Geometrothermodynamics in Hořava–Lifshitz gravity

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Abstract
We investigate the thermodynamic geometries of the most general static, spherically symmetric, topological black holes of the Hořava–Lifshitz gravity. In particular, we show that a Legendre invariant metric derived in the context of geometrothermodynamics for the equilibrium manifold correctly reproduces the phase transition structure of these black holes. Moreover, the limiting cases in which the mass, entropy or Hawking temperature vanish are also accompanied by curvature singularities which indicate the limit of applicability of the thermodynamics and geometrothermodynamics of black holes. The Einstein limit and the case of a black hole with a flat horizon are also investigated.

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1. Introduction

General relativity and quantum mechanics are considered to be the cornerstones of modern physics; however, all attempts to formulate a unified theory of quantum gravity have so far been unsuccessful. Interesting technical results have been obtained in different approaches, but the physical problem remains open due, in part, to the fact that general relativity turned out to be non-renormalizable in the ultraviolet (UV) regime.

A field theoretical model which can be interpreted as a complete theory of gravity in the UV limit was recently proposed by Hořava [1, 2]. The model is closely related to Einstein’s theory but is powercounting renormalizable [3] and non-relativistic in the UV regime. Since
space and time have different scalings at the UV fixed point, i.e. \( x' \rightarrow lx', \ t \rightarrow lt \) where \( z \) is the scaling exponent, the model is usually named Hořava–Lifshitz (HL) theory in the literature [4]. Several variants of the model have been proposed (see [5] for a recent review), but only the consistent extension formulated in [6] is free of known pathologies such as instabilities, strong coupling at low energies or over-constrained evolution. This variant corresponds to the non-projectable version of the original version proposed by Hořava and includes additional terms in the potential term of the Lagrangian.

The field equations of this modified Hořava model have been investigated with certain detail and exact solutions have been found with different types of symmetries [7–10]. This theoretical model of quantum gravity has been extensively investigated in the context of black hole physics and classical cosmology (see, for instance, [11–21] and the references cited therein).

The study of black holes and their thermodynamic properties in the HL theory has been the subject of intensive research in recent years. In particular, the geometry of the thermodynamics of black holes has been considered in several works [22–25] by using different approaches. A first approach consists in introducing in the space of equilibrium states the Weinhold [26] metric which is defined as the Hessian of the internal energy \( U \) of the system

\[
g_{ij}^W = \partial_i \partial_j U(S, N'), \tag{1}
\]

where \( S \) is the entropy and \( N' \) represents the remaining extensive thermodynamic variables of the system. An alternative metric was proposed by Ruppeiner [27, 28] as minus the Hessian of the entropy,

\[
g_{ij}^R = -\partial_i \partial_j S(U, N'), \tag{2}
\]

and is related to the Weinhold metric through the line element relationship \( d\sigma^2_W = T d\sigma^2_R \), where \( T \) denotes the temperature. Both metrics have been applied to study the geometry of the thermodynamics of ordinary systems [29–41]; however, several inconsistencies and contradictions have been found, particularly in the study of black hole thermodynamics [42–50]. More recently, Liu et al [51] proposed to use the metric

\[
g_{ij}^{\text{LLL}} = \partial_i \partial_j \tilde{U}(N^k), \tag{3}
\]

where \( \tilde{U} \) is any of the thermodynamic potentials that can be obtained from \( U \) by means of a Legendre transformation. In the special cases \( \tilde{U} = U \) and \( \tilde{U} = S \), one obtains the Weinhold and Ruppeiner metrics, respectively.

The inconsistencies that appear from using the above metrics are explained in the framework of geometrothermodynamics (GTD) [52] as due to the fact that all those metrics are not invariant with respect to Legendre transformations. In this work, we will use a Legendre invariant metric in the context of GTD to formulate an invariant geometric representation of the thermodynamics of one of the most general black hole solutions known in the HL theory. We also compare our results with those obtained by using the Weinhold and Ruppeiner geometries.

This paper is organized as follows. In section 3, we review the main thermodynamic properties of the topological black hole solutions of the HL gravity. In section 2, we review the formalism of GTD and discuss the non-invariance of some of the known thermodynamic metrics. Section 4 contains the results of analyzing the thermodynamics of the topological black hole solution using the Weinhold and Ruppeiner geometries. In section 5, we investigate the GTD of the topological black hole and show that it agrees with the results following from the analysis of the corresponding thermodynamics. In this section, we also analyze the thermodynamic properties in the Einstein limit of the HL gravity, and the special case of a black hole with a flat horizon. Finally, in section 6, we present the conclusions of our work.
2. Review of GTD

GTD is a theory that has recently been formulated [52] in order to introduce in a consistent manner the Legendre invariance in the geometric description of the space of thermodynamic equilibrium states. This theory has been applied to different thermodynamic systems such as black holes, the ideal gas or the van der Waals gas [53–58]. In all the cases analyzed so far, GTD has delivered consistent results and allows us to describe geometrically the thermodynamic interaction and the phase transitions by means of Legendre invariant metrics.

