The Lee model: a tool to study decays

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Abstract. We describe -in a didactical and detailed way- the so-called Lee model (which shares similarities with the Jaynes-Cummings and Friedrichs models) as a tool to study unstable quantum states/particles. This Lee model is based on Quantum Mechanics (QM) but possesses some of the features of Quantum Field Theory (QFT). The decay process can be studied in great detail and typical QFT quantities such as propagator, one-loop resummation, and Feynman rules can be introduced. Deviations from the exponential decay law as well as the Quantum Zeno effects can be studied within this framework. The survival probability amplitude as a Fourier transform of the energy distribution, the normalization of the latter, and the Breit-Wigner limit can be obtained in a rigorous mathematical approach.

1. Introduction

The study of unstable quantum states is a central problem of Quantum Mechanics (QM) and Quantum Field Theory (QFT) [1].

When considering elementary unstable particles or composite unstable hadrons, QFT represents the correct framework to treat decays. Yet, it is difficult to study the decay law in the context of QFT: the typical textbook treatment deals with scattering processes in which ‘in’ and ‘out’ states are regarded as stable particles (the interaction is eventually switched off in the far past and future) [2]. The discussion of decays (typically the derivation of the decay width formula) is then performed by generalizing/modifying the scattering expressions with \textit{ad hoc} arguments. In the textbook of Ref. [2] it is stated that care is needed for the study of decays, since an unstable state cannot be prepared in the far past.

In this work, we present a useful model -called the Lee model- to treat unstable states in general and decays in particular. This model, originally introduced in Ref. [3], is rooted in QM but -since it contains an infinite (and countless) number of states- it mimics many features of QFT [4]. The Lee model is exactly solvable, thus offers a very useful framework to test QFT ideas (it was indeed developed to investigate some properties of renormalization). Within the Lee model, the time evolution of an unstable state can be evaluated from an initial time \((t = 0)\) to any subsequent time.

Intuitively, the Lee model is based on the generalization of the idea of mixing. By denoting \(|S\rangle\) as the quantum ket describing an unstable quantum state or particle and \(|k\rangle\) as one of the (countless) possible final states of the decay, the interacting Hamiltonian is obtained by coupling them according to the schematic form [5, 6, 7, 8, 9]

\[
\text{Interacting part of the Lee Hamiltonian } \sim \sum_{k} g_{k} \left( |S\rangle \langle k| + |k\rangle \langle S| \right).
\]
Above, $g_k$ modulates the coupling of $|S\rangle$ with each $|k\rangle$ and depends on the underlying interaction responsible for the transition/decay. It is then visible that the Hamiltonian is the ‘sum’ of mixing terms responsible for the $|S\rangle \rightarrow |k\rangle$ transition. In Secs. 2 and 3 we shall derive such Hamiltonian step by step.

As a clarifying introductory example, let us consider a very well known phenomenon, the radioactive α decay of Radon into Polonium, already investigated by M. Sklodowska-Curie over one century ago:

$$^{222}_{86}\text{Rn} \rightarrow ^{218}_{81}\text{Po} + \alpha \quad (2)$$

This decay can be understood as the tunnelling of a preformed α particle inside the nucleus through the potential generated by the interplay of short-range nuclear attraction and long-range Coulomb repulsion. It represents a two-body decay, in which the final state is made of an heavy object, the Polonium, and a light object, the α, which are emitted back-to-back. Hence, in this case, $|S\rangle$ represents the Rn at rest, while the state $|k\rangle$ is one of the possible finale states of the system, describing the joint (and entangled) state of one Polonium atom and one α-particle together. [For instance, $k$ may denote the modulus of the three-momentum $\mathbf{k}$ of the emitted α particle (the momentum of Polonium must be then $-\mathbf{k}$ because of momentum conservation)]. The important point is that $k$ is not fixed, but -in principle- any value $k \geq 0$ is admitted. This, in turn, is due to the fact that an unstable particle- such as the Radon- does not have a definite mass, see the discussion below.

For the reasons described above, the Lee model is very versatile and can be applied to quite different systems. In Table 1 we present some recent works that made use of the Lee model in order to study related but somewhat different topics. As it is visible, one may use the Lee model to describe a concrete physical system (such as the decay of a selected resonance) or to study general properties (such as the finite temperature behavior of resonances in a thermal gas). In Sec. 5 we shall come back to this table by presenting more details.

| Topic                              | Ref  |
|-----------------------------------|------|
| Two-channel decay                 | [4]  |
| Moving unstable state, time dilation | [10, 11] |
| Delta resonance                   | 12   |
| X(3872)                           | 13   |
| Finite Temperature                | 14   |
| Broadening of the spectrum        | 15   |
| QZE and IZE (and fundamental issues) | 16   |

Quite remarkably, very similar models were developed in the literature, such as the Jaynes-Cummings model used in Quantum Optics [17, 18, 19] or the so-called Friedrichs model in mathematical physics [20, 21, 22, 23, 24]. Indeed, both the Jaynes-Cummings Hamiltonian and the Friedrichs model are tailor-made for the decay of one excited atom into a ground state atom and one (or more) photon(s). For that reason, they are typically presented by using photon (or bosonic) creation/annihilation operators. Yet, the situation is very similar to the decay of radon mentioned above, upon consider a photon instead of an α particle. More in general, it is possible to recast the Jaynes-Cummings and the Friedrichs models in the form of the Lee model by an appropriate identification of the initial and final states as well as an appropriate choice of the modulating function.

We now turn back to the general study of an unstable state. Both in QM and in QFT, there is a famous formula that describes the survival probability amplitude of an unstable state $|S\rangle$ [1, 25]:

$$a_S(t) = \int_{-\infty}^{+\infty} dS(E) e^{-iEt} dE . \quad (3)$$
Then, the survival probability, i.e. the probability that the state did not decay yet at the time $t$, reads $p_S(t) = |a_S(t)|^2$. Note, when Eq. (3) applies, a mathematical consequence is that $a_S(t \to \infty) = 0$, i.e., the Poincaré time is infinite.

The quantity $d_S(E)$ is the energy (or mass) probability distribution for the unstable state: $d_S(E)\,dE$ is the probability that the energy (or mass) of the unstable state $S$ is contained in the interval $(E,E + dE)$. A general -and intuitive- property is its normalization:

$$\int_{-\infty}^{+\infty} d_S(E)\,dE = 1.$$  \hspace{1cm} (4)

This is quite obvious from a physical point of view, since $p_S(0) = 1$ (by construction). Yet its mathematical verification is not trivial.

Two basic and general properties of the function $d_S(E)$ have important consequences: (i) the existence of a low-energy threshold, $d_S(E) = 0$ for $E < m_{th}$, implies that for large times $p_S(t)$ is a power function $t^{-\alpha}$ \[1, 26, 27\]; (ii) the finiteness of the mean energy, $\langle E \rangle = \int_{-\infty}^{+\infty} E\,d_S(E)\,dE < \infty$, implies that $p'_S(0) = 0$ and $p_S(t)$ is flat for short times \[1, 4, 28\]. (If $\langle E^2 \rangle$ is also finite, then $p_S(t) \simeq 1 - t^2/\tau^2$, as often presented in many papers). Deviations from the exponential law has been observed at short times by studying the tunneling of sodium atoms in an optical potential [29]. The experimental verification of deviations from the exponential law at long times has been obtained by investigating the fluorescence decays of organic molecules [30]. Recently, these effects were also verified by using optical waveguides and coherent light states [31].

In this work we intend to prove -in a didactical way and by showing all intermediate steps- both Eqs. (3) and (4) by using the Lee model mentioned above. Indeed, to show these equations in a pure QFT treatment is very hard, since one has to go beyond the typical framework used to do QFT calculations. In this sense, the Lee model offers a clear way to see how QFT works for decays.

Conversely, the exponential decay law $p_S(t) = e^{-\Gamma_{BW}t}$, although never exact, is a very good approximation in many physical cases. The corresponding Breit-Wigner (BW) [32] energy distribution is:

$$d_{SBW}^S(E) = \frac{\Gamma_{BW}}{2\pi} \left[ (E - M_{BW})^2 + \Gamma_{BW}^2/4 \right]^{-1}. \hspace{1cm} (5)$$

Indeed, for many unstable states $d_S(E) \simeq d_{SBW}^S(E)$ (close to $M_{BW}$). Yet, $d_{SBW}^S(E)$ is unphysical since it does not fulfill the two conditions (i) and (ii) mentioned above. By solving the Lee model we can see how the BW function emerges and how to calculate the Breit-Wigner mass $M_{BW}$ and decay width $\Gamma_{BW}$.

In addition, we also recall that a non-exponential behavior at short times implies the existence of the so-called Quantum Zeno Effect (QZE) and the Inverse Zeno Effect (IZE). In the former, the slowing down of the decay due to frequent measurements takes place, while in the latter an acceleration of the decay rate is realized [33, 34, 5, 6, 35, 36, 7, 8, 37]. The experimental results of Ref. [38], based on the same setup of Ref. [29], could verify also the QZE and IZE effects by adding intermediate measurements.

