Logarithmic entropy of black hole in gravity with conformal anomaly from quantum tunneling approach

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Abstract – Using the quantum tunneling approach, we are able to derive the entropy with logarithmic term of the static spherically symmetric black hole in semi-classical Einstein equations with conformal anomaly. The results indicate that the logarithmic correction to the Bekenstein-Hawking area entropy can be well-explained by the self-gravitation.

Introduction. – Recently, a static, spherically symmetric black-hole solution in semi-classical gravity theory with conformal anomaly has been presented by Cai, Cao and Ohta in [1]. By studying the thermodynamics properties of the black-hole solution, they found that the black-hole entropy contains the usual Bekenstein-Hawking area term and a logarithmic term. The logarithmic correction to the Bekenstein-Hawking area entropy has been regarded as a universal feature of the quantum theory of gravity [2]. They argued that such a logarithmic term might come from the non-local properties of the effective action of gravity theory with conformal anomaly. This explanation of a logarithmic term in the black-hole entropy is qualitative. Hence, this problem is worthy of further studying.

On the other hand, the studying of the thermodynamics of the Kehagias-Sfetsos (K-S) black hole [3] in Horava gravity shows that the black-hole entropy also has a logarithmic term [4]. By using the tunneling formalism [5], Liu and Lu [6] have been able to explain the logarithmic entropy of the K-S black hole by the self-gravitation effect. This interpretation is quite different from the other approaches [7,8]. For a recent review of the quantum tunneling method, one can refer to [9].

In this paper, we will try to derive the black-hole entropy of the static spherically symmetric black hole in semi-classical Einstein gravity with conformal anomaly by using the quantum tunneling approach. Firstly, we study the tunneling process of massless particles by using the null-geodesic method. By taking energy conservation into account, the tunneling probability is computed in the framework of the WKB approximation, from which the black-hole entropy can be read off. Then, by treating massive particles as de Broglie wave, we are able to calculate the tunneling probability of massive particles, which is the same as that of massless particles. The results indicate that the logarithmic correction to the Bekenstein-Hawking area entropy can be well explained by the self-gravitation. This implies that the self-gravitation effect is probably connected with the non-local property of the quantum theory of gravity. Finally, we show that the black-hole entropy with the logarithmic term can also be derived by studying the fermion tunneling process with the back reaction effect.

Black-hole solutions of semi-classical Einstein equations with conformal anomaly. – In this section, we will firstly give a brief review of static, spherically symmetric black holes in gravity with conformal anomaly. Cai, Cao and Ohta [1] consider the semi-classical Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle,$$  

where $\langle T_{\mu\nu} \rangle$ is the effective energy-momentum tensor by quantum loops. In four spacetime dimensions, the trace anomaly of the energy-momentum tensor is given by [10,11]

$$g^\mu\nu \langle T_{\mu\nu} \rangle = \tilde{\lambda} I_{(4)} - \tilde{\alpha} E_{(4)},$$

where $I_{(4)} = C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$ is the type-B anomaly, $E_{(4)} = \mathcal{R}^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$ is the Gauss-Bonnet term, which is called type-A anomaly [11].

They obtained the static spherically symmetric black-hole solution which is described by the metric [1]

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_2^2,$$  

(3)
where the metric function \( f(r) \) is given by
\[
f(r) = 1 - \frac{r^2}{4\alpha} \left( 1 - \sqrt{1 - \frac{16\alpha GM}{r^3} + \frac{8\alpha Q}{r^4}} \right),
\]
with \( M \) and \( Q \) being the integration constants and \( \alpha = 8\pi G\tilde{a} \). The parameter \( \lambda \) has been set to zero. According to the detailed discussion in [1], the constant \( M \) is just the ADM mass of the black hole and the constant \( Q \) corresponds to the \( U(1) \) charge square of some conformal field theory.

Under the condition of \( G^2 M^2 > (Q - 2\alpha) \), one can find that this black hole has two horizons. The horizons are determined by the equation \( f(r) = 0 \), which gives the locations of the inner and the outer event horizons as \( r_{\pm} = GM \pm \sqrt{G^2 M^2 - (Q - 2\alpha)} \). Then, the ADM mass \( M \), Hawking temperature \( T \) and entropy \( S \) of black hole can be given as [1]
\[
M = \frac{r_+}{2G} \left( 1 + \frac{Q}{r_+} - \frac{\alpha}{r_+^2} \right), \quad T = \frac{r_+}{4\pi(\omega^2 - 4\alpha)} \left( 1 - \frac{Q}{r_+} + 2\alpha \right), \quad S = \frac{A}{4G} - \frac{4\pi\alpha}{G} \ln \frac{A}{A_0},
\]
where \( A = 4\pi r_+^2 \) is the area of the outer horizon and \( A_0 \) is a constant with dimension of area. It can be easily checked that these thermodynamics quantities satisfy the first law of black-hole thermodynamics \( dM = TdS \).

