The Natural Operators Similar to the Twisted Courant Bracket One

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Abstract. Given natural numbers \( m \geq 3 \) and \( p \geq 3 \), all \( \mathcal{M}_f^m \)-natural operators \( A_H \) sending \( p \)-forms \( H \in \Omega^p(M) \) on \( m \)-manifolds \( M \) into bilinear operators \( A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M) \) transforming pairs of couples of vector fields and 1-forms on \( M \) into couples of vector fields and 1-forms on \( M \) are founded. If \( m \geq 3 \) and \( p \geq 3 \), then that any (similar as above) \( \mathcal{M}_f^m \)-natural operator \( A \) which is defined only for closed \( p \)-forms \( H \) can be extended uniquely to the one \( A \) which is defined for all \( p \)-forms \( H \) is observed. If \( p = 3 \) and \( m \geq 3 \), all \( \mathcal{M}_f^m \)-natural operators \( A \) (as above) such that \( A_H \) satisfies the Leibniz rule for all closed 3-forms \( H \) on \( m \)-manifolds \( M \) are extracted. The twisted Courant bracket \( [-, -]^C \) for all closed 3-forms \( H \) on \( m \)-manifolds \( M \) gives the most important example of such \( \mathcal{M}_f^m \)-natural operator \( A \).

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1. Introduction

The “doubled” tangent bundle \( T \oplus T^* \) over \( m \)-dimensional manifolds (\( m \)-manifolds) is full of interest because it has the natural inner product, and the Courant bracket, see [1]. Besides, generalized complex structures are defined on \( T \oplus T^* \), generalizing both (usual) complex and symplectic structures, see e.g. [3,4].

In Sect. 2, the description from [2] of all \( \mathcal{M}_f^m \)-natural bilinear operators

\[ A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M), \]

transforming pairs of couples of vector fields and 1-forms on \( m \)-manifolds \( M \) into couples of vector fields and 1-forms on \( M \) will be shortly cited. The most important example of such \( \mathcal{M}_f^m \)-natural bilinear operator \( A \) is given by the Courant bracket \( [-, -]^C \), see Example 2.2. This Courant bracket was used in [1] to define the concept of Dirac structures being hybrid of both symplectic and Poisson structures.
In Sect. 2 we also deduce that the “trivial” Lie algebroid \( (TM \oplus T^*M, 0, 0) \) is the only \( \mathcal{M}f_m \)-natural Lie algebroid \( (EM, [[-,-]], a) \) with \( EM := TM \oplus T^*M \).

In Sect. 3, using essentially the results from [2], if \( m \geq 3 \) and \( p \geq 3 \), we find all \( \mathcal{M}f_m \)-natural operators \( A \) sending \( p \)-forms \( H \in \Omega^p(M) \) on \( m \)-manifolds \( M \) into bilinear maps

\[
A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \rightarrow \mathcal{X}(M) \oplus \Omega^1(M).
\]

The most important example of such \( A \) is given by the \( H \)-twisted Courant bracket \([-,-]_H\) for all 3-forms \( H \) on \( m \)-manifolds \( M \), see Example 3.2. Properties of \([-,-]_H\) (as the Leibniz rule for closed 3-forms \( H \)) were used in [7,8] to define the concept of exact Courant algebroid.

In Sect. 4, we observe that if \( m \geq 3 \) and \( p \geq 3 \), then any (similar as above) \( \mathcal{M}f_m \)-natural operator \( A \) which is defined only for closed \( p \)-forms \( H \) can be extended uniquely to the one \( A \) which is defined for all \( p \)-forms \( H \).

In Sect. 5, if \( p = 3 \) we extract all \( \mathcal{M}f_m \)-natural operators \( A \) as above satisfying the Leibniz rule

\[
A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3)),
\]

for any closed \( H \in \Omega^3(M), \rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and \( M \in \text{obj}(\mathcal{M}f_m) \).

From now on, \((x^i) (i = 1, \ldots, m)\) denote the usual coordinates on \( \mathbb{R}^m \) and \( \partial_i = \frac{\partial}{\partial x^i} \) are the canonical vector fields on \( \mathbb{R}^m \).

All manifolds considered in this paper are assumed to be finite dimensional second countable Hausdorff without boundary and smooth (of class \( C^\infty \)). Maps between manifolds are assumed to be smooth (of class \( C^\infty \)).

2. The Natural Bilinear Operators Similar to the Courant Bracket

The general concept of natural operators can be found in the fundamental monograph [5]. In the paper, we need two particular cases of natural operators presented in Definitions 2.1 (below) and 3.1 (in the next section).

Let \( \mathcal{M}f_m \) be the category of \( m \)-dimensional \( C^\infty \) manifolds as objects and their immersions of class \( C^\infty \) as morphisms (\( \mathcal{M}f_m \)-maps).

**Definition 2.1.** A natural (called also \( \mathcal{M}f_m \)-natural) operator \( A \) sending pairs of couples of vector fields and 1-forms on \( m \)-manifolds \( M \) into couples of vector fields and 1-forms on \( M \) is a \( \mathcal{M}f_m \)-invariant family of operators (functions)

\[
A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \rightarrow \mathcal{X}(M) \oplus \Omega^1(M),
\]

for all \( m \)-manifolds \( M \), where \( \mathcal{X}(M) \oplus \Omega^1(M) \) is the vector space of couples \((X, \omega)\) of vector fields \( X \) on \( M \) and 1-forms \( \omega \) on \( M \). Such \( \mathcal{M}f_m \)-natural operator \( A \) is called bilinear if \( A \) is bilinear (i.e., \( A(\rho^1, -) \) and \( A(-, \rho^2) \) are linear (over the field \( \mathbb{R} \) of real numbers) functions \( \mathcal{X}(M) \oplus \Omega^1(M) \rightarrow \mathcal{X}(M) \oplus \Omega^1(M) \) for any fixed \( \rho^1, \rho^2 \in \mathcal{X}(M) \oplus \Omega^1(M) \) for any \( m \)-manifold \( M \). Such \( \mathcal{M}f_m \)-natural operator \( A \) is called skew-symmetric if \( A \) is skew-symmetric for any \( m \)-manifold \( M \).
The $\mathcal{M}_m$-invariance of $A$ means that if $(X^1 \ominus \omega^1, X^2 \ominus \omega^2)$ and $(\bar{X}^1 \oplus \bar{\omega}^1, \bar{X}^2 \oplus \bar{\omega}^2)$ are $\varphi$-related by an $\mathcal{M}_m$-map $\varphi : M \to \bar{M}$ (i.e., $\bar{X}^i \circ \varphi = T \varphi \circ X^i$ and $\bar{\omega}^i \circ \varphi = T^T \varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \ominus \omega^1, X^2 \ominus \omega^2)$ and $A(\bar{X}^1 \oplus \bar{\omega}^1, \bar{X}^2 \oplus \bar{\omega}^2)$.

