Hirzebruch classes and motivic Chern classes for singular spaces

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Dedicated to the memory of Shiing-shen Chern and to Friedrich Hirzebruch

Abstract

In this paper we study some new theories of characteristic homology classes of singular complex algebraic (or compactifiable analytic) spaces.

We introduce a motivic Chern class transformation $mC_\ast : K_0(\text{var}/X) \to G_0(X) \otimes \mathbb{Z}[y]$, which generalizes the total $\lambda$-class $\lambda_y(T^*X)$ of the cotangent bundle to singular spaces. Here $K_0(\text{var}/X)$ is the relative Grothendieck group of complex algebraic varieties over $X$ as introduced and studied by Looijenga and Bittner in relation to motivic integration, and $G_0(X)$ is the Grothendieck group of coherent sheaves of $\mathcal{O}_X$-modules.

A first construction of $mC_\ast$ is based on resolution of singularities and a suitable “blow-up” relation. In the (complex) algebraic context this “blow-up” relation follows from work of Du Bois, based on Deligne’s mixed Hodge theory. Other approaches by work of Guillén and Navarro Aznar (using “only” resolution of singularities) or Looijenga and Bittner (using the “weak factorization theorem”) also apply to the compactifiable complex analytic context. A second more functorial construction of $mC_\ast$ is based on some results from the theory of algebraic mixed Hodge modules due to M.Saito.

We define a natural transformation $T_y : K_0(\text{var}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ commuting with proper pushdown, which generalizes the corresponding Hirzebruch characteristic. $T_y$ is a homology class version of the motivic measure corresponding to a suitable specialization of the well known Hodge polynomial. This transformation unifies the Chern class transformation of MacPherson and Schwartz (for $y = -1$), the Todd class transformation in the singular Riemann-Roch theorem of Baum-Fulton-MacPherson (for $y = 0$) and the $L$-class transformation of Cappell-Shaneson (for $y = 1$).

In the simplest case of a normal Gorenstein variety with “canonical singularities” we also explain a relation among the “stringy version” of our characteristic classes, the elliptic class of Borisov-Libgober and the stringy Chern classes of Aluffi and De Fernex-Luperio-Nevins-Urbe.

Moreover, all our results can be extended to varieties over a base field $k$ of characteristic 0.

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Introduction

In this paper we study some new theories of characteristic homology classes of a singular algebraic variety $X$ defined over a base field $k$ of characteristic 0.

Let $K_0(var/X)$ be the relative Grothendieck group of algebraic varieties over $X$ as introduced and studied by Looijenga [Lo] and Bittner [Bi] in relation to motivic integration. Here a variety is a separated scheme of finite type over $\text{spec}(k)$. $K_0(var/X)$ is the quotient of the free abelian group of isomorphism classes of algebraic morphisms $Y \to X$ to $X$, modulo the “additivity” relation generated by

$$[Y \to X] = [Z \to Y \to X] + [Y \setminus Z \to Y \to X]$$ (add)

for $Z \subset Y$ a closed algebraic subvariety of $Y$. Taking $Z = Y_{\text{red}}$ we see that these classes depend only on the underlying reduced spaces. By resolution of singularities, $K_0(var/X)$ is generated by classes $[Y \to X]$ with $Y$ smooth, pure dimension and proper over $X$. Moreover, for any morphism $f : X' \to X$ we get a functorial pushdown:

$$f_! : K_0(var/X') \to K_0(var/X); [h : Z \to X'] \mapsto [f \circ h : Z \to X].$$

We introduce characteristic homology class transformations $mC_*$ and $T_y*$ on $K_0(var/X)$ related by the following commutative diagram:

$$
\begin{array}{ccc}
K_0(var/X) & \xrightarrow{mC_*} & K_0(var/X) \\
\downarrow & & \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
Moreover, the transformation $T_y$ fits into the commutative diagram

$$
\begin{array}{ccc}
F(X) & \xleftarrow{e} & K_0(\text{var}/X) \\
\downarrow{c_* \otimes \mathbb{Q}} & & \downarrow{T_y} \\
H_*(X) \otimes \mathbb{Q} & \xrightarrow{y=-1} & H_*(X) \otimes \mathbb{Q}[y] \\
\downarrow{T_y} & & \downarrow{y=0} \\
H_*(X) \otimes \mathbb{Q} & \xrightarrow{y=1} & TD_*(X) \otimes \mathbb{Q},
\end{array}
$$

(2)

with $c_* \otimes \mathbb{Q}$ the (rationalized) Chern-Schwartz-MacPherson transformation on the group $F(X)$ of algebraically constructible functions \cite{Schwa, MT, BrS, Ken}. Here $mC_0$ is the degree zero component of our \textit{motivic Chern class transformation} $mC_*$, and $e$ is simply given by

$$e([f : Y \to X]) := f_! 1_Y \in F(X),$$

i.e., by taking fiberwise the (topological) Euler characteristic with compact support. Note that the homomorphisms $e$ and $mC_0$ are surjective.

Remark 0.1. Note that in the algebraic context the \textit{Chern-Schwartz-MacPherson transformation} $c_*$ of \cite{Ken} is defined only on spaces $X$ embeddable into a smooth space (e.g., for quasi-projective varieties). But using the technique of “Chow envelopes” as in \cite[sec.18.3]{Ful}, this transformation can uniquely be extended to all (reduced) separated schemes of finite type over $\text{spec}(k)$.

For $X$ a compact complex algebraic variety, we can also construct the following commutative diagram of natural transformations:

$$
\begin{array}{ccc}
K_0(\text{var}/X) & \xrightarrow{sd} & \Omega(X) \\
\downarrow{T_y} & & \downarrow{L}, \\
H_2_*(X, \mathbb{Q})[y] & \xrightarrow{y=1} & H_2_*(X, \mathbb{Q}),
\end{array}
$$

(4)

with $L$ the homology L-class transformation of Cappell-Shaneson \cite{CS1} (as reformulated by Yokura \cite{Y1}). Here $\Omega(X)$ is the abelian group of cobordism classes of selfdual constructible complexes. To make $\Omega(\cdot)$ covariant functorial (correcting \cite{Y1}), we use the definition of a “cobordism” given by Youssin \cite{You} in the more general context of triangulated categories with duality. Note that the homology L-class transformation of Cappell-Shaneson \cite{CS1} is defined only for compact spaces and takes values in usual homology, since its definition is based on a corresponding signature invariant together with the Thom-Pontrjagin construction. Also $sd$ does not need to be surjective.

So $T_y$ unifies the (rationalized) Chern-Schwartz-MacPherson transformation $c_* \otimes \mathbb{Q}$, the Todd transformation $td_*$ of Baum-Fulton-MacPherson (for Borel-Moore homology) or Fulton (for Chow groups) and the L-class transformation of Cappell-Shaneson. However, we point out that our approach does not give a new proof of the existence of these (now) classical transformations, since our approach only gives the corresponding transformations on...
the relative Grothendieck group $K_0(var/X)$. Moreover, the Hirzebruch class $T_y(X) := T_y([id_X])$ of the singular space $X$ specializes, for $y = -1$, to the (rationalized) Chern-Schwartz-MacPherson class $c_*(X) \otimes \mathbb{Q} := c_*(1_X) \otimes \mathbb{Q}$ of $X$, since $e([id_X]) = 1_X$. But, in general we have: for $y = 0$

$$mC_0([id_X]) \neq [O_X] \quad \text{and} \quad T_0(X) \neq td_*(X) := td_*([O_X]),$$

and similarly for $y = 1$

$$sd([id_X]) \neq [IC_X] \quad \text{and} \quad T_1(X) \neq L_*(X) := L_*([IC_X]),$$

with $IC_X$ the middle intersection cohomology complex of Goresky-MacPherson [GM]. This means that our approach based on the “additivity” property picks out new distinguished elements

$$mC_0([id_X]) \in G_0(X) \quad \text{and} \quad sd([id_X]) \in \Omega(X).$$

On the positive side, we can show $mC_0([id_X]) = [O_X]$ if $X$ has at most “Du Bois singularities” (e.g. rational singularities [Kov, Sai5]). And similarly, we conjecture $sd([id_X]) = [IC_X]$ for $X$ a rational homology manifold.

**Remark 0.2.** Maybe here is the right place to explain our notion motivic Chern class transformation for $mC_*$. Of course the notion “motivic (dual) $\lambda$-class transformation” would also be possible by the corresponding normalization condition for $X$ smooth. But we understand our transformations $T_y$ and $mC_*$ by (1) and (2) as natural “motivic liftings” of the Chern-Schwartz-MacPherson transformation $c_*$, with $T_{-1,*(x)}([id_X]) = c_*(X) \otimes \mathbb{Q}$ for any singular $X$.

Our first construction of the motivic Chern class transformation $mC_*$ is based on the following simple description of $K_0(var/X)$ in terms of proper morphisms $[X' \to X]$. Let $Iso^{pr}(var/X)$ be the free abelian group on isomorphism classes of proper morphisms $[X' \to X]$. Then one gets a canonical quotient map

$$Iso^{pr}(var/X) \to K_0(var/X)$$

(commuting with proper pushdown), which is an epimorphism of groups by “additivity”.

**Lemma 0.1.** $K_0(var/X)$ is isomorphic to the quotient of $Iso^{pr}(var/X)$ modulo the “acyclicity” relation

$$[\emptyset \to X] = 0 \quad \text{and} \quad [\hat{X}' \to X] - [\hat{Z}' \to X] = [X' \to X] - [Z' \to X], \quad \text{(ac)}$$

for any cartesian diagram

$$\begin{array}{ccc}
\hat{Z}' & \longrightarrow & \hat{X}' \\
\downarrow & & \downarrow q \\
Z' & \longrightarrow & X,
\end{array}$$

with $q$ proper, $i$ a closed embedding and $q : \hat{X}' \setminus \hat{Z}' \to X' \setminus Z'$ an isomorphism.
For the base field $k = \mathbb{C}$ and $q : X' \to X$ proper we now consider the filtered Du Bois complex $(\Omega^*_X, F)$ of $X'$ as introduced in [DB]. This is a filtered complex of $O_{X'}$-modules with maps differential operators of order at most one, whose graded pieces

$$gr^p_F(\Omega^*_X) \in D^b_{coh}(X)$$

are bounded complexes of $O_{X'}$-modules with coherent cohomology. Here the filtration is decreasing. This filtered Du Bois complex is unique up to isomorphism in a suitable derived category $\mathcal{D}_{diff}(X')$. Here the assumption $k = \mathbb{C}$ is used, since the proof of [DB] depends on Deligne’s theory of mixed Hodge structures. In particular,

$$[gr^p_F(\Omega^*_X)] := \sum_{i} (-1)^i H^i(gr^p_F(\Omega^*_X)) \in G_0(X')$$

is well defined. Moreover, $gr^p_F$ commutes with proper pushdown [DB, prop.1.3]. By the construction of [DB] one has

$$gr^p_F(\Omega^*_X) \simeq 0 \quad \text{for} \quad p < 0 \text{ or } p > \dim(X'),$$

together with a filtration-preserving map from the algebraic De Rham complex (with the “stupied” filtration, compare Section 5)

$$can : (DR(O_{X'}), F) \to (\Omega^*_X, F),$$

which is a filtered quasi-isomorphism for $X'$ smooth. Then the transformation

$$mC_* : Iso^p(var/X) \to G_0(X) \otimes \mathbb{Z}[y];$$

$$[q : X' \to X] \mapsto \sum_{p \geq 0} q_*[gr^p_F(\Omega^*_X)] \cdot (-y)^p$$

satisfies, by [DB, prop.1.3,prop.4.11], the “acyclicity” relation and therefore, by lemma 0.1, induces our motivic Chern class transformation $mC_*$. In particular

$$mC_p([id_X]) = [gr^p_F(\Omega^*_X)] \cdot -p] = (-1)^p \cdot [gr^p_F(\Omega^*_X)] \in G_0(X)$$

and by definition $X$ has at most “Du Bois singularities” if

$$can : O_X = gr^0_F(DR(O_X)) \to gr^0_F(\Omega^*_X)$$

is a quasi-isomorphism.

The more recent results of Guillén and Navarro Aznar [GNA] allow a construction of a Du Bois complex $\Omega^*_X$ as before without the use of (mixed) Hodge theory. Instead, “only” Hironaka’s resolution of singularities is used [GNA], together with a corresponding “Chow lemma” [GNA] for the comparison of different resolutions. So by this approach we can define $mC_*$ as before in the algebraic context over any base field $k$ of characteristic 0.
Similarly it applies to the context of compactifiable complex analytic spaces \cite[thm. 4.1, sec. 4.5-4.7]{GNA}, using Hironaka’s resolution of singularities for analytic spaces together with an “analytic Chow lemma” \cite[lem. on p. 69]{GNA}. Here we work in the category \( AN_\infty \) of compactifiable complex analytic spaces \( X = \bar{X} \setminus \partial X \), with \( \partial X \) a closed analytic subspace of the compact complex analytic space \( \bar{X} \). Then we fix an equivalence class \( \{(\bar{X}, \partial X)\} \) of bimeromorphic equivalent compactifications, and consider only similar equivalence classes of complex analytic morphism \( f : X' \to X \) with a holomorphic extension \( \bar{f} : \bar{X}' \to \bar{X} \) (compare \cite[sec. 4.5-4.7]{GNA}). And for a compact complex analytic space \( X \) we introduce the analytic relative Grothendieck group \( K_0^{an}(X) \) as the quotient of the free abelian group of isomorphism classes of compactifiable morphisms \( Y \to X \) to \( X \), modulo the “additivity” relation \( \text{add} \) for \( Z \subset Y \) a compactifiable inclusion of a closed analytic subspace of \( Y \). Then lemma \[ \text{(1)} \] holds also in this analytic context, with \( Iso^{pr}(an/X) \) the free abelian group on isomorphism classes of proper analytic morphisms \( [X' \to X] \) (i.e. with \( X' \) compact).

Note that the Chern-Schwartz-MacPherson class transformation \( c_* \) and the \( L \)-class transformation of Cappell-Shaneson are also defined for compact spaces in this complex analytic context. The Todd transformation \( td_* : G_0(X) \to H_{2*}(X, \mathbb{Q}) \) can be defined as the composition

\[
G_0(X) \xrightarrow{\alpha} K_0^{top}(X) \xrightarrow{td_*} H_{2*}(X, \mathbb{Q}),
\]

with \( \alpha \) the Riemann-Roch transformation to (periodic) topological K-homology (in even degrees) constructed by Levy \cite{Levy} (generalizing the corresponding transformation of Baum-Fulton-MacPherson \cite{BFM2} for the quasi-projective complex algebraic context) and the topological Todd transformation \( td_* \) constructed in \cite{BFM2, FM}. If we restrict ourself to a compact complex manifold \( X \), then we can also apply the Riemann-Roch transformation to Hodge cohomology

\[
G_0(X) \xrightarrow{\tau} \bigoplus_{k \geq 0} H^k(X, \Omega^k_X)
\]

constructed by O’Brien-Toledo-Tong \cite{OTT}. In any case we also get the commutative diagrams \[ \text{(1)}, \text{(2)} \text{ and } \text{(4)} \] for a compact complex analytic space \( X \).

The simplest approach to our characteristic classes and the corresponding natural transformations comes from a description of \( K_0(var/X) \) (or \( K_0(an/X) \)) in terms of a “blow-up” relation for smooth spaces mapping properly to \( X \), which is due to Looijenga \cite{Lo} and Bittner \cite{Bi}. This presentation is an easy application of the deep “weak factorization theorem” of \cite{AKMW, W}.

Let \( Iso^{pr}(sm/X) \) be the free abelian group on isomorphism classes of proper morphisms \( [X' \to X] \), with \( X' \) smooth (and pure dimensional and/or quasi-projective if one wants). Then one gets a canonical quotient map

\[
Iso^{pr}(sm/X) \to K_0(var/X)
\]
(commuting with proper pushdown), which is an epimorphism of groups by
"additivity" and Hironaka’s resolution of singularities. Then we have by [B3]
thm.5.1] the following basic result:

**Theorem 0.1 (Bittner).** The group \( K_0(\text{var}/X) \) is isomorphic to the quotient of \( \text{Iso}^{pr}(\text{sm}/X) \) modulo the “blow-up” relation

\[ [\emptyset \to X] = 0 \quad \text{and} \quad [\text{Bl}_Y X' \to X] - [E \to X] = [X' \to X] - [Y \to X], \quad (\text{bl}) \]

for any cartesian diagram

\[
\begin{array}{ccc}
E & \xrightarrow{i'} & \text{Bl}_Y X' \\
\downarrow q' & & \downarrow q \\
Y & \xrightarrow{i} & X' & \xrightarrow{f} & X,
\end{array}
\]

with \( i \) a closed embedding of smooth (pure dimensional) spaces and \( f : X' \to X \) proper. Here \( \text{Bl}_Y X' \to X' \) is the blow-up of \( X' \) along \( Y \) with exceptional divisor \( E \). Note that all these spaces over \( X \) are also smooth (and pure dimensional and/or quasi-projective).

\[ \Box \]

**Remark 0.3.** Since “resolution of singularities” and the “weak factorization theorem” can also be used in the complex analytic context, the simple proof of [B3] thm.5.1] implies the counterpart of Theorem 0.1 also for \( K_0(\text{an}/X) \) with \( X \) a compact complex analytic space. Of course here we cannot require \( X' \) to be quasi-projective!

A big advantage of Theorem 0.1 is the fact that the “blow-up” relation (bl) is a statement for smooth (pure dimensional) spaces mapping properly to \( X \). Therefore all our transformations are determined by “functoriality” and “normalization”. Then we only have to check the “blow-up” relation in the following form (and similarly for the complex analytic context):

**Corollary 0.1.** Let \( B_* : \text{var}/k \to \text{group} \) be a functor from the category \( \text{var}/k \) of (reduced) seperated schemes of finite type over \( \text{spec}(k) \) to the category of abelian groups, which is covariantly functorial for proper morphism, with \( B_*(\emptyset) := \{0\} \). Assume we can associate to any (quasi-projective) smooth space \( X \in \text{ob}(\text{var}/k) \) (of pure dimension) a distinguished element \( d_X \in B_*(X) \) such that

1. \( h_* (d_{X'}) = d_X \) for any isomorphism \( h : X' \to X \).
2. \( q_*(d_{\text{Bl}_Y X}) - i_* q'_*(d_E) = d_X - i_*(d_Y) \in B_*(X) \) for any cartesian blow-up diagram as in theorem (\text{[7.7]} with \( f = \text{id}_X \).

