Piateski-Shapiro Primes in a Beatty Sequence

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Abstract

Let $\alpha, \beta$ be real numbers such that $\alpha > 1$ is irrational and of finite type, and let $c$ be a real number in the range $1 < c < \frac{14}{13}$. In this paper, it is shown that there are infinitely many Piatetski-Shapiro primes $p = \lfloor nc \rfloor$ in the non-homogenous Beatty sequence $\left( \lfloor \alpha m + \beta \rfloor \right)_{m=1}^{\infty}$. 
1 Introduction

For fixed real numbers $\alpha, \beta$ the associated non-homogeneous Beatty sequence is the sequence of integers defined by

$$B_{\alpha, \beta} = \left(\lfloor \alpha n + \beta \rfloor \right)_{n=1}^\infty,$$

where $\lfloor t \rfloor$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences are also called generalized arithmetic progressions. It is known that there are infinitely many prime numbers in the Beatty sequence if $\alpha > 0$ (see, for example, the proof of Ribenboim [7, p. 289]). Moreover, if $\alpha \geq 1$, then the counting function

$$\pi_{\alpha, \beta}(x) = \#\{\text{prime } p \leq x : p \in B_{\alpha, \beta}\}$$

satisfies the asymptotic relation

$$\pi_{\alpha, \beta}(x) \sim \frac{x}{\alpha \log x} \quad \text{as } x \to \infty.$$

The Piatetski-Shapiro sequences are sequences of the form

$$N^c = \left(\lfloor n^c \rfloor \right)_{n=1}^\infty \quad (c > 1, \ c \notin \mathbb{N}).$$

Such sequences have been named in honor of Piatetski-Shapiro, who proved [6] that $N^c$ contains infinitely many primes if $c \in (1, \frac{12}{11})$. More precisely, for such $c$ he showed that the counting function

$$\pi^c(x) = \#\{\text{prime } p \leq x : p \in N^c\}$$

satisfies the asymptotic relation

$$\pi^c(x) \sim \frac{x^{\frac{1}{c}}}{c \log x} \quad \text{as } x \to \infty.$$

The admissible range for $c$ in this asymptotic formula has been extended many times over the years and is currently known to hold for all $c \in (1, \frac{242}{205})$ thanks to Rivat and Wu [8]. The same result is expected to hold for all larger values of $c$. We remark that if $c \in (0, 1)$ then $N^c$ contains all natural numbers, hence all primes in particular.
Since both sequences $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{N}^{(c)}$ contain infinitely many primes in the cases described above, it is natural to ask whether infinitely many primes lie in the intersection $\mathcal{B}_{\alpha,\beta} \cap \mathcal{N}^{(c)}$ in some instances. In this paper we answer this question in the affirmative for certain values of the parameters $\alpha, \beta, c$. Our main result is the following quantitative theorem.

**Theorem 1.** Let $\alpha, \beta \in \mathbb{R}$, and suppose that $\alpha > 1$ is irrational and of finite type. Let $c \in (1, \frac{14}{13})$. There are infinitely many primes in both the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ and the Piatetski-Shapiro sequence $\mathcal{N}^{(c)}$. Moreover, the counting function

$$\pi^{(c)}_{\alpha,\beta}(x) = \{ \text{prime } p \leq x : p \in \mathcal{B}_{\alpha,\beta} \cap \mathcal{N}^{(c)} \}$$

satisfies

$$\pi^{(c)}_{\alpha,\beta}(x) = \frac{x^{1/c \alpha}}{\alpha c \log x} + O \left( \frac{x^{1/c \alpha}}{\log^2 x} \right),$$

where the implied constant depends only on $\alpha$ and $c$.

**Remarks.** We recall that the type $\tau = \tau(\alpha)$ of the irrational number $\alpha$ is defined by

$$\tau = \sup \{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \lfloor \alpha n \rfloor = 0 \},$$

where $\lfloor t \rfloor$ denotes the distance from a real number $t$ to the nearest integer. For technical reasons we assume that $\alpha$ is of finite type in the statement of the theorem; however, we expect the result holds without this restriction.

