Well-posedness of stochastic third grade fluid equation

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Abstract

In this paper, we establish the well-posedness for the third grade fluid equation perturbed by a multiplicative white noise. This equation describes the motion of a non-Newtonian fluid of differential type with relevant viscoelastic properties. We are faced with a strongly nonlinear stochastic partial differential equation supplemented with a Navier slip boundary condition. Taking the initial condition in the Sobolev space $H^2$, we show the existence and the uniqueness of the strong (in the probability sense) solution in a two dimensional and non axisymmetric bounded domain.

Key words. Non-Newtonian fluid, stochastic partial differential equation, third grade fluid, well-posedness.

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1 Introduction

This work deals with fluids of differential type, which is a very special class of non-Newtonian fluids. For these fluids, the relation between the shear stress and the shear strain rate is not linear, this means that the viscosity does not satisfy Newton’s law. Considering the velocity field $y$ of the fluid, we introduce the Rivlin-Ericksen kinematic tensors in $\mathbb{A}_n$, $n \geq 1$, defined by

$$A_1(y) = \nabla y + \nabla y^T,$$

$$A_n(y) = \frac{d}{dt} A_{n-1}(y) + A_{n-1}(\nabla y) + (\nabla y)^T A_{n-1}(y), \quad n = 1, 2, \ldots,$$

(1.1)

(1.2)

While a Newtonian fluid is characterized by the relation $T = \nu A_1(y)$, where $T$ is the shear stress tensor and $\nu$ is the viscosity; a third grade fluid is given by the following constitutive equation

$$T = -pI + \nu A_1(y) + T_1 + T_2,$$

(1.3)

where $T$ is the Cauchy stress tensor, $\frac{D}{Dt} = \frac{d}{dt} + y \cdot \nabla$ stands for the material derivative, $p$ is the pressure, $I$ is the unit tensor,

$$T_1 = \alpha_1 A_2 + \alpha_2 A_1^2,$$

$$T_2 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (\text{tr} A_1) A_1,$$

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and $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $\beta_3$ are material moduli. The momentum equations is given by

$$\frac{Dy}{Dt} = \text{div} \, \mathbb{T}.$$  

According to the analysis in [15], [16] in order to allow the motion of the fluid to be compatible with thermodynamic, it should be imposed that

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24 \nu \beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0. \tag{1.4}$$

Setting

$$\beta = \beta_3, \quad A = A_1, \tag{1.5}$$

and denoting $v(y) = y - \alpha_1 \Delta y$, the deterministic equation of motion for a third grade fluid reads

$$\frac{d}{dt}(v(y)) = -\nabla p + \nu \Delta y - (Y \cdot \nabla) v - \sum_j v^j \nabla y^j - (\alpha_1 + \alpha_2) \text{div} \, (A^2) + \beta \text{div} \, (|A|^2 A) + U, \tag{1.6}$$

where $U$ represents a body force. In practice, the non-Newtonian fluids are real fluids present in industry, food processing, biological fluids, etc. (see [16], [15], [17], [18], [20]). In the case where $\beta = 0$, the model reduces to the second grade fluid model which is mathematically more tractable. However the second grade fluid model does not capture important rheological properties as for instance the shear thinning and shear thickening effects, so there is a real need to study the third grade fluid model.

From a theoretical point of view, this model is a strongly nonlinear partial differential equation modelling complex viscoelastic fluids, therefore the small noise perturbations should have relevant impact in the fluid dynamic. It is well known that increasing the typical velocity will increase the Reynolds number: the fluid develops a turbulent behaviour and small disturbances should have strong macroscopic effects on the dynamic. Let us mention the pioneer work [2] on the stochastic Navier-Stokes equations, and [3] (see Lemma 2.2) where the stochastic Navier-Stokes equations were deduced from fundamental principles showing that the stochastic equations are real physical models. There is an extensive literature on the stochastic Eulerian description of a Newtonian fluid (see also [11], [13] and references therein for a stochastic Lagrangian approach).

In this paper, we perturb the deterministic third grade fluid model by a multiplicative white noise

$$d(v(Y)) = (-\nabla p + \nu \Delta Y - (Y \cdot \nabla) v - \sum_j v^j \nabla Y^j + (\alpha_1 + \alpha_2) \text{div} \, (A^2) + \beta \text{div} \, (|A|^2 A) + U) \, dt + \sigma(t, Y) \, dW_t, \tag{1.7}$$

and we extend the deterministic results on the existence and uniqueness obtained in [5] and [6], where the parameter $\beta$ is considered strictly positive, the equation is supplemented with a Navier slip boundary condition and the initial condition is taken in the Sobolev space $H^2$. As far as we know, the stochastic third grade fluid equations are being studied for the first time in this work. Here, we also consider $\beta > 0$ and the initial condition in $H^2$.

Concerning the analysis, we recall that the deterministic strategy in [5] and [6] is based on the deduction of a priori estimates that allow to use compactness theorems in order to pass the nonlinear terms to the limit in the weak sense. However, for the stochastic equation, due to the lack of regularity with respect to time and to the stochastic parameter, we can not use the compactness arguments to pass to the limit in those nonlinear terms. Instead, we will follow the methods introduced in [4], which have been successfully applied to the stochastic second grade fluids in...
More precisely, we consider an appropriate Galerkin basis, and deduce suitable uniform estimates in order to get weak convergence of a subsequence. Next we project the weak limit on the finite $n$–dimensional space. We then show that the difference between the sequence obtained by projecting the weak limit and the finite dimensional Galerkin approximations converges strongly to zero up to a special stopping time. Finally we are able to identify the nonlinear terms of the equation.

The remainder of this paper is structured into four sections and one appendix. Section 2 is devoted to the introduction of the functional settings and appropriate notations. In Section 3 we present some properties of the nonlinear terms of equation (1.7) which will be applied in the next section. The main results concerning the existence and uniqueness of the strong stochastic solution are established in Section 4. Finally, in the Appendix we collect some known inequalities related with the nonlinear terms of the equation that are used throughout the article.

## 2 Functional setting and notations

We consider the stochastic third grade fluid equation (1.7) in a bounded, not axisymmetric and simply connected domain $\mathcal{O}$ of $\mathbb{R}^2$ with a sufficiently regular boundary $\Gamma$, and supplemented with a Navier slip boundary condition, which reads

$$
\begin{aligned}
\left\{ \begin{array}{ll}
d(\nu(Y)) = (-\nabla p + \nu \Delta Y - (Y \cdot \nabla)v - \sum_{j=1}^2 \nu^j \nabla Y^j + (\alpha_1 + \alpha_2) \nabla (A^2) + \beta \nabla \left( |A|^2 A \right) + U) dt + \sigma(t, Y) d\mathcal{W}_t, & \text{in } \mathcal{O} \times (0, T), \\
\nabla Y = 0 & \text{in } \mathcal{O} \times (0, T), \\
Y \cdot n = 0, & \text{on } \Gamma \times (0, T), \\
y(0) = Y_0 & \text{in } \mathcal{O},
\end{array} \right.
\end{aligned}
$$

where $Y$ is the velocity field of the fluid, $\nabla Y$ is its Jacobian matrix, $D(Y) = \nabla Y + (\nabla Y)^\top$, $A = A(Y) = 2D(Y)$, $\nu(Y) = Y - \alpha_1 \Delta Y$ and the constants $\nu$, $\alpha_1$, $\alpha_2$, $\beta$ verify (1.4)-(1.5).

The stochastic perturbation is defined by

$$
\sigma(t, Y) d\mathcal{W}_t = \sum_{k=1}^d \sigma^k(t, Y) d\mathcal{W}_t^k,
$$

where the diffusion coefficient

$$
\sigma(t, Y) = (\sigma^1(t, Y), \ldots, \sigma^d(t, Y))
$$

satisfy suitable growth assumptions that will be defined below, and $\mathcal{W}_t = (\mathcal{W}_t^1, \ldots, \mathcal{W}_t^m)$ is a standard $\mathbb{R}^m$–valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ for $\mathcal{W}_t$. We assume that $\mathcal{F}_0$ contains every $P$–null subset of $\Omega$.

We consider the following Hilbert spaces

$$
\begin{aligned}
H &= \left\{ y \in L^2(\mathcal{O}) \mid \text{div } y = 0 \text{ in } \mathcal{O} \text{ and } y \cdot n = 0 \text{ on } \Gamma \right\}, \\
V &= \left\{ y \in H^1(\mathcal{O}) \mid \text{div } y = 0 \text{ in } \mathcal{O} \text{ and } y \cdot n = 0 \text{ on } \Gamma \right\}, \\
W &= \left\{ y \in V \cap H^2(\mathcal{O}) \mid (n \cdot D(y)) \cdot \tau = 0 \text{ on } \Gamma \right\}.
\end{aligned}
$$

On $H$ we consider the $L^2$–inner product $(\cdot, \cdot)$ and the associated norm $\| \cdot \|_2$. 

[8, 21] (see also 9, 10, 13, 19, 22, 24, 25).
Let us introduce the Helmholtz projector \( P : L^2(\Omega) \rightarrow H \), which is the linear bounded operator characterized by the following \( L^2 \)-orthogonal decomposition
\[
v = PV + \nabla \phi, \quad \phi \in H^1(\Omega).
\]

We define the following inner products
\[
(u, z)_V := (v(u), z) = (u, z) + 2\alpha_1 (Du, Dz), \quad (2.3)
\]
\[
(u, z)_W := (u, z)_V + (Pv(u), Pv(z)), \quad (2.4)
\]
and denote by \( \| \cdot \|_V \) and \( \| \cdot \|_W \) the norms induced by these inner products on \( V \) and \( W \), respectively.

The space \( V \) being a subspace of \( H^1(\Omega) \) is naturally endowed with the norm \( \| \cdot \|_{H^1} \). The norms \( \| \cdot \|_{H^1} \) and \( \| \cdot \|_V \) on the space \( V \) are equivalent. Similarly \( W \subset H^2(\Omega) \) and the norms \( \| \cdot \|_W \) and \( \| \cdot \|_{H^2} \) are equivalent on \( W \).

In addition, through the article the usual norms on the spaces \( L^p(\Omega) \), \( p > 0 \), are denoted by \( \| \cdot \|_p \) and the norms on the Sobolev spaces \( W^{1,p}(\Omega) \) for \( p > 2 \) are denoted by \( \| \cdot \|_{W^{1,p}} \).

Let us introduce the trilinear functional
\[
b(\phi, z, y) = (\phi \cdot \nabla z, y), \quad \forall \phi, z, y \in V. \quad (2.5)
\]

Taking into account that \( \phi \) is divergence free and \( (\phi \cdot n) = 0 \) on \( \Gamma \), a standard integration by parts gives
\[
b(\phi, z, y) = -b(\phi, y, z). \quad (2.6)
\]

Assume that \( \sigma = (\sigma^1, \ldots, \sigma^m) : [0, T] \times V \rightarrow (L^2(\Omega))^m \) is Lipschitz in the second variable and verifies a growth condition, i.e., there exist positive constants \( L, K \) and \( 0 \leq \gamma < 2 \) such that
\[
\| \sigma(t, y) \|_2^2 \leq L(1 + \| y \|_{W^{1,4}}), \quad \forall y \in W^{1,4}(\Omega) \cap V, \quad (2.7)
\]
\[
\| \sigma(t, y) - \sigma(t, z) \|_2^2 \leq K \| y - z \|_V^2, \quad \forall y, z \in V, \ t \in [0, T], \quad (2.8)
\]
where
\[
\| \sigma(t, y) \|_2^2 := \sum_{i=1}^m \| \sigma^i(t, y) \|_2^2.
\]

We also introduce the notation
\[
| (\sigma(t, y), v) | := \left( \sum_{k=1}^m (\sigma^k(t, y), v)^2 \right)^{1/2}, \quad \forall v \in L^2(\Omega).
\]

In addition, we take \( p \geq 6 \) and suppose that the initial condition \( Y_0 \) and the force \( U \) satisfy
\[
Y_0 \in L^p(\Omega, W), \quad \text{there exists } \lambda > 0 \text{ such that } \quad E e^{\lambda \left(t_0 \int_0^T \| U \|^2 ds + \| Y_0 \|^2 \right)} < \infty. \quad (2.9)
\]

Here, we recall the Korn inequality
\[
\| y \|_{W^{1,p}} \leq K_1(p) \left( \| y \|_p + \| A(y) \|_p \right), \quad \forall y \in V, \quad p \geq 2, \quad (2.10)
\]
and the Poincaré inequality
\[ \|y\|_2 \leq \mathcal{P}\|\nabla y\|_2, \quad \forall y \in V. \] (2.11)

For non-axisymmetric bounded domains, we also have the following version of the Korn inequality (see Theorem 3 in [14])
\[ \|\nabla y\|_2 \leq K_2(\mathcal{O})\|A(y)\|_2, \quad \forall y \in V. \] (2.12)

The Sobolev embedding \( H^1(\mathcal{O}) \hookrightarrow L^4(\mathcal{O}) \) and (2.11) give
\[ \|y\|_4 \leq K_3\|\nabla y\|_2, \quad \forall y \in V. \] Combining this inequality with (2.12), we get
\[ \|y\|_4 \leq K_3\|\nabla y\|_2 \leq K_3K_2(\mathcal{O})\|A(y)\|_2. \] (2.13)

Due to the embedding \( L^4(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \), we have
\[ \|y\|_2 \leq C_*\|y\|_4. \] (2.14)

Then (2.13), (2.14) and (2.10) yield the following lemma:

**Lemma 2.1** There exists a positive constant \( K_* \) such that
\[ \|y\|_{W^{1,4}} \leq K_*\|A(y)\|_4, \quad \forall y \in V. \] (2.15)

Let us mention that through the article, we represent by \( C \) a generic constant. Its value can change from line to line. To explicitly write its dependence with respect of some parameters \( \lambda_1, \ldots, \lambda_k \), we also write \( C(\lambda_1, \ldots, \lambda_k) \) instead of \( C \).

