Chiral topological states in Bose-Fermi mixture

Ying-Hai Wu\textsuperscript{1,2}

\textsuperscript{1}School of Physics, Huazhong University of Science and Technology, Wuhan 430074, China
\textsuperscript{2}Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany

Topological states were initially discovered in solid state systems and have generated widespread interests in many areas of physics. The advances in cold atoms create novel settings for studying topological states that would be quite unrealistic in solid state systems. One example is that the constituents of quantum gases can be various types of bosons, fermions, and their mixtures. This work explores interaction-induced topological states in two-dimensional Bose-Fermi mixture. We propose a class of topological states with no fractionalized excitations but maximally chiral edge states. For all previously known topological states, these two features can only be found simultaneously in the integer quantum Hall states of fermions and the $E_8$ state of bosons. The existences of some proposed states in certain continuum and lattice models are corroborated by exact diagonalization and density matrix renormalization group calculations. This work suggests that Bose-Fermi mixture is a very appealing platform for studying topological states.

I. INTRODUCTION

The adventure of topological states begins with the observation of quantum Hall effect in high quality two-dimensional electron gases (2DEGs) in the 1980s [1, 2]. The interests on this topic is further boosted by the discovery of topological insulators in the past decade [3, 4]. These states are so named because they have topology related properties that are insensitive to local perturbations, which is in sharp contrast to the Landau theory of symmetry breaking phases distinguished by local order parameters. It is desirable to classify topological states based on their physical properties and the conditions needed to stabilize them. This is a formidable task that has only be partially accomplished in a few cases.

Two fundamental criteria in classification are fractionalization and symmetry protection. If a topological state has elementary excitations carrying fractional quantum numbers of the underlying constituents, it is called fractionalized. If a topological state can only be stable when certain symmetry constraints are satisfied, it is called symmetry-protected. Integer quantum Hall (IQH) states and time-reversal symmetric topological insulators (TIs) are typical examples of non-fractionalized topological states that can be realized using free fermions. The latter ones require protection of time-reversal symmetry and particle number conservation but the former ones do not. Topological states of free fermions under a variety of symmetry conditions have been fully uncovered [5–7]. One can get a much richer set of phenomena with interactions: topological states that are distinct in free fermions may be connected adiabatically [8–10]; topological states that do not exist in free fermions may be enabled [11, 12]; topological states may arise in some systems composed of bosons [13–15]. Fractional quantum Hall (FQH) states are paradigmatic examples of interaction-induced fractionalized topological states in which the elementary excitations carry fractional charges and obey fractional braid statistics. In addition to the fermionic FQH states that have been studied extensively in 2DEGs, bosonic ones are also well-established in theory and actively pursued in experiments [16–18].

One characteristic signature of topological states is edge states. For IQH states and TIs, their bulk topology are charaterized by the Chern number and the $Z_2$ invariant defined on closed manifolds. Topological non-triviality would lead to gapless edge states when such states are placed on open mainfolds. The bulk-boundary correspondence goes beyond free fermions and also plays an important role in many fractionalized states [19]. The stability of a topological state can be analyzed by studying the robustness of its edge states. IQH states have chiral edge states where boundary excitations only propagate along one direction and backscattering is forbidden. TIs have helical edge states where boundary excitations can propagate along both directions, so time-reversal symmetry and particle number conservation must be imposed to make sure that no backscattering between the counter-propagating modes would occur. The presence of fractionalized excitations can also prevent helical edge states from being gapped out in certain cases [20, 21].

The special properties of topological states make them useful in some technological applications. The precisely quantized Hall conductance of IQH states provides an excellent unit of electrical conductance. The robust edge states may be the foundation of next-generation electronic circuits. The elementary excitations that obey non-Abelian braid statistics in certain systems may serve as qubits in quantum computation to defy errors in a topologically robust way [22]. The exotic properties and potential applications of topological states have inspired people to search for them in cold atoms, photons, and superconducting circuits [23–26]. There have been tremendous advances in cold atoms since the observation of Bose-Einstein condensation. It is possible to do experiments on various types of bosons and fermions with Hamiltonians that can be tailored to a great extent. The time-, position-, energy-, and momentum-resolved measurement techniques can help us to extract information in an unprecedented manner.

