Invariant recurrence relations for $\mathbb{C}P^{N-1}$ models

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Abstract

In this paper, we present invariant recurrence relations for the completely integrable $\mathbb{C}P^{N-1}$ Euclidean sigma model in two dimensions defined on the Riemann sphere $S^2$ when its action functional is finite. We determine the links between successive projection operators, wavefunctions of the linear spectral problem and immersion functions of surfaces in the $\mathfrak{su}(N)$ algebra together with the outlines of the proofs. Our formulation preserves the conformal and scaling invariance of these quantities. Certain geometrical aspects of these relations are described. We also discuss the singularities of the meromorphic solutions of the $\mathbb{C}P^{N-1}$ model and show that they do not affect the invariant quantities. We illustrate the construction procedure through the examples of the $\mathbb{C}P^2$ and $\mathbb{C}P^3$ models.

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1. Introduction

The general properties of nonlinear sigma models and techniques for finding associated surfaces remain among the essential subjects of investigation in modern mathematics and physics. Originally invented by Gell-Mann and Lévy [15] in order to explain the lifetime of charged pions by introducing a new boson field, the models were later refined by Callan, Coleman, Wess and Zumino [5, 6] in the low-energy limit, by including the nonlinearity into the pion field $\Phi$. This approach transformed the pion field into an effective field having a simple Lagrangian density

$$\mathcal{L} = C \partial_\mu \Phi^T \cdot \partial^\mu \Phi,$$

where $C$ is a constant, with the constraint

$$\Phi^T \cdot \Phi = 1.$$
The effective Lagrangian approach (1.1) proved to be a fruitful tool, even in two dimensions. In particular, the two-dimensional sigma models, as well as the surfaces associated with those models, have found many applications to such diverse areas as two-dimensional gravity [7, 16], string theory [38–40], various branches of the quantum field theory [8, 44, 52, 56], theory of the quantum Hall effect [41], statistical physics [34, 42], fluid dynamics [33], biophysics (the Canham–Helfrich membrane model) [9, 30, 37], etc. A survey covering different stages of the development of the theory can be found in several papers (see e.g. [20, 22, 23] and references therein). Most of the above models also include the constant-norm constraint like (1.2), or its complex counterpart, in which the transposition is replaced by Hermitian conjugation. The latter are well known as the $\mathbb{C}P^{N-1}$ sigma models.

Geometrization has proved to be a fruitful approach to completely integrable models, including the Euclidean $\mathbb{C}P^{N-1}$ sigma models in two dimensions. Those models are completely integrable, and they possess the Lax pair as introduced by Zakharov and Mikhailov [32, 53]. The concept of constructing infinitely many surfaces immersed in multidimensional spaces to describe the spectral problem as an immersion problem was first presented by Sym and Tafel (ST) [45–48]. Their formula allows us to express an immersion function of a surface directly in terms of a wavefunction satisfying the associated linear spectral problem. This subject has been developed further by many authors (see e.g. [3, 13, 14, 19, 21, 27, 28, 50, 51] and references therein). More recently, the conservation laws of the considered model have led to the generalized Weierstrass formula for immersion (GWFI) of 2D surfaces, which originated from the work of Konopelchenko [26]. It was shown [17, 18] that, for 2D surfaces immersed in $\mathfrak{su}(N)$ algebras in the case of $\mathbb{C}P^{N-1}$ models, the ST formula coincides with the GWFI. On the basis of this geometrical view, the present authors focus their attention on the case where the model is defined on the Riemann sphere $\mathbb{S}^2$ and its action functional is finite.

The complete set of regular solutions is known for such equations due to Din and Zakrzewski as well as Sasaki, Eells and Wood [10, 12, 43]. In the classical approach to the $\mathbb{C}P^{N-1}$ models [10], new solutions are constructed by multiple application of a ‘creation operator’ $P_+$ to any holomorphic solution or an ‘annihilation operator’ $P_-$ to any antiholomorphic solution.

It seems worthwhile to provide an explicitly invariant formulation of the main ingredients of the theory. Namely the considered models are complex projective. The equations of motion, as well as their integration schemes, are invariant under scaling not only by a constant factor but also by any scalar complex-valued function. For such models, the natural approach seems to be to express all the quantities in the scaling-invariant form and to formulate the theory in terms of invariant projection operators rather than the previously used unnormalized (homogeneous) coordinates. Such a projector approach has the advantage of simplicity and clarity of the governing equations as compared to the description of the model in terms of homogeneous field variables. In this paper, we show that this language is ‘natural’ for the formulation of recurrence operators which have acquired simple closed forms. By the recurrence operators, we mean the operators which raise or lower the index of the projectors present in the sigma model equations. We provide a physical interpretation of the operators, which is that they can play a role similar to that of ‘creation’ and ‘annihilation’ operators in linear systems. A similar procedure is proposed for the wavefunctions of the linear spectral problem and the immersion functions of surfaces in the $\mathfrak{su}(N)$ algebra, expressed in terms of the projection operators. This will enable us to construct the invariant recurrence relations for these functions in an explicit and simple form. It is worth noting that these relations will reveal certain general characteristics of the surfaces associated with sigma models. For example, we show in this paper that such surfaces remain regular even in cases when the vector fields of a model have singularities (poles).
Starting from the invariant Lagrangian density, we recover the well-known equations of motion in the form of conservation laws [32, 53], and then we construct the solutions in a way similar to [10] by means of the appropriate ‘creation’ and ‘annihilation’ operators applied to those projectors. The corresponding recurrence operators are then also derived for the wavefunctions of the spectral problem and for the immersion functions of the surfaces corresponding to those functions (soliton surfaces). Finally, the geometrical characteristics of the surfaces are also expressed in terms of the projectors. We complete our analysis by commenting on the possible behavior of the invariant solutions in a neighborhood of (what used to be) the singularities of homogeneous field coordinates in the $\mathbb{C}P^{N-1}$ model. For a deeper insight into the $\mathbb{C}P^{N-1}$ model theories, we refer the reader to some standard books on the subject [1, 4, 20, 22, 25, 31, 36, 54].

Throughout this paper we use the terms ‘creation’ and ‘annihilation’ operators, suggested by the commonly applied symbols $P_\pm$. However, the reader should bear in mind that the procedure is a walk over a sphere rather than up or down a ladder (for this reason, we retain the quotation marks). The construction of orthogonal functions or projectors is in fact an application of the classical Gram–Schmidt orthogonalization procedure [29] in which the subsequent base vectors are constructed from the derivatives of their predecessors. This aspect will be discussed in more detail later.

This paper is organized as follows. In section 2, we recall the main elements of the $\mathbb{C}P^{N-1}$ theory, which will be the basis for further calculations (including the introduction of the invariant Lagrangian). Section 3 is devoted to a description of the invariant recurrence formulae for the $\mathbb{C}P^{N-1}$ models. The goals are described in figure 1.

