QUIVER W-ALGEBRAS
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Abstract. For a quiver with weighted arrows we define gauge-theory K-theoretic W-algebra generalizing the definition of Shiraishi et al., and Frenkel and Reshetikhin. In particular, we show that the \( q \bar{q} \)-character construction of gauge theory presented by Nekrasov is isomorphic to the definition of the W-algebra in the operator formalism as a commutant of screening charges in the free field representation. Besides, we allow arbitrary quiver and expect interesting applications to representation theory of generalized Borcherds-Kac-Moody Lie algebras, their quantum affinizations and associated W-algebras.

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1. Introduction

Let Γ be a quiver with μ-weighted arrows μ : Γ → \C^×. We construct two-parametric algebra W_{q_1,q_2}(Γ) from the equivariant K-theory on the moduli space of Γ-quiver sheaves on C^2_{q_1,q_2}. If a quiver is simply-laced Dynkin graph, our construction agrees with Frenkel-Reshetikhin definition [1] of W_{q_1,q_2}(g_Γ) on the one hand. On the other hand, it explains that certain observables of the gauge theory, coming from the q_2-lift of the gauge-theory construction of q_1-characters in [2], described in details and called q_1q_2-characters in [3], can be promoted to operator valued currents that form non-commutative associative current algebra W_{q_1,q_2}(g_Γ).

In our constructions the q_1q_2-character currents are valued in the algebra of differential operators in higher times t_{i,1}, ..., t_{i,∞} of the gauge theory [3]. We show that the algebra of these differential operators is equivalent to the Heisenberg algebra of q_1q_2-bosons used by Shiraishi et al. [3] and Frenkel-Reshetikhin [1, 2] to define W_{q_1,q_2}(g_Γ) algebra. Specializing to g_Γ of ADE type (in this case Γ has no loops and hence μ-parameters represent necessarily the trivial class in H_1(Γ, \C^×) and hence are gauged away) we show that the pole cancellation construction of [2] and [3] developed from the cut cancellation construction of [7], is isomorphic to the definition of W_{q_1,q_2}(g_Γ) algebra in [1] as the commutant of screening charges, hence explaining the isomorphism between the gauge theory construction presented in [3] and the algebraic contruction of [1].
The gauge theory definition of $W_{q_1,q_2}(\Gamma)$ algebra is symmetric in exchange $1 \leftrightarrow 2$. However, for the free field realization there is a choice between (12) and (21). The equivalence between the two realizations, transparent from the gauge theory, leads to ‘quantum $q$-geometric’ Langlands equivalence. The duality of $W$-algebras and the connection with geometric Langlands was first found in [8]. The ‘quantum $q$-geometric’ Langlands [1, 9] degenerates to the (CFT) ‘quantum geometric’ Langlands duality $\beta \leftrightarrow \beta^{-1}$ in the limit $q_1 = e^{\epsilon_1}, q_2 = e^{\epsilon_2}$ with $\epsilon_1, \epsilon_2 \to 0$ and $\beta = -\epsilon_1/\epsilon_2$, and further down in the limit $\epsilon_1 = h, \epsilon_2 = 0$ to the ‘geometric’ Langlands duality [10, 12].

For a survey of duality of $W$-algebras and its connection with the geometric Langlands program see [1] and [13] section 8.6.

In the language of complex integrable systems, and in the reverse order, the ‘geometric’ Langlands duality ($\epsilon_1 = h, \epsilon_2 = 0$) is $T$-duality along the fibers of the phase space of Hitchin integrable system. The K-theory lift to $\Gamma$-quiver representations in the category of coherent sheaves on a complex variety $S$ = $S_1 \times S_2$, so that we consider equivariant K-theory on the moduli space $\mathcal{M}(\Gamma, \text{CMod})$ of $\Gamma$-quiver representations in the category of vector spaces $\text{C-Mod}$. To see the $q_2$-parameter one needs to consider the central extension of $U_{q_1}(L\mathfrak{g}_\Gamma)$ to $U_{q_1}(\hat{\mathfrak{g}}_\Gamma)$ (quantum Drinfeld affinization of $\mathfrak{g}_\Gamma$). The central extension is missing in Nakajima’s construction which concerns only the specialization of $U_{q_1}(\hat{\mathfrak{g}}_\Gamma)$ by the trivial center to $U_{q_1}(L\mathfrak{g}_\Gamma)$.

Compared to Nakajima, we replace a point by a complex variety $\mathcal{S}$ and replace $\text{CMod}$ by $\mathcal{O}_S\text{Mod}$, so that we consider equivariant K-theory on the moduli space $\mathcal{M}(\Gamma, \mathcal{O}_S\text{Mod})$ of $\Gamma$-quiver representations in the category of coherent sheaves on a complex variety $\mathcal{S}$. For $\mathcal{S} = \mathbb{C}_{q_1,q_2}$ we recover $W_{q_1,q_2}(\mathfrak{g}_\Gamma)$ from K-theory on $\mathcal{M}(\Gamma, \mathcal{O}_{\mathbb{C}_{q_1,q_2}}\text{Mod})$. As proposed by Nekrasov in [3] using complex 4-dimensional setup this should be equivalent to considering the K-theory on the $q_2$-twisted fiber-parity-inversed total space of the tangent bundle to Nakajima’s quiver variety $\Pi T_{q_2}^*T_{q_1,q_2}^*\mathcal{M}(\Gamma, \text{CMod})$.

We expect that K-theory definition of quiver $W$-algebra $W(\Gamma, \mathbb{C}_{q_1,q_2})$ can be given a more geometric sense in the more general situation when $\mathbb{C}_{q_1,q_2}$ is replaced by a generic complex variety $\mathcal{S}$ factorized into a product $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ and we expect that the $1 \leftrightarrow 2$ duality will be lifted to higher Langlands duality. The relation to cohomological Hall algebra of quiver [21] remains to be clarified.

This paper takes equivariant K-theory as example of generalized cohomological theory corresponding to the supersymmetric 5d theory reduced on $S^1$. However, all constructions

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1The parameters $(q_1^{\pm}, q_2^{\pm})$ are called by $(q, t)$ in [1, 12, 13]. However, the parameter $t$ in Nakajima’s $(q, t)$-characters [20], which grades the cohomological degree, has different meaning from the present $q_2$. **
remain intact if equivariant K-theory is replaced by the ordinary equivariant cohomology (4d theory) or by equivariant elliptic cohomology (6d theory reduced on elliptic curve). Consequently, the geometric construction of K-theoretic W-algebra can be scaled to its Yangian version [1, 15, 22] using cohomology and lifted to the elliptic version [23, 24] using elliptic cohomology.

For $A_r$-quivers the defining relation of the present note between gauge theory and $W(A_r)$-algebra after the $90^\circ$ brane rotation (the exchange between the rank of the gauge group in the quiver nodes and the rank of the quiver [25], equivalently Nahm transform, equivalently fiber-base duality) implies the AGT duality of [26–28]. The invariance under the brane rotation of the gauge theory partition function is clear from the formalism of refined topological vertex [29–31] and was explicitly checked in [32].

We do not restrict $\Gamma$ to be quiver of finite Dynkin type and consequently expect interesting applications to representation theory of (quantum affinization of) generalized Borcherds-Kac-Moody Lie algebras, such as $E_{11}$ symmetry prominently appearing in M-theory or Borcherds Monster Lie algebra for Conway-Norton moonshine. The affine and hyperbolic quivers generate new W-algebras describing affine (such as sinh-Gordon) and hyperbolic quantum 2d Toda models.

Also it would be interesting to interpret the higher times and the meaning of the presented $W(\Gamma)$-symmetry in the context of topological string on toric CY realization of the gauge theory partition function for ADE and affine ADE quivers [33].

Note. The origins of the $q$-Virasoro symmetry the case of single node quiver $\Gamma = A_1 = \bullet$ to the $\text{Vir}_{q_1,q_2} = W_{q_1,q_2}(A_1)$ algebra can be traced to Eynard’s $q$-deformed single matrix model [34–36]. Elliptic version of matrix model is discussed in [37]. It would be interesting to explore $\Gamma$-quiver matrix models beyond Dynkin graphs of finite and affine type [38].

The regularity of $qq$-characters was explained by Nekrasov in the talk in Strings 2014 and multiple other talks. In [39] the regularity for a linear quiver was interpreted in the language of the quantum toroidal algebra $U_{q_1,q_2}(\hat{\mathfrak{sl}}_1)$ which by Nakajima’s construction [40–46] acts on the instanton moduli spaces on $\mathbb{C}^2$ for each individual node.

The relation between quiver gauge theories and W-algebras in terms of Toda conformal blocks for finite ADE quivers also appeared in [47].

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2. Quiver gauge theory

In this section we define the extended partition function $Z$ of quiver gauge theories [4, 48, 49].
2.1. **Quiver.** Let \( \Gamma \) be a quiver with the set of nodes \( \Gamma_0 \) and the set of arrows \( \Gamma_1 \). By \( i, j \in \Gamma_0 \) we label the nodes, and by \( e : i \to j \) we denote an arrow \( e \) from the source \( i = s(e) \) to the target \( j = t(e) \). We allow loops and multiple arrows.

2.2. **Cartan matrix and Kac-Moody algebra.** A quiver \( \Gamma \) defines \(|\Gamma_0| \times |\Gamma_0|\) matrix \((c_{ij})\)

\[
c_{ij} = 2 - \#(e : i \to j) - \#(e : j \to i)
\]

(2.1)

that is called *quiver Cartan matrix* \( c \). By definition, the quiver Cartan matrix \( c \) is symmetric. If there are no single node loops, all diagonal entries of the quiver Cartan matrix are equal to 2 and such Cartan matrix defines Kac-Moody algebra \( g_\Gamma \) with Dynkin graph \( \Gamma \).

2.3. **Quiver sheaves.** Choose the space-time to be a complex variety \( S \) with structure sheaf \( \mathcal{O}_S \), and let \( \text{Coh}(S) \) denote the category of coherent sheaves on \( S \) (the category \( \mathcal{O}_S \text{-Mod} \) of \( \mathcal{O}_S \)-modules).

