On the Geometry of Coset Models with Flux

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Abstract

We study the 3-form flux $H_{\mu\nu\lambda}$ associated with the semi-classical geometry of $G/H$ gauged WZW models. We derive a simple, general expression for the flux in an orthonormal frame and use it to explicitly verify conformal invariance to the leading order in $\alpha'$. For supersymmetric models, we briefly revisit the conditions for enhanced supersymmetry. We also discuss some examples of non-abelian cosets with flux.

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**Introduction**

WZW models and their cosets (gauged WZW) provide examples of string backgrounds where both the exact CFT description and the geometry of the target space are well-known. The coset space $G/H$ is obtained by the identification $g \sim hgh^{-1}$ ($g \in G, h \in H$), hence its geometry is quite different from that of the usual left-coset ($g \sim hg$). The ‘adjoint-coset’ is also required to have non-trivial dilaton and three-form flux ($H_{\mu\nu\lambda}$) on it in order to ensure conformal invariance.

For left-cosets, the invariant one-forms and structure constants offer a clear intuitive picture of the geometry. In Ref. [1], analogous one-forms were introduced for adjoint cosets and were shown to define an orthonormal frame for the metric. The goal of this note is to take advantage of these one-forms to better understand the geometry of the adjoint coset with emphasis on the properties of the flux.

We first derive a simple, general expression for the flux in the orthonormal frame. As a consistency check, we use it to verify conformal invariance to the leading order. We then specialize to supersymmetric cases and comment on the enhancement of world-sheet supersymmetry from $N = 1$ to $N = 2$ in the presence of the flux. Finally, we discuss the conditions for vanishing of the flux and two examples of non-abelian cosets with $\dim(G/H) = 6$. Our result may be useful in the study of how mirror symmetry works [3] (See also [4]) in an NS-NS flux background and the geometric aspects of D-branes in gauged WZW model [5].

**Setup**

We begin with a very brief review of WZW model and its cosets to set up our notations. Let $G$ be a compact, simple Lie group. The Lie algebra of $G$ is written in terms of an orthonormal basis of anti-Hermitian generators as

$$[T_A, T_B] = f_{AB}^C T_C, \quad \text{Tr}(T_A T_B) = -\delta_{AB}. \quad (1)$$

To describe the geometry of the group manifold, we introduce the standard one-forms:

$$g^{-1}dg = E^A T_A, \quad dg^{-1} = \tilde{E}^A T_A$$

$$\tilde{E}^A = C^{AB} E^B, \quad C_{AB} = -\text{Tr}(T_A g T_B g^{-1}), \quad C C^T = 1. \quad (2)$$

1Throughout this paper, we work only in the semi-classical ($\alpha'/R^2 \sim 1/k \ll 1$) limit because the problem of obtaining the exact expression for the flux is quite involved [2].
The WZW model defined for $G$, 
\[ S_G = -\frac{k}{4\pi} \int d^2 z \text{Tr} (g^{-1} \partial g \cdot g^{-1} \bar{\partial} g) + ik \Gamma_{WZ}, \]  
(3)
corresponds to a sigma model on the group manifold with constant dilaton and the following metric and flux 
\[ ds^2 = E_A E_A, \quad H = \frac{1}{6} f_{ABC} E_A E_B E_C. \]  
(4)
More precisely, the metric and the flux should be scaled by the radius square $R^2 = k \psi^2 \alpha'/4$, where the integer $k$ is the level of WZW model and $\psi$ is the highest root of $\text{Lie}(G)$. We will suppress $R^2$ in the following unless its precise value becomes important.

We will consider cosets of type $G/H$, where $\text{rank}(H) = \text{rank}(G)$ and $H$ acts on $G$ as $g \to hgh^{-1}$. We use $(a, b, \cdots)$ indices for $\text{Lie}(H)$ and $(\alpha, \beta, \cdots)$ indices for its orthogonal complement. The coset theory is realized as a gauged WZW theory with the following action and gauge transformation law:
\[ S = S_G + S_A, \]  
(5)
\[ S_A = \frac{k}{2\pi} \int d^2 z \text{Tr} (\bar{A}g^{-1} \partial g - A \bar{\partial} gg^{-1} - \bar{A} A + g^{-1} A \bar{g} \bar{A}) \]  
\[ = -\frac{k}{2\pi} \int d^2 z (\bar{A}_a E^a - A_a \bar{E}^a - A^a (\eta_{ab} - C_{ab}) \bar{A}^b), \]  
\[ g \to u^{-1} gu, \quad A_i \to u^{-1} (A_i + \partial_i) u. \]  
(6)

