A FINITARY HASSE PRINCIPLE FOR DIAGONAL CURVES

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ABSTRACT. We prove a Hasse principle for solving equations of the form \( ax + by + cz = 0 \) where \( x, y, z \) belong to a given finite index subgroup of \( \mathbb{Q}^* \). From this we deduce a Hasse principle for diagonal curves over subfields of \( \mathbb{Q} \) with finitely generated Galois group.

1. Introduction

Let \( a, b, c \) be non-zero rational numbers and \( n \geq 2 \) an integer. Let \( X \) denote the projective curve \( ax^n + by^n + cz^n = 0 \). For \( n = 2 \), the following are equivalent:

1. \( X(\mathbb{Q}_p) \neq \emptyset \) for all \( p \) and \( X(\mathbb{R}) \neq \emptyset \).
2. \( X(\mathbb{Q}) \neq \emptyset \).
3. \( X(\mathbb{Q}) \) is infinite.

The equivalence of (1) and (2) is the Hasse-Minkowski theorem for conics over \( \mathbb{Q} \), while the equivalence of (2) and (3) follows from stereographic projection. For \( n > 2 \), neither equivalence holds in general. Already for \( n = 3 \), the Tate-Shafarevich group gives an obstruction to (1)\( \Rightarrow \) (2); for instance, Selmer showed that \( 3x^3 + 4y^3 + 5z^3 = 0 \) has local solutions for all places of \( \mathbb{Q} \) but no global solution \[^7\] p. 8. For \( a = b = -c = 1 \), Fermat’s Last Theorem shows that (2) does not imply (3) for any \( n \geq 3 \).

We fix once and for all an algebraic closure \( \bar{\mathbb{Q}} \) of \( \mathbb{Q} \). We can view elements of \( X(\mathbb{Q}) \) as elements of \( X(\bar{\mathbb{Q}}) \) which are invariant under the action of \( G_\mathbb{Q} := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \). As \( G_\mathbb{Q} \) is not finitely generated, this can be regarded as an infinitary condition. It turns out that if we replace invariance under \( G_\mathbb{Q} \) by any finite collection of invariance conditions, the equivalence of conditions (1)–(3) as above holds for all \( n \) and all \( a, b, c \).

Let \( \Sigma \subset G_\mathbb{Q} \) be any finite subset. Let

\[ K_\Sigma := \{ x \in \bar{\mathbb{Q}} \mid \sigma(x) = x \ \forall \sigma \in \Sigma \} \]

denote the field of invariants of the closed subgroup \( \langle \Sigma \rangle \) generated by \( \Sigma \). A subfield \( K \) of \( \bar{\mathbb{Q}} \) is of this form if and only its absolute Galois group \( G_K \) is (topologically) finitely generated. We prove the following theorem:

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Theorem 1. Given $a,b,c \in \mathbb{Q}^*$ and $n$ a positive integer, the following conditions on the projective curve $X : ax^n + by^n + cz^n = 0$ are equivalent:

1. $X(\mathbb{Q}_p) \neq \emptyset$ for all $p$ and $X(\mathbb{R}) \neq \emptyset$.
2. $X(K) \neq \emptyset$ for all $K \subset \bar{\mathbb{Q}}$ with $G_K$ finitely generated.
3. $|X(K)| = \infty$ for all $K \subset \bar{\mathbb{Q}}$ with $G_K$ finitely generated.

One can prove that (2) implies (3) in greater generality:

Theorem 2. If $K$ is a field in characteristic zero such that $G_K$ is finitely generated, then $X(K)$ non-empty implies $|X(K)| = \infty$.

The proof of Theorem 2 is purely combinatorial, following the strategy of [4].

The proof that (1) implies (2) is more difficult and depends on the following Hasse principle, unusual in that we need to consider finite combinations of local conditions:

Theorem 3. Let $G$ denote a finite index subgroup of $\mathbb{Q}^*$, and let $a,b,c \in \mathbb{Q}^*$ belong to $\mathbb{Q}^*$. For every set $S$ of places of $\mathbb{Q}$, we define $\mathbb{Q}_S := \prod_{v \in S} \mathbb{Q}_v$ and let $G_S$ denote the closure of $G$ in $\mathbb{Q}_S^*$. Then

\[(1) \quad ax + by + cz = 0\]

has a solution for $x,y,z \in G$ if and only if the same equation has a solution in $G_S$ for all finite $S$.

