IDENTITIES AND QUASIIDENTITIES IN THE LATTICE OF OVERCOMMUTATIVE SEMIGROUP VARIETIES

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ABSTRACT. We describe overcommutative varieties of semigroups whose lattice of overcommutative subvarieties satisfies a non-trivial identity or quasiidentity. These two properties turn out to be equivalent.

1. INTRODUCTION AND SUMMARY

It is generally known that the lattice of all semigroup varieties is a disjoint union of two wide and important sublattices: the ideal of all periodic varieties and the co-ideal of all overcommutative varieties, that is, varieties containing the variety $\text{COM}$ of all commutative semigroups. We denote the lattice of all overcommutative varieties by $\text{OC}$. By $L(V)$ we denote the subvariety lattice of a semigroup variety $V$. Identities and quasiidentities in lattices $L(V)$ were investigated in several papers, see Sections 11 and 12 in the survey [8]. The results of [2] and [7] imply that no non-trivial lattice quasiidentity holds in the lattice of commutative semigroup varieties and hence in the lattice $L(V)$ whenever $V$ is overcommutative. Therefore investigation of identities and quasiidentities in lattices $L(V)$ gives no information about the lattice $\text{OC}$. In view of this fact it is natural to study identities and quasiidentities in lattices of overcommutative subvarieties of overcommutative varieties. For an overcommutative variety $V$, its lattice of overcommutative subvarieties (that is, the interval between $\text{COM}$ and $V$) will be denoted by $L_{\text{OC}}(V)$.

The structure of the lattice $\text{OC}$ has been revealed by Volkov in [11]. We shall give the formulations of the results of this paper in Section 2.

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Basing on the results of [11], Vernikov described overcommutative varieties whose lattice of overcommutative subvarieties is distributive, modular, arguesian, lower or upper semimodular, lower or upper semidistributive or satisfies some other related restrictions [9],[10]. In the present paper we describe overcommutative varieties \( V \) whose lattice \( L_{OC}(V) \) satisfies a non-trivial lattice identity or quasiidentity.

We need the following definitions and notation. Lattices are called [quasi]equationally equivalent if they satisfy the same [quasi]identities. A semigroup variety \( V \) is permutative if it satisfies an identity of the form

\[
    x_1 x_2 \ldots x_n = x_{g(1)} x_{g(2)} \ldots x_{g(n)}
\]

where \( g \) is a non-trivial permutation on the set \( \{1, \ldots, n\} \). The semigroup variety given by an identity system \( \Sigma \) is denoted by \( \text{var } \Sigma \). Put

\[
    \mathcal{LZ} = \text{var } \{xy = x\}, \quad \mathcal{RZ} = \text{var } \{xy = y\}, \\
    \mathcal{X} = \text{var } \{xyzt = xytz, \quad x^2 y^2 = y^2 x^2 = (xy)^2\}.
\]

The variety dual to \( \mathcal{X} \) is denoted by \( \mathcal{X}^\perp \).

The main result of this article is

**Theorem 1.1.** For an overcommutative semigroup variety \( V \), the following are equivalent:

a) the lattice \( L_{OC}(V) \) satisfies a non-trivial lattice identity;

b) the lattice \( L_{OC}(V) \) satisfies a non-trivial lattice quasiidentity;

c) the lattice \( L_{OC}(V) \) is equationally equivalent to a finite lattice;

d) the lattice \( L_{OC}(V) \) is quasiequationally equivalent to a finite lattice;

e) the variety \( V \) is permutative and contains none of the varieties \( \mathcal{LZ}, \mathcal{RZ}, \mathcal{X}, \mathcal{X}^\perp \).

Since every finite lattice has a finite identity basis [4], Theorem 1.1 immediately imply the following

**Corollary 1.2.** If \( V \) is an overcommutative variety and the lattice \( L_{OC}(V) \) satisfies a non-trivial identity then this lattice has a finite identity basis.

