Convergence of Viscosity Solutions of Generalized Contact Hamilton–Jacobi Equations

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Abstract

For any compact connected manifold $M$, we consider the generalized contact Hamiltonian $H(x, p, u)$ defined on $T^*M \times \mathbb{R}$ which is convex in $p$ and monotonically increasing in $u$. Let $u^-_\varepsilon : M \to \mathbb{R}$ be the viscosity solution of the parametrized contact Hamilton–Jacobi equation

$$H(x, d_x u^-_\varepsilon(x), \varepsilon u^-_\varepsilon(x)) = c(H),$$

with $c(H)$ being the Mañé Critical Value. We prove that $u^-_\varepsilon$ converges uniformly, as $\varepsilon \to 0_+$, to a specific viscosity solution $u^-_0$ of the critical equation

$$H(x, d_x u^-_0(x), 0) = c(H),$$

which can be characterized as a minimal combination of the associated Peierls barrier functions.

1. Introduction

For a smooth compact Riemannian manifold $M$ without boundary, the contact Hamiltonian $H$ is a continuous function on the cotangent bundle $T^*M \times \mathbb{R}$, of which the contact Hamilton–Jacobi equation

$$H(x, d_x u(x), u(x)) = c, \quad x \in M$$

(1)

is usually considered (for suitable constant $c \in \mathbb{R}$). This equation, from the view of physics, naturally arises in contact Hamiltonian mechanics [2,3]. A systematic discussion of the viscosity solution for such contact Hamilton–Jacobi equations was also made in [1,10]. Inspired by these works, we propose the following standing assumptions:

(H0) Smoothness $H : (x, p, u) \in T^*M \times \mathbb{R} \to \mathbb{R}$ is $C^2$-smooth.
(H1) **Positive definiteness** For every \((x, p, u) \in T^*M \times \mathbb{R}\), the second order partial derivative \(\partial^2 H / \partial p^2(x, p, u)\) is positive definite as a quadratic form.

(H2) **Superlinearity** For every \((x, u) \in M \times \mathbb{R}\), \(\lim_{|p|_x \to +\infty} H(x, p, u)/|p|_x = +\infty\), with \(|\cdot|_x\) being the norm induced by the Riemannian metric on the fiber \(T^*_x M\).

(H3) **Monotonicity** There exists \(\Delta_1 > 0\), such that \(0 < \partial H / \partial u(x, p, u) \leq \Delta_1\) for all \((x, p, u) \in T^*M \times \mathbb{R}\).

In some works, e.g. [8], (H1)–(H2) are usually called Tonelli conditions. Based on these assumptions, we get a dual Tonelli Lagrangian \(L : TM \times \mathbb{R} \to \mathbb{R}\) by

\[
L(x, \dot{x}, u) = \max_{p \in T^*_x M} \{\langle \dot{x}, p \rangle - H(x, p, u)\},
\]

which is \(C^2\)-smooth, positive definite and superlinear in \(\dot{x} \in T_x M\), but strictly decreasing in \(u \in \mathbb{R}\). In [5,14], the viscosity solution of (1) is interpreted by a variational principle associated with \(L(x, \dot{x}, u)\). In addition, the generalized characteristics decided by the viscosity solution solve the contact Hamilton’s equations

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p}(x, p, u), \\
\dot{p} &= -\frac{\partial H}{\partial x}(x, p, u) - p \frac{\partial H}{\partial u}(x, p, u), \\
\dot{u} &= p \cdot \frac{\partial H}{\partial p}(x, p, u) - H(x, p, u);
\end{align*}
\]

see [1,10] for more details.

In this paper, we consider the contact Hamilton–Jacobi equation satisfying our standing assumptions and parametrized by \(\varepsilon \in [0, 1]\), i.e.

\[
H(x, d_x u_\varepsilon(x), \varepsilon u_\varepsilon(x)) = c(H)
\]

for the Mañé Critical Value

\[
c(H) := \inf\{c \in \mathbb{R}|H(x, d_x u_\varepsilon(x), 0) = c\ \text{admits a viscosity solution}\}.
\]

As (3) obeys the Comparison Principle, for any \(\varepsilon > 0\) the viscosity solution \(u_\varepsilon\) of (3) has to be unique (see [1] or Theorem 3.2 of [5]). Our goal is to study the convergence of \(u_\varepsilon\) as \(\varepsilon \to 0_+\), i.e. the so called vanishing contact structure problem in [5]. A heuristic idea is to show that \(u_\varepsilon\) are equi-Lipschitz and uniformly bounded when \(\varepsilon \in (0, 1]\) and if so, by the Ascoli-Alzelà Theorem and the stability of viscosity solutions [6], each accumulating function of \(u_\varepsilon\) would be a viscosity solution of the conservative Hamilton–Jacobi equation

\[
H(x, d_x u(x), 0) = c(H).
\]

Our first conclusion proves the uniqueness of the accumulating function of \(u_\varepsilon\) by a qualitative expression.
**Theorem 1.1.** (Main 1) Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a Hamiltonian satisfying the standing assumptions. For $\varepsilon > 0$, we denote by $u_\varepsilon : M \to \mathbb{R}$ the unique continuous viscosity solution of (3), then the family $u_\varepsilon$ converges to a unique viscosity solution $u_0$ of (4) as $\varepsilon \to 0$, which is the largest critical subsolution $u : M \to \mathbb{R}$ of (4) such that for every Mather measure $\tilde{\mu}$ (see Definition 5.8 in [7]),

$$\int_{TM} u(y) \cdot \frac{\partial L}{\partial u}(y, v, 0) \, d\tilde{\mu}(y, v) \geq 0.$$ 

The second conclusion we want to display is a dynamic interpretation of previous $u_0$, by using the Peierls barrier functions.

**Theorem 1.2.** (Main 2) The limit function $u_0$, obtained in Theorem 1.1 above, can be characterized in the following way: it is the infimum of the functions $h_\mu^\infty$ defined by

$$h_\mu^\infty(x) := \frac{\int_{TM} h_\infty(y, x) \cdot \frac{\partial L}{\partial u}(y, v, 0) \, d\tilde{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\tilde{\mu}(y, v)}, \quad \forall \, x \in M$$

over all Mather measures $\tilde{\mu}$, where

$$h_\infty : M \times M \to \mathbb{R}$$

via

$$h_\infty(x, y) := \lim_{t \to +\infty} \left( \inf_{\xi \in C^1([0, t], M)} \int_0^t L(\xi(s), \dot{\xi}(s)) \, ds + c(H)t \right)$$

is the Peierls barrier of the conservative Hamiltonian $H(x, p, 0)$.

