Cosine Manifestations of the Gelfand Transform

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Abstract. The goal of the paper is to provide a detailed explanation on how the (continuous) cosine transform and the discrete(-time) cosine transform arise naturally as certain manifestations of the celebrated Gelfand transform. We begin with the introduction of the cosine convolution $\star_c$, which can be viewed as an “arithmetic mean” of the classical convolution and its “twin brother”, the anticonvolution. D’Alembert’s property of $\star_c$ plays a pivotal role in establishing the bijection between $\Delta(L^1(G), \star_c)$ and the cosine class $\text{COS}(G)$, which turns out to be an open map if $\text{COS}(G)$ is equipped with the topology of uniform convergence on compacta $\tau_{ucc}$. Subsequently, if $G = \mathbb{R}, \mathbb{Z}, \mathbb{S}_1$ or $\mathbb{Z}_n$ we find a relatively simple topological space which is homeomorphic to $\Delta(L^1(G), \star_c)$. Finally, we witness the “reduction” of the Gelfand transform to the aforementioned cosine transforms.

Mathematics Subject Classification. 42A38, 43A20, 43A32, 44A15.

Keywords. Gelfand transform, convolution, structure space, cosine class, continuous/discrete cosine transform.

1. Introduction

The current introductory section is divided into three parts. The first one centres around the general framework of our work—we recall such concepts as the structure space of a Banach algebra, the Gelfand transform or the Haar measure of a locally compact group. Since many have already covered these topics at great length, we restrict ourselves to laying out just the “basic facts”, i.e., those that are vital in comprehending further sections of the paper. More
inquisitive Readers are encouraged to study the materials in the bibliography, which are usually referenced in the footnotes.

Second part of the introductory section aims at delivering the context for our research regarding the cosine convolution and the cosine transforms. We strive to convince the Reader that the subject of the paper belongs to an active field of mathematical study and even “creeps into” the neighbouring disciplines such as algorithmics, computational complexity theory, data and image compression or signal processing. Consequently, our paper can be regarded as a theoretical foundation for various applications in engineering and technology.

Last but not least, the third and final part of the introduction serves as a brief summary of further sections. We lay out the structure of the paper in order to “paint the big picture” of our research prior to delving into technical subtleties.

1.1. What is the Gelfand Transform?
Without further ado we begin with the basic principles of the Banach algebra theory. In his tour de force “A Course in Commutative Banach Algebras”\(^1\) Eberhard Kaniuth defines a normed algebra as a normed linear space \((A, \| \cdot \|)\) over the field of complex numbers \(\mathbb{C}\), which is also an algebra and whose norm is submultiplicative, i.e.,

\[
\forall f,g \in A \quad \| f \cdot g \| \leq \| f \| \cdot \| g \|.
\]

If \((A, \| \cdot \|)\) is complete (i.e., it is a Banach space), we say that it is a Banach algebra (a multitude of examples of Banach algebras is provided in Chapter 6 in [8]). Given two Banach algebras \(A_1\) and \(A_2\) we say that a map \(\Psi : A_1 \longrightarrow A_2\) is a Banach algebra homomorphism if it is

- continuous,
- \(\mathbb{C}\)-linear, and
- multiplicative, i.e.,

\[
\forall f,g \in A_1 \quad \Psi(f \cdot g) = \Psi(f) \cdot \Psi(g).
\]

A structure space \(\Delta(A)\) of a Banach algebra \(A\) is the set of all nonzero Banach algebra homomorphisms \(m : A \longrightarrow \mathbb{C}\).\(^2\) Although many authors require the structure space to be defined solely for commutative Banach algebras, we intentionally stick to the general case, i.e., not necessarily commutative. The price we pay for taking such an approach is that \(\Delta(A)\) can be an empty set—this happens, for instance, if \(A\) is the matrix algebra \(M_{n \times n}(\mathbb{C})\).\(^3\)

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\(^1\)See [26], p. 1.

\(^2\)For a thorough exposition of this concept see [13], p. 43 or [17], p. 5 or [26], p. 46.

\(^3\)In a unital Banach algebra \(A\) every multiplicative functional \(m \in \Delta(A)\) determines a maximal ideal \(\ker(m)\) (see Theorem 1.12 in [17], p. 6 or Theorem 2.1.8 in [26], p. 49). Furthermore, if \(\ker(m) = \{0\}\) then \(A = \mathbb{C}\). Consequently, since Proposition 1.4 in [19], p. 360 states that the matrix algebra \(M_{n \times n}(\mathbb{C})\), \(n > 1\) is simple (it has no nonzero, proper ideals), we conclude that \(\Delta(M_{n \times n}(\mathbb{C})) = \emptyset\).
The set $\Delta(A)$ becomes a locally compact space\(^4\) when endowed with the weak* topology\(^5\) and its raison d’être lies in the fact that (if $\Delta(A) \neq \emptyset$) there exists a norm-decreasing, algebra homomorphism $\Gamma : A \rightarrow C_0(\Delta(A))$ given by the formula
\[
\forall m \in \Delta(A) \quad \Gamma(f)(m) := m(f).
\]
The function $\Gamma(f) \in C_0(\Delta(A))$ is commonly referred to as the Gelfand transform of the element $f \in A$ and the notation “$\Gamma(f)$” is usually reduced to $\hat{f}$.

In order to reinforce our intuition regarding the abstract theory above let us focus on a particular instance of a commutative Banach algebra. Let $G$ be a locally compact abelian group and let $\mu$ be its Haar measure,\(^6\) i.e., a nonzero, Borel measure which is
\begin{itemize}
  \item finite on compact sets,
  \item \textit{inner regular}, i.e., for every open set $U$ in $G$ we have
    \[ \mu(U) = \sup \{ \mu(K) : K \subset U, K - \text{compact} \}, \]
  \item \textit{outer regular}, i.e., for every Borel set $A$ in $G$ we have
    \[ \mu(A) = \inf \{ \mu(U) : A \subset U, U - \text{open} \}, \]
  \item \textit{translation-invariant}, i.e., for every $x \in G$ and every Borel set $A$ we have
    \[ \mu(x + A) = \mu(A). \]
\end{itemize}
For every two functions $f, g \in L^1(G)$ we define their convolution $f \ast g$ with the formula
\[
\forall x \in G \quad f \ast g(x) := \int_G f(y)g(x - y)dy,
\]
where the integration is with respect to the Haar measure $\mu$.\(^7\) This operation turns $L^1(G)$ into a commutative Banach algebra (obviously $L^1(G)$ is already a Banach space, so $\ast$ defines the “multiplication” of the elements).

It is one of the gems of abstract harmonic analysis that the structure space $\Delta(L^1(G), \ast)$ is homeomorphic\(^8\) to the dual group $\hat{G}$, i.e., the group of all nonzero, multiplicative homomorphisms (called \textit{characters}) $\chi : G \rightarrow S^1$.

\(^4\)Every topological space that appears in the paper is assumed to be Hausdorff so we refrain from writing that explicitly.

\(^5\)For a detailed exposition of the weak* topology $\tau^*$ see Chapter 3.4 in [9], p. 62 or Chapter 2.4 in [36], p. 62. The fact that $\Delta(A)$ is a locally compact space (under $\tau^*$) can be found as Theorem 2.4.5 in [13], p. 46 or Theorem 2.2.3 in [26], p. 52. The former theorem also establishes the existence of the Gelfand transform—for another point of view see Theorem 1.13 in [17], p. 7.

\(^6\)See [21] for the original paper by Haar as well as [10, 14] or [40] (chapter 2) for a further study of the subject.

\(^7\)Formally, we probably should write “$d\mu(y)$” when integrating with respect to the Haar measure, yet we feel that this is an unnecessary complication of the nomenclature. Hence, we stick to an abbreviated form “$dy$” for the sake of simplicity.

