THE LOCAL GAN-GROSS-PRASAD CONJECTURE FOR SPECIAL ORTHOGONAL GROUPS OVER ARCHIMEDEAN LOCAL FIELDS

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Abstract. In the paper, we prove the local Gan-Gross-Prasad conjecture for special orthogonal groups over archimedean local fields for generic local $L$-parameters. The non-archimedean case was proved by C. Moeglin and J.-L. Waldspurger in [MW12]. When the local $L$-parameters are tempered, the conjecture was proved by Waldspurger in [Wal09] for non-archimedean local fields and by Zhilin Luo in [Luo20] for archimedean local fields.

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References
1. Introduction

The local Gan-Gross-Prasad conjecture for classical groups over any local field \( F \) (\cite{GGPT12}) is an extension to general classical groups of the original Gross-Prasad conjecture for special orthogonal groups in \cite{GP92} and \cite{GP94}.

Let \((V, V')\) be a pair of non-degenerate quadratic spaces defined over a local field \( F \) such that \( V' \) is a subspace of \( V \) and its orthogonal complement \( V^\perp \) in \( V \) is odd dimensional and split. Let \( \tilde{G} = \text{SO}(V) \times \text{SO}(V') \) be the product of two special orthogonal groups, and \( \tilde{H} \) the subgroup of \( \tilde{G} \) defined by the semi-direct product of the image \( \Delta \text{SO}(V') \) of the diagonal embedding of \( \text{SO}(V') \) into \( \tilde{G} \) with the unipotent subgroup \( \tilde{N} \) of \( \text{SO}(V) \) defined by the totally isotropic flag determined by \( V^\perp \). Fix a generic character \( \tilde{\nu} \) of \( \tilde{N} \) which extends to \( \tilde{H} \). For any irreducible Casselman-Wallach representation \( \tilde{\pi} \) of \( \tilde{G} \), define the multiplicity

\[
m(\tilde{\pi}) = \dim \text{Hom}_H(\tilde{\pi}, \tilde{\nu}).
\]

The Multiplicity One Theorem that \( m(\tilde{\pi}) \leq 1 \) was proved in \cite{AGRS10} and \cite{GGPT12} when \( F \) is nonarchimedean and in \cite{SZ12} and \cite{JSZ10} when \( F \) is archimedean. The relevant pure inner forms \( \tilde{G}_\alpha \) of \( \tilde{G} \) are classified by \( \alpha \in H^1(F, \text{SO}(V')) = H^1(F, \tilde{H}) \). We write that \( \tilde{G}_\alpha = \text{SO}(V_\alpha) \times \text{SO}(V'_\alpha) \), where \((V_\alpha, V'_\alpha)\) is a pair of quadratic spaces such that \( V_\alpha = V'_\alpha \oplus V^\perp \). Then one can define \( \tilde{H}_\alpha, \tilde{\nu}_\alpha \) and the multiplicity accordingly. Moreover, \( \tilde{G}_\alpha \) has the same \( L \)-group as \( \tilde{G} \). For every generic \( L \)-parameter \( \tilde{\varphi} : W_F \rightarrow \text{L} \tilde{G} \) where \( W_F \) is the Weil-Deligne group of \( F \), let \( \Pi_{\tilde{\varphi}}(\tilde{G}_\alpha) \) be the associated \( L \)-packet of \( \tilde{G}_\alpha \).

**Conjecture 1.1** (\cite{GGPT12}). For every generic \( L \)-parameter \( \tilde{\varphi} : W_F \rightarrow \text{L} \tilde{G} \), we have

\[
\sum_{\alpha \in H^1(F, \tilde{H})} \sum_{\tilde{\pi} \in \Pi_{\tilde{\varphi}}(\tilde{G}_\alpha)} m(\tilde{\pi}) = 1.
\]

A refinement of Conjecture 1.1 that determines when the multiplicity \( m(\tilde{\pi}) \) is not equal to 0 will be discussed in Conjecture 2.1.

The goal of this paper is to prove Conjecture 1.1, the local Gan-Gross-Prasad conjecture for special orthogonal groups over archimedean local fields for general generic local \( L \)-parameters (Theorem 2.1 in Section 2.4), based on the case of tempered local \( L \)-parameters. When the local \( L \)-parameters are tempered, Conjecture 1.1 was proved by Zhilin Luo in his paper \cite{Luo20}, following the idea of the pioneering work of J.-L. Waldspurger in \cite{Wal09} for nonarchimedean local fields. We also study Conjecture 2.1. We prove that Conjecture 2.1 for generic \( L \)-parameters holds over the complex field \( \mathbb{C} \) (Theorem 2.2 in Section 2.4 or Theorem 4.1). However, for the real field case, we deduce Conjecture 2.1 for general generic \( L \)-parameters from that for tempered \( L \)-parameters (Theorem 2.1 in Section 2.4). While Conjecture 2.1 for tempered \( L \)-parameters is our on-going work joint with Zhilin Luo.

Our proof of Conjecture 1.1 follows philosophically the ideas of C. Moeglin and Waldspurger in \cite{MW12}, which proves the conjecture for general generic local \( L \)-parameters over nonarchimedean local fields, based on the work of Waldspurger (\cite{Wal09}) for tempered local \( L \)-parameters. There are three key steps in their proof: (1) the relation between the local Bessel functionals and the parabolic induction, (2) the mathematical induction to prove the co-dimension one case with the results of the tempered case, and (3) the reduction to the co-dimension one case. More specifically, Moeglin and Waldspurger proved a multiplicity formula that shows the relation between the local Bessel functionals and the parabolic inductions, and with this multiplicity formula, they can reduce the generic co-dimension one case to the tempered case by mathematical induction. With the
same multiplicity formula, they reduce the general case to the co-dimension one case by using a supercuspidal representation in the relevant parabolic induction.

Over archimedean local fields, there are no supercuspidal representations. However, we are able to establish parallel arguments along the lines of Moeglin and Waldspurger by using a principal series representation. Such an idea goes back to the work of Jiang-Sun-Zhu in [JSZ10] on their proof of the uniqueness of Bessel models over archimedean local fields, and has been recently used by H. Xue in his work on the local Gan-Gross-Prasad conjecture for the unitary group case over archimedean local fields ([Xue20]). Another difficulty is the proof of the multiplicity formula that manifests the relation between local Bessel functionals and the parabolic inductions, due to that the distributional analysis involved in the proof over archimedean local fields is much more subtle than that over nonarchimedean local fields. The complicated nature of the parameters makes the orthogonal case more difficult than the unitary case, especially in the analysis of the "descent" on the open orbit in Section 5.3, and our method can be applied to other cases. In order to demonstrate our arguments, a proof of the complex case is given in Section 4 in the language of distribution theory, and a nonzero Bessel functional with required properties in this case is constructed explicitly. From our work in the complex case, we observe that the multiplicity formula can be deduced from the relation between the equivariant tempered distributions on a given manifold and the equivariant tempered distributions on an open submanifold of the manifold.

Although the distributional analysis gets much more involved in the archimedean case, with the results on Schwartz homologies developed by Chen-Sun in [CS20], we are still able to obtain a sufficient condition for the bijectiveness of the restriction map from equivariant distributions on a Nash manifold to that of its open Nash submanifold in Appendix A and Appendix B. This is the main technical ingredient in our proof of the multiplicity formula in the archimedean case, and we use it to prove a technical lemma in Section 5.1. In Section 5.2, we do analysis on the closed orbits and prove a vanishing result which shows the restriction map from the invariant functionals of the space of Schwartz sections on a Nash manifold to that of its open Nash submanifold is bijective. In Section 5.3, we do analysis on the open Nash submanifold, and prove some vanishing results that show how the invariant functional of Schwartz sections on the open orbit descends to Bessel functionals associated to a smaller Gan-Gross-Prasad triple. Our proof of the vanishing results is based on infinitesimal characters of representations, so it is independent on the choice of parabolic subgroups in the even orthogonal case (Remark 3.1). With the results in Section 5.2 and Section 5.3, we can construct a descent map between the spaces of Bessel functionals for two pairs of representations, whose bijectiveness gives a multiplicity formula at the end of Section 5.3. With this multiplicity formula, in Section 5.5, we can use mathematical induction to prove the co-dimension one case by using Luo’s results of the tempered case in [Luo20], the proof of which is similar to that in [MW12] for the nonarchimedean case. Finally, we take parabolic induction of the completed tensor product of one representation and a principal series representation to reduce the general case to the co-dimension one case in Section 5.6.

It is worthwhile to mention that the local Gan-Gross-Prasad conjecture at all local places implies the uniqueness in the statement of the global Gan-Gross-Prasad conjecture over number fields ([GGP12, Conjectures 24.1 and 26.1]). As shown in the work of Jiang and Zhang in [JZ20], the local Gan-Gross-Prasad conjecture at all local places is one of the key ingredients in their proof of irreducibility of the twisted automorphic descents.

The paper is organized as follows. In Section 2, we introduce concepts pertaining to the local Gan-Gross-Prasad conjecture to state our main results, Theorem 2.1 and Theorem 2.2. And in Section 3, we express representations in generic Vogan packets as normalized parabolic inductions
of tensor products of tempered representations and essentially discrete series representations. In Section 4 we will give a constructive proof of the local Gan-Gross-Prasad conjecture over the complex field in terms of equivariant distributions. In Section 5, we prove a technical lemma that implies the vanishing of the Schwartz homologies of certain representations. Then we make use of Schwartz analysis in our proofs of some multiplicity formulas in the co-dimension one case, which are essentially some distributional analysis. Finally we reduce the generic case to the tempered case by using the proved multiplicity formulas. In Appendix A we review the properties of Schwartz inductions and Schwartz homologies discussed in [CS20], and in Appendix B we recall the Harish-Chandra correspondence and use it to parameterize the infinitesimal characters.

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2. The Local Gan-Gross-Prasad Conjecture

In this section, we will define every ingredients of the local Gan-Gross-Prasad conjecture of special orthogonal groups over archimedean local fields and give a complete statement of that.

2.1. The Gan-Gross-Prasad triples. In this subsection, we will define the Gan-Gross-Prasad triples in the setting of special orthogonal groups over an archimedean local field \( F \).

Let \( F = \mathbb{R} \) or \( \mathbb{C} \). For a non-degenerate quadratic space \((V, q)\) over \( F \) and its non-degenerate quadratic subspace \((V', q')\), the pair \((V, V')\) is called admissible if the orthogonal complement \( V'_{\perp} \) of \( V' \) is odd dimensional and split, that is, \( V'_{\perp} = X_r \oplus Y_r \oplus C v_0 \), where \( X_r, Y_r \) are maximal totally isotropic subspaces in \( V'_{\perp} \) and \( v_0 \) is an anisotropic vector orthogonal to \( X_r \oplus Y_r \).

For a nontrivial unitary additive character \( \psi \) of \( F \), define \( \nu \) of \( N \) to be

\[
\nu(n) = r \sum_{i=0}^{r-1} q(v_{-i-1}, nv_i).
\]

For a nontrivial unitary additive character \( \psi \) of \( F \), define \( \nu \) of \( N \) to be

\[

\nu(n) = \sum_{i=0}^{r-1} q(v_{-i-1}, nv_i).
\]

If we denote \( X_i = \langle v_1, \cdots, v_i \rangle \) and \( X_{i+1} = X_r \oplus C v_0 \), we can define a subgroup

\[
S_r = \{ g \in G : (g - 1)X_i \subset X_{i-1}, \ 1 \leq i \leq r + 1 \},
\]

which is often called a Bessel subgroup of \( G \) associated to the pair \((F, v_0)\). And we have

\[
S_r = G' \rtimes N
\]

so \( \nu \) induces a character \( \nu_r \) on \( S_r \), which is called a generic Bessel character.
Define $\tilde{G} = G \times G'$, $\tilde{H} = \Delta G' \times N \subset S_p \times G'$ and $\tilde{\nu}$ is the character of $\tilde{H}$ induced from the character $\nu$ of $N$. Then $(\tilde{G}, \tilde{H}, \tilde{\nu})$ is called the Gan-Gross-Prasad triple associated to the admissible pair $(V, V')$ and the co-dimension of this Gan-Gross-Prasad triple is defined to be $2r + 1$.

2.2. $L$-parameters and Vogan packets. In this subsection, we review local $L$-parameters and local $L$-packets for special orthogonal groups over archimedean local fields, define pure inner forms of special orthogonal groups over archimedean local fields and the local Vogan packet of a $L$-parameter.

Let $G$ be a real reductive group, $K$ is its maximal compact subgroup, and $\mathfrak{g}_C$ is the complexification of the real Lie algebra of $G$. A smooth representation $(\pi, V)$ of $G$ is said to be admissible if the $(\mathfrak{g}_C, K)$-module $V^K$ consisting of the $K$-finite vectors of $V$ is admissible, that is, it is a direct sum of irreducible representations of $K$ with finite multiplicities. We call a $(\mathfrak{g}_C, K)$-module $V^K$ Harish-Chandra if it is admissible and $\mathcal{Z}(\mathfrak{g}_C)$-finite, where $\mathcal{Z}(\mathfrak{g}_C)$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_C)$. A Casselman-Wallach representation $(\pi, V)$ of $G$ is a smooth Fréchet representation of moderate growth whose $(\mathfrak{g}_C, K)$-module is Harish-Chandra (see [Cas89] [Wal94]). The Casselman-Wallach globalization theorem (see [BK14]) gives categorical equivalence between the category of Harish-Chandra $(\mathfrak{g}_C, K)$-modules and Casselman-Wallach representations of $G$.

The local Langlands group $\mathcal{L}_F$ of an archimedean local field $F$ is equal to the Weil-Deligne group

$$\mathcal{W}_F = \begin{cases} \mathbb{C}^\times & \text{when } F = \mathbb{C}, \\ \mathbb{C}^\times \cup \mathbb{C} \times j & \text{when } F = \mathbb{R}, \end{cases}$$

where $j$ satisfies $j^2 = -1$ and $jzj = -z$. A local $L$-parameter $\varphi$ of $G = SO(V, q)$ is a $\tilde{G}$-conjugacy class of smooth morphism

$$\varphi : \mathcal{L}_F \to {}^L G = \tilde{G} \rtimes \text{Gal}(E/F),$$

such that the elements in $\text{Im} \varphi$ are semisimple in ${}^L G$, where $E$ is the splitting field of the non-degenerate quadratic space $(V, q)$. A parameter $\varphi$ is called tempered if $\text{Im} \varphi$ is bounded.

The local Langlands correspondence of real reductive groups was established by Langlands in [Lan88], which asserts that every irreducible finitely generated $(\mathfrak{g}_C, K)$-module of $G$ can be assigned to a local $L$-parameter $\varphi$, and for every $L$-parameter $\varphi$, only finitely many equivalence classes of $(\mathfrak{g}_C, K)$-modules are associated to $\varphi$, and their completions by Casselman-Wallach globalization theorem form the local $L$-packet $\Pi_\varphi(G)$ of the local $L$-parameter $\varphi$.

For $G = SO(V, q)$, the pure inner forms of $G$ are the groups $G_\alpha(\alpha \in H^1(F, G))$ obtained by inner twisting by elements $\alpha$, and they are characterized as special orthogonal groups $G_\alpha = SO(V_\alpha, q_\alpha)$ of quadratic spaces $(V_\alpha, q_\alpha)$ over $F$ satisfying

$$\dim(V) = \dim(V_\alpha), \quad \text{disc}(V) = \text{disc}(V_\alpha).$$

When $F = \mathbb{C}$, the special orthogonal group $G = SO(V, q)$ is the unique pure inner form of itself. When $F = \mathbb{R}$, the pure inner forms of $SO(p, q)$ are $G_\alpha = SO(p_\alpha, q_\alpha)$ where $p + q = p_\alpha + q_\alpha$ and $p - p_\alpha$ is even. It is straightforward that every $G$ has a unique quasisplit pure inner form $G_{\alpha_5}$ up to isomorphism.

