Flavor Oscillations in Field Theory

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Abstract

Neutrino flavor oscillations are discussed in terms of two coupled Dirac fields. The interacting Lagrangian is diagonalized to obtain the exact eigenvalues and eigenfunctions. Flavor wave functions are then derived directly from the quantized neutrino fields. Probability densities obtained by squaring these wave functions upon taking into account the neutrino chirality are in agreement with the standard neutrino oscillation probabilities.

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I. INTRODUCTION

The theory and phenomenology of neutrino flavor mixing has been extensively studied mainly in the framework of quantum mechanics [1–3]. Only very recently, a quantum field theoretical analysis of flavor mixing has been considered using the LSZ formalism [4].

Given the conflicting experimental results which have been obtained from a variety of neutrino observations, it is important to investigate the problem of neutrino propagation in the general context of field theory to reassure ourselves that there are indeed no differences between these results and standard ones obtained from the quantum mechanical treatment. Otherwise said, we need to firmly establish the approximations necessary to derive the expressions used phenomenologically. It is perhaps not superfluous to point out that in the Dirac theory, the contribution of negative energy states becomes substantial, an aspect of the problem totally absent in the quantum mechanical treatment.

Our discussion in the present paper will be through the following Lagrangian

\[ L = \bar{\psi}_e (i \gamma \cdot \partial - m_e) \psi_e + \bar{\psi}_\mu (i \gamma \cdot \partial - m_\mu) \psi_\mu - \delta (\bar{\psi}_e \psi_\mu + \bar{\psi}_\mu \psi_e), \]  

consisting of two coupled Dirac neutrino fields \( \psi_e, \psi_\mu \) with masses \( m_e \) and \( m_\mu \). The interaction is provided through a lepton number violating term with a coupling constant \( \delta \). The model allows for exact diagonalization. Neutrino and anti-neutrino flavor wave functions can be directly obtained as matrix elements of the quantized neutrino fields.

The fully interacting Lagrangian should also include the weak interaction term. It is ignored here (as is done usually), since we are interested in free propagation and possible oscillation in flavor alone. For its production, we simply assume that the neutrino is created through some weak interaction process. Such a breakup is of course an approximation. The effect of parity non-conservation due to the weak interaction term on the other hand, is taken into account by considering the left-handed (right-handed) components of the neutrino (anti-neutrino) flavor wave functions.

The model discussed here represents an example (albeit simple) of an interacting field theory which is exactly solvable. This may be of some help in elucidating the properties
of interacting theories, which can be normally studied only in perturbation theory. For example, the interaction between the electron and muon neutrino fields produces a change in their masses: the “experimental” masses $m_1, m_2$ depend on the “bare” muon and electron neutrino masses $m_e, m_\mu$ and the coupling constant $\delta$.

The paper is organized as follows. In Sec. 2, the standard neutrino oscillation phenomenology is reviewed. In Sec. 3, the two coupled Dirac equations, obtained from the Lagrangian (1.1) are solved. The electron and muon neutrino fields are then quantized in terms of the two free uncoupled fields, which diagonalize Eq. (1.1), as described in Sec. 4. The total conserved charge is the sum of the electron and muon flavor charges, which are not conserved separately. The last section closes with some concluding remarks.

II. STANDARD TREATMENT

We will critically review the standard quantum mechanical treatment of neutrino oscillations to bring out the essential approximations made implicitly and the boundary conditions which are imposed.

A state vector $|\psi>$ is introduced as a linear combination of the flavor eigenstates $|e>$ and $|\mu>$ (assuming just two flavors)

$$|\psi> = C_e|e> + C_\mu|\mu>,$$

$$|\psi> = \begin{pmatrix} C_e \\ C_\mu \end{pmatrix},$$

with $C_e = <e|\psi>$, $C_\mu = <\mu|\psi>$ and

$$|C_e|^2 + |C_\mu|^2 = 1.$$  \hspace{1cm} (2.1c)

$C_e$ and $C_\mu$ then become the amplitudes for detecting an electron neutrino and a muon neutrino respectively.