The manuscript is organized according to the following scheme:

Section 2: we discuss a simplified version of the problem. We start from the two-body mixing problem and generalize it obtaining some general formulas in an heuristic framework.

Section 3: we present the Lee model. As a first step the discrete version is discussed, then the continuous (and correct) Lee Hamiltonian as a limiting case of the discrete Lee Hamiltonian is presented. Some subtle mathematical aspects are dealt with care.

Section 4: this section contains the most important results of this work; we show the validity of Eqs. (3) and (4). We shall also present the Feynman rules for the Lee model in analogy with QFT. As a last step, we discuss the Breit-Wigner limit of Eq. (5).

3
Section 5: we discuss some recent works which made use of the Lee model in different research topics and summarize their findings.

Section 6: conclusions and outlooks are outlined.

2. Heuristic presentation of the problem
In the “baby version” of the problem, let us first consider two quantum states: a quantum state $|S\rangle$ (corresponding to the unstable state that we aim to study) and a quantum state $|K\rangle$ subject to the Hamiltonian
\[ H = H_0 + H_1, \]
(6)
where $H_0$ describes the free (non-interacting) part,
\[ H_0 = M_S |S\rangle \langle S| + M_K |K\rangle \langle K| \]
(7)
with energies (or masses) $M_S$ and $M_K$, and $H_1$ describes the ‘interaction’
\[ H_1 = g (|S\rangle \langle K| + |K\rangle \langle S|) . \]
(8)
In this simple case, the interaction term amounts to a mixing of two states, whose strength is controlled by the coupling constant $g$. In matrix form:
\[ H = \begin{pmatrix} \langle S| & \langle K| \end{pmatrix} \begin{pmatrix} M_S & g \\ g & M_K \end{pmatrix} \begin{pmatrix} |S\rangle \\ |K\rangle \end{pmatrix}, \]
(9)
where the matrix
\[ \Omega = \begin{pmatrix} M_S & g \\ g & M_K \end{pmatrix} \]
(10)
has been introduced. The diagonalization of the system is straightforward:
\[ H = E_1 |E_1\rangle \langle E_1| + E_2 |E_2\rangle \langle E_2| \]
(11)
with
\[ \begin{pmatrix} \langle E_1| \\ \langle E_2| \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} |S\rangle \\ |K\rangle \end{pmatrix}, \]
(12)
or
\[ \begin{pmatrix} |S\rangle \\ |K\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix}. \]
(13)
Plugging Eq. (13) into Eq. (6) and requiring the validity of Eq. (11) lead to the mixing angle as function of $g$ and masses:
\[ \tan 2\theta = -\frac{2g}{M_K - M_S}. \]
(14)
The energies $E_1$ and $E_2$ are the eigenvalues of the matrix $\Omega$:
\[ E_1 = M_S \cos^2 \theta + M_K \sin^2 \theta + g \sin 2\theta = \frac{1}{2} \left[ M_0 + M_K - \sqrt{(M_K - M_0)^2 + g^2} \right], \]
(15)
\[ E_2 = M_S \sin^2 \theta + M_K \cos^2 \theta - g \sin 2\theta = \frac{1}{2} \left[ M_0 + M_K + \sqrt{(M_K - M_0)^2 + g^2} \right]. \]
(16)
The state $|S\rangle$ can be written as the superposition of the energy eigenstates:
\[ |S\rangle = \alpha_1 |E_1\rangle + \alpha_2 |E_2\rangle \]
(17)
with \( \alpha_1 = \cos \theta \) and \( \alpha_2 = -\sin \theta \).

Ergo, the “survival probability amplitude” for the state \(| S \rangle\) is:

\[
a_S(t) = \langle S | U(t) | S \rangle = |\alpha_1|^2 e^{-iE_1t} + |\alpha_2|^2 e^{-iE_2t}
\]  

where

\[
U(t) = e^{-iHt}
\]

is the time-evolution operator. Hence the “survival probability” is

\[
p_S(t) = |a_S(t)|^2 = \cos^4 \theta + \sin^4 \theta + 2 \cos^2 \theta \sin^2 \theta \cos ((E_2 - E_1)t).
\]  

It is clear that after the time \( T = \frac{2\pi}{E_2 - E_1} \) one has \( p_S(T) = 1 \) (in general \( p_S(t+T) = p_S(t) \)). The system is periodic and \( T \) is the Poincaré recurrence time.

Suppose that now, instead of only two states, we have a \( N \)-mixing problem, i.e. we consider the states \{\(| S \rangle, |K_1 \rangle, ..., |K_{N-1} \rangle\} \) with

\[
H = M_S |S \rangle \langle S| + \sum_{j=1}^{N-1} M_K |K_j \rangle \langle K_j| + g_j (|S \rangle \langle K_j| + |K_j \rangle \langle S|).
\]

Thus, there are \( N - 1 \) mixing term, each modelled by the own coupling constant \( g_j \).

(Note, in principle one could also add the mixing terms proportional to \(|K_i \rangle \langle K_j| + |K_j \rangle \langle K_i|\), but this is an unnecessary complication for our purposes.)

By repeating the previous steps one finds the eigenstates \{\(|E_1 \rangle, ..., |E_N \rangle\)\} of the Hamiltonian \( H \), for which

\[
H = \sum_{k=1}^{N} E_k |E_k \rangle \langle E_k|.
\]

Notice that \( k = 1, ..., N \), while in the previous sum \( j = 1, ..., N - 1 \).

The initial state \(| S \rangle\) can be expressed as

\[
| S \rangle = \sum_{k=1}^{N} \alpha_k |E_k \rangle
\]

with the normalization condition

\[
\sum_{k=1}^{N} |\alpha_k|^2 = 1.
\]

Then:

\[
a_S(t) = \langle S | U(t) | S \rangle = \sum_{k=1}^{N} |\alpha_k|^2 e^{-iE_k t}.
\]

Of course, the coefficients \( \alpha_k \) as well as the energies \( E_k \) are function of the parameters of the models: the masses \( M_S \) and \( M_{K,j} \) and the couplings \( g_j \). We do not evaluate them here, since this is the task of the next section. Yet, let us make some simplifying assumptions that allow to illustrate the problem. We assume that

\[
E_k = kb \text{ with } b > 0.
\]

The fact that the minimal energy is \( b > 0 \) is an arbitrary choice (one could anyhow translate the energy to achieve it). In particle physics the minimal energy corresponds to the sum of the rest.
masses of the decay process. Moreover, the maximal energy $E_{\text{max}}$ does not need to be finite. The case $E_{\text{max}} = \infty$ is indeed pretty common.

Then, in this case, it follows that

$$p_S(T = 2\pi/b) = 1, \quad p_S(t + T) = p_S(t).$$

The Poincaré time $T = 2\pi/b$ tends to infinity when $b \to 0$. Thus, in this limit, we really have a genuine “decay” since the original state $|S\rangle$ does not form again at any time $t > 0$. The infinite mixing problem implies that the decay

$$|S\rangle \to |K_j\rangle$$

takes place (where of course all $j$ are in principle admitted).

If we decrease $b$ and increase $N$ such that $Nb = E_{\text{max}}$ (by keeping $E_{\text{max}}$ fixed), the sum of Eq. (25) reduces to an integral:

$$a_S(t) = \langle S | U(t) | S \rangle = \sum_{k=1}^{N} |\alpha_k|^2 e^{-iE_k t} = \sum_{k=1}^{N} \frac{|\alpha_k|^2}{b} e^{-i(kb)t}$$

$$\lim_{b \to 0} = \int_0^{E_{\text{max}}} dmd_S(m)e^{-imt},$$

where the continuous variable $m = kb$ has been introduced and the function $d_S(m)$ is given by

$$d_S(m) = \frac{|\alpha_k=m/b|^2}{b}$$

The normalization condition of Eq. (24) translates into

$$\int_0^{E_{\text{max}}} dmd_S(m) = 1.$$  

Indeed, we have proven in an heuristic way both equations (3) and (4). Even if these arguments do not represent a rigorous proof, they are intuitive and -as a matter of fact- also correct. Our next task is to derive them in a formally correct way.

3. Lee Hamiltonian: definitions and properties

In this Section we introduce the Lee model. We do it in a two-step process. First, we consider the discrete case, and then -as a limiting process of the former- the continuous case. The latter represents the final and correct form of the Lee model.

3.1. Discrete case

The basis of the Hilbert space of our problem is assumed to be given by:

$$|S\rangle : \text{the quantum state corresponding to the unstable state under study},$$

$$|k_n\rangle : \text{an infinite set of quantum states corresponding to decay products}.$$
The state $|S\rangle$ does not represent a stable state (it is not an eigenstate of the Hamiltonian, otherwise the system would be trivial), but transitions to other states are possible. To be more precise, it mixes with an ‘infinity’ of states, denoted as

$$|k_n\rangle \text{ with } k_n = \frac{2n\pi}{L} \text{ and } n = 0, \pm 1, \pm 2, \ldots,$$

(34)

where the quantity $L$ (with the dimension of energy$^{-1}$) can be thought as the dimension of the linear box in which we place our system. $L$ shall be regarded as a large number, and indeed in the end the results should not depend on the box dimension $L$. One already sees that $k_n$ can be interpreted as a ‘momentum’ of the outgoing particles (more details in the following).