Then there is by (1) a unique group homomorphism \( \Phi : \text{Iso}^{pr}(\text{sm}/X) \to B_*(X) \) satisfying the “normalization” condition \( \Phi([f : X' \to X]) = f_* (d_X) \). By (2) and functoriality this satisfies the “blow-up” relation of Theorem (\text{[7.7]} so that there is a unique induced group homomorphism \( \Phi : K_0(\text{var}/X) \to B_*(X) \) commuting with proper pushdown and satisfying the “normalization” condition \( \Phi([\text{id}_X]) = d_X \) for \( X \) (quasi-projective) smooth (and pure dimensional).

\[ \Box \]
As a first example, we get the existence of a unique transformation
\[ mC_0 : K_0(\text{var}/X) \to G_0(X) \]
with \( mC_0([\text{id}_X]) = [\mathcal{O}_X] \) for \( X \) smooth (and pure dimensional). Note that the “blow-up” relation (2) of Corollary follows from the well known relations (compare [FL, (R5) on p.106, prop.4.1 on p.169]):
\[ q'_*[\mathcal{O}_E] = [\mathcal{O}_Y] \quad \text{and} \quad q_*[\mathcal{O}_{Bl_YX}] = [\mathcal{O}_X]. \]

Similarly, the existence of a unique transformation
\[ mC_* : K_0(\text{var}/X) \to G_0(X) \otimes \mathbb{Z}[y] \]
with \( mC_*([\text{id}_X]) = \sum_{i=0}^d [\Lambda^i T^*X] \cdot y^i = \lambda_y([T^*X]) \cap [\mathcal{O}_X] \) for \( X \) smooth and pure \( d \)-dimensional follows directly from [Gr, IV.1.2.1] or [GNA, prop.3.3] (where the last reference also applies to the complex analytic context).

From the “blow-up” relation for \( \lambda_y([T^*X]) \cap [\mathcal{O}_X] \) we can deduce the “blow-up” relation for the Hirzebruch class \( T^*_y(TX) \cap [X] \in H_*(X) \otimes \mathbb{Q}[y] \) in the context of quasi-projective smooth spaces by using “only” the classical Grothendieck-Riemann-Roch theorem for quasi-projective smooth spaces as in [L1, thm.15.2] (instead of the (modified) singular Riemann-Roch transformation as in (II)).

Finally, in Section 4 we deduce from Corollary the existence of the unique selfduality transformation \( sd : K_0(\text{var}/X) \to \Omega(X) \) with \( sd([\text{id}_X]) = [\mathcal{Q}_X[n]] \) for \( X \) smooth and pure \( n \)-dimensional.

In the final section we explain, for \( k \subset \mathbb{C} \), the most powerful and functorial construction of the motivic Chern class transformation \( mC_* \). This was in the beginning our original approach. It is based on some deep results from the theory of algebraic mixed Hodge modules due to M.Saito [Sai1]-[Sai6], which imply the existence of the two natural transformations
\[
\begin{align*}
K_0(\text{var}/X) & \xrightarrow{mH} K_0(MHM(X/k)) \xrightarrow{gr^F_D, DR} H_*(X) \otimes \mathbb{Z}[y, y^{-1}]
\end{align*}
\]
whose composition is our motivic transformation \( mC_* \). Here \( K_0(MHM(X/k)) \) is the Grothendieck group of the abelian category \( MHM(X/k) \) of mixed Hodge modules on \( X \), and \( mH \) is defined by
\[
mH([f : Y \to X]) := [f! \mathbb{Q}^H_Y] \in K_0(MHM(X/k)).
\]
Here \( \mathbb{Q}^H_Y \) is in some sense the “constant Hodge module” on \( Y \). Similarly, \( gr^F_D, DR \) comes form a suitable filtered de Rham complex of the filtered holonomic D-module underlying a mixed Hodge module.

In this paper we focus on the construction of our homology class transformations \( mC_* \) and \( T^*_y \), together with the unification of the transformations \( c_* \otimes \mathbb{Q} \), \( td_* \) and \( L_* \).
In a sequel to this paper we will show that $mC_\ast, T_y\ast$ and the Chern-Schwartz-MacPherson transformation $c_\ast$ are in fact universal additive characteristic classes that can be defined as natural transformations on the relative Grothendieck group $K_0(var/X)$. The appropriate notion of a general (co)homology theory with Chern class operators was already introduced by Levine and Morel \[LM\] in the algebraic context as an oriented cohomology theory or an oriented Borel-Moore weak homology theory. And two of their main results describe

- algebraic $K$-theory $K^0(X) = G_0(X)$ as the universal oriented cohomology theory with a multiplicative formal group law on the category of smooth quasi-projective spaces, and
- Chow groups $A_\ast(X)$ as the universal oriented Borel-Moore weak homology theory with an additive formal group law on $var/k$.

Here we only sketch our main idea in the context of a classical characteristic cohomology class transformation $c_\ast : K^0(X) \to H^{2\ast}(X, R)$ for complex vector bundles (with $R$ a $\mathbb{Q}$-algebra), which is multiplicative

$$cl\ast(V \times W) = cl\ast(V) \times cl\ast(W)$$

and normalized: $cl^0(V) = 1 \in H^0(\cdot, R)$. And we assume that the corresponding transformation (for the base field $k = \mathbb{C}$)

$$\Phi_{cl} : Iso^pro(sm/X) \to H_{BM}^{2\ast}(X, R) ; [f : X' \to X] \mapsto f_\ast(cl\ast(TX') \cap [X'])$$

satisfies the “blow-up” relation (2) of Corollary \[HM\] in the special case $X = \{pt\}$ a point. Then $\Phi = \Phi_{cl}$ commutes with exterior products (by the multiplicativity of $cl\ast$) and factorizes as ring homomorphisms

$$\xymatrix{Iso^pro(sm/\{pt\}) \ar[r] \ar[d] & \Omega^U_\ast \otimes \mathbb{Q} \ar[d]^-\Phi \\
K_0(var/\{pt\}) \ar[r]^-\Phi & R = H_{2\ast}(\{pt\}, R),}$$

with $\Omega^U_\ast$ the cobordism ring of stable almost complex manifolds. But $\Omega^U_\ast \otimes \mathbb{Q}$ is a polynomial ring in the classes of all complex projective spaces. Moreover, the characteristic class transformation $cl\ast$ is uniquely fixed by

$$\Phi([P^n(C)]) := \Phi([P^n(C) \to \{pt\}]) = \int_{[P^n(C) \subseteq \mathbb{P}^n(C)]} (cl\ast(T\mathbb{P}^n(C)) \cap [P^n(C)])$$

for all $n$. But if $\Phi$ also factorizes over $K_0(var/\{pt\})$ then we get by “additivity” and “multiplicativity”:

$$\Phi([P^n(C)]) = 1 + (-y) + \cdots + (-y)^n \quad \text{with} \quad y := 1 - \Phi([\mathbb{P}^1(C)]).$$

So $\Phi$ is a specialisation of the Hirzebruch $\chi_y$-genus corresponding to the Hirzebruch characteristic class $T_y$ \[Hi\] and our Hirzebruch class transformation $T_y\ast$. 
is the most general “additive” one for homology with values in a $\mathbb{Q}$-algebra $R$!

Note that the specialisation $y = 1$ corresponding to the signature genus $\text{sign} = \chi_1$ and the characteristic $L$-class transformation $cl^* = L^* = T^*_1$ is the only one that factorizes by the canonical map $\Omega^*_U \rightarrow \Omega^*_{SO}$ over the cobordism ring $\Omega^*_{SO}$ of oriented manifolds:

$$
\begin{array}{ccc}
\text{Iso}^{pro}(sm/{\{pt\}}) & \longrightarrow & \Omega^*_{SO} \otimes \mathbb{Q} \\
\downarrow & & \downarrow \phi \\
K_0(var/{\{pt\}}) & \longrightarrow & \Phi \longmapsto R = H_{2n}({\{pt\}}, R),
\end{array}
$$

since $[\mathbb{P}^1(C)] = 0 \in \Omega^*_{SO}$!

More generally, $\Omega^*_U$ is a commutative ring generated by the classes of all complex projective spaces together with the classes of the Milnor manifolds $H_{n,m} := \{(x_0, \ldots, x_n), (y_0, \ldots, y_m) \in \mathbb{P}^n(C) \times \mathbb{P}^m(C) | \sum_{i=0}^{m} x_i \cdot y_i = 0\}$ for $0 < m \leq n$. These are algebraic (Zariski trivial) projective bundles over projective spaces. So again any ring homomorphism $\Phi : \text{Iso}^{pro}(sm/{\{pt\}}) \rightarrow R$ to a commutative ring $R$, which factorizes as

$$
\begin{array}{ccc}
\text{Iso}^{pro}(sm/{\{pt\}}) & \longrightarrow & \Omega^*_U \\
\downarrow & & \downarrow \phi \\
K_0(var/{\{pt\}}) & \longrightarrow & \Phi \longmapsto R
\end{array}
$$

has to be a specialisation of the Hirzebruch $\chi_y$-genus.

If one considers more general (co)homology theories $H^*$, it becomes more natural to weaken the normalization condition to

- $cl^*(L) = f(c^1(L))$ for any line bundle $L$, with $f(z) \in H^*({\{pt\}})[[z]]$ a formal power series with $f(0) \in H^*({\{pt\}})$ a unit,

with $cl^*$ a multiplicative transformation on the set of isomorphism classes of vector bundles. Note that this is indeed the case for $cl^* = \sum_{i=0}^{r(k(V))} [A^i V^\vee] \cdot y^i = \lambda_y([V^\vee]) \in K^0(X) \otimes \mathbb{Z}[y, (1 + y)^{-1}]$ (with $V^\vee$ the dual bundle) corresponding to our motivic Chern class transformation $mC^*$ (with $c^1(L) = 1 - [L^\vee]$ for a line bundle $L$ so that $f(z) = 1 + y - y \cdot z$). But for the composed transformation $mC^* : K_0(var/X) \rightarrow G_0(X) \otimes \mathbb{Z}[y] \rightarrow G_0(X) \otimes \mathbb{Z}[y, (1 + y)^{-1}]$
we are not allowed anymore to specialize to the special value $y = -1$! And indeed the corresponding genus $\chi_{-1}$ is just the topological Euler characteristic corresponding to the total Chern class transformation $c^* = c^*$ related to the Chern-Schwartz-MacPherson class transformation

$$c_* : K_0(\text{var} / X) \to A_*(X) \to H^B_{BM} (X, \mathbb{Z}).$$

Here one can deduce this Chern class transformation on the relative Grothendieck group $K_0(\text{var} / X)$ without appealing to MacPherson’s theorem, since the distinguished element

$$d_X := c^* (TX) \cap [X] \in A_*(X)$$

of a smooth space $X$ satisfies the assumptions of Corollary 0.1. Condition (1) follows from the projection formula, and condition (2) is an easy application (by pushing down to $X$) of the classical “blowing up formula for Chern classes” [Ful, thm. 15.4]. In fact this formula is true over any base field $k$. In particular for $k = \mathbb{R}$ we get a commutative diagram of natural transformations:

$$
\begin{array}{ccc}
K_0(\text{var} / X) & \xrightarrow{e_2} & F(\mathbb{R}(X), \mathbb{Z}_2) \\
\downarrow c_* & & \downarrow w_* \\
A_*(X/\mathbb{R}) & \xrightarrow{cl_R} & H^B_{BM} (\mathbb{R}(X), \mathbb{Z}_2),
\end{array}
$$

with $cl_R$ the corresponding fundamental class map of Borel-Haefliger [BH], $F(\mathbb{R}(X), \mathbb{Z}_2)$ the group of real algebraically $\mathbb{Z}_2$-valued constructible functions and $w_*$ the Stiefel-Whitney homology class transformation of Sullivan [Sm] [PaM]. Here $e_2$ is again given by

$$e_2([f : Y \to X]) := f_1 Y(R) \in F(\mathbb{R}(X), \mathbb{Z}_2),$$

i.e. by taking fiberwise the topological mod 2 Euler characteristic with compact support of the corresponding map $f : Y(\mathbb{R}) \to X(\mathbb{R})$ on the set of real points.

We also have a canonical fundamental class map

$$Iso^\text{pro} (\text{sm} / \{\text{spec}(\mathbb{R})\}) \to \Omega^O_* ; [X \to \{pt\}] \mapsto [X(\mathbb{R})]$$

to the cobordism ring $\Omega^O_*$ of unoriented manifolds, which is a polynomial $\mathbb{Z}_2$-algebra generated by the images of all projective spaces and Milnor manifolds (which are defined over $\text{spec}(\mathbb{R})$). So again any ring homomorphism

$$\Phi : Iso^\text{pro} (\text{sm} / \{\text{spec}(\mathbb{R})\}) \to \mathbb{Z}_2$$

which factorizes as

$$
\begin{array}{ccc}
\text{Iso}^\text{pro} (\text{sm} / \{\text{spec}(\mathbb{R})\}) & \longrightarrow & \Omega^O_* \\
\downarrow & & \downarrow \Phi \\
K_0(\text{var} / \{pt\}) & \xrightarrow{\Phi} & \mathbb{Z}_2
\end{array}
$$
has to be the specialisation $\chi_1 \mod 2$ of the Hirzebruch $\chi_y$-genus, since now $[\mathbb{P}^1(\mathbb{R})] = 0 \in \Omega_*^\ast$!

And this is just the mod 2 Euler characteristic corresponding to the total Stiefel-Whitney class transformation $c^* = w^*$ for real vector bundles related to Sullivan’s homology class transformation $w_*$. In this way the relative Grothendieck group $K_0(var/X)$ of algebraic spaces unifies all known functorial homology class transformations

$$mC_\ast, T_y, c_\ast, td_\ast, L_\ast \quad \text{and} \quad w_\ast,$$

and the “additivity” condition “singles these out” of all multiplicative (and normalized) cohomology class transformations $c^*$ for vector bundles. More precisely, this fixes the genus $\Phi$, but in general not the corresponding cohomology class $c^*$. This is related to suitable (co)homology operations on $K^0(\cdot) \otimes R$ or $A_\ast(\cdot) \otimes R$, e.g.:

1. The duality involution $[V] \mapsto [V^\vee]$ on $K^0(\cdot)$, or the Adams operation $\Psi^j$ on $K^0(\cdot) \otimes \mathbb{Z}[1/j]$ (for $j \in \mathbb{Z}$, and compare with [FL]).
2. The Steenrod $p$-th power operation on $A_\ast(\cdot) \otimes F_p$ (for $p \in \mathbb{N}$ a prime number, and compare with [Bro]).

Remark 0.4. The “additivity” of the Euler characteristic (with compact support) is well known. Compare with [Ful, Ex.15.2.10] for a characterization of the arithmetic genus in terms of a different “additivity” property, and with [J] for a characterization of the signature in terms of “Novikov additivity” (as developed in [Si] in the context of “Witt spaces”).

Let us finally remark that most of our constructions would apply to a perfect base field $k$ of positive characteristic, once a suitable version of resolution of singularities is available (e.g. such that the construction of the Du Bois complex from [GNA] applies).

It is a pleasure to thank P.Aluffi for some discussions on this subject. The paper [A1] was a strong motivation for our work, which started with the papers [Y1]–[Y6] of the third author. Some of these papers are partly motivated by the final remark of MacPherson’s survey article:

1. [MP] p.326]: “... It remains to be seen whether there is a unified theory of characteristic classes of singular varieties like the classical one outlined above. ...”.

We hope that our results give some key to MacPherson’s question and answer the following question or problem:

2. [Y4] p.267]: “... Is there a theory of characteristic homology classes unifying the above three characteristic homology classes of possibly singular varieties? ...”.

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2. [Y4] p.267]: “... Is there a theory of characteristic homology classes unifying the above three characteristic homology classes of possibly singular varieties? ...”.
• [Al] p.3367: “... There is a strong motivic feel to the theory of Chern-Schwartz-MacPherson classes, although this does not seem to have yet been congealed into a precise statement in the literature. ...”.

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1 The generalized Hirzebruch theorem

First we recall the classical generalized Hirzebruch Riemann-Roch theorem [Hi] (compare with [Y3, Y4]). Let \( X \) be a smooth complex projective variety and \( E \) a holomorphic vector bundle over \( X \). The \( \chi_y \)-characteristic of \( E \) is defined by

\[
\chi_y(X, E) := \sum_{p \geq 0} \chi(X, E \otimes \Lambda^p T^* X) \cdot y^p
\]

\[
= \sum_{p \geq 0} \left( \sum_{i \geq 0} (-1)^i \dim C \chi^i(X, E \otimes \Lambda^p T^* X) \right) \cdot y^p,
\]

with \( T^* X \) the holomorphic cotangent bundle of \( X \). Then one gets

\[
\chi_y(X, E) = \int_X T^*_y(TX) \cdot ch_{(1+y)}(E) \cap [X] \quad \in \mathbb{Q}[y], \quad \text{(gHRR)}
\]

with \( ch_{(1+y)}(E) := \sum_{j=1}^{rk E} e^{\beta_j(1+y)} \) and \( T^*_y(TX) := \prod_{i=1}^{dim X} Q_y(\alpha_i) \).

