Constructing regular self-similar solutions to the 3D Navier-Stokes equations originating at singular and arbitrary large initial data

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Abstract

Global-in-time smooth self-similar solutions to the 3D Navier-Stokes equations are constructed emanating from homogeneous of degree $-1$ \textit{arbitrary large} initial data belonging only to the closure of the test functions in $L^2_{loc,unif}$.

\textit{Keywords:} Navier-Stokes equations, regularity, self-similar solutions
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1 Introduction

A very simplistic description of the existing global-in-time results for the 3D Navier-Stokes equations is as follows: weak solutions are constructed from arbitrary large (mostly) $L^2$-type initial data, while mild and strong solutions are generated by small initial data in a variety of functional spaces. It is not known if the weak solutions are regular and there are only partial regularity results – Caffarelli, Kohn and Nirenberg [CKN] proved that in a class of weak solutions satisfying localized energy inequality, the so called suitable weak solutions, for every time $T > 0$ the one-dimensional Hausdorff measure of the (possible) singular set in $\Omega \times (0, T)$ is 0 ($\Omega$ denotes the spatial domain). The standard constructions of weak solutions including the original Leray’s construction [L] and the construction of the suitable weak solutions [CKN] start from the initial data belonging to $L^2$ on the whole spatial domain. Recently, Lemarié-Rieusset [L-R1] (henceforth, the references will be to the book [L-R2]) constructed localized weak solutions started from initial data in the closure of the test functions in $L^2_{loc,unif}$.

Scaling-invariant solutions of any physical model are of special interest – heuristically (and sometimes rigorously), they capture the qualitative behavior of the model. It turns out that there is a unique scaling invariance for the Navier-Stokes system: if $u$ is a solution on $\mathbb{R}^3 \times (0, \infty)$ with initial data $u_0$ then $v^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ is a solution corresponding to the initial data $v_0^\lambda(x) = \lambda u_0(\lambda x)$ for any $\lambda > 0$. If the original initial data $u_0$ are homogeneous of degree $-1$, i.e. if $u_0(\lambda x) = \lambda^{-1}u_0(x)$ for all $x$ in $\mathbb{R}^3$ and all $\lambda > 0$, and the problem is considered in a uniqueness class for the Navier-Stokes equations, it follows readily that $u(\lambda x, \lambda^2 t) = \lambda^{-1}u(x, t)$ for all $x$ in $\mathbb{R}^3$, all $t > 0$ and all $\lambda > 0$, i.e. $u$ is scaling-invariant or self-similar. It is worth noting that initial data generating (nontrivial) self-similar solutions are necessarily singular.
– the homogeneity of degree $-1$ implies the existence of a singularity at the origin of the order of $\frac{1}{|x|}$. This prevents obtaining self-similar solutions in the ‘usual’ spaces, e.g., the Lebesgue spaces $L^p$ for $p \geq 3$ or the Sobolev spaces $H^s$ for $s \geq 1/2$. In fact, the $\frac{1}{|x|}$-type singularity is exactly a borderline singularity for the borderline spaces $L^3$ and $H^{1/2}$. The first rigorous construction of self-similar solutions is due to Giga and Miyakawa [GM]. They worked in the Morrey-type spaces of measures and obtained self-similar solutions for the vorticity equation starting from small initial data. More recently, Cannone [C] developed a systematic approach to constructing self-similar solutions in a variety of spaces (including the homogeneous Besov spaces) originating at small initial data. (See also [CK, CMP, CP, P1, P2].)

It is well-known there are no nontrivial self-similar solutions satisfying the Leray’s energy inequality (see, e.g., [GM]). Consequently, the standard constructions of weak solutions are not suitable for generating self-similar solutions. However, the weak solutions constructed in [LR2] satisfy only the localized energy inequality and hence are not a priori incompatible with (nontrivial) self-similarity.

In this paper we obtain existence of regular self-similar solutions emanating from arbitrary large initial data (homogeneous of degree $-1$) belonging only to $L^2_{loc,unif}$. To the best of the author’s knowledge, this is the first time global-in-time existence of smooth solutions to the 3D Navier-Stokes equations is established without any smallness condition on the initial data.

