambda} \right) \, dx.
\]

We now prove (3.11). For each fixed \( j \), let

\[
A_j(y) = A(y) - \langle \nabla A \rangle_{Q_j} y.
\]

We can write

\[
T_A b_k(x) = \sum_j \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A_j(x) - A_j(y)) b_{k,j}(y) \, dy \\
+ \sum_{i=1}^n \int_{\mathbb{R}^n} \Omega(x-y) \frac{x_i - y_i}{|x-y|^{n+1}} \sum_j (\partial_i A(y) - \langle \partial_i A \rangle_{Q_j}) b_{k,j}(y) \, dy \\
:= \sum_j T_A^i b_{k,j}(y) + \sum_{i=1}^n T_A^i b_k(y).
\]

Invoking Minkowski’s inequality, we see that for each \( j \),

\[
\| \{ b_{k,j}(x) \} \|_q \leq \left( (\| b_k \|_q + \lambda) \chi_{Q_j} \right).
\]
By the vector-valued Calderón-Zygmund theory (see [1]), we see that for each fixed $1 \leq i \leq n$,

\begin{equation}
\left| \{ x \in \mathbb{R}^n : \| \{ T_A, \partial_y b(y) \} \|_{L^1} \geq \frac{\lambda}{\sqrt[4n]{4n}} \} \right|
\end{equation}

\begin{align*}
&\lesssim \lambda^{-1} \sum_j \int_Q |\nabla A(y) - \langle \partial_y A \rangle_Q| \| b_{k,j}(y) \|_{L^1} dy \\
&\lesssim \lambda^{-1} \sum_j |Q_j| |\nabla A(y) - \langle \partial_y A \rangle_Q|_{\exp, Q_j} \| b_{k,j} \|_{L^1} \| L_{\log, Q_j} \\
&\lesssim \int_{\mathbb{R}^n} \left\| f_k(x) \right\|_{L^1} \log \left( 1 + \frac{\left\| f_k(x) \right\|_{L^1}}{\lambda} \right) dx.
\end{align*}

where

\( \| h \|_{\exp, Q_j} = \inf \left\{ t > 0 : \frac{1}{|Q_j|} \int_{Q_j} \exp \left( \frac{|h(y)|}{t} \right) dy \leq 2 \right\}, \)

and the second inequality follows from the generalization of Hölder’s inequality (see [26, p.64]), and the last inequality follows from the fact that

\( \| h \|_{L_{\log, Q_j}} \leq \lambda + \frac{\lambda}{|Q_j|} \int_{Q_j} \frac{|h(y)|}{\lambda} \log \left( 1 + \frac{|h(y)|}{\lambda} \right) dy, \)

see [26, p. 69].

It remains to estimate \( T_A^i b_k \). For each fixed \( Q_j \), we choose \( x_j \in 3Q_j \setminus 2Q_j \). By vanishing moment of \( b_{j,k} \), we can write

\begin{align*}
|T_A^i b_{k,j}(x)| &\leq \frac{1}{|x - x_j|^{n+1}} \int_{\mathbb{R}^n} |A_j(x_j) - A_j(y)| |b_{j,k}(y)| dy \\
&+ \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^{n+1}} - \frac{\Omega(x-x_j)}{|x-x_j|^{n+1}} \right| |A_j(x) - A_j(y)| |b_{j,k}(y)| dy \\
&=: T_A^i b_{k,j}(x) + T_H^i b_{k,j}(x).
\end{align*}

By Lemma 3.1 we have that

\begin{align*}
\sum_j |T_A^i b_{k,j}(x)| &\lesssim \sum_j \frac{1}{|x - x_j|^{n+1}} |Q_j|^\frac{1}{4n} \| b_{k,j} \|_{L^1(\mathbb{R}^n)},
\end{align*}

which via Minkowski’s inequality implies that,

\begin{align*}
\left( \sum_k \left( \sum_j |T_A^i b_{k,j}(x)| \right)^{q} \right)^\frac{1}{q} &\lesssim \sum_j \frac{1}{|x - x_j|^{n+1}} |Q_j|^\frac{1}{4n} \left( \sum_k \| b_{k,j} \|_{L^1(\mathbb{R}^n)} \right)^\frac{1}{q} \\
&\lesssim \sum_j \frac{1}{|x - x_j|^{n+1}} |Q_j|^\frac{1}{4n} \| \{ b_{k,j} \} \|_{L^1(\mathbb{R}^n)}.
\end{align*}