If $\alpha$ is a rational number, then the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ is a finite union of arithmetic progressions. In the case, Theorem 1 also holds (in a wider range of $c$) thanks to the work of Leitmann and Wolke [11], who showed that for any coprime integers $a, d$ with $1 \leq a \leq d$ and any real number $c \in (1, \frac{14}{13})$ the counting function

$$\pi^{(c)}(x; d, a) = \# \{ p \leq x : p \in \mathcal{N}^{(c)} \text{ and } p \equiv a \mod d \},$$

satisfies

$$\pi^{(c)}(x; d, a) \sim \frac{x^{1/c \alpha}}{\phi(d) \log(x)} \text{ as } x \to \infty,$$

as $x \to \infty$.  

(1)
where $\phi$ is the Euler function (a more explicit relation than (1) holds in the shorter range $1 < c < \frac{18}{17}$; see Baker et al [1, Theorem 8]).

We also remark that our theorem is only stated for real numbers $\alpha > 1$, for if $\alpha \in (0, 1]$ then the set $B_{\alpha, \beta}$ contains all but finitely many natural numbers.

2 Preliminaries

2.1 Notation

We denote by $\lfloor t \rfloor$ and $\{ t \}$ the integer part and the fractional part of $t$, respectively. As is customary, we put

$$ e(t) = e^{2\pi it} \quad \text{and} \quad \{ t \} = t - \lfloor t \rfloor \quad (t \in \mathbb{R}). $$

Throughout the paper, we make considerable use of the sawtooth function defined by

$$ \psi(t) = t - \lfloor t \rfloor - \frac{1}{2} = \{ t \} - \frac{1}{2} \quad (t \in \mathbb{R}) $$

For the Beatty sequence $B_{\alpha, \beta} = (\lfloor \alpha n + \beta \rfloor)_{n=1}^\infty$ we systematically denote $a = \alpha^{-1}$ and $b = \alpha^{-1}(1 - \beta)$. For the Piatetski-Shapiro sequence $(\lfloor n^{c} \rfloor)_{n=1}^\infty$ we always put $\gamma = 1/c$.

Throughout, the letter $p$ always denotes a prime.

Implied constants in the symbols $O$ and $\ll$ may depend on the parameters $c$ and $A$ (where obvious) but are absolute otherwise. We use notation of the form $m \sim M$ as an abbreviation for $M < m \leq 2M$.

For any set $E$ of real numbers, we denote by $\mathcal{X}_E$ the characteristic function of $E$; that is,

$$ \mathcal{X}_E(n) = \begin{cases} 
1 & \text{if } n \in E, \\
0 & \text{if } n \notin E.
\end{cases} $$
2.2 Discrepancy

The discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, a_2, \ldots, a_M \in [0, 1)$ is defined by

$$D(M) = \sup_{\mathcal{I} \subseteq [0, 1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|,$$

(2)

where the supremum is taken over all intervals $\mathcal{I}$ contained in $[0, 1)$, $V(\mathcal{I}, M)$ is the number of positive integers $m \leq M$ such that $a_m \in \mathcal{I}$, and $|\mathcal{I}|$ is the length of the interval $\mathcal{I}$.

For any irrational number $\theta$ the sequence of fractional parts $\{\lfloor n\theta \rfloor\}_{n=1}^\infty$ is uniformly distributed over $[0, 1)$ (see, e.g., [5, Example 2.1, Chapter 1]). In the special case that $\theta$ is of finite type, the following more precise statement holds (see [5, Theorem 3.2, Chapter 2]).

**Lemma 1.** Let $\theta$ be a fixed irrational number of finite type $\tau$. Then, for every $\theta \in \mathbb{R}$ the discrepancy $D_{\theta, \mu}(M)$ of the sequence $(\lfloor \theta m + \mu \rfloor)_{m=1}^M$ satisfies the bound

$$D_{\theta, \mu}(M) \leq M^{-1/\tau+o(1)} \quad (M \to \infty),$$

where the function implied by $o(\cdot)$ depends only on $\theta$.