To derive the time evolution of the coefficients $C_e(t)$ and $C_\mu(t)$, the state vector $|\psi>$ is written as a superposition of the energy (mass) eigenstates $|\nu_1>$ and $|\nu_2>$

$$|\psi> = C_1|\nu_1> + C_2|\nu_2>,$$

$$|\psi> = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

with $C_1 = <\nu_1|\psi>$ and $C_2 = <\nu_2|\psi>$. The total conserved charge is the sum of the electron and muon flavor charges, which are not conserved separately. The last section closes with some concluding remarks.
\[ |\psi> = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (2.2b) \]

with \( C_1 = <\nu_1 | \psi> \), \( C_2 = <\nu_2 | \psi> \) and

\[ |C_1|^2 + |C_2|^2 = 1, \quad (2.2c) \]

where \( C_1 \) and \( C_2 \) are the amplitudes for finding the neutrino in the energy states \( E_1 \) and \( E_2 \) respectively. These coefficients evolve in time as

\[ C_1(t) = C_1(0)e^{-iE_1t}, \quad C_2(t) = C_2(0)e^{-iE_2t}. \quad (2.3) \]

Introducing the rotation matrix between flavor and mass eigenstates

\[ \begin{pmatrix} |\nu_1> \\ |\nu_2> \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} |e> \\ |\mu> \end{pmatrix}, \quad (2.4) \]

it is easy to see that the following relation between the energy and flavor amplitudes holds

\[ \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} C_e(t) \\ C_\mu(t) \end{pmatrix}. \quad (2.5) \]

Hence, the time evolution of the coefficients \( C_e \) and \( C_\mu \) is given by

\[ \begin{pmatrix} C_e(t) \\ C_\mu(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} C_1(0)e^{-iE_1t} \\ C_2(0)e^{-iE_2t} \end{pmatrix}. \quad (2.6) \]

In Eq. (2.6), the boundary condition that at production we have only a given flavor must be imposed. Suppose for example that at \( t = 0 \) a muon neutrino is produced, i.e.

\[ C_\mu(0) = 1, \quad C_e(0) = 0. \quad (2.7) \]

\( \dot{\text{From Eq. (2.5) at } t = 0 \text{ we obtain}} \]

\[ C_1(0) = -\sin(\theta), \quad C_2(0) = \cos(\theta). \quad (2.8) \]

The time evolution of the flavor amplitudes is obtained by substituting Eq. (2.8) in Eq. (2.6)

\[ C_e(t) = \sin(\theta)\cos(\theta)(e^{-iE_2t} - e^{-iE_1t}), \quad (2.9a) \]
\[ C_\mu(t) = \sin^2(\theta)e^{-iE_1t} + \cos^2(\theta)e^{-iE_2t}. \]  

(2.9b)

Space and therefore momentum is introduced by assuming in Eqs. (2.9) \( E_1^2 = m_1^2 + p^2 \), \( E_2^2 = m_2^2 + p^2 \) and \( x = t \). The probability of finding a given flavor is obtained by squaring Eqs. (2.9)

\[ |C_e|^2 = \sin^2(2\theta)\sin^2[(E_2 - E_1)t/2] \simeq \sin^2(2\theta)\sin^2\left(\frac{(m_1^2 - m_2^2)x}{2p}\right), \]  

(2.10a)

\[ |C_\mu|^2 = 1 - \sin^2(2\theta)\sin^2[(E_2 - E_1)t/2] \simeq 1 - \sin^2(2\theta)\sin^2\left(\frac{(m_1^2 - m_2^2)x}{2p}\right). \]  

(2.10b)

The assumption that the muon neutrino is created with a definite momentum \( p \) is only an approximation as has been pointed out previously [5–8]. It is in contradiction with four momentum conservation, for example for the reaction \( \pi \to \mu \nu \). Each of the possible energy eigenstates has a somewhat different momentum \( p_i \). In the rest frame of the pion, energy conservation dictates that \( (i = 1, 2) \)

\[ M_\pi = \sqrt{M_\mu^2 + p_i^2} + \sqrt{m_i^2 + p_i^2}. \]  

(2.11)

Therefore, if we introduce momentum, i.e. space in Eqs. (2.9) we should write

\[ C_e(x, t) = \sin(\theta)\cos(\theta)(e^{ip_1x}e^{-iE_1t} - e^{ip_2x}e^{-iE_2t}), \]  

(2.12a)

\[ C_\mu(x, t) = \sin^2(\theta)e^{ip_1x}e^{-iE_1t} + \cos^2(\theta)e^{ip_2x}e^{-iE_2t}. \]  

(2.12b)

In the relativistic approximation \( x \simeq t \) the squared moduli of the amplitudes \( C_e(x, t) \), \( C_\mu(x, t) \) reduce to the standard neutrino oscillation probabilities given by Eqs. (2.10), as described in Ref. [3]. The semiclassical approximation that a neutrino moves at a velocity close to \( c \) (assuming very small neutrino masses) “on a classical path” \( (x = ct) \) can be a good approximation if the neutrino travels over a macroscopic distance.