We seek the link between the projectors $P_k$ and $P_{k-1}$, the wavefunctions $\Phi_k$ and $\Phi_{k-1}$ and the immersion functions $X_k$ and $X_{k-1}$. It should be noted that the projectors $P_k$ and $P_{k-1}$ are related to the wavefunctions $\Phi_k$ and $\Phi_{k-1}$ respectively through the concept of Lax pairs. Likewise, the projectors $P_k$ and $P_{k-1}$ are connected with the immersion functions $X_k$ and $X_{k-1}$ respectively through the relation found in [17], which we refer to as the generalized Weierstrass formula involving projectors (GWFP) formula. In our formulation we preserve the conformal invariance of the action functional, projectors, wavefunctions and surfaces under consideration. In this way, given any particular surface associated with $\mathbb{C}P^{N-1}$ models, we are able to construct all the remaining ones and, moreover, do it in a very straightforward way. This procedure, providing a new wavefunction in terms of an old one in addition to the corresponding relation between solutions of the nonlinear problem, may be regarded as a kind of Darboux transformation. The derivation of the ‘creation’ and ‘annihilation’ operators for the projectors
The Lagrangian density $L$ and the covariant derivatives with the usual definition of the scalar product, while $\zeta$ is inconvenient. The common approach to the model theory which constitutes the background of our calculations.

The dynamics of $\mathbb{C}P^{N-1}$ sigma models defined on the Riemann sphere $S^2$ are determined by the stationary points of the action functional (see e.g. [54])

$$ S = \int \int_{S^2} \mathcal{L} \, d\xi \, d\xi' = \frac{1}{4} \int \int_{S^2} (D_\mu z)^\dagger (D_\mu z) \, d\xi \, d\xi', $$

(2.1)

where the Lagrangian density $\mathcal{L}$ is

$$ \mathcal{L} = \frac{1}{4} (D_\mu z)^\dagger (D_\mu z), $$

(2.2)

and the covariant derivatives $D_\mu$ are defined according to the formula

$$ D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z) z, \quad \partial_\mu = \partial_{\xi^\mu}, \quad \mu = 1, 2. $$

(2.3)

The field variables $z = (z_0, \ldots, z_{N-1})$ are the points of the coordinate space which constitute the $(N-1)$-dimensional unit sphere immersed in $\mathbb{C}^N$:

$$ z^\dagger \cdot z = 1, $$

(2.4)

with the usual definition of the scalar product, while $z^\dagger$ is the Hermitian conjugate of $z$. The space of independent variables is two dimensional. Consisting originally of the unit sphere, it is usually converted to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ by stereographic projection. In our paper the independent variables are pairs $(\xi^1, \xi^2) \in \mathbb{R}^2$ or $(\xi, \bar{\xi}) \in \mathbb{C}$, where $\xi = \xi^1 + i \xi^2$ and complex conjugates are marked by a bar over a quantity.

The normalization of $z$ (2.4) imposes a constraint on its components, which makes them inconvenient. The common approach to the $\mathbb{C}P^{N-1}$ models is to describe the models in terms of the homogeneous, unnormalized field variables $f$, such that $z = f/(f^\dagger \cdot f)^{1/2}$. The vector $z$ is determined by the Euler–Lagrange (E-L) equations with the constraints (2.4)

$$ D_\mu D_\mu z + (D_\mu z)^\dagger (D_\mu z) z = 0, \quad z^\dagger \cdot z = 1, $$

(2.5)

whereas the homogeneous variables $f$ satisfy an unconstrained form of the E-L equations

$$ \left( \mathbb{I} - \frac{\bar{f} \otimes f}{f^\dagger \cdot f} \right) \left[ \partial \bar{f} - \frac{1}{f^\dagger \cdot f} ((f^\dagger \cdot \partial f)\partial f + (f^\dagger \cdot \partial f)\partial f) \right] = 0, $$

(2.6)

where $\partial$ and $\bar{\partial}$ denote the derivatives with respect to $\xi$ and $\bar{\xi}$ and $\mathbb{I}$ is the $N \times N$ unit matrix. An important property of these equations is their invariance under scaling by the multiplication of $f$ by an arbitrary scalar function $\varphi(\xi)$.

The E-L equations (2.6) take the elegant form of a conservation law if we express them in terms of Hermitian projection matrices $P : S^2 \to \text{Aut}(\mathbb{C}^N)$:

$$ P = (1/f^\dagger \cdot f) f \otimes f^\dagger, \quad P^2 = P, \quad P^\dagger = P, $$

(2.7)

namely

$$ \partial [\bar{\partial} P, P] + \bar{\partial} [\partial P, P] = 0. $$

(2.8)
In this paper we are going to use the projectors $P$ as our fundamental unknown variables. The advantage of such an approach is the explicit invariance of these variables under scaling with any scalar function of $\xi$. Thus the scaling-invariant Euler–Lagrange equations (2.8) are expressed in terms of scaling-invariant variables. On the other hand, the projectors are obviously subject to another constraint: $P^2 = P$. Due to this constraint we introduce the Lagrange multiplier $\mu = \mu^\dagger \in \text{Aut}(\mathbb{C}^N)$ into the action (2.1), and we get

$$S = \int_S \text{tr}\{\partial P \cdot \bar{\partial} P + \mu \cdot (P^2 - P)\} \, d\xi \, d\bar{\xi}.$$  

(2.9)

By the variation of the action (2.9), we obtain

$$\delta \mu : \quad P^2 - P = 0$$

$$P : \quad 2\bar{\partial}P + \mu \cdot P + P \cdot \mu - \mu = 0.$$  

(2.10)

We eliminate the Lagrange multiplier $\mu$ by multiplying (2.10) from the left and from the right by $P$. Next we subtract the obtained results, which yield equation (2.8) as the E-L equation of action (2.9).

The conservation law (2.8) means that the 1-form

$$dX = i\{-(\bar{\partial}P, P) d\xi + [\bar{\partial}P, P] d\bar{\xi}\}$$

(2.11)

is a closed differential. Hence its integral, independent of a trajectory, may be used to construct the following $N \times N$ matrix in $\text{su}(N)$:

$$X(\xi, \bar{\xi}) = i \int_Y (-(\bar{\partial}P, P) d\xi + [\bar{\partial}P, P] d\bar{\xi}),$$  

(2.12)

which may be regarded as a surface immersed in a real $(N^2 - 1)$-dimensional space [19]. The mapping $X : \mathbb{S}^2 \ni (\xi, \bar{\xi}) \rightarrow X(\xi, \bar{\xi}) \in \text{su}(N)$ is known in the literature [26, 35] as the generalized Weierstrass formula for the immersion of 2D surfaces in $\mathbb{R}^{(N^2 - 1)} \cong \text{su}(N)$. The space is equipped with the scalar product

$$(A, B) = -(1/2) \text{tr}(A \cdot B), \quad A, B \in \text{su}(N),$$

(2.13)

which is used to construct an orthonormal basis (Pauli matrices in three dimensions, Gelfand matrices in eight dimensions, etc; see e.g. [24, 49]).

