EQUIVALENCE OF $\mathbb{Z}_4$-ACTIONS ON HANDLEBODIES OF GENUS $g$

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Abstract. In this paper we consider all orientation-preserving $\mathbb{Z}_4$-actions on 3-dimensional handlebodies $V_g$ of genus $g > 0$. We study the graph of groups $(\Gamma(v), G(v))$, which determines a handlebody orbifold $V(\Gamma(v), G(v)) \simeq V_g/\mathbb{Z}_4$. This algebraic characterization is used to enumerate the total number of $\mathbb{Z}_4$ group actions on such handlebodies, up to equivalence.

1. Introduction

A $G$-action on a handlebody $V_g$, of genus $g > 0$, is a group monomorphism $\phi: G \rightarrow \text{Homeo}^+(V_g)$, where $\text{Homeo}^+(V_g)$ denotes the group of orientation-preserving homeomorphisms of $V_g$. Two actions $\phi_1$ and $\phi_2$ on $V_g$ are said to be equivalent if and only if there exists an orientation-preserving homeomorphism $h$ of $V_g$ such that $\phi_2(x) = h \circ \phi_1(x) \circ h^{-1}$ for all $x \in G$. From [4], the action of any finite group $G$ on $V_g$ corresponds to a collection of graphs of groups. We may assume these particular graphs of groups are in canonical form and satisfy a set of normalized conditions, which can be found in [2].

Let $v = (r, s, t, m, n)$ be an ordered 5-tuple of nonnegative integers. The graph of groups $(\Gamma(v), G(v))$ in canonical form, shown in Figure 1, determines a handlebody orbifold $V(\Gamma(v), G(v))$. The orbifold $V(\Gamma(v), G(v))$ is constructed in a similar manner as described in [2]. Note that the quotient of any $\mathbb{Z}_4$-action on $V_g$ is an orbifold of this type, up to homeomorphism.

An explicit combinatorial enumeration of orientation-preserving $\mathbb{Z}_4$-actions on $V_g$, up to equivalence, is given in [2]. In this work we will be interested in examining the orientation-preserving geometric group actions on $V_g$ for the group $\mathbb{Z}_4$. The case for $\mathbb{Z}_p$, when $p$ is an odd prime is considered in [5] and gives a different result. As we will see, there is exactly one equivalence class of $\mathbb{Z}_4$-actions on the handlebody of genus 2.

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This result coincides with [3]. In this paper we will prove the following main theorem:

**Theorem 1.1.** If $\mathbb{Z}_4$ acts on $V_g$, where $g > 0$, then $V_g/\mathbb{Z}_4$ is homeomorphic to $V(\Gamma(v), G(v))$ for some 5-tuple $v = (r, s, t, m, n)$ of nonnegative integers with $r + s + t + m + n > 0$ and $g + 3 = 4(r + s + m) + 3t + 2n$. The number of equivalence classes of $\mathbb{Z}_4$-actions on $V_g$ with this quotient type is $m$ if $r + s + t = 0$, and $m + 1$ if $r + s + t > 0$.

To illustrate the theorem, let $g = 3$. Then the genus equation becomes $6 = 4(r + s + m) + 3t + 2n$ so that $r + s + m$ must equal 0 or 1, and $(r, s, t, m, n)$ is one of $(0, 0, 2, 0, 0)$, $(1, 0, 0, 0, 1)$, $(0, 1, 0, 0, 1)$, or $(0, 0, 1, 1)$. Applying Theorem 1.1 to these four possibilities shows that there are a total of $1 + 1 + 1 + 1 = 4$ equivalence classes of orientation-preserving $\mathbb{Z}_4$-actions on $V_3$. Some results that follow directly from Theorem 1.1:

**Corollary 1.2.** Every $\mathbb{Z}_4$-action on a handlebody of even genus must have an interval of fixed points and at least two fixed points on the boundary of the handlebody.

**Corollary 1.3.** Every $\mathbb{Z}_4$-action that is free on the boundary of the handlebody will have $t = n = 0$ and $g \equiv 1 \pmod{4}$.