Therefore,

\begin{equation}
\int_{\mathbb{R}^n \setminus E_\lambda} \left( \sum_k \left( \sum_j |T_A^i b_{k,j}(x)| \right)^{q} \right)^\frac{1}{q} dx \\
\lesssim \sum_j \int_{\mathbb{R}^n \setminus 4nQ_j} \frac{|Q_j|^\frac{1}{4n}}{|x - x_j|^{n+1}} dx \| \{ b_{k,j} \} \|_{L^1(\mathbb{R}^n)} \\
\lesssim \| \{ b_k \} \|_{L^1(\mathbb{R}^n)}.
\end{equation}
To estimate $|T_A^1 b_{k,j}(x)|$, we first observe that if $y \in Q_j$ and $x \in 2^{d+1}nQ_j \setminus 2^d nQ_j$ with $d \geq 2$, then by Lemma 3.1

$$|A_j(x) - A_j(y)| \lesssim |x - y| \left( \frac{1}{|I^y_j|} \int_{I^y_j} |\nabla A(y) - \langle \nabla A \rangle_{Q_j}| dy \right)^{\frac{1}{2}}$$

$$\lesssim |x - y| \left( \frac{1}{|I^y_j|} \int_{I^y_j} |\nabla A(z) - \langle \nabla A \rangle_{I^y_j}| dz \right)^{\frac{1}{2}}$$

$$+ |x - y| \|\langle \nabla A \rangle_{Q_j} - \langle \nabla A \rangle_{I^y_j} \right|$$

$$\lesssim d|x - y|.$$ 

This, via the continuity condition (1.8), implies that for each $δ < γ < 1$, 

$$\sum_{d=2}^{\infty} d \int_{2^{d+1}nQ_j \setminus 2^{d} nQ_j} \frac{\Omega(x - y)}{|x - y|^{n+1}} dy \lesssim \frac{\Omega(x - x_j)}{|x - x_j|^{n+1}}. $$

On the other hand, another application of Minkowski’s inequality gives us that

$$\left( \sum_k \left( \sum_j |T_A^1 b_{k,j}(x)| \right)^{q} \right)^{\frac{1}{q}} \lesssim \sum_j \left( \sum_k |T_A^1 b_{k,j}(x)| \right)^{\frac{1}{q}}$$

$$\lesssim \sum_j \int \frac{\Omega(x - y)}{|x - y|^{n+1}} \|A_j(x) - A_j(y)\| \|b_{k,j}(y)\| dy.$$ 

We thus deduce that

$$\left( \sum_k \left( \sum_j |T_A^1 b_{k,j}(x)| \right)^{q} \right)^{\frac{1}{q}} \lesssim \sum_j \|b_{k,j}\| \|L^1(n; \mathbb{R}^n) \| \|b_k\| \|L^1(n; \mathbb{R}^n).$$

Combining the estimates (3.13) and (3.14) yields

$$\left( \sum_k \left( \sum_j |T_A^1 b_{k,j}(x)| \right)^{q} \right)^{\frac{1}{q}} \lesssim \|b_k\| \|L^1(n; \mathbb{R}^n),$$

which, via the estimates (3.12), leads to (3.11) and then completes the proof of Lemma 3.3.

Now let $γ \in (0, 1]$. We know from Theorem 1 in [12] that,

$$T_A^1 f(x) \lesssim \gamma M_A f(x) + M_{L \log L} f(x).$$

For fixed $0 < δ < γ < 1$, dyadic grid $D$ and cube $Q \in D$. Again as in [7],

$$\inf_{c \in C} \left( \frac{1}{|Q|} \int_Q \|\{M_D, γf_k(y)\}\|_{l^q} - c \delta dy \right)^{\frac{1}{q}}$$

$$\lesssim \left( \frac{1}{|Q|} \int_Q \|\{M_D, γf_k(y)\}\|_{l^q} \right)^{\frac{1}{q}}$$

$$\lesssim \left( \frac{1}{|Q|} \int_Q \|\{f_k(y)\}\|_{l^q} \right)^{\frac{1}{q}}.$$
Our desired conclusion about \( T_A \) is bounded from \( L^1(\mathbb{R}^d; \mathbb{R}^n) \) to \( L^{1, \infty}(\mathbb{R}^d; \mathbb{R}^n) \). Recall that by (3.3) \( T_A \) is bounded from \( L^1(\mathbb{R}^d; \mathbb{R}^n) \) to \( L^{1, \infty}(\mathbb{R}^d; \mathbb{R}^n) \). By (2.1) and the argument used in the proof of Lemma 3.2, we get that