In a classical paper [10], a base of vectors $f_i$ was constructed by the Gram–Schmidt orthogonalization procedure, namely by consecutive applications of the contracting operator $P$, defined by

$$P_+ (f) = (1 - P) \cdot \bar{\partial}f,$$  

(2.14)

which we will refer to as a ‘creation operator’, while the inverse operation $P_-$, an ‘annihilation operator’, is defined by

$$P_- (f) = (1 - P) \cdot \partial f.$$  

(2.15)

The usual procedure for constructing the orthogonal basis $f_0, \ldots, f_{N-1}$ includes normalization by setting the first nonzero component of each $f_k$ to one. Unlike the standard creation and annihilation operators known in the literature [2, 29], the operations (2.14, 2.15) leading to the new vectors $f$ are nonlinear. Although the new vectors are obtained from their predecessors by a linear operation of matrix multiplication, the multiplier, which defines the direction of the projection, also depends on the argument (the new direction is the projection of the tangent to the graph $f(\xi, \bar{\xi})$ onto the hyperplane orthogonal to the vector $f(\xi, \bar{\xi})$).

It was shown in [10] that multiple applications of $P_+$ to any holomorphic function lead to an antiholomorphic one after $(N - 1)$ steps, and obviously the application of $P_-$
to an antiholomorphic function yields zero. In this way, we obtain $N$ orthogonal functions $f_0, \ldots, f_{N-1}$, and—as a by-product—$N$ projectors $P_0, \ldots, P_{N-1}$ acting on the orthogonal complements of one-dimensional subspaces in $\mathbb{C}^N$.

In [53] the linear problem containing a spectral parameter $\lambda \in \mathbb{C}$ was found in the form of a system

$$\partial \Phi_k = \frac{2}{1 + \lambda} [\partial P_k, P_k] \Phi_k, \quad \bar{\partial} \Phi_k = \frac{2}{1 - \lambda} [\bar{\partial} P_k, P_k] \Phi_k, \quad k = 0, 1, \ldots, N - 1,$$

(2.16)

where $\lambda \in \mathbb{C}$ is the spectral parameter and $P_k$ is a sequence of rank-1 orthogonal projectors which map onto the direction of $f_k$:

$$P_k = \frac{f_k \otimes f_k^\dagger}{f_k^\dagger \cdot f_k}, \quad f_k = P_k f, \quad P_k^2 = P_k, \quad P_k^\dagger = P_k.$$  

(2.17)

The compatibility condition for equation (2.16) corresponds precisely to the E-L equations (2.8). The same set of equations may be obtained as a geometric condition for the immersion of the surfaces $X_k$ in $\mathbb{R}^{N^2 - 1}$, i.e. as Gauss–Mainardi–Codazzi equations for the surfaces given by (2.12).

An explicit solution, vanishing at complex infinity, was found for equations (2.16) in [11]:

$$\Phi_k = I + \frac{4\lambda}{(1 - \lambda)^2} \sum_{j=0}^{k-1} P_j - \frac{2}{1 - \lambda} P_k,$$

(2.18)

$$\Phi_k^{-1} = I - \frac{4\lambda}{(1 + \lambda)^2} \sum_{j=0}^{k-1} P_j - \frac{2}{1 + \lambda} P_k.$$  

(2.19)

Similarly the integration in (2.12) has explicitly been carried out for the $\mathbb{C}P^{N-1}$ models defined on $S^2$ and having finite action [17]. By choosing the integration constant so that the $X_k$ are traceless, we obtain the following solution:

$$X_k = -i \left( P_k + 2 \sum_{j=0}^{k-1} P_j \right) + \frac{i(1 + 2k)}{N} I, \quad k = 0, 1, \ldots, N - 2.$$  

(2.20)

The above formula will be referred to as the GWFI formula. Finally the Sym–Tafel formula [45–48] yields $X_k$ from $\Phi_k$ as

$$X_k = \alpha(\lambda) \Phi_k^{-1} \partial_x \Phi_k + \frac{(1 + 2k)}{N} I, \quad k = 0, 1, \ldots, N - 2.$$  

(2.21)

For the $\mathbb{C}P^{N-1}$ models, $\alpha(\lambda)$ was found to be equal to $2/(1 - \lambda^2)$. We have found another way of obtaining $X_k$ from $\Phi_k$, from its asymptote at large values of the spectral parameter $\lambda$, namely

$$X_k = i \frac{2k + 1}{N} I + \frac{i}{2} \lim_{\lambda \to \infty} [\lambda (I - \Phi_k)].$$

(2.22)

However, this procedure is obviously limited to the $\mathbb{C}P^{N-1}$ models while the Sym–Tafel formula is universal.

The solutions $z = f/|f|$ have a well-known physical interpretation as localized soliton-like objects, i.e. instantons. As a rule, the holomorphic solution ($k = 0$) is recognized as an instanton, while the antiholomorphic one ($k = N - 1$) as an anti-instanton, and the intermediate solutions ($k = 1, \ldots, N - 2$, possible in the $\mathbb{C}P^{N-1}$ models for $N \geq 2$) describe various mixed instanton–anti-instanton states.
3. Recurrence in the projection space

The recurrence in the projection space is a construction of new projectors in terms of the previous ones. First we look for an operator which transforms each projector \( P_i \) to the next one \( P_{i+1} \) \( (0 \leq i \leq N-2) \). Each of the projectors maps onto a one-dimensional space and altogether they constitute a partition of the identity matrix. In this way, we may systematically build consecutive dimensions in the partition of unity, starting from a holomorphic or antiholomorphic solution of the Euler–Lagrange equations (2.8).

Let \( \Pi_{\pm} \) be operators acting on the projectors in the way

\[
\Pi_{\pm}(P_k) = P_{k \pm 1},
\]

(3.1)

These operators play the role of annihilation and creation operators (respectively) in the space of projectors. However, they are nonlinear and the objects on which they act have to remain normalized to retain their projective character. For this reason they cannot be used to construct the ‘particle number operator’.

It is proven in appendix A that operators (3.1) may be cast into the forms

\[
\Pi_{\pm}(P) = \frac{\partial P \cdot P \cdot \partial P}{tr(\partial P \cdot P \cdot \partial P)} = \frac{\partial P \cdot \partial P \cdot (1 - P)}{tr(\partial P \cdot P \cdot \partial P)},
\]

(3.2)

and

\[
\Pi_{\pm}(P) = \frac{\partial P \cdot P \cdot \partial P}{tr(\partial P \cdot P \cdot \partial P)} = \frac{(1 - P) \cdot \partial P \cdot \partial P}{tr(\partial P \cdot P \cdot \partial P)},
\]

(3.3)

where the traces in the denominators are different from zero unless the whole matrix is zero (which occurs when applying \( \Pi_{\pm} \) to the holomorphic or \( \Pi_{\pm} \) to the antiholomorphic solution). At the end of appendix A, we prove that the resulting matrices \( \Pi_{\pm}(P) \) and \( \Pi_{\pm}(P) \) have the orthogonal projective property \( M^2 = M \) and \( M^1 = M \), provided that the argument \( P \) is a projector mapping onto a one-dimensional subspace. The non-vanishing of the traces in (3.2) and (3.3) is a consequence of the proof (see the comment to (A.9)).

