Determination of the strength of the vector-type four-quark interaction in the entanglement Polyakov-loop extended Nambu–Jona-Lasinio model

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We determine the strength \( G_v \) of the vector-type four-quark interaction in the entanglement Polyakov-loop extended Nambu–Jona-Lasinio (EPNJL) model from the results of recent lattice QCD simulations with two-flavor Wilson fermion. The quark number density is normalized by the Stefan-Boltzmann limit. The strength determined from the normalized quark number density at a baryon chemical potential \( \mu \) and temperature \( T \) (which is higher than the pseudocritical temperature \( T_c \) of the deconfinement transition) is \( G_v = 0.33G_s \), where \( G_s \) is the strength of the scalar-type four-quark interaction. We explore the hadron-quark phase transition in the \( \mu-T \) plane by using the two-phase model in which the quantum hadrodynamics model is used for the hadron phase and the EPNJL model is used for the quark phase. When \( G_v = 0.33G_s \), the critical baryon chemical potential of the transition at zero \( T \) is \( \mu_c \sim 1.6 \) GeV, which accounts for two-solar-mass measurements of neutron stars in the framework of the hadron-quark hybrid star model.

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Introduction. Determining the QCD phase diagram is an important subject in particle and nuclear physics, as well as in cosmology and astrophysics. However, using lattice QCD (LQCD) as a first-principle calculation presents a sign problem at finite baryon chemical potential \( \mu \). Several methods were proposed to resolve this problem, such as the reweighting method [1], the Taylor expansion method [2,3], and the analytic continuation from imaginary \( \mu \) to real \( \mu \) [4–10]. The results are reliable for small \( \mu \); say, \( \mu/T \lesssim 3 \), where \( T \) is the temperature. Very recently, remarkable progress toward larger \( \mu/T \) has been made with the complex Langevin method [11–13] and the Lefschetz thimble theory [14,15]. However, the results are still far from the physics at \( \mu/T = \infty \), as occurs in nuclear matter and neutron stars.

In another important approach, one can consider effective models such as the Polyakov-loop extended Nambu–Jona-Lasinio (PNJL) model [14–22]. The PNJL model can treat the confinement mechanism approximately and the chiral symmetry breaking. As for zero \( T \), the chiral and deconfinement transitions coincide with each other in LQCD simulations, but not in the PNJL model, when the model parameters are set to the realistic transition temperature \( T_c \) [21]. This problem was solved by introducing the Polyakov-loop dependent four-quark interaction to the PNJL model [23,24]. This model is called the entanglement-PNJL (EPNJL) model. The EPNJL model also accounts for the phase structure calculated with LQCD at imaginary \( \mu \) [7,8] and real isospin chemical potential [25]. Very recently, Ishii et al. [26] showed that the EPNJL model reproduces the meson screening masses on temperature, as calculated with LQCD [27]. Since baryons are not treated in NJL-type models, a plausible approach is to take the two-phase model in which two different models are used between the hadron and quark phases to analyze the hadron-quark transition. In NJL-type models for the quark phase, the stiffness of the equation of state (EoS) is sensitive to the strength \( G_v \) of the vector-type four-quark interaction [28,29]. Reference [30] shows that the condition \( G_v \geq 0.03G_s \) is necessary for neutron stars (NSs) to have masses larger than two solar masses (2M⊙), where \( G_s \) is the strength of the scalar-type four-quark interaction. If \( G_v < 0.03G_s \), the EoS for the hadron phase is softened by the hadron-quark transition before exceeding two solar masses. The value of \( G_v \) is thus quite important to explain the observed 2M⊙ NSs [31,32].

