THE $K_t$–FUNCTIONAL FOR THE
INTERPOLATION COUPLE $L^\infty(d\mu; L^1(d\nu))$, $L^\infty(d\nu; L^1(d\mu))$

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§0. Introduction

Let $(A_0, A_1)$ be a compatible couple of Banach spaces in the sense of Interpolation Theory (cf. e.g. [BL]). Then the $K_t$–functional of an element $x$ in $A_0 + A_1$ is defined for all $t > 0$ as follows

$$K_t(x; A_0, A_1) = \inf \{ \|x_0\|_{A_0} + t \|x_1\|_{A_1} \mid x = x_0 + x_1 \}.$$  

For instance the $K_t$–functional of the couple $(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$ is well known. We have

$$K_t(x; L^1(\mathbb{R}), L^\infty(\mathbb{R})) = \int_0^t x^*(s) \, ds = \sup \left\{ \int_E |x(s)| \, ds, |E| = t \right\}.$$ 

See [BL] for more details and references.

Let $(M, \mu)$ and $(N, \nu)$ be measure spaces. In this paper, we study the $K_t$–functional for the couple

(0.1) \quad A_0 = L^\infty(d\mu; L^1(d\nu)), \quad A_1 = L^\infty(d\nu; L^1(d\mu)).

Here, and in what follows the vector valued $L^p$–spaces $L^p(d\mu; L^q(d\nu))$ are meant in Bochner’s sense.

One of our main results is the following, which can be viewed as a refinement of a lemma due to Varopoulos [V].

**Theorem 0.1.** Let $(A_0, A_1)$ be as in (0.1). Then for all $f$ in $A_0 + A_1$ we have

$$\frac{1}{2} K_t(f; A_0, A_1) \leq \sup \left\{ \left( (\mu(E) \vee t^{-1} \nu(F)) \right)^{-1} \int_{E \times F} |f| \, d\mu \, d\nu \right\} \leq K_t(f; A_0, A_1),$$

where the supremum runs over all measurable subsets $E \subset M$, $F \subset N$ with positive and finite measure and $u \vee v$ denotes the maximum of the reals $u$ and $v$.

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1 Partially supported by the N.S.F.
This result is a particular case of Theorem 3.2. below and its corollaries, which give an analogous estimate for the couple

\[ A_0 = L^{p,\infty}(d\mu ; L^q(d\nu)), \quad A_1 = L^{p,\infty}(d\nu ; L^q(d\mu)), \]

when \( 1 \leq q < p \leq \infty \). The preceding Theorem 0.1 corresponds to the case \( p = \infty, q = 1 \).

The paper is organized as follows. In §1, we prove Theorem 0.1 in the finite discrete case, i.e. when \( M \) and \( N \) are finite sets equipped with discrete measures. In §2, we discuss the extension to \( L^{p,\infty}(d\mu ; L^q(d\nu)), L^{p,\infty}(d\nu ; L^q(d\mu)) \), again in the discrete case and treat the case of general measure spaces in §3. Finally in §4 we give some applications.

To describe the main one, let us denote by \( B_1 \) (resp. \( B_0 \)) the space of all bounded operators from \( L^1(d\nu) \) to \( L^1(d\mu) \) (resp. from \( L^\infty(d\nu) \) to \( L^\infty(d\mu) \)). Our results yield a description of the space \((B_0, B_1)_{\theta,q}, 0 < \theta < 1, 1 \leq q \leq \infty, \) obtained by the real (i.e. Lions–Peetre) interpolation method. The result is particularly simple in the case \( q = \infty \). In that case, we prove

**Theorem 0.2.** The space \((B_0, B_1)_{\theta,\infty}\) coincides with the space of all bounded regular operators \( u \) from \( L^{p,1}(d\nu) \) to \( L^{p,\infty}(d\mu) \) with \( p = 1/\theta \).

Here ”regular” is meant in the sense of [MN]. Equivalently \( u \) belongs to \((B_0, B_1)_{\theta,\infty}\) iff there is a positive (i.e. positivity preserving) operator \( v \) from \( L^{p,1}(d\nu) \) to \( L^{p,\infty}(d\mu) \) which dominates \( u \), i.e. such that \(-v \leq u \leq v\). (Note that we implicitly use only real scalars, but this is not essential.) In particular, it follows that \( u \) belongs to \((B_0, B_1)_{\theta,\infty}\) iff \(|u|\) is bounded from \( L^{p,1}(d\nu) \) to \( L^{p,\infty}(d\mu) \), or equivalently iff \(|u|\) is of ”very weak type \((p,p)\)” in the sense of [SW, chapter 3.3]. We refer to [MN] for the definition of the modulus \(|u|\) of a regular operator acting between Banach lattices. We merely recall that if \( u \) is given by a matrix \((a_{ij})\) acting between two sequence spaces, then \(|u|\) corresponds to the matrix \((|a_{ij}|)\). A similar fact holds with ”kernels” instead of matrices.
§1. The $K_t$–functional for the interpolation couple $\ell^\infty_m(\ell^1_n), \ell^\infty_n(\ell^1_m)$

We use the following notation: for $p \in \mathbb{R}, p \geq 1$, let $p^*$ be the conjugate exponent of $p$, $1/p + 1/p^* = 1$, the dot $\cdot$ denotes the pointwise multiplication of matrices $a, b$ according to $(a \cdot b)(i, j) = a(i, j)b(i, j)$. For a set $A$ let $1_A$ be the characteristic function of $A$ (the whole set will always be known from the context).