The main ingredient of GTD is a (2n + 1)-dimensional manifold \( T \) with coordinates \( Z^a = (\Phi, E^a, P^a) \), where \( \Phi \) is an arbitrary thermodynamic potential, \( E^a, a = 1, 2, \ldots, n \), are the extensive variables and \( P^a \) the intensive variables. It is also possible to introduce in a canonical manner the fundamental one-form \( \Theta = d\Phi - \delta_{ab} P^b dE^a, \delta_{ab} = \text{diag}(+1, \ldots, +1) \), which satisfies the condition \( \Theta \wedge (d\Theta)^n \neq 0 \), where \( n \) is the number of thermodynamic degrees of freedom of the system, and is invariant with respect to the Legendre transformations \( (\Phi, E^a, P^a) \rightarrow (\Phi, \tilde{E}^a, \tilde{P}^a) \) with \( \Phi = \tilde{\Phi} - \delta_{ab} \tilde{E}^b P^a, \tilde{E}^a = -\tilde{P}^a \) and \( \tilde{P}^a = \tilde{P}^a \). Moreover, we assume that on \( T \) there exists a metric \( G \) which is also invariant with respect to Legendre transformations. The triad \( (T, \Theta, G) \) defines a Riemannian contact manifold which is called the thermodynamic phase space (phase manifold). The space of thermodynamic equilibrium states (equilibrium manifold) is an \( n \)-dimensional Riemannian submanifold \( \mathcal{E} \subset T \) induced by a smooth map \( \varphi : \mathcal{E} \rightarrow T \), i.e. \( \varphi : (E^a) \mapsto (\Phi, E^a, P^a) \), such that \( \varphi^* (\Theta) = \varphi^* (d\Phi - \delta_{ab} P^b dE^a) = 0 \) holds, where \( \varphi^* \) is the pullback of \( \varphi \). The manifold \( \mathcal{E} \) is naturally equipped with the Riemannian metric \( g = \varphi^* (G) \). The purpose of GTD is to demonstrate that the geometric properties of \( \mathcal{E} \) are related to the thermodynamic properties of a system with the fundamental thermodynamic equation \( \Phi = \Phi (E^a) \).

The nondegenerate Legendre invariant metric [59]

\[
G = \Theta^2 + \left( \chi_{ab} E^a P^b \right) (\eta_{cd} dE^c dP^d),
\]

where \( \chi_{ab} \) is a diagonal constant tensor, has been used extensively to describe second-order phase transitions, especially in the context of black hole thermodynamics. In fact, independently of the value of \( \chi_{ab} \), for a given black hole configuration metric (4) shows the same curvature singularities at those points where the heat capacity indicates the existence of second-order phase transitions. This has been shown in particular for several black holes in diverse theories and dimensions [59]. In the present work, this result is confirmed in the case of HL gravity.

The arbitrariness contained in the choice of the constant tensor \( \chi_{ab} \) has been used to simplify the final form of the metric \( g = \varphi^* (G) \). For instance, if \( \chi_{ab} = \delta_{ab} \), the term \( \varphi^* (\delta_{ab} E^a P^b) \) turns out to be proportional to the thermodynamic potential \( \Phi (E^a) \), by virtue of the Euler identity [59]. However, it is also possible to choose \( \chi_{ab} = \eta_{ab} \) without affecting the Legendre invariance. In this way, we recently found [60, 61] that a slightly generalized metric

\[
G = \Theta^2 + \frac{1}{2} (\delta_{ab} - \eta_{ab}) E^a P^b (\eta_{cd} dE^c dP^d)
\]

(5) can be used to handle in a geometric manner not only second-order phase transitions, but also the thermodynamic limit \( T \rightarrow 0 \). In this work, we consider metric (5) that induces on \( \mathcal{E} \), by means of \( g = \varphi^* (G) \), the thermodynamic metric

\[
g = \frac{1}{2} \left[ E^a \left( \frac{\partial \Phi}{\partial E^a} - \eta_{ab} \delta^{bc} \frac{\partial \Phi}{\partial E^c} \right) \right] (\eta_{cd} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^a \partial E^d} dE^a dE^d).
\]

(6)

Note that in the formalism of GTD the metric

\[
g_0 = \frac{\partial^2 \Phi}{\partial E^a \partial E^b} dE^a dE^b
\]

(7)
is generated as \( g_0 = \psi(G_0) = \psi^*(\delta_\alpha \, dE^\alpha \, d\ell^\beta) \), where the metric \( G_0 \) is not Legendre invariant. This implies that the results obtained by using the metric \( g_0 \) can depend on the choice of thermodynamic potential and, consequently, can lead to contradictory results. In particular, for \( \Phi = U \) (internal energy), we obtain that \( g_0 \) is equivalent to the Weinhold metric, and for \( \Phi = S \), \( g_0 \) is the Ruppeiner metric. If \( \Phi \) is any thermodynamic potential that can be obtained from \( U \) by means of a Legendre transformation, \( g_0 \) turns out to be proportional to the metric \( g^{\text{HL}} \) given in equation (3).