It is shown that the entropy is composed by the Bekenstein-Hawking area term and the logarithmic term. It is argued in [1] that such a logarithmic term coming from non-local trace anomaly represents a universal feature of a full quantum theory of gravity. In the following, we will try to derive this logarithmic term by using the quantum tunneling approach.

Logarithmic entropy from quantum tunneling.

In this section, we will calculate the entropy of the black hole in terms of the tunneling formalism. Three cases are considered respectively, the massless particles tunneling, massive particles tunneling and fermion tunneling.

Massless particles case. To apply the null-geodesics method, it is necessary to choose coordinates which are not singular at the horizon. These coordinates have been systematically studied by Maulik K. Parikh in [12]. Introducing the coordinate transformation
\[
dt = dT - \Lambda(r)dr,
\]
where the function \( \Lambda(r) \) is required to depend only on \( r \) not on \( t \), then, the line element (3) becomes
\[
ds^2 = -f(r)dt^2 + \left( \frac{1}{f(r)} - f(r)\Lambda^2(r) \right)dr^2 + 2f(r)\Lambda(r)dtdr + r^2d\Omega^2.
\]
Restricting the condition
\[
\frac{1}{f(r)} - f(r)\Lambda^2(r) = 1,
\]
one can obtain the line element in the new coordinates
\[
ds^2 = -f(r)dt^2 + 2\sqrt{1 - f(r)}dTdr + dr^2 + r^2d\Omega^2.
\]
The radial null geodesics in the new coordinates system is given by
\[
\frac{dr}{dT} = \pm 1 - \sqrt{1 - f(r)},
\]
where the sign + corresponds the outgoing null geodesics, while the sign – corresponds the ingoing null geodesics.

The imaginary part of the classical action for an outgoing positive energy particle is
\[
\text{Im} S = \text{Im} \int_{r_{in}}^{r_{out}} p_r dr = \text{Im} \int_{r_{in}}^{r_{out}} \int_{0}^{p_r} dp_r dr,
\]
where \( r_{in} = GM + \sqrt{G^2 M^2 - (Q - 2\alpha)} \) and \( r_{out} = G(M - \omega' + \sqrt{G^2 (M - \omega')^2 - (Q - 2\alpha)} \) are the initial and the final radii of the black hole during the tunneling process. Assume that the emitted energy \( \omega' \ll M \). According to energy conservation, the energy of the background spacetime \( M \) becomes \( (M - \omega') \). Then, the outgoing radial geodesic is modified to be
\[
\frac{dr}{dT} = 1 - \frac{r^2}{4\alpha} \left( 1 - \sqrt{1 - \frac{16\alpha G(M - \omega')}{r^3} + \frac{8\alpha Q}{r^4}} \right).
\]
From Hamilton equation \( \frac{dH}{dp_r} \bigg|_{r_{in}} \), the integral can be rewritten as
\[
\text{Im} S = \text{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M - \omega} \frac{dH}{r} d\Omega,
\]
where the energy \( H \) is identified as the mass of black hole \( (M - \omega') \).

In order to perform the integral, one can introduce a new variable \( u \) as
\[
u = \sqrt{\frac{r^2}{4\alpha} \left( 1 - \sqrt{1 - \frac{16\alpha G(M - \omega')}{r^3} + \frac{8\alpha Q}{r^4}} \right)}.
\]
Then, one can easily derive a simple relation
\[
d(M - \omega') = \frac{r}{G} \left( u - \frac{4\alpha u^3}{r^2} \right) du.
\]
The integral becomes
\[
\text{Im } S = \text{Im} \int_{r_{in}}^{r_{out}} \int_{u_{in}}^{u_{out}} \left( \frac{u - 4\alpha r^2 u^3}{G} \right) \frac{du}{1-u} dr
\]
\[
= \text{Im} \int_{r_{in}}^{r_{out}} \int_{u_{in}}^{u_{out}} \frac{4\alpha}{r^2} \left( 1 + u + u^2 \right) - 1 + \left(1 - \frac{4\alpha}{r^2}\right) \frac{1}{1-u} du dr
\]
\[
= -\frac{\pi}{G} \int_{r_{in}}^{r_{out}} \left( r - \frac{4\alpha}{r} \right) dr
\]
\[
= \frac{\pi}{G} \left( \frac{r_{in}^2 - r_{out}^2}{2} - 2\alpha \ln \frac{r_{in}}{r_{out}} \right). \quad (16)
\]

