A remark on the Omori-Yau maximum principle for semi-elliptic operators

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Abstract. We generalize A. Börbény’s condition for the conclusion of the Omori-Yau maximum principle for the Laplace operator on a complete Riemannian manifold to a second-order linear semi-elliptic operator $L$ with bounded coefficients and no zeroth order term. Also, we consider a new sufficient condition for the existence of a tamed exhaustion function.

As a corollary, we show that the existence of a tamed exhaustion function is more general than the hypotheses in the version of the Omori-Yau maximum principle that was given by A. Ratto, M. Rigoli and A.G. Setti.

Keywords: Riemannian manifold; Curvature; Omori-Yau maximum principle

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1. Introduction

Let $(M, \widehat{g})$ be a smooth complete Riemannian manifold of dimension $n$. A second-order linear differential operator $L : C^\infty(M) \to C^\infty(M)$ without zeroth order term can be written as

\[(1.1) \quad Lg = \text{Tr}(A \circ \text{Hess}_g) + \widehat{g}(V, \nabla g),\]

where $A \in \Gamma(\text{End}(TM))$ is self-adjoint with respect to $\widehat{g}$, $\text{Hess}_g \in \Gamma(\text{End}(TM))$ is the Hessian of $g$ in the form defined by $\text{Hess}_g(X) = \nabla_X \nabla g$ for $X \in \Gamma(TM)$, and finally $V \in \Gamma(TM)$. In this article, we will deal with the semi-elliptic case, i.e., $A$ is positive semi-definite at each point, and we always assume that

\[(1.2) \quad \sup_M \text{Tr}(A) + \sup_M |V| < \infty.\]

Definition 1.3. A smooth complete Riemannian manifold $M$ is said to satisfy the Omori-Yau maximum principle for the Laplace operator $\Delta$ (the above semi-elliptic operator $L$, resp.) if for any $C^2$ function $g : M \to \mathbb{R}$ which is bounded from above and for any $\epsilon > 0$ there is a point $x_\epsilon \in M$, such that $|g(x_\epsilon) - \sup_M g| < \epsilon$, $\|\nabla g(x_\epsilon)\| < \epsilon$ and $\Delta g(x_\epsilon) < \epsilon$ ($Lg(x_\epsilon) < \epsilon$, resp.).
For the operator $\Delta$, Definition 1.3 is the well-known Omori-Yau maximum principle for the Laplacian, which was first proven by H. Omori [7] and S.T. Yau [14] when the Ricci curvature is only bounded below. This improve upon by Q. Chen and Y.L. Xin [4] and A. Ratto, M. Rigoli and A.G. Setti [11] when the Ricci curvature decays slower than a certain decreasing function tending to minus infinity. For instance,

**Theorem 1.4.** (Ratto-Rigoli-Setti [11, Theorem 2.3]) Let $o \in M$ be a fixed point and $r(x)$ be the distance function from $o$. Let us assume that away from the cut locus of $o$ we have

\[ \text{Ricc}(\nabla r, \nabla r) \geq -(n-1)BG^2(r), \]

where $B > 0$ is some constant and $G(t)$ on $[0, \infty)$ satisfies

\[ G(0) = 1, \quad G' \geq 0, \quad \int_0^\infty \frac{1}{G(t)} \, dt = \infty, \]

\[ \sqrt{G^{(2k+1)}}(0) = 0 \quad \forall k \geq 0, \quad \limsup_{t \to \infty} \frac{t\sqrt{G(\sqrt{t})}}{\sqrt{G(t)}} < \infty. \]

Then $M$ satisfies the Omori-Yau maximum principle for the Laplacian $\Delta$.

A. Borbély [3, Theorem] obtained the conclusion of the Omori-Yau maximum principle where the Ricci curvature condition (1.5) is replaced by the assumption $\Delta r(x) \leq G(r(x))$ without (1.6). As a corollary, he proved Theorem 1.4 without (1.6) in Ratto-Rigoli-Setti’s condition. Also, G.P. Bessa, S. Pigola, and A.G. Setti [2, Theorem 5.6] proved Borbély’ theorem [3, Theorem] for the $f$-Laplacian $\Delta f$. In this article, we first show that Borbély’ theorem [3, Theorem] is also true for our semi-elliptic operator $L$ by following his method in [3] (see Theorem 1.12).

To state another results, we need the following definitions.

**Definition 1.7.** Let $u$ be a real-valued continuous function on $M$ and let a point $p \in M$.

- a function $u$ is called proper, if the set \( \{ p : u(p) \leq r \} \) is compact for every real number $r$.
- a function $v$ defined on a neighborhood $U_p$ of $p$ is called an upper-supporting function for $u$ at $p$, if the conditions $v(p) = u(p)$ and $v \geq u$ hold in $U_p$.