The article consists of four sections. Sections 2 and 3 contain preliminary results. In Section 4 the proof of Theorem 1.1 is given.

### 2. Subdirect decomposition of the lattice \( OC \)

The aim of this section is to formulate the results of [11]. In order to do this, we need some new definitions and notation. The free semigroup
over the countably infinite alphabet \( X = \{ x_1, x_2, \ldots \} \) is denoted by \( F \). The symbol \( \equiv \) stands for the equality relation on \( F \). Put \( X_m = \{ x_1, \ldots, x_m \} \). Let \( F_m \) be the free semigroup over the set \( X_m \). If \( w \) is a word then we denote the length of \( w \) by \( \ell(w) \) and the number of occurrences of a letter \( x_i \) in \( w \) by \( \ell_x(w) \) or, shortly, by \( \ell_i(w) \). The symmetric group on the set \( \{ 1, \ldots, m \} \) is denoted by \( S_m \). For \( g \in S_m \) and \( 1 \leq i \leq m \), we put \( g(x_i) = x_{g(i)} \) thus identifying \( S_m \) with the symmetric group on \( X_m \). The lattice of all equivalence relations on a set \( A \) is denoted by \( \text{Part}(A) \). A set \( A \) on which a group \( G \) acts is called a \( G \)-set. A \( G \)-set can be considered as a unary algebra with the set \( G \) of operations. This observation, in particular, allows us to consider congruences of \( G \)-sets. The congruence lattice of a \( G \)-set \( A \) is denoted \( \text{Con}(A) \). If \( L \) is a lattice and \( x \in L \) then \( [x] \) (respectively, \( \{ x \} \)) stands for the principal ideal (respectively, co-ideal) generated by the element \( x \). By \( \overline{L} \) we denote the dual lattice to a lattice \( L \).

A **partition** is a sequence of positive integers \( \lambda = (\lambda_1, \ldots, \lambda_m) \) where \( \lambda_1 \geq \cdots \geq \lambda_m \) and \( m \geq 2 \). The set of all partitions is denoted by \( \Lambda \). Let us fix a partition \( \lambda \). We say that \( \lambda \) is a partition of the number \( n \) into \( m \) parts where \( n = \sum_{i=1}^{m} \lambda_i \). The numbers \( \lambda_i \) are called components of \( \lambda \). We consider the set

\[
W_{\lambda} = \{ w \in F_m \mid \ell_i(w) = \lambda_i \text{ for } 1 \leq i \leq m \}
\]

and the group

\[
G_{\lambda} = \{ g \in S_m \mid \lambda_i = \lambda_{g(i)} \text{ for } 1 \leq i \leq m \}.
\]

Every element \( g \in G_{\lambda} \), as a permutation on the alphabet \( X_m \), defines a permutation on the set \( W_{\lambda} \) which renames letters in each word in \( W_{\lambda} \). This means that the group \( G_{\lambda} \) acts on the set \( W_{\lambda} \) and this set is considered as a \( G_{\lambda} \)-set. For an overcommutative variety \( \mathcal{V} \), we define an equivalence relation \( \varphi_{\lambda}(\mathcal{V}) \) on \( W_{\lambda} \) as the restriction to the set \( W_{\lambda} \) of the fully invariant congruence on \( F \) corresponding to \( \mathcal{V} \). Thus a mapping \( \varphi_{\lambda}: \text{OC} \to \text{Part}(W_{\lambda}) \) is defined.

