Non-asymptotic convergence bounds for Wasserstein approximation using point clouds

Abstract

Several issues in machine learning and inverse problems require to generate discrete data, as if sampled from a model probability distribution. A common way to do so relies on the construction of a uniform probability distribution over a set of \( N \) points which minimizes the Wasserstein distance to the model distribution. This minimization problem, where the unknowns are the positions of the atoms, is non-convex. Yet, in most cases, a suitably adjusted version of Lloyd’s algorithm — in which Voronoi cells are replaced by Power cells — leads to configurations with small Wasserstein error. This is surprising because, again, of the non-convex nature of the problem, as well as the existence of spurious critical points. We provide explicit upper bounds for the convergence speed of this Lloyd-type algorithm, starting from a cloud of points sufficiently far from each other. This already works after one step of the iteration procedure, and similar bounds can be deduced, for the corresponding gradient descent. These bounds naturally lead to a modified Poljak-Łojasiewicz inequality for the Wasserstein distance cost, with an error term depending on the distances between Dirac masses in the discrete distribution.

1 Introduction

In recent years, the theory of optimal transport has been the source of stimulating ideas in machine learning and in inverse problems. Optimal transport can be used to define distances, called Wasserstein or earth-mover distances, between probability distributions over a metric space. These distances allows one to measure the closeness between a generated distribution and a model distribution, and they have been used with success as data attachment terms in inverse problems. Practically, it has been observed for several different inverse problems that replacing usual loss functions with Wasserstein distances tend to increase the basin of convergence of the methods towards a good solution of the problem, or even to convexify the landscape of the minimized energy [7, 6]. This good behaviour is not fully understood, but one may attribute it partly to the fact that the Wasserstein distances encodes the geometry of the underlying space. A notable use of Wasserstein distances in machine learning is in the field of generative adversarial networks, where one seeks to design a neural network able to produce random examples whose distribution is close to a prescribed model distribution [2].
When applying gradient descent to the nonconvex optimization problem (2), it is in principle possible to end up on local minima corresponding to a high energy critical points of the Wasserstein distance, regardless of the non-linearity of the map \( \theta \) to \( \mathbb{R}^d \). This can be done by concentrating the support of the measure \( \mu \) on lower-dimensional subspaces of \( \mathbb{R}^d \), as in Remarks 2 and 3. Solving (2) numerically is challenging for several reasons, but in this article we will concentrate on one of them: the non-convexity of the Wasserstein distance under displacement of the measures.

### Non-convexity of the Wasserstein distance under displacements.

It is well known that the Wasserstein distance is convex for the standard (linear) structure of the space of probability measures, meaning that if \( \nu_0 \) and \( \nu_1 \) are two probability measures and \( \nu_t = (1-t)\nu_0 + t\nu_1 \), then the map \( t \in [0,1] \mapsto W_p(\nu_t, \rho) \) is convex. However, in the regression problem (2), the perturbations are Lagrangian rather than Eulerian, and as in Remarks 2 and 3, the ratio between the Wasserstein distance of exponent \( p \) and Wasserstein regression is bounded from below by a positive constant, while \( \lim_{N \to +\infty} \min F_N = 0 \), so that the ratio between \( F_N(Y_N) \) and \( \min F_N \) is arbitrarily large as \( N \to +\infty \). This can be done by concentrating the points \( Y_N \) on lower-dimensional subspaces of \( \mathbb{R}^d \), as in Remarks 2 and 3.

When applying gradient descent to the nonconvex optimization problem (2), it is in principle possible to end up on local minima corresponding to a high energy critical points of the Wasserstein distance, regardless of the non-linearity of the map \( \theta \) to \( \mathbb{R}^d \). Our main theorem, or rather its Corollary 6 shows that if the points of \( Y \) are at distance at least \( \varepsilon > 0 \) from one another, then

\[
F_N(Y) - C \frac{1}{N^{d-1}} \leq N \| \nabla F_N(Y) \|^2.
\]
The uniform optimal quantization problem (4) is a very natural variant of the (standard) optimal quantization problem. Our main result also has implications in terms of the gradient descent, defining the iterates

\[ V_i(Y) = \{ x \in \Omega \mid \forall j \in \{1, \ldots, N\}, \| x - y_i \| \leq \| x - y_j \| \}. \]  

In the previous equation \( \| \nabla F_N(Y) \| \) denotes the Euclidean norm of the vector in \( \mathbb{R}^{Nd} \) obtained by putting one after the other the gradients of \( F_N \) w.r.t. the positions of the atoms \( y_i \). We note that due to the weights \( 1/N \) in the atomic measure \( \delta_Y \), the components of this vector are in general of the order of \( 1/N \), see Proposition[1]. This inequality resembles the Polyak-Łojasiewicz inequality, and shows in particular that if the quantization error \( F_N(Y) = W_2^2(\rho, \delta_Y) \) is large, i.e. larger than \( \varepsilon^{1-d}/N \), then the point cloud \( Y \) is not critical for \( F_N \). From this, we deduce in Theorem[7] that if the points in the initial cloud are not too close to each other at the initialization, then the iterates of fixed step gradient descent converge to points with low energy \( F_N \), despite the non-convexity of \( F_N \).

Relation to optimal quantization. Our main result also has implications in terms of the uniform optimal quantization problem, where one seeks a point cloud \( y = (y_1, \ldots, y_N) \) in \( \mathbb{R}^d \) such that the uniform measure supported over \( Y \), denoted \( \delta_Y \), is as close as possible to the model distribution \( \rho \) with respect to the 2-Wasserstein distance:

\[ \min_{Y \in \mathbb{R}^d} F_N(Y). \]  

The uniform optimal quantization problem (4) is a very natural variant of the (standard) optimal quantization problem, where one does not impose that the measure supported on \( Y \) is uniform:

\[ \min_{Y \in \mathbb{R}^d} G_N(Y), \text{ where } G_N(Y) = \min_{\mu \in \Delta_N} W_2^2(\rho, \sum_{i=1}^N \mu_i \delta_{y_i}), \]  

and where \( \Delta_N \subseteq \mathbb{R}^N \) is the probability simplex. This standard optimal quantization problem is a cornerstone of sampling theory, and we refer the reader to the book of Graf and Luschgy[10] and to the survey by Pagès[15]. The uniform quantization problem (4) is less common, but also very natural. It has been used in imaging to produce stipplings of an image[4,3] or for meshing purposes[9]. A common difficulty for solving (5) and (4) numerically is that the minimized functionals \( F_N \) and \( G_N \) are non-convex and have many critical points with high energy. However, in practice, simple fixed-point or gradient descent strategies behave well when the initial point cloud is not chosen adversely. Our second contribution is a quantitative explanation for this good behaviour in the case of the uniform optimal quantization problem.

Lloyd’s algorithm[12] is a fixed point algorithm for solving approximately the standard optimal quantization problem (5). Starting from a point cloud \( Y^k = (y_1^k, \ldots, y_N^k) \in \mathbb{R}^d \) with distinct points, one defines the next iterate \( Y^{k+1} \) in two steps. First, one computes the Voronoi diagram of \( Y \), a tessellation of the space into convex polyhedra \( V_i(Y^k) \) for \( 1 \leq i \leq N \), where

\[ V_i(Y) = \{ x \in \Omega \mid \forall j \in \{1, \ldots, N\}, \| x - y_i \| \leq \| x - y_j \| \}. \]  

In the second step, one moves every point \( y_i^k \) towards the barycenter, with respect to \( \rho \), of the corresponding cell \( V_i(Y^k) \). This algorithm can also be interpreted as a fixed point algorithm for solving the first-order optimality condition for (5), i.e. \( \nabla G_N(Y) = 0 \). One can show that the energy \( (G_N(Y^k))_{k \geq 0} \) decreases in \( k \). The convergence of \( Y^k \) towards a critical point of \( F_N \) as \( k \to +\infty \) has been studied in[5], but the energy of this limit critical point is not guaranteed to be small.