3. Topological black hole solutions in HL gravity

The HL gravity breaks general 4D covariance and splits it into 3D covariance plus reparametrization invariance of time. It is therefore convenient to formulate it in the \((3 + 1)-\)ADM formalism, where an arbitrary metric can be written in the form

\[
ds^2 = -N^2 \, dt^2 + g_{ij}(dx^i + N^i \, dt)(dx^j + N^j \, dt), \tag{8}\]

where \( N^2 \) is the lapse function and \( N^i \) represents the shift. Then, the HL action is written as [62]

\[
\mathcal{I}_{\text{HL}} = \int \mathcal{L}_{\text{HL}} \, dt \, d^3x, \tag{9}\]

where

\[
\mathcal{L}_{\text{HL}} = \sqrt{\mathcal{E}} \left[ \frac{2}{\kappa} (K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2(\Lambda R - 3\Lambda^2)}{8(1 - 3\lambda)} + \frac{\kappa^2 \mu^2(1 - 4\Lambda)}{32(1 - 3\lambda)} \right] + \frac{\kappa^2}{2\omega^2} \left[ C_{ij} - \frac{\mu \omega^2}{2} R_{ij} \right] \left[ C^{ij} - \frac{\mu \omega^2}{2} R^{ij} \right]. \tag{10}\]

Here \( R_{ij} \) and \( R \) are the 3D Ricci tensor and curvature scalar, respectively. Moreover, the extrinsic curvature \( K_{ij} \) and the Cotton tensor \( C_{ij} \) are given by the expressions

\[
K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_j N_i - \nabla_i N_j \right), \quad C_{ij} = \epsilon^{\alpha\beta\gamma} \nabla_\alpha \left( R^\beta_{\ i} - \frac{1}{4} R \delta^\beta_{\ i} \right), \tag{11}\]

where a dot represents differentiation with respect to the time coordinate. Finally, \( \kappa^2, \lambda, \mu, \omega \) and \( \Lambda \) are constants parameters.

A comparison of the HL action with the Einstein–Hilbert action leads to the conclusion that the speed of light, Newton’s constant and the cosmological constant \( \Lambda \) are given by

\[
c = \frac{\kappa^2 \mu}{\sqrt{1 - 3\lambda}}, \quad G = \frac{\kappa^2 c}{32\pi}, \quad \tilde{\Lambda} = \frac{3}{2}\Lambda. \tag{12}\]

Consider now the spherically symmetric line element

\[
ds^2 = -\tilde{N}^2(r) f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega_k^2, \tag{13}\]

where \( d\Omega_k^2 \) is the line element of the two-dimensional Einstein space with constant curvature \( 2\kappa \). Substituting metric (13) into action (9), we obtain [63]

\[
\mathcal{I}_{\text{HL}} = \frac{\kappa^2 \mu^2 \Omega_k}{8(1 - 3\lambda)} \int \tilde{N} \left[ \frac{(\lambda - 1)}{2} F'^2 - \frac{2\Lambda}{r} F F' + \frac{(2\lambda - 1)}{r^2} F^2 \right] \, dt \, dr, \tag{14}\]

where \( \Omega_k \) is the volume of the two-dimensional Einstein space, the prime denotes the derivative with respect to \( r \) and \( F \) is defined as

\[
F(r) = k - \Lambda r^2 - f(r). \tag{15}\]
The variation of (14) leads to the following set of equations:

\[ \left( \frac{2\lambda}{r} F - (\lambda - 1)F' \right) \tilde{N}' + (\lambda - 1) \left( \frac{2}{r^2} F - F'' \right) \tilde{N} = 0, \]

\[ (\lambda - 1) r^2 F' - 4\lambda r FF' + 2(2\lambda - 1) F^2 = 0, \]

whose solutions are

\[ F(r) = \alpha r^s, \quad \tilde{N} = \gamma r^{1-2s}, \]

where \( \alpha \) and \( \gamma \) are integration constants and \( s \) is given by

\[ s = \frac{2\lambda - \sqrt{2(3\lambda - 1)}}{\lambda - 1}. \]

This solution was obtained recently by Cai, Cao and Ohta (CCO) [64]. In general, the value of \( s \) with a positive sign in front of the square root is also a solution of the above equations. However, in this case the asymptotic properties of the solution are not compatible with the properties of a black hole spacetime. In the allowed interval \( \lambda > 1/3 \), i.e. for \( s \in (-1, 2) \), the above solution is asymptotically anti-de Sitter and describes the gravitational field of a static black hole.

In order to obtain the thermodynamic variables of the CCO black hole, it is necessary to use the canonical Hamilton formulation for the corresponding thermodynamic ensemble [63]. According to this Hamiltonian approach, the mass of the black hole is

\[ M = \frac{c^3 \gamma \Omega l^2}{16\pi G} \left( \frac{1 + s}{2 - s} \right) \left[ \frac{k + \frac{r_+^2}{l^2}}{(\frac{s}{2})^k} \right]^2, \]

where \( l^2 = -1/\Lambda \) represents the radius of curvature. Moreover, the Hawking temperature is given by

\[ T = \frac{\gamma}{4\pi r_+^2} \left( 2 - s \right) \frac{r_+^2}{l^2} - ks \].

