Performance of two redundant quantum channels for single qubits under indefinite causal order

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Abstract. Indefinite causal order has introduced novel procedures to improve the quality of quantum communication. The superposition of paths on a set of consecutive quantum channels has demonstrated enhancement in communication on some of them. This work models a generic quantum channel for single qubits in terms of Kraus operators in the form of Pauli operators. The output state going through the channel can be obtained analytically to then analyse the quality by means of the quantum fidelity.

1. Introduction
Quantum communication exploits quantum channels to maintain or transport quantum states being used in quantum processing. When such channels become imperfect or noisy, information becomes transmitted imperfectly. If transmission is recursively transmitted through similar imperfect channels the quality of information worsens. By introducing the concept of indefinite causal order (ICO), a quantum treat to superpose the order in which quantum information goes through the channels, enhancement has been shown for channels as the depolarizing [1], the teleportation [2] and the dephasing ones [3]. Then, surprisingly we can transmit information with higher quality compared with the redundant definite causal order case.

The aim of this work is to analyse an certain type of noisy channels for a single qubit when two channels are in ICO aided by measurement. Second section presents relevant notation to describe the channel. Third section presents the analysis for the causal redundant case. Fourth section analyse the application of ICO using two channels. Last section sets the conclusions.

2. Noisy channels for a single qubit in terms of Pauli operators
In quantum information, a quantum channel is a medium through a quantum system is transmitted before its usage. Its quantum information is transferred (not necessarily moved) or inclusively teleported to meet certain processing together with other systems. Channel could modify the quantum information in the original state planned to be transmitted due to noise or decoherence. Thus, each channel is considered as a completely positive trace-preserving map (CPTP) between two spaces of operators in order to maintain the properties of the density matrix representing the quantum information involved [4]. In this work, we restrict the state to single quits (dimension 2) whose density matrix has the general form:

\[ \rho = \frac{1}{2}(\sigma_0 + \vec{n} \cdot \vec{\sigma}) \]
in terms of the Pauli operators \( \sigma_0, \sigma = (\sigma_1, \sigma_2, \sigma_3) \). There, \( \vec{n} \) is three-dimensional vector fulfilling \( |\vec{n}| \leq 1 \). If \( |\vec{n}| = 1 \), we have a pure state \( \rho = |\psi\rangle \langle \psi| \) and otherwise a mixed state. In addition, we introduce the form for the output state of a CPTP map through its Kraus operators [5], a set of operators \( K_i, i = 0, 1, 2, 3 \) fulfilling \( \sum_{i=0}^{3} K_i^\dagger K_i = \sigma_0 \) to preserve the unitary trace of \( \Lambda[\rho] \).

If the channel action is local with classical communication (LOCC), it could be expressed in terms of unitary operators \( U_i \) as \( K_i = \sqrt{\alpha_i} U_i \) with \( \sum_{i=0}^{3} \alpha_i = 1 \). A basis change could transform \( K_i \). For qubits, it is easy to solve the linear system \( \sigma_i = TU_i T^\dagger, i = 0, 1, 2, 3 \) to find \( T \) transforming \( \{U_i\} \) into \( \{\sigma_i\} \), the generators of \( su(2) \) algebra together with the identity [6]. Then, for a LOCC, a representation of Kraus operators is: \( K_i = \sqrt{\alpha_i} \sigma_i \). Such representation is easy to manage due to its algebraic properties. Such expressions for the channel output become:

\[
\Lambda[\rho] = \sum_{i=0}^{3} K_i \rho K_i^\dagger \quad \rightarrow \quad \Lambda[\rho] = \sum_{i=0}^{3} \alpha_i \sigma_i \rho \sigma_i^\dagger \quad (2)
\]

3. Redundant usage of two noisy channels in a definite causal order

The fidelity on a noisy channel as depicted in (2) could be analysed for two redundant identical noisy channels as a composition of (2):

\[
(\bigcirc_2 \Lambda)[\rho] \equiv \Lambda[\Lambda[\rho]] = \sum_{i_1,i_2=0}^{3} \alpha_{i_1} \alpha_{i_2} \sigma_i \sigma_j \rho \sigma_i^\dagger \sigma_j^\dagger \quad (3)
\]

Analysing the case \( \alpha_1 = \alpha_2 = \alpha_3 \equiv p, \quad 0 \leq p \leq \frac{1}{3} \) to avoid lots of parameters, the fidelity (assuming \( \rho \) is restricted to pure states) \( F_{\bigcirc N \Lambda} \equiv \text{Tr}(\rho(\bigcirc_N \Lambda)[\rho]) \) for \( N = 1, 2 \) becomes:

\[
F_{\bigcirc 1 \Lambda} = 1 - 2p \quad \text{and} \quad F_{\bigcirc 2 \Lambda} = 1 - 4p + 8p^2 \quad (4)
\]

Figure 1a shows the fidelity for both cases as function of \( p \). Note the crossing in \( F_{\bigcirc 2 \Lambda} = \frac{1}{2} \), the value for the fidelity for the depolarizing channel (black dotted line) in \( p = \frac{1}{4} \) and certain improvement after such value for \( N = 2 \).

4. Two noisy channels in an indefinite causal order

The application of two channels in ICO, requires an additional state to control the ICO:

\[
\rho_c = |\psi_c\rangle \langle \psi_c| = \sum_{i,j=0}^{1} \sqrt{q_i q_j} |i\rangle \langle j| \quad (5)
\]

with \( q_1 = 1 - q_0 \). Figure 2 shows two noisy channels a) - b) in a definite causal order as function of the control state, \( C_1 \) first and then \( C_2 \) if \( |\psi_c\rangle = |0\rangle \) or \( C_2 \) first and then \( C_1 \) if \( |\psi_c\rangle = |1\rangle \); and c) in an arrangement of ICO if \( |\psi_c\rangle \) is a superposition of the two last control states as in (5).

The output states are constructed by considering the Kraus operators of the whole process in terms of the single channel Kraus operators \( W_{i_1,i_2} = K_{i_1} K_{i_2} \otimes |0\rangle_c \langle 0| + K_{i_2} K_{i_1} \otimes |1\rangle_c \langle 1| \) the output for the two channels in ICO \( \Lambda^2[\rho \otimes \rho_c] \) becomes:

\[
\Lambda^2[\rho \otimes \rho_c] = \sum_{i_1,i_2,j=1}^{2} \alpha_{i_j} (q_0 |0\rangle \langle 0| \otimes \sigma_{i_1} \sigma_{i_2} \rho \sigma_{i_2} \sigma_{i_1} + \sqrt{q_0 q_1} |0\rangle \langle 1| \otimes \sigma_{i_1} \sigma_{i_2} \rho \sigma_{i_1} \sigma_{i_2} + q_1 |1\rangle \langle 1| \otimes \sigma_{i_2} \sigma_{i_1} \rho \sigma_{i_1} \sigma_{i_2}) \quad (6)
\]
Figure 1. Fidelity for a) the sequential case ($N = 1, 2$) and the two channels in ICO $N = 2$ for $\alpha_1 = \alpha_2 = \alpha_3 = p$, and b) the general case of two channels in ICO in color inside of the allowed region for $\{\alpha_i | i = 1, 2, 3\}$ and the corresponding statistical distribution for $P_m$.

where $K_j = \sqrt{\alpha_j \sigma_j}$ as in Section 2. Now, we note that control state is entangled with the state being transmitted. We can trace, as in [1], the control state on $\Lambda^2[\rho \otimes \rho_c]$ to calculate the Holevo bound [4, 7] or otherwise to project the control state via measurement on a convenient orthonormal basis $B = \{\psi_m\} = \{\psi_m^\perp\} = \{\psi_m^\perp\}$ where only the first state pretends the enhancement of the transmitted state through the noisy channel. This procedure has been suggested by [2] analyzing this kind of channels in teleportation, combining ICO with projective measurements. We follow this idea here, then, by projecting (6) on $\psi_m$ (it implies not always we will get the enhanced state, instead with certain probability $P_m$) to get the un-normalized output $\Lambda^2[\rho] \equiv \langle \psi_m | \Lambda^2[\rho \otimes \rho_c] | \psi_m \rangle$:

$$\Lambda^2[\rho] = \sum_{i,j=0}^{3} \alpha_i \alpha_j \left( \left( \frac{1}{2} + \left( q_0 - \frac{1}{2} \right) \cos \theta \right) \sigma_i \sigma_j \rho \sigma_j \sigma_i + \sqrt{q_0 q_1} \sin \theta \cos \phi \sigma_i \sigma_j \rho \sigma_i \sigma_j \right) \right)$$

