Tractability of Multivariate Problems for Standard and Linear Information in the Worst Case Setting: Part I

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Abstract

We present a lower error bound for approximating linear multivariate operators defined over Hilbert spaces in terms of the error bounds for appropriately constructed linear functionals as long as algorithms use function values. Furthermore, some of these linear functionals have the same norm as the linear operators. We then apply this error bound for linear (unweighted) tensor products. In this way we use negative tractability results known for linear functionals to conclude the same negative results for linear operators. In particular, we prove that $L^2$-multivariate approximation defined for standard Sobolev space suffers the curse of dimensionality if function values are used although the curse is not present if linear functionals are allowed.

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1 Introduction

The understanding of the intrinsic difficulty of approximation of $d$-variate problems is a challenging problem especially when $d$ is large. We consider algorithms that approximate $d$-variate problems and use finitely many linear functionals: we compare the class $\Lambda^\text{all}$ of arbitrary linear information functionals with the class $\Lambda^\text{std}$ of information functionals that are given by function evaluations at single points.

To find best algorithms for the class $\Lambda^\text{all}$ is usually much easier than for the class $\Lambda^\text{std}$, in particular if the source space is a Hilbert space. This is especially the case for the worst case setting. The state of art may be found in [9], where the reader may find a number of surprising results. For example, there are multivariate problems for which the best rate of convergence of algorithms using $n$ appropriately chosen linear functionals is $n^{-1/2}$ whereas for $n$ function values the best rate can be arbitrarily bad, i.e., like $1/\ln(\ln(\cdots \ln(n)))$, where the number of $\ln$ can be arbitrarily large, see [4] which is also reported in [9] pp. 292-304. Furthermore, the dependence on $d$ may be quite different for the linear and standard classes. There are examples of interesting multivariate problems for which the dependence on $d$ is not exponential for the class $\Lambda^\text{all}$, and is exponential for the class $\Lambda^\text{std}$. The exponential dependence on $d$ is called the curse of dimensionality. On the other hand, for some other multivariate problems there is no difference between $\Lambda^\text{all}$ and $\Lambda^\text{std}$. Examples can be found, in particular, in [2] and [6, 7, 9].

Tractability deals with how the intrinsic difficulty of a multivariate problem depends on $d$ and on $\varepsilon^{-1}$, where $\varepsilon$ is an error threshold. We would like to know when the curse of dimensionality holds and when we have a specific dependence on $d$ which is not exponential. There are various ways of measuring the lack of exponential dependence and that leads to different notions of tractability. In particular, we have polynomial tractability (PT) if the intrinsic difficulty is polynomial in both $d$ and $\varepsilon^{-1}$. We have quasi-polynomial tractability (QPT) if the intrinsic difficulty is at most proportional to $\varepsilon^{-t \ln d}$ for some $t$ independent of $d$ and $\varepsilon$.

Obviously, tractability may depend on which of the classes $\Lambda^\text{std}$ or $\Lambda^\text{all}$ is used. Tractability results for $\Lambda^\text{std}$ cannot be better than for $\Lambda^\text{all}$. The main question is for which multivariate problems they are more or less the same or for which multivariate problems they are essentially different.

These questions were already addressed in [6, 7, 9]. Still, especially the worst case setting is not fully understood. We would like to get a better understanding how the power of the standard class $\Lambda^\text{std}$ is related to the power of the class $\Lambda^\text{all}$ of information. Ideally, we would like to characterize for which multivariate problems the classes $\Lambda^\text{std}$ and $\Lambda^\text{all}$ lead to more or less the same tractability results and for which tractability results are essentially different.

We plan to write a number of papers about this problem under the same title. We present
the first part of this project. We restrict ourselves to linear multivariate problems defined as approximation of a linear continuous operator $S : F \rightarrow G$ for general Hilbert spaces $F$ and $G$. Since we want to study the class $\Lambda^{std}$ we need to assume that function values are well defined and they correspond to linear continuous functionals. This is equivalent to assuming that $F$ is a reproducing kernel Hilbert space.

For the worst case setting and for the class $\Lambda^{all}$, it is known what is the best way to approximate $S$. The intrinsic difficulty of approximating $S$ is defined as the information complexity which is the minimal number of linear functionals which are needed to find an algorithm whose worst case error is at most $\varepsilon\|S\|$. This depends on the eigenvalues of the operator $S^*S : F \rightarrow F$. For the class $\Lambda^{std}$ the situation is much more complex and the information complexity, which is now the minimal number of function values needed to get an error $\varepsilon\|S\|$, depends not only on the eigenvalues of $S^*S$.

Our first result is the construction of continuous linear functionals $I$ which are at most as hard to approximate as $S$ for the class $\Lambda^{std}$. Furthermore, we characterize $I$ for which $\|I\| = \|S\|$. They are of the form

$$I = \langle \cdot, S^*g \rangle_F \quad \text{with} \quad g = \lambda_1^{-1/2}S\eta,$$

where $\lambda_1$ is the largest eigenvalue of $S^*S$ and $\eta$ of norm $1$ belongs to the eigenspace corresponding to $\lambda_1$. Hence, if $\lambda_1$ is of multiplicity $1$ then the choice of $g$ is essentially unique. If $\lambda_1$ is of multiplicity larger than $1$, then the choice of $g$ is not unique and may lead to trivial or hard linear functionals $I$.

For $I$ with $\|I\| = \|S\|$, the information complexity of $I$ for the class $\Lambda^{std}$ is at most equal to the information complexity of $S$. Hence, if $I$ is hard to approximate so is $S$.

The essence of this result is that for approximation of linear functionals over some Hilbert spaces there is a proof technique which allows to find sharp error bounds. This proof technique was developed in [5] and requires that the reproducing kernel of $F_1$ has a so called decomposable part.

We verify how this lower bound on approximating $S$ works for linear $d$-folded (unweighted) tensor product problems. Then the corresponding linear functionals $I$ are also $d$-folded tensor products. We then may apply the existing negative tractability results for $I$ and conclude the same negative tractability results for $S$.

We illustrate our approach for a number of examples. In particular, we consider the Sobolev space $F_d = F_1^\otimes d$ with the reproducing kernel

$$K_d(x, t) = \prod_{j=1}^{d}(1 + \min(x_j, t_j)) \quad \text{for all} \quad x, t \in [0, 1]^d,$$

and $G_d = L_2([0, 1]^d)$. 3
Let \( S_d : F_d \to G_d \) be any non-zero tensor product operator \( S_d = S_1^d \) with \( S_1 : F_1 \to G_1 \). Let \( \{ \lambda_j \} \) be the ordered sequence of eigenvalues of \( S_1^* S_1 \). Let
\[
\text{decay}_\lambda := \sup \{ r > 0 : \lim_{n \to \infty} n^r \lambda_n = 0 \}
\]
denote the polynomial decay of the eigenvalues \( \lambda_n \). If the set of \( r \) above is empty we set \( \text{decay}_\lambda = 0 \).

