Research Article

Existence and Multiplicity of Solutions for a Class of Anisotropic Double Phase Problems

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1Introduction and Statement of Results

In this paper, we deal with the existence and multiplicity of solutions for the following double phase problem

\[
\begin{align*}
-\text{div} \left( |\nabla u|^{p(x)} - 2 \nabla u + a(x) |\nabla u|^{q(x)} - 2 \nabla u \right) &= \lambda f(x, u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \lambda > 0 \) is a real parameter, \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary, \( p(x) = \frac{Np(x)}{p(x) - p(\cdot)} \), \( p(\cdot) \), and \( q(\cdot) \) are Lipschitz continuous in \( \mathbb{R}^N \). Moreover,

\[
\frac{q(x)}{p(x)} < 1 + \frac{1}{N}, \quad a : \Omega \to [0, +\infty), \text{is Lipschitz continuous}
\]

and we also assume that the nonlinearity \( f \) satisfies the following conditions:

\[
(f) f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function and there exists } C_1 > 0 \text{ such that}
\]

\[
|f(x, t)| \leq C_1 \left( 1 + |t|^{\alpha(x)} \right),
\]

for all \( (x, t) \in \Omega \times \mathbb{R} \), where \( \alpha \in C(\Omega), 1 < q^* < \alpha^* \leq \alpha^* < p^*(\cdot), (f(2)) \lim_{t \to 0^+} f(x, t)/|t|^q = 0 \), uniformly for a.e. \( x \in \Omega \).

\[
(f_3) \lim_{t \to \infty, \in} F(x, t)/|t|^{q^*} = +\infty, \text{ uniformly for a.e. } x \in \Omega,
\]

where \( F(x, t) = \int_0^t f(x, s)ds \). There exists a constant \( C_0 > 0 \) such that

\[
G(x, t) \leq G(x, s) + C_0,
\]

for any \( x \in \Omega, 0 < t < s \) or \( s < t < 0 \), where \( G(x, t) = tf(x, t) - q^* F(x, t) (f_*) \). There exists \( T_0 > 0 \) such that \( f(x, t)/|t|^{q^* - 2} \) is nondecreasing in \( t \) when \( t \geq T_0 \) and nonincreasing in \( t \leq -T_0 \) for all \( x \in \Omega \).

\[
(f_4) f(x, t, -f_*) \text{ is equivalent to the following condition (see [1]): } (f_*)
\]

Remark 1. We point out that the condition \((f_4)\) is weaker than \((f_3)\). It is not difficult to check that the condition \((f_3)\) is equivalent to the following condition (see [1]): \((f_*)\)
$G(x, t)$ is increasing in $t \geq T_0$ and decreasing in $t \leq -T_0$ for all $x \in \Omega$.

Hence, $(f^\ast_t)$ implies $(f_t)$.

Similar problems have been investigated and it is well known they have a strong physical meaning because they appear in the models of strongly anisotropic materials, see, e.g., [2, 3]. The energy functionals of the form

$$u \mapsto \int_{\Omega} \mathcal{H}(x, |\nabla u(x)|) \, dx, \quad \mathcal{H}(x, t) = t^{p(x)} + a(x) \theta^q(x),$$  

where the integrand $\mathcal{H}$ switches between two different elliptic behaviors have been intensively studied in recent years, see [2–11]. Recently, Mingione et al. have obtained the regularity theory for minimizers of (5), see, e.g., [7].

When $a(x) = 1$ and $\lambda = 1$, problem $(P_\lambda)$ becomes a $(p(x), q(x))$-Laplacian problem of the form

$$-\Delta_{p(x)} u(x) - \Delta_{q(x)} u(x) = f(x, u), \quad \text{in} \Omega,$$

$$u = 0, \quad \text{on} \partial \Omega,$$

where $-\Delta_{p(x)} := \text{div} \, (|\nabla u|^{p(x)-2} \nabla u)$. In particular, we refer to [9] where the authors proved the existence of one and three nontrivial weak solutions of (6), by the mountain pass theory and Morse theory.

If $p(x) = q(x)$, then $a(x) = 1$. Vetro [12] studied the following Dirichlet boundary value problem involving the $p(x)$-Laplacian-like operator:

$$-\Delta_{p(x)} u(x) + |u(x)|^{p(x)-2} u(x) = \lambda f(x, u), \quad \text{in} \Omega,$$

$$u = 0, \quad \text{on} \partial \Omega,$$

where $-\Delta_{p(x)} := \text{div} \left( \left(1 + \frac{|\nabla u|^{p(x)}}{1 + |\nabla u|^{2p(x)}} \right) |\nabla u|^{p(x)-2} \nabla u \right)$, is the $p(x)$-Laplacian-like. They have established the existence and multiplicity results for the problem (7) when $\lambda$ is sufficiently small.

In the particular case of $p(x) \equiv p, q(x) \equiv q$, such problems have been recently studied in, e.g., [13–16]. The existence and multiplicity of weak solutions of problem $(P_\lambda)$ with $\lambda = 1$ has been established in Liu and Dai [13]. In [15], by using the Morse theory, Perera and Squassina obtained a nontrivial weak solution of problem $(P_\lambda)$. In [14], by utilizing the Nehari method, Liu and Dai obtained three ground state solutions. Usually, the authors in those references considered the nonlinearities $f(x, t)$ satisfying the Ambrosetti-Rabinowitz type condition ([AR] in short): i.e., there exist $L > 0, \theta > q$, such that for $|t| \geq L$ and a.e. $x \in \Omega$,

$$0 < \theta F(x, t) \leq tf(x, t).$$  

Under some appropriate assumptions, one can consider a much weaker condition on $f(x, t)$

$$\lim_{|u| \to +\infty} \frac{F(x, u)}{|u|^q} = +\infty \text{uniformly in } x.$$  

This means that $F$ is $q$-superlinear at infinity. But the (AR) condition is useful and natural to ensure the mountain pass geometry and the Palais-Smale condition (PS) in short. So it has attracted much interest in recent literature, see for example [13, 15, 17–19] and the references therein. However, in this paper, we consider the problem $(P_\lambda)$ in the case when the nonlinearity $F$ is $q'$-superlinear at both infinity and origin (see conditions $(f_1)$ and $(f_2)$). These conditions are weaker than the (AR) condition. For example, Papageorgiou, Vetro, and Vetro [16] investigated the following $(p, 2)$-equation with combined nonlinearities:

$$-\Delta_p u(x) - \Delta u(x) = \lambda f(x, u) + g(x, u), \quad \text{in} \Omega,$$

$$u = 0, \quad \text{on} \partial \Omega,$$

where $\lambda > 0, 2 < p < +\infty, \Omega \subset \mathbb{R}^N$, be a bounded domain with a $C^2$-boundary $\partial \Omega$. Using the critical point theory, critical groups, and flow invariance arguments, the authors obtained at least five nontrivial smooth solutions of (11) when $f$ is $(p-1)$-superlinear near $+\infty$ but does not satisfy the (AR) condition.

