BITWISTOR FORMULATION OF SPINNING PARTICLE

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Abstract

Twistorial formulation of a particle of arbitrary spin has been constructed. Equations of twistor transformation are obtained for massive and massless spinning particles. Twistor space of the massive particle is formed by two twistors and two complex scalars. In the massive case, integral transformations relating twistor fields with usual space–time fields have been constructed.

1 Introduction

Penrose twistors [1, 2] are a powerful tool for the analysis of (super)symmetric models of point–like and extended objects. In the twistor approach mainly massless (super)particles [3] have been considered so far. In twistor formalism massive particle, especially with non–zero spin, is investigated in a rather limited number of paper [1], [4]-[6]. But particles [3] have been considered so far. In twistor formalism massive particle, especially with standard twistor formulation and formulation in (real) space-time produces series of questions.

In the twistor formulation phase space of a massless particle is described by two fundamental relations of twistor transformation are obtained for massive and massless spinning particles. Penrose twistors [1], [2] are a powerful tool for the analysis of (super)symmetric models of point–like and extended objects. In the twistor approach mainly massless (super)particles [3] have been considered so far. In twistor formalism massive particle, especially with non–zero spin, is investigated in a rather limited number of paper [1], [4]-[6]. But particles [3] have been considered so far. In twistor formalism massive particle, especially with standard twistor formulation and formulation in (real) space-time produces series of questions.

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In the twistor formulation phase space of a massless particle is described by two canonically conjugated each other Weyl spinors $\lambda^a$, $\bar{\lambda}_a = (\bar{\lambda}_a^a)$ and $\omega^\alpha$, $\bar{\omega}^a = (\bar{\omega}^a)$ which combined in four–component Penrose twistor $Z_a = (\lambda_a, \bar{\omega}^a)$. In twistor variables Lagrangian of massless particle has the form

$$L = -\frac{1}{2} (\ddot{Z}^a Z_a - \dddot{Z}^a \dot{Z}_a) - N (\dot{\bar{\omega}}^a Z_a + j)$$

(1)

where conjugate twistor is defined as $Z^a = (\bar{\omega}^a, -\bar{\lambda}_a)$ and $N$ is Lagrange multiplier for twistor spin constraint

$$\frac{i}{2} \dot{Z}^a Z_a + j = \frac{i}{2} (\dot{\bar{\omega}}^a \lambda_a - \dot{\bar{\lambda}}_a \omega^a) + j \approx 0.$$  

(2)

Obtained by Noether procedure conserved charges $P_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}$, $M_{a\dot{b}} = i \lambda_{(\dot{a}\dot{b})}$, $\bar{M}_{a\dot{b}} = i \bar{\lambda}_{(\dot{a}\dot{b})}$, $K^{\alpha\dot{\alpha}} = \omega^\alpha \bar{\omega}^{\dot{\alpha}}$, $D = \frac{1}{2} (\dot{\bar{\omega}}^a \lambda_a + \dot{\bar{\lambda}}_a \omega^a)$ corresponding to conformal transformations give for pseudovector Pauli-Lubanski $W_{a\dot{a}} = P_{a\dot{b}} \bar{M}_{\dot{b}a} - P_{\dot{b}a} M_{a\dot{b}}$ following value $W_{a\dot{a}} = (-\frac{i}{2} \dot{Z}^a Z_a) P_{a\dot{a}}$. Thus the classical consideration shows that model with twistor Lagrangian (1) describes massless particles of finite spin ($P^2 = 0, W^2 = 0$). Spin (helicity) is defined by quantity $(-\frac{i}{2} \dot{Z}^a Z_a)$, which equal on classical level constant $j$ due to constraint (2). Therefore constant $j$ is ‘classical helicity’ of massless particle.

