Computer simulation of derivative markets using Black-Scholes model

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Abstract. In recent years diverse mathematical physics models became widely used in practice as a great implement for describing economical and physical phenomena. Modeling of the market of derivatives, in particular case, computation of European option rational price, is carried out with Black-Scholes equation, which in terms of constant volatility and non-dividend situation has the well-known analytical solve. Several generalizations of Black-Scholes model were suggested to reach better conformity between the results of mathematical modeling and the real market figures. In these generalized models volatility depends on the desirable function in a rather complicated way. Surveyed are Leland model, Frey-Patie model and also the model of indefinite volatility, which vary from minimum to maximum value. In order to calculate the numerical solution of this mixed task the scheme with scales, which has the second approximation order on the coordinate and time, is used for the equation of parabolic type. Series of computing experiments which allow to compare the results of analytic and numerical analysis were conducted for the different range of parameters. According to the results of computer simulation, generalized non-linear Black-Scholes equation describes the real market situation with a very close approximation.

1. Introduction

The public interest in trade securities has continuously grown over the past years. Consequently, events influencing stock prices, opinions and speculations on such events and their consequences, and even the daily stock quotes, receive much attention and media coverage. So, because the financial markets on the systems with many participants and on influenced by different stochastic events, it is natural to study these phenomena from a physicists point of view [7]. Statistical physics describes the complex behavior observed in many physical systems in terms of their simple basic constituents and simple interaction laws. Complexity cuises from interaction and disorder, from the cooperation and competition of the basic units.

A derivative is a financial instrument whose value depends on other, more basic underlying variables. Very often, these variables on the prices of other securities (such as stocks, bonds, currencies) with, of course, a series of additional parameters involved in determining of precise dependence. There are the three simplest derivatives on the market: forward and futures contracts, and call and put options. We shall discuss the behavior of options. Options may be written on any kind of underlying assets, such as stocks, bonds, commodities, futures, many indices measuring entire markets, etc. Options give their holder the right to by or sell an
underlying assets in the future at a fixed price. However, they imply an obligation for the writer of the option to deliver or buy the underlying asset.

There are two basic types of options: call option which gives the holder the right to buy, and put option which gives their holder the right to sell the underlying asset in the future at a specified price, the strike price of the option. Options are distinguished as being of European type, if the right to buy or sell can only be exercised at their date of maturity, or of American type if they can be exercised at any time from now until their data of maturity. Options are traded regularly on exchanges.

There is no obligations for the holder to exercise the options while the writer has an obligations. As a consequence of this asymmetry, there is an intrinsic cost (similar to insurance premium) associated with the option which the holder has to pay to the writer. This is different from forwards and futures which carry an obligation for both parties, and where there is no intrinsic cost associated with these contracts. Options can therefore be considered as insurance contracts.

2. Option pricing

The important achievement of Black and Scholes [2] was to determine the real price of option for a certain idealized market. The remarkable feature of the classical Black-Scholes model is its property to price security-based derivatives with only two parameters, the volatility \( \sigma_0 \) of the underlying asset and the risk-free short rate \( r \). However, comparing theoretical Black-Scholes pricing of derivatives with their actually traded prices reveals that the model assumptions are oversimplifying the market mechanism. In practice, the deficiencies of the Black-Scholes model are balanced by reverse-engineering the volatility out of the market data [5].

The implied volatility structure is then used to price derivatives. The disadvantage of this elegant trick is the dependance of the volatility of the underlying on the strike and maturity of its derivative.

To reflect the market properties in more detail several modifications of the Black-Scholes model have been suggested and discussed in recent years. All these models differ from the classical Black-Scholes equation by a non-constant volatility term, which depends on time \( t \), spot price \( S \) of the underlying asset and the second derivative of the function \( V(S,t) \), which is the value of the option. Hence, the basic equation is a non-linear partial differential equation of parabolic type, which can be expressed as

\[
V_t + \frac{1}{2} \sigma^2(t, S, V_{SS}) S^2 V_{SS} + (r - q)SV_S - rV = 0.
\]

with \( \sigma^2(t, S, V_{SS}) \) depending on the particular model, constant short rate and dividend yield \( q \).

The boundary conditions and terminal profile at maturity \( T \) are chosen according to the contemplated derivative of portfolio.