In the case of the uniform quantization problem (4), one can try to minimize the energy \( F_N \) by gradient descent, defining the iterates

\[ Y^{k+1} = Y^k - \tau N \nabla F_N(Y^k), \]  

Figure 1: From left to right, a point cloud \( Y^0 \) in the square \( \Omega = [0; 1] \times [0; 1] \), the associated power cells \( P_i(Y) \) in the optimal transport to the Lebesgue measure on \( \Omega \), the vectors \( -N \nabla F_N(Y^0) = B_N(Y^0) - Y^0 \) followed during the Lloyd step and the positions of the barycenters \( Y^1 = B_N(Y) \).
where $\tau > 0$ is the time step. The factor $N$ in front of $\nabla F_N$ is set as a compensation for the fact that we have, in general, $\nabla F_N(Y) = O(1/N)$. When $\tau = 1$, one recovers a version of Lloyd’s algorithm for the uniform quantization problem, involving barycenters $B_N(Y)$ of Power cells, rather than Voronoi cells, associated to $Y$. More precisely, Proposition 1 proves that $\nabla F_N(Y) = (Y - B_N(Y))/N$ so that $Y^{k+1} = B_N(Y^k)$ when $\tau = 1$. Quite surprisingly, we prove in Corollary 2 that if the points in the initial cloud $Y^0$ are not too close to each other, then the uniform measure over the point cloud $Y^1 = Y^0 - N\nabla F_N(Y^0)$ obtained after only one step of Lloyd’s algorithm is close to $\rho$. This is illustrated in Figure 1. We prove in particular the following statement.

**Theorem** (Particular case of Corollary 4). Let $\rho$ be a probability density over a compact convex set $\Omega \subseteq \mathbb{R}^d$, let $Y^0 = (y_1^0, \ldots, y_N^0) \in \Omega^d$ and assume that the points lie at some positive distance from one another: for some constant $c$,

$$\forall i \neq j, \|y_i - y_j\| \geq cN^{-1/d},$$

corresponding for instance to a point cloud sampled on a regular grid. Then, the point cloud $Y^1 = Y^0 - N\nabla F_N(Y^0)$ obtained after one step of Lloyd’s algorithm satisfies

$$W_2^2(\delta_Y^1, \rho) \leq C_{c,d,\Omega}N^{-1/d},$$

where $C_{c,d,\Omega}$ is a constant depending on $c$, $d$ and $\text{diam}(\Omega)$.

**Outline** In Section 2, we start by a short review of background material on optimal transport and optimal uniform quantization. We then establish our main result (Theorem 3) on the approximation of a measure $\rho$ by barycenters of Power cells. This theorem yields error estimates for one step of Lloyd’s algorithm in deterministic and probabilistic settings (Corollaries 4 and 5). In Section 3, we establish a PolyaŁojasiewicz-type inequality (Corollary 6) for the function $F_N = \frac{1}{2}W_2^2(\rho, \delta_Y)$ introduced in (8), and we study the convergence of a gradient descent algorithm for $F_N$ (Theorem 7). Finally, in Section 4, we report numerical results on optimal uniform quantization in dimension $d = 2$.

## 2 Lloyd’s algorithm for optimal uniform quantization

**Optimal transport and Kantorovich duality** In this section we briefly review Kantorovich duality and its relation to semidiscrete optimal transport. The cost is fixed to $c(x, y) = \|x - y\|^2$, and we assume that $\rho$ is a probability density over a compact convex domain $\Omega$. In this setting, Brenier’s theorem implies that given any probability measure $\mu$ supported on $\Omega$, the optimal transport plan between $\rho$ and $\mu$, i.e. the minimizer $\pi$ in the definition of the Wasserstein distance (1) with $p = 2$, is induced by a transport map $T_{\mu, \rho}: \Omega \to \Omega$, meaning $\pi = (T_{\mu, \rho}, \text{Id})_\# \rho$. One can derive an alternative expression for the Wasserstein distance using Kantorovich duality, which leads to a more precise description of the optimal transport map [13, Theorem 1.39]:

$$W_2^2(\rho, \mu) = \max_{\phi:Y \to \mathbb{R}} \int_{\mathbb{R}^d} \phi(x) \rho(dx) + \int_{\Omega} \phi(y) \mu(dy),$$

(8)

where $\phi^*(x) = \min \phi(x, y) - \phi_1$. When $\mu = \delta_Y$ is the uniform probability measure over a point cloud $Y = (y_1, \ldots, y_N)$ containing $N$ distinct points, we set $\phi_i = \phi(y_i)$ and we define the $i$th Power cell associated to the couple $(Y, \phi)$ as

$$\text{Pow}_i(Y, \phi) = \{x \in \mathbb{R}^d \mid \forall j \in \{1, \ldots, N\}, \|x - y_i\|^2 - \phi_i \leq \|x - y_j\|^2 - \phi_j\}. $$

Then, the Kantorovich dual [13] of the optimal transport problem between $\rho$ and $\delta_Y$ turns into a finite-dimensional concave maximization problem

$$W_2^2(\mu, \rho) = \max_{\phi \in \mathbb{R}^N} \sum_{i=1}^N \phi_i + \int_{\text{Pow}_i(Y, \phi)} \left(\|x - y_i\|^2 - \phi_i\right) d\rho(x),$$

(9)

By Corollary 1.2 in [11], a vector $\phi \in \mathbb{R}^N$ is optimal for this maximization problem if and only if the potential $\phi$ is such that each Power cell contain the same amount of mass, i.e. if

$$\forall i \in \{1, \ldots, N\}, \quad \rho(\text{Pow}_i(Y, \phi)) = \frac{1}{N},$$

(10)

From now on, we denote $P_i(Y) = \text{Pow}_i(Y, \phi) \cap \Omega$, where $\phi \in \mathbb{R}^N$ satisfies (10). The optimal transport map $T_Y$ between $\rho$ and $\delta_Y$ sends every Power cell $P_i(Y)$ to the point $y_i$, i.e. it is defined $\rho$-almost everywhere by $T_Y|_{P_i(Y)} = y_i$. We refer again to the introduction of [11] for more details.
**Optimal uniform quantization** In this article, we study the behaviour of the squared Wasserstein distance between the (fixed) probability density \( \rho \) and a uniform finitely supported measure \( \delta_Y \) where \( Y = (y_1, \ldots, y_N) \) is a cloud of \( N \) points, in terms of variations of \( Y \). As in equation (3), we denote \( F_N = \frac{1}{2} W_2^2(\rho, \cdot) \). Proposition 21 in [14] gives an expression for the gradient of \( F \), and proves its semiconcavity. We recall that \( F \) is called \( \alpha \)-semiconcave, with \( \alpha \geq 0 \), if the function \( F - \frac{\alpha}{2} \| \cdot \|^2 \) is concave. We denote \( \mathbb{D}_N \) the generalized diagonal

\[
\mathbb{D}_N = \{ Y \in (\mathbb{R}^d)^N \mid \exists i \neq j \text{ s.t. } y_i = y_j \}.
\]

**Proposition 1 (Gradient of \( F_N \)).** The function \( F_N \) is \( \frac{1}{N} \)-semiconcave on \( (\mathbb{R}^d)^N \) and is of class \( C^1 \) on \( (\mathbb{R}^d)^N \setminus \mathbb{D}_N \). In addition, for any \( Y \in \mathbb{D}_N \) one has

\[
\forall Y \in (\mathbb{R}^d)^N \setminus \mathbb{D}_N, \nabla F_N(Y) = \frac{1}{N} (Y - B_N(Y)), \text{ where } B_N(Y) = (b_1(Y), \ldots, b_N(Y))
\]

(11) and where \( b_i(Y) \) is the barycenter of the \( i \)th power cell, i.e. \( b_i(Y) = N \int_{P_i(Y)} d\rho(x) \).