Finally, integrating the first law of thermodynamics, \( dM = T dS + \mu_i dQ_i \), for constant values of the additional thermodynamic variables \( Q_i \), the entropy associated with the black hole is obtained as

\[ S = \frac{c^3 \Omega l^2}{4G} \left( \frac{1 + s}{2 - s} \right) \left[ \frac{r_+^2}{l^2} + k \ln \frac{r_+^2}{l^2} \right]. \]

The entropy \( S \) is defined up to an additive constant that can be chosen arbitrarily in order to avoid zero or negative values. Here, \( r_+ \) represents the radius of the exterior horizon which is a function of \( M \) and \( l \) determined by the algebraic equation

\[ r_+^2 - \frac{\kappa}{M l^2} r_+^2 - \frac{\kappa k}{M^2} = 0, \quad \kappa = \frac{\kappa \mu \gamma l^2 \Omega_2^2}{2^3 [3\lambda - 1]^3}. \]

Using expressions (20) and (21), we obtain the heat capacity

\[ C = \left( \frac{\partial M}{\partial r_+} \right) \left( \frac{\partial T}{\partial r_+} \right)^{-1} = \frac{c^3 \Omega l^2}{4G} \left( \frac{1 + s}{2 - s} \right) \left[ \left( k + \frac{r_+^2}{l^2} \right) \left( 2 - s \right) \frac{r_+^2}{l^2} - ks \right]. \]

According to Davies [65–67], second-order phase transitions take place at those points where the heat capacity diverges, i.e. for

\[ \frac{r_+^2}{l^2} = \frac{ks^2}{(s - 1)(2 - s)}. \]
Then, we conclude that phase transitions can occur only for \( k = 1 \) and \( s \in (1, 2) \), and for \( k = -1 \) and \( s \in (-1, 1) \). For all the remaining values of \( k \) and \( s \), the corresponding black hole cannot undergo a phase transition. Note, however, that the phase transition condition (25) must be considered together with the inequality

\[
\frac{r_s^2}{l^2} > \frac{ks}{2 - s}
\]

that follows from the condition \( T > 0 \) from equation (21).

4. Weinhold and Ruppeiner geometries

According to equation (20), the mass of the CCO black hole depends explicitly on the curvature radius \( l \) and the horizon radius \( r_h \), that, in turn according to equation (21), can be considered as an implicit function of the entropy \( S \). Below, for explicit computations, we will use \( r_h \) as a variable and relationship (21) to manipulate the dependence on \( S \). Although it is not possible to write down explicitly the function \( M = M(S, l) \), from the physical point of view it is convenient to consider the mass as depending on the entropy and the curvature radius. Although the entropy is clearly a thermodynamic variable, the thermodynamic nature of the radius of curvature is not so obvious. Nevertheless, a detailed analysis [68] of the thermodynamic properties of AdS black holes reveals that it is indeed possible to consider the cosmological constant as a well-defined thermodynamic variable. Here, we follow this result and assume that the radius of curvature is a thermodynamic variable.

Let us first consider the Weinhold metric (1). Since in the case of black holes the internal energy is represented by the mass \( M \), the Weinhold metric becomes

\[
g_W = M_S dS^2 + 2M_S dS dl + M_l dl^2,
\]

where \( M_S = \partial M / \partial S \), etc. Since the expression for \( M \) as given in equation (20) does not contain \( S \) explicitly, it is necessary to use \( r_h \) as a coordinate and its relation to \( S \) and \( l \) by means of equations (21) and (22). Then, we obtain

\[
g_W = M_S S^2 r_h^2 dS^2 + 2(M_S S + M_S)S r_h dS dl + (M_S S^2 + 2M_S S + M_l) dl^2.
\]

Introducing the thermodynamic equations (20)–(22), we obtain the explicit metric components

\[
g_{r_h r_h} = \frac{c^3 \Omega_4 \gamma}{4 \pi G} \left( \frac{1 + s}{2 - s} \right) l^2 t^{-2s} \left( \frac{r_h^2}{T} \right)^{2-2s} \left( k + \frac{r_h^2}{l^2} \right) \left( k s^2 + (s^2 - 3s + 2) \frac{r_h^2}{l^2} \right),
\]

\[
g_{r_h r_s} = \frac{c^3 \Omega_4 \gamma}{4 \pi G r_h} \left( \frac{1 + s}{2 - s} \right) l^{-2s} \left( \frac{r_h^2}{T} \right)^{-1-2s} \left( k s^2 + (s^2 - 3s + 2) \frac{r_h^2}{l^2} \right) ln \left( \frac{r_h^2}{l^2} \right) + (s - 2) \frac{r_h^2}{l^2} \left[ k^2 \frac{r_h^2}{l^2} - k(s - 2) \right] - k^2 s^2,
\]

\[
g_{r_s r_s} = \frac{c^3 \Omega_4 \gamma}{8 \pi G (k + l^2 r_h^2) r_h} \left( \frac{1 + s}{2 - s} \right) \left[ 2 k^2 \left( k s^2 + (s^2 - 3s + 2) \frac{r_h^2}{l^2} \right) \right] \ln^2 \left( \frac{r_h^2}{l^2} \right) + 4k \left( s - 2 \right) \frac{r_h^2}{l^2} \left[ k^2 \frac{r_h^2}{l^2} - k(s - 2) \right] - k^2 s^2 \ln \left( \frac{r_h^2}{l^2} \right) + 3 \frac{r_h^2}{l^2} + k \left( 11 - 4s \right) \frac{r_h^2}{l^2} + k^2 \left( 2s^2 - 10s + 13 \right) \frac{r_h^2}{l^2} + k^3 (1 + 2s),
\]

