THE UNIVERSAL THEORY OF FIRST ORDER ALGEBRAS AND VARIOUS REDUCTS

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Abstract. First order formulas in a finite relational signature can be considered as operations on the finitary relations of an underlying set, giving rise to multisorted algebras we call first order algebras. We present universal axioms so that an algebra satisfies the axioms iff it embeds into a first order algebra. Importantly, our argument is modular and also works for, e.g., the positive existential algebras (where we restrict attention to the positive existential formulas) and the quantifier-free algebras. We also explain the relationship to theories, and indicate how to add in function symbols.

1. Introduction

Consider the propositional formula \( \varphi \) with two proposition letters \( P \) and \( Q \) given by \( P \land \neg Q \). We can think of such a formula \( \varphi \) as giving rise to an operation on subsets of a set. If \( p \) and \( q \) are subsets of some set \( W \), then \( \varphi(p, q) := p \cap (W - q) \) is also a subset of \( W \). This function \( \varphi : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W) \) accepts as input two subsets of \( W \) and outputs a subset. Indeed, every propositional formula gives rise to a finitary operation \( \mathcal{P}(W)^n \to \mathcal{P}(W) \). In this way we arrive at a functional signature \( \tau \) where there is a function symbol of arity \( n \) for every propositional formula involving \( n \) proposition letters, and we have for every set \( W \) a \( \tau \)-algebra \(^1\) whose underlying set is the powerset \( \mathcal{P}(W) \) and so we may call it a powerset algebra. Of course, we do not work in practice with the full signature \( \tau \), but rather we isolate just some of the operations which compositionally generate all the others in the class of algebras of interest. One convenient choice is \( 0, 1, \lor, \land, \neg \) where 0 and 1 are the constants interpreted by \( \emptyset \) and \( W \) in the powerset algebra determined by \( W \).

Stone’s representation theorem ([9]) gives equational axioms so that a \( \tau \)-algebra satisfies the axioms iff it embeds into a powerset algebra. This result can be understood in two different directions as it were. On the one hand, we can imagine being presented with some axioms (the

\(^1\)By “algebra” I mean a structure in a signature with only function symbols, i.e. a functional signature.
Boolean algebra axioms in this case) and wishing to find some geometrical representation of any algebra that satisfies these axioms. On the other hand, we could begin with a geometric or otherwise natural class of algebras, and then try to axiomatize this class. It is this latter point of view which motivates the present paper.

If we use $K$ to denote the class of powerset algebras, the class $S(K)$ of subalgebras of powerset algebras consists of the $\tau$-algebras whose elements are some of the subsets of some set with the operations interpreted normally. Stone’s result yields an equational axiomatization of $S(K)$. A $\tau$-algebra $M$ is in $S(K)$ iff $M$ satisfies the Boolean algebra axioms.

Analogously, Cayley’s theorem that any group embeds into the group of all permutations on some set can be read in two ways. On the one hand, we are showing that any algebra satisfying certain axioms (the group axioms) is representable in a certain way. On the other hand, we are finding an axiomatization of a certain natural class of algebras (the permutation groups).

If instead of having an operation for every propositional formula we only have one for every positive propositional formula (in other words we look at the operations generated by $0, 1, \lor, \land$), then a version of Stone’s argument works to axiomatize the subalgebras of the powerset algebras in this reduced signature (1). Once again, this result could be expressed as saying that every distributive lattice has a certain geometric representation, but for the purposes of this paper I prefer to think of it as a way of arriving at the distributive lattice axioms. Interestingly, the usual argument for the positive case easily adapts to deal with negation as well, but there is a version of the argument that works with negation that does not seem to adapt easily to deal with the positive case.

Now instead of having an operation for every propositional formula, let us have an operation for every first order formula in a finite relational signature. For example, if $\theta$ is the formula $\exists y[R_1(x, y) \land R_2(x, y)]$, then as an operation on relations $\theta$ accepts as input two binary relations $r_1, r_2 \subseteq W^2$ and outputs the unary relation $\{x \in W \mid \exists y[r_1(x, y) \land r_2(x, y)]\}$ which is a projection of their intersection. For the next example it is important to note that we consider a formula to come specifically equipped with a variable context that contains the free variables of the formula but could contain variables otherwise not explicitly occurring in the formula. Let $\theta$ now be the formula $R(x, y)$ in the variable context $(x, y, z)$. Then as an operation $\theta$ accepts as input a binary relation $r \subseteq W^2$ and outputs the 3-ary relation $\{(x, y, z) \in W^3 \mid (x, y) \in r\}$. We will be calling such an operation a cylindrification. More generally,
let \( \sigma \) be the finite relational signature consisting of the relation symbols \( R_1, \ldots, R_n \) of arities \( m_1, \ldots, m_n \) respectively. Let \( \theta(\bar{x}) \) be a first order \( \sigma \)-formula in the variable context \( x_1, \ldots, x_k \). Let \( W \) be any set. Then \( \theta \) induces a function on relations \( \theta: \mathcal{P}(W^{m_1}) \times \cdots \times \mathcal{P}(W^{m_n}) \rightarrow \mathcal{P}(W^k) \) defined by

\[
\theta(r_1, \ldots, r_n) := \{ \bar{x} \in W^k \mid (W, r_1, \ldots, r_n) \models \theta(\bar{x}) \}
\]

where \( (W, r_1, \ldots, r_n) \) denotes the \( \sigma \)-structure where each relation symbol \( R_i \) is interpreted as \( r_i \) and \( \models \) denotes the usual notion of satisfaction.

We see that the operations arising from first order formulas are a little bit different from those arising from propositional formulas in that now there are different sorts \( \mathcal{P}(W^0), \mathcal{P}(W^1), \mathcal{P}(W^2) \), etc. Instead of a single-sorted algebraic signature, the first order formulas naturally give rise to a multisorted algebraic signature where there is a sort for each natural number. Every set \( W \) gives rise to a multisorted algebra in this signature, with the operations as defined above. We call the algebras that arise in this way first order algebras.

In this paper we present a universal axiomatization of the subalgebras of first order algebras. The argument we use is a generalization of the distributive lattice argument (rather than a version of the Boolean algebra argument) in the sense that it works when dealing with various reducts, and easily adapts to larger signatures. For the classes of algebras \( K \) of interest to us here, a universal axiomatization of the subalgebras of \( K \) is the same thing as a universal axiomatization of the universal theory of \( K \), hence the title of this paper is appropriate.

It is not possible to replace our universal axiomatization with an equational one, nor even a Horn clause one, because the algebras in question are not closed under products. On the other hand, a (reasonable) axiomatization of the Horn clause theory of first order algebras (which is equivalent to the equational theory) is an algebraic version of the completeness theorem for first order logic and has already been noted in the multisorted formalism by Schwartz [8] and Börner [2] and previously in other formalisms (e.g., a cylindric algebra approach in [5] or [1]). The approach to algebraization of logic taken in this paper is called “many-sorted cylindric algebras” in the survey [7].

One highlight of this paper is the introduction of a geometrically intuitive axiom useful for generalizing Stone’s representation theorem to various reducts of first order logic (see axiom (0) discussed in Section 3 below). This axiom allows for a uniform and natural argument in the various reducts. However, at the same time the paper provides, without
using category-theoretic notions, a particularly simple algebraic way of viewing theories, which should be more well-known.

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2. Preliminaries

Instead of dealing with a signature where there is a function symbol for every first order formula, it suffices to deal with a subsignature which will compositionally generate all the operations of interest. There is of course some degree of choice here, and we have generally chosen so as to make our axioms and arguments to follow more conveniently stated. Here is the (largest) signature we will use:

Definition 1. The multisorted signature of first order algebras (with equality) is given as follows.

- We have a sort $n$ for each natural number $n \in \{0, 1, 2, \ldots \}$. The sort $n$ is intended to consist of $n$-ary relations on some set.
- For each function $\alpha : \{1, \ldots, n\} \to \{1, \ldots, k\}$ we have a function symbol $\alpha : n \to k$. These are called substitutions and will correspond to the operations on relations arising from atomic formulas.
- For each $n$ we have a constant symbol $0^n$ belonging to sort $n$, which we may write as $0^n : n$. Likewise we have constant symbols $1^n : n$ and function symbols $\lor^n : n \times n \to n$, $\land^n : n \times n \to n$, and $\neg^n : n \to n$. We usually omit the superscript and write simply $0, 1, \lor, \land, \neg$, leaving the arity implicit to the context.
- For each $n$ we have a function symbol $\exists^n : n + 1 \to n$, which we will generally write $\exists$. This will correspond to projection or existential quantification of the last coordinate.
- For each $n$ with $1 \leq i, j \leq n$ we have a constant $\Delta^n_{i,j} : n$. These will correspond to equality of various coordinates.

