Heun and Mathieu functions as solutions of the Dirac equation

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Abstract
We give examples of where the Heun function exists as solutions of wave equations encountered in general relativity. While the Dirac equation written in the background of Nutku helicoid metric yields Mathieu functions as its solutions in four spacetime dimensions, the trivial generalization to five dimensions results in the double confluent Heun function. We reduce this solution to the Mathieu function with some transformations. We must apply Atiyah-Patodi-Singer spectral boundary conditions to this system since the metric has a singularity at the origin.

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1 Introduction

Most of the theoretical physics known today is described by a rather few number of differential equations. If we study only linear problems, the different special cases of the hypergeometric or the confluent hypergeometric equations often suffice to analyze scores of different phenomena. Both of these equations have simple recursive relations between consecutive coefficients when series solutions are sought. They also have simple integral representations. If the problem is nonlinear then one can often use one of the forms of the Painlevé equation. In the linear case, sometimes it is necessary to work with more complicated equations. Such an equation is the Heun equation, whose special forms we encounter as the Mathieu, Lamé, Coulomb spheroidal equations. These equations do not have two way recursion relations. There are several examples in the literature for this kind of equations [1] [2] [3].

We ”encountered” this equation when we tried to solve the scalar and Dirac equations in the background of the Nutku helicoid instanton [4] [5] [6] [7] [8].

2 Dirac Equation

The original four dimensional Nutku helicoid metric is given as

\[ ds^2 = \frac{a^2}{2} \sinh 2x (dx^2 + d\theta^2) + \frac{2}{\sinh 2x} [(\sinh^2 x + \sin^2 \theta)dy^2 - \sin 2\theta dz^2 + (\sinh^2 x + \cos^2 \theta)dz^2] \]  

where \(0 < x < \infty\), \(0 \leq \theta \leq 2\pi\), \(y\) and \(z\) are along the Killing directions and will be taken to be periodic coordinates on a 2-torus [6]. This is an example of a multi-center metric.

Sucu et al studied the solutions of the Dirac equation in the background of this metric [9]. In their paper they study the spinor field in the background of the Nutku helicoid instanton. They obtain an exact solution, which, however, can be expanded in terms of Mathieu functions [10] [7].

Here we study the same problem in five dimensions. The addition of the time component to the previous metric gives yields, \(ds^2 = -dt^2 + ds_4^2\), resulting in the massless Dirac equation as:

\[ \sqrt{2} \frac{\sqrt{2}}{a\sinh 2x \sqrt{2}} \{(\partial_4 + i \partial_\theta) \Psi_4 + a[\cos(\theta+ix)\partial_y + \sin(\theta+ix)\partial_z]\Psi_4 + i \frac{a\sqrt{\sinh 2x}}{\sqrt{2}} \partial_t \Psi_1 \} = 0, \]  

\[ \sqrt{2} \frac{\sqrt{2}}{a\sinh 2x \sqrt{2}} \{(\partial_4 - i \partial_\theta) \Psi_4 - a[\cos(\theta-ix)\partial_y + \sin(\theta-ix)\partial_z]\Psi_3 + i \frac{a\sqrt{\sinh 2x}}{\sqrt{2}} \partial_t \Psi_2 \} = 0, \]  

(2)  

(3)
\[
\sqrt{2} \frac{1}{a\sqrt{\sinh 2x}} \left\{ (\partial_x - i\partial_\theta + \coth 2x)\Psi_1 - a[\cos(\theta + ix)\partial_y + \sin(\theta + ix)\partial_z] \Psi_2 - i\frac{a\sqrt{\sinh 2x}}{\sqrt{2}} \partial_t \Psi_3 \right\} = 0,
\]

\[
\sqrt{2} \frac{1}{a\sqrt{\sinh 2x}} \left\{ (\partial_x + i\partial_\theta + \coth 2x)\Psi_2 + a[\cos(\theta - ix)\partial_y + \sin(\theta - ix)\partial_z] \Psi_1 - i\frac{a\sqrt{\sinh 2x}}{\sqrt{2}} \partial_t \Psi_4 \right\} = 0.
\]

(4)

(5)

