Decoupling Cross-Quadrature Correlations using Passive Operations

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Quadrature correlations between subsystems of a Gaussian quantum state are fully characterised by its covariance matrix. For example, the covariance matrix determines the amount of entanglement or decoherence of the state. Here, we establish when it is possible to remove correlations between conjugate quadratures using only passive operations. Such correlations are usually undesired and arise due to experimental cross-quadrature contaminations. Using the Autonne–Takagi factorisation, we present necessary and sufficient conditions to determine when such removal is possible. Our proof is constructive and whenever it is possible, we obtain an explicit expression for the required passive operation.

I. INTRODUCTION

The decomposition of Gaussian quantum systems has proven to be a fruitful subject of research. For instance, the textbook examples of Williamson [1, 2] and Braunstein [3] tell us that any Gaussian state can be decomposed through beamsplitters, phase shifters and single-mode squeezers into uncorrelated thermal states. This is useful for designing quantum gates [4]. More generally, instead of demanding the complete diagonalisation of the state, it can also be transformed into another that has specific kinds of correlations. An early example of this are the Simon and Duan et al.’s standard forms [5, 6]: using local squeezing and phase shifts to bring an entangled state into some standard form of correlations. This turned out to be important in advancing our understanding of Gaussian entanglement.

All the transformations above require the use of active operations and bring the state to a form that does not have any cross-quadrature correlations. Active operations are those that require an external source of energy, for example squeezing, while passive operations are those that do not [7]. Active operations are usually more difficult to implement in a real device compared to passive operations which can be implemented almost free of errors using beamsplitters and phase shifts [8]. When restricted to only passive operations, a generic Gaussian state cannot be diagonalised; it can only be brought to standard forms that remains correlated. There exist conditions with which one can check whether a Gaussian state can be diagonalised by a passive operation [2, 9].

Here, instead of requiring the state to be fully diagonalised, we report a necessary and sufficient condition under which the correlations between conjugate quadrature variables can be entirely removed using passive operations only. This is stated in the following theorem.

Theorem 1. Let \( a = [a_1, \ldots, a_n, a_1^†, \ldots, a_n^†] \) be a vector collecting the annihilation and creation operators of \( n \)-modes. Then, given an \( n \)-mode Gaussian state \( \rho \) with complex covariances

\[
S_{jk} = \text{Tr} \left[ \rho (a_j a_k + a_k^† a_j^†) \right] = \begin{bmatrix} X & Y \\ Y^* & X^* \end{bmatrix}_{jk},
\]

and the Autonne–Takagi factorisation of \( Y: Y = Z^† Y_0 Z \), \( S \) can be brought into a cross-quadrature decorrelated form using passive operations if and only if the entries of \( ZXZ^† \) consist of only purely real or purely imaginary numbers. Furthermore, the required passive operation, up to swapping of quadratures, is given by \( Z \).

The crux of the theorem is the diagonalisation of \( Y \), which is given to us by the Autonne–Takagi factorisation [10, 11].

Theorem 2 (Autonne–Takagi factorisation). Let \( Y \) be a complex, square and symmetric matrix. Then there exists a unitary matrix \( Z \) such that \( Y = Z^† Y_0 Z \), with \( Y_0 \) real, non-negative and diagonal.

Essentially, the physical situation of interest is a correlated state with unwanted correlations between some of the conjugate quadratures and we are concerned with the conditions under which these unwanted correlations can be removed using only passive operations. We mean “conjugate quadratures” in a more general sense—any quadrature pairs, \( q_j \) and \( p_k \) with \( j \) not necessarily equal to \( k \) and where \( \{q_j, p_k\} = i \delta_{jk} \). In other words, the theorem identifies those states that are composed of \( q \)-correlations and \( p \)-correlations plus passive operations. As a corollary, it also identifies states which cannot be constructed by passive operations on initially uncorrelated, squeezed or otherwise, single modes. The proof of the theorem is constructive in that the required passive operation is obtained whenever it exists. It turns out to be, up to local rotations, just \( Z \) given by the Takagi’s factorisation, which is very convenient.

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We note that Takagi’s factorisation makes its appearance in multimode quantum optics [12, 13] that resembles the approach we have taken here, but there is one important difference—we consider the application of Takagi directly to the quantum state rather than the decomposition of unitaries for determining supermodes as is the case in multimodal theories.

II. PROOF OF THEOREM 1

In what follows, we proof Theorem \( \mathbb{I} \). We work with the complex covariance matrix which can be obtained from the quadrature covariance matrix by the change of variables [14]

\[
a_j = \frac{q_j + ip_j}{\sqrt{2}} \quad \text{and} \quad a_j^* = \frac{q_j - ip_j}{\sqrt{2}}.
\]

The reason for working in such a basis is twofold. Firstly, the conjugate quadratures have vanishing correlations if and only if both matrices \( X \) and \( Y \) are real. Secondly, passive operations take the simple form

\[
\begin{pmatrix}
E & 0 \\
0 & E^T
\end{pmatrix}
\]

with \( E \) unitary due to the symplectic conditions. A direct calculation shows that the covariance matrix transforms as \( E: (X, Y) \mapsto (EX^T, EY^T) \) under passive operations, whence it follows that the problem of decoupling conjugate variables is reduced to finding a unitary matrix \( E \) such that \( EX^T \) and \( EY^T \) are simultaneously real. We can now proceed to prove the main result.

