On Computing Relevant Features for Explaining NBCs*

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Abstract
Despite the progress observed with model-agnostic explainable AI (XAI), it is the case that model-agnostic XAI can produce incorrect explanations. One alternative are the so-called formal approaches to XAI, that include abductive explanations. Unfortunately, abductive explanations also exhibit important drawbacks, the most visible of which is arguably their size. The computation of relevant features serves to trade off probabilistic precision for the number of features in an explanation. However, even for very simple classifiers, the complexity of computing sets of relevant features is prohibitive. This paper investigates the computation of relevant sets for Naive Bayes Classifiers (NBCs), and shows that, in practice, these are easy to compute. Furthermore, the experiments confirm that succinct sets of relevant features can be obtained with NBCs.

Keywords
Naive Bayes, Explainability, Dynamic Programming

1. Introduction
The advances in Machine Learning (ML) in recent years motivate an ever increasing range of practical applications of Artificial Intelligence (AI) systems. In some domains, the use of AI systems is premised on the availability of mechanisms for explaining the often opaque operation of ML models. Some uses of ML models are deemed high-risk given the impact that their operation can have on people [2]. (Other authors refer to high-stakes applications [3].) For high-risk AI systems, a critical requirement is rigor, either when reasoning about these systems, or when explaining their predictions.

Recent years have witnessed a growing interest in eXplainable AI (XAI) [4, 5, 6, 7, 8, 9]. The best-known XAI approaches can be broadly categorized as model-agnostic methods, that include for example LIME [10], SHAP [11] and Anchor [12], and intrinsic interpretability [3, 8], for which the explanation is represented by the actual (interpretable) ML model. Intrinsic interpretability may not represent a viable option in some uses of AI systems. On the other hand, model-agnostic methods, although locally accurate, can produce explanations that are unsound [13], in addition to displaying several other drawbacks [14, 15, 16, 17]. Unsound explanations are hopeless whenever rigor is a key requirement; thus, model-agnostic explanations ought not be used in high-risk settings. Indeed, it has been reported [13] that an explanation X can be consistent with different predicted classes. For example, for a bank loan application, X might be consistent with an approved loan application, but also with a declined loan application. An explanation that is consistent with both a declined and an approved loan applications offers no insight to why one of the loan applications was declined. There have been recent efforts on rigorous XAI approaches [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40], most of which are based on feature attribution, namely the computation of so-called abductive explanations (AXp’s). However, these efforts have mostly focused on the scalability of computing rigorous explanations, with more recent work investigating input distributions [34]. Nevertheless, another important limitation of rigorous XAI approaches is the often unwieldy size of explanations. Recent work studied probabilistic explanations, as a mechanism to reduce the size of rigorous explanations [41, 42]. Probabilistic explanations have extended model-agnostic approaches [41], and so can suffer from unsoundness. Alternatively, more rigorous approaches to computing probabilistic explanations have been shown to be computationally hard, concretely hard for NP<sup>NP</sup><sup>NP</sup>, and so most likely beyond the reach of modern automated reasoners. This paper builds on recent work [42] on rigorous probabilistic explanations, and investigates their practical scalability. However, instead of considering classifiers represented as boolean circuits (as in [42]), the paper specifically considers the family of naive Bayes classifiers (NBCs). Earlier work showed that rigorous explanations of NBCs, concretely AXp’s, are computed in polynomial time, and that their enumeration is achieved with polynomial delay [43]. Unfortunately, the size of explanations was not investigated in this earlier work. This paper studies probabilistic explanations for the concrete case of NBCs. For the case of categorical features, the paper re-
lates probabilistic explanations of NBCs with the problem of counting the models of (restricted forms) of integer programming constraints, and proposes a dynamic programing based, pseudo-polynomial algorithm for computing approximate (or locally-minimal) explanations. Such approximate explanations offer important theoretical guarantees: i) approximate explanations are not larger than some rigorous explanation; ii) approximate explanations are not smaller than some rigorous probabilistic explanation; and iii) approximate explanations offer strong probabilistic guarantees on their precision. More importantly, the experimental results demonstrate that succinct explanations, with sizes that can be deemed within the grasp of human decision makers [44], can be very efficiently computed with most often a small decrease in the precision of the explanation.

The paper is organized as follows. Section 2 introduces the definitions and notation used throughout the paper. Section 3 summarizes the computation of explanations for NBCs proposed in earlier work [13]. Section 4 details the approach proposed in this paper for computing locally-minimal probabilistic AXp’s. Section 5 presents experimental results confirming that precise short locally-minimal probabilistic AXp’s. Section 6 concludes the paper.

