A SHARP INTEGRAL REARRANGEMENT INEQUALITY FOR
THE DYADIC MAXIMAL OPERATOR AND APPLICATIONS

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Abstract: We prove a sharp integral inequality for the dyadic maximal operator and
give as an application another proof for the computation of its Bellman function of
three variables.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$M_d\phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| du : x \in Q, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$
for $N = 0, 1, 2, \ldots$.

It is well known that it satisfies the following weak type $(1, 1)$ inequality

$$|\{x \in \mathbb{R}^n : M_d\phi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{M_d\phi > \lambda\}} |\phi(u)| du,$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$.

Using this inequality it is not difficult to prove the following known as Doob’s in-

$$\|M_d\phi\|_p \leq \frac{p}{p-1} \|\phi\|_p,$$

for $p > 1$ and $\phi \in L^p(\mathbb{R}^n)$.

It is an immediate result that the weak type inequality (1.2) is best possible, while (1.3) is also sharp (see [1], [2] for general martingales and [16] for dyadic ones).

A way of studying the dyadic maximal operator is by making refinements of the
above inequalities.

The above inequalities hold true even in more general settings. More precisely we
consider a non-atomic probability space $(X, \mu)$ equipped with a tree structure $T$ and define

$$M_T\phi(x) = \sup \left\{ \frac{1}{|I|} \int_I |\phi| d\mu : x \in I \in T \right\}.$$

Concerning (1.2) some refinements have been done in [8] and [9] while for (1.3) the
Bellman function of the dyadic maximal operator has been explicitly computed in [3].

This is given by

$$B_p(f, F) = \sup \left\{ \int_X (M_T\phi)^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F \right\},$$

for $1 

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for $p > 1$ and every $f, F$ such that $0 < f^p \leq F$.

It is proved in [3] that it equals

$$B_p(f, F) = F \omega_p(f^p/F)^p,$$

where $\omega : [0, 1] \to \left[1, \frac{p}{p-1}\right]$ denotes the inverse function $H^{-1}_p$ of $H_p$, which is defined by $H_p(z) = -(p-1)z^p + pz^{p-1}$.

After this evaluation the second task is to find the exact value of the following

$$B_p(f, F, L) = \sup \left\{ \int_X \max(M_T \phi, L)^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F \right\},$$

for $p > 1$, $0 < f^p \leq F$ and $L \geq f$.

It turns out that

$$B_p(f, F, L) =
\begin{cases} 
F \omega_p\left(\frac{pL^{p-1}f - (p-1)L^p}{F}\right)^p, & \text{if } L < \frac{p}{p-1}f \\
L^p + \left(\frac{p}{p-1}\right)^p (F - f^p), & \text{if } L > \frac{p}{p-1}f.
\end{cases}$$

(1.6)

For this evaluation the author in [3] used the result for (1.4) on suitable subsets of $X$ and after some calculus arguments he was able to provide a proof of (1.6).

The Bellman functions have been studied also in [4]. There a more general Bellman function has been computed namely

$$T_p,G,H(f, F, k) = \sup \left\{ \int_{K} G(\mathcal{M}_T \phi) d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int H(\phi) d\mu = F, \right\}$$

for suitable convex, non-negative increasing functions $G, H$.

(1.7)

The approach used in [4] is by proving that $T_p,G,H(f, F, k)$ equals

$$S_p,G,H(f, F, k) = \sup \left\{ \int_0^k G\left(\frac{1}{t} \int_0^t g\right) dt : g : (0, 1] \to \mathbb{R}^+ \text{ non-decreasing, continuous} \right\},$$

with $\int_0^1 g(u) du = f$, $\int_0^1 H(g) dt = F$.

The second step then is to evaluate $T_p,G,H(f, F, k)$ which is as it can be seen in [4] a difficult task. Concerning the first step ($T_p,G,H = S_p,G,H$) the following equality has been proved in [10] stated as

**Theorem A.** If $g, h : (0, 1] \to \mathbb{R}^+$ are non-increasing integrable functions and $G : [0, +\infty) \to [0, +\infty)$ is non-decreasing, then the following is true

$$\sup \left\{ \int_K G(\mathcal{M}_T \phi^*) h(t) dt, \phi^* = g, K \text{ measurable subset of } (0, 1] \right\} = \int_0^k G\left(\frac{1}{t} \int_0^t g(u) du\right) h(t) dt.$$
This is a symmetrization principle that immediately yields the equality $T_{p,G,H} = S_{p,G,H}$ and has several applications in the theory of the dyadic maximal operator.

In this paper our aim is to find another proof of (1.6) by using a variant of Theorem A.

More precisely we will prove the following

**Theorem 1.** The following equality is true

$$\sup \left\{ \int_K G_1(M_T \phi)G_2(\phi)d\mu : \phi^* = g, \ K \text{ measurable subset of} \right.$$  

$$X \text{ with } \mu(K) = k \right\} = \int_0^k G_1 \left( \frac{1}{t} \int_0^t g \right) G_2(g(t))dt,$$

where $G_i : [0, +\infty) \to [0, +\infty), \ i = 1, 2,$

are increasing functions while $g : (0, 1] \to \mathbb{R}$ is non-increasing.

This theorem and some extra effort will shows us the way to provide a simpler proof of (1.6).

We also remark that there are several problems in Harmonic Analysis were Bellman functions arise. Such problems (including the dyadic Carleson imbedding theorem and weighted inequalities) are described in [7] (see also [5], [6]) and also connections to Stochastic Optimal Control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second-order PDEs. The exact evaluation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding Harmonic Analysis problem. Until now several Bellman functions have been computed (see [1], [2], [3], [5], [12], [13], [14], [15]). The exact evaluation of (1.4) has been also given in [11] by L. Slavin, A. Stokolos and V. Vasyunin which linked the computation of it to solving certain PDEs of the Monge-Ampère type and in this way they obtained an alternative proof of the results in [3] for the Bellman functions related to the dyadic maximal operator.

The paper is organized as follows.

In Section 2 we give some preliminaries needed for use in the subsequent sections.

In Section 3 we prove Theorem A while in Section 4 we give a proof that the right side of (1.6) is an upper bound of the quantity: $\int_X \max(M_T \phi, L)^p d\mu$.

In Section 5 we prove the sharpness of the just mentioned result and by this we end the paper.

2. Preliminaries

Let $(X, \mu)$ be a non-atomic probability measure space. A set $T$ of measurable subsets of $X$ will be called a tree if it satisfies conditions of the following

**Definition 2.1.**
i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.

ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that

(a) the elements of $C(I)$ are pairwise disjoint subsets of $I$

(b) $I = \bigcup C(I)$.

iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}(m)$ where $\mathcal{T}(0) = \{X\}$ and $\mathcal{T}(m+1) = \bigcup_{I \in T(m)} \mathcal{C}(I)$.

iv) We have that $\lim_{m \to \infty} \sup_{I \in \mathcal{T}(m)} \mu(I) = 0$. □

Examples of trees are given in [3]. The most known is the one given by the family of all dyadic subcubes of $[0,1]^n$.

The following has been proved in [3].

**Lemma 2.1.** For every $I \in \mathcal{T}$ and every $a$ such that $0 < a < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of disjoint subsets of $I$ such that

$$\mu\left(\bigcup_{J \in \mathcal{F}(I)} J\right) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1-a)\mu(I).$$

□

We will also need the following fact obtained in [7].

**Lemma 2.2.** Let $\phi : (X, \mu) \to \mathbb{R}^+$ and $(A_j)_j$ a measurable partition of $X$ such that $\mu(A_j) > 0 \forall j$. Then if $\int_X \phi d\mu = f$ there exists a rearrangement of $\phi$, say $h(h^* = \phi^*)$ such that

$$\frac{1}{\mu(A_j)} \int_{A_j} h d\mu = f, \text{ for every } j.$$ □

Now given a tree on $(X, \mu)$ we define the associated dyadic maximal operator as follows

$$\mathcal{M}_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\},$$

for every $\phi \in L^1(X, \mu)$.

We will need only the case $k = 1$ of the above theorem in the applications below, and that is what we going to prove. The general case of a given $k \in (0,1]$ is proved in similar ways (see comments at the end of Section [3]).

## 3. The rearrangement inequality

We prove first the following

**Lemma 3.1.** With the notation of Theorem 1 the following inequality holds

$$\int_K G_1(\mathcal{M}_T \phi) G_2(\phi) d\mu \leq \int_0^t G_1 \left( \frac{1}{t} \int_0^t g \right) G_2(g(t)) dt$$
**Proof.** We follow [10]. We set \( v(A) = v_\phi(A) = \int_A G_2(\phi) d\mu \) which is a finite measure (we suppose without loss of generality that \( f_2 = \int G_2(\phi) d\mu < +\infty \)). Then

\[
\int_X G_1(\mathcal{M}_T \phi) G_2(\phi) d\mu = \int_X G_1(\mathcal{M}_T \phi) dv
= \int_{\lambda=0}^{+\infty} v(\{\mathcal{M}_T \phi \geq \lambda\}) dG_1(\lambda)
= \int_{\lambda=f}^{+\infty} + \int_{\lambda=f}^{+\infty} v(\{\mathcal{M}_T \phi \geq \lambda\}) dG_1(\lambda) = I + II,
\]

where

\[
f = \int_X \phi d\mu = I = \int_{\lambda=0}^{f} v(X) dG_1(\lambda) = v(X)[G_1(f) - G_1(0)], \quad \text{and}
\]

\[
(3.1) \quad II = \int_{\lambda=f}^{+\infty} v(\{\mathcal{M}_T \phi \geq \lambda\}) dG_1(\lambda).
\]

We have that

\[
v(\{\mathcal{M}_T \phi \geq \lambda\}) = \int_{\{\mathcal{M}_T \phi \geq \lambda\}} G_2(\phi) d\mu
\]

Set \( a(\lambda) = \mu(\{\mathcal{M}_T \phi \geq \lambda\}) \).

Obviously, since \( G_1 \) is increasing and \( \phi^* = g \) is the decreasing rearrangement of \( \phi \) we must have that

\[
\int_{\{\mathcal{M}_T \phi \geq \lambda\}} G_2(\phi) d\mu \leq \int_0^{a(\lambda)} G_2(g(u))du.
\]

So \( III \) becomes

\[
II \leq \int_{\lambda=f}^{+\infty} \left( \int_0^{a(\lambda)} G_2(g(u))du \right) dG_1(\lambda).
\]

For every \( \lambda > f \) now we consider the unique \( \beta(\lambda) \in (0, 1] \) such that \( \frac{1}{\beta(\lambda)} \int_0^{\beta(\lambda)} g(u) du = \lambda \). (In fact we suppose without loss of \( f \) generality that \( g(0^+) = +\infty \), the other case is treated in a similar way).

Let now \( \lambda > f \).

Because of the weak type \( III \) inequality we must have that

\[
\frac{1}{a(\lambda)} \int_{\{\mathcal{M}_T \phi \geq \lambda\}} \phi d\mu \geq \lambda = \frac{1}{\beta(\lambda)} \int_0^{\beta(\lambda)} g(u) du.
\]

On the other hand

\[
\frac{1}{a(\lambda)} \int_{\{\mathcal{M}_T \phi \geq \lambda\}} \phi d\mu \leq \frac{1}{a(\lambda)} \int_0^{a(\lambda)} g(u) du.
\]

From the above two inequalities we see that

\[
\frac{1}{a(\lambda)} \int_0^{a(\lambda)} g(u) du \geq \frac{1}{\beta(\lambda)} \int_0^{\beta(\lambda)} g(u) du,
\]
and since $g$ is decreasing this means exactly that $a(\lambda) \leq \beta(\lambda)$. So

$$\int_0^{a(\lambda)} g(u) du \leq \int_0^{\beta(\lambda)} g(u) du.$$ 

Thus

$$I I \leq \int_{\lambda=\text{f}}^{+\infty} \left( \int_0^{\beta(\lambda)} G_2(g(u)) du \right) dG_1(\lambda).$$ 

But

$$\int_0^{\beta(\lambda)} G_2(g(u)) du = \int_{\{t : \frac{1}{t} \int_0^t g \geq \lambda\}} G_2(g(u)) du,$$

by the definition of $(\lambda)$. So

$$I I \leq \int_{\lambda=\text{f}} v_g\left(\left\{ t \in (0,1] : \frac{1}{t} \int_0^t g \geq \lambda \right\}\right) G_1(\lambda)$$

where $v_g(B) = \int_B G_2(g(u)) du$, for all Borel subsets $B$ of $(0,1]$. All together we have that

$$\int_X G_1(M_T \phi)G_2(\phi) d\mu = I + I I \leq v(X)[G_1(f) - G_1(0)] + \int_{\lambda=\text{f}}^{+\infty} v_g\left(\left\{ t \in (0,1] : \frac{1}{t} \int_0^t g \geq \lambda \right\}\right) dG_1(\lambda) = \int_{\lambda=\text{f}}^{+\infty} v_g\left(\left\{ t \in (0,1] : \frac{1}{t} \int_0^t g \geq \lambda \right\}\right) dG_1(\lambda) = \int_0^1 G_1\left(\frac{1}{t} \int_0^t g\right) G_2(g(t)) dt$$

and Lemma 3.1 is proved. \hfill \Box

We now proceed to prove Theorem 1. We suppose first that $k = 1$.

Let $g : (0,1] \to \mathbb{R}^+$ be a non-increasing function.

We are going to construct a family $(\phi_a)_{a \in (0,1)}$ of functions defined on $(X, \mu)$, rearrangements of $g$ ($\phi_a^* = g$), such that

$$\limsup_{a \to 0^+} \int_X G_1(M_T \phi_a)G_2(\phi_a) d\mu \geq \int_0^1 G_1\left(\frac{1}{t} \int_0^t g\right) G_2(g(t)) dt.$$

**Proof.** We follow [7]. Let $a \in (0,1)$. By using Lemma 2.1 we choose for every $I \in \mathcal{T}$ a family $\mathcal{F}(I) \subseteq \mathcal{T}$ of disjoint subsets of $I$ such that

$$\sum_{J \in \mathcal{F}(I)} \mu(J) = (1-a)\mu(I).$$

We define $S = S_a$ to be the smallest subset of $\mathcal{T}$ such that $X \subseteq S$ and for every $I \in S$, $\mathcal{F}(I) \subseteq S$. We write for $I \in S$, $A_I = I \setminus \bigcup_{J \in \mathcal{F}(I)} J$. Then if $a_I = \mu(A_I)$ we have because
of (3.3) that \( a_I = a\mu(I) \). It is also clear that

\[
S_a = \bigcup_{m \geq 0} S_{a,(m)}, \quad \text{where} \quad S_{a,(0)} = \{X\} \quad \text{and} \quad S_{a,(m+1)} = \bigcup_{I \in S_{a,(m)}} \mathcal{F}(I).
\]

We define also for \( I \in S_a \), \( \text{rank}(I) = r(I) \) to be the unique integer \( m \) such that \( I \in S_{a,(m)} \).

Additionally, we define for every \( I \in S_a \) with \( r(I) = m \)

\[
\gamma(I) = \gamma_m = \frac{1}{a(1-a)^m} \int_{(1-a)^{m+1}}^1 g(u) du.
\]

We also set for \( I \in S_a \)

\[
b_m(I) = \sum_{S \ni J \subseteq I, r(J) = r(I) + m} \mu(J).
\]

We easily then see inductively that

\[
b_m(I) = (1-a)^m \mu(I).
\]

It is also clear that for every \( I \in S_a \)

\[
I = \bigcup_{S_a \ni J \subseteq I} A_J.
\]

At last we define for every \( m \) the measurable subset of \( X \), \( S_m = \bigcup_{I \in S_{a,(m)}} I \).

Now, for each \( m \geq 0 \), we choose \( \tau_a^{(m)} : S_m \setminus S_{m+1} \rightarrow \mathbb{R} \) such that

\[
\left[ \tau_a^{(m)} \right]^* = \left( g/\left( (1-a)^{m+1}, (1-a)^m \right) \right)^*.
\]

This is possible since \( \mu(S_m \setminus S_{m+1}) = \mu(S_m) - \mu(S_{m+1}) = b_m(X) - b_{m+1}(X) = (1-a)^m - (1-a)^{m+1} = a(1-a)^m \). It is obvious now that \( S_m \setminus S_{m+1} = \bigcup_{I \in S_{a,(m)}} A_I \) and that

\[
\int_{S_m \setminus S_{m+1}} \tau_a^{(m)} d\mu = \int_{(1-a)^{m+1}}^1 g(u) du \Rightarrow \frac{1}{\mu(S_m \setminus S_{m+1})} \int_{S_m \setminus S_{m+1}} \tau_a d\mu = \gamma_m.
\]

Using now Lemma 2.2 we see that there exists a rearrangement of \( \tau_a |_{S_m \setminus S_{m+1}} = \tau_a^{(m)} \) called \( \phi_a^{(m)} \) for which \( \frac{1}{a_I A_I} \int \phi_a^{(m)} = \gamma_m \), for every \( I \in S_{a,(m)} \).

Define now \( \phi_a : X \rightarrow \mathbb{R}^+ \) by \( \phi_a(x) = \phi_a^{(m)}(x) \), for \( x \in S_m \setminus S_{m+1} \). Of course \( \phi_a^* = g \).
Let now $I \in S_{a,(m)}$. Then

$$Av_I(\phi_a) = \frac{1}{\mu(I)} \int_I \phi_a d\mu = \frac{1}{\mu(I)} \sum_{S_a \ni J \subseteq I} \int_{A_J} \phi_a d\mu$$

$$= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{S_a \ni J \subseteq I} \gamma_{m+\ell} a_J$$

$$= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{S_a \ni J \subseteq I} a\mu(J) \frac{1}{a(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}} g(u) du$$

$$= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}} g(u) du \cdot \sum_{S_a \ni J \subseteq I} \mu(J)$$

$$= \frac{1}{(1-a)^m} \sum_{\ell \geq 0} \int_{(1-a)^{m+\ell+1}} g(u) du$$

$$(3.4)$$

$$= \frac{1}{(1-a)^m} \int_0 (1-a)^m g(u) du.$$

Now for $x \in S_m \setminus S_{m+1}$, there exists $I \in S_{a,(m)}$ such that $x \in I$ so

$$M_T(\phi_a)(x) \geq Av_I(\phi_a) = \frac{1}{(1-a)^m} \int_0 (1-a)^m g(u) du =: \theta_m, \quad (3.5)$$

Then for each $a \in (0,1)$ we have that

$$\int_X G_1(M_T\phi_a)G_2(\phi_a)d\mu = \sum_{\ell \geq 0} \int_{S_\ell \setminus S_{\ell+1}} G_1(M_T\phi_a)G_2(\phi_a)d\mu \geq (\text{due to (??)})$$

$$\geq \sum_{\ell \geq 0} G_1(\theta_\ell) \int_{S_\ell \setminus S_{\ell+1}} G_2(\phi_a)d\mu. \quad (3.6)$$

By the construction now of $\phi_a$ we must have that

$$\left( \phi_a \bigg/ S_\ell \setminus S_{\ell+1} \right)^* = \left( g \bigg/ ((1-a)^{\ell+1},(1-a)^\ell) \right)^*.$$
so that (3.6) becomes
\[
\int_X G(M_T \phi_a)G_2(\phi_a)d\mu \geq \sum_{\ell \geq 0} G_1 \left( \frac{1}{(1-a)^\ell} \int_0^{(1-a)^\ell} g(u)du \right) \cdot \int_{(1-a)^{\ell+1}} G_2(g(u))du \\
\geq \sum_{\ell \geq 0} G_1 \left( \frac{1}{(1-a)^\ell} \int_0^{(1-a)^\ell} g(u)du \right) a(1-a)^\ell G_2(g((1-a)^\ell)) \\
(3.7) = \sum_{\ell \geq 0} G_1 \left( \frac{1}{(1-a)^\ell} \int_0^{(1-a)^\ell} g(u)du \right) G_2(g((1-a)^\ell))\left|((1-a)^{\ell+1}, (1-a)^\ell]\right|.
\]

The sum now in (3.7) expresses a Riemman sum of the integral
\[
\frac{1}{t} \int_0^t g(t)G_2(g(t))dt,
\]
so as \(a \to 0^+\), we see that we have the needed inequality. The general case of the sharpness of Lemma 3.1 can be proved across the same lines, integrating \(G_1(M_T \phi_a)\cdot G_2(\phi_a)\) on \(S_{m_a}\) for each \(a\), where \(m_a \in \mathbb{N}\) such that \((1-a)^{m_a+1} < k \leq (1-a)^{m_a}\), thus \((1-a)^{m_a} \to k\), and so by continuity reasons we have the result.

Also the proof of the Lemma 3.1 for general \(k \in (0, 1]\) is proved in a similar way as Lemma 3.1 where we replace the set \(\{x \in X : M_T \phi(x) \geq \lambda\}\) by \(\{x \in K : M_T \phi(x) \geq \lambda\}\).

We now go through the next section.

4. The Bellman function

We prove first the following

**Lemma 4.1.** For every \(f, F\) such that \(0 < f^p \leq F\) and \(L \geq f\) we have that
\[
\int_X \max(M_T \phi, L)^p d\mu \leq \begin{cases} 
F \omega_p \left( \frac{pL^{p-1}f - (p-1)L^p}{F} \right)^p, & \text{if } L < \frac{p}{p-1}f \\
F^p + \left( \frac{p}{p-1} \right)^p(F - f^p), & \text{if } L \geq \frac{p}{p-1}f
\end{cases}
\]
for every \(\phi\) such that, \(\int_X \phi d\mu = f\) and \(\int_X \phi^p d\mu = F\).

**Proof.** We set \(I = \int_X \max(M_T \phi, L)^p d\mu\). Then
\[
I = \int_{\lambda=0}^{+\infty} p\lambda^{p-1} \mu(\{x \in X : \max(M_T \phi(x), L) > \lambda\})d\lambda \\
= \int_{\lambda=0}^{L} + \int_{\lambda=L}^{+\infty} p\lambda^{p-1} \mu(\{x \in X : \max(M_T \phi(x), L) > \lambda\})d\lambda \\
= II + III, \quad \text{where}
\]
\[
II = \int_{\lambda=0}^{L} p\lambda^{p-1}d\lambda = L^p,
\]

\[
III = \int_{\lambda=L}^{+\infty} p\lambda^{p-1} \mu(\{x \in X : \max(M_T \phi(x), L) > \lambda\})d\lambda
\]
since \((X, \mu)\) is a probability space, and
\[
III = \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\mu(\{x \in X : M_T\phi(x) > \lambda\})d\lambda.
\]
By the weak type inequality \((1.2)\) we obtain that
\[
III \leq \int_{\lambda=L}^{+\infty} p\lambda^{p-2}\left(\int_{\{\max(M_T\phi, L) > \lambda\}} d\lambda \right) \left(\lambda^{-1}\int \phi(x) d\mu(x)\right)
\]
\[
= \int_{\lambda=L}^{+\infty} \left(\lambda^{-1}\int \phi(x) d\mu(x)\right) d\lambda.
\]
By using Fubini’s theorem and since \(\max(M_T\phi, L) \geq L\) in all \(X\)
\[
= \int_{X} \phi(x) \left(\lambda^{-1}\int \max(M_T\phi, L) \right) d\mu(x)
\]
\[
= \int_{X} \phi(x) \frac{p}{p-1} [\lambda^{-1}\int \max(M_T\phi, L) d\mu(x)]
\]
\[
(4.1) \quad = \frac{p}{p-1} \int_{X} \phi(x) \max(M_T\phi, L)^{p-1} d\mu(x) - \frac{p}{p-1} L^{p-1} f.
\]
By \((1.1)\) then
\[
III \leq \frac{p}{p-1} \left(\int_{X} \phi(x) d\mu(x)\right)^{1/p} \cdot \left(\int_{X} \max(M_T\phi, L)\right)^{p-1/p} - \frac{p}{p-1} L^{p-1} f \Rightarrow
\]
\[
I \leq \frac{p}{p-1} I^{1/p} I^{(p-1)/p} + L^p - \frac{p}{p-1} L^{p-1} f \Rightarrow
\]
\[
I \leq \frac{p}{p-1} \left(\int_{F} \frac{I}{p} \right)^{p-1/p} + L^p - \frac{p}{p-1} L^{p-1} f \Rightarrow
\]
\[
\Rightarrow pu^{p-1} - (p-1)w^p \geq \frac{pL^{p-1}f - (p-1)L^p}{F},
\]
where \(w = \left(\frac{I}{F}\right)^{1/p}\). This gives
\[
(4.2) \quad - (p-1)\omega^p + pu^{p-1} = H_p(w) \geq \frac{pL^{p-1}f - (p-1)L^p}{F}.
\]
The function \(H_p\) is defined on \(\left[1, \frac{p}{p-1}\right]\) with values on \([0,1]\).

We consider the function \(h : [f, +\infty) \rightarrow \mathbb{R}\) defined by
\[
h(t) = pt^{p-1}f - (p-1)t^p, \quad t \geq f.
\]
Then
\[
h'(t) = p(p-1)t^{p-2}f - p(p-1)t^{p-1}
\]
\[
= p(p-1)(f-t)t^{p-2} < 0 \Rightarrow h \text{ is strictly decreasing in its domain}
\]
Therefore, \(h(t) \leq h(f) = f^p\) for every \(t \geq f\), thus the right hand side of \((4.2)\) is less than \(f^p/F \leq 1\). Say \(b\) the right side of \((4.2)\).
We consider two cases

i) $b > 0$. Then we have that $b \in [0, 1]$ and $H_p(\omega) \geq b$. If $w \leq 1$ then we must have that $I \leq F$ which gives in view of the fact that $\omega_p(b) > 1$, the inequality $I \leq F[\omega_p(b)]^p$, that is our result. We are in case where $b > 0$ or equivalently $L < \frac{p}{p-1}f$. We consider now the case $w > 1$.

Then, since $H_p : [1, \frac{p}{p-1}] \to [0, 1]$ is strictly decreasing we have that

$$H_p(w) \geq b \Rightarrow w \leq \omega_p(b) \Rightarrow \frac{I}{F} \leq [\omega_p(b)]^p$$

$$\Rightarrow I \leq F \omega_p \left( \frac{pL^{p-1}f - (p-1)L^p}{F} \right)^p,$$

in case where $L < \frac{p}{p-1}f$.

That is we proved our Lemma in case where $L < \frac{p}{p-1}f$.

ii) We consider now the second case: $b \leq 0$ that is $L \geq L_0 = \frac{p}{p-1}f$. Then

$$I = \int_X \max(\mathcal{M}_T \phi, L)^p d\mu = L^p + III$$

where as we have seen

$$(4.3) \quad III = \int_{\lambda=L}^{+\infty} p\lambda^{p-2} \left( \int_{\{\mathcal{M}_T \phi > \lambda\}} \phi d\mu \right) d\lambda.$$

Since $L \geq L_0$ we conclude by (4.3) that

$$III \leq \int_{\lambda=L_0}^{+\infty} p\lambda^{p-2} \left( \int_{\{\mathcal{M}_T \phi > \lambda\}} \phi d\mu \right) d\lambda = \int_X \max(\mathcal{M}_T \phi, L_0)^p d\mu - L_0^p.$$

But the case $L_0 = \frac{p}{p-1}f$ was treated in i). Therefore

$$\int_X \max(\mathcal{M}_T \phi, L_0)^p d\mu \leq F \omega_p \left( \frac{pL_0^{p-1}f - (p-1)L_0^p}{F} \right)^p$$

$$= F[\omega_p(0)]^p = F \left( \frac{p}{p-1} \right)^p.$$

Altogether we have that

$$I \leq L^p + F \left( \frac{p}{p-1} \right)^p - L_0^p = L^p + \left( \frac{p}{p-1} \right)^p (F - f^p),$$

that is our result in the second case.

Lemma 4.1 is now proved.

We consider now a non-increasing function $g : (0, 1] \to \mathbb{R}^+$ and the quantities

$$v_g(L) = \frac{1}{t} \max_{t \geq 0} \left( \frac{1}{t} \int_0^t g(L) \right)^p dt \quad \text{and}$$

$$v'_g(L) = \int_0^t g(t) \max_{t \geq 0} \left( \frac{1}{t} \int_0^t g(L) \right)^{p-1} dt.$$
We will prove the following lemma.

**Lemma 4.2.** With the above notation the following equality holds for every \( f : (0, 1] \to \mathbb{R}^+ \),

\[
v_g(L) = L^p - \frac{p}{p-1} f L^{p-1} + \frac{p}{p-1} v'_g(L).
\]

**Proof.** The proof runs as the first part of Lemma 3.1. We discuss the details:

We have that

\[
v_g(L) = \int_{\lambda=0}^L + \int_{\lambda=L}^{+\infty} p\lambda^{p-1} \left\{ t \in (0, 1] : \max \left( \frac{1}{t} \int_0^t g, L \right) \geq \lambda \right\} d\lambda
\]

\[
= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1} \left\{ t \in (0, 1] : \frac{1}{t} \int_0^t g \geq \lambda \right\} d\lambda.
\]

We consider now for each \( \lambda > L \geq f \), the unique \( \beta(\lambda) \in (0, 1] \) such that

\[
\frac{1}{\beta(\lambda)} \int_0^\beta(\lambda) g(u) du = \lambda
\]

(we suppose that \( g(0^+) = +\infty \), without loss of the generality). Therefore,

\[
v_g(L) = L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1} |A_\lambda| d\lambda,
\]

where

\[
A_\lambda = \left\{ t \in (0, 1] : \frac{1}{t} \int_0^t g > \lambda \right\} = (0, \beta(\lambda)).
\]

So

\[
v_g(L) = L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1} \beta(\lambda) d\lambda
\]

\[
= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-2} \left( \frac{1}{\lambda} \int_0^\beta(\lambda) g(u) du \right) d\lambda
\]

\[
= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-2} \left( \int_{\{t : \frac{1}{t} \int_0^t g > \lambda\}} g(u) du \right) d\lambda
\]

by using Fubinis theorem

\[
= L^p + \int_0^1 g(t) \frac{p}{p-1} \left[ \lambda^{p-1} \right]_{\lambda=L}^{\max \left( \frac{1}{t} \int_0^t g, L \right)} d\lambda
\]

\[
= L^p - \frac{p}{p-1} f L^{p-1} + \frac{p}{p-1} v'_g(L),
\]

that is what we wanted to prove. \( \square \)

We proceed now to the last section.
5. Sharpness of Lemma 3.1

We look at the relations (4.1) and (4.4) we suppose that \( L < \frac{p}{p-1} f \).

The first one is an inequality and states that
\[
\int_X \max(M_T \phi, L)^p d\mu \leq L^p - \frac{p}{p-1} L^{p-1} f + \frac{p}{p-1} \int_X \phi \max(M_T \phi, L)^{p-1} d\mu
\]
while the second is an equality stating
\[
\int_0^1 \max \left( \frac{1}{t} \int_0^t g, L \right)^p dt = L^p - \frac{p}{p-1} L^{p-1} f + \frac{p}{p-1} \int_0^1 g(t) \max \left( \frac{1}{t} \int_0^t g, L \right)^p dt.
\]
We fix \( g : (0, 1] \to \mathbb{R}^+ \). By Theorem 1 for
\[
G_1(t) = \max(t, L)^p, \quad t \geq 0
\]
\[
G_2(t) = 1, \quad \text{and} \quad k = 1
\]
we have that
\[
\sup_{\phi^* = g} \int_X \max(M_T \phi, L)^p d\mu = v_g(L)
\]
while for
\[
G_1(t) = \max(t, L)^{p-1}, \quad t \geq 0
\]
\[
G_2(t) = t, \quad \text{and} \quad k = 1
\]
we produce
\[
\sup_{\phi^* = g} \int_X \phi \max(M_T \phi, L)^{p-1} d\mu = v'_g(L).
\]
That is if we leave the \( \phi \)'s to move along the rearrangements of \( g \) in (4.4) we produce the equality (4.4). At the proof of Lemma 3.1 it was used an inequality at another also point, namely that
\[
\int_X \phi \max(M_T \phi, L)^{p-1} d\mu \leq \left( \int_X \phi^p \right)^{1/p} \left( \int_X \max(M_T \phi, L)^p d\mu \right)^{(p-1)/p}.
\]
Totally, inequalities were used there in two places. The first is attained if we use (4.4) and the discussion before. For the second we conclude that we need to find a sequence \( g_n : (0, 1] \to \mathbb{R}^+ \) with \( \int_0^1 g_n(u) du = f \) and \( \int_0^1 g_n^p(u) du = F \) for which
\[
\int_0^1 g_n(t) \max \left( \frac{1}{t} \int_0^t g_n, L \right)^{p-1} dt \approx \left( \int_0^1 g_n^p \right)^{1/p} \cdot \left( \int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L \right) dt \right)^{(p-1)/p}
\]
that is we need equality in a Holder inequality.

Therefore, we are forced to search for a \( g : (0, 1] \to \mathbb{R}^+ \) with
\[
\int_0^1 g(u) du = f \quad \text{and} \quad \int_0^1 g^p(u) du = F
\]
for which
\[
\max \left( \frac{1}{t} \int_0^t g, L \right) = cg(t), \quad \text{for} \quad t \in (0, 1]
\]
where
\[ c = \omega_p \left( \frac{pL^{p-1}f - (p-1)L^p}{F} \right). \]

We state it as

**Lemma 5.1.** There exists \( g : (0, 1] \to \mathbb{R}^+ \) non-increasing, continuous for which the above three equations for the constants \( f, F \) and \( c_0 \) hold, in case where \( L < \frac{p}{p-1}f \).

**Proof.** We set
\[ g(t) = Kt^{-\frac{1}{p} + \frac{1}{c}}, \quad t \in [0, \gamma] \]
\[ = \frac{L}{c}, \quad t \in [\gamma, 1] \]
where \( \gamma \) and \( K \) are such that \( \frac{1}{\gamma} \int_0^\gamma g(u)du = L \) that is
\[ (5.5) \quad Kc\gamma^{-\frac{1}{p} + \frac{1}{c}} = L. \]

It is obvious that \( g \) is continuous and non-increasing. We ask now for the constant \( \gamma \) such that
\[ \int_0^1 g^p(u)du = F \iff \frac{Kp \left[ t^{-\frac{p}{p} + \frac{1}{c}} \right]_t=0}{-p + \frac{p}{c} + 1} + \frac{L^p}{c^p} (1 - \gamma) = F \iff \]
\[ \frac{Kp \gamma^{-\frac{p}{p} + \frac{1}{c} + 1}}{c^p \left( -p + \frac{p}{c} + 1 \right)} + \frac{L^p}{c^p} (1 - \gamma) = F \]
\[ \iff \frac{c^p K \gamma^{-\frac{p}{p} + \frac{1}{c} + 1}}{-(p-1)c^p + pc^{p-1}} + \frac{L^p}{c^p} (1 - \gamma) = F. \]

Since (5.6) holds (5.6) becomes
\[ (5.7) \quad \frac{L^p \cdot \gamma}{-(p-1)c^p + pc^{p-1}} + \frac{L^p}{c^p} (1 - \gamma) = F. \]

By the definition of \( c \) we have that
\[ -(p-1)c^p + pc^{p-1} = \frac{pL^{p-1}f - (p-1)L^p}{F} = b, \]
so (5.7) becomes
\[ \frac{FL^p \cdot \gamma}{pL^{p-1}f - (p-1)L^p} + \frac{L^p}{c^p} (1 - \gamma) = F \iff \]
\[ \iff \gamma = \frac{F - L^p/c^p}{L^p \left( \frac{1}{b} - \frac{1}{c^p} \right)}. \]

We need to see that \( \gamma \in [0, 1] \). Obviously, we have that
\[ L^p \leq \int_X \max(\mathcal{M}_\tau \phi, L)^p d\mu \]
for any \( \phi : \int_X \phi d\mu = f \) and \( \int_X \phi^p d\mu = F \). Additionally
\[
\int_X \max(MT\phi, L)^p d\mu \leq [\omega_p(b)]^p \cdot F = c^p F \Rightarrow F - L^p/c^p \geq 0.
\]
Further \( c \) satisfies \((- (p - 1))c^p + pc^{p-1} = b\) as it is mentioned before \( p(c^p - c^{p-1}) = c^p - b \Rightarrow c^p - b > 0 \Rightarrow \frac{1}{b} - \frac{1}{c^p} > 0.\)

From the above two inequalities we see of course that \( \gamma \geq 0 \)

We prove now that \( \gamma \leq 1 \iff \int_X \gamma d\mu \leq \int_X f d\mu \iff \int_X \gamma \cdot f d\mu \leq \int_X f d\mu \Rightarrow f \leq L^p, \)

which is true because of the fact that always \( L \geq f.\)

We consider now the function \( g \) as defined before with \( \gamma = \frac{F - L^p/c^p}{L^p(\frac{1}{b} - \frac{1}{c^p})} \in [0, 1]. \)

We prove that we additionally have that
\[
\int_0^1 g(u)du = \int_0^{\gamma} Kt^{1+\frac{L}{c}}dt + \frac{L}{c}(1 - \gamma) = f
\]
\( \iff Kc\gamma^{1/c} + \frac{L}{c}(1 - \gamma) = f \)
\( \iff (\text{since } Kc = L\gamma^{1 - \frac{1}{c}}) \)
\[
L\gamma + \frac{L}{c}(1 - \gamma) = f \iff \gamma = \frac{f - L/c}{L(1 - \frac{1}{c})}.
\]

So we need to check that
\[
\frac{f - L/c}{L(1 - \frac{1}{c})} = \frac{F - L^p}{c^p L^p(\frac{1}{b} - \frac{1}{c^p})} \iff
\]
\[
\frac{fc - L}{(c - 1)} = \frac{F c^p - L^p}{L^{p-1}(\frac{c^p}{b} - 1)} \iff
\]

\( (5.8) \)
\[
\frac{c^{p-1}(fc - L)L^{p-1}}{F(c^p - c^{p-1}) - L^p + fL^{p-1}}.
\]

Because now of
\[
c^p - c^{p-1} = \frac{b + c^p}{p}, \quad (5.8) \text{ becomes:}
\]

\( (5.9) \)
But
\[
\frac{F}{p}(-b + c^p) - L^p + fL^{p-1} = \frac{F}{p} \left( -\frac{pL^{p-1}f - (p - 1)L^p}{F} + c^p \right) - L^p + fL^{p-1}
\]
\[
= -L^{p-1}f + \frac{p-1}{p}L^p + \frac{F}{p}c^p - L^p + fL^{p-1}
\]
\[
= \frac{F}{p}c^p - \frac{L^p}{p} = \frac{Fc^p - L^p}{p}.
\]

Now (5.9) is equivalent to
\[
b = \frac{pe^{p-1}(fc - L)L^{p-1}}{Fc^p - L^p} \iff
\]
\[
\iff \frac{pe^p f}{L} - pe^{p-1} = b \left( \frac{Fc^p}{L^p} - 1 \right) \iff \text{(since } pe^{p-1} = b + (p - 1)c^p) \]
\[
\iff \frac{pe^p f}{L} - b - (p - 1)c^p = bF\frac{c^p}{L^p} - b \iff \frac{p\bar{f}}{L} - (p - 1) = b\frac{F}{L^p} \iff
\]
\[
b = \frac{pL^{p-1}f - (p - 1)L^p}{F}
\]
which is true from the definition of \(b\).

