Here, using two real non-zero parameters $\lambda$ and $\mu$, we construct pseudo-Gaussian orthogonal ensembles of a large number $N$ of $n \times n$ (n even and large) real pseudo-symmetric matrices under the metric $\eta$ using $N = n(n + 1)/2$ independent and identically distributed random numbers as their elements and investigate the statistical properties of the eigenvalues. When $\lambda \mu > 0$, we show that the pseudo-symmetric matrix is similar to a real symmetric matrix, consequently, all the eigenvalues are real and so the spectral distributions satisfy Wigner’s statistics. But when $\lambda \mu < 0$ the eigenvalues are either real or complex conjugate pairs. We find that these real eigenvalues display intermediate statistics. We show that the diagonalizing matrices $D$ of these pseudo-symmetric matrices are pseudo-orthogonal under a constant metric $\zeta$ as $D^t \zeta D = \zeta$, and hence they belong to a pseudo-orthogonal group. These pseudo-symmetric matrices serve to represent the parity-time (PT)-symmetric quantum systems having exact (un-broken) or broken PT-symmetry.

I. INTRODUCTION

Eigenvalues of a Hamiltonian of a physical system can also be seen as eigenvalues of the matrix which is obtained in a complete orthonormal basis for the corresponding system. In Random Matrix Theory (RMT)\cite{1,2,3}, the invariance properties of the complex many-body Hamiltonian are seen in a class of matrices and the spectral properties of the complex many-body Hamiltonian are then predicted thereof. RMT has been widely used in the analysis of spectra of various other physical systems, such as strongly correlated systems\cite{4}, quantum spin chains\cite{5}, and disordered quantum systems\cite{6}. RMT has been applied in the investigation of space-domain reactor-noise problems to calculate the probability distribution of reactivities\cite{7}. Moreover, Random matrix theory has also been a natural tool for quantum information theory\cite{8}, where the entanglement spectrum statistics of many-body quantum systems have been investigated in the framework of RMT. The nearest level spacing $\delta \epsilon = |\epsilon_{n+1} - \epsilon_n|$ distribution (NLSD) of ensemble of a $2 \times 2$ real symmetric matrices is well known as\cite{9} $p_W(s) = \frac{\pi s}{2} \exp - \frac{\pi s^2}{4}, s = \delta \epsilon/\langle \delta \epsilon \rangle$; where matrix elements $a, b, c$ are independent and identically distributed (iid) as Gaussian probability function (PDF). This is called the NLSD of Gaussian Orthogonal Ensemble (GOE) due to the orthogonal symmetry of real symmetric matrices. Wigner surmised that even when $n$ becomes large ($n \gg 2$), NLSD $p(s)$ remains to be same as $p_W(s)$. The NLSD $p_W(s)$ is known as the Wigner distribution function. RMT has traditionally originated from the field of nuclear physics, where the Wigner surmise $p_W(s)$ represents the spectral distribution of neutron-nucleus scattering resonances, and NLSD $p(s)$ of nuclear levels of the same angular momentum $J$ and parity $\pi$ display the Wigner’s surmise $p_W(s)$, whereas the mixed levels show the Poisson statistics $p_P(s) = \exp (-s)$. In the case of quantum spin chains, if the Hamiltonian is integrable by Bethe ansatz, NLSD $p(s)$ is given by Poisson distribution $p_P(s)$, and in case of non-integrable by Bethe ansatz, NLSD $p(s)$ is given by Wigner distribution $p_W(s)$. In Anderson model of disordered systems, which undergoes a phase transition between an insu-
lating and a metallic phase as a function of the disorder strength (Anderson metal-insulator transition), in the insulating phase, the eigenenergies are Poisson distributed \( p_P(s) \), and the metallic phase leads to a Wigner distribution \( p_W(s) \) of the energy levels, but at the critical point between the two phases, an intermediate statistics \( p(s) \), which describes most closely both the Wigner’s distribution \( p_W(s) \) (linear repulsion) at small spacings and the Poisson distribution (exponential tail) at large spacings, occurs. Random matrix models to describe such intermediate statistics\(^{10} \) have been proposed. Across the many-body localization transition\(^{13} \) intermediate statistics interpolating between \( p_W(s) \) and \( p_P(s) \) is proposed to be \( p_{MBL}(s) = C_1 s^\gamma e^{-C_2 s^{\gamma_P}}, \gamma_P \leq 1 \), which has been referred as sub-Wigner statistic\(^{12} \). Moreover, topological transitions in a Josephson junction are described by the semi-Poisson distribution\(^{13} \) \( p_{SP}(s) = 4s e^{-2s} \), which is a simpler form of intermediate level spacing distribution \( p_{MBL}(s) \) in the limit \( \gamma_P \to 1 \). The intermediate spectral statistics have also been found to occur in several other systems, such as pseudo-integrable billiards\(^{14} \) and quantum maps\(^{15} \), molecular resonances in Er isotopes\(^{16} \). In\(^{17} \), statistical properties of structured random matrices present the intermediate statistics and it is argued to be more ubiquitous and universal than was considered so far in RMT.

Our motivation stems from conjecture\(^{18} \) that a non-Hermitian complex PT-parity-time)-symmetric Hamiltonians, which have been associated with pseudo-Hermitian Hamiltonians\(^{19} \), connected to their adjoints by a similarity transformation given as \( \eta H \eta^{-1} = \tilde{H} \) under a generalized parity \( \eta \), may also have real discrete spectrum and eigenvalues are either real or complex conjugate pairs below or above a critical value of a real parameter in the potential\(^{18} \).

Random matrix theory of PT-symmetric or pseudo-Hermitian quantum systems has been subjected to great interest of research due to a remarkable surge of interest in PT-symmetric quantum systems and has been investigated extensively in recent years\(^{20–29} \).