Hence, if a muon neutrino is produced at the space-time \( (x \simeq ct = 0) \), the probability of observing an electron neutrino is maximum for those space-time points \( x \simeq ct \) at which \( |C_e|^2 = 1 \). If some matter is present at these points, processes such as \( \nu_e + n \to p + e^- \), will occur, but not processes of the type \( \nu_\mu + n \to p + \mu^- \) indicating therefore flavor oscillations.
III. FIELD THEORETICAL DISCUSSION

Since neutrinos are relativistic particles of spin 1/2, it is important to derive a relativistic equation of motion which can describe such flavor mixing. Energy eigenfunctions can be derived from this equation and one proper way to deal with states of negative energies is to quantize the field. As stated in the Introduction, we will consider the following interacting Lagrangian

$$\mathcal{L} = \bar{\psi}_e (i \gamma \cdot \partial - m_e) \psi_e + \bar{\psi}_\mu (i \gamma \cdot \partial - m_\mu) \psi_\mu - \delta (\bar{\psi}_e \psi_\mu + \bar{\psi}_\mu \psi_e).$$  (3.1)

The parameter $\delta$ is an extra mass (energy) related to the small amplitude that a neutrino can flip flavor. The following two coupled Dirac equations

$$ (i \gamma \cdot \partial - m_e) \psi_e - \delta \psi_\mu = 0, \quad (3.2) $$

$$ (i \gamma \cdot \partial - m_\mu) \psi_\mu - \delta \psi_e = 0, \quad (3.3) $$

describe a neutrino which has some probability to flip flavor. If no flavor flips were possible ($\delta = 0$), the rest energies (masses) of the system would be $m_e$ and $m_\mu$, possibly equal to zero. However, since there is some amplitude that a neutrino, which is produced as an electron neutrino becomes later a muon neutrino, the possible rest energies of the system are not simply $m_e$ and $m_\mu$, but are functions of the flipping energy.

It is easy to see that the conserved current is

$$ J^\mu = \bar{\psi}_e \gamma^\mu \psi_e + \bar{\psi}_\mu \gamma^\mu \psi_\mu = J^\mu_e + J^\mu_\mu. $$  (3.4)

Thus, the separate electron and muon flavor currents are not conserved, only their sum is.

In order to determine the energy eigenvalues and eigenfunctions of the system of equations (3.2) and (3.3) we consider the ansatz

$$ \psi_e = a e^{-iP x}, \quad (3.5) $$

$$ \psi_\mu = b e^{-iP x}, \quad (3.6) $$
where \( P \) is the four-momentum \( P = (E, p) \), which is unknown and is to be determined so that the system of differential equations (3.5) and (3.6) is satisfied. The coefficients \( a \) and \( b \) are Dirac spinors, which can be written as

\[
a = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad (3.7a)
\]

\[
b = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (3.7b)
\]

where \( \chi_{1,2} \) and \( \varphi_{1,2} \) are two component vectors. Substituting Eqs. (3.5), (3.6) into Eqs. (3.2), (3.3), we obtain the system of linear homogeneous equations

\[
E \chi_1 = \sigma \cdot p \chi_2 + m_e \chi_1 + \delta \varphi_1, \quad E \chi_2 = \sigma \cdot p \chi_1 - m_e \chi_2 - \delta \varphi_2, \quad (3.8a)
\]

\[
E \varphi_1 = \sigma \cdot p \varphi_2 + m_\mu \varphi_1 + \delta \chi_1, \quad E \varphi_2 = \sigma \cdot p \varphi_1 - m_\mu \varphi_2 - \delta \chi_2, \quad (3.8b)
\]

where \( \sigma \) are the Pauli matrices.

