Concerning Infeasibility of the Wave Functions of the Universe

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Abstract

Difficulties with finding the general exact solutions to the Wheeler-DeWitt equation, i.e. the wave functions of the Universe, are known and well documented. However, the present paper draws attention to a completely different matter, which is rarely if ever discussed in relation to this equation, namely, the time complexity of the Wheeler-DeWitt equation, that is, the time required to exactly solve the equation for a given universe. As it is shown in the paper, whatever generic exact algorithm is used to solve the equation, most likely such an algorithm cannot be faster than brute force, which makes the wave functions of the Universe infeasible.

Keywords: Quantum gravity, Wheeler-DeWitt equation, Wave functions of the Universe, Time complexity

1 Introduction

Even with the complete lack of observational and experimental evidence that clearly contradicts to either Einstein’s theory of general relativity (GR) or quantum mechanics (QM) and, as a result, with no compelling reason to adopt another theory, the aim of describing the quantum behavior of the gravitational field – called a quantum theory of gravity – is regarded as one of the biggest tasks of the modern physics. This is probably so because it is widely believed that interplay between GR and QM cannot be avoided if we want to understand how the Universe works [1, 2].

In the literature, one can find a number of different types of logic rationalizing the necessity of reconciling the laws of QM with GR (see, for example, papers [3, 4] that catalog most of them). Yet, the simplest (and perhaps the most convincing) reason behind the need for quantization of the gravitational field is the quantum superposition principle resulting from the property of linearity of the solutions to the Schrödinger equation – the central equation of QM. Certainly, if Schrödinger’s equation – established and empirically confirmed for systems made up of a small number of microscopic constituent particles $N$ – continues to be valid even for systems containing a number of those particles as large as (or even larger than) Avogadro’s number $N_A \approx 10^{24}$, then the quantum superposition principle should be applicable to general states of macroscopic systems and, consequently, to the gravitational field produced by macroscopic systems.

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Theoretically however, it is quite conceivable that due to certain additional nonlinear stochastic terms present in the Schrödinger equation, the linear nature of QM breaks down in situations where microscopic systems grow into macroscopic ones (i.e., where the number $N$ grows large) and thus the gravitational field that these systems produce becomes distinguishable; see, for example, papers [5, 6, 7, 8] that discuss in detail this approach called objective collapse theories or the Dynamical Reduction Program (DRP). However, no empirical hint for such a drastic modification of Schrödinger’s equation exists so far. Moreover, a modification of Schrödinger’s equation such as this involves phenomenological parameters, which, if the whole DRP is taken seriously, acquire the status of new constants of nature. Therefore, from a pragmatic point of view, it is hard to accept that in order to avoid one theoretical construct, a quantum theory of gravity, we have to agree to another, the DPR.

Withal, from a point of view of computational complexity theory, the stretch of the applicability of the quantum superposition principle far beyond its established microscopic scale is equivalent to the assumption of Schrödinger’s equation scalability on $N$ – that is, the assertion that Schrödinger’s equation is solvable in a suitably efficient and practical way even when it is applied to an arbitrary physical system with large $N$.

Truly, assume the opposite: Schrödinger’s equation is not scalable on $N$ and hence there is no practical possibility to solve this equation for an arbitrary system with large $N$ in any reasonable amount of time. In such circumstances, Schrödinger’s equation would not be able to actually describe how states of macroscopic systems (possessing $N \gtrsim 10^{24}$ microscopic constituent particles) would change with time. So, in that case, QM as a physical theory would not have the ability to predict for macroscopic systems and, for this reason, the quantum superposition principle could not be practically applicable to the description of states of macroscopic systems.

Thus, if held true, scalability on $N$ of the Schrödinger equation would necessitate the relevance of QM to the description of any macroscopic system including the entire Universe and therefore would lead with necessity to the quantization of the gravitational field. This would imply that it would be possible (albeit in principle) to find the wave functions describing the Universe – i.e., the general exact solutions to the quantum gravitational analogue of the Schrödinger equation (such as the Wheeler-DeWitt equation) – in a reasonable amount time. The aim of the present paper is to evaluate such a claim.

## 2 Quantum model of the Universe

For that purpose, let us examine a customary quantum model of the Universe. Following the accepted quantum cosmological doctrine, the quantum state of the Universe is described by a wave function $\Psi[h_{ab}, \phi]$ (called “the wave function of the Universe”), which is a functional on the three-metric $h_{ab}$ (i.e., metric on the hypersurface) and on the values of nongravitational fields symbolically denoted by $\phi$ [9, 10]. The wave function of the Universe $\Psi[h_{ab}, \phi]$ obeys the Wheeler-DeWitt (WDW) second-order functional differential equation
\[ \mathcal{H}\Psi[h_{ab}, \phi] = 0 \tag{1} \]

where \( \mathcal{H} \) stands for the Hamiltonian operator of the Universe (a first class constraint on the Universe physical states) that involves functional derivatives with respect to the metric components \( h_{ab} \) and the Hamiltonian density \( \mathcal{H}_m \) for nongravitational fields \( \phi \). The WDW equation basically says that the operator \( \mathcal{H} \) acts on the wave functional \( \Psi[h_{ab}, \phi] \), which provides all of the information about the geometry and matter content of the Universe.

