IMPROVED QUARK ACTIONS FOR LATTICE QCD

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I present a brief summary of the status and prospects of improved Wilson-type quark actions for coarse lattice simulations. My conclusions are optimistic.

1 Introduction

In the last few years it has become clear that there are a variety of gluon actions that give accurate results on coarse lattices \((a = 0.2 - 0.4 \text{ fm})\); see various contributions to Lattice 96. Such actions have been constructed within the Symanzik improvement program, eliminating \(O(a^2)\) errors using tadpole improvement 2 at tree- and at one-loop level, as well as within the MCRG (or “perfect action”) approach. There are of course several open problems that should be addressed, 3 but, generally speaking, the errors of these actions on coarse lattices seem quite small, much smaller than those of any improved quark action proposed so far. Given the dramatic increase in cost of a full QCD simulation as the lattice spacing is decreased, it is very important to find improved quark actions that are accurate on coarse lattices. This is the aim I address in this contribution.

Besides the use of improved actions, another tool that seems likely to become a staple of lattice QCD technology, is the use of anisotropic lattices, 4, 5, 6 with smaller temporal than spatial lattice spacing, \(a_0 \equiv a_t < a_s \equiv a_i (i = 1, 2, 3)\). [A lattice with \(\xi = a_s/a_t\) will be referred to as a “\(\xi : 1\) lattice.”] Such lattices have clear advantages in the study of heavy quarks, 7 lattice thermodynamics, and for particles with bad signal/noise properties, like glueballs and P-state mesons. On the classical level anisotropic lattices are as easy to treat as isotropic ones. On the quantum level, however, more coefficients have to be tuned to restore space-time exchange symmetry. Perturbative calculations and preliminary simulations with heavy quarks 8 and glueballs 10 have appeared using improved anisotropic gluon actions; further work is in progress.

\[\text{For example, finding a practical method of non-perturbatively eliminating all } O(a^2) \text{ errors of Symanzik improved gluon actions (not just the violation of rotational symmetry), or, whether the second-order phase transition in the fundamental-adjoint plane of gauge couplings affects the physics of these actions on coarse lattices, in particular the } \eta^{++} \text{ glueball.}\]

\[\text{Important, since NRQCD methods seem to break down for charmonium.}\]
Doubler-Free, Classically Improved Quark Actions

The first step in the Symanzik improvement program is the construction of a classically improved action. Usually this is quite simple; for fermions one has a slight complication due to the doubler problem. Recall that the origin of doublers is that the standard discretization of the continuum derivative $D_\mu$,

$$\nabla_\mu \psi(x) = \frac{1}{2a_\mu} \left[ U_\mu(x) \psi(x+\mu) - U_{-\mu}(x) \psi(x-\mu) \right] = D_\mu \psi(x) + \mathcal{O}(a^2_\mu),$$

(1)

decouples even and odd sites of the lattice. Wilson suggested to avoid this problem by adding a second-order derivative term $\bar{\psi} \sum \Delta_\mu \psi$ to the action,

$$\Delta_\mu \psi(x) = \frac{1}{a_\mu^2} \left[ U_\mu(x) \psi(x+\mu) + U_{-\mu}(x) \psi(x-\mu) - 2\psi(x) \right] = D^2_\mu \psi(x) + \mathcal{O}(a^2_\mu).$$

(2)

If one adds such a term naively, one breaks chiral symmetry at $\mathcal{O}(a)$. But chiral symmetry can be preserved to higher order if one introduces the Wilson term $\sum \Delta_\mu$ by a field transformation. The simplest way to proceed is to start with the continuum action $\bar{\psi} M_c \psi_c \equiv \bar{\psi}(\slashed{\partial} + m_c)\psi$ and perform the field redefinition $\psi_c = \Omega_c \psi$, $\bar{\psi}_c = \bar{\psi} \Omega_c$ with

$$\Omega_c = \Omega_c, \quad \bar{\Omega}_c \Omega_c = 1 - \frac{ra_0}{2} (\slashed{\partial} - m_c).$$

(3)