Fundamental relations of twistor transformation

$$p_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}; \quad \omega^\alpha = \frac{1}{2} x^{\dot{\alpha}\alpha} \lambda_a, \quad \bar{\omega}^a = \frac{1}{2} \bar{\lambda}_{\dot{a}} x^{\alpha\dot{\alpha}}$$

(3)

connect twistor variables and space–time ones $p_{a\mu} = p_{\mu} \sigma^a_{\alpha\dot{a}}$, $x^{\dot{\alpha}\alpha} = x^\mu \sigma^a_{\mu \alpha\dot{a}}$. But in traditional writing of the relations (3) it should be isotropy condition for twistor $\dot{Z}^a Z_a = \ddot{\omega}^a \lambda_a - \dot{\bar{\lambda}}_a \omega^a = 0$. Since ‘twistor $Z_a$ of general form describes classical massless particle

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with helicity $-\frac{i}{2} Z^a Z_a$ (\cite{2}, Sec.6, §2, text after (6.3.9)) the isotropic twistor describes massless particle of zero helicity $j = 0$. Thus basic conditions (3) determine the correspondence between twistor formulation (11) and space–time one of massless particle only at zero spin $j = 0$. Massless particles with nonzero helicity are obtained in quantum spectrum upon the transition from space–time formulation to twistor formulation either by introducing nonzero spin (helicity) constant $j$ directly into the spin constraint $\bar{Z} Z - 2ij \approx 0$ in the twistor action (11) ‘by hand’ or taking into account an ordering ambiguity of quantum variables in the spin constraint. Taking into account basic ideology of twistor approach as alternative for space–time description \cite{2}, from set of formulations of spinning particle it must be space–time formulation corresponding to one (11) at fulfilment of basic relation for twistor transformation which generalize the expressions (3).

Twistor description of massive particle requires with necessity more than one twistor \cite{1}, \cite{4}–\cite{6}. It follows directly from this that time–like momentum of massive particle can be resolved only with using two or more twistor spinors $p_{\alpha i} = \lambda_{\alpha i} \bar{\lambda}_{\alpha i}$. In a minimal case two twistors ($i = 1, 2$) are used. But the complete twistor description of the massive particle of an arbitrary spin has not been accomplished yet. It should include the constraints and the Lagrangian of the massive spinning particle and also a correct chooses of corresponding variables in twistor formulation. In contrast to the massless case where, in principle, the twistor description of a particle of arbitrary helicity is achieved by using only one twistor in the massive case it is necessary to use some spinning variables in addition to the twistor ones \footnote{Otherwise the constraints and the Lagrangian of the system would be nonlinear and rather complicated [private discussion with J.Lukierski]}. In this case we have some analog with case of massless superparticle where for description of nonzero superhelicity it is necessary to use additional variables which together with twistor variables form supertwistor. As in case massless particle it is necessary to have corresponding space–time formulation of massive spinning particle which connected with twistor formulation after using of fundamental relations of twistor transformation. It is important also that massless case and massive one have certain analogies. In ideal massless case can be obtained from massive one in level zero mass $m \to 0$.

In present paper we follow the constructive way to finding of twistor formulation of spinning particle implies using the appropriate space–time formulation. For this aim from all space–time formulations the more appropriate formulation are those in which the spin degrees of freedom are described by means of commuting variables. Also from such formulations there are appropriate ones in which spin variables are spinors (for obtaining arbitrary spins including half–integer ones) and describing of arbitrary spins is realized in uniform way. The formulation relativistic spinning particle with index spinor \cite{7} is more appropriate formulation for these aims. In this formulation uniform Lagrangian describes massive and massless spinning particles. Also spinning particle with index spinor has a some analogy with usual superparticle. Therefore for our aim we can exploit the some elements of transition from space–time formulation of superparticle to twistor one.