The system of Eqs. (3.8) admits non trivial solutions only if

\[
E^4 - E^2(2p^2 + 2\delta^2 + m_e^2 + m_\mu^2) + p^4 + \delta^4 + p^2(2\delta^2 + m_e^2 + m_\mu^2) + m_e^2 m_\mu^2 - 2\delta^2 m_e m_\mu = 0. \quad (3.9)
\]

Solving Eq. (3.9), we obtain \( (p = |p|) \)

\[
E_{1,2} = \pm \sqrt{p^2 + m_{1,2}^2}, \quad (3.10)
\]

with \( m_{1,2} \) given by

\[
m_{1,2} = \frac{1}{2}[(m_e + m_\mu) \pm R], \quad (3.11)
\]

and

\[
R = \sqrt{(m_\mu - m_e)^2 + 4\delta^2}. \quad (3.12)
\]

Therefore, while in the free Dirac equation there are two energies (one positive and one negative) for every possible value of the momentum \( p \), for a system of two coupled Dirac equations, for every possible value of the momentum, there are four possible values of the
energy, two positive and two negative, due to the possibility of (flavor) oscillations. Also, because there is some chance that the neutrino can flip flavor, the rest energies of the electron and muon neutrino system are not simply $m_e, m_\mu$ but are given by Eq. (3.11).

The solutions of Eqs. (3.8) can be written as

$$
\chi_1 = -\frac{\delta(\sigma \cdot \mathbf{p})(m_e + m_\mu)}{Q} \varphi_2, \quad \chi_2 = \frac{\delta(E^2 - \mathbf{p}^2 - \delta^2 - E(m_e + m_\mu) + m_em_\mu)}{Q} \varphi_2,
$$

$$
\varphi_1 = \frac{\delta(E^2 - \mathbf{p}^2 + \delta^2 - \mathbf{p}^2)}{Q} \varphi_2, \quad \varphi_2
$$

with

$$
Q = -E^3 + E^2 m_\mu + E(\delta^2 + \mathbf{p}^2 + m_e^2) + \delta^2 m_e - \mathbf{p}^2 m_\mu - m_e^2 m_\mu.
$$

As stated above, for a given value of the momentum $\mathbf{p}$, there are four different energies $\pm E_{1,2}$ and for each energy, there are two eigenfunctions with different (up and down) spins.

Corresponding to the positive energy solution $E_1 = \sqrt{m_1^2 + \mathbf{p}^2}$, we have the following two solutions

$$
\psi_1(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_1}} \phi_1(s, \mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}} e^{-iE_1 t},
$$

where $s = 1, 2$ is the spin index and $\phi_1(s, \mathbf{p})$ is given by

$$
\phi_1(s, \mathbf{p}) = \frac{1}{\sqrt{1 + M_1^2}} \begin{pmatrix} u_1(s, \mathbf{p}) \\ M_1 u_1(s, \mathbf{p}) \end{pmatrix},
$$

with

$$
M_1 = \frac{m_\mu - m_e + R}{2\delta}.
$$

and $u_1(s, \mathbf{p})$ is the Dirac spinor

$$
u_1(s, \mathbf{p}) = \sqrt{E_1 + m_1} \begin{pmatrix} \chi^{(s)} \\ \frac{\sigma \cdot \mathbf{p}}{E_1 + m_1} \chi^{(s)} \end{pmatrix},
$$

satisfying the Dirac equation $(\gamma^\mu p_\mu - m_1)u_1(s, \mathbf{p}) = 0$, and $\chi^{(s)}$ satisfies the normalization condition

$$
\chi^{(s)} \chi^{(s)} = 1.
$$
For the other positive energy solution \( E_2 = \sqrt{p^2 + m_2^2} \), we have
\[
\psi_2(x, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_2}} \phi_2(s, p)e^{ip \cdot x}e^{-iE_2 t},
\] (3.19)
with \( \phi_2(s, p) \) given by
\[
\phi_2(s, p) = \frac{1}{\sqrt{1 + M_2^2}} \begin{pmatrix} u_2(s, p) \\ M_2 u_2(s, p) \end{pmatrix},
\] (3.20)
and \( M_2 \) defined as
\[
M_2 = \frac{m_\mu - m_e - R}{2 \delta}.
\] (3.21)

We notice here that because \( M_1 M_2 = -1 \) we can write \( \phi_2(s, p) \) in terms of \( M_1 \) as
\[
\phi_2(s, p) = \frac{1}{\sqrt{1 + M_1^2}} \begin{pmatrix} M_1 u_2(s, p) \\ -u_2(s, p) \end{pmatrix},
\] (3.22)
and \( u_2(s, p) \) is the Dirac spinor
\[
u_2(s, p) = \sqrt{E_2 + m_2} \begin{pmatrix} \chi(s) \\ \sigma p \gamma E_2 + m_2 \chi(s) \end{pmatrix},
\] (3.23)
satisfying the Dirac equation \((\gamma^\mu p_\mu - m_2)u_2(s, p) = 0\).