(31)
Moreover, the curvature scalar can be expressed as
\[ R^W = N^W D^W, \quad D^W = (s^2 - s - 2)r_+^6 + k(s^2 + 8s - 8)r_+^4 + k^2(s^2 - 3s + 2)t^4r_+^2 + k^3s^3l^6, \]
(32)
where \( N^W \) is a function of \( r_+ \) and \( l \) which is finite at those points where the denominator vanishes. The singularities are determined by the roots of the equation \( D^W = 0 \). It is easy to see that the solutions of this equation do not coincide with the points where the heat capacity diverges. Figure 1 shows the concrete example of a stable black hole in which a curvature singularity exists at a point where the heat capacity is regular. We conclude that the Weinhold curvature fails to reproduce the phase transition structure of the CCO topological black hole.

We now consider the Ruppeiner metric
\[ g^R = S_{MM} \, dM^2 + 2S_{MI} \, dM \, dl + S_{II} \, dl^2, \]
(33)
which in terms of the coordinates \( r_+ \) and \( l \) becomes
\[ g^R = S_{MM}M_1^2 \, dr_+^2 + 2(S_{MI} + S_{MM}M_1) \, dr_+ \, dl + (S_{MM}M_1^2 + 2S_{MI}M_1 + S_{II}) \, dl^2. \]
(34)
Using the expressions for the thermodynamic variables (20) and (22), we obtain
\[ g^R_{rr_+} = \frac{c^2\Omega_k}{G} \left( \frac{1 + s}{2 - s} \right) \frac{l^2}{r_+^3} \left( k + r_+^2 \frac{l^2}{T^2} \right) \left[ \frac{s^2(k + \frac{r_+^2}{T^2}) + (2 - 3s) \frac{r_+^2}{T^2}}{s(k + \frac{r_+^2}{T^2}) - 2 \frac{r_+^2}{T^2}} \right], \]
(35)
\[ g^R_{lrr_+} = -\frac{c^2\Omega_k}{G} \left( \frac{1 + s}{2 - s} \right) \frac{l}{r_+} \left( k + r_+^2 \frac{l^2}{T^2} \right) \left[ \frac{s^2k(k - \frac{r_+^2}{T^2}) - k(s - 2) \frac{r_+^2}{T^2} - (2s^2 - 7s + 6) \frac{r_+^2}{T^2}}{s(k + \frac{r_+^2}{T^2}) - 2 \frac{r_+^2}{T^2}} \right], \]
(36)
\[ g^R_{ll} = \frac{c^2\Omega_k}{2G} \left( \frac{1 + s}{2 - s} \right) \frac{1}{s(k + \frac{r_+^2}{T^2}) - 2 \frac{r_+^2}{T^2}} \left[ (6s^2 - 22s + 20) \frac{r_+^2}{l^2} + 2ks \frac{r_+^2}{l^2} \left( 11s - 19 \right) \frac{r_+^2}{l^2} + 7k(2s - 1) \frac{r_+^2}{l^2} + k^2(1 + 7s) \right]. \]
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\[ + k \ln \left( \frac{r_+^4}{l^2} \right) \left( s^3 - 6s^2 + 12s - 8 \right) \frac{r_+^4}{l^6} \]
\[ + ks(3s^2 - 12s + 21) \frac{r_+^4}{l^2} + 3k^2s^2(s - 2) \frac{r_+^4}{l^2} + k^3s^3 \]
\[ + k \left[ (20 - 3s^3) \frac{r_+^4}{l^2} - k(20 + 9s^3) \frac{r_+^4}{l^2} + k^2(4 - 9s^3) \frac{r_+^4}{l^2} + k^3s^2(2 - 3s) \right] \].

(37)

From these expressions for the metric functions, it is then straightforward to find the scalar curvature

\[ R^R = \frac{N^R}{D^R}, \quad D^R = (s + 1)l^{10} \left[ A \ln \left( \frac{r_+^4}{l^2} \right) - B \right] \left( k + \frac{r_+^4}{l^2} \right) \left[ s \left( k + \frac{r_+^4}{l^2} \right) - 2 \frac{r_+^4}{l^2} \right]^3, \]

(38)

with

\[ A = kl^6 \left[ \frac{r_+^4}{l^2} (s^3 - 3s^2 + 8s - 4) + \frac{r_+^4}{l^2} (2s^2 - 5s + 2)ks + k^2s^3 \right], \]

(39)

\[ B = kl^6 \left[ \frac{r_+^4}{l^2} (3s^3 - 13s^2 + 16s - 4) + \frac{r_+^4}{l^2} (6s^2 - 15s + 6)ks + 3k^2s^3 \right] + l^6 (2s^2 - 8s + 8). \]

(40)

The curvature singularities are determined by the roots of the equation \( D^R = 0 \) which do not coincide with the points where the heat capacity (24) shows second-order phase transitions. To illustrate the behavior of the curvature, we analyze the particular case with \( k = 1, l = 1 \) and \( s = -1/2 \) for which the temperature is always positive and the heat capacity is a smooth positive function which corresponds to a stable black hole configuration. The numerical analysis of this case is depicted in figure 2.

