A convergent finite element algorithm
for generalized mean curvature flows of closed surfaces

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An algorithm is proposed for generalized mean curvature flow of closed two-dimensional surfaces, which include inverse mean curvature flow, powers of mean and inverse mean curvature flow, etc. Error estimates are proven for semi- and full discretisations for the generalized flow. The algorithm proposed and studied here combines evolving surface finite elements, whose nodes determine the discrete surface, and linearly implicit backward difference formulae for time integration. The numerical method is based on a system coupling the surface evolution to non-linear second-order parabolic evolution equations for the normal velocity and normal vector. Convergence proofs are presented in the case of finite elements of polynomial degree at least two and backward difference formulae of orders two to five. The error analysis combines stability estimates and consistency estimates to yield optimal-order $H^1$-norm error bounds for the computed surface position, velocity, normal vector, normal velocity, and therefore for the mean curvature. The stability analysis is performed in the matrix–vector formulation, and is independent of geometric arguments, which only enter the consistency analysis. Numerical experiments are presented to illustrate the convergence results, and also to report on monotone quantities, e.g. Hawking mass for inverse mean curvature flow. Complemented by experiments for non-convex surfaces.

Keywords: generalized mean curvature flow, inverse mean curvature flow, $H^\alpha$-flow, optimal-order convergence, evolving surface finite elements, linearly implicit BDF methods, energy estimates, monotone quantities

1. Introduction

In this paper we propose and prove convergence of a numerical method for the evolution of a two-dimensional closed surface $\Gamma(t)$ evolving under generalized mean curvature flow. The velocity of the surface $\Gamma(t)$ is given by the velocity law:

$$v = -V(H)n_{\Gamma(t)},$$

(1.1)

here is $H$ the mean curvature of the surface $\Gamma(t)$, $n_{\Gamma(t)}$ denotes the outward unit normal vector, (i.e. using an orientation such that the mean curvature of a sphere is positive), and $V : \mathbb{R} \rightarrow \mathbb{R}$ is a given function.

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Many notable geometric flows fit into this framework, in particular

- mean curvature flow, see Huisken (1984),
- inverse mean curvature flow, see Huisken & Polden (1999); Huisken & Ilmanen (2001),
- powers of mean curvature flow, see Schulze (2005, 2006, 2008, 2002),
- and powers of inverse mean curvature flow, see Gerhardt (2014); Scheuer (2016),
- as well as a logarithmic mean curvature flow, see Alessandroni & Sinestrari (2010); Espin (2020), etc.

There are a few papers which prove theoretical results for the general flow (1.1), see, e.g. Huisken & Polden (1999); Alessandroni & Sinestrari (2010); Espin (2020).

The solution of these flows, and their properties as well, are of theoretical and modelling interest: For example, to prove interesting geometric inequalities, most notably, the weak solvability theory of inverse mean curvature flow was used to prove: the positive mass conjecture Schoen & Yau (1979), and the Riemannian Penrose inequality from general relativity Huisken & Ilmanen (2001). On the modelling side, mean curvature flow is used for various purposes, see the references in Kovács et al. (2019), various types of inverse mean curvature flows are utilised in image processing (Alvarez et al., 1993, equation (23) and Section 8), (Angenent et al., 1998, equation (15)), and Sapiro & Tannenbaum (1994), see as well (Malladi & Sethian, 1995).

A number of numerical methods have been proposed for the above flows. For surfaces finite volume algorithms were introduced by Pasch (1998) for mean curvature and inverse mean curvature flow. A surface finite element based algorithm for the general flow (1.1) was proposed by Barrett, Garcke, and Nürnberg in Barrett et al. (2008). For curves and networks evolving according the general flow they had also proposed a finite element algorithm in Barrett et al. (2007). For inverse mean curvature flow many algorithms have been proposed, which use an equivalent formulation based on a non-linear singular elliptic equation on an unbounded domain derived by Huisken & Ilmanen (2001). This problem represents a level-set formulation for the inverse mean curvature flow. A finite element method for a regularized flow, based on the regularisation of the singular elliptic problem, was proposed and analysed by Feng, Neilan, and Prohl in Feng et al. (2007). They also proved error estimates for the regularized problem. Using the same approach, level-set finite element method was proposed for powers of mean curvature flow by Kröner (2013), the method was analysed by Kröner, Kröner, and Kröner in Kröner et al. (2018), while convergence rates for the semi-discretisation were proven by Kröner (2017), similarly, for inverse mean curvature flow by Kröner (2019).

However, to our knowledge, no convergence results have been proved for evolving surface finite element algorithms for any of the above generalized mean curvature flows. Also, to our knowledge, no convergence results are available for any algorithm for the generalized mean curvature flow (1.1).

The main goals of the present paper are:

- **To propose a finite element algorithm for the generalized mean curvature flow (1.1) of closed two-dimensional surfaces, and to prove optimal-order error estimates for the proposed algorithm, under minor conditions on the function V.**

To achieve these goals, the key idea is to derive non-linear parabolic evolution equations for the normal velocity \( V = V(H) \) and the surface normal \( n \) along the generalized mean curvature flow. In recent previous works for mean curvature flow Kovács et al. (2019) and Willmore flow Kovács et al. (2020), which motivate this approach, the analogous evolution equations for \( H \) and \( n \) were used. In Huisken & Polden (1999); Huisken & Ilmanen (2001, 2008) the authors have already derived evolution
equations for the mean curvature $H$ and the surface normal $n$. The evolution equation for the normal velocity $V$ is first derived here, and until the present work it was not evident that the evolution equations for $V$ and $n$ form a closed system that does not involve further geometric quantities.

A new non-linearity appears with the time derivatives in the evolution equations for $V$ and $n$, but otherwise they are very similar to the evolution equations for $H$ and $n$ for mean curvature flow Kovács et al. (2019). This structural similarity enables us to use many results from Kovács et al. (2019), but there are substantial parts that require a careful and extended analysis.

The system coupling the non-linear evolution equations for $V$ and $n$, the velocity law (1.1), and an ordinary differential equation for the surface evolution is spatially discretized using evolving surface finite elements (of degree at least 2), and using linearly implicit backward difference formulae (of order 2 to 5).

We will prove optimal-order $H^1$-norm semi- and fully discrete error estimates for the surface position $X$ and all variables $v, n, V$ (and hence also for $H$). The convergence proofs clearly separate the issues of stability and consistency, and hold under sufficient regularity assumptions on the solution of generalized mean curvature flow, which excludes the formation of singularities.

For proving convergence of both the semi- and full discretisations, the main issue is to prove stability, that is to bound the errors in terms of consistency defects and errors in the initial values. The stability proofs (as for Kovács et al. (2019) and Kovács et al. (2020)) are performed in the matrix–vector formulation, where the similarity of the coupled system for generalized mean curvature flow and mean curvature flow Kovács et al. (2019) will become more apparent. The main difference is a solution-dependent mass matrix with the time derivatives. A key step in the stability proof is to establish uniform-in-time $W^{1,\infty}$-norm error bounds for all variables, shown via $H^1$-norm error bounds using inverse estimates. Additionally, in order to estimate the terms with the non-linear mass matrix, these $W^{1,\infty}$-bounds are used to prove $h$-uniform upper and lower bounds for the approximation of the mean curvature and some related variables. Due to the mentioned structural similarity, most of the stability proof uses the same techniques as (Kovács et al., 2019, Proposition 7.1), it is based on energy estimates testing with the time-derivative of the errors. The terms involving the solution-dependent mass matrix are estimated using a new technical lemma and a solution-dependent norm equivalence result based on the mentioned upper and lower bounds.

Consistency estimates, i.e. bounding the defects occurring upon inserting appropriate projections of the exact solution into the method, are analogous to the same result for mean curvature flow (Kovács et al., 2019, Lemma 8.1), and we mainly focus on the differences due to the non-linearity.

The paper is organized as follows. Section 2 introduces some basic notations and geometric concepts and is mainly devoted to deriving the evolution equations for the normal velocity and the normal vector along the generalized mean curvature flow. The coupled non-linear system and its weak formulation, which serves as the basis of the algorithm, is presented here. Section 3 describes the evolving surface finite element semi-discretization, the matrix–vector formulation. We also discuss here the similarity of the matrix–vector formulation to that of mean curvature flow. Section 4 describes the linearly implicit time discretisation. Section 5 states the main results of the paper: optimal-order semi- and fully discrete error bounds in the $H^1$-norm for the errors in all variables. In Section 6 we prove the semi-discrete stability result after presenting the required auxiliary results. Section 7 contains the semi-discrete consistency analysis. In Section 8 we combine the results of the previous two sections to prove the semi-discrete convergence theorem. Section 9, 10 and 11 contain the same parts, respectively, for the full discretisation. Section 12 presents numerical experiments illustrating and complementing our theoretical results: reporting on convergence tests, on numerical solutions for various flows also with non-convex initial surfaces, and on the behaviour of monotone quantities, e.g. Hawking mass.
2. Evolution equations for generalized mean curvature flow

2.1 Basic notions and notation

We start by introducing some basic concepts and notations, taking this description almost verbatim from Kovács et al. (2019).

We consider the evolving two-dimensional closed surface \( \Gamma[X] \subset \mathbb{R}^3 \) as the image

\[
\Gamma[X] = \Gamma[X(\cdot, t)] = \{X(p, t) : p \in \Gamma^0\},
\]

of a smooth mapping \( X : \Gamma^0 \times [0, T) \to \mathbb{R}^3 \) such that \( X(\cdot, t) \) is an embedding for every \( t \). Here, \( \Gamma^0 \) is a smooth closed initial surface, and \( X(p, 0) = p \). In view of the subsequent numerical discretization, it is convenient to think of \( X(p, t) \) as the position at time \( t \) of a moving particle with label \( p \), and of \( \Gamma[X] \) as a collection of such particles.

The velocity \( v(x, t) \in \mathbb{R}^3 \) at a point \( x = X(p, t) \in \Gamma[X(\cdot, t)] \) equals

\[
\partial_t X(p, t) = v(X(p, t), t). \tag{2.1}
\]

For a known velocity field \( v \), the position \( X(p, t) \) at time \( t \) of the particle with label \( p \) is obtained by solving the ordinary differential equation (2.1) from 0 to \( t \) for a fixed \( p \).

For a function \( u(x, t) (x \in \Gamma[X], 0 \leq t \leq T) \) we denote the material derivative (with respect to the parametrization \( X \)) as

\[
\partial^* u(x, t) = \frac{d}{dt} u(X(p, t), t) \quad \text{for} \quad x = X(p, t).
\]

On any regular surface \( \Gamma \subset \mathbb{R}^3 \), we denote by \( \nabla_{\Gamma} u : \Gamma \to \mathbb{R}^3 \) the tangential gradient of a function \( u : \Gamma \to \mathbb{R}^\kappa \), and in the case of a vector-valued function \( u = (u_1, u_2, u_3)^T : \Gamma \to \mathbb{R}^3 \), we let \( \nabla_{\Gamma} u = (\nabla_{\Gamma} u_1, \nabla_{\Gamma} u_2, \nabla_{\Gamma} u_3) \). We thus use the convention that the gradient of \( u \) has the gradient of the components as column vectors. We denote by \( \nabla_{\Gamma} : f \) the surface divergence of a vector field \( f \) on \( \Gamma \), and by \( \Delta_{\Gamma} u = \nabla_{\Gamma} : \nabla_{\Gamma} u \) the Laplace–Beltrami operator applied to \( u \); see the review Deckelnick et al. (2005) or (Ecker, 2012, Appendix A) or any textbook on differential geometry for these notions.

We denote the unit outer normal vector field to \( \Gamma \) by \( n : \Gamma \to \mathbb{R}^3 \). Its surface gradient contains the (extrinsic) curvature data of the surface \( \Gamma \). At every \( x \in \Gamma \), the matrix of the extended Weingarten map,

\[
A(x) = \nabla_{\Gamma} n(x),
\]

is a symmetric \( 3 \times 3 \) matrix (see, e.g., (Walker, 2015, Proposition 20)). Apart from the eigenvalue 0 (with eigenvector \( n \)), its other two eigenvalues are the principal curvatures \( \kappa_1 \) and \( \kappa_2 \). They determine the fundamental quantities

\[
H := \text{tr}(A) = \kappa_1 + \kappa_2, \quad |A|^2 = \kappa_1^2 + \kappa_2^2, \tag{2.2}
\]

where \( |A| \) denotes the Frobenius norm of the matrix \( A \). Here, \( H \) is called the mean curvature (as in most of the literature, we do not put a factor 1/2).

2.2 Evolution equations for normal vector and normal velocity of a surface evolving under generalized mean curvature flow

The velocity law of the generalized mean curvature flow is given by

\[
v = -V(H)n, \tag{2.3}
\]
where $V = V(H)$ denotes the normal velocity of the surface depending on the mean curvature $H$.

It is important to observe that (2.3) includes many classical surface flows (with non-exhaustive reference lists):

$$V(H) = H,$$

mean curvature flow, see Huisken (1984),

$$V(H) = - \frac{1}{H},$$

inverse mean curvature flow,

see Huisken & Polden (1999); Huisken & Ilmanen (2001),

$$V(H) = H^\alpha,$$

powers of mean curvature ($\alpha > 0$), see Schulze (2002, 2005, 2006, 2008),

$$V(H) = - \frac{1}{H^\alpha},$$

powers of inverse mean curvature ($\alpha > 0$),

see Gerhardt (2014); Scheuer (2016),

$$V(H) = \frac{H + \tilde{H}}{\log(H + \tilde{H})},$$

a non-homogeneous mean curvature flow ($\tilde{H} > 0$),

see Alessandroni & Sinestrari (2010); Espin (2020),

etc.

Throughout the paper we will assume that along the flow the mean curvature satisfies, for $0 \leq t \leq T$ and $x \in \Gamma[X(\cdot, t)]$,

$$0 < H_0 \leq H(x, t) \leq H_1$$

with constants $H_0, H_1 > 0$. (2.4)

We assume that

$$V$$

is a smooth and strictly monotonic increasing function, (2.5)

hence it has – in particular – the properties:

$$V$$

is invertible such that its inverse $V^{-1}$ is locally Lipschitz, and

its derivative $V'$ is everywhere positive and locally Lipschitz. (2.6)

Similar assumptions were made, e.g., in Espin (2020).

**Remark 2.1** We note here that a positive lower bound on the mean curvature of the initial surface is usually ensured by assuming its strict convexity (i.e. the principal curvatures $\kappa_i$ are all positive). In general such an assumption is not restrictive, since assuming strict convexity of the initial surface is necessary to show the parabolicity of the evolution equations and, hence, existence results for the exemplary generalized mean curvature flows above. This property is conserved along the flow for these problems. See, e.g., (Huisken, 1984, Theorem 3.1) for mean curvature flow, (Huisken & Polden, 1999, Theorem 3.1) for inverse mean curvature flow, (Schulze, 2002, Theorem 1.1), (Schulze, 2005, Theorem 1.1) for the $H^\alpha$-flow, (Gerhardt, 2014, Theorem 1.2) for the $H^{-\alpha}$-flow, and (Alessandroni & Sinestrari, 2010, Theorem 1) for the non-homogeneous flow. (In some of these theorems only weak convexity ($\kappa_i \geq 0$) is needed.) It is worth to note here that the role of the (sufficiently large) constant $\tilde{H}$ is exactly to ensure that the function $(H + \tilde{H})/\log(H + \tilde{H})$ preserves this property of the flow, in contrast to the flow with $H/\log(H)$, cf. Alessandroni & Sinestrari (2010).

The normal velocity $V = V(H)$ and the normal vector $n$ from (2.3) satisfy the following non-linear evolution equations along the generalized mean curvature flow (2.3).
LEMMA 2.1 Assume that the function $V$ satisfies (2.5). For a regular surface $\Gamma[X]$ moving under generalized mean curvature flow, satisfying (2.4), the normal vector $n$ and the normal velocity $V := V(H)$ satisfy the following non-linear strictly parabolic evolution equations:

\begin{align}
(V'(H))^{-1} \partial^* n &= \Delta_{\Gamma[X]} n + |A|^2 n, \\
(V'(H))^{-1} \partial^* V &= \Delta_{\Gamma[X]} V + |A|^2 V.
\end{align}

Proof. By using the normal velocity in the proof of (Huisken, 1984, Lemma 3.3), or see also Ecker (2012), (Barrett et al., 2019, Lemma 2.37), the following evolution equation for the normal vector holds:

$$
\partial^* n = \nabla_{\Gamma[X]} (V(H)) = V'(H) \nabla_{\Gamma[X]} H,
$$

by the velocity law (2.3) and using the chain rule. On any surface $\Gamma$, it holds true that (see (Ecker, 2012, (A.9)) or (Walker, 2015, Proposition 24)):

$$
\nabla_{\Gamma[X]} H = \Delta_{\Gamma[X]} n + |A|^2 n.
$$

This, in combination with the previous equation and noting that, by (2.6), via the monotonicity of $V$ it follows that $V'(H) > 0$. Dividing both sides by $V'(H)$ then gives the stated evolution equation for $n$.

By revising the proof of (Huisken, 1984, Theorem 3.4 and Corollary 3.5), or see Ecker (2012), (Barrett et al., 2019, Lemma 2.39), with the normal velocity $V$ we obtain

$$
\partial^* H = \Delta_{\Gamma[X]} (V(H)) + |A|^2 V(H),
$$

which, again by the chain rule for $\partial^* V = \partial^* (V(H))$, and dividing by $V'(H)$ again, finally yields the evolution equation for $V = V(H)$.

It is instructive to relate the evolution equations for generalized mean curvature flow, Lemma 2.1, with those for standard mean curvature flow, Huisken (1984), or (Kovács et al., 2019, Lemma 2.1). In particular, we point out that the right-hand sides of (2.7) and (2.8) are formally the same (with $V$ instead of $H$) as the right-hand sides of (Kovács et al., 2019, equations (2.4) and (2.5)):

\begin{align}
\partial^* n &= \Delta_{\Gamma[X]} n + |A|^2 n, \\
\partial^* H &= \Delta_{\Gamma[X]} H + |A|^2 H.
\end{align}

The only differences are the non-linear factors on the left-hand sides. This structural similarity already suggests that the approach and numerical analysis presented in Kovács et al. (2019) can be extended to the generalized mean curvature flow, but will require modifications treating the terms involving the non-linear weights in front of the material derivatives.

In this paper, we will address these modifications, and extend the stability and convergence analysis of Kovács et al. (2019) for the generalized mean curvature flow (2.3).

2.3 The evolution equation system for generalized mean curvature flow

The evolution of a surface under generalized mean curvature flow is then governed by the coupled system (2.3), (2.7)–(2.8) together with the ODE (2.1). The numerical method is based on the weak form of the above coupled system which reads, denoting $|A|^2 = |
abla_{\Gamma[X]} n|^2$ and $V' = V'(H)$ (with $H$ obtained.
by inverting $V = V(H)$:

$$
\int_{\Gamma[X]} \nabla_{\Gamma[X]}(V)^T \nabla_{\Gamma[X]} \phi^v + \int_{\Gamma[X]} (V)^T \cdot \phi^v = - \int_{\Gamma[X]} \nabla_{\Gamma[X]}(V \cdot n) \cdot \nabla_{\Gamma[X]} \phi^v - \int_{\Gamma[X]} V \cdot \phi^v, \tag{2.9a}
$$

$$
\int_{\Gamma[X]} (V)^{-1} \partial^* n \cdot \phi^v = - \int_{\Gamma[X]} \nabla_{\Gamma[X]} n \cdot \nabla_{\Gamma[X]} \phi^v + \int_{\Gamma[X]} |A|^2 n \cdot \phi^v, \tag{2.9b}
$$

$$
\int_{\Gamma[X]} (V)^{-1} \partial^* V \phi^v = - \int_{\Gamma[X]} \nabla_{\Gamma[X]} V \cdot \nabla_{\Gamma[X]} \phi^v + \int_{\Gamma[X]} |A|^2 V \phi^v, \tag{2.9c}
$$

for all test functions $\phi^v \in H^1(\Gamma[X])^3$ and $\phi^a \in H^1(\Gamma[X])^3$, $\phi^v \in H^1(\Gamma[X])$, together with the ODE for the positions (2.1). This system is complemented with the initial data $x^0$, $n^0$ and $V^0 = V(H^0)$.

For simplicity, by $\cdot$ we denote both the Euclidean scalar product for vectors, and the Frobenius inner product for matrices (i.e., the Euclidean product with an arbitrary vectorisation).

3. Evolving finite element semi-discretization

3.1 Evolving surface finite elements

We formulate the evolving surface finite element (ESFEM) discretization for the velocity law coupled with evolution equations on the evolving surface, following (almost verbatim) the description in Kovács et al. (2017); Kovács et al. (2019), which is based on Dziuk (1988); Demlow (2009); Kovács (2018). We use simplicial finite elements and continuous piecewise polynomial basis functions of degree $k$, as defined in (Demlow, 2009, Section 2.5).

We triangulate the given smooth initial surface $\Gamma^0$ by an admissible family of triangulations $\mathcal{T}_h$ of decreasing maximal element diameter $h$; see Dziuk & Elliott (2007) for the notion of an admissible triangulation, which includes quasi-uniformity and shape regularity. For a momentarily fixed $h$, we denote by $x^0$ the vector in $\mathbb{R}^{3N}$ that collects all nodes $p_j$ ($j = 1, \ldots, N$) of the initial triangulation. By piecewise polynomial interpolation of degree $k$, the nodal vector defines an approximate surface $\Gamma_h^0$ that interpolates $\Gamma^0$ in the nodes $p_j$. We will evolve the $j$th node in time, denoted $x_j(t)$ with $x_j(0) = p_j$, and collect the nodes at time $t$ in a column vector

$$
x(t) \in \mathbb{R}^{3N}.
$$

We just write $x$ for $x(t)$ when the dependence on $t$ is not important.

By piecewise polynomial interpolation on the plane reference triangle that corresponds to every curved triangle of the triangulation, the nodal vector $x$ defines a closed surface denoted by $\Gamma_h[x]$. We can then define globally continuous piecewise finite element basis functions

$$
\phi_i(x) : \Gamma_h[x] \to \mathbb{R}, \quad i = 1, \ldots, N,
$$

which have the property that on every triangle their pullback to the reference triangle is polynomial of degree $k$, and which satisfy at the nodes $\phi_i(x_j) = \delta_{ij}$ for all $i, j = 1, \ldots, N$. These functions span the finite element space on $\Gamma_h[x]$:

$$
S_h[x] = S_h(\Gamma_h[x]) = \text{span} \{ \phi_1[x], \phi_2[x], \ldots, \phi_N[x] \}.
$$

For a finite element function $u_h \in S_h[x]$, the tangential gradient $\nabla_{\Gamma_h[x]}u_h$ is defined piecewise on each element.
The discrete surface at time $t$ is parametrized by the initial discrete surface via the map $X_h(\cdot,t) : \Gamma_h^0 \rightarrow \Gamma_h[t]$ defined by

$$X_h(p_h,t) = \sum_{j=1}^{N} x_j(t) \phi_j[x(0)](p_h), \quad p_h \in \Gamma_h^0,$$

which has the properties that $X_h(p_j,t) = x_j(t)$ for $j = 1, \ldots, N$, that $X_h(p_h,0) = p_h$ for all $p_h \in \Gamma_h^0$, and

$$\Gamma_h[x(t)] = \Gamma[X_h(\cdot,t)] = \{X_h(p_h,t) \mid p_h \in \Gamma_h^0\}.$$

The discrete velocity $v_h(x,t) \in \mathbb{R}^3$ at a point $x = X_h(p_h,t) \in \Gamma[X_h(\cdot,t)]$ is given by

$$\partial_t X_h(p_h,t) = v_h(X_h(p_h,t),t).$$

In view of the transport property of the basis functions Dziuk & Elliott (2007), $\frac{d}{dt} \langle \phi_j[x(t)](X_h(p_h,t)) \rangle = 0$, the discrete velocity equals, for $x \in \Gamma_h[x(t)]$,

$$v_h(x,t) = \sum_{j=1}^{N} v_j(t) \phi_j[x(t)](x) \quad \text{with} \quad v_j(t) = \dot{x}_j(t),$$

where the dot denotes the time derivative $d/dt$. Hence, the discrete velocity $v_h(\cdot,t)$ is in the finite element space $S_h[x(t)]$, with nodal vector $v(t) = x(t)$.

The discrete material derivative of a finite element function $u_h(x,t)$ with nodal values $u_j(t)$ is

$$\partial^*_h u_h(x,t) = \frac{d}{dt} u_h(X(h,p_h,t)) = \sum_{j=1}^{N} \dot{u}_j(t) \phi_j[x(t)](x) \quad \text{at} \quad x = X_h(p_h,t).$$

### 3.2 ESFEM spatial semi-discretizations

Now we will describe the semi-discretization of the coupled system for generalized mean curvature flow.

The finite element spatial semi-discretization of the weak coupled parabolic system (2.9) reads as follows: Find the unknown nodal vector $x(t) \in \mathbb{R}^{3N}$ and the unknown finite element functions $v_h(\cdot,t) \in S_h[x(t)]^3$ and $n_h(\cdot,t) \in S_h[x(t)]^3$, and $V_h(\cdot,t) \in S_h[x(t)]^3$ such that, by denoting $|A_h|^2 = |\nabla G_h|^2 n_h^2 \cdot \phi_h^2$ and $V_h' = V'(H_h)$ (with $H_h$ obtained by inverting $V_h = V(H_h)$),

$$\int_{\Gamma_h[x]} \nabla G_h[x] \cdot \nabla G_h[x] \phi_h^\ast \ast + \int_{\Gamma_h[x]} v_h \cdot \phi_h^\ast \ast = - \int_{\Gamma_h[x]} \nabla G_h[x] (V_h n_h) \cdot \nabla G_h[x] \phi_h - \int_{\Gamma_h[x]} V_h n_h \cdot \phi_h^\ast \ast, \quad (3.1a)$$

$$\int_{\Gamma_h[x]} (V_h')^{-1} \partial^*_h n_h \cdot \phi_h^0 = - \int_{\Gamma_h[x]} \nabla G_h[x] (V_h n_h) \cdot \nabla G_h[x] \phi_h^0 + \int_{\Gamma_h[x]} |A_h|^2 n_h \cdot \phi_h^0, \quad (3.1b)$$

$$\int_{\Gamma_h[x]} (V_h')^{-1} \partial^*_h V_h \phi_h^V = - \int_{\Gamma_h[x]} \nabla G_h[x] V_h \cdot \nabla G_h[x] \phi_h^V + \int_{\Gamma_h[x]} |A_h|^2 V_h \phi_h^V, \quad (3.1c)$$

for all $\phi_h^\ast \ast \in S_h[x(t)]^3$, $\phi_h^0 \in S_h[x(t)]^3$, and $\phi_h^V \in S_h[x(t)]^3$, with the surface $\Gamma_h[x(t)] = \Gamma[X_h(\cdot,t)]$ given by the differential equation

$$\partial_t X_h(p_h,t) = v_h(X_h(p_h,t),t), \quad p_h \in \Gamma_h^0. \quad (3.2)$$

The initial values for the nodal vector $x$ are taken as the positions of the nodes of the triangulation of the given initial surface $\Gamma^0$. The initial data $n_h^0$ and $V_h^0 = V(H_h^0)$ are determined by Lagrange interpolation of $n^0$ and $H^0$. 

3.3 Matrix–vector formulation

The nodal values of the unknown semi-discrete functions are collected into column vectors \( v = (v_j) \in \mathbb{R}^{3N}, n = (n_j) \in \mathbb{R}^{3N}, \) and \( V = (V_j) \in \mathbb{R}^N. \) We furthermore collect
\[
u := \begin{pmatrix} n \\ V \end{pmatrix} \in \mathbb{R}^{4N}.
\]

We define the surface-dependent mass matrix \( M(x) \) and stiffness matrix \( A(x) \), as well as the solution-dependent mass matrix \( M(x, u) \):
\[
M(x)_{ij} = \int_{\Omega_i} \phi_i(x) \phi_j(x) \quad \text{and} \quad A(x)_{ij} = \int_{\Omega_i} \nabla \phi_i(x) \cdot \nabla \phi_j(x),
\]
\[
M(x, u)_{ij} = \int_{\Omega_i} (V'_h)^{-1} \phi_i(x) \phi_j(x),
\]
for \( i, j = 1, \ldots, N \). The non-linear terms \( f(x, u) = (f_1(x, u), f_2(x, u))^T \) and \( g(x, u) \) are defined by
\[
f_1(x, u)_j = \int_{\Omega_i} |A_h|_{2} (n_h)_j \phi_j(x),
\]
\[
f_2(x, u)_j = \int_{\Omega_i} |A_h|_{2} V_h \phi_j(x),
\]
\[
g(x, u)_j = -\int_{\Omega_i} V_h (n_h)_j \phi_j(x) - \int_{\Omega_i} \nabla \phi_i(x) (V'_h(n_h)) \cdot \nabla \phi_j(x),
\]
for \( j = 1, \ldots, N \) and \( \ell = 1, 2, 3 \). We recall that \( |A_h|_{2} = |\nabla \phi_i(x)|_{H_0}^2 \) and \( (V'_h)^{-1} = 1/V'(H_h) \), with \( H_h \) obtained by inverting \( V_h = V(H_h) \).

We further let, for \( d \in \mathbb{N} \) (with the identity matrices \( I_d \in \mathbb{R}^{d \times d} \))
\[
M^{[d]}(x) = I_d \otimes M(x), \quad A^{[d]}(x) = I_d \otimes A(x), \quad K^{[d]}(x) = I_d \otimes (M(x) + A(x)),
\]
and similarly for \( M(x, u) \). When no confusion can arise, we will write \( M(x) \) for \( M^{[d]}(x) \), \( M(x, u) \) for \( M^{[d]}(x, u) \), \( A(x) \) for \( A^{[d]}(x) \), and \( K(x) \) for \( K^{[d]}(x) \).

Using these definitions (3.1) with (3.2) can be written in the matrix–vector form:
\[
K^{[3]}(x)v = g(x, u),
\]
\[
M^{[4]}(x, u)\ddot{u} + A^{[4]}(x)u = f(x, u),
\]
with (3.2) equivalent to
\[
\dot{x} = v.
\]

We again compare the above matrix–vector formulation (3.4) for generalized mean curvature flow, to the same formulas for standard mean curvature flow (Kovács et al., 2019, equation (3.4)–(3.5)):
\[
K^{[3]}(x)v = g(x, u),
\]
\[
M^{[4]}(x, u)\ddot{u} + A^{[4]}(x)u = f(x, u),
\]
with
\[
\dot{x} = v.
\]
The two formulations are formally the same, the only difference is the solution-dependent mass matrix \( M(x, u) \) in the term with a time derivative of \( u \), which in the case of mean curvature flow is simply \( M(x) \).

(Also note that here \( u \) collects \( n \) and \( V \), whereas for mean curvature flow \( u = (n, H) \).

The stability proof presented in (Kovács et al., 2019, Section 7) will therefore be generalized below to accommodate the use of solution-dependent mass matrices, but we will exploit the similarities of the two problems as most as possible. We will also extend here the estimates of (Kovács et al., 2019, Section 7.1), relating different finite element surfaces, to the solution-dependent case.

**Remark 3.1** Instead of enforcing the velocity law (2.3) via the Ritz projection (3.1a), in Kovács et al. (2020) the velocity law is enforced using the nodal finite element interpolation. That is (3.4a) is replaced by

\[
v = -V \cdot n, \quad \text{with} \quad (V \cdot n)_j = V_j \cdot n_j \quad (j = 1, \ldots, N). \tag{3.6}
\]

The stability proof requires an \( H^1 \)-norm stability for the velocity law, which is rather straightforward for the Ritz projection (see Kovács et al. (2019), or Proposition 6.1 later on), while for the above pointwise velocity it is shown in (Kovács et al., 2020, Part (B) of Proposition 5.1).

### 3.4 Lifts

As in Kovács et al. (2017) and (Kovács et al., 2019, Section 3.4), we compare functions on the exact surface \( \Gamma[X(\cdot, t)] \) with functions on the discrete surface \( \Gamma_h^t[x(t)] \), via functions on the interpolated surface \( \Gamma_h^t[x^*(t)] \), where \( x^*(t) \) denotes the nodal vector collecting the grid points \( x_j^*(t) = X(p_j, t) \) on the exact surface, where \( p_j \) are the nodes of the discrete initial triangulation \( \Gamma_h^0 \).

Any finite element function \( w_h \) on the discrete surface, with nodal values \( w_j \), is associated with a finite element function \( \tilde{w}_h \) on the interpolated surface \( \Gamma_h^t \) with the exact same nodal values. This can be further lifted to a function on the exact surface using the lift operator \( L \), mapping a function on the interpolated surface \( \Gamma_h^t \) to a function on the exact surface \( \Gamma \), provided that they are sufficiently close, see Dziuk (1988); Demlow (2009).

Then the composed lift \( L \) maps finite element functions on the discrete surface \( \Gamma_h^t[x(t)] \) to functions on the exact surface \( \Gamma[X(\cdot, t)] \) via the interpolated surface \( \Gamma_h^t[x^*(t)] \) is denoted by

\[
w_h^t = (\tilde{w}_h)^t.
\]

### 4. Linearly implicit full discretization

Similarly as for mean curvature flow Kovács et al. (2019), for the time discretization of the system of ordinary differential equations (3.4) we use a \( q \)-step linearly implicit backward difference formula (BDF) with \( q \leq 5 \). For a step size \( \tau > 0 \), and with \( t_n = n \tau \leq T \), we determine the approximations to all variables \( x^n \) to \( x(t_n) \), \( v^n \) to \( v(t_n) \), and \( u^n \) to \( u(t_n) \) by the fully discrete system of linear equations

\[
\begin{align*}
K(\tilde{x}^n)v^n &= g(\tilde{x}^n, \tilde{v}^n), \tag{4.1a} \\
M(\tilde{x}^n, \tilde{u}^n)u^n + A(\tilde{x}^n)u^n &= f(\tilde{x}^n, \tilde{u}^n), \tag{4.1b} \\
x^n &= v^n, \tag{4.1c}
\end{align*}
\]

where the discretized time derivatives are given by

\[
\begin{align*}
\dot{x}^n &= \frac{1}{\tau} \sum_{j=0}^q \delta x^{n-j}, \\
\dot{u}^n &= \frac{1}{\tau} \sum_{j=0}^q \delta u^{n-j}, \quad n \geq q.
\end{align*}
\]
and where $\tilde{x}^n$ and $\tilde{u}^n$ are extrapolated values, approximating $x(t_n)$ and $u(t_n)$:

$$\tilde{x}^n = \sum_{j=0}^{q-1} \gamma_j x^{n-1-j}, \quad \tilde{u}^n = \sum_{j=0}^{q-1} \gamma_j u^{n-1-j}, \quad n \geq q.$$  \hspace{1cm} (4.3)

The starting values $x^i$ and $u^i \ (i = 0, \ldots, q-1)$ are assumed to be given. They can be precomputed using either a lower order method with smaller step sizes, or an implicit Runge–Kutta method.

The method is determined by its coefficients, given by $\delta(\zeta) = \sum_{j=0}^q \delta_j \zeta^j = \sum_{j=1}^q \frac{1}{j!(1-\zeta)^j}$ and $\gamma(\zeta) = \sum_{j=0}^{q-1} \gamma_j \zeta^j = (1 - (1 - \zeta)^q)/\zeta$. The classical BDF method is known to be zero-stable for $q \leq 6$ and to have order $q$; see (Hairer & Wanner, 1996, Chapter V). This order is retained by the linearly implicit variant using the above coefficients $\gamma_j$; cf. Akrivis & Lubich (2015); Akrivis et al. (2017).

The analogous linearly implicit backward difference methods were used for mean curvature flow Kovács et al. (2019). Theorem 6.1 in Kovács et al. (2019) proves optimal-order error bounds for the combined ESFEM–BDF full discretization of the mean curvature flow system, for finite elements of polynomial degree $k \geq 2$ and BDF methods of order $2 \leq q \leq 5$.

We note that in the $n$th time step, the method requires solving two linear systems with the symmetric positive definite matrices $K(x^n)$ and $\frac{D}{2}M(x^n) + A(x^n)$.

From the vectors $x^n = (x^n_j)$, $v^n = (v^n_j)$, and $u^n = (u^n_j)$ with $u^n_j = (n^n_j, V^n_j) \in \mathbb{R}^3 \times \mathbb{R}$ we obtain position approximations to $X(\cdot, t_n)$, $\text{id} \Gamma_{X(\cdot, t_n)}$, velocity approximations to $v(\cdot, t_n)$, and approximations to the normal vector and the normal velocity, respectively, at time $t_n$ as

$$X^n_h(p_h) = \sum_{j=1}^N x^n_j \phi_j(x(0)) (p_h) \quad \text{for } p_h \in \Gamma_h^0;$$

$$x^n_h = \text{id} \Gamma_{X^n_h};$$

$$v^n_h(x) = \sum_{j=1}^N v^n_j \phi_j(x^n)[x^n](x) \quad \text{for } x \in \Gamma_h[x^n];$$

$$n^n_h(x) = \sum_{j=1}^N n^n_j \phi_j(x^n)[x^n](x) \quad \text{for } x \in \Gamma_h[x^n];$$

$$V^n_h(x) = \sum_{j=1}^N V^n_j \phi_j(x^n)[x^n](x) \quad \text{for } x \in \Gamma_h[x^n].$$

The approximation $H^n_h$ of the mean curvature is similarly given by the nodal values $H^n = V^{-1}(V^n)$.

In the semi-discrete case, the approximations of the same quantities are given analogously.

5. Main results: error estimates

We will now formulate the main results of this paper, which provide optimal-order error bounds for the finite element semi-discretisation, for finite elements of polynomial degree $k \geq 2$, and of the full discretisation with linearly implicit BDF methods, of order $2 \leq q \leq 5$.

We denote by $\Gamma(t) = \Gamma[X(\cdot, t)]$ the exact surface and by $\Gamma_h(t) = \Gamma[X_h(\cdot, t)] = \Gamma_h[X(\cdot, t)]$ the discrete surface at time $t$. We introduce the notation

$$x^n_h(x, t) = X^n_h(p, t) \in \Gamma_h(t) \quad \text{for } x = X(p, t) \in \Gamma(t).$$
5.1 Convergence of the semi-discretization

Theorem 5.1 Consider the semi-discretization (3.1) of the coupled generalized mean curvature flow problem (2.9) with (2.1), using evolving surface finite elements of polynomial degree \( k \geq 2 \). Let the function \( V \) satisfy (2.6). If the flow (2.3) has a sufficiently regular solution \( (X, v, n, V) \) on some time interval \([0, T]\), and that the flow map \( X(\cdot, t) : \Gamma^0 \to \Gamma(t) \subset \mathbb{R}^3 \) is non-degenerate so that \( \Gamma(t) \) is a regular surface, with mean curvature \( 0 < H_0 \leq H(\cdot, t) \leq H_1 \), on the time interval \( t \in [0, T] \).

Then, there exist constants \( h_0 > 0 \) and \( C > 0 \) such that
\[
\|X^k_h(\cdot, t) - \text{Id}_{\Gamma(t)}\|_{H^1(\Gamma(t))} \leq Ch^k,
\|
v^k_h(\cdot, t) - v(\cdot, t)\|_{H^1(\Gamma(t))} \leq Ch^k,
\|n^k_h(\cdot, t) - n(\cdot, t)\|_{H^1(\Gamma(t))} \leq Ch^k,
\|V^k_h(\cdot, t) - V(\cdot, t)\|_{H^1(\Gamma(t))} \leq Ch^k,
\]
and, since \( V \) is a smooth and invertible function of the mean curvature \( H \), we also obtain
\[
\|H^k_h(\cdot, t) - H(\cdot, t)\|_{H^1(\Gamma(t))} \leq C h^k,
\]
for all \( h < h_0 \). Furthermore, we obtain
\[
\|X^k_h(\cdot, t) - X(\cdot, t)\|_{H^1(\Gamma(t))} \leq C h^k
\]
for all \( h < h_0 \). The constant \( C \) is independent of \( h \), but depends on bounds of higher derivatives of the solution \( (X, v, n, V) \) of the generalized mean curvature flow, and on the length \( T \) of the time interval.

5.2 Convergence of the full discretization

Theorem 5.2 Consider the ESFEM–BDF full discretization (4.1) of the coupled generalized mean curvature flow problem (2.9) with (2.1), using evolving surface finite elements of polynomial degree \( k \geq 2 \) and linearly implicit BDF time discretization of order \( q \) with \( 2 \leq q \leq 5 \). Let the function \( V \) satisfy (2.6). Suppose that the generalized flow admits an exact solution \( (X, v, n, V) \) that is sufficiently smooth on some time interval \( t \in [0, T] \), and that the flow map \( X(\cdot, t) : \Gamma^0 \to \Gamma(t) \subset \mathbb{R}^3 \) is non-degenerate so that \( \Gamma(t) \) is a regular surface, with mean curvature \( 0 < H_0 \leq H(\cdot, t) \leq H_1 \), on the time interval \( t \in [0, T] \).

Then, there exist \( h_0 > 0 \), \( \tau_0 > 0 \), and \( C > 0 \) such that for all mesh sizes \( h \leq h_0 \) and time step sizes \( \tau \leq \tau_0 \) satisfying the step size restriction
\[
\tau \leq C_0 h
\]
(where \( C_0 > 0 \) can be chosen arbitrarily), the following error bounds for the lifts of the discrete position, velocity, normal vector and normal velocity hold over the exact surface: provided that the starting values are sufficiently accurate in the \( H^1 \)-norm at time \( t_i = i\tau \) for \( i = 0, \ldots, q - 1 \), we have at time \( t_n = n\tau \leq T \)
\[
\| (x^k_n)^L - \text{Id}_{\Gamma(t_n)} \|_{H^1(\Gamma(t_n))} \leq C(h^k + \tau^q),
\| (v^k_n)^L - v(\cdot, t_n) \|_{H^1(\Gamma(t_n))} \leq C(h^k + \tau^q),
\| (n^k_n)^L - n(\cdot, t_n) \|_{H^1(\Gamma(t_n))} \leq C(h^k + \tau^q),
\| (V^k_n)^L - V(\cdot, t_n) \|_{H^1(\Gamma(t_n))} \leq C(h^k + \tau^q),
\]
and also
\[
\| (X_h^n)^{-1} - X(\cdot, t_0) \|_{H^1(\Gamma_0)^3} \leq C(h^k + \tau^q), \quad \text{and}
\]
\[
\| (H_h^n)^{-1} - H(\cdot, t_0) \|_{H^1(\Gamma_0)} \leq C(h^k + \tau^q),
\]
where the constant $C$ is independent of $h$, $\tau$ and $n$ with $n \tau \leq T$, but depends on bounds of higher derivatives of the solution $(X, v, n, V)$ of the generalised mean curvature flow, and on the length $T$ of the time interval, and on $C_0$.

Sufficient regularity assumptions are the following: uniformly in $t \in [0, T]$ and for $j = 1, \ldots, q + 1,$
\[
X(\cdot, t) \in H^{k+1}(\Gamma^0), \quad \partial^j_t X(\cdot, t) \in H^1(\Gamma^0),
\]
\[
v(\cdot, t) \in H^{k+1}(\Gamma(X(\cdot, t))), \quad \partial^j_t v(\cdot, t) \in H^2(\Gamma(X(\cdot, t))),
\]
for $u = (n, V), \quad u(\cdot, t), \partial^* u(\cdot, t) \in W^{k+1,\infty}(\Gamma(X(\cdot, t)))^4,$
\[
\partial^* u(\cdot, t) \in H^2(\Gamma(X(\cdot, t)))^4.
\]

For the starting values, sufficient approximation conditions are the following: for $i = 0, \ldots, q - 1,$
\[
\| (x_h^L)^{-1} - \text{id}_{\Gamma(\cdot, t)} \|_{H^1(\Gamma(\cdot, t))} \leq C(h^k + \tau^q),
\]
\[
\| (v_h^L - v(\cdot, t)) \|_{H^1(\Gamma(\cdot, t))} \leq C(h^k + \tau^q),
\]
\[
\| (n_h^L - n(\cdot, t)) \|_{H^1(\Gamma(\cdot, t))} \leq C(h^k + \tau^q),
\]
\[
\| (V_h^L - V(\cdot, t)) \|_{H^1(\Gamma(\cdot, t))} \leq C(h^k + \tau^q),
\]
and in addition, for $i = 1, \ldots, q - 1,$
\[
\tau^{1/2} \left\| \frac{1}{\tau} (X_h^i - X_h^{i-1}) - \frac{1}{\tau} (X(\cdot, t_i) - X(\cdot, t_{i-1})) \right\|_{H^1(\Gamma_0)^3} \leq C(h^k + \tau^q).
\]

In view of Remark 3.1, both of the above theorems hold verbatim if the discretized velocity law is enforced using the nodal finite element interpolation, cf. (3.6), instead of the Ritz map.

It is important to note here that, since both of the above results are shown by extending the techniques of Kovács et al. (2019) to generalized mean curvature flow, the observations (including preservation of mesh admissibility and non-degeneration under the assumed regularity) after Theorem 4.1 and 6.1 from Kovács et al. (2019) hold analogously to Theorem 5.1 and 5.2 here.

6. Semi-discrete stability

6.1 Preparations: relating different surfaces

In our previous work Kovács et al. (2017); Kovács et al. (2019) we proved some technical results relating different finite element surfaces. Here we use the same setting, and briefly (and almost verbatim) recapitulate it below.

The finite element matrices defined in Section 3.3 induce discrete versions of Sobolev norms on the discrete surface $\Gamma_h[x]$. For any nodal vector $w \in \mathbb{R}^N$, with the corresponding finite element function
$w_h \in S_h [x]$, we define the following norms:
\[
\| w \|_{M(x)}^2 = w^T M(x) w = \| w_h \|_{L^2(\Gamma_h[x])}^2,
\]
\[
\| w \|_{A(x)}^2 = w^T A(x) w = \| \nabla \Gamma_h[x] w_h \|_{L^2(\Gamma_h[x])}^2,
\]
\[
\| w \|_{K(x)}^2 = w^T K(x) w = \| w_h \|_{H^1(\Gamma_h[x])}^2.
\]

(6.1)

We also note here that the matrix $M(x, u)$ also generates a solution dependent norm:
\[
\| w \|_{M(x,u)}^2 = w^T M(x,u) w = \int_{\Gamma_h[x]} (V_h')^{-1} |w_h|^2,
\]
equivalent to $\| \cdot \|_{M(x)}$.

For arbitrary nodal vectors $x, y \in \mathbb{R}^{3N}$ defining the discrete surfaces $\Gamma_h[x]$ and $\Gamma_h[y]$, respectively. Their difference is denoted by $e = (e_j)^N_{j=1} = x - y \in \mathbb{R}^{3N}$.

For $\theta \in [0, 1]$ we consider the intermediate surface $\Gamma^\theta_h = \Gamma_h[y + \theta e]$, and for vectors $w, z \in \mathbb{R}^N$, we also consider the corresponding finite element functions on $\Gamma^\theta_h$:
\[
e^\theta_h = \sum_{j=1}^N e_j \phi_j[y + \theta e], \quad w^\theta_h = \sum_{j=1}^N w_j \phi_j[y + \theta e], \quad z^\theta_h = \sum_{j=1}^N z_j \phi_j[y + \theta e].
\]

Figure 1 illustrates the described construction.

Analogous to (Kovács et al., 2019, Section 7), we prove the following lemma relating different quantities on different surfaces, in particular proving a new result which compares solution dependent matrices.

Similarly, for $u = (n, V)^T \in \mathbb{R}^{4N}$, with $V \in \mathbb{R}^N$ defining $V^{-1}(V) = H \in \mathbb{R}^N$ by inverting the function $V$, we consider the corresponding finite element function $(V_h')^\theta$ on $\Gamma^\theta_h$, which appears in the solution-dependent mass matrix $M(x, u)$:
\[
(V_h')^\theta = \sum_{j=1}^N V'(H_j) \phi_j[y + \theta e].
\]

(6.2)

Analogous to (Kovács et al., 2019, Section 7), we prove the following to lemmata relating different quantities on different surfaces, in particular proving a new result which compares solution dependent matrices.
Assuming that \( \| \nabla \Gamma_h y e^0 \|_{L^\infty(I_h[y])} \leq \frac{1}{4} \) Lemma 7.2 of Kovács et al. (2019) (with \( p = 2 \)) shows that the norms \( \| \cdot \|_{M(Y + \theta e)} \) and the norms \( \| \cdot \|_{A(Y + \theta e)} \) are \( h \)-uniformly equivalent for \( 0 \leq \theta \leq 1 \). (6.3)

Under the condition that \( \varepsilon := \| \nabla \Gamma_h y e^0 \|_{L^\infty(I_h[y])} \leq \frac{1}{4} \), using the definition of \( e^0 \) in Lemma 4.1 of Kovács et al. (2017) and applying the Cauchy–Schwarz inequality yields the bounds,

\[
w^T (M(x) - M(y)) z \leq c \varepsilon \| w \|_{M(Y)} \| z \|_{M(Y)},
\]

\[
w^T (A(x) - A(y)) z \leq c \varepsilon \| w \|_{A(Y)} \| z \|_{A(Y)}.
\]

We will also use the bounds with additionally assuming \( z_h \in W^{1,\infty}(I_h[y]) \):

\[
w^T (M(x) - M(y)) z \leq c \| w \|_{M(Y)} \| e \|_{A(Y)},
\]

\[
w^T (A(x) - A(y)) z \leq c \| w \|_{A(Y)} \| e \|_{A(Y)}.
\]

Consider now a continuously differentiable function \( x : [0, T] \to \mathbb{R}^{3N} \) that defines a finite element surface \( I_h^x(t) \) for every \( t \in [0, T] \), and assume that its time derivative \( v(t) = \dot{x}(t) \) is the nodal vector of a finite element function \( v_h(\cdot, t) \) that satisfies

\[
\| \nabla I_h^x(t) v_h(\cdot, t) \|_{L^\infty(I_h^x(t))} \leq K, \quad 0 \leq t \leq T.
\]

With \( e = x(t) - x(s) = \int_s^t v(r) dr \), the bounds (6.4) then yield the following bounds, which were first shown in Lemma 4.1 of Dziuk et al. (2012): for \( 0 \leq s, t \leq T \) with \( K |t - s| \leq \frac{1}{4} \), we have with \( C = cK \)

\[
w^T (M(x(t)) - M(x(s))) z \leq C |t - s| \| w \|_{M(x(t))} \| z \|_{M(x(t))},
\]

\[
w^T (A(x(t)) - A(x(s))) z \leq C |t - s| \| w \|_{A(x(t))} \| z \|_{A(x(t))}.
\]

Letting \( s \to t \), this implies the bounds stated in Lemma 4.6 of Kovács et al. (2017):

\[
w^T \frac{d}{dt} M(x(t)) z \leq C \| w \|_{M(x(t))} \| z \|_{M(x(t))},
\]

\[
w^T \frac{d}{dt} A(x(t)) z \leq C \| w \|_{A(x(t))} \| z \|_{A(x(t))}.
\]

Moreover, by patching together finitely many intervals over which \( K |t - s| \leq \frac{1}{4} \), we obtain that

the norms \( \| \cdot \|_{M(x(t))} \) and the norms \( \| \cdot \|_{A(x(t))} \) are \( h \)-uniformly equivalent for \( 0 \leq t \leq T \). (6.9)

The following new result is a solution dependent variant of the estimates relating mass matrices on different surfaces and with different geometric variables. In both cases we establish the analogons of (6.4)–(6.5). These estimates will play a crucial role in the stability proofs.

**Lemma 6.1** Let \( \varepsilon := \| \nabla \Gamma_h y e^0 \|_{L^\infty(I_h[y])} \) and let \( u = (n, V)^T \) and \( u^* = (n^*, V^*)^T \in \mathbb{R}^{4N} \) such that the corresponding \( (V_h^0)^{-1} \) and \( \left(V_h^*\right)^{-1} \), defined by (6.2), have bounded \( L^\infty(I_h[y]) \) norms. If \( \varepsilon \leq \frac{1}{4} \), then, in the above setting, the following bounds hold:

\[
w^T (M(x, u) - M(y, u)) z \leq c \| \nabla \Gamma_h y e^0 \|_{L^\infty(I_h[y])} \| w \|_{M(Y)} \| z \|_{M(Y)},
\]

\[
w^T (M(x, u) - M(y, u)) z \leq c \| e \|_{A(Y)} \| w \|_{M(Y)} \| z_h \|_{L^\infty(I_h[y])},
\]
and
\[ w^T (M(x, u) - M(x, u^*)) z \leq c \|u_h - u^*_h\|_{L^\infty(I_h[x])} \|w\|_{M(x)} \|z\|_{M(x)}, \tag{iii} \]
\[ w^T (M(x, u) - M(x, u^*)) z \leq c \|u - u^*\|_{M(x)} \|w\|_{M(x)} \|z_h\|_{L^\infty(I_h[x])}. \tag{iv} \]

**Proof.** The first step of the proof is similar to that of (Kovács et al., 2017, Lemma 4.1). Using the fundamental theorem of calculus and the Leibniz formula (Dziuk & Elliott, 2007, Lemma 2.2) we obtain

\[
\begin{aligned}
 w^T (M(x, u) - M(y, u)) z &= \int_{I^h[y]} ((V_h')^1)^{-1} w_h^1 z_h - \int_{I^h[y]} ((V_h')^0)^{-1} w_h^0, \\
 &= \int_0^1 \frac{d}{d\theta} \int_{I^h[y]} ((V_h')^\theta)^{-1} w_h^\theta \cdot e_h^\theta d\theta \\
 &= \int_0^1 \int_{I^h[y]} ((V_h')^\theta)^{-1} w_h^\theta (V_{I^h[x]} \cdot e_h^\theta) z_h^\theta d\theta \\
 &\leq \int_0^1 \|((V_h')^\theta)^{-1}\|_{L^\infty(I_h[x])} \|w_h^\theta\|_{L^2(I_h[x])} \|V_{I^h[x]} \cdot e_h^\theta\|_{L^\infty(I_h[x])} \|z_h^\theta\|_{L^2(I_h[x])} d\theta,
\end{aligned}
\]  

where we have used that the material derivatives of $w_h^\theta$, $z_h^\theta$ and $(V_h')^\theta$ are vanishing with respect to $\theta$. By (Kovács et al., 2019, Lemma 7.2) it follows then, that under the condition

\[ \|V_{I^h[x]} \cdot e_h^\theta\|_{L^\infty(I_h[x])} \leq \frac{1}{2}, \]

the $L^p(I_h^\theta)$ norms and $W^{1,p}(I_h^\theta)$ semi-norms are equivalent for all $0 \leq \theta \leq 1$. This, together with the previous estimates imply the first estimate:

\[ w^T (M(x, u) - M(y, u)) z \leq c \|w\|_{M(y)} \|z\|_{M(y)}, \]

where the constant $c$ depends on the $L^\infty(I_h[x])$ norm of $(V_h')^{-1}$. By interchanging the roles of $V_{I^h[x]} \cdot e_h^\theta$ and $z_h$ in the last estimate of (6.10), and then using the same argument from above, we obtain the second bound.

The third estimate is shown, using Lemma A.1 with $\gamma = -1$ and the local Lipschitz continuity of $V'$, by (momentarily dropping the superscript $^1$ for the finite element functions on $I^h[x]$)

\[
\begin{aligned}
 w(M(x, u) - M(x, u^*)) z &= \int_{I^h[x]} w_h \left( \frac{1}{V'(H_h^\gamma)} - \frac{1}{V'(H_h^\gamma)} \right) z_h \\
 &\leq \int_{I^h[x]} |w_h| \left| \frac{1}{V'(H_h^\gamma)} - \frac{1}{V'(H_h^\gamma)} \right| |z_h| \\
 &\leq c \int_{I^h[x]} |w_h| |V'(H_h^\gamma)|^{-2} |V'(H_h^\gamma) - V'(H_h^\gamma)| |z_h| \\
 &\leq c \int_{I^h[x]} |w_h| |V'(H_h^\gamma)|^{-2} |H_h - H_h^\gamma| |z_h| \\
 &\leq c \|w_h\|_{L^2(I^h[x])} \|H_h - H_h^\gamma\|_{L^2(I^h[x])} \|z_h\|_{L^\infty(I_h[x])},
\end{aligned}
\]  

where the constant $c$ depends on the $L^\infty(I_h[x])$ norm of $((V_h')^\gamma)^{-1}$. 
Combining these inequalities we obtain
\[ w^T (M(x, u) - M(x, u^*)) z \leq c \| u - u^* \|_{M(x)} \| w \|_{M(x)}, \]
where the constant depends on the \( L^\infty (\Gamma_h[x]) \) norms of \( z_h \) and \( ((V^*_h)^{-1}) \). Similarly as before, by interchanging the roles of \( H_h - H_h^* \) and \( z_h \) in the last estimate of (6.11) we obtain the fourth estimate.

\[ \square \]

### 6.2 Errors and defects

We define the nodal vectors \( x^*(t) \in \mathbb{R}^{3N} \), \( v^*(t) \in \mathbb{R}^{3N} \) and \( u^*(t) \in \mathbb{R}^{3N} \) collects the values at the finite element nodes of the exact solution \( X(\cdot, t) \) and \( v(\cdot, t) \), respectively. The vector \( u^*(t) \) contains the nodal values of the finite element function \( u_h^*(t) := R_h u(\cdot, t) \in S_h[x^*(t)] \), that is defined by a Ritz map on the interpolated surface \( I_h[x^*(t)] \):

\[ \int_{I_h[x^*]} \nabla I_h[x^*] u_h \cdot \nabla I_h[x^*] \phi_h + \int_{I_h[x^*]} u_h \cdot \phi_h = \int_{\Gamma[x]} \nabla \Gamma[x] u \cdot \nabla \Gamma[x] \phi_h + \int_{\Gamma[x]} u \cdot \phi_h \quad (6.12) \]

for all \( \phi_h \in S_h[x^*] \), where again \( \phi_h \) denotes the lift to a function on \( \Gamma[x] \).

The nodal vectors \( x^*(t), v^*(t), u^*(t) \) satisfy the equations (3.4) up to some defects \( d_v \) and \( d_u \) that will be studied in Section 7:

\begin{align}
K(x^*)v^* &= g(x^*, u^*) + M(x^*)d_v, \\
M(x^*, u^*)u^* &= f(x^*, u^*) + M(x^*)d_u, \\
\dot{x}^* &= v^*. 
\end{align}

#### 6.2.1 An \( L^\infty \) bound and error estimates for the Ritz map

In the upcoming stability proof, we will need some \( L^\infty \)-norm estimates for the Ritz map of \( V \). This preparatory section is devoted to the proof of these estimates.

Recalling, form (6.12), that \( V^*_h(\cdot, t) = R_h V(\cdot, t) \in S_h[x^*(t)] \) is the Ritz map of \( V(\cdot, t) \), and \( V^*_h(\cdot, t) = R_h V \), then using multiple triangle inequalities and an inverse estimate (Brenner & Scott, 2008, Theorem 4.5.11), and norm equivalences, we obtain, for \( t \in [0, T] \) and \( h \leq h_0 \),

\[ \| (V^*_h)^{-1} - V \|_{L^\infty(\Gamma[x])} \leq c \| \tilde{R}_h V - \tilde{I}_h V \|_{L^\infty(\Gamma[x^*])} + \| I_h V - V \|_{L^\infty(\Gamma[x])} \]

\[ \leq ch^{-d/2} \| \tilde{R}_h V - \tilde{I}_h V \|_{L^2(\Gamma[x^*])} + \| I_h V - V \|_{L^\infty(\Gamma[x])} \]

\[ \leq ch^{-d/2} \left( \| R_h V - V \|_{L^2(\Gamma[x^*])} + \| I_h V - V \|_{L^2(\Gamma[x])} \right) + \| I_h V - V \|_{L^\infty(\Gamma[x])} \]

\[ \leq ch^{-2d/2} \| V \|_{H^2(\Gamma[x])} + ch^2 \| V \|_{W^{2,\infty}(\Gamma[x])}, \quad (6.14) \]

where for the last estimate, we have used the (sup-optimal) error bounds for the Ritz map in the \( H^1 \) norm, see (Kovács, 2018, Theorem 6.2), and error estimates for the interpolation in the \( H^1 \) and the in the \( L^\infty \) norm, see Proposition 2.7 in Demlow (2009), with \( m = 2 \) for \( p = 2 \) and \( p = \infty \) therein, respectively.
Using the equivalence of the $L^\infty$ norms on $I_h^T[x^\dagger]$ and $\Gamma[X]$, via (Kovács et al., 2019, Lemma 7.2), the error bound (6.14) and a reverse triangle inequality, we then immediately obtain, for $t \in [0,T]$ (again omitted as an argument) and $h \leq h_0$,

$$\|V_h^*\|_{L^\infty(I_h^T[x^\dagger])} \leq c\|V_h^*\|_{L^\infty(\Gamma[X])} \leq c\|V_h^*\|_{L^\infty(\Gamma[X])} + c\|V\|_{L^\infty(\Gamma[X])} \leq \chi h^2 - \frac{d}{2} \|V\|_{H^2(\Gamma[X])} + (\chi h^2 + c)\|V\|_{W^2,-(\Gamma[X])} \leq M,$$

(6.15)

with an $M > 0$ independent of $h$ and $t$.

6.2.2 Error equations. Denoting the errors between the nodal values of the numerical solutions and the (approximated) exact solutions by $e_x = x - x^\dagger$, $e_v = v - v^\dagger$ and $e_u = u - u^\dagger$. Subtracting (6.13) from (3.4) yields the error equations:

$$K(x)e_x = -(K(x) - K(x^\dagger))v^\dagger + (g(x,u) - g(x^\dagger,u^\dagger)) - M(x^\dagger)d_v,$$

(6.16a)

$$M(x,u)e_u + A(x)e_u = -(M(x,u) - M(x^\dagger,u^\dagger))u^\dagger - (M(x,u^\dagger) - M(x^\dagger,u^\dagger))u^\dagger - (A(x) - A(x^\dagger))u^\dagger + (f(x,u) - f(x^\dagger,u^\dagger)) - M(x^\dagger)d_u,$$

(6.16b)

$$\dot{e}_x = e_v. \quad \text{(6.16c)}$$

With initial data $e_x(0) = 0$ and $e_v(0) = 0$, but with $e_u(0) \neq 0$ in general.

In the sequel we need the following discrete dual norm

$$\|d\|_{x,v,\chi}^2 := d^T M(x)K(x)^{-1}M(x)d.$$

6.3 Stability estimate

The following stability result holds for the errors in the positions $e_x$, in the velocity $e_v$, and in the geometric variables $e_u = (e_n,e_Y)^T$, provided that the defects are small enough. Later on in Lemma 7.1 we will prove that the assumed defect bounds (6.17) are indeed satisfied with $\kappa = k \geq 2$, provided the exact solution is sufficiently smooth.

The basic idea of the proof of this result is the same as for the stability result for mean curvature flow (Kovács et al., 2019, Proposition 7.1), however there are substantial differences due to the solution-dependent mass matrix which have to be addressed carefully.

**Proposition 6.1** Assume that the function $V$ satisfies the assumptions in (2.6). Assume moreover that, for some $\kappa$ with $1 < \kappa \leq k$, the defects are bounded by

$$\|d_x(t)\|_{x,v(t)} \leq \chi h^\kappa, \quad \text{and} \quad \|d_u(t)\|_{M(x^\dagger(t))} \leq \chi h^\kappa,$$

(6.17)

for $0 \leq t \leq T$, and that also the errors in the initial value satisfy

$$\|e_u(0)\|_{K(x^\dagger(0))} \leq \chi h^\kappa.$$

(6.18)
Then, there exists $h_0 > 0$ such that the following stability estimate holds for all $h \leq h_0$ and $0 \leq t \leq T$:

$$
\|e_x(t)\|_{K(x(t))}^2 + \|e_v(t)\|_{K(x(t))}^2 + \|e_u(t)\|_{K(x(t))}^2 \\
\leq C\|e_u(0)\|_{K(x(0))}^2 + C \max_{0 \leq s \leq t} \|d_v(s)\|_{M(x(s))}^2 + C \int_0^t \|d_u(s)\|_{M(x(s))}^2 ds,
$$

(6.19)

where $C$ is independent of $h$ and $t$, but depends on the final time $T$.

**Proof.** Similarly, as in Kovács et al. (2017); Kovács et al. (2019), the stability proof uses energy estimates in the matrix–vector formulation, and relies mostly on the preparatory results of Section 6.1 (with $x$ and $x^*$ in the role of $x$ and $y$). By testing with the time derivative of the error vectors we obtain uniform-in-time $H^1$-norm error bounds, which allow a control in the $W^{1,\infty}$-norm of the errors via an inverse estimate.

As we have noted before algorithm (3.4) for the generalized mean curvature flow is very similar to that of mean curvature flow Kovács et al. (2019). In particular, the error equations are the same except the two solution-dependent mass terms on the right-hand side of (6.16b). Most steps of the stability proof, proof of Proposition 7.1 in Kovács et al. (2019), can be repeated verbatim. Due to the mentioned similarities, the proof below has the same structure as the proof of Proposition 7.1 in Kovács et al. (2019). Corresponding estimates are carried out in corresponding parts, but we will omit arguments which are exactly the same. A crucial difference is that the estimates for the solution-dependent mass terms require $L^\infty$-norm bounds on the weight function $V^\dagger$ in (3.3).

Throughout the proof we will use the following conventions: References to the proof Proposition 7.1 in Kovács et al. (2019) are abbreviated to Kovács et al. (2019), unless a specific reference therein is given. For example, (i) in part (A) of the proof of Proposition 7.1 of Kovács et al. (2019) is referenced as (Kovács et al., 2019, (A.i)). By $c$ and $C$ we will denote generic $h$-independent constants, which might take different values on different occurrences. In the norms the matrices $M(x)$ and $M(x^*)$ will be abbreviated to $M$ and $M^*$, i.e. we will write $\| \cdot \|_M$ for $\| \cdot \|_{M(x)}$ and $\| \cdot \|_{M^*}$ for $\| \cdot \|_{M(x^*)}$, and similarly for the other norms.

Testing (6.16a) with $e_v$ yields

$$
\|e_v\|_K^2 = e_v^T Ke_v = -e_v^T (K(x) - K(x^*)) v^* \\
+ e_v^T (g(x, u) - g(x^*, u^*)) - e_v^T M^* d_v,
$$

(6.20)

whereas testing (6.16b) with $e_u$ leads to

$$
e_u^T M(x, u) e_u + e_u^T A(x) e_u = -e_u^T (M(x, u) - M(x, u^*)) u^* \\
- e_u^T (M(x, u^*) - M(x^*, u^*)) u^* \\
+ e_u^T (f(x, u) - f(x^*, u^*)) - e_u^T M^* d_u.
$$

(6.21)

**Preparations:** Let $t^* \in (0, T]$ the maximal time such that the following inequalities hold

$$
\|e_x(\cdot, t)\|_{W^{1,\infty}(\overline{I_0} | x^*(t))} \leq h^{(k-1)/2}, \\
\|e_v(\cdot, t)\|_{W^{1,\infty}(\overline{I_0} | x^*(t))} \leq h^{(k-1)/2}, \quad \text{for } t \in [0, t^*], \\
\|e_u(\cdot, t)\|_{W^{1,\infty}(\overline{I_0} | x^*(t))} \leq h^{(k-1)/2},
$$

(6.22)
Observe that \( t^* > 0 \), since the initial values are \( e_*(\cdot, 0) = 0, e_*(\cdot, 0) = 0 \) and \( \|e_u(\cdot, 0)\|_{W^{1,\infty}(I_0^N[\mathbf{x}^*(0)])} \leq Ch^{-1}\|e_u(\cdot, 0)\|_{H^1(I_0^N[\mathbf{x}^*(0)])} \leq Ch^{\kappa-1} \) by an inverse inequality and the error estimates of the Ritz map (Kovács, 2018, Theorem 6.2). We will first prove the stated stability bound in \([0, t^*] \), and at the end of the proof we will show that in fact \( t^* = T \).

Through a series of bounds we now show \( L^\infty \)-norm bounds for \((V'_h)^*\) and \(V'_h\), which are crucial to estimate the solution-dependent mass terms.

It is crucial to note, that the third equation implies that, for \( t \in [0, t^*] \),

\[
\|V_h(\cdot, t) - V_h(\cdot, t)\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \leq \|e_u(\cdot, t)\|_{W^{1,\infty}(I_0^N[\mathbf{x}^*(t)])} \leq h^{(\kappa-1)/2}. \tag{6.23}
\]

We now show the \( L^\infty(I_0^N[\mathbf{x}^*(t)]) \) boundedness of \( V_h(\cdot, t) \). Combining (6.15) and (6.23) yields, for \( \|h \leq h_0 \) and for \( t \in [0, t^*] \) (which is omitted within the norms),

\[
\|V_h\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \leq \|V_h\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} + \|V_h - V_h\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \leq M + h^{(\kappa-1)/2} \leq 2M. \tag{6.24}
\]

Upon recalling that \( V_h = V(H_h) \) and \( V_h^* = V(H_h^*) \), the local Lipschitz continuity of the function \( V^{-1} \) (see (2.6)) implies that, for \( t \in [0, t^*] \) (omitted again),

\[
\begin{align*}
\|V_h - V_h\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} &\leq \|V(H_h) - V(H_h)\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \\
&\leq \left\| \frac{1}{L_{2M}^V} \|V^{-1}(V(H_h)) - V^{-1}(V(H_h^*))\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \right\| \\
&\leq \left\| \frac{1}{L_{2M}^V} \|H_h - H_h^*\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \right\| = c\|eh\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])}, \tag{6.25}
\end{align*}
\]

where \( L_{2M}^V \) depends on the \( L^\infty(I_0^N[\mathbf{x}^*(t)]) \) norms of \( V_h^* \) and \( V_h \), (6.15) and (6.24).

Since the mean curvature \( H \) is assumed to be time-uniformly bounded from above and below, recall (2.4), using (6.25) we are able to derive pointwise \( h \)-uniform bounds on the Ritz map of the mean curvature. By the definition of the lift map the equality \( H_h^*(x, t) = (H_h^*)^T(x^*, t) \), and then by the triangle and reversed triangle inequalities, we obtain (with omitted time arguments)

\[
\begin{align*}
|H_h^*| &\leq |H| + |H - H_h^*|^T |H| &\geq |H| - |H - H_h^*|^T |H|
\end{align*}
\]

\[
\begin{align*}
&\leq H_1 + c\|V - (V_h)^*\|^T |L^\infty(\Omega^N[x^*(0)])| \\
&\leq H_1 + c\|V - (V_h)^*\|^T |L^\infty(\Omega^N[x^*(0)])| \\
&\geq H_0 - c\|V - (V_h)^*\|^T |L^\infty(\Omega^N[x^*(0)])| \\
&\geq H_0 - c\|V - (V_h)^*\|^T |L^\infty(\Omega^N[x^*(0)])| \\
&\geq H_0 - c\|V - (V_h)^*\|^T |L^\infty(\Omega^N[x^*(0)])|,
\end{align*}
\]

where for the last inequalities we have used (6.14). Recalling that the surface dimension \( d \leq 3 \) and using the bounds on the mean curvature (2.4), the function \( H_h^* \) satisfies the bounds, for \( h \leq h_0 \),

\[
0 < \frac{3}{2}H_0 \leq \|H_h(\cdot, t)\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \leq \frac{3}{2}H_1, \quad \begin{aligned}
\text{h-uniformly for } 0 \leq t \leq T. \tag{6.26}
\end{aligned}
\]

A similar argument comparing \( H_h \) with \( H_h^* \) now, using (6.25) instead of (6.14), together with (6.26), yields the bounds, for \( h \leq h_0 \),

\[
0 < \frac{3}{2}H_0 \leq \|H_h(\cdot, t)\|_{L^\infty(I_0^N[\mathbf{x}^*(t)])} \leq 2H_1, \quad \begin{aligned}
\text{h-uniformly for } 0 \leq t \leq t^*. \tag{6.27}
\end{aligned}
\]

Note that (6.26) and (6.27) hold on different time intervals, since the estimates (6.14) and (6.25), respectively, hold over different time intervals.
By (6.26) and (6.27), and using that the function $V' : \mathbb{R} \to \mathbb{R}$ is positive everywhere, the functions $V_h' = V'(H_h)$ and $(V_h')^* = V'(H_h^*)$ satisfy the bounds

$$(V_h')^* \text{ and } V_h' \text{ have } h\text{-uniform positive upper and lower bounds for } 0 \leq t \leq t^*.$$  \hfill (6.28)

The upper and lower bounds take the form on $V'(c_1 H_t)$ and $V'(c_0 H_0)$, with $c_1 = 2, \frac{2}{3}$ and $c_0 = \frac{2}{3}, \frac{1}{2}$, respectively.

Altogether, the main condition in Section 6.1, i.e. the $W^{1,\infty}$ norm boundedness of the position errors $e_*$, is satisfied for $0 \leq t \leq t^*$ via the first bound in (6.22). The results and estimates collected therein will be used throughout the stability analysis, usually with $x$ and $x^*$ in the role of $x$ and $y$, respectively. The assumptions in Lemma 6.1 on the $L^\infty$ boundedness of $(V_h')^{-1}$ and $((V_h')^*)^{-1}$ are also satisfied in view of (6.28).

Energy estimates:

(A) Estimates for the surface PDE: We start with estimates of (6.21).

(i) The definition of $M(x,u)$ and the $h$-uniform bounds of (6.28) yield

$$\dot{e}_u^T M(x,u) \dot{e}_u = \int_{\Gamma_h} (V_h')^{-1} |\dot{e}_u|^2 \geq c ||\dot{e}_u||_M^2.$$

(ii) Exactly as in (Kovács et al., 2019, (A.ii)), by the symmetry of $A$ and (6.8) we obtain

$$\dot{e}_u^T A(x) \dot{e}_u = \frac{1}{2} \dot{e}_u^T \frac{d}{dt} (A(x)) \dot{e}_u + \frac{1}{2} \frac{d}{dt} \left( \dot{e}_u^T A(x) \dot{e}_u \right) \geq -c ||\dot{e}_u||_A^2 + \frac{1}{2} \frac{d}{dt} ||\dot{e}_u||_A^2.$$

(iii) The two (non-linear) mass matrix terms on the right-hand side are estimated using Lemma 6.1 together with (6.22) and (6.28). For the first term, by Lemma 6.1 (iv) (with $\dot{e}_u^*$, $\dot{u}^*$, and $u^*$ in the role of $w$, $z$, and $u$, respectively), together with (6.22), (6.28) and the $W^{1,\infty}$ bound on $\partial_h^* u_h$. Using the norm equivalence (6.3), we altogether obtain

$$-\dot{e}_u^T (M(x,u) - M(x,u^*)) \dot{u}^* \leq c ||\dot{e}_u||_{M^*} ||\dot{u}^*||_{M^*}.$$

For the other term we apply Lemma 6.1, (ii) (with $\dot{e}_u^*$, $\dot{u}^*$, and $u^*$ in the role of $w$, $z$, and $u$, respectively), and obtain

$$-\dot{e}_u^T (M(x,u^*) - M(x^*,u^*)) \dot{u}^* \leq c ||\dot{e}_u||_{A^*} ||\dot{u}^*||_{M^*},$$

where we again used the norm equivalence (6.3).

(iv) The third term on the right-hand side of (6.21) is estimated exactly as in (Kovács et al., 2019, (A.iv))

$$-\dot{e}_u^T (A(x) - A(x^*)) \dot{u}^* \leq -\frac{d}{dt} \left( \dot{e}_u^T (A(x) - A(x^*)) \dot{u}^* \right) + c ||\dot{e}_u||_{A^*} (||\dot{e}_x||_{K^*} + ||\dot{e}_x||_{K^*}).$$

(v) Since the estimates for the non-linear term (Kovács et al., 2019, (A.v)) were shown for a general locally Lipschitz function, they apply for the present case as well (note, however, that $f$ defined in (Kovács et al., 2019, Section 3.3) is different from the one here), and yield

$$\dot{e}_u^T (f(x,u) - f(x^*,u^*)) \leq c ||\dot{e}_u||_{M^*} (||\dot{e}_u||_{K^*} + ||\dot{e}_x||_{A^*}).$$
(vi) The defect term is simply bounded by the Cauchy–Schwarz inequality and a norm equivalence (6.3):

$$-\bar{e}_u^T M(x^*, u^*) d_u \leq c \|e_u\| \|d_u\|_{M^*}.$$  

Since the final estimates (i)–(vi) here are the same as in Kovács et al. (2019, (A.i)–(A.vi)), the rest of Part (A) can be finished exactly as in Kovács et al. (2019), which finally yields

$$\|e_u(t)\|_{K(x^*(t))} \leq c \int_0^t \left( \|e_u(s)\|_{K(x^*(s))}^2 + \|e_v(s)\|_{K(x^*(s))}^2 + \|e_u(s)\|_{K(x^*(s))}^2 \right) ds$$

+ $$c \|e_u(t)\|_{\bar{M}(x^*(t))}^2 + \|e_u(0)\|_{K(x^*(0))}^2 + c \int_0^t \|d_u(s)\|_{\bar{M}(x^*(s))}^2 ds,$$

cf. (Kovács et al., 2019, equation (7.33)).

(B) Estimates for the velocity equation: Since the velocity equation is formally the same here and in Kovács et al. (2019): (6.20) and (Kovács et al., 2019, equation (7.19)) coincide, therefore the analysis in (Kovács et al., 2019, Part (B)) and the obtained result applies to the present situation as well, and yields the estimate:

$$\|e_v(t)\|_{K(x^*(t))}^2 \leq c \|e_u(t)\|_{K(x^*(t))}^2 + c \|e_v(t)\|_{K(x^*(t))}^2 + c \|d_u(t)\|_{x^*(t)}^2.$$  

(C) Combination: Since the two estimates (6.29) from Part (A) and (6.30) from Part (B) are formally the same as the two corresponding estimates in equation (7.33) and (7.34) in Kovács et al. (2019), the proof can be finished exactly as it was done in (Kovács et al., 2019, Part (C)) (combining (6.29), (6.30), and (6.16c), and using Gronwall’s inequality), by which we obtain the stability estimate (6.19) for $0 \leq t \leq t^*$. It remains to show that $t^* = T$ for $h$ sufficiently small. Upon noting that by the assumed defect bounds (6.17) and (6.18), the obtained stability bound (6.19) implies

$$\|e_x(t)\|_{K(x^*(t))} + \|e_v(t)\|_{K(x^*(t))} + \|e_u(t)\|_{K(x^*(t))} \leq C h^\kappa,$$

and therefore by an inverse inequality we obtain, for $t \in [0, t^*],

$$\|e_x(\cdot, t)\|_{W^{1, \infty}(\bar{B}(x^*(t)))} + \|e_v(\cdot, t)\|_{W^{1, \infty}(\bar{B}(x^*(t)))} + \|e_u(\cdot, t)\|_{W^{1, \infty}(\bar{B}(x^*(t)))} \leq \frac{c}{h} \left( \|e_x(t)\|_{K(x^*(t))} + \|e_v(t)\|_{K(x^*(t))} + \|e_u(t)\|_{K(x^*(t))} \right)$$

$$\leq c C h^{\kappa - 1} \leq \frac{1}{2} h^{(\kappa - 1)/2},$$

for sufficiently small $h$. This means that the bounds (6.22) can be extended beyond $t^*$, contradicting the maximality of $t^*$, unless $t^* = T$ already. Therefore we have show the stability bound (6.19) over the whole time interval $[0, T]$. □

7. Consistency estimates for the semi-discretisation

The following estimates for the defect bounds are analogous to (Kovács et al., 2019, Lemma 8.1).

**Lemma 7.1** Assume that the surface $\Gamma[X]$ evolving under generalized mean curvature flow is sufficient regular on the time interval $[0, T]$. Then, there exists a constant $h_0 > 0$ such that for all $h \leq h_0$ and for all
\( t \in [0, T] \), the defects \( d_u(t) \in S_h(\Gamma_h[x_*]) \) and \( d_*u(t) \in S_h(\Gamma_h[x_*]) \) of the \( k \)-th degree finite elements are bounded by

\[
\begin{align*}
\|d_u(t)\|_{\mathcal{X}(\Gamma)} &= \|d_u(\cdot,t)\|_{H^k(\Gamma_h[x_*])} \leq c h^k, \\
\|d_*u(t)\|_{\mathcal{M}(\Gamma)} &= \|d_*(\cdot,t)\|_{L^2(\Gamma_h[x_*])} \leq c h^k.
\end{align*}
\]

The constant \( c > 0 \) is independent of \( h \) and \( t \), but depends on the final time \( T \).

**Proof.** Since (6.13a) is the same equation as equation (7.14a) in Kovács et al. (2019) the proof of the defect bound of \( d_u \) in (Kovács et al., 2019, Lemma 8.1) holds, and gives the stated result.

For the defect in \( u \) the proof is also similar to (Kovács et al., 2019, Lemma 8.1). By (Kovács, 2018, Theorem 6.3), the error in the Ritz map (6.12) and in its material derivative there holds:

\[
\begin{align*}
\|(u_h^*)^t - u(t)\|_{H^1(\Gamma'h[x_*])} &\leq c h^k; \\
\|(\partial^*u_h^*)^t - \partial^*u(t)\|_{H^1(\Gamma'h[x_*])} &\leq c h^k.
\end{align*}
\]  

(7.1)

Note that the first equation implies

\[
\|(V_h^* - V(t))\|_{H^1(\Gamma'h[x_*])} \leq c h^k.
\]  

(7.2)

Using the function \( f(u, \nabla \Gamma u) = |A|^2 u \) we rewrite (6.13) as

\[
\int_{\Gamma_h[x_*]} ((V_h^*)^{-1})_{\partial^* u_h^*} \cdot \phi_h + \int_{\Gamma_h[x_*]} \nabla \Gamma_h[x_*] u_h^* \cdot \nabla \Gamma_h[x_*] \phi_h = \int_{\Gamma_h[x_*]} f(u_h^*, \nabla \Gamma_h[x_*] u_h^*) \cdot \phi_h + \int_{\Gamma_h[x_*]} d_u \cdot \phi_h
\]

for all \( \phi_h \in S_h(\Gamma'h[x_*]) \). Subtracting the weak formulation for the exact solution (2.9) from this equation, we obtain

\[
\int_{\Gamma_h[x_*]} d_u \cdot \phi_h = \left[ \int_{\Gamma_h[x_*]} ((V_h^*)^{-1})_{\partial^* u_h^*} \cdot \phi_h - \int_{\Gamma'[x_*]} (V')^{-1} \partial^* u \cdot \phi_h^c \right] \\
+ \left[ \int_{\Gamma_h[x_*]} \nabla \Gamma_h[x_*] u_h^* \cdot \nabla \Gamma_h[x_*] \phi_h - \int_{\Gamma'[x_*]} \nabla \Gamma'[x_*] u \cdot \nabla \Gamma'[x_*] \phi_h^c \right] \\
- \left[ \int_{\Gamma_h[x_*]} f(u_h^*, \nabla \Gamma_h[x_*] u_h^*) \cdot \phi_h - \int_{\Gamma'[x_*]} f(u, \nabla \Gamma'[x_*] u) \cdot \phi_h^c \right]
\]

for all \( \phi_h \in S_h(\Gamma'h[x_*]) \). The second and the third term on the right-hand side can be estimated exactly as in (Kovács et al., 2019, Lemma 8.1). The critical term in the square brackets can be rewritten as

\[
\begin{align*}
&\int_{\Gamma_h[x_*]} ((V_h^*)^{-1})_{\partial^* u_h^*} \cdot \phi_h - \int_{\Gamma'[x_*]} (V')^{-1} \partial^* u \cdot \phi_h^c \\
= &\left( \int_{\Gamma_h[x_*]} ((V_h^*)^{-1})_{\partial^* u_h^*} \cdot \phi_h - \int_{\Gamma'[x_*]} ((V_h^*)^{-1})_{\partial^* u_h^*} \cdot \phi_h^c \right) \\
+ &\left( \int_{\Gamma'[x_*]} (V')^{-1} (\partial^* u_h^*)^f \cdot \phi_h^c - \int_{\Gamma'[x_*]} (V')^{-1} (\partial^* u_h^*)^f \cdot \phi_h^c \right) \\
+ &\left( \int_{\Gamma'[x_*]} (V')^{-1} (\partial^* u_h^*)^f \cdot \phi_h^c - \int_{\Gamma'[x_*]} (V')^{-1} (\partial^* u_h^*)^f \cdot \phi_h^c \right)
\end{align*}
\]
Using (2.6) and (6.28), (Kovács et al., 2017, Lemma 7.4) implies that the first term is bounded by \( ch^k \| \varphi_h^t \|_{L^2(I[X])} \). Using Cauchy-Schwarz inequality, (7.2) and Lemma A.1 (with \( \gamma = -1 \)) the second term is bounded by \( ch^k \| \varphi_h^t \|_{L^2(I[X])} \). Finally, by Cauchy-Schwarz inequality the last term is bounded by \( ch^k \| \varphi_h^t \|_{L^2(I[X])} \) using (7.1). From here on the proof can be finished exactly as the proof of (Kovács et al., 2017, Lemma 8.1).

\[ \square \]

8. Proof of Theorem 5.1

By the combination of the above stability and consistency results we obtain the following proof for the convergence result.

**Proof of Theorem 5.1.** By definition of the lift operator \( \ell \) we obtain

\[
\begin{align*}
X^t_h - X &= (\hat{X}_h - X^t_h) + ((X^t_h) - X), \\
\nu^t_h - v &= (\hat{\nu}_h - \nu^t_h) + ((\nu^t_h) - X), \\
u^t_h - u &= (\hat{\nu}_h - \nu^t_h) + ((\nu^t_h) - u).
\end{align*}
\]

Using the interpolation error bounds (Demlow, 2009, Proposition 2.7), the last terms on the right-hand side of the first and second equation are bounded in the \( H^1(I) \) norm by \( ch^k \). Whereas the analogous estimate for the second term of the last equation follows by (7.1).

Combining the stability estimate Proposition 6.1 and the defect bounds Lemma 7.1 we obtain

\[
\| e_x \|_{K(x')} + \| e_v \|_{K(x')} + \| e_u \|_{K(x')} \leq c h^k.
\]

By the norm equivalence (Dziuk, 1988, Lemma 3), it follows

\[
\begin{align*}
\| (\hat{X}_h - X^t_h) \|_{H^1(I[X])} &\leq c \| \hat{X}_h - X^t_h \|_{H^1(I[I[X]])} = c \| e_x \|_{H^1(I[I[X]])} = c \| e_x \|_{K(x')} \leq c h^k, \\
\| (\hat{\nu}_h - \nu^t_h) \|_{H^1(I[X])} &\leq c \| \hat{\nu}_h - \nu^t_h \|_{H^1(I[I[X]])} = c \| e_v \|_{H^1(I[I[X]])} = c \| e_v \|_{K(x')} \leq c h^k, \\
\| (\hat{\nu}_h - \nu^t_h) \|_{H^1(I[X])} &\leq c \| \hat{\nu}_h - \nu^t_h \|_{H^1(I[I[X]])} = c \| e_u \|_{H^1(I[I[X]])} = c \| e_u \|_{K(x')} \leq c h^k,
\end{align*}
\]

and hence the theorem is proven.

\[ \square \]

9. Stability of the full discretization

In the following section we will prove a stability result for linearly implicit BDF discretisations. The proof is a time discrete extension of Proposition 6.1, and due to the mentioned structural similarity between (3.4) and (3.5) it is based on (Kovács et al., 2019, Proposition 10.1).

9.1 Auxiliary results by Dahlquist and Nevanlinna & Odeh

We recall two important results that enable us to use energy estimates for BDF methods up to order 5: the first result is from Dahlquist’s \( G \)-stability theory, and the second one from the multiplier technique of Nevanlinna and Odeh.

**Lemma 9.1** (Dahlquist Dahlquist (1978)) Let \( \delta(\zeta) = \sum_{j=0}^q \delta_j \zeta^j \) and \( \mu(\zeta) = \sum_{j=0}^q \mu_j \zeta^j \) be polynomials of degree at most \( q \) (at least one of them of degree \( q \)) that have no common divisor. Let \( \langle \cdot, \cdot \rangle \) denote an inner product on \( \mathbb{R}^N \). If

\[
\text{Re} \frac{\delta(\zeta)}{\mu(\zeta)} > 0 \quad \text{for} \quad |\zeta| < 1,
\]

then there exists a constant \( k > 0 \) such that

\[
\text{Re} \frac{\langle \hat{U}, u \rangle}{\langle \hat{U}, u \rangle} > k \quad \text{for} \quad |\zeta| < 1.
\]
then there exists a symmetric positive definite matrix \( G = (g_{ij}) \in \mathbb{R}^{q \times q} \) such that for all \( w_0, \ldots, w_q \in \mathbb{R}^N \)

\[
\sum_{i=0}^{q} \delta_i \langle w_{q-i}, \sum_{i=0}^{q} \mu_i w_{q-i} \rangle \geq \sum_{i,j=1}^{q} g_{ij} \langle w_i, w_j \rangle - \sum_{i,j=1}^{q} g_{ij} \langle w_{i-1}, w_{j-1} \rangle .
\]

In view of the following result, the choice \( \mu(\xi) = 1 - \eta \xi^2 \) together with the polynomial \( \delta(\xi) \) of the BDF methods will play an important role later on.

**LEMMA 9.2** (Nevanlinna & Odeh (1981)) If \( q \leq 5 \), then there exists \( 0 \leq \eta < 1 \) such that for \( \delta(\xi) = \sum_{i=1}^{q} \frac{\xi^i}{i} \),

\[
\text{Re} \left( \frac{\delta(\xi)}{1 - \eta \xi^2} \right) > 0 \quad \text{for} \quad |\xi| < 1.
\]

The smallest possible values of \( \eta \) are found to be \( \eta = 0, 0.0836, 0.2878, 0.8160 \) for \( q = 1, \ldots, 5 \), respectively.

These results have previously been applied in the error analysis of BDF methods, in particular for mean curvature flow Kovács et al. (2019), and also for various parabolic problems in Akrivis et al. (2017); Akrivis & Lubich (2015); Kovács & Lubich (2018); Kovács & Power Guerra (2016); Lubich et al. (2013); Akrivis et al. (2019), where they were used when testing the error equation with the discretized time derivative of the error. Similarly as in Kovács et al. (2019) and Akrivis et al. (2019), and in contrast to the other references above, these results are used for testing the error equation with the discretized time derivative of the error.

For the six-step BDF method a new and intriguing energy approach was recently introduced in Akrivis et al. (2020).

### 9.2 Errors and defects

We recall from Section 6.2, that the nodal vectors \( x^*(t) \in \mathbb{R}^{3N} \), \( v^*(t) \in \mathbb{R}^{3N} \) and \( u^*(t) \in \mathbb{R}^{3N} \) denote the finite element interpolations of the exact solution \( X(\cdot, t) \) and \( v(\cdot, t) \), respectively, while the vector \( u^*(t) \) contains the nodal values of the Ritz map of \( u = (n, V)^T \) on the interpolated surface \( \Gamma_b[x^*(t)] \). We abbreviate \( x^n = x^*(t_n) \), \( v^n = v^*(t_n) \), and \( u^n = u^*(t_n) \).

We insert these values into the numerical scheme, and obtain defects \( d^n_v, d^n_u, d^n_w \): for \( n \geq q \),

\[
\begin{align}
K(\tilde{x}^n)\nu^n &= g(\tilde{x}^n, \tilde{u}^n) + M(\tilde{x}^n)d^n_v, \\
M(\tilde{x}^n, \tilde{u}^n)\dot{u}^n + A(\tilde{x}^n)u^n &= f(\tilde{x}^n, \tilde{u}^n) + M(\tilde{x}^n)d^n_u, \\
\dot{x}^n &= v^n + d^n_x,
\end{align}
\]

where the backward difference time derivatives and the extrapolated values are given by (4.2) and (4.3).

#### 9.2.1 Two estimates for the extrapolation

Analogously to the time-continuous stability proof, in the proof of fully discrete stability we will need \( L^\infty \)-norm estimates (now) for the extrapolation of the Ritz map of \( u \). These preparatory estimates will play analogous roles as those in Section 6.2.1, and are proved below.

We first derive an estimate for the error in the extrapolation of the Ritz map of the exact normal velocity. Using the Peano kernel representation (see (Gautschi, 1997, Section 3.2.6)) of the extrapolation.
error, analogously to the proof of Lemma 4.3 in Kovács & Lubich (2018), we obtain
\[
\|((V_h^e)^n)^f - ((\tilde{V}_h^e)^n)^f\|_{L^\infty([\Omega])} \leq \left\|((V_h^e)^n)^f - \left(\sum_{j=0}^{q-1} \gamma_j (V_h^e)^{n-1-j}\right)\right\|_{L^\infty([\Omega])} \leq c \tau^q. \tag{9.2}
\]

Furthermore, we prove that the extrapolations at subsequent times \(t_n\) and \(t_{n-1}\) of the Ritz map of \(u(\cdot, t)\) are \(L^\infty\)-norm bounded by \(O(\tau)\), similarly as in the proof of Lemma 8.1 in Akrivis et al. (2019). Namely, the following estimate holds:
\[
\|\tilde{u}^n - \tilde{u}_{n-1}\|_{L^\infty([\Omega])} \leq \sum_{j=0}^{k-1} \gamma_j \int_{t_{n-j-2}}^{t_{n-j-1}} \|\partial_t^q \tilde{R}_h u(\cdot, s)\|_{L^\infty([\Omega])} ds \leq c \tau, \tag{9.3}
\]

where in the last estimate to show the boundedness of \(\partial_t^q \tilde{R}_h u\), we have used a similar argument as (6.14), but here using the error estimates in the material derivative of the Ritz map (Kovács, 2018, Theorem 6.4).

9.2.2 Fully discrete error equations. The errors of the numerical solution \(x^n, v^n\) and \(u^n = (n^n, V^n)^T\) are denoted by
\[
e^n_x = x^n - x^n_n, \quad e^n_v = v^n - v^n_n, \quad e^n_u = u^n - u^n_n,
\]
and we abbreviate
\[
e^n_x = \frac{1}{\tau} \sum_{j=0}^{q} \delta_j e^n_{x,j}, \quad e^n_v = \frac{1}{\tau} \sum_{j=0}^{q} \delta_j e^n_{v,j}, \quad e^n_u = \frac{1}{\tau} \sum_{j=0}^{q} \delta_j e^n_{u,j}. \tag{9.4}
\]

Subtracting (9.1) from (4.1), we obtain the following error equations:
\[
K(\bar{x}) e^n_x = r^n_v, \tag{9.5a}
\]
\[
M(\bar{x}, \bar{v}) e^n_u + A(\bar{x}) e^n_v = r^n_u, \tag{9.5b}
\]
\[
e^n_v = e^n_v - d^n_v, \tag{9.5c}
\]

where the right-hand side terms denote
\[
r^n_v = - (K(\bar{x}) - K(\bar{x}_n)) v^n_n + (g(\bar{x}, \bar{v}_n) - g(\bar{x}_n, \bar{v}_n)) - M(\bar{x}_n) d^n_v,
\]
\[
r^n_u = - (M(\bar{x}, \bar{u}_n) - M(\bar{x}_n, \bar{u}_n)) u^n_n - (M(\bar{x}, \bar{u}_n) - M(\bar{x}_n, \bar{u}_n)) u^n_n - (A(\bar{x}) - A(\bar{x}_n)) u^n_n + (f(\bar{x}, \bar{u}_n) - f(\bar{x}_n, \bar{u}_n)) - M(\bar{x}_n) d^n_u.
\]

9.3 Stability estimate

The following fully discrete stability result is a time discrete analogue of Proposition 6.1, and a generalization of (Kovács et al., 2019, Proposition 10.1) to generalized mean curvature flow.
Proposition 9.1 Assume that the function $V$ satisfies the assumptions in (2.6). Consider the linearly implicit BDF time discretization (4.1) for generalized mean curvature flow, with order $q$ with $2 \leq q \leq 5$. Assume that, for step sizes restricted by $\tau \leq C_0 h$, there exists $\kappa$ with $2 \leq \kappa \leq k$ such that the defects are bounded by

$$\|d^n_u\|_{K(x^n)} \leq ch^\kappa, \quad \|d^n_{\xi}\|_{\mathcal{L}^\kappa_x} \leq ch^\kappa, \quad \|e^n_{\xi}\|_{M(x^n)} \leq ch^\kappa,$$

(9.7)

for $q \tau \leq n \tau \leq T$, and that also the errors of the starting values are bounded by

$$\|e^0_{\xi}\|_{K(x^0)} \leq ch^\kappa, \quad \|e^0_{\xi}\|_{\mathcal{L}^\kappa_x} \leq ch^\kappa, \quad \|e^0_u\|_{M(x^0)} \leq ch^\kappa,$$

(9.8)

for $i = 0, \ldots, q - 1$, and, with the notation $\partial^\tau \xi_i = (\xi_i - \xi_i^{(i-1)}) / \tau$, for $i = 1, \ldots, q - 1$,

$$\tau^{1/2} \|\partial^\tau \xi_i\|_{K(x^n)} \leq ch^\kappa.$$

(9.9)

Then, there exist $h_0 > 0$ and $\tau_0 > 0$ such that the following stability estimate holds for all $h \leq h_0$, $\tau \leq \tau_0$, and with $n \tau \leq T$, satisfying $\tau \leq C_0 h$,

$$\|e^n_{\xi}\|_{K(x^n)}^2 + \|e^n_{u}\|_{\mathcal{L}^\kappa_x}^2 + \|e^n_u\|_{K(x^n)}^2 \leq C \sum_{j=0}^{q-1} \left( \|e^j_{\xi}\|_{K(x^n)}^2 + \|e^j_{u}\|_{\mathcal{L}^\kappa_x}^2 + \|e^j_u\|_{K(x^n)}^2 \right) + C \tau \sum_{i=1}^{q-1} \|\partial^\tau \xi_i\|_{K(x^n)}^2,$$

(9.10)

where $C$ is independent of $h$, $\tau$ and $n$ with $n \tau \leq T$, but depends on the final time $T$.

Proof. The proof of this result is again based on energy estimates testing with the (time-discrete) $e^n_{\xi}$, relying on the $G$-stability theorem of Dahlquist Lemma 9.1, and the multiplier techniques of Nevanlinna and Odeh Lemma 9.2. Just as the semi-discrete stability proof, the proof of this result is also closely related to that of Proposition 10.1 in Kovács et al. (2019).

A key difference in the proofs, as compared to the mean curvature flow Kovács et al. (2019), is that the present proof works with the solution dependent norm $\|\cdot\|_{M(x^n)}$ instead of $\|\cdot\|_{M(\bar{x})}$. This, simple looking, yet crucial difference requires extra care during the stability analysis.

To exploit the analogies between the semi- and fully discrete stability proofs, the organisation of this proof, including the numbering of its parts, will be the same as that of Proposition 6.1 (and also the same as in the proof of Proposition 10.1 in Kovács et al. (2019)). Throughout, the proof we use the same notational and reference conventions as before (with the additional note that the generic constants are here $h$- and $\tau$-independent).

Preparations: Let $t^*$ with $0 < t^* \leq T$ (which a priori might depend on $\tau$ and $h$) be the maximal time such that the following inequalities hold:

$$\|e^n_{\xi}\|_{W^{1,\infty}(I^n_{\bar{x}})} \leq h^{(\kappa - 1)/2},$$

$$\|e^n_{u}\|_{W^{1,\infty}(I^n_{\bar{x}})} \leq h^{(\kappa - 1)/2},$$

$$\|e^n_u\|_{W^{1,\infty}(I^n_{\bar{x}})} \leq h^{(\kappa - 1)/2},$$

(9.11)

Note that the by the smallness condition for the errors in the initial data we have, at least, $t^* = q \tau$. At the end of the proof we will show that in fact $t^* = T$. 

Similarly as in the preparations section of the proof of Proposition 6.1, the bounds (9.11) have the analogous consequences, obtained by the exact same arguments as previously therein. A similar argument was necessary in the proof of Proposition 7.1 and 7.2 in Akrivis et al. (2019).

In particular, the third bound implies, using the equivalence of the $L^\infty$ norms, for $h \leq h_0$ sufficiently small and $n\tau \leq t^*$,

$$
\|\tilde{V}_h^n - (\tilde{V}_h^n)^n\|_{L^\infty(\tilde{t}_n|x^s|)} \leq \sum_{j=0}^{q-1} |\gamma_j| \|\tilde{V}_h^{n-1-j} - (\tilde{V}_h^n)^{n-1-j}\|_{L^\infty(\tilde{t}_n|x^{n-1-j}|)} \leq \sum_{j=0}^{q-1} |\gamma_j| \|e_n^{n-1-j}\|_{W^{1,\infty}(\tilde{t}_n|x^{n-1-j}|)} \leq C_\gamma h^{(k-1)/2},
$$

with $C_\gamma = \sum_{j=0}^{q-1} |\gamma_j| = 2^q - 1$.

The $L^\infty(\tilde{t}_n|x^s|)$ boundedness of $\tilde{V}_h^n$ is obtained by combining (6.15) and (9.12): for $h \leq h_0$ and for $n\tau \leq t^*$,

$$
\|\tilde{V}_h^n\|_{L^\infty(\tilde{t}_n|x^s|)} \leq \|((\tilde{V}_h^n)^n)\|_{L^\infty(\tilde{t}_n|x^s|)} + \|\tilde{V}_h^n - (\tilde{V}_h^n)^n\|_{L^\infty(\tilde{t}_n|x^s|)} \leq C_\gamma M + C_\gamma h^{(k-1)/2} \leq (C_\gamma + 1)M.
$$

Recall that $\tilde{V}_h^n = V(\tilde{H}_h^n)$, which defines the curvature data $\tilde{H}_h^n$ by inverting the function $V$. Note, however, that $\tilde{H}_h^n$ is not an extrapolation for the mean curvature, but merely a suggestive notation, expressing its relation to $\tilde{V}_h^n$.

Then, by the local Lipschitz continuity of the function $V^{-1}$ (see (2.6)), we have, for $n \leq n^*$,

$$
C_\gamma h^{(k-1)/2} \geq \|\tilde{V}_h^n - (\tilde{V}_h^n)^n\|_{L^\infty(\tilde{t}_n|x^s|)} = \|V(\tilde{H}_h^n) - V((\tilde{H}_h^n)^n)\|_{L^\infty(\tilde{t}_n|x^s|)} \geq \frac{1}{L_{nM}^{-1}} \|V^{-1}(V(\tilde{H}_h^n)) - V^{-1}(V((\tilde{H}_h^n)^n))\|_{L^\infty(\tilde{t}_n|x^s|)} \geq \frac{1}{L_{nM}^{-1}} \|\tilde{H}_h^n - (\tilde{H}_h^n)^n\|_{L^\infty(\tilde{t}_n|x^s|)} = c\|e_n^n\|_{L^\infty(\tilde{t}_n|x^s|)}.
$$

where $L_{nM}^{-1}$ depends on the $L^\infty(\tilde{t}_n|x^s|)$ norms of $\tilde{V}_h^n$ and $\tilde{V}_h$, (6.15) and (9.13).

As for the semi-discrete case, since the mean curvature $H$ is assumed to be time-uniformly bounded from above and below (2.4), using the local Lipschitz continuity of $V^{-1}$ as in (9.14) we derive $h$-uniform bounds on the Ritz map of the mean curvature. By the definition of the lift map, we have the equality $\tilde{H}_h^n(x,t) = (\tilde{H}_h^n)^n(x^s,t)$, for any time $t$ and for any $x \in \tilde{t}_n|x^s(t)|$. Then, by the triangle inequality, we obtain

$$
|(\tilde{H}_h^n)^n| \leq |H(\cdot,t_n)| + |H(\cdot,t_n) - ((\tilde{H}_h^n)^n)^\ell| \leq |H(\cdot,t_n)| + |H(\cdot,t_n) - ((\tilde{H}_h^n)^n)^\ell|_{L^\infty(\tilde{t}_n|x(t_n)|)} \leq H_1 + c\|V(\cdot,t_n) - ((\tilde{V}_h^n)^n)^\ell\|_{L^\infty(\tilde{t}_n|x(t_n)|)} + c\|((\tilde{V}_h^n)^n)^\ell - ((\tilde{V}_h^n)^n)^\ell\|_{L^\infty(\tilde{t}_n|x(t_n)|)} \leq H_1 + ch^{2-d/2} + c\tau^d,
$$
and similarly
\[
|\tilde{H}_h^n| \geq |H(\cdot, \tau_n)| - |H(\cdot, \tau_n) - (\tilde{H}_h^n)|
\]
where in both estimates we have used (6.14) and (9.2). The argument is now repeated for \(H_h^n\), now comparing \(\tilde{H}_h^n\) with \((\tilde{H}_h^n)^*\), and using (9.14) instead of (6.14), and (9.2):

\[
|\tilde{H}_h^n| \leq |(\tilde{H}_h^n)^*| + |\tilde{H}_h^n - (\tilde{H}_h^n)^*| \\
\leq |(\tilde{H}_h^n)^*| + |\tilde{H}_h^n - (\tilde{H}_h^n)^*|_{L^2(\Omega[x])} \\
\leq |(\tilde{H}_h^n)^*| + c\|\tilde{\nu}_H^{\ell}\|_{L^2(\Omega[x])} \\
|\tilde{H}_h^n| \geq |(\tilde{H}_h^n)^*| - |\tilde{H}_h^n - (\tilde{H}_h^n)^*| \\
\geq |(\tilde{H}_h^n)^*| - |\tilde{H}_h^n - (\tilde{H}_h^n)^*|_{L^2(\Omega[x])} \\
\geq |(\tilde{H}_h^n)^*| - c\|\tilde{\nu}_H^{\ell}\|_{L^2(\Omega[x])} \\
\leq |(\tilde{H}_h^n)^*| + c h^{(\kappa-1)/2}, \\
\geq |(\tilde{H}_h^n)^*| - c h^{(\kappa-1)/2}.
\]

Altogether, recalling that \(d \leq 3\) and \(\kappa \geq 2\), we obtain the bounds, for \(h \leq h_0\) and \(\tau \leq \tau_0\),
\[
0 < \frac{\tau}{3} H_0 \leq \|\tilde{H}_h^n\|_{L^2(\Omega[x])} \leq \frac{2}{\tau} H_1, \quad h\text{- and } \tau\text{-uniformly for } n\tau \leq T, \quad (9.15)
\]
and
\[
0 < \frac{1}{2} H_0 \leq \|\tilde{H}_h^n\|_{L^2(\Omega[x])} \leq 2 H_1, \quad h\text{- and } \tau\text{-uniformly for } n\tau \leq t^*. \quad (9.16)
\]

By (9.15) and (9.16), and using that the function \(V' : \mathbb{R} \rightarrow \mathbb{R}\) is positive everywhere, the functions \((\tilde{V}_h^n)^* = V'(\tilde{H}_h^n(\cdot, t_n))\) and \((\tilde{V}_h^n)^* = V'(\tilde{H}_h^n(\cdot, t_n))\) satisfy that
\[
(\tilde{V}_h^n)^*(\cdot, t_n) \quad \text{and} \quad \tilde{V}_h^n(\cdot, t_n) \quad \text{have } h\text{- and } \tau\text{-uniform positive upper and lower bounds for } n\tau \leq t^*. \quad (9.17)
\]

**Norm equivalences:** Throughout the stability proof we will additionally need some norm equivalence results. The fact that
\[
\text{the norms } \| \cdot \|_{M(\tilde{x}^n)} \text{ and } \| \cdot \|_{A(\tilde{x}^n)} \text{ are } h\text{- and } \tau\text{-uniformly equivalent for } q \tau \leq n\tau \leq t^*, \quad (9.18)
\]
is proven by the same techniques as (10.12) in Kovács et al. (2019).

As we have already pointed out, the function \(H_h^n\) is given by \(H_h^n = V^{-1}(\tilde{V}_h^n)\) (and not an extrapolation). This then defines \((\tilde{V}_h^n)^* = V'(H_h^n)\), which appears in the solution-dependent norm generated by \(M(\tilde{x}^n, \tilde{u}^n)\):
\[
\|w\|^2_{M(\tilde{x}^n, \tilde{u}^n)} := w^T M(\tilde{x}^n, \tilde{u}^n) w = \int_{\Omega[x]} ((\tilde{V}_h^n)^*)^{-1} |w_h|^2.
\]

In view of (9.17), i.e. \((\tilde{V}_h^n)^*\) is bounded away from zero and bounded from above for \(n \leq n^*\), the matrix \(M(\tilde{x}^n, \tilde{u}^n)\) is indeed generates a norm.

Furthermore, the bounds (9.17) additionally yield that
\[
\text{the norm } \| \cdot \|_{M(\tilde{x}^n, \tilde{u}^n)} \text{ is equivalent to } \| \cdot \|_{M(\tilde{x}^n)} \text{ uniformly in } h \text{ and } \tau \text{ for any } q \tau \leq n\tau \leq t^*. \quad (9.19)
\]
As a final preparatory result, we prove the norm equivalence of the solution-dependent norm at different times, i.e. the solution-dependent analogue of the norm equivalence (9.18). To this end, we start by rewriting

\[ \|w\|_{M(\tilde{x}^n, \tilde{w}^n)}^2 - \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2 = w^T (M(\tilde{x}^n, \tilde{u}^n) - M(\tilde{x}^{n-1}, \tilde{u}^{n-1})) w \]

\[ = w^T (M(\tilde{x}^n, \tilde{u}^n) - M(\tilde{x}^{n-1}, \tilde{u}^n)) w + w^T (M(\tilde{x}^{n-1}, \tilde{u}^n) - M(\tilde{x}^{n-1}, \tilde{u}^{n-1})) w. \]

The first term on the right-hand side is estimated using Lemma 6.1 (i), using (9.11) to ensure the $W^{1,\infty}$ norm boundedness of $\tilde{u}_h^n$, together with an estimate for $\tilde{x}^{n} - \tilde{x}^{n-1}$. In (10.11) of Kovács et al. (2019) it was shown that $\|\tilde{W}_h^n\|_{W^{1,\infty}(I_h[\tilde{x}^n])} \leq K$, where the coefficient functions of $\tilde{W}_h^n$ are given by $\tilde{W}_h^n = (\tilde{x}^n - \tilde{x}^{n-1})/\tau$. Hence, we altogether obtain

\[ w^T (M(\tilde{x}^n, \tilde{u}^n) - M(\tilde{x}^{n-1}, \tilde{u}^n)) w \leq c \tau \|\tilde{W}_h^n\|_{W^{1,\infty}(I_h[\tilde{x}^n])} \|w\|_{M(\tilde{x}^n)}^2, \]

where for the second inequality we have used (9.19), i.e. the equivalence between the norms $\|\cdot\|_{M(\tilde{x}^n, \tilde{w}^n)}$ and $\|\cdot\|_{M(\tilde{x}^n)}$.

The second term on the right-hand side is estimated using Lemma 6.1 (iii). Using the inequality (9.3), we estimate as

\[ w^T (M(\tilde{x}^{n-1}, \tilde{u}^n) - M(\tilde{x}^{n-1}, \tilde{u}^{n-1})) w \]

\[ = w^T (M(\tilde{x}^{n-1}, \tilde{u}^n) - M(\tilde{x}^{n-1}, \tilde{u}_h^n)) w + w^T (M(\tilde{x}^{n-1}, \tilde{u}_h^n) - M(\tilde{x}^{n-1}, \tilde{u}^{n-1})) w \]

\[ \leq c \left( \|\tilde{u}_h^n\|_{L^\infty(I_h[\tilde{x}^{n-1}])} + \|\tilde{u}_h^n - \tilde{u}_h^{n-1}\|_{L^\infty(I_h[\tilde{x}^{n-1}])} + \|\tilde{u}_h^{n-1}\|_{L^\infty(I_h[\tilde{x}^{n-1}])} \right) \|w\|_{M(\tilde{x}^{n-1})}^2 \]

\[ \leq c \left( h^{(\chi-1)/2} + c \tau \right) \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2, \]

where in the last inequality we have again used the norm equivalence (9.19).

The combination of the two above estimates, together with the mild restriction $\tau \leq C_0 h$, yields

\[ \|w\|_{M(\tilde{x}^n, \tilde{w}^n)}^2 \leq \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2 + C_0 h \|w\|_{M(\tilde{x}^n, \tilde{w}^n)}^2 + c \left( h^{(\chi-1)/2} + c C_0 h \right) \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2. \]

Then absorbing the second term to the right-hand side yields

\[ \|w\|_{M(\tilde{x}^n, \tilde{w}^n)}^2 \leq \frac{1 + c h^{(\chi-1)/2} + c h}{1 - c h} \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2 \leq (1 + C h^r) \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2, \]

with $1/2 \leq r := \min \{1, (\chi - 1)/2\}$ and for $h \leq h_0$. For the last inequality here we have used that $(1 - c h)^{-1} \leq 1 + Ch$ for some constant $C > 0$.

By reversing the roles of the arguments, we obtain

\[ (1 - C h^r) \|w\|_{M(\tilde{x}^{n-1}, \tilde{w}^{n-1})}^2 \leq \|w\|_{M(\tilde{x}^n, \tilde{w}^n)}^2. \quad (9.20) \]

Therefore, for sufficiently small $h \leq h_0$ and $\tau \leq \tau_0$ (subject to $\tau \leq C_0 h$), we have that

the norms $\|\cdot\|_{M(\tilde{x}^n, \tilde{w}^n)}$ are $h$- and $\tau$-uniformly equivalent for $q \tau \leq n\tau \leq r^*$. \quad (9.21)
(A) *Estimates for the surface PDE:* In the time-continuous case, we tested the error equation for $e_u$ with the time derivative $\dot{e}_u$. Now, in the time-discrete case, we form the difference of equation (9.5b) for $n$ with $\eta$ times this equation for $n - 1$, for $\eta \in [0,1)$ of Lemma 9.2, and then we test this difference with the discrete time derivative $\dot{e}_u^n$ defined by (9.4). This yields, for $(q + 1)\tau \leq n\tau \leq t^*$,

$$
(e_u^n)^T M(\vec{x}^n, \vec{u}^n)\dot{e}_u^n - \eta (e_u^n)^T M(\vec{x}^{n-1}, \vec{u}^{n-1})\dot{e}_u^{n-1} + (e_u^n)^T A(\vec{x}^n)(e_u^n - \eta e_u^{n-1}) \leq -\eta (e_u^n)^T (A(\vec{x}^n) - A(\vec{x}^{n-1}))e_u^{n-1} + (e_u^n)^T (r_u^n - \eta r_u^{n-1}).
$$

(9.22)

(i) On the left-hand side of (9.22), the first term is

$$
(e_u^n)^T M(\vec{x}^n, \vec{u}^n)\dot{e}_u^n = \|e_u^n\|^2_{M(\vec{x}^n, \vec{u}^n)}.
$$

The second term is bounded by

$$
(e_u^n)^T M(\vec{x}^{n-1}, \vec{u}^{n-1})\dot{e}_u^{n-1} \leq \|e_u^n\|_{M(\vec{x}^{-1}, \vec{u}^{-1})} \|\dot{e}_u^{n-1}\|_{M(\vec{x}^{n-1}, \vec{u}^{n-1})} \leq \frac{1}{2} \|e_u^n\|^2_{M(\vec{x}^{-1}, \vec{u}^{-1})} + \frac{1}{2} \|\dot{e}_u^{n-1}\|^2_{M(\vec{x}^{n-1}, \vec{u}^{n-1})} \leq \frac{1}{2} (1 + ch') \|e_u^n\|^2_{M(\vec{x}^n, \vec{u}^n)} + \frac{1}{2} \|\dot{e}_u^{n-1}\|^2_{M(\vec{x}^{n-1}, \vec{u}^{n-1})},
$$

where for the last inequality we used the bound (9.20) (to raise the superscript from $n - 1$ to $n$). This yields

$$
(e_u^n)^T M(\vec{x}^n, \vec{u}^n)\dot{e}_u^n - \eta (e_u^n)^T M(\vec{x}^{n-1}, \vec{u}^{n-1})\dot{e}_u^{n-1} \geq (1 - \frac{1}{2} \eta (1 + ch')) \|e_u^n\|^2_{M(\vec{x}^n, \vec{u}^n)} - \frac{1}{2} \eta \|\dot{e}_u^{n-1}\|^2_{M(\vec{x}^{n-1}, \vec{u}^{n-1})}.
$$

(9.23)

The above terms pose the requirement to work with the solution-dependent norm $\| \cdot \|_{M(\vec{x}^n, \vec{u}^n)}$, and not with $\| \cdot \|_{M(\vec{x})}$-norms used for mean curvature flow.

(ii) The terms involving the stiffness matrix are estimated exactly as in (Kovács et al., 2019, (A.ii)). We first estimate the third term on the left-hand side of (9.22): the combination of Lemma 9.1 and 9.2 yields

$$
(e_u^n)^T A(\vec{x}^n)(e_u^n - \eta e_u^{n-1}) = \left(\frac{1}{\tau} \sum_{i=0}^{q} \delta e_u^{n-i}\right)^T A(\vec{x}^n)(e_u^n - \eta e_u^{n-1}) \geq \frac{1}{\tau} \left(\sum_{i,j=1}^{q} g_{ij} (e_u^{n-q+i})^T A(\vec{x}^n)e_u^{n-q+j} - \sum_{i,j=1}^{q} g_{ij} (e_u^{n-q+i-1})^T A(\vec{x}^n)e_u^{n-q+j-1}\right).
$$

(9.24)

While for the first term on the right-hand side of (9.22), cf. equation (10.18) in Kovács et al. (2019), we have

$$
-(e_u^n)^T (A(\vec{x}^n) - A(\vec{x}^{n-1}))e_u^{n-1} \leq c\rho \|e_u^n\|^2_{M(\vec{x}^n)} + c\rho^{-1} \|e_u^{n-1}\|^2_{A(\vec{x}^{n-1})} \leq c\rho \|e_u^n\|^2_{M(\vec{x}^n, \vec{u}^n)} + c\rho^{-1} \|e_u^{n-1}\|^2_{A(\vec{x}^{n-1})},
$$

(9.25)

where for the last estimate we have used the norm equivalence (9.19) to measure $e_u^n$ in the solution dependent norm $\| \cdot \|_{M(\vec{x}^n, \vec{u}^n)}$.

We now estimate the remaining terms. Recalling (9.6), the last term on the right-hand side of (9.22)
altogether reads:

\[
\begin{align*}
(e_u^0)^T (r_u^0 - \eta r_u^{a-1}) \\
= - (e_u^0)^T (M(\tilde{x}^a, \tilde{u}^a) - M(\tilde{x}^a, \tilde{u}^a)) u_s^a + \eta (e_u^0)^T (M(\tilde{x}^{a-1}, \tilde{u}^{a-1}) - M(\tilde{x}^{a-1}, \tilde{u}^{a-1})) u_s^{a-1} \\
- (e_u^0)^T (M(\tilde{x}^a, \tilde{u}^a) - M(\tilde{x}^a, \tilde{u}^a)) u_s^a + \eta (e_u^0)^T (M(\tilde{x}^{a-1}, \tilde{u}^{a-1}) - M(\tilde{x}^{a-1}, \tilde{u}^{a-1})) u_s^{a-1} \\
- (e_u^0)^T (A(\tilde{x}^a) - A(\tilde{x}^a)) u_s^a + \eta (e_u^0)^T (A(\tilde{x}^{a-1}) - A(\tilde{x}^{a-1})) u_s^{a-1} \\
+ (e_u^0)^T (f(\tilde{x}^a, \tilde{u}^a) - f(\tilde{x}^a, \tilde{u}^a)) - \eta (e_u^0)^T (f(\tilde{x}^{a-1}, \tilde{u}^{a-1}) - f(\tilde{x}^{a-1}, \tilde{u}^{a-1})) \\
- (e_u^0)^T M(\tilde{x}^a) d_u^a + \eta (e_u^0)^T M(\tilde{x}^{a-1}) d_u^{a-1}.
\end{align*}
\]

(9.26)

The terms not involving the solution-dependent mass matrix are estimated by the exact same techniques as the corresponding terms in Kovács et al. (2019) (see (iv), (v), and (vi) below). On the other hand, the terms in the first two lines require new estimates compared to (Kovács et al., 2019, (A.ii)).

(iii) In the two terms in the first line of (9.26) the position vectors are fixed. Hence, using Lemma 6.1 (iv) we estimate (similarly to the time-continuous case) by

\[
\begin{align*}
- (e_u^0)^T (M(\tilde{x}^a, \tilde{u}^a) - M(\tilde{x}^a, \tilde{u}^a)) u_s^a + \eta (e_u^0)^T (M(\tilde{x}^{a-1}, \tilde{u}^{a-1}) - M(\tilde{x}^{a-1}, \tilde{u}^{a-1})) u_s^{a-1} \\
\leq c\|e_u^0\|_{M(x^a)}\|e_u^{a-1}\|_{M(x^{a-1})} + c\|\tilde{e}_u^0\|_{M(x^a)}\|\tilde{e}_u^{a-1}\|_{M(x^{a-1})} \\
\leq \rho\|e_u^0\|_{M(x^a, \tilde{u}^a)} + c\rho^{-1}\|e_u^0\|_{M(x^a)}^2 + c\rho^{-1}\|\tilde{e}_u^0\|_{M(x^{a-1})}^2,
\end{align*}
\]

(9.27)

where we have used the norm equivalence (9.19), and then Young’s inequality with a small \(\rho > 0\), independent of \(h, \tau, \) and \(n \leq n^*\), which will be chosen later on.

Analogously, using Lemma 6.1 (ii), for the terms in the second line of (9.26) we obtain, with a small \(\rho > 0\),

\[
\begin{align*}
- (e_u^0)^T (M(\tilde{x}^a, \tilde{u}^a) - M(\tilde{x}^a, \tilde{u}^a)) u_s^a + \eta (e_u^0)^T (M(\tilde{x}^{a-1}, \tilde{u}^{a-1}) - M(\tilde{x}^{a-1}, \tilde{u}^{a-1})) u_s^{a-1} \\
\leq c\|e_u^0\|_{M(x^a)}\|\tilde{e}_u^{a-1}\|_{K(x^{a-1})} + c\|\tilde{e}_u^0\|_{M(x^a)}\|\tilde{e}_u^{a-1}\|_{K(x^{a-1})} \\
\leq \rho\|e_u^0\|_{M(x^a, \tilde{u}^a)} + c\rho^{-1}\|\tilde{e}_u^0\|_{K(x^a)}^2 + c\rho^{-1}\|\tilde{e}_u^0\|_{K(x^{a-1})}^2.
\end{align*}
\]

(9.28)

where we have again used the norm equivalence (9.19).

(iv) For the stiffness matrix terms in (9.26), the rather complicated estimates of (Kovács et al., 2019, (A.iv)) can be used verbatim, and they yield the bound, see (10.24)–(10.27) in Kovács et al. (2019):

\[
\begin{align*}
- (e_u^0)^T (A(\tilde{x}^a) - A(\tilde{x}^a)) u_s^a + \eta (e_u^0)^T (A(\tilde{x}^{a-1}) - A(\tilde{x}^{a-1})) u_s^{a-1} \\
\leq c \sum_{j=0}^a \|e_u^0\|_{A(x^{a-j})}^2 + c \sum_{j=0}^a \|\tilde{e}_u^{a-1-j}\|_{K(x^{a-1-j})}^2 + 2 c \sum_{j=0}^{a-1} \|\tilde{e}_u^{a-1-j}\|_{K(x^{a-1-j})}^2.
\end{align*}
\]

(9.29)

(v) The non-linear terms in (9.26) are estimated as (Kovács et al., 2019, (A.v)), with a small \(\rho > 0\), by

\[
\begin{align*}
(e_u^0)^T (f(\tilde{x}^a, \tilde{u}^a) - f(\tilde{x}^a, \tilde{u}^a)) - \eta (e_u^0)^T (f(\tilde{x}^{a-1}, \tilde{u}^{a-1}) - f(\tilde{x}^{a-1}, \tilde{u}^{a-1})) \\
\leq \rho\|e_u^0\|_{M(x^a, \tilde{u}^a)}^2 + c\|\tilde{e}_u^0\|_{K(x^a)}^2 + c\|\tilde{e}_u^{a-1}\|_{K(x^{a-1})}^2 + c\|\tilde{e}_u^{a-1}\|_{K(x^{a-1})}^2.
\end{align*}
\]

(9.30)

where for the final estimate we have used the norm equivalence (9.19).
(vi) Finally, the defect terms are bounded, exactly as (Kovács et al., 2019, (A,v)) but additionally using the norm equivalence (9.19) with a small $\rho > 0$, by

$$- (e^q_0)^T M(x^0) d^0_u + \eta (e^q_0)^T M(x^0) d^{-1}_u \leq \rho \| e^0_u \|^2_{M(x^0, \bar{a})} + c \| d^0_u \|^2_{M(x^0)} + c \| d^{-1}_u \|^2_{M(x^0)}.$$

(9.31)

We substitute the estimates from (i)–(vi) into (9.22), which altogether yields the inequality, for $q + 1 \leq n \leq n^*$,

$$\left( 1 - \frac{1}{2} \eta (1 + c\tau) - c\rho \right) \| e^0_u \|^2_{M(x^0, \bar{a})} - \frac{1}{2} \eta \| e^{q-1}_u \|^2_{M(x^{q-1}, \bar{a})}$$

$$+ \frac{1}{\tau} \sum_{i,j=1}^q g_{ij} (e^{q-i}_u)^T A(x^0) e^{q-j}_u - \sum_{i,j=1}^q g_{ij} (e^{q-i-1}_u)^T A(x^0) e^{q-j-1}_u$$

$$\leq - \sum_{j=0}^{q-1} \sigma_j \partial^T \left( (e^{q-j}_u)^T (A(x^0) - A(x^0)) u^0_u \right)$$

$$+ \eta \sum_{j=0}^{q-1} \sigma_j \partial^T \left( (e^{q-j}_u)^T (A(x^{q-1}) - A(x^{q-1})) u^{q-1}_u \right) + c \varepsilon_n$$

with

$$e_n = \sum_{j=0}^q \| e^{q-j}_u \|^2_{A(x^0)} + \sum_{j=0}^q \| e^{q-j-1}_u \|^2_{A(x^{q-1})} + \sum_{j=0}^{q-1} \| \partial^T e^{q-j-1}_u \|^2_{A(x^{q-1})}$$

$$+ \| d^0_u \|^2_{M(x^0)} + \| d^{-1}_u \|^2_{M(x^0)}.$$

(9.33)

This is exactly the same formula as (10.30)–(10.31) in Kovács et al. (2019), except the solution-dependent norms (instead of $\| \cdot \|_{M(x)}$) on the $e_u$ terms. Therefore, the proof can be finished by the exact same arguments, but using the norm equivalence (9.21). Altogether, using (9.19) in the final steps, we finally obtain

$$\| e^0_u \|^2_{M(x^0)} \leq c \tau \sum_{j=0}^n \| e^j_u \|^2_{M(x^0)} + c \sum_{j=0}^{n-1} \| e^j_u \|^2_{M(x^0)} + c \tau \sum_{j=0}^n \| d^j_u \|^2_{M(x^0)}.$$  

(9.34)

Similarly to the semi-discrete case, Parts (B) and (C) of Kovács et al. (2019), i.e. the estimates for the velocity equation and the combination of the estimates, is repeated verbatim from (Kovács et al., 2019, Proposition 10.1). Similarly, as in the time-continuous case, we show that $\tau^* = T$ by repeating (6.31) in the fully discrete setting.

These final steps complete the proof of the fully discrete stability result.

\[\square\]

10. Consistency estimates for the full discretisation

The following estimates for the defects (9.1) are directly proved by combining the results of Lemma 7.1 and (Kovács et al., 2019, Lemma 11.1).

**Lemma 10.1** Assume that the surface $\Gamma [x]$ evolving under generalized mean curvature flow is sufficient regular on the time interval $[0, T]$. Then, there exists constants $h_0 > 0$, $\tau_0 > 0$, and $c = c(T) > 0$ such that for all $h \leq h_0$ and $\tau \leq \tau_0$, satisfying $0 \leq n \tau \leq T$, the defects $d^n_l \in S_h(\Gamma_l^0[x]^0)$, $d^n_e \in S_h(\Gamma_l^0[x]^0)$.
and $d^n u \in S_h(\Gamma_h[x^n])^4$ of the $k$th-degree finite elements and the $q$-step backward difference formula are bounded as

\[
\|d^n_x\|_{K(\Gamma_h[x^n])} = c \tau^q, \\
\|d^n_x\|_{\mathbb{X}(\Gamma_h[x^n])} = c(h^k + \tau^q), \\
\|d^n_u\|_{M(\Gamma_h[x^n])} = \|d^n_u\|_{L^2(\Gamma_h(x^n))} = c(h^k + \tau^q).
\]

The constant $c$ is independent of $h$, $\tau$ and $n$ with $n \tau \leq T$.

11. Proof of Theorem 5.2

With the stability estimate of Proposition 9.1 and the defect bounds of Lemma 10.1 at hand, the proof is now essentially the same as in Section 8, or cf. (Kovács et al., 2019, Section 12).

Analogously to (8.1), the errors are decomposed using finite element interpolations of $X$ and $v$ and the Ritz map (6.12) for $u$. The decomposed parts are then estimated, as $O(h^k)$ in the $H^1(\Gamma[X])$ norm, either using the stability estimate of Proposition 9.1 together with the defect bounds of Lemma 10.1, or using the interpolation and Ritz map error bounds of Kovács (2018). Altogether proving the stated theorem.

12. Numerical examples

We performed the following numerical experiments for various generalized mean curvature flows:

- Convergence tests for spheres where the exact solutions of generalized mean curvature flows are known, i.e. for inverse mean curvature flow, and generalized mean curvature and generalized inverse mean curvature flow.

- We report on numerical solutions for various flows, and also on geometric quantities which are known to be monotone along their respective flows, e.g. Hawking mass for inverse mean curvature flow.

- We have performed some numerical experiments for some non-convex initial surfaces.

All our numerical experiments use quadratic evolving surface finite elements, and linearly implicit backward difference time discretisation of various orders. The numerical computations were carried out in Matlab. The initial meshes for all surfaces were generated using DistMesh by Persson & Strang (2004), without exploiting any symmetry of the surfaces.

12.1 Convergence tests

Using the algorithm (4.1), i.e. using quadratic evolving surface finite elements for spatial discretisation in combination with a $q$-step linearly implicit BDF method for time integration, we computed approximations to under various generalized mean curvature flows in two dimensions over the time interval $[0, T] = [0, 1]$. The computations are carried out for a sphere (with initial radius $R_0 = 3$), since in these cases the exact solutions are known. Under all these flows spheres remain spherical and only change their radius, expressed by the mean curvature as $R(t) = 2/H(\cdot, t)$. For our computations we used a sequence of time step sizes $\tau_k = \tau_{k-1}/2$ with $\tau_0 = 0.2$, and a sequence of meshes with mesh widths $h_k \approx 2^{-1/2}h_{k-1}$. 
For all experiments we have started the time integration from the nodal interpolations of the exact initial values $n(\cdot, 0)$ and $H(\cdot, 0)$.

**Inverse mean curvature flow.** For an $m$-dimensional sphere $\Gamma[\mathcal{X}(\cdot, t)] = \{R(t)x \mid x \in \Gamma^0\}$, with $\Gamma^0 = \{|x| = 1 \mid x \in \mathbb{R}^{m+1}\}$, from the velocity law (2.3) and the ODE (2.1) we obtain that the radius $R(t)$ of the sphere satisfies the ODE

$$\frac{d}{dt} R(t) = \frac{R(t)}{m}, \quad \text{with initial value } R(0) = R_0,$$

whose solution is given by

$$R(t) = R_0 e^{t/m}, \quad \text{for all } 0 \leq t < \infty. \quad (12.1)$$

**Generalized inverse mean curvature flow** $\alpha > 1$. For a sphere, from the velocity law (2.3) and the ODE (2.1), we now obtain that the radius $R(t)$ satisfies the ODE, with $\alpha > 1$,

$$\frac{d}{dt} R(t) = \left(\frac{R(t)}{m}\right)^\alpha, \quad \text{with initial value } R(0) = R_0,$$

whose solution is given by

$$R(t) = \left(R_0^{1-\alpha} - (\alpha - 1)m^{-\alpha}t\right)^{1-\alpha},$$

with maximal existence time $T_{\text{max}} = \frac{R_0^{1-\alpha}}{(\alpha - 1)m^{-\alpha}}$. \quad (12.4)

We note here that for $0 < \alpha < 1$ a solution of the same form exists for all $0 \leq t < \infty$.

**Generalized mean curvature flow** $\alpha > 0$. For a sphere, from the velocity law (2.3) and the ODE (2.1), we now obtain that the radius $R(t)$ satisfies the ODE, with $\alpha > 0$,

$$\frac{d}{dt} R(t) = -\left(\frac{m}{R(t)}\right)^\alpha, \quad \text{with initial value } R(0) = R_0,$$

whose solution is given by

$$R(t) = \left(R_0^{1+\alpha} - (1 + \alpha)m^{\alpha}t\right)^{1/(1+\alpha)},$$

with maximal existence time $T_{\text{max}} = \frac{R_0^{1+\alpha}}{(1 + \alpha)m^{\alpha}}$. \quad (12.6)

See, e.g., (Schulze, 2002, Beispiel 2.13).

### 12.2 Monotone geometric quantities

We report on surface evolutions under various generalized mean curvature flows (in two dimensions), and on the time-evolution of corresponding monotone geometric quantities for surfaces with non-negative mean curvature. Such quantities and their monotonicity are often used in analysis to prove
inverse mean curvature flow for a sphere in [0, 1]

Fig. 2. Spatial and temporal convergence of the BDF2 / quadratic ESFEM discretisation for the inverse mean curvature flow of a sphere for $T = 1$. 
generalised inverse mean curvature flow for a sphere in $[0,1]$
generalised mean curvature flow for a sphere in $[0, 1]$
various results (e.g. convergence to a round point for $H^\alpha$-flows), see (Huisken & Polden, 1999, Section 6), (Schulze, 2006, Appendix A), and in particular Schnürer (2005), where such quantities are found by the aid of a randomized algorithmic test.

The fact that the algorithm analysed in this paper preserves the monotonicity of such quantities is of interest both from an analytical and a numerical viewpoint.

For inverse mean curvature flow an important monotone quantity is the Hawking mass, see Hawking (1968), or (Huisken & Polden, 1999, Section 6):

$$m_H(\Gamma[X]) = \sqrt{\text{Area}(\Gamma[X])} \left( 1 - \frac{1}{16\pi} \int_{\Gamma[X]} H^2 \right),$$

which is non-decreasing in time, i.e. $\frac{d}{dt} m_H(\Gamma[X]) \geq 0$.

For generalized mean curvature flow ($H^\alpha$-flow), for $1 \leq \alpha \leq 5$, an important monotone quantity is the following (Schulze, 2006, Appendix A), (also expressed using the principal curvatures $\kappa_j$):

$$m_{H^\alpha, \text{flow}}(\Gamma[X]) = \max_{\Gamma[X]} \frac{H^{2\alpha}(2|A|^2 - H^2)}{(H^2 - |A|^2)^2}$$

$$= \frac{(\kappa_1 + \kappa_2)^{2\alpha}(\kappa_1 - \kappa_2)^2}{4\kappa_1^{2\alpha}\kappa_2^{2\alpha}},$$

which is non-increasing in time, i.e. $\frac{d}{dt} m_{H^\alpha, \text{flow}}(\Gamma[X]) \leq 0$.

Let us point out that both monotone quantities can be very easily computed from the geometric quantities obtained from our algorithm.

In Figure 5 we report on the numerical solution and the corresponding monotone quantity for three different generalized flows: inverse mean curvature flow (iMCF), powers of mean curvature flow ($H^\alpha$-flow) with $\alpha = 2$ and $6$ (plotted left to right in the figure). In the experiments we have used time step size $\tau = 0.0125$, and surfaces with degrees of freedom 6242, 4210, and 4242 respectively.

For inverse mean curvature flow and for the $H^\alpha$-flow with $\alpha = 2$ the quantities (12.7) and (12.8), respectively, are known to be monotone (non-decreasing and non-increasing). For $\alpha = 6$, to our knowledge, such a result is an open question. This numerical evidence suggest that it is not monotone. However, we strongly note here, that the small jump at $t \approx 0.25$ is a numerical artefact, which (in our experience) is due to violations of discrete maximum principles (the mean curvature, computed from the normal velocity, becomes non-positive). The algorithm could benefit from applying the techniques of Frittelli et al. (2018).

12.3 Generalized mean curvature flows of non-convex surfaces

To complement our theoretical results, a numerical experiment is presented here for generalized mean curvature flows of two non-convex initial surfaces $\Gamma^0 \subset \mathbb{R}^3$, given by

$$\Gamma^0 = \left\{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + 2x_3\left(x_3^2 - 159/200 - 0.5\right) \right\}, \quad \text{cf. Elliott & Styles (2012)},$$

and

$$\Gamma^0 = \left\{ x \in \mathbb{R}^3 \mid (x_1^2 - 1)^2 + (x_2^2 - 1)^2 + (x_3^2 - 1)^2 - 1.05 \right\}. \quad (12.9)$$
Fig. 5. Surface evolutions (from left to right, inverse mean curvature flow, $H^2$-flow, and $H^6$-flow) at different times (from top to bottom) and corresponding monotone quantities (bottom row)
These initial surfaces have regions with both negative and positive mean curvature, and hence of a class not covered by our theorems. Nevertheless, our algorithm can be still used to compute numerical solutions of generalized flows.

In Figure 6, 7, and 8, we respectively report on the numerical solution to the inverse mean curvature flow and for the $H^\alpha$-flow and $-H^{-\alpha}$-flow (with $\alpha = 2$). For the first two experiments we have used a dumbbell shaped two-dimensional surface with 3538 nodes, while for the third experiment the genus 5 surface with 11424 nodes. For all experiments we have used a time step size $\tau = 0.0015625$.

In the case of inverse mean curvature flow, Figure 6, some slight surface distortions can be observed around the neck (where mean curvature switches sign), this is however not observable for the $-H^{-\alpha}$-flow in Figure 8. The algorithm performs rather robust for the $H^\alpha$-flow, rapidly shrinking towards a point. Note that the different flows are integrated until different final times, see the figures. In the case of Figure 7 also note the rapidly shrinking surface.

![Fig. 6. Inverse mean curvature flow (side view) of a dumbbell in $\mathbb{R}^3$ at different times in $[0,1]$.](image)
Fig. 7. Powers of mean curvature flow with $\alpha = 2$ of a dumbbell in $\mathbb{R}^3$ at different times in $[0, 0.0625]$. 
Fig. 8. Powers of inverse mean curvature flow with $\alpha = 2$ of a genus 5 surface in $\mathbb{R}^3$ at different times in $[0, 0.7]$.

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Appendix A.

**Lemma A.1** For $a, b > 0$ and $\gamma \in \mathbb{R}$ we obtain

$$|a^\gamma - b^\gamma| \leq \begin{cases} C_\gamma \cdot |b|^{\gamma - 1} \cdot |a - b| & \text{for } \gamma < 0, \\
C_\gamma \cdot |a - b|^{\gamma} & \text{for } \gamma \in (0, 1], \\
C_\gamma \cdot (|a - b|^{\gamma} + |b|^{\gamma - 1} \cdot |a - b|) & \text{for } \gamma > 1. \end{cases}$$

In particular, if $|a - b| \ll 1$ is small, it follows that $|a^\gamma - b^\gamma| \ll 1$ is small.

**Proof.** Note that all functions in the statement are homogeneous of degree $\gamma$. So we assume without loss of generality that $b = 1$. By continuity of both sides and compactness it remains to consider the case $|a - 1| < \varepsilon$ small, where the right-hand side becomes small and the case $a \to \infty$, where the left-hand side becomes large. We consider the three different cases:

(i) For large $a > 1$ the term on the left-hand side growth like $a^\gamma \leq a$, since $\gamma < 0$. For $|a - 1| < \varepsilon$ small we have $|a^\gamma - 1| \leq C_\gamma |a - 1|$.

(ii) For large $a > 1$ the term on the left-hand side growth like $|a^\gamma| = |a|^{\gamma}$ for $\gamma > 0$. For $|a - 1| < \varepsilon$ small one obtains $|a^\gamma - 1| \leq C_\gamma |a - 1|^{\gamma}$.

(iii) The case $a > 1$ large is in (ii), whereas the case $|a - 1| < \varepsilon$ is analogous to (i).

□