Moreover, we consider here only a one-dimensional box: $D = 1$. The extension to a 3D case is straightforward and—in many physical cases which embody spherical symmetry— one can reduce a 3D problem into a 1D problem.

Finally, the whole basis of our quantum problem reads:

Basis of the Hilbert space $\mathcal{H}$: $$\{ |S\rangle, |k_0\rangle, |k_1\rangle, |k_{-1}\rangle, \ldots \} \equiv \{ |S\rangle, |k_n\rangle \}$$

(35)

with the usual orthonormal relations:

$$\langle S|S\rangle = 1 \ , \ \langle S|k_n\rangle = 0 \ , \ \langle k_n|k_m\rangle = \delta_{nm} .$$

(36)

The completeness equation is given by:

$$|S\rangle \langle S| + \sum_n |k_n\rangle \langle k_n| = 1_\mathcal{H} .$$

(37)

The Hamiltonian of the system consists of two pieces and is constructed in a similar way as in our ‘baby’ problem of Sec. 2:

$$H = H_0 + H_1 ,$$

(38)

where $H_0$ describes the free (non-interacting) part:

$$H_0 = M_0 |S\rangle \langle S| + \sum_{n=0,\pm 1 \ldots} \omega(k_n) |k_n\rangle \langle k_n| ,$$

(39)

and where $H_1$ mixes $|S\rangle$ with all $|k_n\rangle$:

$$H_1 = \sum_{n=0,\pm 1 \ldots} \frac{gf(k_n)}{\sqrt{L}} (|S\rangle \langle k_n| + |k_n\rangle \langle S|) .$$

(40)

The following comments are in order:

• All the coefficients $M_0, \omega(k_n), gf(k_n)$ are real.
• The Hamiltonian $H$ is Hermitian.
• Dimensions: $M_0$ and $\omega(k_n)$ have dimensions [energy], while $g$ has dimensions [energy$^{+1/2}$]
• The quantity $M_0$ is the bare energy (or mass) of the state $|S\rangle$. Note, introduce the ‘bare mass’ $M_0$ (instead $M_S$) since a dressing process takes place and the mass of the state $S$ is in general shifted by quantum fluctuations.
• The energy $\omega(k_n)$ is the (bare) energy of the state $|k_n\rangle$. 


• The coupling constant $g$ measures the strength of the interaction; the (dimensionless) form factor $f(k_n)$ modulates the interaction. In practice, each mixing $|S\rangle \leftrightarrow |k_n\rangle$ has its own coupling constant $g f(k_n)$.

• The factor $\sqrt{L}$ is introduced for future convenience: it is necessary for a smooth continuous limit $L \to \infty$.

• For simplicity of notations, $\sum_{n=0, \pm 1, \ldots}$ can be also expressed as $\sum_n$ (or as $\sum_k$).

It is also important to discuss the physical interpretation of the set-up. Thinking in terms of unstable particles, the state $|S\rangle$ represents an unstable particle $S$ in its rest frame. That means, the total momentum of $|S\rangle$ vanishes. The state $|k_n\rangle$ represents a possible final state of the decay of $S$. In the simplest case of a two-body decay, the state $|k_n\rangle$ represents two particles emitted by $S$ and flying back-to-back. What we have in mind is a decay of the type

$$S \to \varphi_1 + \varphi_2 .$$

(41)

In the case of a spacial one-dimensional decay, $k_n$ can be interpreted as the momentum of the first emitted particle, while $-k_n$ is the momentum of the second emitted particle. Schematically:

$$|k_n\rangle \equiv |\varphi_1(k_n), \varphi_2(-k_n)\rangle$$

(42)

In this way, the total three-momentum of $|k_n\rangle$ is still zero, as it must. The 3D extension is straightforward.

As possible and clarifying examples of such a process we mention (see also the Radon example in the introduction):

(i) The neutral pion $\pi^0$ decays into two photons: $\pi^0 \to \gamma \gamma$. Then, $\pi^0$ in its rest frame corresponds to $|S\rangle$, while $\gamma \gamma$ corresponds to $|k_n\rangle$ (one photon has momentum $k_n$, the other $-k_n$).

(ii) An excited atom $A^*$ decays into the-ground state atom $A$ emitting a photon $\gamma$: $A^* \to A \gamma$. In this case, $A^*$ is the sate $|S\rangle$, while $|k_n\rangle$ represents the joint system of the ground-state atom $A$ and the photon.

Clearly, a huge numbers of such examples can be presented. In general, it is not necessary to consider only a two-body decay. It is just simpler doing so for obvious reasons. Yet, the important point is that there is an infinity of states of the type $|k_n\rangle$, one for each $k_n = 2\pi n/L$.

While the Lee Hamiltonian has its own validity even without the present analogy to particle physics, it should be actually stressed that this is more then an analogy. One can namely show that a Quantum Field Theory (under certain approximations) reduces to a Lee Hamiltonian [4].

**Function** $\omega(k_n)$: the function $\omega(k_n)$ represents the energy of the state $|k_n\rangle$. In the case of a two-body particle decay, its form is given by

$$\omega(k_n) = \sqrt{k_n^2 + m_1^2} + \sqrt{k_n^2 + m_2^2} ,$$

(43)

where $m_1$ is the mass of the first particle $\varphi_1$ and $m_2$ of the second particle $\varphi_2$. Clearly:

$$\omega(k_n) \geq m_1 + m_2 .$$

(44)

In the two-photon decay such as the process (i) described above, one has $m_1 = m_2 = 0$, hence

$$\omega(k_n) = 2|k_n| .$$

(45)
In the atomic decay (ii) described above, one has $M_0 = M_A + \Delta, m_1 = 0, m_2 = M_A$, hence:

$$\omega(k_n) \simeq |k_n| + M_A.$$  \hfill (46)

In this case, one could also subtract a constant term, $H \to H - M_A \mathcal{1}_H$. Hence,

$$\omega(k_n) \simeq |k_n|$$  \hfill (47)

holds.

**Linear Lee Model (LLM):** A useful model, that we call ‘Linear Lee model’ (LLM) is obtained by making the choice

$$\omega(k_n) = k_n,$$  \hfill (48)

in which the energy function of the state $|k_n\rangle$ has a simple linear form.

Clearly, the fact that negative values of the energy are admitted makes it fundamentally different from Eq. (43). Namely, for $\omega(k_n) = k_n$ there is no minimal energy of the system, which is clearly an unphysical property.

Even when the masses $m_1$ and $m_2$ vanishes, one has $\omega(k_n) = 2|k_n|$, where the modulus is present. Also, we may consider the limiting case of Eq. (47), $\omega(k_n) \simeq |k_n|$ where -again- the modulus is present. Yet, we might imagine the photon is always emitted to the “right”, hence $\omega(k_n) \simeq k_n > 0$. In this respect, Eq. (48) would represent a good (sometimes extremely good) numerically approximation to this problem, but from a fundamental point of view we still violate basic properties of our original system.

Yet, the LLM has some advantages:

(i) It allows in most cases for simple analytic expressions. The corresponding properties are similar also in the case of an arbitrary (and more realistic) function $\omega(k)$.

(ii) The exponential limit for the decay of $S$ can be nicely obtained for $f(k) = 1$, see details later.

(iii) The LLM is nevertheless not trivial. For an arbitrary $f(k)$, one has the all the nontrivial properties that one expects: nonexponential decay law for short and long times as well as non-trivial scattering of the type $|k_n\rangle \to |k_m\rangle$. In this respect, the LLM model is not a defined model as long as $f(k)$ is not determined. Even if extremely simplified, it describes an infinity of possible models.

3.2. Continuous case

Let us now turn to the continuous Lee model. To this end, we perform the limit $L \to \infty$, implying that the variable $k_n$ becomes continuous:

$$k_n = \frac{2\pi n}{L} \to k \in (-\infty, +\infty).$$  \hfill (49)

This limit is however rather subtle and requires some care. When $L$ is sent to infinity, the sum turns to an integral:

$$\sum_n = \frac{L}{2\pi} \sum_n \frac{2\pi}{L} \to \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk = L \int_{-\infty}^{+\infty} \frac{dk}{2\pi},$$  \hfill (50)

where $\delta k = L/2\pi$ has been introduced in order to generate the differential $dk$.