2. Preliminaries

2.1. Classification problems

This paper considers classification problems, which are defined on a set of features (or attributes) \( \mathcal{F} = \{1, \ldots, m\} \) and a set of classes \( \mathcal{K} = \{c_1, c_2, \ldots, c_k\} \). Each feature \( i \in \mathcal{F} \) takes values from a domain \( \mathcal{D}_i \). In general, domains can be categorical or ordinal, with values that can be boolean, integer or real-valued but in this paper we restrict \( \mathcal{K} = \{0, 1\} \), i.e. binary classifiers, and all features are categorical. (Throughout the paper, we also use the notations \( \lor \) and \( \oplus \) to denote, resp. class 0 and class 1.) Feature space is defined as \( \mathcal{F} = \mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_m; |\mathcal{F}| \) represents the total number of points in \( \mathcal{F} \) if none of the features is real-valued. For boolean domains, \( \mathcal{D}_i = \{0, 1\} = \mathcal{B}, i = 1, \ldots, m, \) and \( \mathcal{F} = \mathcal{B}^m \). The notation \( \mathbf{x} = (x_1, \ldots, x_m) \) denotes an arbitrary point in feature space, where each \( x_i \) is a variable taking values from \( \mathcal{D}_i \). The set of variables associated with features is \( \mathcal{X} = \{x_1, \ldots, x_m\} \). Moreover, the notation \( \mathbf{v} = (v_1, \ldots, v_m) \) represents a specific point in feature space, where each \( v_i \) is a constant representing one concrete value from \( \mathcal{D}_i \). An ML classifier \( \mathcal{M} \) is characterized by a (non-constant) classification function \( \kappa : \mathcal{F} \to \mathcal{K} \). An instance (or observation) denotes a pair \((\mathbf{v}, c)\), where \( \mathbf{v} \in \mathcal{F} \) and \( c \in \mathcal{K} \), with \( c = \kappa(\mathbf{v}) \). (We also use the term instance to refer to \( \mathbf{v} \), leaving \( c \) implicit.)

2.2. Formal explanations

We now define formal explanations. In contrast with the well-known model-agnostic approaches to XAI [10, 11, 12, 5], formal explanations are model-precise, i.e. their definition reflects the model’s computed function. Prime implicant (PI) explanations [18] denote a minimal set of literals (relating a feature value \( x_i \) and a constant \( v_i \in \mathcal{D}_i \)) that are sufficient for the prediction. PI-explanations are related with abduction, and so are also referred to as abductive explanations (AXp) [19]. Formally, given \( \mathbf{v} = (v_1, \ldots, v_m) \in \mathcal{F} \) with \( \kappa(\mathbf{v}) = c \), an AXp is any minimal subset \( \mathcal{X} \subseteq \mathcal{F} \) such that,

\[
\forall (\mathbf{x} \in \mathcal{F}). \left( \bigwedge_{i \in \mathcal{X}} (x_i = v_i) \right) \rightarrow (\kappa(\mathbf{x}) = c)
\]

i.e. the features in \( \mathcal{X} \) are sufficient for the prediction when these take the values dictated by \( \mathbf{v} \), and \( \mathcal{X} \) is irreducible. Also, a non-minimal set such that (1) holds is a WeakAXp. AXp’s can be viewed as answering a ‘Why?’ question, i.e. why is some prediction made given some point in feature space. Contrastive explanations [45] offer a different view of explanations, but these are beyond the scope of the paper.

2.3. \( \delta \)-relevant sets

\( \delta \)-relevant sets were proposed in more recent work [42] as a generalized formalization of explanations. \( \delta \)-relevant sets can be viewed as probabilistic PIs, with AXp’s representing a special case of \( \delta \)-relevant sets where \( \delta = 1 \), i.e. probabilistic PIs that are actual PIs. We briefly overview the definitions related with relevant sets. The assumptions regarding the probabilities of logical propositions are those made in earlier work [42]. Let \( \Pr_{\mathbf{x}}(A(\mathbf{x})) \) denote the probability of some proposition \( A \) defined on the vector of variables \( \mathbf{x} = (x_1, \ldots, x_m) \), i.e.

\[
\Pr_{\mathbf{x}}(A(\mathbf{x})) = \frac{|\{\mathbf{x} \in \mathcal{F}; A(\mathbf{x}) = 1\}|}{|\{\mathbf{x} \in \mathcal{F}\}|}
\]

\[
\Pr_{\mathbf{x}}(A(\mathbf{x}) | B(\mathbf{x})) = \frac{|\{\mathbf{x} \in \mathcal{F}; A(\mathbf{x}) = 1, B(\mathbf{x}) = 1\}|}{|\{\mathbf{x} \in \mathcal{F}; B(\mathbf{x}) = 1\}|}
\]

(Similar to earlier work, it is assumed that the features are independent and uniformly distributed [42]. Moreover, the definitions above can be adapted in case some of the features are real-valued. As noted earlier, the paper studies only categorical features.)

**Definition 2.1** (\( \delta \)-relevant set [42]). Consider \( \kappa : \mathcal{B}^m \to \mathcal{K} = \mathcal{B}, \mathbf{v} \in \mathcal{B}^m, \kappa(\mathbf{v}) = c \in \mathcal{B}, \) and \( \delta \in [0, 1], \mathcal{S} \subseteq \mathcal{F} \) is a \( \delta \)-relevant set for \( \kappa \) and \( \mathbf{v} \) if,

\[
\Pr_x(\kappa(\mathbf{x}) = c | \mathbf{x} \in \mathcal{S} = \mathbf{v}) \geq \delta
\]

(3)

(where the restriction of \( \mathbf{x} \) to the variables with indices in \( \mathcal{S} \) is represented by \( \mathbf{x}_\mathcal{S} = (x_i)_{i \in \mathcal{S}} \).)
we use $lPr$ to add a sufficiently large positive threshold getting:

For simplicity, and following the notations used in [43],

Furthermore, it is also common in practice to apply logarithmic transformations on probabilities of $Pr(c,x)$, thus getting:

Therefore, (4) can be rewritten as follows:

For simplicity, and following the notations used in [43], we use $IPr$ to denote the logarithmic probabilities, thus getting:

(Note that also for simplicity, it is common in practice to add a sufficiently large positive threshold $T$ to each probability and then use only positive values.)

**Running Example.** Consider the NBC depicted graphically in Figure 1. The features are the discrete random variables $R_1, R_2, R_3, R_4$ and $R_5$. Each $R_i$ can take values $t$ or $f$ denoting, respectively, whether a listener likes or not that radio station. Random variable $G$ denotes an age class, which can take values $Y$ and $O$, denoting young and older listeners, respectively. The target class $\oplus$ denotes the prediction yes (i.e. the listener likes the radio station) and $\ominus$ denotes the prediction no (i.e. the listener does not like the radio station). Thus, $\mathcal{K} = \{\ominus, \oplus\}$. Let us consider $v = (R_1, R_2, R_3, R_4, R_5) = (t, f, f, f, t)$. We associate $r_i$ to each literal ($R_i = t$) and $\neg r_i$ to literals ($R_i = f$). Using (6), we get the values shown in Figure 2. (Note that to use positive values, we added $T = +4$ to each $IPr(\cdot)$.) As can be concluded, the classifier will predict $\oplus$.

### 2.4. Naive Bayes Classifiers (NBCs)

NBC [46] is a Bayesian Network model [47] characterized by strong conditional independence assumptions among the features. Given some observation $x \in \mathcal{F}$, the predicted class is given by:

Using the Bayes theorem, $Pr(c|x)$ can be computed as follows: $Pr(c|x) = Pr(x|c)/Pr(x)$. In practice, we compute only the numerator of the fraction, since the denominator $Pr(x)$ is constant for every $c \in \mathcal{K}$. Moreover, given the conditional mutual independency of the features, we have:

Furthermore, it is also common in practice to apply logarithmic transformations on probabilities of $Pr(c,x)$, thus getting:

Therefore, (4) can be rewritten as follows:

For simplicity, and following the notations used in [43], we use $IPr$ to denote the logarithmic probabilities, thus getting:

(Recall that the paper considers NBCs with binary classes and categorical data.) Each feature $i \in \mathcal{F}$ has $x_i \in \{1, \ldots, d_i\}$, (i.e. $D_i = \{1, \ldots, d_i\}$). Let,

\[
\nu(x) = w_0 + \sum_{i \in \mathcal{F}} \sigma(x_i, v_i^1, v_i^2, \ldots, v_i^{d_i})
\]

$\sigma$ is a selector function that picks the value $v_i^r$ if $x_i$ takes value $r$. Moreover, let us define the decision function, $\kappa(x) = \oplus$ if $\nu(x) > 0$ and $\kappa(x) = \ominus$ if $\nu(x) \leq 0$.

The reduction of a binary NBC, with categorical features, to an XLC is completed by setting: $w_0 \triangleq IPr(\ominus) - IPr(\oplus)$, $v_i^1 \triangleq IPr(x_i = 1|\oplus) - IPr(x_i = 1|\ominus)$, $v_i^2 \triangleq IPr(x_i = 2|\ominus) - IPr(x_i = 2|\oplus)$, $\ldots$, $v_i^{d_i} \triangleq IPr(x_i = d_i|\ominus) - IPr(x_i = d_i|\oplus)$. Hence, the argmax in (6) is replaced with inequality to get the following:

(1This example of an NBC is adapted from [41], which is initially reported in [48, Ch.10].)
We now describe how AXp’s can be computed for XLCs. alternatively, $\mathcal{S} \subseteq \mathcal{F}$ minimal set of literals

For a given instance $x$, the problem is to find the smallest slack that can be achieved by allowing the feature not in $\mathcal{S}$ to take any value (i.e. universal/free features), given that the literals in $\mathcal{S}$ are fixed by $a$ (i.e. $\bigwedge_{i \in S} (x_i = a_i)$).

Let $v_i^\omega$ denote the smallest (or worst-case) value associated with $x_i$. Then, by letting every $x_i$ take any value, the worst-case value of $\nu(e)$ is,

$$\Gamma^\omega = \nu_0 + \sum_{i \in \mathcal{F}} v_i^\omega$$

Moreover, from (9), we have: $\Gamma = \nu_0 + \sum_{i \in \mathcal{F}} v_i^a$. The expression above can be rewritten as follows,

$$\Gamma^\omega = \nu_0 + \sum_{i \in \mathcal{F}} v_i^a - \sum_{i \in \mathcal{F}} (v_i^a - v_i^\omega) = \Gamma - \sum_{i \in \mathcal{F}} \delta_i = -\Phi$$

Example 1. Figure 3a shows the resulting XLC formulation for the example in Figure 2. We also let $\mathbf{f}$ be associated with value 1 and $\mathbf{t}$ be associated with value 2, and $d_i = 2$.

3.2. Explaining XLC’s

We now describe how AXp’s can be computed for XLCs. For a given instance $x = a$, define a constant slack (or gap) value $\Gamma$ given by,

$$\Gamma \triangleq \nu(a) = \sum_{i \in \mathcal{F}} \sigma(a_i, v_i^1, v_i^2, \ldots, v_i^d)$$

Computing an AXp corresponds to finding a subset-minimal set of literals $\mathcal{S} \subseteq \mathcal{F}$ such that (1) holds, or alternatively,

$$\forall (x \in \mathcal{F}), \bigwedge_{i \in \mathcal{S}} (x_i = a_i) \rightarrow (\nu(x) > 0)$$

Example 1: Figure 3a shows the resulting XLC formula

Figure 2: Deciding prediction for $v = (t, f, f, f, t)$

Figure 3: Values used in the running example (Example 1 and Example 2)
where \( \delta_i \triangleq \nu_i^a - \nu_i^c \), and \( \Phi \triangleq \sum_{i \in F} \delta_i - \Gamma = -\Gamma^\omega \). Recall the goal is to find a subset-minimal set \( S \) such that the prediction is still \( c \) (whatever the values of the other features):

\[
W_0 + \sum_{i \in S} \nu_i^a + \sum_{i \notin S} \nu_i^c = -\Phi + \sum_{i \in S} \delta_i > 0
\]

In turn, (13) can be represented as the following knapsack problem [49]:

\[
\begin{align*}
\min & \quad \sum_{i=1}^m p_i \\
\text{such that} & \quad \sum_{i=1}^m \delta_ip_i > \Phi \\
& \quad p_i \in \{0, 1\}
\end{align*}
\]

where the variables \( p_i \) assigned value 1 denote the indices included in \( S \). Note that, the fact that the coefficients in the cost function are all equal to 1 makes the problem solvable in log-linear time.

**Example 2.** Figure 3b shows the values used for computing explanations for the example in Figure 2. For this example, the sorted \( \delta_i \)'s become \( (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \). By picking \( \delta_1 \), \( \delta_2 \) and \( \delta_3 \), we ensure that the prediction is \( \hat{c} \), independently of the values assigned to features 3 and 4. Thus \( \{1, 2, 5\} \) is an AXp for the NBC shown in Figure 1, with the input instance \( (v_1, v_2, v_3, v_4, v_5) = (t, f, f, f, t) \). (It is easy to observe that \( \kappa((t, f, f, f, t)) = \kappa((t, t, t, t, t)) = \kappa((t, f, f, f, t)) \).

4. \( \delta \)-Relevant Sets for NBCs

This section investigates the computation of \( \delta \)-relevant sets in the concrete case of NBCs.

Observe that Definition 2.2 imposes no restriction on the representation of the classifier that is assumed in earlier work [42], i.e. the logical representation of \( \kappa \) need not be a boolean circuit. As a result, we extend the definitions from earlier work [42], as detailed below.

4.1. Weak, Locally-Minimal & Smallest Probabilistic AXp’s

A weak probabilistic AXp (WeakPAXp) is a set of fixed features for which the conditional probability of predicting the correct class \( c \) exceeds \( \delta \), given \( c = \kappa(v) \). Thus, \( S \subseteq F \) is a WeakPAXp if,

\[
\text{WeakPAXp}(S; F, \kappa, \nu, \delta) := \Pr_{\kappa}((\nu(x) = c | x_S = v_S) \geq \delta)
\]

which means that the fraction of the number of models predicting the target class and consistent with the fixed features (represented by \( S \)), given the total number of points in feature space consistent with the fixed features, must exceed \( \delta \). (The main difference to (3) is that features and classes are no longer required to be boolean. Also, the definition makes explicit the parameterizations assumed.) Moreover, a probabilistic AXp (PAXp) \( \lambda \) is a WeakPAXp that is also subset-minimal,

\[
PAXp(\lambda; F, \kappa, \nu, \delta) := \text{WeakPAXp}(\lambda; F, \kappa, \nu, \delta)
\]

Minimum-size PAXp’s (MinPAXp, or smallest PAXp) generalize Min-\( \delta \)-relevant sets in Definition 2.2. Furthermore, we define an locally-minimal probabilistic AXp (LmPAXp) \( \lambda \) as a WeakPAXp such that the removal of any single feature \( i \) from \( \lambda \) will falsify PAXp(\( \lambda \setminus \{i\} \); \( F, \kappa, \nu, \delta \)). Formally:

\[
\text{LmPAXp}(\lambda; F, \kappa, \nu, \delta) := \text{WeakPAXp}(\lambda; F, \kappa, \nu, \delta) \wedge \forall(\lambda' \subseteq \lambda), \text{WeakPAXp}(\lambda' \setminus \{i\}; F, \kappa, \nu, \delta)
\]

As stated earlier, the main purpose of this paper is to investigate the computation of LmPAXp explanations. The next section introduces a pseudo-polynomial time algorithm for computing LmPAXp’s. Although, LmPAXp are not minimal subset/cardinality, our experiments show that the proposed approach computes (in pseudo-polynomial time) succint [44] and highly precise approximate explanations.

4.2. Counting Models of XLCs

Earlier work [50, 51, 52, 53] proposed the use of dynamic programming (DP) for approximating the number of feasible solutions of the 0-1 knapsack constraint, i.e. the \#knapsack problem. Here we propose an extension of the basic formulation, to allow counting feasible solutions of XLCs.

We are interested in the number of solutions of,

\[
\sum_{j \in F} \sigma(x_j, v_j^1, v_j^2, \ldots, v_j^d) > w_0
\]

where we assume all \( v_j^i \) to be integer-valued, and non-negative (e.g. this is what the translation from NBCs to XLCs yields). Moreover, (18) can be written as follows:

\[
\sum_{j \in F} \sigma(x_j, -v_j^1, -v_j^2, \ldots, -v_j^d) < w_0
\]

which reveals the relationship with the Knapsack constraint.
For each \( j \), let us sort the \(-w_j^i\) in non-decreasing order, collapsing duplicates, and counting the number of duplicates, obtaining two sequences:

\[
\{ w_j^1, \ldots, w_j^d_j \} \\
\{ n_j^1, \ldots, n_j^d_j \}
\]

such that \( w_j^1 < w_j^2 < \ldots < w_j^{d_j} \) and each \( n_j^i \geq 1 \) gives the number of repetitions of weight \( w_j^i \).

**Counting.** Let \( C(k, r) \) denote the number of solutions of (19) when the subset of features considered is \( \{1, \ldots, k\} \) and the sum of picked weights is at most \( r \). To define the solution for the first \( k \) features, taking into account the solution for the first \( k-1 \) features, we must consider that the solution for the first \( k \) features can be obtained due to any of the possible values of \( x_j \). As a result, for an XLC general recursive definition of \( C(k, r) \) becomes,

\[
C(k, r) = \sum_{i=0}^{d_k} n_k^i \times C(k-1, r-w_k^i)
\]

Moreover, \( C(1, r) \) is given by,

\[
C(1, r) = \begin{cases} 
0 & \text{if } r < w_1^1 \\
1 & \text{if } w_1^1 \leq r < w_1^2 \\
1 + n_1^2 & \text{if } w_1^2 \leq r < w_1^3 \\
\vdots \\
\sum_{i=1}^{d_1} n_1^i & \text{if } w_1^{d_1} \leq r
\end{cases}
\]

In addition, if \( r < 0 \), then \( C(k, r) = 0 \), for \( k = 1, \ldots, m \). Finally, the dimensions of the \( C(k, r) \) table are as follows:

1. The number of rows is \( m \).
2. The (worst-case) number of columns is given by:

\[
W' = \sum_{j \in \mathcal{F}} d_j^r \times n_j^r
\]

\( W' \) represents the largest possible value, in theory. However, in practice, it suffices to set the number of columns to \( W = w_0 + T \), which is often much smaller than \( W' \).

**Example 3.** Consider the following problem. There are 4 features, \( \mathcal{F} = \{1, 2, 3, 4\} \). Each feature \( j \) takes values in \( \{1, 2, 3\} \), i.e. \( x_j \in \{1, 2, 3\} \). The prediction should be 1 when the sum of the values of the \( x_j \) variables is no less than 8. We set \( w_0 = -7 \), and get the formulation,

\[
\sum_{j \in \{1,2,3,4\}} \sigma(x_j, 1, 2, 3) > 7
\]

where each \( x_j \) picks value in \( \{1, 2, 3\} \). We translate to the extended knapsack formulation and obtain:

\[
\sum_{j \in \{1,2,3,4\}} \sigma(x_j, -1, -2, -3) < -7
\]

We require the weights to be integer and non-negative, and so we sum to each \( w_j^i \) the complement of the most negative \( w_j^i \) plus 1. Therefore, we add +4 to each \( j \) and +16 to right-hand side of the inequality. Thus, we get

\[
\sum_{j \in \{1,2,3,4\}} \sigma(x_j, 3, 2, 1) < 9
\]

For this formulation, \( x_j = 1 \) picks value 3. (For example, we can pick two \( x_j \) with value 1, but not 3, as expected.)