2.3 Lemmas

The following lemma provides a convenient characterization of the numbers that occur in the Beatty sequence $B_{\alpha, \beta}$.

**Lemma 2.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$. Then

$$n \in B_{\alpha, \beta} \iff \mathcal{X}_\alpha(an + b) = 1$$

where $\mathcal{X}_\alpha$ is the periodic function defined by

$$\mathcal{X}_\alpha(t) = \mathcal{X}_{(0, \alpha]}(\{t\}) = \begin{cases} 1 & \text{if } 0 < \{t\} \leq \alpha, \\ 0 & \text{otherwise}. \end{cases}$$
By a classical result of Vinogradov (see [10, Chapter I, Lemma 12]) we have the following approximation of $X_a$ by a Fourier series.

Lemma 3. For any $\Delta \in \left(0, \frac{1}{8}\right)$ with $\Delta \leq \frac{1}{2}\min\{a, 1-a\}$, there is a real-valued function $\Psi$ with the following properties:

(i) $\Psi$ is periodic with period one;

(ii) $0 \leq \Psi(t) \leq 1$ for all $t \in \mathbb{R}$;

(iii) $\Psi(t) = X_a(t)$ if $\Delta \leq \{t\} \leq a - \Delta$ or if $a + \Delta \leq \{t\} \leq 1 - \Delta$;

(iv) $\Psi(t) = \sum_{k \in \mathbb{Z}} g(k)e(kt)$ for all $t \in \mathbb{R}$, where $g(0) = a$, and the other Fourier coefficients satisfy the uniform bound

$$g(k) \ll \min \{ |k|^{-1}, |k|^{-2}\Delta^{-1} \} \quad (k \neq 0). \quad (3)$$

We need the following well known approximation of Vaaler [9].

Lemma 4. For any $H \geq 1$ there are numbers $a_h, b_h$ such that

$$\left| \psi(t) - \sum_{0 < |h| \leq H} a_h e(th) \right| \leq \sum_{|h| \leq H} b_h e(th), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.$$

Next, we recall the following identity for the von Mangoldt function $\Lambda$, which is due to Vaughan (see Davenport [3, p. 139]).

Lemma 5. Let $U, V \geq 1$ be real parameters. For any $n > U$ we have

$$\Lambda(n) = -\sum_{k \mid n} a(k) + \sum_{cd=n \atop d \leq V} (\log c) \mu(d) - \sum_{k \mid n \atop k \geq 1 \atop c \geq U} \Lambda(c)b(k),$$

where

$$a(k) = \sum_{cd=k \atop c \leq U} \Lambda(c)\mu(d) \quad \text{and} \quad b(k) = \sum_{d \mid k \atop d \leq V} \mu(d)$$

We also need the following standard result; see [4, p. 48].
Lemma 6. For a bounded function $g$ and $N' \sim N$ we have

$$\sum_{N<p<N'} g(p) \ll \frac{1}{\log N} \max_{N_1 \leq 2N} \left| \sum_{N<n\leq N_1} \Lambda(n)g(n) \right| + N^{1/2}.$$ 

We use the following result of Banks and Shparlinski [2, Theorem 4.1].

Lemma 7. Let $\theta$ be a fixed irrational number of finite type $\tau < \infty$. Then, for every real number $0 < \varepsilon < 1/(8\tau)$, there is a number $\eta > 0$ such that the bound

$$\left| \sum_{m \leq M} \Lambda(qm + a) e(\theta km) \right| \leq M^{1-\eta}$$

holds for all integers $1 \leq k \leq M^\varepsilon$ and $0 \leq a < q \leq M^{\varepsilon/4}$ with $\gcd(a, q) = 1$ provided that $M$ is sufficiently large.

We need the following lemma by Van der Corput; see [4, Theorem 2.2].

