Polynomial method to study the entanglement of pure $N$-qubit states

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We present a mapping which associates pure $N$-qubit states with a polynomial. The roots of the polynomial characterize the state completely. Using the properties of the polynomial we construct a way to determine the separability and the number of unentangled qubits of pure $N$-qubit states.

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I. INTRODUCTION

Considerable effort is spent in developing methods for the detection and classification of entangled states. One important aim is to find ways to detect the separability of mixed states consisting of an arbitrary number of subsystems. While a general, easily computable, method to detect the separability of arbitrary mixed multipartite states is still lacking, some partial results exist. Maybe the most famous separability condition for mixed states is the positive partial transposition, also known as Peres-Horodecki criterion [1, 2]. This method is simple and easy to apply, but it can be used to detect only bipartite separability. Therefore various separability conditions which work in an $N$-party setting have been developed. Examples of these are permutation criteria, where the indices of the density matrix are permuted [3], the use of quadratic Bell-type inequalities [4], algorithmic approaches [5], and the use of positive maps [6]. For a more comprehensive list, see [7, 8]. In the case of pure states the situation is simpler. A pure $N$-partite state is separable if and only if all the reduced density matrices of the elementary subsystems describe pure states. Alternatively, in a bipartite case, separability can be determined by calculating the Schmidt decomposition of the state. Unfortunately, the concept of the Schmidt decomposition cannot be straightforwardly generalized to the case of $N$ separate subsystems [9, 10]. In addition to these two well-known methods, various other approaches to the pure state separability have been discussed. A separability condition based on comparing the amplitudes and phases of the components of the state has been discussed in [11, 12]. It has been shown that the separability of pure three-qubit states can be detected by studying two-qubit density operators [13] and expectation values of spin operators [14, 15]. Separability tests based on studying matrices constructed from the components of the state vector, known as coefficient matrices, have gained attention recently [16, 18].

In this article we present a mapping which associates the pure states of an $N$-qubit system with a polynomial. The roots of the polynomial determine the state completely and vice versa. We show that this polynomial establishes a simple way to test the separability of pure $N$-qubit states and to study the number of unentangled particles. The idea to associate a state of a quantum mechanical system with a polynomial is not new. Already in 1932 E. Majorana presented a polynomial, nowadays known as the Majorana polynomial, which he used to show that the states of a spin-$S$ particle can be expressed as a superposition of symmetrized states of $2^S$ spin-$\frac{1}{2}$ systems [19, 20]. This decomposition, the Majorana representation, has been relatively unknown for a long time. However, it has recently found applications in many different fields, such as in studying the symmetries of spinor Bose-Einstein condensates [21, 23], in the context of reference frame alignment [22], in helping to define anticoherent spin states [20], and in calculating the spectrum of the Lipkin-Meshkov-Glick model [27, 28]. It has also been used to give a graphical representation for the states of an $n$-level system [25].

The states of an $N$-qubit quantum register can be viewed as the spin states of a particle with spin $S = (2^N - 1)/2$. Therefore, expressing the pure states of an $N$-qubit system utilizing the Majorana representation requires the use of $2^N - 1$ spin-$\frac{1}{2}$ systems. In the approach we present in this article only $N$ two-level systems are needed to characterize the states of this system. The Majorana representation is useful in studying the behavior of spin states under spin rotations as a spin rotation of a spin-$S$ particle is equivalent with rotating the states of the constituent spin-$\frac{1}{2}$ particles [20]. However, when discussing the states of an $N$-qubit quantum register, this property is not very helpful and therefore the benefits of the Majorana representation cannot fully be taken advantage of. In this case the simplified description presented in this article becomes useful.