As the pure inner forms of $G$ shares the same $L$-group, a parameter $\varphi$ of $G$ can be regarded as a parameter $\varphi$ of any pure inner form $G_\alpha$. A parameter $\varphi$ is called generic if for the quasi-split pure inner form $G_{\alpha_5}$ of $G$, $\Pi_\varphi(G_{\alpha_5})$ contains a generic representation, that is, a member with a nonzero Whittaker model with respect to a certain Whittaker datum for $G_{\alpha_5}$. The Vogan packet $\Pi_{\varphi}^{\text{Vogan}}$ for a generic parameter $\varphi$ is defined by

$$\Pi_{\varphi}^{\text{Vogan}} = \bigoplus_{\alpha \in H^1(F, G)} \Pi_\varphi(G_\alpha).$$
Then we consider the Gan-Gross-Prasad triple \((\tilde{G}, \tilde{H}, \tilde{\nu})\) for an admissible pair \((V, V')\). We call a pure inner form \(\tilde{G}_\alpha\) of \(\tilde{G}\) is relevant if \(\tilde{G}_\alpha = SO(V_\alpha) \times SO(V'_\alpha)\) where \(SO(V'_\alpha)\) is a pure inner form of \(SO(V')\) and \(V_\alpha = V_\alpha \oplus V'^\perp\). The relevant pure inner forms of \(\tilde{G}\) are parameterized by \(H^1(F, SO(V'_\alpha)) = H^1(F, \tilde{H})\). With the flag \(\tilde{F}\) and the anisotropic vector \(v_0\), we are able to define \(\tilde{H}_\alpha\) and \(\tilde{v}_\alpha\), then we have a Gan-Gross-Prasad triple \((\tilde{G}_\alpha, \tilde{H}_\alpha, \tilde{v}_\alpha)\). For a \(L\)-parameter \(\tilde{\varphi} = \varphi \times \varphi'\), \(\tilde{\varphi}\) is called generic if both \(\varphi\) and \(\varphi'\) are generic, and the Vogan packet \(\Pi_{\tilde{\varphi}}^{\text{Vogan}}\) of \(\tilde{\varphi}\) is defined by

\[
\Pi_{\tilde{\varphi}}^{\text{Vogan}} = \prod_{\alpha \in H^1(F, \tilde{H})} \Pi_{\varphi}(G_\alpha) \times \Pi_{\varphi'}(G'_\alpha) = \prod_{\alpha \in H^1(F, \tilde{H})} \{\pi_\alpha \otimes \pi'_\alpha | \pi_\alpha \in \Pi_{\varphi}(G_\alpha), \pi'_\alpha \in \Pi_{\varphi'}(G'_\alpha)\}.
\]

where \(\pi_\alpha \otimes \pi'_\alpha\) is the \(\tilde{G}\)-representation defined by the completed tensor product of \(\pi_\alpha \otimes 1\) and \(1 \otimes \pi'_\alpha\).

### 2.3. Component groups

In this section, we define the component groups for \(L\)-parameters and define some local root numbers. Define a pairing \(B\) of a representation \(M\) of a group \(L\) with sign \(b\) \((b \in \{\pm 1\})\) to be a bilinear map:

\[
B : M \times M \to \mathbb{C}
\]

satisfying

\[
B(ln, ln) = B(m, n), \quad B(m, n) = bB(n, m), \quad \text{for } m, n \in M, \quad l \in L.
\]

In [GGPT12 Section 7], the \(L\)-group of a special orthogonal group \(G = SO(V, q)\) can be embedded into the invariant group of a pairing \(B\) on \(M\) with sign \(b\). If \(\dim V = 2n + 1\), then

\[
^{L}G = Sp(M)
\]

where \(M = \mathbb{C}^{2n}\) and \(b = -1\). If \(\dim(V) = 2n\) and \(\text{disc}(V) \in F^{x2}\) then

\[
^{L}G = SO(M)
\]

where \(M = \mathbb{C}^{2n}\) and \(b = 1\). If \(\dim(V) = 2n\) and \(\text{disc}(V) \notin F^{x2}\) then

\[
^{L}G = O(M)
\]

where \(M = \mathbb{C}^{2n}\) and \(b = 1\).

Hence, we can identify a parameter \(\varphi\) of \(G = SO(V, q)\) as a semisimple representation of \(W_{\varphi}\) on the space \(M\) with a pairing \(B\) of sign \(b\). Let \(S_\varphi\) be the centralizer of the \(Im\varphi\) in \(GL(M)\), and the component group \(S_\varphi\) be the finite group \(\pi_0(S_\varphi)\). For an element \(a \in S_\varphi\), let \(M^a\) be the \((-1)\)-eigenspace for \(a\) on \(M\) which is only dependent on the image of \(a\) in the component group \(S_\varphi\).

For the Gan-Gross-Prasad triple \((\tilde{G}, \tilde{H}, \tilde{\nu})\) for an admissible pair \((V, V')\), we define characters \(\chi_V\) and \(\chi'_{V'}\) of the component groups \(S_\varphi\) and \(S_{\varphi'}\) as

\[
\eta_V(a) = \det(M^a)(-1)^{\dim(M^a)/2} \cdot \det(M')(1)^{\dim(M'/a)/2} \cdot \epsilon(\frac{1}{2}, M^a \otimes M', \psi),
\]

for \(a \in S_\varphi\), and

\[
\eta_{V'}(a') = \det(M)(-1)^{\dim(M'/a')/2} \cdot \det(M')(-1)^{\dim(M)/2} \cdot \epsilon(\frac{1}{2}, M \otimes M', \psi),
\]

for \(a' \in S_{\varphi'}\). For the \(L\)-parameter \(\varphi = \varphi \times \varphi'\) of \(\tilde{G} = SO(V) \times SO(V')\), we have \(S_\varphi = S_\varphi \times S_{\varphi'}\) and let \(\eta_{\tilde{\varphi}} = \eta_V \times \eta_{V'}\), be a character of \(S_{\tilde{\varphi}}\).

Vogan conjectured in [Vog93] that, for a given Whittaker datum in the quasisplit pure inner form \(G_{\alpha,\nu}\) of \(G\), there is a non-degenerate pairing

\[
\Pi_{\tilde{\varphi}}^{\text{Vogan}}(G) \times S_{\varphi} \to \{\pm 1\}.
\]
Therefore, we can parameterize representations in $\Pi^\text{Vogan}(G)$ with characters $\eta_\pi : S_\varphi \to \{\pm 1\}$, which is called Langlands-Vogan parameterization.

2.4. Statements of our results. We are able to state our main results of this article.

**Theorem 2.1.** The Conjecture 1.1 is true over archimedean local fields.

Theorem 2.1 will be proved in Section 4 for complex field case, and in Section 5 for the real field case. Moreover, there is a refinement of Conjecture 1.1 in [GGP12, Conjecture 17.3].

**Conjecture 2.1.** For a Gan-Gross-Prasad triple $(\tilde{G}, \tilde{H}, \tilde{\nu})$ over an archimedean local field $F$ and generic parameters $\tilde{\varphi}$, the unique representations $\tilde{\pi} \in \Pi^\text{Vogan}_\tilde{\varphi}$ such that $m(\tilde{\pi}) = 1$ satisfy

$$\eta_{\tilde{\pi}} = \eta_{\tilde{\nu}}.$$

From [GGP12], the statement of Conjecture 2.1 is independent of the choice of the Whittaker datum in the Langlands-Vogan parameterization. We are going to prove Conjecture 2.1 over the complex field in Section 4. Over the real field, we can deduce Conjecture 2.1 for generic $L$-parameters from that for tempered $L$-parameters in Section 5. Hence we have

**Theorem 2.2.** Over the complex field $\mathbb{C}$, Conjecture 2.1 holds for general generic $L$-parameters. Over the real field $\mathbb{R}$, if Conjecture 2.1 is true for tempered $L$-parameters, then it is true for general generic $L$-parameters.

3. Genericity and irreducibility

In this section, we prove that every representation in generic Vogan packets is isomorphic to a normalized parabolic induction that induces a representation of the Levi subgroup with essentially discrete series representations and tempered representations on the blocks. The nonarchimedean counterpart of this section was proved in [MW12, Section 3].

First of all, we summarize some classical results about the reducibility of parabolically induced representations. Let $G$ be a quasisplit connected real reductive group and let $B = TU$ be a Borel subgroup of $G$, where $T$ is the maximal torus of $B$ and $U$ is the unipotent radical of $B$. Let $A_0$ be the maximal split torus of $T$. Fix a standard parabolic subgroup $P$ of $G$, that is, a parabolic subgroup $P = MN$, with $T \subset M$ and $N \subset U$.

Let $\sigma$ be an irreducible tempered representation of $M$ and choose $\nu \in a_+^*$, the complex dual of the real Lie algebra of the split component $A$ of $M$. Let $I(\nu, \sigma)$ be the normalized induction $I^G_B(\nu \otimes \sigma)$. Assume that $\nu$ is in the positive Weyl chamber. Then $I(\nu, \sigma)$ is called a standard module. Let $J(\nu, \sigma)$ be the unique Langlands quotient of $I(\nu, \sigma)$ from Langlands classification. The following conjecture is proved by Vogan in [Vog78, Theorem 6.2], which is called the archimedean case of the standard model conjecture.

**Proposition 3.1.** If $J(\nu, \sigma)$ is generic, then

$$I(\nu, \sigma) = J(\nu, \sigma).$$

Hence $I(\nu, \sigma)$ is irreducible.

For an archimedean local field $F$, it was proved by Knapp and Stein in [KSS80] with the irreducibility of principal series representation when $F = \mathbb{C}$, by Vogan in his classification of unitary dual of $GL_n(\mathbb{R})$ in [Vog86] when $F = \mathbb{R}$ that

**Proposition 3.2.** Let $G = GL_n(F)$ and $\sigma$ be an irreducible unitary representation of $M$, then $I^G_B(\sigma)$ is irreducible.
Speh and Vogan proved an irreducibility criterion for limits of generalized principal series representations of Lie groups in [SV80, Theorem 6.19]. Since the proof of Speh and Vogan only use the structure of the relevant local L-parameters of the induced data, we can deduce the following irreducibility of the representations in a given local L-packet.

**Proposition 3.3.** Let \( G_0 = SO(V_0, q_0) \) be the special orthogonal group for a non-degenerate quadratic space \((V_0, q_0)\). Suppose that for essentially discrete series representations \( \sigma_i \) of \( GL_{r_i} \) and an irreducible representation \( \pi_0 \) of \( G_0 \) with L-parameter \( \varphi_0 \), the normalized induced representation \( \sigma_1 \times \cdots \times \sigma_i \times \pi_0 \) is an irreducible representation. Then \( \sigma_1 \times \cdots \times \sigma_i \times \pi'_0 \) is an irreducible representation for every \( \pi'_0 \in \Pi^\text{Vogan} \).

**Remark 3.1.** In the literature, \( \sigma_1 \times \cdots \times \sigma_i \times \pi_0 \) represents the normalized induction of \( \sigma_1 \otimes \cdots \otimes \sigma_i \otimes \pi_0 \) from a parabolic subgroup \( P \) of \( G = SO(V, q) \) whose Levi component \( M \) is isomorphic to \( GL_{r_1}(F) \times \cdots \times GL_{r_i}(F) \times SO(V_0, q_0) \), where \( V = V_0 \oplus D_{2r} \) for a hyperbolic space \( D_{2r} \) orthogonal to \( V_0 \). When \( F = \mathbb{R} \), \( \dim V \) is even and the quasi-split inner form of \( G \) is \( SO(n, n - 2) \), the parabolic subgroups with Levi component \( M \) are conjugate to each other only by \( O(V, q) \) but not by \( SO(V, q) \), and the induced representations from different parabolic subgroups may not be isomorphic as representations of \( SO(V, q) \). Therefore, we need to clarify the meaning of our notation \( \sigma_1 \times \cdots \times \sigma_i \times \pi_0 \) in this case.

Let \( V = X_r \oplus Y_r \oplus V_\text{an} \) be a generalized polar decomposition, where \( X_r, Y_r \) are maximal totally isotropic in \( V \) and \( q|_{X_r \times Y_r} \) is non-degenerate, and \( V_\text{an} \) is anisotropic. Then there is a basis \( \{v_i\}_{1 \leq i \leq r} \) of \( X_r \) and a basis \( \{v_{-i}\}_{1 \leq i \leq r} \) of \( Y_r \), such that \( q(v_i, v_{-j}) = \delta_{ij} \). Let \( I_1, \ldots, I_c \) be disjoint subsets of \( \{\pm 1, \pm 2, \ldots, \pm r\} \) and \( \bar{I}_j = \{-i | i \in I_j\} \). The parabolic subgroup associated to \( I_1, I_2, \ldots, I_c \) is the parabolic subgroup stabilizing

\[
X_1 \subset X_2 \subset \cdots \subset X_l,
\]

where \( X_j = \text{Span}\{v_i \in I_k | 1 \leq k \leq j\} \). The Levi component of \( P \) is isomorphic to \( GL_{r_1}(F) \times \cdots \times GL_{r_f}(F) \times GL(V_0, q_0) \), where \( r_i = |I_i| \) and \( V_0 = V_\text{an} \oplus \bigoplus_{j \in \bar{I}_j} I_j(Cv_i \oplus Cv_{-i}) \). For admissible representations \( \tau_i \) of \( GL_{r_i} \) and admissible representation \( \pi_0 \) of \( GL(V_0, q_0) \), denote

\[
(\tau_1, I_1) \times (\tau_2, I_2) \times \cdots \times (\tau_l, I_l) \times \pi_0 = I_P^G(\tau_1 \otimes \cdots \otimes \tau_l \otimes \pi_0),
\]

where \( I_P^G \) is the normalized induction from \( P \) to \( G \). We abbreviate it as \( \tau_1 \times \tau_2 \times \cdots \times \tau_l \times \pi_0 \) when the \( I_j \)’s associated to \( \tau_j \)’s are clear. In the analysis in Section 3, the choice of the parabolic subgroup will not affect the analysis as the induced representations from different parabolic subgroups have the same infinitesimal characters.

With these results we are ready to prove that representations in generic Vogan packets are fully induced. For a generic parameter \( \varphi \) of the Weil-Deligne group \( \mathcal{W}_F \), it can be regarded a semisimple representation with a pairing (see Section 2.3). Then \( \varphi \) can be decomposed as a direct sum of irreducible representations

\[
\varphi = \bigoplus_{i \in I} \varphi_{0,i} + \bigoplus_{j=1}^l (\varphi_j + \varphi_j^\vee)
\]

where \( \varphi_{0,i} \)'s are distinct self-dual irreducible representations, and \( \varphi_j \)'s are not self-dual. One can find the classification of the irreducible pieces in [Kna94].
Let \( \varphi_0 \) be the direct sum of self-dual representations in the decomposition \( \mathfrak{S}_0 \), that is, \( \varphi_0 = \bigoplus_{i \in I} \varphi_{0,i} \). Since each self-dual representation of \( \mathcal{V}_\varphi \) has bounded image, \( \varphi_{0,i} \) are tempered parameters, and thus the parameter \( \varphi_0 \) itself is a tempered parameter. It is well known (see [GGPT2 Section 4]) that the component group \( S_\varphi \) of the parameter \( \varphi \) depends only on \( \varphi_0 \).

