Nonminimal coupling for the gravitational and electromagnetic fields:
Traversable electric wormholes

Alexander B. Balakin, José P. S. Lemos, and Alexei E. Zayats

1Department of General Relativity and Gravitation,
Kazan State University, Kremlevskaya str. 18, Kazan 420008, Russia.
2Centro Multidisciplinar de Astrofísica-CENTRA Departamento de Física, Instituto Superior Técnico-IST,
Universidade Técnica de Lisboa-UTL, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal

We discuss new exact solutions of a three-parameter nonminimal Einstein-Maxwell model. The solutions describe static spherically symmetric objects with and without center, supported by an electric field nonminimally coupled to gravity. We focus on a unique one-parameter model, which admits an exact solution for a traversable electrically charged wormhole connecting two universes, one asymptotically flat the other asymptotically de Sitter ones. The relation between the asymptotic mass and charge of the wormhole and its throat radius is analyzed. The wormhole solution found is thus a nonminimal realization of Wheeler's idea about charge without charge and shows that, if the world is somehow nonminimal in the coupling of gravity to electromagnetism, then wormhole appearance, or perhaps construction, is possible.

PACS numbers: 04.20.Jb, 04.20.Gz
Keywords: nonminimal coupling, traversable wormhole

I. INTRODUCTION

The wormhole concept was invented by Wheeler (see, e.g., [1]) to provide a mechanism for having charge without charge, since in a such a spacetime without a center, the field lines seen in one part of the Universe could thread the throat and reappear in other part. The idea, was further extended by Morris and Thorne to allow, not only field lines, but also observers to travel through the throat [2]. By having invoked an arbitrarily technologically advanced civilization, this work [2] initiated the engineering of wormhole construction, theoretically, and the study of wormholes as topological bridges joining two different spacetimes has since then attracted extraordinary attention in this modern context (see, e.g., [3] and references therein). The main feature of traversable wormhole physics is the fact that the matter threading the wormhole throat should possess exotic properties, one of them being the violation of the null energy condition [4]. To provide the existence of wormholes one should either include some hypothetical forms of matter into the model, or introduce interactions of a new type. Various models, admitting the required violation of the null energy condition, have been considered in the literature, among them thin shells in a cosmological constant background [5, 6], scalar fields [7], wormhole solutions in semiclassical gravity [8], solutions in Brans-Dicke theory [9], wormholes on the brane [10], wormholes supported by matter with an exotic equation of state, namely, phantom energy [11], the generalized Chaplygin gas [12], tachyon matter [13], nonlinear electrodynamics [14], and other cases.

Now, when one considers nonminimal coupling of gravity with vector-type fields, such as Maxwell, Yang-Mills or Proca fields, new interesting possibilities appear. Nonminimal phenomena, i.e., phenomena induced by the interaction between curvature, or gravity, and other fields can be characterized on the one hand by unusual effective energy conditions, and on the other hand, allow one to exclude exotic substrates. In [15] an exact solution of the nonminimal Einstein–Yang-Mills model was obtained which demonstrates that an $SU(2)$ symmetric gauge field nonminimally coupled to curvature can support a traversable wormhole. This is a nonminimal Wu-Yang magnetic wormhole, and the throat joins two asymptotically flat regions. The corresponding spacetime has no center and this model could be an illustration of Wheeler’s idea about “charge without charge” [1]. Following this idea we now intend to consider an electrically charged object in the context of a spacetime without a center. A few wormhole models are known in which the electric charge is spread on the spherical shell [16], or the throat is filled with some nonminimal and ghostlike scalar field [17]. Here, we find an exact solution of a nonminimal traversable wormhole, which contains neither electric charge on spherical shells, either thin or thick, nor scalar fields, nevertheless, possesses a static spherically symmetric electric field which is charged from the point of view of a distant observer. Thus, our goal is twofold. We present explicitly a nonminimal realization of Wheeler’s idea about charge without charge, and we show that, if the world is somehow nonminimal in the coupling of gravity to electromagnetism, then wormhole appearance, or perhaps construction by an absurdly advanced civilization [5], is possible.

* Electronic address: Alexander.Balakin@ksu.ru
† Electronic address: joselemos@ist.utl.pt
‡ Electronic address: Alexei.Zayats@ksu.ru
For this purpose we address a nonminimal Einstein-Maxwell theory, which has been earlier elaborated in detail in both linear (see, e.g., [13] [19]) and nonlinear (see [20]) versions. The model linear in the spacetime curvature and quadratic in the Maxwell tensor [20] contains three nonminimal coupling constants \( q_1, q_2, \) and \( q_3 \). These quantities have the dimensionality of area and characterize the cross terms in the Lagrangian linking the Maxwell field \( F_{ik} \) and terms linear in the Ricci scalar \( R \), Ricci tensor \( R_{ik} \), and Riemann tensor \( R_{ikmn} \), respectively. These parameters are a priori free ones but can acquire specific values in certain effective field theories. The first example of a calculation of the three coupling parameters was based on one-loop corrections to quantum electrodynamics in curved spacetime, a direct and nonphenomenological approach considered by Drummond and Hathrell [21]. In another instance, Buchdahl [22] and then Müller-Hoissen [23] obtained a nonminimal Einstein-Maxwell model from dimensional reduction of the Gauss-Bonnet action. This model contains one coupling parameter.

Based on these works, and specially on [20], solutions, which described nonminimal electrically charged objects with and without centers, characterized by two sets of relations for nonminimal coupling parameters, namely, \( q_1 + q_2 + q_3 = 0 \), \( 2q_1 + q_2 = 0 \), and \( q_1 + q_2 = 0, q_3 = 0 \), were obtained [24]. However, none of these solutions can be used to construct traversable wormholes. In this paper we formulate a new nonminimal model, for which the coupling parameters satisfy the relations, \( 3q_1 + q_2 = 0, q_3 = 0 \). Exact solutions of this model are shown to admit the existence of nonminimal traversable electrically charged wormholes. Comparing these new results with the ones obtained in [15] we would like to emphasize the following features. First, here we deal with an electric field instead of a magnetic one; second, in [15] we used the basic conditions 12\(q_1 + 4q_2 + q_3 = 0, q_3 \neq 0 \); third, the spacetime is now nonsymmetric, i.e., the throat joins one asymptotically flat region to another asymptotically de Sitter region. In [25] the authors have suggested the name “black universes” to nonsymmetric wormhole spacetime configurations for which the first asymptotics is flat and the second one is of cosmological type.

The paper is organized as follows: In Section II, we quote the details of the three-parameter nonminimal Einstein-Maxwell model and formulate the corresponding key equations for the electric and gravitational fields. Furthermore, we introduce a new one-parameter nonminimal model, given by the conditions 3\(q_1 + q_2 = 0, q_3 \neq 0 \), derive a key (cubic) equation for the electric field \( E \) and obtain the metric functions \( \sigma, N \) in terms of \( E \). In Section III, we discuss three solutions of the equation for the electric field. We focus on the Coulombian-type solution: This is the solution to which the others should be compared. In Section IV, we obtain exact solutions with a center. In Section V, we thoroughly discuss wormhole solutions. Section VI contains the conclusions.

