METASTABILITY AND DISPERSIVE SHOCK WAVES IN FERMI-PASTA-ULAM SYSTEM

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Abstract. We show the relevance of the dispersive analogue of the shock waves in the FPU dynamics. In particular we give strict numerical evidences that metastable states emerging from low frequency initial excitations, are indeed constituted by dispersive shock waves travelling through the chain. Relevant characteristics of the metastable states, such as their frequency extension and their time scale of formation, are correctly obtained within this framework, using the underlying continuum model, the KdV equation.

1. Introduction

The Fermi–Pasta–Ulam (FPU) model was introduced in [9] in order to study the rate of relaxation to the equipartition in a chain of nonlinearly interacting particles, starting with initial conditions far away from equilibrium. Both the original and the subsequent numerical experiments show that this transition, if present, takes much longer times than expected, raising the so called FPU problem, or FPU paradox.

Recently there has been a growing interest in the field of continuum limits and modulation equations for lattice models (see e.g., [5, 6, 10, 11, 25], just to cite a few without any claim of completeness). Actually, it is possible to say that one of the starting points of this activity goes back to a paper by Zabusky and Kruskal [27], where the recurrence phenomena observed in the FPU system are related to the soliton behaviour exhibited by the KdV equation, which emerges as a continuum limit of the FPU. Surprisingly, these two viewpoints evolved for several years somewhat independently, with some people working essentially on integrable systems, and other people interested mainly in the nearly integrable aspects of the FPU type models.

In the first community the work [27] was a milestone. A crucial remark in that paper is that the time evolution of a class of initial data for the KdV equation are approximately described by three phases: one dominated by the nonlinear term, one characterized by a balance between the nonlinear term and the dispersive term where some oscillations, approximately described by a train of solitons with increasing amplitude, start to develop, and the last where the oscillations prevail. Since the soliton velocities are related to their amplitude, at a certain time they start to interact. By observing the trajectories of the solitons in the space-time, Kruskal and Zabusky realized that, periodically, they almost overlap and nearly reproduce the initial state. Clearly, such an overlapping is not exact, due to the phase shift given by the nonlinear interaction between solitons. They conjectured that this behaviour gives an explanation of the FPU phenomenology.

As discovered later by Gurevich and Pitaevskii [15] in a different context, the oscillations observed by Zabusky and Kruskal are more properly described in terms of modulated one-phase solutions of KdV equations, and the slow evolution of their wave parameters is governed by the so called Whitham’s equations [26].

The analytic treatment of the dispersive analogue of shock waves is less developed in the non integrable case even if some steps in this direction have been made (see for instance [8]); moreover, recently it has been conjectured that such phenomenon, in its early development, is governed by a universal equation [7].

From the FPU side, due to the original motivations arising from Statistical Mechanics, the core question become the possible persistence of the FPU problem in the thermodynamic limit, i.e.
when the number of particles \( N \) goes to infinity at a fixed specific energy (energy per particle). Izrailev and Chirikov [16] suggested the existence of an energy threshold below which the phenomenon is present; but such a threshold should vanish for growing \( N \) due to the overlapping of resonances; the recurrence in the FPU system is thus explained in terms of KAM theorem (see also [23]). According to Bocchieri et al. [4] instead, the threshold should be in specific energy and independent of \( N \).

Along the lines of the second conjecture, Galgani, Giorgilli and coworkers [2, 3, 13, 18] have recently put into evidence a metastability scenario (see also the previous paper by Fucito et al. [12]): for different classes of initial data, among which those with low frequency initial conditions, the dynamics reaches in a relatively short time a metastable state. Such situation can be characterized in terms of energy spectrum by the presence of a natural packet of modes sharing most of the energy: and the corresponding shape in the spectrum does not evolve for times which appear to grow exponentially with an inverse power of the specific energy. The possible theoretical framework for such experimental results could be a suitably weakened form of Nekhoroshev theory.

