Maximizing the Probability of Fixation in the Positional Voter Model

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Diffusion processes

Natural spread through networks

- propagation of information in social networks
- spread of virus in human population
- spread of mutation in biological networks
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Type of Diffusion process

- Progressive:
  - independent cascade,
  - linear threshold,
  - triggering,...

- Non-Progressive:
  - moran,
  - voter,
  - SIR,
  - SIS,...

This paper:
Non-Progressive model that describes the spread of mutation/novel-trait.
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Positional Voter Model - This Work

**Graph:** Population of \( n \) agents spread over nodes of graph \( G = (V, E, w) \).
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Agents:
- Mutants
- Residents

Nodes:
- Biased
- Unbiased

Example for $\delta=2$
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**Process:**
1. Initially residents occupy all nodes.
2. At \( t = 0 \): random mutation in one agent.
3. For \( t > 0 \): repeat death-Birth steps:
   - Death: Pick a random node \( v \) to update.
   - Birth: Pick an in-neighbor node \( u \) of \( v \) proportionally to its fitness \( f(u|v) \) and edge-weight \( w(u,v) \) to transfer its trait on \( v \).

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| death - Birth | $f(u|v)$ |
|--------------|---------|
| $v$ to $u$   | $1+\delta$ |
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Example for \( \delta = 2 \):
Fixation Probability

Setting Parameters: graph $G$, set of biased nodes $S$, bias $\delta$.

Fixation Probability: The probability $f_p(G^S, \delta)$ that a random mutation leads to fixation.
Positional vs. Standard Voter Model [Liggett 1975]

\[ S = V \implies \text{Positional} = \text{Standard} \]
Optimization Problem: Given a graph $G$ and a budget $k$, which $k$ nodes should we bias with $\delta$ to maximize the fixation probability?

$$S^* = \arg\max_{S, |S| = k} \text{fp}(G^S, \delta)$$
**Fixation Maximization**

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**Results Overview:**

1. **FPRAS:** for $fp(G^S, \delta)$ in undirected graphs.
2. Monotone and non-Submodular in general.
3. NP-hardness of $S^* = \arg \max_{S, |S|=k} fp(G^S, \delta)$.
4. Approximations for undirected graphs with self-loops and $\delta \to \infty$.
5. Optimal Solution in polynomial time for symmetric graphs (i.e. $w(u, v) = w(v, u)$) as $\delta \to 0$. 

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![Graph with fixation probability curves showing the effect of bias $\delta$ on fixation probability.](image)
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   $$S^* = \arg \max_{S, |S|=k} fp(G^S, \delta).$$
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\[\text{Fixation Probability} \]

\[\text{Bias } \delta \to 0 \quad \text{Bias } \delta \to \infty\]
The complexity of computing $\text{fp}(G^S, \delta)$ is OPEN even when $S = V$.

Lemma 1 - Expected Time

For undirected graphs, the expected time to a homogeneous state (all nodes are either mutants or residents) is $O(n^5)$.

Approximations of $\text{fp}(G^S, \delta)$ via monte-carlo simulations in P-time.
Lemma 2 - Monotonicity

Given biased sets $S_1, S_2$ with $S_1 \subseteq S_2$ and $\delta_1, \delta_2 \geq 0$ with $\delta_1 \leq \delta_2$, we have:

$$fp(G^{S_1}, \delta_1) \leq fp(G^{S_2}, \delta_2)$$

| death - Birth | $f(u|v)$ |
|----------------|---------|
| v              | u       | 1+\delta |
| v              | u       | 1        |
| v              | u       | 1        |

Lemma 3 - Non-Submodularity

- $fp(G^S, \delta)$ is not submodular.
- $fp^\infty(G^S)$ is not submodular in general.

Submodular function: $\forall S_1, S_2 \subseteq V \Rightarrow f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2)$
Key Lemma

Lemma 4 - Self-looped Graphs

In undirected graphs with self-loops, if \( \delta \rightarrow \infty \), mutant agents in biased nodes are deathless; reproduce to themselves with probability 1.