Topological states of electrons in solid state systems
depend crucially on gauge fields, but the analogs of such
gauge fields do not appear naturally in the Hamiltoni-
ans for cold atoms. This calls for ingenious methods to
synthesize effective gauge fields [27, 28]. For an atom
under rotation, the Coriolis force in the rotating frame
has the same effect as a magnetic field. This method is
conceptually simple, but it is difficult to get a sufficiently
strong magnetic field (i.e., high rotation frequency). An-
other route that has lead to great success in the past few
years is laser-assisted coupling, which can generate many
types of gauge fields in continuum and optical lattices.
One impressive achievement relevant to this work is the
realization of the Harper-Hofstadter model that describes
the motion of particles in the presence of both magnetic
field and periodic potential [29–36].

This work investigates novel topological states in Bose-
Fermi mixture, which has been successfully prepared in
cold atoms [37, 38] but very difficult to find in solid
state systems. It should be emphasized that the bosons
and fermions are microscopically independent particles
in our systems. This is different from the scenario where
fermions pair up to form bosons [39–42]. For solid state
systems, spin models emerge when the charge degree of
electrons is frozen due to strong correlation. It is known
that some quantum spin liquids are equivalent to bosonic
topological states because spins can be recasted as inter-
acting bosons [13, 15] [44]. The rest of this paper is
organized as follows. In Sec. II, continuum and lattice
models are proposed and analyzed using wave functions
and field theory. In Sec. III, topological states for these mod-
els are defined. In Sec. IV, numerical results are presented
to demonstrate that some proposed states do exist in
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to demonstrate that some proposed states do exist in
these models. In Sec. V, we conclude the paper with an
outlook. The appendices provide some technical details
and additional numerical results.

II. MODELS

The models of our interest describe two-dimensional
(2D) Bose-Fermi mixture in synthetic gauge fields which
act independently on bosons and fermions. The indices
$\sigma, \tau = b, f$ will be used as superscripts or subscripts on
many symbols to indicate bosons and fermions, respec-
tively. The numbers of particles are denoted as $N^\sigma$.

A. Continuum Model

As shown in Fig. 1 (a), the particles experience two
independent magnetic fields $B^\sigma$ generated by synthetic
gauge potentials $A^\sigma$. The single-particle Hamiltonian
for the $\sigma$ particles is

$$H_0^\sigma = \frac{1}{2M^\sigma} (p - A^\sigma)^2$$

where $M^\sigma$ is the mass of the particles. The solutions
to these Hamiltonians are Landau levels (LLs). The specific forms of the single-particle eigenstates depend on
the choices of gauge and boundary condition. For
our purposes, it is useful to employ disk, torus, and
sphere [45, 46]. For an infinite disk with symmetric gauge
$A^\sigma = (-B^\sigma y/2, B^\sigma x/2, 0)$, the lowest LL (LLL) wave
functions are $\psi^\sigma_m(x, y) \sim z^m (z = x + iy$ is the complex
coordinate in two spatial dimensions). The analyticity
of these wave functions make them convenient to use in
theoretical studies. The bulk properties of a system can
be seen more clearly on torus and sphere as they are free
of edges. In both cases, the numbers of magnetic fluxes
will be quantized and denoted as $F^\sigma$.

The particles interact with each other via the contact
interactions

$$\sum_{j<k} \left[ g_{bb} \delta_{bb} (r_j - r_k) + g_{bf} \delta_{bf} (r_j - r_k) \right]$$

and we assume that they only occupy their respective
LLs. The second quantized many-body Hamiltonian is

$$V = \frac{1}{2} \sum_{\sigma, \tau} \sum_{\{m_j\}} V_{\sigma,\tau}^{m_1,2,3,4} C^\dagger_{\sigma,m_1} C^\dagger_{\tau,m_2} C_{\tau,m_3} C_{\sigma,m_4}$$

where $C^\dagger_{\sigma,m} (C^\dagger_{\sigma,\sigma})$ is the creation (annihilation)
operator for the single-particle state with quantum number $m$ in
the LLL of the $\sigma$ particles. The interaction strengths are
chosen to be $g_{bb} = g_{bf} = 4\pi t_2^2$ such that the zeroth order
Haldane pseudopotential is 1 [46] and will be used as the
unit of energy. The explicit forms of $V_{\sigma,\tau}^{m_1,2,3,4} $ on torus,
sphere, and disk are given in Appendix A.