The most important example of such $\mathcal{M}_m$-natural bilinear operator $A$ is given by the (skew-symmetric) Courant bracket $[-, -]^C$ for any $m$-manifold $M$.

**Example 2.2.** On the vector bundle $TM \oplus T^*M$ there exist canonical symmetric and skew-symmetric pairings

$$\langle X^1 \ominus \omega^1, X^2 \ominus \omega^2 \rangle = \frac{1}{2}(i X^2 \omega^1 \pm i X^1 \omega^2)$$

for any $X^1 \ominus \omega^1, X^2 \ominus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where $i$ is the interior derivative. Further, the (skew-symmetric) Courant bracket is given by

$$[X^1 \ominus \omega^1, X^2 \ominus \omega^2]^C = [X^1, X^2] \oplus \left(\mathcal{L}_{X^1} \omega^2 - \mathcal{L}_{X^2} \omega^1 + d \langle X^1 \ominus \omega^1, X^2 \ominus \omega^2 \rangle \right)$$

for any $X^1 \ominus \omega^1, X^2 \ominus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where $[-, -]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative and $d$ is the exterior derivative.

**Theorem 2.3** [2]. If $m \geq 2$, any $\mathcal{M}_m$-natural bilinear operator $A$ in the sense of Definition 2.1 is of the form

$$A(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left( b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle \right)$$

for (uniquely determined by $A$) real numbers $a, b_1, b_2, b_3, b_4$, where $\rho^i = X^i \ominus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ are arbitrary, and where $\langle -,- \rangle_+$ and $\langle -,- \rangle_-$ are as in Example 2.2.

**Corollary 2.4** [2]. If $m \geq 2$, any $\mathcal{M}_m$-natural skew-symmetric bilinear operator $A$ in the sense of Definition 2.1 is of the form

$$A(X^1 \ominus \omega^1, X^2 \ominus \omega^2) = a[X^1, X^2] \oplus (b \mathcal{L}_{X^1} \omega^2 - \mathcal{L}_{X^2} \omega^1 + c d \langle X^2 \ominus \omega^1, X^1 \ominus \omega^2 \rangle)$$

for (uniquely determined by $A$) real numbers $a, b, c$.

Roughly speaking, Corollary 2.4 says that if $m \geq 2$, then any $\mathcal{M}_m$-natural skew-symmetric bilinear operator $A$ in the sense of Definition 2.1 coincides with the one given by Courant bracket $[-, -]^C$ up to three real constants.

**Definition 2.5.** A $\mathcal{M}_m$-natural bilinear operator $A$ in the sense of Definition 2.1 satisfies the Leibniz rule if

$$A(\rho_1, A(\rho_2, \rho_3)) = A(A(\rho_1, \rho_2), \rho_3) + A(\rho_2, A(\rho_1, \rho_3))$$

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and all $m$-manifolds $M$.

Of course, in the case of skew-symmetric bilinear $A$ the Leibniz rule is equivalent to the Jacobi identity $\sum_{cyc} A(\rho_1, A(\rho_2, \rho_3)) = 0$. 


Example 2.6. The (not skew-symmetric) Courant bracket given by

\[ [X^1 \oplus \omega^1, X^2 \oplus \omega^2]_0 \]

\[ := [X^1, X^2] \oplus (\mathcal{L}_{X^1} \omega^2 - i_{X^2} \omega^1), \]

where \( X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \), satisfies the Leibniz rule, see [7,8].

The Courant bracket \([-,-]\) from Example 2.2 does not satisfy the Leibniz rule.

Theorem 2.7 [2]. If \( m \geq 2 \), any \( \mathcal{M}_{f_m} \)-natural bilinear operator \( A \) in the sense of Definition 2.1 satisfying the Leibniz rule is one of the following ones:

\[ A^{(1,a)}(\rho^1, \rho^2) = a[X^1, X^2] \oplus 0, \]

\[ A^{(2,a)}(\rho^1, \rho^2) = a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1} \omega^2 - \mathcal{L}_{X^2} \omega^1)), \]

\[ A^{(3,a)}(\rho^1, \rho^2) = a[X^1, X^2] \oplus a\mathcal{L}_{X^1} \omega^2, \]

\[ A^{(4,a,0)}(\rho^1, \rho^2) = a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1} \omega^2 - i_{X^2} \omega^1)), \]

where \( a \) is an arbitrary real number, and where \( \rho^1 = X^1 \oplus \omega^1 \) and \( \rho^2 = X^2 \oplus \omega^2 \).

Corollary 2.8. If \( m \geq 2 \), the Courant bracket \([-,-]_0 \) from Example 2.6 for \( m \)-manifolds \( M \) is the unique \( \mathcal{M}_{f_m} \)-natural bilinear operator \( A \) in the sense of Definition 2.1 satisfying the conditions:

(A1) \( A(\rho_1, A(\rho_2, \rho_3)) = A(A(\rho_1, \rho_2), \rho_3) + A(\rho_2, A(\rho_1, \rho_3)), \)

(A2) \( \pi A(\rho_1, \rho_2) = [\pi \rho_1, \pi \rho_2], \)

(A3) \( A(\rho_1, \rho_1) = i_0 d \langle \rho_1, \rho_1 \rangle_+, \)

for all \( \rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( m \)-manifolds \( M \), where \( \langle -,- \rangle_+ \) is the pairing of Example 2.2, \( \pi : TM \oplus T^*M \to TM \) is the fibred projection given by \( \pi(v, \omega) = v \) and \( i_0 : T^*M \to TM \oplus T^*M \) is the fibred embedding \( i_0(\omega) = (0, \omega) \).

Consequently, if \( m \geq 2 \), then a \( \mathcal{M}_{f_m} \)-natural bilinear operator \( A \) in the sense of Definition 2.1 satisfying the conditions (A1)–(A3) satisfies the conditions:

(A4) \( \pi_1 \langle \rho_2, \rho_3 \rangle_+ = \langle A(\rho_1, \rho_2), \rho_3 \rangle_+ + \langle \rho_2, A(\rho_1, \rho_3) \rangle_+, \)

(A5) \( A(\rho_1, f \rho_2) = \pi_1(f) \rho_2 + f A(\rho_1, \rho_2) \)

for all \( \rho_1, \rho_2 \in \mathcal{X}(M) \oplus \Omega^1(M) \), all \( f \in \mathcal{C}^\infty(M) \) and all \( m \)-manifolds \( M \) (i.e., putting \([[-,-]] := A \) we get an exact Courant algebroid \( E = (TM \oplus T^*M, [[-,-]], \langle-,- \rangle_+, \pi, i_0) \) in the sense of [8] for any \( m \)-manifold \( M \)).