**Definition 1.8.** A proper continuous function $u : M \to \mathbb{R}$ is called a $\Delta$-tamed exhaustion, if the following condition holds:

1. $u \geq 0$.
2. At all points $p \in M$ it has a $C^2$ smooth, upper-supporting function $v$ at $p$ defined on an open neighborhood $U_p$ such that $\|\nabla v\|_p \leq 1$ and $\Delta v|_p \leq 1$.

H.L. Royden [12] showed that every complete Riemannian manifold satisfying Omori-Yau’s condition (i.e., the Ricci curvature is bounded from below) admits a $\Delta$-tamed exhaustion.
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function. Inspired by Royden’s article [12], K.-T. Kim and H. Lee [6, Theorem 2] proved that the Omori-Yau maximum principle for the Laplacian $\Delta$ when there exists a $\Delta$-tamed exhaustion function. Moreover, they proved that every complete Riemannian manifold satisfying Ratto-Rigoli-Setti’s condition admits a $\Delta$-tamed exhaustion function [6]. Similar to Definition 1.8, we define an $L$-tamed exhaustion function (i.e., we replace $\Delta$ with $L$) [5, Definition 1.4]. Then, using the existence of an $L$-tamed exhaustion function, K. Hong and C. Sung [5, Theorem 2.1] generalized the Omori-Yau maximum principle for the Laplacian $\Delta$ to the operator $L$. In this article, we give a new sufficient condition for the existence of an $L$-tamed exhaustion function (see Theorem 1.13). We prove this result using the ideas adapted from that of the article [6]. Note that Theorem 1.13, together with [5, Theorem 2.1] implies the maximum principle of Omori and Yau for the operator $L$.

As a corollary, we prove that the existence of a $\Delta$-tamed exhaustion is more general than the Ratto-Rigoli-Setti’s condition. Unfortunately, for the operator $L$, the relation between Börbély’s condition (or the existence of an $L$-tamed exhaustion) and Ratto-Rigoli-Setti’s condition remains for further study.

Now, we formulate our main results. Fix $x_\epsilon \in M$. Since $A$, in the notation (1.1), is symmetric, it is diagonalizable at each point in an orthonormal basis, so we can take a normal coordinate $(x_1, \cdots, x_n)$ around $x_\epsilon \in M$ such that $A$ at $x_\epsilon$ is represented as a diagonal matrix, and hence

$$Lh|_{x_\epsilon} = \sum_l a_{ll}(x_\epsilon) \frac{\partial^2}{\partial x_l^2} h|_{x_\epsilon} + \sum l a_l(x_\epsilon) \frac{\partial}{\partial x_l} h|_{x_\epsilon},$$

for a real-valued function $h$ on $M$, where each $a_{ll}(x_\epsilon)$ is nonnegative, and the entries $a_{ll}(x_\epsilon)$ and $|a_l(x_\epsilon)|$ are bounded above as $x_\epsilon$ varies by (1.2). For a notational convenience, let’s introduce a locally-defined differential operator

$$\tilde{\Delta} := a_{11}(x_\epsilon) \frac{\partial^2}{\partial x_1^2} + \cdots + a_{nn}(x_\epsilon) \frac{\partial^2}{\partial x_n^2}, \quad \nabla := a_1(x_\epsilon) \frac{\partial}{\partial x_1} + \cdots + a_n(x_\epsilon) \frac{\partial}{\partial x_n},$$

Put $d_l = a_{ll}(x_\epsilon)$ and $e_l = |a_l(x_\epsilon)|$ for $1 \leq l \leq n$. We may assume that $d_1$ and $e_1$ are the largest of $\{d_1, \cdots, d_n\}$ and $\{e_1, \cdots, e_n\}$ respectively.

Then we have the followings:

**Theorem 1.12.** Let $o \in M$ be a fixed point and $r(x)$ be the distance function from $o$. Let us assume that for all $x \in M$

$$\tilde{\Delta}r(x) \leq G(r(x)),$$

where $r$ is smooth, $r(x) > 1$, and $G(t)$ on $[0, \infty)$ satisfies

$$G \geq 1, \ G' \geq 0, \ \text{and} \ \int_0^\infty \frac{dt}{G(t)} = \infty.$$

Then $M$ satisfies the Omori-Yau maximum principle for the operator $L$. 
Theorem 1.13. Let \( o \in M \) be a fixed point and \( r(x) \) be the distance function from \( o \). Let us assume that for all \( x \in M \)

\[
\bar{\Delta} r(x) \leq G(r(x)),
\]

where \( r \) is smooth, \( r(x) > 1 \), and \( G(t) \) on \([0, \infty)\) satisfies

\[
G \geq 1, \quad G' \geq 0, \quad \int_{0}^{\infty} \frac{dt}{G(t)} = \infty,
\]

\[
\limsup_{t \to +\infty} \frac{t\sqrt{G(\sqrt{t})}}{\sqrt{G(t)}} < +\infty.
\]

Then \( M \) admits an \( L \)-tamed exhaustion function.