That is we derived Lemma [5.1].

We turn now to the case: \(L \geq \frac{p}{p - 1}f\).

For this case we need to construct a sequence \((g_n)_n\) with \(g_n : (0, 1] \to \mathbb{R}^+\) non-increasing and continuous such that
\[
\int_0^1 g_n(u)du = f, \quad \int_0^1 g_n^p(u)du = F \quad \text{and}
\]
\[
\lim_n \int_0^1 \max \left( \frac{1}{t} \int_0^t g_n(u, L) \right)^p dt \geq K^p + \left( \frac{p}{p - 1} \right)^p (F - f^p)
\]
where \(L \geq \frac{p}{p - 1}f\).

We set as before
\[
g(t) = \begin{cases} k_n t^{-1 + \frac{1}{c_n}}, & t \in (0, \gamma_n) \\ \frac{L_n}{c}, & t \in [\gamma_n, 1] \end{cases}
\]
where \(L_n > L_0 = \frac{p}{p - 1}f\),
\[
\gamma_n = \frac{F - L^p/c_n}{L_n^p(\frac{1}{b_n} - \frac{1}{c_n})} = \frac{f - L_n/c_n}{L_n(1 - \frac{1}{c_n})}
\]
where \(c_n = \omega_p(b_n), b_n = \frac{pL^{p-1}f - (p - 1)L^p}{F}\) and \(k_n\) is such that \(k_n c_n \gamma_n^{-1 + \frac{1}{c_n}} = L_n\).
Since $L_n \to L_0$ we have that $b_n \to 0$, $c_n \to \frac{p}{p-1}$ and $\gamma_n \to \frac{f - L_0^p}{p} = 0$,
that is $\gamma_n \searrow 0$.

According to the first case (where $L < \frac{p}{p-1}f$) we have that

$$\int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L_n \right)^p dt = \left[ \omega_p(b_n) \right]^p F \to \left( \frac{p}{p-1} \right)^p F.$$

Now for $L \geq \frac{p}{p-1}f$,

$$\int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L_n \right)^p dt = L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-2} \left( \int_{\left\{ \frac{t}{\lambda} \int_0^t g_n > \lambda \right\}} g_n(u) du \right) d\lambda$$

$$= L^p - L_0^p + \int_{\lambda=L_0}^{+\infty} p\lambda^{p-2} \left( \int_{\left\{ \frac{t}{\lambda} \int_0^t g_n > \lambda \right\}} g_n(u) du \right) d\lambda$$

$$= \int_{\lambda=L_0}^{+\infty} p\lambda^{p-2} \left( \int_{\left\{ \frac{t}{\lambda} \int_0^t g_n > \lambda \right\}} g_n(u) du \right) d\lambda$$

But by definition of $g_n$ we have that

$$\max \left( \frac{1}{t} \int_0^t g_n, L_n \right) = \omega_p(b_n) g_n(t).$$

Thus

$$\int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L_0 \right)^p dt \geq \int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L_n \right)^p dt$$

$$= \left[ \omega_p(b_n) \right]^p \int_0^1 g_n^p(u) du = F[\omega_p(b_n)]^p, \text{ for every } n$$

and so

$$\lim_n \int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L_0 \right)^p dt = F \left( \frac{p}{p-1} \right)^p.$$

At last if

$$a_n(L) = \int_L^{+\infty} p\lambda^{p-2} \left( \int_{\left\{ \frac{t}{\lambda} \int_0^t g_n > \lambda \right\}} g_n(u) du \right) d\lambda$$
satisfies for a given \( L \geq L_0 \)

\[
a_n(L) \leq \int_{\lambda=L_0}^{L} p\lambda^{p-2} \left( \int_{\{t, \frac{1}{t} \int_0^t g_n > L_0\}} g_n(u)du \right) d\lambda
\]

\[
= \left( \int_{\{t, \frac{1}{t} \int_0^t g_n > L_0\}} g_n(u)du \right) \int_{\lambda=L_0}^{L} p\lambda^{p-2} d\lambda
\]

\[
= \tau_L \cdot \int_{\{t, \frac{1}{t} \int_0^t g_n > L_0\}} g_n(u)du.
\]

(5.11)

But

\[
\left| \left\{ t \in (0,1] : \frac{1}{t} \int_0^t g_n \geq L_0 \right\} \right| \leq \left| \left\{ t \in (0,1] : \frac{1}{t} \int_0^t g_n \geq L_n \right\} \right| = \gamma_n,
\]

since \( \gamma_n \) is the unique element of \((0,1]\) such that \( \frac{1}{\gamma_n} \int_0^\gamma g_n = L_n \).

Of course \( \gamma_n \to 0 \), so from (5.11) we deduce that \( a_n(L) \to 0 \), as \( n \to \infty \), thus from (5.10)

\[
\lim_n \int_0^1 \max \left( \frac{1}{t} \int_0^t g_n, L \right)^p dt \geq L^p - L_0^p + \left( \frac{p}{p-1} \right)^p F = L^p + \left( \frac{p}{p-1} \right)^p (F - f^p),
\]

which is the result we needed to prove.

From Lemma 5.1 and the calculations after it’s proof we conclude the sharpness of Lemma 4.1.

Thus we found the Bellman function of three variables of the dyadic maximal operator.

\[\square\]

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