Applications of position-based coding to classical communication over quantum channels

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Abstract

Recently, a coding technique called position-based coding has been used to establish achievability statements for various kinds of classical communication protocols that use quantum channels. In the present paper, not only do we apply this technique in the entanglement-assisted setting in order to establish lower bounds for error exponents, lower bounds on the second-order coding rate, and one-shot lower bounds, but we also demonstrate that position-based coding can be a powerful tool for analyzing other communication settings. In particular, we reduce the quantum simultaneous decoding conjecture for entanglement-assisted or unassisted communication over a quantum multiple access channel to open questions in multiple quantum hypothesis testing. We then determine an achievable rate region for entanglement-assisted or unassisted classical communication over a quantum multiple-access channel, when using a particular quantum simultaneous decoder. The achievable rate regions given in this latter case are generally suboptimal, involving differences of Rényi-2 entropies and conditional quantum entropies.

1 Introduction

Understanding optimal rates for classical communication over both point-to-point quantum channels and quantum network channels are fundamental tasks in quantum Shannon theory (see, e.g., [Hol12, Wat16, Wil16, Wil17b]). Early developments of quantum Shannon theory are based on the assumption that the information is transmitted over an arbitrarily large number of independent and identically distributed (i.i.d.) uses of a given quantum channel. By taking advantage of this assumption, general formulas have been established for capacities of various communication protocols, with or without preshared entanglement. When a sender and receiver do not share entanglement before communication begins, it is known that the Holevo information of a quantum channel is an achievable rate for classical communication [Hol98, SW97]. Regularizing the Holevo information leads to a multi-letter formula that characterizes the capacity for this task. Regarding communication over quantum network channels, an achievable rate region for classical communication over quantum multiple-access channels was given in [Win01] and regularizing it leads to a characterization of the

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capacity region for this task. However, only inner bounds on the capacity region for general broadcast channels are known [YHD11, RSW16, SW15], except when the quantum broadcast channel is a particular kind of degraded channel [WDW16]. When there is entanglement shared between the communicating parties, many results have also been established, including classical communication over point-to-point quantum channels [BSST02, Hol02], quantum multiple-access channels [HDW08, XW13], and quantum broadcast channels [DHL10, WDW16].

Although channel capacity gives a fundamental characterization of the communication capabilities of a quantum channel, many practically important properties of quantum channels are not captured by this quantity. To close this gap, several recent works have focused on the study of refined notions of capacity, including error exponents [BH98, Win99b, Hol00, Hay07, Dal13, DW14, CH16, CHT17] and second-order asymptotics [TH13, Li14, TT15, WRG16, DTW16]. These refined characterizations are of importance for regimes of practical interest, in which a limited number of uses of a quantum channel are available. Complementary to these developments, to go beyond the i.i.d. assumption, many works have been dedicated to the one-shot formalism [WR12, DRRW13, DH13, MW14] and the information-spectrum approach [Hay03, HN03, Hay06, BD06, NH07], with very few assumptions made on the structure of quantum channels.

In a recent work [AJW17], a powerful, yet simple technique called position-based coding was developed in order to give one-shot achievability bounds for various classical communication protocols that use entanglement assistance. This was later extended to the cases of unassisted classical communication and private classical communication [Wil17a].

In the present paper, we use position-based coding to establish several new results. First, we establish lower bounds on the entanglement-assisted error exponent and on the one-shot entanglement-assisted capacity. The latter improves slightly upon the result from [AJW17] and in turn gives a simpler proof of one of the main results of [DTW16], i.e., a lower bound on the second-order coding rate for entanglement-assisted classical communication. We then turn to communication over quantum multiple-access channels when using a quantum simultaneous decoder, considering both cases of entanglement assistance and no assistance. The quantum simultaneous decoding conjecture from [FHS+12, Wil11] stands as one of the most important open problems in network quantum information theory. Here we report progress on this conjecture and connect it to some open questions from [AM14, BHOS15] in multiple quantum hypothesis testing. At the same time, we give a new achievable rate region for entanglement-assisted classical communication over multiple-access channels, where the bounds on achievable rates are expressed as a difference of a Rényi entropy of order two and a conditional quantum entropy.

This paper is organized as follows. We first summarize relevant definitions and lemmas in Section 2. In Section 3, we consider entanglement-assisted point-to-point classical communication. By using position-based coding, we establish a lower bound on the entanglement-assisted error exponent. We also establish a lower bound on the one-shot entanglement-assisted capacity in terms of hypothesis-testing mutual information and state how it is close to a previously known upper bound from [MW14]. Based upon this one-shot bound, we immediately obtain a lower bound on the second-order coding rate for entanglement-assisted communication with a proof that is arguably simpler than that given in [DTW16]. In Section 4, we apply position-based coding to entanglement-assisted classical communication over multiple-access channels and establish an explicit link to multiple quantum hypothesis testing. We give an achievable rate region for i.i.d. channels by using techniques from the theory of quantum typicality. We demonstrate the power of position-based coding technique in unassisted classical communication in Section 4.2, by considering classical
communication over multiple-access channel. We explicitly show how to derandomize a randomness-assisted protocol. In Section 5, we tie open questions in multiple quantum hypothesis testing to quantum simultaneous decoding for the quantum multiple-access channel. Finally, we summarize our main results and open questions in Section 6.

2 Preliminaries

Trace distance, fidelity, and gentle measurement. Let \( \mathcal{D}(\mathcal{H}) \) denote the set of density operators (positive semi-definite operators with unit trace) acting on a Hilbert space \( \mathcal{H} \). The trace distance between two density operators \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) is equal to \( \|\rho - \sigma\|_1 \), where \( \|A\|_1 \equiv \text{Tr}\{\sqrt{A^\dagger A}\} \).

Another quantity to measure the closeness between two quantum states is the fidelity, defined as

\[
F(\rho, \sigma) = \| \sqrt{\rho} \sqrt{\sigma} \|_2^2 \quad \text{[Uhl76]}
\]

Two inequalities relating trace distance and quantum measurement operators are as follows:

**Lemma 1** (Gentle operator [Win99a, ON07]) Consider a density operator \( \rho \in \mathcal{D}(\mathcal{H}) \) and a measurement operator \( \Lambda \) where \( 0 \leq \Lambda \leq I \). Suppose that the measurement operator \( \Lambda \) detects the state \( \rho \) with high probability \( \text{Tr}\{\Lambda \rho\} \geq 1 - \varepsilon \), where \( \varepsilon \in [0, 1] \). Then

\[
\left\| \rho - \sqrt{\Lambda} \sqrt{\rho \sqrt{\Lambda}} \right\|_1 \leq 2\sqrt{\varepsilon} .
\] (2.1)

**Lemma 2** Consider two quantum states \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) and a measurement operator \( \Lambda \) where \( 0 \leq \Lambda \leq I \). Then we have

\[
\text{Tr}\{\Lambda \rho\} \geq \text{Tr}\{\Lambda \sigma\} - \|\rho - \sigma\|_1 .
\] (2.2)

More generally, the same bound holds when \( \rho \) and \( \sigma \) are subnormalized, i.e., \( \text{Tr}\{\rho\}, \text{Tr}\{\sigma\} \leq 1 \).

Information spectrum. The information spectrum approach [Han03, NH07, HN03, DR09] gives one-shot bounds for operational tasks in quantum Shannon theory, with very few assumptions made about the source or channel [NH07, Hay03, HN03, Hay07]. What plays an important role in the information spectrum method is the positive spectral projection of an operator. For a Hermitian operator \( X \) with spectral decomposition \( X = \sum \lambda_i |i\rangle\langle i| \), the associated positive spectral projection is denoted and defined as

\[
\{X \geq 0\} = \sum_{i : \lambda_i \geq 0} |i\rangle\langle i| .
\] (2.3)

Relative entropies and mutual informations. For two states \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \), the quantum Rényi relative entropy of order \( \alpha \), where \( \alpha \in [0, 1) \cup (1, +\infty) \) is defined as [Pet86, TCR09]

\[
D_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha - 1} \log_2 \text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\} .
\] (2.4)

If \( \alpha > 1 \) and \( \text{supp}(\rho) \nsubseteq \text{supp}(\sigma) \), it is set to \( +\infty \). In the limit as \( \alpha \to 1 \), the above definition reduces to the quantum relative entropy [Ume62]

\[
D(\rho\|\sigma) \equiv \text{Tr}\{\rho \log_2 \rho - \log_2 \sigma\} ,
\] (2.5)
which is defined as above when \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \) and it is set to \(+\infty\) otherwise. Using the above definition, we can define the Rényi mutual information for a bipartite state \( \theta_{RB} \) as

\[
I_\alpha(R; B)_\theta \equiv D_\alpha(\theta_{RB} \| \theta_R \otimes \theta_B).
\]  

(2.6)

The \( \epsilon \)-hypothesis testing relative entropy for states \( \rho \) and \( \sigma \) is defined for \( \epsilon \in [0, 1] \) as [BD10, WR12]

\[
D_H^\epsilon(\rho \| \sigma) \equiv -\log_2 \inf_\Lambda \{ \text{Tr}\{\Lambda \sigma\} : 0 \leq \Lambda \leq I \land \text{Tr}\{\Lambda \rho\} \geq 1 - \epsilon\}.
\]  

(2.7)

Similarly, we define the \( \epsilon \)-hypothesis testing mutual information of a bipartite state \( \theta_{RB} \) as

\[
I_H^\epsilon(R; B)_\theta \equiv D_H^\epsilon(\theta_{RB} \| \theta_R \otimes \theta_B).
\]  

(2.8)

The hypothesis testing relative entropy has the following second-order expansion for i.i.d. states [TH13, Li14, DPR16]:

\[
D_H^\epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}) = nD(\rho \| \sigma) + \sqrt{nV(\rho \| \sigma)\Phi^{-1}(\epsilon)} + O(\log n),
\]  

(2.9)

where \( V(\rho \| \sigma) = \text{Tr}\{\rho \log_2 \rho - \log_2 \sigma - D(\rho \| \sigma)^2\} \) is the quantum relative entropy variance and the function \( \Phi(a) \) is the cumulative distribution function for a standard normal distribution:

\[
\Phi(a) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} dx \ e^{-x^2/2}.
\]  

(2.10)

The hypothesis testing relative entropy is relevant for asymmetric hypothesis testing, in which the goal is to minimize the error probability \( \text{Tr}\{\Lambda \sigma\} \) subject to a constraint on the other kind of error probability \( \text{Tr}\{(I - \Lambda) \rho\} \leq \epsilon \). We could also consider symmetric hypothesis testing, in which the goal is to minimize both kinds of error probabilities simultaneously. It is useful for us here to take the approach of [AM14] and consider general positive semi-definite operators \( A \) and \( B \) rather than states \( \rho \) and \( \sigma \). As in [AM14], we can define the error “probability” in identifying the operators \( A \) and \( B \) as follows:

\[
P_\epsilon^s(A, B) \equiv \inf_{T : 0 \leq T \leq I} \text{Tr}\{(I - T)A\} + \text{Tr}\{TB\}
\]  

(2.11)

\[
= \text{Tr}\{A\} - \sup_{T : 0 \leq T \leq I} \text{Tr}\{T(A - B)\}
\]  

(2.12)

\[
= \text{Tr}\{A\} - \text{Tr}\{\{A - B \geq 0\}(A - B)\}
\]  

(2.13)

\[
= \frac{1}{2} (\text{Tr}\{A + B\} - \|A - B\|_1).
\]  

(2.14)

The following lemma allows for bounding \( P_\epsilon^s(A, B) \) from above, and we use it to establish bounds on the error exponent for entanglement-assisted communication.

**Lemma 3 ([ACMnT+07])** Let \( A \) and \( B \) be positive semi-definite operators and \( s \in [0, 1] \). Then the following inequality holds

\[
P_\epsilon^s(A, B) = \frac{1}{2} (\text{Tr}\{A + B\} - \|A - B\|_1) \leq \text{Tr}\{A^sB^{1-s}\}.
\]  

(2.15)
**Weak typicality.** We will use results from the theory of weak typicality in some of our achievability proofs (see, e.g., [Wil16, Wil17b] for a review). Consider a density operator \( \rho_A \) with spectral decomposition: \( \rho_A = \sum_x p_X(x) |x\rangle \langle x|_A \). The weakly \( \delta \)-typical subspace \( T_{A,n}^{\rho,\delta} \) is defined as the span of all unit vectors \( |x^n\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle \) such that the sample entropy \( H(X) = H(A)_\rho \) of their classical label is close to the true entropy \( H(X) \):

\[
T_{A,n}^{\rho,\delta} \equiv \text{span}\{ |x^n\rangle : |\overline{H}(x^n) - H(X)| \leq \delta \},
\]

where \( H(X) \equiv -\frac{1}{n} \log_2(p_X(x^n)) \) and \( H(X) \equiv -\sum_x p_X(x) \log_2 p_X(x) \). The \( \delta \)-typical projector \( \Pi_{A,n}^{\rho,\delta} \) onto the typical subspace of \( \rho \) is defined as

\[
\Pi_{A,n}^{\rho,\delta} \equiv \sum_{x^n \in T_{X,n}^{\delta}} |x^n\rangle \langle x^n|,
\]

where we have used the symbol \( T_{X,n}^{\delta} \) to refer to the set of \( \delta \)-typical sequences:

\[
T_{X,n}^{\delta} \equiv \{ x^n : |\overline{H}(x^n) - H(X)| \leq \delta \}.
\]

Three important properties of the typical projector are as follows:

\[
\begin{align*}
\text{Tr}\{\Pi_{A,n}^{\rho,\delta} \rho^\otimes n\} &\geq 1 - \varepsilon, \\
\text{Tr}\{\Pi_{A,n}^{\rho,\delta}\} &\leq 2^{n[H(A)+\delta]}, \\
2^{-n[H(A)+\delta]} \Pi_{A,n}^{\rho,\delta} &\leq \Pi_{A,n}^{\rho,\delta} \rho^\otimes n \Pi_{A,n}^{\rho,\delta} \leq 2^{-n[H(A)-\delta]} \Pi_{A,n}^{\rho,\delta},
\end{align*}
\]

where the first property holds for arbitrary \( \varepsilon \in (0,1) \), \( \delta > 0 \), and sufficiently large \( n \). We will also need the following ‘projector trick’ inequality [GLM12, FHS+12]:

\[
\Pi_{A,n}^{\rho,\delta} \leq 2^{n[H(A)+\delta]} \rho_A^\otimes n ,
\]

which follows as a consequence of the leftmost inequality in (2.21) and the fact that \( \Pi_{A,n}^{\rho,\delta} \rho^\otimes n \Pi_{A,n}^{\rho,\delta} = \sqrt{\rho \otimes n} \Pi_{A,n}^{\rho,\delta} \sqrt{\rho \otimes n} \leq \rho \otimes n \).