### 2. The Main Theorem

The orbifold fundamental group of $V(\Gamma(v), G(v))$ is an extension of $\pi_1(V_g)$ by the group $G$. We may view the fundamental group as a free product $G_1 * G_2 * G_3 * \cdots * G_{r+s+t+m+n}$, where $G_i$ is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}_4 \times \mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2$. We establish notation similar to [2] and denote the generators of the orbifold fundamental group by $\{a_i : 1 \leq i \leq \cdots \}$. 

![Figure 1. (Γ(v), G(v))](image-url)
Consider the set of pairs \(((\Gamma(v), G(v)), \lambda))\, where \(\lambda\) is a finite injective epimorphism from \(\pi_1^{\text{orb}}(V(\Gamma(v), G(v)))\) onto \(\mathbb{Z}_4\). We say \(\lambda\) is finite injective since the kernel of \(\lambda\) is a free group of rank \(g\). We consider only finite injective epimorphisms such that \(\ker(\lambda) = \text{im}(\nu)\) for some orbifold covering \(\nu : V \to V(\Gamma(v), G(v))\). Since \(V\) is a handlebody with torsion free fundamental group, \(V\) is homeomorphic to a handlebody \(V_g\) of genus \(g = 1 - 4\chi(\Gamma(v), G(v))\). Define an equivalence relation on this set of pairs by setting \(((\Gamma(v), G(v)), \lambda) \equiv ((\Gamma(v), G(v)), \lambda')\) if and only if there exists an orbifold homeomorphism \(h : V(\Gamma(v), G(v)) \to V(\Gamma(v), G(v))\) such that \(\lambda' = \lambda \circ h\). We define the set \(\Delta(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))\) to be the set of equivalence classes \(((\Gamma(v), G(v)), \lambda)\) under this relation.

Denote the set of equivalence classes \(\mathcal{E}(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))\) to be the set \(\{[\phi] \mid \phi : \mathbb{Z}_4 \to \text{Homeo}^+(V_g)\text{ and } V_g/\phi \simeq V(\Gamma(v), G(v))\}\). Note that given any \(\mathbb{Z}_4\)-action \(\phi : \mathbb{Z}_4 \to \text{Homeo}^+(V_g)\), it must be the case that for some \(V(\Gamma(v), G(v))\), \([\phi] \in \mathcal{E}(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))\). The following proposition has a similar proof technique as found in [2].

**Proposition 2.1.** Let \(v = (r, s, t, m, n)\). The set \(\mathcal{E}(\mathbb{Z}_4, V_g, V(\Gamma(v))\) is in one-to-one correspondence with the set \(\Delta(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))\) for every \(g > 0\).

To prove the main theorem, we count the number of elements in the delta set and use the one-to-one correspondence given in Proposition 2.1 to give the total count for the set \(\mathcal{E}(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))\). We resort to the following lemma to help count the number of elements in the delta set. The proof is an adaptation from [2].

**Lemma 2.2.** If \(\alpha\) is an automorphism of \(\pi_1^{\text{orb}}(V(\Gamma(v), G(v)))\), then \(\alpha\) is realizable [\(\alpha = h_*\) for some orientation-preserving homeomorphism \(h : V(\Gamma(v), G(v)) \to V(\Gamma(v), G(v))\)] if and only if

\[
\begin{align*}
\alpha(b_j) &= x_j b_j^\sigma c_j x_j^{-1}, \\
\alpha(c_j) &= x_j b_j^\sigma c_j^\sigma x_j^{-1}, \\
\alpha(d_k) &= y_k d_k^\sigma f_k x_k^{-1}, \\
\alpha(e_l) &= w_l e_l^\sigma d_l x_l^{-1}, \\
\alpha(f_l) &= w_l e_l^\sigma f_l x_l^{-1}, \\
\alpha(g_q) &= z_q g_q x_q^{-1},
\end{align*}
\]
for some \( x_j, y_k, u_l, z_q \in \pi_1^{orb}(V(\Gamma(v), G(v))); \sigma \in \sum_\sigma, \tau \in \sum_\tau, \gamma \in \sum_\gamma, \xi \in \sum_\xi, \varepsilon_j, \delta_k, \varepsilon'_j, \delta'_k \in \{+1, -1\}; \text{ and } 0 \leq v_j < 4, 0 \leq w_l < 2.

