DYADIC BI-PARAMETER SIMPLE COMMUTATOR AND DYADIC LITTLE BMO

IRINA HOLMES, SERGEI TREIL, AND ALEXANDER VOLBERG

Abstract. Let $T$ is a certain tensor product of simple dyadic shifts defined be low. We prove here that for dyadic bi-parameter commutator the following equivalence holds $\|Tb - bT\| \asymp \|b\|_{\text{bmo}}$. This result is well-known for many types of bi-parameter commutators, see [FS], [DLWY] and [DPSK] for more details.

1. Introduction

In this note we are considering a simple dyadic model of bi-parameter commutator. Bi-parameter theory is notoriously more difficult than the more classical one parameter theory of singular integrals. The good place to get acquainted with multi-parameter specifics are the papers of J.-L. Journé [JLJ], [JLJ2] and Muscalu–Pipher–Tao–Thiele [MPTT1], [MPTT2]. The applications to analysis in polydisc can be found in [Ch], [ChF], [ChF2].

It is well known [ChF, ChF2, JLJ, JLJ2] that in the multi-parameter setting all concepts of Carleson measure, $BMO$, John–Nirenberg inequality, Calderón–Zygmund decomposition (used in classical theory) are much more delicate. Paper [MPTT1] develops a completely new approach to prove natural tri-linear bi-parameter estimates on bi-parameter paraproducts, especially outside of Banach range. In [MPTT1] Journé’s lemma [JLJ2] was used, but the approach did not generalize to multi-parameter paraproduct forms. This issue was resolved in [MPTT2], where a simplified method was used to address the multi-parameter paraproducts.

One of the new feature of the multi-parameter theory is captured by several different definitions of $BMO$, see [FS], [Ch]. The necessity of those new effects was discovered first by Carleson [Car], see also [Tao]. The difficulties with multi-parameter theory was highlighted recently by two very different series of papers. One concerns with the Carleson measure and Carleson embedding on polydisc and on multi-tree, see, e.g. [AMP18], [AHMV], [MPV], [MPVZ]. These papers, roughly speaking, are devoted to harmonic analysis (Carleson embedding in particular) on graphs with cycles. This theory is drastically different from the usual one, and it is much more delicate. Another particularity of the multi-parameter theory is highlighted by the repeated commutator story. Lacey and Ferguson [FL] found the characterization of the symbols that give us bounded “small Hankel operators”. It was a breakthrough article that gave a multi-parameter Nehari theorem and a long searched after factorization of bi-parameter Hardy space $H^1$. It was exactly equivalent to a bi-parameter “repeated commutator characterization”. Several papers followed where “bi-” was upgraded to “multi-”, and where repeated commutation was performed with different classical singular integrals: [DP], [FL], [L], [LPPW], [LT].

There are many new and beautiful ideas in the above mentioned papers devoted to this subject. We also want to mention [OPS], where the authors use an argument inspired by Toeplitz operators to show the lower bounds. It assumes the lower norm for the Hilbert transform claimed in [FL], [LPPW] as a black box. But there is a problem with [FL] and with what followed.

That was a big breakthrough in multi-parameter theory. Unfortunately [V] indicated a hole in all the proofs of [FL], [L], and this circle of problems is still unsettled.

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We decided to simplify the problem to the “bare bones”, and to consider commutators with the simplest dyadic singular integral—the dyadic shift of order 1. The repeated commutator problem presented the same type of difficulty as in [FL], and subsequent papers. In our mind, it is very nice to see the difficulty in a dyadic case removed from all technicalities of a continuous case. But here we consider a much simpler problem: the characterization of the boundedness of the usual (not repeated) commutator of multiplication and the tensor product of two dyadic singular integrals. Again, the dyadic singular integrals are the simplest possible, they are the simplest dyadic shifts.

Not surprisingly we are able to solve this problem completely and to get an expected characterization of the symbol of the bounded simple commutator: it is small (dyadic) \( b_{\text{mo}} \). This answer is not surprising at all, as the similar results (for various singular operators) were obtained in [FS], [DLWY], [H], [DPSK].

Notice that for repeated commutator the expected characterization of the symbol of the bounded repeated commutator is a different \( BMO \), namely, it is \( BMO_{\text{ChF}} \): this is Chang–Fefferman \( BMO \) (or product \( BMO \)) studied in [ChF], [ChF2]. The fact that \( BMO_{\text{ChF}} \) is an expected characterization of bounded repeated commutators has at least two “confirmations”. One is the paper of Blasco–Pott [BP], where it is proved that the dyadic \( BMO_{\text{ChF}} \) characterizes the boundedness of repeated commutator with \textit{all dyadic martingale transforms simultaneously}. What one would like to show, that just one interesting transform (dyadic) is enough. The second “confirmation” is that the counterexample found in [Y] seems to be “easily circumvented”. Unfortunately the attempt to do that did not succeed—so far.

Let us finish by writing a simple commutator (we deal with it below) and repeated commutator (that brings so much pain). Let \( T = T_1 \otimes T_2 \), where \( T_1, T_2 \) are two (dyadic) singular integrals, each acting in its own \( L^2(\mathbb{R}) \): \( T_1 \) acts on functions of variable \( x \), \( T_2 \) acts on functions whose variable is called \( y \). Now let \( b(x, y) \) be a symbol. We need to characterize the boundedness of simple commutator \( [T, b] = T_1 b - b T_2 T_1 \) in terms of \( b \).

We do this below. On the other hand, if one is concerned with repeated commutator, then one is considering the following nested commutation: \([T_2; [T_1, b]] = T_2(T_1 b - b T_1) - (T_1 b - b T_1) T_2 = T_2 T_1 b + b T_1 T_2 - T_2 b T_1 - T_1 b T_2\).

2. Plan

Let \( D \) be the usual dyadic lattice on the line and \( D \times D \) be the family of dyadic rectangles on the plane. Haar functions are denoted \( \{h_I\}_{I \in D} \). We consider \( R = I \times J \) and \( h_R = h_I \otimes h_J \). Let \( T \) be the dyadic shift defined by

\[ T h_I = h_{I_+} - h_{I_-}, \text{ if } I \text{ is even; } T h_I = 0, \text{ otherwise,} \]

where we say \( I \in D \) is even if \( |I| = 2^{-2k} \).

