ON NEGATIVITY OF TOTAL k-JET CURVATURE AND AMPLENESS OF THE CANONICAL BUNDLE

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Abstract. A celebrated conjecture of Kobayashi and Lang says that the canonical line bundle $K_X$ of a Kobayashi hyperbolic compact complex manifold $X$ is ample. In this note we prove that $K_X$ is ample if $X$ is projective and satisfies a stronger condition of nondegenerate negative total jet curvature. The main idea is to apply recent results on positivity of direct image sheaves to the Demailly-Semple tower in order to produce pluridifferentials on $X$.

1. Introduction

A complex manifold $X$ is called Kobayashi hyperbolic if an intrinsic pseudometric $d_K$, introduced by S. Kobayashi (see [Kob70], IV. 1) is a metric on $X$. Assume from now on that $X$ is compact, for instance projective. Thanks to the Brody criterion [Brd78], Kobayashi hyperbolicity of $X$ is equivalent to non-existence of entire curves, i.e. nonconstant holomorphic maps $f: \mathbb{C} \to X$.

One of the most interesting and challenging tasks in complex geometry is to describe the precise relations between Kobayashi hyperbolicity and birational geometry of $X$. Recall the two celebrated conjectures.

Conjecture 1. (S. Kobayashi, [Kob70], S. Lang [Lan86]) Let $X$ be a Kobayashi hyperbolic compact complex manifold. Then the canonical line bundle $K_X$ is ample.

Conjecture 2. (M. Green – P. Griffiths, S. Lang [GG80, Lan86]) Let $X$ be a variety of general type. Then there exists a proper subvariety $Z \subset X$ such that for every entire curve $f: \mathbb{C} \to X$ the image $f(\mathbb{C})$ lies in $Z$.

We refer an interested reader to e.g. [GG80, McQ98, DMR10] as well as to expository works [Bru99, Dem12, Siu04, Voi03] for the progress towards the Green-Griffiths-Lang conjecture. As for Conjecture 1, it is obviously true in dimension one and follows from Kodaira-Enriques classification in dimension two. A possible strategy (which works in dimension 3) is to argue by contradiction using the Iitaka fibration (M. Green – P. Griffiths, S. Lang [GG80, Lan86]) Let

$$\begin{align*}
\text{Conjecture 1.} & \quad \text{Let } X \text{ be a Kobayashi hyperbolic compact complex manifold. Then the canonical line bundle } K_X \text{ is ample.}
\end{align*}$$

Note that negativity of the holomorphic sectional curvature is a strictly stronger assumption than hyperbolicity. On the other hand, we can generalize it in the following way. The Kähler metric $\omega$ on $X$ induces a smooth Hermitian metric on the hyperplane line bundle $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ over the projectivized bundle $\pi: \mathbb{P}(T_X) \to X$ of lines in $T_X$. Negativity of the holomorphic sectional curvature $\Theta_\omega(v \otimes v)$ is equivalent to negativity of the curvature form $\Theta_h(\mathcal{O}_{\mathbb{P}(T_X)}(1))$ along the subbundle $V_1 \subset \mathcal{O}_{\mathbb{P}(T_X)}(1)$ of vectors projected to the line $\mathcal{O}_{\mathbb{P}(T_X)}(-1)$ by $\pi_*$. We can proceed inductively starting from $(X_0, V_0) = (X, T_X)$ and construct the $k$-th stage of the Demailly-Semple tower by putting

$$\begin{align*}
(X_k, V_k) := (\mathbb{P}(V_{k-1}), (\pi_{k-1})_*^{-1} \mathcal{O}_{\mathbb{P}(V_{k-1})}(-1))
\end{align*}$$

where $\pi_{k-1,k}: X_k \to X_{k-1}$ is the projection. The varieties $X_k$ are endowed with hyperplane line bundles $\mathcal{O}_{X_k}(1)$. We say that $(X, T_X)$ has negative $k$-jet curvature if there exists a (possibly singular) Hermitian metric $h_k$ on $\mathcal{O}_{X_k}(1)$ such that

$$\Theta_{h_k}(\mathcal{O}_{X_k}(1))|_{V_k} \geq \varepsilon \omega|_{V_k}$$
in the sense of currents for some $\varepsilon > 0$ and a metric $\omega$ on $X_k$. It can be shown that $k$-jet negativity becomes weaker as $k$ increases and together with nondegeneracy condition (see Definition 2.3.1 below) implies Kobayashi hyperbolicity. The examples in [Dem97], Theorem 8.2 show that for every $k_0 \in \mathbb{N}$ there are Kobayashi hyperbolic surfaces which do not admit a $k$-jet negative metric for $k \leq k_0$. Nevertheless, the following has a chance to be true.