\(^8\)See Theorem 3.2.1 in [13], p. 67 or Theorem 2.7.2 in [26], p. 89.
where $S^1$ is the unit circle (with multiplication of elements as the group operation). The homeomorphism $S: \Delta(L^1(G), \star) \rightarrow \hat{G}$ in question assigns a unique character $\chi \in \hat{G}$ to every multiplicative linear functional $m \in \Delta(L^1(G), \star)$ in such a way that
\[
\forall f \in L^1(G) \quad m(f) = \int_G f(x)\overline{\chi(x)}dx,
\]
where $\chi(x)$ is the value of the character $\chi$ at point $x$.

As a consequence of $\Delta(L^1(G), \star)$ and $\hat{G}$ being homeomorphic, the Banach algebra $C_0(\Delta(L^1(G), \star))$ is isometrically isomorphic with $C_0(\hat{G})$.\(^9\) It follows that we may treat the Gelfand transform $\hat{f} \in C_0(\Delta(L^1(G), \star))$ of $f \in L^1(G)$ as an element of $C_0(\hat{G})$ given by the formula
\[
\forall f \in L^1(G) \quad \hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx.
\]
To put it in other words, the Gelfand transform of $f \in L^1(G)$ is the Fourier transform $\hat{f} \in C_0(\hat{G})$.

1.2. Context
The aim of the current subsection is to provide the context for the upcoming sections by giving a brief overview of the present state of the literature. We strive to substantiate the claim that the cosine transform is not only an invaluable tool in the mathematical toolbox (especially for those working in the field of harmonic analysis), but its usage stretches outside of the mathematical realm, to fields such as engineering and technology.

Although the concept of the cosine transform goes as far back as the monumental work “The analytical theory of heat” by Jean Baptiste Fourier,\(^10\) it is widely agreed that the “modern history” of the operator starts roughly 50 years ago. As Nasir Ahmed recollects,\(^11\) in late 60s and early 70s “there was a great deal of research activity related to digital orthogonal transforms” and a large number of transforms were introduced “with claims of better performance relative to others transforms”. Inspired by the Karhunen-Loeve transform, Ahmed came up with the idea of the cosine transform and issued a proposal to the National Science Foundation to study the newly-invented operator. His proposal was dismissed due to the idea being “too simple”, but Ahmed continued to work on the concept with his Ph.D. student T. Natarajan and a colleague at the University of Texas, Dr. K. R. Rao. The team was quickly surprised how well the cosine transform performed relative to other transforms.

\(^9\)See Corollary 2.2.13 in [26], p. 57 for a more general result stating that Banach algebras $C_0(X)$ and $C_0(Y)$ are isometrically isomorphic if and only if locally compact Hausdorff spaces $X$ and $Y$ are homeomorphic.

\(^10\)For an english translation of Fourier’s book with commentary by Alexander Freeman see [18].

\(^11\)See [2].
and published their results in 1974.\footnote{See [4].} 17 years later, Ahmed wrote “Little did we realize at that time that the resulting DCT\footnote{DCT stands for “discrete cosine transform”} would be widely used in the future! It is indeed gratifying to see that the DCT is now essentially a standard in the area of image data compression via transform coding techniques”.

A fleeting glimpse at the literature reveals that Ahmed’s claim is not unsubstantiated. The cosine transform has found applications in:

- algorithmics and computational complexity theory ([5, 7, 11, 20, 23, 29–32, 34, 35, 37, 39]),
- data compression (see [22]),
- image compression and JPEG format (see [1, 3, 38]),
- signal processing (see [25]).

We believe that such broad interest in the subject of cosine transform is not coincidental and reflects the fact that the topic is still within the scope of active mathematical research.

### 1.3. Layout of the Paper

In this final part of the introduction we summarize the structure of the paper. The goal of this quick outline is to facilitate the comprehension of the “big picture” before delving into technical subtleties.

Section 2 begins with the definition of the cosine convolution and what follows is an investigation of its basic features like d’Alembert’s property (see Theorem 2). We proceed with establishing a bijection $\beta_G$ (see Theorem 3) between the structure space $\Delta(L^1(G), \star_c)$ and the cosine class $\mathcal{COS}(G)$, i.e., a family of all nonzero, continuous, bounded functions which satisfy d’Alembert’s functional equation

$$\forall x, y \in G \quad \phi(x)\phi(y) = \frac{\phi(x + y) + \phi(x - y)}{2}.$$

Section 3 adds a topological layer to our analysis—by topologizing the set $\mathcal{COS}(G)$ with the topology of uniform convergence on compacta we discover that $\beta_G$ is in fact an open map. The section goes on to prove that $\mathcal{COS}(G)$ is homeomorphic to

- $\mathbb{R}_+ \cup \{0\}$ if $G = \mathbb{R}$,
- $S^1_+ \cup \{1\}$ if $G = \mathbb{Z}$, where
  $$S^1_+ := \left\{ z \in S^1 : \text{Im}(z) > 0 \right\},$$
- $\mathbb{N}_0$ if $G = S^1$,
- $\mathbb{Z}_{\lceil \frac{n+1}{2} \rceil}$ if $G = \mathbb{Z}_n$ ($x \mapsto \lceil x \rceil$ is the ceiling function).
The focal point of Sect. 4 is the computation of particular instances of (what we call) cosine structure spaces. It turns out (see Theorems 9, 10, 11 and 12) that if \( G = \mathbb{R}, \mathbb{Z}, S^1 \) or \( \mathbb{Z}_n \) then \( \beta_G : \Delta(L^1(G), \star_c) \rightarrow \text{COS}(G) \) is continuous (and thus a homeomorphism in view of what we said earlier). The section concludes with bringing all the pieces of the puzzle as we witness the Gelfand transform manifest itself in the form of the cosine transforms.

Our final remarks take the form of an Epilogue, in which we indicate a possible direction of future research. Naturally, the paper concludes with a bibliography.

2. Cosine Convolution

We begin our story with the definition of the cosine convolution operator \( \star_c : L^1(G) \times L^1(G) \rightarrow \mathbb{C} \) with the formula:

\[
\forall x \in G \quad f \star_c g(x) := \int_G f(y) \cdot \frac{g(x + y) + g(x - y)}{2} dy.
\] (2)

The question that immediately springs to mind when looking at formula (2) is whether the cosine convolution is well-defined? In other words, given two functions \( f, g \in L^1(G) \) does \( f \star_c g(x) \) make sense for at least some values \( x \in G \)? The answer is (luckily) affirmative and in fact, \( f \star_c g \) defines a multiplication on \( L^1(G) \) turning it into a Banach algebra:

**Theorem 1.** If \( f, g \in L^1(G) \) then \( f \star_c g(x) \) is well-defined (i.e., finite for almost every \( x \in G \)) and we have

\[
\|f \star_c g\|_1 \leq \|f\|_1 \|g\|_1.
\]

Consequently, \( (L^1(G), \star_c) \) is a Banach algebra.

We have taken the liberty of omitting the proof of this result as it is almost a verbatim rewrite of the proof of Theorem 1.6.2 in [12], p. 26 where Deitmar shows (with painstaking precision) that \( L^1(G) \) is a Banach algebra under the standard convolution \( \star \) defined by (1). Whoever shall read Deitmar’s reasoning will surely discover that his proof works equally well for the anticonvolution:

\[
\forall x \in G \quad f \star_a g(x) := \int_G f(y)g(x + y) dy.
\]

Thus, if we view the cosine convolution \( \star_c \) as an “arithmetic mean of convolutions”, i.e., \( \frac{\star + \star}{2} \) then Theorem 1 becomes trivial. There is one more issue, however, that we would like to address at this point. Although \( (L^1(G), \star_c) \) is a Banach algebra, it need not be a commutative Banach algebra, as the anticonvolution part need not be commutative:

\[
\forall x \in G \quad f \star_a g(x) = \int_G f(y)g(x + y) dy \int_G f(-(y + x))g(-y) dy \quad \text{with} \quad \begin{cases} y = y' + x, \\ y = y' - x \end{cases} 
\]

\[
= \int_G g(y)f(-x + y) dy = g \star_a f(-x).
\]
In spite of this nagging splinter, we stick to our guns by defining $\Delta(L^1(G), \ast_c)$ as the set of all nonzero Banach algebra homomorphisms $m : (L^1(G), \ast_c) \rightarrow \mathbb{C}$.