Let \( \mathbb{S} = \{ S_d \}_{d=1}^{\infty} \). It is known, see [2], that \( \mathbb{S} \) is quasi-polynomially tractable (QPT) for the class \( \Lambda^{\text{all}} \) iff \( \lambda_2 < \lambda_1 \) and \( \text{decay}_\lambda > 0 \). Furthermore, if \( \lambda_2 \) is positive then \( \mathbb{S} \) is not polynomially tractable (PT). On the other hand, if \( \lambda_2 = \lambda_1 \) then \( \mathbb{S} \) suffers from the curse of dimensionality for the class \( \Lambda^{\text{all}} \) (and obviously also for \( \Lambda^{\text{std}} \)).

For the class \( \Lambda^{\text{std}} \), assume without loss of generality that \( \lambda_2 < \lambda_1 \). Let \( \eta_1 \) be a normalized eigenfunction corresponding to the largest eigenvalue \( \lambda_1 \). We prove that \( \mathbb{S} \) suffers from the curse of dimensionality if
\[
\eta_1 \neq a K_1(\cdot, t) = a (1 + \min(\cdot, t)) \quad \text{for all } a \in \mathbb{R} \text{ and } t \in [0, 1].
\]
This means that we have the curse of dimensionality as long as the eigenfunction of \( S_1 \) corresponding to the largest eigenvalue is not proportional to the univariate reproducing kernel with one argument fixed. We then verify that this assumption holds for multivariate approximation, i.e., for \( S_1 f = f \). This partially solves the open problem 131 from [9] p. 361.

We believe that the assumption (1) is also necessary for the curse. More generally, we believe that for \( \eta_1 = a K_1(\cdot, t) \) for some real \( a \) and \( t \) from the common domain of univariate functions, and of course for \( \lambda_2 = \lambda_1 \) and \( \text{decay}_\lambda > 0 \), we have QPT for the class \( \Lambda^{\text{std}} \) and this holds for any \( K_1 \). But this will be the subject of the next part of our project.

In this paper we discuss only unweighted tensor products and that is why we do not have polynomial tractability (PT) for problems with two positive eigenvalues. PT and other notions of tractability may hold if we consider weighted tensor products with sufficiently decaying weights. This will be also a subject of our next study.

## 2 Relation between Linear Functionals and Operators

Consider a continuous linear and non-zero operator \( S : F \to G \), where \( F \) is a reproducing kernel Hilbert space of real functions \( f \) defined over a common domain \( D \subset \mathbb{R}^d \) for some positive integer \( d \), and \( G \) is a Hilbert space. We approximate \( S \) by algorithms \( A_n \) that use at most \( n \) linear functionals. Without loss of generality we may assume that \( A_n \) is linear, see e.g., [6, 12]. That is,
\[
A_n f = \sum_{j=1}^{n} L_j(f) g_j
\]
for some \( L_j \in F^* \) and \( g_j \in G \). Using the same proof as in [6] p. 345, we may also assume that \( g_j = Sf_j \) for some \( f_j \in F \).

We consider two classes of linear functionals \( L_j \)’s:

- the linear class of information \( \Lambda^{\text{all}} \) which consists of all continuous and linear functionals \( L_j \)’s, i.e., \( \Lambda^{\text{all}} = F^* \), and
- the standard class of information \( \Lambda^{\text{std}} \) which consists of function values, i.e., \( L_j(f) = \langle f, K(\cdot, t_j) \rangle_F = f(t_j) \) for some \( t_j \in D \), where \( K : D \times D \to \mathbb{R} \) is the reproducing kernel of \( F \).

The \( n \)th minimal (worst case) error of approximating \( S \) for the class \( \Lambda \in \{ \Lambda^{\text{std}}, \Lambda^{\text{all}} \} \) is defined as

\[
e_n(S, \Lambda) = \inf_{L_1, \ldots, L_n \in \Lambda, g_1, \ldots, g_n \in G} \sup_{\|f\|_F \leq 1} \|Sf - A_n f\|_G.
\]

For \( n = 0 \), we take \( A_n = 0 \) and then we obtain the initial error which is independent of \( \Lambda \) and given by

\[
e_0(S, \Lambda) = e_0(S) = \|S\|_{F \to G}.
\]

For the class \( \Lambda^{\text{all}} \), it is well known that \( \lim_{n \to \infty} e_n(S, \Lambda^{\text{all}}) = 0 \) iff \( S \) is compact. This is why we always may assume that \( S \) is compact. Then it is known that the \( n \)th minimal errors depend on the eigenvalues of \( W = S^* S : F \to F \).

More precisely, let \((\lambda_j, \eta_j)\) be eigenpairs of \( W \),

\[
W\eta_j = \lambda_j \eta_j, \quad \text{with} \quad \langle \eta_j, \eta_k \rangle_F = \delta_{j,k} \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \cdots.
\]

Observe here that the \( \lambda_j \) are uniquely defined, but the \( \eta_j \) are not unique. Moreover, we formally define \( \lambda_j = 0 \) if the dimension of \( F \) is finite and \( j \) is larger than this dimension. Then

\[
e_n(S, \Lambda^{\text{all}}) = \sqrt{\lambda_{n+1}} \quad \text{for} \ n = 0, 1, \ldots.
\]

Hence \( e_0(S) = \|S\|_{F \to G} = \sqrt{\lambda_1} \), and since \( S \) is non-zero we have \( \lambda_1 > 0 \).

The situation is much more complicated for the class \( \Lambda^{\text{std}} \). Obviously,

\[
e_n(S, \Lambda^{\text{std}}) \geq e_n(S, \Lambda^{\text{all}}) \quad \text{for all} \ n = 0, 1, \ldots,
\]

but it is not clear when the sequences \( e_n(S, \Lambda^{\text{std}}) \) and \( e_n(S, \Lambda^{\text{all}}) \) behave similarly. There are many papers studying the powers of \( \Lambda^{\text{std}} \) and \( \Lambda^{\text{all}} \). The state of art can be found in [9].
In this paper we continue this study and show that the sequence $e_n(S, \Lambda^{\text{std}})$ can behave quite differently than the sequence $e_n(S, \Lambda^{\text{all}})$. This will be done by showing first that many continuous and linear functionals are at most as hard to approximate as $S$ and there are functionals for which we can also match the initial error $e_0(S)$ of $S$.