Conjecture 3. (J.-P. Demailly [Dem97]) The variety $X$ is Kobayashi hyperbolic if and only if there exists $k \in \mathbb{N}$ such that $(X, T_X)$ has negative $k$-jet curvature.

Therefore it is plausible to expect the following generalization of Theorem 1.0.1.

**Question 1.0.1.** Assume that $(X, T_X)$ has nondegenerate negative $k$-jet curvature for some $k \in \mathbb{N}$. Is the canonical line bundle $K_X$ ample?

Observe that in [Dem97] the curvature of $\Theta_{X_k}(1)$ is only partially positive, so it does not a priori say anything about the existence of sections. The methods of [WY16, WY16b] also do not seem to generalize directly to the case $k > 1$. Nevertheless, we are able to answer the Question 1.0.1 under a strong (in fact, strongest possible, see [Dem97], Theorem 6.8 (iii)) positivity assumption on the curvature of $\Theta_{X_k}(1)$.

Our main theorem states as follows.

**Theorem 1.0.2.** Assume that $(X, T_X)$ has nondegenerate negative total $k$-jet curvature, i.e. the line bundle $\Theta_{X_k}(1)$ is big on $X_k$ and the singularity set $\Sigma_{h_k}$ is contained in $X_k^{\text{sing}} \subset X_k$ (see Definition 2.3.1).

Then the canonical line bundle $K_X$ is ample.

Recent works [BD15, Bre16, Den17] show that there exist many examples of varieties satisfying the aforementioned condition, such that general hypersurfaces $H_d \subset X$ of high degree and general complete intersections $H_{d_1, \ldots, d_r} \subset X$ of at least $c \geq \lceil n/2 \rceil$ high-degree hypersurfaces. Moreover, a remarkable result of Demailly (see e.g. [Dem12], Corollary 15.74) asserts that for every variety of general type there exist $m, k \in \mathbb{N}$ and an ample line bundle $A$ such that $\Theta_{X_k}(m) \otimes (-\frac{m}{2k}(1 + \frac{1}{2} + \ldots + \frac{1}{k})A)$ is big, so that $(X, T_X)$ has nonnegative total $k$-jet curvature. We give a converse to this theorem (see Proposition 3.2.1). Note also that from the results of [DR15] it follows that for many hyperbolic varieties (for example, products $C_1 \times \ldots \times C_n$ of genus $g \geq 2$ curves) the intersection of the base loci of jet differentials dominate $X$. Therefore the new methods have to be found to answer the Question 1.0.1 in full generality.

The paper is organized as follows. In Section 2 we recall basic definitions and results on jet bundles, jet differentials and curvature. The proof of Theorem 1.0.2 is given in Section 3. Finally, in Section 4 we discuss versions of Theorem 1.0.2 in the case of a log-pair $(X, D)$ and in the case of singular directed variety $(X, V)$.

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## 2. Jet bundles and curvature

In this section we introduce the definitions and basic results on jet bundles and jet differentials following [Dem97].

### 2.1. Jets of curves and jet differentials

A directed manifold is a pair $(X, V)$ where $X$ is a compact complex manifold of dimension $n$ and $V$ is a linear subspace of generic rank $r$ of the tangent bundle $T_X$. In fact, below we will need only the absolute case, so the reader can suppose that $V = T_X$ and $r = \text{rk}(T_X) = n$. We consider the bundle $p_k : J_k V \to X$ of $k$-jets of parameterized curves in $X$. It is the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ tangent to $V$ with $f \sim g$ if and only if all derivatves $f^{(j)}(0) = g^{(j)}(0)$ for $0 \leq j \leq k$ coincide when computed in some local coordinate chart near $x$. The projection map is defined by $p_k([f]) = f(0)$. Choose local holomorphic coordinates $(z_1, \ldots, z_n)$ on an open set $\Omega \subset X$; then for every $x \in \Omega$ the fibers $J_k V_x$ can be seen as $\mathbb{C}^r$-valued maps