In this case, the DP table size will be \( 4 \times 12 \) even though we are interested in entry \( C(4,8) \). Table 1 shows the DP table, and the number of solutions for the starting problem, i.e. there are 50 combinations of values whose sum is no less than 8.

By default, the dynamic programming formulation assumes that features can take any value. However, the same formulation can be adapted when features take a given (fixed) value. Observe that this will be instrumental for computing \( \text{LmPAXp} \).

Consider that feature \( k \) is fixed to value \( l \), then, the formulation for \( C(k, r) \) becomes:

\[
C(k, r) = n_k^l \times C(k-1, r-w_k^l) = C(k-1, r-\delta)
\]

Given that \( k \) is fixed, then it is the case that \( n_k^l = 1 \).

**Example 4.** For Example 3, assume that \( x_2 = 1 \) and \( x_4 = 3 \). Then, the constraint we want to satisfy is:

\[
\sum_{j \in \{1,3\}} \sigma(x_j, 1, 2, 3) > 3
\]

Following a similar transformation into knapsack formulation, we get

\[
\sum_{j \in \{1,3\}} \sigma(x_j, 3, 2, 1) < 5
\]

After updating the DP table, with fixing features 2 and 4, we get the DP table shown in Table 2. As a result, we can conclude that the number of solutions is 6.

The table \( C(k, r) \) can be filled out in pseudo-polynomial time. The number of rows is \( m \). The number of columns is \( W \) (see (20)). Moreover, the computation of each entry uses the values of at most \( m \) other entries. Thus, the total running time is: \( \Theta(m^4 \times W) \).

| \( k \) | \( r \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 2 | 2 | 0 | 0 | 1 | 3 | 6 | 8 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| 3 | 0 | 0 | 0 | 1 | 4 | 10 | 17 | 23 | 16 | 16 | 16 | 16 | 16 | 16 |
| 4 | 0 | 0 | 0 | 0 | 1 | 5 | 15 | 31 | 50 | 50 | 50 | 50 | 50 | 50 |
4.3. Computing LmPAXp’s

Algorithm 1 depicts our method for computing LmPAXp’s given a prediction function \( \kappa \) of an NBC, an input instance \( v \) and a threshold \( \delta \). The procedure LmPAXp is referred to as a deletion-based algorithm\(^3\); it starts from a set of features \( S \), e.g., initialized to \( F \) and then it iteratively drops features while the updated set \( S \) remains a WeakPAXp. The function is WeakPAXp implements the approach described in the previous section, which measures explanation precision by exploiting a pseudo-polynomial algorithm for model counting.

Hence, it is implicit that the DP table is updated at each iteration of the loop in the LmPAXp procedure. More specifically, when a feature \( i \) is newly set universal, its associated cells \( C(i, r) \) are recalculated such that \( C(k, r) = \sum_{i=1}^{d_i} n_i^k \times C(k-1, r - w_i^r) \); and when \( i \) is fixed, i.e. \( i \in S \), then \( C(i, r) = C(i-1, r - v_i^r) \) where \( v_i^r \equiv \Pi \text{Pr}(v_i = j|c) - \Pi \text{Pr}(v_i = j|\neg c) \). Furthermore, we point out that in our experiment, \( S \) is initialized to an AXp \( X \) that we compute initially for all tested instances using the outlined (polynomial) algorithm in Section 3. It is easy to observe that features not belonging to \( X \) do not contribute in the decision of \( \kappa(v) \) (i.e. their removal does not change the value of \( n_{\partial} \) that equals to zero) and thus can be set universal at the initialisation step, which allows us to improve the performance of Algorithm 1.

Moreover, we apply an heuristic order over \( S \) that aims to remove earlier less relevant features and thus to produce shorter approximate explanations. Typically, we order \( S \) following the increasing order of \( \delta \), values, namely the reverse order applied to compute the AXp. Conducted preliminary experiments using a (naive heuristic) lexicographic order over the features show less succinct explanations.

Finally, notice that Algorithm 1 can be used to compute an AXp, i.e. an LmPAXp with \( \delta = 1 \). Nevertheless, the polynomial time algorithm for computing AXp’s proposed in [43] remains a better choice to use in case of AXp’s than Algorithm 1 which runs in pseudo polynomial time.

**Example 5.** Let us consider again the NBC of the running example (Example 1) and \( v = (\text{t}, \text{f}, \text{f}, \text{f}, \text{t}) \). The corresponding XLC is shown in Figure 3b (Example 2). Also, consider the AXp \( \{1, 2, 5\} \) of \( v \) and \( \delta = 0.85 \). The resulting DP table for \( S = \{1, 2, 5\} \) is shown in Table 3.