Sakai et al. [33] estimated the strength of \( G_v \) from two-flavor LQCD results [4,3] for the deconfinement transition line at the imaginary chemical potential by using the PNJL model with the vector-type four-quark and scalar-type eight-quark interactions in addition to the scalar-type four-quark interaction. Applying the mean-field approximation reduces the scalar-type four- and eight-quark interactions to a scalar-type four-quark interaction with an effective strength \( G_v^0 \). They suggested that \( G_v^0/G_s \approx 0.8 \). A similar analysis based on the nonlocal PNJL model suggests that \( G_v^0/G_s \approx 0.4 \) [34]. Recently, using the PNJL-like model, Steinheimer and Schramm [35] estimated the strength of \( G_v^0 \) from three-flavor LQCD results [36] for the quark number susceptibility, and concluded that \( G_v \) is nearly zero. Meanwhile, Lourenco et al. [37] estimated the strength of \( G_v \) by comparing the PNJL model with the two-phase model for the hadron-quark phase transition and suggested that \( 1.52 \lesssim G_v/G_s \lesssim 3.2 \) [37]. The strength of \( G_v \) is thus undetermined.

Because previous analyses are mainly based on LQCD simulations with the Kogut-Susskind fermion, we determine in this brief report the strength of \( G_v \) in the EPNJL model from the results of recent two-flavor LQCD simulations [3] with the Wilson fermion at \( T > T_c \) and small \( \mu/T \). The quark number density \( n_q \) is sensitive to the strength of \( G_v \), but is \( \mu \) odd.
and thus tiny for small $\mu$. It is then convenient to consider the quark number density normalized by the Stefan-Boltzmann (SB) limit, $n_q/n_{SB}$. The normalized quark number density is $\mu$ even and thus finite even in the limit of $\mu = 0$. It hardly depends on $\mu$ in the region $\mu/T \lesssim 1$ where LQCD data are available. The ratio $n_q/n_{SB}$ is considered to be more reliable in the vicinity of $\mu = 0$, because the results there are obtained with the Taylor-expansion method. Therefore, we consider the ratio $n_q/n_{SB}$ in the limit $\mu \to 0$ to estimate the strength of $G_v$. We show herein that the strength of $G_v$ thus determined is $G_v = 0.33G_s$ and is not so small.

We also draw the hadron-quark transition line in the $\mu$-$T$ plane by using the two-phase model composed of the EPNJL model with the vector-type interaction for the quark phase and the quantum hadrodynamics (QHD) model for the hadron phase. We evaluate the critical baryon chemical potential $\mu_c$ of the transition at $T = 0$ and discuss whether the result for $\mu_c$ is consistent with observations of $2M_N$ NSs.

**EPNJL model for quark phase.** We add the vector-type four-quark interaction to the isospin-symmetric two-flavor EPNJL model $[23, 24]$. The Lagrangian density is

$$L_{EPNJL} = \bar{q}(i\gamma^\mu D_\mu - m_0)q + \tilde{G}_s(\Phi)[(\bar{q}q)^2 + (\bar{q}i\gamma_5\tau q)^2] - \tilde{G}_v(\Phi)(\bar{q}\gamma_\mu q)^2 - U(\Phi[A], \Phi^* [A], T),$$

where $q$ is the quark field, $m_0$ is the current quark mass, and $\tau$ is the isospin matrix. As a characteristic of the EPNJL model, the coupling constants $\tilde{G}_s(\Phi)$ and $\tilde{G}_v(\Phi)$ of the scalar- and vector-type four-quark interactions depend on the Polyakov-loop $\Phi$,

$$\tilde{G}_s(\Phi) = G_s \left[ 1 - \alpha_1 \Phi^2 + \alpha_2 (\Phi^3 + \Phi^* 3^3) \right],$$

$$\tilde{G}_v(\Phi) = G_v \left[ 1 - \alpha_1 \Phi^2 + \alpha_2 (\Phi^3 + \Phi^* 3^3) \right],$$