$$f = (f_1, \ldots, f_r) : (\mathbb{C}, 0) \to \Omega \subset X$$

that are determined by their Taylor expansion at $t = 0$

$$f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1}).$$
Thus in these coordinates the fibers are parameterized by
\[(f'_1(0), \ldots, f'_i(0)); (f''_1(0), \ldots, f''_i(0)); \ldots; (f^{(k)}_1(0), \ldots, f^{(k)}_i(0)) \in (\mathbb{C}^r)^k\]
so that \(J_kV\) is a locally trivial \(\mathbb{C}^r\)-bundle over \(X\). We denote by \(J_kV^{\text{res}}\) the open subset of regular \(k\)-jets
\[J_kV^{\text{res}} = \{ [f] \in J_kV \mid f'(0) \neq 0 \} \]
We define \(\mathbb{G}_k\) to be the group of germs of biholomorphic maps \(\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)\) given by
\[t \mapsto \varphi(t) = a_1t + a_2t^2 + \ldots + a_k t^k, \quad a_1 \in \mathbb{C}, a_i \in \mathbb{C}, 2 \leq i \leq k\]
with the composition taken modulo terms \(t^i\) for \(i > k\). Then \(\mathbb{G}_k\) is a \(k\)-dimensional nilpotent complex Lie group, which admits a fiberwise action on \(J_kV\) (and on \(J_kV^{\text{res}}\) by change of parameter \((f, \varphi) \to f \circ \varphi\). There is an exact sequence
\[1 \to \mathbb{G}_k^\prime \to \mathbb{G}_k \to \mathbb{C}^* \to 1\]
the map \(\mathbb{G}_k \to \mathbb{C}^*\) being \(\varphi \mapsto \varphi'(0)\). The corresponding \(\mathbb{C}^*\)-action is simply the weighted action
\[\lambda \cdot (f', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).\]
Notice also that \(\mathbb{G}_k\) has a representation in \(GL_k(\mathbb{C})\) given in the above basis by
\[\varphi(t) = a_1t + \ldots + a_k t^k \mapsto \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 0 & a_2^2 & \cdots & 2a_1a_k-1 + \cdots \\ 0 & 0 & a_3^2 & \cdots & 3a_1^2a_k-2 + \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_k^2 \end{pmatrix} \in GL_k(\mathbb{C})\]
the \((i, j)\)-th entry of the matrix being
\[p_{ij} = \sum_{l_1+l_2+\ldots+l_i=j} a_{l_1}a_{l_2}\cdots a_{l_i}.\]
We are interested in polynomial functions \(P(f', \ldots, f^{(k)})\) of weighted degree \(l_1 + 2l_2 + \ldots + kl_k = m\) on fibers of \(J_kV\) invariant under the \(\mathbb{G}_k\)-action:
\[P\left( (f \circ \varphi)', (f \circ \varphi)'', \ldots, (f \circ \varphi)^{(k)} \right) = \varphi'(0)^m P(f', f'', \ldots, f^{(k)}).\]
These functions are called invariant jet differentials of order \(k\) and degree \(m\) and form a vector bundle \(E_{k,m}V^* \to X\) [Dem97, Definition 6.7]. Invariant jet differentials can be interpreted as sections of line bundles on a certain compactification of \(J_kV^{\text{res}}/\mathbb{G}_k\); this compactification is introduced in the next subsection.

2.2. Projectivized jet bundles. In this subsection \((X, V)\) is a directed manifold with \(r = \text{rk}(V) \geq 2\).

Definition 2.2.1. The \(k\)-th (stage of) the Demazure–Semple tower of a directed manifold \((X, V)\) is a directed manifold \((X_k, V_k)\) defined inductively by \((X_0, V_0) := (X, V)\) and
\[(X_k, V_k) := (\mathbb{P}(V_{k-1}), (\pi_{k-1,k})^{-1} \mathcal{O}_{X_k}(-1))\]
where \(\pi_{k-1,k} : X_k \to X_{k-1}\) is the projection, \((\pi_{k-1,k})_* = d\pi_{k-1,k} : T_{X_k} \to (\pi_{k-1,k})^{-1} T_{X_{k-1}}\) its differential and \((\pi_{k-1,k})_*^{-1} \mathcal{O}_{X_k}(-1)\) is a linear subspace of \(T_{X_k}\) which can be defined pointwise by
\[V_k = (\pi_{k-1,k})^{-1} \mathcal{O}_{X_k}(-1) = \{ \xi \in T_{X_k,x,[v]} \mid (\pi_{k-1,k})_* \xi \in \mathcal{O}_x \}.\]
We get for every \(k \in \mathbb{N}\) a tower of \(\mathbb{P}^r\)-bundles over \(X\) with \(\dim(X_k) = n + k(r - 1)\) and \(\text{rk}(V_k) = r\).
For all pair of indices \(0 \leq j \leq k\) we have natural morphisms \(\pi_{j,k} : X_k \to X_j\) and their differentials
\[(\pi_{j,k})_* = (d\pi_{j,k})|_{V_k} : V_k \to (\pi_{j,k})^{-1} V_j.\]
The manifolds \(X_k\) carry hyperplane line bundles \(\mathcal{O}_{X_k}(1)\). We introduce the following notation for weighted line bundles:
\[\mathcal{O}_{X_k}(a) := \bigotimes_{i=1}^k \pi_{i,k}^* \mathcal{O}_{X_i}(a_i) \quad a = (a_1, \ldots, a_k) \in \mathbb{Z}^k.\]
For every germ $f: \mathbb{C} \to (X, V)$ of holomorphic curves we can define inductively the $k$-th lifting