Amongst all properties of the cosine convolution we focus on the one bearing a close resemblance with the classical d’Alembert’s functional equation

$$\forall x, y \in G \quad \phi(x) \phi(y) = \frac{\phi(x + y) + \phi(x - y)}{2},$$

whose continuous and bounded (and nonzero) solutions are the functions $x \mapsto \cos(\omega x), \omega \in \mathbb{R}$. After all, it is hardly surprising that the cosine convolution should have so much in common with the cosine function itself! Without further ado, here is d’Alembert’s property of the cosine convolution:

**Theorem 2.** Let $x, y \in G$. If $g \in L^1(G)$ is an even function, then

$$L_y g \ast_c L_x g = g \ast_c \frac{L_{x+y} g + L_{x-y} g}{2},$$

where for every $z \in G$ the operator $L_z : L^1(G) \rightarrow L^1(G)$ is given by

$$\forall u \in G \quad L_z f(u) := f(u - z).$$

**Proof.** First we note that for every $u \in G$ we have

$$\int_G L_y g(v) L_x g(u + v) dv = \int_G g(v) L_{x-y} g(u + v) dv,$$

and analogously

$$\int_G L_y g(v) L_x g(u - v) dv = \int_G g(v) L_{x+y} g(u - v) dv,$$

and

$$\int_G L_y g(v) L_x g(u + v) dv = \int_G g(v) L_{x+y} g(u + v) dv,$$

and

$$\int_G L_y g(v) L_x g(u - v) dv = \int_G g(v) L_{x-y} g(u - v) dv.$$

Summation of equations (4) (and division by 2) yields

$$\forall u \in G \quad \int_G L_y g(v) L_x g(u + v) dv = \int_G g(v) \cdot \frac{L_{x-y} g(u + v) + L_{x-y} g(u - v)}{2} dv$$

$$= g \ast_c L_{x-y} g(u),$$

\[14\text{For completeness we should mention that if we allow for unbounded (yet still continuous) solutions then the family of functions } x \mapsto \cosh(\omega x), \omega \in \mathbb{R} \text{ also satisfies the classical d’Alembert’s functional equation. What is more, apart from the zero function and the two families } x \mapsto \cos(\omega x), x \mapsto \cosh(\omega x), \omega \in \mathbb{R} \text{ there are no other solutions.} \]
while the summation of equations (5) (and again division by 2) yields
\[
\forall u \in G \quad \int_G L_y g(v)L_x g(u - v) dv = \int_G g(v) \cdot \frac{L_{x+y} g(u + v) + L_{x+y} g(u - v)}{2} dv
\]
\[= g \ast c L_{x+y} g(u). \quad (7) \]
Finally, summation of equations (6) and (7) (and division by 2 for the last time) produces the desired d’Alembert’s property (3).

With d’Alembert’s property of the cosine convolution in our toolbox we begin to investigate the structure space \(\Delta(L^1(G), \ast_c)\). Our first goal is to prove that \(\Delta(L^1(G), \ast_c)\) is in bijective correspondence with the cosine class \(\text{COS}(G) := \{ \phi \in C^b(G) : \phi \neq 0, \forall_{x,y \in G} \phi(x)\phi(y) = \frac{\phi(x+y) + \phi(x-y)}{2} \}\).

In order to demonstrate this bijection, we will use the following lemma:

**Lemma 1.** For every \(m \in \Delta(L^1(G), \ast_c)\) there exists an even function \(g_* \in L^1(G)\) such that \(m(g_*) \neq 0\).

**Proof.** Let \(\iota : G \rightarrow G\) be the inverse function \(\iota(x) := -x\). Then
\[
\forall f \in L^1(G) \quad (f \circ \iota) \ast_c f(x) = \int_G f \circ \iota(y) \cdot \frac{f(x+y) + f(x-y)}{2} dy \]
\[= f \ast c f, \quad \text{which leads to} \]
\[
\forall f \in L^1(G) \quad m(f \circ \iota)m(f) = m((f \circ \iota) \ast_c f) = m(f \ast_c f) = m(f)^2. \quad (8) \]
Since \(m\) is nonzero by the definition of \(\Delta(L^1(G), \ast_c)\), then there must exist a function \(f_* \in L^1(G)\) such that \(m(f_*) \neq 0\). Consequently, equations (8) yield \(m(f_* \circ \iota) = m(f_*)\). Finally, we put \(g_* := f + f \circ \iota\) and observe that
\[
m(g_*) = m(f_* + f_* \circ \iota) = m(f_*) + m(f_* \circ \iota) = 2m(f_*) \neq 0, \]
which concludes the proof. \(\square\)

We are now in position to prove a bijective correspondence between \(\Delta(L^1(G), \ast_c)\) and \(\text{COS}(G)\):

**Theorem 3.** If \(m_\phi : L^1(G) \rightarrow \mathbb{C}\) is given by
\[
m_\phi(f) := \int_G f(x) \phi(x) dx \quad (9) \]
for some \(\phi \in \text{COS}(G)\), then \(m_\phi \in \Delta(L^1(G), \ast_c)\). Furthermore, for every \(m \in \Delta(L^1(G), \ast_c)\) there exists a unique \(\phi \in \text{COS}(G)\) such that \(m = m_\phi\).
Proof. Obviously, if $m_{\phi}$ is given by formula (9) then it is a linear functional, which satisfies

$$\forall f \in L^1(G) \ |m_{\phi}(f)| \leq \|\phi\|_{\infty} \|f\|_1,$$

where $\|\cdot\|_{\infty}$ stands for the supremum norm. The inequality above means that $m_{\phi}$ is a bounded (and thus continuous) functional on $L^1(G)$. Thus, in order to conclude that $m_{\phi} \in \Delta(L^1(G), *_c)$ we simply need to demonstrate the multiplicativity of $m_{\phi}$. We have

$$\forall f, g \in L^1(G) \ m_{\phi}(f *_c g) = \int_G f *_c g(x)\phi(x)dx,$$

$$= \int_G \left( \int_G f(y) \cdot \frac{g(x+y) + g(x-y)}{2} dy \right) \phi(x)dx,$$

$$= \int_G \int_G f(y) \cdot \frac{g(x+y) + g(x-y)}{2} \cdot \phi(x)dxdy,$$

$$= \frac{1}{2} \int_G \int_G f(y)g(x+y)\phi(x)dxdy + \frac{1}{2} \int_G \int_G f(y)g(x-y)\phi(x)dxdy,$$

Applying the substitution $x \mapsto x - y$ to the first double integral and the substitution $x \mapsto x + y$ to the second double integral in the last line we obtain

$$\forall f, g \in L^1(G) \ m_{\phi}(f *_c g) \equiv \int_G \int_G f(y)g(x) \cdot \frac{\phi(x-y) + \phi(x+y)}{2} dx dy,$$

$$= \int_G \int_G f(y)g(x) \phi(x)\phi(y)dxdy = m_{\phi}(f)m_{\phi}(g).$$

This implies that $m_{\phi} \in \Delta(L^1(G), *_c)$ and concludes the first part of the proof.

For the second part of the proof, we fix an element $m \in \Delta(L^1(G), *_c)$ and seek to show that there exists a unique $\phi \in CO(S(G))$ such that $m = m_{\phi}$. By Lemma 1 there exists an even function $g_* \in L^1(G)$ such that $m(g_*) \neq 0$. Moreover, by Theorem 2 we know that

$$\forall x, y \in G \ L_yg_* *_c L_xg_* = g_* *_c \frac{L_{x+y}g_* + L_{x-y}g_*}{2}.$$

Applying the functional $m$ to this equation and using its linearity and multiplicativity we obtain

$$\forall x, y \in G \ m(L_yg_*)m(L_xg_*) = m(g_*) \cdot \frac{m(L_{x+y}g_*) + m(L_{x-y}g_*)}{2}. \quad (10)$$

Next, we define the function $\phi : G \rightarrow \mathbb{C}$ by the formula

$$\forall x \in G \ \phi(x) := \frac{m(L_xg_*)}{m(g_*)}, \quad (11)$$

which is

- nonzero, because $\phi(0) = 1,$
• continuous by Lemma 1.4.2 in [13], p. 18, and
• bounded, because \( \|L_x g_*\|_1 = \|g_*\|_1 \) for every \( x \in G \).