**Hayashi-Nagaoka operator inequality.** We repeatedly use the following operator inequality from [HN03] when analyzing error probability:

**Lemma 4** Given operators \( S \) and \( T \) such that \( 0 \leq S \leq I \) and \( T \geq 0 \), the following inequality holds for all \( c > 0 \):

\[
I - (S + T)^{-1/2} S (S + T)^{-1/2} \leq c_1 (I - S) + c_1 T ,
\]

where \( c_1 \equiv 1 + c \) and \( c_1 \equiv 2 + c + c^{-1} \).

### 3 Entanglement-assisted point-to-point classical communication

We begin by defining the information-processing task of point-to-point entanglement-assisted classical communication [BSST02, Hol02, DH13, MW14, DTW16]. Before communication begins, the sender Alice and the receiver Bob share entanglement in whatever form they wish, and we denote their shared state as \( \Psi_{RA} \). Suppose Alice would like to communicate some classical message \( m \)
from a set $\mathcal{M} \equiv \{1, \ldots, M\}$ over a quantum channel $\mathcal{N}_{A' \to B}$, where $M \in \mathbb{N}$ denotes the cardinality of the set $\mathcal{M}$. An $(M, \varepsilon)$ entanglement-assisted classical code, for $\varepsilon \in [0,1]$, consists of a collection $\{E_m^{A \to A'}\}_m$ of encoders and a decoding POVM $\{\Lambda_{RB}^m\}_m$, such that the average error probability is bounded from above by $\varepsilon$:

$$\frac{1}{M} \sum_{m=1}^{M} \text{Tr}\{(I - \Lambda_{RB}^m)\mathcal{N}_{A' \to B}(E_m^{A \to A'}(\Psi_{RA}))\} \leq \varepsilon,$$

(3.1)

For fixed $\varepsilon$, let $M^*(\mathcal{N}, \varepsilon)$ denote the largest $M$ for which there exists an $(M, \varepsilon)$ entanglement-assisted classical communication code for the channel $\mathcal{N}$. Then we define the $\varepsilon$-one-shot entanglement-assisted classical capacity as $\log_2 M^*(\mathcal{N}, \varepsilon)$. We note that one could alternatively consider maximum error probability when defining this capacity. The entanglement-assisted capacity of a channel $\mathcal{N}$ is then defined as

$$C_{EA}(\mathcal{N}) \equiv \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log_2 M^* (\mathcal{N} \otimes n, \varepsilon).$$

(3.2)

3.1 One-shot position-based coding

We now review the method of position-based coding [AJW17] and follow the review by showing how the approach leads to a lower bound on the error exponent for entanglement-assisted communication, a lower bound for one-shot entanglement-assisted capacity, and a simple proof for a lower bound on the second-order coding rate for entanglement-assisted communication. We note that a lower bound for one-shot entanglement-assisted capacity using position-based coding was already given in [AJW17], but the lower bound given here leads to a lower bound on the entanglement-assisted second-order coding rate that is optimal for covariant channels [DTW16].

**Encoding:** Before communication begins, Alice and Bob share the following state:

$$\theta^{\otimes M}_{RA} \equiv \theta_{R_1 A_1} \otimes \cdots \otimes \theta_{R_M A_M},$$

(3.4)

where Alice possesses the $A$ systems and Bob has the $R$ systems. To send message $m$, Alice simply sends the $m$th $A$ system through the channel. So this leads to the following state for Bob:

$$\rho_{RB}^m \equiv \theta_{R}^{\otimes m-1} \otimes \mathcal{N}_{A \to B}(\theta_{R m A_m}) \otimes \theta_{R}^{\otimes M-m}.$$

(3.5)

**Decoding:** Define the following measurement:

$$\Gamma_{RB}^m \equiv I_{R}^{m-1} \otimes T_{RB}^{m} \otimes I_{R}^{M-m},$$

(3.6)

where $T_{RB}^{m} = T_{RB}$ is a “test” or measurement operator satisfying $0 \leq T_{RB} \leq I_{RB}$, which we will specify later. For now, just think of it as corresponding to a measurement that should distinguish well between $\mathcal{N}_{A \to B}(\theta_{RA})$ and $\theta_{R} \otimes \mathcal{N}_{A \to B}(\theta_{A})$. This is important for the following reason:
message $m$ is transmitted and the test is performed on the $m$th $R$ system and the channel output system $B$, then the probability of it accepting is

$$\text{Tr}\{\Gamma^m_{RMB} \rho^m_{RMB}\} = \text{Tr}\{T_{RB}N_{A\rightarrow B}(\theta_{RA})\}. \quad (3.7)$$

If however the test is performed on the $m'$th $R$ system and $B$, where $m' \neq m$, then the probability of it accepting is

$$\text{Tr}\{\Gamma^{m'}_{RMB} \rho^m_{RMB}\} = \text{Tr}\{T_{RB}[\theta_{R} \otimes N_{A\rightarrow B}(\theta_{A})]\}. \quad (3.8)$$

We use these facts in the forthcoming error analysis.

We use a square-root measurement to form a decoding POVM for Bob as follows:

$$\Lambda^m_{RMB} \equiv \left( \sum_{m' = 1}^{M} \Gamma^{m'}_{RMB} \right)^{-1/2} \Gamma^m_{RMB} \left( \sum_{m' = 1}^{M} \Gamma^{m'}_{RMB} \right)^{-1/2}. \quad (3.9)$$

This is called the position-based decoder.

**Error analysis:** The error probability under this coding scheme is the same for each message $m$ (see, e.g., [AJW17, Wil17a]) and is as follows:

$$p_e(m) \equiv \text{Tr}\{(I_{RMB} - \Lambda^m_{RMB}) \rho^m_{RMB}\}. \quad (3.10)$$

Applying Lemma 4 with $S = \Gamma^m_{RMB}$ and $T = \sum_{m' \neq m} \Gamma^{m'}_{RMB}$, we find that this error probability can be bounded from above as

$$\text{Tr}\{(I_{RMB} - \Lambda^m_{RMB}) \rho^m_{RMB}\} \leq c_1 \text{Tr}\{(I_{RMB} - \Gamma^m_{RMB}) \rho^m_{RMB}\} + c_2 \sum_{m' \neq m} \text{Tr}\{\Gamma^{m'}_{RMB} \rho^m_{RMB}\} \quad (3.11)$$

$$= c_1 \text{Tr}\{(I_{RB} - T_{RB})N_{A\rightarrow B}(\theta_{RA})\} + c_2 \sum_{m' \neq m} \text{Tr}\{T_{RB}[\theta_{R} \otimes N_{A\rightarrow B}(\theta_{A})]\} \quad (3.12)$$

$$= c_1 \text{Tr}\{(I_{RB} - T_{RB})N_{A\rightarrow B}(\theta_{RA})\} + c_2(M - 1) \text{Tr}\{T_{RB}[\theta_{R} \otimes N_{A\rightarrow B}(\theta_{A})]\}. \quad (3.13)$$

The same bound applies for both the average and the maximum error probability, due to the symmetric construction of the code.

Our fundamental bound for a test operator $T_{RB}$ is thus as follows and highlights, as in [AJW17], an important connection between quantum hypothesis testing (i.e., the ability to distinguish the states $N_{A\rightarrow B}(\theta_{RA})$ and $\theta_{R} \otimes N_{A\rightarrow B}(\theta_{A})$) and entanglement-assisted communication:

$$p_e(m) \leq c_1 \text{Tr}\{(I_{RB} - T_{RB})N_{A\rightarrow B}(\theta_{RA})\} + c_2(M - 1) \text{Tr}\{T_{RB}[\theta_{R} \otimes N_{A\rightarrow B}(\theta_{A})]\}. \quad (3.14)$$

### 3.2 Lower bounds on one-shot and i.i.d. entanglement-assisted error exponents

We first prove a lower bound on the one-shot error exponent, and then a lower bound for the entanglement-assisted error exponent in the i.i.d. case directly follows.

**Theorem 5** For a quantum channel $N_{A\rightarrow B}$, a lower bound on the one-shot entanglement-assisted error exponent for fixed message size $M$ is as follows:

$$-\log_2 \varepsilon^*(N, M) \geq \sup_{s \in [0, 1]} \left( 1 - s \right) \left[ \sup_{\theta_{RA}} I_s(R; B|N(\theta)) - \log_2 M \right] - 2, \quad (3.15)$$
where $\theta_{RA}$ is a pure bipartite entangled state and $I_s(R; B)_{N(\theta)}$ is the Rényi mutual information defined in (2.6).

**Proof.** Following the position-based encoding and decoding procedure described in Section 3.1 and setting $c = 1$ in (3.14), the error probability for each message can be bounded as

$$p_e(m) = \text{Tr}\{(I_{RM}B - N_{RM}^m B)\rho_{RM}^m B\}$$

$$\leq 4 \text{Tr}\{(I_{RB} - T_{RB})N_{A\rightarrow B}(\theta_{RA})\} + M \text{Tr}\{T_{RB} [\theta_R \otimes N_{A\rightarrow B}(\theta_A)]\}$$  \hspace{1cm} (3.16)

$$= 4 \left[ \text{Tr}\{N_{A\rightarrow B}(\theta_{RA})\} - \text{Tr}\{T_{RB} (N_{A\rightarrow B}(\theta_{RA}) - M [\theta_R \otimes N_{A\rightarrow B}(\theta_A)]\) \right].$$  \hspace{1cm} (3.17)

To minimize the term in the last line above, it is well known that one should take the test operator $T_{RB}$ as follows:

$$T_{RB} = \{N_{A\rightarrow B}(\theta_{RA}) - M [\theta_R \otimes N_{A\rightarrow B}(\theta_A)] \geq 0\}. \hspace{1cm} (3.19)$$

The statement for quantum states is due to [Hel69, Hol73, Hel76] and the extension (relevant for us) to the more general case of positive semi-definite operators appears in [AM14, Eq. (22)] (see also (2.11)–(2.14)). This then leads to the following upper bound on the error probability:

$$p_e(m) \leq 4 \left[ \text{Tr}\{N_{A\rightarrow B}(\theta_{RA})\} - \text{Tr}\{T_{RB} (N_{A\rightarrow B}(\theta_{RA}) - M [\theta_R \otimes N_{A\rightarrow B}(\theta_A)]\) \right]$$  \hspace{1cm} (3.20)

$$= 2 \left[ \text{Tr}\{N_{A\rightarrow B}(\theta_{RA}) + M [\theta_R \otimes N_{A\rightarrow B}(\theta_A)]\} \right.$$  \hspace{1cm} (3.21)

$$- ||N_{A\rightarrow B}(\theta_{RA}) - M [\theta_R \otimes N_{A\rightarrow B}(\theta_A)]||_1 \right]$$

$$\leq 4 \text{Tr}\{N_{A\rightarrow B}(\theta_{RA})^s [M \theta_R \otimes N_{A\rightarrow B}(\theta_A)]^{1-s}\}$$  \hspace{1cm} (3.22)

$$= 4M^{1-s} \text{Tr}\{N_{A\rightarrow B}(\theta_{RA})^s [\theta_R \otimes N_{A\rightarrow B}(\theta_A)]^{1-s}\}$$  \hspace{1cm} (3.23)

$$= 4 \left( 2^{-(1-s)I_s(R; B)_{N(\theta)} - \log_2 M} \right).$$  \hspace{1cm} (3.24)

The first equality is standard, using the relation of the positive part of an operator to its modulus (see, e.g., [AM14, Eq. (23)]). The second inequality is a consequence of [ACMnT+07, Theorem 1], recalled as Lemma 3 in Section 2, and holds for all $s \in [0, 1]$ (see [JOPP11] for a simpler proof of [ACMnT+07, Theorem 1]). The last equality follows from the definition of Rényi mutual information in (2.6). Since this bound holds for an arbitrary $s \in [0, 1]$ and an arbitrary input state $\theta_{RA}$, we can conclude the following bound:

$$p_e(m) \leq 4 \left( 2^{\sup_{s \in [0, 1]} (1-s) \left[ \sup_{\theta_{RA}} I_s(R; B)_{N(\theta)} - \log_2 M \right]} \right).$$  \hspace{1cm} (3.25)

Note that it suffices to take $\theta_{RA}$ as a pure bipartite state, due to the ability to purify a mixed $\theta_{RA}$ and the data-processing inequality for $I_s(R; B)_{N(\theta)}$, holding for all $s \in [0, 1]$ [Pet86]. Finally taking a negative binary logarithm of both sides of (3.25) gives (3.15).

