Deformation of Dyonic Black Holes and Vacuum Geometries in Four Dimensional $N = 1$ Supergravity

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ABSTRACT

We study some aspects of dyonic non-supersymmetric black holes of four dimensional $N = 1$ supergravity coupled to chiral and vector multiplets. The scalar manifold can be considered as a one-parameter family of Kähler manifolds generated by a Kähler-Ricci flow equation. This setup implies that we have a family of dyonic non-supersymmetric black holes deformed with respect to the flow parameter related to the Kähler-Ricci soliton. We mainly consider two types of the black holes, namely extremal dyonic Reissner-Nordström-like and Bertotti-Robinson-like black holes. In addition, the corresponding vacuum structures for such black holes are also discussed in the context of Morse-Bott theory. Finally, we give some simple $\mathbb{C}P^n$-models whose superpotential and gauge couplings have the linear form.
1 Introduction

Some intensive studies have been done by two of the authors in exploring the nature of solitonic solutions such as domain walls of four dimensional $N = 1$ supergravity coupled to chiral multiplets $[1, 2, 3, 4]$. In those papers, we consider in particular BPS domain walls of $N = 1$ supergravity in which the scalar manifold can be viewed as a one-parameter family of Kähler manifolds generated by the Kähler-Ricci flow equation $[6, 7, 8]$. In other words, we have the chiral $N = 1$ theory on Kähler manifolds whose volume is not fixed and so, the manifold evolves with respect to a real parameter, say $\tau$. This setup implies that BPS equations depend on $\tau$ describing a family of BPS domain walls. Moreover, the flow parameter $\tau$ indeed controls the stability of the walls near Lorentz invariant vacua.

In this paper we continue the study by considering another type of solutions, namely black hole solutions. Moreover, those objects belong to the class of solitonic solutions of four dimensional $N = 1$ supergravity coupled to chiral and vector multiplets on a Kähler-Ricci soliton regarded as a one-parameter family of Kähler manifolds. In particular, the black holes are non-supersymmetric and admit a spherical symmetry. Additionally, since the vector multiplets present, so the black holes may have both electric and magnetic charges. Such objects are called dyonic black holes. Similar to the case mentioned above, as consequences of the setup, thus we have a family of non-supersymmetric, dyonic, and spherical symmetric black holes.

In order to obtain such black holes, one has to solve a set of equations of motions such as the Einstein field equation, the gauge field and the scalar field equations of motions by varying the $N = 1$ supergravity action with respect to the metric, gauge fields, and scalar fields on the spherical symmetric metric together with an additional necessary condition, namely the variation of the flow parameter $\tau$ vanishes $[4]$. The analysis of the equations of motions shows that there is an additional potential called black hole potential $[12]$ beside the scalar potential. These potentials play an important role in analyzing the nature of black holes and the corresponding vacuum structures.

Here, we intensely study some aspects of the black holes where the scalar fields are frozen with respect the spacetime coordinates. For such black holes, we refer to as extremal black holes. Moreover, the extremality further follows that there exits a vacuum structure which can be viewed as a submanifold of Kähler manifolds. Unlike domain wall cases in $[1, 2, 3]$, since we couple vector multiplets to the theory, so we have generally a family of submanifolds which depends on electric and magnetic charges, and deformed with respect to the flow parameter $\tau$. The shape of these submanifolds depends on the type of black holes and can be characterized by the dimension and the Morse-Bott index extracting from the Hessian of the potentials which could be both the black hole and the scalar potentials or a potential formed by them $[5]$.

In the present paper, we mainly focus on two kinds of dyonic spherical symmetric black holes. The first kind is called a family of extremal dyonic Reissner-Nordström-like black holes whose backgrounds have to be a family of four dimensional Einstein geometries. The corresponding vacuum structures of the model are the critical points of both the

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2We particularly study properties of BPS domain walls from a chiral multiplet in $[5]$.

3Interested reader can further consult some excellent reviews of the subject, for example in $[9, 10, 11]$.

4This necessary condition could also be applied in more general cases such as non-supersymmetric theories. For example, this has been considered in the case of BPS-like domain walls $[3]$.

5See Section 4 and 5 for details.
black hole and the scalar potentials. In this case, as we will see later, the flow parameter $\tau$ may determine the nature of backgrounds which could be de Sitter (dS), Minkowski, or anti-de Sitter (AdS).

The second kind is called a family of extremal dyonic Bertotti-Robinson-like black holes. Such black holes are occurred near horizon and the geometry is a product of two families of surfaces, namely the families of two dimensional surfaces and two-spheres. We particularly consider the case where the two dimensional surfaces are the family of Einstein surfaces with positive entropy. The vacuum structures for this case can be viewed as the critical points of so called effective black hole potential formed by the black hole and the scalar potentials.

The organization of this paper can be mentioned as follows. In Section 2 we discuss some aspects of four dimensional $N = 1$ supergravity coupled to chiral and vector multiplets. In particular, as mentioned above, the $N = 1$ theory on Kähler-Ricci solitons. Then, in Section 3 we derive a set of equations of motions including the Einstein field equation, the gauge field and the scalar field equations of motions. Section 4 is devoted to exploit some properties of extremal dyonic Reissner-Nordström-like black holes and their vacuum structure. Next, we consider another type of black holes so called extremal dyonic Bertotti-Robinson-like black holes together with their vacuum structure in Section 5. We give some simple models for those types of black holes in Section 6. Finally, we conclude our results in Section 7.

## 2 $N = 1$ Supergravity Coupled To Vector Multiplets on Kähler-Ricci Soliton

In this section we consider some properties of four dimensional $N = 1$ supergravity coupled to vector multiplets on a one-parameter family of Kähler manifolds generated by a Kähler-Ricci soliton satisfying [6, 7]

$$
\frac{\partial g_{ij}}{\partial \tau}(z, \bar{z}; \tau) = -2R_{ij}(z, \bar{z}; \tau), \quad 0 \leq \tau < T ,
$$

(2.1)

where $\tau \in \mathbb{R}$ and the indices $i, j$ run from 1 until the dimension of the Kähler-Ricci soliton. Here, we only consider the "ungauged" case in the sense that we omit the isometries of the Kähler-Ricci soliton.

The $N = 1$ theory contains a gravitational multiplet coupled with $n_v$ vector and $n_c$ chiral multiplets. The gravitational multiplet has a vierbein $e^a_\mu$ and a vector spinor $\psi_\mu$ where $a = 0, ..., 3$ and $\mu = 0, ..., 3$ are the flat and the curved indices, respectively. A vector multiplet consists of a vector $A_\mu$ and its spin-$\frac{1}{2}$ fermionic superpartner $\lambda$, whereas the field ingredients of a chiral multiplet are a complex scalar $z$ and a $\chi$.

The construction of the local $N = 1$ theory on a Kähler-Ricci is as follows. First, we consider the Lagrangian in [13, 14] as the initial Lagrangian at $\tau = 0$. Then, by replacing all couplings that depend on the geometric quantities such as the metric $g_{ij}(0)$

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6Here, we rather use the convention in [13].
by the soliton $g^{ij}(\tau)$, the bosonic parts of the Lagrangian has the form

$$\mathcal{L}^{N=1} = -\frac{M_P^2}{2} R + R_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\Sigma|\mu\nu} + I_{\Lambda\Sigma} F_{\mu\nu}^\Lambda \tilde{F}^{\Sigma|\mu\nu} + g_{\bar{j}j}(z,\bar{z};\tau) \partial_\mu z^i \partial_\nu \bar{z}^j - V(z,\bar{z};\tau), \quad (2.2)$$

where $M_P$ is the Planck mass. The quantity $R$ is the Ricci scalar of the four dimensional spacetime. The scalar fields $(z, \bar{z})$ span a Hodge-Kähler manifold endowed with metric $g_{\bar{j}j}(z,\bar{z};\tau) \equiv \partial_\bar{i} \partial_j K(z,\bar{z};\tau)$ satisfying (2.1) with $K(z,\bar{z};\tau)$ is a real function called the Kähler potential. In this case we have $i, j = 1, ..., n_c$. The gauge field strength and its dual are defined as

$$F_{\mu\nu}^\Lambda \equiv \frac{1}{2} \left( \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu \right),$$

$$\tilde{F}^\Lambda_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^\Lambda_{\rho\sigma}, \quad (2.3)$$

respectively. The gauge couplings $R_{\Lambda\Sigma}$ and $I_{\Lambda\Sigma}$ are given by

$$R_{\Lambda\Sigma} \equiv \text{Re} N_{\Lambda\Sigma},$$

$$I_{\Lambda\Sigma} \equiv \text{Im} N_{\Lambda\Sigma}, \quad (2.4)$$

with $N_{\Lambda\Sigma}$ are arbitrary holomorphic functions, namely $N_{\Lambda\Sigma} \equiv N_{\Lambda\Sigma}(z)$ and $\Lambda, \Sigma = 1, ..., n_c$. Moreover, in order to have a consistent theory the gauge coupling matrix $N$ must be invertible. The $N = 1$ scalar potential $V(z,\bar{z};\tau)$ has the form

$$V(z,\bar{z};\tau) = e^{K(\tau)/M_P^2} \left( g^{\bar{j}j}(\tau) \nabla_i W \nabla_j W - \frac{3}{M_P^2} W W \right), \quad (2.5)$$

where $W$ is a holomorphic superpotential, $K(\tau) \equiv K(z,\bar{z};\tau)$, and $\nabla_i W \equiv \partial_i W + (K_{\bar{i}}(\tau)/M_P^2) W$.

