JOSEFSON-NISSENZWEIG PROPERTY FOR $C_p$-SPACES

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To the memory of our Friend Professor Mati Rubin

Abstract. The famous Rosenthal-Lacey theorem asserts that for each infinite compact space $K$ the Banach space $C(K)$ admits a quotient which is either a copy of $c_0$ or $\ell_2$. The aim of the paper is to study a natural variant of this result for the space $C_p(X)$ of continuous real-valued maps on $X$ with the pointwise topology. Following famous Josefson-Nissenzweig theorem for infinite-dimensional Banach spaces we introduce a corresponding property (called Josefson-Nissenzweig property, briefly, the JNP) for $C_p$-spaces. We prove: For a Tychonoff space $X$ the space $C_p(X)$ satisfies the JNP if and only if $C_p(X)$ has a quotient isomorphic to $c_0$ (with the product topology of $\mathbb{R}^\mathbb{N}$) if and only if $C_p(X)$ contains a complemented subspace, isomorphic to $c_0$. For a pseudocompact space $X$ the space $C_p(X)$ has the JNP if and only if $C_p(X)$ has a complemented metrizable infinite-dimensional subspace. This applies to show that for a Tychonoff space $X$ the space $C_p(X)$ has a complemented subspace isomorphic to $\mathbb{R}^\mathbb{N}$ or $c_0$ if and only if $X$ is not pseudocompact or $C_p(X)$ has the JNP. The space $C_p(\beta\mathbb{N})$ contains a subspace isomorphic to $c_0$ and admits a quotient isomorphic to $\ell_\infty$ but fails to have a quotient isomorphic to $c_0$. An example of a compact space $K$ without infinite convergent sequences with $C_p(K)$ containing a complemented subspace isomorphic to $c_0$ is constructed.

1. Introduction and the main problem

Let $X$ be a Tychonoff space. By $C_p(X)$ we denote the space of real-valued continuous functions on $X$ endowed with the pointwise topology.

We will need the following simple observation stating that each metrizable (linear) quotient $C_p(X)/Z$ of $C_p(X)$ by a closed vector subspace $Z$ of $C_p(X)$ is separable. Indeed, this follows from the separability of metizable spaces of countable cardinality and the fact that $C_p(X)$ has countable cellularity, being a dense subspace of $\mathbb{R}^X$, see [2].

The classic Rosenthal-Lacey theorem, see [24], [16], and [20], asserts that the Banach space $C(K)$ of continuous real-valued maps on an infinite compact space $K$ has a quotient isomorphic to $c_0$ or $\ell_2$, or equivalently, there exists a continuous linear (and open; by the open mapping Banach theorem) map from $C(K)$ onto $c_0$ or $\ell_2$.

This theorem motivates the following natural question for spaces $C_p(X)$.

Problem 1. For which compact spaces $K$ any of the following equivalent conditions holds:

(1) The space $C_p(K)$ can be mapped onto an infinite dimensional metrizable locally convex space under a continuous open linear map.
(2) The space $C_p(K)$ can be mapped onto an infinite dimensional metrizable separable locally convex space under a continuous open linear map.
(3) The space $C_p(K)$ has an infinite dimensional metrizable quotient.
(4) The space $C_p(K)$ has an infinite dimensional metrizable separable quotient.
(5) The space $C_p(K)$ has a quotient isomorphic to a dense subspace of $\mathbb{R}^\mathbb{N}$.

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Note that there is a continuous linear map from a real topological vector space $E$ onto a dense subspace of $\mathbb{R}^N$ if and only if the continuous dual $E'$ is infinite dimensional. Thus when $K$ is infinite, (1) and (2) hold provided we delete “open” in both cases. When we retain “open” and delete “metrizable” in (2), the question is unsolved and more general: For every infinite compact set $K$, does $C_p(K)$ admit an infinite dimensional separable quotient?

In [13] it was shown that $C_p(K)$ has an infinite-dimensional separable quotient algebra if and only if $K$ contains an infinite countable closed subset. Hence $C_p(\beta\mathbb{N})$ lacks infinite-dimensional separable quotient algebras. Nevertheless, as proved in [19, Theorem 4], the space $C_p(K)$ has infinite-dimensional separable quotient for any compact space $K$ isomorphic to the subspace $\ell_1$. In this paper $c_0$ means the subspace $\{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} : x_n \to_n 0\}$ of $\mathbb{R}^\mathbb{N}$ endowed with the product topology.

2. The main results

For a Tychonoff space $X$ and a point $x \in X$ let $\delta_x : C_p(X) \to \mathbb{R}$, $\delta_x : f \mapsto f(x)$, be the Dirac measure concentrated at $x$. The linear hull $L_p(X)$ of the set $\{\delta_x : x \in X\}$ in $\mathbb{R}^{C_p(X)}$ can be identified with the dual space of $C_p(X)$.

Elements of the space $L_p(X)$ will be called finitely supported sign-measures (or simply sign-measures) on $X$.

Each $\mu \in L_p(X)$ can be uniquely written as a linear combination of Dirac measures $\mu = \sum_{x \in F} \alpha_x \delta_x$ for some finite set $F \subset X$ and some non-zero real numbers $\alpha_x$. The set $F$ is called the support of the sign-measure $\mu$ and is denoted by $\text{supp}(\mu)$. The measure $\sum_{x \in F} |\alpha_x| \delta_x$ will be denoted by $|\mu|$ and the real number $\|\mu\| = \sum_{x \in F} |\alpha_x|$ coincides with the norm of $\mu$ (in the dual Banach space $C(\beta X)^*$).

The sign-measure $\mu = \sum_{x \in F} \alpha_x \delta_x$ determines the function $\mu : 2^X \to \mathbb{R}$ defined on the power-set of $X$ and assigning to each subset $A \subset X$ the real number $\sum_{x \in A \cap F} \alpha_x$. So, a finitely supported sign-measure will be considered both as a linear functional on $C_p(X)$ and an additive function on the power-set $2^X$.

The famous Josefson-Nissenzweig theorem asserts that for each infinite-dimensional Banach space $E$ there exists a null sequence in the weak*-topology of the topological dual $E^*$ of $E$ which is of norm one in the dual norm, see for example [5].

We propose the following corresponding property for spaces $C_p(X)$.

**Definition 1.** For a Tychonoff space $X$ the space $C_p(X)$ satisfies the Josefson-Nissenzweig property (JNP in short) if there exists a sequence $(\mu_n)$ of finitely supported sign-measures on $X$ such that $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$, and $\mu_n(f) \to_n 0$ for each $f \in C_p(X)$.

Concerning the JNP of function spaces $C_p(X)$ on compacta we have the following:
(1) If a compact space $K$ contains a non-trivial convergent sequence, say $x_n \to x$, then $C_p(K)$ satisfies the JNP. This is witnessed by the weak* null sequence $(\mu_n)$ of sign-measures $\mu_n := \frac{1}{2}(\delta_{x_n} - \delta_x)$, $n \in \mathbb{N}$.