where $D_\mu = \partial^\mu + iA^\mu$ for $A^\mu = gG^\mu_\alpha(A_\mu)\lambda_\alpha/2 = -ig\delta^{\mu}_\alpha(A_4)\lambda_\alpha/2$, $A^\mu_\alpha$ is the gauge field, $\lambda_\alpha$ is the Gell-Mann matrix, and $g$ is the gauge coupling. Eventually, the NJL sector has five parameters ($m_0, G_s, G_v, \alpha_1, \alpha_2$). In the present parametrization, the ratio $\tilde{G}_v(\Phi)/\tilde{G}_s(\Phi)$ is independent of $\Phi$, and $\tilde{G}_v(\Phi) = \tilde{G}_s(\Phi) = G_s$ at $T = 0$ where $\Phi = 0$. When $\alpha_1 = \alpha_2 = 0$, the EPNJL model reduces to the PNJL model.

In the EPNJL model, only the time component of $A_\mu$ is treated as a homogeneous and static background field. This parameter is governed by the Polyakov-loop potential $U$. The Polyakov-loop $\Phi$ and its conjugate $\Phi^*$ are then obtained in the Polyakov gauge as

$$\Phi = \frac{1}{3} \text{tr}_c(L), \quad \Phi^* = \frac{1}{3} \text{tr}_c(L^\dagger),$$

where $L = \exp[i(A_4/T)] = \exp[i \text{Diag}(A_{41}^1, A_{42}^2, A_{43}^3)/T]$ for the classical variables $A_{4i}^\mu$ satisfying $A_{41}^{\mu_1} + A_{42}^{\mu_2} + A_{43}^{\mu_3} = 0$. We use the logarithm-type Polyakov-loop potential $U$ of Ref. [22]. The parameter set in $U$ is fit to reproduce LQCD data at finite $T$ in the pure gauge limit. The potential $U$ yields the first-order deconfinement phase transition at $T = T_0$. In the pure gauge limit, LQCD reveals a phase transition at $T = 270$ MeV. Thus, the parameter $T_0$ is often set to 270 MeV; however, with this value of $T_0$, the EPNJL model yields a larger $T_c$ for the deconfinement transition than the full-LQCD prediction $T_c = 173 \pm 5$ MeV [38, 40]. We thus rescale $T_0$.

The EPNJL model with $T_0 = 190$ MeV and $\alpha_1 = \alpha_2 = 0.2$ reproduces well the full LQCD results for the deconfinement and chiral transition lines at zero and imaginary $\mu$ [23]. As mentioned above, the parameter $\alpha_3 \equiv G_v/G_s$ is determined from the full LQCD results for $n_q/n_{SB}$ in the limit $\mu \to 0$.

For the NJL sector, we take the same parameter set as in Ref. [23] except for the current quark mass $m_0$. The LQCD simulations of Ref. [3] were done on a $4 \times 16^3$ lattice with two-flavor clover-improved Wilson quark action along the line of constant physics of $m_\pi/m_\rho = 0.65$ and 0.8 for $\pi$- and $p$-meson masses $m_\pi$ and $m_\rho$, respectively. The corresponding vacuum values of $m_\pi$ are 500 MeV and 616 MeV, and the parameters refer to these values are $m_0 = 72$ MeV and 130 MeV.

The mean field approximation to Eq. (1) leads to the thermodynamic potential (per unit volume) of

$$\Omega_{EPNJL} = U_M + U - 2N_1 \int \frac{d^3 p}{(2\pi)^3} \left[ 3E - \frac{1}{\beta} \ln \left[ 1 + 3(\Phi + \Phi^* e^{-\beta(E-\mu_q)})e^{-\beta(E-\mu_q)} + e^{-3\beta(E-\mu_q)} \right] + \frac{1}{\beta} \ln \left[ 1 + 3(\Phi^* + \Phi e^{-\beta(E+\mu_q)})e^{-\beta(E+\mu_q)} + e^{-3\beta(E+\mu_q)} \right] \right],$$

where $E = \sqrt{p^2 + M^2}$, $M = m_0 - 2G_s\sigma$, $\bar{\mu}_q = \mu_q - 2G_vn_q$, $U_M = G_s\sigma^2 - \tilde{G}_v n_q^2$, $N_1$ is the number of flavors, and the quark chemical potential $\mu_q$ is related to the baryon chemical potential $\mu$ as $\mu = 3\mu_q$.