Similarly for the solutions of negative energies \(-E_{1,2}\) we have the following eigenfunctions:
\[
\psi_3(x, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_1}} \phi_3(s, p)e^{-ip \cdot x}e^{iE_1 t},
\] (3.24)
with \( \phi_3(s, p) \) given by
\[
\phi_3(s, p) = \frac{1}{\sqrt{1 + M_2^2}} \begin{pmatrix} -M_2 v_1(s, p) \\ v_1(s, p) \end{pmatrix} = \frac{1}{\sqrt{1 + M_1^2}} \begin{pmatrix} v_1(s, p) \\ M_1 v_1(s, p) \end{pmatrix},
\] (3.25)
and
\[
v_1(s, p) = \sqrt{E_1 + m_1} \begin{pmatrix} \sigma p \gamma E_1 + m_1 \chi(s) \\ \chi(s) \end{pmatrix},
\] (3.26)

\[
\psi_4(x, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_2}} \phi_4(s, p)e^{-ip \cdot x}e^{iE_2 t},
\] (3.27)
with \( \phi_4(s, p) \) given by
\[
\phi_4(s, p) = \frac{1}{\sqrt{1 + M_2^2}} \begin{pmatrix} M_1 v_2(s, p) \\ -v_2(s, p) \end{pmatrix},
\] (3.28)
and
\[
v_2(s, p) = \sqrt{E_2 + m_1} \begin{pmatrix} \sigma p \gamma E_2 + m_2 \chi(s) \\ \chi(s) \end{pmatrix}.
\] (3.29)
IV. FIELD QUANTIZATION, ANTI-COMMUTATION RELATIONS, AND WAVE FUNCTIONS

We introduce the matrix $U$

$$U = \begin{pmatrix} \frac{1}{\sqrt{1+M_1^2}} & \frac{M_1}{\sqrt{1+M_1^2}} \\ \frac{M_1}{\sqrt{1+M_1^2}} & -\frac{1}{\sqrt{1+M_1^2}} \end{pmatrix},$$

(4.1)

to make the transformation on the field

$$\psi_{\nu} = \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

(4.2)

which uncouples the interacting Lagrangian given by Eq. (3.1)

$$\mathcal{L}_D = \tilde{\phi}_1 (i \gamma \cdot \partial - m_1) \phi_1 + \tilde{\phi}_2 (i \gamma \cdot \partial - m_2) \phi_2.$$

(4.3)

Following relations are useful to see that $U$ uncouples Eq. (3.1)

$$\delta(M_1 + M_2) = m_\mu - m_e; \quad R = \delta(M_1 - M_2); \quad M_1 M_2 = -1.$$

(4.4)

Therefore the fields $\phi_1$ and $\phi_2$ describe the “normal modes”. To quantize $\psi_{\nu}$, we expand $\psi_e$ and $\psi_\mu$ in terms of the normal modes (energy eigenfunctions) found in Sec. 3

$$\hat{\psi}_e(x, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{1 + \frac{1}{M_1^2}}} \sum_p \sum_s \left[ b_1(s, p) \frac{u_1(s, p)}{\sqrt{2E_1}} e^{-iE_1 t} + M_1 b_2(s, p) \frac{u_2(s, p)}{\sqrt{2E_2}} e^{-iE_2 t} \right] e^{ip \cdot x} +$$

$$\left[ d_1^\dagger(s, p) \frac{v_1(s, p)}{\sqrt{2E_1}} e^{iE_1 t} + M_1 d_2^\dagger(s, p) \frac{v_2(s, p)}{\sqrt{2E_2}} e^{iE_2 t} \right] e^{-ip \cdot x},$$

(4.5a)

$$\hat{\psi}_\mu(x, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{1 + \frac{1}{M_1^2}}} \sum_p \sum_s \left[ M_1 b_1(s, p) \frac{u_1(s, p)}{\sqrt{2E_1}} e^{-iE_1 t} - b_2(s, p) \frac{u_2(s, p)}{\sqrt{2E_2}} e^{-iE_2 t} \right] e^{ip \cdot x} +$$

$$\left[ M_1 d_1^\dagger(s, p) \frac{v_1(s, p)}{\sqrt{2E_1}} e^{iE_1 t} - d_2^\dagger(s, p) \frac{v_2(s, p)}{\sqrt{2E_2}} e^{iE_2 t} \right] e^{-ip \cdot x},$$