(2) The space $C_p(\beta\mathbb{N})$ does not satisfy the JNP. This follows directly from the Grothendieck theorem, see [4, Corollary 4.5.8].

(3) There exists a compact space $K$ containing a copy of $\beta\mathbb{N}$ but without non-trivial convergent sequences such that $C_p(K)$ satisfies the JNP, see Example 1 below.

Consequently, if compact $K$ contains an infinite convergent sequence $x_n \to x$, then $C_p(K)$ satisfies the JNP with $C_p(Z)$ complemented in $C_p(K)$ and isomorphic to $c_0$, where $Z := \{x\} \cup \{x_n\}_{n \in \mathbb{N}}$. However for every infinite compact $K$ the space $C_p(K)$ contains a subspace isomorphic to $c_0$ but not necessary complemented in $C_p(K)$. Nevertheless, there exists a compact space $K$ without infinite convergent sequences and such that $C_p(K)$ enjoys the JNP (hence contains complemented subspaces isomorphic to $c_0$, as follows from Theorem 1 below).

It turns out that the Josefson-Nissenzweig property characterizes an interesting case related with Problem 1.

**Theorem 1.** For a Tychonoff space $X$ the following conditions are equivalent:

1. $C_p(X)$ satisfies the JNP;
2. $C_p(X)$ contains a complemented subspace isomorphic to $c_0$;
3. $C_p(X)$ contains a quotient isomorphic to $c_0$.

If the space $X$ is pseudocompact, then the conditions (1)–(3) are equivalent to

4. $C_p(X)$ contains a complemented infinite-dimensional metrizable subspace;
5. $C_p(X)$ contains a complemented infinite-dimensional separable subspace;
6. $C_p(X)$ has an infinite-dimensional Polishable quotient.

We recall that a locally convex space $X$ is Polishable if $X$ admits a stronger separable Fréchet (= complete metrizable) locally convex topology. Equivalently, Polishable locally convex spaces can be defined as images of separable Fréchet spaces under continuous linear maps. Clearly, the subspace $c_0$ of $\mathbb{R}^\mathbb{N}$ is Polishable.

A topological space $X$ is pseudocompact if it is Tychonoff and each continuous real-valued function on $X$ is bounded. It is known (see [3]) that a Tychonoff space $X$ is not pseudocompact if and only if $C_p(X)$ contains a complemented copy of $\mathbb{R}^\mathbb{N}$. Combining this characterization with Theorem 1 we obtain another characterization related to Problem 1.

**Corollary 1.** For a Tychonoff space $X$ the following conditions are equivalent:

1. $C_p(X)$ has an infinite-dimensional Polishable quotient;
2. $C_p(X)$ contains a complemented infinite-dimensional Polishable subspace;
3. $C_p(X)$ contains a complemented subspace isomorphic to $\mathbb{R}^\mathbb{N}$ or $c_0$;
4. $X$ is not pseudocompact or $C_p(X)$ has the JNP.

**Corollary 2.** The space $C_p(\beta\mathbb{N})$

1. has a quotient isomorphic to $\ell_\infty$;
2. contains a subspace isomorphic to $c_0$;
3. has no quotient isomorphic to $c_0$;
4. has no Polishable infinite-dimensional quotients;
5. contains no complemented separable infinite-dimensional subspaces.

Indeed, the first claim follows from [3, Proposition], the others follow from Theorem 1 and the statement (1) after Definition 1.
In the final Section 5 we shall characterize Tychonoff spaces whose function space \( C_p(X) \) is Polish and prove the following theorem.

**Theorem 2.** For a Tychonoff space \( X \) the following conditions are equivalent:

1. \( C_p(X) \) is Polishable;
2. \( C_k(X) \) is Polish;
3. \( X \) is a submetrizable hemicompact \( k \)-space.

In this theorem \( C_k(X) \) denotes the space of continuous real-valued functions on \( X \), endowed with the compact-open topology. It should be mentioned that a locally convex space is Polish if and only if it is a separable Fréchet space, by using, for example, the Birkhoff-Kakutani theorem [14, Theorem 9.1].

3. **Proof of Theorem 1**

We start with the following

**Lemma 1.** Let a Tychonoff space \( X \) be continuously embedded into a compact Hausdorff space \( K \). Let \( (\mu_n) \) be a sequence of finitely supported sign-measures on \( X \) (and so, on \( K \)) such that

1. \( \|\mu_n\| = 1 \) for all \( n \in \mathbb{N} \), and
2. \( \mu_n(f) \to_n 0 \) for all \( f \in C(K) \).

Then there exists an infinite subset \( \Omega \) of \( \mathbb{N} \) such that

(a) the closed subspace \( Z = \bigcap_{k \in \Omega} \{f \in C_p(X) : \mu_k(f) = 0\} \) of \( C_p(X) \) is complemented in the subspace \( L = \{f \in C_p(X) : \lim_{k \in \Omega} \mu_k(f) = 0\} \) of \( C_p(X) \);
(b) the quotient space \( L/Z \) is isomorphic to the subspace \( c_0 \) of \( \mathbb{R}^\mathbb{N} \);
(c) \( L \) contains a complemented subspace isomorphic to \( c_0 \);
(d) the quotient space \( C_p(X)/Z \) is infinite-dimensional and metrizable (and so, separable).

**Proof.** (I) First we show that the set \( M = \{\mu_n : n \in \mathbb{N}\} \) in not relatively weakly compact in the dual of the Banach space \( C(K) \). Indeed, assume by contrary that the closure \( \overline{M} \) of \( M \) in the weak topology of \( C(K)^* \) is weakly compact. Applying the Eberlein-Šmulian theorem [11, Theorem 1.6.3], we conclude that \( \overline{M} \) is weakly sequentially compact. Thus \( (\mu_n) \) has a subsequence \( (\mu_{n_k}) \) that weakly converges to some element \( \mu_0 \in C(K)^* \). Taking into account that the sequence \( (\mu_n) \) converges to zero in the weak* topology of \( C(K)^* \), we conclude that \( \mu_0 = 0 \) and hence \( (\mu_{n_k}) \) is weakly convergent to zero in \( C(K)^* \). Denote by \( S \) the countable set \( \bigcup_{n \in \mathbb{N}} \text{supp}(\mu_n) \). The measures \( \mu_n, n \in \mathbb{N} \), can be considered as elements of the unit sphere of the Banach space \( \ell_1(S) \subset C(K)^* \). By the Schur theorem [11, Theorem 2.3.6], the weakly convergent sequence \( (\mu_{n_k}) \) is convergent to zero in the norm topology of \( \ell_1(S) \), which is not possible as \( \|\mu_n\| = 1 \) for all \( n \in \mathbb{N} \). Thus the set \( M \) is not relatively weakly compact in \( C(K)^* \).