Let \( T := T \otimes T \). We are interested in the commutator

\[ T b - b T \]

and in characterization of its boundedness. The ultimate goal is to prove that its boundedness is equivalent to \( b \in b_{\text{mo}}^d \), where \( b_{\text{mo}}^d \) is dyadic little \( b_{\text{mo}} \).

3. The 1-parameter case

**Theorem 3.1** \([T, b] \) is \( L^2 \)-bounded if and only if \( b \in BMO^d \).

Recall that

\[ \|b\|_{BMO^d}^2 = \sup_{I \in D} \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I| \, dx = \sup_{I \in D} \frac{1}{|I|} \sum_{I' \subseteq I} \|b, h_I\|^2. \]

We will make use of the paraproduct decomposition

\[ b(x)f(x) = \pi(b, f)(x) + Z(b, f)(x) + D(b, f)(x), \]

where

\[ \pi(b, f) := \sum_I \langle b, h_I \rangle \langle f \rangle_I h_I, \quad Z(b, f) := \sum_I \langle b, h_I \rangle \langle f, h_I \rangle \hat{1}_I, \quad D(b, f) := \sum_I \langle b \rangle_I \langle f, h_I \rangle h_I. \]

(1)
Above, summation is always over $I \in \mathcal{D}$, and we used the recurring notations

$$
(f, h_I) := \int f h_I \, dx; \quad \langle f \rangle_I := \frac{1}{|I|} \int_I f(x) \, dx; \quad \hat{I}_I := \frac{1}{|I|}.
$$

Key to the proof will be the following:

**Lemma 1** If $[T, b]$ is $L^2$-bounded, then

$$
|\langle b, h_I \rangle|^2 \leq C_0 |I|
$$

for all intervals $I \in \mathcal{D}$.

We prove this result below in section 3.2.

### 3.1. Proof of Theorem 3.1 – Lower bound.

This part shows that if $[T, b]$ is bounded then $b \in \text{BMO}^d$. We split the $Tb - bT$ operator into $D, \pi, Z$ pieces:

$$
[T, b]f = T(bf) - bTf = [T\pi(b, f) - \pi(b, Tf)] + [TD(b, f) - D(b, Tf)] + [TZ(b, f) - Z(b, Tf)].
$$

We consider the function $b$ to be fixed throughout, so we refer to these operators simply as $(T\pi - \pi T)$, $TD - DT$, and $TZ - ZT$.

1. **D part.** We have

$$
(TD - DT)f = \sum_{I \text{even}} (D(b, f), h_I)(h_{I+} - h_{I-}) - \sum_I \langle b \rangle_I (Tf, h_I)h_I.
$$

Now, by definition of $T$,

$$
(Tf, h_I) = \begin{cases} 
(f, h_I)s(I, \hat{I}), & \text{if } I \text{ odd} \\
0, & \text{otherwise},
\end{cases}
$$

where, for every $I \in \mathcal{D}$, $\hat{I}$ denotes the dyadic parent of $I$ and

$$
s(I, \hat{I}) := \begin{cases} 
1, & \text{if } I = \hat{I}_+ \\
-1, & \text{if } I = \hat{I}_-.
\end{cases}
$$

So

$$
(TD - DT)f = \sum_{I \text{even}} \langle b \rangle_I (f, h_I)(h_{I+} - h_{I-}) - \sum_I \langle b \rangle_I (f, h_I)s(I, \hat{I})h_I.
$$

Relabeling the first term over the odd intervals instead,

$$
(TD - DT)f = \sum_{I \text{odd}} \langle b \rangle_I (f, h_I)(h_{I+} - h_{I-}) - \sum_{I \text{odd}} \langle b \rangle_I (f, h_I)s(I, \hat{I})h_I
$$

$$
= \sum_{I \text{odd}} (\langle b \rangle_I - \langle b \rangle_{\hat{I}})(f, h_I)s(I, \hat{I})h_I.
$$

It is easy to see that

$$
\langle b \rangle_I - \langle b \rangle_{\hat{I}} = \frac{s(I, \hat{I})}{\sqrt{|I|}}(b, h_I),
$$

so

$$
(TD - DT)f = \sum_{I \text{odd}} \frac{1}{\sqrt{|I|}}(b, h_I)(f, h_I)h_I.
$$

This convenient formula for this term, combined with Lemma 1 shows that $TD - DT$ is actually *a priori bounded* on all test functions:

$$
\|(TD - DT)f\|_2^2 = \sum_{I \text{odd}} \frac{1}{|I|}|(b, h_I)|^2|\langle f, h_I \rangle|^2 \leq C_0 \sum_{I \text{odd}} |(f, h_I)|^2 \leq \|f\|_2^2.
$$

Therefore we can forget about this term completely, and focus on the remaining two.
2. Z part. This term does not simplify as nicely as the $D$-term:

$$(TZ - ZT)f = \sum_{I} (b, h_I)(f, h_I)T(\mathbf{1}_I) - \sum_{I} (b, h_I)(Tf, h_I)\mathbf{1}_I.$$ 

So now we test our commutator on a Haar function $h_{I_0}$, where $I_0$ is even. In particular, the $Z$-term gives

$$(TZ - ZT)h_{I_0} = (b, h_{I_0})T\mathbf{1}_{I_0} - (b, h_{I_0}^+)\mathbf{1}_{I_0}^- + (b, h_{I_0}^-)\mathbf{1}_{I_0}^+.$$ 

From Lemma [1], we see immediately that the $(TZ - ZT)$ term is a priori uniformly bounded on $h_{I_0}$, so it can be put aside.

3. $\pi$ part. Now we are left with

$$(T\pi - \pi T)f = \sum_{I \in \text{even}} (b, h_I)(f, h_I)(Th_I) - \sum_{I \in \text{odd}} (b, h_I)(Tf, h_I)h_I,$$ 

which we write as

$$(T\pi - \pi T)f = \Sigma_1 + \Sigma_2,$$ 

where

$$(4) \quad \Sigma_1 = -\sum_{I \in \text{even}} (b, h_I)(Tf, h_I)h_I$$ 

$$\Sigma_2 = \sum_{I \in \text{even}} (b, h_I)(f, h_I)(Th_I) - \sum_{I \in \text{odd}} (b, h_I)(Tf, h_I)h_I.$$ 

Now, note that $\Sigma_1$ and $\Sigma_2$ are orthogonal, since $\Sigma_1$ lives in the even space and $\Sigma_2$ lives in the odd space. So, $(T\pi - \pi T)$ being uniformly bounded on $h_0$ means that the terms $\Sigma_1$ and $\Sigma_2$ are individually uniformly bounded on $h_0$.