II. NONMINIMALLY EXTENDED EINSTEIN-MAXWELL THEORY

A. Master equations

Details of three-parameter nonminimal Einstein-Maxwell theory can be found in the papers [20] [24]; here we extract the main elements of this model only. The action functional is of the form

\[
S_{\text{NMEM}} = \int d^4x \sqrt{-g} \left[ \frac{R}{\kappa} + \frac{1}{2} F_{ik} F^{ik} + \frac{1}{2} \chi^{ikmn} F_{ik} F_{mn} \right].
\]

(1)

Here, \( g = \det(g_{ik}) \) is the determinant of the metric tensor \( g_{ik} \), \( R \) is the Ricci scalar, \( \kappa \) is the gravitational constant. The Latin indices run from 0 to 3. The Maxwell tensor \( F_{ik} \) is expressed, as usual, in terms of a potential four-vector \( A_k \)

\[
F_{ik} = \nabla_i A_k - \nabla_k A_i,
\]

(2)

where the symbol \( \nabla_i \) denotes the covariant derivative. The tensor \( \chi^{ikmn} \) is defined as follows (see [20]):

\[
\chi^{ikmn} = \frac{q_1}{2} R (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn},
\]

(3)

where \( R_{ik} \) and \( R_{ikmn} \) are the Ricci and Riemann tensors, respectively, and \( q_1, q_2, q_3 \) are the phenomenological parameters describing the nonminimal coupling of electromagnetic and gravitational fields. The variation of the action functional with respect to potential \( A_i \) yields

\[
\nabla_k H^{ik} = 0, \quad H^{ik} \equiv F^{ik} + \chi^{ikmn} F_{mn},
\]

(4)

where \( H^{ik} \) is the nonminimal excitation tensor [20]. In a similar manner, the variation of the action functional with respect to the metric yields

\[
R_{ik} - \frac{1}{2} R g_{ik} = \kappa T^{(\text{eff})}_{ik}.
\]

(5)

The effective stress-energy tensor \( T^{(\text{eff})}_{ik} \) can be divided into four parts:

\[
T^{(\text{eff})}_{ik} = T^{(M)}_{ik} + q_1 T^{(I)}_{ik} + q_2 T^{(II)}_{ik} + q_3 T^{(III)}_{ik}.
\]

(6)
The first term $T^{(M)}_{ik}$:

$$T^{(M)}_{ik} = \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{in} F_{k}^{n},$$

(7)

is a stress-energy tensor of the pure electromagnetic field. The definitions of other three tensors are related to the corresponding coupling constants $q_1$, $q_2$, $q_3$:

$$T^{(I)}_{ik} = R T^{(M)}_{ik} - \frac{1}{2} R_{ik} F_{mn} F^{mn} + \frac{1}{2} \left[ \nabla_i \nabla_k - g_{ik} \nabla^j \nabla_j \right] [F_{mn} F^{mn}],$$

(8)

$$T^{(II)}_{ik} = -\frac{1}{2} g_{ik} \left[ \nabla_m \nabla_l (F_{mn} F_{l}^{n}) - R_{lm} F_{mn} F_{l}^{n} \right] - F^{in} (R_{il} F_{kn} + R_{kl} F_{in}) - R_{mn} F_{lm} F_{kn} - \frac{1}{2} \nabla^m \nabla_m (F_{in} F_{k}^{n}) + \frac{1}{2} \nabla_l \left[ \nabla_i (F_{kn} F^{ln}) + \nabla_k (F_{in} F^{ln}) \right],$$

(9)

$$T^{(III)}_{ik} = \frac{1}{4} g_{ik} R^{mnl} F_{mn} F_{ls} \frac{3}{4} F^{ls} (F_{i}^{n} R_{kns} + F_{k}^{n} R_{ins}) - \frac{1}{2} \nabla_m \nabla_n \left[ F_{i}^{n} F_{k}^{m} + F_{k}^{n} F_{i}^{m} \right].$$

(10)

One may check directly that the tensor $T^{(eff)}_{ik}$ satisfies the equation $\nabla^k T^{(eff)}_{ik} = 0$.

B. Spherically symmetric model with electric charge

Below we consider the nonminimally extended Einstein-Maxwell equations, given in Eqs. (4)-(10), for the case of a static spherically symmetric spacetime, given by the metric

$$ds^2 = \sigma^2 N dt^2 - \frac{dr^2}{N} - Y^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),$$

(11)

where $N$, $\sigma$, and $Y$ are functions of the radial variable $r$ only satisfying the asymptotic flatness conditions when $r \to +\infty$

$$\sigma(r) \to 1, \quad N(r) \to 1, \quad Y(r) \sim r.$$

(12)

In this paper, asymptotic flatness means that we do not consider the case $Y'(r) = 0$ identically. Let us assume also that the electromagnetic field inherits the static and spherical symmetries. Then the potential four-vector of the electric field $A_i$ and the Maxwell tensor have the form

$$A_i = A_0(r) \delta_i^0, \quad F_{ik} = A'_0(r) \left( \delta_i^r \delta_k^0 - \delta_i^0 \delta_k^r \right),$$

(13)

where a prime denotes the derivative with respect to $r$. To characterize the electric field, it is useful to introduce a new scalar quantity $E(r)$ as

$$E^2(r) = -\frac{1}{2} F_{ik} F^{ik}$$

(14)

Then the electric field squared is $(A'_0)^2$ from which one obtains in turn $F_{r0} = -\sigma(r) E(r)$.

The Maxwell equations (4) give only one nontrivial equation, namely,

$$[Y^2 E(1 + 2 \chi^0 r^0)]' = 0,$$

(15)

which can be integrated immediately to give

$$E = \frac{Q}{Y^2 (1 + 2 \chi^0 r_0)},$$

(16)

where $Q$ is a constant to be associated with an electric charge of the object. Supposing the spacetime to be asymptotically flat, i.e., $R_{iklm}(r \to +\infty) = 0$, one can see that (16) yields asymptotically the Coulomb law $E(r) \to Q/r^2$. The nonminimal excitation tensor $H^{ik}$ given in (4) with $\chi^{kmn}$ given by (3) has only one nonvanishing component $H^{r0}$. Let us introduce an electric excitation scalar as follows

$$D \equiv \sqrt{-\frac{1}{2} H_{ik} H^{ik}} = \sqrt{-H_{r0} H^{r0}} = E(1 + 2 \chi^0 r_0).$$

(17)

According to (16) the electric excitation takes a simple form

$$D = \frac{Q}{Y^2},$$

(18)

i.e., $D$ also tends to zero at $r \to +\infty$. 


C. Key equations

1. Preliminaries

Analogously to the static spherically symmetric model in minimal electrodynamics there are two independent Einstein equations only, e.g.,

\[ G_0^0 - G_{rr} = \kappa \left( T_{0}^{0(\text{eff})} - T_{r}^{r(\text{eff})} \right), \quad G_0^0 = \kappa T_{0}^{0(\text{eff})}. \] (19)

The first one gives the equation linking the function \( \sigma \) with \( E(r) \) and \( Y(r) \),

\[ \frac{\sigma'}{\sigma} = \frac{\kappa(q_1 + q_2 + q_3) \left[ Y^2(EE')' + YY''E^2 \right] + 2\kappa(q_2 + 2q_3)YY'EE' + \kappa q_3 Y^2 E^2 - YY''}{Y \left[ \kappa(q_1 + q_2 + q_3)(YE' + Y' E^2) - Y' \right]}. \] (20)

It is convenient to rewrite the second equation in (19) using (16) as

\[ G_0^0 - \kappa T_{0}^{0(\text{eff})} + 2\kappa E^2 \chi_{r_0} = \kappa E^2 \left( \frac{Q}{\sqrt{r^2 - E^2}} - 1 \right). \] (21)

For generic coupling constants this equation yields

\[ N'Y \left[ \kappa(q_1 + q_2 + q_3)(Y E' + Y' E^2) - Y' \right] + N \left( 2\kappa(q_1 + q_2 + q_3) \left[ Y^2(EE')' + YY''E^2 \right] + 2\kappa E Y YY'(2(q_1 + q_2 + q_3) + q_2 + 2q_3) + \kappa (q_1 + q_2 + q_3) E Y^{2} - Y^{2} - 2YY'' \right) + 
\quad + 1 + \frac{\kappa E^2 Y^2}{2} - \kappa q_1 E^2 - \kappa Q E = 0. \] (22)

