On Variational Properties of Quadratic Curvature Functionals

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Abstract
In this paper, we investigate a class of quadratic Riemannian curvature functionals on closed smooth manifold $M$ of dimension $n \geq 3$ on the space of Riemannian metrics on $M$ with unit volume. We study the stability of these functionals at the metric with constant sectional curvature as its critical point.

Keywords: Quadratic curvature functional, Variational, Transverse-traceless, Conformal variation, Stability

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1 Introduction

Let $M$ be an $n$-dimensional compact and smooth manifold, and $\mathcal{M}$ the space of smooth Riemannian metrics on $M$ with unit volume, i.e. $\mathcal{M} = \{g \in \mathcal{M} : vol(g) = 1\}$, where $\mathcal{M}$ is the space of smooth Riemannian metrics on $M$. A functional $F : \mathcal{M} \to R$ is called Riemannian if it is invariant under the action of the diffeomorphism group.

Recall the decomposition of Riemannian curvature tensor $Rm$

$$Rm = W + \frac{1}{n-2} Ric \otimes g - \frac{1}{(n-1)(n-2)} R g \otimes g,$$  \hspace{1cm} (1.1)

where $W$, $Ric$ and $R$ denote the Weyl curvature tensor, the Ricci tensor and the scalar curvature, respectively, and $\otimes$ the Kulkarni-Nomizu product. From (1.1), we have,

$$|Rm|^2 = |W|^2 + \frac{4}{n-2} |Ric|^2 - \frac{2}{(n-1)(n-2)} R^2.$$  \hspace{1cm} (1.2)

The basic quadratic curvature functionals are

$$W = \int_M |W|^2 dV_g, \quad \rho = \int_M |Ric|^2 dV_g, \quad S = \int_M R^2 dV_g.$$

From the decomposition formula (1.2), one has

$$\mathcal{R} = \int_M |Rm|^2 dV_g = \int_M |W|^2 dV_g + \frac{4}{n-2} \int_M |Ric|^2 dV_g - \frac{2}{(n-1)(n-2)} \int_M R^2 dV_g.$$  \hspace{1cm} (1.2)

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We point out that in dimension three, Weyl tensor vanishes, and in dimension four, the famous Chern-Gauss-Bonnet formula implies that $\mathcal{W}$ can be expressed as a linear combination of $\rho$ and $S$ with the addition of a topological term. There are many results on these quadratic functionals. See [3, 4, 2, 12] for example. In [7], Gursky and Viaclovsky focus attention on a class of general quadratic curvature functionals

$$\tilde{F}_\tau = (Vol)^{\frac{4-n}{2n}} \left( \int_M |Ric|^2 dV_g + \tau \int_M |R|^2 dV_g \right).$$

They investigate rigidity and stability properties of critical points of $\tilde{F}_\tau$ on the space of Riemannian metric space $\mathcal{M}$, and obtain a series of beautiful results.

In this paper, we study a class of more general quadratic curvature functionals:

$$F_{s, \tau} = \int_M |Rm|^2 dV_g + s \int_M |Ric|^2 dV_g + \tau \int_M R^2 dV_g \quad (1.3)$$

on Riemannian metrics space $\mathcal{M}_1$, where $s, \tau$ are some constants.

Actually, $F_{s, \tau}$ have been widely studied. We first introduce the following definition.

**Definition 1.1** Let $M$ be a compact $n$-dimension manifold, a critical metric $g$ for $F_{s, \tau}$ is called a local minimizer if for all metrics $\bar{g}$ in a $C^2, \alpha$-neighborhood of $g$, satisfying:

$$F_{s, \tau}[\bar{g}] \geq F_{s, \tau}[g].$$

We can also define the local maximizer for $F_{s, \tau}$ by the same way.

In [11], Y. Muto studied Riemannian functional $I[g] = \int_M |Rm|^2 dV_g$ on $\mathcal{M}_1$, and proved when $M$ is a $C^\infty$ manifold diffeomorphic to $S^n$, the mapping $I : \mathcal{M}_1/D \to R$ has a local minimum at the Riemannian metric $\bar{g}$ of positive constant sectional curvature, where $I : \mathcal{M}_1/D \to R$ is a mapping deduced from $I : \mathcal{M}_1 \to R$ and $D$ is the diffeomorphism group of $M$ and $\mathcal{M}_1/D$ is the space of orbits generated by $D$ of Riemannian metrics. Recently, S. Maity [10] generalized the result for functional $R_p(g) = \int_M |Rm|^p dV_g$ on $\mathcal{M}_1$, she proved that for a compact Riemannian manifold $(M, g)$, if $g$ is either a spherical space form and $p \in [2, \infty)$, or a hyperbolic manifold and $p \in [\frac{n}{2}, \infty)$, then $g$ is strictly stable for $R_p$. O. Kobayashi [8] investigated variational properties of the conformally invariant functional $v(g) = \frac{2}{n} \int_M |W|^\frac{2}{n}$, and derived that when $n = 4$, $S^2(1) \times S^2(1)$ endowed with the standard Einstein metric $g$ is a strictly stable critical point of $v(g)$. In [12], X. Guo, H. Li and G. Wei developed the result to the case of dimension $n = 6$.

In this paper, we concern the various properties for quadratic curvature functional $F_{s, \tau}$. Before giving our results, we state some background knowledge.

Recall a canonical decomposition of tangent space of $\mathcal{M}$. Define the divergence operator $\delta_g : S^2(M) \to T^*M$ by

$$\delta_g h_{ij} = g^{pq} h_{pj,q}.$$

and $\delta^*_g : T^*M \to S^2(M)$ its $L^2$-adjoint operator

$$(\delta^*_g \omega)_{ij} = -\frac{1}{2}(\omega_{i,j} + \omega_{j,i}),$$
in local coordinates \( \{ e_i \}_{1 \leq i \leq n} \), where \( h_{ij,k} = \nabla_k h_{ij} \) and \( \omega_{i,j} = \nabla_j \omega_i \), \( \nabla \) is the Riemannian covariant derivative of \((M, g)\). For a compact Riemannian manifold \( M \), the tangent space of \( \mathcal{M} \) at \( g \), which is denoted by \( T_g \mathcal{M} \) has the orthogonal decomposition (see [3] Lemma 4.57):

\[
T_g \mathcal{M} = S^2(M) = (\text{Im} \delta_g^* + C^\infty(M) \cdot g) \oplus (\delta_g^{-1}(0) \cap \text{tr}_g^{-1}(0)),
\]

(1.4)

where \( \text{Im} \delta_g^* \) is just the tangent space of the orbit of \( g \) under the action of the group of diffeomorphisms of \( M \). For \( T_g \mathcal{M}_1 = \{ h \in S^2(M) : \int_M \text{tr}_g h dV_g = 0 \} \), we have:

\[
T_g \mathcal{M}_1 = ((\text{Im} \delta_g^* + C^\infty(M) \cdot g) \cap T_g \mathcal{M}_1) \oplus (\delta_g^{-1}(0) \cap \text{tr}_g^{-1}(0)).
\]

(1.5)

**Definition 1.2** Let \( h \in S^2(M) \), then \( h \) is called transverse-traceless (TT for short) if \( \delta_g h = 0 \) and \( \text{tr}_g h = 0 \).