According to the WKB approximation, the tunneling probability for the classically forbidden trajectory is given by
\[
\Gamma = \exp(-2 \text{Im } S) = \exp \left[ \left( \frac{\pi}{G} \frac{r_{out}^2 - r_{in}^2}{G} \right) - \frac{4\pi\alpha}{G} \left( \ln r_{out}^2 - \ln r_{in}^2 \right) \right]. \quad (17)
\]

It is well known that the tunneling probability can also be expressed as the change of entropy in the tunneling process
\[
\Gamma = \exp(\Delta S). \quad (18)
\]

Then, one can read the entropy of the black hole as
\[
S = \frac{A}{4G} - \frac{4\pi\alpha}{G} \ln \frac{A}{A_0}, \quad (19)
\]
which coincides with the thermodynamic entropy given in eq. (5).

**Massive particles case.** To consider the quantum tunneling of massive particles, one should firstly deduce the corresponding radial time-like geodesic. According to the proposal of Zhang and Zhao in ref. [13], if treating the radiative particles as de Broglie wave, the velocity of the massive particle is given by the phase velocity, i.e. the radial time-like geodesic is given by
\[
\dot{r} = v_p = \frac{v_g}{2} = -\frac{g_{TT}}{2g_{rr}}, \quad (20)
\]
where \(v_g\) is the group velocity of the wave package. From the line element (9), the radial time-like geodesic can be expressed as
\[
\dot{r} = \frac{1 - \frac{\gamma^2}{4\alpha}}{2 \sqrt{\frac{\gamma^2}{4\alpha} - \left( \frac{1 - 16\alpha GM}{r^4} + \frac{8\alpha G}{r^2} - \frac{5\alpha}{r^4} \right)}}. \quad (21)
\]

Then, the imaginary part of the action of the radiating massive particle is given by
\[
\text{Im } S = \text{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M - \omega} \frac{dr}{r} dH
\]
\[
= -\text{Im} \int_{r_{in}}^{r_{out}} \int_{u_{in}}^{u_{out}} \frac{r}{G} \left( u^2 - \frac{4\alpha}{r^2} u^3 \right)
\]
\[
\times \left( \frac{1}{u-1} - \frac{1}{u+1} \right) du dr
\]
\[
= -\frac{\pi}{G} \int_{r_{in}}^{r_{out}} \left( r - \frac{4\alpha}{r} \right) dr
\]
\[
= \frac{\pi}{G} \left( \frac{r_{in}^2 - r_{out}^2}{2} - 2\alpha \ln \frac{r_{in}}{r_{out}} \right), \quad (22)
\]
which is just the result (16) of massless particles case. So, we can conclude the thermodynamics entropy with the logarithmic term in gravity with the conformal anomaly can also be explained well by the self-gravitation in the semi-classical tunneling process of massive particles.

**Fermion case.** This subsection is dedicated to the tunneling process of fermion [14] to derive the entropy with logarithmic term. For simplicity, we consider the massless spinor field \(\Psi\) obeys the general covariant Dirac equation [15]
\[
-i\hbar\gamma^a \partial^a \Psi = 0, \quad (23)
\]
where \(\nabla^a\) is the spinor covariant derivative defined by \(\nabla^a = \partial^a + \frac{1}{4} \omega^{ab} \gamma^a \gamma^b\), and \(\omega^{ab}\) is the spin connection, which can be given in terms of the tetrad \(e^b_a\). The \(\gamma\) matrices are selected as
\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},
\end{align*}
\]
where the matrices \(\sigma^k(k=1,2,3)\) are the Pauli matrices. According to the line element (3), the tetrad fields \(e^b_a\) can be selected to be \(e^b_a = \text{diag}(1/\sqrt{f(r)}, \sqrt{f(r)}, 1/r, 1/r \sin \theta)\).