\[
\left| \left\{ x \in \mathbb{R}^n : \| \{ M_{\mathcal{S}, \gamma}(T_A f_k)(x) \} \|_{t_0} > 1 \right\} \right| \\
\leq \sup_{\lambda > 0} \Phi(\lambda) \left| \left\{ x \in \mathbb{R}^n : M_{\mathcal{S}, \gamma}^\mathcal{S}(\{ M_{\mathcal{S}, \gamma}(T_A f_k) \} \|_{t_0})(x) > \lambda \right\} \right| \\
\leq \sup_{\lambda > 0} \Phi(\lambda) \left| \left\{ x \in \mathbb{R}^n : \| T_A f_k \|_{t_0}(x) > \lambda \right\} \right| \\
\leq \sup_{\lambda > 0} \Phi(\lambda) \lambda^{-1} \sup_{s \geq 2^{d + \lambda}} \left| \left\{ x \in \mathbb{R}^n : \| T_A f_k(x) \|_{t_0} > s \right\} \right| \\
\leq \int_{\mathbb{R}^n} \| \{ f_k(x) \} \|_{t_0} \log \left( 1 + \| f_k(x) \|_{t_0} \right) dx,
\]

where the second-to-last inequality follows from the inequality (11) in [12], and the last inequality follows from Lemma 3.3. This, together with the one-third trick and Lemma 3.2 leads to that

\[
\left| \left\{ x \in \mathbb{R}^n : \| T_A f_k(x) \|_{t_0} > \lambda \right\} \right| \\
\lesssim \int_{\mathbb{R}^n} \| \{ f_k(y) \} \|_{t_0} \log \left( 1 + \| f_k(y) \|_{t_0} \right) dy.
\]

**Proof of Theorem 1.1** Let \( q \in (1, \infty) \). Recall that \( T_A \) is bounded on \( L^q(\mathbb{R}^n) \). If we can prove that for all \( x \in \mathbb{R}^n \),

\[
\mathcal{M} T_A f(x) \leq C M_{L, 1} f(x) + T_A^* f(x),
\]

then by Lemma 3.2 and (3.15),

\[
\left| \left\{ x \in \mathbb{R}^n : \| M_{T_A f}(x) \|_{t_0} > 2\lambda \right\} \right| \lesssim \int_{\mathbb{R}^n} \| \{ f_k(y) \} \|_{t_0} \log \left( 1 + \| f_k(y) \|_{t_0} \right) dy.
\]

This, via Theorem 2.3 implies that for bounded functions \( f_1, \ldots, f_N \) with compact supports, there exists a sparse family \( \mathcal{S} \), such that for a. e. \( x \in \mathbb{R}^n \),

\[
\| M_{T_A f}(x) \|_{t_0} \lesssim \mathcal{A}_{\mathcal{S}, 1} \log L(\| f_k \|_{t_0})(x).
\]

Our desired conclusion about \( T_A \) then follows from Lemma 2.1 directly.

We now prove (3.16). Let \( Q \subset \mathbb{R}^n \) be a cube and \( x, \xi \in Q \). Denote by \( B_x \) the ball centered at \( x \) and having diameter \( 20 \text{diam } Q \). As in [18], we can write

\[
|T_A(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \leq |T_A(f\chi_{\mathbb{R}^n \setminus B_x})(\xi)| \leq |T_A(f\chi_{\mathbb{R}^n \setminus B_x})(\xi)|
\]

It is obvious that

\[
|T_A(f\chi_{\mathbb{R}^n \setminus B_x})(x)| \leq T_A^* f(x).
\]

Let \( A_{B_x}(y) = A(y) - \langle \nabla A \rangle_{B_x} y \). We have that

\[
A(\xi) - A(y) - \nabla A(y)(\xi - y) = A_{B_x}(\xi) - A_{B_x}(y) - \nabla A_{B_x}(y)(\xi - y).
\]
A trivial computation then leads to that
\[ |T_A(f \chi_{B_x \setminus 3Q})(\xi)| \lesssim \int_{B_x \setminus 3Q} \frac{|A(\xi) - A(y) - \nabla A(y)(\xi - y)|}{|\xi - y|^{n+1}} |f(y)| dy \]
\[ \lesssim \frac{1}{|B_x|^{1+\frac{3}{n}}} \int_{B_x \setminus 3Q} |A_{B_x}(\xi) - A_{B_x}(y)| |f(y)| dy \]
\[ + \frac{1}{|B_x|} \int_{B_x} |\nabla A(y) - m_{B_x}(\nabla A)| |f(y)| dy = I(\xi) + \Pi(\xi). \]

Note that for \( y \in B_x \setminus 3Q \) and \( \xi \in Q \), \( I_\xi^y \subset B_x \subset 4nI_\xi \). An application of Lemma 3.1 shows that
\[ |A_{B_x}(\xi) - A_{B_x}(y)| \lesssim |B_x|^{\frac{3}{n}}, \]
and so
\[ I(\xi) \lesssim \frac{1}{|B_x|} \int_{B_x} |f(y)| dy \lesssim Mf(x). \]