Dividing equation (10) by \( m(g_*)^2 \) we may rewrite it in the form
\[
\forall x,y \in G \quad \phi(y)\phi(x) = \frac{\phi(x+y) + \phi(x-y)}{2},
\]
which means that \( \phi \in \mathcal{COS}(G) \). We have
\[
\forall f \in L^1(G) \int_G f(x)\phi(x)dx \quad \phi \in \mathcal{COS}(G) \quad \frac{\phi(x+y) + \phi(x-y)}{2} dx
\]
\[
= \int_G f(x) \cdot \frac{m(L_x g_*) + m(L_{-x} g_*)}{2m(g_*)} dx
\]
\[
= \frac{1}{m(g_*)} \cdot m\left( \int_G f(x) \cdot \frac{L_x g_* + L_{-x} g_*}{2} dx \right)
\]
\[
= \frac{1}{m(g_*)} \cdot m(f \ast_c g_*) = \frac{1}{m(g_*)} \cdot m(f)m(g_*) = m(f),
\]
where the third equality holds true due to Lemma 11.45 in [6], p. 427 (or Proposition 7 in [15], p. 123). We have thus proved that \( m = m_\phi \).

Finally, the fact that (11) is a unique \( \phi \) such that \( m = m_\phi \) follows from a technique, which is well-known in the field of variational calculus: suppose that there exist \( \phi_1, \phi_2 \in \mathcal{COS}(G) \) such that \( m = m_{\phi_1} = m_{\phi_2} \). Consequently, we have
\[
\forall f \in L^1(G) \int_G f(x)\left( \phi_1(x) - \phi_2(x) \right) dx = 0. \tag{12}
\]
Assuming the existence of an element \( x_* \in G \) such that \( \phi_1(x_*) \neq \phi_2(x_*) \), we can choose an open neighbourhood \( U_* \) of \( x_* \) such that
• \( \phi_1 - \phi_2 \) is of constant sign on \( U_* \), and
• \( U_* \) is compact (because we constantly work under the assumption that the group \( G \) is locally compact).

It remains to observe that for \( f_* := (\phi_1 - \phi_2)1_{U_*} \) we have
\[
0 \overset{(12)}{=} \int_G f_*(x)\left( \phi_1(x) - \phi_2(x) \right) dx = \int_{U_*} \left( \phi_1(x) - \phi_2(x) \right)^2 dx > 0,
\]
which is a contradiction. This means that \( \phi_1 = \phi_2 \) and concludes the proof. \( \square \)

3. Structure Spaces and Cosine Classes

To summarize the climactic point of the previous section, Theorem 3 establishes that the function \( \beta_G : \Delta(L^1(G), \ast_c) \rightarrow \mathcal{COS}(G) \) given by \( \beta_G(m) := \phi \) (where \( \phi \) is a unique element of \( \mathcal{COS}(G) \) such that \( m = m_\phi \)) is a bijection. Our next task is to prove that \( \beta_G \) is in fact more than just a bijection of two sets.
Before we discuss the topology that we impose on the cosine class $COS(G)$ let us examine its elements a little closer:

**Theorem 4.** For every $\phi \in COS(G)$ there exists a character $\chi_\phi \in \hat{G}$ such that

$$\forall x \in G \; \phi(x) = \frac{\chi_\phi(x) + \overline{\chi_\phi(x)}}{2}.$$  \hfill (13)

In particular, $\|\phi\|_\infty \leq 1$.

**Proof.** By Corollary 1 in [27] there exists a continuous homomorphism $\chi_\phi : G \rightarrow C^\ast$ such that

$$\forall x \in G \; \phi(x) = \chi_\phi(x) + \chi_\phi(x)^{-1}. \hfill (14)$$

Furthermore, since $\phi$ is bounded, then so is $\chi_\phi$. We fix $y_* \in G$ and observe that the multiplicative property

$$\forall x \in G \; \chi_\phi(x) \chi_\phi(y_*) = \chi_\phi(x + y_*)$$

implies

$$\sup_{x \in G} |\chi_\phi(x)||\chi_\phi(y_*)| = \sup_{x \in G} |\chi_\phi(x + y_*)| = \sup_{z \in G} |\chi_\phi(z)|.$$

By the finiteness of $\sup_{x \in G} |\chi_\phi(x)| = \sup_{z \in G} |\chi_\phi(z)|$ we obtain $|\chi_\phi(y_*)| = 1$. Since the element $y_*$ was chosen arbitrarily, then we have established that $\chi_\phi$ is in fact a character, i.e., $\chi_\phi \in \hat{G}$. Formula (13) follows from (14) and the fact that $\chi_\phi(x)^{-1} = \overline{\chi_\phi(x)}$ for every $x \in G$. \hfill $\Box$

Knowing what the elements of $COS(G)$ look like, we intend to show that $\beta_G$ is an open map if the cosine class is endowed with the proper topology.\footnote{As we have already mentioned in the introductory section, $\Delta(L^1(G), \ast_c)$ is endowed with the weak* topology $\tau^\ast$, which makes it a locally compact space.}

What do we mean by “proper”? Well, $COS(G)$ is a subspace of $C^b(G)$ so it is natural to endow it with the topology of uniform convergence on compacta $\tau_{ucc}$.\footnote{For a detailed discussion on the properties of $\tau_{ucc}$ see Chapter 7 in [28], Chapter 46 in [33] or Chapter 43 in [41].} Our next result confirms that this is the right choice:

**Theorem 5.** $\beta_G : (\Delta(L^1(G), \ast_c), \tau^\ast) \rightarrow (COS(G), \tau_{ucc})$ is an open map.

**Proof.** Our task is to prove that the image (under $\beta_G$) of an arbitrary set

$$U := \left\{ m \in \Delta(L^1(G), \ast_c) : \forall n=1, \ldots, N \; |m(f_n) - m_{\phi_*}(f_n)| < \varepsilon \right\},$$

where $\varepsilon > 0$, $m_{\phi_*} \in \Delta(L^1(G), \ast_c)$ and $(f_n)_{n=1}^N \subseteq L^1(G)$ are fixed, is $\tau_{ucc}$-open. We fix $\phi_{\ast *} \in \beta_G(U)$, which means that there exists $\delta \in (0, 1)$ such that

$$\forall n=1, \ldots, N \; |m_{\phi_{\ast *}}(f_n) - m_{\phi_*}(f_n)| < \delta \varepsilon.$$  \hfill (15)

Furthermore, we pick \( K \) to be a compact subset of \( G \) such that
\[
\forall n = 1, \ldots, N, \int_{G \setminus K} |f_n(x)| \, dx \leq \frac{(1 - \delta) \varepsilon}{4}. \tag{16}
\]

Last but not least, we put
\[
V := \left\{ \phi \in \mathcal{COS}(G) : \sup_{x \in K} |\phi(x) - \phi^{**}(x)| < \frac{(1 - \delta) \varepsilon}{2 \max_{n=1, \ldots, N} \|f_n\|_1} \right\},
\]
which is a \( \tau_{ucc} \)-open neighbourhood of \( \phi^{**} \). Finally, we calculate that
\[
\forall \phi \in V, \forall n = 1, \ldots, N, m(\phi(f_n) - m^{**}(f_n)) \leq m(\phi(f_n) - m^{**}(f_n)) + m(\phi^{**}(f_n) - m^{**}(f_n)) \leq |m(\phi(f_n) - m^{**}(f_n)) + \delta \varepsilon|
\]
\[
\leq \int_K |f_n(x)||\phi(x) - \phi^{**}(x)| \, dx + \int_{G \setminus K} |f_n(x)||\phi(x) - \phi^{**}(x)| \, dx + \delta \varepsilon \tag{15}
\]
\[
\leq \int_K |f_n(x)||\phi(x) - \phi^{**}(x)| \, dx + 2 \int_{G \setminus K} |f_n(x)| \, dx + \delta \varepsilon \tag{16}
\]
\[
\leq \phi \in V, (1 - \delta) \varepsilon \frac{1}{2 \max_{n=1, \ldots, N} \|f_n\|_1} \cdot \int_K |f_n(x)| \, dx + (1 + \delta) \varepsilon \frac{1}{2} \leq \varepsilon.
\]