Here \( \beta_j \) are the Chern roots of \( E \) and \( \alpha_i \) are the Chern roots of the tangent bundle \( TX \). Finally \( Q_y(\alpha) \) is the normalized power series

\[
Q_y(\alpha) := \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \quad \in \mathbb{Q}[y][[\alpha]].
\]

So this power series \( Q_y(\alpha) \) specializes to

\[
Q_y(\alpha) = \begin{cases} 
1 + \alpha & \text{for } y = -1, \\
\frac{\alpha}{1-y} & \text{for } y = 0, \\
\frac{\alpha}{\tanh \alpha} & \text{for } y = 1.
\end{cases}
\]
Therefore the modified Todd class $T_y^*(TX)$ unifies the following important characteristic cohomology classes of $TX$:

$$T_y^*(TX) = \begin{cases} c^*(TX) & \text{the total Chern class for } y = -1, \\ t_d^*(TX) & \text{the total Todd class for } y = 0, \\ L^*(TX) & \text{the total Thom-Hirzebruch L-class for } y = 1. \end{cases}$$

Note that $\text{(gHRR)}$ implies for $y = 0$ the classical Hirzebruch Riemann-Roch theorem $\text{[Hi]}$:

$$\chi(X, E) = \int_X t_d^*(TX) \cdot ch^*(E) \cap [X],$$

with $ch^*(E) := \sum_{j=1}^{rk E} e^{\beta_j}$

the classical Chern character. This is a ring homomorphism (for the tensor product)

$$ch^* : K^0(X) \to H^{2*}(X),$$

so that one gets back $\text{(gHRR)}$ by

$$\chi_y(X, E) = \int_X t_d^*(TX) \cdot ch^*(E \otimes \lambda_y(T^*X)) \cap [X]$$

with $\tilde{T}_y^*(TX) := t_d^*(TX) \cdot ch^*(\lambda_y(T^*X))$ corresponding to the unnormalized power series (compare $\text{[HBJ]}$ p.11,p.61)

$$\tilde{Q}_y(\alpha) := \frac{\alpha(1 + ye^{-\alpha})}{1 - e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]], \quad \text{with} \quad \tilde{Q}_y(0) = 1 + y.$$

Then the relation (compare $\text{[HBJ]}$ p.62)

$$Q_y(\alpha) = \tilde{Q}_y(\alpha \cdot (1 + y)) \cdot (1 + y)^{-1} \in \mathbb{Q}[y][[\alpha]]$$

implies for $X$ pure $d$-dimensional:

$$T_y^i(TX) = (1 + y)^{i-d} \cdot \tilde{T}_y^i(TX) \in H^{2i}(X) \otimes \mathbb{Q}[y] \quad \text{and} \quad \langle T_y \rangle_j(X) = (1 + y)^{-j} \cdot \langle \tilde{T}_y \rangle_j(X) \in H_{2j}(X) \otimes \mathbb{Q}[y].$$

Here we use the notation $cl_*(X) := cl^*(TX) \cap [X]$ for a characteristic class $cl^*$ of vector bundles. In particular $(T_y)_*(X)$ and $(\tilde{T}_y)_*(X)$ agree in degree 0 so that

$$\chi_y(X) := \chi_y(X, \mathcal{O}_X) = \int_X (T_y)_*(X) = \int_X (\tilde{T}_y)_*(X).$$

So this “twisting” by powers of $1 + y$ just comes from changing $\tilde{Q}_y$ to the normalized power series $Q_y$, which has the right specialization properties.
And (gHRR') implies (gHRR) by a similar calculation.

As an example, Hirzebruch [Hi, lem.1.8.1] gets by a simple residue calculation the equation

\[ \int_X (T_y)_*(X) = 1 + (-y) + \cdots + (-y)^n \quad \text{for} \quad X = \mathbb{P}^n(\mathbb{C}). \]

The gRRH-theorem for the trivial bundle \( E \) specializes to the calculation of the following important invariants:

\[
\chi_y(X) = \begin{cases} 
  e(X) = \int_X c^*(TX) \cap [X] & \text{the Euler characteristic for} \quad y = -1, \\
  \chi(X) = \int_X td^*(TX) \cap [X] & \text{the arithmetic genus for} \quad y = 0, \\
  sign(X) = \int_X L^*(TX) \cap [X] & \text{the signature for} \quad y = 1,
\end{cases}
\]

corresponding to the Poincaré-Hopf or Gauss-Bonnet theorem \((y = -1)\), the Hirzebruch Riemann-Roch theorem \((y = 0)\) and the Hirzebruch signature theorem \((y = 1)\).

These three invariants and classes have been generalized to a singular complex algebraic variety \( X \) in the following way (where the invariants are only defined for \( X \) compact):

\[ e(X) = \int_X c_*(X), \quad \text{with} \quad c_* : F(X) \to H_*(X) := \begin{cases} 
  A_*(X) & (y = -1) \\
  H_{BM}^2(X, \mathbb{Z}) & \text{(for Borel-Moore homology)} \quad (y = 0)
\end{cases} \]

the Chern class transformation of MacPherson [M1, Ken] from the abelian group \( F(X) \) of complex algebraically constructible functions to homology, where one can use Chow groups \( A_*(\cdot) \) or Borel-Moore homology groups \( H_{BM}^2(\cdot, \mathbb{Z}) \). Then \( c_*(X) := c_*(1_X) \) agrees by [BrS] via “Alexander duality” for compact \( X \) embeddable into a complex manifold with the Schwartz class of \( X \) as introduced before by M.-H. Schwartz [Schwa].

\[ \chi(X) = \int_X td_*(X), \quad \text{with} \quad td_* : G_0(X) \to H_* (X) \otimes \mathbb{Q} \quad (y = 0) \]

the Todd transformation in the singular Riemann-Roch theorem of Baum-Fulton-MacPherson [BFM1, FM] (for Borel-Moore homology) or Fulton [Flu] (for Chow groups). Here \( G_0(X) \) is the Grothendieck group of coherent sheaves. Then \( td_*(X) := td_*(\mathbb{O}_X) \), with \( \mathbb{O}_X \) the class of the structure sheaf.

Finally for compact \( X \) one also has

\[ sign(X) = \int_X L_*(X), \quad \text{with} \quad L_* : \Omega(X) \to H_{2*}(X, \mathbb{Q}) \quad (y = 1) \]

the homology L-class transformation of Cappell-Shaneson [CS1] (as reformulated by Yokura [Y1] and corrected in Section 4). Here \( \Omega(X) \) is the abelian
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group of cobordism classes of selfdual constructible complexes. Then $L_*(X) := L_*(\mathcal{IC}_X)$ is the homology $L$-class of Goresky-MacPherson \cite{GM}, with $\mathcal{IC}_X$ the class of their intersection cohomology complex. For a rational PL-homology manifold $X$, these $L$-classes are due to Thom \cite{Thom} (compare \cite{MS}, sec.20).

Remark 1.1. The discussion above applies to any compact complex manifold $X$, since the generalized Hirzebruch Riemann-Roch theorem is also true in this context by an application of the Atiyah-Singer Index theorem \cite{AS}. Similarly, $\mathrm{gHRR}$ follows as before in the algebraic context over any base field $k$ from the corresponding $\mathrm{HRR}$-theorem \cite{Ful} cor.15.2.1 for Chow-groups $A_*(\cdot) \otimes \mathbb{Q}$ (instead of homology $H^{2*}(\cdot, \mathbb{Q})$). And this HRR-theorem is just the special case of the Grothendieck Riemann-Roch theorem \cite{Ful} thm.15.2 for a constant map $X \to \text{spec}(k)$, with $X$ a smooth complete variety.

All these transformations commute with the corresponding pushdown for proper maps (where all spaces are assumed to be compact in the case of the $L$-class transformation). They are uniquely characterized by this pushdown property and the normalization condition that for $X$ smooth and pure-dimensional one gets back the corresponding classes of $TX$:

$$c_*(X) = c^*(TX) \cap [X], \quad \mathrm{td}_*(X) = Td^*(TX) \cap [X] \quad \text{and} \quad L_*(X) = L^*(TX) \cap [X].$$

Here the uniqueness result follows from resolution of singularities, and in the case of the $L$-class transformation one has to be more careful: This normalization fixes $L_*$ only on the image of the transformation $sd$ from $\mathbb{H}$.

So all these theories have the same formalism, but they are defined on completely different theories! Nevertheless, it is natural to ask for another theory of characteristic homology classes of singular complex algebraic varieties, which unifies the above characteristic homology class transformations (as in \cite{M2, Y3, Y4}). Of course in the smooth case, this is done by the generalized Todd class $T_\mathcal{N}(TX) \cap [X]$ of the tangent bundle. We now explain our solution to this question.

2 Motivic Chern classes for singular varieties

In the following we consider reduced separated schemes of finite type over a base field $k$ of characteristic 0, and for simplicity we just call them algebraic varieties.

Let $K_0(\text{var}/X)$ be the relative Grothendieck group of algebraic varieties over $X$, i.e. the quotient of the free abelian group of isomorphism classes of algebraic morphisms $Y \to X$ to $X$, modulo the “additivity relation” $\text{add}$. These relative groups were introduced by Looijenga in his Bourbaki talk \cite{Lo} about motivic measures and motivic integration, and then further studied by Bittner \cite{Bi}. From our point of view, these are a “motivic version” of the group $F(X)$ of algebraically constructible functions (compare also with \cite{CL}).

In particular, they have the same formalism, i.e. functorial pushdown $f_!$ and pullback $f^*$ for any algebraic map $f : X' \to X$ (which is not necessarily proper),
together with a ring multiplication (with unit \([id_X] = k^*[id_{pt}]\) for \(k: X \to \{pt\}\) a constant map) satisfying the *projection formula*

\[ f_!(\alpha \cdot f^* \beta) = (f \alpha) \cdot \beta \]

and the *base change formula* \(g^*f_! = f'_!g^*\) for any cartesian diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & X' \\
\downarrow{f'} & & \downarrow{f} \\
Y & \xrightarrow{g} & X.
\end{array}
\]

For later use, let us recall the simple definition of the **pullback** and **pushdown** for \(f: X' \to X\), and of **exterior products**:

\[ f_!([h: Z \to X']) = [f \circ h: Z \to X] \quad \text{and} \]

\[ [Z \to X] \times [Z' \to X'] = [Z \times Z' \to X \times X'] \]

Moreover \(f^*([g: Y \to X]) = [g': Y' \to X']\)

is defined by taking fiber products as above.

By these exterior products, \(K_0(var/\{pt\})\) becomes a commutative ring and \(K_0(var/X)\) a \(K_0(var/\{pt\})\)-module such that \(f_!\) and \(f^*\) are \(K_0(var/\{pt\})\)-linear. Moreover, the homomorphism \(e\) from (3) commutes with all these transformation. In particular it is a ring homomorphism.

**Remark 2.1.** If we consider only *proper* morphism for the pushdown, and only *smooth* morphism (of constant relative dimension) for the pullback, then the same formalism applies to the groups \(Iso^{pr}(var/X)\) and \(Iso^{pr}(sm/X)\). But here we only have an *exterior product*, but no ring structure on these groups, since the diagonal morphism \(X \to X \times X\) is not a smooth morphism. But this is true for \(X = \{pt\} = \text{spec}(k)\) a point, so \(Iso^{pr}(var/\{pt\})\) and \(Iso^{pr}(sm/\{pt\})\) become a commutative ring such that the transformations above are linear over these rings. Moreover, the group epimorphisms

\[ Iso^{pr}(sm/X) \to Iso^{pr}(var/X) \to K_0(var/X) \]

commute with these three operations. A similar remark also applies to the compactifiable complex analytic context.

Note that the groups \(Iso^{pr}(var/X)\) and \(Iso^{pr}(sm/X)\) are *graded* by the dimension of the spaces mapping to \(X\) (if we work with pure dimensional spaces), whereas \(K_0(var/X)\) only becomes a *filtered group* with \(F_kK_0(var/X)\) generated by \([X' \to X]\) with \(\text{dim}X' \leq k\).

Let us now explain the simple
Proof of lemma 0.1. The surjective projection \( \pi : Iso^{pr}(\text{var}/X) \rightarrow K_0(\text{var}/X) \) factorizes by definition over the quotient \( Iso^{pr}(\text{var}/X)/(\text{ac}) \). So it is enough to define an inverse map \( \phi : K_0(\text{var}/X) \rightarrow Iso^{pr}(\text{var}/X)/(\text{ac}) \).

And this is induced by the map \( \phi : Iso(\text{var}/X) \rightarrow Iso^{pr}(\text{var}/X)/(\text{ac}) \);

\[
[X' \rightarrow X] \mapsto [\bar{X}' \rightarrow X]_{ac} - [\bar{X}' \setminus X' \rightarrow X]_{ac}
\]

taking the “difference class” for a compactification \( X' \subset \bar{X}' \rightarrow X \) of the corresponding map, with \( X' \subset \bar{X}' \) (Zariski) open and \( \bar{X}' \rightarrow X \) proper (i.e. by Nakayama’s theorem [4]). By using the fiber product of two such compactifications, one gets from the relation \( \text{add} \) that this is well defined. A simple calculation shows that it satisfies also the “additivity relation” \( \text{add} \) for \( Z \subset X' \) closed:

\[
\phi([X' \rightarrow X]) = [\bar{X}' \rightarrow X]_{ac} - [\bar{X}' \setminus X' \rightarrow X]_{ac}
= [\bar{X}' \rightarrow X]_{ac} - [Z \cup \bar{X}' \setminus X' \rightarrow X]_{ac} + [Z \cup \bar{X}' \setminus X' \rightarrow X]_{ac} - [\bar{X}' \setminus X' \rightarrow X]_{ac}
= \phi([X' \setminus Z \rightarrow X]) + \phi([Z \rightarrow X])
\]

For \( X' \rightarrow X \) proper we can take \( X' = \bar{X}' \) so that \( \phi \circ \pi = \text{id} \).

One can avoid the use of Nakayama’s theorem by decomposing \( X' \) first into the disjoint union of quasi-projective (e.g affine) pieces \( X'_i \). For the \( X'_i \) one uses the definition as before (with \( \bar{X}' \rightarrow X \) projective) and then one defines

\[
\phi([X' \rightarrow X]) := \sum_i \phi([X'_i \rightarrow X]),
\]

which is well defined by “additivity”. \( \square \)

Remark 2.2. The proof of Theorem 0.1 given in [3] is of similar nature. But here one takes for \( X' \) smooth a compactification such that \( \bar{X}' \setminus X' \) is a divisor with simple normal crossings. This can be done by Hironaka’s resolution of singularities. Then one has to compare two such compactifications. And for this the “weak factorization theorem” of [21] is used, which relates two such compactifications by a finite sequence of blowing ups and blowing downs along (suitable) smooth centers! Moreover, it is enough to consider the “blow-up” relation only for (pure dimensional) quasi-projective varieties \( X' \) with \( X' \rightarrow X \) a projective morphism.

By the discussion in the introduction we therefore get the

**Theorem 2.1.** There exists a unique group homomorphism \( mC_* \) commuting with pushdown for proper maps:

\[
mC_* : K_0(\text{var}/X) \rightarrow G_0(X) \otimes \mathbb{Z}[y],
\]
satisfying the normalization condition

\[ m_{C_*}([\text{id}_X]) = \sum_{i=0}^{\dim X} [\Lambda^i T^* X] \cdot y^i = \lambda_y([T^* X]) \cap [O_X] \]

for \( X \) smooth and pure-dimensional. Here \( \lambda_y : K^0(X) \to K^0(X) \otimes \mathbb{Z}[y] \) is the total \( \lambda \)-class transformation on the Grothendieck \( K^0(X) \) of coherent locally free sheaves on \( X \), with \( \cap [O_X] : K^0(X) \to G_0(X) \) induced by \( \otimes O_X \), which is an isomorphism for \( X \) smooth.

Remark 2.3. For a compact complex space \( X \) this result is true for \( K_0(\text{an}/X) \), except that \( \cap [O_X] : K^0(X) \to G_0(X) \) needs not to be an isomorphism.

Corollary 2.1. 1. \( m_{C_*} \) is filtration preserving, if \( G_0(X) \otimes \mathbb{Z}[y] \) has the induced filtration coming from the grading with \( y \) of degree one.

2. \( m_{C_0} : K_0(\text{var}/X) \to G_0(X) \) is the unique group homomorphism commuting with pushdown for proper maps and satisfying the normalization condition \( m_{C_0}([\text{id}_X]) = [O_X] \) for \( X \) smooth (and pure-dimensional).

3. \( m_{C_*} \) commutes with exterior products.

4. One has the following Verdier Riemann-Roch formula for \( f : X' \to X \) a smooth morphism (of constant relative dimension):

\[ \lambda_y(T_f^* \cap f^* m_{C_*}([Z \to X])) = m_{C_*} f^*([Z \to X]). \]

Here \( T_f \) is the bundle on \( X' \) of tangent spaces to the fibers of \( f \), i.e. the kernel of the surjection \( df : TX' \to f^* TX \). Moreover \( f^* : G_0(X) \otimes \mathbb{Z}[y] \to G_0(X') \otimes \mathbb{Z}[y] \) is induced from the corresponding pullback of Grothendieck groups by linear extension over \( \mathbb{Z}[y] \). In particular \( m_{C_*} \) commutes with pullback under étale morphisms (i.e. smooth of relative dimension 0).

Proof. 1. follows by induction on \( \dim X \) from resolution of singularities, “additivity” and the normalization condition.

2. follows from the fact that pushdown on \( G_0 \otimes \mathbb{Z}[y] \) is degree preserving.

3. follows from the normalization condition together with

\[ (f \times f')_* \simeq f_* \times f'_* \quad \text{on} \quad K_0(\text{var}/X \times X') \quad \text{and} \quad G_0(X \times X') \otimes \mathbb{Z}[y] \]

for \( f, f' \) proper, and by the multiplicativity of \( \lambda_y((\cdot)^\vee) \):

\[ \lambda_y(T^* (X \times X')) = \lambda_y(T^* X) \times \lambda_y(T^* X'). \]

Compare also with \([KW, KWY]\) for the case of Chern classes.
4. It is enough to prove the claim for $g : Z \to X$ proper with $Z$ smooth (and pure dimensional). Then it follows from the projection formula

$$g'_*(\alpha \cdot g'^* \beta) = (g'_* \alpha) \cdot \beta$$

for $g'$ proper and the base change formula $f^*g_*=g'_*f'^*$ for the cartesian diagram

$$
\begin{array}{c}
Z' \xrightarrow{f'} Z \\
g' \downarrow \quad \quad \downarrow g \\
X' \xrightarrow{f} X
\end{array}
$$

with $g,g'$ proper and $f,f'$ smooth (of constant relative dimension). Here these formulae also hold for $G_0(\cdot) \otimes \mathbb{Z}[y]$:

$$\lambda_y(T^*_f) \cap f^*mC_*([Z \to X]) = \lambda_y(T^*_f) \cap f^*g_*mC_*([id_Z])$$

$$= \lambda_y(T^*_f) \cap g'_*f'^*mC_*([id_Z]) = g'_* (g'^* \lambda_y(T^*_f) \cap f'^*(\lambda_y(T^*Z) \cap [O_Z]))$$

$$= g'_* (\lambda_y(T^*_f) \cup f'^* \lambda_y(T^*Z) \cap [O_{Z'}]) = g'_* (\lambda_y(T^*Z') \cap [O_{Z'}])$$

$$= g'_*mC_*([id_{Z'}]) = mC_*([Z' \to X']) = mC_* f^*([Z \to X]).$$

Of course, we also used $f'^*[O_Z] = [O_{Z'}]$ together with the multiplicativity and functoriality of $\lambda_y((\cdot)^\vee)$. Compare also with [Y5] for the case of Chern classes.