Now, a natural question is whether the results contained in [13] can be generalized to the variable exponents $(p(x), q(x))$ case. Moreover, can we assume that the nonlinearity $f$ satisfies a more natural and weaker $(q-1)$-superlinear condition near $+\infty$ instead of the (AR) condition?

Inspired by the above works, we will answer these questions. For a detailed motivation of our context and additional references, we refer to the introduction of [8, 20]. To the best of our knowledge, there are very few papers related to the existence of solutions of problem $(P_\lambda)$ with variable exponents. This paper was motivated by the interest in applications of the variable exponent Orlicz-Sobolev spaces. Before stating our main results, we introduce some notations.

1.1. Notations and definitions. Throughout this paper, we define the class

$$C_+(\Omega) = \{ p \in C(\Omega), \ p(x) > 1 \text{ for all } x \in \Omega \}.$$  

For any $p \in C_+(\Omega)$, we denote

$$\bar{p}^\ast := \text{ess sup} p(x), \quad \bar{p} := \text{ess inf} p(x),$$  

where $x \in \mathbb{R}^N$.
and we denote by \( p_1 \ll p_2 \) the fact that
\[
\text{ess inf}_{x \in \mathbb{R}^n} (p_2(x) - p_1(x)) > 0. \tag{14}
\]

The letters \( C, C_i, i = 1, 2, \ldots \), denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process. The notion of weak solution for problem \((P_\lambda)\) is that \( u \in W_0^{1,\infty}(\Omega) \) is a solution of \((P_\lambda)\) if
\[
\int_\Omega \left( |\nabla u|^{p(x)} + a(x)|\nabla u|^{q(x)} \right) \nabla u \cdot \nabla v dx = \int_\Omega f(x, u) v dx, \quad \forall v \in W_0^{1,\infty}(\Omega). \tag{15}
\]

It is formulated in a suitable Orlicz-Sobolev space \( W_0^{1,\infty}(\Omega) \) that will be introduced in Section 2. It is easy to see that solutions of \((P_\lambda)\) correspond to the critical points of the energy functional \( I_\lambda \) defined by
\[
I_\lambda(u) = \int_\Omega \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) dx - \lambda \int_\Omega F(x, u) dx, \forall u \in W_0^{1,\infty}(\Omega), \tag{16}
\]
where \( F(x, t) = \int_0^t f(x, s) ds \).

Now, we present the main results of this paper as follows:

**Theorem 2.** Suppose \((f_1) - (f_4)\) are satisfied. Then problem \((P_\lambda)\) has at least one nontrivial weak solution in \( W_0^{1,\infty}(\Omega) \) for all \( \lambda > 0 \).

**Theorem 3.** Suppose \((f_1), (f_3) - (f_5)\) are satisfied. Then there exists \( \lambda_0 > 0 \) such that for all \( \lambda \in (0, \lambda_0) \), problem \((P_\lambda)\) has at least one solution \( u_\lambda \) and
\[
\lim_{\lambda \to 0^+} ||u_\lambda|| = +\infty. \tag{17}
\]

**Theorem 4.** Suppose \((f_1) - (f_5)\) are satisfied. Then problem \((P_\lambda)\) has infinitely many solutions in \( W_0^{1,\infty}(\Omega) \) for all \( \lambda > 0 \).

**Theorem 5.** Suppose \((f_5)\) and the following condition \((f_6) f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function, and there exist positive constants \( d_0, d_1 \) such that
\[
d_0 |t|^{\beta(x)-1} \leq f(x, t) \leq d_1 |t|^{\beta(x)-1}, \tag{18}
\]
for all \( x \in \Omega \) and \( t \geq 0 \), where \( \beta \in C(\Omega) \) such that \( 1 < \beta(x) < p^*(x) \) with \( p^* < p^* \). Then problem \((P_\lambda)\) has infinitely many solutions in \( W_0^{1,\infty}(\Omega) \) for all \( \lambda > 0 \).

**Theorem 6.** Suppose \((f_1), (f_3) - (f_5)\) are satisfied. Then for all \( \lambda > 0 \), problem \((P_\lambda)\) has infinitely many solutions \( \{u_n\}_{n \in \mathbb{N}} W_0^{1,\infty}(\Omega) \) such that \( \lim_{n \to \infty} I_\lambda (u_n) = \infty \).

**Theorem 7.** Suppose \((f_1), (f_3) - (f_5)\) are satisfied. Then for all \( \lambda > 0 \), problem \((P_\lambda)\) has infinitely many solutions \( \{u_n\}_{n \in \mathbb{N}} W_0^{1,\infty}(\Omega) \) such that \( I_\lambda (u_n) < 0, \lim_{n \to \infty} I_\lambda (u_n) = 0 \).

**Remark 8.** Note that our Theorems 2–7 answer the above questions. To be precise, Theorems 2, 4, and 7 extend the main results of [13] to the variable exponents \((p(x), q(x))\) case. Compared with [13], the main difficulty is that since both \( p(x) \) and \( q(x) \) are nonconstant functions, then \((P_\lambda)\) has a more complicated structure, due to its nonhomogeneities and to the presence of the nonlinear term.

**Remark 9.** In Theorem 5, we obtain infinitely many solutions by using Krasnoselskii’s genus theory. Moreover, we consider continuous functions \( f = f(x, u) \) satisfying the growth condition
\[
d_0 |u|^{\beta(x)-1} \leq f(x, u) \leq d_1 |u|^{\beta(x)-1}. \tag{19}
\]

The rest of this paper is organized as follows. In Section 2, we state some preliminary notations and the main lemmas. In Section 3, we prove the Theorems 2 and 3. The proofs of Theorems 4–5 are given in Section 4. By using the Fountain Theorem and the Dual Fountain Theorem, infinitely many pairs of solutions are provided in Section 5.

