Integral representation and sharp asymptotic results for some Heckman-Opdam hypergeometric functions of type BC

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Abstract  
The Heckman-Opdam hypergeometric functions of type BC extend classical Jacobi functions in one variable and include the spherical functions of non-compact Grassmann manifolds over the real, complex or quaternionic numbers. There are various limit transitions known for such hypergeometric functions, see e.g. [dJ], [RKV]. In the present paper, we use an explicit form of the Harish-Chandra integral representation as well as an interpolated variant, in order to obtain limit results for three continuous classes of hypergeometric functions of type BC which are distinguished by explicit, sharp and uniform error bounds. The first limit realizes the approximation of the spherical functions of infinite dimensional Grassmannians of fixed rank; here hypergeometric functions of type A appear as limits. The second limit is a contraction limit towards Bessel functions of Dunkl type.

Key words: Hypergeometric functions associated with root systems, Grassmann manifolds, spherical functions, Harish-Chandra integral, asymptotic analysis, Bessel functions related to Dunkl operators.  
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1 Introduction  
The theory of hypergeometric functions associated with root systems provides a framework which generalizes the classical theory of spherical functions on Riemannian symmetric spaces; see [H], [HS] and [O2] for the general theory, as well as [Sch] and [NPP] for some more recent developments. Here we consider the non-compact Grassmannians $G_{p,q}(F) = G/K$ over one of the (skew-) fields $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, where $G$ is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q,p)$, $SU(q,p)$ or $Sp(q,p)$ with $p > q$, and $K$ is the maximal compact subgroup $K = SO(q) \times SO(p)$, $SU(q) \times U(p)$ or $Sp(q) \times Sp(p)$, respectively. The real rank of $G/K$ is $q$, and the restricted root system $\Delta(g,a)$ is of type $BC$. Let $F_{BC}(\lambda, k; t)$ denote the Heckman-Opdam hypergeometric function associated with the root system

$$R = 2 \cdot BC_q = \{ \pm 2e_i, \pm 4e_i, \pm 2e_i \pm 2e_j : 1 \leq i < j \leq q \} \subset \mathbb{R}^q,$$

with spectral variable $\lambda \in \mathbb{C}^q$ and multiplicity parameter $k$. The spherical functions of $G/K = G_{p,q}(F)$, which are $K$-biinvariant as functions on $G$, are then given by

$$\varphi_{\lambda}^{p}(a_t) = F_{BC}(i\lambda, k; t) \quad (t \in \mathbb{R}^q)$$
with \( \lambda \in \mathbb{C}^g \) and multiplicity
\[
k_p = \frac{(d(p - q)/2, (d - 1)/2, d/2)}{}
\]
corresponding to the roots \( \pm 2e_i, \pm 4e_j \) and \( 2(\pm e_i \pm e_j) \) respectively; here \( d \in \{1, 2, 4\} \) denotes the dimension of \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) over \( \mathbb{R} \); see [R2] and Remark 2.3. of [H]. In [R2], the product formula for spherical functions,
\[
\phi(g) \phi(h) = \int_k \phi(ghk)dk \quad (g, h \in G),
\]
was made explicit in such a way that it could be extended to a product formula for the hypergeometric function \( F_{BC} \) with multiplicity \( k_p \) corresponding to arbitrary real parameters \( p > 2q - 1 \). This led to three continuous series of positive product formulas for \( F_{BC} \) corresponding to \( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \) as well as associated commutative, probability-preserving convolution algebras of measures (hypergroups in the sense of [J]) on the \( BC_q \)-Weyl chamber
\[
C_q = \{ t = (t_1, \ldots, t_q) \in \mathbb{R}^q : t_1 \geq \ldots \geq t_q \geq 0 \}. 
\]
On the other hand, the spherical functions of \( G/K \) have the Harish-Chandra integral representation
\[
\phi^p(\lambda_\ell) = \int_K e^{(\lambda - p)\langle H(\lambda, k) \rangle}dk, \quad \lambda \in \mathfrak{a}_C^* \cong \mathbb{C}^q,
\]
see [He] or [CV] for the general theory and Section 2 for details in our particular case. The Harish-Chandra integral was made explicit by Sawyer [Sa] for the real Grassmannians \( G_{p,q}(\mathbb{R}) \). In the present paper, we extend Sawyer’s representation to general \( F \) and further reduce it to a form which allows an extension from the spherical case with integers \( p \geq 2q \) to a positive integral representation for the three classes of hypergeometric functions \( F_{BC} \) as above, with arbitrary real parameters \( p > 2q - 1 \), the rank \( q \) being fixed. This (in part) generalizes the well-known integral representation of Jacobi functions, which are the hypergeometric functions of type \( BC \) in rank one (see [K1]). We also give an analogous integral representation for the corresponding Heckman-Opdam polynomials.

Our integral representation (Theorem 2.4) for the spherical functions of \( G_{p,q}(F) \) is closely related to those for the spherical functions of the type \( A \) symmetric spaces \( GL(q,F)/U(q,F) \). In particular, we obtain immediately that for \( p \to \infty \), the spherical functions of \( G_{p,q}(F) \) tend to the spherical functions of \( GL(q,F)/U(q,F) \), a result which was proven recently by completely different methods and in more generality in [RKV]; see also the note [K2] for the polynomial case.

As a main result of the present paper, we shall deduce from our explicit integral representation a result on the rate of convergence (Theorem 1.2): the convergence of the bounded hypergeometric functions \( F_{BC} \), with multiplicities depending on \( p \) as above, is of order \( p^{-1/2} \) for \( p \to \infty \), uniformly on the chamber \( C_q \) and locally uniformly in the spectral variable. Moreover, a corresponding result is obtained in the unbounded case. It seems that these results cannot be obtained by the methods of [RKV]. Corresponding results for \( q = 1 \), i.e., for Jacobi functions, can be found in [V2]. We also mention that our convergence results are related to further limits, e.g., to limits in [D] and [SK] for multivariate polynomials as well as to the convergence of (multivariable) Bessel functions of type \( B \) to those of type \( A \) and related results for matrix Bessel functions in [RV2], [RV3]. We point out that our convergence results with error bounds may serve as a basis to derive central limit theorems for random walks on the Grassmannians \( G_{p,q}(F) \) when for fixed rank \( q \), the time parameter of the random walks as well as the dimension parameter \( p \) tend to infinity in a coupled way. For results in this direction we refer to [RV3], [V2].

In generalization of the contraction principle for Riemannian symmetric spaces, Heckman-Opdam hypergeometric functions can be approximated for small space variables and large spectral parameters by corresponding Bessel functions of Dunkl type. This was first proven in [JJ] by an asymptotic analysis of the Cherednik system; see also [RV1]. In the present paper, we shall use the integral representation of Theorem 2.4 in order to obtain this approximation in our series of \( BC \)-cases (which include the spherical functions on Grassmannians), again with an explicit error estimate. For the case
q = 1 and the use of the error estimate in the proof of central limit theorems we refer to [V2] and references cited there.

We finally mention that the Harish-Chandra integral in Proposition 5.4.1 of [HS] for the K-spherical functions of the symmetric spaces $U(p,q)/(U(p) \times SU(q))$ over $\mathbb{C}$ may be used to derive an explicit integral representation for Heckman-Opdam hypergeometric functions of type BC for a different class of parameters as considered here. For such cases, associated convolution structures have been derived in [V3].

The organization of this paper is as follows: In Section 2 we treat the Harish-Chandra integral representation for the spherical functions of $G_{p,q}(\mathbb{F})$ as well as for the associated three continuous series of Heckman-Opdam hypergeometric functions. In Section 3 we deduce the convergence of the spherical functions of $G_{p,q}(\mathbb{F})$ to those of $GL(q,\mathbb{F})/U(q,\mathbb{F})$ as $p \to \infty$. Section 4 is then devoted to precise estimates for the rate $p^{-1/2}$ of convergence. In particular, in order to obtain a uniform rate for $t \in C_q$, we need a technical result on the convex hull of Weyl group orbits of the half sum $\rho$ of roots which will be proven separately in an appendix (Section 6). The quantitative contraction estimates between hypergeometric functions of type BC and Bessel functions of type B will be presented in Section 5.

2 An integral representation for spherical functions on Grassmann manifolds and hypergeometric functions of type BC

In this section, we extend Sawyer’s ([Sa]) integral representation for spherical functions on real Grassmannians and deduce an explicit integral representation (Theorem 2.4) for three continuous series for hypergeometric functions of type BC.

Let $\mathbb{F}$ be one of the (skew-) fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $d = \dim_\mathbb{F} \mathbb{F} \in \{1, 2, 4\}$. On $\mathbb{F}$, we have the standard involution $x \mapsto \overline{x}$ and norm $|x| = (\overline{x}x)^{1/2}$. By $M_{q,p}(\mathbb{F})$ we denote the set of $q \times p$ matrices over $\mathbb{F}$, also viewed as $\mathbb{F}$-linear transformations from $\mathbb{F}^p$ to $\mathbb{F}^q$, which are considered as right $\mathbb{F}$-vector spaces. We write $M_q(\mathbb{F}) = M_{q,q}(\mathbb{F})$.

We consider the Grassmannians $G/K = G_{p,q}(\mathbb{F})$ where $G$ is one of the groups $SO_0(p,q), SU(p,q)$ or $Sp(C,q)$, and $K$ is the maximal compact subgroup $K = SO(p) \times SO(q), SU(p) \times SU(q), Sp(p) \times Sp(q)$, respectively. Note that $G$ is the identity component of $SU(q,p;\mathbb{F})$, where $U(q,p;\mathbb{F})$ is the isometry group for the quadratic form

$$|x_1|^2 + \ldots + |x_q|^2 - |x_{q+1}|^2 - \ldots - |x_{p+q}|^2$$

on $\mathbb{F}^{p+q}$. In the same way, $K$ is a subgroup of $U(q,p;\mathbb{F}) \times U(p,q;\mathbb{F})$ where $U(q,p;\mathbb{F}) = \{X \in M_q(\mathbb{F}) : X^*X = I_q\}$ is the unitary group over $\mathbb{F}$; here $X^* = X^t$ denotes the conjugate transpose. The Lie algebra $\mathfrak{g}$ of $G$ consists of the matrices

$$X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in M_{p+q}(\mathbb{F})$$

with blocks $A = -A^* \in M_q(\mathbb{F})$ and $D = -D^* \in M_p(\mathbb{F})$ satisfying $trA + trD = 0$, as well as $B \in M_{q,p}(\mathbb{F})$. Let $\mathfrak{k}$ be the Lie algebra of $K$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition of $\mathfrak{g}$, with $\mathfrak{p}$ consisting of the $(q,p)$-block matrices

$$X = \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}, \quad X \in M_{q,p}(\mathbb{F}).$$

In accordance with [Sa], we use as a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ the set of matrices $H_k = \begin{pmatrix} I_{(p-q)\times q} & 0_{q\times(p-q)} \\ 0_{(p-q)\times q} & 0_{(p-q)\times(p-q)} \end{pmatrix}$.
where \( t = \text{diag}(t_1, \ldots, t_q) \) is the diagonal matrix corresponding to \( t = (t_1, \ldots, t_q) \in \mathbb{R}^q \). We remark that our present notions are adjusted to those of [Sa] (with \( p \) and \( q \) exchanged), and are slightly different from those used in [R2].

The restricted root system \( \Sigma = \Sigma(g, a) \) of \( g \) with respect to \( a \) consists of the non-zero linear functionals \( \alpha \in a^* \) such that

\[
g_\alpha = \{ X \in g : [H, X] = \alpha(H)X \ \forall \ H \in a \} \neq \{0\}.
\]

In our case, the root system is of type \( B_q \) if \( F = \mathbb{R} \) and of type \( BC_q \) if \( F = \mathbb{C} \) or \( \mathbb{H} \). The multiplicities \( m_\alpha = \dim g_\alpha \) can be found e.g. in table 9 of [OV]. We shall need an explicit description of the root spaces. For this, define \( f_i \in a^* \) by \( f_i(H_i) = t_i, \ i = 1, \ldots, q \). We shall write matrices from \( g \) in \((q, q, p - q)\)-block form. By \( E_{ij} \) we denote a matrix of appropriate size which has entries 0 except position \((i, j)\), where the entry is 1. Notice that \( E_{ij} \cdot \lambda = \lambda \cdot E_{ij} \) for \( \lambda \in F \). The following list of roots is easily verified by block multiplications; in the real case, it matches Theorem 5 of [Sa].