It is not difficult to prove that \( F_N \) admits at least one minimizer, and that this minimizer \( Y \) satisfies the first-order optimality condition \( Y = B_N(Y) \). A point cloud that satisfies this condition is called **critical**.

**Remark 1 (Upper bound on the minimum of \( F_N \)).** We note from [14] Proposition 12 that when \( \rho \) is supported on a compact subset of \( \mathbb{R}^d \), then

\[
\min_{Y \in [\mathbb{R}^d]^N} F_N(Y) = \min_{Y \in (\mathbb{R}^d)^N \setminus \mathbb{D}_N} \frac{1}{2} W_2^2(\rho, \delta_Y) \lesssim \begin{cases} 
N^{-\frac{d}{2}} & \text{if } d > 2 \\
N^{-1} \log N & \text{if } d = 2 \\
N^{-1} & \text{if } d = 1.
\end{cases}
\]

(12) These upper bounds may not be tight, in particular when \( \rho \) is separable (see Appendix E).

**Remark 2 (High energy critical points).** On the other hand, since \( F_N \) is not convex, this first-order condition is not sufficient to have a minimizer of \( F_N \). For instance, if \( \rho \equiv 1 \) on the unit square \( \Omega = [0, 1]^2 \), one can check that the point cloud

\[
Y_N = \left( \left( \frac{1}{2N^2} \cdot \frac{1}{2} \right), \left( \frac{3}{2N^2} \cdot \frac{1}{2} \right), \ldots, \left( \frac{2N-1}{2N^2} \cdot \frac{1}{2} \right) \right)
\]

is a critical point of \( F_N \) but not a minimizer of \( F_N \). In fact, this critical point becomes arbitrarily bad as \( N \to +\infty \) in the sense that

\[
\lim_{N \to +\infty} \frac{F_N(Y_N)}{\min F_N} = +\infty.
\]

On the other hand, we note that the point cloud \( Y_N \) is highly concentrated, in the sense that the distance between consecutive points in \( Y_N \) is \( \frac{1}{2N} \), whereas in an evenly distributed point cloud, one would expect the minimum distance between points to be of order \( N^{-1/d} \).

**Gradient descent and Lloyd’s algorithm** One can find a critical point of \( F_N \) by following the discrete gradient flow of \( F_N, \) defined in (7), starting from an initial position \( Y^0 \in (\mathbb{R}^d)^N \setminus \mathbb{D}_N \). Thanks to the expression of \( \nabla F_N \) given in Proposition 1, the discrete gradient flow may be written as

\[
Y^{k+1} = Y^k + \tau_N (B_N(Y^k) - Y^k),
\]

(13) where \( \tau_N \) is a fixed time step. For \( \tau_N = 1 \), one recovers a variant of Lloyd’s algorithm, where one moves every point to the barycenter of its Power cell \( P_i(Y^k) \) at each iteration:

\[
Y^{k+1} = B_N(Y^k)
\]

(14)

We can state the following result about Lloyd’s algorithm for the uniform quantization problem, whose proof is postponed to the appendix.

**Proposition 2.** Let \( N \) be a fixed integer and \( (Y^k)_{k \geq 0} \) be the iterates of (14), with \( Y^0 \not\in \mathbb{D}_N \). Then, the energy \( k \mapsto F_N(Y^k) \) is decreasing, and \( \lim_{k \to +\infty} \| \nabla F_N(Y^k) \| = 0 \). Moreover, the sequence \( (Y^k)_{k \geq 0} \) belongs to a compact subset of \( (\mathbb{R}^d)^N \setminus \mathbb{D}_N \) and every limit point of a converging subsequence of it is a critical point for \( F_N \).
Experiments suggest that following the discrete gradient flow of $F_N$ does not bring us to high energy critical points of $F_N$, such as those described in Remark 2, unless we started from an adversely chosen point cloud. The following theorem and its corollaries, the main results of this article,backs up this experimental evidence. It shows that if the point cloud $Y$ is not too concentrated, then the uniform measure over the barycenters of the power cells, $\delta_{B_N(Y)}$, is a good quantization of the probability density $\rho$, i.e. it bounds the quantization error after one step of Lloyd’s algorithm [14].

We will now prove an upper bound on the sum of the diameters of the cells

\[ \text{We now estimate the transport cost between} \]

\[ \text{The Minkowski sum in the left-hand side contains in particular the product of a} \]

\[ \text{Let} \]

\[ \text{Using that the Power cells} \]

\[ \text{For any index} \]

\[ \text{Indeed, let} \]

\[ \text{Expanding the squares and subtracting} \]

\[ \text{We will now prove an upper bound on the sum of the diameters of the cells} \]

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Note that we used $\rho(P(Y)) = \frac{1}{N}$. On the other hand, the transport cost associated with indices in $I_c(Y)$ can be bounded using (17) and $\text{diam}(P(Y)) \leq \text{diam}(\Omega)$:

$$\sum_{i \in I_c(Y)} \int_{P(Y)} \|x - y_i\|^2 d\rho(x) \leq \frac{1}{N} \sum_{i \in I_c(Y)} \text{diam}(P(Y))^2$$

$$\leq \frac{1}{N} \text{diam}(\Omega) \sum_{i \in I_c} \text{diam}(P(Y))$$

$$\leq \frac{2^{d-1}}{\omega_d} \left( \text{diam}(\Omega) + 1 \right)^{d-1} \varepsilon^{1-d} \frac{1}{N} + \text{diam}(\Omega)^2 \left( 1 - \frac{\text{Card} I_c}{N} \right). \quad \Box$$

In conclusion, we obtain the desired estimate:

$$W_2^2(\rho, \delta_{B_N(Y)}) \leq \frac{2^{d-1}}{\omega_d} \left( \text{diam}(\Omega) + 1 \right)^{d-1} \varepsilon^{1-d} \frac{1}{N} + \text{diam}(\Omega)^2 \left( 1 - \frac{\text{Card} I_c}{N} \right).$$

This theorem could be extended mutatis mutandis to the case where $\rho$ is a general probability measure (i.e. not a density). However, this would imply some technical complications in the definition of the barycenters $b_i$ by introducing a disintegration of $\rho$ with respect to the transport plan $\pi$.

**Consequence for Lloyd’s algorithm** ([14]). In the next corollary, we assume that any pair of distinct points in $Y_N \in (\mathbb{R}^d)^N$ is bounded from below by $\varepsilon_N \geq C N^{-\beta}$, implying that $I_c(Y_N) = N$. This corresponds to the value one could expect for a point set uniformly sampled from a set with Minkowski dimension $\beta$. When $\beta > d - 1$, the corollary asserts that one step of Lloyd’s algorithm is enough to approximate $\rho$, in the sense that the uniform measure $\delta_{B_N(Y)}$ over the barycenters converges towards $\rho$ as $N \to +\infty$.