**Example 1** (Action of \( \Pi_{\pm} \) in \( \mathbb{C}P^2 \)). A projector corresponding to the holomorphic Veronese solution of the Euler–Lagrange equations (2.6), which itself is a solution of equation (2.8) reads [55]

\[
P_0 = \frac{2}{(|\xi|^2 + 1)^2} \begin{pmatrix} \frac{1}{\sqrt{2} \xi} & \sqrt{2} \xi & \sqrt{2} \xi \xi^2 \\ 2|\xi|^2 & \frac{\xi^2}{\sqrt{2} |\xi|^2} & \sqrt{2} |\xi|^2 \xi^2 \\ \xi^2 & \sqrt{2} |\xi|^2 & |\xi|^4 \end{pmatrix}.
\]

(3.4)

An action of the ‘creation operator’ \( \Pi_+ \), (3.3) converts it into a projector corresponding to a mixed solution

\[
P_1 = \frac{1}{(|\xi|^2 + 1)^2} \begin{pmatrix} 2|\xi|^2 & \sqrt{2}(|\xi|^2 - 1)\xi & -2\xi^2 \\ \sqrt{2}(|\xi|^2 - 1)\xi & (|\xi|^2 - 1)^2 & -\sqrt{2}(|\xi|^2 - 1)\xi \\ -2\xi^2 & -\sqrt{2}(|\xi|^2 - 1)\xi & 2|\xi|^2 \end{pmatrix}.
\]

(3.5)

This procedure can be repeated once to yield a projector mapping onto the direction of the antiholomorphic solution of (2.6). Further application of the creation operator, i.e. on the antiholomorphic projector, yields an indeterminate expression of the form 0/0 since an action of the \( \partial \) operator on an antiholomorphic function yields zero both in the numerator and the denominator of (3.3).

These operators will further be used to construct the corresponding ‘creation’ and ‘annihilation’ operators for the wavefunctions \( \Phi_k \) and for the immersion functions \( X_k \).
The corresponding recurrence relations for the wavefunctions $\Phi_k$ may be obtained from the solution of the spectral problem (2.16). The relations are more conveniently expressed in terms of an auxiliary function
\[ \Psi_k = (1 - \lambda)^2 (1 - \Phi_k). \]  
(3.6)
As in the case of the projection matrices $P_k$, the ‘creation/annihilation’ operators $\Lambda_\pm$ raise or lower the index of $\Psi_k$ by 1. The operators, which depend on the spectral parameter $\lambda$, read
\[ \Lambda_-(\Psi(\lambda)) = \frac{1}{2}[(1 + \lambda)\Psi(\lambda) - (1 - \lambda)\Psi(-\lambda)] + 2(1 + \lambda)\Pi_{\pm}\left(\frac{1}{2}[\Psi(\lambda) + \Psi(-\lambda)]\right), \]  
(3.7)
and
\[ \Lambda_+(\Psi(\lambda)) = \frac{1}{2}[(1 - \lambda)\Psi(\lambda) + (1 + \lambda)\Psi(-\lambda)] + 2(1 - \lambda)\Pi_{\pm}\left(\frac{1}{2}[\Psi(\lambda) + \Psi(-\lambda)]\right), \]  
(3.8)
where $\Psi(-\lambda)$ may also be expressed in terms of $\Psi(\lambda)$ if we make use of the symmetry $\Phi^{-1}(\lambda) = \Phi(1/\lambda)$, namely
\[ \Psi(-\lambda) = -(1 + \lambda)^2\Psi(\lambda)(1 - \lambda)^2\Psi(\lambda)^{-1}. \]  
(3.9)
A simple proof of these formulae may be found in appendix B.

**Example 2** (Action of $\Lambda_\pm$ in $CP^3_\lambda$). For the $CP^3$ model, the wavefunction for the spectral problem whose compatibility condition is (2.8) may be constructed according to (2.18). If we use that equation with $k = 0$, we obtain the wavefunction $\phi_0$. The auxiliary function $\Psi_0$ may be obtained from it as $(1 - \lambda)^2(1 - \Phi_0)$. It reads
\[ \Psi_0 = \frac{2(1 - \lambda)}{(\xi^2 + 1)^3} \begin{pmatrix} 1 & \sqrt{3}\xi & \sqrt{3}\xi^2 & \xi^3 \\ \sqrt{3}\xi & \sqrt{3}\xi^2 & \sqrt{3}\xi^3 \\ \sqrt{3}\xi^2 & \sqrt{3}\xi^3 & \xi^4 \\ \xi^3 & \sqrt{3}\xi^4 & \sqrt{3}\xi^5 & |\xi|^6 \end{pmatrix}. \]  
(3.10)
An action of the operators $\Lambda_\pm$ on $\Psi_0$ yields the next $\Psi$, i.e. $\Psi_1 = (1 - \lambda)^2(1 - \Phi_1)$, where $\Phi_1$ is another wavefunction, whose spectral problem (2.16) yields equation (2.8) as compatibility condition, with $P_1$ instead of $P_0$. The new wavefunction $\Phi_k$ may also be constructed in terms of the projectors according to (2.18) with $k = 1$. The new $\Psi_1$ has the form
\[ \Psi_1 = -\frac{2}{(\xi^2 + 1)^3} \begin{pmatrix} 3(\lambda - 1)|\xi|^2 + 2\lambda & \sqrt{3}[2(\lambda - 1)|\xi|^2 + \lambda + 1]\xi \\ \sqrt{3}[2(\lambda - 1)|\xi|^2 + \lambda + 1]\xi & 4(\lambda - 1)|\xi|^4 + 2(\lambda + 2)|\xi|^2 + \lambda - 1 \\ 3(\lambda - 1)|\xi|^2 + 2\lambda & \sqrt{3}[2(\lambda - 1)|\xi|^4 + (\lambda + 5)|\xi|^2 + 2(\lambda - 1)]\xi \\ (3 - \lambda)|\xi|^3 & \sqrt{3}[2|\xi|^2 + \lambda - 1]\xi \end{pmatrix}. \]  
(3.11)
Such an action of the nonlinear operator $\Lambda_\pm$ may be repeated by applying it consecutively to $\Psi_1$ and $\Psi_2 = \Lambda_+(\Psi_1)$. Further application of the operator yields a trivial result. Inversely, we can go down the ladder by applying $\Lambda_-$ to $\phi_1$, $\phi_2$ and $\phi_1$.

Although the usual creation and annihilation operators have a well-defined interpretation for wavefunctions, our nonlinear operators cannot be interpreted that way.

Finally, the recurrence relations may be constructed for the immersion functions $X_k$. In this case, the value of the index $k$ appears in the formulae explicitly. Note that in principle the explicit use of $k$ can be eliminated from (3.12) by expressing $k$ in terms of $\text{tr}(X^2)$, which is
equal to \((2k + 1)^2/N - (4k + 1)\). However, this does not make much sense as the immersion functions \(X_i\) are only well defined for \(k = 0, \ldots, N - 1\).