### 1.1. A Brief Review of Relevant Works

The first result about the vanishing contact limit of solutions to the contact Hamilton–Jacobi equation was achieved by the group of Davini--Itturiaga--Fathi--Zavidovique [7], for $H(x, p, u)$ linearly increasing in $u$. Such a kind of Hamiltonians $H : T^*M \times \mathbb{R} \to \mathbb{R}$ is called discounted, and the associated (2) is called conformally symplectic in [4,12].

Later, a similar result was proved for contact Hamiltonians by the group of Chen--Cheng--Ishii--Zhao [5], by proposing a reasonable asymptotic condition of $H(x, p, u)$, as an extension of the idea in [7]. Precisely, they required the following (D2) assumption:

The system

$$\tilde{G}(x, p) := \frac{H(x, p, 0) - c(H)}{\partial_u H(x, p, 0)}$$

is convex and coercive in variable $p$. 


For $C^1$-smooth $H(x, p, u)$, this (D2) assumption guarantees the system
\[
\dot{H}(x, p, \epsilon u) := \frac{H(x, p, \epsilon u) - c(H)}{\partial_u H(x, p, 0)}
\]
is linearly asymptotic to $\dot{G}(x, p)$ as $\epsilon \to 0$, and the convexity of (D2) admits the Comparison Principle (see Theorem 3.2 of [5]). Nonetheless, in these two works, the regularity of $H(x, p, u)$ is lower than ours.

Benefiting from the Aubry–Mather theory of contact Hamilton’s equations developed in [13,14], our Theorem 1.1 and Theorem 1.2 generalize the results of [5] for contact $C^2$-Tonelli Hamiltonians by removing (D2). Moreover, our theorems recover the same conclusions as in [7] for systems satisfying the standing assumptions.

Corollary 1.3. (Discounted Equation) For the system satisfying $\partial_u H(x, p, u) \equiv 1$ for all $(x, p, u) \in T^*M \times \mathbb{R}$, the parametrized viscosity solutions $u_\epsilon$ converges to a unique $u_0$ of (4), which can be characterized in the following ways:

- it is the largest critical subsolution $u : M \to \mathbb{R}$ such that for every projected Mather measure $\mu$ (see Definition 5.8 in [7]), it holds that
  \[
  \int_M u(y) \, d\mu \leq 0;
  \]

- it is the infimum over all projected Mather measures $\mu$ of the functions $h_\mu^{\infty}$ defined by
  \[
  h_\mu^{\infty}(x) = \int_M h^{\infty}(y, x) \, d\mu(y), \quad \forall x \in M,
  \]
  where $h^{\infty}(y, x)$ is the Peierls barrier function of the conservative Hamiltonian $H(x, p, 0)$.

1.2. Organization of the Article

The paper is organized as follows: first, we exhibit some conclusions about the viscosity solutions of the contact Hamilton–Jacobi equation in Section 2, which will be used in the proof of our main theorems. Second, we prove the convergence of the viscosity solutions of (3) in Section 3. Finally, in Section 4, we present an alternative interpretation of the limit solution $u_0$ by using the language of the Peierls barrier function, and explore the dynamic differences between our limit $u_0$ and the discounted limit in [7]. For the consistency and the readability of the proof, we move some technical proofs to “Appendix 5”.
2. Preliminary: Contact Hamiltonian and Its Variational Principle

In this section, we provide some conclusions for the contact Hamiltonian satisfying the standing assumptions, which can be interpreted as a contact version of the Aubry–Mather theory. Recall that $M$ is a connected and compact manifold without boundary, so we denote by $x \in M$ the coordinate of the configuration space and $p \in T^*_x M$ the 1-form of the vector space $T_x M$. The Legendre transformation of $H(x, p, u)$ w.r.t. $p$ gives the Lagrangian

$$L(x, \dot{x}, u) = \max_{p \in T^*_x M} \{ \langle \dot{x}, p \rangle - H(x, p, u) \},$$

of which the maximizer is achieved as $\dot{x} = \partial_p H(x, p, u)$. Indeed, due to the convexity of $H(x, p, u)$ in $p$-variable,

$$L : T^* M \times \mathbb{R} \to TM \times \mathbb{R}, \quad \text{via} \quad (x, p, u) \mapsto (x, \partial_p H(x, p, u), u)$$

is a diffeomorphism. Due to the conclusion in [13], there exists an implicit backward Lax-Oleinik operator $T_t^-$ defined by

$$T_t^- \phi(x) = \inf_{\gamma \in C^0([0, t], M)} \left\{ \phi(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), T_s^- \phi(\gamma(s))) + c(H) \, ds \right\}$$

for any $\phi(\cdot) \in C^0(M, \mathbb{R})$ and $t \geq 0$.

**Lemma 2.1.** [13] For contact Hamiltonian $H(x, p, u)$ satisfying the standing assumptions, and any $\phi \in C^0(M, \mathbb{R})$, $T_t^- \phi(x)$ uniformly converges as $t \to +\infty$, to a viscosity solution $u^-(x) \in \text{Lip}(M, \mathbb{R})$ of

$$H(x, d_x u^-, u^-) = c(H), \quad \text{a.e.} \quad x \in M. \quad (10)$$

Moreover, $u^-(x)$ satisfies that

- for any $x, y \in M, s < t \in \mathbb{R}$ and any piecewise $C^1$-continuous curve $\gamma$ connecting them, we have

$$u^-(y) - u^-(x) \leq \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau), u^-(\gamma(\tau))) + c(H) \, d\tau;$$

- for any $x \in M$, there exists a backward calibrated curve $\gamma^-_x : (-\infty, 0] \to M$ ending with it, such that for all $s < t \leq 0$,

$$u^-(\gamma^-_x(t)) - u^-(\gamma^-_x(s)) = \int_s^t L(\gamma^-_x(\tau), \dot{\gamma}^-_x(\tau), u^-(\gamma^-_x(\tau))) + c(H) \, d\tau.$$

Such a viscosity solution is also called a weak KAM solution [8].

**Lemma 2.2.** (Lemma 4.3 of [14]) For any $x \in M$, the viscosity solution $u^-$ of (10) is always differentiable along the interior of the backward calibrated curve $\gamma^-_x$ ending with it, namely, $u^-$ is differentiable on the set $\{\gamma^-_x(\tau) | t \in (-\infty, 0)\}$. 
2.1. Viscosity Solutions of Parametrized Contact Hamiltonians

Now we parameterize the Hamiltonian by $H(x, p, \varepsilon u)$ with $\varepsilon \in (0, 1]$. Due to Lemma 2.1, we get a family of viscosity solutions $\{u_\varepsilon^-\}$ of (3), which satisfies

**Lemma 2.3.** $\{u_\varepsilon^-\}$ is uniformly bounded, and equi-Lipschitz for all $\varepsilon \in (0, 1]$ with the Lipschitz constant $\kappa$ depending only on $H(x, p, 0)$.

