Real-Smooth Hypersurfaces in $\mathbb{C}^{N+1}$

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Abstract. We define Pseudo-Weighted Fischer Spaces of Normalizations with respect to Pseudo-Weighted Versions of the Fischer Decomposition, which are defined iteratively with respect to a well-chosen System of Pseudo-Weights. In particular, we construct a formal normal form for a class of Real-Smooth Hypersurfaces in $\mathbb{C}^{N+1}$.

1. Introduction and Main Result

The problem of finding normal forms is well known in Complex Analysis\cite{17}. The formal constructions of normal forms\cite{1, 2, 6, 10, 11, 13, 18} represent useful procedures in order to understand fundamental problems in Complex Analysis, such as the local equivalence problem\cite{7, 12}. Several problems, including classification problems related to such Domains, are reduced to the study of the Real-Hypersurfaces, which are generally considered Smooth, or equivalently Formal. The Real-Hypersurfaces are Real Submanifolds of codimension 1 in Complex Space. They represent boundaries of Domains in Complex Spaces.

In the equidimensional case, such procedures (see \cite{21}) is based on imposing (formally) normalization conditions in the local defining equations determining simultaneously the formal (holomorphic) equivalence, aiming to simplify the local defining equations and especially to find invariants (like Huang-Yin\cite{12}). Such construction may be simple, like Moser-Webster’s Normal Form\cite{18} which is algebraic, or more complicated as the author’s Normal Forms\cite{1, 2}. Furthermore, such construction may be considered in Almost-Complex Spaces without any assumption of integrability. First steps, towards such normal forms, have been done by my supervisor\cite{21}, constructing an analogue of Chern-Moser’s Normal Form. Regarding the convergence or the divergence of the normal forms, there are recommended Gong-Stolovitch\cite{10, 11} and Lamel-Stolovitch\cite{16} to the reader for further reading.

In the non-equidimensional case, such procedure (see \cite{21}) aims to provide classifications of the formal (holomorphic) mappings between Models in Complex Spaces of different dimensions. It is based on compositions using suitable automorphisms of such Models, in order to find normal forms for possibly formal mappings, but sometimes it suffices to work with such mappings just of class $C^2$ or $C^3$ like Faran and Huang, in order to obtain simple classifications, which are directly motivated by the classification of the Proper Holomorphic Mappings between unit balls in Complex Spaces.

In this paper, we construct a formal normal form for a large class of Real-Formal Hypersurfaces in Complex Space. This normal form may be seen as an alternative to the previous constructed normal forms from Zaitsev\cite{21} and Chern-Moser\cite{6}. The imposed normalizations iteratively cover sums of homogeneous terms respecting a system of weights or pseudo-weighted motivated by\cite{2}. In particular, we obtain the 2-jet determination of the automorphisms of a real-smooth strongly pseudoconvex in $\mathbb{C}^{N+1}$, in the coordinates

\[(w, z) = (w; z_1, z_2, \ldots, z_N) \in \mathbb{C}^{N+1}.\]

Let $M \subset \mathbb{C}^{N+1}$ be a real-formal hypersurface defined as follows

\[(1.1) \quad \text{Im} w = (\text{Re} w)^{m} P(z, \overline{z}) + O(k_0 + 1),\]

satisfying the following non-degeneracy condition

\[(1.2) \quad \sum_{k, l=1}^{N} \frac{\partial P(z, \overline{z})}{\partial z_k} a_{k l} z_l \rightarrow a_{k l} = 0, \quad \text{for all } k, l = 1, \ldots, N,\]

such that it holds:

\[(1.3) \quad \text{Deg} (P) + s = k_0, \quad \text{for Deg} (P), m \in \mathbb{N}.\]

Then, in order to construct normal forms, we consider the following Model

\[(1.4) \quad \text{Im} w = (\text{Re} w)^{s} P(z, \overline{z}),\]

when \[(1.1)\] holds, but it is required by \[(2)\] to make homogeneous this Model \[(1.4)\], regardless of its non-triviality. Then, we use by \cite{2, 4} the following system of pseudo-weights, according to the following notations

\[(1.5) \quad x = \text{Re} w, \quad P(z, \overline{z}) = \sum_{\alpha_k + \beta_l = k_0 - s} P_{\alpha_k \beta_l} z_k \overline{z_l}^{\beta_l}.\]

Keywords: C-R. Geometry, Equivalence Problem, Normal Norm, Real-Hypersurface, Transversality, Algebraicity, Formal Power Series.