Here $r$ is a free parameter, to be fixed later. The transformed fermion operator reads

$$\bar{\Omega}_c M_c \Omega_c = m_c (1 + \frac{1}{2} r a_0 m_c) + \slashed{\partial} - \frac{1}{2} r a_0 \left( \sum \mu \right) \left( D^2_\mu + \frac{1}{2} \sigma \cdot F \right),$$

(4)

where we used $\slashed{\partial}^2 = \sum \mu D^2_\mu + \frac{1}{2} \sigma \cdot F$. Here $\sigma \cdot F \equiv \sum \mu \sigma_{\mu \nu} F^\nu_{\mu}$ is the clover term, containing the field strength $F^\nu_{\mu}$. The above continuum action still has (slightly hidden) chiral symmetry. We can now discretize this action by replacing $\slashed{\partial}$, $D^2_\mu$ and $F^\nu_{\mu}$ by suitable lattice versions, differing at $\mathcal{O}(a^n)$, say, from the former. Let us call the action so obtained $M$. It will not have a doubler problem. It will break chiral symmetry, however; classically at $\mathcal{O}(a_0 a^n)$, on the quantum level at $\mathcal{O}(a_0 a g^2)$. $M$ will correctly give on-shell quantities up to $\mathcal{O}(a^n)$ errors at tree level. Off-shell quantities can also be obtained with such errors, by simply undoing the field transformation on the lattice.

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The Jacobian of a field transformation matters only at the quantum level, where, in the case at hand, its leading effect is to renormalize the gauge coupling.
For \( n = 4 \) one obtains the D234 actions on an anisotropic lattice,

\[
M_{\text{D234}} = m_c (1 + \frac{1}{2} r a_0 m_c) + \sum_{\mu} \gamma_{\mu} \nabla_{\mu} (1 - b_{\mu} a_{\mu}^2 \Delta_{\mu})
- \frac{1}{2} r a_0 \left( \sum_{\mu} \Delta_{\mu} + \frac{1}{2} \sigma \cdot F \right) + \sum_{\mu} c_{\mu} a_{\mu}^3 \Delta_{\mu}^2
\]

where \( b_{\mu} = \frac{1}{6} \) and \( c_{\mu} = \frac{r a_0}{24 a_{\mu}} \). The SW (or clover) action is the \( n = 2 \) case, corresponding to \( b_{\mu} = c_{\mu} = 0 \). The Wilson action is obtained from the SW action by ignoring the clover term \( \sigma \cdot F \).

A generic property of improved actions is the existence of unphysical branches in their (free) dispersion relations, arising from higher order temporal derivatives in the action. We will refer to such branches as ghost branches. The SW action has no ghost branches for \( r = 1 \), which is therefore the canonical choice in this case. In general one can not eliminate all ghost branches. The D234 action as derived above with \( \mathcal{O}(a^4) \) errors has three ghost branches (except for \( r = 2 \), when there are two). It is not clear if this is really a problem, but we have first considered slightly modified D234 actions that have only one ghost branch, at the expense of introducing small \( a^3 \) or \( a_0^3 \) errors for isotropic, respectively, anisotropic lattices. The former case, the “isotropic D234” action corresponds to choosing \( r = \frac{2}{3} \) and \( c_{\mu} = \frac{1}{12} \). The latter, the anisotropic “D234i(\( \frac{2}{3} \))” action, has \( r = \frac{2}{3}, c_0 = \frac{1}{12} \) (and \( c_i = \frac{r a_0}{24 a_i} \) as originally derived).

For more details about improved quark actions, including plots of various dispersion relations, we refer the reader to ref. On-Shell Quantum Improvement

3 On-Shell Quantum Improvement

For both gluon and quark actions the largest error at \( \mathcal{O}(a^2) \) is the violation of rotational symmetry, which already exists at the classical level. In fact, in the case of gluons no significant differences have so far been found between actions that correct only the violation of rotational symmetry and others that also take into account \( a^3 \) terms that are only necessary on the quantum level. Whereas there is some hope that all (two) terms necessary for the \( \mathcal{O}(a^2) \) on-shell improvement of a gluonic action can eventually be tuned non-perturbatively, this is clearly impossible for a Wilson-type quark action; there are just too many terms. But it is not unreasonable to hope that, as for glue, the largest \( a^2 \) errors of quark actions are the violations of rotational symmetry.