## 2. Preliminaries

In order to discuss the problem \((P_\lambda)\), we need some theories on generalized Orlicz spaces and Sobolev spaces. For more details, we refer to the references [20–23]. The variable exponent Lebesgue space \( L^{p(x)}(\Omega) \) is defined by
\[
L^{p(x)}(\Omega) = \left\{ u \text{ is a measurable real valued function} \left| \int_\Omega |u(x)|^{p(x)} dx < +\infty \right. \right\}, \tag{20}
\]
endowed with the Luxemburg norm
\[
||u||_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \frac{|u(x)|^{p(x)}}{\lambda} dx \leq 1 \right\}. \tag{21}
\]

Note that, if \( p \) is a constant function, the Luxemburg norm \( ||u||_{p(x)} \) coincide with the standard norm \( ||u||_p \) of the Lebesgue space \( L^p(\Omega) \). Then, \( L^{p(x)}(\Omega), ||u||_{p(x)} \) becomes a Banach space, and we call it the variable exponent Lebesgue space. It is easy to check that the embedding \( L^{p_r(x)}(\Omega) \hookrightarrow L^{p'_r(x)}(\Omega) \) is continuous, where \( 0 < [\Omega] < \infty \) and \( p_1, p_2 \) are variable exponents such that \( p_1 \leq p_2 \) in \( \Omega \).

The following property of spaces with variable exponent is essentially due to Fan and Zhao [24].

**Lemma 10.** The space \( L^{p(x)}(\Omega), ||u||_{p(x)} \) is a separable, uniformly convex Banach space, and its dual space is \( L^{p'(x)}(\Omega) \) where \((1/p(x)) + (1/p'(x)) = 1\). For any \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{p'(x)}(\Omega) \), we have
\[
\left| \int_\Omega u v dx \right| \leq \left( \frac{1}{p'} + \frac{1}{0} \right) \|u\|_{\bar{p}'(x)} \|v\|_{\bar{p}'(x)}. \tag{22}
\]

The Musielak-Orlicz space \(L^{\mathcal{H}}(\Omega)\) is defined by

\[
L^{\mathcal{H}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} \mathcal{H}(x, |u|) dx < +\infty \right\},
\]

endowed with the norm

\[
\|u\|_{\mathcal{H}} = \inf \left\{ \lambda > 0 : \mathcal{H}\left(\frac{u}{\lambda}\right) \leq 1 \right\}, \tag{24}
\]

where \(\mathcal{H}\) is defined in (5). The space \(L^{\mathcal{H}}(\Omega)\) is a separable, uniformly convex, and reflexive Banach space. We denote by \(L^{q(x)}(\Omega)\) the space of all measurable functions \(u : \Omega \to \mathbb{R}\) with the seminorm

\[
\|u\|_{q(x)} := \left( \int_{\Omega} a(x) |u|^{q(x)} dx \right)^{1/q(x)} < \infty. \tag{25}
\]

It is easy to check that the embeddings

\[
\begin{align*}
L^{q(x)}(\Omega) &\hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \cap L^{q_{\ast}(x)}(\Omega), \tag{26}
\end{align*}
\]

are continuous. Since \(\rho\mathcal{H}(u)\|u\|_{\mathcal{H}} = 1\) whenever \(u \neq 0\), we have

\[
\begin{align*}
\min \left\{ \|u\|_{p(x)}^{p(x)}, \|u\|_{q(x)}^{q(x)} \right\} &\leq \|u\|_{p(x)}^{p(x)} + \|u\|_{p(x)}^{p(x)}, \\
&\leq \max \left\{ \|u\|_{p(x)}^{p(x)}, \|u\|_{q(x)}^{q(x)} \right\}, \quad \forall u \in L^{\mathcal{H}}(\Omega). \tag{27}
\end{align*}
\]

The related Sobolev space \(W^{1,\mathcal{H}}(\Omega)\) is defined by

\[
W^{1,\mathcal{H}}(\Omega) := \left\{ u \in L^{\mathcal{H}}(\Omega) : \|\nabla u\|_{\mathcal{H}} \right\}, \tag{28}
\]

equipped with the norm

\[
\|u\|_{\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}, \tag{29}
\]

where \(\|\nabla u\|_{\mathcal{H}} = \|\nabla u\|_{\mathcal{H}}\). The completion of \(C_{0}^{\infty}(\Omega)\) in \(W^{1,\mathcal{H}}(\Omega)\) is denoted by \(W_{0}^{1,\mathcal{H}}(\Omega)\) and it can be equivalently renormed by

\[
\|u\| := \|\nabla u\|_{\mathcal{H}}. \tag{30}
\]

via a Poincaré-type inequality, cf ([6], Proposition 2.18(iv)), under assumption (2). The spaces \(W^{1,\mathcal{H}}(\Omega)\) and \(W_{0}^{1,\mathcal{H}}(\Omega)\) are uniformly convex, and hence reflexive, Banach space. By (27), We have

\[
\min \left\{ \|u\|_{p(x)}, \|u\|_{q(x)} \right\} \leq \|\nabla u\|_{p(x)} + \|\nabla u\|_{q(x)} \leq \max \left\{ \|u\|_{p(x)}, \|u\|_{q(x)} \right\}, \quad \forall u \in W^{1,\mathcal{H}}(\Omega). \tag{31}
\]

We point out that if \(r \in C_{c}(\Omega)\) and \(r(x) \leq p^{\ast}(x)\) for all \(x \in \Omega\), then \(W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r^{\ast}}(\Omega)\) is continuous. This embedding is compact if

\[
\inf_{x \in \Omega} \{p^{\ast}(x) - r(x)\} > 0. \tag{32}
\]

Let us now define \(J(\cdot) : W_{0}^{1,\mathcal{H}}(\Omega) \to \mathbb{R}\) as

\[
J(u) = \int_{\Omega} \left( \frac{1}{p(x)} \|\nabla u\|_{p(x)}^{p(x)} + a(x) \|u\|_{p(x)}^{q(x)} \right) dx, \tag{33}
\]

and we denote the derivative operator by \(A\), that is

\[
A = J' : W_{0}^{1,\mathcal{H}}(\Omega) \to (W_{0}^{1,\mathcal{H}}(\Omega))^*, \tag{34}
\]

with

\[
\langle A(u), v \rangle = \int_{\Omega} \left( \|\nabla u\|_{p(x)}^{p(x)} + a(x) \|u\|_{q(x)}^{q(x)} \right) \nabla u \cdot \nabla v dx, \quad u, v \in W_{0}^{1,\mathcal{H}}(\Omega). \tag{35}
\]

Here, \((W_{0}^{1,\mathcal{H}}(\Omega))^*\) denotes the dual space of \(W_{0}^{1,\mathcal{H}}(\Omega)\), and \(\langle \cdot, \cdot \rangle\) denotes the paring between \(W_{0}^{1,\mathcal{H}}(\Omega)\) and \((W_{0}^{1,\mathcal{H}}(\Omega))^*\). In the following lemma, we summarize some properties of \(A\), useful to study our problem. When \(p(x) \equiv p, q(x) \equiv q\), we refer to ([13], Proposition 3.1).

