Multiparticle solutions to Einstein’s equations

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In this letter we present the first multiparticle solutions to Einstein’s field equations in the presence of matter. These solutions are iteratively obtained via the perturbiner method, which can circumvent gravity’s infinite number of vertices with the definition of a multiparticle expansion for the inverse spacetime metric as well. Our construction provides a simple layout for the computation of tree level field theory amplitudes in \( D \) spacetime dimensions involving any number of gravitons and matter fields, with or without supersymmetry.

\section{I. OVERVIEW}

Gravity is still in many ways the least understood of the fundamental forces of nature, arguably at the macroscopic but definitely at the microscopic level. The former is splendidly described by the general theory of relativity, while the latter hopefully tangible by the long sought for theory of quantum gravity.

As a first approximation, the Einstein-Hilbert action can be seen as a common denominator in this range of scales. It yields as classical equations of motion Einstein’s field equations, and offers a natural path for a (quantum) field theory of gravitons, the messengers of gravity.

From this field theory perspective, gravity contains an infinite number of vertices and is, in fact, non-renormalizable. Even at tree level, the computation of graviton scattering amplitudes quickly becomes impractical using standard Feynman diagrams (e.g. \cite{1}).

Modern scattering-amplitude techniques have overcome this problem. Among them the Britto-Cachazo-Feng-Witten (BCFW) recursion \cite{2,12} and the double copy \cite{5–8} are most successful. At their core, they are connected by a simple fact: cubic vertices are enough to describe any tree level graviton amplitude. In BCFW we see this via onshell recursions. In the double copy, graviton amplitudes are recast as two copies of gluon amplitudes with a special trivalent configuration using the color-kinematic duality \cite{9}. Indeed, pure graviton amplitudes have been shown to be recursively described by a cubic action with auxiliary fields that is classically equivalent to the Einstein-Hilbert action \cite{10}. This strictification has been formally demonstrated in \cite{11} using \( \mathfrak{g}_\infty \) algebras.

Our universe, on the other hand, is not pure gravity. Our interest resides in the study of interactions between gravitons and matter particles. In this case, results using BCFW recursions (e.g. \cite{12,13}, double copy (e.g. \cite{14}) or other diagrammatic techniques (\cite{15}), are considerably scarce, subject to different subtleties/limitations that have so far eluded a more systematic and practical output. At the dawn of gravitational waves detection and black hole observation, any advance in the understanding and formulation of the scattering of gravitons by matter is very welcome. This letter is a step in this direction.

To our avail, the tree level information of a given field theory can be elegantly extracted from its classical equations of motion \cite{16}. This idea was further explored in \cite{17,18}, and later streamlined by the perturbiner method \cite{19,20}. As it turns out, there is an inspired multiparticle ansatz for the solution of classical equations of motion that can be used to define an offshell recursion for tree level amplitudes in terms of Berends-Giele currents \cite{21}. This method recently regained interest \cite{22–24} and has been since explored in different contexts \cite{25,30}. Rather surprisingly, perturbiner methods have never been fully applied to gravity, except for the very early analysis of the self-dual case in \cite{20} and a simplified version for conformal supergravity amplitudes in \cite{31}. Naively, a proper recursive solution cannot be defined in a theory with an infinite number of vertices. As we will show, however, there is a way around this obstacle in gravity.

In this work we propose a series of multiparticle solutions to Einstein’s field equations based on the perturbiner method. These solutions encompass a broad class of interesting cases and can be applied to any two-derivative matter field theory coupled to gravity.

We can then define \( n \)-point tree level scattering amplitudes between gravitons and matter particles using a similar prescription to the super Yang-Mills case \cite{22}. In this prescription, diffeomorphism invariance is manifest, with a clear decoupling of pure gauge states. In addition, the analysis of the soft limit behavior at leading order is surprisingly transparent. First, we discuss pure gravity with a subsequent coupling to bosonic matter. We then recast the Einstein-Hilbert action in terms of the vielbein and the spin connection, in order to introduce the coupling to fermionic matter and, consequently, supersymmetry. Our results are agnostic to the number of spacetime dimensions and can be easily automated. Whether or not there is an underlying worldsheet description, they provide a compact and efficient computation of the scattering of gravitons and matter at tree level.