The three Eqs. (16), (20) and (22) form the key system of equations determining the four unknown functions \( \sigma, N, E \) and \( Y \). One of these functions is arbitrary. Generally, one has \( \sigma'(r) \) and \( N'(r) \) given in Eqs. (20) and (22), and put both into (16). This procedure yields

\[ 2NE \left[ \kappa ^2(q_1 + q_2 + q_3)^2(3q_1 + q_2)YE^2(EE'YY' - EE'YY'' - YY'E' + EE'Y'^2) - 
\quad - \kappa (q_1 + q_2 + q_3)(3q_1 + q_2 - q_3)(E')^2 Y^3 - EE'Y'^2 + Y'E^2) - 
\quad - \kappa (q_1 + q_2 + q_3)(3q_1 + q_2 - q_3)(Q)^2 E^3 Y^2 + \kappa q_3 q_5 Y^3 (2q_1 + q_2 + q_3) - 
\quad - \kappa E^2 E'(q_1 + q_2 + q_3)(Y^2 - 2Y(3q_1 + q_2)) + \kappa Q YEE'q_1 + q_2 + q_3) - \kappa q_1 YEE'q_2 + q_3)q_1 Y^2 + q_2 + q_3) + 
\quad + \kappa ^2 Q Y'E'(q_1 + q_2 + q_3)(q_1 - q_2 - q_3) - \kappa Y'^E'(q_2 + 2q_3)(Y^3 - 2q_2 + q_3) + 2\kappa QY'E^2(q_2 + 2q_3) + 
\quad + Y'E' + 2q_3)Y^2 - 2q_1) + 2\kappa QY'E^2(q_2 + 2q_3) + 
\quad + Y'E' - 2q_3) - QY'E \right] + \frac{\kappa E^2 Y^2}{2} - \kappa q_1 E^2 - \kappa Q E = 0. \] (23)

thus giving \( N \) as a function of \( E, E', E'', Y, Y' \) and \( Y'' \). In principle, putting \( N \) from Eq. (23) in Eq. (22) one can obtain a key equation linking \( E(r) \) and \( Y(r) \). This equation happens to be nonlinear and contains third order derivatives \( E'' \) and \( Y'' \). Although there are no explicit exact solutions for arbitrary \( q_1, q_2, q_3 \), one can estimate the behavior of the integral curves using a decomposition with respect to \( Y^{-1} \). For instance, when the model is asymptotically flat, the decompositions start as

\[ E = \frac{Q}{Y^2} \left( 1 + \frac{4q_3 M}{Y^3} - \frac{\kappa q_3 Y^2}{4Y^2} \right) + \ldots, \] (24)

\[ \sigma = Y' \left[ 1 + \frac{\kappa (10q_1 + 6q_2 + 3q_3) Q^2}{4Y^4} \right] + \ldots, \] (25)

\[ N = \frac{1}{Y^2} \left[ 1 + \frac{2M Y}{Y^2} + \frac{\kappa q^2}{Y^4} \right] + \ldots, \] (26)

where the constant \( M \) is an asymptotic mass of the object.

There are two special cases arising when the left-hand side of Eq. (23) vanishes so that the function \( N \) cannot be found from this equation directly. The first one is the model with \( q_1 + q_2 + q_3 = 0 \) and \( q_3 = 0 \), which has been studied in [21], the second relates to the model with \( 3q_1 + q_2 = 0 \) and \( q_3 = 0 \). Since the model with \( q_1 + q_2 + q_3 = 0 \) and \( q_3 = 0 \) has been studied in [21] we briefly mention that this one-parameter model can be characterized by the relationships \( q_1 = -q, q_2 = q, q_3 = 0 \), for some \( q \), and (23) is satisfied, when the right-hand side of this equation also vanishes, i.e.,

\[ \kappa q E^4 (Y^2 + 2q) - 2\kappa q E^2 - E Y^2 + Q = 0. \] (27)
The function $\sigma$ can be readily found as
\[ \sigma = Y' \exp(-\kappa q E^2). \] (28)

When the function $Y$ coincides with the radius $r$, i.e., $Y(r) = r$, we obtain indeed the model which has been investigated in the paper [24]. On the other hand the model with $3q_1 + q_2 = 0$ and $q_3 = 0$ is also very interesting and will be studied in detail here.

2. One-parameter model with $3q_1 + q_2 = 0$ and $q_3 = 0$

The one-parameter model with $3q_1 + q_2 = 0$ and $q_3 = 0$ is characterized by the relationships $q_1 = -q$, $q_2 = 3q$, $q_3 = 0$, for some $q$. The susceptibility tensor is now proportional to the difference of the Riemann ($R_{ikmn}$) and Weyl ($C_{ikmn}$) tensors
\[ \chi_{ikmn} = 3q (R_{ikmn} - C_{ikmn}). \] (29)

Equation (23) is satisfied when
\[ - [Y' - 2\kappa q(YEE' + Y' E^2)] [\kappa q E^3(Y^2 + 6q) - 4\kappa q^2 E^2 - Y^2 + Q] = 0. \] (30)

Again, one has two variants to obtain exact solutions. The first one can be realized, when the first bracket vanishes, and we obtain
\[ Y' - 2\kappa q(YEE' + Y' E^2) = 0 \quad \Rightarrow \quad Y'' = \frac{\text{const}}{1 - 2\kappa q E^2}. \] (31)

The minimal limit $q \to 0$ relates to the model $Y \to \text{const}$ instead of $Y \to r$, and we do not consider here such a model. When the second bracket in (30) vanishes, we obtain the following cubic equation for the electric field
\[ \kappa q E^3(Y^2 + 6q) - 4\kappa q^2 E^2 - Y^2 + Q = 0. \] (32)

This equation differs from (27) by the numerical multipliers in front of $q$. Surprisingly, the equation for the function $\sigma(r)$ [see Eq. (20)] can be now explicitly resolved in terms of $E$ and $Y$ as
\[ \sigma = Y' - 2\kappa q (E^2 Y' + Y E'). \] (33)

In order to find the function $N(r)$ we consider Eq. (22), which gives
\[ N = \frac{1}{\sigma^2 Y} \left\{ \int dr \sigma \left( 1 + \frac{\kappa E^2 Y^2}{2} + \kappa^2 E^2 - \kappa Q E \right) + \text{const} \right\}. \] (34)

Thus, any solution $E(Y)$ of the cubic equation (32) for the electric field gives us a new exact solution of this nonminimal model, since (33) gives immediately the function $\sigma(r)$ and (34) gives the function $N(r)$ in quadratures.