More precisely, for any $g \in G$ with $\|g\|_G = 1$ define

$$I_g f = \langle f, S^* g \rangle_F$$

for all $f \in F$.

Note that $S^* g \in F$ and therefore $I_g$ is a continuous linear functional. The $n$th minimal error of approximating $I_g$ is defined as for $S$, this time with $G$ replaced by $\mathbb{R}$. Clearly, $e_0(I_g) = \|S^* g\|_F$.

**Theorem 1.**

For any $g \in G$ with $\|g\|_G = 1$ we have

$$e_n(I_g, \Lambda^{\text{std}}) \leq e_n(S, \Lambda^{\text{std}})$$

for all $n \in \mathbb{N}_0$.

Furthermore,

$$e_0(I_g) = e_0(S) \quad \text{and} \quad \|g\|_G = 1 \quad \text{iff} \quad g = \lambda_1^{-1/2} S \eta,$$

where $\eta$ is any element of norm 1 with $S^* S \eta = \lambda_1 \eta$.

**Proof.** Take an arbitrary linear algorithm $A_n f = \sum_{j=1}^n f(t_j) S f_j$ for approximating $S$. Define

$$B_n f = \sum_{j=1}^n f(t_j) \langle f_j, S^* g \rangle_F$$

as a linear algorithm for approximating $I_g$. Then

$$I_g f - B_n f = \left\langle f - \sum_{j=1}^n f(t_j) f_j, S^* g \right\rangle_F = \left\langle S f - \sum_{j=1}^n f(t_j) S f_j, g \right\rangle_G = \langle S f - A_n f, g \rangle_G.$$

Therefore

$$|I_g f - B_n f| \leq \|S f - A_n f\|_G \|g\|_G = \|S f - A_n f\|_G.$$

Taking the supremum over the unit ball of $F$ and then the infimum over $t_j$’s and $f_j$’s, we conclude that

$$e_n(I_g, \Lambda^{\text{std}}) \leq e_n(S, \Lambda^{\text{std}}),$$

as claimed.
Let $m$ be the multiplicity of the largest eigenvalue $\lambda_1$, i.e., span($\eta_1, \eta_2, \ldots, \eta_m$) is the eigenspace of $W$ for the eigenvalue $\lambda_1$. Take now $g = \lambda_1^{-1/2} S \eta$ for any $\eta \in \text{span}(\eta_1, \eta_2, \ldots, \eta_m)$ with $\|\eta\|_F = 1$. Then we have $S^* g = \lambda_1^{-1/2} W \eta = \lambda_1^{1/2} \eta$ and

$$e_0(I_g) = \|S^* g\|_F = \sqrt{\lambda_1} = \|S\|_{F \to G} = e_0(S).$$

Furthermore,

$$\|g\|_G^2 = \left\langle \lambda_1^{-1/2} S \eta, \lambda_1^{-1/2} S \eta \right\rangle_G = \lambda_1^{-1} \left\langle \eta, W \eta \right\rangle_F = \lambda_1^{-1} \lambda_1 \|\eta\|^2_F = 1.$$

We need to show that $e_0(I_g) = e_0(S)$ holds only for such $g$. Take then any $g$ from $G$ such that $\|g\|_G = 1$ and $\|S^* g\|_F = \lambda_1^{1/2}$. We can represent

$$g = \alpha S \eta + h,$$

where $\eta \in \text{span}(\eta_1, \eta_2, \ldots, \eta_m)$ with $\|\eta\|_F = 1$, $\alpha \in \mathbb{R}$ and $h$ is orthogonal to $S \eta_j$ for all $j = 1, 2, \ldots, m$, i.e.,

$$0 = \left\langle h, S \eta_j \right\rangle_G = \left\langle S^* h, \eta_j \right\rangle_F \quad \text{for} \quad j = 1, 2, \ldots, m.$$ 

Since $\|S \eta\|_G = \lambda_1^{1/2}$ we have

$$1 = \|g\|^2_G = \alpha^2 \lambda_1 + \|h\|^2_G.$$

On the other hand,

$$1 = \frac{1}{\lambda_1} \|S^* g\|^2_F = \frac{1}{\lambda_1} \|\alpha S^* S \eta + S^* h\|^2_F = \frac{1}{\lambda_1} \|\alpha \lambda_1 \eta + S^* h\|^2_F = \frac{1}{\lambda_1} \left(\alpha^2 \lambda_1^2 + \|S^* h\|^2_F\right) = \alpha^2 \lambda_1 + \frac{1}{\lambda_1} \|S^* h\|^2_F.$$

We now analyze $\|S^* h\|_F$. Note that $\|S^* h\|^2_F = \left\langle h, SS^* h \right\rangle_G$. Let

$$G = \overline{SS^*(G)} \oplus [SS^*(G)]^\perp.$$

Hence, for any $h$ from $G$ we have $h = h_1 + h_2$ with $h_1 \in \overline{SS^*(G)}$ and $h_2$ orthogonal to $SS^*(G)$. Then

$$\|S^* h\|^2_F = \left\langle h_1 + h_2, SS^* h_1 \right\rangle_G = \left\langle h_1, SS^* h_1 \right\rangle_G.$$ 

(2)

Let $k = \sup\{j : \lambda_j > 0\}$. Hence, if all $\lambda_j > 0$ then $k = \infty$, otherwise $k$ is the number of positive eigenvalues $\lambda_j$. Clearly, $k \geq m$. 7
We know that \( S^* S \eta_j = \lambda_j \eta_j \). Then \( S \eta_j \neq 0 \) for all finite \( j \) which are at most \( k \). Multiplying the last equation by \( S \) we obtain

\[
SS^*(S \eta_j) = \lambda_j (S \eta_j).
\]

Hence, \((\lambda_j, \lambda_j^{-1/2} S \eta_j)\) is an eigenpair of \(SS^*\) and

\[
\langle \lambda_j^{-1/2} S \eta_j, \lambda_i^{-1/2} S \eta_i \rangle_G = \frac{\langle S \eta_j, S \eta_i \rangle_G}{\sqrt{\lambda_j \lambda_i}} = \frac{\langle \eta_j, S^* S \eta_i \rangle_F}{\sqrt{\lambda_j \lambda_i}} = \delta_{i,j}.
\]