Let \( r_j = \dim \varphi_j \) and \( r = \sum_{j=1}^l r_j \). Under the local Langlands correspondence of the general linear groups (see [Kna94]), \( \varphi_j \) corresponds to \( |\det(\cdot)|^{s_j} \tau_j \), where \( \tau_j \) is equal to the character \( \rho_{m_j}(z) = (z^{m_j})^{m_j} \) of \( GL_1(F) \) or a discrete series representation of \( GL_2(\mathbb{R}) \). We may assume that \( Re(s_1) \geq \cdots \geq Re(s_l) > 0 \), then under the local Langlands correspondence of the general linear group \( GL_r(F) \), the parameter

\[
\varphi^+ = \bigoplus_{j=1}^l \varphi_j
\]

corresponds to the unique quotient \( \tau \) of

\[
|\det(\cdot)|^{s_1} \tau_1 \times \cdots \times |\det(\cdot)|^{s_l} \tau_l.
\]

The decomposition (3.1) can be simplified as

(3.2)

\[
\varphi = \varphi_0 \oplus \varphi^+ \oplus (\varphi^+)^\vee.
\]

It is well known (see [GGPT2 Section 4]) that the component group \( S_\varphi \) of the parameter \( \varphi \) depends only on \( \varphi_0 \). And there is commutative diagram

\[
\begin{array}{ccc}
\Pi^{\text{Vogan}}_{\varphi_0} & \longrightarrow & \text{Hom}(S_{\varphi_0}, \{\pm 1\}) \\
\downarrow & & \downarrow \\
\Pi^{\text{Vogan}}_{\varphi} & \longrightarrow & \text{Hom}(S_{\varphi}, \{\pm 1\})
\end{array}
\]

where the first vertical arrow maps every tempered representation \( \pi_0 \in \Pi^{\text{Vogan}}_{\varphi_0} \) to the unique quotient \( \pi \times \pi_0 \), and the horizontal arrows are defined by Langlands-Vogan parameterization. Then each \( \pi \in \Pi^{\text{Vogan}}_{\varphi} \) is the unique quotient of a normalized induction

\[
|\det(\cdot)|^{s_1} \tau_1 \times \cdots \times |\det(\cdot)|^{s_l} \tau_l \times \pi_0,
\]

Let \( \lambda_1 > \cdots > \lambda_t > 0 \) be all possible values of \( Re(s_i)(1 \leq i \leq l) \). And for \( 1 \leq c \leq t \), let

\[
\sigma_c = |\det(\cdot)|^{i_{m(s_{i+1})}} \tau_{i+1} \times \cdots \times |\det(\cdot)|^{i_{m(s_{i+c})}} \tau_{i+c},
\]

where \( s_j(j = i_c + 1, \cdots, i_c + k_c) \) are all \( s_j \) such that \( Re(s_j) = \lambda_c \). From Proposition 3.2, \( \sigma_c \) is an irreducible tempered unitary representation.

When \( G \) is quasisplit and \( \pi \) is generic, from Proposition 3.3, this induced representation

\[
I_\pi = |\det(\cdot)|^{s_1} \tau_1 \times \cdots \times |\det(\cdot)|^{s_l} \tau_l \times \pi_0 = |\det(\cdot)|^{\lambda_1} \sigma_1 \times \cdots \times |\det(\cdot)|^{\lambda_t} \sigma_t \times \pi_0
\]

is irreducible. In general, as there exists a generic representation in \( \Pi^{\text{Vogan}}_{\varphi} \), from Proposition 3.3, we have the irreducibility of the induced representation \( I_\pi \), and thus \( \pi = I_\pi \).

**Proposition 3.4.** For a generic \( L \)-parameter \( \varphi = \varphi_0 \oplus \varphi^+ \oplus (\varphi^+)^\vee \), every representation \( \pi \) in \( \Pi^{\text{Vogan}}_{\varphi} \) can be expressed as

\[
\pi = \tau \times \pi_0
\]

where \( \pi_0 \) is a tempered representation with \( L \)-parameter \( \varphi_0 \) and \( \tau \) is the representation of \( GL_r(F) \) with \( L \)-parameter \( \varphi^+ \). Moreover,

\[
\tau = |\det(\cdot)|^{s_1} \tau_1 \times \cdots \times |\det(\cdot)|^{s_l} \tau_l
\]
where \( \text{Re}(s_1) \geq \text{Re}(s_2) \geq \cdots \geq \text{Re}(s_l) > 0 \) and \( \tau_i \) is a discrete series representation of \( GL_r(G) \).

In particular,

**Corollary 3.1.** When \( F = \mathbb{C} \), representations in generic Vogan packets are principal series representations.

4. Conjectures \([10, 14]\) and \([2.1]\) over the complex field

The co-dimension one case of the local Gan-Gross-Prasad conjecture for special orthogonal groups over the complex field was proved by Jan Möllers in \([\text{Möll17}]\). And in this section, we generalize his results and give the proof for the general case by constructing a nonzero Bessel functional using the method in \([\text{GSS19}]\).

For the Gan-Gross-Prasad triple \((\widetilde{G}, \widetilde{H}, \widetilde{\nu})\) for an admissible pair \((V, V')\). We first show that we can choose the Borel subgroup \( \widetilde{B} \) of \( \widetilde{G} \) appropriately such that the double coset \( \widetilde{B}\widetilde{H} \) is open dense in \( \widetilde{G} \) and the intersection \( \widetilde{B} \cap \widetilde{H} \) is the trivial group.

Recall that in the notations in Section 2.1 \( P \) is the parabolic subgroup of \( G = \text{SO}(V, q) \) stabilizing the flag \( F \) with the Levi decomposition \( P = MN \). The Levi part \( M \) can be decomposed as \( A \times G^+ \), where \( A \cong (\mathbb{C}^\times)^r \) and \( G^+ = \text{SO}(V' \oplus \mathbb{C}v_0) \). Let \( \widetilde{G}^+ = \text{SO}(V' \oplus \mathbb{C}v_0) \times \text{SO}(V') \) and \( \widetilde{H}^+ \) is the image of the diagonal embedding from \( G^+ = \text{SO}(V') \) into \( \widetilde{G}^+ \). Then the group \( \widetilde{P} = P \times G^+ \) is a parabolic subgroup of \( \widetilde{G} \), with Levi part \( \widetilde{M} = (A \times 1)\widetilde{G}^+ \) and unipotent radical \( \widetilde{N} = N \times 1 \). Let \( \widetilde{P} \) be the opposite of the parabolic subgroup \( \widetilde{P} \) and \( \widetilde{N} \) be the unipotent radical of \( \widetilde{P} \). Then \( \widetilde{N}\widetilde{M}\widetilde{N} \) is open dense in \( \widetilde{G} \) and the multiplication map from \( \widetilde{N} \times \widetilde{M} \times \widetilde{N} \) to \( \widetilde{N}\widetilde{M}\widetilde{N} \) is an isomorphism.

From \([\text{Möll17}]\) Section 6.2.4, \( \widetilde{G}^+ \) has a Borel subgroup \( \widetilde{B}^+ \) such that the double coset \( \widetilde{B}^+\widetilde{H}^+ \) is open dense and \( \widetilde{B}^+ \cup \widetilde{H}^+ = 1 \). Notice that \( \widetilde{H} = \widetilde{H}^+\widetilde{N} \), so if we choose \( \widetilde{B} = \widetilde{N}(A \times 1)\widetilde{B}^+ \), we have

\[
\widetilde{B} \cap \widetilde{H} = \widetilde{N}(A \times 1)\widetilde{B}^+ \cap \widetilde{H}^+\widetilde{N} = (A \times 1)\widetilde{B}^+ \cap \widetilde{H}^+ = 1
\]

and \( \widetilde{B}\widetilde{H} = \widetilde{N}(A \times 1)(\widetilde{B}^+\widetilde{H}^+)\widetilde{N} \) is open dense in \( \widetilde{N}\widetilde{M}\widetilde{N} \), which implies \( \widetilde{B}\widetilde{H} \) is open dense in \( \widetilde{G} \).

For a character \( \sigma \) of \( \widetilde{B} \) that trivialize the unipotent radical of \( \widetilde{B} \), from \([\text{DC91}]\) Remark 2.1.4, as \( P \setminus G \) is compact, \( \text{Hom}_{\widetilde{H}}(I_{\widetilde{B}}^{\widetilde{G}}(\sigma), \widetilde{\nu}) \) is equal to the space of \( (\widetilde{B} \times \widetilde{H}, \delta^{{\widetilde{B}}}_{\widetilde{B}}\sigma \times \widetilde{\nu}) \)-equivariant distributions. Then we construct such a distribution by “extending” an equivariant tempered measure on the open double coset \( \widetilde{B}\widetilde{H} \) to \( \widetilde{G} \) with results of Gourevitch, Sahi and Sayag in \([\text{GSS19}]\).

Since \( \widetilde{B} \cap \widetilde{H} = 1 \), we can construct a \( (\widetilde{B} \times \widetilde{H}, \delta^{{\widetilde{B}}}_{\widetilde{B}}\sigma \times \widetilde{\nu}) \)-equivariant measure \( \mu \) on \( \widetilde{B}\widetilde{H} \) as

\[
(4.3) \quad \mu = \delta^{-\frac{1}{2}}_{\widetilde{B}}\sigma^{-1}(h)\widetilde{\nu}(h)d\widetilde{h}d\widetilde{b}d\widetilde{a}.
\]

Recall that \( \widetilde{\nu} \) is a unitary character and \( \delta^{{\widetilde{B}}}_{\widetilde{B}}\sigma \) is a multiplicative character of \( \widetilde{B} \) which is trivial in the unipotent radical of \( \widetilde{B} \) so it is of moderate growth. This implies that \( \mu \) is a tempered measure, that is, a measure that yields finite value on Schwartz functions of \( \widetilde{B}\widetilde{H} \).

Because \( \widetilde{B} \) is solvable, from \([\text{GSS19}]\) Corollary 2.10, there exists a character \( \chi \) of \( \widetilde{B} \times \widetilde{H} \) and a \( (\widetilde{B} \times \widetilde{H}, \chi) \)-equivariant regular function \( f \) of \( \widetilde{G} \) such that \( \widetilde{B}\widetilde{H} = \{ g \in \widetilde{G} | f(g) \neq 0 \} \). For \( \text{Re}(\lambda) \gg 0 \)

the measure \( \mu(\lambda) = |f|^{\lambda} \) can be extended by 0 to a \( (\widetilde{B} \times \widetilde{H}, |\chi|^\lambda \otimes (\delta^{{\widetilde{B}}}_{\widetilde{B}}\sigma \times \widetilde{\nu})) \)-equivariant distribution \( \mu(\lambda) \) on \( \widetilde{G} \). From \([\text{GSS19}]\) Lemma 4.1, \( \mu(\lambda) \) can be meromorphically continued to \( \lambda \in \mathbb{C} \). In
consequence, the leading term of the Laurent series of $\mu(\lambda)$ is a nonzero $(\tilde{B} \times \tilde{H}, \delta_\tilde{B}^\lambda \tilde{B} \times \tilde{V})$-equivariant distribution on $G$, which corresponds to a nonzero element in $\text{Hom}_{\tilde{H}}(I^\tilde{G}_{\tilde{B}}(\tilde{\sigma}), \tilde{\nu})$.

For every principal series representation $\tilde{\pi} = I^\tilde{G}_{\tilde{B}}(\tilde{\sigma})$ of $\tilde{G}$, by combining the above construction with \cite[Theorem A]{JSZ10}, we can conclude that

$$m(\tilde{\pi}) = \dim \text{Hom}_{\tilde{H}}(I^\tilde{G}_{\tilde{B}}(\tilde{\sigma}), \tilde{\nu}) = 1.$$ 

For a generic parameter $\tilde{\varphi}$ of $\tilde{G}$, since there is no discrete series representation of special orthogonal groups over the complex field, each $L$-packet $\Pi_{\tilde{\varphi}}(G, SO)$ contains only one representation. Besides, there is only one pure inner form of $SO(V, q)$, so the Vogan packet $\Pi^{\text{Vogan}}_\tilde{\varphi}$ contains only one representation. From Corollary \ref{cor:5.1} this representation is equal to $\pi \hat{\otimes} \pi'$ where $\pi$ and $\pi'$ are principal series representations of $G$ and $G'$ respectively, and from \cite[Appendix 2.3]{War12}, $\pi \hat{\otimes} \pi'$ is a principal series representation. Therefore, Conjecture \ref{conj:1.1} and Conjecture \ref{conj:2.1} are proved over the complex field.

**Theorem 4.1.** Conjectures \ref{conj:1.1} and \ref{conj:2.1} hold over the complex field $\mathbb{C}$.

5. **Conjectures \ref{conj:1.1} and \ref{conj:2.1} over the real field**

In this section, we prove the generic part of the conjectures based on the tempered part over the real field. We construct a descent map between Bessel functionals for representations of two Gan-Gross-Prasad triples to manifest the relation between local Bessel functionals and parabolic inductions. We will first prove a technical lemma in Section 5.1. Then we review our basic settings and introduce the notations we would use in our proof. Take the generic parameter $\tilde{\varphi}$ of $\tilde{G}$, which corresponds to a nonzero element in $\text{Hom}_{\tilde{H}}(I^\tilde{G}_{\tilde{B}}(\tilde{\sigma}), \tilde{\nu})$. Then we apply the multiplicity formula to prove the Theorem 2.1 and 2.2 following Moeglin and Waldspurger’s idea in \cite{MW12}.

We work in the category $\mathcal{R}(G)$ of smooth Fréchet representations of moderate growth for an almost linear Nash group $G$. A representation $\pi$ in $\mathcal{R}(G)$ is called Casselman-Wallach if the $(g_C, K)$-module $\pi^K$ is Harish-Chandra (see Section 2.2). The reason for not working in the category of Casselman-Wallach representations is that the representation $”^u \pi_0|_{SO(V', \delta_0)}”$ in Section 5.2 is not admissible. When no confusion is possible, we do not distinguish a representation with its underlying space.

For the Gan-Gross-Prasad triple $(\tilde{G}, \tilde{H}, \tilde{\nu})$ of an admissible pair $(V, V')$, a Casselman-Wallach representation $\pi$ of $SO(V)$ and a Casselman-Wallach representation $\pi'$ of $SO(V')$, we denote $m(\pi, \pi')$ to be the dimension of the space of Bessel functionals $\text{Hom}_{S_r}(\pi \hat{\otimes} \pi', \nu_r)$ of the pair $(\pi, \pi')$, where $S_r$ is the Bessel subgroup and $\nu_r$ is the generic Bessel character defined in Section 2.1. Then we have

$$m(\pi, \pi') = \dim \text{Hom}_{S_r}(\pi \hat{\otimes} \pi', \nu_r) = \dim \text{Hom}_{\tilde{H}}(\pi \hat{\otimes} \pi', \tilde{\nu}) = m(\pi \otimes \pi').$$

For a closed subgroup $P$ of $G$, we denote by $\text{S-Ind}^P_G \sigma$ the Schwartz induction of $\sigma$ from $H$ to $G$, which is defined in Appendix A.1. One can check that $\text{S-Ind}^P_G \sigma$ is canonically isomorphic to the unnormalized smooth induction when $P \backslash G$ is compact. We denote by $I^P_G(\sigma)$ the normalized induction of $\sigma$ from $P$ to $G$, which is isomorphic to $\text{S-Ind}^P_G(\delta_\sigma^\frac{1}{2} \delta_P^\frac{1}{2} \sigma)$. Here $\delta_P, \delta_G$ are the modular characters of $P$ and $G$ respectively.