A clear difference between gluon and Wilson-type quark actions emerges at \( \mathcal{O}(a) \). No such terms are present for gluons, but for quarks we have to introduce Wilson and clover terms to eliminate doublers without introducing classical
\( \mathcal{O}(a) \) errors. In this case these two terms are also the only ones that can exist at the quantum level at this order. The coefficient of one of these terms, usually chosen to be the Wilson term, can be adjusted at will by a field redefinition. The other one then has to be tuned to eliminate \( \mathcal{O}(a) \) quantum errors.

Note that the \( \mathcal{O}(a) \) terms break chiral but not rotational symmetry; exactly the opposite behavior of the leading \( a^2 \) terms. This qualitative difference allows one to tune both, even non-perturbatively, by demanding the restoration of chiral symmetry at zero quark mass for the former (how to implement this in practice has recently been shown in important work by Lüscher et al\(^1\)), that of rotational symmetry for the latter.

So far we have discussed isotropic lattices. The anisotropic case is more complicated. However, after considering the most general field redefinitions up to \( \mathcal{O}(a) \), one easily sees that only two more parameters have to be tuned for on-shell improvement of a quark action up to \( \mathcal{O}(a) \). One already appears at \( \mathcal{O}(a^0) \), namely, a “bare velocity of light” that has to be tuned to restore space-time exchange symmetry (by, say, demanding that the pion have a relativistic dispersion relation for small masses and momenta). The other is at \( \mathcal{O}(a) \); the two terms that now have to be tuned at this order can be chosen to be the temporal and spatial parts of the clover term.

### 4 Quenched Simulation Results with Tadpole Improved Actions

We would now like to use simulation results for various tadpole improved actions in an attempt to disentangle the effects of the \( \mathcal{O}(a) \) and \( \mathcal{O}(a^2) \) terms, as a handle on how well tadpole improvement (TI) estimates the coefficients of these terms. To alleviate the problem of (absolute) scale setting, we will concentrate on dimensionless quantities and compare results obtained with the Wilson, SW and D234 actions with the same improved gluon actions. Specifically, we consider:

(a) The “effective velocity of light”, \( c(p) \), of various hadrons, defined by

\[
c(p)^2 p^2 = E(p)^2 - E(0)^2.
\]

(b) \( J \), a dimensionless measure of the vector versus pseudo-scalar meson mass relation, defined by \( J = \frac{m_V}{m_P} \left( \frac{m_N}{m_P} \right)^2 \) at \( m_V/m_P = m_K/m_K = 1.8 \).

(c) The nucleon over rho mass ratio, \( m_N/m_\rho \) (defined by extrapolation of lattice data to \( m_\rho/m_\pi = 5.58 \)).

(d) The quenched hyper-fine splitting (HFS) of charmonium.

(e) The rho mass in units of the string tension, \( m_\rho/\sqrt{\sigma} \).

\[^{d}\text{As an aside we remark that } J \text{ is one of the most accurately known numbers in quenched continuum QCD, } J = 0.39(1) \text{ (see below). It also seems to be a very sensitive indicator of quenching errors, since in nature } J = 0.48(2).\]

4
Table 1: $c(p)^2$ for mesons with momentum $p = 2\pi/aL$, $aL \approx 2.0$ fm at $m_\rho/m_\pi \approx 1.4$, calculated for various actions on tadpole and one-loop improved quenched glue.

| $\beta$ | $\pi$ | $\rho$ | $\pi$ | $\rho$ | $\pi$ | $\rho$ |
|---------|-------|--------|-------|--------|-------|--------|
| 6.8     | 0.95(2) | 0.93(3) | 0.83(2) | 0.75(4) | 0.63(2) | 0.48(3) |
| 7.1     | 0.94(3) | 0.96(5) | —      | —      | 0.74(3) | 0.55(4) |
| 7.4     | 0.99(4) | 1.00(6) | —      | —      | —      | —      |