### 2 Twistor formulation of massive spinning particle

We take index spinor formulation \cite{7} of spinning particle as starting point for obtaining twistor formulation. In index spinor formalism spinning particle is described with space–time vector $x^\mu$ and commuting Weyl spinor $\zeta^\alpha$. In first order formalism its Lagrangian
has the form \[7\]

\[
L = -\frac{1}{2}p_{\alpha \dot{\alpha}} \dot{\Pi}^{\alpha \dot{\alpha}} + \frac{1}{2}V(p_{\alpha \dot{\alpha}}p^{\alpha \dot{\alpha}} - 2m^2) - \Lambda(\zeta^a p_{\alpha \dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} - j),
\]

where the bosonic ‘superform’ is \( \Pi^{\alpha \dot{\alpha}} = \dot{\Pi}^{\alpha \dot{\alpha}} d\tau \equiv dx^{\alpha \dot{\alpha}} + 2i\dot{\zeta}^{\dot{\alpha}} d\zeta^{\alpha} - 2id\zeta^{\alpha} \dot{\zeta}^{\dot{\alpha}} ; \) \( p_\mu \) is momentum vector of particle with mass \( m \) and real scalars \( V \) and \( \Lambda \) are Lagrange multipliers. After quantization the wave function of the spinning particle \( \Psi(\zeta, \bar{\zeta}) = e^{-i\mu \zeta^2} \Phi(\zeta) \) is expressed by holomorphic polynomial \( \Phi(\zeta) = \zeta^{\alpha_1} \ldots \zeta^{\alpha_\delta} \phi_{\alpha_1 \ldots \alpha_{\delta}} \) on spinor \( \zeta \). The spin (helicity) of particle \( s \) is equal to the constant \( j \) renormalized by ordering constants.\(^3\)

Twistor formulation of massive spinning particle, which corresponds to the space–time formulation \([4]\), has been constructed in \([9]\). For this we introduce pure gauge variables in initial system which are Lorentz harmonics \([10], [11], [12]\). In what follows after canonical transformation we exclude space–time variables by means of gauge fixing and remain only with twistor variables. Canonical transformations and gauge fixing conditions have physical meaning. In particular they produce the fundamental conditions of twistor transformations relating the space–time formulation and twistor one.

The massive spinning particle in the twistor formulation is described by the variables \( \lambda^a, \bar{\lambda}_{\dot{a}} = (\lambda^a), \omega^a, \bar{\omega}_\dot{a} = (\omega^a); \xi^i, \bar{\xi}_\dot{i} = (\xi^i); i = 1, 2 \). The connection of twistor variables with the space–time ones is defined by the conditions of twistor transformation

\[
p_{\alpha \dot{\alpha}} = \lambda^i_{\alpha} \bar{\lambda}_{\dot{i} \dot{\alpha}} \quad (5)
\]

\[
\bar{\omega}^i_{\dot{a}} = \frac{1}{2} \bar{\lambda}_{\dot{a} i} x^{\alpha \dot{\alpha}} + i\xi^i \zeta^\alpha, \quad \omega^i_{\alpha} = \frac{1}{2} x^{\alpha \dot{\alpha}} \lambda^i_{\alpha} - i\xi^i \bar{\zeta}^{\dot{\alpha}}
\]

\[
\xi^i = \zeta^a \bar{\lambda}^i_{\dot{a}}, \quad \bar{\xi}^i = \bar{\lambda}^i_{\dot{a}} \bar{\zeta}^{\dot{a}}
\]

The twistor phase space of massive spinning particle is subjected to the set of the first class constraints

\[
M \equiv \lambda^a \bar{\lambda}_{\dot{a}i} + \bar{\lambda}^i_{\dot{a}} \lambda_{\dot{a}i} + 4m \approx 0, \quad D_i^j \equiv \frac{1}{2} (\bar{\omega}^i_{\dot{a}} \lambda^a - \bar{\lambda}^i_{\dot{a}} \omega^a) + \bar{\xi}^i \xi^j \approx 0, \quad S \equiv \bar{\xi}^i \xi^j - j \approx 0.
\]