We end this section with the Young’s inequality
\[ uz \leq \frac{1}{r}u^r + \frac{1}{s}z^s, \quad \forall u, z \geq 0, \quad s, r > 0 \text{ such that } \frac{1}{r} + \frac{1}{s} = 1. \] (2.16)

Accordingly, for real numbers \( \gamma, a, b, x \) such that \( 0 \leq \gamma < a \) and \( b, x \geq 0 \), we have the algebraic relation
\[ \forall \delta > 0, \quad bx^\gamma \leq C(\gamma, a, b, \delta) + \delta x^a, \] (2.17)
that will be used several times.

### 3 Preliminary results

We consider the following auxiliary modified Stokes problem with Navier boundary condition
\[
\begin{cases}
\tilde{f} - \alpha_1 \Delta \tilde{f} = f - \nabla p, & \text{div} \tilde{f} = 0 \quad \text{in } \mathcal{O}, \\
\tilde{f} \cdot n = 0, & (n \cdot D(\tilde{f})) \cdot \tau = 0 \quad \text{on } \Gamma.
\end{cases}
\] (3.1)

We recall from [9] that assuming \( f \in H^m(\mathcal{O}), m = 0, 1 \), the problem (3.1) has a solution \((\tilde{f}, p) \in H^{m+2}(\mathcal{O}) \times H^{m+1}(\mathcal{O})\) verifying
\[ \|\tilde{f}\|_{H^2} \leq C\|f\|_2. \] (3.2)
According to the definition of the inner product  \( (\cdot, \cdot)_V \), we have
\[
(\tilde{f}, h)_V = (f, h), \quad \forall h \in V. \tag{3.3}
\]

In the next two lemmas, we establish properties of the nonlinear terms that will be useful in Section 4 to identify the weak limits of the nonlinear terms of the equation. Let us introduce the operators
\[
S(y) := \beta \left( |A(y)|^2 A(y) \right), \tag{3.4}
\]
\[
N(y) := \alpha_1 (y \cdot \nabla A(y) + (\nabla y)^	op A(y) + A(y) \nabla y) - \alpha_2 (A(y))^2. \tag{3.5}
\]

**Lemma 3.1** For any \( y, \tilde{y}, \phi \in W \), we have
\[
|\langle \text{div} (S(y) - S(\tilde{y}), \phi) \rangle| \leq C\|y\|^2_V \|y - \tilde{y}\|_W \|\phi\|_W + C\|\tilde{y}\|_W \|A|^2 - |\hat{A}|^2\|_2 \|\phi\|_W. \tag{3.6}
\]
where \( A = A(y) \) and \( \hat{A} = A(\tilde{y}) \).

**Proof.** Using the Hölder inequality, and the Sobolev injections \( H^1(O) \hookrightarrow L^p(O) \) for \( p < \infty \) and \( H^2(O) \hookrightarrow L^\infty(O) \), we derive
\[
|\langle \text{div} (S(y) - S(\tilde{y}), \phi) \rangle| = \beta \left( |A|^2 A - |\hat{A}|^2 \hat{A} \right) \cdot \nabla \phi
= \beta \left( |A|^2 (A - \hat{A}) + \hat{A}(|A|^2 - |\hat{A}|^2) \right) \cdot \nabla \phi
\leq C\|A\|^2_4 \|A(y - \tilde{y})\|_4 \|\nabla \phi\|_4 + C\|\hat{A}\|_4 \|A|^2 - |\hat{A}|^2\|_2 \|\nabla \phi\|_4
\leq C\|y\|^2_H \|y - \tilde{y}\|_{H^1} \|\phi\|_{H^2} + C\|\tilde{y}\|_{H^2} \|A|^2 - |\hat{A}|^2\|_2 \|\phi\|_{H^2}. \tag{3.7}
\]

**Lemma 3.2** For any \( y, \tilde{y}, \phi \in W \), the following inequality holds
\[
|\langle \text{div} (N(\tilde{y}) - N(y), \phi) \rangle| \leq C \left( |A(y - \tilde{y})| \sqrt{|A|^2 + |\hat{A}|^2} \right) \|\phi\|_V
+ C\|y - \tilde{y}\|_V (\|y\|_W + \|\tilde{y}\|_W) \|\phi\|_W. \tag{3.8}
\]
where \( A = A(y) \) and \( \hat{A} = A(\tilde{y}) \).

**Proof.** Here we apply the same reasoning that is done in [5] to show the property \( (A.13) \).
\[
\langle \text{div} (N(\tilde{y}) - N(y), \phi) \rangle = \langle N(\tilde{y}) - N(y), \nabla \phi \rangle = \frac{1}{2} \langle N(\tilde{y}) - N(y), A(\phi) \rangle
= -\frac{\alpha_2}{2} \int_O (A^2 - \hat{A}^2) \cdot A(\phi) - \frac{\alpha_1}{2} \int_O (y \cdot \nabla A - \tilde{y} \cdot \nabla \hat{A}) \cdot A(\phi)
= \frac{\alpha_1}{2} \int_O \left( (\nabla y)^	op A + A \nabla y - (\nabla \tilde{y})^	op \hat{A} - \hat{A} \nabla \tilde{y} \right) \cdot A(\phi) = I_1 + I_2 + I_3. \tag{3.9}
\]
\[ |I_1| \leq C \left( |A(y - \hat{y})| \sqrt{|A|^2 + |\hat{A}|^2} \right)_2 \|A(\phi)\|_2. \] (3.10)

Next, we use the properties of the trilinear form, as well as the Hölder inequality, and the Sobolev injections \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) and \( H^1(\Omega) \hookrightarrow L^\infty(\Omega) \) in order to deduce that
\[
|I_2| \leq \left| b(y, A, A(\phi)) + b(\hat{y}, \hat{A}, A(\phi)) \right|
\leq \left| b(y, A - \hat{A}, A(\phi)) + b(y - \hat{y}, \hat{A}, A(\phi)) \right|
= \left| b(y, A(\phi), A - \hat{A}) + b(y - \hat{y}, \hat{A}, A(\phi)) \right|
\leq \|y\|_\infty \|A(\phi)\|_{H^1} \|A - \hat{A}\|_2 + \|y - \hat{y}\|_4 \|\hat{A}\|_{H^1} \|A(\phi)\|_4
\leq C \|y - \hat{y}\|_{H^1} (\|y\|_{H^2} + \|\hat{y}\|_{H^2}) \|\phi\|_{H^2}.
\] (3.11)

For \( I_3 \) we have the same estimate as for \( I_1 \), namely
\[
|I_3| \leq C \left( |A(y - \hat{y})| \sqrt{|A|^2 + |\hat{A}|^2} \right)_2 \|A(\phi)\|_2. \] (3.12)

\[ \blacksquare \]

## 4 Existence of strong solution

This section establishes the main results of the article. More precisely the solution of the equation is constructed through the finite dimensional Galerkin approximation method. We first deduce key uniform estimates for the finite dimensional approximations in order to get a weakly convergent sequence. Next, with the help of a suitable stopping time, and using the structure of the equation, we improve the convergence results. Finally, with these new convergence results, we will be able to identify the nonlinear terms of the equation.

Let us to introduce the notion of the solution.

**Definition 4.1** Let \( U \in L^2(\Omega \times (0, T), L^2(\Omega)) \) and \( Y_0 \in L^2(\Omega, W) \). Then a stochastic process \( Y \in L^2(\Omega, L^\infty(0, T; W)) \) is a strong solution of (2.1), if the following equation holds
\[
(v(Y(t)), \phi) = \int_0^t \left( -2\nu (D(Y(t)), D(\phi)) + ((Y \cdot \nabla)\phi, v(Y)) - \sum_j (v_j(Y) \nabla Y^j, \phi) \right) ds
- \int_0^t \left( (\alpha_1 + \alpha_2) (A^2) + \beta (|A|^2 A), \nabla \phi \right) ds
+ (v(Y(0)), \phi) + \int_0^t (U(s), \phi) ds + \int_0^t (\sigma(s, Y(s)), \phi) d\mathcal{W}_s
\] (4.1)

for a.e. \((\omega, t) \in \Omega \times [0, T]\) and for all \( \phi \in V \), where the stochastic integral is defined by
\[
\int_0^t (\sigma(s, Y(s)), \phi) d\mathcal{W}_s = \sum_{k=1}^m \int_0^t (\sigma^k(s, Y(s)), \phi) d\mathcal{W}^k_s.
\]
Now we state the main result.

**Theorem 4.2** Assume (2.7)–(2.9). Then there exists a unique solution $Y$ to equation (2.1) which belongs to $L^p(Ω, L^∞(0, T; W))$.

In order to show the existence of the solution, we apply the Galerkin’s approximation method for an appropriate basis. We recall that the injection operator $I : W \hookrightarrow V$ being a compact operator guarantees the existence of a basis $\{e_i\} \subset W$ of eigenfunctions to the problem

$$(v, e_i)_W = λ_i (v, e_i)_V, \quad \forall v \in W; \quad i \in \mathbb{N},$$

which is an orthonormal basis in $V$ and an orthogonal basis in $W$. In addition the sequence $\{λ_i\}$ of the corresponding eigenvalues fulfills the properties: $λ_i > 0$, $\forall i \in \mathbb{N}$, and $λ_i \to \infty$ as $i \to \infty$.

Since the ellipticity of the equation (4.2) increases the regularity of their solutions (see [7]), we may consider $\{e_i\} \subset H^4$.

We consider the finite dimensional space $W_n = \text{span} \{e_1, \ldots, e_n\}$, and introduce the Faedo-Galerkin approximation of the system (2.1). Namely, we look for a solution to the following stochastic differential equation

$$
\begin{align*}
\begin{cases}
d(υ_n, φ) &= (νΔY_n - (Y_n \cdot \nabla)υ_n - \sum_j υ_j n ∇Y_n^j + (α_1 + α_2) \text{div} (A_n^2)) dt + (σ(t, Y_n), φ) dW_t, \\
Y_n(0) &= Y_{n,0},
\end{cases}
\end{align*}
$$

where

$$Y_{n,0} = \sum_{j=1}^{n} c_j^0(t)e_j.$$ 

Here $Y_{n,0}$ denotes the projection of the initial condition $Y_0$ onto the space $W_n$, $υ_n = Y_n - α_1 ΔY_n$ and $A_n = ∇Y_n + (∇Y_n)^\top$.

Due to the relation (4.2), the sequence $\{e_j = \frac{1}{√λ_j}e_j\}$ is an orthonormal basis for $W$ and

$$Y_{n,0} = \sum_{j=1}^{n} (Y_0, e_j)_V e_j = \sum_{j=1}^{n} (Y_0, e_j)_W e_j,$$

The Parseval’s identity yields

$$\|Y_n(0)\|_V \leq \|Y_0\|_V \quad \text{and} \quad \|Y_n(0)\|_W \leq \|Y_0\|_W.$$

The equation (4.3) can be written as a system of stochastic ordinary differential equations in $\mathbb{R}^n$ with locally Lipschitz nonlinearities. From classical results there exists a local-in-time solution $Y_n$ that is an adapted stochastic process with values in $C([0, T_n], W_n)$.

The existence of a global-in-time solution follows from the uniform estimates on $n = 1, 2, \ldots$, that will be deduced in the next lemma (a similar reasoning can be found in [1], [9], [21]).
Lemma 4.3  Let us assume (2.4), (2.5). Then the problem (4.3) admits a unique solution $Y_n \in L^2(\Omega, L^\infty(0,T;W))$, which verifies the following estimates

\[
\mathbb{E} \sup_{s \in [0,t]} \|Y_n(s)\|^2_V + 8\nu \mathbb{E} \int_0^t \|DY_n\|^2_V ds + \frac{\nu}{4} \mathbb{E} \int_0^t \|A_n\|^4_V ds \\
\leq C \left(1 + \mathbb{E} \|Y_0\|^2_V + \mathbb{E} \|U\|^2_{L^2(0,T;L^2(\Omega))}\right), \quad \forall t \in [0,T],
\]

where $C$ is a positive constant independent of $n$.

