Gauge invariance, infrared/collinear singularities and tree level matrix element for $e^+e^- \rightarrow \nu_e \bar{\nu}_e \gamma \gamma$.

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Abstract

One of the necessary steps in constructing high precision option of KKMC was to install the double bremsstrahlung matrix element for the process $e^+e^- \rightarrow \nu_e \bar{\nu}_e$ into the scheme of Coherent Exclusive Exponentiation. The process is also interesting because of gauge cancellation of contributions for photon emission from incoming fermion lines and $t$-channel $W$. The QED U(1) gauge properties require terms of the triple and quartic gauge couplings to be taken into considerations as well. Thanks to expansion starting from the approximation of contact interaction, good example to study the internal structure of the amplitude is available.

In the developed scheme, natural separation of the complete amplitude into gauge invariant parts is straightforward. Each part has well defined physical interpretation, which after partial integration over phase space provides terms: infrared singular, leading log, next-to-leading-log, etc. Contributions related to triple and quartic gauge coupling of $W$ (extracted with the help of expansion around contact $W$-interaction), have been ordered as well. The separation is also helpful, to define extrapolation/reduction procedure of CEEX exponentiation for the $\nu_e$ channel.

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1 Introduction

Higher order radiative corrections are usually necessary to obtain from Standard Model high precision results for phenomenologically important quantities. The techniques of direct calculations, lead to expressions, of often hundreds, thousands or even million’s of terms. These are difficult to control analytically and/or numerically. This is worrisome, because to obtain phenomenologically sound results third order effects are mandatory, see e.g. [1]. This is clearly outside the reach of presently available methods of direct perturbative calculations. There is no doubt, that resummation is necessary of at least some contributions from orders higher than the second one.

In case of electroweak processes at LEP techniques based on exclusive exponentiation of QED effects turned out to be powerful and enabled high precision predictions for a wide game of processes, such as Bhabha scattering, production of heavy bosons $W$ or $Z$ and lepton pairs. The underlying method, originating from pioneering work of Yennie Frautschi and Suura [2] turned out to be realizable [3–7] in practice, thanks to accumulated experience and ever increasing computer power.

One of the necessary elements in approach based on exponentiation is rigorous study of matrix elements obtained from perturbative calculation. In fact it is not enough to calculate predictions at as high order of perturbation expansion as possible, but also to carefully separate results into infrared singular and remaining finite parts. Thanks to the properties of QED, results of explicit perturbative calculations are not necessary to obtain singular and leading terms of every order. These leading parts of amplitudes can be combined with the phase space into the module of the low level Monte Carlo generator or in general into multi-dimensional distribution which can be understood as lowest order of improved perturbative expansion. Later, finite parts of the matrix elements can be added order by order. In case of Monte Carlo algorithms it can be done with the help of the correcting weight, which can be shown to be positive and bound from above.

Details of such a rigorous scheme can be found in refs. [7, 8]. It improves significantly the convergence of perturbative expansion; final states with arbitrary number of photons are present already at the lowest level of expansion. This allows for predictions with realistic experimental cut-offs included. The solution based on separation performed at the amplitude level, is specially useful. It opens the way for easy implementation of all sort of interferences, also convergence of expansion is particularly fast in this case. The underlying exponentiation scheme is called Coherent Exclusive Exponentiation (CEEX).

The following point is of practical importance. In case of exponentiation, already at the lowest order of expansion, configurations with multiple real photons are present. That is why, it may happen, that for particular event, there is more explicit photons in final state, than in expression directly available from standard perturbative expansion. Reduction/extrapolation methods are then necessary. We will not elaborate much on theoretical aspects of this point here. However let us stress that if sufficiently high order of perturbation expansion is available, dependence on the choice of reduction procedure or extrapolation is dropping out. Particularly bad choice may nonetheless degrade convergence of expansion. It is thus of the importance to provide results of perturbative
calculations in a form as convenient for extrapolation procedure as possible. Comparisons of amplitudes calculated at different orders of perturbative expansion can provide a useful hint.

In the present paper, we heavily rely on ref. [9]. We will assume certain level of familiarity of the reader with that reference. Also, let us note that spin amplitudes for the process $e^+e^- \rightarrow \nu_e \bar{\nu}_e \gamma \gamma$ are well defined within the Standard Model and known since long, see e.g. [10]. We could profit in our work from the ready to use computer codes such as [11] available for numerical cross checks of our results.

Our paper is organized as follows. Section 2 is devoted to the case of single bremsstrahlung $e^+e^- \rightarrow \nu_e \bar{\nu}_e \gamma$, some basic elements of spin amplitude techniques, useful in the more complex case of double bremsstrahlung are presented there as well. In section 3, we provide the main results, in particular we explicitly identify gauge invariant parts of the amplitudes. We stress points which will be useful for extrapolation schemes used in CEEX exponentiation as well. We keep our discussion, having in mind future applications in spin-amplitude automated programs. In section 4 we discuss issues related to extrapolation procedure in more detail. Finally, Section 5 closes the paper.

2 Amplitude for one real photon and notation

Let us start with the well-known and straightforward to calculate by any method $O(\alpha)$ spin amplitude for the $e^+e^- \rightarrow \nu_e \bar{\nu}_e \gamma$ single-photon bremsstrahlung process, see fig 1. We will recall it nonetheless here, to define framework for our discussion. We will use conventions of refs. [8,12]. Let us recall here only the most important notations. The four momenta $p_a, p_b, p_c, p_d, k_1$ denote respectively momenta of incoming electron, positron, outcoming neutrino, antineutrino and finally photon. The indices for the spin states for the fermions are denoted respectively as $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ and for photon $\sigma_1$. The photon polarization vector is denoted as $\epsilon_{\sigma_1}$. The gauge transformation in our case reduces to the replacement $\epsilon_{\sigma_1} \rightarrow \epsilon_{\sigma_1} + x k_1$ (with arbitrary coefficient $x$), there will be no external line bosons, and incoming fermions lead to the trivial phases only. With these notations the first-order matrix element\(^1\) obtained from the Feynman diagrams depicted in fig. 1.

\(^1\) $M_{1(I)} (p_{\lambda \sigma_1})$ The subscripts 1 and $\{I\}$ denote respectively, that the amplitudes are of the first order and are included as part of the initial state bremsstrahlung. This spurious notation is however convenient for the reader interested in ref. [9].
can be written in a rather straightforward way:

\[
M_{1(l)}(p_{k1}) = e Q_e \bar{v}(p_b, \lambda_b) M_{(I)}^{bd}(l) \frac{p_{a} + m - k_{1}}{-2 k_{1} p_{a}} \phi_{\sigma_{1}}^{*}(k_{1}) u(p_{a}, \lambda_{a})
+ e Q_e \bar{v}(p_b, \lambda_b) \phi_{\sigma_{1}}^{*}(k_{1}) \frac{-p_{b} + m + k_{1}}{-2 k_{1} p_{b}} M_{(I)}^{ac} u(p_{a}, \lambda_{a})
+ e \bar{v}(p_b, \lambda_b) M_{(I)}^{bd,ac} u(p_{a}, \lambda_{a}) \frac{\phi_{\sigma_{1}}^{*}(k_{1}) \cdot (p_{c} - p_{a} + p_{b} - p_{d})}{(t_a - M_W^2)(t_b - M_W^2)}
+ e \bar{v}(p_b, \lambda_b) g_{\lambda_{c},\lambda_{d}}^{W_{\mu\nu}} \phi_{\sigma_{1}}^{*}(k_{1}) v(p_{d}, \lambda_{d}) \bar{u}(p_{c}, \lambda_{c}) g_{\lambda_{c},\lambda_{a}}^{W_{\mu\nu}} \phi_{\sigma_{1}}^{*}(k_{1}) u(p_{a}, \lambda_{a})
+ e \bar{v}(p_b, \lambda_b) g_{\lambda_{c},\lambda_{d}}^{W_{\mu\nu}} \phi_{\sigma_{1}}^{*}(k_{1}) v(p_{d}, \lambda_{d}) \bar{u}(p_{c}, \lambda_{c}) g_{\lambda_{c},\lambda_{a}}^{W_{\mu\nu}} \phi_{\sigma_{1}}^{*}(k_{1}) u(p_{a}, \lambda_{a})
\]

(1)
or, equivalently:

\[ M_{1_{\{I\}}}^{(p_{k1})} = M^0 + M^1 + M^2 + M^3 \]

\[ M^0 = eQ_e \bar{v}(p_b, \lambda_b) M^{bd}_{\{I\}} \frac{p_a + m - \not{k_1}}{-2k_1 p_a} \varphi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \]

\[ + eQ_e \bar{v}(p_b, \lambda_b) \varphi_{\sigma_1}^*(k_1) - \frac{p_b + m + \not{k_1}}{-2k_1 p_b} M^{ac}_{\{I\}} u(p_a, \lambda_a) \]

\[ M^1 = M^{1'} + M^{1''} \]

\[ M^{1'} = + e \bar{v}(p_b, \lambda_b) M^{bd,ac}_{\{I\}} u(p_a, \lambda_a) e_{\sigma_1}^*(k_1) \cdot (p_c - p_a) \frac{1}{t_a - M^2_W t_b - M^2_W}, \]

\[ M^{1''} = + e \bar{v}(p_b, \lambda_b) M^{bd,ac}_{\{I\}} u(p_a, \lambda_a) e_{\sigma_1}^*(k_1) \cdot (p_b - p_d) \frac{1}{t_a - M^2_W t_b - M^2_W}, \]

\[ M^2 = + e \bar{v}(p_b, \lambda_b) g^{W_{ev}}_{\lambda_b, \lambda_d} \varphi_{\sigma_1}^*(k_1) v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) g^{W_{ev}}_{\lambda_c, \lambda_a} \not{k_1} u(p_a, \lambda_a) \frac{1}{t_a - M^2_W t_b - M^2_W}, \]

\[ M^3 = - e \bar{v}(p_b, \lambda_b) g^{W_{ev}}_{\lambda_b, \lambda_d} \not{k_1} v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) g^{W_{ev}}_{\lambda_c, \lambda_a} \varphi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \frac{1}{t_a - M^2_W t_b - M^2_W}. \]

where, the part of the amplitude, consisting of bosonic couplings \((g^{\lambda f}_{\lambda} \text{ denote coupling constant of } Z \text{ with fermion } f \text{ and handedness } \lambda, \text{ in electric charge units})\), final state fermion spinors and boson propagators reads as

\[ M^{xy}_{\{I\}} = ie^2 (R_Z + R_W) = ie^2 \sum_{B=W,Z} \Pi^\mu (X) G^B_{\epsilon, \mu} (G^B_{f, \nu})_{[cd]} \]

with

\[ G^B_{\epsilon, \mu} = \gamma^\mu \sum_{\lambda=\pm} \frac{1}{2} (1 + \lambda \gamma_5) g^B_{\lambda} \]

\[ (G^B_{f, \nu})_{[cd]} = \bar{u}(p_c, \lambda_c) G^B_{f, \nu} v(p_d, \lambda_d) \]

\[ \Pi^\mu_{B=Z} (X) = X^2 - M^2_Z + i \Gamma_Z X^2 / M_Z \]

\[ \Pi^\mu_{B=W} (X) = \frac{g^\mu}{t - M^2_W} . \]