Proof. By Theorem 2.7, the conditions (A1) and (A2) imply that \( A = A^{(1,1)} \) or \( A = A^{(2,1)} \) or \( A = A^{(3,1)} \) or \( A = A^{(4,1,0)} \). On the other hand if \( \rho_1 = X \oplus \omega, \) then \( i_0 \langle \rho_1, \rho_1 \rangle_+ = 0 \oplus di_X \omega \) and \( A^{(1,1)}(\rho_1, \rho_1) = 0 \oplus 0 \) and \( A^{(2,1)}(\rho_1, \rho_1) = 0 \oplus 0 \) and \( A^{(3,1)}(\rho_1, \rho_1) = 0 \oplus \mathcal{L}_X \omega \) and \( A^{(4,1,0)}(\rho_1, \rho_1) = 0 \oplus di_X \omega \). Then \( A = A^{(4,1,0)} \).

Corollary 2.9. If \( m \geq 2 \), any \( \mathcal{M}_{f_m} \)-natural Lie algebra brackets on \( \mathcal{X}(M) \oplus \Omega^1(M) \) [i.e., \( \mathcal{M}_{f_m} \)-natural skew-symmetric bilinear operator satisfying the
Lemma 2.10. 

$M$ must be $T$ which is fibre linear is the constant multiple of the fibre projection $\pi$. We get following two Lie algebra brackets:

\[
[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_1 = [X^1, X^2] \oplus 0, \\
[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_2 = [X^1, X^2] \oplus (L_{X^1}\omega^2 - L_{X^2}\omega^1).
\]

At the end of this section we are going to describe completely all Lie algebroids $(TM \otimes T^*M, [[-,-]], a)$ which are invariant with respect to immersions $(\mathcal{M}f_m\text{-maps})$. The concept of Lie algebroids can be found in the fundamental book [6].

Of course, the anchor $a : TM \otimes T^*M \to TM$ for all $m$-manifolds $M$ must be $\mathcal{M}f_m$-natural transformation [i.e., $Tf \circ a = a \circ (Tf \otimes T^*f)$] for any $\mathcal{M}f_m$-map $f : M \to M^1$ and fibre linear. By Corollary 2.9, $[[-, -]] = \mu([-[-,-]])_1$ or $[[-, -]]_2 = \mu([-[-,-]])_2$ for some $\mu \in \mathbb{R}$.

**Lemma 2.10.** Any $\mathcal{M}f_m$-natural transformation $a : TM \otimes T^*M \to TM$ which is fibre linear is the constant multiple of the fibre projection $\pi : TM \oplus T^*M \to TM$.

**Proof.** Clearly, $a$ is determined by the values $<\eta, a_x(v, \omega) > \in \mathbb{R}$ for all $\omega, \eta \in T^*_xM$, $v \in T_xM$, $x \in M$, $M \in \text{Obj}(\mathcal{M}f_m)$. By the standard chart arguments, we may assume $M = \mathbb{R}^m$, $x = 0$, $\eta = d_0 x^1$. We can write $<d_0 x^1, a_0(v, \omega) >= \sum_i \alpha_i v^i + \sum_j \beta_j \omega_j$, where $v^i$ are the coordinates of $v$ and $\omega_j$ are the coordinates of $\omega$, and where $\alpha_i$ and $\beta_j$ are the real numbers determined by $a_0$. Then using the invariance of $a_0$ with respect to the maps $(\tau^1 x^1, ..., \tau^m x^m)$ for $\tau^1 > 0, ..., \tau^m > 0$ we deduce that $\alpha_2 = \cdots = \alpha_m = 0$ and $\beta_1 = \cdots = \beta_m = 0$. Then the vector space of all $a$ in question is at most 1-dimensional. Thus the dimension argument completes the proof. □

So, $a = k\pi$ for some real number $k$. It must be $a([X^1 \oplus 0, X^2 \oplus 0]) = [a(X^1 \oplus 0), a(X^2 \oplus 0)]$ for any vector fields $X^1$ and $X^2$ on $M$. This gives the condition $k\mu[X^1, X^2] = k^2[X^1, X^2]$. Then $k\mu = k^2$, and then $(k = 0$ and $\mu$ arbitrary) or $(k \neq 0$ and $\mu = k$). Consider two cases:

1. $[[-, -]] = \mu([-[-,-]])_1$. Let $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$. It must be $[[\rho^1, f\rho^2]] = a(\rho^1(f)\rho^2) + f[[\rho^1, \rho^2]]$. Considering the $\Omega^1(M)$-parts of both sides of this equality we get $0 = kX^1(f)\omega^2 + 0$ for any vector fields $X^1, X^2$ on $M$ any map $f : M \to \mathbb{R}$ and any $\omega^1, \omega^2 \in \Omega^1(M)$. Then $k = 0$. Then considering the $X(M)$-parts we get $\mu[X^1, fX^2] = f\mu[X^1, X^2]$. Then $\mu X^1(f)X^2 = 0$ for all vector fields $X^1$ and $X^2$ on $M$ and all maps $f : M \to \mathbb{R}$, i.e., $\mu = 0$.

2. $[[-, -]] = \mu([-[-,-]])_2$. Let $\rho^1 = 0 \oplus \omega^1$ and $\rho^2 = X^2 \oplus 0$. It must be $[[\rho^1, f\rho^2]] = a(\rho^1(f)\rho^2) + f[[\rho^1, \rho^2]]$. Considering the $\Omega^1(M)$-parts of both sides of this equality we get $-\mu L_{fX^2}\omega^1 = -\mu fL_{X^2}\omega^1$. Then $\mu = 0$ or $d_i f_{X^2}\omega^1 + i_{fX^2}d\omega^1 = f d_i X^2\omega^1 + f i_{X^2}d\omega^1$. Putting $\omega^1 = dg$ we get $\mu = 0$ or $d(i_{fX^2}dg) = f d_i X^2 dg$. Then $\mu = 0$ or $d(fX^2 g) = f dg(X^2 g)$. Then $\mu = 0$ or $X^2(g)df = 0$ for any $X^2, g, f$ in question. Putting $X^2 = \frac{\partial}{\partial x^1}$ and $f = g = x^1$ we get $\mu = 0$ or $dx^1 = 0$. Then $\mu = 0$, and then $k = \mu = 0$.

On the other hand one can directly show that $(TM \oplus T^*M, 0([-[-,-]])_1, 0\pi)$ is a Lie algebroid. Thus we have
Proposition 2.11. If $m \geq 2$, $(TM \otimes T^*M, 0, 0)$ is the only invariant with respect to $\mathcal{M}f_m$-maps Lie algebroid $(EM, [[-,-]], a)$ with $EM = TM \oplus T^*M$.