\section{II. FIELD EQUATIONS AND GRAVITONS}

Einstein’s field equations without cosmological constant can be cast as

\[ R_{\mu\nu} = \frac{\kappa}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \]
The field equations (1) are covariant under general co-
metric (with inverse $\kappa R R$ curvature and $\eta$ around a given background, i.e. the gravitons. These
which can be used to analyze linearized solutions
isfies $\eta$ and $\Gamma$ transforming as
with $k$ to work with
the graviton field, $g$, multiplied by the gravitational constant $\kappa$.
In the absence of matter, (1) reduces to
$\delta h g = 0$, (5)

which can be used to analyze linearized solutions
around a given background, i.e. the gravitons. These
single-particle solutions around flat space (with metric $\eta_{\mu\nu}$) are given by
$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu} e^{ik \cdot x}$, (6)

with $k \cdot x = k_\lambda x^\lambda$. The graviton polarization, $h_{\mu\nu}$, satisfies $\eta^{\mu\nu} h_{\mu\nu} = 0$. There is also a residual
gauge transformation of the form $\delta h_{\mu\nu} = k_\mu \lambda_\nu + k_\nu \lambda_\mu$, with $k \cdot \lambda = 0$.

III. MULTIPARTICLE SOLUTIONS AND RECURRENCES

We can now look at the multiparticle solutions of the graviton field, $g_{\mu\nu}(x)$, satisfying (5). Consider
$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_P H_{P\mu\nu} e^{ikP \cdot x}$, (7)

where $H_{P\mu\nu}$ represents the multiparticle currents. The word $P$ denotes a sequence of ordered letters,
$P = p_1 \ldots p_n$, where $p_i$ is a single-particle label, with $k_P \equiv k_{p_1} + \ldots + k_{p_n}$.

In order to find the solutions for $H_{P\mu\nu}$, we have also to work with $g^{\mu\nu}(x)$ satisfying $g^{\mu\nu}g_{\mu\nu} = \delta^{\mu\nu}$. For the expansion
$g^{\mu\nu}(x) = \eta^{\mu\nu} - \sum_P I_P^{\mu\nu} e^{ikP \cdot x}$, (8)

the inverse identity implies that the currents $I_P^{\mu\nu}$ are constrained to be
$I_P^{\mu\nu} = \eta^{\mu\nu} \eta^{\rho\sigma} H_{P\rho\sigma} - \eta^{\rho\sigma} \sum_{P=Q\cup R} I_Q^{\rho\sigma} H_{R\rho\sigma}$, (9)

where the sum goes over all deshuffles of $P$ into ordered words $Q, R$ (see e.g. [28]). Although not explicitly,
$I_P^{\mu\nu} = I_P^{\nu\mu}$ and this can be recursively demonstrated order by order in the sub-deshuffles.

Multiparticle currents with one-letter words are simply associated to their single-particle equivalents (polarizations): $H_{P\mu\nu} = h_{P\mu\nu}$ and $I_P^{\mu\nu} = \eta^{\rho\sigma} h_{P\rho\sigma}$.

To every $x$-dependent object we will associate a multiparticle expansion. For example, the Christoffel symbol can be expressed as $\Gamma_{P\mu\nu} = \sum_P I_P^{\mu\nu} e^{ikP \cdot x}$, with
$\Gamma_{P\mu\nu} \equiv \frac{1}{2} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$. (3)

The parameter of general coordinate transformations may too be cast as a multiparticle expansion as
$\lambda^\mu = -i \sum_P \alpha_P^{\mu\nu} e^{ikP \cdot x}$. (11)

This way, the gauge transformation (9) implies that
$\delta H_{P\mu\nu} = \sum_{P=Q\cup R} \alpha_Q^{\rho\sigma} (k_Q H_{R\rho\sigma} + k_Q H_{R\rho\sigma} + k_R H_{P\rho\sigma})$
$+ k_P \Lambda_{P\rho\nu} + k_P \Lambda_{P\rho\nu}$. (12)

We will choose the gauge $\eta^{\mu\nu} \Gamma_{\mu\nu\rho} = 0$. This is simpler than the de Donder gauge $g^{\mu\nu} \Gamma_{\mu\nu\rho} = 0$, because its multiparticle version does not involve deshuffles, being neatly expressed as
$\eta^{\mu\nu} \Gamma_{\mu\nu\rho} = i \eta^{\mu\nu} (k_{P\mu} H_{P\rho\nu} - \frac{1}{2} k_{P\mu} H_{P\mu\nu}) = 0$. (13)