3. Dimensionless quantities and equations

In order to analyze Eqs. (32)-(34), and in particular the cubic equation (32) for the electric field, let us introduce the following dimensionless quantities (see [24])
\[ r_Q = \sqrt{\kappa Q^2/2}, \quad E_Q = Q/r_Q^2, \quad a = \frac{2q}{r_Q^2}, \quad \rho = \frac{r}{r_Q}, \quad y = \frac{Y}{r_Q}, \quad Z = \frac{E}{E_Q}. \] (35)

In these terms the Coulombian branch of Eq. (32) corresponds to the solution with the asymptotic behavior $Z \sim \rho^{-2}$ at $\rho \to +\infty$. Then the key equations (32)-(34) take the form
\[ a Z^3(y^2 + 3a) - 4a Z^2 - Z y^2 + 1 = 0, \] (36)
\[ \sigma = \frac{dy}{d\rho} - 2a \left( Z^2 \frac{dy}{d\rho} + y Z \frac{dZ}{d\rho} \right), \] (37)
\[ N = \frac{1}{\sigma^2 y} \left\{ \int d\rho \sigma \left( 1 + Z^2 y^2 + a Z^2 - 2Z \right) + \text{const} \right\}. \] (38)

The dimensionless parameter $a$ is a guiding parameter for the key equation (36). When $a = 0$, then $q = 0$ and the model is minimal.
Before we enter into the really interesting solutions, we mention the solutions with \(a < 0\). The expression \(y^2 + 3a\), the coefficient of \(Z^3\) in the Eq. (36), can vanish, when \(a < 0\); thus, the corresponding solution \(Z(y)\) is singular. The equation \(y = \sqrt{-3a}\) describes a vertical asymptote. There exists one critical value of the guiding parameter \(a\), namely, \(a = -16/9\), for which the point characterizing by the condition \(
abla_{Q} \frac{dZ}{dQ} = \infty\), belongs to this vertical asymptote. The typical behavior of electric field at \(a < 0\) is presented on Fig. [1]. Anyway, we will consider regular solutions only, thus assuming that \(a > 0\) from now on.

![Plots of the rescaled electric field](image)

**FIG. 1**: Plots of the rescaled electric field \(Z(y) = E/E_Q\) as function of \(y = Y/r_Q\) for \(a < 0\). All curves have a vertical asymptote at \(y = \sqrt{-3a}\). (a) When \(-\infty < a < -16/9\), the curve crosses the asymptote twice. (b) When \(a = -16/9\), the curve touches the asymptote at one point. (c) When \(-16/9 < a < 0\), the curve does not cross the asymptote. For all cases the function \(Z(y)\) is infinite, the singularity point being situated at \(y = 0\), when \(a = 0\) only. When \(a \to -0\), the vertical asymptote coincides with the coordinate line \(y = 0\) and the right branch of the curve (c) converts into the standard Coulombian curve \(Z(y) = 1/y^2\).

## III. ELECTRIC FIELD: THE COULOMBIAN AND NON-COULOMBIAN SOLUTIONS

Now we study the \(a > 0\) cases. The Coulombian solution for the electric field is the basic solution. Indeed, there are three branches of solutions to Eq. (36) with \(a > 0\). One of them \(Z_+(y,a)\) can be identified as the Coulombian branch, the other two can be indicated as the first and second non-Coulombian solutions, \(Z_-(y,a)\) and \(Z_+(y,a)\), respectively. Let us study first the Coulombian solution. From Eq. (36) this solution is

\[
Z_+^{-1}(y,a) = \frac{y^2}{3} + \sqrt{\frac{16a}{3}} \left[ 1 + \frac{y^4}{12a} \right] \cos \left\{ \frac{1}{3} \arccos \left[ \frac{\sqrt{243a}}{256} \left( \frac{2y^6}{81a^2} + \frac{y^2}{9a} - 1 \right) \right] \right\}.
\]

Clearly, \(Z_+(y,a) \to y^{-2}\), when \(y \to \infty\) for arbitrary \(a\), thus, this solution can indeed be referred to as the Coulombian branch. This solution remains real when the modulus of the arccosine argument does not exceed unity, i.e., \(y\) belongs to the interval given by the inequality

\[
\sqrt{\frac{243a}{256}} \left| \frac{2y^6}{81a^2} + \frac{y^2}{9a} - 1 \right| \leq \left( \frac{y^4}{12a} \right)^{3/2}.
\]

In particular, when \(a \leq 256/243\), then this interval is \(0 \leq y < \infty\). At \(y = 0\) this solution is not singular, since

\[
Z_+(0,a) = \frac{1}{4} \sqrt{\frac{3}{a}} \cos^{-1} \left\{ \frac{1}{3} \arccos \left( -\sqrt{\frac{243a}{256}} \right) \right\}, \quad 0 < a \leq \frac{256}{243}.
\]

The function \(Z_+(0,a)\) as a function of the guiding parameter \(a\) has a minimum at \(a = 8/9\), the minimal value being \(Z(0)_{\text{min}} = 3/4\). The behavior of the Coulombian branch of the solution for \(E(r)\) can be illustrated in a figure. In Fig. [2] we plot the rescaled electric field \(Z(y)\) as a function of the rescaled radius \(y\), for different values of \(a\). The Coulombian branch \(Z_+(y,a)\) is represented by the middle line.

The second solution \(Z_-(y,a)\) of the cubic equation (36) can be represented in the form

\[
Z_-^{-1}(y,a) = \frac{y^2}{3} - \sqrt{\frac{16a}{3}} \left[ 1 + \frac{y^4}{12a} \right] \cos \left\{ \frac{1}{3} \arccos \left[ \frac{\sqrt{243a}}{256} \left( \frac{1 - y^2}{3a} - \frac{2y^6}{81a^2} \right) \right] \right\}.
\]
FIG. 2: Plots of the rescaled electric field $Z(y)$ displaying the behavior of the Coulombian branch and the two non-Coulombian branches for $a > 0$. The interval $0 < a < +\infty$ is split into three regions with respect to two specific values of $a$, $a = 1$ and $a = 256/243$. The curve $Z(y)$ is also drawn for the cases $a = 1$ and $a = 256/243$. Dashed lines indicate upper ($Z = 1/\sqrt{a}$) and lower ($Z = -1/\sqrt{a}$) horizontal asymptotes. (a) When $0 < a < 1$, $Z_+(y,a) > 1/\sqrt{a}$. (b) When $a = 1$, the second non-Coulombian branch degenerates into a straight line $Z_+(y,1) = 1$. (c) When $1 < a < 256/243$, $Z_+(y,a) < 1/\sqrt{a}$. (d) When $a = 256/243$, the second non-Coulombian branch has a common point with the Coulombian one at $y = 0$. (e) When $a > 256/243$, these branches have a common point at $y > 0$.

At $y \to \infty$ this curve tends to the horizontal asymptote $Z = -1/\sqrt{a}$, and in this sense it is non-Coulombian, we call it the first non-Coulombian solution. The solution is real for arbitrary (positive) value of the guiding parameter $a$, see the lower curves of the Fig. 2a-e. At the point $y = 0$, the center, this solution is also nonsingular

$$Z_-(0) = -\frac{1}{4} \sqrt{\frac{3}{a}} \cos^{-1} \left\{ \frac{1}{3} \arccos \left( \frac{243a}{256} \right) \right\}, \quad a < 256/243,$$

$$Z_-(0) = -27/64, \quad a = 256/243,$$

$$Z_-(0) = -\frac{1}{4} \sqrt{\frac{3}{a}} \cosh^{-1} \left\{ \frac{1}{3} \arccosh \left( \frac{243a}{256} \right) \right\}, \quad a > 256/243.$$

The third solution of the cubic equation (36) is of the form

$$Z_+^{-1}(y,a) = \frac{y^2}{3} + \frac{16a}{3} \left[ 1 + \frac{y^4}{12a} \right] \sin \left\{ \frac{1}{3} \arcsin \left[ \sqrt{\frac{243a}{256}} \left( 1 - \frac{y^2}{12a} - \frac{2y^6}{81a^2} \right) \right] \right\}. \quad (44)$$

For $y \to \infty$ this curve tends to the horizontal asymptote $Z = 1/\sqrt{a}$, thus the solution is also non-Coulombian, it is the second non-Coulombian solution. Nevertheless, now the curve behaves analogously to the Coulombian one, for instance, the solution is real for $0 \leq y < +\infty$ when $a \leq 256/243$, see the upper curves of the Figs. 2a-e. The value at the center is

$$Z_+(0,a) = \frac{1}{4} \sqrt{\frac{3}{a}} \sin^{-1} \left\{ \frac{1}{3} \arcsin \left( \frac{243a}{256} \right) \right\}, \quad a \leq 256/243. \quad (45)$$

Clearly, $Z_+(0) = Z_+(0) = 27/32$, when $a = 256/243$, i.e., the curves $Z_+$ and $Z_+$ contact at $y = 0$ at this value of the guiding parameter $a$, see Fig. 2. When $a > 256/243$ the curves $Z_+$ and $Z_+$ contact at $y > 0$, the inverse function $y(Z)$ being smooth in the vicinity of this point. Thus, when $a > 256/243$ one has that $y_{\text{min}} > 0$, and there is the possibility of finding a throat for a traversable wormhole.
Let us also mention that \( Z_+(0, a) > Z_+(\infty, a) \) when \( a < 1 \), and \( Z_+(0, a) < Z_+(\infty, a) \) when \( a > 1 \) (see Figs. 2a, c). In this connection we would like to make the following remarks: The cubic equation \((36)\) differs from the analogous equation discussed in \([24]\) \([\text{see (27)}]\) by the coefficients in front of \( Z^3 \) and \( Z^2 \). This leads to the following differences: \( (i) \) In contrast to \([24]\) the curves \( Z_+ \) and \( Z_- \) do not intersect at \( a = 1 \). \( (ii) \) There are no bubbles in the vicinity of the center in the framework of this new model. \( (iii) \) The branches \( Z_+ \) and \( Z_- \) can contact at \( y = 0 \), it is possible, when \( a = 256/243 \).

IV. EXACT SOLUTIONS: SOLUTIONS WITH A CENTER

A. Behavior of the function \( \sigma(\rho) \)

Based on the study of the cubic equation for the electric field, we consider the solutions for the metric functions \( \sigma(r) \) and \( N(r) \). We focus here on the solutions with center, i.e., on the cases, when the point \( Y = 0 \) is accessible. The solution with center for the nonminimal electric field with Coulombian asymptote exists, when \( 0 < a \leq 256/243 \) (see Figs. 2a-d). Of course, solutions with center with non-Coulombian asymptotes exist for other \( a \).

Let us consider the behavior of the function \( \sigma(\rho) \) for the case \( 0 < a \leq 256/243 \). We can put now \( Y = r \) and, respectively, \( y = \rho = \frac{y}{r_q} \), keeping in mind the asymptotic behavior \((12)\). Then the formula \((37)\) takes the form

\[
\sigma = 1 - 2a \left( Z^2 + \rho Z \frac{dZ}{d\rho} \right).
\]

According to Fig. 2a-d, \( Z(0) \) and \( \frac{dZ(0)}{d\rho} \) are finite and the value of the function \( \sigma(\rho) \) at the origin reads

\[
\sigma(0, a) = 1 - 2a Z^2(0, a).
\]

Substituting \((41)\) into \((47)\) we obtain for \( 0 < a \leq \frac{256}{243} \)

\[
\sigma(0, a) = 1 - \frac{3}{8} \cos^{-2} \left( \frac{1}{3} \arccos \left( \frac{-243a}{256} \right) \right).
\]

The value \( \sigma(0, a) \) as a function of the guiding parameter \( a \) is steadily decreasing with increasing \( a \). When \( a \to 0 \), this function tends to 1/2, when \( a = 256/243 \) then \( \sigma(0, a) = -1/2 \), and when \( a = 8/9 \) the function \((48)\) takes the value zero. When \( 8/9 \leq a \leq 256/243 \), the function \( \sigma(0, a) \) is nonpositive. Thus taking into account the asymptotic condition \( \sigma(\infty, a) = 1 > 0 \) there is at least one root \( \rho_s \geq 0 \) of the equation \( \sigma(\rho) = 0 \). When \( a < 8/9 \), the value \( \sigma(0, a) \) is positive.

B. Exact solution for \( a = 1 \)

When \( a = 1 \) Eq. \((36)\) admits the solution \( Z = 1 \), which it is one of the curves from the family \( Z_+(y, a) \), namely, \( Z_+(y, 1) \). Thus, in this case the cubic equation \((36)\) splits, so that the Coulombian solution \( Z_+ \) and the first non-Coulombian solution \( Z_- \) can be obtained as solutions of the corresponding quadratic equations. This procedure yields,

\[
Z_+(y, 1) = \frac{\sqrt{13 + 2y^2 + y^4} + 1 - y^2}{2(3 + y^2)}, \quad Z_+(0, 1) = \frac{1 + \sqrt{13}}{6}, \quad (49)
\]

\[
Z_-(y, 1) = -\frac{\sqrt{13 + 2y^2 + y^4} + 1 - y^2}{2(3 + y^2)}, \quad Z_-(0, 1) = \frac{1 - \sqrt{13}}{6}. \quad (50)
\]

We now study the behavior of the functions \( \sigma(\rho) \) and \( N(\rho) \) for this model with \( a = 1 \) which is of interest for illustrative purposes. In this case the Coulombian branch of the electric field is described by \((49)\), thus the metric function \( \sigma(\rho) \) is of the form

\[
\sigma(\rho) = \frac{(6 + 34\rho^2)\sqrt{13 + 2\rho^2 + \rho^4} - 39 + 82\rho^2 + 10\rho^4 + 10\rho^6 + \rho^8}{(3 + \rho^2)^3 \sqrt{13 + 2\rho^2 + \rho^4}}. \quad (51)
\]

The plot of the function \( \sigma(\rho) \) is represented in Fig. 3. At the origin \( \sigma(0, 1) = -\frac{\sqrt{13} - 2}{9} \approx -0.178 \) is negative. This function has only one root \( \rho_s = 0.289 \). At \( \rho = 1.706 \) it reaches a maximum, \( \sigma_{\text{max}} = 1.067 \), and then tends to one asymptotically.
FIG. 3: Plot of the metric function $\sigma(\rho)$ at $a = 1$. This function has only one root $\rho_s = 0.289$.

FIG. 4: Plots of the function $N(\rho)$ for $a = 1$ for various values of the rescaled asymptotic mass $m = M/r_Q$. We split the interval $0 < m < +\infty$ into three regions with respect to two specific values of mass: $m = m^* = 0.7913$ and $m = m_0 = 0.7924$. All curves except (b) possess a vertical asymptote at $\rho = \rho_s = 0.289$. When $m = m_0$ [plot (d)] $N(0)$ is finite, contrary to the other cases.

The constant in (38) is connected with the asymptotic mass $M$ defined by $M = \frac{1}{2} \lim_{r \to \infty} [r(1 - g_{00})]$, where $g_{00} = \sigma^2 N$. Taking into account this condition one obtains

$$g_{00} = 1 - \frac{2m - J(\rho)}{\rho}, \quad N = \frac{1}{\sigma^2} \left[ 1 - \frac{2m - J(\rho)}{\rho} \right],$$

$$J(\rho) = \int_{\rho}^{\infty} \sigma \left( 1 - \frac{\sigma}{2(3 + \rho^2)} \left( 19 + 23\rho^2 + 5\rho^4 + \rho^6 - (5 + 2\rho^2 + \rho^4)\sqrt{13 + 2\rho^2 + \rho^4} \right) \right) d\rho,$$

where $m \equiv M/r_Q$. The value of the integral $J(\rho)$ at $\rho = 0$ is finite and equal to $J(0) = 1.5848$. Depending on the value of mass $m$ there are three variants of the behavior of the functions $g_{00}(\rho)$ and $N(\rho)$ in the vicinity of $\rho = 0$. They are: (i) $m < m_0 = \frac{1}{2} J(0) = 0.7924$. The function $g_{00}$ tends to positive infinity, when $\rho \to 0$ [see Fig. 5a-c]. (ii) $m = m_0$. The function $g_{00}$ takes the negative finite value $g_{00}(0) = -0.0096$ [see Fig. 5d]. (iii) $m > m_0$. The function $g_{00}$ tends to negative infinity, when $\rho \to 0$ [see Fig. 5e]. Plots of the function $N(\rho)$ and of the function $g_{00}(\rho) = \sigma^2 N$ for $a = 1$ are given in Figs. 4 and 5, respectively.