$$f_{[k]}(t) := (f_{[k-1]}(t), [f_{[k-1]}(t)])$$

of $f: \mathbb{C} \to (X, V)$ to $f_{[k]}: \mathbb{C} \to (X_k, V_k)$. Denote by $X_k^{reg}$ the set of points of $X_k$ which can be reached by liftings of regular germs of curves. We have the following result \cite[Theorem 6.8]{Dem97}.

**Proposition 2.2.1.** Let $(X, V)$ be a directed manifold, $r = \text{rk}(V) \geq 2$. Consider the bundle $J_kV^{reg}$ of regular $k$-jets of curves $f: \mathbb{C} \to X$. Then there exists a holomorphic embedding $J_kV^{reg} \to GL_k \times X^{reg}$. In other words, the manifold $X_k$ is a relative compactification of $J_kV^{reg} / GL_k$ over $X$ and $X_k^{sing} = X_k \setminus X_k^{reg}$ is a vertical divisor in $X_k$. Moreover, we have a direct image formula

$$\pi_{0,k}, \Theta_{X_k}(m) \simeq E_{k,m}V^*$$

and for every line bundle $L \to X$ we have an identification

$$H^0(X_k, \Theta_{X_k}(m) \otimes \pi_{0,k}^*L) = H^0(X, E_{k,m}V^* \otimes L).$$

More precisely, for every $a = (a_1, \ldots, a_n) \in \mathbb{Z}^k$ we have

$$(\pi_{0,k}), \Theta_{X_k}(a) = \mathcal{O}(F_aE_{k,m}V^*)$$

where $F_aE_{k,m}V^*$ is the subbundle of polynomials $P(f', f'', \ldots, f^{(k)}) \in E_{k,m}V^*$ involving only monomials $(f^{(\bullet)})^l$ such that

$$l_{s+1} + 2l_{s+2} + \ldots + (k-s)l_k \leq a_{s+1} + \ldots + a_k$$

for every $0 \leq s \leq k-1$.

We obtain a filtration $F_a$ on $E_{k,m}V^*$ with the associated graded object described in \cite{Dem97}, §12.

**Proposition 2.2.2.** The graded object of the above filtration on $E_{k,m}V^*$ is a direct sum of irreducible $GL(V)$-representations and is isomorphic to

$$\text{Gr}^*(E_{k,m}V^*) = \left( \bigoplus_{l \in \mathbb{N}^k, l_1 + 2l_2 + \ldots + kl_k = m} S^{l_1}V^* \otimes S^{l_2}V^* \otimes \cdots \otimes S^{l_k}V^* \right) G_k.$$

### 2.3. Curvature of jet bundles and hyperbolicity.

**Definition 2.3.1.** Let $(X, V)$ be a directed manifold. We say that $(X, V)$ has negative $k$-jet curvature if there exist a singular Hermitian metric on the line bundle $\Theta_{X_k}(1)$ such that

$$\Theta_{h_k}(\Theta_{X_k}(1)|_{V_k}) \geq \varepsilon \omega|_{V_k} \quad V_k \subset T_{X_k}$$

in the sense of currents for some metric $\omega$ on $X_k$ and $\varepsilon > 0$. We say that $(X, V)$ has negative total $k$-jet curvature if

$$\Theta_{h_k}(\Theta_{X_k}(1)) \geq \varepsilon \omega.$$

Finally, we say that $(X, V)$ has nondegenerate negative (total) $k$-jet curvature if in addition the degeneration set $\Sigma_{h_k}$ of the metric $h_k$ lies in $X_k^{sing}$.

A connection between hyperbolicity and $k$-jet negativity is given by the Ahlfors-Schwarz lemma \cite[Theorem 7.8]{Dem97}.