3. Hirzebruch classes for singular varieties

We continue to work with reduced algebraic varieties over a base field $k$ of characteristic 0. In the following, the homology group $H_* (\cdot)$ is either the Chow group or the Borel-Moore homology group (in even degrees) in case $k = \mathbb{C}$.

Let us start with a reformulation by Yokura [Y4] of the *singular Riemann-Roch theorem* of Baum-Fulton-MacPherson (for Borel-Moore homology) and Fulton (for Chow groups). Consider the group homomorphism

$$td_{1+y} : G_0(X) \otimes \mathbb{Z}[y] \to H_* (X) \otimes \mathbb{Q}[y, (1+y)^{-1}],$$

$$td_{1+y}([F]) := \sum_{i \geq 0} td_i([F]) \cdot (1+y)^{-i}, \quad \text{(gBFM)}$$

with $td_i$ the degree $i$ component of the transformation $td_*$, which is linearly extended over $\mathbb{Z}[y]$. Since $td_*d$ is degree preserving, this new transformation also commutes with proper pushdown (which again is defined by linear extension over $\mathbb{Z}[y]$). By [Y2], we have the

**Lemma 3.1.** Assume $X$ is smooth (and pure dimensional). Then

$$td_{1+y} (\lambda_y(T^*X) \cap [O_X]) = T^*_y(TX) \cap [X] \in H_* (X) \otimes \mathbb{Q}[y],$$
Hirzebruch classes and motivic Chern classes

\[ \text{Let us sketch the simple proof. Since } X \text{ is smooth, the singular Riemann-Roch theorem reduces to the classical Grothendieck Riemann-Roch theorem, i.e.} \]

\[ \text{with } \text{ch}^*: K^0(X) \to H_*(X) \otimes \mathbb{Q}, \]

\[ \text{and the claim follows from (11). In particular, the “twisting” (by powers of } 1 + y \text{) form } \text{td}^* \text{ to } \text{td}^*(1+y) \text{ is only used to get the right normalization in lemma 3.1!} \]

Remark 3.1. The same reasoning applies also to the compactifiable complex analytic context with the Todd transformation (6), since for \( X \) smooth the following diagram commutes by [Levy]:

\[
\begin{array}{cccc}
K^0(X) & \xrightarrow{\text{can}} & K^0_{\text{top}}(X) \\
\downarrow \cap [O_X] & & \downarrow \cap [X] \\
G_0(X) & \xrightarrow{\alpha} & K^0_{\text{top}}(X).
\end{array}
\]

Here \( \text{can} \) is the canonical map induced from taking the underlying topological complex vector bundle of a holomorphic vector bundle, and \( PD \) is the Poincaré duality for topological \( K \)-theory.

Now we are ready to introduce our second motivic transformation.

**Theorem 3.1.** There exist unique group homomorphisms \( T_y, \tilde{T}_y \) commuting with pushdown for proper maps:

\[ T_y, \tilde{T}_y : K_0(\text{var}/X) \to H_*(X) \otimes \mathbb{Q}[y], \]

satisfying the normalization condition

\[ T_y([id_X]) = T_y^*(TX) \cap [X], \quad \tilde{T}_y([id_X]) = \tilde{T}_y^*(TX) \cap [X] \]

for \( X \) smooth and pure-dimensional.

**Proof.** The natural transformation \( T_y \) is given as the composition

\[ T_y := \text{td}_{(1+y)} \circ mC_* : K_0(\text{var}/X) \to H_*(X) \otimes \mathbb{Q}[y] \subset H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]. \]

And similarly

\[ \tilde{T}_y := \text{td} \circ mC_* : K_0(\text{var}/X) \to H_*(X) \otimes \mathbb{Q}[y]. \]

The normalization condition follows from lemma 3.1. Since \( \text{Iso}^{op}(sm/X) \to K_0(\text{var}/X) \) is surjective, we get uniqueness together with \( T_y(K_0(\text{var}/X)) \subset H_*(X) \otimes \mathbb{Q}[y]. \) \( \square \)
By the same argument one gets from functoriality and normalization the commutativity of the diagrams \([\mathbf{2}]\) and \([\mathbf{4}]\).

**Remark 3.2.** The same result and proof also apply to the compactifiable complex analytic context.

Before we state some corollaries, let us introduce the following filtration on \(H_*(X) \otimes \mathbb{Q}[y]\):

\[ F_k (H_*(X) \otimes \mathbb{Q}[y]) := F_k H_*(X) \otimes F_k \mathbb{Q}[y], \]

where each factor has its canonical filtration coming from the natural grading. In particular, any evaluation homomorphism \(H_*(X) \otimes \mathbb{Q}[y] \to H_*(X)\) for a \(y \in \mathbb{Q}\) is then also filtration preserving.

**Corollary 3.1.**

1. \(T_y\) is filtration preserving.

2. \(T_y, \tilde{T}_y\) commute with exterior products.

3. One has the following Verdier Riemann-Roch formula for \(f : X' \to X\) a smooth morphism (of constant relative dimension):

\[ T_y^*(T_f) \cap f^* T_y^*([Z \to X]) = T_y^* f^*([Z \to X]), \]

and similarly for \(\tilde{T}_y\). In particular \(T_y, \tilde{T}_y\) commute with pullback under étale morphisms. \(\square\)

Let us recall our definition

\[ T_y(X) := T_y([id_X]) \in H_*(X) \otimes \mathbb{Q}[y] \]
\[ \chi_y(X) := T_y([X \to \{pt\}]) \in \mathbb{Z}[y] \subset \mathbb{Q}[y] \]

for the Hirzebruch class and characteristic of the singular space \(X\) (and similarly for the motivic Chern class), so that

\[ T_{-1}([id_X]) = c_*(X) \otimes \mathbb{Q} \quad \text{for any singular } X. \]

It seems that our Hirzebruch class \(T_y(X)\) corresponds (for \(k = \mathbb{C}\)) to a similar class announced some years ago by Cappell and Shaneson \([CS2]\) and \([Sh\text{ sec.4}].\) If this is the case, then there is a mistake in their announcement, because one of their statements can be interpreted as claiming that \(T_{0*}([id_X])\) is the singular Todd class \(td_*(X)\) for any singular \(X\). But this is not the case, since

\[ mC_0([id_X]) \neq [\mathcal{O}_X] \in G_0(X) \quad \text{and} \quad T_{0*}([id_X]) \neq td_*(X) \]

for some singular spaces \(X\).

**Example 3.1.** Let \(X\) be a singular curve (i.e. \(\dim X = 1\)) such that \(X\) is not maximal (maximal is sometimes also called weakly normal), but the weak normalization \(X_{\text{max}}\) is smooth. Then the canonical projection \(\pi : X' := X_{\text{max}} \to X\)
is not an isomorphism, but nevertheless a topological homeomorphism. By “additivity” one gets
\[ \pi_*([id_X]) = [id_X] \in K_0(var/X) \] so that
\[ T_0\ast(X) = \pi_*T_0\ast(X') = \pi_*td\ast(X') = td\ast([O_X]) \] But by assumption \[ \pi_*([O_X]) = [O_X] + n \cdot [pt] \] with \( n > 0 \) so that
\[ T_0\ast(X) = td\ast(X) + n \cdot [pt] \neq td\ast(X) \in H_\ast(X) \otimes \mathbb{Q} \].

One gets similar examples in any dimension by taking the product with a projective space.

Taking a complete singular curve \( X \) over \( k = \mathbb{C} \) such that the normalization \( \pi : X' := X_{nor} \to X \) is not a topological homeomorphism, one gets in the same way examples of singular \( X \) with
\[ T_1\ast(X) \neq L_\ast(X) := L_\ast([IC_X]) \] Note that the normalization map \( \pi \) is a small resolution of singularities so that \( \pi_\ast(IC_{X'}) = IC_X \). To distinguish between these characteristic classes, we call (for later reasons) \( T_0\ast(X) \) the Hodge-Todd class and \( T_1\ast(X) \) the Hodge L-class of \( X \).

**Example 3.2.** Assume that the algebraic variety \( X \) has at most “Du Bois singularities” in the sense that
\[ can : O_X = gr^0_F(DR(O_X)) \to gr^0_F(\Omega^\ast_X) \] is a quasi-isomorphism. For example \( X \) has only “rational singularities” [Kov, Sai5], e.g. \( X \) is a toric variety. Then
\[ mC_0([id_X]) = [O_X] \in G_0(X) \] and therefore \( T_0\ast(X) = td\ast(X) \).

Using the “additivity” in \( K_0(var/X) \) and the natural transformation \( T_y\ast \), one gets similar additivity properties of the Todd class \( td\ast(X) = T_0\ast(X) \) for \( X \) smooth (or with a most “Du Bois singularities”), which seem to be new and do not follow directly from the original definition.

**Example 3.3.** 1. By the gHRR-theorem we have \( T_y\ast([\mathbb{P}^1 \to pt]) = \chi_y(\mathbb{P}^1) = 1 - y \) so that
\[ \chi_y(\mathbb{A}^1) = -y \quad \text{and} \quad \chi_y(\mathbb{P}^n) = 1 - y + \cdots + (-y)^n \] by “additivity” and “multiplicativity” for exterior products.
2. $T_{y*}$ becomes multiplicative in Zariski locally trivial bundles, e.g. if $E \to X$ is an algebraic vector bundle of rank $r + 1$, then the corresponding projective bundle $\mathbb{P}(E) \to X$ is Zariski locally trivial so that

$$T_{y*}([\mathbb{P}(E) \to X]) = T_{y*}([id_X]) \cdot (1 - y + \cdots + (-y)^r) \in H_*(X) \otimes \mathbb{Q}[y].$$

3. Let $\pi : X' \to X$ be the blow-up of an algebraic variety $X$ along an algebraic subvariety $Y$ such that the inclusion $Y \to X$ is a regular embedding of pure codimension $r + 1$ (e.g. $X$ and $Y$ are smooth). Then $\pi$ is an isomorphism over $X \setminus Y$ and a projective bundle over $Y$ corresponding to the normal bundle of $Y$ in $X$ of rank $r + 1$. So by “additivity” one gets

$$T_{y*}([X' \to X]) = T_{y*}([id_X]) + T_{y*}([Y \to X]) \cdot (-y + \cdots + (-y)^r)$$

in $H_*(X) \otimes \mathbb{Q}[y]$. In particular $T_{y*}([X' \to X]) = T_{y*}([id_X])$, which is a homology class version of the birational invariance of the arithmetic genus $\chi_0$. More generally, by pushing down to a point we get for $X$ complete the blow-up formula

$$\chi_p(X') = \chi_p(X) + \chi_p(Y) \cdot (-y + \cdots + (-y)^r) \in \mathbb{Q}[y]. \quad (14)$$

Note that in case 3. one also has $\pi_*td_*(X') = td_*(X)$, by functoriality of $td_*$ and the relation $[R\pi_*\pi^*\mathcal{O}_X] = [\mathcal{O}_X]$ for such a blow-up. Using the “weak factorization theorem” [AKMW, W], we get the following result, which seems to be new and was motivated by a corresponding study of Aluffi about Chern classes [A1], and of Borisov-Libgober about elliptic classes [BL2].

**Corollary 3.2.** Let $\pi : Y \to X$ be a resolution of singularities. Then the class

$$\pi_* (Td^*(TY) \cap [Y]) = \pi_* T_0([id_Y]) \in H_*(X) \otimes \mathbb{Q}$$

is independent of $Y$.

**Proof.** Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be two resolution of singularities, together with a resolution of singularities of the fiber-product $Z \to Y \times_X Y'$ so that we get induced birational morphisms $p : Z \to Y$ and $p' : Z \to Y'$.

By the “weak factorization theorem” [AKMW, W], this map $p$ (or $p'$) can be decomposed as a finite sequence of projections from smooth spaces lying over $Y$ (or $Y'$), which are obtained by blowing up or blowing down along smooth centers. By the birational invariance above we get $\pi_*td_*(Z) = td_*(X)$ (or $\pi'_*td_*(Z) = td_*(X')$), from which the claim follows. \qed

**Remark 3.3.** In the framework of motivic integration [DL, Lo, Craw], it is natural to localize the $K_0(var/\{pt\})$-module $K_0(var/X)$ with respect to the class of the affine line $[k^1 \to \{pt\}] =: \mathbb{L}$. Here the module structure comes from pullback along a constant map $X \to \{pt\}$. Then $mC_*$, $T_{y*}$ and $\mathbb{L}$ induce similar transformations on $M(var/X) := K_0(var/X)[\mathbb{L}^{-1}]$:

$$mC_* : M(var/X) \to G_0(X) \otimes \mathbb{Z}[y, y^{-1}],$$

$$T_{y*} : M(var/X) \to H_*(X) \otimes \mathbb{Q}[y, y^{-1}].$$
since $\chi_y(L) = -y$ is invertible. Note that the original transformations $mC_*$, $T_{y*}$ and $\tilde{T}_{y*}$ are ring homomorphisms on a point space $\{pt\}$, and module homomorphisms over any space $X$, by the multiplicativity with respect to exterior products. Similarly, these extend by Corollary 2.1.1, Corollary 3.1.1 and (11) to transformations

$$mC^\wedge : \hat{M}(\text{var}/X) \to G_0(X) \otimes \mathbb{Z}[y][[y^{-1}]],$$

$$T_{y*}^\wedge, \tilde{T}_{y*}^\wedge : \hat{M}(\text{var}/X) \to H_*(X) \otimes \mathbb{Q}[y][[y^{-1}]],$$

of the corresponding completions (for $k \to -\infty$) with respect to the dimension filtration $F_k M(\text{var}/X)$ of $M(\text{var}/X)$. Here

$$F_k M(\text{var}/X) \text{ is generated by } [X' \to X] L^{-n} \text{ with } \dim(X') - n \leq k.$$ 

This completion $\hat{M}(\text{var}/X)$ also comes up in the context of motivic integration. In the absolute case $X = \{pt\}$ it was introduced by Kontsevich in his study of the arc-space $L(X)$ of $X$ as the value group of a “motivic measure” $\hat{\mu}$ on a suitable Boolean algebra of subsets of $L(X)$. This allows one (compare [DL, 4.4] and [V]) to introduce new invariants for $X$ pure-dimensional, but maybe singular, as the value of

$$\hat{\mu}(L(X)) \in \hat{M}(\text{var}/\{pt\}) \to R$$

under a suitable homomorphism to a ring $R$. Instead of $\hat{\mu}(L(X))$, one can also use related “motivic integrals” over $L(X)$. By our work, one can now introduce similar characteristic classes by using a “relative motivic measure” $\tilde{\mu}_X$ with values in $\hat{M}(\text{var}/X)$ [Lo, sec.4], and the same for “motivic integrals” (compare also with [Y6]).

**Example 3.4.** Let $Y$ be a pure-dimensional manifold and $D = \sum_{i=1}^r a_i D_i$ be an effective normal crossing divisor (e.g. $a_i \in \mathbb{N}_0$) on $Y$, with smooth irreducible components $D_i$. Then one can introduce and evaluate a motivic integral of the following type (compare [DL] [Lo] [Craw] [V]):

$$\int_{L(Y)} L^{-\text{ord}(D)} d\tilde{\mu}_Y = \sum_{I \subseteq \{1, \ldots, r\}} [D_I \to Y] \cdot \prod_{i \in I} \frac{L - 1}{\frac{L}{a_i + 1} - 1} = \sum_{I \subseteq \{1, \ldots, r\}} [D_I \to Y] \cdot \prod_{i \in I} \left(\frac{L - 1}{\frac{L}{a_i + 1} - 1} - 1\right).$$

(16)

Here we use the notation:

$$D_I := \bigcap_{i \in I} D_i \quad \text{(with } D_{\emptyset} := Y), \quad D_I^c := D_I \setminus \bigcup_{i \in \{1, \ldots, r\} \setminus I} D_i.$$
the following "stringy invariant" of \( X \) motivic integration says (as an application of the "transformation rule"), that simplicity we do not consider \( Q \) of Stein and [DL, Lo, Craw]. So this is an intrinsic invariant of \( X \) does not depend on the choice of the resolution (compare with [V, (7.7),(7.8)], i.e. this resolution of singularities \( \pi \) developed as the corresponding geometric series in \( \hat{D} := D \). But \( D \) is a closed smooth submanifold of \( Y \) so that \( \text{cl}_s([D \to Y]) \) is just the pushforward to \( Y \) of the corresponding characteristic (homology) class
\[
(17) \quad \text{cl}_s((D \to Y)) = \text{cl}^s(TD_I) \cap [D_I] \quad \text{for} \quad \text{cl}_s = mC_*, T_y*, \hat{T}_y*.
\]

Here the factor \((L^{a_i+1} - 1)^{-1} \in \mathbb{Z}[y][[y^{-1}]]\) in (16) has to be developed as the corresponding geometric series in \( \hat{M}(\text{var}/\{pt\}) \), and similarly for the factor \(((y)^{a_i+1} - 1)^{-1} \in \mathbb{Z}[y][[y^{-1}]]\) in (17). Moreover one gets the last equality in (16) by multiplying out the following products:
\[
(18) \quad \prod_{i=1}^{r} \left( b_i \cdot [D_i \to Y] + [Y \setminus D_i \to Y] \right) = \prod_{i=1}^{r} \left( (b_i - 1) \cdot [D_i \to Y] + [id_Y] \right) \in \hat{M}(\text{var}/Y),
\]
with \( b_i := (L - 1)(L^{a_i+1} - 1)^{-1} \in \hat{M}(\text{var}/\{pt\}) \). Recall that multiplication in \( \hat{M}(\text{var}/Y) \) is induced from taking the fiber product over \( Y \).