Lemma 8. Let $f$ be three times continuously differentiable on a subinterval $I$ of $(N, 2N]$. Suppose that for some $\lambda > 0$, the inequalities

$$\lambda \ll |f''(t)| \ll \lambda \quad (t \in I)$$

hold, where the implied constants are independent of $f$ and $\lambda$. Then

$$\sum_{n \in I} e(f(n)) \ll N^{1/2} + \lambda^{-1/2}.$$ 

We also need the following two lemmas for the bounds of certain type I and II sums. The two lemmas can be derived by revising the last three lines from the proofs of Baker et al [11 Lemma 24] and [11 Lemma 25], optimizing the ranges of $K$ and $L$. Specifically we replace $1/3$ and $2/3$ into $3/7$ and $4/7$, respectively.

Lemma 9. Suppose $|a_k| \leq 1$ for all $k \sim K$. Fix $\gamma \in (0,1)$ and $m, h, d \in \mathbb{N}$. Then for any $K \ll N^{3/7}$ the type I sum

$$S_I = \sum_{k \sim K} \sum_{l \sim L} a_k e(mk^\gamma l^\gamma + klh/d)$$

is.
satisfies the bound

\[ S_I \ll m^{1/2} N^{3/7 + \gamma/2} + m^{-1/2} N^{1-\gamma/2}. \]

Lemma 10. Suppose \(|a_k| \leq 1\) and \(|b_l| \leq 1\) for \((k, l) \sim (K, L)\). Fix \(\gamma \in (0, 1)\) and \(m, h, d \in \mathbb{N}\). For any \(K\) in the range \(N^{3/7} \ll K \ll N^{1/2}\), the type II sum

\[ S_{II} = \sum_{k \sim K} \sum_{l \sim L} a_k b_l e(mk^\gamma l^\gamma + klh/d) \]

satisfies the bound

\[ S_{II} \ll m^{-1/4} N^{1-\gamma/4} + m^{1/6} N^{16/21 + \gamma/6} + N^{11/14}. \]

Finally, we use the following lemma, which provides a characterization of the numbers that occur in the Piatetski-Shapiro sequence \(\mathcal{X}^{(c)}\).

Lemma 11. A natural number \(m\) has the form \(\lfloor n^c \rfloor\) if and only if \(\mathcal{X}^{(c)}(m) = 1\), where \(\mathcal{X}^{(c)}(m) = \lfloor -m^\gamma \rfloor - \lfloor -(m + 1)^\gamma \rfloor\). Moreover,

\[ \mathcal{X}^{(c)}(m) = \gamma m^{\gamma - 1} + \psi(-m^\gamma) - \psi(-(m + 1)^\gamma) + O(m^{\gamma - 2}). \]

In particular, for any \(c \in (1, \frac{243}{205})\) the results of [5] yield the estimate

\[ \pi^{(c)}(x) = \sum_{p \leq x} \mathcal{X}^{(c)}(p) = \frac{x^\gamma}{c \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right). \quad (4) \]

3 Construction

In what follows, we use \(\tau\) to denote the (finite) type of \(\alpha\).

To begin, we express \(\pi^{(c)}_{\alpha, \beta}(x)\) as a sum with the characteristic functions of the Beatty and Piatetski-Shapiro sequences; using Lemmas 2 and 11 we have