This article is organized as follows. In Sec. II we introduce a mapping between the pure states of an $N$-qubit quantum register and polynomials. We argue that the roots of a polynomial determine a unique state and vice versa. In Sec. III we calculate the polynomial of separable pure states and derive a necessary and sufficient condition for the separability of an arbitrary pure $N$-qubit state. We also briefly discuss the generalization of the polynomial approach to systems containing $N$ copies of an $h$-level system. In Sec. IV we show how the polynomial can be used to study the number of entangled qubits. In Sec. V we present the conclusions.
II. CHARACTERISTIC POLYNOMIAL

We denote the basis of the qubit $j$ by $\{|0\rangle_j, |1\rangle_j\}$, so the basis vectors of an $N$-qubit quantum register can be chosen as $|i_1\cdots i_{N-1}\rangle = |i_1\rangle_0 \otimes |i_2\rangle_1 \otimes \cdots \otimes |i_{N-1}\rangle_{N-1}$, where every $i_j \in \{0,1\}$. Each natural number $0 \leq i \leq 2^N - 1$ can be written using binary notation as $i = \sum_{j=0}^{N-1} i_j 2^j$, where $i_j \in \{0,1\}$. Using this we can associate the basis vector $|i_1\cdots i_{N-1}\rangle$ with $|i\rangle_d$. Here the subscript $d$ shows that decimal notation is used to label the basis states. Let

$$\phi = \sum_{i=0}^{2^N-1} C_i |i\rangle_d, \quad C_i \in \mathbb{C},$$  \hspace{1cm} (1)

be some, possibly unnormalized, state vector of an $N$-qubit system. We associate this vector with the polynomial

$$P(\phi; x) \equiv \sum_{i=0}^{2^N-1} C_i x^i,$$  \hspace{1cm} (2)

which we call the characteristic polynomial of $\phi$. By the fundamental theorem of algebra, this polynomial can be written in a unique way as

$$P(\phi; x) = C_k \prod_{j=0}^{k-1} (x - x_j),$$  \hspace{1cm} (3)

where $\{x_j \mid j = 0, 1, \ldots, k - 1\}$ are the roots and $k$ is the degree of $P(\phi; x)$. If $k = 0$ we define $\prod_{j=0}^{k-1} (x - x_j) = 1$. The set of vectors $\{c \phi \mid c \in \mathbb{C}, c \neq 0\}$ determines a unique set of roots and each set of roots $\{x_0, x_1, \ldots, x_{k-1}\}$ determines the vector $\phi$ up to normalization and phase. Therefore we have a bijective map between the pure states of an $N$-qubit quantum register and the roots of complex polynomials of degree $k \leq 2^N - 1$ [31]. Explicitly, the components of $\phi$ are determined by the roots through the formula

$$C_i = (-1)^{k-i} \sum_{j_0 < j_1 < \cdots < j_{k-1}} x_{j_0} x_{j_1} x_{j_2} \cdots x_{j_{k-1}},$$  \hspace{1cm} (4)

where $i = 0, 1, 2, \ldots, k - 1$ and we have chosen $C_k = 1$. The roots contain the same amount of information on the system as the state vector $\phi$. In particular, all the entanglement properties of $\phi$ are encoded in the set of roots corresponding to $\phi$. With the help of the roots the state $\phi$ can be given a geometrical representation as $2^N - 1$ points on the Bloch sphere, see Ref. [30].

III. SEPARABLE PURE $N$-QUBIT STATES

In this section we show how the separability of $\phi$ can be detected with the help of $P(\phi; x)$. In order to do so, we first calculate the characteristic polynomial of a separable state. Any separable pure state $\phi_s$ can be written as

$$\phi_s = \bigotimes_{j=0}^{N-1} \phi_j,$$

where $\phi_j$ is a $d$-qubit state.

Assume that $|l\rangle_d$ is a basis state of an $L$-qubit system and that $|m\rangle_d$ is that of an independent $M$-qubit system. Using the binary expressions for $l$ and $m$ it is easy to see that

$$|l\rangle_d |m\rangle_d = |l + 2^L m\rangle_d$$  \hspace{1cm} (6)

holds for the tensor product of $|l\rangle_d$ and $|m\rangle_d$. Here and in what follows we omit the tensor product symbol. Let $\xi^L$ and $\xi^M$ be states of $L$-qubit and $M$-qubit quantum registers, respectively. Then we can write $\xi^L = \sum_{i=0}^{2^L-1} \xi^L_i |i\rangle_d$ and $\xi^M = \sum_{i'=0}^{2^M-1} \xi^M_{i'} |i'\rangle_d$. If $\phi \in (\mathbb{C}^2)^{L+M}$ can be written as $\phi = \xi^L \xi^M$, then