On the other hand, by the generalization of Hölder’s inequality and the John-Nirenberg inequality, we deduce that
\[ \Pi(\xi) \lesssim \|f\|_{L_{\log, B_x}} \lesssim M_{L_{\log}} L f(x). \]

Therefore,
\[ (3.19) \quad |T_A(f \chi_{B_x \setminus 3Q})(\xi)| \lesssim M_{L_{\log}} L f(x). \]

To estimate \( |T_A(f \chi_{R^d \setminus B_x})(x) - T_A(f \chi_{R^d \setminus B_x})(\xi)| \), we employ the ideas used in [10, 13]. Write
\[ \left| \frac{\Omega(x - y)}{|x - y|^{n+1}} (A(x) - A(y) - \nabla A(y)(x - y)) \right| \]
\[ - \frac{\Omega(\xi - y)}{|\xi - y|^{n+1}} (A(\xi) - A(y) - \nabla A(y)(\xi - y)) \]
\[ \lesssim \left| \frac{\Omega(x - y)}{|x - y|^{n+1}} - \frac{\Omega(\xi - y)}{|\xi - y|^{n+1}} \right| |A_{B_x}(\xi) - A_{B_x}(y) - \nabla A_{B_x}(y)(\xi - y)| \]
\[ + \frac{\Omega(x - y)}{|x - y|^{n+1}} |A_{B_x}(x) - A_{B_x}(\xi) - \nabla A_{B_x}(y)(x - \xi)| \]
\[ := G(x, \xi) + H(x, \xi). \]

Another application of Lemma 3.1 gives us that for \( q \in (n, \infty) \),
\[ |A_{B_x}(x) - A_{B_x}(\xi)| \lesssim |x - \xi| \left( \frac{1}{|I_\xi^y|} \int_{I_\xi^y} |\nabla A(z) - (\nabla A)_{B_x}| \, dz \right)^{1/q} \]
\[ \lesssim |x - \xi| \left( 1 + \left| (\nabla A)_{B_x} - (\nabla A)_{I_\xi} \right| \right) \]
\[ \lesssim |x - \xi| \left( 1 + \log \frac{\ell(Q)}{|x - \xi|} \right). \]
A trivial computation leads to that if $\xi \in Q\setminus \{x\}$, then
\[
\int_{\mathbb{R}^n \setminus B_x} H(x, \xi)|f(y)|dy \lesssim |x - \xi| \left( 1 + \log \left( \frac{\ell(Q)}{|x - \xi|} \right) \right) \int_{\mathbb{R}^n \setminus B_x} \frac{|f(y)|}{|x - y|^{n+1}} dy \\
+ |x - \xi| \int_{\mathbb{R}^n \setminus B_x} \frac{|\nabla A(y) - \langle \nabla A \rangle_{B_x}|}{|x - y|^{n+1}} |f(y)|dy \\
\lesssim |x - \xi| \left( 1 + \log \left( \frac{\ell(Q)}{|x - \xi|} \right) \right) Mf(x) \\
+ + M_{L \log Lf(x)} \lesssim M_{L \log Lf(x)}.
\]

For each $y \in 2^k B_x \setminus 2^{k-1} B_x$ with $k \in \mathbb{Z}$, we have by Lemma 4.1 that
\[
|A_{B_x}(\xi) - A_{B_x}(y) - \nabla A_{B_x}(y)(\xi - y)| \lesssim (k + |\nabla A(y) - \langle \nabla A \rangle_{B_x}|).
\]

This, in turn leads to that
\[
\int_{\mathbb{R}^n \setminus B_x} G(x, \xi)|f(y)|dy \lesssim \sum_{k=1}^{\infty} \int_{2^k B_x \setminus 2^{k-1} B_x} \frac{\Omega(x - y)}{|x - y|^{n+1}} - \frac{\Omega(\xi - y)}{|\xi - y|^{n+1}} \\
\times (k + |\nabla A(y) - \langle \nabla A \rangle_{B_x}|) |f(y)|dy \\
\lesssim M_{L \log Lf(x)}.
\]

Therefore, for each $\xi \in Q$,
\begin{equation}
T_A(f_{\chi_{\mathbb{R}^n \setminus B_x}}(x) - T_A(f_{\chi_{\mathbb{R}^n \setminus B_x}})(\xi)) \lesssim M_{L \log Lf(x)}.
\end{equation}

Combining the estimates (3.18)-(3.20) leads to that
\[
\text{ess sup}_{\xi \in Q} |T_A(f_{\chi_{\mathbb{R}^n \setminus 3Q}})(\xi)| \leq CM_{L \log Lf(x)} + T_A^*f(x).
\]

