The Minimum Rank Problem: a counterexample

Swastik Kopparty
Computer Science and Artificial Intelligence Laboratory
MIT
Cambridge, MA, U.S.A.
E-mail: swastik@mit.edu

and

K. P. S. Bhaskara Rao
Department of Mathematics and Computer Science
Indiana State University
Terre Haute, IN 47802 U.S.A.
E-mail: bkopparty@isugw.indstate.edu

Abstract

We provide a counterexample to a recent conjecture that the minimum rank of every sign pattern matrix can be realized by a rational matrix. We use one of the equivalences of the conjecture and some results from projective geometry. As a consequence of the counterexample we show that there is a graph for which the minimum rank over the reals is strictly smaller than the minimum rank over the rationals. We also make some comments on the minimum rank of sign pattern matrices over different subfields of $\mathbb{R}$.

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1 Introduction

The main reference for this paper is [1] in which the conjecture and its equivalences appear.

A matrix whose entries are from the set $\{+,-,0\}$ is called a sign pattern matrix. A matrix with real entries is called a real matrix and a matrix with rational entries is called a rational matrix. For a real matrix $B$, $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of $B$ by $+$ (respectively, $-$, $0$). If $A$ is a sign pattern matrix and $F$ is a subfield of $\mathbb{R}$, the sign pattern class of $A$ over $F$ is defined by

$$Q_F(A) = \{ B : B \text{ is a matrix with entries in } F \text{ and } \text{sgn}(B) = A \}.$$
For a sign pattern matrix $A$ and a subfield $\mathbb{F}$ of $\mathbb{R}$, the minimum rank of $A$ over $\mathbb{F}$, denoted $\text{mr}_\mathbb{F}(A)$, is defined as

$$\text{mr}_\mathbb{F}(A) = \min_{B \in \mathbb{Q}_\mathbb{F}(A)} \{ \text{rank } B \}.$$ 

In [1], the authors made the following basic conjecture:

For any $m \times n$ sign pattern matrix $A$, $\text{mr}_\mathbb{R}(A) = \text{mr}_\mathbb{Q}(A)$. They showed that the conjecture holds in certain special cases.

In [1], it was also shown that the above conjecture is equivalent to another conjecture, namely,

For any real matrices $D$, $C$, and $E$, with $DC = E$, there are rational matrices $D^*$, $C^*$, and $E^*$ such that $\text{sgn}(D^*) = \text{sgn}(D)$, $\text{sgn}(C^*) = \text{sgn}(C)$, $\text{sgn}(E^*) = \text{sgn}(E)$, and $D^*C^* = E^*$.

In the following section, we shall give an example to show that this conjecture is not true. We also show that there is a graph for which the minimum rank over the reals is strictly smaller than the minimum rank over the rationals. In the last section we make some comments on the minimum rank of sign pattern matrices over different subfields of $\mathbb{R}$.

## 2 The Counterexample

Consider a configuration $C$ (from [2], p.92) of nine points and nine lines given by $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, $I$, and nine lines $ABE$, $ADG$, $AHI$, $BCH$, $BGI$, $CEG$, $CFI$, $DEI$, $DFH$ as drawn in Figure 1 below starting with a regular pentagon.

Let $\ell_1, \ell_2, \ldots, \ell_9$ be the nine lines in Figure 1 and let the equation of $\ell_i$ be $a_i x + b_i y + c_i = 0$. Let the nine points (with real coordinates) be $(x_i, y_i)$, $i = 1, 2, \ldots, 9$.

Let $D$ be the $9 \times 3$ matrix whose $i^{th}$ row is $(a_i, b_i, c_i)$ and $C$ be the $3 \times 9$ matrix whose $j^{th}$ column is the transpose of the row $(x_i, y_i, 1)$. Let $DC = E$. $E$ is a $9 \times 9$ matrix whose $(i, j)^{th}$ element is 0 if the $j^{th}$ point is on the $i^{th}$ line and $\neq 0$ if the $j^{th}$ point is not on the $i^{th}$ line. The incidences of the 9 points on the 9 lines are exactly dictated by the zero and nonzero elements of $E$.

The result on p.93 of [2] states that (the incidence structure) $C$ cannot be realized with nine points with rational coordinates. Suppose now that there are rational matrices $D^*$, $C^*$, and $E^*$ such that $D^*C^* = E^*$ and the zero non-zero pattern of $E^*$ is same as the zero non-zero pattern of $E$. Since the third row of $C^*$ has nonzero elements, by dividing each column of $C^*$ and the corresponding column of $E^*$ by a nonzero rational number we may assume that the third row of $C^*$ has all 1’s. Now, let the $j^{th}$ column of $C^*$ be the transpose of $(x^*_j, y^*_j, 1)$. If $D^*$ is the $9 \times 3$ matrix whose $i^{th}$ row is $(a^*_i, b^*_i, c^*_i)$, then the $j^{th}$ point $(x^*_j, y^*_j)$ will be on the line $a^*_i x + b^*_i y + c^*_i = 0$ if and only if $(x_j, y_j)$ is on $\ell_i$ for $i = 1, 2, \ldots, 9$. This is because $D^*C^* = E^*$ and $E^*$ and $E$ have the same zero non-zero pattern. Hence $(a^*_i, b^*_i)$
for \(i = 1, 2, \ldots, 9\) will be nine points with rational coordinates with the same structure of \(C\).

Hence there are no rational matrices \(D^*, C^*, E^*\) such that \(D^*C^* = E^*\) and \(E^*\) has the same zero pattern as \(E\). Hence there are no rational matrices \(D^*, C^*\) and \(E^*\) such that \(D^*C^* = E^*\) and \(\text{sgn}(D^*) = \text{sgn}(D), \text{sgn}(C^*) = \text{sgn}(C), \text{sgn}(E^*) = \text{sgn}(E)\).