**Proposition 2.1** ([11]). Every mapping \( \varphi_{\lambda} \) is a homomorphism of the lattice \( \text{OC} \) onto the lattice \( \overline{\text{Con}(W_{\lambda})} \). These homomorphisms are components of an embedding

\[
\varphi = (\varphi_{\lambda})_{\lambda \in \Lambda}: \text{OC} \to \prod_{\lambda \in \Lambda} \overline{\text{Con}(W_{\lambda})}
\]

which decomposes the lattice \( \text{OC} \) into a subdirect product of the lattices \( \overline{\text{Con}(W_{\lambda})}, \ \lambda \in \Lambda \). \[\square\]
One can generalize Proposition 2.1 in order to obtain a subdirect decomposition of the lattice $L_{OC}(\mathcal{V})$. As a surjective homomorphism, $\varphi_\lambda$ maps principal ideals to principal ideals, so

$$\varphi_\lambda(L_{OC}(\mathcal{V})) = (\varphi_\lambda(\mathcal{V}))_{\text{Con}(W_\lambda)} = [\varphi_\lambda(\mathcal{V})]_{\text{Con}(W_\lambda)}.$$ 

The co-ideal $[\varphi_\lambda(\mathcal{V})]_{\text{Con}(W_\lambda)}$ is isomorphic to the congruence lattice of the factor $G_\lambda$-set $W_\lambda/\varphi_\lambda(\mathcal{V})$. Thus we have

**Corollary 2.2** ([11]). For any variety $\mathcal{V} \in \text{OC}$, the homomorphism $\varphi|_{L_{OC}(\mathcal{V})}$ defines a decomposition of the lattice $L_{OC}(\mathcal{V})$ into a subdirect product of the lattices $\text{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))$. □

Another result we need is

**Proposition 2.3** ([11]). Every lattice $\text{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))$ can be embedded into $L_{OC}(\mathcal{V})$. □

3. **Preliminaries on semigroup identities**

In this section we study some equational properties of the varieties $L\mathcal{Z}$, $R\mathcal{Z}$, $\mathcal{X}$ and $\mathcal{X}$. The following two lemmas and their duals give the solution of word problem in these varieties. For the varieties $L\mathcal{Z}$ and $R\mathcal{Z}$ it is generally known and evident.

**Lemma 3.1.** An identity $u = v$ holds in $L\mathcal{Z}$ if and only if the words $u$ and $v$ start with the same letters. □

An identity $u = v$ is called balanced if $\ell_x(u) = \ell_x(v)$ for every $x \in X$. All identities satisfied by overcommutative varieties are balanced. A letter $x$ in a word $w \in F$ is called simple if $\ell_x(w) = 1$ and multiple otherwise.

**Lemma 3.2.** An identity $u = v$ holds in $\mathcal{X}$ if and only if it is balanced and at least one of the following holds:

1. $u \equiv v \in X$;
2. $u$ and $v$ have equal first letters and equal second letters;
3. $u$ and $v$ have equal first letters and their second letters are multiple;
4. the first and the second letters in $u$ and $v$ are multiple.

**Proof.** Let us denote by $\alpha$ the fully invariant congruence on $F$ corresponding to $\mathcal{X}$ and by $\beta$ the set of all balanced identities $u = v$ (considered as pairs of words) satisfying one of the conditions (i)–(iv). We must prove that $\alpha = \beta$.

First, one can prove that $\alpha \subseteq \beta$. The identity $xyzt = xytz$ satisfies (ii) while the identities $x^2y^2 = y^2x^2 = (xy)^2$ satisfy (iv), so all these
identities belong to $\beta$. Straightforward verification shows that $\beta$ is a fully invariant congruence on $F$. This implies the desired inclusion.

It remains to verify that $\beta \subseteq \alpha$. We shall prove that a balanced identity $u = v$ holds in $X$ in each of the cases (i)–(iv). The case (i) is trivial. The identity $xyzt = xytz$ implies every identity of the kind (1) with $g(1) = 1$ and $g(2) = 2$. Identifying and renaming letters in the latter identity, one can obtain every identity with the property (ii). In the rest of the proof we suppose that the identity $u = v$ is written in the form $xya = ztb$ where $x, y, z, t \in X$ and $a, b \in F$ (the letters $x, y, z, t$ are not assumed to be distinct). Consider the case (iv). Suppose that $x \equiv z \equiv t$ and $x \nmid y$, that is $u \equiv xya$ and $v \equiv x^2b$. Since $y$ is multiple, there exist balanced identities of the form $xya = (xy)^2c$ and $x^2b = x^2y^2c$ for some $c \in F$. These identities satisfy (ii), so they hold in $X$. Hence we have