Initially, the ensembles proposed were restricted to the case of \( 2 \times 2 \) pseudo-Hermitian matrices\(^{20} \), and pseudo-Hermitian random matrix models were approached in the \( N \times N \) case a few years later\(^{21–23} \) and further, a general formalism for pseudo-Hermitian random matrix models has been laid down in\(^{25} \). The level-spacing distribution of the pseudo-Hermitian Dicke model near the integrable limit is close to the Poisson distribution, while it is Wigner distribution for the ranges of the parameters for which the Hamiltonian is nonintegrable\(^{26} \). In Marinello et al.\(^{27} \) investigated the statistical properties of eigenvalues of pseudo-Hermitian random matrices to find that spectrum splits into separated sets of real and complex conjugate eigenvalues, the real ones show characteristics of an intermediate incomplete spectrum, and on the other hand, the complex ones show repulsion compatible with cubic-order repulsion. Concerning pseudo-Hermitian random matrices, the collection of work by Pato et al.\(^{28} \) is worth mentioning. Moreover, a recent study\(^{29} \) on level statistics of real eigenvalues in non-Hermitian systems serves as effective tools for detecting quantum chaos, many-body localization, and real-complex transitions in non-Hermitian systems with symmetries.

Real non-symmetric matrices \( H_{n \times n} \) may have both real and complex conjugate eigenvalues. It can be shown that any real square matrix which is diagonalizable is pseudo-symmetric\(^{19} \), \( \eta^{-1} H \eta = \tilde{H} \) under the metric \( \eta = (DD^t)^{-1} \) or some other secular metric (constant matrix). Here \( D \) is the diagonalizing matrix of \( H \), and \( t \) is the transpose operation. As per the theory of matrices, a square matrix with distinct eigenvalues is always diagonalizable (det \( |D| \neq 0 \)). Thus the real number of eigenvalues of \( N, n \times n \) random matrices will have an interesting statistical distribution. The number of real eigenvalues of a real Gaussian random \( n \times n \) matrix is found to be \( \sqrt{2n/\pi} \) when \( n \) is large, and distribution \( D(\bar{\epsilon}) \) of real eigenvalues as an involved analytic function\(^{31} \) of \( \bar{\epsilon} \) and \( n \).
Pseudo-symmetric matrices, a form of more general pseudo-Hermitian matrices, with some of the eigenvalues as real can represent PT-symmetric quantum systems having broken PT-symmetry, while pseudo-symmetric matrices with all the eigenvalues as real can represent the systems with exact (unbroken) PT-symmetry, and in more general way, these matrices can made to represent the both the scenario: unbroken PT-symmetry and broken PT-symmetry, under the change of characteristic parameter of the system. In [12], we have studied the spectral distributions of real eigen-values of the pseudo-symmetric matrices where eigenvalues are either real or complex conjugate pairs, with \( N[\frac{n(n + 1)}{2} \leq N \leq n^2] \) iid random numbers as their elements to find the NLSDs as semi-Poisson and sub-Wigner distributions (intermediate statistics). Here, we focus on pseudo-symmetric matrices that can have eigenvalues both all or some of them real, thus presenting the spectral distributions of PT-symmetric quantum systems in both regimes. We construct a set of real pseudo-symmetric matrices containing two real parameters \( \lambda \) and \( \mu \), which are in a hidden way similar and not similar to a real symmetric matrix, and investigate the spectral distributions of the ensemble of pseudo-symmetric matrices using \( N = n(n + 1)/2 \) Gaussian iid random numbers as their elements, called as pseudo-Gaussian Orthogonal Ensemble (pGOE) owing the pseudo-orthogonal symmetry of these pseudo-symmetric matrices. We find that, when \( \lambda \mu > 0 \), NLSDs \( p(s) \) come out to be Wigner’s surmise as,

\[
p_{W}(s) = \frac{\pi}{2} s \exp(-bs^2), \quad 0 < c < 2, \quad (1)
\]

and distribution of eigenvalues \( D(\bar{\epsilon}) \) are semi-circle law as,

\[
D(\bar{\epsilon}) = \frac{2}{\pi} \sqrt{1 - \bar{\epsilon}^2}; \quad \bar{\epsilon} = \epsilon/\epsilon_{\text{max}}, \quad (2)
\]

where \( \epsilon \) are the eigenvalues of the matrix.

For \( \lambda \mu < 0 \), spacing distribution is found to be the intermediate statistics, which fits well to the sub-Wigner form [12,27] given as

\[
p_{abc}(s) = a s \exp(-bs^c), \quad 0 < c < 2, \quad (3)
\]

and distribution of eigenvalues \( D(\bar{\epsilon}) \) for some of these ensemble can be fitted to

\[
D(\bar{\epsilon}) = A \left( \tanh \left( \frac{\bar{\epsilon} + B}{C} \right) - \tanh \left( \frac{\bar{\epsilon} - B}{C} \right) \right). \quad (4)
\]

The fitted parameters \( a, b, c; \) and \( A, B, C \) are real and do depend on \( n \). Importantly, in complete parametric space of \( \lambda \) and \( \mu \) \( (\lambda, \mu \in \mathbb{R} \neq 0) \), both the type of statistics, Wigner’s surmise and intermediate statistics can be seen to occur.

The paper is organized as follows: in sec. II, we construct sets of real pseudo-symmetric matrices and we prove their (hidden) similarity to real symmetric matrices (when \( \lambda \mu > 0 \)) and hence the reality of their eigenvalues. The constructed matrices for the case \( \lambda \mu < 0 \) have eigenvalues as both real and complex conjugate pairs. We also discuss the pseudo-orthogonal property for the diagonalizing matrices of these pseudo-symmetric matrices and show that these matrices form the pseudo-orthogonal group. In sec. IV, we investigate the spectral statistics for the ensemble of constructed pseudo-symmetric random matrices followed by a description of the unfolding procedure in sec. III. In sec. V, we derive the NLSD \( p(s) \) for an ensemble of \( 2 \times 2 \) pseudo-symmetric matrices discussed in sec. II. Finally, we consider the more general form of constructed pseudo-symmetric matrices to find Wigner’s distribution and intermediate statistics and then we conclude the present work.