That means that \( \lambda_j^{-1/2} S \eta_j \)'s are orthonormal in \( G \). Since \( SS^*(G) \subset S(F) \), the \( \lambda_j^{-1/2} S \eta_j \)'s build a complete orthonormal system of \( SS^*(G) \) and, when we return to (2), we may write

\[
h_1 = \sum_{j=1}^k \langle h_1, \lambda_j^{-1/2} S \eta_j \rangle_G \lambda_j^{-1/2} S \eta_j
\]

and then

\[
\langle h_1, SS^* h_1 \rangle_G = \sum_{j=1}^k \langle h_1, \lambda_j^{-1/2} S \eta_j \rangle_G^2 \lambda_j.
\]

Since \( 0 = \langle h, S \eta_j \rangle_G = \langle h_1, S \eta_j \rangle_G \) for all \( j \leq m \), we conclude that

\[
\langle h_1, SS^* h_1 \rangle_G = \sum_{j=m+1}^k \langle h_1, \lambda_j^{-1/2} S \eta_j \rangle_G^2 \lambda_j \leq \lambda_{m+1} \sum_{j=m+1}^k \langle h_1, \lambda_j^{-1/2} S \eta_j \rangle_G^2 \leq \lambda_{m+1}^2 \langle h \rangle_G^2.
\]

From this, we get

\[
1 = \alpha^2 \lambda_1 + \frac{1}{\lambda_1} \| S^* h \|_G^2 \leq \alpha^2 \lambda_1 + \frac{\lambda_{m+1}}{\lambda_1} \| h \|_G^2 = \alpha^2 \lambda_1 + \| h \|_G^2 - \left( 1 - \frac{\lambda_{m+1}}{\lambda_1} \right) \| h \|_G^2
\]

\[
= 1 - \left( 1 - \frac{\lambda_{m+1}}{\lambda_1} \right) \| h \|_G^2.
\]

Since \( \lambda_{m+1} < \lambda_1 \) we conclude that \( h = 0 \) and \( g = \alpha S \eta \) with \( \alpha^2 \lambda_1 = 1 \). This completes the proof.

For any \( g \) from \( G \) the linear functional \( I_g \) can be also written as

\[
I_g f = \langle S f, g \rangle_G \quad \text{for all } f \in F.
\]

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**Example 2.** As an example, if we take $G = L_2([0,1]^d)$ then

$$I_g f = \int_{[0,1]^d} (Sf)(x) g(x) \, dx.$$ 

Furthermore, if we additionally assume that $F$ is continuously embedded in $L_2([0,1]^d)$ and take $S = \text{APP}_d$ as multivariate approximation, $\text{APP}_d f = f$ for all $f \in F$, then

$$I_g f = \int_{[0,1]^d} f(x) g(x) \, dx.$$ 

If $g \equiv 1$ then

$$\text{INT}_d f = I_g f = \int_{[0,1]^d} f(x) \, dx$$

is multivariate integration, and

$$e_0(\text{INT}_d) \leq e_0(\text{APP}_d).$$

This relation between multivariate integration and approximation has been used in many papers. For some spaces the norm of multivariate integration and approximation is the same. This is the case for Korobov spaces and some Sobolev spaces as will be reported later.

However, in general, the norm of multivariate integration is smaller and sometimes exponentially smaller than the norm of multivariate approximation. This is the case for some other Sobolev spaces. For instance, this holds for the space $F = F_d$ with the reproducing kernel

$$K_d(x, y) = \prod_{j=1}^d (1 + \min(x_j, y_j)) \quad \text{for all} \quad x_j, y_j \in [0,1].$$

It is known, see [9] pp. 353 and 411, that

$$e_0(\text{INT}_d) = (4/3)^{d/2} = (1.3333\ldots)^{d/2} \quad \text{and} \quad e_0(\text{APP}_d) = (1.35103388\ldots)^{d/2}.$$ 

Hence,

$$\frac{e_0(\text{APP}_d)}{e_0(\text{INT}_d)} = (1.013\ldots)^{d/2}.$$ 

Although 1.013\ldots is barely larger than one, the ratio of the initial errors for multivariate approximation and integration goes to infinity exponentially fast with $d$.

As we shall see, the multiplicity of the largest eigenvalue for this multivariate approximation is $m = 1$. Therefore, in order to match the norm of multivariate approximation we must
use a weighted integration problem $I_g(f) = (f, g)_{L_2}$ with $g = \lambda_1^{-1/2} \eta_1$ (or $g = -\lambda_1^{-1/2} \eta_1$) which for our example of $F_d$ is not equal to the constant function 1. In Section 5 we will show that $g(x) = \prod_{j=1}^d g_1(x_j)$ for $x = [x_1, \ldots, x_d] \in [0, 1]^d$. We find it interesting to know the “most difficult” integration problem $I_g$ (with $\|g\|_2 = 1$) for a Hilbert space of functions and hence present the graph of the function $g_1$ in Figure 1. The same $g$ is also the unique (up to a multiplicative constant) function that maximizes $\|g\|_2$ and hence solves an important optimization problem.

![Figure 1: Density $g$ for which $\|I_g\| = \|\text{APP}_1\|$](image)

We now show that the choice of $\eta$ in $g = \lambda_1^{-1/2} \eta$ may be important if the multiplicity of $\lambda_1$ is larger than 1. That is, it may happen that for some such $g$ the functional $I_g$ is trivial and for some other $g$, it may be very difficult.

**Example 3.**

Let $F_1$ be the space of functions $f : [0, 1] \to \mathbb{R}$ that are constant over $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$. That is, for $f \in F_1$ there are real $a$ and $b$ such that $f(x) = a$ for all $x \in [0, \frac{1}{2}]$ and $f(x) = b$ for all $x \in (\frac{1}{2}, 1]$.

We equip $F_1$ with the $L_2$ norm which can be written (for the space $F_1$) as

$$\|f\|_{F_1} = \frac{1}{\sqrt{2}} \left( f^2(0) + f^2(1) \right)^{1/2}.$$
We define $F = G = F^\otimes d_1$ as the $d$ folded tensor product of the space $F_1$. The space $F$ consists of piecewise constant functions over $2^d$ subintervals of volume $2^{-d}$ which are a partition of the cube $[0,1]^d$. The space $F_d$ is also equipped with the $L_2$ norm. Clearly, $\dim(F) = 2^d$.