Before going further, let us introduce some notation involving the substitutions. A function $\alpha : \{1, \ldots, n\} \to \{1, \ldots, k\}$ can also be described as a sequence of length $n$ with repetition allowed taken from a $k$-element set. Thus, $\alpha$ gives a way of transforming any $k$-tuple into an $n$-tuple. Let $W$ be a set. Define $\alpha_{\text{tuple}} : W^k \to W^n$ to be the obvious function induced on tuples. In detail, $\alpha_{\text{tuple}}(x_1, \ldots, x_k) := (x_{\alpha(1)}, \ldots, x_{\alpha(n)})$. This function on tuples in turn induces a function on relations of particular interest, the inverse image. I.e., define $\alpha_{\text{relation}} : \mathcal{P}(W^n) \to \mathcal{P}(W^k)$ by $\alpha_{\text{relation}}(r) := \{\vec{x} \mid \alpha_{\text{tuple}}(\vec{x}) \in r\}$. An atomic formula like $R(x, x, y, x)$ in the variable context $(x, y, z)$ corresponds to the substitution $\alpha_{\text{tuple}}(xyz) = xxxy$. Both give rise to the
same operation on relations which accepts as input a 4-ary relation \( r \) and outputs a 3-ary relation \( \{(x, y, z) \mid (x, x, y, x) \in r\} \). We often omit the superscripts and simply write \( \alpha \) for both \( \alpha \)-tuple and \( \alpha \)-relation. We generally use lower case Greek letters near the beginning of the alphabet to denote substitutions, e.g. \( \alpha, \beta, \gamma \).

**Definition 2.** We say a **cylindrification** is a substitution where as a tuple of symbols \( \alpha(x_1, \ldots, x_k) \) is a subtuple of \( (x_1, \ldots, x_k) \). I.e., the function \( \alpha: \{1, \ldots, n\} \to \{1, \ldots, k\} \) is increasing. We often use lowercase \( c \) to denote cylindrifications. A collection \( c_1, \ldots, c_m \) of cylindrifications are called **partitioning cylindrifications** when they take the form \( c_i(\bar{x}_1 \cdots \bar{x}_m) = \bar{x}_i \). I.e., we have \( c_i: k_i \to n \) where \( n = k_1 + \cdots + k_m \), and the function \( c_i: \{1, \ldots, k_i\} \to \{1, \ldots, n\} \) is given by \( c_i(l) = l + \sum_{j=1}^{i-1} k_j \).

We use \( \text{id} \) to denote the identity substitution \( \text{id}: n \to n \) for each \( n \). Let \( \alpha: n \to k \) and \( \beta: k \to m \) be two composable substitutions. Note that \( (\beta \circ \alpha)_{\text{relation}} = \beta_{\text{relation}} \circ \alpha_{\text{relation}} \). To be able to say this axiomatically, we need different notation for the two compositions. We shall use \( (\beta \circ \alpha)(r) \) for the former and \( \beta(\alpha(r)) \) for the latter.

Now we define the classes of algebras of interest to us.

**Definition 3.** A **first order algebra** (with equality) is an algebra in the multisorted signature specified in Definition 1 that arises from some set \( W \) in the following way:

- The interpretation of the sort \( n \) is \( P(W^n) \), the collection of all \( n \)-ary relations on \( W \).
- The interpretation of a substitution \( \alpha: n \to k \) is \( \alpha_{\text{relation}}: P(W^n) \to P(W^k) \) as defined above.
- The Boolean operations \( 0, 1, \lor, \land, \neg \) are interpreted as usual on each sort.
- Projection \( \exists^n: n + 1 \to n \) is interpreted as expected. In detail, \( \exists^n(r) := \{(x_1, \ldots, x_n) \mid \exists y((x_1, \ldots, x_n, y) \in r)\} \)
- The constant \( \Delta^n_{i,j}: n \) is interpreted as \( \Delta^n_{i,j} := \{(x_1, \ldots, x_n) \mid x_i = x_j\} \)

We will have occasion to look at various reducts of our signature, and the corresponding reducts of the first order algebras are given appropriate names. E.g., the **positive existential algebras** (without equality) are the reducts of the first order algebras to the signature not containing negation (for any sort), and not containing the constants
for equality, but otherwise containing all the symbols. The positive quantifier-free algebras (without equality) are when we restrict attention to just the substitutions and the lattice operations 0, 1, \lor, \land for each sort.

Just as all first order formulas can be constructed from the atomic formulas using the Boolean connectives and existential quantification, so too is every operation on relations arising from a first order formula equivalent to a term in our signature when looking at the first order algebras. Similarly, there are terms for every positive existential formula in the positive existential algebras, etc.

3. Positive Quantifier-free Algebras

The core of our argument can already be illustrated with the positive quantifier-free algebras, where our signature is restricted to the substitutions and the lattice operations 0, 1, \lor, \land for each sort. We begin by presenting the universal axioms which we will see axiomatize the subalgebras of the positive quantifier-free algebras — this is our goal in this section. Note that I have placed a list of all the axioms considered in this paper (for the various reducts) at the end of the paper for ease of reference. Also note that each of these “axioms” is actually an axiom schema.

The Positive Quantifier-free Axioms

0) When \(c_1, \ldots, c_m\) are partitioning cylindrifications of arities \(c_i: k_i \to (k_1 + \cdots + k_m)\) we write: For all \(r_1, s_1: k_1\), and all \(r_2, s_2: k_2, \ldots\), and all \(r_m, s_m: k_m\) we have

\[
\text{If } \bigvee_{i=1}^m c_i(s_i) \geq \bigwedge_{i=1}^m c_i(r_i), \text{ then } s_i \geq r_i \text{ for some } i = 1, \ldots, m.
\]

1) 0, 1, \lor, \land form a (bounded) distributive lattice in each sort. In particular, each sort comes with a partial order \(\leq\) defined by \(r \leq s\) just in case \(r = r \land s\) or equivalently \(s = r \lor s\).

2) Substitutions preserve 0, 1, \lor, \land. E.g., when \(\alpha: k \to n\) we write: For all \(r, s: k\) we have \(\alpha(r \land^k s) = \alpha(r) \land^n \alpha(s)\).

3) When \(\alpha: k \to n\) and \(\beta: n \to m\) we write: For all \(r: k\) we have

\[
(\beta \circ \alpha)(r) = \beta(\alpha(r)).
\]

4) For each identity substitution \(\text{id}: n \to n\) we write: For all \(r: n\) we have

\[
\text{id}(r) = r.
\]

Remark 4. Here are some notes on the axioms, and intuitive explanations.
**FIRST ORDER ALGEBRAS**

- Intuitively, axiom (0) says that if you have a union of “orthogonal cylinders” covering a “rectangle”, then one of the cylinders has width larger than the width of the corresponding side of the rectangle. Note that the instances of axiom (0) are not Horn clauses (they are universal implications where the conclusion is a disjunction of atomic formulas).
- Axioms (1)-(4), which are equations, axiomatize the Horn clause theory of positive quantifier-free algebras. When axiomatizing the algebras of larger signatures, we will see that the difference between the universal and the Horn clause theories is still just axiom (0).
- Axiom (4) is redundant in the context of axioms (0), (1), and (3). However, I include it because we will have occasion to omit axiom (0) when considering the Horn clause theory. To see this redundancy, note that id: \( n \to n \) just by itself is trivially a partitioning cylindrification. Thus by axiom (0) we have \( \text{id}(s) \geq \text{id}(r) \) implies \( s \geq r \). But axiom (3) gives \( \text{id}(\text{id}(t)) = \text{id}(t) \), and so we can get \( \text{id}(t) = t \).
- It is straightforward to check that the axioms are all true in positive quantifier-free algebras. For illustration, let us verify axiom (0). Suppose that \( s_i \not\geq r_i \) for each \( i = 1, \ldots, m \). Then we may introduce \( \bar{x}_i \in r_i - s_i \), and so \( (\bar{x}_1 \cdots \bar{x}_m) \in \bigwedge_i c_i(r_i) - \bigvee_i c_i(s_i) \).
- It follows from axioms (0), (1), and (3) that everything in sort zero is either 0 or 1. To see this, note that \( \text{id}_1, \text{id}_2: 0 \to 0 \) are partitioning cylindrifications. Then given any element \( r \) of sort zero, we have
  \[
  \text{id}_1(r) \vee \text{id}_2(0) = r \vee 0 \geq 1 \land r = \text{id}_1(1) \land \text{id}_2(r)
  \]
  So either \( r \geq 1 \) (and hence \( r = 1 \)) or \( 0 \geq r \) (and hence \( r = 0 \)).
- We may consider \( 0 \not\geq 1 \) in sort zero to be a special case of axiom (0), because the empty collection of cylindrifications trivially forms a partitioning cylindrification (of sort zero). If this offends the reader’s sensibilities, then they may specifically add an axiom asserting that \( 0 \not\geq 1 \) in sort zero. Taken together with the previous remark, we see that an algebra satisfying axioms (0), (1), and (3) will have exactly two elements in sort zero.

The main result of this section and essentially this paper as a whole is the following theorem.