(II) By the Grothendieck theorem [11, Theorem 5.3.2] there exist a number \( \epsilon > 0 \), a sequence \( (m_n) \subset \mathbb{N} \) and a sequence \( (U_n) \) of pairwise disjoint open sets in \( K \) such that \( |\mu_{m_n}(U_n)| > \epsilon \) for any \( n \in \mathbb{N} \). Clearly, \( \lim_{n \to \infty} \mu_k(U_n) = 0 \) for any \( k \in \mathbb{N} \), since

\[
\sum_{n \in \mathbb{N}} |\mu_k(U_n)| = |\mu_k| \left( \bigcup_{n \in \mathbb{N}} U_n \right) \leq |\mu_k|(K) = 1.
\]

Thus we can assume that the sequence \( (m_n) \) is strictly increasing.

For some strictly increasing sequence \( (m_k) \subset \mathbb{N} \) we have \( U_{n_k} \cap \text{supp}(\mu_{m_{n_i}}) = \emptyset \) for all \( k, i \in \mathbb{N} \) with \( k > i \).

Put \( \nu_k = \mu_{m_{n_k}} \) and \( W_k = U_{n_k} \) for all \( k \in \mathbb{N} \). Then
(A1) \( \nu_k(f) \to k 0 \) for every \( f \in C(K) \);
(A2) \( |\nu_k(W_k)| > \epsilon \) for every \( k \in \mathbb{N} \);
(A3) \( |\nu_k(W_n)| = 0 \) for all \( k, n \in \mathbb{N} \) with \( k < n \).

(III) By induction we shall construct a decreasing sequences \((N_k)\) of infinite subsets of \( \mathbb{N} \) with \( \min N_k < \min N_{k+1} \) for \( k \in \mathbb{N} \) such that \( |\nu_n|(W_m) \leq \epsilon/3^k \) for every \( k \in \mathbb{N}, m = \min N_k, n \in N_k \) and \( n > m \). Let \( N_0 = \mathbb{N} \). Assume that for some \( k \in \mathbb{N} \) an infinite subset \( N_{k-1} \) of \( \mathbb{N} \) has been constructed. Let \( F \) be a finite subset of \( N_{k-1} \) with \(|F| > 3^k/\epsilon \) and min \( F > \min N_{k-1} \). For every \( i \in F \) consider the set

\[ \Lambda_i = \{ n \in N_{k-1} : |\nu_n|(W_n) \leq \epsilon/3^k \} . \]

For every \( n \in N_{k-1} \) we get \( |\nu_n|(X) \geq \sum_{i \in F} |\nu_n|(W_i) \). Hence there exists \( i \in F \) such that

\[ |\nu_n|(W_i) \leq 1/|F| \leq \epsilon/3^k . \]

Thus \( N_{k-1} = \bigcup_{i \in F} \Lambda_i \), so for some \( m \in F \) the set \( \Lambda_m \) is infinite. Put

\[ N_k = \{ n \in \Lambda_m : n > m \} \cup \{ m \} . \]

Then \( \min N_{k-1} < \min F \leq m = \min N_k \) and \( |\nu_n|(W_m) \leq \epsilon/3^k \) for \( n \in N_k \) with \( n > m \).

(IV) Let \( i_k = \min N_k, \lambda_k = \nu_{i_k} \) and \( V_k = W_{i_k} \) for \( k \in \mathbb{N} \). Then

(B1) \( \lambda_k(f) \to k 0 \) for every \( f \in C(K) \);
(B2) \( |\lambda_k(V_k)| > \epsilon \) for every \( k \in \mathbb{N} \);
(B3) \( |\lambda_k(V_i)| = 0 \) and \( |\lambda_l(V_k)| \leq \epsilon/3^k \) for all \( k, l \in \mathbb{N} \) with \( k < l \).

Clearly, the set

\[ \Omega = \{ n \in \mathbb{N} : \mu_n = \lambda_k \text{ for some } k \in \mathbb{N} \} \]

is infinite. Put

\[ Z = \bigcap_{n \in \mathbb{N}} \{ f \in C_p(X) : \lambda_n(f) = 0 \} \]

and \( L = \{ f \in C_p(X) : \lambda_n(f) \to n 0 \} \). Clearly, \( Z \) and \( L \) are subspaces of \( C_p(X) \) and \( Z \) is closed in \( L \) and in \( C_p(X) \). The linear operator

\[ S : L \to c_0, \ S : f \mapsto \lambda_n(f) , \]

is continuous and \( \ker S = Z \).

We shall construct a linear continuous map \( P : c_0 \to L \) such that \( S \circ P \) is the identity map on \( c_0 \). For every \( k \in \mathbb{N} \) there exists a continuous function \( \varphi_k : K \to [-1, 1] \) such that

\[ \varphi_k(s) = \lambda_k(V_k)/|\lambda_k(V_k)| \]

for \( s \in V_k \cap \text{supp}(\lambda_k) \) and \( \varphi_k(s) = 0 \) for \( s \in (K \setminus V_k) \). Then

\[ \lambda_k(\varphi_k) = |\lambda_k(V_k)| > \epsilon , \]

\( \lambda_n(\varphi_k) = 0 \) for all \( n, k \in \mathbb{N} \) with \( n < k \) and

\[ |\lambda_n(\varphi_k)| \leq |\lambda_n|(V_k) \leq \epsilon/3^k \]

for all \( n, k \in \mathbb{N} \) with \( n > k \).

(V) Let \( (x_n) \in c_0 \). Define a sequence \( (x'_n) \in \mathbb{R}^\mathbb{N} \) by the recursive formula

\[ x'_n := [x_n - \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k)]/\lambda_n(\varphi_n) \text{ for } n \in \mathbb{N} . \]

We shall prove that \( (x'_n) \in c_0 \).
First we show that \( \sup_n |x'_n| < \infty \). Since \((x_n) \in c_0\), there exists \( m \in \mathbb{N} \) such that \( \sup_{n \geq m} |x_n| < \epsilon \). Put
\[
M_n = \max\{2, \max_{1 \leq k < n} |x'_k|\}
\]
for \( n \geq 2 \). For every \( n > m \) we get
\[
|x'_n| = |x_n - \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k)|/\lambda_n(\varphi_n) \leq [\epsilon + M_n \sum_{1 \leq k < n} \epsilon/3^k]/\epsilon \leq 1 + M_n/2 \leq M_n.
\]
Hence \( M_{n+1} = \max\{M_n, |x'_n|\} \leq M_n \) for \( n > m \). Thus \( d := \sup_n |x'_n| \leq M_{m+1} < \infty \).

