ON THE CONSTELLATIONS OF WEIERSTRASS POINTS

FERNANDO TORRES

Abstract. We prove that the constellation of Weierstrass points characterizes the isomorphism-class of double coverings of curves of genus large enough.

1. Let $X$ be a projective, irreducible, non-singular algebraic curve defined over an algebraically closed field $k$ of characteristic $p$. Let $n \geq 1$ be an integer and $C_X$ a canonical divisor of $X$. The pluricanonical linear system $|nC_X|$ defines a nondegenerate morphism

$$
\pi_n : X \to \mathbb{P}^{N(n)},
$$

where $N(1) = g - 1$, and $N(n) = (2n - 1)(g - 1) - 1$ for $n \geq 2$. To any $P \in X$ we then associate the sequence of multiplicities

$$
\{v_P(\pi_n^*(H)) : H \text{ hyperplane } \subseteq \mathbb{P}^{N(n)}\} = \{\epsilon_0(P) < \epsilon_1(P) < \ldots < \epsilon_{N(n)}(P)\}.
$$

Such a sequence is the same for all but finitely many points (cf. [F-K, III.5], [Lak, Prop.3], [S-V, §1]). These finitely many points are the so called $n$-Weierstrass points of $X$. There exists a divisor $W_n$ on $X$ whose support is the set of $n$-Weierstrass points and satisfies the property below. Let denote by $v_P(W_n)$ the coefficient of $W_n$ in $P$ (called the $n$-Weierstrass weight at $P$). Then

$$
\omega_n := \deg(W_n) = \sum_P v_P(W_n) = \sum_{i=0}^{N(n)} \epsilon_i(2g - 2) + n(2g - 2)(N(n) + 1),
$$

where $\epsilon_0 < \epsilon_1 < \ldots < \epsilon_{N(n)}$ denotes the sequence at a generic point ([F-K, III.5], [Lak, Thm.6], [S-V, p.6]). One has $\epsilon_i(P) \geq \epsilon_i$ for each $i$ and for each $P$ ([F-K, III.5], [Lak, Prop.3], [S-V, p.5]).

Let $P_1, \ldots, P_{\omega_n}$ be the $n$-Weierstrass points (counted with multiplicity according to their $n$-Weierstrass weights).

Definition. The orbit $\mathcal{O}_n(X)$ of $(\pi_n(P_1), \ldots, \pi_n(P_{\omega_n})) \in (\mathbb{P}^{N(n)})^{\omega_n}$ under the action of the product of the symmetric group $S_{\omega_n}$ and the projective linear group $PGL(N(n) + 1)$ is called the constellation of $n$-Weierstrass points of $X$. $\mathcal{O}_1(X)$ is called the constellation of Weierstrass points of $X$.

Let $X, X_1$ be curves as above of genus $g$. Then clearly $\mathcal{O}_n(X) = \mathcal{O}_n(X_1)$ if $X$ is isomorphic to $X_1$. In this note we are interested in the converse:

$$(P) \quad \mathcal{O}_n(X) = \mathcal{O}_n(X_1) \quad \text{for some } n \geq 1 \Rightarrow X \text{ is isomorphic to } X_1?$$

This problem was studied by Pflaum [Pf] for $p = 0$. He showed that $(P)$ is true in the following cases:

(i) If $n \geq 2$;
(ii) If $n = 1$, and $2 \leq g \leq 15$;

The author is supported by a grant from the International Atomic Energy Agency and UNESCO.
(iii) If \( n = 1 \), and \( X \) and \( X_1 \) are \( \gamma \)-hyperelliptic curves (that is double coverings of curves of genus \( \gamma \)) with \( g \geq 2 \) and \( \gamma \in \{0, 1, 2\} \).