**Lemma 11** (see [19], Lemma 3.4). Under the condition (2), \(A\) is a mapping of type \((S_{1})\), that is, if \(u_{n} \to u\) in \(W_{0}^{1,\mathcal{H}}(\Omega)\) and \(\lim sup \langle A(u_{n}) - A(u), u_{n} - u \rangle \leq 0\), then \(u_{n} \rightharpoonup u\) in \(W_{0}^{1,\mathcal{H}}(\Omega)\).

**Lemma 12** (see [19], Lemma 3.2). Under the condition \((f_{1}), I_{\lambda}\) is well defined on \(W_{0}^{1,\mathcal{H}}(\Omega)\), and \(I_{\lambda} \in C^{1}(W_{0}^{1,\mathcal{H}}(\Omega), \mathbb{R})\) with Fréchet derivative given by

\[
\begin{align*}
\langle I_{\lambda}'(u), v \rangle &= \int_{\Omega} \left( \|\nabla u\|_{p(x)}^{p(x)} + a(x) \|u\|_{q(x)}^{q(x)} \right) \nabla u \cdot \nabla v dx, \quad u, v \in W_{0}^{1,\mathcal{H}}(\Omega). \tag{35}
\end{align*}
\]

Firstly, we show the functional \(I_{\lambda}\) satisfies the \((C)_{c}\) condition.

**Lemma 13.** If hypotheses \((f_{1}), (f_{2})\), and \((f_{4})\) hold, then \(I_{\lambda}\) satisfies the \((C)_{c}\) condition.
Proof. For every \( c \in \mathbb{R} \), let \( \{ u_n \} \subset W_0^{1,\infty}(\Omega) \) be a \((C)_c\) sequence, that is,

\[
I_\lambda(t_n u_n) \to c, \text{ and } \| I_\lambda'(u_n) \| (W_0^{1,\infty}(\Omega))' \to 0 \text{ as } n \to \infty.
\]

(36)

We claim that \( \{ u_n \} \) is bounded in \( W_0^{1,\infty}(\Omega) \). In fact, suppose by contradiction that \( \| u_n \| \to +\infty \), as \( n \to \infty \). Let \( v_n = u_n / \| u_n \|, n \geq 1 \). Up to a subsequence, we may assume that

\[
\begin{align*}
&v_n \to v, \text{ a.e. in } \Omega, \\
v_n \rightharpoonup^* v, \text{ weakly in } W_0^{1,\infty}(\Omega), \\
v_n \to v, \text{ strongly in } L^q(\Omega), \\
v_n \to v, \text{ strongly in } L^{\infty}(x)(\Omega).
\end{align*}
\]

(37)

We know that \( v \) satisfies the following alternative: \( v = 0 \) or \( v \neq 0 \). In what follows, we will show that under the condition \( \| u_n \| \to +\infty \), \( v \) satisfies neither \( v = 0 \) nor \( v \neq 0 \). This is a contradiction. Thus, \( \{ u_n \} \) is bounded.

If \( v = 0 \), then \( v_n \rightharpoonup 0 \) a.e. \( x \in \Omega \), as \( n \to \infty \). Since \( I_\lambda(t_n u_n) \) is continuous in \( t \in [0,1] \), for each \( n \), there exists \( t_n \in [0,1] (n = 1,2,\ldots) \) such that

\[
I_\lambda(t_n u_n) = \max_{t \in [0,1]} I_\lambda(t u_n).
\]

(38)

It is easily seen that \( t_n > 0 \) and \( I_\lambda(t_n u_n) \geq c > 0 = I_\lambda(0u_n) \). If \( t_n < 1 \), then \( (d/dt)I_\lambda(t u_n) \big|_{t=t_n} = 0 \), which implies

\[
\left< I_\lambda'(t_n u_n), t_n u_n \right> = 0.
\]

(39)

Moreover, if \( t_n = 1 \), then, from (36) we have \( \left< I_\lambda'(u_n), u_n \right> = o_n(1) \). So, we always have

\[
\left< I_\lambda'(t_n u_n), t_n u_n \right> = o_n(1).
\]

(40)

Let \( \gamma_k \) be a sequence of positive real numbers such that \( \gamma_k > 1 \) for any \( k \) and \( \lim_{k \to \infty} \gamma_k = +\infty \). Then

\[
\| y_k v_n \| = \gamma_k > 1 \text{ for any } k \text{ and } n.
\]

Fix \( k \), using \((f_1),(37)\), and the Lebesgue dominated convergence theorem we deduce that

\[
\lim_{n \to \infty} \int_\Omega F(x, y_k v_n) dx = 0.
\]

(41)

Recall that \( \| u_n \| \to +\infty \) as \( n \to \infty \). So, we have \( \| u_n \| \gamma_k \) or \( \| u_n \| \gamma_k < 1 \) for \( n \) large enough. Hence, from (31) and (38), we deduce that

\[
I_\lambda(t_n u_n) \geq I_\lambda \left( \frac{y_k}{\| u_n \|} u_n \right) = I_\lambda(y_k v_n)
\]

\[
= \int_{\Omega} \left( \frac{y_k^{p(x)}}{p(x)} |\nabla v_n|^{p(x)} + \frac{y_k^{q(x)}}{q(x)} a(x)|\nabla v_n|^{q(x)} - \lambda F(x, y_k v_n) \right)
\]

\[
	imes dx \geq \frac{y_k^q}{q^*} - \lambda \int_{\Omega} F(x, y_k v_n) dx,
\]