Proof. For each $n \in \mathbb{N}$, we define the following sequence of stopping times

\[
\tau^n_M = \inf\{t \geq 0 : \|Y_n(t)\|^2_V \geq M\} \land T_n, \quad M \in \mathbb{N}.
\]

Let us set

\[
f(Y_n) := \nu \Delta Y_n - (Y_n \cdot \nabla)v_n - \sum_j v_n^j \nabla Y_n^j + (\alpha_1 + \alpha_2) \text{div} \left(A_n^2 + \beta \text{div} (|A_n|^2 A_n) + U\right).
\]

Using (2.4), and considering in (4.3) the test functions $\phi = e_i$, $i = 1, \ldots, n$, we write

\[
d(Y_n, e_i)_V = (f(Y_n), e_i) dt + (\sigma(t, Y_n), e_i) dW_t.
\]

Applying the Itô formula, we deduce

\[
d(Y_n, e_i)^2_V = 2(Y_n, e_i)_V (f(Y_n), e_i) dt + 2(Y_n, e_i)_V (\sigma(t, Y_n), e_i) dW_t + (\sigma(t, Y_n), e_i)^2 dt.
\]

Summing over $i = 1, \ldots, n$, we derive

\[
d\|Y_n\|^2_V = 2(f(Y_n), Y_n) dt + 2(\sigma(t, Y_n), Y_n) dW_t + \sum_{i=1}^n (\sigma(t, Y_n), e_i)^2 dt.
\]

We have

\[
(f(Y_n), Y_n) = -2\nu \|DY_n\|^2_V - ((Y_n \cdot \nabla)v_n + \sum_j v_n^j \nabla Y_n^j, Y_n)
\]

\[
+ ((\alpha_1 + \alpha_2) \text{div} (A_n^2), Y_n) + (\beta \text{div} (|A_n|^2 A_n), Y_n) + (U, Y_n)
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5.
\]

By the symmetry of the trilinear functional (2.5), we obtain

\[
I_2 = -((Y_n \cdot \nabla)v_n + \sum_j v_n^j \nabla Y_n^j, Y_n) = -b(Y_n, v_n, Y_n) = b(Y_n, Y_n, v_n)
\]

\[
= -b(Y_n, v_n, Y_n) + b(Y_n, v_n, Y_n) = 0.
\]

Taking into account the boundary conditions $Y_n = (Y_n \cdot \tau)$, $(n \cdot A_n) \cdot \tau = 0$ on $\Gamma$ and the symmetry of $\nabla Y_n$, the divergence theorem gives

\[
I_4 = \int_{\Gamma} \beta \text{div} (|A_n|^2 A_n) \cdot Y_n = \beta \int_{\Gamma} |A_n|^2 (Y_n \cdot \tau) (n \cdot A_n) \cdot \tau - \beta \int_{\Gamma} |A_n|^2 A_n \cdot \nabla Y_n
\]

\[
= -\frac{\beta}{2} \|A_n\|^4_V.
\]
Taking $\varepsilon = \frac{\beta}{4}$ in (A.1), we obtain

$$|I_5| \leq \frac{\beta}{4} \|A_n\|_2^2 + \frac{(\alpha_1 + \alpha_2)^2}{4\beta} \|A_n\|_2^2. \quad (4.12)$$

In addition, we have

$$|I_5| = |(U, Y_n)| \leq \frac{1}{2} \|Y_n\|_2^2 + \frac{1}{2} \|U\|_2^2. \quad (4.13)$$

Therefore, introducing (4.10)-(4.13) in (4.8), we derive

$$d \|Y_n\|_V^2 + \frac{\beta}{2} \|A_n\|_4^4 dt + 4\nu \|DY_\gamma\|_2^2 dt \leq \frac{(\alpha_1 + \alpha_2)^2}{8\beta} \|A_n\|_2^2 dt$$

$$+ \left( \|U\|_2^2 + \|Y_n\|_2^2 \right) dt + 2 (\sigma(t, Y_n), Y_n) dW_t + \sum_{i=1}^{n} \| \sigma(t, Y_n), e_i \|_V^2 dt. \quad (4.14)$$

We write

$$d \|Y_n\|_V^2 + \frac{\beta}{2} \|A_n\|_4^4 dt + 4\nu \|DY_\gamma\|_2^2 dt \leq C(\beta, \alpha_1, \alpha_2) \|Y_n\|_V^2 dt$$

$$+ \|U\|_2^2 dt + 2 (\sigma(t, Y_n), Y_n) dW_t + \sum_{i=1}^{n} \| \sigma(t, Y_n), e_i \|_V^2 dt. \quad (4.15)$$

Denoting by $\tilde{\sigma}_n$ the solution of the generalized Stokes problem (3.1) for $f = \sigma(t, Y_n)$, we have

$$(\tilde{\sigma}_n, e_i)_V = (\sigma(t, Y_n), e_i) \quad \text{for } i = 1, \ldots, n,$$

then (2.7), (2.15) and Young’s inequality give

$$\sum_{i=1}^{n} \| \sigma(t, Y_n), e_i \|_V^2 \leq C \| \tilde{\sigma}_n \|_V^2 \leq C \| \sigma(t, Y_n) \|_V^2 \leq C(1 + \|Y_n\|_V^2.$$}

$$\leq CL + C(L(K_\gamma)\|A_n\|_4^4 \leq C(L, \gamma, \beta) + \frac{\beta}{4} \|A_n\|_4^4. \quad (4.16)$$

For any $t \in [0, T]$, integrating the inequality (4.15) on $(0, s)$, $s \in [0, \tau^*_M \wedge t]$ and using (4.16), we derive

$$\|Y_n(s)\|_V^2 + \frac{\beta}{4} \int_{0}^{s} \|A_n\|_4^4 dr + 4\nu \int_{0}^{s} \|DY_\gamma\|_2^2 dr \leq \|Y_n(0)\|_V^2 + C(\beta, \alpha_1, \alpha_2, T)$$

$$+ C(\beta, \alpha_1, \alpha_2) \int_{0}^{s} \|Y_n\|_V^2 dr + \int_{0}^{s} \|U\|_2^2 dr + 2 \int_{0}^{s} (\sigma(t, Y_n), Y_n) dW_t. \quad (4.17)$$

On the other hand, the Burkholder-Davis-Gundy inequality, (2.7), (2.15) and the Young inequality yield

$$\mathbb{E} \sup_{s \in [0, \tau^*_M \wedge t]} \left| \int_{0}^{s} (\sigma(r, Y_n), Y_n) dW_r \right| \leq C \left( \int_{0}^{\tau^*_M \wedge t} |(\sigma(s, Y_n), Y_n)|^2 ds \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_{0}^{\tau^*_M \wedge t} L(1 + \|Y_n\|^2_{W_{1,4}}) \|Y_n\|^2_{V} ds \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0, \tau^*_M \wedge t]} \|Y_n\|_V^2 + C^2 LT + \frac{1}{4} \mathbb{E} \int_{0}^{\tau^*_M \wedge t} C^2 L(K_\gamma) \|A_n\|_4^4 ds$$

$$\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0, \tau^*_M \wedge t]} \|Y_n\|_V^2 + C(L, T, K_\gamma, \beta) + \frac{\beta}{16} \int_{0}^{\tau^*_M \wedge t} \|A_n\|_4^4 ds. \quad (4.18)$$
Taking the supremum on \( s \in [0, \tau_M^n \wedge t] \) and the expectation in (4.17) and incorporating the estimate (4.18), we obtain

\[
\frac{1}{2} \mathbb{E} \sup_{s \in [0, \tau_M^n \wedge t]} \| Y_n(s) \|_{V}^2 + 4\nu \mathbb{E} \int_0^{\tau_M^n \wedge t} \| D Y_n \|_{2}^2 \, ds + \frac{\beta}{8} \mathbb{E} \int_0^{\tau_M^n \wedge t} \| A_n \|_{4}^4 \, dr \\
\leq C(\beta, \alpha_1, \alpha_2, T) + \mathbb{E} \| Y_0 \|_{V}^2 + \mathbb{E} \int_0^t \| U \|_{2}^2 \, ds + C(\beta, \alpha_1, \alpha_2) \mathbb{E} \int_0^t \sup_{r \in [0, \tau_M^n \wedge s]} \| Y_n(r) \|_{V}^2 \, ds. \tag{4.19}
\]

Then the function

\[
f(t) = \frac{1}{2} \mathbb{E} \sup_{s \in [0, \tau_M^n \wedge t]} \| Y_n(s) \|_{V}^2 + 4\nu \mathbb{E} \int_0^{\tau_M^n \wedge t} \| D Y_n \|_{2}^2 \, ds + \frac{\beta}{4} \mathbb{E} \int_0^{\tau_M^n \wedge t} \| A_n \|_{4}^4 \, ds
\]

fulfils the Gronwall’s inequality

\[
f(t) \leq C + \mathbb{E} \| Y_0 \|_{V}^2 + \mathbb{E} \int_0^t \| U \|_{2}^2 \, ds + C \int_0^t f(s) \, ds,
\]

which implies

\[
\mathbb{E} \sup_{s \in [0, \tau_M^n \wedge t]} \| Y_n(s) \|_{V}^2 + 8\nu \mathbb{E} \int_0^{\tau_M^n \wedge t} \| D Y_n \|_{2}^2 \, ds + \frac{\beta}{4} \mathbb{E} \int_0^{\tau_M^n \wedge t} \| A_n \|_{4}^4 \, ds \\
\leq C \left( 1 + \mathbb{E} \| Y_0 \|_{V}^2 + \mathbb{E} \| U \|_{2(0,t;L^2(C))}^2 \right). \tag{4.20}
\]

Then there exists a constant \( C \) independent of \( M \) and \( n \) such that

\[
\mathbb{E} \sup_{s \in [0, \tau_M^n \wedge t]} \| Y_n(s) \|_{V}^2 \leq C, \quad \forall t \in [0, T]. \tag{4.21}
\]

Let us fix \( n \in \mathbb{N} \), writing

\[
\mathbb{E} \sup_{s \in [0, \tau_M^n \wedge T]} \| Y_n(s) \|_{V}^2 = \mathbb{E} \left( \sup_{s \in [0, \tau_M^n \wedge T]} 1_{\{ \tau_M^n < T \}} \| Y_n(s) \|_{V}^2 \right) + \mathbb{E} \left( \sup_{s \in [0, \tau_M^n \wedge T]} 1_{\{ \tau_M^n \geq T \}} \| Y_n(s) \|_{V}^2 \right) \\
\geq \mathbb{E} \left( \max_{s \in [0, \tau_M^n]} 1_{\{ \tau_M^n < T \}} \| Y_n(s) \|_{V}^2 \right) \geq M^2 P(\tau_M^n < T), \tag{4.22}
\]

we deduce that \( P(\tau_M^n < T) \leq \frac{\mathbb{E} \sup_{s \in [0, \tau_M^n \wedge T]} \| Y_n(s) \|_{V}^2}{M^2} \). This means that \( \tau_M^n \to T \) in probability, as \( M \to \infty \). Then there exists a subsequence \( \{ \tau_M^{n_k} \} \) of \( \{ \tau_M^n \} \) (that may depend on \( n \)) such that

\[
\tau_M^{n_k} \to T \quad \text{a.e. as} \quad k \to \infty.
\]

Since \( \tau_M^{n_k} \leq T_n \leq T \), we deduce that \( T_n \to T \), so \( Y_n \) is a global-in-time solution of the stochastic differential equation (4.3). In addition for fixed \( n \), the monotonicity of the sequence \( \{ \tau_M^n \} \) allows to apply the monotone convergence theorem in order to pass to the limit, as \( M \to \infty \), in the inequality (4.20) in order to obtain (4.4).

\[
\mathbb{E} \sup_{s \in [0, t]} \| Y_n(s) \|_{V}^2 + 8\nu \mathbb{E} \int_0^t \| D Y_n \|_{2}^2 \, ds + \frac{\beta}{4} \mathbb{E} \int_0^t \| A_n \|_{4}^4 \, ds \\
\leq C \left( 1 + \mathbb{E} \| Y_0 \|_{V}^2 + \mathbb{E} \| U \|_{2(0,t;L^2(C))}^2 \right). \tag{4.23}
\]
This inequality gives
\[
\mathbb{E} \int_0^t \|A_n\|^4_4 \, ds \leq C(\beta, \alpha_1, \alpha_2) \left( 1 + \mathbb{E} \|Y_0\|^2_\nu + \mathbb{E} \|U\|^2_{L^2([0,t];L^2(\Omega))} \right), \quad \forall t \in [0,T],
\]
that together with Lemma 2.1 yields
\[
\mathbb{E} \|Y_n\|^4_{L^4([0,t];W^{1,4}(\mathcal{O}))} \leq \mathbb{E} \int_0^t \|Y_n\|^4_{W^{1,4}} \, ds \leq (K_\ast)^4 \mathbb{E} \int_0^t \|A_n\|^4_4 \, ds
\]
\[
\leq C(\beta, \alpha_1, \alpha_2) \left( 1 + \mathbb{E} \|Y_0\|^2_\nu + \mathbb{E} \|U\|^2_{L^2([0,t];L^2(\Omega))} \right), \quad \forall t \in [0,T].
\]
(4.25)

The Hölder’s inequality also gives
\[
\mathbb{E} \|Y_n\|^4_{L^4([0,t];W^{1,4}(\mathcal{O}))} \leq C(\beta, \alpha_1, \alpha_2) \left( 1 + \mathbb{E} \|Y_0\|^2_\nu + \mathbb{E} \|U\|^2_{L^2([0,t];L^2(\Omega))} \right)^{\frac{4}{5}}, \quad \forall t \in [0,T].
\]
(4.26)

\[\Box\]

**Lemma 4.4** Assume (2.7) - (2.9). Then we have
\[
\mathbb{E} e^{\frac{\lambda}{4} t} \|Y_n\|^4_{W^{1,4}} \, ds < C \mathbb{E} e^{\lambda \left( \int_0^t \|U\|^2_{2} \, ds + \|Y_0\|^2_\nu \right)}, \quad \forall t \in [0,T],
\]
(4.27)
where \(C\) is a positive constant independent of \(n\), and \(K_\ast\) is defined by (2.13).

