Cavity-Mediated Strong Matter Wave Bistability in a Spin-1 Condensate

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We study matter wave bistability in a spin-1 Bose-Einstein condensate dispersively coupled to a unidirectional ring cavity. A unique feature is that the population exchange among different modes of matter fields are accomplished via the spin-exchange collisions. We show that the interplay between the atomic spin mixing and the cavity light field can lead to a strong matter wave nonlinearity, making matter wave bistability in a cavity at the single-photon level achievable.

FIG. 1: Schematic diagram showing the system under consideration.

Our model — a spinor BEC with hyperfine spin $F = 1$ confined in a unidirectional ring cavity — is depicted schematically in Fig. 1. At zero temperature we assume single-mode approximation (SMA) that atoms in different spin states can be described by the same spatial wave function $\phi(r)$, then each spin component can be associated with an annihilation operator $\hat{c}_\alpha$ ($\alpha = \pm, 0$). A weak external magnetic field may be applied to break the de-
by introducing a decay rate \( \gamma \) to treat the leakage of cavity photons phenomenologically and around 50 Hz for the atomic transition frequency is below 10 Hz for \(^{23}\text{Na} \). For simplicity, we also assume that the coupling strength between the cavity field and spin-\( \pm \) atoms are the same. We will treat the leakage of cavity photons phenomenologically by introducing a decay rate \( \kappa \) with typical values \( \sim 1 \) MHz. By contrast, the time scale for the atomic spin-mixing dynamics is much longer — the measured population among different spin states through the atomic field operators read

\[
\hat{a} = \frac{\varepsilon_p}{\kappa - i} \left( \bar{\delta}_c - U_0 \left( \hat{c}^\dagger \hat{c}_+ + \hat{c}^\dagger \hat{c}_- \right) \right),
\]

The corresponding Heisenberg equations of motion for the atomic field operators read

\[
i \dot{\hat{c}}_{\pm} = \left[ \hat{c}_{\pm}, \hat{H}_0 \right] + U_0 \hat{c}^\dagger \hat{a} \hat{c}_{\pm}, \quad i \dot{\hat{c}}_0 = \left[ \hat{c}_0, \hat{H}_0 \right].
\]

Combining Eqs. (1) and (2), in the bad cavity limit one can find the effective Hamiltonian \( \hat{H}_{\text{eff}} \) which satisfies

\[
i \dot{\hat{c}}_{\alpha} = \left[ \hat{c}_{\alpha}, \hat{H}_{\text{eff}} \right]
\]

\[
\hat{H}_{\text{eff}} = H_0 - \frac{\varepsilon_p^2}{\kappa} \tan^{-1} \left[ \frac{\bar{\delta}_c - U_0 \left( \hat{c}^\dagger \hat{c}_+ + \hat{c}^\dagger \hat{c}_- \right) \kappa}{\kappa} \right].
\]

In the following we adopt a mean-field treatment by replacing the operators \( \hat{a} \) and \( \hat{c}_0 \) with the corresponding \( \mathcal{C} \) numbers \( \alpha = \langle \hat{a} \rangle \) and \( \alpha_0 = \sqrt{N_0} \exp(-i\alpha_0) \), where \( N_\alpha \) and \( \theta_\alpha \) represent the number and phase of the bosonic field for the particles in the spin component \( \alpha \). We do not consider the anti-ferromagnetic atoms (\(^{23}\text{Na} \)) with \( \bar{\lambda}_n > 0 \). The ferromagnetic case is not qualitatively different. In the absence of the cavity field, the equilibrium solutions \( (x_0, \theta_0) \) have been studied in [13, 25] and others. Besides the phase-independent solutions of \( x_0 = 0 \) and \( x_0 = 1 - |m| \),
for which the relative phase \( \theta_0 \) is not well-defined, the spinor condensate system supports at most one phase-dependent solution with \( \theta_0 = 0 \) or \( \pi \). The presence of the cavity field dramatically changes this property. The phase diagram identifying different types of solutions is mapped out in the parameter space of \( \eta^2 \) and \( \delta_c \), as shown in Fig. 2. We can see that, in certain parameter regime, the number of different phase-dependent solutions can be more than one, different solution regimes of the coupling system can be crossed by varying \( \delta_c \) and \( \eta^2 \). Since these two parameters are directly related to the pump laser, this means that the dynamical properties of the system can be easily manipulated by tuning the intensity or frequency of the pump laser field.

![Fig. 2](image1)

**FIG. 2:** (Color online) Phase diagram in the parameter space of \( \delta_c \) and \( \eta^2 \) for different type of solutions: (a) \( \theta = 0 \); (b) \( \theta = \pi \). Different regions are differentiated by their colors and are labeled with the numbers of corresponding solutions. In the black region, no physical phase-dependent solutions can be found. The dimensionless parameters are estimated to be \( \lambda_a = 10^{-3} \), \( \tilde{q} = 2\lambda_a \) and \( \tilde{U}_0 = -5 \). The other parameters are set as \( m = 0 \) and \( N = 1000 \). The red dashed line in (a) correspond to \( \eta^2 = 0.8 \).