On an Einstein manifold, by [9], we have the Lichnerowicz Laplacian

\[
\triangle_L h_{ij} = \triangle h_{ij} + 2R_{ikjl} h_{kl} - \frac{2}{n} R h_{ij},
\]

(1.6)

where \( \triangle \) is the rough Laplacian, and \( \triangle_L \) maps the space of transverse-traceless tensors to itself (see [7] also). By use of (1.6) we can get following Propositions 1.1-1.2 easily.

**Proposition 1.1** Let \((M, g)\) be a compact manifold with constant sectional curvature \( 1 \). Then the least eigenvalue of the Lichnerowicz Laplacian on TT-tensors is \( 4n \).

**Proof.** By (1.6), the Lichnerowicz Laplacian on TT tensor \( h \) is

\[
\triangle_L h = \triangle h - 2nh.
\]

Then by use of the inequality

\[
0 \leq \int_M |h_{ij,k} + h_{jk,i} + h_{ki,j}|^2 dV_g,
\]

exchanging covariant derivatives and integrating by parts, we have that the least eigenvalue of the Lichnerowicz Laplacian on TT-tensors is \( 4n \). 

**Proposition 1.2** Let \((M, g)\) be a compact manifold with constant sectional curvature \( -1 \). Then the least eigenvalue of the Lichnerowicz Laplacian on TT-tensors is bounded below by \( -n \).

**Proof.** Make use of the inequality

\[
\int_M |h_{ij,k} - h_{ik,j}|^2 dV_g \geq 0,
\]

we can get the result in the same way.

It can be easily to see (Corollary 2.2 below) that the metric \( g \) with constant sectional curvature must be a critical point of \( F_{s,\tau} \). In this paper we study the stability of \( F_{s,\tau} \) at the critical point \( g \) with constant sectional curvature. We have the following.
Theorem 1.1 Let \((M, g)\) be a \(n\)-dimensional compact manifold with constant sectional curvature \(\lambda = 1\), then restricted to transverse-traceless variations, the following hold:

1. If \(s > -4\), \(\tau < \frac{6n-12}{n(n-1)}\), the second variation of \(F_{s,\tau}\) on \((M, g)\) is non-negative. Therefore \(F_{s,\tau}\) gets its local minimum in TT directions;
2. If \(s < -4\), \(\tau > \frac{6n-12}{n(n-1)}\), the second variation of \(F_{s,\tau}\) on \((M, g)\) is non-positive. Therefore \(F_{s,\tau}\) gets its local maximum in TT directions.

Theorem 1.2 Let \((M, g)\) be a \(n\)-dimensional compact manifold with constant sectional curvature \(\lambda = -1\), then restricted to transverse-traceless variations, the following hold:

1. If \(s > -4\), \(\tau > \frac{6n-12}{n(n-1)}\), the second variation of \(F_{s,\tau}\) on \((M, g)\) is non-negative. Therefore \(F_{s,\tau}\) gets its local minimum in TT directions;
2. If \(s < -4\), \(\tau < \frac{6n-12}{n(n-1)}\), the second variation of \(F_{s,\tau}\) on \((M, g)\) is non-positive. Therefore \(F_{s,\tau}\) gets its local maximum in TT directions.

Theorem 1.3 Let \((M, g)\) be a compact manifold with constant sectional curvature \(\lambda = 0\). Then the second variation of \(F_{s,\tau}\) at \(g\) is non-negative as \(s > -4\) and non-positive as \(s < -4\) when the variation is restricted in TT directions.

Theorem 1.4 Let \((M, g)\) be a compact manifold with constant sectional curvature \(\lambda = 0\). Then the second variation of \(F_{s,\tau}\) at \((M, g)\) is nonnegative as \(s + 4\tau > \frac{4}{n}(\tau - 1)\) and non-positive as \(s + 4\tau < \frac{4}{n}(\tau - 1)\) when the variation is restricted in the conformal directions.

We denote \(M_1([g])\) the space of unit volume metrics conformal to \(g\).

Theorem 1.5 Let \((M, g)\) be an \(n\)-dimensional compact manifold with constant sectional curvature \(\lambda = 1\). Then the following hold:

1. If \(n = 4\), \(s + 3\tau < -1\), then \(F_{s,\tau}\) attains a local minimizer at \(g\) in \(M_1([g])\). If \(s + 3\tau < -1\), then \(g\) is a local maximizer in \(M_1([g])\).
2. If \(n = 3\), \(\tau < 1\) and \(s > -\frac{8}{3}\tau - \frac{4}{3}\), or \(\tau > 1\) and \(s > -\frac{12}{5}\tau - \frac{8}{5}\), then \(F_{s,\tau}\) attains a local minimizer at \(g\) in \(M_1([g])\). If \(\tau < 1\) and \(s < -\frac{12}{5}\tau - \frac{8}{5}\), or \(\tau > 1\) and \(s < -\frac{8}{5}\tau - \frac{4}{3}\), then \(g\) is a local maximizer in \(M_1([g])\).
3. When \(n \geq 5\), then \(F_{s,\tau}\) attains a local minimizer at \(g\) in \(M_1([g])\) if

\[
\begin{align*}
\begin{cases}
\tau > \frac{2}{(n-1)(n-2)} \\
\frac{4n-2}{3n-4} \tau - \frac{4}{3n-4}
\end{cases}
\quad \text{or} \quad
\begin{cases}
\tau < \frac{2}{(n-1)(n-2)} \\
\frac{2n(n-1)}{3n-4} \tau - \frac{8}{3n-4}
\end{cases}
\end{align*}
\]

and \(g\) is a local maximizer in \(M_1([g])\) if

\[
\begin{align*}
\begin{cases}
\tau > \frac{2}{(n-1)(n-2)} \\
\frac{2n(n-1)}{3n-4} \tau - \frac{8}{3n-4}
\end{cases}
\quad \text{or} \quad
\begin{cases}
\tau < \frac{2}{(n-1)(n-2)} \\
\frac{4n-2}{3n-4} \tau - \frac{4}{3n-4}
\end{cases}
\end{align*}
\]
Theorem 1.6 Let \((M, g)\) be an \(n\)-dimensional compact manifold with constant sectional curvature \(\lambda = -1\), the following hold:

