KINKS AND SOLITONS IN LINEAR AND NONLINEAR–DIFFUSION KELLER–SEGEL TYPE MODELS WITH LOGARITHMIC SENSITIVITY

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ABSTRACT. This paper deals with the existence of traveling waves type patterns in the case of the Keller–Segel model with logarithmic sensitivity. The cases in which the diffusion is linear and nonlinear with flux-saturated (of the relativistic heat equation-type) are fully analyzed by comparing the difference between both cases. Moreover, special attention is paid to traveling waves with compact support or with support in the semi-straight line. The existence of these patterns is rigorously proved and the differences between both cases (linear or nonlinear diffusion) are analyzed.

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1. INTRODUCTION

The aim of this paper is to study traveling wave patterns of the Keller–Segel model with logarithmic sensitivity, both in the case of linear diffusion and in the case of flux–saturated non–linear diffusion, focusing our analysis on the so-called relativistic heat equation. Our objective is to prove the existence of traveling waves of soliton type with compact support in both cases of linear and non-linear diffusion.

Chemotaxis refers to the motion of the species up or down a chemical concentration gradient. Examples of this biological process are the propagation of traveling bands of bacterial toward the oxygen [1, 2] or the outward propagation of concentric ring waves by the E. Coli [16, 17]. The prototypical chemotaxis model was proposed by Keller and Segel [31] and in its general

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form reads as
\[
\begin{align*}
\partial_t u(t, x) &= \partial_x \left\{ u(t, x) \Phi \left( \frac{\partial_x u(t, x)}{u(t, x)} \right) - a u(t, x) \partial_x f(S) \right\}, \quad x \in \mathbb{R}, t > 0, \\
\delta \partial_t S(t, x) &= \gamma \partial_{xx}^2 S(t, x) + k(u, S), \quad x \in \mathbb{R}, t > 0, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

The function \( u = u(t, x) \) refers to the cell density at position \( x \) and time \( t \), whereas \( S(t, x) \) means the density of the chemoattractant. Then, the above system consists in two coupled equations in terms of \( u \) and \( S \). The parameter \( a \geq 0 \) measures the strength of the chemical signal and is called as the chemotactic coefficient. We take also \( \delta \) and \( \gamma \) positive numbers where \( \gamma \) is the chemical diffusion coefficient. In the classical Keller–Segel model, the function \( \Phi \) is taken to be the identity map in order to have a classical diffusion in the first term of (1). Moreover, \( f \) refers the chemosentivity function describing the signal mechanism and \( k(u, S) \) characterizes the chemical growth and degradation.

The chemosensitivity function \( f \) can be chosen in different ways. The linear law agrees with \( f(S) = S \), the logarithmic law is \( f(S) = \log(S) \) or the receptor law refers to \( f(S) = S^m/(1 + S^m) \) for \( m \in \mathbb{N} \). The system with linear law and \( k(u, S) = S - u \) is called as the minimum chemotaxis model (see [26, 30]). The second one referring to the logarithmic law follows from the Weber–Frechner law, see [3, 8, 27, 31] for some applications. Although initially the Keller–Segel model was motivated by chemotaxis processes, the field of application of these models is increasingly wide and covers fields of population dynamics, biopolymers or cross-diffusion in quantum mechanics, among others. We refer to [39] for a survey concerning the logarithmic law.

Then, here we will focus on the case of logarithmic sensitivity, meaning \( f(S) = \log(S) \) and where \( k(u, S) = u - \lambda S \) with \( \lambda \geq 0 \). In this way, (1) agrees with
\[
\begin{align*}
\partial_t u(t, x) &= \partial_x \left\{ u(t, x) \Phi \left( \frac{\partial_x u(t, x)}{u(t, x)} \right) - a \frac{\partial_x S(t, x)}{S(t, x)} u(t, x) \right\}, \quad x \in \mathbb{R}, t > 0, \\
\delta \partial_t S(t, x) &= \gamma \partial_{xx}^2 S(t, x) - \lambda S(t, x) + u(t, x), \quad x \in \mathbb{R}, t > 0, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

We will assume that \( \Phi \) verifies
\[(H1) \quad \Phi \in C^2(\mathbb{R}), \quad \Phi(-s) = -\Phi(s), \quad \Phi'(s) > 0, \quad \forall s \in \mathbb{R}.
\]

We will initially assume that \( \Phi(s) = s \) that corresponds to a linear diffusion (of the Laplacian type) with respect to mass density (which is the classical Keller–Segel application), or more specifically \( \Phi(s) = \mu s \), where \( \mu \) represents the viscosity coefficient.

As we have mentioned before, we will give a comparative between the Keller–Segel with linear diffusion and the flux–limited diffusion. More specifically, we will deal with the so-called relativistic heat equation, which corresponds to the choice

\[
\Phi(s) = \mu \frac{s}{\sqrt{1 + (\frac{s}{c})^2}}
\]

where \( \mu \) and \( c \) are positive parameters.

Let us remark that functions \( \Phi \) satisfying
\[
\lim_{s \to \infty} \Phi(s) = \infty,
\]
(which is verified for the linear diffusion) does not seem to have new phenomena except for a casuistic complication. On the other hand, if
\[(H2) \quad \lim_{s \to \infty} \Phi(s) < \infty,
\]
we speak of a flux–saturated process on which we will analyze the qualitative differences with linear diffusion, although the casuistry is similar. In the relativistic heat equation the limit in \((H2)\) is exactly the parameter \(c\), which expresses the growth rate of the support of the solution for such specific flux–saturated equation. We will refer to both hypothesis \((H1)-(H2)\) as \((H)\). In addition to the relativistic heat equation, we should mention that the case of Larson operators, which are defined by 

\[
\Phi(s) = \mu s \sqrt{1 + (\mu c)^p |s|^p},
\]

can be analyzed with exactly the same techniques used in this paper, as in general limiters verifying \(\Phi(s) - c = O(s^{-p})\) as \(s \to \infty\), with \(p \in (1, \infty)\).

In addition to the complexity in the analysis introduced by the flux-saturated operators that we study in this paper, there are many other operators of this type that would require an adaptation or extension of the techniques used here, among them we mention the Wilson operator, which corresponds to \(p = \infty\) in the Larson case, and is defined by

\[
\phi(s) = \mu \frac{s}{1 + (\mu) |s|}.
\]

These are just some classical examples of diffusion by flux–saturated mechanisms, an extensive review can be seen in [19]. Note that the above flux-saturated operators verify a sublinear growth property, that is, there exist \(a, b \in \mathbb{R}^+\) such that \(|\Phi(s)| \leq a|s| + b\). Models with flux–saturated have been studied in various contexts and from diverse perspectives, from analytical studies, the appropriate concept of entropy solutions, hydrodynamic limits, etc., we refer to [4, 5, 6, 9, 10, 11, 12, 13, 14, 18, 21, 22, 23, 24, 25, 28, 33, 34, 35, 37, 38] for some references.

In this work, we will focus on the study of traveling waves solutions, that is

\[
u(t,x) = u(x - \sigma t), \quad \text{and} \quad S(t,x) = S(x - \sigma t), \tag{3}
\]

with \(\sigma > 0\), for some profiles \(u, S : \mathbb{R} \to [0, \infty)\). The search for traveling waves solutions is crucial to understand the mechanisms behind various propagating wave patterns. The analysis of traveling waves solutions for Keller-Segel type systems has been widely carried out from various perspectives and techniques (variational or dynamic systems), see [29, 31, 32, 36, 15] and the references therein.

We will show that the Keller–Segel system together with a flux–saturated mechanism exhibits diverse properties with respect to the classical one. Here, we will study both cases and show their differences. We refer to Figure 1 for the shapes of traveling waves solutions in the cases of classical diffusion and flux–saturated mechanisms. Note that the difference between the patterns with compact support in Figure 1 is mainly due to the associated flux–saturated mechanisms, where there are jumps in the connection with zero and these jumps have infinite slopes at both ends of the support, regardless of the parameter values. As we will see, there are other types of more classical traveling waves solutions with support in the straight or in the semi-straight line, although we believe that those that have compact support have a special interest in physics or biology problems.

Let us briefly explain the idea of the work. Assume that we have a solution of type \(\mathbf{3}\), hence \(\mathbf{2}\) agrees with

\[
-\sigma u' = \left( u \Phi \left( \frac{u'}{u} \right) - a \frac{S'}{S} u \right)', \tag{4}
\]

\[
-\sigma \delta S' = \gamma S'' - \lambda S + u, \tag{5}
\]

where \(u', S'\) and \(S''\) represent the derivative with respect to the new variable \(s = x - \sigma t\). Although we have expressed the previous system taking into account all its terms, our study will focus on the case where \(\delta = 0\). Under a suitable change of variables that is presented in Proposition 2.1 in the case \(\delta = 0\), the above system of ordinary differential equations is related to

\[
u' = w \Phi^{-1} (av - \sigma) - wv, \tag{6}
\]
with some initial conditions. Throughout this paper we will analyze the coupled system (6)–(7) and later come back to the original variables \( u \) and \( S \) in order to transfer and interpret the results obtained there. We will separate in two cases: first we will assume that \( \Phi = \text{Id} \) giving rise to a linear diffusion and later we will deal with the nonlinear flux–saturated case for \( \Phi \) satisfying the hypothesis \((H)\). The shape of the profiles \( u \) and \( S \) strongly depends on the previous cases.