We conclude that \( V \) is a \( \tau_{ucc} \)-open neighbourhood of (an arbitrarily chosen) \( \phi^{**} \) and \( V \subset \beta_G(U) \). Thus \( \beta_G \) is an open map. \( \Box \)

The question of continuity of \( \beta_G \) is more subtle. In fact, we do not know whether this function is continuous for an arbitrary locally compact abelian group \( G \), but it turns out to be true for very important particular cases. We will come back to this issue in the next section. However, before we do that we wish to investigate the cosine classes a little further.

Our goal is to “compute”\(^\text{17}\) the cosine classes \( \mathcal{COS}(G) \) if \( G = \mathbb{R}, \mathbb{Z}, S^1 \) and \( \mathbb{Z}_n \). We will refer to these families as the canonical cosine classes since the four groups \( \mathbb{R}, \mathbb{Z}, S^1 \) and \( \mathbb{Z}_n \) play a fundamental role in commutative harmonic analysis.

\(^{17}\)By “computing” the cosine class \( \mathcal{COS}(G) \) we mean “finding a (relatively simple) topological space \( T \) which is homeomorphic to \( \mathcal{COS}(G) \)”. 

It is well-known in the literature\(^{18}\) that
\[
\hat{\mathbb{R}} = \left\{ x \mapsto e^{2\pi iyx} : y \in \mathbb{R} \right\},
\]
\[
\hat{\mathbb{Z}} = \left\{ k \mapsto z^k : z \in S^1 \right\},
\]
\[
\hat{S}^1 = \left\{ x \mapsto e^{2\pi ikx} : k \in \mathbb{Z} \right\},
\]
\[
\widehat{\mathbb{Z}}_n = \left\{ k \mapsto e^{\frac{2\pi ilk}{n}} : l \in \mathbb{Z}_n \right\},
\]
so by Theorem 4 we have
\[
\text{COS}(\mathbb{R}) = \left\{ x \mapsto \frac{e^{2\pi iyx} + e^{-2\pi iyx}}{2} = \cos(2\pi yx) : y \in \mathbb{R}_+ \cup \{0\} \right\},
\]
\[
\text{COS}(\mathbb{Z}) = \left\{ k \mapsto \frac{z^k + z^{-k}}{2} : z \in S^1_+ \cup \{1\} \right\},
\]
where \(S^1_+ := \left\{ z \in S^1 : \text{Im}(z) > 0 \right\}\),
\[
\text{COS}(S^1) = \left\{ x \mapsto \frac{e^{2\pi ikx} + e^{-2\pi ikx}}{2} = \cos(2\pi kx) : k \in \mathbb{N}_0 \right\},
\]
\[
\text{COS}(\mathbb{Z}_n) = \left\{ k \mapsto \frac{e^{\frac{2\pi ilk}{n}} + e^{-\frac{2\pi ilk}{n}}}{2} = \cos \left( \frac{2\pi lk}{n} \right) : l \in \mathbb{Z}_{\lceil \frac{n+1}{2} \rceil} \right\}.
\]

We go on to prove that \((\text{COS}(\mathbb{R}), \tau_{ucc})\) is homeomorphic to \(\mathbb{R}_+ \cup \{0\}\) but first we need the following technical lemma:

**Lemma 2.** If \(y_\ast \in \mathbb{R}_+ \cup \{0\}\) and \((y_n) \subset \mathbb{R}_+ \cup \{0\}\) is an unbounded sequence, then for every \(\varepsilon > 0\) the sequence
\[
n \mapsto \sup_{x \in [0, \varepsilon]} |\cos(2\pi y_n x) - \cos(2\pi y_\ast x)|
\]
does not converge to zero.

**Proof.** If \(y_\ast = 0\) then for any \(y_n \geq 1\) the function \(x \mapsto \cos(2\pi y_n x)\) has period \(T_n = \frac{1}{y_n} \leq 1\) so
\[
\forall y_n \geq 1 \sup_{x \in [0, 1]} |\cos(2\pi y_n x) - 1| \geq 2,
\]
which ends the proof. Therefore, suppose that \(y_\ast \neq 0\) and put \(K := [0, \frac{1}{8y_\ast}]\). Then
\[
\forall x \in K \cap [0, \varepsilon] \cos(2\pi y_\ast x) \geq \frac{\sqrt{2}}{2}
\]

\(18\)See Proposition 7.1.6 in [12], p. 106 or Theorem 4.5 in [17], p. 89.
whereas for any \( y_n = \max\left(4y_*, \frac{1}{2\varepsilon}\right) \) the function \( x \mapsto \cos(2\pi y_n x) \) has period \( T_n = \frac{1}{y_n} \leq \min\left(\frac{1}{4y_*}, 2\varepsilon\right) \) and thus attains the value \(-1\) on \( K \cap [0, \varepsilon] \). Consequently, we have

\[
\forall y_n \geq \max\left(4y_*, \frac{1}{2\varepsilon}\right) \sup_{x \in K \cap [0, \varepsilon]} |\cos(2\pi y_n x) - \cos(2\pi y_* x)| \geq 1 + \frac{\sqrt{2}}{2},
\]

which concludes the proof. \(\square\)

**Theorem 6.** The function \( \mathcal{H}_\mathbb{R} : (COS(\mathbb{R}), \tau_{ucc}) \rightarrow \mathbb{R}_+ \cup \{0\} \) given by the formula

\[
\mathcal{H}_\mathbb{R}(x \mapsto \cos(2\pi yx)) := y
\]

is a homeomorphism.

*Proof.* It is easy to check that \( \mathcal{H}_\mathbb{R} \) is a bijection, so we focus on the topological properties of this function. By Proposition 1.2 in [24], p. 152 the space \((COS(\mathbb{R}), \tau_{ucc})\) is second-countable and thus, by Theorem 1.6.14 in [16], p. 53 it is sequential. This means that for \( \mathcal{H}_\mathbb{R} \) to be a homeomorphism it is necessary and sufficient that \( \mathcal{H}_\mathbb{R}^{-1} \) satisfies

\[
y_n \rightarrow y_* \iff \mathcal{H}_\mathbb{R}^{-1}(y_n) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{R}^{-1}(y_*).
\]

First, suppose that \((y_n) \subset \mathbb{R}_+ \cup \{0\}\) is a sequence convergent to \( y_* \). Then

\[
\forall x \in \mathbb{R} \quad |\cos(2\pi y_n x) - \cos(2\pi y_* x)| = 2\pi|x| \left|\int_{y_*}^{y_n} \sin(2\pi yx)dy\right| \leq 2\pi|x||y_n - y_*|,
\]

so for every compact \( K \subset \mathbb{R} \) we have

\[
\sup_{x \in K} |\cos(2\pi y_n x) - \cos(2\pi y_* x)| \rightarrow 0
\]
as \( n \rightarrow \infty \). Consequently, \( \mathcal{H}_\mathbb{R}^{-1}(y_n) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{R}^{-1}(y_*) \) as desired.