$$xya = (xy)^2c = x^2y^2c = x^2b$$

in $X$. The same arguments show that $X$ satisfies $u = v$ whenever $y \equiv z \equiv t$ and $x \nmid y$ (one should use the identity $(xy)^2 = y^2x^2$ rather than $(xy)^2 = x^2y^2$ in this case). Therefore in the general case $X$ satisfies

$$xya = x^2c = xtd = t^2e = ztb$$

where $c, d, e \in F$ whenever these identities are balanced. Of course, such words $c, d,$ and $e$ exist, so we are done in the case (iv). In the case (iii) the identity $u = v$ is $xya = xtb$ where $y$ and $t$ are multiple. We may suppose that the letter $x$ is simple, because otherwise the property (iv) holds. In particular, $x \nmid y$ and $x \nmid z$. The variety $X$ satisfies $xya = xy^2c$ and $xtb = xt^2d$ ($c, d \in F$) whenever these identities are balanced (the case (ii)). Furthermore, $X$ satisfies $y^2c = z^2d$ (the case (iv)), so it satisfies $xya = xy^2c = xt^2d = xtb$. □

For a non-negative integer $k$, consider the variety

$$P_k = \text{var}\{x_1 \ldots x_k yzt_1 \ldots t_k = x_1 \ldots x_k yzt_1 \ldots t_k\}.$$ 

This variety satisfies every balanced identity of the form $acb = adb$ where $\ell(a) = \ell(b) = k$.

**Lemma 3.3** ([6]). Every permutative variety is contained in $P_k$ for some $k$. □

**Lemma 3.4.** Any overcommutative permutative variety $V$ such that $LZ \not\subseteq V$ satisfies the identity

$$x^n y^n z^n = y^n x^n z^n$$

for any sufficiently large $n$.
Proof. Being permutative, the variety $\mathcal{V}$ is contained in $\mathcal{P}_k$ for some $k$ by Lemma 3.3. Lemma 3.1 and the fact that $\mathcal{LZ} \not\subseteq \mathcal{V}$ imply that the variety $\mathcal{V}$ satisfies an identity $xa = yb$ where $x \neq y$. The identity $xa = yb$ is balanced because $\mathcal{V}$ is overcommutative. We may suppose that $a$ and $b$ contain only the letters $x$ and $y$. If this is not the case then we identify all other letters with $x$. Assume that $n \geq k + \ell(a) = k + \ell(b)$. We are going to prove that $\mathcal{V}$ satisfies all identities of the form $cz^n = dz^n$ where $c$ and $d$ contain only the letters $x$ and $y$ and $\ell_x(c) = \ell_x(d) = \ell_y(c) = \ell_y(d) = n$. This would imply the statement of the lemma we prove as a partial case. Take the greatest common prefix $e$ of the words $c$ and $d$. There are words $c'$ and $d'$ with $c \equiv exc'$ and $d \equiv eyd'$. If $\ell(e) \geq k$ then the identity $cz^n \equiv dzn$ holds in $\mathcal{V}$ because $\mathcal{V} \subseteq \mathcal{P}_k$ and $n > k$. Suppose that $0 \leq \ell(e) \leq k$. To prove that $\mathcal{V}$ satisfies $cz^n = dz^n$ in this case, we use inverse induction by $\ell(e)$. As the induction base we take the case $\ell(e) = k$ which has already been considered. Now we shall prove the statement for $\ell(e) < k$ assuming that it is proved for greater $\ell(e)$. Put $p = \ell_x(e) + \ell_x(b)$ and $q = \ell_y(e) + \ell_y(a)$. The inequality $n \geq k + \ell(a) = k + \ell(b)$ imply $n > p$ and $n > q$. The variety $\mathcal{V}$ satisfies