For the cases \( n = 2, g = 2; n = 1, g = 3, 4; \) and \( n = 1, X, X_1 \) hyperelliptic curves the proof is given by a direct application of the definition above. For the remaining cases Pflaum used the following argument. By means of a lower bound on the number \( W_n \) of \( n \)-Weierstrass points, he stated (Corollary 2.6, loc. cit.) a sufficient condition to have an affirmative answer to the problem (this condition holds regardless the characteristic of the field). Define the number \( N(g, n) \) by \( N(6, 1) = 25, N(g, 1) = \operatorname{max}\{3g + 6, 4g - 4\} \) for \( g \neq 6 \), and \( N(g, n) = 4n(g - 1) \) for \( n \geq 2 \). Pflaum then showed that \( (P) \) is true provided

\[
W_n > N(g, n).
\]

A way of getting a lower bound on \( W_n \) is by bounding from above \( v_p(W_n) \), and then by using \( (\mathbf{I}) \). One has

\[
v_p(W_n) \geq \sum_{i=1}^{N(n)} (\epsilon_i(P) - \epsilon_i)
\]

for each \( P \), and equality holds if

\[
\det \left( \begin{pmatrix} \epsilon_i(P) \\ \epsilon_j \end{pmatrix} \right) \equiv 0 \pmod{p}
\]

([Lak, p.239], [S-V, Thm.1.5]). A curve \( X \) is called classical (with respect to the pluri-canonical linear system \( |nC_X| \) if \( \epsilon_i = i \) for each \( i \). This is the case if \( p = 0 \) or \( p > n(2g - 2) \), and here one also has the equality in \( (3) \) for each \( P \) ([F-K, III.5], [S-V, Corollary 1.8], [Lak, Thm.11]).

Let \( X \) and \( X_1 \) be classical curves of genus \( g \), and suppose that we have equality in \( (3) \) for each point of \( X \) and \( X_1 \). If \( n = 1 \) and \( 5 \leq g \leq 15 \), Pflaum showed \( (2) \) by direct computations. If \( n \geq 2 \), then Homma and Ommori ([H-O]) noticed that \( v_p(W_n) \leq g(g + 1)/2 \). Hence by means of \( (\mathbf{I}) \) Pflaum obtained \( (2) \). Now if \( X \) and \( X_1 \) are \( \gamma \)-hyperelliptic curves of genus large enough with \( \gamma \in \{1, 2\} \) and if \( p = 0 \) or \( p > 2g - 2 \), by using results due to Garcia ([3]) and Kato ([K]) Pflaum can bound from above \( v_p(W_1(P)) \) and obtain \( (2) \).

The aim of this note is to extend Pflaum’s result (iii) above to \( \gamma \)-hyperelliptic curves of genus large enough \( (\gamma \geq 3) \) and whenever \( p = 0 \) or \( p > 2g - 2 \). We will show that such curves satisfy \( (2) \) and to do that we use some results concerning Weierstrass weights in [101] and [102]. We show

**Theorem.** Let \( X \) and \( X_1 \) be \( \gamma \)-hyperelliptic curves of genus \( g \geq 9\gamma - 17 + \frac{43\gamma - 20}{2\gamma + \gamma - 1} \) with \( \gamma \geq 3 \). Assume that \( p = 0 \) or \( p > 2g - 2 \). Then \( X \) and \( X_1 \) are isomorphic provided \( \mathcal{O}_1(X) = \mathcal{O}_1(X_1) \).

In general one cannot expect to fulfil condition \( (2) \) for \( 0 < p \leq 2g - 2 \), because in this case there exist curves with small number of Weierstrass points. For instance there exist curves with just one Weierstrass point (see [Lak, §6]).

2. Let

\[
\pi : X \to \tilde{X}
\]

be a double covering of curves of genus \( g \) and \( \gamma \geq 3 \) respectively. Let \( P \in X \), and set \( w(P) := v_p(W_1) \). The key point of the proof is the fact that we can bound from above \( w(P) \) by considering the following three cases:
(I) \(P\) is a ramified point of \(\pi\) such that \(\pi(P)\) is a Weierstrass point of \(\tilde{X}\).

(II) \(P\) is a ramified point of \(\pi\) such that \(\pi(P)\) is not a Weierstrass point of \(\tilde{X}\).

(III) \(P\) is not a ramified point of \(\pi\).