So let us first plug in our test function into $\Sigma_1$:

$$\Sigma_1 h_{I_0} = \sum_{I \subset I_0, I \in \text{even}} (b, h_I)h_{I_0}^+(I)h_I - \sum_{I \subset I_0, I \in \text{even}} (b, h_I)h_{I_0}^-(I)h_I,$$ 

where for dyadic intervals $I \subset J$, $h_J(I)$ denotes the constant value that $h_J$ takes on the interval $I$. Then

$$\|\Sigma_1 h_{I_0}\|_2^2 = \frac{2}{|I_0|} \sum_{I \subset I_0, I \in \text{even}} |(b, h_I)|^2 \lesssim C$$ 

is uniformly bounded. Combined with (2), we get

$$(5) \quad \sum_{I \subset I_0, I \in \text{even}} |(b, h_I)|^2 \lesssim C|I_0|,$$ 

for all dyadic intervals $I_0$. Technically, so far this is only true for even $I_0$, but if $I_0$ is odd, the same quickly follows by splitting into the subintervals of the even halves $I_0^\pm$ and then using (2) once more.

Since (5) is only half the battle, we must look at $\Sigma_2$ now, which is also uniformly bounded on our test functions. We will split $\Sigma_2$ into its own two pieces:

$$\Sigma_2 = \Sigma_{21} - \Sigma_{22},$$ 

where

$$\Sigma_{21} f = \sum_{I \in \text{even}} (b, h_I)(f, h_I)(Th_I)$$ 

and

$$\Sigma_{22} f = \sum_{I \in \text{odd}} (b, h_I)(Tf, h_I)h_I.$$

For $f = h_{I_0}$,

$$\Sigma_{21} h_{I_0} = \sum_{I \in \text{odd}} (b, h_I)h_{I_0}(\hat{I})s(I, \hat{I})h_I,$$
so
\[
\|\Sigma_{21}h_{I_0}\|_2^2 = \frac{1}{|I_0|} \sum_{I \in I_0^+} |(b, h_I)|^2 = \frac{1}{|I_0|} \sum_{I \in I_0^-} |(b, h_I)|^2,
\]
which we know to be uniformly bounded already from (5). This means that the last remaining term, \(\Sigma_{22}\) is again uniformly bounded on our test functions:
\[
\Sigma_{22}h_{I_0} = \sum_{I \in I_0^+} (b, h_I)h_{I_0^+}(I)h_I - \sum_{I \in I_0^-} (b, h_I)h_{I_0^-}(I)h_I.
\]

So
\[
\|\Sigma_{22}h_{I_0}\|_2^2 = \frac{2}{|I_0|} \sum_{I \in I_0^+} |(b, h_I)|^2
\]
is also uniformly bounded. Then the odd counterpart of (5):
\[
\sum_{I \subset I_0, I \text{odd}} |(b, h_I)|^2 = \sum_{I \in I_0^+} |(b, h_I)|^2 + |(b, h_{I_0^+}(I))|^2 + |(b, h_{I_0^-}(I))|^2 \lesssim C|I_0|
\]
follows again by combination with (2).

So now we have
\[
\sum_{I \subset I_0} |(b, h_I)|^2 \lesssim C|I_0|
\]
for all \(I_0 \in D\), or that \(b \in BMO^d\). \(\square\)

3.2. Proof of Lemma 1

The boundedness of \(Tb - bT\) implies uniform boundedness of
\[
((Tb - bT)h_I, h_K) \text{ for all } I, K \in D.
\]

Now
\[
(Tb - bT)h_I, h_K = (bh_I, T^*h_K) - (b, Th_I), h_K)
\]
since
\[
T^*h_I = \begin{cases} s(J, \hat{J})h_J, & \text{if } J \text{ is odd;} \\ 0, & \text{if } J \text{ is even.} \end{cases}
\]

First, in (10), take \(I\) to be even and \(K = I_\pm\) (a dyadic child of \(I\)). Then \(T^*h_K = s(K, I)h_I\), so
\[
((Tb - bT)h_I, h_{I_\pm}) = (bh_I, s(K, I)h_I) - (b(h_{I_+} - h_{I_-}), h_K) = s(K, I)((b_I - \hat{b}_K) = \frac{\pm 1}{\sqrt{|I|}}(b, h_I),
\]
which proves (2) for all even \(I\). Now take in (10) an odd \(I\) and \(K = I^{(2)}\), the dyadic grandparent of \(I\) which will then also be odd. In this case, we have
\[
((Tb - bT)h_I, h_{I^{(2)}}) = (bh_I, T^*h_{I^{(2)}}) = (bh_I, \pm h_{I^{(3)}}) = \pm \frac{1}{\sqrt{8|I|}}(b, h_I),
\]
and this proves (2) for all odd \(I\). \(\square\)
3.3. Proof of Theorem 3.1 – Upper bound.

The proof that \(|T, b|\) is bounded if \(b \in BMO^d\) is completely standard and well-known, but we include it here for completeness. We show that each of the terms \((TD - DT), (T \pi - T)\) and \((TZ - ZT)\) is bounded. The easiest term is \((TD - DT)\), whose \(L^2\) norm can just be computed to be

\[
\| (TD - DT) \|_2^2 = \sum_{I \text{odd}} \frac{1}{|I|} |(b, h_I)(f, h_I)|^2 \lesssim \|b\|_{BMO^d} \|f\|_2^2,
\]

since obviously \(|(b, h_I)| \leq \|b\|_{BMO^d} |I|\) for all \(I\). For the other terms we can use \(H^1_d \sim BMO^d\) duality, which gives us that

\[
| (b, \Phi) | \lesssim \|b\|_{BMO^d} \|S_d \Phi\|_1,
\]

where \(S_d\) is the dyadic square function:

\[
S_d^2 f := \sum_I |(f, h_I)|^2 \mathbf{1}_I.
\]

