Ab Initio Approach to $s$-Shell Hypernuclei $^3_\Lambda$H, $^4_\Lambda$H, $^4\Lambda$He and $^5\Lambda$He with a $\Lambda N - \Sigma N$ Interaction

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Variational calculations for $s$-shell hypernuclei are performed by explicitly including $\Sigma$ degrees of freedom. Four sets of $YN$ interactions (SC97d(S), SC97e(S), SC97f(S) and SC89(S)) are used. The bound-state solution of $^5\Lambda$He is obtained and a large energy expectation value of the tensor $\Lambda N - \Sigma N$ transition part is found. The internal energy of the $^4\Lambda$He subsystem is strongly affected by the presence of a $\Lambda$ particle with the strong tensor $\Lambda N - \Sigma N$ transition potential.

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Few-body calculations for $s$-shell hypernuclei with mass number $A = 3 - 5$ are important not only to explore exotic nuclear structure, including the strangeness degrees of freedom, but also to clarify the characteristic features of the hyperon-nucleon ($YN$) interaction. Although several interaction models have been proposed, the detailed properties (e.g. $^3S_0$ or $^3P_0$ phase shift, strength of $\Lambda N - \Sigma N$ coupling term) of the $YN$ interaction are different among the models. The observed separation energies ($B_s$’s) of light $\Lambda$ hypernuclei are expected to provide important information on the $YN$ interaction, because the relative strength of the spin-dependent term or of the $\Lambda N - \Sigma N$ coupling term is affected from system to system.

Recently, few-body studies for $A = 3, 4$ hypernuclei have been conducted using modern $YN$ interactions. According to these developments, the Nijmegen soft core (NSC) model 97f (or 97e) seems to be compatible with the experimental $B_s$’s, though the calculated $B_s$ for $^4\Lambda$H* or $^5\Lambda$He* is actually slightly smaller than the experimental value. These few-body calculations, however, have not yet reached a stage to calculate $B_s(\Lambda\Lambda\Lambda)$.

If one constructs a phenomenological central $\Lambda N$ potential, which is consistent with the experimental $B_s(\Lambda\Lambda\Lambda)$, $B_s(\Lambda\Lambda\Lambda)$, $B_s(\Lambda\Lambda\Lambda)$, $B_s(\Lambda\Lambda\Lambda)$ and $B_s(\Lambda\Lambda\Lambda)$ values as well as the $\Lambda p$ total cross section, that kind of potential would overestimate the $B_s(\Lambda\Lambda\Lambda)$ value due to the difficulty of performing a complete five-body treatment. Only one attempt was made, using a variational Monte Carlo calculation with the NSC99 $YN$ interaction. Though NSC99 well reproduces both the experimental $B_s(\Lambda\Lambda\Lambda)$ and $B_s(\Lambda\Lambda\Lambda)$ values as well as the experimental $\Lambda p$ total cross section, a bound-state solution of $\Lambda\Lambda\Lambda$ was not found. In view of the aim to pin down a reliable $YN$ interaction, a systematic study for all $s$-shell hypernuclei is desirable.

The $NN$ tensor interaction due to a one-pion-exchange mechanism is the most important ingredient for the binding mechanisms of light nuclei. More than a third, or about one half, of the interaction energy comes from the tensor force for the $^4\Lambda$He. Since the pion-(or kaon-) exchange also induces the $\Lambda N - \Sigma N$ transition for the $YN$ sector, both the $NN$ and $\Lambda N - \Sigma N$ tensor interactions may also play important roles for light hypernuclei. If this is the case, the structure of the core nucleus (e.g. $^4\Lambda$He) in the hypernucleus ($^5\Lambda$He) would be strongly influenced by the presence of a $\Lambda$ particle.

The purpose of this letter is twofold: First is to perform an $ab$ initio calculation for $^5\Lambda$He as well as $A = 3, 4$ hypernuclei explicitly including $\Sigma$ degrees of freedom. Second is to discuss the structural aspects of $^4\Lambda$He with an appropriate $YN$ interaction which is consistent with all of the $s$-shell hypernuclear data.