(1) \( \alpha = \pm f_i, \ 1 \leq i \leq q \). The root space \( g_\alpha \) is given by \( g_\alpha = \{ X_{ir}^{\pm}(\lambda) : \lambda \in F, \ r = 1, \ldots, p - q \} \) with

\[
X_{ir}^{\pm}(\lambda) = \begin{pmatrix}
0 & 0 & \lambda E_{ir} \\
\mp \lambda E_{ri} & 0 & \pm \lambda E_{ir}
\end{pmatrix}.
\]

The multiplicity of \( \alpha \) is \( m_\alpha = d(p - q) \).

(2) \( \alpha = \pm (f_i - f_j), \ 1 \leq i < j \leq q \). In this case, \( g_\alpha = \{ Y_{ij}^{\pm}(\lambda) : \lambda \in F \} \) with

\[
Y_{ij}^{\pm}(\lambda) = \begin{pmatrix}
\pm (\lambda E_{ij} - \bar{\lambda} E_{ji}) & \lambda E_{ij} + \bar{\lambda} E_{ji} & 0 \\
\lambda E_{ij} + \bar{\lambda} E_{ji} & \pm (\lambda E_{ij} - \bar{\lambda} E_{ji}) & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

The multiplicity is \( m_\alpha = d \).

(3) \( \alpha = \pm (f_i + f_j), \ 1 \leq i < j \leq q \). Here \( g_\alpha = \{ Z_{ij}^{\pm}(\lambda) : \lambda \in F \} \) with

\[
Z_{ij}^{\pm}(\lambda) = \begin{pmatrix}
\pm (\lambda E_{ij} - \bar{\lambda} E_{ji}) & -\lambda E_{ij} + \bar{\lambda} E_{ji} & 0 \\
-\lambda E_{ij} + \bar{\lambda} E_{ji} & \pm (\lambda E_{ij} - \bar{\lambda} E_{ji}) & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Again, the multiplicity is \( m_\alpha = d \).

(4) \( \alpha = \pm 2f_i, \ 1 \leq i \leq q \). This family of roots occurs only for \( F = \mathbb{C}, \mathbb{H} \). The root spaces are given by \( g_\alpha = \{ \lambda \cdot W_{i}^{\pm} : \lambda \in F, \bar{\lambda} = -\lambda \} \) with

\[
W_{i}^{\pm} = \begin{pmatrix}
E_{ii} & 0 & \mp E_{ii} \\
0 & 0 & 0 \\
\pm E_{ii} & 0 & -E_{ii}
\end{pmatrix}.
\]

In order to obtain a unified notion, we consider \( \alpha = \pm 2f_i \) also a root if \( F = \mathbb{R} \), with multiplicity zero. Then \( m_\alpha = d - 1 \) for \( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \).

In our unified notion, \( \Sigma \) is of type \( BC_q \) in all cases, with the understanding that 0 may occur as a multiplicity on the long roots. As usual, we choose the positive subsystem

\[
\Sigma^+ = \{ f_i, 2f_i, 1 \leq i \leq q \} \cup \{ f_i \pm f_j, 1 \leq i < j \leq q \}
\]

Then the weighted half sum of positive roots is

\[
\rho^{BC} = \rho^{BC}(p) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \sum_{i=1}^{q} \left( \frac{d}{2} (p + q + 2 - 2i) - 1 \right) f_i.
\]
Let 
\[ n = \sum_{\alpha \in \Sigma^+} g_\alpha \]
and \( N = \exp n, A = \exp a \). Then \( A \) is abelian, \( N \) is nilpotent, and \( G = KAN \) is an Iwasawa decomposition of \( G \). The spherical functions of \( G/K \) are given by the Harish-Chandra integral formula
\[ \varphi^\lambda(a_t) = \int_K e^{(i\lambda - \rho_{BC})}H(a_t k))dk, \quad \lambda \in \mathfrak{a}^*_C \]
where \( H(g) \in A \) denotes the unique abelian part of \( g \in G \) in the Iwasawa decomposition \( G = KAN \) (see e.g. [GV]), and
\[ a_t = \exp(H_t) = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & I_{n-q} \end{pmatrix} \]
with \( \cosh t = \text{diag}(\cosh t_1, \ldots, \cosh t_q) \), \( \sinh t = \text{diag}(\cosh t_1, \ldots, \cosh t_q) \).

We shall identify \( \mathfrak{a}^*_C \) with \( \mathbb{C}^q \) via \( \lambda \mapsto (\lambda_1, \ldots, \lambda_q) \) for \( \lambda \in \mathfrak{a}^*_C \) given by \( \lambda(H_t) = \sum_{r=1}^q \lambda_r t_r \), \( \lambda_r \in \mathbb{C} \).

In order to state a more explicit form of the Harish-Chandra integral above, we need some further notation. For a square matrix \( A = (a_{ij}) \) over \( \mathbb{F} \) we denote by \( \Delta_r(A) = \det((a_{ij})_{1 \leq i,j \leq r}) \) the \( r \)-th principal minor of \( A \). Here, for \( \mathbb{F} = \mathbb{H} \), the determinant is understood in the sense of Dieudonné, i.e. \( \det(A) = (\det_{\mathbb{C}}(A))^{1/2} \), when \( X \) is considered as a complex matrix.

We introduce the usual power functions on the cone
\[ \Omega_q = \{ x \in M_q(\mathbb{F}) : x = x^*, x \text{ strictly positive definite} \}, \]
(c.f. [FK], Chap.VII.1.): For \( \lambda \in \mathbb{C}^q \equiv \mathfrak{a}^*_C \) and \( x \in \Omega_q \), we define
\[ \Delta_\lambda(x) = \Delta_1(x)^{\lambda_1} \cdots \Delta_{q-1}(x)^{\lambda_{q-1}} \cdot \Delta_q(x)^{\lambda_q}. \]
We also define the projection matrix
\[ \sigma_0 := \begin{pmatrix} I_q \\ 0_{(p-q) \times q} \end{pmatrix} \in M_{p,q}(\mathbb{F}). \]

The following result generalizes Theorem 16 of [Sa].

2.1 Theorem. For the Grassmannian \( G_{p,q}(\mathbb{F}) \), the spherical functions (2.2) are given by
\[ \varphi^\lambda(a_t) = \int_K \Delta_{(i\lambda - \rho_{BC})/2}(x_t(k))dk, \quad \lambda \in \mathbb{C}^q \]
where for \( k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in K \) with \( u \in U(q,\mathbb{F}), v \in U(p,\mathbb{F}), \)
\[ x_t(k) := (\cosh t u + \sinh t \sigma_0^* v \sigma_0) \ast (\cosh t u + \sinh t \sigma_0^* v \sigma_0) \in \Omega_q. \]

Proof. We closely follow [Sa]. Let
\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix} I_q & 0_{q \times (p-q)} & J_q \\ I_q & 0_{q \times (p-q)} & -J_q \\ 0_{(p-q) \times q} & 0_{(p-q) \times (p-q)} & 0_{(p-q) \times q} \end{pmatrix} \text{ with } J_q = (\delta_{i,q+1-j})_{i,j} \in M_q(\mathbb{F}). \]

Notice that \( S^* S = I_{p+q} \). Using the explicit form of the root spaces above, one checks that \( S^* X S \) is strictly upper triangular for each \( X \in n \). Thus for \( n \in N \), the matrix \( S^* n S \) is upper triangular with entries 1 in the diagonal. Furthermore,
\[ S^* \exp(H_t) S = \text{diag}(e^{t_1}, \ldots, e^{t_n}, 1, \ldots, 1, e^{-t_q}, \ldots, e^{-t_1}) \]

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with \( p - q \) entries 1. Consider \( g = k \exp(H_t)n \in KAN \) and let \( 1 \leq r \leq q \). As in the proof of Proposition 14 of [Sa], we calculate the principal minors
\[
\Delta_r(S^*g^*S) = \Delta_r((S^*nS)^*(S^* \exp(2H_t)S)S^*nS) = e^{2(t_1 + \ldots + t_r)}.
\]

Writing \( g = k \exp(H_t)n \) in \((p,q)\)-block form as \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), the upper left \( q \times q \)-block of \( S^*g^*gS \) becomes
\[
(A + B\sigma_0)^*(A + B\sigma_0) \quad \text{with} \quad \sigma_0 = \begin{pmatrix} I_q \\ 0_{(p-q)\times q} \end{pmatrix} \in M_{p,q}(\mathbb{F}).
\]
Thus
\[
t_r = \frac{1}{2} \log \frac{\Delta_r((A + B\sigma_0)^*(A + B\sigma_0))}{\Delta_r-1((A + B\sigma_0)^*(A + B\sigma_0))},
\]
with the agreement \( \Delta_0 := 1 \). Notice that this generalizes Proposition 14 of [Sa], and that the arguments of \( \Delta_r \) and \( \Delta_r-1 \) belong to the cone \( \Omega_q \), because \( gS \) is non-singular.

Now consider \( g = a_tk \) with \( k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in K \). We have
\[
a_tk = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} & 0 & 0 \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} & 0 & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \cdot \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \cosh \frac{t}{2} \cdot u & \sinh \frac{t}{2} \cdot \sigma_0^* \cdot v \end{pmatrix}.
\]

By (2.5), this gives
\[
e^{\lambda(H(a_tk))} = \prod_{r=1}^{q} \left( \frac{\Delta_r(x_t(k))}{\Delta_r-1(x_t(k))} \right)^{-r/2} = \Delta_{\lambda/2}(x_t(k)),
\]
which proves the statement. \( \square \)

For \( p \geq 2q \) we may reduce the integral in Theorem 2.1 by techniques from [R1], [R2]. For this, consider the ball
\[
B_q = \{ w \in M_q(\mathbb{F}) : w^*w < I \},
\]
where \( A < B \) means that \( B - A \) is (strictly) positive definite, as well as the probability measure \( m_p \) on \( B_q \) given by
\[
dm_p(w) = \frac{1}{\kappa_{pd/2}} \cdot \Delta(I - w^*w)^{pd/2-\gamma} dw,
\]
where
\[
\gamma := d\left(q - \frac{1}{2}\right) + 1,
\]
\( \Delta \) denotes the determinant on the open cone \( \Omega_q \), \( dw \) is the Lebesgue measure on the ball \( B_q \), and
\[
\kappa_{pd/2} = \int_{B_q} \Delta(I - w^*w)^{pd/2-\gamma} dw.
\]
Notice that \( m_p \) is a probability measure on \( B_q \).

By \( U_0(q,\mathbb{F}) \) we denote the identity component of \( U(q,\mathbb{F}) \). Notice that \( U(q,\mathbb{F}) = U_0(q,\mathbb{F}) \) for \( \mathbb{F} = \mathbb{C}, \mathbb{H} \), while \( U_0(q,\mathbb{R}) = SO(q) \). With these notions, we obtain the following integral representation:

2.2 Corollary. Let \( p \geq 2q \) be an integer. Then the spherical functions (2.2) can be written as
\[
\varphi^p_{\lambda}(a_t) = \int_{U_0(q,\mathbb{F}) \times B_q} \Delta_{(\lambda - \rho \cdot Bc)/2}(g_t(u,w)) dm_p(w) du
\]
(2.8)
where $du$ denotes the normalized Haar measure on $U_0(q)$, and

$$g_t(u,w) = u^{-1}(\cosh t + \sinh t w)^*(\cosh t + \sinh t w)u.$$  

The same formula holds with the argument $g_t(u,w)$ replaced by

$$\widetilde{g}_t(u,w) = u^{-1}(\cosh t + \sinh t w)(\cosh t + \sinh t w)^* u.$$  

**Proof.** In a first step, we replace the integral over $K$ in Theorem 2.1 by an integral over $U_0(q,F) \times U(p,F)$. This is achieved in the same way as for the integral (2.5) in [R2]: it is important in this context that the argument $x_t(k)$ depends only on the upper left $q \times q$-block of $v$. Lemma 2.1 of [R2] then gives the first formula with the argument $(\cosh t u + \sinh t w)^*(\cosh t u + \sinh t w)$ instead of $g_t(u,w)$, which is then obtained by a change of variables $w \mapsto wu$.

