Discrete Linear Geometry on Non-square Grid

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Abstract. We define the algebraic discrete geometry to hexagonal grid system on a plane. Since a hexagon is an element for tiling on a plane, hexagons are suitable as elements of discrete objects. For the description of linear objects in a discrete space, algebraic discrete geometry provides a unified treatment employing double Diophantus equations. Furthermore, we develop an algorithm for the polygonalisation of discrete objects on the hexagonal grid system.

1 Introduction

This paper deals with algebraic discrete geometry on hexagonal grid systems [1–8]. In the following, we first derive a set of inequalities for the parameters of a Euclidean linear manifold from sample points in the hexagonal grid system and an optimisation criterion with respect to this set of constraints for the recognition of the Euclidean line on the hexagonal grid system. Second, using this optimisation problem, we prove uniqueness and ambiguity theorems for the reconstruction of a Euclidean line on the hexagonal grid system. Finally, we develop a polygonalisation algorithm for the boundary of a discrete shape from a sequence of hexagonal grids.

Algebraic discrete geometry [9–13] allows us to describe linear manifolds, which are collections of unit elements, in two-dimensional discrete space as double Diophantus inequalities. For the reconstruction of a smooth boundary from sample points, polygonalisation on a plane is the first step. Following polygonalisation, we estimate the geometric features of a figure, such as the normal vector at each point on the boundary, and the length and area of planar shapes. There are basically three types of model for the expression of a linear manifold in the grid space, supercover, standard, and naive models [13]. We deal with the supercover model for the hexgonal grid system on a plane.

A hexagon on a plane has both advantages and disadvantages as an elemental cell of discrete objects [1–6]. The area encircled by a hexagon is closer to the area encircled by a circle than is the area encircled by a square. Although the dual lattice of a square grid is a square grid, the dual grid of a hexagonal grid...
is a triangular grid \[14\]. Therefore, for multi-resolution analysis, we are required to prepare two types of grid. From the application in omni-directional imaging systems in computer vision and robot vision \[15,16\], the spherical camera model has recently been of wide concern. Although the square grid yields uniform tiling on a plane, it is not suitable as a grid element on the sphere. The hexagonal grid system provides a uniform grid on both a sphere \[15,17–19\] and a plane \[1–6\].

In refs. \[20–22\] a linear-programming-based method for the recognition of linear manifolds for the square grid system has been proposed. This method is based on the mathematical property that a point set determines a system of linear inequalities for the parameters of a linear manifold, and the recognition process for a linear manifold is converted to the computation of the feasible region for this system of inequalities. The other class for the recognition of a linear manifold is based on the binary relation among local configurations in $3 \times 3$ pixel regions, since the geometrical properties of the discrete linear manifold are characterised by a sequence of $3 \times 3$ pixel regions \[7,8,14\]. Our method proposed in this paper is based on the former method for the derivation of constraints on parameters of the Euclidean line that passes through hexagonal grids.

2 Hexagonal Grid System on a Plane

We first define hexagonal grids on a two-dimensional Euclidean plane $(x, y)$.

**Definition 1.** We call the region

$$
\begin{align*}
&y_0 - 1 \leq y \leq y_0 + 1, \\
&2x_0 + y_0 - 2 \leq 2x + y \leq 2x_0 + y_0 + 2, \\
&2x_0 - y_0 - 2 \leq 2x - y \leq 2x_0 - y_0 + 2, \\
&x_0 = 3\alpha, y_0 = 2\beta \text{ or } x_0 = 3(\alpha + \frac{1}{2}), y_0 = 2\beta + 1,
\end{align*}
$$

(1)

the hexagonal grid centred at $x_0 = (x_0, y_0)^\top$. Simply, we call it the hexel at $x_0$.

The supercover \[12,13\] in the hexagonal grid is defined as follows.

**Definition 2.** The supercover in the hexagonal grid system is a collection of all hexagonal grids that cross a certain line.

