Multicritical Nishimori point in the phase diagram of the $\pm J$ Ising model on a square lattice

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Abstract

We investigate the critical behavior of the random-bond $\pm J$ Ising model on a square lattice at the multicritical Nishimori point in the $T$-$p$ phase diagram, where $T$ is the temperature and $p$ is the disorder parameter ($p = 1$ corresponds to the pure Ising model). We perform a finite-size scaling analysis of high-statistics Monte Carlo simulations along the Nishimori line defined by $2p - 1 = \text{Tanh}(1/T)$, along which the multicritical point lies. The multicritical Nishimori point is located at $p^* = 0.89081(7)$, $T^* = 0.9528(4)$, and the renormalization-group dimensions of the operators that control the multicritical behavior are $y_1 = 0.655(15)$ and $y_2 = 0.250(2)$; they correspond to the thermal exponent $\nu \equiv 1/y_2 = 4.00(3)$ and to the crossover exponent $\phi \equiv y_1/y_2 = 2.62(6)$.

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I. INTRODUCTION

The $\pm J$ Ising model on a square lattice represents an interesting theoretical laboratory, in which one can study the effects of quenched disorder and frustration on the critical behavior of two-dimensional (2D) spin systems. It is defined by the lattice Hamiltonian

$$\mathcal{H} = -\sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y, \quad \text{(1)}$$

where $\sigma_x = \pm 1$, the sum is over pairs of nearest-neighbor sites of a square lattice, and the exchange interactions $J_{xy}$ are uncorrelated quenched random variables, taking values $\pm J$ with probability distribution

$$P(J_{xy}) = p \delta(J_{xy} - J) + (1 - p) \delta(J_{xy} + J). \quad \text{(2)}$$

In the following we set $J = 1$ without loss of generality. For $p = 1$ we recover the standard Ising model, while for $p = 1/2$ we obtain the bimodal Ising spin-glass model. The $\pm J$ Ising model is a simplified model\textsuperscript{1} for disordered spin systems showing glassy behavior in some region of their phase diagram. The random nature of the short-ranged interactions is mimicked by nearest-neighbor random bonds. The 2D $\pm J$ Ising model is also interesting for the description of quantum Hall transitions\textsuperscript{2,3,4} and for its applications in coding theory\textsuperscript{5,6,7,8}. 

FIG. 1: (Color online) Phase diagram of the square-lattice $\pm J$ Ising model in the $T$-$p$ plane.
The $T$-$p$ phase diagram of the $2D \pm J$ Ising model is sketched in Fig. 1 (it is symmetric for $p \to 1-p$ and thus we only report it for $1-p < 1/2$). It has been investigated and discussed in several works, see, e.g., Refs. 2,5,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29. For sufficiently small values of $1-p$, which is the probability of antiferromagnetic bonds, the model presents a paramagnetic phase and a ferromagnetic phase, separated by a transition line. The paramagnetic-ferromagnetic (PF) transition line starts at the Ising point $X_{\text{Is}} = (T = T_{\text{Is}}, p = 1)$, where $T_{\text{Is}} = 2/\ln(1 + \sqrt{2}) = 2.26919...$ is the critical temperature of the 2D Ising model, and extends up to the multicritical Nishimori point (MNP) at $X_{\text{MNP}} = (T^*, p^*)$, with $T^* \approx 0.95$ and $p^* \approx 0.89$. Along this line, the critical behavior is analogous to that observed in 2D randomly dilute Ising (RDI) models. It is controlled by the pure Ising fixed point and disorder is marginally irrelevant, giving rise to universal logarithmic corrections, as shown in Refs. 33,34. As argued in Refs. 35,36,37, the MNP is located along the so-called Nishimori line ($N$ line)$^8,38$ defined by the equation

$$\tanh \beta = 2p - 1,$$

(3)

where $\beta \equiv 1/T$. As a consequence of the inequality$^{38}$

$$||\langle \sigma_x\sigma_y \rangle_T|| \leq ||\langle \sigma_x\sigma_y \rangle_{TN(p)}||$$

(4)

(the angular and the square brackets refer respectively to the thermal average and to the quenched average over the bond couplings $\{J_{xy}\}$, while the subscripts indicate the temperature of the thermal average), ferromagnetism can only exist in the region $p \geq p^*$, and the system is maximally magnetized along the $N$ line. This implies that the PF boundary lies in the region $p \geq p^*$. At the MNP the transition line is predicted to be parallel to the $T$ axis. Then, it reaches the $T = 0$ axis at $X_c = (0, p_c)$. As a consequence of inequality (4), $p_c$ must satisfy the inequality

$$p_c \geq p^*.$$
below the MNP, although it is not exact. Indeed, numerical analyses\textsuperscript{5,10,12,16,22,24} clearly support a reentrant phase transition line with $p_c > p^*$. The difference is however quite small, $p_c - p^* \approx 0.006$. The critical behavior along the transition line connecting the MNP to the $T = 0$ axis is an open issue. Even though it separates a paramagnetic phase from a ferromagnetic phase, it seems unlikely that such transitions belong to the same universality class as the PF transitions that occur on the line connecting the Ising point to the MNP.

The glassy transitions at $T = 0$ and $p < p_c$ are expected to belong to the same universality class as that of the bimodal model with $p = 1/2$, see, e.g., Ref. 39 and references therein. It is worth noting that the point $X_c = (0, p_c)$ is a multicritical point: it is connected to three phases and it is the intersection of two different transition lines, the PF line at $T > 0$ and the glassy line at $T = 0$. For $T = 0$ the critical point $X_c$ separates a ferromagnetic phase from a glassy phase, while for $T > 0$ the transition line separates a ferromagnetic from a paramagnetic phase. The behavior in a neighborhood of the multicritical point $X_c$ depends on the nature of the transition. If the PF transition and the glassy transition are effectively decoupled, we expect a phase diagram like that reported in Fig. 1. On the other hand, if the critical modes are coupled at $X_c$, all transition lines should be tangent at the multicritical point; therefore the PF line should be tangent to the glassy transition line $T = 0$. Moreover, in this case the magnetic critical behavior at $T = 0$ should differ from that at $T > 0$ along the transition line from the MNP to $X_c$.