(4.5b)

where the number operators $b_i$ and $d_i$ (i=1,2) satisfy the anti-commutation relations

$$\{b_i(s, p), b_j^\dagger(s', p')\} = \delta_{ij} \delta_{pp'} \delta_{ss'},$$

(4.6a)

$$\{d_i(s, p), d_j^\dagger(s', p')\} = \delta_{ij} \delta_{pp'} \delta_{ss'},$$

(4.6b)
and all the other anti-commutators are zero. \( b_{1,2}(s, p) \) annihilates the normal mode of positive energy \( E_{1,2} \) and spin \( s \); \( d_{1,2}^{\dagger} \) creates the anti-normal mode of positive energy \( E_{1,2} \) and spin \( s \). The vacuum state is defined by

\[
b_{i}|0> = d_{i}|0> = 0. \tag{4.7}
\]

The total charge operator is

\[
Q = Q_{e} + Q_{\mu} = \int \psi_{e}^{\dagger} \psi_{e} d^{3}x = \int (\psi_{e}^{\dagger} \psi_{e} + \psi_{\mu}^{\dagger} \psi_{\mu}) d^{3}x = \sum_{p} \sum_{s} [b_{1}^{\dagger}(s, p)b_{1}(s, p) + b_{2}^{\dagger}(s, p)b_{2}(s, p) - d_{1}^{\dagger}(s, p)d_{1}(s, p) - d_{2}^{\dagger}(s, p)d_{2}(s, p)]. \tag{4.8}
\]

The following relations hold

\[
[Q, b_{1}^{\dagger}(s, p)] = b_{1}^{\dagger}(s, p) \quad [Q, d_{1}^{\dagger}(s, p)] = -d_{1}^{\dagger}(s, p). \tag{4.9}
\]

Hence, for a given value of the momentum \( p \) and spin \( s \), there are four possible normal mode states

\[
\begin{align*}
b_{1}^{\dagger}(s, p)|0> &= |1_{ps}>, \quad b_{2}^{\dagger}(s, p)|0> &= |2_{ps}>, \tag{4.10a} \\
d_{1}^{\dagger}(s, p)|0> &= |-1_{ps}>, \quad d_{2}^{\dagger}(s, p)|0> &= |-2_{ps}>. \tag{4.10b}
\end{align*}
\]

These states differ for the charge (±1, ±2)

\[
\begin{align*}
Q|1_{ps} > &= |1_{ps} >, \quad Q|2_{ps} > &= |2_{ps} >, \tag{4.11a} \\
Q|-1_{ps} > &= |-1_{ps} >, \quad Q|-2_{ps} > &= |-2_{ps} >. \tag{4.11b}
\end{align*}
\]

The above states allow us to construct wave functions in space-time. For example, the wave function associated with the state \( |1_{ps} > \) is

\[
\psi_{\nu}(x, t) = \begin{pmatrix} \psi_{e}(x, t) \\ \psi_{\mu}(x, t) \end{pmatrix} = \begin{pmatrix} < 0|\hat{\psi}_{e}(x, t)|1_{ps} > \\ < 0|\hat{\psi}_{\mu}(x, t)|1_{ps} > \end{pmatrix} = \left( \frac{1}{\sqrt{1+M_{1}^{2}}} \right) \left( \frac{1}{\sqrt{V}} \right) \frac{1}{\sqrt{2E_{1}}} u_{1}(s, p) e^{ip \cdot x} e^{-iE_{1}t}. \tag{4.12}
\]

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This being a plane wave, gives a stationary probability of finding a neutrino at a given space-time point. However in any location inside the volume \( V \) there is a probability equal to \( \frac{1}{1+M_1^2} \) of finding the neutrino in the electron flavor and probability equal to \( \frac{M_1^2}{1+M_1^2} \) of finding it in the muon flavor. Similar considerations can be applied for the other three states \( |2p, s\rangle, |-1p, s\rangle, \) and \( |-2p, s\rangle \). Therefore, these states represent states of mixed flavor at any given space-time point. To obtain states for which we have only one flavor at a given space-time point we need a superposition of states. A general state of positive charge, momentum \( p \) and spin \( s \) is given by

\[
|\phi_+> = [A b_1^+(s, p) + B b_2^+(s, p)]|0>,
\]

(4.13)

where \( A \) and \( B \) specify the amount of each normal mode state of positive energy present in the state \( |\phi_+> \).