**Proof.** Let us consider the inequality (4.17) and write
\[
\|Y_n(t)\|^2_\nu + \int_0^t \|A_n\|^4_4 \, ds + 4 \nu \int_0^t \|DY_n\|^2_2 \, ds \leq \|Y_n(0)\|^2_\nu + \int_0^t \|U\|^2_2 \, ds
\]
\[
+ C(\beta, \alpha_1, \alpha_2, T) + C(\beta, \alpha_1, \alpha_2) \int_0^t \|Y_n\|^2_\nu \, ds + 2 \int_0^t (\sigma(s, Y_n), Y_n) \, dW_s.
\]
(4.28)

Multiplying by \(\frac{\lambda}{2}\) and knowing that \(W^{1,4}(\mathcal{O}) \subset H^1(\mathcal{O})\), we deduce
\[
\frac{\lambda}{2} \|Y_n(t)\|^2_\nu + \int_0^t \|A_n\|^4_4 \, ds + 2 \nu \int_0^t \|DY_n\|^2_2 \, ds \leq \frac{\lambda}{2} \left( \|Y_n(0)\|^2_\nu + \int_0^t \|U\|^2_2 \, ds \right)
\]
\[
+ C(\beta, \alpha_1, \alpha_2, T, K_\ast) + C(\beta, \alpha_1, \alpha_2) \int_0^t \|Y_n\|^2_{W^{1,4}} \, ds + \lambda \int_0^t (\sigma(s, Y_n), Y_n) \, dW_s.
\]

The Korn inequality (2.13) gives
\[
\frac{\lambda \beta}{8(K_\ast)^4} \|Y_n\|^4_{W^{1,4}} \leq \frac{\lambda \beta}{8} \|A_n\|^4_4;
\]
to we have
\[
\frac{\lambda \beta}{8(K_\ast)^4} \int_0^t \|Y_n\|^2_{W^{1,4}} \, ds \leq \frac{\lambda}{2} \left( \|Y_n(0)\|^2_\nu + \int_0^t \|U\|^2_2 \, ds \right) + C(\beta, \alpha_1, \alpha_2, T, K_\ast)
\]
\[
+ C(\beta, \alpha_1, \alpha_2) \int_0^t \|Y_n\|^2_{W^{1,4}} \, ds + \lambda \int_0^t (\sigma(s, Y_n), Y_n) \, dW_s.
\]
(4.29)
Let us notice that with the help of (2.7), the Sobolev embedding $W^{1,4}(\mathcal{O}) \to H$ and the Young's inequality, for any $\delta > 0$, we can verify that
\[
\int_0^t \lambda^2 (\sigma(s, Y_n, Y_n))^2 ds \leq \int_0^t \lambda^2 \|\sigma(s, Y_n)\|_{L^2}^2 ds \leq \int_0^t \lambda^2 L(1 + \|Y_n\|_{H^{1,4}}) \|Y_n\|_{L^2}^2 ds
\]
\[
\leq \int_0^t \lambda^2 L \|Y_n\|_{W^{1,4}}^2 ds + \int_0^t \lambda^2 L \|Y_n\|_{H^{1,4}}^2 ds
\]
\[
\leq C(\lambda, L, \delta, T) + \frac{\delta}{2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds;
\]
which implies
\[
-\frac{\delta}{2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds - C(\lambda, L, \delta, T) \leq -\int_0^t \lambda^2 (\sigma(s, Y_n, Y_n))^2 ds.
\]
Adding this relation to (4.29), we write
\[
\frac{\lambda \beta}{8(K_\mathcal{O})^2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds - \frac{\delta}{2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds \leq \frac{\lambda}{2} \left( \|Y_n(0)\|_{V}^2 + \int_0^t \|U\|_{L^2}^2 ds \right)
\]
\[
+ C(\beta, \alpha_1, \alpha_2, \delta, T) + C(\beta, \alpha_1, \alpha_2) \int_0^t \|Y_n\|_{H^{1,4}}^2 ds
\]
\[
+ \lambda \int_0^t (\sigma(s, Y_n), Y_n) dW_s - \int_0^t \lambda^2 (\sigma(s, Y_n, Y_n))^2 ds.
\]
(4.30)
Once again, the Young inequality gives
\[
C(\beta, \alpha_1, \alpha_2) \int_0^t \|Y_n\|_{W^{1,4}}^2 ds \leq C(\beta, \alpha_1, \alpha_2, \delta) + \frac{\delta}{2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds.
\]
Introducing this estimate in (4.30) and next taking $\delta = \frac{\lambda^2}{16(K_\mathcal{O})^2}$, it follows that
\[
\frac{\lambda \beta}{16(K_\mathcal{O})^2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds \leq \frac{\lambda}{2} \left( \|Y_n(0)\|_{V}^2 + \int_0^t \|U\|_{L^2}^2 ds \right) + C(\beta, \alpha_1, \alpha_2, T)
\]
\[
+ \lambda \int_0^t (\sigma(s, Y_n), Y_n) dW_s - \int_0^t \lambda^2 (\sigma(s, Y_n, Y_n))^2 ds.
\]
(4.31)
Now, we take the exponential, the expectation and the Hölder inequality in order to deduce that
\[
\mathbb{E} \left( \frac{\lambda^2}{16(K_\mathcal{O})^2} \int_0^t \|Y_n\|_{W^{1,4}}^4 ds \right) \leq C(\beta, \alpha_1, \alpha_2, T) \left( \mathbb{E} \left( \|Y_n\|_{V}^2 + \int_0^t \|U\|_{L^2}^2 ds \right) \right)^{\frac{1}{2}}
\]
\[
\cdot \left( \mathbb{E} \left( \int_0^t (2\lambda^2 (\sigma(s, Y_n), Y_n) dW_s - \frac{\delta}{2} \int_0^t (2\lambda^2 (\sigma(s, Y_n, Y_n))^2 ds) \right) \right)^{\frac{1}{2}}.
\]
Since the stochastic process inside the second expectation is a supermartingale its expectation is less or equal to 1, hence we obtain (4.32).

\textbf{Lemma 4.5} Assume (2.7) - (2.9). Then the unique solution $Y_n$ of the problem (4.3) verifies the following uniform estimate
\[
\mathbb{E} \sup_{s \in [0,t]} \|Y_n(s)\|_{W^1}^p \leq C, \quad \forall t \in [0, T],
\]
(4.32)
where $C$ is a positive constant independent of $n$. \hfill $\blacksquare$
The Itô formula gives
\[ \lambda d\nu = (f_n, e_i) \nu dt + (\tilde{\sigma}_n, e_i) \nu dW_t. \]

Multiplying by \( \lambda \) and using the estimates (A.4)-(A.6) as in [6], page 373, for (2.14) and the Young’s inequality allow to verify that
\[ \| \tilde{\sigma}_n \|_W = (\tilde{\sigma}_n, e_i) \nu \| dW_t. \]

Now, multiplying by \( \lambda \) and using (4.12), we obtain
\[ d\| Y_n \|_W^2 = 2\| f_n, Y_n \| W dt + 2\| \tilde{\sigma}_n, Y_n \| W dW_t + \| (\tilde{\sigma}_n, e_i) \|_W^2 dt. \]

The Itô formula gives
\[ d\| Y_n \|_W^2 = 2\| f_n, Y_n + (f_n, \nabla V(Y_n)) \| dt + \| (\sigma_n, Y_n) \|_W^2 dt + C(y, Y_n) dW_t. \]

Let us recall from (4.9)-(4.13) that
\[ 2(f_n, Y_n) \leq -4\nu \| \nabla Y_n \|_W^2 - \frac{\beta}{2} \| A_n \|_W^4 + \frac{\alpha_1 + \alpha_2}{2\beta} \| A_n \|_W^2 + \| Y_n \|_W^2 + \| U \|_W^2. \]

On the other hand, considering the Sobolev inequality
\[ \| y \|_6 \leq C_1 \| y \|_{H^1}, \quad \forall y \in H^1, \]
and using the estimates (A.4)-(A.6) as in [6], page 373, for \( \epsilon = \min \left\{ \frac{1}{\| \nabla V(Y_n) \|_W}, \frac{\alpha_1}{\| \nabla \|_W}, \frac{\alpha_2}{\| \nabla \|_W} \right\} \) we derive
\[ 2(f_n, \nabla V(Y_n)) \leq -\frac{\beta}{2} \| A_n \|_W^4_1 - \frac{\alpha_1 \beta}{2} \| A_n \|_W^2 \| \nabla A_n \|_W^2 + \frac{\alpha_2 \beta}{4} \| \nabla (|A_n|^2) \|_W^2 + C(\nu, \beta, \alpha_1, \delta) \| Y_n \|_W^2 + C(\beta, \alpha_1) \| Y_n \|_W \| Y_n \|_W^2 + 2\delta \| Y_n \|_{W^{1,4}} + \| U \|_W^2. \]
Therefore, we have
\[
2(f_n, P_v(Y_n)) \leq \frac{\beta}{2} \|A_n\|^2 + \frac{\alpha_1 \beta}{2} \|A_n\| \|\nabla A_n\|^2 + \frac{\alpha_1 \beta}{4} \|\nabla (\|A_n\|^2)\|^2
+ C(\nu, \beta, \alpha_1, \delta) \|Y_n\|^4_{W^{1.4}} + 2\delta \|Y_n\|^4_{W^{1.4}} + 2\delta \|Y_n\|^4_{W^{1.4}} + \|U\|^2.
\]
(4.38)

Now, we choose \(\delta\) such that \(2D_1 := 4\delta \leq \frac{\lambda_0}{\|n(K_n)^T\|}\) and introduce the function
\[
\xi_1(t) = e^{-2D_1\int_0^t \|\nabla \theta\|^2 \nu ds}.
\]
We apply the Itô formula to determine the differential of the product \(\xi_1(t)\|Y_n(t)\|^2_W\), namely from the equation (4.39) we derive
\[
d \left( \xi_1(t) \|Y_n\|^2_W \right) = \xi_1(t) \left[ 2(f_n, Y_n) + 2(f_n, P_v(Y_n)) \right] dt + \xi_1(t) \|\tilde{\sigma}_n\|^2_W dt
+ \xi_1(t) \left[ 2(\sigma_n, Y_n) + 2(\sigma_n, P_v(Y_n)) \right] dW_t
- 2D_1 \xi_1(t) \|Y_n\|^2_W \|Y_n\|^4_{W^{1.4}} dt.
\]
(4.39)

Using the Itô formula once again for the function \(\theta(x) = x^p\), and integrating on \([0, s], s \leq t \wedge \tau_M, t \in [0, T]\), we deduce
\[
\left( \xi_1(s) \|Y_n\|^2_W \right)^p = \|Y_n(0)\|_{W^p}^p + p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \left[ 2(f_n, Y_n) + 2(f_n, P_v(Y_n)) \right] dr
+ p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \|\tilde{\sigma}_n\|^2_W dr
+ p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \left[ 2(\sigma_n, Y_n) + 2(\sigma_n, P_v(Y_n)) \right] dW_r
- 2D_1 p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \|Y_n\|^2_W \|Y_n\|^4_{W^{1.4}} dr
+ 2p(p-1) \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-2} (\xi_1(r))^2 \left[ (\sigma_n, Y_n) + (\sigma_n, P_v(Y_n)) \right]^2 dr.
\]
(4.40)

Next, using (4.39) and (4.38) to estimate the right hand side, we obtain
\[
\left( \xi_1(s) \|Y_n\|^2_W \right)^p \leq \|Y_n(0)\|_{W^p}^p + p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \left[ C(\nu, \beta, \alpha_1, \delta) \|Y_n\|^4_{W^{1.4}} + 2\|U\|^2 \right] dr
+ p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \left[ D_1 \|Y_n\|^4_{W^{1.4}} + D_1 \|Y_n\|^4_{W^{1.4}} \right] dr
+ p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \|\tilde{\sigma}_n\|^2_W ds
+ 2p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \left[ (\sigma_n, Y_n) + (\sigma_n, P_v(Y_n)) \right] dW_r
- 2D_1 p \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-1} \xi_1(r) \|Y_n\|^2_W \|Y_n\|^4_{W^{1.4}} dr
+ 2p(p-1) \int_0^s \left( \xi_1(r) \|Y_n\|^2_W \right)^{p-2} (\xi_1(r))^2 \left[ (\sigma_n, Y_n) + (\sigma_n, P_v(Y_n)) \right]^2 dr.
\]
(4.41)
Introducing these estimates in (4.42), we deduce
\[
\left( \xi_1(s) \| Y_n \|_{W_{1,1}^2}^2 \right)^p \leq \| Y_n(0) \|_{W_{1,1}^2}^p + pC(\nu, \beta, \alpha_1, \delta) \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^p dr \\
+ pD_1 \int_0^s \left( \xi_1(r) \right)^p \| Y_n \|_{W_{1,1}^4}^2 dr \\
+ 2p \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-1} \xi_1(r) \| U \|_2^2 dr \\
+ p \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-1} \xi_1(r) \| \sigma_n \|_{W_{1,1}^2}^2 dr \\
+ 2p \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-1} \xi_1(r) \| \bar{\sigma}_n \|_{W_{1,1}^2}^2 dr \\
+ 2p(p-1) \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-2} \left( \xi_1(r) \right)^2 \left[ (\sigma_n, Y_n) + (\sigma_n, \mathbb{P}v(Y_n)) \right]^2 dW_r,
\]
(4.42)

Taking into account that \( \| \bar{\sigma}_n \|_{W_{1,1}^2} \leq C \| \sigma \|_{W_{1,1}^2}^2 \), using (2.7), the embedding \( W \hookrightarrow W^{1,4}(\mathcal{O}) \) and the Young’s inequality, we infer that
\[
p \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-1} \xi_1(r) \| \bar{\sigma}_n \|_{W_{1,1}^2}^2 dr \\
+ 2p(p-1) \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-2} \left( \xi_1(r) \right)^2 \left[ (\sigma_n, Y_n) + (\sigma_n, \mathbb{P}v(Y_n)) \right]^2 dr \\
\leq C(p, T) + C(p) \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^p dr.
\]

On the other hand, the Young’s inequality (2.10) with \( r = \frac{p}{p-1} \) and \( 0 \leq \xi_1(t) \leq 1 \) give
\[
2p \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-1} \xi_1(r) \| U \|_2^2 dr \leq 2(p-1) \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^p dr + 2 \int_0^s \| U \|_2^{2p} dr.
\]