As one of the first attempts to join again in an original way the two different aspects of the FPU paradox and of its integrable aspects, we recall the paper [20]; in that work, the approaching of the FPU chain to the shock is observed, but unfortunately the range of specific energy considered is too high to have the metastability (see the higher part of the Fig.1, left panel).

Some recent papers (see e.g., [21]) recovered the spirit of the work by Zabukii and Kruskal: exploiting the typical time scales of the KdV equation, it is possible to recover those of the FPU system, in particular the time scale of creation of the metastable states; the length of the natural packet is also obtained and its spectrum is expected to be connected to that of solitons of the underlying KdV.

Given all the elements we have put into evidence from the literature, it is natural to make a further step. Our claim is that, concerning the metastability scenario, among the several aspects of the KdV equation, precisely the dispersive analogue of the shock wave is the relevant one.

Thus the aim of this paper is to show, at least numerically, that the metastable state is completely characterized by the formation and persistence of the modulated solutions of KdV equation and by their self interactions. We are indeed able to put into evidence exactly such phenomena directly in the FPU dynamics.

In fact, we observe the formation of dispersive shock waves in the FPU chain: correspondingly the metastable state sets in. Exploiting the scaling properties of the KdV equation one obtains for the spatial frequency (wave number) of the dispersive waves, which can be naturally thought as an upper bound for the natural packet, the dependence on the specific energy \( \epsilon \), as observed in the numerical experiments: \( \omega \sim \epsilon^{1/4} \).

Similarly we can also explain the dependence of the formation time of the metastable state on the number \( N \) of particles and on the specific energy \( \epsilon \): \( t \sim N\epsilon^{-1/2} + \epsilon^{-3/4} \) when energy is initially placed on a low frequency mode.

Although this is a kind of preliminary and qualitative investigation, we think in this way to give a unified and synthetic picture of the metastability scenario in the FPU system, providing the precise dynamical mechanism leading to its creation.

In what follows, after a short description of the FPU model, we recall the main facts about the metastability scenario in the FPU (see Sect. 2), and some elements of the Gurevich–Pitaevskii phenomenon for KdV equation (see Sect. 3); we then show in Sect. 4 the deep relations between the two aspects.

1.1. The model. In the original paper [9], Fermi, Pasta and Ulam considered the system given by the following Hamiltonian

\[
H(x, y) = \sum_{j=1}^{N} \left[ \frac{1}{2} y_j^2 + V(x_{j+1} - x_j) \right], \quad V(s) = \frac{1}{2} s^2 + \frac{\alpha}{3} s^3 + \frac{\beta}{4} s^4,
\]
describing the one–dimensional chain of \( N + 2 \) particles with fixed ends, which are represented by \( x_0 = x_{N+1} = 0 \). For the free particles, \( x_1, \ldots, x_N \) are the displacements with respect to the equilibrium positions. This is the FPU \( \alpha, \beta \)–model.

The normal modes are given by
\[
x_j = \sqrt{\frac{2}{N+1}} \sum_{k=1}^{N} q_k \sin \frac{j k \pi}{N+1}, \quad y_j = \sqrt{\frac{2}{N+1}} \sum_{k=1}^{N} p_k \sin \frac{j k \pi}{N+1},
\]
\((q_k, p_k)\) being the new coordinates and momenta. The quadratic part of the Hamiltonian in the normal coordinates is given the form
\[
H_2 = \sum_{j=1}^{N} E_j, \quad E_j = \frac{1}{2} \left( p_j^2 + \omega_j^2 q_j^2 \right), \quad \omega_j = 2 \sin \frac{j \pi}{2(N+1)},
\]
\(E_j\) being the harmonic energies and \(\omega_j\) the harmonic frequencies.

Given the following limit, representing the equipartition,
\[
E_j = \frac{1}{T} \int_{0}^{T} E_j(t) dt \rightarrow \epsilon := \frac{E_j}{N} \quad \forall j,
\]
the fundamental question of the FPU problem is how long it takes to be reached, if ever.