For the \((T \pi - T)\) term: \((T \pi - T)f = \Pi_1 f - \Pi_2 f\), where

\[
\Pi_1 f = \sum_{I \text{even}} (b, h_I)(f, h_I)(Th_I); \quad \Pi_2 f = \sum_I (b, h_I)(Tf, h_I).
\]

Then for \(f, g \in L^2(\mathbb{R})\), we have \(|(\Pi_1 f, g)| \leq \|b\|_{BMO^d} \|S_d \Phi_1\|_1\)

\[
\Phi_1 = \sum_{I \text{even}} (f, h_I)(T^* g, h_I)h_I,
\]

so

\[
S_d^2 \Phi_1 = \sum_{I \text{even}} |(f, h_I)|^2 |(T^* g, h_I)|^2 \mathbf{1}_I \leq (M_d f)(S_d^2 T^* g),
\]

where \(M_d\) is the dyadic maximal function. Then (since \(M^d, S^d\) and \(T\) are bounded), \(\|S_d \Phi_1\|_1 \leq \|f\|_2 \|g\|_2\) and \(|(|\Pi_1 f, g)| \lesssim \|b\|_{BMO^d} \|f\|_2 \|g\|_2\) for all \(f, g \in L^2(\mathbb{R})\). So we have bounded the first term \(\Pi_1\). Similarly, \(|(\Pi_2 f, g)| = |(b, \Phi_2)|\) where

\[
\Phi_2 = \sum_I (T f, h_I)h_I.
\]

Then

\[
S_d^2 \Phi_2 = \sum_I (T^* f, h_I)h_I \lesssim M_d^2 (T^* f) S_d^2 g,
\]

giving again \(|(\Pi_2 f, g)| \lesssim \|b\|_{BMO^d} \|S_d \Phi_2\|_1 \lesssim \|b\|_{BMO^d} \|f\|_2 \|g\|_2\).

The term \((TD - DT)\) follows very similarly. \(\square\)

4. The bi-parameter commutator

Now let us work in \(\mathbb{R}^2 = \mathbb{R} \otimes \mathbb{R}\) and consider \([T, b]\), where \(T = T \otimes T\) and \(b(x, y)\) is an \(\mathbb{R}^2\)-function. We will denote \(T = T_1 \otimes T_2\), where \(T_1\) means \(T\) acting only on the \(i\)th variable. Dyadic intervals in first coordinate will be called only by letters \(I, I', I'', K\), in the second coordinate by \(J, J', J'', L\). Dyadic rectangles \(R \in \mathcal{D}^2\) are of the form \(R = I \times J\), with \(I, J \in \mathcal{D}\). We will show

**Theorem 4.1** \([T, b]\) is \(L^2\)-bounded if and only if \(b \in \text{bmo}^d\).

Recall that the space \(\text{bmo}^d\), or dyadic \(\text{little bmo}\), is defined by

\[
\|b\|_{\text{bmo}^d} := \sup_{R \in \mathcal{D}^2} \frac{1}{|R|} \left( \int_R |b(x, y) - (b)_R|^2 d(x, y) \right).
\]

In terms of Haar functions, this is the same as

\[
\|b\|_{\text{bmo}^d} = \sup_{R_0 = I_0 \times J_0} \left\{ \frac{1}{|I_0||J_0|} \sum_{I \subseteq I_0} \sum_{J \subseteq J_0} \|(b, h_I \otimes h_J)\|^2 + \frac{1}{|I_0|} \sum_{I \subseteq I_0} \|(b, h_I \otimes \mathbf{1}_{J_0})\|^2 + \frac{1}{|J_0|} \sum_{J \subseteq J_0} \|(b, \mathbf{1}_{I_0} \otimes h_J)\|^2 \right\}
\]
Above we had $D, \pi, Z$ parts of the commutator. Now we will have the following parts:

$$DD, D\pi, \pi D, \pi \pi, \pi Z, Z\pi, ZD, ZD, ZZ,$$
detailed below. As before, we consider $b$ fixed so the term $T\pi\pi(b, f) - \pi\pi(b, Tf)$ will simply be denoted $T\pi\pi - \pi\pi T$, for example.

1. $\pi\pi$ term:

$$(T\pi\pi - \pi\pi T)f = \sum_{I, J, even} (b, h_I \otimes h_J)(f)_{I \times J}(T_1 h_I) \otimes (T_2 h_J) - \sum_{I, J} (b, h_I \otimes h_J)(T_f)_{I \times J} h_I \otimes h_J.$$  

2. $ZZ$ term:

$$(TZZ - ZZT)f = \sum_{I, J} (b, h_I \otimes h_J)(f, h_I \otimes h_J)(T_1 \hat{1}_I) \otimes (T_2 \hat{1}_J) - \sum_{I, J, odd} (b, h_I \otimes h_J)(T_f, h_I \otimes h_J) s(I, \hat{1}_I) \hat{1}_I \otimes \hat{1}_J.$$  

3. $\pi Z$ term:

$$(T\pi Z - \pi Z T)f = \sum_{I, J, even} (b, h_I \otimes h_J)(f, \hat{1}_I \otimes h_J)(T_1 \hat{1}_I) \otimes (T_2 \hat{1}_J) - \sum_{I, J, odd} (b, h_I \otimes h_J)(T_f, \hat{1}_I \otimes h_J) \hat{1}_I \otimes h_J.$$  

4. $Z \pi$ term:

$$(TZ\pi - Z\pi T)f = \sum_{I, J, even} (b, h_I \otimes h_J)(f, \hat{1}_I \otimes h_J)(T_1 \hat{1}_I) \otimes (T_2 h_J) - \sum_{I, J, odd} (b, h_I \otimes h_J)(T_f, \hat{1}_I \otimes h_J) \hat{1}_I \otimes h_J.$$  

5. $\pi D$ term:

$$(T\pi D - \pi DT)f = \sum_{I, J, even} (b, h_I \otimes \hat{1}_J)(f, \hat{1}_I \otimes h_J)(T_1 \hat{1}_I) \otimes (T_2 h_J) - \sum_{I, J, odd} (b, h_I \otimes \hat{1}_J)(T_f, \hat{1}_I \otimes h_J) h_I \otimes h_J.$$  

