Lie symmetries
of difference equations

D. Levi*, J. Negro** and M.A. del Olmo**

*Departamento di Fisica, Universitá Roma Tre and INFN–Sezione di Roma Tre
Via della Vasca Navale 84, 00146 Roma, Italy

**Departamento de Física Teórica, Universidad de Valladolid,
E-47011, Valladolid, Spain.
e-mail:levi@amaldi.fis.uniroma3.it, jnegro@fta.uva.es, olmo@fta.uva.es

Abstract

The discrete heat equation is worked out in order to illustrate the search of symmetries of difference equations. It is paid an special attention to the Lie structure of these symmetries, as well as to their dependence on the derivative discretization. The case of $q$–symmetries for discrete equations in a $q$–lattice is also considered.

1 Introduction

As it is well known Lie point symmetries were introduced by Lie for solving differential equations, providing one of the most efficient methods for obtaining exact analytical solutions of partial differential equations [1]. The interest for discrete systems in the last years has led to extend the Lie method to the case of discrete equations [2]–[4].

A general difference equation, involving one scalar function $u(x)$ of $p$ independent variables $x = (x_1, x_2, \ldots, x_p)$ evaluated at a finite number of points on a lattice will be written in the form

$$E(x, T^a u(x), T^{b_i} \Delta_{x_i} u(x), T^{c_{ij}} \Delta_{x_i} \Delta_{x_j} u(x), \ldots) = 0,$$

where the shift operators $T^a, T^{b_i}, T^{c_{ij}}$ are defined by

$$T^a u(x) := \left\{ T^{a_1}_{x_1} T^{a_2}_{x_2} \ldots T^{a_p}_{x_p} u(x) \right\}_{a_1=1}^{n_1} a = (a_1, a_2, \ldots, a_p), \quad i = 1, 2, \ldots, p,$$

1Talk delivered by J. Negro at the DI-CRM Workshop held in Prague, 18-21.6.2000
with \(a_i, m_i, n_i, (m_i \leq n_i)\), fixed integers,
\[
T^a_{x_i} u(x) = u(x_1, x_2, \ldots, x_{i-1}, x_i + a_i \sigma_i, x_{i+1}, \ldots, x_p),
\]
and \(\sigma_i\) is the positive lattice spacing in the uniform lattice of the variable \(x_i\) \((i = 1, \ldots, p)\).

The other shift operators \(T^b_i\), \(T^{c_{ij}}\) are defined in a similar way. The difference operators \(\Delta_{x_i}\) are defined so that in the continuous limit turn into partial derivatives.

In the following we will make use of the approach presented in [3], based on the formalism of continuous evolutionary vector fields \([2]\). The infinitesimal symmetry vectors in evolutionary form for the difference equation of order \(N\) given in (1.1) take the general expression
\[
X_e \equiv Q \partial u = \left( \sum_i \xi_i(x, T^a u, \sigma_x, \sigma_t) T^b_i \Delta_{x_i} u - \phi(x, T^c u, \sigma_x, \sigma_t) \right) \partial u, \tag{1.2}
\]
where \(\xi_i(x, T^a u, \sigma_x, \sigma_t)\) and \(\phi(x, T^c u, \sigma_x, \sigma_t)\) are operator valued functions which in the continuous limit become the functions \(\xi_i(x, u)\) and \(\phi(x, u)\), respectively, giving rise to Lie point symmetries.

The vector fields \(X_e\) generate the symmetry algebra of the discrete equation (1.1), whose elements transform solutions \(u(x)\) of the equation into solutions \(\tilde{u}(x)\). The \(N\)-th prolongation of \(X_e\) must verify the invariance condition
\[
pr^N X_e |_{E=0} = 0. \tag{1.3}
\]
The prolongation \(pr^N X_e\) is
\[
pr^N X_e = \sum_a T^a Q \partial_{T^a u} + \sum_b b_i T^b Q \partial_{T^b \Delta_{x_i} u} + \sum_{c_{ij}} T^{c_{ij}} Q^{x_{ij}} \partial_{T^{c_{ij}} \Delta_{x_i \Delta_{x_j} u}} + \ldots \tag{1.4}
\]
where summations in (1.4) are over all the sites present in (1.1). The symbols \(Q^{x_{ij}}, Q^{x_{ij} x_{jk}}, \ldots\) are total variations of \(Q\), i.e., \(Q^{x_{ij}} = \Delta_{x_i}^T Q\), \(Q^{x_{ij} x_{jk}} = \Delta_{x_i}^T \Delta_{x_j}^T Q\), \ldots, defined by
\[
\Delta_{x_i}^T f(x, u(x), \Delta_x u(x), \ldots) = \frac{1}{\sigma} [f(x+\sigma, u(x+\sigma), (\Delta_x u)(x+\sigma), \ldots) - f(x, u(x), \Delta_x u(x), \ldots)],
\]
while the partial variation \(\Delta_{x_i}\) is
\[
\Delta_x f(x, u(x), \Delta_x u(x), \ldots) = \frac{1}{\sigma} [f(x+\sigma, u(x), (\Delta_x u)(x), \ldots) - f(x, u(x), \Delta_x u(x), \ldots)]. \tag{1.5}
\]