**Lemma.** Let \(X\) be a \(\gamma\)-hyperelliptic curve of genus \(g\) \((\gamma \geq 3)\). Let assume that \(p = 0\) or \(p > 2g - 2\), and let \(P \in X\).

(i) If \(P\) is as in (I), then
\[
w(P) \leq c_1 := \frac{(g - 2\gamma)}{2} + 2\gamma^2.
\]

(ii) If \(P\) is as in (II), then
\[
w(P) \leq c_2 := \frac{(g - 2\gamma)}{2} + 4\gamma - 4.
\]

(iii) If \(P\) is as in (III) and \(g \geq 2\gamma\), then
\[
w(P) \leq c_3 := \max\{2(\gamma - 1)g - (\gamma - 1)(2\gamma + 1), 2\gamma g - 2\gamma(4\gamma - 1)\}.
\]

3. **Proof of the Theorem.** Since \(g \geq 2\gamma + 2\), every ramified point of \(\pi\) is a Weierstrass point (see §4). Let \(t\) denote the number of points of type (I). Then by the lemma we have:
\[
deg(W_1) = g^3 - g \leq tc_1 + (2g - 4\gamma + 2 - t)c_2 + (W_1 - 2g + 4\gamma - 2)c_3
\]
\[
= (c_1 - c_2)t + (2g - 4\gamma + 2)c_2 + (W_1 - 2g + 4\gamma - 2)c_3.
\]

Then by noticing that \(t \leq \min\{\gamma^3 - \gamma, 2g - 4\gamma + 2\}\), we find
\[
(W_1 - 2g + 4\gamma - 2)c_3 \geq 6\gamma g^2 - 16\gamma^2 g + 16\gamma^3 - 4\gamma^2 - 2\gamma.
\]

If \(g \geq 6\gamma^2 - \gamma + 1\), then \(c_3 = 2\gamma(g - 4\gamma + 1)\) and from (I) we get
\[
W_1 \geq 5g - 1 + \frac{24\gamma^2 - 10\gamma - 4}{g - 4\gamma + 1} > N(g, 1).
\]

Now suppose that \(c_3 = (2\gamma - 1)g - (2\gamma^2 - \gamma - 1)\). From (I) we find that \(W_1 > N(g, 1)\) if
\[
(2\gamma - 1)(2\gamma + 2)g - (36\gamma^3 - 50\gamma^2 + 34\gamma - 6) + \frac{24\gamma^5 - 40\gamma^4 + 22\gamma^3 + 4\gamma}{(2\gamma - 1)g - (2\gamma^2 - \gamma - 1)} > 0.
\]

This is satisfied provided \(g \geq 9\gamma - 17 + \frac{43\gamma - 20}{2\gamma^2 + \gamma - 1}\).

4. **Proof of the Lemma.** First we recall some properties of Weierstrass semigroups.

Let \(P \in X\). In the case of 1-Weierstrass points, the set
\[
G(P) := \{\epsilon_i(P) + 1 : 0 \leq i \leq g - 1\}
\]
is the complement (or the gaps) of a semigroup \(H(P)\), the so called Weierstrass semigroup at \(P\). \(H(P)\) looks like
\[
H(P) = \{0 < m_1(P) < \ldots < m_g(P) = 2g < 2g + 1 < \ldots\},
\]
and it is satisfied the following property (\cite{E}, \cite{Oliv}, Thm. 1.1(ii)). Let \(\ell_i(P) := \epsilon_i(P) + 1\).

Then
\[
\ell_i(P) \leq 2i - 2, \quad \text{for} \quad i = 2, \ldots, g - 1, \quad \ell_g(P) \leq 2g - 1,
\]
provided \( m_1(P) \geq 3 \). Then if \( X \) is classical and if we have equality in (3), \( w(P) \) can be computed by the formula

\[
w(P) = \frac{3g^2 + g}{2} - \sum_{m \in H(P), m \leq 2g} m.
\]

Now in case of \( \gamma \)-hyperelliptic curves, \( P \) a ramified point of \( \pi \) and \( p > 2 \), \( H(P) \) fulfil the following properties ([To1, Lemma 3.4]):

(A) \( \gamma = \#\{\ell \in G(P) : \ell \text{ even}\} \).