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We define
\[ \text{wt}\{x\} = k_0, \quad \text{wt}\{x_k\} = 1, \quad \text{wt}\{\bar{x}_k\} = 1, \quad \text{for all } k = 1, \ldots, N. \]

We define
\[ \text{wt}\{z^\alpha \bar{x}^\beta\} = \alpha + \beta, \quad \text{for all } \alpha, \beta \in \mathbb{N}. \]

Similarly, we define
\[ \text{wt}\{z^\alpha x^\beta\} = \alpha + \beta, \quad \text{for all } \alpha, \beta \in \mathbb{N} \text{ with } \alpha \neq 0. \]

Similarly, we define
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eq 0. \]

Next, we define
\[ \text{wt}\{x^N z^\alpha \bar{x}^\beta\} = \begin{cases} N + \alpha + \beta, & \text{for all } N, \alpha, \beta \in \mathbb{N} \text{ with } \alpha + \beta < k_0 - s \text{ and } \alpha \neq 0 \text{ or } \beta \neq 0, \\ N + s + \alpha + \beta, & \text{for all } N, \alpha, \beta \in \mathbb{N} \text{ with } \alpha + \beta = k_0 - s \text{ and } \alpha \neq 0 \text{ and } \beta \neq 0, \\ (N - (s - 1)\beta) k_0, & \text{for all } N, \alpha, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 - s \text{ and } (N - (s - 1)\beta) k_0 \geq 0. \end{cases} \]

extending (1.7), (1.8) and (1.9).

Now, in order to provide further definitions, we observe that the homogeneous polynomial $L$ of degree $k_0 - s$, is defined by monomials $z^\alpha x^\beta$, with $a + b = k_0 - s$ with $a, b \in \mathbb{N}^*$.

Then, we define
\[ \text{wt}\{x^N z^\alpha \bar{x}^\beta\} = (N - (s - 1)\beta) k_0, \quad \text{for all } N, \alpha, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 - s \text{ and } (N - (s - 1)\beta) k_0 \geq 0. \]

Therefore, it is required to define
\[ \begin{align*}
\text{wt}\{x^N z^\alpha \bar{x}^\beta\} &= (N - (s - 1)\beta) k_0 + c, \quad \text{for all } N, c, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 - s \text{ and } (N - (s - 1)\beta) k_0 \geq 0, \\
\text{wt}\{x^N z^\alpha \bar{x}^\beta\} &= (N - (s - 1)\beta) k_0 + c, \quad \text{for all } N, c, \beta, a, b \in \mathbb{N} \text{ with } a + b = k_0 - s \text{ and } (N - (s - 1)\beta) k_0 \geq 0.
\end{align*} \]

Otherwise, when $(N - (s - 1)\beta) k_0 \leq 0$, we define
\[ \text{wt}\{x^N z^\alpha \bar{x}^\beta\} = (N - (s - 1)\beta) k_0 + (a + b) (\beta - \beta'), \]
for all $N, a, b \in \mathbb{N}$ with $a + b = k_0$, and $\beta' \in \mathbb{N}$ maximal such that $(N - (s - 1)\beta') k_0 \geq 0$.

Clearly, the best definition is attained when the right-hand side of (1.13) is minimal, more precisely when $\beta' \in \mathbb{N}$ is maximal satisfying the above property, because there exist more evaluations available.

We denote them by
\[ \text{wt}_{N,a,b}\{x^N z^\alpha \bar{x}^\beta\}, \quad \text{where } N, a, b \in \mathbb{N} \text{ and } a, b \neq 0. \]

Therefore, the best definition is clearly the following
\[ \text{wt}\{x^N z^\alpha \bar{x}^\beta\} = \text{Min}\left( \text{wt}_{N,a,b}\{x^N z^\alpha \bar{x}^\beta\} \right), \quad \text{where } N, a, b \in \mathbb{N} \text{ and } a, b \neq 0. \]

Now, the Model (1.3) becomes pseudo-weighted-homogeneous in respect to the system of pseudo-weights from (1.7), (1.8), (1.9), (1.10), (1.11), (1.13), (1.14), (1.15). Then, we consider the following weighted Fischer Decompositions
\[ z^I = (x + ix^s P(z, x)) A_I(z, \bar{x}, x) + B_{a}(z, \bar{x}, x), \quad \text{where } \text{tr} \left( B_I(z, \bar{x}, x) \right) = 0; \]
\[ x^I z^J = (x + ix^s P(z, \bar{x})) A_{J}(z, \bar{x}, x) + B_{J}(z, \bar{x}, x), \quad \text{where } \text{tr} \left( B_{J}(z, \bar{x}, x) \right) = 0, \]
for all $I, J \in \mathbb{N}^N$ such that $|I| = k$ and $|J| = k - 2$, for all $l = 1, \ldots, N$ and $k \in \mathbb{N} - \{1\}$, where $\text{tr}$ is the associated pseudo-weighted differential operator, for all $I, J \in \mathbb{N}^N$ such that $|I| = k$ and $|J| = k - 1$, for all $l = 1, \ldots, N$ and $k \in \mathbb{N} - \{1\}$, according to the following standard notations
\[ I = (i_1, i_2, \ldots, i_N), \quad |I| = i_1 + i_2 + \cdots + i_N; \]
\[ J = (j_1, j_2, \ldots, j_N), \quad |J| = j_1 + j_2 + \cdots + j_N. \]