We remark that the above proof bears some similarities to that for [MD09, Lemma 3.1], which in turn bears similarities to the approach from [Hay07].

Applying the above result in the i.i.d. case for a memoryless channel $N_{A\rightarrow B}^{\otimes n}$ leads to the following:
Proposition 6 For a quantum channel $N_{A \rightarrow B}$, a lower bound on the entanglement-assisted error exponent $E_{EA}(N, R)$ (defined in (3.3)) for fixed rate $R \geq 0$ is as follows:

$$E_{EA}(N, R) \geq \sup_{s \in [0,1]} (1-s) \left[ \sup_{\theta_{RA}} I_s(R; B)_{N(\theta)} - R \right], \quad (3.26)$$

where $\theta_{RA}$ is a pure bipartite entangled state and $I_s(R; B)_{N(\theta)}$ is the Rényi mutual information.

Proof. A proof follows by plugging in the memoryless channel $N_{A \rightarrow B}^{\otimes n}$ into (3.15), considering that

$$\sup_{\theta_{RA}^n} I_s(R^n; B^n)_{N_{A \rightarrow B}^{\otimes n}} \geq \sup_{\theta_{RA}^n} I_s(R^n; B^n)_{N^{\otimes n}(\theta)^{\otimes n}} \quad (3.27)$$

$$= n \sup_{\theta_{RA}} I_s(R; B)_{N(\theta)}, \quad (3.28)$$

and setting the number of messages to be $M = 2^n R$. The last line above follows from the additivity of the Rényi mutual information for tensor-power states. Applying definitions, we arrive at (3.26).

More specifically, plugging in to (3.25), we find that the bound on the error probability becomes

$$p_e(m) \leq 4 \left( 2^{-(1-s)n} |I_s(R; B)_{N(\theta)} - R| \right), \quad (3.29)$$

holding for all $s \in [0,1]$ and states $\theta_{RA}$. After taking a negative logarithm, normalizing by $n$, and taking the limit as $n \to \infty$, we arrive at (3.26). \[\blacksquare\]

### 3.3 Lower bounds on one-shot entanglement-assisted capacity and entanglement-assisted second-order coding rate

By using position-based coding, here we establish a lower bound on the one-shot entanglement-assisted capacity. Note that a similar lower bound was established in [AJW17], but the theorem below allows for an additional parameter $\eta \in (0, \varepsilon)$, which is helpful for giving a lower bound on the entanglement-assisted second-order coding rate.

Theorem 7 Given a quantum channel $N_{A \rightarrow B}$ and fixed $\varepsilon \in (0, 1)$, the $\varepsilon$-one-shot entanglement-assisted capacity of $N_{A \rightarrow B}$ is bounded as

$$\log_2 M^*(N,\varepsilon) \geq \max_{\theta_{RA}} I_H^{\varepsilon-\eta}(R; B)_{N(\theta)} - \log_2(4\varepsilon/\eta^2), \quad (3.30)$$

where $\eta \in (0, \varepsilon)$ and the hypothesis testing mutual information is defined in (2.8).

Proof. The idea is to use the same coding scheme described in Section 3.1 and take the test operator $T_{RB}$ in Bob's decoder to be $Y_{RB}^*$, where $Y_{RB}^*$ is the optimal measurement operator for $I_{H}^{\varepsilon-\eta}(R; B)_{N(\theta)}$, with $\eta \in (0, \varepsilon)$. Then, starting from the upper bound on the error probability in (3.14), the error analysis reduces to

$$\text{Tr}\left\{ (I_{RM} - \Lambda_{RM}) \rho_{RM}^m \right\} \leq c_I \text{Tr}\left\{ (I_{RM} - Y_{RB}^*) [N_{A \rightarrow B}^{\otimes n}(\theta_{RA})] \right\} + c_I M^2 \text{Tr}\left\{ Y_{RB}^* [\theta_{RA} \otimes N_{A \rightarrow B}^{\otimes n}(\theta_{A})] \right\} \quad (3.31)$$

$$\leq c_I (\varepsilon - \eta) + c_I M^2 I_{H}^{\varepsilon-\eta}(R; B)_{N(\theta)}. \quad (3.32)$$
The second inequality follows from the definition of quantum hypothesis testing relative entropy, which gives that

\[
\text{Tr} \left\{ \Upsilon^*_{RB} [N_{A \to B}(\theta_{RA})] \right\} \geq 1 - (\varepsilon - \eta),
\]

\[
\text{Tr} \left\{ \Upsilon^*_{RB} [\theta_R \otimes N_{A \to B}(\theta_A)] \right\} = 2^{-I^\varepsilon\eta_H(R;B)_{N(\theta)}}.
\]

(3.33)

(3.34)

To make the error \( p_e(m) \leq \varepsilon \), we set \( c = \frac{\eta}{2\varepsilon - \eta} \) for \( \eta \in (0, \varepsilon) \) [TH13, WR12], and this leads to

\[
\log_2 M = I^\varepsilon\eta_H(R;B)_{N(\theta)} - \log_2 (4\varepsilon/\eta^2).
\]

(3.35)

The inequality in the theorem follows after maximizing \( I^\varepsilon\eta_H(R;B)_{N(\theta)} \) with respect to all input states \( \theta_{RA} \). □

**Comparison to upper bound.** The authors of [MW14] established the following upper bound on one-shot entanglement-assisted capacity:

\[
I^\varepsilon_H(R;B)_{N(\theta)} \geq \max_{\theta_{RA}} \min_{\sigma_B} D^\varepsilon_H(N_{A \to B}(\theta_{RA}) \parallel \theta_R \otimes \sigma_B)
\]

\[
\geq \log_2 M^*(N, \varepsilon).
\]

(3.36)

(3.37)

Thus, there is a sense in which the upper bound from [MW14] is close to the lower bound in (3.30), in the sense that the two bounds differ by terms of order \( \log \varepsilon \).

**Lower bound on second-order coding rate.** To get a lower bound on the entanglement-assisted second-order coding rate for an i.i.d. channel \( N^{\otimes n} \), evaluate the formula \( I^\varepsilon\eta_H(R;B)_{N(\theta)} \) for an i.i.d. state \( N(\theta)^{\otimes n} \), pick \( \eta = 1/\sqrt{n} \) and \( n \) large enough such that \( \varepsilon - \eta > 0 \), and use the second-order expansions for \( D^\varepsilon_H \) in (2.9). We then recover one of the main results of [DTW16]:

\[
\log_2 M^*(N, \varepsilon) \geq nI(R;B)_{N(\theta)} + \sqrt{nV(R;B)_{N(\theta)}\Phi^{-1}(\varepsilon)} + O(\log n).
\]

(3.38)

Interestingly, this is achievable for maximal error in addition to average error due to the above analysis. Additionally, it does seem that this approach for arriving at a lower bound on the entanglement-assisted second-order coding rate is much simpler than the previous approach developed in [DTW16].

### 4 Classical communication over quantum multiple-access channels

We now establish a link between communication over a quantum multiple-access channel and multiple quantum hypothesis testing. One advantage of this development is the reduction of the communication problem to a hypothesis testing problem, which is perhaps simpler to state and could also be considered a more fundamental problem than the communication problem. Later, in Section 5, we discuss the relation of the quantum simultaneous decoding conjecture from [FHS+12, Will11] to open questions in multiple quantum hypothesis testing from [AM14, BHIOS15] (here we note that the solution of the multiple Chernoff distance conjecture from [Li16] does not evidently allow for the solution of the quantum simultaneous decoding conjecture). We point the reader to [Dut11, DF13] for further discussions and variations of the quantum simultaneous decoding conjecture.
We begin by considering the case of entanglement-assisted communication and later consider unassisted communication. We first define the information-processing task for entanglement-assisted classical communication over quantum multiple-access channels (see also [HDW08, XW13]). Consider the scenario in which two senders Alice and Bob would like to transmit classical messages to a receiver Charlie over a two-sender single-receiver quantum multiple-access channel $N_{AB\rightarrow C}$. Alice and Bob choose their messages from message sets $L$ and $M$. The cardinality of the sets $L$ and $M$ are denoted as $L$ and $M$, respectively. Suppose that Alice and Bob each share an arbitrary entangled state with Charlie before communication begins. Let $\Phi_{RA'}$ denote the state shared between Charlie and Alice, and let $\Psi_{SB'}$ denote the state shared between Charlie and Bob.

Let $L, M \in \mathbb{N}$ and $\varepsilon \in [0, 1]$. An $(L, M, \varepsilon)$ entanglement-assisted multiple-access classical communication code consists of a set $\{E_{RA'}^l, F_{B'}^m, A_{RSC}^{l,m}\}$ of encoders and a decoding POVM, such that the average error probability is bounded from above by $\varepsilon$:

$$\frac{1}{LM} \sum_{l=1}^{L} \sum_{m=1}^{M} p_e(l, m) \leq \varepsilon ,$$

(4.1)

where the error probability for each message pair is given by

$$p_e(l, m) \equiv \text{Tr}\{(I_{RSC} - A_{RSC}^{l,m})N_{AB\rightarrow C}(E_{RA'}^l(\Phi_{RA'}) \otimes F_{B'}^m(\Psi_{SB'}))\} .$$

(4.2)

### 4.1 One-shot position-based coding scheme

We now describe and analyze a position-based coding scheme for entanglement-assisted communication over a quantum multiple-access channel, in which the decoding POVM is a quantum simultaneous decoder.

**Encoding:** Before communication begins, suppose that Alice and Charlie share $L$ copies of the same bipartite state: $\theta_{RA}^{\otimes L} \equiv \theta_{R_1A_1} \otimes \cdots \otimes \theta_{R_LA_L}$. Similarly, suppose that Bob and Charlie share $M$ copies of the same bipartite state: $\gamma_{SB}^{\otimes M} \equiv \gamma_{S_1B_1} \otimes \cdots \otimes \gamma_{S_MB_M}$. To send message $(l, m) \in L \times M$, Alice sends the $l$th $A$ system of $\theta_{RA}^{\otimes L}$ and Bob sends the $m$th $B$ system of $\gamma_{SB}^{\otimes M}$ over the quantum multiple-access channel $N_{AB\rightarrow C}$. So this leads to the following state for Charlie:

$$\rho_{R^lS^mC}^{l,m} = \theta_R^{\otimes (l-1)} \otimes \gamma_S^{\otimes (m-1)} \otimes N_{AB\rightarrow C}(\theta_{R_1A_1} \otimes \gamma_{S_mB_m}) \otimes \theta_R^{\otimes (L-l)} \otimes \gamma_S^{\otimes (M-m)} .$$

(4.3)

**Decoding:** To decode the message transmitted, Charlie performs a measurement on the systems $R^L, S^M$, and the channel output $C$ to determine the message pair $(l, m)$ that Alice and Bob transmitted. Consider the following measurement operator:

$$\Gamma_{R^lS^mC}^{l,m} \equiv T_{R^lS^mC} ,$$

(4.4)

where identity operators are implicit for all of the $R$ and $S$ systems besides $R_l$ and $S_m$ and $T_{RSC}$ is a measurement operator satisfying $0 \leq T_{RSC} \leq I_{RSC}$. Let us call the action of performing the measurement $\{\Gamma_{R^lS^mC}^{l,m}, I_{R^lS^mC} - \Gamma_{R^lS^mC}^{l,m}\}$ “checking for the message pair $(l, m)$.” If Charlie checks for message pair $(l, m)$ when indeed message pair $(l, m)$ is transmitted, then the probability of incorrectly decoding is

$$\text{Tr}\{(I - \Gamma_{R^lS^mC}^{l,m})\rho_{R^lS^mC}^{l,m}\} = \text{Tr}\{(I - T_{RSC})N_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\} .$$

(4.5)
The equality above follows by combining (4.3) and (4.4) and applying partial traces. There are three other kinds of error probabilities to consider. If message pair \((l, m)\) was transmitted and Charlie checks for message pair \((l', m)\) for \(l' \neq l\), then the probability of decoding as \((l', m)\) is

\[
\text{Tr}\{T_{RLSM}^{l,m} \rho_{RLSM}^{l,m}\} = \text{Tr}\{T_{RSC}N_{AB} \rightarrow C(\theta_R \otimes \theta_A \otimes \gamma_{SB})\}. \tag{4.6}
\]

If message pair \((l, m)\) was transmitted and Charlie checks for message pair \((l', m')\) for \(m' \neq m\), then the probability of decoding as \((l, m)\) is

\[
\text{Tr}\{T_{RLSM}^{l',m'} \rho_{RLSM}^{l,m}\} = \text{Tr}\{T_{RSC}N_{AB} \rightarrow C(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)\}. \tag{4.7}
\]

If message pair \((l, m)\) was transmitted and Charlie checks for message pair \((l', m')\) for \(l' \neq l\) and \(m' \neq m\), then the probability of decoding as \((l', m')\) is

\[
\text{Tr}\{T_{RLSM}^{l',m'} \rho_{RLSM}^{l,m}\} = \text{Tr}\{T_{RSC}N_{AB} \rightarrow C(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)\}. \tag{4.8}
\]

The above observations are helpful in the forthcoming error analysis.