**Corollary 4** (Quantization by barycenters, asymptotic case). Assume $\varepsilon_N \geq C \cdot N^{-1/\beta}$ with $C, \beta > 0$. Then, with $\alpha = 1 - d/\beta$

$$\forall Y \in (\mathbb{R}^d)^N \setminus D_{\varepsilon_N}, \quad W_2^2(\rho, \delta_{B_N(Y)}) \leq \frac{C d \varepsilon N^{1-\alpha}}{C d - 1} N^{-\alpha}, \quad (18)$$

and in particular, if $\beta > d - 1$,

$$\lim_{N \to +\infty} \max_{Y \in (\mathbb{R}^d)^N \setminus D_{\varepsilon_N}} W_2^2(\rho, \delta_{B_N(Y)}) = 0. \quad (19)$$

**Remark 3** (Optimality of the exponent when $\beta = d$). There is no reason to believe that the exponent in the upper bound (18) is optimal in general. However, it seems to be optimal in a “worst-case sense” when $\beta = d$. More precisely, we show the following result in Appendix 3 for any $\delta \in (0, 1)$, and for every $N = n^d$ ($n \in \mathbb{N}$) there exists a sequence of separable probability densities $\rho_N$ over $X = [-\varepsilon, 1]^d$ ($\rho_N$ is a truncated Gaussian distributions, whose variance converges to zero slowly as $N \to +\infty$) such that if $Y_N$ is a uniform grid of size $n \times n \times n = N^d$ in $X$, then

$$W_2^2(\delta_{B_N(Y_N)}, \rho_N) \geq C N^{-\frac{2-\delta}{d}}$$

where $C$ is independent of $N$. On the other hand, in this setting every point in $Y_N$ is at distance at least $C N^{-1/d}$ from any other point in $Y_N$. Applying Corollary 4 with $\beta = d$, i.e. $\alpha = \frac{1}{d}$, we get

$$W_2^2(\delta_{B_N(Y_N)}, \rho_N) \leq C N^{1-\frac{2}{d}}.$$ 

Comparing this upper bound on $W_2^2(\delta_{B_N(Y_N)}, \rho_N)$ with the above lower bound, one sees that is is not possible to improve the exponent.

**Remark 4** (Optimality of (19)). The assumption $\beta > d - 1$ for (19) is tight: if $\rho$ is the Lebesgue measure on $[0,1]^d$, it is possible for to construct a point cloud $Y_N$ with $N$ points on the $(d-1)$-cube $(\frac{1}{2}) \times [0,1]^{d-1}$ such that distinct point in $Y_N$ are at distance at least $\varepsilon_N \geq C \cdot N^{-1/(d-1)}$. Then, the barycenters $B_N(Y_N)$ are also contained in the cube, so that $W_2^2(\rho, \delta_{B_N(Y_N)}) \geq \frac{1}{12}$.

The next corollary is a probabilistic analogue of Corollary 4 assuming that the initial point cloud $Y$ is drawn from a probability density $\sigma$ on $\Omega$. Note that $\sigma$ can be distinct from $\rho$. The proof of this corollary relies on McDiarmid’s inequality to quantify the proportion of $\varepsilon$-isolated points in a point cloud that is drawn randomly and independently from $\sigma$. The proof of this result is in Appendix 5.
We note that this inequality ensures that when
\[ ε \leq \frac{1}{N} \] 
where \( C \) is a positive constant. This inequality has been originally used by Polyak to prove convergence of gradient descent towards the global minimum of \( F \). Note in particular that such an inequality implies that any critical point of \( F \) is a global minimum of \( F \). By Remark 2, \( F_N \) has critical points that are not minimizers, so that we cannot expect the standard PŁ inequality to hold. What we get is a similar inequality relating \( F_N(Y) \) and \( \| \nabla F_N(Y) \| \) but with a term involving the minimum distance between the points in place of \( \min Y \).

**Corollary 5** (Quantization by barycenters, probabilistic case). Let \( σ \in L^∞(Ω) \) and let \( X_1, \ldots, X_N \) be i.i.d. random variables with distribution \( σ \in L^∞(R^d) \). Then, there exists a constant \( C > 0 \) depending only on \( \| σ \|_{L^∞} \) and \( d \), such that for \( N \) large enough,

\[ P \left( W_2^2 \left( \frac{1}{N} \sum_{i=1}^N δ_{X_i}, ρ \right) \leq N^{-\frac{1}{d+1}} \right) \geq 1 - e^{-CN^{\frac{3d+3}{d+1}}} \]

### 3 Gradient flow and a Polyak-Łojasiewicz-type inequality

Theorem 3 can be interpreted as a modified Polyak-Łojasiewicz-type (PŁ for short) inequality for the function \( F_N \). The usual PŁ inequality for a differentiable function \( F : R^D \rightarrow R \) is of the form

\[ \forall Y \in R^D, \quad F(Y) - \min F \leq C\| \nabla F(Y) \|^2, \]

where \( C \) is a positive constant. This inequality has been originally used by Polyak to prove convergence of gradient descent towards the global minimum of \( F \). Note in particular that such an inequality implies that any critical point of \( F \) is a global minimum of \( F \). By Remark 2, \( F_N \) has critical points that are not minimizers, so that we cannot expect the standard PŁ inequality to hold. What we get is a similar inequality relating \( F_N(Y) \) and \( \| \nabla F_N(Y) \| \) but with a term involving the minimum distance between the points in place of \( \min Y \).

**Corollary 6** (Polyak-Łojasiewicz-type inequality). Let \( Y \in (R^d)^N \setminus D_N(ε) \). Then,

\[ F_N(Y) - C_{d, l} \left( \frac{1}{N} \right)^{d-1} \leq N\| \nabla F_N(Y) \|^2 \quad (20) \]

We note that when \( ε \simeq \left( \frac{1}{N} \right)^{1/d} \), the term \( \frac{1}{N} \left( \frac{1}{N} \right)^{d-1} \) in (20) has order \( \left( \frac{1}{N} \right)^{1/d} \). On the other hand, as recalled in Remark 1, \( \min F_N \simeq \left( \frac{1}{N} \right)^{2/d} \) when \( d > 2 \). Thus, we do not expect (20) to be tight.

**Convergence of a discrete gradient flow** The modified Polyak-Łojasiewicz inequality (20) suggests that the discrete gradient flow (13) will bring us close to a point cloud with low Wasserstein distance to \( ρ \), provided that can guarantee that the the points clouds \( Y^k \) remain far for generalized diagonal during the iterations. We prove in Lemma 5 in Appendix B that if \( Y^{k+1} = Y^k - τ_N \nabla F_N(Y^k) \) and \( τ_N \in (0, 1) \), then

\[ \forall i \neq j, \quad \| y_i^{k+1} - y_j^{k+1} \| \geq (1 - τ_N) \| y_i^k - y_j^k \|. \quad (21) \]

We note that this inequality ensures that \( Y^k \) never touches the generalized diagonal \( D_N \), so that the gradient \( \nabla F_N(Y^k) \) is well-defined at each step. Combining this inequality with Theorem 3, one can
Then, let Remark 5

As we mentioned in the introduction, uniform optimal quantization allows to distribute in $Y$ is a normalization constant. On the left column of this figure, the initial point clouds distributed in $\mathbb{R}^d$. We now consider a toy model where we approximate a Gaussian density with small variance $\sigma$. Darker $[4, 3]$. On figure 3, we plotted the point clouds obtained after a single Lloyd step toward the convergence, density representing the image on the left (Puffin), starting from regular grids. The observed rate of discrepancy between the exponent of $N$ as a function of $N$ the number of points, showing that $W_2^2(\rho, \delta_{B_N}) \simeq N^{-1.00}$.

Figure 3: Optimal quantization of a density $\rho$ corresponding to a gray-scale image (Wikimedia Commons, CC BY-SA 3.0). (Left) We display the point clouds obtained after one step of Lloyd’s algorithm, starting from a regular grid of size $N \in \{3750, 7350, 15000, 43350\}$. (Right) Quantization error $W_2^2(\rho, \delta_{B_N})$ as a function of $N$ the number of points, showing that $W_2^2(\rho, \delta_{B_N}) \simeq N^{-1.00}$.

actually prove that if the points in the initial cloud $Y_N^0$ are not too close to each other, then a few steps of gradient discrete gradient descent leads to a discrete measure $Y_N^{k}$ that is close to the target $\rho$. Precisely, we arrive at the following theorem, proved in Appendix D.