\[ \pi^{(c)}_{\alpha, \beta}(x) = \sum_{p \leq x} \mathcal{X}_\alpha(ap + b) \mathcal{X}^{(c)}(p). \]
In view of the properties (i)–(iii) of Lemma 3 it follows that
\[ \pi_{\alpha,\beta}(x) = \sum_{p \leq x} \Psi(ap + b)X(p) + O(V(I, x)) \] (5)
holds with some small \( \Delta > 0 \), where \( V(I, x) \) is the number of primes \( p \in \mathcal{N} \) not exceeding \( x \) for which
\[ \{ap + b\} \in I = [0, \Delta) \cup (\alpha - \Delta, \alpha + \Delta) \cup (1 - \Delta, 1); \]
that is,
\[ V(I, x) = \sum_{p \leq x} X_I(\{ap + b\})X(p). \]
By Lemma 11 we see that
\[ V(I, x) = \gamma V_1(x) + V_2(x) + O(1), \]
where
\[ V_1(x) = \sum_{p \leq x} X_I(\{ap + b\})p^{-1}, \]
\[ V_2(x) = \sum_{p \leq x} X_I(\{ap + b\})(\psi(-p^\gamma) - \psi(-(p + 1)^\gamma)). \]
Using (4) we immediately derive the bound
\[ V_2(x) \ll \sum_{p \leq x} (\psi(-p^\gamma) - \psi(-(p + 1)^\gamma)) \ll \frac{x^\gamma}{\log^2 x}. \]
To bound \( V_1(x) \) we split the sum over \( n \leq x \) into \( O(\log x) \) dyadic intervals of the form \( (N, 2N] \) with \( N \ll x \) and apply Lemma 8 obtaining that
\[ V_1(x) \ll \log x \cdot \max_{N \leq x} \left( \frac{1}{\log N} \max_{N_1 \leq 2N} \left| \sum_{N < n < N_1} \Lambda(n)X_I(\{an + b\})n^{-1} \right| + N^{1/2} \right) \]
\[ \ll x^{\gamma - 1} \log x \cdot \max_{N \leq x} \max_{N_1 \leq 2N} \left| \sum_{N < n < N_1} X_I(\{an + b\}) \right| + x^{1/2} \log x. \]
Since \(|\mathcal{I}| = 4\Delta\), it follows from the definition (2) and Lemma 1 that

\[ V_1(x) \ll \Delta x^\gamma \log x + x^{\gamma - \frac{1}{2} + o(1)} \quad (x \to \infty). \]

Therefore,

\[ V(\mathcal{I}, x) \ll \Delta x^\gamma \log x + \frac{x^\gamma}{\log^2 x}. \quad (6) \]

Now let \( K \geq \Delta^{-1} \) be a large real number, and let \( \Psi_K \) be the trigonometric polynomial defined by

\[ \Psi_K(t) = \sum_{|k| \leq K} g(k)e^{kt}. \quad (7) \]

Using (3) it is clear that the estimate

\[ \Psi(t) = \Psi_K(t) + O(K^{-1} \Delta^{-1}) \quad (8) \]

holds uniformly for all \( t \in \mathbb{R} \). Combining (8) with (5) and taking into account (6) we derive that

\[ \pi^{(c)}(x) = \sum_{p \leq x} \Psi_K(ap + b)X^{(c)}(p) + O(E(x)), \]

where

\[ E(x) = \Delta x^\gamma \log x + \frac{x^\gamma}{\log^2 x} + \frac{x^{\gamma - A/2}}{\log x} \sum_{p \leq x} X^{(c)}(p). \]

For fixed \( A \in (0, 1) \) we put

\[ \Delta = x^{-A/2} \quad \text{and} \quad K = x^A. \]

Note that our previous application of Lemma 3 to deduce (5) is justified. Use these values of \( \Delta \) and \( K \) along with (4) we obtain that

\[ E(x) \ll x^{\gamma - A/2} \log x + \frac{x^\gamma}{\log^2 x} + \frac{x^{\gamma - A/2}}{\log x} \ll \frac{x^\gamma}{\log^2 x}. \]

Using the definition (7) it therefore follows that

\[ \pi^{(c)}_{\alpha, \beta}(x) = \sum_{p \leq x} \sum_{|k| \leq x^A} g(k)e(kap + kb)X^{(c)}(p) + O\left(\frac{x^{\gamma}}{\log^2 x}\right). \quad (9) \]
Next, using Lemma 11 we express the double sum in (9) as $\sum_1 + \sum_{2,1} + \sum_{2,2}$ with

$$\sum_1 = g(0) \sum_{p \leq x} \mathcal{X}(c)(p),$$
$$\sum_{2,1} = \sum_{k \neq 0 \atop |k| \leq x^A} g(k) \sum_{p \leq x} e(kap + kb)(\gamma p^{\gamma-1} + O(p^{\gamma-2})), $$
$$\sum_{2,2} = \sum_{k \neq 0 \atop |k| \leq x^A} g(k) \sum_{p \leq x} e(kap + kb)\{\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\}. $$

Recalling that $g(0) = \alpha^{-1}$ we have

$$\sum_1 = \alpha^{-1} \sum_{p \leq x} \mathcal{X}(c)(p) = \frac{x^\gamma}{\alpha c \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right),$$

which provides the main term in our estimation of $\pi^{(c)}_{\alpha,\beta}(x)$.

