Dynamical algebras in the 1+1 Dirac Oscillator and the Jaynes-Cummings model

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We study the algebraic structure of the one-dimensional Dirac oscillator, by extending the concept of spin symmetry to a noncommutative case. An SO(4) algebra is found connecting the eigenstates of the Dirac oscillator, in which the two elements of Cartan subalgebra are conserved quantities. Similar results are obtained in the Jaynes-Cummings model.

I. INTRODUCTION

The Dirac equation is a cornerstone of modern physics [1]. It is consistent with both the principles of quantum mechanics and the theory of special relativity, and plays an essential role in the field of high energy physics [2]. The Dirac oscillator (DO), obtained by the substitution $p \rightarrow p - i \beta m \omega x$ into the free Dirac equation [3, 4], has become the paradigm for the construction of covariant quantum models with some well determined nonrelativistic limit [5]. It is used in various branches of physics, such as nuclear physics [6] and subnuclear physics [7], and recently simulated in quantum optics [8–11] and classical microwave setups [5, 12]. What is particularly noteworthy is that, the simulation in quantum optics is based on the equivalence between the DO and the Jaynes-Cummings (JC) model, which is the simplest soluble model in light-matter interaction within the frame of completely quantized theory [13].

On the other hand, the concepts of dynamical symmetry and algebraic structure are essential and prevalent both in classical and quantum mechanics [14, 15]. As the two simple examples, the Hydrogen atom and harmonic oscillator in nonrelativistic quantum mechanics [14, 15], the energy levels of a quantum system may be derived based on its algebraic structure. Algebraic properties of the DO has been studied from different perspectives [16, 17] shortly after the original work of Moshinsky [3]. Very recently, Zhou et al. [18] construct shift operators by using the matrix-diagonalizing technique [19].

In this work, we reexamine the algebraic structures of the one-dimensional (1D) DO and the JC model. The motivation for this research is that, the eigenstates of the JC model (and similarly, the DO) are determined by two good quantum numbers, but only one pair of raising and lowering operators are obtained in Ref. [18]. Therefore, our aim is to derive the two conserved quantities accompanied their shift operators, which describe the full algebraic structure of the two models.

Our work begins with the 1D DO. It can be obtained by linearizing the quadratic form $E^2 = m^2 + p^2 + m^2 \omega^2 x^2 - \beta m \omega$, as the original approach to derive the Dirac Hamiltonian [1], with $\omega$ and $\beta$ being the frequency and one of the Dirac matrices. We choose $\hbar = c = 1$ in this paper. This implies that, the 1D DO and two-dimensional (2D) free Dirac system may share similar features, whereas difference comes from the commutator $[x, p] = i$.

The Dirac Hamiltonian, with scalar and vector potentials of equal magnitude, certainly including the free Dirac equation, has the spin or pseudospin symmetry corresponding to the same or opposite sign [20]. These systems are shown to have the same dynamical symmetries with their nonrelativistic counterparts [21–25]. The key step to reveal the symmetries is to derive the conserved quantities with the aid of an unitary operator [21–23]. Zhang et al. propose a 2D version of the unitary operator [24], and show it can be used to deal with noncentral systems [25].

In this work, we extend the unitary for 2D Dirac equation to a noncommutative case. Namely, one of the momentum in the unitary is replaced by a coordinate operator. By using such unitary, one can derive conserved a number operator and a spin of the 1D DO, the shift operators for which can be constructed simultaneously. With these conserved quantities and shift operators, a SO(4) algebraic structure of the system is revealed, which is said to be a dynamical symmetry in broad sense [26]. Similar results are obtained in the JC model by a similar procedure.

