Irreducible triangulations of the Möbius band

María-José Chávez
Departamento de Matemática Aplicada I,
Universidad de Sevilla, Spain
mjchavez@us.es

Serge Lawrencenko
Faculty of Control and Design
Russian State University of Tourism and Service
Lyubertsy, Moscow Region 140000, Russia
lawrencenko@hotmail.com

Antonio Quintero
Departamento de Geometría y Topología
Universidad de Sevilla, Spain
quintero@us.es

María Trinidad Villar
Departamento de Geometría y Topología
Universidad de Sevilla, Spain
villar@us.es

May 7, 2014

This work has been partially supported by PAI FQM-164; PAI FQM-189;
MTM 2010-20445.

Abstract
A complete list of irreducible triangulations is identified on the Möbius band.

1
Keywords: triangulation of surface, irreducible triangulation, Möbius band.

MSC[2010]: 05C10 (Primary), 57M20, 57N05 (Secondary).

1 Introduction

Let $S \in \{S_g, N_k\}$ be the closed orientable surface $S_g$ of genus $g$ or the closed non-orientable surface $N_k$ of non-orientable genus $k$. In particular, $S_0$ is the sphere and $N_1$ is the projective plane. Let $D$ be an open disk in $S$, so that the boundary $\partial(S - D) (=\partial D)$ is homeomorphic to a circle. In particular, $S_0 - D$ is the disk and $N_1 - D$ is the Möbius band. We use the notation $\Sigma$ whenever we assume the general case: $\Sigma \in \{S, S - D\}$.

If a graph $G$ is 2-cell embedded in $\Sigma$, the components of $\Sigma - G$ are called faces. A triangulation of $\Sigma$ with a simple graph $G$ (without loops or multiple edges) is a 2-cell embedding $T : G \to \Sigma$ in which each face is bounded by a 3-cycle (that is, a cycle of length 3) of $G$ and any two faces are either disjoint, share a single vertex, or share a single edge. We denote by $V = V(T)$, $E = E(T)$, and $F = F(T)$ the sets of vertices, edges, and faces of $T$, respectively. The cardinality $|V(T)|$ is called the order of $T$. By $G(T)$ we denote the graph $(V(T), E(T))$ of triangulation $T$. Two triangulations $T_1$ and $T_2$ are called isomorphic if there is a bijection, called an isomorphism, $\varphi : V(T_1) \to V(T_2)$, such that $uvw \in F(T_1)$ if and only if $\varphi(u)\varphi(v)\varphi(w) \in F(T_2)$. Throughout this paper we distinguish triangulations only up to isomorphism. For $\Sigma = S - D$, let $\partial T$ ($=\partial D$) denote the boundary cycle of $T$. The vertices and edges of $\partial T$ are called boundary vertices and boundary edges of $T$.

A triangulation is called irreducible if no edge can be shrunk without producing multiple edges or changing the topological type of the surface. The term “irreducible triangulation” is more accurately introduced in Section 2. The irreducible triangulations of $\Sigma$ form a basis for the family of all triangulations of $\Sigma$, in the sense that any triangulation of $\Sigma$ can be obtained from a member of the basis by repeatedly applying the splitting operation (introduced in Section 2) a finite number of times. Barnette and Edelson [2] and independently Negami [9] have proved that for every closed surface $S$ the basis of irreducible triangulations is finite. At present such bases are known for seven closed surfaces: the sphere (Steinitz and Rademacher [10]), projective plane (Barnette [11]), torus (Lawrencenko [6]), Klein bottle
(25 Lawrencenko and Negami’s [8] triangulations plus 4 more irreducible triangulations found later by Sulanke [12]) as well as $S_2, N_3$, and $N_4$ (Sulanke [13, 14]). Boulch, Colin de Verdière, and Nakamoto [3] have established upper bounds on the order of an irreducible triangulation of $S-D$. In this paper we obtain a complete list of irreducible triangulations of $N_1-D$.

2 Preliminaries

Let $T$ be a triangulation of $\Sigma$. An unordered pair of distinct adjacent edges $vu$ and $vw$ of $T$ is called a corner of $T$ at vertex $v$, denoted by $\langle u, v, w \rangle$. The splitting of a corner $\langle u, v, w \rangle$, denoted by $\text{sp} \langle u, v, w \rangle$, is the operation which consists of cutting $T$ open along the edges $vu$ and $vw$ and then closing the resulting hole with two new triangular faces, $v'v''u$ and $v'v''w$, where $v'$ and $v''$ denote the two images of $v$ appearing as a result of cutting. Under this operation, vertex $v$ is extended to the edge $v'v''$ and the two faces having this edge in common are inserted into the triangulation. Especially in the case $\{ \Sigma = S-D \wedge uv \in E(T) \wedge v \in V(\partial T) \}$, the operation $\text{sp} \langle u, v \rangle$ of splitting a truncated corner $\langle u, v \rangle$ produces a single triangular face $uv'v''$, where $v'v'' \in E(\partial (\text{sp} \langle u, v \rangle(T)))$.