For the proof of the second equation, notice that for $a := \cosh t + \sinh t \cdot w \in M_q(F)$, the matrices $a^*a$ and $aa^*$ have the same eigenvalues with the same multiplicities. Therefore, $a^*a = vaa^*v^*$ with some $v \in U(q,F)$ for $F = \mathbb{R}, \mathbb{C}$. In fact, this also valid for $\mathbb{H}$. To check this, write $a \in M_q(\mathbb{H})$ as $a = a_1 + ja_2$ for complex matrices $a_1, a_2 \in M_q(\mathbb{C})$, and form

$$\chi_a := \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} \in M_{2q}(\mathbb{C}).$$

The mapping $\chi : M_q(\mathbb{H}) \to M_{2q}(\mathbb{C}), a \mapsto \chi_a$ is a *-homomorphism of algebras, and $\chi_a^* \chi_a$ and $\chi_a^* \chi_a$ have the same eigenvalues as $a^*a$ and $aa^*$ respectively with the doubled multiplicities; see the survey [Zh]. Thus, $a^*a$ and $aa^*$ have the same eigenvalues with the same multiplicities, and hence $a^*a = vaa^*v^*$ with some $v \in U(q,F)$.

Using $a^*a = vaa^*v^*$ for some $v \in U(q,F)$, we see that for each fixed $w \in B_q$

$$\int_{U_0(q,F)} \Delta_{(i\lambda - \rho)/2}(\widetilde{g}(t,u,w)) 
\int_{U_0(q,F)} \Delta_{(i\lambda - \rho)/2}(u^* \widetilde{g}(t,u,w)) 
\int_{U_0(q,F)} \Delta_{(i\lambda - \rho)/2}(g(t,u,w)) du.$$  

This yields the second equation. \hfill \Box

We now identify $t \in C_q$ with the matrices $a_t \in G$ as above and regard the spherical functions $\varphi^p_\lambda$ above as functions on the Weyl chamber $C_q$. With this agreement we now extend the integral representation (2.8) above from integer parameters $p \geq 2q$ to arbitrary real parameters $p \geq 2q - 1$. For this we fix $F$ (and thus $d = 1, 2, 4$) and define the functions

$$\varphi^p_\lambda(t) := F_{BC}(i\lambda, k_p; t) \quad (t \in C_q, \lambda \in \mathbb{C}^q)$$  

with

$$k_p = (d(p-q)/2, (d-1)/2, d/2),$$  

which are analytic in $p$ with $\text{Re}\, p > q$. Note that for integers $p$, the functions $\varphi^p_\lambda$ are precisely the spherical functions (2.2). For the extension of the integral representation, we shall employ Carleson’s theorem on analytic continuation which we recapitulate from [Ti], p.186:

**2.3 Theorem.** Let $f(z)$ be holomorphic in a neighbourhood of $\{z \in \mathbb{C} : \text{Re}\, z \geq 0\}$ satisfying $f(z) = O(e^{c|z|})$ on $\text{Re}\, z \geq 0$ for some $c < \pi$. If $f(z) = 0$ for all nonnegative integers $z$, then $f$ is identically zero for $\text{Re}\, z > 0$.

We shall prove:

**2.4 Theorem.** Let $p \in \mathbb{R}$ with $p > 2q - 1$. Then the functions (2.9) satisfy

$$\varphi^p_\lambda(t) = \int_{B_q \times U_0(q,F)} \Delta_{(i\lambda - \rho_{BC})/2}(g_t(u,w)) 
\int_{U_0(q,F)} \Delta_{(i\lambda - \rho_{BC})/2}(g_t(u,w)) \, dm_p(w) du$$  

for all $\lambda \in \mathbb{C}^q$ and $t \in C_q$, where again the argument $g_t$ may be replaced by $\widetilde{g}_t$ as in Corollary 2.2.
Proof. We first observe that both sides of (2.10) are analytic in $p$ and $\lambda$. In order to employ Carleson’s theorem to extend (2.8) to $p \in [2q - 1, \infty]$, we need a suitable exponential growth bound on $F_{BC}$ w.r.t. $p$ in some right half plane. Such exponential estimates are available only for real, nonnegative multiplicities; see Proposition 6.1 of [O2], [Sch], and Section 3 of [RKV]. We thus proceed in two steps and closely follow the proof of Theorem 4.1 of [R2], where a product formula is obtained by analytic continuation. We first restrict our attention to a discrete set of spectral parameters $\lambda$ for which $F_{BC}$ is a (renormalized) Jacobi polynomi and where the growth condition is easily checked. Carleson’s theorem then leads to (2.10) for this discrete set of parameters $\lambda$ and all $p \in [2q - 1, \infty]$. In a further step we fix $p \in [2q - 1, \infty]$ and extend (2.10) to all $\lambda \in \mathbb{C}^q$.

Let us go into details. We need some notation and facts from [O2] and [HS]. For $p$ the details. We may therefore apply Carleson’s theorem to (2.14), and this proves (2.10) for $\lambda \in P_+$. According to Eq. (4.4.10) of [HS], we have for $k \geq 0$ and $\lambda \in P_+$ the connection

$$F_{BC}(\lambda + \rho(k), k; t) = c(\lambda + \rho(k), k)P_{\lambda}(k; t)$$

(2.13)

where the $P_{\lambda}$ are the Heckman-Opdam Jacobi polynomials associated with $BC_q$. We also consider the specific multiplicities $k_p := (d(p - q)/2, (d - 1)/2, d/2)$ and the associated half sums $\rho(k_p) = \rho_{BC}$ as in (2.1). With these notations we obtain from (2.13) and (2.9) that the integral representation (2.10) can be written as

$$P_{\lambda}(k_p, t) = \frac{1}{c(\lambda + \rho(k_p), k_p)} \cdot \frac{1}{\kappa_{pd/2}} \int_{B_q \times U_0(q, \mathbb{R})} \Delta_{\lambda/2}(g_t(u, w)) \Delta(I - w^*w)^{pd/2 - \gamma} dw du.$$  

(2.14)

Exactly as in the proof of Theorem 4.1 of [R2], it is now checked that both sides of (2.14) are, as functions of $n$, of polynomial growth in the half-plane $\{p \in \mathbb{C} : \text{Re}(pd/2) > \gamma - 1\}$; we omit the details. We may therefore apply Carleson’s theorem to (2.14), and this proves (2.10) for $p$ with $\text{Re}(pd/2) > \gamma - 1$ and all spectral parameters of the form $-i(\lambda + \rho(k_p))$ with $\lambda \in P_+$.

We next fix $p \in \mathbb{R}$ with $p > 2q - 1$ (in which case $k_p$ is nonnegative) and extend (2.10) with respect to the spectral parameter $\lambda$. According to Proposition 6.1 of [O2],

$$|F_{BC}(\lambda, k_p; t)| \leq |W|^{1/2} e^{\max_{w \in W} \text{Re}(w \lambda, t)}$$

where $W$ is the Weyl group of $BC_q$. Let $C_0^q$ be the interior of $C_q$ and $H' := \{\lambda \in \mathbb{C}^q : \text{Re} \lambda \in C_0^q\}$. Then

$$\text{Re}(w \lambda, t) \leq \text{Re}(\lambda, t) \quad \text{for} \quad \lambda \in H', \ t \in C_q, \ w \in W.$$  

(2.16)

Now fix $t \in C_q$ and $p$ as above, and choose a vector $a \in C_0^q$ sufficiently large. Then (2.10) for the spectral parameter $\lambda + \rho(k_p)$ is equivalent to

$$e^{-(\lambda, a + t)} q_{-i(\lambda + \rho(k_p))}(t) = \int_{B_q \times U_0(q, \mathbb{R})} e^{-(\lambda, a + t)} \cdot \Delta_{\lambda/2}(g_t(u, w)) \ dm_p(w) du.$$  

(2.17)
The left hand side remains bounded for \( \lambda \in H' \). Moreover, for \( a \in C_q^0 \) sufficiently large,
\[
\sup_{(u,w)\in U_0(q,F)\times B_q;\lambda \in H'} |e^{-(\lambda, a+t)} \cdot \Delta_{\lambda/2}(g_t(u, w))| < \infty,
\]
which proves that also the right hand side remains bounded for \( \lambda \in H' \). By a \( q \)-fold application of Carleson’s theorem we thus may extend the preceding equation from \( \lambda \in P_+ \) to \( \lambda \in H' \). A classical analytic continuation now finishes the proof.

The above proof reveals in particular the following integral representation for Heckman-Opdam polynomials of type BC:

**2.5 Corollary.** Let \( k_p = (d(p-q)/2, (d-1)/2, d/2) \) with \( p \in \mathbb{R} \), \( p > 2q - 1 \). Then the Heckman-Opdam polynomials of type BC\(_q \) with multiplicity \( k_p \) have the integral representation

\[
P_\lambda(k_p, t) = \frac{1}{c(\lambda + \rho(k_p), k_p)} \int_{B_q \times U_0(q,F)} \Delta_{\lambda/2}(g_t(u, w)) dm_p(w) du \quad \text{for } t \in \mathbb{C}^q.
\]

Here \( \lambda \in P_+ = C_q \cap 2\mathbb{Z}^q \) and

\[
g_t(u, w) = u^{-1}(\cosh \xi + w^* \sinh \xi)(\cosh \xi + \sinh \xi)w u.
\]

**2.6 Remark.** For the limit case \( p = 2q - 1 \), a degenerate version of the integral representation (2.10) is available. For this we follow Section 3 of [R1].

We fix the dimension \( q \) and consider the matrix ball \( B_q := \{w \in M_q(F) : w^* w < I_q\} \) as above as well as the ball \( B := \{y \in F^q : \|y\| = \left(\sum_{j=1}^q y_j^2\right)^{1/2} < 1\} \) and the sphere \( S := \{y \in F^q : \|y\| = 1\} \).

By Lemma 3.7 and Corollary 3.8 of [R1], the mapping

\[
P(y_1, \ldots, y_q) := \left(\begin{array}{c} y_1 \\ y_2(I_q - y_1^*y_1)^{1/2} \\ \vdots \\ y_q(I_q - y_{q-1}^*y_{q-1})^{1/2} \cdot (I_q - y_q^*y_q)^{1/2} \end{array}\right), \quad y_1, \ldots, y_q \in B
\]

(2.15)
establishes a diffeomorphism \( P : B^q \to B_q \). The image of the measure \( dm_p(w) \) under \( P^{-1} \) is given by

\[
\frac{1}{\kappa_{pd/2}} \prod_{j=1}^q (1 - \|y_j\|^2)^{d(p-q-j+1)/2-1} dy_1 \ldots dy_q.
\]

(2.16)

Thus for \( p > 2q - 1 \), the integral representation (2.10) may be rewritten as

\[
\varphi^{P}_\lambda(t) = \frac{1}{\kappa_{pd/2}} \int_{B^q} \int_{U_0(q,F)} \Delta_{(i\lambda - \rho_{BC})/2}(g_t(u, P(y))) \prod_{j=1}^q (1 - \|y_j\|^2)^{d(p-q-j+1)/2-1} dy_1 \ldots dy_q dw
\]

(2.17)

where \( dy_1, \ldots, dy_q \) means integration w.r.t. the Lebesgue measure on \( F^q \). Moreover, for \( p \leq 2q - 1 \), (2.17) and continuity lead to the following degenerated product formula:

\[
\varphi^{2q-1}_\lambda(t) = \frac{1}{\kappa_{(2q-1)d/2}} \int_{B^{q-1}} \int_{S} \int_{U_0(q,F)} \Delta_{(i\lambda - \rho_{BC})/2}(g_t(u, P(y))) \cdot 
\cdot \prod_{j=1}^{q-1} (1 - \|y_j\|^2)^{d(q-j)/2-1} dy_1 \ldots dy_{q-1} d\sigma(y_q) dw
\]