Note that for illustrating small tables, we set the number of decimal places to zero (greater number of decimal places, i.e. 1,2, etc, were tested and return the results). (Also, note that the DP table reports "-" if the cell is not calculated during the running of Algorithm 1.) Moreover,

\(^3\)This sort of algorithm can be traced at least to the work of Valaan[55], but some authors [56] argue that it is also implicit in works from the 19th century [57].
Thus, we get \( \sum \) subsets considered. It must also be the case that if \( 2 \) is strictly less than \( \) proper subset of another AXp; but this would contradict the number of models must be \( 4 \) (i.e. \( 8 \)), and since \( \) at iteration#1, feature 1 is tested and since \( \) feature 2, the precision of \( \mathcal{S} \), \( \mathcal{S} \) is WeakPAXp's. The evaluation aims at assessing not only the succinctness and precision of computed explanations but also the scalability of our solution.

5. Experimental Results

This section evaluates the algorithm proposed for computing LmPAXp's. The evaluation aims at assessing not only the succinctness and precision of computed explanations but also the scalability of our solution.

5.1. Experimental setup

The benchmarks used in the experiments comprise publicly available and widely used datasets that originate from UCI ML Repository\(^4\) and Penn ML Benchmarks\(^5\). The number of training data (resp. features) in the target datasets varies from 336 to 14113 (resp. 10 to 37) and on average is 3999.1 (resp. 20.0). All the NBCs are trained using the learning tool \textit{scikit-learn}\(^6\). The data split for training and test data is set to 80% and 20%, respectively. Model accuracies are above 80% for the training accuracy and above 75% for the test accuracy.

A prototype implementation of the proposed approach for computing relevant sets is developed in Python. To compute AXp's, we use the Perl script implemented by \cite{43}. The prototype implementation was tested with varying the threshold \( \delta \in \{0.90, 0.93, 0.95, 0.98\} \). When

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\(^4\)https://archive.ics.uci.edu
\(^5\)https://epistasslab.github.io/pmlb/
\(^6\)https://scikit-learn.org

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\begin{algorithm}
\begin{algorithmic}
\State\textbf{Input:} Classifier \( \kappa \), instance \( \nu \), threshold \( \delta \)
\State\textbf{Output:} LmPAXp \( \mathcal{S} \)
\Procedure{LmPAXp}{\( \kappa \), \( \nu \), \( \delta \)}
\State \( \mathcal{S} \leftarrow \{1, \ldots, m\} \)
\For{\( i \in \{1, \ldots, m\} \)}
\If{\text{isWeakPAXp}(\mathcal{S} \setminus \{i\}, \kappa(\nu) = c, \delta)}
\State \( \mathcal{S} \leftarrow \mathcal{S} \setminus \{i\} \)
\EndIf
\EndFor
\State\textbf{return} \( \mathcal{S} \)
\EndProcedure
\end{algorithmic}
\end{algorithm}
converting probabilities from real values to integer values, the selected number of decimal places is 3. (As outlined earlier, we observed that there is a negligible accuracy loss from using three decimal places.) In order to produce explanations of size admissible for the cognitive capacity of human decision makers [44], we selected three different target sizes for the explanations to compute: 9, 7 and 4, and we compute a LmPAXp for the input instance when its AXp \( \mathcal{X} \) is larger than the target size (recall that \( \mathcal{S} \) is initialized to \( \mathcal{X} \)); otherwise we consider the AXp succinct and the explainer returns \( \mathcal{X} \). For example, assume the target size is 7, an instance \( v_1 \) with an AXp \( \mathcal{S}_1 \) of 5 features and an second instance \( v_2 \) with an AXp \( \mathcal{S}_2 \) of 8 features, then for \( v_1 \) the output will be \( \mathcal{S}_1 \) and for \( v_2 \) the output will be a subset of \( \mathcal{S}_2 \).

For each dataset, we run the explainer on 200 instances
randomly picked from the test data or on all instances if there are less than 200.

The experiments are conducted on a MacBook Air with a 1.1GHz Quad-Core Intel Core i5 CPU with 16 GByte RAM running macOS Monterey.

### 5.2. Results

Table 7 summarizes the results of our experiments for \( \delta = 0.95 \) and target size 9 and 7. The complete results of the empirical evaluation are reported in [1]. (Note that the same observations are perceived on the results obtained with the remaining parameters, i.e. \( \delta \in \{0.90, 0.93, 0.98\} \) and \( \text{LmPAXp} \leq 4 \).) As can be observed for all considered settings, the locally-minimal explanations are succinct, in particular the average sizes of the explanations are invariably lower than the target sizes. Moreover, these explanations offer strong guarantees of precision, as their average precisions are strictly greater than \( \delta \) with significant gaps (e.g. above 97%, in column \( \text{LmPAXp} \leq 9 \) and target size 9 and 7). The complete results obtained with the remaining parameters, i.e. \( \text{agarius or mushroom} \) and the average is 0.33 seconds for all tested instances across all datasets and all settings. Furthermore, we point out that the implemented prototype was tested with 4 decimal places to assess further the scalability of the algorithm on larger DP tables, and the results show that computing \( \text{LmPAXp} \)'s is still feasible, e.g. with \( \text{agarius} \) the average runtime when the target size set to 7 is 10.08 seconds.