**Theorem 7.** Let $0 < \alpha < \frac{1}{d(d-1)}$, $\varepsilon N \geq N^{-\frac{d}{2} - \alpha}$, and $Y_N^0 \in \Omega \setminus \mathbb{D}_{\varepsilon N}$. Let $(Y_N^{k})_k$ be the iterates of (13) starting from $Y_N^0$ with timestep $0 < \tau_N < 1$. We assume that $\lim_{N \to \infty} \tau_N = 0$ and we set

$$ k_N = \left[ \frac{1}{d\tau_N} \ln(F_N(Y_N^0)N_{\varepsilon}^{-k-1}) \right]. $$

Then,

$$ W_2^2(\rho, \delta_{Y_N^{k}}) = O_{N \to \infty} \left( W_2^2(\rho, \delta_{Y_N^{0}})^{-\frac{d}{2}} \cdot N_{\varepsilon}^{-\frac{d}{2} + \alpha(1 - \frac{1}{2})} \right). \tag{22} $$

**Remark 5.** Note that the exponential behavior implied by (21) and Lemma 3 is coherent with the estimates that are known in the absolutely continuous setting for the continuous gradient flow. When transitioning from discrete measures to probability densities, lower bounds on the distance between points become upper bounds on the density. The gradient flow $\mu_t = \frac{1}{2} \nabla_{\rho} W_2^2(\rho, \mu_t)$ has an explicit solution $\mu_t = \sigma_{1-\varepsilon - t}$, where $\sigma$ is a constant-speed geodesic in the Wasserstein space with $\sigma_0 = \mu_0$ and $\sigma_1 = \rho$. In this case, a simple adaptation of the estimates in Theorem 2 in [17] shows the bound $\|\mu_t\|_{L^\infty} \leq e^{d\|\mu_0\|_{L^\infty}}$. Still in this absolutely continuous setting, it is possible to remove the exponential growth if the target density is also bounded, as a consequence of displacement convexity [13, Theorem 2.2]. There seems to be no discrete counterpart to this argument, explaining in part the discrepancy between the exponent of $N$ in (22) with the one obtained in Corollary 4.

### 4 Numerical results

In this section, we report some experimental results in dimension $d = 2$.

**Gray-scale image** As we mentioned in the introduction, uniform optimal quantization allows to sparsely represent a (gray scale) image via points, clustered more closely in areas where the image is darker [3][3]. On figure 3 we plotted the point clouds obtained after a single Lloyd step toward the density representing the image on the left (Puffin), starting from regular grids. The observed rate of convergence, $N^{-1.50}$, is coherent with the theoretical estimate $\log(N)/N$ of Remark 1.

**Gaussian density with small variance** We now consider a toy model where we approximate a gaussian density truncated to the unit square $\Omega = [0, 1]^2$, $\rho(x, y) = \frac{1}{2} e^{-8((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2)}$ where $Z$ is a normalization constant. On the left column of this figure, the initial point clouds $Y_N^0$ are randomly distributed in $[0, 1]^2$. The three point clouds represented above are obtained after one step of Lloyd’s algorithm [14]. The red curve displays in a log-log scale the mean values of $F_N(B_N(Y_N))$ over a hundred random point clouds, for $N \in \{400, 961, 1600, 2500\}$. In this case, we observe a decrease rate $N^{-0.95}$ with respect to the number of points, similar to the case of the gray scale picture.
However, an interesting phenomena occurs when the initial point cloud $Y_0^N$ is aligned on a axis-aligned grid. The pictures in the right column of Fig. 2 where computed starting from such a grid with $N \in \{400, 961, 1600, 2500\}$ points. As in the randomly initialized case, we represented the values of $F_N(B_N(Y_N))$ in log-log scale. The corresponding discrete probability measure $\delta_{B_N(Y_N)}$ seems to converge to $\rho$ as $N \to \infty$, but with a much worse rate for these "low" values of $N$: $F_N(B_N(Y_N)) \simeq N^{-0.8}$. In this specific setting, with a separable density and an axis-aligned grid $Y_0$, the power cells are rectangles and a single Lloyd step brings us to a critical point of $F_N$. Thanks to this remark, it is possible to estimate the approximation error from the one-dimensional case. In fact, Appendix E shows that for any $\delta \in (0, 1)$, there exists variances $\sigma_N = \sigma_N(\delta)$ such that the approximation error $W_2^2(\rho_{\sigma_N}, \delta_{B_N})$ is of order $N^{-2\delta - 1}$. On the other hand, for a fixed $\sigma$, the approximation error is of order $N^{-1}$, to be compared with the bound $\log(N)/N$ for general measures.

5 Discussion

We have studied the problem of minimizing the Wasserstein distance between a fixed probability measure $\rho$ and a uniform measure over $N$ points $\delta_Y$, parametrized by the position of the points $Y = (y_1, \ldots, y_N)$. The main difficulty is the nonconvexity of the Wasserstein distance $F_N : Y \in (\mathbb{R}^d)^N \mapsto \frac{1}{2} W_2^2(\rho, \delta_Y)$, which we tackled by means of a modified Polyak-Łojaciewicz inequality \cite{20}. One limitation of our work is that the terms replacing $\min F_N$ in the Polyak-Łojaciewicz inequality \cite{20} does not match the theoretical bounds recalled in Remark \cite{1}. Future work will concentrate on bridging that gap, but also on deriving consequences for the algorithmic resolution of Wasserstein regression problems $\min_{\theta} W_2^2(\rho, T_{\theta}#\mu)$, starting with the case where $\theta \mapsto T_{\theta}$ is linear.

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A Proof of Proposition 2

Given $Y = (y_1, \ldots, y_N) \in (\mathbb{R}^d)^N \setminus \mathbb{D}_N$, one has for any $i \in \{1, \ldots, N\}$,

$$
\int_{P_i(Y)} \|x - y_i\|^2 d\rho(x) = \int_{P_i(Y)} \|x - b_i(Y) + b_i(Y) - y_i\|^2 d\rho(x)
= \int_{P_i(Y)} \|x - b_i(Y)\|^2 d\rho(x) + \frac{1}{N} \|b_i(Y) - y_i\|^2.
$$

Summing these equalities over $i$ and remarking that the map $T_Y$ defined by $T_Y|_{P_i(Y)} = y_i$ is an optimal transport map between $\rho$ and $\delta_{Y_i}$, we get

$$
\frac{1}{N} \|B_N(Y) - Y\|^2 = W_2^2(\rho, y_i) - \sum_{i} \int_{P_i(Y)} \|x - b_i(Y)\|^2 d\rho(x)
\leq W_2^2(\rho, \delta_Y) - W_2^2(\rho, \delta_{B_N(Y)}).
$$

Thus, with $Y^{k+1} = B_N(Y^k)$, we have

$$
N \|\nabla F_N(Y^{k+1})\|^2 = \frac{1}{N} \|Y^{k+1} - Y^k\|^2 \leq 2(F_N(Y^k) - F_N(Y^{k+1})).
$$

This implies that the values of $F_N(Y^k)$ are decreasing in $k$ and, since they are bounded from below, that $\|\nabla F_N(Y^k)\| \to 0$ since $\sum_k \|\nabla F_N(Y^k)\|^2 < +\infty$. The sequence $(Y^{k})_k$ can be easily seen to be bounded, since $F_N(Y^k)$ is bounded, which implies a bound on the second moment of $\delta_{Y^k}$.