For the most part, difficulties with finding the WDW general exact solutions (essential because implications for the meaning of the wave functions of the Universe must be derived from the exact solutions to the WDW equation) are known and well documented (see, for example, the paper \[11\], which in detail reviews such difficulties). Here, however, we want to draw attention to a completely different matter, which is rarely if ever discussed in relation to the WDW equation – the time complexity of this equation, i.e., the time (or the number of elementary operations) required to exactly solve this equation for a given universe.

### 3 Time complexity of the Wheeler-DeWitt equation

Obviously, the amount of time needed to exactly solve the WDW equation is subject to an algorithm used for solving the equation. So therefore, let us figure out how fast such an algorithm might be in principle.

Assume that \( A(\Psi[h_{ab}, \phi]) \) is the generic exact algorithm capable of finding the set of the general exact solutions \( \Psi[h_{ab}, \phi] \) to the WDW equation with an arbitrary Hamiltonian constraint \( \mathcal{H} \) (i.e., for an arbitrary physical universe with any geometry and any matter content). If this algorithm \( A(\Psi[h_{ab}, \phi]) \) were to exist, it would be also capable of exactly solving the Schrödinger equation for an arbitrary matter source. Let us show this.

Consider the ansatz

\[ \Psi[h_{ab}, \phi] = \exp \left( \frac{i}{\hbar} S[h_{ab}, \phi] \right) = \exp \left( \frac{i}{\hbar} (MS_0 + S_1 + M^{-1}S_2 + \ldots) \right) \tag{2} \]

where \( M = (32\pi G)^{-1}c^2 \) is the parameter, with respect to which the semiclassical expansion is performed in \[2\]. Now suppose that the Hamiltonian constraint \( \mathcal{H} \) is such that the wave function of the Universe \( \Psi[h_{ab}, \phi] \) is of the special form

\[ \Psi[h_{ab}, \phi] = \frac{1}{D[h_{ab}]} \exp \left( \frac{i}{\hbar} MS_0[h_{ab}] \right) \psi[h_{ab}, \phi] \tag{3} \]

where \( \psi[h_{ab}, \phi] \) is the wave functional at the order \( M^0 \) in the expansion \[2\].
\[ \psi[h_{ab}, \phi] = D[h_{ab}] \exp \left( \frac{i}{\hbar} S_1[\phi] \right) . \] (4)

Inserting (3) into the WDW equation we find that \( \psi[h_{ab}, \phi] \) obeys the functional Schrödinger equation in its local form for quantum fields propagating on the classical spacetimes described by \( S_0[h_{ab}] \):

\[ i\hbar \frac{\delta \psi[h_{ab}, \phi]}{\delta \tau} = H_m \psi[h_{ab}, \phi] , \] (5)

where \( \tau \) is the time functional (defined on the configuration space) generating the time parameter \( t \) in each classical spacetime (see [12] for a review of the semiclassical approximation to quantum gravity in the canonical framework).

Unlike the Schrödinger equation for a \( N \)-particle system, where a wave function \( \psi(\phi) \) evolves against a classical background potential \( U(\phi) \), the functional equation (5) is a “second-quantized” equation, therefore its eigenstate \( \psi[h_{ab}, \phi] \) is a wave functional on the configuration space, whose points are field configurations. Nevertheless, in the non-relativistic limit and for the negligible gravitational field (\( c = \infty \) and \( G = 0 \)), the field Hamiltonian \( H_m \) will take the form of an expression for the expectation value of the \( N \)-particle system’s energy, with \( \psi[h_{ab}, \phi] \) playing the role of the wave function \( \psi(\phi) \) of the system. It follows, then, that in the limits \( c = \infty \) and \( G = 0 \), by finding the solution \( \Psi[h_{ab}, \phi] \) to the WDW equation, the generic algorithm \( A(\Psi[h_{ab}, \phi]) \) yields the solution \( \psi(\phi) \) to the Schrödinger equation as well.

When evaluating the difficulty of a particular Hamiltonian, the amount of computation (measured as the time or the number of elementary operations required to verify the energy of the ground state of the given \( N \)-particle system) defines the complexity of the Hamiltonian. If for the given system the ground state energy can be verified in the time polynomial in \( N \), then the system’s Hamiltonian \( H_m \) will take the form of an expression for the expectation value of the \( N \)-particle system’s energy, with \( \psi[h_{ab}, \phi] \) playing the role of the wave function \( \psi(\phi) \) of the system. It follows, then, that in the limits \( c = \infty \) and \( G = 0 \), by finding the solution \( \Psi[h_{ab}, \phi] \) to the WDW equation, the generic algorithm \( A(\Psi[h_{ab}, \phi]) \) yields the solution \( \psi(\phi) \) to the Schrödinger equation as well.