(2.18)

where \( \sigma \in M^1(S) \) is the uniform distribution on the sphere \( S \) and

\[
\kappa_{(2q-1)d/2} = \int_{B^{q-1}} \int_{S} \prod_{j=1}^{q-1} (1 - \|y_j\|^2)^{d(q-j)/2-1} dy_{q-1} d\sigma(y_q).
\]

Notice that the \( \varphi^{2q-1}_\lambda \) are the spherical functions of the Grassmannian \( G_{2q-1,q}(F) \).
3 The connection with spherical functions of type \(A_{q-1}\)

We shall compare the spherical functions of the Grassmannians \(\mathcal{G}_{p,q}(\mathbb{F})\) with the spherical functions of the symmetric space \(\mathcal{P}_q(\mathbb{F}) = G/K\) with \(G = GL(q,\mathbb{F}), K = U(q,\mathbb{F})\). It is well-known that \(G\) has the Iwasawa decomposition \(G = K\mathcal{A}\mathcal{N}\) where \(A = \exp a, a = \{H_i = t_i, t = (t_1, \ldots, t_q) \in \mathbb{R}^q\}\) and \(N\) is the unipotent group consisting of all upper triangular matrices with entries 1 in the diagonal. The restricted root system \(\Delta(g,a)\) is of type \(A_{q-1}\), with a positive subsystem given by

\[
\Delta_+ = \{f_i - f_j : 1 \leq i < j \leq q\}.
\]

Here the multiplicity is \(m_\alpha = d\) for all \(\alpha \in \Delta_+\) and the weighted half-sum of positive roots is

\[
\rho^A = \sum_{i=1}^q d \left( q + 1 - 2i \right) f_i.
\]

Again, \(a^\circ\) may be identified with \(\mathbb{C}^q\) via \(\lambda \mapsto (\lambda_1, \ldots, \lambda_q)\) for \(\lambda \in a^\circ\), given by \(\lambda(H_t) = \sum_{r=1}^q \lambda_r t_r\), \(\lambda_r \in \mathbb{C}\). We briefly recall the further well-known calculation, which is similar to the Grassmannian case. For \(g = k \exp(H_t)n \in K\mathcal{A}\mathcal{N}\) one obtains \(\Delta_r(g^*g) = e^{2(t_1 + \cdots + t_r)}\) and thus

\[
t_r = \frac{1}{2} \log \frac{\Delta_r(g^*g)}{\Delta_{r-1}(g^*g)} \quad (r = 1, \ldots, q).
\]

If \(g = a_t k\) with \(a_t = \exp(H_t) = e^{t_1} k\) and \(k \in K\), then \(g^*g = k^{-1} e^{2t} k\). The spherical functions of \(G/K = \mathcal{P}_q(\mathbb{F})\) are given by

\[
\varphi^A_\lambda(e^t) = \int_K e^{(i\lambda - \rho^A)(H(a_t,k))} dk, \quad \lambda \in \mathbb{C}^q.
\]

The above considerations lead to the known integral representation

\[
\varphi^A_\lambda(e^t) = \int_{U(q,\mathbb{F})} \Delta((i\lambda - \rho^A)/2)(u^{-1} e^{2t} u) du = \int_{U_0(q,\mathbb{F})} \Delta((i\lambda - \rho^A)/2)(u^{-1} e^{2t} u) du.
\]

We also remark that the functions \(\varphi^A_\lambda\) can be written in terms of the Heckman-Opdam hypergeometric function \(F_A\) associated with the root system \(2A_{q-1} = \{ \pm 2(e_i - e_j) : 1 \leq i < j \leq q\}\), as follows:

\[
\varphi^A_\lambda(e^t) = e^{(t - \pi(t,\lambda))} F_A(\pi(\lambda), d/2; \pi(t)) \quad (\lambda \in \mathbb{C}^q, t \in \mathbb{R}^q).
\]

Here \(\pi\) denotes the orthogonal projection from \(\mathbb{R}^q\) onto \(\mathbb{R}^q_0 := \{ t \in \mathbb{R}^q : t_1 + \cdots + t_q = 0\}\); see Eq. (6.7) of [RKV], and note our rescaling of the root system by the factor 2.

We compare (3.2) with the integral (2.8) for the spherical functions of \(\mathcal{G}_{p,q}(\mathbb{F})\) and, more generally, with representation (2.10) for the hypergeometric functions \(\varphi^p_{\lambda - ip^{BC}}\). As for \(p \to \infty\) the probability measures \(m_p\) on \(B_q\) tend weakly to the point measure at the zero matrix, we obtain:

3.1 Corollary. The spherical functions of \(\mathcal{G}_{p,q}(\mathbb{F})\), and more generally, the hypergeometric functions \(\varphi^p_{\lambda - ip^{BC}}\) with \(p \in \mathbb{R}, p > 2q - 1\) are related to the spherical functions of \(\mathcal{P}_q(\mathbb{F})\) by

\[
\lim_{p \to \infty} \varphi^p_{\lambda - ip^{BC}}(t) = \varphi^A_{\lambda - ip^{A}}(\cosh t) \quad (t \in \mathbb{R}^q).
\]

This result was already obtained in Corollary 6.1 of [RKV] by completely different methods, namely as a special case of a general limit transition for hypergeometric functions of type BC. However, the approach in [RKV] seems not suitable to gain information on the rate of convergence. In the following section, we study the integral representations (3.2) and (2.8) (or (2.10) for continuous \(p\)) in order to derive precise estimates on the rate of convergence.
4 The rate of convergence for $p \to \infty$

The main result of this section is Theorem 4.2. It sharpens the qualitative limit of Corollary 3.1 for the Heckman-Opdam hypergeometric functions $\varphi_p^A$ by a precise estimate of the approximation error. Again, $p > 2q - 1$ varies and the rank $q$ as well as the dimension $d = 1, 2, 4$ of $F$ are fixed. For convenience, we consider the type $A$ spherical functions $\varphi^A_\lambda$ as functions on $\mathbb{R}^q$ and study

$$\psi_\lambda(t) := \varphi^A_\lambda(\cosh t) = \int_{U_0(q,F)} \Delta((\lambda - \rho^A)/2)(u^{-1} \cosh^2 t u) du. \quad (4.1)$$

We write

$$\|\lambda\|_1 := |\lambda_1| + \ldots + |\lambda_q| \quad \text{for } \lambda \in \mathbb{C}^q;$$

$$\tilde{t} := \min(t_1, 1) \geq 0 \quad \text{for } t = (t_1, \ldots, t_q) \in \mathbb{C}^q.$$

The action of the Weyl group $W$ of type $BC_q$ extends in a natural way to $\mathbb{C}^q$. We write $\rho := \rho^{BC}(p)$ for the half sum \([2.1]\). Moreover, $\text{co}(W, \rho) \subset \mathbb{R}^q$ denotes the convex hull of the $W$-orbit of $\rho$.

Let us recapitulate the following known properties of $\varphi^p_\lambda$:

4.1 Lemma. (1) For all $t \in \mathbb{C}^q$, $\lambda \in \mathbb{C}^q$, and $p \in \mathbb{R}$ with $p \geq q$,

$$\left| \varphi^p_{\lambda - ip}(t) \right| \leq \varepsilon_{\max w \in W} \text{Im} \langle \lambda, t \rangle.$$

(2) $\varphi^p_\lambda$ is bounded if and only if $\text{Im} \lambda \in \text{co}(W, \rho)$. Moreover, in this case $\|\varphi^p_\lambda\|_\infty = 1$.

(3) If $\lambda$ is purely imaginary, then $\varphi^p_\lambda$ is real-valued and strictly positive on $\mathbb{C}^q$.

Proof. (1) follows from Corollary 3.4 of \([RKV]\). For part (2) we refer to Theorem 5.4 of \([R2]\) and Theorem 4.2 of \([NPP]\) (the proof of the only-if-part in \([R2]\) contains a gap). Part (3) follows from Lemma 3.1. of \([Sch]\). \(\square\)

Notice that by Corollary 3.1 the same estimates as in Lemma 4.1 hold for the function $\psi_{\lambda - ip^A}(t)$.

The following theorem is the main result of this section:

4.2 Theorem. There exists a universal constant $C = C(\mathbb{R}, q)$ as follows:

(1) For all $p > 2q - 1$, $t \in \mathbb{C}^q$ and $\lambda \in \mathbb{C}^q$,

$$\left| \varphi^p_{\lambda - ip}(t) - \psi_{\lambda - ip^A}(t) \right| \leq C \cdot \frac{\|\lambda\|_1 \cdot \tilde{t}}{p^{1/2}} \cdot e^{\max w \in W} \text{Im} \langle \lambda, t \rangle.$$

(2) Let $p > 2q - 1$, $t \in \mathbb{C}^q$, and $\lambda \in \mathbb{C}^q$ such that $\text{Im} \lambda - \rho$ is contained in $\text{co}(W, \rho)$, i.e., $\varphi^p_{\lambda - ip}$ is bounded on $\mathbb{C}^q$. Then

$$\left| \varphi^p_{\lambda - ip}(t) - \psi_{\lambda - ip^A}(t) \right| \leq C \cdot \frac{\|\lambda\|_1 \cdot \tilde{t}}{p^{1/2}}.$$

In particular, for these spectral parameters $\lambda$ the order of convergence is uniform of order $p^{-1/2}$ in $t \in \mathbb{C}^q$. 11
We briefly discuss this result in the rank-one case \( q = 1 \). Here the Heckman-Opdam functions \( \varphi^\rho_\lambda \) are Jacobi functions \( \varphi^{(\alpha,\beta)}_\lambda \) as studied in Koornwinder \[K1\]. More precisely,

\[
\varphi^\rho_\lambda(t) = \varphi^{(\alpha,\beta)}_\lambda(t) \quad \text{with} \quad \alpha = dp/2, \quad \beta = d/2 - 1, \quad d = 1, 2, 4
\]

and \( \rho = \alpha + \beta + 1 = d(p + 1)/2 \). Furthermore,

\[
\psi_\lambda(t) = e^{i\lambda\min(t,1)} = (\cosh t)^{i\lambda}
\]

independently of \( d \), and \( \rho^A = 0 \). Thus, Theorem 4.2 implies for \( q = 1 \) the following

**4.3 Corollary.** There exists a constant \( C > 0 \) as follows:

1. For \( \beta = -1/2, 0, 1 \), all \( t \in [0, \infty[ \), \( \alpha > 0 \), and \( \lambda \in \mathbb{C} \),

\[
\left| \varphi^{(\alpha,\beta)}_\lambda -i\rho (t) - (\cosh t)^{i\lambda} \right| \leq C \cdot \frac{|\lambda| \min(t,1)}{\sqrt{\alpha}} \cdot e^{i\lambda t}. \]

2. Let \( \beta = -1/2, 0, 1 \), \( t \in [0, \infty[ \), \( \alpha > 0 \), and \( \lambda \in \mathbb{C} \) with \( \Im \lambda \in [0, 2\rho] \). Then

\[
\left| \varphi^{(\alpha,\beta)}_\lambda -i\rho (t) - (\cosh t)^{i\lambda} \right| \leq C \cdot \frac{|\lambda| \min(t,1)}{\sqrt{\alpha}}. \]

**4.4 Remarks.** (1) For \( \Im \lambda = 0 \) and all \( \beta \geq -1/2 \), Corollary 4.3(2) was proven in \[V2\]. The proof there relies on the well-known integral representation for the Jacobi functions for \( \alpha \geq \beta \geq -1/2 \) and is similar to that given here. Corollary 4.3(2) for \( \Im \lambda = 0 \) is used in \[V2\] to derive a central limit theorem for the hyperbolic distances of radial random walks on hyperbolic spaces from their starting point when the number of time steps as well as the dimensions of the hyperbolic spaces tend to infinity. Similar results can be derived from Theorem 4.2 for \( q \geq 2 \).

(2) Corollary 4.3 corresponds to the convergence of the well-known one-dimensional Jacobi convolutions \(*_{(\alpha,\beta)}\) to a semigroup convolution on \([0, \infty[ \) where the multiplicative functions of the limit semigroup are precisely the functions \( t \mapsto (\cosh t)^{i\lambda} \); i.e., the convergence of the convolution structures \(*_{(\alpha,\beta)}\) for \( \alpha \to \infty \) corresponds to the convergence of the multiplicative functions. The same picture appears for \( q > 1 \); see \[R2\] for the explicit convolution and \[RKV\] for the corresponding limit transition. In \[K2\], a corresponding result for polynomials was derived.