It is a striking fact that it does not suffice to check solvability in $G_S$ for singleton sets $S = \{v\}$—see Proposition 5 below. We remark also that solving (1) in $G$ is equivalent to solving it in any coset of $G$. Richard Rado [9] considered which systems of homogeneous linear equations have the property that for every finite partition of $\mathbb{N}$, the system can be solved with all variables belonging to a single part of the partition. In the case of a single equation (1), the system satisfies this property if and only if $a + b = 0$, $b + c = 0$, $c + a = 0$, or $a + b + c = 0$. In these special cases, therefore, Theorem 3 follows directly from Rado’s theorem. This corresponds to the fact that Theorem 2 can be deduced from Ramsey theory, while the general case of Theorem 1 requires the circle method.

2. The Circle Method and Multiplicative Functions on $\mathbb{Q}$

In this section, we apply the circle method to prove Theorem 3. We begin with some preliminary lemmas.

We fix a finite index subgroup $G \subset \mathbb{Q}^*$ and non-zero $a,b,c \in \mathbb{Q}$ such that $ax + by + cz = 0$ has a solution in $G_S$ for all finite sets $S$ of places of $\mathbb{Q}$. We can freely replace $a$, $b$, or $c$ by any element in its $G$-coset, and we are free to multiply all three of them by a common non-zero rational number.

Lemma 4. For all integers $D > 0$, there exist elements $x,y,z \in G$ and $w \in \mathbb{Q}^*$ such that $a' := wax$, $b' := wby$, $c' := wcz$ satisfy the following properties:

(a) $\min(a',b',c') < 0$, 

\[\text{Lemma 4. For all integers } D > 0, \text{ there exist elements } x,y,z \in G \text{ and } w \in \mathbb{Q}^* \text{ such that } a' := wax, b' := wby, c' := wcz \text{ satisfy the following properties:} \]

(a) $\min(a',b',c') < 0$,
Proof. The proof consists of a series of steps in which we replace \( a, b, \) and \( c \) by \( wax, wby, \) and \( wcz \) respectively, with the goal that at the end of the process, the resulting triple \( a, b, c \) satisfies properties (a)–(f).

Let \( \mathbb{P} \) denote the set of all prime numbers and \( \mathbb{P}_0 \) the set of prime divisors of \( D \). Let \( Q := \mathbb{Q}^\times / G \), and define

\[
\phi = (\phi_1, \phi_2): \mathbb{P} \setminus \mathbb{P}_0 \to Q \times (\mathbb{Z}/D\mathbb{Z})^\times,
\]

where \( \phi_1 \) denotes the restriction of the quotient map \( \mathbb{Q}^\times \to Q \) to \( \mathbb{P} \setminus \mathbb{P}_0 \) and \( \phi_2 \) denotes the restriction of \( \mathbb{Z} \to \mathbb{Z}/D\mathbb{Z} \) to \( \mathbb{P} \setminus \mathbb{P}_0 \). Let \( S \) be the union of all finite subsets of the form \( \mathbb{P} \cup \{\infty\} \setminus \phi^{-1}(Q') \) where \( Q' \) is a subgroup of \( Q \times (\mathbb{Z}/D\mathbb{Z})^\times \). Thus \( S \) is finite, and if \( p \not\in S \) and \( M \) is a given integer, then there exists a product \( m \) of primes \( > M \) such that \( p(pm) = 1 \).

By hypothesis, equation (1) has a solution \( (x_S, y_S, z_S) \) in \( G_S \). Let \( x_v \) denote the \( v \)-component of \( x_S \) for \( v \in S \) and likewise for \( y_v, z_v \). As \( ax_{\infty} + by_{\infty} + cz_{\infty} = 0 \), it follows that replacing \( a, b, c \) by \( ax, by, cz \), where \( x, y, z \) are sufficiently close to \( x_{\infty}, y_{\infty}, z_{\infty} \), the resulting triple satisfies properties (a) and (b).

Choose \( k \) to be a positive integer larger than

\[
\max_{p \in S} \max(v_p(x_p), v_p(y_p), v_p(z_p)) + v_p(D)
\]

and choose \( x, y, z \in G \) such that for all \( p \in S \setminus \{\infty\},
\[
v_p(x_p - x), v_p(y_p - y), v_p(z_p - z) > k,
\]

and \( ax, by, \) and \( cz \) are neither all positive nor all negative. Multiplying each of these by

\[
w := \prod_{p \in S} p^{-\min(v_p(ax), v_p(by), v_p(cz))},
\]

we obtain \( wax, wby, wcz \) which add to \( 0 \) (mod \( D \)) and to zero (mod \( p \)) for each \( p \in S \). Moreover, for each \( p \), all three belong to \( \mathbb{Z}_p^* \), and at least one of the three belongs to \( \mathbb{Z}_p^* \); as they sum to zero (mod \( p \)), at least two are units. Replacing \( a, b, c \) by \( wax, wby, wcz \), the resulting triple now satisfies properties (a)–(c), and at most one of \( v_p(a), v_p(b), v_p(c) \) is positive for \( p \in S \).