Figure 1(a) shows an example of the supercover in the hexagonal grid. Since the vertices of a hexagon are $(x_0 - 1, y_0)^\top$, $(x_0 + 1, y_0)^\top$, $(x_0 - \frac{1}{2}, y_0 + 1)^\top$, $(x_0 + \frac{1}{2}, y_0 + 1)^\top$, $(x_0 - \frac{1}{2}, y_0 - 1)^\top$, and $(x_0 + \frac{1}{2}, y_0 - 1)^\top$, the distances from these vertices to the centre of the hexagon are

$$
D = \{d_i\}_{i=1}^6 = \left\{ \frac{x_0a + y_0b + \mu - a}{\sqrt{a^2 + b^2}}, \frac{x_0a + y_0b + \mu + a}{\sqrt{a^2 + b^2}}, \frac{x_0a + y_0b + \mu + b - \frac{1}{2}a}{\sqrt{a^2 + b^2}}, \frac{x_0a + y_0b + \mu + b + \frac{1}{2}a}{\sqrt{a^2 + b^2}}, \frac{x_0a + y_0b + \mu - b - \frac{1}{2}a}{\sqrt{a^2 + b^2}}, \frac{x_0a + y_0b + \mu - b + \frac{1}{2}a}{\sqrt{a^2 + b^2}} \right\}.
$$

(2)
Fig. 1. Line on the hexagonal grid system. (a) Supercover in the hexagonal grids. (b) Bubble in the hexagonal grids. (c) Local relation between a line and a bubble on hexagonal grid system.

Therefore, if a line crosses a hexagon, we have the relations

\[
\begin{align*}
\min\{d_i\} & \leq 0 \leq \max\{d_i\} \\
\iff -\max\{|a|, \frac{1}{2}|a| + |b|\} & \leq x_0a + y_0b + \mu \leq \max\{|a|, \frac{1}{2}|a| + |b|\} \\
\iff 0 & \leq |x_0a + y_0b + \mu| \leq \max\{|a|, \frac{1}{2}|a| + |b|\}.
\end{align*}
\tag{3}
\]

These relations lead to the next theorem.

**Theorem 1.** Setting \(a\) and \(b\) to be integers, the supercover of a line \(L : ax + by + \mu = 0\) on the hexagonal grid is a collection of hexagons that satisfy the relations

\[
\mathcal{L} = \{(x, y)^T \mid x = 3\alpha, y = 2\beta, \ |ax + by + \mu| \leq \max\{|a|, \frac{1}{2}|a| + |b|\}\} \\
\cup\{(x, y)^T \mid x = 3(\alpha + \frac{1}{2}), y = 2\beta + 1, \ |ax + by + \mu| \leq \max\{|a|, \frac{1}{2}|a| + |b|\}\} \tag{4}
\]

for integers \(\alpha\) and \(\beta\).

The following is a geometric definition of the bubble \([10–13]\) in the hexagonal grid system.

**Definition 3.** A bubble in the hexagonal grid system is three connecting hexagons that share a vertex, as shown in Fig. 1(b).

Figure 1(c) shows that a supercover in the hexagonal grid system contains bubbles if a line crosses a pair of edges that share a vertex. Considering this geometric property of a bubble, we derive mathematical conditions of a line whose supercover contains bubbles.

First, we sort the elements of \(D = \{d_i\}_{i=1}^6\), which are defined by (2), as