The Hamiltonian ($H$) of a system comprising nucleons and a hyperon ($\Lambda$ or $\Sigma$) is given by 2 $\times$ 2 components as

$$H = \begin{pmatrix} H_A & V_{\Lambda - \Sigma} \\ V_{\Lambda - \Sigma}^T & H_\Sigma \end{pmatrix},$$

where $H_A(H_\Sigma)$ operates on the $(\Lambda-(\Sigma-\Sigma))$ component and

$$V_{\Lambda - \Sigma} = \sum_{i=1}^{A-1} \langle N\Lambda - N\Sigma \rangle_i.$$

We employ the G3RS potential for the $NN$ interaction and the SC97d(S), SC97e(S), SC97f(S) or SC89(S) potential for the $YN$ interaction, where all interactions have tensor and spin-orbit components in addition to the central one. We omit small nonstatistical correction terms ($L \cdot S$ terms) in the G3RS $NN$ interaction and odd partial-wave components in each interaction in order to focus on the main part of the interaction in the even parity state. The calculated binding energies for light nuclei ($^2H$, $^3H$, $^4\Lambda$He and $^4\Lambda$He) are 2.28, 7.63, 6.98 and 24.57 MeV, respectively. The $YN$ interactions have Gaussian form factors whose parameters are set to...
reproduce the low-energy $S$ matrix of the corresponding original Nijmegen $YN$ interactions\cite{20}. These Gaussian form factors help to save significant computer time.

The binding energies of various systems are calculated in a complete $A$-body treatment. The variational trial function must be flexible enough to incorporate both the explicit $\Sigma$ degrees of freedom and higher orbital angular momenta. The trial function is given by a combination of basis functions:

\[
\Psi_{JM\Sigma T\tau} = \sum_{k=1}^{N} c_k \varphi_k, \quad \text{with} \quad \varphi_k = A \{ G(\mathbf{x}; A_k) | \theta_{L_k} (\mathbf{x}; u_k, K_k) \chi_{S_k} | J M \eta_{T M T \tau} \} .
\]

(3)

Here, $A$ is an antisymmetrizer acting on nucleons and $\chi_{S_k}$ ($\eta_{T M T \tau}$) is the spin (isospin) function. $\eta_{T M T}$ has two components: upper (lower) component refers to the $\Lambda$-($\Sigma$)-component. The abbreviation $\mathbf{x} = (x_1, \cdots , x_{A-1})$ is a set of relative coordinates. A set of linear variational parameters ($c_1, \cdots , c_N$) is determined by the Ritz variational principle.

A spatial part of the basis function is constructed by the correlated Gaussian(CG) multiplied by the orbital angular momentum part $\theta_L (\mathbf{x})$, expressed by the global vector representation\cite{21}. CG is defined by

\[
G(\mathbf{x}; A_k) = \exp \left\{ -\frac{1}{2} \sum_{i<j}^A \alpha_{kij} (r_i - r_j)^2 \right\} = \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^{A-1} (A_k)_{ij} x_i \cdot x_j \right\} .
\]

(4)

The $(A-1) \times (A-1)$ symmetric matrix $(A_k)$ is uniquely determined in terms of the interparticle correlation parameter ($\alpha_{kij}$). The GVR of $\theta_{L_k} (\mathbf{x}; u_k, K_k)$ takes the form

\[
\theta_{L_k} (\mathbf{x}; u_k, K_k) = \varphi_k^{2K_k + L_k} Y_{L_k} (\hat{\mathbf{v}}_k), \quad \text{with} \quad \varphi_k = \sum_{i=1}^{A-1} (u_k)_i x_i .
\]

(5)

The $A_k$ and $u_k$ are sets of nonlinear parameters which characterize the spatial part of the basis function. Allowing the factor $\varphi_k^{2K_k}$ ($K_k \neq 0$) is useful to improve the short-range behavior of the trial function. The value of $K_k$ is assumed to take 0 or 1. The variational parameters are optimized by a stochastic procedure. The above form of the trial function gives accurate solutions. The reader is referred to Refs. \cite{14, 21} for details and recent applications. For the spin and isospin parts, all possible configurations are taken into account.

Table \ref{table1} lists the results of the $\Lambda$ separation energies. The scattering lengths of the $^1S_0(a_0)$ and $^3S_1(a_1)$ states for each $YN$ interaction are also listed in Table \ref{table1} where the interactions are given in increasing order of $|a_1|$ (and in decreasing order of $|a_0|$). The $SC97(S)$ interaction produces no or very weakly bound state for $^3\Lambda^+\,$, $^3\Lambda^+\,$ or $^5\Lambda^+\,$. For the $SC97d(S)$, the $B_A(\Lambda^{+})$ value is about $2 - 3$ MeV. This is a first \textit{ab initio} calculation to produce the bound state of $\Lambda^{+}$ with explicit $\Sigma$ degrees of freedom.

![FIG. 1: Density distributions of $N$, $\Lambda$ and $\Sigma$ for $^3\Lambda^+\,$](image)

The order of the spin doublet structure of the $A = 4$ system is correctly reproduced for all $YN$ interactions; the ground (excited) state has spin-parity, $J^P = 0^+(1^+)$ for both isodoublet hypernuclei $^6\Lambda^+\,$ and $^6\Lambda^+\,$. Although the strengths of the $^1S_0$ and $^3S_1$ interactions of the $SC97d(S)$ are almost the same as each other, the energy-level of the $0^+$ state is clearly lower than that of the $1^+$ state. All of the $A = 3$ bound states given in Table \ref{table1} have $J^P = 1^+$, in agreement with experiment. No other bound state has been obtained for all of the $YN$ interactions. For the $SC97e(S)$, the differences between the calculated and experimental $B_A$ values are the smallest among the $YN$ interactions employed in the present study.