(B) \( H(\pi(P)) = \{\frac{h}{2} : h \in H(P), h \text{ even}\} \).

(Note that property (B) implies \( h \leq 2\gamma + 2 \) for \( h \in H(P), h \text{ even} \). In particular if \( X \) is classical and \( g \geq 2\gamma + 2 \), then each ramified point of \( \pi \) is a Weierstrass point of \( X \).)

**Proof of (i).** Follows from property (A) above and [To2, Lemma 3.1.2(ii)].

**Proof of (ii).** Let \( P \in X \) be as in (II). Since \( p = 0 \) or \( p > 2g - 2 > 2\gamma - 2 \), then \( \tilde{X} \) is also a classical curve. Thus from properties (A) and (B) we have that all the even positive non-gaps of \( H \) belong to the following set:

\[
\{2\gamma + 2i : i \in \{1, \ldots, g - \gamma\}\}.
\]

Hence

\[
(*) \quad \sum_{h \in H(P), h \text{ even}, h \leq 2g} h = g^2 + g - \gamma^2 - \gamma.
\]

Let denote by \( u_\gamma < \ldots < u_1 \) the \( \gamma \) odd non-gaps at \( P \) in \([1, 2g - 1]\). According to (3) and (\( \ast \)), an upper bound for \( w(P) \) corresponds to a lower bound for \( \sum_{i=1}^{\gamma} u_i \). By [To1, Lemma 2.1], \( u_\gamma \geq 2g - 4\gamma + 1 \). If \( u_\gamma \geq 2g - 2\gamma - 1 \), any odd number in \([2g - 2\gamma + 1, 2g - 1]\) could be an odd non-gap at \( P \). Hence in this case we have

\[
\sum_{i=1}^{\gamma} u_i \geq 2\gamma g - \gamma^2 - 2\gamma.
\]

If \( u_\gamma \leq 2g - 2\gamma - 3 \) (then \( \gamma \geq 2 \)), it is easy to see that the minimum for \( \sum_{i=1}^{\gamma} u_i \) is reached for the sequence \( 2g - (2i + 5), 2g - 1, i = 1, \ldots, \gamma - 1 \). Hence in this case we have

\[
\sum_{i=1}^{\gamma} u_\gamma \geq 2\gamma g - \gamma^2 - 4\gamma + 4.
\]

Then since \( \gamma > 1 \) from (\( \ast \)), the last inequality and (3) we obtain (ii).

**Proof of (iii).** Let \( P \in X \) and suppose that \( P \) is not a ramified point of \( \pi \).

**Claim.** Let \( h \) be a non-gap at \( P \). Then \( h \geq g - 2\gamma + 1 \).

**Remark.** Let \( f \in k(X) \) and denote by \( O(f) \) the degree of \( f \). Then \( O(f) \) is even provided \( O(f) < g + 1 - 2\gamma \) and \( g \geq 4\gamma + 2 \). For \( p = 0 \) this is a result due to Farkas ([4, Thm.2(iii)]) (see also [F-K, Thm.V.1.9], Accola [A, Lemma 4]) and in general is due to Stichtenoth [St, Satz 2]. The claim follows from this result but with an extra hypothesis on \( g \). We will see that in the case that \( f \) has just one pole one can avoid such a hypothesis. The claim is a particular case of [To1, Corollary 3.3(ii)], and for the sake of completeness we state a proof of it.
Proof. (Claim.) Suppose that \( h < g - 2\gamma + 1 \). Consider \( K' := k(\tilde{X}).k(f) \), with \( \text{div}_\infty(f) = hP \). Then by Castelnuovo’s inequality concerning subfields of \( k(\tilde{X}) \) (see [C], [Sti1]) we must have \( K' = k(\tilde{X}) \). Thus there exists \( \tilde{f} \in K(\tilde{X}) \) such that \( f = \tilde{f} \circ \pi \) and we would have that \( P \) is a totally ramified point of \( \pi \), a contradiction.