Introducing these estimates in (4.42), we deduce
\[
\left( \xi_1(s) \| Y_n(s) \|_{W_{1,1}^2}^2 \right)^p \leq \| Y_0 \|_{W_{1,1}^2}^p + C \left( \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^p dr + \int_0^s \| Y_n \|_{W_{1,1}^4}^2 dr + \int_0^s \| U \|_2^{2p} dr + 1 \right) \\
+ 2p \int_0^s \left( \xi_1(r) \| Y_n \|_{W_{1,1}^2}^2 \right)^{p-1} \xi_1(r) \left[ (\sigma(r, Y_n), \mathbb{P}v(Y_n) + Y_n) \right] dW_r.
\]
(4.43)
The Burkholder-Davis-Gundy inequality, \((2.7)\) and the Young’s inequality yield

\[
\mathbb{E} \sup_{s \in [0, \tau_n^M \land t]} \left| \int_0^s (\xi_1(r) \| Y_n \|^2_W)^{p-1} \xi_1(r) \left( (\sigma(r, Y_n), \mathbb{P}_t(Y_n) + Y_n) dW_r \right) \right| \\
\leq C \mathbb{E} \left( \int_0^{\tau_n^M \land t} (\xi_1(s) \| Y_n \|^2_W)^{p-1} \xi_1(s) \left( \| \sigma(s, Y_n) \|^2 \| Y_n \|^2_W \right) ds \right)^\frac{1}{p} \\
\leq C \sqrt{L} \mathbb{E} \left( \int_0^{\tau_n^M \land t} (\xi_1(s) \| Y_n \|^2_W)^{2p} + \int_0^{\tau_n^M \land t} (\xi_1(s) \| Y_n \|^2_W)^{4p-2} ds \right)^\frac{1}{p} \\
\leq C \sqrt{2LT} + C \sqrt{2L} \mathbb{E} \left( \int_0^{\tau_n^M \land t} (\xi_1(s) \| Y_n \|^2_W)^{2p} ds \right)^\frac{1}{p} \\
\leq C_{\|L\|, \|\eta\|, p} \mathbb{E} \int_0^{\tau_n^M \land t} (\xi_1(s) \| Y_n \|^2_W)^{p} ds, \quad (4.44)
\]

for any \(\eta > 0\). Here we take \(\eta = \frac{1}{2}\). Considering the supremum on \(s \in [0, \tau_n^M \land t]\) and the expectation in \((4.44)\), with the help of \((4.44)\) we derive the following Gronwall’s inequality

\[
\frac{1}{2} \mathbb{E} \sup_{s \in [0, \tau_n^M \land t]} \left( \xi_1(s) \| Y_n(s) \|^2_W \right)^p \leq \| Y_0 \|^2_W + C \left( \int_0^t \mathbb{E} \sup_{r \in [0, \tau_n^M \land s]} \left( \xi_1(r) \| Y_n(r) \|^2_W \right)^p ds \right) + \mathbb{E} \int_0^{\tau_n^M \land t} \| Y_n \|^2_{W^1.4} ds + \mathbb{E} \int_0^t \| U \|^2_{2^p} ds + 1.
\]

Therefore, we obtain

\[
\mathbb{E} \sup_{s \in [0, \tau_n^M \land t]} \left( \xi_1(s) \| Y_n(s) \|^2_W \right)^p \leq C \left( 1 + \mathbb{E} \int_0^t \| U \|^2_{2^p} ds + \mathbb{E} \int_0^t \| Y_n \|^4_{W^1.4} ds \right). \quad (4.45)
\]

The estimates \((4.5)\) and \((2.9)\) yield

\[
\mathbb{E} \sup_{s \in [0, \tau_n^M \land t]} \left( \xi_1(s) \| Y_n(s) \|^2_W \right)^p \leq C
\]

with \(C\) independent of \(n\) and \(M\). We verify that for \(n\) fixed, \(\tau_n^M \to T\) in probability, as \(M \to \infty\). Then, there exists a subsequence \(\{\tau_{n_M}^n\}\) of \(\{\tau_n^M\}\) (that may depend on \(n\)) such that \(\tau_{n_M}^n \to T\) for a.e. \(\omega \in \Omega\), as \(k \to \infty\). Using the monotone convergence theorem, we pass to the limit in \((4.45)\) as \(k \to \infty\), deriving the estimate

\[
\mathbb{E} \sup_{s \in [0, t]} \left( \xi_1(s) \| Y_n(s) \|^2_W \right)^p \leq C.
\]
The Hölder inequality gives
\[
E \sup_{s \in [0,t]} \| Y_n(s) \|_W^p \leq E \left( \sup_{s \in [0,t]} \| \xi_1(s) \|_W^p \right)^{\frac{p}{2}} \left( \| \xi_1(t) \|_W^{-p} \right)^{\frac{1}{2}} 
\]
\[
\leq \left( E \| \xi_1 \|_W^p \right)^{\frac{p}{2}} \left( \| \xi_1(t) \|_W^{-p} \right)^{\frac{1}{2}} \leq \sqrt{C} \left( E e^{2\mu D} \int_0^t \| Y_n \|_{W_1,4}^4 ds \right)^{\frac{1}{2}}.
\]

Using Lemma 4.4, we deduce (4.32).

4.1 Proof of Theorem 4.2

In order to show the existence of the solution to the system (2.1) it is convenient to write the equation (2.1) in the following form (see [5], page 3)
\[
d(\nu(Y)) = (-\nabla p + \nu \Delta Y - (Y \cdot \nabla) Y + \text{div} N(Y) + \text{div} S(Y) + U) dt + \sigma(t, Y) dW_t, \quad (4.46)
\]
with the operators $S$ and $N$ defined in (A.7)-(A.8). The corresponding finite dimensional approximation reads
\[
d(\nu(Y_n)) = (-\nabla p_n + \nu \Delta Y_n - (Y_n \cdot \nabla) Y_n + \text{div} N(Y_n) + \text{div} S(Y_n) + U) dt + \sigma(t, Y_n) dW_t \quad (4.47)
\]

The proof of Theorem 4.2 is splitted into five steps.

Step 1. Convergences related with the projection operator. Let $P_n : W \to W_n$ be the orthogonal projection defined by
\[
P_n y = \sum_{j=1}^n \hat{c}_j \hat{e}_j \quad \text{with} \quad \hat{c}_j = (y, \hat{e}_j)_W, \quad \forall y \in W,
\]
where $\{\hat{e}_j = \frac{1}{\sqrt{\lambda_j}} e_j\}_{j=1}^\infty$ is the orthonormal basis of $W$. It is easy to check that
\[
P_n y = \sum_{j=1}^n c_j e_j \quad \text{with} \quad c_j = (y, e_j)_V, \quad \forall y \in W.
\]

By Parseval’s identity we have that
\[
||P_n y||_V \leq ||y||_V, \quad \forall y \in V,
\]
\[
||P_n y||_W \leq ||y||_W \quad \text{and} \quad P_n y \to y \quad \text{strongly in } W, \quad \forall y \in W.
\]

Considering an arbitrary $Z \in L^q(\Omega \times (0,T); W)$, we have
\[
||P_n Z||_W \leq ||Z||_W \quad \text{and} \quad P_n Z(\omega, t) \to Z(\omega, t) \quad \text{strongly in } W,
\]
which are valid for $P$-a.e. $\omega \in \Omega$ and a.e. $t \in (0,T)$. Hence Lebesgue’s dominated convergence theorem and the inequality
\[
||Z||_V \leq C ||Z||_W \quad \text{for any } Z \in W
\]
implies

$$P_nZ \rightarrow Z \quad \text{strongly in} \quad L^2(\Omega \times (0, T), W),$$
$$P_nZ \rightarrow Z \quad \text{strongly in} \quad L^2(\Omega \times (0, T), V). \quad (4.48)$$

**Step 2. Passing to the limit in the weak sense.** From Lemma 4.5, we have

$$E \sup_{t \in [0, T]} \|Y_n(t)\|^q_{W} \leq C. \quad (4.49)$$

Then there exists a subsequence of $Y_n$, still denoted by $Y_n$ such that

$$Y_n \rightharpoonup Y \quad \ast\text{-weakly in} \quad L^q(\Omega, L^\infty(0, T; W)). \quad (4.50)$$

Moreover, we have

$$P_nY \rightarrow Y \quad \text{strongly in} \quad L^q(\Omega \times (0, T), W). \quad (4.51)$$

Let us notice

$$|(S(y), \phi)| \leq C\|y\|_W^3\|\phi\|_2 \quad \text{for any} \quad y \in W \quad \text{and} \quad \phi \in H,$$

which implies that $S : W \rightarrow H^*$ and

$$\|S(y)\|_{H^*} \leq C\|y\|_W^3, \quad \forall y \in W.$$ 

Therefore

$$\|S(Y_n)\|_{L^2(\Omega, L^2(0, T; H^*))}^2 = \mathbb{E} \int_0^T \|S(Y_n)\|_{H^*}^2 \leq C\mathbb{E} \sup_{t \in [0, T]} \|Y_n(t)\|_W^q < C. \quad (4.52)$$

We also have

$$|\text{div} (S(y), \phi)| \leq C\|y\|_W^3\|\phi\|_V \quad \text{for any} \quad y \in W \quad \text{and} \quad \phi \in V,$$

then

$$\|\text{div} S(y)\|_{V^*} \leq C\|y\|_W^3, \quad \forall y \in W;$$

and

$$\|\text{div} S(y)\|_{L^2(\Omega, L^2(0, T; V^*))}^2 \leq \|\text{div} S(y)\|_{L^2(\Omega, L^2(0, T; V^*))}^2 < C. \quad (4.53)$$

The operator $N$ verifies

$$|(N(y), \phi)| \leq C\|y\|_W^2\|\phi\|_2 \quad \text{for any} \quad y, \phi \in W \quad \text{and} \quad \phi \in V,$$

In addition

$$|\text{div} N(y), \phi)| \leq C\|y\|_W^2\|\phi\|_W \quad \text{for any} \quad y, \phi \in W \quad \text{and} \quad \phi \in V,$$

which imply

$$\|N(y)\|_{L^2(\Omega, L^2(0, T; H^*))}^2 < C, \quad (4.54)$$

and

$$\|\text{div} N(y)\|_{L^2(\Omega, L^2(0, T; V^*))}^2 \leq \|\text{div} N(y)\|_{L^2(\Omega, L^2(0, T; V^*))}^2 < C. \quad (4.55)$$
Let us introduce the operator $B$, defined by

$$B(y) := -(y \cdot \nabla)y.$$ 

We have

$$|(B(y), \phi)| \leq C\|y\|_V^2 \|\phi\|_V, \tag{4.56}$$

then

$$\|B(Y_n)\|_{L^2(\Omega, L^2(0, T, V^*))} \leq C_1 \|Y_n\|_{L^4(\Omega, L^\infty(0, T, V^*))}^2 < C. \tag{4.57}$$

The diffusion operator is bounded. Then there exist operators $N^*(t)$, $S^*(t)$, $\sigma^*(t)$ and a subsequence on $(n)$, such that as $n \to \infty$ we have

$$B(Y_n) \to B^*(t) \text{ weakly in } L^2(\Omega \times (0, T), V^*),$$
$$N(Y_n) \to N^*(t) \text{ weakly in } L^2(\Omega \times (0, T), H^*),$$
$$\text{div } N(Y_n) \to \text{div } N^*(t) \text{ weakly in } L^2(\Omega \times (0, T), V^*),$$
$$S(Y_n) \to S^*(t) \text{ weakly in } L^2(\Omega \times (0, T), H^*),$$
$$\text{div } S(Y_n) \to \text{div } S^*(t) \text{ weakly in } L^2(\Omega \times (0, T), V^*),$$
$$\sigma(t, Y_n) \to \sigma^*(t) \text{ weakly in } L^2(\Omega \times (0, T), (L^2(\Omega))'). \tag{4.58}$$

Therefore, passing to the limit with respect to the weak topology, as $n \to \infty$, all terms in the equation (4.3), we derive that the limit function $Y$ satisfies the stochastic differential equation

$$d (\nu Y, \phi) = [(\nu \Delta Y + U, \phi) + (B^*(t), \phi) + (\text{div } N^*(t), \phi) + \text{div } S^*(t), \phi)] dt + (\sigma^*(t), \phi) dW_t, \forall \phi \in V. \tag{4.59}$$

**Step 3. Passing to the limit in the strong sense up to a stopping time.** Let us introduce the following convenient sequence $(\tau_M)$, $M \in \mathbb{N}$, of stopping times

$$\tau_M = \inf\{t \geq 0 : \|Y(t)\|_W \geq M\} \wedge T.$$

**Proposition 4.6** Let $Y_n$ be the solution of (4.47) and $P_n Y$ the orthogonal projection of the weak limit $Y$ on the space $W_n$. Then for $M$ fixed we have

$$\mathbb{E} \left( \xi_2(t \wedge \tau_M) \|P_n Y(t \wedge \tau_M) - Y_n(t \wedge \tau_M)\|^2 \right) + \frac{\beta}{2} \mathbb{E} \int_0^{t \wedge \tau_M} \xi_2(s) \|D(P_n Y - Y_n)\|^2 ds + \frac{\beta}{4} \mathbb{E} \int_0^{t \wedge \tau_M} \xi_2(s) \int_\Omega (|A_n|^2 + |A|^2)|A(Y_n - Y)|^2 ds \to 0, \quad \text{as} \quad n \to \infty, \tag{4.60}$$

where

$$\xi_2(t) = e^{-D_3 t - 2D_4 \int_0^t \|Y\|_W^4 ds}$$

and $D_3$, $D_4$ are specific constants to be defined later on.

**Proof.** Taking the difference between the equations (4.3) and (4.50), we write

$$d (Y_n - P_n Y, e_i)_V = [(\nu \Delta (Y_n - Y), e_i) + (B(Y_n) - B^*(t), e_i)] dt + [(\text{div } N(Y_n) - \text{div } N^*(t), e_i) + (\text{div } S(Y_n) - \text{div } S^*(t), e_i)] dt + (\sigma(t, Y_n) - \sigma^*(t), e_i) dW_t, \tag{4.61}$$
which holds for any $e_i \in W$, $i = 1, \ldots, n$.

