Quantum Potts Models on the Sierpiński Pyramid

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Phase transition of the two- and three-state quantum Potts models on the Sierpiński pyramid are studied by means of a tensor network framework, the higher-order tensor renormalization group method. Critical values of the transverse magnetic field and the magnetic exponent $\beta$ are evaluated. Despite the fact that the Hausdorff dimension of the Sierpiński pyramid is exactly two (= log\textsubscript{4} 4), the obtained critical properties show that the effective dimension is lower than two.

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I. INTRODUCTION

Dimensionality plays an important role in quantum phenomena. In one dimension, Luttinger liquid behavior is common to a variety of quantum systems\textsuperscript{1,2}. In two dimensions, topological phases can play another important role, as has been widely studied for the electronic structure of Graphene and related materials. In phase transitions and critical phenomena, the lattice dimension is a key component for the universality\textsuperscript{3}, in addition to a symmetry in the local degrees of freedom and interaction range. For the classification of criticality, quantum models, such as the transverse-field Ising model, have been extensively studied on regular lattices in various spatial dimensions, including hyperbolic lattices which have infinite effective dimension\textsuperscript{4}.

Fractal lattices can bring novel views on dimensional studies in critical phenomena since their Hausdorff dimensions can be non-integer. Quantum systems with fractal geometry have been occasionally studied, partially because they appear in a variety of physical phenomena. For example, fractal nature arises when a quantum phase transition occurs in solids\textsuperscript{5}. Besides, fractal structures have been inspiring scientist from various fields of physics. Amongst them all, we mention that it is even possible to fabricate artificial fractals, such as Sierpiński triangle, on a solid surface\textsuperscript{6}. One also encounters presence of fractals in quantum gravity\textsuperscript{7}.

In this article we focus on quantum lattice models on the Sierpiński fractals. Figure\textsuperscript{1} shows small finite size lattices with the geometry of the Sierpiński triangle (left) and the pyramid (right). Both of these lattices can be constructed recursively. In the case of the Sierpiński triangle, an elementary unit consists of 3 lattice sites that are located at the vertices of a triangle, and the extension of the system is performed by connecting 3 units so that the adjacent units are connected through an additional bond. In the case of Sierpiński pyramid, the elementary unit is a tetrahedron that consists of 4 lattice sites, and 4 units are connected through 6 additional bonds. Repeating such extension process recursively, one can construct a fractal lattice of an arbitrary size. The coordination number is 3 in the case of the Sierpiński triangle thus created, and is 4 for the Sierpiński pyramid.

There is another type of Sierpiński triangle and pyramid, where elementary units are joined so that each site is shared by adjacent units. Under this extension scheme, the coordination number of the Sierpiński triangle is 4, and that of the pyramid is 6. In order to avoid any confusion, we refer to these “site-sharing fractals” as the B-type, and the “bond-sharing fractals” shown in Fig.\textsuperscript{1} as the A-type in the following. It should be noted that the fractal dimension in the thermodynamic (i.e., large system-size) limit does not depend on the choice of the A- or B-type.

As a typical example of quantum lattice models, the $q$-state Potts models, which include the transverse-field Ising (TFI) model as the case $q = 2$, have been studied on the Sierpiński triangle. A method of analysis is the conventional real-space renormalization group (RSRG). Kubica and Yoshida obtained the critical indices $\nu = 0.7196$ and $\nu = 0.6213$, respectively, for the cases $q = 2$ and $q = 3$\textsuperscript{8}. The same result was also reported by Xu et al.\textsuperscript{9}. Another choice of the numerical method is the quantum Monte-Carlo (QMC) simulation, where Yi obtained $\nu = 0.66 \pm 0.05$ for the case $q = 2$\textsuperscript{10}. They also reported $\nu = 0.53 \pm 0.03$ for the case $q = 3$\textsuperscript{11}. Independently, Yoshida and Kubica reported $\nu = 0.76 \pm 0.01$ for $q = 2$\textsuperscript{12}. The Table I summarizes these results, including the recent study performed by the higher-order
TABLE I: Critical exponents of the quantum Potts models $(q = 2$ and $q = 3$) on the Sierpiński triangle. The lattice A- or B-type are shown as the superscripts of the method used to avoid any other confusions.