For fixed $N$, since all atoms of $\delta_{Y^k}$ have mass $1/N$, this implies that all points $y^k_i$ belong to a same fixed compact ball. If $\rho$ itself is compactly supported, we can also prove that all points $Y^{k+1} = B_N(Y^k)$ are contained in a compact subset of $(\mathbb{R}^d)^N \setminus \mathbb{D}_N$, which means obtaining a lower bound on the distances $|b_i(Y) - b_j(Y)|$ for arbitrary $Y$. This lower bound can be obtained in the following way: since $\rho$ is absolutely continuous it is uniformly integrable which means that for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any set $A$ with Lebesgue measure $|A| < \delta$ we have $\rho(A) < \varepsilon$. We claim that we have $|b_i(Y) - b_j(Y)| \geq r := (2R)^{1-d} \delta(\frac{1}{2N})$, where $R$ is such that $\rho$ is supported in a ball $B_R$ of radius $R$. Indeed, it is enough to prove that every barycenter $b_i(Y)$ is at distance at least $r/2$ from each face of the convex polytope $P_i(Y)$. Consider a face of such a polytope and suppose, by simplicity, that it lies on the hyperplane $\{x_d = 0\}$ with the cell contained in $\{x_d \geq 0\}$. Let $s$ be such that $\rho(P_i(Y) \cap \{x_d > s\}) = \rho(P_i(Y) \cap \{x_d < s\}) = \frac{1}{2N}$. Then since the diameter of $P_i(Y) \cap B_R$ is smaller than $2R$, the Lebesgue measure of $P_i(Y) \cap \{x_d < s\}$ is bounded by $(2R)^{d-1}s$, which provides $s \geq r$ because of the definition of $r$. Since at least half of the mass (according to $\rho$) of the cell $P_i(Y)$ is above the level $x_d = s$ the $x_d$-coordinate of the barycenter is at least $r/2$. This shows that the barycenter lies at distance at least $r/2$ from each of its faces.

As a consequence, the iterations $Y^k$ of the Lloyd algorithm lie in a compact subset of $(\mathbb{R}^d)^N \setminus \mathbb{D}_N$, on which $F_N$ is $C^1$. This implies that any limit point must be a critical point.

We do not discuss here whether the whole sequence converges or not, which seems to be a delicate matter even for fixed $N$. It is anyway possible to prove (but we do not develop the details here) that the set of limit points is a closed connected subset of $(\mathbb{R}^d)^N$ with empty interior, composed of critical points of $F_N$ all lying on a same level set of $F_N$.

B Proof of Corollary 5

Given $Y = (y_1, \ldots, y_N) \in (\mathbb{R}^d)^N$, we denote

$$
I_\varepsilon(Y) = \{i \in \{1, \ldots, N\} \mid \forall j \neq i, \|y_i - y_j\| \geq \varepsilon\}.
$$

We call points $y_i$ such that $i \in I_\varepsilon(Y)$ $\varepsilon$-isolated, and points $y_i$ such that $i \not\in I_\varepsilon(Y)$ $\varepsilon$-connected.

**Lemma 1.** Let $X_1, \ldots, X_N$ be independent, $\mathbb{R}^d$-valued, random variables. Then, there is a constant $C_d > 0$ such that

$$
\mathbb{P}(\{\|\kappa(X_1, \ldots, X_N) - \mathbb{E}(\kappa)\| \geq \eta\}) \leq e^{-N\eta^2/C_d}.
$$


We conclude by noting that

\[|\kappa(x_1, \ldots, x_i, \ldots, x_N) - \kappa(x_1, \ldots, \hat{x}_i, \ldots, x_N)|,\]

we first note that at most \(c_d\) points may become \(\varepsilon\)-isolated when removing \(x_i\). To prove this, we remark that if a point \(x_j\) becomes \(\varepsilon\)-isolated when \(x_i\) is removed, this means that \(|x_i - x_j| \leq \varepsilon\) and \(|x_j - x_k| > \varepsilon\) for all \(k \neq \{i, j\}\). The number of such \(j\) is bounded by \(c_d\). Symmetrically, there may be at most \(c_d\) points becoming \(\varepsilon\)-connected under addition of \(\hat{x}_i\). Finally, the point \(x_i\) itself may change status from \(\varepsilon\)-isolated to \(\varepsilon\)-connected. To summarize, we obtain that with \(C_d = 2c_d + 1\),

\[|\kappa(x_1, \ldots, x_i, \ldots, x_N) - \kappa(x_1, \ldots, \hat{x}_i, \ldots, x_N)| \leq \frac{1}{N} C_d.\]

The conclusion then directly follows from McDiarmid’s inequality.

Lemma 2. Let \(\sigma \in L^\infty(\mathbb{R}^d)\) be a probability density and let \(X_1, \ldots, X_N\) be i.i.d. random variables with distribution \(\sigma\). Then,

\[\mathbb{E}(|\kappa(X_1, \ldots, X_N)|) \geq (1 - \|\sigma\|_{L^\infty} \omega_d d^d)^{N-1}.\]

Proof. The probability that a point \(X_i\) belongs to the ball \(B(X_j, \varepsilon)\) for some \(j \neq i\) can be bounded from above by \(\sigma(B(X_j, \varepsilon)) \leq \|\sigma\|_{L^\infty} \omega_d d^d\), where \(\omega_d\) is the volume of the \(d\)-dimensional unit ball. Thus, the probability that \(X_i\) is \(\varepsilon\)-isolated is larger than

\[(1 - \|\sigma\|_{L^\infty} \omega_d d^d)^{N-1}.\]

We conclude by noting that

\[\mathbb{E}(\kappa(X_1, \ldots, X_N)) = \frac{1}{N} \sum_{1 \leq i \leq N} \mathbb{P}(X_i \text{ is } \varepsilon\text{-isolated}).\]

Proof of Corollary 3. We apply the previous Lemma 2 with \(\varepsilon_N = N^{-\beta}\) and \(\beta = d - \frac{1}{2}\). The expectation of \(\kappa(X_1, \ldots, X_N)\) is lower bounded by:

\[\mathbb{E}(\kappa(X_1, \ldots, X_N)) \geq \left(1 - N^{-\beta} \|\sigma\|_{L^\infty} \omega_d \right)^{N-1}

\geq 1 - CN^{1 - \frac{d}{2}}\]

for large \(N\), since \(\beta < d\). By Lemma 1 for any \(\eta > 0\),

\[\mathbb{P}(\kappa(X_1, \ldots, X_N) \geq 1 - CN^{1 - \frac{d}{2}} - \eta) \geq 1 - e^{-K N \eta^2},\]

for constants \(C, K > 0\) depending only on \(\|\sigma\|_{L^\infty}\) and \(d\). We choose \(\eta = N^{-\frac{1}{2d-1}}\), so that \(\eta\) is of the same order as \(N^{1 - \frac{d}{2}}\) since \(1 - \frac{d}{2} = -\frac{1}{2d-1}\). Thus, for a slightly different \(C\),

\[\mathbb{P}(\kappa(X_1, \ldots, X_N) \geq 1 - C \eta) \geq 1 - e^{-K N \eta^2}.\]

Now, for \(\omega_1, \ldots, \omega_N\) such that

\[\kappa(X_1(\omega_1), \ldots, X_N(\omega_N)) \geq 1 - C \eta,\]

Theorem 3 yields:

\[W_2^2(\delta_{BN(X(\omega))}, \rho) \lesssim \frac{N^{d-1}}{N} + \eta \lesssim N^{-\frac{1}{2d-1}}\]

and such a disposition happens with probability at least

\[1 - e^{-KN \eta^2} = 1 - e^{-KN \frac{1}{2d-1}}.\]
Proof of Corollary 6

We first note that by Proposition 1, we have \( \|\nabla F_N(Y)\|^2 = \frac{1}{N^d} \|B_N(Y) - Y\|^2 \). We then use \( W^2(\delta_B(Y), \delta_Y) \leq \frac{1}{N^d} \|B_N(Y) - Y\|^2 \) and
\[
W^2(\rho, \delta_Y) \leq 2(W^2(\rho, \delta_B(Y)) + 2N\|\nabla F_N(Y)\|^2).
\]
Thus, using Theorem 3 to bound \( W^2(\rho, \delta_B(Y)) \) from above, we get the desired result.