Spinors \( \lambda^i \) and \( \omega^a \) are components of the twistors \( Z^i_a = (\lambda^i_a, \omega^a); \) \( \bar{Z}_{\dot{a} i} = (\bar{\lambda}_{\dot{a} i}, \bar{\omega}^i_{\dot{a}}) \). Introducing in a standard way conjugate twistors \( \bar{Z}_{\dot{a} i} = (\bar{Z}_{\dot{a} i}, \bar{Z}^i_a) \), \( \bar{Z}^i_a = \bar{g}^{\dot{a} b} \bar{Z}_{\dot{b} i} \), the kinetic terms are rewritten as

\[-\frac{1}{2} p_{\alpha \dot{\alpha}} \Pi^{\alpha \dot{\alpha}} = \frac{1}{2} (\bar{Z}^i_a \dot{Z}_a^i - dZ^i_a d\bar{Z}_a^i + i(d\xi^i d\bar{\xi}^i)).
\]

The twistor mass–shell constraints \([8]\) take the form \( M = \bar{Z}^i_a I^a_{\dot{b}} Z^j_\dot{b} + \bar{Z}_a^i I_{ab} \bar{Z}_b^a + 4m \approx 0 \) where \( I_{ab}, I^a_{\dot{b}} \) are asymptotic twistors \([2]\). Also the constraints \([9]\) are represented as covariant contractions of twistors \( D_i^j = \frac{1}{2} \bar{Z}^i_a Z^j_a + \bar{\xi}^i \xi^j \approx 0 \). Thus the twistor formulation of massive spinning particle is described by Lagrangian

\[
L = \frac{1}{2} \left( \bar{Z}^i_a \dot{Z}_a^i - \bar{Z}_a^i \dot{Z}^i_a \right) - i \left( \bar{\xi}^i \xi^j - \bar{\xi}_\dot{i} \xi^\dot{j} \right) - \Lambda_m M - \Lambda_j^i D_i^j - \Lambda_s S,
\]

where \( \Lambda_m, \Lambda_j^i, \) and \( \Lambda_s \) are Lagrange multipliers for constraints \([8], [9]\).\(^3\)

\(^3\)When potential term \(-\Lambda(\zeta^2 \bar{\zeta}^2 - j)\) in Lagrangian \([4]\) is absent after quantization the wave function is expressed by function which is polynomial series on \( \zeta, \bar{\zeta} \) with non-fixed degree of homogeneity

\[
\Phi(\zeta, \bar{\zeta}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \zeta^{\alpha_1} \ldots \zeta^{\alpha_n} \bar{\zeta}^{\dot{\alpha}_1} \ldots \bar{\zeta}^{\dot{\alpha}_m} \phi_{\alpha_1 \ldots \alpha_n \dot{\alpha}_1 \ldots \dot{\alpha}_m}.
\]

In this case we have in spectrum the set of particles with all arbitrary spins (helicities). Thus index spinor variables have the role analogical to role of the spinor variables in unfolded formulation of higher spin field theory \([8]\).
3 Twistorial formulation of massless particle

In level of zero mass $m \to 0$ we can leave only one spinor $\lambda_\alpha$ which resolves the light–like
vector of four–momentum $p_\mu$. The equation of twistor transformation (5)–(7) for massless
particle take the form

$$p_{\alpha\dot{a}} = \lambda_\alpha \dot{\lambda}_{\dot{a}}; \quad \dot{\omega}^\alpha = \frac{1}{2} \lambda_\alpha x^{\dot{a}\alpha} + i \xi \zeta^\alpha; \quad \xi = \zeta^\alpha \lambda_\alpha$$

(11)

and c. c. Here $\lambda_\alpha$, $\omega^\dot{a}$ and c. c. are spinor components of twistors $Z_a = (\lambda_\alpha, \omega^\dot{a})$ and
$\bar{Z}^a = (\bar{\omega}^\alpha, -\bar{\lambda}_{\dot{a}})$ as whereas commuting complex scalar $\xi$, $\bar{\xi} = (\bar{\xi})$ is corresponding spin
variable in twistor formalism. The phase space of the massless particle is subject to the first
class constraints

$$D_0 \equiv i(\bar{\omega}^\alpha \lambda_\alpha - \bar{\lambda}_{\dot{a}} \omega^\dot{a}) + 2\bar{\xi} \dot{\xi} = i \bar{Z}^a Z_a + 2 \bar{\xi} \dot{\xi} \approx 0, \quad S \equiv \bar{\xi} \xi - j \approx 0.$$ (12)

These constraints follow from the constraints of massive twistorial particle \[9\] in case of
using one twistor and zero mass.

