Rankin-Cohen Operators for Jacobi and Siegel Forms

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Abstract

For any non-negative integer \( v \) we construct explicitly \( \lfloor \frac{v}{2} \rfloor + 1 \) independent covariant bilinear differential operators from \( J_{k,m} \times J_{k',m'} \) to \( J_{k+k'+v,m+m'} \). As an application we construct a covariant bilinear differential operator mapping \( S_k^{(2)} \times S_{k'}^{(2)} \) to \( S_{k+k'+v}^{(2)} \). Here \( J_{k,m} \) denotes the space of Jacobi forms of weight \( k \) and index \( m \) and \( S_k^{(2)} \) the space of Siegel modular forms of degree 2 and weight \( k \). The covariant bilinear differential operators constructed are analogous to operators already studied in the elliptic case by R. Rankin and H. Cohen and we call them Rankin-Cohen operators.

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1 Introduction and results

It has been known for some time how to obtain an elliptic modular form from the derivatives of $N$ elliptic modular forms. The case $N = 1$ has already been studied in detail by R. Rankin in 1956 [10]. For $N = 2$ H. Cohen has constructed certain covariant bilinear operators which he used to obtain modular forms with interesting Fourier coefficients [5]. Later, these operators were called Rankin-Cohen operators by D. Zagier who studied their algebraic relations [11].

The main result of this paper is the explicit description of covariant bilinear operators for Jacobi forms and Siegel modular forms of degree 2. Since they are generalisations of the Rankin-Cohen operators in the elliptic case we will also call them Rankin-Cohen operators.

The main theorem reads

**Theorem 1.1** Let $f$ and $f'$ be Jacobi forms of weight and index $k,m$ and $k',m'$, respectively. For any $X \in \mathbb{C}$ and any non-negative integer $v$ define

$$[f, f']_{X,v} = \sum_{r+s+p \leq v} \sum_{i+j=v-2|v/2|} C_{r,s,p}(k,k') \cdot D_{r,s,i,j}(m,m',X) \cdot \left( L_m^p(\partial^i_x f) L_{m'}^{i'}(\partial^j_x f') \right)$$

where

$$D_{r,s,i,j}(m,m',X) = m'^i (-m')^j (1 + mX)^r (1 - m'X)^s,$$

$$C_{r,s,p}(k,k') = \frac{(\alpha + r + s + p)_{s+p}}{r!} \cdot \frac{(\beta + r + s + p)_{r+p}}{s!} \cdot \frac{(-\gamma + r + s + p)_{r+s}}{p!},$$

where $(x)_m = \prod_{0 \leq i \leq m-1} (x-i)$, $[x]$ denotes the largest integer $\leq x$, and, where $L_m(f) = (8\pi i m \partial f_\tau - \partial^2 f_{\tau})/2$ for $f$ a Jacobi form of index $m$.

Then $[f, f']_{X,v}$ is a Jacobi form of weight $k + k' + v$ and index $m + m'$ and, even more, a Jacobi cusp form for $v > 1$.

Let us remark that some special cases of Theorem 1.1 have already been considered in the literature: Firstly, the bilinear operator $[\cdot, \cdot]_{X,1}$, which actually does not depend on $X$, has already been shown to map two Jacobi forms to a Jacobi form [7, Theorem 9.5]. Secondly, one bilinear operator for each even $v$ has already been constructed in ref. [2]: up to a scalar multiple this operator is equal to

$$\left( \frac{d}{dX} \right)^{v/2} [f, f']_{X,v} \quad (v \in 2\mathbb{N} = \{0, 2 \ldots \}).$$

For fixed $v$ and $k,m$ and $k',m'$ large enough the operators $[\cdot, \cdot]_{X,v}$ ($X \in \mathbb{C}$) span a vector space of dimension $\left\lceil \frac{v}{2} \right\rceil + 1$. This shows that the space of such Rankin-Cohen operators is, in general, at least $\left\lceil \frac{v}{2} \right\rceil + 1$ dimensional. A result of Böcherer [3], obtained by using Maaß operators, shows that this dimension actually equals $\left\lceil \frac{v}{2} \right\rceil + 1$ in general (cf. Theorem 2.3).
One of the applications of our result is to Siegel modular forms of degree 2: the bilinear operators $[,]_{0,v}$ with even $v$ can be lifted to bilinear covariant differential operators for Siegel modular forms of degree 2.

More precisely one has the following theorem.

**Theorem 1.2** Let $F$ and $F'$ be Siegel modular forms of degree 2 and weight $k$ and $k'$, respectively. Define, for any non-negative integer $l$,

$$[F,F']_l = \sum_{r+s+p=l} C_{r,s,p}(k,k') \mathbb{D}^p(D^r(F) D^s(F'))$$

with $C_{r,s,p}(k,k')$ as in Theorem 1.1 and $\alpha = k - 3/2$, $\beta = k' - 3/2$, $\gamma = k + k' - 3/2$ and

where $\mathbb{D} = \frac{\partial^2}{\partial \tau_1 \partial \tau_2} - \frac{\partial^2}{\partial z^2}$ with $Z = \left( \begin{array}{cc} \tau_1 & z \\ z & \tau_2 \end{array} \right)$ the variable in $\mathbb{H}_2$.

Then $[F,F']_l$ is a Siegel modular form of degree 2 and weight $k + k' + 2l$ and, even more, a Siegel cusp form for $l > 0$.