Let \( \text{Coh}(S)_\Gamma = \text{Rep}(\Gamma, \text{Coh}(S)) \) be the category of representations of quiver \( \Gamma \) in \( \text{Coh}(Q) \). We call \( \text{Rep}(\Gamma, \text{Coh}(S)) \) by \( \Gamma \)-quiver gauge theory on \( S \): each node \( i \) is sent to a sheaf \( Y_i \) on \( S \) and each arrow \( e : i \to j \) is sent to an element of \( \text{Hom}_{\mathcal{O}_S}(Y_i, Y_j) \).

In the context of \( N = 2 \) gauge theories, a sheaf \( Y_i \) represents gauge connection in the \( i \)-th vector multiplet, and an element in \( \text{Hom}_{\mathcal{O}_S}(Y_i, Y_j) \) represents field in the \( i \to j \) bi-fundamental hypermultiplet.

2.4. **Moduli space.** Let

\[
\mathcal{M}(\Gamma, S) = \text{Coh}(S)_\Gamma / \text{Aut}(\text{Coh}(S)_\Gamma)
\]

(2.2)

be the moduli space of \( \Gamma \)-quiver sheaves on \( S \).

Let \( \gamma = \text{ch} \mathcal{Y} \) denote the Chern character of the collection \( \mathcal{Y} = (Y_i)_{i \in \Gamma_0} \) so that \( \gamma = (\gamma_i)_{i \in \Gamma_0} \) with \( \gamma_i = \text{ch} Y_i \in H^\bullet(S) \). The Chern character \( \gamma_i \) characterizes the topological class of sheaf \( Y_i \).

The total moduli space \( \mathcal{M}(\Gamma, S) \) of \( \Gamma \)-quiver sheaves on \( S \) is a disjoint union over topological sectors

\[
\mathcal{M}(\Gamma, S) = \bigsqcup_\gamma \mathcal{M}(\Gamma, S)_\gamma
\]

(2.3)

Algebraically, the moduli space \( \mathcal{M}(\Gamma, S) \) is a derived stack with virtual tangent bundle \( T \mathcal{M}(\Gamma, S) \) at \( \mathcal{Y} \) given by

\[
T_\mathcal{Y} \mathcal{M}(\Gamma, S) = \text{Coh}(S)_\Gamma(\mathcal{Y}, \mathcal{Y})[1]
\]

(2.4)

More explicitly

\[
T_\mathcal{Y} \mathcal{M}(\Gamma, S)^\bullet = \bigsqcup_{(i \to j) \in \Gamma_1} \text{Ext}^\bullet_{\mathcal{O}_S}(Y_i, Y_j) \oplus \bigsqcup_{i \in \Gamma_0} \text{Ext}^{i+1}_{\mathcal{O}_S}(Y_i, Y_i)
\]

(2.5)

2.5. **Universal sheaf.** Let \( \mathcal{Y} = (\hat{Y}_i)_{i \in \Gamma_0} \) denote the *universal sheaf* over \( \mathcal{M}(\Gamma, S) \times S \) that is associated to the family of sheaves \( Y_i \) on \( S \) parametrized by \( \mathcal{M}(\Gamma, S) \).
2.6. **Equivariant version.** Suppose we are given an equivariant action of a complex group $T$ on the sheaves $\text{Coh}(S)$. Then quiver gauge theory can be defined $T$-equivariantly. In particular, group $T$ acts on the moduli space $\mathcal{M}(\Gamma, S)$ of $T$-equivariant $\Gamma$-quiver sheaves on $S$.

2.7. **Partition function.** Define partition function $Z_T(\Gamma, S)_\gamma$ in topological sector $\gamma$ be the $T$-equivariant index (holomorphic equivariant Euler characteristic) of the structure sheaf on the moduli space of $\Gamma$-quiver sheaves on $S$ of charge $\gamma$

$$Z_T(\Gamma, S)_\gamma = \sum_{n \in \mathbb{Z}} (-1)^n \text{ch}_T H^n(\mathcal{M}(\Gamma, S)_\gamma, \mathcal{O}_{\mathcal{M}(\Gamma, S)_\gamma})$$  \hspace{1cm} (2.6)

The total partition function is the sum over the charges

$$Z_T(\Gamma, S) = \sum_\gamma q^\gamma Z_T(\Gamma, S)_\gamma$$  \hspace{1cm} (2.7)

This partition function in the context of $\mathcal{N} = 2$ gauge theories is known under the name *K-theoretic* Nekrasov partition function, or the partition function of the 5d quiver gauge theory reduced on $S^1$ \[^{[49]}\] where $q$ is the coupling constant.

We can write the partition function using the notation of the derived pushforward $\pi_t = \sum (-1)^i \pi_! \pi^*$ for the projection (integration) map $\pi : \mathcal{M}(\Gamma, S) \to \text{point}$

$$Z_T(\Gamma, S) = \text{ch}_T \pi_t q^\gamma$$  \hspace{1cm} (2.8)

By definition, the partition function $Z_{T,\gamma}$, being a character of a virtual representation in $\text{Rep}(T)$, can be evaluated on an element $t \in T$. In the context of Nekrasov’s partition function the element $t$ comprises all *equivariant parameters*.

2.8. **Fundamental matter.** Since quiver $\Gamma$ is arbitrary, unlike \[^{[2,7]}\] where $\Gamma$ was of finite or affine type, in the present formalism (anti) fundamental matter for a node $i$ is treated simply as bi-fundamental arrow between the node $i$ and another *frozen node* which is represented in constant sheaves with gauge coupling constant $q$ turned off.

2.9. **Local observables.** Let $o \in S$ be a $T$-invariant point on space-time $S$ and let $i_o : o \to S$ be the inclusion map that naturally induces $i_o : \mathcal{M}(\Gamma, S) \to \mathcal{M}(\Gamma, S) \times S$.

We define *observable sheaves* $(Y_i)_{i \in \Gamma_0}$ over the moduli space $\mathcal{M}(\Gamma, S)$ as the pullback of the universal sheaf $(\hat{Y}_i)_{i \in \Gamma_0}$ from $\mathcal{M}(\Gamma, S) \times S$ to $\mathcal{M}(\Gamma, S)$ by the inclusion $i_o$

$$Y_i = i_o^* \hat{Y}_i$$  \hspace{1cm} (2.9)

Let $Y_i^{[p]}$ be the $p$-th Adams operation applied to $Y_i$. The sheaves $(Y_i^{[p]})_{i \in \Gamma_0, p \in \mathbb{Z}_{\geq 0}}$ generate the *ring of observables* (using the direct sum modulo equivalence from exact sequences as the addition and the tensor product as the multiplication) which is a subring in the $T$-equivariant $K$-theory of sheaves on $\mathcal{M}(\Gamma, S)$.
2.10. Extended partition function. We fix quiver $\Gamma$ and the space-time $S$ and drop the symbols from the notations.

Associated to the local observables $(Y^{[p]}_{i})_{i\in\Gamma_{0}, p\in\mathbb{Z}_{\geq 0}}$ introduce parameters, called higher times $t = (t_{i,p})_{i\in\Gamma_{0}, p\in\mathbb{Z}_{\geq 1}}$ and Chern–Simons levels $(\kappa_{i})_{i\in\Gamma_{0}}$. We treat higher times $t_{i,p}$ as the conjugate variable to $Y^{[p]}_{i}$ in the sense of the generating function \[ Z_{T}(t) = \text{ch}_{T} q^{\gamma} \prod_{i\in\Gamma_{0}} \left( \det Y_{i} \right)^{\kappa_{i}} \exp \left( \sum_{p=1}^{\infty} t_{i,p} Y_{i}^{[p]} \right) \] (2.10)

2.11. Localization. Suppose that space of the $T$-fixed points in $M^{T}$ is a discrete set of points $M^{T}$ with inclusion $i_{T} : M^{T} \hookrightarrow M$. Then the generating function (2.10) can be computed by localization formula

\[ Z_{T}(t) = \sum_{\mathfrak{m}^{T}} q^{\gamma} \exp \left( \sum_{p=1}^{\infty} \frac{1}{p} \text{ch}_{T}(T^{\vee}_{\mathfrak{m}^{T}} M^{[p]}) \right) \prod_{i\in\Gamma_{0}} \text{ch}_{T}[\det i_{T}^{*} Y_{i}]^{\kappa_{i}} \exp \left( \sum_{p=1}^{\infty} t_{i,p} \text{ch}_{T}(i_{T}^{*} Y_{i})^{[p]} \right) \] (2.11)

3. Quiver gauge theory on $\mathbb{C}^{2}$

In this section we specialize to the space-time $S = \mathbb{C}^{2}$ with marked point $o \in S$ and the natural action of complex group $GL(2)$ on $S$ by with fixed point $o$.

3.1. Automorphism group $GL(Q)$. We denote by $Q$ the fiber of the cotangent bundle to $S$ at $o$

\[ Q = T^{\vee}_{o} S \] (3.1)

Then $Q$ is the defining module for the group of its automorphisms $GL(Q)$. We split $Q = Q_{1} \oplus Q_{2}$ with respect to Cartan torus

\[ T_{Q} \cong GL(Q_{1}) \times GL(Q_{2}) \subset GL(Q) \] (3.2)

and define $q_{1}, q_{2}$ to be corresponding characters

\[ q_{1} = \text{ch} Q_{1}, \quad q_{2} = \text{ch} Q_{2}, \quad Q = \text{ch}_{T} Q = q_{1} + q_{2}, \quad q = q_{1} q_{2} = \text{ch} \Lambda^{2} Q \] (3.3)

Remark. The parameters $(q_{1}, q_{2})$ are exponentiated $\varepsilon$-parameters of the gauge theory $[7, 50]$.  

3.2. Automorphism groups $GL(N)$ and $GL(M)$. Let $O(S) = \mathbb{C}[z_{1}, z_{2}]$, the ring of polynomials in two variables, be the coordinate ring of $S = \mathbb{C}^{2}$. The space of sections of a coherent sheaf on $S = \mathbb{C}^{2}$ is a $O(S)$-module. Then $\Gamma$-quiver gauge theory on $S$ is identified with representation of $\Gamma$ in $O(S)$-modules. We can take $\mathbb{C}^{2} \cong \mathbb{P}^{2} \setminus \mathbb{P}^{1\infty}$ and fix framing at the $\mathbb{P}^{1\infty}$.