Similarly, a general state of negative charge, momentum \( p \) and spin \( s \) is given by

\[
|\phi_-> = [C d_1^+(s, p) + D d_2^+(s, p)]|0>,
\]

(4.14)

where \( C \) and \( D \) specify the amount of each normal mode of negative energy present in the state \( |\phi_-> \). The matrix element

\[
<0|\hat{\psi}_e(x,t)|\phi_+> = \psi_e(x,t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{1+M_1^2}} \left[ A u_1(s, p) e^{-iE_1t} + M_1 B u_2(s, p) e^{-iE_2t} \right] e^{i p \cdot x},
\]

(4.15a)

gives the probability amplitude of finding a neutrino of momentum \( p \) and spin \( s \) at the space-time point \( (x, t) \) with the electron flavor. In the same way, the matrix element

\[
<0|\hat{\psi}_\mu(x,t)|\phi_+> = \psi_\mu(x,t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{1+M_1^2}} \left[ M_1 A u_1(s, p) e^{-iE_1t} - B u_2(s, p) e^{-iE_2t} \right] e^{i p \cdot x},
\]

(4.15b)

is the probability amplitude for the muon flavor.

Similarly the matrix elements \( <0|\hat{\psi}_e(x,t)|\phi_-> \) and \( <0|\hat{\psi}_\mu(x,t)|\phi_-> \) give the probability amplitudes for finding an electron anti-neutrino flavor and a muon anti-neutrino flavor respectively.
To take into account the fact that neutrinos (anti-neutrinos) are created with negative (positive) chiralities, we define the “observable wave functions” as

\[
\psi_{eL}(x, t) = (1 - \gamma_5)\psi_e(x, t), \quad \psi_{\mu L}(x, t) = (1 - \gamma_5)\psi_\mu(x, t)
\]

where \(\psi_e(x, t)\) and \(\psi_\mu(x, t)\) are given by Eqs. (4.15).

If \(p\) is for example along the z-axis, we can choose \(\chi^{(s)}\) as

\[
\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

and the left-handed Dirac spinor is

\[
\psi_{eL}(x, t) = \psi_{\mu L}(x, t) = \begin{pmatrix} \begin{pmatrix} \chi^{(2)} \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \chi^{(2)} \end{pmatrix} \end{pmatrix} \frac{\sqrt{E + m}}{\sqrt{1 + \frac{p}{E + m}}}
\]

In the limit \(E \gg m\), \((1 - \gamma_5)u(p, s = 1) \simeq 0\).

For \(p\) along an arbitrary direction \(\hat{n}\)

\[
p = p[\hat{e}_x \sin(\theta) \cos(\phi) + \hat{e}_y \sin(\theta) \sin(\phi) + \hat{e}_z \cos(\theta)],
\]

we can choose the spinors \(\chi^{(s)}\) as

\[
\chi^{(1)} = \begin{pmatrix} e^{-\frac{i}{2} \phi} \cos\left(\frac{\theta}{2}\right) \\ e^{\frac{i}{2} \phi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \quad \chi^{(2)} = \begin{pmatrix} -e^{-\frac{i}{2} \phi} \sin\left(\frac{\theta}{2}\right) \\ e^{\frac{i}{2} \phi} \cos\left(\frac{\theta}{2}\right) \end{pmatrix},
\]

and the left-handed spinor is still given by Eq. (4.18).

Hence, the observable flavor neutrino wave functions are

\[
\psi_{eL}(x, t) = \frac{1}{\sqrt{V}} \frac{e^{i p \cdot x}}{\sqrt{1 + M_1^2}} [A \frac{\sqrt{E_1 + m_1}}{\sqrt{2E_1}} (1 + \frac{p}{E_1 + m_1}) e^{-iE_1 t} + \\
M_1 B \frac{\sqrt{E_2 + m_2}}{\sqrt{2E_2}} (1 + \frac{p}{E_2 + m_2}) e^{-iE_2 t}]) \begin{pmatrix} \chi^{(2)} \\ -\chi^{(2)} \end{pmatrix},
\]

\[
\psi_{\mu L}(x, t) = \frac{1}{\sqrt{V}} \frac{e^{i p \cdot x}}{\sqrt{1 + M_1^2}} [AM_1 \frac{\sqrt{E_1 + m_1}}{\sqrt{2E_1}} (1 + \frac{p}{E_1 + m_1}) e^{-iE_1 t} + \\
-B \frac{\sqrt{E_2 + m_2}}{\sqrt{2E_2}} (1 + \frac{p}{E_2 + m_2}) e^{-iE_2 t})] \begin{pmatrix} \chi^{(2)} \\ -\chi^{(2)} \end{pmatrix}.
\]
The coefficients $A$ and $B$ in Eqs. (4.21) are determined through the initial boundary conditions. Suppose that at $t = 0$