It follows from (2.20) that the projectors \(P_k\) may be expressed as

\[
P_k = X_k^2 - 2i \left( \frac{2k + 1}{N} - 1 \right) X_k - \frac{2k + 1}{N} \left( \frac{2k + 1}{N} - 2 \right) \mathbb{I},
\]  
(3.12)

which allows us to write the ‘annihilation’ operator as

\[
X_{k-1}(X_k) = X_k + i[\Pi_-(P_k) + P_k] - (2i/N)\mathbb{I},
\]

From (3.12). Similarly, the ‘creation’ operator may be defined by

\[
X_{k+1}(X_k) = X_k - i[\Pi_+(P_k) + P_k] + (2i/N)\mathbb{I}.
\]

Thus, surfaces associated with the \(\mathbb{C}P^{N-1}\) models satisfy a cubic equation (3.15). This imposes constraints on the eigenvalues of \(X_k\) but it does not mean that the surfaces are cubics, as (3.15) in a matrix equation.

**Example 3** (Action of \(X_k\) in \(\mathbb{C}P^2\), this time ‘descending’ the ladder). The surface \(X_1\), whose condition of immersion in \(\mathbb{R}^8\) is equation (2.8) for \(P = P_1\), may be written in the matrix form as

\[
X_1 = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{i}{(\xi|\xi|^2 + 1)^2} \begin{pmatrix} 2 & \sqrt{2}\xi & 0 \\ \sqrt{2}\xi & (\xi|\xi|^2 + 1) & \sqrt{2}\xi \\ 0 & \sqrt{2}\xi & 2|\xi|^2 \end{pmatrix}.
\]

(3.16)

If we apply the operator \(X_-\) to (3.16), then we obtain a matrix (after some simplification)

\[
X_0 = \frac{1}{3} i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{i}{(\xi|\xi|^2 + 1)^2} \begin{pmatrix} 1 & \sqrt{2}\xi & \xi^2 \\ \sqrt{2}\xi & 2|\xi|^2 & \sqrt{2}|\xi|^2 \xi \\ \xi^2 & \sqrt{2}|\xi|^2 \xi & |\xi|^4 \end{pmatrix}.
\]

(3.17)

This is the matrix form of a two-dimensional surface immersed in \(\mathbb{R}^8\) representing the soliton whose condition for immersion in \(\mathbb{R}^4\) (the Gauss-Mainardi-Codazzi equations) is (2.8) for the projector \(P = P_0\).

The surface \(X_1\) may in turn be obtained by a similar procedure performed on the surface \(X_0\), whose condition for immersion is (2.8) for the projector \(P_2\) which maps onto the direction of the antiholomorphic solution of (2.6).

We may also express each of the projection operators \(P_k\) as a linear function of the surfaces. However, this requires the knowledge of the immersion functions \(X_0, \ldots, X_{k-1}\), thus making the recurrence more involved. Namely from the equations of the surfaces in terms of the projectors (2.20), we obtain the relation

\[
P_k = i \sum_{j=1}^{k} (-1)^{k-j} (X_j - X_{j-1}) + (-1)^{k-j}iX_0 + \frac{1}{N}\mathbb{I},
\]

(3.18)

which may be used to construct the recurrence relations involving all the lower index \(X_j, j = 0, \ldots, k - 1\). In a similar way a downward recurrence might be obtained, involving all the higher index \(X_i\).

Equation (3.12) directly follows from equations (2.20). A short derivation of that equation is given in appendix C.
4. Geometrical aspects of the $\mathbb{C}P^{N-1}$ model

Let us now explore certain geometrical characteristics of the surfaces immersed in the $su(N)$ algebra and express them in terms of the projectors $P_k$. These geometrical properties include the Gaussian curvature, the mean curvature vector, the topological charge, the Willmore functional and the Euler–Poincaré character (see e.g. [31, 35, 54]). Under the assumption that the $\mathbb{C}P^{N-1}$ model is defined on the Riemann sphere $S^2$ and the associated action functional of this model is finite, we can show that the surfaces are conformally parametrized. The proof is similar to that given in [18]. In appendix D we demonstrate that whenever the equation of motion (2.8) is satisfied, the holomorphic quantity is

$$J_k = (g_k)_{11} = -\frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)) = -\frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)) = 0,$$

and its respective complex conjugate is

$$\bar{J}_k = (g_k)_{22} = -\frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)) = -\frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)) = 0.$$

The first fundamental form $I_k$ becomes

$$I_k = 2(g_k)_{12} \frac{d\xi}{d\bar{\xi}},$$

where the nonzero components of the induced metric $(g_k)_{ij}$ on the surfaces $X_k$ are given by

$$(g_k)_{12} = -\frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)) = -\frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)) = \frac{1}{2} \text{tr}((\partial P_k) \cdot (\partial P_k)).$$

Here the index inside the parentheses in $g$ refers to the number of the surface, while the other two indices denote the appropriate components of the metric tensor.

It follows from the Bonnet theorem that the surfaces $X_k$ are determined uniquely up to Euclidean motions by their first fundamental forms (4.3) and their second fundamental forms

$$II_k = (\Gamma_k)_{11} \frac{d\xi}{d\bar{\xi}} + 2 \frac{\partial \bar{\partial} X_k}{d\xi} d\xi + \frac{\partial^2 X_k}{d\xi} d\xi + \frac{\partial^2 X_k}{d\xi} d\xi$$

$$= -\text{tr}((\partial P_k) \cdot (\partial P_k)) \frac{[\partial P_k, \partial P_k]}{\text{tr}((\partial P_k) \cdot (\partial P_k))} \frac{d\xi^2}{d\bar{\xi}^2} + 2 \frac{\partial \bar{\partial} X_k}{d\xi} d\xi + \frac{\partial^2 X_k}{d\xi} d\xi,$$

where the immersion function $X_k$ is expressed in terms of the projectors $P_k$ by formula (2.20), and the nonzero Christoffel symbols of the second kind are given by

$$(\Gamma_k)_{11} = \partial \ln (g_k)_{12} = \partial \ln \left[\text{tr}((\partial P_k) \cdot (\partial P_k))\right],$$

$$(\Gamma_k)_{22} = \partial \ln (g_k)_{12} = \partial \ln \left[\text{tr}((\partial P_k) \cdot (\partial P_k))\right].$$

Since the projectors $P_0, \ldots, P_{N-1}$ are uniquely determined by the surfaces $X_k$ (see (3.12) and (3.18)), it follows that the projectors $P_k$ are determined (to that extent) by the fundamental forms (4.3) and (4.5). When $J_k = 0$, the Gaussian curvatures $K_k$ and the mean curvature vectors $\mathcal{H}_k$ (written as a matrix) take the simple form

$$K_k = -\frac{1}{(g_k)_{12}} \frac{\partial \bar{\partial} \ln (g_k)_{12}}{\partial \bar{\partial} X_k} = -\frac{2 \partial \bar{\partial} \ln \left[\text{tr}((\partial P_k) \cdot (\partial P_k))\right]}{\text{tr}((\partial P_k) \cdot (\partial P_k))},$$

and

$$\mathcal{H}_k = \frac{2}{(g_k)_{12}} \frac{\partial \bar{\partial} X_k}{\partial \bar{\partial} X_k} = -\frac{4i}{\text{tr}((\partial P_k) \cdot (\partial P_k))} [\partial P_k, \partial P_k].$$

Example 4 (Geometrical properties of the family generated by the Veronese solutions in $\mathbb{C}P^2$). We may easily determine the geometrical characteristics of the holomorphic Veronese solution of (2.6), the corresponding projector solutions of (2.8) and the solutions obtained from
them by application of the ‘creation’ operator (3.3). The first fundamental form is completely determined by

\[ (g_0)_{12} = \frac{1}{(|\xi|^2 + 1)^2} = (g_2)_{12}, \quad (g_1)_{12} = \frac{2}{(|\xi|^2 + 1)^2}, \]

where the index inside the parentheses in \( g \) is 0 for the holomorphic solution, 1 for the mixed solution and 2 for the antiholomorphic solution.