Take, for example, the Ising Hamiltonian function \( H(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_N) \) describing the energy of configuration of a set of \( N \) spins \( \sigma_j \hbar = 2\phi_j \in \{-\hbar, +\hbar\} \) in classical Ising models of a spin glass:

\[ H(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_N) = - \sum_{j<k} A_{jk} \sigma_j \sigma_k - B \sum_j C_j \sigma_j \quad (A_{jk}, B, C_j = \text{const}) \] . (6)

Considering that all problems in the NP class can be mapped to the Schrödinger equation with the quantum version of the Ising Hamiltonian \( H(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_N) \), in which spins \( \sigma_j \) have been merely replaced by quantum operators, Pauli spin-1/2 matrices \( \sigma_j^i \),

\[ H(\sigma_1^\downarrow, \ldots, \sigma_j^\downarrow, \ldots, \sigma_N^\downarrow) \psi(\phi) = 0 \] , (7)
the Hamiltonian $H(\sigma_1^z, \ldots, \sigma_j^z, \ldots, \sigma_N^z)$ is NP-complete (see paper [13], which provides Ising formulations for many NP-complete and NP-hard problems, including all of Karp’s 21 NP-complete problems, for details).

In this way, we can conclude that as long as the generic algorithm $A(\Psi[h_{ab}, \phi])$ can solve the Schrödinger equation recovered, in the weak field approximation, from the WDW equation, this algorithm can resolve any NP-complete problem.

On the other hand, every NP-complete problem can be exactly solved by brute force, that is, by the generic and exact algorithm of enumerating and checking all possible candidate solutions to the problem. Obviously, brute force is not efficient: For example, deciding whether there is a spin configuration $\phi = (\phi_1, \ldots, \phi_j, \ldots, \phi_N)$ of a spin glass with zero energy by exhaustively checking all possible spin configurations may well require $O^*(2^N)$ elementary operations (where the $O^*$ notation is used that suppresses all factors polynomial in $N$).

Consequently, the question becomes, is it possible that the generic exact algorithm $A(\Psi[h_{ab}, \phi])$ can be significantly faster than brute force? The short answer is no, according to what now seems true, it is not possible.

Indeed, assume that the algorithm $A(\Psi[h_{ab}, \phi])$ is a sub-exponential time algorithm (i.e., faster than brute force). It follows then $A(\Psi[h_{ab}, \phi])$ can solve any NP-complete problem in sub-exponential time. This means that many computational problems known to be solved in exponential time $O^*(2^N)$ can be improved to $O^*(c^N)$ with some $c < 2$. However, such an improvement would be highly surprising since it would refute the Exponential Time Hypothesis (ETH) – the widely believed conjecture based on strong evidence that the low bound $O^*(2^N)$ matches the running time of the best possible algorithms for many computational problems (see [14, 15] for the survey of results obtained in the field of exact exponential time algorithms under the assumption of the ETH).

Therefore, almost certainly no generic algorithm for finding the general exact solutions to the WDW equation can be significantly faster than brute force. So, it is extremely probable that the wave functions of the Universe are infeasible, i.e., they cannot be calculated in any reasonable time.

4 Concluding remarks

Let us dwell for a moment on this conclusion trying to think about what it means.

Does it mean that even if we manage to overcome all the difficulties arise in merging QM with GR and finally build a definitive self-consistent theory of quantum gravity, we will know nothing more about the Universe, gravitating bodies, the motion of planets and stars than before having this theory since it will not be able to make practically meaningful predictions?

Paradoxically, the answer should be yes (the only way to dodge such an answer would be equality of the NP class to the P class of computational problems solvable in polynomial time; yet, this
equality is prevalently believed to be not true).

The conclusion about infeasibility of the wave functions of the Universe also casts doubt on the definition of the notion of an “element of physical reality” introduced by EPR paper [16]. In its original interpretation, this notion states that if without in any way disturbing a system we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. As follows, the definition of value definiteness and physical reality is rendered based on the ability to predict. On the other hand, the ability to predict means the quality of a theory that permits us to calculate the value of the system’s quantity in a reasonable amount of time. Thus, does this imply that if the theory did not allow calculating values of the given quantity in reasonable time, then this theoretical quantity would not have a counterpart in physical reality? Particularly, does this imply that the wave functions of the Universe do not correspond to any element of physical reality, inasmuch as they cannot be calculated in any reasonable time?

Furthermore, if such a ‘computational amendment’ to the definition of an element of physical reality is important and physically meaningful, should we then exclude infeasible, i.e., practically useless, solutions from all the equations of physical theories?

Clearly, those are difficult questions disturbing the very fundamentals of physical science. However, without comprehensive answers to them the further progress in quantum gravity, in all likelihood, seems impossible.

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