For a $\Gamma$-quiver sheaf $\mathcal{Y}$, let $n \in \mathbb{Z}_{> 0}$ denote the rank and let $k \in \mathbb{Z}_{\geq 0}$ denote the instanton charge

\[ n(\mathcal{Y}) = \text{ch}_{0} \mathcal{Y}, \quad k(\mathcal{Y}) = -\text{ch}_{2} \mathcal{Y} \] (3.4)

Let $N = (N_{i})_{i\in\Gamma_{0}}$ be the framing space associated to $\Gamma_{0}$ part of quiver (nodes). To each node $i$ we associate the framing $N_{i} \cong \mathbb{C}^{n_{i}}$ for the respective sheaf $\mathcal{Y}_{i}$ on $S$. Let

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\footnotetext[2]{Lefschetz - Grothendieck-Hirzebruch-Riemann-Roch - Atiyah-Singer formula}
GL(N) = \prod_{i \in I_0} GL(N_i) be the respective group of automorphisms and let T_N be a Cartan torus of GL(N). Let N be the character of N

N = \text{ch}_{T_N} N = \nu_1 + \cdots + \nu_n \hspace{1cm} (3.5)

Remark. The parameters (\nu_{i, \alpha})_{i \in \Gamma_0, \alpha \in [1 \ldots n]} are the exponentiated Coloumb parameters of the gauge theory.

Let M = (M_e)_{e \in \Gamma_1} be the framing space associated to \Gamma_1 part of quiver (arrows). To each individual arrow e we associate 1-dimensional mass-twisting space M_e \simeq C. Let GL(M) = \prod_{e \in \Gamma_1} GL(M_e) be the respective group of automorphisms and let T_M be a Cartan torus in GL(M). Let M be the character of M

M = \text{ch}_{T_M} M = \mu \hspace{1cm} (3.6)

Remark. The parameters \mu_e are exponentiated mass parameters of the bifundamental fields e : i \rightarrow j of the gauge theory.

If we assign a multiplicity m_e to an arrow e \in \Gamma_1, then the mass-twisting space is M_e \simeq C^{m_e} and its character

M = \text{ch}_{T_M} M = \mu_1 + \ldots + \mu_{m_e} \hspace{1cm} (3.7)

Since in our formalism GL(M_e) is reduced to its Cartan torus T_{M_e} = (C^\times)^{m_e}, the formalism where an arrow e : i \rightarrow j is assigned a multiplicity m_e is equivalent to the formalism where this arrow is replaced by m_e individual arrows i \rightarrow j.

3.3. Complete group of equivariance. For \Gamma-quiver gauge theory on S = C^2 we denote by

T = T_Q \times T_N \times T_M \hspace{1cm} (3.8)

the Cartan torus in the automorphism group of the moduli group \mathcal{M}(S, \Gamma).

3.4. Fundamental matter as background of higher times. Alternatively, fundamental matter can be realized as a background in higher times. To add fundamental multiplet with mass \mu to node i it is sufficiently to additively modify the times to

\[ t_{i,p} \rightarrow t_{i,p} + \frac{1}{p} \frac{q^p}{(1 - q_1^p)(1 - q_2^p)} \mu^{-p} \hspace{1cm} (3.9) \]

To simplify presentation we don’t keep track of the fundamental matter since it is a particular case of the higher times theory. In the operator formalism, on the other hand, this shift is imposed by additional vertex operators introduced in Sec. 3.21

3.5. Localization in quiver theory on C^2. The localization formula \[2.11\] to the T-fixed point set \mathcal{M}^T in the moduli space of \Gamma-quiver sheaves on S = C^2 can be explicitly computed \[49-51\].

The T-fixed sheaves split into the direct sum of 1-dimensional T-fixed ideal sheaves, which are classified as T_Q-fixed ideals in O(S) \simeq C[z_1, z_2] where the fixed point o \in C^2 is the origin o = (0, 0). A T_Q-fixed ideal in C[z_1, z_2] of ch_2 = -k is labelled by a partition (\lambda) = \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \geq 0 \ldots of total size |\lambda| = \sum_{i=1}^{\infty} \lambda_i = k. Each box s = (s_1, s_2) in the partition \lambda with s_1 \in [1 \ldots \infty] and s_2 \in [1 \ldots \lambda_{s_1}] is associated to the monomial z_1^{s_1-1} z_2^{s_2-1}. The ideal I_\lambda \subset O(S) = I_0 is O(S)-generated by all monomials outside of the partition \lambda.

Let K_\lambda = I_\lambda/I_\alpha be generated by the monomials in the partition \lambda.
A $T_N \times T_Q$-fixed $\mathcal{O}(S)$-module $Y_S = \mathcal{Y}(S)$ of rank $n$ splits into direct sum of $T_Q$-fixed ideals

$$Y_S = \bigoplus_{\alpha \in [1, \ldots, n]} \mathcal{L}_{\alpha} \otimes N_{\alpha}$$

(3.10)

and let $Y \equiv Y_o = i_o^* Y_S$. Then we have in K-theory by localization to $i_o : o \leftrightarrow S$

$$[Y_S] = [Y_o]/[\Lambda Q]$$

(3.11)

where $\Lambda Q = \sum_i (-1)^i \Lambda^i Q$ and the division is in formal series. Then

$$[Y_o] = [N] - [\Lambda Q][K]$$

(3.12)

3.6. **Cotangent moduli space.** From (2.5) we find the K-theory class $[T^\vee_Y \mathcal{M}]$ at fixed point $\hat{Y} \in \mathfrak{M}^T$

$$[T^\vee_Y \mathcal{M}] = \frac{1}{[\Lambda Q^\vee]} \left( \sum_{(i, j) \in \Gamma_1} [M^\vee_e][Y_o][Y^\vee_o] - \sum_{i \in \Gamma_0} [Y_o][Y^\vee_o] \right)$$

(3.13)

3.7. **Two commutative reductions.** Since the space-time $S$ is a product $S = S_1 \times S_2$ the reduction from $Y_S$ to $Y_o$ can be done in two steps in two ways, either first project along $S_2$ and then along $S_1$ (left path) or first project along $S_1$ and then along $S_2$ (right path)

$$[X] := [Y_{S_1}] \quad [Y] := [Y_{S_2}] =: [\check{X}]$$

(3.14)

so that it holds

$$[Y_o] = [\Lambda Q_1][X], \quad [Y_o] = [\Lambda Q_2][\check{X}]$$

(3.15)

3.8. **Quantum $q$-geometric Langlands Duality.** The exchange $1 \leftrightarrow 2$ in the above diagram leads to the quantum $q$-geometric Langlands duality $q_1 \leftrightarrow q_2$. See the section [1] for references.

3.9. **Intermediate reduction.** The class $[T^\vee_Y \mathcal{M}]$ at fixed point $\hat{Y} \in \mathfrak{M}^T$ in the equation (3.13) of the partition function can be expressed in terms of $[X] \equiv [Y_{S_1}]$

$$[T^\vee_Y \mathcal{M}] = \frac{[\Lambda Q_1]}{[\Lambda Q_2^\vee]} \left( - \sum_{(i, j) \in \Gamma_0 \times \Gamma_0} [X^\vee_e] c^+_ij [X]_j \right)$$

(3.16)

and the K-theory valued half Cartan matrix $c^+_ij$ defined as

$$[c^+_ij] := \delta_{ij} - \sum_{e : j \to i} [M^\vee_e]$$

(3.17)

with Chern character

$$\text{ch}[c^+_ij] = \delta_{ji} - \sum_{e : j \to i} \mu_e^{-1}$$

(3.18)
3.10. The set of eigenvalues. Then the Chern characters $X = \text{ch} X$ at $T$-fixed point $\lambda$ can be explicitly described. Let

$$\mathcal{X}_i = \{x_{i, \alpha, s_1}\}_{\alpha \in [1, \ldots, n_i], s_1 \in [1, \ldots, \infty]}, \quad \mathcal{X} = \bigsqcup_{i \in \Gamma_0} \mathcal{X}_i$$

be the set of characters of the monomials associated to boxes $(s_1, \lambda, s_1 + 1)$ that generate $(\mathcal{Y}_S)$ as $\mathcal{O}(S_2)$-module so that

$$x_{i, \alpha, s_1} = \nu_{i, \alpha} q_1^{s_1-1} q_2^\lambda$$

and $X_i = \sum_{x \in \mathcal{X}_i} x$.

Let $i: \mathcal{X} \to \Gamma_0$ be the node label so that $i(x) = i$ for $x \in \mathcal{X}_i$.

3.11. The partition function. In terms of $x$-variables the extended partition function (2.11) is

$$Z_T(t) = \sum_{X \in \mathfrak{G}^T} \exp \left( - \sum_{(x_L, x_R) \in \mathcal{X} \times \mathcal{X}} \sum_{p=1}^{\infty} \frac{1 - q_1^p}{p} (c_{i(x_L), i(x_R)})^{[p]} x_L^{-p} x_R^{-p} \right) \times \exp \left( \sum_{x \in \mathcal{X}} \left( -\frac{K_i(x)}{2} (\log_q x - 1) \log_q x + \log q_i(x) \log_q x + \sum_{p=1}^{\infty} \frac{1 - q_1^p}{p} t_i(x, p) x^p \right) \right)$$

where $\tilde{x}_{i, \alpha, s_1} = \nu_{i, \alpha} q_1^{s_1-1}$ denotes ground configuration of the empty partition $\lambda = 0$, so that the $\log_q x$ counts the size of the partition $\lambda$ equal to the instanton charge $k$.

Remark. In the limit $q_2 \to 1$ the extended partition function is dominated by the critical set $\mathcal{X}_{\text{crit}}$ determined in [2] and the variables $x \in \mathcal{X}_{\text{crit}}$ satisfy the Bethe ansatz equations.