Then we review our basic settings and introduce the notations we would use in our proof. Take the Gan-Gross-Prasad triple $(\tilde{G}', \tilde{H}', \tilde{\nu}')$ for an admissible pair $(V', V_0)$. Let the orthogonal complement
\(V_0^+ = (X'_r \otimes Y'_r) \oplus \mathbb{C}v'_0\) for totally isotropic subspaces \(X'_r, Y'_r\) and anisotropic vector \(v'_0\), and we denote \(X'_{r+1} = X'_r \oplus \mathbb{C}v'_0\). Now we construct an anisotropic vector \(v\) such that \((v, v) = -(v'_0, v'_0)\) and \(v\) is orthogonal to \(V'\), and define \(V = V' \oplus \mathbb{C}v\). Then we have
\[
V = V' \oplus \mathbb{C}v_0 = (X_{r+1} \oplus Y_{r+1}) \oplus V_0.
\]
where \(X_{r+1} = X_r \oplus \mathbb{C}(v + v'_0)\) and \(Y_{r+1} = Y_r \oplus \mathbb{C}(v - v'_0)\). By taking \(G = SO(V, q)\), we obtain a co-dimension one Gan-Gross-Prasad triple \((\tilde{G}, \tilde{H}, \tilde{\nu})\), where \(\tilde{G} = G \times G'\) is the image \(\Delta G'\) of the diagonal embedding of \(G'\) into \(\tilde{G}\), and \(\tilde{\nu}\) is the trivial character of \(\tilde{H}\). Let \(P\) be the parabolic subgroup of \(G\) stabilizing \(X_{r+1}\), and \(P = MN\) be the Levi decomposition of \(P\) where \(M = GL(X_{r+1}) \times G_0\). From the calculation in [MW12, Section 6.3.5], \(PG'\) is open dense in \(G\). Take \(P' = P \cap G'\). Then \(P'\) is the subgroup of \(G'\) consisting of elements \(p'\) that satisfy
\[
p'X'_r \subset X'_r, \quad p'(v_0 + v'_0) \subset X'_r \oplus \mathbb{C}v'_0.
\]
that is,
\[
p'X'_r \subset X'_r, \quad (p' - 1)v_0 \subset X'_r.
\]
Therefore, \(P'\) can be decomposed as
\[
(R_{r, 1} \times G_0) \rtimes N',
\]
where \(G_0 = SO(V_0, q_0)\) is the mirabolic subgroup of \(GL(X_{r+1})\) that consists of \(m'\) satisfying
\[
(m' - 1)(v_0 + v'_0) \subset X'_r, \quad m'X'_r \subset X'_r,
\]
and \(N'\) is the unipotent subgroup of \(G'\) satisfying
\[
(g - 1)X'_{r+1} = 0, \quad (g - 1)V_0 \subset X'_{r+1}.
\]
Then there is a decomposition
\[
S'_r = (N_{r+1} \times G_0) \rtimes N' \subset P',
\]
where \(N_{r+1}\) is the subgroup of \(GL_r = GL(X'_{r+1})\) consisting of \(g \in GL(X'_{r+1})\) such that
\[
(g - 1)X'_{r+1} \subset X'_r, \quad \text{for } i = 0, 1, \ldots, r.
\]
Let \(\psi_{r+1}\) be the (unitary) character of \(N_{r+1}\) obtained by the restriction of \(\nu'\) to \(N_{r+1}\), then we have
\[
\text{S-Ind}_{S'_r} \pi' = \text{S-Ind}_{(N_{r+1} \times G_0) \rtimes N'} \pi' = \text{S-Ind}_{N_{r+1} \rtimes \psi_{r+1}} \pi' = \text{S-Ind}_{N_{r+1}} \pi'.
\]
Let \(\pi'(\text{resp. } \pi_0, \sigma)\) be a Casselman-Wallach representation of \(G'(\text{resp. } G_0, GL_{r+1})\). For \(s \in \mathbb{C}\), let \(\pi_s\) be the representation \(|\det(\cdot)|^s \sigma \rtimes \pi_0 = I_G^\mathbb{C}(\det(\cdot)|^s \sigma \rtimes \pi_0)\).

When \(\sigma\) is a principal series, our construction of the descent map \(F_{s}(s \in \mathbb{C})\) is the composite \(F_{2,s} \circ F_{1,s}\) of
\[
F_{1,s} : \text{Hom}_{G'}(\pi_s \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{G'}(\text{S-Ind}_{\mathbb{C}}^G(\det(\cdot)|^s \delta_p^G \sigma|_{R_{r, 1}} \otimes \pi_0) \otimes \pi', \mathbb{C})
\]
constructed in Section 5.2 and
\[
F_{2,s} : \text{Hom}_{G'}(\text{S-Ind}_{\mathbb{C}}^G(\det(\cdot)|^s \delta_p^G \sigma|_{R_{r, 1}} \otimes \pi_0) \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{S'}(\pi' \otimes \pi_0, \pi'_s)
\]
constructed in Section 5.3. And we prove that for general \(s\), that is, for \(s \in \mathbb{C} - E\) where \(E\) is a countable subset of \(\mathbb{C}\), \(F_{1,s}\) and \(F_{2,s}\) are isomorphisms in Section 5.2 and 5.3, which implies that \(m(\pi'_s, \pi_0) = m(\pi_s, \pi'_s)\) for general \(s\). In order to use the mathematical induction as in [MW12], the multiplicity formulas should be proved under some stronger conditions (which will be called as "Condition (P)" in the rest of this section) when \(\pi_0, \pi'_s\) are in generic Vogan packets, that is,
(1) for general \( s \), if \( \sigma \) is a principal series representation;
(2) for \( Re(s) \geq s' \), if \( r = 0 \) and \( \sigma = sgn^m(m = 0, 1) \) of \( GL_1(\mathbb{R}) \), where \( sgn(x) = 1 \) when \( x > 0 \) and \( sgn(x) = -1 \) when \( x < 0 \);
(3) for \( Re(s) + \frac{\rho}{\sigma} \geq s' \), if \( r = 1 \) and \( \sigma \) is the discrete series representation \( D_n \) of \( GL_2(\mathbb{R}) \), which is the unique subrepresentation of \( |\cdot|^\sigma sgg^{n+1} \times |\cdot|^s \).

Here \( s' \) is a real number associated to \( \pi' \) defined by (6.7) in Section 5.1.

When \( r = 0 \) and \( \sigma = sgn^m(m = 0, 1) \), we would construct the same descent map as the principal series case, which is an isomorphism when \( Re(s) \geq s' \).

When \( r = 1 \) and \( \sigma \) is the discrete series representation \( D_n \) of \( GL_2(\mathbb{R}) \), we have to construct \( F_{2,n} \) differently to ensure it is bijective when \( Re(s) + \frac{\rho}{\sigma} \geq s' \), and in this case the decent map "descends" Bessel functionals of the pair \( (\sigma \times \pi_0, \pi') \) to Bessel functionals of the pair \( (\pi', \chi \times \pi_0) \), where \( \chi \) is the character of \( GL_1(\mathbb{R}) \) defined by \( \chi(x) = |x|^\sigma sgn^{n+1}(x) \).

5.1. A technical lemma. In this subsection, we apply the results in Appendix A and Appendix B to prove Lemma 5.1 our main technical lemma. We would freely use the notations about Schwartz homologies in Appendix A and Harish-Chandra parameters in Appendix B.

Lemma 5.1. Let \( p, q, l \) be non-negative integers and \( n = \lfloor \frac{p+q}{2} \rfloor \). Suppose we have a representation \( \pi_0 \in \mathcal{R}(SO(p,q)) \), an irreducible Casselman-Wallach representation \( \tau \) of \( GL_1(\mathbb{R}) \) with a Harish-Chandra parameter \( (s_1, \cdots, s_l) \) and an irreducible Casselman-Wallach representation \( \pi' \) of \( SO(p+l, q+l) \) with a Harish-Chandra parameter \( (\lambda_1, \cdots, \lambda_n) \), suppose we have \( s_1 \neq \pm \lambda_i \) for \( 1 \leq i \leq n \), then the Schwartz homologies of \( (\tau \times \pi_0) \otimes \pi' \) vanish.

Proof. From Corollary A.2 it suffices to find an element \( z \in \mathcal{Z}(\mathfrak{g}_C) \) such that \( \chi_{\pi' \tau}(z) \neq 0 \) and \( z \cdot v = 0 \), for every \( v \in \tau \times \pi_0 \).

Let \( \mathfrak{h} \) the Cartan subalgebra of \( GL_1(\mathbb{R}) \times SO(p,q) \). With the parameterization of the complex dual \( \mathfrak{h}^*_C \) of \( \mathfrak{h} \) in Appendix B.1 we define a polynomial \( f \) in the polynomial algebra \( P(\mathfrak{h}^*_C) \) by

\[
f(\lambda_1, \cdots, \lambda_n) = \prod_{i=1}^{n} (\lambda_i^2 - s_i^2)
\]

and \( f \) is invariant under \( W_G \), so \( f \in P(\mathfrak{h}^*_C)^{W_G} \). Then we take the \( z \in \mathcal{Z}(\mathfrak{g}_C) \) corresponding to \( f \) under Harish-Chandra isomorphism.

On the one hand, for every irreducible component \( \pi_0 \) of \( \pi_0 \), suppose the Harish-Chandra conjugacy class of \( \pi_0 \) is \( (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) \), then the Harish-Chandra conjugacy class for every irreducible component \( \pi_\tau \) of \( \tau \times \pi_0 \) is \( [(s_1, s_2, \cdots, s_l, \lambda_1, \lambda_2, \cdots, \lambda_{n-1})] \).

Then \( z \cdot v = 0 \) for every \( v \in \pi_\tau \), so \( z \cdot v = 0 \) for every \( v \in \tau \times \pi_0 \). On the other hand, since

\[
\chi_{\pi' \tau}(z) = \prod_{i=1}^{n} (-\lambda_i^2 + s_i)(-\lambda_i^2 - s_i)
\]

and \( s_i \neq \pm \lambda_i \) for \( i = 1, \cdots, n \), we have \( \chi_{\pi' \tau}(z) \neq 0 \).

Therefore, the Schwartz homologies of \( (\tau \times \pi_0) \otimes \pi' \) vanish. \( \square \)
For a Casselman-Wallach representation $\pi'$ in the general packets of $G' = SO(V', q')$, from Proposition 3.4, $\pi'$ can be expressed as the parabolic induction

$$|\det(\cdot)|^{s'_1} \tau_1 \times \cdots \times |\det(\cdot)|^{s'_r} \tau_r \times \pi_0',$$

where $Re(s'_1) \geq \cdots \geq Re(s'_r) > 0$, $\tau_j$ are isomorphic to $sgn^{m_j}(m_j' = 0, 1)$ of $GL_1(\mathbb{R})$ or discrete series representations $D_{\alpha_j'}$ of $GL_2(\mathbb{R})$ and $\pi_0'$ is tempered. Then the Harish-Chandra conjugacy class $[v_{\pi'}]$ can be expressed as follows

$$[v_{\pi'}] = [(v_1, \cdots, v_l, v_{\pi'}),]$$

where $v_{\pi_0'}$ is a Harish-Chandra parameter of $\pi_0'$ and

$$v_j = \begin{cases} (s'_j) & \text{if } \tau_j = sgn^{m_j'}, \\ (s'_j + \frac{n_j}{2}, s'_j - \frac{n_j}{2}) & \text{if } \tau_j = D_{\alpha_j'} . \end{cases}$$

Now define the complex number $s'$ by

$$s' = \max\{\sup\{Re(s'_j) + \frac{n_j}{2}\}, 0\}.$$  

In particular, $s'$ is equal to zero when $\pi'$ is tempered. It is worthwhile to mention that the $s'$ defined above depends only on the parameter $\varphi^+$ in (3.3) from Section 2.4. And from Lemma 5.1 we have

**Corollary 5.1.** With the notations in Proposition 5.1 if $s_1 > s'$, then the Schwartz homologies of $(\tau \times \pi_0) \otimes \pi'$ vanish.

By combining Lemma 5.1 with Proposition 3.2 and Proposition 3.3.

**Corollary 5.2.** With the notations in Proposition 5.1 and let $\rho$ be a finite-dimensional representation of $GL_1(\mathbb{R})$. Then the Schwartz homologies of $(|\det(\cdot)|^s \tau \otimes \pi_0) \otimes \pi'$ vanish for general $s$.

### 5.2. Construction and bijectiveness of $F_{1,s}$.

In this subsection, we first construct $F_{1,s}$ and then do the Schwartz analysis on the closed orbits to obtain a vanishing result in Proposition 5.2 which implies $F_{1,s}$ is isomorphic under Condition(P).

On the one hand, if we denote $\pi_s$ to be the representation $|\det(\cdot)|^s \tau \otimes \pi_0$, we can express $\pi_s$ as the space of Schwartz sections on $P \backslash G$, which gives

$$\pi_s \otimes \pi' = I_G^G(|\det(\cdot)|^s \otimes \pi_0) \otimes \pi'$$

$$= S \text{-Ind}_P^G(|\det(\cdot)|^s \delta_P^1 \otimes \pi_0) \otimes \pi'$$

$$= \Gamma^S(P \backslash G, |\det(\cdot)|^s \delta_P^1 \otimes \pi_0) \otimes \pi'.$$

Here $\delta_P$ is the modular character of $P$, and we have $\delta_P(p) = |\det(m)|^{d-1-r}$ where $d = \dim V$ and $p = (m \times g_0) \times n \in P$. For simplicity, we regard $\delta_P$ as a character of $GL_{r+1}$.

On the other hand, we have

$$S \text{-Ind}_P^G(|\det(\cdot)|^s \delta_P^1 \sigma|_{R_{r+1}} \otimes \pi_0) \otimes \pi' = \Gamma^S(P' \backslash G', |\det(\cdot)|^s \delta_P^1 \sigma|_{R_{r+1}} \otimes \pi_0) \otimes \pi'$$

$$= \Gamma^S(P \backslash PG', |\det(\cdot)|^s \delta_P^1 \sigma \otimes \pi_0) \otimes \pi'.$$

Since $P' \backslash PG'$ is an open orbit of $P \backslash G$, there is a natural embedding

$$i_{U,X} : \Gamma^S(P' \backslash PG', |\det(\cdot)|^s \delta_P^1 \sigma \otimes \pi_0) \rightarrow \Gamma^S(P \backslash G, |\det(\cdot)|^s \delta_P^1 \sigma \otimes \pi_0) \equiv \Gamma^S(P \backslash G, |\det(\cdot)|^s \sigma \otimes \pi_0)$$

\[ (5.8) \]
defined by the extension by zero. With this embedding \( i_{t, X} \), we define

\[
F_{1,s} : \text{Hom}_{G'}(\pi, \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{G'}(\text{S-Ind}^G_{P'}([\det()]^{\ast} \delta_{\sigma}^{\hat{\rho}} |_{R_{\eta_0} \otimes \pi_0} \otimes \pi', \mathbb{C})
\]

induced by \( i_{t, X} \). We will prove that \( F_{1,s} \) is an isomorphism under Condition(P) using Schwartz homologies. For this purpose, we first analyze the structure of \( G' \)-orbits of \( P \backslash G \).

Let \( X = P \backslash G \), then \( X \) is the topological subspace of the Grassmannian manifold \( \text{Gr}(r + 1, V) \) consisting of totally isotropic subspaces \( V_{r+1} \) of dimension \( r + 1 \) in \( V \) and it can be regarded as a \( G' \)-space under the right action of \( G' \). Note that \( \mathcal{U} = P \backslash PG' \) is the open orbit of \( X \) consisting of \( V_{r+1} \) that is not contained in \( V' \) and \( Z = X \setminus \mathcal{U} \) is the closed subspace of \( X \) consisting of \( V_{r+1} \) that is contained in \( V' \).