The expression

Since the action is quadratic in the non-propagating gauge field, it is easy to integrate out the gauge field and find \[ G_{MN} = G^{(0)}_{MN} + 2(M^{-1})_{ab} E^a_M \bar{E}^b_N, \]  
(7)
\[ B_{MN} = B^{(0)}_{MN} + 2(M^{-1})_{ab} E^a_M \bar{E}^b_N, \]  
(8)
\[ e^{-2\phi} = \det M, \]  
(9)
where $M_{ab} \equiv \delta_{ab} - C_{ab}$. Although $G_{MN}$ and $B_{MN}$ carry $d_G = \dim(G)$ indices, they actually depend only on the ‘coset directions,’ as can be seen from the existence of the $d_H = \dim(H)$ null vectors
\[ Z_a^M = E_a^M - \bar{E}_a^M = M_{ab} E_b^M - C_{ab} E_\beta^M \implies G_{MN} Z_a^M = 0. \]  
(10)
Removal of $d_H$ degrees of freedom and gauge-invariant way can be made clear with the help of the one forms \[ \]

\[ H_\alpha = E_\alpha + E_a (M^{-1})_{ab} C_{b\alpha} \quad (Z_a \cdot H_\alpha = 0). \]  

(11)

As shown in [1], these one-forms define an orthonormal frame, i.e.,

\[ ds^2 = H_\alpha H_\alpha. \]  

(12)

It is natural to write down the flux also in this frame. A lengthy but straightforward computation using the basic identities,

\[ dC_{AB} = -C_{AD} f_{DBC} E_C, \]  

(13)

\[ f_{ABC} = C_{AD} C_{BE} C_{CF} f_{DEF}, \]  

(14)

\[ f_{ACD} f_{BCD} = c G \delta_{AB}, \; f_{acd} f_{bcd} = c^H \delta_{ab}, \]  

(15)

\[ f_{AB} [C f_{BDE}] = 0, \]  

(16)

\[ f_{ab\gamma} = 0, \]  

(17)

shows that the flux also takes a very simple form in this frame,

\[ H = \frac{1}{6} \left\{ f_{\alpha\beta\gamma} + 3 A_{[\alpha\beta\gamma]} \right\} H_\alpha \wedge H_\beta \wedge H_\gamma, \]

\[ A_{\alpha\beta\gamma} = f_{\alpha\beta\gamma} (M^{-1})_{ab} C_{b\gamma}. \]  

(18)

This expression is the starting point of our discussion in what follows.

It is useful to note that the gauge transformation \[ \]

\[ g_0(x) \rightarrow h(f^m(x)) g_0(x) h(f^m(x))^{-1}, \]  

(19)

where $h(y^m), (m = 1, \cdots, d_H)$ define a coordinate system on $H$. The functions $f^m(x)$ shift the gauge slice from the original one without inducing a coordinate change. Upon this type of gauge transformation, the one-forms $E_a$, $E_\alpha$ and $H_\alpha$ transform as

\[ E_a \rightarrow Q_{ab} (E_b - e_c M_{cb}), \] \[ E_\alpha \rightarrow Q_{\alpha\beta} (E_\beta + e_c C_{c\beta}), \] \[ H_\alpha \rightarrow Q_{\alpha\beta}(x) H_\beta, \]  

(20)  

(21)

where $Q_{AB} = -\text{Tr}(T_A h T_B h^{-1})$, and $h^{-1} dh = e_a T_a$. Clearly, the change of gauge slice results in a local Lorentz transformation on $H_\alpha$.  

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**Conformal invariance**

The leading order conformal invariance condition for a sigma model is well known to be

\[ R_{MN} - \frac{1}{4} H_{MIJ} H_{N}^{IJ} + 2 \nabla_{M} \nabla_{N} \phi = 0, \tag{22} \]

\[ \nabla^{M} (e^{-2\phi} H_{MIJ}) = 0, \tag{23} \]

\[ e^{2\phi} \nabla^{2} (e^{-2\phi}) - \frac{1}{6} H^{2} = \mathcal{L}. \tag{24} \]

For WZW or coset models, the constant \( \mathcal{L} \) on the RHS of the third equations equals \( 2(\Delta d)/3\alpha' \), where \( (\Delta d) \) is the deviation of the ‘dimension of the target space’ (more precisely, the central charge) from an integer value.

For a WZW model, it follows straight from \( dE_A = -\frac{1}{2} f_{ABC} E_B \wedge E_C \) that

\[ 4R_{AB} = H_{ACD} H_{BCD} = f_{ACD} f_{BCD} = c_G \delta_{AB}, \tag{25} \]

\[ H^{2} = \frac{c_G d_G}{R^2} = \frac{4c_G d_G}{k\psi^2 \alpha'}. \tag{26} \]

At a large \( k \), the value of \( H^2 \) agrees with the central charge of the WZW model at level \( k \) subtracted from its value in the \( k \to \infty \) limit (Recall \( c = \frac{k\psi^2 d_G}{k\psi^2 + c_G} \)). Eq. (23) follows from Jacobi identity for the structure constants.