**Proof.** The proof is postponed to “Appendix 5”.

**Lemma 2.4.** For any $\varepsilon \in (0, 1]$ and any $x \in M$, the backward calibrated curve $\gamma_{x, \varepsilon}^- : (-\infty, 0] \to M$ associated with $u_\varepsilon^-$ has a uniformly bounded velocity, i.e. there exists a constant $K > 0$ depending only on $H(x, p, 0)$, such that

$$|\dot{\gamma}_{x, \varepsilon}^-(t)| \leq K, \quad \forall \ t \in (-\infty, 0].$$

**Proof.** Due to Lemma 2.2, \[
\dot{\gamma}_{x, \varepsilon}^-(t) = \frac{\partial H}{\partial p}(\gamma_{x, \varepsilon}^-(t), d_x u_\varepsilon^-(\gamma_{x, \varepsilon}^-(t)), \varepsilon u_\varepsilon^-(\gamma_{x, \varepsilon}^-(t))), \quad t \in (-\infty, 0].
\]

Note that $u_\varepsilon^-$ is proved to be uniformly bounded and equi-Lipschitz due to Lemma 2.3, then $\dot{\gamma}_{x, \varepsilon}^-$ is also uniformly bounded.

3. Convergence of the Viscosity Solutions of Contact Hamilton–Jacobi Equations

In this section we prove Theorem 1.1, namely, $u_\varepsilon^-$ converges as $\varepsilon \to 0_+$, to a particular solution $u_0^-$ of (4). Due to the stability of viscosity solutions, any accumulating function $u_0^-$ of $u_\varepsilon^-$ as $\varepsilon \to 0_+$ will be a viscosity solution of (4).

If we could find a unique dynamic interpretation of $u_0^-$, it has to be unique. For readability we decompose the proof into a list of progressive propositions.

**Proposition 3.1.** For any Mather measure $\tilde{\mu}$, and any accumulating function $u_0^-(x)$ of $\{u_\varepsilon^-\}$ as $\varepsilon \to 0_+$, it holds that

$$\int_{TM} \frac{\partial L}{\partial u}(x, v, 0) \cdot u_0^-(x) \, d\tilde{\mu} \geq 0. \quad (11)$$

**Proof.** By the Birkhoff Ergodic Theorem, for any ergodic Mather measure $\tilde{\mu}$ there exists a generic Euler–Lagrange orbit $(\gamma(s), \dot{\gamma}(s)) := \varphi_{L,0}^s(\gamma(0), \dot{\gamma}(0))$ such that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T L(\gamma(t), \dot{\gamma}(t), 0) \, dt = \int_{TM} L(x, v, 0) \, d\tilde{\mu} = -c(H).$$

Let $u_0^-$ be the uniform limit of a sequence $u_{\varepsilon_n}^-$ with $\varepsilon_n \to 0_+$, then

$$u_{\varepsilon_n}^-(\gamma(T)) - u_{\varepsilon_n}^-(\gamma(0)) \leq \int_0^T L(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}^-(\gamma(s))) + c(H) \, ds$$
Convergence of Viscosity Solutions

\[ L(\gamma(s), \dot{\gamma}(s), 0) + c(H)ds \]

\[ + \int_0^T L(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}(\gamma(s))) - L(\gamma(s), \dot{\gamma}(s), 0)ds. \]

Note that

\[ \int_0^T L(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}(\gamma(s))) - L(\gamma(s), \dot{\gamma}(s), 0)ds \]

\[ = \int_0^1 \frac{d}{d\tau} L(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}(\gamma(s))) d\tau ds \]

\[ = \int_0^1 \int_0^T \frac{\partial L}{\partial u}(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}(\gamma(s))) \cdot \varepsilon_n u_{\varepsilon_n}(\gamma(s)) ds d\tau. \]

We derive

\[ \frac{1}{T} \left( u_{\varepsilon_n}(\gamma(T)) - u_{\varepsilon_n}(\gamma(0)) \right) \leq \frac{1}{T} \int_0^T L(\gamma(s), \dot{\gamma}(s), 0)ds + c(H) \]

\[ + \int_0^1 \frac{1}{T} \int_0^T \frac{\partial L}{\partial u}(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}(\gamma(s))) \cdot \varepsilon_n u_{\varepsilon_n}(\gamma(s)) ds d\tau. \]

By the Birkhoff Ergodic Theorem,

\[ \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{\partial L}{\partial u}(\gamma(s), \dot{\gamma}(s), \varepsilon_n u_{\varepsilon_n}(\gamma(s))) \cdot \varepsilon_n u_{\varepsilon_n}(\gamma(s)) ds \]

\[ = \int_{TM} \frac{\partial L}{\partial u}(x, \nu, \tau \varepsilon_n u_{\varepsilon_n}(x)) \cdot \varepsilon_n u_{\varepsilon_n}(x) d\tilde{\mu}. \]

Hence,

\[ 0 = \lim_{T \to +\infty} \frac{1}{T \varepsilon_n} \left( u_{\varepsilon_n}(\gamma(T)) - u_{\varepsilon_n}(\gamma(0)) \right) \]

\[ \leq \int_0^1 \left( \int_{TM} \frac{\partial L}{\partial u}(x, \nu, \tau \varepsilon_n u_{\varepsilon_n}(x)) \cdot u_{\varepsilon_n}(x) d\tilde{\mu} \right) d\tau. \]

By the Dominated Convergence Theorem, as \( n \) tends to \( +\infty \) we derive

\[ \int_{TM} \frac{\partial L}{\partial u}(x, \nu, 0) u_0(x) d\tilde{\mu} \geq 0. \]

Due to the Ergodic Decomposition Theorem, each Mather measure is a convex combination of ergodic Mather measures. So the assertion holds for any Mather measure.

This proposition inspires us to define the following set:
Definition 3.2. Suppose $\tilde{\mathcal{M}}$ consists of all the Mather measures, then we can denote by $\mathcal{F}_-$ the set of all $c(H)$-viscosity subsolution $u : M \to \mathbb{R}$ of (4) such that

$$
\int_{TM} \frac{\partial L}{\partial u}(x, v, 0) \cdot u(x) \, d\tilde{\mu} \geq 0, \quad \forall \tilde{\mu} \in \tilde{\mathcal{M}}.
$$

(12)

Remark 3.3. Recall that Proposition 3.1 indicates that $\mathcal{F}_-$ is nonempty since any accumulating function $u^-_0$ of $\{u^-_\varepsilon\}$ as $\varepsilon \to 0_+$ is an element of $\mathcal{F}_-$.