6. $Z D$ term:

$$(TZD - ZDT)f = \sum_{I, J, even} (b, h_I \otimes \hat{1}_J)(f, h_I \otimes h_J)(T_1 \hat{1}_I) \otimes (T_2 h_J) - \sum_{I, J, odd} (b, h_I \otimes \hat{1}_J)(f, h_I \otimes h_J) s(I, \hat{1}_I) \hat{1}_I \otimes h_J.$$  

7. $D \pi$ term:

$$(TD\pi - D\pi T)f = \sum_{I, J, even} (b, \hat{1}_I \otimes h_J)(f, h_I \otimes \hat{1}_J)(T_1 \hat{1}_I) \otimes (T_2 h_J) - \sum_{I, J, odd} (b, \hat{1}_I \otimes h_J)(T_f, h_I \otimes \hat{1}_J) h_I \otimes h_J.$$  

8. $DZ$ term:

$$(TDZ - DZT)f = \sum_{I, J, even} (b, \hat{1}_I \otimes h_J)(f, h_I \otimes h_J)(T_1 \hat{1}_I) \otimes (T_2 \hat{1}_J) - \sum_{I, J, odd} (b, \hat{1}_I \otimes h_J)(f, h_I \otimes h_J) s(I, \hat{1}_I) \hat{1}_I \otimes \hat{1}_J.$$  

9. $DD$ term:

$$(TDD - DDT)f = \sum_{I, J, odd} [(b)_{I \times J} - (b)_{I \times J}](f, h_I \otimes h_J)s(I, \hat{1}_I) \hat{1}_I \otimes h_J.$$  

We also have a key lemma for this proof:

**Lemma 2** If $[T, b]$ is bounded, then:

$$|(b, h_I \otimes h_J)|^2 \lesssim |I||J|,$$

$$|(b, \hat{1}_I \otimes h_J)|^2 \lesssim |J|,$$

$$|(b, h_I \otimes \hat{1}_J)|^2 \lesssim |I|,$$

for all $I, J \in \mathcal{D}$. 
Above and from here on out, the symbol $\lesssim$ means that the quantity is bounded by some constant multiple of the $L^2$-norm of $[T,b]$. We prove this lemma in section 4.2.

Remark that this Lemma is the full bi-parameter equivalent of (1). Simply testing on $h_{I_0} \otimes h_{J_0}$, where both $I_0, J_0$ are even (the simple analog of the one-parameter case) will indeed give us that

$$\langle b \rangle_{J \times J} - \langle b \rangle_{I \times J} \lesssim 1, \forall I, J \text{ both odd},$$

which does bound the $DD$ term above – as detailed in Section 4.2. However, even having (1) for all $I, J$ is not enough for our purposes, because in two parameters

$$\langle b \rangle_{J \times J} - \langle b \rangle_{I \times J} = (b, h_I \otimes h_J) h_I (I) h_J (J) + (b, \hat{1}_I \otimes h_J) h_I (I) + (b, h_I \otimes \hat{1}_J) h_I (I).$$

So Lemma 2 is much stronger.

4.1. Proof of Theorem 4.1 - Lower Bound.

First of all, it is easy to see that Lemma 2 automatically bounds the operators $DD, ZZ, ZD, \text{and } DZ$ for all test functions. So these operators may be discarded. The remaining five operators, we now test on a Haar function $h_{I_0} \otimes h_{J_0}$. This yields ten operators, grouped by the various parity restrictions on $I_0$ and $J_0$:

**Group 1:** Terms that require both $I_0, J_0$ be even:

$$\sum_{I \subseteq I_0^+, J \subseteq J_0^+} (b, h_I \otimes h_J) h_{I_0^+} (I) h_{J_0^+} (J) h_I (I) h_J (J) + \sum_{I \subseteq I_0^+, J \subseteq J_0^-} (b, h_I \otimes h_J) h_{I_0^-} (I) h_{J_0^-} (J) h_I (I) h_J (J)$$

$$+ \sum_{I \subseteq I_0^-} (b, h_I \otimes h_J) h_{I_0^-} (I) h_{J_0^+} (J) h_I (I) h_J (J) - \sum_{I \subseteq I_0^-} (b, h_I \otimes h_J) h_{I_0^-} (I) h_{J_0^-} (J) h_I (I) h_J (J).$$

**Group 2:** Terms that require $I_0$ be even:

$$\sum_{I \subseteq I_0^+, J \subseteq J_0^-} (b, h_I \otimes h_J) h_{I_0^+} (I) h_{J_0^-} (J) h_I (I) h_J (J)$$

$$+ \sum_{I \subseteq I_0^-} (b, h_I \otimes h_J) h_{I_0^-} (I) h_{J_0^-} (J) h_I (I) h_J (J).$$
Group 3: Term that requires $J_0$ be even:

$$
\pi D_1 = \sum_{I_{\text{even}}, I \subseteq J_0} (b, h_I \otimes \mathbf{1}_{J_0}) h_{I_0}(I)(T_1 h_I) \otimes (T_2 h_{J_0}).
$$

Group 4: Terms with no restrictions on the parity of $I_0, J_0$:

$$
\pi p_1 = \sum_{I_{\text{even}}, J \subseteq J_0} (b, h_I \otimes h_J) h_{I_0}(I) h_{J_0}(J)(T_1 h_I) \otimes (T_2 h_J).
$$

$$
\pi Z_1 = \sum_{I_{\text{even}}, J \subseteq J_0} (b, h_I \otimes h_J) h_{I_0}(I)(T_1 h_I) \otimes (T_2 \mathbf{1}_{J_0}).
$$

$$
Z p_1 = \sum_{J_{\text{even}}, J \subseteq J_0} (b, h_{I_0} \otimes h_J) h_{J_0}(J)(T_1 \mathbf{1}_{I_0}) \otimes (T_2 h_J).
$$

I. Take $I_0, J_0$ both odd. In this case, we only have the three terms in Group 4 above, the terms with no restrictions on the parity of $I_0, J_0$. Note that all three of these terms are mutually orthogonal:

$$
(17) \quad \pi p_1 : (I \subseteq I_0) \times (J \subseteq J_0)
$$

$$
(18) \quad \pi Z_1 : (I \subseteq I_0) \times (J \supset J_0)
$$

$$
Z p_1 : (I \supset J_0) \times (J \subseteq J_0),
$$

so all three are individually bounded on the test function. Looking at the $L^2$-norm of $\pi p_1$, for example, shows that

$$
\sum_{I_{\text{even}}, I \subseteq J_0} |(b, h_I \otimes h_J)|^2 \lesssim |I_0||J_0|,
$$

for all $I_0, J_0$ both odd.