If trajectory \( X_t = (X_0, X_1, ..., X_t) \) hits \( S \), mutants fixate.

\[
\begin{array}{|c|c|}
\hline
\text{death - Birth} & f(u|v) \\
\hline
\rightarrow u & 1 + \delta \\
\rightarrow v & 1 \\
\rightarrow u & 1 \\
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\hline
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\]
Theorem 5 - NP-hard

Maximizing $fp(G^S, \delta)$ with $|S| = k$, is NP-hard.
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Maximizing $fp(G_S^S, \delta)$ with $|S| = k$, is NP-hard.

Proof.

Reduction from Vertex Cover in regular graphs, which is NP-hard. On undirected $d$-regular graphs with self-loops:

$$fp^\infty(G_S^S) = \frac{|S| + d}{n + d} \iff S \text{ is a vertex-cover}.$$

$$fp^\infty(G_S^S) = \frac{2}{4} + \frac{3}{1+3} = \frac{3.5}{4}$$

$$fp^\infty(G_S^S) < \frac{3.5}{4}$$
Lemma 6 - Submodularity

For undirected graphs with self-loops \( f_{p^\infty}^S(G_S) \) is submodular;
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For undirected graphs with self-loops $f^\infty(p)$ is submodular.

Proof.

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$X_t$ hits $S \implies$ mutants fixate

$fp^\infty(G^{S_1}) + fp^\infty(G^{S_2}) \geq fp^\infty(G^{S_1 \cup S_2}) + fp^\infty(G^{S_1 \cap S_2})$
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\( X_t \) hits \( S \) \( \implies \) mutants fixate

\[
\frac{fp^\infty (G^{S_1})}{fp^\infty (G^{S_2})} \geq \frac{fp^\infty (G^{S_1 \cup S_2})}{fp^\infty (G^{S_1 \cap S_2})}
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Lemma 6 - Submodularity

For undirected graphs with self-loops \( f_p^\infty(G^S) \) is submodular;

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\[ X_t \text{ hits } S \implies \text{mutants fixate} \]

\[
\frac{f_p^\infty(G^{S_1}) + f_p^\infty(G^{S_2})}{\geq} \frac{f_p^\infty(G^{S_1 \cup S_2}) + f_p^\infty(G^{S_1 \cap S_2})}{\ }
\]
Corollary - Approximations

In undirected graphs with self-loops, $f_{p}^{\infty}(G^{S})$ is:

- **Monotone**
- **Submodular**

$(1-1/e)$ greedy approximation algorithm [Nemhauser1978]
Theorem 7 - Optimal Solution

For symmetric graphs \((w(u, v) = w(v, u))\), when \(\delta \to 0\), finding \(S^* = \arg \max_{S, |S| = k} fp(G^S, \delta)\) can be solved in P-time.
Theorem 7 - Optimal Solution

For symmetric graphs \((w(u, v) = w(v, u))\), when \(\delta \to 0\), finding \(S^* = \arg \max_{S, |S| = k} \text{fp}(G^S, \delta)\) can be solved in P-time.

Proof.

Using the Taylor expansion of \(\text{fp}(G^S, \delta)\) around \(\delta = 0\), that is:

\[
\frac{\text{fp}(G^S, 0)}{1/n} + \delta \cdot \frac{\text{fp}'(G^S, 0)}{\text{Maximize this}} + O(\delta^2)
\]

By solving a linear system of \(n^2\) unknowns we can find the optimal \(S^* = \arg \max_{S, |S| = k} \text{fp}'(G^S, 0)\).
FPRAS: for $\text{fp}(G^S, \delta)$ in undirected graphs.

2 Monotone and not Submodular in general.

3 NP-hardness of $S^* = \arg \max_{S, |S|=k} \text{fp}(G^S, \delta)$.

4 Approximation Algorithm for undirected graphs with self-loops and $\delta \to \infty$:
   Monotone + Submodular $\to 1 - \frac{1}{e}$ greedy apx. algorithm.

5 Optimal Solution in polynomial time for symmetric graphs (i.e. $w(u, v) = w(v, u)$) as $\delta \to 0$.

Thank you!