For the reverse implication (i.e., \("\iff\"\)) we suppose that \((y_n) \subset \mathbb{R}_+ \cup \{0\}\) is a sequence such that \( \mathcal{H}_\mathbb{R}^{-1}(y_n) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{R}^{-1}(y_*) \) for some \( y_* \in \mathbb{R}_+ \cup \{0\} \). By Lemma 2 the sequence \((y_n)\) is necessarily bounded, so using the Bolzano-Weierstrass theorem there exists a convergent subsequence \((y_{n_k})\). If \( y_{**} \) denotes the limit of this subsequence, then by the first part of the reasoning we have \( \mathcal{H}_\mathbb{R}^{-1}(y_{n_k}) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{R}^{-1}(y_{**}) \). Since \( \tau_{ucc} \) is a Hausdorff topology then it follows that \( \mathcal{H}_\mathbb{R}^{-1}(y_{**}) = \mathcal{H}_\mathbb{R}^{-1}(y_*), \) which in turn implies the equality \( y_{**} = y_* \). Since the reasoning works for an arbitrary choice of the subsequence we have \( y_n \rightarrow y_* \), which concludes the proof. \(\square\)

Let us prove a corresponding result for the cosine class \(COS(\mathbb{Z})\):

**Theorem 7.** The function \( \mathcal{H}_\mathbb{Z} : (COS(\mathbb{Z}), \tau_{ucc}) \rightarrow \mathbb{S}_1^+ \cup \{1\} \) given by the formula

\[
\mathcal{H}_\mathbb{Z}(k \mapsto \frac{z^k + z^{-k}}{2}) := z
\]
is a homeomorphism.
Proof. Arguing as in Theorem 6 it is enough to prove that
\[ z_n \rightarrow z_\ast \iff \mathcal{H}_\mathbb{Z}^{-1}(z_n) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{Z}^{-1}(z_\ast). \]

First, suppose that \((z_n) \subset S_+^1 \cup \{1\}\) is a sequence convergent to \(z_\ast\). Then
\[
\forall k \in \mathbb{Z}, \quad \left| \frac{z_n^k + z_n^{-k}}{2} - \frac{z_\ast^k + z_\ast^{-k}}{2} \right| \leq \frac{|z_n^k - z_\ast^k| + |z_n^{-k} - z_\ast^{-k}|}{2} \leq k \cdot \frac{|z_n - z_\ast| + |z_n^{-1} - z_\ast^{-1}|}{2} = k |z_n - z_\ast|,
\]
so for every compact (i.e., finite) \(K \subset \mathbb{Z}\) we have
\[
\sup_{k \in K} \left| \frac{z_n^k + z_n^{-k}}{2} - \frac{z_\ast^k + z_\ast^{-k}}{2} \right| \rightarrow 0
\]
as \(n \rightarrow \infty\). Consequently, \(\mathcal{H}_\mathbb{Z}^{-1}(z_n) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{Z}^{-1}(z_\ast)\) as desired.

For the reverse implication (i.e., \(\iff\)) we suppose that \((z_n) \subset S_+^1 \cup \{1\}\) is a sequence such that \(\mathcal{H}_\mathbb{Z}^{-1}(z_n) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{Z}^{-1}(z_\ast)\) for some \(z_\ast = e^{2\pi i \alpha_\ast} \in S_+^1 \cup \{1\}\). Moreover, let \((\alpha_\ast) \subset [0, \frac{1}{2})\) be such that \(z_n = e^{2\pi i \alpha_n}\). Since the sequence \((\alpha_n)\) is bounded, then using the Bolzano-Weierstrass theorem there exists a convergent subsequence \((\alpha_{n_k})\). If \(\alpha_{n_k}\) denotes the limit of this subsequence, then \(z_{n_k} = e^{2\pi i \alpha_{n_k}} \rightarrow z_\ast = e^{2\pi i \alpha_\ast}\) and by the first part of the reasoning we have \(\mathcal{H}_\mathbb{Z}^{-1}(z_{n_k}) \rightarrow_{\tau_{ucc}} \mathcal{H}_\mathbb{Z}^{-1}(z_\ast)\). We conclude the proof just as in Theorem 6. \(\square\)

In a similar vein (with even simpler proofs) one can prove analogous results for the remaining two cosine classes:

Theorem 8. The functions
\[
\mathcal{H}_{\mathbb{S}^1}: (COS(S^1), \tau_{ucc}) \rightarrow \mathbb{N}_0, \quad \mathcal{H}_{\mathbb{S}^1}(x \mapsto \cos(2\pi k x)) := k,
\]
\[
\mathcal{H}_{\mathbb{Z}^1_n}: (COS(\mathbb{Z}_n), \tau_{ucc}) \rightarrow \mathbb{Z}_{\left\lfloor \frac{n+1}{2} \right\rfloor}, \quad \mathcal{H}_{\mathbb{Z}^1_n}(k \mapsto \cos \left( \frac{2\pi l k}{n} \right)) := l,
\]
are homeomorphisms.

4. Canonical Cosine Structure Spaces and Transforms

In the previous section we have investigated the canonical cosine classes and found relatively simple spaces to which they are homeomorphic. We have also shown (see Theorem 5) that \(\beta_G: \Delta(L^1(G), \tau^*) \rightarrow (COS(G), \tau_{ucc})\) is always an open map. This raises a natural question: can we compute the canonical cosine structure spaces \(\Delta(L^1(\mathbb{R}), \ast_c), \Delta(L^1(\mathbb{Z}), \ast_c), \Delta(L^1(S^1), \ast_c)\) and \(\Delta(L^1(\mathbb{Z}_n), \ast_c)\)?\(^{19}\) A major part of the present section is devoted to answering this question affirmatively.

Theorem 9. \((\Delta(L^1(\mathbb{R}), \ast_c), \tau^*)\) is homeomorphic to \(\mathbb{R}_+ \cup \{0\}\).

\(^{19}\)Again, by “compute” we mean “find a topological space \(T\), which is homeomorphic to \(\Delta(L^1(G), \ast_c)\)."
Proof. By Theorem 5 we know that \( \beta_{\mathbb{R}} : (\Delta(L^1(\mathbb{R}), \tau^*), \tau^*) \rightarrow (\text{COS}(\mathbb{R}), \tau_{\text{ucc}}) \) is an open map and by Theorem 6 the function \( \mathcal{H}_{\mathbb{R}} : (\text{COS}(\mathbb{R}), \tau_{\text{ucc}}) \rightarrow \mathbb{R}_+ \cup \{0\} \) is a homeomorphism. Consequently, it suffices to prove that \( \mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}} \) is continuous. To this end we fix \( y_* \in \mathbb{R}_+ \cup \{0\} \) as well as its arbitrary open neighbourhood

\[
U_\varepsilon := \left\{ y \in \mathbb{R}_+ \cup \{0\} : |y - y_*| < \varepsilon \right\}
\]

where \( \varepsilon > 0 \). Our task is to prove that

\[
(\mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}})^{-1}(U_\varepsilon) = \left\{ m \in \Delta(L^1(\mathbb{R}), \star_c) : |\mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}}(m) - y_*| < \varepsilon \right\}
\]

is weak* open and we do it by fixing an arbitrary element \( m_{**} \in (\mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}})^{-1}(U_\varepsilon) \) and constructing a weak* open set \( W_{**} \) such that

\[
m_{**} \in W_{**} \subset (\mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}})^{-1}(U_\varepsilon).
\]

To begin with, since \( m_{**} \in (\mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}})^{-1}(U_\varepsilon) \) then \( y_{**} := \mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}}(m_{**}) \) satisfies \( |y_{**} - y_*| < \delta \varepsilon \) for some \( \delta \in [0, 1) \). Further reasoning depends on whether \( y_{**} \) is zero or not:

- If \( y_{**} \neq 0 \) then we define a function \( g : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R} \) with the formula

\[
g(z) := \frac{1}{2\pi y_{**}} \cdot \left( \frac{\pi}{2} \cdot \text{sinc} \left( \frac{\pi}{2} \cdot z \right) - 1 \right).
\]

Its crucial property is that there exists \( \eta > 0 \) such that

\[
\forall z \in \mathbb{R}_+ \cup \{0\} \quad |g(z)| < \eta \implies |z - 1| < \frac{(1 - \delta)\varepsilon}{y_{**}}. \tag{17}
\]