The second fundamental form is determined by the Christoffel symbols. They have the same values for all three surfaces as the constant factor 2 in (4.9) does not influence the logarithmic derivative in (4.6).

The nonzero Christoffel symbols read (with the same convention about the indices)

\[ (\Gamma_0)^1_{11} = (\Gamma_1)^1_{11} = (\Gamma_2)^1_{11} = -\frac{2\xi^2}{|\xi|^2 + 1}, \quad (\Gamma_0)^2_{22} = (\Gamma_1)^2_{22} = (\Gamma_2)^2_{22} = -\frac{2\xi^2}{|\xi|^2 + 1}. \]

(4.10)

The Gaussian curvature may be obtained in a straightforward way from (4.7) as

\[ K_0 = K_2 = 2, \quad K_1 = 1. \]

(4.11)

Hence all these surfaces have a constant positive Gaussian curvature.

The mean curvature is given by a rather complicated traceless matrix expression (or a vector expression if we decompose the matrix in the basis of the Gelfand matrices). In the matrix form, we get e.g. for the surface corresponding to the holomorphic solution

\[ \mathcal{H}_0 = \frac{4i}{(|\xi|^2 + 1)^2} \begin{pmatrix} 1 - 2|\xi|^2 & \sqrt{2}(2 - |\xi|^2)\xi & 3\xi^2 \\ \sqrt{2}(2 - |\xi|^2)\xi & -(|\xi|^4 - 4|\xi|^2 + 1) & \sqrt{2}(2|\xi|^2 - 1)\xi \\ 3\xi^2 & \sqrt{2}(2|\xi|^2 - 1)\xi & \xi^2(|\xi|^2 - 2) \end{pmatrix}. \]

(4.12)

However, the mean curvature proves to be a vector of constant norm, namely the squares of the norms calculated according to (2.13) are

\[ (\mathcal{H}_0, \mathcal{H}_0) = 4, \quad (\mathcal{H}_1, \mathcal{H}_1) = (\mathcal{H}_2, \mathcal{H}_2) = 16. \]

(4.13)

This result may be used to calculate the Willmore functional. The Willmore functionals (also called the total squared mean curvature vectors) are defined by

\[ W_k = \frac{1}{2} \int_{\Omega} ||\mathcal{H}_k||^2 \sqrt{\text{det}(g_k)} \, d\xi \, d\xi^2 = \int_{\Omega} \text{tr}((\tilde{\nabla} P_k, \tilde{\nabla} P_k)^2) \, d\xi \, d\xi^2, \]

(4.14)

where \( \Omega \subset \mathbb{C} \) is an open connected and simply connected set, while the norm is defined in terms of the scalar product in the usual way: \( ||\cdot|| = (\cdot, \cdot)^{1/2} \). Hence, in our example, if we take \( \Omega = S^2 = \mathbb{C} \cup \{\infty\} \) in (4.14) (i.e. we integrate over the whole Riemann sphere), we obtain from (4.13) and (4.9)

\[ W_0 = 4W_1 = W_2 = 4\pi. \]

(4.15)

In a similar way we may calculate a few other global characteristics of the soliton surfaces defined by the immersion functions \( X_k \). In particular, a significant quantity which characterizes solutions satisfying the \( \mathbb{C}P^{N-1} \) model equations (2.8) are the topological charges associated with the surfaces \( X_k [10] \).

\[ Q_k = \frac{1}{\pi} \int_{\Sigma^2} \hat{a} \hat{\bar{a}} \ln |f_k| \, d\xi \, d\xi^2, \]

(4.16)

which may be transformed into

\[ Q_k = -\frac{1}{\pi} \int_{\Sigma^2} \text{tr}(P_k \cdot [\tilde{\nabla} P_k, \tilde{\nabla} P_k]) \, d\xi \, d\xi^2. \]

(4.17)
Integral (4.17) exists and is a topological invariant of the surfaces given by (3.13) or (3.14). It is an integer which globally characterizes the surfaces $X_k$.

In the case of compact oriented and connected surfaces $X_k$, other topological invariants, the Euler–Poincaré characters, are given by

$$
\Delta_k = \frac{i}{2\pi} \int_{S^2} K_k (g_k)_{12} \, d\xi \, d\bar{\xi} = -\frac{i}{2\pi} \int_{S^2} \partial \bar{\partial} \ln (g_k)_{12} \, d\xi \, d\bar{\xi}
$$

If we know the projector $P_k$ explicitly, the calculation of $\Delta_k$ is straightforward.

Example 5. In the particular case where $N = 3$ (the $\mathbb{C}P^2$ model), (4.17) and (4.18) turn into

$$
Q_0 = 2, \quad Q_1 = 0, \quad Q_2 = -2
$$

for the topological charges, and

$$
\Delta_0 = \Delta_1 = \Delta_2 = 2
$$

for the Euler–Poincaré characters.

The result (4.19) is in accordance with the values of the topological charge obtained in [10]. The value of the topological charge distinguishes the instantons ($Q = 2$ for a one-instanton state in $\mathbb{C}P^2$, $Q = -2$ from the anti-instanton state, which produces the same winding over the target sphere but in the opposite direction). The fact that all surfaces possess the same value of the Euler–Poincaré character, equal to 2, and positive Gaussian curvatures $K_k > 0$ means that all surfaces are homeomorphic to spheres.

5. Singularities of the $\mathbb{C}P^{N-1}$ model

In what follows we do not impose the assumption that the action functional of the $\mathbb{C}P^{N-1}$ model (2.2) is finite.

The E-L equations of the $\mathbb{C}P^{N-1}$ model (2.6) are autonomous; hence, they do not have fixed singularities at finite points. On the other hand, as nonlinear equations, they might in principle have movable singularities. Let us limit ourselves to solutions without branch points or essential singularities. The scaling invariance puts limits on the singular behavior of such solutions: the singularities disappear in the invariant description. The following statements directly follow from the scaling invariance.

(i) If the $j$th component of a homogeneous field coordinate $f$ has a pole of order $p$ greater than or equal to the order of the other poles at a point $\xi_0$, then the solution may be multiplied by $(\xi - \xi_0)^p$. This yields a solution $f$ of the E-L equations, which constitutes the same solution in the invariant variables. An appropriate multiplication by a product of such factors can always be performed if the number of poles is finite. Also, in many cases with an infinite number of poles, we can build a holomorphic function which would regularize the solution making use of the Weierstrass theorem (provided that the poles have no finite accumulation point). This multiplication makes the solution regular, and we will refer to the procedure as regularization.