3.12. Reflection of the index. Let $X^{[p]}$ be $p$-th Adams operation applied to an object $X$. Then the following reflection equation holds

$$\exp \left( \sum_{p=1}^{\infty} \frac{1}{p} ([X^p]^{[p]}) \right) = (-1)^{rk X} [\det X] \exp \left( \sum_{p=1}^{\infty} \frac{1}{p} ([X^p]^{[p]}) \right)$$

3.13. The ordered partition function. Pick an order $\succ$ on the set $\mathcal{X}$. For example, an order can be chosen by taking $|q_1| \ll |q_2| < 1$ and $|\nu| \simeq |\mu| \simeq 1$. Then define $x_L \succ x_R$ if $|x_L| > |x_R|$. The sum over all pairs $(x_L, x_R) \in \mathcal{X} \times \mathcal{X}$ in the partition function (3.21) can be transformed to the sum over pairs $(x_L \succ x_R)$, over pairs $(x_L \prec x_R)$ and the diagonal pairs $(x_L = x_R)$. The diagonal part gives $(q, \nu, \mu, t)$-independent factor that we omit. The sum over pairs $(x_L \succ x_R)$ and $(x_L \prec x_R)$ can be combined together using the reflection equation (3.22)

$$Z_T(t) = \sum_{X \in \mathfrak{G}^T} \exp \left( \sum_{(x_L \succ x_R) \in \Lambda^2 X} -(c_{i(x_L), i(x_R)})^{[0]} \beta \log x_R x_L^{-1} - \sum_{p=1}^{\infty} \frac{1 - q_1^p}{p} (c_{i(x_L), i(x_R)})^{[p]} x_R^{-p} x_L^{-p} \right) \times \exp \left( \sum_{x \in \mathcal{X}} \left( -\frac{K_i(x)}{2} (\log_q x - 1) \log_q x + \log q_i(x) \log_q x + \sum_{p=1}^{\infty} \frac{1 - q_1^p}{p} t_i(x, p) x^p \right) \right)$$
where $\beta = -\frac{\log q_1}{\log q_2} = -\frac{\epsilon_2}{\epsilon_1}$ and the mass deformed Cartan matrix is
\[
c_{ij} = c_{ij}^+ + c_{ij}^-, \quad c_{ij}^- = q^{-1}(c_{ji}^+)\rangle, \quad c_{ij} = (\delta_{ji} - \sum_{e:j \to i} \mu_e^{-1}) + (q^{-1}\delta_{ji} - \sum_{e:i \to j} q^{-1}\mu_e) \quad (3.24)
\]

3.14. **Extended partition function is a state.** The exponentiated sum over the pairs $(x_L, x_R)$ in the equation (3.23) suggests a natural way to present the extended partition function $Z_T(t)$ as a state $|Z_T\rangle$ in the infinite-dimensional $T$-character valued Fock space $\text{ch Rep}_T[[t]]$. The Fock space $\text{ch Rep}_T[[t]]$ is Verma module for the Heisenberg algebra $\mathbf{H}$ generated by the operators $(\partial_{i,p})_{i \in \Gamma_0, p \in [0,\infty]}$ and $t = (t_{i,p})_{i \in \Gamma_0, p \in [0,\infty]}$ over $\text{ch Rep}_T$ with canonical commutators
\[
[\partial_{i,m}, t_{j,n}] = \delta_{ij}\delta_{mn} \quad (3.25)
\]
where
\[
t_{i,0} = \log q_2 q_i \quad (3.26)
\]
The elements of the Fock space $\text{ch Rep}_T[[t]]$ are formal $t$-series valued in the ring of $T$-characters. The $t$-constants are lowest-weight states (vacua); they are annihilated by all lowering operators $\partial_{i,p}$. A state in the Fock space $\text{ch Rep}_T[[t]]$ can be obtained by an action of an operator in the algebra $\mathbf{H}$ on the vacuum $|1\rangle$

3.15. **Free bosons and vertex operators.** The state $|Z_T\rangle$ can be presented as
\[
|Z_T\rangle = \sum_{x \in \mathcal{X}} \prod_{x \in \mathcal{X}} S_{i(x), x}|1\rangle \quad (3.27)
\]
where $\prod_{x \in \mathcal{X}}$ denotes the $\triangleright$-ordered product over $x \in \mathcal{X}$ of the vertex operators
\[
S_{i,x} =: \exp \left( \sum_{p \ge 0} s_{i,-p}x^p + s_{i,0}\log x + \tilde{s}_{i,0} + \sum_{p > 0} s_{i,p}x^{-p} \right) : \quad (3.28)
\]
Here the free field modes or oscillators are
\[
s_{i,-p} = (1 - q_1^p)t_{i,p}, \quad s_{i,0} = t_{i,0}, \quad s_{i,p} = -\frac{1}{p} \frac{1}{1 - q_2^p c_{ij}} \partial_{j,p} \quad (3.29)
\]
with commutation relations
\[
[s_{i,p}, s_{j,p'}] = -\delta_{p+p',0} \frac{1}{p} \frac{1 - q_1^p}{1 - q_2^p c_{ij}}, \quad p > 0 \quad (3.30)
\]
The conjugate zero mode $\tilde{s}_{i,0}$ satisfies
\[
[\tilde{s}_{i,0}, s_{j,p}] = -\beta\delta_{0,p} c_{ij}^{[0]} \quad (3.31)
\]
The normal product notation: $e^{A_1}e^{A_2}$, where operators $A_1, A_2$ are linear in the free fields, means that all operators $(s_{i,p})_{p \ge 0}$ are placed to the left of $(s_{i,p})_{p > 0}$ and $\tilde{s}_{i,0}$.

The relations (3.31) and (3.30), and the relation $e^{A_1}e^{A_2} = e^{[A_1,A_2]}e^{A_2}e^{A_1}$ for central $[A_1,A_2]$, imply that the operator-state representation of the partition function (3.27) is equivalent to the quiver gauge theory definition (3.23) if gauge theory couplings $\kappa_i$ and $q_i$ are evaluated as
\[
\kappa_i = -(c_{ij}^-)^{[0]} n_i, \quad \log q_2 q_i = \beta + t_{i,0} + (c_{ij}^-)^{[\log q_2]} n_j - (c_{ij}^-)^{[0]} \log q_2 ((-1)^{p_i}, \nu_j) \quad (3.32)
\]
where
\[
(c^{-1}_{ij})^{\log q_2} = \delta_{ij} \log q_2 q^{-1} - \sum_{e\in i\to j} \log q_2 (q^{-1} \mu_e)
\] (3.33)

3.16. Screening charges. The configuration sets \(X \in \mathcal{M}^T\) are described by the partitions, which are explicitly collections of constrained sequences
\[
(\lambda_{i,\alpha,1} \geq \lambda_{i,\alpha,2} \geq \cdots \geq 0 = 0 = \cdots)_{i\in \Gamma_0, \alpha \in [1\ldots n_i]}
\] (3.34)

Let \(X_0\) be the ground configuration with all \(\lambda_{i,\alpha,*} = 0\).

Let \(Z_{X_0}\) be the set of collections of arbitrary integer sequences terminating by zeroes
\[
(\lambda_{i,\alpha,s_1} ? \lambda_{i,\alpha,s_2} ? \ldots ? = 0 = 0 = \ldots)_{i\in \Gamma_0, \alpha \in [1\ldots n_i]}
\] (3.35)

Then \(\mathcal{M}^T \subset Z_{X_0}\). It turns out that the summation over \(\mathcal{M}^T\) in (3.27) can be extended to the whole \(Z_{X_0}\) without changing the result because
\[
\prod_{x\in X} S_{i(x),x}|1\rangle = 0 \quad \text{if} \quad X \in Z_{X_0} \quad \text{but} \quad X \notin \mathcal{M}^T
\] (3.36)
due to the zero factors in the normal ordering product of vertex operators \(S_{i,x}\) for the sequences \((x_{i,\alpha,s_1} = \nu_{i,\alpha,q_1^{s_1-1,q_2^{\lambda_{i,\alpha,s_1}}})\) where \(\lambda\) does not satisfy the constraint (3.34).

Therefore
\[
|Z_T\rangle = \sum_{X \in Z_{X_0}} \prod_{x\in X} S_{i(x),x}|1\rangle
\] (3.37)

For every point \(\hat{x}_{i,\alpha,s_1} = \nu_{i,\alpha,q_1^{s_1-1}}\) in the ground configuration \(X_0\) define the operator called screening charge
\[
S_{i,\hat{x}} = \sum_{s_2 \in \mathbb{Z}} S_{i,q_2^{s_2}x}
\] (3.38)

Then the state \(|Z_T\rangle\) is obtained by applying to the vacuum the ordered product of \(S_i\) operators
\[
|Z_T\rangle = \prod_{\hat{x} \in X_0} S_{i(\hat{x}),\hat{x}}|1\rangle
\] (3.39)

The partition function of plain, not \(t\)-extended theory, can be interpreted as the projection
\[
\langle 1|Z_T\rangle = \langle 1| \prod_{\hat{x} \in X_0} S_{i(\hat{x}),\hat{x}}|1\rangle
\] (3.40)

3.17. The Ward identities. The sum representation of the operator \(S_{i,\hat{x}}\) in (3.38) is explicitly invariant under the \(\mathbb{Z}\)-translational symmetry \(s_2 \to s_2 + \mathbb{Z}\) (change of variables). Hence the representation of the partition function (3.39) is invariant under the \(\mathbb{Z}^{X_0}\) symmetry that shifts the summation variables \(s_2\) for each \(\hat{x}_{i,\alpha,s_1}\). In the \(S\) space-time picture the variation \(s_2 \to s_2 + 1\) amounts to the \(z_2\) multiplicative change of variables in the \(z_1, z_2\)-mode expansion \(\phi_{i,\alpha,s_1} z_1^{s_1} \to z_2 \phi_{i,\alpha,s_1} z_1^{s_1}\) where \(\phi\) is in the sheaf \(\mathcal{Y}\). The shift \(s_2 \to s_2 + 1\) adds one box to the partition, or equivalently one instanton to the gauge field on the space-time.
3.18. **The Y-operators.** In [2, 7] the \( (Y_{i,x})_{i \in \Gamma_0} \) observables were introduced in the K-theory of the moduli space \( \mathcal{M}^T \) of the quiver gauge theory

\[
Y_{i,x} := \exp \left( - \sum_{p=1}^{\infty} \frac{x^{-p}}{p} Y_i^p \right) \tag{3.41}
\]

The expectation value of the observable \( Y_{i,x} \) in the plain (not \( t \)-extended) theory is computed by the pushforward integration over the moduli space \( \mathcal{M}^T \) \( (2.10) \)

\[
\langle Y_{i,x} \rangle := \text{ch}_T \pi_! q^\gamma Y_{i,x} \tag{3.42}
\]

It is natural to lift the \( Y_{i,x} \) observables to the \( t \)-extended theory by giving them the operator definition:

\[
Y_{i,x} = q_1^{\tilde{\rho}_i} : \exp \left( \sum_{p > 0} y_{i,-p} x^p + y_{i,0} + \sum_{p > 0} y_{i,p} x^{-p} \right) : \tag{3.43}
\]

where \( \tilde{\rho}_i := \sum_{j \in \Gamma_0} c_{ij}^{[0]} \) are components of the Weyl vector in the basis of simple roots. If the quiver is the affine type, we put \( \tilde{\rho}_i = 0 \).