1. If \(n = 4\), \(s \geq 3\tau > -1\), then \(\mathcal{F}_{s, \tau}\) attains a local minimizer at \(g\) in \(\mathcal{M}_1([g])\). If \(s + 3\tau < -1\), then \(g\) is a local maximizer in \(\mathcal{M}_1([g])\).

2. If \(n = 3\), \(\tau < 1\) and \(s > -\frac{8}{3}\tau - \frac{4}{3}\), or \(\tau > 1\) and \(s > -\frac{12}{3}\tau - \frac{8}{3}\), then \(\mathcal{F}_{s, \tau}\) attains a local minimizer at \(g\) in \(\mathcal{M}_1([g])\). If \(\tau < 1\) and \(s < -\frac{12}{3}\tau - \frac{8}{3}\), or \(\tau > 1\) and \(s < -\frac{8}{3}\tau - \frac{4}{3}\), then \(g\) is a local maximizer in \(\mathcal{M}_1([g])\).

3. When \(n \geq 5\), then \(\mathcal{F}_{s, \tau}\) attains a local minimizer at \(g\) in \(\mathcal{M}_1([g])\) if

\[
\begin{align*}
\begin{cases}
\tau < \frac{2}{(n-1)(n-2)} & \text{for } 1 < n < 4, \\
-\frac{4(n-1)}{n} \tau - \frac{4}{n} < s < -n \tau - \frac{2}{n-1}.
\end{cases}
\end{align*}
\]

and \(g\) is a local maximizer in \(\mathcal{M}_1([g])\) if

\[
\begin{align*}
\begin{cases}
\tau < \frac{2}{(n-1)(n-2)} & \text{for } 1 < n < 4, \\
s > -n \tau - \frac{2}{n-1} \quad \text{or} \quad \tau > \frac{2}{(n-1)(n-2)} & \text{for } 1 < n < 4,
\end{cases}
\end{align*}
\]

2 The first variation and Euler-Lagrange equation

Suppose \((M, g)\) is a \(n\)-dimensional Riemannian manifold. We choose a local orthonormal frame \(\{e_1, e_2, \ldots, e_n\}\), in accordance with the dual coframe \(\{\omega^1, \omega^2, \ldots, \omega^n\}\). Throughout this paper, we always adopt the moving frame notation with respect to a chosen local orthonormal frame, and also the Einstein summation convention. Let \(g \in \mathcal{M}\) be an arbitrary fixed metric and \(g = g_{ij}\omega^i \otimes \omega^j\), \(g^{ij} = (g_{ij})^{-1}\). Given a tensor, we raise or lower an index by contracting the tensor with the metric tensor \(g\).

We denote \(\nabla\) as the covariant derivative, and write \(R_{ij,k} = \nabla_k R_{ij}\), \(R_{ij,kl} = \nabla_l \nabla_k R_{ij}\), the Laplacian \(\Delta R_{ij} = g^{kl} R_{ij,kl}\) and so on. For any \((0, 2)\) tensor \(S\), the Ricci identities can be expressed as

\[
S_{ij,kl} - S_{ijkl} = S_{pj} R_{pijkl} + S_{ip} R_{pijkl}.
\]

Let \(g(t) \in \mathcal{M}_1\) be a smooth variation of \(g\) with \(g(0) = g\) which can be represented locally as \(g_{ij}(x^1, \ldots, x^n; t)\). We define a tensor field \(h \in S^2(M)\) with \(h := \frac{d}{dt} g(t)\). For convenience, we write \((\cdot)'\) to stand for \(\frac{d}{dt}\). Then we have the following formulae

\[
(g_{ij})' = h_{ij}, \quad (g^{ij})' = -h^{ij}.
\]

Proposition 2.1 Let \(g(t) \in \mathcal{M}_1\) is a smooth variation \((2.1)\), then

\[
\int_M tr(g(t))h dV_g = 0, \quad \int_M \left( g^{ij} \frac{d^2}{dt^2} g_{ij} - |h|^2 + \frac{1}{2} H^2 \right) dV_g = 0,
\]

where \(|h|^2 = h^{ij} h_{ij}\), \(H = tr(g(t)) h = g^{ij} h_{ij}\).

Proof. Recall \(g(t) \in \mathcal{M}_1\), we have \(\int_M dV_g = 1\), then we have

\[
0 = \frac{d}{dt} \left( \int_M dV_g \right) = \int_M \frac{d}{dt} (dV_g) = \int_M \frac{1}{2} g^{ij} (g_{ij})' dV_g = \frac{1}{2} \int_M tr(g(t)) h dV_g.
\]
Differentiating the above equality, we get
\[
\int_M \left( g_{ij} \frac{d^2}{dt^2} g_{ij} - |h|^2 + \frac{1}{2} (\text{tr}_g h)^2 \right) dV_g = 0.
\]
This proves (2.2).

From (2.1), we get the variation of the Christoffel symbols
\[
\frac{d}{dt} \Gamma^k_{ij} = \frac{1}{2} g^{kl} (h^i_{l,j} + h^j_{k,i} - h^k_{i,j}).
\] (2.3)

By use of (2.3), we can get the following variational formulae of Riemannian curvature tensor, Ricci curvature tensor and scalar curvature directly.