More specifically, in the case of a linear diffusion, i.e. \( \Phi = \text{Id} \), hence the system (6)–(7) is not singular and classical theory for ODEs gives us existence and uniqueness of solution. Moreover, analyzing the phase diagram and coming back to \( u \) and \( \tilde{S} \) we are able to find different profiles with and without compact support. We refer to Figure 5 which illustrates the shapes of the profiles. The existence of the diverse types of solutions strongly depends on the parameters \( a \) and \( \sigma \). This is the main goal of Section 3.

On the other hand, by virtue of the hypothesis \((H)\) for \( \Phi \), one has that system (6)–(7) is singular at the boundary. Indeed, note that \( \Phi^{-1} \) is defined only in \((-c,c)\) as a consequence of \((H)\) and this gives us a boundary for the solutions of (6)–(7). Moreover, \( \Phi^{-1}(\pm c) = \pm \infty \) which implies an infinite derivative of \( w \) on the boundary. That amounts to have the solutions described in Figure 8.

The main goal of Section 4 is to analyze the existence of the different types of traveling wave solutions in the case of the so-called relativistic heat equation (see Figure 1). The existence of traveling waves were analyzed for the case of flux–saturated mechanisms for the first time in [20], while in the case of flux–saturated Keller–Segel in [7].

Finally, the results obtained in this work can be summarized in the following (formal) theorem, which will be developed and specified in the following sections.

**Theorem 1.1.** There are traveling wave type profiles with compact support or with support in the semi-straight line for the Keller-Segel model with logarithmic sensitivity both in the case of linear diffusion and nonlinear flux–saturated mechanisms. In the latter case, the solutions must be understood in an entropic sense and present a jump with an infinite slope when extended by zero.

This work is organized as follows. Section 2 aims to give the equivalent equations for traveling waves solutions in terms of \((w,v)\). In Section 3 we analyze the classical Keller–Segel equations by getting traveling waves solutions with the shapes described in the upper part of Figure 1.
Finally, Section 4 deals with the Keller–Segel system with a flux–limited diffusion getting the solutions presented in at the bottom of Figure 1.

Let us establish the notation that will be used throughout the paper. We will denote
\[ u(s_0) = \lim_{s \to s_0} u(s), \]
and
\[ u(s_±_0) = \lim_{s \to s_±_0} u(s). \]
even when the values are not finite, we still use the right hand notation when the left hand one was not clear.

2. The equations for traveling waves solutions

The idea of this section is to explore the existence of solutions of traveling waves type of the system (2), by analyzing (4)–(5).

To make lighter the writing, we assume \( \delta = 0 \), searching for solutions of the system
\[
\begin{align*}
\left( u \Phi \left( \frac{u'}{u} \right) - a \frac{S'}{S} u + \sigma u \right)' &= 0, \\
\gamma S'' - \lambda S + u &= 0.
\end{align*}
\]

We will look for solutions \( u(s) \) that are positive and bounded, and then \( S(s) \) is defined by (9).

Let us consider solutions of (8) in a distributional framework in the sense of
\[ \int_{\mathbb{R}} \left( u \Phi \left( \frac{u'}{u} \right) - a \frac{S'}{S} u + \sigma u \right) \psi' \, ds = 0, \]
for any test function \( \psi \in C_0^\infty(\mathbb{R}) \). We observe that if \( u \) is positive, then from (9) we deduce that \( S > 0 \). Furthermore, in the case that \( u \) is bounded, then \( S \in C^1 \), since we are working in the one–dimensional case. Therefore, \( \frac{S'}{S} \in L^1_{loc}(\mathbb{R}) \). The term \( u \Phi \left( \frac{u'}{u} \right) \) is the product of a bounded and integrable function \( u \in BV_{loc}(\mathbb{R}) \), by a semilinear function \( \Phi \), where \( u' \) makes sense as the Radon–Nikodym derivative of \( u \). Then, the function
\[ s \in \text{supp } u \mapsto u(s) \Phi \left( \frac{u'(s)}{u(s)} \right) - a \frac{S'(s)}{S(s)} u(s) + \sigma u(s), \]
is a \( L^1_{loc}(\mathbb{R}) \) function. Therefore, the fact that \( u \) is a distributional solution of (8) implies the existence of a constant \( k \) such that
\[ u(s) \Phi \left( \frac{u'(s)}{u(s)} \right) - a \frac{S'(s)}{S(s)} u(s) + \sigma u(s) = k. \]

In the case that \( \text{supp } u \neq \mathbb{R} \), we can deduce that \( k = 0 \), since (10) is defined for every test function \( \psi \in C_0^\infty(\mathbb{R}) \). If \( \text{supp } u = \mathbb{R} \), we look for traveling wave profiles for which \( u(s) \to 0 \), as \( s \to +\infty \) or \( s \to -\infty \), which again provides us that \( k = 0 \).

Consequently, we look for functions \( u > 0 \) and \( S > 0 \) that are maximal solutions in an interval \( (s_-, s_+) \), with \( -\infty \leq s_- < s_+ \leq +\infty \), such that
\[
\begin{align*}
\Phi \left( \frac{u'}{u} \right) - a \frac{S'}{S} + \sigma &= 0, \\
\gamma S'' - \lambda S + u &= 0.
\end{align*}
\]

In the following proposition, we arrive at an equivalent system to (11)–(12) which will help us on the study of the existence of solutions.

**Proposition 2.1.** Let \( g : (-c, c) \to \mathbb{R} \) given by \( \Phi(g(y)) = y \) with \( y \in \mathbb{R} \), that is \( g = \Phi^{-1} \) in the sense of the composition of applications. Then, the solutions of (8)–(9) can be obtained by solving the system
\[ w' = w g(av - \sigma) - w v, \]
\[
\gamma v' = -\gamma v^2 - w + \lambda,
\]
where
\[
w(s) = \frac{u(s)}{S(s)} \quad \text{and} \quad v(s) = \frac{S'(s)}{S(s)},
\]
(14)

Proof. The calculations made previously allow us to see that \(w, v\) defined by (15) give rise to (13) and (14).

To go back and recover the solutions of (8)–(9), we take \(s_0 \in \mathbb{R}, u_0 > 0, S_0 > 0, S'_0 \in \mathbb{R}\) and solve the initial values problem consisting of (13)–(14) with initial data \(w(s_0) = u_0 S_0, v(s_0) = S'_0 S_0\), where \(S(s) = S_0 \exp(\int_{s_0}^{s} v(\delta)d\delta)\) and \(u(s) = w(s)S(s)\).

\[\Box\]

3. Linear diffusion

This section aims to analyze the existence of traveling waves solutions for the classical Keller–Segel model with logarithmic sensitivity interacting with linear diffusion. Recall that throughout the paper we assume \(\delta = 0\), for the sake of simplicity. Therefore, the system under study in this section is
\[
\begin{align*}
\partial_t u &= \partial_x \left( \mu \partial_x u - \alpha \frac{\partial_x S}{S} u \right), \\
0 &= \gamma \partial_{xx} S - \lambda S + u.
\end{align*}
\]
Moreover, we assume that \(\sigma\) (the velocity of the traveling wave), \(\alpha, \gamma\) and \(\lambda\) are positive real numbers. Then, after the change of variable to traveling waves coordinates \((t,x) \rightarrow x - \sigma t\), system (16) becomes:
\[
\begin{align*}
w' &= w \left\{ (a-1)v - \sigma \right\}, \\
v' &= \frac{\lambda}{\gamma} - v^2 - \frac{1}{\gamma} w.
\end{align*}
\]
(17) (18)

The main difference between \(a < 1\) and \(a \geq 1\) is the behavior of the fixed points and the sign of the derivative \((w,v)'\). In what follows, we will analyze the system for each value of \(a\). After showing the phase diagram for every case, we shall focus on the continuation of the solutions together with an asymptotic analysis. Finally, we need to come back to the original variables \(u\) and \(S\) via Proposition 2.1.

3.1. Analysis of equilibrium points. In this section, we explore a local analysis of the solutions around the fixed points by studying the linearized equation around them.

From now on define \(v_*\) as
\[
v_* := \sqrt{\frac{\lambda}{\gamma}},
\]
(19)
and
\[
\sigma_* := |1 - a| \sqrt{\frac{\lambda}{\gamma}} = |1 - a|v_*.
\]
(20)
The value of \(\sigma_*\) will determine the different scenarios for the solutions. This will be analyzed in the following proposition.

**Proposition 3.1.** Define
\[
(w_1, v_1) = (0, v_*), \quad (w_2, v_2) = (0, -v_*), \quad \text{and} \quad (w_3, v_3) = \left( \lambda - \frac{\gamma \sigma^2}{(a-1)^2}, \frac{\sigma}{a-1} \right),
\]
where \(v_*\) is defined in (19). Hence,

1. If \(a > 0\) and \(\sigma_* < \sigma\), then (17)–(18) has two fixed points given by \((w_i, v_i)\), with \(i = 1, 2\). The point \((w_1, v_1)\) is a stable point, whereas \((w_2, v_2)\) is a saddle point. The stable manifold associated to the saddle point is generated by \((\gamma((1+a)v_* + \sigma), 1)\), and the unstable one is generated by \((0, 1)\).
(2) If \(a \in (0, 1)\) and \(\sigma < \sigma_\ast\), then \([17]-[18]\) has three fixed points given by \((w_i, v_i)\), with \(i = 1, 2, 3\). The point \((w_1, v_1)\) is a stable point, \((w_2, v_2)\) is an unstable point and \((w_3, v_3)\) is a saddle point. Both the stable and unstable manifold associated to the saddle point are one–dimensional.