**Theorem 5.** Axioms (0)-(4) axiomatize the subalgebras of the positive quantifier-free algebras.
A basic step in our proof of Theorem 5 is the observation that a prime filter in any one of the sorts gives rise to a morphism from an abstract algebra that satisfies the axioms to a concrete positive quantifier-free algebra. If you think of the abstract algebra as a theory, and the morphism to a concrete algebra as a model of this theory, then intuitively Lemma 6 says that any prime filter in sort $n$ is the type of an $n$-tuple in some model. In fact, we can take the model to just consist of this $n$-tuple. Lemma 8 will allow us to realize finitely many types at once, and so then by the compactness theorem we will be able to realize all types at once, yielding an embedding.

**Lemma 6.** Let $F$ be a prime filter on sort $n$ of some algebra $L$ satisfying the axioms (1), (2), and (3). Let $F_1, \ldots, F_n$ be distinct symbols. Let $W = \{F_1, \ldots, F_n\}$. Let $A(W)$ denote the positive quantifier-free algebra generated by the set $W$. Define a function (on each sort) $\varphi: L \to A(W)$ by putting

$$
\alpha(F_1, \ldots, F_n) \in \varphi(r) \iff \alpha(r) \in F
$$

Then $\varphi$ is a morphism.

**Proof.** Note that the definition makes sense because every tuple (of any length) from $W$ can be expressed as $\alpha(\bar{F})$ for a unique substitution $\alpha$. Further, if $\alpha(\bar{F})$ has length $k$, then $\alpha: k \to n$, and so $\alpha(r): n$.

- $\varphi(0) = \emptyset$ for each sort because $\alpha(0) = 0 \notin F$
- $\varphi(1) = W^k$ because $\alpha(1) = 1 \in F$

$$
\alpha(\bar{F}) \in \varphi(r) \cup \varphi(s) \iff \alpha(r) \in F \text{ or } \alpha(s) \in F
$$

$$
\iff \alpha(r \lor s) \in F
$$

$$
\iff \alpha(\bar{F}) \in \varphi(r \lor s)
$$

$$
\alpha(\bar{F}) \in \varphi(r) \cap \varphi(s) \iff \alpha(r) \in F \text{ and } \alpha(s) \in F
$$

$$
\iff \alpha(r \land s) \in F
$$

$$
\iff \alpha(\bar{F}) \in \varphi(r \land s)
$$

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2Axiom (1) ensures that each sort is a distributive lattice, and so it makes sense to speak of a prime filter on a sort.
\[ \alpha(\bar{F}) \in \beta^{relation}(\varphi(r)) \iff \beta(\alpha(\bar{F})) \in \varphi(r) \]
\[ \iff (\alpha \circ \beta)(\bar{F}) \in \varphi(r) \]
\[ \iff \alpha(\beta(r)) = (\alpha \circ \beta)(r) \in F \]
\[ \iff \alpha(\bar{F}) \in \varphi(\beta(r)) \]

Using axiom (4), we can make sure to separate any two distinct points using such a morphism. Let \( r \neq s \) in the same sort. Then there is a prime filter \( F \) containing say \( r \) and not \( s \). \( \bar{F} = \text{id}(\bar{F}) \in \varphi(r) - \varphi(s) \) because \( \text{id}(r) = r \in F \) and \( \text{id}(s) = s \notin F \). Taking a product of a bunch of such morphisms, we actually get an embedding of an algebra satisfying axioms (1)-(4) into a product of positive quantifier-free algebras. Hence the following proposition.

**Proposition 7.** Axioms (1)-(4) axiomatize the class of subalgebras of products of positive quantifier-free algebras. Thus, we have found an equational axiomatization of the Horn clause theory of positive quantifier-free algebras.

Observe that this does not automatically give us the universal theory, because the positive quantifier-free algebras are not closed under products (even just look at the zero sort and observe that there must be exactly two elements in it). Note that whereas \( \mathcal{P}(\bigcup W_i) = \prod \mathcal{P}(W_i) \) it is not the case that \( \mathcal{P}((\bigcup W_i)^n) = \prod \mathcal{P}(W_i^n) \).

In order to have an embedding into an actual positive quantifier-free algebra instead of a product of them, we show that we can deal with all the prime filters at once by satisfying a certain first order theory.

Given an algebra \( L \) that satisfies axioms (0)-(4), let us introduce a relation symbol \( r \) of arity \( n \) for each element \( r \) of \( L \) of each sort \( n \). We also introduce constants \( F_1, \ldots, F_n \) for each prime filter \( F \) of \( L \) on sort \( n \). Let \( T \) be the first order theory in this language with the following axiom schemata:

\[ A) \pm r(\bar{F}) \text{ according to whether } r \in F \]

\[ ^{3}\text{We take this as an assumption in this paper. The existence of prime filters extending filters is known \((10)\) to be equivalent to the compactness theorem for first order logic (modulo ZF say), and we also freely use the compactness theorem. This property is implied by the axiom of choice, but is known to be strictly weaker \((4)\). It is of course also known to be independent of ZF \((3)\). We also will use the property in the following form: When we have a filter disjoint from an ideal in a distributive lattice, we may introduce a prime filter extending the filter and not containing anything from the ideal.} \]
B) The **morphic conditions**, i.e.

i) \[ \forall \bar{x} \neg 0(\bar{x}) \]. We have such a sentence for the 0 of each sort.

ii) \[ \forall \bar{x} 1(\bar{x}) \]

iii) \[ \forall \bar{x} (r \lor s)(\bar{x}) \iff (r(\bar{x}) \lor s(\bar{x})) \]. Note that in \((r \lor s)(\bar{x})\) the “\(\lor\)” is an operation of the algebra, while in \((r(\bar{x}) \lor s(\bar{x}))\) the “or” is a logical symbol of the ambient first order logic. We have such a sentence for every pair \((r, s)\) of elements from the same sort.

iv) \[ \forall \bar{x} (r \land s)(\bar{x}) \iff (r(\bar{x}) \land s(\bar{x})) \]

v) \[ \forall \bar{x} (\alpha(r))(\bar{x}) \iff r(\alpha^\text{tuple}(\bar{x})) \]. We have such a sentence for every substitution \(\alpha : k \to n\) and element \(r \in L\) of sort \(k\).

It is straightforward to verify that a model of the morphic conditions of \(T\) is (essentially) the same thing as a morphism from the algebra \(L\) to a positive quantifier-free algebra. Item (A) of \(T\) ensures that this morphism is 1-1 (on each sort). To show that this theory is satisfiable, we use the compactness theorem and show that the theory is finitely satisfiable. Thus, it suffices to find a model satisfying all of item (B) but the instances of item (A) involving only finitely many prime filters \(F^1, \ldots, F^m\). (We use superscript here to avoid confusion of the prime filters with the constants associated to each of them.)

The key idea for how to proceed is to assemble these finitely many prime filters into one master prime filter on a larger sort. We formalize this in the following lemma.

**Lemma 8.** Let \(L\) be an algebra that satisfies axioms (0), (1), and (2). Let \(k_1 + \cdots + k_m = n\). Let \(c_i : k_i \to n\) be partitioning cylindrifications. Let \(F^i\) be prime filters on sort \(k_i\) respectively. Then there is a prime filter \(G\) on sort \(n\) such that for each \(i = 1, \ldots, m\), for all \(r \in L\) of sort \(k_i\) we have \(c_i(r) \in G\) iff \(r \in F^i\).

**Proof.** Let \(A\) be the distributive lattice which is sort \(n\) of \(L\). Define

\[ G_F := \{ z \in A \mid z \geq \bigwedge_{i=1}^m c_i(r_i) \text{ for some } r_i \in F^i \} \]

and

\[ G_I := \{ z \in A \mid \bigvee_{i=1}^m c_i(s_i) \geq z \text{ for some } s_i \not\in F^i \} \]

We claim that \(G_F\) is a filter, \(G_I\) is an ideal, and they are disjoint. First we show they are disjoint. Suppose, to get a contradiction, that \(\bigvee_i c_i(s_i) \geq z \geq \bigwedge_i c_i(r_i)\) where \(s_i \not\in F^i\) and \(r_i \in F^i\). Then \(s_i \geq r_i\) for some \(i\) by axiom (0), implying that \(s_i \in F^i\) because \(F^i\) is upward-closed, but as noted \(s_i \not\in F^i\), and we have a contradiction.
Next, observe that $1 \in G_F$ since $1 \in F^i$ for each $i$ and $1 \geq \bigwedge_i c_i(1)$. Similarly $0 \in G_I$. It follows at once that $0 \not\in G_F$ and $1 \not\in G_I$ because $G_F$ and $G_I$ are disjoint.

It is obvious from the definitions that $G_F$ is upward-closed and $G_I$ is downward closed.