**Proposition 2.3.1.** Let $(X, V)$ be a compact directed manifold. Suppose that $(X, V)$ has a metric with negative $k$-jet curvature. Then for every entire curve $f: \mathbb{C} \to (X, V)$ the $k$-th lifting $f_{[k]}: \mathbb{C} \to X_k$ is such that $f_{[k]}(\mathbb{C}) \subset X_k^{sing}$. In particular, if $(X, V)$ has a metric with nondegenerate negative $k$-jet curvature then $(X, V)$ is hyperbolic.

In particular, negativity of total $k$-jet curvature is equivalent to bigness of $\Theta_{X_k}(1)$, i.e. existence of many invariant jet differentials. In this case we have the following fundamental theorem \cite[Corollary 7.9]{Dem97} which establishes the existence of many algebraic differential equations satisfied by every entire curve $f: \mathbb{C} \to (X, V)$.
Theorem 2.3.1. (M. Green – P. Griffiths, J.-P. Demailly, Y.-T. Siu – S.-K. Yeung) Let \((X, V)\) be a compact directed manifold. Suppose that there exist \(k, m \in \mathbb{N}\) and an ample line bundle \(A\) such that \(H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}) = H^0(X, E_{k,m} V^* \otimes A^{-1})\) has nonzero sections \(\sigma_1, \ldots, \sigma_N\). Denote by \(Z\) the base locus of these sections. Then for every entire curve \(f : \mathbb{C} \to (X, V)\) the image of its \(k\)-th lifting lies in \(Z: f|_\mathbb{C}(\mathbb{C}) \subset Z\). In other words, every entire curve \(f : \mathbb{C} \to (X, V)\) satisfies the algebraic differential equations \(\sigma_1(f^*, f'', \ldots, f^{(k)}) = \ldots = \sigma_N(f^*, f'', \ldots, f^{(k)}) = 0\).

3. The main result

In this section we prove our main Theorem 1.0.2. The proof is divided into three steps. Below we assume the following to hold.

Assumption 3.0.1. Let \(X\) be a smooth complex projective variety. We assume that there exists \(k \in \mathbb{N}\) such that \(\mathcal{O}_{X_k}(1)\) is endowed with a Hermitian metric \(h_k\) with degeneration set \(\Sigma_{h_k} \subset X_k^{\operatorname{sing}}\) and curvature current \(\Theta_{h_k}(\mathcal{O}_{X_k}(1)) \geq \varepsilon \omega\) for a metric \(\omega\) on \(X_k\) and \(\varepsilon > 0\).

3.1. Step 1: the canonical line bundle is nef. We present here a proposition from [Dem97], 8.1, which implies nefness of \(K_X\).

Proposition 3.1.1. Let \(X\) be a complex projective variety. Suppose that \((X, T_X)\) has negative \(k\)-jet curvature. Then there exist a constant \(\varepsilon > 0\) such that for every closed irreducible curve \(C \subset X\) such that \(\nu|_{\mathcal{C}}(C) \not\in \Sigma_{h_k}\) the following inequality holds:

\[-\chi(C) = 2g(C) - 2 \geq \varepsilon \deg(C) + \sum_{t \in C} (m_{k-1}(t) - 1) > 0\]

where \(\nu: \mathcal{C} \to C\) is the normalization and \(m_k(t)\) is the multiplicity at point \(t\) of the lifting \(\nu|_{\mathcal{C}}: \mathcal{C} \to P_k T_X = X_k\).

Proof. We have a lifting \(\nu_k: \mathcal{C} \to X_k\) and the derivative gives a canonical map

\[\nu'_{[k-1]}: T_C \to \nu|_{\mathcal{C}} \mathcal{O}_{X_k}(-1).\]

Let \(t_j \in \mathcal{C}\) be the singular points of \(\nu|_{\mathcal{C}}\) and let \(m_j = m_{k-1}(t_j)\) be the corresponding multiplicities. Then \(\nu'_{[k-1]}\) vanishes to order \(m_j - 1\) at \(t_j\) so we obtain

\[T_C \simeq \nu|_{\mathcal{C}}^* \mathcal{O}_{X_k}(-1) \otimes \mathcal{O}_C \left( - \sum_j (m_j - 1) t_j \right).\]

We take a metric \(h_k\) with negative \(k\)-jet curvature and degeneration set \(\Sigma_{h_k}\). From the nondegeneracy assumption it follows that \(\nu|_{\mathcal{C}}(C) \not\in \Sigma_{h_k}\) so \(\int_C \Theta_{h_k}(\nu|_{\mathcal{C}}^* \mathcal{O}_{X_k}(1))\) is well-defined. The Gauß–Bonnet formula yields

\[2g(C) - 2 = \int_C \Theta(\Omega^1_C) = \sum_j (m_j - 1) + \int_C \Theta_{h_k}(\nu|_{\mathcal{C}}^* \mathcal{O}_{X_k}(1)).\]

