ENERGY DECAY FOR THE DAMPED WAVE EQUATION ON AN UNBOUNDED NETWORK

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Abstract. We study the wave equation on an unbounded network of \( N, N \in \mathbb{N}^* \), finite strings and a semi-infinite one with a single vertex identified to 0. We consider continuity and dissipation conditions at the vertex and Dirichlet conditions at the extremities of the finite edges. The dissipation is given by a damping constant \( \alpha > 0 \) via the condition \( \sum_{j=0}^{N} \partial_x u_j(0,t) = \alpha \partial_t u_0(0,t) \). We give a complete spectral description and we use it to study the energy decay of the solution. We prove that for \( \alpha \neq N+1 \) we have an exponential decay of the energy and we give an explicit formula for the decay rate when the finite edges have the same length.

1. Introduction. In the study of wave propagation in physical multi-structures such as networks, graphs or trees, a lot of questions are asked and considered from different points of view. The approaches used depend on the structures properties (open sets, networks, compact, unbounded, etc...) and also on the conditions imposed in the mathematical formulation of the physical problem. One of the most interesting question is about the energy of a wave that propagates along a network of strings (bounded or unbounded). The knowledge of the behaviour of the local or global energy gives a direct response to the question of stabilization or the existence of an equilibrium state of the considered system. For a network with a finite number of edges, say \( N \geq 2 \), many developments on the previous question were obtained in different contexts. For compact (or bounded) networks, a deep investigation was done and a big amount of answers were given. In [1, 3, 4, 5, 22, 11, 15], the authors considered the wave equation \( \partial_t^2 u - c^2 \Delta u = 0 \) on bounded networks with a variety of boundary (external) and vertex (internal) conditions. Namely the external conditions are Dirichlet, Neumann or Robin conditions. At the internal vertices, the continuity is imposed and a damping or Kirchhoff condition is considered. A formulation of the problem in a simple context is the following: Given \( N \geq 2 \) positive numbers \( \ell_1, \ldots, \ell_N \) and a real constant \( \alpha \), we let \( \Gamma \) be a network of \( N \) edges \( e_1, \ldots, e_N \) with lengths \( \ell_1, \ldots, \ell_N \) respectively. We consider the case where all of

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the edges are connected at a single vertex $v$. We identify each edge $e_j$ to the interval $[0, \ell_j]$ and the common vertex to 0. The free endpoints of the network are the points $x = \ell_j, j = 1, \ldots, N$. The propagation of a wave in this structure with speed $c = 1$ is mathematically formulated by the system

$$\begin{cases}
\partial^2_{tt}u_j - \partial^2_{xx}u_j = 0, & \forall x \in [0, \ell_j], j = 1, \ldots, N \\
u_j(0, t) = u_0(0, t), & \forall 1 \leq i, j \leq N, t \geq 0 \\
B.C., \quad I.C. \\
\sum_{j=1}^{N} \partial_x u_j(0, t) = \alpha \partial_t u_1(0, t), \forall t \geq 0
\end{cases} \quad (1)$$

where B.C. are the conditions imposed at the points $x = \ell_j$ while I.C. are the initial data given at $t = 0$. The internal condition $\sum_{j=1}^{N} \partial_x u_j(0, t) = \alpha \partial_t u_1(0, t), \forall t \geq 0$ is a dissipation condition at $x = 0$ when $\alpha > 0$, a conservation condition when $\alpha = 0$ and an amplification one if $\alpha < 0$. The solution to (1) is a vector valued function $u(x, t) = (u_1(x, t), \ldots, u_N(x, t))$ lying in some functional space $X$. The most frequently asked question is about the comparison between the norm in $X$ of the solution at time $t$ and the norm of the given initial data. The energy is in some sense this considered norm. The stabilization of the system (1) and some more complicated configurations has been considered in a lot of papers [1, 3, 5, 22, 15], . . . . The decay of the energy and its rate have been established under some assumptions on edge lengths and on the damping amount term $\alpha$. Negative answer for the finite-time stabilization was given in [1] if $\alpha$ takes some particular values related to the number of edges where the damping is acting.

The aim of this paper is to investigate the situation where the network is unbounded. More precisely we consider the configuration where an infinite edge $e_0$, identified to the interval $e_0 := [0, +\infty]$ is attached to a compact network with $N$ finite edges. We put a modified Kirchhoff condition at the vertex 0. Precisely this is

$$\sum_{j=0}^{N} \partial_x u_j(0, t) = \alpha \partial_t u_0(0, t), \forall t \geq 0 \quad (2)$$

When $\alpha \geq 0$, the energy is a non increasing function of $t$. The case $\alpha = 0$ is a conservative one and was considered in [6]. In that paper the authors have studied the local energy behaviour. By the way of two different approaches, they have proved that the local energy is exponentially decaying and the exact decay rate was given. It equals $\gamma := \frac{N}{2L} \ln \left( \frac{N+1}{N-1} \right)$ where $L$ is the total length of the compact part of the network, i.e $L = \sum_{j=1}^{N} \ell_j$. Let’s stress that the given results were obtained for a network with equal edge lengths. In a recent work by Assel-Khenissi-Royer [7], the case of a network with different edge lengths $\ell_1, \ldots, \ell_N$ is studied and some results on the energy decay are obtained. Namely, if the ratios $\frac{\ell_i}{\ell_j}, i \neq j$ are not rational numbers, then exponential decay fails. Otherwise, the exponential decay takes place with a rate that can be computed.

In this paper, we deal with a damping acting at the vertex 0 via a constant $\alpha > 0$ and given by the condition (2). To our best knowledge this is the first work studying the energy decay in this unbounded configuration. Thus, in order to
obtain a complete and clear description and to go further with explicit formulae, we study the case of a network with \( N \geq 2 \) finite edges with equal lengths and one infinite edge attached to them at the vertex 0. We transform the problem into a Cauchy initial value problem and we compute the spectrum of the associated linear operator. We write the wave propagator \( \mathcal{U}(t) \) in terms of the resolvent and by a contour deformation technique we demonstrate an asymptotic expansion of \( \mathcal{U}(t) \). We use this expansion to prove the main result of this paper, namely that for \( \alpha \neq N + 1 \) the energy decays exponentially and the exact decay rate is obtained.

The paper is organized as follows. In section 2, we introduce the problem, the functional spaces and we prove the existence, uniqueness and regularity of the solution using the semigroup theory. In section 3, we investigate the spectral properties of the linear operator found in section 2. We compute explicitly the point spectrum and the resolvent. In section 4, the results of Section 3 are used for the study of the wave propagator \( \mathcal{U}(t) \). Using the Laplace transform with a contour deformation and the residue theorem, we find an asymptotic expansion of \( \mathcal{U}(t) \) when \( t \) tends to \( +\infty \) in terms of spectral projections. Finally, we exploit the information on \( \mathcal{U}(t) \) to prove the exponential decay and to give its rate.

