TOWARDS INTEGRABILITY OF THE QUARTIC ANALOGUE OF THE KONTSEVICH MODEL

JÖRG SCHÜRMANN AND RAIMAR WULKENHAAR

Abstract. We consider an analogue of Kontsevich’s matrix Airy function where the cubic potential \( \text{Tr}(\Phi^3) \) is replaced by a quartic term \( \text{Tr}(\Phi^4) \). Cumulants of the resulting measure are known to decompose into cycle types for which a recursive system of equations can be established. We develop a new, purely algebraic geometrical solution strategy for these equations, based on properties of Cauchy matrices. We explicitly solve the two most difficult equations and reduce the solution of all other equations to a combinatorial challenge. Thereby we provide strong evidence that the quartic analogue of the Kontsevich model is integrable.

1. Introduction

Guided by uniqueness of quantum gravity in two dimensions, Witten conjectured in [Wit91] that the generating function of intersection numbers of tautological characteristic classes on the moduli space of stable complex curves has to satisfy the PDE of the Korteweg-de Vries hierarchy. The conjecture was proved a few months later by Kontsevich in his seminal paper [Kon92]. Kontsevich understood that critical graphs of the canonical Strebel differential [Str84] on a punctured curve give a cell-decomposition of the moduli space of punctured curves, which can be organised into a novel type of matrix model (the ‘matrix Airy function’) with covariance

\[
\langle \Phi(e_{jk})\Phi(e_{lm}) \rangle_c = \frac{\delta_{kl}\delta_{jm}}{\lambda_j + \lambda_k}
\]

(where \( e_{jk} \) denotes the standard matrix basis and \( \delta_{kl} \) the Kronecker symbol) and tri-valent vertices. The \( \lambda_j \) are Laplace transform parameters of the lengths \( L_j \) of critical trajectories of the Strebel differentials, and the generically simple zeros of the Strebel differential correspond to tri-valent vertices. Kontsevich went on to establish that the logarithm of the partition function of his matrix model is the \( \tau \)-function for the KdV-hierarchy, thereby proving that his matrix model is integrable.

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These \( \lambda_j \) will be denoted by \( E_j \) in this paper.
The same covariance (up to normalisation)
\[
\langle \Phi(e_{jk})\Phi(e_{lm}) \rangle_c = \frac{\delta_{kl}\delta_{jm}}{N(E_j + E_k)}
\]
arises in quantum field theory models on noncommutative geometries \cite{GS06}, where the \( E_k \) are the spectral values (‘energy levels’) of the Laplace operator. These are models for scalar fields with cubic self-interaction. From a quantum field theoretical point of view one would be more interested in a quartic self-interaction, which e.g. is characteristic to the Higgs field. Such quartic models have been understood in \cite{GW05b} at the level of formal power series. Later in \cite{GW09,GW14} exact equations between correlation functions in the quartic (matrix) model were derived. These equations share many aspects with a universal structure called topological recursion \cite{EO07}.

Such recursions typically rely on the initial solution of a non-linear problem (for the Kontsevich model achieved in \cite{MS91}). For the quartic model, the corresponding equation (for the planar two-point function of cycle type \((0,1)\)) is given in \cite{GHW19} below. Its solution succeeded in \cite{GHW19}, via a larger detour. It was assumed that \cite{GHW19} converges for \( N \to \infty \) to an integral equation with Hölder-continuous measure. The special case of constant measure was solved in \cite{PW19} with help from computer algebra. Its structure suggested a conjecture for the general case which was proved in \cite{GHW19} by residue theorem and Lagrange-Bürmann resummation.

This paper provides a novel algebraic geometrical solution strategy for the non-linear equation \cite{GHW19} and the affine equation \cite{GHW19} (which determines the planar two-point function of cycle type \((2,0)\)). We (re)prove that these cumulants are compositions of rational functions with a preferred inverse of another rational function
\[
R(z) = z - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\rho_k}{z + \varepsilon_k}.
\]
This rationality is strong support for the conjecture that the quartic analogue of the Kontsevich model is integrable. It will be a fascinating long-term project to make the integrability precise and to possibly relate it, similar to the Kontsevich model, to intersection numbers of tautological characteristic classes on some moduli space, as well as to systems of non-linear evolution equations.

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2. Matrix integrals

Let \( X_N \) be the real vector space of self-adjoint \( N \times N \)-matrices and \((E_1, \ldots, E_N)\) not necessarily distinct positive real numbers. By the Bochner-Minlos theorem \([\text{Boc33}]\), combined with the Schur product theorem \([\text{Sch11}, \S4]\), there is a unique probability measure \( d\mu_0(\Phi) \) on the dual space \( X'_N \) with

\[
\exp \left( -\frac{1}{2N} \sum_{k,l=1}^{N} \frac{M_{kl} M_{lk}}{E_k + E_l} \right) = \int_{X'_N} d\mu_0(\Phi) e^{i\Phi(M)}
\]

for any \( M = M^* = \sum_{k,l=1}^{N} M_{kl} e_{kl} \in X_N \), where \((e_{kl})\) is the standard matrix basis. The linear forms extend via \( \Phi(M_1 + iM_2) := \Phi(M_1) + i\Phi(M_2) \) to arbitrary complex \( N \times N \)-matrices. This allows us to evaluate \( \Phi(e_{jk}) \) and to identify the covariance

\[
\int_{X'_N} d\mu_0(\Phi) \Phi(e_{jk})\Phi(e_{lm}) = \frac{\delta_{kl} \delta_{jm}}{N(E_j + E_k)}.
\]

We are going to deform the Gaussian measure (1) by a quartic potential,

\[
d\mu_\lambda(\Phi) := \frac{d\mu_0(\Phi)}{\int_{X'_N} d\mu_0(\Phi)} \mathcal{P}_4(\Phi, \lambda),
\]

\[
\mathcal{P}_4(\Phi, \lambda) = \exp \left( -\frac{\lambda N}{4} \mathrm{Tr}(\Phi^4) \right) := \exp \left( -\frac{\lambda N}{4} \sum_{j,k,l,m=1}^{N} \Phi(e_{jk})\Phi(e_{kl})\Phi(e_{lm})\Phi(e_{mj}) \right),
\]

for some \( \lambda > 0 \). This matrix measure is the quartic analogue of the Kontsevich model \([\text{Kon92}]\) in which the deformation is given by the cubic term

\[
\mathcal{P}_3(\Phi, \lambda) = \exp \left( -\frac{\lambda N}{3} \mathrm{Tr}(\Phi^3) \right) := \exp \left( -\frac{\lambda N}{3} \sum_{k,l,m=1}^{N} \Phi(e_{kl})\Phi(e_{lm})\Phi(e_{mk}) \right).
\]

The cubic measure was designed to prove Witten’s conjecture \([\text{Wit91}]\) that intersection numbers of tautological characteristic classes on the moduli space of stable complex curves are related to the KdV hierarchy. Kontsevich proved that \( \log \int_{X'_N} d\mu_0(\Phi) \mathcal{P}_3(\Phi, \frac{i}{2}) \), viewed as function of \( t_k = -(2k - 1)!! \sum_{j=1}^{N} E_j^{-(2k+1)} \), is the generating function of these intersection numbers.

We are interested in moments of the measure (2),

\[
\langle e_{k_1 l_1} \cdots e_{k_n l_n} \rangle := \int_{X'_N} d\mu_\lambda(\Phi) \Phi(e_{k_1 l_1}) \cdots \Phi(e_{k_n l_n}) = \left. \frac{\partial^n Z(M)}{\partial M_{k_1 l_1} \cdots \partial M_{k_n l_n}} \right|_{M=0},
\]

\[
Z(M) = \int_{X'_N} d\mu_\lambda(\Phi) e^{i\Phi(M)}.
\]

(3)
As explained in Appendix A (see also [McC12, Spe83]), the moments (3) decom- pose into cumulants
\[
\left\langle \prod_{i=1}^{n} e_{k_i l_i} \right\rangle = \sum_{\text{partitions } \pi \text{ of } \{1, \ldots, n\}} \prod_{\beta \in \pi} \left\langle \prod_{i \in \beta} e_{k_i l_i} \right\rangle_c. \tag{4}
\]

For a quartic potential (2), moments and cumulants are only non-zero if \(n\) is even and every block \(\beta\) is of even length. The structure of the Gaußian measure (1) (together with the invariance of a trace under cyclic permutations) implies that \(\left\langle e_{k_1 l_1} \ldots e_{k_n l_n} \right\rangle_c\) is only non-zero if \((l_1, \ldots, l_n) = (k_{\sigma(1)}, \ldots, k_{\sigma(n)})\) is a permutation of \((k_1, \ldots, k_n)\), and in this case the cumulant only depends on the cycle type \(\sigma\) in the symmetric group \(S_n\) (see Appendix A) with \(b \geq 1\) the number of cycles of length \(n_i > 0, n_1 + \cdots + n_b = n\):

\[
N^{n_1 + \cdots + n_b} \left\langle \left( e_{k_1^1 l_1^1} e_{k_1^2 l_1^2} \cdots e_{k_1^{n_1} l_1^{n_1}} \right) \cdots \left( e_{k_b^1 l_1^1} e_{k_b^2 l_1^2} \cdots e_{k_b^{n_b} l_1^{n_b}} \right) \right\rangle_c
= N^{2-b} G_{|k_1^1 \ldots k_1^{n_1}| \ldots |k_b^1 \ldots k_b^{n_b}|} \tag{5}
\]