\[
d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5 \leq d_6.
\]
If a line crosses a pair of edges that share a vertex, the relations
\[
d_1 \leq 0 \leq d_2 \leq d_3 \leq d_4 \leq d_5 \leq d_6
\]
(5)
or
\[
d_1 \leq d_2 \leq d_3 \leq d_4 \leq 0 \leq d_6
\]
(6)
are satisfied. These conditions are equivalent to
\[
d_1 \leq 0 \leq d_2, \quad d_5 \leq 0 \leq d_6.
\]
(7)
Therefore, setting
\[
C = \{-a, a, b - \frac{1}{2} a, b + \frac{1}{2} a, -b - \frac{1}{2} a, -b + \frac{1}{2} a\}
\]
\[
= \{c_i \mid i = 1, \ldots, 6, c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5 \leq c_6\},
\]
(9)
Equations (7) and (8) are expressed as
\[
c_1 \leq ax + by + \mu \leq c_2,
\]
(10)
\[
c_5 \leq ax + by + \mu \leq c_6.
\]
(11)
Furthermore, Definition 3 implies that the centre of a hexagonal grid \((x, y)^\top\) satisfies the conditions
\[
x = 3\alpha, y = 2\beta,
\]
(12)
or
\[
x = 3(\alpha + \frac{1}{2}), y = 2\beta + 1,
\]
(13)
for integers \(\alpha\) and \(\beta\). This algebraic relation implies the following system of inequalities:
\[
c_1 \leq 3a\alpha + 2b\beta + \mu \leq c_2,
\]
(14)
\[
c_5 \leq 3a\alpha + 2b\beta + \mu \leq c_6,
\]
(15)
\[
c_1 \leq 3a(\alpha + \frac{1}{2}) + b(2\beta + 1) + \mu \leq c_2,
\]
(16)
\[
c_5 \leq 3a(\alpha + \frac{1}{2}) + b(2\beta + 1) + \mu \leq c_6.
\]
(17)
These relations lead to the conclusion that the supercover of line \(ax + by + \mu = 0\)
contains bubbles in the hexagonal grid system if a pair of integers \((\alpha, \beta)\) satisfies
Eq. (14), (15), (16), or (17). The analysis above leads to the next theorem.

**Theorem 2.** If the relations
\[
\frac{\mu - c_1 - m}{\gcd(3|a|, 2|b|)} \in \mathbb{Z} \quad \lor \quad \frac{\mu + b + \frac{3}{2} a - c_1 - n}{\gcd(3|a|, 2|b|)} \in \mathbb{Z}
\]
\[
\lor \quad \frac{\mu - c_5 - n}{\gcd(3|a|, 2|b|)} \in \mathbb{Z} \quad \lor \quad \frac{\mu + b + \frac{3}{2} a - c_5 - n}{\gcd(3|a|, 2|b|)} \in \mathbb{Z}
\]
(18)
are satisfied, the supercover of line \( ax + by + \mu = 0 \) contains bubbles, where
\[
0 \leq m + c_1 \leq c_2, \ 0 \leq n + c_5 \leq c_6, \ m + c_1 \in \mathbb{Z}, \ n + c_5 \in \mathbb{Z},
\]
\[
C = \{-a, a, b - \frac{1}{2}a, b + \frac{1}{2}a, -b - \frac{1}{2}a, -b + \frac{1}{2}a\} = \{c_i \mid i = 1, \cdots, 6, c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5 \leq c_6\}.
\]

Now, we show two examples for the supercover in the hexagonal grid system.

**Example 1.** For line \( 2x + 7y - 1 = 0 \), we have
\[
C = \{-2, 2, 6, 8, -8, -6\} = \{-8, -6, -2, 2, 6, 8\} = \{c_i \mid i = 1, \cdots, 6, c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5 \leq c_6\},
\]
\[
gcd(3|a|, 2|b|) = gcd(6, 14) = 2.
\]

From this relation, we can select \( m \) and \( n \) from \( m = 0, 1, 2 \), and \( n = 0, 1, 2 \), respectively. If we select \( m = 1 \), we have the relation
\[
\frac{\mu - c_1 - m}{gcd(3|a|, 2|b|)} = 3 \in \mathbb{Z}.
\]

Therefore, the supercover of line \( 2x + 7y - 1 \) contains bubbles, as shown in Fig. 2(a).

**Example 2.** For line \( 2x - 3y + 1 = 0 \), we have
\[
C = \{-2, 2, -4, -2, 2, 4\} = \{-4, -2, -2, 2, 2, 4\} = \{c_i \mid i = 1, \cdots, 6, c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5 \leq c_6\},
\]
\[
gcd(3|a|, 2|b|) = gcd(6, 6) = 6.
\]