Finally, suppose $z, z' \in G_F$. Let $z \geq \bigwedge_i c_i(r_i)$ and $z' \geq \bigwedge_i c_i(r'_i)$ where $r_i, r'_i \in F^i$. Then $z \wedge z' \geq \bigwedge_i c_i(r_i \wedge r'_i)$ by axiom (2), and $r_i \wedge r'_i \in F^i$ for each $i$. So $z \wedge z' \in G_F$. The argument that $z, z' \in G_I$ implies $z \vee z' \in G_I$ is similar.

Because $G_F$ and $G_I$ are disjoint, we may introduce a prime filter $G$ such that $G_F \subseteq G$ and $G \cap G_I = \emptyset$. This $G$ is a prime filter satisfying the desired property. If $r \in F^i$, then $c_i(r) = c_i(r) \wedge \bigwedge_{j \neq i} c_j(1) \in G_F$, and so $c_i(r) \in G$. If $r \notin F^i$, then $c_i(r) = c_i(r) \vee \bigvee_{j \neq i} c_j(0) \in G_I$, and so $c_i(r) \notin G$.

Armed with this lemma, we may return to showing that the theory $T$ is finitely satisfiable. Given our finitely many prime filters $F^1, \ldots, F^m$, we may introduce by Lemma 8 a prime filter $G$ such that $c_i(r) \in G \iff r \in F^i$ for each $i$. Introduce distinct symbols $G_1, \ldots, G_n$, where $n = k_1 + \cdots + k_m$, the sum of the arities of the $F^i$. Let $W = \{G_1, \ldots, G_n\}$. We will interpret the constants corresponding to each prime filter $F^i$ by $c_i(G)$ respectively. By Lemma 6 we know that $\varphi: L \rightarrow A(W)$ defined by

\[
\alpha(G) \in \varphi(r) \iff \alpha(r) \in G
\]

is a morphism. Further,

\[
c_i(G) \in \varphi(r) \iff c_i(r) \in G \iff r \in F^i
\]

as desired. So $\varphi$ yields the desired model of the small portion of $T$ we gave ourselves. We have finished the proof of Theorem 5.

4. Adding Negation, Projection, Equality

4.1. Negation. It is relatively easy to extend the results of Section 3 to algebras with negation, yielding an axiomatization of the subalgebras of the quantifier-free algebras.

Theorem 9. Axioms (0)-(6) axiomatize the subalgebras of the quantifier-free algebras. (Axioms (5) and (6) are given below.)

Examining the argument of Section 3, the only place where there needs to be significant change is for Lemma 6 where we need to now also verify that the function defined is morphic for negation. In other words, we want $\varphi(\neg r) = \neg \varphi(r)$. I.e., we want $\alpha(\neg r) \in F \iff \alpha(r) \notin F$.
Axioms for Negation

5) When \( \alpha: k \to n \) is a substitution we write: For all \( r: k \) we have
\[
\alpha(\neg r) = \neg \alpha(r).
\]

6) For each sort \( n \), we write: For all \( r: n \) we have
\[
r \lor \neg r = 1 \quad \text{and} \quad r \land \neg r = 0.
\]

Axiom (6) ensures that for any prime filter \( F \), \( \neg r \in F \) iff \( r \notin F \).

Then using axiom (5) we have \( \alpha(\neg r) \in F \) iff \( \neg \alpha(r) \in F \) iff \( \alpha(r) \notin F \). It is easy to check that these axioms are true in quantifier-free algebras.

So we get a modified version of Lemma 6.

**Lemma 10.** Let \( F \) be a prime filter in sort \( n \) of some algebra \( L \) satisfying the axioms (1)-(3), and (5)-(6). Let \( F_1, \ldots, F_n \) be distinct symbols. Let \( W = \{F_1, \ldots, F_n\} \). Let \( A(W) \) denote the quantifier-free algebra generated by the set \( W \). Define a function (on each sort) \( \varphi: L \to A(W) \) by putting
\[
\alpha(F) \in \varphi(r) \iff \alpha(r) \in F
\]

Then \( \varphi \) is a morphism.

As before we get the following proposition:

**Proposition 11.** Axioms (1)-(6) axiomatize the class of subalgebras of products of quantifier-free algebras. Thus, we have found an equational axiomatization of the Horn clause theory of quantifier-free algebras.

The argument of Lemma 8 remains unchanged, but the theory \( T \) discussed in the surrounding argument changes in a very minor way: we must add preservation of negation to the morphic conditions. In detail, we add the following sentences to \( T \):

**B) vi) \( \forall \vec{x} \ (\neg r)(\vec{x}) \iff \neg (r(\vec{x})) \)**

The addition of this to the theory \( T \) does not change the rest of the argument because Lemma 10 handles negation.

4.2. **Projection.** Adding projection takes more work than adding negation. As in the quantifier-free case, our argument below works whether negation is present or not, and so we will obtain the following two theorems. (The new axioms (7)-(10) are presented later in this section.)

**Theorem 12.** Axioms (0)-(4) and (7)-(10) axiomatize the subalgebras of the positive existential algebras.

**Theorem 13.** Axioms (0)-(10) axiomatize the subalgebras of the first order algebras.
For definiteness we will speak as if negation is not present. That is, the signature under consideration includes the substitutions, the lattice operations, and the projections. Our general approach is to find a 1-1 function from the abstract algebra satisfying the axioms to a concrete positive existential algebra which is not quite a morphism because there are not enough witnesses, but then we modify the function to obtain an actual embedding by adding witnesses.

Often when dealing with a projection \( \exists: n+1 \to n \) we wish to also speak of the associated cylindrification \( c: n \to n+1 \) given by \( c(i) = i \), i.e. \( c(\bar{x}y) = \bar{x} \). The operations \( \exists \) and \( c \) form a Galois connection, which is a special case of how direct image and inverse image form a Galois connection. However, we present the situation equationally with the following axioms, which imply more than just this Galois connection.

**Axioms for Projection**

7) \( \exists \) preserves 0 and \( \lor \)

8) For each projection \( \exists: (n+1) \to n \) and associated cylindrification \( c: n \to (n+1) \) we write: For all \( r: (n+1) \) we have

\[
r \leq c(\exists(r))
\]

9) For each projection \( \exists: (n+1) \to n \) and associated cylindrification \( c: n \to (n+1) \) we write: For all \( r: (n+1) \) and all \( s: n \) we have

\[
\exists(r \land c(s)) = \exists(r) \land s.
\]

10) Let \( \alpha_i: k_i \to m \) be substitutions for \( i = 1, \ldots, n \). Let \( \beta_i: (k_i+1) \to (m+n) \) be the substitutions defined by \( \beta_i(\bar{x}y_1 \cdots y_n) = \alpha_i(\bar{x})y_i \). Then we write: For all \( r_1: k_1+1, \ldots, \) and all \( r_n: k_n+1 \) we have

\[
\exists^{(n)}(\bigwedge_{i=1}^n \beta_i(r_i)) = \bigwedge_{i=1}^n \alpha_i(\exists(r_i))
\]

where \( \exists^{(n)} \) means we apply projection \( n \) times.

**Remark 14.** Here are some notes on the axioms for projection, and intuitive explanations.

- It is straightforward to check that these axioms are all true in the positive existential algebras. For illustration, consider axiom (10). Intuitively, this axioms says that casting an ensemble for a theatrical production involving \( n \) roles is equivalent to finding a good actor for each role, as long as you do not care about how the team works together. A tuple \( \bar{x} \) is in \( \exists^{(n)}(\bigwedge_{i=1}^n \beta_i(r_i)) \) iff there are \( y_1, \ldots, y_n \) such that for each \( i = 1, \ldots, n \) we have \( \alpha_i(\bar{x})y_i = \beta_i(\bar{x}y) \in r_i \). However, since each \( y_i \) occurs on its own, this is equivalent to saying that for each \( i = 1, \ldots, n \) there is some \( y_i \) with \( \alpha_i(\bar{x})y_i \in r_i \), which is to say \( \bar{x} \) is in \( \bigwedge_{i=1}^n \alpha_i(\exists(r_i)) \).
Note that \( r \leq s \) implies \( \exists(r) \leq \exists(s) \) follows from axiom (7). Of course the substitutions are also increasing in this way because of axiom (2).

A subtle point is that typically \( \exists(c(s)) = s \), as seems apparent from axiom (9), but not always. If we consider the algebra generated by the empty set, and \( s = 1 \) in sort zero, then \( c(s) = 1 = 0 \) in sort one, and \( \exists(c(s)) = 0 \neq 1 = s \). On the other hand, we do always have \( \exists(c(s)) \leq s \).

**Definition 15.** Let \( L \) be an algebra in the positive existential signature, and let \( A(W) \) be the positive existential algebra generated by some set \( W \). An **almost morphism** is a function (on each sort) \( \varphi : L \to A(W) \) such that

1. \( \varphi \) is morphic for the substitutions and the lattice operations
2. \( \exists(\varphi(r)) \subseteq \varphi(\exists(r)) \)

I.e., an almost morphism is a morphism except for the possibility it might not satisfy \( \exists(\varphi(r)) \supseteq \varphi(\exists(r)) \).