Let $(M, \mu)$ be the measure space consisting of the atoms $\{1, \ldots, m\}$ of positive masses $\mu_1, \ldots, \mu_m$ and $(N, \nu)$ the measure space consisting of the atoms $\{1, \ldots, n\}$ of positive masses $\nu_1, \ldots, \nu_n$. We equip $M \times N$ with the product measure $\mu \times \nu$.

Besides the $\ell^p$–norms

$$\|a\|_{\ell^p} = \left( \sum_{(i,j) \in M \times N} |a(i, j)|^p \mu_i \nu_j \right)^{1/p},$$

we introduce in this section for $m \times n$–matrices $a$ the norms

$$\|a\| = \|a\|_{\ell^\infty_m(\ell^1_n)} = \max \left( \sum_{j \in N} |a(i, j)| \nu_j \right),$$

$$\|a\|^T = \|a^T\|_{\ell^\infty_n(\ell^1_m)} = \max \left( \sum_{i \in M} |a(i, j)| \mu_i \right).$$

It is straightforward to see that for $E \subset M, F \subset N$,

$$\|1_{E \times F} \cdot a\|_{\ell^1} \leq \min \{\mu(E)\|a\|, \nu(F)\|a\|^T\}.$$

Therefore we have for matrices $b, c$ and any $t > 0$

$$(1.1) \quad \|1_{E \times F} \cdot (b + c)\|_{\ell^1} \leq \left( \mu(E) \lor t^{-1} \nu(F) \right) \left( \|b\| + t \|c\|^T \right).$$

If we introduce the norm

$$\|a\|_t = \sup_{E, F} \left( \mu(E) \lor t^{-1} \nu(F) \right)^{-1} \|1_{E \times F} \cdot a\|_{\ell^1},$$

then (1.1) means that the $K_t$–functional for the interpolation couple $\ell^\infty_m(\ell^1_n), \ell^\infty_n(\ell^1_m)$

$$K_t(a) = \inf \{\|b\| + t \|c\|^T \mid a = b + c\}$$

is an upper bound for this norm

$$\|a\|_t \leq K_t(a).$$

In the following we prove an upper estimate of $K_t(.)$ by the functional $\|\cdot\|_t$. 

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Proposition 1.1. Let \( t > 0 \). For any matrix \( a \) with \( \| a \|_t \leq 1 \) we can split \( M \times N \) into a disjoint union \( M \times N = A \cup B \), \( A \cap B = \emptyset \), such that

\[
\| 1_A \cdot a \| \leq 1, \quad \| 1_B \cdot a \|_\top \leq 1/t,
\]

hence

\[
K_t(a) \leq 2 \| a \|_t.
\]

Proof. We proceed by induction on \( m \). For \( m = 1 \) suppose, without loss of generality, that

\[
|a(1,1)| \geq |a(1,2)| \geq \ldots \geq |a(1,n)|.
\]

If \( t^{-1}\nu(N) \leq \mu(\{1\}) = \mu_1 \) we put \( k = n \), if not, let \( k \), \( 0 \leq k < n \), be as large as possible such that \( t^{-1}\nu(\{1, \ldots, k\}) \leq \mu_1 \). For \( E = \{1\} \) and \( F = \{1, \ldots, k\} \) we have in any case

\[
(\mu(E) \vee t^{-1}\nu(F))^{-1} \| 1_{E \times F} \cdot a \|_t = \sum_{j \leq k} |a(1,j)| \nu_j \leq 1.
\]

If \( k < n \) then \( t^{-1}\nu(\{1, \ldots, k+1\}) > \mu_1 \), so we obtain for \( E = \{1\} \) and \( F = \{1, \ldots, k+1\} \)

\[
(\mu(E) \vee t^{-1}\nu(F))^{-1} \| 1_{E \times F} \cdot a \|_t = \mu_1 t \frac{\sum_{j \leq k+1} |a(1,j)| \nu_j}{\sum_{j \leq k+1} \nu_j} \leq 1.
\]

By (1.3) this yields

\[
|a(1,n)| \mu_1 \leq \ldots \leq |a(1,k+1)| \mu_1 \leq 1/t.
\]

Take now \( A = \{1\} \times \{1, \ldots, k\} \), \( B = (M \times N) \setminus A \), and (1.2) is fulfilled.

Let us assume the truth of the proposition for \( 1, \ldots, m-1 \). Without loss of generality, we can suppose the sums over \( M \)

\[
\sigma_j = \sum_{i \leq m} |a(i,j)| \mu_i
\]

to be in descending order \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \). If \( t^{-1}\nu(N) \leq \mu(M) \) let \( k = n \), if not, let \( k \), \( 0 \leq k < n \), be the largest number such that \( t^{-1}\nu(\{1, \ldots, k\}) \leq \mu(M) \). For \( E=M \) and \( F=\{1, \ldots, k\} \) we have

\[
\| 1_{E \times F} \cdot a \|_t \leq \mu(M).
\]

If we permute the rows such that

\[
\tau_i = \sum_{j \leq k} |a(i,j)| \nu_j
\]
are in descending order $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_m$ then it follows from (1.6) that

\[(1.7) \quad \tau_m \leq 1.\]

By the induction hypothesis on \(\{1, \ldots, m-1\} \times \{1, \ldots, k\}\) we find a splitting

\[
\{1, \ldots, m-1\} \times \{1, \ldots, k\} = A_1 \cup B_1, \quad A_1 \cap B_1 = \emptyset,
\]

such that

\[
\|1_{A_1} \cdot a\| \leq 1, \quad \|1_{B_1} \cdot a\|^\top \leq 1/t.
\]

For the wanted splitting of \(M \times N\) we only have to put

\[
A = A_1 \cup (\{m\} \times \{1, \ldots, k\}), \quad B = (M \times N) \setminus A.
\]

Indeed, by the induction hypothesis on \(A_1\) and (1.7)

\[
\|1_A \cdot a\| \leq 1,
\]

and we are done for \(k = n\) which implies \(B = B_1\). If \(k < n\) we argue as in case \(m = 1\)
(using \(E = M\) and \(F = \{1, \ldots, k+1\}\)) and show that

\[
\sigma_n \leq \ldots \leq \sigma_{k+1} \leq 1/t.
\]

This yields together with the hypothesis on \(B_1\)

\[
\|1_B \cdot a\|^\top \leq 1/t.
\]

**Remark 1.2.** There is another “norm”

\[
|a|_t = \sup \{|\mu(E)^{-1}\|1_{E \times F} \cdot a\|_\ell^t, \ | t^{-1} \nu(F) \leq \mu(E)\}
\]

closely related to \(\|a\|_t\). This functional \(|a|_t\) has the drawback that for small values of \(t\) or for strongly varying masses the supremum may be on an empty set, in that case we put \(|a|_t = 0\). In any case,

\[
|a|_t \leq \|a\|_t \leq K_t(a).
\]

But for uniform masses, say \(\mu_i = \nu_j = 1\), and for \(t \geq 1\), where \(|a|_t\) is indeed a norm, one can construct for a matrix \(a\) with \(|a|_t \leq 1\) almost along the same lines as above a splitting \(M \times N = A \cup B\), \(A \cap B = \emptyset\), such that (denoting by \([t]\) the integer part of \(t\))

\[
\|1_A \cdot a\| \leq 1, \quad \|1_B \cdot a\|^\top \leq 1/[t],
\]

hence

\[
K_t(a) \leq (1 + t/[t]) |a|_t < 3 |a|_t,
\]

an inequality due, for \(t = 1\), to Varopoulos [V], cf. also [BF].
§2. The $K_t$–functional for the interpolation couple $\ell^p_M(\ell^q_N), \ell^p_N(\ell^q_M)$

For any Bochner–measurable function $f$ on a measure space $(\Omega, \Sigma, \mu)$ with values in a Banach space $X$ and for $p > 0$ let

$$\| f \|_{L_p,\infty(X)} = \inf \{ C | t^p \mu \{ \| f \| > t \} \leq C^p \text{ for all } t > 0 \},$$

or more generally by using the nonincreasing, equimeasurable rearrangement $f^*$ of $\| f \|$

$$\| f \|_{L_p,\infty(X)} = \sup_{t>0} t^{1/p} f^*(t), \quad \| f \|_{L_p,q(X)} = \left( \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q}.$$ 

In general, these quantities are not norms but can be replaced for $p > 1$, $q \geq 1$ by equivalent norms. We do not use this and refer to [BS], Lemma IV.4.5, p. 219.

We start with the case of the interpolation couple $\ell^p_m(\ell^1_n), \ell^p_n(\ell^1_m)$ on the finite measure spaces $(M, \mu), (N, \nu)$ from section 1 and consider the functionals

$$\| a \| = \| a \|_{\ell^p,\infty(\ell^1_n)} \text{ and } \| a \|^\top = \| a^\top \|_{\ell^p,\infty(\ell^1_m)}.$$ 

We have for $p > 1$ and $E \subset M$, $F \subset N$

$$\| 1_{E \times F} \cdot a \|_{\ell^1} \leq p^* \min \{ \mu(E)^{1/p} \| a \|, \nu(F)^{1/p} \| a \|^{\top} \}. \tag{2.3}$$

We introduce the norm $\| . \|_{p,t}$ as follows

$$\| a \|_{p,t} = \sup_{E,F} \left( \mu(E)^{1/p^*} \vee t^{-1} \nu(F)^{1/p^*} \right)^{-1} \| 1_{E \times F} \cdot a \|_{\ell^1}. \tag{2.4}$$

As in section 1 we want to compare the $K_t$–functional $K_{p,t}(\cdot)$ for the interpolation couple $\ell^p_m(\ell^1_n), \ell^p_n(\ell^1_m)$ with the norm $\| . \|_{p,t}$. In view of (2.3) we have a lower estimate of $K_{p,t}(a)$

$$\| a \|_{p,t} \leq p^* K_{p,t}(a). \tag{2.5}$$

Now we establish the corresponding upper estimate.

**Proposition 2.1.** Let $t > 0$, $p > 1$. For any matrix $a$ with $\| a \|_{p,t} \leq 1$ we can split $M \times N$ into a disjoint union $M \times N = A \cup B$, $A \cap B = \emptyset$, such that

$$\| 1_A \cdot a \| \leq 1, \quad \| 1_B \cdot a \|^{\top} \leq 1/t,$$

hence

$$K_{p,t}(a) \leq 2 \| a \|_{p,t}.$$
Proof. Similar to the proof of Proposition 1.1. For $m = 1$ take $k = n$ if $\nu(N) \leq \mu_1 t^{p^*}$, if not, suppose $|a(1, 1)| \geq |a(1, 2)| \geq \ldots \geq |a(1, n)|$ and choose $k$, $0 \leq k < n$, as large as possible such that $\nu(\{1, \ldots, k\}) \leq \mu_1 t^{p^*}$. Inserting $E = \{1\}$ and $F = \{1, \ldots, k\}$ we have

$$\mu(\{1\})^{1/p} \sum_{j \leq k} |a(1, j)| \nu_j \leq 1,$$

and for $E = \{1\}$ and $F = \{1, \ldots, l\}$, $l > k$,

$$\nu(\{1, \ldots, l\})^{1/p} |a(1, l)| \mu_1 \leq 1/t.$$

So we can put $A = \{1\} \times \{1, \ldots, k\}$, $B = (M \times N) \setminus A$ to obtain (2.5).

For $m > 1$ let us put $k = n$ if $\nu(N) \leq \nu(M) t^{p^*}$, if not, choose the largest $k$, $0 \leq k < n$, with $\nu(\{1, \ldots, k\}) \leq \nu(M) t^{p^*}$. Taking $E = M$ and $F = \{1, \ldots, k\}$ we have

$$\|1_{E \times F} \cdot a\|_{\ell^t} \leq \mu(M)^{1/p^*},$$

consequently (we use the notations $\tau_m$, $\sigma_l$ of the preceding proof)

(2.6) $$\mu(\{1, \ldots, m\})^{1/p} \tau_m \leq 1,$$

and for $E = M$ and $F = \{1, \ldots, l\}$, $l > k$,

(2.7) $$\nu(\{1, \ldots, l\})^{1/p} \sigma_l \leq 1/t.$$

The induction hypothesis on $\{1, \ldots, m-1\} \times \{1, \ldots, k\}$ together with (2.6) and (2.7) yields the result.

Remark 2.2. The inequality (2.4) is meaningless for $p = 1$, but the corresponding norm

$$\|a\|_{\ell^t} = \sup_{E, F} (1 \wedge t) \|1_{E \times F} \cdot a\|_{\ell^t} = (1 \wedge t) \|a\|_{\ell^t(M \times N)}$$

is proportional to the norm $\|a\|_{\ell^t(M \times N)}$.

Remark 2.3. For $M = N = \{1, \ldots, m\}$, $\mu_i = \nu_j = 1$, and for $t \geq 1$ one can prove with only slight changes the equivalence of $\|\cdot\|_{p, t}$ and

$$\|a\|_{p, t} = \sup \{\mu(E)^{-1/p^*} \|1_{E \times F} \cdot a\|_{\ell^t} : t^{-1} \nu(F)^{1/p^*} \leq \mu(E)^{1/p^*}\},$$

namely, with constants independent of $m$ and $t$

$$(1/p^*) \|a\|_{p, t} \leq (1/p^*) \|a\|_{p, t} \leq K_{p, t}(a) \leq 3 \|a\|_{p, t}.$$

We leave the details to the reader. From the example $M = \{1\}$, $N = \{1, \ldots, n\}$, the $\nu_j$ and $\mu_1$ equal to 1 and $a \equiv 1$, where $\|a\|_{p, 1} = 1$, but $K_{p, 1}(a) = n^{1/p}$, we see that there should be a restriction $M = N$ in order to have the occurring constants independent of $n$. 