Lemma 3.4. The set $\mathcal{F}_-$ is uniformly bounded from above, i.e.

$$
\sup\{u(x) | \forall x \in M, u \in \mathcal{F}_-\} < +\infty.
$$

Proof. Recall that the set of $c(H)$-viscosity subsolutions of (4) is equi-Lipschitz with a Lipschitz constant $\kappa$ (see Proposition 2.3 in [7]). For any $u \in \mathcal{F}_-$, we have

$$
\min_{x \in M} u = \frac{\int_{TM} \frac{\partial L}{\partial u}(x, v, 0) \cdot \min_{x \in M} u \, d\tilde{\mu}}{\int_{TM} \frac{\partial L}{\partial u}(x, v, 0) \, d\tilde{\mu}} \leq \frac{\int_{TM} \frac{\partial L}{\partial u}(x, v, 0) \cdot u(x) \, d\tilde{\mu}}{\int_{TM} \frac{\partial L}{\partial u}(x, v, 0) \, d\tilde{\mu}} \leq 0.
$$

Therefore, $\max_{x \in M} u \leq \max u - \min u$. Due to the equi-Lipschitzness and the compactness of $M$, we have $\max u - \min u \leq \kappa \, \text{diam}(M) < +\infty$. \(\square\)

As $\mathcal{F}_-$ is now upper bounded, we can define a supreme subsolution by

$$
u^*_0 := \sup_{u \in \mathcal{F}_-} u.
$$

Later we will see that $\nu^*_0$ is actually a viscosity solution of (4) and the unique accumulating function of $\{u^-_\varepsilon\}$ as $\varepsilon \to 0_+$.

Proposition 3.5. Let $\omega$ be any subsolution of (4). For any $x \in M$, we have

$$
u^-_\varepsilon(x) \geq \omega(x) + \int_{TM} \omega(y) \int_0^1 \frac{\partial L}{\partial u}(y, v, \tau \varepsilon u^-_\varepsilon(y)) \, d\tau \, d\tilde{\mu}_x^\varepsilon(y, v),
$$

(14)

where $\tilde{\mu}_x^\varepsilon(\cdot, \cdot)$ is a finite measure defined by

$$
\int_{TM} f(y, v) d\tilde{\mu}_x^\varepsilon(y, v) = \varepsilon \int_{-\infty}^0 f(y^-_{x,\varepsilon}(t), \dot{y}^-_{x,\varepsilon}(t)) \exp \left( -\varepsilon \int_0^t \int_0^1 \frac{\partial L}{\partial u} \left( y^-_{x,\varepsilon}(s), \dot{y}^-_{x,\varepsilon}(s), \tau \varepsilon u^-_\varepsilon(y^-_{x,\varepsilon}(s)) \right) \, d\tau \, ds \right) \, dt
$$

(15)

for any $f(\cdot, \cdot) \in C_c(TM, \mathbb{R})$. 


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Proof. Due to Lemma 2.2, for any \( x \in M \) and any \( \varepsilon \in (0, 1] \), there exists a backward calibrated curve \( \gamma_{x, \varepsilon}^- : (-\infty, 0] \rightarrow M \) ending with \( x \). Moreover, the viscosity solution \( u_\varepsilon^- (\cdot) \) is differentiable along \( \gamma_{x, \varepsilon}^- \) for all \( t \in (-\infty, 0) \), which implies that

\[
\frac{d}{dt} u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) = L \left( \gamma_{x, \varepsilon}^- (t), \dot{\gamma}_{x, \varepsilon}^- (t), \varepsilon u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \right) + c(H) \\
= L(\gamma_{x, \varepsilon}^- (t), \dot{\gamma}_{x, \varepsilon}^- (t), 0) \\
+ \varepsilon \int_0^1 u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \frac{\partial L}{\partial u} \left( \gamma_{x, \varepsilon}^- (t), \dot{\gamma}_{x, \varepsilon}^- (t), \tau \varepsilon u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \right) d\tau \\
+ c(H).
\]

Recall that \( \omega \) is a subsolution of (4). Then,

\[
H(x, d_x \omega(x), 0) \leq c(H).
\]

As in [9], for any \( \delta > 0 \), there exists a smooth function \( w_\delta \) such that

\[
\| \omega - \omega_\delta \| < \delta \quad \text{and} \quad H(x, d_x \omega_\delta(x), 0) < c(H) + \delta.
\]

Combining the previous two inequalities we get

\[
\frac{d}{dt} \omega_\delta(\gamma_{x, \varepsilon}^- (t)) \leq \frac{d}{dt} u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \\
- \varepsilon u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \int_0^1 \frac{\partial L}{\partial u} \left( \gamma_{x, \varepsilon}^- (t), \dot{\gamma}_{x, \varepsilon}^- (t), \tau \varepsilon u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \right) d\tau + \delta.
\]

For brevity, we denote that

\[
\alpha_{x, \varepsilon}(t) := \int_0^1 \frac{\partial L}{\partial u} \left( \gamma_{x, \varepsilon}^- (t), \dot{\gamma}_{x, \varepsilon}^- (t), \tau \varepsilon u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \right) d\tau.
\]

Then

\[
\frac{d}{dt} \left( u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) e^{-\varepsilon \int_0^t \alpha_{x, \varepsilon}(s) ds} \right) \geq e^{-\varepsilon \int_0^t \alpha_{x, \varepsilon}(s) ds} \left( \frac{d}{dt} \omega_\delta(\gamma_{x, \varepsilon}^- (t)) - \delta \right).
\]