(3) If \(a > 1\) and \(\sigma < \sigma_\ast\), then \([17]-[18]\) has three fixed points given by \((w_i, v_i)\), with \(i = 1, 2, 3\). The points \((w_1, v_1)\) and \((w_2, v_2)\) are saddle points, whereas \((w_3, v_3)\) is a stable point or a stable focus. The stable manifold associated to \((w_1, v_1)\) is generated by \(\sigma < \sigma\) and the unstable one is generated by \(\gamma((1+a)\sigma + \sigma), 1\). Moreover, the stable manifold associated to \((w_2, v_2)\) is generated by \(\gamma((1+a)\sigma + \sigma), 1\), and the unstable one is generated by \((0, 1)\).

**Proof.** The linearized problem associated to such system is given by

\[
(w, v)' = A(w_i, v_i)(w, v),
\]

for \((w_i, v_i)\) a fixed point, where

\[
A(w, v) = \begin{pmatrix}
(a-1)v - \sigma & (a-1)w \\
\frac{1}{\gamma} & -2v
\end{pmatrix}.
\]

The eigenvalues are given by the solutions of

\[(a-1)v - \sigma - x(-2v - x) + \frac{(a-1)w}{\gamma} = 0.
\]

In the case of \((w_1, v_1)\), we have that the eigenvalues are

\[x_1 = (a-1)v_\ast - \sigma, \quad \text{and} \quad x_2 = -2v_\ast,
\]

where \(v_\ast > 0\). Then, since \(x_2 < 0\) we have that if \(a-1)v_\ast - \sigma < 0\) we obtain that \((w_1, v_1)\) is stable. In the case that \((a-1)v_\ast - \sigma > 0\) we have a saddle point.

Let us analyze now the fixed point \((w_2, v_2)\). The associated eigenvalues read as

\[x_1 = -(a-1)v_\ast - \sigma, \quad \text{and} \quad x_2 = 2v_\ast.
\]

In this case \(x_2\) is positive. Hence if \((1-a)v_\ast < \sigma\), \((w_2, v_2)\) is a saddle point. Otherwise, it is an unstable point.

Finally, in the case that \(a \neq 1\), there exists \((w_3, v_3)\) such that \([17]-[18]\) holds. The associated eigenvalues to this point are described by

\[
x_1 = -\frac{\sigma}{a-1} - \sqrt{\frac{\sigma^2}{(a-1)^2} - \frac{(a-1)}{\gamma}\left(\lambda - \frac{\gamma\sigma^2}{(a-1)^2}\right)},
\]

\[
x_2 = -\frac{\sigma}{a-1} + \sqrt{\frac{\sigma^2}{(a-1)^2} - \frac{(a-1)}{\gamma}\left(\lambda - \frac{\gamma\sigma^2}{(a-1)^2}\right)}.
\]

Note that since we are studying solutions with \(w > 0\), we consider such fixed point only in the case when \(w_3 > 0\). That correspond to

\[\lambda - \frac{\gamma\sigma^2}{(a-1)^2} > 0,
\]

which agrees with \(\sigma_\ast > \sigma\). In the case that \(a > 1\) we find that \(x_1 < 0\) and \(x_2 < 0\). Moreover, if

\[\frac{\sigma^2}{(a-1)^2} - \frac{(a-1)}{\gamma}\left(\lambda - \frac{\gamma\sigma^2}{(a-1)^2}\right) > 0,
\]

we obtain a stable point. Otherwise, it is a stable focus. On the other hand, if \(a < 1\) we have that \(x_2 > 0\) and \(x_1 < 0\), finding a saddle point. Finally, the generators of the stable and unstable manifold associated to the saddle points are given through the associated eigenfunctions. \(\square\)
Remark 3.2. From the previous proposition, note that if \( a < 1 \) and \( \sigma = \sigma_* \), then \((w_3, v_3) = (w_2, v_2)\). Moreover, one gets that \((w_1, v_1)\) is a stable point. However, for \((w_3, v_3) = (w_2, v_2)\) one gets a zero eigenvalue and another positive one.

In the case that \( a > 1 \) and \( \sigma = \sigma_* \), then \((w_3, v_3) = (w_1, v_1)\). In such a case, \((w_2, v_2)\) is a saddle point. However, the point \((w_3, v_3) = (w_1, v_1)\) is degenerate as it has a zero eigenvalue and a negative one.

3.2. Phase space diagram. Here, we will give a formal discussion about the monotonicity of the solutions and show the phase diagram depending also on the choice of the parameters.

The isocline map depends on the position (in case it exists) of the vertical line \( v = \frac{\sigma}{a-1} \) and the parabola \( w = \lambda - \gamma v^2 \), with \( w \geq 0 \). This can be described via \( v_* \) and \( \sigma_* \) defined in (19)–(20).

First, consider small chemotactic sensitivity agreeing with \( a < 1 \). That corresponds to the upper part of Figure 2 and, following its notation, we consider Case A, for \( 0 < \sigma < \sigma_* \), and Case B, for \( \sigma > \sigma_* \). In both cases, \((w_1, v_1)\) and \((w_2, v_2)\) are fixed points. Moreover, if \( 0 < \sigma < \sigma_* \) (Case A), we have an extra fixed point given by the intersection between the parabola and the vertical line: this is \((w_3, v_3)\) defined in Proposition 3.1. Note also that limit cycles are not allowed here. Indeed, for Case B we have that the region under the parabola is positively invariant. Therefore, either the solution enters the area under the parabola and the component of \( v \) of the system changes from decreasing to increasing, or the solution never touches the parabola and \( v \) is always decreasing. In Case A, the region under the parabola can be divided in two small regions described in Figure 2 (A). There, the region to the left-hand side of the vertical line is positively invariant and a limit cycle cannot touch it. On the other hand, the region to the right-hand side of the vertical line is negative invariant. Finally, putting together all the previous arguments we find the phase diagrams described in the upper part of Figure 2.

The case \( a = 1 \) is very special since we do not have any vertical line: it corresponds to Proposition 3.1. Indeed, the system reduces to

\[
\begin{align*}
w' &= -\sigma w, \\
v' &= \frac{\lambda}{\gamma} - v^2 - \frac{w}{\gamma}.
\end{align*}
\]

As in the previous case, the area under the parabola remains positively invariant. We refer the reader to Figure 2 C for its phase diagram.

Finally, if we have a large chemotactic sensitivity parameter, i.e., \( a > 1 \), we get again two possibilities: either the vertical line intersects the parabola. This is described at the bottom of Figure 2. We denote then Cases D when \( 0 < \sigma < \sigma_* \) and E when \( \sigma > \sigma_* \). Here, we also have that \((w_1, v_1)\) and \((w_2, v_2)\) are fixed points for both cases, and \((w_3, v_3)\) only for Case D. The non existence of limit cycles in these cases is not clear here. For Case E, we can work as for \( a < 1 \) getting the phase diagram of Figure 2. However, Case D is not clear and limit cycles around \((w_3, v_3)\) may appear.

3.3. Unbounded solutions of the \((w, v)\) system. In this section, we study the unbounded solutions (the upper blue lines) in Figure 3. To do so, first we shall assume that the initial condition \((w_0, v_0)\) satisfies \( v_0 > v_* \) and \( w_0 > 0 \), where \( v_* \) is defined in (19). That will be presented in the following proposition:

Proposition 3.3. Let \( a > 0 \), \( v_0 > v_* \) and \( w_0 > 0 \), where \( v_* \) is defined in (19). Consider \((w, v)\) the maximal solution to (17)–(18), with initial data \((w(s_0), v(s_0)) = (w_0, v_0)\), defined in \((s_-, s_+)\). Then, we have \( s_\in \mathbb{R} \), \( v(s_-) = +\infty \), and

\[
\begin{align*}
w(s_-) &= +\infty, & \text{for } a < 1, \\
w(s_-) &\in \mathbb{R}^+, & \text{for } a = 1, \\
w(s_-) &= 0, & \text{for } a > 1.
\end{align*}
\]

Proof. Notice that since \( v_0 > v_* \) the solution in \((s_-, s_0)\) lies outside the parabola, which implies that \( v \) decreases. On the other hand, the sign of \( w \) depends on \( a \) and \( \sigma \). We refer to Figure 2.
Figure 2. Top (A and B): $a < 1$. Center (C): $a = 1$. Bottom (D and E): $a > 1$. Left (A and D): $\sigma < \sigma_\star$. Right (B and E): $\sigma > \sigma_\star$. The red curves determine the changes of direction of $(v, w)$ and the black arrows refer to such directions.

First, let us prove that $s_- \in \mathbb{R}$ by a reductio ad absurdum argument. In this way, assume that $s_- = -\infty$ and we will arrive to a contradiction. By the monotonicity of $v$, we have two possibilities:

$$\lim_{s \to -\infty} v(s) = L_1 > v_\star, \quad \text{or} \quad \lim_{s \to -\infty} v(s) = +\infty.$$ 

Consider the first one. Hence, there exists a sequence $s_n \to -\infty$ such that

$$\lim_{n \to +\infty} v'(s_n) = 0,$$

and using the equation for $v$ we arrive at

$$\lim_{n \to +\infty} v'(s_n) = \frac{\lambda}{\gamma} - L_1^2 < 0,$$

getting a contradiction. Hence, now assume

$$\lim_{s \to -\infty} v(s) = +\infty.$$ 

In this case, one has that $\frac{1}{v(s)}$ tends to 0 as $s \to -\infty$. As a consequence, there exists $s_n \to -\infty$ such that

$$\lim_{n \to +\infty} \left( \frac{1}{v(s_n)} \right)' = 0,$$
which agrees with
\[
0 = -\lim_{n \to +\infty} \frac{v'(s_n)}{v(s_n)^2} = \lim_{n \to +\infty} \left\{ 1 + \frac{w(s_n)}{\gamma v^2(s_n)} - \frac{\lambda}{\gamma v^2(s_n)} \right\} = \lim_{n \to +\infty} \left\{ 1 + \frac{w(s_n)}{\gamma v^2(s_n)} \right\} > 1,
\]
getting again a contradiction. Then, we can conclude that \( s_− \in \mathbb{R} \).