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Remark 2.4. If we replace $\|a\|$ by $\|a\|_{\ell^p_m(\ell^1_n)}$ and $\|a\|^\top$ by $\|a\|^\top_{\ell^p_m(\ell^1_n)}$ then

$$\|1_{E \times F} \cdot a\|_{\ell^1} \leq C(p, q) \min \{\mu(E)^{1/p}\|a\|, \nu(F)^{1/p}\|a\|^\top\}$$

for the constant $C(p, q) = (p^*/q^*)^{1/q}$. Therefore the $K_t$–functional for the interpolation couple $\ell^p_m(\ell^1_n), \ell^q_m(\ell^1_n)$ is greater than $C(p, q)^{-1}\|\cdot\|_{p,t}$. But there is no upper bound of this $K_t$–functional by $\|\cdot\|_{p,t}$ independent of $m$ and $n$.

Take e.g. $M = N = \{1, \ldots, n\}$, $\mu_i = \nu_j = 1$ and

$$a = \begin{pmatrix} 1 & 2^{-1/p} & \cdots & n^{-1/p} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

then $\|a\|_{p,t} \leq p^*$, $K_t(a) \sim (\log n)^{1/q}$.

If we now put for $1 \leq p, q \leq \infty$

$$\|a\| = \|a\|_{\ell^p_m(\ell^q_n)}, \quad \|a\|^\top = \|a\|^\top_{\ell^p_m(\ell^q_n)},$$

define the $K_t$–functional $K_{p,q,t}(\cdot)$ for the interpolation couple $\ell^p_m(\ell^q_n), \ell^p_m(\ell^q_n)$ and let, with the abbreviation $\alpha = 1/q - 1/p$,

$$\|a\|_{p,q,t} = \sup_{E,F} (\mu(E)^\alpha \vee t^{-1} \nu(F)^\alpha)^{-1}\|1_{E \times F} \cdot a\|_{\ell^q},$$

then we derive from Proposition 2.1 by $q$-convexification (cf. [LT]) the following theorem.

Theorem 2.5. Let $t > 0$, $1 \leq q < p \leq \infty$. Denoting $C(p, q) = (1 - q/p)^{1/q}$ we can estimate the $K_t$–functional $K_{p,q,t}(\cdot)$ for the interpolation couple $\ell^p_m(\ell^q_n), \ell^p_m(\ell^q_n)$ by

$$C(p, q) \|a\|_{p,q,t} \leq K_{p,q,t}(a) \leq 2 \|a\|_{p,q,t}.$$

More precisely, for $\|a\|_{p,q,t} \leq 1$ we find $A, B$ with $M \times N = A \cup B$, $A \cap B = \emptyset$, and

$$\|1_A \cdot a\| \leq 1, \quad \|1_B \cdot a\|^\top \leq 1/t.$$

Remark 2.6. An analogous result can be stated for measure spaces $M = N$ with equal atoms $\mu_i = \nu_j = 1$ and $t \geq 1$ for the norm

$$\|a\|_{p,q,t} = \sup \{\mu(E)^{-\alpha}\|1_{E \times F} \cdot a\|_{\ell^1} \mid t^{-1} \nu(F)^\alpha \leq \mu(E)^\alpha\}, \quad (\alpha = 1/q - 1/p)$$

(cf. Remark 2.3).
Theorem 2.7. The statement (2.10) remains true for arbitrary discrete measure spaces.

Proof. Let $M$, $N$ be arbitrary discrete measure spaces. It is very easy to check that for an element $a = (a(i,j))$ we have

$$
\|a\|_\ell^p_\infty(\ell^q_M) = \sup \|1_{A\times B} \cdot a\|_\ell^p_\infty(\ell^q_N),
$$

where the sup runs over all finite subsets $A \subset M$, $B \subset N$. Hence the result follows by a simple pointwise compactness argument left to the reader.
§3. The $K_t-$functional for the couple $L^{p,\infty}(d\mu;L^q(\nu)), L^{p,\infty}(d\nu;L^q(d\mu))$

Now we generalize the results from the previous sections to arbitrary measure spaces and treat the generic case $(M,\mu) = (N,\nu) = (\mathbb{R},\lambda)$ of non-atomic measure spaces, where $\lambda$ is the Lebesgue-measure. For $a = a(x,y) \in L_q^{loc}(\mathbb{R}^2)$ let us define the functionals

$$\|a\| = \|a\|_{L^{p,\infty}(L_q^\infty)}, \quad \|a\|^T = \|a^T\|_{L^{p,\infty}(L_q^\infty)}.$$  