Recently, Ref. 17 put forward an interesting conjecture concerning the location of the MNP in a general class of models in generic dimension. In the case of the 2D $\pm J$ model it predicts the MNP at

$$X_e \equiv (T_e = 0.956729..., p_e = 0.889972...).$$

The available numerical results show that Eq. (6) is a very good approximation of the location of the MNP; for example, the transfer-matrix calculations reported in Refs. 16, 18 and 20 give $p^* = 0.8907(2)$, $0.8906(2)$, $0.8905(5)$, respectively. Actually, since the small difference $p^* - p_e \approx 0.0006$ corresponds at best to approximately three error bars, these numerical works do not conclusively rule it out.\textsuperscript{9} The conjecture has also been tested on hierarchical lattices, where it has been found that it is not exact, although discrepancies are numerically small\textsuperscript{40,41} also in this case.

In this paper we consider the square-lattice $\pm J$ model, determine the location of the
MNP, and study the critical behavior in its vicinity. For this purpose, we perform high-statistics Monte Carlo (MC) simulations along the \( N \) line close to the MNP. We consider lattices of size \( L^2 \) with \( 6 \leq L \leq 64 \). A detailed finite-size scaling (FSS) analysis allows us to determine the location of the MNP quite precisely. We obtain

\[
X_{\text{MNP}} = [T^* = 0.9528(4), p^* = 0.89081(7)].
\]

We determine the renormalization-group (RG) dimensions \( y_1 \) and \( y_2 \) of the relevant operators that control the RG flow close to the MNP. We obtain \( y_1 = 0.655(15) \) and \( y_2 = 0.250(2) \), corresponding to the temperature and crossover exponents \( \nu \equiv 1/y_2 = 4.00(3) \) and \( \phi \equiv y_1/y_2 = 2.62(6) \), respectively. Our results confirm that \( X_e \) defined in Eq. (6) is a very good approximation of the MNP location: indeed, \( p^* - p_e = 0.00084(7) \). However, they also show that the conjecture of Ref. 17 leading to \( X_e \) is not exact.

The paper is organized as follows. In Sec. II we summarize the theoretical results, focusing in particular on the FSS behavior expected at the MNP. In Sec. III we present the FSS analysis of high-statistics MC simulations along the \( N \) line. In Sec. IV we summarize our results and draw our conclusions. In App. A we report some notations.

**II. FINITE-SIZE SCALING AT THE MULTICRITICAL POINT**

In the absence of external fields, the critical behavior at the MNP is characterized by two relevant RG operators. The singular part of the disorder-averaged free energy in a volume \( L^d \) can be written as

\[
F_{\text{sing}}(T, p, L) = L^{-d} f(u_1 L^{y_1}, u_2 L^{y_2}, \{u_i L^{y_i}\}), \quad i \geq 3,
\]

where \( y_1 > y_2 > 0 \), \( y_i < 0 \) for \( i \geq 3 \), \( u_i \) are the corresponding scaling fields, \( u_1 = u_2 = 0 \) at the MNP, and \( d \) is the space dimension (\( d = 2 \) in the present case). In the infinite-volume limit and neglecting scaling corrections due to irrelevant scaling fields, we have

\[
F_{\text{sing}}(T, p) = |u_2|^{d/y_2} f_\pm(u_1 |u_2|^{-\phi}), \quad \phi \equiv y_1/y_2 = 1,
\]

where the functions \( f_\pm(x) \) apply to the parameter regions in which \( \pm u_2 > 0 \). Close to the MNP, the transition lines correspond to constant values of the product \( u_1 |u_2|^{-\phi} \) and thus, since \( \phi > 1 \), they are tangent to the line \( u_1 = 0 \).
The scaling fields $u_i$ are analytic functions of the model parameters $T$ and $p$. Using symmetry arguments, Refs. 36,37 showed that one of the scaling axes is along the $N$ line, i.e., that the $N$ line is either tangent to the line $u_1 = 0$ or to $u_2 = 0$. Since the $N$ line cannot be tangent to the transition lines at the MNP and these lines are tangent to $u_1 = 0$, the first possibility is excluded. Thus, close to the MNP the $N$ line corresponds to $u_2 = 0$. Thus, we identify \cite{36,37}

$$u_2 = \tanh \beta - 2p + 1.$$  \hspace{1cm} (10)

As for the scaling axis $u_1 = 0$, $\epsilon \equiv 6 - d$ expansion calculations predict it\cite{37} to be parallel to the $T$ axis. The extension of this result to lower dimensions suggests

$$u_1 = p - p^*.$$  \hspace{1cm} (11)

Note that, if Eq. (11) holds, only the scaling field $u_2$ depends on the temperature $T$. We may then identify $\nu = 1/y_2$, and rewrite Eq. (9) as

$$F_{\text{sing}}(T, p) = |t|^{2\nu} f_\pm (g|t|^{-\phi}),$$  \hspace{1cm} (12)

where $t \equiv (T - T^*)/T^*$, $g \equiv p - p^*$, and $\phi$ is the crossover exponent.

These results give rise to the following predictions for the FSS behavior around $T^*$, $p^*$. Let us consider a RG invariant quantity $R$, such as $R_\xi \equiv \xi/L$, $U_4$, $U_{22}$, which are defined in the App. A and called phenomenological couplings. In the FSS limit $R$ obeys the scaling law

$$R = R(u_1 L^{y_1}, u_2 L^{y_2}, \{u_i L^{y_i}\}), \hspace{1cm} i \geq 3.$$  \hspace{1cm} (13)

Neglecting the scaling corrections which vanish in the limit $L \to \infty$, we expand in the neighborhood of the MNP:

$$R = R^* + b_{11} u_1 L^{y_1} + b_{21} u_2 L^{y_2} + \ldots.$$  \hspace{1cm} (14)

Along the $N$ line, the scaling field $u_2$ vanishes, so that we can write

$$R_N = R^* + b_{11} u_1 L^{y_1} + \ldots,$$  \hspace{1cm} (15)

where the subscript $N$ indicates that $R$ is restricted to the $N$ line. Let us now consider the derivative of $R$ with respect to $\beta \equiv 1/T$. Differentiating Eq. (14), we obtain

$$R' = b_{11} u_1' L^{y_1} + b_{21} u_2' L^{y_2} + \ldots$$  \hspace{1cm} (16)
If Eq. (11) holds, then \( u'_1 = 0 \), so that
\[
R' = b_{21} u'_2 L^{\eta_2} + \cdots
\] (17)

This result gives us a method to verify the conjecture of Ref. 37: once \( y_1 \) has been determined from the scaling behavior of a RG invariant quantity \( R \) close to the MNP, it is enough to check the scaling behavior of \( R' \). If \( R' \) scales as \( L^\zeta \) with \( \zeta < y_1 \), the conjecture is confirmed and \( \zeta \) provides an estimate of \( y_2 \). Along the \( N \) line the magnetic susceptibility is expected to behave as
\[
\chi_N = e L^{2-\eta} \left( 1 + e_1 u_1 L^{\eta_1} + \cdots \right).
\] (18)

Let us mention that the general features of the MNP are expected to be independent of \( d \). In three dimensions they have been accurately verified in Refs. 42,43.