B. Lattice Model

As shown in Fig. 1 (b), the particles reside on a square
lattice with gauge fields encoded in some complex hop-
pings of the Harper-Hofstadter type. The numbers of
lattice sites along the two directions are denoted as $N_x$
and $N_y$. The single-particle Hamiltonian is

$$H_0 = - \sum_{\sigma, m, n} t_{\sigma, m, n}^b \left( C^\dagger_{\sigma,m,n} C_{\sigma,m+1,n} + \text{H.c.} \right)$$

$$- \sum_{\sigma, m, n} t_{\sigma, m, n}^f \left( e^{i\phi_m} C^\dagger_{\sigma,m,n} C_{\sigma,m,n+1} + \text{H.c.} \right)$$

where $C^\dagger_{\sigma,m,n} (C^\dagger_{\sigma,\sigma,m})$ is the creation (annihilation)
operator for the $\sigma$ particles on the lattice site labeled by
The particles experience two independent magnetic fields. The hoppings along the $x$ direction are real and those along the $y$ direction are complex with certain phases. The presence of these phases change the translational symmetry of the lattice. It is useful to introduce magnetic unit cells for these phases change the translational symmetry of the lattice model. The hoppings along the $x$ direction are complex with certain phases; the composite fermion theory [52]:

\[
\prod_{j<k}(z_j^b - z_k^b)^2 \prod_{j<k}(z_j^f - z_k^f) \prod_{j}^{N_b}(z_j^b - z_k^f) \prod_{j}^{N_f}(z_j^b - z_k^f) \tag{6}
\]

has the highest density so it will be taken as the ground state. The first part has a power 2 instead of 1 because $\prod_{j<k}(z_j^b - z_k^b)$ does not obey Bose statistics even though the vanishing condition is met. The maximal single-particle angular momentum of bosons is $2N_b + N_f - 2$ and that of fermions is $N_b + N_f - 1$, so the magnetic fields $B^\sigma$ should be chosen independently.

The factor $\prod_{j<k}^{N_b}(z_j^b - z_k^b)^2$ in Eq. 6 is the bosonic Laughlin 1/2 state [51], which may be replaced by the composite fermion states at filling factor $\mu/(\mu + 1)$ ($\mu \in \mathbb{N}$) [17] (the Laughlin 1/2 state is reproduced at $\mu = 1$) to construct a class of states

\[
\Psi^{b}_{\mu+1} (\{z^b\}) = \mathcal{P} \chi_\mu (\{z^b\}) \prod_{j<k}^{N_b}(z_j^b - z_k^b) \tag{7}
\]

To express these wave functions compactly, we define $\chi_\mu$ as the fermionic IQH state at filling factor $\mu$ ($\chi_\mu = \prod_{j<k}(z_j^f - z_k^f)$). The first part is

\[
\Psi^{b}_{\mu+1} (\{z^b\}) = \mathcal{P} \chi_\mu (\{z^b\}) \prod_{j<k}^{N_b}(z_j^b - z_k^b) \tag{8}
\]

where $\mathcal{P}$ is the LLL projection operator and Eq. 7 becomes

\[
\mathcal{P} \chi_\mu (\{z^b\}) \chi_1 (\{z^f\}) \prod_{j<k}^{N_b}(z_j^b - z_k^b) \prod_{j}^{N_b}(z_j^b - z_k^f) \prod_{j}^{N_f}(z_j^b - z_k^f) \tag{9}
\]

The physical picture for these states is provided by the composite fermion theory [52]: $\prod_{j}^{N_b}(z_j^b - z_k^f)$ dresses each boson with one flux from the other bosons; $\prod_{j}^{N_b}(z_j^b - z_k^f)$ dresses each boson (fermion) with one

**III. TOPOLOGICAL STATES**

In this section, we propose a class of topological states and analyze their properties in detail using wave functions in the continuum and effective field theory. The reason for us to work in the continuum is that previous experiences accumulated when studying FQH states can provide valuable guides. The continuum wave functions can be transformed to lattices [47–49], but the results are not as easy to analyze as those in continuum.

**A. Wave Function**

The analyticity of the LLL single-particle wave functions on disk geometry makes it ideal for constructing many-body wave functions. It is known that contact interactions and its derivatives often have exact zero-energy eigenstates when they are projected to the LLL [46, 50]. If Eq. 2 were to have any zero-energy eigenstates, their wave functions must vanish when the distance between any two particles goes to zero (all other particles are kept at fixed positions), otherwise the energy expectation value would be infinite. For all states that satisfy this condition,

\[
\prod_{j<k}^{N_b}(z_j^b - z_k^b)^2 \prod_{j<k}^{N_f}(z_j^f - z_k^f) \prod_{j}^{N_b}(z_j^b - z_k^f) \prod_{j}^{N_f}(z_j^b - z_k^f) \tag{6}
\]
flux from the fermions (bosons); the flux-attached composite fermions form IQH states \( \chi_\mu (\{z^b\}) \) and \( \chi_1 (\{z^f\}) \) in effective magnetic fields.