The solutions of (1.3) gives the symmetries of equation (1.1) when using the difference operator (1.3). The determining equations for \(\xi_i\) and \(\phi\) are obtained by considering linearly independent the expressions in the discrete derivatives \(T^a \Delta_{x_i} u, T^b \Delta_{x_i} \Delta_{x_j} u, \ldots\). The Lie commutators of the vector fields \(X_e\) are obtained by commuting their first prolongations and projecting onto the symmetry algebra \(G\).

Since we will restrict here to linear equations we can assume that the evolutionary vectors (1.2) have the form \(X_e = (\hat{X} u) \partial_u\), where
\[
\hat{X} = \sum_i \xi_i(x, T^a, \sigma_x, \sigma_t) \Delta_{x_i} - \phi(x, T^a, \sigma_x, \sigma_t).
\]
The operators $\hat{X}$, in general, may span only a subalgebra of the whole Lie symmetry algebra $[5]$. In Ref. $[6]$ the symmetries were obtained using the above mentioned difference operator

$$\Delta_{x_i} \equiv \Delta_{x_i}^+ = \frac{T_{x_i} - 1}{\sigma_i},$$

(1.6)

which, when $\sigma_i \to 0$, goes into the standard right derivative with respect to $x_i$. Since other definitions of the difference operator can be introduced $[7]$, one would like to show that the algebraic structure of the symmetries is independent on the choice undertaken. In this work we will also use the left derivative

$$\Delta_{x_i}^- = \frac{1 - T_{x_i}^{-1}}{\sigma_i},$$

(1.7)

and the symmetric derivative (which goes into the derivative with respect to $x_i$ up to terms of order $\sigma_i^2$)

$$\Delta_{x_i}^s = \frac{T_{x_i} - T_{x_i}^{-1}}{2\sigma_i}.$$  (1.8)

In the following we will see that by an appropriate definition of the Leibniz rule we can construct Lie symmetries, in principle, for any difference operator. In Section $[8]$ we will introduce this procedure on the example of the discrete heat equation. We shall study separately the cases of the discrete derivatives (1.6)–(1.8), as well as that of a $q$–derivative. In this way we will get different representations of the same Lie algebra. We conclude with some remarks and comments.

## 2 Discrete heat equation

Let us consider the second order difference equation

$$(\Delta_t - \Delta_{xx})u(x) = 0,$$

as a discretization of the heat equation. Since it is linear, we can consider an evolutionary vector field of the form

$$X_e \equiv Q\partial_u = (\tau \Delta_t + \xi \Delta_x u + fu)\partial_u,$$

(2.1)

where $\tau$, $\xi$ and $f$ are (operator valued) functions of $x, t, T_x, T_t, \sigma_x$ and $\sigma_t$. The determining equation is

$$\Delta_t^T Q - \Delta_{xx}^T Q|_{\Delta_{xx}u = \Delta_t u} = 0,$$

whose explicit expression is

$$\Delta_t (\xi \Delta_x u) + \Delta_t (\tau \Delta_t u) + \Delta_t (fu) - [\Delta_{xx} (\xi \Delta_x u) + \Delta_{xx} (\tau \Delta_t u) + \Delta_{xx} (fu)]|_{\Delta_{xx}u = \Delta_t u} = 0.$$  (2.2)
Only when expression (2.2) is developed one needs to apply a Leibniz rule and, hence, the results will depend from the definition of the corresponding discrete derivative. We propose a Leibniz rule having the form

\[ \Delta_x (f(x)g(x)) = f(x)\Delta_x g(x) + D_x(f(x))g(x), \]  

(2.3)

where \( D_x(f(x)) = [\Delta_x, f(x)] \) is a function of \( x, T_x \) and \( \sigma_x \) (similarly for \( D_t(f(t)) \)).