**Proposition 2.2** The variations of Riemannian curvature tensor, Ricci curvature tensor and scalar curvature are expressed as
\[
(R^l_{ijk})' = \frac{1}{2} g^{pl} \left( h_{ip,kj} + h_{kp,ij} - h_{ik,pj} + h_{ip,ik} + h_{ij,pk} \right),
\]
\[
(R_{ijk})' = h_{il} R^l_{ijk} + \frac{1}{2} \left( h_{il,kj} + h_{kl,ij} - h_{ik,lj} - h_{il,jk} - h_{jl,ik} + h_{ij,lk} \right),
\]
\[
(R_{ik})' = \frac{1}{2} \left( h^j_{i,kj} + h^j_{k,ij} - \Delta h_{ik} - H_{ik} \right),
\]
\[
R' = -h^{ij} R_{ij} + h^{ij}_{,ij} - \Delta H.
\]

Now, we compute the first variation of the quadratic curvature functional \( F_{s,\tau} \) restricting on Riemannian metrics space \( \mathcal{M}_1 \) and derive its Euler-Lagrange equation. At first, we compute the first variations of \( R, \rho, \) and \( S \), respectively.

By Proposition 2.2, we have
\[
\frac{d}{dt} |Rm|^2_a = \frac{d}{dt} \left( g^{pi} g^{qk} g^{rj} g^{sk} R_{pqrs} R_{ijk} \right)
= -R^l_{ijk} R_{lij} h^{pl} - R^l_{ij} R_{lijk} h^{qi} - R^l_{ik} R_{lijk} h^{qj} - R^l_{ij} R_{lijk} h^{sk} + 2 R^l_{ijk} \frac{d}{dt} R_{lijk}
= -R^l_{ijk} R_{lijk} h^{pl} - R^l_{ij} R_{lijk} h^{qi} - R^l_{ik} R_{lijk} h^{qj} - R^l_{ij} R_{lijk} h^{sk} + 2 R^l_{ijk} h_{lq} R^q_{ijk}
+ R^{i,j,k} \left( h_{il,kj} + h_{kl,ij} - h_{ik,lj} - h_{il,jk} - h_{jl,ik} + h_{ij,lk} \right)
= -R^l_{ijk} R_{lijk} h^{pl} - R^l_{ij} R_{lijk} h^{qi} - R^l_{ik} R_{lijk} h^{qj} - R^l_{ij} R_{lijk} h^{sk}
+ 2 R^{i,j,k} h_{lq} R^q_{ijk} + 4 R^{i,j,k} h_{ij,lk},
\]
\[
\frac{d}{dt} |\text{Ric}|^2 = \frac{d}{dt} \left( g^{pi} g^{qk} R_{pq} R_{ik} \right)
= -2 h^{pi} R^k_{pi} R_{ik} + 2 R^r_{ik} \frac{d}{dt} R_{ik}
= -2 h^{pi} R^k_{pi} R_{ik} + R^{i,j,k} \left( h^j_{i,kj} + h^j_{k,ij} - \Delta h_{ik} - H_{ik} \right),
\]
\[
\frac{d}{dt} R^2 = 2 R \left( -h^{ij} R_{ij} + h^{ij}_{,ij} - \Delta H \right).
\]
Integrating by parts, we get

\[
\frac{d}{dt} R = \int_M \left( \frac{d}{dt} |Rm|^2 + \frac{1}{2} |Rm|^2 H \right) dV_g
\]

\[
= \int_M \left( - R_i^{jk} R_{ijkl} h^{pl} - R_i^{lk} R_{ijkl} h^{pj} - R_i^{lk} R_{ijkl} h^{pj} - R_i^{k} R_{ijkl} h^{rj} - R_i^{j} R_{ijkl} h^{rk} + 2 R_i^{jk} h_{ij} R^a_{ij} + 4 R_i^{jk} h_{ij,lk} + \frac{1}{2} |Rm|^2 H \right) dV_g
\]

\[
= \int_M \left( - 2 R_i^{pl} R_{pljk} + 2 R_i^{ij} - 4 \Delta R_i^{ij} - 4 R_i^{pl} R_{iplj} + 4 R_i^{pi} R_i^{pl} + \frac{1}{2} |Rm|^2 g_{ij} \right) h^{ij} dV_g,
\]

\[
\frac{d}{dt} \rho = \int_M \left( \frac{d}{dt} |\text{Ric}|^2 + \frac{1}{2} |\text{Ric}|^2 H \right) dV_g
\]

\[
= \int_M \left( - 2 h^{pl} R_i^{p,kl} + R_i^{kl} \left( h_i^{l,kl} + h_i^{l,kj} - \Delta g_{lk} - H_{lk} \right) + \frac{1}{2} |\text{Ric}|^2 H \right) dV_g
\]

\[
= \int_M \left( - \Delta R_i^{ij} - 2 R_i^{pl} R_{iplj} + R_i^{ij} - \frac{1}{2} (\Delta R) g_{ij} + \frac{1}{2} |\text{Ric}|^2 g_{ij} \right) h^{ij} dV_g,
\]

\[
\frac{d}{dt} S = \int_M \left( \frac{d}{dt} |R|^2 + \frac{1}{2} |R|^2 H \right) dV_g
\]

\[
= \int_M \left( 2 R_i^{l,kl} R_{ijkl} + R_i^{rkl} - \frac{1}{2} (\Delta R) g_{ij} + \frac{1}{2} R_i^{r,kl} + 2 \frac{1}{2} |R|^2 g_{ij} \right) h^{ij} dV_g
\]

Lemma 2.1 \((\mathbb{E})\) The gradients of the functionals \(R\), \(\rho\), \(S\), and \(F_{s,\tau}\) are given by the following equations.

\[
(\nabla R)_{ij} = -2 R_i^{pl} R_{pljk} + 2 R_i^{ij} - 4 \Delta R_i^{ij} - 4 R_i^{pl} R_{iplj} + 4 R_i^{pi} R_i^{pl} + \frac{1}{2} |Rm|^2 g_{ij},
\]

\[
(\nabla \rho)_{ij} = -\Delta R_i^{ij} - 2 R_i^{pl} R_{iplj} + R_i^{ij} - \frac{1}{2} (\Delta R) g_{ij} + \frac{1}{2} |\text{Ric}|^2 g_{ij},
\]

\[
(\nabla S)_{ij} = 2 R_i^{ij} - 2 (\Delta R) g_{ij} - 2 R_R i^{ij} + \frac{1}{2} R_i^{i,ij} + \frac{1}{2} R^2 g_{ij},
\]

\[
(\nabla F_{s,\tau})_{ij} = -2 R_i^{pl} R_{pljk} + 2 R_i^{ij} - 4 \Delta R_i^{ij} - 4 R_i^{pl} R_{iplj} + 4 R_i^{pi} R_i^{pl} + \frac{1}{2} |Rm|^2 g_{ij}
\]

\[
+ s \left( - \Delta R_i^{ij} - 2 R_i^{pl} R_{iplj} + R_i^{ij} - \frac{1}{2} (\Delta R) g_{ij} + \frac{1}{2} |\text{Ric}|^2 g_{ij} \right)
\]

\[
+ \tau \left( 2 R_i^{ij} - 2 (\Delta R) g_{ij} - 2 R_R i^{ij} + \frac{1}{2} R^2 g_{ij} \right). \tag{2.5}
\]