The composite fermion interpretation goes beyond the ground states. Since the composite fermions are taken as non-interacting objects, the excitations of the IQH states \( \chi_\mu (\{z^b\}) \) and \( \chi_1 (\{z^f\}) \) would give us the excitations of Eq. 9. This is the case for both bulk and edge excitations as illustrated schematically in Fig. 1 at \( \mu = 1 \). If one composite fermion in \( \chi_\mu (\{z^b\}) \) \( \chi_1 (\{z^f\}) \) is excited from an occupied LL to an empty LL, one type B (type F) neutral excitation is created. If the magnetic flux for bosons is increased (decreased) by one unit, one type I quasihole excitation is created. If one fermion is removed, we have a total of \( \mu + 1 \) edge modes, so we have a total of \( \mu + 1 \) edge modes.

The charges of quasihole and quasiparticle deserve special attention. As the gauge potentials for bosons and fermions are independent, we should introduce two kinds of charges for the particles. The bosons (fermions) have unit (zero) charge with respect to a composite fermion in \( \chi_\mu (\{z^b\}) \) \( \chi_1 (\{z^f\}) \) factor contributes \( \mu \) edge modes and the \( \chi_1 (\{z^f\}) \) factor contributes 1 edge mode, so we have a total of \( \mu + 1 \) edge modes.

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### B. Field Theory

The topological field theory for Eq. 9 is the Chern-Simons theory with Lagrangian density [53]

\[
\mathcal{L}_1 = \frac{1}{4\pi\hbar} \epsilon^{\lambda \mu \nu} K_{IJ} a_{I\lambda} \partial_\mu a_{J\nu} - j_{I\lambda} a_{I\lambda}
\]  

(11)

where \( a_{I\lambda} \) (\( I = 1, 2, \lambda = 0, x, y \)) are gauge fields, \( j_{I\lambda} \) is the excitation current, and \( K \) is an integer matrix. For \( \mu = 1 \) and 2, the \( K \) matrix is

\[
K = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]  

(12)

and

\[
K = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

respectively. In general, \( K \) is a \((\mu+1)\)-dimensional matrix in which \( \mu \) diagonal elements are 2, 1 diagonal element is 1, and all off-diagonal elements are 1. When the system is defined on a torus, the number of degenerate ground states is \( |\det K| \). For topologically ordered states, we have \( |\det K| > 1 \) because fractionalized excitations lead to multiple degenerate ground states. However, one can prove by recursion that the \( K \) matrices for all \( \mu \) have unit determinant so these states do not possess fractionalized excitations.

If a system is described by \( \mathcal{L}_1 \) in the bulk, it would have gapless edge modes when placed on an open manifold. The edge physics is captured by the Lagrangian density

\[
\mathcal{L}_{\text{edge}} = \frac{1}{4\pi\hbar} (K_{IJ} \partial_\mu \phi_I \partial_\mu \phi_J - V_{IJ} \partial_\mu \phi_I \partial_\mu \phi_J)
\]  

(14)

where \( \phi_I \) is a chiral boson field and \( V_{IJ} \) depends on the microscopic details at the edge. The propagating directions of the edge modes are determined by the signs of the eigenvalues of the \( K \) matrix. For the \((\mu+1)\)-dimensional \( K \) matrix, we have \( \mu + 1 \) branches of chiral edge modes so they cannot be gapped out by perturbations at the edge.

The charges of quasiparticles and quasiholes can be derived by coupling the excitation currents to two probing gauge fields \( \vec{A}^b \) and \( \vec{A}^f \). This operation adds the Lagrangian density

\[
\mathcal{L}_2 = -\frac{\epsilon^{\lambda \mu \nu}}{2\pi\hbar} (t^b_I A^b_{I\lambda} \partial_\mu a_{I\nu} + t^f_I A^f_{I\lambda} \partial_\mu a_{I\nu})
\]  

(15)

to Eq. 11, where \( t^b = (1, \cdots, 1, 0)^T \) and \( t^f = (0, \cdots, 0, 1)^T \) are termed charge vectors. An excitation is labeled by an integer vector \( l \) and its charge with respect to \( \vec{A}^\sigma \) is \( |l^\sigma|^T K^{-1} l \). If \( l \) is chosen to be \((\pm 1, 0, \cdots, 0)^T\) for type I quasiparticle/quasihole and \((0, \cdots, 0, \pm 1)^T\) for type II quasiparticle/quasihole, the values of \( Q^b_{II} \) computed from wave functions can be reproduced. The braid statistics angle of two excitations labeled by \( l_1 \) and \( l_2 \) is \( \theta_{12} = \pi |l_1|^T K^{-1} l_1 \) if \( l_1 = l_2 \) and \( \theta_{12} = 2\pi |l_1|^T K^{-1} l_2 \) otherwise. It is easy to verify that there is no fractional braid statistics between any pairs of excitations, as we expect for systems with non-degenerate ground state on torus.