The final-state spinors are explicitly included, and Fierz transformation is applied for the part of W exchange. The W coupling constant reads

\[ g^{W_{ev}}_{\lambda_e, \lambda_d} = \frac{1}{\sqrt{2 \sin \theta_W}} \delta^\lambda_e_\lambda_d \delta^\lambda_c_+ . \]

Only for the W contribution, the superscripts \(xy\) in \(M_{\{I\}}\) have the meaning, they define the momentum transfer in the W propagator \(\Pi^\mu (X)\): for \(xy = ac\) the transfer \(^2\) is

\(^2\)Transfers can be expressed also as \(t_a = (p_a - k_1 - p_d)^2\) and \(t_b = (p_a - k_1 - p_c)^2\), this make difference if extrapolation procedures are used for the configurations off mass shell where \(p_a + p_b \neq p_c + p_d + k_1\), otherwise \(M^{1'} = M^{1''}\) of course.
\[ t_a = (p_a - p_c)^2, \quad \text{for } bd \text{ it is } t_b = (p_b - p_d)^2. \] If both are explicitly marked, then the expression

\[ M_{\{I\}}^{bd,ac} = i e^2 G_{\nu, \mu}^{W} (G^{W, \nu})_{[cd]} \]  

(6)
is used. For that parts of formula (2) \( W \) propagators are explicitly given. The notations \( R_Z \) and \( R_W \) will be used later, see respectively formula (24) and (27).

Let us start now to rewrite expression (2). It is straightforward to notice that the first term \( M^0 \) can be split into soft IR parts proportional to \( (\not p + m) \) and non-IR parts proportional to \( k_1 \). The non-IR parts are individually gauge invariant by construction. The soft part of \( M^0 \), with \( Z \) couplings only, is gauge invariant as well.

Employing the completeness relations of eq. (A14) from ref. [8] we obtain the different form of (2):

\[
M_{\{I\}}^{(p_{k_1})} = -i e Q_e \frac{2k_1 p_a}{2k_1 p_b} \sum_{\rho} \mathfrak{B} \left[ \frac{p_{k_1 p_a}}{[\rho_{\sigma_1}]} \right]_{[cd]} U \left[ \frac{p_{k_1 p_a}}{[\rho_{\sigma_1}]} \right] + i e Q_e \frac{2k_1 p_b}{2k_1 p_b} \sum_{\rho} \mathfrak{B} \left[ \frac{p_{k_1 p_b}}{[\rho_{\sigma_1}]} \right]_{[cd]} V \left[ \frac{p_{k_1 p_b}}{[\rho_{\sigma_1}]} \right]_{[cd]}
\]

\[
+ \sum_{\rho} \mathfrak{B} \left[ \frac{p_{k_1 p_a}}{[\rho_{\sigma_1}]} \right]_{[cd]} U \left[ \frac{p_{k_1 p_b}}{[\rho_{\sigma_1}]} \right] - \frac{e Q_e}{2k_1 p_b} \sum_{\rho} \mathfrak{B} \left[ \frac{k_1 p_b}{[\rho_{\sigma_1}]} \right]_{[cd]} V \left[ \frac{k_1 p_b}{[\rho_{\sigma_1}]} \right]_{[cd]}
\]

(7)

The terms \( M^1 \) to \( M^3 \) correspond to the last three lines\(^3\) of eq. (11). These contributions are also IR-finite. In the next step let us remove the sum in the first two terms thanks to the diagonality of \( U \) and \( V \) (ref. [8]). The matrices \( \mathfrak{B} \) are also defined in this reference. We obtain

\[
M_{\{I\}}^{(p_{k_1})} = \mathfrak{s}_{\{I\}}^{(k_1)} \mathfrak{B} \left[ \frac{p_{k_1}}{[\lambda]} \right] + \left( r_{\{I\}}^{B^0} + M^1 \right) + \left( r_{\{I\}}^{B^0} + M^1 \right) + \left( M^2 + M^3 \right)
\]

\[
r_{\{I\}}^{B^0} = - \frac{e Q_e}{2k_1 p_a} \sum_{\rho} \mathfrak{B} \left[ \frac{p_{k_1 p_a}}{[\rho_{\sigma_1}]} \right]_{[cd]} U \left[ \frac{p_{k_1 p_a}}{[\rho_{\sigma_1}]} \right]
\]

\[
r_{\{I\}}^{B^0} = \frac{e Q_e}{2k_1 p_b} \sum_{\rho} \mathfrak{B} \left[ \frac{k_1 p_b}{[\rho_{\sigma_1}]} \right]_{[cd]} V \left[ \frac{k_1 p_b}{[\rho_{\sigma_1}]} \right]
\]

\[
r_{\{I\}}^{B^0} = \frac{e Q_e}{2k_1 p_a} \sum_{\rho} \mathfrak{B} \left[ \frac{k_1 p_a}{[\rho_{\sigma_1}]} \right]_{[cd]},
\]

\[
r_{\{I\}}^{A^0} = \mathfrak{s}_{\{I\}}^{(k_1)} = \frac{b_{\sigma_1}^{(k_1, p_a)}}{2k_1 p_a} + \frac{e Q_e}{2k_1 p_b} b_{\sigma_1}^{(k_1, p_b)} \]

(8)

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\(^3\)The term \( M^1 + M^2 + M^3 \) originates from the \( WW\gamma \) vertex

\[ -ie \left[ g_{\mu\nu} (p - q)_\rho + g_{\rho\nu} (q - r)_\mu + g_{\mu\rho} (r - p)_\nu \right] \]

where all momenta are outgoing, and indices on outgoing lines are paired with momenta as \( p^\mu, q^\nu, r^\rho \); \( M^1 \) originates from the term where \( g^{\mu\nu} \) connects the \( e^- - \nu_e, e^+ - \bar{\nu}_e \) fermion lines.
The soft part is now clearly separated from the remaining non-IR part, used in the CEEX exponentiation for construction of $\mathcal{O}(\alpha)$ corrections. We have ordered the expression, with the help of expansion similar to the contact interaction for $W$ propagator as well. In $\mathfrak{B}^r_{[1]}$ we use an auxiliary fixed transfer $t_0$, independent of the place where the photon is attached to the fermion line. In fact $t_0$ is arbitrary and the choice $t_0 = 0$ could be used as well\(^4\). With the help of $\mathfrak{B}$ we provide the residual contribution calculated as a difference of the expression calculated with the true $t$-transfers ($t_a$ or $t_b$) and the auxiliary $t_0$ one. Note that $\mathfrak{B} = \mathfrak{B} + \mathfrak{B}$. Each of the contribution to the sum given in the first equation of \(\Box\) is independently gauge invariant.

We can see, that it was possible to separate the complete spin amplitude for the process $e^+ e^- \rightarrow \overline{\nu}_e \nu_e \gamma$ into six individually QED gauge invariant parts. This conclusion is rather straightforward to check, replacing photon polarization vector with its four-momentum. Each of the obtained parts has rather well defined physical interpretation. It is also easy to verify that the gauge invariance of each part can be easily preserved to the case of the extrapolation, when because of additional photons, condition $p_a + p_b = p_c + p_d + k_1$ is not valid. Let us elaborate on this point a bit more.

- The first term of the soft photon type $\mathfrak{s}_s^{[1]}(k_1) \hat{\mathcal{B}}^r_{[1]}$ is gauge invariant thanks to invariance of the standard ISR soft factor $\mathfrak{s}_s^{[1]}(k_1)$. It is also of the universal form, identical for the diagrams with $s$-channel $Z$-exchange as well as $t$-channel $W$.
- The next two terms $(r_{\{1\}}^{B'} + \mathcal{M}^1)$ and $(r_{\{1\}}^{B} + \mathcal{M}^1)$ originate only from diagrams of $t$-channel $W$ exchange. For the gauge invariance to hold, the $t$-channel transfers have to be $t_a = (p_a - p_c)^2$, $t_b = (p_a - k_1 - p_c)^2$ for the first term (and $t_a = (p_b - k_1 - p_d)^2$, $t_b = (p_b - p_d)^2$ for the second one\(^5\).
- The consecutive two terms $r_{\{1\}}^{A'}(p_{k_1})$ and $r_{\{1\}}^{A''}(p_{k_1})$ are again of the same universal form as for any $s$-channel process and gauge invariant by construction. These are also the terms which lead to leading-log (but not infrared) singular terms, after phase space integration. In the corresponding Feynman diagrams the photon polarization vector and its momentum stand side by side.
- Finally, for the last expression $(\mathcal{M}^2 + \mathcal{M}^3)$ to be gauge invariant it is enough that for the two terms choices for transfers $t_a$, $t_b$ are identical; the same reduction procedure\(^6\) is used.