Thus in twistor variables the massless particles with nonzero helicity is described by
Lagrangian

$$L = \frac{1}{2}(\bar{Z}^a \dot{Z}_a - \dot{\bar{Z}}^a Z_a) - i(\dot{\xi} \xi - \dot{\bar{\xi}} \bar{\xi}) - \Lambda D_0 - \Lambda_\alpha S .$$ (13)

This system is the space–time formulation \[4\] of the spinning particle – reformed in the
twistor approach. Thus after using conditions \[11\] kinetic terms of the Lagrangian \[4\]
transform in kinetic terms of the Lagrangian \[13\]–$\frac{1}{2} p_{\alpha\dot{a}} \Pi^\alpha^{\dot{a} a} = \frac{1}{2}(\bar{Z}^a dZ_a - d\bar{Z}^a Z_a) + i(d\xi \xi - \dot{\bar{\xi}} d\bar{\xi})$. Resolving the condition $p_{\alpha\dot{a}} = \lambda_\alpha \dot{\lambda}_{\dot{a}}$ and using the equations \[11\] for
calculating $\xi$, $\bar{\xi}$, the spin constraint $\zeta \bar{p}^\alpha - j \approx 0$ of the Lagrangian \[4\] takes the form of the
constraint $S \approx 0$ in twistor variables.

We can exclude the variables $\xi$ and $\bar{\xi}$ with the help of first class constraint $\xi \xi - j \approx 0$.
After that massless particle is described by Lagrangian \[4\]. Due to the spin constraint
\[\frac{1}{2} \bar{Z}^a Z_a + j \approx 0 \text{ in Lagrangian } \[4\] the twistor $Z_a$ is nonisotropic (even on classical level)
and describes massless particle with nonzero helicity. We obtain the twistor formulation
with the nonisotropic twistor upon transition from the space–time formulation because
of the presence of the second terms in the incidence conditions \[11\]. The presence in the
incidence conditions, written in real (not complex!) space–time, of the second terms for
spinning case is well known \[2\]. Real rays which correspond to the twistor $Z = (\lambda, \omega)$
with the incidence conditions \[11\] (nonisotropic twistor) form the Robinson congruence.

Note that the conditions \[11\] for definition of spinors $\omega$ can be represented as twistorial
shift $\omega^\dot{a} \to \omega^\dot{a} - i \xi \zeta^\dot{a}$, $\bar{\omega}^\alpha \to \bar{\omega}^\alpha + i \xi \zeta^\alpha$ along spinor $\zeta$. Since $\zeta$ and $\lambda$ are orthogonal,
$\zeta \lambda \neq 0$, we obtain variation of helicity $\frac{i}{2}(\bar{\lambda}_\alpha \omega^\dot{a} - \omega^\dot{a} \lambda_\alpha) \to \frac{i}{2}(\bar{\lambda}_\alpha \omega^\dot{a} - \omega^\dot{a} \lambda_\alpha) + j$. This
twistorial shift is distinguished from twistor shift \[13\] $\omega^\dot{a} \to \omega^\dot{a} + l \bar{\lambda}_{\dot{a}}$, $\bar{\omega}^\alpha \to \bar{\omega}^\alpha + l \lambda^\alpha$
along spinor $\lambda$ where $l$ is a length constant. That shift results in a modification of particle
(or string) interactions with background fields \[13\] and does not produce any change of
helicity since under this shift $\bar{\lambda}_\alpha \omega^\dot{a} - \omega^\dot{a} \lambda_\alpha = \text{inv}$.