In this gauge, the multiparticle currents of the Ricci tensor, $R_{P\mu\nu}$, are computed to be
$R_{P\mu\nu} = \frac{ie}{\kappa} H_{P\mu\nu} - i \sum_{P=Q\cup R} I_Q^{\rho\sigma} (k_P \Gamma_{R\rho\nu\sigma} - k_P \Gamma_{P\rho\nu\sigma} - k_P \Gamma_{R\rho\nu\sigma})$
$- \eta^{\rho\sigma} \sum_{P=Q\cup R} (Q_{\rho\mu\nu} \Gamma_{Q,\rho\mu\nu} - Q_{\rho\mu\nu} \Gamma_{Q,\rho\mu\nu})$
$+ \eta^{\rho\sigma} \sum_{P=Q\cup R} I_Q^{\rho\sigma} (\Gamma_{R\rho\nu\sigma} - \Gamma_{R\rho\nu\sigma})$
$+ \eta^{\rho\sigma} \sum_{P=Q\cup R} I_Q^{\rho\sigma} (\Gamma_{S\rho\mu\sigma} - \Gamma_{S\rho\mu\sigma} - \Gamma_{S\rho\mu\sigma})$
$- \sum_{P=Q\cup R, S, T} I_Q^{\rho\sigma} (\Gamma_{S\rho\mu\sigma} \Gamma_{T\rho\mu\sigma} - \Gamma_{S\rho\mu\sigma} \Gamma_{T\rho\mu\sigma})$. (14)

where $s_p \equiv \eta^{\mu\nu} k_{P\mu} k_{P\nu}$ denotes the generalized Mandelstam variables. The recursion relation for $H_{P\mu\nu}$ is then obtained using equation (15), i.e. $R_{P\mu\nu} = 0$.

IV. TREE LEVEL AMPPLITUDES

Motivated by the Berends-Giele prescription [22], the tree level amplitude for the scattering of $n$ gravitons is defined as
$M_n \equiv \kappa \lim_{s_{2 \ldots n} \to 0} s_{2 \ldots n} h_{1 \mu_1 \nu_1} I_{\mu_1 \ldots \nu_1}$
$= \kappa \lim_{s_{2 \ldots n} \to 0} s_{2 \ldots n} h_{1 \mu_1} h_{2 \ldots n \nu_1}$, (15)

on the support of momentum conservation. Whenever convenient, we will raise or lower spacetime indices using the flat metric.
By construction, \( H_{\mu \nu} \) is symmetric in the exchange of any two single-particle labels. This symmetry is lifted to the amplitude \( \mathcal{M}_n \), which is also symmetric in the exchange of any two graviton legs, although only \( n - 1 \) are manifest through \( H_{2 \ldots n} \).

The particle in the first leg can be thought of as an off-shell leg in the multiparticle recursion, then placed onshell in the definition of the amplitude in \( M_n \). In order to see this, we can examine the residual gauge transformations preserving \( \mathcal{M}_n \). They lead to a recursion for the currents \( A_{\mu \nu} \) in \( M_n \), given by

\[
A_{\mu \nu} = -\frac{k^\mu}{s^\nu} \sum_{P=Q,R} \Lambda_{\rho \sigma}^P (k_{\rho \sigma} H_{R \rho \sigma} + k_{R \rho} H_{R \rho \nu}).
\]

It is then just an algebraic step to show the invariance of \( \mathcal{M}_n \) under \( \Lambda \) with multiparticle parameters \( k \).

The three-point amplitude is given by the well-known result

\[
\mathcal{M}_3(h_1, h_2, h_3) = \frac{4}{3} h_1 h_2 h_3 \sqrt{s} V_{\mu \nu} V_{\nu \rho} V_{\rho \sigma},
\]

in terms of the three-point Yang-Mills vertex

\[
V_{\mu \nu} = (k_1^\mu - k_2^\mu) \eta^\nu + (k_2^\mu - k_3^\mu) \eta^\nu + (k_3^\mu - k_1^\mu) \eta^\nu.
\]

The current \( H_{\mu \nu} \) effectively describes interactions with vertices from three to five points, as can be seen from the number of deshuffles in \( \mathcal{M}_n \). The four- or higher-point amplitudes will not be explicitly displayed here, as their size grows rapidly due to the nested deshuffles. We found it easier to perform most of the cross-checks numerically, since it is straightforward to implement the recursions for \( H_{\mu \nu} \) computationally.