Here we consider the case with the pump intensity \( \eta^2 \) fixed, by varying the cavity-pump detuning \( \delta_c \), the equilibrium properties of the system are changed, as shown in the red-dashed line in Fig. 2(a). The corresponding phase-dependent fixed points are derived and the results are shown in Fig. 3. The system exhibits typical bistable behavior: For certain values of \( \delta_c \), it supports three stationary solutions. A standard linear stability analysis shows that in the region with three solutions, two of these are dynamically stable and the third one is dynamically unstable. Further insights can be gained by examining the corresponding contour plot of \( H \) (Fig. 4 lower panel). The unstable fixed points correspond to the saddle points in the contour plots.

Let us now return to Fig. 3. As indicated in the upper panel of Fig. 3 both the cavity field and the atoms exhibit bistable behavior. The mean cavity photon number involved is always less than unity. Remarkably, such a small number of photons affect the whole condensate and lead to complete population redistribution among different internal atomic spin states, which can be readily observed in experiment. This behavior can be understood as following: The collective nature of the condensate greatly enhances the atom-photon coupling such that a single photon gives rise to a significant atomic phase shift, which in turn strongly modifies the population distribution among the spin states. Bistability results from the nonlinear feedback between photons and atoms.

Let us now return to Fig. 4. As indicated in the upper panel of Fig. 3 both the cavity field and the atoms exhibit bistable behavior. The mean cavity photon number involved is always less than unity. Remarkably, such a small number of photons affect the whole condensate and lead to complete population redistribution among different internal atomic spin states, which can be readily observed in experiment. This behavior can be understood as following: The collective nature of the condensate greatly enhances the atom-photon coupling such that a single photon gives rise to a significant atomic phase shift, which in turn strongly modifies the population distribution among the spin states. Bistability results from the nonlinear feedback between photons and atoms.

It is interesting to compare our study with the experi-
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1. P. Meystre, Atom Optics (Springer-Verlag, New York, 2001).
2. H. M. Gibbs, Controlling Light with Light (Academic, Orlando, Fl., 1985).
3. C. J. Hood et al., Phys. Rev. Lett. 80, 4157 (1998).
4. A. Boca et al., Phys. Rev. Lett. 93, 235603 (2004).
5. V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne, Science 209, 547 (1980).
6. F. Brennecke et al., Nature 450, 268 (2007).
7. Y. Colombe et al., Nature 450, 272 (2007).
8. S. Slama et al., Phys. Rev. Lett. 98, 053603 (2007).
9. S. Gupta et al., Phys. Rev. Lett. 99, 213601 (2007).
10. K. W. Murch et al., Nature Phys. 4, 561 (2008).
11. F. Brennecke et al., Science 322, 235 (2008).
12. M. G. Moore and P. Meystre, Phys. Rev. A 59, R1754 (1999); M. G. Moore, O. Zobay and P. Meystre, Phys. Rev. A 60, 1491 (1999).
13. P. Horak, S. M. Barnett, and H. Ritsch, Phys. Rev. A 61, 033609 (2000); P. Horak and H. Ritsch, Phys. Rev. A 63, 023603 (2001).
14. J. Larson et al., Phys. Rev. Lett. 100, 050401 (2008).
15. J. M. Zhang et al., Phys. Rev. A 79, 033401 (2009).
16. T.-L. Ho, Phys. Rev. Lett. 81, 742 (1998); T. Ohmi and K. Machida, J. Phys. Soc. Jap. 67, 1822 (1998).
17. C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. 81, 5257 (1998); H. Pu et al., Phys. Rev. A 60, 1463 (1999); H. Pu, S. Raghavan, and N. P. Bigelow, Phys. Rev. A 61, 023602 (2000).
18. D. R. Romano and E. J. V. de Passos, Phys. Rev. A 70, 043614 (2004).
19. W. Zhang et al., Phys. Rev. A 72, 013602 (2005).
20. J. Stenger et al., Nature 396, 345 (1998).
21. T. Kuwamoto et al., Phys. Rev. A 69, 063604 (2004).
22. H. Schmaljohann et al., Phys. Rev. Lett. 92, 040402 (2004).
23. M.-S. Chang et al., Nature Phys. 1, 111 (2005).
24. A. T. Black et al., Phys. Rev. Lett. 99, 070403 (2007); Y. Liu et al., Phys. Rev. Lett. 102, 125301 (2009).
25. W. Zhang, S. Yi, and L. You, New J. Phys. 5, 77 (2003).
26. The dimensionless parameters are estimated with experimentally accessible parameters: $\kappa \sim 2\pi \times 20$ KHz [8] and $N\Delta \sim 2\pi \times 20$ Hz for sodium atoms with a typical density $10^{14}$ cm$^{-3}$ [24], $q \sim 2\pi \times 40$ Hz and $U_0 \sim 2\pi \times 100$ Hz.
27. Z. P. Karkuszewski, K. Sacha, and A. Smerzi, Eur. Phys. J. D 21, 251 (2002); B. Wu and J. Liu, Phys. Rev. Lett. 96, 020405 (2006).
28. H. Y. Ling et al., Phys. Rev. A 63, 053810 (2001); H. Y. Ling, Phys. Rev. A 65, 013608 (2001).
29. G. Szirmai, D. Nagy, and P. Domokos, Phys. Rev. Lett. 102, 080401 (2009).