But combining with Lemma 2 we see that this is actually true for all $I_0, J_0$, meaning that $\pi p_1$ is uniformly bounded on all test functions $h_{I_0} \otimes h_{J_0}$ and this term may be discarded. The same holds similarly for the remaining terms $\pi Z_1$ and $Z p_1$ in Group 4. With all Group 4 terms now being discarded, we move on.

II. Take $I_0$ even, $J_0$ odd. In this case we only have the term $\pi D_1$ in Group 2. Computing the $L^2$-norm here gives

$$
\sum_{J_{\text{even}}, J \subseteq J_0} |(b, \mathbf{1}_{I_0} \otimes h_J)|^2 \lesssim |J_0|,
$$

for all odd $J_0$.

Again, combined with Lemma 2 this actually holds for all $J_0$, meaning that $\pi D_1$ is uniformly bounded on all test functions $h_{I_0} \otimes h_{J_0}$, and may be discarded.

III. Take $I_0$ odd, $J_0$ even. This case is symmetric to Case II, and yields that the term $\pi D_1$ in Group 3 is uniformly bounded on all test functions $h_{I_0} \otimes h_{J_0}$, and may be discarded.

IV. Take $I_0, J_0$ both even. The last case remaining, where we only need to look at the five terms in Group 1. But happily, these are also mutually orthogonal and therefore individually bounded:

$$
(19) \quad \pi p_2 : (I \subseteq I_0^+) \times (J \subseteq J_0^+)
$$

$$
(20) \quad \pi Z_2 : (I \subseteq I_0^+) \times (J \supset J_0^+)
$$

$$
(21) \quad \pi Z_2 : (I \supset J_0^+) \times (J \subseteq J_0^+)
$$

$$
(22) \quad \pi D_2 : (I \subseteq I_0^+) \times (J \supset J_0^+)
$$

$$
Z p_2 : (I_0^+) \times (J \subseteq J_0^+)
$$

Computing the $L^2$-norm of $\pi p_2$ gives

$$
\sum_{I \subseteq I_0, J \subseteq J_0} |(b, h_I \otimes h_J)|^2 \lesssim |I_0||J_0|,
$$
the $L^2$-norm of $\pi D 2$ gives
\[
\sum_{I \subseteq J_0} |(b, h_I \otimes \tilde{i}_{J_0})|^2 \lesssim |J_0|,
\]
and finally the $L^2$-norm of $D\pi 2$ gives
\[
\sum_{J \subseteq J_0} |(b, \tilde{i}_{J_0} \otimes h_J)|^2 \lesssim |J_0|.
\]
These mean exactly that $b \in bmo^d$.

4.2. Proof of Lemma 2

I. Testing on $h_I \otimes h_J$. If $[T, b]$ is bounded, then $\|[T, b]h_R\| \lesssim 1$ for all rectangles $R \in \mathcal{D}$. In particular,
\[
\|[T, b]h_{I \times J}, h_{K \times L}\| \lesssim 1, \forall I, J, K, L \in \mathcal{D}.
\]
Further expanding this:
\[
(23) \quad \left| bh_I \otimes h_J, (T^*_I h_K) \otimes (T^*_J h_L) \right| - \left| b \ (T^*_I h_I) \otimes (T^*_J h_J), h_K \otimes h_L \right| \lesssim 1, \forall I, J, K, L \in \mathcal{D}.
\]

Ia. Take $I, J$ both even and $K = I_\pm, L = J_\pm$ (children of $I, J$) – so both $K, L$ are odd. Then both terms of (23) are present, and it becomes exactly (11).

Ib. Take $I, J$ both odd (so second term in (23) is 0), and $K = I^{(2)}, L = J^{(2)}$. Both $K, L$ are then even, and $T^*_I h_K = \pm h_{I(3)}, T^*_J h_L = \pm h_{J(3)}$. Equation (23) then becomes
\[
\left| bh_I \otimes h_J, h_{I(3)} \otimes h_{J(3)} \right| \lesssim 1.
\]
But since $h_{I(3)}, h_{J(3)}$ are constant on $I, J$, respectively:
\[
h_{I(3)}(I) = \pm \frac{1}{\sqrt{|I(3)|}}; \quad h_{J(3)}(J) = \pm \frac{1}{\sqrt{|J(3)|}},
\]
we now have (3) for the case when $I, J$ are both odd:
\[
\|[b, h_I \otimes h_J]|^2 \lesssim |I||J|, \forall I, J \text{ both odd}.
\]

Ic. Take $I$ even, $J$ odd (so second term in (23) is 0), and $K = I_\pm, L = J^{(2)}$ (both odd). Then $T^*_I h_K = \pm h_I$, and $T^*_J h_L = \pm h_{J(3)}$. Equation (23) becomes
\[
\left| bh_I \otimes h_J, h_I \otimes h_{J(3)} \right| \lesssim 1,
\]
which yields (9) for the case when $I$ is even and $J$ is odd:
\[
\|[b, \tilde{i}_I \otimes h_J]|^2 \lesssim |J|, \forall I \text{ even, } J \text{ odd}.
\]

Id. Take $I$ odd, $J$ even (so second term in (23) is 0), and $K = I^{(2)}, L = J_\pm$ (both odd). Symmetrically with case Ic, this will give (10) for the case of $I$ odd and $J$ even:
\[
\|[b, h_I \otimes \tilde{i}_J]|^2 \lesssim |I|, \forall I \text{ odd, } J \text{ even}.
\]

Ie. Take $I, J$ both even, and $K = I$ (even), $L = J_\pm$ (odd). Then the first term in (23) is 0, and we have
\[
\left| bh_{I_\pm} \otimes h_{J_\pm}, (h_{I_\pm} - h_{J_\pm}), h_I \otimes h_{J_\pm} \right| \lesssim 1.
\]
This gives us (10) for the case when $I, J$ are both odd.
If. Symmetrically, take $I, J$ both even, $K = I_\pm$ (odd), $L = J$ (even), and we have (9) for the case when $I, J$ are both odd.