Soft limit

The soft limit of graviton amplitudes has a universal behavior \( \mathcal{M}_n \), constituting a natural test for our proposal in \( \mathcal{M}_n \). As it turns out, its soft limit analysis is very simple at leading order.

We will take \( h_{\mu \nu} \) as the soft graviton and parametrize its momentum as \( k_1^\mu = \tau q^\mu \), with \( q^2 = 0 \) and parameter \( \tau \to 0 \). In the soft limit, we can directly identify the dominant contributions in \( H_{2 \ldots n} \), for they come from the poles of the generalized Mandelstam variables with \( (n - 2) \) momenta. For example, \( s_{2 \ldots n} = \tau (q + k_2)^2 = 2 \tau (q \cdot k_2) \),

\[
s_{2 \ldots n} = (\tau q + k_2)^2 = 2 \tau (q \cdot k_2),
\]

is attached to the multiparticle current with \( (n - 2) \) particles, \( H_{3 \ldots n} \). We can then reexamine the recursion of the \( (n - 1) \)-particle currents \( H_{2 \ldots n} \) and readily express it as

\[
s_{23 \ldots n} H_{23 \ldots n} = k_2 k_3 \ldots k_n H_{3 \ldots n} + \text{sym}(2, 3, \ldots, n) + \mathcal{O}(\tau^0),
\]

where \( \text{sym}(2, 3, \ldots, n) \) takes care of the symmetrization of the single-particle labels.

In terms of the amplitude, this parametrization leads to the leading order contribution

\[
\lim_{\tau \to 0} \mathcal{M}_n = \frac{1}{\tau} \left( \sum_{a=2}^n \frac{k_{a} k_{a,\mu} k_{a,\nu}}{2 (q \cdot k_a)} \right) M_{n-1} (h_2, \ldots, h_a),
\]

manifesting the universal Weinberg pole. Since diffeomorphism invariance is inherited in our results, subleading soft limits should be directly reproduced \( \mathcal{M}_n \).

V. MATTER COUPLED TO GRAVITY

The matter contributions to \( \mathcal{M}_n \) come from the energy-momentum tensor

\[
T_{\mu \nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu \nu}} S_{\text{matter}},
\]

where \( S_{\text{matter}} \) is the matter action. In terms of a multiparticle expansion, we have \( T_{\mu \nu} = \sum_{l} T_{\mu \nu} e^{ik_{l} x} \), where the form of the currents \( T_{\mu \nu} \) is particular to the model. In order for this to make sense, the single-particle solutions of the free equations of motion associated to the matter action must be described in terms of plane waves. These are our asymptotic states.

The recursion relations for the currents \( H_{\mu \nu} \) are obtained by plugging the corresponding multiparticle expansions in \( \mathcal{M}_n \). The result is

\[
R_{\mu \nu} = \frac{1}{2} (Q_{\mu} \eta_{\rho \sigma} R_{\nu \rho \sigma} + \kappa T_{\mu \nu})
\]

+ \( \frac{1}{2} \sum_{P=Q,R} (H_{Q \mu} \eta_{\rho \sigma} - \eta_{\rho \sigma} I^p_{Q}) R_{\rho \sigma} \)

\[
- \frac{1}{2} \sum_{P=Q,R,J} H_{Q \mu, J \rho \sigma} R_{S \rho \sigma},
\]

where \( R_{\rho \sigma} \) is defined in \( \mathcal{M}_n \). Naturally, we recover \( T_{\mu \nu} = 0 \) when \( T_{\mu \nu} = 0 \).

The amplitude prescription is the same of \( M_n \), but now we are able to describe the scattering of matter bosons and gravitons.