Remark: It is clearly possible to consider also the case with periodic boundary conditions. In the present paper we focus our attention on the case of fixed ends. We also point out that, although all the pictures shown but the first one refer to the \( \alpha \)–model, the main qualitative aspects are still valid in the \( \beta \) one.

Remark: All the numerical result present here, with the exception of those in Fig. 3, are obtained using a second order symplectic integrator (Verlet’s algorithm), with a typical time step of 0.05.

2. Metastability in the FPU chain

Concerning the FPU dynamics, many of the recent numerical evidences \cite{2,3,13,18} support a metastability scenario according to which, below a certain threshold in specific energy, the system is characterized by a metastable state; it remains frozen in such a state for a time which grows rapidly as the specific energy goes to zero.

We will concentrate here on initial data with the whole energy in the lowest frequency normal mode. In order to better illustrate this aspect of the dynamics, i.e. to show the time scales of these metastable states and their dependence on the relevant parameters \( E \) and \( N \), we give in Fig. 1, left panel, a typical picture. Following \cite{2}, one considers the energies \( E_s = \sum_{j=1}^{s} E_j \) of packets of modes, and defines a corresponding critical time \( t_s \), as the time when the \( s^{th} \) packet has lost a fixed fraction \( \gamma \) of the energy it has to lose to reach equipartition. For every fixed specific energy, the different critical times for all the possible packets are drawn in the picture. From its qualitative aspect, which turns out to be practically independent of the value of \( \gamma \), one clearly recognizes two different time scales, the first one giving the creation time of the so called natural packet, which then survive up to the second time scale characterized by the reaching of the equipartition. It has been estimated, for the \( \alpha, \beta \)–model, a power law scaling of the type \( t \sim \epsilon^{-3/4} \) for the creation of the natural packet, formed with normal modes with frequencies up to \( \omega \sim \epsilon^{1/4} \), with no dependence from \( N \) in the range \( 7 - 1023 \) (see \cite{2}, Fig. 7).

As we said, the qualitative aspects of this figure (Fig. 1, left panel) are present both in \( \alpha \)–model and \( \beta \)–model, for different number of particles, and for initial data involving one or more low frequency modes.

A second important point is the following: as observed from the very beginning by Giorgilli and coworkers, the existence of these natural packets appears to be an integrable phenomenon. It is indeed possible to produce the corresponding picture (figure not shown) for the Toda lattice, which is integrable, and one obtains again the straight branch of points corresponding to the creation of the packets, while the second branch is completely missing, i.e. in the Toda chain these packets
Figure 1. Metastability. Left panel: natural packet phenomenon. $\beta$–model, with $\beta = 1/10$, $N = 255$, initial energy on mode 1; the threshold $\gamma$ is fixed to 0.1. Abscissa: critical time $t_\gamma$ for every packet; ordinate: specific energy (from [18]). Right panel: exponentially long relaxation times. $\beta$–model, with $\beta = 1/10$. Time needed by $n_{\text{eff}}$ to overcome the threshold $0.5$ vs. the specific energy power to $-1/4$, for fixed $N = 255$ and $\epsilon$ in the range $[0.0089, 7.7]$, in semi-log scale; every point is obtained averaging over 25 orbits with different phases. The straight line is the best fit using all the points with $\epsilon \leq 1$ (from [3, 18]).

seem to be stable structures. The lower is the energy, the longer the FPU takes to manifest its non integrability and to behave differently from the Toda system.

A quantitative estimate of the second time scale related to the equipartition is given in [3] (but see also [19]). Using, as usual in this type of investigation, the spectral entropy indicator $n_{\text{eff}} := \frac{S}{N}$, where $S := -\sum_{j=1}^{N} e_j \ln e_j$, and $e_j := \frac{E_j}{\sum_{i=1}^{N} E_i}$, it is possible to estimate the effective number of normal modes involved in the dynamics. Plotting the time necessary for $n_{\text{eff}}$ to overcome a fixed threshold close to its saturating value, as a function of the specific energy, one obtains a reasonable estimate of the relaxation times to equipartition, that is an estimate for the metastability times. The corresponding results are shown in Fig. 1: the numerical evidence supports the exponential scaling, with a possible law of the type $T \sim \exp(\epsilon^{-1/4})$, for the $\beta$–model.