These \(N\)-rescaled cumulants are further expanded as formal power series
\[
G_{\ldots} = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} G^{(g)}_{\ldots} \text{ in } N^{-2} \text{ so that}
\]

\[
N^{n_1 + \cdots + n_b} \left\langle \left( e_{k_1^1 l_1^1} e_{k_1^2 l_1^2} \cdots e_{k_1^{n_1} l_1^{n_1}} \right) \cdots \left( e_{k_b^1 l_1^1} e_{k_b^2 l_1^2} \cdots e_{k_b^{n_b} l_1^{n_b}} \right) \right\rangle_c
= \sum_{g=0}^{\infty} N^{2-b-2g} G^{(g)}_{|k_1^1 \ldots k_1^{n_1}| \ldots |k_b^1 \ldots k_b^{n_b}|} \tag{6}
\]

This grading \((g, b)\) of the elementary building blocks \(G^{(g)}_{|k_1^1 \ldots k_1^{n_1}| \ldots |k_b^1 \ldots k_b^{n_b}|}\) of moments fits with the combinatorics of ribbon graphs (with 4-valent vertices) on a connected oriented compact topological surface of genus \(g \geq 0\) with \(b \geq 1\) boundary components (and \(n_i\) labels on the \(i^{th}\) boundary component) and Euler characteristic \(\chi = 2 - b - 2g\) (see e.g. [GW05a, §3] for the particular case of 4-valent vertices, and compare also with [Kon92] or [Eyn16, §2 and §6]). Note that the moments are related to ribbon graphs on possibly non-connected oriented compact topological surfaces (see e.g. [LZ04, §3, Prop.3.8.3]).

Starting point for the investigation of cumulants are equations of motion for \(Z(M)\):

**Lemma 1.** The Fourier transform \(Z(M)\) of the measure (2) satisfies

\[
\frac{1}{i} \frac{\partial Z(M)}{\partial M_{ab}} = \frac{i M_{ab}}{N(E_a + E_b)} - \frac{\lambda}{i^3} \sum_{k,l=1}^{N} \frac{\partial^3 Z(M)}{\partial M_{ak} \partial M_{kl} \partial M_{lb}}. \tag{7}
\]

**Proof.** This follows from basic properties of the Gaußian measure (1). The derivative \(\frac{1}{i} \frac{\partial}{\partial M_{ab}}\) applied to \(Z(M)\) produces a factor \(\Phi(e_{ab})\) under the integral. Moments of \(d\mu_0(\Phi)\) are by (2) a sum over pairings. This means that \(\Phi(e_{ab})\) is paired in all possible ways with a \(\Phi(e_{cd})\) contained in \(\exp(i\Phi(M))\) or in \(P_4(\Phi, \lambda)\).
Every such pair contributes a factor \( \frac{\delta_{ab}\delta_{bc}}{N(E_a + E_b)} \), and summing over all pairings is the same as taking the derivative, thus producing a term

\[
\frac{1}{N(E_a + E_b)} \left( iM_{ba} - \lambda N \sum_{k,l=1}^{N} \Phi(e_{ak})\Phi(e_{kl})\Phi(e_{lb}) \right)
\]

under the integral. The triple product of \( \Phi(e_{\cdot}) \) is written as a third derivative with respect to the corresponding entries of \( M \).

The Kontsevich model [Kon92] with cubic deformation \( P_3(\Phi, \lambda) \) is governed by the equation of motion

\[
\frac{1}{i} \frac{\partial Z(M)}{\partial M_{ab}} = \frac{iM_{ba}Z(M)}{N(E_a + E_b)} - \frac{\lambda}{i^2(E_a + E_b)} \sum_{k=1}^{N} \frac{\partial^2 Z(M)}{\partial M_{ak}\partial M_{kb}}.
\]

For \( N = 1 \) this is essentially the ODE

\[
f''(x) + 2cf'(x) = xf(x)
\]

solved by the Airy function \( f(x) = e^{-cx}Ai(x + c^2) \), hence the title of [Kon92]. Its quartic analogue is the matrix version of the ODE

\[
f'''(x) + 3cf'(x) = xf(x),
\]

which does not seem to have a name.

Another equation of motion will be necessary for subsequent investigations outlined in sec. 6:

**Lemma 2.** The Fourier transform \( Z(M) \) of the measure (2) satisfies

\[
\frac{1}{N} \frac{\partial Z(M)}{\partial E_a} = \left( \sum_{k=1}^{N} \frac{\partial^2}{\partial M_{ak}\partial M_{ka}} + \frac{1}{N} \sum_{k=1}^{N} G_{[ak]} + \frac{1}{N^2} G_{[a][a]} \right) Z(M).
\]  

Proof. Application of \( \frac{1}{N} \frac{\partial}{\partial E_a} - \sum_{k=1}^{N} \frac{\partial^2}{\partial M_{ak}\partial M_{ka}} - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{(E_a + E_k)} \) to the left hand side of (1) yields zero so that it gives

\[
\frac{1}{N} \frac{\partial}{\partial E_a} (d\mu_0(\Phi)) = d\mu(\Phi) \sum_{k=1}^{N} \left( \frac{1}{N(E_a + E_k)} - \Phi(e_{ka})\Phi(e_{ak}) \right)
\]

when applying it to the right hand side. Apply this identity to (2) to get

\[
\frac{1}{N} \frac{\partial}{\partial E_a} (d\mu_\lambda(\Phi)) = d\mu_\lambda(\Phi) \sum_{k=1}^{N} \left( \frac{1}{N(E_a + E_k)} - \Phi(e_{ka})\Phi(e_{ak}) \right)
\]

\[
- d\mu_\lambda(\Phi) \int_{X'_N} d\mu_\lambda(\Phi) \sum_{k=1}^{N} \left( \frac{1}{N(E_a + E_k)} - \Phi(e_{ka})\Phi(e_{ak}) \right).
\]

Multiplying with \( e^{i\Phi(M)} \) and integrating over \( X'_N \) gives with (36) the assertion.
The equations of motion (7) and (8) induce identities between cumulants. Some of them are derived in Appendix B, for others see \[GW14\]. Taking also the grading by \((g,b)\) into account, one can establish a partial order in the homogeneous building blocks \(G_{(g)}\). The least element is the planar two-point function \(G_{(0)}\), which is the dominant part (at large \(N\)) of the cumulant of length 2 and cycle type \((0,1)\) (i.e. one cycle \(ab\) of length 2). It satisfies a closed non-linear equation for it alone, given in (9) below. Any other homogeneous building block of (6) satisfies an affine equation with inhomogeneity that depends only on functions of strictly larger topological Euler characteristic \(\chi = 2 - b - 2g\), which are known by induction. Similar recursive systems have been identified in many areas of mathematics. Their common universal structure has been axiomatised under the name topological recursion \[EO07\], since the recursion is by the topological Euler characteristic. Starting from a few initial data called the spectral curve, topological recursion constructs a hierarchy of differential forms and understands them as spectral invariants of the curve. A prominent example is the Kontsevich model \[Kon92\] whose topological recursion is described e.g. in \[Eyn16, \S 6\]. Other classes of examples are the one- and two-matrix models \[CEO06\], Mirzakhani’s recursions \[Mir06\] for the volume of moduli spaces of Riemann surfaces, and recursions in Hurwitz theory \[BM08\] and Gromov-Witten theory \[DBMN+17\].

3. The planar two-point function

The two-point function \(G_{[ab]}\) is the cumulant of length 2 and cycle type \((0,1)\) (i.e. one cycle \(ab\) of length 2), see Appendix A. We reprove in Appendix B that the planar two-point function \(G_{(0)}^{(0)}\) (of degree or genus \(g = 0\)) satisfies

\[
\left( E_a + E_b + \frac{\lambda}{N} \sum_{k=1}^{N} G_{[ab]}^{(0)} \right) G_{[ab]}^{(0)} = 1 + \frac{\lambda}{N} \sum_{k=1}^{N} \frac{G_{[kb]}^{(0)} - G_{[ab]}^{(0)}}{E_k - E_a}.
\]

This equation was first established in \[GW09\]; equation (42) which involves all \(G_{[ab]}^{(g)}\) was obtained in \[GW14\].