II. PSEUDO-ORTHOGONAL GROUP OF NEW REAL PSEUDO-SYMMETRIC MATRICES

Let \( M \) be a real symmetric square matrix of dimension \( n \). Let us define the \( n \times n \) (\( n \) even) Pauli like block matrices \( \Sigma_{r} \) using identity matrices \( I_{n/2 \times n/2} \) and null matrices \( O_{n/2 \times n/2} \),
\[
\Sigma_1(\lambda) = \begin{pmatrix} O & \lambda I \\ I & O \end{pmatrix}, \quad \Sigma_2(\lambda) = \begin{pmatrix} O & -i\lambda I \\ iI & O \end{pmatrix}, \quad \Sigma_3(\lambda) = \begin{pmatrix} \lambda I & O \\ O & -I \end{pmatrix},
\]

where \( \lambda \in \mathbb{R}_{\neq 0} \). Now let us construct the set of matrices \( Q_k(\lambda) = \Sigma_k(\lambda)M\Sigma_k(\lambda) \) and \( R_k(\lambda) = \Sigma_k(\lambda)M\Sigma_k^{-1}(\lambda) \), where \( k = 1, 2, 3 \). The more generalized form of these matrices, \( Q_k(\lambda, \mu) = \Sigma_k(\lambda)M\Sigma_k(\mu) \) and \( R_k(\lambda, \mu) = \Sigma_k(\lambda)M\Sigma_k(\mu) \) for \( k = 1, 2, 3 \) are also constructed. The matrices \( Q_k(\lambda) \) and \( R_k(\lambda) \) are pseudo-symmetric under the constant metric \( \eta_1 \) and \( \eta_2 \) respectively for all \( \lambda \neq 1 \) and for \( \lambda = 1 \), these matrices turn to be the real symmetric matrices. Similarly, the generalized matrices \( Q_k(\lambda, \mu) \) and \( R_k(\lambda, \mu) \) are pseudo-symmetric under the constant metric \( \eta_3 \) and \( \eta_4 \) respectively for all \( \lambda, \mu \) \((\lambda = \mu \neq 1)\). The matrices \( \eta_n \) are given as,

\[
\eta_1 = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \quad \eta_4 = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}. \tag{5}
\]

Since the matrices \( Q_k(\lambda) \) and \( Q_k(\lambda, \mu) \) turn out to be real symmetric matrices for the case \( k = 3 \) for all \( \lambda, \mu \), so we have not included in our investigation. For brevity, we denote the pseudo-symmetric matrices \( Q_k(\lambda) \), \( R_k(\lambda) \), \( Q_k(\lambda, \mu) \) and \( R_k(\lambda, \mu) \) as \( Q_k \), \( R_k \), \( Q_k \) and \( R_k \) in rest of the paper unless stated otherwise. In the following section, we shall establish the pseudo orthogonal group formed by their diagonalizing matrices along with the general consideration associated with pseudo-symmetric matrices.

### A. PSEUDO-SYMMETRY:

Pseudo-symmetry of these real non-symmetric matrices under constant metrics \( \eta_n \) can be seen as follow,

\[
\eta_1 Q_k \eta_1^{-1} = \eta_1 \Sigma_k(\lambda)M\Sigma_k(\lambda) \eta_1^{-1} = \Sigma_k^t(\lambda)M\Sigma_k(\lambda) = \Sigma_k(\lambda)M^t \Sigma_k^t(\lambda) = Q_k^t
\]

\[
\eta_2 R_k \eta_2^{-1} = \eta_2 \Sigma_k(\lambda)M\Sigma_k^{-1}(\lambda) \eta_2^{-1} = (\Sigma_k^{-1}(\lambda))^tM\Sigma_k^t(\lambda) = (\Sigma_k^{-1}(\lambda))^tM^t \Sigma_k^t(\lambda) = R_k^t
\]

\[
\eta_3 Q_k \eta_3^{-1} = \eta_3 \Sigma_k(\lambda)M\Sigma_k(\mu) \eta_3^{-1} = (\Sigma_k(\mu))^tM(\Sigma_k(\lambda))^t = Q_k^t
\]

\[
\eta_4 R_k \eta_4^{-1} = \eta_4 \Sigma_k(\lambda)M\Sigma_k^{-1}(\mu) \eta_4^{-1} = (\Sigma_k^{-1}(\mu))^tM\Sigma_k^t(\lambda) = (\Sigma_k^{-1}(\mu))^tM^t \Sigma_k^t(\lambda) = R_k^t. \tag{6}
\]

### B. REAL SPECTRUM:

Reality of the spectrum of these real pseudo-symmetric (non-symmetric) can be proved as follow,

(i) \( Q_k = \Sigma_k(\lambda)M\Sigma_k(\lambda) = \Sigma_k^{-1}(\lambda)\Sigma_k(\lambda)\Sigma_k(\lambda)M\Sigma_k(\lambda) \Rightarrow Q_k \) and \( \Sigma_k\Sigma_k M \) are similar matrices. So \( \text{eig}(Q_k) = \text{eig}(\Sigma_k\Sigma_k M) = \lambda \text{ eig}(M) \). Hence \( Q_k \) matrices will have all the eigenvalues as real.

(ii) \( R_k = \Sigma_k(\lambda)M\Sigma_k^{-1}(\lambda) = \Sigma_k(\lambda)M\Sigma_k^{-1}(\lambda) \Rightarrow R_k \) and \( M \) are similar matrices. So \( \text{eig}(R_k) = \text{eig}(M) \). Hence \( R_k \) matrices will have all the eigenvalues as real.