Yet, the subtle piece is to note that the kets must change. Namely, in terms of continuous variables it must be

$$\langle k_1|k_2 \rangle = \delta(k_1 - k_2).$$  \hfill (51)
To this end, let us write down the following $L$-dependent representation of the Dirac-delta function:

$$
\delta_L(k_n) = \begin{cases} 
0 & \text{for } n \neq 0 \\
\frac{L}{2\pi} & \text{for } n = 0
\end{cases}.
$$

(52)

Clearly:

$$
\delta_L(0) = \frac{L}{2\pi}.
$$

(53)

As a proof that $\delta_L(k_n)$ has the desired properties we evaluate

$$
\sum_n \delta k \delta_L(k_n) = 1 \forall L \rightarrow \sum_n \delta k \delta_L(k_n) u(k_n) = u(0),
$$

(54)

as it should. Hence, in the limit $L \rightarrow \infty$ one obtains:

$$
\sum_n \delta k \delta_L(k_n) u(k_n) \rightarrow \int_{-\infty}^{+\infty} dk \delta(k) u(k) = u(0),
$$

(55)

showing that we have obtained a representation of the Dirac function as:

$$
\delta(k) = \lim_{L \rightarrow \infty} \delta_L(k_n).
$$

(56)

We can verify the results also by using the standard integral representation

$$
\delta_L(k_n) = \int_{-L/2}^{L/2} dx e^{ik_n x} = \begin{cases} 
0 & \text{for } n \neq 0 \\
\frac{L}{2\pi} & \text{for } n = 0
\end{cases}.
$$

(57)

Finally, the link between $|k_n⟩$ and $|k⟩$ is given by:

$$
|k_n⟩ = \sqrt{\frac{2\pi}{L}} |k⟩.
$$

(58)

In fact, in this way:

$$
\langle k_1 | k_2 \rangle = \lim_{L \rightarrow \infty} \frac{L}{2\pi} \langle k_{n_1} | k_{n_2} \rangle = \lim_{L \rightarrow \infty} \left\{ \frac{L}{2\pi} = \delta_L(0) \text{ for } n_1 = n_2 \right\} = \delta(k_1 - k_2)
$$

(59)

as expected.

Note, the extension to $D = 3$ is straightforward:

$$
\sum_k \rightarrow V \int \frac{d^3k}{(2\pi)^3},
$$

(60)

where

$$
V = L^3 \text{ and } \left| \vec{k}_{\text{discrete}} \right| \rightarrow (2\pi)^{3/2} / \sqrt{V} \left| \vec{k}_{\text{cont}} \right|
$$

(61)

is the link between discrete and continuous kets.

It is also quite peculiar that the very dimension of the ket has changed in the passage:

$$
\dim[|k_n⟩] = [\text{Energy}^0] \quad (\text{dimensionless})
\dim[|k⟩] = [\text{Energy}^{-1/2}]
$$

(62)
Then, the continuous Hilbert space is given by:

\[ \mathcal{H} = \{ |S\rangle, |k\rangle \} \]  

(63)

with

\[ \langle S | S \rangle = 1, \quad \langle S | k \rangle = 0, \quad \langle k_1 | k_2 \rangle = \delta (k_1 - k_2) . \]  

(64)

We also check the completeness relation:

\[ \sum_n \delta k \left( \sqrt{\frac{L}{2\pi}} |k_n\rangle \langle k_n| \sqrt{\frac{L}{2\pi}} \right) = 1 \]  

(65)

\[ \sum_{k=\pm\infty} dk |k\rangle \langle k| = 1_{\mathcal{H}} . \]  

(66)

Finally, we are ready to present the Lee Hamiltonian in the continuous limit:

\[ H = H_0 + H_1 \]  

(67)

where

\[ H_0 = M \langle S | S \rangle + \int_{-\infty}^{+\infty} dk \omega(k) |k\rangle \langle k| , \]

\[ H_1 = \int_{-\infty}^{+\infty} dk \frac{g f(k)}{\sqrt{2\pi}} (|S\rangle \langle k| + |k\rangle \langle S|) . \]  

(68)

One can verify that the dimensions are preserved, as it must. For instance:

\[ \text{dim} [dk \omega(k) |k \rangle \langle k|] = \text{dim} [dk] \text{dim}[\omega(k)] \text{dim}^2[|k\rangle] \]  

(69)

\[ = |\text{Energy}| |\text{Energy}| |\text{Energy}^{-1}| = |\text{Energy}| \]  

(70)

[The continuous LLM is obtained upon setting \( \omega(k) = k \). The framework for the study of the time evolution is ready, see Sec. 4.]

As the last step of this subsection, we show how the results can be easily generalized to the 3D case, in which the final state \( |k\rangle \) describes the final state. The Lee Hamiltonian \( H = H_0 + H_1 \) reads:

\[ H_0 = M \langle S | S \rangle + \int d^3k \omega(k) |k\rangle \langle k| , \]

\[ H_1 = \int d^3k \frac{g f(k)}{(2\pi)^{3/2}} (|S\rangle \langle k| + |k\rangle \langle S|) . \]  

(71)

Quite interestingly, the Lee Hamiltonian in the 3D case can be reduced to the 1D case when \( \omega(k) = \omega(k) \) and \( f(k) = f(k) \) with \( k = |k| \). We then introduce the ket \( |k\rangle \) as

\[ |k\rangle \equiv \frac{|k \text{ with } k = |k|\rangle}{\sqrt{4\pi k^2}} . \]  

(72)

In practice, we identify all the different \( |k\rangle \) with the same modulus as a unique state. These states are of course different, but for the purpose of the study of the decay they can be seen as
referring to the same final state. Then:

\[ H_0 = M \langle S | S \rangle + \int_0^{\infty} dk \omega(k) |k\rangle \langle k| \]  
\[ H_1 = \int_0^{\infty} dk \frac{g(k)}{\sqrt{2\pi}} \frac{krF(k)}{\sqrt{2\pi}} (\langle S | k \rangle + \langle k | S \rangle) \]  

which takes the usual 1D form where, however, the modulation function has been rescaled and the integral is confined between 0 and \(\infty\). Indeed, this is formalism applies to the majority of decays (including the example of Radon mentioned in the Introduction), since in reality decays take place in a 3D world. Moreover, this setup guarantees that the decay law is not exponential.

4. Survival amplitude, propagator, and spectral function

4.1. Time-evolution operator

The Schrödinger equation (in natural units) can be univocally solved for a certain given initial state

\[ i \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle \]  

\[ |\psi(0)\rangle = \beta_S |S\rangle + \sum_n \beta_n |k_n\rangle \overset{L \to \infty}{=} \beta_S |S\rangle + \int_{-\infty}^{+\infty} dk \beta(k) |k\rangle \]  

with:

\[ \beta(k) \overset{L \to \infty}{=} \sqrt{\frac{L}{2\pi}} \beta_n \]  

Hence:

\[ 1 = |\beta_S|^2 + \sum_n |\beta_n|^2 \overset{L \to \infty}{=} |\beta_S|^2 + \int_{-\infty}^{+\infty} dk |\beta(k)|^2 \]  

In particular, we shall be interested to the case \(\beta_S = 1\) and \(\beta_n = \beta(k) = 0\).

A formal solution to the time evolution is obtained by introducing the time-evolution operator:

\[ U(t) = e^{-iHt} \]  

out of which

\[ |\psi(t)\rangle = U(t) |\psi(0)\rangle \]  

The time-evolution operator can be expressed in terms of a Fourier transform:

\[ U(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \frac{1}{E - H + i\varepsilon} e^{-iEt} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dEG(E)e^{-iEt} \]  

for \(t > 0\)

where \(\varepsilon\) is an infinitesimal number and where the ‘propagator operator’

\[ G(E) = \frac{1}{E - H + i\varepsilon} \]  

has been introduced. For \(t > 0\) one can formally close the integral in the lower half complex plane. In fact:

\[ e^{-iEt} = e^{-it} \text{Re} E e^{t} \text{Im} E \]
means that for \( t > 0 \) one should consider \( \text{Im} E < 0 \) in such a way that the \( e^{-iEt} \) goes to zero when \( |E| \to \infty \). Then, the residue theorem assures that Eq. (81) is correct. This equality holds also at the level of operators since it is valid for any eigens state of \( H \) (see the next section for an explicit example).

The propagator operator \( G(E) \) can be expanded as:

\[
G(E) = \frac{1}{E - H + i\varepsilon} = \frac{1}{E - H_0 - H_1 + i\varepsilon} \\
= \frac{1}{(E - H_0 + i\varepsilon)} \left( 1 - \frac{1}{E - H_0 + i\varepsilon} H_1 \right) \\
= \frac{1}{(1 - \frac{1}{E - H_0 + i\varepsilon} H_1)} (E - H_0 + i\varepsilon) \\
= \sum_{n=0}^{\infty} \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right)^n \frac{1}{E - H_0 + i\varepsilon} \tag{84}
\]

where we have used that \((AB)^{-1} = B^{-1}A^{-1}\) (\(A, B\) arbitrary operators on \(\mathcal{H}\)).

4.2. Survival probability’s amplitude of \( S \)

We are interested in the evaluation of the survival probability amplitude

\[
a_S(t) = \langle S | U(t) | S \rangle \ . \tag{85}
\]

out of which the survival probability of the state \( S \) reads:

\[
p_S(t) = |a_S(t)|^2 \ . \tag{86}
\]

Trivial limit: Let us first consider a trivial example: \( H = H_0 \) (this corresponds to the limit \( g \to 0\), no interaction and no decay).