To give a meaning to the term \(k = a\) in (9) the following is assumed. Let \(E_1, \ldots, E_d\) be the distinct entries in the tuple \((E_k)\), which occur with multiplicities \(r_1, \ldots, r_d\), with \(N = r_1 + \cdots + r_d\). We assume that for some neighbourhoods \(U_k \subset \mathbb{C}\) of \(E_k\) there is a holomorphic function \(G^{(0)} : \bigcup_{k,l=1}^{d} (U_k \times U_l) \to \mathbb{C}\) which interpolates \(G_{[ab]}^{(0)} = G^{(0)}(E_a, E_b)\) and satisfies the natural holomorphic extension

\[
\left( \zeta + \eta + \frac{\lambda}{N} \sum_{k=1}^{d} r_k G^{(0)}(\zeta, E_k) \right) G^{(0)}(\zeta, \eta) = 1 + \frac{\lambda}{N} \sum_{k=1}^{d} \frac{r_k G^{(0)}(E_k, \eta) - G^{(0)}(\zeta, \eta)}{E_k - \zeta}
\]

of (9), for \((\zeta, \eta) \in \bigcup_{k,l=1}^{d} (U_k \times U_l)\). Equation (9) is understood as the limit \(\zeta \to E_a\) and \(\eta \to E_b\) of (10) when taking multiplicities into account. Then:
Theorem 3. Construct 2d functions \( \{ \varepsilon_k(\lambda), \varrho_k(\lambda) \}_{k=1}^{d} \), with \( \lim_{\lambda \to 0}(\varepsilon_k, \varrho_k) = (E_k, r_k) \), as implicitly defined solution of the system

\[
E_k = R(\varepsilon_k), \quad r_k = \varrho_k R'(\varepsilon_k), \quad \text{where} \quad R(z) = z - \frac{\lambda}{N} \sum_{j=1}^{d} \frac{\varrho_j}{\varepsilon_j + z}.
\]

Then (10) is solved by \( G^{(0)}(\zeta, \eta) = G^{(0)}(R^{-1}(\zeta), R^{-1}(\eta)) \), where \( G^{(0)} : \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is the rational function

\[
G^{(0)}(z, w) = \frac{1 - \frac{\lambda}{N} \sum_{k=1}^{d} (R(z) - R(\varepsilon_k))(R(\varepsilon_k) - R(-w)) \prod_{j=1}^{d} \frac{R(w) - R(-\hat{\varepsilon}_k)}{R(w) - R(\varepsilon_j)} \prod_{j=1}^{d} \frac{R(w) - R(-\hat{\varepsilon}_j)}{R(w) - R(\varepsilon_j)} \prod_{j=1}^{d} \frac{R(w) - R(-\hat{\varepsilon}_k)}{R(w) - R(\varepsilon_j)}}{R(w) - R(-z)}
\]

which is symmetric, \( G^{(0)}(z, w) = G^{(0)}(w, z) \). Here \( z \in \{ u, \hat{u}, \ldots, \hat{u}^d \} \) is the list of the different roots of \( R(z) = R(u) \), and the correct branch of \( R^{-1} \) is chosen by the implicitly defined solutions above (i.e. \( \varepsilon_k \in R^{-1}(\mathcal{U}_k) \) for this branch).

Theorem 3 was already proved in [GHW19]. There, inspired by the solution of a particular case [PW19], equation (10) was interpreted as an integral equation for a Dirac measure. Approximating the Dirac measure by a Hölder-continuous function allowed to employ boundary values techniques for sectionally holomorphic functions. Residue theorem and Lagrange-Bürmann resummation gave a solution formula whose limit back to Dirac measure was arranged into (12).

In this paper we provide a new, much more elementary and also shorter proof of Theorem 3, which solely needs properties of Cauchy matrices established by Schechter [Sch59]:

Proposition 4 (Sch59). For two d-tuples \( (a_1, \ldots, a_d) \) and \( (b_1, \ldots, b_d) \), with all \( a_i, b_j \) distinct, consider the \( d \times d \)-matrix \( H = (\frac{1}{x-a_i} \frac{1}{y-b_j})_{kl} \). Let \( A(x) := \prod_{i=1}^{d} (x - a_i) \) and \( B(y) := \prod_{j=1}^{d} (y - b_j) \). Then the inverse of \( H \) is given by

\[
(H^{-1})_{kl} = (a_l - b_k) A_l(b_k) B_k(a_l),
\]

where \( A_l, B_k \) are the Lagrange interpolation polynomials

\[
A_l(x) = \frac{A(x)}{(x - a_l) A'(a_l)}, \quad B_k(y) = \frac{B(y)}{(y - b_k) B'(b_k)}.
\]

Moreover, the row sums and column sums of \( H^{-1} \) are given by

\[
\sum_{j=1}^{d} (H^{-1})_{kj} = -\frac{A(b_k)}{B'(b_k)}, \quad \sum_{i=1}^{d} (H^{-1})_{il} = \frac{B(a_l)}{A'(a_l)}.
\]
and one has, for all $j = 1, \ldots, d$,
\[
\sum_{k=1}^{d} \frac{A(b_k)}{(b_k - a_j)B'(b_k)} = 1 \quad \text{and} \quad \sum_{l=1}^{d} \frac{B(a_l)}{(a_l - b_j)A'(a_l)} = 1.
\] (16)

4. Proof of Theorem 3

We are going to construct a non-constant rational function $R \in \mathbb{C}(z)$ viewed as a branched cover $R : \hat{C} = \mathbb{C} \cup \{\infty\} \to \hat{C} = \mathbb{C} \cup \{\infty\}$ of Riemann surfaces (with $z = \text{id}_\mathbb{C}$ the standard coordinate on $\mathbb{C}$) via the following:

**Ansatz 5.** A branched cover $R : \hat{C} \to \hat{C}$ is supposed to be determined by conventions (i)--(vi) and an algebraic relation (vii):

(i) $R$ has degree $d + 1$.
(ii) All branch points of $R$ do not belong to $R^{-1}(\{E_1, \ldots, E_d\})$.
(iii) Without loss of generality, $R(\infty) = \infty$ with residue $-1$ in the sense that $\text{Res}_{z \to \infty} R(z)dz = -1$.
(iv) For every $k = 1, \ldots, d$, distinguish any of the $d+1$ distinct points of the fibre $R^{-1}(E_k)$ as $\varepsilon_k$. Take any connected neighbourhood $U_k \subset \mathbb{C}$ of $E_k$ for which $R^{-1}(U_k)$ has $d+1$ connected components, and let $V_k$ be the connected component of $R^{-1}(U_k)$ which contains $\varepsilon_k$. Then the choice of $\{\varepsilon_k\}$ determines a holomorphic function $G^{(0)} : \bigcup_{k,l=1}^{d} (V_k \times V_l) \to \mathbb{C}$ by $G^{(0)}(z,w) = G^{(0)}(R(z),R(w))$, where $G^{(0)} : \bigcup_{k,l=1}^{d} (U_k \times U_l) \to \mathbb{C}$ satisfies (10).
(v) For any $w \in \bigcup_{j=1}^{d} V_j$, let $\hat{w}^1, \ldots, \hat{w}^d$ be the $d$ other distinct preimages of $R(w) \in \bigcup_{j=1}^{d} U_j$ under $R$, i.e. $R(\hat{w}^k) = R(w)$. Assume that
\[
\infty \neq R(-\hat{w}^l) \quad \text{and} \quad R(-\hat{w}^l) \neq R(-\hat{w}^{l'})
\]
for all $l, l' = 1, \ldots, d$ with $l \neq l'$ and $w$ close to some $\varepsilon_k$.
(vi) For any $w$ close to some $\varepsilon_k$, $G^{(0)}(-\hat{w}^l, w)$ is defined and finite for all $l = 1, \ldots, d$. This is the case e.g. if $-\hat{w}^l \in \bigcup_{j=1}^{d} V_j$ for all $l = 1, \ldots, d$, or if $G^{(0)}$ extends to a suitable rational function on $\hat{C} \times \hat{C}$.
(vii) For any $z \in V_1$ one has
\[
R(z) + \frac{\lambda}{N} \sum_{k=1}^{d} r_k G^{(0)}(z, \varepsilon_k) + \lambda \sum_{k=1}^{d} \frac{r_k}{R(\varepsilon_k) - R(z)} = -R(-\hat{z})
\] (17)

With the properties (iv) and (vii) in this Ansatz 5 we turn (10) into
\[
(R(w) - R(-\hat{z})) G^{(0)}(z,w) = 1 + \frac{\lambda}{N} \sum_{k=1}^{d} r_k \frac{G^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k) - R(z)},
\] (18)

where $(z,w) \in \bigcup_{k,l=1}^{d} (V_k \times V_l)$. Next, setting $z = -\hat{w}^l$ in (18) for $l = 1, \ldots, d$ and a given $w$ close to some $\varepsilon_k$, requirements (v) and (vi) of Ansatz 5 give (by
\( \infty \neq R(-\hat{w}^j) \) and \( \mathcal{G}^{(0)}(-\hat{w}^j, w) \) is defined and finite) the \( d \) equations

\[
\frac{\lambda}{N} \sum_{k=1}^{d} r_k \frac{\mathcal{G}^{(0)}(\varepsilon_k, w)}{R(-\hat{w}^j) - R(\varepsilon_k)} = 1 .
\] (19)

This identifies \( \frac{\lambda}{N} r_k \mathcal{G}^{(0)}(\varepsilon_k, w) \) as row sums of the inverse of a Cauchy matrix.