2. Description of the problem. Let \( N \) be a positive integer and let us denote by \( \Gamma \) a network of strings with \( N \) finite edges \( \{e_j\}_{j=1}^{N} \) and one infinite edge \( e_0 \), all of them connected at a single vertex. If we denote by \( \{\ell_j\}_{j=1}^{N} \) the lengths of the finite edges then we may identify each edge \( e_j \) to the interval \([0, \ell_j]\), the vertex to 0 and the infinite edge to the semi-infinite interval \([0, +\infty[\).

We consider the wave equation \( \partial_t^2 u(x,t) - \Delta u(x,t) = 0 \) on the network \( \Gamma \) subject to a damping condition at the vertex 0 with initial data at \( t = 0 \), Dirichlet boundary conditions at the finite edges endpoints \( \ell_j \) and continuity condition at the vertex 0. The solution \( u \) of this problem is a collection of \( N + 1 \) functions \( u_0, u_1, \ldots, u_N \) each of them is a solution of the wave equation on the corresponding edge and which are coupled through the vertex conditions : \( \sum_{j=0}^{N} \partial_x u_j(0, t) = \alpha \partial_t u_0(0, t), \forall t \geq 0 \) where \( \alpha \) is a strictly positive constant.

The problem is formulated by the following system :

\[
\begin{align*}
(\partial_t^2 - \partial_x^2)u_j &= 0, \forall(x, t) \in [0, \ell_j] \times [0, \infty), 1 \leq j \leq N \\
(\partial_t^2 - \partial_x^2)u_0 &= 0, \forall(x, t) \in [0, \infty) \times [0, \infty) \\
\sum_{j=0}^{N} \partial_x u_j(0, t) &= \alpha \partial_t u_0(0, t), \forall t \geq 0 \\
u_j(x, 0) &= u_j^0(x), \partial_t u_j(x, 0) = u_j^1(x), \forall x \in \Gamma, 1 \leq j \leq N \\
u_j(\ell_j, t) &= 0, \forall t \in [0, \infty), j = 1 \leq j \leq N \\
u_j(0, t) &= u_0(0, t), \forall t \geq 0, 1 \leq j \leq N
\end{align*}
\]

(3)

The functions \( u_j^0, u_j^1, j = 0, \ldots, N \) are the initial data. Figure 1 describes the setting of the problem (3).

In order to guarantee the wellposedness of the problem, we introduce the functional spaces. For \( k = 1, 2 \) we denote by \( H^k(\Gamma) \) the space

\[
H^k(\Gamma) = H^k(0, +\infty) \times \prod_{j=1}^{N} H^k(0, \ell_j)
\]
where $H^k(I)$ is the standard Sobolev space over the interval $I$.

$$V = \left\{ (\phi_0, \phi) \in H^1(\Gamma), \left\{ \begin{array}{l} \phi_j(\ell_j) = 0 \\
\phi_j(0) = \phi_0(0), \forall 1 \leq j \leq N \end{array} \right. \right\}$$

and

$$X = \left[ V \times \left( L^2(0, +\infty) \times \prod_{j=1}^{N} L^2(0, \ell_j) \right) \right]$$

where $L^2$ is the space of square integrable functions. The space $X$ has the structure of a Hilbert space when equipped with the inner product

$$\langle (y, v), (z, w) \rangle_X = \int_0^{+\infty} \left( \frac{dy_0}{dx} \frac{dz_0}{dx} + v_0 w_0 \right) dx + \sum_{j=1}^{N} \int_0^{\ell_j} \left( \frac{dy_j}{dx} \frac{dz_j}{dx} + v_j w_j \right) dx. \quad (4)$$

We associate to the system (3) the linear operator $A$ defined by its domain

$$D(A) = \{ (u, v) \in (V \cap (H^2(\Gamma))) \times V; \sum_{j=0}^{N} \frac{du_j}{dx}(0) = \alpha v_0(0) \}$$

and acting on $D(A)$ by $A(u, v) = (v, u'')$, $\forall (u, v) \in D(A)$. Here $u''$ stands for the second derivative of $u$ with respect to $x$ in $[0, +\infty]$ or $[0, \ell_j]$, $j = 1, \ldots, N$. To be more precise, the operator $A$ has the operator valued matrix representation $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ and thus if we set $\Psi = (u, \partial_t u)^t$, then $u$ is a solution of (3) if and only if $\Psi$ is a solution of the time evolution problem

$$\begin{cases}
\dot{\Psi}(t) = A\Psi(t) \\
\Psi(0) = (f, g)^t
\end{cases} \quad (5)$$

for some given initial data $(f, g) \in V \times X$ that we can write in terms of the initial conditions satisfied by $u$. This correspondence between (3) and (5) allows us to use the abstract semigroup theory and especially the Hille-Yosida Theorem (see [16]).

**Theorem 2.1.** The operator $A$ with domain $D(A)$ generates a $C^0$-semigroup of contractions on the space $X$. 
Proof. We start by proving that $A$ is dissipative. Let $(u, v) \in D(A)$. To simplify the writing we put $\ell_0 = +\infty$. We have

$$\langle A(u, v), (u, v) \rangle_X = \sum_{j=0}^{N} \int_{0}^{\ell_j} \left( v'_j \overline{u'_j} + u'_j \overline{v'_j} \right) dx$$

$$= -\sum_{j=0}^{N} v_j(0) \overline{u'_j(0)} - \sum_{j=0}^{N} \int_{0}^{\ell_j} \left( v'_j \overline{u''_j} - u'_j \overline{v''_j} \right) dx$$

$$= -\alpha |v_0(0)|^2 - \sum_{j=0}^{N} \int_{0}^{\ell_j} \left( v'_j \overline{u''_j} - u'_j \overline{v''_j} \right) dx$$

(6)

The real part of the preceding is $-\alpha |v_0(0)|^2$ which is negative due to the prescribed sign of the damping coefficient $\alpha$. Now we proceed to prove that $A$ is maximal dissipative. For this to be done, we let $\lambda$ be a positive number and we claim that $\lambda I - A$ is surjective from $D(A)$ to $X$. We take $(f, g) \in X$ and we look for $(u, v) \in D(A)$ such that $(f, g) = (\lambda I - A)(u, v)$. This writes as a system

$$\left\{ \begin{array}{l}
\lambda u_j - v_j = f_j \\
\lambda v_j - u'_j = g_j
\end{array} \right. \quad \text{for } j = 0, \ldots, N. \quad (7)$$