(iii) \( Q_k(\lambda, \mu) = \Sigma_k(\lambda)M\Sigma_k(\mu) = \Sigma_k^{-1}(\mu)[\Sigma_k(\mu)\Sigma_k(\lambda)M]\Sigma_k(\mu), \Rightarrow \) so the matrices \( Q_k \) are similar to \( \Sigma_k(\mu)\Sigma_k(\lambda)M \). For \( \lambda \mu > 0 \), we can write, \( \Sigma_k(\mu)\Sigma_k(\lambda)M = \text{sgn}(\mu)J_k^2(\mu, \lambda)M = \text{sgn}(\mu)J_k(\mu, \lambda)[J_k(\mu, \lambda)M] \Sigma_k(\mu, \lambda) \). Where \( J_k^2(\mu, \lambda) = \Sigma_k(\mu)\Sigma_k(\lambda) \). So the pseudo-symmetric matrices \( Q_k \) are
similar to the real symmetric matrices $J_k(\mu, \lambda)M J_k(\mu, \lambda)$ for $\lambda \mu > 0$. Hence $Q_k(\mu, \lambda)$ will have all real eigenvalues for $\lambda \mu > 0$ and partially real for $\lambda \mu < 0$.

(iv) $R_k(\mu, \lambda) = \Sigma_k(\lambda) M \Sigma_k^{-1}(\mu)$ \(= \Sigma_k(\lambda)[M \Sigma_k^{-1}(\mu) \Sigma_k(\lambda)] \Sigma_k^{-1}(\lambda)\). For $\lambda \mu > 0$, we can write, $M \Sigma_k^{-1}(\mu) \Sigma_k(\lambda) = \text{sgn}(\mu) M K_k^2(\mu, \lambda) = \text{sgn}(\mu) K_k^{-1}(\mu, \lambda)[K_k(\mu, \lambda) M K_k(\mu, \lambda)] K_k(\mu, \lambda)$ where, $K_k^2(\mu, \lambda) = \Sigma_k^{-1}(\mu) \Sigma_k(\lambda)$. So finally the matrices $R_k$ are similar to the symmetric matrices $K_k(\mu, \lambda) M K_k(\mu, \lambda)$ for $\lambda \mu > 0$. Hence $R_k(\mu, \lambda)$ will display both type of scenarios: all and some of the eigenvalues as real for $\lambda \mu > 0$ and $\lambda \mu < 0$ respectively.

C. PSEUDO-ORTHOGONALITY:

A real square matrix $A$ is said to be pseudo-orthogonal under a metric $\zeta$, if

$$A^t \zeta A = \zeta.$$  \(7\)

Let $A$ be $G^{-1}BG$, where $B$ is an orthogonal matrix as $BB^t = I$. Eq.(7) follows,

$$A^t \zeta A = G^t B^t (G^{-1})^t \zeta G^{-1} BG \quad 8$$

Let $(G^{-1})^t \zeta G^{-1} = I \implies \zeta = G^t G$. \(9\)

Hence, the matrix $A(= G^{-1}BG)$ is pseudo-orthogonal under the metric $\zeta = G^t G$ as $A^t \zeta A = \zeta$.

(i) Let the diagonalizing matrix of $M$ be $D$ as $D^t MD = E$, where $E$ is the diagonal matrix. Since $M$ is a real symmetric matrix ($M = M^t$), so the diagonalizing matrix $D$ is orthogonal as $DD^t = I$. We can find the diagonalizing matrix of $Q_k$ as,

$$Q_k = \Sigma_k M \Sigma_k = \Sigma_k^{-1} (\Sigma_k \Sigma_k M) \Sigma_k$$

$$\Sigma_k Q_k \Sigma_k^{-1} = \lambda M$$

$$D^{-1} \Sigma_k Q_k \Sigma_k^{-1} D = \lambda E$$

$$(\Sigma_k^{-1} D \Sigma_k)^{-1} Q_k (\Sigma_k^{-1} D \Sigma_k) = \lambda \Sigma_k^{-1} E \Sigma_k = \lambda E'$$ \(10\)

The diagonal matrices $E$ and $E'$ are the same except for the arrangement of elements along the diagonal. Since $\text{eig}(Q_k) = \text{eig}(\lambda M)$, so the matrices $D_k = \Sigma_k^{-1} D \Sigma_k$ are the diagonalizing matrices for the pseudo-symmetric matrices $Q_k$. Hence, the non-symmetric matrices $Q_k(\neq Q_k^t)$ are pseudo-orthogonal under the constant matrices $\zeta_k = \Sigma_k \Sigma_k^t$. Here, the matrices $\zeta_k$ turn out to be the same as $\eta_1$, so the diagonalizing matrix for the matrix $Q_k$, which is pseudo-symmetric under the constant matrix $\eta_1$, is pseudo-orthogonal under the same constant metric $\eta_1$.

(ii) Now, on reconsidering the Eq.(7) for a real square matrix $A = GBG^{-1}$,

$$A^t \zeta A = (G^{-1})^t B^t G^t \zeta G B G^{-1} \quad 11$$

Let $G^t \zeta G = I \implies \zeta = (G G^t)^{-1}$. \(12\)

Hence, the matrix $A(= GBG^{-1})$ is pseudo-orthogonal under the metric $\zeta = (G G^t)^{-1}$ as $A^t \zeta A = \zeta$.