(42)

for any \( n \) large enough. By combing this inequality with (41), as \( n, k \to +\infty \), we have

\[
\limsup_{n \to \infty} I_\lambda(t_n u_n) = +\infty.
\]

(43)

On the other hand, using condition \((f_\beta)\) and (40), for all \( n \) large enough, we obtain

\[
I_\lambda(t_n u_n) = I_\lambda(t_n u_n) - \frac{1}{q^*} \left< t_n' u_n, t_n u_n \right> + o(1)
\]

\[
= \int_{\Omega} \left( \frac{1}{p(x)} - \frac{1}{q^*} \right) |\nabla (t_n u_n)|^{p(x)} dx + \frac{1}{q^*} \left< t_n' u_n, t_n u_n \right>
\]

\[
+ \lambda \int_{\Omega} \frac{1}{q^*} f(x, t_n u_n) t_n u_n - F(x, t_n u_n) dx
\]

\[
\leq I_\lambda(u_n) - \frac{1}{q^*} \left< I_\lambda'(u_n), u_n \right> + \frac{\lambda C_0 |\Omega|}{q^*} \to c + \frac{\lambda C_0 |\Omega|}{q^*}, \text{ as } n \to \infty.
\]

From (43) and (44), we obtain a contradiction. This shows that \( v \neq 0 \), and thus,

\[
v_n \to v(x) \neq 0 \text{ a.e. in } \Omega.
\]

(45)

Let \( \Omega_v := \{ x \in \Omega : v(x) \neq 0 \} \). It implies that

\[
\| u_n(x) \| \to +\infty, \text{ in } \Omega_v, \text{ as } n \to \infty.
\]

(46)

Using condition \((f_\beta)\), we obtain

\[
\lim_{n \to \infty} \frac{F(x, u_n(x))}{\| u_n(x) \|^q} = \lim_{n \to \infty} \frac{F(x, u_n(x))}{\| u_n(x) \|^q} = +\infty, \quad x \in \Omega.
\]

(47)

Also by \((f_1)\) and \((f_\beta)\), we can get a constant \( C_2 > 0 \) such that

\[
F(x, t) \geq -C_2, \quad \forall (x, t) \in \bar{\Omega} \times \mathbb{R}.
\]

(48)
Thus, we get
\[
\frac{F(x, u_n) + C_2}{\|u_n\|^q} \geq 0. 
\] (49)

From (31), we see that
\[
c = I_1(u_n(x)) + o_n(1) \\
= \int_\Omega \left( \frac{1}{p(x)} |\nabla u_n|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u_n|^{q(x)} - \lambda F(x, u_n) \right) \\
- \frac{1}{p} \left[ \int_\Omega \lambda F(x, u_n) dx + o_n(1) \right] \\
\geq \frac{1}{p} \|u_n\|^p - \int_\Omega \lambda F(x, u_n) dx + o_n(1), 
\] (50)

which implies
\[
\left[ \frac{\|u_n\|^p}{\lambda} - \frac{c}{\lambda} + o_n(1) \right] \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty. 
\] (51)

Similarly, from (31), we also get
\[
c = I_1(u_n(x)) + o_n(1) \leq \frac{1}{\rho \lambda} \|u_n\|^\rho - \int_\Omega \lambda F(x, u_n) dx + o_n(1), 
\] (52)

which implies
\[
\|u_n\|^{\rho} \geq \rho^{-1} c + \lambda \rho \int_\Omega F(x, u_n) dx - o_n(1) > 0, 
\] (53)

for \( n \) large enough.

We claim that \( \Omega_2 \equiv 0 \). Indeed, suppose by contradiction \( |\Omega_2| \neq 0 \), then by (47)–(53) and Fatou’s lemma, we obtain
\[
+\infty \geq \liminf_{n \rightarrow +\infty} \int_\Omega \left( \frac{F(x, u_n(x))}{\|u_n(x)\|^q} |v_n(x)|^q + \frac{C_2}{\|u_n\|^q} \right) dx \\
\geq \liminf_{n \rightarrow +\infty} \int_\Omega \left( \frac{F(x, u_n(x))}{\|u_n(x)\|^q} |v_n(x)|^q + \frac{C_2}{\|u_n\|^q} \right) dx \\
= \liminf_{n \rightarrow +\infty} \int_\Omega \left( \frac{F(x, u_n(x))}{\|u_n(x)\|^q} |v_n(x)|^q + \frac{C_2}{\|u_n\|^q} \right) dx \\
\leq \frac{1}{\rho} \|v_n\|^{\rho} + \lambda \rho \int_\Omega F(x, u_n(x)) dx - o_n(1) = \frac{1}{\rho} \|v_n\|^{\rho}, 
\] (54)

which yields a contradiction. Therefore the sequence \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \). Thus, there is a subsequence (which we still denote by \( \{u_n\} \)) that converges weakly to some \( u \in W_0^{1,p}(\Omega) \) and strongly in \( L^{\alpha q}(\Omega) \). It is easy to check from (f.1) and Hölder’s inequality that
\[
\left| \int_\Omega f(x, u_n(u_n - u)) dx \right| \leq C\|1 + |u_n|^{\alpha - 1}\|u_n - u\|_\alpha \longrightarrow 0. 
\] (55)

Then
\[
\langle A(u_n), u_n - u \rangle = \left\langle I'(u_n), u_n - u \right\rangle + \lambda \int_\Omega f(x, u_n(u_n - u)) dx \longrightarrow 0. 
\] (56)

So \( u_n \longrightarrow u \) follows from Lemma 11.

\section{3. Proofs of Theorems 2 and 3}

First, we will show the functional \( I_1 \) satisfies the mountain pass geometry [25].

\textbf{Lemma 14.} Assume hypotheses \((f_1) - (f_2)\) hold. Then the functional \( I_1 \) satisfies the following properties:

(i) There exist \( \rho, \delta > 0 \) such that \( I_1(u) \geq \delta \) for any \( u \in W_0^{1,p}(\Omega) \) with \( \|u\| = \rho \).

(ii) There exists a \( \eta \in W_0^{1,p}(\Omega) \setminus B_\rho \) such that \( I_1(\eta) \leq 0 \).

