Abstract

Let $M_1$ and $M_2$ be $n$-dimensional connected orientable finite-volume hyperbolic manifolds with geodesic boundary, and let $\varphi : \pi_1(M_1) \rightarrow \pi_1(M_2)$ be a given group isomorphism. We study the problem whether there exists an isometry $\psi : M_1 \rightarrow M_2$ such that $\psi_* = \varphi$. We show that this is always the case if $n \geq 4$, while in the 3-dimensional case the existence of $\psi$ is proved under some (necessary) additional conditions on $\varphi$. Such conditions are trivially satisfied if $\partial M_1$ and $\partial M_2$ are both compact.

MSC (2000): 30F40 (primary), 57N16 (secondary).

Let $M_1$ and $M_2$ be connected orientable finite-volume hyperbolic $n$-manifolds with geodesic boundary. Suppose $n \geq 3$ and let $\varphi : \pi_1(M_1) \rightarrow \pi_1(M_2)$ be an isomorphism of abstract groups. We determine necessary and sufficient conditions for $\varphi$ to be induced by an isometry $\psi : M_1 \rightarrow M_2$. When this is the case, we say that $\varphi$ is geometric (see Section 1 for a more detailed definition). Mostow-Prasad’s rigidity theorem ensures geometricity of $\varphi$ whenever the boundary of $M_i$ is empty for $i = 1, 2$.

Building on a result of Floyd [1], we will extend Mostow-Prasad’s result to the non-empty boundary case, following slightly different strategies according to the dimension of the manifolds involved.

If $M_1$ and $M_2$ are 3-dimensional hyperbolic manifolds with non-empty geodesic boundary, applying Mostow-Prasad’s rigidity theorem to their doubles, i.e. to the manifolds obtained by mirroring $M_1$ and $M_2$ in their boundary, we will show that $\varphi$ is geometric provided it is induced by a homeomorphism, rather than an isometry. A result of Marden and Maskit [7] will then be applied to relate the existence of a homeomorphism inducing $\varphi$ to the behaviour of $\varphi$ with respect to the peripheral subgroups of $\pi_1(M_1)$ and $\pi_1(M_2)$ (see below for a definition).

*This research was partially supported by the University of Melbourne.
If \( \dim(M_1) = \dim(M_2) \geq 4 \), the existence of an isometry \( \psi : M_1 \to M_2 \) such that \( \psi_* = \varphi \) will be proved by a more direct argument using results from [12].

1 Preliminaries and statement

In this section we list some preliminary facts about the topology and geometry of orientable finite-volume hyperbolic \( n \)-manifolds with geodesic boundary and we state our main theorem and its corollaries. From now on we will always suppose \( n \geq 3 \). Moreover, all manifolds will be connected and orientable. We omit all proofs about the basic material addressing the reader to [2, 5, 6].

Before going into the real matter, we devote the first paragraph to give a formal definition of the notion of geometric isomorphism between fundamental groups of hyperbolic manifolds. To this aim we will need to spell out in detail some well-known elementary results in the theory of fundamental groups.

**Homomorphisms between fundamental groups** If \( \varphi, \varphi' : G \to H \) are group homomorphisms, we say that \( \varphi' \) is conjugated to \( \varphi \) if there exists \( h \in H \) such that \( \varphi'(g) = h\varphi(g)h^{-1} \) for every \( g \in G \). Let \( X \) be a manifold and \( x_0, x_1 \) be points in \( X \). Then there exists an isomorphism \( \pi_1(X, x_0) \cong \pi_1(X, x_1) \) which is canonical up to conjugacy. It follows that an abstract group \( \pi_1(X) \) is well-defined and for any \( x_0 \in X \) there exists a preferred conjugacy class of isomorphisms between \( \pi_1(X) \) and \( \pi_1(X, x_0) \).

If \( f : X \to Y \) is a continuous map between manifolds, then \( f \) determines a well-defined conjugacy class of homomorphisms \( f_* \in \text{Hom}(\pi_1(X), \pi_1(Y))/\pi_1(Y) \). If a homomorphism \( \varphi : \pi_1(X) \to \pi_1(Y) \) is given, we say that \( \varphi \) is induced by \( f \) if \( \varphi \) belongs to \( f_* \); if so, with an abuse we will write \( \varphi = f_* \), rather than \( [\varphi] = f_* \).

**Definition 1.1.** Let \( M_1 \) and \( M_2 \) by hyperbolic manifolds with geodesic boundary and \( \varphi : \pi_1(M_1) \to \pi_1(M_2) \) be a group isomorphism. Then \( \varphi \) is geometric if \( \varphi = \psi_* \) for some isometry \( \psi : M_1 \to M_2 \).