III. MONTE CARLO RESULTS

A. Simulation details

In the following we present a FSS analysis of high-statistics MC data along the \( N \) line defined by
\[
\beta = \beta_N(p) \equiv -\frac{1}{2} \ln \left( \frac{1-p}{p} \right).
\] (19)

We performed MC simulations on square lattices of linear size \( L \) with periodic boundary conditions, for several values of \( L, L = 6, 8, 12, 16, 24, 32, 48, 64 \). Most simulations were performed close to the MNP, for values of \( p \) in the range \( 0.8895 \leq p \leq 0.8920 \), which includes the value \( p_e = 0.889972 \ldots \).

We used a standard Metropolis algorithm up to \( L = 24 \), while for \( L \geq 32 \) we supplemented the updating method with the random-exchange technique\(^{44} \) (see also Sec. 3 in Ref. 45 for a discussion of the random-exchange method in a disordered system). In order to determine MC estimates at \( p \) and \( \beta = \beta_N(p) \), we considered \( N_T \) systems at the same value of \( p \) and at inverse temperatures \( \beta_{\min} \equiv \beta_1, \ldots, \beta_{N_T} = \beta_N(p) \). The chosen values of \( \beta \) were equally spaced, i.e. \( \beta_{i+1} - \beta_i = \Delta \beta \), with a constant \( \Delta \beta \) (typically, \( \Delta \beta \approx 0.06, 0.04, 0.03 \) for \( L = 32, 48 \), and 64). The spacing \( \Delta \beta \) was chosen such that the acceptance probability was significantly larger than zero, while \( \beta_{\min} \) was chosen to have a sufficiently fast thermalization at \( \beta = \beta_{\min} \). The elementary unit of the algorithm consisted in \( N_{ex} \) Metropolis
sweeps for each configuration followed by an exchange move. We considered all pairs of configurations corresponding to nearby temperatures and proposed a temperature exchange with acceptance probability

\[ P = \exp\{\beta_i - \beta_{i+1} (E_i - E_{i+1})\}, \tag{20} \]

where \( E_i \) is the energy of the system at inverse temperature \( \beta_i \). In our MC runs we chose \( N_{ex} = 20 \). In our simulations we used multispin coding (details can be found in Ref. 46).

In Table I we report the parameters of our simulations performed with the random-exchange algorithm: here \( N_{run} \) is the number of Metropolis sweeps per sample and temperature, while \( N_{therm} \) is the corresponding number of Metropolis sweeps discarded for thermalization. We also report the range of the exchange probability, which depends on the temperatures \( \beta_i \) and \( \beta_{i+1} \) considered.

For every disorder sample we performed a MC run of \( N_{run} \) Metropolis sweeps, collecting \( N_{meas} \) measures of the quantities defined in App. A. We used \( N_{meas} = 400 \) for \( L \leq 24 \) and \( N_{meas} = 100 \) for \( L \geq 32 \). In order to obtain equilibrated data, we discarded a fraction of the measures which is determined by using the following procedure. We divided the measures into \( N_B \) parts (typically \( N_B = 10, 20 \)) of length \( l = N_{meas}/N_B \). Then, we considered the disorder-averaged susceptibilities

\[ \chi_b(t) = \left[ \frac{1}{l} \sum_{i=tl}^{(t+1)l-1} \chi(i) \right], \quad t = 0 \ldots N_B - 1. \tag{21} \]

Starting from random infinite-temperature spin configurations, \( \chi_b(t) \) increases with \( t \). When \( t \) is sufficiently large, \( \chi_b(t) \) becomes constant within error bars, thus signalling that thermalization has been reached. We considered the susceptibility because it is expected to be particularly sensitive to thermalization. Whenever one determines disorder averages of functions of thermal averages one should perform a bias correction; for this purpose we used the results of Ref. 47.

MC results are reported in Tables II and III. To obtain small statistical errors, we generated a large number of samples \( N_s \): \( N_s = 10^6 \) in all cases, except for the run with \( L = 32 \) and \( p = 0.891 \), where \( N_s = 4 \times 10^6 \). An important check of our simulations is given by the comparison with the exact behavior of the energy density along the \( N \) line,\(^{38}\)

\[ E_N(p) = \frac{1}{V} [\langle \mathcal{H} \rangle_{T_N(p)}] = 2 - 4p. \tag{22} \]
TABLE I: Parameters of our random-exchange MC runs. $N_{\text{run}}$ is the number of Metropolis sweeps per configuration and sample, $N_{\text{therm}}$ is the corresponding number of Metropolis sweeps discarded for thermalization.

| $L$ | $p$     | $\beta_{\text{min}}$ | $N_T$ | acc. range | $N_{\text{run}}/10^3$ | $N_{\text{therm}}/10^3$ |
|-----|---------|-----------------------|-------|------------|------------------------|------------------------|
| 32  | 0.8895  | 0.3228                | 13    | 4% − 56%  | 240                    | 48                     |
| 32  | 0.889972| 0.3252                | 13    | 4% − 56%  | 240                    | 48                     |
| 32  | 0.8905  | 0.3279                | 13    | 4% − 56%  | 240                    | 48                     |
| 32  | 0.8910  | 0.3305                | 13    | 4% − 57%  | 240                    | 72                     |
| 32  | 0.8915  | 0.3331                | 13    | 4% − 57%  | 240                    | 48                     |
| 48  | 0.889972| 0.285228              | 20    | 4% − 57%  | 400                    | 80                     |
| 48  | 0.8905  | 0.2879                | 20    | 4% − 57%  | 400                    | 80                     |
| 48  | 0.8910  | 0.2905                | 20    | 4% − 57%  | 400                    | 80                     |
| 48  | 0.8915  | 0.3310                | 25    | 13% − 66% | 320                    | 64                     |
| 64  | 0.889972| 0.265228              | 27    | 4% − 57%  | 900                    | 180                    |
| 64  | 0.8906  | 0.268442              | 27    | 4% − 57%  | 600                    | 180                    |
| 64  | 0.8909  | 0.2700                | 27    | 4% − 58%  | 600                    | 300                    |
| 64  | 0.8909  | 0.2700                | 27    | 4% − 58%  | 1200                   | 240                    |
| 64  | 0.8912  | 0.271529              | 27    | 4% − 58%  | 600                    | 240                    |