Let now \( X \) be a normal pure-dimensional algebraic variety which is Gorenstein, i.e., such that the canonical divisor \( K_X \) is a Cartier divisor. Take a resolution of singularities \( \pi : Y \to X \) such that the “discrepancy divisor” \( D := K_Y - \pi^*K_X \) is a divisor with normal crossing. Assume \( X \) has only log-terminal singularities, i.e. this \( D \) is for one (and then for any) such resolution an effective divisor (so that \( X \) has already canonical singularities, since for simplicity we do not consider \( \mathbb{Q} \)-divisors). Then one of the main results from motivic integration says (as an application of the “transformation rule”), that the following “stringy invariant” of \( X \):
\[
(19) \quad \mathcal{E}_{st}(X) = \pi_* \left( \int_{\mathcal{L}(Y)} L^{-\text{ord}(D)} d\mu_Y \right) \in \hat{M}(\text{var}/X)
\]
does not depend on the choice of the resolution (compare with [V (7.7),(7.8)] and [DL, Lo, Craw]). So this is an intrinsic invariant of \( X \), and we can introduce the corresponding stringy characteristic homology class \( \text{cl}^{st}_s(X) \) of \( X \) for \( \text{cl}_s = mC_*, T_y*, \hat{T}_y* \), by
\[
(20) \quad \text{cl}^{st}_s(X) = \pi_* \mathcal{E}_{st}(X) := mC_*, T_y*, \hat{T}_y*, \pi_* \left( \int_{\mathcal{L}(Y)} L^{-\text{ord}(D)} d\mu_Y \right) \in \hat{M}(\text{var}/X).
\]
Then the stringy Hirzebruch classes $T_{yst}^a(X)$ and $\tilde{T}_{yst}^a(X)$ interpolate in the following sense between the elliptic class $Ell(X)$ of $X$ defined by Borisov-Libgober [BL1] [BL2] and the stringy $E$-function $E_{st}(X)$ of Batyrev [Bat] for $X$ complex algebraic and complete:

$$\lim_{\tau \to i\infty} Ell(X)(z, \tau) = y^{-1/2 \cdot \dim X} : \tilde{T}_{yst}^a(X) \quad \text{for} \quad y = e^{2\pi iz},$$

and

$$\chi_{st}^a(X) : = \int_{[X]} T_{-yst}^a(X)$$

$$= \int_{[X]} \tilde{T}_{-yst}^a(X) = E_{st}(X)(y, 1)$$

for $X$ (complex algebraic and complete). Here the equality (21) for the elliptic class of $X$ in the sense of Borisov-Libgober [BL2] def.3.2,rem.3.3, $Ell(X) := Ell_{orb}(X, E, G)$ with $G = \{id\}$ and $E = \emptyset$, follows directly from (17) and the calculation given in [BL2] p.14/15. Similarly, (20) follows from (17) and the definition of $E_{st}(X)(u, v) := E_{st}(X, \emptyset)(u, v)$ given in [Bat] def.3.7, if one takes $(u, v) = (y, 1)$ (compare with (41) in Section 5).

So these stringy Hirzebruch classes are “in between” the elliptic class and the stringy $E$-function, and as suitable limits they are “weaker” than these more general invariants. But they have the following good properties of both of them:

- The stringy Hirzebruch classes come from a functorial “additive” characteristic homology class.

- The stringy $E$-function comes from the “additive” $E$-polynomial defined by Hodge theory on $K_0(var/\{pt\})$, which doesn’t have a homology class version (compare with section 5).

- The elliptic class is a homology class, which doesn’t come from an “additive” characteristic class (of vector bundles), since the corresponding elliptic genus is more general than the Hirzebruch $\chi_y$-genus, which is the most general “additive” genus of such a class (as explained in the introduction).

Finally the stringy Hirzebruch class $T_{yst}^a(X)$ specializes for $y = -1$ in the following way to the stringy Chern class $c_{yst}^a(X)$ of $X$ as introduced in [A2] [FLNU]:

$$\lim_{y \to -1} T_{yst}^a(X) = c_{yst}^a(X) \in A_*(X) \otimes \mathbb{Q}.$$  (23)

This follows directly from (17) and the left square of the commutative diagram (2), if one compares this with [A2] sec.5.5,6.5 and [FLNU] sec.3,def.4.1.

If we specialize in (17) to $y = 0$, then we get by “additivity”:

$$\lim_{y \to 0} T_{yst}^a(X) = \pi_*(Td^*(TY) \cap [Y])$$  (24)
so that the well-definedness of this limit is just a special case of Corollary 5.2.

Assume now that \( \pi : Y \to X \) is a crepant resolution, i.e. all multiplicities \( a_i \) of the “discrepancy divisor” \( D \) are zero. Then one gets in the same way:

\[
T^{\text{st}}_{y^*}(X) = \pi_* (T^*_y(T^*Y) \cap [Y])
\]

(25)

and

\[
c^{\text{st}}(X) = \lim_{y \to -1} T^{\text{st}}_{y^*}(X) = \pi_* (c^*(T^*Y) \cap [Y]).
\]

(26)

In particular the right hand side does not depend on the choice of a crepant resolution (compare [A1, cor.1.2] and [FLNU, prop.4.4]).

4 Comparison with functorial \( L \)-classes

In this section we work in the algebraic context over the base field \( k = \mathbb{C} \), or in the complex analytic context with compact spaces, and explain the relation of our motivic class \( T_1 \), to the \( L \)-class transformation of Cappell-Shaneson [CS1].

Let \( X \) be a complex analytic (algebraic) space with \( A := D^b_c(X) \) the bounded derived category of complex analytically (algebraically) constructible complexes of \( \mathbb{K} \)-vector spaces (for \( k \) a subfield of \( \mathbb{R} \), compare [KS, Sch]). So we consider bounded sheaf complexes \( F \), which have locally constant cohomology sheaves with finite dimensional stalks along the strata of a complex analytic (algebraic) Whitney stratification \( X_\bullet \) of \( X \). In the algebraic context, or for a compact analytic space \( X \), such a stratification has only finitely many strata. Then \( A \) is a triangulated category with translation functor \( T_A = T = [1] \) and a duality \( D_A \) in the sense of [You, Ba1, Ba2, GN] induced by the Verdier duality functor (compare [Sch1, chap.4] and [KS, chap.VIII]):

\[
D_X := \text{Rhom}(\cdot, k!\mathbb{K}_{\text{pt}}) : D^b_c(X) \to D^b_c(X),
\]

with \( k : X \to \{\text{pt}\} \) a constant map. Here a triangulated category \( B \) with translation functor \( T_B \) has a duality \( D_B \) in the sense of [You, Ba1, Ba2, GN], if

1. \( D_B : B \to B \) is a contravariant functor with \( D_B \circ T_B = T_B^{-1} \circ D_B \), which preserves distinguished triangles,

2. one has a biduality isomorphism \( \text{can} : \text{id} \cong D_B \circ D_B \) such that \( \text{can}_{T_B(M)} = T_B(\text{can}_M) \) (i.e. \( \text{can} \) commutes with translation) and \( D_B(\text{can}_M) \circ \text{can}_{D_B(M)} = \text{id}_{D_B(M)} \) for any \( M \in \text{ob}(B) \).

For the biduality isomorphism \( \text{can} \) in the case of the Verdier duality functor \( D_X \) compare with [Sch, cor.4.2.2] and [KS, prop.3.4.3,prop.8.4.9].

Remark 4.1. Our notion of a duality \( D_B \) in a triangulated category \( B \) corresponds to a \( \delta = 1 \)-duality in the sense of [Ba1, Ba2, GN]. These authors also consider a \( \delta = -1 \)-duality corresponding to the case that \( D_B \) maps distinguished triangles to the negative of distinguished triangles.
A constructible complex $\mathcal{F} \in \text{ob}(\mathcal{D}^b_c(X))$ is called selfdual (and similarly in the more general context of a triangulated category with duality), if there is an isomorphism

$$d : \mathcal{F} \xrightarrow{\sim} D_X(\mathcal{F}) .$$

The pair $(\mathcal{F}, d)$ is called symmetric or skew-symmetric (in [You] this is called “even” or “odd”), if

$$D_X(d) \circ \text{can} = d \quad \text{or} \quad D_X(d) \circ \text{can} = -d .$$

Note that a skew-symmetric pair $(\mathcal{F}, d)$ for the biduality isomorphism can is just the same as a symmetric pair $(\mathcal{F}, d)$ for the biduality isomorphism $-\text{can}$.

Finally an isomorphism or isometry of selfdual objects $(\mathcal{F}, d)$ and $(\mathcal{F}', d')$ is an isomorphism $u$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{u} & \mathcal{F}' \\
\downarrow d & & \downarrow d' \\
D_X(\mathcal{F}) & \xleftarrow{\sim} & D_X(\mathcal{F}') .
\end{array}$$

We consider now the complex algebraic context, or a compact analytic space $X$ so that the isomorphism classes of such (skew-)symmetric selfdual complexes form a set. This becomes a monoid with addition induced by the direct sum. Then the Witt group $W_\pm(X)$ ($W_-(X)$, resp.) of such symmetric (skew-symmetric, resp.) selfdual complexes on $X$ is the quotient of this monoid with respect to the submonoid of neutral symmetric (skew-symmetric, resp.) selfdual complexes in the sense of [Ba1, Ba2]. Here $(\mathcal{F}, d)$ is called neutral, if there exists an isomorphism of distinguished triangles (for some $\mathcal{L} \in \text{ob}(\mathcal{D}^b_c(X))$):

$$\begin{array}{ccc}
D_X(\mathcal{L})[-1] & \xrightarrow{\text{can}} & \mathcal{L} \\
\downarrow & & \downarrow \\
D_X(D_X(\mathcal{L})) & \xrightarrow{\text{can}} & D_X(\mathcal{L}) \text{.}
\end{array}$$

Then $W_\pm(X)$ is indeed an abelian group, since $(\mathcal{F}, d) \oplus (\mathcal{F}, -d)$ is always neutral! These Witt groups in the sense of Balmer are different (!) from a corresponding notion introduced by Youssin [You], based on his notion of an elementary cobordism in the context of a triangulated category with duality. We will call these groups cobordism groups $\Omega_\pm(X)$ of selfdual constructible complexes on $X$. Youssin’s starting point is the notion of an octahedral diagram in a triangulated category with duality $D$, i.e. a diagram $(\text{Oct})$ of the following form:
Here the morphism marked by [1] are of degree one, the triangles marked + are commutative, and the ones marked d are distinguished. Finally the two composite morphisms from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) (via \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \)) have to be the same, and similarly for the two composite morphisms from \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \) (via \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \)).

Application of the duality functor \( D \) and a rotation by 180° about the axis connecting upper-left and lower-right corner induces another octahedral diagram \( (RD \cdot Oct) \) such that \( RD \) applied to \( (RD \cdot Oct) \) gives the octahedral diagram \( (D^2 \cdot Oct) \) which one gets from \( (Oct) \) by application of \( D^2 \) (compare with [You, p.387/388] for more details). Then the octahedral diagram \( (Oct) \) is called symmetric or skew-symmetric (in [You] this is called “even” or “odd”), if there is an isomorphism \( d : (Oct) \to (RD \cdot Oct) \) of octahedral diagrams such that

\[
RD(d) \circ \text{can} = d \quad \text{or} \quad RD(d) \circ \text{can} = -d
\]

as maps of octahedral diagrams \( (Oct) \to (RD \cdot Oct) \) (compare [You, def.6.1]). Note that this induces in particular (skew-)symmetric dualities \( d_1 \) and \( d_2 \) of the corners \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), and \( (Oct, d) \) is called an elementary cobordism between \( (\mathcal{F}_1, d_1) \) and \( (\mathcal{F}_2, d_2) \). This notion is a symmetric relation [You, rem.6.2]. Similarly, \( (\mathcal{F}, d) \) is elementary cobordant to itself (use the octahedral diagram \( (Oct) \) with \( \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{H}_1 = \mathcal{H}_2, \mathcal{G}_1 = 0 = \mathcal{G}_2 \) and the corresponding isomorphism induced by \( d \)). \((\mathcal{F}, d)\) and \((\mathcal{F}', d')\) are called cobordant (compare [CS1, p.528]), if there is a sequence

\[
(\mathcal{F}, d) = (\mathcal{F}_0, d_0), (\mathcal{F}_1, d_1), \ldots, (\mathcal{F}_m, d_m) = (\mathcal{F}', d')
\]

with \((\mathcal{F}_i, d_i)\) elementary cobordant to \((\mathcal{F}_{i+1}, d_{i+1})\) for \( i = 0, \ldots, m - 1 \). This cobordism relation is then an equivalence relation.

The cobordism group \( \Omega_+(X) \) (\( \Omega_-(X) \), resp.) of selfdual constructible complexes on \( X \) is the quotient of the monoid of isomorphism classes of symmetric (skew-symmetric, resp.) selfdual complexes by this cobordism relation. These
are again monoids, since this relation commutes with direct sums. But a neutral selfdual constructible complex \((\mathcal{F}, d)\) in the sense of Balmer given by a distinguished triangle \(\xymatrix{ & \mathbb{L} \ar[r]^-d & \mathcal{F} \ar[l]^-\alpha} \) induces a (skew-)symmetric octahedral diagram of the following type (with isomorphism \(d\) induced by the isomorphism of this distinguished triangle):

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \longrightarrow & [1] \\
\downarrow & & \downarrow \\
\mathbb{L} & \longrightarrow & \mathcal{F}
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
[1] & + & D(\mathbb{L}) \\
\downarrow & & \downarrow \\
\mathcal{F} & \longrightarrow & \mathbb{L} & \longrightarrow & [1]
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
\mathbb{L} & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
D(\mathbb{L}) & + & [1]
\end{array}
\end{array}
\quad \begin{array}{ccc}
\begin{array}{ccc}
D(\mathbb{L}) & + & \mathbb{L} \\
\downarrow & & \downarrow \\
\mathcal{F} & \longrightarrow & \mathbb{L} & \longrightarrow & [1]
\end{array}
\end{array}
\]

So a neutral selfdual constructible complex \((\mathcal{F}, d)\) is elementary corbordant to \(0\). But then \(\Omega_{\pm}(X)\) is also an abelian group together with a canonical group epimorphism (compare \([\text{Ba1}, \text{introduction}]\) and \([\text{Ba2}, \text{rem.3.25}]\)):

\[
W_{\pm}(X) \to \Omega_{\pm}(X) \to 0 .
\]

Consider now a proper algebraic (or holomorphic) map \(f : X \to Y\), with \(X, Y\) compact in the analytic context. Then \(Rf_{\ast} \simeq Rf_{!}\) maps \(D^b_c(X)\) to \(D^b_c(X)\) (compare \([\text{Sch1}, \text{chap.4}]\) and \([\text{KS}, \text{chap.VIII}]\)). Moreover, the adjunction isomorphism (\([\text{Sch1}, p.120]\), \([\text{KS}, \text{prop.3.1.10}]\)):

\[
Rf_{\ast} \text{Rhom}(\mathcal{F}, f'^{-1} \mathbb{K}_{\text{pt}}) \simeq \text{Rhom}(Rf_{\ast} \mathcal{F}, k^\ast \mathbb{K}_{\text{pt}})
\]

induces the isomorphism

\[
Rf_{\ast} D_X \cong D_Y Rf_{!} \simeq D_Y Rf_{\ast}
\]  

so that \(Rf_{\ast}\) commutes with Verdier-duality (compare with \([\text{GN}, \text{def.1.8}]\) for the abstract notion of a duality preserving functor between triangulated categories with duality). In particular \(Rf_{\ast}\) maps selfdual constructible complexes on \(X\) to selfdual constructible complexes on \(Y\) inducing group homomorphisms

\[
f_{\ast} : \Omega_{\pm}(X) \to \Omega_{\pm}(Y); \quad [(\mathcal{F}, d)] \mapsto [(Rf_{\ast} \mathcal{F}, Rf_{\ast}(d))].
\]

Before we can compare this with the corresponding notion from \([\text{CS II, Y1}]\), we have to explain the relation between duality \(D = D_X\) and translation \(T = [1]\). By the equality \(D_X \circ T = T^{-1} \circ D_X\) one gets the translated duality \(T^{2n} D_X := T^{2n} \circ D_X\) \((n \in \mathbb{Z})\) with the biduality isomorphism

\[
can : id \cong D_X \circ D_X \simeq D_X \circ T^{-2n} \circ T^{2n} \circ D_X \simeq T^{2n} D_X \circ T^{2n} D_X .
\]
Then the translation \( T^n \) \((n \in \mathbb{Z})\) induces an isomorphism of cobordism groups (and similarly for Witt groups)

\[
T^n : \Omega(X) := \Omega(X, D_X) := \Omega_+(X, D_X) \oplus \Omega_-(X, D_X) \xrightarrow{\sim} \Omega(X, T^{2n}D_X); \\
[(F, d)] \mapsto [(T^n(F), T^n(d))],
\]

(29)

with

\[
T^n(d) : T^n(F) \xrightarrow{\sim} T^n \circ D_X(F) = T^{2n} \circ T^{-n} \circ D_X(F) = T^{2n}D_X(T^n(F)).
\]

Note that this isomorphism is parity preserving (or changing) depending on \( n \) even (or odd). This follows from the anti-commutative diagram (compare \([KS, rem.1.10.16]\)):

\[
\begin{array}{ccc}
\text{Hom}(F[1], G[1]) & \xrightarrow{\sim} & \text{Hom}(F, G[1])[-1] \\
\downarrow & & \downarrow \\
\text{Hom}(F[1], G)[1] & \xrightarrow{\sim} & \text{Hom}(F, G).
\end{array}
\]

**Remark 4.2.** We only need to consider even shifts \( T^{2n}D_X \) of the Verdier duality. Note that \( T \) maps by definition distinguished triangles to the negative of distinguished triangles so that \( T \circ D_X \) is a \( \delta = -1 \)-duality in the sense of \([Ba1, Ba2, GN]\).

In the following we identify the shifted cobordism groups by the isomorphism (29). Note that \([CS1, Y1]\) use the shifted duality \( T^{2n}D_X \) with \( n(X) \) the complex dimension of \( X \) in their definition of selfdual constructible complexes. Then the following diagram becomes commutative under these identifications:

\[
\begin{array}{ccc}
\Omega(X) & \xrightarrow{T^n(X)} & \Omega(X, T^{2n}(X)D_X) \\
f_* & & f_* \\
\Omega(Y) & \xrightarrow{T^n(Y)} & \Omega(Y, T^{2n}(Y)D_Y),
\end{array}
\]

with \( f_* \) on the right hand side induced by \( Rf_*[n(Y)−n(X)] \) as in \([CS1\ prop.4.1]\) and \([Y1\ prop.1.6]\).