For a coset space, the computation is somewhat more involved. As usual, the metric connection is derived from

\[ dH_\alpha = -\frac{1}{2} f_{\alpha\beta\gamma} + A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} - A_{\alpha\gamma\beta} H_\beta \wedge H_\gamma - (M^{-1})_{ab} f_{\alpha\beta b} H_\beta \wedge E_a. \tag{27} \]

The last term ensures that the spin-connection \( \omega_{\alpha\beta} \) transform inhomogeneously under a local Lorentz transformation. It also produces many non-tensor terms in the intermediate steps of the computation of the curvature tensor. This complication can be avoided by using the gauge transformation (20) to set \( E_a = 0 \). This can be always done at any point on the coset space, although care should be taken to include the derivatives of \( E_a \), which do not vanish in general. In this special gauge, the connection is given by

\[ \omega_{\alpha\beta} = -\frac{1}{2} (f_{\alpha\beta\gamma} - A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} - A_{\alpha\gamma\beta}) H_\beta \wedge H_\gamma \equiv \omega_{\alpha\beta\gamma} H_\gamma, \tag{28} \]

and the components of its derivatives that are relevant in computing \( R_{\alpha\beta} \) are

\[ d(\omega_{\alpha\beta\gamma}) = \left\{ \frac{1}{2} (A_{\alpha\beta\gamma|\delta} - A_{\beta\gamma\alpha|\delta} + A_{\alpha\gamma\beta|\delta}) + \Delta \omega_{\alpha\beta\gamma|\delta} \right\} H_\delta, \]

\[ A_{\alpha\beta\gamma|\delta} = A_{\alpha\beta\sigma} (A_{\sigma\delta\gamma} + f_{\sigma|\delta}) + f_{\alpha\beta\gamma} (M^{-1})_{ab} C_{bc} f_{\delta\gamma}, \]

\[ 2\Delta \omega_{\alpha\beta\gamma|\delta} = -(M^{-1})_{ab} f_{\alpha\beta b} f_{\gamma|\delta}. \tag{29} \]
Using these results and the basic properties (13)-(17), it is straightforward to verify the conformal invariance conditions (25) including the precise value of $Ł$.

**N = 2 Supersymmetry**

It is well-known [7, 8] that supersymmetry of $N = 1$ $G/H$ coset is enhanced to $N = 2$ when $T \equiv \text{Lie}(G) - \text{Lie}(H)$ decomposes as $T = T_+ \oplus T_-$, where $T_\pm$ are complex conjugate representations of $H$ with $[T_+, T_+] \subset T_+$, $[T_-, T_-] \subset T_-$. In complex notation, closure under commutation implies that $f_{ijk} = 0 = ar{f}_{ijk}$ and $f_{ija} = 0 = ar{f}_{ija}$. It follows that the $(3,0)$ and $(0,3)$ components of the flux vanish. This fact is in agreement with a related analysis [9] of supersymmetry enhancement of sigma models in the presence of the flux; in Ref. [9], it was shown that in order for an $N = 1$ supersymmetric sigma model to have an extra supersymmetry, the target space should be complex and the $(3,0)$ and $(0,3)$ components of the flux should vanish.

**Examples**

Given the formula for the flux (18), it is natural to ask what are the conditions for a $G/H$ coset to have non-vanishing flux. First, we note that the flux cannot vanish when $f_{\alpha\beta\gamma} \neq 0$. The reason is that $f_{\alpha\beta\gamma}$ and $A_{[\alpha\beta\gamma]}$ are orthogonal to each other ($f_{\alpha\beta\gamma}A_{\alpha\beta\gamma} = 0$) as follows from (15) and (17), and therefore cannot cancel each other. For $N = 2$ supersymmetric cosets (Kazama-Suzuki models), all such examples have been classified in Ref. [10]. The simplest among them is $SO(5)/SU(2) \times U(1)$ where $su(2)$ is embedded along a pair of long roots in $so(5)$.

For cosets with $f_{\alpha\beta\gamma} = 0$, it remains to determine when $A_{[\alpha\beta\gamma]}$ also vanishes. To our knowledge, the full answer to this question is not known. In the literature, all known examples with $f_{\alpha\beta\gamma} = 0$ and $A_{[\alpha\beta\gamma]} \neq 0$ are abelian cosets (i.e., the subset $H$ is abelian) [11, 12, 13, 14, 15]. Several non-abelian cosets with $f_{\alpha\beta\gamma} = A_{[\alpha\beta\gamma]} = 0$ are also known [17, 6, 18, 19, 20, 21, 22, 23].

Using our formula (18) and a gauge choice similar to that of [6], we have computed the flux for the two Kazama-Suzuki models of dimension 6: $SU(4)/SU(3) \times U(1)$ and $SO(5)/SO(3) \times SO(2)$. It turns out that $A_{[\alpha\beta\gamma]}$ vanishes for the former and not for the latter. It would be interesting to develop a systematic method to determine whether a given coset with $f_{\alpha\beta\gamma} = 0$ has vanishing flux. Algebraic CFT description of coset models may turn out to be useful in that direction.

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2See [16] for an example of $(G \times G')/H$ coset that is rather different from the $G/H$ cosets considered here.
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