If \( V_0' = V_0 \oplus C \cdot v_0' \) is anisotropic, then \( X_0' \) is maximal isotropic in \( V' \). In this case, \( Z \) is empty. If \( V_0' \) is isotropic, then there is \( v_0 \in V_0 \) such that \((v_0, v_0) = -(v_0', v_0)\). In this case, \( Z \) is the disjoint union of two connected \( G' \)-orbits \( \{\mathbb{C}(v_0 + v_0')\} \) and \( \{\mathbb{C}(v_0 - v_0')\} \) when \( \dim V' = 2 \). If \( V_0' \) is isotropic and \( \dim V' > 2 \), then \( Z \) is a single \( G' \)-orbit \( P \backslash P \eta G' \), where \( \eta \) is an element in \( G \) satisfying \( P \backslash P \eta = [X_0' \oplus \mathbb{C}(v_0 + v_0')] \). Then \( \eta^{-1} P \eta \) is the parabolic subgroup of \( G \) stabilizing \( X_0' \oplus \mathbb{C}(v_0 + v_0') \) and \( Q' = \eta^{-1} P \eta \cap G' \) is the parabolic subgroup of \( G' \) stabilizing \( X_0' \oplus \mathbb{C}(v_0 + v_0') \). The Levi decomposition \( Q' = M' \backslash N' \) indicates that \( M' \) is isomorphic to \( GL(X_0' \oplus \mathbb{C}(v_0 + v_0')) \times SO(V_0', q_0) \).

The complexified conormal bundle \( N'_{Z/X} \) has dimension \( r + 1 \), and gives rise to a representation of \( Q' \) on its fiber over \([X_0' \oplus \mathbb{C}(v_0 + v_0')]\) such that \( N' Q' \) and \( SO(V_0', q_0) \) acts trivially and \( GL(X_0' \oplus \mathbb{C}(v_0 + v_0')) \) acts as the standard representation \( \rho \) on the fiber.

Let \( \Gamma_S' (X, |\det()|^{s} \delta_{\hat{\rho}}^{\frac{1}{2}} \otimes \pi_0) = \Gamma_S (X, \mathcal{E}) / \Gamma_S (\mathcal{U}, \mathcal{E}) \), where \( \mathcal{E} \) is the tempered bundle

\[
P' \backslash (G' \times ([\det()]^{s} \delta_{\hat{\rho}}^{\frac{1}{2}} \otimes \pi_0)).
\]

**Proposition 5.1.** The Schwartz homologies of

\[
\Gamma_S' (X, |\det()|^{s} \delta_{\hat{\rho}}^{\frac{1}{2}} \otimes \pi_0) \tilde{\otimes} \pi'
\]

vanish under Condition (P).

**Proof.** From Proposition A.6, \( \Gamma_S' (X, |\det()|^{s} \delta_{\hat{\rho}}^{\frac{1}{2}} \otimes \pi_0) \) has a complete decreasing filtration

\[
\Gamma_S' (X, |\det()|^{s} \delta_{\hat{\rho}}^{\frac{1}{2}} \otimes \pi_0)
\]

whose graded pieces are isomorphic to

\[
\Gamma^S (Z, \text{Sym}^k N'_{Z/X} \otimes \mathcal{E}|_{Z})
\]

\[
\cong \left\{ \begin{array}{ll}
I_{Q'}' (|\det()|^{s + \frac{1}{2}} (\sigma \otimes \text{Sym}^k \rho) \otimes \eta \pi_0|_{SO(V_0', q_0)}) & \text{if } V_0' \text{ is anisotropic,} \\
(\text{I}_{Q'} (|\det()|^{s + \frac{1}{2}} (\sigma \otimes \text{Sym}^k \rho) \otimes \eta \pi_0|_{SO(V_0', q_0)}) \otimes \pi') & \text{if } \dim V \neq 2, \text{ and } V_0' \text{ is isotropic,} \\
\text{I}_{Q'} (|\det()|^{s + \frac{1}{2}} (\sigma \otimes \text{Sym}^k \rho) \otimes \eta \pi_0|_{SO(V_0', q_0)}) & \text{otherwise,}
\end{array} \right.
\]

for \( k = 0, 1, \ldots \), where \( \eta \pi \) is the representation of \( \eta^{-1} G \eta \) defined by \( \eta \pi (g_0) = \pi_0 (\eta g' \eta^{-1}) \). The factor \( |\det()|^{\frac{1}{2}} \) comes from the fact that \( \delta_{\eta^{-1} \rho (p')} / \delta_{\rho (p')} = |\det(m')|^{\frac{1}{2}} \), for every \( p' = (m' \times g_0') \times n' \in Q' \).

Then we use our technical lemma to prove that, under Condition(P), the Schwartz homologies of

\[
(5.9) \quad I_{Q'} (|\det()|^{s + \frac{1}{2}} \sigma \otimes \text{Sym}^k \rho) \otimes \eta \pi_0|_{SO(V_0', q_0)} \tilde{\otimes} \pi'
\]

vanish.
If \( \sigma \) is the principal series representation with a Harish-Chandra parameter \((s_1, s_2, \cdots, s_l)\), then the parameters of the irreducible components of \( \sigma \otimes \text{Sym}^k \rho \) are \([(s_1 + a_1, \cdots, s_l + a_l)]\) from Proposition [3.3] where \( a_i \) are integers such that \( \sum_{i=1}^l a_i = k \). Hence, for general \( s, s + s_1 + k + \frac{1}{2} \neq \pm \lambda' \) for \( i = 1, \cdots, n \) and non-negative integer \( k \). From Lemma [5.1] and Proposition [B.2] the Schwartz homologies of

\[
I_{Q'}^r((|\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \text{Sym}^k \rho) \otimes^G \pi_0|_{SO(V_{0}', \langle \phi \rangle)}) \otimes \pi'
\]

vanish for general \( s \).

If \( r = 0 \) and \( \sigma = \text{sgn}^m(m = 0, 1) \), the Harish-Chandra conjugacy classes of the irreducible representation \( |\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \text{Sym}^k \rho \) is \([(s + k + \frac{1}{2})]\). If \( \text{Re}(s) \geq s' \), we have \( \text{Re}(s) + \frac{n}{2} > s' \), then from Corollary 5.1 the Schwartz homologies of

\[
I_{Q'}^r((|\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \text{Sym}^k \rho) \otimes^G \pi_0|_{SO(V_{0}', \langle \phi \rangle)}) \otimes \pi'
\]

vanish.

If \( r = 1 \) and \( \sigma \) is the discrete series representation \( D_n \) of \( GL_2(\mathbb{R}) \), the Harish-Chandra parameters of for irreducible components of \( |\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \text{Sym}^k \rho \) are

\[
[(s + \frac{n}{2} + a_1 + \frac{1}{2}, s - \frac{n}{2} + a_2 + \frac{1}{2})]
\]

such that \( a_1, a_2 \) are integers satisfying \( a_1 + a_2 = k \). If \( \text{Re}(s) + \frac{n}{2} \geq s' \), we have \( \text{Re}(s) + \frac{n}{2} + \frac{1}{2} > s' \), from Corollary 5.1 and Proposition [B.2] the Schwartz homologies of

\[
I_{Q'}^r((|\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \text{Sym}^k \rho) \otimes^G \pi_0|_{SO(V_{0}', \langle \phi \rangle)}) \otimes \pi'
\]

vanish.

Finally, from Corollary [A.1] we can conclude that

\[
\Gamma^S_{\phi}(\mathcal{X}, |\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \pi_0) \otimes \pi'
\]

vanish under Condition (P).

From the long exact sequence of the Schwartz homologies for the short exact sequence

\[
0 \to \Gamma^S(U, \mathcal{E}) \otimes \pi' \to \Gamma^S(\mathcal{X}, \mathcal{E}) \otimes \pi' \to \Gamma^S_{\phi}(\mathcal{X}, |\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \pi_0) \otimes \pi' \to 0,
\]

the above proposition implies \( F_{1,s} \) is isomorphic under Condition (P).

5.3. Construction and bijectiveness of \( F_{2,s} \). In this subsection, we construct \( F_{2,s} \) and do Schwartz analysis on the open orbit to obtain vanishing results in Proposition [5.2] and Proposition [5.3] which shows that \( F_{2,s} \) is isomorphic under Condition (P).

**Proposition 5.2.** If \( \sigma \) is a principal series representation of \( GL_{r+1} \), there is a \( R_{r,1} \)-homomorphism

\[
T_\rho : \text{S-Ind}_{V_{r+1}}^{R_{N_{r+1}}}(\psi_{r+1} \otimes \pi_0) \to |\det(\cdot)|^{s+\frac{1}{2}} \sigma \otimes \pi_0|_{R_{r,1}}
\]

that induces an isomorphism

\[
H_0^S(G', \text{S-Ind}_{V_{r+1}}^{R_{N_{r+1}}}(\psi_{r+1} \otimes \pi_0) \otimes \pi') \cong H_0^S(G', \text{S-Ind}_{V_{r+1}}^{R_{N_{r+1}}}(\psi_{r+1} \otimes \pi_0) \otimes \pi')
\]

for general \( s \).

**Proof.** The proof for [Xue20] Proposition 5.1] over the complex field can be generalized to principal series representations over the real field word by word. Hence, for a principal series representation \( \sigma, |\det(\cdot)|^{s+\frac{1}{2}} \sigma \) has a \( R_{r,1} \)-subrepresentation isomorphic to \( \text{S-Ind}_{V_{r+1}}^{R_{N_{r+1}}}(\psi_{r+1}) \), and their quotient has a \( R_{r,1} \)-stable complete filtration whose graded pieces \( \sigma_n \) satisfy

\[
H_0^S(G', \text{Ind}_{V_{r+1}}^{R_{N_{r+1}}}(\sigma_n) \otimes \pi') = 0
\]
for general \( s \) (see [Xue20, (6.2)]) and \( k = 0, 1, \ldots \). Therefore the proposition follows from Corollary A.4 and the long exact sequence.

When \( \sigma \) is a principal series representation, the above proposition shows that \( T_p \) induces a map (5.10)

\[
\text{Hom}_G(S-\text{Ind}^G_{B_r}(|\det(\cdot)|^s \delta^{1/2}_P \sigma \otimes \pi_0) \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_G(S-\text{Ind}^G_{B_{r,1}}(S-\text{Ind}^G_{N_{r,1}} \psi_{r+1} \otimes \pi_0) \otimes \pi', \mathbb{C}),
\]

which is isomorphic for general \( s \). In particular, if \( r = 0 \), \( B_{r,1} \) is the trivial group, and thus the map in (5.10) is the identity map. Recall that \( S-\text{Ind}^G_{B_{r,1}} \psi_{r+1} = S-\text{Ind}^G_{S'} \psi_r \). Since \( \delta_P |_{S'} = 1 \), from Proposition A.4 and Corollary A.3, we have

\[
\text{Hom}_G(S-\text{Ind}^G_{B_r}(S-\text{Ind}^G_{S'}(\psi_r \otimes \pi_0 \otimes \pi'|_{S'}), \mathbb{C})
\]

\[
= \text{Hom}_G(S-\text{Ind}^G_{S'}(\pi_0 \otimes \pi'_{\tau}, \mathbb{C})
\]

\[
= \text{Hom}_{S'}(\pi_0 \otimes \pi'_{\tau}, \nu')
\]

(5.11)

Now we can define

\[
F_{2,s} : \text{Hom}_G(S-\text{Ind}^G_{B_r}(|\det(\cdot)|^s \delta^{1/2}_P \sigma \otimes \pi_0) \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{S'}(\pi_0 \otimes \pi'_{\tau}, \nu')
\]

to be the composition of (5.10) and (5.11) when \( \sigma \) is a principal series (including the case when \( r = 0 \) and \( \sigma = \rho_m \)), and \( F_{2,s} \) is an isomorphism under Condition(P).

When \( r = 1 \) and \( \sigma \) is the discrete series representation \( D_n \) of \( GL_2(\mathbb{R}) \), let \( \chi_2 \) be the character of \( GL_1(\mathbb{R}) \) defined by \( \chi_2(x) = |x|^s \delta^{1/2}_P \sigma \otimes \pi_0) \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{S'}(\pi_0 \otimes \pi'_{\tau}, \nu')
\]

\[
T_g : S-\text{Ind}^G_{R_{1,1}} \chi_2 \rightarrow |\det(\cdot)|^s \delta^{1/2}_P (\rho_{\text{det}}(\cdot)) |_{R_{1,1}}.
\]

Denote \( GL_2 = GL_2(\mathbb{R}) \), \( w_2 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \in GL_2 \), and \( B_2 \) is the (upper-triangular) Borel subgroup of \( GL_2 \) with Levi decomposition \( B_2 = T_2 N_2 \). For a character \( \chi_1 \) of \( GL_1(\mathbb{R}) \), \( \chi_1 \otimes 1 \) can be regarded as a character of \( T_2 \) and also as a character of \( B_2 \) that trivialize \( N_2 \). Let \( \mathcal{X} = B_2 \setminus GL_2 \), \( \mathcal{U} = B_2 \setminus B_2 w_2 B_2 \subset \mathcal{X} \), and \( \mathcal{Z} = B_2 \setminus B_2 \). Recall that \( R_{1,1} \) is the mirabolic subgroup of \( GL_2 \) consisting of matrices with \((0,1)\) on the last row. And we have the following lemmas.

**Lemma 5.2.** Extension by zero gives a natural embedding

\[
\Gamma^S(\mathcal{U}, \chi_1 \otimes 1) \rightarrow \Gamma^S(\mathcal{X}, \chi_1 \otimes 1),
\]

and \( \Gamma^S(\mathcal{X}, \chi_1 \otimes 1)/\Gamma^S(\mathcal{U}, \chi_1 \otimes 1) \) has a complete decreasing filtration \( \Gamma^S_Z(\mathcal{X}, \chi_1 \otimes 1)_k \) with graded pieces isomorphic to

\[
\chi_1(\det(\cdot)) \text{sgn}^k(\det(\cdot)) |_{R_{1,1}},
\]

for \( k = 0, 1, \ldots \).

**Proof.** This lemma follows from the Borel’s Lemma.

**Lemma 5.3.** We have an \( R_{1,1} \)-isomorphism

\[
\Gamma^S(\mathcal{U}, \chi_1 \otimes 1) \cong S-\text{Ind}^G_{R_{1,1}} \chi_1.
\]
\[
\Gamma^S(U, \chi_1 \otimes 1) = \Gamma^S(B_2 \backslash B_2 w_2 B_2, \chi_1 \otimes 1) = \Gamma^S(T_2 \backslash B_2, 1 \otimes \chi_1) = \Gamma^S(\mathbb{R}^\times \times 1 \backslash R_{1, 1}, 1) = S\text{-Ind}_{\mathbb{R}^\times \times 1}^{R_{1, 1}}. \]

\[\square\]

Recall the modular character \(\delta_F((m \times g_0) \times n) = |\det(m)|^{d-1-r} = |\det(m)|^{d-2} \cdot \sigma\), and it is the unique subrepresentation of

\[
\delta_F^\pi(\det(\cdot)) = S\text{-Ind}^{GL_2}_{B_2}(\cdot | s + \frac{d-r-1}{2} \cdot \text{sgn}(n+1) \otimes | \cdot |^{s+\frac{d-r-3}{2}}),
\]

so it is isomorphic to the unique quotient \(\pi_I/F_n\) of

\[
\pi_I = S\text{-Ind}^{GL_2}_{B_2}(\cdot | s + \frac{d-r-1}{2} \otimes | \cdot |^{s+\frac{d-r-3}{2}}),
\]

where \(\chi_1 = | \cdot |^{-n+1} \cdot \text{sgn}^{-n+1}\) and \(F_n\) is the unique subrepresentation of \(\pi_I\) which has dimension \(n\). Now we define a finite-dimensional representation \(F' = \chi_2^1(\det(\cdot)) \otimes F_n\). It is the unique subrepresentation of \(\pi_{I'} = S\text{-Ind}^{GL_2}_{B_2}(\chi_1 \otimes 1) = \chi_2^1(\det(\cdot))\pi_I\). It is clear that their quotient \(D' = \pi_{I'}/F' \cong \chi_2^1(\det(\cdot))\sigma'\).