We claim that

\[
W_{**} := \left\{ m \in \Delta(L^1(\mathbb{R}), \star_c) : |m \left( \mathbbm{1}_{[0, \frac{1}{y_{**}}]} \right) - m_{**} \left( \mathbbm{1}_{[0, \frac{1}{y_{**}}]} \right)| < \eta \right\}
\]

is the desired weak* open neighbourhood of \( m_{**} \). Indeed, we have

\[
\forall m \in \Delta(L^1(\mathbb{R}), \star_c) \quad m \left( \mathbbm{1}_{[0, \frac{1}{y_{**}}]} \right) - m_{**} \left( \mathbbm{1}_{[0, \frac{1}{y_{**}}]} \right) = \int_0^{\frac{1}{y_{**}}} \cos(2\pi yx) - \cos(2\pi y_{**}x)dx
\]

\[
= \int_0^{\frac{1}{y_{**}}} \frac{1}{2\pi y_{**}} \cdot \left( \frac{\pi}{2} \cdot \cos \left( \frac{\pi}{2} \cdot y \right) \right) - \cos(x)dx
\]

\[
= \frac{1}{2\pi y_{**}} \cdot \left( \frac{\pi}{2} \cdot \text{sinc} \left( \frac{\pi}{2} \cdot \frac{y}{y_{**}} \right) - 1 \right)
\]

\[
= g \left( \frac{y}{y_{**}} \right).
\]

where \( y = \mathcal{H}_{\mathbb{R}} \circ \beta_{\mathbb{R}}(m) \). If \( m \in W_{**} \) then \( |g \left( \frac{y}{y_{**}} \right)| < \eta \), so by (17) we have \( |y - y_{**}| < (1 - \delta)\varepsilon \). Finally, we have

\[
\forall m \in W_{**} \quad |y - y_*| \leq |y - y_{**}| + |y_{**} - y_*| < \delta \varepsilon + (1 - \delta)\varepsilon = \varepsilon,
\]
Its crucial property is that there exists $\eta > 0$.

**Theorem 10.** $(\ell^1(\mathbb{Z}), \ast_c, \tau^*)$ is homeomorphic to $S^1_+ \cup \{1\}$.

**Proof.** As in Theorem 9 we argue that it is sufficient to prove that $\mathcal{H}_\ell \circ \beta_\ell$ is continuous so we choose an arbitrary open neighbourhood

$$W_\varepsilon := \left\{ z \in S^1_+ \cup \{1\} : |z - z_*| < \varepsilon \right\}$$

of a fixed element $z_* \in S^1_+ \cup \{1\}$. Our task is to prove that

$$(\mathcal{H}_\ell \circ \beta_\ell)^{-1}(W_\varepsilon) = \left\{ m \in \Delta(\ell^1(\mathbb{Z}), \ast_c) : |\mathcal{H}_\ell \circ \beta_\ell(m) - z_*| < \varepsilon \right\}$$

is weak* open and we do it by fixing an arbitrary element $m_{**} \in (\mathcal{H}_\ell \circ \beta_\ell)^{-1}(U_\varepsilon)$ and constructing a weak* open set $W_{**}$ such that

$$m_{**} \in W_{**} \subset (\mathcal{H}_\ell \circ \beta_\ell)^{-1}(U_\varepsilon).$$

To begin with, since $m_{**} \in (\mathcal{H}_\ell \circ \beta_\ell)^{-1}(U_\varepsilon)$ then $z_{**} := \mathcal{H}_\ell \circ \beta_\ell(m_{**})$ satisfies $|z_{**} - z_*| < \delta \varepsilon$ for some $\delta \in (0, 1)$. We define a function $g : S^1_+ \cup \{1\} \rightarrow \mathbb{R}$ with the formula

$$g(z) := \frac{z + z^{-1}}{2} - \frac{z_{**} + z_{**}^{-1}}{2}.$$

Its crucial property is that there exists $\eta > 0$ such that

$$\forall z \in S^1_+ \cup \{1\} \ |g(z)| < \eta \implies |z - z_{**}| < (1 - \delta)\varepsilon. \quad (19)$$

We claim that

$$W_{**} := \left\{ m \in \Delta(\ell^1(\mathbb{Z}), \ast_c) : \left| m(\mathbb{1}_{\{1\}}) - m_{**}(\mathbb{1}_{\{1\}}) \right| < \eta \right\}$$
is the desired weak* open neighbourhood of $m_{**}$. Indeed, we have
\[
\forall m \in \Delta(L^1(S^1), \tau^*) \quad m \left( \mathbf{1}_{(1)} \right) - m_{**} \left( \mathbf{1}_{(1)} \right) = \sum_{k \in \mathbb{Z}} \mathbf{1}_{(1)}(k) \cdot \left( \frac{z^k + z^{-k}}{2} - \frac{z_{**}^k + z_{**}^{-k}}{2} \right) = \frac{z + z^{-1}}{2} - \frac{z_{**} + z_{**}^{-1}}{2} = g(z),
\]
where $z = \Delta_{S^1} \circ \beta_{S^1}(m)$. If $m \in W_{**}$ then $|g(z)| < \eta$, so by (19) we have $|z - z_{**}| < (1 - \delta)\varepsilon$. Finally, we have
\[
\forall m \in W_{**} \quad |z - z_{**}| \leq |z - z_{**}| + |z_{**} - z| < \delta \varepsilon + (1 - \delta)\varepsilon = \varepsilon,
\]
which proves that $m_{**} \in W_{**} \subset (\Delta_{S^1} \circ \beta_{S^1})^{-1}(U_{\varepsilon})$. We conclude the reasoning as in Theorem 9. \hfill \square

**Theorem 11.** $(\Delta(L^1(S^1), \tau^*), \tau^*)$ is homeomorphic to $\mathbb{N}_0$.

**Proof.** As in Theorem 9 we argue that it is sufficient to prove that $\Delta_{S^1} \circ \beta_{S^1}$ is continuous. Since the topology on $\mathbb{N}_0$ is discrete, then we have to show that
\[
\{ m_* \} := (\Delta_{S^1} \circ \beta_{S^1})^{-1}(\{ k_* \})
\]
is weak* open for every $k_* \in \mathbb{N}_0$. Further reasoning depends on whether $k_*$ is zero or not:

- If $k_* \neq 0$ then we define a function $g : \mathbb{N}_0 \rightarrow \mathbb{R}$ with the formula
  \[
g(k) := \frac{1}{2\pi k_*} \cdot \left( \pi \cdot \text{sinc} \left( \frac{\pi}{2} \cdot \frac{k}{k_*} \right) - 1 \right).
  \]
  Its crucial property is that there exists $\eta > 0$ such that
  \[
  \forall k \in \mathbb{N}_0 \quad |g(k)| < \eta \implies k = k_*.
  \]
  We claim that
  \[
  W_* := \left\{ m \in \Delta(L^1(S^1), \tau^*) : m \left( \mathbf{1}_{[0, \frac{1}{4k_*}]} \right) - m_* \left( \mathbf{1}_{[0, \frac{1}{4k_*}]} \right) < \eta \right\}
  \]
is the desired weak* open neighbourhood of $m_*$. Indeed, we have
  \[
  \forall m \in \Delta(L^1(S^1), \tau^*) \quad m \left( \mathbf{1}_{[0, \frac{1}{4k_*}]} \right) - m_* \left( \mathbf{1}_{[0, \frac{1}{4k_*}]} \right) = \int_0^{\frac{1}{4k_*}} \cos(2\pi k x) - \cos(2\pi k_* x) \, dx = \frac{1}{2\pi k_*} \cdot \int_0^{\frac{1}{2k_*}} \cos \left( \frac{k}{k_*} \cdot x \right) - \cos(x) \, dx = \frac{1}{2\pi k_*} \cdot \left( \frac{\pi}{2} \cdot \text{sinc} \left( \frac{\pi}{2} \cdot \frac{k}{k_*} \right) - 1 \right) = g(k),
  \]
where $k = \Delta_{S^1} \circ \beta_{S^1}(m)$. If $m \in W_*$ then $|g(k)| < \eta$, so by (20) we have $k = k_*$. This proves that
\[
\{ m_* \} = W_* = (\Delta_{S^1} \circ \beta_{S^1})^{-1}(\{ k_* \}).
\]
• If \( k_\ast = 0 \) then we define a function \( g : \mathbb{N}_0 \rightarrow \mathbb{R} \) with the formula
\[
g(k) := \text{sinc}(2\pi k) - 1.
\]
Its crucial property is that there exists \( \eta > 0 \) such that
\[
\forall k \in \mathbb{N}_0 \ |g(k)| < \eta \implies k = 0.
\] (21)