**Simplest case of** $e^+ e^- \rightarrow \nu_\mu \overline{\nu}_\mu$ 

Let us finish this Section with the discussion of the $Z$ exchange part of the amplitude \(\Box\) in simple language of spinors and four-vectors. This part of the amplitude is important

\(^4\)The choice is nonetheless important from the point of view of efficiency, it affects size of corrections in CEEX expansion scheme. Condition that $\lim_{k_1 \rightarrow 0} \ t_{a/b} = t_0$ is desirable.

\(^5\)It is interesting to realize that only part of the diagram is involved in the cancellation, namely fermion and boson lines, from which emission of the photon takes place. This observation will become useful in case of double bremsstrahlung amplitudes.

\(^6\)Mechanism of gauge cancellation is fulfilled already at the level of this part of bosonic interaction alone. This observation will be useful in study of double bremsstrahlung amplitudes.
because it will define the framework for our main results collected in Section 3.

\[ M_{1(I)}^{Z} (p_{k_1}) = e Q_e \, \bar{v}(p_b, \lambda_b) \, M_{(I)} \frac{\not{p}_a + m - \not{k}_1}{2k_1 p_a} \, \phi^{*}_{\sigma_1}(k_1) \, u(p_a, \lambda_a) \]
\[ + e Q_e \, \bar{v}(p_b, \lambda_b) \, \phi^{*}_{\sigma_1}(k_1) \frac{\not{p}_b + m + \not{k}_1}{2k_1 p_b} \, M_{(I)} \, u(p_a, \lambda_a) \]  

(9)

the superscript \(ac\) or \(bd\) can be dropped in \(M_{(I)}\) as it does not depend on the \(t\) transfer of \(W\) propagator, then \(M_{(I)} = R_Z\) of \([24]\), see later in the text.

The gauge invariance of the two sub-parts proportional to \(k_1\) is straightforward to see, because the expression \(k_1 \, \phi^{*}_{\sigma_1}(k_1)\) alone is gauge invariant thanks to \(k_1 k_1 = 0\). These parts of the amplitude do not contribute to infrared singularity, however do contribute to the big logarithm related to collinear singularity (once amplitudes are squared and integrated over the phase space). That is why, we will call these parts of the amplitude as infrared finite collinear singular. The remaining part of the amplitude:

\[ M_{1(I)}^{Z - ir} (p_{k_1}) = e Q_e \, \bar{v}(p_b, \lambda_b) \, M_{(I)} \frac{\not{p}_a + m}{2k_1 p_a} \, \phi^{*}_{\sigma_1}(k_1) \, u(p_a, \lambda_a) \]
\[ + e Q_e \, \bar{v}(p_b, \lambda_b) \, \phi^{*}_{\sigma_1}(k_1) \frac{\not{p}_b + m}{2k_1 p_b} \, M_{(I)}^{ac} \, u(p_a, \lambda_a) \]  

(10)

factorizes thanks to the orthogonality for Dirac spinors:

\[ \not{p}_a + m = \sum_{\lambda} u(p_a, \lambda) \bar{u}(p_a, \lambda) \]
\[ - \not{p}_b + m = - \sum_{\lambda} v(p_b, \lambda) \bar{v}(p_b, \lambda). \]  

(11)

into gauge invariant soft photon factor and Born amplitude:

\[ M_{1(I)}^{Z - ir} (p_{k_1}) = s_{\sigma_1}^{(I)}(k_1) \, \bar{v}(p_b, \lambda_b) \, M_{(I)}^{bd} \, u(p_a, \lambda_a); \]

\[ s_{\sigma_1}^{(I)}(k_1) = \frac{e Q_e}{2k_1 p_b} \, \bar{u}(p_b, \lambda) \, \phi^{*}_{\sigma_1}(k_1) \, u(p_a, \lambda_a) + \frac{e Q_e}{2k_1 p_b} \, \bar{v}(p_b, \lambda_b) \, \phi^{*}_{\sigma_1}(k_1) \, v(p_b, \lambda) \]
\[ = \frac{-e Q_e}{2k_1 p_a} b_{\sigma_1}(k_1, p_a) \delta_{\lambda \lambda_0} + \frac{e Q_e}{2k_1 p_b} b_{\sigma_1}(k_1, p_b) \delta_{\lambda \lambda_0} \]  

(12)

The gauge invariance, takes place in case of \(Z\) exchange (and also \(W\) exchange if approximation of contact interaction is used) because then \(M_{(I)} = M_{(I)}^{bd} = M_{(I)}^{ac}\), also two parts of \(s_{\sigma_1}^{(I)}(k_1)\) are diagonal respectively in indices \(\lambda \lambda_{a,b}\). The Born level spin amplitude factorizes out, and the gauge dependent soft factors for emission from electron and positron lines, can be summed to gauge invariant \(s_{\sigma_1}^{(I)}(k_1)\). For the explicit definition of \(b_{\sigma_1}(k_1, p_b)\), see e.g. formula (231) of ref. [8]. This part of the amplitude is infrared singular. We will use the factorization of the soft factors explained here, also later in the paper.
Note that in case of single photon $Z$ exchange amplitude, we got only three gauge invariant parts: infrared-singular, and two others contributing to collinear-singular terms (after phase space integration). The residual (contributing only non enhanced terms after phase space integration) terms are absent. Note however, that they are present in case of the $W$ exchange diagrams.

Finally let us comment that similar pattern of amplitude separation into gauge invariant parts can be observed for $W^\pm \rightarrow l\nu\gamma$ in [13].

### 3 Double bremsstrahlung

In the present section we will study the amplitudes for double bremsstrahlung in $e^+e^- \rightarrow \nu\bar{\nu}e$ production process. There are two classes of diagrams in this case, the first one with $Z$ boson exchange in $s$ channel and the second one with $t$ channel $W$ exchange. Similarly as in previous section and single bremsstrahlung we will look if gauge invariant parts of the complete amplitude can be defined. We will be also interested, if this can be done in a semi-automatic way, directly from the Feynman rules.

The presentation of the complete amplitudes is rather difficult because of their length. To avoid lengthy formulas, we will start with largely incomplete set of diagrams, nonetheless sufficient to localize some gauge invariant group of terms. Once localized, it will be hidden under symbol $L^b_a$ and, to the remaining part contributions from the next diagrams will be added. Again gauge invariant group of terms will be searched for. This procedure will be repeated until the complete list of diagrams of our process will be exhausted. The choice for the first diagram in this procedure is motivated by its particular (unique) form, later it is motivated by the form of the gauge dependent rest remaining from the previous step. For short hand notations we will use extended subscripts and superscripts for $L^b_a$. For example we will use symbol $L^{k_1,k_2}_a(n)$, to denote contribution for the diagram with: first photon of momentum $k_1$ and second $k_2$, attached to incoming $e^-$ line. The number $n$ in bracket (if present) will denote that it is only a part of the contribution from the particular Feynman diagram (or diagrams). There will be often bar over this number to point that the particular part is gauge dependent.

Let us start our iteration with diagrams involving double fermion propagator, that is diagrams where two photons are attached either to incoming electron or to incoming positron. These are the only diagrams with $k_1 \cdot k_2$ term in fermion propagators. Our first aim will be thus to localize the parts which are gauge invariant by themselves, and include this $k_1 \cdot k_2$ term. Let us consider the eight diagrams with the photon lines attached either to electron or positron line, see fig 2. Explicitly, we will write down the part of the amplitude corresponding to the incoming electron line only. In fact the diagrams with $Z$ and $W$ exchange are quite similar:
Figure 2: Double emission from electron

\[ L_{e^-}^{k_1, k_2} = (eQ_e)^2 \bar{\nu}(p_b, \lambda_b) R_B \left( \frac{\hat{p}_a + m - \hat{k}_1 - \hat{k}_2}{-2k_1 p_a - 2k_2 p_a - 2k_1 k_2} \varphi_{\sigma_2}^*(k_2) \frac{\hat{p}_a + m - \hat{k}_1}{-2k_1 p_a} \varphi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) 
+ \frac{\hat{p}_a + m - \hat{k}_1 - \hat{k}_2}{-2k_1 p_a - 2k_2 p_a - 2k_1 k_2} \varphi_{\sigma_1}^*(k_1) \frac{\hat{p}_a + m - \hat{k}_2}{-2k_2 p_a} \varphi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right) \]

Expression \( R_B \) describing final state neutrino interaction and \( Z \) or \( W \) exchange, will be defined later, respectively in formulas (24) and (27). We can separate formula (13) into the following parts

\[ L_{e^-}^{k_1, k_2} = L_{e^-}^{k_1, k_2}(1) + L_{e^-}^{k_1, k_2}(2) + L_{e^-}^{k_1, k_2}(3) + L_{e^-}^{k_1, k_2}(4), \]

where:

\[ L_{e^-}^{k_1, k_2}(1) = (eQ_e)^2 \bar{\nu}(p_b, \lambda_b) R_B \left( \frac{-\hat{k}_2}{-2k_1 p_a - 2k_2 p_a - 2k_1 k_2} \varphi_{\sigma_2}^*(k_2) \frac{-\hat{k}_1}{-2k_1 p_a} \varphi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) 
+ \frac{-\hat{k}_1}{-2k_1 p_a - 2k_2 p_a - 2k_1 k_2} \varphi_{\sigma_1}^*(k_1) \frac{-\hat{k}_2}{-2k_2 p_a} \varphi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right) \]

is gauge invariant by construction, thanks to terms \( \hat{k}_1 \varphi_{\sigma_1}^*(k_1) \) and \( \hat{k}_2 \varphi_{\sigma_2}^*(k_2) \) which are gauge invariant by themselves. This is similar to the case of single bremsstrahlung. The
second part

\[ L_{e^{-,k_2}}^{k_1,k_2}(2) = (eQ_e)^2 \bar{v}(p_b, \lambda_b) R_B \left( \frac{\not{p}_a + m - \not{k}_2}{-2k_1p_a - 2k_2p_a - 2k_1k_2 - 2k_2p_a} \phi_{\sigma_1}^*(k_1) \frac{-\not{k}_2}{-2k_2p_a} \phi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right. \]

\[ + \left. \frac{-\not{k}_2}{-2k_1p_a - 2k_2p_a - 2k_1k_2 - 2k_2p_a} \phi_{\sigma_1}^*(k_1) \frac{-\not{p}_a + m}{-2k_2p_a} \phi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right) \]

\[ - \not{k}_2 \left( \frac{1}{-2k_1p_a - 2k_2p_a - 2k_1k_2 - 2k_2p_a} \right) \phi_{\sigma_2}^*(k_2) \frac{-\not{p}_a + m}{-2k_1p_a} \phi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \]

\[ + (1 \leftrightarrow 2) \]