5. GTD of the CCO black holes

The formalism of GTD is invariant and, consequently, we can choose any arbitrary thermodynamic potential \( \Phi \) in any arbitrary representation to describe the thermodynamics of a black hole. Take, for instance, \( \Phi = M \) for the CCO topological black holes presented in section 3. The coordinates of the five-dimensional phase manifold can be chosen as \( Z^A = (M, S, I, T, L) \), where \( T \) is the temperature dual to \( S \) and \( L \) is the dual of the curvature radius \( I \). The fundamental one-form is then \( \Theta \equiv dM - TdS - Ldl \) and the Legendre invariant metric (5) is written as

\[ G = \Theta^2 + ST (-dSdT + dl dL). \]

(41)

The smooth map \( \varphi : E \to T \) or in coordinates \( \varphi : (S, I) \mapsto [M(S, I), S, I, T(S, I), L(S, I)] \) determines the equilibrium manifold \( E \) with the metric

\[ g^{GTD} = g^\varphi (G) = S M_S (-M_{SS} dS^2 + M_{II} dl^2), \]

(42)

on which the first law of thermodynamics \( dM = TdS + L dl \) and the equilibrium conditions

\[ T = \frac{\partial M}{\partial S} = M_S, \quad L = \frac{\partial M}{\partial l} = M_I. \]

(43)
Figure 2. Thermodynamic curvature of the Ruppeiner metric. This case corresponds to the following choice: $k = 1$, $l = 1$, $s = -1/2$, $y = 1$ and $\Omega_1 e^3/(4G) = 1$. A singularity exists at $r_+ \approx 1.18$ which does not correspond to a phase transition.

hold. As mentioned above, the fundamental equation $M = M(S, l)$ cannot be written explicitly and so we use $r_+$ instead of $S$ as a coordinate. Then,

$$g^{\text{GTD}} = \frac{c^3 \Omega_1^2 r_+^2 (s - 1)^2}{16 \pi^2 G} \left[ (s - 2) k_+^2 + ks \right] S(r_+, l) \left[ (s - 1)(s - 2) k_+^2 + ks^2 \right]
$$

Using the expressions for the mass and the entropy, we obtain

$$g^{\text{GTD}} = \frac{c^3 \Omega_1^2 r_+^2 (s - 1)^2}{16 \pi^2 G} \left[ (s - 2) k_+^2 + ks \right] S(r_+, l) \left[ (s - 1)(s - 2) k_+^2 + ks^2 \right]
$$

where

$$H(r_+, l) = \frac{2k^2 \ln \left( \frac{L}{r_+} - 1 \right)}{2(k + \frac{r_+}{L})}.$$ (46)

The curvature scalar corresponding to metric (44) is found to be

$$R^{\text{GTD}} = \frac{N^{\text{GTD}}}{D^{\text{GTD}}},$$ (47)

$$D^{\text{GTD}} = \left( k^2 + \frac{r_+^4}{L^2} \right)^2 \left( k + \frac{r_+^2}{L^2} \right)^3 \left( k^2 + k \ln \frac{r_+^2}{L^2} \right)^3 \times \left[ (2 - s) \frac{r_+^2}{L^2} - ks \right] \left[ (s - 1)(s - 2) \frac{r_+^2}{L^2} + ks^2 \right]^2,$$ (48)

where $N^{\text{GTD}}$ is a function of $r_+$ and $l$ that is finite at those points where the denominator vanishes. There are several curvature singularities in this case. The first one occurs if...
Figure 3. Thermodynamic curvature of the GTD metric. This case corresponds to the following choice: $k = 1$, $l = 1$, $s = -1/2$, $\gamma = 1$ and $\Omega c^3/(4G) = 1$. The expression $S^3 R^{GTD}$ is plotted to avoid the unphysical singularity at $S = 0$. 

$k + r_+^2/l^2 = 0$ and corresponds to the limit $M \to 0$, as follows from equation (20). A second singularity is located at the roots of the equation $r_+^2 + k \ln r_+^2 = 0$ and can be interpreted from equation (22) as the limit $S \to 0$. Moreover, according to equation (21), the singularity situated at $r_+^2/l^2 = ks/(2 - s)$ corresponds the limit $T \to 0$. Finally, if $(s - 1)(s - 2)r_+^2/l^2 + ks^2 = 0$, a singularity occurs that, according to equation (24), coincides with the limit $C \to \infty$, i.e. with the points where second-order phase transitions take place. Clearly, the singularities at which the mass, entropy or temperature vanish must be considered as unphysical and indicate the limit of applicability of the thermodynamics of black holes. The thermodynamic curvature in GTD for the case $k = 1$, $l = 1$ and $s = -1/2$ shows a singularity at the value $r_+^2 + \ln r_+^2 = 0$, i.e. for $r_+ \approx 0.75$, which corresponds to the limit $S \to 0$. Figure 3 illustrates the behavior of the curvature $S^3 R^{GTD}$ to avoid the unphysical singularity as $S \to 0$. We see that in the analyzed interval, no curvature singularities appear. This is in accordance with the behavior of the heat capacity, which in the same interval is free of phase transitions (see figure 1). A curvature singularity can be observed for $r_+ \to 0$ which indicates the break down of the black hole configuration and, consequently, of its thermodynamics. We conclude that the curvature obtained within the formalism of GTD correctly describes the thermodynamic behavior of topological black holes in HL gravity.