Setting \( a_j = R(-\hat{w}^j) \) and \( b_l = R(\varepsilon_l) \) in the first identity (15) in Proposition 4 we conclude (since the \( a_j, b_i \) for \( j, i = 1, \ldots, d \) are pairwise distinct by requirement \( (v) \) of Ansatz 5):

**Corollary 6.** With Ansatz 5 one has

\[
\frac{\lambda}{N} r_k \mathcal{G}^{(0)}(\varepsilon_k, w) = -\frac{\prod_{j=1}^{d} (R(\varepsilon_k) - R(-\hat{w}^j))}{\prod_{j=1, j \neq k}^{d} (R(\varepsilon_k) - R(\varepsilon_j))} .
\] (20)

Inserted back into (18) expresses \( \mathcal{G}^{(0)}(z, w) \) in terms of \( R \). The result simplifies:

**Lemma 7.** With Ansatz 5 one has

\[
\mathcal{G}^{(0)}(z, w) = \frac{1}{(R(w) - R(\varepsilon_k))} \prod_{j=1}^{d} \frac{R(z) - R(-\hat{w}^j)}{R(z) - R(\varepsilon_j)} .
\] (21)

**Proof.** This follows from (16) for \((d+1)\)-tuples with index 0 prepended. Setting \( b_0 = R(z) \), \( b_k = R(\varepsilon_k) \), \( a_l = R(-\hat{w}^j) \) for \( k, l = 1, \ldots, d \), then the case \( j = 0 \) of the first identity (16) reads (independent of \( a_0 \))

\[
\frac{\prod_{j=1}^{d} (R(z) - R(-\hat{w}^j))}{\prod_{j=1}^{d} (R(z) - R(\varepsilon_j))} + \sum_{k=1}^{d} \frac{\prod_{j=1}^{d} (R(\varepsilon_k) - R(-\hat{w}^j))}{\prod_{j=1, j \neq k}^{d} (R(\varepsilon_k) - R(\varepsilon_j))} = 1 .
\]

Equation (21) results from this identity when inserting (20) into (18). \( \square \)

**Lemma 8.** With Ansatz 5 the rational function \( R \in \mathbb{C}(z) \) is necessarily given by

\[
R(z) = z + c_0 - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\theta_k}{\varepsilon_k + z} \quad \text{for some} \ c_0 \in \mathbb{C} , \quad \theta_k = \frac{r_k}{R'(\varepsilon_k)} .
\] (22)

**Proof.** Comparing the limit \( z \to \varepsilon_k \) of (21) with (20) shows that \( R \) has a simple pole at every \( -\varepsilon_k \) with

\[
R'(\varepsilon_k) \ \text{Res} \ \frac{R(z)}{z - \varepsilon_k} dz = -\frac{\lambda r_k}{N} \neq 0 .
\]

By construction, \( R \) has also a pole at \( \infty \). Since \( R \) has degree \( d + 1 \) by (i) in Ansatz 5 \{\(-\varepsilon_1, \ldots, -\varepsilon_d, \infty\)\} is already the complete list of poles (i.e. preimages of \( \infty \)) of \( R \). Moreover, the pole at \( \infty \) has to be simple with \( \lim_{z \to \infty} \frac{R(z)}{z} = 1 \) by (iii) in Ansatz 5. Therefore, \( R(z) - z - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\theta_k}{\varepsilon_k + z} \) is a bounded holomorphic function on \( \hat{\mathbb{C}} \), which by Liouville’s theorem is a constant \( c_0 \). \( \square \)
Corollary 9. For \( u \in \bigcup_{j=1}^{d} V_j \) one has an equality of rational functions in \( z \):

\[
R(z) - R(u) = (z - u) \prod_{k=1}^{d} \frac{(z - \hat{u}^k)}{z + \varepsilon_k},
\]

where \( \hat{u}^k \) are the other preimages of \( R(u) \) under \( R \).

Proof. Both sides are a rational function \( r(z) \), with zeros only in \( u, \hat{u}^1, \ldots, \hat{u}^d \) and poles only in \( -\varepsilon_1, \ldots, -\varepsilon_d, \infty \), all of which are simple. So they differ by a constant factor, which has to be 1 because both sides satisfy \( \lim_{z \to \infty} \frac{r(z)}{z} = 1 \). \( \square \)

Proposition 10. With Ansatz 5 the two-point function is symmetric, \( G^{(0)}(w, z) = G^{(0)}(z, w) \).

Proof. Inserting (23) into (21) gives for \( z, w \in \bigcup_{j=1}^{d} V_j \):

\[
G^{(0)}(z, w) = \frac{\prod_{k=1}^{d} (\varepsilon_k - z) \prod_{l=1}^{d} (z - \hat{u}^l)}{(z + w) \prod_{k=1}^{d} (-z - \hat{u}^k) \prod_{l=1}^{d} (-\varepsilon_l - \hat{u}^l)} \frac{\prod_{k=1}^{d} \varepsilon_k - \hat{u}^k}{\prod_{l=1}^{d} (-\varepsilon_l - \hat{u}^l)} \frac{\prod_{l=1}^{d} \varepsilon_k + \varepsilon_l}{\prod_{l=1}^{d} (-\varepsilon_l + \hat{u}^l)} \frac{\prod_{k=1}^{d} \varepsilon_k - \hat{u}^l}{\prod_{l=1}^{d} (\varepsilon_k - \hat{u}^l)} \frac{\prod_{k=1}^{d} \varepsilon_k - \hat{u}^k}{\prod_{l=1}^{d} (\varepsilon_k - \hat{u}^l)}.
\]

We prove that (ii), (iv), (v), (vi) and (vii) of Ansatz 5 are automatic. We start with

Proposition 11. Relation (vii) of Ansatz 5 is consistent provided that \( c_0 = 0 \).

Proof. With Lemma 8 and Lemma 7 both a consequence of Ansatz 5 each side of (17) is a rational function, and all poles are simple. For the term \( R_{\varepsilon_k}^{(-\varepsilon_k)} - R(z) \) this follows from the assumption (ii) of Ansatz 5. We show that both sides of (17) have the same simple poles with the same residues. Then by Liouville’s theorem their difference is a constant, which is easy to control.

First, it follows from (22) and (21) that both sides of (17) approach \( z \) for \( z \to \infty \). Near \( \infty \) the difference between both sides of (17) is \( \pm 2c_0 \), which shows that \( c_0 = 0 \) in (22) is necessary.

Next, (22) shows that the only other poles of the right hand side of (17) are simple and located at \( z = \varepsilon_k \) with residue \(-\frac{\lambda}{N} q_k \). The same simple poles with the same residues are produced by \( \frac{\lambda}{N} \sum_{k=1}^{d} R_{\varepsilon_k}^{(-\varepsilon_k)} - R_{\varepsilon_k}^{(-\varepsilon_k)} \) on the left hand side, taking \( r_k / R'(\varepsilon_k) = q_k \) into account.

But the left hand side of (17) could also have poles at \( z = -\varepsilon_j \) and \( z = -\varepsilon_m \) (see (22) and (24)). We have \( \text{Res}_{z \to -\varepsilon_j} R(z) dz = -\frac{\lambda}{N} q_j \). Setting \( w \to \varepsilon_l \) in (21), then with \( \lim_{z \to -\varepsilon_j} R(z) dz = -\frac{\lambda}{N} q_j \) for any \( k, l \) (here (v) of Ansatz 5 is used) one easily
finds that $G^{(0)}(-\varepsilon_j, \varepsilon_l)$ is regular for $j \neq l$ and that $\text{Res}_{z \to -\varepsilon_j} \frac{1}{N} r_j G^{(0)}(z, \varepsilon_j) dz = \frac{1}{N} \frac{r_j}{N(\varepsilon_j)}$, which thus cancels $\text{Res}_{z \to -\varepsilon_j} R(z) dz = -\frac{1}{N} \theta_j$.

Finally, from (21) we conclude

$$\text{Res}_{z \to \varepsilon_m} G^{(0)}(z, \varepsilon_k) dz = \frac{1}{(R(\varepsilon_k) - R(-\varepsilon_m^j)) R'(\varepsilon_m^j)} \prod_{i=1}^{d} (R(\varepsilon_m) - R(-\varepsilon_i^j))$$

$$= \frac{1}{r_m} r_k R'(\varepsilon_m^j) \left( R(\varepsilon_k) - R(-\varepsilon_m^j) \right) \prod_{i=1, i \neq m}^{d} (R(\varepsilon_m) - R(-\varepsilon_i^j)),$$

where (20), the symmetry $G^{(0)}(\varepsilon_m, \varepsilon_k) = G^{(0)}(\varepsilon_k, \varepsilon_m)$ and again (20) have been used. The first identity (16) for $b_k = R(\varepsilon_k)$ and $a_j = R(-\varepsilon_m^j)$ gives

$$\text{Res}_{z \to \varepsilon_m^j} \sum_{k=1}^{d} r_k G^{(0)}(z, \varepsilon_k) dz = \frac{r_m}{R'(\varepsilon_m^j)}$$

which precisely cancels $\text{Res}_{z \to \varepsilon_m^j} \sum_{k=1}^{d} \frac{r_k}{R'(\varepsilon_m^j)} dz = -\frac{r_m}{R'(\varepsilon_m^j)}$.