The first model, which incorporates transaction costs into the Black-Scholes equation, was suggested by Leland [4]. Leland includes round-trip transaction costs \( k \) and a fixed time interval \( \delta t \) between successive adjustments of the portfolio into the Black-Scholes model and deduces for convex payoffs

\[
\sigma^2_{Le}(t, S, V_{SS}) = \sigma_0^2 \cdot (1 + A \cdot \text{sign}(V_{SS})),
\]

where \( \sigma_0 \) is the volatility of the underlying asset and \( A = \sqrt{\frac{2}{\pi \sigma_0 \sqrt{\delta t}}} \) is called Leland number.

A second class of non-linear BlackScholes-type equations arises from the assumption that the market is not perfectly liquid and trading actions of a large investor will affect the price of the underlying asset [3]. As pointed out by these authors the resulting feedback causes a non-linear volatility term. For example, Frey and Patie derive the following volatility term
\[
\sigma_{FP}^2(t, S, V_{SS}) = \frac{\sigma_0^2}{(1 - \rho \cdot \lambda(S)SV_{SS})^2},
\]  
(3)

where \(\sigma_0\) is the volatility of the asset, \(\rho\) is a parameter measuring the market liquidity and \(\lambda(S)\) describes the liquidity profile in dependence of the asset price.

The next non-linear model was derived by Avellaneda, Levy and Paras [1]. This model studies uncertain volatility models, in which the volatility is not known precisely but is assumed to live between extreme values \(\sigma_{\text{min}}\) and \(\sigma_{\text{max}}\).

\[
\sigma_{ALP+}^2(t, S, V_{SS}) = \begin{cases} 
\sigma_{\text{max}}^2 & : V_{SS} \leq 0, \\
\sigma_{\text{min}}^2 & : V_{SS} > 0.
\end{cases}
\]
(4)

Because of the non-linear nature of all these models numerical methods are mandatory to price derivatives and portfolios.

3. Transformation and Boundary Conditions

For numerical investigations it necessary to transform equation 1 to the new unknown function and variables using the following formula,

\[
x = \ln \frac{S}{K}, \tau = \frac{1}{2} \sigma_0^2 (T - t), u(x, \tau) = e^{-x} \frac{V(S, t)}{K},
\]
(5)

where \(K > 1\) can be any value and \(\sigma_0\) is a model-dependent parameter. Of course, typical choises are the strike price \(K\) of the derivative-hence \(x = 0\) corresponds to \(S\) being at-the-money and the volatility \(\sigma_0\) of the underlying. Time is reversed, so that the terminal payoff becomes an initial condition. Equation 1 is transformed into

\[-u_\tau + \tilde{\sigma}^2(\tau, x, u_x, u_{xx}) \cdot (u_x + u_{xx}) + \frac{2r}{\sigma_0^2} u_x = 0.
\]
(6)

In the new coordinates the volatility terms read as

\[
\tilde{\sigma}_{Le}^2 = 1 + A \cdot \text{sign}(u_x + u_{xx}),
\]
(7)

\[
\tilde{\sigma}_{FP}^2 = (1 - \rho \cdot \lambda(Ke^x) \cdot (u_x + u_{xx}))^{-2},
\]
(8)

\[
\tilde{\sigma}_{ALP+}^2 = \begin{cases} 
\sigma_{\text{max}}^2 & : u_x + u_{xx} \leq 0, \\
\sigma_{\text{min}}^2 & : u_x + u_{xx} > 0.
\end{cases}
\]
(9)

We denote the transformed payoff profile by \(\Lambda(x)\) in the following. Thus, the initial condition for Equation 1 is given by

\[u(x, 0) = \Lambda(x).
\]
(10)

The payoff can easily be determined from the payoff in \((S, t)\) coordinates and subsequent transformation. For call option:

\[\Lambda_C(x) = (1 - e^{-x})^+.\]
(11)

For put option:

\[\Lambda_P(x) = (e^{-x} - 1)^+.
\]
(12)

The admissible domain \(S \in [0, \infty)\) of the underlying asset is transformed in the domain \(x \in (-\infty, \infty)\), which has to be truncated to a numerically feasible, finite interval \(x \in [A, B]\).
The boundary conditions at \( x = A \) and \( x = B \) for \( \tau > 0 \) are given by functions \( \alpha_C(\tau, A) = 0 \) and \( \beta_C(\tau, B) \), which are derived by asymptotic considerations in \((S, t)\) coordinates for extreme values \( S \to 0 \) and \( S \to \infty \) and subsequent transformation. For example, the boundary conditions for a call option:

\[
\alpha_C(\tau, A) = 0, \quad \beta_C(\tau, B) = 1 - \exp \left( -\frac{2r}{\sigma^2_0} \tau - B \right). \tag{13}
\]