A. Massive scalar

Our first example is the massive scalar coupled to gravity and otherwise free, with equation of motion

\[
(g^\mu \nu \partial_\mu \partial_\nu - m^2) \phi = g^\mu \nu \Gamma_\rho \partial_\mu \phi,
\]

and covariantly conserved energy-momentum tensor

\[
T_{\mu \nu} = -\partial_\nu \phi \partial_\mu \phi + \frac{1}{2} g_{\mu \nu} (g^{\rho \sigma} \partial_\rho \phi \partial_\sigma \phi + m^2 \phi^2).
\]

Equation \( \mathcal{M}_n \) leads to the following recursion for the scalar multiparticle currents, \( \Phi_\rho \),

\[
(s_p + m^2) \Phi_\rho = \sum_{P=Q,R} \left( T_{\mu \nu} k_{R_\mu} k_{R_\nu} + \eta_{\nu \rho} I_{Q} \Gamma_{Q \mu \nu} k_{R_\nu} \right) \Phi_\rho
\]

\[
- \eta_{\nu \rho} I_{Q} \Phi_\rho S_\rho (\Gamma_{Q \mu \nu} k_{S_{\mu \rho}} + \Gamma_{R \rho \sigma} k_{S_{\rho \sigma}})
\]

\[
+ \sum_{P=Q,R,J,S,T} T_{\mu \nu} I_{Q} \Gamma_{S_{\mu \rho}} k_{T_\rho} \Phi_\rho.
\]
Similarly, equation (24) leads to

\[ T_{\mu\nu} = \sum_{p=q,R} \{ k_{Q}\mu k_{R}\nu + \frac{1}{4}\eta_{\mu\nu}[m^2 - (k_{Q} \cdot k_{R})] \} \Phi Q \Phi R + \frac{1}{2} \sum_{p=q,R,J,S} H_{Q\mu\nu} \Phi P_{\nu S}[m^2 - (k_{Q} \cdot k_{R})] \]

These quantities are then used to compute the tree level scattering of gravitons and massive scalars. For example, the four point amplitude with two gravitons \((h_1, h_2)\) and two massive scalars \((3, 4)\) is given by

\[ A_4 = 2\kappa^2(k_3 \cdot h_1, h_2, k_1) - \frac{1}{2} \kappa^2 s_{34}(h_1, h_2) \]

VI. FERMIONS AND SUPERSYMMETRY

In order to consider fermions, we turn the local Lorentz group into a gauge symmetry. The spacetime metric \(g_{\mu\nu}\) is mapped to the (local) flat metric, \(\eta_{ab}\), using the vielbein, \(e_a^\mu\) (with inverse \(e_\mu^a\)), such that \(g_{\mu\nu} = \eta_{ab}e_a^\mu e_b^\nu\). The gauge field of the Lorentz symmetry is the spin connection, \(\omega^a_{\mu\nu}\), and the flattened Riemann tensor, \(R^{\mu\nu}_{ab} \equiv \eta^{\rho\sigma}e^\nu_\rho e^\mu_\sigma R_{\rho\sigma\mu\nu}\), can be seen as its field strength, given by

\[ R^{\mu\nu}_{ab} = \partial_\mu \omega^\nu_{ab} + \eta_{cd} \omega^\mu_{ac} \omega^\nu_{bd} - (\mu \leftrightarrow \nu), \]

with scalar curvature \(R = e^a_\mu e^\nu_a R^{\mu\nu}_{ab}\).

Spinor couplings to the curved background are implemented by replacing spacetime derivatives by their Lorentz-covariant version. Given a spinor \(\psi\), its covariant derivative is defined as

\[ D_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega^a_\mu \Gamma_{ab} \psi, \]

where \(\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]\) and \(\Gamma_a\) denotes the usual gamma matrices satisfying \((\Gamma_a, \Gamma_b) = 2\eta_{ab}\).

Next, we rewrite Einstein’s field equations in terms of the vielbein and, independently, the spin connection. This is known as the Palatini variation. In the presence of matter, they take the form

\[ R^a_\mu = \kappa T^a_\mu + \frac{1}{2} \epsilon^a_\mu R, \]

\[ \omega^{ab}_\mu = \kappa W^{ab}_\mu + \frac{1}{2} \epsilon^{a\mu} \{(\delta_\mu e^b_\rho) - \Gamma^b_{\rho\mu}\epsilon^{ab}\}, \]

where \(R^a_\mu \equiv e^c_a T^c_\mu, \{ab\} = ab - ba\), and \(\Gamma^a_{\mu\nu}\) is the zero-torsion Christoffel symbols in (3). The matter tensors are defined as

\[ T^a_\mu \equiv \epsilon^a_\mu S_{\text{matter}}, \]

\[ W^{a\mu}_{\nu} \equiv \epsilon^{a\mu}_{\nu}\delta^\nu_\lambda S_{\text{matter}}, \]

with \(\epsilon = \det \epsilon^a_\mu\), and

\[ P^{a\mu,cd}_{\nu} \equiv \eta^{b\mu}[e^b_\nu e^c_\mu + \frac{1}{2} \eta^{b\nu} g_{\mu\nu} + \frac{2}{(D-2)} e^b_\nu e^c_\mu]. \]