(16)

is also gauge invariant, but it need to be checked by direct calculation. This contribution, like the previous one, is free of infrared singularity. In definition of \( L_{e^{-,k_2}}^{k_1,k_2}(2) \) we had to introduce subtraction; terms proportional to \( \not{k}_2 \left( \frac{1}{-2k_2p_a} \right) \) and \( \left( \not{p}_a + m \right) \left( \frac{1}{-2k_2p_a} \right) \)

The subtraction terms are added back to (14) as formulas (17) and (18), but with the opposite sign of course. It is important to realize, that these subtraction terms are defined uniquely by the \( Z \) exchange part of the amplitude for single photon emission\(^7\), see formula (9). It has to be multiplied by the soft photon factor for the second photon and separated into infrared finite and infrared singular parts, as explained in section 2.

\[ L_{e^{-,k_2}}^{k_1,k_2}(3) = (eQ_e)^2 \bar{v}(p_b, \lambda_b) R_B \left( \frac{\not{p}_a + m}{-2k_1p_a - 2k_2p_a} \phi_{\sigma_2}^*(k_2) \frac{-\not{p}_a + m}{-2k_2p_a} \phi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \right. \]

\[ + \left. \frac{-\not{p}_a + m}{-2k_1p_a - 2k_2p_a} \phi_{\sigma_1}^*(k_1) \frac{-\not{p}_a + m}{-2k_2p_a} \phi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right) \]

\[ = (eQ_e)^2 \bar{v}(p_b, \lambda_b) R_B \left( \frac{\not{p}_a + m}{-2k_2p_a} \phi_{\sigma_2}^*(k_2) \frac{-\not{p}_a + m}{-2k_1p_a} \phi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \right. \]

\[ + \left. \frac{-\not{p}_a + m}{-2k_2p_a} \phi_{\sigma_1}^*(k_1) \frac{-\not{p}_a + m}{-2k_1p_a} \phi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right) \]

(17)

The next term \( L_{e^{-,k_2}}^{k_1,k_2}(4) \) is, also free of \( k_1k_2 \) and its numerator is linear in the photon momentum

\[ L_{e^{-,k_2}}^{k_1,k_2}(4) = (eQ_e)^2 \bar{v}(p_b, \lambda_b) R_B \left( \frac{-\not{k}_2}{-2k_2p_a} \phi_{\sigma_2}^*(k_2) \frac{-\not{p}_a + m}{-2k_1p_a} \phi_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \right. \]

\[ + \left. \frac{-\not{k}_1}{-2k_1p_a} \phi_{\sigma_1}^*(k_1) \frac{-\not{p}_a + m}{-2k_2p_a} \phi_{\sigma_2}^*(k_2) u(p_a, \lambda_a) \right) \]

(18)

\(^7\)The subtraction term for the \( W \) exchange differs only by the replacement \( R_B = R_W \).
The complete contribution from diagrams of fig. 2, formula (14), is not gauge invariant. The last two terms are gauge dependent, but are also relatively short. The first one (17) has a structure of Born amplitude multiplied by soft photon factors, the second one (18), has a structure of soft photon emission for one of the two photons only, see discussion at the end of Section 2.

Once we have completed the diagrams with two photon lines attached to one fermion line, let us turn to another group of diagrams, where one of the photons is attached to electron and another one to positron line, see Fig. 3. Note that for subgroup of diagrams with $Z$ boson exchange only, these are the last contributing diagrams:

$$L^k_{k_1, k_2} = (eQ_e)^2 \left( \bar{v}(p_b, \lambda_b) \, \sigma^*(k_2) \left[ \begin{array}{c} \frac{k_2}{-2k_2p_b} \mathcal{R}_B \frac{p_a + m - k_1}{-2k_1p_a} \, \sigma^*_1(k_1) \\ \bar{v}(p_b, \lambda_b) \, \sigma^*_1(k_1) \left[ \begin{array}{c} \frac{k_1}{-2k_1p_b} \mathcal{R}_B \frac{p_a + m - k_2}{-2k_2p_a} \, \sigma^*_2(k_2) \\ \bar{v}(p_b, \lambda_b) \, \sigma^*_2(k_2) \end{array} \right] u(p_a, \lambda_a) \right] \right)$$

(19)

Figure 3: Single emission from electron and positron

As in the previous case the expression for $L^k_{k_1, k_2}$ can be easily separated into parts,

$$L^k_{k_1, k_2} = L^k_{k_1, k_2} + L^k_{k_1, k_2} + L^k_{k_1, k_2}$$

(20)

The first part $L^k_{k_1, k_2}$ is by construction gauge invariant, it is also the only part from this group of diagrams with numerator proportional both to the momenta of $k_1$ and $k_2$:

$$L^k_{k_1, k_2} = (eQ_e)^2 \left( \bar{v}(p_b, \lambda_b) \, \sigma^*(k_2) \left[ \begin{array}{c} \frac{k_2}{-2k_2p_b} \mathcal{R}_B \frac{p_a + m - k_1}{-2k_1p_a} \, \sigma^*_1(k_1) \\ \bar{v}(p_b, \lambda_b) \, \sigma^*_1(k_1) \left[ \begin{array}{c} \frac{k_1}{-2k_1p_b} \mathcal{R}_B \frac{p_a + m - k_2}{-2k_2p_a} \, \sigma^*_2(k_2) \\ \bar{v}(p_b, \lambda_b) \, \sigma^*_2(k_2) \end{array} \right] u(p_a, \lambda_a) \right] \right)$$

(21)
The second term \( I_{e^-,e^+}^{k_1,k_2}(2) \) has two contributions, of numerators linear either in \( k_1 \) or \( k_2 \), it reads:

\[
I_{e^-,e^+}^{k_1,k_2}(2) = (eQ_e)^2 \left( \bar{v}(p_b, \lambda_b) \, \gamma_{\sigma_2}(k_2) \frac{-p_b + m}{-2k_2p_b} \, R_B \frac{-k_1}{-2k_1p_a} \, \phi_{\sigma_1}(k_1) \, u(p_a, \lambda_a) \right)
+ \bar{v}(p_b, \lambda_b) \, \phi_{\sigma_1}(k_1) \frac{k_1}{-2k_1p_b} \, R_B \frac{p_a + m}{-2k_2p_a} \, \phi_{\sigma_2}(k_2) \, u(p_a, \lambda_a)
+ \bar{v}(p_b, \lambda_b) \, \phi_{\sigma_2}(k_2) \frac{k_2}{-2k_2p_b} \, R_B \frac{p_a + m}{-2k_2p_a} \, \phi_{\sigma_1}(k_1) \, u(p_a, \lambda_a)
+ \bar{v}(p_b, \lambda_b) \, \phi_{\sigma_1}(k_1) \frac{-p_b + m}{-2k_1p_a} \, R_B \frac{k_2}{-2k_2p_a} \, \phi_{\sigma_2}(k_2) \, u(p_a, \lambda_a)
\]  

Finally the third one \( I_{e^-,e^+}^{k_1,k_2}(3) \) is free of the \( k_1 \) or \( k_2 \) in the numerator.

\[
I_{e^-,e^+}^{k_1,k_2}(3) = (eQ_e)^2 \left( \bar{v}(p_b, \lambda_b) \, \gamma_{\sigma_2}(k_2) \frac{-p_b + m}{-2k_2p_b} \, R_B \frac{p_a + m}{-2k_2p_a} \, \phi_{\sigma_1}(k_1) \, u(p_a, \lambda_a) \right)
+ \bar{v}(p_b, \lambda_b) \, \phi_{\sigma_1}(k_1) \frac{-p_b + m}{-2k_1p_b} \, R_B \frac{p_a + m}{-2k_2p_a} \, \phi_{\sigma_2}(k_2) \, u(p_a, \lambda_a)
\]  

To complete the sub-set of diagrams for double bremsstrahlung from initial state, the contribution of the double emission from positron line should be added. We will omit explicit formulas here, the explicit expressions for \( I_{e^+,e^-}^{k_1,k_2}(1), I_{e^+,e^-}^{k_1,k_2}(2), I_{e^+,e^-}^{k_1,k_2}(3), I_{e^+,e^-}^{k_1,k_2}(4) \) can be obtained from \( I_{e^-,e^+}^{k_1,k_2}(1), I_{e^-,e^+}^{k_1,k_2}(2), I_{e^-,e^+}^{k_1,k_2}(3), I_{e^-,e^+}^{k_1,k_2}(4) \) by analogy or explicit calculation.