**Natural compactification of hyperbolic manifolds** Let \( N \) be a complete finite-volume hyperbolic \( n \)-manifold with (possibly empty) geodesic boundary (from now on we will summarize all this information saying just that \( N \) is hyperbolic). Then \( \partial N \), endowed with the Riemannian metric it inherits from \( N \), is a hyperbolic \( (n-1) \)-manifold without boundary (completeness of \( \partial N \) is obvious, and the volume of \( \partial N \) is proved to be finite in [5]). It is well-known [5] that \( N \) consists of a compact portion together with some cusps based on Euclidean \( (n-1) \)-manifolds. More precisely, the \( \varepsilon \)-thin part of \( N \) (see [10]) consists of cusps of the form \( T \times [0, \infty) \),...
where $T$ is a compact Euclidean $(n-1)$-manifold with (possibly empty) geodesic boundary such that $(T \times [0, \infty)) \cap \partial N = \partial T \times [0, \infty)$. A cusp based on a closed Euclidean $(n-1)$-manifold is therefore disjoint from $\partial N$ and is called *internal*, while a cusp based on a Euclidean $(n-1)$-manifold with non-empty boundary intersects $\partial N$ in one or two internal cusps of $\partial N$, and is called a *boundary cusp*. This description of the ends of $N$ easily implies that $N$ admits a natural compactification $\overline{N}$ obtained by adding a closed Euclidean $(n-1)$-manifold for each internal cusp and a compact Euclidean $(n-1)$-manifold with non-empty geodesic boundary for each boundary cusp. When $n = 3$, $\overline{N}$ is obtained by adding to $N$ some tori and some closed annuli. In this case we denote by $A_N$ the family of added closed annuli, and we observe that no annulus in $A_N$ lies on a torus in $\partial \overline{N}$. Note also that $A_N = \emptyset$ if $\partial N$ is compact.

A loop $\gamma \in \pi_1(N)$ will be called an *annular cusp loop* if it is freely-homotopic to a loop in some annulus of $A_N$.

**Main result** We are now ready to state our main result.

**Theorem 1.2.** Let $N_1$ and $N_2$ be hyperbolic $n$-manifolds, and let $\varphi : \pi_1(N_1) \to \pi_1(N_2)$ be a group isomorphism. If $n = 3$, suppose also that the following condition holds:

- $\varphi(\gamma)$ is an annular cusp loop in $\pi_1(N_2)$ if and only if $\gamma$ is an annular cusp loop in $\pi_1(N_1)$.

Then $\varphi$ is geometric.

Theorem 1.2 readily implies the following corollaries:

**Corollary 1.3.** Let $N_1$ and $N_2$ be hyperbolic 3-manifolds with compact geodesic boundary and let $\varphi : \pi_1(N_1) \to \pi_1(N_2)$ be an isomorphism. Then $\varphi$ is geometric.

**Corollary 1.4.** Let $N$ be a hyperbolic $n$-manifold, let $\text{Iso}(N)$ be the group of isometries of $N$ and let $\text{Out}(\pi_1(N)) := \text{Aut}(\pi_1(N))/\pi_1(N)$ be the group of the outer isomorphisms of $\pi_1(N)$. If $n = 3$, suppose also that the boundary of $N$ is compact. Then there is a natural isomorphism $\text{Iso}(N) \cong \text{Out}(\pi_1(N))$.

**Proof:** Let $h : \text{Iso}(N) \to \text{Out}(\pi_1(N))$ be the map defined by $h(\psi) = \psi_*$. Then $h$ is a well-defined homomorphism. Injectivity of $h$ is a well-known fact, while surjectivity of $h$ is an immediate consequence of Theorem 1.2 and Corollary 1.3. \qed

3
Universal covering and action at the infinity  Let $N$ be a $n$-dimensional hyperbolic manifold and let $\pi : \tilde{N} \to N$ be the universal covering of $N$. By developing $\tilde{N}$ in $\mathbb{H}^n$ we can identify $\tilde{N}$ with a convex polyhedron of $\mathbb{H}^n$ bounded by a countable number of disjoint geodesic hyperplanes $S_i$, $i \in \mathbb{N}$. For any $i \in \mathbb{N}$ let $S_i^+$ denote the closed half-space of $\mathbb{H}^n$ bounded by $S_i$ and containing $\tilde{N}$, let $S_i^-$ be the closed half-space of $\mathbb{H}^n$ opposite to $S_i^+$ and let $\Delta_i$ be the internal part of the closure at infinity of $S_i^-$. Of course we have $\tilde{N} = \bigcap_{i \in \mathbb{N}} S_i^+$, so denoting by $\tilde{N}_\infty$ the closure at infinity of $\tilde{N}$ we obtain $\tilde{N}_\infty = \partial \mathbb{H}^n \setminus \bigcup_{i \in \mathbb{N}} \Delta_i$.

The group of the automorphisms of the covering $\pi : \tilde{N} \to N$ can be identified in a natural way with a discrete torsion-free subgroup $\Gamma$ of $\text{Iso}^+(\mathbb{H}^n)$ such that $\gamma(\tilde{N}) = \tilde{N}$ for any $\gamma \in \Gamma$ and $N \cong \tilde{N}/\Gamma$. Also recall that there exists an isomorphism $\pi_1(N) \cong \Gamma$, which is canonical up to conjugacy. Let $\Lambda(\Gamma)$ denote the limit set of $\Gamma$ and let $\Omega(\Gamma) = \partial \mathbb{H}^n \setminus \Lambda(\Gamma)$. Kojima has shown in [5] that $\Lambda(\Gamma) = \tilde{N}_\infty$, so the round balls $\Delta_i$, $i \in \mathbb{N}$ previously defined actually are the connected components of $\Omega(\Gamma)$. A subgroup of $\Gamma$ is called peripheral if it is equal to the stabilizer of one of the $\Delta_i$’s.