The above procedure actually gives a real \(12 \times 12\) matrix \(B = \begin{bmatrix} I_3 & C \\ D & E \end{bmatrix}\), such that \(\text{rank}(B) = 3\), for which there is no rational matrix \(F\) such that \(\text{rank}(F) = 3\) and \(F\) and \(B\) have the same zero non-zero pattern.

If \(A = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix}\), then \(A\) is a \(24 \times 24\) symmetric real matrix for which there is no rational matrix \(A^*\) such that \(\text{sgn}(A) = \text{sgn}(A^*)\) and \(\text{rank}(A) = \text{rank}(A^*)\). This in turn gives us a bipartite graph \(G\) on 24 points (with 12 points on each side) whose incidence matrix has the zero non-zero pattern of \(A\). For this graph the minimum rank over the rationals is strictly more than that over the reals (this rank being 6).

Note that in [1] it was shown that for every real matrix \(B\) of rank 2 there is a rational matrix \(F\) of rank 2 such that \(B\) and \(F\) have the same sign pattern.

### 3 General Results

Incidence structures with properties such as that of Figure 1 were first constructed systematically by Maclane [3] using the “von Staudt algebra of throws”. Theorem 3 of that paper states:

**Theorem 1** (Maclane [3]) Let \(K\) be a finite algebraic field over the field of rational numbers. Then there exists a matroid \(M\) of rank 3 which can be represented by a matrix with elements of \(K\), while any other representation of \(M\) by a matrix of elements in a number-field \(K_1\) requires \(K_1 \supset K\).

Using this theorem along with the argument of the previous section gives us the following general result.

**Theorem 2** Let \(K\) be a subfield of \(\mathbb{R}\), finite and algebraic over \(\mathbb{Q}\). Then there exists a sign pattern matrix \(A\), such that for any field \(K_1 \subset \mathbb{R}\) with \(K \not\subset K_1\), \(\text{mr}_K(A) < \text{mr}_{K_1}(A)\).

In contrast, the situation completely changes for purely transcendental extensions.

**Theorem 3** Let \(F\) be a subfield of \(\mathbb{R}\), and let \(\alpha \in \mathbb{R}\) be transcendental over \(F\). Then for any sign pattern matrix \(A\), \(\text{mr}_{F(\alpha)}(A) = \text{mr}_{F}(A)\).

**Proof** It is clear that for any sign pattern matrix \(A\), \(\text{mr}_{F(\alpha)}(A) \leq \text{mr}_{F}(A)\). To prove the reverse inequality, it suffices to show that for any matrix \(M\) with entries in \(F(\alpha)\), there exists a matrix \(M^*\) with entries in \(F\) such that \(\text{rank}(M^*) \leq \text{rank}(M)\) and \(\text{sgn}(M^*) = \text{sgn}(M)\).
Let $M$ be an $m \times n$ matrix with entries in $\mathbb{F}(\alpha)$. By multiplying $M$ by a suitable element of $\mathbb{F}[\alpha]$, it suffices to prove the theorem when $M$ has entries in $\mathbb{F}[\alpha]$ (which is isomorphic to a polynomial ring, since $\alpha$ is transcendental over $\mathbb{F}$). For each $i \in [m], j \in [n]$, let $M_{ij} = P_{ij}(\alpha)$, where $P_{ij}$ is a polynomial with coefficients in $\mathbb{F}$. As $\alpha$ is transcendental, $P_{ij}(\alpha) = 0$ if and only if $P_{ij}$ is the zero polynomial. Thus we may pick $\beta \in \mathbb{F}$ sufficiently close to $\alpha$, so that for each $i, j$, $P_{ij}(\beta)$ has the same sign as $P_{ij}(\alpha)$. Now let $g : \mathbb{F}[\alpha] \to \mathbb{F}$ be the substitution homomorphism (of rings) with $g(\alpha) = \beta$. Define $M^*$ to be the matrix whose $(i, j)$ entry is $g(M_{ij}) = P_{ij}(\beta)$.

By construction, $\text{sgn}(M^*) = \text{sgn}(M)$. Let $r = \text{rank}(M)$. Consider any $S \subseteq [m], T \subseteq [n]$ with $|S| = |T| = r + 1$. We know that the $S \times T$ minor of $M$ vanishes. Thus the determinant $|\{(M_{ij})_{i\in S, j\in T}\}| = 0$. The corresponding minor of $M^*$ equals the determinant $|\{(g(M_{ij}))_{i\in S, j\in T}\}|$ which, using the fact that $g$ is a homomorphism, equals the determinant $g(|\{(M_{ij})_{i\in S, j\in T}\}|) = g(0) = 0$. Thus we have shown that any $(r + 1) \times (r + 1)$ minor of $M^*$ also vanishes, which gives us the result. ■

References

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[2] Branko Grunbaum, Convex Polytopes *Graduate Texts in Mathematics*, Springer(1967).

[3] Saunders Maclane, Some Interpretations of Abstract Linear Dependence in Terms of Projective Geometry, *American Journal of Mathematics*, Vol. 58, No. 1. (Jan., 1936), pp. 236-240.
For the configuration mentioned, as above, the nine points are A, B, C, D, E, F, G, H and I, and the nine lines are ABEF, ADG, AHI, BCH, BGI, CEG, CFI, DEI and DFH drawn on the plane, starting with the regular pentagon for which G, E, F, H are four of the vertices.

Figure 1: