TAUTOLOGICAL RELATIONS IN MODULI SPACES OF WEIGHTED POINTED CURVES

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Abstract. Pandharipande-Pixton have used the geometry of the moduli space of stable quotients to produce relations between tautological Chow classes on the moduli space $M_g$ of smooth genus $g$ curves. We study a natural extension of their methods to the boundary and more generally to Hassett’s moduli spaces $\overline{M}_{g,w}$ of stable nodal curves with weighted marked points.

Algebraic manipulation of these relations brings them into a Faber-Zagier type form. We show that they give Pixton’s generalized FZ relations when all weights are one. As a special case, we give a formulation of FZ relations for the $n$-fold product of the universal curve over $M_g$.

1. Introduction

1.1. Moduli spaces of curves with weighted markings.

1.1.1. Definition. As a GIT variation of the Deligne-Mumford moduli space of stable marked curves, for any $n$-tuple $w = (w_1, \ldots, w_n)$ with $w_i \in \mathbb{Q} \cap [0,1]$ Hassett [1] has defined a moduli space $\overline{M}_{g,w}$, parametrizing nodal semi-stable curves $C$ of arithmetic genus $g$ with $n$ numbered marked points $(p_1, \ldots, p_n)$ in the smooth locus of $C$ satisfying two stability conditions:

1. The points in a subset $S \subseteq \{1, \ldots, n\}$ are allowed to come together if and only if $\sum_{i \in S} w_i \leq 1$.
2. $\omega_C(\sum_{i=1}^n w_i p_i)$ is ample.

The second condition implies that the total weight plus the number of nodes of every genus 0 component of $C$ must be strictly greater than 2.

The main cases we have in mind are the usual moduli space $\overline{M}_{g,n}$ of marked curves, which occurs when all the weights are equal to 1, the case when $\sum_{i=1}^n w_i \leq 1$, which is a desingularization of the $n$-fold product of the universal curve over $\overline{M}_g$, and the case when $g = 0$, $w_1 = 1$, $w_2 = 1$ and $\sum_{i=3}^n w_i \leq 1$, which gives the Losev-Manin spaces [2]. Moduli spaces mixed pointed curves also naturally appear when studying moduli spaces of stable quotients [3].

We will use various abbreviations for the weight data, like $(w, 1^m)$ for the data with first entries given by $w$ and further $m$ entries of 1.

1.1.2. Tautological classes. Using the universal curve $\pi : \overline{C}_{g,w} \to \overline{M}_{g,w}$, the $n$ sections $s_i : \overline{M}_{g,w} \to \overline{C}_{g,w}$ corresponding to the markings and the relative dualizing sheaf $\omega_\pi$, we can define $\psi$ and $\kappa$ classes:

$$\psi_i = c_1(s_i^* \omega_\pi)$$
$$\kappa_i = \pi_*(c_1(\omega_\pi)^{i+1})$$
Notice that in the case of $\overline{M}_{g,n}$ the definition of $\kappa$ classes is different from the usual definition as in [4].

Each subset $S \subseteq \{1, \ldots, n\}$ defines a diagonal class $D_S$ as the class of the locus where all the points of $S$ coincide. By Condition (1) the class $D_S$ is zero if and only if $\sum_{i \in S} w_i > 1$.

As $\overline{M}_{g,n}$ the moduli space $\overline{M}_{g,w}$ is stratified according to the topological type of the curve and each stratum, indexed by a dual graph $\Gamma$ (see [5, Appendix A] for a description of dual graphs of strata of $\overline{M}_{g,n}$), is the image of a clutching map

$$\xi_\Gamma : \prod_i \overline{M}_{g_i,w_i} \to \overline{M}_{g,w}.$$ 

Here points which are glued together have weight 1. The map $\xi_\Gamma$ is finite of degree $|\text{Aut}(\Gamma)|$.

1.2. Formulation of the relations. To state the Faber-Zagier-type relations on $\overline{M}_{g,w}$ we need to introduce several formal power series.

The hypergeometric series $A$ and $B$ already appeared in the original FZ relations. They are defined by

$$A(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left( \frac{t}{72} \right)^i = 1 + O(t^1), \quad B(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} \left( \frac{t}{72} \right)^i.$$ 

We will actually not directly use $B$ in the definition of the relations but $A$ and a family $C_i$ of series strongly related to $A$ and $B$ which were already used in [6] for the proof of the equivalence between stable quotient and FZ relations on $M_g$. They are defined recursively by

$$C_1 = C = \frac{B}{A}, \quad C_{i+1} = \left( 12t^2 \frac{d}{dt} - 4it \right) C_i.$$ 

Notice that $C_i$ is a multiple of $t^{i-1}$.

As in [6] these series appear in the study of the two variable functions

$$\Phi(t, x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1-it} \frac{(-1)^d x^d}{d!} \frac{t^d}{t^d}$$

$$= \frac{1}{2} + \delta = \gamma = \sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i-1)} t^{2i-1} + \log(\Phi),$$

where the Bernoulli numbers $B_k$ are defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.$$ 

1. The $\kappa$ classes we use here appear naturally when pushing forward powers of $\psi$ classes along maps forgetting points of small weight whereas the usual $\kappa$ classes are convenient when studying push-forwards of powers of $\psi$ classes along maps forgetting points of weight equal to 1. The fact that we will mainly consider the first kind of push-forwards explains our choice of $\kappa$ classes.

2. To determine the number $|\text{Aut}(\Gamma)|$ of automorphisms of $\Gamma$ the graph $\Gamma$ should be regarded as a collection of distinct half-edges of which some are glued together. For example when $n = 0$ and $\Gamma$ consists of exactly one vertex and one edge, there is exactly one non-trivial automorphism, which interchanges the two half-edges; accordingly the map $\xi_\Gamma : \overline{M}_{g-1,2} \to \overline{M}_g$ is a double cover.
These two-variable functions appear both in the localization formula for stable quotients (see Section 3 and [6]) and the $S$-matrix in the equivariant genus 0 Gromov-Witten theory of $\mathbb{P}^1$ (see Section 5).

Various linear bracket operators are used to insert Chow classes into the power series in $t$. We define $\{k\}_\kappa$, $\{k\}_\psi$ and $\{k\}_D$ for $S \subseteq \{1, \ldots, n\}$ by

$$\{k\}_\kappa = \kappa_k t^k,$$

$$\{k\}_\psi = \psi_k t^k,$$

$$\{k\}_D = \begin{cases} \psi_i^{k+1}, & \text{if } S = \{i\} \\ (-1)^{|S|} \psi_i^{k-|S|+1} t^k, & \text{else,} \end{cases}$$

and linear continuation. It will moreover be useful to define brackets modified by a sign $\zeta \in \{\pm 1\}$, denoted by $\{\zeta k\}_\kappa$, $\{\zeta k\}_\psi$ and $\{\zeta k\}_D$ respectively, by composing the usual bracket operator with the ring map induced by $t \mapsto \zeta t$. For a power series $F$ in $t$ we will use the notation $[F]_{t^i}$ for the $t^i$ coefficient of $F$.

**Proposition 1.** For any codimension $r$ and the choice of a subset $S \subseteq \{1, \ldots, n\}$ such that $3r \geq g + 1 + |S|$ the class

$$\left[ \sum_{\zeta, \Gamma \to \{\pm 1\}} \frac{1}{|\text{Aut}(\Gamma)|} \xi_{\Gamma^*} \left( \prod_v -\log(A) \right)_{\kappa(v)} \prod_{e \in E} \xi_e \prod_{P \in S_v} \psi_i \right]_{t^r}$$

in $A^*(\overline{M}_{g,w})$ is zero, where the sum is taken over all dual graphs $\Gamma$ of $\overline{M}_{g,w}$ with vertices colored by $\zeta$ with $+1$ or $-1$, the class $\psi_i^{(v)}$ is the $i$-th $\kappa$ class in the factor corresponding to $v$ and $S_v$ is $S$ restricted to the markings at $v$. The edge term $\Delta_e$ depends only on the $\psi$ classes $\psi_1, \psi_2$ and colors $\zeta_i = \zeta(v_i) \in \{\pm 1\}$ at the vertices $v_1, v_2$ joined by $e$ and is defined by

$$2t(\psi_1 + \psi_2)\Delta_e = (\zeta_1 + \zeta_2)\{A^{-1}\}_\psi (A^{-1})_\psi + \zeta_1 \{C\}_\psi + \zeta_2 \{C\}_\psi.$$

To see that the series $\Delta_e$ is well-defined one can use the identity $A(t)B(-t) + A(-t)B(t) + 2 = 0$ [7]. The proof of Proposition 1 gives an alternative more geometric proof.

In the case of $\overline{M}_{g,n}$ the relations of Proposition 1 are a reformulation of the part of Pixton’s generalized FZ relations [7] with empty partition $\sigma$ and coefficients $a_i$ only valued in $\{0, 1\}$. The set $S$ corresponds to the set of all $i$ such that $a_i = 1$.

To obtain a set of relations analogous to Pixton’s relations we need to take the closure of the relations of Proposition 1 under multiplication with $\psi$ and $\kappa$ classes and push forward under maps forgetting marked points of weight 1. See for this also the discussion in Section 3 and [8, Section 3.5].

In total this gives a proof of [7, Conjecture 1] in Chow. We therefore have verified Pixton’s remark [7] that it should be possible to adapt the stable quotients method to prove that his generalized relations hold. In cohomology [7, Conjecture 1] has already been established by a completely different method in [8].

**1.3. Plan of Paper.** Section 2 introduces stable quotient moduli spaces of $\mathbb{P}^1$ with weighted marked points, slightly generalizing the usual moduli spaces of stable quotients. We define them, sketch their existence and review structures on them. In Section 3 we review the virtual localization formulas both for stable quotients and stable maps to $\mathbb{P}^1$. Section 4 contains a proof of Proposition 1 restricted to
powers of the universal curve over $M_g$. The proof in this case is much simpler than the general case but many parts of it can be referred to later on. Section 5 provides calculations of localization sums using Givental’s method necessary in the proof of the general relations. Finally Section 6 contains the proof of Proposition 1.

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2. Moduli spaces of stable quotients

2.1. Introduction. The proof of Proposition 1 is based on the geometry of stable quotients. These moduli spaces give an alternative compactification of the space of maps from curves to Grassmannians and were first introduced in [3]. We will need a combination of these spaces with Hassett’s spaces of weighted stable curves. These spaces are different from the $\epsilon$-stable quotient spaces introduced by Toda [10], where the stability conditions on quotient sheaf instead of the points varies. Similar spaces $\overline{M}_{g,w}(\mathbb{P}^m, d)$ in Gromov-Witten theory have been studied in [11], [12] and [13].

The moduli space $\overline{Q}_{g,w}(\mathbb{P}^1, d)$ parametrises nodal curves $C$ of arithmetic genus $g$ with $n$ markings $p_i$ weighted by $w$ together with a quotient sequence

$$0 \to S \to \mathcal{O}_C \otimes \mathbb{C}^2 \to Q \to 0$$

satisfying several conditions:

- The underlying curve with weighted marked points satisfies all the properties of being stable but possibly the ampleness condition (2).
- $S$ is locally free of rank 1.
- $Q$ has degree $d$.
- The torsion of $Q$ is outside the nodes and the markings of weight 1.
- $\omega_C (\sum_{i=1}^n w_i p_i) \otimes S^{\otimes(-\epsilon)}$ is ample for any $0 < \epsilon \in \mathbb{Q}$.

Isomorphisms of stable quotients are defined by isomorphisms of weighted stable curves such that the kernels of the quotient maps are related via pull-back by the isomorphism. In Section 2.3 we give a precise definition for families and sketch a proof that $\overline{Q}_{g,w}(\mathbb{P}^1, d)$ is a proper Deligne-Mumford stack.

2.2. Structures. There is a universal curve $\overline{C}_{g,w}(\mathbb{P}^1, d)$ over $\overline{Q}_{g,w}(\mathbb{P}^1, d)$ with a universal quotient sequence

$$0 \to S \to \mathcal{O}_{\overline{C}_{g,w}(\mathbb{P}^1, d)} \otimes \mathbb{C}^2 \to Q \to 0.$$
also consider more generally stable quotients to quotients over $S$

Definition. An object $(C,s_1,...,s_n,\mathcal{O}_C^{m+1} \to \mathcal{Q})$ in the stack $\mathcal{Q}_{g,w}(\mathbb{P}^n, d)$ over a scheme $S$ is a proper, flat morphism $\pi : C \to S$ together with $n$ sections $s_i$ and a quotient sequence of quasi-coherent sheaves on $C$ flat over $S$

$$0 \to S \to \mathcal{O}_C^{m+1} \to \mathcal{Q} \to 0$$

such that

1. The fibers of $\pi$ over geometric points are nodal connected curves of arithmetic genus $g$.
2. For any $S \subset \{1,\ldots,n\}$ such that the intersection of $s_i$ for all $i \in S$ is nonempty we must have $\sum_{i \in S} w_i \leq 1$ and if in addition the intersection touches the singular locus of $\pi$ we must have $\sum_{i \in S} w_i = 0$.
3. $S$ is locally free of rank 1.
4. $\mathcal{Q}$ is locally free outside the singular locus of $\pi$ and of degree $d$.
5. $\omega_{\pi}(\sum_{i=1}^n w_i s_i) \otimes \mathcal{S}^\otimes (-\varepsilon)$ is $\pi$ relatively ample for any $\varepsilon > 0$.

Two families $(C, s_1,\ldots,s_n,\mathcal{O}_C^{m+1} \to \mathcal{Q})$, $(C', s'_1,\ldots,s'_n,\mathcal{O}_{C'}^{m+1} \to \mathcal{Q'})$ of stable quotients over $S$ are isomorphic if there exists an isomorphism $\phi : C \to C'$ over $S$ mapping $s_i$ to $s'_i$ for all $i \in \{1,\ldots,n\}$ and such that $S$ and $\phi^*(\mathcal{S}')$ coincide when viewed as subsheaves of $\mathcal{O}_C^{m+1}$.

Notice that there is no condition on the sections of weight 0. Therefore the space $\mathcal{Q}_{g,w,0}(\mathbb{P}^n, d)$ is isomorphic to the $f$-fold power of the universal curve of $\mathcal{Q}_{g,w}(\mathbb{P}^n, d)$ over $\mathcal{Q}_{g,w}(\mathbb{P}^m, d)$.

In order to prove that $\mathcal{Q}_{g,w}(\mathbb{P}^n, d)$ is a Deligne-Mumford stack it is enough to show that it is locally isomorphic to a product of universal curves over $\mathcal{Q}_{g,1}(\mathbb{P}^m, d)$ since this is a Deligne-Mumford stack as shown in [3] (by realizing it as a stack quotient of a locally closed subscheme of a relative Quot scheme).

\footnote{The only components which are stable in the Gromov-Witten but not in the stable quotients theory are non-contracted components of genus 0 with exactly one node and no marking.}
Lemma 1. For each point \((C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to Q)\) in \(\overline{Q}_{g,w}(\mathbb{P}^m, d)\) there exists an open neighborhood which is isomorphic to an open substack of \(\overline{Q}_{g,w}(\mathbb{P}^m, d)\) where \(w' = \{0, 1\}\) valued.

Proof. The argument is the same as in [11] Corollary 1.18. □

Separatedness follows from the following lemma, which is analogous to [12] Proposition 1.3.4.

Lemma 2. The diagonal \(\Delta : \overline{Q}_{g,w}(\mathbb{P}^m, d) \to \overline{Q}_{g,w}(\mathbb{P}^m, d) \times \overline{Q}_{g,w}(\mathbb{P}^m, d)\) is representable, finite and separated.

Proof. We proceed as in [12] Proposition 1.3.4.

Let \((C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to Q)\) and \((C', s'_1, \ldots, s'_n, \mathcal{O}_{C'}^{m+1} \to Q')\) be two stable quotients over a base scheme \(S\). We need to show that the category

\[
\text{Isom}((C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to Q), (C', s'_1, \ldots, s'_n, \mathcal{O}_{C'}^{m+1} \to Q'))
\]

is represented by scheme finite and separated over \(S\).

The images \(T\) and \(T'\) of the maps \((\mathcal{O}_C^{m+1})^s \to S^s\) and \((\mathcal{O}_{C'}^{m+1})^s \to (S')^s\) are given by \(T = S^s(-D), T' = (S')^s(-D')\) for effective divisors \(D, D'\) of some degree \(d' \leq d\) on \(C\) respectively \(C'\). From this we can construct \(d'\) additional sections \(s_{n+1}, \ldots, s_{n+d'}\) for \(C\) and \(s'_{n+1}, \ldots, s'_{n+d'}\) for \(C'\). As in [14] Proposition 1.3.4] at least étale locally \(d - d'\) further sections \(s_{n+d'}, \ldots, s_{n+d}\) for \(C\) and \(s'_{n+d'}, \ldots, s'_{n+d}\) for \(C'\) can be constructed by choosing suitable hyperplanes \(H_i\) in \(\mathbb{C}^{m+1}\) and marking sections at which the quotient is locally free and the quotient sequence coincides with the sequence corresponding to the inclusion of \(H_i\) in \(\mathbb{C}^{m+1}\).

By the definition of stable quotients the resulting marked curves \((C, s_1, \ldots, s_{n+d})\) and \((C', s'_1, \ldots, s'_{n+d})\) are \((w, \varepsilon^d)\)-stable. This gives a closed embedding

\[
\bigcup_{\sigma \in S_{d'}} \text{Isom}((C, s_1, \ldots, s_{n+d}), (C', s'_1, \ldots, s'_n, s'_{n+\sigma(1)}', \ldots, s'_{n+\sigma(d')}, s_{n+d+1}', \ldots, s_{n+d}')).
\]

Since by [11] the right hand side is a scheme finite and separated over \(S\) so is the left hand side. □

Lemma 3. There is a surjective comparison map \(c : \overline{M}_{g,w}(\mathbb{P}^m, d) \to \overline{Q}_{g,w}(\mathbb{P}^m, d)\).

Proof. This is similar to [10] Lemma 2.23. □

Lemma 4. For \(w' \leq w\) there is a surjective reduction morphism

\[
\rho_{w'w} : \overline{Q}_{g,w}(\mathbb{P}^m, d) \to \overline{Q}_{g,w'}(\mathbb{P}^m, d).
\]

Proof. This follows from Lemma 3 and the corresponding result for stable maps (see [12] Proposition 1.2.1). □

Since \(\overline{Q}_{g,w}(\mathbb{P}^m, d)\) is proper for \(w = 1^n\) the preceding lemma implies that \(\overline{Q}_{g,w}(\mathbb{P}^m, d)\) is proper in general.

\[4\)Actually \((m + 1)(d - d')\]
The virtual localization formulas ([13]) for $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^m, d)$ and $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^m, d)$ are the main tool we use to derive stable quotient relations. The existence of the necessary virtual fundamental class $[\overline{\mathcal{M}}_{g,w}(\mathbb{P}^m, d)]_{vir}$ for $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^m, d)$ has been shown in [11] and [12]. For the existence of a 2-term obstruction theory of $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^m, d)$ the same arguments as in [3] Section 3.2] can be used. They depend on existence of a $\nu$-relative 2-term obstruction theory of the Quot scheme and the non-singularity of the Hassett moduli spaces of weighted curves. We will now follow Section 7 of [3].

In this paper we will only look at torus actions on moduli spaces of stable quotients or stable maps of $\mathbb{P}^1$ which are induced from the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $([x_0 : x_1], \lambda) \mapsto [x_0 : \lambda x_1]$. By $s$ we will denote the pull-back of the equivariant class $s \in A^*_\mathbb{C}(\text{pt})$ defined as the first equivariant Chern class of the trivial rank 1 bundle on a point space with weight one action of $\mathbb{C}^*$ on it.

The equivariant cohomology of $\mathbb{P}^1$ is generated as an algebra over $\mathbb{Q}[s]$ by the equivariant classes $[0], [\infty]$ of the two fixed points 0 and $\infty$. These classes satisfy (among others) the relation $[0] = [\infty] = s$. Localizing by $s$, the classes $[0]$ and $[\infty]$ give a basis of $\mathbb{Q}[s, s^{-1}]$ as a free $\mathbb{Q}[s, s^{-1}]$-module.

### 3.1. Fixed loci

The fixed loci for the action of $\mathbb{C}^*$ on $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^1, d)$ and $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^1, d)$ are very similar. They are both given by the data of

1. a graph $\Gamma = (V, E)$,
2. a coloring $\zeta : V \to \{\pm 1\}$,
3. a genus assignment $\xi : V \to \mathbb{Z}_{\geq 0}$,
4. a map $d : V \cup E \to \mathbb{Z}_{\geq 0}$,
5. a point assignment $p : \{1, \ldots, n\} \to V$,

such that $\Gamma$ is connected and contains no self-edges, two vertices directly connected by an edge do not have the same color $\zeta$, $d = h^1(\Gamma) + \sum_{v \in V} g(v)$,

$$d = \sum_{i \in V \cup E} d(i), \quad d|_E \geq 1,$$

and one further condition which depends on whether we look at $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^1, d)$ or $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^1, d)$.

To state the stable quotient condition we need the following definition: A vertex $v \in V$ is called non-degenerate if the inequality

$$2g(v) - 2 + n(v) + \varepsilon d(v) > 0$$

holds for any $\varepsilon > 0$. Here $n(v)$ is the number of edges at $v$ plus the number of preimages under $p$ weighted by the corresponding weight.

Then for the combinatorial data on the stable quotients side we demand each vertex to be non-degenerate, whereas for the stable maps data the additional condition is $d|_V = 0$.

In the combinatorial data for $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^1, d)$ the vertices $v$ of $\Gamma$ correspond to curve components contracted to the fixed point of $\mathbb{P}^1$ specified by $\zeta$, 0 for $\zeta(v) = 1$ and $\infty$ for $\zeta(v) = -1$. The edges correspond to multiple covers of $\mathbb{P}^1$ ramified at 0 and $\infty$ with degree given by the map $d$. For $\overline{\mathcal{M}}_{g,w}(\mathbb{P}^1, d)$ the vertices of $\Gamma$ correspond to components $C$ of the curve over which the subsheaf $S$ is an ideal sheaf of the trivial
subsheaf given by one of the two 1 dimensional fixed subspaces of $\mathbb{C}^2$. $\zeta$ specifies which subspace of $\mathbb{C}^2$ was chosen and $-d$ the degree of $S$. Edges correspond to multiple covers of $\mathbb{P}^1$ ramified in the two torus-fixed points.

The fixed locus corresponding to the combinatorial data is, up to a finite map, isomorphic to the product

$$\prod_{v \in V} \mathcal{M}_{g(v),(w(v),\varepsilon^d(v))}/S_d(v),$$

where $w(v)$ is a $(|p^{-1}(v)| + |\{e \text{ adjacent to } v\}|)$-tuple, which we will index by $p^{-1}(v) \cup \{e \text{ adjacent to } v\}$, such that $w(v)_i = w_i$ if $i \in p^{-1}(v)$ and $w(v)_e = 1$ for adjacent edges $e$. The symmetric group $S_d(v)$ permutes the $\varepsilon$-stable points. The product should be taken only over all non-degenerate vertices.

3.2. The formula. The virtual localization formula expresses the virtual fundamental class as a sum of the (virtual) fundamental classes of the fixed loci $X$ weighted by the inverse of the equivariant Euler class of the corresponding virtual normal bundle $N_X$. In order to make sense of this inverse it is necessary to localize the the equivariant cohomology ring by $s$.

The inverse of the Euler class of $N_X$ is in both cases a product

$$\text{Cont} := \prod_v \text{Cont}(v) \prod_{e \text{ edge}} \text{Cont}(e),$$

for certain vertex and edge contributions depending only on the data of the graph corresponding to the vertex or edge. The contributions $\text{Cont}(e)$ and $\text{Cont}(v)$ for $v$ degenerate are pulled back from the equivariant cohomology of a point. We will not need to know the exact form of the edge and unstable vertex contributions here apart from the fact that the contribution of a vertex $v$ with $d(v) = g(v) = 0$ and $|w(v)| = 1$ is equal to 1.

The non-degenerate vertex contribution is pulled back from

$$A^*_C(\mathcal{M}_{g(v),(w(v),\varepsilon^d(v))}/S_d(v)) \otimes \mathbb{Q}[s,s^{-1}].$$

It is given by

$$\text{Cont}(v) = (\zeta(v)s)^{g(v)-d(v)-1} \sum_{j=0}^{\infty} \frac{c_j(F_d(v))}{(\zeta(v)s)^j} \prod_{e \text{ adjacent to } v} \frac{\zeta(v)s}{\omega^v_e - \psi^v_e},$$

where $F_d$ is the K-theory class $F_d = \mathbb{E}^* - B_d - C$, $\prod_e$ denotes a product over edges adjacent to $v$ and

$$\omega^v_e = \frac{\zeta(v)s}{d(e)}.$$
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over a marked curve \((C, p_i)\). The bundle \(B_d\) can also be thought as an \(S_d\) invariant bundle living on \(\overline{M}_{g,(w,\epsilon^d)}\). In \([3]\) the Chern classes of this lifted bundle have been computed:

\[
c(-B_d) = \prod_{i=n+1}^{n+d} \frac{1}{1 - \psi_i - \sum_{j=n+1}^{n+i} D_{ij}}.
\]

Notice the similarity to the \(\Phi\) function. The Chern classes for \(B_d\) on \(\overline{M}_{g,(w,\epsilon^d)/S_d}\) can be calculated by push-forward along the finite projection map and dividing by the degree \(d!\).

3.3. Comparison. The \(\mathbb{C}^*\) actions on \(\overline{M}_{g,w}(\mathbb{P}^1, d)\) and \(\overline{Q}_{g,w}(\mathbb{P}^1, d)\) are compatible with the comparison map \(c : \overline{M}_{g,w}(\mathbb{P}^1, d) \to \overline{Q}_{g,w}(\mathbb{P}^1, d)\) and hence maps fixed loci to fixed loci. The image of a locus corresponding to some combinatorial data is given by contracting all degenerate vertices, adding the values of \(d\) of the contracted vertices and edges to the degree of the image vertex of the contraction, and replacing the point assignment \(p\) by its composition with the contraction.

The Gromov-Witten stable quotient comparison says that the contribution of a stable quotient fixed locus to the virtual fundamental class of \(\overline{Q}_{g,w}(\mathbb{P}^1, d)\) is the sum of push-forwards of the contributions of all stable maps loci in the preimage of \(c\) to the virtual fundamental class of \(\overline{M}_{g,w}(\mathbb{P}^1, d)\). This in particular implies that

\[
c_*(\overline{Q}_{g,w}(\mathbb{P}^1, d)^{vir}) = [\overline{M}_{g,w}(\mathbb{P}^1, d)]^{vir}.
\]

4. THE OPEN LOCUS

We will first restrict ourselves to the special case of \(n\)-fold tensor powers \(M_{g|n}\) of the universal curve \(C_g\) over \(M_g\). This case occurs when the weights are sufficiently small (i.e. \(\sum_{i=1}^{n} w_i < 1\)) and we restrict ourselves to the locus corresponding to smooth curves.

4.1. Statement of the stable quotient relations. Let us define the bracket operator \(\{\}\) on \([\mathbb{Q}[t,p_1,\ldots,p_n]]\) in terms of the operators from the introduction by

\[
\{f(t)\} = - \{f(t)\},
\]

\[
\{f(t) \prod_{i \in S} p_i\} = \{f(t)\} D_S
\]

\[
\{p_i f(t,p_1,\ldots,p_n)\} = \{p_i f(t,p_1,\ldots,p_n)\},
\]

if \(e > 0\). For example \(\{t^2 p_1^3\} = t^2 p_1^3\) and \(\{t^2 p_1 p_2\} = -D_{12} p_1 = -D_{12} p_2\).

Proposition 2. The relations given by

\[
\sum_{\zeta \in \{\pm 1\}} \zeta^{-1} \left[ \exp \left( -\frac{1}{2} \zeta p + \{\exp(-pD)\gamma(\zeta,t,x)\} \right) \right]_{t^p x^d p^n} = 0,
\]

with the differential operator \(D = tx\frac{d}{dx}\) and \(p = p_1 + \cdots + p_n\) hold in \(M_{g|n}\) under the condition \(g - 2d - 1 + |a| < r\).
4.2. Proof of the stable quotient relations. We generalize here the localization method of \[6\].

For each \( i \in \{1, \ldots, n\} \) we can define a class \( s_i \in A^1(M_{g[n]}(\mathbb{P}^1, d)) \) as the pull-back of \( c_1(S) \) from the universal curve \( C_{g[n]}(\mathbb{P}^1, d) \) via the \( i \)-th section.

For given nonnegative integers \( a_i \) let us look at the class
\[
s^a = \prod_{i=1}^{n} (-s_i)^{a_i} \in A^{|a|}(M_{g[n]}(\mathbb{P}^1, d)).
\]

The strategy is to study the \( \mathbb{C}^* \) action from Section \[3\] to lift \( s^a \) to equivariant cohomology and to write down the localization formula calculating the push-forward of this class to the equivariant cohomology of \( M_{g[n]} \). As we have seen the general form of the localization formula is a sum of contributions from the fixed loci. For each fixed locus a class from its equivariant Chow ring localized by the localization variable \( s \) pushed forward via the inclusion of the fixed locus. We get the relations from the fact that the rational functions in the localization variable we obtain must actually be polynomials in \( s \) after summing over all fixed loci.

Remark 1. In \[3\] and \[6\] the same strategy was pursued but other related classes were chosen to be pushed forward. In similar spirit we could add factors of the form \( \pi^* (s_{n+1} c_1(\omega_\pi)^d) \), where \( \pi : Q_{g[n+1]}(\mathbb{P}^1, d) \to Q_{g[n]}(\mathbb{P}^1, d) \) is the forgetful map and \( \omega_\pi \) is the relative dualising sheaf, to the class we are pushing forward. However because of the commutative diagram
\[
\begin{array}{ccc}
Q_{g[n+1]}(\mathbb{P}^1, d) & \xrightarrow{\pi} & Q_{g[n]}(\mathbb{P}^1, d) \\
\downarrow \nu & & \downarrow \nu \\
M_{g[n+1]} & \xrightarrow{\pi} & M_{g[n]}
\end{array}
\]
and the fact that \( c_1(\omega_\pi) = \nu^*(\psi_{n+1}) \) these will be contained in the completed set of stable quotient relations\[6\].

In \[6\] only factors with \( a = 1 \) were used to derive the FZ relations on \( M_\nu \). The results from Section \[4, 5, 2\] imply that allowing higher values for \( a \) would not have led to more stable quotient relations.

The action of \( \mathbb{C}^* \) on \( \mathbb{P}^1 \) is induced by the action of \( \mathbb{C}^* \) on \( \mathbb{C}^2 \) given by \( (z_0, z_1, \lambda) \mapsto (z_0, \lambda z_1) \). This naturally induces \( \mathbb{C}^* \) actions not only on \( Q_{g[n]}(\mathbb{P}^1, d) \) but also on the universal curve \( C_{g[n]}(\mathbb{P}^1, d) \) and the universal sheaf \( S \). This gives a natural lift of the \( s_i \) to equivariant cohomology and therefore also a natural lift of \( s^a \). We will not choose this lift but instead
\[
s^a = \prod_{i=1}^{n} \left(-s_i - \frac{1}{2}a_i\right)^{a_i} \in A^{|a|}(M_{g[n]}(\mathbb{P}^1, d)),
\]
where the \( s_i \) are the natural lifts.

Let us consider the localization formula for this equivariant lift applied to the push-forward
\[
\nu_* (s^a \cap [M_{g[n]}(\mathbb{P}^1, d)]^{vir}) \in A_{2g+2d-2+n-|a|}(M_{g[n]}).
\]

\[To see this for more than one factor (say \( m \) factors) one needs to first interpret the product as a class on the \( m \)-fold tensor power \( X \) of \( Q_{g[n+1]}(\mathbb{P}^1, d) \) over \( Q_{g[n]}(\mathbb{P}^1, d) \) and use the birational map from \( Q_{g[n+m]}(\mathbb{P}^1, d) \) to \( X \).
Let us shorten this by writing $\nu^\text{vir}_* (s^a)$ for this push-forward after capping with the virtual fundamental class.

Because we assume the strict inequality $\sum_{i=1}^n w_i < 1$ there are only two fixed loci with respect to the torus actions in the description of Section 3.1. Concretely in this special case they correspond to elements $(C, p_1, \ldots, p_n)$ of $M_{g|n}$ with quotient sequence

$$0 \to \mathcal{O}_C(-p_{n+1} - \cdots - p_{n+d}) \to \mathcal{O}_C \to \mathcal{O}_C^2 \to Q \to 0$$

with $\mathcal{O}_C \to \mathcal{O}_C^2$ induced from one of the two torus invariant inclusions of $C$ as a coordinate axis in $\mathbb{C}^2$. Here $\mathcal{O}_C(-p_{n+1} - \cdots - p_{n+d})$ is an ideal sheaf of $\mathcal{O}_C$ of degree $d$. Both fixed point loci can be identified with $M_{g|n+d}/S_d$ where the symmetric group $S_d$ permutes the last $d$ markings.

Since the graphs corresponding to the fixed point loci have each only one vertex and no edge we can calculate the inverse of the equivariant Euler class of the normal bundle to each fixed locus using (1). It is given by

$$(\zeta s)^{g-d-1} \sum_{j=0}^\infty \frac{c_j(\mathbb{F}_d)}{(\zeta s)^t},$$

where $\zeta$ is $+1$ and $-1$ for 0 and $\infty$ respectively.

Applying the fixed point formula we obtain for the $s^c$ part of the push-forward (2)

$$[\nu^\text{vir}_* (s^a)]_{s^c} = \frac{1}{d!} \sum_{\zeta} \epsilon_* \left[ \prod_{i=1}^n \left( -s_i t - \frac{1}{2} \zeta \right) \sum_{j=0}^\infty \zeta^j \sum_{d=0}^c \frac{c_j(\mathbb{F}_d)}{d!} \right],$$

where $\epsilon : M_{g|n+d} \to M_{g|n}$ forgets the last $d$ markings. We have here by abuse of notation denoted similarly defined classes on $M_{g|n+d}$ with the same name as on $M_{g|n}(\mathbb{P}^1, d)$. The expression $(-s_i t - \frac{1}{2} \zeta)$ comes from the fact that the torus acts trivially on the subspace of $\mathbb{C}^2$ given by 0 and with weight 1 on the subspace given by $\infty$. The equivariant lift of the $s_i$ was chosen in order to have this symmetric expression.

Since the push-forward must be an honest equivariant class the classes (2) must be zero if $c < 0$.

Let us package these relations into a power series. We have

(3)

$$\prod_{i=1}^n a_i! \cdot [\nu^\text{vir}_* (s^a)]_{s^c} = \sum_{\zeta} \zeta^{g-d-1} \epsilon_* [\exp(-T_1)T_2]_{t^a d^b p^s},$$

with

$$T_1 = \sum_{i=1}^n \left( s_i t + \frac{1}{2} \zeta \right) p_i,$$

$$T_2 = \sum_{j=0}^\infty \sum_{d=0}^c (\zeta t)^j c_j(\mathbb{F}_d) \frac{x^d}{d!}.$$
we can rewrite $T_1$ as

$$T_1 = \sum_{i=1}^{n} \sum_{j=n+1}^{n+d} D_{ij} p_i + \frac{1}{2} \zeta \sum_{i=1}^{n} p_i.$$ 

4.2.1. Relations between diagonal classes. In order to better understand $c(\mathbb{F}_d)$ let us collect the universal relations between classes in $A^*(M_{g|n})$ here.

The basic relations are

$$D_{ij} \psi_i = D_{ij} \psi_j = -D^2_{ij}$$
$$D_{ij} D_{ik} = D_{ijk}$$

for pairwise different $i, j, k$ (compare to [16]). Let $D_{S,a} \in A^a(M_{g|n})$ by defined for any $S \subset \{1, \ldots, n\}$ and $a \geq |S| - 1$ by

$$D_{S,a} = \begin{cases} \psi_a^{|S|} & \text{if } S = \{i\} \\ (-1)^{|S| - 1} D_S \psi_a^{|S| - 1} & \text{else, where } i \in \{1, \ldots, d\}. \end{cases}$$

Then $D_{S,a} D_{T,b} = D_{S,T,a+b}$ if $S \cap T \neq \emptyset$.

**Lemma 5.** Each monomial $M$ in diagonal and cotangent line classes in $A^*(M_{g|n})$ can be written as a product

$$M = \prod_{S \in P} D_{S,a(S)+|S| - 1}$$

for some partition $P$ of $\{1, \ldots, d\}$ and a function $a : P \to \mathbb{Z}_{\geq 0}$. This product decomposition is unique if we only use the above relations between diagonal and cotangent line classes.

**Remark 2.** If the partition $P$ is the one element set partition, we say that $M$ is connected. We call the factors of the decomposition (or just the elements of $P$) the connected components of $M$.

The push-forward under the forgetful map $\pi : M_{g|n} \to M_{g|n-1}$ is given by

$$\pi_* D_{S,a} = \begin{cases} 0 & \text{if } n \notin S \\ \kappa_{a-1} & \text{if } |S \cap \{n\}| = 1 \\ -D_{S \setminus \{n\}, a-1} & \text{else}. \end{cases}$$

Here and in the rest of this article $\kappa_{-1}$ is defined to be zero.

4.2.2. Simplest relations. Let us first consider the stable quotient relations in the case of $a = 0$. Then they are simply

$$0 = \sum_{\zeta} \zeta^{g-d-1} \varepsilon_* \left[ \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} (\zeta t)^j \varepsilon_j(\mathbb{F}_d) \frac{2^d}{d!} \right]^{\nu_x d}$$

for

$$r > g - 1 - 2d.$$ 

By the definition of $\mathbb{F}_d$ as a K-theoretic difference of $E^*$, a trivial rank 1 bundle and $\mathbb{B}_d$ the inner sum breaks into two factors. The part corresponding to $E^*$ is
pulled back from $M_{g,n}$ and does not depend on $d$. Using Mumford’s formula \[17\] we can therefore rewrite the relations as:
\[
\sum_{\zeta} \zeta^{g-d-1} \left[ \exp \left( - \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} (t\zeta)^{2i-1} \right) \varepsilon_s \sum_{d=0}^\infty \sum_{j=0}^\infty (\zeta t)^j c_j (-B_d)^{\frac{d}{d!}} \right] \varepsilon^r \varepsilon^{x_d}
\]
vanishes if \[1\] holds.

To deal with the second factor we formally expand it as a power series in $x$, $t$ and the classes $D_{S,a}$ for various $S \subset \{n+1, \ldots, n+d\}$ and $a \in \mathbb{Z}_{\geq 0}$. By the exponential formula and the facts from Section 4.2.1 we have that
\[
\varepsilon_s \sum_{d=0}^\infty \sum_{j=0}^\infty (\zeta t)^j c_j (-B_d)^{\frac{d}{d!}} = \exp \left( \varepsilon_s \sum_{d=1}^\infty \sum_{r=0}^\infty S_d^r D_{\{n+1, \ldots, n+d\}, r} (\zeta t)^r \frac{x^d}{d!} \right)
\]
where
\[
\log \left( \sum_{d=0}^\infty \sum_{i=1}^d \frac{1}{1-it} \right) \frac{x^d}{d!} = \sum_{d=1}^\infty \sum_{r=0}^\infty S_d^r \varepsilon^r \frac{x^d}{d!}.
\]

With $\varepsilon_s D_{\{n+1, \ldots, n+d\}, r} = (-1)^{d-1} \kappa_{r-d}$ and noticing the similarities between the series defining $S_d^r$ and $\log(\Phi)$ we obtain
\[
\varepsilon_s \sum_{d=0}^\infty \sum_{j=0}^\infty (\zeta t)^j c_j (-B_d)^{\frac{d}{d!}} = \exp \left( -\{\log(\Phi(\zeta t))\}_x \right)
\]
and so the stable quotient relations in the case $a = 0$ are
\[
\left[ \sum_{\zeta} \zeta^{g-1} \exp \left( -\{\gamma(\zeta t)\}_x \right) \right] \varepsilon^r \varepsilon^{x_d} = 0
\]
under Condition \[1\].

4.2.3. General relations. We will investigate how monomials in the $s_i$ affect the push-forward under $\varepsilon$ of monomials supported only on the last $d$ points.

Notice that for each partition $P$ of $\{n+1, \ldots, n+d\}$ we have
\[
(5) \quad \exp \left( - \sum_{i=1}^n s_i \right) = \prod_{S \in P} \exp \left( - \sum_{i=1}^n \sum_{j \in S} D_{ij} \right),
\]
and each factor is pulled back via the map forgetting all points in $\{n+1, \ldots, n+d\}$ not in $S$.

Moreover notice that if $M$ is a connected monomial supported in the last $d$ marked points with $\varepsilon_s M = [-\{f(t)\}_x]_x = [\{f(t)\}_\Delta]_x$, we have
\[
(6) \quad \varepsilon_s ((-s_i p_i)^c x^d M)]_{x^d} = \varepsilon_s ((-d D_{i,n+1} p_i)^c x^d M)]_{x^d} = [\{(p_i D)^c (x^d f(t))\}_\Delta]_{x^d}.
\]

From \[5\], \[6\] and the identity
\[
\exp(pD) \exp(f(t,x)) = \exp(\exp(pD)f(t,x))
\]
we obtain the general stable quotient relations
\[
\sum_{\zeta} \zeta^{g-1} \left[ \exp \left( -\frac{1}{2} \zeta p + \{\exp(pD)\gamma(\zeta t,x)\}_x \right) \right] \varepsilon^r \varepsilon^{x_d p^a} = 0,
\]
if
\[ r > g - 1 - 2d + |a|. \]

4.3. Evaluation of the relations.

4.3.1. Minor simplification. Notice that in the relations of Proposition 2 the summands in the $\zeta$ sum are equal up to a sign
\[ (-1)^{g-1+r+|a|}. \]
Therefore the relations are in fact zero if $g + r + |a| \equiv 0 \pmod{2}$, and in the case
\[ g - 1 + r + |a| \equiv 0 \pmod{2} \]
we can reformulate them to
\[
\left[ \exp \left( -\frac{1}{2}p + \{\exp(pD)\gamma(t,x)\} \right) \right]_{t,x^d p^n} = 0.
\]

4.3.2. Differential algebra. In this section we will establish that it is enough to consider the stable quotient relations in the case that $a_i < 2$ for all $i \in \{1, \ldots, m\}$.

The series $\delta = D\gamma - \frac{1}{2}$ satisfies the differential equation
\[ D\delta = -\delta^2 + x + \frac{1}{4}, \]
as can be derived from the differential equation
\[ D(\Phi - D\Phi) = -x\Phi, \]
which is satisfied by $\Phi$ as seen by looking at its series definition.

Reformulating the relations with $\delta$ gives
\[
\left[ \exp \left( -\{\gamma\}_\kappa + \sum_{i=1}^{\infty} \frac{1}{i!} \{p^i D^{i-1}\delta\} \right) \right]_{t,x^d p^n} = 0
\]
if (7) and (8) hold.

Let us consider
\[ G = \frac{\partial^2}{\partial p_j^2} F, \]
with
\[ F = \exp \left( -\{\gamma\}_\kappa + \sum_{i=1}^{\infty} \frac{1}{i!} \{p^i D^{i-1}\delta\} \right). \]
We have that
\[
G = \frac{\partial^2}{\partial p_j^2} \sum_{i=1}^{\infty} \frac{1}{i!} \{p^i D^{i-1}\delta\} \Delta_j + \left( \frac{\partial}{\partial p_j} \sum_{i=1}^{\infty} \frac{1}{i!} \{p^i D^{i-1}\delta\} \Delta_j \right)^2.
\]
We can move the bracket out of the square in the second summand since the part of the brackets in both factors which is not annihilated by $\frac{\partial}{\partial p_j}$ contains at least a factor $p_j$. So we obtain
\[
G = \{\exp(pD)D\delta + (\exp(pD)\delta)^2\} \Delta_j = \{\exp(pD)(D\delta + \delta^2)\} \Delta_j
\]
\[ = \left\{ \exp(pD) \left( x + \frac{1}{4} \right) \right\} \Delta_j = \frac{1}{4} + x \{\exp(pt)\} \Delta_j, \]
where the operator \( \{ \} \Delta_j \) is defined by \( \{ f \}_\Delta_j = p_j^{-1} \{ p_j f \}_\Delta \). Therefore we have that
\[
\frac{\partial^2}{\partial p_j^2} F = \left( \frac{1}{4} + x \{ \exp(pt) \}_\Delta \right) F
\]
and can express the relations for \( a_j \geq 2 \) in terms of lower relations multiplied by monomials in cotangent line and diagonal classes.

4.3.3. Substitution. The differential equation satisfied by \( -t\gamma \) has been studied by Ionel in [18]. In [6] her results were extended to give formulas for \( D^i\gamma \). We will collect some of their results on \( \gamma \) and its derivatives here.

With the new variables
\[
u = \frac{t}{\sqrt{1+4x}}, \quad y = \frac{-x}{1+4x}
\]
one can write
\[
\gamma = \frac{1}{t} (t\gamma)(0, x) + \frac{1}{4} \log(1 + 4y) + \sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j} u^k y^j,
\]
for some coefficients \( c_{k,j} \), which are defined to vanish outside the summation region above. Furthermore, we have for the derivatives of \( \delta \)
\[
D^{i-1}\delta = (1 + 4y)^{-\frac{i}{2}} \left( \sum_{j=0}^{i-1} b^j_i u^{i-1} y^j - \sum_{k=0}^{\infty} \sum_{j=0}^{k+i} c_{k,j} u^{k+i} y^j \right) = (1 + 4y)^{-\frac{i}{2}} \delta_i,
\]
for some coefficients \( b^j_i, c_{k,j} \).

We will also need a result by Ionel relating coefficients of a power series \( F \) in \( x \) and \( t \) before and after the variable substitution:
\[
[F]_{t^r x^d} = (-1)^d \left( (1 + 4y)^{\frac{d^2 - 2}{2}} F \right)_{u^r y^d}
\]
Let us now apply these formulas to the relations. Using the fact that \( \kappa - 1 = 0 \) we get
\[
\left[ (1 + 4y)^e \exp \left( - \sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j} u^k y^j \right) + \sum_{i=1}^{\infty} \frac{1}{i!} \{ p^i \delta_i \}_\Delta \right]_{u^r y^d p^i} = 0,
\]
under conditions (7) and (8) with the exponent
\[
e = \frac{r + 2d - 2}{2} - \frac{\kappa_0}{4} - \frac{|a|}{2} = \frac{r + 2d - 1 - g - |a|}{2}.
\]
4.3.4. Extremal coefficients. It is noticeable that in the series
\[
\sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j} u^k y^j, \quad \sum_{j=0}^{i-1} b^j_i u^{i-1} y^j - \sum_{k=0}^{\infty} \sum_{j=0}^{k+i} c_{k,j} u^{k+i} y^j
\]
appearing in the formulas for \( \gamma \) and \( D^{i-1}\delta \) the \( y \) degree is bounded from above by the \( u \) degree. We will only be interested in the extremal coefficients. Here the \( A \) and \( C_i \) come into the picture.

In [18] it has been proven that
\[
\log(A(t)) = \sum_{k=1}^{\infty} c_{k,k} t^k
\]
and in [6] it is shown that
\[ 2^i C_i(t) = b_i^{-1}t^{i-1} - \sum_{k=0}^{\infty} c_{k,k+i} t^{k+i}. \]

We see that in the exponential factor of the relations for each summand the \( y \) degree is bounded from above by the \( u \) degree. Furthermore the exponent \( e \) of the \((1 + 4y)\) factor is integral and positive by (8) and (7). This implies that the relation is zero unless
\[ 3r \geq g + 1 + |a| \]
holds. With the following lemma we can extract the extremal part of the relations.

**Lemma 6.** Fix any \( \mathbb{Q} \) algebra \( A \), any \( F \in A[y] \) and any \( c \in \mathbb{Z}_{\geq 0} \). The relations
\[ [(1 + 4y)^dF]_{y^d} = 0 \]
for all \( d > c \) imply \( F = 0 \).

**Proof.** The relations can be rewritten to
\[ \left[ \left( \frac{1}{y} + 4 \right)^d F \right]_{y^0} = 0. \]
The lemma follows by the fact that \( F \) is a polynomial in \( y \) and linear algebra. \( \square \)

Using the lemma and the formulas for the extremal coefficients we obtain that the relations
\[ \exp(-\{\log(A)\}_\kappa) \exp \left( \sum_{i=1}^{\infty} \frac{1}{i!} \{p^i C_i\}_\Delta \right) \]
hold under the conditions (8) and (9).

Ignoring terms of higher order in the \( p_i \) the second factor can be rewritten
\[ \exp \left( \sum_{i=1}^{\infty} \frac{1}{i!} \{p^i C_i\}_\Delta \right) \equiv \exp \left( \sum_{\emptyset \neq S \subseteq \{1, \ldots, n\}} \{p^S C_i |_S\}_\Delta \right) \]
\[ \equiv \sum_{S \subseteq \{1, \ldots, n\}} p^S \sum_{P \vdash S} \prod_{i \in P} \{C_i |_i\} D_i. \]

Thus we finally obtain that the FZ type relations
\[ \left[ \exp(-\{\log(A)\}_\kappa) \sum_{P \vdash S} \prod_{i \in P} \{C_i |_i\} D_i \right]_{z^r} = 0, \]
hold for any \( S \subseteq \{1, \ldots, n\} \) if (8) and (9).

### 5. Localization sums

In the derivation of the more general stable quotient relations we will need to deal with two types of localization sums related to the nodes and marked points respectively. To keep the length of the proof of the stable quotient relations more reasonable we will deal with them here. The sums are genus independent and have been studied more generally by Givental [19]. We apply here his method in a special case. See also [20] and [21].
Let $N^*_\mathbb{C}(\mathbb{P}^1)$ be the Novikov ring of $\mathbb{P}^1$ with values in $\mathbb{C}[s, s^{-1}]$. We define a formal Frobenius manifold structure on $X = A^*_\mathbb{C}(\mathbb{P}^1) \otimes N^*_\mathbb{C}(\mathbb{P}^1)$ over $N^*_\mathbb{C}(\mathbb{P}^1)$ using equivariant Gromov-Witten theory. The $N^*_\mathbb{C}(\mathbb{P}^1)$-module $X$ is free of dimension two with idempotent basis $\{\phi_0, \phi_{\infty}\}$ for $\phi_i = [i]/e_i$, where $e_i$ is the equivariant Euler class of the tangent space of $\mathbb{P}^1$ at $i$. We denote the corresponding coordinate functions by $y_0, y_{\infty}$. This gives a basis $\{\frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_{\infty}}\}$ of the space of vector fields on $X$. The metric is given in this basis by

$$g = \begin{pmatrix} s^{-1} & 0 \\ 0 & -s^{-1} \end{pmatrix} = \begin{pmatrix} e_0^{-1} & 0 \\ 0 & e_{\infty}^{-1} \end{pmatrix}.$$ 

The primary equivariant Gromov-Witten potential is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d=0}^{\infty} x^d \int_{[0,n,(\mathbb{P}^1,d)]_{vir}} \prod_{i=1}^{n} ev_i^* (y_0 \phi_0 + y_{\infty} \phi_{\infty}) = \frac{y_0^3}{6s} - \frac{y_{\infty}^3}{6s} + xe^w,$$

where we have set $w = (y_0 - y_{\infty})/s$ (compare to [22]). Thus we have

$$\alpha = \left( \frac{dy_0 + \frac{x}{e_0} e^w dw}{e_0} e^w dw, \frac{dy_{\infty} + \frac{x}{e_{\infty}} e^w dw}{e_{\infty}} e^w dw \right),$$

for the matrix of one forms $\alpha$ defined by

$$\alpha = \sum_i A_i dy_i$$

from the matrices $A_i$ defined by the quantum product

$$\frac{\partial}{\partial y_i} \star \frac{\partial}{\partial y_a} = \sum_b [A_i]_{ab} \frac{\partial}{\partial y_b}.$$

Using $\alpha$ we can compactly write down the differential equation for the $S$-matrix

$$(td - \alpha) S = 0,$$

with initial condition $S(0, 0) = \text{Id}$.

If we set

$$S = \begin{pmatrix} S_0^0 & S_{0\infty}^0 \\ S_{0\infty}^0 & S_{\infty\infty}^0 \end{pmatrix}$$

this gives the system

$$td S_i^j = S_i^j dy_j + \sum_k S_k^i \frac{x}{e_k} e^w dw,$$

with solution

$$S_i^j = e^{\frac{w}{s}} \left( \left( \frac{1 + \zeta_i \zeta_j}{2} - tx \frac{d}{dx} \right) \Phi \right) \left( - \frac{t}{e_i} \frac{x e^w}{e_i} \right),$$

with the signs $\zeta_i = \frac{\Phi}{s} \in \{\pm 1\}$.

There is a set of canonical coordinates $u^i$ on $X$ defined by the localization sums

$$u^i = \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} x^d \sum_{\Gamma \in G_{0,n+2}(\mathbb{P}^1,d)} \text{Cont} \left( e_i \frac{\partial^2 F_0}{(\partial y_i)^2} \right),$$

where $G_{0,n+2}(\mathbb{P}^1, d)$ is the set of $n + 2$-point (each of weight 1) degree $d$ localization graphs with the first two points on a single component contracted to $i$. By the
Gromov-Witten stable quotient comparison and since there is only one fixed locus on the stable quotient side we have

\[ u^i(0, 0) = e_i e_i^{−d−1} \sum_{d=1}^{\infty} x^d \int_{\overline{M}_{0, 2|d}} \sum_{j=0}^{\infty} \frac{c_j(\mathbb{P}_d)}{e_i}, \]

where \( \overline{M}_{0, 2|d} = \overline{M}_{0, (2^d)} \) is a Losev-Manin space. Since the Hodge bundle is trivial on \( \overline{M}_{0, 2|d} \) this simplifies to

\[ u^i(0, 0) = e_i e_i^{−d−1} \sum_{d=1}^{\infty} x^d \int_{\overline{M}_{0, 2|d}} \sum_{j=0}^{\infty} \frac{c_j(-\mathbb{P}_d)}{e_i}. \]

Recalling the definition of \( \mathbb{P}_d \) we can write the integrand on the right hand side as a sum of monomials in \( \psi \) and diagonal classes. The integral of such a monomial vanishes unless it is the diagonal where all \( d \) points come together. The constants \( -C_d^{-1} \) which are defined from \( \log(\Phi) \) by

\[ \log(\Phi(t, x)) = \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} C_d^{r} t^r \sum_{d=1}^{\infty} C_d^{r} t^r \frac{x^d}{dt} \]

exactly count these contributions. So we get

\[ (10) \quad u^i(0, 0) = -\sum_{d=1}^{\infty} C_d^{−1} x^d \frac{e_i}{dt} e_i^{−2d+1}. \]

Using the \( S \)-matrix and the \( u^i \) we can now calculate the series we are interested in. We will not regard \( S \) and the \( u^i \) outside \( (0, 0) \), so let us write from now on \( S_i^j \) for \( S_i^j(0, 0) \) and \( u^i \) for \( u^i(0, 0) \).

The first series \( P^i_{ij} \) will be needed to deal with stable quotient localization chains containing one of the weight 1 marked points. We have

\[ P^i_{ij}(t, x) := \frac{1 + \zeta_i \zeta_j}{2} \sum_{d=1}^{\infty} \sum_{\Gamma \in G^{i}_{E, 2}(\mathbb{P}^1, d)} e_i e_j e^{u_i/\omega_{\Gamma, 1}} e_j e^{u_j/\omega_{\Gamma, 2}} \text{Cont}_{\Gamma} \left( \frac{\partial F_0}{\partial y_i \partial y_j} \right) = e^{u_i/\omega_{\Gamma, 1}} e_j e^{u_j/\omega_{\Gamma, 2}} \text{Cont}_{\Gamma} \left( \frac{\partial F_0}{\partial y_i \partial y_j} \right), \]

where \( G^{i}_{E, 2}(\mathbb{P}^1, d) \) is the set of all localization graphs with the first marking on a valence 2 vertex at \( i \) and the second marking at \( j \) and \( \omega_{\Gamma} \) is the \( \omega \) as in \( \mathbb{1} \) corresponding to the flag at the first marking.

The second series is needed to deal with chains at the nodes of the curve. We have

\[ E^i_{ij}(t_1, t_2, x) := \sum_{d=1}^{\infty} x^d \sum_{\Gamma \in G^{i}_{E, 2}(\mathbb{P}^1, d)} e_i e_j e^{u_i/\omega_{\Gamma, 1}} e_j e^{u_j/\omega_{\Gamma, 2}} \text{Cont}_{\Gamma} \left( \frac{\partial F_0}{\partial y_i \partial y_j} \right) \]

\[ = \frac{s}{t_1 + t_2} \left( \frac{\zeta_i + \zeta_j}{2} - e^{u_i/\omega_{\Gamma, 1}} + e^{u_j/\omega_{\Gamma, 2}} \right) \sum_{\mu} S_{1}^{\mu}(-t_1) \zeta_{\mu} S_{2}^{\mu}(-t_2), \]

where \( G^{i}_{E, 2}(\mathbb{P}^1, d) \) is the set of all localization graphs with both markings on valence 2 vertices, the first marking mapped to \( i \), the second marking to \( j \), and \( \omega_{\Gamma, 1}, \omega_{\Gamma, 2} \) are \( \omega \) as in Section \( \mathbb{2} \) corresponding to the flag at the first and second marking respectively.
Let us now simplify the expressions for \( P_{ij}(t,x) \) and \( E_{ij}(t_1,t_2,x) \) using the explicit \( S \)-matrix and canonical coordinates at 0. For \( P_{ij}(t,x) \) we obtain

\[
P_{ij}(t,x) = \exp \left( -\sum_{d=1}^{\infty} C_{d-1}^{-1} t^{-1} x^d \frac{d}{dt} \right) \left( \left( \frac{1 + \zeta_i \zeta_j}{2} - t \zeta_i \zeta_j x \frac{d}{dx} \right) \Phi \left( \frac{t}{e_i}, \frac{x}{e_i^2} \right), \right)
\]

where \( \Phi' \) is defined by

\[
\Phi'(t,x) = \exp \left( -\sum_{d=1}^{\infty} C_{d-1}^{-1} t^{-1} x^d \frac{d}{dt} \right) \Phi(t,x) = \exp \left( \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} C_{d} t^r x^d \right).
\]

Similarly we obtain

\[
t_1 + t_2 E_{ij}(t_1,t_2,x) = \frac{\zeta_i + \zeta_j}{2} + \Phi' \left( \frac{t_1}{e_i}, \frac{x}{e_i} \right) \Phi' \left( \frac{t_2}{e_j}, \frac{x}{e_j} \right) \left( \zeta_i \delta \left( \frac{t_1}{e_i}, \frac{x}{e_i} \right) + \zeta_j \delta \left( \frac{t_2}{e_j}, \frac{x}{e_j} \right) \right).
\]

6. THE GENERAL CASE

We now extend the relations of Section 4 to the boundary and allow nearly arbitrary weights. More precisely we will assume that:

\[
(11) \quad \text{If for some } S \subset \{1, \ldots, n\} \text{ we have } \sum_{i \in S} w_i = 1, \text{ we must have } |S| = 1.
\]

We still obtain relations for any weight data since one can always modify \( w \) a bit such that the moduli space \( \overline{M}_{g,w} \) does not change (whereas \( Q_{g,w}(\mathbb{P}^1, d) \) will change). It is even not necessary to allow that there exist points of weight 1 at all but it is interesting to see different ways of obtaining the same relations.

6.1. Statement of the stable quotient relations. Let \( G \) be the set of stable graphs describing the strata on \( M_{g,w} \). The data of \( G \) in particular includes a map \( p \) from \( \{1, \ldots, n\} \) to the set of vertices as in Section 3.1. Let us assume that the first \( n' \) points are of weight different from 1 and the remaining \( n'' \) points are of weight 1.

**Proposition 3.** For a codimension \( r \), a degree \( d \), an \( n \)-tuple \( a \) such that \( g - 2d - 1 + |a| < r - |E| \), in \( A'(\overline{M}_{g,w}) \) the relation

\[
0 = \left[ \sum_{G \in G} \frac{1}{\text{Aut}(G)} \xi_G \left( \prod_v \text{Vertex}^{3^{\Delta(v)}} \prod_e \text{Edge}^{3^{\Delta(v)}} \right) \prod_{i=n'+1}^n \text{Point}^{3}(\psi_i \zeta(p(i)), t, p_i) \right]_{\text{tr} - |E| x^d p^n}
\]

holds, where \( \text{Vertex}^{3^\zeta} \) is a product

\[
\text{Vertex}^{3^\zeta} = \zeta^{g(v)-1} \exp \left( -\frac{1}{2} \zeta p(v) + \{\exp(p(v)D) \gamma(t \zeta, x)\}_{\Delta(v)} + V^{\zeta} \right)
\]
in $A^*(\overline{M}_{g,w})[x,t,p]$, where $p(v) = \sum_{i=1}^{n'} p_{i(v)} \; \{ \Delta(v) \}$ is defined identically to $\{ \Delta \}$ but $\kappa_j$ is replaced by $\kappa_j^{(v)}$ — the $\kappa_j$ class at $v$ — and

$$V^\xi = - \sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i - 1)} \frac{1}{t^2} \sum_{\Delta \in \mathcal{D}G} \frac{1}{\text{Aut}(\Delta)})^\xi_{\Delta,s} \left( \psi^2 - \psi h \right)^{(\zeta t)^{2i-1}}.$$

Here $\mathcal{D}G \subseteq G$ is the set of graphs corresponding to divisor classes, i.e., graphs with only one edge, and $\psi_a, \psi_b$ are the two cotangent line classes corresponding to the unique node corresponding to a divisor. The edge and point series are given by

$$t(\psi_1 + \psi_2)_{\text{Edge}}^3 = (\Phi'(t \zeta t \psi_1) \Phi'(t \zeta t \psi_2))^{-1} \left( \frac{\zeta_1 + \zeta_2}{2} + \zeta_1 \delta(t \zeta t \psi_1, x) + \zeta_2 \delta(t \zeta t \psi_2, x) \right),$$

where $\Phi'$ is from Section 5, and

$$\text{Point}^3(t, p) = 1 + p \delta(t, x).$$

6.2. Proof of the stable quotient relations. In Section 4.2 we looked at the integrand $s^b$ coming from powers of the pulled back first Chern class of the universal sheaf $\mathcal{S}$ over $\mathcal{G}_{g,n}(\mathbb{P}^1, d)$. This works also well for $\overline{M}_{g,w}(\mathbb{P}^1, d)$ if no point is of weight 1. For the points of weight 1 we however need to choose different classes since by the stability conditions the torsion of $\mathcal{S}$ must be away from the sections of points of weight 1. Instead we will pull back classes from $\mathbb{P}^1$ via evaluation maps.

For an $n$-tuple $a$, which can be split into an $n'$-tuple $a'$ and an $n''$-tuple $a''$, we will thus study the equivariant class

$$\varphi(a) := \tilde{s}^n \prod_{i=n'+1}^n \text{ev}^*_i \left( \left( \frac{[0] + [\infty]}{2} \right)^{a_i} \right) \in A^*[\ast]_{\mathcal{C}}(\overline{M}_{g,w}(\mathbb{P}^1, d),$$

where $[0]$ and $[\infty]$ are the equivariant classes of 0 and $\infty$ respectively, and $\tilde{s}^n$ is the equivariant class from Section 4.2. Since $A^*[\ast]_{\mathcal{C}}(\mathbb{P}^1) \otimes \mathbb{Q}[s, s^{-1}]$ is a two dimensional $\mathbb{Q}[s, s^{-1}]$ module we do not lose any relations by considering only the case when $a''$ is $\{0, 1\}$ valued.

As before we consider the $s^c$ part of the push-forward of $\varphi(a)$ for $c < 0$ when using the virtual localization formula.

In order to gain overview over the in $d$ monotonously growing number of fixed loci we sort them according to the stratum of $\overline{M}_{g,w}$ they push forward to. Let us consider stable graphs $\Gamma = (V, E)$ of $\overline{M}_{g,w}$ together with a coloring $\zeta : V \to \{ \pm 1 \}$ and a degree assignment $d : V \cap E \cap \{ n'+1, \ldots, n \} \to \mathbb{Z}_{\geq 0}$. This data records coloring and the $d_i$ from a stable quotient graph, the total degrees of the chains which destabilize to a node or a weight 1 marked point when pushing forward to $\overline{M}_{g,w}$. To get back to a stable quotient graph one needs to choose for each edge and each weight 1 marked point of degree $d$ a splitting $d = d_{e_1} + d_{e_2} + \cdots + d_{e_{\ell-1}} + d_{e_{\ell}}$ of $d$ corresponding to the degrees on the chain (see Figure 1). Because of the conditions on the coloring of a stable graph there is a mod $2$ condition on the length $\ell$ of the chains depending on the color at the connection vertices or $a$. If we had not imposed (11) we would also have chains for each set of marked points of total weight exactly 1.
The fixed loci corresponding to such a decorated stable graph are, up to a finite map, isomorphic to products

\[ \prod_v M_{g(v), w(v), \epsilon_d(v)} / S_d \]

times a number of Losev-Manin factors \( M_{0,2} / S_d \) corresponding to the vertices inside the chains.

For the localization calculation we will also have to consider the pull-back of the integrand \( \varphi(a) \) to each fixed locus. The factors \( ev_i^*(\{[0] + [\infty]\}) \) of \( \varphi(a) \) merely restrict the coloring of the stable quotient graphs and their contribution is pulled back from the equivariant cohomology of a point. The other factors \( s_i^{a_i} \) need to be partitioned along the factors of the stable quotient fixed locus. However it is easy to see that in order for the contribution to be nonzero \( s_i^{a_i} \) has to land at the factor corresponding to the vertex \( p(i) \).

After all these preconsiderations let us write down the localization formula. It will be the most convenient to write it in a power series form. We have

\[ \nu^{\text{vir}}_* \varphi(a) = \left[ \sum_{\Gamma, \xi, d} \frac{1}{\text{Aut}(\Gamma)} \xi_\Gamma \epsilon_s \left( \prod_v \text{Vertex}_v^1 \prod_e \text{Edge}_e^1 \prod_{i=1}^n \text{Point}_i^1 \right) \right] \times \mathbb{P}^n, \]

where \( \nu^{\text{vir}}_* \) again denotes push-forward using \( \nu \) after capping with the virtual fundamental class and the vertex, edge and point series still need to be defined.

The vertex series \( \text{Vertex}_v^1 \) is given by

\[ \text{Vertex}_v^1 = (\zeta(v)s)^{g(v)-1}\exp(-T_1)T_2 \in \bigoplus_{d=0}^{\infty} \mathbb{A}_c^+ \left( \prod_v M_{g(v), w(v), \epsilon_d(v)} \right) \otimes \mathbb{Q}[[x, p, s, s^{-1}]] \]

with

\[ T_1 = \sum_{n' \geq 1 \in p^{-1}(v)} sp_{n'} + \frac{1}{2} \zeta(v) \sum_{n' \geq 1 \in p^{-1}(v)} sp_{n'} \]

\[ T_2 = \sum_{j=0}^{\infty} \sum_{d=0}^{\infty} (\zeta(v)s)^{d-j} c_j(F_d) \frac{x^d}{(\zeta(v)s)^{2d}d!}. \]

For the edge series \( \text{Edge}_e^1 \) we have

\[ \text{Edge}_e^1 = \sum_{d=1}^{\infty} x^d \sum_{\Gamma_d=(V_e,E_e)} \frac{1}{\text{Aut}(\Gamma_d^e)} \]

\[ \prod_{v} \text{Cont}(f) \prod_{e} \text{Cont}(v), \]
where $\Gamma_d^i$ is a stable quotient localization graph of $\overline{Q}_{0,2}(\mathbb{P}^1, d)$ with color of the two vertices determined by $\zeta$ at the two vertices adjacent to $e$. The variables $\zeta_i$, $d_i$, $\psi_i$ for $i \in \{1, 2\}$ denote the color, the weight and the $\psi$ classes of the two marked points corresponding to the edge respectively. The contributions $\text{Cont}(f)$ and $\text{Cont}(v)$ are contributions to the calculation of the integrals

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^1, d)} \text{ev}_1^*(\phi_1) \text{ev}_2^*(\phi_2)$$

as in Section 5. By the stable quotients Gromov-Witten comparison we can replace $\overline{Q}_{0,2}(\mathbb{P}^1, d)$ everywhere in this paragraph by $\overline{M}_{0,2}(\mathbb{P}^1, d)$ and in then $E_{ij}$ is very similar to $E_{ij}(t_1, t_2, x)$ from Section 5.

Finally the point series $\text{Point}_1^i$ is similarly given by

$$\text{Point}_1^i = \left(\frac{s}{2}\zeta_i\right)^{a_i} + \sum_{d=1}^{\infty} \sum_{\Gamma_d^i=(V_i, E_i)} 1 \frac{1}{\text{Aut}(\Gamma_d^i)} \prod_{f \text{ edge}} \text{Cont}(f) \prod_{v \text{ vertex}} \text{Cont}(v),$$

where $\zeta_i$ is $\zeta$ at the vertex with $i$, $\Gamma_d^i$ is a stable quotient localization graph of $\overline{Q}_{0,2}(\mathbb{P}^1, d)$ with color of the first vertex determined by $\zeta_i$. Here the contributions $\text{Cont}(f)$ and $\text{Cont}(v)$ are contributions to the calculation of the integrals

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^1, d)} \text{ev}_1^*(\phi_1) \text{ev}_2^*(\phi_2 \left( [0] + [\infty] \right)^{a_i}) \right).$$

The first summand corresponds to the case when the length of the chain is 0. Its form comes from the identity $[0] - [\infty] = s$ and the fact that $[0]$ vanishes when at a vertex at $\infty$ and vice versa.

6.2.1. Pushing forward. Next we will study the $\varepsilon$-push-forward in the formula for $\nu^*_a \varphi(a)$. For this we want to replace the $\psi$ classes in the edge and point terms by $\psi$ classes pulled back via $\varepsilon$, since the other $\psi$ classes are already pull-backs. For a fixed localization graph $\varepsilon$ is a composition of local maps, one for each vertex in the graph, of the form $M_{g(v), (w(v), x^d(v))} \rightarrow M_{g(v), w(v)}$, and one for each edge and marked point which contracts products of factors of the form $M_{0,2[d]}$.

Let us first look at the push-forward local to a vertex. For this we only need to look at the product of the vertex term with the factors of the form

$$\frac{1}{\omega_i - \psi_i}$$

from the adjacent edge and point series. Let us simplify the notation for this local discussion. With $d$ we denote the degree at this vertex, with $n = n' + n''$ the number of marked points with weight different or equal to 1 respectively, and with $w$ the weights at the vertex. We will index the marked points by the set $\{1, \ldots, n\} \cup \{1, \ldots, d\}$. Hopefully the non-empty intersection of these sets will not cause any confusion.

The basic pull-back formula is that

$$\psi_i = \varepsilon^*(\psi_i) + \Delta_{i1}$$

\footnote{which is here a chain similar to the first in Figure \[\text{1}\].}
in the case that \( d = 1 \). Here \( \Delta_{i_1} \) is the boundary divisor of curves who have one irreducible component of genus 0 containing only \( i \) and the weight \( \varepsilon \) point. This generalizes to the formula
\[
\psi_i^k = \varepsilon^*(\psi_i^k) + \varepsilon^*(\psi_i^{k-1}) \sum_{\emptyset \neq S \subset \{1, \ldots, d\}} \Delta_{iS},
\]
where \( \Delta_{iS} \) is the boundary divisor of curves who have one irreducible component containing only \( i \) and the weight \( \varepsilon \) points indexed by \( S \). Thus
\[
\frac{1}{\omega_i - \psi_i} = \frac{1}{\omega_i - \varepsilon^*(\psi_i)} \left( 1 + \omega_i^{-1} \sum_{\emptyset \neq S \subset \{1, \ldots, d\}} \Delta_{iS} \right).
\]

Modulo factors pulled back via \( \varepsilon \) the most general classes we will need to push forward are
\[
\prod_{i=1}^{n'} D_{iS_i} \prod_{i=n'+1}^n \Delta_{iT_i} M
\]
for subsets \( S_i, T_i \subset \{1, \ldots, d\} \) and a monomial \( M \) in \( \psi \) and diagonal classes of the \( d \) points of weight \( \varepsilon \). In order that this class is not zero the \( S_i \) and \( T_i \) must be pairwise disjoint. Moreover for each \( i \in \{n'+1, \ldots, n\} \) the monomial \( M \) must contain the diagonal corresponding to \( T_i \) as a connected factor.

Therefore for a monomial \( M = \prod_{i \in P} M_i \) corresponding to a set partition \( P \vdash \{1, \ldots, d\} \) we obtain
\[
\varepsilon_* \left( \prod_{i=n'+1}^n \left( 1 + \sum_{\emptyset \neq S \subset \{1, \ldots, d\}} \omega_i^{-1} \Delta_{iS} \right) \exp \left( \sum_{i=1}^{n'} s_{iP_i} \right) M \right) = \prod_{i \in P} \left( \delta_i \sum_{j=n'+1}^n \omega_j^{-1} + \varepsilon_{is} \left( \exp \left( \sum_{j=1}^{n'} s_{jP_j} \right) M_i \right) \right),
\]
where here \( \delta_i \) is one if \( M_i \) is a diagonal class and zero otherwise and the \( \varepsilon_i \) are forgetful maps \( \varepsilon_i : \overline{\mathcal{M}}_{g,(w,\varepsilon^{(i)})} \to \overline{\mathcal{M}}_{g,w} \). Notice that the push-forward is a product where each factor is dependent only on one factor of \( M \).

This allows us to use the arguments from Section 4.2.2 to calculate the necessary \( \varepsilon \)-push-forwards modulo classes pulled back via \( \varepsilon \). We obtain
\[
\sum_{d=0}^\infty \varepsilon_* \sum_{i=n'+1}^n \left( 1 + \sum_{\emptyset \neq S \subset \{1, \ldots, d\}} \omega_i^{-1} \Delta_{iS} \right) \exp \left( \sum_{i=1}^{n'} s_{iP_i} \right) \prod_{i=n'+1}^n \exp \left( \frac{u^c}{\omega_i} - \log(\Phi') \left( \frac{\psi_i}{\zeta(v) s} \cdot \frac{x}{\zeta(v) s} \right) \right) \exp \left( \left\{ \exp(-pD) \log(\Phi) \left( \frac{t}{\zeta(v) s} \cdot \frac{x}{\zeta(v) s} \right) \right\}_\Delta \right)_{t=1},
\]
\[
(12)
\]
\[8\]To think of \( m_i \) as living on \( \overline{\mathcal{M}}_{g,(w,\varepsilon^{(i)})} \) one needs to choose a bijection \( i \to \{1, \ldots, |i|\} \) but the \( \varepsilon_i \)-push-forward is independent of that choice.
where \( u^e \) is as in \([10]\) and \( p = p_1 + \cdots + p_n \). Here we rather artificially introduced a variable \( t \) to make use of the bracket notation. Notice that in the first factor of this formula the factor corresponding to some \( i \in \{n' + 1, \ldots, n\} \) depends only on \( \omega_i \), which depends on the degree splitting of the chain, whereas the second factor is independent of the degree splittings.

Now we can again step back from the vertex and look at the global picture. With \([12]\) we can give a new formula for \( \nu^e_{\text{ir}}(a) \)

\[
\nu^e_{\text{ir}}(a) = \left[ \sum_{\Gamma \leq \nu} \frac{1}{\text{Aut}(\Gamma)} \xi_{\nu} \left( \prod_v \text{Vertex}^2_v \prod_e \text{Edge}^2_e \prod_{i=1}^{n} \text{Point}^i_t \right) \right] \cdot w^a
\]

with new vertex, edge and point series. The term \( \text{Vertex}^2_v \) already has the form as in the stable quotient relations \( \nu^e_{\text{ir}}(a) \)

\[
\text{Vertex}^2_v = (\zeta(v,s))^d(v)^{-1} \exp \left( -\frac{1}{2} \zeta(v,s)p(v) + \left\{ \exp(p(v)sD) \log(\Phi) \right\} \right),
\]

where

\[
V' = \sum_{j=0}^{\infty} (\zeta(s))^{-j} c_j(\mathbb{R}) \in A^*(\overline{M}_{g,w})[[x,s,s^{-1}]].
\]

Furthermore the edge and point series are given by

\[
\text{Edge}^2_e = \Phi^{-1} \left( \frac{\psi_1}{\zeta_1}, x \frac{x}{\zeta_1} \right) \text{Edge}^{-1} \left( \frac{\psi_2}{\zeta_2}, x \frac{x}{\zeta_2} \right) = \sum_{d=1}^{\infty} x^d \sum_{\Gamma_d^d = (V_d,E_d)} \frac{1}{\text{Aut}(\Gamma_d^d)} \cdot \frac{\zeta_d}{\omega_{\Gamma_d^d,1} - \psi_1} \cdot \frac{\zeta_{d^2}}{\omega_{\Gamma_d^d,2} - \psi_2} \cdot \prod_{f \in \text{edge}} \text{Cont}(f) \prod_{v \in \text{vertex}} \text{Cont}(v)
\]

and

\[
\text{Point}^i_t = \Phi^{-1} \left( \frac{\psi_1}{\zeta_i}, x \frac{x}{\zeta_i} \right) = \sum_{d=0}^{\infty} x^d \sum_{\Gamma_t^i = (V_t,E_t)} \frac{1}{\text{Aut}(\Gamma_t^i)} \cdot \frac{\zeta_t}{\omega_{\Gamma_t^i,1} - \psi_i} \cdot \prod_{f \in \text{edge}} \text{Cont}(f) \prod_{v \in \text{vertex}} \text{Cont}(v).
\]

Using Mumford’s formula (pulled back from \( \overline{M}_g \) to \( \overline{M}_{g,w} \)) we find

\[
V' = -\sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i - 1)} \kappa_{2i-1} (\zeta(s))^{1-2i}
\]

\[
-\sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i - 1)} s^{2i} \sum_{\Delta \in D_{\psi}} \frac{1}{\text{Aut}(\Delta)} \xi_{\Delta,s} \cdot \left( \frac{\psi_{a,1}^{2i-1} + \psi_{b,1}^{2i-1}}{\psi_a + \psi_b} \right) (\zeta(s))^{1-2i}.
\]

Notice that the edge series \( \text{Edge}^2_e \) modulo a slight change in notation and the \( \Phi^{-1} \) factors is the series \( E_{g,j}(\psi_1, \psi_2, x) \) from Section \([5]\) where \( i \) and \( j \) correspond to the color of the vertices \( e \) connects. Similarly we identify the point term \( \text{Point}^i_t \) up to the \( \Phi^{-1} \) factor with

\[
2^{-a_i} \left( s^a_{\psi_1} P^0(\psi_1, x) + (-s)^{n_i} P(\psi_1, x) \right).
\]
We finally obtain the relations by taking the $s^e$ part of $\nu_i^{vir} \varphi(a)$ for $c < 0$. So we replace everywhere $s$ by $t^{-1}$, $x$ by $xt^2$ and $p(v)$ by $p(v)t^{-1}$. After dividing out a common factor of $t^e$ for

\[ e = \sum_v (-g(v) + 1) + 2d + |a| = -g + 1 + |E| + 2d + |a| \]

and introducing variables $p_i$ for the $a''_i$ we arrive at the stable quotient relations of Section 6.1.

6.3. Evaluation of the relations.

6.3.1. Minor simplification. With the same proof as in Section 4.3.2 the stable quotient relations are implied from the stable quotient relations in the case that the $n'$-tuple $a'$ is only $\{0,1\}$-valued. The same holds trivially for $a''$ because of the form of the point term.

Furthermore the relations in the case that a point is of weight 1 and a point is of weight slightly smaller than 1 are the same. This is because a point $i$ of weight slightly smaller than 1 is not allowed to meet any other point, therefore the contribution of that point, which is solely in the vertex contribution, is

\[ \exp\left(-\frac{1}{2} \zeta(p(i))p_i + p_iD\gamma(t\zeta(p(i))\psi_1, x)\right) = 1 + p_i\zeta(p(i))\delta(t\zeta(p(i))\psi_1, x) \]

modulo $(p_i^2)$. This is the point term after suitably renaming $p_i$.

Therefore we can from now on treat points of weight 1 the same way as points of weight slightly less than 1.

6.3.2. Edge terms. The factor $\exp(V')$ of the vertex contribution to the stable quotient relations contains intersections of classes supported on divisor classes of $\overline{M}_{g(v),w}$. We want to reformulate the relations such that the vertex term only contains $\kappa$, diagonal and $\psi$ classes corresponding to the markings. Some excess intersection calculations will be necessary to deal with $\exp(V')$.

Proposition 4. The set of stable quotient relations is equivalent to the following set of relations: Under the conditions of Proposition 3 it holds

\[ 0 = \left[ \sum_{\zeta \in \Gamma \rightarrow \{\pm 1\}} \frac{1}{\text{Aut}(\Gamma)} \zeta_{\Gamma} \left( \prod_v \text{Vertex}_{\zeta}^{4\zeta(v)} \prod_e \text{Edge}_{\zeta}^{4\zeta(\psi_1),\zeta(\psi_2)} \right) \right] t^{\gamma - |E|} x^4 p^n, \]

with

\[ \text{Vertex}_{\zeta}^{4\zeta(v)} = \zeta^{\delta(v)-1} \exp\left(\frac{1}{2} \zeta p(v) + \{\exp(p(v)D)\gamma(t\zeta, x)\}_{\Delta(v)}\right), \]

where $p(v) = \sum_{i \in p(v)} p_i$, and

\[ t(\psi_1 + \psi_2)\text{Edge}_{\zeta}^{4\zeta(v)} = \frac{\zeta_1 + \zeta_2}{2} \exp(-\gamma'(t\zeta_1\psi_1) - \gamma'(t\zeta_2\psi_2)) + \zeta_1\delta(t\zeta_1\psi_1) + \zeta_2\delta(t\zeta_2\psi_2), \]

where $\gamma'$ is defined in the same way from $\Phi'$ as $\gamma$ is from $\Phi$:

\[ \gamma' = \sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i - 1)} t^{2i-1} + \log(\Phi') \]
Remark 3. As in Section 4.3.2 we can also write

\[ \text{Vertex}^4_{v} = \zeta^{(v)-1} \exp \left( -\{\gamma\}_{x_{(e)}} + \sum_{i=1}^{\infty} \frac{\gamma_i}{i!} (p_{(e)} D^{i-1}) \zeta_{\Delta_{(e)}} \right). \]

The power \( \zeta^i \) appears because of the \( t \) in \( D = tx \). 

The proof of Proposition 11 depends on the following lemma.

**Lemma 7.** For a polynomial \( f \) in two variables we have

\[ \exp \left( \sum_{\Delta \in DG} \frac{1}{|\text{Aut}(\Delta)|} \xi_{\Delta, *} (f(\psi_a, \psi_b)) \right) = \sum_{\Gamma \in \mathcal{G}} \frac{1}{|\text{Aut}(\Gamma)|} \xi_{\Gamma, *} \left( \prod_{e} \exp \left( -f(\psi_1^{(e)}, \psi_2^{(e)})(\psi_1^{(e)} + \psi_2^{(e)}) \right) - 1 \right), \]

where \( \psi_i^{(e)} \) are the two cotangent line classes belonging to edge \( e \).