Since $\tilde{N}_\infty = \Lambda(\Gamma)$, we have that $\tilde{N}$ is the intersection of $\mathbb{H}^n$ with the convex hull of $\Lambda(\Gamma)$, so $\tilde{N}$ is the convex core (see [10]) of the hyperbolic manifold $\mathbb{H}^n/\Gamma$. This implies that $N$ uniquely determines $\Gamma$ up to conjugation by elements in $\text{Iso}^+(\mathbb{H}^n)$, that $\Gamma$ is geometrically finite and that $N$ is homeomorphic to the manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.

Parabolic subgroups of $\Gamma$  Let $\Gamma'$ be a subgroup of $\Gamma$. We say that $\Gamma'$ is maximal parabolic if it is parabolic (i.e. all its non-trivial elements are parabolic) and it is maximal with respect to inclusion among parabolic subgroups of $\Gamma$. If $\Gamma'$ is a maximal parabolic subgroup of $\Gamma$, then there exists a point $q \in \partial \mathbb{H}^n$ such that $\Gamma'$ equals the stabilizer of $q$ in $\Gamma$. Then $\Gamma'$ can be naturally identified with a discrete subgroup of $\text{Iso}^+(\mathbb{E}^{n-1})$, so by Bieberbach’s Theorem [8] $\Gamma'$ contains an Abelian subgroup $H$ of finite index. If $k$ is the rank of $H$, we say that $\Gamma'$ is a rank-$k$ parabolic subgroup of $\Gamma$. Now it is shown in [5] that if $i \neq j$, then $\overline{\Delta_i \cap \Delta_j}$ is either empty or consists of one point $p$ whose stabilizer is a rank-$(n-2)$ parabolic subgroup of $\Gamma$. Moreover, any maximal rank-$(n-2)$ parabolic subgroup of $\Gamma$ is the stabilizer of a point $p$ which lies on the boundary of two different $\Delta_i$’s. On the other hand, the intersection of $\tilde{N}$ with a horoball centered at a point with rank-$(n-2)$ parabolic stabilizer projects onto a boundary cusp of $N$, and any boundary cusp of $N$ lifts to the intersection of $\tilde{N}$ with a horoball centered at a point with rank-$(n-2)$ parabolic stabilizer. It follows that there is a natural correspondence between the boundary cusps of $N$ and the conjugacy classes of rank-$(n-2)$ maximal parabolic subgroups of $\Gamma$.

We shall see that rank-1 maximal parabolic subgroups of $\Gamma$ play a special role in the proof of our main theorem. Since any parabolic subgroup of $\Gamma$ corresponds to a cusp of $N$, we have that if $n \geq 4$ then $\Gamma$ does not contain rank-1 maximal parabolic
subgroups, while when \( n = 3 \) the elements of rank-1 maximal parabolic subgroups of \( \Gamma \) correspond to the annular cusp loops previously defined. For later purpose we point out the following:

**Remark 1.5.** For any \( k \in \mathbb{N} \) let \( H_k \) be the stabilizer of \( \Delta_k \) in \( \Gamma \). If \( i \neq j \), then either \( \overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset \) and \( H_i \cap H_j = \emptyset \), or \( \overline{\Delta_i} \cap \overline{\Delta_j} = \{ p \} \) and \( H_i \cap H_j \) is the rank-(\( n - 2 \)) parabolic stabilizer of \( p \) in \( \Gamma \).

## 2 Some preliminary lemmas

The following result is a slight generalization of Lemma 5.1 in [4], which is due to J.P. Otal. Notations are kept from the preceding section.

**Lemma 2.1.** Let \( j : S^{n-2} \to \Lambda(\Gamma) \) be a topological embedding. Then \( \Lambda(\Gamma) \setminus j(S^{n-2}) \) is path connected if and only if \( j(S^{n-2}) = \partial \Delta_l \) for some \( l \in \mathbb{N} \).