D Proof of Theorem 7

Lemma 3. Let \( Y^0 \in (\mathbb{R}^d)_N \setminus \mathbb{D}_{N,\varepsilon_N} \) for some \( \varepsilon_N > 0 \). Then, the iterates \((Y^k)_{k \geq 0}\) of (13) satisfy for every \( k \geq 0 \), and for every \( i \neq j \)
\[
\|y^k_i - y^k_j\| \geq (1 - \tau_N)^k \varepsilon_N
\]  
(23)
Proof. We consider the distance between two trajectories after \( k \) iterations: \( e_k = \|y^k_i - y^k_j\| \). Assuming that \( e_k > 0 \), the convexity of the norm immediately gives us:
\[
eq k \geq (y^k_i - y^k_j) \cdot (y^{k+1}_i - y^{k+1}_j - (y^k_i - y^k_j))
\]
\[
eq k \geq \tau_N \left( \frac{y^k_i - y^k_j}{\|y^k_i - y^k_j\|} \right) \cdot (b^k_i - b^k_j) \geq \tau_N \|y^k_i - y^k_j\|
\]
where we denoted \( b^k_i := b_i(Y^k_N) \) the barycenter of the \( i \)th Power cell \( P_i(Y^k_N) \) in the tessellation associated with the point cloud \( Y^k_N \). Since each barycenter \( b^k_i \) lies in its corresponding Power cell, the scalar product \( (y^k_i - y^k_j) \cdot (b^k_i - b^k_j) \) is non-negative: Indeed, for any \( i \neq j \),
\[
\|y^k_i - b^k_i\|^2 - \|y^k_j - b^k_j\|^2 \leq \phi^k_i - \phi^k_j
\]
Summing this inequality with the same inequality with the roles of \( i \) and \( j \) reversed, we obtain:
\[
(y^k_i - y^k_j) \cdot (b^k_i - b^k_j) \geq 0
\]
thus giving us the geometric inequality \( e_{k+1} \geq (1 - \tau_N)e_k \). Since \( Y^0_N \) was chosen in \( \Omega_N \setminus \mathbb{D}_{N,\varepsilon_N} \), this yields \( e_k \geq (1 - \tau_N)^k e_0 \) and inequality (23).

Lemma 4. For any \( k \geq 0 \)
\[
F_N(Y^k_N) \leq F_N(Y^0_N) + 2C_{d,\Omega}(1 - \eta_N)\frac{\varepsilon^1_{N}}{N} \frac{A^1_{N} - \eta^1_{N}}{A_N - \eta_N}
\]  
(24)
where we denote \( \eta_N = 1 - \frac{\tau_N}{2}(2 - \tau_N) \) and \( A_N = (1 - \tau_N)^{1-d} \).
Proof. This is obtained in a very similar fashion as Lemma 3. For any \( k \geq 0 \), the semi-concavity of \( F_N \) yields the inequality:
\[
F_N(Y^{k+1}_N) - \frac{\|Y^{k+1}_N\|^2}{2N} - \left( F_N(Y^k_N) - \frac{\|Y^k_N\|^2}{2N} \right) \leq \left( -\frac{B^k_N}{N} \right) \cdot (Y^{k+1}_N - Y^k_N)
\]
with \( B^k_N := B_N(Y^k_N) \) in accordance with the previous proof.
Rearranging the terms,
\[
F_N(Y^{k+1}_N) - F_N(Y^k_N) \leq -\tau_N \left( 1 - \frac{\tau_N}{2} \right) \frac{\|B^k_N - Y^k_N\|^2}{N}
\]
\[
= -\tau_N \left( 1 - \frac{\tau_N}{2} \right) W^2(\delta_B(Y^k_N), \delta_Y^k)
\]
\[
\leq \tau_N \left( 1 - \frac{\tau_N}{2} \right) \left( -\frac{1}{2} W^2(\delta_Y^k, \rho) + W^2(\rho, \delta_B^k) \right)
\]
14
We make use here of the notation from Section 3: From the proof, one can see that the dependence of \( \sigma \) on \( k \) and \( N \) and we simply iterate on \( k \) to end up with the bound claimed in Lemma 4.

**Proof of Theorem 7.** To conclude, we simply make (order 1) expansions of the terms in 24. The definition of \( k_N \) in Theorem 7, although convoluted, was made so that both terms in the right-hand side of this inequality, \( F_N(Y_N^0)\eta_N^{k_N} \) and \( (1 - \eta_N)\frac{e^{-\frac{1}{2}d}}{A_N} \eta_N^{k_N} \), have the same asymptotic decay to 0 (as \( N \to +\infty \)): With the notations of the previous proposition, we have for fixed \( N \):

\[
W_2^2(\mu, \delta_{Y_N^{k_N}}) \leq W_2^2(\mu, \delta_{Y_N^0}) \eta_N^{k_N} + 2C_d,\Omega (1 - \eta_N) \frac{e^{-\frac{1}{2}d}}{A_N} \eta_N^{k_N} \frac{1}{N^{\varepsilon - 1}} \tag{25}
\]

We make use here of the notation from Section 3:

\[
T_N = k_N \tau_N = \left[ \frac{1}{d} \ln(F_N(Y_N^0)N^{-\frac{d-1}{2}}) \right]
\]

to clear this expression a bit, and, because of the assumption \( \lim_{N \to \infty} \tau_N = 0 \), we may write:

\[
\frac{A_N^{k_N} - \eta_N^{k_N}}{N^{\varepsilon - 1}} = \frac{e^{(d-1)T_N}}{N^{\varepsilon - 1}} + o_{N \to \infty} \left( \frac{T_N}{(N^{\varepsilon - 1})^{\frac{1}{2}}} \right)
\]

as well as \( \eta_N^{k_N} = e^{-T_N} + o_{N \to \infty} \left( \frac{T_N}{(N^{\varepsilon - 1})^{\frac{1}{2}}} \right) \), and substituting \( T_N \):

\[
W_2^2(\mu, \delta_{Y_N^{k_N}}) \lesssim W_2^2(\mu, \delta_{Y_N^0}) \frac{d-1}{N^{\varepsilon - 1}} + o_{N \to \infty} \left( \frac{T_N}{(N^{\varepsilon - 1})^{\frac{1}{2}}} \right) \leq W_2^2(\mu, \delta_{Y_N^0}) \frac{d-1}{N^{\varepsilon - 1}} + \alpha(1 - \frac{1}{2}) \]

**E  Case of a low variance Gaussian in Section 4**

Here, we consider \( \rho_\sigma \) the probability measure obtained by truncating and renormalizing a centered normal distribution with variance \( \sigma \) to the segment \([-1, 1]\). We first show that for any \( N \in \mathbb{N} \) and \( \delta \in (0, 1) \), we can find a small \( \sigma_{N, \delta} \) such that the Wasserstein distance between \( \rho_{\sigma_{N, \delta}} \) and its best \( N \)-points approximation of is at least \( CN^{-2(\delta-\delta)} \).