We can find the diagonalizing matrix of pseudo-symmetric matrices $R_k$ as,

$$R_k = \Sigma_k M \Sigma_k^{-1} \implies D^{-1} \Sigma_k R_k \Sigma_k D = E$$

$$(\Sigma_k D \Sigma_k^{-1})^{-1} R_k (\Sigma_k D \Sigma_k^{-1}) = \Sigma_k E \Sigma_k^{-1} = E'' \quad 13$$

Again, the diagonal matrices $E$ and $E''$ are the same except for the arrangement of elements along the diagonal. Since $\text{eig}(R_k) = \text{eig}(\lambda M)$, so the matrices $D_k = \Sigma_k D \Sigma_k^{-1}$ are the diagonalizing matrices for the pseudo-symmetric matrices $R_k$. Hence, the matrices $R_k(\neq R_k^t)$ are pseudo-orthogonal under the constant matrices $\zeta_k = (\Sigma_k \Sigma_k^{-1})^{-1}$. The metrics $\zeta_k$ are turn out to be same as $\eta_2$. Similarly, the diagonalizing matrices of general pseudo-symmetric matrices $Q_k(\lambda, \mu)$ and $R_k(\lambda, \mu)$ can be found as $\Sigma_k^{-1}(\lambda) D \Sigma(\mu)k$ and $\Sigma(\lambda) D \Sigma^{-1}(\mu)k$ respectively, these in turn are pseudo-orthogonal under constant metrics $\Sigma_k(\lambda)(\Sigma_k(\mu))^t$ and $\Sigma_k(\lambda)(\Sigma_k(\mu))^t)^{-1}$ respectively.
D. PSEUDO-ORTHOGONAL GROUP:

On re-writing the condition for pseudo-orthogonality (7) as \( \zeta^{-1}A^\dagger \zeta = A^{-1} \implies A^\# = A^{-1} \), where \( ^\# \) denotes distortion from orthogonality.

(i) Pseudo-orthogonal matrices \( D_k \) are closed under multiplication,

\[
(D_k D_m)^\# = \eta_1^{-1} (D_k D_m) \eta_1 = \eta_1^{-1} (\Sigma_m)^t D^t (\Sigma_m)^t \eta = + \Sigma_m^{-1} D^t \Sigma_m \Sigma_k^{-1} D^t \Sigma_k = (D_k D_m)^{-1} \tag{14}
\]

(ii) If \( D_k \) is pseudo-orthogonal under \( \eta_1 \), then \( D_k^{-1} \) is also pseudo-orthogonal under the same metric \( \eta_1 \),

\[
(D_k^{-1})^\# = \eta_1^{-1} (D_k^{-1}) \eta_1 = \eta_1^{-1} \Sigma_k^{-1} (D^t)^{-1} \Sigma_k \eta_1 = \Sigma_k D^t \Sigma_k^{-1} = D_k \tag{15}
\]

(iii) Pseudo-orthogonal matrices \( D_k \) are associative under multiplication, and associativity of pseudo-orthogonal matrices \( (D_k (D_m D_o))^\# = (D_k (D_m D_o))^{-1} \) can be proved trivially.

(iv) The identity matrix would act as the unit element of this symmetry transformation group. Similarly, pseudo-orthogonal group structure can be shown for other sets of pseudo-orthogonal matrices found in sub-sec. II-C.

III. UNFOLDING THE SPECTRUM TO FIND THE SPECTRAL DISTRIBUTIONS

To find universal statistical properties of the system, the eigenvalues of random matrices need to be normalized, also called as unfolding\[31\], to separate the average behavior of the non-universal spectral density from the universal spectral fluctuations. Unfolding the spectrum is essentially the local re-normalization of eigenvalues in such a way that their mean density of eigenvalues is equal to unity. In this paper, the unfolded eigenvalues \( \epsilon_i \) are obtained as

\[
\epsilon_i = N(\epsilon_i). \tag{16}
\]

where \( \epsilon_i \) are the true eigenvalues of the random matrices, and \( N(\epsilon) \) is cumulative mean density.

We will now prove that with respect to a fixed metric, \( \eta_1 \) (4), pseudo-orthogonal matrices, let us take, \( D_k = \Sigma_k^{-1} D \Sigma_k \) (sub-sec. II-C) form a group under matrix multiplication.

IV. SPECTRAL DISTRIBUTIONS FOR PSEUDO-GAUSSIAN ORTHOGONAL ENSEMBLE OF \( N \times N \) REAL RANDOM MATRICES \( Q_k(\lambda) \), AND \( R_k(\lambda) \): WIGNER’S SURMISE

The non-symmetric matrices \( Q_k(\lambda) \) and \( R_k(\lambda) \) constructed in sec. II are pseudo-symmetric matrices under generalized \( \eta \) having all the eigenvalues as real, which can represent the systems having exact PT-symmetry\[18\]. Here, we propose to investigate the spectral distributions \( p(s) \) and \( D(\epsilon) \) of pseudo-Gaussian orthogonal ensemble of real random matrices arising from pseudo-symmetric matrices \( Q_k(\lambda) \) and \( R_k(\lambda) \) for the parameter \( \lambda \in \mathbb{R} \neq 0 \). We do so by considering 5000 sampling of matrices \( Q_k = \Sigma_k(\lambda) M \Sigma_k(\lambda) \), and \( R_k = \Sigma_k(\lambda) M \Sigma_k^{-1}(\lambda) \) of dimension 100 \( \times \) 100, where the real symmetric square matrix \( M \) is having \( N = n(n+1)/2 \) iid random numbers under Gaussian probability distribution with zero mean and variance 1. We find the NLSDs \( (p(s)) \)
for the ensemble of these pseudo-symmetric random matrices numerically after unfolding the spectrum (17) and $p(s)$ of unfolded energy levels turns out to be Wigner’s surmise $p_W(s)$ as in Eq. (1). In Fig. 1(a), we have plotted the numerically obtained NLSD histogram $p(s)$ of unfolded energy levels against Wigner’s surmise $p_W(s)$.