**Proof:** Suppose that \( j(S^{n-2}) = \partial \Delta_0 \). Using the upper half-space model of hyperbolic space, we identify \( \partial \mathbb{H}^n \) with \( (\mathbb{R}^{n-1} \times \{ 0 \}) \cup \{ \infty \} \) in such a way that \( \Delta_0 \) corresponds to \( \mathbb{H} = \{(x,0) \in \mathbb{R}^{n-1} \times \{ 0 \} : \ x_{n-1} > 0 \} \). Now let \( p_1, p_2 \in \Lambda(\Gamma) \setminus \partial \Delta_0 \) and let \( \alpha : [0,1] \to (\mathbb{R}^{n-1} \times \{ 0 \}) \setminus \mathbb{H} \) be the straight Euclidean segment which joins \( p_1 \) to \( p_2 \). If \( \{a_i, b_i\} \subset [0,1], \ i \geq 1 \) is the set of the connected components of \( \alpha^{-1}(\Omega(\Gamma)) \), then, up to reordering the \( \Delta_i \)'s with \( i \geq 1 \), we have \( \alpha([a_i, b_i]) \subset \overline{\Delta_i} \). Let \( r_i \) be the Euclidean radius of \( \Delta_i \). Since \( \partial \Delta_i \) can touch \( \partial \Delta_0 \) at most in one point, for any \( i \geq 1 \) there exists a path \( \beta_i : [a_i, b_i] \to \partial \Delta_i \) with \( \beta_i(a_i) = \alpha(a_i) \), \( \beta_i(b_i) = \alpha(b_i) \) and \( \text{length}(\beta_i) \leq 2\pi r_i \). Now let \( \alpha_i \) be the path inductively defined as follows: \( \alpha_0 = \alpha, \alpha_{i+1}(t) = \beta_{i+1}(t) \) if \( t \in [a_{i+1}, b_{i+1}] \) and \( \alpha_{i+1}(t) = \alpha_i(t) \) if \( t \in [0, a_{i+1}] \cup [b_{i+1}, 1] \). The path \( \alpha_i \) is obviously continuous for any \( i \in \mathbb{N} \). Moreover, since \( \lim_{i \to \infty} r_i = 0 \), the sequence of paths \( \{\alpha_i, i \in \mathbb{N}\} \) uniformly converges to the desired continuous path \( \alpha_\infty : [0,1] \to \Lambda(\Gamma) \setminus \partial \Delta_0 \).

Suppose now that \( \Lambda(\Gamma) \setminus j(S^{n-2}) \) is path connected. The Jordan-Brouwer separation theorem implies that \( \partial \mathbb{H}^n \setminus j(S^{n-2}) = A_1 \cup A_2 \), where the \( A_i \)'s are disjoint open subset of \( \partial \mathbb{H}^n \) with \( \partial A_i = j(S^{n-2}) \) for \( i = 1, 2 \) (since we are not assuming that \( j \) is tame, at this stage we are not allowed to claim that the \( A_i \)'s are topological balls). Our hypothesis now forces \( A_k \cap \Lambda(\Gamma) = \emptyset \) for some \( k \in \{1, 2\} \), so \( A_k \subset \Delta_l \) for some \( l \in \mathbb{N} \). Moreover, since \( \partial A_k = j(S^{n-2}) \subset \Lambda(\Gamma) \), it is easily seen that \( j(S^{n-2}) = \partial \Delta_l \), and we are done. \( \square \)

Form now on let \( N_1 \) and \( N_2 \) be hyperbolic \( n \)-manifolds, let \( \pi_i : \mathbb{H}^n \supset \widetilde{N}_i \to N_i \) be the universal covering of \( N_i \) and let \( \Gamma_i \) be a discrete subgroup of \( \text{Iso}^+(\mathbb{H}^n) \) such that \( N_i \cong \widetilde{N}_i/\Gamma_i \). Let also \( \varphi : \Gamma_1 \to \Gamma_2 \) be a group isomorphism satisfying the condition of Theorem 1.3.2. If \( f : N_1 \to N_2 \) is a continuous map, it is easily seen that
\( \varphi \) is induced by \( f \) if and only if \( f \) admits a continuous lift \( \tilde{f} : \tilde{N}_1 \to \tilde{N}_2 \) such that \( f \circ \gamma = \varphi(\gamma) \circ \tilde{f} \) for every \( \gamma \in \Gamma_1 \).

**Lemma 2.2.** There exists a homeomorphism \( \hat{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2) \) such that \( \hat{\varphi}(\gamma(x)) = \varphi(\gamma)(\hat{\varphi}(x)) \) for any \( x \in \Lambda(\Gamma_1), \gamma \in \Gamma_1 \).

**Proof:** For any group \( G \), let us denote by \( \overline{G} \) the completion of \( G \) (see [1] for a definition). Recall that \( G \) acts in a natural way on \( \overline{G} \) as a group of homeomorphisms. It is proved in [1] that any group isomorphism \( \psi : G_1 \to G_2 \) induces a homeomorphism \( \psi : \overline{G}_1 \to \overline{G}_2 \) such that \( \psi(g(x)) = \psi(g)(\psi(x)) \). Moreover, if \( G \) is a geometrically finite subgroup of \( \text{Iso}^+(\mathbb{H}^n) \) then there exists a natural continuous surjection \( p_G : \overline{G} \to \Lambda(G) \) which is 2-to-1 onto points with rank-1 parabolic stabilizer, and injective everywhere else (this was shown in [1] under the assumption \( n = 3 \), but as it was observed in [13] the proof in [1] actually works in any dimension).