All runs give estimates of $E_N(p)$ which are consistent with Eq. (22). For example, we obtain $E_N(p)/(2 - 4p) = 1.00001(1), 1.00000(1)$ for $L = 32, p = 0.891$ and $L = 64, p = 0.8909$, respectively.

**B. Results**

MC estimates of the RG invariant quantities $R_\xi, U_4, U_{22}$ along the $N$ line are shown in Fig. 2. There is clearly a crossing point at $p \approx 0.891$. The raw data already indicate that $p^* > p_e = 0.889972..$, where $p_e$ is the value conjectured in Ref. 17. Their difference can hardly be explained in terms of scaling corrections. Indeed, the crossing point $p_{\text{cross}}(L, \kappa)$ of the data corresponding to lattice sizes $L$ and $\kappa L$ scales for $L \to \infty$ as

$$p_{\text{cross}}(L, \kappa) - p^* \sim L^{-y_1 - \omega},$$

(23)
TABLE II: MC data for $L = 6, 8, 12, 16$ along the $N$ line. For all runs the number of samples is $N_s = 10^6$.

| $L$ | $p$    | $R_\xi$ | $U_4$   | $U_{22}$ | $U_d$   | $\chi$ | $R'_\xi$ |
|-----|--------|---------|---------|----------|---------|--------|----------|
| 6   | 0.8895 | 0.9806(8)| 1.1316(2)| 0.08302(17) | 1.04856(7) | 26.17(8) | 6.532(6) |
|     | 0.889972 | 0.9886(8) | 1.1295(2) | 0.08176(17) | 1.04776(7) | 26.270(8) | 6.456(6) |
|     | 0.8905 | 0.9979(9) | 1.1272(2) | 0.08032(17) | 1.04687(6) | 26.382(8) | 6.371(6) |
|     | 0.891 | 1.0068(8) | 1.1250(2) | 0.07899(17) | 1.04604(6) | 26.486(8) | 6.290(6) |
|     | 0.8915 | 1.0158(8) | 1.1229(2) | 0.07765(17) | 1.04521(6) | 26.590(8) | 6.210(6) |
|     | 0.892 | 1.0251(8) | 1.1207(2) | 0.07632(16) | 1.04438(6) | 26.695(8) | 6.129(6) |
| 8   | 0.8895 | 0.9741(7) | 1.1327(2) | 0.08455(17) | 1.04810(6) | 44.150(13) | 13.471(11) |
|     | 0.889972 | 0.9837(7) | 1.1301(2) | 0.08301(16) | 1.04713(6) | 44.366(13) | 13.307(11) |
|     | 0.8905 | 0.9945(7) | 1.1274(2) | 0.08132(16) | 1.04609(6) | 44.604(13) | 13.126(11) |
|     | 0.891 | 1.0049(7) | 1.1249(2) | 0.07973(16) | 1.04512(6) | 44.829(13) | 12.953(11) |
|     | 0.8915 | 1.0156(7) | 1.1223(2) | 0.07811(16) | 1.04418(6) | 45.054(12) | 12.787(11) |
|     | 0.892 | 1.0265(8) | 1.1198(2) | 0.07654(16) | 1.04323(6) | 45.278(12) | 12.615(11) |
| 12  | 0.8895 | 0.9630(7) | 1.1350(2) | 0.08672(17) | 1.04831(6) | 91.98(3) | 36.01(3) |
|     | 0.889972 | 0.9754(7) | 1.1317(2) | 0.08464(17) | 1.04707(6) | 92.61(3) | 35.56(3) |
|     | 0.8905 | 0.9893(7) | 1.1281(2) | 0.08239(17) | 1.04572(6) | 93.31(3) | 35.02(2) |
|     | 0.891 | 1.0030(7) | 1.1247(2) | 0.08028(16) | 1.04444(6) | 93.97(3) | 34.52(2) |
|     | 0.8915 | 1.0176(8) | 1.1214(2) | 0.07821(16) | 1.04322(6) | 94.63(3) | 34.04(2) |
|     | 0.892 | 1.0309(8) | 1.1182(2) | 0.07618(16) | 1.04202(6) | 95.29(3) | 33.56(2) |
| 16  | 0.8895 | 0.9554(7) | 1.1373(2) | 0.08840(16) | 1.04891(7) | 154.69(5) | 71.09(4) |
|     | 0.889972 | 0.9701(7) | 1.1333(2) | 0.08587(16) | 1.04740(6) | 156.02(5) | 70.15(4) |
|     | 0.8905 | 0.9870(7) | 1.1289(2) | 0.08311(16) | 1.04577(6) | 157.52(5) | 69.11(4) |
|     | 0.891 | 1.0033(7) | 1.1248(2) | 0.08051(16) | 1.04428(6) | 158.92(5) | 68.10(4) |
|     | 0.8915 | 1.0199(7) | 1.1208(2) | 0.07804(15) | 1.04280(6) | 160.31(4) | 67.06(4) |
|     | 0.892 | 1.0367(8) | 1.1170(2) | 0.07562(15) | 1.04140(6) | 161.68(4) | 66.03(4) |
TABLE III: MC data for $L = 24, 32, 48, 64$ along the $N$ line. The number of samples is $N_s = 10^6$, except for the run with $L = 32$ and $p = 0.891$. In this case $N_s = 4 \times 10^6$.