Way 1:

\[
a_S(t) = \langle S | U(t) | S \rangle = \langle S | e^{-iH_0t} | S \rangle = e^{-iM_0t} \ \tag{87}
\]

\[
p_S(t) = 1 \ \text{(stable state)}. \tag{88}
\]

Way 2:

\[
a_S(t) = \langle S | U(t) | S \rangle = \langle S | i \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE \frac{1}{E - H_0 + i\varepsilon} e^{-iEt} | S \rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \frac{1}{E - M_0 + i\varepsilon} e^{-iEt} \ . \tag{89}
\]

The latter integral can be solved by using the Jordan lemma (close down and pick up the pole for \( E = M_0 - i\varepsilon \). Note, as discussed above, one is obliged to close down for \( t > 0 \)). Then, by using the residue theorem we obtain:

\[
a_S(t) = \frac{i}{2\pi} (-1)2\pi i e^{-i(M_0-\varepsilon)t} \ , \tag{90}
\]

where the extra-factor \((-1)\) comes from the fact that the path is followed clockwise. Finally, by sending \( \varepsilon \to 0 \) we get the expected result:

\[
a_S(t) = e^{-iM_0t} \to p_S(t) = 1 \ . \tag{91}
\]
In passing by, we note that the object
\[ G^\text{free}_S(E) = G^{(0)}_S(E) = \langle S | \frac{1}{E - H_0 + i\varepsilon} | S \rangle = \frac{1}{E - M_0 + i\varepsilon} \] (92)
is called the free propagator of the state \( S \).

**Evaluation of \( a(t) \) in the full case.** In the full case one proceeds as follow. The survival amplitude \( a_S(t) \) takes the form
\[ a_S(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE G_S(E) e^{-iEt} \] (93)
where
\[ G_S(E) = \langle S | G(E) | S \rangle = \langle S | \frac{1}{E - H + i\varepsilon} | S \rangle \] (94)
is the full propagator of \( S \).

It is now necessary to evaluate \( G_S(E) \) explicitly. As a first step, we use Eq. (84) obtaining the expansion:
\[ G_S(E) = \langle S | G(E) | S \rangle = \sum_{n=0}^{\infty} \langle S \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right)^n \frac{1}{E - H_0 + i\varepsilon} S \rangle = \sum_{n=0}^{\infty} G^{(n)}_S(E) \] (95)
with
\[ G^{(n)}_S(E) = \langle S \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right)^n \frac{1}{E - M_0 + i\varepsilon} \rangle \] (96)

Let us evaluate the first three terms:
\[ n = 0 \to G^{(0)}_S(E) = \langle S | 1 | S \rangle \frac{1}{E - M_0 + i\varepsilon} = \frac{1}{E - M_0 + i\varepsilon} , \] (97)
\[ n = 1 \to \langle S \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right) \frac{1}{E - M_0 + i\varepsilon} | S \rangle = 0 , \] (98)
\[ n = 2 \to G^{(1)}_S(E) = \langle S \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right)^2 \frac{1}{E - M_0 + i\varepsilon} \rangle \] (99)
\[ = \frac{1}{E - M_0 + i\varepsilon} \langle S | H_1 \frac{1}{E - H_0 + i\varepsilon} H_1 | S \rangle \frac{1}{E - M_0 + i\varepsilon} = \frac{1}{(E - M_0 + i\varepsilon)^2} \] (100)

The recursive quantity is \( \Pi(E) \):
\[ \Pi(E) = - \langle S | H_1 \frac{1}{E - H_0 + i\varepsilon} H_1 | S \rangle . \] (102)

We introduce \( 1_H = | S \rangle \langle S | + \int_{-\infty}^{+\infty} dk |k\rangle \langle k| \) two times, obtaining:
\[ \Pi(E) = - \langle S | H_1 1_H \frac{1}{E - H_0 + i\varepsilon} 1_H H_1 | S \rangle = \] (103)
\[ = - \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dq \langle S | H_1 |k\rangle \langle k| \frac{1}{E - H_0 + i\varepsilon} |q\rangle \langle q| H_1 | S \rangle \] (103)
\[ = - \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dq \frac{g f(k)}{\sqrt{2\pi}} \delta(k - q) \frac{g f(q)}{E - \omega(k) + i\varepsilon} \sqrt{2\pi} \] (103)
\[ = - \int_{-\infty}^{+\infty} dk \frac{g^2 f(k)^2}{2\pi E - \omega(k) + i\varepsilon} = g^2 \Sigma(E) . \] (104)
where we have used that

$$\langle S | H_1 | k \rangle = \frac{g f(k)}{\sqrt{2\pi}}. \quad (105)$$

Summarizing:

$$\Pi(E) = g^2 \Sigma(E) = -\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{g^2 f(k)^2}{E - \omega(k) + i\varepsilon}. \quad (106)$$

Going further, we get:

$$G_S^{(3)}(E) = \langle S | \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right)^3 | S \rangle \frac{1}{E - M_0 + i\varepsilon} = \frac{1}{E - M_0 + i\varepsilon} \langle S | H_1 | E - H_0 + i\varepsilon | H_1 \rangle \frac{1}{E - H_0 + i\varepsilon} \frac{1}{E - H_0 + i\varepsilon} | S \rangle \frac{1}{E - M_0 + i\varepsilon}$$

$$= 0. \quad (107)$$

Namely, one can again insert 1_{H_1}, but an odd number of $H_1$ implies that this amplitude vanishes. In general:

$$G_S^{(2n+1)}(E) = 0, \ n = 0, 1, 2, \ldots. \quad (108)$$

Next, by properly inserting 1_{H_1} two times:

$$G_S^{(4)}(E) = \langle S | \left( \frac{1}{E - H_0 + i\varepsilon} H_1 \right)^4 | S \rangle \frac{1}{E - M_0 + i\varepsilon}$$

$$= \frac{1}{(E - M_0 + i\varepsilon)^2} \left[ \int_{-\infty}^{+\infty} dk_1 \frac{g f(k_1)}{\sqrt{2\pi}} \frac{1}{E - \omega(k_1) + i\varepsilon} \frac{1}{\sqrt{2\pi}} \right] \frac{1}{E - M_0 + i\varepsilon} \times \left[ \int_{-\infty}^{+\infty} dk_2 \frac{g f(k_2)}{\sqrt{2\pi}} \frac{1}{E - \omega(k_2) + i\varepsilon} \frac{1}{\sqrt{2\pi}} \right]$$

$$= \frac{\Pi(E)^2}{(E - M_0 + i\varepsilon)^2}. \quad (109)$$

Putting all the pieces together:

$$G_S^{(2n)}(E) = \frac{[-\Pi(E)]^n}{(E - M_0 + i\varepsilon)^{n+1}}. \quad (110)$$

Finally:

$$G_S(E) = \sum_{n=0}^{\infty} G_S^{(n)}(E) = \sum_{n=0}^{\infty} G_S^{(2n)}(E) = \sum_{n=0}^{\infty} \frac{[-\Pi(E)]^n}{(E - M_0 + i\varepsilon)^{2n+1}}$$

$$= \frac{1}{(E - M_0 + i\varepsilon)} \sum_{n=0}^{\infty} \frac{[-\Pi(E)]^n}{(E - M_0 + i\varepsilon)^n} = \frac{1}{(E - M_0 + i\varepsilon)} \frac{1}{1 + \frac{\Pi(E)}{E - M_0 + i\varepsilon}}$$

$$= \frac{1}{E - M_0 + i\varepsilon}. \quad (113)$$
Figure 1. Schematic presentation of the sum leading to the dressed propagator. In the last part the Bethe-Salpeter resummation is depicted.

The sum in Eq. (113) is shown in Fig. 1. It is interesting to notice that the very same result can be obtained in an elegant way by using the Bethe-Salpeter formalism [39] (see also Fig. 1):

\[
G_S(E) = \frac{1}{E - M_0 + i\varepsilon} - \frac{1}{E - M_0 + i\varepsilon} \Pi(E) G_S(E) ,
\]

hence

\[
G_S(E) \left( 1 + \frac{\Pi(E)}{E - M_0 + i\varepsilon} \right) = \frac{1}{E - M_0 + i\varepsilon} .
\]

then

\[
G_S(E) = \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} .
\]

At this point we can identify Feynman rules for the Lee model:

\[
\begin{align*}
\text{bare } S & \text{ propagator } \rightarrow \frac{1}{E - M_0 + i\varepsilon} ; \\
\text{bare } k & \text{ propagator (}k\text{ fixed)} \rightarrow \frac{1}{E - \omega(k) + i\varepsilon} ; \\
\text{kS vertex } & \rightarrow gf(k) ; \\
\text{internal } k & \text{ line (}k\text{ not fixed)} \rightarrow -\Pi(E) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{g^2 f(k)^2}{E - \omega(k) + i\varepsilon} .
\end{align*}
\]

Note, the latter expression can be understood as resulting from \( gf(k) \) at each vertex, by the \( k \)-propagator in the middle, and by an overall integration \( \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \) due to the fact that \( k \) is not fixed (just as a loop in QFT).