3. The Natural Operators Similar to the Twisted Courant Bracket

Definition 3.1. A $\mathcal{M}f_m$-natural operator $A$ sending $p$-forms $H \in \Omega^p(M)$ on $m$-manifolds $M$ into bilinear operators

$$A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$A : \Omega^p(M) \to Lin_2((\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)), \mathcal{X}(M) \oplus \Omega^1(M))$$

for all $m$-manifolds $M$, where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbb{R}$) functions $U \times V \to W$ for any real vector spaces $U, V, W$.

The $\mathcal{M}f_m$-invariance of $A$ means that if $H^1 \in \Omega^p(M)$ and $H^2 \in \Omega^p(M)$ are $\varphi$-related and $(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$ are $\varphi$-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$, then so are $A_{H^1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A_{H^2}(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

Example 3.2. The most important example of $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p = 3$ is given by the $H$-twisted Courant bracket

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_H := [X^1, X^2] \oplus (\mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 + i_{X^1} d\omega^2)$$

for all 3-forms $H \in \Omega^3(M)$ and all $m$-manifolds $M$. We call this $\mathcal{M}f_m$-natural operator the twisted Courant bracket $\mathcal{M}f_m$-natural operator.

Example 3.3. The operator given by $[-,-]_{dH}$ for all $H \in \Omega^2(M)$ and all $m$-manifolds $M$ is a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p = 2$.

The main result of this section is the following

Theorem 3.4. Assume $m \geq 3$. Then we have:

1. Any $\mathcal{M}f_m$-natural operator $A$ in the sense of Definition 3.1 for $p = 2$ such that $A_H = A_{H + dH}$ for any $H \in \Omega^2(M)$ and any $H^1 \in \Omega^1(M)$ is of the form

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- + ci_{X^1} i_{X^2} dH \right),$$

for (uniquely determined by $A$) reals $a, b_1, \ldots, c$, where 2-forms $H \in \Omega^2(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.
2. Any $\mathcal{M}f_m$-natural operator (not necessarily satisfying $A_H = A_{H+dH^1}$) in the sense of Definition 3.1 for $p = 3$ is of the form

$$A_H(\rho^1, \rho^2) = a[X^1, X^2] + \langle b_1 L_{X^2} \omega^1 + b_2 L_{X^1} \omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- + c_i X^i X^2 H \rangle,$$

for (uniquely determined by $A$) reals $a, b_1, \ldots, c$, where 3-forms $H \in \Omega^3(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.

3. If $p \geq 4$, any $\mathcal{M}f_m$-natural operator (not necessarily satisfying $A_H = A_{H+dH^1}$) in the sense of Definition 3.1 is of the form

$$A_H(\rho^1, \rho^2) = a[X^1, X^2] + \langle b_1 L_{X^2} \omega^1 + b_2 L_{X^1} \omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- \rangle$$

for (uniquely determined by $A$) reals $a, b_1, \ldots, b_4$, where $p$-forms $H \in \Omega^p(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.

Proof. Clearly, $A_0$, where 0 is the zero $p$-form, can be treated as the bilinear operator in the sense of Definition 2.1. Then $A_0$ is described in Theorem 2.3. So we can replace $A$ by $A - A_0$. In other words, we have assumption $A_0 = 0$.

By the invariance of $A$, it is determined by the values $A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}$ for all $H \in \Omega^p(\mathbb{R}^m)$, $X^i \oplus \omega^i \in \mathcal{X}(\mathbb{R}^m) \oplus \Omega^1(\mathbb{R}^m)$. Put

$$A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0} = \langle A_{H}^1(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, A_{H}^2(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0} \rangle,$$

where $A_{H}^1(...)|_0 \in T_{0} \mathbb{R}^m$ and $A_{H}^2(...)|_0 \in T^{*} \mathbb{R}^m$. Then $A$ is determined by

$$\langle A_{H}^1(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta \rangle \in \mathbb{R}$$

and $\langle A_{H}^2(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \mu \rangle \in \mathbb{R}$

for all $H \in \Omega^p(\mathbb{R}^m)$, $X^i \oplus \omega^i \in \mathcal{X}(\mathbb{R}^m) \oplus \Omega^1(\mathbb{R}^m)$, $\eta \in T_{0} \mathbb{R}^m$, $\mu \in T_{0} \mathbb{R}^m$, $i = 1, 2$.

By the non-linear Peetre theorem, see [5], $A$ is of finite order. It means that there is a finite number $r$ such that from $(j^r_x H = j^r_x \mathcal{H}, j^r_x (\rho^i) = j^r_x (\rho^i), i = 1, 2)$ it follows $A_H(\rho^1, \rho^2)_{|x} = A_{\mathcal{H}}(\rho^1, \rho^2)_{|x}$. So, we may assume that $H, X^1, X^2, \omega^1, \omega^2$ are polynomials of degree not more than $r$.

Using the invariance of $A$ with respect to the homotheties and the bilinearity of $A_H$ (for given $H$) we obtain homogeneity condition

$$\left\langle A_{\left(\frac{1}{r} \text{id} \right)} H \left( t \left(\frac{1}{r} \text{id} \right)_* X^1 \oplus t \left(\frac{1}{r} \text{id} \right)_* \omega^1, t \left(\frac{1}{r} \text{id} \right)_* X^2 \right. \right. \right.$$

$$+ \left. \left. t \left(\frac{1}{r} \text{id} \right)_* \omega^2 \right)_{|0}, \eta \right\rangle = t \left\langle A_{H}^1(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta \right\rangle .$$

Then, by the homogeneous function theorem, since $A$ is of finite order and regular and $A_0 = 0$ and $p \geq 2$, we have $\langle A_{H}^1(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta \rangle = 0$. 

Using the same arguments we get homogeneity condition
\[
\left\langle A^2_{\left(\frac{1}{t}id\right)_{\ast}} H \left( t \left( \frac{1}{t}id \right)_{\ast} X^1 \oplus t \left( \frac{1}{t}id \right)_{\ast} X^2 \right. \right. \\
\left. \left. \oplus t \left( \frac{1}{t}id \right)_{\ast} \omega^2 \right) \right|_0, \mu \right\rangle = t^3 \left\langle A^2_{\left(\frac{1}{t}id\right)_{\ast}} \left( X^1 \oplus \omega^1, X^2 \oplus \omega^2 \right) \right|_0, \mu \right\rangle.
\]