Under the inverse operation, shrinking the edge $v'v''$, denoted by $\text{sh} \langle v'v'' \rangle$, this edge collapses to a single vertex $v$, the faces $v'v''u$ and $v'v''w$ collapse to the edges $vu$ and $vw$, respectively. Therefore $\text{sh} \langle v'v'' \rangle \circ \text{sp} \langle u, v, w \rangle(T) = T$. It should be noticed that in the case $\{ \Sigma = S-D \wedge v'v'' \in E(\partial(T)) \}$, there is only one face incident with $v'v''$, and only that single face collapses to an edge under $\text{sh} \langle v'v'' \rangle$. Clearly, the operation of splitting doesn’t change the topological type of $\Sigma$. We demand that the shrinking operation must preserve the topological type of $\Sigma$ as well; moreover, multiple edges must not be created in a triangulation. A 3-cycle of $T$ is called nonfacial if it doesn’t bound a face of $T$. In the case in which an edge $e \in E(T)$ occurs in some nonfacial 3-cycle, if we still insist on shrinking $e$, multiple edges would be produced, which would expel $\text{sh} \langle e \rangle$ from the class of triangulations. An edge $e$ is called shrinkable or a cable if $\text{sh} \langle e \rangle$ is still a triangulation of $\Sigma$; otherwise the edge is called unshrinkable or a rod. The subgraph of $G(T)$ made up of all cables is called the cable-subgraph of $G(T)$.

The only impediment to edge shrinkability in a triangulation $T$ of a closed surface $S$ is identified in [1 2 10]: an edge $e \in E(T)$ is a rod if and only if $e$ satisfies the following condition:
(2.1) $e$ is in a nonfacial 3-cycle of $G(T)$.

The impediments to edge shrinkability in a triangulation $T$ of a punctured surface $S - D$ are identified in [3]: an edge $e \in E(T)$ is a rod if and only if $e$ satisfies either condition (2.1) or the following condition:

(2.2) $e$ is a chord of $D$—that is, the end vertices of $e$ are in $V(\partial D)$ but $e \notin E(\partial D)$.

A triangulation is said to be irreducible if it is free of cables or in other words, each edge is a rod. For instance, a single triangle is the only irreducible triangulation of the disk $S_0 - D$ although its edges don’t meet either of conditions (2.1) and (2.2). Thus, we have yet one more impediment to edge shrinkability:

(2.3) $e$ is a boundary edge in the case the boundary cycle is a 3-cycle.

Although condition (2.3) is a specific case of condition (2.1) (unless $S = S_0$) and is not explicitly stated in [3], it deserves especial mention.

3 The structure of irreducible punctured triangulations

In the remainder of this paper we assume that $S \neq S_0$. Let $T$ be an irreducible triangulation of $S - D$. Let us restore the disk $D$ in $T$, add a vertex $p$ in $D$ and join $p$ to the vertices in $\partial D$. We thus obtain a triangulation, $T^*$, of the closed surface $S$. In this setting we call $D$ the patch, call $p$ the central vertex of the patch, and say that $T$ is obtained from the corresponding triangulation $T^*$ of $S$ by the patch removal. Notice that $T^*$ may turn out to be an irreducible triangulation of $S$, but not necessarily.

A vertex of a triangulation $R$ of $S$ is called a pylonic vertex if that vertex is incident with all cables of $R$. A triangulation that has at least one cable and at least one pylonic vertex is called a pylonic triangulation. It should be noticed that there exist triangulations of the torus with exactly one cable, and thereby with two different pylonic vertices; however, if a pylonic triangulation $R$ has at least two cables, $R$ has a unique pylonic vertex.

**Lemma 1.** If $T^*$ has at least two cables, then the central vertex $p$ of the patch is the only pylonic vertex of $T^*$.

**Proof.** Using the assumption that $T$ is irreducible and the fact that each cable of $T^*$ fails to satisfy condition (2.1), it can be easily seen that in the
case $T^*$ is not irreducible, all cables of $T^*$ have to be entirely in $D \cup \partial D$ and, moreover, there is no cable that is entirely in $\partial D$. In particular, we observe that any chord of $D$ is a rod in $T$ because it meets condition (2.2), and is also a rod in $T^*$ because it meets condition (2.1).

4 Irreducible triangulations of the Möbius band

Barnette’s theorem [1] states that there exist two irreducible triangulations of $N_1$; those are presented in Figure 1: $P_1$ and $P_2$. (For each hexagon identify each antipodal pair of vertices in the boundary to obtain an actual triangulation of $N_1$.) By repeatedly applying the splitting operation to $P_1$ and $P_2$, we can generate all triangulations of $N_1$. Sulanke [11] has generated by computer all triangulations of $N_1$ with up to 19 vertices; in particular, among them there are 20 triangulations with up to 8 vertices. Independently, the authors of the present paper have identified the same list of 20 triangulations by hand (Figure 1), using the automorphisms of $P_1$ and $P_2$. An automorphism of a triangulation $P$ is an isomorphism of $P$ with itself. The set of all automorphisms of $P$ forms a group, called the automorphism group of $P$ (denoted $\text{Aut}(P)$).