Using the general rule (2.3) for an arbitrary discrete derivative we obtain from (2.2), equating to zero the coefficients of \( \Delta_x u, \Delta_t u, \Delta_x u \) and \( u \), respectively, the following set of determining equations

\[ D_x(\tau) = 0, \]
\[ D_t(\tau) - 2D_x(\xi) = 0, \]
\[ D_t(\xi) - D_{xx}(\xi) - 2D_x(f) = 0, \]
\[ D_t(f) - D_{xx}(f) = 0 \]

(2.4)

where \( D_{xx}(f) = D_x(D_x(f)) \). Next, starting from (2.4) we will study separately the cases for \( \Delta^+ \) and \( \Delta^- \).

2.1 Symmetries for right (left) discrete derivatives

Choosing as in Ref. [5, 6] the derivative \( \Delta^+ \) and, consequently, the Leibniz rule

\[ \Delta^+(fg) = f\Delta^+g + \Delta^+(f)Tg, \]

we get from (2.4),

\[ \Delta^+_x \tau = 0, \]
\[ (\Delta^+_x \tau)T_t - 2(\Delta^+_x \xi)T_x = 0, \]
\[ (\Delta^+_x \xi)T_t - (\Delta^+_x \xi)T^2_x - 2(\Delta^+_x f)T_x = 0, \]
\[ (\Delta^+_x f)T_t - (\Delta^+_x f)T^2_x = 0. \]

(2.5)

The solution of (2.5) gives

\[ \tau = t^{(2)} \tau_2 + t \tau_1 + \tau_0, \]
\[ \xi = \frac{1}{2} x(\tau_1 + 2t\tau_2)T_tT_x^{-1} + t\xi_1 + \xi_0, \]
\[ f = \frac{1}{4} x^{(2)} \tau_2 T^2_x T^{n-2} + \frac{1}{2} t \tau_2 T_t + \frac{1}{2} x\xi_1 T_t T_x^{-1} + \gamma, \]

(2.6)

where \( \tau_0, \ \tau_1, \ \tau_2, \ \xi_0, \ \xi_1 \) and \( \gamma \) are arbitrary functions of \( T_x, T_t \), and the spacings \( \sigma_x \) and \( \sigma_t \). The notation \( x^{(n)}, t^{(n)} \) is for Pochhammer symbols; for instance

\[ x^{(n)} = x(x - \sigma_x) \ldots (x - (n-1)\sigma_x). \]
By a suitable choice of the functions $\tau_i$, $\xi_i$, and $\gamma$ we get the following symmetries \[\begin{align*}
P_0 &= (\Delta_t u) \partial_u, \\
P_1 &= (\Delta_x u) \partial_u, \\
W &= u \partial_u, \\
B &= (2tT_t^{-1} \Delta_x u + xT_x^{-1} u) \partial_u, \\
D &= (2tT_t^{-1} \Delta_t u + xT_x^{-1} \Delta_x u + \frac{1}{2} u) \partial_u, \\
K &= (t^2 T_t^{-2} \Delta_t u - \sigma_t T_t^{-2} \Delta_t u + txT_t^{-1} T_x^{-1} \Delta_x u \\
&\quad + \frac{1}{4} x^2 T_x^{-2} u - \frac{1}{4} \sigma_x xT_x^{-2} u + \frac{1}{2} t T_t^{-1} u) \partial_u.
\end{align*}\] Let us note that the above discrete symmetries have a well defined limit when $\sigma_x, \sigma_t \to 0$, which leads to the symmetries of the continuous heat equation. Also, it can be checked that with this choice, the symmetries (2.7) close a 6-dimensional Lie algebra isomorphic to the symmetry algebra of the continuous heat equation valid for any value of $\sigma_x, \sigma_t$.

A second choice for the discrete derivative is $\Delta^-$. The Leibniz rule becomes
\[
\Delta^-(fg) = f\Delta^- g + \Delta^-(f) T^{-1} g.
\]
It gives the same results (2.6) and (2.7) provided we make the substitution $T \to T^{-1}$.