We claim that
\[
W_\ast := \left\{ m \in \Delta(L^1(S^1), \star_c) : |m(1_{S^1}) - m_\ast(1_{S^1})| < \eta \right\}
\]
is the desired weak* open neighbourhood of \( m_\ast \). Indeed, we have
\[
\forall m \in \Delta(L^1(S^1), \star_c) m(1_{S^1}) - m_\ast(1_{S^1}) = \int_0^1 \cos(2\pi kx) - 1 \, dx = \text{sinc}(2\pi k) - 1 = g(k).
\]

If \( m \in W_\ast \) then \( |g(k)| < \eta \), so by (21) we have \( k = 0 \). This proves that
\[
\{m_\ast\} = W_\ast = (\mathcal{F}_{S^1} \circ \beta_{S^1})^{-1}(\{0\}),
\]
which concludes the proof.

\begin{theorem}
(\Delta(\ell^1(\mathbb{Z}_n), \star_c), \tau^\ast) is homeomorphic to \( \mathbb{Z}_{\lceil \frac{n+1}{2} \rceil} \), where \( x \mapsto \lceil x \rceil \) is the ceiling function.
\end{theorem}

\begin{proof}
As in Theorem 11 our task is to prove that
\[
\{m_\ast\} := (\mathcal{F}_{\mathbb{Z}_n} \circ \beta_{\mathbb{Z}_n})^{-1}(\{k_\ast\})
\]
is a weak* open neighbourhood of \( m_\ast \) for every \( k_\ast \in \mathbb{Z}_n \). We define a function \( g : \mathbb{Z}_{\lceil \frac{n+1}{2} \rceil} \rightarrow \mathbb{R} \) with the formula
\[
g(k) := \cos\left(\frac{2\pi k}{n}\right) - \cos\left(\frac{2\pi k_\ast}{n}\right).
\]
Its crucial property is that there exists \( \eta > 0 \) such that
\[
\forall k \in \mathbb{Z}_n \ |g(k)| < \eta \implies k = k_\ast.
\] (22)

We claim that
\[
W_\ast := \left\{ m \in \Delta(\ell^1(\mathbb{Z}_n), \star_c) : |m(1_{\mathbb{Z}_n}) - m_\ast(1_{\mathbb{Z}_n})| < \eta \right\}
\]
is the desired weak* open neighbourhood of \( m_\ast \). Indeed, we have
\[
\forall m \in \Delta(\ell^1(\mathbb{Z}_n), \star_c) m(1_{\mathbb{Z}_n}) - m_\ast(1_{\mathbb{Z}_n}) = \cos\left(\frac{2\pi k}{n}\right) - \cos\left(\frac{2\pi k_\ast}{n}\right) = g(k),
\]
where \( k = \mathcal{F}_{\mathbb{Z}_n} \circ \beta_{\mathbb{Z}_n}(m) \). If \( m \in W_\ast \) then \( |g(k)| < \eta \), so by (22) we have \( k = k_\ast \). This proves that
\[
\{m_\ast\} = W_\ast = (\mathcal{F}_{\mathbb{Z}_n} \circ \beta_{\mathbb{Z}_n})^{-1}(\{k_\ast\}),
\]
which concludes the proof.
\end{proof}
The last four theorems can be summarized as follows:

**Theorem 13.** The cosine structure space $\Delta(L^1(G), \star_c)$ is homeomorphic to:
- $\mathbb{R}_+ \cup \{0\}$ if $G = \mathbb{R}$,
- $S^1_+ \cup \{1\}$ if $G = \mathbb{Z}$,
- $\mathbb{N}_0$ if $G = S^1$,
- $\mathbb{Z}_{\lceil n+1 \rceil}$ if $G = \mathbb{Z}_n$.

It is high time we reaped what we have sown and enjoyed the fruits of our labour. Due to Theorem 13 we know that
- $C_0(\Delta(L^1(\mathbb{R}), \star_c))$ is homeomorphic to $C_0(\mathbb{R}_+ \cup \{0\})$,
- $C_0(\Delta(\ell^1(\mathbb{Z}), \star_c))$ is homeomorphic to $C_0(S^1_+ \cup \{1\})$,
- $C_0(\Delta(L^1(S^1), \star_c))$ is homeomorphic to $C_0(\mathbb{N}_0)$,
- $C_0(\Delta(\ell^1(\mathbb{Z}_n), \star_c))$ is homeomorphic to $C_0(\mathbb{Z}_{\lceil n+1 \rceil}) = C(\mathbb{Z}_{\lceil n+1 \rceil}) = C(\mathbb{Z}_{\lceil n+1 \rceil})$.

Hence, the Gelfand transform $\hat{f} \in C_0(\Delta(L^1(G), \star_c))$ of a function $f \in L^1(G)$ manifests itself as
- the **classical cosine transform**
  \[
  \forall y \in \mathbb{R}_+ \cup \{0\} \quad \hat{f}(y) = \int_{\mathbb{R}} f(x) \cos(2\pi y x) dx,
  \]
  if $G = \mathbb{R}$,
- the **discrete-time cosine transform**
  \[
  \forall z \in S^1_+ \cup \{1\} \quad \hat{f}(z) = \sum_{k \in \mathbb{Z}} f(k) \cdot \frac{z^k + z^{-k}}{2},
  \]
  if $G = \mathbb{Z}$,
- the $k$-th cosine coefficient in the Fourier series
  \[
  \forall k \in \mathbb{N}_0 \quad \hat{f}(k) = \int_0^1 f(x) \cos(2\pi k x) dx,
  \]
  if $G = S^1$,
- the **discrete cosine transform**
  \[
  \forall l \in \mathbb{Z}_{\lceil n+1 \rceil} \quad \hat{f}(l) = \sum_{k=1}^{n} f(k) \cos \left( \frac{2\pi lk}{n} \right),
  \]
  if $G = \mathbb{Z}_n$.

**Epilogue**

Our journey has come to an end and it is instructive to pause one last time and, with the benefit of hindsight, reflect on how far we have travelled and what lies ahead. Last section taught us that $\Delta(L^1(G), \star_c)$ is (homeomorphic to) a relatively simple topological space if $G = \mathbb{R}, \mathbb{Z}, S^1$ or $\mathbb{Z}_n$. As a result,
we rediscovered the cosine transforms as special manifestations of the Gelfand transform. However, one would be wrong thinking that the topic has been exhausted. Four homeomorphisms, which appear in Theorem 13, force all four functions $\beta_R, \beta_Z, \beta_S_1$ and $\beta_Z_n$ to be homeomorphisms as well. This raises the very natural question: is it true that $\beta_G : \Delta(L^1(G), \ast_c) \rightarrow \text{COS}(G)$ is a homeomorphism for every locally compact abelian group $G$? Unfortunately, despite our best efforts we were not able to answer that question. Thus, we leave it as an open problem with the intention of stimulating future research in the fascinating field of cosine transforms.

Acknowledgements

We would like to express our deep gratitude for the anonymous Reviewer, whose comments and suggestions helped us improve the quality of the paper. We feel privileged to be provided with such an insightful feedback and constructive criticism.

Funding Not applicable.

Availability of data and material Not applicable.

Declarations

Code availability Not applicable.

Conflict of interest The author declares that he has no conflict of interest.

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Received: July 16, 2021.
Accepted: January 29, 2022.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.