To bound $\sum_{2,1}$ we follow the method used above to bound $V(I, x)$ and use partial summation together with (3) to conclude that

$$\sum_{2,1} \ll x^{\gamma-1} \log x \sum_{k \neq 0 \atop |k| \leq x^A} \frac{1}{|k|} \max_{N \leq x} \left(\frac{1}{\log N} \max_{N' \leq 2N} \left| \sum_{N \leq n \leq N'} \Lambda(n) e(k\alpha^{-1} n) \right| + 1 \right)$$

Assuming as we may that $0 < A < 1/(8\tau)$, by Lemma 7 it follows that there exists $\eta \in (0, 1)$ such that the bound

$$\max_{N \leq x} \left(\frac{1}{\log N} \max_{N' \leq 2N} \left| \sum_{N \leq n \leq N'} \Lambda(n) e(k\alpha^{-1} n) \right| \right) \ll x^{1-\eta}$$

holds uniformly for $|k| \leq x^A, k \neq 0$. Consequently, we derive the bound

$$\sum_{2,1} \ll \left(x^{\gamma-1} x^{1-\eta} + x^{\gamma-1}\right) \log^2 x \ll \frac{x^\gamma}{\log^2 x},$$

which is acceptable.
To complete the proof it suffices to show that $\sum_{2,2} \ll x^{\gamma}/\log^2 x$. To accomplish this task we use the method in [4, pp. 47–53]. Denote

$$\sum_3 = \sum_{p \leq x} e(kap + kb)\{\psi(-(p + 1)\gamma) - \psi(-p\gamma)\}.$$ 

It is enough to show that the bound $\sum_3 \ll x^{\gamma-\varepsilon}$ holds with some $\varepsilon > 0$ uniformly for $k$, for then we have by (3):

$$\sum_{2,2} \ll \sum_{k \neq 0} \frac{1}{|k|} \cdot x^{\gamma-\varepsilon} \ll x^{\gamma-\varepsilon} \log x \ll \frac{x^{\gamma}}{\log^2 x}.$$ 

By Lemma 4 for any $H \geq 1$ we can write

$$\sum_3 = \sum_4 + O(\sum_5),$$

where

$$\sum_4 = \sum_{p \leq x} \sum_{0 < |h| \leq H} a_h (e(kap + kb + h(p + 1)\gamma) - e(kap + kb + hp\gamma)),
$$

$$\sum_5 = \sum_{n \leq x} \sum_{|h| \leq H} b_h (e(kan + kb + h(n + 1)\gamma) + e(kan + kb + hn\gamma)).$$

with some numbers $a_h, b_h$ that satisfy $a_h \ll |h|^{-1}$ and $b_h \ll H^{-1}$. Thus, it suffices to show that the bounds $\sum_4 \ll x^{\gamma-\varepsilon}$ and $\sum_5 \ll x^{\gamma-\varepsilon}$ hold with an appropriate choice of $H$. To this end, we put

$$H = x^{1-\gamma+2\varepsilon}.$$

First, we consider $\sum_5$. The contribution from $h = 0$ is

$$2 \sum_{n < x} b_0 e(kan + kb) \ll b_0 |ka|^{-1} \ll 1.$$ 

Suppose that $N \leq x$ and $N_1 \sim N$. We denote

$$S_j = \sum_{N < n \leq N_1} \sum_{0 < |h| \leq H} b_h e(kan + kb + h(n + j)\gamma).$$
To bound the part that $h \neq 0$, it is suffices to show that $S_j \ll x^{1-\varepsilon}$ for $j = 0$ or $1$. By a shift of $n$, we have

$$S_j \ll \sum_{N < n \leq N_1} \sum_{0 < h \leq H} e(kan + hn^\gamma).$$

Using Lemma 8 with the choice of $\lambda = hN^\gamma - 2$, we obtain

$$S_j \ll H^{-1} \sum_{0 < h \leq H} \left( N(hN^\gamma - 2)^{1/2} + (hN^\gamma - 2)^{-1/2} \right)$$

$$\ll (x^{1-\gamma+2\varepsilon})^{1/2}x^{\gamma/2} + (x^{1-\gamma+2\varepsilon})^{-1/2}x^{1-\gamma/2} \ll x^{1/2+2\varepsilon}.$$