$$\phi = \sum_{i=0}^{2^L-1} \sum_{i'=0}^{2^M-1} \xi^L_i \xi^M_{i'} |i\rangle_d |i'\rangle_d$$  \hspace{1cm} (7)$$

$$= \sum_{i=0}^{2^L-1} \sum_{i'=0}^{2^M-1} \xi^L_i \xi^M_{i'} |i + 2^L i'\rangle_d,$$  \hspace{1cm} (8)

where we have used Eq. (9). Consequently, the characteristic polynomial of $\phi$ becomes

$$P(\phi; x) = \sum_{i=0}^{2^L-1} \sum_{i'=0}^{2^M-1} \xi^L_i \xi^M_{i'} x^{2^L i} x^{2^M i'},$$

$$= \left( \sum_{i=0}^{2^L-1} \xi^L_i x^i \right) \left( \sum_{i'=0}^{2^M-1} \xi^M_i (x^{2^L})^{i'} \right)$$

$$= P(\xi^L; x) P(\xi^M; x^{2^L}).$$  \hspace{1cm} (9)

Therefore, if the state of the quantum register is the product of an $L$-qubit state and an $M$-qubit state, the characteristic polynomial factorizes as the product of the polynomials of the two states. In the polynomial of the $M$-qubit state the variable $x$ is replaced by $x^{2^L}$. Using Eq. (9) it is easy to calculate the characteristic polynomial $P(\phi_s; x)$ of a separable state $\phi_s \equiv \phi_0 \phi_1 \cdots \phi_{N-1}$ given by Eq. (5). By defining $\phi_{j:N} \equiv \phi_j \phi_{j+1} \cdots \phi_{N-1}$, so that $\phi_{j:N} = \phi_j \phi_{j+1};N$, and using Eq. (9) repeatedly
we get

\[ P(\phi_0, x) = P(\phi_0, x)P(\phi_1, x^2) \]

\[ = P(\phi_0, x)P(\phi_1, x^2)P(\phi_2, x^4) \]

\[ = \cdots \]

\[ = N_{-1} \sum_{j=0}^{N-1} P(\phi_j, x^{2^j}) \]

\[ = \prod_{j=0}^{N-1} (a_j + b_j x^{2^j}). \quad (10) \]

We see that the characteristic polynomial of a separable state can always be written in the form of (10). On the other hand, there always exists a separable state whose characteristic polynomial is given by Eq. (10), namely the state \( \phi_s \). From the definition of \( P(\phi, x) \) it follows that if \( P(\phi, x) = P(\phi, x) \), then necessarily \( \phi = \phi_s \). Therefore \( \phi_s \) is the unique vector which gives rise to the polynomial of Eq. (10). In conclusion, a pure \( N \)-qubit state \( \phi \) is separable if and only if \( P(\phi, x) \) can be written as in Eq. (10). The roots of this equation are

\[ x_{jm} = \left(-\frac{a_j}{b_j}\right)^{1/2^j} e^{i\frac{2\pi j m}{2^j}}, \quad m = 0, 1, \ldots, 2^j - 1, \quad (11) \]

where \( b_j \) has to be nonzero. If \( b_j \) is zero the degree of the polynomial is decreased by \( 2^j \) from the maximal degree \( 2^N - 1 \).

The separability of a state \( \phi \) can be determined by calculating the roots of \( P(\phi, x) \) and checking if they are of the form given by Eq. (11). These calculations can in practice turn out to be very complicated. It may be computationally demanding to achieve accurate enough results in order to reliably see how the roots are distributed in the complex plane. This is partly related to the fact the degree of the polynomial \( P(\phi, x) \) can be \( 2^N - 1 \), which grows rapidly with \( N \), rendering the calculation of roots time-consuming for large \( N \). However, we will show next that the roots of \( P(\phi, x) \) can be expressed in a simple way in terms of the components \( C_i \) of the state vector if \( \phi \) is separable. Let \( \phi_s \) be the separable state given by Eq. (10). When this vector is written in the form \( \phi_s = \sum_{i=0}^{2^N-1} C_i |i\rangle_d \), the components \( C_i \) are easily obtained by noting that \( i_j = 0 \) \( (i_j = 1) \) corresponds to \( a_j \) \( (b_j) \):

\[ C_i = \prod_{j=0}^{N-1} [(1 - i_j)a_j + i_j b_j]. \quad (12) \]

Here we have used the binary form of \( i \), that is, we have written \( i = \sum_{j=0}^{N-1} i_j 2^j \). We assume that \( C_k \neq 0 \), \( C_{k+1} = \cdots = C_{2^N-1} = 0 \), so that the degree of \( P(\phi_s, x) \) is \( k \).