II. Testing on $h_I \otimes T_J^* h_J$. Running through the same process with the test function $h_I \otimes T_J^* h_J$ instead of $h_I \otimes h_J$, we get

\[
(24) \quad \left| \frac{\tilde{h}_I \otimes T_J^* h_J, (T_I^* h_K) \otimes (T_J^* h_L)}{0 \text{ unless } I \text{ odd}} - \left( b \left( T_I^* h_I \right) \otimes (T_J^* h_J), h_K \otimes h_L \right) \right| \lesssim 1, \forall I, J, K, L \in \mathcal{D}.
\]

Note that, in one parameter,

$$TT^* h_I = h_I - h_{s(I)}, \forall I \text{ odd}; 0 \text{ otherwise}.$$ 

where $s(I)$ denotes the dyadic sibling of $I$.

IIa. Take $I$ even, $J$ odd, and $K = \hat{I}$ (odd), $L = J$ (odd). Then both terms of (24) are present, and give

$$\left| \frac{1}{|I|^2} \left( b, h_I \otimes \hat{I}_J \right) - s(I, \hat{I}) \frac{1}{|I|} \left( b, (h_{I, -} - h_{I, -}) \otimes \hat{I}_J \right) \right| \lesssim 1.$$ 

Since $I_\pm$ and $J$ are odd, we already know from case Ie that

$$\left| s(I, \hat{I}) \frac{1}{|I|} \left( b, (h_{I, -} - h_{I, -}) \otimes \hat{I}_J \right) \right| \lesssim 1,$$

so we now have (11) for $I, J$ both even.

IIb. Take $I$ even, $J$ odd, and $K = I_\pm$ (odd), $L = J$ (odd). Then the first term in (24) is 0, and the second term is

$$\left| \frac{1}{|I|^2} \left( b(h_{I, +} - h_{I, -}), (h_J - s(J)) \otimes h_{I_\pm} \otimes h_{J_\pm} \right) \right| = \frac{1}{|I|^2} \left( b, \hat{I}_J \otimes \hat{I}_{J_\pm} \right) \lesssim 1,$$

so now we have (9) for the case where $I$ is odd and $J$ is even.

IIc. Take $I$ even, $J$ odd, and $K = I_\pm$ (odd), $L = J_\pm$ (even). Again the first term in (24) is 0, and we have

$$\left| \frac{1}{|I| |J|} \left( b(h_{I, +} - h_{I, -}), (h_J - s(J)) \otimes h_{I_\pm} \otimes h_{J_\pm} \right) \right| = \frac{1}{|I|^2 |J|} \left( b, h_{I_\pm} \otimes h_{J_\pm} \right) \lesssim 1,$$

which gives us (8) for $I$ odd and $J$ even.

III. Testing on $T^*_I h_I \otimes h_J$. This case is symmetrical to case II, and it gives us (9) for $I, J$ both even, (11) for $I$ even and $J$ odd, and (8) for $I$ even and $J$ odd.

At this point (9) and (11) are fully proved, and all that is left is (8) for $I, J$ both even. This final situation is dealt with below.

IV. Testing on $T^*_I h_I \otimes T^*_J h_J$. In this case we have

$$\left| \frac{b(T^*_I h_I) \otimes (T^*_I h_J), (T^*_I h_K) \otimes (T^*_I h_L)}{0 \text{ unless } I, J \text{ both odd}} - \left( b(T^*_I h_I) \otimes (T^*_J h_J), h_K \otimes h_L \right) \right| \lesssim 1, \forall I, J, K, L \in \mathcal{D}.$$ 

In the equation above, take $I, J$ both odd, and $K = I_\pm, L = J_\pm$ (both even), so we only have the second term above:

$$\left| \frac{b(h_I - s(I)), (h_J - s(J)) \otimes h_{I_\pm} \otimes h_{J_\pm}}{0 \text{ unless } I, J \text{ both odd}} = \frac{1}{|I| |J|} \left( b, h_{I_\pm} \otimes h_{J_\pm} \right) \right| \lesssim 1,$$

which gives us the last piece: (8) for $I, J$ both even. \qed
4.3. Proof of Theorem 4.1 - Upper Bound.

Again, the proof that \(|T, b\) is bounded if \(b \in bmo^d\) is completely standard, but we include it here for completeness. We will show that each term in the paraproduct decomposition of \(|T, b\) is bounded. Some of these will pair with \textit{product} \(BMO\) and others with \textit{little} \(bmo\). The easiest one is again the \(DD\) term, since its \(L^2\)-norm can simply be computed:

\[
\|(TDD - DDT)\|_2^2 \leq \sum_{I, J} |\langle b \rangle|_{I \times J}^2 \| (f, h_I \otimes h_J) \|_2^2 \lesssim \|b\|_{bmo^d} \|f\|_2^2.
\]

Of the remaining eight terms, four will “pair” with \textit{product} \(BMO^d\), and four will “pair” with \(bmo^d\).

4.3.1. \textit{Product BMO}^d terms: \(\pi \pi, ZZ, \pi Z, Z \pi\). For these terms, we use the fact that if \(b \in bmo^d\) then \(b \in BMO^d\), and the bi-parameter \(H^1_1 - BMO^d\) duality: \(|\langle b, \Phi \rangle| \lesssim \|b\|_{BMO^d} \|S_d \Phi\|_1\), where \(S_d\) now denotes the bi-parameter dyadic square function: \(S_d^2 f = \sum_{I, J} |(f, h_I \otimes h_J)|^2 1_I \otimes 1_J\).