From here onward, the perturbative method goes as usual. We define the multiparticle expansion for the vielbein and its inverse analogously to the metric expansions in (7) and (8).

\[ e^a_\mu = \delta^a_\mu + \sum P E^a_\mu e^{ik_\nu, \chi}, \]

\[ e^a_\mu = \delta^a_\mu - \sum \frac{P F^a_\mu e^{ik_\nu, \chi}}{2}. \]

The mixed Kronecker deltas \(\delta^a_\mu\) and \(\delta^a_\mu\) indicate that the vielbeins are expanded around flat space. The inverse relations \(e^a_\mu e^b_\nu = \delta^a_\nu\) and \(e^a_\nu e^c_\mu = \delta^a_\mu\) constrain \(F^a_\mu\) to satisfy

\[ F^a_\mu = \delta^a_\mu e^b_\nu F^b_\nu - \delta^a_\mu \sum_{p=Q,R} E^a_\nu F^b_\nu, \]

\[ = \delta^a_\mu e^b_\nu F^b_\nu - \delta^a_\mu \sum_{p=Q,R} E^b_\nu F^b_\nu. \]
The proof of equivalence between the first and second lines follows the same logic of \( I_P^{\mu\nu} = I_P^{\mu\nu} \) after equation 9. We can then use general coordinate transformations and local Lorentz symmetry to fix a convenient gauge. We found the simplest one to be

\[
\begin{align*}
(\eta^{\mu\nu} \eta_{ab} - \frac{1}{2} \delta^{\mu} \delta^{\nu} \delta_{ab}) \partial_\nu e^b_{\mu} &= 0, \quad (41a) \\
\delta^T_{ab} e_{\mu b} - \delta^T_{a b} e_{\mu a} &= 0. \quad (41b)
\end{align*}
\]

The first equation is a truncated version of 13, while the second is known as symmetric gauge. In terms of the multiparticle currents, this gauge has a simple realization and does not involve deshuffles. The single-particle polarizations, \( e_{\mu a} \), satisfy \( k^\mu e^a_{\mu} = \delta^a_{\mu} e_{\mu a} = 0 \), with residual gauge symmetry

\[
\delta e_{p a a} = \delta^a_{\mu} (k_{p \mu} \lambda_{p \mu} + k_{p \nu} \lambda_{p \nu})
\]

and \( k_{p} \cdot \lambda_{p} = 0 \).

With these choices, the recursion for \( E_{P \mu}^a \) can be written as

\[
s_p E_{P \mu} = \kappa (\delta^T_{ab} \delta^F_{\mu} + \frac{k^a}{(2-\delta_p)} \delta^F_{\mu} b) T_{P \mu} - ik \delta^T_{ab} (k_{P \mu} W^a_{P \mu} - k_{P \nu} W^b_{P \mu}) \\
+ \frac{k^a}{2} \delta^T_{ab} \sum_{P=Q} (k_{P \mu} F_{P \mu}^a (k_{R \mu} E_{P \mu}^b - k_{R \mu} E_{R \mu}^b) - k_{P \mu} F_{Q \mu}^a (k_{R \mu} E_{P \mu}^b - k_{R \mu} E_{R \mu}^b) - (a \leftrightarrow b)) \\
+ \frac{1}{2} \sum_{P=Q} [(k_{P \mu} \delta^T_{ab} - k_{P \mu} \delta^F_{ab}) F_{P \mu}^a F_{P \mu}^b - (k_{P \mu} E_{P \mu}^a - k_{P \mu} E_{P \mu}^b) (F_{Q \mu}^a \eta_{Q \mu} - F_{Q \mu}^b \eta_{Q \mu}^{a b}) (k_{R \mu} E_{P \mu}^c - k_{R \mu} E_{R \mu}^c)] \\
+ \frac{1}{2} \delta^T_{ab} \sum_{P=Q} [k_{P \mu} \delta^T_{ab} - k_{P \mu} \delta^F_{ab}) F_{P \mu}^a F_{P \mu}^b - (k_{P \mu} E_{P \mu}^a - k_{P \mu} E_{P \mu}^b) (F_{Q \mu}^a \eta_{Q \mu} - F_{Q \mu}^b \eta_{Q \mu}^{a b}) (k_{R \mu} E_{P \mu}^c - k_{R \mu} E_{R \mu}^c)] \\
+ \frac{k^a}{2} \delta^T_{ab} \sum_{P=Q} (k_{P \mu} E_{P \mu}^a - k_{P \mu} E_{P \mu}^b) F_{P \mu}^a S_{P \mu}^{b c} (k_{T \sigma} E_{T \sigma c} - k_{T \rho} E_{T \sigma c}), \quad (43)
\]