Now we show that \( x'_n \rightarrow_n 0 \). Given any \( \delta > 0 \), find \( v \in \mathbb{N} \) such that \( d < 3^v \delta \). Since \((x_n) \in c_0\) and \( \lambda_n(f) \rightarrow_n 0 \) for any \( f \in C(K) \), there exists \( m > v \) such that for every \( n \geq m \)
\[
|x_n| < \delta \epsilon \quad \text{and} \quad d \sum_{1 \leq k \leq n} |\lambda_n(\varphi_k)| < \delta \epsilon.
\]
Then for \( n \geq m \) we obtain
\[
|x'_n| \leq \left[ |x_n| + \sum_{1 \leq k \leq v} |x'_k| |\lambda_n(\varphi_k)| + \sum_{v < k < n} |x'_k| |\lambda_n(\varphi_k)| \right]/\lambda_n(\varphi_n) \leq \delta \epsilon + \sum_{1 \leq k \leq v} d |\lambda_n(\varphi_k)| + \sum_{v < k < n} d |\lambda_n(\varphi_k)| / \epsilon \leq \delta + \delta + \sum_{v < k < n} d/3^k < 2\delta + d/3^v < 3\delta.
\]

Thus \((x'_n) \in c_0\).

Clearly, the operator
\[
\Theta : c_0 \rightarrow c_0, \quad \Theta : (x_n) \mapsto (x'_n),
\]
is linear and continuous. We prove that \( \Theta \) is surjective. Let \( (y_n) \in c_0 \). Set \( t = \sup_n |y_n| \). Let
\[
x_n = \sum_{k=1}^n \lambda_n(\varphi_k) y_k \quad \text{for} \quad n \in \mathbb{N}.
\]
First we show that \((x_n) \in c_0\). Given any \( \delta > 0 \), find \( v \in \mathbb{N} \) with \( \epsilon t < 3^v \delta \). Clearly, there exists \( m > v + 2 \) such that \( |y_n| < \delta \) and \( \sum_{k=1}^v |\lambda_n(\varphi_k)| < \delta \) for \( n \geq m \). Then for every \( n \geq m \) we obtain
\[
|x_n| \leq \sum_{k=1}^n |\lambda_n(\varphi_k)| |y_k| \leq \sum_{k=1}^v t |\lambda_n(\varphi_k)| + \sum_{v < k < n} t |\lambda_n(\varphi_k)| + |\lambda_n(\varphi_n)| |y_n| < \delta + \sum_{v < k < n} t \epsilon/3^k + \|\lambda_n\| |y_n| < \delta + t \epsilon/3^v + \delta < 3\delta.
\]

Thus \((x_n) \in c_0\). Clearly, \( \Theta((x_n)_{n \in \mathbb{N}}) = (y_n)_{n \in \mathbb{N}} \); so \( \Theta \) is surjective.

(VI) The operator
\[
T : c_0 \rightarrow C_p(X), \quad T : (x_n) \mapsto \sum_{n=1}^\infty x_n \varphi_n |X|
\]
is well-defined, linear and continuous, since the functions \( \varphi_n, n \in \mathbb{N} \), have pairwise disjoint supports and \( \varphi_n(X) \subset [-1, 1], n \in \mathbb{N} \). Thus the linear operator
\[
\Phi = T \circ \Theta : c_0 \rightarrow C_p(X)
\]
is continuous.
Let $x = (x_k) \in c_0$ and $x' = (x'_k) = \Theta(x)$. Then
\[
\Phi(x) = T(x') = \sum_{k=1}^{\infty} x'_k \varphi_k | X.
\]

Using (B3) and the definition of $\Theta$, we get for every $n \in \mathbb{N}$
\[
\lambda_n(\Phi(x)) = \sum_{k=1}^{\infty} x'_k \lambda_n(\varphi_k) = x'_n \lambda_n(\varphi_n) + \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k) = (x_n - \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k)) + \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k) = x_n;
\]
so $\lambda_n(\Phi(x)) \to_n 0$. This implies that $\Phi(x) \in L$ and $S \circ \Phi(x) = x$ for every $x \in c_0$. Therefore the operator $P := \Phi \circ S: L \to L$ is a continuous linear projection with $\ker P = \ker S = Z$. Thus the subspace $Z$ is complemented in $L$. Since $S \circ \Phi$ is the identity map on $c_0$, the map $S: L \to c_0$ is open. Indeed, let $U$ be a neighbourhood of zero in $L$; then $V = \Phi^{-1}(U)$ is a neighbourhood of zero in $c_0$ and
\[
V = S \circ \Phi(V) \subset S(U).
\]

Thus the quotient space $L/Z$ is topologically isomorphic to $c_0$ and $\Phi(c_0)$ is a complemented subspace of $L$, isomorphic to $c_0$. In particular, $Z$ has infinite codimension in $L$ and in $C_p(X)$.

(VII) Finally we prove that the quotient space $C_p(X)/Z$ is first countable and hence metrizable. Let
\[
U_n = \{ f \in C_p(X) : |f(x)| < 1/n \text{ for every } x \in \bigcup_{k=1}^{n} \text{supp}(\lambda_k) \}, n \in \mathbb{N}.
\]

The first countability of the quotient space $C_p(X)/Z$ will follow as soon as for every neighbourhood $U$ of zero in $C_p(X)$ we find $n \in \mathbb{N}$ with $Z + U_n \subset Z + U$. Clearly we can assume that
\[
U = \bigcap_{x \in X} \{ f \in C_p(X) : |f(x)| < \delta \}
\]
for some finite subset $F$ of $X$ and some $\delta > 0$.

By the continuity of the operator $\Phi: c_0 \to C_p(X)$, there exists $n \in \mathbb{N}$ such that for any $y = (y_k) \in c_0$ with
\[
\max_{1 \leq k \leq n} |y_k| \leq 1/n
\]
we get $\Phi(y) \in \frac{1}{2}U$. Replacing $n$ by a larger number, we can assume that $\frac{1}{n} < \frac{1}{2}\delta$ and
\[
F \cap \bigcup_{k=1}^{\infty} \text{supp}(\lambda_k) \subset \bigcup_{k=1}^{n} \text{supp}(\lambda_k).
\]