The Lagrangian (2.2) is invariant under the following supersymmetry transformations up to three-fermion terms

$$\delta \psi_{1\nu} = M_P \left( D_\nu \epsilon_1 + \frac{i}{2} e^{K(\tau)/2M_P^2} W \gamma_\nu \epsilon_1 + \frac{i}{2M_P} Q_\nu(\tau) \epsilon_1 \right),$$

$$\delta \lambda^A_1 = \frac{1}{2} (F^\Lambda_{\mu\nu} - i \tilde{F}_{\mu\nu}^\Lambda) \gamma^{\mu\nu} \epsilon_1,$$

$$\delta \chi^i = i \partial_\nu z^i \gamma^\nu \epsilon_1 + N^i(\tau) \epsilon_1,$$

$$\delta e^a_\nu = -\frac{i}{M_P} \left( \tilde{\psi}_{1\nu} \gamma^a \epsilon_1 + \bar{\psi}_1 \gamma^a \epsilon_1 \right),$$

$$\delta A^A_\mu = \frac{1}{2} \lambda^A \gamma_\mu \epsilon_1 + \frac{i}{2} \bar{\epsilon}_1 \gamma_\mu \lambda^A,$$

$$\delta z^i = \bar{\chi}^i \epsilon_1,$$

where $N^i(\tau) \equiv e^{K(\tau)/2M_P^2} g^{\bar{j}j}(\tau) \nabla_j W$, $g^{\bar{j}j}(\tau)$ is the inverse of $g_{\bar{j}j}(\tau)$, and the $U(1)$ connection $Q_\nu(\tau) \equiv - (K_{\bar{i}}(\tau) \partial_\nu z^i - K_i(\tau) \partial_\nu \bar{z}^i)$. In addition, we have also introduced $\epsilon_1 \equiv \epsilon_1(x,\tau)$.
3 Dyonic Black Holes: The Equations of Motions

In this section we mainly discuss some aspects of the equation of motion derived from the Lagrangian (2.2), namely the Einstein equation together with the equation of motion of the vector and the scalar fields. In particular, the four dimensional metric ansatz is set to be static and spherical symmetric.

Let us first consider the equations of motion of the fields which can be obtained by varying the action related to the Lagrangian (2.2) with respect to \( g_{\mu\nu}, A_\mu^\Lambda, \) and \( z^i \) and also, applying the necessary condition that the variation of the flow parameter \( \tau \) vanishes. The results are the Einstein field equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = g^{ij}(\tau) \left( \partial_{\mu} z^i \partial_{\nu} z^j + \partial_{\nu} z^i \partial_{\mu} z^j \right) - g^{ij}(\tau) g_{\mu\nu} \partial_\rho z^i \partial_\rho z^j + 4 \mathcal{R}_{\Lambda\Sigma} F^{\Lambda}_{\mu\rho} F^\Sigma_{\nu\sigma} g^{\rho\sigma} - g_{\mu\nu} \mathcal{R}_{\Lambda\Sigma} F^{\Lambda}_{\rho\sigma} F^\Sigma_{\nu\sigma} + g_{\mu\nu} V(\tau),
\]

where \( V(\tau) \equiv V(z, \bar{z}; \tau) \), the gauge field equation of motion

\[
\partial_\nu \left( \varepsilon^{\mu\nu\rho\sigma} \sqrt{-g} \, G_{\Lambda|\rho\sigma} \right) = 0,
\]

and the scalar field equation of motion

\[
\frac{g^{ij}(\tau)}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu z^j \right) + \bar{\tilde{\alpha}} g_{\nu j}(\tau) \partial_\nu \bar{z}^i \partial_\rho z^i = \partial_\tau \mathcal{R}_{\Lambda\Sigma} F^\Lambda_{\mu\rho} F_\Sigma^{\mu\nu} + \partial_\tau \mathcal{I}_{\Lambda\Sigma} F^\Lambda_{\rho\sigma} F_\Sigma^{\mu\nu} - \partial_\tau V(\tau),
\]

where \( g \equiv \det(g_{\mu\nu}) \). Moreover, we also have the Bianchi identities

\[
\partial_\nu \left( \varepsilon^{\mu\nu\rho\sigma} \sqrt{-g} \, F^\Lambda_{\rho\sigma} \right) = 0,
\]

from the definition of \( F^\Lambda_{\rho\sigma} \).

Then, our interest is to solve the equations \((3.1)-(3.3)\) on the background where the four dimensional metric ansatz admits static and spherical symmetric given by \(\Sigma\)

\[
ds^2 = e^{A(r, \tau)} dt^2 - e^{B(r, \tau)} dr^2 - e^{C(r, \tau)} (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( A(r, \tau), B(r, \tau), \) and \( C(r, \tau) \) are generally assumed to be \( \tau \)-dependent functions. In other words, the above ansatz defines a one-parameter family of spherical static metrics with respect to \( \tau \). It is worth mentioning that in general a redefinition of the radial coordinate \( r \) cannot be employed since all of the functions vary with respect to \( \tau \) and in addition, the ansatz \((3.5)\) may have a singularity at finite \( \tau \). However, there are two particular situations where one can redefine \( r \). The first case is when all functions do not depend on \( \tau \), while the second case is that we can do the redefinition of \( r \) at a particular value of \( \tau \). The latter case is so because we have a one-parameter family of spherical symmetric geometries in which there possibly exists a topological changes of the shape of geometries described by the ansatz \((3.5)\) with respect to the flow parameter \( \tau \). We will back to this issue in Section \(\Sigma\).

Let us first investigate the gauge field equation of motions \((3.2)\) together with the Bianchi identities \((3.4)\). We simply take a case where the nonzero electromagnetic field

\footnote{The metric ansatz here is similar to \cite{15} for the \( N = 2 \) supergravity case and all of them depend only on the radial coordinate \( r \), namely \( A(r), B(r), \) and \( C(r) \).}
strength components are $F^A_{01}$ and $F^A_{23}$. Then, solving the Bianchi identities \((3.4)\) we find $F^A_{01} = F^A_{01}(r)$ and $F^A_{23} = F^A_{23}(\theta)$. These imply that the solution of \((3.2)\) on the spacetime metric \((3.5)\) has to be

$$F^A_{01} = \frac{1}{2} e^{\frac{1}{2}(A+B)-C} (R^{-1})^{\Lambda\Sigma}(\mathcal{I}_{2\Sigma} g^\Lambda - q_\Lambda) ,$$

$$F^A_{23} = -\frac{1}{2} g^A \sin \theta ,$$

\((3.6)\)

where $q_\Lambda$ and $g^A$ are the electric and magnetic charges, respectively [15]. In addition, the charges $q_\Lambda$ and $g^A$ are real constants.

Inserting the metric ansatz \((3.5)\) together with \((3.6)\) into the equations of motion \((3.1)\) and \((3.3)\), we obtain the following equations

$$-e^{-B} \left( C'' + \frac{3}{4} C'^2 - \frac{1}{2} C' B' \right) + e^{-C} = e^{-B} g_{ij}(\tau) z^{i' \bar{z} j'} + e^{-2 C} V_{\text{BH}} + V(\tau) ,$$

$$-\frac{1}{2} C' \left( \frac{1}{2} C' + A' \right) + e^{B-C} = -g_{ij}(\tau) z^{i' \bar{z} j'} + e^B \left( e^{-2 C} V_{\text{BH}} + V(\tau) \right) ,$$

$$-\frac{1}{2} e^{-B} \left( A'' + C'' + \frac{1}{2} (A' + C') (A' - B') + \frac{1}{2} C'^2 \right) = e^{-B} g_{ij}(\tau) z^{i' \bar{z} j'} - e^{-2 C} V_{\text{BH}} + V(\tau) ,$$

$$g_{ij}(\tau) \bar{z}^{i''} + \bar{\partial}_k g_{ij}(\tau) \bar{z}^{i' \bar{z} j'} + \frac{1}{2} (A' - B' + 2 C') g_{ij}(\tau) \bar{z}^{i' j'} = e^B \left( e^{-2 C} \partial_\tau V_{\text{BH}} + \partial_\tau V(\tau) \right) ,$$

\((3.7)\)

where we have also assumed $z^i = z^i(r, \tau)$ and $z^{i'} \equiv \partial z^i / \partial r$. The scalar function $V_{\text{BH}}$ is defined as

$$V_{\text{BH}} \equiv -\frac{1}{2} (g \ q) \mathcal{M} \left( \begin{array}{c} g \\ q \end{array} \right) ,$$

\((3.8)\)

which is called the black hole potential [12] where

$$\mathcal{M} = \left( \begin{array}{ccc} R + \mathcal{I} R^{-1} \mathcal{I} & -\mathcal{I} R^{-1} \\ -R^{-1} \mathcal{I} & R^{-1} \end{array} \right) .$$

\((3.9)\)

The function $V(\tau)$ is the scalar potential \((2.5)\). Note that in this case we have $V_{\text{BH}} \geq 0$, while $V(\tau) \in \mathbb{R}$ and is assumed to be well-defined for finite $\tau \neq \tau_0$ where $\tau_0$ is a singular point of the geometric flow \((2.1)\). In order to have a well-defined theory, one has to set the matrix \((3.9)\) to be invertible. In this paper we assume that $V_{\text{BH}} = 0$ if all charges vanish.