The hypothesis on curvature implies

\[\Theta_{h_k}(\nu|_{\mathcal{C}}^* \mathcal{O}_{X_k}(1))(\xi) \geq \varepsilon' |\xi|^2_{h_k} \geq \varepsilon'' (|\xi|_{\omega}^2\).\]

for some \(\varepsilon', \varepsilon'' > 0\) and smooth Hermitian metrics \(\omega_k\) on \(X_k\). As \(\pi_{0,k} \circ \nu|_{\mathcal{C}} = \nu\) we infer that \(\Theta_{h_k}(\nu|_{\mathcal{C}}^* \mathcal{O}_{X_k}(1)) \geq \varepsilon \omega\) and therefore

\[\int_C \Theta_{h_k}(\nu|_{\mathcal{C}}^* \mathcal{O}_{X_k}(1)) \geq \frac{\varepsilon''}{2\pi} \int_C \omega = \varepsilon \deg(C)\]

with \(\varepsilon = \varepsilon''/2\pi\). Proposition 3.1.1 now follows.

\[\square\]

Corollary 3.1.1. In the assumptions of 3.0.1, the canonical line bundle \(K_X\) is nef.

Proof. Indeed, by Proposition 3.1.1 we have the inequality \(\chi(C) \leq 0\) for every \(C \subset X\) which shows that \(X\) has no rational curves, therefore \(K_X\) is nef by the Cone Theorem (see e. g. [Deb01], Theorem 6.1).
3.2. Step 2: the canonical line bundle is big. This is the core of the proof. First of all, we state the bigness criterion of Campana and Păun (CamP16, Theorem 1.2).

Theorem 3.2.1. (F. Campana – M. Păun) Let $X$ be a complex projective variety such that $K_X$ is pseudoeffective. If there exists a big line bundle $L \to X$ and $m \in \mathbb{N}$ such that

$$H^0(X, \Omega^1_X(D)^{\otimes m} \otimes L^{-1}) \neq 0$$

then the canonical line bundle $K_X + D$ of the pair $(X, D)$ is big.

Thus in our case we need to construct a nontrivial morphism $L \to (\Omega^1_X)^{\otimes m}$ from a big line bundle $L$ to the sheaf of pluridifferentials $(\Omega^1_X)^{\otimes m}$. To do so, we will use the filtration structure on $\mathcal{O}(E_k, m)$ as follows (see CamP15, Theorem 4.5 for the version for Green-Griffiths jets).

Proposition 3.2.1. Let $X$ be a projective variety with the canonical line bundle $K_X$ pseudoeffective. Suppose that there exists $a = (a_1, \ldots, a_k) \in \mathbb{N}^k$ such that $\mathcal{O}(E_k, a)$ is big. Then the canonical line bundle $K_X$ is big.

Proof. Let us denote the direct image sheaf $(\pi_{0,k})_* \mathcal{O}_X(a)$ by $\mathcal{O}_a$ with $a = (a_1, \ldots, a_k) \in \mathbb{N}^k$ such that $\mathcal{O}_a$ is big. Take also an ample line bundle $A$ on $X$. By Proposition 2.2.1 the direct image $(\pi_{0,k})_*$ gives the identification

$$H^0(X, \mathcal{O}_a(k \cdot a) \otimes A^{-1}) = H^0(X, \text{Sym}^k(\mathcal{O}_a) \otimes A^{-1})$$

for all $k \in \mathbb{N}$. Since $\mathcal{O}_a$ is big, the Kodaira lemma gives a nonzero section $s \in H^0(X, \text{Sym}^k(\mathcal{O}_a) \otimes A^{-1})$ for some $k \in \mathbb{N}$ or, equivalently, a sheaf injection

$$0 \to A \to \text{Sym}^k(\mathcal{O}_a) \to \bigotimes_k \mathcal{O}_k \mathcal{O}_{\text{Sym}^1}.$$

By Proposition 2.2.1 there exists a filtration on $\mathcal{O}(E_k, m \Omega^1_X)$ such that the graded object is a direct sum of irreducible $GL(T_X)$-representations contained in $(\Omega^1_X)^{\otimes l}$ for $l \leq |a|$. By induction on $k$, we can endow $k$-th tensor power of $\mathcal{O}(E_k, m \Omega^1_X)$ with a filtration such that its graded pieces are contained in $(\Omega^1_X)^{\otimes |a|}$. Taking the induced morphism of graded sheaves, we obtain a nontrivial morphism $A \to (\Omega^1_X)^{\otimes |a|}$ for some $l \in \mathbb{N}$. Applying Theorem 3.2.1 with $D = \emptyset$, $L = A$ and $m = lk$ we obtain bigness of $K_X$. \qed

Recall that by Step 1 the canonical line bundle $K_X$ is nef. Therefore an application of Proposition 3.2.1 with $a = (0, \ldots, 1)$ and completes Step 2 of the proof.