Let $f \in U_n$. Choose a function $h \in C_p(K)$ such that $h(x) = f(x)$ for every
\[
x \in F \setminus \bigcup_{k=1}^{\infty} \text{supp}(\lambda_k)
\]
and $h(x) = 0$ for every $x \in \bigcup_{k=1}^{n} \text{supp}(\lambda_k)$. Put $g = h|X$. Then $g \in L$, since $\lambda_k(g) = \lambda_k(h) \to_k 0$. Put $y = S(g)$ and $\xi = \Phi(y)$. Since $g(x) = 0$ for
\[
x \in \bigcup_{k=1}^{n} \text{supp}(\lambda_k),
\]
we get $\xi = \Phi(y) \in V = S \circ \Phi(V) \subset S(U)$.
we have $|λ_k(g)| = 0 < \frac{1}{n}$ for $1 \leq k \leq n$, so $\max_{1 \leq k \leq n} |y_k| < \frac{1}{n}$. Hence $ξ = Φ(y) \in \frac{1}{2} U$, so

$$\max_{x \in F} |ξ(x)| < \frac{1}{2} δ.$$  

For $ς = g - ξ$ we obtain

$$S(ς) = S(g) - S(Φ Φ(g)) = S(g) - S(g) = 0,$$

so $ς \in Z$. Moreover $f - ξ \in U$. Indeed, we have

$$|f(x) - ξ(x)| = |f(x) - g(x) + ξ(x)| = |ξ(x)| < δ$$

for $x \in F \setminus \bigcup_{k=1}^{∞} \text{supp}(λ_k)$ and

$$|f(x) - ξ(x)| = |f(x) - g(x) + ξ(x)| \leq |f(x)| + |g(x)| + |ξ(x)| < \frac{1}{n} + 0 + \frac{δ}{2} < δ$$

for every $x \in \bigcup_{k=1}^{n} \text{supp}(λ_k)$. Thus $f = ξ + (f - ξ) \in Z + U$, so $U_n \subset Z + U$. Hence $Z + U_n \subset Z + U$. □

**Lemma 2.** Let $X$ be a Tychonoff space. Each metrizable continuous image of $C_p(X)$ is separable.

**Proof.** It is well-known [10] 2.3.18 that the Tychonoff product $\mathbb{R}^X$ has countable cellularity, which means that $\mathbb{R}^X$ contains no uncountable family of pairwise disjoint non-empty open sets. Then the dense subspace $C_p(X)$ of $\mathbb{R}^X$ also has countable cellularity and so does any continuous image $Y$ of $C_p(X)$. If $Y$ is metrizable, then $Y$ is separable according to Theorem 4.1.15 in [10]. □

**Lemma 3.** Let $X$ be a pseudocompact space. A closed linear subspace $S$ of $C_p(X)$ is separable if and only if $S$ is Polishable.

**Proof.** If $S$ is Polishable, then $S$ is separable, being a continuous image of a separable Fréchet locally convex space. Now assume that $S$ is separable. Fix a countable dense subset $\{f_n\}_{n \in \mathbb{N}}$ in $S$ and consider the continuous map

$$f : X \to \mathbb{R}^N, \quad f : x \mapsto (f_n(x))_{n \in \mathbb{N}}.$$  

By the pseudocompactness of $X$ and the metrizability of $\mathbb{R}^N$, the image $M := f(X)$ is a compact metrizable space. The continuous surjective map $f : X \to M$ induces an isomorphic embedding

$$C_p f : C_p(M) \to C_p(X), \quad C_p f : φ \mapsto φ \circ f.$$  

So, we can identify the space $C_p(M)$ with its image $C_p f(C_p(M))$ in $C_p(X)$. We claim that $C_p(M)$ is closed in $C_p(X)$. Given any function $φ \in C_p(X) \setminus C_p(M)$, we should find a neighborhood $O_φ \subset C_p(X)$ of $φ$, which is disjoint with $C_p(M)$.

We claim that there exist points $x, y \in X$ such that $f(x) = f(y)$ and $φ(x) \neq φ(y)$. In the opposite case, $φ = ψ \circ f$ for some bounded function $ψ : M \to \mathbb{R}$. Let us show that the function $ψ$ is continuous. Consider the continuous map

$$h : X \to M \times \mathbb{R}, \quad h : x \mapsto (f(x), φ(x)).$$  

The pseudocompactness of $X$ implies that the image $h(X) \subset M \times \mathbb{R}$ is a compact closed subset of $M \times \mathbb{R}$. Let $pr_M : h(X) \to M$ and $pr_\mathbb{R} : h(X) \to \mathbb{R}$ be the coordinate projections. It follows that

$$pr_\mathbb{R} \circ h = φ = ψ \circ f = ψ \circ pr_M \circ h,$$

which implies that $pr_\mathbb{R} = ψ \circ pr_M$. The map $pr_M : h(X) \to M$ between the compact metrizable spaces $h(X)$ and $M$ is closed and hence is quotient. Then the continuity of the map $pr_\mathbb{R} = ψ \circ pr_M$ implies the continuity of $ψ$. Now we see that the function $φ = ψ \circ f$ belongs to the
subspace $C_p(M) \subset C_p(X)$, which contradicts the choice of $\varphi$. This contradiction shows that $\varphi(x) \neq \varphi(y)$ for some points $x, y \in X$ with $f(x) = f(y)$. Then

$$O_\varphi := \{ \phi \in C_p(X) : \phi(x) \neq \phi(y) \}$$

is a required neighborhood of $\varphi$, disjoint with $C_p(M)$.

Therefore the subspace $C_p(M)$ of $C_p(X)$ is closed and hence $C_p(M)$ contains the closure $S$ of the dense set $\{ f_n \}_{n \in \mathbb{N}}$ in $S$. Since the space $C_p(M)$ is Polishable, so is its closed subspace $S$.

Now we are at the position to prove the main Theorem

**Proof of Theorem**

First, for a Tychonoff space $X$ we prove the equivalence of the conditions:

1. $C_p(X)$ satisfies the JNP;
2. $C_p(X)$ contains a complemented subspace isomorphic to $c_0$;
3. $C_p(X)$ has a quotient isomorphic to $c_0$.

The implication (1) $\Rightarrow$ (2) follows from Lemma applied to the Stone-Čech compactification $K = \beta X$ of $X$. The implication (2) $\Rightarrow$ (3) is trivial.

To prove the implication (3) $\Rightarrow$ (1), assume that $C_p(X)$ has a quotient isomorphic to $c_0$. Then it admits an open continuous linear operator $T : C_p(X) \to c_0$. Let

$$\{ e_n^* \}_{n \in \mathbb{N}} \subset c_0^*$$

be the sequence of coordinate functional. By definition of $c_0$, $e_n^*(y) \to_n 0$ for every $y \in c_0$. For every $n \in \mathbb{N}$ consider the linear continuous functional

$$\lambda_n \in C_p(X)^*, \ \lambda_n : f \mapsto \lambda_n(f),$$

which can be thought as a finitely supported sign-measure on $X$. It follows that for every $f \in C_p(X)$ we have $\lambda_n(f) = e_n^*(T f) \to_n 0$. If $\| \lambda_n \| \not\to_n 0$, then we can find an infinite subset $\Omega \subset \mathbb{N}$ such that $\inf_{n \in \Omega} \| \lambda_n \| > 0$. For every $n \in \Omega$ put

$$\mu_n := \frac{\lambda_n}{\| \lambda_n \|} \in C_p(X)^*$$

and observe that the sequence $\{ \mu_n \}_{n \in \mathbb{N}}$ witnesses that the function space $C_p(X)$ has the JNP.