Now \( \varphi \) induces by hypothesis a bijective correspondence between rank-1 maximal parabolic subgroups of \( \Gamma_1 \) and rank-1 maximal parabolic subgroups of \( \Gamma_2 \). Using this fact it is easily seen that there exists a unique bijective map \( \hat{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2) \) such that \( \hat{\varphi}(\gamma(x)) = \varphi(\gamma)(\hat{\varphi}(x)) \). Since \( \overline{\Gamma_1} \) and \( \Lambda(\Gamma_i) \) are Haussdorff compact spaces for \( i = 1, 2 \), the map \( \hat{\varphi} \) is a homeomorphism, and we are done. \( \square \)

**Corollary 2.3.** \( \partial N_1 = \emptyset \) if and only if \( \partial N_2 = \emptyset \).

**Proof:** Since \( \Gamma_i \) is geometrically finite, the boundary of \( N_i \) is empty if and only if \( \Lambda(\Gamma_i) \) is homeomorphic to \( S^{n-1} \). Lemma 2.2 provides a homeomorphism between \( \Lambda(\Gamma_1) \) and \( \Lambda(\Gamma_2) \), and the conclusion follows at once. \( \square \)

If \( \partial N_1 = \partial N_2 = \emptyset \), Mostow-Prasad’s rigidity theorem applies ensuring geometricity of \( \varphi \). Then from now on we shall assume that both \( N_1 \) and \( N_2 \) have non-empty boundary.

**Lemma 2.4.** The isomorphism \( \varphi \) satisfies the following conditions:

1. \( \varphi(H) \) is a peripheral subgroup of \( \Gamma_2 \) if and only if \( H \) is a peripheral subgroup of \( \Gamma_1 \); if so we also have \( \hat{\varphi} \) is a peripheral subgroup of \( \Gamma_2 \);

2. \( \varphi(\gamma) \) is a parabolic element of \( \Gamma_2 \) if and only if \( \gamma \) is a parabolic element of \( \Gamma_1 \).

**Proof:** Let \( \hat{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2) \) be the homeomorphism constructed in Lemma 2.2 and let \( H = \text{stab}(\Delta) \) be a peripheral subgroup of \( \Gamma_1 \), where \( \Delta \) is a component of \( \Omega(\Gamma_1) \). By Lemma 2.1, \( \Lambda(\Gamma_1) \setminus \Lambda(H) = \Lambda(\Gamma_1) \setminus \partial \Delta \) is path connected, so \( \Lambda(\Gamma_2) \setminus \hat{\varphi}(\Lambda(H)) = \hat{\varphi}(\Lambda(\Gamma_1) \setminus \Lambda(H)) \) is also path connected, and \( \hat{\varphi}(\Lambda(H)) \) is equal to \( \Lambda(K) \) for some peripheral subgroup \( K \) of \( \Gamma_2 \). Let \( K = \text{stab}(\Delta') \), where \( \Delta' \) is a component of \( \Omega(\Gamma_2) \). Now let \( h \) be a loxodromic element of \( H \) with fixed points \( p_1, p_2 \) in \( \Lambda(H) \). Since \( \hat{\varphi} \)
is $\varphi$-equivariant, we have that $\varphi(h)$ is a loxodromic element of $\Gamma_2$ with fixed points $\hat{\varphi}(p_1), \hat{\varphi}(p_2)$ which lie in $\Lambda(K)$. Since the boundaries of two different components of $\Omega(\Gamma_2)$ can intersect at most in one point, it easily follows that $\varphi(h) \in \text{stab}(\Delta^i) = K$. Now $H$ is generated by its loxodromic elements, so $\varphi(H)$ is contained in $K$. On the other hand, the same argument applied to $\varphi^{-1}$ shows that $\varphi^{-1}(K)$ is contained in a peripheral subgroup of $\Gamma_1$, say $H'$, with $H \subset H'$. Now Remark 1.3 implies that $H = H'$, so $\varphi(H) = K$ and point (1) is proved.

To prove point (2), we observe that the $\varphi$-equivariance of $\hat{\varphi}$ implies that for any $\gamma \in \Gamma_1$ the fixed points of $\varphi(\gamma)$ are exactly the images under $\hat{\varphi}$ of the fixed points of $\gamma$. This implies that the number of fixed points of $\varphi(\gamma)$ on $\Lambda(\Gamma_2)$ equals the number of fixed points of $\gamma$ on $\Lambda(\Gamma_1)$, so $\varphi(\gamma)$ is parabolic if and only if $\gamma$ is. □

3 The $n$-dimensional case, $n \geq 4$

The next proposition easily implies Theorem 1.2 under the assumption that the dimension of $N_1$ and $N_2$ is at least 4.

**Proposition 3.1.** Let $n \geq 4$. Then there exists a conformal map $f : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ such that $f \circ \gamma = \varphi(\gamma) \circ f$ for any $\gamma \in \Gamma_1$.