The variables of the twistorial formulation \[13\], coordinates of twistor $Z_a = (\lambda_\alpha, \omega^\dot{a})$
and complex scalar $\xi$, may be combined in quantity $Z_A = (Z_\alpha; \xi) = (\lambda_\alpha, \omega^\dot{a}; \xi)$ which has five of complex components and can be called as ‘bosonic supertwistor’. Introducing conjugate quantities $\bar{Z}_A = (\bar{Z}_\dot{a}; \bar{\xi}) = (\bar{\lambda}_{\dot{a}}, \bar{\omega}^\alpha; \bar{\xi})$, $\bar{Z}_A = g^{AB} \bar{Z}_B = (\bar{Z}_\dot{a}; -2i \bar{\xi}) = (\bar{\omega}^\alpha, -\bar{\lambda}_{\dot{a}}; -2i \bar{\xi})$ we see that Lagrangian \[13\] without last term $-\Lambda_\alpha (\bar{\xi} \xi - j)$ takes the
form $\frac{1}{2}(\bar{Z}^A \dot{Z}_A - \dot{\bar{Z}}^A Z_A) - \Lambda \bar{Z}^A Z_A$. 

\[4\]
In the variables $\alpha$, $\beta$, $\gamma$, $\delta$ introduced by $\lambda_1 = i(\alpha + \beta)$, $\lambda_2 = i(\gamma + \delta)$, $\omega^1 = \alpha - \beta$, $\omega^2 = \gamma - \delta$ the norm of 'bosonic super-twistor' $rac{i}{2} \bar{Z}^A Z_A = \frac{i}{2} (\bar{\omega}^a \lambda_a - \bar{\lambda}_a \omega^a) + \bar{\xi} \xi$ takes the form $\frac{i}{2} \bar{Z}^A Z_A = \bar{\beta} \beta + \bar{\delta} \delta + \bar{\xi} \xi - \bar{\alpha} \alpha - \gamma \gamma$ and is quadratic Hermitian form of signature $(+++--)$. Thus the Lagrangian [13] without last term describing the set of particles with all possible helicities is invariant under global transformations of group $U(3, 2)$ which play 'bosonic superconformal' group in twistor formalism. At transition to space–time formulation [1] these $U(3, 2)$–transformations reduce to nonlinear transformation including usual the conformal transformations, the 'bosonic supersymmetric' transformations and 'bosonic superboosts'.

4 Wave function of the twistorial spinning particle

Canonical quantization a la Dirac of massive spinning particle in the twistor formulation has been carry out in [9]. The wave function is $(2J + 1)$–component field $\Psi_M(\lambda, \bar{\lambda})$, defined up to local transformations acting on index $M = -J, -J + 1, ..., J$

$$\Psi'_M(\lambda') = D_{MN}^I(h)\Psi_N(\lambda),$$

where $h \in SU(2)$ and $\lambda^i_{\alpha} = h^i_{\alpha} \lambda_{\alpha}$. The $D_{MN}^I$ is matrix of $SU(2)$–transformations of weight $J$. Thus the wave function is defined in fact on homogeneous space $M = G/H = SL(2, C)/SU(2)$. In form of $SU(2)$-index $i = 1, 2$ the index $M$ is collective index $M = (i_1 \ldots i_{2J})$. Then the wave function which represented twistor field of massive spinning particle is

$$\Psi_M(\lambda, \bar{\lambda}) = \Psi_{i_1 \ldots i_{2J}}(\lambda, \bar{\lambda}),$$

The wave function is completely symmetric $\Psi_{i_1 \ldots i_{2J}} = \Psi_{(i_1 \ldots i_{2J})}$.