### 4 Extremal Dyonic Reissner-Nordström-like Black Holes

In this section we consider a case where the scalar fields $z^i$ are frozen with respect to the radial coordinate $r$, namely $z^{i'} = 0$. In particular, the geometry of black holes is taken to be similar to the Reissner-Nordström spacetime, namely a family of extremal dyonic Reissner-Nordström-like spacetimes which are non-supersymmetric. We organize this section into two parts. First, we consider some general aspects of dyonic Reissner-Nordström-like black holes in various backgrounds. Then, in the second part the relation between vacuum structures and Morse theory will be discussed in detail.
4.1 General Picture

Within the ansatz (3.5), in order to obtain the extremal generalized (dyonic) Reissner-Nordström-like black hole one has to set

\[
\frac{z_i}{z_i}' = 0, \quad \partial_i V_{\text{BH}} = 0, \quad \partial_i V(\tau) = 0.
\]

(4.1)

The second equation in (4.1) defines critical points of the black hole potential \( V_{\text{BH}} \) in which we have generally \( \tilde{z}_0^i(g, q) \). However, there is also a possible case where

\[
\partial_i N_{\Lambda \Sigma} = 0 \implies \partial_i V_{\text{BH}} = 0,
\]

(4.2)

which means that means that the critical points of the black hole potential \( V_{\text{BH}} \) are also the critical points of the gauge couplings \( N_{\Lambda \Sigma} \), namely \( \tilde{z}_0^i \). Meanwhile, the third equation in (4.1) implies \( \tilde{z}_0^i(\tau) \). We will discuss some aspects of (4.1) in the next subsection.

In addition, in this region the function \( C(r, \tau) \) simply takes the form

\[
C(r, \tau) = 2 \ln r + \ln \hat{\sigma}(\tau),
\]

(4.3)

where \( \hat{\sigma}(\tau) \) is an arbitrary function of the flow parameter \( \tau \) and related to the area deformation of the two-sphere \( S^2 \) since the terms \( d\theta^2 + \sin^2 \theta d\phi^2 \) in the ansatz (3.5) describe the two-sphere \( S^2 \simeq \mathbb{CP}^1 \). In other words, \( 2n_c \)-dimensional Kähler-Ricci solitons described by (2.1) may induce the area deformation of \( S^2 \) which might also cause a topological change of the black hole geometry. However, the form of \( C(r, \tau) \) in (4.3) shows that we always have

\[
\hat{\sigma}(\tau) > 0, \quad \text{for all } \tau.
\]

(4.4)

We have two examples as follows. First, if \( \hat{\sigma}(\tau) \) is fixed for all \( \tau \), say \( \hat{\sigma}(\tau) = 1 \), then there is no area deformation of \( S^2 \) caused by the soliton (2.1). Second, if we take

\[
\hat{\sigma}(\tau) = 1 - 4\tau,
\]

(4.5)

then the \( 2n_c \)-dimensional Kähler-Ricci soliton described by (2.1) induces a two dimensional Kähler-Ricci soliton on \( S^2 \) which diverges at \( \tau = 1/4 \). For \( \tau < 1/4 \), one has \( S^2 \) with the constant \( \Lambda_2 = 2 \) and the signature of the metric (3.5) is \((+, -, -, -)\). On the other hand, for \( \tau > 1/4 \), the flow changes to \( \hat{S}^2 \) with new constant \( \hat{\Lambda}_2 = -2 \) and the signature of the metric (3.5) becomes \((+, +, +, -)\). Therefore, the condition (4.4) forbids the latter situation and we only have \( \tau < 1/4 \).

Let us first mention the case for uncharged black holes, namely \( g^\Lambda = q_\Lambda = 0 \) for all \( \Lambda \), which are useful for our analysis in this section.

**Lemma 4.1.1.** For uncharged extremal black holes satisfying (4.3), the black hole geometry is a family of four dimensional Einstein manifolds.

The above Lemma further restricts the solution of (3.7) in order to get a consistent theory. Now, we discuss the solution of the equation of motions (3.7) in the extremal condition (4.1) by writing down the following results.
Theorem 4.1.2. Suppose \( p_0 \equiv (z_0(g, q; \tau), \bar{z}_0(g, q; \tau)) \) is a vacuum satisfying (4.1). Then, at \( p_0 \) the solution of the equation of motions (3.7) has the form

\[
ds^2 = \Delta(r, \tau) \, dt^2 - \Delta(r, \tau)^{-1} \, dr^2 - \hat{\sigma}(\tau) r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]

(4.6)

with

\[
\Delta(r, \tau) = \hat{\sigma}(\tau)^{-1} - \frac{2M(\tau)}{r} + \frac{V_{\text{BH}}^0(\tau)}{\hat{\sigma}^2(\tau) r^2} - \frac{1}{3} V_0(\tau) r^2,
\]

(4.7)

where we have defined

\[
V_{\text{BH}}^0(\tau) \equiv V_{\text{BH}}(p_0), \quad V_0(\tau) \equiv V(p_0; \tau).
\]

(4.8)

Proof. Inserting (4.1) and (4.3) into (3.7) and after some computations we finally obtain a general form

\[
\Delta(r, \tau) = a(\tau) - \frac{2M(\tau)}{r} + b(\tau) \left[ \frac{V_{\text{BH}}^0(\tau)}{\hat{\sigma}^2(\tau) r^2} - \frac{1}{3} V_0(\tau) r^2 \right].
\]

(4.9)

Then, using Lemma 4.1.1 and by simply taking \( b(\tau) = 1 \) for all \( \tau \) we have \( a(\tau) = \hat{\sigma}(\tau)^{-1}. \)

It is easy to see that the metric (4.6) does not preserve supersymmetry because the supersymmetry transformations (2.6) do not vanish on the background (4.6). Note that \( M(\tau) \) is the black hole’s mass and \( M(\tau) \geq 0 \) which evolves generally with respect to \( \tau \). On the other side, the black hole potential \( V_{\text{BH}}^0(\tau) > 0 \) for all \( \tau \) and \( V_{\text{BH}}^0(\tau) = 0 \) when all charges vanish.

Some comments are as follows. In general, the cosmological constant \( V_0(\tau) \) may deform with respect to \( \tau \) for \( \tau \neq \tau_0 \). We have three different backgrounds, namely de Sitter (dS) for \( V_0(\tau) > 0 \), Minkowskian for \( V_0(\tau) = 0 \), and anti-de Sitter (AdS) for \( V_0(\tau) < 0 \). However, as we shall see in Section 6 there exists a model where \( V_0(\tau) \) does not depend on \( \tau \) at a particular point \( \tau_0 \). In addition, the metric (4.6) generally becomes Einstein in asymptotic region, namely around \( r \rightarrow +\infty \), for finite \( M(\tau), V_{\text{BH}}^0(\tau) \) and \( V_0(\tau) \).

Let us turn our attention to discuss some aspects of black hole thermodynamics. Suppose there exists an event horizon radius \( r_+ \equiv r_+(\tau) \) of the metric (4.6), such that \( \Delta(r_+, \tau) = 0 \). Then the mass of a black hole is given by

\[
M(r_+, \tau) = \frac{1}{2} r_+ \left( \hat{\sigma}(\tau)^{-1} + \frac{V_{\text{BH}}^0(\tau)}{\hat{\sigma}^2(\tau) r_+^2} - \frac{1}{3} V_0(\tau) r_+^2 \right).
\]

(4.10)

Its entropy has the form [16]

\[
S(\tau) = \frac{A_h(\tau)}{4} = \pi r_+^2,
\]

(4.11)

and \( A_h(\tau) \) is the area of a black hole. The Hawking temperature for the horizon is determined by

\[
T_+ = \frac{\kappa(r_+)}{2\pi},
\]

(4.12)

\(^8\)Such a situation also occurs in the case of domain walls [11 2].
where $\kappa(r_+)$ is the surface gravity at the horizon

$$\kappa(r_+) = \frac{1}{2} |\Delta'(r_+, \tau)| . \quad (4.13)$$

As mentioned above, we have $V_0(\tau) \in \mathbb{R}$ for $\tau \neq \tau_0$. Thus, the metric (4.6) have three backgrounds which can be split into the following cases.

### 4.1.1 The $V_0(\tau) > 0$ case

In this case we have a situation which is the case of dS backgrounds. Here, it has in general three horizons with radii denoted by $\hat{r}_c, \hat{r}_+, \hat{r}_-$,

$$\hat{r}_c \equiv \hat{r}_c(\tau), \quad \hat{r}_+ \equiv \hat{r}_+(\tau), \quad \hat{r}_- \equiv \hat{r}_-(\tau), \quad (4.14)$$

and $\hat{r}_c > \hat{r}_+ \geq \hat{r}_-$ which are the positive roots of (4.7)\footnote{The fourth root of \(4.7\) is negative and therefore, has no physical meaning.}. The radii have three physical meanings: $\hat{r}_c$ is the radius of the cosmological horizon, $\hat{r}_+$ is the radius of the event horizon, and $\hat{r}_-$ is the radius of the Cauchy horizon \cite{17}. For such a situation, the necessary and sufficient conditions are

$$0 \leq V_{BH}^0(\tau) < \frac{1}{4V_0(\tau)},$$

$$M_e(\hat{r}_e, \tau) \leq M(\hat{r}_+, \tau) < M_e(\hat{r}_0, \tau), \quad (4.15)$$

where

$$M_e(\hat{r}_e, \tau) = \frac{1}{3\hat{\sigma}(\tau)} (2\hat{\sigma}(\tau)V_0(\tau))^{-1/2} \left[ 1 - (1 - 4V_0(\tau)V_{BH}^0(\tau))^{1/2} \right]^{1/2} \left[ 2 + (1 - 4V_0(\tau)V_{BH}^0(\tau))^{1/2} \right], \quad (4.16)$$

is the mass of the black hole whose two horizons coalesce, namely

$$\hat{r}_+ = \hat{r}_- \equiv \hat{r}_e = (2\hat{\sigma}(\tau)V_0(\tau))^{-1/2} \left[ 1 - (1 - 4V_0(\tau)V_{BH}^0(\tau))^{1/2} \right]^{1/2}, \quad (4.17)$$

for some finite $\tau$. Meanwhile, $M_e(\tau)$ is defined as

$$M_e(\hat{r}_0, \tau) = \frac{1}{3\hat{\sigma}(\tau)} (2\hat{\sigma}(\tau)V_0(\tau))^{-1/2} \left[ 1 + (1 - 4V_0(\tau)V_{BH}^0(\tau))^{1/2} \right]^{1/2} \left[ 2 - (1 - 4V_0(\tau)V_{BH}^0(\tau))^{1/2} \right], \quad (4.18)$$

is again the mass of the black hole whose two horizons coincide with

$$\hat{r}_c = \hat{r}_+ \equiv \hat{r}_0 = (2\hat{\sigma}(\tau)V_0(\tau))^{-1/2} \left[ 1 + (1 - 4V_0(\tau)V_{BH}^0(\tau))^{1/2} \right]^{1/2}, \quad (4.19)$$

for some finite $\tau$. In general, if $\hat{r}_c > \hat{r}_+$ and $\hat{r}_+ \neq \hat{r}_-$, then $\hat{r}_+ > \hat{r}_e > \hat{r}_-$. The conditions in (4.15) are the generalization of the previous results in \cite{17} \cite{18} \cite{19}.