| $L$ | $p$     | $R_\xi$ | $U_4$   | $U_{22}$ | $U_d$   | $\chi$ | $R_\xi^2$ |
|-----|---------|---------|---------|----------|---------|---------|-----------|
| 24  | 0.8895  | 0.9423(7) | 1.1411(2) | 0.09071(18) | 1.05037(7) | 321.10(10) | 182.16(12) |
|     | 0.889972 | 0.9615(7) | 1.1356(2) | 0.08730(17) | 1.04833(7) | 324.95(9)  | 179.74(11) |
|     | 0.8905  | 0.9831(7) | 1.1299(2) | 0.08373(16) | 1.04613(6) | 329.20(9)  | 176.77(11) |
|     | 0.891   | 1.0042(7) | 1.1247(2) | 0.08054(16) | 1.04411(6) | 333.15(9)  | 173.76(11) |
|     | 0.8915  | 1.0265(8) | 1.1194(2) | 0.07724(15) | 1.04217(6) | 337.14(10) | 170.92(11) |
|     | 0.892   | 1.0486(8) | 1.1145(2) | 0.07412(15) | 1.04038(6) | 341.06(9)  | 168.12(10) |
| 32  | 0.8895  | 0.9319(6) | 1.1443(2) | 0.09279(18) | 1.05152(7) | 538.29(16) | 351.4(3)   |
|     | 0.889972 | 0.9533(7) | 1.1380(2) | 0.08878(18) | 1.04922(7) | 546.13(17) | 346.3(3)   |
|     | 0.8905  | 0.9808(7) | 1.1306(2) | 0.08423(17) | 1.04640(7) | 555.14(17) | 334.0(3)   |
|     | 0.891   | 1.0058(4) | 1.12439(10) | 0.080358(8) | 1.04404(3) | 563.43(8)  | 334.02(14) |
|     | 0.8915  | 1.0315(8) | 1.1184(2) | 0.07651(16) | 1.04189(6) | 571.52(16) | 328.6(3)   |
|     | 0.892   | 1.0587(8) | 1.11249(18) | 0.07292(15) | 1.03958(5) | 579.70(16) | 322.2(2)   |
| 48  | 0.889972 | 0.9430(7) | 1.1413(2) | 0.09079(17) | 1.05050(7) | 1134.1(3)  | 864.7(8)   |
|     | 0.8905  | 0.9758(7) | 1.1321(2) | 0.08498(17) | 1.04715(7) | 1158.4(3)  | 847.8(8)   |
|     | 0.891   | 1.0088(7) | 1.1237(2) | 0.07984(16) | 1.04390(6) | 1181.8(3)  | 830.2(7)   |
|     | 0.8915  | 1.0438(8) | 1.11581(18) | 0.07487(15) | 1.04094(6) | 1204.9(3)  | 814.2(7)   |
| 64  | 0.889972 | 0.9311(6) | 1.1449(2) | 0.09308(17) | 1.05186(7) | 1900.0(6)  | 1641.3(1.6)|
|     | 0.8906  | 0.9792(7) | 1.1313(2) | 0.08460(17) | 1.04670(6) | 1961.0(6)  | 1596.9(1.6)|
|     | 0.8909  | 1.0037(7) | 1.1252(2) | 0.08072(17) | 1.04444(6) | 1990.3(6)  | 1580.2(1.5)|
|     | 0.8912  | 1.0286(8) | 1.1194(2) | 0.07711(16) | 1.04226(6) | 2019.1(6)  | 1560.2(1.6)|
FIG. 2: (Color online) MC data of $U_4$, $U_{22}$ and $R_{\xi} \equiv \xi/L$ vs $p$. The dashed lines connecting the data at given $L$ are drawn to guide the eye. The dotted vertical line corresponds to $p = p_c = 0.889972$. 
where $\omega > 0$ is the exponent associated with the leading irrelevant operator. Since, as we shall see, $y_1 \approx 0.6$, the approach is reasonably fast, so that our data, that correspond to lattice sizes between 6 and 64, should be able to detect a drift due to scaling corrections. The very good stability of the results excludes a delayed approach to $p_c$.

In order to estimate precisely $p^*$, $T^*$, and $y_1$, we perform a FSS analysis of the phenomenological couplings $R_\xi \equiv \xi/L$, $U_4$, $U_{22}$, and $U_d$, which are defined in App. A and are generically denoted by $R$. Since we vary $p$ and $\beta$ along the $N$ line, close to the MNP we expect

$$R = f_R[(p - p^*)L^{y_1}],$$

(24)

with $f_R(0) = R^*$. This functional form relies on the property that $u_2 = 0$ along the $N$ line. Since our data are sufficiently close to the MNP, the product $(p - p^*)L^{y_1}$ is small. We can thus expand $f_R(x)$ in powers of $x$. Thus, we fit the numerical data to

$$R = R^* + \sum_{n=1}^{n_{\text{max}}} a_n(p - p^*)^n L^{ny_1},$$

(25)

keeping $R^*$, the coefficients $\{a_n\}$, $p^*$, and $y_1$ as free parameters. Here we neglect scaling corrections. To monitor their role, we repeat the fits several times, each time only including data satisfying $L \geq L_{\text{min}}$. Fits with $n_{\text{max}} = 1$ have a large $\chi^2$/DOF (DOF is the number of degrees of freedom of the fit), indicating that the range of values of $p$ we are considering is too large to allow for a linear approximation of the scaling function $f_R(x)$. Fits with $n_{\text{max}} = 2$ have instead a good $\chi^2$ for $L_{\text{min}} \geq 6$ ($U_4$), 12 ($R_\xi$), and 16 ($U_{22}$ and $U_d$). We also perform fits with $n_{\text{max}} = 3$, but we do not observe significant differences: for $U_d$ and $L_{\text{min}} = 6, 12$ we obtain $\chi^2$/DOF = 2148/39, 34/27 with $n_{\text{max}} = 2$, and 2147/38, 34/26 for $n_{\text{max}} = 3$. Clearly, a parabolic approximation is fully adequate. Beside fitting separately each observable, we also perform combined fits of three different phenomenological couplings. The results are reported in Table IV. In the case of $U_{22}$, $U_4$, and $R_\xi$ all estimates of $p^*$ show a systematic downward trend with $L_{\text{min}}$, with $0.89080 \lesssim p^* \lesssim 0.89083$ for $L_{\text{min}} = 32$. Fits of $U_d$ (note that this quantity is statistically more precise than the other ones, see the Tables II and III, which explains the somewhat larger $\chi^2$/DOF of the fits) show instead a different behavior and suggest a somewhat larger value of $p^*$, $p^* \approx 0.89087$. Similar trends are observed in the estimates of $y_1$, which in most of the cases increases with $L_{\text{min}}$ and varies essentially in the range $0.65 \lesssim y_1 \lesssim 0.67$ with a statistical error of $\pm 0.01-0.02$. 