### 3.1 Diagrams with \( Z \) exchange

Before going to the more complex case of \( W \) exchange, where complications due to \( t \) dependence of \( W \) propagator occur, let us concentrate on the simpler one, \( Z \) exchange alone. The diagrams discussed so far, represent then the complete gauge invariant amplitude for the process \( e^+e^- \to \nu_\mu \bar{\nu}_\mu \gamma\gamma \). In such a sub-group of diagrams (for the process \( e^+e^- \to \nu_\mu \bar{\nu}_\mu \gamma\gamma \)) symbol \( R_{B=Z} \) represents always

\[
R_Z = \left( \gamma^\mu (v^1 + a\gamma^5) \right) \alpha_\beta \left( \bar{u}(p_\mu, \lambda_\mu) \gamma_\mu (v^1 + a\gamma^5) u(p_\lambda, \lambda_\lambda) \right) BW_Z((p_\mu + p_\lambda)^2)
\]  

which is a constant algebraic expression, independent on photon momenta and identical for all diagrams listed. The \( Z \) boson propagator \( BW_Z((p_\mu + p_\lambda)^2) \) depends on the invariant mass of the outcoming neutrinos only. The bi-spinor indices of \( \gamma^\mu, \gamma^\mu\gamma^5 \) matrices which enter into the matrix products of formulae such as \([13]\) to \([23]\) are explicitly given as \( \alpha_\beta \). The complete amplitude reads:

\[
\mathcal{M} = I_{e^-,e^+}^{k_1,k_2} + I_{e^+,e^-}^{k_1,k_2} + I_{e^-,e^+}^{k_1,k_2}
= I_{e^-,e^+}^{k_1,k_2}(1) + I_{e^-,e^+}^{k_1,k_2}(2) + I_{e^-,e^+}^{k_1,k_2}(3) + I_{e^-,e^+}^{k_1,k_2}(4) + I_{e^+,e^-}^{k_1,k_2}(1) + I_{e^+,e^-}^{k_1,k_2}(2) + I_{e^+,e^-}^{k_1,k_2}(3) + I_{e^+,e^-}^{k_1,k_2}(4) + I_{e^+,e^-}^{k_1,k_2}(1) + I_{e^+,e^-}^{k_1,k_2}(2) + I_{e^+,e^-}^{k_1,k_2}(3).
\]
Where \( L_{k_1,k_2}^{k_1,k_2} \) is given by formula (13) (or if separated into parts by (14)) and \( L_{e^-,e^+}^{k_1,k_2} \) by (19) (or (20)). For \( L_{e^-,e^+}^{k_1,k_2} \) the expressions of \( L_{e^-,e^+}^{k_1,k_2} \) can be used, with appropriate changes of signs momenta etc.

The formula for the complete spin amplitude (\( Z \) exchange only) can be easily reordered into consecutive contributions \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots \), each gauge invariant, and each expressed as group of \( L \)'s in the bracket or just individual \( L \):

\[
\mathcal{M} = M_{2(I)}^{Z}(p_{k_1 k_2})
= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7
= L_{e^-,e^+}^{k_1,k_2}(1) + L_{e^-,e^+}^{k_1,k_2}(2) + L_{e^+,e^-}^{k_1,k_2}(1) + L_{e^+,e^-}^{k_1,k_2}(2) + L_{e^-,e^+}^{k_1,k_2}(1)
+ (L_{e^-}^{k_1,k_2}(4) + L_{e^+}^{k_1,k_2}(4) + L_{e^-}^{k_1,k_2}(4)) + (L_{e^-}^{k_1,k_2}(3) + L_{e^+}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(3))
\]

(26)

As one can see, the sum of terms \( \mathcal{M}_1 \) to \( \mathcal{M}_5 \), contributing to \( \beta_2^2 \) of the CEEX exponentiation scheme (these terms are not infrared singular at all), is gauge invariant and clearly separated from the rest. It can be even further sliced into five parts, each individually gauge invariant. The last two terms, \( \mathcal{M}_6 \) and \( \mathcal{M}_7 \) corresponds respectively to \( \beta_1^1 \) and \( \beta_0^0 \) (multiplied by one or two soft photon factors) and can be obtained from lower order of perturbation expansion. It is rather straightforward to see, that the term \( \mathcal{M}_7 \) consist of Born level amplitude multiplied by soft factors corresponding to emission of two photons.

The term \( \mathcal{M}_6 \) can be seen as consisting of two factors; for one of the photons soft factor, and for the other one term of \( \beta_1^1 \). To see it better it is convenient to order the expression accordingly to terms proportional either to \( \vec{k}_1 \) or \( \vec{k}_2 \).

Note also, that for the each of the parts to be gauge invariant, it is not necessary that four-momentum conservation is fulfilled. That is why, the separation is easily adaptable to extrapolation procedure as used in KKMC.

This completes our discussion of results for \( s \)-channel exchange of \( Z \). Let us now turn to the contributions related to the \( t \)-channel \( W \)-exchange.

### 3.2 Diagrams with \( W \) exchange

First, let us note that all formulae presented so far are valid for the diagrams involving \( W \) exchange as well. The difference is that instead of formula (24) for \( \mathcal{R}_B \) one should use:

\[
\mathcal{R}_W = \left( \gamma_\mu (1 - \gamma^5) v(p_d, \lambda_d) \right)_\alpha \left( \bar{u}(p_c, \lambda_c) \gamma_\mu (1 - \gamma^5) \right)_\beta B W_W(t)
\]

(27)

The spinorial form of this expression is universal, and as in the case of \( Z \) exchange, in all places the same expression is to be used. The difference lies in \( t \) dependence of \( W \) propagator, the transfer will depend on the way how the photon lines are attached to the fermion ones. Nonetheless in some groups of terms gauge cancellation do occur anyway in the same way as before. If we recall the part of the \( W \) exchange amplitude, written in
analogy to (26) as:

\[ \mathcal{M}_W^A = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \bar{\mathcal{M}}_6 + \bar{\mathcal{M}}_7 
= L_{e^-}^{k_1,k_2}(1) + L_{e^+}^{k_1,k_2}(2) + L_{e^-}^{k_1,k_2}(1) + L_{e^-}^{k_1,k_2}(2) + L_{e^-}^{k_1,k_2}(1) 
+ (L_{e^-}^{k_1,k_2}(4) + L_{e^+}^{k_1,k_2}(4) + L_{e^-}^{k_1,k_2}(2)) + ((L_{e^-}^{k_1,k_2}(3) + L_{e^+}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(3)) \] (28)

then the parts \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4 \) and \( \mathcal{M}_5 \) remain gauge invariant. Only the last two terms will need contributions from diagrams with triple and quartic gauge boson couplings for the gauge invariance to hold. To visualize this point the bar sign is now placed over \( \bar{\mathcal{M}}_6 \) and \( \bar{\mathcal{M}}_7 \). Note, that as already pointed in the previous subsection these are the contributions, which could be obtained (extra soft photon factors are only needed) from the results of the calculation at lower perturbative order if the complications due to variation of \( W \) exchange transfers were not taken into account.

Figure 4: Single and double emission from \( W \)

Figure 5: Four boson coupling and coupling for unphysical \( \chi \) field.
Let us continue with the second part of our discussion now. The two (now gauge dependent), contributions $\mathcal{N}_6$ and $\mathcal{N}_7$ will be first completed with diagrams from the left-hand side of fig. Note that these diagrams are the last ones with photon line attached to incoming electron/positron, thus the last ones contributing collinear and/or soft singularities. The contribution to the scattering amplitude from these new diagrams reads:

$$L_{e^{-},W}^{k_{1},k_{2}}=(eQ_{e})^2BW_{W}((p_{c}+k_{2}-p_{a})^2)BW_{W}((p_{c}+k_{2}+k_{1}-p_{a})^2)$$

$$\left(\bar{v}(p_{b},\lambda_{b})(1-\gamma^{5})\gamma^{\mu}v(p_{d},\lambda_{d})\right)$$

$$\left[g_{\mu\nu}(p-q)_{\rho}+g_{\nu\rho}(q-k_{1})_{\mu}+g_{\mu\rho}(k_{1}-p)_{\nu}\right](\epsilon_{\sigma_{1}}^{*}(k_{1}))^{\rho}$$

$$\bar{u}(p_{c},\lambda_{c})(1-\gamma^{5})\gamma^{\nu}\frac{p_{a}+m-k_{2}}{-2k_{2}p_{a}}g_{\sigma_{2}}(k_{2})u(p_{a},\lambda_{a})$$

$$+(1\leftrightarrow2)$$

Here $p = p_{d} - p_{b} = -(p_{e} - p_{a} + k_{2})$ and $q = p_{e} - p_{a} + k_{2} = -(p_{d} - p_{b} + k_{1})$. Similarly one can write contribution $L_{e^{-},W}^{k_{1},k_{2}}$ for the two diagrams with emission from positron and $W$, we will omit the corresponding formulae. As before, we separate $L_{e^{-},W}^{k_{1},k_{2}} = L_{e^{-},W}^{k_{1},k_{2}}(k^{0}) + L_{e^{-},W}^{k_{1},k_{2}}(k^{1})$ into parts, $(k^{1})$ marks contribution where only $k_{2}$ is taken from the fermionic propagator and $(k^{0})$ marks the rest. The explicit formulas are:

$$L_{e^{-},W}^{k_{1},k_{2}}(k^{0})=(eQ_{e})^2BW_{W}((p_{c}+k_{2}-p_{a})^2)BW_{W}((p_{c}+k_{2}+k_{1}-p_{a})^2)$$

$$\left(\bar{v}(p_{b},\lambda_{b})(1-\gamma^{5})\gamma^{\mu}v(p_{d},\lambda_{d})\right)$$

$$\left[g_{\mu\nu}(p-q)_{\rho}+g_{\nu\rho}(q-k_{1})_{\mu}+g_{\mu\rho}(k_{1}-p)_{\nu}\right](\epsilon_{\sigma_{1}}^{*}(k_{1}))^{\rho}$$

$$\bar{u}(p_{c},\lambda_{c})(1-\gamma^{5})\gamma^{\nu}\frac{p_{a}+m-k_{2}}{-2k_{2}p_{a}}g_{\sigma_{2}}(k_{2})u(p_{a},\lambda_{a})$$

$$+(1\leftrightarrow2)$$

and

$$L_{e^{-},W}^{k_{1},k_{2}}(k^{1})=(eQ_{e})^2BW_{W}((p_{c}+k_{2}-p_{a})^2)BW_{W}((p_{c}+k_{2}+k_{1}-p_{a})^2)$$

$$\left(\bar{v}(p_{b},\lambda_{b})(1-\gamma^{5})\gamma^{\mu}v(p_{d},\lambda_{d})\right)$$

$$\left[g_{\mu\nu}(p-q)_{\rho}+g_{\nu\rho}(q-k_{1})_{\mu}+g_{\mu\rho}(k_{1}-p)_{\nu}\right](\epsilon_{\sigma_{1}}^{*}(k_{1}))^{\rho}$$

$$\bar{u}(p_{c},\lambda_{c})(1-\gamma^{5})\gamma^{\nu}\frac{-k_{2}}{-2k_{2}p_{a}}g_{\sigma_{2}}(k_{2})u(p_{a},\lambda_{a})$$

$$+(1\leftrightarrow2)$$

Let us start with the second one, which can be easily transformed (with help of Dirac
equation) into

\[ L_{e^{-},W}^{k_{1},k_{2}}(k^{1}) = (eQ_{e})^{2} BW_{W} ((p_{c} + k_{2} - p_{a})^{2}) BW_{W} ((p_{c} + k_{2} + k_{1} - p_{a})^{2}) \]

\[ \left( \bar{v}(p_{b}, \lambda_{b}) (1 - \gamma^{5})\gamma^{\mu} \right. v(p_{d}, \lambda_{d}) \]
\[ \left[ g_{\mu\nu}(p_{d} - p_{b} - p_{c} + p_{a} - k_{2})_{\nu} + g_{\nu\rho}(-2k_{1})_{\mu} + g_{\mu\rho}(2k_{1} - p_{a})_{\nu} \right] (\epsilon_{\sigma_{1}}(k_{1}))^{\rho} \quad (32) \]
\[ \left( \bar{u}(p_{c}, \lambda_{c}) (1 - \gamma^{5})\gamma^{\mu} \right. \frac{-k_{2}}{-2k_{2}p_{a}} \phi_{\sigma_{2}}^{*}(k_{2}) u(p_{a}, \lambda_{a}) \]
\[ + (1 \leftrightarrow 2) \]