In the next step we will prove that if
\[
\lim_{s \to s_-} w(s) = +\infty,
\]
(21)
hence
\[
\lim_{s \to s_-} v(s) = +\infty.
\]
(22)
As a consequence and using that \( s_- \in \mathbb{R} \), we will have that (22) always happens. We work again with a \textit{reductio ad absurdum argument}. Assume that
\[
\lim_{s \to s_-} v(s) = L_1 > v_*
\]
occurs. Here, we will use the graph system associated to \([17]-[18]\). Denote \( V(x) = v(w^{-1}(x)) \). Hence
\[
\lim_{x \to +\infty} V(x) = L_1,
\]
and we deduce that there exists a sequence \( x_n \to +\infty \) such that
\[
\lim_{n \to +\infty} V'(x_n) = 0.
\]
By using the equations for \((w, v)\), we have that
\[
V'(x) = \frac{\lambda - \gamma V^2(x) - x}{\gamma x \{(a - 1)V(x) - \sigma\}},
\]
and then
\[
\lim_{x \to +\infty} V'(x) = \lim_{x \to +\infty} \frac{1}{(a - 1)V(x) - \sigma} \left\{ \frac{1}{\gamma} + \frac{\lambda - \gamma V^2(x)}{\gamma x} \right\} \neq 0,
\]
getting a contradiction. Then, we can conclude that if (21) happens, (22) also does. Since \( s_- \in \mathbb{R} \), we achieve that in any case (22) occurs. That concludes the first part of the lemma.

In order to have the behavior of \( w \) at \( s_- \), we need to use the graph system for \( W(x) = w(v^{-1}(x)) \), that satisfies
\[
W'(x) = \frac{\gamma W(x) \{(a - 1)x - \sigma\}}{\lambda - W(x) - \gamma x^2}.
\]
(23)
Here, we will use a comparison argument. Consider first the case \( a < 1 \), where we have that \( w' < 0 \) and \( v' < 0 \), for \( s \in (s_-, s_0) \). Take \( \overline{x} \) such that \( \overline{x} \leq v(s) \) for any \( s \in (s_-, s_0) \). Our goal will be to prove that
\[
\lim_{s \to s_-} w(s) = +\infty.
\]
(24)
Note that the graph system is well-defined in \( x \in [v_0, +\infty) \), since \( v \) is monotone. From (23) one has that \( W' > 0 \), and then \( W(x) \geq W(\overline{x}) \), for any \( x \in [v_0, +\infty) \). Moreover
\[
W'(x) \geq \alpha \frac{W(x)}{x},
\]
for some \( \alpha > 0 \). Hence, considering \( y \) the solution to
\[
y'(x) = \alpha \frac{y(x)}{x},
\]
with \( y(v_0) = W(\overline{x}) \), we find that \( W(x) \geq y(x) \). Consequently, one has that \( W(x) \geq W(\overline{x})x^\alpha \), achieving (24).

The case \( a = 1 \) is very special since the equation for \( w \) can be integrated: note that it does not depend on \( v \). Hence, one has \( w(s) = w_0 e^{-\sigma(s-s_0)} \).

The last case \( a > 1 \) can be treated in a similar manner. Note that in this case one finds \( \overline{x} < s_0 \) such that \( v(s) > \frac{\sigma}{\alpha - 1} \). Then, for any \( s \in (s_-, \overline{x}) \) one has that \( v \) decreases and \( w \) increases.
Hence, we can set $\sigma$ such that $\sigma < \sigma_s$ for any $s \in (s_-, \sigma)$. We can define the graph system for $x \in [v(\sigma), +\infty)$, and we can check
\[
\frac{W'(x)}{W(x)} \leq \frac{-\alpha}{x},
\]
for some $\alpha > 0$. By using again the comparison principle one finds $W(x) \leq Cx^{-\alpha}$, concluding the proof.

The analysis for solutions with initial data $v_0 < -v_*$ is completely symmetric. In fact, it is enough to make an investment in the path of $s$. Then, we achieve the following result.

**Proposition 3.4.** Let $a > 0$, $v_0 < -v_*$ and $w_0 > 0$, where $v_*$ is defined in (19). Consider $(w, v)$ the maximal solution to (17)-(18), with initial data $(w_0, v_0)$, defined in $(s_-, s_+)$. Then, we have $s_+ \in \mathbb{R}$, $v(s_+) = -\infty$, and

\[
\begin{align*}
w(s_+) &= +\infty, \text{ for } a < 1, \\
w(s_+) &\in \mathbb{R}^+, \text{ for } a = 1, \\
w(s_+) &= 0, \text{ for } a > 1.
\end{align*}
\]

### 3.4. Behavior of the $(w, v)$ system

Here, we shall prove that there exists each of the solutions (the blue curve) of Figure 3. There, the initial condition is taking to be in the right hand side of the parabola. The proof will have three parts. First, we will check the existence of $w_0 = w_0^*$ such that we have a unique solution ending in $(w_2, v_2)$ for Cases B, C, D and E; and ending in $(w_3, v_3)$ for Case A. The idea of the proof is the uniqueness of the stable manifold associated to each saddle point. Later, we shall analyze the solutions with either $w_0 > w_0^*$ or $w_0 < w_0^*$. At the end of this section, we will study the special case in which the starting point is a saddle fixed point by studying solutions with initial data $v_0 < -v_*$. More specifically, we will focus on $a < 1$ and $\sigma < \sigma_*$ corresponding to Figure 3-A.

By using the uniqueness of the stable manifold associated to a saddle point, we are able to prove the existence of $w_0^*$ in Figure 3.

**Proposition 3.5.** Let $a > 0$ and $v_0 > v_*$, where $v_*$ is defined in (19). There exists $w_0^* > 0$ and a maximal solution to (17)-(18), with initial data $(w_0^*, v_0)$, defined in $(s_-, +\infty)$ with $s_- \in \mathbb{R}$, satisfying the following.

- **Asymptotic behavior at $s_-$:** We have that
  \[
  \lim_{s \to s_-} v(s) = +\infty,
  \]
  for any $a > 0$, and
  \[
  \begin{align*}
  &\lim_{s \to s_-} w(s) = +\infty, \text{ for } a < 1, \\
  &\lim_{s \to s_-} w(s) \in \mathbb{R}^+, \text{ for } a = 1, \\
  &\lim_{s \to s_-} w(s) = 0, \text{ for } a > 1.
  \end{align*}
  \]

- **Asymptotic behavior at $+\infty$:** We find that
  \[
  \lim_{s \to +\infty} (w(s), v(s)) = (w_3, v_3),
  \]
  for $a < 1$ and $\sigma < \sigma_*$. Otherwise,
  \[
  \lim_{s \to +\infty} (w(s), v(s)) = (w_2, v_2).
  \]

**Remark 3.6.** Here we are considering $\sigma \neq \sigma_*$. By virtue of Remark 3.2, we can prove also the existence of such $w_0^*$, but not the uniqueness: we can not ensure that there is a unique curve ending at $(w_3, v_3)$.\
Proof. Let us explain the existence of \( w^*_0 \). It appears since the fixed point \((w_3, v_3)\), for the case \( a < 1 \) and \( \sigma < \sigma_* \), or \((w_2, v_2) = (0, -v_*)\), for the other cases described in Proposition 3.1, are saddle points.

Let us focus on the case of having the saddle point \((w_2, v_2)\), which is the case of Figure 2: B, C, D and E. There, the stable manifold is generated by the vector \((\gamma((1 + a)v_* + \sigma), 1)\). Moreover, the derivative at the point \( v = -v_* \) of the parabola \( w = -\gamma v^2 + \lambda \) is \( 2\gamma v_* \). Since the vector \((\gamma(2v_* + \sigma), 1)\) is steepest than the parabola at that point \((2\gamma v_*, 1)\), one can consider \((0, -v_*)\) as the starting point of the time reversed system and it will not enter in the parabola. The same argument of Lemma 3.3 can be applied here to prove that \( s_- \in \mathbb{R} \) and

\[
\lim_{s \to s_-} v(s) = +\infty.
\]

Hence, we have that such solution must intersect the line \( v = v_0 \). In this way, we find the existence of such \( w^*_0 \), see Figure 3. Moreover, from Lemma 3.3 one finds the behavior at \( s_- \). Note that by the continuation argument, one has that \( s_+ = +\infty \).

As a consequence of the uniqueness of solutions and Propositions 3.3 and 3.4 we get the following asymptotic behavior for the solutions with \( w_0 > w^*_0 \), where \( w^*_0 \) is defined in Proposition 3.5.