For $j = 0, \ldots, N$ we substitute the first equation in the system into the second one. This yields $\lambda^2 u_j - u''_j = g_j + \lambda f_j$. We multiply this identity by $\overline{w}_j$ where $w \in V$ is an arbitrary function then we integrate over the intervals $(0, \ell_j), j = 0, \ldots, N$ and we take the sum over $j$. We get

$$\sum_{j=0}^{N} \int_{0}^{\ell_j} (\lambda^2 u_j - u''_j) \overline{w}_j dx = \sum_{j=0}^{N} \int_{0}^{\ell_j} (g_j + \lambda f_j) \overline{w}_j dx.$$  

(8)

After one integration per parts, the left hand side of (8) takes the form

$$\sum_{j=0}^{N} \int_{0}^{\ell_j} \left( \lambda^2 u_j \overline{w}_j + u'_j \overline{w}_j' \right) dx + \alpha (\lambda u_0(0) - f_0(0)) \overline{w}_0(0).$$  

(9)

Thus we reformulate the identity (8) as a variational problem of finding $u$ such that $b(u, w) = F(w)$ for all $w \in V$ where

$$b(u, w) = \sum_{j=0}^{N} \int_{0}^{\ell_j} \left( \lambda^2 u_j \overline{w}_j + u'_j \overline{w}_j' \right) dx + \alpha \lambda u_0(0) \overline{w}_0(0)$$

and

$$F(w) = \sum_{j=0}^{N} \int_{0}^{\ell_j} (g_j + \lambda f_j) \overline{w}_j dx + \alpha f_0(0) \overline{w}_0(0).$$

It is straightforward to see that $b$ is a continuous and coercive bilinear form on $V$ and that $F$ is a continuous linear form on $V$. Therefore, by the Lax-Milgram theorem the variational problem has a unique solution $u$ in $V$. Going back to the equations $\lambda^2 u_j - u''_j = g_j + \lambda f_j, j = 0, \ldots, N$, we claim that $u$ is in the space $H^2(0, +\infty) \times \prod_{j=1}^{N} H^2(0, \ell_j)$ and so the couple $(u, v)$ is a solution of (7) in $D(A)$. According to the Lumer-Philips Theorem [9], the operator $A$ is maximal dissipative. \hfill \square

Hence we have the following existence and regularity result:
Theorem 2.2. Let $\alpha > 0$. For any $(u^0, u^1) \in X$, there exists a unique mild solution $(u, \partial_t u) \in C([0, +\infty), X)$ to (3). Moreover, for any $(u^0, u^1) \in D(A)$, there exists a unique strong solution $(u, \partial_t u) \in C([0, +\infty), D(A))$ to (3).

The aim of this paper is study the decay of the energy of the solution $u$.

Definition 2.3. Let $u = (u_0, u_1, \ldots, u_N)$ be the unique solution of (3). We define:

(i) the global energy $E_u(t)$ of $u$ by

$$E_u(t) = \frac{1}{2} \left( \int_0^t \left( |\partial_t u_0(x, t)|^2 + |\partial_x u_0(x, t)|^2 \right) dx + \sum_{j=1}^N \int_0^t \left( |\partial_t u_j(x, t)|^2 + |\partial_x u_j(x, t)|^2 \right) dx \right)$$

(ii) the local energy $E_u^R(t)$ of $u$ at time $t$ by

$$E_u^R(t) = \frac{1}{2} \left( \int_0^t \int_0^R \left( |\partial_t u_0(x, t)|^2 + |\partial_x u_0(x, t)|^2 \right) dx + \sum_{j=1}^N \int_0^t \left( |\partial_t u_j(x, t)|^2 + |\partial_x u_j(x, t)|^2 \right) dx \right)$$

where $R$ is a positive number.

The global and the local energy of $u$ are defined for all $t \geq 0$ and one can see that $E_u^R(t) \leq E_u(t), \forall t \geq 0$ and that $\lim_{R \to +\infty} E_u^R(t) = E_u(t)$. The first result on the behaviour of the energy for large $t$ is the following.

Proposition 1. Let $\alpha \in \mathbb{R}$ and $(u^0, u^1) \in D(A)$. Let $u$ be the solution of the problem (3). Then for any $t \geq 0$ we have

$$E_u(t) - E_u(0) = -\alpha \int_0^t |\partial_s u_0(0, s)|^2 ds. \tag{12}$$

Proof. Let $u = (u_0, u_1, \ldots, u_N)$ be the solution of (3) and let $t \geq 0$. According to the fundamental identity $E_u(t) - E_u(0) = \int_0^t \frac{dE_u}{ds} ds$ one needs to calculate the derivative of the functional $E_u$ with respect to $t$. Denoting again $\ell_0 = +\infty$, we have:

$$\frac{dE_u}{dt}(t) = \sum_{j=0}^{N} \int_0^{\ell_j} \left( \partial_t^2 u_j \partial_t u_j + \partial_{tx}^2 u_j \partial_x u_j \right) dx$$

$$= \sum_{j=0}^{N} \left[ \int_0^{\ell_j} \partial_t^2 u_j \partial_t u_j dx + [\partial_x u_j \partial_t u_j]_{0}^{\ell_j} - \int_0^{\ell_j} \partial_{tx}^2 u_j \partial_x u_j dx \right]. \tag{13}$$

Using the relation $\partial_t^2 u_j = \partial_{tx}^2 u_j, \forall j = 0, \ldots, N$, the boundary conditions and the conditions satisfied by $u$ at the vertex 0, yields

$$\frac{dE_u}{dt}(t) = -\alpha |\partial_t u_0(0, t)|^2. \tag{14}$$
This ends the proof.

The previous result shows that the decay of the energy is governed by the sign of $\alpha$. The energy of the solution is non increasing for $\alpha \geq 0$. Without the dissipation condition at the vertex 0, i.e when $\alpha = 0$, the problem is conservative and the global energy of the solution is constant for all time $t \geq 0$ and equals to the one of the initial data. However, if $\alpha > 0$ the energy is decaying and for fixed $t > 0$ its dissipated amount is given by $\alpha\|\partial_t u_0(0,.)\|_{L^2(\Omega,\lambda)}^2$. Our aim in the rest of this paper is to study the decaying behaviour of the energy when $t$ tends to infinity. We will prove under some assumption on $\alpha$ that the energy is exponentially decaying and the rate at which this happens will be given. The main tool will be the full spectral analysis of the resolvent associated to the operator $A$ and an explicit computation of its spectrum.

3. Spectral analysis. In this section we give the spectral properties of the operator $A$. We describe the spectrum and investigate the properties of the resolvent.

3.1. Computation of the point spectrum. According to the m-dissipativity of the operator $A$ the spectrum is located in the set $\mathbb{C}^- := \{ \lambda \in \mathbb{C}; \Re(\lambda) \leq 0 \}$. We look here for the eigenvalues of finite multiplicities with negative real parts. Thus let $\lambda \in \mathbb{C}^-$ and let’s find a non trivial $(u, v) \in D(A)$ such that $A(u, v) = \lambda(u, v)$.