This contribution can be separated even further:

\[ L_{e^{-},W}^{k_{1},k_{2}}(k^{1}) = L_{e^{-},W}^{k_{1},k_{2}}(1) + L_{e^{-},W}^{k_{1},k_{2}}(2) + L_{e^{-},W}^{k_{1},k_{2}}(3) \quad (33) \]

where

\[ L_{e^{-},W}^{k_{1},k_{2}}(1) = (eQ_{e})^{2} BW_{W} ((p_{c} + k_{2} - p_{a})^{2}) BW_{W} ((p_{c} + k_{2} + k_{1} - p_{a})^{2}) \]

\[ \left( \bar{v}(p_{b}, \lambda_{b}) (1 - \gamma^{5})\gamma^{\mu} \right. v(p_{d}, \lambda_{d}) \]
\[ \left( \bar{u}(p_{c}, \lambda_{c}) (1 - \gamma^{5})\gamma^{\mu} \right. \frac{-k_{2}}{-2k_{2}p_{a}} \phi_{\sigma_{2}}^{*}(k_{2}) u(p_{a}, \lambda_{a}) \]
\[ (p_{d} - p_{b} - p_{c} + p_{a} - k_{2}) \cdot \epsilon_{\sigma_{1}}(k_{1}) \]
\[ + (1 \leftrightarrow 2) \]

(34)

and the second term

\[ L_{e^{-},W}^{k_{1},k_{2}}(2) = (eQ_{e})^{2} BW_{W} ((p_{c} + k_{2} - p_{a})^{2}) BW_{W} ((p_{c} + k_{2} + k_{1} - p_{a})^{2}) \]

\[ \left( \bar{v}(p_{b}, \lambda_{b}) (1 - \gamma^{5})\gamma^{\mu} \right. v(p_{d}, \lambda_{d}) \]
\[ 2 \left[ - (\epsilon_{\sigma_{1}}(k_{1}))_{\nu}(k_{1})_{\mu} + (\epsilon_{\sigma_{1}}(k_{1}))_{\mu}(k_{1})_{\nu} \right] \]
\[ \left( \bar{u}(p_{c}, \lambda_{c}) (1 - \gamma^{5})\gamma^{\mu} \right. \frac{-k_{2}}{-2k_{2}p_{a}} \phi_{\sigma_{2}}^{*}(k_{2}) u(p_{a}, \lambda_{a}) \]
\[ + (1 \leftrightarrow 2) \]

(35)

is gauge invariant by itself. The third one is explicitly less divergent in collinear configuration:

\[ L_{e^{-},W}^{k_{1},k_{2}}(3) = - (eQ_{e})^{2} BW_{W} ((p_{c} + k_{2} - p_{a})^{2}) BW_{W} ((p_{c} + k_{2} + k_{1} - p_{a})^{2}) \]

\[ \left( \bar{v}(p_{b}, \lambda_{b}) (1 - \gamma^{5})\phi_{\sigma_{1}}(k_{1}) \right. v(p_{d}, \lambda_{d}) \]
\[ \left. \bar{u}(p_{c}, \lambda_{c}) (1 - \gamma^{5})\phi_{a} \right. \frac{-k_{2}}{-2k_{2}p_{a}} \phi_{\sigma_{2}}^{*}(k_{2}) u(p_{a}, \lambda_{a}) \]
\[ + (1 \leftrightarrow 2) \]

(36)
Let us now turn to the other part of the amplitude, and again present it in a form of the sum:

\[
L_{e^-W}^{k_1,k_2}(5) = L_{e^-W}^{k_1,k_2}(4) + L_{e^-W}^{k_1,k_2}(5) + L_{e^-W}^{k_1,k_2}(6)
\]

where first term reads

\[
L_{e^-W}^{k_1,k_2}(4) = (eQ_c)^2 BW_W\left((p_c + k_2 - p_a)^2\right) BW_W\left((p_c + k_2 + k_1 - p_a)^2\right)
\]

\[
\left(\bar{v}(p_b, \lambda_b) (1 - \gamma^5)\gamma^\mu v(p_d, \lambda_d) \right. \\
\left. 2\left[-(\epsilon^*_{\sigma_1}(k_1))_\mu (k_1)_\mu - (p_b)_\mu + (\epsilon^*_{\sigma_1}(k_1))_{(k_1)_\mu}\right]ight) \\
\left. \bar{u}(p_c, \lambda_c) (1 - \gamma^5)\gamma^\nu \frac{\not{p}_a + m}{-2k_2p_a} \not{g}_{\sigma_2}(k_2) u(p_a, \lambda_a)\right)
\]

+ (1 \leftrightarrow 2),

the second one

\[
L_{e^-W}^{k_1,k_2}(5) = (eQ_c)^2 BW_W\left((p_c + k_2 - p_a)^2\right) BW_W\left((p_c + k_2 + k_1 - p_a)^2\right)
\]

\[
\left(\bar{v}(p_b, \lambda_b) (1 - \gamma^5)\gamma^\mu v(p_d, \lambda_d) \right. \\
\left. 2\left[-(\epsilon^*_{\sigma_1}(k_1))_\mu (k_1)_\mu - (p_b)_\mu + (\epsilon^*_{\sigma_1}(k_1))_{(k_1)_\mu}\right]ight) \\
\left. \bar{u}(p_c, \lambda_c) (1 - \gamma^5)\gamma^\nu \frac{\not{p}_a + m}{-2k_2p_a} \not{g}_{\sigma_2}(k_2) u(p_a, \lambda_a)\right)
\]

+ (1 \leftrightarrow 2),

and the third

\[
L_{e^-W}^{k_1,k_2}(6) = (eQ_c)^2 BW_W\left((p_c + k_2 - p_a)^2\right) BW_W\left((p_c + k_2 + k_1 - p_a)^2\right)
\]

\[
\left(\bar{v}(p_b, \lambda_b) (1 - \gamma^5)\gamma^\mu v(p_d, \lambda_d) \right. \\
\left. 2\left[-(\epsilon^*_{\sigma_1}(k_1))_\mu (k_1)_\mu + (\epsilon^*_{\sigma_1}(k_1))_{(k_1)_\mu}\right]ight) \\
\left. \bar{u}(p_c, \lambda_c) (1 - \gamma^5)\gamma^\nu \frac{\not{p}_a + m}{-2k_2p_a} \not{g}_{\sigma_2}(k_2) u(p_a, \lambda_a)\right)
\]

+ (1 \leftrightarrow 2).

The last two terms can be modified further, and some terms neglected. One can check that these terms contribute at the level of \(\frac{m}{\sqrt{s}}\) only, and that is why, we will exclude them from explicit considerations. After these simplifications, we finally obtain gauge invariant by itself

\[
L_{e^-W}^{k_1,k_2}(5) = (eQ_c)^2 BW_W\left((p_c + k_2 - p_a)^2\right) BW_W\left((p_c + k_2 + k_1 - p_a)^2\right)
\]

\[
\left(\bar{v}(p_b, \lambda_b) (1 - \gamma^5)\gamma^\mu v(p_d, \lambda_d) \right. \\
\left. 2\left[-(\epsilon^*_{\sigma_1}(k_1))_\mu (k_1)_\mu + (\epsilon^*_{\sigma_1}(k_1))_{(k_1)_\mu}\right]ight) \\
\left. \bar{u}(p_c, \lambda_c) (1 - \gamma^5)\gamma^\nu \frac{\not{p}_a + m}{-2k_2p_a} \not{g}_{\sigma_2}(k_2) u(p_a, \lambda_a)\right)
\]

+ (1 \leftrightarrow 2).
and now explicitly less divergent in collinear configuration term

\[ L_{\text{coll}}^{k_1,k_2}(5) = (eQ_e)^2 BW_W\left((p_c + k_2 - p_a)^2\right) BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \]

\[
\left( \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \right. \\
\left. \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \right) \]

\[ (p - q) \cdot \epsilon^*_{\sigma_1}(k_1) (p' - q') \cdot (\epsilon^*_\sigma_2(k_2)) + (1 \leftrightarrow 2) \]
amplitude take a form

\[
L_{W_{1,2}}(2) = (eQ_e)^2 BW_W \left( (p_c - p_a)^2 \right) BW_W \left( (p_c + k_1 - p_a)^2 \right) \\
BW_W \left( (p_c + k_1 + k_2 - p_a)^2 \right) \\
\left( \bar{v}(p_b, \lambda_b) \left( 1 - \gamma^5 \right) \gamma^\mu v(p_d, \lambda_d) \right. \\
\left. \bar{u}(p_c, \lambda_c) \left( 1 - \gamma^5 \right) \gamma^\nu u(p_a, \lambda_a) \right) \\
2 \left[ - (\epsilon^*_{\sigma_1}(k_1))_\nu(k_1)_\mu + (k_1)_\nu(\epsilon^*_{\sigma_1}(k_1))_\mu \right] \\
(p' - q') \cdot (\epsilon^*_{\sigma_2}(k_2)) \\
+ (1 \leftrightarrow 2),
\]

and

\[
L_{W_{1,2}}(3) = (eQ_e)^2 BW_W \left( (p_c - p_a)^2 \right) BW_W \left( (p_c + k_1 - p_a)^2 \right) \\
BW_W \left( (p_c + k_1 + k_2 - p_a)^2 \right) \\
\left( \bar{v}(p_b, \lambda_b) \left( 1 - \gamma^5 \right) \gamma^\mu v(p_d, \lambda_d) \right. \\
\left. \bar{u}(p_c, \lambda_c) \left( 1 - \gamma^5 \right) \gamma^\nu u(p_a, \lambda_a) \right) \\
2 \left[ - (\epsilon^*_{\sigma_2}(k_2))_\nu(k_2)_\mu + (k_2)_\nu(\epsilon^*_{\sigma_2}(k_2))_\mu \right] \\
(p - q) \cdot (\epsilon^*_{\sigma_1}(k_1)) \\
+ (1 \leftrightarrow 2).
\]