We turn our attention to $M_{T_A^*}$. Again, it suffices to verify that for bounded functions $f_1, \ldots, f_N$ with compact supports, there exists a sparse family $\mathcal{S}$, such that for a.e. $x \in \mathbb{R}^n$,
\begin{equation}
\|\{T_A^*f_k(x)\}\|_{\mathcal{S}} \lesssim A_{S, L \log L}(\|f_k\|_W)(x),
\end{equation}

which, by Theorem 2.3, can be reduced to proving that
\begin{equation}
M_{T_A^*}f(x) \leq CM_{L \log Lf(x)} + T_A^*f(x).
\end{equation}

Let $Q \subset \mathbb{R}^n$ be a cube and $x, \xi \in Q$. Write
\[
|T_A^*(f_{\chi_{\mathbb{R}^n \setminus 3Q}})(\xi)| \leq |T_A^*(f_{\chi_{\mathbb{R}^n \setminus B_x}})(x) - T_A^*(f_{\chi_{\mathbb{R}^n \setminus B_x}})(\xi)| \\
+ |T_A^*(f_{\chi_{\mathbb{R}^n \setminus B_x}})(x) + |T_A^*(f_{\chi_{B_x \setminus 3Q}})(\xi)| \\
\lesssim \sup_{\epsilon > 0} |T_A, \epsilon(f_{\chi_{\mathbb{R}^n \setminus B_x}})(x) - T_A, \epsilon(f_{\chi_{\mathbb{R}^n \setminus B_x}})(\xi)| \\
+ T_A^*f(x) + M_{L \log Lf(x)}.
\]
A straightforward computation leads to that for each \( \epsilon > 0 \),
\[
|T_{A, \epsilon}(f \chi_{\mathbb{R}^n \setminus B_\epsilon})(x) - T_{A, \epsilon}(f \chi_{\mathbb{R}^n \setminus B_\epsilon})(\xi)| \\
\lesssim \int_{|x-y| > \epsilon, |\xi-y| < \epsilon} \frac{|\Omega(x-y)|}{|x-y|^{n+1}} |A(x) - A(y) - \nabla A(y)(x-y)||f\chi_{\mathbb{R}^n \setminus B_\epsilon}(y)|dy \\
+ \int_{|x-y| \leq \epsilon, |\xi-y| > \epsilon} \frac{|\Omega(\xi-y)|}{|\xi-y|^{n+1}} |A(\xi) - A(y) - \nabla A(y)(\xi-y)||f\chi_{\mathbb{R}^n \setminus B_\epsilon}(y)|dy \\
+ \int_{\mathbb{R}^n \setminus B_\epsilon} \frac{\Omega(x-z)}{|x-z|^{n+1}} A(x) - A(y) - \nabla A(y)(x-y)\|f(y)\|dy \\
= E_1 + E_2 + E_3.
\]
As in the proof of (3.20), we know that
\[
E_3 \lesssim M_L \log Lf(x).
\]
On the other hand, as in (3.19), we deduce that
\[
E_1 \lesssim \int_{\epsilon < |x-y| \leq 2\epsilon} \frac{|\Omega(x-y)|}{|x-y|^{n+1}} |A(x) - A(y) - \nabla A(y)(x-y)||f(y)||dy \\
\lesssim M_L \log Lf(x),
\]
and
\[
E_2 \lesssim \int_{\epsilon < |\xi-y| \leq 2\epsilon} \frac{|\Omega(\xi-y)|}{|\xi-y|^{n+1}} |A(\xi) - A(y) - \nabla A(y)(\xi-y)||f(y)||dy \\
\lesssim M_L \log Lf(x).
\]
(3.22) now follows from the estimates for \( E_1, E_2 \) and \( E_3 \). This completes the proof of Theorem 1.1 \( \square \)

**Example 3.4.** Let us consider the operator
\[
T_A f(x) = \int_{\mathbb{R}} \frac{A(x) - A(y) - A'(y)(x-y)}{(x-y)^2} f(y)dy.
\]
For \( A \) on \( \mathbb{R} \) such that \( A' \in \text{BMO}(\mathbb{R}) \), \( T_A \) is bounded on \( L^p(\mathbb{R}, w) \) for \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}) \). Now let \( \delta \in (0, 1/2) \) and
\[
f(x) = x^{-1+\delta} \chi_{(0,1)}(x), \quad w(x) = |x|^{(p-1)/(1-\delta)}.
\]
It is well known that \( |w|_{A_p} \approx \delta^{-p+1} \) (see [2] [5]). Also, \( \|f\|_{L^p(\mathbb{R}^n, w)} = \delta^{-1} \). Let \( A(y) = y \log(|y|) \). We know that \( A'(y) = 1 + \log |y| \in \text{BMO}(\mathbb{R}) \). A straightforward computation leads to that for \( x \in (0, 1) \),
\[
T_A f(x) = \int_0^1 x \frac{\log x - y \log y - (1 + \log y)(x-y)}{(x-y)^2} y^{-1+\delta} dy \\
= x \int_0^1 \frac{\log x - y \log y}{(x-y)^2} y^{-1+\delta} dy - \int_0^1 \frac{1}{x-y} y^{-1+\delta} dy \\
= x^{-1+\delta} \int_0^1 \frac{\log y}{(1-t)^2} t^{-1+\delta} dt - x^{-1+\delta} \int_0^1 \frac{1}{1-t} t^{-1+\delta} dt.
\]
Recall that for $t \in (0, 1) \cup (1, \infty)$, $\log \frac{1}{t} \geq 1 - t$. Therefore, for $x \in (0, 1)$,