From this relation, we can select \( m \) and \( n \) from \( m = 0, 1, 2 \) and \( n = 0, 1, 2 \), respectively. Here, plugging all combinations of \( m \) and \( n \) to Eq. (18) of theorem 2, we have the relations
\[
\frac{\mu - c_1 - m}{gcd(3|a|, 2|b|)} = \frac{5 \ 2 \ 1}{6 \ 3 \ 2},
\]
\[
\frac{\mu + b + \frac{3}{2}a - c_1 - n}{gcd(3|a|, 2|b|)} = \frac{5 \ 2 \ 1}{6 \ 3 \ 2},
\]
\[
\frac{\mu - c_5 - n}{gcd(3|a|, 2|b|)} = \frac{-1 \ -1 \ -1}{6 \ -3 \ -2},
\]
\[
\frac{\mu + b + \frac{3}{2}a - c_5 - n}{gcd(3|a|, 2|b|)} = \frac{-1 \ -1 \ -1}{6 \ -3 \ -2}.
\]

Since all combinations of \( m \) and \( n \) yield noninteger, the supercover of line \( 2x - 3y + 1 \) does not contain any bubbles, as shown in Fig. 2(b).
3 Reconstruction of Euclidean Line

In this section, we develop an algorithm for the reconstruction of the Euclidean line \([12, 20–22]\) from sample hexels. For integers \(\alpha_i\) and \(\beta_i\), setting

\[
P = \{(x_i, y_i)^\top | x_i = 3\alpha_i, y_i = 2\beta_i, i = 1, 2, \ldots, N\}
\]  

or

\[
P = \{(x_i, y_i)^\top | x_i = 3(\alpha_i + \frac{1}{2}), y_i = 2\beta_i + 1, i = 1, 2, \ldots, N\}
\]

to be the centroids of the hexels, for a pair of positive integers \(a\) and \(b\), we have four cases:

- **case1**: \(a \geq 2b > 0, 0 \leq |ax_i + by_i + \mu| \leq a\),
- **case2**: \(a \geq 2b > 0, 0 \leq |ax_i - by_i + \mu| \leq a\),
- **case3**: \(0 < a < 2b, 0 \leq |ax_i + by_i + \mu| \leq \frac{1}{2}a + b\),
- **case4**: \(0 < a < 2b, 0 \leq |ax_i - by_i + \mu| \leq \frac{1}{2}a + b\).

Equations (23) and (24) are derived from Eqs. (12) and (13), respectively.

Here, we show the reconstruction algorithm for case 1. Assuming that all sample hexels are elements of the supercover of line \(ax + by + \mu = 0\) for \(a \geq 0\) and \(b \geq 0\), we have the relations

\[
\begin{cases}
-a \leq ax_i + by_i + \mu \leq a, \\
a - 2b \geq 0, \\
a, b > 0, \\
i = 1, 2, \ldots, N,
\end{cases}
\]
\[ \begin{align*}
\left\{ \begin{array}{l}
(x_i + 1)a + y_i b + \mu \geq 0 \\
-(x_j - 1)a - y_j b - \mu \geq 0
\end{array} \right. \\
\iff \left\{ \begin{array}{l}
a - 2b \leq 0 \\
a, b > 0 \\
i \neq j, i, j = 1, 2, \ldots, N,
\end{array} \right.
\end{align*} \]

\[ \begin{align*}
\left\{ \begin{array}{l}
-(x_i + 1)a - y_i b \leq \mu \leq -(x_j - 1)a - y_j b, \\
(x_i - x_j + 2)a + (y_i - y_j)b \geq 0
\end{array} \right. \\
\iff \left\{ \begin{array}{l}
a - 2b \geq 0, \\
a, b > 0 \\
i \neq j, i, j = 1, 2, \ldots, N,
\end{array} \right. \\
\end{align*} \]  

for

\[ X_{ij} = x_i - x_j + 2, \quad Y_{ij} = y_i - y_j. \]

Then, Eq. (29) becomes

\[ \begin{align*}
\left\{ \begin{array}{l}
-(x_i + 1)a - y_i b \leq \mu \leq -(x_j - 1)a - y_j b \\
X_{ij}a + Y_{ij}b \geq 0 \\
a - 2b \geq 0 \\
a, b > 0 \\
i \neq j, i, j = 1, 2, \ldots, N.
\end{array} \right.
\]

This expression allows us to use the algorithm derived in the ref. [23].

### 4 Polygonalisation from Hexels

Using the optimisation procedure for the recognition of a Euclidean line from a collection of hexels, in this section, we develop an algorithm for the polygonalisation of the discrete boundary of a binary shape [12].