Let $S : F \to G$ be the identity operator. Then $\lambda_j = 1$ for all $j = 1, 2, \ldots, 2^d$ and any nonzero function from $F$ is an eigenfunction of $W = S^*S = I$. Clearly, $\lambda_{2^d+1} = 0$ and therefore $e_n(S, \Lambda^{\text{all}}) = 1$ for all $n < 2^d$ and $e_{2^d}(S, \Lambda^{\text{all}}) = 0$.

Obviously, it also proves that $e_n(S, \Lambda^{\text{std}}) = 1$ for $n < 2^d$ since $1 = e_n(S, \Lambda^{\text{all}}) \leq e_n(S, \Lambda^{\text{std}}) \leq e_0(S) = 1$.

For any $g$ of norm 1 we have $e_0(I_g) = e_0(S) = 1$. We now show that $e_n(I_g, \Lambda^{\text{std}})$ very much depends on the choice of $g$.

Suppose we take $g = 2^d/2$ over $[0, \frac{1}{2}]^d$ and $g = 0$ otherwise. Then

$$I_g(f) = 2^{d/2} \int_{[0,\frac{1}{2}]^d} f(x) \, dx = 2^{-d/2} f(0)$$

is a trivial linear functional which can be solved exactly by using one function value at 0. Hence, $e_n(I_g, \Lambda^{\text{std}}) = 0$ for all $n \geq 1$.

In this case, the bound $0 = e_n(I_g, \Lambda^{\text{std}}) \leq e_n(S, \Lambda^{\text{std}}) = 1$ is useless.

Take now $g = 1$ over the cube $[0,1]^d$. Then

$$I_g(f) = \int_{[0,1]^d} f(x) \, dx = \frac{1}{2^d} \sum_{j = [j_1, j_2, \ldots, j_d] \in \{0,1\}^d} f(t_{j_1}, t_{j_2}, \ldots, t_{j_d}),$$

where $t_0 = 0$ and $t_1 = 1$. We prove that the $n$th minimal error for $n \leq 2^d$ is

$$e_n(I_d, \Lambda^{\text{std}}) = (1 - n 2^{-d})^{1/2}.$$
All linear algorithms must approximate $I_g(f)$ by zero and therefore their worst case error is at least $(1 - n 2^{-d})^{1/2}$. The last bound is sharp if we take sample points $x_1, x_2, \ldots, x_n$ at disjoint sub-cubes, as claimed.

Hence, in this case we have

$$(1 - n 2^{-d})^{1/2} = e_n(I_g, \Lambda_{\text{std}}) \leq e_n(S, \Lambda_{\text{std}}) = 1.$$  

The bound is quite sharp as long as $n$ is much smaller than $2^d$. 

For general spaces, we will use Theorem 1 for $I_g$ with $g = \lambda^{-1/2} S h_1$. For the standard class of information $\Lambda_{\text{std}}$, using lower bounds results for $I_g$ from [7], we obtain lower bounds results for $S$. In this way we show, in particular, that we have sometimes the curse of dimensionality for $\Lambda_{\text{std}}$ which is not present for the class $\Lambda_{\text{all}}$.

### 3 Tractability Notions

We need to recall the definition of the information complexity for the so-called normalized error criterion. It is defined as the minimal number of linear functionals from the class $\Lambda$ which are needed to reduce the initial error by a factor $\varepsilon \in (0, 1)$, where $\Lambda \in \{ \Lambda_{\text{std}}, \Lambda_{\text{all}} \}$. That is,

$$n(\varepsilon, S, \Lambda) = \min\{ n : e_n(S, \Lambda) \leq \varepsilon e_0(S) \}.$$  

For the class $\Lambda_{\text{all}}$, we obviously have

$$n(\varepsilon, S, \Lambda_{\text{all}}) = \min\{ n : \lambda_{n+1} \leq \varepsilon^2 \lambda_1 \}.$$  

Unfortunately, there is no such or similar formula for the class $\Lambda_{\text{std}}$.

Assume now that we have a sequence

$$S = \{ S_d \}_{d=1}^{\infty}$$  

of continuous linear non-zero operators $S_d : F_d \to G_d$, where $F_d$ is a reproducing kernel Hilbert space of real function defined over $D_d \subset \mathbb{R}^d$ and $G_d$ is a Hilbert space. In this case, we want to verify how the information complexity $n(\varepsilon, S_d, \Lambda)$ depends on $\varepsilon^{-1}$ and $d$.

In this paper we will use only a few tractability notions which are defined as follows. We say that

1. **Efficiency:**
   - If $n(\varepsilon, S_d, \Lambda)$ is bounded above by a polynomial in $d$, then $S$ is efficient for $\varepsilon$.

2. **Tractability:**
   - If $n(\varepsilon, S_d, \Lambda)$ is bounded above by a polynomial in $\varepsilon^{-1}$, then $S$ is tractable for $\varepsilon$.

3. **Curse of dimensionality:**
   - If $n(\varepsilon, S_d, \Lambda)$ grows exponentially in $d$ or $\varepsilon^{-1}$, then $S$ exhibits the curse of dimensionality.

4. **Exponential growth:**
   - If $n(\varepsilon, S_d, \Lambda)$ grows exponentially in $d$, then $S$ exhibits exponential growth.

In this paper, we will focus on the tractability notion and apply it to various settings, including the curse of dimensionality.
• $S$ suffers from the curse of dimensionality for the class $\Lambda \in \{\Lambda^{std}, \Lambda^{all}\}$ iff there are positive numbers $c$ and $C$ as well as $\varepsilon_0 \in (0, 1)$ such that

$$n(\varepsilon, S_d, \Lambda) \geq C (1 + c)^d$$

for all $\varepsilon \in (0, \varepsilon_0]$ and infinitely many $d$.

• $S$ is quasi-polynomially tractable (QPT) for the class $\Lambda$ iff there are non-negative numbers $C$ and $t$ such that

$$n(\varepsilon, S_d, \Lambda) \leq C \exp\left( t (1 + \ln \varepsilon^{-1})(1 + \ln d) \right) \text{ for all } \varepsilon \in (0, 1), d = 1, 2, \ldots.$$ 

The infimum of numbers $t$ satisfying the bound above is called the exponent of QPT and denoted by $t^*$.

• $S$ is polynomially tractable (PT) for the class $\Lambda$ iff there are non-negative numbers $C, p, q$ such that

$$n(\varepsilon, S_d, \Lambda) \leq C \varepsilon^{-p} d^q \text{ for all } \varepsilon \in (0, 1), d = 1, 2, \ldots.$$ 

Clearly, PT implies QPT. More about these and other tractability concepts can be found in [6, 7, 9].