This writes as a system \[
\begin{cases}
  v = \lambda u \\
  u'' = \lambda v
\end{cases}
\] The problem is therefore reduced to a list of second order differential equations involving the functions $u_j, j = 0, \ldots, N$ satisfying the boundary conditions at $\ell_j, j = 1, \ldots, N$ and coupled through the continuity and dissipation conditions at the vertex 0. More precisely, we have to find $N+1$ functions $u_0, \ldots, u_N$ such that $(u_0, \ldots, u_N) \not\equiv 0_{R^{N+1}}$,

\[
 u_j''(x) - \lambda^2 u_j(x) = 0, \quad \forall x \in [0, \ell_j], j = 0, \ldots, N \tag{15}
\]

and $(u, \lambda u) \in D(A)$. It’s clear that $\lambda = 0$ is not an eigenvalue since the only corresponding solution in $D(A)$ is $(0,0)$. Let $\lambda \neq 0$. For every $j = 0, \ldots, N$ a fundamental system of solutions to equation (15) is $\{e^{\lambda x}, e^{-\lambda x}\}$, so there exist two constants $\alpha_j, \beta_j$ such that $u_j(x) = \alpha_j e^{\lambda x} + \beta_j e^{-\lambda x}$, for $j = 1, \ldots, N, x \in [0, \ell_j]$ and a constant $\alpha_0$ such that $u_0(x) = \alpha_0 e^{\lambda x}, \forall x \in [0, +\infty[$. For $1 \leq j \leq N$, the Dirichlet conditions at $x = \ell_j$ gives $\beta_j = -e^{2\lambda \ell_j} \alpha_j$ and the continuity condition at the vertex 0 yields $\alpha_0 = (1-e^{2\lambda \ell_j})\alpha_j$.

Writing the dissipation condition at the vertex 0 gives

\[ (1-\alpha)\alpha_0 + \sum_{j=1}^{N} (1+e^{2\lambda \ell_j})\alpha_j = 0. \]

The computation of the eigenvalues is equivalent to finding the values of $\lambda \in \mathbb{C}^-$ for which the linear system

\[
\begin{dcases}
\sum_{j=1}^{N} (1+e^{2\lambda \ell_j})\alpha_j + (1-\alpha)\alpha_0 = 0 \\
(1-e^{2\lambda \ell_j})\alpha_j - \alpha_0 = 0, \quad j = 1, \ldots, N
\end{dcases} \tag{16}
\]

has non trivial solution $(\alpha_1, \ldots, \alpha_N, \alpha_0) \in \mathbb{C}^{N+1}$.
Lemma 3.1. Let \( c = 1 - \alpha \) and \( a_j = 1 - e^{2\lambda \ell_j}, b_j = 1 + e^{2\lambda \ell_j} \) for \( j = 1, \ldots, N \). The determinant of the system (16) is given by

\[
\delta_N := c \prod_{j=1}^{N} a_j + \sum_{j=1}^{N} b_j \left( \prod_{k \neq j}^{N} a_k \right).
\]

(17)

Proof. For two given sets \( A = \{a_1, \ldots, a_N\} \) and \( B = \{b_1, \ldots, b_N\} \) of complex numbers, we denote by \( A_k = A \setminus \{a_k\}, B_k = B \setminus \{b_k\} \) and \( \delta(A, B) \) the determinant

\[
\delta(A, B) := \begin{vmatrix}
    a_1 & 0 & \cdots & \cdots & 0 & -1 \\
    0 & a_2 & 0 & \cdots & \cdots & 0 & -1 \\
    \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\
    0 & 0 & \cdots & 0 & a_N & -1 \\
    b_1 & b_2 & \cdots & \cdots & b_N & c
\end{vmatrix}.
\]

We have recursively

\[
\delta(A, B) = a_1 \delta(A_1, B_1) + (-1)^{N+2} b_1 (-1)^{N+2} \prod_{j=2}^{N} a_j
\]

\[
= a_1 a_2 \delta((A_1)_2, (B_1)_2) + b_1 \prod_{k \neq 1}^{N} a_k + b_2 \prod_{k \neq 2}^{N} a_k
\]

\[
= c \prod_{j=1}^{N} a_j + \sum_{j=1}^{N} b_j \left( \prod_{k \neq j}^{N} a_k \right).
\]

We want to find the eigenvalues of \( A \) so we need to factorize \( \delta_N \) then we look for its roots. The dependence of \( \delta_N \) on \( \lambda \) is made through the coefficients \( a_j, b_j, j = 1, \ldots, N \). The formula (17) becomes

\[
\delta_N = c \prod_{j=1}^{N} (1 - e^{2\lambda \ell_j}) + \sum_{j=1}^{N} (1 + e^{2\lambda \ell_j}) \left( \prod_{k \neq j}^{N} (1 - e^{2\lambda \ell_j}) \right)
\]

\[
= e^{\lambda L} (-2)^N \left[ c \prod_{j=1}^{N} \sinh(\lambda \ell_j) - \sum_{j=1}^{N} \cosh(\lambda \ell_j) \right] \times \prod_{k \neq j}^{N} \sinh(\lambda \ell_k)
\]

(18)

where \( L = \sum_{j=1}^{N} \ell_j \) is the total length of the bounded component of the network.

To go further on in the factorization of \( \delta_N \) we need to make some assumptions on the lengths \( \ell_j, j = 1, \ldots, N \). We assume for instance that there exists a constant \( \ell > 0 \) such that \( \ell_j = \ell, \forall j = 1, \ldots, N \). The cases where the edge lengths satisfy the more general condition \( \ell_j / \ell_k \in \mathbb{Q} \) or the irrationality condition \( \ell_j / \ell_k \notin \mathbb{Q} \) will be considered in a further paper.
Proposition 2. Let $N \in \mathbb{N}$ such that $N \geq 2, \alpha > 0$ and $r = \frac{1 - \alpha}{N}$. We assume that $\ell_j = \ell, \forall j = 1, \ldots, N$ with $\ell > 0$. Then:

(1) If $\alpha = N + 1$, the operator $A$ has a set of eigenvalues equal to $\left\{ \frac{in\pi}{\ell}; n \in \mathbb{Z}^* \right\}$.