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TABLE IV: Estimates of $p^*$ and $y_1$ obtained by performing a fit to Eq. (25) with $n_{\text{max}} = 2$. DOF is the number of degrees of freedom in the fit and $L_{\text{min}}$ is the minimum lattice size included in the fit.

|       | $L_{\text{min}}$ | $\chi^2$/DOF | $p^*$      | $y_1$  |
|-------|------------------|---------------|------------|--------|
| $U_d$ | 12               | 34/27         | 0.890860(7)| 0.658(7)|
|       | 16               | 19/21         | 0.890877(8)| 0.656(9)|
|       | 24               | 17/15         | 0.890881(11)| 0.647(11)|
|       | 32               | 14/9          | 0.890871(14)| 0.664(16)|
| $U_{22}$ | 12           | 41/27         | 0.890895(12)| 0.639(11)|
|       | 16               | 11/21         | 0.890857(13)| 0.647(13)|
|       | 24               | 6/15          | 0.890831(17)| 0.663(18)|
|       | 32               | 3/9           | 0.890813(21)| 0.672(25)|
| $U_4$ | 12               | 19/27         | 0.890882(9)| 0.647(9)|
|       | 16               | 9/21          | 0.890865(10)| 0.650(10)|
|       | 24               | 6/15          | 0.890850(13)| 0.656(14)|
|       | 32               | 3/9           | 0.890834(17)| 0.669(19)|
| $R_\xi$ | 12             | 21/27         | 0.890835(8)| 0.647(8)|
|       | 16               | 12/21         | 0.890820(9)| 0.650(9)|
|       | 24               | 7/15          | 0.890805(12)| 0.653(12)|
|       | 32               | 5/9           | 0.890795(15)| 0.664(17)|
| $R_\xi, U_4, U_{22}$ | 12         | 107/85        | 0.890864(5)| 0.646(5)|
|       | 16               | 44/67         | 0.890844(6)| 0.650(6)|
|       | 24               | 27/49         | 0.890826(8)| 0.656(8)|
|       | 32               | 14/31         | 0.890812(10)| 0.668(11)|
| $R_\xi, U_4, U_d$ | 12           | 92/85         | 0.890858(4)| 0.651(4)|
|       | 16               | 64/67         | 0.890855(5)| 0.653(5)|
|       | 24               | 54/49         | 0.890847(7)| 0.652(7)|
|       | 32               | 37/31         | 0.890836(9)| 0.665(10)|
These tiny discrepancies indicate that scaling corrections are not negligible if compared with our small statistical errors. In order to estimate their quantitative role, we also perform fits in which scaling corrections are taken into account. Thus, we fit the MC data to

$$R = R^* + \sum_{n=1}^{n_{\text{max}}} a_n (p - p^*)^n L^{ny_1} + L^{-\omega} \sum_{k=0}^{k_{\text{max}}} b_k (p - p^*)^k L^{ky_1}. \quad (26)$$

Results for $k_{\text{max}} = 0$ and 1 have both a good $\chi^2$/DOF, even for $L_{\text{min}} = 6$. In the following we present results corresponding to $k_{\text{max}} = 1$, since this choice allows us to take into account the scaling corrections that affect the determination of both $p^*$ and $y_1$. The correction-to-scaling exponent $\omega$ is not known and thus we keep it as a free parameter. Our results are reported in Table V. Because of the large number of parameters this fit gives stable results only for $L_{\text{min}} = 6, 8$ (for $U_4$ this is not even the case). For larger values of $L_{\text{min}}$ errors are so large to make the results meaningless. The results are fully consistent. First of all, they predict $\omega \gtrsim 1$. Thus, corrections to scaling decay reasonably fast, indicating that the systematic error should be reasonably estimated by considering data in our range $6 \leq L \leq 64$. Second, fits that involve $U_d$ give estimates of $\omega$ that are significantly larger. This is consistent with the results reported in Table IV: fits involving $U_d$ show a small dependence on $L_{\text{min}}$, suggesting that $U_d$ is less affected by scaling corrections. The estimates of $p^*$ obtained in fits of $\xi/L$ and $U_{22}$ show a trend that is opposite to that observed in fits without corrections, indicating that the correct value for $p^*$ belongs to the range of values that occur in the two types of fits: values smaller than 0.89070 are not consistent with our data. To quote a final result, let us note that the fits with $L_{\text{min}} = 8$ reported in Table V give (including the statistical error) $0.89074 \lesssim p^* \lesssim 0.89089$. A conservative estimate is therefore

$$p^* = 0.89081(7). \quad (27)$$

This result is fully consistent with those obtained in the fits without scaling corrections. Using Eq. (3) we obtain

$$\beta^* = 1.0495(4), \quad T^* = 0.9528(4). \quad (28)$$

Note that the conjectured value$^{17} p_e = 0.889972\ldots$ is excluded, the difference $p^* - p_e = 0.00084(7)$ corresponding to 12 error bars.