By writing \( k = \sum_{j=0}^{N-1} k_j 2^j \) we see that if \( k_j = 1 \), then \( (k - 2^j)l = k_l - \delta_{jl}, \ l = 0, 1, \ldots, N - 1 \). Using this and Eq. (12) it is easy to see that now \( a_j/b_j = C_{k-2^j}/C_k \).

On the other hand, if \( k_j = 0 \), then \( (k + 2^j)l = k_l + \delta_{jl} \) and Eq. (12) gives \( b_j/a_j = C_{k+2^j}/C_k = 0 \). Summarizing,

\[ \frac{a_j}{b_j} = \frac{C_{k-2^j}}{C_k} \quad \text{if} \quad k_j = 1, \]

\[ b_j = 0 \quad \text{if} \quad k_j = 0. \quad (13) \]

Using Eq. (11) we immediately see that the \( k \) roots of \( P(\phi_s, x) \) are

\[ x_{jm} = \left(-\frac{C_{k-2^j}}{C_k}\right)^{1/2^j} e^{i\frac{2\pi j m}{2^j}}, \quad m = 0, 1, \ldots, 2^j - 1, \quad (14) \]

where \( j \) takes those values for which \( k_j = 1 \). On the other hand, if the roots and their multiplicities are known, the polynomial can be determined up to a multiplying constant. In particular, if \( x = 0 \) is a root, then its multiplicity has to be equal to the lowest power of the polynomial.

In conclusion, an arbitrary pure state \( \phi \) is separable if and only if

\begin{enumerate}
\item[(Ia)] All the numbers \( x_{jm} \) given by Eq. (14) are roots of \( P(\phi, x) \).
\item[(Ib)] The number of \( x_{jm} \) equaling zero is equal to the lowest power of \( P(\phi, x) \).
\end{enumerate}

An alternative formulation is that \( \phi \) is separable if and only if the quantity

\[ S(\phi) \equiv \sum_{j, k_j = 1}^{2^j - 1} |P(\phi, x_{jm})| \quad (15) \]

equals zero and Condition (Ib) holds. Note that if \( k = 0 \) the state is separable.

If a state is found to be separable, then the one-particle states it consists of can be explicitly constructed with the help of the ratios \( a_j/b_j \) given by Eq. (13). We now present some examples of the detection of separability of states with several freely varying components.

### A. Example 1

As the first example we consider a state defined as

\[ \xi^N = C_0 |0\rangle_d + C_1 |1\rangle_d + \cdots + C_{k-2} |k-2\rangle_d + C_k |k\rangle_d, \quad (16) \]

where \( C_0, C_k \neq 0 \) and \( k \) is odd. Since \( k \) is odd \( k_0 = 1 \) and Eq. (14) shows that \( x_{00} = C_{k-1}/C_k = 0 \). Because \( P(\xi^N; x_{00}) = C_0 \neq 0 \), \( \xi \) cannot be a separable state. In a three-qubit case we see that, for example,

\[ \xi^3 = C_0 |000\rangle + C_1 |100\rangle + C_2 |010\rangle + C_3 |110\rangle + C_4 |001\rangle + C_5 |101\rangle + C_7 |111\rangle \quad (17) \]

where \( C_0C_7 \neq 0 \) cannot be separable.