This paper is organised as follows. In §2 we recall some results on Jacobi forms which will be needed in the proof of Theorems 1.1. Section 3 contains the definition of Siegel modular forms and their relation to Jacobi forms in the degree 2 case. In §4 we give a (combinatoric) proof of Theorem 1.1 and §5 contains a proof using generating functions. In §6 we prove Theorem 1.2 using Theorem 1.1 and in §7 we give a second, independent proof using theta series with spherical coefficients and a general result of Ibukiyama (cf. Theorem 3.6). We conclude with several remarks and some open questions in section §8. In particular we discuss the uniqueness of the Rankin-Cohen operators for Siegel modular forms of degree 2.

## 2 Jacobi forms

In this section we recall a few general results about Jacobi forms. We first give the definition of Jacobi forms and the heat operator (as a general reference for Jacobi forms we refer to [7]).

Denote by $\mathbb{H}$ the complex upper half plane and define, for holomorphic functions $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ and integers $k$ and $m$,

$$(f|_{k,m}M)(\tau, z) = (c\tau + d)^{-k} e^{2\pi im\frac{cz}{c\tau + d}} f \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right),$$

$$(f|_mY)(\tau, z) = e^{2\pi im(\lambda^2 \tau + 2\lambda z)} f(\tau, z + \lambda \tau + \nu)$$

where $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})$ and $Y = (\lambda, \nu) \in \mathbb{Z}^2$.

Using these slash actions the definition of Jacobi forms is as follows.

**Definition 2.1** A Jacobi form of weight $k$ and index $m$ $(k, m \in \mathbb{N})$ is a holomorphic function $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying

$$(f|_{k,m}M)(\tau, z) = f(\tau, z), \quad (f|_mY)(\tau, z) = f(\tau, z)$$

$$\text{for } \tau \in \mathbb{H}, \ z \in \mathbb{C}, \ M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}) \text{ and } Y = (\lambda, \nu) \in \mathbb{Z}^2.$$
for all $M \in SL(2, \mathbb{Z})$ and $Y \in \mathbb{Z}^2$ and such that it has a Fourier expansion of the form

$$f(\tau, z) = \sum_{n = 0}^{\infty} c(n, r)q^n \zeta^r, \quad r \in \mathbb{Z}, r^2 \leq 4nm$$

where $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi iz}$. If $f$ has a Fourier expansion of the same form but with $r^2 < 4nm$ then $f$ is called a Jacobi cusp form of weight $k$ and index $m$.

We denote by $J_{k,m}$ the (finite dimensional) vector space of all Jacobi forms of weight $k$ and index $m$ and by $J_{k,m}^{cusp}$ the vector space of all Jacobi cusp forms of weight $k$ and index $m$.

Our main result (Theorem 1.1) involves the heat operator which has already been studied in [7] to connect Jacobi forms and elliptic modular forms and in ref. [2, 3] in the context of bilinear differential operators (cf. the remark after Theorem 1.1).

**Definition 2.2** For any non-negative integer $m$ the heat operator $L_m$ is defined by

$$L_m(f) = \left(8\pi im \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right)(f) \quad (f \in J_{k,m}).$$

Finally, let us mention a result of Böcherer [1].

**Theorem 2.3** For fixed $v$ and $k, m, k', m'$ large enough the vector space of all covariant bilinear differential operators mapping $J_{k,m} \times J_{k',m'}$ to $J_{k+k'+v,m+m'}$ has dimension $\lfloor v/2 \rfloor + 1$.

Note that Theorem 1.1 describes a basis of this space explicitly.

### 3 Siegel modular forms

In this section we recall a few basic facts about Siegel modular forms and, in particular, the construction of Siegel modular forms using theta series with spherical coefficients (Theorem 3.5). Furthermore, we describe the connection between Siegel modular forms of degree 2 and Jacobi forms (Theorem 3.2). Finally, we mention (a special case of) a result of Ibukiyama (Theorem 3.6) which will be needed in the proof of Theorem 3.7 in section 5. The reader may take ref. [8] as a general reference for Siegel modular forms.

For any holomorphic function $f$ on the Siegel upper half plane $\mathbb{H}_n$, i.e. the space of complex symmetric $n \times n$ matrices with positive definite imaginary part, and $M \in \text{Sp}(2n, \mathbb{Z})$ define

$$(f|_M)(Z) = f(MZ) \det(CZ + D)^{-k}$$

where $Z \in \mathbb{H}_n$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $n \times n$ matrices $A, B, C, D$, and, where $MZ = (AZ + B)(CZ + D)^{-1}$.

Then the definition of Siegel modular forms is given as follows.
Definition 3.1 Let $n$ be a positive integer greater than 1. Then a holomorphic function $f$ on $\mathbb{H}_n$ is called a Siegel modular form of degree $n$ and weight $k$ if

$$(f|_M^k)(Z) = f(Z)$$

for all $M \in \text{Sp}(2n, \mathbb{Z})$.

We denote the space of all Siegel modular forms of degree $n$ and weight $k$ by $S(n)_k$.

Note that for $n = 1$ one has to add a further condition on $f$ in order to obtain the usual definition of modular forms.

The connection between Jacobi forms and Siegel modular forms of degree two becomes clear by the following theorem (see e.g. [7, Theorem 6.1]).

Theorem 3.2 Let $F$ be a Siegel modular form of degree 2 and weight $k$ and write the Fourier development of $F$ with respect to $\tau_2$ in the form

$$F(Z) = F(\tau_1, z, \tau_2) = \sum_{m=0}^{\infty} f_m(\tau_1, z)e^{2\pi im\tau_2}$$

with $Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix}$ the variable in $\mathbb{H}_2$.