The modified version of Lemma 6 is as follows.

**Lemma 16.** Let \( F \) be a prime filter in sort \( n \) of some algebra \( L \) satisfying the axioms (1), (2), (3), and (8). Let \( F_1, \ldots, F_n \) be distinct symbols. Let \( W = \{F_1, \ldots, F_n\} \). Let \( A(W) \) denote the positive existential algebra generated by the set \( W \). Define a function (on each sort) \( \varphi : L \to A(W) \) by putting

\[
\alpha(\bar{F}) \in \varphi(r) \iff \alpha(r) \in F
\]

Then \( \varphi \) is an almost morphism.

**Proof.** The new thing we need to verify is that \( \exists(\varphi(r)) \subseteq \varphi(\exists(r)) \). Let \( \alpha(F) \in \exists(\varphi(r)) \). We may introduce \( \beta(\bar{F}) \) such that \( c(\beta(\bar{F})) = \alpha(F) \) and \( \beta(F) \in \varphi(r) \). Thus, \( \beta(r) \in F \). We want to check that \( c(\beta(F)) \in \varphi(\exists(r)) \), i.e. \( (\beta \circ c)(\exists(r)) \in F \). Well,

\[
(\beta \circ c)(\exists(r)) = \beta(c(\exists(r))) \\
\geq \beta(r) \\
\in F
\]

Of course, Lemma 16 does not immediately yield an axiomatization of the Horn clause theory, but rather the following lemma.

**Lemma 17.** Let \( L \) be an algebra satisfying axioms (1)-(4) and (8). Let \( r \neq s \) be two distinct elements in the same sort. Then there is an
almost morphism \( \varphi \) from \( L \) to a positive existential algebra such that \( \varphi(r) \neq \varphi(s) \).

Similarly, from Lemma 16 we may repeat our now familiar argument involving satisfying the theory \( T \), except that now the morphic conditions become the almost morphic conditions. That is, instead of adding negation we add

B) vi) \( \forall \bar{x} \ (\exists y(r(\bar{x}y)) \Rightarrow (\exists(r))(\bar{x})) \)

The argument does not lead us to the finish line, rather it yields the following lemma:

**Lemma 18.** Let \( L \) be an algebra satisfying axioms (0)-(4), and (8). Then there is a 1-1 almost morphism from \( L \) to some positive existential algebra.

To go further, we need a way of turning an almost morphism into an actual morphism. The following lemmas help us accomplish this.

**Lemma 19.** Let \( L \) be an algebra that satisfies the axioms (1), (2), (7), and (9). Let \( \exists(r) \) be in some prime filter \( F \) on sort \( n \). Then there is some prime filter \( G \) on sort \( n + 1 \) such that \( r \in G \) and for all \( u \) in sort \( n \) we have \( c(u) \in G \) iff \( u \in F \).

**Proof.** We use the same approach as in the proof of Lemma 8. Let \( A \) be the distributive lattice which is sort \( n + 1 \) of \( L \). Define

\[
G_F := \{ z \in A \mid z \geq r \land c(u) \text{ for some } u \in F \}
\]

and

\[
G_I := \{ z \in A \mid c(u) \geq z \text{ for some } u \notin F \}
\]

Then as in Lemma 8 we have that \( G_F \) is a filter, \( G_I \) is an ideal, and they are disjoint. These things are easy to check, and we here only deal with disjointness for illustration. Suppose, to get a contradiction, that \( c(u) \geq z \geq r \land c(t) \) where \( t \in F \) and \( u \notin F \). Then by axioms (7) and (9) we get

\[
\begin{align*}
  u &\geq \exists(c(u)) \\
  &\geq \exists(r \land c(t)) \\
  &= \exists(r) \land t \\
  \in F
\end{align*}
\]

putting \( u \in F \), a contradiction.

So we may introduce a prime filter \( G \) extending \( G_F \) and disjoint from \( G_I \). This works. \(\square\)
Given a prime filter $G$ on sort $n + 1$, note that $c^{-1}(G) := \{ u \mid c(u) \in G \}$ is always a prime filter on sort $n$ such that $u \in c^{-1}(G)$ iff $c(u) \in G$.

The above lemma asserts that given any prime filter $F$ on sort $n$ and element $r$ with $\exists(r) \in F$, there is some prime filter $G$ such that $r \in G$ and $c^{-1}(G) = F$.

Recall that when $L$ is an algebra in the positive existential signature, an almost morphism from $L$ to a positive existential algebra is (essentially) the same thing as a model of the almost morphic conditions. Note that every tuple $(a_1, \ldots, a_n)$ from a model $M$ of the almost morphic conditions gives rise to a prime filter $p(\bar{a}) := \{ r \mid M \models r(\bar{a}) \}$ of $L$ on sort $n$.

If $M_1$ and $M_2$ are models of the almost morphic conditions, then we say that $M_2$ has witnesses over $M_1$ when $M_1$ is a substructure of $M_2$, written $M_1 \subseteq M_2$, and whenever $(a_1, \ldots, a_n)$ is a tuple from $M_1$ and $G$ is a prime filter of $L$ on sort $n + 1$ with $c^{-1}(G) = p(\bar{a})$, then there is some element $b$ in $M_2$ such that $\bar{a}b$ weakly realizes $G$, i.e. $M_2 \models r(\bar{a}b)$ for every $r \in G$. Note that I say “weakly realizes” instead of “realizes” because we don’t require that $M_2 \models \neg r(\bar{a}b)$ when $r \notin G$.

**Lemma 20.** Let $L$ be an algebra that satisfies axioms (1)-(3), (7)-(10). Let $M_1$ be a model of the almost morphic conditions. Then there is some model $M_2 \supseteq M_1$ of the almost morphic conditions which has witnesses over $M_1$.

**Proof.** Consider the following first order theory $U$, the signature for which contains a relation symbol for each element of $L$ with arity corresponding to its sort, and also some constants as indicated below:

A) The almost morphic conditions.
B) The literal diagram of $M_1$. I.e. $\pm r(\bar{a})$ according to whether this is true in $M_1$.
C) For each prime filter $G$ in sort $n + 1$ of $L$, for each $n$-tuple $\bar{a}$ from $M_1$ with $p(\bar{a}) = c^{-1}(G)$, and for each $r \in G$, we write

$$r(\bar{a}y_{G,\bar{a}})$$

where $y_{G,\bar{a}}$ is a new constant.

Items (A) and (B) of the theory ensure that a model satisfies the almost morphic conditions and is a superstructure of $M_1$. Item (C) ensures that a model will have witnesses over $M_1$.

Given finitely much of items (B) and (C), only finitely many elements of $M_1$ appear. Collect them together in one big tuple $\bar{a}$, say of length $m$, without duplicates. We will be satisfying all of item (A). Let $(G_1, \bar{a}_1), \ldots, (G_n, \bar{a}_n)$ be the tuple/prime filter pairs that occur in item (C). We may assume $n \geq 1$ (otherwise we can let $M_2 = M_1$). Finitely
many of the elements \( r \in G_i \) will occur, but we will actually be ensuring 
things work for all \( r \in G_i \), for each \( i \). We have \( c^{-1}(G_i) = p(\bar{a}_i) \) where 
\( \alpha_i(\bar{a}) = \bar{a}_i \) for some substitution \( \alpha_i \). Let \( k_i \) denote the length of \( \bar{a}_i \). 
Define substitutions \( \beta_0(\bar{a}y_1 \cdots y_n) = \bar{a} \) and \( \beta_i(\bar{a}y) = \bar{a}_iy_i \) for \( 1 \leq i \leq n \).

Now I claim that there is a prime filter \( H \) on sort \( m+n \) such that 

I) For all \( r \in \text{sort } m \) we have \( \beta_0(r) \in H \) iff \( r \in p(\bar{a}) \), and 

II) For each \( i = 1, \ldots, n \), for each \( r \in \text{sort } k_i+1 \) we have \( r \in G_i \) 
implies \( \beta_i(r) \in H \).

Suppose for now that there is such a prime filter. Then we define a 
model \( M_2^- \) with underlying set \( \{ H_1, \ldots, H_{m+n} \} \) as follows:

\[
M_2^- \models r(\gamma(H)) \iff \gamma(r) \in H
\]

We interpret the constants \( a_1, \ldots, a_m \) by \( H_1, \ldots, H_m \), and the constants 
\( y_{G_1, \bar{a}_1}, \ldots, y_{G_n, \bar{a}_n} \) by \( H_{m+1}, \ldots, H_{m+n} \). Of course item (A) is satisfied 
by Lemma 18. Now consider item (C). Let \( r \in G_i \). We want \( M_2^- \models r(\bar{a}_i y_{G_i, \bar{a}_i}) \), i.e. \( \beta_i(r) \in H \) (because \( \beta_i(\bar{a} \bar{y}) = \bar{a}_i y_i \)). But this is implied 
by \( r \in G_i \), according to (II).