$$|T_A f(x)| \geq x^{-1+\delta} \int_0^1 \frac{\log \frac{1}{x} - 1 + t}{(1-t)^2} t^{-1+\delta} dt \geq x^{-1+\delta} \int_0^1 \left( \frac{\log \frac{1}{x} - 1}{1-t} \right) t^{-1+\delta} dt \geq x^{-1+\delta} \int_0^1 \left( \frac{1}{t} - 1 \right) t^{-1+\delta} dt \geq (\delta^{-2} - \delta^{-1}) x^{-1+\delta} \geq \frac{1}{2} \delta^{-2} x^{-1+\delta}.
$$

Therefore,

$$\|T_A f\|_{L^p(\mathbb{R}, w)} \geq \frac{1}{2} t^{-2} \|f\|_{L^p(\mathbb{R}, w)}.$$ 

This shows that the conclusion in Theorem 1.1 is sharp when $p \in (1, 2]$.

4. PROOF OF THEOREM 1.3

We begin with a lemma.

**Lemma 4.1.** Let $\beta \in [0, \infty)$, $S$ be a sparse family and $A_{S,L, (\log L)^\beta}$ be the associated sparse operator. Then for $p \in (1, \infty)$, $\epsilon \in (0, 1)$ and weight $u$,

$$\|A_{S,L, (\log L)^\beta} f\|_{L^p(\mathbb{R}, u)} \lesssim p^{1+\beta} \|f\|_{L^p(\mathbb{R}, M_{L,(\log L)^{p-1+\epsilon}})}.$$ 

**Proof.** Denote by $A_{S,L, (\log L)^\beta}^*$ the adjoint operator of $A_{S,L, (\log L)^\beta}$. Then for suitable functions $f$ and $g$, and any $s \in (1, \infty)$,

$$\left| \int_{\mathbb{R}^n} A_{S,L, (\log L)^\beta}^* f(x) h(x) dx \right| \leq \sum_{Q \in S} |Q| |\langle f \rangle_Q| Q \|h\|_{L^s(\mathbb{R}, M_{L,(\log L)^{p-1}})}$$

$$\lesssim \frac{1}{(s-1)^\beta} |Q| |\langle f \rangle_Q| \left( \frac{1}{|Q|} \int_Q |g(y)|^s dy \right)^{1 \over s}.$$

Repeating the argument used in the proof of Theorem 1.7 in [20], we deduce that for $p \in (1, \infty)$, $\epsilon \in (0, 1)$ and weight $u$,

$$\|A_{S,L, (\log L)^\beta}^* f\|_{L^{p'}(\mathbb{R}^n, (M_{L,(\log L)^{p-1+\epsilon}})^{1-p'})} \lesssim_n p^{1+\beta} \|f\|_{L^{p'}(\mathbb{R}^n, u^{1-p'})}.$$ 

This, via a duality argument, shows that

$$\|A_{S,L, (\log L)^\beta} f\|_{L^p(\mathbb{R}^n, u)} \lesssim_n p^{1+\beta} \left( \frac{1}{\epsilon} \right)^{1 \over p} \|f\|_{L^p(\mathbb{R}^n, M_{L,(\log L)^{p-1+\epsilon}})}.$$ 

This completes the proof of Lemma 4.1. □

The following Theorem is an improvement of Lemma 4.1 in [19], and the proof here is of independent interest.