If \( a, b, \) or \( c \) fails to be \( p \)-integral for some \( p \not\in S \), by definition of \( S \), there exists \( m \in \mathbb{N} \) such that \( pm \in G, \, pm \equiv 1 \) (mod \( D \)), and all prime factors of \( m \) are as large as we may wish. In particular, we may assume that for each prime factor \( q \) of \( m \), \( q \neq p, q \not\in S \), and \( v_q(a) = v_q(b) = v_q(c) = 0 \). Multiplying by \( pm \) eliminates a factor of \( p \) from the denominator of the desired element, \( a, b, \) or \( c \), without changing the residue class (mod \( D \)) or the sign of the given element or introducing a common prime factor of any two elements of the set. Continuing this process as long as necessary, we can assume that

(b) \( \max(a', b', c') > 0 \),
(c) \( a' + b' + c' \equiv 0 \) (mod \( D \)),
(d) \( a', b' \) and \( c' \) are pairwise relatively prime,
(e) \( a', b', c' \in \mathbb{Z} \),
(f) \( a'b'c' \) is even.
the resulting elements satisfy (a)–(e). If $a$, $b$, and $c$ are all odd, then $D$ is odd as well, so $2^k \equiv 1 \pmod{D}$ for some positive integer $k$ divisible by $|Q|$; replacing $a$ by $2^k a$, we obtain a new triple $a, b, c$ satisfying properties (a)–(f).

\textbf{Lemma 5.} Let $D$ be a positive integer. Let $a, b, c$ be integers satisfying conditions (a)–(f). There exists a constant $\epsilon > 0$ and for every prime $p$ a constant $d_p > \max(1, 1 - 3/p)$ such that for every finite set $S$ of primes not dividing $D$, the number of solutions of (1) in $x, y, z \in (1 + D\mathbb{Z}) \cap [0, N]$ such that $xyz$ is not divisible by any prime in $S$ is at least

$$N^2 \epsilon \prod_{p \in S} d_p$$

for all $N$ sufficiently large.

\textit{Proof.} By conditions (a)–(c), the intersection of $ax + by + cz = 0$ with the cube $[0, N]^3$ is a non-trivial polygonal region which up to homothety is independent of $N$. The intersection of $ax + by + cz = 0$ with $(1 + D\mathbb{Z})^3$ is the translate of a 2-dimensional lattice. If $\Lambda$ is a lattice and $R$ is a polygonal region, then

$$|\Lambda \cap (v + tR)| = \frac{\text{Area}(R)}{\text{Coarea}(\Lambda)} t^2 + O(t).$$

Thus, the number of solutions of (1) in $x, y, z \in (1 + D\mathbb{Z}) \cap [0, N]$ is of the form $AN^2 + O(N)$. By condition (d), for each $p \in S$, the conditions $p|\alpha$, $p|\gamma$, and $p|\beta$ each define a sublattice of $\Lambda$ of index $p$, so the subset $\Lambda_p$ of $\Lambda$ satisfying the condition $p \nmid xyz$ is the union of $p^2 \alpha_p$ cosets of $p\Lambda$, where $\alpha_p > 1 - 3/p$. By condition (f), $\alpha_2 > 0$ if $2 \in S$.

Thus, $\bigcap_{p \in S} \Lambda_p$ is the union of $\prod_{p \in S} p^2 \alpha_p$ cosets of $(\prod_{p \in S} p)\Lambda$. The lemma now follows from (2). \qed

Let $X, Y$, and $Z$ be finite sets of integers. The number of solutions of (1) with $x \in X$, $y \in Y$, and $z \in Z$ can be written

$$\left| \int_0^1 \sum_{x \in X} e(axt) \sum_{y \in Y} e(boy) \sum_{z \in Z} e(czt) \, dt \right|$$

where $e(t) := e^{2\pi it}$.