(2) If $\alpha > N + 1$, the operator $A$ has two sequences of eigenvalues given by

\[ \left( \frac{in\pi}{\ell} \right)_{n \in \mathbb{Z}^*} \quad \text{and} \quad \left( \frac{\text{Arcoth}(r)}{\ell} + \frac{in\pi}{\ell} \right)_{n \in \mathbb{Z}}. \]

(3) If $\alpha < N + 1$, the operator $A$ has two sequences of eigenvalues given by

\[ \left( \frac{in\pi}{\ell} \right)_{n \in \mathbb{Z}^*} \quad \text{and} \quad \left( \frac{\text{Argth}(r)}{\ell} + \frac{i(2n + 1)\pi}{2\ell} \right)_{n \in \mathbb{Z}}. \]

In the previous relations, Arcoth and Argth are the reciprocal functions of the hyperbolic cotangent function and of the hyperbolic tangent function respectively.

Proof. The determinant $\delta_N$ factorizes in this case as

\[ \delta_N = (-2)^N e^{\lambda N \ell} (\sinh(\lambda \ell))^{N-1} [c \sinh(\lambda \ell) - N \cosh(\lambda \ell)]. \]

Thus $\delta_N = 0$ iff $\sinh(\lambda \ell) = 0$ or $c \sinh(\lambda \ell) = N \cosh(\lambda \ell)$. For $\sinh(\lambda \ell) = 0$, the roots are $\lambda_n := \frac{in\pi}{\ell}, n \in \mathbb{Z}$. We already know that $\lambda = 0$ is not an eigenvalue.

If $\sinh(\lambda \ell) \neq 0$, then the other roots of $\delta_N$ are the solutions of the equation

\[ e^{2\lambda \ell} = \frac{\alpha - (N + 1)}{\alpha + N - 1}. \]  \hspace{1cm} (19)

The denominator in the left hand side is positive since $\alpha > 0$ and $N \geq 2$. A direct computation of the solutions of (19) with a discussion on the sign of $\alpha - (N + 1)$ give the result. \hfill \square

The eigenvectors associated with the eigenvalues given in the last proposition are explicitly computed and we have the following result:

Proposition 3. Let $N \in \mathbb{N}, N \geq 2, \alpha > 0$ and $r = \frac{1 - \alpha}{N}$. We assume that $\ell_j = \ell, \forall j = 1, \ldots, N$ with $\ell > 0$. Then

(1) Let $n \in \mathbb{Z}^*$ and $\lambda_n = \frac{in\pi}{\ell}$. Then $\lambda_n$ is a simple eigenvalue of $A$ and an associated eigenvector is $(\phi_n, \lambda_n \phi_n)^t$ where $\phi_n(x) = \sin(\lambda_n x) \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \forall x \in \Gamma.$
(2) If $\alpha > N + 1$, then $\forall n \in \mathbb{Z}$, $\mu_n := \frac{\text{Arcosh}(r)}{\ell} + \frac{i\pi n}{\ell}$ is a simple eigenvalue of $A$ and an associated eigenvector is $(\psi_n, \mu_n \psi_n)$ where

$$\psi_n(x) = (\cosh(\mu_n x) - r \sinh(\mu_n x)) \left( \begin{array}{c} 0 \\ 1 \\ \vdots \\ 1 \end{array} \right) + \left( \begin{array}{c} e^{\mu_n x} \\ 0 \\ \vdots \\ 0 \end{array} \right), \forall x \in \Gamma.$$

(3) If $\alpha < N + 1$, then $\forall n \in \mathbb{Z}$, $\mu_n := \frac{\text{Argth}(r)}{\ell} + i\frac{(2n + 1)\pi}{2\ell}$ is a simple eigenvalue of $A$ and an associated eigenvector is $(\psi_n, \mu_n \psi_n)$ where

$$\psi_n(x) = (\cosh(\mu_n x) - r \sinh(\mu_n x)) \left( \begin{array}{c} 0 \\ 1 \\ \vdots \\ 1 \end{array} \right) + \left( \begin{array}{c} e^{\mu_n x} \\ 0 \\ \vdots \\ 0 \end{array} \right), \forall x \in \Gamma.$$

Proof. For $\lambda_n = \frac{i\pi n}{\ell}, n \in \mathbb{Z}^*$ to be an eigenvalue of $A$ with domain $D(A)$ the solution of $A(u,v) = \lambda(u,v)$ must satisfy $u_0 = 0$. The other components of $u$ are easily calculated using the differential equations, the Dirichlet condition at $x = \ell$ and the two conditions at $x = 0$.

For $\mu_n, n \in \mathbb{Z}$, the computation of the coefficients $\alpha_j, \beta_j, j = 1, \ldots, N$ and $\alpha_0$ using the conditions at $x = \ell$ and $x = 0$ gives, using $r = \frac{1 - \alpha}{N}$

$$\alpha_0 \in \mathbb{C}, \quad \alpha_j = \frac{1 - r}{2} \alpha_0, \quad \beta_j = \frac{1 + r}{2} \alpha_0, \forall j = 1, \ldots, N.$$

Remark 1. 1. In the particular case when $N = 1$, the equation $\delta_1 = 0$ reduces to $c \sinh(\lambda \ell) = \cosh(\lambda \ell)$. Thus we have

- if $\alpha = 2$, then $A$ has no eigenvalues of finite multiplicities.
- if $\alpha > 2$, then $A$ has its eigenvalues in the set

$$\left\{ \frac{1}{2\ell} \ln \left( \frac{\alpha - 2}{\alpha} \right) + \frac{i\pi n}{\ell}, n \in \mathbb{Z} \right\}.$$

- if $\alpha < 2$, then $A$ has its eigenvalues in the set

$$\left\{ \frac{1}{2\ell} \ln \left( \frac{2 - \alpha}{\alpha} \right) + \frac{i(2n + 1)\pi}{2\ell}, n \in \mathbb{Z} \right\}.$$
2. In the particular case when $\alpha = 1$ and $N \geq 2$, the eigenvalues are purely imaginary and are given by the sequence $\left(\frac{i\pi n}{2\ell}\right)_{n \in \mathbb{Z}^*}$.

### 3.2. Computation of the resolvent.

Let $\lambda \in \mathbb{C}$ such that $\lambda$ is in the resolvent set of $A$. We denote by $R(\lambda) = (\lambda I - A)^{-1}$ the resolvent of $A$ defined as an operator from $(H^1 \times L^2)(\Gamma)$ to $D(A)$. We will calculate $R(\lambda)$ and use it to study the wave semigroup associated to $A$ by Theorem (2.1).

Let $(f, g) \in H^1 \times L^2$. We look for $(\phi, \psi) \in D(A)$ such that $(\lambda I - A)(\phi, \psi) = (f, g)$ is true. This relation can be written as system

\[
\begin{align*}
\left\{ \begin{array}{l}
-\Delta \phi + \lambda^2 \phi &= g + \lambda f \\
\lambda \phi - \psi &= f 
\end{array} \right. 
\end{align*}
\]

in which the first equation has to be understood as a collection of the $N + 1$ differential equations $-\phi_{j''} + \lambda^2 \phi_j = g_j + \lambda f_j$ while the second one is $\psi_j = \lambda \phi_j - f_j$, for $j = 0, \ldots, N$.