Remark 3.2.1. If the degeneracy locus $\Sigma$ of the metric on $\mathcal{O}_X(a)$ does not dominate $X$ then $K_X$ is automatically pseudoeffective. Indeed, we only need to note that by Proposition 3.1.1 $X$ is not covered by rational curves, so $K_X$ is pseudoeffective by BDPP13.

3.3. Step 3: the canonical line bundle is ample. This part of the proof is completely standard (see e. g. AWY16, DT16, HW10). Now $K_X$ is big and nef, so by the Base Point-Free theorem it is semiample, i. e. there exists a birational contraction $c : X \to X'$ such that $K_X = c^* A$ where $A$ is ample on $X'$. Every irreducible component of $\text{Exc}(c) = \text{Null}(K_X)$ must be covered by rational curves contracted by $c$.

On the other hand, by Proposition 3.1.1 there are no rational curves on $X$. Thus $\text{Exc}(c) = \text{Null}(K_X)$ is empty. Then by Nakai – Moishezon criterion $K_X$ is ample, as desired. Alternatively, we could use the following result (Tak08), Theorem 1.1 (ii).

Theorem 3.3.1. (S. Takayama) Let $X$ be a smooth projective variety with $K_X$ big. Then every irreducible component of the non-ample locus $\text{N\text{A}mp}(K_X)$ is uniruled.

Again, by Proposition 3.1.1 and Theorem 3.3.1 the canonical line bundle $K_X$ is ample. The proof of Theorem 1.0.2 is now complete.

4. Concluding remarks

4.1. The logarithmic case. Consider a pair $(X, D)$ where $D$ is a normal crossings divisor. To study hyperbolicity of the open variety $X \setminus D$ we can use the techniques of logarithmic jet bundles and jet differentials as developed in DL01. In particular we can consider the log-directed manifold $(X, D, T_X(D))$ and construct the Demailly – Semple tower

$$(X_k, D_k, V_k) \to (X_{k-1}, D_{k-1}, V_{k-1}) \to \ldots \to (X, D, T_X(D)).$$
It is possible to extend Conjecture 1.0.1 and Question 4.2.1 to the logarithmic setting (see [LZ17] for recent progress towards Conjecture 1). By adapting the arguments in Section 3 we can prove the following version of our main Theorem 1.0.2.

**Theorem 4.1.1.** Let $X$ be a smooth projective variety and $D$ a normal crossings divisor on $X$. Suppose that the log-directed manifold $(X, D, T_X(D))$ has a metric with negative total $k$-jet curvature such that the degeneration set $\Sigma_h$ does not dominate $X$. Then the canonical line bundle $K_X + D$ of the pair $(X, D)$ is big. If, moreover, we have $\Sigma_h \subset X_k^{\text{sing}} \cup D_k$ and $D$ does not contain rational curves then $K_X + D$ is ample.

4.2. The case of a singular directed variety. Consider now the general case of a directed variety $(X, V)$ where $V \subset T_X$ is a holomorphic subbundle outside of a codimension $\geq 2$ analytic subvariety $\text{Sing}(V)$. In this case the canonical sheaf can be defined by $K_V = i_* \det i^*(\mathcal{O}(V^*))$ for $i : X \setminus \text{Sing}(V) \to X$ an inclusion. This definition is standard in foliation theory. On the other hand, a more refined notion of the canonical sheaf $\mathcal{K}_V$ was introduced by Demailly (see e.g. [Dem15], Definition 2.10). This version of the canonical sheaf better reflects the impact that the singularities of $V$ have on the geometry of $(X, V)$.

A possible generalization of Question 4.2.1 is the following.

**Question 4.2.1.** Suppose that $r = \text{rk}(V) \geq 2$ and $(X, V)$ has a metric with nondegenerate negative (total) $k$-jet curvature. Is the canonical sheaf $\mathcal{K}_V$ big in the sense of Demailly?

We hope to be able to address this question in a forthcoming paper.