**Proof.** Formally expanding the left hand side using the intersection formulas, for example described in [5, Appendix A], we can write it as a sum over stable graphs \((\Gamma, E)\). Let us look at the term corresponding to a given graph \( \Gamma \). By contracting all but one edge of \( \Gamma \) one obtains a divisor graph. This process gives a map \( e_{\Gamma} : E \to DG \). Counting the preimages of \( e_{\Gamma} \) gives a map \( m_{\Gamma} : DG \to \mathbb{N}_0 \). In the formal expansion of the exponential on the left hand side each term also corresponds to a function \( \sigma : DG \to \mathbb{N}_0 \).

A term contributes to a graph \( \Gamma \) if and only if \( m_{\Gamma} \leq \sigma \), of the \( |\sigma| \) intersections \( |m_{\Gamma}| = |E| \) are transversal and the others are excess. In addition, a contributing term of the left hand side determines a partition \( \mathbf{p} \) indexed by \( E \) of \( \sigma = \sum_{e \in E} p_e \) such that \( p_e(\Delta) = 0 \) unless \( e_{\Gamma}(e) = \Delta \).

With this we can explicitly write down \( |\text{Aut}(\Gamma)| \) times the \( \Gamma \)-contribution as

\[ \xi_{\Gamma, * \sum_{\sigma \geq m_{\Gamma}} \sum_{\mathbf{p}} \prod_{\Delta \in DG} \frac{1}{\sigma(\Delta)!} \left( \sigma(\Delta) \right) \prod_{e} f(\psi_1^{(e)}, \psi_2^{(e)}) p_e(e_{\Gamma}(e)) \left( -\left( \psi_1^{(e)} + \psi_2^{(e)} \right) \right) p_e(e_{\Gamma}(e)) - 1 \]

\[ = \xi_{\Gamma, * \sum_{\sigma \geq m_{\Gamma}} \sum_{\mathbf{p}} \prod_{e} \frac{1}{(p_e(\sigma(\Delta)))} f(\psi_1^{(e)}, \psi_2^{(e)}) \left( -\left( \psi_1^{(e)} + \psi_2^{(e)} \right) \right) p_e(e_{\Gamma}(e)) - 1 \]

\[ = \xi_{\Gamma, * \sum_{\sigma \geq m_{\Gamma}} \sum_{\mathbf{p}} \prod_{e} \frac{1}{(p_e(\sigma(\Delta)))} (\psi_1^{(e)} + \psi_2^{(e)}) \left( -\left( \psi_1^{(e)} + \psi_2^{(e)} \right) \right) - 1 \]

\[ = \xi_{\Gamma, * \sum_{\sigma \geq m_{\Gamma}} \sum_{\mathbf{p}} \prod_{e} \exp \left( -f(\psi_1^{(e)}, \psi_2^{(e)})(\psi_1^{(e)} + \psi_2^{(e)}) \right) - 1 \]

Here the factor \( \sigma(\Delta)! \) comes from the exponential and

\[ \prod_{\Delta \in DG} \left( \frac{\sigma(\Delta)}{\mathbf{p}(\Delta)} \right) \]

comes from the choice of which intersections are excess.

Summing the contributions for all \( \Gamma \) finishes the proof. \( \square \)
Proof of the proposition. We will apply the lemma in the case that

\[ f(x_1, x_2) = - \sum_{i \geq 1} \frac{B_{2i} x_1^{2i-1} + x_2^{2i-1}}{2i(2i - 1) x_1 + x_2}, \]

but now we also need to take care of the coloring of the vertices.

For each graph \( \Gamma \in G \) with a coloring \( \zeta : \Gamma \to \{ \pm 1 \} \) we can construct a new graph \( \Gamma_{\text{red}} \), its reduction, by contracting all edges of \( \Gamma \) connecting two vertices of the same color. The induced coloring on \( \Gamma_{\text{red}} \) satisfies the property that neighboring vertices are differently colored. Let us call such a graph reduced. Having the same reduction also defines an equivalence relation on \( G \).

The idea is now to apply lemma \( \text{[7]} \) to each vertex of each graph \( \Gamma \). In this way one gets terms at each specialization \( \Gamma' \) of \( \Gamma \) in the same equivalence class of \( \Gamma \).

Let us collect all the different contributions at a graph \( \Gamma' \) coming from graphs \( \Gamma \). Recall the pull-back formula for the \( \kappa \) classes

\[ p_v^*(\xi_v^* \kappa_i) = \kappa_i + \sum_e \psi_e^i, \]

where \( p_v \) denotes the projection map to the factor corresponding to each vertex \( v \) and the sum is over all outgoing edges at \( v \). This implies that the contributions at \( \Gamma' \) all have the same vertex contribution up a sign and a factor

\[ \exp(-\gamma'(t\zeta_1 \psi_1^{(e)}) - \gamma'(t\zeta_2 \psi_2^{(e)})) \]

for each edge of \( \Gamma \) not in \( \Gamma' \). The the edge terms corresponding to common edges do exactly coincide. The different factors split into a product over the connected components of the graph obtained by removing the edges which need to be contracted to obtain \( \Gamma_{\text{red}} \).

So let us look at just one connected component \( \Gamma'' \subset \Gamma' \setminus \Gamma_{\text{red}} \). We have to sum over the possibilities \( E \subseteq E(\Gamma'') \) of contracting edges in \( \Gamma'' \). We thus have the contribution

\[
\sum_{E(\Gamma'')} = E \prod_{e \in F} \text{Edge}_e^{3\zeta, -\zeta} \\
\prod_{e \in E} \exp(-f(t\zeta_1 \psi_1^{(e)}, t\zeta_2 \psi_2^{(e)})(t\zeta_1 \psi_1^{(e)} + t\zeta_2 \psi_2^{(e)})) - 1 \exp(-\gamma'(t\zeta_1 \psi_1^{(e)}) - \gamma'(t\zeta_2 \psi_2^{(e)})) \\
= \prod_{e \in E(\Gamma'')} \left( \text{Edge}_e^{3\zeta, -\zeta} + \right.
\exp(-f(t\zeta_1 \psi_1^{(e)}, t\zeta_2 \psi_2^{(e)})(t\zeta_1 \psi_1^{(e)} + t\zeta_2 \psi_2^{(e)})) - 1 \exp(-\gamma'(t\zeta_1 \psi_1^{(e)}) - \gamma'(t\zeta_2 \psi_2^{(e)}))) \\
= \prod_{e \in E(\Gamma'')} \text{Edge}_e^{4\zeta, -\zeta}.
\]

Because of

\[ \text{Edge}_e^{3\zeta, -\zeta} = \text{Edge}_e^{4\zeta, -\zeta} \]

we can replace \( \text{Edge}_e^3 \) by \( \text{Edge}_e^4 \) also for the edges connecting differently colored vertices. \( \square \)

\( ^9 \gamma' \) appears here instead of \( \gamma \) because \( \kappa_{-1} = 0 \) while \( \psi_{-1} \) is not defined.
6.3.3. Variable transformations. Using the results of Section 4.3.3 we can give a new formulation of the stable quotient relations.

We have

$$0 = \left[ \sum_{\Gamma \in G} \frac{1}{\text{Aut}(\Gamma)} \sum_{\chi: \Gamma \to \{\pm 1\}} \prod_{v} \text{Vertex}_{v}^{5\chi(v)} \prod_{e} \text{Edge}_{e}^{5\chi(v_{1}), \chi(v_{2})} \right]_{u^{r-|E|}y^{p}}$$

with

$$\text{Vertex}_{v}^{5\chi} = c^{g(v)-1} \chi \exp \left( - \sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j} u_{\chi}^{k} y_{j}^{k} \right)$$

and

$$u(\psi_{1} + \psi_{2})\text{Edge}_{e}^{5\chi_{1}, \chi_{2}} = \frac{\chi_{1} + \chi_{2}}{2} \exp \left( - \sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j} (u_{\chi_{1}} y_{j}^{k}) + (u_{\chi_{2}} y_{j}^{k}) \right)$$

and the exponent

$$e_{\Gamma} = \frac{r + 2d - 2 - \kappa_{0}}{2} = \frac{|a|}{2} = \frac{r - g + 2d - 1 - |a|}{2}$$

under the condition of Proposition 3 on \(r\).

We can assume that \(e_{\Gamma}\) is integral because otherwise the relation is zero since the term corresponding to a coloring \(\chi\) and the opposite coloring \(-\chi\) exactly cancel each other in this case.

Next we look as in Section 4.3.4 at the extremal coefficients of this series and obtain the FZ relations of Proposition 1.

6.4. Final remarks. Let \(R(g, w, r; S)\) denote the relation on \(\overline{M}_{g,w}\) of Proposition 1 in codimension \(r\) corresponding to \(S \subset \{1, \ldots, n\}\) viewed as a class in the formal strata algebra, i.e. the formal \(\mathbb{Q}\)-algebra generated by the symbols

$$\xi_{\Gamma^{*}} \left( \prod_{v} M_{v} \right),$$

where \(\Gamma\) is a stable graph of \(\overline{M}_{g,w}\) and the \(M_{v}\) are formal monomials in \(\kappa, \psi\) and diagonal classes, modulo the relations given by the formal multiplication rules for boundary strata described in [5, Appendix A] and the relations between diagonal and \(\psi\) classes from 4.2.1. One can describe formal analogs of the push-forwards and pull-backs along the forgetful, gluing and weight reduction maps.

By the way we have constructed the stable quotient relations, for \(w' \leq w\) the push-forward of \(R(g, w, r; S)\) via the weight reduction map is \(R(g, w', r; S)\). Therefore the relations of Proposition 4 are (up to a constant factor) the push-forward of a subset of Pixton’s generalized FZ relations.

As mentioned in the introduction more relations than in Proposition 1 can be obtained by taking for a partition \(\sigma\) with no part equal to 2 (mod 3) the class

$$R(g, (w, 1^{\ell(\sigma)}), r - ||\sigma/3|| \cdot \bar{S}) \prod_{i} \psi_{n+i}^{\sigma_{i}/3} + 1$$

in \(A^{r+\ell(\sigma)}(\overline{M}_{g,(w, 1^{\ell(\sigma)})})\), where \(\bar{S}\) equals \(S\) on the first \(n\) markings and is given by the remainders when dividing the parts of \(\sigma\) by 3 on the other markings, and pushing
this class forward to $\overline{M}_{g,w}$ under the forgetful map. For explicitly calculating this push-forward it is better to use the usual $\kappa$ classes $\tilde{\kappa}_i = \pi_*(c_1(\omega_\pi(D))^{i+1})$, which are related to the $\kappa$ classes we have used in this article by $\tilde{\kappa}_i = \kappa_i + \sum_{j=1}^n \psi_j$, in order to use Faber’s formula for the push-forward of monomials in cotangent line classes [3]. Let us call these relations $R(g,w,r; \sigma,S)$. As in [7] even more generally one can look at the $\mathbb{Q}$-vector space $R_{g,w}$ generated by the relations obtained by choosing a boundary stratum corresponding to a dual graph $\Gamma$, taking a FZ relation $R(g_i,w_i,r; \sigma,S)M_i$ for any $r$, $S$, $\sigma$ and monomial $M_i$ in the diagonal and cotangent line classes on one of the components, arbitrary tautological classes on the other components and pushing this forward along $\xi_\Gamma$. Because of the compatibility with the birational weight reduction maps [7, Proposition 1] implies that the system $R_{g,w}$ of $\mathbb{Q}$-vector spaces cannot be tautologically enlarged, i.e. it is closed under formal push-forward and pull-back along forgetful and gluing maps as well as multiplication with arbitrary tautological classes.

As in [6] we have thrown away many of the stable quotient relations: We looked only at the extremal relations in Sections 4.3.4 and 6.3.3. However one should expect that these additional relations can also be expressed in terms of FZ relations.

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