(ii) The regularization of a field coordinate $f$ through the multiplication by a singularity-removing factor $(\xi - \xi_0)^p$ would introduce zeros at the point $\xi_0$ in all the components which were regular or had poles of lower order than $p$. In such a case, the usual normalization of $f$ by setting its first component to 1 may be impossible.
(iii) In the $\mathbb{C}P^{N-1}$ model, we also consider functions which are not holomorphic as the E-L equations (2.6) depend on both $f$ and $f'$. To perform a singularity analysis in such cases, both independent variables are extended to separate complex planes, and the field coordinates $f$ intrinsically become functions of two complex variables. However, all the previous and further considerations hold, with the modification that $\xi - \xi_0$ is replaced by some function $F(\xi, \bar{\xi})$ which would vanish at the line of singularity (except that the class of exceptions is richer in two dimensions than in one).

(iv) To summarize, the regularization leaves invariant

- the E-L equations in both forms (2.6), (2.8) and the action functional (2.1);
- the projectors $P_k$, $k = 0, \ldots, N - 1$, as well as any projection operators in the algebra $\mathfrak{su}(N)$ of anti-Hermitian matrices (or $\mathfrak{i}\mathfrak{su}(N)$ in the case of Hermitian matrices);
- the surfaces $X_k$ with all their induced metrics $(g_{kl})$ and curvature properties $\kappa_k$ and $\mathcal{H}_k$;
- the ‘creation’ and ‘annihilation’ operators $\Pi_{\pm}$, $\Lambda_{\pm}$ and $\chi_{\pm}$.

The classical operators $P_-$ and $P_+$ are covariant in the sense that $P_{\pm}(f_k(\xi - \xi_0)p) = (\xi - \xi_0)pP_{\pm}(f_k)$, which allows for regularization of the Din–Zakrzewski procedure [10].

6. Summary and concluding remarks

The objective of this paper was to provide an invariant description of recurrence relations for the completely integrable $\mathbb{C}P^{N-1}$ sigma models defined on the Riemann sphere $S^2$ when its action functional is finite. We have determined the connection between successive projector operators, wavefunctions of the linear spectral problem and immersion functions which immerse the surfaces in the $\mathfrak{su}(N)$ algebra in such a way that they preserve conformal invariance. Through this link, we have found explicit expressions for these quantities and established a commutative diagram for them. An advantage of the presented approach is that, without reference to any additional considerations, the recurrence relations give a very useful tool for constructing each successive surface associated with the $\mathbb{C}P^{N-1}$ sigma model from the knowledge of the previous one. We have also analyzed the asymptotic properties of the solutions of the $\mathbb{C}P^{N-1}$ model in neighborhoods of zeros and poles (excluding branch points and essential singularities) and demonstrated that the singularity structures of the meromorphic solutions of the model do not influence the above-mentioned invariant quantities. Consequently, we have shown that the surfaces associated with the $\mathbb{C}P^{N-1}$ model are regular. Furthermore, we provide a certain geometrical setting which allows us to obtain explicit formulas in terms of the projector $P_k$ for the Gaussian and mean curvatures, the Willmore functional, the Euler–Poincaré character and the topological charge of the considered surfaces. This allows us to study certain global properties of the surfaces as illustrated by concrete examples of surfaces associated with the $\mathbb{C}P^2$ and $\mathbb{C}P^3$ models. In particular, we have shown that for the Veronese vectors we obtain constant positive Gaussian curvatures as expected.

It may be worthwhile to extend the investigation of surfaces to the case of the sigma models defined on other homogeneous spaces via Grassmannian models. This case can lead to different classes and more diverse types of surfaces than those investigated in this paper, including those with a constant negative Gaussian curvature. These types of surfaces immersed in Lie algebras are known to have many fundamental applications in physics, chemistry and biology (see e.g. [9, 30, 34, 37, 39, 42]). This task will be undertaken in a future work.
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Appendix A. Derivations of the recurrence relations for the projection operators (3.2) and (3.3)

To construct the recursion operator, we start with the $P_{\pm}$ operators (3.1), which raise or lower the index of the homogeneous field coordinates $f_k$ by 1.

The $k$th coordinate $f_k$ may be regained from the respective projector $P_k$ by an extraction of its first column

$$f_k = \frac{1}{(P_k)_{11}} P_k \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

(A.1)

where $(P_k)_{11}$ is the first row–first column element of the matrix $P_k$. The first row of its Hermitian conjugate is obtained similarly by multiplying on the left by $(1, 0, \ldots, 0)$.

This equation yields $f_k$ with the first component of $f_k$ normalized to 1. For the sake of simplicity, the derivation will be done for that case. If $(P_k)_{11} = 0$, the first component of $f_k$ is zero. In that case we can get the $f_k$ by extracting another column of $P_k$, which is done by multiplying with a vector having 1 at the other position (and zeros elsewhere).

Substituting (A.1), together with its Hermitian conjugate, into (2.15) and (2.14), we obtain the nonlinear ‘creation operator’ $\Pi \ast$ for the projectors $P_k$:

$$\Pi \ast (P) = \frac{(I - P) \cdot \partial P \cdot I_0 \cdot \partial P \cdot (I - P)}{[\partial P \cdot (I - P) \cdot \partial P]_{11}},$$

(A.2)

where $[\cdot]_{11}$ denotes the leftmost-uppermost element of the matrix while

$$I_0 = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}.$$ 

(A.3)

In the transition from (A.1) to (A.2), we used the scaling invariance to get rid of the factor $(P_k)_{11}$.

This operator may be further simplified if we use the following property of projectors:

$$(I - P) \cdot \partial P = (\partial P) \cdot P, \quad (\partial P) \cdot (I - P) = P \cdot \partial P.$$ 

(A.4)

Identity (A.4) yields equation (3.3) in a straightforward way if we note that any Hermitian projection operator $P$ mapping onto a one-dimensional space and satisfying $\text{tr}(P) = 1$ may be represented as $U^{-1}I_0U$, where $U$ is a unitary matrix (the diagonalized $P$ has only one nonzero element, equal to 1 and this 1 may always be placed at the upper-left corner as in the $I_0$ matrix). Moreover, by direct calculation,

$$P_{11} = (U^{-1})_{11} U_{11} = U_{11}^2 \quad \text{(A.5)}$$

as $U^{-1} = U^\dagger$. Hence

$$P \cdot I_0 \cdot P = U^{-1} \cdot I_0 \cdot U \cdot I_0 \cdot U^{-1} I_0 \cdot U = P_{11} P.$$

(A.6)
as we have
\[ \| 0 \cdot M \cdot \| 0 = M_{11} \| 0 \]  \quad (A.7)
for any matrix \( M \). Equation (A.6) yields the numerator of (3.3) up to a constant factor. The denominator immediately follows from the normalization \( \text{tr}(\Pi_1(P)) = \text{tr}(P) = 1 \), provided that the matrix is nonzero (see below for the proof that its trace is also nonzero).