We now take Charlie’s position-based quantum simultaneous decoder to be the following square-root measurement:

\[
\Lambda_{RLSM}^{l,m} = \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma_{RLSM}^{l',m'} \right)^{-1/2} \Gamma_{RLSM}^{l,m} \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma_{RLSM}^{l',m'} \right)^{-1/2}. \tag{4.9}
\]

**Error analysis:** Due to the code construction, the error probability under the position-based coding scheme is the same for each message pair \((l, m)\):

\[
p_e(l, m) = \text{Tr}\{(I - \Lambda_{RLSM}^{l,m} \rho_{RLSM}^{l,m})\}. \tag{4.10}
\]

Applying Lemma 4 and (4.5)–(4.8), we arrive at the following upper bound on the decoding error probability:

\[
p_e(l, m) \leq c_1 \text{Tr}\{(I - \Gamma_{RLSM}^{l,m} \rho_{RLSM}^{l,m})\} + c_{11} \sum_{(l', m') \neq (l, m)} \text{Tr}\{\Gamma_{RLSM}^{l',m'} \rho_{RLSM}^{l,m}\} \tag{4.11}
\]

\[
= c_1 \text{Tr}\{(I - \Gamma_{RLSM}^{l,m} \rho_{RLSM}^{l,m})\} + c_{11} \sum_{l' \neq l} \text{Tr}\{\Gamma_{RLSM}^{l',m'} \rho_{RLSM}^{l,m}\}
+ c_{11} \sum_{m' \neq m} \text{Tr}\{\Gamma_{RLSM}^{l,m} \rho_{RLSM}^{l,m'}\} \tag{4.12}
\]

\[
= c_1 \text{Tr}\{(I - T_{RSC})N_{AB} \rightarrow C(\theta_{RA} \otimes \gamma_{SB})\} + c_{11} \left[ (L - 1) \text{Tr}\{T_{RSC}N_{AB} \rightarrow C(\theta_R \otimes \theta_A \otimes \gamma_{SB})\} + (M - 1) \text{Tr}\{T_{RSC}N_{AB} \rightarrow C(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)\} + (L - 1)(M - 1) \text{Tr}\{T_{RSC}N_{AB} \rightarrow C(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)\} \right]. \tag{4.13}
\]

The error probability associated with \(c_1\) is the probability of incorrectly decoding when Charlie checks for message pair \((l, m)\). The error probabilities associated with \(c_{11}\) are the probabilities of decoding as some other message pair when message pair \((l, m)\) is transmitted. There are \(L - 1\) possibilities for Charlie to erroneously decode Alice’s message and correctly decode Bob’s message, \(M - 1\) possibilities to erroneously decode Bob’s message and correctly decode Alice’s message, and \((M - 1)(L - 1)\) possibilities of incorrectly decoding both Alice and Bob’s messages.
Fundamental one-shot bound for quantum simultaneous decoding. Thus, our fundamental bound on the decoding error probability for a position-based entanglement-assisted coding scheme is as follows:

\[
p_e(l,m) \leq c_1 \text{Tr} \left( (I - T_{RSC}) N_{AB \rightarrow C}(\theta_{RA} \otimes \gamma_{SB}) \right) + c_1 \left[ (L-1) \text{Tr} \left( T_{RSC} N_{AB \rightarrow C}(\theta_{R} \otimes \theta_{A} \otimes \gamma_{SB}) \right) \\
+ (M-1) \text{Tr} \left( T_{RSC} N_{AB \rightarrow C}(\theta_{RA} \otimes \gamma_{S} \otimes \gamma_{B}) \right) \\
+ (L-1)(M-1) \text{Tr} \left( T_{RSC} N_{AB \rightarrow C}(\theta_{R} \otimes \theta_{A} \otimes \gamma_{S} \otimes \gamma_{B}) \right) \right].
\] (4.14)

Interestingly, this bound is the same for all message pairs, and thus holds for maximal or average error probability. We also remark that the above inequality forges a transparent link between communication over multiple-access channels and multiple quantum hypothesis testing, a point that we will return to in Section 5.

Generalization to multiple senders. The above bound can be extended as follows for an entanglement-assisted quantum multiple-access channel \( N_{A_1 \cdots A_K \rightarrow C} \) with \( K \) senders and a single receiver:

\[
p_e(m_1, \ldots, m_K) \leq c_1 \text{Tr} \left\{ (I - T_{R_{1 \cdots R_K C}}) N_{A_1 \cdots A_K \rightarrow C} \left( \bigotimes_{k=1}^{K} \theta_{R_k A_k} \right) \right\} \\
+ c_1 \sum_{J \subset [K]} \prod_{j \in J} (M_j - 1) \text{Tr} \left\{ T_{R_{1 \cdots R_K C}} N_{A_1 \cdots A_K \rightarrow C} \left( \bigotimes_{j \in J} \theta_{R_j} \otimes \theta_{A_j} \otimes \bigotimes_{l \in J^c} \theta_{R_l A_l} \right) \right\}.
\] (4.15)

In the above, \( m_1, \ldots, m_K \) are the messages for senders 1 through \( K \), respectively, chosen from respective message sets of size \( M_1, \ldots, M_K \). The states \( \theta_{R_1 A_1}, \ldots, \theta_{R_K A_K} \) are entangled states shared between the receiver and senders 1 through \( K \), with the receiver possessing all of the \( R \) systems. Finally, \( T_{R_{1 \cdots R_K C}} \) is a test operator satisfying \( 0 \leq T_{R_{1 \cdots R_K C}} \leq I_{R_{1 \cdots R_K C}} \) and \( J \) is a proper subset of \([K] \equiv \{1, \ldots, K\}\). The above bound is derived by using position-based coding as described above and a square-root measurement that generalizes (4.9). We omit the straightforward proof.

4.2 Unassisted classical communication over multiple-access channels

The position-based coding technique is not only a powerful tool for entanglement-assisted classical communication protocols, but also for those that do not employ entanglement assistance or any other kind of assistance. This was shown explicitly for the single-sender, single-receiver case in [Wil17a]. We now demonstrate this point further by considering unassisted classical communication over a classical-input quantum-output multiple-access channel \( N_{XY \rightarrow C} \). We do so by first considering classical communication assisted by shared randomness, such that we can employ a position-based coding scheme, and then we derandomize the protocol to obtain a codebook for unassisted communication.

The classical-classical-quantum channel that we consider can be written in fully quantum form as

\[
N_{XY' \rightarrow C}(\omega_{X'|Y'}) = \sum_{x,y} \langle x|X'|y|Y'\omega_{X'|Y'}|x\rangle_{X'}|y\rangle_{Y'} \rho_C^{x,y}.
\] (4.16)
Before communication begins, Alice and Charlie share randomness in the form of the following classical–classical state:

$$
\rho_{XX'} \equiv \sum_x p_X(x) \langle x | X \otimes | x | X' \rangle .
$$

(4.17)

Similarly Bob and Charlie also share randomness represented by the following classical–classical state:

$$
\sigma_{YY'} \equiv \sum_y p_Y(y) \langle y | Y \otimes | y | Y' \rangle .
$$

(4.18)

We demonstrate the procedure of derandomization by proving the following theorem:

**Theorem 8** There exists an unassisted, simultaneous decoding protocol for classical communication over a quantum multiple-access channel with the following upper bound on its average error probability, holding for all $T_{XYC}$ such that $0 \leq T_{XYC} \leq I_{XYC}$:

$$
\frac{1}{LM} \sum_{l,m} p_e(l,m) \leq c_I \text{Tr} \{ (I - T_{XYC}) \mathcal{N}_{XY'\rightarrow C}(\rho_{XX'} \otimes \sigma_{YY'}) \}
$$

$$
+ c_{II} \left[ (L - 1) \text{Tr} \{ T_{XYC} \mathcal{N}_{XY'\rightarrow C}(\rho_{XX'} \otimes \rho_{XX'} \otimes \sigma_{YY'}) \} 
\right.
$$

$$
+ (M - 1) \text{Tr} \{ T_{XYC} \mathcal{N}_{XY'\rightarrow C}(\rho_{XX'} \otimes \sigma_Y \otimes \sigma_{YY'}) \}
$$

$$
+ (L - 1) (M - 1) \text{Tr} \{ T_{XYC} \mathcal{N}_{XY'\rightarrow C}(\rho_{XX} \otimes \rho_{XX} \otimes \sigma_Y \otimes \sigma_{YY'}) \} \right] ,
$$

(4.19)

where $L$ is the number of messages for the first sender and $M$ is the number of messages for the second sender. A generalization of this statement holds for multiple senders, with an upper bound on the average error probability given by the right-hand side of (4.15), but with all of the $R$ and $A$ systems being classical.

**Proof.** The position-based coding scheme operates exactly as specified in Section 4.1, with the states $\theta_{RA}$ and $\gamma_{SB}$ replaced by $\rho_{XX'}$ and $\sigma_{YY'}$, respectively, the channel $\mathcal{N}_{AB\rightarrow C}$ replaced by $\mathcal{N}_{XY'\rightarrow C}$, and the test operator $T_{RSC}$ replaced by $T_{XYC}$. The same error analysis then leads to the following bound on the error probability when decoding the message pair $(l,m)$:

$$
p_e(l,m) \leq c_I \text{Tr} \{ (I - T_{XYC}) \mathcal{N}_{XY'\rightarrow C}(\rho_{XX'} \otimes \sigma_{YY'}) \}
$$

$$
+ c_{II} \left[ (L - 1) \text{Tr} \{ T_{XYC} \mathcal{N}_{XY'\rightarrow C}(\rho_{XX'} \otimes \rho_{XX'} \otimes \sigma_{YY'}) \} 
\right.
$$

$$
+ (M - 1) \text{Tr} \{ T_{XYC} \mathcal{N}_{XY'\rightarrow C}(\rho_{XX'} \otimes \sigma_Y \otimes \sigma_{YY'}) \}
$$

$$
+ (L - 1) (M - 1) \text{Tr} \{ T_{XYC} \mathcal{N}_{XY'\rightarrow C}(\rho_{XX} \otimes \rho_{XX} \otimes \sigma_Y \otimes \sigma_{YY'}) \} \right] .
$$

(4.20)

**Derandomization:** Extending the development in [Wil17a], first notice that we can rewrite
the four trace terms in (4.20) as follows:

\[
\begin{align*}
\text{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\rightarrow C}(\rho_{XX'} \otimes \sigma_{YY'})\} &= \text{Tr}\{T_{XYC} \sum_{x,y} p_X(x)p_Y(y)|xy\rangle\langle xy| \otimes \rho_{x,y}^{x,y} \}, \\
&= \sum_{x,y} p_X(x)p_Y(y) \text{Tr}\{Q_{C}^{x,y} \bar{\rho}_{C}^{x,y} \}, \\
\text{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\rightarrow C}(\rho_{X} \otimes \rho_{X'} \otimes \sigma_{YY'})\} &= \sum_{x,y} p_X(x)p_Y(y) \text{Tr}\{Q_{C}^{x,y} \bar{\rho}_{C}^{y} \}, \\
\text{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\rightarrow C}(\rho_{XX'} \otimes \sigma_{Y} \otimes \sigma_{Y'})\} &= \sum_{x,y} p_X(x)p_Y(y) \text{Tr}\{Q_{C}^{x,y} \bar{\rho}_{C}^{x} \}, \\
\text{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\rightarrow C}(\rho_{X} \otimes \rho_{X'} \otimes \sigma_{Y} \otimes \sigma_{Y'})\} &= \sum_{x,y} p_X(x)p_Y(y) \text{Tr}\{Q_{C}^{x,y} \bar{\rho}_{C}^{x} \}, 
\end{align*}
\]

where we define the following averaged output states:

\[
\begin{align*}
\bar{\rho}_{C} &\equiv \sum_{x,y} p_X(x)p_Y(y)\rho_{x,y}^{x,y}, \\
\bar{\rho}_{C}^{y} &\equiv \sum_{x} p_X(x)\rho_{x,y}^{x,y}, \\
\bar{\rho}_{C}^{x} &\equiv \sum_{y} p_Y(y)\rho_{x,y}^{x,y},
\end{align*}
\]

and the measurement operator

\[
Q_{C}^{x,y} \equiv \langle x, y|_{XY} T_{XYC}|x, y\rangle_{XY}.
\]

Thus, in the case that the code is randomness-assisted, it suffices to take the test operator \(T_{XYC}\) to have the following form:

\[
T_{XYC} = \sum_{x,y} |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes Q_{C}^{x,y},
\]

because, as we will show, the upper bound on the average error probability does not change when doing so. Then we can rewrite the decoding POVM in (4.9) as follows:

\[
\Gamma_{X_{L}Y_{M}C}^{L_{l},M_{m}} \equiv T_{X_{l}Y_{m}C} = \sum_{x^{L},y^{M}} |x^{L}\rangle\langle x^{L}|_{X_{L}} \otimes |y^{M}\rangle\langle y^{M}|_{Y_{M}} \otimes Q_{C}^{x^{L},y^{M}},
\]

where we use the resolution of the identity \(I_{X} = \sum_{x} |x\rangle\langle x|_{X}\) to expand the implicit identity
operators and we employ the notation $x^L \equiv x_1 \cdots x_L$ and $y^M \equiv y_1 \cdots y_M$. Then this implies that
\[
\left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma_{L}^{l',m',X_L,Y_M} \right)^{-1/2} = \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} \sum_{x_L,y_M}^{n} \left| x^L \right\rangle \left\langle x^L \right|_{X_L} \otimes \left| y^M \right\rangle \left\langle y^M \right|_{Y_M} \otimes Q_{C}^{x^L,y^M} \right)^{-1/2}
\]
\[
= \left( \sum_{x_L,y_M}^{n} \left| x^L \right\rangle \left\langle x^L \right|_{X_L} \otimes \left| y^M \right\rangle \left\langle y^M \right|_{Y_M} \right)^{-1/2}
\]
\[
= \sum_{x_L,y_M}^{n} \left| x^L \right\rangle \left\langle x^L \right|_{X_L} \otimes \left| y^M \right\rangle \left\langle y^M \right|_{Y_M} \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_{C}^{x^L,y^M} \right)^{-1/2} .
\]