For the class $\Lambda^{all}$, tractability notions depend on the decay of the eigenvalues $\lambda_{d,j}$ of the operator $W_d = S_d^* S_d : F_d \to F_d$. Necessary and sufficient conditions can be found in the works cited above. Again for the class $\Lambda^{std}$, no such conditions are known and they cannot depend only on the eigenvalues $\lambda_{d,j}$.

### 4 Linear Tensor Product Problems

From now on we study a sequence

$$\mathbb{S} = \{S_d\}_{d=1}^{\infty}$$

de a sequence of tensor product problems. Hence the spaces $F = F_d = F_1^{\otimes d}$ and $G = G_d = G_1^{\otimes d}$ as well as $S = S_d = S_1^{\otimes d}$ are given by tensor products of $d$ copies of $F_1$ and $G_1$ as well as a continuous linear operator $S_1 : F_1 \to G_1$, respectively, where $F_1$ is a reproducing kernel Hilbert space of real univariate functions defined over $D_1 \subset \mathbb{R}$ and $G_1$ is a Hilbert space. Then $F_d$ is a space of $d$-variate real functions defined on $D_d = D_1 \times D_1 \times \cdots \times D_1$ ($d$ times).

An important example is given by multivariate approximation. That is, we now take $G_d = L_2([0, 1]^d)$ and $S_1 = \text{APP}_1 : F_1 \to G_1$ with the embedding operator $\text{APP}_1 f = f$. Then
$S_d = \text{APP}_d : F_d \to G_d$ is also the embedding operator $\text{APP}_df = f$ for all $f \in F_d$. In this case, we denote

$$S = \text{APP}, \quad \text{where} \quad \text{APP} = \{\text{APP}_d\}_{d=1}^\infty.$$  

If $K_1$ is the reproducing kernel of $F_1$ then $F_d$ is a reproducing kernel Hilbert space whose kernel is

$$K_d(x, y) = \prod_{j=1}^{d} K_1(x_j, y_j) \quad \text{for all} \quad x = [x_1, x_2, \ldots, x_d], \quad y = [y_1, y_2, \ldots, y_d] \in D_d.$$

For the class $\Lambda_{\text{all}}$, it is well known that the eigenpairs $(\lambda_{d,j}, \eta_{d,j})$ of $W_d = S_d^*S_d : F_d \to F_d$ are given in terms of the eigenpairs $(\lambda_j, \eta_j)$ of the univariate operator $W_1 = S_1^*S_1 : F_1 \to F_1$. As before we assume that $\lambda_1 \geq \lambda_2 \geq \cdots$, $\langle \eta_j, \eta_k \rangle_{F_1} = \delta_{j,k}$ and that $S_1$ is non-zero. This means that $\lambda_1 > 0$. For the operator $W_d$ we have

$$\{\lambda_{d,j}\}_{j=1}^\infty = \{\lambda_1, \lambda_2, \ldots, \lambda_{d,j_{1,j_2,\ldots,j_d=1}}\}^\infty_{j=1}.$$

Similarly, the eigenfunctions of $W_d$ are of product form

$$\{\eta_{d,j}\}_{j=1}^\infty = \{\eta_{j_1} \otimes \eta_{j_2} \otimes \cdots \otimes \eta_{j_d}\}_{j_1,j_2,\ldots,j_d=1}^\infty,$$

where

$$[\eta_{j_1} \otimes \eta_{j_2} \otimes \cdots \otimes \eta_{j_d}](x) = \prod_{k=1}^{d} \eta_{jk}(x_k) \quad \text{for all} \quad x = [x_1, \ldots, x_d] \in D_d.$$

Then $\|S_d\|_{F_d \to G_d} = \|W_d\|_{F_d \to F_d}^{1/2} = \lambda_1^{d/2}$. Hence, the initial error is

$$e_0(S_d) = \lambda_1^{d/2}.$$

If $\lambda_2 = \lambda_1$ then we have at least $2^d$ eigenvalues of $W_d$ equal to $\lambda_1^d$, and therefore

$$n(\varepsilon, S_d) \geq 2^d \quad \text{for all} \quad \varepsilon \in (0, 1) \quad \text{and} \quad d = 1, 2, \ldots.$$  

In this case $S = \{S_d\}_{d=1}^\infty$ suffers from the curse of dimensionality. On the other hand, it is proved in [2], see also [9] p. 112, that $S$ is QPT for the class $\Lambda_{\text{all}}$ iff $\lambda_2 < \lambda_1$ and

$$\text{decay}_\lambda := \sup\{ r > 0 : \lim_{n \to \infty} n^r \lambda_n = 0 \} > 0.$$  

(3)

If the last conditions hold then the exponent of QPT is

$$t^* = \max \left( \frac{2}{\text{decay}_\lambda}, \frac{2}{\ln \frac{\lambda_1}{\lambda_2}} \right).$$  

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Note that for $\lambda_2 = 0$ we have $t^* = 0$. In this case, $S_d$ is a continuous linear functional and $n(\varepsilon, S_d, \Lambda_{\text{all}}) = 1$ for all $\varepsilon \in [0, 1)$ and all $d = 1, 2, \ldots$. If $\lambda_2 > 0$ then the exponent of QPT is positive and in this case it is also known that the problem $S$ is not PT for the class $\Lambda_{\text{all}}$.

We now turn to the class $\Lambda_{\text{std}}$. Without loss of generality we assume that $\lambda_2 < \lambda_1$ since otherwise $S$ suffers from the curse of dimensionality also for the class $\Lambda_{\text{std}}$. Then the choice of the element $g$ for which $e_0(I_g) = e_0(S)$ in Theorem 1 is essentially unique and we take $g = g_d$ with $g_d = \lambda_{d,1}^{-1/2}S_d\eta_{d,1}$. We have $S_d^*g_d = \lambda_{d,1}^{1/2}\eta_{d,1}$ which is a tensor product since

$$S_d^*g_d = (\lambda_{1}^{1/2}\eta_1) \otimes \cdots \otimes (\lambda_1^{1/2}\eta_1) \ (d \text{ times}).$$

Let

$$I_1 f = \left<f, \lambda_{1}^{1/2}\eta_1\right>_{F_1} \quad \text{for all} \quad f \in F_1.$$

Then

$$I_d = I_1 \otimes \cdots \otimes I_1 \ (d \text{ times})$$

is a linear tensor product functional. We have $I_d = I_{g_d}$ and $e_0(I_d) = e_0(S_d)$. Let

$$\mathbb{I} = \{I_d\}_{d=1}^\infty.$$

Theorem 1 yields that

$$n(\varepsilon, I_d, \Lambda_{\text{std}}) \leq n(\varepsilon, S_d, \Lambda_{\text{std}}) \quad \text{for all} \quad \varepsilon \in (0, 1), \ d = 1, 2, \ldots.$$

This implies the following corollary

**Corollary 4.**

If one of the tractability notions does not hold for $\mathbb{I}$ then it also does not hold for $S$. □

We now illustrate Corollary 4 for two examples for which $\mathbb{I}$ is multivariate integration and for which it is known that multivariate integration suffers from the curse of dimensionality.