Corollary 1.17. The existence of a \( \Delta \)-tamed exhaustion function for the conclusion of the Omori-Yau maximum principle for the Laplacian \( \Delta \) is more general than the hypothesis in Theorem 1.4.

Remark 1.18. There are some other sufficient conditions under which the Omori-Yau maximum principle for the Laplacian \( \Delta \) holds \([8, 9, 13]\). Recently, G.P. Bessa and L.F. Pessoa \([1, \text{Theorem 1}]\) present a sufficient condition for the conclusion of the Omori-Yau maximum principle for a second-order linear semi-elliptic operator with bounded first-order coefficients and no zeroth order term. However, all the sufficient conditions are not the existence of a tamed exhaustion function.

2. Proof of Theorem 1.12

The proof is similar to the method in the article \([3]\). Let \( U = \sup g \). We may assume that \( g < U \) at every point of \( M \); otherwise, \( g \) has its maximum at some point and that point directly satisfies the Omori-Yau maximum principle for a semi-elliptic operator \( L \).

Define the function \( F(t) \) as

\[
F(t) = e^{\int_{0}^{t} \frac{1}{G(s)} ds}.
\]

Then

\[
F' = \frac{F}{G}.
\]

Since \( G \geq 1 \) on \([0, \infty)\), we have \( F \geq 1 \), and \( F' > 0 \). Hence the function \( F \) is strictly increasing, and \( \lim_{t \to \infty} F(t) = \infty \). Since the set \( \{x \in M : r(x) \leq 1\} \) is compact, we have

\[
U - \sup \{g(x) : r(x) \leq 1\} > 0.
\]

For any positive constant \( \epsilon < \min \{1, U - \sup \{g(x) : r(x) \leq 1\} \} \), we define the function \( h_\lambda : M \to \mathbb{R} \) as

\[
h_\lambda(x) = \lambda F(r(x)) + U - \epsilon.
\]

Then

\[
h_\lambda(x) > g(x) \quad \text{if } r(x) \leq 1 \text{ and } \lambda \geq 0.
\]
Because, for all \( x \in M \), \( F(r(x)) \geq 1 \) and \( U > g(x) \). If \( \lambda > \epsilon \), then we have
\[
 h_\lambda(x) > g(x) \text{ for all } x \in M.
\]
Define \( \lambda_0 \) as
\[
 \lambda_0 = \inf \{ \lambda : h_\lambda(x) > g(x) \text{ for all } x \in M \}.
\]
Then clearly, \( \lambda_0 > 0 \). Furthermore, we can obtain \( h_{\lambda_0}(x) \geq g(x) \) for all \( x \in M \), i.e., there is a point \( x_\epsilon \in M \) such that \( h_{\lambda_0}(x_\epsilon) = g(x_\epsilon) \). Let assume that to the contrary \( h_{\lambda_0}(x) > g(x) \) for all \( x \in M \). Then we will show that there is a constant \( \lambda' \) with \( \lambda_0 > \lambda' \) such that \( h_{\lambda'}(x) > g(x) \) for all \( x \in M \). This is a contradiction to the definition of \( \lambda_0 \).

Let \( \lambda_0 > \lambda_1 \). Because \( \lim_{r \to -\infty} F(r) = \infty \), there is a sufficiently large positive number \( r_0 \) such that \( h_{\lambda_1}(x) > U > g(x) \) for \( r(x) > r_0 \). Also, because the set \( \{ x \in M : r(x) \leq r_0 \} \) is compact, the statement \( h_{\lambda_0}(x) > g(x) \) for all \( x \in M \) implies that there is a constant \( \lambda_2 \) with \( \lambda_0 > \lambda_2 \) such that \( h_{\lambda_2}(x) > g(x) \) for \( r(x) \leq r_0 \). Now, let \( \lambda' = \max\{\lambda_1, \lambda_2\} \). Then, for \( \lambda_0 > \lambda' \), we have \( h_{\lambda'}(x) > g(x) \) for all \( x \in M \). Moreover, by (2.1) and \( \lambda_0 > 0 \), we have \( r(x_\epsilon) > 1 \).