$$cz^n \equiv exc'z^n = eydx^n - p, y^n, z^n = eyd'z^n \equiv dzn$$

by the induction assumption, as was to be proved. □

Lemma 3.5. Any overcommutative permutative variety $\mathcal{V}$ such that $\mathcal{LZ}, \mathcal{X} \not\subseteq \mathcal{V}$ satisfies the identity

$$xtx^{n-1}y^n z^n = yty^{n-1}x^n z^n$$

for any sufficiently large $n$.

Proof. By Lemma 3.3 we have $\mathcal{V} \subseteq \mathcal{P}_k$ for some $k$. By Lemma 3.4 the variety $\mathcal{V}$ satisfies

$$x^m y^n z^m = y^m x^m z^m$$

for some $m \geq k$. The variety $\mathcal{V}$ satisfies a balanced identity $u = v$ which fails in $\mathcal{X}$. According to Lemma 3.2, there are four possible cases.

Case 1. The first letters in $u$ and $v$ coincide, the second letters are distinct and at least one of the second letters is simple. Identifying all
letters in \( u = v \) except this simple letter, we obtain an identity of the form
\[
xyx^{p+q-1} = x^{p+1}yx^{q-1}
\]
for some \( p \) and \( q \). This identity implies \( xyx^{pr+q-1} = x^{pr+1}yx^{q-1} \) for all positive integers \( r \), so \( p \) can be replaced by \( pr \) in (5). This allows us to suppose that \( p \geq k \). Let us take \( n \) with \( n \geq m + k \) and \( n \geq p + q \). The variety \( \mathcal{V} \) satisfies
\[
xtx^{n-1}y^n z^n = x^{p+1}tx^{n-p-1}y^n z^n
\]
by (5)
\[
= x^m y^m z^m tx^{n-m} y^{n-m} z^{n-m}
\]
because \( \mathcal{V} \subseteq \mathcal{P}_k \)
\[
= y^m x^m z^m tx^{n-m} y^{n-m} z^{n-m}
\]
by (4)
\[
= y^{p+1}ty^{n-p-1}x^n z^n
\]
because \( \mathcal{V} \subseteq \mathcal{P}_k \)
\[
= yty^{n-1}x^n z^n
\]
by (5).

**Case 2.** The first letters in \( u \) and \( v \) are distinct and at least one of these letters is simple. Identifying all letters in \( u = v \) except this simple letter we obtain \( yx^{p+q} = x^p yx^q \) for some positive \( p \) and non-negative \( q \). This identity implies \( xyx^{p+q} = x^{p+1}yx^q \), so we return to the Case 1.

**Case 3.** The second letters in \( u \) and \( v \) coincide and are simple while the first letters are distinct and multiple. Let us write the identity \( u = v \) in the form \( xtu' = ytv' \). We may suppose that \( u' \) and \( v' \) contain only the letters \( x \) and \( y \) because all other letters can be identified with \( x \). Put \( p = \ell_x(u') \) and \( q = \ell_y(u') \). Let us take \( n \) with \( n \geq k + p \), \( n \geq q \), \( n \geq k + m \), and \( n \geq m + 1 \). We have that
\[
xtx^{n-1}y^n z^n = xtx^k u' x^{n-k-p} y^{n-q} z^n
\]
because \( \mathcal{V} \subseteq \mathcal{P}_k \)
\[
= ytx^k v' x^{n-k-p} y^{n-q} z^n
\]
because \( xtu' = ytv' \)
\[
= ytx^m y^m z^m x^{n-m} y^{n-m-1} z^{n-m}
\]
because \( \mathcal{V} \subseteq \mathcal{P}_k \)
\[
= yty^m x^m z^m x^{n-m} y^{n-m-1} z^{n-m}
\]
by (4)
\[
= yty^{n-1}x^n z^n
\]
because \( \mathcal{V} \subseteq \mathcal{P}_k \)
holds in the variety \( \mathcal{V} \).

**Case 4.** The first letters in \( u \) and \( v \) are distinct and multiple, the second letters are distinct, and at least one of the second letters is simple. Identifying the first letters in the words \( u \) and \( v \), we return to the Case 1. \( \square \)
4. Proof of Theorem 1.1

The proof follows the scheme \( a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \rightarrow a \). The implications \( a \rightarrow b \) and \( d \rightarrow c \) are obvious. The implication \( c \rightarrow a \) holds because every finite lattice satisfies a non-trivial identity (see [3, Lemma V.3.2], for instance). It remains to verify the implications \( b \rightarrow e \rightarrow d \).

\( b \rightarrow e \) Arguing by contradiction, suppose that the property \( e \) fails. We shall prove that every finite lattice can be embedded into one of the lattices \( \text{Con}(W_\lambda/\varphi_\lambda(V)) \). Hence every finite lattice can be embedded into \( \text{Loc}(V) \) by Proposition 2.3. Since every non-trivial lattice quasiidentity fails in some finite lattice [1], this will give us the contradiction we need. There are three cases to consider.

Case 1. The variety \( V \) is not permutative. Consider the partition \( \lambda = (1, \ldots, 1) \). For this partition we have \( G_\lambda = S_n \). The corresponding \( G_\lambda \)-set \( W_\lambda \) is regular (i.e., it is transitive and any non-unit element of \( G_\lambda \) has no fixed points). In this case \( \text{Con}(W_\lambda/\varphi_\lambda(V)) \cong \text{Sub}(S_n) \) where \( \text{Sub}(G) \) is the subgroup lattice of a group \( G \) (see [5, Lemma 4.20]). Since the variety \( V \) is not permutative, the congruence \( \varphi_\lambda(V) \) is the equality relation on \( W_\lambda \), so \( W_\lambda/\varphi_\lambda(V) = W_\lambda \). We have obtained that \( \text{Con}(W_\lambda/\varphi_\lambda(V)) \cong \text{Sub}(S_n) \). Every finite lattice can be embedded into a lattice \( \text{Sub}(S_n) \) for some \( n \) [7], so we are done.

Case 2. The variety \( V \) contains one of the subvarieties \( \mathcal{LZ} \) and \( \mathcal{RZ} \). By duality principle, we may suppose that \( \mathcal{LZ} \subseteq V \). Consider the partition \( \lambda = (m, m - 1, \ldots, 2, 1) \) for an arbitrary \( m \geq 2 \). The group \( G_\lambda \) is trivial, whence \( \text{Con}(W_\lambda/\varphi_\lambda(V)) = \text{Part}(W_\lambda/\varphi_\lambda(V)) \). Since \( \mathcal{LZ} \subseteq V \), the variety \( V \) satisfies no identity \( u = v \) where the first letters in \( u \) and \( v \) are distinct. In particular, \( (x_i a, x_j b) \not\in \varphi_\lambda(V) \) whenever \( x_i a, x_j b \in W_\lambda \) and \( i \neq j \). Hence the set \( W_\lambda/\varphi_\lambda(V) \) contains at least \( m \) elements. Any finite lattice can be embedded into any sufficiently large finite partition lattice [7], so it can be embedded into some of the lattices \( \text{Con}(W_\lambda/\varphi_\lambda(V)) \).