Let \( g \geq 2\gamma \). We have

\[
(7) \quad w(P) = \sum_{i=0}^{g-2\gamma+1} (\ell - i),
\]

and then we consider two cases:

(a) There exists \( \ell \in G(P) \cap [g-2\gamma+1, g] \).

(b) \([g-2\gamma+1, g] \cap \mathbb{N} \subseteq H(P)\).

In the first case we have \( \ell_{g-2\gamma+1} = g - 2\gamma + j \) with \( j \in \{1, \ldots, \gamma\} \). Then from (7) we get

\[
w(P) \leq (2\gamma - 1) + \sum_{i=g-2\gamma+2}^{g-1} (i - 2) + (g - 1) = (2\gamma - 1)g - (\gamma - 1)(2\gamma + 1).
\]

In the second case, due to the semigroup property of \( H(P) \), \( w(P) \) reaches its maximum whenever \( G(P) = \{1, \ldots, g-2\gamma, 2g-6\gamma+2, \ldots, 2g-4\gamma+1\} \). Then from (7) we find

\[
w(P) \leq 2\gamma g - 2\gamma(4\gamma - 1).
\]

This finish the proof of the lemma.

References

[A] Accola, R.D.M.: *Strongly branched coverings of closed Riemann surfaces*, Proc. Amer. Math. Soc. 26 (1970), 315–322.

[B] Buchweitz, R.O.: “Über deformationen monomialer kurvensingularitäten und Weierstrasspunkte auf Riemannschen flächen”, Thesis, Hannover 1976.

[C] Castelnuovo, G.: *Sulle serie algebriche di gruppi di punti appartenenti ad una curva algebrica*, Rendiconti della Reale Accademia dei Lincei (5) 15 (1906), 337–344. Reprinted in Memoria scelta, Zanichelli, Bologna, 509–517, 1937.

[F] Farkas, H.M.: *Remarks on automorphisms of compact Riemann surfaces*, Ann. of Math. Stud. 78 (1974), 121–144.

[F-K] Farkas, H.M.; Kra, I.: “Riemann surfaces”, Grad. Texts in Math. 71 (second edition) Springer-Verlag 1992.

[G] Garcia, A.: *Weights of Weierstrass points in double covering of curves of genus one or two*, Manuscripta Math. 55 (1986), 419–432.

[H-O] Homma, M.; Ommori, S.: *On the weight of higher order Weierstrass points*, Tsukuba J. Math. 8 (1984), 189–198.

[K] Kato, T.: *Non-hyperelliptic Weierstrass points of maximal weight*, Math. Ann. 239, (1979), 141–147.

[Lak] Laksov, D.: *Weierstrass points on curves*, Astérisque 87-88. (Société Mathématique de France, Paris, 1981), 221–247.

[Oliv] Oliveira, G.: *Weierstrass semigroups and the canonical ideal of non–trigonal curves*, Manuscripta Math. 71 (1991), 431–450.

[Pf] Pfalz, U.: *The canonical constellations of k-Weierstrass points*, Manuscripta Math. 59 (1987), 21–34.

[St] Stichtenoth, H.: *s-Erweiterungen algebraischer Funktionenkörper*, Arch. Math. 43 (1984), 27–31.

[St1] Stichtenoth, H.: *Die Ungleichung von Castelnuovo*, J. Reine Angew. Math. 348 (1984), 197–202.
[S-V] Stöhr, K.O.; Voloch, J.F.: *Weierstrass points and curves over finite fields*, Proc. London Math. Soc. (3), **52** (1986), 1–19.

[To1] Torres, F.: *On certain N-sheeted coverings of curves and numerical semigroups which cannot be realized as Weierstrass semigroups*, Comm. Algebra **23** (11) (1995), 4211–4228.

[To2] Torres, F.: *Remarks on numerical semigroups*, alg-geo e-print 9512012.

ICTP, Mathematics Section, P.O. Box 586, 34100, Trieste - Italy

E-mail address: feto@ictp.trieste.it