Setting $P$ to be a digital curve which is described a sequence of hexels, our problem is described as follows.

**Problem 1.** Let $P$ be the digital boundary curve of an object on the hexagonal grid system. Compute a partition of $P$, such that $P = \cup_{i=1}^{n} P_i$, $P_i = \{p_{ij}\}_{j=1}^{n(i)}$ for $p_{ij} = (x_{ij}, y_{ij})^{T}$ and $|P_i \cap P_{i+1}| = \varepsilon$ for an appropriate small integer $\varepsilon$.

We solve the problem using the minimisation problem.

**Problem 2.** Compute the number of polygonal edges $n$ and triplets of parameters $\{(a_i, b_i, \mu_i)\}_{i=1}^{n}$ for edges that minimise the criterion

\[ z = \sum_{i=1}^{n} (|a_i| + |b_i| + \mu_i) \]  

(32)
**Fig. 3.** Configuration of hexels and dual hexels. (a) Sequence of hexels, (b) Sequence of dual hexels in hexels, and (c) Local configuration of a line, hexels and dual hexels.

with respect to the system of inequalities,

\[ |a_i x_{ij} + b_i y_{ij} + \mu_i| \leq \max\{|a_i|, \frac{1}{2} |a_i| + |b_i|\}, \quad x_{ij} = 3\alpha_{ij}, y_{ij} = 2\beta_{ij} \quad (33) \]

or

\[ |a_i x_{ij} + b_i y_{ij} + \mu_i| \leq \max\{|a_i|, \frac{1}{2} |a_i| + |b_i|\}, \quad x_{ij} = 3(\alpha_i + \frac{1}{2}), y_{ij} = 2\beta_{ij} + 1, \quad (34) \]

where \( \alpha_{ij}, \beta_{ij} \in \mathbb{Z} \).

The following is an incremental algorithm for this minimisation problem.

---

**step 1:** Input the centroids of hexels, say, \( \mathbf{P} = \{\mathbf{p}_i | i = 0, 1, 2, \ldots, n\} \).

**step 2:** Set \( head = 0, j = 0 \).

**step 3:** Set \( tail = head + 3 \).

**step 4:** Set \( \mathcal{L}_j = \{\mathbf{p}_i\}_{head}^{tail} \).

**step 5:** If a line \( l_j \) which passes through \( \mathcal{L}_j = \{\mathbf{p}_i\}_{tail}^{head} \),

then set \( tail = tail + 1 \) and go to step 3.

**step 6:** If \( j = 0 \), then set \( j = j + 1, head = tail \) and go to step 2.

**step 7:** If \( j > 0 \), then compute the common point of \( l_{j-1} \) and \( l_j \),

and set it as \( A_{j-1} \).

**step 8:** If \( A_{j-1} \) exists and it is included in \( \mathcal{L}_j \) or \( \mathcal{L}_{j-1} \), then go to step 10.

**step 9:** Set \( head = head - 1 \) and go to step 3.

**step 10:** Output \( \mathcal{L}_{j-1} \) and \( l_{j-1} \).

**step 11:** If \( tail < n \), then set \( head = tail \) and \( j = j + 1 \), and go to step 3.

**step 12:** If \( tail = n \), then stop.

---

To remove the bubbles from the supercover of a line, we introduce a dual hexel. For hexel ABCDEF in Fig. 3(b), we define the dual hexel A'B'C'D'E'F', as shown in Fig. 3(b). As shown in Fig. 3(c), if line \( L \) passes through points A'B'C'D'E'F' without crossing with vertices A, B, C, D, E, and F, the supercover of \( L \) is bubble-free.
Fig. 4. Relation between the feasible regions and the number of dual hexels. (a), (b), and (c) are configurations of dual hexels in the connected hexels. (d), (e), and (f) are feasible regions of parameters that define Euclidean lines for the hexel configurations of (a), (b), and (c), respectively.