Integrating both sides, we get

\[
u_\varepsilon^- (\gamma_{x, \varepsilon}^- (t)) \exp \left( - \varepsilon \int_0^t \alpha_{x, \varepsilon}(s) ds \right) \bigg|_{-T}^0
\geq \int_{-T}^0 \exp \left( - \varepsilon \int_0^t \alpha_{x, \varepsilon}(s) ds \right) \cdot \left( \frac{d}{dt} \omega_\delta(\gamma_{x, \varepsilon}^- (t)) - \delta \right) dt,
\]

which can be transferred into

\[
u_\varepsilon^- (x) - u_\varepsilon^- (\gamma_{x, \varepsilon}^- (-T)) \exp \left( \varepsilon \int_{-T}^0 \alpha_{x, \varepsilon}(s) ds \right)
\geq \omega_\delta(x) - \left( \omega_\delta(\gamma_{x, \varepsilon}^- (-T)) + \delta T \right) \exp \left( \varepsilon \int_{-T}^0 \alpha_{x, \varepsilon}(s) ds \right).
\]
\[- \int_{-T}^{0} (\omega_{\delta}(\gamma_{x,\varepsilon}(t)) - \delta t) \frac{d}{dt} \exp \left( - \varepsilon \int_{0}^{t} \alpha_{x,\varepsilon}(s) \, ds \right) \, dt \]

\[= \omega_{\delta}(x) - (\omega_{\delta}(\gamma_{x,\varepsilon}(-T)) + \delta T) \exp \left( \varepsilon \int_{-T}^{0} \alpha_{x,\varepsilon}(s) \, ds \right) \]

\[+ \varepsilon \int_{-T}^{0} \omega_{\delta}(\gamma_{x,\varepsilon}^{-}(t) - \delta t) \cdot \alpha_{x,\varepsilon}(t) \cdot \exp \left( - \varepsilon \int_{0}^{t} \alpha_{x,\varepsilon}(s) \, ds \right) \, dt. \quad (16)\]

Taking \( \delta \to 0_{+} \), we derive that

\[u_{\varepsilon}^{-}(x) - u_{\varepsilon}^{-}(\gamma_{x,\varepsilon}^{-}(-T)) \geq \omega(x) - \omega(\gamma_{x,\varepsilon}^{-}(-T)) \cdot \exp \left( \varepsilon \int_{-T}^{0} \alpha_{x,\varepsilon}(s) \, ds \right) \]

\[+ \varepsilon \int_{-T}^{0} \omega(\gamma_{x,\varepsilon}^{-}(t)) \cdot \alpha_{x,\varepsilon}(t) \cdot \exp \left( - \varepsilon \int_{0}^{t} \alpha_{x,\varepsilon}(s) \, ds \right) \, dt. \]

By Lemmas 2.2 and 2.3, there exists a constant \( a > 0 \) such that \( \alpha_{x,\varepsilon}(t) < -a \) for all \( t < 0 \). Then

\[\lim_{T \to +\infty} \exp \left( \int_{-T}^{0} \alpha_{x,\varepsilon}(s) \, ds \right) = 0.\]

Taking \( T \to +\infty \) of (16), we derive

\[u_{\varepsilon}^{-}(x) \geq \omega(x) + \varepsilon \int_{-\infty}^{0} \omega(\gamma_{x,\varepsilon}(t)) \cdot \alpha_{x,\varepsilon}(t) \cdot \exp \left( - \varepsilon \int_{0}^{t} \alpha_{x,\varepsilon}(s) \, ds \right) \, dt,\]

which completes the proof. \( \Box \)

As we can see from (15), for any \( x \in M \) and \( \varepsilon \in (0, 1] \), the Radon measure \( \tilde{\mu}_{x}^{\varepsilon} \) is uniformly bounded, since \( \alpha_{x,\varepsilon}(t) < -a < 0 \) for all \( t < 0 \). The following conclusion implies that if \( \tilde{\mu}_{x}^{\varepsilon} \) weakly converges to \( \tilde{\mu}_{x} \), then \( \tilde{\mu}_{x} \) has to be a rescaled Mather measure

**Lemma 3.6.** Any weak limit \( \hat{\mu}_{x} \) of the normalized measure

\[\hat{\mu}_{x}^{\varepsilon} := \left. \frac{\tilde{\mu}_{x}^{\varepsilon}}{\int_{TM} d\tilde{\mu}_{x}^{\varepsilon}} \right|_{\varepsilon \to 0_{+}} \tag{17}\]

is a Mather measure. Accordingly, \( \mu_{x} := (\pi)^{*}\hat{\mu}_{x} \) is a projected Mather measure, where \( \pi : TM \to M \) is the standard projection.

**Proof.** For brevity, let us denote that

\[\beta_{x,\varepsilon}(t) := \exp \left( - \varepsilon \int_{0}^{t} \alpha_{x,\varepsilon}(s) \, ds \right).\]

For any sequence \( \varepsilon \to 0_{+} \) such that \( \hat{\mu}_{x}^{\varepsilon} \) weakly converges to a \( \hat{\mu}_{x} \), it suffices to show \( \hat{\mu}_{x} \in \mathfrak{M} \) by the following two steps (due to Theorem B in [11]):
First, we show \( \tilde{\mu}_x \) is a closed measure, which is equivalent to showing that for any \( \phi(\cdot) \in C^1(M, \mathbb{R}) \),
\[
\lim_{\varepsilon \to 0^+} \varepsilon \int_{-\infty}^0 \frac{d}{dt} \phi(y_{x,\varepsilon}(t)) \beta_{x,\varepsilon}(t) \, dt = 0.
\]
Taking integration by parts for the left side, we get
\[
\varepsilon \beta_{x,\varepsilon}(t) \phi(y_{x,\varepsilon}^{-}(t)) \Big|_{-\infty}^{0} - \varepsilon \int_{-\infty}^0 \phi(y_{x,\varepsilon}^{-}(t)) \, d\beta_{x,\varepsilon}(t),
\]
which tends to zero as \( \varepsilon \to 0^+ \) since there exists a \( \delta > 0 \) such that for any \( x \in M \), \( \varepsilon \in (0, 1] \) and \( t \in (-\infty, 0] \), \( \frac{\partial L}{\partial u} (y_{x,\varepsilon}^{-}(t), \dot{y}_{x,\varepsilon}^{-}(t), u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t))) \leq -\delta < 0 \).

Next, we will show that
\[
\lim_{\varepsilon \to 0^+} \varepsilon \int_{-\infty}^0 \left[ L(y_{x,\varepsilon}^{-}(t), \dot{y}_{x,\varepsilon}^{-}(t), \varepsilon u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t))) + c(H) \right] \beta_{x,\varepsilon}(t) \, dt = 0. \tag{18}
\]
Recall that for any backward calibrated curve \( y_{x,\varepsilon}^{-} \), Lemma 2.2 tells us \( u_{\varepsilon}^{-} \) is differentiable along it for all \( t \in (-\infty, 0) \). Therefore, we have
\[
\frac{d}{dt} u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t)) = L(y_{x,\varepsilon}^{-}(t), \dot{y}_{x,\varepsilon}^{-}(t), \varepsilon u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t)))
\]
\[
+ H(y_{x,\varepsilon}^{-}(t), d_x u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t)), \varepsilon u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t)))
\]
\[
= L(y_{x,\varepsilon}^{-}(t), \dot{y}_{x,\varepsilon}^{-}(t), \varepsilon u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t))) + c(H), \tag{19}
\]
which implies
\[
\lim_{\varepsilon \to 0^+} \varepsilon \int_{-\infty}^0 \frac{d}{dt} u_{\varepsilon}^{-}(y_{x,\varepsilon}^{-}(t)) \beta_{x,\varepsilon}(t) \, dt = 0,
\]
since \( \tilde{\mu}_x \) proves to be a closed measure so (18) is obtained. Due to Theorem B of [11], the normalized measure \( \tilde{\mu}_x \) is indeed a Mather measure. With the help of the standard projection \( \pi : TM \to M \), we get \( \mu_x := \pi_\# \tilde{\mu}_x \) is a projected Mather measure.