Let $j \in \{0, 1, \ldots, N\}$ and $h_j = \lambda f_j + g_j$. In the following we give the solutions of the inhomogeneous second order differential equation $-\phi_{j''} + \lambda^2 \phi_j = h_j$ under the condition that $(\phi, \lambda \phi - f)$ is in the domain $D(A)$.

**Theorem 3.2.** Let $\lambda \in \mathbb{C}^+ =: \{\lambda \in \mathbb{C}; \Re \lambda > 0\}$ and $(f, g) \in X$. The resolvent equation $R(\lambda)(f, g) = (\phi, \psi)$ has a unique solution $(\phi, \psi)$ in $D(A)$. Moreover, $(\phi, \psi)$ is given by

\[
\phi_0(x) = \frac{1}{\lambda} \left[ \left( \frac{\alpha \sinh(\lambda \ell)}{C_{\alpha}(\lambda)} f_0(0) - \frac{\tilde{H}(\lambda)}{C_{\alpha}(\lambda)} \right) e^{\lambda x} + \int_0^x (\lambda f_0(s) + g_0(s)) \sinh(\lambda (s - x)) ds \right],
\]

\[
\psi_0(x) = \lambda \phi_0(x) - f_0(x), \quad \forall x \in [0, +\infty],
\]

\[
\phi_j(x) = \frac{1}{\lambda} \left[ \left( \tilde{H}(\lambda) - \alpha \sinh(\lambda \ell) f_0(0) \right) \frac{\sinh(\lambda (x - \ell))}{\sinh(\lambda \ell) C_{\alpha}(\lambda)} \right. \\
\left. - \left( \lambda \tilde{f}_j(\lambda) + \tilde{g}_j(\lambda) \right) \frac{\sinh(\lambda x)}{\sinh(\lambda \ell)} \right] \\
+ \int_0^x (\lambda f_j(s) + g_j(s)) \sinh(\lambda (s - x)) ds
\]

and

\[
\psi_j(x) = \lambda \phi_j(x) - f_j(x), \quad \forall x \in [0, \ell], \quad j = 1, \ldots, N.
\]

Here we have made the notations $\tilde{F}(\lambda) = \sum_{j=1}^N \tilde{f}_j(\lambda), \tilde{G}(\lambda) = \sum_{j=1}^N \tilde{g}_j(\lambda)$ and $\tilde{H}(\lambda) = \lambda \tilde{F}(\lambda) + \tilde{G}(\lambda)$ with

\[
\tilde{f}_j(\lambda) = \int_0^\ell f_j(s) \sinh(\lambda (s - \ell)) ds
\]

and

\[
\tilde{g}_j(\lambda) = \int_0^\ell g_j(s) \sinh(\lambda (s - \ell)) ds
\]

and $C_{\alpha}(\lambda) = (\alpha - 1) \sinh(\lambda \ell) + N \cosh(\lambda \ell)$.
Proof. Let \((f, g) \in X\). From (20) we see that the equation \(R(\lambda)(\phi, \psi) = (f, g)\) is equivalent to the system
\[
\begin{align*}
\phi &= R(\lambda)(\lambda f + g) \\
\psi &= \lambda \phi - f
\end{align*}
\]
with \(R(\lambda) = (-\Delta + \lambda^2 I)^{-1}\). We denote by \(h = \lambda f + g\) so the equation for \(\phi\) writes as \(-\Delta \phi + \lambda^2 \phi = h\) and we look for \(\phi \in V\) such that \((\phi, \lambda \phi - f) \in D(A)\).

Let \(0 \leq j \leq N\). A computation of the solutions of the equations \(-\phi''_j + \lambda^2 \phi_j = h_j\) gives for \(1 \leq j \leq N\)
\[
\phi_j(x) = \alpha_j e^{\lambda x} + \beta_j e^{-\lambda x} + \frac{1}{\lambda} \int_0^x h_j(s) \sinh(\lambda(s-x)) ds,
\]
and for \(j = 0\)
\[
\phi_0(x) = \alpha_0 e^{\lambda x} + \frac{1}{\lambda} \int_0^x h_0(s) \sinh(\lambda(s-x)) ds.
\]
The continuity condition at the single vertex gives
\[
\beta_j = \alpha_0 - \alpha_j, \quad \text{for} \quad j = 1, \ldots, N.
\]
Thus we need to calculate \(\alpha_j, j = 0, \ldots, N\) and the solutions \(\phi_j\) will be in the form
\[
\phi_j(x) = 2\alpha_j \sinh(\lambda x) + \alpha_0 e^{-\lambda x} + \frac{1}{\lambda} \int_0^x h_j(s) \sinh(\lambda(s-x)) ds.
\]
When \(1 \leq j \leq n\), the Dirichlet condition at the edges endpoints \(x = \ell\) gives
\[
2\alpha_j \sinh(\lambda \ell) + \alpha_0 e^{-\lambda \ell} = -\frac{1}{\lambda} \int_0^\ell h_j(s) \sinh(\lambda(s-\ell)) ds.
\]
The dissipation condition \(\sum_{j=0}^N \phi'_j(0) = \alpha(\lambda \phi_0(0) - f_0(0))\) gives
\[
\sum_{j=1}^N \alpha_j = \frac{\alpha + N - 1}{2} \alpha_0 - \frac{\alpha}{2\lambda} f_0(0).
\]
Taking the sum over \(j\) in (26) and using (27) one obtains
\[
\alpha_0 = -\frac{\tilde{H}(\lambda)}{\lambda C_\alpha(\lambda)} + \frac{\alpha \sinh(\lambda \ell)}{\lambda C_\alpha(\lambda)} f_0(0)
\]
where
\[
\tilde{H}(\lambda) = \int_0^\ell \sum_{j=1}^N h_j(s) \sinh(\lambda(s-\ell)) ds
\]
and
\[
C_\alpha(\lambda) = (\alpha - 1) \sinh(\lambda \ell) + N \cosh(\lambda \ell).
\]
It is not difficult to see the \(C_\alpha(\lambda)\) does not vanish when \(\lambda \in \mathbb{C}^+\).