II. CONSERVED ANGULAR MOMENTUM OF THE 2D FREE DIRAC EQUATION

Let us begin with the 2D free Dirac Hamiltonian

$$H = \tilde{\alpha} \cdot \tilde{p} + \beta \tilde{m},$$  

(1)

where the Dirac matrices are conveniently defined in terms of the Pauli matrices $\tilde{\alpha} = (\sigma_1, \sigma_2)$ and $\beta = \sigma_3$. In matrix form, it reads

$$H = \begin{pmatrix} m & B^\dagger \\ B & -m \end{pmatrix},$$  

(2)

with $B = p_1 - i p_2$ and $B^\dagger = p_1 + i p_2$. The spin-orbit coupling leads to the usual orbital angular momentum $l = x_2 p_2 - x_1 p_1$ does not commute with the Hamiltonian (1). A deformed orbit momentum defined in Ref. [24]

$$L = U^\dagger U$$  

(3)

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can be easily proved to be conserved, when the unitary
\[ U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \]  
transforming the number operator into a conserved one as
\[ \mathcal{N} = U\mathcal{N}U^\dagger = \begin{pmatrix} N & 0 \\ 0 & N+1 \end{pmatrix} \]  
It commutes with the Hamiltonian \( \mathcal{H} \). In addition, the spin operator is invariant under the unitary, that is
\[ s = UsU^\dagger, \]
where \( s = \frac{\mathbf{p}}{\hbar} \).

**Spectrum of the 1D DO.**– Under the inverse transformation of the unitary (8), the Hamiltonian becomes
\[ U^\dagger \mathcal{H} U = \begin{pmatrix} m & \sqrt{2m\omega N} \\ \sqrt{2m\omega N} & -m \end{pmatrix}. \]

It can be diagonalized in the diagonalized in the two-dimensional subspace with the same quantum number \( N \), that is
\[ e^{i\frac{\pi}{2} \theta_N} U^\dagger \mathcal{H} U e^{-i\frac{\pi}{2} \theta_N} = \sqrt{2m\omega N} + m^2 \sigma_3, \]
where \( \theta_N \) is a operator function defined by \( \tan \theta_N = \sqrt{2\omega N/m} \). Here, we remark that, the above form in (12) is different with the Foldy-Wouthuysen transformation [28].

The eigenstates of the DO can be obtained directly as
\[ |\varphi^n_\pm\rangle = U e^{-i\frac{\pi}{2} \theta_N} |\pm\rangle \otimes |n\rangle, \]
where \(|\pm\rangle\) and \(|n\rangle\) are eigenstates of the spin and the harmonic oscillator respectively, satisfying \( \sigma_3 |\pm\rangle = \pm |\pm\rangle \) and \( N |n\rangle = n |n\rangle \). Here, the quantum number \( n = 0,1,2\cdots \) for the plus sign (+), but \( n = 1,2\cdots \) for the minus sign (−). They satisfy
\[ \mathcal{H} |\varphi^n_\pm\rangle = \mathcal{E}^n_\pm |\varphi^n_\pm\rangle, \]
and the corresponding eigenenergies are
\[ \mathcal{E}^n_\pm = \pm \sqrt{2m\omega n + m^2}. \]

The eigenstates in matrix form are given by
\[ |\varphi^n_+\rangle = \begin{pmatrix} \cos \theta_n |n\rangle \\ \sin \theta_n |n-1\rangle \end{pmatrix}, \quad |\varphi^n_-\rangle = \begin{pmatrix} -\sin \theta_n |n\rangle \\ \cos \theta_n |n-1\rangle \end{pmatrix}, \]
where \( \theta_n \) is the eigenvalue of \( \theta_N \) satisfying \( \tan \theta_n = \sqrt{2m\omega n/m} \).

**IV. SO(4) Algebra of the 1D DO**

Now, we investigate the full algebraic structure of the 1D DO. The diagonalized form of the Hamiltonian (12) indicates that there is a conserved quantity in addition
to the deformed number operator, which is given by
\[ \Sigma_3 = U e^{-i\tilde{\varphi}_N} \sigma_3 e^{i\tilde{\varphi}_N} U^\dagger, \] (17)
And the deformed number operator (9) equals
\[ \mathcal{N} = U e^{-i\tilde{\varphi}_N} N e^{i\tilde{\varphi}_N} U^\dagger, \] (18)
as \([N, \theta_N] = 0\). The shift operators of the two conserved quantities can be obtained as
\[ b = U e^{-i\tilde{\varphi}_N} a e^{i\tilde{\varphi}_N} U^\dagger, \quad b^\dagger = U e^{-i\tilde{\varphi}_N} a^\dagger e^{i\tilde{\varphi}_N} U^\dagger, \] (19)
\[ \Sigma_\pm = U e^{-i\tilde{\varphi}_N} \sigma_\pm e^{i\tilde{\varphi}_N} U^\dagger, \] (20)
where \(\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2\). They satisfy the commutation relations of the harmonic oscillator and the Pauli matrices as
\[ [\mathcal{N}, b] = -b, \quad [\mathcal{N}, b^\dagger] = b^\dagger, \quad [b, b^\dagger] = 1, \] (21)
\[ [\Sigma_3, \Sigma_\pm] = \pm 2\Sigma_\pm, \quad [\Sigma_+, \Sigma_-] = \Sigma_3. \] (22)