| $q$ | Method | $\nu$  | $\beta$  | $\gamma$  |
|-----|--------|-------|-------|-------|
| 2   | RSRG$^A$ [8, 9] | 0.7196 | -     | -     |
|     | QMC$^B$ [10]    | 0.66(5) | 0.19(2) | 1.45(5) |
|     | QMC$^C$ [12]    | 0.76(1) | 0.18   | 1.60   |
|     | HOTRG$^A$ [13]  | -     | -     | -     |
| 3   | RSRG$^A$ [8, 9] | 0.6213 | -     | -     |
|     | QMC$^B$ [11]    | 0.53(3) | 0.145(10) | 1.24(3) |

TABLE II: Critical exponents of the quantum Potts models $(q = 2$ and $q = 3$) on the Sierpiński pyramid, including $\beta$ we have calculated by means of the HOTRG method in this work.

| $q$ | Method | $\nu$  | $\beta$  | $\gamma$  |
|-----|--------|-------|-------|-------|
| 2   | RSRG$^A$ [8] | 0.6174 | -     | -     |
|     | QMC$^B$ [10] | 0.62(5) | 0.25(2) | 1.55(5) |
|     | QMC$^C$ [12] | 0.66(5) | -     | -     |
|     | HOTRG$^A$ [13] | -     | -     | -     |
| 3   | RSRG$^A$ [8] | 0.5390 | -     | -     |
|     | QMC$^B$ [11] | 0.43(2) | 0.15(1) | 1.18(5) |
|     | HOTRG$^A$ [13] | -     | -     | -     |

tensor renormalization group (HOTRG) method [13].

Relatively less is known on the Sierpiński pyramid. By means of RSRG, the exponents $\nu = 0.6174$ and $\nu = 0.5390$ are known, respectively, for the case $q = 2$ and $q = 3$ [8, 9]. By QMC simulations, Yi reported $\nu = 0.62 \pm 0.05$ [10] and $\nu = 0.43 \pm 0.02$ [11], respectively, for the cases $q = 2$ and $q = 3$. Independently, Yoshida and Kubica obtained $\nu = 0.660 \pm 0.005$ for $q = 2$, and they conjectured that the quantum $q = 3$ Potts model exhibits a discontinuous (first-order) phase transition on the Sierpiński pyramid [12]. These results for the pyramid are summarized in Table II.

Along with these studies of phase transitions on Sierpiński fractals, it turned out that the definition of the effective dimension is not straightforward. The calculated exponents do not agree with the hyper-scaling hypothesis if the Hausdorff dimension of the fractal lattice is considered as the effective spatial dimension. A similar discrepancy is also reported for the classical Ising model defined on fractal lattices [12–10]. For the purpose of getting better insight into the effective dimension, we perform a precise numerical study for the $q = 3$ quantum Potts model on the Sierpiński triangle and pyramid by means of the HOTRG method.

The structure of this article is as follows: In the next section we introduce the Potts models on the Sierpiński pyramid. In Section III we show the numerical results. Conclusions are summarized in the last section.

II. QUANTUM POTTS MODEL

The quantum $q$-state Potts model is described by the lattice Hamiltonian

$$\hat{H} = -J \sum_{\langle i,j \rangle} \delta (\hat{s}_i, \hat{s}_j) - h \sum_i \sum_{k=1}^{q-1} (\hat{\Gamma}_i)^k,$$  

(1)

where $\hat{s}_i$ is the diagonal operator, whose eigenvalues are integers from 1 to $q$, on the lattice site labeled by $i$ [10, 11, 17]. The summation of the first term on the r.h.s. runs over pairs of neighboring sites, it is denoted by $\langle i,j \rangle$, and $J$ parameterizes the ferromagnetic interaction. We assume that the system is on a sufficiently large Sierpiński pyramid. The second term represents the quantum flipping effect by means of the transverse field, which is parameterized by the constant magnetic field $h$. The matrix representation of the operator $\hat{\Gamma}_i$ is nothing but the shift matrix

$$\Gamma_i = \begin{pmatrix} 0 & I_{q-1} \\ 1 & 0 \end{pmatrix},$$

(2)

where $I_{q-1}$ is an identity matrix of the dimension $q-1$. In the case of $q = 2$, the Hamiltonian $\hat{H}$ in Eq. (1) coincides with the transverse-field Ising model if rescaling $J \rightarrow 2J$. In the following we consider the cases $q = 2$ and 3 only. We focus on the ground-state phase transition of the system, which is located at a certain value of the transverse field $h = h_c$.