![NLSD histogram](image)

FIG. 1: (a): NLSD histogram for the pseudo-Gaussian orthogonal ensemble (pGOE) of 5000, $100 \times 100$ real pseudo-symmetric matrices $Q_2(\lambda)$ for $\lambda = 0.5$, plotted against the Wigner’s surmise $p_W(s)$, which excellently fits the numerically computed histogram. These results are insensitive to parameters $\lambda$. (b): Histograms for the average density of eigenvalues $D(\bar{\epsilon})$ for the ensemble of 1000 pseudo-symmetric matrices $Q_1(0.5)$ of order $100 \times 100$ under the Gaussian PDF, plotted against the semi-circle law (2). This is the universality for the cases of other pGOE of the matrices $Q_2(\lambda)$, $R_{k=1,2,3}(\lambda)$.

| $\lambda$ | $\mu$ | Real spectrum | $a$ | $b$ | $c$ | Fitted NLSD |
|-----------|-------|--------------|----|----|----|-------------|
| 0.6       | 1.0   | Complete     | $\pi/2$ | $\pi/4$ | 2 | Wigner      |
| 0.8       | 1.0   | Complete     | $\pi/2$ | $\pi/4$ | 2 | Wigner      |
| 1.0       | 1.0   | Complete     | $\pi/2$ | $\pi/4$ | 2 | Wigner      |
| -1.0      | -1.0  | Complete     | $\pi/2$ | $\pi/4$ | 2 | Wigner      |
| -1.0      | 1.0   | Partial      | 6.96  | 2.65  | 0.81 | Sub-Wigner  |
| -0.9      | 1.0   | Partial      | 6.42  | 2.56  | 0.82 | Sub-Wigner  |
| -0.8      | 1.0   | Partial      | 4.16  | 2.03  | 1.00 | Sub-Poisson |
| -0.7      | 1.0   | Partial      | 2.89  | 1.56  | 1.28 | Sub-Wigner  |
| -0.6      | 1.0   | Partial      | 2.68  | 1.45  | 1.37 | Sub-Wigner  |

TABLE I: The parameters of the fitted NLSD function $p_{NLSD}(s) = a \exp(-bs^c)$ (3) to the numerically computed NLSD histograms for the ensembles of pseudo-symmetric matrices $Q_k(\lambda, \mu)$ along with corresponding statistics are listed here.

have also been observed for the ensembles of other pseudo-symmetric random matrices $Q_2(\lambda)$ and $R_{k=1,2,3}(\lambda)$ for all $\lambda \in \mathbb{R} \neq 0$. Though the distribution of eigenvalues $D(\bar{\epsilon})$ deviates from semicircle law (2) under the change of parameter $\lambda$, however NLSD after unfolding the spectrum remains invariant as Wigner’s surmise under the change of parameter $\lambda$. Ensembles of real pseudo-symmetric matrices with all real eigenvalues having $\mathcal{N} = n$ iid number of matrix elements display the Poisson NLSD. Here, we have considered the pseudo-symmetric matrices with all real eigenvalues having $\mathcal{N} = n(n+1)/2$, which is very large compared to $\mathcal{N} = n$. Hence the number $\mathcal{N}$ of iid matrix elements is very crucial in observing Wigner’s statistics.

V. DERIVATION OF NLSD $p(s)$ FOR PSEUDO-GAUSSIAN ORTHOGONAL ENSEMBLE OF $2 \times 2$ REAL RANDOM MATRICES

Let us take a $n = 2$ case of the pseudo-symmetric matrix $Q_1(\lambda) = \Sigma_1(\lambda)M \Sigma_1(\lambda)$ made up of three $(\mathcal{N} = 2(2+1)/2 = 3)$ iid elements $a_{11}, a_{12}, a_{22}$, which is pseudo-symmetric under the constant metric $\eta_1$,

$$Q_1(\lambda) = \begin{pmatrix} \lambda a_{22} & \lambda^2 a_{12} \\ a_{12} & \lambda a_{11} \end{pmatrix} \eta_1 = \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad (18)$$

as $\eta_1 Q_1^2(\eta_1)^{-1} = Q_1^f$. Its eigenvalues $\epsilon_{1,2} = \lambda(a_{11} + a_{22} \mp \frac{1}{2} \sqrt{4a_{12}^2 + (a_{11} - a_{22})^2})$ are uncondition-
as we get the nearest level-spacing distribution \( P \) where \( \kappa \) of eigenvalues as, 

\[
\kappa \Sigma \text{II}, \quad \text{the diagonalising matrix}
\]

Here as \( \Sigma \) dotted) along with the Wigner’s surmise \( p_W(s) \) (blue-circles). The \( Q_1(\lambda) \) for \( \lambda = 1 \) turns out to be a real-symmetric matrix, hence we recover the expected Wigner’s distribution (1) from Eq. (23). In Fig. 2, we have plotted the normalized nearest level-spacing distribution (NLSD) \( p(s) \) (23) for different values of \( \lambda = 1 \) (blue-solid), \( \lambda = 0.9 \) (black-dotted), \( \lambda = 1.1 \) (red-dashed). As discussed in sec. II, for \( \lambda = 1 \) matrix \( Q_1(\lambda) \) becomes a symmetric matrix, hence we recover the expected Wigner’s distribution (1) from Eq. (23). In Fig. 2, we have plotted the normalized nearest level-spacing distribution (NLSD) \( p(s) \) (23) for different values of \( \lambda = 0.9 \) (black-dotted), \( \lambda = 1.0 \) (blue-solid), \( \lambda = 1.1 \) (red-dashed) along with the \( p_W(s) \) (blue-circles). \( p(s) \) for \( 2 \times 2 \) case of matrix \( Q_1 \) depends upon the parameter \( \lambda \) in sharp contrast to its \( n \times n \) results discussed in sec. III. It is clear that in large \( n (n \to \infty) \) limit, system-specific effects vanish in NLSD \( p(s) \). Similar results have also been observed for the \( n = 2 \) case of other pseudo-symmetric matrices discussed in sec. II. Sev-

\[
D = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \zeta = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 \end{pmatrix} \quad (19)
\]

where, \( \theta = \frac{1}{2} \tan^{-1} \frac{2a_{12}-a_{11}}{a_{22}} \implies \theta \in (-\pi/4: \pi/4) \).