$$
\psi_{\mu L}(\mathbf{x}, t = 0) = 0,
$$

we have only the electron flavor present. The other one is obtained by the normalization condition

$$
\int_V d^3\mathbf{x} |\psi_{eL}(\mathbf{x}, t = 0)|^2 = 1.
$$

By imposing the boundary conditions given by Eq. (4.22) and Eq. (4.23) we obtain the following flavor wave functions

$$
\psi_{eL}(\mathbf{x}, t) = e^{i\mathbf{p}_1 \cdot \mathbf{x}} \frac{1}{\sqrt{V}} \frac{1}{1 + M_1^2} \left[ e^{-iE_1 t} + M_1^2 e^{iE_1 t} \right] \begin{pmatrix} \chi^{(2)} \\ -\chi^{(2)} \end{pmatrix},
$$

$$
\psi_{\mu L}(\mathbf{x}, t) = e^{i\mathbf{p}_2 \cdot \mathbf{x}} \frac{M_1}{\sqrt{V}} \frac{1}{1 + M_1^2} \left[ e^{-iE_2 t} - e^{iE_2 t} \right] \begin{pmatrix} \chi^{(2)} \\ -\chi^{(2)} \end{pmatrix}.
$$

Hence the probability densities of finding the electron and muon neutrino flavor are given respectively by

$$
\rho_e(t) = \frac{1}{V} \left[ 1 - \left( \frac{2M_1}{1 + M_1^2} \right)^2 \sin^2 \frac{(E_2 - E_1)t}{2} \right],
$$

$$
\rho_{\mu}(t) = \frac{1}{V} \left( \frac{2M_1}{1 + M_1^2} \right)^2 \sin^2 \frac{(E_2 - E_1)t}{2}.
$$

The coefficient $[2M_1/(1 + M_1^2)]^2$ is equivalent to $\sin^2(2\theta)$ in Eqs. (2.10). Therefore the field theory treatment reduces to the standard quantum mechanical treatment described in Sec. 2.

As another example, we consider the case in which the neutrino eigenfunctions of different masses have different momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ with

$$
E_1 = \sqrt{\mathbf{p}_1^2 + m_1^2}, \quad E_2 = \sqrt{\mathbf{p}_2^2 + m_2^2}.
$$

The neutrino flavor wave functions $\psi_{\nu L}(\mathbf{x}, t)$ are therefore

$$
\psi_{\nu L}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \frac{M_1}{1 + M_1^2} \left[ e^{-iE_1 t} e^{i\mathbf{p}_1 \cdot \mathbf{x}} - e^{iE_2 t} e^{i\mathbf{p}_2 \cdot \mathbf{x}} \right] \begin{pmatrix} \chi^{(2)} \\ -\chi^{(2)} \end{pmatrix}.
$$
\[
\psi_{\mu L}(x,t) = -\frac{1}{\sqrt{V}} \frac{1}{1 + M_1^2} \left[ M_1^2 e^{-iE_1 t} e^{i\mathbf{p}_1 \cdot \mathbf{x}} + e^{iE_2 t} e^{i\mathbf{p}_2 \cdot \mathbf{x}} \right] \begin{pmatrix} \chi^{(2)}_1 \\ -\chi^{(2)}_1 \end{pmatrix},
\]
where we have assumed that at a given space-time point \((x, t) = (0, 0)\), we have only the muon flavor.

The probability densities of finding at the space-time point \((x, t)\) the electron and muon neutrino flavors reduce to Eqs. (2.10) in the relativistic approximation \(|x| \approx t\).

V. CONCLUSIONS

We have discussed an explicit model of neutrino flavor mixing in the framework of quantum field theory. In this model, the equations of motion for the interacting fields are solved directly and the system is diagonalized in terms of the two uncoupled free fields \(\phi_1\) and \(\phi_2\) of mass \(m_1\) and \(m_2\) respectively. We notice here that because we can directly diagonalize the Lagrangian we do not need to write the interacting fields in terms of the free asymptotic fields \(\psi_{0e}, \psi_{0\mu}\) of mass \(m_e\) and \(m_\mu\) respectively. We have also derived neutrino flavor wave functions in such a way that the total flavor charge is constant. The probability densities, derived from these wave functions, are in agreement with the standard neutrino oscillation probabilities, if we take into account the neutrino chirality.

Also, since explicit plane wave solutions for all normal modes have been obtained, wave packets corresponding to these can be constructed via standard techniques [9].

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