### 2.2 Symmetries for symmetric discrete derivatives

Next, let us consider the case of the symmetric derivative $\Delta^s$ (1.8). The commutator
\[
[\Delta^s_x, x] = \frac{T_x + T_x^{-1}}{2}
\]
can always rewritten by introducing a function $\beta^s_x = \beta^s(T_x) = 2(T_x + T_x^{-1})^{-1}$ as
\[
[\Delta_x, x \beta_x] = 1.
\]
This fact will help us in the computation of general commutators, so
\[
[\Delta^s_x, f(x) \beta^s_x] = (\Delta^s_x f(x)) T_x \beta_x + \left(\frac{T_x^{-1} f(x) - f(x)}{\beta^s_x} \right) \beta^s_x \Delta^s_x.
\]
From relation (2.9) we get the explicit Leibniz rule
\[
\Delta^s_x (f(x) g(x)) = f(x) \Delta^s_x g(x) + \frac{1}{\sigma_x} \left((T_x^{-1} - 1) f(x) \right) \left(T_x - (\beta^s_x)^{-1}\right) + (\Delta^s_x(f(x)) T_x) g(x).
\]
This formula allows us to write explicitly the determining equations (2.4). Their solution is given by \[\begin{align*}
\tau^s &= t^2 \tau_2 + t \tau_1 + \tau_0, \\
\xi^s &= \frac{1}{2} x \left(2t \tau_2 + \tau_1 + \sigma_t T_t^{-1} \beta^s_{\tau_2}\right) (\beta^s_{\tau_1})^{-1} \beta^s_x + t \xi_1 + \xi_0, \\
f^s &= \frac{1}{4} x^2 \tau_2 (\beta^s_{\tau_2})^2 (\beta^s_{\tau_1})^{-2} + \frac{1}{4} x \xi_1 \beta^s_x (\beta^s_{\tau_1})^{-1} + \frac{1}{4} t \sigma_x \tau_2 T_x^{-1} (\beta^s_{\tau_1})^3 (\beta^s_{\tau_0})^{-2} + \frac{1}{2} t \tau_2 (\beta^s_{\tau_0})^{-1} + f_0.
\end{align*}\]
where \( \tau_2, \tau_1, \tau_0, \xi_1, \xi_0 \) and \( f_0 \) are arbitrary functions of \( T_x, T_t, \sigma_x \) and \( \sigma_t \). Now, from these solutions and (2.1) we obtain, with a suitable choice of \( \tau_2, \tau_1, \tau_0, \xi_1, \xi_0 \) and \( f_0 \), the following symmetries

\[
\begin{align*}
P_0^s &= (\Delta^s_x u) \partial_u, \\
P_1^s &= (\Delta^s_x u) \partial_u, \\
W^s &= u \partial_u, \\
B^s &= (2t\xi_1^s \Delta^s_x u + x\beta^s_x u) \partial_u, \\
D^s &= (2t\beta^s_t \Delta^s_t u + x\beta^s_x \Delta^s_x u + \frac{1}{2}u) \partial_u, \\
K^s &= ((t^2(\beta^s_t)^2 - t\sigma_x^2(\beta^s_x)^3 \Delta^s_t u + tx\beta^s_t \beta^s_x \Delta^s_x u \\
&\quad - \frac{1}{4}x\sigma_x^2(\beta^s_x)^3 \Delta^s_x u + \frac{1}{4}x^2(\beta^s_x)^2 u + \frac{1}{2}t\beta^s_x u) \partial_u.
\end{align*}
\]

These symmetries close the same 6–dimensional Lie algebra generated by the operators (2.7), and have a well defined continuous limit.

### 2.3 Symmetries for \( q \)–derivatives

In the following we shall extend the preceding method to the case of \( q \)–derivatives and \( q \)–symmetries. We deal briefly with a \( q \)–discretized heat equation where the \( q \)–difference operator is defined by

\[
\Delta^q_x = \frac{1}{(q_x - 1)x}(T_x - 1),
\]

with the help of a \( q \)–shift operator

\[
T_x = e^{q_x x\partial_x}, \quad T_x f(x) = f(q_x x).
\]

In this case we have the commutator

\[
[\Delta^q_x, x] = T_x.
\]

If we look for a function \( \beta_x(T_x) \) satisfying

\[
[\Delta^q_x, \beta_x x] = 1,
\]

a formal solution is

\[
\beta_x(T_x) = (q_x - 1) x\partial_x \frac{T_x}{T_x - 1}.
\]

Thus, we can perform a change of basic operators

\[
\{x, T_x\} \rightarrow \{\bar{x}, \Delta^q_x\}, \quad \bar{x} = \beta_x(T_x) x,
\]

so that, formally, we can express any function \( f(x, T_x) \) as

\[
f(x, T_x) = \tilde{f}(\bar{x}, \Delta^q_x).
\]
In this way the determining equations (2.4) take the form
\[
\bar{\tau}_x = 0, \\
\bar{\tau}_t - 2\bar{\xi}_x = 0, \\
\bar{\xi}_t - \bar{\xi}_{\bar{x}\bar{x}} - 2\bar{f}_x = 0, \\
\bar{f}_t - \bar{f}_{\bar{x}\bar{x}} = 0.
\]