Table \ref{table1} lists the probability, $P_{\Sigma}$ (in percentage), of finding a $\Sigma$ particle in the system. The sizable amount of $P_{\Sigma}(\Lambda^{+})$'s is obtained. This implies that the $\Lambda - \Sigma$ coupling plays an important role, even for the $^3\Lambda^+\,$, despite a large excitation energy of the core nucleus, $^4\Lambda$ (with the isospin 1), in the $\Sigma$-component. For the $A = 4$ system, the $P_{\Sigma}$'s of the $0^+$ state are about $1 - 2\%$, except for the $SC89(S)$, while the $P_{\Sigma}$'s of the $1^+$ state are nearly equal to or smaller than that of the $0^+$ state.

Figure \ref{figure1} displays the density distributions for $^5\Lambda^+\,$ using $SC97e(S)$, and of $N$, $\Lambda$ and $\Sigma$ from the center-of-mass (CM) of $^4\Lambda$. Figure \ref{figure1} also shows the $\Lambda$-distribution obtained from the Isle $\Lambda - \alpha$ potential\cite{22}. The experimental pionic decay width of $^5\Lambda$ suggests that the $\Lambda$-distribution should spread over a rather outer region compared to the distribution of the $\alpha$, as was discussed in Ref. \cite{22}. The present curve of the $\Lambda$-distribution is similar to that obtained by the Isle potential. The $\Sigma$-distribution has a shape similar to the $N$-distribution. The root-mean-square (rms) radii of $N$, $\Lambda$ and $\Sigma$ from the CM of the $^4\Lambda$ are 1.5, 2.9 and 1.6 fm, respectively.
where \( \mu \) is the reduced mass of the \( Y+c \) system and \( \mu_Y \) is the mass of the core nucleus.

The kinetic energy of the relative motion between the \( Y \) and \( c \) is given by

\[
T_{Y\rightarrow c} = \frac{\mu_Y}{\mu} \left( \frac{p_Y^2}{2m_Y} - \frac{2}{(A-1)m_N} \right),
\]

where \( \mu_Y = \frac{(A-1)m_N+m_Y}{A-1} \) is the reduced mass for the \( Y+c \) system and \( \mu_Y \) is the dynamical mass of the relative coordinate between \( Y \) and \( c \) (Y = \( \Lambda \) or \( \Sigma \)).

Table I lists the energy expectation values of the kinetic and potential energy terms for \( ^4\text{He} \). The contributions from the spin-orbit and the Coulomb potentials are not shown in the table, though the calculations include them. Here, \( T_c \) is the kinetic energy of the core nucleus (c) subtracted by the CM energy of c:

\[
T_c = \sum_{i=1}^{A-1} \frac{p_i^2}{2m_N} - \frac{(\sum_{i=1}^{A-1} p_i)^2}{2(A-1)m_N}. \tag{6}
\]

The kinetic energy of the relative motion between the \( Y \) and the CM of c is given by

\[
T_{Y\rightarrow c} = \frac{\mu_Y}{\mu} \left( \frac{p_Y^2}{2m_Y} - \frac{2}{(A-1)m_N} \right) \tag{7}
\]

where \( \mu_Y = \frac{(A-1)m_N+m_Y}{A-1} \) is the reduced mass for the \( Y+c \) system and \( \mu_Y \) is the dynamical mass of the relative coordinate between \( Y \) and \( c \) (Y = \( \Lambda \) or \( \Sigma \)).

Each potential part (V) takes account of a summation over appropriate particle pairs (see Eq. (9) for example). The energy expectation values of the first three columns in Table I are written as

\[
\langle V \rangle = \langle \Psi_\Lambda | V | \Psi_\Lambda \rangle + \langle \Psi_\Sigma | V | \Psi_\Sigma \rangle \tag{8}
\]

where the upper (lower) component of the \( \Psi_{J\text{MTM}} \) is denoted by \( \Psi_\Lambda \) (\( \Psi_\Sigma \)).