Figure 2. a) - b) Two communication channels in a definite causal order as function of the control state $|0\rangle$ or $|1\rangle$, and c) in ICO as a superposition of the previous cases.
and using the expressions for the probability of success and the fidelity: $P_m = \text{Tr}(\Lambda^2[\rho])$ and $F_2 = P_m^{-1}\text{Tr}(\rho \Lambda^2[\rho])$ (assuming $\rho$ is a pure state), we get the function $G_2^k$:

$$G_2^k = \frac{3}{i,j=0} \alpha_i \alpha_j \left( \frac{1}{2} + (q_0 - \frac{1}{2}) \cos \theta \right) S_{i,j}^k + \sqrt{q_0 q_1} \sin \theta \cos \phi T_{i,j}^k \right) \rightarrow P_m = G_2^0, F_2 = G_2^0 \quad (8)$$

where: $S_{i,j}^k = \text{Tr}(\rho^k \sigma_i \sigma_j \rho \sigma_i \sigma_j)$ and $T_{i,j}^k = \text{Tr}(\rho^k \sigma_i \sigma_j \rho \sigma_i \sigma_j)$, which have interesting properties of reduction when we introduce (1).

It has been demonstrated in teleportation for the case $\alpha_i = p, i=1,2,3$ [2, 8] that despite there are solutions for all $\theta, \phi$ in the maximization of $F_2 = 1$, the maximum obtained together for $P_m = \frac{1}{3}$ occurs if $|\psi_m\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ ($\theta = \frac{\pi}{4}, \phi = 0$). In addition, those outcomes are independent from the state being transmitted. Then, the same is true for LOOC channels. In fact, (8) for $F_2$ for the case $\alpha_1 = \alpha_2 = \alpha_3 = p, \ 0 \leq p \leq \frac{1}{3}$ becomes:

$$P_m = 1 - 6p^2 \quad , \quad F_2 = \frac{1 - 4p + 6p^2}{1 - 6p^2} \quad (9)$$

$F_2$ is shown in the Figure 1a (blue line) together with the sequential cases. Notably, apart from the obvious case $\alpha_0 = 1$, the case $\alpha_0 = 0$ exhibits a perfect transmission, $\rho_{out} = \text{Tr}_m^{-1}\Lambda^2[\rho] = \rho$ (unfortunately $P_m$ reduces when $p$ rises).

More generally, we can numerically optimize (7) on $\theta, \phi, q_0$ ranges through the allowed region for $\alpha_i, i=1,2,3$ to get the best $F_2$ (Figure 1b in color) showing that the worst case corresponds to the previous case $\alpha_1 = \alpha_2 = \alpha_3 = p, \ 0 \leq p \leq \frac{1}{3}$ (on the central line of that region) for $p = \frac{1}{3}(3 - \sqrt{3})$ with $F_2 = 1/\sqrt{3} \approx 0.58$, the minimum of $F_2$ in (9). In this case, $P_m$ exhibits a peaked symmetric statistical distribution around its mean $P_m = \frac{1}{3}$ (upper inset). Moreover, it is easy to demonstrate if $q_0 = \frac{1}{3}(1 - \cos \theta)$ then on the frontal face $\alpha_0 = 0$: $F_2 = 1, P_m = \sum_{i=1}^{3} \alpha_i^2$.

5. Conclusions

In this work we have characterized quantum channels in terms of the Pauli operators. We made an analysis on the application of two channels both sequentially and in ICO supported by measurement. The fidelity was obtained for the process showing advantages in the applying of ICO regarding the redundant scenario. It was shown that a perfect fidelity in a noisy channel can be reached by applying two of them in ICO, nevertheless, the probability of measurement is not outstanding. An analysis regarding the use of more than two channels in ICO is required to seek the improvement in the success probability but maintaining the perfect fidelity.

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