The two conserved quantities and their shift operators constitute two independent Lie algebras. The harmonic oscillator operators construct a SU(1,1) algebra [18, 19] as
\[ K_3 = \mathcal{N} + \frac{1}{2}, \quad K_+ = b^\dagger \xi(K_3), \quad K_- = \xi(K_3)b, \] (23)
with \(\xi(K_3) = \sqrt{\mathcal{N} + 1}\), and they satisfy commutation relations
\[ [K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_3. \] (24)
And, the deformed Pauli matrices construct a SU(2) algebra naturally as
\[ S_3 = \frac{\Sigma_3}{2}, \quad S_\pm = \Sigma_\pm, \] (25)
satisfying
\[ [S_3, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_3. \] (26)
The non hermitian generators in the two Lie algebras can be rewritten as Hermitian ones as
\[ K_1 = \frac{1}{2i}(K_+ + K_-), \quad K_2 = \frac{1}{2i}(K_+ - K_-), \] (27)
\[ S_1 = \frac{1}{2}(S_+ + S_-), \quad S_2 = \frac{1}{2i}(S_+ - S_-). \] (28)
The two algebras are decoupled, as
\[ [S_i, K_j] = 0, \] (29)
with \(i, j = 1, 2, 3\).

The above results show that the 1D DO has a SO(4) algebraic structure, the six generators of which are defined as
\[ I_i = \tilde{K}_i + S_i, \quad R_i = \tilde{K}_i - S_i, \quad \text{with } i = 1, 2, 3, \] (30)
where \(\tilde{K}_1 = iK_1, \quad \tilde{K}_2 = iK_2, \quad \tilde{K}_3 = K_3\). The commutation relations of the SO(4) algebra are
\[ [I_i, I_j] = i\epsilon_{ijk} I_k, \quad [I_i, R_j] = i\epsilon_{ijk} R_k, \quad [R_i, R_j] = i\epsilon_{ijk} I_k, \] (31)
with \(i, j, k = 1, 2, 3\). The two conserved Hermitian operators \(I_3\) and \(R_3\), satisfying \([I_3, R_3] = 0\), form the Catan subalgebra. The Hilbert space of the 1D DO provides a representation of the SO(4) Lie algebra, in which the \(I_3\) and \(R_3\) are diagonal. One can derive their matrix elements directly as
\[ A_1|\varphi_n^+\rangle = \frac{1}{2}[(n+1)|\varphi_{n+1}^+\rangle + n|\varphi_{n-1}^+\rangle] + (-1)^\alpha \frac{1}{2}|\varphi_n^\mp\rangle, \] (32)
\[ A_2|\varphi_n^+\rangle = \frac{1}{2}[(n+1)|\varphi_{n+1}^\mp\rangle - n|\varphi_{n-1}^\mp\rangle] \pm (-1)^\alpha \frac{1}{2}|\varphi_n^\pm\rangle, \] (33)
\[ A_3|\varphi_n^\pm\rangle = [n+1 \pm (-1)^\alpha \frac{1}{2}]|\varphi_n^\pm\rangle, \] where \(A = I\) with \(\alpha = 0\) and \(A = R\) with \(\alpha = 1\).