Forward direction. Suppose \( S \) is the covariance matrix of a state \( \rho \) whose cross-quadrature correlations can be removed by a passive operation \( Q \). In other words, after applying \( Q \), the cross-quadrature correlations \( \{q_j, p_k\} = 0 \), where to simplify notations, we use \( \{q_j, p_k\} \) to mean \( \text{Tr}[\rho(q_j p_k + p_j q_k)] \). In the complex representation, denoting the transformed matrix as \( X_1 = QXQ^\dagger \) and \( Y_2 = QYQ^T \), the transformed covariance matrix has entries

\[
[X_1]_{jk} = \{a_j, a_k\} = \frac{\{q_j, q_k\} + \{p_j, p_k\}}{2},
\]

\[
[Y_2]_{jk} = \{a_j, a_k\} = \frac{\{q_j, q_k\} - \{p_j, p_k\}}{2},
\]

which are purely real. Since \( Y_2 \) is a real symmetric matrix, it has a spectral decomposition \( Y_2 = R_2^T Y_1 R_1^T \) [13], where \( R_1 \) is a real orthogonal matrix and \( Y_1 \) is a real (but not necessarily positive) diagonal matrix whose entries are the eigenvalues of \( Y_2 \). To obtain the Takagi decomposition, consider a passive unitary (but not necessarily real) transformation \( R: (a_j, a_j^*) \mapsto (ia_j, -ia_j^*) \) on \( Y_1 \) for every \( j \in J \) where \( J \) is the set containing all index \( j \) for which \( [Y_1]_{jj} \) is negative. This corresponds to a rotation of the quadratures \( R: (q_j, p_j) \mapsto (p_j, -q_j) \) for \( j \in J \). In matrix form, \( R \) is diagonal with entries

\[
[R]_{jk} = \begin{cases} 
1 & \text{for } j = k \neq J, \\
i & \text{for } j = k \in J, \\
0 & \text{for } j \neq k.
\end{cases}
\]

Applying this to \( Y_1 \) brings it to a non-negative diagonal matrix \( Y_0 = R Y_1 R^T \) since

\[
R : \{a_j, a_j\} \mapsto \begin{cases} 
-\{a_j, a_j\} & \text{for } \{a_j, a_j\} < 0, \\
\{a_j, a_j\} & \text{for } \{a_j, a_j\} \geq 0.
\end{cases}
\]

Putting everything together, we arrive at the Takagi decomposition of \( Y \) as

\[
Y = Q^\dagger R_1^T R_1 Y_0 R^T R_1^T Q^*.
\]

Then \( X \) transforms as

\[
ZXZ^\dagger = RR^T QXQ^\dagger R^T R^T,
\]

where \( X_0 \) is a real (symmetric) matrix since both \( X_1 \) and \( R_1 \) are real. Finally, the transformation \( R \) puts an \( i \) on the rows of \( X_0 \) for \( j \in J \) and a \( -i \) on the columns for \( j \in J \) which completes the proof.

Reverse direction. Reversing the forward proof, given \( ZXZ^\dagger \) consists of only purely real or imaginary numbers, we can construct a transformation \( R \) that makes both \( X \) and \( Y \) simultaneously real. When \( X \) and \( Y \) are simultaneously real, it follows from direct substitution that the covariance matrix has no cross-quadrature correlations.

What does this mean? It means that we have a way of testing if the correlations between conjugate variables can be removed—diagonalise \( Y \) to obtain the matrix \( Z \) using the Autonne–Takagi factorisation and subsequently compute \( ZXZ^\dagger \). If any of the entries are neither purely

![FIG. 1. The output state with quadrature covariance matrix given by (2), has cross-quadrature correlations that cannot be removed by passive operations. AM: Amplitude modulator. RNG: Gaussian random number generator with variance 1. R(\frac{\pi}{4}): \pi/4 phase shifter. SQZ: 3dB squeezer.](image-url)
real nor purely imaginary, then the correlations cannot be decoupled. If all the entries are purely real, then $Z$ is the passive operation that we are after. If some entries are purely imaginary then in addition to $Z$, additional local rotations $R$ are required.