Let us consider now the rational function

$$R(z) = z - \frac{1}{N} \sum_{i=1}^{d} \frac{\theta_k}{\varepsilon_k + z}$$

from equation (22) with $c_0 = 0$. We are interested in the real and complex solutions $\{\varepsilon_k, \theta_k\}_{k=1,\ldots,d}$ (depending on $\lambda$) of the $2d$ equations

$$0 = R(\varepsilon_l) - E_l = \varepsilon_l - E_l - \frac{1}{N} \sum_{k=1}^{d} \frac{\theta_k}{\varepsilon_k + \varepsilon_l} =: f_l(\varepsilon_l, \theta_1, \ldots, \varepsilon_d, \theta_d, \lambda), \quad (26a)$$

$$0 = R'(\varepsilon_l) - \frac{\theta_l}{\theta_l} = 1 - \frac{\theta_l}{\theta_l} + \frac{1}{N} \sum_{k=1}^{d} \frac{\theta_k}{(\varepsilon_k + \varepsilon_l)^2} =: g_l(\varepsilon_l, \theta_1, \ldots, \varepsilon_d, \theta_d, \lambda) \quad (26b)$$

for $l = 1, \ldots, d$ from Theorem [3] for the given positive real numbers $E_l > 0$ and $r_l > 0$ from Section [2]. In the following we only use that all these $E_l, r_l$ are positive (but not that $r_l$ is an integer counting the multiplicity of $E_l$). So we consider for $K = \mathbb{R}$ or $K = \mathbb{C}$ the real or complex algebraic subset

$$Z(K) := \{(\varepsilon_1, \theta_1, \ldots, \varepsilon_d, \theta_d, \lambda) \in U(K) \mid f_l = 0, g_l = 0, l = 1, \ldots, d\}$$

in

$$U(K) := \{(\varepsilon_1, \theta_1, \ldots, \varepsilon_d, \theta_d, \lambda) \mid \varepsilon_k + \varepsilon_l \neq 0, \theta_l \neq 0, k, l = 1, \ldots, d\} \subset K^{2d+1},$$
i.e. in the complement of the corresponding central hyperplane arrangement in \( \mathbb{K}^{2d+1} \). Note that \((\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, 0) \in Z(\mathbb{K})\) iff \(\varepsilon_l = E_l\) and \(\varrho_l = r_l\) for all \(l = 1, \ldots, d\), and this real point belongs to the chamber

\[
U_+(\mathbb{R}) := \{(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) \mid \varepsilon_l > 0, \ \varrho_l > 0, \ l = 1, \ldots, d\} \subset U(\mathbb{R}) \subset \mathbb{R}^{2d+1}.
\]

Note that the complex dimension of any irreducible component of \(Z(\mathbb{C})\) is at least \(1 = (2d + 1) - 2d\), since we are considering \(2d\) equations in a Zariski-open subset of \(\mathbb{C}^{2d+1}\).

**Lemma 12.** \(Z(\mathbb{K})\) is a one-dimensional (real or complex) algebraic submanifold of \(U(\mathbb{K})\) near the reference point \((E_1, r_1, \ldots, E_d, r_d, 0)\), with the projection

\[pr : \mathbb{K}^{2d+1} \supset Z(\mathbb{K}) \to \mathbb{K}\]

onto the last \(\lambda\)-coordinate a submersion near \((E_1, r_1, \ldots, E_d, r_d, 0)\) in the Zariski topology. In particular, the reference point \((E_1, r_1, \ldots, E_d, r_d, 0)\) only belongs to one irreducible component of \(Z(\mathbb{C})\), which is of dimension one. Moreover, the map (of pointed sets)

\[pr : Z(\mathbb{K}) \to \mathbb{K} \quad \text{with} \quad pr(E_1, r_1, \ldots, E_d, r_d, 0) = 0\]

becomes locally near \((E_1, r_1, \ldots, E_d, r_d, 0)\) a (real or complex) analytic isomorphism onto an open interval or disc around \(\lambda = 0 \in \mathbb{K}\), fitting with the description given in Theorem 3 in terms of the implicit function theorem.

**Proof.** The claim follows from

\[
\left( \frac{\partial f_l}{\partial \varepsilon_k}(E_1, r_1, \ldots, E_d, r_d, 0), \frac{\partial f_l}{\partial \varrho_k}(E_1, r_1, \ldots, E_d, r_d, 0) \right) = (\delta_{lk}, 0)
\]

and

\[
\left( \frac{\partial g_k}{\partial \varepsilon_l}(E_1, r_1, \ldots, E_d, r_d, 0), \frac{\partial g_k}{\partial \varrho_l}(E_1, r_1, \ldots, E_d, r_d, 0) \right) = \left( 0, \delta_{lk} \cdot \frac{1}{r_l} \right)
\]

for \(l, k = 1, \ldots, d\). \(\Box\)

**Remark 13.** Let us rewrite for a fixed \(\lambda \in \mathbb{C}\) the equations \[(26a)\] and \[(26b)\] in terms of the \(d\) polynomials

\[F_l(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d) := f_l(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) \prod_{k=1}^{d} (\varepsilon_k + \varepsilon_l)
\]

of degree \(d + 1\) and the \(d\) polynomials

\[G_l(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d) := g_l(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d, \lambda) \cdot \varrho_l \prod_{k=1}^{d} (\varepsilon_k + \varepsilon_l)^2
\]

of degree \(2d + 1\) (for \(l = 1, \ldots, d\)), so that:

\[Z(\mathbb{C}) \cap \{pr = \lambda\} \subset \{(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d) \in \mathbb{C}^{2d} \mid F_l = 0, G_l = 0, l = 1, \ldots, d\} \times \{\lambda\}.
\]

If for a given \(\lambda \in \mathbb{C}\) the set

\[\{(\varepsilon_1, \varrho_1, \ldots, \varepsilon_d, \varrho_d) \in \mathbb{C}^{2d} \mid F_l = 0, G_l = 0, l = 1, \ldots, d\}
\]
is finite, then one gets by the affine Bezout inequality [Sch95, Thm 3.1] the upper estimate $(d+1)^d(2d+1)^d$ for the number of solutions of the equations (26a) and (26b) (for this $\lambda$).

Let us come back to the rational function $R(z)$ from (25) for the case of positive real $E_1 > 0$ and $r_1 > 0$ for $l = 1, \ldots, d$ related to the solution of Theorem 3 as discussed before. Then

$$R'(z) = 1 + \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\eta_k}{(\varepsilon_k + z)^2} > 0$$

for all $\lambda \geq 0$ and $z \in \mathbb{R} \setminus \{-\varepsilon_1, \ldots, -\varepsilon_d\}$.

**Lemma 14.** $R^{-1}(E_k)$ consists for all $E_k > 0$ of $d+1$ different real points so that assumption (ii) of Ansatz 5 holds. Moreover we can choose $\mathcal{U} = \mathcal{U}_1 = \cdots = \mathcal{U}_d$ as a small simply connected open neighbourhood of $(R(0), \infty) \subset \mathbb{R}$ in $\mathbb{C}$, with $\mathcal{V} = \mathcal{V}_1 = \cdots = \mathcal{V}_d$ also containing $(0, \infty)$. By shrinking of $\mathcal{U}$ we can even assume that $\mathcal{V} \subset \{z \in \mathbb{C} | \text{Re}(z) > 0\}$. Then the assumptions (v) and (vi) of Ansatz 5 hold for all $w \in \mathcal{V}$ with $\mathcal{V}$ small enough, as well as the assumption (iv) with $\mathcal{G}^{(0)}$ as in (21) resp. (24).

**Proof.** If we order the numbering of the $\varepsilon_i$ as $\varepsilon_i < \varepsilon_{i+1}$ for $i = 0, \ldots, d$ with $\varepsilon_0 := -\infty$ and $\varepsilon_{d+1} := +\infty$, then

$$R : (-\varepsilon_{i+1}, -\varepsilon_i) \to \mathbb{R}$$

is for all $i = 0, \ldots, d$ strictly monotone increasing and injective by the estimate $R'(z) > 0$ above and the intermediate value theorem. This proves the first claim. Similarly, $R^{-1}(R(w))$ consists for any $w \in \mathcal{V} \cap (0, \infty)$ of $d+1$ real points which we can order as $\hat{w}^j \in (-\varepsilon_{i+1}, -\varepsilon_i)$. Therefore, $\hat{w}^j \in (0, \infty)$ and all $R(-\hat{w}^j)$ are distinct for $w \in \mathcal{V} \cap (0, \infty)$, because $R$ is injective on $(-\varepsilon_1, \infty)$. Moreover, $R$ is an injective immersion in $(-\varepsilon_1, \infty)$, which is an open condition so that also the second claim follows. Finally the assumption (iv) with $\mathcal{G}^{(0)}$ as in (24) follows from $0 \neq z + w$ for all $z, w \in \mathcal{V}$, since $\text{Re}(z + w) = \text{Re}(z) + \text{Re}(w) > 0$. \hfill \Box