Next, we have to show that \( h_{\lambda_0} \) is smooth at \( x_\epsilon \). Since \( h_{\lambda'}(x) = \lambda F(r(x)) + U - \epsilon \), it is enough to show that \( r \) is smooth at \( x_\epsilon \). Note that \( r \) is a Lipschitz function and is smooth on \( M \setminus \{p, C_p\} \), where \( C_p \) is the cut locus of \( p \). Suppose that \( x_\epsilon \in C_p \). Then we have two possibilities (P. Peter [10] Lemma 8.2): either there are two distinct minimizing geodesic segments \( \gamma_1, \gamma_2 : [0, t_0] \to M \) joining \( p \) to \( x_\epsilon \), or there is a geodesic segment \( \gamma : [0, t_0] \to M \) from \( p \) to \( x_\epsilon \) along which \( x_\epsilon \) is conjugate to \( p \). Notice that
\[
 t_0 = r(\gamma_i(t_0)) = r(x_\epsilon) \text{ for } i = 1 \text{ or } 2.
\]
We consider the first case. Let \( w = \gamma'_1(t_0) \) and \( v = \gamma'_2(t_0) \). Since \( \gamma_1 \) and \( \gamma_2 \) are distinct segments, we have \( w \neq v \). For \( i = 1 \) or \( 2 \), the functions \( t \to r(\gamma_i(t)) \) are differentiable on \( (0, t_0) \) and they have a left-derivative at \( t_0 \). Note that \( g \) is \( C^2 \) smooth on \( M \). From the definition of \( \lambda_0 \), \( h_{\lambda_0} \geq g \), and \( h_{\lambda_0}(x_\epsilon) = g(x_\epsilon) \) we obtain
\[
(2.2) \quad \liminf_{s \to 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq D_v g(x_\epsilon),
\]
where \( D_v g(x_\epsilon) \) denotes the directional derivative of \( g \) at the point \( x_\epsilon \) in the direction of \( v \). Furthermore, since \( h_{\lambda_0} \) has a directional derivative at \( x_\epsilon \) in the direction of \( -v \), we have
\[
 -\lambda_0 F'(t_0) = -\lambda_0 F'(r(x_\epsilon)) = D_{-v} h_{\lambda_0}(x_\epsilon) \geq D_{-v} g(x_\epsilon) = -D_v g(x_\epsilon).
\]
This yields
\[
(2.3) \quad D_v g(x_\epsilon) \geq \lambda_0 F'(r(x_\epsilon)).
\]
Hence, by (2.2) and (2.3), we get the following inequality
\[
(2.4) \quad \liminf_{s \to 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq \lambda_0 F'(r(x_\epsilon)).
\]
Taking \( h_{\lambda_0} = \lambda_0 r(x) + U - \epsilon \), i.e., \( F(r(x)) = r(x) \) with \( r(x) > 1 \). Recall that \( \lambda_0 > 0 \). Then, by (2.4), we can get

\[
(2.5) \quad \liminf_{s \to 0^+} \frac{r(\gamma_2(t_0 + s)) - r(\gamma_2(t_0))}{s} \geq 1.
\]

The inequality (2.5) will lead to a contradiction. Since \( \gamma_1 \) and \( \gamma_2 \) are different segments, by connecting from the point \( \gamma_1(t_0 - s) \) to the point \( \gamma_2(t_0 + s) \) with a geodesic segment, there is a constant \( c \) with \( 0 < c < 1 \) such that, for a sufficiently small \( s > 0 \), the distance \( d(\gamma_1(t_0 - s), \gamma_2(t_0 + s)) < c2s \). Thus there is a constant \( c' \) with \( 0 < c' < 1 \) depending only on the angle of \( v \) and \( w \) such that

\[
(2.6) \quad r(\gamma_2(t_0 + s)) < t_0 + c's,
\]

for a sufficiently small \( s > 0 \). Because \( r(\gamma_2(t_0)) = t_0 \). By plugging (2.0) to (2.5), we have a contradiction.

From now, let’s consider the second case. Since \( \gamma \) is distance minimizing between \( p \) and \( x_\epsilon \), \( r \) is smooth at \( \gamma(t) \) for \( 0 < t < t_0 \). Let \( m(t) = \Delta r(\gamma(t)) \). Then \( m(t) \) is also smooth for \( 0 < t < t_0 \). Because \( \gamma(t_0) \) is conjugate to \( p = \gamma(0) \) along \( \gamma \). By a simple calculation, we get

\[
(2.7) \quad \lim_{t \to t_0^-} m(t) = -\infty.
\]

Because \( \lambda_0 F'(r(x_\epsilon)) > 0 \). By (2.3), we get \( D_v g(x_\epsilon) > 0 \), i.e., \( \nabla g(x_\epsilon) \neq 0 \). Hence the level surface \( H = \{ x \in M : g(x) = g(x_\epsilon) \} \) is a \( C^2 \) smooth hypersurface near \( x_\epsilon \). Denote by \( H_s \) the surface parallel to \( H \) and passing through the point \( \gamma(t_0 - s) \) for some \( s > 0 \). Since \( H \) is \( C^2 \) smooth near \( x_\epsilon \), the surface \( H_s \) is also \( C^2 \) smooth near \( \gamma(t_0 - s) \) for a sufficiently small \( s > 0 \). Therefore, by (2.7), for some sufficiently small \( s \), the trace of the second fundamental form of \( H_s \) at \( \gamma(t_0 - s) \) in the direction of \( \gamma'(t_0 - s) \) is greater than \( m(t_0 - s) \), where \( m(t_0 - s) \) is the trace of the second fundamental form of the geodesic sphere \( B(p, t_0 - s) \) at \( \gamma(t_0 - s) \) with respect to the normal vector \( \gamma'(t_0 - s) \). This implies that, for a sufficiently close to \( \gamma(t_0 - s) \), there has to be a point \( q_s \in H_s \), that lies inside \( B(p, t_0 - s) \), i.e.,