Let us now explain an important example of a (skew-)symmetric selfdual constructible complex.

**Example 4.1.** Assume \( Z \) is a complex manifold of pure dimension \( n \) so that the complex orientation of \( Z \) induces an isomorphism \( k^p \mathcal{K}_Z \cong \mathbb{K}_Z[2n] \) (compare \([KS\ sec.III.3.3]\)). Let \( \mathcal{L} \) be a Poincaré local system on \( Z \), i.e. a locally constant sheaf of finite rank with a symmetric nondegenerate bilinear pairing

\[ \phi : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{K}_Z. \]
For example we can take $\mathcal{L} = \mathbb{K}_Z$ with the obvious pairing given by multiplication. Then $\phi$ induces an isomorphism

$$d : \mathcal{L} \xrightarrow{\sim} \hom(\mathcal{L}, \mathbb{K}_Z) \simeq \Rhom(\mathcal{L}, \mathbb{K}_Z) = D_Z(\mathcal{L})[-2n].$$

In particular $d = d'[n]$ induces a (skew-)symmetric selfduality of $\mathcal{L}[n]$.

Assume moreover that $Z$ is a locally closed constructible subset of the analytic space $X$, with $i : \bar{Z} \to X$ the closed inclusion of the complex analytic closure $\bar{Z}$ of $Z$. Then $d$ induces a (skew-)symmetric selfduality of the twisted intersection cohomology complex $IC^\bullet_{\bar{m}}(\bar{Z}, \mathcal{L})[n]$ on $\bar{Z}$ (compare [Sch1, sec.6.02], with $\bar{m}$ the middle perversity). Here we use the convention that $IC^\bullet_{\bar{m}}(\bar{Z}, \mathcal{L})$ gives back $\mathcal{L}$ by restriction to the open set $Z$ of $\bar{Z}$. By proper pushdown along the map $i$ we finally get the (skew-)symmetric selfdual complex $Ri^*IC^\bullet_{\bar{m}}(\bar{Z}, \mathcal{L})[n]$ on $X$.

We also have to point out that [CS1, Y1] use a slightly weaker notion of “elementary cobordism” for the definition of their cobordism groups of selfdual constructible complexes. In terms of the unshifted Verdier duality functor, this is defined as follows: Start with morphisms

$$\mathcal{G} \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{H}$$

in $D^b_c(X)$ with $v \circ u = 0$. Suppose that there is an isomorphism $\mathcal{H} \simeq D_X(\mathcal{G})$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{v} & \mathcal{H} \\
i \downarrow & & \downarrow i \\
D_X(\mathcal{F}) & \xrightarrow{D_X(u)} & D_X(\mathcal{G})
\end{array}$$

Then the morphisms $u$ and $v$ can be included into an octahedral diagram (Oct) with $X_2 =: C_{u,v}$ an "iterated cone" in the sense of [CS1] (compare [You, lem.5.3]). Moreover one can lift the duality $d : \mathcal{F} \xrightarrow{\sim} D_X(\mathcal{F})$ to a duality

$$d : C_{u,v} \xrightarrow{\sim} D_X(C_{u,v}),$$

and $(\mathcal{F}, d)$ is by their definition elementary cobordant to $(C_{u,v}, d)$.

But the induced duality of $C_{u,v}$ is non-canonical and therefore also non-functorial, since “the cone” of a morphism is not canonically defined in a triangulated category. In particular, it is not clear in general, if the corresponding octahedral diagram (Oct) can be chosen to be selfdual and (skew-)symmetric (compare [You, p.390])! This is indeed the case if $\hom(\mathcal{G}[1], \mathcal{H}) = 0$, e.g. if $\mathcal{G}$ and $\mathcal{H}$ are perverse sheaves with respect to the middle perversity $t$-structure.
Jean-Paul Brasselet, Jörg Schürmann and Shoji Yokura

On the other hand, it is clear by definition, that an elementary cobordism in the sense of [You] is functorial, and that it induces an elementary cobordism in the sense of [CS1]. Therefore the results of [CS1, sec.5] can be reformulated as in [Y1, cor.2.3]:

**Theorem 4.1 (Cappell-Shaneson).** For a compact complex analytic (or algebraic) space $X$ there is a homology $L$-class transformation

$$L_* : \Omega(X) = \Omega^+(X) \oplus \Omega^-(X) \to H_2^*(X, \mathbb{Q})$$

as in [4], which is a group homomorphism functorial for the pushdown $f_*$ induced by a holomorphic (or algebraic) map. The degree of $L_0((\mathcal{F}, d))$ is the signature of the induced pairing

$$H^0(X, \mathcal{F}) \otimes_\mathbb{R} \mathbb{R} \times H^0(X, \mathcal{F}) \otimes_\mathbb{R} \mathbb{R} \to \mathbb{R}$$

(by definition this is 0 for a skew-symmetric pairing). Moreover, for $X$ smooth of pure dimension $n$ one has the normalization

$$L_* ((\mathbb{K}_X[n], d)) = L^*(TX) \cap [X] ,$$

with $(\mathbb{K}_X[n], d)$ as in example 4.1.

If $h : X' \to X$ is an isomorphism of purely $n$-dimensional complex manifolds, then

$$h_* ((\mathbb{K}_{X'}[n], d)) = (\mathbb{K}_X[n], d)$$

(with $d$ as in example 4.1). Consider now the complex algebraic context, or the analytic context with $X$ compact. Then there is a unique group homomorphism

$$sd : Iso^{pro}(sm/X) \to \Omega(X)$$

(30)

satisfying the normalization condition

$$sd([f : X' \to X]) = f_* ((\mathbb{K}_{X'}[n], d))$$

for $X'$ purely $n$-dimensional. But contrarily to what is claimed in [Y1], this transformation $sd$ is in general not surjective!

Before we explain this, we have to recall another important result from [CS1, You]. Let

$$(^m D^{\leq 0}_c(X), ^m D^{\geq 0}_c(X))$$

be the perverse $t$-structure on $D^b_c(X)$ with respect to the middle perversity $m$. (compare [KS] chapter X and [Sch1] chapter 6). The Verdier duality $D_X$ interchanges $^m D^{\leq 0}_c(X)$ and $^m D^{\geq 0}_c(X)$ [KS prop.10.3.5] so that the corresponding perverse cohomology functor $^m H^0 = \tau^{\geq 0} \tau^{\leq 0}$ [KS def.10.1.9] commutes with Verdier duality. Then one has by [You Ex.6.6] (and compare with [CS1 lem.3.3]) the important
Lemma 4.1 (Youssin). Assume \((\mathcal{F}, d)\) is a (skew-)symmetric selfdual constructible complex on \(X\). Then \((\mathcal{F}, d)\) is elementary cobordant to the (skew-)symmetric selfdual constructible complex \((\mathcal{M}^0(\mathcal{F}), \mathcal{M}^0(d))\).

Consider a proper holomorphic map \(f : X' \to X\) of pure-dimensional complex manifolds such that all higher direct image sheaves \(R^i f_* \mathbb{K}_{X'}\) \((i \in \mathbb{Z})\) are locally constant. Let \(n(X), n(X')\) be the complex dimensions of \(X\) and \(X'\). Then

\[
\mathcal{M}^0(Rf_* \mathbb{K}_{X'}[n(X')]) \simeq R^i f_* \mathbb{K}_{X'}[n(X)] \quad \text{for } i = n(X) - n(X').
\]

Note that \(i\) is just the complex fiber dimension of \(f\). Of course, in general the higher direct image sheaves \(R^i f_* \mathbb{K}_{X'}\) are only constructible for such a proper holomorphic map \(f\). But assume that \(X = \mathbb{P}^1(\mathbb{C})\) so that

\[
f : f^{-1}(D_*^+(0)) \to D_*^+(0) := \{z \in \mathbb{C} | 0 < |z| < r\}
\]
satisfies the assumption above for \(r\) small enough. Then the monodromy automorphism of the complexification \((-) \otimes_K \mathbb{C}\) of the local system

\[
\mathcal{M}^0(Rf_* \mathbb{K}_{X'}[n(X')])|D_*^+(0)
\]
has only roots of unity as eigenvalues by the well known “monodromy theorem”.

Take now an irreducible Poincaré local system \(\mathcal{L}\) on \(\mathbb{C}^*\), whose corresponding monodromy automorphism (of the complexification) has not roots of unity as eigenvalues (e.g. a suitable rotation of \(\mathcal{L}_z := \mathbb{C} = \mathbb{R}^2\) for \(\mathbb{K} = \mathbb{R}\) and \(z \in \mathbb{C}^*\)). Then

\[
(\mathcal{L} \mathcal{C}^n_m(\mathbb{P}^1(\mathbb{C}), \mathcal{L})[1], d) \not\in \text{sd}(\text{Is}(\text{sm})(\mathbb{P}^1(\mathbb{C})))
\]
since this is a simple (selfdual) perverse sheaf, and \(\Omega(X)\) is freely generated by the isomorphism classes of such simple (selfdual) perverse sheaves \([\text{You}, \text{cor}.7.5]\).

The mistake of [Y1] can already be explained for a holomorphic submersion \(f : X' \to X\) of compact pure-dimensional complex manifolds. Then all higher direct image sheaves \(R^i f_* \mathbb{K}_{X'}\) \((i \in \mathbb{Z})\) are locally constant by the “Ehresmann fibration theorem” (compare also with [Sch1, Ex.4.1.2]). Then one gets by Lemma 4.1 and 4.

\[
f_*([(\mathbb{K}_{X'}[n(X')], d')]) = [(R^i f_* \mathbb{K}_{X'}[n(X)], d)]
\]
for \(i = n(X) - n(X')\) and a suitable induced duality \(d\). And in general

\[
[(R^i f_* \mathbb{K}_{X'}[n(X')], d)] \neq rk \cdot [(\mathbb{K}_{X}[n(X)], d)]
\]
with \(rk\) the rank of the local system \(R^i f_* \mathbb{K}_{X'}\), contrarily to what is claimed in [Y1] p.1011. If \(X\) is connected and this local system is already constant, then one gets for \(\mathbb{K} = \mathbb{R}\) by [CS1] prop.4.4:

\[
[(R^i f_* \mathbb{R}_{X'}[n(X')], d)] = \text{sign}(F_y) \cdot [(\mathbb{R}_{X}[n(X)], d)]
\]
(32)
with sign($F_y$) the signature of the fiber $F_y := f^{-1}(\{y\})$ for $y \in X$. In particular theorem \ref{thm:mult} implies in this case the “multiplicativity theorem” of Chern, Hirzebruch and Serre (compare \cite{HBJ}, p.42):

$$\text{sign}(X') = \text{sign}(F_y) \cdot \text{sign}(X).$$

But if the corresponding local system $R^if_*\mathbb{K}_{X'}$ is not constant, then one gets in general no equality of the form

$$[(R^if_*\mathbb{K}_{X'}[n(X')], d)] = m \cdot [(\mathbb{K}_{X}[n(X)], d)]$$

for some $m \in \mathbb{Z}$. Consider for example a compact complex (algebraic) surface $X'$ with sign($X'$) \neq 0, which fibers by a holomorphic (algebraic) submersion over a compact connected holomorphic curve $X$. Then

$$\text{sign}(X') = \deg L_0([(R^1f_*\mathbb{K}_{X'}[2], d)]) \neq m \cdot \deg L_0([(\mathbb{K}_{X}[1], d)]) = 0.$$  

Examples of such surfaces are due to Atiyah \cite{At} and Kodaira.

Now we are ready for the main result of this section.

**Theorem 4.2.** Consider the complex algebraic context. Then the self-duality transformation $sd: \text{Iso}_\text{pro}(\mathfrak{sm}/X) \to \Omega(X)$ satisfies the “blow-up relation” \ref{thm:bl} of Theorem \ref{thm:mult} for smooth pure-dimensional spaces with $f = \text{id}_X$. It induces therefore by Corollary \ref{cor:bl} a unique group homomorphism

$$sd: K_0(\var/X) \to \Omega(X)$$

commuting with proper pushdown and satisfying the normalization condition

$$sd([\text{id}_X]) = [(\mathbb{K}_X[n], d)]$$

for $X$ smooth and purely n-dimensional. The same is true in the analytic context for

$$sd: K_0(\text{an}/X) \to \Omega(X)$$

with $X$ compact.

**Proof.** Let us consider the blow-up diagram

$$E \xrightarrow{\epsilon'} B\text{ly}X = X' \xrightarrow{q'} \xrightarrow{q} Y \xrightarrow{i} X,$$

with $i$ a closed embedding of smooth pure dimensional spaces. Here $B\text{ly}X \to X$ is the blow-up of $X$ along $Y$ with exceptional divisor $E$. Let $d(E), d(X), d(Y)$ be the complex dimension of the corresponding manifolds, with $m := d(X) - d(Y)$ the complex codimension of $Y$ in $X$. The case $m = 0$ is obvious, since $Y$ is a
union of irreducible components of $X$, with $E = \emptyset$ and $X' = X \backslash Y$. So we can assume $m > 0$. Then

$$q': E = \mathbb{P}(N_Y X) \to Y$$

is just the projection of the projective bundle $\mathbb{P}(N_Y X)$ corresponding to the normal bundle $N_Y X$ of $Y$ in $X$. In particular all higher direct image sheaves $R^k q'_* (\mathbb{K}_E)$ are constant:

$$R^k q'_* (\mathbb{K}_E) = \begin{cases} \mathbb{K}_Y & \text{for } k = 0, 2, \ldots, 2m - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover this is already true for $\mathbb{K} = \mathbb{Z}$ so that

$$q'_* ([(\mathbb{K}_E [n(E)],[d])] = \epsilon \cdot [(\mathbb{K}_Y [n(Y)],[d])]$$

for a suitable (locally constant) $\epsilon \in \{-1, 0, 1\}$. But then we can assume $\mathbb{K} = \mathbb{R}$ so that by Lemma 4.1 and (32):

$$q'_* ([(\mathbb{K}_E [n(E)],[d])]) = \text{sign}(\mathbb{P}^{m-1}(\mathbb{C})) \cdot [(\mathbb{K}_Y [n(Y)],[d])]. \quad (33)$$

Let $j : X \backslash Y \to X$ be the open inclusion of the complement of $Y$, and consider the distinguished triangle:

$$\mathbb{K}_X \xrightarrow{ad_q} Rq_* \mathbb{K}_X' \to \mathcal{K} \to [1].$$

Then $ad_q : \mathbb{K}_X \xrightarrow{\sim} R^0 q_* \mathbb{K}_X'$ is an isomorphism, with $j^* \mathcal{K} = 0$ so that $Ri_* i^* \mathcal{K} \simeq \mathcal{K}$. Pulling back this triangle by $i^*$ we get the distinguished triangle

$$\mathbb{K}_Y \xrightarrow{ad_{q'}} Rq'_* \mathbb{K}_E \to i^* \mathcal{K} \to [1],$$

with $ad_{q'} : \mathbb{K}_Y \xrightarrow{\sim} R^0 q'_* \mathbb{K}_E$ an isomorphism. In particular

$$\mathcal{H}^k (i^* \mathcal{K}) = \begin{cases} R^k q'_* (\mathbb{K}_E) = \mathbb{K}_Y & \text{for } k = 2, \ldots, 2m - 2, \\ 0 & \text{otherwise.} \end{cases}$$

But then $\text{Hom}(\mathcal{K}, \mathbb{K}_X [1]) = 0$ so that the triangle defining $\mathcal{K}$ splits:

$$Rq_* \mathbb{K}_X' \simeq \mathbb{K}_X \oplus \mathcal{K}.$$
for a suitable (locally constant) $\epsilon \in \{-1,0,1\}$. But then we can assume $\mathbb{K} = \mathbb{R}$ so that by Lemma 4.1 and [CS1] cor.4.8:

$$q_*([([K_{X'}[n'(X')], d])]) = ([K_X[n(X)], d]) + \text{sign}(E_y) \cdot i_*([([K_Y[n(Y)], d])]).$$  \hfill (34)

Here

$$E_y := q^{-1}N(y)/q^{-1}L(y) \cong q^{-1}N(y) \cup q^{-1}L(y)$$

for $y \in Y$ is obtained from the inverse image of a “normal slice” $N(y)$ to $Y$ at $y$ by collapsing the boundary to a point, or equivalently, by attaching the cone on the boundary (compare [CS1, p.522, thm.4.7]). Note that the arguments of [CS1] work for our proper map $q$ without assuming $X'$ to be compact!

But in our case we can take

$$y = pt = 0 \in N(y) = (N_YX)_y = \mathbb{C}^m \subset \mathbb{P}(\mathbb{C}^m \oplus \mathbb{C}) = \mathbb{P}^m(\mathbb{C}),$$

with $L(y) \subset \mathbb{C}^m$ a sphere around $y = pt = 0$. And to calculate $\text{sign}(E_y)$, we make the following trick: By “Novikov additivity” of the signature (compare [J, thm.(A)] and [Si, prop.3.1]) we get

$$\text{sign}(E_y) = \text{sign}(Bl_{(pt)}\mathbb{P}^m(\mathbb{C})) - \text{sign}(\mathbb{P}^m(\mathbb{C})),$$

with $Bl_{(pt)}\mathbb{P}^m(\mathbb{C})$ the blow-up of $\mathbb{P}^m(\mathbb{C})$ along the point $y = pt = 0$. But by the blow-up formula for the signature, i.e. equation (14) with $y = 1$, we also have

$$\text{sign}(E_y) = \text{sign}(\mathbb{P}^{m-1}(\mathbb{C})) - 1.$$

So we finally get

$$q_*([([K_{X'}[n'(X')], d])]) = ([K_X[n(X)], d]) + (\text{sign}(\mathbb{P}^{m-1}(\mathbb{C})) - 1) \cdot i_*([([K_Y[n(Y)], d])]),$$  \hfill (35)

which together with (33) implies the “blow-up formula” (2.) of Corollary 0.1.

Remark 4.3. Note that our proof of Theorem 4.2 does not apply to the Witt group $W_\pm(X)$, instead of the cobordism group $\Omega_\pm(X)$, since it depends on Lemma 4.1.