### IV. NUMERICAL RESULTS

In this section, we use exact diagonalization and density matrix renormalization group (DMRG) [54–56] to study the many-body Hamiltonians Eqs. 3 and 5 numerically. Exact diagonalization is relatively straightforward and will be used in Sections IV A and IV B. DMRG is more involved and will only be used in Section IV B.
The first signature to be confirmed is the ground state degeneracy on torus. The ground states occur at magnetic fluxes $F^b = (\mu + 1)N^b/\mu + N^f$ and $F^f = N^b + N^f$. The Hamiltonian conserves a special momentum $Y$ (see Appendix A for details). For both $\mu = 1$ and $2$, there is a unique ground state on torus as shown in Fig. 2, in consistency with the absence of fractionalized excitations. The validity of the wave functions for neutral excitations have been tested on sphere. The ground states occur at angular momentum $m$ relative to the ground state, we have $N^b + N^f = 0$ since there is no edge excitation. By assigning non-zero values to $N^b$ and $N^f$, the edge spectrum versus angular momentum can be constructed (see Table I and Fig. 4 for some examples). This procedure gives us a counting $1, 2, 5, 10, \cdots$, which is confirmed in Fig. 4 by both the energy spectrum on disk and the entanglement spectrum on sphere [57]. The edge physics at $\mu = 2$ is more complicated and will be discussed in Appendix B.

There are two issues regarding the energy spectrum on disk. The number of single-particle states is finite on torus and sphere, but is in principle infinite on disk. To get a finite dimensional many-body Hilbert space, we need to introduce cutoffs (also denoted as $F^\sigma$) such that the particles can only occupy the states with $m \leq F^\sigma$. The conservation of the total angular momentum $L_z$ gives natural cutoffs, but it is helpful to use smaller values to further reduce the Hilbert space dimension. The energy spectrum in Fig. 4 (a) is computed using $F^b = 24$ and $F^f = 16$, which are sufficiently large in the sense that the first few eigenvalues would not change visibly if $F^\sigma$ increases. The existence of multiple degenerate states is a considerable challenge for sparse matrix diagonalization. At angular momentum 3 relative to the ground state, we can get 10 zero-energy states for the $N^b = N^f = 4$ system but not for the $N^b = N^f = 5$ system. This issue can be resolved by adding a small perturbation to the Hamiltonian to slightly break the 10 fold degeneracy.

![Figure 2](image1.png)

![Figure 3](image2.png)

**FIG. 2.** Energy spectra of the continuum model on torus. The system parameters are given $[N^b, N^f, F^b, F^f]$ in each panel. The energy levels are labeled by the total momentum $Y$. There is a unique ground state in both cases.

**FIG. 3.** Energy spectra of the continuum model on sphere. The system parameters are given $[N^b, N^f, F^b, F^f]$ in each panel. The energy levels are the total angular momentum $L$ and its $z$-component $L_z$ (chosen to be 0 in both panels). The dots represent counterparts of the wave functions Eq. 9 on sphere. The overlaps with the exact eigenstates are $0.9991, 0.9433, 0.9882, 0.9939, 0.9945, 0.9934$ [from $L = 0$ to 7 in (a)] and $0.9982, 0.9955, 0.9850, 0.9796, 0.9863$ [from $L = 0$ to 4 in (b)].

| label | occupation numbers | $L_{edge}$ | $H_{edge}$ |
|-------|--------------------|------------|------------|
| 1(a)  | $N_{m=1}^\sigma = 1$ | 1          | $v_l$      |
| 1(b)  | $N_{m=1}^{\sigma\up} = 1$ | 1          | $v_{li}$   |
| 2(a)  | $N_{m=2}^\sigma = 1$ | 2          | $2v_{li}$  |
| 2(b)  | $N_{m=2}^{\sigma\up} = 1$ | 2          | $2v_{li}$  |
| 2(c)  | $N_{m=1}^\sigma = 1$ | 1          | $v_l + v_{li}$ |
| 2(d)  | $N_{m=2}^{\sigma\up} = 1$ | 2          | $2v_{li}$  |
| 2(e)  | $N_{m=1}^{\sigma\up} = 2$ | 2          | $2v_{li}$  |