We employ the ansatz for the spin-up spinor field \(\Psi\) as follows:
\[
\Psi = (A(t,r,\theta,\phi), 0, B(t,r,\theta,\phi), 0)^T \times \exp \left( \frac{i}{\hbar} I(t,r,\theta,\phi) \right). \quad (24)
\]
In order to apply the WKB approximation, we can insert the ansatz for spinor field \(\Psi\) into the general covariant Dirac equation. Dividing by the exponential term and neglecting the terms with \(\hbar\), one can arrive at the following
four equations:

\[
\begin{align*}
\frac{iA}{\sqrt{f(r)}} \partial_t I + B \sqrt{f(r)} \partial_r I &= 0, \\
B \left( \partial_t I + \frac{i}{\sin \theta} \partial_\theta I \right) &= 0, \\
A \sqrt{f(r)} \partial_t I - \frac{iB}{\sqrt{f(r)}} \partial_r I &= 0, \\
A \left( \partial_t I + \frac{i}{\sin \theta} \partial_\theta I \right) &= 0.
\end{align*}
\]  

(25)

Note that although \( A \) and \( B \) are not constant, their derivatives and the components \( \omega_\mu \) are all of the factor \( \hbar \), so can be neglected to the lowest order in the WKB approximation. The second and fourth equations indicate that

\[
\partial_\theta I + \frac{i}{\sin \theta} \partial_r I = 0.
\]

(26)

From the first and third equations one can see that these two equations have a non-trivial solution for \( A \) and \( B \) if and only if the determinant of the coefficient matrix vanishes. Then we can get

\[
\frac{(\partial_r I)^2}{f(r)} - f(r)(\partial_r I)^2 = 0.
\]

(27)

By separating the variables of action as \( I = -\omega t + R(r) + \Theta(\theta, \phi) \), one can get

\[
\text{Im} R(\pm)(r) = \pm \text{Im} \int \frac{\omega}{f(r)} \, dr = \pm \pi \frac{\omega}{f(\omega)}
\]

(28)

where the signs \(+/-\) represent the outgoing/ingoing fermions and the residue theorem has been used to perform the integral. The tunneling probability of fermions from inside to outside the event horizon is given by

\[
\Gamma = \exp \left\{ -2 \text{Im} R_+ \right\} = \exp \left\{ -\frac{4\pi \omega}{f(\omega)} \right\}.
\]

(29)

From the tunneling probability, the fermionic spectrum of the Hawking radiation can be deduced following the standard arguments [16,17], which results in a pure thermal spectrum. Then, the Hawking temperature can be determined as

\[
T = \frac{f(\omega)}{4\pi} \left( 1 - \frac{\omega}{\omega^2} - \frac{2\alpha}{\omega} \right),
\]

(30)

which coincides with the result (5) calculated from the surface gravity.

Now, we study the back reaction of radiating fermions to the spacetime by considering energy conservation. When a fermion with energy \( \omega_i \) radiates from the black hole, the mass \( M \) of the black hole should be replaced by \((M - \omega_i)\) due to energy conservation. Then the tunneling probability should be modified as

\[
\Gamma_i = \exp \left\{ \frac{-\omega_i}{T(M - \omega_i)} \right\},
\]

(31)

where the black-hole mass \( M \) should be replaced by \((M - \omega_i)\) in the expression of the Hawking temperature. Then, for a continuous process of lots of fermions radiating from the black hole, the total tunneling probability is given by

\[
\Gamma = \prod_i \Gamma_i = \exp \left\{ \prod_i \frac{-\omega_i}{T(M - \omega_i)} \right\}
\]

\[
= \exp \left\{ -\int_0^\omega \frac{d\omega_i}{T(M - \omega_i)} \right\}.
\]

(32)

After performing the integral, one can obtain

\[
\Gamma = \exp \left\{ \frac{(\pi/G)^2}{2} - \frac{(\pi/G)^2}{2} \frac{4\pi \alpha}{G} \left( \ln r_{out}^2 - \ln r_{in}^2 \right) \right\},
\]

(33)

which also coincides with the result (17). Then, one can conclude that the entropy with logarithmic term can also be derived by considering the fermion tunneling with back reaction.

**Conclusion.** In this paper, we have studied the quantum tunneling process of the static spherically symmetric black hole in semi-classical Einstein gravity with conformal anomaly [1]. This black hole exhibits a peculiar property that there exists a logarithmic term in its entropy. The tunneling probabilities for the massless and the massive particles are calculated by using the null-geodesic method and the phase velocity method, from which the black-hole entropy can be directly read off. This results show that the logarithmic correction to Bekenstein-Hawking area entropy can be well explained by the self-gravitation in the tunneling formalism. It implies that self-gravitation effect is probably connected with the non-local property of quantum theory of gravity. Finally, we show that the black-hole entropy with the logarithmic term can also be derived by studying the fermion tunneling process with the back reaction effect.

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