**Proposition 8.** For any \( \sigma > 0 \), consider \( \rho_\sigma \) defined as the truncated centered Gaussian density, where \( m_\sigma \) is taken so that \( \rho_\sigma \) has unit mass. Then, for every \( \delta \in (0, 1) \), there exists a constant \( C > 0 \) and a sequence of variances \( \{\sigma_N\}_{N \in \mathbb{N}} \) such that

\[
\forall Y \in (\mathbb{R}^d)^N \setminus \mathbb{D}_N, \quad W_2^2(\delta_{B_N(Y)}, \rho_{\sigma_N}) \geq CN^{-2(\delta-\delta)}
\]

From the proof, one can see that the dependence of \( \sigma_N \) on \( N \) is logarithmic.

**Proof.** We denote \( g : x \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}} \) the density of the centered Gaussian distribution and \( F_g \) its cumulative distribution function, so that

\[
m_\sigma^{-1} = \int_{-1}^{1} e^{-\frac{|x|^2}{2\sigma^2}} dx = \sigma \sqrt{2\pi} \int_{-1/\sigma}^{1/\sigma} g(y) dy = \sqrt{2\pi} \sigma (F_g(1/\sigma) - F_g(-1/\sigma)) \tag{26}
\]
Note that, whenever $\sigma \to 0$, we have $\sigma m_\sigma \to \sqrt{2\pi}$. We denote by $F_\sigma : [-1,1] \to [0,1]$ the cumulative distribution function of $\rho_\sigma$. Given any point cloud $Y = (y_1, \ldots, y_N)$ such that $y_1 \leq \ldots \leq y_N$, the Power cells $P_i(Y)$ is simply the segment

$$P_i(Y) = [F_\sigma^{-1}(i/N), F_\sigma^{-1}((i+1)/N)].$$

Since these segments do not depend on $Y$, we will denote them $(P_i)_{1 \leq i \leq N}$. Finally, defining $b_i = N \int_{P_i} x d\rho_\sigma(x)$ as the barycenter of the $i$th power cell and $\delta_B = \frac{1}{N} \sum_i \delta_{b_i}$, we have

$$W_2^2(\delta_B, \rho_\sigma) = \sum_{i=1}^{N} \int_{P_i} (x - b_i)^2 d\rho_\sigma(x)$$

$$\geq \rho_\sigma(-1) \sum_{i=1}^{N} \int_{P_i} (x - b_i)^2 dx$$

$$\geq C \rho_\sigma(-1) \sum_{i=1}^{N} (F_\sigma^{-1}((i+1)/N) - F_\sigma^{-1}(i/N))^3,$$

where we used that $\rho_\sigma$ attains its minimum at $\pm 1$ to get the first inequality. We now wish to provide an approximation for $F_\sigma^{-1}(t)$, $t \in [0,1]$. We first note, using Taylor’s formula, that we have

$$F_\sigma^{-1}(t) = \sigma F_g^{-1}\left(\frac{1}{\sigma}\right) + t \left[ F_g\left(\frac{1}{\sigma}\right) - F_g\left(-\frac{1}{\sigma}\right)\right]$$

$$= \sigma F_g^{-1}\left(\frac{1}{\sigma}\right) + \frac{t}{\sqrt{2\pi} \sigma m_\sigma},$$

$$= -1 + \sigma (F_g^{-1})'(t) \left(\frac{1}{\sigma}\right) + \frac{t^2}{2\sigma^2 m_\sigma} + \frac{\sigma}{2} (F_g^{-1})''(t) \left(\frac{1}{\sigma}\right),$$

for some $s \in [F_g(-\frac{1}{\sigma}), F_g(\frac{1}{\sigma})]$. But,

$$(F_g^{-1})'(t) = \frac{1}{g \circ F_g^{-1}(t)} = \frac{1}{\sqrt{2\pi} \sigma} |F_g^{-1}(t)|,$$

$$(F_g^{-1})''(t) = -\frac{g' \circ F_g^{-1}(t)}{(g \circ F_g^{-1}(t))^3} = 2\pi F_g^{-1}(t) |F_g^{-1}(t)|,$$

and we see that

$$\left|F_\sigma^{-1}(t) - \left(-1 + \frac{t}{m_\sigma} e^{\frac{1}{2\pi^2}}\right)\right| \leq e^{\frac{1}{2\pi^2}} \frac{t^2}{2\sigma^2 m_\sigma^2}.$$

Therefore, if we denote $\varepsilon(\sigma, t)$ the second-order error in the above formula, i.e. $\varepsilon(\sigma, t) = e^{\frac{1}{2\pi^2}} \frac{t^2}{2\sigma^2 m_\sigma^2}$, the size of the first Power cell $P_0(Y)$ is of order:

$$F_\sigma^{-1}(1/N) - F_\sigma^{-1}(0) = \frac{1}{N m_\sigma} e^{\frac{1}{2\pi^2}} + O\left(\varepsilon\left(\sigma, \frac{1}{N}\right)\right).$$

We will choose $\sigma_N$ depending on $N$ in order for the first term in the left-hand side to dominate the second one:

$$\varepsilon\left(\sigma_N, \frac{1}{N}\right) = o\left(\frac{1}{N m_\sigma} e^{\frac{1}{2\pi^2}}\right).$$

In this way, we have

$$\left(F_\sigma^{-1}(1/N) - F_\sigma^{-1}(0)\right)^3 \rho_\sigma(-1) \geq c \frac{1}{N^3 m_\sigma^2} e^{\frac{3}{2\pi^2}} m_\sigma e^{-\frac{1}{2\pi^2}}$$

$$= c \frac{1}{N^3 m_\sigma^2} e^{\frac{1}{2\pi^2}}.$$

We now choose $\sigma = \sigma_N$ such that $e^{\frac{1}{2\pi^2}} = N^\alpha$ for an exponent $\alpha$ to be chosen. We need $\alpha > 0$ so that $\sigma_N \to 0$. This last condition and (26) implies that $m_{\sigma_N}$ is of order $\sqrt{\log N}$. This means that the condition (28) is satisfied if $\alpha < 1$ and $N$ large enough.
The sum in (27) is lower bounded by its first term, (29), and we get
\[ W_2^2(\delta_B, \rho_\sigma) \geq c \frac{1}{N^{3m_\sigma^2}} e^{\frac{1}{N}} \geq C \left( \frac{N^{2\alpha-3}}{\ln(N)} \right) \]
for some constant \( C > 0 \), since \( \sigma \) depends logarithmically on \( N \). Finally, if we want this last expression to be larger than \( N^{-(2-\delta)} \) we can take for instance \( 2\alpha > 1 + \delta \) and \( N \) large enough.

The following corollary, whose proof can just be obtained by adapting the above proof to a simple multi-dimensional setting where measures and cells “factorize” according to the components, confirms the facts observed in the numerical section (Section 4), and the sharpness of our result (Remark 4).

**Corollary 9.** Fix \( \delta \in (0, 1) \). Given any \( n \in \mathbb{N} \), consider an axis-aligned discrete grid of the form \( Z_N = Y_1 \times \ldots \times Y_d \) in \( \mathbb{R}^d \), with \( N = \text{Card}(Z_N) = n^d \), where each \( Y_j \) is a subset of \( \mathbb{R} \) with cardinal \( n \). Finally, define \( \sigma_N := \sigma_n,\delta \) as in Proposition 8. Then we have
\[ W_2^2(\delta_{B_N(Z_N)}, \rho_{\sigma_N} \otimes \cdots \otimes \rho_{\sigma_N}) \geq C N^{-\frac{(2-\delta)}{d}}, \]
where the constant \( C \) is independent of \( N \).