Note that \( \Sigma_l \) is the permutation group on \( l \) letters.

Note that from [1], a generating set for the automorphisms of the orbifold fundamental group \( \pi_1^{orb}(V(\Gamma(v), G(v))) \) is the set of mappings \( \{\rho_j(x), \lambda_j(x), \mu_j(x), \omega_i, \sigma_i, \phi_i\} \) whose definitions may be found in [1]. The first five maps are realizable. The realizable \( \phi_i \)'s are of the form found in Lemma 2.2 and will be used in the remaining arguments of this paper.

**Lemma 2.3.** Let \( v = (r, s, t, m, n) \) with \( m > 0 \) and let \( \lambda_1, \lambda_2 : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be two finite injective epimorphisms such that there exists a \( j \) with \( \lambda_1(f_j) \) being a generator of \( \mathbb{Z}_4 \) and \( \lambda_2(f_j) \) is not a generator of \( \mathbb{Z}_4 \) for all \( i \). Then \( \lambda_1 \) and \( \lambda_2 \) are not equivalent.

**Proof.** Let \( \lambda_1, \lambda_2 : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be two finite injective epimorphisms such that \( \lambda_1 \) sends \( f_j \) to a generator of \( \mathbb{Z}_4 \) for some \( j \) and \( \lambda_2 \) does not send \( f_i \) to a generator of \( \mathbb{Z}_4 \) for all \( i \). We may assume that \( \lambda_2(f_i) = 0 \) for all \( i \) by composing \( \lambda_2 \) with the realizable automorphism \( \prod_\phi_i \), where \( \phi_i \) sends the generator \( f_i \) to the element \( e_i \), \( f_i \) and leaves all other generators fixed. Note that \( w_i = 0 \) if \( \lambda_2(f_i) = 0 \) and \( w_i = 1 \) if \( \lambda_2(f_i) = 2 \). To show that \( \lambda_1 \) and \( \lambda_2 \) are not equivalent we will consider the element \( f_j \) such that \( \lambda_1(f_j) \) generates \( \mathbb{Z}_4 \). For contradiction, assume that \( \lambda_1 \) is equivalent to \( \lambda_2 \). Then by Lemma 2.2 there exists a realizable automorphism \( \alpha \) such that \( \alpha(f_j) = w^m_i f_i \), where \( w \in \pi_1^{orb}(V(\Gamma(v), G(v))) \) and \( 0 \leq w < 2 \). Hence \( \lambda_1(f_j) = w \lambda_2(e_m) \), where \( \lambda_2(e_m) \) is a multiple of 2, and hence \( \lambda_1(f_j) \) is a multiple of 2. This is impossible since \( \lambda_1(f_j) \) is a generator of \( \mathbb{Z}_4 \). Therefore \( \lambda_1 \) and \( \lambda_2 \) cannot be equivalent, proving the lemma.

**Lemma 2.4.** Let \( v = (r, s, t, m, n) \) and let \( \lambda : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be a finite injective epimorphism. There exists a finite injective epimorphism \( \tilde{\lambda} : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) equivalent to \( \lambda \) such that the following hold:

1. \( \tilde{\lambda}(a_1) = \cdots = \tilde{\lambda}(a_s) = 1. \)
2. \( \tilde{\lambda}(b_1) = \cdots = \tilde{\lambda}(b_s) = 1. \)
3. \( \tilde{\lambda}(c_i) = 0 \) for all \( 1 \leq i \leq s. \)
4. \( \tilde{\lambda}(d_1) = \cdots = \tilde{\lambda}(d_t) = 1. \)
5. \( \tilde{\lambda}(e_1) = \cdots = \tilde{\lambda}(e_m) = 2. \)
6. \( \tilde{\lambda}(f_i) = 1 \) for all \( i \leq k \) some \( 0 \leq k \leq m. \)
7. \( \tilde{\lambda}(g_i) = 0 \) for all \( k < i \leq m. \)
8. \( \tilde{\lambda}(h_i) = \cdots = \tilde{\lambda}(d_m) = 2. \)
Proof. Let $\lambda : \pi_1^{orb}(V(\Gamma(v), G(v))) \to \mathbb{Z}_4$ be a finite injective epimorphism. Properties (5) and (8) must occur since $\lambda$ is finite injective. Property (4) follows by composing $\lambda$ with the realizable automorphism $\prod \phi_i$, where $\phi_i$ sends the generator $d_i$ to the element $d_i^{\phi_i}$ and leaves all other generators fixed. Note that $\varepsilon_i = 1$ if $\lambda(d_i) = 1$ and $\varepsilon_i = -1$ if $\lambda(d_i) = 3$. Property (2) follows by a similar technique. Assuming property (2) holds, property (3) follows by composing $\lambda$ with the realizable automorphism $\prod \phi_i$, where $\phi_i$ sends the generator $c_i$ to the element $b_i^{-\lambda(c_i)}c_i$ and leaves all other generators fixed. To show properties (6) and (7) hold, we may compose $\lambda$ with the realizable automorphism $\prod \phi_i$, where $\phi$ sends the generator $f_i$ to the element $e_i^{\phi_i}f_i$ and leaves all other generators fixed. Note that $z_i = 1$ if $\lambda(f_i) = 2$, $z_i = 2$ if $\lambda(f_i) = 0$ or $\lambda(f_i) = 1$, and $z_i = -1$ if $\lambda(f_i) = 3$. Furthermore, composing $\lambda$ with the realizable automorphisms $\omega_{ij}$ we may interchange $f_i$ as needed so that the first $k$ generators map to 1 and the last $m - k$ generators map to 0. Finally, to prove property (1) we may assume that there exists an element $x \in G_j$ (where $G_j$ is either $\mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}$, or $\mathbb{Z}_2 \times \mathbb{Z}$) such that $\lambda(x) = 1$. Note that we may compose $\lambda$ with a realizable automorphism that sends $x$ to $x^{-1}$ if needed. Now compose $\lambda$ with the realizable automorphism $\prod \rho_i(x^{-\lambda(a_i)+1})$. It may be shown that $(\lambda \circ \alpha)(a_i) = 1$ for all $i$. \hfill $\square$

Proposition 2.5. Let $v = (r, s, t, m, n)$ with $m > 0$ and let $\lambda, \lambda' : \pi_1^{orb}(V(\Gamma(v), G(v))) \to \mathbb{Z}_4$ be two finite injective epimorphisms that satisfy the conclusion of Lemma 2.4, where $\lambda(f_i) = 1$ for all $1 \leq i \leq k$ and $\lambda'(f_i) = 1$ for all $1 \leq i \leq k'$. Then $\lambda$ is equivalent to $\lambda'$ if and only if $k = k'$.

Proof. For a contradiction, assume that $\lambda$ is equivalent to $\lambda'$ and $k \neq k'$. Without loss of generality we may assume that $k > k'$. Hence, $\lambda$ maps at least one more generator $f_i$ to 1 as does $\lambda'$. This would mean that there must exist a realizable automorphism $\alpha$ such that $(\lambda \circ \alpha)(f_{k'+1}) = 0$. By Lemma 2.2, this is impossible. Thus, $k = k'$. For the reverse implication suppose that $k = k'$. Then $\lambda = \lambda'$, proving the proposition. \hfill $\square$

We will now prove the main theorem.

Proof of Theorem 2.2. Define $\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))$ to be the set of equivalence classes $[(\Gamma(v), G(v), \lambda)]$ such that $\lambda(f_i) = 0$ for all $1 \leq i \leq m$. Define $\Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))$ to be the set of equivalence classes $[(\Gamma(v), G(v), \lambda)]$ such that $\lambda(f_i) = 1$ for at least one $i$ such that $1 \leq i \leq m$. By Lemma 2.3, the delta set $\Delta(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))$ is the disjoint union $\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v))) \bigcup \Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))$. Hence, the order of the delta set is the sum of the orders of the two sets $\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))$ and $\Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(v), G(v)))$. Applying
Lemma 2.4 and Proposition 2.5, $|\Delta_1(Z_4, V_g, V(\Gamma(v), G(v)))| = m$ and $|\Delta_0(Z_4, V_g, V(\Gamma(v), G(v)))| = 1$. Hence by Proposition 2.1, the theorem follows.

\[ \square \]

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