**Proposition 3.7.** Let \( a > 0 \), \( v_0 > v_* \) and \( w_0 > w^*_0 \), where \( v_* \) in defined in (19) and \( w^*_0 \) is defined in Proposition 3.5. Then, any maximal solution (17) - (18), with initial data \((w_0, v_0)\), defined in \((s_-, s_+)\) satisfies the following:

1. We have that \( s_-, s_+ \in \mathbb{R} \).
2. Moreover, we find

\[
\lim_{s \to s_-} v(s) = +\infty, \quad \text{and} \quad \lim_{s \to s_+} v(s) = -\infty,
\]

Figure 3. Top (A and B): \( a < 1 \). Center (C): \( a = 1 \). Bottom (D and E): \( a > 1 \). Left (A and D): \( \sigma < \sigma_* \). Right (B and E): \( \sigma > \sigma_* \). The blue curves refer to the solutions with initial data \((v_0, w_0)\) with \( v_0 > v_* \) and where \( w_0 \) depends on the position with respect to \( w^*_0 \).
for any \( a > 0 \), and

\[
\lim_{s \to s_-} w(s) = \lim_{s \to s_+} w(s) = +\infty, \text{ for } a < 1, \\
\lim_{s \to s_-} w(s), \lim_{s \to s_+} w(s) \in \mathbb{R}^+, \text{ for } a = 1, \\
\lim_{s \to s_-} w(s) = \lim_{s \to s_+} w(s) = 0, \text{ for } a > 1.
\]

The solutions starting from \( w_0 < w_0^* \) will enter in the parabola and, hence, we can check that they are bounded close to \( s_+ \). Indeed, for cases A, B, C and E of Figure 2 we can prove that they converge to a fixed point. Since for case D we can not ensure the non existence of limit cycles around \((w_3, v_3)\), we can not ensure the convergence to such point. However, the solution will be bounded.

**Proposition 3.8.** Let \( a > 0, v_0 > v_* \) and \( w_0 < w_0^* \), where \( v_* \) is defined in (19) and \( w_0^* \) is defined in Proposition 3.5. Then, any maximal solution (17)–(18), with initial data \((w_0, v_0)\), defined in \((s_-, s_+)\) satisfies the following:

1. We have that \( s_- \in \mathbb{R}, \) and \( s_+ = +\infty. \)

2. Moreover, the following assertions hold true:
   - **Asymptotic behavior at \( s_- \):** We have
     \[
     \lim_{s \to s_-} v(s) = +\infty,
     \]
     for any \( a > 0 \), and
     \[
     \lim_{s \to s_-} w(s) = +\infty, \text{ for } a < 1, \\
     \lim_{s \to s_-} w(s) \in \mathbb{R}^+, \text{ for } a = 1, \\
     \lim_{s \to s_-} w(s) = 0, \text{ for } a > 1.
     \]

   - **Asymptotic behavior at \( +\infty \):** We find that the solution is bounded and satisfies
     \[
     -v_* < \liminf_{s \to +\infty} v(s) \leq \limsup_{s \to +\infty} v(s) < v_*, \quad \text{and} \quad \liminf_{s \to +\infty} w(s) > 0,
     \]
     for \( a > 1 \) and \( \sigma < \sigma_* \). Otherwise
     \[
     \lim_{s \to +\infty} (w(s), v(s)) = (w_1, v_1).
     \]

**Proof.** From Lemma 3.3 one finds that \( s_- \in \mathbb{R} \) and the mentioned behavior at \( s_- \).

By the uniqueness of solution, we have that such solution must enter in the parabola as \( s \) approaches \( s_+ \). Once it enters, one has that \( v' > 0 \).

In the case \( a > 1 \) and \( \sigma < \sigma_* \), which correspond to Case D of Figure 2, we can not ensure the non existence of limit cycles. What we know is that the solution is bounded as it approaches \( s_+ \) and then we can continue it having \( s_+ = +\infty \). Moreover, we get that

\[
-v_* < \liminf_{s \to +\infty} v(s) \leq \limsup_{s \to +\infty} v(s) < v_*, \quad \text{and} \quad \liminf_{s \to +\infty} w(s) > 0.
\]

On the other hand, from the phase portrait one has that the solution must converges to a fixed point: \((w_1, v_1)\). By the continuation principle one finally achieves that \( s_+ = +\infty \), concluding the proof.

Hence, the previous Propositions 3.5, 3.7 and 3.8 yield the following theorem.

**Theorem 3.9.** Let \( a > 0 \) and \( v_0 > v_* \), where \( v_* \) is defined in (19). Consider any maximal solution (17)–(18), with initial data \((w_0, v_0)\), defined in \((s_-, s_+)\). Then, there exists \( w_0^* > 0 \) such that the following is satisfied:
• If \( w_0 > w_0^* \), then \( s_-, s_+ \in \mathbb{R} \). Moreover, we find
\[
\lim_{s \to s_-} v(s) = +\infty, \quad \text{and} \quad \lim_{s \to s_+} v(s) = -\infty,
\]
for any \( a > 0 \), and
\[
\begin{align*}
\lim_{s \to s_-} w(s) &= \lim_{s \to s_+} w(s) = +\infty, \quad \text{for } a < 1, \\
\lim_{s \to s_-} w(s) &= \lim_{s \to s_+} w(s) \in \mathbb{R}^+, \quad \text{for } a = 1, \\
\lim_{s \to s_-} w(s) &= \lim_{s \to s_+} w(s) = 0, \quad \text{for } a > 1.
\end{align*}
\]

• If \( w_0 = w_0^* \), then \( s_- \in \mathbb{R} \) and \( s_+ = +\infty \). We have
\[
\lim_{s \to s_-} v(s) = +\infty,
\]
for any \( a > 0 \), and
\[
\begin{align*}
\lim_{s \to s_-} w(s) &= +\infty, \quad \text{for } a < 1, \\
\lim_{s \to s_-} w(s) &= \mathbb{R}^+, \quad \text{for } a = 1, \\
\lim_{s \to s_-} w(s) &= 0, \quad \text{for } a > 1.
\end{align*}
\]
Moreover, we find
\[
\lim_{s \to +\infty} (w(s), v(s)) = (w_3, v_3),
\]
for \( a < 1 \) and \( \sigma < \sigma_* \), and otherwise one gets
\[
\lim_{s \to +\infty} (w(s), v(s)) = (w_2, v_2).
\]

• If \( w_0 < w_0^* \), then \( s_- \in \mathbb{R} \) and \( s_+ = +\infty \). We have
\[
\lim_{s \to s_-} v(s) = +\infty,
\]
for any \( a > 0 \), and
\[
\begin{align*}
\lim_{s \to s_-} w(s) &= +\infty, \quad \text{for } a < 1, \\
\lim_{s \to s_-} w(s) &= \mathbb{R}^+, \quad \text{for } a = 1, \\
\lim_{s \to s_-} w(s) &= 0, \quad \text{for } a > 1.
\end{align*}
\]
Moreover, we find that the solution is bounded and satisfies
\[
-v_* < \liminf_{s \to +\infty} v(s) \leq \limsup_{s \to +\infty} v(s) < v_*, \quad \text{and} \quad \liminf_{s \to +\infty} w(s) > 0,
\]
for \( a > 1 \) and \( \sigma < \sigma_* \), and otherwise
\[
\lim_{s \to +\infty} (w(s), v(s)) = (w_1, v_1).
\]

Finally, let us focus on \( a < 1 \) and \( \sigma < \sigma_* \). From the phase diagram in Figure 2 we find interesting solutions when considering \( v_0 < -v_* \) in such a case. Those solutions are described in Figure 1.

**Theorem 3.10.** Let \( a < 1 \), \( \sigma < \sigma_* \), \( w_0 > 0 \) and \( v_0 < -v_* \), where \( v_* \) and \( \sigma_* \) are defined in (19)–(20). Consider the maximal solution \((w, v)\) to (17)–(18), with initial data \((w(s_0), v(s_0)) = (w_0, v_0)\), defined in \((s_- , s_+)\). Then, there exists \( w_0^* \) satisfying the following.

1. **Asymptotic behavior at \( s_+ \):** We have that \( s_+ \in \mathbb{R} \) and
\[
\lim_{s \to s_+} v(s) = -\infty, \quad \text{and} \quad \lim_{s \to s_+} w(s) = +\infty.
\]

2. **Asymptotic behavior at \( s_- \):**
   • If \( w_0 > w_0^* \) then \( s_- \in \mathbb{R} \) and
\[
\begin{align*}
\lim_{s \to s_-} v(s) &= +\infty, \quad \text{and} \quad \lim_{s \to s_-} w(s) = +\infty.
\end{align*}
\]
Figure 4. Case $a \in (0, 1)$ and $\sigma < \sigma_\ast$. The blue curves refer to the solutions with initial data $(v_0, w_0)$ with $v_0 < -v_\ast$ and where $w_0$ depends on the position with respect to $w\ast_0$.

- If $w_0 = w\ast_0$ then $s_\ast = -\infty$ and
  \[
  \lim_{s \to -\infty} (w(s), v(s)) = (w_3, v_3).
  \]
- If $w_0 < w\ast_0$ then $s_\ast = -\infty$ and
  \[
  \lim_{s \to -\infty} (w(s), v(s)) = (w_2, v_2).
  \]

Proof. The asymptotic behavior at $s_\ast$ is achieved as a consequence of Lemma 3.4.

The existence of $w\ast_0$ comes from the uniqueness of the stable manifold associated to the saddle point $(w_3, v_3)$. This is similar to the proof of the existence of $w\ast_0$ in Proposition 3.5. By the continuation of the solutions one gets that $s_\ast = -\infty$ in such a case.

In the case that $w_0 > w\ast_0$, then the solution can not enter in the parabola by the uniqueness of solution. Then, a similar scenario to Proposition 3.7 occurs here obtaining the announced result.