Setting \((x_i, y_i)^\top\) to be the centroid of a hexel, the vertices of the dual hexel are \((x_i - \frac{1}{2}, y_i)^\top\), \((x_i - \frac{1}{2}, y_i + \frac{1}{3})^\top\), \((x_i, y_i + \frac{2}{3})^\top\), \((x_i + \frac{1}{2}, y_i + \frac{1}{3})^\top\), \((x_i + \frac{1}{2}, y_i - \frac{1}{3})^\top\), \((x_i, y_i - \frac{2}{3})^\top\), and \((x_i - \frac{1}{2}, y_i - \frac{1}{3})^\top\). Therefore, the bubble-free supercover defined by hexels whose centroids are \(\{p_i = (x_i, y_i)^\top\}_{i=1}^{n-1}\) is

\[
0 \leq |ax_i + by_i + \mu_i| \leq \max\{\frac{2|b|}{3}, \frac{|a| + |b|}{2}\}, \ i = 0, 1, 2, ..., n - 1. \tag{35}
\]

Figures 4 and 5 show the transition of the feasible regions of the parameters of lines for the dual hexel and hexels, respectively. In Fig. 4, (a), (b), and (c) are configurations of dual hexels in the connected hexels. (d), (e), and (f) are feasible regions of parameters that define Euclidean lines for the hexel configurations of (a), (b), and (c), respectively. Furthermore, in Fig. 5, (a), (b), and (c) are configurations of dual hexels in the connected hexels. (d), (e), and (f) are feasible regions of parameters that define Euclidean lines for the hexel configurations of (a), (b), and (c), respectively.

5 Numerical Examples

For error analysis, we have evaluated the total areas encircled by the reconstructed curves and the total lengths of reconstructed curves for the circles whose radius are from 10 to 1000.
Fig. 5. Relation between the feasible regions and the number of hexels. (a), (b), and (c) are configurations of dual hexels in the connected hexels. (d), (e), and (f) are feasible regions of parameters that define Euclidean lines for the hexel configurations of (a), (b), and (c), respectively.

Fig. 6. Error analysis of polygonalisation from hexels. (a) Error analysis for the polygons reconstructed from hexels (b) Error analysis for the polygons reconstructed from dual hexels.

Figure 6(a) and 6(b) show the results of error analysis for the polygons reconstructed from hexels and dual hexels, respectively. These results show that by increasing the resolution, the reconstructed curves converge to the original curves.

Figures 7 and 8 show the results of polygonalisation from digital curves of hexels and dual hexels, respectively. The order of the dual-hexel sequence is the same as that of the original hexels on the curve. The numbers of edges of the
Fig. 7. Polygonalisation from hexels. Figure shows a polygon reconstructed from hexels. The number of edges of the polygon is 130.
**Fig. 8.** Polygonalisation from hexels. Figure shows a polygon reconstructed from dual hexels. The number of edges of the polygon is 260. This polygon is bubble-free. Furthermore, edges of this polygon pass through the inner areas of hexels.
polygon in Figs. 7 and 8 are 130 and 260, respectively. Furthermore, the edges of the polygon in Fig. 8 pass through the inner areas of the original hexel and there is no bubble of three hexels. These results show the validity of our bubble removal strategy introduced in the previous section.

6 Conclusions

We defined algebraic discrete geometry to hexagonal grids on a plane. Using algebraic discrete geometry, we developed an algorithm for the polygonalisation of discrete objects on a hexagonal-grid plane.

In this paper, we aimed to formulate the recognition of a linear manifold from a discrete point set as a nonlinear optimisation problem. We dealt with a supercover model on a plane and in a space. We first derived a set of inequalities for the parameters of a Euclidean linear manifold from sample points and an optimisation criterion with respect to this set of constraints for the recognition of the Euclidean linear manifold. Second, using this optimisation problem, we develop an algorithm for the computation of parameters of the Euclidean linear manifold from hexels on a plane.

Liu [2,3] introduced algorithms for the generation of sequences of connected hexagons from a line and a circle, respectively. In this paper, we dealt with three problems

1. Generation of a sequence of hexels from piecewise linear curves.
2. Generation of a sequence of hexels that contain given hexels.
3. Reconstruction of a piecewise linear curve from a sequence of connected hexels.

Therefore, our algorithm generally resolves the problems dealt with by Liu [2,3].

Most of this paper is based on the research conducted by Troung Kieu Linh while she was at the School of Science and Technology, Chiba University.

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