Now we can finish this section with the new

**Proof of Theorem 3.** We have seen that equation (10) can be solved by Ansatz 5 to $\mathcal{G}^{(0)}(R(z), R(w)) = \mathcal{G}^{(0)}(z, w)$, where $R$ is given in (25) and $\mathcal{G}^{(0)}$ in (21) resp. (24). This solution depends on the choice of preimages $\varepsilon_k \in R^{-1}(E_k)$ made in (iv) of Ansatz 5. Any solution $\{\varepsilon_1, \ldots, \varepsilon_d\}$ of the system of equations (11) provides a solution of (21), if also the assumptions (iv), (v) and (vi) of Ansatz 5 hold. Theorem 3 selects one particular solution of (10) which satisfies the assumptions (ii), (iv), (v) and (vi) of Ansatz 5 by Lemma 14. Hence also relation (vii) of Ansatz 5 holds by Proposition 11. The choice $\lim_{\lambda \to 0} \varepsilon_k = E_k$ and $\lim_{\lambda \to 0} \theta_k = r_k$ is made to recover in the limit $\lambda \to 0$ the moments of the Gaussian measure (1).
It remains to show (12). On the right hand side of (18) we use the the symmetry \( G^{(0)}(\varepsilon_k, w) = G^{(0)}(w, \varepsilon_k) \) from Proposition 10 and express \( G^{(0)}(w, \varepsilon_k) \) as (21) for \( w \mapsto \varepsilon_k \) and \( z \mapsto w \). Dividing by \( (R(w) - R(z)) \) gives (12). \( \square \)

5. The Planar 1 + 1-Point Function

The 1 + 1-point function \( G^{(0)}_{[a|b]} \) is the cumulant of length 2 and cycle type \((2,0)\) (i.e. two cycles \( a \) and \( b \) of length 1), see Appendix A. We derive in Appendix B its equation of motion (43) whose restriction to the planar sector (of degree or genus \( g = 0 \)) reads

\[
(E_a + E_a)G^{(0)}_{[a|b]} = -\frac{\lambda}{N} \sum_{k=1}^{N} G^{(0)}_{[ak]} G^{(0)}_{[a|b]} + \frac{\lambda}{N} \sum_{k=1}^{N} \frac{G^{(0)}_{[k|b]} - G^{(0)}_{[a|b]}}{E_k - E_a} + \frac{\lambda}{N} \frac{G^{(0)}_{[b|b]} - G^{(0)}_{[a|b]}}{E_b - E_a}.
\]

(28)

We interpret this equation as evaluation \( G^{(0)}_{[a|b]} = G^{(0)}(\varepsilon_a | \varepsilon_b) \) of a function \( G^{(0)}(z|w) \) which satisfies

\[
(R(z) - R(-z))G^{(0)}(z|w) - \frac{\lambda}{N} \sum_{k=1}^{d} r_k G^{(0)}(\varepsilon_k|w) = \frac{\lambda}{N} \frac{G^{(0)}(z, w) - G^{(0)}(w, w)}{R(z) - R(w)}.
\]

(29)

The identity (17) was decisive here, and multiplicities \( r_k \) of the \( E_k = R(\varepsilon_k) \) were admitted.

Let \( z \in \{0, \pm \alpha_1, \ldots, \pm \alpha_d\} \) be the list of roots of

\[
0 = R(z) - R(-z) = 2z - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z} - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{\varrho_k}{-\varepsilon_k + z},
\]

with the convention \( \alpha_i > 0 \) and \( \alpha_i \neq \alpha_j \) for \( i \neq j \). Since here all \( \varepsilon_k > 0 \) and \( \varrho_k > 0 \) are positive real numbers, we can argue as for the rational function \( R \) that also the rational function \( \tilde{R} \) with \( \tilde{R}(z) := R(z) - R(-z) \) maps each of the 2d + 1 connected components of \( \mathbb{R} \setminus \{-\varepsilon_1, \ldots, \varepsilon_d\} \) bijectively onto \( \mathbb{R} \). So the odd function \( \tilde{R} \) has indeed 2d + 1 different real roots \( \{0, \pm \alpha_1, \ldots, \pm \alpha_d\} \) of the equation \( \tilde{R} = 0 \).

Since \( G^{(0)}(\alpha_k|w) \) must be regular\(^3\) for any \( w > 0 \), evaluation at \( z = \alpha_k \) produces \( d \) equations

\[
\frac{\lambda}{N} \sum_{l=1}^{d} \frac{r_l G^{(0)}(\varepsilon_l|w)}{R(\alpha_k) - R(\varepsilon_l)} = \frac{\lambda}{N} \frac{G^{(0)}(\alpha_k, w) - G^{(0)}(w, w)}{R(\alpha_k) - R(w)}.
\]

(30)

\(^3\) Be careful to distinguish \( G^{(0)}(z|w) \) from \( G^{(0)}(z, w) \).

\(^4\) Regularity of \( G^{(0)}(\alpha_k|w) \) is here a technical assumption which is justified by viewing (29) as limiting case of singular integral equations of Carleman type. Their solutions are regular for any \( z, w > 0 \).
This equation is immediately solved via Proposition 4, which thus expresses \( G^{0}(z|w) \) in terms of \( G^{0}(z, w) \) already known from Theorem 3 (here the \( R(\alpha_k) \) and \( R(\varepsilon_l) \) are pairwise distinct, since \( R \) is injective on \( (-\varepsilon_1, \infty) \)):

**Proposition 15.** Equation (29) has the solution

\[
G^{0}(z|w) = \frac{\lambda}{R(z) - R(-z)} \left\{ \frac{G^{0}(z, w) - G^{0}(w, w)}{R(z) - R(w)} \right\} - \sum_{k,l=1}^{d} \frac{(R(\alpha_l) - R(\varepsilon_k))A_l(R(\varepsilon_k))E_k(R(\alpha_l))}{R(z) - R(\varepsilon_k)} \frac{G^{0}(\alpha_l, w) - G^{0}(w, w)}{R(\alpha_l) - R(w)},
\]

where \( A_l(x) = A(x) = \prod_{k=1}^{d} (x - R(\alpha_k)) \) and \( E_j(y) = \prod_{k=1}^{d} (y - R(\varepsilon_k)) \).

Proof. Equation (30) is with Proposition 4 solved by

\[
\frac{r_k}{N} G^{0}(\varepsilon_k|w) = \sum_{i=1}^{d} (R(\alpha_i) - R(\varepsilon_k)) A_i(R(\varepsilon_k)) E_i(R(\alpha_i)) \frac{G^{0}(\alpha_i, w) - G^{0}(w, w)}{R(\alpha_i) - R(w)}.
\]

Inserting this back into (30) confirms the assertion. □

Properties of the solution will be studied elsewhere.

6. Outlook

We have developed a new algebraic geometrical solution strategy for the two initial cumulants of the quartic analogue of the Kontsevich model. We will now argue that the same method will also solve all other affine equations of the recursive system. The first step is to extend the procedure outlined in Appendix B to cumulants of any length \( 2p \). We know from [GW14] that only ‘basic’ cycle types \((\ell_1, \ell_2, 0, \ldots, 0)\) with \( \ell_1 + 2\ell_2 = 2p \) need to be discussed. All other cycle types reduce to polynomials in cumulants of basic cycle type and fractions \( \frac{1}{E_i - E_j} \).

For cycle type \((0, \ldots, 0, 1)\) these polynomials have been explicitly constructed in [dJHW19], and the same combinatorial structure (‘Catalan tables’) should give the general case.

The affine equations for cumulants of basic cycle type fall into two classes. Equations for \((\ell_1, \ell_2, 0, \ldots, 0) \mapsto (\ell_1 + 2, \ell_2 - 1, 0, \ldots, 0)\) are of the form as (29),

\[
(R(z) - R(-z))f(z, \bar{w}) - \frac{\lambda}{N} \sum_{j=1}^{d} \frac{r_j f(\varepsilon_j, \bar{w})}{R(\varepsilon_j) - R(z)} = h(z, \bar{w}),
\]

(32)
where $\vec{w} = (w_1, \ldots, w_{2p-1})$ and $h(z, \vec{w})$ is known by induction. This equation is immediately solved via [Sch59], as in Proposition 15, to

$$f(z, \vec{w}) = \frac{h(z, \vec{w}) - \sum_{k,l=1}^{d} \frac{(R(\alpha_l) - R(\varepsilon_k))A_l(R(\varepsilon_k))E_k(R(\alpha_l))}{R(z) - R(\varepsilon_k)} h(\alpha_l, \vec{w})}{R(z) - R(-z)}.$$  

Equations for $(\ell_1, \ell_2, 0, \ldots, 0) \mapsto (\ell_1, \ell_2 + 1, 0, \ldots, 0)$ are of the form

$$(R(w) - R(-z))f(z, w, \vec{w}) - \frac{\lambda}{N} \sum_{j=1}^{d} \frac{r_j f(\varepsilon_j, w, \vec{w})}{R(\varepsilon_j) - R(z)} = h(z, w, \vec{w}).$$  

where $\vec{w} = (w_1, \ldots, w_{2p-2})$ and $h(z, w, \vec{w})$ is known by induction. Here the other equation of motion (8) is essential. The inhomogeneity $h(z, w, \vec{w})$ thus contains derivatives of cumulants with respect to the $E_k$, reexpressed via chain rule as derivatives with respect to $z, w, \vec{w}$. Details are not yet worked out. Since (33) has the same form as (18), the same discussion combined with [Sch59] solves (33) to

$$f(z, w, \vec{w}) = \frac{h(z, w, \vec{w}) - \sum_{k,l=1}^{d} \frac{(R(-\hat{w}^l) - R(\varepsilon_k))W_l(R(\varepsilon_k))E_k(R(-\hat{w}^l))}{R(z) - R(\varepsilon_k)} h(-\hat{w}^l, w, \vec{w})}{R(w) - R(-z)},$$

where

$$W_i(x) = \frac{W(x)}{(x - R(-\hat{w}^i))W'(R(-\hat{w}^i))} \quad \text{and} \quad W(x) := \prod_{k=1}^{d} (x - R(-\hat{w}^k)).$$

Although many details remain to be elaborated, our work strongly suggests that the quartic analogue of the Kontsevich model is exactly solvable in terms of rational functions. This rationality is a strong signal of integrability. Apart from understanding the derivatives contributing to $h(z, w, \vec{w})$ in (33), these details include a characterisation of poles, residues, branch points and symmetries. Here a considerable combinatorial challenge will be to master the inhomogeneities $h(z, w, \vec{w})$ and $h(z, w, \vec{w})$ in all affine equations.