The proof starts off by showing that a modified local-in-time part of the construction of weak solutions in [LR2] yields “partially self-similar” solutions on $(0,T)$. Partial self-similarity is then used to infer spatial regularity on $(0,T)$ via Caffarelli-Kohn-Nirenberg estimate on the size of a singular set and local in space-time Serrin’s $L^p_t L^q_x$-criterion. Spatial continuity in turn leads to full self-similarity on $(0,T)$. This
interplay between self-similarity and regularity is exploited further until fully self-similar, $C_x^\infty C_t^\infty$-solutions on $\mathbb{R}^3 \times (0, \infty)$ are constructed.

2 Preliminaries

We consider the 3D Navier-Stokes equations (NSE) with the unit viscosity and the zero external force,

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \quad (1)$$
$$\nabla \cdot u = 0$$

where $u$ denotes the velocity of the fluid and $p$ the pressure. The spatial domain will be $\mathbb{R}^3$.

In what follows, a projected form of the equation will be useful:

$$u_t - \Delta u + P \nabla \cdot (u \otimes u) = 0 \quad (2)$$
$$\nabla \cdot u = 0$$

($P$ denotes the Leray projector). Let us mention here that if a distributional solution $u$ of (2) on $\mathbb{R}^3 \times (0, T)$ is uniformly locally square-integrable, then there exists $p$ in $\mathcal{D}'(\mathbb{R}^3 \times (0, T))$ such that

$$P \nabla \cdot (u \otimes u) = \nabla \cdot (u \otimes u) + \nabla p$$

in the sense of distributions, i.e. $(u, p)$ is a distributional solution of (1).

Lemarié-Rieusset defined a localized version of weak solutions, the so called local Leray solutions. The definition is as follows \cite{L-R2}.
Let $u_0 \in L^2_{\text{loc,unif}}$ with $\nabla \cdot u_0 = 0$. A local Leray solution $u$ corresponding to the initial data $u_0$ is a locally square-integrable distributional solution of (1) on $\mathbb{R}^3 \times (0, T)$ with the following properties:

i) $u \in \cap_{t<T} L^\infty \bigl((0, t), L^2_{\text{unif,loc}}\bigr)$

ii) $\sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{|x-x_0| \leq 1} |\nabla \otimes u|^2 \, dx \, ds < \infty$ for all $t < T$

iii) $\lim_{t \to 0^+} \int_K |u - u_0|^2 \, dx = 0$ for any compact subset $K$ of $\mathbb{R}^3$

iv) $u$ is a suitable weak solution in the sense of Caffarelli, Kohn and Nirenberg – in particular, $u$ satisfies the localized energy inequality.

The following result can be found in [L-R2] – the idea of the proof is to use the localized energy inequality as the main tool in constructing the solutions originating at the $L^2_{\text{loc,unif}}$-initial data. Let $E$ denote the closure of the test functions in the norm of $L^2_{\text{loc,unif}}$.

**Theorem 1.** [L-R2] i) Let $u_0 \in L^2_{\text{loc,unif}}$ be divergence free. Then there exists a local Leray solution $u$ on $\mathbb{R}^3 \times (0, T)$ for some $T > 0$.

ii) If in addition $u_0 \in E$, then $u \in \cap_{t<T} L^\infty \bigl((0, t), E\bigr)$ and $\lim_{t \to 0^+} \|u-u_0\|_{L^2_{\text{loc,unif}}} = 0$.

The construction is based on deriving some (uniform in $\epsilon$) local energy-type estimates for solutions of the family of mollified Navier-Stokes equations

\[
(u_\epsilon)_t - \Delta u_\epsilon + P \nabla \cdot ((u_\epsilon * \rho_\epsilon) \otimes u_\epsilon) = 0
\]

\[
\nabla \cdot u_\epsilon = 0
\]

supplemented by $u_\epsilon(0, \cdot) = u_0$ for all $\epsilon > 0$. A family of (standard) mollifiers $\{\rho_\epsilon\}_{\epsilon>0}$ is defined as follows. Let $\rho$ be a non-negative test function normalized such that $\int \rho \, dx = 1$. Then $\rho_\epsilon$ is given by $\rho_\epsilon(x) = \frac{1}{\epsilon^3} \rho \left( \frac{x}{\epsilon} \right)$ for all $x$ in $\mathbb{R}^3$. We will come back to this family of smooth approximating solutions in the following section.
Remark 2. It was shown in [L-R2] that local-in-time local Leray solutions can be extended to global-in-time local Leray solutions; however, we will not need that result here.