Then, for each non-negative integer $m$, the function $f_m$ is a Jacobi form of weight $k$ and index $m$.

Remark 3.3 Note that for general degree $n$ a Siegel cusp form is a Siegel modular form which is contained in the kernel of the Siegel operator $\Phi$ (for more details see e.g. [8]). In the case of degree 2 a Siegel cusp form $F$ is a Siegel modular form whose Jacobi-Fourier expansion is of the form $F = \sum_{m>0} f_m(\tau_1, z)e^{2\pi im\tau_2}$, i.e. the first coefficient in the Jacobi-Fourier expansion is identically zero.

To recall some facts about theta series with spherical coefficients we introduce the notion of spherical polynomials first.

Definition 3.4 A spherical polynomial $P$ of weight $w$ in a matrix variable $X \in M_{m,n}$ is a polynomial (in the matrix elements of $X$, i.e. a polynomial in $\mathbb{C}[x_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$) satisfying

1) $P(XA) = \det(A)^w P(X)$ for all $A \in M_{n,n}$,

2) $\Delta P = \sum_{i,j} (\partial^2_{x_{ij}}) P = 0$.

Then one has (see e.g. [8, p. 161])

Theorem 3.5 Let $P$ be a spherical polynomial of degree $k$ in a matrix variable $X \in M_{m,n}$ and let $S \in M_{m,m}(\mathbb{Z})$ be a symmetric, positive, even and unimodular matrix. Then the function

$$\theta_{S,P}(Z) = \sum_{G \in M_{m,n}(\mathbb{Z})} P(S^{1/2}G)e^{\pi i \text{tr}(G^tSGZ)}$$

is a Siegel modular form of degree $n$ and weight $m/2 + k$. 

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Finally, let us mention (a special case of) a result of Ibukiyama (Corollary 2 (2) of ref. [2] with \( r = 2 \)) which we use to prove Theorem 1.2 in §4.

**Theorem 3.6** Let \( P \) be a spherical polynomial of even weight \( d \) in the matrix variable \((X, X')^t \in M_{m+m',n}\) which can be written as \( P(X, X') = \tilde{Q}(X^t X, X'^t X') \) for some polynomial \( \tilde{Q} \). Set \( D = \tilde{Q}(\delta_{\nu,\mu}, \delta'_{\nu',\mu'}) \) where \( \delta_{\nu,\mu} = (1+\delta_{\nu,\mu})\frac{\partial}{\partial z^\nu} \) and \( \delta'_{\nu',\mu'} = (1+\delta_{\nu',\mu'})\frac{\partial}{\partial z'^{\nu'}} \) with \( Z = (z_{\nu,\mu}) \) and \( Z' = (z'_{\nu',\mu'}) \) for \( 1 \leq \nu, \mu, \nu', \mu' \leq n \). Then, for any two Siegel modular forms \( F(Z) \) and \( F'(Z') \) of degree \( n \) and weight \( k + k' + d \), respectively, the function \( D(F(Z)F'(Z'))|_{Z=Z'} \) is a Siegel modular form of degree \( n \) and weight \( k + k' + d \).

Note that Theorem 3.6 essentially says that if a bilinear differential operator maps all pairs of theta series to theta series with spherical coefficients then it even maps all pairs of Siegel modular forms to Siegel modular forms (cf. the discussion at the end of §6).

### 4 A combinatorial proof of Theorem 1.1

Let us now give the proof of our main theorem which will, in particular, imply Theorem 1.2 (see §3 for details).

To prepare the proof of Theorem 1.1 we need the following lemmata.

**Lemma 4.1** Let \( f \) be a holomorphic function on \( \mathbb{H} \) and \( g \) a holomorphic function on \( \mathbb{H} \times \mathbb{C} \). Then, for each non-negative integer \( r \), one has

\[
 L_r^m(fg) = \sum_{j=0}^{r} (8\pi im)^{r-j} \binom{r}{j} \left( \partial_{\tau}^{r-j} f \right) (L^j_m g)
\]

where \( \tau \) is the variable in \( \mathbb{H} \).

**Proof.** We prove the formula by induction. Firstly, note that for \( r = 1 \) one has \( L_1(fg) = (8\pi im\partial_{\tau} - \partial_{\tau}^2)(fg) = 8\pi im(\partial_{\tau} f)g + 8\pi imf(\partial_{\tau} g) - f \partial_{\tau}^2 g = 8\pi im(\partial_{\tau} f)g + fL_1 g \).

Secondly, assume that the formula is valid for some \( r \). Then we find

\[
 L_{m+1}^r(fg) = L_m \left( \sum_{j=0}^{r} (8\pi im)^{r-j} \binom{r}{j} \left( \partial_{\tau}^{r-j} f \right) (L^j_m g) \right)
 = \sum_{j=0}^{r} \left( (8\pi im)^{r-j+1} \binom{r}{j} \left( \partial_{\tau}^{r-j+1} f \right) (L^j_m g) \right) + \left( (8\pi im)^{r-j} \binom{r}{j} \left( \partial_{\tau}^{r-j} f \right) (L^{j+1}_m g) \right)
 = \sum_{j=0}^{r+1} (8\pi im)^{r+1-j} \binom{r+1}{j} \left( \partial_{\tau}^{r+1-j} f \right) (L^j_m g)
\]

and the lemma becomes obvious. \( \Box \)

The second lemma we will need is Lemma 3.1 of ref. [3] (note that the normalisation of \( L_m \) in loc. cit. differs from ours by a factor of \((2\pi i)^2\)).
Lemma 4.2 Let $f$ be a holomorphic function on $\mathbb{H} \times \mathbb{C}$. Then, for any non-negative integer $r$, one has

$$L^r_m(f)_{k+2r,m} M = \sum_{j=0}^{r} (8\pi m)^{r-j} \frac{r!}{j!} \left( \frac{k - 3/2 + r}{c + d} \right)^{r-j} L^j_m(f|_{k,m} M)$$

for all $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2,\mathbb{Z})$.

Furthermore, one has

$$L_m(f)_{m} Y = L_m(f|_{m} Y)$$

for all $Y \in \mathbb{Z}^2$.

Proof. The proof of the first formula is a simple exercise and can be found in [2]. The second formula becomes obvious after a short calculation using only the definition of the $|mY$ action.

The third lemma is essentially equivalent to Theorem 9.5 of ref. [7].

Lemma 4.3 Let $f$ and $f'$ be holomorphic functions on $\mathbb{H} \times \mathbb{C}$. Then, for $z \in \mathbb{C}$, one has

$$(m'(\partial_z f') - mf (\partial_z f'))|_{k+k'+1,m+m'} M = m'(\partial_z (f|_{k,m} M) - m(f|_{k,m} M) (\partial_z f')|_{k',m'} M)$$

$$(m'(\partial_z f') - mf (\partial_z f'))|_{m+m'} Y = m'(\partial_z (f|_{m} Y) - m(f|_{m} Y) (\partial_z f)|_{m'} Y)$$

for all $M \in \text{SL}(2,\mathbb{Z})$ and all $Y \in \mathbb{Z}^2$.

Proof. The two formulas can be obtained by a straightforward calculation.

We are now ready to prove our main theorem.

Proof of Theorem 1.1. The definition of Jacobi forms contains essentially two parts: the invariance under the slash actions and an expansion condition. If $f$ and $f'$ are Jacobi forms then a simple computation shows that the latter condition is always satisfied for $[f,f']_{X,v}$ and, even more, that $[f,f']_{X,v}$ satisfies the expansion condition of a Jacobi cusp form for $v > 1$. Therefore, it only remains to check the invariance under the slash actions.

We consider first the case of even $v = 2v'$. Since, by Lemma 4.2, the $|mY$ action commutes with $L_m$ we only have to show that

$$([f,g]_{X,v})|_{k+k'+v,m+m'} M = [f|_{k,m} M, g|_{k',m'} M]_{X,v}$$

for all $M \in \text{SL}(2,\mathbb{Z})$. Using Lemma 4.1 and 4.3 to calculate the left hand side we obtain that this equation is equivalent to

$$\sum_{r+s+p=v'} C_{r,s,p} r! s! p! \left( \frac{\alpha + r'}{\beta + s} \frac{\beta + s}{\gamma + v' + j'} \frac{\gamma + v' + j'}{p - j} \right) \times (1 + m X)^{s} (1 - m X)^{r} m^{r-j'} m^{s+p-j'-j''} (1 + m/m')^{p-j} = \delta_{v'+j'+v''} C_{v',j'+v''} (1 + m X)^{v''} (1 - m X)^{j'}.$$
Some simple manipulations show that this equation is equivalent to

\[
\sum_{r+s+p=v'} \frac{(v'-j'!)(v'-j''!)(v'-p)!}{(r-j'!)(s-j''!)(v'-j)!} \left( \frac{-(\gamma + v')}{v' - p} \right) \left( \frac{\gamma + v' + j' + j''}{p - j} \right) \\
\times m^{r-j'} m^{s+p-j''} (1 + m/m')^{p-j} (1 + mX)^{s-j'} (1 - m'X)^{r-j''}
\]

\[= \delta_{v',j'+j''} \left( \frac{-(\gamma + v')}{v' - j} \right). \]

To show this equality we view it as an equation between polynomials in \( \gamma \) (of degree \( v' - j \)). It is easy to see that both sides coincide for \( \gamma + v' + j' + j'' = 0 \). Therefore, it is enough to show that both sides agree for all \( \gamma = -v' - x \) (\( 0 \leq x < v' - j \)). Note that for these values of \( \gamma \) the right hand side obviously vanishes. With \( A = \frac{m(1-m'X)}{m(1+mX)} \) the left hand side becomes

\[
c \sum_{r,p} \left( \frac{v'-p-j'-j''}{r-j'} \right) \left( \frac{x-j'-j''}{v'-p-j'-j''} \right) \left( \frac{j'+j''-x}{p-j} \right) A^r (A+1)^{p-j} \\
= c' \sum_{r,v'} \sum_{r,p} \left( \frac{j'+j''-v'}{x+p-v'} \right) \left( \frac{v'-j-j''-v'}{v'-r-p-j''} \right) \left( \frac{v'-j'-j''}{r-j'} \right) A^r
\]

for some (non-zero) factors \( c \) and \( c' \). The product of the three binomial coefficients is the factor in front of

\[Z^{x+p-v'} \cdot Z^{v'-r-p-j''} \cdot Z^{r-j'} = Z^{x-j'-j''} \]

in

\[(1 + Z)^{j'+j''-v'} \cdot (1 + Z)^{v'-j-j''-v'} \cdot (1 + Z)^{v'-j'} = 1.\]

Hence we find that our expression is zero unless \( x = j' + j'' \). However, in this case our expression can only be non-zero if \( v' = j + j' + j'' \) which is not allowed since \( x = j' + j'' \) has to be strictly less than \( v' - j = j' + j'' = x \). Hence also the left hand side is equal to zero for \( \gamma = -v' - x \) (\( 0 \leq x < v' - j \)) so that we have proved the desired equality. This proves the theorem for even \( v = 2v' \).

For odd \( v = 2v' + 1 \) note that the summand for fixed \( r,s,p \) is equal to

\[C_{r,s,p}(1 + mX)^s (1 - m'X)^s L_{m+m'}^p \left( (m' \partial_z L_{m'}^s (f)) L_{m'}^s (f') - mL_{m'}^s (f) (\partial_z L_{m'}^s (f')) \right). \]

The expression inside the \( L_{m+m'}^p \) is exactly of the form considered in Lemma \[4.3\] so that we obtain

\[((f, f')|_{X,v})|_{m+m'} Y = [f|_{m'} Y, f'|_{m'} Y]|_{X,v}.\]

Therefore, it only remains to show the invariance with respect to the other slash action as in the even case. Note, however, that using Lemma \[4.3\] we have to check exactly the same combinatorial identity as in the case of even \( v \) but with \( \gamma = k + k' - 1/2 \) instead of \( \gamma = k + k' - 3/2 \). This completes the proof of Theorem \[4.1\]. \( \square \)
5 A proof of Theorem 1.1 using generating functions

In this section we give a second, independent proof of Theorem 1.1. Instead of proving the theorem directly we use some results of ref. [3] (Theorem 3.1 and Corollary 3.1) on generating functions. Let us first recall these results.

**Theorem 5.1** Let $\tilde{f}(\tau, z; W)$ be a formal power series in $W$, i.e. $\tilde{f}$ can be written as

$$\tilde{f}(\tau, z; W) = \sum_{\nu=0}^{\infty} \chi_{\nu}(\tau, z) W^{\nu},$$

satisfying the functional equation

$$\tilde{f}(a\tau + b, c\tau + d, z; W) = (c\tau + d)^{K} e^{2\pi i M(e^{\frac{x^2}{\tau + y^2}} - e^{\frac{\ell W}{\tau + d}})} \tilde{f}(\tau, z; W)$$

for some integers $K$ and $M$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. Furthermore, assume that the coefficients $\chi_{\nu}$ are holomorphic functions on $\mathbb{H} \times \mathbb{C}$ with a Fourier expansion of the form

$$\chi_{\nu}(\tau, z) = \sum_{\substack{r, n \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) q^n \xi^r \quad (q = e^{2\pi i \tau}, \xi = e^{2\pi i z})$$

satisfying $\chi_{\nu}|_m Y = \chi_{\nu}$ for all $Y \in \mathbb{Z}^2$.

Then, for each non-negative integer $\nu$, the function $\zeta_{\nu}$ defined by

$$\zeta_{\nu}(\tau, z) = \sum_{j=0}^{\nu} \frac{(-(K - 3/2 + \nu))_{\nu-j}}{j!} L_{M}^{2j}(\chi_{\nu-j})$$

is a Jacobi form of weight $K + 2\nu$ and index $M$.

An immediate consequence of this theorem is the following corollary [3, Corollary 3.1].

**Corollary 5.2** Let $f(\tau, z)$ be a Jacobi form of weight $k$ and index $m$. Then

$$\tilde{F}(\tau, z; W) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!(K - 3/2 + \nu)!} L_{M}^{\nu}(f) W^{\nu}$$

satisfies the functional equation

$$\tilde{F}(a\tau + b, c\tau + d, z; W) = (c\tau + d)^{k} e^{2\pi i m(e^{\frac{x^2}{\tau + y^2}} - e^{\frac{\ell W}{\tau + d}})} \tilde{F}(\tau, z; W)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$.

Using these two results we are now able to prove Theorem 1.1.

**Proof of Theorem 1.1** Let $f$ and $f'$ be Jacobi forms of weight and index $k, m$ and $k', m'$, respectively. Denote by $\tilde{f}(\tau, z; W)$ and $\tilde{f}'(\tau, z; W)$ the formal power series associated to $f$ and $f'$ as in Corollary 5.2, respectively. Then, for any fixed complex number $X$, the formal power series $F_{X}(\tau, z; W)$ defined by

$$F_{X}(\tau, z; W) = \tilde{f}(\tau, z; (1 + m' X) W) \tilde{f}'(\tau, z; (1 - m X) W)$$

is a Jacobi form of weight $k + k'$ and index $m + m'$.
satisfies, by Corollary 5.2, the functional equation stated in Theorem 5.1 with \( K = k + k' \) and \( M = m + m' \). Furthermore, it is simple to check that its coefficients satisfy the expansion condition assumed in Theorem 5.1 and, by Lemma 4.2, are invariant under the \( |mY \) action. Hence the function \( \tilde{F}_X(\tau, z; W) \) satisfies all assumptions of Theorem 5.1 so that the corresponding functions \( \zeta_\nu(\tau, z) \) are Jacobi forms of weight \( k + k' + 2\nu \) and index \( m + m' \). It is now a simple exercise to see that these functions are just constant multiples of the Rankin-Cohen operators \([f, f']_{X, 2\nu}\). This proves Theorem 1.1 for even \( v = 2\nu \).