We thus see that a complete solution of the quartic analogue of the Kontsevich model can succeed solely via inversion [Sch59] of Cauchy matrices, without any reference to topological recursion [EO07]. But to make integrability precise it seems highly desirable to clarify the relationship between our approach and topological recursion. Comparing (29) with the equation in [Eyn16, §6.4] for $\Omega_{0,2}(z, w)$ in the Kontsevich model gives a first hint that the classical spectral curve of the quartic model could be a rational plane algebraic curve (i.e. with the corresponding Riemann surface given by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) realised as

$$E(x(z), y(z)) = 0 \quad \text{with} \quad x(z) = R(z), \quad y(z) = -R(-z).$$
The inhomogeneities \( h(z, w), \tilde{h}(z, w) \) in [32] and [33] seem to be, however, very different.

Many questions arise: Can the inversion by Cauchy matrices be related to the recursion kernel of topological recursion? What corresponds to the Bergmann kernel? Is there any symplectic invariance? Can we access the partition function? Is its logarithm a \( \tau \)-function for a Hirota equation? What hierarchy of evolution equations arises? Can the logarithm of the partition function be understood as a formal power series with rational coefficients? Are these rational coefficients of topological significance? Could they be intersection numbers of tautological characteristic classes on a moduli space? Is there a geometric construction, analogous to critical trajectories of Strebel differentials, which relates them to ribbon graphs with 4-valent vertices?

**Appendix A. Decomposition of moments via cumulants**

The moments [3] decompose into cumulants (see e.g. [McC12, Spe83]),

\[
\left\langle \prod_{i=1}^{\pi} e_{k_i l_i} \right\rangle = \sum_{\pi \text{ partitions of } \{1, \ldots, n\}} \prod_{\beta \in \pi} \left\langle \prod_{i \in \beta} e_{k_i l_i} \right\rangle_c.
\]

(34)

There is a similar formula expressing the cumulants in terms of moments [Spe83 Eq.(1.2)], related to [34] via Möbius inversion on the partially ordered set of (partitions of) subsets of indices (the partition lattice of \([N] \times [N]\))

\[ I := \{k_1 l_1, \ldots, k_n l_n\} \subset [N] \times [N], \]

with \(|I| = n\) and \([N] := \{1, \ldots, N\}\). For a quartic potential [2], moments and cumulants are only non-zero if \(n\) is even and every block \(\beta\) is of even length. For example,

\[
\langle e_{k_1 l_1} e_{k_2 l_2} e_{k_3 l_3} e_{k_4 l_4} \rangle = \langle e_{k_1 l_1} e_{k_2 l_2} e_{k_3 l_3} e_{k_4 l_4} \rangle_c + \langle e_{k_1 l_1} e_{k_2 l_2} \rangle_c \langle e_{k_3 l_3} e_{k_4 l_4} \rangle_c
\]

\[+ \langle e_{k_1 l_1} e_{k_3 l_3} \rangle_c \langle e_{k_2 l_2} e_{k_4 l_4} \rangle_c + \langle e_{k_1 l_1} e_{k_4 l_4} \rangle_c \langle e_{k_2 l_2} e_{k_3 l_3} \rangle_c. \]

Note that in our context the moments \(\langle \prod_{i=1}^{n} e_{k_i l_i} \rangle\) are invariant under permutations of \(I := \{k_1 l_1, \ldots, k_n l_n\}\) so that they only depend on the subset \(I \subset [N] \times [N]\), but not on the choice of a labelling \(I = \{k_1 l_1, \ldots, k_n l_n\} \simeq [n]\). By [Spe83 Eq.(1.2)] the same is then true for the cumulants, i.e. \(\langle \prod_{i \in \beta} e_{k_i l_i} \rangle_c\) only depends on the subset \(\{k_i l_i | i \in \beta\} \subset [N] \times [N]\).

Next, the structure of the Gaussian measure [3] (together with the invariance of a trace under cyclic permutations) implies that the cumulant \(\langle \prod_{i=1}^{n} e_{k_i l_i} \rangle_c\) corresponding to \(I = \{k_1 l_1, \ldots, k_n l_n\}\) with \(|I| = n > 0\) is only non-zero if \(I\) has a permutation \(\sigma\) with \(pr_2 = pr_1 \circ \sigma\). Here

\[ pr_i : [N] \times [N] \supset I \rightarrow [N] \]
is the projection onto the corresponding factor for \( i = 1, 2 \). By choosing a labelling

\[
I := \{k_1 l_1, \ldots, k_n l_n\} \simeq [n]
\]
as before, this corresponds to a permutation \( \sigma \) in the symmetric group \( S_n \), with \((l_1, \ldots, l_n) = (k_{\sigma(1)}, \ldots, k_{\sigma(n)})\).

Therefore, the cumulant \( \left< \prod_{i=1}^n e_{k_i l_i} \right>_c \) only depends on \( I \) and the conjugacy class of a permutation in \( S_n \) (corresponding to the permutation \( \sigma \) of \( I \) with \( p r_2 = p r_1 \circ \sigma \)), which is again independent of the choice of the labelling of \( I \). In fact such conjugacy classes in \( S_n \) just correspond to the different cycle types of a permutation in the symmetric group \( S_n \). The cycle type of \( \sigma \) is the \( n \)-tuple \((\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))\) where \( \ell_k(\sigma) \) is the number of cycles of length \( k \) in \( \sigma \), with \( \sum_{i=1}^n i \ell_i(\sigma) = n \). The number of cycles in a permutation \( \sigma \) is \( b(\sigma) = \sum_{i=1}^n \ell_i(\sigma) \).

The number of different cycle types is the partition number \( p(n) \), and there are \( \frac{n!}{\prod_{i=1}^n i^{\ell_i(\sigma)} \ell_i(\sigma)!} \) permutations with the same cycle type \((\ell_1, \ldots, \ell_n)\).

Conversely, the \( l \)-indices of a non-vanishing cumulant \( \langle e_{k_1 l_1} \cdots e_{k_n l_n} \rangle_c \) are completely determined by the cycle type and the information which \( k \)'s belong in which cyclic order to the same cycle. If, after renaming the \( k \)'s, \((k_1^1, \ldots, k_n^1)\) belong to one cycle, \((k_1^2, \ldots, k_n^2)\) belong to another cycle, and so on up to the \( b \)th cycle, this information uniquely encodes a cumulant (with \( n = n_1 + \cdots + n_b \))

\[
N^n \langle (e_{k_1^1} e_{k_2^1} e_{k_3^1} \cdots e_{k_n^1}) \cdots (e_{k_1^b} e_{k_2^b} e_{k_3^b} \cdots e_{k_n^b}) \rangle_c =: N^{2-b} G_{|k_1^1 \ldots k_n^1| \ldots |k_1^b \ldots k_n^b|}.
\]

(35)

The power series expansion of the Fourier transform \( \mathcal{Z}(M) \) into moments (3) can be compared with the insertion of (35) into (34). The first terms are:

\[
\mathcal{Z}(M) = 1 - \frac{1}{N^2} \sum_{j,k=1}^{N} \left\{ \frac{N}{2} G_{|jk|} M_{jk} M_{kk} + \frac{1}{2} G_{|jk|} M_{jj} M_{kk} \right\}
\]

\[
+ \frac{1}{N^4} \sum_{j,k,l,m=1}^{N} \left\{ \frac{N}{4} G_{|jklm|} M_{jk} M_{kl} M_{lm} M_{mj} + \frac{1}{3} G_{|jklm|} M_{jj} M_{kl} M_{lm} M_{mk} \right\}
\]

\[
+ \frac{1}{8} G_{|jklm|} M_{jk} M_{kl} M_{lm} M_{ml} + \frac{1}{4N} G_{|jklm|} M_{jj} M_{kk} M_{lm} M_{ml}
\]

\[
+ \frac{1}{24N^2} G_{|jklm|} M_{jj} M_{kk} M_{ll} M_{mm} + \frac{N^2}{8} G_{|jklm|} M_{jj} M_{kk} M_{ll} M_{mm}
\]

\[
+ \mathcal{O}(M^6).
\]

(36)
Appendix B. Equations for the second cumulant

We derive here equations for the two non-vanishing second-order cumulants $G_{[ab]} = \frac{1}{N} \langle e_{ab} e_{ca} \rangle_c$ of cycle type $(0, 1)$ (i.e. one cycle $ab$ of length 2) and $G_{[a][b]} = \langle e_{aa} e_{bb} \rangle_c$ of cycle type $(2, 0)$ (i.e. two cycles $a$ and $b$ of length 1).