The last step follows from the fact that $\{|x\rangle\}_x$ and $\{|y\rangle\}_y$ form orthonormal bases. Therefore, the decoding POVM for the randomness-assisted protocol can be decomposed as
\[
\Lambda_{X_L,Y_M}^{l,m} = \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma_{X_L,Y_M}^{l',m'} \right)^{-1/2} \Gamma_{X_L,Y_M}^{l,m} \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma_{X_L,Y_M}^{l',m'} \right)^{-1/2}
\]
\[
= \sum_{x_L,y_M}^{n} \left| x^L \right\rangle \left\langle x^L \right|_{X_L} \otimes \left| y^M \right\rangle \left\langle y^M \right|_{Y_M} \Omega_{C}^{x^L,y^M} ,
\]
where
\[
\Omega_{C}^{x^L,y^M} = \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_{C}^{x^L,y^M} \right)^{-1/2} Q_{C}^{x^L,y^M} \left( \sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_{C}^{x^L,y^M} \right)^{-1/2}.
\]

By definition, the output state of Charlie in (4.3), for our case of interest, can be written as
\[
\rho_{X_L,Y_M}^{l,m} = \sum_{x_L,y_M}^{n} p_{X_L}(x^L)p_{Y_M}(y^M) \left| x^L \right\rangle \left\langle x^L \right|_{X_L} \otimes \left| y^M \right\rangle \left\langle y^M \right|_{Y_M} \rho_{C}^{x^L,y^M} .
\]

By combining (4.37) and (4.39), we find that the average error probability for the code can be rewritten as
\[
\frac{1}{LM} \sum_{l=1}^{L} \sum_{m=1}^{M} \operatorname{Tr} \{ (I_{X_L,Y_M} - \Lambda_{X_L,Y_M}^{l,m}) \rho_{X_L,Y_M}^{l,m} \}
\]
\[
= \frac{1}{LM} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{x_L,y_M}^{n} p_{X_L}(x^L)p_{Y_M}(y^M) \operatorname{Tr} \{ (I_{C} - \Omega_{C}^{x^L,y^M}) \rho_{C}^{x^L,y^M} \}
\]
\[
= \sum_{x_L,y_M}^{n} p_{X_L}(x^L)p_{Y_M}(y^M) \left[ \frac{1}{LM} \sum_{l=1}^{L} \sum_{m=1}^{M} \operatorname{Tr} \{ (I_{C} - \Omega_{C}^{x^L,y^M}) \rho_{C}^{x^L,y^M} \} \right].
\]

Suppose now that there exists a randomness-assisted position-based code that has an average error probability $\leq \varepsilon$. By the above equalities and since the average can never exceed the maximum, we know there must exist a particular choice of $x^L, y^L$ such that
\[
\frac{1}{LM} \sum_{l=1}^{L} \sum_{m=1}^{M} \operatorname{Tr} \{ (I - \Omega_{C}^{x^L,y^M}) \rho_{C}^{x^L,y^M} \} \leq \varepsilon .
\]
Thus for an unassisted multiple-access classical communication protocol, if we choose \( \{x_l\}_{l=1}^L \) as Alice’s codebook and \( \{y_m\}_{m=1}^M \) as Bob’s codebook, and the POVM \( \{Q_{x_i}^{x_iy_m}\} \) as Charlie’s decoder, an upper bound on the average probability error is given by

\[
\frac{1}{LM} \sum_{l=1}^L \sum_{m=1}^M p_e(l, m) = \frac{1}{LM} \sum_{l=1}^L \sum_{m=1}^M \Tr\{(I - Q_{x_i}^{x_iy_m})\rho_C^{x_iy_m}\} \leq \varepsilon .
\]  

(4.43)

This proves the statement of the theorem after considering the upper bound in (4.20). ■

### 4.3 Achievable rate region for i.i.d. case

We now demonstrate rates that are achievable when using a particular quantum simultaneous decoder. The rates are bounded by terms which consist of the difference of a Rényi entropy of order two and a conditional quantum entropy. Although these rates are suboptimal when compared to what is achievable by using successive decoding [Win01, HDW08], we nevertheless think that the following coding theorem represents progress toward finding a quantum simultaneous decoder.

Before we state the theorem, we require the following definition: A rate pair \((R_1, R_2)\) is achievable for communication over a quantum multiple access channel if there exists a \((2^{n[R_1-\delta]}, 2^{n[R_2-\delta]}, \varepsilon)\) code for communication over \(\mathcal{N}_{AB\rightarrow C}^{\otimes n}\) for all \(\varepsilon \in (0, 1)\) and sufficiently large \(n\).

**Theorem 9** An achievable rate region \((R_1, R_2)\) for entanglement-assisted classical communication over a quantum multiple-access channel \(\mathcal{N}_{AB\rightarrow C}\) is as follows:

\[
R_1 \leq \tilde{I}(S; CR)_\omega ,
\]

(4.44)

\[
R_2 \leq \tilde{I}(R; CS)_\omega ,
\]

(4.45)

\[
R_1 + R_2 \leq \tilde{I}(RC; S)_\omega ,
\]

(4.46)

where \(\omega_{RSC} \equiv \mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\) and \(\theta_{RA}\) and \(\gamma_{SB}\) are quantum states. Here we define the following mutual-information-like quantities:

\[
\tilde{I}(R; CS)_\omega \equiv H_2(SC)_\omega - H(SC|R)_\omega ,
\]

(4.47)

\[
\tilde{I}(S; CR)_\omega \equiv H_2(RC)_\omega - H(RC|S)_\omega ,
\]

(4.48)

\[
\tilde{I}(RS; C)_\omega \equiv H_2(C)_\omega - H(C|RS)_\omega ,
\]

(4.49)

where \(H_2(A) \equiv -\log_2 \Tr\{\rho_A^\delta\}\) is the Rényi entropy of order two.

**Proof.** In our setting, Alice and Bob use an i.i.d. channel \(\mathcal{N}_{A\rightarrow BC}^{\otimes n}\). In order to bound the error probability, we replace each state in (4.14) by its \(n\)-copy version. Defining \(\omega_{RSC} \equiv \mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\), we choose the test operator \(T\) to be the following ‘coated’ typical projector:

\[
T_{R^nS^nC^n} \equiv (\Pi_{R^n}^{\omega_R} \otimes \Pi_{S^n}^{\omega_S} \otimes \Pi_{C^n}^{\omega_C})(\Pi_{R^n}^{\omega_R} \otimes \Pi_{S^n}^{\omega_S} \otimes \Pi_{C^n}^{\omega_C}) ,
\]

(4.50)

where \(\Pi_{R^n}^{\omega_R}, \Pi_{S^n}^{\omega_S},\) and \(\Pi_{C^n}^{\omega_C}\) are the typical projectors corresponding to the respective states \(\omega_R, \omega_S,\) and \(\omega_{RSC}\). Applying (4.14), we find the following upper bound on the error probability when decoding the message pair \((l, m)\):

\[
p_e(l, m) \leq c_1 \Tr\{\left(I - T_{R^nS^nC^n}\right)\omega_{RSC}^{\otimes n}\} + c_1 \left[ L \Tr\{T_{R^nS^nC^n}[\mathcal{N}_{AB\rightarrow C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})]^{\otimes n}\} + M \Tr\{T_{R^nS^nC^n}[\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)]^{\otimes n}\} + LM \Tr\{T_{R^nS^nC^n}[\mathcal{N}_{AB\rightarrow C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)]^{\otimes n}\}\right] .
\]

(4.51)
We evaluate each term sequentially. To give an upper bound on the first term, consider the following chain of inequalities, which holds for sufficiently large $n$:

$$\begin{align*}
\text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) & (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{RSC}^{\otimes n} \} \\
\geq & \text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{RSC}^{\otimes n} (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) - \omega_{RSC}^{\otimes n} \} \text{ for } n \geq \frac{1}{2} - \epsilon - 2\sqrt{2\epsilon} .
\end{align*}$$

The first inequality follows from Lemma 2. The second inequality follows from (2.19), [HDW08, Eq. (81)], and the application of Lemma 1. We then obtain an upper bound on the first term:

$$\begin{align*}
\text{Tr}\{ (I - T_{R^n S^n C^n}) \omega_{RSC}^{\otimes n} \} \leq & \epsilon + 2\sqrt{2\epsilon} .
\end{align*}$$

Now we consider the second term in (4.51):

$$\begin{align*}
L \text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) & (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{SC}^{\otimes n} \} \\
\leq & L 2^{n[H(RSC)]+\delta} \text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{RSC}^{\otimes n} (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{SC}^{\otimes n} \} \\
= & L 2^{n[H(RSC)]+\delta} \text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{RSC}^{\otimes n} (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{SC}^{\otimes n} \} \\
\leq & L 2^{n[H(SC|R)]+\delta} \text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{SC}^{\otimes n} \} \\
\leq & L 2^{n[H(SC|R)]+\delta} \text{Tr}\{ (\omega_{SC}^{\otimes n})^2 \} \\
= & 2^{nR_1} 2^{n[H(SC|R)]+\delta} 2^{-nH_2(SC)} - 2^{nR_1} 2^{n[H(SC|R)]+\delta} 2^{-nH_2(SC)} - R_1 - 2\delta).
\end{align*}$$

The first inequality follows from the application of the projector trick inequality from (2.22) to the state $\omega_{RSC}^{\otimes n}$. The first equality follows from cyclicity of trace. The second inequality follows from the right-hand side of (2.21), the fact that $\theta_R = \omega_R$, and the inequality $\Pi_{R^n}^{\omega,\delta} \leq I_{R^n}$. The third inequality follows from a partial trace over the $R^n$ systems. The fourth inequality follows because

$$\begin{align*}
\text{Tr}\{ (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta} \otimes \Pi_{C^n}^{\omega,\delta}) \omega_{RSC}^{\otimes n} \} \leq & \text{Tr}\{ (\omega_{SC}^{\otimes n})^2 \Pi_{R^n}^{\omega,\delta} \Pi_{S^n}^{\omega,\delta} \Pi_{C^n}^{\omega,\delta} \omega_{RSC}^{\otimes n} \} = \text{Tr}\{ (\omega_{SC}^{\otimes n})^2 \Pi_{R^n}^{\omega,\delta} \} \leq \text{Tr}\{ (\omega_{SC}^{\otimes n})^2 \},
\end{align*}$$

which is a consequence of the facts that $\Pi_{R^n}^{\omega,\delta} \leq I_{R^n}$ and $\omega_{SC}^{\otimes n} \Pi_{R^n}^{\omega,\delta} \omega_{SC}^{\otimes n}$ and $(\omega_{SC}^{\otimes n})^2$ are positive semi-definite. Finally, the second equality follows from the fact $L = 2^{nR_1}$ and the definition of Rényi entropy of order two.

Following a similar analysis, we obtain the following upper bounds for the other two terms:

$$\begin{align*}
M \text{Tr}\{ T_{R^n S^n C^n} \omega R_A \otimes \gamma_S \otimes \gamma_B \} \leq & 2^{-n[H_2(RC)_{\omega} - H(RC|S)_{\omega} - R_2 - 2\delta] ,} \\
LM \text{Tr}\{ T_{R^n S^n C^n} \omega R_A \otimes \gamma_S \otimes \gamma_B \} \leq & 2^{-n[H_2(C)_{\omega} - H(C|RS)_{\omega} - (R_1 + R_2) - 3\delta] .}
\end{align*}$$

Taking the sum of the upper bounds for the above four terms, we find the following upper bound on the error probability when decoding the message pair $(l, m)$:

$$\begin{align*}
p_e(l, m) \leq & \epsilon + 2\sqrt{2\epsilon} + 2^{-n[H_2(SC)_{\omega} - H(SC|R)_{\omega} - R_1 - 2\delta] + 2^{-n[H_2(RC)_{\omega} - H(RC|S)_{\omega} - R_2 - 2\delta] + 2^{-n[H_2(C)_{\omega} - H(C|RS)_{\omega} - (R_1 + R_2) - 3\delta] .}
\end{align*}$$
Thus, if the rate pair \((R_1, R_2)\) satisfies the following inequalities (related to those in the statement of the theorem)

\[
\begin{align*}
R_1 + 3\delta &\leq \tilde{I}(S; CR)_{\omega}, \\
R_2 + 3\delta &\leq \tilde{I}(R; CS)_{\omega}, \\
R_1 + R_2 + 4\delta &\leq \tilde{I}(RC; S)_{\omega},
\end{align*}
\]

then the error probability can be made arbitrarily small with increasing \(n\). However, since \(\delta\) can be taken arbitrarily small after the limit of large \(n\), we conclude that the rate region given in the statement of the theorem is achievable.

Our results can be easily extended to the multiple-sender scenario, which we state below without explicitly writing down a proof (note that the proof is a straightforward generalization of the above analysis for two senders).

**Theorem 10** An achievable rate region \((R_1, R_2, ..., R_K)\) for entanglement-assisted classical communication over quantum channel \(N_{A_1A_2...A_K\rightarrow C}\) is given by the following:

\[
\sum_{j\in J} R_j \leq \tilde{I}(S(J); CS(J^c))_{\omega}, \quad \text{for every } J \subset [K],
\]

where \(\omega_{CS...SK} = N_{A_1...A_K\rightarrow C}(\theta_{A_1S_1} \otimes \cdots \otimes \theta_{A_KS_K})\). Here we define mutual-information-like quantities

\[
\tilde{I}(S(J); CS(J^c))_{\omega} = H_2(S(J^c)C)_{\omega} - H(S(J^c)C|S(J))_{\omega},
\]

where \(H_2(A)_{\rho} = -\log_2 \text{Tr} \{\rho_A^2\}\) is the Rényi entropy of order two.