Then summing over $N$, adding the part that $h = 0$ from (10) and recalling that $\gamma > 1/2$, we see that the bound

$$\sum_5 \ll x^{1/2+2\varepsilon} \log x + 1 \ll x^{\gamma-\varepsilon}$$

holds if the parameter $\varepsilon$ is sufficiently small, which we can assume.

To bound $\sum_4$ we apply Lemma 6 and split the sum into $O(\log x)$ dyadic intervals of $(N, N_1]$ to derive the bound

$$\sum_{N < p \leq N_1} \sum_{0 < |h| \leq H} a_h (e(kap + kb + h(p + 1)^\gamma) - e(kap + kb + hp^\gamma))$$

$$\ll \frac{N^{\gamma-1}}{\log N} \max_{N_2 \leq 2N} \left| \sum_{1 \leq h \leq H} \sum_{N < n \leq N_2} \Lambda(n)e(kan + kb + hn^\gamma) \right| + N^{1/2}.$$

Summing over $N$ and taking into account that $\gamma > 1/2$, we obtain the desired bound $\sum_4 \ll x^\gamma/\log^2 x$ (hence also $\sum_3 \ll x^\gamma/\log^2 x$) provided that

$$\sum_{1 \leq h \leq H} \sum_{N < n \leq N_2} \Lambda(n)e(kan + kb + hn^\gamma) \ll x^{1-\varepsilon}. \quad (11)$$

Using Lemma 5 we can express the sum on the left side of (11) as

$$\sum_{1 \leq h \leq H} \left( -S_{1,h} + S_{2,h} - S_{3,h} \right),$$

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where

\[ S_{1,h} = \sum_{m \leq UV} \sum_{N/m \leq n \leq N_2/m} \tilde{a}(m)e(kamn + kb + hm^\gamma n^\gamma), \]

\[ S_{2,h} = \sum_{m \leq V} \sum_{N/m \leq n \leq N_2/m} \mu(m)(\log n)e(kamn + kb + hm^\gamma n^\gamma), \]

\[ S_{3,h} = \sum_{V < n \leq N} \sum_{N/n \leq m \leq N / n} \tilde{b}(n)\Lambda(m)e(kamn + kb + hm^\gamma n^\gamma), \]

and the functions \( \tilde{a} \) and \( \tilde{b} \) are given by

\[ \tilde{a}(m) = \sum_{c \leq U} \Lambda(c)\mu(d) \quad \text{and} \quad \tilde{b}(n) = \sum_{d \mid n} \mu(d). \]

To establish (11) it suffices to show that

\[ \sum_{1 \leq h \leq H} S_{j,h} \ll x^{1-\varepsilon} \quad (j = 1, 2, 3). \]  

(12)

We turn to the problem of bounding \( S_{1,h}, S_{2,h} \) and \( S_{3,h} \). The sum \( S_{2,h} \) is of type I, and \( S_{3,h} \) is of type II. To bound \( S_{1,h} \) we write it in the form \( S_{4,h} + S_{5,h} \), where \( S_{4,h} \) is a type I sum and \( S_{5,h} \) is a type II sum. To simplify the calculation, we take

\[ V = N^{3/7} \quad \text{and} \quad U = N^{1/7}. \]

Since \( V \ll N^{3/7} \), we apply Lemma 9 to bound the sum \( S_{2,h} \).