By the Lagrangian multiplier method, a Riemannian metric \(g \in \mathcal{M}_1\) is critical for \(F_{s,\tau}|_{\mathcal{M}_1}\) if and only if it satisfies the equation

\[
(\nabla F_{s,\tau})_{ij} = cg_{ij} \tag{2.6}
\]

for some constant \(c\). The Euler-Lagrange equations for \(F_{s,\tau}|_{\mathcal{M}_1}\) is obtained after a simple computation.

**Theorem 2.1** Let \(M\) be a compact \(n\)-dimensional Riemannian manifold. Then the Euler-Lagrange
The equations of $\mathcal{F}_{s,\tau}|\mathcal{M}_1$ are
\[-(4 + s)\triangle R_{ij} + (2 + s + 2\tau)R_{ij} + \frac{2-2\tau}{n}\triangle R g_{ij} - 2R_i{}^p{}_{kl}R_{jpkl} - (4 + 2s)R_{ij} + 4R_{ij} - 4\triangle R_{ij} - 4R_{ijkl} + 4R_{jpj} + 4R_{ij} + 1 = 0,\]
\[n - 4)(|Rm|^2 + s|Ric|^2 + \tau|R|^2) - (4 + ns + 4(n - 1)\tau)\triangle R = 2nc.\]

In fact, (2.8) comes from (2.6) by taking trace on both sides. Substituting (2.8) into (2.6) we can get (2.7) directly.

From Theorem 2.1, we know that any compact Riemannian manifold $(M, g)$ with constant sectional curvature is a critical metric for $\mathcal{F}_{s,\tau}$ on $\mathcal{M}_1$. However, Einstein metric can not always be a critical point of these functionals.

**Corollary 2.1** ([3, 5]) Restricting on $\mathcal{M}_1$, an Einstein metric $g$ is a critical point of $\mathcal{F}_{s,\tau}$ if and only if the metric $g$ satisfies
\[R_i{}^p{}_{kl}R_{jpkl} = \frac{1}{n}|Rm|^2 g_{ij}.\]

**Corollary 2.2** Any metric with constant sectional curvature is a critical point of $\mathcal{F}_{s,\tau}$ restricted on $\mathcal{M}_1(M^n)$.

### 3 Second variations on constant sectional curvature manifolds

In this section, we derive the second variations of $\mathcal{F}_{s,\tau}|\mathcal{M}_1$ at the metric with constant sectional curvature. Suppose $(M, g)$ has constant sectional curvature, then for some constant $\lambda$,
\[R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk}).\]

From (3.1), we also have
\[R_{ij} = (n - 1)\lambda g_{ij}, \quad R = n(n - 1)\lambda.\]

Following (2.5), the first variation of $\mathcal{F}_{s,\tau}$ is
\[
\frac{d}{dt} \mathcal{F}_{s,\tau} = \int_M \left(-2R_i{}^p{}_{kl}R_{jpkl} + 2R_{ij} - 4\triangle R_{ij} - 4R_{ijkl} + 4R_{jpj} + \frac{1}{2}|Rm|^2 g_{ij} + s\left(-\triangle R_{ij} - 2R_{ijkl} + R_{ij} - \frac{1}{2}(\triangle R)g_{ij} + \frac{1}{2}|Ric|^2 g_{ij}\right) + \tau\left(2R_{ij} - 2(\triangle R)g_{ij} - 2R_{ij} + \frac{1}{2}R^2 g_{ij}\right)\right)h^{ij}dV_g.
\]
For convenience, we write \( G = \nabla F_{s, \tau} \),

\[
G_{ij} = -2R_{ij}^{pk} R_{jplk} + 2R_{ij} - 4\triangle R_{ij} - 4R_{ij}^{pl} R_{ij}^{pk} + 4R_{jp} R_{ij}^{pk} + \frac{1}{2} |Rm|^2 g_{ij}
\]

\[
+ s \left( - \triangle R_{ij} - 2R^{pl} R_{jpl} + R_{ij} - \frac{1}{2} (\triangle R) g_{ij} + \frac{1}{2} |Ric|^2 g_{ij} \right)
\]

\[
+ \tau \left( 2R_{ij} - 2(\triangle R) g_{ij} - 2R R_{ij} + \frac{1}{2} R^2 g_{ij} \right).
\]

(3.4)

Then, we rewrite the first variation of \( F_{s, \tau} \) (3.3) as

\[
\frac{d}{dt} F_{s, \tau} = \int_M G_{ij} h^{ij} dV_g.
\]

(3.5)

Differentiating equality (3.4) again, by use of Proposition 2.1, we have

\[
\frac{d^2}{dt^2} \bigg|_{t=0} F_{s, \tau} = \frac{d}{dt} \bigg|_{t=0} \int_M G_{ij} h^{ij} dV_g
\]

\[
= \int_M \frac{d}{dt} \bigg|_{t=0} (G_{ij}) h^{ij} dV_g + \int_M \left( -2c|\mathbf{h}|^2 + c g^{ij} \frac{d^2}{dt^2} \bigg|_{t=0} (g_{ij}) \right) dV_g + \int_M \frac{1}{2} c R^2 dV_g
\]

\[
= \int_M \frac{d}{dt} \bigg|_{t=0} (G_{ij}) h^{ij} dV_g - \int_M c|\mathbf{h}|^2 dV_g.
\]

(3.6)

Nextly, we concern the second variations on TT directions at some critical metric with constant sectional curvature.

### 3.1 Transverse-traceless variations

According to (3.4) and (3.6), we compute \( \frac{d}{dt} \bigg|_{t=0} (G_{ij}) \) at the metric \( g \) with constant sectional curvature.