Case 3. The variety \( V \) contains one of the subvarieties \( \mathcal{X} \) and \( \overline{\mathcal{X}} \), say, \( \mathcal{X} \subseteq V \). Consider the same partition \( \lambda \) as in Case 2. Again we have \( \text{Con}(W_\lambda/\varphi_\lambda(V)) = \text{Part}(W_\lambda/\varphi_\lambda(V)) \). Since \( \mathcal{X} \subseteq V \), Lemma 3.2 implies that the variety \( V \) satisfies no identity \( u = v \) where the first letters in \( u \) and \( v \) are distinct and the second letter in \( u \) is simple. In particular, \( (x_i x_m a, x_j x_m b) \not\in \varphi_\lambda(V) \) whenever \( x_i x_m a, x_j x_m b \in W_\lambda \) and \( i \neq j \). Hence the set \( W_\lambda/\varphi_\lambda(V) \) contains at least \( m - 1 \) elements, so we are done, as in Case 2.
e) → d). Let \( \mathcal{V} \) be an overcommutative variety satisfying e). Consider the subdirect decomposition of the lattice \( L_{OC}(\mathcal{V}) \) given by Corollary 2.2. We will prove that the cardinalities of the subdirect multipliers \( \text{Con}(W_\lambda/\varphi_\lambda(\mathcal{V})) \) are bounded. This implies that there exist only finite number of non-isomorphic lattices among these multipliers. The lattice \( L_{OC}(\mathcal{V}) \) is quasiequationally equivalent to the direct product of these distinct multipliers because quasiidentities are preserved under taking sublattices and direct products. Therefore the implication will be proved.

Let us fix a partition \( \lambda \). The variety \( \mathcal{V} \) is contained in \( \mathcal{P}_k \) for some \( k \) by Lemma 3.3. We may assume that \( \lambda \) is a partition of a number greater than \( 2k + 1 \). Indeed, there is only a finite number of other partitions and existence of an upper bound for \( |\text{Con}(W_\lambda/\varphi_\lambda(\mathcal{V}))| \) does not depend on them. By Lemmas 3.4, 3.5 and their duals the variety \( \mathcal{V} \) satisfies the identities (2), (3), and their duals for some \( V \). Consider the following two restrictions on a word \( w \in W_\lambda \):

(i) there are no letters \( x \) in the words \( L(w) \) and \( R(w) \) with \( \ell_x(w) \geq n+2k \) and \( x \neq x_1, x \neq x_2 \) (recall that \( \ell_1(w) \geq \ell_2(w) \geq \ell_x(w) \) for any \( x \in X \setminus \{x_1, x_2\} \), so this property trivially holds whenever \( \ell_2(w) < n+2k \);

(ii) there are no letters \( x \) in the words \( L(w) \) and \( R(w) \) with \( \ell_x(w) = i \in I \) and \( x \not\in Y_i \).

Let us prove that, for any \( w \in W_\lambda \), there exist \( w' \in W_\lambda \) with the property (i) and such that \( w = w' \) in \( \mathcal{V} \). This means that each \( \varphi_\lambda(\mathcal{V}) \)-class contains a word with the property (i). Consider an occurrence in \( L(w) \) of a letter \( x \) with \( \ell_x(w) > n+2k, x \neq x_1, \) and \( x \neq x_2 \). There are words \( d \) and \( e \) with \( L(w) \equiv dxe \). Since \( \ell_1(w) \geq \ell_2(w) \geq \ell_x(w) \geq n+2k \), we have

\[ \ell_x(M(w)), \ell_1(M(w)), \ell_2(M(w)) \geq n. \]
Hence there exists a balanced identity of the form $M(w) = x^{n-1}x_1^n x_2^nf$ for some word $f$. The variety $\mathcal{V}$ satisfies
\[
w \equiv L(w)M(w)R(w) \\
\equiv dxeM(w)R(w) \\
dxex^{n-1}x_1^n x_2^nfR(w) \quad \text{because } \mathcal{V} \subseteq \mathcal{P}_k \\
dx_1x^{n-1}x_1^n x_2^nfR(w) \quad \text{by (2) if } e \text{ is empty} \\
or \text{ by (3) otherwise.}
\]
The word $w'' \equiv dx_1ex^{n-1}x_1^n x_2^nfR(w)$ is such that $L(w'') \equiv dx_1e$, $R(w'') \equiv R(w)$, and $(w, w'') \in \varphi(\mathcal{V})$. We have excluded one occurrence of the letter $x$ in $L(w)$. Repeating this procedure one can exclude all occurrences in $L(w)$ of letters $x$ with $\ell_x(w) > n + 2k$ except $x_1$ and $x_2$. Dually, one can exclude all occurrences of such letters in $R(w)$.