**Proof of Theorem 1.1.** Now we are ready to prove \( u_0^- \) is the unique accumulating function of \( \{u_{\varepsilon}^-\} \) as \( \varepsilon \to 0^+ \). First, for any accumulating function \( u_0^- \), due to Proposition 3.1, \( u_0^- \in \mathcal{F}_- \). That implies \( u_0^- \leq u_0^* \). On the other side, due to Proposition 3.5, if we take \( \omega \in \mathcal{F}_- \), then
\[
u_0^- = \omega(x) + \int_{TM} \omega(y) \frac{\partial L}{\partial \mu}(y, v, 0) \, d\hat{\mu}_x(y, v)
\]
\[
= \omega(x) + \int_{TM} \omega(y) \frac{\partial L}{\partial \mu}(y, v, 0) \, d\hat{\mu}_x(y, v) \cdot \int_{TM} d\hat{\mu}_x,
\]
\[
\geq \omega(x), \tag{20}
\]
since \( \hat{\mu}_x \) is a Mather measure due to Lemma 3.6. As a result, we get \( u_0^- \geq \sup_{\omega \in \mathcal{F}_-} \omega = u_0^* \). Combining these two parts we get \( u_0^- = u_0^* \). \qed
4. Peierls Barrier’s Interpretation of the Limit Solution

In this section, we will give a different characterization of $u_0^*$, by using the Peierls barrier $h^\infty(\cdot, \cdot)$ and the set of Mather measures $\hat{\mathcal{M}}$. In Section 3 we have proved the convergence of $u_\varepsilon$ to $u_0^*$ as $\varepsilon \to 0+$. We will show the equivalence of $u_0^*$ to

$$\hat{u}_0^*(x) = \inf_{\hat{\mu} \in \hat{\mathcal{M}}} \frac{\int_{TM} h^\infty(y, x) \cdot \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}, \quad \forall x \in M. \quad (21)$$

**Proof of Theorem 1.2.** This proof can be divided into two steps. First, for any $\omega \in \mathcal{F}_-$, we have

$$\omega(x) - \omega(y) \leq h^\infty(y, x), \quad \forall x, y \in M.$$  

This implies that

$$\omega(x) = \frac{\int_{TM} \omega(x) \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)} \leq \frac{\int_{TM} h^\infty(y, x) \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)} + \frac{\int_{TM} h^\infty(y, x) \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)} \leq \frac{\int_{TM} h^\infty(y, x) \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}, \quad \forall \hat{\mu} \in \hat{\mathcal{M}}.$$  

By taking the infimum over $\hat{\mu} \in \hat{\mathcal{M}}$ for the right side, we get

$$\omega(x) \leq \hat{u}_0^*(x), \quad \forall x \in M,$$

then, by taking the supremum over $\omega \in \mathcal{F}_-$ for the left side, we get $u_0^* \leq \hat{u}_0^*$. To show that $\hat{u}_0^* \leq u_0^*$, we first show that $\hat{u}_0^*$ is a viscosity subsolution of (4). Due to the Ergodic Decomposition Theorem, any $\hat{\mu} \in \hat{\mathcal{M}}$ can be represented as a convex combination of a family of ergodic measure $\hat{\mu}_i \in \mathrm{ex}(\hat{\mathcal{M}})$. On the other side, for any fixed $y \in M$, $h^\infty(y, \cdot)$ is a viscosity solution of (4), which is definitely a subsolution (shown in Proposition 5.3 in [7]). By the convexity of $H(x, p, 0)$ in $p$-variable and the equi-Lipschitz continuity of all the subsolutions, it follows that

$$h_\mu^\infty(x) := \frac{\int_{TM} h^\infty(y, x) \cdot \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}{\int_{TM} \frac{\partial L}{\partial u}(y, v, 0) \, d\hat{\mu}(y, v)}, \quad \forall x \in M$$

is a subsolution as well. By Proposition 2.1 in [7], we infer that an infimum of $h_\mu^\infty$ over $\hat{\mu} \in \hat{\mathcal{M}}$ is still a subsolution. Thus $\hat{u}_0^*$ is a subsolution of (4). By Proposition 5.5 of [7], we just need to show $\hat{u}_0^* \leq u_0^*$ on the projected Aubry set $\hat{A} := \{x \in M|h^\infty(x, x) = 0\}$, then $\hat{u}_0^*(x) \leq u_0^*(x)$ for all $x \in M$. Recall that for any $y \in \hat{A}$ fixed, the function $-h^\infty(\cdot, y)$ is a viscosity subsolution of (4), then

$$\omega(x) := -h^\infty(x, y) + \inf_{\hat{\mu} \in \hat{\mathcal{M}}} \frac{\int_{TM} h^\infty(z, y) \frac{\partial L}{\partial u}(z, v, 0) \, d\hat{\mu}(z, v)}{\int_{TM} \frac{\partial L}{\partial u}(z, v, 0) \, d\hat{\mu}(z, v)}$$
Convergence of Viscosity Solutions

\[ = -h^\infty(x, y) + \hat{u}_0(y) \]

is a subsolution as well. For any \( \tilde{\mu} \in \tilde{\mathcal{M}} \), we have

\[
\int_{TM} \omega(x) \frac{\partial L}{\partial u}(x, v, 0) d\tilde{\mu}(x, v) = -\int_{TM} h^\infty(x, y) \frac{\partial L}{\partial u}(x, v, 0) d\tilde{\mu}(x, v) + \hat{u}_0(y) \int_{TM} \frac{\partial L}{\partial u}(x, v, 0) d\tilde{\mu}(x, v) \geq 0,
\]
due to \( \frac{\partial L}{\partial u}(x, v, 0) < 0 \) for all points in \( TM \times \mathbb{R} \) and the definition of \( \hat{u}_0 \). That implies \( \omega \in \mathcal{F}_- \). Therefore, \( \omega(x) \leq u_0^*(x) \) for all \( x \in M \). In particular, we have that

\[ u_0^*(y) \geq \omega(y) = -h^\infty(y, y) + \hat{u}_0(y) = \hat{u}_0(y), \quad \forall y \in A. \]