For the case of odd \( v = 2\nu + 1 \) consider the function \( \tilde{G}_X(\tau, z; W) \) defined by

\[
\tilde{G}_X(\tau, z; W) = m'(\partial_z \tilde{f}(\tau, z; (1 + m'X)W)) (\tilde{f}(\tau, z; (1 - mX)W)) - m(\tilde{f}(\tau, z; (1 + m'X)W)) (\partial_z \tilde{f}(\tau, z; (1 - mX)W))
\]

where \( X \) is a fixed complex number. Using again Lemma 4.2 and Corollary 5.2 we find that the function \( \tilde{G}_X(\tau, z; W) \) satisfies the functional equation of Theorem 5.1 with \( K = k + k' + 1 \) and \( M = m + m' \). By the same calculation as in the case of even \( v \) the coefficients of \( \tilde{G}_X(\tau, z; W) \) satisfy the expansion condition and are, by Lemma 4.2, invariant under the \( |m+mY \) action. Therefore, the corresponding functions \( \zeta_\nu(\tau, z) \) are Jacobi forms of weight \( k + k' + 2\nu + 1 \) and index \( m + m' \). These functions are just constant multiples of \([f, f']_{X, 2\nu+1}\). This completes the proof of Theorem 1.1.

\[
\square
\]

6 \quad A proof of Theorem 1.2 using Theorem 1.1

In this section we give a proof of Theorem 1.2 using Theorem 1.1.

Proof of Theorem 1.2. First we recall some well known facts about Siegel modular forms of degree 2 which we need in the proof. Using

\[
Z = \begin{pmatrix} \tau_1 & z & \tau_2 \\ z & \tau_2 \end{pmatrix}
\]

for a variable in \( \mathbb{H}_2 \) the \( \left| \frac{k}{M} \right| \) action for the whole group \( \text{Sp}(4, \mathbb{Z}) \) (introduced in \( \S 3 \)) is generated by the following three slash actions (see \( \text{e.g.} \) page 73 of [4]):

\[
f|_A(\tau_1, z, \tau_2) = (c\tau_1 + d)^{-k} f(a\tau_1 + b, c\tau_1 + d, \frac{z}{c\tau_1 + d}, \tau_2 - \frac{cz^2}{c\tau_1 + d})
\]

\[
f|_{(\lambda, \nu)}(\tau_1, z, \tau_2) = f(\tau_1, z + \lambda\tau_1 + \nu, \tau_2 + 2\lambda z + \lambda^2 \tau_1)
\]

\[
f|_I(\tau_1, z, \tau_2) = f(\tau_2, z, \tau_1)
\]

where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is in \( \text{SL}(2, \mathbb{Z}) \) and \( \lambda, \nu \in \mathbb{Z} \). Here we have identified, for any function \( f \) on \( \mathbb{H}_2 \), \( f(\tau_1, z, \tau_2) \) with \( f(Z) \).

For the proof of the theorem note that a simple calculation, using only the definition of the slash actions, shows that the operator \( D \) commutes with the \( |_{(\lambda, \nu)} \) and \( |_I \) action, respectively. Secondly, it is essential to realise that for a Jacobi form \( f \) of index \( m \)

\[
D(fq^m) = L_m(f)q^m
\]
where $\tilde{q} = e^{2\pi i \tau_2}$ and $f$ is considered to be a function of $\tau_1$ and $z$.

Consider now two Siegel modular forms $F$ and $F'$ of weight $k$ and $k'$, respectively. By Theorem 3.2 we can write $F$ and $F'$ as

$$F = \sum_{m \geq 0} f_m \tilde{q}^m, \quad F' = \sum_{m' \geq 0} f_m' \tilde{q}^{m'}$$

where $f_m$ and $f_m'$ are Jacobi forms of weight $k$ and $k'$ and index $m$ and $m'$, respectively. Then

$$[F, F']_l = \sum_{m, m' \geq 0} [f_m, f_m']_{X=0,2l} \tilde{q}^{m+m'}.$$  

By Theorem 1.1 the functions $[f_m, f_m']_{X=0,2l}$ are Jacobi forms of weight $k + k' + 2l$ and index $m + m'$ so that the right hand side looks like the expansion of a Siegel form of degree 2 and of weight $k + k' + 2l$. In order to show that this is indeed the case the only additional property to be checked is that $[F, F']_l$ is symmetric in $\tau_1$ and $\tau_2$. This, however, is obvious from the definition of $[F, F']_l$ and the fact that $F$ and $F'$ are themselves Siegel modular forms. Hence, we have proved that $[\cdot, \cdot]_l$ maps $S_k^{(2)} \times S_{k'}^{(2)}$ to $S_{k+k'+2l}^{(2)}$. Finally, note that the first coefficient in the Jacobi-Fourier expansion of $[F, F']_l$ is $[f_0, f_0']_{X=0,2l}$ which is identically zero for $l > 0$. This implies that the image of $[\cdot, \cdot]_l$ is contained in the space of Siegel cusp forms for $l > 0$ (cf. Remark 3.3).

\[\square\]

## 7 A proof of Theorem 1.2 using theta series

In this section we give a proof of Theorem 1.2 using theta series with spherical coefficients and Theorem 3.6. This proof also shows that the Rankin-Cohen operator $[\cdot, \cdot]_l$ is (up to multiplication by a constant) the only covariant bilinear differential operator for Siegel modular forms of degree 2 which can be written in terms of the differential operator $D$.

The proof does not use Theorem 1.1 but instead Theorem 3.6 which essentially allows to assume that $F$ and $F'$ both can be written as theta series with harmonic coefficients. The proof is of independent interest since part of the calculations are valid for arbitrary degree $n$ of the Siegel modular forms involved.

Firstly, we define the differential operator $D$ for general degree $n$

$$D = \det(\partial_{\nu,\mu})$$

where $\partial_{\nu,\mu} = (1 + \delta_{\nu,\mu}) \frac{\partial}{\partial z_{\nu,\mu}}$ with $Z = (z_{\nu,\mu})$ the usual variable in the Siegel half plane $\mathbb{H}_n$. This operator has already been considered in the context of Siegel modular forms (see e.g. [8]).