**Proof:** Let $\Delta^1$ be a connected component of $\Omega(\Gamma_1)$, and $H_1$ be the stabilizer of $\Delta^1$ in $\Gamma_1$. By Lemma 2.3 the group $H_2 = \varphi(H_1)$ is a peripheral subgroup of $\Gamma_2$. Let now $\Delta^2$ be the $H_2$-invariant component of $\Omega(\Gamma_2)$, i.e. the unique component of $\Omega(\Gamma_2)$ whose boundary is equal to $\Lambda(H_2)$. By Lemma 2.3 (1), the homeomorphism constructed in Lemma 2.2 restricts to a homeomorphism $\hat{\varphi}|_{\partial \Delta^1} : \partial \Delta^1 \to \partial \Delta^2$ such that $\hat{\varphi}|_{\partial \Delta^1} \circ \gamma = \varphi(\gamma) \circ \hat{\varphi}|_{\partial \Delta^1}$ for every $\gamma \in H_1$. Let now $S^1, S^2$ be the hyperplanes of $\mathbb{H}^n$ bounded respectively by $\partial \Delta^1$ and $\partial \Delta^2$. Then $S^k/H_k$ is isometric to a component of the geodesic boundary of $N_k$ for $k = 1, 2$, so it is a finite-volume complete hyperbolic $(n-1)$-manifold without boundary. Since $n \geq 4$, Mostow-Prasad’s rigidity theorem applies providing an isometry $g : S^1 \to S^2$ whose continuous extension to $\partial \Delta^1$ is equal to $\hat{\varphi}|_{\partial \Delta^1}$. Let now $p_k, k = 1, 2$ be the orthogonal projection of $S^k$ onto $\Delta^k$, i.e. the function which maps a point $q \in S^k$ to the point $p \in \Delta^k$ such that the geodesic ray $(q, p)$ is orthogonal to $S^k$. The map $g' : \Delta^1 \to \Delta^2$ defined by $g' = p_2 \circ g \circ p_1^{-1}$ is conformal, and its continuous extension to $\partial \Delta^1$ is equal to $\hat{\varphi}|_{\partial \Delta^1}$.

By repeating the construction described above for each component of $\Omega(\Gamma_1)$, we can construct a conformal map $t : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$. This map is a homeomorphism, since it admits a continuous inverse which can be constructed from the isomorphism $\varphi^{-1} : \Gamma_2 \to \Gamma_1$. We want now to show that for any $\gamma \in \Gamma_1$, we have $t \circ \gamma = \varphi(\gamma) \circ t$. Let $\Delta$ be a component of $\Omega(\Gamma_1)$. By the very definition of $t$ it follows that $t(\Delta)$ is
the unique component of $\Omega(\Gamma_2)$ which is bounded by $\tilde{\varphi}(\partial \Delta)$, so

$$\partial(\varphi(\gamma)(t(\Delta))) = \varphi(\gamma)(\partial(t(\Delta))) = \varphi(\gamma)(\tilde{\varphi}(\partial \Delta)) = \tilde{\varphi}(\partial(\gamma(\Delta))) = \partial(t(\gamma(\Delta))).$$

This shows that both $t \circ \gamma$ and $\varphi(\gamma) \circ t$ map $\Delta$ onto the same component $\Delta'$ of $\Omega(\Gamma_2)$. Moreover, the continuous extensions of $t \circ \gamma$ and $\varphi(\gamma) \circ t$ to $\partial \Delta$ are respectively equal to $\tilde{\varphi} \circ \gamma$ and $\varphi(\gamma) \circ \tilde{\varphi}$, which are in turn equal to each other because of the $\varphi$-equivariance of $\tilde{\varphi}$. Being conformal, the maps $t \circ \gamma$ and $\varphi(\gamma) \circ t$ must then be equal on $\Delta$, and this proves the required $\varphi$-equivariance of $t$.

Now let $f : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ be defined by $f(x) = t(x)$ if $x \in \Omega(\Gamma_1)$, and $f(x) = \tilde{\varphi}(x)$ if $x \in \Lambda(\Gamma_1)$. To conclude the proof we only have to observe that since $f$ is $\varphi$-equivariant and conformal on $\Omega(\Gamma_1)$, a result of Tukia [12] ensures that $f$ is a coformal map.

We can now conclude the proof of Theorem 1.2 under the assumption that the dimension of $N_1$ and $N_2$ is greater than 3. Let $\tilde{\psi}$ be the unique isometry of $\mathbb{H}^n$ whose continuous extension to $\partial \mathbb{H}^n$ is equal to $f$. The $\varphi$-equivariance of $f$ readily implies that $\tilde{\psi}(\gamma(x)) = \varphi(\gamma)(\tilde{\psi}(x))$ for every $x \in \mathbb{H}^n, \gamma \in \Gamma_1$. If we identify $N_i$ with the convex core of the manifold $\mathbb{H}^n/\Gamma_i$ for $i = 1, 2$, then $\tilde{\psi}$ induces an isometry $\psi : N_1 \to N_2$ with $\psi_* = \varphi$.

4 The 3-dimensional case

As briefly explained in the introduction, the 3-dimensional case needs a different approach.