In order to compare our approach with other separability tests, we now check the separability of \( \xi^3 \) using an
alternative method. There exist various (partial) multipartite separability criteria for mixed states (see, for example, \[14, 15\]). While these are useful when mixed states are studied, in the case of pure states the most convenient separability check is usually the standard method of calculating the reduced single-qubit density matrices of the \(N\)-qubit state. This view is supported by the fact that alternative pure state separability tests require examining the properties of matrices that are higher dimensional than the two-by-two dimensional reduced single-qubit density matrices \[14, 15\] or require the calculation of the expectation values of operators expressed as tensor products of the Pauli spin matrices \[14, 15\]. This results in a complex calculation if a state that contains many freely varying components, such as \(\xi^3\), is studied. For these reasons we now examine the separability of \(\xi^3\) by using the method of partial traces. Here and in what follows we denote the reduced single-qubit density matrix pertaining to qubit \(j\) by \(\rho_j\). Now the indexing of qubits runs from 0 to \(N - 1\). The vector \(\xi^3\) is separable if and only if any two of the three density matrices \(\rho_0, \rho_1, \rho_2\) describe pure states. The state \(\rho_2\) is pure if and only if \(\det(\rho_2) = 0\), so if \(\det(\rho_2) \neq 0\) for at least one \(j\), then \(\xi^3\) is entangled. As an example we determine \(\det(\rho_0)\). A simple calculation shows that the single-qubit reduced density matrix of the first qubit is

\[
\rho_0 = \left( |C_0|^2 + |C_2|^2 + |C_4|^2 \right) \left( |C_0C_1^* + C_2C_3^* + C_4C_5^*| C_0C_1 + C_2C_3 + C_4C_5 |C_1|^2 + |C_3|^2 + |C_5|^2 \right).
\]

Using the inequality \(\text{Re}(C) \leq |C|\), where \(C\) is an arbitrary complex number, it can be shown that the following inequality holds for the determinant of \(\rho_0\)

\[
\det(\rho_0) \geq |C|^2 \left( |C_0|^2 + |C_2|^2 + |C_4|^2 + (|C_0C_3| - |C_1C_2|)^2 + (|C_0C_5| - |C_1C_4|)^2 + (|C_2C_5| - |C_3C_4|)^2 \right).
\]

This is bounded below by \(|C_0C_2|^2 > 0\), confirming the aforementioned result concerning the separability of \(\xi^3\). Therefore a necessary condition for the separability of \(\xi^3\) can be straightforwardly obtained using partial traces. However, the polynomial method provides a simpler separability test in the present example. Even more so if instead of \(\xi^3\) the separability of the \(N\)-qubit state \(\xi^N\) is studied.

### B. Example 2

In the second example we choose \(\xi^N\) such that the degree of \(P(\xi^N; x)\) is \(k = 2^N - 2\). Then \(k_i = 1 - \delta_{j0}\). We assume that \(C_{2^N-1-j}(= C_{k-2^N-1}) = 0\), from which it follows that \(x_{(N-1)m} = 0\) for \(m = 0, 1, \ldots, 2^N-1\). According to Condition (Ib) the lowest order of the polynomial has to be at least \(2^{N-1}\) for the state to be a product state. Thus, if \(C_i \neq 0\) for at least one \(i\) such that \(0 \leq i < 2^{N-1}, i \neq 2^{N-1} - 2\), then \(\xi^N\) must be entangled.

In the case of a three-qubit system this result means that

\[
\xi^3 = C_0|000\rangle + C_1|100\rangle + C_3|110\rangle + C_4|011\rangle, \quad C_6 \neq 0,
\]

cannot be a product state if \(C_0, C_1, C_3\) is nonzero. If \(C_4 = 0\) we have \(x_{10} = x_{11} = 0\), which means that all \(x_{jm}\) are equal to zero. Then \(\xi^3\) cannot be separable unless all \(C_i\) except \(C_6\) are zero. The reduced single-qubit density matrices \(\rho_0, \rho_1, \rho_2\) can be straightforwardly calculated and are not presented here. The determinant of \(\rho_2\) is

\[
\det(\rho_2) = \left( |C_0|^2 + |C_1|^2 + |C_3|^2 \right) \left( |C_4|^2 + |C_5|^2 + |C_6|^2 \right)
\]

\[
- |C_0C_4^* + C_1C_5^*|^2
\]

\[
\geq |C_0|^2( |C_1|^2 + |C_3|^2)
\]