V. JC MODEL

We now turn to the JC model, the simplest completely quantized model in light-matter interaction, in which a two-level atom (matter) is coupled with a quantized mode (light) of an optical cavity. Its has many applications not only in the field of quantum optics [29] but also in the field of solidstate quantum information circuits [30], both experimentally and theoretically. The Hamiltonian reads
\[ \mathcal{H}_{JC} = \omega a^\dagger a + \frac{\Omega}{2} \sigma_z + J(a^\dagger \sigma^- + a\sigma^+) \] (34)
where \(\Omega\) is the level splitting of the two-level system, \(a (a^\dagger)\) is the destruction (creation) operator of a single bosonic mode with frequency \(\omega\), \(J\) is the coupled coefficient. It in matrix form is given by
\[ \mathcal{H}_{JC} = \begin{pmatrix} \omega a^\dagger a + \frac{\Omega}{2} & J a^\dagger \sigma^- \\ J a \sigma^- & \omega a^\dagger a - \frac{\Omega}{2} \end{pmatrix}. \] (35)

Following a similar procedure for the DO, one can define conserved number operator
\[ \mathcal{N}_{JC} = \mathcal{V} \mathcal{N} \mathcal{V}^\dagger = \begin{pmatrix} N & 0 \\ 0 & N-1 \end{pmatrix} \] (36)
by the unitary
\[ \mathcal{V} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{N}} a^\dagger \end{pmatrix}. \] (37)
The Hamiltonian can be diagonalized in two steps. First, by the unitary, it is transformed into

$$V^\dagger H_{JC} V = \left( \begin{array}{cc} \omega N + \frac{\Omega}{2} & J\sqrt{N+1} \\ J\sqrt{N+1} & \omega(N+1) - \frac{\Omega}{2} \end{array} \right).$$

(37)

Second, one can choose a $2 \times 2$ operator in the subspace with the same quantum number as

$$e^{i\frac{2}{\omega} \phi N} V^\dagger H_{JC} V e^{-i\frac{2}{\omega} \phi N}$$

$$= \omega(N + \frac{1}{2}) + \sqrt{J^2(N+1) + \frac{(\Omega - \omega)^2}{4}} \sigma_3,$$

(38)

where $\phi$ is a operator function defined by $\tan \phi = 2J\sqrt{N+1}/(\Omega - \omega)$. Hence, its eigenvalues are

$$\varepsilon^\pm_n = \omega(n + \frac{1}{2}) \pm \sqrt{J^2(n+1) + \frac{(\Omega - \omega)^2}{4}},$$

(39)

corresponding to the eigenstates

$$|\psi^\pm_n\rangle = V e^{-i\frac{2}{\omega} \phi N} |\pm\rangle \otimes |n\rangle,$$

(40)

where $n = 0, 1, 2, \ldots$. Two conserved quantities and their shift operators can be constructed as

$$N_{JC} = V e^{-i\frac{2}{\omega} \phi N} N e^{i\frac{2}{\omega} \phi N} V^\dagger,$$

$$b_{JC} = V e^{-i\frac{2}{\omega} \phi N} a e^{i\frac{2}{\omega} \phi N} V^\dagger,$$

$$b_{JC}^\dagger = V e^{-i\frac{2}{\omega} \phi N} a^\dagger e^{i\frac{2}{\omega} \phi N} V^\dagger,$$

$$\Sigma_{JC}^{(3)} = V e^{-i\frac{2}{\omega} \phi N} \sigma_3 e^{i\frac{2}{\omega} \phi N} V^\dagger,$$

$$\Sigma_{JC}^{(\pm)} = V e^{-i\frac{2}{\omega} \phi N} \sigma_\pm e^{i\frac{2}{\omega} \phi N} V^\dagger,$$

(41)

Follow the same steps to study the DO, one can show a SO(4) algebraic structure in the JC model.

VI. SUMMARY

By extending the approach to study the 2D Dirac system with a spin symmetry, we present a unitary which transforms the number operator to a conserved quantity of the 1D DO. Based on this result, one can diagonalize the DO Hamiltonian, and obtain the two conserved quantities together with their shift operators. These operators show an SO(4) algebra connecting the eigenstates of the Dirac oscillator. By a similar procedure, we also show the SO(4) algebra in the JC model.

Further researches on this topic in the two following directions would be interesting. First, whether one can extend the conserved number operator to the two- or three-dimensional Dirac oscillators, and derive a deformed isotropic harmonic oscillator, is a natural question. Second, it is fascinating to consider the algebra of the multiphoton JC Model [31]. More specifically, we look forward to obtain a polynomial generalization of the SO(4) algebra [25] in it.

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