Then, if \( p = 2 \), by the homogeneous function theorem and the bilinearity of \( A_H \) and the assumptions \( A_0 = 0 \) and \( A_H = A_{H+\ell H} \), the value \( \left\langle A^2_{\left(\frac{1}{t}id\right)_{\ast}} \left( X^1 \oplus \omega^1, X^2 \oplus \omega^2 \right) \right|_0, \mu \rangle \) depends quadrilinearly on \( X^1 |_0, X^2 |_0, j^1_0(H - H|_0) \) and \( \mu \), only. By \( m \geq 3 \) and the regularity of \( A \), we may assume that \( X^1 |_0, X^2 |_0 \) and \( \mu \) are linearly independent. Then by the invariance we may assume \( X^1 |_0 = \partial_1 |_0, X^2 |_0 = \partial_2 |_0, \mu = \partial_3 |_0 \). Then \( A \) is determined by the values \( \left\langle A^2_{x^{i_1, dx^{i_2} \wedge dx^{i_3}}} \left( \partial_1 \oplus 0, \partial_2 \oplus 0, \partial_3 |_0 \right) \right\rangle \) for all \( i_1 = 1, \ldots, m \) and \( i_2, i_3 \) with \( 1 \leq i_2 < i_3 \leq m \). Then using the invariance of \( A \) with respect to \( \tau |_0 \) for \( \tau^i > 0 \) we deduce that only \( v := \left\langle A^2_{x^{i_1, dx^{i_2} \wedge dx^{i_3}}} \left( \partial_1 \oplus 0, \partial_2 \oplus 0, \partial_3 |_0 \right) \right\rangle, \)
\( w := \left\langle A^2_{x^{i_1, dx^{i_2} \wedge dx^{i_3}}} \left( \partial_1 \oplus 0, \partial_2 \oplus 0, \partial_3 |_0 \right) \right\rangle, \)
\( z := \left\langle A^2_{x^{i_1, dx^{i_2} \wedge dx^{i_3}}} \left( \partial_1 \oplus 0, \partial_2 \oplus 0, \partial_3 |_0 \right) \right\rangle \) may be not-zero. But \( x^1 dx^2 \wedge dx^3 = -x^2 dx^1 \wedge dx^3 + d(...) \). So, \( v = -w \). Similarly, \( v = -z \). Therefore the vector space of all \( A \) in question with \( A_0 = 0 \) and \( A_H = A_{H+\ell H} \) is at most one-dimensional. The part (1) of the theorem is complete. If \( p = 3 \), then (by almost the same arguments as for \( p = 2 \)) \( A \) is determined by the values \( \left\langle A^2_{dx^{i_1, dx^{i_2} \wedge dx^{i_3}}} \left( \partial_1 \oplus 0, \partial_2 \oplus 0, \partial_3 |_0 \right) \right\rangle \in \mathbb{R} \) for all \( i_1, i_2, i_3 \) with \( 1 \leq i_1 < i_2 < i_3 \leq m \). Then using the invariance with respect to \( \left( \tau^1 x^1, \ldots, \tau^m x^m \right) \) for \( \tau^i > 0 \) we deduce that only the value \( \left\langle A^2_{dx^{i_1, dx^{i_2} \wedge dx^{i_3}}} \left( \partial_1 \oplus 0, \partial_2 \oplus 0, \partial_3 |_0 \right) \right\rangle \in \mathbb{R} \) may be not-zero. Therefore the vector space of all \( A \) in question with \( A_0 = 0 \) is one-dimensional (generated by the natural operator \( 0 \oplus i X_1 i X_2 H \)).

If \( p \geq 4 \), then (similarly as for \( p = 2 \)) \( \left\langle A^2_{\left(\frac{1}{t}id\right)_{\ast}} \left( X^1 \oplus \omega^1, X^2 \oplus \omega^2 \right) \right|_0, \mu \rangle = 0 \).

Theorem 3.4 is complete. \( \square \)

**Corollary 3.5.** If \( m \geq 3 \), any \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 for \( p = 3 \) such that \( A_H \) is skew-symmetric for any \( H \in \Omega^3(M) \) and any \( m \)-manifold \( M \) is of the form
\[
A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \\
\oplus \left( b(L X^1, \omega^2 - L X^2 \omega^1) + cd \left( X^2 \oplus \omega^1, X^1 \oplus \omega^2 \right) \right) \plus Sign \sum_{i=1}^{3} \omega^i \wedge i X_i H)
\]
for (uniquely determined by \( A \)) real numbers \( a, b, c, e \).

Roughly speaking, Corollary 3.5 says that any \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 such that \( A_H \) is skew-symmetric for any \( H \in \Omega^3(M) \) and any \( m \)-manifold \( M \) coincides with the “skew-symmetrization” of the twisted Courant bracket \( \mathcal{M}f_m \)-natural operator up to four real constants \( a, b, c, e \).

**Corollary 3.6.** If \( m \geq 3 \), then the twisted Courant bracket \( \mathcal{M}f_m \)-natural operator from Example 3.2 is the unique \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 for \( p = 3 \) satisfying the following properties:
(B1) \( A_0(\rho_1, \rho_2) = [\rho_1, \rho_2]_0 \),
(B2) \( A_H(X \oplus 0, Y \oplus 0) = [X,Y] \oplus i_X i_Y H \)
for all closed \( H \in \Omega^3_{cl}(M) \), all \( \rho_1, \rho_2, X \oplus 0, Y \oplus 0 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( m \)-manifolds \( M \). where \([-,-]_0 \) is the \( \mathcal{M}f_m \)-natural bilinear operator given by the (not skew-symmetric) Courant bracket as in Example 2.6.

Proof. Clearly, the twisted Courant bracket \( \mathcal{M}f_m \)-natural operator satisfies (B1) and (B2). Consider \( A \) in question satisfying (B1) and (B2). Then by Theorem 3.4, there exist uniquely determined reals \( a, b_1, ..., c \) such that for all \( H \in \Omega^3(M) \) and \( m \)-manifolds \( M \)

\[
A_H(\rho^1, \rho^2) = a[X^1, X^2] + \sum b_i L_{X^i} \omega^1 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + c i_{X^1} i_{X^2} H,
\]

where \( \rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \) are arbitrary. Putting \( \omega^1 = \omega^2 = 0 \) we get \( A_H(\rho^1, \rho^2) = a[X^1, X^2] + c i_{X^1} i_{X^2} H \). Then condition (B2) implies \( c = 1 \). Putting \( H = 0 \) we get

\[
A_0(\rho^1, \rho^2) = a[X^1, X^2] + \sum b_i L_{X^i} \omega^1 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- \]

for all \( \rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( m \)-manifolds \( M \). But \( A_0 \) is a \( \mathcal{M}f_m \)-natural bilinear operator in the sense of Definition 2.1. Then \( a, b_1, b_2, b_3, b_4 \) are uniquely determined because of Theorem 2.3. Then \( a, b_1, ..., c \) are uniquely determined. So, \( A \) is uniquely determined by conditions (B1) and (B2). \( \square \)

4. The Natural Operators Similar to the Twisted Courant Bracket and Defined for Closed \( p \)-Forms Only

In the previous section, we considered \( \mathcal{M}f_m \)-natural operators \( A \) which are defined for all \( p \)-forms \( H \). In this section, we observe what happens if \( A \) are defined for closed \( p \)-forms \( H \), only. We start with the following

Definition 4.1. A \( \mathcal{M}f_m \)-natural operator \( A \) sending closed \( p \)-forms \( H \in \Omega^p_{cl}(M) \) on \( m \)-manifolds \( M \) into bilinear operators

\[
A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),
\]

is a \( \mathcal{M}f_m \)-invariant family of regular operators (functions)

\[
A : \Omega^p_{cl}(M) \to \text{Lin}_2((\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)), \mathcal{X}(M) \oplus \Omega^1(M)),
\]

for all \( m \)-manifolds \( M \).