**Example 5. Korobov Space**

As in [2, 9], let $F_1$ be a Korobov space whose reproducing kernel is

$$K_1(x, y) = 1 + 2\beta \sum_{k=1}^\infty \frac{\cos(2\pi(x - y))}{k^{2\alpha}} \quad \text{for all} \quad x, y \in [0, 1]$$

for some $\beta \in (0, 1]$ and $\alpha > \frac{1}{2}$. This corresponds to the norm

$$\|f\|_{F_1}^2 = |\hat{f}(0)|^2 + \beta^{-1} \sum_{h \in \mathbb{Z} \setminus \{0\}} |h|^{2\alpha} |\hat{f}(h)|^2;$$
for Fourier coefficients $\hat{f}(h)$ of $f$. We take $G_1 = L_2([0, 1])$ and $S_1 f = f$, hence we consider the approximation problem $APP$.

In this case we know that

$$
\lambda_1 = 1, \quad \lambda_2 = \beta \quad \text{and} \quad \eta_1 \equiv 1.
$$

Hence, for $\beta = 1$ multivariate approximation suffers from the curse of dimensionality for $\Lambda^{\text{all}}$ (and of course for $\Lambda^{\text{std}}$), and for $\beta < 1$, multivariate approximation is QPT with the exponent

$$
t^* = \max \left( \frac{1}{\alpha}, \frac{2}{\ln \beta^{-1}} \right).
$$

Note that $g_1 = \lambda_1^{-1/2} S_1 \eta_1 \equiv 1$. Therefore $g_d \equiv 1$ and

$$
I_d(f) = \int_{[0,1]^d} f(x) \, dx \quad \text{for all} \quad f \in F_d
$$

is multivariate integration. From Theorem 16.16 on p. 457 in [7] which is based on [3, 5], we know that multivariate integration suffers from the curse of dimensionality. So does multivariate approximation due to Corollary 4.

**Example 6. Sobolev Space**

We now take the Sobolev space $F_1$ of absolutely continuous functions on $[0, 1]$ whose first derivatives are square integrable with the inner product

$$
\langle f, u \rangle_{F_1} = \int_0^1 f(x) u(x) \, dx + \int_0^1 f'(x) u'(x) \, dx.
$$

This space has the intriguing reproducing kernel

$$
K_1(x, t) = \frac{1}{\sinh(1)} \cosh(1 - \max(x, t)) \cosh(\min(x, t)) \quad \text{for all} \quad x, t \in [0, 1],
$$

see [1]. We consider the $L_2$ approximation problem, as in Example 4. Hence we have $S_1 f = f$ and $G_1 = L_2([0, 1])$.

In this case we have for $d = 1$ that $\lambda_1 = 1$ is of multiplicity 1, and $\eta_1 \equiv 1$. The second largest eigenvalue satisfies the condition

$$
\lambda_2 = \max_{f \in F_1, \int_0^1 f(x) \, dx = 0} \frac{\int_0^1 f^2(x) \, dx}{\int_0^1 f(x)^2 \, dx + \int_0^1 [f'(x)]^2 \, dx} = \frac{1}{1 + \mu_2},
$$

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where
\[ \mu_2 = \min_{f \in F_1} \frac{\int_0^1 [f'(x)]^2 \, dx}{\int_0^1 f^2(x) \, dx}. \]

It is known, see e.g. [10], that \( \mu_2 = \pi^2 \). Hence, have
\[ \lambda_2 = \frac{1}{1 + \pi^2} = 0.091999668 \ldots. \]

It is well known that \( \lambda_j = \Theta(j^{-2}) \) so that \( \text{decay}_\lambda = 2 \). This implies that multivariate approximation for tensor products \( F_d = F_1^{\otimes d} \) and \( G_d = L_2([0, 1]^d) \) is QPT for \( \Lambda^{\text{all}} \) with the exponent
\[ t^* = 1. \]

Due to the form of \((\lambda_1, \eta_1)\), the linear functional \( I_d \) corresponds to multivariate integration. It is known that multivariate integration suffers from the curse, see [11] which is also reported in [7] pp. 605-606. Hence, multivariate approximation also suffers the curse of dimensionality for the class \( \Lambda^{\text{std}} \) due to Corollary 4. \qed

Tractability of tensor product functionals \( I \) was thoroughly studied in [5], see also Chapters 11 and 12 of [7]. In particular, for many spaces \( F_1 \) the problem \( I \) suffers from the curse of dimensionality for the class \( \Lambda^{\text{std}} \). This holds if the reproducing kernel \( K_1 \) of \( F_1 \) has a decomposable part and the univariate function \( \eta_1 \) has non-zero components with respect to the decomposable part. If this is the case then \( S \) also suffers from the curse of dimensionality for the class \( \Lambda^{\text{std}} \) although we may have QPT for the class \( \Lambda^{\text{all}} \). We will mention more specific results in the next section.

## 5 Sobolev Space

We now consider tensor product problems \( S \) defined as in the previous section for the space \( F_1 \) taken as a Sobolev space of univariate real functions defined over \([0, 1]\). More precisely, let \( F_1 \) be the space of absolutely continuous functions defined over \([0, 1]\) and whose first derivatives belong to \( L_2([0, 1]) \). The space \( F_1 \) has the reproducing kernel
\[ K_1(x, y) = 1 + \min(x, y) \quad \text{for all } x, y \in [0, 1], \tag{4} \]
and the inner product for \( f, h \in F_1 \) is
\[ \langle f, h \rangle_{F_1} = f(0) h(0) + \int_0^1 f'(x) h'(x) \, dx. \]

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For the tensor product space $F_d = F_1^\otimes d$ of $d$ copies of $F_1$, the inner product for $f, h \in F_d$ is now of the form

$$\langle f, h \rangle_{F_d} = f(0)h(0) + \sum_{\emptyset \neq u \subseteq \{1, 2, \ldots, d\}} \int_{[0,1]^{|u|}} \partial^{|u|} f(x_u, 0) \partial^{|u|} h(x_u, 0) \, dx_u,$$

where $\partial x_u = \prod_{j \in u} \partial x_j$, $dx_u = \prod_{j \in u} dx_j$ and $(x_u, 0)$ is a $d$ dimensional vector with components $x_j$ for $j \in u$ and 0 otherwise.