\textbf{Lemma 6.} If $|\alpha_x| = |\beta_y| = |\gamma_z| = 1$ for all $x, y, z$, then

$$\left| \int_0^1 \sum_{x \in X} \alpha_x e(axt) \sum_{y \in Y} \beta_y e(byt) \sum_{z \in Z} \gamma_z e(czt) \right| \leq \sup_t \left| \int_0^1 \sum_{x \in X} \alpha_x e(axt) \right| |Y|^{1/2} |Z|^{1/2}.$$
Proof. By Hölder and Cauchy-Schwartz,
\[
\left| \int_0^1 \sum_{x \in X} \alpha_x e(a(x) t) \sum_{y \in Y} \beta_y e(b(y) t) \sum_{z \in Z} \gamma_z e(c(z) t) \right|
\leq \left\| \sum_{x \in X} \alpha_x e(a(x) t) \right\|_\infty \left\| \sum_{y \in Y} \beta_y e(b(y) t) \right\|_2 \left\| \sum_{z \in Z} \gamma_z e(c(z) t) \right\|_2
= \sup_{t \in [0,1]} \left| \sum_{x \in X} \alpha_x e(a(x) t) \right| |Y|^{1/2} |Z|^{1/2}.
\]

\[\square\]

Corollary 7. If \(\delta > 0\), \(X' \subset X\) has at least \((1 - \delta)|X|\) elements, and \(|\alpha_x| = |\beta_x| = |\gamma_x| = 1\) for all \(x \in X\), then
\[
\left| \int_0^1 \sum_{x \in X} \alpha_x e(a(x) t) \sum_{y \in Y} \beta_y e(b(y) t) \sum_{z \in Z} \gamma_z e(c(z) t) dt \right|
- \int_0^1 \sum_{x \in X} \alpha_x e(a(x) t) \sum_{y \in X'} \beta_y e(b(y) t) \sum_{z \in Z} \gamma_z e(c(z) t) dt \right| \leq 3\delta|X|^2.
\]

Regarding the characters \(f \in \mathbb{Q}^\times\) as functions on \(\mathbb{Q}^\times\) and therefore on \(X\), we can write
\[(4) \sum_{x \in X \cap G} e(a(x)) = \frac{1}{|Q|} \sum_{f \in \mathbb{Q}^\times} \sum_{x \in X} f(x) e(a(x)),\]

and likewise for \(\sum_{y \in X \cap G} e(b(y))\) and \(\sum_{z \in X \cap G} e(c(z))\).

Every complex character \(\chi: \mathbb{Q}^\times \rightarrow U(1)\) defines a homomorphism \(\mathbb{Q}^\times \rightarrow U(1)\) and hence a strictly multiplicative function on \(\mathbb{N}\). For each such function \(f\) there is at most one pair \((\psi, t)\) consisting of a primitive Dirichlet character \(\psi\) and a real number \(t\) such that
\[(5) \sum_p \frac{1 - \text{Re}(f(p) \bar{\psi}(p)p^{-it})}{p} < \infty,\]

where the sum is taken over rational primes. Following terminology of Granville and Soundararajan [2], we will say that \(f\) is pretentious if such a pair exists.

If \(f\) takes values in a finite subgroup of \(U(1)\) (as in our case, where \(f\) arises from a homomorphism \(Q \rightarrow U(1)\)), and if \((\psi, t)\) satisfies [5], then \(t = 0\). By a theorem of Halász [10, III.4 Theorem 4], for any multiplicative function \(f\) which takes values in the unit disk,
\[(6) \sum_{n=1}^N f(n) = o(N)\]

unless \(f\) satisfies [11] for some \(t\) with \(\psi = 1\). In our setting, this means [6] holds unless \(f(p) = 1\) outside a set \(\mathbb{P}_f\) of primes with
\[
\sum_{p \in \mathbb{P}_f} \frac{1}{p} < \infty.
\]
We denote by $Q_{\text{pre}}^*$ the set of pretentious elements of $Q^*$. For each $f \in Q_{\text{pre}}^*$ there exists a unique primitive Dirichlet character $\psi$ such that $f$ satisfies (5) with $t = 0$. We define $\mathbb{P}_G$ to be the union of all the sets $\mathbb{P}_f \psi^{-1}$ where $f \in Q_{\text{pre}}^*$ and $\psi$ is the primitive character associated to $f$. Again, 

$$\sum_{p \in \mathbb{P}_G} \frac{1}{p} < \infty.$$ 

We define $D := D_G$ to be the least common multiple of the conductors of all characters $\psi$ associated with $f \in Q_{\text{pre}}^*$.