The operator \( Y_{i,x} \) is an element of the Heisenberg algebra \( \mathbf{H} \). The oscillators \( y_{i,p} \) are expressed in terms of \( t_{i,p} \) and \( \partial_{i,p} \)

\[
(p > 0) \quad y_{i,-p} = (1 - q_1^p)(1 - q_2^p)(\tilde{c}_{ij}^{-p})_{ij} t_{j,p}, \quad y_{i,0} = -c_{ij}^{[0]} t_{j,0} \log q_2 \quad y_{i,p} = -\frac{1}{p} \partial_{i,p} \tag{3.44}
\]

or equivalently terms of the free field \( s_{i,p} \)

\[
(y_{i,p} = 0) \quad y_{i,p} = (1 - q_2^{-p}) c_{ij}^{[p]} s_{j,p}, \quad y_{i,0} = (\log q_2^{-1}) c_{ij}^{[0]} s_{j,0} \tag{3.45}
\]

where \( \tilde{c}_{ij} \) is the inverse to the mass-deformed Cartan matrix \( c_{ij} \) defined in \( (3.24) \).

The definition \( (3.44) \) and the definition \( (2.10) \) imply

\[
\langle Y_{i,x} \rangle = \langle 1| \sum_{x \neq x'} Y_{i,x} S_{i,x'} |1 \rangle \tag{3.46}
\]

3.19. **The OPE of Y and S.** The commutation relations between \( y_{i,p} \) and \( s_{j,p'} \) are

\[
[y_{i,p}, s_{j,p'}] = -\frac{1}{p} (1 - q_1^p) \delta_{p+p',0} \delta_{ij}, \quad [s_{i,0}, y_{j,0}] = -\delta_{ij} \log q_1 \tag{3.47}
\]

Then \( (3.36) \) can be also seen from the commutation relations \( (3.37) \) and normal ordering because at \( |x| > |x'| \) we have

\[
Y_{i,x} S_{i,x'} = \frac{1 - x'/x}{1 - q_1 x'/x} : Y_{i,x} S_{i,x'} : \quad Y_{i,x} S_{j,x'} = : Y_{i,x} S_{j,x'} : \quad i \neq j \tag{3.48}
\]

Therefore at each fixed point configuration \( x \in \mathcal{X} \in \mathcal{M}^T \)

\[
\langle 1| \sum_{x' \in \mathcal{X}} S_{i(x),x'} |1 \rangle = q_1^{\tilde{\rho}_i} \left( \prod_{x' \in \mathcal{X}_i} \frac{1 - x'/x}{1 - q_1 x'/x} \right) \langle 1| \sum_{x' \in \mathcal{X}} S_{i(x),x'} |1 \rangle \tag{3.49}
\]

like in the definition that was given in [2]. The observable \( Y_{i,x} \) is not regular in the \( \mathbb{C}_x^\times \) because of the possible poles at points \( x = q_1 x' \).
3.20. The commutator of $Y$ and $S$. The commutation relations (3.47) also imply for $|x'| > |x|

\[ S_{i,x} Y_{i,x} = q_1^{-1} \frac{1 - x/x'}{1 - q_1^{-1} x/x'} : S_{i,x} Y_{i,x} : \]

(3.50)

Therefore (3.48) (3.50) imply the non-zero radial-ordered commutator

\[ [Y_{i,x}, S_{i,x'}] = (1 - q_1^{-1}) \delta(q_1 x'/x) : S_{i,x'} Y_{i,x} : \]

(3.51)

where by definition $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

The fact that observable $Y_{i,x}$ has singularities at $x = q_1 x'$ in (3.50) is equivalent to the presence of the $\delta(q_1 x'/x)$ in the radial ordered commutator between $Y_{i,x}$ and $S_{i,x'}$. This is a general statement implied by Cauchy integral formula and familiar from the formalism of radial quantization in CFT.

3.21. The $\mathcal{V}$-operators. We introduce another kind of vertex operator to reproduce the fundamental matter contribution in gauge theory. As explained before, this contribution is given by shift of the time variables (3.9), which can be implemented by the operator

\[ \mathcal{V}_{i,x} = \exp \left( \sum_{p \neq 0} v_{i,p} x^{-p} \right) . \]

(3.52)

The corresponding free field is explicitly written

\[ (p > 0) \quad v_{i,-p} = - \left( \hat{c}_{i,p} \right) , \quad v_{i,p} = \frac{1}{p(1 - q_1)(1 - q_2^p)} \partial_{i,p} . \]

(3.53)

We remark a simple relation to the $y$-operators (3.44)

\[ v_{i,p} = - \frac{1}{(1 - q_1)(1 - q_2^p)} y_{i,p} . \]

(3.54)

Then the OPE of $V$ and $S$ operators are given by

\[ \mathcal{V}_{i,x} S_{i,x'} = \left( \frac{x'}{x} ; q_2 \right)^{-1}_\infty : \mathcal{V}_{i,x} S_{i,x'} : , \quad S_{i,x'} \mathcal{V}_{i,x} = \left( \frac{q_2 x'}{x} ; q_2 \right)_\infty : \mathcal{V}_{i,x} S_{i,x'} : \]

(3.55)

corresponding to the fundamental and antifundamental hypermultiplet contributions. Thus the extended partition function in the presence of (anti)fundamental matters is obtained by inserting the $\mathcal{V}$-operators

\[ |Z_T \rangle = \left( \prod_{x \in \mathcal{X}_i} \mathcal{V}_{i(x),x} \right) \left( \prod_{\hat{x} \in \mathcal{X}_0} \mathcal{S}_{i(\hat{x}),\hat{x}} \right) \left( \prod_{x \in \hat{\mathcal{X}}_i} \mathcal{V}_{i(x),x} \right) |1 \rangle \]

(3.56)

where $\mathcal{X}_i = \{ \mu_{i,f} \}_{i \in \Gamma_0, f \in [1 \ldots n_i]}$ and $\hat{\mathcal{X}}_i = \{ \hat{\mu}_{i,f} \}_{i \in \Gamma_0, f \in [1 \ldots \hat{n}_i]}$ are sets of fundamental and antifundamental mass parameters. This $\mathcal{V}$-operator creates a singularity on the curve at $x = \mu_{i,f}$. Then the plain partition function ($t = 0$) is given as a correlator as shown in (3.40),

\[ Z_T (t = 0) = \langle 1 | Z_T \rangle = \langle 1 | \left( \prod_{x \in \mathcal{X}_i} \mathcal{V}_{i(x),x} \right) \left( \prod_{\hat{x} \in \mathcal{X}_0} \mathcal{S}_{i(\hat{x}),\hat{x}} \right) \left( \prod_{x \in \hat{\mathcal{X}}_i} \mathcal{V}_{i(x),x} \right) |1 \rangle . \]

(3.57)
Here we describe the construction of regular observables $T$ of the extended gauge theory and explain isomorphism with Shiraishi et al. [3] and Frenkel-Reshetikhin [1] definition of $W_{q_1,q_2}$ algebra as commutant of screening charges in the Heisenberg algebra $H$, and define K-theoretical quiver W-algebra for $S = \mathbb{C}_{q_1,q_2}$.

4.1. **Pole cancellation in $T$: $A_1$-example.** Consider simplest quiver $\Gamma = A_1$ for example. In [2] in the study of the $q_2 = 1$ limit of the gauge theory partition function, motivated by cut-crossing story of [7], it was suggested to consider the observable

$$T_{1,x} = Y_{1,x} + Y_{1,q^{-1}x}$$

for its virtue of being regular function in $\mathbb{C}_x$. This is the simplest example of $q$-character representing the $T$-matrix of Baxter coming from $U_q(\hat{sl}_2)$-integrable system and Baxter equation.

In fact, the same observable $T_{1,x}$ remains regular function of $x$ for generic $q_2$. Indeed, in the operator formalism we find

$$Y_{1,x}S_{1,x'} = \frac{1 - x'/x}{1 - q_1x'/x} : Y_{1,x}S_{1,x'} :$$

$$Y_{1,q^{-1}x}^{-1}S_{1,x''} = \frac{1 - qq_1x''/x}{1 - qx''/x} : Y_{1,q^{-1}x}^{-1}S_{1,x''} :$$

so the potential singularity in the first line is for $x' = q_1^{-1}x$ and in the second line for $x'' = q^{-1}x$. Therefore, the two singularities have chance to cancel at $x' = q_2 x''$. Recall, that the state $|Z_T\rangle$ is obtained with the sums (3.38) and there is internal symmetry for the shift of the summation indexing variable (see Ward identity in Sec. 3.17) so that for every term $S_{1,x''}$ there is a term with $S_{1,x'}$ with $x' = q_2 x''$.