\[
+ |C_4|^2( |C_5|^2 + |C_6|^2) + (\bar{C}_0C_5 - |C_1C_4|)^2.
\]

where we have obtained a lower bound for the determinant in the same fashion as in the previous example. We reproduce the earlier result that \(\xi^3\) is necessarily entangled if \(C_1, C_2\) or \(C_3\) is nonzero. In order to determine the separability conditions in the case \(C_4 = 0\) one has to calculate \(\det(\rho_0)\) and repeat the above calculation for this quantity. The result agrees with the one obtained using the polynomial approach, that is, if \(\xi^3\) is separable and \(C_4 = 0\), then only \(C_6\) can be nonzero. We see that also in this case the polynomial approach provides an easier way to check the separability than the method of partial traces.

### C. Example 3

As the final example we study a state given by

\[
\xi^N = \sum_{i=1}^{2^N-1} |i\rangle_d + e^{i\theta} \sum_{i=0}^{2^{N-1}-1} |4i\rangle_d
\]

\[
= \sum_{i=0}^{2^N-1} |i\rangle_d + (e^{i\theta} - 1) \sum_{i=0}^{2^{N-1}-1} |4i\rangle_d.
\]

Now \(C_{2(N-1)-2}/C_{2N-1} = 1\) for all \(j\), so Eq. (11) gives

\[
x_{jm} = e^{i(2m+1)\pi/3}, \quad m = 0, 1, \ldots, 2^j - 1,
\]

where \(j = 0, 1, 2, \ldots, N - 1\). Using the sum formula of geometric series we find that the characteristic polynomial can be written as

\[
P(\xi^N; x) = \frac{x^{2^N-1} - 1}{x - 1} + (e^{i\theta} - 1)\frac{x^{2^N-1} - 1}{x^4 - 1}.
\]

It is easy to see that for \(j = 2, 3, \ldots, N - 1\)

\[
P(\xi^N; x_{jm}) = 0, \quad m = 0, 1, \ldots, 2^j - 1,
\]
while
\[ P(\xi^N; x_{00}) = P(\xi^N; x_{10}) = P(\xi^N; x_{11}) = 2^{N-2}(e^{i\theta} - 1). \]  
(27)
The state \( \xi^N \) is separable if and only if \( \theta = 2\pi n \) for some integer \( n \). If \( \xi^N \) is separable Eq. (13) shows that
\[ \xi^N = \otimes_{j=1}^{N-1}(|0_j + |1_j). \]
Now the \( N \) reduced single-qubit density matrices of \( \xi^N \) can be straightforwardly determined. Lengthy calculation shows that
\[ \det(\rho_0) = \det(\rho_1) = 2^{2N-3}(1 - \cos \theta) \]
and \( \det(\rho_j) = 0 \) when \( j = 2, 3, \ldots, N - 1 \), confirming the earlier result. In the present example the polynomial method does not seem to provide as obvious calculational simplification as in the previous two examples.

D. Generalization to \( h \)-level systems

We now briefly discuss a generalization of the separability test to a system consisting of \( N \) copies of an \( h \)-level system. We write the basis of a single \( h \)-level system as \{0\}_h, \{1\}_h, \ldots, \{h - 1\}_h \} and choose the basis vectors for the \( N \)-partite system as \(|i\>_d = |i_0 i_1 \cdots i_{N-1}\>_h \), where \( i = \sum_{j=0}^{N-1} i_j h^j \) and \( i_j \in \{0, 1, 2, \ldots, h-1\} \). An arbitrary pure state can be expressed as
\[ \phi = \sum_{i=0}^{h^{N-1}-1} C_i |i\>_d. \]
(28)
Let \( \phi^h = \phi^y_0 \phi^h_1 \cdots \phi^h_{N-1} \) be a separable state where \( \phi^h_j = a_j |0\>_h + b_j |1\>_h + c_j |2\>_h + \cdots + q_j |h-1\>_h \). A straightforward calculation shows that
\[ P(\phi^h; x) = \prod_{j=0}^{N-1} \left( a_j + b_j x^{h^j} + \cdots + q_j x^{(h-1)h^j} \right). \]  
(29)
In order to establish a separability test, one has to express the roots of this polynomial in terms of the coefficients \( C_0, C_1, \ldots, C_{h^{N-1}} \). This is possible but complicated if \( 2 < h \leq 6 \). If \( h \geq 6 \), the roots cannot be calculated analytically and therefore cannot be written using the coefficients \( C_i \). Thus the separability test can be extended to systems containing less than six levels, but it is more complicated to apply than in the two-level case. An extension is not feasible if the number of levels is equal to or larger than six.