Finally, the survival amplitude can be expressed as

\[
a_S(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE G_S(E) e^{-iEt} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} e^{-iEt} .
\]

This expression is an important intermediate result for the study of time evolution of the unstable state \( S \), but it is not yet in the desired form of Eq. (3). In order to achieve that, additional steps are required.
4.3. Definition of the spectral function

Let us denote the basis of eigenstates of the Hamiltonian $H$ as $|m\rangle$ with

$$H |m\rangle = m |m\rangle \quad \text{for} \quad m \geq m_{th} \quad (\text{the low-energy threshold}) \ .$$

The existence of a minimal energy $m_{th}$ is a general physical requirement. The states $|m\rangle$ form an orthonormal basis of the Hilbert space

$$\mathcal{H} = \{|m\rangle \quad \text{with} \quad m \geq m_{th}\} \ .$$

The standard relations hold:

$$1_\mathcal{H} = \int_{m_{th}}^{+\infty} dm \ |m\rangle \langle m| \ ,$$

$$\langle m_1 | m_2 \rangle = \delta(m_1 - m_2) \ .$$

The link between the old basis $\{|S\rangle, |k\rangle\}$ (the eigenstates of $H_0$) and the new basis $\{|m\rangle\}$ (the eigenstates of $H$) is not trivial. Yet, for our purposes, the only requirement is that this basis of eigenstates of $H$ exists. Indeed, this property follows from the fact that the Hamilton operator is Hermitian.

Now, the state $|S\rangle$ can be expressed in terms of the basis $\{|m\rangle\}$ as

$$|S\rangle = \int_{m_{th}}^{\infty} \alpha_S(m) |m\rangle \quad \text{with} \quad \alpha_S(m) = \langle S|m\rangle \ .$$

The quantity

$$d_S(m) = |\alpha_S(m)|^2 = |\langle S|m\rangle|^2$$

is called the spectral function (or energy/mass distribution) of the state $S$ (in agreement with the heuristic discussion of Sec. 2).

The normalization of the state $|S\rangle$ implies the normalization of $d_S(m)$:

$$1 = \langle S|S\rangle = \int_{m_{th}}^{\infty} d_S(m) dm \ .$$

The simple intuitive interpretation is that $d_S(m) dm$ represents the probability that the state $S$ has an energy (or mass) between $m$ and $m + dm$.

As a consequence, the time-evolution can be easily evaluated by inserting $1_\mathcal{H} = \int_{-\infty}^{+\infty} dm \ |m\rangle \langle m|$ two times:

$$a_S(t) = \langle S| U(t) |S\rangle = \langle S| e^{-iHt} |S\rangle = \int_{m_{th}}^{\infty} d_S(m) e^{-imt} \ .$$

This is all “general, nice, and beautiful”, but does not help us further as long as we do not know how to calculate $d_S(m)$ in the framework of the Lee model (or of any other model that we might use). This is fortunately possible by using the propagator

$$G_S(E) = \frac{1}{E - M_0 + \Pi(E) + i\epsilon} \ ,$$

which can be re-expressed as (inserting $1_\mathcal{H} = \int_{m_{th}}^{+\infty} dm \ |m\rangle \langle m|$ two times):

$$G_S(E) = \langle S| \frac{1}{E - H + i\epsilon} |S\rangle = \langle S| 1_\mathcal{H} \frac{1}{E - H + i\epsilon} 1_\mathcal{H} |S\rangle$$

$$= \int_{m_{th}}^{\infty} dm_1 \int_{m_{th}}^{\infty} dm_2 \langle m_1 | \frac{1}{E - H + i\epsilon} |m_2\rangle \alpha_S^*(m_1) \alpha_S(m_2)$$

$$= \int_{m_{th}}^{+\infty} dm \frac{d_S(m)}{E - m + i\epsilon} \ .$$
Then, we obtain
\[
G_S(E) = \frac{1}{E - M_0 + \Pi(E)} + i\varepsilon = \int_{m_{th}}^{+\infty} \frac{dS(m)}{E - m + i\varepsilon} .
\] (132)

Eq. (132) can be considered as the formal definition of the spectral function \(dS(m)\). Its physical meaning can be understood by noticing that the dressed propagator \(G_S(E)\) has been rewritten as the ‘sum’ of free propagators, whose weight function is \(dS(m)\). (In fact, \(\left| S \right\rangle\) is not an eigenstate of \(H\) as soon as \(H_1 \neq 0\)). Moreover, we expect that \(dS(m) \geq 0\) and that the normalization
\[
\int_{-\infty}^{+\infty} dm dS(m) = 1
\] (133)
holds. The proof of the latter is indeed not trivial when starting from Eq. (132) (see the next subsection), but its physical and intuitive justifications should be evident.

Let us turn to the evaluation of \(dS(E)\). First, let us consider the case \(g = 0\). In this limit, it is clear that:
\[
dS(E) = \delta(E - M_0) .
\] (134)
Namely, if the state \(\left| S \right\rangle\) is an eigenstate of the Hamiltonian, the mass distribution is a delta-function peaked at \(M_0\).

When the interaction is switched on, we evaluate the imaginary part of Eq. (132):
\[
\text{Im} G_S(E) = \int_{-\infty}^{+\infty} \frac{-\varepsilon}{(E - m)^2 + \varepsilon^2} dS(m) = -\pi dS(E).
\] (135)

Hence:
\[
dS(E) = -\frac{\text{Im} G_S(E)}{\pi} = \frac{1}{\pi (E - M_0 + \text{Re} \Pi(E))^2 + (\text{Im} \Pi(E))^2}.
\] (136)
This is a very important result of the present work, since it links the spectral function \(dS(E)\) in terms of the (calculable) propagator \(G_S(E)\).

Once the spectral function \(dS(E)\) is known, the survival amplitude can be expressed as its Fourier transform:
\[
a_S(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE G_S(E) e^{-iEt} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} \frac{dS(m)}{E - m + i\varepsilon} e^{-iEt}
\]
\[
= \int_{-\infty}^{+\infty} dS(m) dm \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \frac{1}{E - m + i\varepsilon} e^{-iEt} = \int_{-\infty}^{+\infty} dm dS(m) e^{-imt}
\]
\[
= \int_{-\infty}^{+\infty} dEdS(E) e^{-iEt}.
\] (137)

This is Eq. (3) what we wanted to show: q.e.d.

### 4.4. Proof of the normalization of the spectral function

We now prove that the spectral function \(dS(E)\) calculated through Eq. (136) is correctly normalized to 1. Of course, this is compelling since \(a_S(0) = 1\) is the starting point of our analysis. Yet, the mathematical proof presented below (and based on Ref. [40]) requires some care.

First, we note that a low-energy threshold \(m_{th}\) (hence a minimal energy) is present in all physical system
\[
\text{Im} \Pi(E) = 0 \text{ for } E < m_{th}
\] (138)
(th stays for threshold).

Then, we first show the normalization under the assumption of a ‘strong’ requirement:

\[ \text{Im } \Pi(E) = 0 \text{ for } E > \Lambda . \]  \hspace{1cm} (139)

This is not valid in general, but it allows for a simpler proof of the normalization of \( d_S(E) \). The real part of the loop \( \text{Re } \Pi(E) \) can be calculated from the dispersion relation

\[ \text{Re } \Pi(E) = \frac{1}{\pi} PP \int_{m_{th}}^{\infty} \frac{\text{Im } \Pi(m)}{E - m} dm = \frac{1}{\pi} PP \int_{m_{th}}^{\Lambda} \frac{\text{Im } \Pi(m)}{E - m} dm , \]  \hspace{1cm} (140)

where \( PP \) stays for principal part, out of which one can see that \( \text{Re } \Pi(E) \) goes to zero as \( 1/E \) for \( E \gg \Lambda \). Hence, taking the limit \( E \to \infty \), one gets

\[ \lim_{E \to \infty} \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} = \frac{1}{E} \]  \hspace{1cm} (141)

\[ = \lim_{E \to \infty} \int_{m_{th}}^{\Lambda} \frac{dS(m)}{E - m + i\varepsilon} = \frac{1}{E} \int_{m_{th}}^{\Lambda} dmdS(m) \]  \hspace{1cm} (142)

Then:

\[ \int_{m_{th}}^{\Lambda} dmdS(m) = 1 \]  \hspace{1cm} (143)

follows.