For a put option:

\[
\alpha_P(\tau, A) = \exp \left( -\frac{2r}{\sigma^2_0} \tau - A \right), \quad \beta_P(\tau, B) = 0. \tag{14}
\]

Summarizing, depending on the chosen model and the payoff \( \Lambda \) together with proper boundary data \( \alpha \) and \( \beta \) the following problem has to be solved numerically

\[
\begin{aligned}
-u_{\tau} + \frac{2}{\sigma^2_0} u_{xx} & + \frac{2r}{\sigma^2_0} u_x = 0, \quad x \in [A, B], \quad \tau \in \left[0, \frac{\sigma^2_0 T}{2}\right], \\
\end{aligned}
\]

\[
\begin{aligned}
u(x, 0) &= \Lambda(x), \\
u(A, \tau) &= \alpha(\tau, A), \\
u(B, \tau) &= \beta(\tau, B).
\end{aligned}
\]

This problem is solved by the implicit difference scheme \([6]\).

4. Results of the Computational Experiment

As a result of the computational experiments, data were obtained that allowed us to compare the various volatility description models with the classical Black-Scholes model. In order to get a more precise picture, we considered European type call option. As it can be seen in figure 1, the Leland model with a small Leland number estimates the option price much cheaper than the classical model. In the case of a large Leland number, the difference turns to be insignificant.

![Figure 1](image.png)

Figure 1. Green line - Leland model with \( A = 0.9 \). Blue line - classical Black-Scholes model. Orange line - Leland model with \( A = 0.3 \)

On the contrary, the Frey-Pattie model appraises the option at a higher price than the classical model, even with low volatility of figure 2. If the volatility figure of the market is high, a double difference becomes apparent.
Figure 2. Green line - Frey-Pattie model with $\rho = 0.01$. Blue line - classical Black-Scholes model. Orange line - Frey-Pattie model with $\rho = 0.04$.

The Avellaneda model with a small volatility range showed a value that is almost similar to the classical Black-Scholes model (this can be observed in figure 3). However, as the volatility range increases, the evaluations of an option differ from each other one and a half times.

Figure 3. Orange line - Avellaneda model with $\sigma_{\text{min}} = 0.15$, $\sigma_{\text{max}} = 0.25$. Blue line - classical Black-Scholes model. Green line - Avellaneda model with $\sigma_{\text{min}} = 0.05$, $\sigma_{\text{max}} = 0.35$. Red line - Avellaneda model with $\sigma_{\text{min}} = 0.03$, $\sigma_{\text{max}} = 0.5$.

5. Conclusions

In this paper we have considered a nonlinear generalization of the classical Black-Scholes equation. Three models were used to describe volatility: the Leland model, the Frey-Patie model and the Avellaneda model. This equation was solved numerically using differences scheme with scales. This allowed us to conduct a series of computational experiments for European type call option and to compare the various models describing volatility with the classical Black-Scholes equation. Obvious results were obtained for different volatility values. The calculations showed that non-linear generalizations describe the dynamics of the market more precisely when it is believed that the market is volatile.

References

[1] Avellaneda M, Levy A and Paras A (1995) Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance*, 2(2), pp. 73-88.
[2] Black F S and Scholes M (1973) The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), pp. 637-654.

[3] Frey R and Patie P (2002) Risk management for derivatives in illiquid markets: a simulation-study. In: K. Sandmann and P. Schonbucher (Eds.), *Advances in Finance and Stochastics* (Berlin: Springer) pp. 137-159.

[4] Leland H E (1985) Option pricing and replication with transactions costs, *The Journal of Finance*, 40. p. 1283-1301.

[5] Pascal Heider (2010) Numerical Methods for Non-Linear BlackScholes Equations, *Applied Mathematical Finance*, 17:1, 59-81.

[6] Samarskii A A *The theory of difference schemes* (2001) (New York: Basel). Marcel Dekker, Inc. p. 761.

[7] Voit J (2005) *The Statistical Mechanics of Financial Markets*. (New York: Springer) p. 378.