These functionals are used to define the $K_t-$functional $K_{p,q,t}(\cdot)$ of the interpolation couple $L^{p,\infty}(dx;L^q(dy)), L^{p,\infty}(dy;L^q(dx))$ in the obvious manner. For $t > 0$, $1 \leq q < p \leq \infty$ and $\alpha = 1/q - 1/p$, let us introduce the norm

$$\|a\|_{p,q,t} = \sup_{E,F} \left( \lambda(E)^\alpha \vee t^{-1} \lambda(F)^\alpha \right)^{-1} \left( \int_{E \times F} |a(x,y)|^q \, dx \, dy \right)^{1/q}.$$  

Obviously (c.f. Theorem 2.5) for $C(p,q) = (1-p/q)^{1/q}$

$$C(p,q) \|a\|_{p,q,t} \leq K_{p,q,t}(a).$$  

We will now prove the counterpart

$$K_{p,q,t}(a) \leq 2 \|a\|_{p,q,t}.$$  

**Theorem 3.1.** For any $t > 0$ we have the inequality

\begin{equation}
C(p,q) \|a\|_{p,q,t} \leq K_{p,q,t}(a) \leq 2 \|a\|_{p,q,t}
\end{equation}

between the norm $\|\cdot\|_{p,q,t}$ defined above and the $K_t-$functional of the interpolation couple $L^{p,\infty}(dx;L^q(dy)), L^{p,\infty}(dy;L^q(dx))$.

**Proof.** Let us call countably simple a function $a: \mathbb{R}^2 \to \mathbb{R}$ of the form $a = \sum a_{ij} 1_{A_i \times B_j}$ where $(A_i)$ and $(B_j)$ are countable measurable partitions of $\mathbb{R}$. One can easily check that the subset of countably simple functions is dense in $L^{p,\infty}(dx;L^q(dy)) + L^{p,\infty}(dy;L^q(dx))$, hence it suffices to check the inequality (3.1) for those functions. But in that case (3.1) follows from Theorem 2.7.

(This argument presupposes that the functional $K_{p,q,t}$ is continuous with respect to the norm $\|\cdot\|_{p,q,t}$, but this can be checked using the equivalent renorming of weak $L^p$, conditional expectations and a weak compactness argument.)

**Corollary 3.2.** By combination of Theorems 2.7 and 3.1 we obtain the inequality (3.1) for all measure spaces.
Remark 3.3. In the case of non–atomic measure spaces \((M, \mu)\), and \((N, \nu)\) of infinite total measures \(\mu(M) = \nu(N) = \infty\) the difference between \(\| a \|_{p,q,t}^p\) and
\[
\| a \|_{p,q,t} = \sup \left\{ \mu(E)^{1-\alpha} \| 1_{E \times F} a \|_{L^q(M \times N)} \mid t^{-1} \nu(F)^\alpha \leq \mu(E)^\alpha \right\} \quad (\alpha = 1/q - 1/p)
\]
disappears completely. Obviously
\[
\| a \|_{p,q,t} = \| a \|_{p,q,t}.
\]
If the (non–atomic) measure spaces \((M, \mu)\), \((N, \nu)\) have the same finite total measure, say 1, the equivalence of these two norms with constants independent of \(t\) can be established for \(t \geq 1\). The example of the function \(a \equiv 1\) on \([0, 1] \times [0, 1]\) (equipped with the Lebesgue measure) gives for \(0 < t \leq 1\)
\[
\| a \|_{p,q,t} = t \quad \text{and} \quad \| a \|_{p,q,t} = \frac{t^{p/q}}{q}.
\]
This leads for \(0 < t \leq 1\) to the inequality
\[
\sup_{a \in L^1(M \times N)} \| a \|_{p,q,t} \geq t^{-q/p - q}.
\]
But we have indeed

Proposition 3.4. For non–atomic measure spaces \((M, \mu)\), \((N, \nu)\) with the same finite total measure, say \(\mu(M) = \nu(N) = 1\), \(1 \leq q < p \leq \infty\), \(\alpha = 1/q - 1/p\), we have for \(0 < t \leq 1\)
\[
\sup_{a \in L^1(M \times N)} \| a \|_{p,q,t} = t^{-q/p - q}.
\]

Proof. Let \(E, F\) with \(t^{-1} \nu(F)^\alpha > \mu(E)^\alpha \) and
\[
t \nu(F)^{-\alpha} \left( \int_{E \times F} |a(x,y)|^q \, dx \, dy \right)^{1/q} > 1.
\]
In order to construct \(\tilde{E}\) and \(\tilde{F}\), \(t^{-1} \nu(\tilde{F})^\alpha \leq \mu(\tilde{E})^\alpha\) with
\[
\mu(\tilde{E})^{-\alpha} \left( \int_{E \times F} |a(x,y)|^q \, dx \, dy \right)^{1/q} > t^{q/p - q},
\]
we can suppose that \(t^{-1} \nu(F)^\alpha > \mu(M)^\alpha\) and \(E = M\) in (3.2). If not, \(\tilde{F} = F\) and any \(\tilde{E} \supset E\) with \(\mu(\tilde{E}) = t^{-1/\alpha} \nu(F)\) verify (3.3).
By [BS], Theorem II, 2.7, p. 51, we find for all \(0 < \lambda \leq 1\) an \(F_\lambda \subset N\), \(\nu(F_\lambda) = \lambda \nu(F)\), such that
\[
t \nu(F_\lambda)^{-\alpha} \left( \int_{E \times F_\lambda} |a(x,y)|^q \, dx \, dy \right)^{1/q} > \lambda^{1/p}.
\]
If we put \(\tilde{E} = M\) and \(\tilde{F} = F_\lambda\) for
\[
\lambda = \frac{\mu(M)}{\nu(F)} t^{1/\alpha} \geq t^{1/\alpha},
\]
then (3.2) is obvious.
4. Applications