Table 2: $J$ and $m_N/m_\rho$ for various tadpole improved actions. The quenched continuum limit of $m_N/m_\rho = 1.29(2)$. The data were partially reanalyzed.

| $\beta$ | $a$(fm) | $D234$ | $SW$ | Wilson |
|---------|---------|--------|------|--------|
| 6.8     | 0.40    | 0.386(5) | 0.345(4) | 0.314(3) |
| 7.1     | 0.33    | 0.381(6) | 0.350(4) | 0.318(3) |
| 7.4     | 0.24    | 0.395(10) | 0.371(5) | 0.335(5) |
| 7.75    | 0.18    | —      | 0.386(9) | 0.350(6) |

Obviously, $c(p)$ is a measure of $O(a^2)$ violations of rotational symmetry, and therefore essentially independent of the clover coefficient. The HFS and the rho mass, on the other hand, clearly depend strongly on the clover coefficient. They also have some dependence on the $O(a^2)$ terms (see below). For $J$ and $m_N/m_\rho$, it seems (see also table 2) that they have a significant dependence on both $O(a)$ and $O(a^2)$ terms, though their dependence on the clover coefficient is much weaker than that of $m_\rho/\sqrt{\sigma}$ (certainly on finer lattices). Data for $c(p)$ are given in table 1. They clearly demonstrate that with TI rotational symmetry is restored to high accuracy for the D234 action. Similar conclusions also hold for anisotropic lattices, even for masses in the charmonium range. In table 2 we show results for $J$ and $m_N/m_\rho$ in figure 1 for the HFS. For $m_\rho/\sqrt{\sigma}$ we present the results in words: Whereas the scaling violations of the rho mass obtained with SW are much smaller (consistent with $O(a^2)$) than those with the Wilson action, the former are almost identical to that of the isotropic D234 action. That they are almost identical is presumably to some extent an accident; what is significant, is that these scaling violations are almost 30% on the coarsest lattice ($a = 0.4$ fm). For the SW action this is not surprising, since other quantities obtained with this action have similarly large errors. It is surprising, though, for the D234 action, where $c(p)$, $J$, and $m_N/m_\rho$ have errors of only a couple percent, even on the coarsest lattice. Note
that the D234 HFS, on the other hand, seems to have similarly large errors as the rho mass on coarse lattices (much smaller scaling errors, though, than the HFS of the SW action).

It is certainly very suggestive that \( m_\rho / \sqrt{\sigma} \) and the HFS are exactly the quantities that depend most strongly, by far, on the value of the clover coefficient. Also, Lüscher et al.\(^\text{13}\) have recently found, for the case of the SW action on Wilson glue, that the non-perturbative clover coefficient is significantly larger than the tadpole estimate (which was used in the simulations described above), even on a relatively fine \( a = 0.1 \) fm lattice.

Although the data summarized above have some uncertainties that prevent them from being conclusive, it seems likely that the true clover coefficients of the (isotropic and anisotropic) D234 actions are significantly larger than the values used above. With the correct values the D234 actions can give accurate results for all the indicators of scaling violations (a)–(e), already on coarse lattices (certainly, the rho mass and HFS will increase, which is what we want).

### 5 Conclusions

The use of the tadpole improved SW action on improved glue is a large step forward compared to the use of the Wilson quark action. With the SW action accurate (quenched) continuum extrapolations have been performed from data in the range \( a = 0.15 - 0.4 \) fm. Further significant improvements are possible with the D234 action. For both SW and D234 actions non-perturbative
\(\mathcal{O}(a)\) tuning should be performed with improved glue. The same applies to the determination of current renormalization constants. The methods can also be used on anisotropic lattices. Tuning the additional coefficients might be significantly more complicated in practice, but seems to be within reach of present technology. Actions and currents with no \(\mathcal{O}(a)\) and only small \(\mathcal{O}(a^2)\) errors should give accurate results on coarse isotropic and anisotropic lattices. Quenched QCD should essentially be “solved” within the next few years (to the extent that it makes sense), and there finally is hope for realistic simulations of full QCD, heavy and heavy-light mesons, as well as glueballs and hybrids.

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