**Proposition 3.1** If \( g \) has constant sectional curvature, satisfying (3.1) and (3.2), then

\[
(R_{ij,k})' = (R_{ij}')_k - \lambda (n-1) h_{ij,k},
\]

\[
(R_{ij,kl})' = (R_{ij}')_{kl} - \lambda (n-1) h_{ij,kl},
\]

\[
(\triangle R_{ij})' = (\triangle R_{ij}') - \lambda (n-1) \triangle h_{ij},
\]

\[
(\triangle R)' = \triangle (\text{tr} Ric') - \lambda (n-1) \triangle H.
\]

From Proposition 3.1 we calculate the second variations on TT direction.

\[
\int_M (R_{ij}^{plk} R_{jplk})' h^{ij} dV_g = \int_M \left( (R_{ij}^{plk})' R_{jplk} + R_{ij}^{plk} (R_{jplk})' \right) h^{ij} dV_g
\]

\[
= \int_M \left( - h^{pm} R_{lmpk} R_{jplkl} - h^{kn} R_{ipkl} R_{jplml} - h^{ks} R_{ipks} R_{jplkl} \right)
\]

\[
+ 2 R_{jkpl}(R_{ijkl})' h^{ij} dV_g
\]

\[
= \int_M \left( (4 - 2n) \lambda^2 h_{ij} + 2 R_{jplkl}(h_{iq} R_{qplkl}
\]

\[
+ \frac{1}{2} (h_{pi,kl} + h_{li,pk} - h_{pl,ik} - h_{pi,kl} - h_{ki,pl} + h_{pk,il})) \) h^{ij} dV_g
\]

\[
= \int_M (2(n+1) \lambda^2 |\mathbf{h}|^2 - 2 \lambda \triangle h) dV_g.
\]
By the same way, we can get the following formulae. Here we omit the detailed calculation.

\[
\begin{align*}
\int_M (\Delta R_{ij})' h^{ij} dV_g &= \int_M \left( -\frac{1}{2} h \Delta^2 h + \lambda h \Delta h \right) dV_g, \\
\int_M (R_{ij})' h^{ij} dV_g &= 0, \\
\int_M (R'_i R_{jj})' h^{ij} dV_g &= \int_M \left( \lambda^2 (n^2 - 1) |h|^2 - \lambda(n-1) h \Delta h \right) dV_g, \\
\int_M (R'^d R_{ip} j')' h^{ij} dV_g &= \int_M \left( (n^2 - 1) \lambda^2 |h|^2 - \frac{1}{2} \lambda(n-2) h \Delta h \right) dV_g, \\
\int_M (\|Rm\|^2 g_{ij})' h^{ij} dV_g &= \int_M 2\lambda^2 n(n-1) |h|^2 dV_g, \\
\int_M (\Delta R g_{ij})' h^{ij} dV_g &= 0, \\
\int_M (\|Ric\|^2 g_{ij})' h^{ij} dV_g &= \int_M \lambda^2 n(n-1) |h|^2 dV_g, \\
\int_M (R R_{ij})' h^{ij} dV_g &= \int_M \left( \lambda^2 n^2 (n-1) |h|^2 - \frac{1}{2} \lambda n(n-1) h \Delta h \right) dV_g, \\
\int_M (R^2 g_{ij})' h^{ij} dV_g &= \int_M \lambda^2 n^2 (n-1)^2 |h|^2 dV_g.
\end{align*}
\]

Following (3.6) and the above equations, we can get the second variations of \( \mathcal{F}_{s, \tau} \) in TT direction.

**Theorem 3.1** Let \((M, g)\) be a compact Riemannian manifold with constant curvature satisfying (3.1) and \( h \) a TT tensor. Then the second variation of \( \mathcal{F}_{s, \tau} \) is

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{F}_{s, \tau} = \int_M \left( 2(\Delta_L + 2(n-1)\lambda)(\Delta_L + (n+2)\lambda)h \\
+ \frac{1}{2}(\Delta_L + 2(n-1)\lambda)(s\Delta_L + (n-1)(4s + 2n\tau)\lambda) \right) dV_g
\]

\[
= \int_M \left( \left( \Delta_L + 2(n-1)\lambda \right) \left( \frac{4 + s}{2} \Delta_L + \lambda(2n+4 + (n-1)(2s+n\tau)\lambda) \right) h, h \right) dV_g. \tag{3.7}
\]

When \( \lambda = 0 \), (3.7) becomes

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{F}_{s, \tau} = \int_M 2(1 + \frac{s}{4}) h \Delta^2 h dV_g. \tag{3.8}
\]

By (3.3), we can get the following corollary easily.

**Corollary 3.1** Let \((M, g)\) be a compact manifold with constant sectional curvature \( \lambda = 0 \). Then the second variation of \( \mathcal{F}_{s, \tau} \) at \( g \) is non-negative as \( s > -4 \) and non-positive as \( s < -4 \) when the variation is restricted on TT directions.

Now, we focus on the case \( \lambda \neq 0 \) and finish the proof of Theorems 1.1 and 1.2. Firstly, if \( \lambda > 0 \), we set \( \lambda = 1 \). We then have

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{F}_{s, \tau} = \int_M \left( \left( \Delta_L + 2(n-1) \right) \left( \frac{4 + s}{2} \Delta_L + (2n+4 + (n-1)(2s+n\tau)) \lambda \right) h, h \right) dV_g. \tag{3.9}
\]
Proof of Theorem 1.1. By Proposition 1.1, we know the least eigenvalue of the Lichnerowicz Laplacian on TT tensors is $4n$. Let $-\triangle L h_{ij} = \lambda_L h_{ij}$, and rewrite (3.9) as

$$\frac{d^2}{dt^2} \bigg|_{t=0} F_{s,\tau} = \int_M \left( (\lambda_L - 2(n-1)) \left( 4 + s \frac{\lambda_L - 2n - 4 - (n-1)(2s + n\tau)}{2} \right) \right) |h|^2 dV_g. \quad (3.10)$$

The first term $\lambda_L - 2(n-1) > 0$. We consider the second term. If $s > -4$, $\tau < \frac{6n-12}{n(n-1)}$,

$$\frac{4 + s}{2} \lambda_L - (2n + 4) - (n-1)(2s + n\tau) > \frac{4 + s}{2} 4n - (2n + 4) - (n-1)(2s + n\tau) > 6n - 12 - n(n-1)\tau > 0,$$

So, in this case we have \( \frac{d^2}{dt^2} \bigg|_{t=0} F_{s,\tau} \geq 0 \). If $s < -4$, $\tau > \frac{6n-12}{n(n-1)}$,

$$\frac{4 + s}{2} \lambda_L - (2n + 4) - (n-1)(2s + n\tau) < \frac{4 + s}{2} 4n - (2n + 4) - (n-1)(2s + n\tau) < 6n - 12 - n(n-1)\tau < 0.$$

Therefore, in this case we have \( \frac{d^2}{dt^2} \bigg|_{t=0} F_{s,\tau} \leq 0 \).