Let us denote by $B(X,Y)$ the space of bounded operators between two Banach spaces $X$, $Y$. Let $(M, \mu)$, $(N, \nu)$ be arbitrary measure spaces. Let $B_0 = B(L^\infty(d\nu), L^\infty(d\mu))$ and $B_1 = B(L^1(d\nu), L^1(d\mu))$. Clearly we may view $(B_0, B_1)$ as a compatible couple in the sense of interpolation theory by identifying an element of $B_0$ or $B_1$ with a linear operator from $L^\infty(d\nu) \cap L^1(d\nu)$ into $L^\infty(d\mu) + L^1(d\mu)$. For simplicity we will assume (although this is inessential) that all the spaces we consider are over the field of real scalars.

Let $L$, $\Lambda$ be real Banach lattices and let $u : L \to \Lambda$ be a bounded operator. Then $u$ is called regular if there is a positive operator $v : L \to \Lambda$ such that

$$|u(x)| \leq v(|x|) \quad \text{for all } x \in L.$$  

We define

$$\|u\|_r = \inf \{\|v\| \}$$

where the infimum runs over all such dominating operators $v$. We denote by $B_r(L, \Lambda)$ the space of all regular operators equipped with this norm. If $\Lambda$ is Dedekind complete in the sense of [MN] then $B_r(L, \Lambda)$ is a Banach lattice and we have simply

$$\|u\|_r = \|u\|_{B_r(L, \Lambda)}.$$ 

This applies in particular when $\Lambda$ is a Lorentz space, as in the situation we consider below. We should mention that any bounded operator between $L^\infty$-spaces (or $L^1$-spaces) is automatically regular, so that $B_0 = B_r(L^\infty(d\nu), L^\infty(d\mu))$ and $B_1 = B_r(L^1(d\nu), L^1(d\mu))$.

**Theorem 4.1.** For all $u$ in $B_0 + B_1$, let

$$\|u\|_t = \sup \left\{ \left( \mu(E) \vee t^{-1} \nu(F) \right)^{-1} \left| \langle u(1_F), 1_E \rangle \right| \right\} \quad \text{where the supremum runs over all measurable subsets } E \subset M, F \subset N \text{ with positive and finite measure.}$$

Then for all $t > 0$

$$\frac{1}{2} K_t(u; B_0, B_1) \leq \|u\|_t \leq K_t(u; B_0, B_1).$$

**Proof.** Assume first that $(M, \mu)$ and $(N, \nu)$ are purely atomic measure spaces each with only finitely many atoms. Then $B_0$ and $B_1$ can be identified (via their kernels) respectively with $L^\infty(d\mu ; L^1(d\nu))$ and $L^\infty(d\nu ; L^1(d\mu))$. Therefore, in that case the nontrivial part of Theorem 4.1 is but a reformulation of Proposition 1.1.

The general case can be deduced from this using conditional expectations and a simple weak compactness argument. We leave the details to the reader.
Let \((A_0, A_1)\) be any compatible couple of Banach spaces. We recall that \((A_0, A_1)_{\theta,q}\) is defined for \(0 < \theta < 1, 1 \leq q \leq \infty\) as the space of all \(x\) in \(A_0 + A_1\) such that

\[
\|x\|_{\theta,q} = \left( \int_0^\infty \left( t^{-\theta} K_t(x; A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q} < \infty
\]

with the usual convention when \(q = \infty\). When equipped with the norm \(\| \cdot \|_{\theta,q}\), the space \((A_0, A_1)_{\theta,q}\) is a Banach space. We refer to [BL] for more informations.

**Theorem 4.2.** Let \(0 < \theta < 1\) and \(\theta = 1/p\). We have

\[
(B_0, B_1)_{\theta,\infty} = B_r \left( L^{p,1}(d\nu), L^{p,\infty}(d\mu) \right)
\]

with equivalent norms.