This finishes the proof. \( \square \)

4.1. A Comparison with the Discounted System

In this part we make a comparison of the vanishing viscosity limit of solutions between the discounted Hamiltonians and the general contact Hamiltonians. For two different Hamiltonians satisfying our standing assumptions,

( Discounted ) \( F(x, p, \varepsilon u) = H_0(x, p) + \varepsilon u \)

and

( Contact ) \( G(x, p, \varepsilon u) = H_0(x, p) + \varepsilon uH_1(x, p) + \varepsilon^2 H_2(x, p, \varepsilon u) \),

the convergence of associated viscosity solutions \( u_{F, \varepsilon}^- \) (resp. \( u_{G, \varepsilon}^- \)) will be quite different as \( \varepsilon \to 0_+ \), once \( H_1(x, p) \) doesn’t equal to a constant. We will explain this point by the following example:

**Example.** Suppose \( (x, p) \in T^*\mathbb{T} \) and we take

\[ H_0(x, p) = \frac{1}{2} p(p + 2V(x)). \quad (22) \]

Notice that for this system there exists a unique variational minimal periodic orbit \( x(t) \in \mathbb{T} \) with

\[ \dot{x}(t) = V(x(t)), \quad \forall t \in \mathbb{R}, \]

once \( V \) keeps positive but not constant.

Due to Corollary 1.3, the discounted vanishing of \( F(x, p, \varepsilon u) \) gives us the limit by

\[ u_F^-(x) = \frac{1}{T_p} \int_0^{T_p} h^\infty(x(t), x) \, dt \]
\[ \frac{1}{2\pi} \int_0^{2\pi} h^\infty(\theta, x) \frac{2\pi}{T_p \cdot V(\theta)} \, \mathrm{d}\theta, \]  
where \( T_p \) is the period of \( x(t) \). Actually, \( \frac{2\pi}{T_p V(\theta)} \) is a density function on \( \mathbb{T} \). However, if we take \( H_1(x, p) = V(x) \), the contact vanishing of \( G(x, p, \varepsilon u) \) give us the limit by

\[ u_{G}^-(x) = \frac{\int_0^{T_p} h^\infty(x(t), x) f(x(t)) \, \mathrm{d}t}{\int_0^{T_p} f(x(t)) \, \mathrm{d}t} = \frac{1}{2\pi} \int_0^{2\pi} h^\infty(\theta, x) \, \mathrm{d}\theta. \]  

We have freedom to choose \( V(\cdot) \) such that \( u^-_f(x) \neq u^-_G(x) \).

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**Appendix A: The Proof of Lemma 2.3**

For any \( \varepsilon \in (0, 1] \) and \( \phi \in C^0(M, \mathbb{R}) \), we have

\[ T_t^{\varepsilon-}\phi(x) = \inf_{\gamma(t) = x} \{ \phi(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), \varepsilon T_s^{\varepsilon-}\phi(\gamma(s))) + c(H) \, \mathrm{d}s \}, \]

implicitly defined, with the infimum taken among all piecewise \( C^1 \) curve. Furthermore, for each \( \phi \in C(M, \mathbb{R}) \),

\[ \lim_{t \to +\infty} T_t^{\varepsilon-}\phi(x) = u^-_\varepsilon(x), \]

where \( u^-_\varepsilon \) is the unique weak KAM solution of \( H(x, \partial_x u^-_\varepsilon, \varepsilon u^-_\varepsilon) = c(H) \), see Theorem 1.4 in [13] and Appendix B in [14]. Now we are ready to give a proof of Lemma 2.3.

**Lemma A.1.** For any \( \phi \in C(M, \mathbb{R}) \) with \( \|\phi\| \leq 1 \), the family \( \{ T_t^{\varepsilon-}\phi(\cdot)|\varepsilon \in (0, 1], t \geq 1 \} \) is uniformly bounded.
**Proof.** We claim that \( \{ T_t^{\epsilon - \phi} \epsilon \in (0, 1], t \geq 1 \} \) is uniformly bounded from below. Without loss of generality, we assume \( T_t^{\epsilon - \phi} (x) < 0 \) for some \( \epsilon \in (0, 1] \) and \((x, t) \in M \times [1, +\infty)\), otherwise 0 would be a lower bound of the family. Let \( \gamma_{x, \epsilon} : [0, t] \to M \) be the associated minimizer of \( T_t^{\epsilon - \phi} (x) \). Then, there are two

**Case I** There exists \( s_0 \in [0, t) \) such that \( T_{s_0}^{\epsilon - \phi} (\gamma_{x, \epsilon}(s_0)) = 0 \) and \( T_{s}^{\epsilon - \phi} (\gamma_{x, \epsilon}(s)) \leq 0, s \in (s_0, t] \).

**Case II** \( T_{t}^{\epsilon - \phi} (\gamma_{x, \epsilon}(s)) < 0 \) for all \( s \in [0, t] \).

For Case I, due to (H3), \( L \) is strictly decreasing of \( u \). Therefore,

\[
T_t^{\epsilon - \phi} (x) = T_{s_0}^{\epsilon - \phi} (\gamma_{x, \epsilon}(s_0)) + \int_{s_0}^{t} L(\gamma_{x, \epsilon}(s), \dot{\gamma}_{x, \epsilon}(s), \epsilon T_s^{\epsilon - \phi} (\gamma_{x, \epsilon}(s))) \, ds + c(H) \, ds \\
\geq \int_{s_0}^{t} L(\gamma_{x, \epsilon}(s), \dot{\gamma}_{x, \epsilon}(s), 0) + c(H) \, ds \\
\geq h^{t - s_0}(\gamma_{x, \epsilon}(s_0), x) + c(H)(t - s_0).
\]

It is well known that \( h^{t - s_0}(\gamma_{x, \epsilon}(s_0), x) + c(H)(t - s_0) \) is uniformly bounded from below (see Lemma 5.3.2 in [8] for instance). Moreover, the lower bound can be made independent of the selection of \( \epsilon \in (0, 1] \) and \((x, t - s_0) \in M \times (0, +\infty)\).

For Case II,

\[
T_t^{\epsilon - \phi} (x) = \phi(\gamma_{x, \epsilon}(0)) \geq \min_{x \in M} \phi(x) + \int_{0}^{t} L(\gamma_{x, \epsilon}(s), \dot{\gamma}_{x, \epsilon}(s), 0) + c(H) \, ds \\
\geq -1 + h^{t}(\gamma_{x, \epsilon}(0), x) + c(H)t.
\]

Hence, \( \{ T_t^{\epsilon - \phi} (\cdot) \epsilon \in (0, 1] \} \) is bounded from below. Still the lower bound is independent of the selection of \( \epsilon \in (0, 1] \) and \((x, t) \in M \times [1, +\infty)\).