Secondly, we study the action of $D$ on theta series.

**Lemma 7.1** Let $A$ be a symmetric matrix in $M_{n,n}$ and $Z$ a variable in $\mathbb{H}_n$. Then one has

$$D \left( e^{\pi i \text{tr}(AZ)} \right) = (2\pi i)^n \det(A) \ e^{\pi i \text{tr}(AZ)}$$

**Proof.** The equality follows directly from

$$\partial_{\nu,\mu} \left( e^{\pi i \text{tr}(AZ)} \right) = (2\pi i)A_{\nu,\mu} \ e^{\pi i \text{tr}(AZ)}$$
with $A = (A_{\nu,\mu})$.

The proof of Theorem 3.4 will follow directly from the following Proposition and Theorem 3.5.

**Proposition 7.2** Let $F$ and $F'$ be Siegel modular forms of degree 2 and weight $k$ and $k'$, respectively. Assume that $F$ and $F'$ can be written as theta series with harmonic coefficients. Then $[F, F']_1$ (defined in Theorem 7.2) is a Siegel modular form of degree 2 and weight $k + k' + 2l$.

**Proof.** Let $F$ and $F'$ be Siegel modular forms of degree $n = 2$ and weight $k$ and $k'$, respectively which can be written as theta series with spherical coefficients. Then one has

$$F = \sum_{G \in M_{2m,n}(\mathbb{Z})} Q(S^{1/2}G) e^{\pi i \text{tr}(G^t S G)Z}, \quad F' = \sum_{G' \in M_{2m',n}(\mathbb{Z})} Q'(S'^{1/2}G') e^{\pi i \text{tr}(G'^t S' G')Z}$$

for some symmetric, positive, even and unimodular matrices $S$ and $S'$, and spherical polynomials $Q$ and $Q'$ of weight $d$ and $d'$, respectively and $k = m + d$ and $k' = m' + d'$.

Then, from the very definition of $[\cdot, \cdot]_1$ and Lemma 7.1, we have

$$[F, F']_1 = (2\pi i)^d \sum_{G, G'} e^{\pi i \text{tr}((G^t G + G'^t G')Z)} P(S^{1/2}G, S'^{1/2}G')$$

with $P(X, X') = Q(X) Q'(X') P(X, X')$ where

$$P(X, X') = \sum_{r+s+p=l} C_{r, s, v}(k, k') \det(X^t X)^r \det(X'^t X')^s \det(X^t X + X'^t X)^p.$$

We will now show that the polynomial $P$ is spherical of weight $w = d + d' + 2l$ in the matrix variable

$$Y = \begin{pmatrix} X \\ X' \end{pmatrix}.$$

If this is the case Theorem 3.5 implies that $[F, F']_1$ is a Siegel modular form of degree 2 and weight $k + k' + 2l$.

Note that $P$ clearly satisfies the first property in the definition of a spherical polynomial with $w = d + d' + 2l$. Therefore, we only have to show that

$$\Delta_Y \tilde{P} = (\Delta_{S^{1/2}G} + \Delta_{S'^{1/2}G'}) \tilde{P} = 0.$$

To calculate $\Delta_Y \tilde{P}$ we use a change of basis such that $S$ and $S'$ become equal to $I_{2m}$ and $I_{2m'}$, respectively. To show that $\tilde{P}$ is spherical we need to know the following expressions which can easily be obtained by a straightforward calculation (here we use that $n = 2$ for the first time)

$$\Delta_G(\det(G^t G)) = 4(m - 1/2) \text{tr}(G^t G),$$

$$\langle \nabla_G(\det(G^t G)) \rangle^2 = 4 \text{tr}(G^t G) \det(G^t G),$$

$$\Delta_G(\det(G^t G + G'^t G')) = 4(m - 1/2) \text{tr}(G^t G) + 4k \text{tr}(G'^t G').$$
\[
(\nabla_G (\det(G^I G + G'^I G')))^2 = 4 \text{tr}(G^I G + G'^I G') (\det(G^I G) - \det(G'^I G')) + 4 \text{tr}(G'^I G') \det(G^I G + G'^I G'),
\]
\[
\nabla_G (\det(G^I G)) \cdot \nabla_G (\det(G^I G + G'^I G')) = 4 \text{tr}(G^I G) \det(G^I G),
\]
\[
\nabla_G (Q(G)) \cdot \nabla_G (\det(G^I G)) = 2d Q(G) \text{tr}(G^I G),
\]
\[
\nabla_G (Q(G)) \cdot \nabla_G (\det(G^I G + G'^I G')) = 2d Q(G) \text{tr}(G^I G + G'^I G').
\]

Note the for deriving the last two expressions one has to use the second property in the definition of spherical polynomials with

\[
A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.
\]

Let us, for the moment, use \( D = \det(G^I G), \ D' = \det(G'^I G') \) and \( D'' = \det(G^I G + G'^I G') \). Then the last equalities imply that

\[
\frac{1}{4} (\Delta_G + \Delta_{G'}) = (Q(G)Q'(G')D^s D'^s D''^p) \quad =
\]
\[
r(\alpha + r) \text{tr}(G^I G) + Q(G)Q'(G') D^r D'^r D''^p
\]
\[
+ s(\beta + s) \text{tr}(H^I H) + Q(G)Q'(G')D^s D'^s D''^{p-1}
\]
\[
+ p(\gamma + l + p) \text{tr}(G^I G + H^I H) + Q(G)Q'(G')D^s D'^s D''^{p-1}.
\]