Figures (a) and (b) show the $T$ dependence of $n_q/n_{SB}$ in the limit $\mu_q \to 0$ for $m_0 = 72$ MeV and $m_0 = 130$ MeV, respectively. In model calculations, $n_q$ is divided by the SB limit in the continuum theory. In LQCD simulations, $n_q$ is normalized by the lattice SB limit to eliminate finite-volume effects. The dotted and solid lines represent the EPNJL results with $G_c = 0$ and $G_v = 0.33G_s$, respectively. In the region $1 < T/T_c \lesssim 1.2$, the $n_q/n_{SB}$ depends weakly on the strength of $G_c$. It is thus not easy to precisely determine the strength for $T$ near $T_c$. This implies that the phase-transition line is not a good quantity to determine the strength. One can see from the region $T/T_c \gtrsim 1.2$ that $G_c = 0.33G_s$ is the best value to explain the LQCD results. Good consistency results for both $m_0 = 72$ and 130 MeV; therefore, the ratio $\alpha_3 \equiv G_v/G_s$ depends only weakly on the value of $m_0$.

The dashed line represents the result of the EPNJL model with $G_v = 0.33G_s$ in which $m_0$ is set to the physical value 5.5 MeV. The dashed line is consistent with the solid line and with LQCD data for $T/T_c \gtrsim 1.7$, where $m_0/T$ is negligibly small. This result means that the strength of $G_v$ is clearly determined from LQCD data for $T/T_c \gtrsim 1.7$, even if $m_0$ is larger than 5.5 MeV in the LQCD calculations.
QCD model for hadron phase. We now explore the hadron-quark transition by using the value of $G_c$ determined above. Because the EPNJL model is designed to treat the deconfinement transition only approximately, the hadron degrees of freedom are not correctly included in the model. We thus use the two-phase model in which the transition line is determined from the Gibbs criteria. For the hadron phase, we use the quantum QHD model of Ref. [41]. The Lagrangian density is

\[ \mathcal{L}_{\text{QHD}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_N - g_\sigma \varphi - g_\omega \gamma^\mu \omega_\mu)\psi + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{4} (\partial^\mu \omega^\nu - \partial^\nu \omega^\mu)(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) - U_{\text{QHD}}, \]

\[ U_{\text{QHD}} = \frac{1}{2} m_\sigma^2 \varphi^2 + \frac{1}{3} g_3 \varphi^3 + \frac{1}{4} g_3 \varphi^4 - \frac{1}{2} m_\omega^2 \omega^2 \omega_\mu, \]  

where $\psi$, $\varphi$, $\omega_\mu$, $m_N$, $m_\sigma$, and $m_\omega$ are nucleon (N), $\sigma$-meson and $\omega$-meson fields, and their masses, respectively, whereas $g_\sigma$, $g_\omega$, $g_2$, and $g_3$ are $\sigma$-N, $\omega$-N, and higher-order couplings, respectively. The mean field approximation to Eq. (6) thus yields the following thermodynamic potential (per unit volume):

\[ \Omega_{\text{QHD}} = U_{\text{QHD}}(\varphi, \omega) - 2 \sum_{N=p,n} \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{\beta} \ln \left[ 1 + e^{-\beta(E_N - \mu^\prime)} \right] + \frac{1}{\beta} \ln \left[ 1 + e^{-\beta(E_N + \mu^\prime)} \right] \right], \]

for $E_N = \sqrt{\mathbf{p}^2 + m_N^2}$ with $m_N^+ = m_N + g_\mu \varphi$, $\mu^\prime = \mu - g_\omega \omega_0$. The meson fields were replaced by constant values in Eq. (8), so that the spatial components of $\omega_\mu$ and all the kinetic terms vanished. Unlike in Eq. (5), the vacuum contribution term is not included in Eq. (8), because the effects were already included in the physical hadron masses and couplings in the Lagrangian [6]. We use the NL3 set [41] as the parameter set of the QHD model. For the quark phase, we use the EPNJL model with $m_0 = 5.5$ MeV and $G_c = 0.33G_s$.