Next, we recall a definition of (space-time) singular/regular points of a weak solution $u$ in the sense of [CKN]. A point $(x, t)$ in $\mathbb{R}^3 \times (0, T)$ is singular if $\|u\|_{L^\infty(D)} = \infty$ for any space-time neighborhood $D$ of $(x, t)$. A point is regular if it is not singular, i.e. if there exists a space-time neighborhood $D$ such that $\|u\|_{L^\infty(D)} < \infty$.

The main partial regularity result is given by the following theorem.

Theorem 3. [CKN] Let $u$ be a suitable weak solution on $\mathbb{R}^3 \times (0, T)$. Then the one-dimensional Hausdorff measure of the possible singular set in $\mathbb{R}^3 \times (0, T)$ is 0. In particular, the singular set can not contain a smooth space-time curve.

Remark 4. The original statement is somewhat stronger – it states that the one-dimensional parabolic Hausdorff measure of the singular set is 0.

At first, it may be puzzling that local boundness is in this context identified as “regularity”. However, the following result due to Serrin [S] implies smoothness in $x$ at any regular point $(x, t)$.

Theorem 5. [S] Let $u$ be a weak solution in some open space-time region $D$ such that $u \in L^p_x L^q_t(D)$ for a pair of exponents $(p, q)$ satisfying the Foias-Prodi-Serrin condition $\frac{2}{p} + \frac{2}{q} < 1$. Then $u$ is of class $C^\infty$ in $x$, and each derivative is bounded on compact subregions of $D$. 
3 The construction

Since our construction of global-in-time regular self-similar solutions will originate at the construction of local-in-time local Leray solutions, the first thing to check is whether the requirements on the initial data in Theorem 1 are compatible with the desired self-similarity of solutions. As already pointed out in the introduction, any nontrivial self-similar solution must emanate from nontrivial initial data homogeneous of degree $-1$ and hence possessing a $\frac{1}{|x|}$-type singularity at the origin. This is not in conflict with being in $L^2_{loc,unif}$, and decay at infinity, i.e. a possibility of an approximation with the test functions, is also compatible with the homogeneity of degree $-1$. A simple example of a family of homogeneous of degree $-1$ initial data satisfying the conditions of Theorem 1 is the following:

$$u^\alpha_0(x) = \alpha \left( \frac{x_2 - x_3}{|x|^2}, \frac{x_3 - x_1}{|x|^2}, \frac{x_1 - x_2}{|x|^2} \right)$$

for $\alpha$ in $\mathbb{R}$. Notice that this family generates arbitrary large initial data in $L^2_{loc,unif}$.

The second imminent problem in constructing any type of invariant weak solutions is that the weak solutions are not known to be unique, so although the initial data may be compatible with the invariance, it is a priori not clear if it is possible to construct any invariant solutions at all. One way out is to show that a particular construction of weak solutions will preserve the invariance – this was shown by Brandolese in the case of a special rotational invariance in the $\text{[CKN]}$-construction of suitable weak solutions $\text{[B1]}$ (as pointed out in $\text{[B2]}$, the same argument works for any rotational invariance, and also for most of the standard constructions of weak solutions). The trick is to choose a test function in the definition of the smooth approximating solutions to be compatible with the invariance – in this case any radial test function will do. The consequence is the invariance of the approximating solutions which is then preserved.
in the limit. The same trick does not seem to work for the self-similarity. It turns out that in order to construct self-similar approximating solutions, the test function would have to satisfy a certain degree of homogeneity which contradicts the very nature of a test function, i.e. having a compact support. What will happen here is that although the approximating solutions will not be self-similar, just enough self-similarity will be recovered in the limit to push the construction through.

**Theorem 6.** Let \( u_0 \in E \) be divergence free and homogeneous of degree \(-1\). Then there exists a self-similar solution \( u \) of (2) on \( \mathbb{R}^3 \times (0, \infty) \) corresponding to the initial data \( u_0 \) with the following properties:

\[\begin{align*}
&i) \lim_{t \to 0^+} \|u(t) - u_0\|_{L^2_{\text{loc,unif}}} = 0 \\
&ii) u \in C_\infty^\infty C_t^\infty \text{ and hence a classical solution on } \mathbb{R}^3 \times (0, \infty)
\end{align*}\]

\[\|u(t)\|_{L^\infty} = \|u(1)\|_{\infty} \frac{1}{\sqrt{t}} < \infty \text{ for all } t \in (0, \infty).\]