Let us now discuss the behaviour of the transformation $sd$ under smooth pullback. Let $f : X' \to X$ be a smooth morphism of complex algebraic varieties (or compact complex spaces) of constant relative dimension $d(f)$. Then one has a duality isomorphism ([KS prop.3.3.2], [Sch1 rem.4.2.3]):

$$f^t \simeq f^*[2d(f)].$$
and $f^*$ maps $D^b_c(X)$ to $D^b_c(X)$. By the canonical isomorphism ([KS prop.3.1.13], [Sch1 cor.4.2.2])

$$f^! \circ D_X \simeq D_{X'} \circ f^*$$

we get

$$(T^d(f^*) \circ D_X \simeq (T^{-d}(f^!) \circ D_X \simeq (T^{-d}(f^*) D_{X'}) \circ f^* \simeq D_{X'} \circ (T^d(f^*))$$

so that $T^d(f^*) = f^*[d(f)]$ commutes with duality. And similarly

$$f^! \circ (T^{2n(X)} D_X) \simeq (T^{2n(X')} D_{X'}) \circ f^*$$

$$\simeq (T^{2n(X')} D_{X'}) \circ (T^{2d}(f^*)) \simeq (T^{2n(X')} D_{X'}) \circ f^!,$$

with $n(X), n(X')$ the corresponding complex dimension of $X, X'$. By our identification of twisted cobordism groups, this corresponds to the following commutative diagram:

$$\Omega(X') \xrightarrow{T^{n(X')}} \Omega(X, T^{2n(X')} D_{X'})$$

$$\xrightarrow{f^*[d(f)]}$$

$$\Omega(X) \xrightarrow{T^{n(X)}} \Omega(X, T^{2n(X)} D_X).$$

Note that $f^*[d(f)]$ is parity preserving (or changing) depending on $d(f)$ even (or odd). If moreover $X$ (and therefore also $X'$) is smooth of pure dimension, then one gets by definition

$$f^*[d(f)]([[K_X[n(X)], d]]) = [[K_{X'}[n(X')], d]].$$

And this implies the

**Corollary 4.1.** Let $f : X' \to X$ be a smooth morphism of complex algebraic varieties (or compact complex spaces) of constant relative dimension $d(f)$. Then the following diagram commutes:

$$K_0(var/X) \xrightarrow{f^*} K_0(var/X')$$

$$\xrightarrow{sd}$$

$$\Omega(X) \xrightarrow{sd} \Omega(X').$$

If moreover $X$ and $X'$ are compact, then the following Verdier Riemann-Roch type diagram

$$K_0(var/X) \xrightarrow{f^*} K_0(var/X')$$

$$\xrightarrow{L_{* \circ sd}}$$

$$H_{2*}(X, \mathbb{Q}) \xrightarrow{L^*_{(T^*_f) \circ f^*}} H_{2*}(X', \mathbb{Q})$$

is also commutative. \qed
Remark 4.4. We expect a similar Verdier Riemann-Roch theorem for the $L$-class transformation $L_*$, but at the moment we have no proof or reference for it.

Let us consider a fixed complex analytic (algebraic) Whitney stratification $X_\bullet$ of a closed subset $X$ in a complex manifold $M$, with $f : X' := X \cap M' \to X$ the inclusion of the intersection with a closed complex submanifold $M'$ of $M$ which is transversal to $X_\bullet$. Then $X'$ gets an induced complex Whitney stratification $X'_\bullet$. Let $D^b_c(X_\bullet)$ be the bounded derived category of sheaf complexes which are constructible with respect to this stratification (and similarly for $X'_\bullet$, compare [Sch1, chap.4]). Then we get by [KS, cor.5.4.11] as before a commutative diagram:

\[
\begin{array}{ccc}
\Omega(X'_\bullet) & \xrightarrow{\sim} & \Omega(X'_\bullet, T^{2n(X')}D_{X'}) \\
\downarrow f^*[d(f)] & & \downarrow f' \\
\Omega(X_\bullet) & \xrightarrow{\sim} & \Omega(X_\bullet, T^{2n(X)}D_X),
\end{array}
\]

with $-d(f) := n(M) - n(M')$ the complex codimension of $M'$.

And again it is natural to ask for $X, X'$ compact a corresponding Verdier Riemann-Roch theorem for the $L$-class transformation $L_*$, with

\[T_f := -N_{M'}M|X' \in K^0(X')\]

(compare with [Sch2] for the corresponding result in the context of the Chern-Schwartz-MacPherson transformation $c_\bullet$).

At least in the case of a trivial normal bundle $N_{M'}M|X'$, this is true by [CSI, thm.5.1]. In fact, this property for “oriented stratified spaces” together with the description of the degree of $L_0$ as in Theorem 4.1 characterizes the $L$-class transformation $L_*$ of Cappell-Shaneson uniquely [CSI, thm.5.1]. But here it is important to go outside the realm of complex analytic stratifications!

Let us finish this section with the multiplicativity of our transformations with respect to exterior products. Consider two complex algebraic (or compact analytic) spaces $X$ and $X'$. Then one has by [Sch1, cor.2.0.4] a natural isomorphism

\[(D_X\mathcal{F}) \boxtimes (D_{X'}\mathcal{F}') \simeq D_{X \times X'}(\mathcal{F} \boxtimes \mathcal{F}')\]  

for $\mathcal{F} \in D^b_c(X)$ and $\mathcal{F}' \in D^b_c(X')$. Assume now that $(\mathcal{F}, d)$ is also selfdual. Then

\[\mathcal{F} \boxtimes (\cdot) : D^b_c(X') \to D^b_c(X \times X')\]

commutes with duality by the isomorphism

\[\mathcal{F} \boxtimes (D_X\mathcal{F}') \xrightarrow{d \boxtimes \text{id}} (D_X\mathcal{F}) \boxtimes (D_{X'}\mathcal{F}') \simeq D_{X \times X'}(\mathcal{F} \boxtimes \mathcal{F}').\]
And similarly for a selfdual \((\mathcal{F}', d')\). In this way we get a bilinear pairing:

\[
\times: \Omega(X) \times \Omega(X') \to \Omega(X \times X') : \\
[(\mathcal{F}, d)] \times [(\mathcal{F}', d')] \mapsto [(\mathcal{F} \boxtimes \mathcal{F}', d \boxtimes d')] .
\]

(38)

Here \((\mathcal{F} \boxtimes \mathcal{F}', d \boxtimes d')\) is symmetric if \((\mathcal{F}, d)\) and \((\mathcal{F}', d')\) are of the same parity. Compare also with [GN] for the corresponding result for Witt groups of abstract triangulated categories with duality.

If moreover \(X\) and \(X'\) are smooth and pure dimensional, then one gets by definition

\[
[(\underline{k}_X[n(X)], d)] \times [(\underline{k}_X'[n(X')], d)] = [(\underline{k}_{X \times X'}[n(X \times X')], d)] .
\]

And this implies the

**Corollary 4.2.** Consider two complex algebraic (or compact analytic) spaces \(X\) and \(X'\). Then the following diagram commutes:

\[
\begin{array}{ccc}
K_0(var/X \times K_0(var/X') & \longrightarrow & K_0(var/X \times X') \\
Ω(X) \times Ω(X') & \longrightarrow & Ω(X \times X').
\end{array}
\]

If moreover \(X\) and \(X'\) are compact, then the following diagram

\[
\begin{array}{ccc}
K_0(var/X \times K_0(var/X') & \longrightarrow & K_0(var/X \times X') \\
H_{2*}(X, \mathbb{Q}) \times H_{2*}(X', \mathbb{Q}) & \longrightarrow & H_{2*}(X \times X', \mathbb{Q})
\end{array}
\]

is also commutative.

**Remark 4.5.** We expect a similar “multiplicativity” for the \(L\)-class transformation \(L_*\), but at the moment we have no proof or reference for it.

## 5 Hodge theoretic definition of motivic Chern classes

In this section we explain another construction of the motivic Chern class transformation \(mC_*\) with the help of some fundamental results from the theory of algebraic mixed Hodge modules due to M.Saito [Sai1]-[Sai6]. This is the most functorial (but also the most difficult) approach to our motivic classes. In fact this is the way we found them first! Moreover, this functorial approach is needed,
if one wants to extend the “generalized Verdier Riemann-Roch theorem” and
the theory of “Milnor classes for local complete intersections” [Sch2] from the
context of Chern classes to the context of our new motivic characteristic classes.

Since this theory of algebraic mixed Hodge modules is a very complicated
(and far away from geometry), we reduce our construction to a few formal
properties, together with a simple and instructive explicit calculation for the
normalization condition, all of which are contained in the work of M.Saito.

Let us assume that our base field is $k = \mathbb{C}$. To motivate the following
constructions, let us first recall the definition of the Hodge characteristic
transformation $Hc : K_0(\text{var/pt}) \to \mathbb{Z}[u, \bar{v}]$. By the now classical theory of Deligne
[Del1, Del2, St], the cohomology groups $V = H^i_c(X_{\text{an}}, \mathbb{Q})$ of a complex algebraic
variety have a canonical functorial mixed Hodge structure, which includes in
particular the following data on the finite dimensional rational vector space $V$:

- A finite increasing (weight) filtration $W$ of $V$ with $W_i = \{0\}$ for $i << 0$
  and $W_i = V$ for $i >> 0$.
- A finite decreasing (Hodge) filtration $F$ of $V \otimes \mathbb{C}$ with $F_p = V$
  for $p << 0$ and $F_p = \{0\}$ for $p >> 0$.

These filtrations have to satisfy some additional properties, which imply that the
transformation of taking suitable graded vector spaces $gr^W_i, gr^p_F$ and $gr^p_F gr^W_i$
for $i, p \in \mathbb{Z}$ induce corresponding transformations on the Grothendieck group
$K^0(MHS)$ of the abelian category of (rational) mixed Hodge structures, i.e.
morphism of mixed Hodge structures are “strictly stable” with respect to the
filtrations $F$ and $W$. Assume $Y$ is a closed algebraic subset of $X$ with open
complement $U := X \setminus Y$. Then the maps in the long exact cohomology sequence

$$\cdots \to H^i_c(U_{\text{an}}, \mathbb{Q}) \to H^i_c(X_{\text{an}}, \mathbb{Q}) \to H^i_c(Y_{\text{an}}, \mathbb{Q}) \to \cdots$$

are morphisms of mixed Hodge structures so that the function

$$X \mapsto Hc(X) := \sum_{i, p, q \geq 0} (-1)^i (-1)^{p+q} \cdot \dim_C (gr^p_F gr^W_q H^i_c(X_{\text{an}}, \mathbb{C})) u^p \bar{v}^q$$

satisfies the “additivity property” (add). In this way we get the Hodge characteristic (compare [St]):

$$Hc : K_0(\text{var/\{pt\}}) \to \mathbb{Z}[u, \bar{v}], [X \to \{pt\}] \mapsto Hc(X).$$

Note that most references do not (!) include the sign-factor $(-1)^{p+q}$ in their
definition of the Hodge characteristic, which is then called the $E$-polynomial:

$$E(X)(u, \bar{v}) := Hc(X)(-u, -\bar{v}).$$

Our sign convention fits better with the following normalization for $X$ smooth
and complete. Specializing further, one gets also the (compare [Lo])
• **Hodge filtration characteristic** corresponding to \((u, v) = (y, -1)\):

\[
Hfc(X) := \sum_{i, p \geq 0} (-1)^i \dim_{\mathbb{C}} (gr_p^F H^i_c(X^{an}, \mathbb{C})) (-y)^p.
\]

• **Weight filtration characteristic** corresponding to \((u, v) = (w, w)\):

\[
wfc(X) := \sum_{i, q \geq 0} (-1)^i \cdot \dim_{\mathbb{C}} (gr_W^q H^i(X^{an}, \mathbb{Q})) (-w)^q.
\]

• **Euler characteristic** (with compact support) corresponding to \((u, v) = (-1, -1)\):

\[
e(X) := \sum_{i \geq 0} (-1)^i \cdot \dim_{\mathbb{C}}(H^i_c(X^{an}, \mathbb{Q})).
\]

So these specializations fit into the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z}[u, v] & \xrightarrow{u=y} & \mathbb{Z}[y] \\
\downarrow{u=w} & & \downarrow{v=-1} \\
\mathbb{Z}[w] & \xrightarrow{v=w} & \mathbb{Z}.
\end{array}
\]

Finally, this classical Hodge theory \([\text{De}1, \text{De}2, \text{Sr}]\) implies for \(X\) smooth complete (of pure dimension \(d\)) the “purity result” \(gr^W_{p+q} H^i(X^{an}, \mathbb{C}) = 0\) for \(p + q \neq i\), together with\):

\[
h^{p,q}(X) := \sum_{i \geq 0} (-1)^i (1)^{p+q} \cdot \dim_{\mathbb{C}} (gr_p^F gr^W_{p+q} H^i_c(X^{an}, \mathbb{C}))
\]

\[
= \dim_{\mathbb{C}} (gr_p^F H^{p+q}(X^{an}, \mathbb{C})) = \dim_{\mathbb{C}} H^q(X^{an}, \Lambda^p T^* X^{an})
\]

\[
= \dim_{\mathbb{C}} H^q(X, \Lambda^p T^* X).
\]

Here the last two equalities follow from GAGA, the degeneration of the “Hodge to de Rham spectral sequence”

\[
E_1^{p,q} = H^q(X^{an}, \Lambda^p T^* X^{an}) \rightarrow H^{p+q}(X^{an}, \Lambda^* T^* X^{an})
\]

at \(E_1\) and the “holomorphic Poincaré lemma”

\[
H^{p+q}(X^{an}, \mathbb{C}) \cong H^{p+q}(X^{an}, \Lambda^* T^* X^{an}).
\]

So the holomorphic de Rham complex \(DR(O_{X^{an}}) := [\Lambda^* T^* X^{an}]\) (with \(O_{X^{an}}\) in degree zero) is a resolution of the constant sheaf \(\mathbb{C}\) on \(X^{an}\), and the “stupid decreasing filtration”

\[
F^p DR(O_{X^{an}}) := [0 \rightarrow \cdots \rightarrow 0 \rightarrow \Lambda^p T^* X^{an} \rightarrow \cdots \Lambda^d T^* X^{an}] \quad (42)
\]
induces the Hodge filtration $F$ on $H^*(X^{an}, \mathbb{C})$.

In particular $T_y([X \to \{pt\}]) = Hfc([X \to \{pt\}])$ for $X$ smooth and complete by (gHRR). But these classes $[X \to \{pt\}]$ generate $K_0(var/\{pt\})$ so that we get the following Hodge theoretic description for any $X$:

$$
\chi_y(X) = T_y([X \to \{pt\}]) = \sum_{i,p \geq 0} (-1)^i \dim C \left( gr^p_F H_c^i(X^{an}, \mathbb{C}) \right) (-y)^p .
$$

(43)

And exactly this description can be generalized to the context of relative Grothendieck groups $K_0(var/X)$ using the machinery of mixed Hodge modules of M.Saito. But before we explain this, let us point out another remark. All our characteristics above are indeed ring homomorphisms on $K_0(var/\{pt\})$, because this is the case for $\chi_y = Hfc$ (for example by remark 3.3). Such ring homomorphism are called “characteristics” \cite{DL,Lo} or sometimes also “motivic measures” \cite{LL}, and there are much more examples known. In this sense our transformation $T_y$ is certainly a motivic characteristic class since it is a homology class version of the motivic characteristic $Hfc$, just like the Chern-Schwartz-MacPherson (class) transformation $c_*$ is a homology class version of the Euler-Poincaré characteristic $e$.

**Remark 5.1.** Our motivic characteristic classes are only group homomorphisms, because the corresponding homology theories are only groups for a singular space $X$. But they commute with exterior products, so that they are ring homomorphisms for $X = \{pt\}$. But this is in general no longer true for $X$ smooth, even if one has then a corresponding ring structure on the (co)homology. This is closely related to a corresponding Verdier Riemann-Roch formula for the diagonal embedding $d : X \to X \times X$ (compare \cite{Sch2,Yell}).

One can ask if also the other characteristic $Hc$ or $wc$ can be “lifted up” to such a homology class transformation. But here the answer will be no (cf. \cite{jo}!)

**Example 5.1.** Assume that there is a functorial transformation

$$
T_{u,v} : K_0(var/X) \to H_*(X) \otimes \mathbb{Q}[u,v]
$$

commuting with proper pushdown, and also with pullback $f^*$ for a finite smooth morphism $f : X' \to X$ between smooth varieties, such that for $X = \{pt\}$ we get back the Hodge characteristic $Hc$. Let $d$ be the degree of such a covering map $f$. Then it follows (with $k : X \to \{pt\}$ a constant map):

$$
Hc(X') = T_{u,v}([X' \to \{pt\}]) = T_{u,v}(k_*f_*f^*[id_X]) = k_*f_*f^*T_{u,v}([id_X])
$$

$$
= k_* (d \cdot T_{u,v}([id_X])) = d \cdot k_*T_{u,v}([id_X]) = d \cdot Hc(X) = Hc(d \cdot [id_{pt}]) \cdot Hc(X)
$$

So (as usual), the transformation $Hc$ has then to be multiplicative in such finite coverings. But this is not the case. Let $X' \to X$ be such a finite covering of degree $d > 1$ over an elliptic curve $X$. Then $X'$ is also an elliptic curve so that

$$
Hc(X) = Hc(X') = (1 + u)(1 + v) \neq 0 .
$$
Note that the same argument applies also to the weight characteristic, with \( wc(X) = wc(X') = (1+w)^2 \neq 0 \). Of course, everything goes well in the context of \( e \) and \( Hfc \), since both are zero for an elliptic curve!

Let us now formulate those results about algebraic mixed Hodge modules, which we need for our application to the motivic Chern class transformation \( mC_\ast \). All these results are contained in the deep and long work of M. Saito. Since most readers will not be familiar with this theory, we present them in an axiomatic way pointing out the similarities to constructible functions \( F(X) \) and motivic Grothendieck groups \( K_0(var/X) \).

Let \( k \) be a subfield of \( \mathbb{C} \). Then we work in the category of reduced separated schemes of finite type over \( \text{spec}(k) \), which we also call “spaces” or “varieties”, with \( pt = \text{spec}(k) \).

MHM1: To such a space \( X \) one can associate an abelian category of algebraic mixed Hodge modules \( MHM(X/k) \), together with a functorial pullback \( f^\ast \) and pushdown \( f_! \) on the level of derived categories \( D^bMHM(X/k) \) for any (not necessarily proper) map \( \text{[Sai2, sec.4]} \) (and compare also with \( \text{[Sai5, Sai6]} \)). These transformations are functors of triangulated categories.