We have the following corollary of Theorem 3.4.

Corollary 4.2. Assume \( m \geq 3 \). Then we have:

1. If \( p = 3 \), any \( \mathcal{M}f_m \)-natural operator in the sense of Definition 4.1 is of the form

\[
A_H(\rho^1, \rho^2) = a[X^1, X^2] + \sum b_i L_{X^i} \omega^1 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + c i_{X^1} i_{X^2} H,
\]
for uniquely determined by $A$ reals $a, b_1, ..., c$, where closed 3-forms $H \in \Omega^3_{\text{cl}}(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.

2. If $p \geq 4$, any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 is of the form

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- \right)$$

for uniquely determined by $A$ reals $a, b_1, ..., b_4$, where closed $p$-forms $H \in \Omega^p_{\text{cl}}(M)$, pairs $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.

**Proof.** Let $A$ be a $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 for $p$. Define a $\mathcal{M}f_m$-natural operator $A^1$ in the sense of Definition 3.1 for $p - 1$ by

$$A^1_H = A_{d \tilde{H}}.$$  

Then $A^1_{H + dH_1} = A^1_{H}$ for any $\tilde{H} \in \Omega^{p-1}(M)$ and $H_1 \in \Omega^{p-2}(M)$.

If $p = 3$, then by Theorem 3.4, $A^1$ is of the form

$$A^1_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + ci X^i i X^2 d \tilde{H} \right)$$

for uniquely determined reals $a, b_1, ..., c$ and all $\tilde{H} \in \Omega^2(M)$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. Then

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + ci X^i i X^2 H \right)$$

for all exact 3-forms $H$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. But by the locality of $A$ and the Poincare lemma we may replace the phrase “all exact 3-forms” by “all closed 3-forms”.

If $p \geq 4$, then by Theorem 3.4, $A^1$ is of the form

$$A^1_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + ci X^i i X^2 d \tilde{H} \right)$$

for uniquely determined reals $a, b_1, ..., c$ (with arbitrary $c$ if $p = 4$ and with $c = 0$ if $p \geq 5$) and all $\tilde{H} \in \Omega^{p-1}(M)$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. The condition $A^1_H = A^1_{H + dH_1}$ implies $ci X^i i X^2 dH_1 = 0$ for any $H_1 \in \Omega^{p-2}(M)$. If $p = 4$, putting $X^2 = \partial_1, X^3 = \partial_2$ and $H_1 = x^1 dx^2 \wedge dx^3$, we get $c(-dx^3) = 0$, i.e., $c = 0$. If $p \geq 5$, then $c = 0$, see above. Next, we proceed similarly as in the case $p = 3$. \(\square\)

The above corollary and Theorem 3.4 imply

**Theorem 4.3.** If $m \geq 3$ and $p \geq 3$ then any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 can be extended uniquely to a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1.

Roughly speaking, if $m \geq 3$ and $p \geq 3$, then any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 can be treated as the $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1, and vice-versa.
5. The Natural Operators Similar to the Twisted Courant Bracket and Satisfying the Leibniz Rule for Closed 3-Forms

**Definition 5.1.** A $\mathcal{M}_{f_m}$-natural operator $A$ in the sense of Definition 3.1 (or equivalently in the sense of Definition 4.1) satisfies the Leibniz rule for closed $p$-forms if

$$A_H(p_1, A_H(p_2, p_3)) = A_H(A_H(p_1, p_2), p_3) + A_H(p_2, A_H(p_1, p_3))$$

for all $p_1, p_2, p_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$, all closed $p$-forms $H \in \Omega^p_{cl}(M)$ and all $m$-manifolds $M$.

**Example 5.2.** The twisted Courant bracket $\mathcal{M}_{f_m}$-natural operator presented in Example 3.2 satisfies the Leibniz rule for closed 3-forms, see [3,8].

**Theorem 5.3.** If $m \geq 3$, any $\mathcal{M}_{f_m}$-natural operator $A$ in the sense of Definition 3.1 (or equivalently of Definition 4.1) for $p = 3$ satisfying the Leibniz rule for closed 3-forms is one of the $\mathcal{M}_{f_m}$-natural operators:

\[
\begin{align*}
A_H^{(1,a)}(\rho_1, \rho_2) &= \rho_1 \wedge \rho_2 \\
A_H^{(2,a)}(\rho_1, \rho_2) &= \rho_1 \wedge (\mathcal{L}_X \omega) + \rho_2 (\mathcal{L}_X \omega) \\
A_H^{(3,a)}(\rho_1, \rho_2) &= a[X^1, X^2] + \rho_1 \wedge (\mathcal{L}_X \omega) \\
A_H^{(4,a,c)}(\rho_1, \rho_2) &= a[X^1, X^2] + \rho_1 \wedge (\mathcal{L}_X \omega) + c_1 dX^1 \wedge \omega + c_2 dX^2 \wedge \omega + e \omega_1 \wedge \omega_2 \\
\end{align*}
\]

where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$, and $a$ and $e$ are arbitrary real numbers.