\[
\sum_{1 \leq h \leq H} S_{2,h} \ll \sum_{1 \leq h \leq H} \log N \left| \sum_{m \leq V} \sum_{N/m \leq n \leq N_2/m} e(kamn + hm^\gamma n^\gamma) \right| \]
\[
\ll \sum_{1 \leq h \leq H} \log N \left( h^{1/2} N^{3/7 + \gamma/2} + h^{-1/2} N^{1-\gamma/2} \right) \]
\[
\ll x^{27/14 - \gamma + 3\varepsilon} + x^{3/2 - \gamma + \varepsilon} \ll x^{1-\varepsilon}
\]

if assuming \( \gamma > \frac{13}{14} \).
The sum $S_{3,h}$ can be split into $\ll \log^2 N$ subsums of the form
\[
\sum_{X \leq m \leq 2X} \sum_{Y \leq n \leq 2Y} \sum_{N \leq mn \leq N_1} a(m)\beta(n)e(k\alpha^{-1}mn + hm^\gamma n^\gamma).
\]

It suffices to consider the special case that $V < Y \leq N^{1/2}$ and $N^{1/2} < X \leq N/V$. Applying Lemma 10 (taking into account the estimates $\alpha(m) \ll N^{\varepsilon/2}$ and $\beta(n) \ll N^{\varepsilon/2}$) each subsum is
\[
\ll \left(h^{-1/4}N^{1-\gamma/4} + h^{1/6}N^{16/21+\gamma/6} + N^{11/14}\right) N^\varepsilon.
\]

Therefore, the bound
\[
\sum_{1 \leq h \leq H} S_{3,h} \ll \left(H^{3/4}N^{1-\gamma/4} + H^{7/6}N^{16/21+\gamma/6} + HN^{11/14}\right) N^\varepsilon
\]
\[
\ll \left((x^{1-\gamma+2\varepsilon})^{3/4} - 1 - \gamma/4 + (x^{1-\gamma+2\varepsilon})^{7/6}x^{16/21+\gamma/6} + (x^{1-\gamma+2\varepsilon})^{11/14}\right) x^\varepsilon
\]
\[
\ll \left(x^{7/4-\gamma} - x^{27/14-\gamma} + x^{25/14-\gamma}\right) x^{4\varepsilon} \ll x^{1-\varepsilon}
\]
under our hypothesis that $\gamma > \frac{13}{14}$.

Finally, to derive the required bound $S_{1,h} \ll x^{1-\varepsilon}$ we write
\[
S_{1,h} = S_{4,h} + S_{5,h},
\]
where
\[
S_{4,h} = \sum_{m \leq V} \sum_{N/m \leq n \leq N_2/m} a(m)e(kamn + kb + hm^\gamma n^\gamma),
\]
\[
S_{5,h} = \sum_{V < m \leq U/V} \sum_{N/m \leq n \leq N_2/m} a(m)e(kamn + kb + hm^\gamma n^\gamma).
\]

Since $a(m) \leq \log m$ the methods used above to bound $S_{2,h}$ and $S_{3,h}$ can be applied to $S_{4,h}$ and $S_{5,h}$, respectively, to see that the bounds
\[
\sum_{1 \leq h \leq H} S_{j,h} \ll x^{1-\varepsilon} \quad (j = 4, 5)
\]
hold under our hypothesis that $\gamma > \frac{13}{14}$. This establishes (13), and the theorem is proved.
4 Remarks

We note that both [1, Theorem 7] and [1, Theorem 8] can be improved using Lemma 9 and Lemma 10 instead of [1, Lemma 24] and [1, Lemma 25], respectively. The range of $c$ in [1, Theorem 7] can be extended from $(1, \frac{147}{145})$ to $(1, \frac{571}{561})$, with a small improvement of 0.004. For [1, Theorem 8], the range of $c$ is improved from $(1, \frac{18}{17})$ to $(1, \frac{14}{13})$ and the error term is improved from $O(x^{17/39+7\gamma/13+\epsilon})$ to $O(x^{3/7+7\gamma/13+\epsilon})$.

It would be interesting to see whether the range of $c$ in the statement of Theorem 1 can be improved using more sophisticated methods to improve our type II estimates. With more work, it should be possible to remove our assumption that $\alpha$ is of finite type. For the sake of simplicity, these ideas have not been pursued in the present paper.

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