(3) There are similar limit results to those of Theorem 4.2 for Dunkl-type Bessel functions of types A and B, and for Bessel functions on matrix cones with applications in probability; see \[RV2\], \[RV3\].

We now turn to the proof of Theorem 4.2. In fact, our main result is essentially a consequence of Lemma 4.1 and the following technical variant of Theorem 4.2.

**4.5 Theorem.** For each \( n \in \mathbb{N} \) there is a constant \( C = C(F, q, n) \) such that for all \( p > 2q-1 \), \( t \in C_q \) and \( \lambda \in \mathbb{C}^q \),

\[
\left| \varphi^\rho_{\lambda - ip}(t) - \psi_{\lambda - ip^A}(t) \right| \leq C \cdot \left( \psi^{\rho^B} + \psi^{\rho^C} \right) \cdot \frac{|\lambda| \min(t,1)}{\sqrt{\alpha}} \cdot \frac{t^{p/2}}{p^{1/2}}. (4.2)
\]

Notice that the functions \( \varphi, \psi \) on the right side take positive values by Lemma 4.1. In fact, Theorem 4.2(1) follows immediately from Lemma 4.1(1) and Theorem 4.5 with \( n = 1 \). For the proof of Theorem 4.2(2), consider \( \lambda \in \mathbb{C}^q \) with \( \Im \lambda - \rho \in \cos(W, \rho) \). As \( \varphi^\rho_\lambda \) is \( W \)-invariant in the spectral variable \( \lambda \) and as the mapping \( \lambda \mapsto -\lambda \) is an element of \( W \), we may assume without loss of generality
that $\text{Im} \lambda - \rho \in -C_q$. Now choose $\epsilon_0 = \epsilon_0(q) > 0$ according to the following Lemma 4.6 and choose $n \in \mathbb{N}$ such that $\epsilon := (2n - 1)^{-1} \leq \epsilon_0$. Lemma 4.6 for $y := \text{Im} \lambda - \rho$ thus implies that

$$\frac{2n}{2n - 1} \text{Im} \lambda - \rho = (1 + \epsilon)\text{Im} \lambda - \rho = (1 + \epsilon) y + \epsilon \rho \in \text{co}(W, \rho).$$

This fact, Lemma 4.1(2), and Theorem 4.5 then lead to Theorem 4.2(2) as claimed.

4.6 Lemma. For each dimension $q$ there exists a constant $\epsilon_0 = \epsilon_0(q) > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, all $\rho$ in the interior of $C_q$, and all $y \in \text{co}(W, \rho) \cap (-C_q)$,

$$(1 + \epsilon) y + \epsilon \rho \in \text{co}(W, \rho).$$

The proof of this lemma will be postponed to an appendix at the end of this paper. We here only mention that for $q = 1, 2$ the lemma can be easily checked with $\epsilon_0 = 1$ at hand of a picture, but for $q \geq 3$ the situation is more complicated, and the lemma is then no longer true with $\epsilon_0 = 1$.

We now turn to the technical proof of Theorem 4.5. We decompose it into several steps. We first recall the integral representation (2.10),

$$\varphi_{\lambda-i\rho}(t) = \int_{B_q} \int_{U_0(q, F)} \Delta_{i\lambda/2}(\tilde{g}_t(u, w)) \, dm_p(w) du$$

(4.3)

with the probability measure $dm_p$ as in Section 2 and

$$\tilde{g}_t(u, w) = u^*(\cosh t + \sinh t w)(\cosh t + \sinh t w)^* u.$$  

(4.4)

In order to analyze the principal minors $\Delta_1, \ldots, \Delta_q$ appearing in the definition of the power function $\Delta_{i\lambda/2}$, we use the singular values $\sigma_1(a) \geq \sigma_2(a) \geq \ldots \geq \sigma_q(a)$ of a matrix $a \in M_q$ ordered by size, i.e., the square roots of the eigenvalues of $a^*a$. We need the following known estimates for singular values:

4.7 Lemma. For all matrices $a_1, a_2 \in M_q(F)$ and $i = 1, \ldots, q$,

$$|\sigma_i(a_1 + a_2) - \sigma_i(a_1)| \leq \sigma_1(a_2) \quad \text{and} \quad \sigma_1(a_1 \cdot a_2) \leq \sigma_i(a_1) \sigma_1(a_2).$$

Proof. For $F = \mathbb{R}, \mathbb{C}$ we refer to Theorem 3.3.16 of [HI]. The case $F = \mathbb{H}$ can be reduced to $F = \mathbb{C}$ by the same arguments as in the second part of the proof of Corollary 2.2.

4.8 Lemma. For $t \in C_q$, $w \in B_q$, $u \in U_0(q, F)$ and $r = 1, \ldots, q$,

$$\frac{\Delta_r(\tilde{g}_t(u, w))}{\Delta_r(\tilde{g}_t(u, 0))} \in [(1 - \overline{t} \sigma_1(w))^{2r}, (1 + \overline{t} \sigma_1(w))^{2r}], \quad \text{with} \quad \overline{t} := \min(t_1, 1).$$

Proof. We write the matrix $\tilde{g}_t(u, w)$ as

$$\tilde{g}_t(u, w) = b(I + \bar{w})(I + \bar{w}^*) b^*$$

(4.5)

with

$$b := u^* \cosh t, \quad \bar{w} := (\cosh t)^{-1} \sinh t \cdot w = \tanh t \cdot w$$

The inequalities of Lemma 4.7 imply for $i = 1, \ldots, q$ that

$$|1 - \sigma_i(I + \bar{w})| = |\sigma_i(I) - \sigma_i(I + \bar{w})| \leq \sigma_1(\bar{w}) = \sigma_1(tanh t \cdot w) \leq \sigma_1(tanh t) \cdot \sigma_1(w).$$
As $0 \leq \tanh x \leq \min(x, 1)$ for $x \geq 0$ and $x \mapsto \tanh x$ is increasing, we conclude that
\[ \sigma_1(\tanh t) = \min(t_1, 1) = \tilde{t} \]
and thus
\[ |1 - \sigma_1(I + \tilde{w})| \leq \tilde{t} \cdot \sigma_1(w) \in [0, 1]. \] (4.6)
This implies for $i = 1, \ldots, q$ that
\[ (1 - \tilde{t} \sigma_1(w))^2 \leq \sigma_1(I + \tilde{w})^2 \leq (1 + \tilde{t} \sigma_1(w))^2. \] (4.7)
This leads to the matrix inequality
\[ (1 - \tilde{t} \sigma_1(w))^2 I \leq (I + \tilde{w})(I + \tilde{w}^*) \leq (1 + \tilde{t} \sigma_1(w))^2 I, \]
and thus
\[ (1 - \tilde{t} \sigma_1(w))^2 bb^* \leq b(I + \tilde{w})(I + \tilde{w}^*)b^* \leq (1 + \tilde{t} \sigma_1(w))^2 bb^*. \]
As for Hermitian matrices $a, b$ with $0 \leq a \leq b$ the determinants satisfy $0 \leq \Delta(a) \leq \Delta(b)$, we finally obtain
\[ \Delta_r(b(I + \tilde{w})(I + \tilde{w}^*)b^*) \in [(1 - \tilde{t} \sigma_1(w))^{2r} \Delta_r(bb^*), (1 + \tilde{t} \sigma_1(w))^{2r} \Delta_r(bb^*)] \] (4.8)
as claimed.

For the next step in the proof of Theorem 4.5 we use the integral representation (4.1),
\[ \psi_{\lambda-i\rho^A}(t) = \int_{U_0(q, F)} \Delta_{\lambda/2}(u^{-1}(\cosh t)^2 u) du = \int_{B_q} \int_{U_0(q, F)} \Delta_{\lambda/2}(\tilde{g}(u, 0)) dm_p(w) du. \] (4.9)

Using Lemma 4.8 we estimate the difference of the integrands in (4.3) and (4.9). We shall obtain the following result.

4.9 Lemma. Let $t \in \mathbb{R}^q$ and $\lambda \in \mathbb{C}^q$. Then for all $n \in \mathbb{N}$,
\[ |\varphi_{\lambda-i\rho^A}^{p}(t) - \psi_{\lambda-i\rho^A}(t)| \leq 8q||\lambda||_1 \tilde{t} \cdot \left( \frac{1}{K_{pd/2}} \int_{B_q} \sigma_1(w)^{2n} \Delta(I - w^*w)^{pd/2-\gamma-2n} dw \right)^{1/2n} \]
\[ \cdot \left( e^{\frac{2n}{2n-1}} t \Delta r(\tilde{g}(u, 0)) - e^{\frac{2n}{2n-1}} t \Delta r(\tilde{g}(u, 0)) \right) \]
Proof. We write the difference
\[ D := |\Delta_{\lambda/2}(\tilde{g}(u, 0)) - \Delta_{\lambda/2}(\tilde{g}(u, 0))| \]
of the integrands in (4.3), (4.9) as $D = |e^\alpha - e^\beta|$ with
\[ \alpha := \alpha(t, \lambda, u, w) = \frac{i}{2} \sum_{r=1}^{q} (\lambda_r - \lambda_{r+1}) \cdot \ln \Delta_r(\tilde{g}(u, w)) \]
and
\[ \beta := \beta(t, \lambda, u) = \frac{i}{2} \sum_{r=1}^{q} (\lambda_r - \lambda_{r+1}) \cdot \ln \Delta_r(\tilde{g}(u, 0)) \]
with the agreement $\lambda_{q+1} = 0$. We further write the functions $\alpha, \beta$ as $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. By elementary calculus, we obtain
\[ |e^\alpha - e^\beta| = |e^{\alpha_1 + i\alpha_2} - e^{\beta_1 + i\beta_2}| \leq |e^{i\alpha_2}| \cdot |e^{\alpha_1} - e^{\beta_1}| + e^{\beta_1} \cdot |e^{i\alpha_2} - e^{i\beta_2}| \]
\[ \leq |e^{\alpha_1} - e^{\beta_1}| + \sqrt{2} \cdot |e^{\beta_1}||\alpha_2 - \beta_2| \]
\[ \leq |\alpha_1 - \beta_1| \cdot (e^{\alpha_1} + e^{\beta_1}) + \sqrt{2}(e^{\alpha_1} + e^{\beta_1})|\alpha_2 - \beta_2| \]
\[ \leq 2 \cdot |\alpha - \beta| \cdot (e^{\alpha_1} + e^{\beta_1}). \] (4.10)
We have
\[ |\alpha - \beta| \leq \|\lambda\|_1 \cdot \max_{r=1,\ldots,q} |\ln \Delta_r(\bar{g}(t, u, w)) - \ln \Delta_r(\bar{g}(t, u, 0))|. \]
Hence we obtain from Lemma 4.8 together with the elementary inequality
\[ |\ln(1 + z)| \leq \frac{|z|}{1 - |z|} \quad \text{for} \quad |z| < 1 \quad (4.11) \]
and with \( \tilde{t} \in [0, 1] \) that
\[ |\alpha - \beta| \leq \|\lambda\|_1 \cdot 2q \cdot \frac{\tilde{t} \sigma_1(w)}{1 - \sigma_1(w)} \leq \|\lambda\|_1 \cdot 2q \cdot \frac{\sigma_1(w)}{1 - \sigma_1(w)}. \]
Furthermore, as \( 1 \geq \sigma_1(w) \geq \ldots \geq \sigma_q(w) \geq 0 \) for \( w \in B_q \), we have
\[ \frac{1}{1 - \sigma_1(w)} \leq \frac{2}{1 - \sigma_1(w)^2} \leq \frac{2}{\prod_{r=1}^{q} 1 - \sigma_r(w)^2} = \frac{2}{\Delta(I - w^*w)}. \quad (4.12) \]
We thus conclude that
\[ D \leq 2(e^{\alpha_1} + e^{\beta_1})|\alpha - \beta| \leq 8q(e^{\alpha_1} + e^{\beta_1})\|\lambda\|_1 \cdot \frac{\sigma_1(w)}{\Delta(I - w^*w)}. \]
By this estimate and Hölder's inequality we obtain
\[ \left| \varphi^{\rho}_{\lambda - i\rho}(t) - \psi_{\lambda - i\rho}(t) \right| \leq \quad (4.13) \]
\[ \leq 8q\|\lambda\|_1 \tilde{t} \cdot \frac{\sigma_1(w)}{\Delta(I - w^*w)} \int_{B_q \times U_0(q, F)} \frac{dm_p(w)}{\lambda} du \]
\[ \leq 8q\|\lambda\|_1 \tilde{t} \cdot \left( \frac{\sigma_1(w)^{2n}}{\Delta(I - w^*w)^{2n}} \int_{B_q} dm_p(w) \right)^{1/2n} \times \]
\[ \times \left[ \left( \int_{B_q \times U_0(q, F)} e^{2n\varrho \alpha_1} dm_p(w) du \right)^{\frac{2n-1}{2n}} + \left( \int_{B_q \times U_0(q, F)} e^{2n\varrho \beta_1} dm_p(w) du \right)^{\frac{2n-1}{2n}} \right]. \]
In view of (4.3) and (4.9), the \([\ldots]\)-term in the last two lines is equal to
\[ \varphi^{\rho}_{\frac{2n-1}{2n} i\lambda - i\rho}(t)^{\frac{2n-1}{2n}} + \psi^{\rho}_{\frac{2n-1}{2n} i\lambda - i\rho}(t)^{\frac{2n-1}{2n}}, \]
and the lemma follows.