Now going back to (26) we get, for \(j = 1, \ldots, N\),
\[
\alpha_j = \frac{e^{-\lambda \ell} \tilde{H}(\lambda)}{2\lambda \sinh(\lambda \ell) C_\alpha(\lambda)} - \frac{\tilde{H}_j(\lambda)}{2\lambda \sinh(\lambda \ell)} - \frac{\alpha e^{-\lambda \ell}}{2\lambda C_\alpha(\lambda)^2} f_0(0).
\]
\(\square\)
4. **Energy decay.** From the results given in Theorem (3.2) the resolvent $\mathcal{R}$ is a holomorphic function on the open set $\mathbb{C}^+ = \{ \lambda \in \mathbb{C}; \Re \lambda > 0 \}$. We already know from (9) that $\mathcal{R}$ is related to the dissipative semigroup $\mathcal{U}(t)$ by the Laplace transform

$$
\mathcal{R}(\lambda) = \int_0^{+\infty} e^{-\lambda t} \mathcal{U}(t) dt, \quad \forall \lambda \in \mathbb{C}^+. \quad (30)
$$

Let $\chi \in \mathcal{C}_c^0(0, +\infty)$. We denote by $\mathcal{R}_\chi(\lambda) = \chi \mathcal{R}(\lambda) \chi$ the cut-off resolvent of $A$. The function $\chi$ is a cut-off on the unbounded edge $\epsilon_0$ of $\Gamma$. Moreover, using the semigroup theory, the solution $\Psi = (u, \partial_t u)$ of the Cauchy problem (5) is given by $\Psi(t) = \mathcal{U}(t)(f, g), \forall t \geq 0$ where $(f, g)$ are the given initial data. Our aim is to find the decay properties of the wave group for large $t$ and then to use them for the energy decay of the solution of (3). Hence we may use an inversion formula for the Laplace transform. Using the fact that $A$ is a dissipative operator that generates a contraction semi-group, we may use the Hille-Yosida theorem (see [16]) and can write $\mathcal{U}(t)$ as a contour integral of $\mathcal{R}(\lambda)$. More precisely If we let $\gamma$ be a strictly positive number then we have

$$
\mathcal{U}(t) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \mathcal{R}(\lambda) d\lambda, \quad \forall t \geq 0. \quad (31)
$$

For $(f, g) \in D(A)$, the solution $u = (u_0, \ldots, u_N)$ of (3) and its derivative $\partial_t u$, with initial data $(u^0, u^1) = (f, g)$ are given $\forall x \in \Gamma, t \geq 0$ by

$$
\begin{align*}
\begin{cases}
  u(x, t) = &\frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \phi(x, \lambda) d\lambda \\
  \partial_t u(x, t) = &\frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} (\lambda \phi(x, \lambda) - f(x)) d\lambda.
\end{cases}
\end{align*} \quad (32)
$$

The function $\phi$ in the formulae for $u$ and $\partial_t u$ is the first component of the solution of the resolvent equation given by (3.2). We have added the variable $\lambda$ to $\phi$ in order to mark its dependence on it. The approach we will use to study the decay of the energy of $u$ is to make a contour deformation in the formulae (32) and then we apply the residue theorem. To achieve this, we need more precise informations on the resolvent $\mathcal{R}$ namely its singularities and useful estimates on the norm of $\mathcal{R}(\lambda)$ as an operator from $X$ to $D(A)$.

**Theorem 4.1.** Let $\alpha > 0$ and $N \in \mathbb{N}, N \geq 2$. Let $r = \frac{1 - \alpha}{N}$. The cut-off resolvent $\mathcal{R}_\chi$ defined on $\mathbb{C}^+$ extends as a meromorphic function on $\mathbb{C}$. If we still denote by $\mathcal{R}_\chi$ this extension, then the singularities of $\mathcal{R}_\chi$ are poles of multiplicity one and are given by the set $\mathcal{P}$ where

(i) if $\alpha = N + 1$, then $\mathcal{P} = \left\{ \frac{i\pi}{\ell}, n \in \mathbb{Z}^* \right\}$.

(ii) if $\alpha > N + 1$, then $\mathcal{P} = \left\{ \frac{i\pi}{\ell}, n \in \mathbb{Z}^* \right\} \cup \left\{ \frac{\text{Arc} \text{coth}(r)}{\ell} + \frac{i\pi}{\ell}, n \in \mathbb{Z} \right\}$.

(iii) if $\alpha < N + 1$, then $\mathcal{P} = \left\{ \frac{i\pi}{\ell}, n \in \mathbb{Z}^* \right\} \cup \left\{ \frac{\text{Ar} \text{g} \text{th}(r)}{\ell} + \frac{i(2n + 1)\pi}{2\ell}, n \in \mathbb{Z} \right\}$.

**Proof.** We use the results given by Theorem (3.2). It can be easily seen that $\phi$ and $\psi$ depend holomorphically on $\lambda$ except when $C_\alpha(\lambda)$ or $\sinh(\lambda \ell)$ vanish. It is straightforward that the roots of $C_\alpha(\lambda)$ and $\sinh(\lambda \ell)$ are given by the set $\mathcal{P}$ depending on the comparison of $\alpha$ and $N + 1$. All of these roots are simple. \[\square\]
**Definition 4.2.** Let \( \alpha > 0 \) such that \( \alpha \neq N + 1 \). The spectral abscissa of the operator \( A \) is defined as

\[
\omega(\alpha) := \sup \{ \Re(\lambda), \lambda \in \mathcal{P}, \lambda \notin i\mathbb{R} \}.
\]

**Remark 2.** It is clear that \( \omega(\alpha) \) is a non positive number whenever it exists. In the case considered in this paper we have

\[
\omega(\alpha) = \begin{cases} 
\text{Arcoth}(r) & \text{if } \alpha > N + 1 \\
\text{Argth}(r) & \text{if } \alpha < N + 1
\end{cases}
\]

In the two cases, the spectral abscissa is a negative number. In the case \( \alpha = N + 1 \) or \( \alpha = 1 \), \( \omega(\alpha) \) does not exist. One might consider in this case that the spectrum is purely imaginary so the exponential decay of the energy does not take place. This case will be studied in a further paper.

**Notation.** From now on, we will denote by \( \gamma_\alpha = -\omega(\alpha) \) whenever it exists.

The main result of this section is the following theorem.

**Theorem 4.3.** Let \( N \in \mathbb{N}, N \geq 2 \) and \( \alpha > 0 \) such that \( \alpha \notin \{1, N + 1\} \). Let \( r = \frac{1 - \alpha}{N} \). For any \( \gamma > \gamma_\alpha \) and for \( t \) large enough, the wave group \( U(t) \) has the following asymptotic expansion in the space \( \mathcal{L}(D(A), X) \)

\[
\chi U(t)\chi = \sum_{n \in \mathbb{Z}} e^{\gamma n} P_{\lambda_n} \chi + e^{-\gamma_\alpha t} \sum_{n \in \mathbb{Z}} \nu e^{\gamma n} \chi P_{\mu_n} \chi + O(e^{-\gamma t})
\]

where \( \nu = 1 \) if \( \alpha > N + 1 \) and \( \nu = e^{i\pi/2} \) if \( \alpha < N + 1 \). The operators \( P_{\lambda_n} \) and \( \Pi_{\mu_n} \) are the Riesz projections onto the spectral eigenspaces associated respectively to \( \lambda_n \) and \( \mu_n \) and are given by the formulae

\[
P_{\lambda_n} (\text{resp } \Pi_{\mu_n}) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_n| = \varepsilon} \mathcal{R}(\lambda) d\lambda,
\]

with \( \varepsilon \leq \min \left( \frac{\pi}{2\ell}, \frac{\gamma_\alpha}{2} \right) \) and \( \gamma_\alpha = -\omega(\alpha) \) where

\[
\omega(\alpha) = \begin{cases} 
\text{Arcoth}(r) & \text{if } \alpha > N + 1 \\
\text{Argth}(r) & \text{if } \alpha < N + 1
\end{cases}
\]