If we solve for \(\Psi_1\) and \(\Psi_2\) and replace them in the latter equations, we get two equations which has only \(\Psi_3\) and \(\Psi_4\) in them. If we take \(\Psi_i = e^{i(k_xt + k_yy + k_zz)}\Psi_1(x, \theta)\), the resulting equations read:

\[
\left\{ \partial_{xx} + \partial_{\theta\theta} + \frac{a^2k^2}{2} \{ \cos[2(\theta + \phi)] - \cosh 2x \} + 2a^2k^2 \sinh 2x \right\} \Psi_{3,4} = 0. \quad (6)
\]

If we assume that the result is expressed in the product form \(\Psi_i = T_1(x)T_2(\theta)\), the angular part is expressible in terms of Mathieu functions.

\[
T_2(\theta) = Se \left[ \eta, -\frac{a^2k^2}{4}, \arccos \left( \frac{1 + \cos(\theta + \phi)}{2} \right) \right] + So \left[ \eta, -\frac{a^2k^2}{4}, \arccos \left( \frac{1 + \cos(\theta + \phi)}{2} \right) \right]. \quad (7)
\]

Here \(\eta\) is the separation constant and it is equal to the square of an integer because of the periodicity of the solution.

The equation for \(T_1\) reads:

\[
\left\{ \partial_{xx} - \frac{a^2k^2}{2} \cosh 2x + 2a^2k^2 \sinh 2x - \eta \right\} T_1 = 0 \quad (8)
\]

The solution of this equation is expressed in terms of double confluent Heun functions \([1]\).

\[
T_1(x) = H_D \left[ 0, \frac{a^2k^2}{2} + \eta, 4a^2k^2, \frac{a^2k^2}{2} - \eta, \tanh x \right] + H_D \left[ 0, \eta + \frac{a^2k^2}{2}, 4a^2k^2, \frac{a^2k^2}{2} - \eta, \tanh x \right] \times \int \frac{-dx}{H_D \left[ 0, \eta + \frac{a^2k^2}{2}, 4a^2k^2, \frac{a^2k^2}{2} - \eta, \tanh x \right]^2}. \quad (9)
\]

We only take the first function and discard the second solution.
There is an obstruction in odd Euclidean dimensions that makes us use the Atiyah-Patodi-Singer (APS) spectral boundary conditions [11]. These boundary conditions can also be used in even Euclidean dimensions if we want to respect the charge conjugation and the $\gamma^5$ symmetry [12] [13] [8]. Just to show the differences with the four dimensional solution, we attempt to write this expression in terms of Mathieu functions. This can be done after few transformations. We define $A = 2a^2k^2, B = -\eta, C = -\frac{a^2k^2}{2}$, and use the transformation $z = e^{-2x}$. Then the differential operator is expressed as

$$O = 4z^2\partial_{zz} + 4z\partial_z + A'z + B + C'\frac{1}{z}. \quad (10)$$

Here $A' = \frac{C-A}{2}, C' = \frac{C+A}{2}$. $\sqrt{\frac{C}{A}}u = z, w = \frac{1}{2}(u + \frac{1}{u})$ and set $E = \sqrt{A'C'}$ we get,

$$O = (w^2 - 1)\partial_{ww} + w\partial_w + \frac{E}{2}w + \frac{B}{4}. \quad (11)$$

The solution of this equation is expressible in terms of Mathieu functions given as:

$$R(z) = Se(-B, E, \arccos \sqrt{\frac{w+1}{2}}) + So(-B, E, \arccos \sqrt{\frac{w+1}{2}}) \quad (12)$$

Although both the radial and the angular part can be written in terms of Mathieu functions, the constants are different, modified by the presence of the new $-2a^2k^2t^2$ term, which makes the summation of these functions to form the propagator quite difficult.

### 3 Laplacian

The Laplacian can be solved by separation of variables method. The solution of the angular part can be expressed in terms of Mathieu functions. The solution of the radial part is can be written in terms of Double confluent Heun functions, which can be reduced to the modified Mathieu function after performing similar transformation as in the spinor case [8].

### 4 Conclusion

Here we related solutions of the Dirac equation in the background of the Nutku helicoid solution in five dimensions to the Double confluent Heun
function. The solution can be also expressed in terms of the Mathieu function at the expense of using a transformation. Often increasing the number of dimensions of the manifold results in higher functions as solutions. Here we call a function of a higher type if it has more singularities. In this respect Heun function belongs to a higher form than the hypergeometric function. APS Boundary conditions should be used in five dimensional case. They can also be used in four dimensional case for physical purposes. With work increasing on higher dimensions, it should not be far when we will encounter Heun functions more often in theoretical physics literature.

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