Eq. (4.2) in Theorem 1.5 is now a consequence of Lemma 4.9 and the following result:

4.10 Lemma. For each \( n \in \mathbb{N} \) there is a constant \( C = C(F, q, n) > 0 \) such that for all \( p \geq 2q, \)
\[ R(p) := \int_{B_q} \frac{\sigma_1(w)^{2n}}{\Delta(I - w^*w)^{2n}} dm_p(w) \leq \frac{C}{p^n}. \]

Proof. We transform the integral in the lemma. The diffeomorphism \( P : B_q \rightarrow B_q \) introduced in Remark 2.6 where \( B \) is the ball \( B := \{ y \in F^q : \|y\|_2 < 1 \} \). We recall from [RH] that for \( w = P(y_1, \ldots, y_q) \), one has \( \Delta(I - w^*w) = \prod_{j=1}^{q} (1 - \|y_j\|_2^2) \). With (2.16) in mind, we obtain
\[ R(p) = \frac{1}{\kappa_{pd/2}} \cdot \int_{B_q} \sigma_1(P(y_1, \ldots, y_q))^{2n} \cdot \prod_{j=1}^{q} (1 - \|y_j\|_2^2)^{d(p-q-j+1)/2-1-2n} du(y_1, \ldots, y_q). \quad (4.14) \]
Moreover, the $j,j$-element $(ww^*)_{jj}$ of $ww^*$ satisfies

$$(ww^*)_{jj} = y_j(I - y_jy_j^*)/(I - y_j^*y_j)^{1/2} \cdot (I - y_j^*y_{j-1})^{1/2}(I - y_{j-1}y_{j-1})^{1/2} \cdots (I - y_1y_1)$$

As the hermitian matrix $I - y^*y$ has eigenvalues in $[0,1]$, it follows readily that $0 \leq (ww^*)_{jj} \leq \|y_j\|^2$ and hence

$$\sigma_1(w)^2 \leq \sum_{j=1}^q (ww^*)_{jj} \leq \sum_{j=1}^q \|y_j\|^2.$$

Therefore,

$$\sigma_1(w)^{2n} \leq C \cdot \sum_{j=1}^q \|y_j\|^{2n}$$

with some constant $C > 0$. This leads to the estimate

$$R(p) \leq \frac{C}{\kappa_{pd/2}} \sum_{j=1}^q \|y_j\|^{2n} \cdot \prod_{r=1}^q (1 - \|y_r\|^{2p})^{(p-q-r+1)/2 - 1/2} \cdot \|y_j\|^{-\alpha} \cdot \frac{\Gamma(\alpha) \Gamma(\frac{dq}{2})}{\Gamma(\alpha + \frac{dq}{2})}$$

Using polar coordinates, we obtain for $y = y_r$ and arbitrary $\alpha > 0$ that

$$\int_B (1 - \|y\|^2)\alpha - 1 \ dx = \omega_{dq} \cdot \frac{\Gamma(\alpha) \Gamma(\frac{dq}{2})}{\Gamma(\alpha + \frac{dq}{2})}$$

and

$$\int_B \|y\|^{2n} (1 - \|y\|^2)\alpha - 1 \ dx = \omega_{dq} \cdot \frac{\Gamma(\alpha) \Gamma(n + \alpha) (\alpha) \Gamma(n + \alpha + \frac{dq}{2})}{\Gamma(n + \alpha + \frac{dq}{2})}$$

with the surface measure $\omega_{dq} := vol(S^{dq-1})$ of the unit sphere in $\mathbb{R}^{dq}$ as normalization constant. These formulas yield that

$$\kappa_{pd/2} = \int_{B^{dq}} \prod_{r=1}^q (1 - \|y_r\|^{2p})^{(p-q-r+1)/2 - 1/2} \cdot \|y_j\|^{-\alpha} \cdot \frac{\Gamma(\alpha) \Gamma(\frac{dq}{2})}{\Gamma(\alpha + \frac{dq}{2})} \prod_{r=1}^q \Gamma(\alpha) \Gamma(\frac{dq}{2})$$

and

$$I_j(p) := \frac{1}{\kappa_{pd/2}} \cdot \int_{B^{dq}} \|y_j\|^{2n} \cdot \prod_{r=1}^q (1 - \|y_r\|^{2p})^{(p-q-r+1)/2 - 1/2} \cdot \|y_j\|^{-\alpha} \cdot \frac{\Gamma(\alpha) \Gamma(\frac{dq}{2})}{\Gamma(\alpha + \frac{dq}{2})} \prod_{r=1}^q \Gamma(\alpha) \Gamma(\frac{dq}{2})$$

From the asymptotics of the gamma function we obtain for $p \to \infty$ the asymptotic equality

$$I_j(p) \sim \frac{\Gamma(n + \frac{dq}{2})}{\Gamma(\alpha)} \cdot \left(\frac{dp}{2}\right)^{-\alpha} \cdot \left(\frac{(p-q-1)}{2}\right)^{-\alpha} \cdot \left(\frac{(p-r+1)}{2}\right)^{-\alpha} \cdot \left(\frac{(p-q+1)}{2}\right)^{-\alpha}$$

This implies that $R(p)$ is of order $O(p^{-\alpha})$ for $p \to \infty$. 

The proof of Theorem 4.3 is now complete.
5 Convergence to Bessel functions of type B

In this section we consider the Heckman-Opdam function $\varphi^p_{\lambda}$ for fixed $p \in \mathbb{R}$ with $p \geq 2q-1$ in a scaling limit. More precisely, we use the integral representation of Theorem 2.4 in order to derive convergence of the rescaled functions $\varphi^p_{n\lambda-\mu}(t/n)$ for $n \to \infty$ to Dunkl-type Bessel functions associated with root system $B_q$. While such asymptotics are well-known in a general context from the asymptotics of the hypergeometric system, we here obtain a precise estimate for the rate of convergence.

To explain the result, let us first recall some facts on Bessel functions from [FK],[Ka] and [R1].

5.1 Multivariate Bessel functions. Let $m = (m_1, \ldots, m_q)$ be a partition of length $q$ with integers $m_1 \geq m_2 \geq \ldots \geq m_q \geq 0$ and let $|m| := m_1 + \ldots + m_q$. For $x \in \mathbb{C}$ and a parameter $\alpha > 0$, the generalized Pochhammer symbol is given by

$$ (x)_m^{\alpha} = \prod_{j=1}^{q} (x - \frac{1}{\alpha}(j-1))_{m_j}. \quad (5.1) $$

For $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with $d = \dim_{\mathbb{R}} F$ and partitions $m$, the spherical polynomials $\Phi_m$ are defined by

$$ \Phi_m(x) = \int_{U_q} \Delta_m(uxu^{-1})du \quad \text{for } x \in M_q(F) $$

where $\Delta_m$ is the power function of Eq. (2.4). We also consider the renormalized polynomials $Z_m = c_m \cdot \Phi_m$ with certain normalization constants $c_m > 0$ which are characterized by the formula

$$ (\text{tr } x)^k = \sum_{|m| = k} Z_m(x) \quad \text{for } k \in \mathbb{N}_0, x \in M_q(F). \quad (5.2) $$

By construction, the $\Phi_m$ and $Z_m$ are invariant under conjugation by $U(q,F)$ and thus depend only on the eigenvalues of their argument. More precisely, for a Hermitian matrix $x \in M_q(F)$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$, we have $Z_m(x) = C_m^\alpha(\xi)$ where the $C_m^\alpha$ are the Jack polynomials of index $\alpha := 2/d$; see Section XI of [FK] and references cited there. The Jack polynomials are homogeneous of degree $|m|$ and symmetric in their arguments.

Following Kaneko [Ka] (see also Section 2.2 of [R1]) we define Bessel functions in two arguments

$$ J_{\mu}(\xi, \eta) := \sum_{m} \frac{(-1)^{|m|}}{(\mu)_m^{2d}|m|!} \cdot \frac{C_m^\alpha(\xi)C_m^\alpha(\eta)}{C_m^\alpha(1, \ldots, 1)} \quad (5.3) $$

for $\mu \in \mathbb{C}$ with $(\mu)_m^{2d} \neq 0$ for all partitions $m$ and with fixed parameter $\alpha := 2/d$. A comparison of (5.3) with the explicit form of the Dunkl-type Bessel functions $J^B_k$ associated with root system $B_q$ in [BF] shows that the Bessel function $J_{\mu}$ can be expressed in terms of $J^B_k$ as

$$ J_{\mu} \left( \frac{\xi^2}{2}, \frac{\eta^2}{2} \right) = J^B_k(\xi, i\eta), $$

with the multiplicity parameter $k := k(\mu, d) := (\mu - (q-1)d/2 - 1/2, d/2)$. For the details see Section 4.3 of [R1] and [Ol] for the general context.

For certain indices $\mu$, the Bessel functions $J_{\mu}$ appear as the spherical functions of the Euclidean-type symmetric spaces $G_0/K$ where $K = U(p,F) \times U(q,F)$ and $G_0 = K \times M_{p,q}(F)$ is the Cartan motion group associated with the Grassmannian $G_{p,q}(F)$. The double coset space $G_0/K$ is naturally identified with the Weyl chamber $C_q$, with $t \in C_q$ corresponding to the double coset of $(I_p, I_q, L) \in G_0$. So we may consider biinvariant functions on $G_0$ as functions on $C_q$. It is well known (see Section 4 of [R1]) that the spherical functions of $(G_0, K)$ are given in terms of the Bessel function $J_{\mu}$ as follows:
5.2 Proposition. The spherical functions of \((G_0, K)\) are given by the Dunkl-type Bessel functions

\[
\varphi^p_\lambda(t) := J_k^p(t, i\lambda) = J_{\mu}(\frac{\lambda^2}{2}, \frac{t^2}{2}), \quad \lambda \in \mathbb{C}^q
\]

with \(\mu := pd/2\) and \(k\) as in Section 5.1. Moreover, \(\varphi^p_{\lambda}\) is bounded precisely for \(\lambda \in \mathbb{R}^q\).