For that we start from (7) with $Z(M)$ given by (36), apply $\frac{N(E_a + E_b)}{i} \frac{\partial}{\partial M_{ab}}$ and put $M = 0$. For $a \neq b$ this gives the following result (the underlining should be ignored for the moment; we explain it later):

\[
(E_a + E_b) G_{[ab]} = 1 - \lambda \left\{ \frac{1}{N^2} \sum_{k,l=1}^{N} G_{[bk]kl] + \frac{1}{N} \sum_{k=1}^{N} (G_{[ab]} G_{[ak]} + G_{[ab]} G_{[bk]}) \right. \\
+ \frac{1}{N^2} (G_{[ab]} G_{[ab]} + G_{[abb]} + G_{[baa]} + G_{[ab]} (G_{[a][a]} + G_{[a][b]} + G_{[b][b]})) \\
+ \frac{1}{N^3} \sum_{k=1}^{N} (G_{[b][bak]} + G_{[a][bak]} + G_{[b][bak]} + G_{[ab][bk]} + G_{[ab][ak]}) \\
+ \frac{1}{N^4} (G_{[a][a][a]} + G_{[a][b][a]} + G_{[b][b][a]}) \right\} . \tag{37}
\]

Next, we set $b \equiv a$ in (7) for $Z(M)$ given by (36), apply $\frac{N^2(E_a + E_b)}{i} \frac{\partial}{\partial M_{aa}}$ for $a \neq b$ and obtain for $M = 0$ (ignore again the underlining):

\[
(E_a + E_a) G_{[a][b]} = -\lambda \left\{ G_{[b][a]} G_{[ab]} + \frac{1}{N^2} \sum_{k,l=1}^{N} G_{[b][a][kl]} \right. \\
+ \frac{1}{N} \sum_{k=1}^{N} (G_{[b][ka]} + G_{[b][a]} + G_{[a][b]} G_{[a][b]} + G_{[a][b]} G_{[b][k]}) \\
+ \frac{1}{N^2} (G_{[b][aaa]} + G_{[a][aab]} + G_{[a][aab]} + G_{[b][aab]} + G_{[b][bab]} + 3G_{[a][b]} G_{[a][a]}) \\
+ \frac{1}{N^3} \sum_{k=1}^{N} (G_{[a][a][k]} + G_{[a][b][a]} + G_{[b][b][k]} G_{[a][a]} + \frac{1}{N^4} G_{[b][a][a][a]} \right\} . \tag{38}
\]

Equations (37) and (38) are the analogues of Dyson-Schwinger equations in quantum field theory. In this form they provide little information because the right hand sides are too complicated. We will now establish from the equations of motion (7) two other identities which collect the underlined terms in (37) and (38) into a function of the left hand sides.

To establish the identities, set $b \mapsto k$ in (7) and apply $\frac{N(E_a + E_b)}{i} \frac{\partial}{\partial M_{ab}}$. Next, set $a \mapsto k$ in (7) and apply $\frac{N(E_a + E_b)}{i} \frac{\partial}{\partial M_{ak}}$. Take the difference of both equations and sum over $k$:

\[
-N \sum_{k=1}^{N} (E_a - E_b) \frac{\partial^2 Z(M)}{\partial M_{ak} \partial M_{kb}} = \sum_{k=1}^{N} \left( M_{ka} \frac{\partial Z(M)}{\partial M_{kb}} - M_{kb} \frac{\partial Z(M)}{\partial M_{ak}} \right) . \tag{39}
\]
This is a Ward-Takahashi identity first discovered in [DGMR07]. The strategy which we follow here was suggested in [GW14]. We insert (36) into (39) and evaluate the derivatives for $a \neq b$:

$$
\frac{1}{N} \sum_{k=1}^{N} \left\{ (G_{[kk]} - G_{[ak]}) M_{bk} M_{ka} + \frac{1}{N} (G_{[kk]} - G_{[a|k]}) M_{ba} M_{kk} \right\} \\
= \frac{1}{N} \sum_{k=1}^{N} (E_a - E_b) \left\{ \frac{1}{N} \sum_{l=1}^{N} G_{[k|al]} + G_{[ak]} G_{[lk]} \right. \\
+ \left. \frac{1}{N^2} (G_{[b|ak]} + G_{[a|bak]} + G_{[bl|ak]}) \right\} M_{bk} M_{ka} \\
+ \frac{1}{N^2} \sum_{k=1}^{N} (E_a - E_b) \left\{ G_{[b|ak]} + \frac{1}{N} \sum_{l=1}^{N} G_{[k|al]} + G_{[ab]}(G_{[b|k]} + G_{[a|k]}) \right. \\
+ \left. \frac{1}{N^2} (G_{[b|k|ab]} + G_{[a|k|ab]}) \right\} M_{ba} M_{kk} + \mathcal{O}(M^4) .
$$

(40)

Applying $N \frac{\partial^2}{\partial M_{ba} \partial M_{pp}}$ or $N^2 \frac{\partial^2}{\partial M_{ba} \partial M_{pp}}$ for $a \neq p \neq b$ gives two independent equations. We make the crucial assumption that they extend continuously to $p = a$ and $p = b$. After exchanging $p \leftrightarrow b$, these equations read

$$
- \frac{G_{[pb]} - G_{[ab]}}{E_p - E_a} = \frac{1}{N} \sum_{k=1}^{N} G_{[b|akp]} + G_{[ap]} G_{[b|p]} + \frac{1}{N^2} (G_{[b|ap]} + G_{[a|bp]} + G_{[bp|ab]}) ,
$$

(41a)

$$
- \frac{G_{[p|b]} - G_{[a|b]}}{E_p - E_a} = G_{[b|bpa]} + \frac{1}{N} \sum_{k=1}^{N} G_{[b|akp]} + G_{[ap]} G_{[b|p]} + G_{[ap]} G_{[a|b]} \\
+ \frac{1}{N^2} (G_{[b|ap]} + G_{[a|bp]}) ;
$$

(41b)

they hold for $p \neq a$. We make another crucial assumption, that (41a) and (41b) extend continuously to $p = a$. Then, summing (41a) over $p$ collects the double-underlined terms in (37) into $- \frac{1}{N} \sum_{p=1}^{N} \frac{G_{[p|b]} - G_{[a|b]}}{(E_p - E_a)}$, and the case $p = b$ of (41b) collects the single-underlined terms in (37). Similarly, summing (41b) over $p$ collects the single-underlined terms in (38) into $- \frac{1}{N} \sum_{p=1}^{N} \frac{(G_{[p|b]} - G_{[a|b]})}{(E_p - E_a)}$, and the case $p = b$ of (41a) collects the double-underlined terms in (38):

$$
(E_a + E_b) G_{[ab]} = 1 - \frac{\lambda}{N} \sum_{p=1}^{N} G_{[ab]} G_{[ap]} + \frac{\lambda}{N} \sum_{p=1}^{N} G_{[pb]} - G_{[ab]} \\
- \frac{\lambda}{N^2} \left( - \frac{G_{[b|b]} - G_{[a|b]}}{E_b - E_a} + G_{[b|ab]} + G_{[b|aa]} + G_{[ab]} G_{[a|a]} \right) \\
+ \frac{1}{N} \sum_{p=1}^{N} G_{[ap]} G_{[ab]} - \frac{\lambda}{N^2} G_{[a|a][ab]} ,
$$

(42)
\[
(E_a + E_a)G_{[a][b]} = \frac{\lambda}{N} \sum_{p=1}^{N} G_{[ap]}G_{[a][b]} + \frac{\lambda}{N} \sum_{p=1}^{N} \frac{G_{[p][b]} - G_{[a][b]}}{E_p - E_a} + \lambda \frac{G_{[b][b]} - G_{[a][a]}}{E_b - E_a}
- \frac{\lambda}{N^2} \left( G_{[b][a][a]} + G_{[a][a][b]} + 3G_{[a][b]}G_{[a][a]} + \frac{1}{N} \sum_{p=1}^{N} G_{[a][b][ap]} \right)
- \frac{\lambda}{N^4} G_{[b][a][a][a]} .
\] (43)

These identities have been found in [GW14] (by a faster, but less elementary approach).

Identities of such type can be solved by a further expansion of all arising functions \(G_\ldots\) as formal power series in \(N^{-2}\),

\[
G_\ldots = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} G^{(g)}_\ldots ,
\] (44)

together with the convention that \(\frac{1}{N} \sum_{1}^{N}\) is of order \(N^0\). For the degree or genus \(g = 0\) we thus obtain from (42) the closed equation (9) for \(G^{(0)}_{[a][b]}\), whereas the restriction of (43) to the degree or genus \(g = 0\) is (28). Both equations have been solved in this paper.

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Mathematisches Institut der WWU, Einsteinstr. 62, 48149 Münster, Germany, E-mail addresses: jschuerm@uni-muenster.de, raimar@math.uni-muenster.de