**TABLE I.** Edge states at relative angular momentum 1 and 2. The states are labeled for reference in the main text. The unspecified occupation numbers are all zero.
The entanglement spectrum is an alternative method for probing edge states. For this calculation, the sphere is divided into two hemispheres separated by a virtual edge along its equator \([58–60]\). The reduced density matrix, \(\rho\) of the southern hemisphere is written as \(\exp(-H)\). The low-energy part of the entanglement Hamiltonian \(H\) captures the edge excitations at the equator. When the eigenvalues \(\xi\) of \(H\) are plotted versus the good quantum numbers of the southern hemisphere (the numbers of particles and the \(z\) component angular momentum), the edge state counting 1, 2, 5, 10 is revealed. It is instructive to inspect some entanglement levels in greater detail. The relative angular momenta \(\Delta L_z\) of entanglement levels are defined with respect to \(L_z^S = -30\). The edge states have been labeled in Table 1 and their energy values are computed using Eq. 17. The splitting between the two levels at \(\Delta L_z = 1\) means that \(v_1\) and \(v_{11}\) are different. The five levels at \(\Delta L_z = 2\) can be divided into three groups: 2(a) and 2(b) are degenerate, 2(d) and 2(e) are degenerate, and 2(c) is separated from others. This is supported by the entanglement spectrum, but the levels are only quasi-degenerate since the dispersions of the actual edge modes are not perfectly linear as in Eq. 17.

B. Lattice Model

We first impose PBCs along both directions of the lattice to check the ground state degeneracy. This can be done if the number of sites along the \(x\) direction is a common multiple of \(q^b\) and \(q^f\). The numbers of energy bands are \(q^b\) and the numbers of states in each energy band are \(N_x N_y / q^b\). In analogy to the continuum LL on torus, we have the relations \(N_x N_y / q^b = (\mu + 1) N^b + N^f\) and \(N_x N_y / q^f = \mu N^b + N^f\). The system parameters need to satisfy two more constraints to stabilize the topological states of our interest. One is that the lowest energy band should be sufficiently flat (so the single-particle contributions are not important) and the other is that \(N_x / N_y\) should not be too large or too small (so the system is not in the thin torus limit \([61–63]\)). These conditions can be met using \(N^b = N^f\), \(q^b = 6\), \(q^f = 9\) at \(\mu = 1\), but no suitable parameters are found at \(\mu = 2\).

The full problem contains multiple energy bands so exact diagonalization cannot be performed on any systems with reasonable sizes. To this end, we project the Hamiltonian to the lowest bands of the lattice model in the same spirit as the LLL approximation adopted in the continuum model. The term “lowest bands” should be
The bosonic and fermionic hopping terms in Eq. 4 give us two independent sets of energy bands (see Fig. 5), and the interactions in Eq. 5 are projected to the lowest bands of each species. When the particles are treated together in the full system, it is only translationally invariant with respect to the super magnetic unit cell that contains multiple bosonic (fermionic) magnetic unit cells. This means that the bosonic and fermionic lowest bands are folded to produce some bands for the full system. The energy levels are labeled by the lattice momenta \( 2\pi \text{LCM}(q_x,q_y)K_x/N_x \) and \( 2\pi K_y/N_y \) defined with respect to the super magnetic unit cell. The two energy spectra in Fig. 6 both have a unique ground state.

The lattice model can be studied using DMRG without lowest band projection. The DMRG algorithm is a variational method within the class of matrix product states. It is designed for one-dimensional (1D) systems and prefers open rather than periodic boundary condition. To apply DMRG in 2D models, a 2D lattice should be mapped to a 1D chain as in Fig. 7 (a). This process poses substantial numerical challenge because many short-range hoppings in 2D are converted to long-range hoppings in 1D. The 2D lattice is chosen to be open (periodic) along the \( x \) (\( y \)) direction, so there is no long-range hopping between the two edges of the cylinder. An important difference between DMRG and exact diagonalization is the choice of \( U_{bh} \). Since the computational cost of DMRG depends on the Hilbert space dimension on each site, the hard-core condition \( U_{bh} = \infty \) is adopted such that the dimension assumes the minimal value 2. It would be impossible to do the calculation if the number of bosons on each lattice site has no upper bound (other than the total numbers of bosons).