Indeed, we find for the first term the normal ordered expression

$$: Y_{1,x}S_{1,q^{-1}x} : = : \exp \left( -\frac{1}{2} s_{1,0} \log q_2 + s_{1,0} \log(xq_1^{-1}) + \bar{s}_{1,0} + \sum_{p \neq 0} (q^p + 1 - q^{-p})s_{1,p}x^{-p} \right) :$$

and for the second term the normal ordered expression

$$q_1 : Y_{1,q^{-1}x}^{-1}S_{1,q^{-1}x} : = : \exp \left( +\frac{1}{2} s_{1,0} \log q_2 + s_{1,0} \log(xq^{-1}) + \bar{s}_{1,0} + \sum_{p \neq 0} (q^p - 1 - q_2^{-p})s_{1,p}x^{-p} \right) :$$

which are exactly identical. The respective residues in the prefactors (4.2) are $(1 - q_1^{-1})$ and $(q_1^{-1} - 1)$ which respectively cancel each other.

This computation proves regularity in $x \in \mathbb{C}_x$ of the state $|Z_T\rangle$ of higher $t$-extended gauge theory in $A_1$ example

$$\partial_\bar{z} T_{1,x} |Z_T\rangle = 0$$

We adopted the normalizations and the zero modes to the conventions of the present paper in which the $T$-observables have the simplest canonical form.
4.2. **Commutator of $T$ and $S$ vanishing: $A_1$-example.** An exactly equivalent presentation of the regularity of $T_{1,x}$ is the statement that

$$[T_{1,x}, S_{1,x'}] = 0$$  \hspace{1cm} (4.6)

where $S_{1,x'}$ is the screening charge \textcolor{red}{[3.38]} defined as the summation over the $q_2^Z$ shifts. Indeed,

$$[Y_{1,x}, S_{1,x'}] = (1 - q_1^{-1}) \delta(q_1\frac{x'}{x}) : Y_{1,x} S_{1,x'} :$$

$$[Y_{1,x}, S_{1,x'}]^{-1} = (q_1^{-1} - 1) \delta(q_1\frac{x'}{x}) : Y_{1,x} S_{1,x'} :$$  \hspace{1cm} (4.7)

The total sum is $q_2$-difference which cancels after summation over $q_2^Z$ shifts entering definition of screening charge \textcolor{red}{[3.38]}. This is the consequence of the Ward identity in Sec. 3.17.

4.3. **$W$-algebra of $A_1$-quiver.** Consequently the operator $T_{1,x}$ can be moved in the position in the radial-ordered operator-state presentation of the extended gauge theory partition state \textcolor{red}{(3.39)}

$$T_{1,x}|1\rangle = T_{1,x} S_{1,x'} S_{1,x''} \ldots |1\rangle = S_{1,x'} T_{1,x} S_{1,x''} \ldots |1\rangle = S_{1,x'} S_{1,x''} T_{1,x} \ldots |1\rangle$$  \hspace{1cm} (4.8)

The operators $S_{i,x'}$ can be thought as exponentiated Hamiltonians of the $q_1, q_2$-deformed CFT. The commutant of the Hamiltonians is the conserved current $T_{1,x}$ which is regular

$$\partial_{\bar{x}} T_{1,x} = 0$$  \hspace{1cm} (4.9)

Consequently, $T_{1,x}$ has well defined, time-radial independent, modes

$$T_{1,x} = \sum_{p \in \mathbb{Z}} T_{1,[p]} x^{-p}$$  \hspace{1cm} (4.10)

We define algebra $W_{q_1,q_2}(A_1)$ to be the subalgebra in $H$ generated by the modes of the conserved current $T_{1,x}$. This definition is in the exact agreement with Shiraishi et al. [5] and Frenkel-Reshetikhin [1].

4.4. **$W$-algebra of quiver: definition.** The definition of the state $|Z_T\rangle$ \textcolor{red}{(3.39)} implies that the current $T_{i,x}$ is regular

$$\partial_{\bar{x}} T_{i,x} |Z_T\rangle = 0$$  \hspace{1cm} (4.11)

if it commutes with all screening operators:

$$[T_{i,x}, S_{j,x'}] = 0 \quad j \in \Gamma_0, \quad x' \in \mathcal{X}_j$$  \hspace{1cm} (4.12)

This explains isomorphism between the gauge theoretic construction of $q_1q_2$-characters \textcolor{red}{[2, 3]} and definition of $W_{q_1,q_2}$-algebras \textcolor{red}{[1, 5, 9]} as the algebra generated by currents $T_{i,x}$ which are defined as commutants of screening charges $S_j$ in the vertex operator algebra defined by the free fields from Heisenberg algebra $H$ and expressed as

$$T_{i,x} = Y_{i,x} + \ldots$$  \hspace{1cm} (4.13)

We define in the same way the $W$-algebra $W(\Gamma, S)$ for generic quiver $\Gamma$ with generalized even symmetric Borcherds-Kac-Moody-Cartan matrix, mass deformed by $\mu : \Gamma_1 \to \mathbb{C}^x$, as in equation \textcolor{red}{(3.24)}, and for $S = \mathbb{C}_{q_1,q_2}$ as the algebra generated by currents $T_{i,x}$ commuting with all screening charges $(S_{j,x})_{j \in \Gamma_0}$, or equivalently, regular on the higher times extended gauge
theory state (4.11). We expect to generalize the definition for more general and possibly higher dimensional varieties \( S \).

5. Examples

We consider a few examples to illustrate the equivalence between gauge-theory formalism [2, 3, 7] and the operator formalism [1, 5, 9, 15].

5.1. Commutator of \( T \) and \( S \) vanishing: general quiver, local reflection. Suppose that there is no edge loop from a node \( i \) to itself and consider

\[
T_{i,x} = Y_{i,x} + Y_{i,q^{-1}x}^{-1} \prod_{e:i \rightarrow j} Y_{j,\mu e^{-1}x} \prod_{e:j \rightarrow i} Y_{j,q^{-1}\mu e x} : + \ldots
\]  

(5.1)

The vanishing of commutator

\[
[T_{i,x}, S_{i,x'}] = 0
\]  

(5.2)

follows from (3.51) and the relation

\[
q_1 : Y_{i,q^{-1}x}^{-1} \left( \prod_{e:i \rightarrow j} Y_{j,\mu e^{-1}x} \prod_{e:j \rightarrow i} Y_{j,q^{-1}\mu e x} \right) S_{i,q^{-1}x} : = : Y_{i,x} S_{i,q^{-1}x} : 
\]  

(5.3)

Indeed, this relation is equivalent to

\[
q_{i}^{-1} : Y_{i,x} Y_{i,q^{-1}x} \left( \prod_{e:i \rightarrow j} Y_{j,\mu e^{-1}x} \prod_{e:j \rightarrow i} Y_{j,q^{-1}\mu e x} \right)^{-1} : = : S_{i,q^{-1}x} S_{i,q^{-1}x}^{-1} : 
\]  

(5.4)

which simply expresses the defining relation (3.45) between \( Y_{i,x} \) and \( S_{j,x} \) in the exponentiated form: the field \( y_i(x) \) (of the Cartan weight type) is the \( q_2 \)-derivative of the field \( s_i(x) \) (of the Cartan root type). Namely, the relation (5.4) is the identity

\[
c_{ij}^{[0]} c_{jk}^{[0]} \log q_2 s_{k,0} + \sum_{p \neq 0} (1-q_2^{-p}) q^{p} c_{ij}^{[p]} c_{jk}^{[p]} s_{k,p} x^{-p} = (\log(xq^{-1})-\log(xq^{-1})) s_{i,0} + \sum_{p \neq 0} (q^{p} - q_1^{-p}) s_{i,p} x^{-p}
\]  

(5.5)

thanks to the definition of the \( \mu \)-dependent Cartan matrix \( c_{ij} \) (3.24) and its inverse \( \tilde{c}_{ij} \) so that \( c_{ij}^{[p]} c_{jk}^{[p]} = \delta_{ij} \).

This leads to

\[
: Y_{i,q^{-1}x}^{-1} \prod_{e:i \rightarrow j} Y_{j,\mu e^{-1}x} \prod_{e:j \rightarrow i} Y_{j,q^{-1}\mu e x} : S_{i,x'} : = (q_1^{-1} - 1) \delta \left( \frac{x'}{x} \right) : Y_{1,xq^{-1}} S_{1,x'} :
\]  

(5.6)

and therefore (5.2) holds at the level of the first two terms.

The second term contains the \( Y_{j} \) fields for the nodes \( j \) linked to the node \( i \). This term might give potential singularities, or, equivalently, \( \delta \)-functions in the commutators associated to the \( S_{j} \) operators. Then one needs to continue to apply Weyl reflections to generate terms which cancel the singularities.

The algebraic structure is associated to highest weight Verma module of the generalized Borcherds-Kac-Moody algebra \( \mathfrak{g}_\Gamma \).
If $\Gamma$ is of finite Dynkin type the process terminates, the associated Verma module is finite-dimensional. The finite-dimensional case was studied in details in [9].

More generally, for infinite-dimensional Verma module, the recursive algorithm is also applicable which builds a tree starting from the root node $Y_{i,x}$. The vertices of the tree are monomials in the $T_{i,x}$ current, and the edges are colored by the nodes $i$ of the quiver. Two monomials are linked by edge of color $i$ if they are related by the local reflection move (5.1). The algorithm can be computerized.

Alternatively, Nekrasov presented closed formula [3] which expresses the $q_1q_2$-characters in terms of geometry of Nakajima’s quiver variety [16, 40]. This formula can be thought as $q_2$-deformation of original Nakajima’s construction of $q$-characters of $U_q(L\mathfrak{g})$ from the $q$-equivariant K-theory on the quiver variety $\mathcal{M}^\text{Nak}_{w,v} = T^*_q\mathcal{M}_{w,v}$ [20]. The formula in [3] amounts to replacing Euler characteristic of $T^*_q\mathcal{M}_{w,v}$ by the $q_2$-equivariant Euler class of the tangent bundle to $T^*_q\mathcal{M}_{w,v}$, so effectively to the integration over $\Pi T^*_q T^*_q \mathcal{M}_{w,v}$.

Here $w : \Gamma_0 \to \mathbb{Z}$ labels the components of the highest weight in the basis of fundamental weights, and $v : \Gamma_0 \to \mathbb{Z}$ labels the components of a positive root in the basis of simple roots which is added to the highest weight to get a weight at the level $\sum_{i \in \Gamma_0} v_i$ in the Verma module.