In addition to the special value of the mass $m = m_0$, there exists another special value $m^* = 0.7913$, which can be obtained as follows. The function $\sigma(\rho)$ takes zero value at $\rho = \rho_s$ (see Fig. 3), and $N(\rho) = g_{00}/\sigma^2$ generally becomes infinite at this point. Nevertheless, the possibility exists to make $N(\rho_s)$ finite, when $\rho_s$ is a double root of the function $g_{00}$. This situation takes place, when $m = m^*$, see Fig. 5b for $N(\rho)$ and Fig. 5h for $g_{00}(\rho)$. However, for both cases the Kretschmann scalar $K = R_{ikmn} R^{ikmn}$ is infinite at this point, $K \propto (\rho - \rho_s)^{-6}$ when $m \neq m^*$ and $K \propto (\rho - \rho_s)^{-2}$ when $m = m^*$. Hence, the sphere with the radius $\rho = \rho_s$ is a singularity. In addition there exists a second singularity at the origin. It is of a Schwarzschild type, when $m \neq m_0$, and becomes a conic singularity,
FIG. 5: Plots of the function $g_{00}(\rho) = \sigma^2 N$ for $a = 1$ for various values of the rescaled asymptotic mass $m = M/r_Q$. We split the interval $0 < m < +\infty$ into three regions with respect to two specific values of mass: $m = m^* = 0.7913$ and $m = m_0 = 0.7924$. (a) When $m < m^*$, the function $g_{00}$ is positive and tends to infinity at origin. (b) When $m = m^*$, the curve touches the horizontal axis at $\rho = \rho_s = 0.289$. (c) When $m^* < m < m_0$, the function $g_{00}$ is negative between these points. (d) When $m = m_0$, the curve intersects the horizontal axis at one point; the function $g_{00}$ tends to a finite negative value at origin. (e) When $m > m_0$, the function $g_{00}$ tends to negative infinity at origin.

V. EXACT SOLUTIONS: WORMHOLES (SOLUTIONS WITHOUT A CENTER)

A. The expression for $Y$

According to Fig. 2, when $a > 256/243$, the value $Y = 0$ is inaccessible for the Coulombian branch of the electric field. It means that $Y(r)$ reaches a minimal value $Y_{\text{min}}(a)$, for which the Coulombian branch smoothly joins with the second non-Coulombian one. Therefore we are in the presence of a wormhole. The corresponding surface of minimal area $4\pi Y_{\text{min}}^2$ is its throat.

In order to describe this object let us introduce a new radial coordinate $r$, which covers the entire range $(-\infty; +\infty)$. Without loss of generality, we can take as usual the throat to occur at $r = 0$

$$Y'(0) = 0, \quad Y''(0) > 0.$$  \hspace{1cm} (53)

Also we assume that positive values of $r$ are associated with the Coulombian branch of the electric field, while the range $(-\infty; 0)$ corresponds to the non-Coulombian one we are studying. We have to supplement the asymptotic form [12] for the function $Y(r)$ with a condition at negative infinity

$$\lim_{r \to -\infty} Y(r) = +\infty,$$  \hspace{1cm} (54)

which correlates with Fig. 2.
B. Radius of the throat

Further analytical progress is possible if we fix an explicit expression for this function, satisfying both the asymptotic condition (54) and the throat conditions (53). The simplest function of such a type is

\[ Y(r) = \sqrt{r^2 + r_Q^2 b^2}, \]

or, after rescaling (35)

\[ y(\rho) = \frac{Y}{r_Q} = \sqrt{\rho^2 + b^2}, \]

where \( b\) is the dimensionless radius of the throat. The value of \( b\) can be found as follows: In Eq. (39) the argument of the function arc cosine takes the maximum value +1 corresponding to \( y \to \infty\). The minimal value \( -1\) relates to \( y = y_{\text{min}} = b\). Thus the value of the throat’s radius satisfies the equation

\[ b^8 + 3ab^6 + \frac{61}{4}ab^4 + \frac{27}{2}a^2b^2 - \frac{a^2}{4}(243a - 256) = 0. \]

(57)

Let us mention that the argument of the arc sine in Eq. (44) takes the value +1, and \( Z^* (b, a) = Z(b, a)\), as it should be at the junction point. Equation (57) is quartic with respect to \( b^2\). For \( a > 256/243\) this equation admits only one positive solution. The value of the throat’s radius as a function of the nonminimal parameter \( a\) is presented in Fig. 6. When \( a = 256/243\), \( b\) vanishes. When \( a \gg 1\), this function behaves as \( b(a) = 3 \left(\frac{a}{7}\right)^{1/3} - \frac{5}{12} + o(1)\).

![FIG. 6: Plot of the dimensionless radius of the wormhole throat \( b\) as a function of the parameter \( a\). It starts at \( a = 256/243\), i.e., if the parameter \( a\) is less than this value a throat does not exist.](image)

For illustration, it is convenient to consider the value \( a = 4\), since \( b(4) = 2\) and \( Z^{-1}(b, 4) = 4\) [see (39)]. Substituting \( y(\rho)\) taken from (56) into the cubic equation (36) we obtain a plot of \( Z(\rho, 4)\) presented in Fig. 7. This curve is nonsymmetric: at \( \rho \to +\infty\) the function \( Z(\rho)\) tends to zero (Coulombian branch), at \( \rho \to -\infty\) it tends to 1/2 (second non-Coulombian branch).

C. Asymptotic behavior

When \( \rho \to +\infty\) the solution for the electric field behaves according to the Coulombian law. In this limit the functions \( \sigma(\rho) \) and \( N(\rho) \) go to one [according to (12), (25), and (26)], and we deal with the traditional asymptotically flat Minkowski spacetime. When \( \rho \to -\infty\) the electric field \( Z(\rho)\) tends to the asymptotic value \( 1/\sqrt{a} \neq 0\). In this case \( \sigma(\rho)\) tends to one as well, since in (37) \( \frac{dy}{d\rho} \to -1\), while \( N(\rho)\) becomes infinite

\[ N(\rho) \sim -\frac{\rho^2}{3a}. \]

(58)

Thus, for \( \rho \to -\infty\) the metric of this spacetime is asymptotically de Sitter, and the Riemann tensor in this limit may be decomposed as

\[ R_{\mu\nu}^{ik} = -K (\delta_{\mu}^i \delta_{\nu}^k - \delta_{\mu}^k \delta_{\nu}^i) + o(r^{-1}), \quad K = \frac{r_Q^2}{3a} = \frac{1}{6q}. \]

(59)

The effective \( \Lambda\)-term induced by the electric field nonminimally coupled to gravity is thus equal to \( \Lambda_{\text{eff}} = 3K = 1/(2q)\). In contrast to the electric strength \( E\), the electric excitation \( D\) [see Eq. (17)], being equal to

\[ D = \frac{Q}{Y^2} = \frac{2}{\kappa Q (\rho^2 + b^2)^2}, \]

(60)

tends to zero both for \( \rho \to +\infty\) and \( \rho \to -\infty\). Similar situations, where the excitation tensor was equal to zero, while the electric, magnetic, or other fields did not vanish, were considered earlier in [23].
FIG. 7: Plot of the rescaled electric field $Z(\rho)$ for $a = 4$. The region $\rho > 0$ is the Coulombian branch [$Z(+\infty) = 0$], the region of negative values of $\rho$ is the second non-Coulombian one [$Z(-\infty) = 1/2$]. The junction point $\rho = 0$ is the point of inflection on this curve.