Now we define a \(R_{1, 1}\)-homomorphism

\[T' : S\text{-Ind}_{\mathbb{R}^\times \times 1}^{R_{1, 1}} 1 \to D'\]

by the composition of

\[S\text{-Ind}_{\mathbb{R}^\times \times 1}^{R_{1, 1}} 1 = \Gamma^S(U, \chi_1 \otimes 1) \to \Gamma^S(U, \chi_1 \otimes 1) \to \pi_{I'} \to \pi_{I'}/F' = D'.\]

**Lemma 5.4.** The \(R_{1, 1}\)-homomorphism \(T'\) is injective.

**Proof.** Suppose \(T'\) is not injective then there exist \(\tilde{f} \in \Gamma^S(U, \chi_1 \otimes 1)\) such that the image \(\tilde{f}_G\) of \(\tilde{f}\) in \(\pi_{I'}\) is contained in \(F'\).

On the one hand, \(f(x) = \tilde{f}(w_2 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\) is a Schwartz function. For \(\theta \in (0, \pi)\), we have

\[
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1/\sin \theta & \cos \theta \\ \sin \theta & \cos \theta \end{pmatrix} w_2 \begin{pmatrix} 1 & -\cot \theta \\ 0 & 1 \end{pmatrix}.
\]

So

\[
\tilde{f} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \chi_1(1/\sin \theta)f(-\cot \theta).
\]

Therefore, for every positive integer \(l\), we have

\[
\left. \left( \frac{d}{d\theta} \right)^l \tilde{f}_G \right|_{\theta = 0} = \lim_{\theta \to 0} \frac{\chi_1(1/\sin \theta)f(-\cot \theta)}{\theta^l} = \lim_{\theta \to 0} (\cot \theta)^{-\frac{l+1}{2}}f(-\cot \theta) \cdot \frac{(\tan \theta)^l}{\theta^l} = 0.
\]

On the other hand, from [God74 Section 2.3], the space of \(F'\) is generated by

\[\varphi_{-n+1}, \varphi_{-n+3}, \ldots, \varphi_{-3}, \varphi_{-1}, \varphi_{n-1}, \varphi_{n+1} \ldots \]

\[\ldots, \varphi_{n-3}, \varphi_{n-1}, \varphi_{-1}, \varphi_{-3}, \ldots \].
where \( \varphi_k \in \text{S-Ind}_{B_2}^{GL_2}(\chi_1 \otimes 1) \), and

\[
\varphi_k \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = e^{ik\theta},
\]

which determines \( \varphi_k \) uniquely due to the Iwasawa decomposition \( GL_2(\mathbb{R}) = B_2 \cdot SO(2, \mathbb{R}) \).

So \( \tilde{f}_G \) is a linear combination of \( \varphi_k \), that is, there is a nonzero \( n \)-tuple \( (\lambda_1, \cdots, \lambda_n) \) such that

\[
\tilde{f}_G = \sum_{k=1}^n \lambda_k \varphi_k.
\]

Then we have

\[
\left( \frac{d}{d\theta} \right)^l \tilde{f}_G \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \bigg|_{\theta = 0} = \sum_{k=0}^{n-1} \lambda_k ((2k - n - 1)i)^l
\]

so there exists a positive integer \( l \) such that \( \left( \frac{d}{dt} \right)^l (\tilde{f}_G \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}) \bigg|_{\theta = 0} \) does not equal to 0, which leads to a contradiction. Therefore, the \( R_{1,1} \)-homomorphism \( \Gamma^S(U, \chi_1 \otimes 1) \to D' \) is injective.

**Lemma 5.5.** The finite-dimensional representation \( F' \) is isomorphic to \( \chi_1(\det(\cdot)) \text{Sym}^{n-1}(\mathbb{C}^2) \) as \( GL_2 \)-modules.

**Proof.** By the action of \( O(2, \mathbb{R}) \) and \( \mathbb{R} \times I_2 \) on \( \varphi_k \), we have

\[
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \varphi_k = e^{ik\theta} \varphi_k,
\]

and

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \varphi_k = \varphi_{-k},
\]

and

\[
\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \varphi_k = \chi_1(a) \varphi_k.
\]

Therefore, from (5.12) and the "unitarian trick", \( F' \) is isomorphic to \( \text{Sym}^{n-1}(\mathbb{C}^2) \) as representation of \( SL_2(\mathbb{R}) \), where \( \mathbb{C}^2 = \mathbb{C}v_1 \oplus \mathbb{C}v_2 \) is the standard representation of \( SL_2(\mathbb{R}) \). We may also write

\[
\varphi_k = (v_1 + iv_2)^{k-1} \otimes (v_1 - iv_2)^{n-1-k} \in \text{Sym}^{n-1}(\mathbb{C}^2).
\]

From (5.13), \( F' \) is isomorphic to \( \text{Sym}^{n-1}(\mathbb{C}^2) \) as a representation of \( SL_2^+(\mathbb{R}) \), where \( \mathbb{C}^2 \) is the standard representation of \( SL_2^+(\mathbb{R}) \). From (5.14), \( F' \cong \chi_1(\det(\cdot)) \text{Sym}^{n-1}(\mathbb{C}^2) \).

**Lemma 5.6.** The irreducible components of the \( R_{1,1} \)-composition series of \( F' \) are

\[
|\det(\cdot)|^k \text{sgn}^k, \text{ for } k = 0, 1, \cdots, k - 1.
\]

**Proof.** This lemma follows from the fact that the restriction of \( F' \cong \chi_1(\det(\cdot)) \text{Sym}^{n-1}(\mathbb{C}^2) \) to \( \mathbb{R}^\times \times 1 \subset M_2 \) can be decomposed as \( n \) characters, that is,

\[
| \cdot |^0 \text{sgn}^0, | \cdot |^{-1} \text{sgn}^1, \cdots, | \cdot |^{-(n-1)} \text{sgn}^{n-1}.
\]
Proposition 5.3. When $\sigma$ is a discrete series representation $D_n$ of $GL_2(\mathbb{R})$, there is a $R_{1,1}$-homomorphism $T_d : S-\text{Ind}_{R_{1,1}}^{R_{1,1}} \chi \to |\det(\cdot)|^{s + \frac{d + 20 - 3}{2}} \text{sgn}^{n+1}(\cdot)$ such that $T_d$ induces an isomorphism

$$H^*_0(G', S-\text{Ind}_{R_{1,1}}^{R_{1,1}} \chi) \otimes \pi_0 \otimes \pi' \cong H^*_0(G', S-\text{Ind}_{R_{1,1}}^{R_{1,1}} (|\det(\cdot)|^{s + \frac{d + 20 - 3}{2}} \pi)| \otimes \pi_0 \otimes \pi').$$

for $\Re(s) + \frac{n}{2} \geq s'$, where

$$\chi(\cdot) = | \cdot |^{s + \frac{d + 20 - 3}{2}} \text{sgn}^{n+1}(\cdot).$$

Proof. We first compute the structure of $D'/\Gamma^S(\mathcal{U}, \chi_1 \otimes 1)$. On the one hand, from Lemma 5.2, $\pi'_{1,1} / \Gamma^S(\mathcal{U}, \chi_1 \otimes 1) = \Gamma^S(\mathcal{X}, \chi_1 \otimes 1) / \Gamma^S(\mathcal{U}, \chi_1 \otimes 1)$ has a complete decreasing filtration $\Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1)_k$ with graded pieces isomorphic to

$$|\det(\cdot)|^k \text{sgn}^k(\chi(\det(\cdot))) \chi_1(\det(\cdot))| \otimes \sigma_1 | R_{1,1}, \text{ for } k = 0, 1, \cdots.$$ 

On the other hand, from Lemma 5.8 the finite dimensional representation $F'$ in $\pi'_I = \Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1)$ has a $R_{1,1}$-composition series with irreducible pieces

$$\sigma'_k = |\det(\cdot)|^k \text{sgn}^k(\det(\cdot)) \chi_1(\det(\cdot))| \otimes \sigma_1 | R_{1,1}, \text{ for } k = 0, 1, \cdots, n - 1,$$

So the projection to the quotient $W = \Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1) / \Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1)_n$ gives an isomorphism between $F'$ and $W$. Therefore, we have

$$\Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1) = F' \oplus \Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1)_n.$$

The quotient $D'/\Gamma^S(\mathcal{U}, \chi_1 \otimes 1) = (\pi'_{I,1} / \Gamma^S(\mathcal{U}, \chi_1 \otimes 1)) / F' \cong \Gamma^S_\mathcal{X}(\mathcal{X}, \chi_1 \otimes 1)_n$ has a complete decreasing filtration with graded pieces isomorphic to

$$|\det(\cdot)|^k \text{sgn}^k(\det(\cdot)) \chi_1(\det(\cdot))| \otimes \sigma_1 | R_{1,1}, \text{ for } k = n, n + 1, \cdots,$$

that is,

$$(\ast) |\det(\cdot)|^k \text{sgn}^k(\det(\cdot)) \chi_1(\det(\cdot))| \otimes \sigma_1 | R_{1,1}, \text{ for } k = 1, 2, \cdots, n.$$

We can define a $R_{1,1}$-homomorphism $T_d$ from

$$S-\text{Ind}_{R_{1,1} \times 1}^{R_{1,1}} \chi_2 \cong \chi_2(\det(\cdot)) S-\text{Ind}_{R_{1,1} \times 1}^{R_{1,1}} 1$$

to

$$\delta_{\text{Sp}}(|\det(\cdot)|^k \text{sgn}^k(\det(\cdot)) \chi_1(\det(\cdot))| R_{1,1}, \text{ for } k = 1, \cdots, n,$$

induced by $T'$ and its cokernel has complete decreasing filtration with graded pieces isomorphic to

$$\sigma_k = |\det(\cdot)|^{k + s + \frac{d + 20 - 3}{2}} \text{sgn}^{k}(\det(\cdot))| R_{1,1}, \text{ for } k = 1, \cdots, n.$$

Recall the totally isotropic flag $\mathcal{X}' \subset \mathcal{X}$ in Section 2.1. Let $P'_I$ be the subgroup of $G'$ stabilizing $\mathcal{X}'$ with Levi decomposition $P'_I = M'_I \times N'_I$ where $M'_I \cong \mathbb{R}^\times \times G'_0$. Since for $t \in \mathbb{C}$,

S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} |\det(\cdot)|^t \text{sgn}^{m}(\det(\cdot))| R_{1,1} \cong S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} |\cdot|^t \text{sgn}^{m}(\cdot) \otimes 1,

one has

S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} (|\det(\cdot)|^t \text{sgn}^{m}(\det(\cdot))| R_{1,1} \otimes \pi_0) \cong S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} (|\cdot|^t \otimes S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} \pi_0).

And for every irreducible component $\pi_0'$ of $S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} \pi_0$, the Harish-Chandra parameter for

$$S-\text{Ind}_{R_{1,1}}^{G'_{1,1}} (|\cdot|^t \otimes \pi_0') = |\cdot|^t \delta_{\text{Sp}}^{-\frac{d}{2}} \times \pi_0'.$$
The above proposition shows that $T_d$ induces an isomorphism

$$\text{Hom}_{G'}(\text{S-Ind}^G_{P', \chi_2} \otimes \pi_0 \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{G'}(\text{S-Ind}^G_{P, \chi_2} \otimes \pi_0 \otimes \pi', \mathbb{C})$$

when $s + \frac{n}{2} \geq s'$. Let $G = SO(V^-, q^-) \subset G'$ where $V^- = X'_- \oplus Y'_- \oplus V_0$ is the orthogonal complement of $v'_0$ in $V'$. Denote $P^- = ((\mathbb{R}^s \times 1) \times G_0) \rtimes N' \subset G^-$, which is the parabolic subgroup of $G^-$ that stabilizes $X'$. Then from Proposition [A.4] and Corollary [A.3] we have

$$\text{Hom}_{G'}(\text{S-Ind}^G_{P', \chi_2} \otimes \pi_0 \otimes \pi', \mathbb{C}) = \text{Hom}_{P^-}(\delta^{\frac{n}{2}} \otimes \chi_2 \otimes \pi_0 \otimes \pi' |_{P^-}, \mathbb{C})$$

and when $\sigma$ is a principal series representation, the descent map

$$F_s : \text{Hom}_{G'}(\pi_0 \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{G}(\pi_0 \otimes \pi', \mathbb{C})$$

and when $\sigma$ is a discrete series representation, the descent map

$$F_s : \text{Hom}_{G'}(\pi_0 \otimes \pi', \mathbb{C}) \rightarrow \text{Hom}_{G}(\pi_0 \otimes \pi', \mathbb{C})$$

From the bijectiveness of the descent map $F_s$ under Condition(P), we have the following multiplicity formulas.

**Theorem 5.1.**

1. If $\sigma$ is a principal series representation,

$$m(\pi_s, \pi') = m(\pi', \pi_0)$$

for general $s$.

2. If $\sigma$ is a discrete series representation of $GL_1(\mathbb{R})$,

$$m(\pi_s, \pi') = m(\pi', \pi_0)$$

for $\text{Re}(s) \geq s'$.

3. If $\sigma$ is the discrete series representation $D_\pi$, then

$$m(\pi_s, \pi') = m(\pi', \hat{\pi})$$

for $\text{Re}(s) + \frac{n}{2} \geq s'$, where $\hat{\pi} = \chi_{2}(n) \cdot | \frac{d-2}{4} | \pi_0 = | \cdot |^{s+\frac{n}{2}} \text{sgn}^{n+1} \times \pi_0$. 


5.5. Proofs for co-dimension one cases. In this subsection, we will make use of the multiplicity formula to prove the co-dimension one case of the local Gan-Gross-Prasad conjecture for generic parameters of special orthogonal groups over the real field by reducing it to the tempered case using mathematical induction.

Let \((\tilde{G}, \tilde{H}, \tilde{v})\) be a co-dimension one Gan-Gross-Prasad triple, where \(\tilde{G} = SO(V, q) \times SO(V', q')\). For a generic \(L\)-parameter \(\varphi \times \varphi' \) of \(\tilde{G}\), and \(\pi \times \pi' = \pi \otimes \pi' \in \Pi_{\varphi}^{\text{temp}}\). Let \(\varphi = \varphi_0 \oplus \varphi^+ \oplus (\varphi^+)^{\vee}\) and \(\varphi' = \varphi'_0 \oplus \varphi'^+ \oplus (\varphi'^+)^{\vee}\) be the decompositions in (5.2), and let \(\tau\) and \(\tau'\) be representations corresponding to the \(L\)-parameter \(\varphi^+\) and \(\varphi'^+\) respectively. From Proposition 3.4

\[
\tau = \tau \times \pi_0, \quad \tau' = \tau' \times \pi'_0
\]

where \(\pi_0, \pi'_0\) are tempered representations with \(L\)-parameter equal to \(\varphi_0\) and \(\varphi'_0\) respectively.

We write

\[
\tau = |\det(\cdot)|^{s_1}\tau_1 \times |\det(\cdot)|^{s_2}\tau_2 \times \cdots \times |\det(\cdot)|^{s_l}\tau_l
\]

\[
\tau' = |\det(\cdot)|^{s'_1}\tau'_1 \times |\det(\cdot)|^{s'_2}\tau'_2 \times \cdots \times |\det(\cdot)|^{s'_l}\tau'_l
\]

where \(Re(s_1) \geq Re(s_2) \geq \cdots \geq Re(s_l) > 0\) and \(Re(s'_1) \geq \cdots \geq Re(s'_l) > 0\). Here \(\tau_i\) are representations of \(GL_{n_i}(\mathbb{R})\), such that when \(r_i = 2\), \(\tau_i \cong D_{n_i}\) for a positive integer \(n_i\), when \(r_i = 1\), \(\tau_i \cong \text{sgn}^{n_i}\) and we define \(n_i = 0\). And we define \(r'_i, n'_i\) similarly based on \(\tau'_i\).