Therefore, we can give a set of solutions that have a similar appearance to the classical symmetries
\[
P^q_0 = (\Delta^q_t u)\partial_u, \\
P^q_1 = (\Delta^q_x u)\partial_u, \\
W^q = u\partial_u, \\
B^q = (2\beta_t t\Delta^q_x u + \beta_x x u)\partial_u, \\
D^q = (2\beta_t t\Delta^q_x u + \beta_x x\Delta^q_x u + \frac{1}{2} u)\partial_u, \\
K^q = (\gamma_t t^2\Delta^q_x u + \beta_x \beta_t tx\Delta^q_x u + \frac{1}{4}\gamma_x x^2 u + \frac{1}{2}\beta_t t u)\partial_u,
\]

where \(\gamma_x = \left[\frac{(q-1)x\partial_x}{T_{x-1}}\right] \left[\frac{(q-1)(-1+x\partial_x)}{q^xT_{x-1}}\right]\) (for \(\gamma_t\) replace \(x\) by \(t\)).

### 3 Conclusions

We remark that the key point in obtaining the explicit determining equations (2.4) and, consequently, the discrete symmetries is the use of the Leibniz rule defined in (2.3). Of course, this approach is not the only possibility; in fact, it must be checked whether it works correctly or not for each case under study. The choice of a commutator in order to define the Leibniz rule implicitly leads us to Lie symmetries, since the natural algebraic structure will be given also in terms of commutators.

Some of the above result deserves some comments. The symmetries associated to the symmetric derivatives (Section 2.2) include functions \(\beta^*_t (\beta^*_x)\) of \(T_t (T_x)\) that can only be understood as infinite series expansions. Therefore, not all the symmetries (2.10) have a local character, in the sense that they are not (finite) polynomials in the operators \(T_t^{\pm 1}, T_x^{\pm 1}\). Note that although these discrete symmetries give rise to the classical symmetries in the limit \(\sigma_x \to 0, \sigma_t \to 0\), one of them, \(K^q\), also includes surprisingly a term in \((\Delta_x)^2\), which vanishes in the continuous limit since it contains a factor \(\sigma^2_t\).

Something similar happens with the \(q\)-symmetries (Section 2.3): they have also a highly non-local character. The origin of this unpleasant feature is that the basic commutator \([\Delta^q_x, \beta_x x] = 1\) needs a non-local function \(\beta_x(T_x)\). If we want to investigate local symmetries it is necessary to use another commutator free of this problem. For instance, we could take as starting point the \(q\)-commutator: \([\Delta^q_x, xT_{x-1}^{-1}]_{q_x} = 1\), where
$[A, B]_{q_x} = AB - q_x^{-1}BA$. However, this treatment would lead us to a $q$–algebra which is out of our present scope.

Let us insist that the procedure here exposed can be straightforwardly applied to other discretizations such as the wave equation or even equations including a potential term as long as we keep inside the field of linear equations. Non-linear equations needs additional improvements in order to have reasonable determining equations.

**Acknowledgments**

This work has been partially supported by DGES of the Ministerio de Educaci´on y Cultura of Spain under Projects PB98-0360 and the Junta de Castilla y Le´on (Spain).

**References**

[1] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York 1991).

[2] D. Levi and P. Winternitz, Phys. Lett. A **152**, 335 (1991); J. Math. Phys. **34**, 3713 (1993).

[3] S.Maeda, Math. Japan **25**, 405 (1980); **26**, 85 (1981); IMA J. Appl, Math. **38**, 129 (1987).

[4] G.R.W. Quispel, H.W. Capel and R. Sahadevan, Phys. Lett. **170A**, 379 (1992).

[5] D. Levi, L. Vinet and P. Winternitz, J. Phys. A: Math. Gen **30**, 633 (1997).

[6] R. Floreanini, J. Negro, L.M. Nieto and L. Vinet, Lett. Math. Phys. **36**, 351 (1996).

[7] W.E. Milne, *Numerical Calculus* (Princeton Univ. Press, Princeton 1949).

[8] D. Levi, J. Negro and M.A. del Olmo, *Discrete derivatives and symmetries of difference equations*, nlin.SI/0010044.

[9] R. Floreanini and L. Vinet, Lett. Math. Phys. **32**, 37 (1994).

[10] J. Negro and L.M. Nieto, J. Phys. A **29**, 1107 (1996).