**Lemma 4.1.** There exists a homeomorphism $g : N_1 \to N_2$ such that $\varphi = g_*$.  

**Proof:** Let $M_i = (\mathbb{H}^3 \cup \Omega(\Gamma_i))/\Gamma_i$ for $i = 1, 2$. By Lemma 2.4 and Remark 1.5 we can apply Theorem 1 of [7] to $\varphi$, obtaining a homeomorphism $g' : M_1 \to M_2$ inducing $\varphi$ (note that our definition of geometric is stronger than the one in [7]). Now $N_i$ is canonically embedded in $M_i$ in such a way that $M_i \setminus N_i$ is an open collar of $\partial M_i$. This implies that $g'$ can be isotoped to a $g'' : M_1 \to M_2$ such that $g''(N_1) = N_2$ and $g = g''|_{N_1}$ is the required homeomorphism. 

**Remark 4.2.** If $N_1$ and $N_2$ have compact geodesic boundary, then Lemma 4.1 can also be deduced by the following result of Johannson [3, 9]: Any homotopy equivalence between compact orientable boundary-irreducible anannular Haken 3-manifolds can be homotoped to a homeomorphism.
We can now conclude the proof of Theorem 1.2 in the case when $N_1$ and $N_2$ are 3-dimensional manifolds. Let $g : N_1 \to N_2$ be the homeomorphism constructed in Lemma 4.1, let $D(N_i)$ be the double of $N_i$ for $i = 1, 2$ and let $D(g) : D(N_1) \to D(N_2)$ be the homeomorphism obtained by doubling $g$. By Mostow-Prasad’s rigidity theorem, $D(g)$ is homotopic to an isometry $h : D(N_1) \to D(N_2)$. Since $\partial N_2 = g(\partial N_1)$ and $h(\partial N_1)$ are embedded totally geodesic homotopic surfaces in $N_2$, we get that $h(\partial N_1) = \partial N_2$, so $h(N_1) = N_2$. Moreover, $h_* = g_*$ on $\pi_1(D(N_1))$, and the inclusion of $\pi_1(N_i)$ in $\pi_1(D(N_i))$ is injective for $i = 1, 2$, so $h_* = g_* = \varphi$ on $\Gamma_1$. In conclusion, we have shown that $h|_{N_1} : N_1 \to N_2$ is an isometry inducing $\varphi$, so $\varphi$ is geometric.

**Counterexamples in the non-compact boundary case** We now show that the conclusions of Corollaries 1.3 and 1.4 are no longer true if we consider hyperbolic 3-manifolds with non-compact geodesic boundary. More precisely, we will prove the following:

**Proposition 4.3.** There exist hyperbolic 3-manifolds with non-compact geodesic boundary $N_1, N_2$ such that:

1. $\pi_1(N_1) \cong \pi_1(N_2)$ but $N_1$ is not homeomorphic to $N_2$;

2. $\text{Out}(\pi_1(N_i)) \not\cong \text{Iso}(N_i)$ for $i = 1, 2$.

**Proof:** We will give an explicit construction of $N_1$ and $N_2$. Let $O \subset \mathbb{H}^3$ be the regular ideal octahedron and let $v_1, \ldots, v_6$ be the vertices of $O$ as shown in Fig. 4. We denote by $F_{ijk}$ the face of $O$ with vertices $v_i, v_j, v_k$. Let $g : F_{134} \to F_{156}$ be the unique orientation-reversing isometry such that $g(v_1) = v_6$, and $h_1, h_2 : F_{542} \to F_{362}$ be the unique orientation-reversing isometries such that $h_1(v_5) = v_6$, $h_2(v_5) = v_3$. We now define $N_1$ to be the manifold obtained by gluing $O$ along $g$ and $f_1$, and $N_2$ to be the manifold obtained by gluing $O$ along $g$ and $f_2$. Since all the dihedral angles of $O$ are right, it is easily seen that the metric on $O$ induces a complete finite-volume hyperbolic structure on the $N_i$’s such that the shadowed faces in Fig. 4 are glued along their egdes to give a non-compact totally geodesic boundary.

Now the natural compactification of $N_i$ is homeomorphic to the genus-2 handlebody for $i = 1, 2$, so $\pi_1(N_1) \cong \pi_1(N_2) \cong \mathbb{Z} \ast \mathbb{Z}$. Moreover, the boundary of $N_1$ is homeomorphic to the 2-punctured torus, while the boundary of $N_2$ is homeomorphic to the 4-punctured sphere, so $N_1$ is not homeomorphic to $N_2$. This proves point (1).

In order to prove point (2), we only have to observe that the group of the outer isomorphisms of $\mathbb{Z} \ast \mathbb{Z}$ is of infinite order, while the group of isometries of any complete finite-volume hyperbolic $n$-manifold with geodesic boundary has a finite number of elements. \qed
Figure 1: The manifolds $N_1, N_2$ and $N_3$ are obtained by gluing in pairs the non-shadowed faces of the regular ideal octahedron along suitable isometries.