The two last terms \ref{46} \ref{47} are partially gauge independent, respectively for the polarization vector of the first and second photon. The remaining fully gauge dependent parts of the amplitude read:

\[
L_{W_{1,2}}(4) = -(eQ_e)^2 BW_W \left( (p_c - p_a)^2 \right) BW_W \left( (p_c + k_1 - p_a)^2 \right) \\
BW_W \left( (p_c + k_1 + k_2 - p_a)^2 \right) \\
\left( \bar{v}(p_b, \lambda_b) \left( 1 - \gamma^5 \right) k_2 \cdot v(p_d, \lambda_d) \right. \\
\left. \bar{u}(p_c, \lambda_c) \left( 1 - \gamma^5 \right) \epsilon^*_{\sigma_1}(k_1) u(p_a, \lambda_a) \right) \\
(p' - q') \cdot (\epsilon^*_{\sigma_2}(k_2)) \\
+ (1 \leftrightarrow 2),
\]

and

\[
L_{W_{1,2}}(5) = (eQ_e)^2 BW_W \left( (p_c - p_a)^2 \right) BW_W \left( (p_c + k_1 - p_a)^2 \right) \\
BW_W \left( (p_c + k_1 + k_2 - p_a)^2 \right) \\
\left( \bar{v}(p_b, \lambda_b) \left( 1 - \gamma^5 \right) \epsilon^*_{\sigma_2}(k_2) \cdot v(p_d, \lambda_d) \right. \\
\left. \bar{u}(p_c, \lambda_c) \left( 1 - \gamma^5 \right) k_1 \cdot u(p_a, \lambda_a) \right) \\
(p - q) \cdot (\epsilon^*_{\sigma_1}(k_1)) \\
+ (1 \leftrightarrow 2).
\]
Finally

\[ L_{W, W}^{k_1, k_2}(6) = (eQ_e)^2 BW_W ((p_c - p_a)^2) BW_W ((p_c + k_1 - p_a)^2) \]

\[ BW_W ((p_c + k_1 + k_2 - p_a)^2) \]

\[ \left( \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \right) \]

\[ + \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \]

\[ + (1 \leftrightarrow 2). \]

At the last step, let us turn to contributions from diagrams presented in fig. 5. The diagram with contribution from quatric gauge coupling reads:

\[ L_{W, W}^{k_1, k_2} = (eQ_e)^2 BW_W ((p_c - p_a)^2) BW_W ((p_c + k_2 + k_1 - p_a)^2) \]

\[ \left( \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \right) \]

\[ + \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \]

\[ + (1 \leftrightarrow 2). \]

It is convenient to write it as a sum of two parts

\[ L_{W, W}^{k_1, k_2} = L_{W, W}^{k_1, k_2}(1) + L_{W, W}^{k_1, k_2}(2) \]

where

\[ L_{W, W}^{k_1, k_2}(1) = 2(eQ_e)^2 BW_W ((p_c - p_a)^2) BW_W ((p_c + k_2 + k_1 - p_a)^2) \]

\[ e^{\ast}_{\sigma_1}(k_1) \cdot e^{\ast}_{\sigma_2}(k_2) \]

\[ \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \]

and

\[ L_{W, W}^{k_1, k_2}(2) = (eQ_e)^2 BW_W ((p_c - p_a)^2) BW_W ((p_c + k_2 + k_1 - p_a)^2) \]

\[ \left( \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \right) \]

\[ + \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \]

\[ + (1 \leftrightarrow 2). \]

Contribution from the diagram involving internal \( \chi \) line reads:

\[ L_{W, \chi}^{k_1, k_2} = (eQ_e)^2 M_W^2 BW_W ((p_c - p_a)^2) BW_W ((p_c + k_2 + k_1 - p_a)^2) \]

\[ \left( \bar{v}(p_b, \lambda_b) (1 - \gamma^5) \gamma^\mu v(p_d, \lambda_d) \right) \]

\[ \bar{u}(p_c, \lambda_c) (1 - \gamma^5) \gamma^\mu u(p_a, \lambda_a) \]

\[ + (1 \leftrightarrow 2). \]
This closes the list of all diagrams entering the complete spin amplitude for the process $e^+e^-\rightarrow \nu\bar{\nu}e^+\gamma\gamma$. The contributing terms were obtained from the Feynman rules, and were grouped on the basis of rather straightforward rules; gauge symmetry and nature of singularities in infrared and collinear limits (phase space integration was not necessary).

The complete gauge invariant part of the spin amplitude of $W$ exchange can be now written as:

$$\mathcal{M}_W = \mathcal{M}_W^A + \mathcal{M}_W^b,$$

where $\mathcal{M}_W^A$ (technically identical to the amplitude of $Z$ exchange) was given by formula (28), and new part, specific to the $W$ bosonic interactions, reads:

$$\mathcal{M}_W^B = L_{e^+W}^{k_1,k_2}(1) + L_{e^-,W}^{k_1,k_2}(2) + L_{e^+,W}^{k_1,k_2}(3) + L_{e^-,W}^{k_1,k_2}(4) + L_{e^+,W}^{k_1,k_2}(5) + L_{e^+,W}^{k_1,k_2}(6) +$$

$$L_{e^+,W}^{k_1,k_2}(1) + L_{e^-,W}^{k_1,k_2}(2) + L_{e^-,W}^{k_1,k_2}(3) + L_{e^-,W}^{k_1,k_2}(4) + L_{e^-,W}^{k_1,k_2}(5) + L_{e^-,W}^{k_1,k_2}(6) +$$

$$L_{W,W}^{k_1,k_2}(1) + L_{W,W}^{k_1,k_2}(2) + L_{W,W}^{k_1,k_2}(3) + L_{W,W}^{k_1,k_2}(4) + L_{W,W}^{k_1,k_2}(5) + L_{W,W}^{k_1,k_2}(6) +$$

$$L_{W,W}^{k_1,k_2}(1) + L_{W,W}^{k_1,k_2}(2) + L_{W,W}^{k_1,k_2}(3) +$$

We can now write the complete spin amplitude, of $W$ interactions, as a sum of gauge invariant parts:

$$\mathcal{M}_W = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 +$$

$$\mathcal{M}_8 + \mathcal{M}_9 + \mathcal{M}_{10} + \mathcal{M}_{11},$$

where

$$\mathcal{M}_1 = L_{e^-}(1)$$
$$\mathcal{M}_2 = L_{e^-}(2)$$
$$\mathcal{M}_3 = L_{e^-}(1)$$
$$\mathcal{M}_4 = L_{e^+}(2)$$
$$\mathcal{M}_5 = L_{e^-}(1)$$
$$\mathcal{M}_6 = L_{e^-}^{k_1,k_2}(4) + L_{e^+}^{k_1,k_2}(4) + L_{e^-}^{k_1,k_2}(2) + L_{e^-}^{k_1,k_2}(4) + L_{e^-}^{k_1,k_2}(1) + L_{e^+}^{k_1,k_2}(1)$$
$$\mathcal{M}_7 = L_{e^-}^{k_1,k_2}(3) + L_{e^+}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(4) + L_{e^+}^{k_1,k_2}(4) + L_{e^-}^{k_1,k_2}(4) + L_{e^-}^{k_1,k_2}(4)$$
$$\mathcal{M}_8 = L_{e^-}^{k_1,k_2}(2)$$
$$\mathcal{M}_9 = L_{e^+}^{k_1,k_2}(2)$$
$$\mathcal{M}_{10} = L_{e^-}^{k_1,k_2}(5) + L_{e^+}^{k_1,k_2}(5) + L_{W,W}^{k_1,k_2}(2) + L_{W,W}^{k_1,k_2}(3)$$
$$\mathcal{M}_{11} = L_{e^-}^{k_1,k_2}(3) + L_{e^+}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(3) + L_{e^-}^{k_1,k_2}(6) + L_{e^+}^{k_1,k_2}(6) + L_{W,W}^{k_1,k_2}(4) + L_{W,W}^{k_1,k_2}(5) + L_{W,W}^{k_1,k_2}(6) +$$

$$L_{W,W}^{k_1,k_2}(2) + L_{W,W}^{k_1,k_2}(3) +$$

(59)
The matching of the $\bar{L}_b^a(\bar{n})$ terms into gauge invariant parts $\mathcal{M}_i$ of the amplitude is straightforward and based on the type of singularities present/absent in the particular group. Each of the listed below contributions $\mathcal{M}_1-\mathcal{M}_{11}$ can be given some physical interpretation. In some cases, appearance of such parts may seem rather unexpected. In brackets we provide symbols such as (IA), they denote the name of variables used in KKMC [7] Monte Carlo, as keys for the parts of the amplitude:

- $\mathcal{M}_1$ (IA), contribution of the infrared non-singular contributions of double emission from electron line, part with straightforward gauge cancellation within the terms originating from diagram of two photons attached to the same incoming electron line.