Note that the starting point in our geometrothermodynamical analysis is the metric $G$ of the phase space. The particular metric (5) induces the thermodynamic metric (6) whose determinant becomes proportional to $\partial^2 \Phi/\partial E^1 \partial E^1$. Furthermore, it is known that the scalar curvature always contains terms with the determinant of the metric in their denominator. It can therefore be expected that there exist true curvature singularities at those points where $\partial^2 \Phi/\partial E^1 \partial E^1 = 0$ (if those zeros are not canceled by the zeros of the numerator). We use in this work the $M$-representation, in which $\Phi = M$, and choose $E^1 = S$ so that the singularities are expected at $\partial^2 M/\partial S^2 = M_{SS} = 0$. On the other hand, in this representation the heat
capacity can be written as $C = M_S / M_{SS}$ with divergencies at $M_{SS} = 0$. It follows that the curvature singularities can coincide with the points where the heat capacity diverges. This, however, cannot be stated as a general result because the roots of $M_{SS} = 0$ can be canceled by the zeros of the numerator.

5.1. The Einstein limit of the CCO black holes

In the limit $\lambda \to 1$, the first three terms of the Lagrangian density (10) can be reduced to the Einstein–Hilbert Lagrangian density with cosmological constant. However, the remaining quadratic term in (10) represents an additional topological contribution to Einstein’s gravity with cosmological constant. Consequently, as shown in [64], the properties of the black holes that follow in this ‘Einstein limit’ are different from the well-known asymptotically de Sitter black holes of general relativity. In fact, the CCO topological black holes reduce in the Einstein limit to a single black hole configuration with $s = 1/2$, whereas the corresponding thermodynamic variables are written as

$$M = \frac{c^3 \Omega_k l^2}{16\pi G} \left( k + \frac{r^2}{l^2} \right)^2, \quad (49)$$

$$S = \frac{c^3 \Omega_k l^2}{4G} \left( r_+^2 + k \ln \frac{r_+^2}{l^2} \right), \quad (50)$$

$$T = \frac{\gamma}{8\pi r_+} \left( 3 \frac{r_+^2}{l^2} - k \right) \quad (51)$$

and

$$C = \left( \frac{\partial M}{\partial r_+} \right) \left( \frac{\partial T}{\partial r_+} \right)^{-1} = \frac{c^3 \Omega_k l^2}{2G} \left( k + \frac{r_+^2}{l^2} \right) \left( 3 \frac{r_+^2}{l^2} - k \right). \quad (52)$$

It follows that the second-order phase transitions take place at the points where the condition $3r_+^2/l^2 + k = 0$ is satisfied.

Introducing expressions (49) and (50) into metric (44), we obtain

$$g^{GTD} = \frac{1}{32} \left( \frac{c^3 \Omega_k l^2}{4G} \right)^2 \left( \frac{r_+^2}{l^2} + k \ln \frac{r_+^2}{l^2} \right) \left( k - 3 \frac{r_+^2}{l^2} \right) \frac{l^2}{r_+^2} \left( A_1 \, dr_+^2 + 2A_2 \, dl \, dr_+ + A_3 \, dl^2 \right),$$

with

$$A_1 = \frac{l^2}{r_+^2} \left( k + \frac{r_+^2}{l^2} \right) \left( k + 3 \frac{r_+^2}{l^2} \right), \quad (54)$$

$$A_2 = k \frac{l}{r_+} \left( \ln \frac{r_+^2}{l^2} - 1 \right) \left( k + 3 \frac{r_+^2}{l^2} \right), \quad (55)$$

$$A_3 = \frac{1}{k + \frac{r_+^2}{l^2}} \left[ k^2 \left( k + 3 \frac{r_+^2}{l^2} \right) \left( \ln \frac{r_+^2}{l^2} - 2 \right) \ln \frac{r_+^2}{l^2} - 6 \frac{l^4}{k^4} \left( k + \frac{r_+^2}{l^2} \right) - k^2 \left( k + \frac{r_+^2}{l^2} \right) \right]. \quad (56)$$

The corresponding scalar curvature can be expressed as

$$R^{GTD} = \frac{\mathcal{N}^{GTD}}{D^{GTD}}, \quad (57)$$
We can see that the roots of the equation \( 3r_c^2/l^2 + k = 0 \) determine curvature singularities which coincide with the points where second-order phase transitions occur (\( C \to \infty \)). Additional singularities occur if \( r_c^2/l^2 + k = 0, r_c^2/l^2 + k \ln(r_c^2/l^2) = 0 \) or \( 3r_c^2/l^2 - k = 0 \) which correspond to the limits \( M \to 0, S \to 0 \) or \( T \to 0 \), respectively.