**Example 4.4.** Let $N_3$ be the hyperbolic manifold with non-compact geodesic boundary obtained by gluing the faces of $O$ along $h_2$ and $g'$, where $g': F_{134} \to F_{156}$ is the unique orientation-reversing isometry such that $g'(v_1) = v_5$. As before, the natural compactification of $N_3$ is the genus-2 handlebody, so $\pi(N_3) \cong \pi(N_2) \cong \mathbb{Z} \ast \mathbb{Z}$. Moreover, with some effort one could show that $\partial N_2$ is homeomorphic but not isometric to $\partial N_3$, and $N_2$ and $N_3$ are not homeomorphic to each other.

**A more general construction** We now briefly describe a different method of constructing homotopically-equivalent non-homeomorphic hyperbolic 3-manifolds with non-compact geodesic boundary. To this aim we first recall that Thurston’s hyperbolization theorem for Haken manifolds [11] gives necessary and sufficient topological conditions on a manifold to be hyperbolic with geodesic boundary:

**Theorem 4.5.** Let $\overline{M}$ be a compact orientable manifold with non-empty boundary, let $\mathcal{T}$ be the set of boundary tori of $\overline{M}$ and let $\mathcal{A}$ be a family of disjoint closed annuli in $\partial \overline{M} \setminus \mathcal{T}$. Then $M = \overline{M} \setminus (\mathcal{T} \cup \mathcal{A})$ is hyperbolic if and only if the pair $(\overline{M}, \mathcal{A})$ satisfies the following conditions:
the components of $\partial M$ have negative Euler characteristic;

- $\overline{M} \setminus \mathcal{A}$ is boundary-irreducible and geometrically atoroidal;

- the only proper essential annuli contained in $M$ are parallel in $\overline{M}$ to the annuli in $\mathcal{A}$.

Using Theorem 4.5 we will now prove the following:

**Proposition 4.6.** Let $N$ be a hyperbolic 3-manifold with non-empty geodesic boundary, and suppose that at least one component of $\partial N$ is not a 3-punctured sphere. Then there exists a hyperbolic 3-manifold with geodesic boundary which is homotopically equivalent but not homeomorphic to $N$.

**Proof:** Let $\overline{N}$ be the natural compactification of $N$ obtained by adding to $N$ a family $\mathcal{A}_N$ of closed annuli and a family $\mathcal{T}_N$ of tori. Let also $\{\alpha_1, \ldots, \alpha_k\}$ be a non-empty family of disjoint essential non-parallel loops on $\partial N$ (such a family always exists because of the assumption on $\partial N$). Let $\mathcal{A}'$ be the family of annuli in $\partial \overline{N} \setminus \mathcal{T}_N$ obtained by adding to $\mathcal{A}_N$ closed regular neighbourhoods in $\partial N$ of the $\alpha_i$’s. It is easily seen that the pair $(\overline{N}, \mathcal{A}')$ satisfies the conditions of Theorem 4.5, so $N' = N \setminus (\bigcup_{i=1}^{k} \alpha_i)$ is hyperbolic. Of course $N'$ is homotopically equivalent to $N$, but $\partial N'$ is not homeomorphic to $\partial N$, so a fortiori $N$ and $N'$ are not homeomorphic to each other. \hfill \Box

**References**

[1] W.J. Floyd, *Group completions and limit sets of Kleinian groups*, Invent. Math. **57** (1980), 205-218.

[2] R. Frigerio, C. Petronio, *Construction and recognition of hyperbolic manifolds with geodesic boundary*, math.GT/0109012, to appear in Trans. Amer. Math. Soc.

[3] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, 761. Springer, Berlin, 1979.

[4] L. Keen, B. Maskit, C. Series, *Geometric finiteness and uniqueness for Kleinian groups with circle packing limit sets*, J. Reine Angew. Math. **436** (1993), 209-219.

[5] S. Kojima, *Polyhedral decomposition of hyperbolic 3-manifolds with totally geodesic boundary*, “Aspects of low-dimensional manifolds, Kinokuniya, Tokyo”, Adv. Stud. Pure Math. **20** (1992), 93-112.
[6] S. Kojima, *Geometry of hyperbolic 3-manifolds with boundary*, Kodai Math. J. 17 (1994), 530-537.

[7] A. Marden, B. Maskit, *On the isomorphism theorem for Kleinian groups*, Invent. Math. 51 (1979), 9-14.

[8] J. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

[9] G.A. Swarup, *On a theorem of Johannson*, J. London Math. Soc. (2) 18 (1978), 560-562.

[10] W.P. Thurston, “The geometry and topology of 3-manifolds”, mimeographed notes, Princeton, 1979.

[11] W.P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) 6 (1982), 357-381.

[12] P. Tukia, *On isomorphisms of geometrically finite Moebius groups*, Inst. Hautes Études Sci. Publ. Math. 61 (1985), 171-214.

[13] P. Tukia, *A remark on a paper by Floyd*, in “Holomorphic functions and moduli, Vol. II” (Berkeley, CA, 1986), 165–172, Math. Sci. Res. Inst. Publ., 11, Springer, New York, 1988.

Scuola Normale Superiore
Piazza dei Cavalieri 7
56127 Pisa, Italy
frigerio@sns.it