Let us finally estimate $y_1$. Fits with scaling corrections give results that decrease with $L_{\text{min}}$, while fits without scaling corrections give estimates that have the opposite trend.
TABLE V: Estimates of \( p^\ast \), \( y_1 \), and \( \omega \) obtained by performing a fit to Eq. (26) with \( n_{\text{max}} = 2 \) and \( k_{\text{max}} = 1 \). DOF is the number of degrees of freedom in the fit and \( L_{\text{min}} \) is the minimum lattice size included in the fit.

| \( L_{\text{min}} \) | \( \chi^2/\text{DOF} \) | \( p^\ast \) | \( y_1 \) | \( \omega \) |
|---------------------|------------------|-----------|--------|--------|
| \( \xi/L \)         |                  |           |        |        |
| 6                   | 9.9/36           | 0.89077(2)| 0.663(16) | 1.18(31) |
| 8                   | 7.3/30           | 0.89079(2)| 0.654(13) | 1.95(68) |
| \( U_{22} \)        |                  |           |        |        |
| 6                   | 7.6/36           | 0.89077(3)| 0.663(21) | 1.23(25) |
| 8                   | 7.1/30           | 0.89078(4)| 0.660(23) | 1.45(46) |
| \( U_d \)           |                  |           |        |        |
| 6                   | 31/36            | 0.89090(1)| 0.655(8)  | 2.67(16) |
| 8                   | 20/30            | 0.89088(1)| 0.654(8)  | 4.15(68) |
| \( \xi/L, U_4, U_{22} \) |          |           |        |        |
| 6                   | 77/114           | 0.89082(1)| 0.657(8)  | 1.59(17) |
| 8                   | 47/96            | 0.89081(1)| 0.657(10) | 1.64(32) |
| \( \xi/L, U_4, U_d \) |          |           |        |        |
| 6                   | 128/114          | 0.890868(5)| 0.652(5) | 3.04(12) |
| 8                   | 81/96            | 0.890860(5)| 0.651(5) | 5.25(71) |

Comparing all results, we infer \( 0.64 \lesssim y_1 \lesssim 0.67 \), so that we arrive at the final estimate

\[
y_1 = 0.655(15). \tag{29}
\]

The fits that we have reported also allow us to estimate the critical-point value \( R^\ast \) of the phenomenological couplings. We obtain:

\[
R^\ast_\xi = 0.996(2), \tag{30}
\]

\[
U_4^\ast = 1.1264(6), \tag{31}
\]

\[
U_{22}^\ast = 0.0817(5), \tag{32}
\]

\[
U_d^\ast = 1.0447(3). \tag{33}
\]

Of course, the estimate of \( U_d^\ast \) is consistent with the relation \( U_d^\ast = U_4^\ast - U_{22}^\ast \). Note that these results are not very much different from those of the pure 2D Ising values that apply along the PF line from the pure Ising point at \( p = 1 \) to the MNP, which are\(^{48} R^\ast_\xi = 0.9050488292(4) \), \( U_4^\ast = U_d^\ast = 1.167923(5) \), \( U_{22}^\ast = 0 \). In particular, the estimate (32) of \( U_{22}^\ast \) is quite small, indicating that the violations of self-averaging are small.
### TABLE VI: Estimates of the exponent $\zeta = y_2$ and $\zeta = \eta$ obtained by performing a fit to Eq. (35) with $n_{\text{max}} = 2$.

|       | $L_{\min}$ | $\chi^2/\text{DOF}$ | $\zeta$     |
|-------|------------|----------------------|-------------|
| $R'_\xi$ | 8          | 20/29                | 0.252(2)    |
|       | 12         | 16/24                | 0.251(2)    |
|       | 16         | 13/19                | 0.250(2)    |
|       | 24         | 12/14                | 0.249(3)    |
| $U'_4$   | 8          | 19/29                | 0.250(1)    |
|       | 12         | 17/24                | 0.249(1)    |
|       | 16         | 14/19                | 0.249(1)    |
|       | 24         | 13/14                | 0.250(2)    |
| $Z$     | 8          | 147/29               | 0.173(3)    |
|       | 12         | 38/24                | 0.175(3)    |
|       | 16         | 21/19                | 0.176(4)    |
|       | 24         | 11/14                | 0.178(5)    |
|       | 32         | 6/9                  | 0.179(5)    |

Let us now consider the derivatives $R'$ of the phenomenological couplings. Close to the MNP, $R'$ is expected to behave as

$$R' = L^\xi f_{R'}[(p - p^*)L^{y_1}],$$

where we have used the fact that along the $N$ line $u_2 = 0$. If Eq. (11) holds, we have additionally $\zeta = y_2$. To determine $\zeta$ we perform analyses analogous to those used before to determine $p^*$ and $y_1$. We expand $f_{R'}(x)$ in powers of $x$ and thus fit $R'$ to

$$\ln R' = \zeta \ln L + \sum_{n=0}^{n_{\text{max}}} a_n (p - p^*)^n L^{y_1}. $$

We always fix $y_1$ to the value (29) and $p^*$ to the value (27), including in the final error the variation of $y_1$ and $p^*$ within one error bar. As in the fits of $R$, we check the role of $n_{\text{max}}$. A significant improvement in the quality of the fit is observed by changing $n_{\text{max}}$ from 1 to 2, while no significant change is obtained by increasing it to 3. Therefore, we fix $n_{\text{max}} = 2$. 

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The results are reported in Table VI. They are very stable and show a very small dependence on $L_{\text{min}}$, of the order of the statistical error. As a final result we quote $\zeta = 0.250(2)$. This result is significantly smaller than $y_1$ and thus confirms the multicritical nature of the MNP and the arguments reported in Sec. II. Therefore $\zeta$ should be identified with $y_2$, so that

$$y_2 = 0.250(2), \quad \nu \equiv \frac{1}{y_2} = 4.00(3).$$

(36)

The corresponding crossover exponent, cf. Eq. (9), is

$$\phi \equiv \frac{y_1}{y_2} = 2.62(6).$$

(37)

The same analysis used to estimate $y_2$ can be employed to determine $\eta$. Instead of $\chi$, we consider

$$Z \equiv \chi/\xi^2 \sim L^{-\eta}$$

(38)

which has smaller statistical errors. We fit $Z$ to

$$\ln Z = -\eta \ln L + \sum_{n=0}^{n_{\text{max}}} a_n (p - p^*)^n L^{ny_1}.$$  

(39)

As before, we fix $y_1$ and $p^*$, set $n_{\text{max}} = 2$, and repeat the fit several times, each time considering only data satisfying $L \geq L_{\text{min}}$. The final results are reported in Table VI. A good $\chi^2$ is obtained only for $L_{\text{min}} \geq 16$. The corresponding fits give $\eta \approx 0.175-0.180$ with a slight upward trend. This effect may be real and due to scaling corrections. Therefore, we also fit $\ln Z$ to

$$\ln Z = -\eta \ln L + \sum_{n=0}^{n_{\text{max}}} a_n (p - p^*)^n L^{ny_1} + L^{-\omega \sum_{k=0}^{k_{\text{max}}} a_k (p - p^*)^k L^{ky_1}},$$