Since \( Q_1 = D_1(\lambda E) D_1^{-1}(10) \), so we can write,

\[
a_{11} = (\epsilon_1 + \epsilon_2) - \frac{(\epsilon_1 - \epsilon_2) \cos(2\theta)}{2\lambda},
\]

\[
a_{12} = \frac{(\epsilon_1 - \epsilon_2) \sin(2\theta)}{2\lambda},
\]

\[
a_{22} = (\epsilon_1 + \epsilon_2) + \frac{(\epsilon_1 - \epsilon_2) \cos(2\theta)}{2\lambda} \quad (20)
\]

The probability distribution function of matrix elements is given as

\[
P(Q_1) = A \exp\left(-\frac{\text{tr}(Q_1 Q_1^t)}{2\sigma^2}\right) \quad (21)
\]

\[
P(\epsilon_1, \epsilon_2) = A(\epsilon_1 - \epsilon_2) I_0 \left(\frac{(\kappa - 2)(\epsilon_1 - \epsilon_2)^2}{16}\right) \exp\left(-\frac{(\kappa + 2)(\epsilon_1 - \epsilon_2)^2}{16} - \frac{(\epsilon_1 + \epsilon_2)^2}{4}\right) \quad (22)
\]

where \( \kappa = \lambda^2 + \lambda^4 \). Defining \( \epsilon_1 - \epsilon_2 = \delta \epsilon \) and \( \epsilon_1 + \epsilon_2 = T \), integrating w.r.t. \( T \) from \( -\infty \) to \( \infty \), we get the nearest level-spacing distribution \( P(\delta \epsilon) \). Further, defining \( s = \delta \epsilon / <\delta \epsilon> \) and using the normalization as \( <\delta \epsilon> = 1 \), we find the normalized nearest level-spacing distribution (NLSD) \( p(s) \) as

\[
p(s, \lambda) = \frac{\sqrt{2\kappa}}{\pi} \gamma^2 s \exp\left(\frac{(\kappa + 2)s^2 \gamma^2}{4\pi}\right) I_0 \left(-\frac{(\kappa - 2)s^2 \gamma^2}{4\pi}\right) \quad (23)
\]

where, \( \gamma = E((\kappa - 2)/\kappa) \) which is the complete elliptic integral. As discussed in sec. II, for \( \lambda = 1 \) matrix \( Q_1(\lambda) \) becomes a symmetric matrix, hence we recover the expected Wigner’s distribution (1) from Eq. (23). In Fig. 2, we have plotted the normalized nearest level-spacing distribution (NLSD) \( p(s) \) (23) for different values of \( \lambda = 0.9 \) (black-dotted), \( \lambda = 1.0 \) (blue-solid), \( \lambda = 1.1 \) (red-dashed) along with the \( p_W(s) \) (blue-circles). \( p(s) \) for \( 2 \times 2 \) case of matrix \( Q_1 \) depends upon the parameter \( \lambda \) in sharp contrast to its \( n \times n \) results discussed in sec. III. It is clear that in large \( n (n \to \infty) \) limit, system-specific effects vanish in NLSD \( p(s) \). Similar results have also been observed for the \( n = 2 \) case of other pseudo-symmetric matrices discussed in sec. II. Sev-
eral other works on ensembles of $2 \times 2$ pseudo-Hermitian matrices are also worth mentioning here\textsuperscript{20}.

VI. SPECTRAL DISTRIBUTIONS FOR PSEUDO-GAUSSIAN ORTHOGONAL ENSEMBLE OF $N \times N$ REAL RANDOM MATRICES $Q_k(\lambda, \mu)$ AND $R_k(\lambda, \mu)$: WIGNER’S SURMISE TO INTERMEDIATE STATISTICS

In this section, we consider the general form of pseudo-symmetric matrices $Q_k(\lambda)$ and $R_k(\lambda)$: $Q_k(\lambda, \mu)$ and $R_k(\lambda, \mu)$. As discussed in sec. II, for $\lambda \mu > 0$ the eigenvalues of these matrices, are all real, hence their statistics are again Wigner’s distribution as found for the pseudo-orthogonal ensemble of matrices $Q_k$ and $R_k$ in sec. III. In

![Graph](image-url)

**FIG. 3:** NLSD histogram for pGOE of 5000, 100 $\times$ 100 real pseudo-symmetric matrices $Q_k(\lambda, \mu)$ for $\lambda = -0.9$, and $\mu = 1.0$, plotted against fitted sub-Wigner distribution $p(s) = a s \exp(-b s^c)$. This seems to be the universality for other pseudo symmetric matrices $Q_k(\lambda, \mu)$ and $R_k(\lambda, \mu)$, however, it deviates from Eq. (4) by peak-ing around $\bar{\varepsilon} = 0$ as the value of product $\lambda \mu$ increases negatively.

![Graph](image-url)

**FIG. 4:** Histograms for the distribution of eigenvalues $D(\bar{\varepsilon})$ for pGOE of the pseudo-symmetric matrices $Q_k(\lambda, \mu)$ and $R_k(\lambda, \mu)$ in exact (all the eigenvalues are real) and broken PT-symmetry (partial eigenvalues are real) phase which indicates the occurrence of Wigner’s distribution in unbroken phase, and intermediate statistics in case of broken phase. The spectral distributions of these matrices may be seen as making the transition from Wigner distribution (1,2) to intermediate statistics (3, 4) as the product $\lambda \mu$ changes from a positive value to negative ($\lambda \mu \neq 0$). Fig. 5 shows such a transition from Wigner’s distribution to sub-Wigner statistics through semi-Poisson distribution, for the ensemble of matrices $Q_k(\lambda, \mu)$ under the change of parameter $\lambda = 1.0$ (dashed-red).