As a summary, the family \( \{ T_t^{\epsilon - \phi} (\cdot) \epsilon \in (0, 1], t \geq 1 \} \) is uniformly bounded from below.

We claim \( \{ T_t^{\epsilon - \phi} (\cdot) \epsilon \in (0, 1], t \geq 1 \} \) is uniformly bounded from above. Without loss of generality, we assume \( T_t^{\epsilon - \phi} (x) > 0 \) for some \( \epsilon \in (0, 1] \) and \((x, t) \in M \times [1, +\infty)\), otherwise 0 is a upper bound of \( \{ T_t^{\epsilon - \phi} (\cdot) \epsilon \in (0, 1], t \geq 1 \} \).

Let \( \beta : [0, t] \to M \) be the associated minimizer of \( h^{t}(\gamma_{x, \epsilon}(0), x) \), i.e.,

\[
h^{t}(\gamma_{x, \epsilon}(0), x) = \int_{0}^{t} L(\beta(s), \dot{\beta}(s), 0) \, ds.
\]

There are also two

**Case I'** \( T_s^{\epsilon - \phi} (\beta(s)) > 0 \) for each \( s \in [0, t] \). Hence,

\[
T_t^{\epsilon - \phi} (x) \leq \phi(\beta(0)) + \int_{0}^{t} L(\beta(s), \dot{\beta}(s), \epsilon T_s^{\epsilon - \phi} (\beta(s))) + c(H) \, ds \\
\leq \max_{x \in M} \phi(x) + \int_{0}^{t} L(\beta(s), \dot{\beta}(s), 0) + c(H) \, ds
\]
\[ \leq 1 + h^t(\gamma_{x, \varepsilon}(0), x) + c(H)t. \]

Since \( t \geq 1 \), \( h^t(\gamma_{x, \varepsilon}(0), x) + c(H)t \) is bounded from above. Hence, \( \{ T^0_t \varepsilon \phi(\cdot) | \varepsilon \in (0, t], t \geq 1 \} \) is uniformly bounded from above.

**Case II'** There exists \( s_1 \in [0, t) \) such that \( T^0_{s_1} \varepsilon \phi(\beta(s_1)) = 0 \) and \( T^0_s \varepsilon \phi(\beta(s)) > 0 \), \( s \in (s_1, t) \). Then,

\[
T^0_t \varepsilon \phi(x) \leq T^0_{s_1} \varepsilon \phi(\beta(s_1)) + \int_{s_1}^t L(\dot{\beta}(s), \dot{\epsilon}(s), \varepsilon \phi(\beta(s))) + c(H) \, ds
\]

\[
= \int_{s_1}^t L(\dot{\beta}(s), \dot{\epsilon}(s), 0) + c(H) \, ds
\]

\[
= h^{t-s_1}(\beta(s_1), x) + c(H)(t - s_1).
\]

If \( t - s_1 \geq \frac{1}{2} \), then \( h^{t-s_1}(\beta(s_1), x) + c(H)(t - s_1) \) is bounded from above. If not, then \( s_1 > \frac{1}{2} \). Note that

\[
h^t(\gamma_{x, \varepsilon}(0), x) = h^{s_1}(\gamma_{x, \varepsilon}(0), \beta(s_1)) + h^{t-s_1}(\beta(s_1), x).
\]

We derive that

\[
h^{t-s_1}(\beta(s_1), x) + c(H)(t - s_1) = \left( h^t(\gamma_{x, \varepsilon}(0), x) + c(H)t \right)
\]

\[
- \left( h^{s_1}(\gamma_{x, \varepsilon}(0), \beta(s_1)) + c(H)s_1 \right),
\]

Note that the first term is bounded from above \( (t \geq 1) \) and the second term is bounded from below \( (s_1 > 1/2) \), see Lemma 5.3.2 in [8]. Hence, \( \{ T^0_t \varepsilon \phi(\cdot) | \varepsilon \in (0, 1], t \geq 1 \} \) is uniformly bounded from above.

**Lemma A.2.** The family \( \{ u^-_\varepsilon \} \) is uniformly bounded for all \( \varepsilon \in (0, 1] \).

**Proof.** Due to the boundedness of \( \{ T^0_t \varepsilon \phi(\cdot) | \varepsilon \in (0, 1], t \geq 1 \} \), there exists a \( K > 0 \) such that for \( \phi \in C(M, \mathbb{R}) \) satisfying \( \| \phi \| \leq 1 \),

\[ |T^0_t \varepsilon \phi(x)| \leq K, (x, t) \in M \times [1, +\infty) \text{ and } \varepsilon \in (0, 1]. \]

Since \( \lim_{t \rightarrow +\infty} T^0_t \varepsilon \phi(x) = u^-_\varepsilon(x) \) is uniquely established, we get \( |u^-_\varepsilon(x)| \leq K \) for all \( \varepsilon \in (0, 1] \), immediately.

**Lemma A.3.** The map \( x \mapsto u^-_\varepsilon(x) \) is equi-Lipschitz for \( \varepsilon \in (0, 1] \).

**Proof.** Let \( x, y \in M \) and \( \alpha : [0, d_R(x, y)] \rightarrow M \) be a geodesic connecting \( x \) and \( y \). Note that \( \sqrt{\langle \alpha, \alpha \rangle} \) is the Riemannian metric on \( M \). We derive that

\[
u^-_\varepsilon(y) - u^-_\varepsilon(x) \leq \int_0^{d_R(x, y)} L(\alpha(s), \dot{\alpha}(s), \varepsilon u^-_\varepsilon(\alpha(s))) + c(H) \, ds
\]

\[
\leq \int_0^{d_R(x, y)} L(\alpha(s), \dot{\alpha}(s), 0) + \Delta \cdot K + c(H) \, ds
\]
\[ \geq (C_1 + \Delta \cdot K + c(H))d_R(x, y), \]

where \( C_1 \) is a uniform constant such that

\[ L(x, v, 0) \leq C_1, \quad \forall \|v\|_R \leq 1. \]

By switching the role of \( x \) and \( y \), we derive

\[ u^-_\varepsilon(x) - u^-_\varepsilon(y) \leq (C_1 + \Delta \cdot K + c(H))d_R(x, y), \]

and then

\[ |u^-_\varepsilon(x) - u^-_\varepsilon(y)| \leq (C_1 + \Delta \cdot K + c(H))d_R(x, y), \]

This finishes the proof. \( \square \)

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