It remains to consider the case when $\| \lambda_n \| \to_n 0$. We are going to prove that the assumption $\| \lambda_n \| \to_n 0$ leads to a contradiction.

First we show that the union $S := \bigcup_{n \in \mathbb{N}} \text{supp}(\lambda_n)$ is bounded in $X$ in the sense that for any continuous function $\varphi : X \to [0, +\infty]$ the image $\varphi(S)$ is bounded in $\mathbb{R}$. To derive a contradiction, assume that for some function $\varphi \in C_p(X)$ the image $\varphi(S)$ is unbounded. Then we can find an increasing number sequence $\{ n_k \}_{k \in \mathbb{N}}$ such that

$$\max \varphi(\text{supp}(\lambda_{n_k})) > 3 + \max \varphi(\text{supp}(\lambda_{n_i}))$$

for any $i < k$.

For every $k \in \mathbb{N}$ choose a point $x_k \in \text{supp}(\lambda_{n_k})$ with

$$\varphi(x_k) = \max \varphi(\text{supp}(\lambda_{n_k})).$$

It follows that $\varphi(x_k) > 3 + \varphi(x_i)$ for every $i < k$. Since the space $X$ is Tychonoff, for every $k \in \mathbb{N}$ we can find an open neighborhood $U_k \subset \{ x \in X : |\varphi(x) - \varphi(x_k)| < 1 \}$ of $x_k$ such that $U_k \cap \text{supp}(\lambda_{n_k}) = \{ x_k \}$. Also find a continuous function $\psi_k : X \to [0, 1]$ such that $\psi_k(x_k) = 1$ and $\psi_k(X \setminus U_k) \subset \{ 0 \}$.

Inductively, choose a sequence of positive real numbers $\{ r_k \}_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$

$$r_k |\lambda_{n_k}(\{ x_k \})| > 1 + \sum_{i < k} \sum_{x \in \text{supp}(\lambda_{n_k}) \setminus \{ x_k \}} r_i \psi_i(x) |\lambda_{n_k}(\{ x \})|. $$
Since the family \((U_k)_{k \in \mathbb{N}}\) is discrete, the function
\[
\psi : X \to \mathbb{R}, \quad \psi : x \mapsto \sum_{k=1}^{\infty} r_k \psi_k(x)
\]
is well-defined and continuous. It follows that for every \(k \in \mathbb{N}\) and \(i > k\) we have \(U_i \cap \text{supp}(\lambda_{n_k}) = \emptyset\) and hence
\[
|\lambda_{n_k}(\psi)| = |\sum_{i \leq k} r_i \lambda_{n_k}(\psi_i)| \geq r_k |\lambda_{n_k}(\psi)| - \sum_{i < k} r_i |\lambda_{n_k}(\psi_i)| \geq
\]
\[
\geq r_k |\lambda_{n_k}(\{x_k\})| - \sum_{i < k} \sum_{x \in \text{supp}(\lambda_{n_k}) \setminus \{x_k\}} r_i \psi_i(x) : |\lambda_{n_k}(\{x\})| > 1.
\]
But this contradicts \(\lambda_n(\psi) \to_n 0\). This contradiction shows that the set \(S = \bigcup_{k \in \mathbb{N}} \text{supp}(\lambda_k)\) is bounded in \(X\) and so is its closure \(\bar{S}\) in \(X\).

Consider the space
\[
C_p(X|\bar{S}) = \{ f|\bar{S} : f \in C_p(X) \} \subset \mathbb{R}^\mathbb{S}
\]
and observe that the restriction operator \(R : C_p(X) \to C_p(X|\bar{S}; X), R : f \mapsto f|\bar{S}\), is continuous and open. Each sign-measure \(\lambda_n\) has support
\[
\text{supp}(\lambda_n) \subset S \subset \bar{S}
\]
and hence can be considered as a linear continuous functional on \(C_p(X|\bar{S})\). Then the operator \(\bar{T} : C_p(X) \to c_0, \bar{T} : f \mapsto (\lambda_n(f))_{n \in \mathbb{N}}\), is well-defined and linear. Since \(T = \bar{T} \circ R\) is open and continuous, so is the operator \(\bar{T} : C_p(X|\bar{S}) \to c_0\). Let \(C(X|\bar{S})\) denote the space \(C_p(X|\bar{S})\) endowed with the sup-norm
\[
\|f\|_\infty = \sup_{x \in \bar{S}} |f(x)|.
\]
This norm is well-defined since the set \(\bar{S}\) is bounded in \(X\). The completion \(\overline{C(X|\bar{S})}\) of the normed space \(C(X|\bar{S})\) can be identified with a closed subspace of the Banach space \(C_b(\bar{S})\) of bounded continuous functions on \(\bar{S}\), endowed with the sup-norm. It follows from \(\|\lambda_n\| \to_n 0\) and \(\lambda_n(f) \to_n 0\) for all \(f \in C(X|\bar{S})\) that \(\lambda_n(f) \to_n 0\) for all
\[
f \in \overline{C(X|\bar{S})} \subset C_b(\bar{S}).
\]
So, \(\Lambda : \overline{C(X|\bar{S})} \to c_0, \Lambda : f \mapsto (\lambda_n(f))_{n \in \mathbb{N}}\), is a well-defined continuous operator such that \(T = \Lambda \circ R\). It follows that the operator
\[
\Lambda : \overline{C(X|\bar{S})} \to (c_0, \| \cdot \|)
\]
to \(c_0\) endowed with its standard norm \(\|x\| = \sup_{n \in \mathbb{N}} |e_n(x)|\) has closed graph and hence is continuous and open (being surjective). Then the image \(\Lambda(B_1)\) of the unit ball
\[
B_1 = \{ f \in \overline{C(X|\bar{S})} : \|f\|_\infty < 1\}
\]
contains some closed \(\varepsilon\)-ball \(B_\varepsilon := \{ x \in c_0 : \|x\| \leq \varepsilon\}\) in the Banach space \((c_0, \| \cdot \|)\). Since \(\|\lambda_n\| \to_n 0\), we can find \(n \in \mathbb{N}\) such that \(\|\lambda_n\| < \varepsilon\). Next, find an element \(y \in B_\varepsilon \subset c_0\) such that \(\|y\| = e_n(\varepsilon)\). Since \(y \in B_\varepsilon \subset \Lambda(B_1)\), there exists a point \(x \in B_1\) such that \(\Lambda(x) = y\). Then
\[
\varepsilon = e_n(\varepsilon) = \lambda_n(x) \leq \|\lambda_n\| \cdot \|x\| < \varepsilon
\]
and this contradiction completes the proof of the implication \((3) \Rightarrow (1)\).