**Proof.** Let \( \gamma > \gamma_\alpha \) and \( a_m = \frac{(3m + 1)\pi}{3\ell} \) for \( m \in \mathbb{N} \). We can write (31) as

\[
\chi U(t)\chi = \lim_{m \to +\infty} \frac{1}{2\pi} \int_{\gamma - ia_m}^{\gamma + ia_m} e^{\gamma \mathcal{R}_\lambda(\lambda)} d\lambda, \quad \forall t \geq 0
\]

and the same is true for (32). We will use the residue theorem, so let \( Y_m \) be the simple closed contour defined as the rectangle defined by the four points \( A_m = (\gamma, \gamma - ia_m), B_m = (\gamma, \gamma + ia_m), C_n = (\gamma, -\gamma + ia_m), D_m = (\gamma, -\gamma - ia_m) \) and oriented in the direct anticlockwise direction \( (A_m \to B_m \to C_m \to D_m \to A_m) \). Using the residue theorem applied to the meromorphic operator valued function
\[ \lambda \mapsto e^{\lambda t} R(\lambda) \] on the contour \( \gamma_m \) for fixed \( m \in \mathbb{N} \), yields
\[
2i\pi \left[ \sum_{|k|<m} \text{Res} \left( e^{\lambda t} R(\lambda), \lambda_k \right) + \sum_{|k|<m} \text{Res} \left( e^{\lambda t} R(\lambda), \mu_k \right) \right] \\
= i \int_{-a_m}^{a_m} e^{\gamma t} e^{iy} R(\gamma + iy) \, dy - i \int_{-\gamma}^{\gamma} e^{ia \gamma} e^{\lambda x} R(x + ia \gamma) \, dx \\
- i \int_{-a_m}^{a_m} e^{-\gamma t} e^{iy} R(\gamma + iy) \, dy \\
+ \int_{-\gamma}^{\gamma} e^{-ia \gamma} e^{\lambda x} R(x - ia \gamma) \, dx \] (34)

Let \( k \in \mathbb{Z}, |k| < m \). By choosing \( \varepsilon < \min \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \) then the circle \( \{ \lambda \in \mathbb{C} : |\lambda - \lambda_k| = \varepsilon \} \) makes of \( \lambda_k \) an isolated simple pole of the resolvent \( R(\lambda) \). The same holds true for \( \mu_k \). Thus, from the contour integral representation of the Riesz projections (see [10]) we obtain
\[
\text{Res} \left( e^{\lambda t} R(\lambda), \lambda_k \right) = e^{\lambda k t} \frac{1}{2i\pi} \int_{|\lambda - \lambda_k| = \varepsilon} R(\lambda) \, d\lambda = e^{\lambda k t} P \lambda_k = e^{\lambda k t} P \lambda_k 
\] (35)
and
\[
\text{Res} \left( e^{\lambda t} R(\lambda), \mu_k \right) = e^{\mu k t} P \mu_k = \nu e^{-\gamma t} e^{i \frac{\pi}{4}} P \mu_k.
\]

If we pass to the limit \( m \to +\infty \) in (34) we will have to study the behaviour of the contour integrals \((1), \ldots, (4))\) therein. According to (35), when \( m \) tends to \( +\infty \) we find from (34)
\[
\chi U(t) \chi = \sum_{n \in \mathbb{Z}^+} e^{i\lambda n t} P \lambda_n \chi + \sum_{n \in \mathbb{Z}^-} \nu e^{-\gamma n t} e^{i \frac{\pi}{4}} \chi P \mu_n \chi \\
+ \chi L_{\pm} \chi - \chi K \chi 
\] (36)

where
\[
L_{\pm} = \mp \lim_{m \to +\infty} \int_{-\gamma}^{\gamma} e^{\pm ia \gamma} e^{\lambda x} R(x + ia \gamma) \, dx \\
K = \lim_{m \to +\infty} \frac{1}{2\pi} \int_{-a_m}^{a_m} e^{-\gamma t} e^{i \gamma t} R(\gamma + iy) \, dy.
\]

We claim that \( L_{\pm} = 0 \) in \( L(D(A), X) \) and there exists a positive constant \( C \) such that
\[
\| K \|_{L(D(A), X)} \leq Ce^{-\gamma t}. 
\] (37)

Let \( (f, g) \in D(A) \) and \( 0 \leq j \leq N \).

For the first claim, we use integration per parts in (21) and (22) and hence we get
\[
\lambda \phi_j(x) = \lambda K_1(x, \lambda) \phi_0(0) + K_2^j(x, \lambda) 
\] (38)
where
\[
K_1(x, \lambda) = \frac{\alpha \sinh(\lambda t) - N \cosh(\lambda t)}{C_{\alpha}(\lambda)} e^{\lambda x} - \cosh(\lambda x),
\]
Let $u$ be the solution of (3). There exists a positive constant $E_{\infty}$ such that for $R > 0$ large enough, the local energy of $u$ satisfies

$$E^R_u(t) - E_{\infty} \leq e^{-\gamma t} E^R_u(0) + O(e^{-\gamma t}), \quad \forall \gamma > \gamma_\alpha \text{ and } t \to \infty.$$ 

Proof. We take the initial data $(u^0, u^1) \in D(A)$ such that $u^0_0$ is compactly supported since we deal with the local energy. By the semigroup theory, we have $\Psi(t) := (u(t), \partial_t u(t)) = U(t)(u^0, u^1)$ for all $t \geq 0$. Where if we put

$$\hat{P}(u^0, u^1) = \sum_{n \in \mathbb{Z}^+} e^{-\frac{i\alpha t}{m}} P_{\lambda_n}(u^0, u^1)$$

then we obtain

$$\|\Psi(t) - \hat{P}(u^0, u^1)\| \leq e^{-\gamma_\alpha t}\|\Psi(0)\| + O(e^{-\gamma t}), \forall \gamma > \gamma_\alpha.$$ 

The limit energy $E_{\infty}$ is the norm of the projection of the initial conditions on the eigenspaces associated to the purely imaginary eigenvalues of the operator $A$. \qed
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