For $h: \mathbb{N} \to \mathbb{C}$, $\alpha \in \mathbb{R}$, and $n \in \mathbb{N}$, we define 

$$S_{h,n}(\alpha) := \sum_{x=1}^{n} e(\alpha x) h(x).$$ 

Lemma 8. Let $f: \mathbb{N} \to \mathbb{C}$ be the restriction of a homomorphism $Q^* \to U(1)$ with finite image, $g: \mathbb{Z} \to \mathbb{C}$ a periodic function, and $\alpha \in \mathbb{R}$. If $f$ is not pretentious, then 

$$S_{fg,n}(\alpha) = o(n).$$ 

Proof. We claim that for all $\epsilon > 0$, there exists $m$ such that for all $n$ and all fractions $\beta = r/s$ in lowest terms with $m < s < n/m$, we have 

(7) $$|S_{fg,n}(\beta)| \leq \epsilon n.$$ 

Indeed, if $g(x)$ is periodic with period $D$, it can be written as a linear combination of $e(\gamma x)$, $\gamma \in D^{-1} \mathbb{Z}$. The denominator of $\beta + \gamma$, written as a fraction in lowest terms, lies in $(m/D, Dn/m)$. By [8, Theorem 1], this implies (7) if $m/D$ is sufficiently large.

If $\beta = r/s$ with $s \leq m$, then $S_{fg,\beta}$ is a linear combination of sums of the form $S_{f,\beta+\gamma}$, where there are only finitely many possibilities for $\beta + \gamma \pmod{1}$. For each possibility, $e((\beta + \gamma)x)$ is periodic of some period $k$ and can therefore be written as a linear combination of (not necessarily primitive) $(\mod{k})$ Dirichlet characters. By (6), 

$$S_{f,1}(n) = o(n),$$ 

so for $n$ sufficiently large, we have 

(8) $$|S_{fg,n}(\beta)| \leq \frac{\epsilon n}{m}.$$ 

To deal with $\alpha \notin \mathbb{Q}$, we follow [8, §6]. For each $\alpha$, we choose the rational value $\beta = r/s$ with $s < n/m$ which is closest to $\alpha$. Thus, 

$$|\alpha - \beta| \leq \frac{m}{ns}.$$ 

Summing by parts, we have 

$$S_{fg,n}(\alpha) = \sum_{x=1}^{n} e((\alpha - \beta)x) e(\beta x) f(x) g(x)$$ 

$$= e((\alpha - \beta)n) S_{fg,n}(\beta) + \sum_{y=1}^{n-1} e((\alpha - \beta)y)(1 - e(\alpha - \beta)) S_{fg,y}(\beta).$$
If $s \geq m$, by (7),
\[|S_{fg,n}(\alpha)| \leq |S_{fg,n}(\beta)| + |\alpha - \beta| \sum_{1 \leq y \leq n/m} |S_{fg,y}(\beta)| + |\alpha - \beta| \sum_{n/m \leq y \leq n} |S_{fg,y}(\beta)|\]
\[\leq \epsilon n + \frac{1}{n} \left( \frac{n}{m} \right)^2 + \frac{n^2}{m} \epsilon \leq \left( \frac{1}{m} + 2 \epsilon \right)n.\]

If $s < m$, by (5),
\[|S_{fg,n}(\alpha)| \leq |S_{fg,n}(\beta)| + |\alpha - \beta| \sum_{1 \leq y \leq n/m} |S_{fg,y}(\beta)| + |\alpha - \beta| \sum_{n/m \leq y \leq n} |S_{fg,y}(\beta)|\]
\[\leq \epsilon n + \frac{m}{n} \left( \frac{n}{m} \right)^2 + \frac{m^2}{n} \epsilon \leq \left( \frac{1}{m} + 2 \epsilon \right)n.\]

Either way, sending $\epsilon \to 0$ and $m \to \infty$, we get the lemma. \(\square\)

We can now prove Theorem 3.

**Proof.** Applying Lemma 4 with $D = D_G$, we may assume $a, b, c$ satisfy conditions (a)–(f). Given $\delta > 0$, let $T(\delta)$ denote the smallest integer such that
\[\sum_{p \in \mathbb{P}_G \cap [T(\delta), \infty)} \frac{1}{p} < \delta.\]

Let $\mathcal{X}$ consist of all integers congruent to 1 (mod $D$) and not divisible by any prime $p \in \mathbb{P}_G \cap [2, T(\delta)]$. Let $\mathcal{X}'$ denote the set of elements of $\mathcal{X}$ divisible by no prime in $\mathbb{P}_G$. Let $X_N := \mathcal{X} \cap [1, N]$ and $X'_N := \mathcal{X}' \cap [1, N]$. By construction,
\[|(X_N \cap G) \setminus (X'_N \cap G)| \leq |X_N \setminus X'_N| < \delta N\]
for $N$ sufficiently large. Moreover,
\[f(x) = g(y) = h(z) = 1\]
for all $f, g, h \in Q^*_\text{pre}$ and $x, y, z \in X'_N$.