Edge states on lattice can be probed using entanglement spectrum. The lattice is divided into left and right halves separated by a virtual edge in the middle. The reduced density matrix of the right half is used to define the entanglement spectrum similar to what was done on sphere. The 1,2,5 counting appears in Fig. 7 (b) when the levels are plotted versus the good quantum numbers of the right half (the numbers of particles and the momentum along \( y \)). The entanglement spectrum on lattice is not as good as that on sphere: there are non-universal levels that do not represent edge states and the degeneracy at \( K_y^R = 7 \) is not very good. To demonstrate that the edge physics is not disproved by these imperfections, we perform finite size scaling analysis of entanglement gaps in Fig. 7 (c) and level splittings in Fig. 7 (d). The entanglement gap of a sector is the separation between the top level in the 1,2,5 counting and the lowest non-universal level. The level splittings are the separations between the two pairs of levels that would be degenerate for perfectly linear edge modes. It is difficult to be very conclusive, but the results suggest that the entanglement gaps remain finite and the level splittings decay to zero as \( N_y \to \infty \).

V. CONCLUSIONS

In summary, we have proposed a class of topological states for 2D Bose-Fermi mixture in synthetic gauge fields. The analysis based on wave functions and field theory reveal that these states have no fractionalized excitations but maximally chiral edge states. For all previously known topological states, these two features only appear simultaneously in the IQH states of free fermions and the \( E_0 \) state of interacting bosons. It is corroborated by numerical calculations that some proposed states can be realized in simple continuum and lattice models. This paper demonstrates that mixing bosons and fermions can potentially lead to a plethora of topological states. The three parts in Eq. 7 may be replaced to construct other wave functions: the first part could be non-Abelian bosonic FQH states or composite fermion liquid, the second part could be any fermionic FQH states, and the third part could have another power. One may use more sophisticated approaches such as conformal field theory and parton theory to construct Bose-Fermi topological states [64, 65]. These intriguing questions are left for future works.
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Appendix A: Hamiltonian Matrix Elements

The magnetic length for the $\sigma$ component is $\ell_\sigma = \sqrt{\hbar/B^\sigma}$. The coefficients $V_{m_{1}m_{2}m_{4}m_{3}}^{\sigma\tau\tau\sigma}$ are

$$
\int d^2r_1 d^2r_2 \left[ \psi_{m_{1}}^{F^\sigma} (r_1) \right]^* \left[ \psi_{m_{2}}^{F^\tau} (r_2) \right]^* 4\pi f_{\delta^G}(r_1-r_2) \psi_{m_{4}}^{E^\tau} (r_2) \psi_{m_{3}}^{F^\sigma} (r_1)
$$

(A1)

1. Torus

A torus can be constructed from a rectangle by imposing periodic boundary conditions (PBCs) in both directions. If the torus is spanned by the vectors $\mathbf{L}_1 = L_1 \hat{e}_x$ and $\mathbf{L}_2 = L_2 \hat{e}_y$, the reciprocal lattice vectors are $\mathbf{G}_1 = 2\pi \hat{e}_x/L_1$ and $\mathbf{G}_2 = 2\pi \hat{e}_y/L_2$. In the Landau gauge $A^\sigma = (0, B^\sigma x, 0)$, the LLL single-particle wave functions on a torus with $F^\sigma$ magnetic fluxes are

$$
\psi_{m}^{F^\sigma} = \frac{1}{(\sqrt{\pi}L_2 \ell_\sigma)^{1/2}} \sum_k \exp \left\{ -\frac{i}{2} \left[ \frac{x}{\ell_\sigma} - \frac{2\pi \ell_\sigma}{L_2} (m + kF^\sigma) \right]^2 + \frac{2\pi y}{L_2} (m + kF^\sigma) \right\}
$$

(A2)

The magnetic length is related to the lengths of torus via $L_1L_2 = 2\pi F^b \ell_b^2 = 2\pi F^f \ell_f^2$. The coefficients $V_{m_{1}m_{2}m_{4}m_{3}}^{\sigma\tau\tau\sigma}$ are

$$
\frac{1}{F^G} \sum_{m_{1}} \sum_{m_{2}} \sum_{q_{1},q_{2}} \exp \left\{ -\frac{q_{1}^2}{4} (\ell_\sigma^2 + \ell_\tau^2) + i2\pi q_1 \left[ \frac{(m_{1} - q_{2}/2)}{F^\sigma} - \frac{(m_{2} + q_{2}/2)}{F^\tau} \right] \right\} \delta_{m_{1}+m_{2},m_{3}+m_{4}}
$$