\textbf{Proof.} Let us check (i). For any \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \) and \( \varepsilon > 0 \) small, it follows from (f.1) – (f.2) that there exists \( C_\varepsilon > 0 \) such that
\[
F(x, t) \leq \varepsilon |t|^q + C_\varepsilon |t|^m, \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}^N. 
\] (57)

Thus, for \( u \in W_0^{1,p}(\Omega) \) and \( \|u\| \leq 1 \), we have
\[
I_1(u) = \int_\Omega \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} - \lambda F(x, u) \right) dx, \\
\geq \int_\Omega \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) dx - \lambda \int_\Omega (\varepsilon |u|^q + C_\varepsilon |u|^m) dx, \\
\geq \frac{1}{q} \|u\|^{q} - \lambda C_\varepsilon \varepsilon \|u\|^{q} + \lambda C_\varepsilon C_4 \|u\|^m, 
\] (58)

by the Sobolev embedding \( W_0^{1,p}(\Omega) \hookrightarrow L^{\alpha q}(\Omega) \) and \( W_0^{1,p}(\Omega) \hookrightarrow L^{\alpha m}(\Omega) \). Since \( q^* < \alpha \) and \( \varepsilon \) arbitrarily small, there exist \( \rho > 0 \) and \( \delta > 0 \) such that \( I_1(u) \geq \delta \) for \( \|u\| = \rho \). Hence item (i) holds.

Let us check (ii). From (f.3), for any \( M > 0 \), we can choose a constant \( C_\delta > 0 \) such that
\[
F(x, t) \geq M |t|^q - C_\delta, \quad \forall (x, t) \in \Omega \times \mathbb{R}. 
\] (59)
Then, for \( \omega \in W^{1,\infty}_0(\Omega) \) and \( t > 0 \), we deduce that

\[
\lim_{t \to +\infty} \frac{I(t\omega)}{pt^r} \leq \lim_{t \to +\infty} \frac{\int_{\Omega} \left( \frac{1}{p(x)} |\nabla (tu)|^{p(x)} + \frac{a(x)}{q(x)} |\nabla \omega|^{q(x)} \right) dx}{pt^r} - \lambda \int_{\Omega} \left( M|tu|^{r'} - C_5 \right) dx,
\]

\[
\leq \lim_{t \to +\infty} \frac{1}{p} \int_{\Omega} \left( \frac{1}{p(x)} |\nabla \omega|^{p(x)} + \frac{a(x)}{q(x)} |\nabla \omega|^{q(x)} - t^r \lambda M|\omega|^{r'} + \lambda C_5 \right) dx
\cdot |dx| \leq \int_{\Omega} \left( \frac{1}{p(x)} |\nabla \omega|^{p(x)} + \frac{a(x)}{q(x)} |\nabla \omega|^{q(x)} - \lambda M|\omega|^{r'} \right) dx.
\]

If \( M \) is large enough such that

\[
\int_{\Omega} \left( \frac{1}{p(x)} |\nabla \omega|^{p(x)} + \frac{a(x)}{q(x)} |\nabla \omega|^{q(x)} - \lambda M|\omega|^{r'} \right) dx < 0,
\]

(61)

(62)

conclusion (ii) follows.

**Proof of Theorem 2.** Since the functional \( I_1 \) has the mountain pass geometry and satisfies the \( (C)_\epsilon \) condition, the mountain pass theorem [25] gives that there exists a critical point \( u \in W^{1,\infty}_0(\Omega) \). Moreover, \( I(u) = c \geq 0 > I(0) \), so \( u \) is a nontrivial solution.

**Lemma 15.** Assume \((f_1)\) holds. Then there exist positive constants \( m_1 \) and \( \rho_1 \) such that \( \lim_{\lambda \to 0^+} m_1 = +\infty \) and \( I_1 \geq m_1 > 0 \) when \( \|u\| = \rho_1 \).

**Proof.** Let \( u \in W^{1,\infty}_0(\Omega) \) with \( \|u\| > 1 \). It follows from \((f_1)\) that there exists \( C_6 > 0 \) such that

\[
|F(x, t)| \leq C_6 \left( |t|^{p(x)} + 1 \right),
\]

(63)

for all \((x, t) \in \Omega \times \mathbb{R}, q^* < a(x) < (p^*)^\sigma\). Hence, we obtain

\[
I_1(u) \geq \int_{\Omega} \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla \omega|^{q(x)} \right) dx
\cdot \left( |u|^{q(x)} + 1 \right) dx - \lambda \int_{\Omega} \left( |u|^{q(x)} + 1 \right) dx
\cdot \lambda \geq \frac{1}{q} |u|^{q'} - \lambda C_7 |u|^{\sigma} - \lambda C_6 |\Omega|.
\]

Let \( \rho_1 = \lambda^{1-\sigma} \) where \( \sigma \in (0, 1/(a^* - p^*)) \). Hence, we get \( \rho_1 > 1 \) for \( \lambda \) small enough. Therefore, substituting \( \|u\| = \rho_1 = \lambda^{1-\sigma} \) in (63), we see that

\[
I_1(u) \geq \frac{1}{\rho_1} - \lambda^{1-\sigma} - C_5 \lambda^{1-\sigma} - \lambda C_6 |\Omega|.
\]

(64)

(65)

(66)

Taking the limit \( \lambda \to 0^+ \) in (66) and using Lemma 15, one has \( \lim_{\lambda \to 0^+} \|u_\lambda\| = +\infty \).

**4. Proofs of Theorems 4 and 5**

**Lemma 16.** Assume the hypotheses \((f_1) - (f_3)\) hold. Then the functional \( I_1 \) satisfies the following properties:

(i) There exist constants \( \rho, \delta > 0 \), such that \( I_1(u) \geq \delta \) for any \( u \in W^{1,\infty}_0(\Omega) \) with \( \|u\| = \rho \)

(ii) For each finite dimensional subspace \( \bar{X} \subset W^{1,\infty}_0(\Omega) \), there exists an \( R = R(\bar{X}) \) such that \( I_1 \leq 0 \) on \( \bar{X} \setminus B_R(\bar{X}) \).