### 5.2. The limiting black hole with a flat horizon

According to [63], the thermodynamics of a CCO black hole with a flat horizon \( (k = 0) \) is described by the following variables:

\[
M = \frac{c^3 \gamma \Omega_k l^{2-2s}}{16 \pi G} \left( \frac{1 + s}{2 - s} \right) \left( \frac{r_+}{l} \right)^{2(2-s)}, \tag{59}
\]

\[
S = \frac{c^3 \Omega_k l^2}{4G} \left( \frac{1 + s}{2 - s} \right) \left( \frac{r_+}{l} \right)^2, \tag{60}
\]

\[
T = \frac{\gamma l^{-2s}}{4\pi} (2 - s) \left( \frac{r_+}{l} \right)^{2-2s}, \tag{61}
\]

and

\[
C = \left( \frac{\partial M}{\partial r_+} \right)^{-1} \left( \frac{\partial T}{\partial r_+} \right)^{-1} = \frac{c^3 \Omega_k}{4G} \left( \frac{1 + s}{2 - s} \right) \frac{r_+^2}{s-1} = \frac{S}{s-1}. \tag{62}
\]

From the expression for the heat capacity, we see that this black hole is free of phase transitions. Using relations (59) and (60), the thermodynamic metric (44) is written as

\[
g_{\text{GTD}}^{\text{flat}} = \frac{(s-2)r_+^6}{8} \left( \frac{c^3 \gamma \Omega_k}{4\pi G} \right)^2 \left( \frac{1 + s}{s-2} \right)^2 \left( \frac{r_+}{l} \right)^{-4s-2} \times [2(s-1)(s-2)l^{4(s+1)} dr_+^2 - 3l^{-2(3+2s)} r_+^2 dl^2], \tag{63}
\]

for which we find that the curvature \( R = 0 \), indicating that no phase transition structure exists. This is in accordance with the result obtained above from the study of the heat capacity.

According to GTD, a flat equilibrium manifold is a consequence of the lack of thermodynamic interaction. This can be understood in the following way. For this special case, equation (60) indicates that the horizon radius is

\[
r_+ = \left[ \frac{4G}{\Omega_k c^3} \left( \frac{2 - s}{1 + s} \right) S \right]^\frac{1}{2s-2}, \tag{64}
\]

so that equation (59) generates the explicit fundamental equation

\[
M = \frac{\gamma}{4\pi} \left( \frac{4G}{\Omega_k c^3} \right)^{1-s} \left( \frac{2 - s}{1 + s} \right)^{1-s} S^{2-s} \left( \frac{l}{r_+} \right), \tag{65}
\]

which, in turn, can be rewritten as

\[
(2 - s) \ln S = \ln M + \ln l^2 + \ln S_0, \quad S_0 = \frac{4\pi}{\gamma} \left( \frac{c^3 \Omega_k}{4G} \right)^{1-s} \left( \frac{1 + s}{2 - s} \right)^{1-s}. \tag{66}
\]
So we see that the entropy function can be separated in the extensive variables $M$ and $l$. On the other hand, all thermodynamic potentials that possess the property of being separable have been shown [55] to correspond to systems with no thermodynamic interaction and zero thermodynamic curvature. This is an indication that the statistical internal structure of a CCO black hole with a flat horizon is equivalent to that of an ideal gas which is the main example of a system with no intrinsic thermodynamic interaction. We note that in this limiting case, Weinhold and Ruppeiner geometries are flat too, also indicating that there exists a statistical analogy between a black hole with a flat horizon and an ideal gas.

6. Conclusions

In this work, we applied the formalism of geometrothermodynamics (GTD) to describe the thermodynamics of the Cai–Cao–Ohta (CCO) topological black holes in the Hořava–Lifshitz model of quantum gravity. In the thermodynamic phase manifold, we introduce a particular Riemannian metric whose main property is its invariance with respect to Legendre transformations, i.e. its geometric characteristics are independent of the choice of thermodynamic potential. This is a property which holds in ordinary thermodynamics and we assume it as valid in GTD too. The Legendre invariant metric induces in a canonical manner a thermodynamic metric in the equilibrium manifold which is defined as a submanifold of the thermodynamic phase manifold.

We used the expressions of the main thermodynamic variables of the CCO black holes in order to compute the explicit form of the thermodynamic metric of the equilibrium manifold. The corresponding thermodynamic curvature turned out to be nonzero in general, indicating the presence of thermodynamic interaction. Moreover, it was shown that the phase transitions which are characterized by divergencies of the heat capacity are described in GTD by curvature singularities in the equilibrium manifold. We also studied the thermodynamics of the CCO black holes by using Weinhold and Ruppeiner geometries and found that they fail to describe the corresponding phase transition structure. These results are in agreement with a recent work by Janke, Johnston and Kenna [24] in which the GTD of the Kehagias–Sfetsos [69] black hole is investigated.

It was found that the geometrothermodynamic equilibrium manifold of the CCO black holes presents additional curvature singularities which correspond to the vanishing of the mass, entropy and Hawking temperature. We interpret in general the vanishing of these thermodynamic variables as an indication of the limit of applicability of black hole thermodynamics. So we conclude that the formalism of GTD breaks down, with curvature singularities, exactly at those points where black hole thermodynamics fails.

In the context of the GTD of the CCO black holes, we also analyzed the limit of Einstein gravity and of black holes with flat horizons. In both cases, we obtained results which are consistent with the thermodynamics of the respective black hole configurations. It turned out that the equilibrium manifold of black holes with flat horizons is flat. The flatness of the equilibrium manifold is interpreted as a consequence of the lack of intrinsic thermodynamic interaction. This property resembles the statistical behavior of an ideal gas.

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