**Proof.** Let $A$ be a $\mathcal{M}_{f_m}$-natural operator in the sense of Definition 3.1 for $p = 3$ such that $A_H$ satisfies the Leibniz rule for any closed $H \in \Omega^3_{cl}(M)$. By Theorem 3.4, $A$ is of the form

$$A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2]$$

$$\oplus (b_1 \mathcal{L}_{X^1} \omega^1 + b_2 \mathcal{L}_{X^2} \omega^2 + c_1 dX^1 \wedge \omega^1 + c_2 dX^2 \wedge \omega^2 + e \omega_1 \wedge \omega_2),$$

for (uniquely determined by $A$) real numbers $a, b_1, b_2, c_1, c_2, e$. Then for any $X^1, X^2, X^3 \in \mathcal{X}(M)$ and $\omega^1, \omega^2, \omega^3 \in \Omega^1(M)$ we have

\[
\begin{align*}
A_H(X^1 \oplus \omega^1, A_H(X^2 \oplus \omega^2, X^3 \oplus \omega^3)) &= a^2[X^1, X^2, X^3] \oplus \Omega, \\
A_H(A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2), X^3 \oplus \omega^3) &= a^2[X^1, X^2, X^3] \oplus \Theta, \\
A_H(X^2 \oplus \omega^2, A_H(X^1 \oplus \omega^1, X^3 \oplus \omega^3)) &= a^2[X^2, X^1, X^3] \oplus T, \\
\end{align*}
\]
where
\[
\Omega = b_1 \mathcal{L}_{a[X^2,X^3]}\omega^1 + c_1 di_a[X^2,X^3]\omega^1 + e_i X^1 i_a[X^2,X^3]H \\
+ b_2 \mathcal{L}_{X^3}(b_1 \mathcal{L}_{X^3} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^3 + c_1 di_{X^3}\omega^2 + c_2 di_{X^2}\omega^3 + e_i X^2 i_X H) \\
+ c_2 di_{X^1}(b_1 \mathcal{L}_{X^3} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^3 + c_1 di_{X^3}\omega^2 + c_2 di_{X^2}\omega^3 + e_i X^2 i_X H),
\]
\[
\Theta = b_2 \mathcal{L}_{a[X^1,X^2]}\omega^3 + c_2 di_{a[X^1,X^2]}\omega^3 + e_i a[X^1,X^2]i_X H \\
+ b_1 \mathcal{L}_{X^3}(b_1 \mathcal{L}_{X^3} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + c_1 di_{X^3}\omega^1 + c_2 di_{X^1}\omega^2 + e_i X^1 i_X H) \\
+ c_1 di_{X^2}(b_1 \mathcal{L}_{X^3} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^2 + c_1 di_{X^3}\omega^1 + c_2 di_{X^1}\omega^2 + e_i X^1 i_X H),
\]
\[
T = b_1 \mathcal{L}_{a[X^1,X^3]}\omega^2 + c_1 di_{a[X^1,X^3]}\omega^2 + e_i X^2 i_a[X^1,X^3]H \\
+ b_2 \mathcal{L}_{X^2}(b_1 \mathcal{L}_{X^3} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^3 + c_1 di_{X^3}\omega^1 + c_2 di_{X^1}\omega^3 + e_i X^1 i_X H) \\
+ c_2 di_{X^2}(b_1 \mathcal{L}_{X^3} \omega^1 + b_2 \mathcal{L}_{X^1} \omega^3 + c_1 di_{X^3}\omega^1 + c_2 di_{X^1}\omega^3 + e_i X^1 i_X H).
\]

The Leibniz rule of \( A_H \) is equivalent to \( \Omega = \Theta + T \).

Putting \( H = 0 \), we are in the situation of Theorem 2.7. Then by Theorem 2.7 (i.e., by Theorem 3.2 in [2]) we get \( (b_1, b_2, c_1, c_2) = (0, 0, 0, 0) \) or \( (b_1, b_2, c_1, c_2) = (0, a, 0, 0) \) or \( (b_1, b_2, c_1, c_2) = (-a, a, 0, 0) \) or \( (b_1, b_2, c_1, c_2) = (-a, a, a, 0) \). More, \( A_0 \) for such \( (b_1, b_2, c_1, c_2) \) satisfies the Leibniz rule.

Therefore (as \( c_2 = 0 \)) the Leibniz rule of \( A_H \) is equivalent to the equality
\[
e ai X^1 i_{[X^2,X^3]}H + b_2 \mathcal{L}_{X^1 i_{X^2}i_{X^3}}H \\
= e ai_{[X^1,X^2]}i_{X^3}H + b_1 e \mathcal{L}_{X^3 i_{X^1}i_{X^2}H} + c_1 ed i_{X^3}i_{X^1}i_{X^2}H \\
+ e ai_{X^2}i_{[X^1,X^3]}H + b_2 e \mathcal{L}_{X^2 i_{X^1}i_{X^3}}H.
\]

If \( (b_1, b_2, c_1, c_2) = (0, 0, 0, 0) \), the above equality is equivalent to
\[
e ai X^1 i_{[X^2,X^3]}H = e ai_{[X^1,X^2]}i_{X^3}H + e ai_{X^2}i_{[X^1,X^3]}H.
\]

Putting \( X^1 = \partial_1 \), \( X^2 = \partial_2 \) and \( X^3 = \partial_3 \) and \( H = x^2 dx^1 \land dx^2 \land dx^3 \) (it is closed) we have \( [X^2,X^3] = 0 \), \( [X^1,X^2] = 0 \), \( [X^1,X^3] = 0 \), \( \mathcal{L}_{X^2 i_{X^1}i_{X^3}}H = \mathcal{L}_{\partial_2} x^2 dx^2 = dx^2 \) and \( \mathcal{L}_{X^3 i_{X^2}i_{X^3}}H = \mathcal{L}_{\partial_1} (-x^2 dx^1) = 0 \). Hence \( e ad x^2 = 0 \). So, \( a = 0 \) or \( e = 0 \).

If \( (b_1, b_2, c_1, c_2) = (-a, a, 0, 0) \), the above equality is equivalent to
\[
e ai X^1 i_{[X^2,X^3]}H + e a \mathcal{L}_{X^1 i_{X^2}i_{X^3}}H \\
= e ai_{[X^1,X^2]}i_{X^3}H - e a \mathcal{L}_{X^3 i_{X^1}i_{X^2}H} + e ai_{X^2}i_{[X^1,X^3]}H + e a \mathcal{L}_{X^2 i_{X^1}i_{X^3}}H.
\]