The previous descriptions are concerned with a kind of more global picture: concentrating on a single evolution at a fixed specific energy, it is possible to have a better insight to the approach to equipartition looking at the time evolution of the distribution of energy among the modes. We plot in Fig. 2 some frames to give an idea: the thick symbols represent the average over all the times, while the thin ones are the averages over a short time window. In the upper left panel, in the first stage of the evolution, one observes the small but regular transfer of energy towards the higher frequencies. In the subsequent panel, as this process goes on, some modulations on the shape of the spectrum appear. At longer times the spread to the higher part slows down, i.e. the slope of the straight line converges to a certain value (see the lower left panel), and the amplitude of the modulating bumps increases and their position slowly moves leftward. In the last frame we see quite well the spectral structure of the metastable state: in the low frequency region the slow motion of the bumps produces a plateau once one look at the average over all times; and in the rest of the spectrum essentially nothing happens a part from an extremely slow emergence of narrow peaks. Such a situation remain practically frozen for times growing exponentially with a suitable power of the inverse of the specific energy.
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Figure 2. Time averaged distribution of energy among the harmonic modes at different times. α-model, with \( \alpha = 1/4 \), \( N = 127 \), specific energy \( \epsilon = 0.001 \), initially on mode 1. Abscissa: \( k/N \), \( k \) being the mode number. Ordinates: the energy of the modes averaged up to time \( t \) reported in the label (thick points: average from the beginning; thin points: average over a time window given by the last 100 time units).

3. The dispersive analogue of the shock waves

This section is a brief summary of some well-known facts about dispersive shock waves. For our purposes it is sufficient to discuss the case of the KdV equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \mu^2 \frac{\partial^3 u}{\partial x^3} = 0
\]

which is known to give a good approximation the continuum limit of the FPU model for long-wave initial conditions.

If the parameter \( \mu \) is equal to zero it is well-known that, if the initial data is decreasing somewhere, after a finite time \( t_0 \), the gradient \( u_x(x,t) \) of the solution becomes infinite in a point \( x_0 \) (point of gradient catastrophe). For monotone initial data, the solution is given in implicit form by the formula \( x = tu + f(u) \), (where \( f(u) \) is the inverse function of the initial datum \( u(x,0) \)); consequently, \( u_x = 1/(f'(u) + t) \) and the time of gradient catastrophe is

\[
t_0 = \max_{u \in \mathbb{R}} f'(u).
\]

For \( \mu \neq 0 \), far from the point of gradient catastrophe, due to the small influence of the dispersive term, the solution has a behaviour similar to the previous case, while, in a neighbourhood of the singular point, where the influence of the dispersion cannot be neglected anymore, modulated oscillations of wave number of order \( 1/\mu \) develop (see Fig.3).

According to the theory of Gurevich and Pitaevskii, these oscillations can be approximately described in terms of one-phase solutions of the KdV equation

\[
u(x,t) = u_2 + a \cosh^2 \left( \sqrt{\frac{u_3 - u_1}{3}} (x - ct); m \right),
\]
that slowly vary in the time.

The wave parameters of the solution (4) depend on the functions $u_j$ which evolve according to an hyperbolic system of quasi-linear PDEs: the Whitham’s equations [26]. Moreover, for generic initial data, the region filled by the oscillations grows as $t^{3/2}$ [1, 15, 22]. Outside this region the evolution of the solution is governed by the dispersionless KdV equation.

Inside the oscillation region, two limiting situations arise when the values of two of the three parameters $u_j$ coincide:

- $m \approx 0, a \approx 0$: in this case we have an harmonic wave with small amplitude which corresponds to the small oscillations that one can observe near the trailing edge of the oscillation zone;
- $m = 1$: in this case we have a soliton which one can observe near the leading edge of the oscillation zone.

Since the Whitham’s equations are hyperbolic, further points of gradient catastrophe could appear in their solutions. As a consequence also multiphase solutions of KdV equation could enter into the picture.

It turns out that, for $\mu \to 0$, the plane $(x, t)$ can be divided in domains $D_g$ where the principal term of the asymptotics of the solution is given by modulated $g$-phase solutions [17]. For monotonically decreasing initial data there exists rigorous theorems which provide an upper bound to the number $g$ [14]. Unfortunately for more general initial data, such as those we need, the analytical description is less developed.

4. Dispersive shock waves in the FPU

The connection between the FPU dynamics, when initial conditions involve only low frequency modes, and KdV equation has been already pointed out in the sixties by Zabusky and Kruskal [27]. And in many other subsequent papers it is heuristically deduced that the KdV (respectively mKdV) may be viewed as continuum limit for FPU $\alpha$–model (respectively $\beta$–model).

In this section we will first show in qualitative way how, in this framework, the metastability is deeply related with the presence of the dispersive shock waves; we will then give some more detail on the continuum model involved.

4.1. Metastability and dispersive shock waves. As we described in Sec. 2, the metastability in the FPU model is a property well visible in the energy spectrum (see Fig. 2). We show now that it can be also detected on the dynamics of the particles.

In order to clarify this point we show in Fig. 4 some frames of the time evolution of positions and velocities of the chain. We choose the orbit whose spectrum is depicted in Fig. 2.
In the panel referring to time $t = 6 \times 10^3$, it is possible to see that the emergence of the bump in the energy spectrum corresponds the formation of the dispersive shock waves in the profile of the velocity. When the wave train has completely filled the chain, the evolution of the energy spectrum stops: the system has reached the metastable state (second panel).

It is worth to point out that the wave number of the modulated waves is exactly of the same order of the bump; this fact suggests that their slow motion, described in Sec. 2, could be explained in terms of the evolution of the wave parameters (see Eq. 4) given by the Whitham’s equations.

Thus the central point is that in the first stage of the dynamics, the evolution may be divided in three regimes:

**dispersionless regime:**

the corresponding continuum model is dominated by the non dispersive term, with the string approaching the shock and the almost regular energy transfer towards higher frequency modes, as shown in Fig. 2, panel a.

**mixed regime:**

in the corresponding PDE is visible the effect of the dispersive term, which prevents the shock and creates the typical spatial oscillations in the chain; correspondingly the energy spectrum exhibits the formation of the bumps located at the correct frequency and at its higher harmonics. The part of the string non occupied by the growing dispersive wave is still governed by the dispersionless part and for this reason the flow of energy slows down and eventually stops when the train involves the whole chain (see Fig. 2, panel b, and Fig. 4, panel a).

**dispersive regime:**

the dispersive shock wave fills the lattice and travels through it: the metastable state is formed (see Fig. 2, panel c, and Fig. 4, panel b). In the underlying integrable continuum limit one has the elastic interactions of the solitons in the head of the train, together with the recurrence phenomena already observed in [27]: this motivates the long time persistence of the natural packet (see Fig. 2, panel d).

Such a correspondence between the evolution of the energy spectrum and the appearance of dispersive waves and their growth up to the occupation of the whole chain, has been observed in all our numerical experiments: it appears to be independent of the number of particles $N$, of the specific energy (in the range for which one has metastability) and of the particular choice of the initial low frequency mode excited. We are working on a more systematic and quantitative comparison.
Figure 5. Apparent elastic interaction of dispersive wave trains, explained in terms of the two decoupled KdV equations for $\xi$ and $\eta$. $\alpha$–model, with $\alpha = 1/4$, $N = 255$, $\epsilon = 0.001$, initial energy on mode 2. Abscissae: $k/N$, $k$ being the mode number. In the velocity subpanel one observes two wave trains travelling in opposite directions without interaction. The same two trains appear respectively in the $\xi$ and $\eta$ subpanels.

In the remaining part of this section we will show more precisely that the above mentioned dispersive and non dispersive regimes admit a clear understanding in terms of the underlying continuum model.