By combining Theorems 8 and 9, we obtain the following rate region that is achievable for unassisted classical communication over a quantum multiple-access channel when using a quantum simultaneous decoder:

**Theorem 11** For a classical-input quantum-output channel \(N_{X'Y'\rightarrow C}\), an achievable rate region \((R_1, R_2)\) for unassisted multiple-access classical communication is given by the following:

\[
\begin{align*}
R_1 &\leq \tilde{I}(X; CY)_{\omega}, \\
R_2 &\leq \tilde{I}(Y; CX)_{\omega}, \\
R_1 + R_2 &\leq \tilde{I}(XY; C)_{\omega},
\end{align*}
\]

where \(\omega_{XYC} = N_{X'Y'\rightarrow C}(\rho_{XX'} \otimes \sigma_{YY'})\) and the information quantities are defined as in (4.47)–(4.49).

In Appendix A, we give an alternate achievable rate region which is generally different from the above one.

**Remark 12** There is a generalization of Theorem 11 to a classical-input quantum-output multiple-access channel that has more than two senders. The rate region is similar to that given in Theorem 10, but all of the states \(\theta_{A_1S_1}, \ldots, \theta_{A_KS_K}\) are taken as classical–classical states.

**Remark 13** To be clear, the quantum simultaneous decoding conjecture from [FHS+12, Wil11] is the statement that the Rényi entropies of order two in Theorem 10 and Remark 12 can be replaced by quantum entropies. It has been solved in the case of two senders [FHS+12, XW13] but not for three or more.
5 Quantum simultaneous decoding for multiple-access channels and multiple quantum hypothesis testing

In this section, we establish explicit links between the quantum simultaneous decoding conjecture from [FHS+12, Wil11] and open questions from [AM14, BHOS15] in multiple quantum hypothesis testing. Recall that the general goal of quantum hypothesis testing is to minimize the error probability in identifying quantum states. In binary quantum hypothesis testing, one considers two hypotheses: the null hypothesis is that a quantum system is prepared in the state $\rho$, and the alternative hypothesis is that the quantum system is prepared in the state $\sigma$, where $\rho, \sigma \in D(H)$.

Operationally, the discriminator receives the state $\rho$ with probability $p \in (0, 1)$ and the state $\sigma$ with probability $1 - p$, and the task is to determine which state was prepared, by means of some quantum measurement $\{T, I - T\}$, where the test operator $T$ satisfies $0 \leq T \leq I$. There are two kinds of errors: a Type I error occurs when the state is identified as $\sigma$ when in fact $\rho$ was prepared and a Type II error is the opposite kind of error. The error probabilities corresponding to the two types of errors are as follows:

$$\alpha(T, \rho) \equiv \text{Tr}\{(I - T)\rho\}, \quad (5.1)$$
$$\beta(T, \sigma) \equiv \text{Tr}\{T\sigma\}. \quad (5.2)$$

As in information theory, quantum hypothesis testing has been studied in the asymptotic i.i.d. setting. In the setting of symmetric hypothesis testing, we are interested in minimizing the overall error probability

$$P_e^s(pp, (1 - p)\sigma) \equiv \inf_{T: 0 \leq T \leq I} p\alpha(T, \rho) + (1 - p)\beta(T, \sigma) \quad (5.3)$$

$$= \frac{1}{2} \left( \text{Tr}\{p\rho + (1 - p)\sigma\} - \|p\rho - (1 - p)\sigma\|_1 \right). \quad (5.4)$$

In the i.i.d. setting, $n$ quantum systems are prepared as either $\rho^{\otimes n}$ or $\sigma^{\otimes n}$, and the goal is to determine the optimal exponent of the error probability, defined as

$$\lim_{n \to \infty} -\frac{1}{n} \log_2 P_e^s(pp^{\otimes n}, (1 - p)\sigma^{\otimes n}). \quad (5.5)$$

One of the landmark results in quantum hypothesis testing is that the optimal exponent is equal to the quantum Chernoff distance [ACMnT’07, NS09]:

$$\lim_{n \to \infty} -\frac{1}{n} \log_2 P_e^s(pp^{\otimes n}, (1 - p)\sigma^{\otimes n}) = C(\rho, \sigma) \equiv \sup_{s \in [0,1]} -\log_2 \text{Tr}\{\rho^s\sigma^{1-s}\}. \quad (5.6)$$

This development can be generalized to the setting in which $\rho, \sigma, p,$ and $1 - p$ can be replaced by positive semi-definite operators $\neq 0$ and positive constants. Indeed, for positive semi-definite $A$ and $B$, we can define

$$P_e^s(A, B) \equiv \inf_{T: 0 \leq T \leq I} \text{Tr}\{(I - T)A\} + \text{Tr}\{TB\} \quad (5.7)$$

$$= \frac{1}{2} \left( \text{Tr}\{A + B\} - \|A - B\|_1 \right). \quad (5.8)$$
Then for positive constants $K_0, K_1 > 0$, we find that
\[
\lim_{n \to \infty} -\frac{1}{n} \log_2 P_e^*(K_0 A^\otimes n, K_1 B^\otimes n) = C(A, B) \equiv \sup_{s \in [0,1]} -\log_2 \text{Tr}\{A^s B^{1-s}\}.
\] (5.9)

In the setting of asymmetric hypothesis testing, we are interested in the optimal exponent of the Type II error $\beta(T, \sigma)$, under a constraint on the Type I error, i.e., $\alpha(T, \sigma) \leq \varepsilon$, with $\varepsilon \in [0, 1]$. That is, we are interested in the following quantity, now known as hypothesis testing relative entropy:
\[
D^\varepsilon_H(\rho || \sigma) \equiv \inf_T \{\beta(T, \sigma) : 0 \leq T \leq I \wedge \alpha(T, \sigma) \leq \varepsilon\}.
\] (5.10)

The optimal exponential decay rate in the asymmetric setting is given by the quantum Stein’s lemma [HP91, ON00], which establishes the following for all $\varepsilon \in (0, 1)$:
\[
\lim_{n \to \infty} \frac{1}{n} D^\varepsilon_H(\rho^\otimes n || \sigma^\otimes n) = D(\rho || \sigma),
\] (5.11)
giving the quantum relative entropy its fundamental operational interpretation.

As we can see from [AJW17] and our developments in Section 3, position-based coding forges a direct connection between single-sender single-receiver communication and binary quantum hypothesis testing. Specifically, the Chernoff distance from symmetric hypothesis testing gives a lower bound on the entanglement-assisted error exponent; while the application of the results from asymmetric hypothesis testing leads to a lower bound on the one-shot entanglement-assisted capacity and in turn on the second-order coding rate for entanglement-assisted communication.

In what follows, we discuss the generalization of both asymmetric and symmetric hypothesis testing to multiple quantum states and their connections to quantum simultaneous decoding.

### 5.1 Symmetric multiple quantum hypothesis testing and quantum simultaneous decoding

We now tie one version of the quantum simultaneous decoding conjecture to [AM14, Conjecture 4.2], which has to do with distinguishing one state from a set of other possible states. To recall the setting of [AM14, Conjecture 4.2], suppose that a state $\rho$ is prepared with probability $p \in (0, 1)$ and with probability $1 - p$ one state $\sigma_i$ of $r$ states is prepared with probability $q_i$, where $i \in \{1, \ldots, r\}$. The goal is to determine whether $\rho$ was prepared or whether one of the other states was prepared, and the error probability in doing so is equal to
\[
P_e^*(p \rho, (1-p) \sum_{i=1}^r q_i \sigma_i).
\] (5.12)

The measurement operator that achieves the minimum error probability is equal to
\[
\left\{p \rho - (1-p) \sum_{i=1}^r q_i \sigma_i \geq 0\right\}.
\] (5.13)

As usual, we are interested in the i.i.d. case, in which $\rho$ and $\sigma_i$ are replaced by $\rho^\otimes n$ and $\sigma_i^\otimes n$ for large $n$, and [AM14, Conjecture 4.2] states that
\[
\lim_{n \to \infty} \frac{1}{n} \log_2 P_e^*(p \rho^\otimes n, (1-p) \sum_{i=1}^r q_i \sigma_i^\otimes n) = \min_i C(\rho, \sigma_i).
\] (5.14)
We now propose a slight generalization of [AM14, Conjecture 4.2] and (5.9), in which $\rho$ and $\sigma_i$ are replaced by positive semi-definite operators and $p$ and $1-p$ are replaced by positive constants:

**Conjecture 14** Let $\{A, B_1, \ldots, B_r\}$ be a set of positive semi-definite operators with trace strictly greater than zero, and let $K_0, \ldots, K_r$ be strictly positive constants. Then the following equality holds

$$
\lim_{n \to \infty} -\frac{1}{n} \log_2 P_e^* \left( K_0 A^\otimes n, \sum_{i=1}^r K_i B_i^\otimes n \right) = \min_i C(A, B_i),
$$

where $P_e^*$ is defined in (5.7)–(5.8) and $C(A, B_i)$ in (5.9).

To see how Conjecture 14 is connected to quantum simultaneous decoding, recall from (4.14) our fundamental bound on the error probability when simultaneously decoding the message pair $(l, m)$:

$$
p_e(l, m) \leq 4 \text{Tr}\{(I - T_{RSC}) N_{AB \to C}(\theta_{RA} \otimes \gamma_{SB})\} + 4L \text{Tr}\{T_{RSC} N_{AB \to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})\}
+ M \text{Tr}\{T_{RSC} N_{AB \to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)\} + LM \text{Tr}\{T_{RSC} N_{AB \to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)\},
$$

where we have set $c = 1$ and used that $L - 1 < L$ and $M - 1 < M$. Now applying this bound to the i.i.d. case and setting $L = 2^{nR_1}$ and $M = 2^{nR_2}$, we find that the upper bound becomes

$$
p_e(l, m) \leq 4 \left[ \text{Tr}\{(I - T) \rho^\otimes n\} + \text{Tr}\{T (B_1^\otimes n + B_2^\otimes n + B_3^\otimes n)\} \right],
$$

where

$$
\rho \equiv N_{AB \to C}(\theta_{RA} \otimes \gamma_{SB}),
B_1 \equiv 2^{R_1} N_{AB \to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB}),
B_2 \equiv 2^{R_2} N_{AB \to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B),
B_3 \equiv 2^{R_1+R_2} N_{AB \to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B).
$$

To minimize the upper bound on the error probability, we should pick the test operator $T$ as follows:

$$
T \equiv \{\rho^\otimes n - (B_1^\otimes n + B_2^\otimes n + B_3^\otimes n) \geq 0\},
$$

and then the upper bound becomes

$$
p_e(l, m) \leq 2 \left( \text{Tr}\{\rho^\otimes n + B_1^\otimes n + B_2^\otimes n + B_3^\otimes n\} - \|\rho^\otimes n - (B_1^\otimes n + B_2^\otimes n + B_3^\otimes n)\|_1 \right)
= 4 P_e^* (\rho^\otimes n, B_1^\otimes n + B_2^\otimes n + B_3^\otimes n).
$$

The test operator $T$ given in (5.22) was previously realized in [Wil10] and [HC17] to be relevant as a quantum simultaneous decoder in the context of unassisted classical communication over a quantum multiple access channel.

Now applying Conjecture 14 (provided it is true), we would find that the error probability $p_e(l, m)$ is bounded from above as $p_e(l, m) \lesssim e^{-nE(R_1, R_2)}$, with the error exponent $E(R_1, R_2)$ equal
Thus the rate region \((R_1, R_2)\) would be achievable as long as \(E(R_1, R_2) > 0\). Now using the fact that, for a bipartite state \(\rho_{AB}\)
\[
\lim_{s \to 1} I_s(A; B|\rho) = I(A; B|\rho),
\]
we would then find that the following rate region is achievable:
\[
\begin{align*}
R_1 & \leq I(R; CS|\omega), \\
R_2 & \leq I(S; CR|\omega), \\
R_1 + R_2 & \leq I(RS; C|\omega),
\end{align*}
\]
where \(\omega_{RSC} = N_{AB \to C}(\theta_{RA} \otimes \gamma_{SB})\) and the above approach would solve the quantum simultaneous decoding conjecture. The method clearly extends to more than two senders. We remark that aspects of the above approach were discussed in the recent work [HC17] for the case of unassisted classical communication over a quantum multiple access channel, but there the connection to [AM14, Conjecture 4.2] or Conjecture 14 was not realized, nor was the entanglement-assisted case considered.

We end this section by noting that [AM14, Theorem 4.3] offers several suboptimal upper bounds on the error probability \(P_e(\rho^{\otimes n}, B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n})\), which in turn could be used to establish suboptimal achievable rate regions for the quantum multiple-access channel. However, here we refrain from the details of what these regions would be, except to say that they would be in terms of the negative logarithm of the fidelity, replacing \(I_s\) in (5.27).