MHM2 Let \( i : Y \rightarrow X \) be the inclusion of a closed subspace, with open complement \( j : U := X \setminus Y \rightarrow X \). Then one has for \( M \in D^bMHM(X/k) \) a distinguished triangle \( \text{[Sai2 (eq.(4.4.1),p.321)]} \)

\[
ji^!j^*M \rightarrow M \rightarrow ii^*M \rightarrow [1]
\]

MHM3: For all \( p \in \mathbb{Z} \) one has a functor of triangulated categories

\[
gr_p DR : D^bMHM(X/k) \rightarrow D^b_{coh}(X)
\]

commuting with proper pushdown (compare with \( \text{[Sai1 sec.2.3]}, \text{[Sai2 p.273]}, \text{[Sai3 eq.(1.3.4),p.9, prop.2.8]} \) and also with \( \text{[Sai4, Sai5]} \)). Here \( D^b_{coh}(X) \) is the bounded derived category of sheaves of \( \mathcal{O}_X \)-modules with coherent cohomology sheaves. Moreover, \( gr_p DR(M) = 0 \) for almost all \( p \) and \( M \in D^bMHM(X/k) \) fixed \( \text{[Sai1 prop.2.2.10, eq.(2.2.10.5)]} \) and \( \text{[Sai3 lem.1.14]} \).

MHM4: There is a distinguished element \( Q_{pt}^H \in MHM(\{pt\}/k) \) such that

\[
gr_p DR(Q_{pt}^H) \simeq \Lambda^pT^*X[-p] \in D^b_{coh}(X)
\]

for \( X \) smooth and pure dimensional \( \text{[Sai2]} \). Here \( Q_{pt}^H := k^*Q_{pt}^H \) for \( k : X \rightarrow \{pt\} \) a constant map, with \( Q_{pt}^H \) viewed as a complex concentrated in degree zero.
Moreover, the functoriality of pushdown and pullback (MHM1): For functoriality:

This finally is nothing else than the asked additivity property of a closed subspace, with open complement \(f\) induces under a triangle (with \(k\) functions, we get a group homomorphism commuting with pushdown (compare also with [Lo, sec. 4]):

By (MHM3) we get a group homomorphism commuting with proper pushdown:

\[
gr^F_p DR : K_0(MHM(X/k)) \to G_0(X) \otimes \mathbb{Z}[y, y^{-1}];
[\mathcal{M}] \mapsto \sum_p [gr^F_p DR(\mathcal{M})] \cdot (-y)^p.
\]

(44)

And as for the map \(e\) from motivic Grothendieck groups to constructible functions, we get a group homomorphism commuting with pushdown (compare also with [Lo, sec. 4]):

\[
mH : K_0(var/X) \to K_0(MHM(X/k)), [f : X' \to X] \mapsto [f_! Q_X^H].
\]

Indeed, the “additivity relation” (add) follows from (MHM2) together with the functoriality of pushdown and pullback (MHM1): For \(i : Y \to X'\) the inclusion of a closed subspace, with open complement \(j : U \to X'\), the distinguished triangle (with \(k\) the constant map on \(X'\))

\[
\xymatrix{x \ar[r]^(0.4)j & y^\ast & z^\ast \ar[l]_f}
\]

induces under \(f\) the distinguished triangle (with \(f : X' \to X\) as before)

\[
\xymatrix{x \ar[r]^{j_i} & y^\ast & z^\ast \ar[l]_f}
\]

It translates in the corresponding Grothendieck group into the relation

\[
[f_! k^\ast Q^H] = [f_! j_! j^\ast k^\ast Q^H] + [f_! i_! i^\ast k^\ast Q^H].
\]

This finally is nothing else than the asked additivity property

\[
[f_! Q^H] = [(f \circ j)_! Q^H] + [(f \circ i)_! Q^H] \in K_0(D^b MHM(X/k))
\]

Moreover, \(mH\) commutes with pushdown for a map \(f : X' \to X\) again by functoriality:

\[
mH(f_![g : Y \to X']) = mH([f \circ g : Y \to X]) = [(f \circ g)_! Q^H_Y]
= [f_! g_! Q^H_Y] = f_! mH([g : Y \to X]).
\]

By (MHM4) we get for \(X\) smooth and pure dimensional:

\[
gr^F_p DR \circ mH([id_X]) = \sum_{i=0}^{dim X} [\Lambda^i T^\ast X] \cdot y^i \in G_0(X) \otimes \mathbb{Z}[y, y^{-1}].
\]
Corollary 5.1. The motivic Chern class transformation $mC_*$ of theorem 2.1 is given as the composition

$$mC_* = gr^F_* DR \circ mH : K_0(var/X) \to G_0(X) \otimes \mathbb{Z}[y] \subset G_0(X) \otimes \mathbb{Z}[y, y^{-1}].$$

□

Remark 5.2. The definition of $MHM(X/k)$ and therefore also the transformations $mH$ and $gr^F_* DR$ depend a priori on the embedding $k \subset \mathbb{C}$. By the uniqueness statement of Theorem 2.1 and 3.1 this is not the case for the transformations $mC_*$ and $T_y^*$, i.e. they are independent of the choice of the embedding $k \subset \mathbb{C}$, if their definition is based on Corollary 5.1.

Let us now explain a little bit of the definition of the abelian category $MHM(X/k)$ of algebraic mixed Hodge modules on $X/k$. Its objects are special tuples $(M, F, K, W)$, which for $X$ smooth are given by

- $(M, F)$ an algebraic holonomic filtered $D$-module $M$ on $X$ with an exhaustive, bounded from below and increasing (Hodge) filtration $F$ by algebraic $\mathcal{O}_X$-modules such that $gr^F_* M$ is a coherent $gr^F_\mathcal{O}_X$-module. In particular, the filtration $F$ is finite, which will imply the last claim of (MHM3). Here the filtration $F$ on the sheaf of algebraic differential operators $\mathcal{D}_X$ on $X$ is the order filtration, and one can work either with left or right $D$-modules. For singular $X$ one works with suitable local embeddings into manifolds and corresponding filtered $D$-modules with support on $X$ (compare [Sai1, Sai2, Sai4]).

- $K \in D^b_c(X(\mathbb{C})^{an}, \mathbb{Q})$ is an algebraically constructible complex of sheaves of $\mathbb{Q}$-vector spaces (with finite dimensional stalks, compare for example with [Sch1]) on the associated analytic space $X(\mathbb{C})^{an}$ corresponding to the induced algebraic variety $X(\mathbb{C}) := X \otimes_k \mathbb{C}$ over $\mathbb{C}$, which is perverse with respect to the middle perversity $t$-structure. $F$ is called the underlying rational sheaf complex.

- In addition one fixes a quasi-isomorphism $\alpha$ between $K(\mathbb{C}) := K \otimes_\mathbb{Q} \mathbb{C}$ and the holomorphic de Rham complex $DR(M(\mathbb{C})^{an})$ associated to the induced $\mathcal{D}_{X(\mathbb{C})}$-module $M(\mathbb{C}) := M \otimes_k \mathbb{C}$.

- $W$ is finally a finite increasing (weight) filtration of $(M, F)$ and $K$, compatible in the obvious sense with the quasi-isomorphism $\alpha$ above.

These data have to satisfy a long list of properties which we do not recall here (since it is not important for us). In particular, one gets the equivalence [Sai2, eq.(4.2.12),p.319])

$$MHM(\{pt\}/\mathbb{C}) \simeq \{(graded) polarizable mixed \mathbb{Q}-Hodge structures\}$$

$$F^{-p} \leftrightarrow F_p \quad (46)$$

between the category of algebraic mixed Hodge modules on $pt = \text{spec}(\mathbb{C})$, and the category of (graded) polarizable mixed $\mathbb{Q}$-Hodge structures. Of course, one has to switch the increasing $D$-module filtration $F^p$ to the decreasing Hodge
filtration by $F^{-p} \leftrightarrow F_{p}$ so that $gr_{F}^{-p} \simeq gr_{F}^{p}$. For elements in $MHM(\{pt\}/k)$, the corresponding Hodge filtration is already defined over $k$ (compare [Sai6, sec.1.3]).

The distinguished element $Q_{H}^{H}(\{pt\}/k)$ of (MHM4) is given by

$$Q_{H}^{H}(\{pt\}/k) = ((k, F), (Q, W)) \text{ with } gr_{F}^{i} = 0 = gr_{W}^{i} \text{ for } i \neq 0$$

(47)

and $\alpha : k \otimes C \simeq Q \otimes C$ the obvious isomorphism (compare [Sai6, sec.1.3]).

The functorial pullback and pushdown of (MHM1) corresponds under the forget functor ([Sai2, thm.0.1,p.222])

$$rat : D^{b}MHM(X/k) \to D^{b}_{c}(X(\mathbb{C})^{an}, Q), ((M, F), K, W) \mapsto K$$

(48)

to the classical corresponding (derived) functors $f^{*}$ and $f_{!}$ on the level of algebraically constructible sheaf complexes, with $rat(Q_{X}^{H}) \simeq Q_{X(\mathbb{C})}^{an}$. So by [H],

one should think of an algebraic mixed Hodge module as a kind of “(perverse) constructible Hodge sheaf”! But one has to be very careful with this analogy. $Q_{X}^{H}$ is in general a highly complicated complex in $D^{b}MHM(X/k)$, which is impossible to calculate explicitly. But if $X$ is smooth and pure $d$-dimensional, then $Q_{X(\mathbb{C})}^{an}[d]$ is a perverse sheaf and $Q_{X}^{H}[d] \in MHM(X/k)$ a single mixed Hodge module (in degree 0), which is explicitly given by ([Sai2, eq.(4.4.2),p.322]):

$$Q_{X}^{H}[d] \simeq ((O_{X}, F), (Q_{X(\mathbb{C})}^{an}, [d], W),$$

(49)

with $F$ and $W$ the trivial filtration $gr_{F}^{i} = 0 = gr_{W}^{i} \text{ for } i \neq 0$. Here we use for the underlying D-module the description as the left D-module $O_{X}$, which maybe is more natural at this point.

The distinguished triangle (MHM2) is a “lift” of the corresponding distinguished triangle for constructible sheaves. Similarly, by taking a constant map $f : X \to pt$ we get by (MHM1) and (46) a functorial (rational) mixed Hodge structure on

$$rat \left(R^{i}f^{*}Q_{X}^{H}\right) \simeq H_{c}^{i}(X(\mathbb{C})^{an}, Q),$$

whose Hodge numbers are easily seen to be the same as those coming from the mixed Hodge structure of Deligne [De1, De2, Sr] (both have the same additivity property so that one only has to compare the case $X$ smooth, which follows from the constructions). In fact, even the Hodge structures are the same by a deep theorem of M.Saito [Sai5, cor.4.3].

Let us finally explain (MHM3) and (MHM4) for the case $X$ smooth and pure $d$-dimensional. The de Rham functor $DR$ factorizes as (compare [Sai1, Sai3, Sai5])

$$DR : D^{b}MHM(X/k) \to D^{b}F_{coh}(X, Diff) \to D^{b}_{c}(X(\mathbb{C})^{an}, \mathbb{C})$$

(50)

with $D^{b}F_{coh}(X, Diff)$ the “bounded derived category of filtered differential complexes” on $X$ with coherent cohomology sheaves. Here the objects are
bounded complexes \((L^\bullet, F)\) of \(\mathcal{O}_X\)-sheaves with an increasing (bounded from below) filtration \(F\) by such sheaves, whose morphisms are “differential operators” in a suitable sense. In particular
\[
gr^F_p(L^\bullet) \in D^b(X, \mathcal{O}_X)
\]
becomes an \(\mathcal{O}_X\)-linear complex with coherent cohomology. Moreover, the morphisms of mixed Hodge modules are “strict” with respect to the Hodge filtration \(F\) (and the weight filtration \(W\)) so that \(gr^F_p DR\) induces the corresponding transformation of \((\text{MHM}3)\). Finally, \(DR(Q^Y_X)\) is given by the usual de Rham complex \(\Lambda^* T^* X\) with the induced increasing filtration
\[
F^p DR(Q^Y_X) := [F^p O_X \to F^p+1 O_X \otimes \Lambda^1 T^* X \to \cdots \to F^p+q O_X \otimes \Lambda^q T^* X],
\]
with \(F^p O_X\) in degree zero and \(F\) the trivial filtration with \(gr^F_i = 0\) for \(i \neq 0\).

Let us switch to the corresponding decreasing filtration (with \(gr^F_i \simeq \cdots \simeq gr^F_{-1} \simeq O_X\)):
\[
F^p DR(Q^Y_X) := [F^p O_X \to F^{p-1} O_X \otimes \Lambda^1 T^* X \to \cdots \to F^{p-d} O_X \otimes \Lambda^d T^* X].
\]
Then the de Rham complex \(DR(\mathcal{O}_X)\) with the stupid filtration \(\sigma^p\) as before in (12):
\[
F^p DR(\mathcal{O}_X) := [0 \to \cdots \to \Lambda^p T^* X \to \cdots \Lambda^d T^* X],
\]
becomes a filtered subcomplex. And one trivially checks that the inclusion induces on the associated graded complexes the isomorphism
\[
gr^p F DR(\mathcal{O}_X) \simeq \Lambda^p T^* X[-p] \simeq gr^p F DR(Q^Y_X) \simeq gr^p F DR(Q^Y_X).
\]
In this way one finally also gets (MHM4).

Remark 5.3. The use of the transformation \(gr^p F DR\) of (MHM3) in the context of \(\text{characteristic classes of singular spaces}\) is not new. It was already used by Totaro [13] in his study of the relation between \(\text{Chern numbers}\) for singular complex varieties and \(\text{elliptic homology}\).

But he was interested in characteristic numbers and classes invariant under \(\text{small resolution}\), and not in \(\text{functoriality}\) as in our paper. So he works with the counterpart \(IC^H_X \subset MHM(X/\mathbb{C})\) of the intersection cohomology complex instead of the constant “Hodge sheaf” \(Q^H_X \subset D^b MHM(X/\mathbb{C})\) as used in this paper. He then also applied the \(\text{singular Riemann-Roch transformation} td\) of Baum-Fulton-MacPherson to associate to a singular complex algebraic variety \(X\) of dimension \(n\) some natural homology classes \(\chi^{n-k}_p(X) \in H^{BM}_{2k}(X, \mathbb{Q})\) for \(p \in \mathbb{Z}\).

In our notation, the corresponding total homology class \(\chi^{n-*}_p(X) \in H^{BM}_{2*}(X, \mathbb{Q})\) is given by evaluating
\[
\text{td}_{(1+y)} \circ gr^F_p DR(\mathcal{I} \mathcal{C}^H_X) \in H^{BM}_{2*}(X, \mathbb{Q})[y, (1+y)^{-1}]
\]
at \(y = 0\). Here it is important to work with the transformation
\[
\text{td}_{(1+y)} \circ gr^F_p DR : K_0(MHM(X/\mathbb{C})) \to H^{BM}_{2*}(X, \mathbb{Q})[y, y^{-1}, (1+y)^{-1}].
\]
This allows one to use more general coefficients like \(IC^H_X \subset MHM(X/\mathbb{C})\), which are a priori not in the image of \(mH\).
Remark 5.4. The announcement [CS2] and [Sh, sec. 4] suggests for a pure $d$-dimensional compact complex algebraic variety $X$ the following relation between the Hodge theoretical classes and the topological $L$-class $L_*(X) = L_*(IC_X)$ of Goresky-MacPherson:

\[
( \text{td}(1+y) \circ gr^F_{-p} DR(Q^H_X)[-d])_{|y=1} = L_*(X).
\]  

At least the equality of their degrees follows from the work of Saito, i.e., the description of the signature of the global Intersection (co)homology in terms of Hodge numbers (as in the Hodge index theorem for smooth Kähler manifolds).

Note that one has a natural morphism $Q^H_X \to IC^H_X[-d]$ in $D^b MHM(X/\mathbb{C})$, which is an isomorphism for $X$ a rational homology manifold. So (52) would imply our conjecture that for a rational homology manifold $T_1(X) = L_*(X)$, and more generally it would explain the difference between $T_1(X)$ and $L_*(X)$.

Similarly, (52) implies that $L_*(X)$ is in the image of the cycle map from the Chow group $A_*(X)$ to homology.

As explained before, it is in general impossible to calculate $gr^F_{-p} DR(Q^H_X)$ explicitly for a singular space $X$. But by comparing our different definitions of $mC_*(X) = mC_*(id_X)$ in terms of the Du Bois complex and in terms of mixed Hodge modules, we get at least (for all $p \in \mathbb{Z}$):

\[
[gr^F_{-p} DR(Q^H_X)] = [gr^F_{-p} (\Omega^*_X)] \in G_0(X).
\]  

In the work of M. Saito [Sai5] the reader will find a deeper identification of the underlying filtered complexes, which is of course much stronger than the equality above on the level of elements in the Grothendieck group. Here we finally state only the following results from [Sai5] for a complex algebraic variety $X$ of dimension $n$:

1. $gr^F_{-p} DR(Q^H_X) \simeq 0 \in D^b_{coh}(X)$ for $p < 0$ and $p > n$.

2. Let $X'$ be a resolution of singularities of the union of the $n$-dimensional irreducible components of $X$, with $\pi : X' \to X$ the induced proper map. Then

\[
gr^F_{-n} DR(Q^H_X) \simeq \pi_* \Lambda^n T^* X'[-n].
\]

Note that $R^i \pi_* \Lambda^n T^* X' = 0$ for $i > 0$ by the Grauert-Riemenschneider vanishing theorem.

3. $h^i(gr^F_0 DR(Q^H_X)) \simeq 0$ for $i < 0$, and

\[
h^0(gr^F_0 DR(Q^H_X)) \simeq O_X^{wn},
\]

with $O_X^{wn}$ the coherent structure sheaf of the weak normalization $X_{max}$ of $X$ (whose underlying space is identified with $X$). One gets in particular natural morphisms

\[
O_X \to O_X^{wn} \to gr^F_0 DR(Q^H_X)
\]

in $D^b_{coh}(X)$. And in this language $X$ has at most “Du Bois singularities” if the composed map $O_X \to gr^F_0 DR(Q^H_X)$ is a quasi-isomorphism.
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