Let us consider some cases as follows. For $\hat{r}_+ = \hat{r}_-$, we have a black hole which is referred to as a cold black hole with vanishing Hawking temperature (4.12) at $\hat{r}_c$ \cite{18}. Note that the Hawking temperature is non-zero at the outer horizon, i.e., at $\hat{r}_c$, namely

$$T_c = \frac{\kappa(\hat{r}_c)}{2\pi}. \quad (4.20)$$
If \( \hat{r}_c = \hat{r}_+ > \hat{r}_- \), then we have a special solution called a family of dyonic Nariai-like black holes. Let us first assume that such a situation occurs at finite \( \tau = \tau_n \neq \tau_0 \). In this case, the radial coordinate \( r \) is no longer suitable to describe the region around \( \hat{r}_c = \hat{r}_+ \). Then, one has to set the spacetime coordinate transformation \[ r = \hat{r}_0 + \varepsilon \cos \chi, \quad \psi = \Delta_0 \varepsilon t, \] (4.21) evaluated at \( \tau = \tau_n \) with \( 0 < \varepsilon \ll 1 \) and \[ \Delta_0 \equiv \Delta_0(\tau_n) = \frac{2V_0(\tau_n) \left(1 - 4V_0(\tau_n)V_{\text{BH}}^0(\tau_n)\right)^{1/2}}{\left[1 + \left(1 - 4V_0(\tau_n)V_{\text{BH}}^0(\tau_n)\right)^{1/2}\right]}, \] (4.22)

where we have also assumed that \( \hat{\sigma}(\tau_n) \) is finite. So, we obtain \( \hat{r}_c = \hat{r}_0 + \varepsilon \) at \( \chi = 0 \) and \( \hat{r}_+ = \hat{r}_0 - \varepsilon \) at \( \chi = \pi \). Next, inserting (4.21) into the metric (4.6) and taking the limit \( \varepsilon \to 0 \), the resulting metric has the form \[ ds^2 = -\frac{1}{\Delta_0} (d\chi^2 - \sin^2 \chi d\psi^2) - \frac{\hat{\sigma}(\tau_n)}{\Sigma_0} (d\theta^2 + \sin^2 \theta d\phi^2), \] (4.23)

where \( \Delta_0 + \Sigma_0 = 2V_0(\tau_n) \). If we have \( \hat{r}_c = \hat{r}_+ \) for some finite \( \tau \), then we just replace \( \tau_n \) by \( \tau \) in (4.22) and (4.23). Since \( \hat{\sigma}(\tau) > 0 \) for all \( \tau \), the topology is \( S^{1,1} \times S^2 \) with different radii. In this case the Hawking temperature vanishes at \( \hat{r}_0 \), but it is non-zero at \( \hat{r}_- \). The metric (4.23) is again a generalization of those in the literatures, for example, in [20, 21, 22].

Another interesting aspect is a case where \( \Delta(r, \tau) \) has a triple root. Such an object is an ultracold black holes with vanishing Hawking temperature [18]. In this case, \( M(\tau), \ V_{\text{BH}}^0(\tau), \) and \( V_0(\tau) \) can be written as functions of the root \[^{10}\].

Finally, we consider a case where \( \Delta(r, \tau) \) does not have double roots, but it must have two distinct roots, namely \( \hat{r}_c \) and \( \hat{r}_+ \) such that this black hole has the same Hawking temperature at \( \hat{r}_c \) and \( \hat{r}_+ \). So, for the case at hand, we have the condition

\[ |\Delta'(\hat{r}_c, \tau)| = |\Delta'(\hat{r}_+, \tau)|, \]

which follows

\[ M^2(\tau) = \frac{V_{\text{BH}}^0(\tau)}{\hat{\sigma}^3(\tau)}. \] (4.25)

Such a black hole is called lukewarm black hole [18] because the Hawking temperature does not vanish at \( \hat{r}_c, \hat{r}_+ \) and \( \hat{r}_- \).

### 4.1.2 The \( V_0(\tau) = 0 \) case

In this case we have Minkowskian backgrounds. Here, the function \[ (4.17) \] has at most two different positive roots describing the radii of two horizons

\[ \hat{r}_+ \equiv \hat{r}_+(\tau) = \hat{\sigma}(\tau)M(\tau) \pm \left[\hat{\sigma}^2(\tau)M^2(\tau) - \hat{\sigma}(\tau)^{-1}V_{\text{BH}}^0(\tau)\right]^{1/2}, \] (4.26)

\[^{10}\]The form of the functions is similar to the simplest case in [18]. See [18] for details.
where $\hat{r}_+$ and $\hat{r}_-$ are the radii of the event and Cauchy horizons, respectively. Our interest is to take the case where the radii (4.26) are real, and therefore we obtain the condition

$$M^2(\tau) \geq \frac{V_{0(BH)}^0(\tau)}{\hat{\sigma}^3(\tau)},$$

where $M(\tau)$ is the black hole’s mass given by

$$M(\tau) \equiv M(\hat{r}_+, \tau) = \frac{1}{2} \hat{r}_+ \left( \hat{\sigma}(\tau)^{-1} + \frac{V_{0(BH)}^0(\tau)}{\hat{\sigma}^2(\tau) \hat{r}_+^2} \right).$$ (4.28)

If the equality holds in (4.27), then two horizons coincide. Similar as the previous case, we assume that such a situation occurs at finite $\tau = \tau_d \neq \tau_0$. Then, defining the new radial coordinate at $\tau = \tau_d$, namely $r = \rho + M(\tau_d)$, the metric (4.6) in this case becomes simply

$$ds^2 = \hat{\sigma}(\tau_d)^{-1} \left( 1 + \frac{M(\tau_d)}{\rho} \hat{\sigma}(\tau_d) \right)^{-2} dt^2 - \hat{\sigma}(\tau_d) \left( 1 + \frac{M(\tau_d)}{\rho} \hat{\sigma}(\tau_d) \right)^2 \left[ d\rho^2 + \hat{\sigma}(\tau_d) \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

(4.29)

for finite $M(\tau_d)$. If the equality holds for some finite $\tau$, then we just replace $\tau_d$ by $\tau$.

Near $\rho \to +\infty$, the metric (4.29) turns into a flat metric, while near horizon, i.e. $\rho \to 0$, the geometries become $AdS_{1,1} \times S^2$ which are conformally flat since $\hat{\sigma}(\tau) > 0$ for all $\tau$. In other words, we have a family of conformally flat spacetimes called Bertotti-Robinson spacetimes [23] [11].

### 4.1.3 The $V_0(\tau) < 0$ case

Here, we have AdS backgrounds. For the case at hand, there exists generally two distinct positive roots $\hat{r}_+$ and $\hat{r}_-$ of (4.7) representing the radii of the event and Cauchy horizons, respectively. This is possible if the mass of black holes $M(\tau)$ satisfying

$$M(\tau) \geq M_e(\hat{r}_e, \tau),$$

where

$$M_e(\hat{r}_e, \tau) = \frac{1}{3 \hat{\sigma}(\tau)} (-2 \hat{\sigma}(\tau) V_0(\tau))^{-1/2} \left[ (1 - 4 V_0(\tau) V_{BHH}^0(\tau))^{1/2} - 1 \right]^{1/2} \left[ 2 + (1 - 4 V_0(\tau) V_{BHH}^0(\tau))^{1/2} \right],$$ (4.31)

is the mass of the black hole whose two horizons coalesce with

$$\hat{r}_+ = \hat{r}_- \equiv \hat{r}_e = (-2 \hat{\sigma}(\tau) V_0(\tau))^{-1/2} \left[ (1 - 4 V_0(\tau) V_{BHH}^0(\tau))^{1/2} - 1 \right]^{1/2}.$$ (4.32)

The equations (4.31) and (4.32) are the generalization of the previous results studied, for example, in [18] [25].

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Footnote 11: For a review, see for example in [24].
4.2 Vacuum Geometry and Morse Theory

In this subsection we draw our attention to discuss a vacuum geometry defined in (4.1) and (4.2) that describes non-supersymmetric vacua related to the black hole geometry (4.6). The first step is to introduce some definitions as follows.