**Proof.** Recall the construction of local-in-time local Leray solutions (3). It is clear that (3) will be scaling invariant if and only if the smoothed-out velocity

\[ w_\epsilon(x, t) = (u_\epsilon * \rho_\epsilon)(x, t) = \int u_\epsilon(x - y, t) \rho_\epsilon(y) \, dy \]

exhibits the same scaling as the original velocity, i.e. if and only if \( w_\epsilon(\lambda x, \lambda^2 t) = \lambda^{-1} w_\epsilon(x, t) \) for all \( \lambda > 0 \). A simple computation yields \( w_\epsilon(\lambda x, \lambda^2 t) = \lambda^{-1} \int u_\epsilon(x - z, t) \lambda^3 \rho_\epsilon(\lambda z) \, dz \) which is equal to \( \lambda^{-1} w_\epsilon(x, t) \) if and only if \( \rho_\epsilon(\lambda z) = \lambda^{-3} \rho_\epsilon(z) \) for all \( \lambda > 0 \), i.e. if and only if \( \rho_\epsilon \) is homogeneous of degree \(-3\). This is not possible since \( \rho_\epsilon \) is just a rescaled test function and hence have a compact support. Consequently, an approximating solution \( u_\epsilon \) will not be self-similar.

However there is some symmetry. For \( 0 < \lambda \leq 1, 0 < \epsilon \leq \epsilon_0 \), define \( \rho_\epsilon^\lambda(x) = \lambda^3 \rho_\epsilon(\lambda x) \) and \( u_\epsilon^\lambda(x, t) = \lambda u_\epsilon(\lambda x, \lambda^2 t) \) for all \( x \in \mathbb{R}^3 \) and all \( t \in (0, T) \). A straightforward calculation reveals that \( u_\epsilon \) solves (3) if and only if \( u_\epsilon^\lambda \) solves the following
system:

\[
(u^\lambda_\varepsilon)_t - \Delta u^\lambda_\varepsilon + P \nabla \cdot ((u^\lambda_\varepsilon \ast \rho^\lambda_\varepsilon) \otimes u^\lambda_\varepsilon) = 0
\]

\[
\nabla \cdot u^\lambda_\varepsilon = 0
\]

supplemented by \( u^\lambda_\varepsilon(0, \cdot) = u_0 \) (utilizing the homogeneity of \( u_0 \)). Since \( \rho^\lambda_\varepsilon \neq \rho_\varepsilon \) for any \( \lambda \neq 1 \), the approximating solutions \( u^\lambda_\varepsilon \) (for \( \lambda \neq 1 \)) and \( u_\varepsilon \) will differ as well. What saves the day is a simple observation that \( \rho^\lambda_\varepsilon = \rho_\varepsilon^\lambda \) and hence \( u^\lambda_\varepsilon = u_\varepsilon^\lambda \). Thus, \( \{u_\varepsilon\} \) is simply a subfamily of \( \{u^\lambda_\varepsilon\} \). However, we need to be somewhat careful here. Since a weak solution \( u \) in Theorem 1 is obtained through a sequence, say \( \{u_\varepsilon^n\} \), it is not possible to infer simultaneous convergence for \textit{continuum} many lambda (and in particular, for \( \lambda \) in \( (0, 1) \)). What we can do is to “precondition” the sequence for a dense set. Let \( D_2 \) denote the set of all dyadics in \( (0, 1) \). Playing with indices in \( D_2 \), we can construct (via a diagonalization procedure) a sequence \( \{u_\varepsilon^n\} \) with a property that \( \{u^\lambda_\varepsilon\} \) converges to a solution \( u \) for all \( \lambda \) in \( D_2 \cup \{1\} \).

Recall that one of the convergences obtained in the proof of Theorem 1 is a strong convergence of \( \varphi u_\varepsilon^n \) to \( \varphi u \) in \( L^p((0, T), L^2(\mathbb{R}^3)) \) for any test function \( \varphi \in \mathcal{D}(\mathbb{R}^3 \times (0, T)) \), and any \( p < \infty \). Fix a ball \( B \) centered at the origin. Then, for any \( \lambda \) in \( D_2 \cup \{1\} \), a.e. \( t \),

\[
\int_B u^\lambda_\varepsilon^n(x, t)g(x) \, dx = \int_B u^\lambda_\varepsilon^n(x, t)g(x) \, dx \to \int_B u(x, t)g(x) \, dx
\]