\[
(2.8) \quad r(q_s) < t_0 - s.
\]

Since \( H_s \) is parallel to \( H \), we also have a point on \( q \in H \) such that the distance \( d(q_s, q) = s \). By (2.8), we have

\[
(2.9) \quad r(q) < t_0 = r(x_\epsilon).
\]

Since \( F \) is strictly increasing, we get

\[
h_{\lambda_0}(q) = \lambda_0 F(r(q)) + U - \epsilon < \lambda_0 F(r(x_\epsilon)) + U - \epsilon = h_{\lambda_0}(x_\epsilon) = g(x_\epsilon) = g(q).
\]

This is a contradiction to the fact that \( h_{\lambda_0}(x) \geq g(x) \) for all \( x \in M \). Therefore, the function \( r \) must be smooth at \( x_\epsilon \).

By the definition of \( F \), \( F \geq 1, G \geq 1 \), and \( G' \geq 0 \), we have

\[
0 < F' = \frac{F}{G} \quad \text{and} \quad F'' = \frac{F'}{G} - \frac{FG'}{G^2} = \frac{F}{G} - \frac{FG'}{G^2} \leq \frac{F}{G^2}.
\]
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Because $\lambda_0 > 0$, $F \geq 1$, and $g(x_\epsilon) = \lambda_0 F(r(x_\epsilon)) + U - \epsilon < U$. We have

(2.10) $0 < -\lambda_0 F(r(x_\epsilon)) + \epsilon = U - g(x_\epsilon) < \epsilon.$

Hence

(2.11) $\lambda_0 \leq \frac{\epsilon}{F(r(x_\epsilon))} \leq \epsilon.$

Recall the notations (1.9), (1.10) and (1.11). Since $h_\lambda_0(x) \geq g(x)$ for all $x \in M$, and $h_\lambda_0(x_\epsilon) = g(x_\epsilon)$,

we have

(2.12) $\nabla h_\lambda_0(x_\epsilon) = \nabla g(x_\epsilon)$ and $L h_\lambda_0(x_\epsilon) \geq Lg(x_\epsilon)$.

Note that $\|\nabla r\| = 1$. By (2.9), (2.11) and $G \geq 1$, the first equality of (2.12) yields

(2.13) $\|\nabla g(x_\epsilon)\| = \|\lambda_0 F'(r(x_\epsilon))\nabla r(x_\epsilon)\| < \frac{\epsilon}{F(r(x_\epsilon))} G(r(x_\epsilon)) \leq \epsilon.$

Also, by (1.2), (2.9), (2.11), (2.13), $G \geq 1$, and $\widetilde{\Delta} r \leq G$, the second inequality of (2.12) yields

$Lg(x_\epsilon) \leq Lh_\lambda_0(x_\epsilon) = \sum_i a_{ti}(x_\epsilon) \frac{\partial^2}{\partial x_i^2} h_\lambda_0|_{x_\epsilon} + \sum_i a_{ti}(x_\epsilon) \frac{\partial}{\partial x_i} h_\lambda_0|_{x_\epsilon}$

\[ \leq \lambda_0 \left( F'(r(x_\epsilon)) \widetilde{\Delta} r(x_\epsilon) + F''(r(x_\epsilon)) \nabla r(x_\epsilon) \cdot \nabla r(x_\epsilon) \right) + e_1 \epsilon \]

\[ < \frac{\epsilon}{F(r(x_\epsilon))} \left( \frac{F(r(x_\epsilon))}{G(r(x_\epsilon))} G(r(x_\epsilon)) + d_1 \frac{F(r(x_\epsilon))}{G(r(x_\epsilon))^2} \right) + e_1 \epsilon \]

\[ \leq \epsilon(1 + d_1 + e_1). \]

If we replace $\epsilon$ with $\epsilon(1 + d_1 + e_1)$, then the above inequality, (2.10), and (2.13) show that the point $x_\epsilon$ satisfies the conditions in Definition 1.3.