Proposition 5.4. With the above notations, there are two tempered representations \(\tau_{\text{temp}}\) and \(\tau'_{\text{temp}}\) of some general linear groups over \(\mathbb{R}\) such that

\(\text{(5.19)}\)

\[m(\tau \times \pi_0, \tau' \times \pi'_0) = m(\tau_{\text{temp}} \times \pi_0, \tau'_{\text{temp}} \times \pi'_0)\]

for every \(\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}\) and every \(\pi'_0 \in \Pi_{\varphi'_0}^{\text{Vogan}}\).

Proof. We prove proposition by mathematical induction on

\[
N(\tau, \tau') = \sum_{Re(s_i) \neq 0} r_i + \sum_{Re(s'_i) \neq 0} r'_i.
\]

If \(N(\tau, \tau') = 0\), we take \(\tau_{\text{temp}} = \tau\) and \(\tau'_{\text{temp}} = \tau'\) and then (5.19) holds for every \(\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}\), \(\pi'_0 \in \Pi_{\varphi'_0}^{\text{Vogan}}\).

If we change the order of the normalized induction, we may assume

\[
Re(s_1) + \frac{n_1}{2} \geq Re(s_2) + \frac{n_2}{2} \geq \cdots \geq Re(s_l) + \frac{n_l}{2} \geq 0,
\]

\[
Re(s'_1) + \frac{n'_1}{2} \geq Re(s'_2) + \frac{n'_2}{2} \geq \cdots \geq Re(s'_l) + \frac{n'_l}{2} \geq 0.
\]

Suppose the proposition is true when \(N(\tau, \tau') \leq k\), then when \(N(\tau, \tau') = k + 1\), we consider the following cases.

Case 1: If \(l \neq 0\) and \(Re(s_1) + \frac{n_1}{2} \geq Re(s'_1) + \frac{n'_1}{2}\), then let \(\tau = |\det(\cdot)|^{s_1}\tau_1 \times \cdots \times |\det(\cdot)|^{s_l}\tau_l\).

If \(\dim \tau_1 = 1\), from part (2) of Theorem 5.4, we have

\[
m(\tau \times \pi_0, \tau' \times \pi'_0) = m(\tau \times \pi_0, \tau' \times \pi'_0),
\]

for every \(\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}\) and \(\pi'_0 \in \Pi_{\varphi'_0}^{\text{Vogan}}\). Now we define \(\tau^{(s)} = |\cdot|^s \times \tau\), then from part (1) of Theorem 5.4 for general \(s\), and thus for some \(s \in i\mathbb{R}\), we have

\[
m(\tau' \times \pi'_0, \tau'' \times \pi''_0) = m(\tau^{(s)} \times \pi_0, \tau' \times \pi'_0).
\]
for every $\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$ and $\pi'_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$.

If $\dim \tau_1 = 2$, let $\hat{\tau} = \lfloor \cdot \cdot \cdot \lfloor \cdot \cdot \cdot \lfloor \cdot \cdot \cdot \lfloor \cdot \cdot \cdot \lfloor \cdot \cdot \cdot$ and from part (3) of Theorem 5.1 we have

$$m(\tau \times \pi_0, \tau' \times \pi'_0) = m(\tau' \times \pi'_0, \hat{\tau} \times \pi_0),$$

for every $\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$ and $\pi'_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$. Now we define $\tau^{(s)} = | \cdot \cdot \cdot |^{s} \times \hat{\tau}$, then from part (1) of Theorem 5.1 for general $s$, and thus for some $s \in \mathbb{R}$, we have

$$m(\tau' \times \pi'_0, \hat{\tau} \times \pi_0) = m(\tau^{(s)} \times \pi_0, \tau' \times \pi'_0),$$

for every $\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$ and $\pi'_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$.

As $N(\tau, \tau') \leq N(\tau, \tau') - 1 = k$, we have tempered representation $\tau_{\text{temp}}$ and $\tau'_{\text{temp}}$ such that

$$m(\tau^{(s)} \times \pi_0, \tau' \times \pi'_0) = m(\tau_{\text{temp}} \times \pi_0, \tau'_{\text{temp}} \times \pi'_0)$$

for every $\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$ and $\pi'_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$, which implies

$$m(\tau \times \pi_0, \tau' \times \pi'_0) = m(\tau_{\text{temp}} \times \pi_0, \tau'_{\text{temp}} \times \pi'_0)$$

for every $\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$ and $\pi'_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$.

Then the pair $(\tau^{(s'_0)}, \tau)$ belongs to Case 1 and $N(\tau^{(s'_0)}, \tau) = N(\tau, \tau') = k + 1$, so there exists $\tau'_{\text{temp}}$ and $\tau_{\text{temp}}$ such that

$$m(\tau^{(s'_0)} \times \pi'_0, \tau \times \pi_0) = m(\tau'_{\text{temp}} \times \pi'_0, \tau_{\text{temp}} \times \pi_0)$$

for every $\pi_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$ and $\pi'_0 \in \Pi_{\varphi_0}^{\text{Vogan}}$, which implies

$$m(\tau \times \pi_0, \tau' \times \pi'_0) = m(\tau'_{\text{temp}} \times \pi'_0, \tau_{\text{temp}} \times \pi_0).$$

By combining both cases, we can conclude that the proposition is true when $N(\tau, \tau') = k + 1$. Finally the proposition follows from mathematical induction. \qed

With this proposition, we are able to reduce the co-dimensional one case of local Gan-Gross-Prasad conjecture for generic $L$-parameters of special orthogonal groups over archimedean local fields to that for tempered $L$-parameters.

Let $\varphi_{\text{temp}}^+$ and $\varphi'_{\text{temp}}^+$ be the $L$-parameter of $\tau_{\text{temp}}$ and $\tau'_{\text{temp}}$ respectively. And we define tempered parameters

$$\varphi_{\text{temp}} = \varphi_0 \oplus \varphi_{\text{temp}}^+ \odot (\varphi_{\text{temp}}^+)^\vee$$

$$\varphi'_{\text{temp}} = \varphi'_0 \oplus \varphi'_{\text{temp}}^+ \odot (\varphi'_{\text{temp}}^+)^\vee.$$
Then from the tempered case proved in [Luo20], we have
\[
\sum_{\pi \otimes \pi' \in \Pi^{Vogan}_{\rho \times \rho'}} m(\pi, \pi') = \sum_{\pi_0 \otimes \pi'_0 \in \Pi^{Vogan}_{\rho_0 \times \rho'_0}} m(\tau \otimes \pi_0, \tau' \otimes \pi'_0) \\
= \sum_{\pi_0 \otimes \pi'_0 \in \Pi^{Vogan}_{\rho_0 \times \rho'_0}} m(\tau_{temp} \otimes \pi_0, \tau'_{temp} \otimes \pi'_0) \\
= \sum_{\pi_{temp} \otimes \pi'_{temp} \in \Pi^{Vogan}_{\rho_{temp} \times \rho'_{temp}}} m(\pi_{temp}, \pi'_{temp}) \\
= 1.
\]
which proves the co-dimension one case of Theorem 2.1.

Now we assume Conjecture 2.1 for tempered parameters. Then for every generic $\tilde{L}$-parameter $\tilde{\phi}$ of $\tilde{G}$, from Proposition 5.4, there exists $\tau_{temp}$ and $\tau'_{temp}$ such that
\[
m(\tau \otimes \pi_0, \tau' \otimes \pi'_0) = m(\tau_{temp} \otimes \pi_0, \tau'_{temp} \otimes \pi'_0).
\]
for every $\pi_0 \in \Pi^{Vogan}_{\rho_0}$ and every $\pi'_0 \in \Pi^{Vogan}_{\rho'_0}$.

On the one hand, the unique pair $(\pi_{temp}, \pi'_{temp}) = (\tau_{temp} \otimes \pi_0, \tau'_{temp} \otimes \pi'_0) \in \Pi^{Vogan}_{\rho_{temp} \times \rho'_{temp}}$ such that
\[
m(\pi_{temp}, \pi'_{temp}) = 1
\]
satisfy
\[
\eta_{\pi_{temp}} = \eta_{\tau_{temp}}, \eta_{\pi'_{temp}} = \eta_{\tau'_{temp}}.
\]
On the other hand, from [GGP12, Proposition 5.1], we have
\[
\eta_{\rho \times \rho} = \eta_{\rho_{temp}}, \eta_{\rho' \times \rho'} = \eta_{\rho'_{temp}},
\]
and for $\pi = \tau \otimes \pi_0$ and $\pi' = \tau' \otimes \pi'_0$, we have
\[
\eta_{\pi} = \eta_{\pi_0} = \eta_{\pi_{temp}}, \eta_{\pi'} = \eta_{\pi'_0} = \eta_{\pi'_{temp}}.
\]
Hence
\[
m(\pi, \pi') = 1
\]
if and only if
\[
\eta_{\pi} = \eta_{\rho}, \eta_{\pi'} = \eta_{\rho'}.
\]
Therefore, the co-dimension one case of Theorem 2.2 is proved.

5.6. Reduction to co-dimension one cases. Finally, we will prove the reduction of the general case to the co-dimension one case.

Let $(\tilde{G}', \tilde{H}', \tilde{\nu}')$ be a Gan-Gross-Prasad triple and use the notations in the introductory part of this section. Let $\sigma$ be a principal series of $GL_{r+1}$, then for every irreducible Casselman-Wallach representation $\pi'$ in generic $L$-packets of $G'$ and irreducible Casselman-Wallach representation $\pi_0$ in the generic $L$-packets of $G_0$, from [SV80, Theorem 1.1] and Langlands classification, for general $s$, $\pi_s = |\det(\cdot)|^s \sigma \otimes \pi_0$ is irreducible.

From Theorem 5.1 for general $s \in \mathbb{C}$, we have
\[
m(\pi_s, \pi'_s) = m(\pi', \pi_0).
\]
We choose $s$ such that $\pi_s$ is irreducible and the above equation holds for all $\pi' \in \Pi_{\nu_0}^{Vogan}$ and $\pi_0 \in \Pi_{\nu_0}^{Vogan}$. Then with the same arguments as the last subsection, the local Gan-Gross-Prasad conjecture for $(\tilde{G}', \tilde{H}', \tilde{\nu}')$ can be reduced to the local Gan-Gross-Prasad conjecture for $(\tilde{G}, \tilde{H}, \tilde{\nu})$ which is a co-dimension one Gan-Gross-Prasad triple. Therefore, Theorem 2.1 and Theorem 2.2 are proved over the real field.

**Appendix A. Schwartz inductions and Schwartz homologies**

In this appendix, we introduce Schwartz inductions and Schwartz homologies, our main technical tools. We work in the setting of almost linear Nash groups $G$ (see [Sun15]) and in the category $\mathcal{R}(G)$ of smooth Fréchet representations of moderate growth. A representation $\pi$ in $\mathcal{R}(G)$ is called Casselman-Wallach if the $(g_C, K)$-module $\pi_K$ is Harish-Chandra (see Section 2.2).

### A.1. Schwartz inductions

For an almost linear Nash group $G$ and a Casselman-Wallach representation $(\pi, V)$, the Casselman-Wallach globalization theorem implies that $V$ is a nuclear Fréchet space (see [BK14]), and the completed tensor product with a nuclear Fréchet space is an exact functor in the category of nuclear Fréchet spaces ([CHM00, Lemma A.3]). Then we have

**Proposition A.1.** The completed tensor product with a Casselman-Wallach representation is an exact functor in $\mathcal{R}(G)$.

In order to define the Schwartz induction in the sense of [DC91, Section 2], we let $H$ be a Nash subgroup of $G$, $(\sigma, V) \in \mathcal{R}(H)$ and $I$ be the continuous map

\[
(A.20) \quad I : S(G, V) \to C^\infty(G, V), \quad f \mapsto \int_H \sigma(h)f(h^{-1}g)dh.
\]

The Schwartz induction $S\text{-}\text{Ind}_G^H V$ is defined to be the image of this map. Chen and Sun used in [CS20] another way to give the Schwartz induction with the Schwartz sections of the tempered bundle $H\backslash G \times V$, where the right action of $H$ is given by $h \cdot (g, v) = (h^{-1}g, h^{-1}v)$, and from [CS20, Propositions 6.7, 6.11], we have

**Proposition A.2.** There is an isomorphism

\[
S\text{-}\text{Ind}_G^H V \cong \Gamma^S(H\backslash G, V) \in \mathcal{R}(G),
\]

where $\Gamma^S(H\backslash G, V)$ stands for the space of Schwartz sections of the tempered bundle $H\backslash G \times V$.

From [DC91, Proposition 2.2.7], the Schwartz induction $S\text{-}\text{Ind}_G^H$ is an exact functor from $\mathcal{R}(H)$ to $\mathcal{R}(G)$, and there is a straightforward transitivity of Schwartz inductions proved in [DC91, Lemma 2.1.6], that is,

**Proposition A.3** (Transitivity). For $H_1 \subset H_2 \subset H_3$, and $V \in \mathcal{R}(H_1)$, we have

\[
S\text{-}\text{Ind}_{H_2}^{H_3} \left( S\text{-}\text{Ind}_{H_1}^{H_2} V \right) = S\text{-}\text{Ind}_{H_1}^{H_3} V.
\]

And Chen and Sun proved in [CS20, Proposition 7.4] that the completed tensor product commutes with the Schwartz induction:

**Proposition A.4.** Let $W \in \mathcal{R}(H)$ and $V \in \mathcal{R}(G)$. Assume that one of $W$ and $V$ is nuclear. Then

\[
S\text{-}\text{Ind}_G^H (W \hat{\otimes} V|_H) = (S\text{-}\text{Ind}_H^G W) \hat{\otimes} V.
\]
A.2. Schwartz homologies. The Schwartz homology $H^S_i(G, V)$ is defined to be the left derived functors of the coinvariant functor $V \to V_G$, where $V_G = V / \sum_{g \in G} \langle g - 1 \rangle V$. In particular, $H^S_0(G, V) = V_G$. Like other kinds of homologies, the Schwartz homologies also produces a long exact sequence from a short exact sequence.

**Proposition A.5** (Long exact sequence). Given a short exact sequence of representations in $\mathcal{R}(G)$

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

Then there is a long exact sequence

$$\cdots \to H^S_{i+1}(G, V_3) \to H^S_i(G, V_1) \to H^S_i(G, V_2) \to H^S_i(G, V_3) \to \cdots$$

Because for every $f \in \text{Hom}_G(V, \mathbb{C})$, we have $f(\sum_{g \in G} (g - 1) V) = \sum_{g \in G} f((g - 1) V) = 0$. Then we have

(A.21) $\text{Hom}_G(H^S_0(G, V), \mathbb{C}) = \text{Hom}_G(V, \mathbb{C})$.

Suppose that we have $V, W \in \mathcal{R}(G)$ and a $G$-homomorphism $f : V \to W$ that induces an isomorphism of

$$H^S_0(G, f) : H^S_0(G, V) \to H^S_0(G, W).$$

Then the induced map

$$D_f : \text{Hom}_G(W, \mathbb{C}) \to \text{Hom}_G(V, \mathbb{C})$$

is an isomorphism as well.

For instance, if $W$ is equal to the space of left equivariant Schwartz functions on a Nash $G$-manifold $\mathcal{X}$ and $V$ is equal to the space of left equivariant Schwartz functions on a open $G$-submanifold $\mathcal{U}$ of $\mathcal{X}$ and $f$ is the extension by zero, then the map $\text{Hom}_G(W, \mathbb{C})$ to $\text{Hom}_G(V, \mathbb{C})$ induced from $f$ is the restriction map from the space of left equivariant tempered distributions on $\mathcal{X}$ to the space of left equivariant tempered distributions on $\mathcal{U}$, and the restriction map is an isomorphism of $G$-modules if we assume $H^S_0(G, f)$ is an isomorphism. Because $f$ is injective, the long exact sequence gives a sufficient condition for $H^S_0(G, f)$ to be bijective, that is, $H^S_i(G, W/V) = 0$ for $i = 0, 1$.