Let $\Sigma(X)$ denote the number of solutions of $ax + by + cz = 0$ with $x, y, z \in X$.

By (3) and (4), $\Sigma(X_N \cap G)$ is given by
\[(9)\]
\[|Q|^{-3} \sum_{f,g,h \in Q^*_\text{pre}} \int_{0}^{1} \left( \sum_{x \in X_N} f(x)e(axt) \right) \left( \sum_{y \in X_N} g(y)e(byt) \right) \left( \sum_{z \in X_N} h(z)e(czt) \right) dt.\]

By Lemma 5 and Lemma 8, if $f$ is not pretentious, the summand is $o(N^2)$. The same is true if $g$ or $h$ is not pretentious.

By construction, for $f, g, h \in Q^*_\text{pre}$, we have $f(x) = g(y) = h(z) = 1$ for all $x, y, z \in X'_N$, so by (3),
\[\Sigma(X'_N) = \int_{0}^{1} \left( \sum_{x \in X'_N} f(x)e(axt) \right) \left( \sum_{y \in X'_N} g(y)e(byt) \right) \left( \sum_{z \in X'_N} h(z)e(czt) \right) dt.\]
Applying Corollary \[4\] twice, we have
\[
\left| \int_0^1 \left( \sum_{x \in X_N} f(x) e(axt) \right) \left( \sum_{y \in X_N} g(y) e(byt) \right) \left( \sum_{z \in X_N} h(z) e(czt) \right) dt - \Sigma(X_N) \right|
\]
\[
\leq \left| \int_0^1 \left( \sum_{x \in X_N} f(x) e(axt) \right) \left( \sum_{y \in X_N} g(y) e(byt) \right) \left( \sum_{z \in X_N} h(z) e(czt) \right) dt - \Sigma(X_N') \right|
\]
\[
+ |\Sigma(X_N') - \Sigma(X_N)|
\]
\[
\leq 6\delta|X_N|^2.
\]
Combining this with \[9\], we obtain
\[
\left| \Sigma(X_N \cap G) - \frac{|Q_{p \in G}^*|^3}{|Q^3|} \Sigma(X_N) \right| = O(\delta N^2).
\]
Since \(\Sigma_{p \in G} p^{-1} < \infty\), Lemma \[5\] implies
\[
\limsup \frac{\Sigma(X_N)}{N^2} > 0.
\]
It follows that by choosing \(\delta\) sufficiently small, we can guarantee
\[
\limsup \frac{\Sigma(X_N \cap G)}{N^2} > 0.
\]
\[\square\]

We remark that the method of proof applies equally to the problem of solving the linear equation \(ax + by + cz = 0\) where \(x \in X, y \in Y,\) and \(z \in Z\), where \(X, Y,\) and \(Z\) are possibly distinct finite index subgroups of \(\mathbb{Q}^\times\).

We conclude this section with a proposition showing that the equation \(1\) with \(x, y, z \in G\) does not satisfy the naive Hasse principle.

**Proposition 9.** There exists a finite index subgroup \(G\) of \(\mathbb{Q}^\times\) and non-zero \(a, b, c \in \mathbb{Z}\) such that \(ax + by + cz = 0\) has no solution in \(G\) but does have a solution in the completion of \(G\) in \(\mathbb{Q}_v^\times\) for each place \(v\) of \(\mathbb{Q}\).

**Proof.** We define
\(G := \{3^n 5^m x \mid m, n \in \mathbb{Z}, m \equiv n \pmod{4}, x \in \mathbb{Q}^\times \cap \mathbb{Z}_3 \cap \mathbb{Z}_5, x \equiv 1 \pmod{15}\}\).

Thus \(G\) is of index \(4 \cdot \phi(15) = 32\) in \(\mathbb{Q}^\times\). It is dense in \(\mathbb{Q}_v^\times\) for \(v \notin \{3, 5\}\) and for \(v = p \in \{3, 5\}\) its closure in \(\mathbb{Q}_p^\times\) is
\[G_{(v)} = p^\mathbb{Z} \{ x \in \mathbb{Z}_p^\times \mid x \equiv 1 \pmod{p} \} \].

However, \(G_{\{3, 5\}}\) is not the product \(G_{\{3\}} \times G_{\{5\}}\); rather, it is
\[(x_3, x_5) \in G_{\{3\}} \times G_{\{5\}} \mid v_3(x_3) \equiv v_5(x_5) \pmod{4}\].