**Definition 4.2.1.** A submanifold $\mathcal{M}_v$ that describes critical points of the scalar potential $V(\tau)$, is defined as

$$\mathcal{M}_v \equiv \{ \hat{p}_0 \in \mathcal{M} \mid \partial_i V(\hat{p}_0) = 0 \} \subset \mathcal{M}, \quad (4.33)$$

where $\mathcal{M}$ is a Kähler manifold generated by (2.1) and $\hat{p}_0 \equiv (\hat{z}_0(\tau), \bar{\hat{z}}_0(\tau))$. $\mathcal{M}_v$ is characterized by the quantities $(m_v, \lambda_v)$ where $m_v \equiv m_v(\tau)$ and $\lambda_v \equiv \lambda_v(\tau)$ are the dimension and the index of $\mathcal{M}_v$, respectively.

Note that the index $\lambda_v$ is the Morse-Bott index related to the number of negative eigenvalues of the Hessian matrix of $V(\tau)$, whereas the dimension $m_v$ corresponds to the number of zero negative eigenvalues of the Hessian matrix of $V(\tau)$ \[12\]. Both quantities depend on the flow parameter $\tau$ related to the Kähler-Ricci soliton (2.1). The trivial case for Definition 4.2.1 is when $V(\tau)$ is constant. In a free chiral theory, such a case implies $\mathcal{M}_v \simeq \mathcal{M}$.

**Definition 4.2.2.** A submanifold $\mathcal{M}_b$ that represents critical points of the black hole potential $V_{BH}$, is defined as

$$\mathcal{M}_b \equiv \{ \tilde{p}_0 \in \mathcal{M} \mid \partial_i V_{BH}(\tilde{p}_0) = 0 \} \subset \mathcal{M}, \quad (4.34)$$

where $\tilde{p}_0 \equiv (\tilde{z}_0, \bar{\tilde{z}}_0)$. $\mathcal{M}_b$ has quantities $(m_b, \lambda_b)$ where $m_b \equiv m_b(g, q)$ and $\lambda_b \equiv \lambda_b(g, q)$ are the dimension and the index of $\mathcal{M}_b$, respectively, and $\mathcal{N}_{\Lambda \Sigma}$ are the holomorphic gauge couplings.

The pair $(g, q)$ are the magnetic and electric charges, respectively given in (3.6). The trivial case for Definition 4.2.2 is when the gauge couplings $\mathcal{N}_{\Lambda \Sigma}$ are constants for all $\Lambda, \Sigma$. As mentioned in the preceding subsection, (4.34) has two possible cases. In general we have $\tilde{z}_0(g, q)$, whereas we get $\tilde{z}_0$ if they can be viewed as critical points of the gauge couplings $\mathcal{N}_{\Lambda \Sigma}$, namely (4.2) is satisfied.

Thus, from the above statements we can write down the following results.

**Theorem 4.2.3.** A vacuum submanifold $\mathcal{M}_0$ of $\mathcal{M}$ describing (4.1) has the form

$$\mathcal{M}_0 \equiv \{ p_0 \in \mathcal{M} \mid \partial_i V(p_0) = \partial_i V_{BH}(p_0) = 0 \} \simeq \mathcal{M}_b \cap \mathcal{M}_v, \quad (4.35)$$

whose dimension and Morse index are respectively given by $(m_0, \lambda_0)$. Then, we have the following cases:

1. For both nonconstant $V(\tau)$ and $V_{BH}$, i) we have $p_0 \equiv (z_0(g, q; \tau), \bar{z}_0(g, q; \tau))$ and the vacuum submanifold $\mathcal{M}_0$ has $m_0 \leq m_b$, $m_0 \leq m_v$, and $\lambda_0 \neq (\lambda_b, \lambda_v)$; or ii) we have empty set.

\[12\] If $m_v$ is the number of zero negative eigenvalues of the Hessian matrix of $V(\tau)$, then in general $\text{dim}_\mathbb{R} \mathcal{M}_v \leq m_v$. Here, we particularly take $\text{dim}_\mathbb{R} \mathcal{M}_v = m_v$. 

2. For nonconstant \( V(\tau) \), but \( V_{\text{BH}} \) is constant (or the gauge couplings \( \mathcal{N}_{\Lambda \Sigma} \) are constant) or \( V_{\text{BH}} = 0 \) (or no magnetic and electric charges present) for all \( z \), then \( M_o \simeq M_v \) with \( \lambda_o = \lambda_v \) and \( m_o = m_v \).

3. If \( V(\tau) \) is constant for all \( z \), but nonconstant \( V_{\text{BH}} \), then \( M_o \simeq M_b \) with \( \lambda_o = \lambda_b \) and \( m_o = m_b \).

4. If both \( V(\tau) \) and \( V_{\text{BH}} \) are constant for all \( z \), then \( M_o \simeq M \).

Proof. First of all, in the case 1.i), one has to solve the equations \( \tilde{z}_0(g, q) = z_0(\tau) \) for some \( z \) which results that some pre-coefficients in the gauge couplings \( \mathcal{N}_{\Lambda \Sigma} \) depend on the charges \( (g, q) \) and the flow parameter \( \tau \). So, this concludes \( z_0(g, q; \tau) \). Next, we consider some situations for case 1.i) as follows. Firstly, suppose we have \( m_b < m_v \) and \( M_b \subset M_v \) with \( \lambda_b \neq \lambda_v \), then \( M_o \simeq M_b \) and \( m_o = m_b < m_v \), while the index \( \lambda_o = (\lambda_b, \lambda_v) \) since we have two different indices \( \lambda_b \) and \( \lambda_v \). Secondly, let \( m_v < m_b \) and \( M_v \subset M_b \) with \( \lambda_b \neq \lambda_v \), then it follows \( M_o \simeq M_v \) and \( m_o = m_v < m_b \) and \( \lambda_o = (\lambda_b, \lambda_v) \). Finally, if \( M_o \subset M_b \) and \( M_o \subset M_v \) with \( \lambda_b \neq \lambda_v \), then \( M_o \simeq M_b \cap M_v \) is nonempty and \( m_o < m_b \) and \( m_o < m_v \) and \( \lambda_o = (\lambda_b, \lambda_v) \). The case 1.ii) exists if \( \tilde{z}_0(g, q) \neq z_0(\tau) \) for all \( z \). The case 2.) follows from Definition 4.2.1 because the second derivative of \( V_{\text{BH}} \) vanishes for all \( z^i \), while the case 3.) is coming from Definition 4.2.2 because the second derivative of \( V(\tau) \) is trivial for all \( z^i \).

The last case is trivial.

We write some remarks here. There is a special case mentioned above, namely, when the condition (4.2) is fulfilled. In this case, we only have \( p_0 \equiv (z_0(\tau), \tilde{z}_0(\tau)) \). Precisely, this can be easily seen in a theory where the gauge couplings simply have the form

\[
\mathcal{N}_{\Lambda \Sigma} = f(z)\delta_{\Lambda \Sigma},
\]

for arbitrary holomorphic function \( f(z) \). Then, the condition (4.2) becomes \( \partial_i f(z) = 0 \).

For example, if \( f(z) = a_0 + b_i z^i \) with \( b_0, b_i \in \mathbb{R} \), then the vacua demand \( b_i = 0 \) but \( b_0 \neq 0 \) for all \( z^i \), and \( \partial_i V(\tau) = 0 \) for some \( z^i \). Here, the second derivative of \( V_{\text{BH}} \) vanishes for all \( z^i \) and thus, we obtain \( M_o \simeq M_v \).

Also, we have

Corollary 4.2.4. For both nonconstant \( V(\tau) \) and \( V_{\text{BH}} \), the vacuum submanifold \( M_o \) has the dimension \( m_o(g, q; \tau) \) and the index \( \lambda_o(g, q; \tau) \).

Proof. Since \( z_0(g, q; \tau) \), then we have \( m_o(g, q; \tau) \) and \( \lambda_o(g, q; \tau) \).

Some comments are as follows. Firstly, in general we have a situation where \( z_0(g, q; \tau) \) which is referred to as a dynamic case. Such a situation occurs if the scalar potential \( V(\tau) \) at least is nonconstant. If \( V(\tau) = 0 \) for all \( z \) and \( \tau \), then \( (m_o, \lambda_o) \) are independent of \( \tau \) which is referred to as a static case. Secondly, if \( M_o \) is an empty set, then we may possibly have a case discussed in Section 5.

In the following we consider a special case. Let the charges \( (g, q) \) be fixed, and so only the Kähler-Ricci flow (2.1) plays an important role in controlling the nature of the vacuum geometry \( M_o \). In the case at hand, we obtain

Theorem 4.2.5. Let \( M_o \) be an \( m_o \)-dimensional submanifold of \( M \) with index \( \lambda_o \) in \( 0 \leq \tau < \tau_1 \) and \( \tau_1 < \tau_0 \) where \( \tau_0 \) is the internal singularity of the Kähler-Ricci flow (2.1). Then we have the following cases.
1. $M_\circ$ deforms to an $\tilde{m}_\circ$-dimensional submanifold $\tilde{M}_\circ \subseteq M$ of the index $\tilde{\lambda}_\circ$ in $\tau_1 \leq \tau < \tau_0$.

2. $M_\circ$ deforms to an $\hat{m}_\circ$-dimensional submanifold $\hat{M}_\circ \subseteq M$ of the index $\hat{\lambda}_\circ$ in $\tau > \tau_0$.

The proof of the above statements is similar to the case of domain walls studied in [1, 4]. We have assumed that the Kähler-Ricci soliton (2.1) is well defined everywhere for $\tau \geq 0$ and $\tau \neq \tau_0$.