IV. NUMBER OF UNENTANGLED QUBITS

Entangled states can be classified based on the number of unentangled one-qubit states. The state \( \phi \) is said to contain \( n \) unentangled qubits if it can be written as a product of \( n \) single-qubit states \( \phi_i \) and an \( (N - n) \)-qubit state \( \phi^{N-n} \). In order to study the number of unentangled particles, we determine the characteristic polynomial of a state which separates as a product of a one-qubit state and an \( (N-1) \)-qubit state. We write \( \phi = \phi_j \phi^{N-1} \), where \( \phi_j = a_j |0\>_j + b_j |1\>_j \) is the state of the qubit \( j \) and \( \phi^{N-1} \) gives the state of the rest of the qubits. As before, the degree of the polynomial is denoted by \( k \). Using Eq. (30) we see that the characteristic polynomial of the basis states reads
\[ P(|i\>_d; x) = P(|i_0 i_1 \cdots i_{N-1}\>_d; x) = x^{i_0 2^j} x^{i_1 2^j} x^{i_2 2^j} \cdots x^{i_{N-1} 2^{N-1}}. \]  
(30)
We write the \((N - 1)\)-qubit state as
\[ \phi^{N-1} = \sum_{i \in \{0, 1\}, i \neq j} C_{i_0 \cdots i_{j-1} i_{j+1} \cdots i_{N-1}} |i_0 i_1 i_{j+1} \cdots i_{N-1}\>_h, \]
so using Eq. (30) we find that
\[ P(\phi; x) = b_j (x^{2^j} - x^{2^{jm}}) \sum_{i \in \{0, 1\}, i \neq j} C_{i_0 \cdots i_{j-1} i_{j+1} \cdots i_{N-1}} x^{i_0 2^{j+1} + \cdots + i_{j-1} 2^{j-1} + i_{j+1} 2^{j+1} + \cdots + i_{N-1} 2^{N-1}} \]  
(32)
where we have assumed that \( b_j \neq 0 \), which is equivalent to \( k_j = 1 \). We have also written \( a_j + b_j x^{2^j} = b_j (x^{2^j} - x^{2^{jm}}) \). Note that \( x^{2^j} \) is independent of \( m \). If \( b_j = 0 \), we get an expression which is obtained by multiplying the sum of Eq. (32) by \( a_j \). Equation (32) shows that the polynomial \( P(\phi; x)/(x^{2^j} - x^{2^{jm}}) \) contains only those powers of \( x \) which do not have \( 2^j \) in their binary representation and that \( x^{2^{jm}} \) is a root of \( P(\phi; x) \) for each \( m = 0, 1, \ldots, 2^j - 1 \). Therefore, if \( k_j = 1 \), necessary conditions for the qubit \( j \) to be unentangled with respect to the rest of the qubits are
\[ \text{(IIa)} \quad P(\phi; x_{jm}) = 0 \quad \text{for every} \quad m = 0, 1, \ldots, 2^j - 1. \]
\[ \text{(IIb)} \quad \text{\( 2^j \) does not appear in the binary representations} \quad \text{of the exponents of} \quad x \quad \text{in} \quad P(\phi; x)/(x^{2^j} - x^{2^{jm}}). \]
If \( k_j = 0 \) there is only one condition, namely,
\[ \text{(IIc)} \quad \text{\( 2^j \) does not appear in the binary representations} \quad \text{of the exponents of} \quad x \quad \text{in} \quad P(\phi; x). \]
It is easy to see that these are also sufficient conditions. The number of unentangled qubits can be obtained by checking Conditions (IIa) and (IIb) for every qubit \( j \) for which \( k_j = 1 \) and Condition (IIc) for the rest of the qubits. It is possible to extract information about the number of unentangled qubits without using Conditions (IIb) and (IIc), namely, an upper bound for this quantity can be obtained by adding to the number of qubits for which (IIa) holds the number of indices \( j \) for which \( k_j = 0 \). This corresponds to assuming that either (IIb) or (IIc) holds for every qubit.