Now, the equation
\[63x + 30y + 25z = 0\]
has solutions in \(G_{\{3\}}\) (for instance \((-5, 3, 9)\)), but all such solutions satisfy
\[v_3(x) = v_3(y) - 1 = v_3(z) - 2.\]
It also has solutions in $G_{(5)}$ (for instance $\langle 25, -45, -9 \rangle$), but all such solutions satisfy
\[ v_5(x) = v_5(y) + 1 = v_5(z) + 2. \]
Therefore, there are no solutions in $G_{(3,5)}$ and, a fortiori, no solutions in $G$. \hfill \Box

### 3. Points on Diagonal Curves

This section gives a proof of Theorem 1. It is easy to see that $G_K$ finitely generated implies $K^*/(K^*)^n$ finite (see, e.g., [3]). We begin by proving Theorem 2.

**Proof.** Suppose $ax^n + by^n + cz^n = 0$ has a non-trivial solution $(\alpha, \beta, \gamma) \in K$. Replacing $a, b, c$ by $a' := ax^n, b' := b'y^n, c' := c'z^n$ respectively, it suffices to prove that the projective curve $X': a'x^n + b'y^n + c'z^n = 0$ has infinitely many points in $K$ such that $x \neq 0$, $y \neq 0$, and $z \neq 0$. Since there are only finitely many points of $X'$ for which any of the coordinates is zero, it suffices to prove $X'(K)$ is infinite. The advantage of $X'$ over $X$ is that $a' + b' + c' = 0$. Let $E \subset K$ be a number field containing $a', b', c'$. As $E$ is infinite, we can find pairwise distinct $p, q, r \in E^*$ such that $a'p + b'q + c'r = 0$ and an infinite sequence $h_1, h_2, \ldots \in E$ such that all finite linear combinations of the $h_i$ with coefficients in $\{p, q, r\}$ are distinct from one another. For each positive integer $k$, the map $f_k:\{p, q, r\}^k \to E$ defined by
\[ f_k(x_1, \ldots, x_k) = h_1x_1 + \cdots + h_kx_k \]
is injective and takes only non-zero values.

Let $H := (K^*)^n \cap E^*$. Let $m$ denote the index of $H$ in $E^*$, which is finite. For every positive integer $k$ the coset decomposition of $E^*$ induces via $f_k$ a partition of $\{p, q, r\}^n$ into $m$ subsets. By the Hales-Jewett theorem, if $k$ is sufficiently large, there exist $k$ functions $g_1, \ldots, g_k: \{1, 2, 3\} \to \{p, q, r\}$ such that for each $i$, either $g_i$ is constant or
\[ (g_i(1), g_i(2), g_i(3)) = (p, q, r), \]
and the three terms
\[ f_k(g_1(j), \ldots, g_k(j)), \quad j = 1, 2, 3, \]
lie in the same part of the partition. If $I \subset \{1, \ldots, k\}$ denotes the set of indices $i$ for which $g_i$ is constant, we set
\[ A = \sum_{i \in I} g_i(1)h_i, \quad B = \sum_{i \in \{1, \ldots, n\} \setminus I} h_i, \]
and then $A + Bp, A + Bq, A + Br$ all belong to the same part of the partition, i.e., to the same coset of $H$. If $C$ belongs to the inverse coset, then
\[ (C(A + Bp), C(A + Bq), C(A + Br)) \in (E^*)^n \times (E^*)^n \times (E^*)^n. \]
Thus,
\[ ((C(A + Bp))^{1/n}, (C(A + Bq))^{1/n}, (C(A + Br))^{1/n}) \]
lies on \(X'(E) \subset X'(K)\). 

Now we prove Theorem 1.

**Proof.** By Theorem 2 it suffices to prove that (1) \(\iff\) (2). For \(Q_v\) any completion of \(Q\) (i.e., \(\mathbb{R}\) or \(\mathbb{Q}_p\) for some \(p\)), we fix an algebraic closure of \(\mathbb{Q}_v\). The algebraic closure \(\mathbb{Q}_v^{\text{cl},v}\) of \(Q_v\) is (non-canonically) isomorphic to \(\mathbb{Q}\). Fixing an isomorphism \(i_v: \mathbb{Q} \to \mathbb{Q}_v^{\text{cl},v}\), the restriction map defines an injective homomorphism \(G_{Q_v} \to \text{Gal}(\mathbb{Q}_v^{\text{cl},v}/\mathbb{Q})\) and via \(i_v\) we obtain an injection \(j_v: G_{Q_v} \to G_Q\). As a topological group, \(G_{Q_v}\) is finitely generated; this is trivial if \(v\) is archimedean and well-known (see, e.g., [1, 5, 6, 11]) in the non-archimedean case. The invariant field \(K_v\) of \(\mathbb{Q}\) by \(j_v(G_{Q_v})\) is isomorphic via \(i_v\) to a subfield of \(\mathbb{Q}_v\), so (2) implies that \(X(K_v)\), and therefore \(X(Q_v)\), is non-empty.