5 Extremal Dyonic Bertotti-Robinson-like Black Holes

This section is devoted to discuss a case which is similar to the model in [15] which is called a family of extremal dyonic Bertotti-Robinson-like black holes which are also non-supersymmetric. In the case at hand, the scalar fields $z^i$ are frozen with respect to $r$ near the horizon of black holes. This section is organized into two parts. The first part is to consider general aspects of extremal Bertotti-Robinson-like black holes near the horizon, whereas in the second part we discuss the relation between effective vacuum structure and Morse theory in detail.

5.1 General Description

As mentioned above, near the horizon we have $z^{i'} = 0$. Furthermore, the metric (3.5) in this limit becomes a product of two families of surfaces, namely $M^{1,1} \times S^2$, where $M^{1,1}$ and $S^2$ are families of two dimensional surfaces and two-spheres, respectively. Thus, the functions in (3.5) has the form

$$\frac{1}{2} e^{-B} \left( A'' + \frac{1}{2} A' (A' - B') \right) = \ell,$$

$$C = \ln r_h,$$  \hspace{1cm} (5.1)

assuming $\tau \neq \tau_0$. The quantity $r_h \equiv r_h(g, q; \tau)$ is the radius of $S^2$, while the first equation in (5.1) determine the shape of $M^{1,1}$ with $\ell \equiv \ell(g, q; \tau)$.

Then, in the near-horizon limit the equations in (3.7) read

$$\frac{1}{r_h^2} = \frac{1}{r_h^4} V_{BH}^h + V_h(\tau),$$

$$\ell = \frac{1}{r_h^4} V_{BH}^h - V_h(\tau),$$

$$\left( \frac{1}{r_h^4} \frac{\partial V_{BH}}{\partial z^i} + \frac{\partial V(\tau)}{\partial z^i} \right) (p_h) = 0,$$  \hspace{1cm} (5.2)

and $p_h \equiv (z_h, \bar{z}_h)$ where we have defined

$$V_{BH}^h \equiv V_{BH}(p_h),$$

$$V_h(\tau) \equiv V(p_h; \tau),$$

$$\lim_{r \to r_h^i} z^i_h \equiv z^i_h.$$  \hspace{1cm} (5.3)

\footnote{An extremal dyonic Bertotti-Robinson black hole appears as a non-supersymmetric black hole in four dimensional $N = 2$ supergravity with FI terms [15]. In our case, the FI terms are replaced by the scalar potential (2.5).}
The solutions of (5.2) are given by
\[ r_h^2 = V^h_\text{eff}(\tau), \]
\[ \ell^{-1} = \frac{V^h_\text{eff}(\tau)}{\sqrt{1 - 4V_{\text{BH}}V(\tau)}}, \] (5.4)
\[ \frac{\partial V_\text{eff}}{\partial z^i}(p_h) = 0, \]
where
\[ V_\text{eff}(\tau) \equiv \frac{1 - \sqrt{1 - 4V_{\text{BH}}V(\tau)}}{2V(\tau)} \] (5.5)
is called the effective black hole potential [15] and
\[ V^h_\text{eff}(\tau) \equiv V_\text{eff}(p_h; \tau). \] (5.6)
The last equation in (5.4) shows that near the horizon the scalars \( z_h \) can be viewed as the critical points of \( V_\text{eff} \) in the scalar manifold \( M \) and moreover, we have \( z_h \equiv z_h(g, q; \tau) \). In this case the black hole entropy simply takes the form [16]
\[ S(\tau) = \frac{A_h(\tau)}{4} = \pi r_h^2 = \pi V^h_\text{eff}(\tau). \] (5.7)

Some comments are in order. As mentioned in the preceding section, the black hole potential \( V_{\text{BH}} \geq 0 \), while the scalar potential \( V(\tau) \) is not necessarily positive. Therefore, the effective potential \( V_\text{eff}(\tau) \) takes the real value with necessary condition
\[ V_{\text{BH}}V(\tau) < \frac{1}{4}, \] (5.8)
while the equal sign is forbidden by the regularity of the first order derivative of the effective black hole potential (5.5). We additionally have
\[ \lim_{V \to 0} V_\text{eff}(\tau) = V_{\text{BH}}, \]
\[ \lim_{V_{\text{BH}} \to 0^+} V_\text{eff}(\tau) = 0. \] (5.9)

Using the above results, we can then write the following statement:

**Theorem 5.1.1.** For \( B = \pm A \), the entropy (5.7) is strictly positive, namely \( S(\tau) > 0 \), if and only if \( M^{1,1} \) is a family of Einstein surfaces where

1. for \( B = -A \), \( M^{1,1} \simeq \text{AdS}_{1,1} \).
2. for \( B = A \), \( M^{1,1} \simeq \text{dS}_{1,1} \).

**Proof.** Using the fact (5.8), it is straightforward from the second equation in (5.4) that \( \ell > 0 \) ensures \( V_\text{eff}(\tau) > 0 \) and vice versa for all \( \tau \) and \( \tau \neq \tau_0 \). For \( B = -A \), we get the constant \( \Lambda_2 = -\ell \) and so \( M^{1,1} \simeq \text{AdS}_{1,1} \). On the other hand, in \( B = A \) case we have \( \Lambda_2 = \ell \) and so \( M^{1,1} \simeq \text{dS}_{1,1} \). \( \square \)

Let us make some remarks here. As observed in [15], in the case at hand the spacetime is not conformally flat since the radius of \( \text{AdS}_{1,1} \) given by \( r_a \equiv \ell^{-1/2} \) does not equal to the radius of \( S^2 \), namely \( r_h \). Such a situation also occurs in the case of \( \text{dS}_{1,1} \) and the cases considered in subsection 4.1.1. Next, the positivity of the entropy (5.7) restricts \( r_h^2 > 0 \) for all \( \tau \) and \( \tau \neq \tau_0 \) which means that the Kähler-Ricci flow keeps the topology of \( S^2 \) for all \( \tau \) and \( \tau \neq \tau_0 \).
5.2 Effective Vacuum Geometry and Morse Theory

First of all, let us introduce the definition of an effective vacuum geometries as follows.

**Definition 5.2.1.** A submanifold $M_e$ defined as

$$M_e \equiv \{ p_h \in M \mid \partial_i V_{\text{eff}}(p_h) = 0 \} \subset M,$$

is called the effective vacuum manifold.

It is important to notice that the points $p_h \neq \hat{p}_0$ where $\hat{p}_0$ is given by (4.35) and thus, $p_h$ are not the critical points of the scalar potential (2.5) and the black hole potential (3.8).

We have then

**Theorem 5.2.2.** For $S(\tau) > 0$, the effective vacuum manifold $M_e$ can be split into the following cases:

1. For nonzero $V_{\text{eff}}(\tau)$, $M_e$ has the dimension $m_e(g, q; \tau)$ and the Morse index $\lambda(g, q; \tau)$.

2. If $V(\tau) \to 0$, then $M_e \simeq M_b$ where $M_b$ described in Definition 4.2.2.

**Proof.** For case 1, since $z_h \equiv z_h(g, q; \tau)$, so from the Hessian of $V_{\text{eff}}(\tau)$ we have the dimension $m_e(g, q; \tau)$ and the Morse index $\lambda_e(g, q; \tau)$. Then, it is straightforward using (5.9) to prove the case 2.

We write some remarks as follows. If both $V_{\text{BH}}$ and $V(\tau)$ are constant and nonzero, then $M_e \simeq M$ which is the trivial case. From (5.2) and (5.4), one can then show that if $V_{\text{BH}} \to 0^+$, then we have a singularity which shows that $S(\tau) = 0$ is naturally forbidden in the model.

In particular, by fixing the charges $(g, q)$ one can then reproduce the case considered in Theorem 4.2.5 because only the Kähler-Ricci soliton determines properties of the effective vacuum manifold $M_e$.

6 Simple Models

In this section we consider some simple models in which the ingredients can be mentioned as follows. The Kähler potential of the theory has the form

$$K(\tau) = \sigma(\tau) \ln(1 + |z|^2),$$

where $|z|^2 \equiv \delta_{ij} z^i \bar{z}^j$ and

$$\sigma(\tau) = 1 - 2(n_c + 1)\tau,$$

which means that the Kähler-Ricci equation is $\mathbb{CP}^{n_c}$ for $\tau < 1/2(n_c + 1)$ and then becomes $\mathbb{CP}^{\infty}$ for $\tau > 1/2(n_c + 1)$ [1]. The gauge couplings and the superpotential are given by

$$\mathcal{N}_{\Lambda\Sigma}(z) = (b_0 + b_i z^i)\delta_{\Lambda\Sigma},$$

$$W(z) = a_0 + a_i z^i,$$

(6.3)
respectively, with \(a_0, a_i, b_0, b_i \in \mathbb{R}\). In addition, we set \(\dot{\sigma}(\tau) = 1\) for all \(\tau\). The black hole potential and the scalar potential are

\[
V_{\text{BH}} = \left( b_0 + b_i x^i + \frac{(b_j y^j)^2}{(b_0 + b_i x^i)} \right) g^2 - \frac{2 b_j y^j}{(b_0 + b_i x^i)} g q + \frac{q^2}{(b_0 + b_i x^i)},
\]

\[
V(\tau) = (1 + x^2 + y^2)^{1+\sigma(\tau)/M_P^2} \left[ \frac{a^2}{\sigma(\tau)} + \frac{1}{M_P^2} \left( 2 + \frac{\sigma(\tau)}{M_P^2} \right) \left( (a_0 + a_i x^i)^2 + (a_i y^i)^2 \right) - \frac{2a_0}{M_P^2} (a_0 + a_i x^i) - \frac{1}{M_P^2} \left( 3 + \frac{\sigma(\tau)}{M_P^2} \right) \left( (a_0 + a_i x^i)^2 + (a_i y^i)^2 \right) \right],
\]

(6.4)

respectively, where we have introduced coordinates \(x^i, y^i \in \mathbb{R}\) such that \(z^i = x^i + iy^i\) and defined some quantities

\[
g^2 \equiv \delta_{\Lambda \Sigma} g^\Lambda g^\Sigma, \quad g q \equiv g^\Lambda q_\Lambda, \\
q^2 \equiv \delta_{\Lambda \Sigma} q_\Lambda q_\Sigma, \quad a^2 \equiv \delta^{ij} a_i a_j \\
x^2 \equiv \delta_{ij} x^i x^j, \quad y^2 \equiv \delta_{ij} y^i y^j.
\]

(6.5)

### 6.1 Simple Models for Reissner-Nordström-like Black Holes

#### 6.1.1 The First Case

Let us first begin to construct the case where at the vacua we have \(b_i = 0\) for all \(z\) coming from the first derivative of the gauge couplings, and \(\partial_i V(p_0; \tau) = 0\). For the case at hand, the black hole potential \(V_{\text{BH}}\) at \(p_0\) becomes

\[
V_{\text{BH}}^0 = b_0 g^2 + \frac{q^2}{b_0},
\]

(6.6)

which is positive with \(b_0 > 0\). It then follows that the vacuum geometry is nothing but \(M_o\) with Kähler potential given in (6.1).