The table also reports the number of explanations being shorter than or of size equal to the target size over the total number of tested instances. We observe that for both settings \( \text{LmPAXp} \leq 9 \) and \( \text{LmPAXp} \leq 7 \) and for the majority of datasets and with a few exceptions the fraction is significantly high, e.g. varying for 96% to 100% for \( \text{adult} \) dataset. However, in our assessment we observed that for \( \text{LmPAXp} \leq 4 \) despite the poor percentage of wins for some datasets, it is the case that the average lengths of computed explanations are close to 4 (see Table 13 in [1]).

Overall, the experiments demonstrate that our approach efficiently computes succinct and provably precise explanations for NBCs. The results also showcase empirically the advantage of the algorithm, i.e. in practice one may rely on the computation of \( \text{LmPAXp} \)'s, which pays off in terms of (1) performance, (2) succinctness and (3) sufficiently high probabilistic guarantees of precision.

| Dataset     | (m #I) | \( \text{NBC} \) | \( \text{AXp} \) | \( \text{LmPAXp} \leq 9 \) | \( \text{LmPAXp} \leq 7 \) |
|-------------|--------|-----------------|-----------------|-----------------|-----------------|
|             |        | A%              | Length          | Length          | Precision       | W%              | Time             | Length          | Precision       | W%              | Time             |
| adult       | (13 200) | 81.37           | 6.8±1.2         | 6.8±1.2         | 99.99±0.2       | 100 0.074       | 5.9±1.0         | 98.87±1.8       | 99 0.058        |                  |                  |
| agarius     | (23 200) | 95.41           | 10.3±2.5        | 6.9±3.1         | 97.62±2.1       | 95 0.954        | 5.3±3.2         | 96.59±1.6       | 92 1.273        |                  |                  |
| chess       | (37 200) | 88.34           | 12.1±3.7        | 7.7±3.8         | 98.51±1.4       | 68 0.404        | 5.5±4.4         | 97.90±0.9       | 64 0.483        |                  |                  |
| vote        | (17 81)  | 89.66           | 5.3±1.4         | 5.3±1.4         | 100±0.0         | 100 0.000       | 5.3±1.3         | 99.93±0.3       | 100 0.008       |                  |                  |
| kr-vs-kp    | (37 200) | 88.07           | 12.2±3.9        | 7.3±3.9         | 98.29±1.4       | 64 0.416        | 6.0±4.3         | 97.89±1.1       | 64 0.453        |                  |                  |
| mushroom    | (23 200) | 95.51           | 10.7±2.3        | 6.5±2.6         | 97.35±1.8       | 96 1.011        | 5.1±2.5         | 96.52±1.0       | 90 1.130        |                  |                  |
| threeOf9    | (10 103) | 83.13           | 4.2±0.4         | 4.2±0.4         | 100±0.0         | 100 0.000       | 4.2±0.4         | 100±0.0         | 100 0.000       |                  |                  |
| x6          | (10 176) | 81.36           | 4.5±0.9         | 4.5±0.8         | 100±0.0         | 100 0.000       | 4.5±0.8         | 100±0.0         | 100 0.000       |                  |                  |
| mamo        | (14 53)  | 80.21           | 4.9±0.8         | 4.9±0.7         | 100±0.0         | 100 0.000       | 4.9±0.7         | 100±0.0         | 100 0.000       |                  |                  |
| tumor       | (16 104) | 83.21           | 5.3±0.9         | 5.3±0.8         | 100±0.0         | 100 0.000       | 5.2±0.6         | 99.83±0.7       | 100 0.012       |                  |                  |
6. Conclusion
This paper builds on recent work on computing rigorous probabilistic explanations [42], and investigates the concrete case of NBCs. The paper proposes a pseudo-polynomial algorithm for computing the number of points in feature space predicting a specific class, and relates this problem with that of computing a rigorous probabilistic explanation. Furthermore, the paper proposes an algorithm for computing locally minimal probabilistic explanations, which offers strong guarantees in terms of precision. The experimental results confirm that short and precise probabilistic explanations can be efficiently computed in the case of NBCs.

Two lines of future work can be envisioned. One line is to investigate the complexity of explaining multi-class NBCs and extend the approach for computing locally minimal probabilistic explanations for multi-class Naive Bayes models. Furthermore, one might be interested in computing smallest probabilistic explanations instead of approximates. Hence, another line of research is to devise a logical (Satisfiability Modulo Theories, SMT) encoding for computing cardinality minimal probabilistic explanations.

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