Now, we release the ‘strong’ assumption (139), but we assume that \( \text{Im } \Pi(E) \) goes to zero sufficiently fast as function of \( E \) for \( E \to \infty \). We rewrite:

\[ \int_{m_{th}}^{\infty} \frac{dS(m)}{E - m + i\varepsilon} = \int_{m_{th}}^{\sqrt{M_0E}} \frac{dS(m)}{E - m + i\varepsilon} + \int_{\sqrt{M_0E}}^{\infty} \frac{dS(m)}{E - E' + i\varepsilon} = I_1 + I_2 . \]  \hspace{1cm} (144)

We have divided the integral into two pieces by setting the division at \( \sqrt{M_0E} \). The result, of course, does not depend on this choice (if on takes, for instance, \( 2\sqrt{M_0E} \), nothing changes). This separation is useful. Namely, the large-\( E \) limit of the first integral is easily taken, because no pole is present in the integration (in fact, \( E \) is surely larger than \( \sqrt{M_0E} \) in the large-\( E \) limit):

\[ \lim_{E \to \infty} I_1(E) = \lim_{E \to \infty} \int_{m_{th}}^{\sqrt{M_0E}} dm \frac{dS(m)}{E - m + i\varepsilon} = \frac{1}{E} \int_{m_{th}}^{\infty} dmdS(m) \]  \hspace{1cm} (145)

Then, the second integral takes the form:

\[ I_2(E) = \int_{\sqrt{M_0E}}^{\infty} dm \frac{dS(m)}{E - m + i\varepsilon} = \int_{\sqrt{M_0E}}^{\infty} dm \frac{1}{E - m + i\varepsilon} \frac{1}{\pi} \frac{\text{Im } \Pi(m)}{(m - M_0 + \text{Re } \Pi(m))^2 + (\text{Im } \Pi(m))^2} . \]  \hspace{1cm} (146)

It is then clear that \( \text{Im } I_2 = d_S(E) \), which is very small for large \( E \). Next, the real part of \( I_2 \) reads

\[ \text{Re } I_2(E) = PP \int_{\sqrt{M_0E}}^{\infty} dm \frac{1}{\pi} \frac{1}{E - m + i\varepsilon} \frac{\text{Im } \Pi(m)}{(m - M_0 + \text{Re } \Pi(E'))^2 + (\text{Im } \Pi(E'))^2} . \]  \hspace{1cm} (147)

We assume that \( \text{Im } \Pi(m) \) goes to zero sufficiently fast for \( m \to \infty \) in such a way that \( \text{Re } I_2 \) vanishes. One has:

\[ \lim_{E \to \infty} I_2(E) = 0 . \]  \hspace{1cm} (148)
Finally:

\[ \int_{m_{th}}^{\infty} m \, dm \, dS(m) = \int_{m_{th}}^{\infty} dE \, dS(E) = \int_{-\infty}^{+\infty} dE \, dS(E) = 1 \]  

(149)
is proven, which corresponds to our second goal mentioned in the introduction, the verification of Eq. (4): q.e.d.

### 4.5. The Breit-Wigner limit

As a last point we discuss the Breit-Wigner limit [19, 32]. To this end, we use the LLM discussed in Sec. 3, \( \omega(k) = k \), together with the modulation function

\[ f(k) = \theta(M_0 + \Lambda - k)\theta(k - (M_0 - \Lambda)). \]

(150)

In this way, the unstable state \( |S\rangle \) couples in a limited window of energy to the final states of the type \( |k\rangle \) (see also [15] for details).

The self-energy \( \Sigma(E) \) reads

\[ \Sigma(E) = \frac{g^2}{2\pi} \ln \left( \frac{E - M_0 + \Lambda}{E - M_0 - \Lambda} \right), \]

(151)

whose real and imaginary parts are

\[ \text{Re} \, \Sigma(E) = \frac{g^2}{2\pi} \ln \left| \frac{E - M_0 + \Lambda}{E - M_0 - \Lambda} \right|, \]

(152)

\[ \text{Im} \, \Sigma(E) = \begin{cases} \frac{g^2}{2} & \text{for } M_0 - \Lambda < E < M_0 + \Lambda \\ 0 & \text{otherwise} \end{cases}. \]

(153)

When \( \Lambda \) is not infinite, deviations both at short and long times occur. Yet, in the limit \( \Lambda \to \infty \) one recovers the pure exponential decay. Namely:

\[ \text{Re} \, \Sigma(E) = 0 \]  
\[ \text{Im} \, \Sigma(E) = \frac{g^2}{2} \] for each \( E \).

(154)

The propagator reduces exactly to the BW form

\[ G_S(E) = \frac{1}{E - M_{BW} + i\Gamma_{BW}/2} \]

(155)

with

\[ M_{BW} = M_0 \] and \( \Gamma_{BW} = g^2. \)

(156)

The survival probability amplitude of the state \( |S\rangle \) is also in this case the usual exponential form

\[ a_S(t) = \langle S | e^{-iHt} | S \rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE G_S(E)e^{-iEt} = \]

\[ = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \frac{1}{E - M_{BW} + i\Gamma_{BW}/2} e^{-iEt} = e^{-i(M_{BW} - i\Gamma_{BW}/2)t} \]

(157)

where the pole \( E = M_{BW} - i\Gamma_{BW}/2 \) is picked up when performing the integration.

The spectral function reads

\[ d_S(E) = -\frac{\text{Im} \, G_S(E)}{\pi} = \frac{1}{2\pi} \frac{\Gamma_{BW}}{(E - M_{BW})^2 + \Gamma_{BW}^2/4} = d_{BW}^S(E), \]

(158)
i.e. the usual Breit-Wigner form introduced already in the introduction. The survival probability amplitude can be also calculated by using Eq. (3) obtaining

\[ a_s(t) = \langle S | e^{-iHt} | S \rangle = \int_{-\infty}^{+\infty} dE dS_{BW}(E)e^{-iEt} = e^{-i(M_{BW}-i\Gamma_{BW}/2)t}, \] (159)

out of which

\[ p_S(t) = e^{-\Gamma_{BW}t}. \] (160)

Quite interestingly, in the BW case it is also possible to evaluate the time-evolution operator applied to the unstable state \(|S⟩[15]:\n
\[ e^{-iHt} |S⟩ = e^{-i(M_{BW}-i\Gamma_{BW}/2)t} |S⟩ + \int_{-\infty}^{+\infty} db(k, t) |k⟩ \] (161)

with

\[ b(k, t) = \frac{g}{\sqrt{2\pi}} \frac{e^{-ikt} - e^{-i(M_{BW}-i\Gamma_{BW}/2)t}}{k - M_{BW} + i\Gamma_{BW}/2}. \] (162)

Obviously, the probability that the decay has occurred is

\[ w(t) = \int_{-\infty}^{+\infty} dk |b(k, t)|^2 = 1 - p(t) = 1 - e^{-\Gamma_{BW}t}. \] (163)

In the end, note that in the BW limit we could describe the evolution of the state \(|S⟩\) by an “effective” non-Hermitian Hamiltonian

\[ H_{eff,S} = \left( M_0 - \frac{i\Gamma}{2} \right) |S⟩⟨S|. \] (164)

Yet, such as an expression -although useful in some cases- should be regarded with due care.

4.6. Breit-Wigner approximation: mass and width

Let us now consider the case in which we do not have exactly a BW spectral function, but it is still possible to show how the latter emerges as an approximation.

Let us consider the propagator

\[ G_S(E) = \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} = \frac{1}{E - M_0 + \text{Re} \Pi(E) + i \text{Im} \Pi(E) + i\varepsilon}. \] (165)

The (renormalized) nominal BW mass of the state \(|S⟩\) is defined as the solution of the

\[ M_{BW} - M_0 + g^2 \text{Re} \Sigma(M_{BW}) = 0. \] (166)

By expanding the real part of \(G_{S}^{-1}(E)\) around \(M_{BW}\), we obtain

\[
G_S(E) \approx \frac{1}{(E - M_{BW} + \Pi(E) + i\varepsilon)} \frac{1}{(1 + g^2 \left( \frac{\partial \text{Re} \Sigma(E)}{\partial E} \right)_{E=M_{BW}} + ... + i \text{Im} \Pi(E) + i\varepsilon)} \\
\approx \frac{1}{1 + g^2 \left( \frac{\partial \text{Re} \Sigma(E)}{\partial E} \right)_{E=M_{BW}} + i \Gamma_{BW}} \frac{1}{E - M_{BW} + i\Gamma_{BW}}
\] (167)
Hence, the Breit-Wigner approximation of the propagator emerges as

$$G_{BW}^S(E) = Z_{BW} \frac{1}{E - M + i \Gamma_{BW}/2}$$

(168)

where

$$Z_{BW} = \left(1 + g^2 \left( \frac{\partial \text{Re} \Sigma(E)}{\partial E} \right)_{E=M_{BW}} \right)^{-1}$$

(169)

is the normalization constant. The decay width $\Gamma_{BW}$ is given by (an extension of) the Fermi golden rule

$$\Gamma_{BW} = \frac{g^2}{1 + g^2 \left( \frac{\partial \text{Re} \Sigma(E)}{\partial E} \right)_{E=M_{BW}}} \text{Im} \Sigma(M_{BW})$$

(170)

$$= \frac{g^2}{1 + g^2 \left( \frac{\partial \text{Re} \Sigma(E)}{\partial E} \right)_{E=M}} \left(\frac{d}{dk}\right)_{k=k_M} f^2(k_M)$$

where $k_M$ is given by

$$\omega(k_M) = M_{BW}.$$  

(171)

Using the approximation in Eq. (168), the survival probability amplitude is given by

$$a_S(t) = \langle S | e^{-iHt} | S \rangle = e^{-i(M_{BW} - i\Gamma_{BW}/2)t}$$

(172)

and the survival probability takes the usual form $p_S(t) = |a_S(t)|^2 \simeq |Z_{BW}|^2 e^{-\Gamma_{BW}t}$. One can then also see that the exponential limit is recovered, but there is a constant $|Z_{BW}|^2$ which differs from 1 in front of it (this fact also implies that for short times deviations from the exponential decay are present; for a detailed discussion of this point see Ref. [7]).