5.2. Higher weight currents. Conjecturally, quiver W-algebra is completely generated by the fundamental currents

$$T_{i,x} = Y_{i,x} + \ldots$$ (5.7)

However, higher weight currents $T^w_{w,x}$ can be defined where to each node $i$ we assign vector space $W$ of dimension $w$ and character

$$W_i = \sum_{i=1}^w w_{i,\omega}, \quad w_{i,\omega} \in \mathbb{C}^\times$$ (5.8)

with the first term

$$T^w_{w,x} = \prod_{i \in \Gamma_0} \prod_{\omega=1}^w Y_{i,xw_{i,\omega}} : + \ldots$$ (5.9)

In the finite-dimensional and irreducible modules of higher weights can be found in the tensor product of the $i$-fundamental modules with weights $w_i = 1, w_j \neq i = 0$. In the not $q$-deformed case, usually the tensor product of fundamental modules decomposes into several irreducible components. For example, for $\mathfrak{sl}_2$ we have $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^3 \oplus \mathbb{C}^1$. This does not hold after $q$-deformation. For generic weights $w$ the tensor product is irreducible.

5.3. Higher weight current in the $A_1$ example. In the example of $A_1$ quiver the higher weight current $T^w_{w,x}$ with $w_1 \in \mathbb{Z}_{>0}$ for generic weights $(w_{1,1}, \ldots, w_{1,w_1})$ contains $2^w$ terms [3] coming from the cohomologies of Nakajima’s quiver variety which in this case are $\Pi_{v \leq w} T^*_q \Gr(w, v)$.

This higher weight character current $T^w_{w,x}$ is elementary to compute from the free-field formalism and normal ordering given the fundamental current $T_{1,x}$ in equation (4.1).

Consider the product

$$T_{1,w_1x}T_{1,w_2x} = (Y_{1,xw_1} + Y_{1,q^{-1}w_1x}) (Y_{1,xw_2} + Y_{1,q^{-1}w_2x})$$ (5.10)
The normal ordering is computed using the commutator from (3.44)

\[ [y_{i,p}, y_{j,-p}] \stackrel{p \geq 0}{=} -\frac{1}{p}(1 - q_1^p)(1 - q_2^p)c_{ji}^{-p} \tag{5.11} \]

with the result

\[
T_{1,w_1x}T_{1,w_2x} = f\left(\frac{w_2}{w_1}\right)^{-1}\left( : Y_{1,xw_1} Y_{1,xw_2} : + S(\frac{w_1}{w_2}) : Y_{1,xw_1}^{-1} Y_{1,q^{-1}xw_2}^{-1} : + : Y_{1,xw_1}^{-1} Y_{1,q^{-1}xw_2}^{-1} : \right) \tag{5.12}
\]

where the scalar prefactor

\[
f(w) = \exp \left( \sum_{p=1}^{\infty} \frac{1}{p} \frac{(1 - q_1^p)(1 - q_2^p)}{1 + q^p} w^p \right) \tag{5.13}\]

is in agreement with the function \( f(w) \) generating the commutation relations for \( W_{q_1,q_2}(A_1) \) current \( T_1(x) \) in Shiraishi et al. [5] and the permutation factor \( S(u) \) is in agreement with formulae for higher \( q q \)-characters in [3]

\[
S(w) = \frac{(1 - q_1 w)(1 - q_2 w)}{(1 - q w)(1 - w)} \tag{5.14}\]

which comes from the equivariant Euler characteristic (or its K-theory version) of the fixed point in \( \Pi T_{q_2} T^*_{q_1} \mathbb{P}^1 \) where \( T^*_{q_1} \mathbb{P}^1 = \mathfrak{M}^{\text{Nak}}_{w=2,v=1} \). This relation leads to

\[
f \left( \frac{w_2}{w_1} \right) T_{1,w_1x}T_{1,w_2x} = f \left( \frac{w_1}{w_2} \right) T_{1,w_1x}T_{1,w_2x} = \frac{(1 - q_1)(1 - q_2)}{1 - q} \left( \delta \left( \frac{w_1}{w_2} \right) - \delta \left( \frac{w_2}{w_1} \right) \right), \tag{5.15}\]

which determines the algebraic relation for the modes \( T_{1,[n]} \). We remark \( f(w)f(qw) = S(w) \).

The degree \( w \) current is similarly computed

\[
T_{1,x}^{[w]} = : Y_{1,w_1x} Y_{1,w_2x} \cdots Y_{1,w_nx} : + \cdots
= \sum_{I \cup J = \{1 \ldots n\} \subseteq I, J} \prod_{i \in I} S\left( \frac{w_i}{w_j} \right) \prod_{i \in I} Y_{1,w_i x} \prod_{j \in J} Y_{1,w_j x}^{-1}, \tag{5.16}\]

in agreement with [3].

The \( S \) factor becomes trivial in the limit \( q_2 \to 1 \) and the ordinary formulae for \( q \)-character is recovered [2, 15].

5.4. Degeneration and derivatives. By definition, vertex operator algebra involves expressions in fields and their derivatives. Hence we shall expect appearance of the derivatives when two vertex operators fuse.
So consider slightly more general situation of $W$-algebra currents with local structure

\[
\mathcal{Y}_{i,x}\mathcal{Y}_{i,u_x} = + \mathcal{S}(u) : \frac{\mathcal{Y}_{i,x}}{u_{i,xq}^{-1}} \left( \prod_{e_i \rightarrow j} \mathcal{Y}_{j,\mu q^{-1}} \prod_{e_j \rightarrow i} \mathcal{Y}_{j,\mu q^{-1}} \right) + \mathcal{S}(u^{-1}) : \frac{\mathcal{Y}_{i,x}}{u_{i,xq}^{-1}} \left( \prod_{e_i \rightarrow j} \mathcal{Y}_{j,\mu q^{-1}} \prod_{e_j \rightarrow i} \mathcal{Y}_{j,\mu q^{-1}} \right) + \mathcal{Y}_{i,xq}^{-1} \mathcal{Y}_{i,u_x q^{-1}} \left( \prod_{e_i \rightarrow j} \mathcal{Y}_{j,\mu q^{-1}} \prod_{e_j \rightarrow i} \mathcal{Y}_{j,\mu q^{-1}} \right) : (5.17)
\]

Taking the collision limit $u \rightarrow 1$, this yields a derivative term

\[
\mathcal{Y}_{i,x}^2 + \mathcal{Y}_{i,x} \left( \prod_{e_i \rightarrow j} \mathcal{Y}_{j,\mu q^{-1}} \prod_{e_j \rightarrow i} \mathcal{Y}_{j,\mu q^{-1}} \right) \times \left( c(q_1, q_2) = \frac{(1 - q_1)(1 - q_2)}{1 - q} x \partial_x \log \left( \frac{\mathcal{Y}_{i,x} \mathcal{Y}_{i,x q^{-1}}}{\prod_{e_i \rightarrow j} \mathcal{Y}_{j,\mu q^{-1}} \prod_{e_j \rightarrow i} \mathcal{Y}_{j,\mu q^{-1}}} \right) \right) + \mathcal{Y}_{i,x q^{-1}}^{-2} \left( \prod_{e_i \rightarrow j} \mathcal{Y}_{j,\mu q^{-1}} \right) \prod_{e_j \rightarrow i} \mathcal{Y}_{j,\mu q^{-1}}^2 : (5.18)
\]

where the coefficient $c(q_1, q_2)$ is determined by

\[
c(q_1, q_2) = \lim_{u \rightarrow 1} \left( \mathcal{S}(u) + \mathcal{S}(u^{-1}) \right) = \frac{1 - 6q_1q_2 + q_1^2q_2^2 + (1 + q_1q_2)(q_1 + q_2)}{(1 - q_1q_2)^2} \rightarrow 2. (5.19)
\]

5.5. **Edge loop: $\hat{A}_0$-example.** Consider an example of a single node with a loop edge. This corresponds to $\mathcal{N} = 2^*$ theory in 4d. Let $n$ be the gauge group rank and $\mu$ be the (exponentiated) adjoint mass. The Cartan matrix is $\mathcal{(0)}$ and the mass deformed Cartan matrix is

\[
c = 1 + q^{-1} - \mu^{-1} - q^{-1} \mu (5.20)
\]

The quantum affinization of the respective algebra by Nakajima’s quiver construction is $\text{U}_{q, \mu}(L \hat{g}_1)$ [43 45] with $q$-character given by the sum over all partitions [2]. Here we consider $q_2$-deformation to recover $W$-algebra of $\hat{A}_0$-quiver.

We need the commutation relation for the oscillator (3.44)

\[
[y_{1,p}, y_{1,p'}] = -\delta_{p+p',0} \frac{1}{p} \frac{(1 - q_1^p)(1 - q_2^p)}{(1 - \mu^p)(1 - q^p \mu^{-p})} (5.21)
\]

Using this oscillator, we construct $W_{q_1,q_2}(\mathfrak{g}_\Gamma)$ algebra associated with the affine quiver $\Gamma = \hat{A}_0$.