The Itô's formula gives

$$d(Y_n - P_nY, e_i)_V = 2(Y_n - P_nY, e_i)_V [(\nu \Delta(Y_n - Y), e_i) + \langle B(Y_n) - B^*(t), e_i \rangle] dt$$

$$+ 2(Y_n - P_nY, e_i)_V [\langle \text{div}(N(Y_n) - \text{div} N^*(t), e_i) + \langle \text{div}(S(Y_n) - \text{div} S^*(t), e_i) \rangle dt$$

$$+ 2(Y_n - P_nY, e_i)_V (\sigma(t, Y_n) - G^*(t), e_i) dW_t + |(\sigma(t, Y_n) - \sigma^*(t), e_i)|^2 dt.$$ 

Summing on $i = 1, \ldots, n$, we obtain

$$d \left( ||Y_n - P_nY||_V^2 \right) + 4\nu ||D(Y_n - P_nY)||_V^2 dt$$

$$= 2\nu \langle \Delta(P_nY - Y), Y_n - P_nY \rangle dt + 2\langle B(Y_n) - B^*(t), Y_n - P_nY \rangle dt$$

$$+ 2 \left[ \langle \text{div}(N(Y_n) - N^*(t), Y_n - P_nY) + \langle \text{div}(S(Y_n) - S^*(t), Y_n - P_nY) \rangle dt$$

$$+ \sum_{i=1}^n (\sigma(t, Y_n) - \sigma^*(t), e_i)^2 dt + 2 (\sigma(t, Y_n) - \sigma^*(t), Y_n - P_nY) dW_t. \tag{4.62}$$

Now, we write each term in the right hand side of this equation in a convenient form

$$\langle \text{div}(S(Y_n) - S^*(t)), Y_n - P_nY \rangle$$

$$= \langle \text{div}(S(Y_n) - S(Y)), Y_n - P_nY \rangle + \langle \text{div}(S(Y) - S^*(t)), Y_n - P_nY \rangle$$

$$= \langle \text{div}(S(Y_n) - S(Y)), Y_n - Y \rangle + \langle \text{div}(S(Y_n) - S(Y)), Y_n - P_nY \rangle$$

$$+ \langle \text{div}(S(Y) - S^*(t)), Y_n - P_nY \rangle = g_1^2(t) + g_2^2(t) + g_3^2(t). \tag{4.63}$$

Due to relation (A.9), we have

$$g_1^2(t) = -\frac{\beta}{4} \int_O (|A_n|^2 - |A|^2)^2 - \frac{\beta}{4} \int_O (|A_n|^2 + |A|^2)|A(Y_n - Y)|^2. \tag{4.64}$$

Using the inequalities (1.63) and (1.64), the equation (4.62) can be written as

$$d \left( ||Y_n - P_nY||_V^2 \right) + 4\nu ||D(Y_n - P_nY)||_V^2 dt$$

$$+ \frac{\beta}{2} \int_O (|A_n|^2 - |A|^2)^2 dt + \frac{\beta}{2} \int_O (|A_n|^2 + |A|^2)|A(Y_n - Y)|^2 dt$$

$$= 2\nu \langle \Delta(P_nY - Y), Y_n - P_nY \rangle dt + 2\langle B(Y_n) - B^*(t), Y_n - P_nY \rangle dt$$

$$+ 2 \left[ \langle \text{div}(N(Y_n) - N^*(t), Y_n - P_nY) + \langle \text{div}(S(Y_n) - S^*(t), Y_n - P_nY) \rangle dt$$

$$+ \sum_{i=1}^n (\sigma(t, Y_n) - \sigma^*(t), e_i)^2 dt + 2 (\sigma(t, Y_n) - \sigma^*(t), Y_n - P_nY) dW_t. \tag{4.65}$$

We also have

$$\langle \text{div}(N(Y_n) - N^*(t)), Y_n - P_nY \rangle$$

$$= \langle \text{div}(N(Y_n) - N(Y)), Y_n - Y \rangle + \langle \text{div}(N(Y_n) - N(Y)), Y_n - P_nY \rangle$$

$$+ \langle \text{div}(N(Y) - N^*(t)), Y_n - P_nY \rangle = h_1^2(t) + h_2^2(t) + h_3^2(t). \tag{4.66}$$

Applying Lemma A.4 with $3\epsilon = \frac{\beta}{8}$, we have

$$h_1^2(t) \leq \frac{\beta}{8} \int_O |A(Y_n - Y)|^2 \left( |A|^2 + |A_n|^2 \right) + C_1 ||Y_n - Y||_{V^2}$$

$$+ \frac{C}{1 - \lambda} \left( \lambda \frac{\beta}{8} \int_O |Y_n - P_nY|^2_{H^2} \right) ||Y_n - Y||_{H^2}^2 + \frac{C}{1 - \lambda} \left( \lambda \frac{\beta}{8} \int_O |P_nY - Y|^2_{H^2} \right) ||Y||_{H^2}^2. \tag{4.67}$$
for any $\lambda \in ]0,1[$. Let us set
\[ h_n^4(t) = \frac{C}{1 - \lambda} \frac{\lambda^{\frac{n}{2}}}{\|P_n Y - Y\|_{H^1}^2} \|Y\|_{H^2}. \]

Proceeding analogously with the convective term, we deduce
\[
(B(Y_n) - B^*(t), Y_n - P_n Y) = (B(Y_n) - B(Y), Y_n - Y) + (B(Y_n) - B(Y), Y - P_n Y) + (B(Y) - B^*(t), Y_n - P_n Y) = b_n^1(t) + b_n^2(t) + b_n^3(t).
\]

In addition
\[
|b_n^3(t)| \leq C_{22} \|Y\|_{W} \|Y_n - Y\|_{V}^2.
\]

Denoting by $\tilde{\sigma}_n, \tilde{\sigma}$ and $\tilde{\sigma}^*$ the solutions of the Stokes system (3.1) for $f = \sigma(t, Y_n)$, $f = \sigma(t, Y)$ and $f = \sigma^*(t)$, respectively, we have
\[
(\sigma(t, Y_n) - \sigma^*(t), e_i) = (\tilde{\sigma}_n - \tilde{\sigma}^*, e_i)_V, \quad i = 1, 2, \ldots, n.
\]

Then
\[
\sum_{i=1}^n |(\sigma(t, Y_n) - \sigma^*(t), e_i)|^2 = \|P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*\|_V^2.
\]

The standard relation $x^2 = (x - y)^2 - y^2 + 2xy$ allows to write
\[
\|P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*\|_V^2 = \|P_n \tilde{\sigma}_n - P_n \tilde{\sigma}\|_V^2 - \|P_n \tilde{\sigma} - P_n \tilde{\sigma}^*\|_V^2 + 2(P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*, P_n \tilde{\sigma} - P_n \tilde{\sigma}^*)_V.
\]

From the properties of the solutions of the Stokes system (3.1) and (2.8), we have
\[
\|P_n \tilde{\sigma}_n - P_n \tilde{\sigma}\|_V^2 \leq \|\tilde{\sigma}_n - \tilde{\sigma}\|_V^2 \leq \|\sigma(t, Y_n) - \sigma(t, Y)\|_V^2 \leq K \|Y_n - Y\|_{V}^2,
\]

then
\[
\|P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*\|_V^2 \leq K \|Y_n - Y\|_{V}^2 - \|P_n \tilde{\sigma} - P_n \tilde{\sigma}^*\|_V^2 + 2(P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*, P_n \tilde{\sigma} - P_n \tilde{\sigma}^*)_V \leq 2K \|Y_n - P_n Y\|_{V}^2 + C \|P_n Y - Y\|_{V}^2 - \|P_n \tilde{\sigma} - P_n \tilde{\sigma}^*\|_V^2 + 2(P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*, P_n \tilde{\sigma} - P_n \tilde{\sigma}^*)_V.
\]

Let us set $D_3 := 2(K + 2C_1)$ and $D_4 := 2C_2$. The positive constants $K$, $C_1$ and $C_2$ and in (2.8), (4.67) and (4.69) are independent of $n$. We introduce the auxiliary function
\[
\xi_2(t) = e^{-D_3 t - 2D_4 \int_0^t \|Y\|_{W} \, ds}.\]
Now, applying the Itô formula and using the equality (4.65), we get
\[
d (\xi_2(t)|Y_n - P_n Y|^2) + 4\nu \xi_2(t)|D(Y_n - P_n Y)|^2 dt
\]
\[
+ \frac{\beta}{2} \xi_2(t)||A_n|^2 - |A|^2||^2 dt + \frac{\beta}{2} \xi_2(t)\sqrt{\|A_n|^2 + |A|^2} |A(Y_n - Y)|^2 dt
\]
\[
= 2\nu \xi_2(t)(\Delta(Y_n - Y), Y_n - P_n Y) dt
\]
\[
+ 2\xi_2(t)\langle B(Y_n) - B^*(t), Y_n - P_n Y \rangle dt
\]
\[
+ 2\xi_2(t)\langle \text{div} N(Y_n) - \text{div} N^*(t), Y_n - P_n Y \rangle dt + 2\xi_2(t)(g_n^2(t) + g_n^3(t)) dt
\]
\[
+ \xi_2(t) \sum_{i=1}^n \|\sigma(t, Y_n) - \sigma^*(t, e_i)\|^2 dt
\]
\[
+ 2\xi_2(t)(\sigma(t, Y_n) - \sigma^*(t, Y_n - P_n Y)) \, dW_t
\]
\[- D_s \xi_2(t)||P_n Y - Y||^2 dt - 2D_s \xi_2(t)||Y||_V ||Y_n - P_n Y||^2 dt.
\]
Incorporate in this equation the relations (4.66), (4.67), (4.68), (4.69) and (4.70), we deduce
\[
d (\xi_2(t)|Y_n - P_n Y|^2) + 4\nu \xi_2(t)|D(Y_n - P_n Y)|^2 dt + \frac{\beta}{2} \xi_2(t)||A_n|^2 - |A|^2||^2 dt
\]
\[
+ \frac{\beta}{4} \xi_2(t)\sqrt{\|A_n|^2 + |A|^2} |A(Y_n - Y)|^2 dt + \xi_2(t)||P_n \tilde{\sigma} - P_n \tilde{\sigma}^*||^2 dt
\]
\[
\leq 2\nu \xi_2(t)(\Delta(Y_n - Y), Y_n - P_n Y) dt + \xi_2(t)\frac{2C}{1 - \lambda} \frac{\|P_n Y - Y_n\|_{H^{\lambda+\frac{3}{2}}}^2}{\|Y\|_{H^2}^2} \|P_n Y - Y_n\|_{H^{\lambda+\frac{3}{2}}} dt
\]
\[
+ 2\xi_2(t)\left[b_n^2(t) + b_n^3(t) + b_n^4(t) + h_n^3(t) + h_n^4(t) + g_n^2(t) + g_n^3(t)\right] dt
\]
\[
+ \xi_2(t)\left[C(1 + ||Y||_V) ||P_n Y - Y||_V^2 + 2(P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*, P_n \tilde{\sigma} - P_n \tilde{\sigma}^*)_V\right] dt
\]
\[
+ 2\xi_2(t)(\sigma(t, Y_n) - \sigma^*(t, Y_n - P_n Y)) \, dW_t
\]
Integrating over the time interval \((0, t \wedge T), t \in [0, T],\) and taking the expectation, we derive
\[
\mathbb{E} (\xi_2(t \wedge T)|P_n Y(t \wedge T) - Y_n(t \wedge T)|^2) + 4\nu \mathbb{E} \int_0^{t \wedge T} \xi_2(s)||D(P_n Y - Y_n)||^2 ds
\]
\[
+ \frac{\beta}{2} \mathbb{E} \int_0^{t \wedge T} \xi_2(s)||A_n|^2 - |A|^2||^2 ds + \frac{\beta}{4} \mathbb{E} \int_0^{t \wedge T} \xi_2(s)\sqrt{\|A_n|^2 + |A|^2} |A(Y_n - Y)|^2 ds
\]
\[
+ \mathbb{E} \int_0^{t \wedge T} \xi_2(s)||P_n \tilde{\sigma} - P_n \tilde{\sigma}^*||^2 ds
\]
\[
\leq 2\nu \mathbb{E} \int_0^{t \wedge T} \xi_2(s)(\Delta(Y - P_n Y), Y_n - P_n Y) ds
\]
\[
+ 2\nu \mathbb{E} \int_0^{t \wedge T} \xi_2(s)\left[b_n^2(s) + b_n^3(s) + h_n^3(s) + h_n^4(s) + g_n^2(s) + g_n^3(s)\right] ds
\]
\[
+ \mathbb{E} \int_0^{t \wedge T} \xi_2(s)\left[C(1 + M)||P_n Y - Y||_V^2 + 2(P_n \tilde{\sigma}_n - P_n \tilde{\sigma}^*, P_n \tilde{\sigma} - P_n \tilde{\sigma}^*)_V\right] ds
\]
\[
+ \mathbb{E} \int_0^{t \wedge T} \xi_2(s)\frac{2C}{1 - \lambda} \frac{\|P_n Y - Y_n\|_{H^{\lambda+\frac{3}{2}}}^2}{\|Y\|_{H^2}^2} \|P_n Y - Y_n\|_{H^{\lambda+\frac{3}{2}}} ds
\]
\[
= J_1^2 + J_2^2 + J_3^3 + J_4^4.
\]
Here, we assume that
\[
r_n(t) = J_1^2 + J_2^2 + J_3^3 + J_4^4 \to 0.
\]
This result will be proved in a lemma at the end of this proposition.