for all \( g \) in \( L^2(B) \). On the other hand, it is easily seen that

\[
\int_B u^\lambda_\varepsilon^n(x, t)g(x) \, dx = \int_B \lambda u_\varepsilon^n(\lambda x, \lambda^2 t)g(x) \, dx \to \int_B \lambda u(\lambda x, \lambda^2 t)g(x) \, dx
\]

for all \( g \) in \( L^2(B) \). Uniqueness of a limit coupled with \( 5 \) and \( 6 \) yields that for any \( \lambda \) in \( D_2 \), a.e. \( t \),

\[
\int_B [u(x, t) - \lambda u(\lambda x, \lambda^2 t)] g(x) \, dx = 0
\]
for all \( g \) in \( L^2(B) \).

Now, fix \( \lambda \) in \( D_2 \) and consider

\[
f(t) = \int_B \left[u(x, t) - \lambda u(\lambda x, \lambda^2 t)\right] g(x) \, dx
\]

for \( t \) in \((0, T)\). Since \( f \) is continuous (by the weak time continuity of \( u \) with values in \( L^2(B) \)) and vanishes \( a.e. \), \( f \) is identically zero on \((0, T)\). Next, for a fixed \( t \) in \((0, T)\) consider

\[
g(\lambda) = \int_B \left[u(x, t) - \lambda u(\lambda x, \lambda^2 t)\right] g(x) \, dx
\]

for \( \lambda \) in \((0, 1)\). This function is also continuous and vanishes on a dense set \((D_2) – \) hence vanishes for all \( \lambda \) in \((0, 1)\). Finally, since the ball \( B \) was arbitrary,

\[
u(x, t) = \lambda u(\lambda x, \lambda^2 t) \tag{7}
\]

for any \( \lambda \) in \((0, 1)\), any \( t \) in \((0, T)\), and \( a.e. \, x \) in \( \mathbb{R}^3 \). \textit{A priori}, the \( a.e. \, x \)-set depends on both \( \lambda \) and \( t \). One can actually show more uniformity, but this will suffice for our construction.

Notice that we can extend \( (7) \) to any \( \lambda > 0 \) provided that we restrict \( t \) to \((0, \min\{T, \frac{T}{\lambda^2}\})\).

Next, we show that the singular set of \( u \) on \( \mathbb{R}^3 \times (0, T) \) is empty. In what follows, it will be convenient to define a family of parabolic space-time cylinders centered at a space-time point \((x, t)\). For \( r > 0 \), define \( C_{x,t}(r) \) by

\[
C_{x,t}(r) = \{(y, \tau) \in \mathbb{R}^3 \times (0, \infty) | |\tau - t| < \frac{1}{2} r^2, |(x, \tau) - (y, \tau)| < r\}.
\]

We argue by contradiction. Suppose there exists a singular point \((x, t)\) of \( u \) in \( \mathbb{R}^3 \times (0, T) \) and consider a family of parabolic cylinders \( \{C_{x,t}(r)\} \) for \( 0 < r \leq \min\{\sqrt{t}, \sqrt{T-t}\} = R_{t,T} \). Since \((x, t)\) is a singular point, \( \|u\|_{L^\infty(C_{x,t}(r))} = \infty \) for
all $r$. Fix any $\lambda$ in $\left(0, \sqrt{\frac{T}{t+\frac{1}{2}R^2_{t,T}}} \right)$, and then fix any $r$, $0 < r \leq R_{t,T}$. For this choice of $\lambda$, $u(\lambda y, \lambda^2 \tau) = \lambda^{-1} u(y, \tau)$ for any $\tau$ in $(t - \frac{1}{2}r^2, t + \frac{1}{2}r^2)$, a.e. $y$ in $\mathbb{R}^3$, and hence a.e. $(y, \tau)$ in $C_{x,t}(r)$. This implies that

$$
\|u\|_{L^\infty(C_{x,\lambda^2 t}(\lambda r))} = \lambda^{-1}\|u\|_{L^\infty(C_{x,t}(r))} = \infty
$$

for any $r$, $0 < r \leq R_{t,T}$. In other words, we constructed a family of parabolic neighborhoods of a point $(\lambda x, \lambda^2 t)$ on which $u$ blows-up and so $(\lambda x, \lambda^2 t)$ is also a singular point. The argument can be repeated for any $\lambda$ in $\left(0, \sqrt{\frac{T}{t+\frac{1}{2}R^2_{t,T}}} \right)$ and hence we generated a smooth space-time curve of singular points passing through the point $(x, t)$ contained in $\mathbb{R}^3 \times (0, T)$. Since local Leray solutions are suitable, this contradicts Theorem 3 and thus the singular set must be empty.