A. Example 1

As an example of the use of this method we consider the state given by Eq. (29). Now \( k = 2^N - 1 \) and therefore
$k_j = 1$ for every $j$. Equations (26) and (27) together with Condition (IIa) show that the number of unentangled qubits is at most $N - 2$ ($N$ if $\theta \neq 2\pi n$ ($\theta = 2\pi n$). In order to simplify the polynomial $P(\xi; x)$ we note that

$$x^{2N} - 1 = (x^2 - 1)(x^2 + 1)(x^4 + 1) \cdots (x^{2N-1} + 1).$$

(33)

With the help of this and Eq. (25) we get

$$P(\xi^N; x) = (x + 1)(x^2 + 1)(x^4 + 1) \cdots (x^{2N-1} + 1)$$

$$+ (e^{i\theta} - 1)(x^2 + 1)(x^4 + 1) \cdots (x^{2N-1} + 1)$$

(34)

Now $x^{2j} - (x_{jm})^{2j} = x^{2j} + 1$ when $j \geq 2$ and using the above equation one can see that Condition (IIb) holds for $j = 2, 3, \ldots, N - 1$ regardless of the value of $\theta$. Furthermore, (IIb) holds for every $j$ if $\theta = 2\pi n$. In conclusion, the qubits $j = 2, 3, \ldots, N - 1$ are always unentangled with respect to the rest of the qubits and if $\theta = 2\pi n$ the state is separable. The same result can be obtained using the reduced single-qubit density matrices $\rho_j$. The number of unentangled qubits is equal to the number of $\rho_j$ for which $\det(\rho_j) = 0$. The values of these determinants have been presented in Example 3 and reproduce the aforementioned result.

A necessary step in the calculation of the number of unentangled qubits is to apply Condition (IIa) to all qubits $j$ for which $k_j = 1$. This is equivalent to checking the separability of the state. In addition to this, Conditions (IIb) and (IIc) have to be controlled. On the other hand, in the case of single-qubit reduced density matrices $\rho_j$, the determination of the number of unentangled qubits does not require any additional operations in comparison with testing the separability. In both cases $\det(\rho_j)$ has to be calculated. This suggests that the method of reduced single-qubit density matrices is preferable if the number of unentangled qubits is studied.

V. CONCLUSIONS

We have defined a mapping which associates pure $N$-qubit states with a polynomial. The roots of this polynomial determine the state completely and vice versa. The structure of the polynomial is inspired by the one used in the Majorana representation [19, 20]. The separability of a state can be studied by examining the properties of the roots of the corresponding polynomial. In particular, we have presented a method which establishes a necessary and sufficient condition for a given pure $N$-qubit state $\phi$ to be separable. This method provides a new point of view to the pure state separability and gives an alternative to the conventional separability test of calculating the reduced single-qubit density matrices of the state. The separability of $\phi$ can be determined by checking whether the numbers $x_{jm}$, defined in equation (14), are roots of the polynomial $P(\phi; x)$ of equation (2). Both the numbers $x_{jm}$ and the polynomial $P(\phi; x)$ can be easily obtained as a function of the components of the state $\phi$. We have illustrated through examples that in some cases the polynomial separability test is easier and faster to apply than the method of reduced single-qubit density matrices. We have also shown how the number of unentangled qubits can be obtained with the help of the polynomial $P(\phi; x)$. It seems, however, that for this task the method of single-qubit density matrices is preferable.

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[31] We could naturally define the characteristic polynomial as 
P(φ; x) = \sum_{i=0}^{2^N−1} C_i g_i x^i

where \{g_i\} is a set of 2^N arbitrarily chosen nonzero complex numbers. However, in order to simplify the ensuing calculations we choose 
g_i = 1 for each i. The choice \(g_i = \left(\frac{2^N−1}{i}\right)^{1/2}\) corresponds to the Majorana representation, see Refs. [19, 20].