Proof of Theorem 1.2. If $\lambda < 0$, we set $\lambda = -1$. We have

$$\frac{d^2}{dt^2} \bigg|_{t=0} F_{s,\tau} = \int_M \left( (\triangle L - 2(n-1)) \left( 4 + s \frac{\lambda_L - 2n + 4 + (n-1)(2s + n\tau)}{2} \right) \right) |h|^2 dV_g. \quad (3.11)$$

By Proposition 1.2, the first term $\lambda_L + 2(n-1) > 0$. We now consider the second term. If $s > -4$, and $\tau > \frac{6n-12}{n(n-1)}$,

$$\frac{4 + s}{2} \lambda_L + (2n + 4) + (n-1)(2s + n\tau) > \frac{4 + s}{2} n + 2n + 4 + (n-1)(2s + n\tau) > -6n + 12 + n(n-1)\tau > 0.$$

In this case we have \( \frac{d^2}{dt^2} \bigg|_{t=0} F_{s,\tau} \geq 0 \). By the same way we know \( \frac{d^2}{dt^2} \bigg|_{t=0} F_{s,\tau} \geq 0 \) when $s < -4$, $\tau < \frac{6n-12}{n(n-1)}$.

3.2 Conformal variations

Now, we consider the conformal variations of the functionals $F_{s,\tau}$ at a Riemannian metric with constant sectional curvature satisfying (3.1). Let $M_1[g]$ denote the space of unit volume metrics conformal to $g$. The tangent space of $M_1[g]$ consists of functionals with mean value zero.
Proposition 3.2 If $g$ has constant sectional curvature, satisfying (3.1) and (3.2), and let $h = fg,$

$$(R_{ij})' = -\frac{1}{2}(n - 2)f_{,ij} - \frac{1}{2}\triangle f g_{ij},$$

$$(R_{ij,kl})' = (R'_{ij})_{,kl} - \lambda(n - 1)f_{,kl}g_{ij},$$

$$(\triangle R_{ij})' = (\triangle R'_{ij}) - \lambda(n - 1)\triangle f g_{ij},$$

$$(\triangle R)' = -(n - 1)\triangle^2 f - \lambda n(n - 1)\triangle f.$$

Following proposition (3.2), we have,

$$\int_M (R_{ijkl}^{pl})' f g^{ij} dV_g = \int_M \left((R_{ijkl}^{pl})' R_{ijkl} + R_{ijkl}^{pl} (R_{ijkl})'ight) h^{ij} dV_g,$$

$$= \int_M \left(- f g^{mn} R_{ijkl} R_{mpkl} - f g^{kl} R_{ijkl} R_{jpml} - f g^{ls} R_{lpkl} R_{jpkl}ight.$$

$$+ 2 R_{jkpl} (R_{ijkl})') f g^{ij} dV_g,$$

$$= \int_M \left(- 3 |Rm|^2 f^2 + 2 f R_{ijkl} (R'_{ijkl}) dV_g,$$

$$= \int_M \left(- 3 |Rm|^2 f^2 + 2 \lambda f (g_{ij} g^{kl} - g^{il} g^{lj}) R_{ijkl} R'_{ijkl} dV_g,$$

$$= \int_M \left(- 3 |Rm|^2 f^2 + 4 \lambda f g_{ij} ((g^{kl} R_{ijkl})' + f g^{kl} R_{ijkl}) dV_g,$$

$$= \int_M \left(- 2 \lambda^2 n(n - 1) f^2 - 4 \lambda(n - 1) f \triangle f dV_g.$$
Using the above equalities, we get the conformal variations for $F_{s, \tau}$ at metrics with constant sectional curvature.

**Theorem 3.2** Let $(M, g)$ be a compact Riemannian manifold with constant curvature satisfying (3.1) and $h = fg$ with $\int_M f dV_g = 0$. Then the second variation of $F_{s, \tau}$ is

$$
\frac{d^2}{dt^2} \bigg|_{t=0} F_{s, \tau} = \int_M \left(2(n-1)f \Delta^2 f + 8\lambda(n-1)f \Delta f - 2\lambda^2(n^2-n)(n-4)f^2 + s\left(\frac{1}{2}n(n-1)f \Delta^2 f - \frac{1}{2}\lambda(n-1)(n^2-10n+8)f \Delta f - \lambda^2 n(n-1)^2(n-4)f^2\right) + \tau\left(2(n-1)^2 f \Delta^2 f - \lambda n(n-1)^2(n-6)f \Delta f - \lambda^2 n(n-1)^2(n-4)f^2\right)\right) dV_g. \tag{3.12}
$$

When $\lambda = 0$, (3.12) becomes

$$
\frac{d^2}{dt^2} \bigg|_{t=0} F_{s, \tau} = \int_M \left(\frac{1}{2}(n-1)(sn + 4(n-1)\tau + 4)f \Delta^2 f dV_g. \tag{3.13}
$$

We then have

**Corollary 3.2** Let $(M, g)$ be a compact manifold with constant sectional curvature $\lambda = 0$. Then the second variation of $F_{s, \tau}$ on $(M, g)$ is nonnegative as $s + 4\tau > \frac{4}{n}(\tau - 1)$ when the variation is restricted on the conformal direction.

Next we consider the case $\lambda \neq 0$ and finish the proof of Theorems 1.5 and 1.6. Firstly, if $\lambda > 0$, we set $\lambda = 1$. We then have

$$
\frac{d^2}{dt^2} \bigg|_{t=0} F_{s, \tau} = \int_M \left(\frac{1}{2}(n-1)(sn + 4(n-1)\tau + 4)f \Delta^2 f dV_g. \tag{3.14}
$$

**Proposition 3.3** ([1]) If $(M^n, g)$ is a compact manifold satisfying $\text{Ric} \geq (n-1) \cdot g$, then the lowest non-trivial eigenvalue satisfies $\lambda_1 \geq n$, and the equality holds if and any if $(M^n, g)$ is isometric to $(S^n, g_s)$.