3. Proof of Theorem 1.13

The proof is similar to the method in the article [6]. Let $o \in M$ be a fixed point and $r(x)$ be the distance function from $o$. Define a function $u : M \to \mathbb{R}$ by

$u(x) = \int_0^{r(x)^2} G(s)^{-1} ds.$

Assume that a smooth complete Riemannian manifold satisfies the assumption (1.14). Then we will prove that $u$ is an $L$-tamed exhaustion function. We divided into two cases.

First case. Assume that $o$ has no cut points in $M$.

By the definition, the function $u$ is an exhaustion function for $M$. We have to show that, for certain positive constants $C$ and $C_1$, $\|\nabla u\| < C$ and $Lu < C_1$ outside a ball of a certain
radius with center \(x_\varepsilon\). Let \(\phi(t) = \exp\{\int_0^t G(s)^{-1}ds\}\) and \(B(x_\varepsilon, r) = \{x \in M \mid \text{dist}(x, x_\varepsilon) < r\}\). Then \(u(x) = \log \phi(r(x)^2)\). By a direct calculation, one gets

\[
(3.1) \quad \nabla u = \nabla \log \phi(r^2) = 2r \nabla r \frac{\phi'(r^2)}{\phi(r^2)} = 2r \nabla r G(r^2)^{-1}.
\]

By (1.16), there is a positive constant \(C\) such that

\[
(3.2) \quad rG(r)G(r^2)^{-1} < \frac{C}{4}.
\]

Moreover, by (1.15), we have

\[
(3.3) \quad \sup_{[0, \infty)} G(r)^{-1} = (\inf_{[0, \infty)} G(r))^{-1} \leq 1.
\]

By plugging (3.2) to (3.1), we have

\[
\|\nabla u\| < \frac{1}{2} \|\nabla r\| CG(r)^{-1}.
\]

By the assumption (1.15), we have

\[
(3.4) \quad \|\nabla u\| < \frac{C}{2}.
\]

By (1.12) and (3.4), one gets

\[
(3.5) \quad \|\tilde{\nabla} u\| < e_1 \frac{C}{2}.
\]

By the assumption (1.14), there exits \(r_0 > 1\) such that

\[
(3.6) \quad \tilde{\Delta} u < \frac{C}{2} d_1^2 + \frac{C}{2} \text{ on } M \setminus B(x_\varepsilon, r_0).
\]

Thus, by (3.5) and (3.6), we have

\[
Lu = \tilde{\Delta} u + \tilde{\nabla}_1 u < \frac{C}{2} (d_1^2 + 1 + e_1) \text{ on } M \setminus B(x_\varepsilon, r_0).
\]
Hence this yields

\begin{equation}
L | \delta < F
\end{equation}

If we replace \( \frac{C}{t}(d^2 + 1 + e_1) \) with \( C \), then \( u \) satisfies the additional conditions for an \( L \)-tamed exhaustion function.

**Second case.** Assume that the cut locus of \( o \) is nonempty.

Let \( x_\epsilon \) be a cut point of \( o \) and let \( F(t) = \log \phi(t^2) \) for \( t > 0 \). We choose a point \( \hat{x}_\epsilon \) outside of cut locus of \( o \) such that \( \text{dist}(x_\epsilon, \hat{x}_\epsilon) < 1 \) and \( r(\hat{x}_\epsilon) > r(x_\epsilon) \). Denote by \( B(y, r) = \{ x \in M \mid \text{dist}(x, y) < r \} \). Take \( \eta, \delta > 0 \) such that \( B(x_\epsilon, \eta) \cap B(\hat{x}_\epsilon, \delta) = \emptyset \) and \( B(\hat{x}_\epsilon, \delta) \) does not have cut point of \( o \).

Now, we present several functions to find an upper supporting function for \( u \).

For a \( U \subset B(x_\epsilon, \eta) \), we define a smooth map \( T : U \to B(\hat{x}_\epsilon, \delta) \) with \( T_x(x_\epsilon) = \hat{x}_\epsilon \), and it is translation sending \( x_\epsilon \) to \( \hat{x}_\epsilon \) in a coordinate chart including both \( B(x_\epsilon, \eta) \) and \( B(\hat{x}_\epsilon, \delta) \) and satisfying \( r(T(x)) \geq r(x) \). Also, we define a \( C^2 \) function \( \lambda \) such that \( \lambda(x_\epsilon) = 1, \nabla \lambda(x_\epsilon) = 0, \Delta \lambda(x_\epsilon) = 0 \) and

\[ \lambda(x)r(T(x)) \geq r(x) + r(\hat{x}_\epsilon) - r(x_\epsilon) \text{ on } U. \]