Finally consider item (B). We show that for any tuple \( \gamma(\bar{a}) \) assembled 
from \( \bar{a} \), and for any \( r \) of the appropriate sort, we have \( M_2^- \models r(\gamma(\bar{a})) \) 
iff \( M_1 \models r(\gamma(\bar{a})) \). Note that 

\[
\gamma(\bar{a}) = \gamma(\beta_0(\bar{a} \bar{y})) = (\beta_0 \circ \gamma)(\bar{a} \bar{y})
\]

and so by the definition of \( M_2^- \), we have \( M_2^- \models r(\gamma(\bar{a})) \) iff \( \beta_0(\gamma(r)) = (\beta_0 \circ \gamma)(r) \in H \). By (I), this is equivalent to \( \gamma(r) \in p(\bar{a}) \), i.e. \( M_1 \models (\gamma(r))(\bar{a}) \). Since \( M_1 \) itself satisfies the almost morphic conditions, this 
is equivalent to \( M_1 \models r(\gamma(\bar{a})) \), as desired.

Now we show that we can get such an \( H \). We use an argument 
similar to that of Lemma 8 or Lemma 19. Let \( A \) be the distributive 
lattice which is sort \( m+n \) of \( L \). Define 

\[
H_F := \{ z \in A \mid z \geq \beta_0(r) \wedge \bigwedge_{i=1}^{n} \beta_i(r_i) \text{ for some } r \in p(\bar{a}) \text{ and } r_i \in G_i \}
\]

and 

\[
H_I := \{ z \in A \mid \beta_0(r) \geq z \text{ for some } r \not\in p(\bar{a}) \}
\]

Then \( H_F \) is a filter, \( H_I \) is an ideal, and they are disjoint. The main 
thing to check is the disjointness. Suppose, to get a contradiction, that 

\[
\beta_0(s) \geq z \geq \beta_0(r) \wedge \bigwedge_{i=1}^{n} \beta_i(r_i)
\]

where \( s \not\in p(\bar{a}), r \in p(\bar{a}), \) and \( r_i \in G_i \) for each \( i \). Observe that \( \ell(\bar{u}) = \beta_0 \), 
and so by repeated use of the facts that \( \exists \ell(c(t)) \leq t \) and \( \exists \) is increasing,
we get
\[ s \geq \exists^n(\beta_0(s)) \geq \exists^n(\beta_0(r) \land \bigwedge_{i=1}^n \beta_i(r_i)) \]

By repeated use of axiom (9), the right hand side becomes
\[ r \land \exists^n\left(\bigwedge_{i=1}^n \beta_i(r_i)\right) \]

Putting this all together with axiom (10), we see that
\[ s \geq r \land \bigwedge_{i=1}^n \alpha_i(\exists(r_i)) \]

To get that \( s \in p(\bar{a}) \), a contradiction, we will show that \( \alpha_i(\exists(r_i)) \in p(\bar{a}) \) for each \( i \). By axiom (8), \( c(\exists(r_i)) \geq r_i \in G_i \), so \( \exists(r_i) \in p(\bar{a}_i) \) (recall that \( c^{-1}(G_i) = p(\bar{a}_i) \) by assumption). Then, as \( M_1 \) satisfies the almost morphic conditions, and \( \alpha_i(\bar{a}) = \bar{a}_i \), we get that \( \alpha_i(\exists(r_i)) \in p(\bar{a}) \).

A prime filter \( H \) which extends \( H_F \) and is disjoint from \( H_I \) is as desired. \( \square \)

**Lemma 21.** Let \( L \) be an algebra that satisfies axioms (1)-(3) and (7)-(10). Let \( \varphi \) be an almost morphism from \( L \) to some positive existential algebra. Then there is a morphism \( \varphi^+ \) from \( L \) to some positive existential algebra such that \( \ker(\varphi^+) \subseteq \ker(\varphi) \). In particular, if \( \varphi \) is 1-1, then \( \varphi^+ \) is an embedding.

**Proof.** The almost morphism \( \varphi \) gives rise to a model \( M_1 \) which satisfies the almost morphic conditions. By Lemma 20 we may introduce a model \( M_2 \supseteq M_1 \) of the almost morphic conditions which has witnesses over \( M_1 \). Continuing in this way, we get a sequence
\[ M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \]

of length \( \omega \) where \( M_{n+1} \) has witnesses over \( M_n \). Let \( M^+ \) be the union of this chain of models. Since the almost morphic conditions are of a form preserved by unions of chains, we get that \( M^+ \) models them too.

Further, if \( M^+ \models (\exists(r))(\bar{a}) \), then \( M_n \models (\exists(r))(\bar{a}) \) for some \( n \). By Lemma 19 we may introduce a prime filter \( G \) such that \( p(\bar{a}) = c^{-1}(G) \) and \( r \in G \). Since \( M_{n+1} \) has witnesses over \( M_n \), there is some element \( b \in M_{n+1} \) such that \( M_{n+1} \models r(\bar{a}b) \). Thus, \( M^+ \models r(\bar{a}b) \). In summary, \( M^+ \models (\exists(r))(\bar{a}) \) implies \( M^+ \models \exists y(r(\bar{a}y)) \). Thus, the function given by \( \varphi^+(r) := r_M^+ \) is a morphism from \( L \) to the positive existential algebra generated by the underlying set of \( M^+ \).
Finally, suppose \( \varphi^+(r) = \varphi^+(s) \). We show that \( \varphi(r) = \varphi(s) \). If there were \( \bar{a} \) from \( M_1 \) with \( \bar{a} \in \varphi(r) - \varphi(s) \), then \( \bar{a} \in \varphi^+(r) - \varphi^+(s) \) as well, because \( M_1 \subseteq M^+ \).

From Lemma [17] and Lemma [21] we get the following proposition:

**Proposition 22.** Axioms (1)-(4) and (7)-(10) equationally axiomatize the subalgebras of products of the positive existential algebras.

Theorem [12] follows immediately from Lemma [18] and Lemma [21].

4.3. **Equality.** If we wish to add equality, we may do so (modularly) with the following axioms.

**Axioms for Equality**

11) (a) \( \Delta_{i,i}^n = 1 \)
(b) \( \Delta_{i,j}^n = \Delta_{j,i}^n \)
(c) \( \Delta_{i,j}^n \wedge \Delta_{j,k}^n \leq \Delta_{i,k}^n \)

12) When \( \alpha, \beta : k \to n \) are substitutions of matching arities we write:
\[
\alpha(r) \wedge \bigwedge_{l=1}^{k} \Delta_{\alpha(l),\beta(l)}^n = \beta(r) \wedge \bigwedge_{l=1}^{k} \Delta_{\alpha(l),\beta(l)}^n
\]

13) For each substitution \( \alpha : k \to n \) we write:
\[
\alpha(\Delta_{i,j}^k) = \Delta_{\alpha(i),\alpha(j)}^n
\]

It is straightforward to check that these equational axioms are all true in the concrete algebras where \( \Delta_{i,j}^n \) is interpreted as the \( n \)-ary relation which holds of an \( n \)-tuple iff the \( i \)th and \( j \)th coordinates are equal. Axiom (11) corresponds to the usual properties of an equivalence relation. Axiom (12) is algebraically saying the obvious fact that
\[
\{ \bar{x} | \alpha(\bar{x}) \in r \text{ and } \alpha(\bar{x}) = \beta(\bar{x}) \} = \{ \bar{x} | \beta(\bar{x}) \in r \text{ and } \alpha(\bar{x}) = \beta(\bar{x}) \}
\]

Finally, to make sense of axiom (13), recall that the \( i \)th coordinate of \( \alpha(\bar{x}) \) is \( x_{\alpha(i)} \). So, \( \alpha(\bar{x}) \in \Delta_{i,j}^k \) iff \( x_{\alpha(i)} = x_{\alpha(j)} \).

Let \( F \) be a prime filter on sort \( n \). Our basic strategy is the same – get a modified version of Lemma [3] by having \( F \) correspond to a tuple \( (F_1, \ldots, F_n) \) – except that now we may have to identify certain of the \( F_i \). By axiom (11), we may put
\[
F_i = F_j \iff \Delta_{i,j}^n \in F
\]

But then we may have \( \alpha(F) = \beta(F) \) for distinct substitutions \( \alpha \) and \( \beta \). The upshot of axiom (12) is that this won’t matter: If \( \alpha(F) = \beta(F) \)
then $\alpha(r) \in F \iff \beta(r) \in F$. To see this, observe that

$$\alpha(F) = \beta(F) \iff F_{\alpha(l)} = F_{\beta(l)} \text{ for each } l = 1, \ldots, k$$

$$\iff \bigwedge_{l=1}^{k} \Delta_{\alpha(l),\beta(l)}^{n} \in F$$

So if $\alpha(r) \in F$ and $\alpha(F) = \beta(F)$, then we get

$$\beta(r) \geq \beta(r) \land \bigwedge_{l=1}^{k} \Delta_{\alpha(l),\beta(l)}^{n}$$

$$= \alpha(r) \land \bigwedge_{l=1}^{k} \Delta_{\alpha(l),\beta(l)}^{n}$$

$$\in F$$

and so $\beta(r) \in F$.