The first (second) term of each element \((T_c), (T_{Y\rightarrow c}) \) or \((V_{NN}) \) in Table I represents the first (second) term of Eq. (7). The energy of the \( ^4\text{He} \) subsystem changes a lot from that of the isolated one,

\[
\Delta E_c = \langle T_c \rangle + \langle V_{NN} \rangle - \langle T_c \rangle - \langle V_{NN} \rangle \approx 4.7 \text{MeV} \tag{9}
\]

This difference is considerably large despite the fact that the rms radius of N from the CM of \( ^4\text{He} \) for the \( ^4\Lambda \text{He} \) hardly changes from that for \( ^4\text{He} \). On the other hand, the tensor \( \Lambda N - \Sigma N \) transition part has a surprisingly large energy expectation value (about \(-20 \text{MeV} \)). This large coupling energy makes \( ^4\Lambda \text{He} \) bound in spite of both the energy loss of \( \Delta E_c \) and the extremely high energy of the \( \Sigma \)-component (\( \Delta E_{\Sigma} \sim 600 \text{MeV} \)).

The calculated wave function is divided into orthogonal components according to the orbital angular momentum \( (L) \), the total spin \( (S) \), the core nucleus spin \( (S_c) \) and the core nucleus isospin \( (T_c) \). Table IV displays the probability of each component for \( ^4\Lambda \text{He} \). The table also lists the probability of \( S \)-state or of \( D \)-state for \( ^4\Lambda \text{He} \).

The sizable amount of probability of the \( \Sigma \)-component is found in the \( D \)-state while the sum of \( S \)-state probabilities in the \( \Lambda \)-component is slightly smaller that for \( ^4\text{He} \). Moreover, though the presence of a \( \Lambda \) in \( ^4\text{He} \) with the strong tensor \( \Lambda N - \Sigma N \) transition potential influences the structure of the \( D \)-state component and reduces the energy expectation value of the tensor \( \Lambda N \) interaction, the large coupling energy \( (V_{\Lambda - \Sigma}) \) of the tensor part bears the bound state of \( ^4\Lambda \text{He} \) instead.

In summary, we have made a systematic study of all s-shell hypernuclei based on ab initio calculations using \( Y N \) interactions with an explicit \( \Sigma \) admixture. The bound-state solution of \( ^4\Lambda \text{He} \) was obtained. As the Ref. claimed, though there is none of the interaction models to describe very precisely the experimental \( B_\Lambda \)'s, the five-body calculation convinced us that the anomalous bind-
TABLE III: Energy expectation values of the kinetic and potential energy terms for $^5$He, given in units of MeV. The SC97e(S) YN interaction is used. For each potential part, a summation over appropriate particle pairs is taken into account (see Eq. [10] for example) and two central ($^3E$ and $^5E$) and a tensor ($^7E$) components are listed separately. The first (second) term of each element ($T_e$), ($T_{Y-c}$) or ($V_{NN}$) represents the first (second) term of Eq. [1] ($O = T_e, T_{Y-c}$ or $V_{NN}$). The energy expectation values of ($T_e$) and three ($V_{NN}$)'s for isolated $^4$He are 84.86, $-33.22$, $-33.05$ and $-43.93$ MeV, respectively.

| ($T_e$) | ($T_{Y-c}$) | ($V_{NN}$) | ($V_{NN}$) | 2($V_{NN}$) - ($V_{NN}$) | ($V_{NN}$) |
|--------|-------------|-------------|-------------|--------------------------|-------------|
| 83.43 + 2.74 | 9.11 + 3.88 | -33.14 - 0.35 | -3.97 | -0.02 | 0.07 |
| 82.73 + 2.14 | 9.11 + 3.88 | -33.14 - 0.35 | 2.95 | -1.02 | 1.56 |
| 82.73 + 2.14 | 9.11 + 3.88 | -33.14 - 0.35 | 2.95 | -1.02 | 1.56 |
| 82.73 + 2.14 | 9.11 + 3.88 | -33.14 - 0.35 | 2.95 | -1.02 | 1.56 |

TABLE IV: Probability, given in percentage, of each component with the total orbital angular momentum ($L$), total spin ($S$), core nucleus spin ($S_c$) and core nucleus isospin ($T_c$) in Λ- or in Σ-component for $^5$He. The SC97e(S) YN interaction is used. The probability in $S$- or in $D$-state for $^4$He is also listed.

| $L = 0$ | $L = 2$ |
|---------|---------|
| $S = \frac{1}{2}$ | $S = \frac{3}{2}$ |
| $S_c = 0$ | $S_c = 1$ |
| $S_c = 1$ | $S_c = 2$ |
| $S_c = 2$ | $S_c = 2$ |

$^5$He

$\langle T_c = 0 \rangle \otimes \Lambda$ | 89.14 | 0.03 | 0.19 | 3.74 | 5.36 |

$\langle T_c = 1 \rangle \otimes \Sigma$ | 0.10 | 0.09 | 1.34 | $\sim 0$ | 0.01 |

$^4$He | 89.56 | 10.44 |

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