The relation of the usual space–time spin–tensor fields $\Phi_{a_1 \ldots a_{2J}}(x)$ with the twistor fields [15] is established by means of an integral transformation in the following way. One constructs $SU(2)$-invariant expressions contracting the twistor fields $\Psi_{i_1 \ldots i_{2J}}(\lambda, \bar{\lambda})$ with twistor spinors $\lambda^i_{a_1}, \ldots, \lambda^i_{a_{2J}}$. Obtained expressions being Lorentz spin–tensors are defined on a homogeneous space $SL(2, C)/SU(2)$. After integration with an invariant measure $d^3 \lambda$ of space $SL(2, C)/SU(2)$ with the standard Fourier exponent $e^{i x^\mu p_\mu}$ where $p_\mu = -\frac{1}{2} p_{\alpha a} \sigma_\mu^{\dot{\alpha}a} = -\frac{1}{2} \lambda^{i} \sigma_\mu \bar{\lambda}_i$ we obtain usual space–time fields

$$\Phi_{a_1 \ldots a_{2J}}(x) = \int d^3 \lambda e^{-\frac{i}{2} x^\mu \lambda^k \sigma_\mu \lambda^i_{a_1} \ldots \lambda^i_{a_{2J}}} \Psi_{i_1 \ldots i_{2J}}(\lambda, \bar{\lambda}).$$

These fields are totally symmetric in spinor indices $\Phi_{a_1 \ldots a_{2J}}(x) = \Phi_{(a_1 \ldots a_{2J})}(x)$ and give us standard $(2J + 1)$-component field description of massive spin $J$. Due to the presence of the exponent in the integral representation for the fields [15] they satisfy automatically massive Klein–Gordon equation $(\partial^\mu \partial_\mu - m^2) \Phi_{a_1 \ldots a_{2J}}(x) = 0$.

We can obtain different, but equivalent descriptions of free massive particles of spin $J$. The $SU(2)$-invariants can be constructed by contraction the twistor fields $\Psi_{i_1 \ldots i_{2J}}(\lambda, \bar{\lambda})$ only with twistor spinors $\bar{\lambda}^i_{\dot{a}}$. In this way we obtain the space–time fields only with primed spinor indices. When constructing the $SU(2)$-invariants, the part of the $SU(2)$-indices in the twistor field $\Psi_{i_1 \ldots i_{2J}}(\lambda, \bar{\lambda})$ can be contracted with spinors $\lambda^i_{\dot{a}}$ whereas the remaining ones are contracted with $\bar{\lambda}^i_{\dot{a}}$. As result we obtain the spin–tensor field $\Phi_{a_1 \ldots a_n \hat{a}_{n+1} \ldots \hat{a}_{2J}}(x)$ with $n$ unprimed spinor indices and $2J - n$ primed ones. By construction this field is

\[\text{Analogous integral transformations for massless twistor fields have been considered in [12, 13, 15.}\]
symmetric with respect all unprimed and all primed indices and satisfy massive Klein–Gordon equation and transversal condition \( \partial^\mu \sigma_\mu^{\alpha_1 \ldots \alpha_{n+1}} \Phi_{\alpha_1 \ldots \alpha_n \tilde{\alpha}_{n+1} \ldots \tilde{\alpha}_{2J}}(x) = 0 \). These fields at fixed \( J \), describing the massive spin \( J \) particle, are related to each other by Fierz–Pauli equations

\[
i \partial^\mu \sigma_\mu^{\tilde{\alpha}_1 \beta \alpha_{n} \ldots \alpha_n} \Phi_{\alpha_1 \ldots \alpha_{n-1} \tilde{\alpha}_{2J-n+1} \ldots \alpha_{2J}} = m \Phi_{\alpha_1 \ldots \alpha_{n-1} \tilde{\alpha}_{2J-n+1} \ldots \alpha_{2J}}
\]

\[
i \partial^\mu \sigma_\mu^{\beta \alpha_n \alpha_{n+1} \ldots \alpha_{2J}} = m \Phi_{\alpha_1 \ldots \alpha_{n-1} \tilde{\alpha}_{2J-n+1} \ldots \alpha_{2J}}.
\]

United the fields \( \Phi_{\alpha_1 \ldots \alpha_n \tilde{\alpha}_{n+1} \ldots \tilde{\alpha}_{2J}} \) with fixed \( J \) in a unique fields we obtain description of spin \( J \) massive particle in Bargman–Wigner or Rarita-Schwinger formalism.

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