Now we are in a position to obtain the with-equality version of Lemma 6 using axiom (13) for the preservation of the $\Delta_{i,j}^{n}$. For definiteness, we state the lemma for positive quantifier-free algebras with equality.

**Lemma 23.** Let $L$ be an algebra that satisfies axioms (1)-(3), (11)-(13). Let $F$ be a prime filter on sort $n$. Let $F_{1}, \ldots, F_{n}$ be symbols such that $F_{i} = F_{j}$ iff $\Delta_{i,j}^{n} \in F$. Let $W = \{F_{1}, \ldots, F_{n}\}$. Let $A(W)$ be the positive quantifier-free algebra with equality generated by $W$. Then $\varphi: L \to A(W)$ defined by

$$\alpha(F) \in \varphi(r) \iff \alpha(r) \in F$$

is a morphism.

**Proof.** As observed above, this definition of $\varphi$ is unambiguous by axiom (12).

The new thing we have to check is that

$$\varphi(\Delta^{k}_{i,j}) = \{\alpha(F) \mid F_{\alpha(i)} = F_{\alpha(j)}\}$$

Well,

$$\varphi(\Delta^{k}_{i,j}) = \{\alpha(F) \mid \alpha(\Delta^{k}_{i,j}) \in F\}$$

$$= \{\alpha(F) \mid \Delta_{\alpha(i),\alpha(j)}^{n} \in F\}$$

$$= \{\alpha(F) \mid F_{\alpha(i)} = F_{\alpha(j)}\}$$

$\square$
5. Theories

We now consider how formulas and theories may be understood in the context of our algebraic approach. Our discussion will also help explain the value of axiom (0) in letting us have a uniform argument for the various reducts.

Let us say a first order formula \( \alpha \) in some relational signature \( \sigma \) is an element of the free algebra (in the first order algebra signature) generated by the symbols of \( \sigma \). Let us use the notation \( F_{\sigma} \) to refer to this free algebra. Positive existential formulas, quantifier-free formulas, etc. are defined correspondingly. For an example, let \( \sigma \) consist of a unary relation symbol \( R \) and a binary relation symbol \( S \). Let \( \alpha : 2 \rightarrow 2 \) be the substitution \( \alpha(xy) = yx \). Then

\[
R, S, \alpha(S), \exists(\alpha(S)), R \land \exists(\alpha(S))
\]

are all formulas.

This way of viewing formulas does away with bound/free variables and the associated “alphabetic variants”, but of course a formula up to logical equivalence may have more than one syntactic representation in this formalism as well (e.g., \( \alpha(\alpha(S)) \) and \( S \) are logically equivalent). Also note that the variable context has now become an intrinsic part of the formula (its arity).

A first order \( \sigma \)-structure is a morphism from \( F_{\sigma} \) to some first order algebra \( M \). This is the same as a function which assigns to every relation symbol of \( \sigma \) a relation on the underlying set of \( M \). Let us use \( K \) to denote the class of concrete algebras for the kind of logic under consideration (i.e., \( K \) could be the first order algebras, or the positive existential algebras, etc.). Then a \( \sigma \)-structure for whatever logic is under consideration is a morphism from \( F_{\sigma} \) to an algebra \( M \in K \).

A first order \( \sigma \)-structure is a morphism from \( F_{\sigma} \) to some first order algebra \( M \). This is the same as a function which assigns to every relation symbol of \( \sigma \) a relation on the underlying set of \( M \). Let us use \( K \) to denote the class of concrete algebras for the kind of logic under consideration (i.e., \( K \) could be the first order algebras, or the positive existential algebras, etc.). Then a \( \sigma \)-structure for whatever logic is under consideration is a morphism from \( F_{\sigma} \) to an algebra \( M \in K \).

Given any collection \( T \) of identities of formulas (i.e., pairs of formulas from the same sort), a structure \( f : F_{\sigma} \rightarrow M \) is a model of \( T \) means that \( f(r) = f(s) \) for each pair \( (r, s) \in T \). A (partial) theory \( T \) is an “implicationally closed” collection of identities in the sense that if every model \( f \) of \( T \) satisfies \( f(r) = f(s) \), then also \( (r, s) \in T \). Every theory is in particular a congruence relation on \( F_{\sigma} \). Thus, we have an associated quotient \( F_{\sigma}/T \), which could be called the theory too. A morphism \( F_{\sigma}/T \rightarrow M \in K \) is the same thing as a model of \( T \). The algebras that arise as quotients in this way are exactly the subalgebras of the products of the concrete algebras, i.e., \( SP(K) \).

We include the easy verification of this fact for illustrative purposes. First let \( Q = F_{\sigma}/T \) be such a quotient. We now prove that \( Q \) must satisfy the equational theory of \( K \) (which is equivalent to the Horn
clause theory for the \( K \) of present interest). Let \( \varphi(\bar{r}) = \chi(\bar{r}) \) be an equation true in all members of \( K \). Let \( \bar{r} \) be some tuple from \( Q \).

Let \( \bar{r} \) be some tuple from \( Q \). Then let \( f : Q \rightarrow M \in K \) be any model of \( T \). Of course we must have \( \varphi(f(\bar{r})) = \chi(f(\bar{r})) \). Since \( f \) is a morphism, this yields \( f(\varphi(\bar{r})) = f(\chi(\bar{r})) \). This works for any model \( f \), and so by the assumption that \( T \) is implicationally closed, we get that \( Q \models \varphi(\bar{r}) = \chi(\bar{r}) \) too. Since \( Q \) satisfies the equational theory of \( K \), by Proposition 7 or its analogue, we get that \( Q \in SP(K) \).

Conversely, let \( Q \in SP(K) \). Specifically let \( Q \subseteq \prod_{i \in I} M_i \) where the \( M_i \) are in \( K \). Introduce a signature \( \sigma \) with a symbol for each element of \( Q \). Then \( F_\sigma/T = Q \) for some congruence \( T \). We claim \( T \) is a theory, i.e. is implicationally closed. Suppose \( r, s \in Q \) with \( f(r) = f(s) \) for all morphisms \( f : Q \rightarrow M \in K \). Then in particular for the projections \( \pi_i : Q \rightarrow M_i \) we have \( \pi_i(r) = \pi_i(s) \). So \( r = s \).

We may thus say that theories are simply subalgebras of products of the concrete algebras in question (with specified generators). The usual notion of theory is an (implicationally closed) collection of sentences (identities of the form \( \varphi = 1 \) in sort zero). In the first order case, where universal quantification and the biconditional are present, this agrees with the notion of theory described above, essentially because \( r = s \) in a first order algebra iff \( \forall^{(n)}(r \leftrightarrow s) = 1 \) (where \( r \) and \( s \) are in sort \( n \)). Intuitively speaking, in the first order signature, the zero sort controls all the sorts. For the reducts this is not true. To illustrate this point, and to help explain the value of axiom (0), we now show that first order theories with exactly two elements in sort zero are the ones in \( S(K) \), but importantly that this characterization does not hold for the reducts.

When considered as a collection of sentences, a first order theory is said to be \textbf{complete} when every sentence or its negation (but not both) is in the theory. Translating this to the quotient view of theories, this says that there are exactly two elements in sort zero. Of course any subalgebra of a first order algebra is going to be a theory with exactly two elements of sort zero. But the converse is true as well, \textit{when negation and projection are present}. We check that axiom (0) follows from the Horn clause theory of first order algebras together with the assumption that there are exactly two elements of sort zero.

Consider an algebra satisfying the Horn clause theory of first order algebras and having exactly two elements in sort zero (0 and 1). Then for any element \( t \) of sort \( k \) we have \( t \neq 0 \) iff \( \exists^{(k)}(t) = 1 \). To see this, observe that \( t = 0 \iff \exists^{(k)}(t) = 0 \) and \( \exists^{(k)}(t) = 0 \iff t = 0 \) are both Horn clauses true of first order algebras (let’s say are “true Horn
clauses"). Let $s_i \not\geq r_i$ for each $i = 1, \ldots, m$, where $r_i, s_i : k_i$, and let $c_i : k_i \to (k_1 + \cdots + k_m) = n$ be partitioning cylindrifications. Since $t \land \neg u = 0 \Rightarrow u \geq t$ is a true Horn clause, we get that $r_i \land \neg s_i \not= 0$ for each $i$. Thus, $\exists^{(k_i)}(r_i \land \neg s_i) = 1$. Another true Horn clause is

$$\bigwedge_{i=1}^{m} c_i(t_i) = 1 \Rightarrow \exists^{(n)} \bigwedge_{i=1}^{m} c_i(t_i) = 1$$

Thus, we get in our case

$$\exists^{(n)} \bigwedge_{i=1}^{m} c_i(r_i \land \neg s_i) = 1$$

So

$$\bigwedge_{i=1}^{m} c_i(r_i \land \neg s_i) \not= 0$$

which simplifies to $\bigwedge_{i=1}^{m} c_i(r_i) \land \neg \bigvee_{i=1}^{m} c_i(s_i) \not= 0$. So $\bigwedge_{i=1}^{m} c_i(r_i) \not\leq \bigvee_{i=1}^{m} c_i(s_i)$.