In the latter case $w_0 < w\ast_0$, the solution will converge to the unstable point $(w_2, v_2)$ (which is a stable point of the time reversed system). By the continuation principle one gets again $s_\ast = -\infty$.

3.5. Going back to the solutions $(u, S)$ of the original system. Different values of $a$ will exhibit diverse scenarios. In particular, we will have the distinct types of solutions A1, A2, A3 and A4, for the original variables $(u, S)$, defined as follows

- **Type A1:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type A1 if $s_-, s_+ \in \mathbb{R}$ and $f(s_-) = f(s_+) = 0$.
- **Type A2:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type A2 if $s_- \in \mathbb{R}$ and $s_+ = +\infty$. Moreover, $f$ satisfies $f(s_-) = 0$, and
  \[
  \lim_{s \to +\infty} f(s) = 0.
  \]
- **Type A3:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type A3 if $s_- \in \mathbb{R}$ and $s_+ = +\infty$. Moreover, $f$ satisfies $f(s_-) = 0$, and
  \[
  \lim_{s \to +\infty} f(s) = +\infty.
  \]
- **Type A4:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type A4 if $s_- = -\infty$ and $s_+ \in \mathbb{R}$. Moreover, $f$ satisfies $f(s_+) = 0$, and
  \[
  \lim_{s \to -\infty} f(s) = +\infty.
  \]

See Figure 5 referring to the different type of solutions.

Finally, we will recover the solution $u(t, x) = u(s)$ and $S(t, x) = S(s)$ to (2) via Proposition 2.1. First, we need to introduce two preliminary results.
Figure 5. Types of solutions: A1, A2, A3, A4 in the case of linear diffusion Keller–Segel system with logarithmic sensitivity.

Lemma 3.11. Let \((w, v)\) be solutions to \((17)-(18)\) defined in \((s_0, s_+).\) If \(s_+ < +\infty,\) then we have

\[
\begin{align*}
\lim_{s \to s_+} S(s) &= 0, \\
\lim_{s \to s_+} u(s) &= 0.
\end{align*}
\]

Proof. Analyzing the different possibilities in terms of \(a,\) we can deduce that in each of the cases \(\lim_{s \to s_+} v(s) = -\infty.\)

From here and from equality \(v = S'/S\) we deduce

\[
S(s) = C_0 e^{\int_{s_0}^s v(\tau) d\tau},
\]

for some \(s_0, C_0\) associated to the initial data. Let us see that \(v\) is not integrable and, therefore, \(\int_{s_0}^{s_+} v(\tau) d\tau = -\infty.\) To demonstrate this fact, we distinguish two cases based on the value of the parameter \(a.\)

In the case \(a \geq 1,\) just make the limit by using L’Hôpital rule (note that \(v'(s) < 0)\)

\[
\lim_{s \to s_+} \frac{s^+ - s}{1/v(s)} = \lim_{s \to s_+} \frac{v^2(s)}{v'(s)} = \lim_{s \to s_+} \frac{1}{1 + \frac{w(s)}{\gamma v^2(s)} - \frac{\lambda}{\gamma v^2(s)}} = 1,
\]

where we have used that \(w(s) \to w_+ = 0,\) since for \(a \geq 1\) we have \(w(s) \to w_+,\) as \(s \to s_+,\) where \(w_+ \in [0, +\infty).\)

In the case \(0 < a < 1,\) the limit \(w_+ = \infty\) and we need to analyze the behaviour of \(\frac{w(s)}{v^2(s)}.\) Take now \(z = -v,\) which implies \(z(s) \to \infty,\) as \(s \to s_+.\) We can build a solution of

\[
\begin{align*}
w' &= w((1 - a)z - \sigma), \\
z' &= z^2 + \frac{w}{\gamma} - \frac{\lambda}{\gamma}
\end{align*}
\]

using the associated graph system. That is, define a function \(W : (z_0, +\infty) \to \mathbb{R},\) such that \(W(z(s)) = w(s).\) This takes the form

\[
W' = W \frac{(1 - a)z - \sigma}{z^2 + \frac{W}{\gamma} - \frac{\lambda}{\gamma}}.
\]
Let $W(z) = z^2R(z)$. Then, $R(s) > 0$ verifies
\[
\frac{zR'}{R} = \frac{(1 - a) - \frac{z}{a}}{1 + \frac{R}{\gamma} - \frac{\lambda}{\gamma^2z^2}} - 2 \leq \frac{(1 - a) - \frac{z}{a}}{1 - \frac{\lambda}{\gamma^2z^2}} - 2.
\]

Then, for $z_1$ large enough and $z \geq z_1$ we have
\[
\frac{zR'}{R} \leq 1.
\]

Therefore, we find a bound for $R$
\[
R(z) \leq R(z_1)\frac{z_1}{z} \to 0,
\]
or, equivalently,
\[
\frac{W(z)}{z^2} \to 0,
\]
which concludes the proof for the case $0 < a < 1$. \hfill \Box

In a symmetric way, we find also the following result for the case $s_- > -\infty$.

**Lemma 3.12.** Let $(w, v)$ be solutions to (17)–(18) defined in $(s_-, s_+)$. If $s_- > -\infty$, then we have
\[
\lim_{s \to s_-} S(s) = 0, \quad \lim_{s \to s_-} u(s) = 0.
\]

By using Lemmas 3.11 and 3.12, we achieve the following result concerning the traveling waves solutions coming from Propositions 3.5, 3.7, and 3.8.

**Theorem 3.13.** Let $a > 0$, $v_0 > v_*$, where $v_*$ is defined in (19), and $w_0^*$ defined in Proposition 3.7. For any $\sigma > 0$, there exists $u(t, x) = u(s)$ and $S(t, x) = S(s)$, with $s = x - \sigma t$, traveling waves solution to (2), with initial data $(u_0, S_0)$, verifying the following.

1. If $\frac{w_0}{S_0} > w_0^*$, then $u$ and $S$ are of Type A1.
2. If $\frac{w_0}{S_0} = w_0^*$, then $u$ and $S$ are of Type A2.
3. If $\frac{w_0}{S_0} < w_0^*$ and $av_* < \sigma$ then $S$ is of Type A3 and $u$ is of Type A2.
4. If $\frac{w_0}{S_0} < w_0^*$ and $av_* > \sigma$, then $S$ and $u$ are of Type A3.

**Proof.** From (15), we recover the value of $u$ and $S$ as
\[
u(s) = C_0 w(s) e^{\int_{s_0}^{s} v(\tau) d\tau},
\]
and
\[
S(s) = C_0 e^{\int_{s_0}^{s} v(\tau) d\tau},
\]
for some $C_0 > 0$ associated to the initial data.

From Propositions 3.5, 3.7, and 3.8 we get the existence of a maximal solution $(w, v)$ defined in $(s_-, s_+)$. If $\frac{w_0}{S_0} \leq w_0^*$, then $s_+ = +\infty$, and in the other case $s_+ \in \mathbb{R}$. Moreover, $s_- \in \mathbb{R}$. Hence, $u$ and $S$ are defined in $(s_-, s_+)$. By using Lemma 3.12 one achieves
\[
\lim_{s \to s_-} u(s) = 0,
\]
and
\[
\lim_{s \to s_-} S(s) = 0,
\]
for any $a > 0$.

- Case $\frac{w_0}{S_0} > w_0^*$. The behavior at $s_+$ in this case can be obtaining using Lemma 3.11
- Case $\frac{w_0}{S_0} = w_0^*$. From Proposition 3.3 one has that $s_+ = +\infty$ and
\[
\lim_{s \to +\infty} (w(s), v(s)) = (w_3, v_3),
\]
if \( a < 1 \) and \( \sigma < |1 - a|v_* \), or
\[
\lim_{s \to +\infty} (w(s), v(s)) = (w_2, v_2),
\]
in other case. Hence, since \( v_2 \) and \( v_3 \) are negative numbers one finds
\[
\lim_{s \to +\infty} S(s) = \lim_{s \to +\infty} C_0 e^{\int_{s_0}^{s} v(\tau) d\tau} = 0,
\]
and then
\[
\lim_{s \to +\infty} u(s) = 0.
\]
- Case \( \frac{w_0}{S_0} < w_*^0 \). From Proposition 3.8 one finds that \( s_+ = +\infty \) and
\[
\lim_{s \to +\infty} (w(s), v(s)) = (w_2, v_2),
\]
if \( a < 1 \) or \( a > 1 \) and \( \sigma < \sigma_* \). Since \( v_2 \) is a positive number one achieves
\[
\lim_{s \to +\infty} S(s) = \lim_{s \to +\infty} C_0 e^{\int_{s_0}^{s} v(\tau) d\tau} = +\infty.
\]
Moreover, since \( w_2 = 0 \), there is a competition between \( w \) and \( S \). Note that
\[
\lim_{s \to +\infty} \int_{s_0}^{s} v(\tau) d\tau = \lim_{s \to +\infty} \frac{v(s)}{(a - 1)v - \sigma} = \frac{v_*}{(a - 1)v_* - \sigma} < 0.
\]
Then
\[
\lim_{s \to +\infty} \left\{ \ln(w(s)) + \int_{s_0}^{s} v(\tau) d\tau \right\} = \lim_{s \to +\infty} \ln(w(s)) \left\{ 1 + \frac{v_*}{(a - 1)v_* - \sigma} \right\}.
\]
Hence, if
\[
1 + \frac{v_*}{(a - 1)v_* - \sigma} < 0,
\]
we find
\[
\lim_{s \to +\infty} u(s) = +\infty.
\]
In the case that
\[
1 + \frac{v_*}{(a - 1)v_* - \sigma} > 0,
\]
we get
\[
\lim_{s \to +\infty} u(s) = 0.
\]

The solutions constructed in Theorem 3.10 gives us solutions of Types A1 and A4.