5.2 Asymmetric hypothesis testing with composite alternative hypothesis

We now tie the quantum simultaneous decoding problem to a different open question in asymmetric hypothesis testing. Recall our fundamental upper bound from (4.14) on the error probability for classical communication over a quantum multiple-access channel, as applied for the i.i.d. case:
\[
\begin{align*}
p_e(l, m) & \leq c_1 \text{Tr}((I - T)^{\otimes n} \rho) + c_2 \text{Tr}(T[B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}]),
\end{align*}
\]
where \( L = 2^{nR_1} \), \( M = 2^{nR_2} \), and the state \( \rho \) and the positive semi-definite operators \( B_1 \), \( B_2 \), and \( B_3 \) are given by

\[
\rho = \mathcal{N}_{AB \to C}(\theta_{RA} \otimes \gamma_{SB}) , \\
B_1 = 2^{R_1}\mathcal{N}_{AB \to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB}) , \\
B_2 = 2^{R_2}\mathcal{N}_{AB \to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B) , \\
B_3 = 2^{R_1+R_2}\mathcal{N}_{AB \to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B) .
\]

Rather than try to minimize the overall error probability as we did in the previous section, we could try to minimize all of the other error probabilities subject to a constraint on the error probability \( \text{Tr}\{(I - T)\rho^\otimes n\} \). Thus, we seek a test operator \( T \) which is capable of discriminating the state \( \rho^\otimes n \) from the operator \( B_1^\otimes n + B_2^\otimes n + B_3^\otimes n \). This kind of task is formally called asymmetric hypothesis testing with composite alternative hypothesis. The problem of a composite null hypothesis is that considered in the context of the quantum Sanov theorem, which was solved in [BDK+05] (see also [Hay02]), and finds application in communication over compound channels [BB09, Mos15] (see also [DD07, Hay09]).

The following open question, strongly related to a question from [BHOS15], is relevant for asymmetric hypothesis testing with composite alternative hypothesis:

**Question 15** Consider a quantum state \( \rho \in \mathcal{D}(\mathcal{H}) \), a positive integer \( r > 1 \), and a finite set of positive semi-definite operators \( \mathcal{B} \equiv \{ B_i : 1 \leq i \leq r \} \), for which \( \text{supp}(\rho) \subseteq \text{supp}(B_i) \ \forall B_i \in \mathcal{B} \)

\[
\min_i D(\rho \| B_i) > 0 .
\]

What is the most general form that \( \rho \) and \( \mathcal{B} \) can take such that the following statement is true? For all \( \varepsilon \in (0, 1) \), \( \delta > 0 \), and sufficiently large \( n \), there exists a binary test \( \{ T, I - T \} \) such that the Type I error is bounded from above by \( \varepsilon \):

\[
\text{Tr}\{(I - T)\rho^\otimes n\} \leq \varepsilon ,
\]

and for all \( B_i \in \mathcal{B} \), the exponential decay rate of the Type II error is bounded from below as follows:

\[
- \frac{1}{n} \log_2 \text{Tr}\{TB_i^\otimes n\} \geq \left[ \min_i D(\rho \| B_i) \right] - \delta .
\]

Below we prove the following special case:

**Theorem 16** The statement at the end of Question 15 is true when the set \( \mathcal{B} \) forms a commuting set of operators (each of which need not commute with \( \rho \)).

**Proof.** To this end, we employ the notion of a relative typical projector, which was used in [BSS03] to establish an alternate proof of the quantum Stein lemma (see also [BLW15] for a different use of relative typical projectors). Let \( B_i = \sum_y f_i(y)\phi_{i,y}^\dagger\phi_{i,y} \) denote a spectral decomposition of \( B_i \). For a state \( \rho \) and positive semi-definite operator \( B_i \), define the relative typical subspace \( T_{\rho \| B_i}^{\delta, n} \) for \( \delta > 0 \) and integer \( n \geq 1 \) as

\[
T_{\rho \| B_i}^{\delta, n} \equiv \text{span}\left\{ |\phi_{i,n}^y\rangle : -\frac{1}{n} \log_2(f_i^n(y^n)) + \text{Tr}\{\rho \log_2 B_i\} \leq \delta \right\} ,
\]

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where
\[ y^n \equiv y_1 \cdots y_n, \] (5.41)
\[ f_i^n (y^n) \equiv \prod_{j=1}^{n} f_Y (y_j), \] (5.42)
\[ |\phi_{y^n}^i \rangle \equiv |\phi_{y_1}^i \rangle \otimes \cdots \otimes |\phi_{y_n}^i \rangle. \] (5.43)

Let \( \Pi_{\rho|B_i, \delta}^n \) denote the projection operator corresponding to the relative typical subspace \( T_{\rho|B_i}^n \). The critical properties of the relative typical projector are as follows:

\[
\text{Tr} \{ \Pi_{\rho|B_i, \delta}^n \} \geq 1 - \varepsilon, \quad 2^{-n[ - \text{Tr} \{ \rho \log_2 B_i \} + \delta \} \Pi_{\rho|B_i, \delta}^n \Pi_{\rho|B_i, \delta}^n \leq 2^{-n[ - \text{Tr} \{ \rho \log_2 B_i \} - \delta]} \Pi_{\rho|B_i, \delta}^n, \]
(5.44)

with the first inequality holding for all \( \varepsilon, \delta > 0 \), and sufficiently large \( n \).

The main idea for the proof under the stated assumptions is to take the test operator \( T \) as
\[
T = \Pi_{\rho|B_r, \delta}^n \cdots \Pi_{\rho|B_1, \delta}^n \Pi_{\rho|B_r, \delta}^n \cdots \Pi_{\rho|B_1, \delta}^n, \]
(5.46)
where \( \Pi_{\rho, \delta}^n \) is the typical projector for \( \rho \). Then we find that for all \( \varepsilon, \delta > 0 \), and sufficiently large \( n \)
\[
\text{Tr} \{ \rho^\otimes n \} \geq \text{Tr} \{ \Pi_{\rho|B_1, \delta}^n \} - \sum_{i=1}^{r} \left\| \Pi_{\rho|B_i, \delta}^n \rho^\otimes n \Pi_{\rho|B_i, \delta}^n - \rho^\otimes n \right\|_1 \geq 1 - \varepsilon - 2r\sqrt{\varepsilon},
\]
(5.47)
which follows by applying Lemmas 1 and 2 and properties of typicality and relative typicality. To handle the other kind of error, consider that, from the assumption, all of the projectors \( \Pi_{\rho|B_i, \delta}^n \) commute, so that
\[
\text{Tr} \{ T B_i^\otimes n \} = \text{Tr} \{ \Pi_{\rho|B_r, \delta}^n \cdots \Pi_{\rho|B_1, \delta}^n \Pi_{\rho|B_r, \delta}^n \cdots \Pi_{\rho|B_1, \delta}^n B_i^\otimes n \} \]
(5.49)
\[
\leq 2^{-n[ - \text{Tr} \{ \rho \log_2 B_i \} - \delta]} \text{Tr} \{ \rho^\otimes n \Pi_{\rho|B_1, \delta}^n \cdots \Pi_{\rho|B_r, \delta}^n \Pi_{\rho|B_1, \delta}^n \cdots \Pi_{\rho|B_r, \delta}^n \} \]
(5.51)
\[
\leq 2^{-n[D(\rho||B_i) + \delta]} \] (5.52)
\[
= 2^{-n[D(\rho||B_i) - 2\delta]}, \]
(5.55)

The statement of the theorem follows by setting \( \varepsilon' \equiv \varepsilon + 2r\sqrt{\varepsilon} \) and \( \delta' \equiv 2\delta \), considering that we have shown the existence of a test \( T \) for which
\[
\text{Tr} \{ \rho^\otimes n \} \geq 1 - \varepsilon', \quad \text{Tr} \{ T B_i^\otimes n \} \leq 2^{-n[D(\rho||B_i) - \delta']},
\]
(5.56)
and it is possible to satisfy this for any choice of \( \varepsilon', \delta' > 0 \) by taking \( n \) sufficiently large. (Note that for the bound on the second kind of error probability to be decaying exponentially, we
require $\delta' > 0$ to be small enough so that $\min_i D(\rho\|B_i) > \delta'$. We remark that this conclusion is actually stronger than what is stated in Question 15 because here we conclude for all $B_i \in \mathcal{B}$, that

$$-\frac{1}{n} \log_2 \text{Tr}\{TB_i^{\otimes n}\} \geq D(\rho\|B_i) - \delta' \geq \left[\min_i D(\rho\|B_i)\right] - \delta'.$$

(5.57)

This concludes the proof. ■

To see how Question 15 is related to quantum simultaneous decoding of the multiple-access channel, consider that for $\rho$, $B_1$, $\ldots$, $B_3$ as defined in (5.33)–(5.36), the inequality

$$\min_i D(\rho\|B_i) > 0$$

(5.58)

is equivalent to the following set of inequalities:

$$R_1 < I(R; CS)_{\omega},$$

(5.59)

$$R_2 < I(S; CR)_{\omega},$$

(5.60)

$$R_1 + R_2 < I(RS; C)_{\omega},$$

(5.61)

where $\omega_{RSC} = \mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})$. This equivalence holds because $\min_i D(\rho\|B_i) > 0$ is equivalent to the following three inequalities:

$$D(\rho\|B_1) > 0, \quad D(\rho\|B_2) > 0, \quad D(\rho\|B_3) > 0,$$

(5.62)

and

$$D(\rho\|B_1) = D(\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\|2^{R_1}\mathcal{N}_{AB\rightarrow C}(\theta_R \otimes \theta_A \otimes \gamma_{SB}))$$

(5.63)

$$= D(\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\|\mathcal{N}_{AB\rightarrow C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})) - R_1,$$

(5.64)

$$= I(R; CS)_{\omega} - R_1,$$

(5.65)

$$D(\rho\|B_2) = D(\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\|2^{R_2}\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B))$$

(5.66)

$$= D(\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\|\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)) - R_2,$$

(5.67)

$$= I(S; CR)_{\omega} - R_2,$$

(5.68)

$$D(\rho\|B_3) = D(\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\|2^{R_1+R_2}\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \theta_A \otimes \gamma_{SB}))$$

(5.69)

$$= D(\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \gamma_{SB})\|\mathcal{N}_{AB\rightarrow C}(\theta_{RA} \otimes \theta_A \otimes \gamma_{SB})) - (R_1 + R_2),$$

(5.70)

$$= I(RS; C)_{\omega} - (R_1 + R_2).$$

(5.71)

Thus, if such a sequence of test operators existed as stated in Question 15, then the error probability $p_e(l, m)$ when decoding a multiple-access channel could be bounded from above as

$$p_e(l, m) \leq c_1 \text{Tr}\{(I - T)\rho^{\otimes n}\} + c_2 \left[\text{Tr}\{T [B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}]\}\right]$$

(5.72)

$$\leq c_1 \epsilon + 3c_2 \left[2^{n[\min_i D(\rho\|B_i) - \delta]}\right],$$

(5.73)

where $\rho$, $B_1$, $\ldots$, $B_3$ are defined in (5.33)–(5.36). Then by choosing the rates $R_1$ and $R_2$ to satisfy

$$R_1 + 2\delta \leq I(R; CS)_{\omega},$$

(5.74)

$$R_2 + 2\delta \leq I(S; CR)_{\omega},$$

(5.75)

$$R_1 + R_2 + 2\delta \leq I(RS; C)_{\omega},$$

(5.76)
which is equivalent to \( \min_i D(\rho\|B_i) \geq 2\delta \), we would have

\[
p_e(l, m) \leq c_1 \varepsilon + 3c_2 2^{-n\delta},
\]

and we could thus make the error probability as small as desired by taking \( n \) sufficiently large. Since \( \delta > 0 \) is arbitrary, we could then say that the rate region in (5.29)–(5.31) would be achievable. If the statement at the end of Question 15 holds for the states given above, then the method would clearly lead to a quantum simultaneous decoder for more than two senders, by a straightforward generalization of the above approach.

The authors of [BP10] considered a similar problem in asymmetric hypothesis testing for a specific family of states with certain permutation symmetry. We should point out that in [BP10], the lower bound for the exponential rate of the Type II error is given by a regularized version of \( \min_i D(\rho\|\sigma_i) \). If a similar result held, along the lines stated in Question 15 and related to the conjecture in [BHOS15], without the need for regularization and for the operators in (5.33)–(5.36) (and more general ones relevant for more senders), then the developments in the present paper would immediately give bounds for the performance of quantum simultaneous decoding for the quantum multiple-access channel.

6 Conclusion

In this paper, we apply position-based coding to establish bounds of various quantities for classical communication. For entanglement-assisted classical communication over point-to-point quantum channels, we establish lower bounds on the one-shot error exponent, the one-shot capacity, and the second-order coding rate. We also give an achievable rate region for entanglement-assisted classical communication over multiple-access quantum channels. Furthermore, we explicitly show how to derandomize a randomness-assisted protocol (for multiple-access channel) to one without assistance from any extra resources. Our results indicate that position-based coding can be a powerful tool in achievability proofs of various communication protocols in quantum Shannon theory (for recent applications of position-based coding to private classical communication, see [Wil17a]). We finally tied some open questions in multiple quantum hypothesis testing to the quantum simultaneous decoding conjecture. Thus, we have shown that open problems in multiple quantum hypothesis testing are fundamental to the study of classical information transmission over quantum multiple-access channels.