Very often (see e.g. Ref. [41] and refs. therein) one extends the propagator to the complex plane upon considering $E \to z \in \mathbb{C}$:

$$G(z) = \frac{1}{z - M_0 + \Pi(z)}$$

(173)

where the loop function on the complex plane reads

$$\Pi(z) = \frac{1}{\pi} \int_{m_{th}}^{\infty} \frac{\text{Im} \Pi(m)}{m - z} \, dm.$$  

(174)

Next, one searches for the pole(s) of $G(z)$ in the complex plane in the II-Riemann sheet

$$z_{\text{pole}} - M_0 + \Pi_{II}(z_{\text{pole}}) = 0$$

(175)

where the loop on the second Riemann sheet is given by:

$$\Sigma_{II}(z) = \Sigma(z) + 2i \text{Im} \Sigma(z),$$

(176)

with $\text{Im} \Sigma(z)$ being the imaginary part of the loop analytically continued to the whole complex plane.

Typically, there is one dominating pole close to the real axis, for which the mass $M_{\text{pole}}$ and the width $\Gamma_{\text{pole}}$ of the unstable state are defined as

$$z_{\text{pole}} = M_{\text{pole}} - i\Gamma_{\text{pole}}/2.$$  

(177)
When considering $z$ close to the pole one can write

$G_S(z) \simeq \frac{Z_{\text{pole}}}{z - z_{\text{pole}}}$  \hspace{1cm} (178)

where $Z_{\text{pole}}$ is the residue of the pole. The evaluation of the survival probability amplitude under the assumption that a single pole dominates leads to

$$a_S(t) = \langle S | e^{-iHt} | S \rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE G_S(E) e^{-iEt} =$$

$$\simeq \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE \frac{Z_{\text{pole}}}{E - z_{\text{pole}}} e^{-iEt} = Z_{\text{pole}} e^{-i(M_{\text{pole}} - i\Gamma_{\text{pole}}/2)t} \hspace{1cm} (179)$$

hence $p_S(t) \simeq |Z_{\text{pole}}|^2 e^{-\Gamma_{\text{pole}}t}$. The form is identical to the BW one, but the numerical results for masses and decays are not exactly equal (they converge to the same values in the small-width limit).

The pole mass and width are typically preferable from a theoretical point of view than the BW mass and width since the position of the pole is process independent [41]. Yet, both of them are commonly used in practice [42].

5. Applications of the Lee model

The Lee model has been commonly employed to describe various problems in different area of physics. In connection to decays of quantum states and their connection to the QZE and IZE effects, it has been used in e.g. Refs. [5, 6, 7, 8, 21, 35, 36] and references therein.

Already in the Introduction we have introduced Table 1, in which recent and quite different applications of the Lee model have been listed. Some comments and details concerning those works are in order:

(i) In the first entry of Table 1 the extension to two decay channel is mentioned. This is quite important since the majority of unstable states has more than a single decay channel. The extension of the Lee model in this case is simple: we couple the state $|S\rangle$ to two sets of final states $|k, 1\rangle$ and $|k, 2\rangle$. The Hamiltonian reads

$$H_0 = M |S\rangle \langle S| + \sum_{i=1,2} \int_{-\infty}^{+\infty} dk \omega_i(k) |k, i\rangle \langle k, i| , \quad H_1 = \sum_{i=1,2} \int_{-\infty}^{+\infty} dk \frac{g_i f_i(k)}{\sqrt{2\pi}} (|S\rangle \langle k, i| + \text{h.c.}) \hspace{1cm} (180)$$

In particular, the BW limit in the LLM (by repeating the steps of Sec. 4.5) implies that the partial decay widths are $\Gamma_1 = g_1^2$ and $\Gamma_2 = g_2^2$ and the survival probability reads $p_S(t) = e^{-\Gamma_{\text{BW}}t}$

with $\Gamma_{\text{BW}} = \Gamma_1 + \Gamma_2$.

In the presence of two decay channels, it is useful to introduce the quantity $h_i(t) = h_i(t)dt$ is the probability that the state $|S\rangle$ decays in the $i$-th channel between $t$ and $t + dt$. In the BW limit $h_i(t) = \Gamma_i e^{-\Gamma t}$ and the ratio $h_1(t)/h_2(t)$ is a constant equal to $\Gamma_1/\Gamma_2$. However, when deviations from the exponential decay are considered, the ratio $h_1(t)/h_2(t)$ is in general not a constant, but shows sizable departures from $\Gamma_1/\Gamma_2$ [4]. In fact, this ratio presents large and irregular oscillations which persist for a long time, even in the regime in which the decay law $p_S(t)$ is very well approximated by an exponential function. In this sense, this ratio represents a novel tool to detect deviations from the exponential decay. For a very recent discussion of this problem by using a QM model, see Ref. [43].

(ii) The decay of the unstable $S$ is calculated -as usual- in the rest frame of the particle $S$. A very interesting question is the evaluation of the survival probability in the case in which the
particle is moving. By denoting $p$ as the modulus of the three-momentum of $S$, one obtains

$$a_p^S(t) = \int_{-\infty}^{+\infty} dS(E)e^{-i\sqrt{E^2+p^2}t}dE.$$  

(181)

For the derivation of this result by using the Lee model see Refs. [10, 11]. (For the discussion of this topic, see also Refs. [44, 45, 46, 47].) In particular, one finds that the usual dilation formula is not reobtained. In the BW limit one finds that

$$|a_p^S(t)|^2 \neq e^{-\frac{E}{\gamma}t}$$  

(182)

where $\gamma = \sqrt{p^2 + M_{BW}^2}/M_{BW}$ is the Lorentz factor (for the explicit expression of $p_S(t)$ in this case, see Ref. [10]). Obviously, $e^{-\frac{E}{\gamma}t}$ is normally used in practice. Indeed, very small deviations from it are present. It should be also stressed that there is no violation of special relativity but that care is needed when an unstable state with nonzero momentum is defined, see details in Ref. [10].

(iii-iv) In the third and fourth entries of Table 1, two resonances are mentioned: the baryon $\Delta$ [12] and the enigmatic $X(3872)$ state [13] (for the role of loops in the latter see also Ref. [48]). In general, one can use similar techniques for any resonance.

(v) Extension to finite temperature. The Lee model can be also used at finite temperature in order to study how to threat unstable resonances in a thermal gas. This has been recently accomplished in Ref. [14] where the so-called ‘phase-shift’ formula for the proper description of resonances at a given temperature could be proven to be exact within the Lee model.

(vi-vii) The Lee model has been utilized in Ref. [15] to study the broadening of the energy spectrum of an unstable state if the measurement is performed early enough. In a further extension, a discussion of fundamental properties -such as the Zeno effect induced by imperfect measurements and the possible connection to the Many World Interpretation of QM-can be found in Ref. [16].

Finally, a remark concerning the connection of the lee model to QFT is necessary. In Ref. [4] the comparison of the Lee model with QFT approaches is presented. In particular, it is shown that the QFT counterpart is given by the interaction Lagrangian

$$\mathcal{L} = gS\varphi^2 ,$$  

(183)

which describes the two-decay process $S \rightarrow \varphi\varphi$ (see [49] for technical details of the QFT treatment). Hence, the field $S$ corresponds to the ket $|S\rangle$ described by the Lee Hamiltonian and the two-state $\varphi\varphi$ corresponds to $|k\rangle$. It is then also possible to verify that deviations form the exponential decay are realized in QFT as well [25]. Yet, even if the Lee model presents many features of QFT, it is not QFT. The issue is that in a genuine QFT approach also transitions of the type $S\varphi^2 \equiv |Sk\rangle \rightarrow |0\rangle$ (the perturbative vacuum) and vice-versa are possible, which are however not included in the Lee model. Moreover, QFT allows for an arbitrary number of $S$ and $\varphi$ fields (and not necessarily 1 and 2, as in the Lee approach). An additional subtle but important problem concerns the identification of the real vacuum of the theory (which is not the perturbative vacuum) [2, 50]. This step is necessary for a proper introduction of an unstable state in QFT. Future developments in this direction can be quite promising.

6. Conclusions

In this work we have described the Lee model by paying attention to many technical details. To this end, we have introduced it as a limiting process of a discrete Lee model. We have shown how the survival decay amplitude can be properly derived within this framework as the Fourier
transform of the spectral function. The latter emerges as the imaginary part of the propagator of the unstable state under study and turns out to be normalized to unity, as we have proven by a detailed analysis.

Moreover, we have also shown how the BW limit emerges as a particular approximation of the spectral function and how the BW mass and widths are correctly defined. In addition, the pole and mass and width have been also be introduced.

The Lee model is a very versatile approach that can be used to test and discuss many different physical situations which ranges from QM systems to purely QFT ones, as we have illustrated in Sec. 6. Hopefully, the detailed presentation of this work may help to initiate new studies that make use of this useful and beautiful model in various areas of physics.

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