For the implication (1) \(\implies\) (2), we define \(G = \mathbb{Q}^* \cap (K^*)^n\), so \(G\) is of finite index in \(\mathbb{Q}^*\). We apply Theorem 3 to \(G\). In particular, \(G \supset (\mathbb{Q}^*)^n\), so by weak approximation, for any finite set \(S\) of places \(v\), the closure \(G_S\) of \(G\) in \(\mathbb{Q}^*\) contains \(\prod_{v \in S}(\mathbb{Q}_v^*)^n\).

In particular, if \(X(Q_v)\) has a point \((x_v : y_v : z_v)\) for each \(v\), then \(au + bv + cw = 0\) has a solution in \(G_S\) for all \(S\) and therefore in \(\mathbb{Q}\) itself, namely \(u_v = x_v^n, v_v = y_v^n, w_v = z_v^n\). 

**Corollary 10.** If \(X\) is a diagonal curve, then \(X(K)\) is infinite for all \(K \subset \mathbb{Q}\) with \(G_K\) finitely generated if and only if \(X(\mathbb{A}_Q) \neq \emptyset\), where \(\mathbb{A}_Q\) denotes the ring of adeles.

**Proof.** The only additional point to check is that for any \(a, b, c \in \mathbb{Q}^*\), there exists a finite set \(S\) of places of \(Q\), including \(\infty\), such that \(X\) has a point over \(\mathbb{Z}_p\) for all \(p \nmid S\). If \(p\) is sufficiently large, \(a, b,\) and \(c\) are \(p\)-adic units, so \(X\) has good reduction (mod \(p\)), and the reduction is a curve of genus \(\frac{(n-1)(n-2)}{2}\). If \(p > (n-1)^2(n-2)^2\), the Weil bound implies that \(X\) has at least one points over \(\mathbb{F}_p\), and Hensel’s lemma implies that any such point lifts to a \(\mathbb{Z}_p\)-point.

**Question 11.** Is it always true that for \(X\) a non-singular curve over a number field \(E\), there exists an \(\mathbb{A}_E\)-point on \(X\) if and only if for all \(K \subset \mathbb{Q}\) with \(G_K\) finitely generated, \(X(K)\) is infinite?

The circle method offers the hope of giving an affirmative answer to this question for some non-diagonal curves. We hope to treat this matter in a subsequent paper.

**References**

[1] Diekert, Volker: Über die absolute Galoisgruppe dyadischer Zahlkörper. *J. Reine Angew. Math.* **350** (1984), 152–172.
[2] Granville, Andrew; Soundararajan, K.: Large character sums: pretentious characters and the Pólya-Vinogradov theorem. *J. Amer. Math. Soc.* **20** (2007), no. 2, 357–384.

[3] Im, Bo-Hae; Larsen, Michael: Generalizing a theorem of Richard Brauer. *J. Number Theory* **128** (2008), no. 12, 3031–3036.

[4] Im, Bo-Hae; Larsen, Michael: Some applications of the Hales-Jewett theorem to field arithmetic. *Israel J. Math.* **198** (2013), no. 1, 35–47.

[5] Jannsen, Uwe: Über Galoisgruppen lokaler Körper. *Invent. Math.* **70** (1982/83), no. 1, 53–69.

[6] Jannsen, Uwe; Wingberg, Kay: Die Struktur der absoluten Galoisgruppe \( p \)-adischer Zahlkörper. *Invent. Math.* **70** (1982/83), no. 1, 71–98.

[7] Knapp, Anthony W.: Elliptic curves. Mathematical Notes, 40. Princeton University Press, Princeton, NJ, 1992.

[8] Montgomery, H. L.; Vaughan, R. C.: Exponential sums with multiplicative coefficients. *Invent. Math.* **43** (1977), no. 1, 69–82.

[9] Rado, Richard: Studien zur Kombinatorik. *Math. Z.* **36** (1933), no. 1, 424–470.

[10] Tenenbaum, Gérard: Introduction to analytic and probabilistic number theory. Cambridge Studies in Advanced Mathematics, 46. Cambridge University Press, Cambridge, 1995.

[11] Wingberg, Kay: Der Eindeutigkeitssatz für Demuškinformationen. *Invent. Math.* **70** (1982/83), no. 1, 99–113.

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