Let $\mu$ be a non-zero eigenvalue of $\Delta$. We consider the following polynomial

$$
P_1(\mu) = (n-1)(\mu - n)\left(\frac{ns - 4\tau + 4n\tau + 4}{2}\mu + (n-4)(n^2\tau + ns - n\tau - s + 2)\right). \tag{3.15}
$$

Since the variation is restricted to $M_1[g]$, we should to prove that the second variation of the functional on functions with mean value zero is non-negative (or non-positive in the maximizing case).

**Proof of Theorem 1.5.** By Proposition (3.3), the second term $\mu - n \geq 0$. Then the sign of the second variations for $F_{s, \tau}$ is determined by the third term. If $n = 4$,

$$
P_1(\mu) = 6(\mu - 4)(s + 3\tau + 1)\mu. \tag{3.16}
$$
Form (3.16), the first part of Theorem 1.5 can get easily.

If \( n = 3 \),

\[
P_1(\mu) = 2(\mu - 3)(\frac{3s + 8\tau + 4}{2} - (6\tau + 2s + 2)).
\] (3.17)

The functional will be minimizing if \( \frac{3s + 8\tau + 4}{2} \) > 0 and

\[
\frac{3s + 8\tau + 4}{2} \mu - (6\tau + 2s + 2) \geq \frac{5}{2} s + 6\tau + 4 > 0.
\]

Then \( P_1(\mu) \geq 0 \) if the following hold:

\[
\begin{cases}
\tau < 1 & \text{or} & \tau > \frac{12}{5} \tau - \frac{8}{5} \\
s > -\frac{8}{3} \tau - \frac{4}{3}
\end{cases}
\]

Similarly we can get \( P_1(\mu) \leq 0 \) if the following hold:

\[
\begin{cases}
\tau < 1 & \text{or} & \tau > \frac{12}{5} \tau - \frac{8}{5} \\
s > -\frac{8}{3} \tau - \frac{4}{3}
\end{cases}
\]

When \( n \geq 5 \), the functional will be minimizing if \( ns - 4\tau + 4n\tau + 4 > 0 \) and

\[
\frac{ns - 4\tau + 4n\tau + 4}{2} n + (n - 4)(n^2\tau + ns - n\tau - s + 2) > 0.
\]

Then \( P_1(\mu) \geq 0 \) if the following hold

\[
\begin{cases}
\tau > \frac{2}{(n-1)(n-2)} & \text{or} & \tau < \frac{2}{3(3n-4)} \\
s > -\frac{8}{3n-4} \tau - \frac{4}{3n-4}
\end{cases}
\]

And we can get \( P_1(\mu) \leq 0 \) if the following hold

\[
\begin{cases}
\tau > \frac{2}{(n-1)(n-2)} & \text{or} & \tau < \frac{2}{3(3n-4)} \\
s < -\frac{8}{3n-4} \tau - \frac{4}{3n-4}
\end{cases}
\]

Finally we consider the case \( \lambda < 0 \). We set \( \lambda = -1 \).

\[
\frac{d^2}{dt^2} F_{s,\tau} = \int_M \left( (n-1)(\triangle - n) \left( \frac{ns - 4\tau + 4n\tau + 4}{2} \right) + (n-4)(n^2\tau - n\tau - s + 2) \right) f, f \rangle dV_g. \] (3.18)

Let \( \mu \) is a non-zero eigenvalue of \( \triangle \), and consider the following polynomial

\[
P_2(\mu) = (n-1)(\mu + n) \left( \frac{ns - 4\tau + 4n\tau + 4}{2} - (n - 4)(n^2\tau + ns - n\tau - s + 2) \right). \] (3.19)

Since the first term of \( P_2(\mu) \) is always nonnegative, then we just consider the second term. Let \( n = 3 \), then \( P_2(\mu) \) is nonnegative if

\[
\begin{cases}
\tau > 2 & \text{or} & \tau < 2 \\
s > -\frac{8}{3} \tau - \frac{4}{3} & \text{or} & s > -3\tau - 2.
\end{cases}
\]

And \( P_2(\mu) \) is non-positive if

\[
\begin{cases}
\tau > 2 & \text{or} & \tau < 2 \\
s < -3\tau - 2 & \text{or} & s < -\frac{8}{3} \tau - \frac{4}{3}.
\end{cases}
\]

When \( n \geq 4 \), do it in the same way as Theorem 1.5. Then we finish the proof of Theorem 1.6.
References

[1] Aubin T. Some nonlinear problems in Riemannian geometry. Berlin: Springer-Verlag, 1998
[2] Berger M. and Ebin D. Some decompositions of the space of symmetric tensors on a Riemannian manifold. J. Diff. Geom. 1969, 3: 379-392
[3] Besse A. L. Einstein Manifolds. Berlin: Springer-Verlag, 1987
[4] Blair D. E. Spaces of metrics and curvature functionals, Handbook of differential geometry, North-Holland, Amsterdam, Vol(I): 153-185, 2000
[5] Catino G. Some rigidity results on critical metrics for quadratic functionals. Calc. Var. Partial Differential Equations, 2015, 54(3):2921-2937
[6] Guo X., Li H. and Wei G. On variational formulas of a conformally invariant functional. Results. Math. 2015, 67: 49-70
[7] Gursky M. and Viaclovsky J. Rigidity and stability of Einstein metrics for quadratic curvature functionals. Journal Für die Reine und Angewandte Mathematik, 2015, 700: 37-91
[8] Kobayashi O. On a conformally invariant functional of the space of Riemannian metrics. J. Math. Soc. Japan, 1984, 37(3): 373-389
[9] Lichnerowicz A. Propagaturs et commutateurs en relativité gén érale (French). Inst. Hautes Études Sci. Publ. Math. 1961,10: 56 pp
[10] Maity S. On the stability of the $L^p$-norm of the Riemannian curvature tensor. Proc. Indian Acad. Sci.(Math. Sci.), 2014, 124(3): 383-409
[11] Muto Y. Curvature and critical Riemannian metric. J. Math. Soc. Japan 1974, 26(4): 686-697
[12] Viaclovsky J. Critical Metrics for Riemannian Curvature Functionals. Geometric analysis, IAS/Park City Math. Ser., 22, Amer. Math. Soc., Providence, RI, 2016, 197-274.

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