Since \( r(\hat{x}_\epsilon) > r(x_\epsilon) \) and \( r \geq 0 \), we get \( \lambda(x) > 0 \). Finally, for \( x \in U \), we define a function

\[
H(x) = \begin{cases} 
L(x) + (1/2)F''(r(x_\epsilon))\lambda(x)(r(T(x)) - r(\hat{x}_\epsilon))^2 & \text{when } F''(r(x_\epsilon)) > 0, \\
L(x) - (1/2)F''(r(\hat{x}_\epsilon))(r(T(x)) - r(\hat{x}_\epsilon))^2 & \text{when } F''(r(x_\epsilon)) < 0, \\
L(x) + (1/2)Q(r(x_\epsilon))(r(T(x)) - r(\hat{x}_\epsilon))^2 & \text{when } F''(r(x_\epsilon)) = 0,
\end{cases}
\]

where \( L(x) = -F'(r(\hat{x}_\epsilon))(r(T(x)) - r(\hat{x}_\epsilon)) + F'(r(x_\epsilon))(\lambda(x)r(T(x)) - r(\hat{x}_\epsilon)) \) and \( Q(r(x_\epsilon)) = \sup \{ F''(t) \mid t \in (r(x_\epsilon) - 1, r(x_\epsilon) + 1) \} \). Note that we choose \( \hat{x}_\epsilon \) as close to \( x_\epsilon \) such that \( \text{sign}[F''(r(\hat{x}_\epsilon))] = \text{sign}[F''(r(x_\epsilon))] \). Therefore, \( H(x) - L(x) \geq 0 \).

Let \( v(x) = F(r \circ T(x)) + F(r(x_\epsilon)) - F(r(\hat{x}_\epsilon)) + H(x) \). Then one gets \( v(x_\epsilon) = F(r(x_\epsilon)) = u(x_\epsilon) \). Since \( F'(r(x))\nabla r(x) = \nabla u(x) = G(r(x)^2)^{-1}2r(x)\nabla r(x) \) and the inequality \((3.2)\), we get

\begin{equation}
0 < F'(r(x)) = G(r(x)^2)^{-1}2r(x) < \frac{C}{2}G(r(x))^{-1}.
\end{equation}

By a direct calculation, we have, for \( x \in U \),

\[ v(x) - H(x) + L(x) - u(x) = F'(r(x_\epsilon))(\lambda(x)r(T(x)) - r(\hat{x}_\epsilon) - (r(x) - r(x_\epsilon))) \geq 0. \]

This yields

\[ v(x) - u(x) \geq H(x) - L(x) \geq 0. \]

Hence \( v \) is an upper supporting function for \( u \) at the point \( x_\epsilon \).

Since \( \nabla H|_{x_\epsilon} = \nabla L|_{x_\epsilon}, ||\nabla \lambda|_{x_\epsilon}|| = 0, \lambda(x_\epsilon) = 1 \) and \( ||\nabla (r \circ T)\|| = 1 \), we have

\[
\|\nabla v|_{x_\epsilon}|| \leq |F'(r(x_\epsilon))|(||\nabla \lambda|_{x_\epsilon}||r(\hat{x}_\epsilon) + |\lambda(x_\epsilon)||\nabla r(\hat{x}_\epsilon)|_{x_\epsilon}||) = |F'(r(x_\epsilon))| = \|\nabla u|_{x_\epsilon}|| < \frac{C}{2}.
\]
By our assumption \((1.2)\), the above inequality implies that

\[
\|\tilde{\nabla}_1 v\| < e_1 \frac{C}{2}.
\]

Notice that

\[
\tilde{\Delta}(r \circ T(x))|_{x_\epsilon} = \|DT\|^2 \tilde{\Delta}r|_{x_\epsilon} = n \tilde{\Delta}r|_{x_\epsilon},
\]

where \(\text{dim } M = n\). By a simple calculation, we have

\[
F''(r(x))\nabla r(x) = 2G(r(x)^2)^{-1}(-2r(x)^2G(r(x)^2)^{-1} + 1)\nabla r(x)
\]

and hence

\[
F''(r(x)) = 2G(r(x)^2)^{-1}(-2r(x)^2G(r(x)^2)^{-1} + 1) < 2G(r(x)^2)^{-1}.
\]

Using \(\|\nabla (r \circ T)\| = 1\), \(\|\tilde{\nabla} (r \circ T)\| \leq d_1\), \((3.7)\), \((3.9)\) and \((3.10)\), we have

\[
\tilde{\Delta}v|_{x_\epsilon} \leq d_1^2 F''(r(x)) + F'(r(x)) \tilde{\Delta}(r \circ T)|_{x_\epsilon} + \tilde{\Delta}H|_{x_\epsilon}
\]

\[
\leq \begin{cases} 
F'(r(x)) \tilde{\Delta}(r \circ T)|_{x_\epsilon} + d_1^2 (F''(r(x)) + F''(r(x))) & \text{if } F''(r(x)) > 0, \\
F'(r(x)) \tilde{\Delta}(r \circ T)|_{x_\epsilon} & \text{if } F''(r(x)) < 0, \\
F'(r(x)) \tilde{\Delta}(r \circ T)|_{x_\epsilon} + d_1^2 (F''(r(x)) + Q(r(x))) & \text{if } F''(r(x)) = 0,
\end{cases}
\]

\[
< (1/2)CG(r(x)^2)^{-1}n \tilde{\Delta}r|_{x_\epsilon} + 4d_1^2 G(r(x)^2)^{-1}.
\]