4.2. The continuum model. As observed by Bambusi and Ponno [21], the FPU, with periodic boundary conditions is well approximated, for low frequency initial data, by a pair of evolutionary PDEs whose resonant normal form, in the sense of canonical perturbation theory, is given by two uncoupled KdV:

\begin{align}
\xi_t &= -\xi_x - \frac{1}{24N^2}\xi_{xxx} - \sqrt{\frac{\epsilon}{2}}\xi_x , \\
\eta_t &= \eta_x + \frac{1}{24N^2}\eta_{xxx} + \sqrt{\frac{\epsilon}{2}}\eta_x ,
\end{align}

using the rescaling with respect to the original FPU variables ($x_{\text{FPU}} = N x_{\text{KdV}}$, $t_{\text{FPU}} = N t_{\text{KdV}}$); $\xi$ and $\eta$ can be written in terms of two functions $q_x$ and $p$ that, a part from a constant factor $\alpha$, respectively interpolates the discrete spatial derivative $q_j - q_{j-1}$ and the momenta $p_j$:

\begin{align*}
\xi(x, t) &\simeq \alpha(q_x + p) \\
\eta(x, t) &\simeq \alpha(q_x - p)
\end{align*}

Taking into account that, in our case, we deal with fixed boundary conditions, we must add to the equations (5,6) the following boundary conditions:

\begin{align*}
\xi(0, t) &= \eta(0, t) \\
\xi(L, t) &= \eta(L, t).
\end{align*}

In other words the variables $\xi$ and $\eta$ exchange their role at the boundary. Moreover, if $\eta(x, t)$ satisfies (5) then $\eta(-x, t)$ satisfies (6) and vice-versa.

As a consequence, we can substitute the evolution of $(\xi(x, t), \eta(x, t))$ with the evolution of a single function $u(x, t)$ defined in terms of $\xi$ and $\eta$ in the following way

\begin{align*}
u(x, t) &= \xi(x, t) & x &\in [0, L] \\
u(x, t) &= \eta(2L - x, t) & x &\in [L, 2L],
\end{align*}

governed by the KdV equation

\begin{align}
u_t &= -\nu_x - \frac{\mu^2}{24} \nu_{xxx} - \sqrt{\frac{\epsilon}{2}}\nu_{xx} ,
\end{align}
Figure 6. Evolution of the chain. $\alpha$–model, with $\alpha = 1/4$, $N = 511$, $\epsilon = 0.001$, initial energy in equipartition on modes 2 and 3. Abscissae: the index particle $j$. Ordinates: $\xi(j)$ in each left sub-panel (full circles), $\eta^*(j) = \eta(N - j)$ in each right sub-panel (empty circles). The dynamics develop three wave trains; in these frames one observes the bigger one flowing in the left direction and passing from the $\eta^*$ subpanel to the $\xi$ one, while the smaller ones go out the $\xi$ subpanel and reenter in the other one: the two variables satisfy indeed a common KdV equation on a circle.

and satisfying periodic boundary conditions.

These facts are clearly visible in Fig. 5 and Fig. 6. In the first one, in order to show the role of the variables $(\xi, \eta)$, we give three almost consecutive frames of the evolution of a chain, with energy initially placed on the second normal mode. As expected, one has the generation of two trains of dispersive waves (the number of trains generated is exactly $k$, when the initially excited normal mode is the $k^{th}$ one). Looking at the velocity sub-panel (the lower left one) of each frame, a sort of elastic interaction appears between the trains: they approach each other in frame (a), they interact in frame (b) and they come out unmodified in frame (c). But looking at the sub-panels of $\xi$ and $\eta$ one realizes that such “elastic interaction” is the superposition of two independent, in the normal form approximation, dynamics.

Moreover, the relation between $(q, p)$ and $(\xi, \eta)$ explains why the dispersive shock is well visible in the velocities, which are directly a linear combination of the variables $\xi$ and $\eta$; the configurations instead are approximately obtained by means of an integration which has a smoothing effect.