A.3. Borel’s lemma. The Borel’s Lemma depicts the structure of $W/V$ in a more general context. Let $\mathcal{X}$ be a Nash manifold, $Z$ be a closed Nash manifold of $\mathcal{X}$, and $\mathcal{U}$ be the open complement of $Z$ in $\mathcal{X}$. For a tempered bundle $\mathcal{E}$ over $\mathcal{X}$, the extension by zero gives a natural injection $\Gamma^S(\mathcal{U}, \mathcal{E}) \to \Gamma^S(\mathcal{X}, \mathcal{E})$. As defined [CS20 Section 6.1], the complexification of the conormal bundle $N^\vee_{Z/\mathcal{X}}$ is a tempered bundle over $Z$. From [CS20 Propositions 8.2, 8.3], we have a proposition that gives a complete decreasing filtration of $\Gamma^S_Z(\mathcal{X}, \mathcal{E}) = \Gamma^S(\mathcal{X}, \mathcal{E}) / \Gamma^S(\mathcal{U}, \mathcal{E})$:

**Proposition A.6** (Borel’s lemma). There is a complete decreasing filtration on $\Gamma^S_Z(\mathcal{X}, \mathcal{E})$, denoted by $\Gamma^S_Z(\mathcal{X}, \mathcal{E})_k$, satisfying

$$\Gamma^S_Z(\mathcal{X}, \mathcal{E}) = \lim_{\leftarrow} \Gamma^S_Z(\mathcal{X}, \mathcal{E})_k$$

and the graded pieces are isomorphic to

(A.22) $\Gamma^S_Z \left( Z, \text{Sym}^k N^\vee_{Z/\mathcal{X}} \otimes \mathcal{E} \bigg|_Z \right), \quad k = 0, 1, 2, \cdots$

Moreover, if $\mathcal{X}$ is a $G$-Nash manifold, $Z$ is stable under the action of $G$ and $\mathcal{E}$ is a tempered $G$-bundle, then this filtration is stable under $G$.
In order to apply the Borel’s lemma to prove that the Schwartz homology $H^S_i(G, V_n) = 0$, we need the following property about the relation between Schwartz homologies and the inverse limits, which follows from the exactness of the inverse limits in inverse systems with Mittag-Leffler conditions, and is elaborated in [Xue20 Proposition 2.13].

**Proposition A.7.** For representation $V \in \mathcal{R}(G)$, assume that $V$ admits a complete decreasing filtration $V_n$, for $n = 0, 1, \cdots$, and that for some $i$, the Schwartz homologies $H^S_i (G,V/V_n)$ are finite-dimensional for all $n$. Then the canonical map

$$H^S_i (G, V) \to \varprojlim_{n} H^S_i (G,V/V_n)$$

is an isomorphism.

Suppose that a representation $V$ has a complete decreasing filtration of subspaces $V_n$ such that $H^S_i(G,V_n/V_{n+1}) = 0$ for $i = 0, 1, \cdots$. Then from the long exact sequence and mathematical induction, we have $H^S_i(G,V_n/V_{n+1}) = 0$ for $i = 0, 1, \cdots$, and hence

(A.23) $H^S_i (G, V) = \varprojlim_{n} H^S_i (G,V/V_n) = 0$.

We call the Schwartz homologies of a representation $V$ of $G$ vanish if and only if $H_i(G,V) = 0$, for $i = 0, 1, \cdots$. The arguments above can be summarized as

**Corollary A.1.** Suppose that $V \in \mathcal{R}(G)$ has a complete decreasing filtration such that the Schwartz homologies of the graded pieces vanish. Then the Schwartz homologies of $V$ vanish.

**A.4. Vanishing theorem.** For $V, V' \in \mathcal{R}(G)$, if there is a non-degenerate $G$-invariant bilinear map $B : V \times V' \to \mathbb{C}$, we call $V$ and $V'$ are contragredient to each other and $(V, V')$ is a contragredient pair of $G$. Suppose $V$ has an infinitesimal character $\chi_V$, that is, a character of $\mathcal{Z}(g_C)$ such that $z.v = \chi(z)v$ for every $z \in \mathcal{Z}(g_C)$ and $v \in V$, then

**Proposition A.8.** The representation $V'$ has an infinitesimal character $\chi_{V'}$ which is defined by

(A.24) $\chi_{V'}(z) = \chi_V(\overline{z})$

for every $z \in \mathcal{Z}(g_C)$, where $z \mapsto \overline{z}$ is the automorphism of the universal enveloping algebra $U(g_C)$ such that $\overline{X} = -X$ for $X \in g_C$.

**Proof.** For every $v \in V$, $v' \in V'$ and $z \in \mathcal{Z}(g_C)$, we have

$$B(v, zv') = B(\overline{z}v, v') = \chi_V(\overline{z})B(v, v') = B(v, \chi_{V'}(z)v').$$

And as the bilinear form is non-degenerate, we have

$$z.v' = \chi_{V'}(z)v'.$$

□

We are going to give a sufficient condition about when the Schwartz homologies of $\pi \hat{\otimes} \sigma$ vanish, where $\pi, \sigma \in \mathcal{R}(G)$. The following theorem is the counterpart of the vanishing theorem of continuous cohomologies, which can be proved by the bridge between Schwartz homologies and continuous cohomologies built by the $(g_C,K)$-homologies and cohomologies and Poincaré duality when $G$ is connected. And it is also proved directly in [Xue20 Proposition 2.7].

**Corollary A.2** (Vanishing theorem). Let $(\pi, V)$ and $(\sigma, W)$ be representations in $\mathcal{R}(G)$, suppose that there are two different complex number $c_1 \neq c_2$ and an element $z \in \mathcal{Z}(g_C)$ such that $z.v = c_1 v$ and $\overline{z}.w = c_2 w$, for every $v \in V$, $w \in W$, then the Schwartz homologies of $V \hat{\otimes} W$ vanish.
A.5. **Shapiro’s lemma.** Chen and Sun proved in [CS20, Proposition 7.5] the following lemma which shows a profound relation between Schwartz inductions and Schwartz homologies.

**Proposition A.9** (Shapiro’s lemma). Let $H$ be a closed Nash subgroup of $G$ and $V \in \mathcal{R}(H)$. Then

$$H^i_s(G, (S-\text{Ind}_H^G(V \otimes \delta_H)) \otimes \delta_G^{-1}) = H^i_s(H, V)$$

for all $i \geq 0$.

In particular, when $i = 0$, it gives an isomorphism between coinvariants (constructed in [CS20, Propositions 6.9, 6.12])

(A.25) \[T_H : (\text{Ind}_H^G(V \otimes \delta_H) \otimes \delta_G^{-1})_G \cong V_H,\]

which induces an isomorphism

(A.26) \[\text{Hom}_G(\text{S-Ind}_H^G(V \otimes \delta_H \otimes \delta_G^{-1}|_H), \mathbb{C}) = \text{Hom}_H(V, \mathbb{C}).\]

By combining these arguments with Proposition A.4 we have

**Corollary A.3.** Let $H$ be a Nash subgroup of $G$ and $V \in \mathcal{H}$. For a Casselman-Wallach representation $W$ of $G$, the map $T_H$ induces an isomorphism

$$\text{Hom}_G(\text{S-Ind}_H^G(V \otimes \delta_H \otimes \delta_G^{-1}|_H) \otimes W, \mathbb{C}) = \text{Hom}_H(V \otimes W, \mathbb{C}).$$

**Appendix B. Harish-Chandra parameters**

We combine the Harish-Chandra’s parameterization of infinitesimal characters and the vanishing theorem in Corollary [A.2] to obtain a sufficient condition for vanishing of Schwartz homologies of a representation. Because the results in this subsection are classical results, we work in the language of real reductive groups and admissible representations.

**B.1. Definitions.** Harish-Chandra gave a full classification of discrete series representations for a connected semisimple group $G$ with a maximal compact subgroup $K$. The group $G$ has a discrete series representation if and only if $G$ and $K$ have the same rank, equivalently, the maximal torus $T$ of $K$ is a compact Cartan subgroup of $G$. In [HC65] and [HC66], Harish-Chandra classified discrete series representations with Harish-Chandra parameters $v_\pi$ in $L + \rho$ such that $v_\pi$ is not orthogonal to any roots of $G$, where $L$ is the weight lattice of $G$ contained in the complex dual $\mathfrak{t}_C^*$ of the Lie algebra of $T$ and $\rho$ is the Weyl vector, that is, the half sum of all positive roots in the root system of $G$. For $v_\pi \in L + \rho$ that is orthogonal to a root of $G$, it also corresponds to a finite set of irreducible representations, and these representations are called limits of discrete series representations of $G$. For simplicity, in this article, a limit of discrete series representations means a discrete series representation or a limit of discrete series representations. For limits of discrete series representations $\pi_1$ and $\pi_2$ of $G$, $\pi_1$ and $\pi_2$ have the same infinitesimal character if and only if $v_{\pi_1}$ is conjugate to $v_{\pi_2}$ by $W_G$, the Weyl group of $G$. Moreover, the infinitesimal character $\chi_{\pi_1}$ of $\pi_1$ and the character $\chi_{v_\pi}$ of the polynomial algebra $P(t_C^*)^{W_G}$ defined by the evaluation at $v_{\pi_1}$ are related to each other through the equation

$$\chi_{\pi_1}(z) = \chi_{v_\pi}(HC(z))$$

for every $z \in Z(\mathfrak{g}_C)$, where $HC$ is the Harish-Chandra isomorphism

$$HC : U(\mathfrak{g}_C)^G = Z(\mathfrak{g}_C) \rightarrow S(t)^{W_G} = P(t^*)^{W_G}.$$ And the correspondence between $v_{\pi_1}$ and $\chi_{\pi_1}$ is called the Harish-Chandra correspondence.
For $G = GL_n(\mathbb{R})$, elements in the dual $\mathfrak{a}_c^*$ of Cartan subalgebra of $\mathfrak{g}_C$ can be parametrized by

$$(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{C}^n$$

and the Weyl group $W_G$ is isomorphic to the permutation group $S_n$ that permutes the $n$-entries. By the Langlands classification, each irreducible admissible representation $\pi$ is a subquotient of some

$$| \cdot |^{s_1 \operatorname{sgn} m_1} \times \cdots \times | \cdot |^{s_n \operatorname{sgn} m_n}$$

where $s_i \in \mathbb{C}$, $m_i = 0, 1$ and $\operatorname{sgn}$ is the sign representation of $GL_1(\mathbb{R})$ such that $\operatorname{sgn}(x) = 1$ when $x > 0$ and $\operatorname{sgn}(x) = -1$ when $x < 0$. Then define the Harish-Chandra conjugacy class $[v_n]$ of $\pi$ as the $W_G$-conjugacy class of

$$(s_1, s_2, \cdots, s_n) \in \mathfrak{a}_c^*.$$ 

The Harish-Chandra correspondence gives a one to one map from the $W_G$-conjugacy classes of the Harish-Chandra parameters to the set of infinitesimal characters of irreducible admissible representations of $G$, which shows $[v_n]$ is well defined. And an element $v_\pi \in [v_n]$ is called a Harish-Chandra parameter.

We now come back to the case when $G = SO(V, q)$, where $(V, q)$ is a non-degenerate quadratic space over $\mathbb{R}$ with a decomposition $V = X_r \oplus Y_r \oplus V_{an}$ such that $X_r, Y_r$ are maximal totally isotropic subspaces of $V$ and $V_{an}$ is anisotropic. Let $A_0$ be a maximal torus of $G\mathcal{L}(r)$ and $T$ be a maximal torus of $SO(V_{an})$. When $\dim V = 2n + 1$, the weight space $\mathfrak{h}_c^* = \mathfrak{a}_0^* + \mathfrak{t}_c^*$ are parameterized as

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{C}^n,$$

where the action of $W_G$ is generated by switching two entries and changing signs. When $\dim V = 2n$ the weight space is parameterized as

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{C}^n,$$

and the action of $W_G$ is generated by switching two entries and changing signs for two entries. From Langlands classification, each irreducible admissible representation $\pi$ is isomorphic to a subquotient of

$$(B.27) \quad | \cdot |^{s_1 \operatorname{sgn} m_1} \times \cdots \times | \cdot |^{s_n \operatorname{sgn} m_n} \times \pi_{an}$$

where $s_i \in \mathbb{C}$, $m_i = 0, 1$ and $\pi_{an}$ is a finite-dimensional representation of $SO(V_{an})$. In this case, the Harish-Chandra conjugacy class $[v_\pi]$ is defined as the $W_G$-conjugacy class of $(v_\tau, v_{\pi_{an}})$, where $v_\tau = (s_1, s_2, \cdots, s_l)$ is a Harish-Chandra parameter of $\tau = | \cdot |^{s_1 \operatorname{sgn} m_1} \times \cdots \times | \cdot |^{s_l \operatorname{sgn} m_l}$ and $v_{\pi_{an}}$ is the sum of the highest weight of $\pi_{an}$ with the Weyl vector of $SO(V_{an})$. Then the Harish-Chandra correspondence gives that for admissible representations $\pi$ and $\pi'$, they have the same infinitesimal characters if and only if they have the same Harish-Chandra conjugacy class. And an element $v_\pi$ in the conjugacy class $[v_n]$ is called a Harish-Chandra parameter.

### B.2. Relations with the vanishing theorem.

For an irreducible admissible representation $\pi$ of $SO(V, q)$ such that $\sigma$ is a subquotient of

$$I = | \cdot |^{s_1 \operatorname{sgn} m_1} \times \cdots \times | \cdot |^{s_n \operatorname{sgn} m_n} \times \pi_{an}$$

as in (B.27). We take

$$I^\vee = | \cdot |^{-s_1 \operatorname{sgn} m_1} \times \cdots \times | \cdot |^{-s_n \operatorname{sgn} m_n} \times \pi_{an}^\vee.$$ 

And there is a non-degenerate $P$-invariant form of the induced data of $I$ and $I^\vee$, which gives a non-degenerate $G$-invariant bilinear form between $I$ and $I^\vee$ by integrating over the maximal compact
subgroup $K$ of $G$. Therefore there is a unique irreducible subquotient $σ^\vee$ of $I^\vee$ such that $B$ induces a non-degenerate bilinear form of $σ$ and $σ^\vee$. Then Proposition A.8 implies that

Proposition B.1. The Harish-Chandra conjugacy class $[v_{π^\vee}]$ of the contragredient representation $π^\vee$ of $π$ is the same as $[-v_{π}]$, where $v_{π}$ is a Harish-Chandra parameter of $π$.

From the above discussions, $χ_{π}$ does not equal to $χ_{π^\vee}$ if and only if $[v_{π}] \neq [-v_{π}]$. When $π$ and $π'$ are Casselman-Wallach, from Corollary A.2 we have

Corollary B.1. If $[v_{π}] \neq [-v_{π}]$, then the Schwartz homologies of $π\otimes π'$ vanish.

B.3. Tensor products with finite-dimensional representations. Finally, we review the Kostant’s results about the composition series of the tensor product of a finite-dimensional representation with an irreducible Casselman-Wallach representation. It follows from [Kos75, Section 1.3] that

Proposition B.2. The tensor product of an irreducible finite-dimensional representation with an irreducible Casselman-Wallach representation has a finite composition series.

And [Kos75, Corollary 5.6] shows that

Proposition B.3. For an irreducible admissible representation $π$ and an irreducible finite-dimensional representation $π'$, the irreducible components of the tensor product of $π'$ and $π$ consists of representations with Harish-Chandra conjugacy classes equal to $[v_{π} + μ]$, where $μ$ is a weight of $π'$ and $v_{π}$ is a Harish-Chandra parameter of $π$.

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