The spherical functions of \((G_0, K)\) with dimension parameters \(p \geq 2q\) admit a Harish-Chandra integral representation which can be extended by Carlson’s theorem to all real parameters \(p > 2q - 1\) and thus to the corresponding indices \(\mu\). This leads to the following

5.3 Proposition. For all real parameters \(p > 2q - 1\) and all \(t \in C_q\) and \(\lambda \in \mathbb{C}^q\),

\[
\varphi^p_\lambda(t) = \int_{B_q} \int_{U_0(q, F)} e^{-i Re tr(w^* u_\lambda)} dm_p(w) du
\]

with the probability measure \(m_p \in M^1(B_q)\) of Eq. (2.6). Moreover, for \(p = 2q - 1\) and with the notations of Remark 2.6,

\[
\varphi^p_\lambda(t) = \frac{1}{\kappa(2q-1)/2} \int_{B^{q-1} \times S} \int_{U_0(q, F)} e^{-i Re tr(P(y^*) u_\lambda)} q^{-1} \prod_{j=1}^{q-1} (1 - \|y_j\|^2) dy_1 \ldots dy_{q-1} d\sigma(y_q) du.
\]

Proof. For \(p > 2q - 1\) and \(\lambda \in C_q\), the first formula is immediate by a combination of the integral representations (3.12) and (4.4) in [R1] (in the latter, integration over \(U(q, F)\) may be replaced by integration over \(U_0(q, F)\)). The general case \(\lambda \in \mathbb{C}^q\) then follows by analytic continuation.

The singular limit case \(p = 2q - 1\) can be derived in the same way as in [R1]; see also Remark 2.6. We omit the details.

A comparison of these integral representations for the Bessel functions \(\varphi^p_{\lambda}\) with the integral representation for the Heckman-Opdam functions \(\varphi^p_{\lambda}\) of Section 2 leads to the following theorem, which is the main result of this section.

5.4 Theorem. For each compact subset \(K \subset \mathbb{R}^q\) there exists a constant \(C = C(K) > 0\) such that for all \(p \in \mathbb{R}\) with \(p \geq 2q - 1\), all \(\lambda \in \mathbb{R}^q\), \(t \in K\), and all \(n \in \mathbb{N}\),

\[
|\varphi^p_{n \lambda - ip}(t/n) - \varphi^p_{\lambda}(t)| \leq C \cdot \frac{\|\lambda\|_1}{n}.
\]

Here again, \(\|\lambda\|_1 = |\lambda_1| + \ldots + |\lambda_q|\).

Proof. We only give a proof for the non-degenerate case \(p > 2q - 1\). The case \(p = 2q - 1\) follows in the same way from (5.5) and Remark 2.6.

We substitute \(w \mapsto -u^* w^*\) in the integral (5.5) and obtain

\[
\varphi^p_{\lambda}(t) = \int_{B_q} \int_{U_0(q, F)} e^{-i Re tr(u^* w^* t u_\lambda)} dm_p(w) du.
\]

Moreover, denoting the trace of the upper left \(r \times r\)-block of a \(q \times q\)-matrix by \(tr_r\), we have

\[
Re tr(u^* w^* t u_\lambda) = \frac{1}{2} \sum_{r=1}^q (u^* (\langle t w \rangle^* + \langle t w \rangle u)_{rr} \cdot \lambda_r
\]

\[
= \sum_{r=1}^q tr_r(u^* (\langle t w \rangle^* + \langle t w \rangle u) - tr_{r-1}(u^* (\langle t w \rangle^* + \langle t w \rangle u)) \cdot \lambda_r / 2
\]

\[
= \sum_{r=1}^q tr_r(u^* (\langle t w \rangle^* + \langle t w \rangle u) \cdot (\lambda_r - \lambda_{r-1}) / 2.
\]
with $\lambda_{q+1} := 0$. Thus,
\[
\bar{\varphi}_p^\lambda(t) = \int_{U_0(q,\mathbb{F}) \times B_q} \prod_{r=1}^q \exp(i \cdot tr_r(u^*((tw)^* + tw)u) \cdot (\lambda_r - \lambda_{r+1})/2) \, dm_p(w) \, du.
\]
Further, according to Theorem 2.4
\[
\bar{\varphi}_{n \lambda - ip}^p(t/n) = \int_{U_0(q,\mathbb{F}) \times B_q} \prod_{r=1}^q \Delta_r(g_{t/n}(u, w))^{in(\lambda_r - \lambda_{r+1})/2} \, dm_p(w) \, du
\]
with the positive definite matrix
\[
g_{t/n}(u, w) = u^* (\cosh(t/n) + \sinh(t/n) \cdot w) (\cosh(t/n) + \sinh(t/n) \cdot w) u.
\]
Using the well-known estimate
\[
\left| \prod_{r=1}^q a_r - \prod_{r=1}^q b_r \right| \leq \sum_{r=1}^q |a_r - b_r| \quad \text{for } a_r, b_r \in \{z \in \mathbb{C} : |z| = 1\},
\]
we obtain
\[
C := |\bar{\varphi}_{n \lambda - ip}^p(t/n) - \bar{\varphi}_\lambda^p(t)|
\leq \sum_{r=1}^q \int_{U_0(q,\mathbb{F}) \times B_q} \left| \Delta_r(g_{t/n}(u, w))^{in(\lambda_r - \lambda_{r+1})/2}
- \exp(i \cdot tr_r(u^*((tw)^* + tw)u) \cdot (\lambda_r - \lambda_{r+1})/2) \right| dm_p(w) \, du.
\]
Further, by the inequality
\[
|e^{ix} - e^{iy}| \leq \sqrt{2} \cdot |x - y| \quad \text{for } x, y \in \mathbb{R},
\]
we obtain
\[
C \leq \frac{1}{\sqrt{2}} \sum_{r=1}^q |\lambda_r - \lambda_{r+1}| \cdot C_r
\] with
\[
C_r := \int_{U_0(q,\mathbb{F}) \times B_q} \left| n \ln \Delta_r(g_{t/n}(u, w)) - tr_r(u^*((tw)^* + tw)u) \right| dm_p(w) \, du.
\]
We now write $g_{t/n}(u, w) = I + A/n + H/n^2$ with $A := u^*((tw)^* + tw)u$ and some Hermitian matrix $H = H(u, w, t, n)$ which stays in a compact subset of $M_q$ for $(u, w, t, n) \in U_0(q,\mathbb{F}) \times B_q \times K \times \mathbb{N}$. Therefore,
\[
n \ln \Delta_r(g_{t/n}(u, w)) = n \ln \Delta_r(I + A/n + H/n^2) = n \ln(1 + tr_r(A)/n + h/n^2)
\]
with some constant $h = h(u, w, t, n) \in \mathbb{C}$ which remains bounded for the arguments under consideration. Using the power series for $\ln(1 + z)$, we get
\[
n \ln \Delta_r(g_{t/n}(u, w)) - tr_r(A) = O(1/n) \quad \text{for } n \to \infty,
\]
uniformly in $u, w$ and $t \in K$. This yields the assertion.

5.5 Remarks. (1) Similar to the results in Section 4, Theorem 5.4 can be extended from $\lambda \in \mathbb{R}^q$ to $\lambda \in \mathbb{C}^q$ with suitable exponential bounds on the right side of the estimate.
We collect some simple facts on the extreme points of $K_C$ with $W_R$ in this appendix. We start with some general facts, where we assume $R$ is a crystallographic root system of rank $q$ in a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ with Weyl group $W$. We fix a closed Weyl chamber $C_q$ for $R$ and denote by $\alpha_1, \ldots, \alpha_q \subset R$ the simple roots associated with $C_q$. We further introduce the dual cone

$$C_q^+ := \{ x \in V : \langle x, y \rangle \geq 0 \}.$$ 

It is well-known (see e.g. Lemma IV.8.3. of [Hu]) that for each $q \geq 1$, the weights $\lambda_1, \ldots, \lambda_r \in V$ defined by $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ with $\alpha_j^\vee = 2\alpha_j/\langle \alpha_i, \alpha_i \rangle$. Then both $x$ and $y$ can be written as linear combinations of the $\lambda_i$ with non-negative coefficients (see [Hu], Section 13.1). By our assumption on $R$ and Section 13 of [Hu], the weights $\lambda_i$ satisfy $\langle \lambda_i, \lambda_j \rangle > 0$ for all $i, j$. We therefore obtain that $\langle x, y \rangle > 0$.

6 Appendix: On convex hulls of Weyl group orbits

In this appendix we present a proof of Lemma 4.6. We start with some general facts, where we assume $R$ is a crystallographic root system of rank $q$ in a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ with Weyl group $W$. We fix a closed Weyl chamber $C_q$ for $R$ and denote by $\alpha_1, \ldots, \alpha_q \subset R$ the simple roots associated with $C_q$. We further introduce the dual cone

$$C_q^+ := \{ x \in V : \langle x, y \rangle \geq 0 \}.$$ 

It is well-known (see e.g. Lemma IV.8.3. of [Hel]) that for each $q \geq 1$, the weights $\lambda_1, \ldots, \lambda_r \in V$ defined by $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ with $\alpha_j^\vee = 2\alpha_j/\langle \alpha_i, \alpha_i \rangle$. Then both $x$ and $y$ can be written as linear combinations of the $\lambda_i$ with non-negative coefficients (see [Hu], Section 13.1). By our assumption on $R$ and Section 13 of [Hu], the weights $\lambda_i$ satisfy $\langle \lambda_i, \lambda_j \rangle > 0$ for all $i, j$. We therefore obtain that $\langle x, y \rangle > 0$.

6.1 Lemma. Suppose that $R$ is irreducible.

1. Let $x, y \in C_q \setminus \{0\}$. Then $\langle x, y \rangle > 0$.
2. There exists a constant $\epsilon_0 > 0$ such that the ball $B_{\epsilon_0}(0) = \{ x \in V : ||x|| < \epsilon_0 \}$ is contained in $co(W.x)$ for each $x \in C_q$ with $||x|| = 1$.

Proof. (1) Let $\lambda_1, \ldots, \lambda_q \in V$ denote the fundamental weights associated with $\alpha_1, \ldots, \alpha_q$, defined by $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ with $\alpha_j^\vee = 2\alpha_j/\langle \alpha_i, \alpha_i \rangle$. Then both $x$ and $y$ can be written as linear combinations of the $\lambda_i$ with non-negative coefficients (see [Hu], Section 13.1). By our assumption on $R$ and Section 13 of [Hu], the weights $\lambda_i$ satisfy $\langle \lambda_i, \lambda_j \rangle > 0$ for all $i, j$. We therefore obtain that $\langle x, y \rangle > 0$.

(2) Let $C_q^1 := \{ x \in C_q : ||x|| = 1 \}$ and consider the continuous mapping $(x, y) \mapsto \langle x, y \rangle$ on the compact set $C_q^1 \times C_q^1$. By part (1), there exists some $\epsilon_0 > 0$ such that $\langle x, y \rangle > \epsilon_0$ for all $x, y \in C_q^1$.

Now fix $x \in C^1_q$. We claim that $B_{\epsilon_0}(0) \subseteq co(W.x)$. For this, let $z \in B_{\epsilon_0}(0) \cap C_q$. Then for each $y \in C_q^1$, we have $\langle z, y \rangle < \epsilon < \langle x, y \rangle$. This shows that $x - z \in C_q^+$ and $z \in x - C_q^+$. In view of (6.1), we thus obtain $B_{\epsilon_0}(0) \cap C_q \subseteq co(W.x) \cap C_q$.

The claim is now immediate.

We now fix some $\rho \in C_q$ and consider the compact convex set $K := co(W, \rho) \cap C_q$.

We collect some simple facts on the extreme points of $K$. 

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6.2 Lemma. (1) The topological boundary \( \partial C_q \) of \( C_q \) is contained in the union of the reflecting hyperplanes \( H_{\alpha_1}, \ldots, H_{\alpha_q} \), associated with the simple reflections \( \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_q} \), and \( C_q \) is the intersection of \( q \) closed half-spaces.

(2) The closed cone \( \rho - C_q^+ \) is also the intersection of \( q \) closed half-spaces corresponding to hyperplanes \( H_{\alpha_1}^+, \ldots, H_{\alpha_q}^+ \).

(3) \( K \) is a compact convex polytope which is obtained as the intersection of \( 2q \) closed half-spaces.

Moreover, if \( x \) is an extreme point of \( K \), then \( x = 0 \) or \( x \in \partial C_q \cap \partial(\text{co}(W,\rho)) \).

(4) If \( x \in K \) is an extreme point different from 0 and \( \rho \), then there exists \( k \in \{1, \ldots, q-1\} \) such that \( x \) is contained in the \( q \)-fold intersection of \( k \) hyperplanes \( H_{\alpha_j} \) and \( q-k \) hyperplanes \( H_{\alpha_j}^+ \).

Proof. (1) See Section 10.1 of [Hu].