**Theorem 3.14.** Let \( a < 1 \), \( \sigma < \sigma_* \), \( v_0 < -v_* \), where \( v_* \) and \( \sigma_* \) are defined in (19)–(20), and \( w_*^0 \) defined in Proposition 3.10. There exists \( u(t, x) = u(s) \) and \( S(t, x) = S(s) \), with \( s = x - \sigma t \), for any \( \sigma > 0 \), which is a traveling waves solutions to (2), with initial data \((u_0, S_0)\), verifying the following:

1. If \( \frac{w_0}{S_0} > w_*^0 \), then \( u \) and \( S \) are of Type A1.
2. If \( \frac{w_0}{S_0} \leq w_*^0 \), then \( u \) and \( S \) are of Type A4.
3.6. Solitons for both density and chemoattractant. Note that the first part of Theorem 3.13 corresponds to the existence of soliton-type patterns for both the density and the chemoattractant. Here, we will concretize the study of such solutions and, in particular, focus on the behavior of the derivative of \( u \).

Consider \( a > 0, v_0 > v_* \) and \( w_0 > w_0^* \), where \( v_* \) is defined in \((19)\) and \( w_0^* \) is defined in Proposition 3.5. Note that, as it was pointed out in Proposition 2.1, we can recover the information on the original variables \((u, S)\) solutions of \((8)–(9)\), by taking \( s_0 \in \mathbb{R}, u_0 > 0, S_0 > 0, S_0' \in \mathbb{R} \) and solving \((13)–(14)\) with initial data \( w(s_0) = u_0 S_0, v(s_0) = S_0' S_0 \), where \( S(s) = S_0 \exp(\int_{s_0}^{s} v(\delta) d\delta) \) and \( u(s) = w(s) S(s) \).

Then, as a consequence of Theorem 3.13 we deduce the existence of \((u, S) : (s_-, s_+) \to (0, +\infty)\) with \(-\infty < s_- < s_+ < +\infty\) and such that
\[
  u(s_-) = S(s_-) = 0 = u(s_+) = S(s_+).
\]

Thus, by using that \( S \) is a solution of a boundary problem, the strong maximum principle assures us that
\[
  S'(s_-) > 0, \quad S'(s_+) < 0.
\]

On the other hand, we can use \((11)\) with \( \Phi(s) = s \) to write
\[
  u' = \left( a \frac{S'}{S} - \sigma \right) u.
\]

Integrating this equation we find
\[
  u(s) = u_0 \left( \frac{S(s)}{S_0} \right)^a e^{-\sigma(s-s_0)}, \quad (28)
\]
with \( u_0 = u(s_0) \). Note that the term \( e^{-\sigma(s-s_0)} \) causes a lateral displacement of \( u \) with respect to \( S \), as can be seen in Figure 6.

Hence, thanks to \((28)\), we deduce the following assertions:

- If \( 0 < a < 1 \), then \( u'(s_-) = \infty \) and \( u'(s_+) = -\infty \);
- If \( a = 1 \), then \( u'(s_-) > 0 \) and \( u'(s_+) < 0 \);
- If \( a > 1 \), then \( u'(s_-) = u'(s_+) = 0 \),

which is reproduced in the following Figure 6.

**Figure 6.** This figure represents the different patterns, depending on the parameter \( a \), of the soliton type for both the density and the chemoattractant.
4. Relativistic flux saturated operators with logarithmic sensitivity

In this section we are going to obtain soliton-type traveling waves for the flux–saturated equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left( \mu \frac{u \frac{\partial u}{\partial x}}{\sqrt{u^2 + \mu^2 (\frac{\partial u}{\partial x})^2}} - a \frac{\partial S}{\partial x} u \right), \\
0 &= \gamma \frac{\partial^2}{\partial x^2} S - \lambda S + u.
\end{align*}
\]

(29)

Here, we shall focus on this system, which is the logarithmic Keller–Segel model associated with the relativistic heat equation. However, most of the results and techniques used in this case can be extended to other flux saturated mechanisms under certain conditions that will be pointed out through this section. The advantage of this particular case is that most of the calculations can be done explicitly.

System (29) fits the case of the general equation (2), where

\[
\Phi(s) = \mu \frac{s}{\sqrt{1 + \mu^2 s^2}}.
\]

(30)

Note that the flux is bounded and then it is in particular a bounded sublinear function, that is, it satisfies \(\textbf{(H2)}\). By virtue of Propostion 2.1, the existence of traveling waves can be achieved by analyzing the following system

\[
\begin{align*}
\frac{\mathrm{d} w}{\mathrm{d} s} &= w \left( g (a v - \sigma) - v \right), \\
\frac{\mathrm{d} v}{\mathrm{d} s} &= \lambda \frac{\gamma}{v} - v^2 - \frac{1}{\gamma} w.
\end{align*}
\]

(31)

where \(g = \Phi^{-1}\). In our case \(g : (-c,c) \to \mathbb{R}\) is defined as \(g(y) = \frac{cy}{\mu \sqrt{c^2 - y^2}}\). Thus, (31) is defined for \(w > 0\) and \(\frac{\sigma - c}{a} < v < \frac{\sigma + c}{a}\).

Note that in the search of the fixed points for (31), one obtains that the equation \(\lambda - \gamma v^2 - w = 0\) could have one, two or three different solutions. That gives rise to a very varied casuistry.

4.1. Moving solutions with flux saturated mechanisms. This section aims to study the existence of some special patterns for (29) as well as its dynamic properties. We shall focus on the most relevant shapes for us that are the solutions of soliton types.

In the following theorem, we show the existence of \((w,v)\) solutions of (31) joining the boundary points \(\sigma - c a \), \(\sigma + c a \). Indeed, we achieve singular solutions due to the behavior of \(w'\) in those points.

**Theorem 4.1.** For any \(v_0 \in \left(\frac{\sigma - c}{a}, \frac{\sigma + c}{a}\right)\), there exists \(w_0^*\) such that every maximal solution of (31) with initial data \((v_0, w_0)\) is defined on an interval \((s_-, s_+)\), with \(-\infty < s_- < s_+ < +\infty\), for any \(w_0 > w_0^*\). This maximal solution verifies

i) \(v(s_-) = \frac{\sigma + c}{a}, \quad v(s_+) = \frac{\sigma - c}{a}\),

ii) \(v \in C^1[s_-, s_+]\) and \(v'(s) < 0\),

iii) \(w(s_-)\), and \(w(s_+) \in (0, +\infty)\),

iv) \(w'(s_-) = +\infty, \quad w'(s_+) = -\infty\).

The type of solutions defined in the statement of Theorem 4.1 can be synthesized in Figure 7.

To prove Theorem 4.1 we need to introduce a previous result on the properties of the graph system associated with (31). Let \((v, w)\) be a solution of (31) defined on an interval \(I\). We define \(w(s) = W(v(s))\), for each \(s \in I\). In this way, \(W\) verifies

\[
W' = W \frac{g(av - \sigma) - v}{-v^2 - \frac{W}{\gamma} + \frac{\lambda}{\gamma}}.
\]

(32)

Let us introduce the following result about the solution \(W\) of (32).
Figure 7. Graph of the solution described by Theorem 4.1. First, we represent the $v$ component by placing $w$ below. The graph on the right hand side reflects the trace in the $(v, w)$ plane. Note that the solution has infinite slopes at the ends of the interval support.

Lemma 4.2. Let $v_0 \in (\frac{\sigma - c}{a}, \frac{\sigma + c}{a})$. There exists $w^*_0$ for which the solution of (32) satisfying $W(v_0) = w_0$ is defined in $(\frac{\sigma - c}{a}, \frac{\sigma + c}{a})$, for any $w_0 > w^*_0$, and

$$\lambda < \inf \left\{ W(v) : v \in \left( \frac{\sigma - c}{a}, \frac{\sigma + c}{a} \right) \right\}. \quad (33)$$

Furthermore, the solution of (32) verifies

$$\lim_{v \to \frac{\sigma - c}{a}} W(v) = (0, \infty), \quad (34)$$
$$\lim_{v \to \frac{\sigma + c}{a}} W'(v) = \pm \infty. \quad (35)$$

Proof. Setting $Y(v) = \frac{1}{W(v)}$ in (32), we find

$$Y' = Y^2 \frac{g(av - \sigma) - v}{1/Y - Y(v^2 - \frac{\sigma}{a})}. \quad (36)$$

We look for $\epsilon > 0$ such that the solution $Y(v)$ of (36) with $Y(v_0) = Y_0$, for $Y_0$ small enough, is defined in $(\frac{\sigma - c}{a}, \frac{\sigma + c}{a})$ and, furthermore, $Y(v) < \epsilon$.