**Proof.** As in the proof of Lemma 14, it is immediate to see that the case (i) is true. Let \( \epsilon \in \bar{X} \) and \( \|\epsilon\| = 1 \) be fixed. From (59), we obtain
for all norms on $\tilde{X}$ are equivalent. Then, we can choose $M$ large enough such that $1/p^* - \lambda MC_4 t_q^* + \lambda C_5 |\Omega|$, 

\begin{equation}
\lim_{n \to \infty} \left< I_\lambda'(u_n), u_n - u \right> = 0. \tag{70}
\end{equation}

It is easy to check from $(f_5)$ and Hölder's inequality that

\begin{equation}
\left| \int_\Omega f(x, u_n)(u_n - u)dx \right| \leq C_{11} \|u_n\|^{\beta(x)-1} \|u_n\|_{\beta(\cdot)} \longrightarrow 0, \quad \text{as } n \to \infty,
\end{equation}

where $\beta'(\cdot) = \beta(\cdot)/\beta(\cdot) - 1$. Then

\begin{equation}
\left< A(u_n), u_n - u \right> = \left< I'(u_n), u_n - u \right> + \lambda \int_\Omega f(x, u_n)(u_n - u)dx \longrightarrow 0, \quad \text{as } n \to \infty. \tag{72}
\end{equation}

So $u_n \to u$ follows from Lemma 11.

Proof of Theorem 5. Consider $K$ is a $k$-dimensional linear subspace $K \subset C_0^\infty(\Omega)$ of $W_0^{1,p}(\Omega)$. We invoke Clark's Theorem in [25], Theorem 9.1. The next result is a compactness result on problem $(P_\lambda)$ which we will use later.

**Lemma 17.** Assume that condition $(f_5)$ holds, then

(i) $I_{\lambda}$ is bounded from below

(ii) $I_{\lambda}$ satisfies the (PS) condition.

**Proof.** (i) Using $(f_5)$, and for $\|u\| > 1$, $\lambda > 0$, we obtain

\begin{equation}
I_{\lambda}(u) \geq \int_\Omega \left( \frac{1}{P(x)} |\nabla u|^{p(x)} + \frac{a(x)}{q(x)} |\nabla u|^{q(x)} \right) \cdot dx - \lambda \frac{d_0}{\beta} \int_\Omega |u|^{\beta(x)} dx \geq \frac{1}{q} \|u\|^{\beta} - C_{10} \|u\|^{\beta^*}.
\end{equation}

Hence, $I_{\lambda}$ is coercive following immediately from the above expression and $\beta^* < p^*$. Therefore, $I_{\lambda}$ is bounded from below.

(ii) Suppose $\{u_n\}$ is a $(PS)_\epsilon$ sequence for $I_{\lambda}$. Thus $I_{\lambda}(u_n) \to c$ and $I_{\lambda}'(u_n) \to 0$ in $(W_0^{1,p}(\Omega))'$ as $n \to +\infty$. It follows from (i) that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Up to a subsequence, we may assume that

\begin{equation}
u_n \to u, \quad \text{a.e. in } \Omega, \quad u_n \to u, \quad \text{weakly in } W_0^{1,p}(\Omega), \quad u_n \to u, \quad \text{strongly in } L^{\beta(\cdot)}(\Omega).
\end{equation}

Since $I_{\lambda}'(u_n) \to 0$ and $u_n - u \to 0$ in $W_0^{1,p}(\Omega)$, (see [26], Proposition 3.5), we get that

5. **Proofs of Theorems 6 and 7**

In this section, we will show that $(P_\lambda)$ has infinitely many pairs of solutions by using the Fountain Theorem and Dual Fountain Theorem. Firstly, we need to recall some
preliminary results. Since \( W_0^{1,\infty}(\Omega) \) is a reflexive and separable Banach space, there are \( e_j \in W_0^{1,\infty}(\Omega) \) and \( e_j^* \in (W_0^{1,\infty}(\Omega))^* \) such that
\[
W_0^{1,\infty}(\Omega) = \text{span} \{ e_j : j = 1, 2, \ldots \},
\]
\[
(W_0^{1,\infty}(\Omega))^* = \text{span} \{ e_j^* : j = 1, 2, \ldots \}.
\]
(75)

Then, we define
\[
X_j = \text{span} \{ e_j \}, Y_k = \oplus_{j=1}^k X_j, Z_k = \overline{\text{span}}_{\infty} X_j.
\]
(76)

We will apply the following Fountain Theorem ([25], Theorem 3.6).

**Lemma 18.** Assume that \( X \) is a Banach space, and let \( \varphi \in C^1(X, \mathbb{R}) \) be an even functional. If, for every \( k \in \mathbb{N} \), there exists \( \rho_k > r_k > 0 \) such that
\[
(A_1) \quad b_k := \inf_{u \in Z_k} \varphi(u) \to +\infty, k \to +\infty,
\]
\[
(A_2) \quad a_k := \max_{u \in Y_k} \varphi(u) \leq 0,
\]
\[
(A_3) \quad \varphi \text{ satisfies the (C) condition for every } c > 0.
\]

Then \( \varphi \) has an unbounded sequence of critical values.

To prove Theorems 6 and 7, the following lemma is needed.

**Lemma 19.** Assume that \( \alpha(x) \in C_1(\overline{\Omega}) \), \( q^- < \alpha(x) < (p^+)^- \), for any \( x \in \Omega \). Let
\[
\beta_k = \sup_{\|u\| = 1} \|u\|_{L^{p^+}(\Omega)},
\]
(78)
then
\[
\lim_{k \to +\infty} \beta_k = 0.
\]

Proof. Obviously, \( 0 < \beta_{k+1} < \beta_k \) and so \( \beta_k \to \beta \geq 0 \). Let \( u_k \in Z_k \) satisfy
\[
\|u_k\| = 1, \quad 0 \leq \|u_k\|_{L^{p^+}(\Omega)} < \frac{1}{k}.
\]
(79)

Then, there exists a subsequence of \( \{ u_k \} \) (which we still denote by \( \{ u_k \} \)) such that \( u_k \to u \), and
\[
\langle e_j^*, u_k \rangle = \lim_{k \to +\infty} \langle e_j^*, u_k \rangle = 0, \quad j = 1, 2, \ldots,
\]
(80)
We are now ready to prove the Theorem 7.