**Theorem 4.2.** Let $S$ be a sparse family and $\beta \in [0, \infty)$, $A_{S,L, (\log L)^\beta}$ be the associated sparse operator. Let $\epsilon \in (0, 1]$ and $u$ be a weight. Then for each $\lambda > 0$,

$$u\{x \in \mathbb{R}^n : A_{S,L, (\log L)^\beta} f(x) > \lambda \} \lesssim \frac{1}{e^{1+\beta}} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\beta} \left( e + \frac{|f(x)|}{\lambda} \right) M_{L,(\log L)^\beta} u(x) dx.$$
Proof. By the well known one-third trick, we may assume that $S \subset \mathcal{D}$ for some dyadic grid $\mathcal{D}$. Now let $M_{\mathcal{D}^q, L(\log L)^q}$ be the maximal operator defined by

$$M_{\mathcal{D}^q, L(\log L)^q}f(x) = \sup_{Q \subseteq \mathcal{D}^q} \|f\|_{L(\log L)^q, Q}.$$  

Decompose the set $\{x \in \mathbb{R}^n : M_{\mathcal{D}^q, L(\log L)^q}f(x) > 1\}$ as

$$\{x \in \mathbb{R}^n : M_{\mathcal{D}^q, L(\log L)^q}f(x) > 1\} = \bigcup Q_j,$$

with $Q_j$ the maximal cubes in $\mathcal{D}$ such that $\|f\|_{L(\log L)^q, Q_j} > 1$. Therefore,

$$1 < \frac{1}{|Q_j|} \int_{Q_j} |f(y)| \log^\beta \left(e + |f(y)|\right) \, dy \lesssim 2^n.$$

Let

$$f_1(y) = f(y)\chi_{\mathbb{R}^n \setminus \bigcup Q_j}(y); f_2(y) = \sum_j f(y)\chi_{Q_j}(y);$$

and

$$f_3(y) = \sum_j \|f\|_{L(\log L)^q, Q_j} \chi_{Q_j}(y).$$

It is obvious that $\|f_1\|_{L^{\infty}(\mathbb{R}^n)} \lesssim 1$. Thus, by Lemma 4.1,

$$\|u\|_{L^{\infty}(\mathbb{R}^n)} \lesssim 1$$

(4.1) \quad \begin{align*}
\|u\|_{L^{\infty}(\mathbb{R}^n)} & \lesssim \|A_S, L(\log L)^q f_1(x) > 1/2\| \lesssim \|A_S, L(\log L)^q f_1\|_{L^q(\mathbb{R}^n, u)} \\
& \lesssim q^{q(1+\beta)} \left(\frac{1}{\epsilon}\right)^{\frac{q}{q-1}} \int_{\mathbb{R}^n} |f_1(y)|^q M_{L(\log L)^q} f_1 \, dy \\
& \lesssim \frac{1}{\epsilon^{1+\beta}} \int_{\mathbb{R}^n} |f_1(y)| M_{L(\log L)^q} u(y) \, dy,
\end{align*}

if we choose $q = 1 + \epsilon/2$.

Now let $E = \bigcup_j \bigcup Q_j$, and $u_E(y) = u(y)\chi_{\mathbb{R}^n \setminus E}(y)$. We can verify that

$$\|u\|_{L^{\infty}(\mathbb{R}^n)} \lesssim \sum_j \inf_{z \in Q_j} M_{u(z)}|Q_j| \lesssim \int_{\mathbb{R}^n} |f(y)| \log^\beta \left(e + |f(y)|\right) M_{u(y)} \, dy$$

and for each $j$ and $\gamma \in [0, \infty)$

$$\sup_{y \in Q_j} M_{L(\log L)^\gamma} u_E(y) \approx \sup_{z \in Q_j} M_{L(\log L)^\gamma} u_E(z).$$

Note that $\|f_3\|_{L^{\infty}(\mathbb{R}^n)} \lesssim 1$ and

$$\|f_3\|_{L^1(\mathbb{R}^n, M_{L(\log L)^\gamma} u_E)} \lesssim \sum_j \inf_{z \in Q_j} M_{L(\log L)^\gamma} u_E(z) |Q_j| \|f\|_{L(\log L)^q, Q_j}$$

$$\lesssim \int_{\mathbb{R}^n} |f(y)| \log^\beta \left(e + |f(y)|\right) M_{L(\log L)^\gamma} u_E(y) \, dy.$$

If we can prove that for $x \in \mathbb{R}^n \setminus E$,

$$A_S, L(\log L)^q f_2(x) \lesssim A_S, L(\log L)^q f_3(x),$$

(4.3) [\text{Insert Equation Here}]
then by Lemma 4.1 and the inequality (4.3),

\( u \{ x \in \mathbb{R}^n \setminus E : \mathcal{A}_S, L(\log L)^\beta f_2(x) > 1 \} \)

\[ \lesssim \| \mathcal{A}_S, L(\log L)^\beta f_3 \|_{L^q(\mathbb{R}^n, \gamma)}^q \]