Let us check that for each $\epsilon > 0$ we can find $\delta > 0$ such that if $Y_0 < \epsilon$, then $Y(v) < \delta$. Setting $\varepsilon > 0$, in a neighborhood of $v_0$, which we denote by $\tilde{I}$, we have $Y(v) < \epsilon$ and, therefore, we can obtain

$$\left| \frac{Y''(v)}{Y(v)} \right| \leq Y(v) \frac{f(v)}{\frac{1}{\gamma} - \epsilon \alpha}, \quad (37)$$

where $f(v) = |g(av - \sigma)| + |v|$, and

$$\alpha = \sup \left\{ v^2(s) + \frac{\lambda}{\gamma}, v \in \left( \frac{\sigma - c}{a}, \frac{\sigma + c}{a} \right) \right\}.$$

We choose $\epsilon$ such that $\frac{1}{\gamma} - \epsilon \alpha > 0$. Using that $Y(v) < \epsilon$, Gronwall lemma applied to (37) leads to

$$Y(v) \leq Y_0 e^{\frac{\epsilon}{\frac{1}{\gamma} - \epsilon \alpha} \left( \int_{v_0}^v f(s) ds \right)} \leq Y_0 e^{\frac{\epsilon}{\frac{1}{\gamma} - \epsilon \alpha} \|f\|_{L^1}},$$

for any $v \in \tilde{I}$. Let $\delta < \epsilon$ such that

$$\frac{\epsilon}{\delta e \frac{1}{\gamma} - \epsilon \alpha} \|f\|_{L^1} < \epsilon.$$
and $Y_0 < \delta$. Then we have $Y(v) < \epsilon$, for $v \in \tilde{I}$.

Finally, a standard argument for the prolongation of solutions in differential equations allows us to ensure that $\tilde{I} = (\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$, since $\delta < \varepsilon$. As a consequence, we obtain that $Y(v) < \delta$, for any $v \in (\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$.

Hence, we can come back to the original function $W(v) = \frac{1}{v(v)}$, and setting $\varepsilon < \frac{1}{\lambda}$, it is enough to take $w_0^* = \frac{1}{\lambda}$ to deduce the statement of the theorem.

It remains to check the behavior at the boundary of definition of $W$, that is, (34) and (35). By using (33), we get that $W$ is uniformly positive and the denominator is uniformly negative. As a consequence, we find (35), taking into account (32). From this it follows that $W$ is monotone near the extremes and therefore it reaches a global maximum in the interval $(\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$, which is bounded, obtaining then that (34) holds.

With the help of the preceding result we can assemble the proof of Theorem 4.1.

**Proof of Theorem 4.1.**

Let $v_0 \in (\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$, we take $w_0^*$ achieved in Lemma 4.2. Then, we consider $w_0 > w_0^*$. The solution of the initial value problem is obtained by integrating the differential equation

$$v' = -v^2 - \frac{W(v)}{\gamma} + \frac{\lambda}{\gamma}, \quad v(0) = v_0,$$

which is defined in a maximal interval $(s_1, s_2)$. Now, let us choose $w : (s_1, s_2) \to (0, \infty)$ defined by $w(s) = W(v(s))$, where $W : (\frac{\alpha - c}{a}, \frac{\alpha + c}{a}) \to (0, \infty)$ is given by Lemma 4.2. Using that this function is bounded together with (33), we obtain the existence of two constants $0 < \varepsilon_1 < \varepsilon_2$ such that

$$-\varepsilon_2 < v'(s) < -\varepsilon_1, \quad s \in (s_1, s_2).$$

From this, we deduce that $s_1$ and $s_2$ are finite, and that $v$ is decreasing, having limits at the ends of the interval. A prolongation argument provides the first statement i) of the theorem.

To prove ii), we just have to check that the limits $\lim_{s \to +\infty} v'(s)$ exist, which can be deduced from (38) and from the asymptotic behavior of $W$ near the extremes of the interval, which have been shown in Lemma 4.2.

To deduce iii) we just need to keep in mind that $w(s) = W(v(s))$ and (34).

Finally, using the chain rule and taking into account ii) and (35), we can prove that iv) holds.

**Remark 4.3.** In this result it has been crucial that $f$ is integrable, which is equivalent to that $g$ is integrable in $(-c, c)$. Therefore, it is possible to use other types of limiters that verify this condition of integrability without a modification of the arguments of Theorem 4.1.

The following result focuses on the existence of singular solutions joining $\frac{\alpha - c}{a}$ with $\frac{\alpha + c}{a}$ located inside the parabola, see Figure 8.

**Theorem 4.4.** Assume $[\frac{\alpha - c}{a}, \frac{\alpha + c}{a}] \subset (-v^*, v^*)$. There exist $w^*_1$ such that any maximal solution of (22), with initial data $(v_0, w_0)$, is defined on an interval $(s_-, s_+)$, $-\infty < s_- < s_+ < +\infty$, for any $v_0 \in (\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$ and $0 < w_0 < w^*_1$. This solution verifies

i) $v(s_-) = \frac{\alpha - c}{a}$, $v(s_+) = \frac{\alpha + c}{a}$,

ii) $v \in C^1[s_-, s_+]$ and $v'(s) > 0$,

iii) $w(s_-)$ and $w(s_+) \in (0, +\infty)$,

iv) $w(s_-) = \infty$, $w(s_+) = \infty$.

From the hypotheses of Theorem 4.3 we can deduce that $w(s) < \lambda - \gamma v^2(s)$. Hence, similarly to Theorem 4.1 we can synthesize the characteristics of the solutions in Figure 8.

The proof of Theorem 4.4 is similar to that of Theorem 4.1. Indeed, his proof is based on the following result that it is the equivalent of Lemma 4.2.

**Lemma 4.5.** Assume $[\frac{\alpha - c}{a}, \frac{\alpha + c}{a}] \subset (-v^*, v^*)$. There exist $w^*_1 > 0$ such that every solution of (22) that satisfies $W(v_0) = w_0$ is defined in $(\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$, for any $v_0 \in (\frac{\alpha - c}{a}, \frac{\alpha + c}{a})$ and $0 < w_0 < w^*_1$. 

Furthermore

\[ W(v) < \lambda - \gamma v^2, \quad \text{for all } v \in \left( \frac{\sigma - c}{a}, \frac{\sigma + c}{a} \right), \]  

(40)

and

\[ \lim_{v \to \pm \infty} W(v) \in (0, \infty), \]  

(41)

\[ \lim_{v \to \pm \infty} W'(v) = \mp \infty. \]  

(42)

Proof. The proof of this lemma is very similar to Lemma 4.2 here one must work directly with (32). Let \( \varepsilon > 0 \) and \( M > 0 \) such that if \( 0 < W < \varepsilon \), then \( M < \frac{\lambda - W}{\gamma} \). Setting \( 0 < \delta < \varepsilon \) and \( 0 < w_0 < \delta \), then the solution of (32) with \( W(v_0) = w_0 \) verifies \( W(v) < \varepsilon \) over an interval \( \tilde{I} \) containing \( v_0 \). Using the Gronwall lemma, we obtain

\[ W(v) < \delta e \left( \frac{1}{M} \int_{v_0}^{v} |f(s)| ds \right) \leq \delta e \left( \frac{\|f\|_1}{M} \right). \]

Assuming \( \delta e \left( \frac{\|f\|_1}{M} \right) < \varepsilon = w_1^*, \) we can deduce that \( \tilde{I} = I \), and we conclude with the same arguments as those of the Lemma 4.2 proof. Similarly, (41) and (42) are proved in the same way as their equivalents in Lemma 4.2. \( \square \)

4.2. Consequences on solutions in the system \((u, S)\). In this subsection, we revert our results to the original context of problem (29), which adapts to the case (2). The search for traveling waves

\[ u(t, x) = u(x - \sigma t) \quad \text{and} \quad S(t, x) = S(x - \sigma t), \]

leads to

\[ \Phi \left( \frac{u'}{u} \right) - a \frac{S'}{S} + \sigma = 0, \]  

(43)

\[ \gamma S'' - \lambda S + u = 0. \]  

(44)

where \( \Phi \) is the map (30). Now we choose \( s_0 \in [s_-, s_+] \) and \( S_0 \in (0, \infty) \), then we need to solve

\[ \begin{cases} S'(s) = v(s)S(s), & S(s_0) = S_0, \\ u(s) = S(s)w(s). \end{cases} \]  

(45)

Recall that we are using \( v = \frac{S'}{S} \) and \( w = \frac{S}{S} \).

Let us obtain a representation of the solution of (29) in Figure 9. Note that in Figure 9 the component \( S \) is logarithmically concave in \([s_-, s_+]\) and has two discontinuities in the second
Figure 9. Graphs of the solution \((u, S)\) of (29) obtained by applying (45) to the solution of Theorem 4.1. The function \(u\) must be continued by zero outside \([s_-, s_+]\) while \(S\) has to be continued in a \(C^1\) sense by solutions of the equation \(\gamma S'' - \lambda S = 0\).

 derivative (\(S\) is not a function \(C^2\)) at \(s_-\) and \(s_+\). About the \(u\) component of the solution, we only know its behavior around the extremes, which presents a saturation front.

On the other hand, if \(c > a \sqrt{\frac{\sigma}{\gamma}}\) holds, we can use Theorem 4.4 provided that \(\sigma < ca \sqrt{\frac{\sigma}{\gamma}}\), obtaining a solution qualitatively similar to Figure 10.

Figure 10. Graphs of the solution \((u, S)\) of (29) obtained by applying (45) to the solution of Theorem 4.4.

Note that here the \(S\) component is logarithmically convex in \([s_-, s_+]\) and also has two singularities in its second derivative (\(S\) is not \(C^2\)) at \(s_-\) and \(s_+\). Again, we only know from the \(u\) component of the solution its behavior around the extremes of the interval. In this case, the shape of the flux saturation is the reverse of the previous one.

Finally, let us mention that the previous arguments can be extended to more general flux saturated mechanics which are presented in the introduction by assuming that \(\Phi^{-1}\) is integrable. However, there the calculations are less explicit than in this particular case. We refer to Remark 4.3 for more details.

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