where we have used the multiparticle currents of \( T_{P \mu}^a \), \( W_{P \mu}^{ab} \) and \( \omega_{P \mu}^{ab} \), respectively \( T_{P \mu}^a \), \( W_{P \mu}^{ab} \) and \( \omega_{P \mu}^{ab} \). The \( n \)-point tree level amplitudes are defined as

\[
M_n \equiv \kappa \lim_{s_2 \ldots n \rightarrow 0} s_{2 \ldots n} e_{1 \mu}^a E_{2 \ldots n \mu}. \quad (44)
\]

Similarly to 13, the invariance of this amplitude under the residual gauge transformations has a straightforward demonstration in the gauge 41.

Now that we are able to consistently account for fermionic degrees of freedom in the multiparticle solutions, it is just a small step to consider supersymmetric field theories, in particular supergravity.

**VII. FINAL REMARKS**

In this letter we have found multiparticle solutions to Einstein’s field equations, with a compact recursive definition for graviton multiparticle currents in \( D \)-dimensional Minkowski space. These currents can then be used to compute any tree level scattering between gravitons and matter particles, with or without supersymmetry.

The key insight is the recursive definition of the inverse metric \( g^{\mu\nu} \) in 15–19, with an analogous expression for the inverse vielbein \( e^a_{\mu} \) in 29–40. Effectively, this recursion works as a truncation of the gravity action. It is yet another way of seeing that the infinite number of graviton vertices, though required by diffeomorphism invariance, play no role at the tree level dynamics.

The practical appeal of our formulas is that they can be easily computerized. For pure gravity amplitudes, this might not present an advantage over current methods, in particular the double copy construction using color-kinematics duality 7,8, and BCFW on-shell recursion 12,18. Nevertheless, for the mixed scattering of gravitons and matter particles, the ingredients presented here constitute a versatile tool for computing tree level amplitudes in a broad class of theories. Our results do not require an underlying worldsheet theory (a stringy origin) and can be applied to more general field configurations. We believe our solutions can become a robust standard for such amplitudes, with a reliable framework to be used to test both (1) the extension of current methods and (2) possible new techniques for tree level scattering.

Towards a more efficient algorithm implementation, the recursions we presented can be recast in a “color-stripped” form. The name is inherited from the Yang-Mills perturbiner, where such a construction is more natural due to the color structure. For the graviton, a color-stripped perturbiner is not limited by the ordered words in the multiparticle expansion. In this case, the word splitting needed to recursively define the currents is greatly simplified (see e.g. 28). For
example, the color-dressed deshuffle $P = Q \cup R$ of a $n$-letter word leads to $(2^n - 2)$ pairs of ordered words. This is to be compared with mere $(n-1)$ pairs of the deconcatenation $P = QR$ in the color-stripped form. We just have to be careful to properly symmetrize the final amplitudes with respect to the graviton legs, but this is computationally much less costly.

A more immediate extension of our results would be to consider multiparticle expansions in curved space. While this cannot be efficiently developed in general backgrounds, we found that the perturbiner extension to (anti) de Sitter spaces leads to an intuitive recursive definition of Witten diagrams [14] for different matter fields. This is being explored in an ongoing project [15].

Finally, a quick comment on loop computations. The perturbiner method seems to be intrinsically classical: it consists of solving equations of motion. However, recent results using homotopy algebras in quantum field theory [16] have uncovered that loop-level scattering amplitudes can also be recursively computed. The natural question then is: can we reformulate these recursions as solutions of some quantum equation of motion? Although this speculation looks a bit far-fetched, based on very preliminary investigations we think the answer might be affirmative.

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