(2) This follows from (1) and the definition of the dual cone.

(3) The first statement is clear by (1), (2) and [6.1]. For the second statement, consider some extreme point \( x \) of \( K = C_q \cap (\rho - C_q^+) \). If \( x \) is contained in the interior of \( C_q \), then it is easily checked that \( x \) has to be an extreme point of the cone \( \rho - C_q^+ \) which implies \( x = \rho \). Moreover, if \( x \) is contained in the interior of \( \rho - C_q^+ \) then by the same reasons, \( x \) has to be an extreme point of \( C_q \) and hence \( x = 0 \). This yields the assertion.

(4) This follows from (3).

\[ \square \]

6.3 Lemma. Let \( W_1, W_2 \) be reflection groups acting on \( V_1 \) and \( V_2 \) respectively. Let \( \rho_i \in V_i \) and \( a_i \in \text{co}(W_i,\rho_i) \) for \( i = 1, 2 \). Then \( (a_1, a_2) \in V_1 \times V_2 \) satisfies \( (a_1, a_2) \in \text{co}((W_1 \times W_2)(\rho_1, \rho_2)) \).

Proof. For \( i = 1, 2 \), we have \( a_i = \sum w_i \in W_i \lambda_{w_i}^i w_i \rho_i \) with \( \lambda_{w_i}^i \geq 0 \) and \( \sum w_i \in W_i \lambda_{w_i}^i = 1 \). Therefore,

\[
(a_1, a_2) = \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \lambda_{w_1}^1 \lambda_{w_2}^2 \cdot (w_1 \rho_1, w_2 \rho_2)
\]

as claimed.

\[ \square \]

We finally turn to the proof of Lemma 4.6. As for Weyl groups of type \( B \) the mapping \( x \mapsto -x \) on \( \mathbb{R}^q \) corresponds to the action of some Weyl group element, Lemma 4.6 is a consequence of part (1) of the following result.

6.4 Proposition. Consider a root system \( R \) of rank \( q \) in a Euclidean space \( V \) with reflection group \( W \subset O(V) \) and a fixed closed Weyl chamber \( C_q \) in one of the following cases:

(1) \( R = B_q \) and \( V = \mathbb{R}^q \), or

(2) \( R = A_q \) and the symmetric group \( W = S_{q+1} \) acts either on \( V = \mathbb{R}^{q+1} \) or \( V = (1, \ldots, 1)^\perp \subset \mathbb{R}^{q+1} \) in a non-effective or effective way.

Then there exists some \( \epsilon_0 > 0 \) (depending on \( R \)) such that for all \( 0 \leq \epsilon \leq \epsilon_0 \), \( \rho \in C_q \), and \( y \in \text{co}(W,\rho) \cap C_q \),

\[
(1 + \epsilon)y - \epsilon\rho \in \text{co}(W,\rho).
\]

Notice that for fixed \( y \), the point \( (1 + \epsilon)y - \epsilon\rho = y + \epsilon(y - \rho) \) is opposite to \( \rho \) with respect to \( y \) on the line through \( y \) and \( \rho \), with distance \( \epsilon\|y - \rho\| \) from \( y \). In case \( \epsilon = 1 \), it is obtained from \( \rho \) by reflection in \( y \).

For the root systems \( A_1, B_1 \) and \( B_2 \) the maximal parameter is \( \epsilon_0 = 1 \) while in the reduced \( A_2 \)-case the maximal parameter is \( \epsilon_0 = 1/2 \). In fact, the cases \( A_1, B_1 \) are trivial, while the cases \( A_2, B_2 \) follow easily from the following diagrams:
Proof of Proposition 6.4. For the proof of the general case, we fix $\rho \in C_q$ and consider

$$K := \text{co}(W, \rho) \cap C_q$$

as well as for $\epsilon > 0$, its image $K_\epsilon := \varphi_\epsilon(K)$ under the affine mapping

$$\varphi_\epsilon : y \mapsto (1 + \epsilon)y - \epsilon \rho.$$

Clearly, $K_\epsilon$ is again compact and convex, and $\varphi_\epsilon$ maps extreme points of $K$ onto extreme point of $K_\epsilon$. For the proof of Proposition 6.4 it suffices to prove that extreme points of $K$ are mapped to points in $\text{co}(W, \rho)$ for $\epsilon \in [0, \epsilon_0]$ with $\epsilon_0 > 0$ sufficiently small. For the proof of this statement, we may assume that in addition $\|\rho\|_2 = 1$ holds, and that, by a continuity argument, $\rho$ is contained in the interior of $C_q$.

We prove Proposition 6.4 by induction on $q$ first for the $A_q$-cases and then for $B_q$, where the $A$-cases are used. The proposition is clear for $A_1$ and $B_1$. Let $y \in K$ be an extreme point. By Lemma 6.2(4), we have 3 cases of extreme points:

- If $y = \rho$, then $\varphi_\epsilon(\rho) = \rho$, and the claimed statement is trivial.
- Moreover, if $y = 0$, then $\varphi_\epsilon(0) = -\epsilon \rho$, and the statement follows in all cases with $\epsilon_0 > 0$ as in Lemma 6.1(2).

We now turn to the third case. Assume first that $S_{q+1}$ acts on the vector space $V_q := (1, \ldots, 1)^\perp \subset \mathbb{R}^{q+1}$ where $C_q$ is the closed Weyl chamber associated with the simple roots

$$\alpha_1 := e_1 - e_2, \alpha_2 := e_2 - e_3, \ldots, \alpha_q := e_q - e_{q+1},$$

and $e_1, \ldots, e_{q+1}$ is the standard basis of $\mathbb{R}^{q+1}$. We first study the extreme point $x_0 \in C_q \cap \text{co}(W, \rho)$ contained in the intersection of the hyperplanes $H_{\alpha_1}, \ldots, H_{\alpha_{q-1}} \subset V_q$ and the hyperplane

$$H := \{x \in V_q : \langle x, e_{q+1} \rangle = \langle \rho, e_{q+1} \rangle\}$$

which contains the $q$ affinely independent points $\rho, \sigma_{\alpha_1}(\rho), \ldots, \sigma_{\alpha_{q-1}}(\rho)$ (notice that $\rho$ is in the interior of $C_q$). We observe that $S_q$ acts on $H$ by permutations of the first $q$ components. We now identify $H$ with the vector space $V_{q-1} \subset \mathbb{R}^q$ via the affine mapping

$$(x_1, \ldots, x_q, \rho_{q+1}) \mapsto (x_1 - \rho_{q+1}/q, \ldots, x_q - \rho_{q+1}/q).$$

In terms of this identification, the action of $S_q$ on $H$ is just the usual action of $S_q$ on $V_{q-1}$ with the simple reflections $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_{q-1}}$. We now regard the points $\rho, x_0, \varphi_\epsilon(x_0), \sigma_{\alpha_1}(\rho), \ldots, \sigma_{\alpha_{q-1}}(\rho) \in H$ as
We now consider the extreme point $x_0$ which is contained in the intersection of the hyperplanes $H_{\alpha_1},\ldots,H_{\alpha_k-1},H_{\alpha_k+1},\ldots,H_{\alpha_q}$ and the hyperplane $H := \{(x_1,\ldots,x_{q+1}) \in \mathbb{R}^{q+1} : x_1 + \ldots + x_k = S, x_{k+1} + \ldots + x_{q+1} = -S\} \subset V_q$.

$H$ contains the affinely independent $q$ points $\rho,\sigma_{\alpha_1}(\rho),\ldots,\sigma_{\alpha_{k-1}}(\rho),\sigma_{\alpha_k+1}(\rho),\ldots,\sigma_{\alpha_q}(\rho)$. We write $H$ as $H := H_1 \times H_2$ with $H_1 := \{(x_1,\ldots,x_k) \in \mathbb{R}^k : x_1 + \ldots + x_k = S\}$ and $H_2 := \{(x_{k+1},\ldots,x_{q+1}) \in \mathbb{R}^{q+1-k} : x_{k+1} + \ldots + x_{q+1} = -S\}$ where the group $S_k \times S_{q+1-k}$ acts on $H$. We now identify $H_1$ with $V_{k-1} \subset \mathbb{R}^k$ via the affine mapping

$$p_1 : (x_1,\ldots,x_k) \mapsto (x_1 - S/k,\ldots,x_k - S/k),$$

and $H_2$ with $V_{q-k} \subset \mathbb{R}^{q+1-k}$ via

$$p_2 : (x_{k+1},\ldots,x_{q+1}) \mapsto (x_{k+1} + S/(q+1-k),\ldots,x_{q+1} + S/(q+1-k)).$$

In terms of this identification of $H$ with $V_{k-1} \times V_{q-k}$, the action of $S_k \times S_{q+1-k}$ above on $H$ is just the usual action of $S_k \times S_{q+1-k}$ on $V_{k-1} \times V_{q-k}$. We now consider the Weyl chamber $C_{k-1} \subset V_{k-1}$ associated with the reflections $\sigma_{\alpha_1},\ldots,\sigma_{\alpha_{k-1}}$. We see that $p_1(\rho) \subset C_{k-1}$, and that the points

$$p_1(\rho),p_1(x_0),p_1(\varphi_\epsilon(\rho)),\sigma_{\alpha_1}(p_1(\rho)),\ldots,\sigma_{\alpha_{k-1}}(p_1(\rho)) \in V_{k-1}$$

are related in a way such that we may apply the induction assumption for $A_{k-1}$. We conclude that $p_1(\varphi_\epsilon(\rho))$ is contained in $co(S_k,p_1(\rho))$ for sufficiently small $\epsilon > 0$. In the same way, $p_2(\varphi_\epsilon(\rho)) \in co(S_{q+1-k},p_2(\rho))$ for sufficiently small $\epsilon > 0$. In view of Lemma 6.3 we conclude that there exists some $\epsilon_0 > 0$ such that $\varphi_\epsilon(\rho) \in co(S_k \times S_{q+1-k},\rho) \subset co(S_{q+1},\rho)$ for $0 \leq \epsilon \leq \epsilon_0$ as claimed.

We next study the extreme points $x_0$ with the property that for some $k \in \{1,\ldots,q-1\}$, the point $x_0$ is contained in the $k$ reflecting hyperplanes $H_{\alpha_{j_1}},\ldots,H_{\alpha_{j_k}}$ with $1 \leq j_1 < \cdots < j_k \leq q+1$ as well as in the $k$-dimensional affine subspace $H \subset V_q$ which is spanned by the $k+1$ affinely independent points $\rho,\sigma_{\alpha_{j_1}}(\rho),\ldots,\sigma_{\alpha_{j_k}}(\rho)$. As in the preceding case, we split the problem into several lower dimensional problems which can be handled separately by induction. Again, by Lemma 6.3 we obtain some $\epsilon_0 > 0$ such that $\varphi_\epsilon(\rho) \in co(S_{q+1},\rho)$ for $\epsilon \leq \epsilon_0$. This completes the proof for the effective $A_q$-case, where $S_{q+1}$ acts on $V_q$. This result implies also immediately the non-effective $A_q$-case, where $S_{q+1}$ acts on $\mathbb{R}^{q+1}$.

We finally consider the case $B_q$ for $q > 1$. We assume that $C_q$ is the Weyl chamber associated with the simple roots

$$\alpha_1 := e_1 - e_2,\alpha_2 := e_2 - e_3,\ldots,\alpha_{q-1} := e_{q-1} - e_q,\alpha_q = e_q.$$

We here immediately study the general case where for some $k \in \{1,\ldots,q-1\}$, the extreme point $x_0$ is contained in the $k$ reflecting hyperplanes $H_{\alpha_{j_1}},\ldots,H_{\alpha_{j_k}}$ with $1 \leq j_1 < \cdots < j_k \leq q+1$ as well as in the affine subspace $H \subset \mathbb{R}^{q+1}$ of dimension $k$ which is spanned by the $k+1$ points $\rho,\sigma_{\alpha_{j_1}}(\rho),\ldots,\sigma_{\alpha_{j_k}}(\rho)$. As in the preceding case, we split the problem into several lower dimensional problems which can be handled either as a lower-dimensional $B$-case or as a known $A$-case. The proof is again completed by induction and by use of Lemma 6.3.
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