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A Alternate achievable rate region for unassisted classical communication over quantum multiple-access channels

In this appendix, we prove an alternate achievable rate region that is generally different from that given in Theorem 11. Before doing so, we review the notion of conditional typicality (see, e.g., [Wil16, Wil17b] for a review). Consider an ensemble \( \{ p_X(x), \rho_x \}_{x \in X} \) of states. Suppose that each state \( \rho_x \) has the following spectral decomposition:

\[
\rho_x = \sum_y p_{Y|X}(y|x) |y_x \rangle \langle y_x |. \tag{A.1}
\]

Consider a density operator \( \rho_{x^n} \) that is conditioned on a classical sequence \( x^n \equiv x_1 \cdots x_n \):

\[
\rho_{x^n} \equiv \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}. \tag{A.2}
\]

We define the weak conditionally typical subspace as the span of vectors (conditioned on the sequence \( x^n \)) such that the sample conditional entropy \( H(y_n|x_n) \) of their classical labels is close to the true conditional entropy \( H(Y|X) \) of the distribution \( p_{Y|X}(y|x)p_X(x) \):

\[
T_{\delta}^{Y^n|x^n} \equiv \text{span} \left \{ |y_{x^n}^n \rangle : \left| H(y_n|x^n) - H(Y|X) \right| \leq \delta \right \}, \tag{A.3}
\]

where

\[
H(y_n|x_n) \equiv - \frac{1}{n} \log_2 \left( p_{Y|X}(y_n|x_n) \right), \tag{A.4}
\]

\[
H(Y|X) \equiv - \sum_x p_X(x) \sum_y p_{Y|X}(y|x) \log_2 p_{Y|X}(y|x). \tag{A.5}
\]

The projector \( \Pi_{\rho_{x^n},\delta} \) onto the weak conditionally typical subspace of \( \rho_{x^n} \) is as follows:

\[
\Pi_{\rho_{x^n},\delta} \equiv \sum_{y^n \in T_{\delta}^{Y^n|x^n}} |y_{x^n}^n \rangle \langle y_{x^n}^n |, \tag{A.6}
\]

where we have again overloaded the symbol \( T_{\delta}^{Y^n|x^n} \) to refer to the set of weak conditionally typical sequences:

\[
T_{\delta}^{Y^n|x^n} \equiv \left \{ y^n : \left| H(y^n|x^n) - H(Y|X) \right| \leq \delta \right \}. \tag{A.7}
\]

The three important properties of the weak conditionally typical projector are as follows:

\[
E_X^n \{ \text{Tr} \{ \Pi_{\rho_{X^n},\delta} \rho_{X^n} \} \} \geq 1 - \epsilon, \tag{A.8}
\]

\[
\text{Tr} \{ \Pi_{\rho_{x^n},\delta} \} \leq 2^{n[H(Y|X)+\delta]}, \tag{A.9}
\]

\[
2^{-n[H(Y|X)+\delta]} \Pi_{\rho_{x^n},\delta} \leq \Pi_{\rho_{x^n},\delta} \rho_{x^n} \Pi_{\rho_{x^n},\delta} \leq 2^{-n[H(Y|X)-\delta]} \Pi_{\rho_{x^n},\delta}, \tag{A.10}
\]

where the first property holds for arbitrary \( \epsilon \in (0,1), \delta > 0 \), and sufficiently large \( n \), and the expectation is with respect to the distribution \( p_{X^n}(x^n) \).

We also require the following Gentle Operator Lemma for ensembles:
Lemma 17 Given an ensemble $\{p_X(x), \rho_x\}$ with expected density operator $\rho \equiv \sum_x p_X(x)\rho_x$ and $\varepsilon \in [0,1]$, suppose that the operator $\Lambda$ such that $I \geq \Lambda \geq 0$ succeeds with high probability on the state $\rho$:

$$\text{Tr} \{\Lambda \rho\} \geq 1 - \varepsilon.$$ (A.11)

Then the subnormalized state $\sqrt{\Lambda} \rho_x \sqrt{\Lambda}$ is close in expected trace distance to the original state $\rho_x$:

$$\mathbb{E}_X \left\{ \|\sqrt{\Lambda} \rho_X \sqrt{\Lambda} - \rho_X\|_1 \right\} \leq 2\sqrt{\varepsilon}.$$ (A.12)

Theorem 18 The following rate region is achievable when using a quantum simultaneous decoder for the multiple-access channel $x, y \rightarrow \rho_{x,y}$:

$$R_1 \leq H_2(B|Y)_\rho - H(B|XY)_\rho,$$ (A.13)

$$R_2 \leq H_2(B|X)_\rho - H(B|XY)_\rho,$$ (A.14)

$$R_1 + R_2 \leq H_2(B)_{\rho} - H(B|XY)_\rho,$$ (A.15)

where the entropies are with respect to the following state:

$$\rho_{X,Y,B} \equiv \sum_{x,y} p_X(x)p_Y(y)|x\rangle\langle x| \otimes |y\rangle\langle y| \otimes \rho_{x,y}.$$ (A.16)

**Proof.** Suppose that the channel is a ccq channel of the form $x, y \rightarrow \rho_{x,y}$ and that the two senders have independent distributions $p_X(x)$ and $p_Y(y)$. These distributions induce the following states:

$$\rho_x \equiv \sum_y p_Y(y)\rho_{x,y},$$ (A.17)

$$\rho_y \equiv \sum_x p_X(x)\rho_{x,y},$$ (A.18)

$$\rho \equiv \sum_{x,y} p_X(x)p_Y(y)\rho_{x,y}.$$ (A.19)

**Codeword Selection.** Senders 1 and 2 choose codewords $\{x^n(l)\}_{l \in \{1,\ldots,L\}}$ and $\{y^n(m)\}_{m \in \{1,\ldots,M\}}$ independently and randomly according to the distributions $p_X^n(x^n)$ and $p_Y^n(y^n)$.

**POVM Construction.** Let $\Pi^n_{\rho_{x^n,y^n,\delta}}$ be the conditionally typical projector for the tensor-product state $\rho_{x^n,y^n}$ defined as the output of the $n$ channels when codewords $x^n$ and $y^n$ are input. In what follows, we make the following abbreviations:

$$\Pi^n_{x^n,y^n} \equiv \Pi^n_{\rho_{x^n,y^n,\delta}}.$$ (A.20)

The detection POVM $\{\Lambda_{l,m}\}$ has the following form:

$$\Lambda_{l,m} \equiv \left( \sum_{l',m'} \Pi'_{l,m} \right)^{-\frac{1}{2}} \Pi'_{l,m} \left( \sum_{l',m'} \Pi'_{l,m} \right)^{-\frac{1}{2}},$$ (A.21)

$$\Pi'_{l,m} \equiv \Pi^n_{x^n(l),y^n(m)}.$$ (A.22)

Observe that the operator $\Pi'_{l,m}$ is a positive semi-definite operator and thus $\{\Lambda_{l,m}\}$ is a valid POVM element.
Error Analysis. The average error probability of the code has the following form:
\[
\overline{p}_e \equiv \frac{1}{LM} \sum_{l,m} \text{Tr} \left\{ (I - \Lambda_{l,m}) \rho_{X^n(l),Y^n(m)} \right\} .
\] (A.23)

We instead analyze the expectation of the average error probability, where the expectation is with respect to the random choice of code:
\[
\mathbb{E}_{X^n,Y^n} \{ \overline{p}_e \} \equiv \mathbb{E}_{X^n,Y^n} \left\{ \frac{1}{LM} \sum_{l,m} \text{Tr} \left\{ (I - \Lambda_{l,m}) \rho_{X^n(l),Y^n(m)} \right\} \right\} = \frac{1}{LM} \sum_{l,m} \mathbb{E}_{X^n,Y^n} \left\{ \text{Tr} \left\{ (I - \Lambda_{l,m}) \rho_{X^n(l),Y^n(m)} \right\} \right\} .
\] (A.24)

Due to the symmetry of the code construction (the fact that the expectation \( \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ (I - \Lambda_{l,m}) \rho_{X^n(l),Y^n(m)} \right\} \} \) is independent of the particular message pair \( (l, m) \)), it suffices to analyze the expectation of the average error probability for the first message pair \( (1, 1) \):
\[
\mathbb{E}_{X^n,Y^n} \{ \overline{p}_e \} = \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ (I - \Lambda_{1,1}) \rho_{X^n(1),Y^n(1)} \right\} \} .
\] (A.26)

We now begin our error analysis. Choosing
\[
S = \Pi'_{1,1},
\] (A.27)
\[
T = \sum_{(l,m) \neq (1,1)} \Pi'_{l,m},
\] (A.28)
we can apply Lemma 4 to bound the expected error probability as
\[
\mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ (I - \Lambda_{1,1}) \rho_{X^n(1),Y^n(1)} \right\} \} \leq 2 \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ (I - \Pi'_{1,1}) \rho_{X^n(1),Y^n(1)} \right\} \} + 4 \sum_{(l,m) \neq (1,1)} \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{l,m} \rho_{X^n(1),Y^n(1)} \right\} \} .
\] (A.29)

We first consider bounding the first term on the right-hand side above. Consider from typicality that
\[
\mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{1,1} \rho_{X^n(1),Y^n(1)} \right\} \} \geq 1 - \varepsilon
\] (A.30)
This bound then implies that
\[
\mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ (I - \Pi'_{1,1}) \rho_{X^n(1),Y^n(1)} \right\} \} \leq \varepsilon.
\] (A.31)
The bound in (A.29) reduces to the following one after applying the above:
\[
\overline{p}_e \leq 2\varepsilon + 4 \sum_{(l,m) \neq (1,1)} \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{l,m} \rho_{X^n(1),Y^n(1)} \right\} \} .
\] (A.32)

We can expand the term inside the doubly-indexed sum on the right-hand side above:
\[
\sum_{(l,m) \neq (1,1)} \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{l,m} \rho_{X^n(1),Y^n(1)} \right\} \} = \sum_{l \neq 1} \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{l,1} \rho_{X^n(1),Y^n(1)} \right\} \}
+ \sum_{m \neq 1} \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{1,m} \rho_{X^n(1),Y^n(1)} \right\} \} + \sum_{l \neq 1, m \neq 1} \mathbb{E}_{X^n,Y^n} \{ \text{Tr} \left\{ \Pi'_{l,m} \rho_{X^n(1),Y^n(1)} \right\} \} .
\] (A.33)
We begin by bounding the first term on the right-hand side above. Consider the following chain of inequalities:

\[
\sum_{l \neq 1} \mathbb{E} X^n, Y^n \left\{ \text{Tr} \left\{ \Pi_{l,1}' \rho X^n(1), Y^n(1) \right\} \right\} \\
= \sum_{l \neq 1} \mathbb{E} X^n, Y^n \left\{ \text{Tr} \left\{ \Pi X^n(l), Y^n(1) \rho X^n(1), Y^n(1) \right\} \right\} \\
\leq 2^{n[H(B|XY)+\delta]} \sum_{l \neq 1} \mathbb{E} X^n, Y^n \left\{ \text{Tr} \left\{ \rho X^n(l), Y^n(1) \rho X^n(1), Y^n(1) \right\} \right\} \\
= 2^{n[H(B|XY)+\delta]} \sum_{l \neq 1} \mathbb{E} Y^n \left\{ \text{Tr} \left\{ \mathbb{E} X^n(l) \left\{ \rho X^n(l), Y^n(1) \right\} \mathbb{E} X^n(1) \left\{ \rho X^n(1), Y^n(1) \right\} \right\} \right\} \\
= 2^{n[H(B|XY)+\delta]} \sum_{l \neq 1} \mathbb{E} Y^n \left\{ \text{Tr} \left\{ \rho Y^n(1) \right\} \right\} \\
= 2^{n[H(B|XY)+\delta]} \sum_{l \neq 1} \left[ \sum_y p_Y(y) \text{Tr} \left\{ \rho_y^2 \right\} \right]^n \\
\leq 2^{n[H(B|XY)+\delta]} 2^{-nH_2(B|Y)} L, \quad (A.40)
\]

where \( H_2(B|Y) = -\log \sum_y p_Y(y) \text{Tr} \left\{ \rho_y^2 \right\} \).

The same kind of analysis gives bounds for the other terms:

\[
\sum_{m \neq 1} \mathbb{E} X^n, Y^n \left\{ \text{Tr} \left\{ \Pi_{1,m}' \rho X^n(1), Y^n(1) \right\} \right\} \leq 2^{n[H(B|XY)+\delta]} 2^{-nH_2(B|X)} M, \quad (A.41)
\]

\[
\sum_{l \neq 1, m \neq 1} \mathbb{E} X^n, Y^n \left\{ \text{Tr} \left\{ \Pi_{l,m}' \rho X^n(1), Y^n(1) \right\} \right\} \leq 2^{n[H(B|XY)+\delta]} 2^{-nH_2(B)} LM, \quad (A.42)
\]

where \( H_2(B|X) = -\log \sum_x p_X(x) \text{Tr} \left\{ \rho_x^2 \right\} \) and \( H_2(B) = -\log \text{Tr} \left\{ \rho^2 \right\} \) (these are conditional Rényi entropies of order two).

Combining everything together, we get the following bound on the expectation of the average error probability:

\[
\mathbb{E} X^n, Y^n \left\{ \tilde{f}_n \right\} \leq 2\varepsilon + 4 \left( 2^{n[H(B|XY)+\delta]} 2^{-nH_2(B|Y)} L \\
+ 2^{n[H(B|XY)+\delta]} 2^{-nH_2(B|X)} M \right) + 2^{n[H(B|XY)+\delta]} 2^{-nH_2(B)} LM. \quad (A.43)
\]

Thus, we can choose the message sizes to be as follows:

\[
L = 2^{n[R_1-3\delta]}, \quad (A.44)
\]

\[
M = 2^{n[R_2-3\delta]}, \quad (A.45)
\]

so that the expectation of the average error probability vanishes whenever the rates \( R_1 \) and \( R_2 \) obey the following inequalities:

\[
R_1 - \delta < H_2(B|Y) - H(B|XY), \quad (A.46)
\]

\[
R_2 - \delta < H_2(B|X) - H(B|XY), \quad (A.47)
\]

\[
R_1 + R_2 - 4\delta < H_2(B) - H(B|XY), \quad (A.48)
\]

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Thus, there exists a particular code with vanishing average error probability, and given that \( \delta > 0 \) is an arbitrarily small positive number, the bounds in the statement of the theorem follow.

We end this appendix by remarking that the scheme clearly generalizes to more than two senders.

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