Here we have used \( \alpha = k - 3/2, \beta = k' - 3/2 \) and \( \gamma = k + k' - 3/2 \). With these expressions it is clear that the equation \( \Delta_X \tilde{P} = 0 \) is certainly satisfied if the coefficients \( C_{r,s,p} \) obey

\[
0 = (r + 1)(\alpha + r + 1) C_{r+1,s,p}(k,k') + (p + 1)(\gamma + l + p + 1) C_{r,s,p+1}(k,k')
\]
\[
0 = (s + 1)(\beta + s + 1) C_{r,s+1,p}(k,k') + (p + 1)(\gamma + l + p + 1) C_{r,s,p+1}(k,k').
\]

It is simple to check that these conditions are indeed satisfied by the \( C_{r,s,p} \) given in the theorem. \( \square \)

**Remark 7.3** Note that a covariant bilinear differential operator which can be written in terms of the operator \( \mathbb{D} \) is, by the recursion relations obtained at the end of the proof of Proposition 7.2, equal to a multiple of \([\cdot, \cdot]\). Furthermore, the recursion relations imply the explicit form of the combinatorial factors \( C_{r,s,p} \).

**Proof of Theorem 7.3.** Firstly, note that the proof of Proposition 7.2 implies for \( Q = Q' = 1 \) that the polynomial \( \tilde{P} = P \) is spherical of weight \( 2l \). Secondly, it can be written as

\[
P(X, X') = \tilde{Q}(X^I X, X'^I X')
\]

with \( \tilde{Q}(a, b) = \sum_{r+s+p=l} C_{r,s,v}(k, k') \det(a + b)^p \det(a)^r \det(b)^s \) so that it satisfies the assumptions of Theorem 3.6. Finally, note that \([F, F']_l \) can be written as

\[
[F, F']_l = \mathcal{D}(F(Z)F'(Z'))|_{Z=Z'}
\]

where \( \mathcal{D} = \tilde{Q}(\partial_{v,\mu}, \partial'_{v',\mu'}) \). This implies the desired result. \( \square \)
8 Concluding remarks and open questions

Let us end with some remarks and mention some open questions.

1. It is obvious from the proofs of the Theorems 1.1 and 1.2 that they also hold true for the case of Jacobi forms on \( \Gamma \times \mathbb{Z}^2 \subset \text{SL}(2, \mathbb{Z}) \times \mathbb{Z}^2 \) and Siegel forms on \( \Gamma' \subset \text{Sp}(4, \mathbb{Z}) \) if \( \Gamma \) and \( \Gamma' \) are finite index subgroups of \( \text{SL}(2, \mathbb{Z}) \) and \( \text{Sp}(4, \mathbb{Z}) \), respectively.

2. In the generic case, i.e. where \( k \) and \( k' \) are large enough, the dimension of the space of Rankin-Cohen operators for Jacobi forms is given by Böcherer’s result so that Theorem 1.1 describes a basis of this space explicitly. This implies that, for \( k \) and \( k' \) large enough, the dimension of the space of Rankin-Cohen operators from \( S_k^{(2)} \times S_k^{(2)} \) to \( S_{k+k'+v}^{(2)} \) is one dimensional for even \( v \) and zero otherwise. (This can be verified by noting that any Rankin-Cohen operator for Siegel modular forms induces a Rankin-Cohen operator for Jacobi forms via the Jacobi-Fourier expansion (Theorem 3.2). Hence it is enough to show that there is only one (up to multiplication by a constant) Rankin-Cohen operator for Jacobi forms for even \( v \) and none for odd \( v \) which can be ‘lifted’ to a Rankin-Cohen operator for Siegel modular forms. In the general case, this can indeed be done using Theorem 1.1.)

3. Using the relation between Jacobi forms and modular forms of half-integral weight one can show that the operators \( \left( \frac{d}{dX} \right)^{v/2} [\cdot, \cdot]_{X,v} \ (v \in 2\mathbb{N}) \) can be obtained from the Rankin-Cohen operators for elliptic modular forms (for more details see [2]). It seems that this is the only Rankin-Cohen operator for Jacobi forms for which such a result holds true.

4. The operators \( \left( \frac{d}{dX} \right)^{v/2} [\cdot, \cdot]_{X,v} \ (v \in 2\mathbb{N}) \) can be used to define generalised Rankin-Cohen algebras [4] which have very similar properties to the Rankin-Cohen algebras in the elliptic case considered in ref. [11].

5. It would be interesting to understand how our constructions (via generating functions or theta series) can be generalised to higher Jacobi and Siegel modular forms and multilinear differential operators. We hope to discuss this in a future publication.

6. Is it possible to obtain the explicit formulae for the Rankin-Cohen operators in the case of Siegel modular forms from the representation theory of \( \text{Sp}(4, \mathbb{R}) \)? In this context the Rankin-Cohen operators can be viewed as certain projection operators.

7. Is it possible to obtain the dimension of the space of covariant bilinear operators from \( S_k^{(2)} \times S_{k'}^{(2)} \) to \( S_{k+k'+v}^{(2)} \) not using Jacobi forms? Of course one possibility would be to use theta series and rephrase the question in terms of the theory of invariants.

8. Is there any connection between covariant bilinear operators for Jacobi forms and/or Siegel modular forms and automorphic pseudodifferential operators like in the case of elliptic modular forms considered in ref. [1]?
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