Let us define

\[ a_n(t) = E (\xi_2(t \wedge \tau_M)\|P_n Y(t \wedge \tau_M) - Y_n(t \wedge \tau_M)\|^2) + 4\nu E \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s)\|D(P_n Y - Y_n)\|_2^2 ds \]

\[ + \frac{\beta}{2} E \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s)\|A_n\|^2 - |A|^2\|^2_2 ds \]

\[ + \frac{\beta}{4} E \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s)\|\sqrt{|A_n|^2 + |A|^2} |A(Y_n - Y)\|_2^2 ds \]

\[ + E \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s)\|P_n \tilde{\sigma} - \tilde{P}_n \tilde{\sigma}\|^2_1 ds. \]  

(4.74)

Taking into account (4.71), (4.73) and the concavity of the function \( x \rightarrow x^{\frac{2(\lambda+1)}{\lambda+3}} \), \( \lambda \in [0,1] \), we derive

\[ a_n(t) \leq r_n(t) + \frac{2C}{1-\lambda} E \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s) e^{\lambda^{-1} s} \|P_n Y - Y_n\|_{H^2} \|Y\|^2_{H^2} ds \]

\[ \leq r_n(t) + M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s \wedge \tau_M)\|P_n Y(s \wedge \tau_M) - Y_n(s \wedge \tau_M)\|_{H^2}^{\frac{4(\lambda+1)}{\lambda+3}} ds \]

\[ \leq r_n(t) + M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} \int_0^t 1_{[0,\tau_M]}(s)\xi_2(s \wedge \tau_M)\|P_n Y(s \wedge \tau_M) - Y_n(s \wedge \tau_M)\|_{H^2}^2 \int_0^t a_n(s) ds \]

\[ \leq r_n(t) + M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} \int_0^t a_n(s) ds \]

\[ \leq r_n(t) + M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} \int_0^t a_n(s) ds \]

(4.75)

which yields

\[ \limsup_{n \to \infty} a_n(t) \leq \limsup_{n \to \infty} r_n(t) + M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} \int_0^t \limsup_{n \to \infty} a_n(s) ds \]

\[ \leq r_n(t) + M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} \int_0^t \limsup_{n \to \infty} a_n(s) ds \]

(4.76)

Denoting

\[ f(t) := \int_0^t \limsup_{n \to \infty} a_n(s) ds, \]

and knowing that \( \lim_{n \to \infty} r_n(t) = 0 \), (4.76) can be written as

\[ f'(t) \leq M^{\frac{4}{\alpha+3}} \frac{2C}{1-\lambda} e^{\lambda^{-1} t} \int_0^t f(s) ds \]

(4.77)

Here, we can proceed as in [5] in order to verify that \( f \equiv 0 \). Since \( f(0) = 0 \) and

\[ \left( f(t) \right)^{\frac{\lambda+3}{4}} \leq \frac{2C}{\lambda+3} M^{\frac{4}{\alpha+3}} e^{\frac{\lambda+3}{4} t}, \]

we have

\[ f(t) \leq \left( \frac{2C}{\lambda+3} M^{\frac{4}{\alpha+3}} e^{\frac{\lambda+3}{4} t} \right)^{\frac{1}{\lambda+3}}. \]
Considering $T_0 = \frac{\lambda_0}{4M^\frac{1}{n+1}}$, we have \( \frac{\lambda_0}{4M^\frac{1}{n+1}} t \leq \frac{1}{2} \). Taking $\lambda \to 1$, we get $f(t) = 0$, $\forall t \in [0,T_0]$. By an extension argument, we obtain $f(t) = 0$, $\forall t \in [0,T]$.

\[
\mathbb{E} \left( \xi(t \land \tau_M) \right) \mathbb{P}(P_0(t \land \tau_M) - Y_n(t \land \tau_M) \| Y_n \|^2) + 4\nu \mathbb{E} \int_0^{t \land \tau_M} \xi(s) \| D(P_n Y - Y_n) \|^2 ds
\]

\[
+ \frac{\beta}{2} \mathbb{E} \int_0^{t \land \tau_M} \xi(s) \int_{\Omega} (|A_n|^2 - |A|^2)^2 ds + \frac{\beta}{4} \mathbb{E} \int_0^{t \land \tau_M} \xi(s) \int_{\Omega} (|A_n|^2 + |A|^2)^2 ds
\]

\[
+ \mathbb{E} \int_0^{t \land \tau_M} \xi(s) \| P_n \bar{\sigma} - P_n \tilde{\sigma} \|^2 \rightarrow 0, \quad \text{as } n \to \infty.
\]

**Lemma 4.7** Let $J_n^1(t)$, $J_n^2(t)$, $J_n^3(t)$ be the terms introduced in (4.71). Then for all $t \in [0,T]$, $J_n^i(t) \to 0$, for $i = 1, 2, 3$.

**Proof.** Using (4.49)–(4.50) and the properties of the projection $P_n$, we have

\[
|J_n^1(t)| = |2\nu \mathbb{E} \int_0^{t \land \tau_M} \xi(s) (1_{[0,\tau_M]}(s) \Delta(Y - P_n Y, P_n Y - Y_n) ds |
\]

\[
\leq C \| P_n Y - Y \|_{L^2(\Omega \times (0,T), L^2)} \| P_n Y - Y_n \|_{L^2(\Omega \times (0,T), L^2)}
\]

\[
\leq C \| P_n Y - Y \|_{L^2(\Omega \times (0,T), W)} \left( \| Y \|_{L^2(\Omega \times (0,T), W)} + \| Y_n \|_{L^2(\Omega \times (0,T), L^2)} \right)
\]

which goes to zero, as $n \to \infty$, by (4.51).

\[
J_n^2(t) = 2 \mathbb{E} \int_0^{t \land \tau_M} \xi(s) \left[ b^2_n(s) + b^2_n(s) + h^2_n(s) + h^2_n(s) + g^2_n(s) + g^2_n(s) \right] ds
\]

From (4.50), (4.52) and (4.51), we deduce

\[
\left| 2 \mathbb{E} \int_0^{t \land \tau_M} \xi(s) b^2_n(s) ds \right| = \left| 2 \mathbb{E} \int_0^{t \land \tau_M} \xi(s) (B(Y_n) - B(Y), P_n Y - Y) \right|
\]

\[
\leq C \mathbb{E} \sup_{t \in [0,T]} \| Y_n \|_{L^2}^\frac{1}{2} \mathbb{E} \| P_n Y - Y \|_{L^2(\Omega \times (0,T), W)}^2 \rightarrow 0, \quad \text{as } n \to \infty.
\]

Convergences (4.50) and (4.51) give that

\[ P_n Y - Y_n \to 0 \quad \text{weakly in } L^2(\Omega \times (0,T), W), \]

then for any operator $R \in L^2(\Omega \times (0,T), W^*)$ we have

\[ \mathbb{E} \int_0^T \langle R, P_n Y - Y_n \rangle ds \rightarrow 0, \quad \text{as } n \to \infty. \]

The function $1_{[0,\tau_M]}(s) \xi_2(s)$ is bounded, then

\[ \| 1_{[0,\tau_M]}(s) \xi_2(s) (B(Y) - B^*) \|_{L^2(\Omega \times (0,T), W^*)}^2 \leq C \left( \| B(Y) \|_{L^2(\Omega \times (0,T), W^*)}^2 + \| B^* \|_{L^2(\Omega \times (0,T), W^*)}^2 \right) \leq C, \]
by (4.19), (4.57) and (4.58). Therefore, as \( n \to \infty \), we have

\[
2\mathbb{E} \int_0^{\tau_M} \xi_2(s)h_n^2(s) \, ds = 2\mathbb{E} \int_0^{\tau_M} \left( 1_{[0,\tau_M]}(s)\xi_2(s)(B(Y) - B^*(s)) \right) P_n Y - Y_n \, ds \to 0.
\]

Using the same reasoning, we show that

\[
2\mathbb{E} \int_0^{\tau_M} \xi_2(s)h_n^2(s) \to 0, \quad 2\mathbb{E} \int_0^{\tau_M} \xi_2(s)h_n^2(s) \to 0.
\]

By the definition of the stopping time \( \tau_M \), we have \( 1_{[0,\tau_M]}(s)\xi_2(s)\|Y\|_W^2 \leq M \tau_M \), so

\[
\left| 2\mathbb{E} \int_0^{\tau_M} \xi_2(s)h_n^2(s) \right| \leq \frac{C}{1 - \lambda} \frac{\tau_M}{\tau_M} \left| 2\mathbb{E} \int_0^T \left( 1_{[0,\tau_M]}(s)\xi_2(s)\|P_n Y - Y\|_W^{4(\lambda + 1)} \right) \right|
\]

\[
\leq \frac{C}{1 - \lambda} \frac{\tau_M}{\tau_M} \left| 2\mathbb{E} \int_0^T \|P_n Y - Y\|_W^{4(\lambda + 1)} \right|
\]

\[
\leq C(M, \lambda) \|P_n Y - Y\|_{L^2(\Omega \times (0, T), Y)} \to 0.
\]

Similarly we verify that the remaining terms in \( J_n^2(t) \) converges to 0, as well as \( J_n^3(t) \) converges to 0, as \( n \to \infty \).

From (4.60), the following strong convergences hold

\[
\lim_{n \to \infty} \mathbb{E} \left( \xi_2(\tau_M)\|P_n Y(\tau_M) - Y_n(\tau_M)\|_W^2 \right) = 0, \quad (4.78)
\]

\[
\lim_{n \to \infty} \mathbb{E} \int_0^{\tau_M} \xi_2(s)\|D(P_n Y - Y_n)\|_2^2 \, ds = 0, \quad (4.79)
\]

\[
\lim_{n \to \infty} \mathbb{E} \int_0^{\tau_M} \xi_2(s)\|A_n - |A||_2^2 \, ds = 0, \quad (4.80)
\]

\[
\lim_{n \to \infty} \int_0^{\tau_M} \xi_2(s)\|\sqrt{A_n^2 + |A|^2} A(Y_n - Y)\|_2^2 \, ds = 0, \quad (4.81)
\]

\[
\lim_{n \to \infty} \int_0^{\tau_M} \xi_2(s)\|P_n \tilde{\sigma} - P_n \tilde{\sigma}^*\|_2^2 \, ds = 0, \quad (4.82)
\]

for each \( M \in \mathbb{N} \). Since there exists a strictly positive constant \( \mu \), such that \( \mu \leq 1_{[0,\tau_M]}(s)\xi_2(s) \leq 1 \), it follows from the Korn inequality (2.12) and (4.51) that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^{\tau_M} \xi_2(s)\|D(P_n Y - Y_n)\|_2^2 \, ds = 0 \quad \text{implies} \quad \lim_{n \to \infty} \mathbb{E} \int_0^{\tau_M} \|Y - Y_n\|_W^2 \, ds = 0. \quad (4.83)
\]

In addition, we have

\[
\lim_{n \to \infty} \mathbb{E} \int_0^{\tau_M} \|A_n - |A||_2^2 \, ds = 0, \quad (4.84)
\]

\[
\lim_{n \to \infty} \mathbb{E} \int_0^{\tau_M} \|\sqrt{A_n^2 + |A|^2} A(Y_n - Y)\|_2^2 \, ds = 0. \quad (4.85)
\]
which implies 

\[ \mathbb{E} \int_0^{\tau_M} \| \tilde{\sigma} - \tilde{\sigma}^* \|^2_V \, ds = 0. \]  

(4.86)

Step 4. Identification of \( B^*(t) \) with \( B(Y) \), \( \text{div} \, N^*(t) \) with \( \text{div} \, N(Y) \), \( \text{div} \, S^*(t) \) with \( \text{div} \, S(Y) \) and \( \sigma^*(t) \) with \( \sigma(t, Y) \) on \([0, \tau_M]\) for each \( M \).

Now, we are able to show that the limit function \( Y \) satisfies equation \( (4.1) \). Integrating equation \( (4.59) \) on the time interval \((0, \tau_M \land t)\), we derive

\[ (v(Y(\tau_M \land t)), \phi) - (v(Y_0), \phi) = \int_0^{\tau_M \land t} \left[ (v\Delta Y + U, \phi) + (B^*(s), \phi) + (\text{div} \, N^*(s), \phi) \right. \]

\[ \left. + \langle \text{div} \, S^*(s), \phi \rangle \right] \, ds + \int_0^{\tau_M \land t} (\sigma^*(s), \phi) \, dW_s \]  

(4.87)

for any \( \phi \in V \). From \( (4.50) \) it follows that

\[ 1_{[0, \tau_M]}(t)\tilde{\sigma} = 1_{[0, \tau_M]}(t)\tilde{\sigma}^* \quad \text{a.e. in } \Omega \times (0, T), \]

which implies

\[ 1_{[0, \tau_M]}(t)\sigma(t, Y) = 1_{[0, \tau_M]}(t)\sigma^*(t) \quad \text{a.e. in } \Omega \times (0, T) \]  

(4.88)

by \( (4.1) \). Since \( B(Y_n) - B(Y) = (Y_n \cdot \nabla)(Y_n - Y) + (Y_n - Y) \cdot \nabla Y \), we verify that

\[ \| B(Y_n) - B(Y) \|_V \leq C (\| Y_n \|_V + \| Y \|_V) \| Y_n - Y \|_V. \]

Then for any \( \varphi \in L^\infty(\Omega \times (0, T), V) \), using \( (4.40), (4.50) \)

\[ \left| \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) \langle B(Y_n) - B(Y), \varphi \rangle \, ds \right| \]

\[ \leq C \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) (\| Y_n \|_V + \| Y \|_V) \| Y_n - Y \|_V \| \varphi \|_V \, ds \]

\[ \leq C \| \varphi \|_{L^\infty(\Omega \times (0, T), V)} \mathbb{E} \int_0^T (\| Y_n \|_V + \| Y \|_V) \| Y_n - Y \|_V \, ds \]

\[ \leq C \| \varphi \|_{L^\infty(\Omega \times (0, T), V)} \left( \mathbb{E} \int_0^{\tau_M} \| Y_n - Y \|_V^2 \, ds \right)^{1/2} \to 0, \quad \text{as } n \to \infty. \]