(40)

fixing $y_1$ and $p^*$, and keeping $\omega$ as a free parameter. For $n_{\text{max}} = 2$ and $k_{\text{max}} = 1$, we obtain a good $\chi^2$ for any $L_{\text{min}} \geq 6$. The corresponding estimates of $\eta$ are: $\eta = 0.182(10)$ ($L_{\text{min}} = 6$) and $\eta = 0.181(10)$ ($L_{\text{min}} = 8$). The central estimates are quite close to those obtained in fits without scaling corrections, indicating that scaling corrections are small. We take as our final estimate

$$\eta = 0.180(5),$$

(41)

which includes all results without scaling corrections and is consistent with the fits in which scaling corrections are taken into account.
IV. CONCLUSIONS

In this paper we have investigated the critical behavior of the square-lattice $\pm J$ Ising model close to the MNP. Our main results are the following:

(i) We have obtained an accurate estimate of the location of the MNP: $p^* = 0.89081(7)$, $T^* = 0.9528(4)$. The conjectured value $p_c = 0.889972\ldots$, put forward in Ref. 17, is a very good approximation, but it is not exact: $p^* - p_c = 0.00084(7)$.

(ii) We have computed the RG dimensions of the relevant operators at the MNP, obtaining $y_1 = 0.655(15)$ and $y_2 = 0.250(2)$. It is tempting to conjecture that $y_2 = 1/4$ exactly. Note also that $y_1$ is consistent with 2/3, though in this case the precision of the result is not good enough to put this conjecture on firm grounds. The above estimates of the RG dimensions give $\nu \equiv 1/y_2 = 4.00(3)$ and $\phi \equiv y_1/y_2 = 2.62(6)$.

(iii) We have computed the critical exponent $\eta$ that controls the critical behavior of the magnetic correlations, obtaining $\eta = 0.180(5)$.

Our estimate of $p^*$ is significantly more precise than those obtained in previous works. By using transfer-matrix methods Refs. 16,18,20,28 obtained $p^* = 0.8907(2), 0.8906(2), 0.8905(5), 0.889(2)$, respectively. We also mention the results $p^* = 0.8872(8)$ obtained by means of an off-equilibrium MC simulation,\textsuperscript{21} and $p^* = 0.886(3)$ from the analysis of high-temperature expansions.\textsuperscript{25} Concerning the critical exponents at the MNP, we mention the square-lattice results $y_1 = 0.676(14), 0.667(13), 0.752(17), 0.75(7), 0.76(5)$ respectively from Refs. 10,16,18,2,25. The estimate\textsuperscript{11} $y_1 = 0.671(9)$ has been obtained on the triangular and honeycomb lattices. Moreover, we mention the result\textsuperscript{10} $y_1 = 0.658(13)$ obtained from a model with Gaussian distributed couplings. The most recent results are clearly consistent with our estimate. As for $y_2$ (or equivalently $\nu = 1/y_2$), Refs. 10,16, 2 report $\nu \approx 3$, $\nu = 4.0(5)$, and $\nu = 2.4(3)$, which are not far from our much more precise result (but the result of Ref. 2 is clearly inconsistent with the quoted errors). Finally, we quote $\eta = 0.183(3)$,\textsuperscript{10} and $\eta = 0.1848(3), 0.1818(2)$ (statistical errors only) obtained in Ref. 10 respectively for the $\pm J$ model and for the model with Gaussian distributed couplings. They are fully consistent with our result.

It is interesting to compare the phase diagram of the two-dimensional $\pm J$ Ising model, shown in Fig. 1, with that of the three-dimensional $\pm J$ Ising model sketched in Fig. 3.
Recent high-statistics numerical studies of the $\pm J$ Ising model on a simple cubic lattice have shown that: (i) the transitions along the PF line belong to the 3D randomly dilute Ising (RDI) universality class,\textsuperscript{46} with critical exponents\textsuperscript{47} $\nu = 0.683(2)$ and $\eta = 0.036(1)$; (ii) this line extends up to a magnetic-glassy multicritical point (MGP) located along the $N$ line, at\textsuperscript{42} $p^* = 0.76820(4)$, where the relevant RG dimensions are given by $y_1 = 1.02(5)$ and $y_2 = 0.61(2)$ (corresponding to the thermal and crossover exponents $\nu = 1.64(5)$ and $\phi = 1.67(10)$); (iii) the critical behavior along the transition line separating the paramagnetic and the spin-glass phase is independent of $p$, and belongs to the Ising spin-glass universality class\textsuperscript{49} with the correlation-length critical exponent $\nu = 2.53(8)$.

APPENDIX A: NOTATIONS

The two-point correlation function is defined as

$$G(x) \equiv \langle \sigma_0 \sigma_x \rangle,$$

where the angular and the square brackets indicate respectively the thermal average and the quenched average over disorder. We define the magnetic susceptibility $\chi \equiv \sum_x G(x)$ and the correlation length $\xi$

$$\xi^2 \equiv \frac{\tilde{G}(0) - \tilde{G}(q_{\text{min}})}{\tilde{q}_{\text{min}}^2 \tilde{G}(q_{\text{min}})},$$

\text{(A2)}
where \( q_{\text{min}} \equiv (2\pi/L,0) \), \( \hat{q} \equiv 2\sin q/2 \), and \( \tilde{G}(q) \) is the Fourier transform of \( G_1(x) \). We also consider quantities that are invariant under RG transformations in the critical limit. Beside the ratio

\[
R_\xi \equiv \xi/L,
\]

we consider the quartic cumulants

\[
U_4 \equiv \frac{[\mu_4]}{[\mu_2]^2}, \quad U_{22} \equiv \frac{[\mu_2^2] - [\mu_2]^2}{[\mu_2]^2}, \quad U_d \equiv U_4 - U_{22},
\]

where

\[
\mu_k \equiv \langle \left( \sum_x \sigma_x \right)^k \rangle.
\]

The quantities \( R_\xi, U_4, U_{22}, \) and \( U_d \) are also called phenomenological couplings. Finally, we consider the derivatives

\[
R'_\xi \equiv \frac{dR_\xi}{d\beta}, \quad U'_4 \equiv \frac{dU_4}{d\beta},
\]

which can be computed by measuring appropriate expectation values at fixed \( \beta \) and \( p \).

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