Table I, we have listed out the values of the fitted parameter for NLSD $p(s) = a s \exp(-b s^c)$ for some case of $\lambda$ and $\mu$ such that $\lambda \mu > 0$. For $\lambda \mu < 0$, the spectrum of these matrices is partially real, and we find the spectral distributions as intermediate statistics, which are sub-Wigner $p(s) = a s \exp(-b s^c)$, $0 < c < 2$, $c \neq 1$ and semi-Poisson $p(s) = a s \exp(-b s)$ depending upon the parameters, as observed in\textsuperscript{[23]} In Fig. 3, we have plotted the NLSD histogram for the pseudo-Gaussian orthogonal ensemble of 5000, 100 $\times$ 100 matrices $Q_1(\lambda, \mu)$ for $\lambda = -0.9$ and keeping $\mu = 1$, against the fitted NLSD $p(s) = a s \exp(-b s^c)$, for the parameters $a = 6.96$, $b = 2.65$, $c = 0.81$, which is a sub-Wigner distribution. The distribution of eigenvalues $D(\bar{\varepsilon})$ for most of the ensembles (Table I) of these matrices fits well to the empirical form (4)\textsuperscript{[12,13]} as shown in Fig. 4 for pGOE of matrices $Q_k(-0.9, 1)$, however $D(\bar{\varepsilon})$ deviates from Eq. (4) by sharp rise in number of eigenvalues\textsuperscript{[14]} near $\bar{\varepsilon} = 0$ as product $\lambda \mu$ increases on negative scale.

A PT-symmetric system can be represented by these pseudo-symmetric matrices $Q_k(\lambda, \mu)$ and $R_k(\lambda, \mu)$ in exact (all the eigenvalues are real) and broken PT-symmetry (partial eigenvalues are real) phase which indicates the occurrence of Wigner’s distribution in unbroken phase, and intermediate statistics in case of broken phase. The spectral distributions of these matrices may be seen as making the transition from Wigner distribution (1,2) to intermediate statistics (3, 4) as the product $\lambda \mu$ changes from a positive value to negative ($\lambda \mu \neq 0$). Fig. 5 shows such a transition from Wigner’s distribution to sub-Wigner statistics through semi-Poisson distribution, for the ensemble of matrices $Q_k(\lambda, \mu)$ under the change of parameter $\lambda = 1.0$ (dashed-red),
-1.0 (dot-dashed-orange), -0.8 (solid-blue), -0.6 (dotted-black), while $\mu$ is fixed at 1. These features in spectral distribution are also found for the pGOE of other sets of pseudo-symmetric matrices $Q_2(\lambda, \mu)$ and $R_{k=1,2,3}(\lambda, \mu)$.

![FIG. 5: Spacing distribution $p_{abc}(s) = a \exp(-bs^c)$ obtained by fitting to the numerically computed NLSD histograms for the pGOE of pseudo-symmetric matrices $Q_1(\lambda, \mu)$ under the change of parameter $\lambda = 1.0$ (dashed-red), $\lambda = -1.0$ (dot-dashed-orange), $\lambda = -0.8$ (solid-blue), $\lambda = -0.6$ (dotted-black), while $\mu$ is fixed at 1, presents the transition from Wigner’s surmise ($\lambda \mu > 0$) to intermediate statistics ($\lambda \mu < 0$). The corresponding fitted parameters $a$, $b$, and $c$ are listed in Table I. This is the typical universality for other pGOE of pseudo-symmetric matrices $Q_2(\lambda, \mu)$ and $R_{k=1,2,3}(\lambda, \mu)$.](image)

**VII. CONCLUSIONS**

In this article, the pseudo-symmetric matrices $Q_k(\lambda)$, $R_k(\lambda)$, $Q_k(\lambda, \mu; \lambda \mu > 0)$ and $R_k(\lambda, \mu; \lambda \mu > 0)$ discussed in sec. II are new and most interestingly similar to real symmetric matrices in a hidden way and hence their eigenvalues are purely real giving rise to Wigner’s surmise yet again. We claim that the similarity of a pseudo-symmetric matrix to a real matrix is new and thought-provoking. These matrices may be found interesting in general matrix theory as a new type. Here, the pseudo-Gaussian orthogonal ensemble of these random matrices with $N = n(n+1)/2$ number of iids has thrown an interesting surprise wherein both the spectral distributions of nearest level spacing and eigenvalues follow Wigner’s surmise. This provides the insight that Wigner’s surmise is the outcome of matrices whose all eigenvalues are real. These eigenvalues can even unconventionally come from non-symmetric (pseudo-symmetric) matrices as against the conventional real symmetric ones. But when the spectrum splits into separated sets of real and complex conjugate eigenvalues for the pseudo-symmetric matrices $Q_k(\lambda, \mu; \lambda \mu < 0)$ and $R_k(\lambda, \mu; \lambda \mu < 0)$, spectral distributions display the intermediate statistics. Since the pseudo symmetric matrices $Q_k(\lambda, \mu)$ and $R_k(\lambda, \mu)$ can represent the unbroken and broken phase of a PT-symmetric quantum system for $\lambda \mu > 0$ and $\lambda \mu < 0$ respectively, thus indicating the connection of unbroken PT-symmetry phase to Wigner’s distribution and broken PT-symmetry phase to intermediate statistics.

More importantly, we have proved that the diagonalizing matrices $D$ of these pseudo-symmetric matrices are pseudo-orthogonal under a constant metric $\zeta$ as $D^T \zeta D = \zeta$, form a pseudo-orthogonal group, and more investigations in this direction are welcome.

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