Now assuming that the space \(X\) is pseudocompact, we shall prove that the conditions \((1)\)–\((3)\) are equivalent to
(4) $C_p(X)$ contains a complemented infinite-dimensional metrizable subspace;

(5) $C_p(X)$ contains a complemented infinite-dimensional separable subspace;

(6) $C_p(X)$ has an infinite-dimensional Polishable quotient.

It suffices to prove the implications $(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1)$. The implication $(2) \Rightarrow (4)$ is trivial and $(4) \Rightarrow (5) \Rightarrow (6)$ follow from Lemmas 2 and 3 respectively.

$(6) \Rightarrow (1)$: Assume that the space $C_p(X)$ contains a closed subspace $Z$ of infinite codimension such that the quotient space $E := C_p(X)/Z$ is Polishable. Denote by $\tau_p$ the quotient topology of $C_p(X)/Z$ and by $\tau_0 \supset \tau_p$ a stronger separable Fréchet locally convex topology on $E$. Denote by $\tau_\infty$ the topology of the quotient Banach space $C(X)/Z$. Here $C(X)$ is endowed with the sup-norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$ (which is well-defined as $X$ is pseudocompact).

The identity maps between $(E, \tau_0)$ and $(E, \tau_\infty)$ have closed graphs, since $\tau_p \subseteq \tau_0 \cap \tau_\infty$. Using the Closed Graph Theorem we infer that the topologies $\tau_0$ and $\tau_\infty$ are equal. Let $G$ be a countable subset of $C(X)$ such that the set $\{g + Z : g \in G\}$ is dense in the Banach space $C(X)/Z$. Then the set

$$G + Z = \{g + z : g \in G, \ z \in Z\}$$

is dense in $C(X)$. Let $(g_n)_{n \in \mathbb{N}}$ be a linearly independent sequence in $G$ such that its linear span $G_0$ has $G_0 \cap Z = \{0\}$ and $G_0 + Z = G + Z$. Let $f_1 = g_1$ and $\nu_1 \in C_p(X)^*$ with $\nu_1|Z = 0$ such that $\nu_1(f_1) = 1$.

Assume that for some $n \in \mathbb{N}$ we have chosen

$$(f_1, \nu_1), \ldots, (f_n, \nu_n) \in C_p(X) \times C_p(X)^*$$

such that

$$\text{lin}\{f_1, \ldots, f_n\} = \text{lin}\{g_1, \ldots, g_n\}$$

and

$$\nu_j|Z = 0, \ \nu_j(f_i) = \delta_{j,i} \text{ for all } i, j \in \{1, \ldots, n\}.$$  

Put

$$f_{n+1} = g_{n+1} - \sum_{i=1}^{n} \nu_i(g_{n+1})f_i.$$  

Then

$$\text{lin}\{f_1, \ldots, f_{n+1}\} = \text{lin}\{g_1, \ldots, g_{n+1}\}$$

and $\nu_j(f_{n+1}) = 0$ for $1 \leq j \leq n$. Let $\nu_{n+1} \in C_p(X)^*$ with $\nu_{n+1}|Z = 0$ such that $\nu_{n+1}(f_i) = 0$ for $1 \leq i \leq n$ and $\nu_{n+1}(f_{n+1}) = 1$.

Continuing on this way we can construct inductively a biorthogonal sequence $((f_n, \nu_n))_{n \in \mathbb{N}}$ in $C_p(X) \times C_p(X)^*$ such that $\text{lin}\{f_n : n \in \mathbb{N}\} = \text{lin}\{g_n : n \in \mathbb{N}\}$ and $\nu_n|Z = 0, \nu_n(f_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$. Then $\text{lin}\{f_n : n \in \mathbb{N}\} + Z$ is dense in $C_u(X)$. Let $\mu_n = \nu_n/\|\nu_n\|$ for $n \in \mathbb{N}$. Then $\|\mu_n\| = 1$ and $\mu_n(f_m) = 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

We prove that $\mu_n(f) \rightarrow 0$ for every $f \in C_p(X)$. Given any $f \in C(X)$ and $\varepsilon > 0$, find $m \in \mathbb{N}$ and $g \in \text{lin}\{f_1, \ldots, f_m\} + Z$ with $d(f, g) < \varepsilon$; clearly $d(f, g) = \|f - g\|_\infty$. Then $\mu_n(g) = 0$ for $n > m$, so

$$|\mu_n(f)| = |\mu_n(f - g)| \leq \|\mu_n\|\|f - g\|_\infty < \varepsilon$$

for $n > m$. Thus $\mu_n(f) \rightarrow 0$, which means that the space $C_p(X)$ has the JNP.  

$\square$
4. An example of Plebanek

In this section we describe the following example suggested to the authors by Grzegorz Plebanek [23].

Example 1 (Plebanek). There exists a compact Hausdorff space $K$ such that

1. $K$ contains no nontrivial converging sequences but contains a copy of $\beta\mathbb{N}$;
2. the function space $C_p(K)$ has the JNP.

We need some facts to present the construction of the space $K$. By definition, the asymptotic density of a subset $A \subset \mathbb{N}$ is the limit

$$d(A) := \lim_{n \to \infty} \frac{A \cap [1, n]}{n}$$

if this limit exists. The family $\mathcal{Z} = \{ A \subset \mathbb{N} : d(A) = 0 \}$ of sets of asymptotic density zero in $\mathbb{N}$ is an ideal on $\mathbb{N}$. Recall the following standard fact (here $A \subset^* B$ means that $A \setminus B$ is finite).

Fact 1: For any countable subfamily $\mathcal{C} \subset \mathcal{Z}$ there is a set $B \in \mathcal{Z}$ such that $C \subset^* B$ for all $C \in \mathcal{C}$.

Let $\mathfrak{A} = \{ A \subset \mathbb{N} : d(A) \in \{0, 1\} \}$ be the algebra of subsets of $\mathbb{N}$ generated by $\mathcal{Z}$. We now let $K$ be the Stone space of the algebra $\mathfrak{A}$ so we treat elements of $K$ as ultrafilters on $\mathfrak{A}$. There are three types of such $x \in K$:

1. $\{ n \} \in x$ for some $n \in \mathbb{N}$; then $x = \{ A \in \mathfrak{A} : n \in A \}$ is identified with $n$;
2. $x$ contains no finite subsets of $\mathbb{N}$ but $Z \in x$ for some $Z \in \mathcal{Z}$;
3. $Z \notin x$ for every $Z \in \mathcal{Z}$; this defines the unique

$$p = \{ A \in \mathfrak{A} : d(A) = 1 \} \in K.$$ 

To see that $K$ is the required space it is enough to check the following two facts.

Fact 2. The space $K$ contains no nontrivial converging sequence.