Putting \( X^1 = \partial_1 \), \( X^2 = \partial_2 \) and \( X^3 = \partial_3 \) and \( H = x^2 dx^1 \land dx^2 \land dx^3 \) we have (see above) \( [X^2,X^3] = 0 \), \( [X^1,X^2] = 0 \), \( [X^1,X^3] = 0 \), \( \mathcal{L}_{X^2 i_{X^1}i_{X^3}}H = dx^2 \), \( \mathcal{L}_{X^1 i_{X^2}i_{X^3}}H = 0 \) and \( \mathcal{L}_{X^3 i_{X^1}i_{X^2}H} = \mathcal{L}_{\partial_1} (-x^2 dx^3) = 0 \). Then \( e ad x^2 = 0 \). So, \( a = 0 \) or \( e = 0 \).
If \((b_1, b_2, c_1, c_2) = (-a, a, a, 0)\), the above equality is equivalent to
\[
e a \sum \left\{ i_{X^1} i_{[X^2, X^3]} H + \mathcal{L}_{X^1} i_{X^2} i_{X^3} H \right\} = e a d i_{X^1} i_{X^2} i_{X^3} H,
\]
where \(\sum\) is the cyclic sum \(\sum_{cyc} (X^1, X^2, X^3)\). Then \(e\) is arbitrary real number because from \(dH = 0\) it follows
\[
\sum \left\{ i_{X^1} i_{[X^2, X^3]} H + \mathcal{L}_{X^1} i_{X^2} i_{X^3} H \right\} = di_{X^1} i_{X^2} i_{X^3} H.
\]
Indeed, using \(dH = 0\) and \(i_{[X^1, X^2]} = \mathcal{L}_{X^1} i_{X^2} - i_{X^1} \mathcal{L}_{X^2}\) and the well-known formula expressing \(dH(X^1, X^2, X^3, X^4)\), we have
\[
\sum \left\{ i_{X^1} i_{[X^2, X^3]} H + \mathcal{L}_{X^1} i_{X^2} i_{X^3} H \right\} = 6 \sum \{ H(X^2, X^3, X^1, X^4) + X^1 H(X^3, X^2, X^4) \}
\]
\[
- 24 dH(X^1, X^2, X^3, X^4) + 6 X^4 H(X^3, X^2, X^1) = i_{X^4} di_{X^1} i_{X^2} i_{X^3} H.
\]
Summing up, given a real number \(a \neq 0\) we have \((b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, 0)\) or \((b_1, b_2, c_1, c_2, e) = (-a, a, 0, 0, 0)\), or \((b_1, b_2, c_1, c_2, e) = (-a, a, a, 0, 0)\), where \(e\) may be arbitrary real number. If \(a = 0\) we have \((b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, e)\), where \(e\) may be arbitrary. Theorem 5.3 is complete. \(\square\)

**Corollary 5.4.** If \(m \geq 3\), then the twisted Courant bracket \(\mathcal{M}_f_{m}\)-natural operator from Example 3.2 is the unique \(\mathcal{M}_f_{m}\)-natural operator \(A\) in the sense of Definition 3.1 for \(p = 3\) satisfying the following conditions:

\((C1)\) \(A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3))\),

\((C2)\) \(A_H(X \oplus 0, Y \oplus 0) = [X, Y] \oplus i_{X} i_{Y} H\)

for all \(\rho_1, \rho_2, \rho_3, X \oplus 0, Y \oplus 0 \in \mathcal{X}(M) \oplus \Omega^1(M)\), all closed \(H \in \Omega^3_{cl}(M)\) and all m-manifolds \(M\).

**Proof.** Indeed, the condition \((C1)\) and Theorem 5.3 imply that \(A = A^{(1,a)}\)
or \(A = A^{(2,a)}\) or \(A = A^{(3,a)}\) or \(A = A^{(4,a,e)}\) for some real numbers \(a\) and \(e\). Then \((C2)\) implies that \(A = A^{(4,a,e)}\) and \(a = 1\) and \(e = 1\) because \(A^{(1,a)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0\) and \(A^{(2,a)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0\) and \(A^{(3,a)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0\) and \(A^{(4,a,e)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus e i_{X} i_{Y} H\). \(\square\)

**Corollary 5.5.** If \(m \geq 3\), any \(\mathcal{M}_f_{m}\)-natural operator \(A\) in the sense of Definition 3.1 for \(p = 3\) such that \(A_H\) is a Lie algebra bracket (i.e., it is skew-symmetric, bilinear and satisfying the Leibniz rule) for all closed 3-forms \(H \in \Omega^3_{cl}(M)\) and all m-manifolds \(M\) is one of the \(\mathcal{M}_f_{m}\)-natural operators:

\[
A^{(1,a)}_H(\rho_1, \rho_2) = a[X^1, X^2] \oplus 0,
\]

\[
A^{(2,a)}_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1} \omega^2 - \mathcal{L}_{X^2} \omega^1)),
\]

\[
A^{(4,0,e)}_H(\rho^1, \rho^2) = 0 \oplus ei_{X^1} i_{X^2} H,
\]

where \(\rho^1 = X^1 \oplus \omega^1\) and \(\rho^2 = X^2 \oplus \omega^2\), and \(a\) and \(e\) are arbitrary real numbers.
Proof. It follows from Theorem 5.3. □

**Corollary 5.6.** If \( m \geq 3 \), any \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 for \( p = 3 \) satisfying the Leibniz rule for all 3-forms \( H \) (or for all closed 3-forms and at least one non-closed 3-form) is one of the \( \mathcal{M}f_m \)-natural operators:

\[
\begin{align*}
A_H^{(1, a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus 0, \\
A_H^{(2, a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(L_{X^1}\omega - L_{X^2}\omega)), \\
A_H^{(3, a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (aL_{X^1}\omega^2), \\
A_H^{(4, a, 0)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(L_{X^1}\omega^2 - i_{X^2}d\omega)), \\
A_H^{(4, 0, e)}(\rho_1, \rho_2) &= 0 \oplus ei_{X^1}i_{X^2}H,
\end{align*}
\]

where \( \rho^1 = X^1 \oplus \omega^1 \) and \( \rho^2 = X^2 \oplus \omega^2 \), and \( a \) and \( e \) are arbitrary real numbers.

**Proof.** It follows from Theorem 5.3 and its proof. □

**Remark 5.7.** It is well-known that given closed 3-form \( H \in \Omega^3_{cl}(M) \) on an \( m \)-manifold \( M \), the twisted Courant bracket \([-, -]_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M)\) is bilinear and satisfies the properties (A1)–(A5) from Corollary 2.8 for all \( \rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( f \in C^\infty(M) \), see [3,8], but \([-, -]_H \neq [-, -]_0 \) if \( H \neq 0 \). Is it a contradiction with the uniqueness from Corollary 2.8? No, it is not. Indeed, \([-, -]_H \) is not extendable to a \( \mathcal{M}f_m \)-natural bilinear operator in the sense of Definition 2.1 because it is invariant only with respect to \( \mathcal{M}f_m \)-maps \( \varphi : M \to M \) preserving \( H \), in fact.

**Remark 5.8.** By Corollary 5.5, given a closed 3-form \( H \) on \( M \), the skew-symmetric bracket \([X^1 \oplus \omega^1, X^2 \oplus \omega^2]_H \) := \( 0 \oplus i_{X^1}i_{X^2}H \) satisfies the Leibniz rule. One can easily directly verify that \((TM \oplus T^*M, e[[-, -]_H, 0\pi])\) for arbitrary fixed \( e \in \mathbb{R} \) and closed 3-form \( H \) is a Lie algebroid canonically depending on \( H \). So, if we have a closed 3-form \( H \) on a \( m \)-manifold \( M \), we can construct canonical (in \( H \)) Lie algebroids (\( EM, [[-, -]_H, a[H]] \)) with \( EM = TM \oplus T^*M \) different than the one from Proposition 2.11.

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