Lemma 2. ([11]). There are precisely one (up to isomorphism) triangulation of $N_1$ with 6 vertices, three with 7 vertices, and sixteen with 8 vertices. They are shown in Figure 1, in which the bold edges indicate the cable-subgraphs of the triangulations.

Theorem 3. There are precisely six non-isomorphic irreducible triangulations of the Möbius band, namely $M_1$ to $M_6$, shown in Figure 2 in which the left and right sides of each rectangle are identified with opposite orientation to obtain an actual triangulation of the Möbius band.

Proof. Observe that in Figure 1 only the following three non-irreducible members have a pyloric vertex: $P_3$ and $P_4$ with pyloric vertex 6′′, and $P_{19}$ with pyloric vertex 7′′. It can be easily proved that if a triangulation of $N_1$ has at least two cables but has no pyloric vertex, then no pyloric vertex can be created under further splitting of the triangulation. On the other hand,
Figure 1: All projective plane triangulations with up to 8 vertices.
it can be easily seen that any one splitting applied to the pylonic triangulations $P_3$, $P_4$, or $P_{19}$ destroys their pylonicity. Therefore, by Lemma 1, each irreducible triangulation of $N_1 - D$ is obtainable either by removing a vertex from an irreducible triangulation in $\{P_1, P_2\}$, or by removing the pylonic vertex from a pylonic triangulation in $\{P_3, P_4, P_{19}\}$. It is known [4, 5, 7] that $\text{Aut}(P_1)$ acts transitively on the vertex set $V(P_1)$, while under the action of $\text{Aut}(P_2)$ the set $V(P_2)$ breaks into two orbits as follows: $\text{orbit}_1 = \{1, 2, 3, 7\}$, $\text{orbit}_2 = \{4, 5, 6\}$. Therefore, all irreducible triangulations of $N_1 - D$ are covered by the followings: $M_1 = P_1$ minus vertex 1 (subtracted with the incident edges and faces), $M_2 = P_2$ minus vertex 1, $M_3 = P_2$ minus vertex 4, $M_4 = P_4$ minus vertex $6''$, $M_5 = P_3$ minus vertex $6''$, $M_6 = P_{19}$ minus vertex $7''$. To see that these triangulations are pairwise non-isomorphic, observe that they have different vertex degree sequences except for the pair $\{M_3, M_4\}$; however, all boundary vertices have degree 5 in $M_3$ but not all in $M_4$. 

References

[1] D. Barnette (D. W. Barnette). Generating the triangulations of the projective plane. J. Comb. Theory, Ser. B, 33 (1982), 222–230.
[2] D. W. Barnette, A. L. Edelson. All 2-manifolds have finitely many minimal triangulations. *Isr. J. Math.*, 67 (1989), No. 1:123–128.

[3] A. Boulch, É. Colin de Verdière, A. Nakamoto. Irreducible triangulations of surfaces with boundary. *Graphs Comb.*, DOI 10.1007/s00373-012-1244-1 (2012).

[4] B. Chen, J. H. Kwak, S. Lawrencenko. Weinberg bounds over nonspherical graphs. *J. Graph Theory*, 33 (2000), No. 4:220-236.

[5] B. Chen, S. Lawrencenko. Structural characterization of projective flexibility. *Discrete Math.*, 188 (1998), No. 1-3:233–238.

[6] S. A. Lavrenchenko (S. Lawrencenko). Irreducible triangulations of the torus. *J. Sov. Math.*, 51 (1990), No. 5:2537–2543; translation from *Ukr. Geom. Sb.*, 30 (1987), 52–62.

[7] S. A. Lavrenchenko (S. Lawrencenko), Number of triangular packings of a marked graph on a projective plane. *J. Sov. Math.*, 59 (1992), No. 2:741–749; translation from *Ukr. Geom. Sb.*, 32 (1989), 71–84.

[8] S. Lawrencenko, S. Negami. Irreducible triangulations of the Klein bottle. *J. Comb. Theory, Ser. B*, 70 (1997), No. 2:265–291.

[9] S. Negami. Diagonal flips in pseudo-triangulations on closed surfaces. *Discrete Math.*, 240 (2001), No. 1–3:187–196.

[10] E. Steinitz, H. Rademacher, Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie. Berlin: Springer, 1976. [Reprint of the original 1934 edition.]

[11] T. Sulanke, Counts of triangulations of surfaces, electronic only (2005). [http://hep.physics.indiana.edu/~tsulanke/graphs/surfttri/counts.txt](http://hep.physics.indiana.edu/~tsulanke/graphs/surfttri/counts.txt)

[12] T. Sulanke, Note on the irreducible triangulations of the Klein bottle. *J. Comb. Theory, Ser. B*, 96 (2006), No. 6:964-972.

[13] T. Sulanke, Generating irreducible triangulations of surfaces. arXiv e-print service, Cornell University Library, Paper No. arXiv:math/0606687v1, 11 p., electronic only (2006).
[14] T. Sulanke, Irreducible triangulations of low genus surfaces, arXiv e-print service, Cornell University Library, Paper No. arXiv:math/0606690v1, 10 p., 1 fig., 5 tabs., electronic only (2006).