(A3)

where $\mathbf{q} = q_{1}\mathbf{G}_1 + q_{2}\mathbf{G}_2$, $F^G$ is the greatest common divisor of $F^b$ and $F^f$, and $\delta_{i,j}^F$ is a generalized Kronecker delta function defined as

$$
\delta_{i,j}^F = 1 \text{ iff } i \text{ mod } F^G = j \text{ mod } F^G
$$

(A4)

The many-body eigenstates are labeled by the total momentum $Y \equiv (\sum_{\sigma=b,f} \sum_{i=1}^{N_{\sigma}} m_{\sigma}^i) \text{ mod } F^G$.

2. Sphere

A magnetic monopole at the center of a sphere creates a uniform magnetic field through its surface. The LLL single-particle wave functions on a sphere with $F^\sigma$ magnetic fluxes are [66]

$$
\psi_{m}^{F^\sigma} (\theta, \phi) = \left[ \frac{F^\sigma + 1}{4\pi} \left( \frac{F^\sigma}{F^\sigma - m} \right) \right]^{1/2} u^{F^\sigma/2 + m} v^{F^\sigma/2 - m}
$$

(A5)

where $\theta$ and $\phi$ are the azimuthal and radial angles, $u = \cos(\theta/2)e^{i\phi/2}$, $v = \sin(\theta/2)e^{-i\phi/2}$ are spinor coordinates, and $m$ is the $z$ component angular momentum. The magnetic length is related to the radius of the sphere via $R = \ell_b \sqrt{F^b/2} = \ell_f \sqrt{F^f/2}$. The coefficients $V_{m_{1}m_{2}m_{4}m_{3}}^{\sigma\tau\tau\sigma}$ are

$$
\frac{1}{R^2} S_{m_{1}m_{2}} S_{m_{3}m_{4}} \delta_{m_{1}+m_{2},m_{3}+m_{4}}
$$

(A6)
FIG. B1. (a) Energy spectrum of the continuum model at $\mu = 2$ on disk. The system parameters are given as $[N_b, N_f, F_b, F_f]$ in each panel. The dots represent the wave functions in Eq. 9 with their composite fermion configurations given in panel (b).

where $S_{m_1 m_2}$ and $S_{m_3 m_4}$ are defined by

$$\psi_{m_1}^{F_\sigma} \psi_{m_2}^{F_\tau} = (-1)^{F_\sigma - F_\tau} \psi_{m_1 + m_2}^{F_\sigma + F_\tau} S_{m_1 m_2}$$

$$\psi_{m_3}^{F_\sigma} \psi_{m_4}^{F_\tau} = (-1)^{F_\sigma - F_\tau} \psi_{m_3 + m_4}^{F_\sigma + F_\tau} S_{m_3 m_4}$$

(A7)

The many-body eigenstates are labeled by the total angular momentum $L$ and its z component $L_z$.

3. Disk

The LLL single-particle wave functions on a disk with magnetic length $\ell_\sigma$ are

$$\psi_m^{F_\sigma}(x,y) = \frac{z^m \exp[-|z|^2/(4\ell_\sigma^2)]}{\ell_\sigma^{m+1} \sqrt{2\pi^m m!}}$$

(A9)

The coefficients $V^{F_\sigma F_\tau}_{m_1 m_2 m_3 m_4}$ are

$$\frac{2g_\sigma g_\tau^{m_2+m_4} g_\sigma^{m_1+m_3}}{(g_\sigma^2 + g_\tau^2)^{\sum_i (m_i/2)+1}} \frac{\Gamma \left[ \sum_i (m_i/2) + 1 \right]}{\prod_i m_i!} \delta_{m_1+m_2, m_3+m_4}$$

(A10)

where $g_\sigma, \tau = \ell_\sigma, \tau / \ell_b$ and $\Gamma(x)$ is the gamma function.

Appendix B: Edge Physics at $\mu = 2$

The edge states at $\mu = 2$ is more complicated than the $\mu = 1$ case. For the disk geometry, the numbers of states in each LL are infinite. The $\chi_2$ state is ambiguous for a finite size system on disk because one has the freedom to decide the numbers of particles in each LL. The true ground state on disk can only be determined after the confinement potential is specified, but we have not studied such cases for simplicity. For each valid finite size representation of $\chi_2$, we can construct the wave function Eq. 9 and compare it with exact diagonalization result. The two examples in Fig. B1 demonstrate that the wave functions are accurate approximations of the exact eigenstates.

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