In this case, the local pole cancellation structure is

\[
\mathcal{Y}_{1,x} + \mathcal{S}(\mu^{-1}) : \mathcal{Y}_{1,q^{-1} x}^{-1} \mathcal{Y}_{1,\mu^{-1} x} \mathcal{Y}_{1,\mu q^{-1} x} : (5.22)
\]
and the holomorphic current can be characterized by a single partition \[ \lambda \]

\[
T_{1,x} = Y_{1,x} + S(\mu^{-1}) : Y_{1,q^{-1}x} Y_{1,\mu^{-1}x} Y_{1,\mu q^{-1}x} : + \cdots
\]

\[
= \sum_{\lambda} \tilde{Z}_{\lambda} : \prod_{s \in \partial_{+}\lambda} Y_{1,q x / \tilde{x}(s)} \prod_{s \in \partial_{-}\lambda} Y_{1,x / \tilde{x}(s)} : (5.23)
\]

where \( \partial_{+}\lambda \) and \( \partial_{-}\lambda \) are the outer and inner boundary of the partition \( \lambda \), and we define

\[
\tilde{x}(s) = (\mu^{-1} q)^{s_{1}-1} \mu^{s_{2}-1} q
\]

The combinatorial weight \( \tilde{Z}_{\lambda} \) obeys

\[
\frac{\tilde{Z}_{\lambda'}}{\tilde{Z}_{\lambda}} = \frac{(1 - \mu^{-1} q_{1})(1 - \mu^{-1} q_{2})}{(1 - q_{1})(1 - q_{2}^{-1})} \left. \frac{\tilde{Y}_{q_{1}x} \tilde{Y}_{q_{2}x}}{\tilde{Y}_{q_{1}x} \tilde{Y}_{q_{2}x}} \right|_{x = \tilde{x}_{k}} (5.25)
\]

where \( \lambda' \) is the shifted partition \( \lambda_{k} \rightarrow \lambda_{k} + 1 \), and we define the “dual” function \( \tilde{Y}_{x} \)

\[
\tilde{Y}_{x} = \prod_{k=1}^{\mid \lambda' \mid} \frac{1 - \tilde{x}_{k}/x}{1 - \tilde{q}_{1} \tilde{x}_{k}/x} (5.26)
\]

with the “dual” parameters

\[
\tilde{q}_{1} = \mu^{-1} q, \quad \tilde{q}_{2} = \mu, \quad \tilde{\mu} = q_{2}, \quad \tilde{x}_{k} = \tilde{x}(k; \lambda_{k} + 1) (5.27)
\]

Here \( \tilde{Y}' \) is evaluated with the shifted configuration \( \lambda' \). Although this dual function also has poles, such a singularity is cancelled in the following combination

\[
\tilde{Y}_{x} + \frac{(1 - \tilde{\mu}^{-1} \tilde{q}_{1})(1 - \tilde{\mu}^{-1} \tilde{q}_{2})}{(1 - \tilde{\mu}^{-1} \tilde{q})(1 - \tilde{\mu}^{-1})} \tilde{Y}_{q_{1}x} \tilde{Y}_{q_{2}x} \tilde{Y}_{\tilde{q}_{1}x} \tilde{Y}_{\tilde{q}_{2}x} (5.28)
\]

This expression is equivalent to the original one \( (5.22) \) in particular for the rank one theory. Again, operator formalism of W-algebra is equivalent to \( q_{2} \)-deformation of Nakajima’s construction \( \cite{3} \).

5.6. W-algebra of hyperbolic quiver example. We consider examples of the hyperbolic quiver, where the determinant of the corresponding Cartan matrix is negative. The simplest example is the quiver having a single node with two loop edges:

\[
c = -(2)
\]

Let \( \mu_{1,2} \) be the mass parameter associated with the edges, and the mass deformed Cartan matrix is given by

\[
c = 1 + q^{-1} - \mu_{1}^{-1} - \mu_{1} q^{-1} - \mu_{2}^{-1} - \mu_{2} q^{-1} (5.30)
\]

Since the local pole cancellation occurs in the following combination

\[
Y_{1,x} + S(\mu_{1}^{-1})S(\mu_{2}^{-1}) : Y_{1,q^{-1}x} Y_{1,\mu_{1}^{-1}x} Y_{1,\mu_{1} q^{-1}x} Y_{1,\mu_{2}^{-1}x} Y_{1,\mu_{2} q^{-1}x} : (5.31)
\]
the first few terms of the holomorphic current are given by

\[
T_{1,x} = Y_{1,x} + S(\mu_1^{-1})S(\mu_2^{-1}) : Y_{1,q^{-1}x}^{-1} Y_{1,\mu_1^{-1}x} Y_{1,\mu_1 \mu_2 q^{-1}x} Y_{1,\mu_2 q^{-1}x} : \\
+ (S(\mu_1^{-1})S(\mu_2^{-1}))^2 \left[ S(\mu_1^2 q^{-1})S(\mu_2 \mu_1^{-1})S(\mu_1 \mu_2^{-1}) : Y_{1,\mu_1^{-2}x} Y_{1,\mu_1 q^{-1}x} Y_{1,\mu_1^{-1} \mu_2^{-1}x} Y_{1,\mu_1 \mu_2^{-1} \mu_2 q^{-1}x} Y_{1,\mu_2^{-1}x} Y_{1,\mu_2 q^{-1}x} : \right. \\
\left. + S(\mu_1^{-2} q)S(\mu_1^{-1} \mu_2^{-1} q)S(\mu_1 \mu_2) : Y_{1,\mu_1^{-1} \mu_2^{-1}x} Y_{1,\mu_1^2 q^{-2}x} Y_{1,\mu_1 \mu_2 q^{-1}x} Y_{1,\mu_2^{-1}x} Y_{1,\mu_2 q^{-1}x} : \right] \\
+ (1 \leftrightarrow 2) \right] + \cdots
\]

(5.32)

We can see a cancellation of factors, which is similar to \( \hat{A}_0 \) theory, and thus there is no colliding term, e.g. \( Y_{1,*} \), in a numerator.

Next example is a rank two quiver with three arrows:

\[
c = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}
\]

(5.33)

Let us assign three mass parameters \( \mu_{1,2,3} \) to the arrows, and then the local cancellation is

\[
Y_{1,x} + : \frac{Y_{2,\mu_1^{-1}x} Y_{2,\mu_2^{-1}x} Y_{2,\mu_3^{-1}x}}{Y_{1,q^{-1}x}} : \quad Y_{2,x} + : \frac{Y_{1,\mu_1 q^{-1}x} Y_{1,\mu_2 q^{-1}x} Y_{1,\mu_3 q^{-1}x}}{Y_{2,q^{-1}x}} : \quad Y_{3,x} + : \frac{Y_{1,\mu_4 q^{-1}x} Y_{1,\mu_5 q^{-1}x} Y_{1,\mu_6 q^{-1}x}}{Y_{3,q^{-1}x}} : \quad Y_{4,x} + : \frac{Y_{1,\mu_7 q^{-1}x} Y_{1,\mu_8 q^{-1}x} Y_{1,\mu_9 q^{-1}x}}{Y_{4,q^{-1}x}} : \quad \cdots
\]

(5.34)

The holomorphic current becomes

\[
T_{1,x} = Y_{1,x} + : \frac{Y_{2,\mu_1^{-1}x} Y_{2,\mu_2^{-1}x} Y_{2,\mu_3^{-1}x}}{Y_{1,q^{-1}x}} : \\
+ \left[ S(\mu_1 \mu_2^{-1})S(\mu_1 \mu_3^{-1}) : \frac{Y_{1,\mu_1^{-1} \mu_2 q^{-1}x} Y_{1,\mu_1^{-1} \mu_3 q^{-1}x} Y_{2,\mu_3^{-1}x} Y_{2,\mu_3 q^{-1}x} :} {Y_{2,\mu_1^{-1}q^{-1}x}} \right. \\
\left. + S(\mu_2 \mu_3^{-1})S(\mu_2) : \frac{Y_{1,\mu_2^{-1} \mu_1 q^{-1}x} Y_{1,\mu_2^{-1} \mu_3 q^{-1}x} Y_{2,\mu_1^{-1}x} Y_{2,\mu_3^{-1}x} :} {Y_{2,\mu_2^{-1}q^{-1}x}} \right] \\
\left. + S(\mu_3 \mu_1^{-1})S(\mu_3) : \frac{Y_{1,\mu_3^{-1} \mu_1 q^{-1}x} Y_{1,\mu_3^{-1} \mu_2 q^{-1}x} Y_{2,\mu_1^{-1}x} Y_{2,\mu_2^{-1}x} :} {Y_{2,\mu_3^{-1}q^{-1}x}} \right] \\
+ \cdots
\]

(5.35)

The other current \( T_{2,x} \) is obtained in the same way. Similarly it is expected that there is no collision term in these holomorphic currents.
6. Applications

6.1. Toda scaling limit. In the scaling limit \( q_1 \to 1, q_2 \to 1, q_i \to 1 \), and \( \log q_2, \log q_i \) are finite, the free field commutation relations (3.30) (3.31) turn into

\[
[s_i, p, s_j, p'] = -\delta_{p+p', 0} \beta c_{ij}^{[0]}, \quad p > 0
\]

(6.1)

This limit was studied in details in section 4.1 of [1].

In terms of the parameter \( b^2 = -\beta \) the vertex operator (3.28) can be written as

\[
S_i(x) = e^{b\phi_i(x)}
\]

(6.2)

where \( \phi_i(x) \) is the free boson that takes value in the Cartan of \( g_\Gamma \) with canonical commutation relations defined by the bilinear form with matrix \( (c_{ij}) \) in the basis of simple roots. Hence \( S_i(x) \) are vertex primary operators of Kac-Moody \( g_\Gamma \)-Toda field theory on punctured disc \( \mathbb{C}_x \).

In the same scaling limit we find from (3.45) that the field \( y_i(x) = -\epsilon_2 b_\epsilon c_{ij} \partial \phi_j(x) \) is also primary.

For example, in the \( \epsilon_2 \)-expansion of \( T_1(x) \) for \( A_1 \)-quiver

\[
T_1(x) = e^{y(x)} + e^{-y(xq^{-1})} = 2 + \frac{1}{4} \epsilon^2 b^2 (\partial \phi)^2 - (b + b^{-1}) \partial^2 \phi + \ldots
\]

(6.3)

we find the stress-energy Virasoro current of the free field \( \phi \) with background charge and the central charge

\[
c = 1 + 6(b + b^{-1})^2
\]

(6.4)

6.2. Affine type. If \( g_\Gamma \) is of affine type, the \( g_\Gamma \)-Toda is affine Toda. For example, the scaling limit of the W-algebra defined by the quiver

\[
\bullet \leftrightarrow \bullet \quad c = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}
\]

(6.5)

with \( g_\Gamma = A_1^{(1)} \) describes quantum \( \sin(h) \)-Gordon theory on punctured disc \( \mathbb{C}_x \).

6.3. Nahm transform. The \( g_\Gamma \)-Toda theory specializes to the finite Toda if \( g_\Gamma \) is of finite type. For \( sl_r \)-quiver with \( n \) colors at each node the \( sl_r \)-Toda is Nahm dual to the \( sl_n \)-Toda proposed in [26, 27].

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