D. Horizons and singularities

If we intend to consider traversable wormholes only, we have to require the spacetime metric to possess no singularities. This means that the function $\sigma(\rho)$ should not vanish anywhere, because, when $\sigma(\rho) = 0$, we have a singularity, as mentioned previously. Therefore, for the model considered in [24] with $q_1 = -q$, $q_2 = q$, $q_3 = 0$ traversable wormholes are absent, since in the throat $\sigma = 0$ (see Eq. (28)). In contrast to [24], for our case ($q_1 = -q$, $q_2 = 3q$, $q_3 = 0$) we have,

$$\sigma(0) = -2abZ(0) \frac{dZ(0)}{d\rho} > 0,$$

(61)

because $\frac{dZ}{d\rho} < 0$ at $\rho = 0$ (see, e.g., Fig. 7). However a singularity can be located at $\rho \neq 0$. In order to find the range for the parameter $a$ for which the function $\sigma(\rho)$ is positive everywhere, let us consider the situation where the function $\sigma(\rho)$ given by (37) with $Z(\rho)$ from (36) has a double root [in this case the curve of $\sigma(\rho)$ touches the horizontal axis]. Numerical calculations show that this is possible for two values of the parameter $a$: $a = a_1 \approx 1.072$ and $a = a_2 \approx 39.380$. When $256/243 < a < a_1$, the equation $\sigma(\rho,a) = 0$ has two positive roots; when $a > a_2$ the two corresponding roots are negative. When $a_1 < a < a_2$, the equation $\sigma(\rho,a) = 0$ has no real roots, i.e., $\sigma > 0$ for any value of $r$. Figure 8 illustrates these features for a typical value $a = 4$, which is in the interval $1.072 < a < 39.380$.

FIG. 8: Plot of the metric function $\sigma(\rho)$ for $a = 4$. This function is positive and tends to unity at both infinities. The curve is nonsymmetric with respect to the vertical axis.

Now, since the wormhole spacetime metric is an asymptotically de Sitter one, it possesses at least one horizon (the discussion of some aspects of wormhole physics in a cosmological context can be found, e.g., in [5, 6]). Thus, our wormhole configuration may be considered as traversable, if, in addition to the requirement of the absence of singularities, we suppose that horizons of the metric are out of a traveler way through the throat. For the metric function $N(\rho)$ it means that in the vicinity of the throat $N(\rho)$ is positive. To illustrate the behavior of the functions $g_{00}(\rho)$ and $N(\rho)$ let us consider again the model with $a = 4$, see Fig. 9. When the dimensionless mass parameter $m$ is less than a critical value $\tilde{m}$ (being equal to 1.0081 for $a = 4$), an event horizon is located behind the throat, hence in this case the wormhole throat is traversable. When $m > \tilde{m}$, a horizon is situated in front of the throat, and a traveler cannot go through, it is a nontraversable wormhole.

E. Comparisons

It is of interest to briefly compare our electric wormhole solutions with the magnetic nonminimal Wu-Yang wormhole solutions studied in [15]. The magnetic nonminimal traversable wormhole joins two asymptotically
FIG. 9: Plots of $g_{00}$ (upper row) and $N(\rho)$ (lower row) for the value $a = 4$, when there is only one horizon. We split the interval $0 < m < +\infty$ into two regions with respect to a specific value of the rescaled mass $\tilde{m} = 1.0081$: (a) When $m < \tilde{m}$, the curves cross the horizontal axis at negative values of $\rho$, so that the horizon is located behind the throat. (b) When $m = \tilde{m}$, then $g_{00}(0) = 0$ and $N(0) = 0$, thus, the horizon is located on the throat. (c) When $m > \tilde{m}$, the horizon is located in front of the throat. Plot (a) illustrates the case in which the horizon belongs to an asymptotically de Sitter region of the combined spacetime, it is a standard cosmological apparent horizon. In this sense such a wormhole is traversable.

Minkowskian flat regions and the magnetic gauge field is symmetric with respect to wormhole throat, i.e., this field asymptotically vanishes in each region of the combined spacetime. On the other hand, the electric nonminimal wormhole joins an asymptotically flat region and an asymptotically de Sitter region, for which the corresponding electric field tends to a constant value. This introduces a new effective, nonminimally induced, cosmological constant $\Lambda_{\text{eff}} = 1/(2q)$.

VI. CONCLUSIONS

We have presented new exact solutions of a nonminimal Einstein-Maxwell model with coupling parameters satisfying the relations $q_1 = -q$, $q_2 = 3q$, $q_3 = 0$, with $q$ arbitrary, which describe spherically symmetric electrically charged objects with and without center. The electric field of the objects is characterized by the following features: (i) it satisfies a cubic key equation and splits into three branches, one of them has Coulombian asymptotics, $E(r) \to Q/r^2$; (ii) when the guiding nonminimal dimensionless parameter $a = 4q/\kappa Q^2$ is positive, the electric field is described by a smooth function finite everywhere; (iii) when $a \geq 256/243$ the Coulombian branch of the electric field conjugates with the non-Coulombian branch with constant asymptotics $E = 1/\sqrt{\kappa q}$, the position of the junction point depends on the parameter $a$ and coincides with geometrical center for $a = 256/243$.

There are wormhole solutions. Indeed, when the nonminimal guiding parameter $a$ is in the interval $1.072 < a < 39.380$ there are no spacetime singularities and the nonminimal field configuration without center is a wormhole. This is an explicit example of a nonminimal traversable electrically charged wormhole joining two regions of spacetime, an asymptotically flat region and an asymptotically de Sitter region. The spacetime of this wormhole has at least one horizon, which, depending on the value of the rescaled asymptotic mass $m$, can be situated in front of the throat, just on the throat and behind it. When $m$ does not exceed some critical value (e.g., $m < 1.0081$ for $a = 4$), the horizon is located behind the wormhole throat, i.e., in the asymptotically de Sitter region of the combined spacetime. Such a configuration is thus a traversable wormhole supported by an electric field nonminimally coupled to gravity. In this manner we have presented explicitly a nonminimal realization of Wheeler’s idea about charge without charge, and showed that, if the world is nonminimal in the coupling of gravity to electromagnetism, then wormhole appearance, or perhaps construction by an absurdly advanced civilization, is possible.

Perhaps, a next natural step is to consider nonminimal Einstein-Maxwell models with an a priori cosmological constant $\Lambda$. It is expect that one can also find solutions of our nonminimal Einstein-Maxwell model. These solution would describe electric or magnetic wormholes joining different combinations of asymptotically de Sitter, anti de Sitter, and Minkowski regions. For instance, an anti de Sitter-Minkowski electric nonminimal wormhole is expected to be free of horizons.

Acknowledgments

This work was partially supported by the Russian Foundation for Basic Research, Grants Nos. 08-02-00325-a and 09-05-99015, and by FCT - Portugal through Projects Nos. PTDC/FIS/098962/2008 and
[1] J. A. Wheeler, *Geometrodynamics* (Academic Press, New York, 1962).

[2] M. S. Morris and K. S. Thorne, *Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity*, Am. J. Phys. 56, 395 (1988).

[3] M. Visser, *Lorentzian Wormholes: from Einstein to Hawking* (AIP Press, New York, 1995).