When $S$ corresponds to a pure state, the matrix $Z$ gives the passive operation required to create it from a product of independent squeeze-states. However if $S$ is mixed, our result implies that it is sometimes impossible to create by passive operations on any independent states, or even on states possessing only $q$-correlations and $p$-correlations. One example is the state with quadrature covariance matrix

$$S = \begin{pmatrix} 3 & 0.5 & 1 & 0 \\ 0.5 & 0.75 & 0.5 & 0 \\ 1 & 0.5 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which can be created by the scheme in Fig. The squeezing operation “locks in” the cross-quadrature correlations and makes it impossible to be removed using passive operations only.

### III. TWO-MODE EXAMPLE

We illustrate our result by working through an example. Consider a two-mode Gaussian state having the following quadrature covariance matrix

$$S = \begin{pmatrix} m & 0 & c & 0 \\ 0 & m & 0 & -c \\ c & 0 & n & s \\ 0 & -c & n & s \end{pmatrix}$$

with all $m$, $n$, $c$ and $s$ positive. We want to determine if this state can be brought into a cross-quadrature decorrelated form. The basis transformation (II) represented by the unitary matrix

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}$$

transforms the quadrature covariance matrix into the complex covariance matrix

$$S = LS^\dagger = \begin{pmatrix} m & 0 & c & 0 \\ 0 & n & c & is \\ 0 & c & m & 0 \\ c & -is & 0 & n \end{pmatrix},$$

which identifies $X$ and $Y$ as

$$X = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & c \\ c & is \end{pmatrix}.$$

The Autonne–Takagi factorisation of $Y = Z^\dagger Y_0 Z$ is given by

$$Z = e^{i\pi/4} \begin{pmatrix} -i\sqrt{1-t} & \sqrt{1-t} \\ \sqrt{1-t} & i\sqrt{1-t} \end{pmatrix}$$

and

$$Y_0 = \frac{1}{2} \begin{pmatrix} \sqrt{4c^2 + s^2} - s & 0 \\ 0 & \sqrt{4c^2 + s^2} + s \end{pmatrix}$$

with $t = (1 + s/\sqrt{4c^2 + s^2})/2$. Finally, we find

$$ZX^\dagger = \begin{pmatrix} n(1-t) + mt & -i\sqrt{1-t}(m-n) \\ i\sqrt{1-t}(m-n) & nt + m(1-t) \end{pmatrix}$$

has entries that are all purely real or purely imaginary which means that the state $S$ can be brought to a cross-quadrature decorrelated form.

The passive operation that does this is $R^\dagger Z$ where

$$R = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

so that

$$R^\dagger ZX^\dagger R = \begin{pmatrix} n(1-t) + mt & \sqrt{1-t}(m-n) \\ \sqrt{1-t}(m-n) & nt + m(1-t) \end{pmatrix}$$

is a purely real matrix. This can be factorised as

$$R^\dagger Z = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} \sqrt{1-t} & -e^{i\pi/2} \\ -e^{-i\pi/2} & \sqrt{1-t} \end{pmatrix} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

which is realised by a beamsplitter of transmissivity $t$ and three phase shifts: $\pi/4$ and $-3\pi/4$ at the outputs and $-\pi/2$ at the input port.

The expert reader might have recognised that the state $S$ can in fact be cross-quadrature decorrelated through the simpler transformation

$$R^\dagger Z = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

requiring just two phase shifts. This shows that when it is possible to decorrelate the conjugate quadratures, the procedure we presented is not the only way to do so. The condition that $Y$ be diagonalised can be relaxed—all we need to decouple $q$ and $p$ for $Y$ to be transformed into a real matrix after applications of the passive operation—this real matrix need not be diagonal or non-negative. In terms of implementations, this would mean that the required operation might be simpler, for instance we can do away with the beamsplitter in the example considered.

### IV. DISCUSSIONS

An immediate application of the theorem is to the calculation of the “squeezing of formation” [10]. This quantity measures how much squeezing is required to create
a given state and indicates the degree of nonclassicality of the state. The squeezing of formation is invariant under passive operations because these transformations do not require any squeezing, which means that the result of this work can be used to simplify complicated states to a form in which the squeezing of formation can be directly calculated.

There is also an interesting connection with the generation of cluster states. A cluster state has multiple quantum modes with correlations between each mode \(17\). Many of these can be shown to possess correlations only between the \(q\)'s and between the \(p\)'s, such as the two-dimensional square cluster. However, in real devices for generating cluster states there are imperfections which give rise to correlations between \(q\) and \(p\). This implies that the theorem might be useful for identifying if an ideal cluster state can be recovered using only passive operations.

What can be said about a state whose cross-quadrature correlations cannot be removed by passive operations? While most theoretical work on Gaussian quantum in-

formation consider cross-quadrature decorrelated states, almost every state realised experimentally would have some cross-quadrature correlations that cannot be decoupled using only passive operations. However, if we are also allowed to add correlated noise in the form of random Gaussian quadrature displacements, then any state can be cross-quadrature decorrelated. One obvious question is then: what is the least amount of noise required to achieve such decorrelation?

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