We split the \((T \pi \pi - \pi \pi T) f\) term into \(\Pi_1 f - \Pi_2 f\), where

\[
\Pi_1 f = \sum_{I, J} \langle b, h_I \otimes h_J \rangle \langle f \rangle_{I \times J} T(h_I \otimes h_J); \quad \Pi_2 f = \sum_{I, J} \langle b, h_I \otimes h_J \rangle \langle T f \rangle_{I \times J} h_I \otimes h_J.
\]

For the first term: \((\Pi_1 f, g) = (b, \Phi_1)\), where \(\Phi_1 = \sum_{I, J} \langle f \rangle_{I \times J} \langle (T^* g, h_I \otimes h_J) h_I \otimes h_J, f, g \in L^2(\mathbb{R})\). Then

\[
S_d^2 \Phi_1 = \sum_{I, J} \langle f \rangle_{I \times J}^2 \langle (T^* g, h_I \otimes h_J) \rangle^2 \hat{1}_{I \times J} \leq [M_d^2 f] [S_d^2 (T^* g)],
\]

where \(M_d\) now denotes the bi-parameter dyadic maximal function. Then

\[
|(\Pi_1 f, g)| = |\langle b, \Phi_1 \rangle| \lesssim \|b\|_{BMO^d} \|S_d \Phi_1\|_1 \lesssim \|b\|_{BMO^d} \|M_d f\|_2 \|S_d (T^* g)\|_2 \lesssim \|b\|_{BMO^d} \|f\|_2 \|g\|_2.
\]

The other term \(\Pi_2\), as well as the terms in \((T ZZ - ZZ T)\) follow similarly.

For the \(\pi Z\) and \(Z \pi\) terms, we appeal to the mixed square functions introduced in [HPW]:

\[
[SM]^2 f(x, y) := \sum_I M_{d_{x,y}}^2 f_I(y) \hat{1}_I(x), \quad [MS]^2 f(x, y) := \sum_J M_{d_{x,y}}^2 f_J(x) \hat{1}_J(y),
\]

where \(M_{d_i}\) denote the dyadic maximal function in parameter \(i\) only, and for every \(I, J\):

\[
f_I(y) := \int_R f(x, y) h_I(x) \, dx; \quad f_J(x) := \int_R f(x, y) h_J(y) \, dy.
\]

More general forms of these mixed functions were proved to be bounded in the weighted setting in [HPW].

Now, looking at our term \((T \pi Z - \pi Z T) f = \Pi Z_1 f - \Pi Z_2 f\), where

\[
\Pi Z_1 f = \sum_{I, J} \langle b, h_I \otimes h_J \rangle \langle f, \hat{1}_I \otimes h_J \rangle T(h_I \otimes \hat{1}_J); \quad \Pi Z_2 f = \sum_{I, J} \langle b, h_I \otimes h_J \rangle \langle T f, \hat{1}_I \otimes h_J \rangle h_I \otimes \hat{1}_J.
\]

For the first term, \((\Pi Z_1 f, g) = (b, \Phi_1)\), where \(\Phi_1 = \sum_{I, J} \langle f_I \rangle \langle (T^* g) \rangle_{I \otimes J} h_I \otimes h_J\). Then

\[
(25) \quad S_d^2 \Phi_1 = \sum_{I, J} |\langle f_I \rangle|^2 |\langle (T^* g) \rangle_{I \otimes J}|^2 \hat{1}_I \otimes \hat{1}_J \leq \left( \sum_I M_{d_{x,y}}^2 \langle T^* g \rangle_I(y) \hat{1}_I(x) \right) \left( \sum_J M_{d_{x,y}}^2 f_J(x) \hat{1}_J(y) \right) = [SM]^2 (T^* g) [MS]^2 (f).
\]

So

\[
|(\Pi Z_1 f, g)| \lesssim \|b\|_{BMO^d} \|S_d \Phi_1\|_1 \lesssim \|b\|_{BMO^d} \|[SM]^2 (T^* g)\|_2 \|[MS]^2 (f)\|_2 \lesssim \|b\|_{BMO^d} \|f\|_2 \|g\|_2.
\]

The other term \(\Pi Z_2\), as well as the terms in \((Z \pi - \pi Z)\) follow similarly.
4.3.2. Little $bmo^d$ terms: $\pi D$, $D\pi$, $ZD$, $DZ$. For these terms we appeal to the dyadic square functions in one-parameter

$$S_{d_1}^2 f(x, y) = \sum_i f_i^2(x) \hat{1}_i(x); \quad S_{d_2}^2 f(x, y) = \sum_j f_j^2(x) \hat{1}_j(y),$$

and the duality with $bmo^d$:

$$|(b, \Phi)| \lesssim \|b\|_{bmo^d} \|S_d \Phi\|_1.$$ 

These were all treated in the weighted situation in [HPW]. For example for the $\pi D$ term we have $(T_{\pi D} - \pi DT)f = \pi D_1 f - \pi D_2 f$ where

$$\pi D_1 f = \sum_{i,j}(b, h_i \otimes \hat{1}_j)(f, \hat{1}_i \otimes h_j)\mathbf{T}(h_i \otimes h_j); \quad \pi D_1 f = \sum_{i,j}(b, h_i \otimes \hat{1}_j)(T_f, \hat{1}_i \otimes h_j)h_i \otimes h_j.$$ 

For the first term, $(\pi D_1 f, g) = (b, \Phi_1)$ where $\Phi_1 = \sum_{i,j}(f, \hat{1}_i \otimes h_j)(T^* g, h_i \otimes h_j)h_i \ast \hat{1}_j$. Then

\begin{equation}
S_{d_1}^2 \Phi_1 = \sum_i \left( \sum_j |(f_j)_i|^2 \hat{1}_j \right)^2 \hat{1}_i 
\end{equation}

\begin{equation}
\leq \sum_i \left( \sum_j |(f_j)_i|^2 \hat{1}_j \right) \left( \sum_j |(T^* g, h_i \otimes h_j)^2 \hat{1}_j \right) - \hat{1}_i 
\end{equation}

$$\leq \sum_i M_{d_1}^2 f(x) \hat{1}_i(y) \cdot \sum_{i,j} |(T^* g, h_i \otimes h_j)^2 \hat{1}_i \otimes \hat{1}_j| = \|MS\|^2 f \cdot S_{d_1}(T^* g).$$

So

$$|(\pi D_1 f, g)| \lesssim \|b\|_{bmo^d} \|S_{d_1} \Phi_1\|_1 \lesssim \|b\|_{bmo^d} \|MS\|^2 \|f\|_2 \|g\|_2.$$ 

The other term $\pi Z_2$, as well as the terms of $(TD\pi - D\pi T)$, $(TZD - ZDT)$ and $(TDZ - DZT)$ follow similarly. 

\[ \square \]

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