**Proof.** Consider an operator \(u : L^{p,1}(d\nu) \rightarrow L^{p,\infty}(d\mu)\). Let us define for \(\theta = 1/p\)

\[
[u]_p = \sup \left\{ \nu(F)^{-\theta} \mu(E)^{\theta-1} \left| \langle u(\varphi 1_F), \psi 1_E \rangle \right| \right\}
\]

where the supremum runs over all \(\varphi\) (resp. \(\psi\)) in the unit ball of \(L^\infty(d\nu)\) (resp. \(L^\infty(d\mu)\)) and over all measurable subsets \(E \subset M, F \subset N\) with finite positive measure. By a well known property of the Lorentz spaces \(L^{p,1}(d\nu)\) and \(L^{p^{-1},1}(d\mu)\) (cf. [SW, Theorem 3.13]) there is a positive constant \(C_p\) depending only on \(p, 1 < p < \infty\), such that we have

\[
C_p^{-1} [u]_p \leq \|u\|_{L^{p,1}(d\nu) \rightarrow L^{p,\infty}(d\mu)} \leq C_p [u]_p,
\]

for all \(u : L^{p,1}(d\nu) \rightarrow L^{p,\infty}(d\mu)\). If \(u\) is positive, (4.1) can be simplified. In particular we have

\[
[u]_p = \sup_{E \subset M} \sup_{F \subset N} \left\{ \nu(F)^{-\theta} \mu(E)^{\theta-1} \left( |u|(1_F), 1_E \right) \right\}
\]

Assume \(\nu(F) = t \mu(E)\) then \(\nu(F)^{-\theta} \mu(E)^{\theta-1} = t^{-\theta} \mu(E)^{-1}\), hence

\[
[u]_p = \sup_{t>0} t^{-\theta} \|u\|_t.
\]

Therefore by Theorem 4.1 \(\|u\|_{(B_0, B_1)_{\theta,\infty}}\) is equivalent to \([u]_p\) or equivalently to the norm of \(u\) in \(B_r \left( L^{p,1}(d\nu), L^{p,\infty}(d\mu) \right)\). This yields immediately the announced result.

**Remark 4.1.** We refer to [P1] for the analogue of Theorem 4.2 for the complex interpolation method.
We conclude this paper with an application to $H^p$–spaces in the framework already considered in [P3]. Let $(A_0, A_1)$ be a compatible couple of Banach spaces and $S_0 \subset A_0$, $S_1 \subset A_1$ be closed subspaces. Following the terminology in [P2], we will say that $(S_0, S_1)$ is $K$–closed in $(A_0, A_1)$ if there is a constant $C$ such that for all $x$ in $S_0 + S_1$ and all $t > 0$ we have

$$K_t(x; S_0, S_1) \leq C K_t(x; A_0, A_1).$$

Let $(T, m)$ be the unit circle equipped with the normalized Lebesgue measure. Let $B$ be a Banach space. We will denote for $1 \leq p \leq \infty$ by $H^p(dm)$ (resp. $H^p(dm; B)$) the subspace of $L^p(dm)$ (resp. $L^p(dm; B)$) of all the functions $f$ such that $\hat{f}(n) = 0$, $\forall n < 0$.

Let $(M, \mu)$ be any measure space. Consider the couple $X_0 = L^1(d\mu; L^\infty(dm)), X_1 = L^1(dm; L^\infty(dm))$ and the subspaces

$$Y_0 = L^1(d\mu; H^\infty(dm)), Y_1 = H^1(dm; L^\infty(dm)).$$

It is proved in [P3, Lemma 2] that $(Y_0, Y_1)$ is $K$–closed in $(X_0, X_1)$. By the simple duality principle emphasized in [P2] (cf. Proposition 1.11 and Remark 1.12 in [P2]) this implies that a similar property holds for the orthogonal subspaces $Y_0^\perp$, $Y_1^\perp$. More precisely, consider the subspaces

$$S_0 = L^\infty(d\mu; H^1(dm)), S_1 = H^\infty(dm; L^1(dm))$$

of the spaces

$$A_0 = L^\infty(d\mu; L^1(dm)), A_1 = L^\infty(dm; L^1(dm)).$$

Then by this duality principle, $(S_0, S_1)$ is $K$–closed in $(A_0, A_1)$. Therefore, our computation of the $K_t$–functional for the couple $(A_0, A_1)$ (cf. Theorem 0.1 above) is applicable to the couple $(S_0, S_1)$. Taking for simplicity $M = N$ equipped with the counting measure, we obtain:

**Theorem 4.3.** There is a numerical constant $C$ with the following property. Let $(f_n)$ be a sequence in $H^1(dm)$ and let $t > 0$. Assume that for all subsets $E \subset N$ and all measurable subsets $F \subset T$ we have

$$\sum_{n \in E} \int_F |f_n| dm \leq |E| \vee t^{-1} m(F).$$

Then there is a decomposition $f_n = g_n + h_n$ with $g_n \in H^1(dm)$, $h_n \in H^\infty(dm)$ such that

$$\sup_{n \in N} \|g_n\|_{H^1(dm)} \leq C \quad \text{and} \quad \left\| \sum_{n \in N} |h_n| \right\|_{L^\infty(dm)} \leq C t^{-1}.$$
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