Now we shall prove that every identity $u = v$ such that $u, v \in W_\lambda$ is equivalent to an identity $u' = v'$ where $u'$ and $v'$ satisfy (ii). Since
\[
\ell(L(u)) + \ell(L(v)) + \ell(R(u)) + \ell(R(v)) = 4k,
\]
the words $L(u)$, $L(v)$, $R(u)$, and $R(v)$ contain at most $4k$ distinct letters. Therefore, for $1 \leq i < n + 2k$, they contain at most $4k$ distinct letters $x$ with $\ell_x(u) = i$. Consider any element $g \in G_\lambda$ which maps, for every $1 \leq i < n + 2k$, all letters $x$ in $L(u), L(v), R(u), R(v)$ with $\ell_x(w) = i$ to the set $Y_i$. To obtain the identity $u' = v'$, one may take $u' \equiv g(u)$ and $v' \equiv g(v)$.

Combining the statements in the previous two paragraphs, we conclude that every identity $u = v$ with $u, v \in W_\lambda$ is equivalent within the variety $\mathcal{V}$ to an identity $u' = v'$ where $u'$ and $v'$ satisfy (i) and (ii). This statement may be reformulated in terms of $G$-sets. To do this, denote by $A$ the set of $\varphi(\mathcal{V})$-classes of all words in $W_\lambda$ satisfying (i) and (ii). We have proved that every congruence on $W_\lambda/\varphi(\mathcal{V})$ is generated by some subset of $A \times A$. The $\varphi(\mathcal{V})$-class of $w$ is defined by $L(w)$ and $R(w)$ and does not depend on $M(w)$. Conditions (i) and (ii) mean that $L(w)$ and $R(w)$ for all such $w$ may contain at most $4k(2n + k - 1) + 2$ distinct letters in common: at most $4k$ letters $x$ with $\ell_x(w) = i$ for every $1 \leq i < 2n + k$ and at most $2$ letters $x$ with $\ell_x(w) \geq 2n + k$.

Hence $|A| \leq N$ where $N = (4k(2n + k - 1) + 2)^{2k}$. Therefore
\[
|\text{Con}(W_\lambda/\varphi(\mathcal{V}))| \leq 2^{4kN} \leq 2^{N^2}.
\]
This upper bound does not depend on the partition $\lambda$.

Theorem 1.1 is proved. \qed
Remark 4.1. The proof of the implication $e) \rightarrow d$) bases on the fact that the cardinalities of the lattices $\text{Con}(W_\lambda/\varphi_\lambda(V))$ are bounded whenever $V$ satisfies $e)$. However the cardinalities of the sets $W_\lambda/\varphi_\lambda(V)$ can be unbounded. For example, put

$$V = \text{var}\{x^2y = yx^2, \ xyz = xzy\}.$$  

For the partition $\lambda = (1^{\text{n times}})$, it is easy to verify that the set $W_\lambda/\varphi_\lambda(V)$ contains exactly $n$ elements.

Remark 4.2. The variety $LZ$ is generally known to be an atom of the lattice of all semigroup varieties. Consequently it would be possible to conjecture that the lattice $L_{\text{OC}}(\text{COM} \lor LZ)$ is small in a sense. Surprisingly, this conjecture is very far from the real situation. The proof of Theorem 1.1 shows that this lattice contains an isomorphic copy of every finite lattice (see Case 2 in the proof of the implication $b) \rightarrow e$)).

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