**References**

[BD15] Damian Brotbek and Lionel Darondeau. Complete intersection varieties with ample cotangent bundles. ArXiv e-prints 1511.04709.

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun and Thomas Peternell. The pseudo-effective cone of a singular variety. J. Algebraic Geom. 22 (2013), no. 2, 201–248.

[Br78] Robert Brody. Compact manifolds and hyperbolicity. Trans. Amer. Math. Soc., 235:213–219, 1978.

[Bro16] Damian Brotbek. On the hyperbolicity of general hypersurfaces. ArXiv e-prints 1604.00311.

[Bru99] Marco Brunella. Courbes entières et feuilletages holomorphes. Enseign. Math. (2) 45 (1999), no. 1-2, 195–216.

[CamP16] Frédéric Campana and Mihai Păun. Positivity properties of the bundle of logarithmic tensors on compact Kähler manifolds. Compositio Math., 152 (2016), 2550–2570.

[Deb01] Olivier Debarre. Higher-dimensional algebraic geometry. Universitext. Springer-Verlag, New York, 2001. xiv+233 pp.

[Dem12] Jean-Pierre Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In Algebraic geometry—Santa Cruz 1995, volume 62 of Proc. Sympos. Pure Math., pages 285–360. Amer. Math. Soc., Providence, RI, 1997.

[Dem15] Jean-Pierre Demailly. Hyperbolic algebraic varieties and holomorphic differential equations. Available at https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/viasm2012expanded.pdf

[Dem17] Jean-Pierre Demailly. Recent progress towards the Kobayashi and Green-Griffiths-Lang conjectures. Available at https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/takagi16_jpd.pdf

[Den17] Ya Deng. On the Diverio – Trapani conjecture. ArXiv e-prints 1703.07560.

[DMR10] Simone Diverio, Joël Merker and Erwan Rousseau. Effective algebraic degeneracy. Invent. Math. 180 (2010), no. 1, 161–223.

[DL01] Gerd-Eberhard Dethloff and Steven S. Y. Lu. Logarithmic jet bundles and applications. Osaka J. Math. 38 (2001), no. 1, 185–237.

[DR15] Simone Diverio and Erwan Rousseau. The exceptional set and the Green-Griffiths locus do not always coincide. Enseign. Math. 61 (2015), no. 3-4, 417–452.

[DT16] Simone Diverio and Stefano Trapani. Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle. ArXiv e-prints 1606.01381.

[GG80] Mark Green and Phillip Griffiths. Two applications of algebraic geometry to entire holomorphic mappings. In The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pages 41–74. Springer, New York-Berlin, 1980.

[GRE65] Hans Grauert and Helmut Reckziegel. Hermitesche Metriken und normale Familien holomorpher Abbildungen. Math. Z. 89 1965 108–125.

[HLM10] Gordon Heier, Steven S. Y. Lu and Bun Wong. On the canonical line bundle and negative holomorphic sectional curvature. Math. Res. Lett. 17 (2010), no. 6, 1101–1110.

[Kob70] Shoshichi Kobayashi. Hyperbolic manifolds and holomorphic mappings, volume 2 of Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1970.

[Lan86] Serge Lang. Hyperbolic and Diophantine analysis. Bull. Amer. Math. Soc. (N.S.), 14(2):159–205, 1986.
[LZ17] Steven S. Y. Lu and De-Qi Zhang. Positivity criteria for log canonical divisors and hyperbolicity. *J. Reine Angew. Math.* 726 (2017), 173–186.

[McQ98] Michael McQuillan. Diophantine approximations and foliations. *Inst. Hautes Études Sci. Publ. Math.* No. 87 (1998), 121–174.

[Siu04] Yum-Tong Siu. Hyperbolicity in complex geometry. In *The legacy of Niels Henrik Abel*, pages 543–566. Springer, Berlin, 2004.

[Tak08] Shigeharu Takayama. On the uniruledness of stable base loci. *J. Differential Geom.* 78 (2008), no. 3, 521–541.

[Voi03] Claire Voisin. On some problems of Kobayashi and Lang; algebraic approaches. In *Current developments in mathematics*, 2003, 53–125, Int. Press, Somerville, MA, 2003.

[WY16] Damin Wu and Shing-Tung Yau. Negative holomorphic curvature and positive canonical bundle. *Invent. Math.* 204 (2016), no. 2, 595–604.

[WY16b] Damin Wu and Shing-Tung Yau. A remark on our paper “Negative holomorphic curvature and positive canonical bundle”. *Comm. Anal. Geom.* 24 (2016), no. 4, 901–912.