In order to analyze the ground-state properties of the system, we use the HOTRG method [13]. As it has been done in the previous study on the Sierpiński triangle [13], we introduce the Trotter-Suzuki decomposition [18–20] of the imaginary-time path integral representation of the thermal density matrix. The first and the second terms on the r.h.s. of Eq. (1) are treated as a non-commuting pair of operators $\hat{H}_i$ and $\hat{H}_i$. Typically, we choose the imaginary time step $\Delta \tau = 0.01$. The classical lattice system obtained through this decomposition naturally has a network structure that consists of a local weight represented by 6-leg tensors, where two legs correspond to the imaginary-time degrees of freedom. Since the tensor network is highly anisotropic, a couple of tensors stack along the imaginary time direction is grouped in advance [14] (we stacked 7 tensors at most). Thus stacked tensors are then renormalized forming another tensor. After such pre-processing, we started the conventional HOTRG procedure [13], which expands the corresponding tensors recursively by alternating the three space and one imaginary time directions. Details of the numerical implementation can be found in Ref. [14].

In order to detect the phase transition, we calculate the expectation value of local magnetization

$$M = \frac{q}{q-1} \left[ \langle \delta (\hat{s}, 1) \rangle - \frac{1}{q} \right]$$

(3)
avering inner spin operators $\hat{s}$ represented by impurity tensors [15]. Such an observation of $M$ can be performed by keeping the renormalized expression of $\hat{s}$ in each of the renormalization-group (RG) transformations [13]. During the HOTRG calculations, we kept $\chi = 40$ for the block-spin states at most.

III. NUMERICAL RESULTS

Let us recall that the Sierpiński pyramid (of the A-type) has the identical coordination number as the square lattice. Figure 2 shows the calculated ground-state magnetization $M$ with respect to the transverse field $h$ for $q = 2$ and $q = 3$ Potts models on the Sierpiński pyramid. For comparison, the values of $M$ calculated on the square lattice are also plotted. In the small $h$-region, where $M$ is close to the unity, the value of $M$ is insensitive to the global structure of the lattice. This is because when the correlation length is small, the effect of the coordination number is dominant. On the other hand, as $h$ increases, the difference between the Sierpiński pyramid and the square lattice becomes clearer. The correlation length grows to infinity toward the critical field $h_c$.

In order to analyze the critical singularity in the magnetization $M$, we evaluate the critical exponent $\beta$ by fitting the calculated data with the scaling formula

$$M = C (h_c - h)^{\beta},$$

where $h_c$ is the critical value of the transverse field, and $C$ is another fitting constant. On the Sierpiński pyramid, we obtained $\beta = 0.232$ and $h_c = 1.358$ for the case $q = 2$ as a result of the fitting. For the case $q = 3$, we calculated $\beta = 0.154$ and $h_c = 0.832$. To confirm these estimated values, we plot $M^{1/\beta}$ in Fig. 3 assuming these obtained exponents. Evidently, below the critical field $h_c$, we observed a linear decrease of $M^{1/\beta}$ for both cases.

We have calculated the ground-state magnetization $M$ of the quantum Potts models on the Sierpiński pyramid (of the A-type) for the cases $q = 2$ and $q = 3$. The estimated critical field $h_c$ and the exponent $\beta$ are listed in Table III, together with the related values reported so far [10, 11, 13]. It is obvious that the value of $h_c$ is consistent with the Monte Carlo studies [10, 11] on the B-type pyramid. This fact suggests that the effective dimension is insensitive to the choice of lattice from A- or B-type. The calculated $\beta$ shows that the effective dimension of the Sierpiński pyramid is less than two, when the critical universality is considered. This is in accordance with previous studies [8, 12].

The calculation of thermodynamic functions, such as internal energy and entropy would provide further information on the dimensionality and the scaling relation on the Sierpiński pyramid [21]. In order to increase the variety of the fractal lattice structure is another direction of the future study. For example, considering a higher-dimensional generalization of the Sierpiński pyramid for the purpose of finding out the upper critical (spatial) dimension is a reachable study with the use of the current computational resources. Evaluation of the entanglement entropy would be a challenging task within the numerical tensor-network frameworks.
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