So, we could have presented an axiomatization of the universal theory of first order algebras by just taking the Horn clause theory and adding to it the axiom that there are exactly two elements in sort zero. However, this would not have yielded results uniformly for the reducts as well. There is a model of the Horn clause theory of positive existential algebras which has exactly two elements of sort zero, but fails to satisfy axiom (0). To see this, consider the (partial) first order theory in a language with three unary relation symbols $R, A, \text{and } B$ generated by the following sentences:

i) $\exists x(A(x) \land B(x))$
ii) $\forall x(R(x) \iff A(x)) \lor \forall x(R(x) \iff B(x))$

Let $T$ be the associated subalgebra of a product of first order algebras. Consider the positive existential reduct of $T$, and then consider the subalgebra generated by $R, A, \text{and } B$. Call it $T_0$, and note that $T_0$ is itself a subalgebra of a product of positive existential algebras. Note that $X \in T_0$ iff there is some positive existential formula $\varphi$ such that $\varphi(R, A, B) = X$. Because we’re dealing with unary relation symbols, and projections of conjunctions of some of $R, A, B$ are predictably 1, we in fact may assume that $\varphi$ is positive quantifier-free. Every element of sort zero in $T_0$ is obtained by projecting an element of sort one. One can check the only possible values are 0 and 1 (and 0 \not= 1 because our theory has a model). On the other hand, letting $c_1(xy) = x$ and $c_2(xy) = y$, we have $c_1(A) \land c_2(B) \leq c_1(R) \lor c_2(R)$ (i.e. $A(x) \land B(y) \models R(x) \lor R(y)$), but $A \not\leq R$ and $B \not\leq R$, violating axiom (0).
It is easy to give a theory in the quantifier-free signature which does not satisfy axiom (0) and still has exactly two elements in sort zero, because there are no functions going from the higher sorts to sort zero in this case. So any violation of axiom (0) not involving sort zero yields an example. For instance, consider the quantifier-free algebra $A(W)$ generated by a set $W$ of one element. Then the product $L := A(W) \times A(W)$ has a “diamond” for each sort. Let us use $0$, $a$, $b$, and $1$ to denote the elements of $L$ in sort one. Let $c_1(xy) = x$ and $c_2(xy) = y$ be partitioning cylindrifications. Then $c_1(a) \land c_2(b)$ is the bottom element in sort two. Thus, $c_1(b) \lor c_2(0) \geq c_1(a) \land c_2(b)$. However, $b \not\geq a$ and $0 \not\geq b$, violating axiom (0).

6. Dealing with Function Symbols

We briefly indicate how to deal with function symbols. Let $\tau$ be a fixed functional signature. We have terms $\alpha(x_1, \ldots, x_n)$ defined as usual (elements of the free $\tau(\bar{x})$-algebra where the $\bar{x}$ are extra constant symbols). From these we obtain “term-tuples”

$$\alpha(\bar{x}) = (\alpha_1(\bar{x}), \ldots, \alpha_k(\bar{x}))$$

where each $\alpha_i(\bar{x})$ is a term.

The term-tuples induce operations on tuples of a $\tau$-algebra $W$ in the obvious way. Given a tuple $\bar{x} \in W^n$, we get $\alpha(\bar{x}) = (\alpha_1(\bar{x}), \ldots, \alpha_k(\bar{x})) \in W^k$. The inverse images of these are operations on relations going in the reverse direction $\alpha: P(W^k) \to P(W^n)$. The substitutions are obtained as a special case for any signature $\tau$, and when $\tau$ is the empty signature, they are the only term-tuples. The multisorted signature of interest to us now has an operation of arity $\alpha: k \to n$ for each such term-tuple, and the concrete algebras of interest are the ones that arise from considering the relations on a $\tau$-algebra.

With respect to axiomatization, if equality is not present, we need only change axioms (2), (3), and (5) by expanding their scope to include all term-tuples (not just substitutions).

As for equality, let us have a constant $\Delta_{\alpha,\beta}$ of sort $n$ for each pair of term-tuples $\alpha, \beta: k \to n$. The intended interpretation is $\Delta_{\alpha,\beta} := \{\bar{x} | \alpha(\bar{x}) = \beta(\bar{x})\}$. In the special case $\alpha(\bar{x}) = x_i$ and $\beta(\bar{x}) = x_j$, we get $\Delta_{\alpha,\beta} = \Delta^n_{i,j}$.

Then we rewrite axioms (11)-(13) as follows:

- $\Delta_{\alpha,\alpha} = 1$
- $\Delta_{\alpha,\beta} = \Delta_{\beta,\alpha}$
- $\Delta_{\alpha,\beta} \land \Delta_{\beta,\gamma} \leq \Delta_{\alpha,\gamma}$
- $\alpha(r) \land \Delta_{\alpha,\beta} = \beta(r) \land \Delta_{\alpha,\beta}$
To these we also add
\[ \Delta_{\alpha,\beta} = \bigwedge_{i=1}^{k} \Delta_{\alpha_i,\beta_i} \text{ where } \alpha = (\alpha_1, \ldots, \alpha_k) \text{ and } \beta = (\beta_1, \ldots, \beta_k) \]
\[ \Delta_{\alpha,\beta} \leq \Delta_{\alpha\gamma,\beta\gamma} \]

So how do the proofs get modified? The only essential change is with the analogues of Lemma 6. We want to have a prime filter \( F \) on sort \( n \) of an abstract algebra satisfying the axioms give rise to a morphism to a concrete algebra. Instead of letting \( W = \{ F_1, \ldots, F_n \} \), we let \( W \) be the free \( \tau \)-algebra with \( F_1, \ldots, F_n \) as generators. In the special case we have no function symbols, i.e. \( \tau \) is empty, we get back the old \( W \).

We define the morphism as before: \( \alpha(F) \in \varphi(r) \iff \alpha(r) \in F \), except that now \( \alpha \) may range over all the term-tuples, not just the substitutions.

When equality is present, we additionally identify certain elements of this free algebra by saying \( \alpha(F) = \beta(F) \iff \Delta_{\alpha,\beta} \in F \). The additional axioms ensure that this makes sense and in fact gives a congruence relation.

Finally, the fixed functional signature \( \tau \) can also be taken to be multisorted, with only minor modifications to our argument.

### 7. The Axioms

For ease of reference, here is a list of the main axioms considered:

0) When \( c_1, \ldots, c_m \) are partitioning cylindrifications we write:

\[ \text{If } \bigvee_{i=1}^{m} c_i(s_i) \geq \bigwedge_{i=1}^{m} c_i(r_i), \text{ then } s_i \geq r_i \text{ for some } i = 1, \ldots, m. \]

1) \( 0, 1, \lor, \land \) form a (bounded) distributive lattice in each sort.
2) Substitutions preserve \( 0, 1, \lor, \land \)
3) \( (\beta \circ \alpha)(r) = \beta(\alpha(r)) \)
4) \( \text{id}(r) = r \)
5) \( \alpha(\neg r) = \neg \alpha(r) \)
6) \( r \lor \neg r = 1, r \land \neg r = 0 \)
7) \( \exists \) preserves \( 0 \) and \( \lor \)
8) \( r \leq c(\exists(r)) \)
9) \( \exists(r \land c(s)) = \exists(r) \land s \)
10) Let \( \alpha_i(\bar{x}) \) be substitutions for \( i = 1, \ldots, n \). Define \( \beta_i(\bar{y}y_1 \cdots y_n) = \alpha_i(\bar{x})y_i \). Then we write

\[ \exists^{(n)}(\bigwedge_{i=1}^{n} \beta_i(r_i)) = \bigwedge_{i=1}^{n} \alpha_i(\exists(r_i)) \]
where \( \exists^{(n)} \) means we apply projection \( n \) times.

11) (a) \( \Delta_{i,i}^n = 1 \)
    (b) \( \Delta_{i,j}^n = \Delta_{j,i}^n \)
    (c) \( \Delta_{i,j}^n \land \Delta_{j,k}^n \leq \Delta_{i,k}^n \)

12) When \( \alpha, \beta : k \rightarrow n \) are substitutions of matching arities we write:

\[
\alpha(r) \land \bigwedge_{l=1}^k \Delta_{\alpha(l),\beta(l)}^n = \beta(r) \land \bigwedge_{l=1}^k \Delta_{\alpha(l),\beta(l)}^n
\]

13) For each substitution \( \alpha : k \rightarrow n \) we write:

\[
\alpha(\Delta_{i,j}^k) = \Delta_{\alpha(i),\alpha(j)}^n
\]

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