Taking into account \( (4.58)_1 \) and that the space \( L^\infty(\Omega \times (0, T), V) \) is dense in \( L^2(\Omega \times (0, T), V) \), we obtain

\[ 1_{[0, \tau_M]}(s)B^*(s) = 1_{[0, \tau_M]}(s)B(Y) \quad \text{a.e. in } \Omega \times (0, T). \]  

(4.89)

From \( (4.8) \), we have

\[ |\langle \text{div} \, (N(Y_n) - N(Y)), \phi \rangle| \leq C \| A(Y_n - Y) \|_V \| A_n \|^2 + \| A \|^2 \| \phi \|_V \]

\[ + C \| Y_n - y \|_V (\| Y_n \|_W + \| Y \|_W) \| \phi \|_W. \]  

(4.90)
Therefore

\[
\left| \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) \langle \text{div}(N(Y_n) - N(Y)), \phi \rangle \, ds \right|
\]

\[
\leq C \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) \left\| A(Y_n - Y) \sqrt{|A_n|^2 + |A|^2} \right\|_2 \, ds
\]

\[
+ C \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) ||Y_n - y||_V (||Y_n||_W + ||Y||_W) \, ds
\]

\[
\leq C||\phi||_{L^\infty(\Omega \times (0, T), W)} \mathbb{E} \int_0^T \left\| A(Y_n - Y) \sqrt{|A_n|^2 + |A|^2} \right\|_2 \, ds
\]

\[
+ C||\phi||_{L^\infty(\Omega \times (0, T), W)} \mathbb{E} \int_0^T (||Y_n||_W + ||Y||_W) ||Y_n - Y||_V \, ds
\]

\[
\leq C||\phi||_{L^\infty(\Omega \times (0, T), W)} \left( \mathbb{E} \int_0^T ||Y_n - Y||_V^2 \, ds \right)^{1/2} \to 0, \quad \text{as } n \to \infty.
\]

Therefore

\[
1_{[0, \tau_M]}(s) \text{div}(N^*(s)) = 1_{[0, \tau_M]}(s) \text{div}(N(Y)) \quad \text{a.e. in } \Omega \times (0, T). \quad (4.91)
\]

Using the same reasoning, we show

\[
1_{[0, \tau_M]}(s) \text{div}(S^*(s)) = 1_{[0, \tau_M]}(s) \text{div}(S(Y)) \quad \text{a.e. in } \Omega \times (0, T). \quad (4.92)
\]

Namely, from \(A.9\), we have

\[
|\langle \text{div}(S(Y) - S(Y_n)), \phi \rangle| \leq C \|Y\|_W^2 \|Y - Y_n\|_V \|\phi\|_W + C \|Y_n\|_W \|\|A\|^2 - |A_n|^2\|_2 \|\phi\|_W, \quad (4.93)
\]

and \((4.83)\) and \((4.84)\) gives

\[
\left| \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) \langle \text{div}(S(Y) - S(Y_n)), \phi \rangle \, ds \right|
\]

\[
\leq C \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) C \|Y\|_W^2 \|Y - Y_n\|_V \|\phi\|_W \, ds
\]

\[
+ C \mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) \|Y_n\|_W \|\|A\|^2 - |A_n|^2\|_2 \|\phi\|_W
\]

\[
\leq C||\phi||_{L^\infty(\Omega \times (0, T), W)} \mathbb{E} \int_0^T \|Y\|_W \|Y_n - Y\|_V \, ds
\]

\[
+ C||\phi||_{L^\infty(\Omega \times (0, T), W)} \mathbb{E} \int_0^T \|Y_n\|_W \|\|A\|^2 - |A_n|^2\|_2 \, ds
\]

\[
\leq C(M)||\phi||_{L^\infty(\Omega \times (0, T), W)} \mathbb{E} \int_0^T \|Y_n - Y\|_V \, ds
\]

\[
+ C(M)||\phi||_{L^\infty(\Omega \times (0, T), W)} \mathbb{E} \int_0^T \|\|A\|^2 - |A_n|^2\|_2 \to 0, \quad \text{as } n \to \infty.
\]
By introducing identities (4.88), (4.89), (4.91) and (4.92) in equation (4.87), it follows that

\[
(\nu (Y(\tau_M \wedge t)), \phi) - (\nu (Y_0), \phi) = \int_0^{\tau_M \wedge T} \left[ (\nu \Delta Y + U, \phi) + \langle B(Y) + \text{div}(N(Y)) + \text{div}(S(Y)), \phi \rangle \right] ds + \int_0^{\tau_M \wedge T} (\sigma(s, Y), \phi) dW_s.
\] (4.94)

Reasoning as in (4.22) we have \( \tau_M \to T \) a.e. in \( \Omega \), as \( M \to \infty \). We can pass to the limit in each term of equation (4.94) in \( L^1(\Omega \times (0, T)) \), as \( M \to \infty \), by applying the Lebesgue dominated convergence theorem and the Burkholder-Davis-Gundy inequality for the last (stochastic) term, deriving an equivalent formulation of equation (4.1) a.e. in \( \Omega \times (0, T) \).

**Step 5. Uniqueness.** In order to prove uniqueness, we take two solutions \( Y_1 \) and \( Y_2 \), and consider the difference \( Y = Y_1 - Y_2 \). Using similar arguments as in the previous steps, introducing the function

\[
\xi_3(t) = e^{-\frac{1}{2} D_3 t - D_4} \| Y_1 \|_W \Delta t,
\]

we show that

\[
\mathbb{E} (\xi_3(t) || Y(t) ||_Y^2) + 4\nu \mathbb{E} \int_0^t \xi_3(s) || D(Y) ||^2_Y ds + \beta \mathbb{E} \int_0^t \xi_3(s) \int_\Omega \langle |A(Y_1)|^2 - |A(Y_2)|^2 \rangle^2 ds + \frac{\beta^2}{4} \mathbb{E} \int_0^t \xi_3(s) \int_\Omega \langle |A(Y_1)|^2 + |A(Y_2)|^2 \rangle |A(Y)|^2 \rangle ds = 0 \quad \text{for a.e. } t \in [0, T].
\]

Therefore, for a.e. \( t \in [0, T] \), we have

\[
\mathbb{E} (\xi_3(t) || Y(t) ||_Y^2) = 0.
\]

Since \( \xi_3 \) is a positive function, we deduce that for a.e. \( t \in [0, T] \)

\[
Y_1(t) = Y_2(t), \quad P - \text{a.s.}
\]

\[\square\]

**A Appendix**

In this appendix, we collect important inequalities from [6] related with the nonlinear terms that we apply throughout the article.

**Lemma A.1** For any \( \epsilon > 0 \) and \( y \in W \), we have

\[
| (\alpha_1 + \alpha_2) \int_\Omega \text{div}(A^2) \cdot y | \leq \epsilon \| A^2 \|^2_2 + \frac{(\alpha_1 + \alpha_2)^2}{16\epsilon} \| A \|^2_2,
\] (A.1)

where \( A = A(y) \).
Lemma A.3

For any \( A \) where \( A \) due to the boundary conditions \( y = (y \cdot \tau) \) and \( (n \cdot A) \cdot \tau = 0 \) on \( \Gamma \), we obtain

\[
(n \cdot A^2) \cdot y = (y \cdot \tau)(n \cdot A^2) \cdot \tau = (y \cdot \tau)(n \cdot A) \cdot \tau = (y \cdot \tau)[((n \cdot A) \cdot n)((n \cdot A) \cdot \tau) + ((n \cdot A) \cdot \tau)((\tau \cdot A) \cdot \tau)] = 0.
\]

Using the symmetry of \( A \), we derive

\[
(\alpha_1 + \alpha_2) \int_\Omega \text{div}(A^2) \cdot y = -\frac{1}{2}(\alpha_1 + \alpha_2) \int_\Omega A^2 \cdot A.
\]

Therefore, the H"older and the Young inequalities give \((A.1)\).

Proof. Integrating by parts gives

\[
(\alpha_1 + \alpha_2) \int_\Omega \text{div}(A^2) \cdot y = (\alpha_1 + \alpha_2) \int_\Gamma (n \cdot A^2) \cdot y - (\alpha_1 + \alpha_2) \int_\Omega A^2 \cdot \nabla y.
\]

Due to the boundary conditions \( y = (y \cdot \tau) \) and \( (n \cdot A) \cdot \tau = 0 \) on \( \Gamma \), we obtain

\[
(\alpha_1 + \alpha_2) \int_\Omega \text{div}(A^2) \cdot y = \frac{1}{2}(\alpha_1 + \alpha_2) \int_\Omega A^2 \cdot A.
\]

Considering a small change in estimate (35) of \([6]\), we collect the following estimates.

Lemma A.2 (see \([6]\), relations (33)-(36)) For each \( y \in W \) and any \( \epsilon, \delta > 0 \), the following estimates are valid

\[
(\text{div} (|A|^2 A), \mathbb{P}v(y)) \leq \frac{1}{2} ||A||_{\infty}^4 + \frac{\alpha_1}{2} |||A|||\nabla A||^2 - \frac{\alpha_1}{4} ||\nabla(|A|^2)||^2 + 3\epsilon ||A||\nabla^2 y||^2_2 + 5\epsilon ||A||_{12}^2 + 3\epsilon ||y||_{H^1}^2 + C(\epsilon) ||y||_{H^2}^2, \quad (A.4)
\]

\[
(\alpha_1 + \alpha_2)(\text{div} (A^2), \mathbb{P}v(y)) \leq \epsilon |||A|||\nabla^2 y||^2_2 + C(\epsilon) ||y||_{H^2}^2, \quad (A.5)
\]

\[
-((y \cdot \nabla)v + \sum_j v_j \nabla y_j, \mathbb{P}v(y)) \leq 4\epsilon |||A|||\nabla^2 y||^2_2 + C(\epsilon, \delta) ||y||_{H^1}^2 + C(\epsilon) ||y||_{H^2}^2 + \delta ||y||_{W^{1,4}}^4, \quad (A.6)
\]

where \( A = A(y) \) and \( v = v(y) \).

Let us introduce the operators

\[
S(y) := \beta (|A(y)|^2 A(y)), \quad (A.7)
\]

\[
N(y) := \alpha_1 (y \cdot \nabla A(y)) + (\nabla y)^T A(y) + A(y) \nabla y - \alpha_2 (A(y))^2. \quad (A.8)
\]

Lemma A.3 For any \( y, \hat{y} \in W \), we have

\[
\langle \text{div}(S(y)) - S(y), \hat{y} - y \rangle = -\frac{\beta}{4} \int_\Omega (|\hat{A}|^2 - |A|^2)^2 - \frac{\beta}{4} \int_\Omega (|\hat{A}|^2 + |A|^2)|A(\hat{y} - y)|^2, \quad (A.9)
\]

where \( A = A(y) \) and \( \hat{A} = A(\hat{y}) \).

Proof. Integrating by parts, we write

\[
(\text{div}(S(y)) - S(y), \hat{y} - y) = \beta \int_\Gamma (n \cdot (|\hat{A}|^2 \hat{A} - |A|^2 A)) \cdot (\hat{y} - y)
\]

\[
-\frac{\beta}{2} \int_\Omega (|\hat{A}|^2 \hat{A} - |A|^2 A) \cdot A(\hat{y} - y) = I_{11} + I_{12}. \quad (A.10)
\]
Using the boundary conditions, we deduce that

\[ I_{11} = \beta \int_{\Gamma} ((\hat{y} - y) \cdot \tau) \left[ |\hat{A}|^2 (u \cdot \hat{A}) \cdot \tau - |A|^2 (u \cdot A) \cdot \tau \right] = 0. \]  

(A.11)

Standard algebraic computations yield

\[ I_{12} = -\frac{\beta}{2} \int_{\Omega} (|\hat{A}|^2 \hat{A} - |A|^2 A) \cdot \hat{A}(\hat{y} - y) \]

\[ = -\frac{\beta}{2} \int_{\Omega} (|\hat{A}|^2 \hat{A} - |A|^2 A) \cdot \hat{A}(\hat{y} - y) \]

\[ = -\frac{\beta}{4} \int_{\Omega} (|\hat{A}|^2 - |A|^2)^2 - \frac{\beta}{4} \int_{\Omega} (|\hat{A}|^2 + |A|^2) |A(\hat{y} - y)|^2. \]  

(A.12)

Lemma A.4 For any \( y, \hat{y} \in W \), the following estimate holds

\[ (\text{div} (N(\hat{y}) - N(y)), \hat{y} - y) = (N(\hat{y}) - N(y), \nabla (\hat{y} - y)) \]

\[ \leq 3 \epsilon \int_{\Omega} |A(\hat{y} - y)|^2 \left( |A|^2 + |\hat{A}|^2 \right) + \frac{C}{\epsilon} \int_{\Omega} |\nabla (\hat{y} - y)|^2 \]

\[ + \frac{C}{1 - \lambda} \left( \frac{\lambda - 1}{H^{(v)} H^{(w)}} \|\hat{y} - y\|_{H^{(v)}} \|y\|_{H^{(w)}} \right) \quad \text{for any} \quad \epsilon > 0, \lambda \in ]0, 1[, \]

(A.13)

where \( A = A(y) \) and \( \hat{A} = A(\hat{y}) \).

Proof. The divergence theorem gives

\[ (\text{div} (N(\hat{y}) - N(y)), \hat{y} - y) = (N(\hat{y}) - N(y), \nabla (\hat{y} - y)) - \int_{\Gamma} \left[ (N(\hat{y}) - N(y)) u \cdot (\hat{y} - y) \right]. \]  

(A.14)

The relation (A.13) is proved in [5] for the case \( \Omega = \mathbb{R}^2 \) (domain without boundary), in [6] it is verified that the boundary term vanishes.

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