First of all, we simply take \(z_0 = 0\). In this case we find

\[
\partial_i V(0; \tau) = -\frac{2}{M_P^2} a_0 a_i = 0,
\]

(6.7)

which can be split into two cases as follows. The first case is \(a_i = 0\) for all \(i\) and \(a_0 \neq 0\). The scalar potential (2.5) then becomes

\[
V(0; \tau) = -\frac{3a_0^2}{M_P^2},
\]

(6.8)

which is independent of \(\tau\). The Hessian matrix of the scalar potential (2.5) is simply diagonal matrix

\[
\partial_i \partial_j V(0; \tau) = -\frac{2\sigma(\tau)}{M_P^4} \delta_{ij} |a_0|^2.
\]

(6.9)

Thus we have isolated unstable dyonic black holes with AdS backgrounds whose Morse index is \(2n_c\) for \(\tau < 1/2(n_c + 1)\). Then, they change to 0 for \(\tau > 1/2(n_c + 1)\) describing stable dyonic black holes as the geometry evolves with respect to the Kähler-Ricci
equation. On the other hand, the ground states turn out to be degenerate Minkowskian spacetime if \( a_0 = 0 \).

The second case is \( a_i \neq 0 \) for some \( i \) and \( a_0 = 0 \). The scalar potential (2.5) has then the form

\[
V(0; \tau) = \sigma(\tau)^{-1} a^2 ,
\]

where \( a^2 \equiv \delta^{ij} a_i a_j \). The non-zero components of the Hessian matrix of the scalar potential (2.5) given by

\[
\partial_i \partial_j V(0; \tau) = \left( 1 + \frac{2\sigma(\tau)}{M_P^2} \right) a^2 \delta_{ij} + \left( 1 - \frac{2}{M_P^2} \right) a_i a_j .
\]

In this case are, for \( a^2 \neq 0 \) we have dyonic black holes with three different backgrounds, namely dS backgrounds for \( \tau < 1/2(n_c + 1) \) and AdS backgrounds for \( \tau > 1/2(n_c + 1) \), while Minkowski backgrounds occur around \( \tau \to \pm \infty \). Note that the latter case happens also for \( a^2 = 0 \). These situations are strong evidences of our previous statements that the parameter \( \tau \) indeed controls the shape of the black hole geometry. In addition, the black holes are stable if all eigenvalues of (6.11) are positive.

Finally, we turn to discuss a dynamic case in which for the sake of simplicity we consider it near the origin, namely \( z \approx 0 \) for finite \( \tau \). In addition, it is convenient to take \( n_c = 1 \) and \( 0 < a_0 \ll a_1 \). Then, by setting \( \partial_i V(\tau) = 0 \), we get

\[
x_0 \approx \frac{2\sigma(\tau) a_0}{M_P^2} \frac{a_1}{a} \left( 2 + \frac{\sigma(\tau)}{M_P^2} \right)^{-1} ,
\]

\[
y_0 = 0 ,
\]

and the scalar potential (2.5) is given by

\[
V(x_0; \tau) \approx \sigma(\tau)^{-1} a^2 - \frac{3a_0^2}{M_P^2} - \frac{4\sigma(\tau)}{M_P^4} a_0^2 \left( 2 + \frac{\sigma(\tau)}{M_P^2} \right)^{-1} .
\]

So, in the case at hand the origin is the singular point at \( \tau = 1/4 \).

### 6.1.2 The Second Case

Here, we consider the case where in general the vacua are defined by

\[
\partial_i V_{BH} = 0 \quad \text{and} \quad \partial_i N_{\Lambda \Sigma} \neq 0 ,
\]

\[
\partial_j V(\tau) = 0 .
\]

In addition, we simply consider \( n_c = 1 \), but \( n_v > 1 \). Then, we take \( a_0 = 0 \), while \( a_1, b_0, b_1 \) are non-zero. After some computations, we find

\[
x_0 = - \frac{b_0}{b_1} + \frac{1}{b_1 g^2} \sqrt{g^2 q^2 - (gq)^2} ,
\]

\[
y_0 = \frac{gq}{b_1 g^2} ,
\]

where

\[
b_1 = \left( \sqrt{\frac{(gq)^2 g^{-4} + \left( -b_0 + g^{-2} \sqrt{g^2 q^2 - (gq)^2} \right)^2}{\frac{1}{2} \left( 5 + 2\sigma(\tau) M_P^{-2} - \sigma(\tau)^{-1} M_P^2 \right) (2 + \sigma(\tau) M_P^{-2}) - 1}} \right)^{1/2} .
\]
The model is consistent if
\[ \tau < \frac{1}{4} (1 - M_P^2) . \] (6.17)

Inserting (6.15) to the black hole potential in (6.4), then it simply has the form
\[ V_{\text{BH}}^0 = 2 \sqrt{g^2 q^2 - (g q)^2}, \] (6.18)
which is positive since
\[ g^2 q^2 > (g q)^2 . \] (6.19)

### 6.2 Simple Models for Bertotti-Robinson-like Black Holes

Taking \( n_c = 1 \) and \( n_v > 1 \), the simplest case of the model is when \( a_0 = a_1 = 0 \), namely \( W(z) = 0 \). Then, the vacua are nothing but the critical points of \( V_{\text{BH}} \). If \( b_1 = 0 \) for all \( z \), then the vacuum geometry is described by the Kähler potential (6.1) for \( n_c = 1 \). The black hole potential becomes simply (6.6) and the positivity of the entropy (5.7) implies \( b_0 > 0 \). On the other side, if \( b_1 \neq 0 \) for all \( z \), then we regain (6.15) but (6.16) and (6.17) do not exist. In this case, the black hole potential is given by (6.18) satisfying (6.19) since the entropy (5.7) is strictly positive.

Next, we consider a case where \( W(z) = a_0 \neq 0 \) and \( b_0 \gg b_1 > 0 \). In addition, we set \( g q = 0 \). Then, by simply taking that the effective vacuum exists near the origin, namely \( z_h \approx 0 \), it results
\[ x_h \approx \frac{3b_1}{4b_0} (q^2 - b_0^2 g^2) \left[ 3b_1^2 g^2 + \frac{\sigma(\tau)}{M_P^2} (b_0^2 g^2 + q^2) + \frac{b_0}{6a_0^2} \sigma(\tau) M_P^2 \right]^{-1}, \]
\[ y_h = 0 , \] (6.20)

with \( \tau \neq 1/4 \). The effective black hole potential (5.5) has the form
\[ V_{\text{eff}}(\tau) \approx \frac{M_P^2}{6a_0^2} \left[ \sqrt{1 + \frac{12a_0^2}{M_P^2} (b_0 + b_1 x_h)^{-1} \left( (b_0 + b_1 x_h)^2 g^2 + q^2 \right) - 1} \right] , \] (6.21)
which is already positive.

### 7 Conclusions

So far, we have discussed some aspects of non-supersymmetric spherical black holes of four dimensional \( N = 1 \) supergravity coupled to \( n_v \) vector- and \( n_c \) chiral multiplets. Moreover, the scalar manifold can be viewed as a one-parameter family of Kähler manifolds generated by the Kähler-Ricci flow (2.1) which may also induces a family of four dimensional spherical black holes given by the ansatz metric (3.5). Such black holes belong to the class of solutions of the set of equations of motions such as the Einstein field equation (3.1), the gauge field equation of motions (3.2), and the scalar field field equation of motions (3.3) which can be derived by varying the action related to the Lagrangian (2.2) with respect to \( g_{\mu\nu}, A_\mu^A, \) and \( z^i \) and correspondingly, taking the necessary condition, namely the variation of the flow parameter \( \tau \) vanishes. For the case at hand, the coordinate redefinition cannot generally be employed since the ansatz (3.5) varies with respect to the flow parameter \( \tau \).
which may have a singularity at finite $\tau$ causing the topological change. Therefore, we can only do the coordinate redefinition at a particular value of $\tau$.

In particular, two models of extremal dyonic black holes have been considered, namely Reissner-Nordström-like and Bertotti-Robinson-like black holes. Our results can be mentioned as follows.

In order to obtain the extremal Reissner-Nordström-like black holes, the first step is to take the condition (4.1) in which it can be split into two statements as follows. Firstly, the scalar fields $z^i$ have to be frozen with respect to the radial coordinate $r$, namely $z^{ii} = 0$. Secondly, the points $z^i_0(g,q;\tau)$ are the critical points of both the black hole potential $V_{BH}$ and the scalar potential $V(\tau)$.