[4] D. Hochberg and M. Visser, *Geometric structure of the generic static traversable wormhole throat*, Phys. Rev. D 56, 4745 (1997);
D. Hochberg and M. Visser, *Dynamic wormholes, antitrapped surfaces, and energy conditions*, Phys. Rev. D 58, 044021 (1998).

[5] J. P. S. Lemos, F. S. N. Lobo, and S. Quinet de Oliveira, *Morris-Thorne wormholes with a cosmological constant*, Phys. Rev. D 68, 064004 (2003).

[6] S.-W. Kim, *Schwarzschild-De Sitter type wormhole*, Phys. Lett. A 166, 13 (1992);
M. S. R. Delgaty and R. B. Mann, *Traversable wormholes in (2+1) and (3+1) dimensions with a cosmological constant*, Int. J. Mod. Phys. D 4, 231 (1995); L.-X. Li, *Two open universes connected by a wormhole: exact solutions*, Journ. Geom. Phys. 40, 154 (2001);
M. Cataldo, S. del Campo, P. Mirino, and P. Salgado, *Evolving Lorentzian wormholes supported by phantom matter and cosmological constant*, Phys. Rev. D 79, 024005 (2009).

[7] H. G. Ellis, *Ether flow through a drainhole: A particle model in general relativity*, J. Math. Phys. (N.Y.) 14, 104 (1973);
K. A. Bronnikov, *Scalar-tensor theory and scalar charge*, Acta Phys. Pol. B 4, 251 (1973);
C. Barceló and M. Visser, *Traversable wormholes from massless conformally coupled scalar fields*, Phys. Lett. B 466, 127 (1999); C. Barceló and M. Visser, *Scalar fields, energy conditions and traversable wormholes*, Classical Quantum Gravity 17, 3843 (2000); S. V. Sushkov and S.-W. Kim, *Wormholes supported by a kink-like configuration of a scalar field*, Classical Quantum Gravity 19, 4909 (2002).

[8] S. V. Sushkov, *A selfconsistent semiclassical solution with a throat in the theory of gravity*, Phys. Lett. A 164, 33 (1992); D. Hochberg, A. Popov, and S. V. Sushkov, *Self-consistent wormhole solutions of semiclassical gravity*, Phys. Rev. Lett. 78, 2050 (1997).

[9] K. K. Nandi, B. Bhattacharjee, S. M. K. Alam, and J. Evans, *Brans-Dicke wormholes in the Jordan and Einstein frames*, Phys. Rev. D 57, 823 (1998).

[10] L. A. Anchordoqui and S. E. Perez Bergliaffa, *Wormhole surgery and cosmology on the brane: The world is not enough*, Phys. Rev. D 62, 067502 (2000);
K. A. Bronnikov and S.-W. Kim, *Possible wormholes in a brane world*, Phys. Rev. D 67, 064027 (2003);
M. La Camera, *Wormhole solutions in the Randall–Sundrum scenario*, Phys. Lett. B 573, 27 (2003);
F. S. N. Lobo, *General class of braneworld wormholes*, Phys. Rev. D 75, 064027 (2007).

[11] S. V. Sushkov, *Wormholes supported by a phantom energy*, Phys. Rev. D 71, 043520 (2005); F. S. N. Lobo, *Phantom energy traversable wormholes*, Phys. Rev. D 71, 084011 (2005); N. S. Kardashev, I. D. Novikov, and A. A. Shatskii, *Magnetic tunnels (wormholes) in astrophysics*, Astronomy Reports 50, 601 (2006).

[12] F. S. N. Lobo, *Chaplygin traversable wormholes*, Phys. Rev. D 73, 064028 (2006).

[13] A. Das and S. Kar, *The Ellis wormhole with “tachyon matter”*, Classical Quantum Gravity 22, 3045 (2005).

[14] A. V. B. Arellano and F. S. N. Lobo, *Evolving wormhole geometries within nonlinear electrodynamics*, Classical Quantum Gravity 23, 5811 (2006); A. V. B. Arellano and F. S. N. Lobo, *Non-existence of static, spherically symmetric and stationary, axisymmetric traversable wormholes coupled to nonlinear electrodynamics*, Classical Quantum Gravity 23, 7229 (2006).

[15] A. B. Balakin, S. V. Sushkov, and A. E. Zayats, *Nonminimal Wu-Yang wormhole*, Phys. Rev. D 75, 084042 (2007).

[16] E. F. Eiroa and G. E. Romero, *Linearized stability of charged thin shell wormholes*, Gen. Relativ. Gravit. 36, 651 (2004); F. Schein and P. C. Aichelburg, *Traversable wormholes in geometries of charged shells*, Phys. Rev. Lett. 77, 4130 (1996); E. F. Eiroa and C. Simeone, *Some general aspects of thin-shell wormholes with cylindrical symmetry*, arXiv:0912.5496[gr-qc].

[17] S.-W. Kim and H. Lee, *Exact solutions of a charged wormhole*, Phys. Rev. D 63, 064014 (2001); K. A. Bronnikov and S. Grinyok, *Charged wormholes with nonminimally coupled scalar fields, existence and stability*, gr-qc/0205131; J. A. Gonzalez, F. S. Guzman, and O. Sarbach, *On the instability of charged wormholes supported by a ghost scalar field*, Phys. Rev. D 80, 024023 (2009).

[18] V. Faraoni, E. Gunzig, and P. Nardone, *Conformal transformations in classical gravitational theories and in cosmology*, Fundam. Cosmol. Phys. 20, 121 (1999).

[19] F. W. Hehl and Yu. N. Obukhov, *How does the electromagnetic field couple to gravity, in particular to metric, non-metricity, torsion, and curvature?,* Lect. Notes Phys. 562, 479 (2001).

[20] A. B. Balakin and J. P. S. Lemos, *Non-minimal coupling for the gravitational and electromagnetic fields: A general system of equations*, Classical Quantum Gravity 22, 1867 (2005).

[21] I. T. Drummond and S. J. Hathrell, *QED vacuum polarization in a background gravitational field and its effect on the velocity of photons*, Phys. Rev. D 22, 343 (1980).

[22] H. A. Buchdahl, *On a Lagrangian for non-minimally coupled gravitational and electromagnetic fields*, J. Phys. A 12,
[23] F. Müller-Hoissen, *Non-minimal coupling from dimensional reduction of the Gauss-Bonnet action*, Phys. Lett. B 201, 325 (1988).

[24] A. B. Balakin, V. V. Bochkarev, and J. P. S. Lemos, *Nonminimal coupling for the gravitational and electromagnetic fields: Black hole solutions and solitons*, Phys. Rev. D 77, 084013 (2008).

[25] K. A. Bronnikov and J. C. Fabris, *Regular phantom black holes*, Phys. Rev. Lett. 96, 251101 (2006);
K. A. Bronnikov, H. Dehnen, and V. N. Melnikov, *Regular black holes and black universes*, Gen. Relativ. Gravit. 39, 973 (2007).

[26] A. C. Eringen and G. A. Maugin, *Electrodynamics of Continua* (Springer-Verlag, New York, 1989).

[27] A. B. Balakin, H. Dehnen, and A. E. Zayats, *Nonminimal monopoles of the Dirac type as realization of the censorship conjecture*, Phys. Rev. D 79, 024007 (2009);
A. B. Balakin and A. E. Zayats, *Non-minimal Wu-Yang monopole*, Phys. Lett. B 644, 294 (2007).

[28] A. B. Balakin and A. E. Zayats, *Curvature coupling in Einstein-Yang-Mills theory and non-minimal self-duality*, Gravit. Cosmol. 12, 302 (2006);
A. B. Balakin, H. Dehnen, and A. E. Zayats, *Nonminimal isotropic cosmological model with Yang-Mills and Higgs fields*, Int. J. Mod. Phys. D 17, 1255 (2008).