It was proved in [7] pp.195-200, see also [8], that for any linear non-zero tensor product functional its information complexity (for $\Lambda_{\text{std}}$) is 1 or it is exponentially large in $d$. Furthermore, the information complexity is 1 only for trivial cases when the linear tensor product functional is of the form

$$a^d f(t, t, \ldots, t) = \langle f, h_d \rangle_{F_d} \quad \text{with} \quad h_d(x) = \prod_{j=1}^{d} a \left(1 + \min(x_j, t)\right)$$

for some non-zero real $a$ and for some $t \in [0, 1]$. Applying this results for $I$ we see that as long as

$$\eta_1 \neq a(1 + \min(\cdot, t)) \quad \text{for all} \quad a \in \mathbb{R} \text{ and } t \in [0, 1] \quad (5)$$

then $I$ as well as $S$ suffer from the curse of dimensionality for the class $\Lambda_{\text{std}}$. We summarize the results from the last two sections in the following theorem.

**Theorem 7.**

Consider a linear non-zero tensor product problem $S$, as defined in this section.

- Let $\lambda_2 = \lambda_1$. Then $S$ suffers from the curse of dimensionality for $\Lambda_{\text{all}}$ (and $\Lambda_{\text{std}}$).
- Let $\lambda_2 < \lambda_1$. Then $S$ is QPT for $\Lambda_{\text{all}}$ iff (3) holds.
- Let $0 < \lambda_2 < \lambda_1$. Then $S$ is not PT for the class $\Lambda_{\text{all}}$.
- Let $\lambda_2 < \lambda_1$. If (5) holds then $S$ suffers from the curse of dimensionality for $\Lambda_{\text{std}}$.

In general, the assumption (5) used in the last part of Theorem 7 is needed. Indeed, if (5) does not hold then we may have $S_d$ as a linear tensor product functional of the form $S_d f = a^d f(t, t, \ldots, t)$ with a nonzero real $a$ and $t \in [0, 1]$. Then $S$ is trivial since $n(\varepsilon, S_d, \Lambda_{\text{std}}) = 1$ for all $\varepsilon \in [0, 1)$ and all $d$.

We now verify the assumptions (3) and (5) for multivariate approximation $\text{APP}$.
The eigenpairs \((\lambda_j, \eta_j)\) were found in [13], see also [9] pp. 409-411. We have \(\lambda_j = \alpha_j^{-2}\), where \(\alpha_j \in ((j - 1)\pi, j\pi)\) is the unique solution of the nonlinear equation
\[
\cot x = x \quad \text{for} \quad x \in ((j - 1)\pi, j\pi),
\]
and
\[
\eta_j(x) = \beta_j \cos(\alpha_j x - \alpha_j) \quad \text{for all} \quad x \in [0, 1],
\]
where
\[
\beta_j = \left(\cos^2(\alpha_j) + \frac{\alpha_j}{2} \left(\alpha_j - \frac{1}{2} \sin(2\alpha_j)\right)\right)^{-1/2}.
\]
For \(j = 1\) and \(j = 2\) the numerical computation yields
\[
\lambda_1 = 1.35103388\ldots, \\
\lambda_2 = 0.08521617\ldots.
\]
Clearly, \(\alpha_j = \pi j (1 + o(1))\) as \(j\) tends to infinity. This shows that
\[
\lambda_j = \frac{1 + o(1)}{\pi^2 j^2} \quad \text{as} \quad j \to \infty.
\]
Therefore decay \(\lambda = 2\). Hence (3) holds and \(\text{APP}\) is QPT for the class \(\Lambda^{\text{all}}\).

The assumption (5) holds if for all \(a \in \mathbb{R}\) and \(t \in [0, 1]\) we can find \(x \in [0, 1]\) such that
\[
\cos(\alpha_1 x - \alpha_1) \neq a(1 + \min(x, t)).
\]
For \(t = 0\), the right hand side is constant, whereas the left hand side varies. For \(t > 0\), the right hand side is constant over \([0, t]\), whereas the left hand side varies. Therefore (5) also holds. We summarize this in the following corollary.

**Corollary 8.**

Consider the multivariate approximation problem \(\text{APP}\) for the Sobolev spaces studied in this section. Then

- \(\text{APP}\) is QPT but not PT for the class \(\Lambda^{\text{all}}\) with the exponent of QPT
  \[t^* = 1.\]
- \(\text{APP}\) suffers from the curse of dimensionality for the class \(\Lambda^{\text{std}}\).
We add in passing that a similar analysis can be done if the reproducing kernel (4) is replaced by
\[ K_{1,a}(x, t) = 1 + \frac{1}{2} ((|x - a| + |t - a| - |x - t|) \quad \text{for all } x, t \in [0, 1], \]
for any \( a \in [0, 1] \) or by
\[ K_1(x, t) = \min(x, t) \quad \text{for all } x, t \in [0, 1]. \]
For these variants of the Sobolev spaces, Corollary 8 is valid.

**Remark 9.** We conclude this paper with a comment on the rates of convergence and tractability notions for \( \Lambda_{\text{all}} \) and \( \Lambda_{\text{std}} \). In [4], the \( L_2 \) approximation problem was studied. It was shown that there are classes \( F \) for which the best rate of convergence of algorithms using \( n \) appropriately chosen linear functionals is \( n^{-1/2} \) whereas for \( n \) function values the best rate can be arbitrarily bad. If the best rate for \( \Lambda_{\text{all}} \) is faster than \( n^{-1/2} \) than we still do not know whether the rates for \( \Lambda_{\text{std}} \) and \( \Lambda_{\text{all}} \) always coincide.

For the examples in this section the rates for \( \Lambda_{\text{std}} \) and \( \Lambda_{\text{all}} \) are basically (up to log terms) \( n^{-1} \) but tractability properties for \( \Lambda_{\text{all}} \) and \( \Lambda_{\text{std}} \) are quite different. Hence, even if the rates are the same, tractability properties can be quite different for \( \Lambda_{\text{all}} \) and \( \Lambda_{\text{std}} \).

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