Fig. 6 illustrates the role of fixed boundary conditions, with the two KdV equations of the normal form for the periodic case, degenerating into a single KdV. In the three consecutive frames it is perfectly clear that the big wave train supported by the $\eta$ variable flows into the equation for $\xi$, and the two smaller trains, initially present in the $\xi$ subpanel, go out through the left border and re-enter from the right side of the $\eta$ part. The true equation is the periodic KdV 7 for the variable $u$.

The recurrence appearing in the dispersive regime, and well visible in our numerical experiments (figures not shown), may be seen as the effect of the quasi-periodic motion on a torus of the underlying integrable system. On the other hand, following Kruskal and Zabusky, it also has a natural explanation in terms of the elastic scattering of solitons. Clearly, a more refined description of this phenomenon should also take into account the effect of the interactions of the small oscillations near the trailing edge. Indeed it could be interesting to study the analytical aspects of the interaction of dispersive wave trains, where the higher genus modulated solutions of KdV equation could play a role.

4.3. The relevant scalings. In the previous section we showed that the continuum limit for the FPU model with fixed ends, $N$ particles, specific energy $\epsilon$ and an initial datum of low frequency,
is given by the single KdV equation 7, which in the moving frame is

\[ u_t = -\frac{1}{24N^2} u_{xxx} - \sqrt{\frac{\epsilon}{2}} u_x. \]

Rescaling the space and time variables according to the following rule

\[ x \rightarrow N \epsilon^{1/4} x, \]
\[ t \rightarrow N \epsilon^{3/4} t, \]

we get an equation independent of \( N \) and \( \epsilon \):

\[ u_t = -\frac{1}{\sqrt{2}} u_x - \frac{1}{24} u_{xxx}. \]

We have seen that the metastable state is characterized by its size in the modal energy spectrum, and by its time of creation and destruction.

Let us start with the first aspect. Following the numerical evidence we claim that the initial wave number of the dispersive wave provides a good estimate, or at least an upper bound, for the width of the packet of modes involved in the metastability. Denoting by \( \omega^* \) the relevant wave number for equation 11, the corresponding wave number \( \omega_{KdV} \) for equation 8 is

\[ \omega_{KdV} = N \epsilon^{1/4} \omega^*, \]

due to the spatial rescaling 9.

Recalling the spatial scaling in the continuum limit process \((x_{FPU} = Nx_{KdV})\), one remains with the experimentally observed law

\[ \omega_{FPU} \sim \epsilon^{1/4}, \]

without any dependence on \( N \).

Let us now consider the second point: the time scales. As pointed out in the previous discussion, the time of formation of metastable state is characterized by two different regimes: one governed by dispersionless KdV, where the effect of the nonlinearity prevails and drive the dynamics towards a shock, and one governed by the full KdV equation, where the dispersion prevents the shock and generates the dispersive train. Therefore we can approximately estimate the time of formation of the metastable state as the sum of the two contribution

\[ t_{FPU} \simeq N(t_1 + t_2); \]

\( t_1 \) is the time of validity of the first regime, that we estimate from above with the time \( t_s \) of formation of the shock in the dispersionless KdV, while \( t_2 \) is the time needed by the dispersive wave to occupy the whole chain in equation 8 (\( t_2^* \) in equation 11), and the factor \( N \) is again due to the rescaling in the continuum limit process \((t_{FPU} = Nt_{KdV})\).

For the equation \( u_t = \sqrt{\epsilon} u_x \), one has \( t_s = \epsilon^{-1/2} F(u_0) \), where \( F(u_0) \) is a function of the initial data (see, e.g., the formula 3).

Since \( t_2 = N^{-1} \epsilon^{-3/4} t_2^* \), due to the time rescaling 10, we obtain the law

\[ t_{FPU} \lesssim N \epsilon^{-1/2} + \epsilon^{-3/4}; \]

we remark that such an estimate improves as \( N \) grows because in that case the first regime extends very close to the shock time due to the delay in the dispersive effect. Such a scaling fits with the experiments [2, Fig. 8].

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