Next, we focus on the following family of polynomials
\[ \{\bar{B}_{J}(z, \bar{x}, x), \quad B_I(z, \bar{x}, x), \quad B_I(z, \bar{x}, x), \quad B_{J}(z, \bar{x}, x)\}_{I,J \in \mathbb{N}^N - \{1\}}, \]
which are linearly independent, and therefore we can apply the methods from [2]. Then, it has sense to consider the following iterative Spaces of Fischer Normalizations denoted as follows
\[ \mathcal{F}_p, \quad \text{where } p \in \mathbb{N}^* - \{1\}, \]
which consist in real-valued polynomials $P(z, x, x) = P_0(z, x, x)$ of weighted degree $p$ in $(z, \bar{x}, x)$ such that:
\[ P_k(z, \bar{x}, x) = P_{k+1}(z, \bar{x}, x) (x + i \langle z, x \rangle) + R_{k+1}(z, \bar{x}, x), \quad \text{for all } k = 0, \ldots, [\frac{p}{2}], \]
where we have
\[ R_{k+1}(z, \bar{x}, x) \in \bigcap_{1 \leq |I| \leq k, \quad 1 \leq |J| \leq k - 1} \ker \left( B_{J}(z, \bar{x}, x) \right) \cap \ker \left( B_I(z, \bar{x}, x) \right) \cap \ker \left( B_I(z, \bar{x}, x) \right) \cap \ker \left( B_I(z, \bar{x}, x) \right). \]
We obtain:

**Theorem 1.1.** Let \( M \subset C^{N+1} \) be a real-formal hypersurface defined as follows

\[
(1.20) \quad \text{Im} w = (\text{Re} w)^m P(z, \bar{w}) + \sum_{k \geq 3} \phi_k (z, \bar{w}, \text{Re} w),
\]

where \( \phi_k (z, \bar{w}, \text{Re} w) \) is a polynomial of degree \( k \) in \( (z, \bar{w}, x) \) for all \( k \geq 3 \).

Then, there exists a unique formal transformation of the following type

\[
(1.21) \quad (F(z, w), G(z, w)) = (z + O(2), w + O(2)),
\]

which transforms \( M \) into the following normal form \( M' \subset C^{N+1} \) defined as follows

\[
(1.22) \quad \text{Im} w' = (\text{Re} w')^m P'(z', \bar{w'}) + \sum_{k \geq 3} \phi'_k (z', \bar{w'}, \text{Re} w'),
\]

where \( \phi'_k (z', \bar{w'}, \text{Re} w') \) is a polynomial of degree \( k \) in \( (z', \bar{w'}, \text{Re} w') \) for all \( k \geq k_0 + 1 \), respecting the following normalizations

\[
(1.23) \quad \phi'_k \in F_k \quad \text{such that} \quad \phi^* \left( P_{k-1} (z, \bar{w}, \text{Re} w) \right) = 0, \quad \text{for all} \quad k \geq k_0 + 1,
\]

given the following assumptions

\[
(1.24) \quad \text{Re} \left( \frac{\partial^{k_0} G(z, w)}{\partial w^{k_0}} (0, 0) \right) = 0, \quad \text{Im} \left( \frac{\partial^{l} F(z, w)}{\partial w^{l}} (0, 0) \right) = 0, \quad \text{for all} \quad l = 1, \ldots, N.
\]

This formal normal form (1.22) is inductively constructed according to the iterative procedure, similar as in [2], computations and the applied strategy described as follows:

2. **Proof of Theorem 1.1**

We take a formal holomorphic change of coordinates as in (1.21), but which sends \( M \subset C^{N+1} \), defined by (1.20), into \( M' \subset C^{N+1} \), defined by (1.22), obtaining the following equation

\[
(2.1) \quad \sum_{m, n \geq 0} \text{Im} G_{m,n}(z) \left( \text{Re} w + i (\text{Re} w)^s P(z, \bar{z}, \text{Re} w) \right)^n \frac{P \left( \sum_{m, n \geq 0} F_{m,n}(z) \left( \text{Re} w + i (\text{Re} w)^s P(z, \bar{z}, \text{Re} w) \right)^n \right)}{\sum_{m, n \geq 0} F_{m,n}(z) \left( \text{Re} w + i (\text{Re} w)^s P(z, \bar{z}, \text{Re} w) \right)^n} + \sum_{k \geq k_0 + 1} \phi'_k \left( \sum_{m, n \geq 0} F_{m,n}(z) \left( \text{Re} w + i (\text{Re} w)^s P(z, \bar{z}, \text{Re} w) \right)^n \right),
\]

and

\[
\text{Re} \sum_{m, n \geq 0} F_{m,n}(z) \left( \text{Re} w + i (\text{Re} w)^s P(z, \bar{z}, \text{Re} w) \right)^n.
\]
Next, by eventually compositing with a linear automorphism of the Model \( M \), we can assume that we can work with a formal equivalence like \( (2.1) \), concluding the following important equation

\[
(2.2) \quad \sum_{m+2n=T} \frac{G_{m,n}(z) - \overline{G_{m,n}(z)}}{2\sqrt{-1}} (\text{Re} w + i (\text{Re} w)^s P(z, z))^n \\
\quad \| (\varphi'_l - \varphi_T) (z, \overline{z}, \text{Re} w) \\
+ (\text{Re} w + i (\text{Re} w)^s P(z, z)) \left( \sum_{m+2n=T} F_{m,n}(z) (\text{Re} w + i (\text{Re} w)^s P(z, z))^n, z \right)
\]

\[
+ \left( \sum_{m+2n=T} \frac{G_{m,n}(z) + \overline{G_{m,n}(z)}}{2} (\text{Re} w + i (\text{Re} w)^s P(z, z))^n \left( \sum_{m+2n=T} F_{m,n}(z) (\text{Re} w + i (\text{Re} w)^s P(z, z))^n, z \right) \\
\quad + \left( \sum_{m+2n=T} F_{m,n}(z) (\text{Re} w + i (\text{Re} w)^s P(z, z))^n, z \right) \right),
\]

where \( \ldots \) contains terms already determined (of the same pseudo-weight) normalized according to a natural induction process depending on the natural number \( T \geq k_0 + 1 \), computing \( (2.2) \) in respect to the normalizations \( (1.23) \) and \( (1.24) \).

Then, \( (2.2) \) impplies

\[
(2.3) \quad \sum_{m+2n=T} \frac{G_{m,n}(z) - \overline{G_{m,n}(z)}}{2\sqrt{-1}} (\text{Re} w + i (\text{Re} w)^s P(z, z))^n \\
\quad \| (\varphi'_l - \varphi_T) (z, \overline{z}, \text{Re} w) \\
+ (\text{Re} w + i (\text{Re} w)^s P(z, z)) \left( \sum_{m+2n=T} F_{m,n}(z) (\text{Re} w + i (\text{Re} w)^s P(z, z))^n, z \right)
\]

for all \( T \geq k_0 + 1 \), where \( \ldots \) contains terms already determined.

Now, it remains to study the linear independence of the polynomials from \( (1.18) \) with respect to \( (1.5) \) and we write with respect to the previous system of pseudo-weights. Then, in order to analyse the linear independence of \( (1.18) \), we write

\[
B_l(x, z) = z^l - (\text{Re} w + i (\text{Re} w)^s P(z, z)) A_l(x, z), \\
\tilde{B}_{jl}(x, z) = \overline{z}^j - (\text{Re} w + i (\text{Re} w)^s P(z, z)) \tilde{A}_{jl}(x, z),
\]

respecting \( (1.14) \).

Now, we analyse the pure terms in \( z \) in \( (2.4) \). It is clear that \( z^l \) is the only pure term as component of the first polynomial in \( (2.3) \). Any linear combination among the first class of polynomials in \( (2.4) \) indicates linear independence.

Next, we analyse the second class of polynomials in \( (2.4) \). Then, \( (1.10) \) may provide a term which can cancel \( \overline{z}^j z^l \) or not. In the second situation, it provides a pure term multiplied with \( x \).

Now, it becomes clear the linear independence of the polynomials considered in \( (1.10) \) with respect to a lexicographic order, because any linear combination with polynomials from \( (1.10) \) is trivial. The proof if completed, because \( (1.10) \) uniquely computes \( (1.24) \).

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