The ‘annihilation’ operator \( \Pi_- \) is obtained from \( \Pi_+ \) by exchanging the partial derivatives \( \partial = \bar{\partial} \).

The projective property of the resulting operators \( \Pi_-(P) \) and \( \Pi_+(P) \) (3.2, 3.3) may be proven by means of the same unitary conversion of \( P \). Let us check the square of \( \Pi_+(P) \) (3.3):
\[
[\Pi_+(P)]^2 = \frac{\partial P \cdot P \cdot \bar{\partial} P \cdot \bar{\partial} P}{[\text{tr}(\partial P \cdot P \cdot \bar{\partial} P)]^2}.
\]
If the numerator of (A.8) is a zero matrix, then the projective property is trivial. If the numerator is a nonzero matrix, then, according to (A.7), its central part which begins and ends with \( \| 0 \) is a diagonal matrix with only one nonzero element in the top-left position. Hence it is equal to its trace multiplied by \( \| 0 \):
\[
\| 0 \cdot U \cdot \bar{\partial} P \cdot \partial P \cdot U^{-1} \cdot \| 0 = \text{tr}(\| 0 \cdot U \cdot \bar{\partial} P \cdot \partial P \cdot U^{-1} \cdot \| 0)\| 0
= \text{tr}(U^{-1} \cdot \| 0 \cdot U \cdot \bar{\partial} P \cdot \partial P)\| 0 = \text{tr}(P \cdot \bar{\partial} P \cdot \partial P)\| 0
= \text{tr}(\partial P \cdot P \cdot \bar{\partial} P)\| 0.
\]
It follows from (A.9) that the trace of \( \partial P \cdot P \cdot \bar{\partial} P \) is nonzero whenever the matrix is nonzero. Otherwise the matrix would be nilpotent, but this is impossible for a nonzero Hermitian matrix.

Combining (A.9) with the rest of equation (A.8) we eventually obtain
\[
\Pi_+(P) \cdot \Pi_+(P) = \text{tr}(\partial P \cdot P \cdot \bar{\partial} P)\partial P \cdot U^{-1} \cdot \| 0 \cdot U \cdot \bar{\partial} P \cdot \partial P / [\text{tr}(\partial P \cdot P \cdot \bar{\partial} P)]^2 = \Pi_+(P).
\]
(A.10)
The same property obviously holds for \( \Pi_-(P) \). Q.E.D.

Appendix B. Derivation of the recurrence relations for the wavefunctions (3.7) and (3.8)

From the solutions of the spectral problem in terms of the projection operators (2.18), we obtain a formula for \( \Phi_k = (1 - \lambda)^2 (k - \Phi_k) \), \( k = 1, \ldots, N - 1 \).
\[
\Psi_k(\lambda) - \Psi_{k-1}(\lambda) = 2(1 - \lambda) \left( P_k - \frac{1 + \lambda}{1 - \lambda} P_{k-1} \right).
\]
(B.1)
If we combine the solution for \( \Phi(\lambda) \) with that for \( [\Phi(\lambda)]^{-1} = \Phi(-\lambda) \), we simply obtain, for \( l = 0, \ldots, N - 1 \),
\[
\Psi_l(\lambda) + \Psi_l(-\lambda) = 4P_l.
\]
(B.2)
Substituting (B.2) for \( l = k \) and \( l = k - 1 \) into (B.1) immediately yields the ‘annihilation operator’ (3.7) if we solve (B.1) for \( \Phi_{k-1} \) and express \( P_{k-1} \) as \( \Pi_-(P_k) \). The same equations (B.1), (B.2) yield the ‘creation operator’ if we solve (B.1) for \( \Psi_k \) while expressing \( P_k \) as \( \Pi_+(P_{k-1}) \).
Appendix C. Derivation of equation (3.12) used in the recurrence relations for the immersion functions (3.13) and (3.14)

We square equation (2.20), bearing in mind that the projectors \( P_0, \ldots, P_k \) are mutually orthogonal and we obtain

\[
X_k \cdot X_k = \left[ \frac{2(2k + 1)}{N} - 1 \right] P_k + 4 \left[ \frac{(2k + 1)}{N} - 1 \right] \sum_{j=0}^{k-1} P_j - \frac{(2k + 1)^2}{N^2} I. \tag{C.1}
\]

This equation may be combined with \( X_k \) multiplied by an appropriate factor, as in (3.12), to get rid of the sum of the lower operators \( P_0 + \cdots + P_{k-1} \). The solution for \( P_k \) is precisely what was found for equation (3.12).

Appendix D. Derivation of the fact that the holomorphic functions \( J_k \) vanish when the \( \mathbb{C}P^{N-1} \) model is defined on \( S^2 \) and its action functional is finite

To prove the vanishing of the holomorphic quantities \( J_k \) and \( \bar{J}_k \), it is sufficient to consider the orthogonality condition for the operator \( P_\pm^k \cdot f \) in the specific case where \( i = k \) and \( j = k + 2 \) [54]:

\[
\left( P_\pm^k \cdot f \right) \cdot \left( P_\pm^{k+2} \cdot f \right) = 0. \tag{D.1}
\]

Here \( 0 \leq k \leq N-2 \) for the operator \( P_+ \) or \( 1 \leq k \leq N-1 \) for the operator \( P_- \). Using the notation \( f_k = P_\pm^k \cdot f \), we get

\[
0 = f_k^\dagger \cdot \left( P_\pm^k \cdot f_k \right) = f_k^\dagger \cdot \left( \mathbb{I} - \left( P_\pm f_k \right) \otimes \left( P_\pm f_k \right)^\dagger \right) \cdot \partial_\pm (P_\pm f_k) = f_k^\dagger \cdot \partial_\pm (P_\pm f_k), \tag{D.2}
\]

where the symbol \( \partial_\pm \) represents the holomorphic derivative \( \partial \) and \( \partial_- \) represents the antiholomorphic derivative \( \bar{\partial} \). Since \( f_k^\dagger \cdot (P_\pm f_k) = 0 \), this implies that

\[
f_k^\dagger \cdot \partial_\pm (P_\pm f_k) = -\partial_\pm f_k^\dagger \cdot (P_\pm f_k). \tag{D.3}
\]

The right-hand side of equation (D.2) can be written in terms of the holomorphic function \( J_k \):

\[
0 = -\partial_\pm f_k^\dagger \cdot (\mathbb{I} - P_k) \cdot \partial_\pm f_k = -(\partial_\pm P_k \cdot P_k)_{11} = -(P_k)_{11} J_k.
\]

\[
0 = -\partial_- f_k^\dagger \cdot (\mathbb{I} - P_k) \cdot \partial_- f_k = -(\partial_- P_k \cdot P_k)_{11} = -(P_k)_{11} \bar{J}_k. \tag{D.4}
\]

Since \( (P_k)_{11} \neq 0 \), we get \( J_k = 0 \). Hence \( J_k \) and \( \bar{J}_k \) vanish identically. The version with the operator \( P_+ \) works for holomorphic and mixed solutions, while the version with the operator \( P_- \) works for antiholomorphic and mixed solutions. Q.E.D.

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