Then, the second step is that the function $C(r,\tau)$ in the region has the form (1.3) which shows that the Kähler-Ricci soliton indeed induce the area deformation of $S^2$ described by $\delta(\tau)$. However, to get a consistent picture we should have $\delta(\tau) > 0$ for all $\tau$. Finally, the background geometries of black holes should be a family of four dimensional Einstein manifolds, namely de Sitter, Minkowski, and anti-de Sitter, which can be seen from the uncharged black holes (Lemma 4.1.1).

Using the above setup, one can then obtain the extremal Reissner-Nordström-like black holes given by Theorem 4.1.2. In the de Sitter backgrounds, namely $V_0(\tau) > 0$, we have a rich and complicated structure. The black holes have generally three horizons with radii $\hat{r}_c, \hat{r}_+, \hat{r}_-$, and $\hat{r}_c > \hat{r}_+ \geq \hat{r}_-$ satisfying the conditions (4.15). The minimal black hole mass in the case at hand is $M_e(\hat{r}_e,\tau)$ given by (4.16) describing black holes with $\hat{r}_+ = \hat{r}_-$. Meanwhile, the maximal mass is $M_c(\hat{r}_c,\tau)$ representing black holes whose two horizons coincide, namely $\hat{r}_c = \hat{r}_+$. Such black holes are referred to as a one-parameter family of Nariai-like black holes whose metric has the form (4.23) and the topology is $S^{1,1} \times S^2$ with different radii. On the other side, in Minkowski or anti-de Sitter backgrounds we only have two horizons with radii $\hat{r}_+, \hat{r}_-$, and $\hat{r}_+ \geq \hat{r}_-$. These horizons exist if the conditions (1.27) (for Minkowski backgrounds) and (4.30) (for anti-de Sitter backgrounds) are satisfied.

Furthermore, as mentioned above, we have a class of vacua spanned by $z^i_0(g,q;\tau)$ which are the critical points of both the black hole potential $V_{BH}$ and the scalar potential $V(\tau)$. Such vacua can be viewed as a vacuum submanifold $M_o$ whose dimension and Morse index are respectively given by $(m_o, \lambda_o)$. There are five possible cases for $M_o$ as follows (Theorem 4.2.3). In the case when both $V_{BH}$ and $V(\tau)$ are nonconstant we have two possible cases, namely the empty set and the vacuum submanifold $M_o \simeq M_b \cap M_v$ with $(m_o(g, q, \tau), \lambda_o(g, q, \tau))$ and $m_o \leq m_b$, $m_o \leq m_v$. The other three cases are in the following: Firstly, when $V(\tau)$ is nonconstant but $V_{BH}$ is constant for all $z$, so we have $M_o \simeq M_v$ with $\lambda_o = \lambda_v$ and $m_o = m_v$. Secondly, if $V(\tau)$ is constant for all $z$ but $V_{BH}$ is nonconstant, then $M_o \simeq M_b$ with $\lambda_o = \lambda_b$ and $m_o = m_b$. The last case is the trivial case, $M_o \simeq M$.

In the extremal Bertotti-Robinson-like black holes, we assume that near the horizon the scalar fields satisfy $z^{ii} = 0$ and the near-horizon geometries have the form $M^{1,1} \times S^2$, where $M^{1,1}$ and $S^2$ are families of two dimensional surfaces and two-spheres, respectively. We particularly consider a case where in this limit the functions $B(r,\tau) = \pm A(r,\tau)$ and the function $C(r,\tau)$ has the form given in (5.1). Imposing the positivity of the entropy (5.7) one then finds that for $B(r,\tau) = -A(r,\tau)$ we have $M^{1,1} \simeq AdS_{1,1}$, while $B(r,\tau) = A(r,\tau)$ we get $M^{1,1} \simeq dS_{1,1}$ (Theorem 5.1.1).

In this model, the class of vacua is referred to as the effective vacuum submanifold.
$M_\epsilon$ spanned by $z_h^i(g, q; \tau)$ which can be viewed as the critical points of the effective scalar potential defined in (5.3). The effective vacuum submanifold $M_\epsilon$ has the dimension $m_\epsilon(g, q; \tau)$ and the Morse index $\lambda_\epsilon(g, q; \tau)$. In the case where $V(\tau) \to 0$, then we get $M_\epsilon \simeq M_\delta$ where $M_\delta$ described in Definition 4.2.2.

We have also considered some simple models where the scalar manifold is diffeomorphic to $\mathbb{CP}^{n_\epsilon}$ for $\tau < 1/2(n_\epsilon + 1)$ and both the gauge couplings and the superpotential have the linear form given in (5.3). For the Reissner-Nordström-like black holes, we gave two cases. The first case is simply setting $b_i = 0$ for all $i$ and taking the origin $z_0 = 0$ and its neighborhood $z_0 \approx 0$. At the origin we finds that in $a_i = 0$ for all $i$ and $a_0 \neq 0$ case one gets isolated unstable dyonic black holes with AdS backgrounds for $\tau < 1/2(n_\epsilon + 1)$ and then, turn to stable dyonic black holes with AdS backgrounds for $\tau > 1/2(n_\epsilon + 1)$. The other possible case at the origin is $a_i \neq 0$ for some $i$ and $a_0 = 0$ such that $a^2 \neq 0$. In this case we have dyonic black holes with dS backgrounds for $\tau < 1/2(n_\epsilon + 1)$, while they change to dyonic black holes with AdS backgrounds for $\tau > 1/2(n_\epsilon + 1)$. On the other hand, dyonic black holes with Minkowski backgrounds occur around $\tau \to \pm \infty$ or $a^2 = 0$. Next, around the origin we simplify the model by taking $n_\epsilon = 1$ and $0 < a_0 \leq a_1$. So, after some computations one gets $y_0 = 0$ and $x_0(\tau)$ given in (6.12).

The second case of the Reissner-Nordström-like black holes is for $b_i = 0$ for all $i$. Similar as above, it is convenient to set $n_\epsilon = 1$ and $a_0 = 0$ but the other pre-coefficients are nonzero. In the case at hand, we have $(x_0(g, q; \tau), y_0(g, q; \tau))$ because $b_1(g, q; \tau)$ given respectively in (6.15) and (6.16). This model exists only for $\tau < \frac{1}{4}(1 - M_\delta^2)$.

Finally, we gave some examples of the Bertotti-Robinson-like black holes. The simplest case is when the superpotential $W(z)$ vanishes for all $z$ which follows $V(\tau) = 0$ for all $z$. Thus, the vacua can be regarded as the critical points of $V_{\text{BH}}$. Firstly, by taking simply $b_1 = 0$ the vacua is described by $\mathbb{CP}^1$ for $\tau < 1/4$ and then becomes $\mathbb{CP}^1$ for $\tau > 1/4$. The entropy is positive if $b_0 > 0$. Secondly, for $b_1 \neq 0$ we obtain (6.15) and $b_1$ is arbitrary. The entropy of the model is indeed positive since $g^2 q^2 > (gq)^2$.

Next, we consider another model in which we set $W(z) = a_0 \neq 0$, $b_0 \gg b_1 > 0$, and $gq = 0$. The effective vacua indeed exist around the origin and have the form given in (6.20) and the entropy of the model is already positive ensured by (6.21).

Acknowledgement

It is pleasure to acknowledge H. Alatas for useful discussions. This work is supported by Riset KK ITB 2009 No. 243/K01.7/PL/2009 and ITB Alumni Association (HR IA-ITB) 2009.

A Convention and Notation

The aim of this appendix is to assemble our conventions in this paper. The spacetime metric is taken to have the signature $(+, -, -, -)$ while the Riemann tensor is defined to be $-\tilde{R}^\mu_{\nu\rho\lambda} = \partial_\rho \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\rho} + \Gamma^\sigma_{\nu\lambda} \Gamma^\mu_{\sigma\rho} - \Gamma^\sigma_{\nu\rho} \Gamma^\mu_{\sigma\lambda}$. The Christoffel symbol is given by $\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho})$ where $g_{\mu\nu}$ is the spacetime metric.

We supply the following indices:
$$\mu, \nu = 0, \ldots, 3$$ label curved four dimensional spacetime indices

$$i, j, k = 1, \ldots, n_c$$ label the number of chiral multiplets

$$\bar{i}, \bar{j}, \bar{k} = 1, \ldots, n_c$$ label conjugate indices of $$i, j, k$$

$$\Lambda, \Sigma, \Gamma = 1, \ldots, n_v$$ label the number of vector multiplets

Some quantities on a Kähler manifold are given as follows. $$g_{\bar{i}j}$$ denotes the metric of the Kähler manifold whose Levi-Civita connection is defined as $$\Gamma_{i j}^l = g^{l k} \partial_i g_{jk}$$ and its conjugate $$\Gamma_{i j}^{\bar{l}} = g^{\bar{l} k} \partial_{\bar{i}} g_{\bar{k} j}$$. Then the curvature of the Kähler manifold are defined as

$$R_{i j k}^l \equiv \partial_{\bar{j}} \Gamma_{i k}^l \ ,$$  \hspace{1cm} (A.1)

while the Ricci tensor has the form

$$R_{k j} = R_{j k} \equiv R_{i j k}^i = \partial_{\bar{j}} \Gamma_{i k}^i = \partial_k \partial_{\bar{j}} \ln \left( \det(g_{\bar{i}j}) \right) \ .$$  \hspace{1cm} (A.2)

Finally, the Ricci scalar can be written down as

$$R \equiv g^{\bar{i}j} R_{i j} \ .$$  \hspace{1cm} (A.3)

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