Proof. In fact we check that every infinite $X \subset K$ contains an infinite set $Y$ such that $\bar{Y}$ is homeomorphic to $\beta\mathbb{N}$. Note first that for every $Z \in \mathcal{Z}$, the corresponding clopen set

$$\hat{Z} = \{ x \in K : Z \subset x \},$$

is homeomorphic to $\beta\mathbb{N}$ because $\{ A \in \mathfrak{A} : A \subset Z \} = P(Z)$.

For an infinite set $X \subset K$, we have two cases:

Case 1, $X \cap \mathbb{N}$ is infinite. There is an infinite $Z \subset X \cap \mathbb{N}$ having density zero. Then every subset of $Z$ is in $\mathfrak{A}$, which implies that $\overline{Z} \cong \beta\mathbb{N}$.

Case 2, $X \cap (K \setminus \mathbb{N})$ is infinite. Let us fix a sequence of different $x_n \in X \cap (K \setminus \mathbb{N})$ such that $x_n \notin p$ for every $n$. Then for every $n$ we have $Z_n \in x_n$ for some $Z_n \in \mathcal{Z}$. Take $B \in \mathcal{Z}$ as in Fact 1. Then $B \in x_n$ because $x_n$ is a nonprincipal ultrafilter on $\mathfrak{A}$ so $A_n \setminus B \notin x_n$. Again, we conclude that $\{ x_n : n \in \mathbb{N} \}$ is $\beta\mathbb{N}$.

$$\nabla$$

Fact 3: If $\nu_n = \frac{1}{n} \sum_{k \leq n} \delta_k$ and $\mu_n = \frac{1}{2}(\nu_n - \delta_p)$ for $n \in \mathbb{N}$, then $\nu_n(f) \to_n \delta_p(f)$ and $\mu_n(f) \to_n 0$ for every $f \in C(K)$.

Proof. Observe $\nu_n(A) \to_n d(A)$ for every $A \in \mathfrak{A}$ since elements of $\mathfrak{A}$ have asymptotic density either 0 or 1. This means that, when we treat $\nu_n$ as measures on $K$ then $\nu_n(V)$ converges to $\delta_p(V)$ for every clopen set $V \subset K$. This implies the assertion since every continuous function on $K$ can be uniformly approximated by simple functions built from clopens.
5. Proof of Theorem 2

Let us recall that a topological space \( X \) is called

- **submetrizable** if \( X \) admits a continuous metric;
- **hemicompact** if \( X \) has a countable family \( K \) of compact sets such that each compact subset of \( X \) is contained in some compact set \( K \in K \);
- a **\( k \)-space** if a subset \( F \subset X \) is closed if and only if for every compact subset \( K \subset X \) the intersection \( F \cap K \) is closed in \( K \).

In order to prove Theorem 2 we should check the equivalence of the following conditions for every Tychonoff space \( X \):

1. \( X \) is a submetrizable hemicompact \( k \)-space;
2. \( C_k(X) \) is Polish;
3. \( C_p(X) \) is Polishable.

(1) \( \Rightarrow \) (2): If \( X \) is a submetrizable hemicompact \( k \)-space, then \( X = \bigcup_{n \in \omega} X_n \) for some increasing sequence \( (X_n)_{n \in \omega} \) of compact metrizable spaces such that each compact subset of \( X \) is contained in some compact set \( X_n \). Then the function space \( C_k(X) \) is Polish, being topologically isomorphic to the closed subspace

\[
\{ (f_n)_{n \in \omega} \in \prod_{n \in \omega} C_k(X_n) : \forall n \in \omega \ f_{n+1} | X_n = f_n \}
\]

of the countable product \( \prod_{n=1}^\infty C_k(X_n) \) of separable Banach spaces.

(2) \( \Rightarrow \) (1): If the function space \( C_k(X) \) is Polish, then by Theorem 4.2 in [21], \( X \) is a hemicompact \( k \)-space. Taking into account that the space \( C_p(X) \) is a continuous image of the space \( C_k(X) \), we conclude that \( C_p(X) \) has countable network and by [2] I.1.3, the space \( X \) has countable network. By [21, 2.9], the space \( X \) is submetrizable.

The implication (2) \( \Rightarrow \) (3) follows from the continuity of the identity map \( C_k(X) \to C_p(X) \).

(3) \( \Rightarrow \) (2): Assume that the space \( C_p(X) \) is Polishable and fix a stronger Polish locally convex topology \( \tau \) on \( C_p(X) \). Let \( C_\tau(X) \) denote the separable Fréchet space \( (C_p(X), \tau) \). By \( \tau_k \) denote the compact open topology of \( C_k(X) \). Taking into account that the space \( C_p(X) \) is a continuous image of the Polish space \( C_\tau(X) \), we conclude that \( C_p(X) \) has countable network and by [2] I.1.3, the space \( X \) has countable network and hence is Lindelöf. By the normality (and the Lindelöf property) of \( X \), each closed bounded set in \( X \) is countably compact (and hence compact). So \( X \) is a \( \mu \)-space. By Theorem 10.1.20 in [22] Theorem 10.1.20 the function space \( C_k(X) \) is barrelled. The continuity of the identity maps \( C_k(X) \to C_p(X) \) and \( C_\tau(X) \to C_p(X) \) implies that the identity map \( C_k(X) \to C_\tau(X) \) has closed graph. Since \( C_k(X) \) is barrelled and \( C_\tau(X) \) is Fréchet, we can apply the Closed Graph Theorem 4.1.10 in [22] and conclude that the identity map \( C_k(X) \to C_\tau(X) \) is continuous.

Next, we show that the identity map \( C_\tau(X) \to C_k(X) \) is continuous. Given any compact set \( K \subset X \) and any \( \varepsilon > 0 \) we have to find a neighborhood \( U \subset C_\tau(X) \) of zero such that

\[
U \subset \{ f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon) \}.
\]

The continuity of the restriction operator \( R : C_p(X) \to C_p(K), R : f \mapsto f|K \), and the continuity of the identity map \( C_\tau(X) \to C_p(X) \) imply that the restriction operator \( R : C_\tau(X) \to C_p(K) \) is continuous and hence has closed graph. The continuity of the identity map \( C_k(K) \to C_p(K) \) implies that \( R \) seen as an operator \( R : C_\tau(X) \to C_k(K) \) still has closed graph. Since the spaces \( C_\tau(X) \) and \( C_k(K) \) are Fréchet, the Closed Graph Theorem 1.2.19 in [22] implies that the restriction operator \( R : C_\tau(X) \to C_k(K) \) is continuous. So, there exists a neighborhood
$U \subset C_\tau(X)$ of zero such that
$$R(U) \subset \{ f \in C_k(K) : f(K) \subset (-\varepsilon, \varepsilon) \}.$$  
Then $U \subset \{ f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon) \}$ and we are done. Hence $\tau = \tau_k$ is a Polish locally convex topology as claimed.

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