Proof of Theorem 7. According to the Dual Fountain Theorem ([25], Theorem 3.18), it suffices to prove that for every \( k \geq k_0 \), there exist \( \rho_k > r_k > 0 \) such that

\[
(B_1) \quad a_k := \max_{\|u\| = r_k} I_k(u) < 0, \\
(B_2) \quad b_k := \inf_{\|u\| = r_k} I_k(u) \geq 0, \\
(B_3) \quad d_k := \inf_{u \in Z_k} I_k(u) \to 0, \quad k \to +\infty. \\
(B_4) \quad I_k \text{ satisfies the } (C)_c^* \text{ condition for every } c \in \mathbb{R}.
\]

By \((f_j)\), there exists \( C_{17} > 0 \) such that

\[
|F(x, t)| \leq C_{17}(|t| + |t|^{\alpha(x)})
\]

for all \((x, t) \in \Omega \times \mathbb{R} \), \( q^* - \alpha(x) < (p^*)^- \). Now we define the function \( \Psi_1, \Psi_2 : X \to \mathbb{R} \) by

\[
\Psi_1(u) = \int_\Omega \lambda C_{17} |u|^{\alpha(x)} dx, \\
\Psi_2(u) = \int_\Omega \lambda C_{17} |u| dx.
\]

By the definition of \( \Psi_1, \Psi_2 \), we have \( \Psi_i(0) = 0, \ i = 1, 2 \), and they are weakly-strongly continuous. Consider

\[
\xi_k = \sup_{u \in Z_k, \|u\| \leq 1} |\Psi_1(u)|, \ \zeta_k = \sup_{u \in Z_k, \|u\| \leq 1} |\Psi_2(u)|. 
\]

From the compact embedding \( \dot{W}_0^1(\Omega) \hookrightarrow L^{\alpha(\cdot)}(\Omega) \) and Lemma 19, we have

\[
\lim_{k \to +\infty} \xi_k = \lim_{k \to +\infty} \zeta_k = 0.
\]

Let \( \omega \in Z_k \) and \( \|\omega\| = 1, \ 0 < t < \rho_k \). Then, from (89) and (90), we obtain

\[
I_k(t\omega) = \int_\Omega \left( \frac{1}{p(x)} |\nabla (t\omega)|^{p(\cdot)} + \frac{a(x)}{q(x)} |\nabla (t\omega)|^{\alpha(\cdot)} - \lambda F(x, t\omega) \right) dx \\
\geq -\lambda \int_\Omega F(x, t\omega) dx \\
\geq -\rho_k^{q^*} \Psi_1(\omega) - \rho_k \Psi_2(\omega) \\
\geq -\rho_k^{q^*} \xi_k - \rho_k \zeta_k.
\]

Passing the limit in the above inequality, as \( k \to +\infty \), we achieve that

\[
\lim_{k \to +\infty} d_k \geq 0,
\]

which, together with (88), implies that \( \lim_{k \to +\infty} d_k = 0 \).

(B_4) Let \( \{u_n\} \) be any sequence in \( W_0^{1,\varphi}(\Omega) \) such that

\[
u_n \in Y_n, \\
I_{\lambda_k}(u_n) \to c > 0, \\
\|I_{\lambda_k}'(u_n)\| \to 0, \quad n \to \infty
\]

as \( n \to \infty \).

Then similar to the proof of Lemma 13, we see that \( \{u_n\} \) is bounded in \( W_0^{1,\varphi}(\Omega) \). Thus, there is a subsequence (which we denote by \( \{u_{n_k}\} \)) that converges weakly...
to some \( u \in W_0^{1,\infty}(\Omega) \) and strongly in \( L^{\infty}(\Omega) \). It is easy to check from (f) and Hölder’s inequality that
\[
\int_{\Omega} f(x, u_{n_k}) (u_{n_k} - u) \, dx \leq C\|1 + |u_{n_k}|^{(c-1)}\|_\infty \|u_{n_k} - u\|_\infty \longrightarrow 0.
\]
(96)

**Claim 21.** \( \lim_{k \to \infty} \langle I'_\lambda(u_{n_k}), u_{n_k} - u \rangle = 0. \)

If Claim 21 holds true, then
\[
\langle A(u_{n_k}), u_{n_k} - u \rangle = \langle I'_\lambda(u_{n_k}), u_{n_k} - u \rangle + \lambda \int_{\Omega} f(x, u_{n_k}) (u_{n_k} - u) \, dx \longrightarrow 0.
\]
(97)

So \( u_{n_k} \longrightarrow u \) follows from Lemma 11. Hence, \( I_\lambda \) satisfies the \( (C)_\epsilon^* \) condition. In order to prove Claim 21, we invoke \( W_0^{1,\infty}(\Omega) = \cup_n Y_n = \text{span}\{e_n : n = 1, 2, \ldots\} \) to choose \( v_n \in Y_n \) such that \( v_n \rightharpoonup u \) strongly in \( W_0^{1,\infty}(\Omega) \). Since \( I'_{\lambda,v_n}(u_{n_k}) \longrightarrow 0 \) and \( u_{n_k} - v_n \rightharpoonup 0 \) in \( Y_n \), (see [26], Proposition 3.5), we get that
\[
\lim_{k \to \infty} \langle I'_\lambda(u_{n_k}), u_{n_k} - v_n \rangle = 0.
\]
(98)

Hence, we obtain
\[
\lim_{k \to \infty} \langle I'_\lambda(u_{n_k}), u_{n_k} - u \rangle = \lim_{k \to \infty} \langle I'_\lambda(u_{n_k}), u_{n_k} - v_n \rangle + \lim_{k \to \infty} \langle I'_\lambda(u_{n_k}), v_n - u \rangle = 0.
\]
(99)

Therefore, the Claim holds true and we conclude that \( I'_\lambda(u_{n_k}) \longrightarrow I'_\lambda(u) \) as \( k \longrightarrow +\infty \). We next show that \( I'_\lambda(u) = 0 \).

To see this, taking \( \omega_j \in Y_j \), we have
\[
\langle I'_\lambda(u), \omega_j \rangle = \langle I'_\lambda(u) - I'_\lambda(u_{n_k}), \omega_j \rangle + \langle I'_\lambda(u_{n_k}), \omega_j \rangle = \langle I'_\lambda(u) - I'_\lambda(u_{n_k}), \omega_j \rangle + \langle I'_{\lambda,v_n}(u_{n_k}), \omega_j \rangle.
\]
(100)

We pass limit in the right side of (100) as \( k \longrightarrow +\infty \) to obtain
\[
\langle I'_\lambda(u), \omega_j \rangle = 0, \quad \text{for all} \quad \omega_j \in Y_j.
\]
(101)

Therefore, \( I_\lambda \) satisfies the \( (C)_\epsilon^* \) condition for every \( c \in \mathbb{R} \). The proof is complete.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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