Now, it is standard to conclude infinite spatial regularity. Fix any space-time point $(x, t)$. Since $(x, t)$ is a regular point, there exists a (bounded) neighborhood $D$ of $(x, t)$ such that $\|u\|_{L^\infty(D)} < \infty$. This implies that $u \in L^p_x L^q_t(D)$ for any $1 \leq p, q \leq \infty$, and in particular $u$ satisfies the condition of Theorem 3. It follows that $u \in C^\infty_x(D)$ and that each derivative is bounded on compact subregions of $D$.

Let us note that the spatial continuity of $u$ now implies that given $\lambda > 0$,

$$
u(x, t) = \lambda u(\lambda x, \lambda^2 t)
$$

for all $t$ in $(0, \min\{T, \frac{T}{\lambda^2}\})$ and all $x \in \mathbb{R}^3$.

Infinite time regularity in general does not follow from the emptiness of the singular set. However, a restricted self-similarity will help. More precisely, we can write $u(x, t) = \frac{1}{\sqrt{t}} U_T \left( \frac{x}{\sqrt{t}} \right)$ where $U_T(\cdot) = \sqrt{T} \frac{\sqrt{T}}{2} u \left( \sqrt{T} \frac{\cdot}{2}, \frac{T}{2} \right)$ for all $(x, t)$ in $\mathbb{R}^3 \times (0, T)$. Since we know that $u \in C^\infty_x$, and it is clear that $u \in C^\infty_x$ if and only if $U_T \in C^\infty$, it follows that $U_T$ is infinitely smooth. The infinite time regularity is now readily seen.
To summarize, $u$ is in $C_2^\infty C_t^\infty$ (with the corresponding infinitely smooth pressure $p$) and hence a classical solutions on $\mathbb{R}^3 \times (0, T)$ which is self-similar on $(0, T)$, i.e. for a given $(x, t)$ in $\mathbb{R}^3 \times (0, T),$

$$u(\lambda x, \lambda^2 t) = \lambda^{-1} u(x, t)$$

for all $\lambda$ in $\left(0, \sqrt{\frac{T}{t}}\right)$ (the corresponding pressure scales as $p(\lambda x, \lambda^2 t) = \lambda^{-2} p(x, t)$).

Finally, we extend $(u, p)$ to the whole space-time by self-similarity. It is clear that the extension will inherit infinite space and time regularity and will solve the equations (1) in the classical sense (pointwise).

One can now study various norms of $u$ – we consider the $L^\infty$-norm here. Recall that $u \in \cap_{t<T} L^\infty((0, t), E)$ which paired with the time continuity yields $u(t)$ in $E$ for any $t$ in $(0, T)$. In particular, $u\left(\frac{T}{2}\right)$ is in $E$ and hence it decays to 0 as $|x| \to \infty$. Since we know $u\left(\frac{T}{2}\right)$ is continuous, it follows $\|u\left(\frac{T}{2}\right)\|_{L^\infty} < \infty$. Self-similarity implies

$$\|u(t)\|_{L^\infty} = \sqrt{\frac{T}{t}} \|u\left(\frac{T}{2}\right)\|_{L^\infty} \frac{1}{\sqrt{t}}$$

for all $t$ in $(0, \infty)$. This is informative at both ends. It gives a blow-up rate as $t \to 0^+$ compatible with the $\frac{1}{|x|}$-type singularity of the initial data, and also a generic decay rate as $t \to \infty$.

The convergence to the initial data in $L^2_{loc, unif}$ follows from Theorem [1].

Remark 7. It is worth noting that most of the functional spaces utilized in the study of the 3D NSE on the whole space are embedded in $L^2_{loc, unif}$ – $u$ in $L^2_{loc}$ gives the bilinear term $\nabla \cdot (u \otimes u)$ at least a distributional meaning, and uniformity corresponds to the (desirable) translational invariance of a norm.
Remark 8. It is clear that the proof of Theorem 6 implies infinite space and time smoothness of any self-similar suitable weak solution (regardless of its construction).

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