\[ \lesssim q^{q(1+\beta)} \left( \frac{1}{\epsilon} \right)^{\frac{\beta}{q'}} \| f_3 \|_{L^q(\mathbb{R}^n, M_{L(\log L)^{q-1+\epsilon/2}})}^q \]

\[ \lesssim q^{q(1+\beta)} \left( \frac{1}{\epsilon} \right)^{\frac{\beta}{q'}} \int_{\mathbb{R}^n} |f(y)| \log^\beta (e + |f(y)|) M_{L(\log L)^{q-1+\epsilon/2}} u_B(y) \, dy \]

\[ \lesssim \frac{1}{\epsilon^{1+\beta}} \int_{\mathbb{R}^n} |f(y)| \log^\beta (e + |f(y)|) M_{L(\log L)} u_B(y) \, dy, \]

again we choose \( q = 1 + \epsilon / 3 \). Our desired estimate for now follows from (4.1), (4.2) and (4.4) directly.

We now prove (4.3). For each fixed \( x \in \mathbb{R}^n \setminus E \) and each cube \( I \in \mathcal{D} \) containing \( x \), note that \( I \cap Q_j \neq \emptyset \) implies that \( Q_j \subset I \). Thus, for each \( \lambda > 0 \), a straightforward computation tells us that

\[ \int_I \frac{|f_3(y)|}{\lambda} \log^\beta \left( e + \frac{|f_3(y)|}{\lambda} \right) \, dy \]

\[ = \sum_{j : Q_j \subset I} \int_{Q_j} \frac{|f_3(y)|}{\lambda} \log^\beta \left( e + \frac{|f_3(y)|}{\lambda} \right) \, dy \]

\[ \lesssim \sum_{j : Q_j \subset I} \frac{\| f \|_{L(\log L)^\beta, Q_j}}{\lambda} \log^\beta \left( e + \frac{\| f \|_{L(\log L)^\beta, Q_j}}{\lambda} \right) \]

\[ \times \int_{Q_j} \frac{|f(y)|}{\| f \|_{L(\log L)^\beta, Q_j}} \log^\beta \left( e + \frac{|f(y)|}{\| f \|_{L(\log L)^\beta, Q_j}} \right) \, dy \]

\[ \lesssim \sum_{j : Q_j \subset I} \frac{|Q_j|}{\lambda} \frac{\| f \|_{L(\log L)^\beta, Q_j}}{\lambda} \log^\beta \left( e + \frac{\| f \|_{L(\log L)^\beta, Q_j}}{\lambda} \right), \]

since for \( t_1, t_2 \in [0, \infty) \),

\[ \log(e + t_1 t_2) \lesssim \log(e + t_1) \log(e + t_2). \]

On the other hand,

\[ \int_I \frac{|f_3(y)|}{\lambda} \log^\beta \left( e + \frac{|f_3(y)|}{\lambda} \right) \, dy \]

\[ = \sum_{j : Q_j \subset I} \int_{Q_j} \frac{|f_3(y)|}{\lambda} \log^\beta \left( e + \frac{|f_3(y)|}{\lambda} \right) \, dy \]

\[ = \sum_{j : Q_j \subset I} |Q_j| \frac{\| f \|_{L(\log L)^\beta, Q_j}}{\lambda} \log^\beta \left( e + \frac{\| f \|_{L(\log L)^\beta, Q_j}}{\lambda} \right). \]

Therefore, for each \( x \in \mathbb{R}^n \setminus E \) and \( I \in \mathcal{D} \) containing \( x \),

\[ \| f_2 \|_{L(\log L)^\beta, I} \lesssim \| f_3 \|_{L(\log L)^\beta, I}. \]

The inequality (4.3) follows directly. This completes the proof of Theorem 4.2. \( \square \)

**Proof of Theorem 4.3.** We only consider \( T_A \). The argument for \( T^*_A \) is similar. Applying the ideas used in [17, p.608] (see also the proof of Corollary 1.3 in [19],...
These estimates extend and improve the main results in [13] and [12].

\[ \lambda > \]

Moreover, for each fixed \( \lambda > 0 \),

This, along with the inequality (3.17), leads to our desired conclusion for \( T_A \).

\[ \square \]

**Remark 4.3.** Let \( \epsilon \in (0, 1] \) and \( u \) a weight. By Lemma 4.1, Theorem 4.2, the estimates (5.17) and (5.21), we know that for \( p \in (1, \infty) \),

\[ \| T_A f_k \|_{L^p(\mathbb{R}^n)} + \| T_A^* f_k \|_{L^p(\mathbb{R}^n)} \leq \lambda \]

Moreover, for each fixed \( \lambda > 0 \),

\[ \| \{ T_A f_k(x) \} \|_{L^1(\mathbb{R}^n, w)} \]

These estimates extend and improve the main results in [13] and [12].

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