Let \(2a\) be the distance to a closest cut point of \(o\). Because the point \(x_\epsilon\) is a cut point of \(o\), by \((3.2)\) and \((3.3)\), we get

\[
2aG(r(x_\epsilon)^2)^{-1} \leq r(x_\epsilon)G(r(x_\epsilon)^2)^{-1} < \frac{C}{4} G(r(x_\epsilon)^2)^{-1} \leq \frac{C}{4}
\]

and

\[
G(r(x_\epsilon)^2)^{-1} < \frac{C}{8a}.
\]

By plugging \((3.12)\) to \((3.11)\), our assumption \((1.14)\) tells us that, for \(r > 1\),

\[
\tilde{\Delta}v|_{x_\epsilon} < \frac{C}{2} n + \frac{C}{2a} d_1^2.
\]

Therefore, by \((3.8)\) and \((3.13)\), we obtain, for \(r > 1\),

\[
Lv|_{x_\epsilon} < \frac{C}{2} (n + \frac{d_1^2}{a} + e_1).
\]

So \(u\) satisfies the conditions for an \(L\)-tamed exhaustion function.

Altogether, we can conclude that \(u\) must be an \(L\)-tamed exhaustion function for \(M\).
4. Proof of Corollary 4.1

**Corollary 4.1.** (Borbély [3, Corollary]) Let \( o \in M \) be a fixed point and \( r(x) \) be the distance function from \( o \). Let us assume that away from the cut locus of \( o \) we have

\[
\text{Ricc}(\nabla r, \nabla r) \geq -G^2(r),
\]

where \( G(t) \) on \([0, \infty)\) satisfies

\[
G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} = \infty.
\]

Then we have

\[
\Delta r(x) < (\sqrt{n - 1} + 1)G(r(x))
\]

for all \( x \in M \), where \( r \) is smooth, \( r(x) > 1 \) and \( \dim M = n \).

By Corollary 4.1 and Theorem 1.13 Ratto-Rigoli-Setti’s condition without \( \sqrt{G}^{(2k+1)}(0) = 0 \ \forall k \geq 0 \) in (1.6) implies the existence of a \( \Delta \)-tamed exhaustion function.

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**References**

1. G.P. Bessa and L.F. Pessoa, *Maximum principle for semi-elliptic trace operators and geometric applications*, arXiv:1208.1322.
2. G.P. Bessa, S. Pigola, and A.G. Setti, *Spectral and stochastic properties of the f-laplacian, solutions of PDEs at infinity and geometric applications*, Rev. Mat. Iberoam. 29 (2013), no. 2, 579–610.
3. A. Borbély, *A remark on Omori-Yau maximum principle*, Kuwait J. Sci. 39(2A) (2012), 45–56.
4. Q. Chen and Y.L. Xin, *A generalized maximum principle and its applications in geometry*, Amer. J. Math. 114 (1992), 355–366.
5. K. Hong and C. Sung *An Omori-Yau maximum principle for semi-elliptic operators and Liouville-type theorems*, Diff. Geom. and its Appl. 31:4 (2013), 533–539.
6. K.-T. Kim and H. Lee, *On the Omori-Yau almost maximum principle*, J. Math. Anal. Appl. 335 (2007), 332–340.
7. H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan 19 (1967), 205–211.
8. S. Pigola, M. Rigoli, and A.G. Setti, *A remark on the maximum principle and stochastic completeness*, Proc. Amer. Math. Soc. 131(4) (2002), 1283–1288.
9. S. Pigola, M. Rigoli, and A.G. Setti, *Maximum principles on Riemannian manifolds and applications*, Memoirs Amer. Math. Soc. 174 (2005), no.882, x+99.
10. P. Peter, *Riemannian Geometry*, volume 171 of Graduate Texts in Mathematics, Springer-Verlag, New York, (1998)
11. A. Ratto, M. Rigoli, and A.G. Setti, *On the Omori-Yau maximum principle and its application to differential equations and geometry*, J. Func. Anal. 134 (1995), 486–510.
12. H.L. Royden, *The Ahlfors-Schwarz lemma in several complex variables*, Comment. Math. Helv. **55**(4) (1980), 547–558.

13. K. Takegoshi, *A volume estimate for strong subharmonicity and maximum principle on complete Riemannian manifolds*, Nagoya Math. J. **151** (1998), 25–36.

14. S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201–228.