The Gibbs criteria dictate that the phase with higher pressure occurs between two phases. At $T = \mu = 0$, the pressure $P_{\text{QHD}} = -\Omega_{\text{QHD}}$ for the hadron phase is zero by definition, whereas the pressure $P_{\text{EPNJL}} = -\Omega_{\text{EPNJL}}$ for the quark phase is finite because of the vacuum term. To eliminate the ambiguity due to the vacuum term, we replace $P_{\text{EPNJL}}$ by

\[ \tilde{P}_{\text{EPNJL}}(T, \mu) = P_{\text{EPNJL}}(T, \mu) - P_{\text{EPNJL}}(0, 0) - B, \]

which introduces the bag constant $B$ with $\tilde{P}_{\text{EPNJL}} = -B$ at $T = \mu = 0$. The value of $B$ is determined to reproduce the LQCD prediction of the pseudocritical temperature of the deconfinement transition at $\mu = 0$.

Figure 2 shows the phase diagram in the $\mu$-$T$ plane for the hadron-quark phase transition. The two-phase model with $G_c = 0.33G_s$ (solid line) shows that the critical baryon chemical potential of the transition at $T = 0$ is $\mu_c \sim 1.6$ GeV. This value is just above the lower bound $\mu_c \sim 1.6$ GeV to account for the observations of 2$M_\odot$ NSs [30]. When $G_c = 0$, the critical value at zero $T$ is shifted down to $\mu_c \sim 1.3$ GeV, as shown by the dotted line. The contribution of the vector-type four-quark interaction is thus quite significant.

Summary. We determined the strength $G_c$ of the vector-type four-quark interaction in the EPNJL model by using the results of LQCD simulations with two-flavor clover-improved Wilson quark action at small $\mu/T$. The results indicate that $G_c/G_s \sim 0.33$ best reproduces LQCD data for the normalized quark number density $n_q/n_{SB}$ for small $\mu$ and $T/T_c > 1.2$. The value of $G_c$ appears to be almost independent of the current quark mass because the EPNJL model with $G_c/G_s = 0.33$ simultaneously accounts for two types of LQCD data: one with $m_\pi/m_\rho = 0.65$ and the other with $m_\pi/m_\rho = 0.8$.

The ratio $G_c/G_s = 0.33$ is consistent with the result $G_c/G_s = 0.4$ obtained from the phase diagram for imaginary $\mu$ with the nonlocal PNJL model [34] and is not far from the ratio $G_c/G_s = 0.5$ calculated with a local version of the gluon exchange interaction model [42].

Using $G_c = 0.33G_s$, we explored the hadron-quark phase transition in the $\mu$-$T$ plane. The critical baryon chemical potential of the transition at $T = 0$ is $\mu_c \sim 1.6$ GeV and is just
above the lower bound $\mu_c \sim 1.6$ GeV to account for observations of $2M_\odot$ NSs. We therefore conclude that the QCD phase diagram drawn with the present two-phase model is consistent with LQCD data at small $\mu/T$ and with observations of $2M_\odot$ NSs at $\mu/T = \infty$.

To obtain more robust information on $G_v$, we plan to analyze the $\mu$ dependence of the ratio $m_\pi/n_{SB}$ more precisely for imaginary $\mu$ by using LQCD simulations with two-flavor Wilson fermion.

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Fig. 2: Phase diagram in $\mu$-$T$ plane for hadron-quark phase transition. The solid and dotted lines show the results of the two-phase model with $G_v = 0.33G_s$ and $G_v = 0$, respectively. In the EP-NJL model, we use $m_0 = 5.5$MeV. We take the approximation of $\phi = \phi^*$.