- $\mathcal{M}_2$ (IV2) contribution of the infrared non-singular contributions of double emission from electron line, part with non-straightforward gauge cancellation within the terms originating from diagram of two photons attached to the same incoming electron line. Part of the diagram contribution had to be subtracted; more precisely expression without $k_1 k_2$ product in electron propagator. This subtraction term is recupered in $\mathcal{M}_6$ and $\mathcal{M}_7$.

- $\mathcal{M}_3$ (IA), $\mathcal{M}_4$ (IV1) as in previous two cases but for emission from positron line.

- $\mathcal{M}_5$ (I8) infrared non-singular contributions of single emission from electron- and another single emission from positron line. This contribution is gauge invariant by construction.

- $\mathcal{M}_6$ (I9X), (I9Y), (I9Z), (I9T), part of the amplitude with infrared factor for one photon, and for the second one infrared non-singular gauge invariant contribution. For the diagrams with $W$ exchange contribution from diagram with photon emission from $W$ need to be taken. For the gauge cancellation to hold, relation between $t$-channel transfers in $W$ propagators and momenta multiplying photon polarization vector need to be fulfilled. Nonetheless certain freedom in choice is left. It was useful in construction of extrapolation procedures\(^8\).

- $\mathcal{M}_7$ (IVI) part of the amplitude with infrared factors for both photons. For the diagrams with $W$ exchange contribution from diagrams with single and double emission of photons from $W$ needs to be taken, also part of the diagram with quartic gauge coupling was needed here.

- $\mathcal{M}_8$ (I71), (I72) part of the amplitude with infrared non-singular contribution of emission from electron for one photon and for another one part of emission from $W$ which is self gauge-conserving.

- $\mathcal{M}_9$ (I71), (I72) as in previous case but for emission from positron.

---

\(^8\)Identical condition, also originating directly from Ward-identities, need to be preserved in $\mathcal{M}_{10}$ and the similar one in $\mathcal{M}_{11}$. 
- $\mathcal{M}_{10}$ (I9s1), (I9s2) part of the amplitude with infrared factor for one photon and for another one part of emission from $W$ which is self gauge-conserving part.

- $\mathcal{M}_{11}$ (I9), (I9B), (I10) all remaining parts, they turn out to be free of singularities both in collinear and soft limits.

- Let us comment that in the limit $M_W \to \infty$ all contributions from $\mathcal{M}_6$ to $\mathcal{M}_{11}$ disappear. In this limit amplitudes for $s$-channel $Z$ exchange and $t$-channel $W$ nearly coincide. The only remaining difference is the coupling constants and hard interaction part of the amplitude given respectively by formulas (24) and (27). This is an extension of similar observation of reference [14] instrumental in construction of extrapolation procedures of ref. [9], to the case beyond real photon interactions with fermions only.

- Let us point that in many places we have used separation of the $WW\gamma$ vertex into three parts; (i) the one with the $g_{\mu\nu}$ tensor along line connecting fermion lines, (ii) the part internally preserving gauge symmetry, (iii) the remaining part which we often could reduce significantly with the help of Dirac equation (for the fermion lines connected with the $WW\gamma$ vertex by $W$ propagator.

- Finally let us note, that the above separation into gauge invariant parts can be continued even further. For example it is rather easy to separate $\mathcal{M}_6$ into four parts. For each, emissions of individual photons are attributed either to electron or positron line.

Let us note that we have not exploited to the end the properties of $\mathcal{M}_{11}$. It was not interesting from the point of view of our main purpose, which is implementation of the matrix element to the environment of Coherent Exclusive Exponentiation. Also in case of $\mathcal{M}_{11}$, contrary to the cases $\mathcal{M}_1$ to $\mathcal{M}_{10}$, similarities with first order results could not be seen. This is rather natural, as for example quatric gauge couplings are absent in first order. In this case hint on pattern of constructing amplitudes of even higher order using iteration techniques could not be found. To this end, discussion of the amplitudes of triple photon emission would be needed. If conclusive, it would point to solutions beyond next-to-leading-log approximation, thus beyond imminent interest of the present paper.

The gauge invariance was not the only element of the criterium which was used here to split amplitude into gauge invariant parts. Equally important was that the two main sources of the radiation, incoming beams, form the unambiguous frame with respect to which, photon energy and the angles of photons with respect to fermions could be defined. That is why there was no need to make any reference to the regulators, singular terms could be localized already at the amplitude level and in fully differential manner, with no need to partially integrate phase space. The expansion in the contact interaction for $W$ propagator enabled to place the gauge cancellation effects of emission from $t$-channel $W$ within the frame of ISR radiation. Also relation between amplitude for double and single photon emission had to be exploited to close down window for ambiguities. Once these assumptions and properties were exploited, the solution seem to be unique, up to
may be grouping or further splitting of the obtained parts. Confirmation, whether this is accidental property and observation which hold for the particular cases and up to the second order only, may require calculation to be extended to at least third order.

4 Some points on extrapolation

Let us summarize here some specific issues related to extrapolation procedure of CEEX scheme described in detail in ref. [8] for purely s-channel hard process. One of the important property of perturbation expansion, rearranged to improve convergence into exclusive exponentiation, is that parts of the amplitudes need to be appropriately shifted between the rearranged orders of expansion. We will concentrate on issues related to real bremsstrahlung only. In particular parts of the higher order terms (directly calculated in a standard way) which are already available at lower level of CEEX perturbation expansion need to be localized and subtracted in a clear way. Only remaining residual parts, called $\beta^0, \beta^1, \beta^2$ etc. [2] will be indeed the term of the given newly rearranged order. The use of $\beta$ functions is unambiguous if sufficiently high order of standard perturbative calculation is available to calculate matrix element, for the configuration with all real photons. However it is not always the case, practical solutions for exponentiation require definition of methods how to calculate matrix elements for the kinematical configuration with large number of real photons, using results of first (or second) order of perturbation expansion only.

There are several rules which extrapolation procedure must fulfill. Already the lowest order must include all terms with the highest power of infrared singularity and for all kinematical configurations of arbitrary number of real photons and in a fully exclusive manner. Then, first order provide all terms with next to highest power of infrared singularity, etc. Let us stress that reduction/extrapolation procedure of exponentiation offers some freedom of choice. This freedom can be used to further improve convergence of perturbation expansion. The best guidance is of course comparison with result of even higher order of expansion to minimize their contribution. If such results are not available, higher order leading log results can be used instead. Finally, let us stress that if sufficiently high order of perturbation expansion is available, dependence on particular choice of extrapolation drops out and unique result, identical to the one of direct perturbation expansion without any reordering, will be obtained. Unfortunately this is not expected to be the case in foreseeable future.

In case of diagrams with Z exchange the question of choice of extrapolation procedure is straightforward. Inspection of first order (formula 2) and second order (formula 26) amplitudes points to the following solution: the terms $\mathcal{M}_1$ to $\mathcal{M}_5$ of (26) should contribute to $\beta^2$, whereas the last two terms $\mathcal{M}_6$ and $\mathcal{M}_7$ can be directly obtained from the lower order. The $\mathcal{M}_6$ can be obtained from $\beta^1$ by multiplication with the soft photon factor for the other photon. The $\beta^1$ can be identified as this part of $\mathcal{M}^0$ (see formula 2) which is proportional to $k_1$. The $\mathcal{M}_7$ can be obtained from the lowest order Born spin amplitude $\beta^0$ of Z exchange by multiplication with two soft photon factors, for each of
the bremsstrahlung photons, exactly as it should be in exponentiation prescription. The factorization properties can be easily seen if rather trivial manipulation on Dirac algebra is performed.

In case of diagrams with $W$ exchange, the question of choice of extrapolation procedure is slightly more complex, because of dependence of the transfers used in $W$ propagators on photon momenta. That is also the reason why triple and quartic gauge couplings need to be included in the considerations. If the kinematical configurations of more than two explicit hard photons are taken, then the transfers calculated for $W$ propagators can be defined in several ways. Our choice used at present in KKMC [7] is inspired by leading log considerations. For lowest order ($\beta^0$) and if there was no addition photons, transfer $t_0$, can be calculated either as (i) $t_0 = (p_c - p_a)^2$ or (ii) $t_0 = (p_d - p_b)^2$. If there is a photon collinear to $p_b$ the first choice is closer to the transfer dominating higher order (i.e. single bremsstrahlung) spin amplitude. In general the choice (i) is thus more favored if total four-momentum carried out by the sum of all photons is pointing rather into direction of $p_b$ than $p_a$. Otherwise the second choice is better. In case of single (or double) photon emission the choice how the transfers are calculated is basically the same. The only difference is, that the photons explicitly included in the particular contribution to $\beta^1$ or $\beta^2$, should contribute to the sum of photons mentioned above. The choice which pair of four momenta ($p_a, p_c$ or $p_b, p_d$) is used in calculation for transfers, must be taken in calculation of algebraic expressions originating from direct $W$ interaction with photons, for gauge invariance to hold.

5 Summary

We have presented complete results for the spin amplitudes of $e^+e^- \rightarrow \nu_e \bar{\nu}_e \gamma \gamma$ process. Using gauge transformation, as well as expansion with respect to contact approximation for $W$ exchange, we were able to identify gauge invariant parts of amplitudes of the well defined physical properties. In particular the parts proportional to inverse of photon energies (i.e. corresponding to infrared singularity), remaining parts proportional to inverse of the product of fermion and photon momenta (i.e. of the type of collinear singularity) as well as residual finite parts could be grouped together in a rather natural way. By comparison with amplitudes for the diagrams involving $s$-channel $Z$ exchange we were able to observe certain pattern of universality for many of those terms.

Let us stress, that some of the results presented here, could be expected from the properties of U(1) gauge symmetry and the corresponding Ward identities. They are known already since a long time, also in the context of QCD. The purpose of the present paper is mainly technical. We illustrate the scheme of step-by-step gauge cancellations and how they work for spin amplitude techniques. Finally, we show, how they helped to develop extrapolation procedures used in the KK Monte Carlo in case of neutrino channel.

Let us point to the similar observation [15] as ours, in case of the single loop corrections, again for $e^+e^- \rightarrow \nu_e \bar{\nu}_e$ process and also for $e^+e^- \rightarrow \nu_e \bar{\nu}_e H$. 

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