Pair tunneling, phase separation and dimensional crossover in imbalanced fermionic superfluids in a coupled array of tubes

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We study imbalanced fermionic superfluids in an array of one-dimensional tubes at the incipient dimensional crossover regime, wherein particles can tunnel between neighboring tubes. In addition to single-particle tunneling (ST), we consider pair tunneling (PT) that incorporates the interaction effect during the tunneling process. We find that with an increase of PT strength, a system of low global polarization evolves from a structure with a central Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state to one with a central BCS-like fully-paired state. For the case of high global polarization, the central region exhibits pairing zeros embedded in a fully paired order. In both cases, PT enhances the pairing gap, suppresses the FFLO order, and leads to spatial separation of fully paired and fully polarized regions, the same as in higher dimensions. Thus, we show that PT beyond second-order ST processes is of relevance to the development of signatures characteristic of the incipience of the dimensional crossover.

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I. INTRODUCTION

Superconductivity and ferromagnetism are two ubiquitous but competing phenomena in condensed matter systems. Spin imbalance and magnetic fields induced by ferromagnetism tend to suppress Cooper pairing, which is responsible for superconductivity. For more than four decades, an interesting phase, the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state \[1, 2\], has been suggested as the concurrence of both ferromagnetic and oscillatory superconducting orders \[3, 4\], but its direct confirmation is still elusive. Recently, owing to the capability of controlling particle densities, tuning interactions, and cooling into quantum degeneracy \[5, 6\], cold atomic systems have become a promising platform for searching for FFLO order \[7, 8\]. In experiments of trapped Fermi gases, density profiles that reflect the interplay of spin imbalance (ferromagnetic order) and Cooper pairing have been observed \[9, 10\]. In addition, experiments have also revealed a significant dimensional dependence of the profiles: in three dimensions (3D) a fully paired profile takes place at the trap center and a polarized profile does off center \[9, 10\], while in 1D the central region is always polarized \[11\]. These observations agree with theoretical studies of the FFLO state in 1D and the trap-induced phase separation in 3D \[12\text{-}19\], but the marked difference between these two limits also raises the need for understanding the intermediate regime. Several works have focused on the dimensional crossover regime of various kinds of continuous systems \[20\text{-}25\] or Hubbard lattices \[26\text{-}27\], but with a different emphasis than the present work.

In this paper, we study a realizable system of a two-dimensional optical lattice array of one-dimensional tubes, subject to a global trapping potential \[11\text{-}28, 29\]. The incipient dimensional crossover regime of this system, which can be experimentally accessed by gradually lowering the lattice depth, is modeled by incorporating the kinetics of single-particle tunneling (ST) as well as a key ingredient representing the tunneling of paired opposite-spin atoms—pair tunneling (PT)—between neighboring tubes (as illustrated in Fig. 1). The ST leads to an interesting magnetic compressible-incompressible phase transition analogous to that in the Bose-Hubbard model (discussed in Ref. \[25\]), but is not responsible for certain observed signatures in the profiles at the dimensional crossover regime (which will be shown below). By considering PT, we are able to describe the incipient evolution of profiles from 1D toward 3D and obtain the emerging signatures of the dimensional crossover at various global polarizations, such as the inversion of the fully paired and polarized centers as well as the growing spatial separation between the fully paired and fully polarized regions.

The paper is outlined as follows. In Sec. II we discuss the microscopic physical cause of the PT and evaluate its strength using a two-channel model. In Sec. III we construct a model Hamiltonian and apply a Bogoliubov-de Gennes (BdG) treatment to solve for the density and pairing profiles of the system. In Sec. IV we present
the results and discuss their physical meanings associated with PT. Finally, we summarize our work in Sec. V.

II. PHYSICS OF PAIR TUNNELING

For free fermions in a lattice potential, the intersite kinetics is well described by ST processes of strength $t_1$ [30]. For an attractively interacting case where two opposite-spin atoms form a pair of binding energy $\epsilon_b$, the kinetics of the pair tunneling would, in principle, be incorporated as a process in which the two atoms of a pair split, separately tunnel to the other site, and rebind [see Fig. 2(a)]. This is contained in the ST model as a second-order process of strength $t_1^2/\epsilon_b$ and accounts for the Josephson phenomena in the presence of superfluid orders, cf. [22].

However, in cold-atom experiments, the interaction is induced via a Feshbach resonance [31], which is controlled by the tuning of a magnetic field affecting the hyperfine energy splittings. Because the field is applied throughout the system, the interaction that leads to pairing exists on and between lattice sites. Therefore, we expect that atoms that remain paired during the whole tunneling event can be another viable process [see Fig. 2(b)]. Such a process can be described as tunneling of the paired atoms, with strength denoted as $t_2$.

One can estimate $t_2$ around the Feshbach resonant regime using a two-channel model [6, 7, 31] that incorporates atomic and molecular degrees of freedom, $\psi_\sigma$ and $\phi$, respectively. In optical lattices [32, 33], the partition function of the system is

$$Z = \int D\{\psi_{\sigma i}, \bar{\psi}_{\sigma i}\} D\{\phi_i, \bar{\phi}_i\} e^{-\int d\tau(S_a+S_m)},$$

where $S_a$ contains terms associated only with the atomic degrees of freedom, including the atomic tunneling as well as any bare interatomic interaction, and

$$S_m = -t_m \sum_{\langle ij \rangle} \bar{\phi}_i \phi_j - \mu_m \sum_i \bar{\phi}_i \phi_i + U_{am} \sum_i (\bar{\phi}_i \psi_{\uparrow i} \psi_{\downarrow i} + H.c.)$$

involves the molecular tunneling $t_m$, molecular chemical potential $\mu_m$, and atom-molecule coupling $U_{am}$. Here we assume the intermolecular interaction is weak such that a mean-field approximation $\bar{\phi}_i \phi_i \phi_j \phi_j \rightarrow \langle \bar{\phi}_i \phi_i \rangle \phi_j \phi_j$ can be applied to incorporate the interaction as effective contributions to the chemical potentials. We integrate out the molecular variable $\phi$ in Eq. (1) and obtain

$$Z = \int D\{\psi_{\sigma i}, \bar{\psi}_{\sigma i}\} e^{-\int d\tau(S_a+S'_a)},$$

where $S'_a$ is expanded as

$$S'_a = \frac{U_{am}^2}{\mu_m} \left[ -\sum_i \bar{\psi}_{\uparrow i} \bar{\psi}_{\downarrow i} \psi_{\uparrow i} \psi_{\downarrow i} - \frac{t_m}{\mu_m} \sum_{\langle ij \rangle} \bar{\psi}_{\uparrow i} \bar{\psi}_{\downarrow i} \psi_{\uparrow j} \psi_{\downarrow j} + O \left( \frac{t_m^2}{\mu_m^2} \right) \right].$$

The first term in Eq. (4) can be treated as a resonant contribution to the inter-atomic interaction\(^1\), while the second one appears as PT. Therefore, we obtain

$$t_2 = \frac{U_{am}^2}{\mu_m^2} t_m.$$  

By considering the tunneling strength in optical lattices given by $\frac{4}{\sqrt{\pi} E_R (V_0/E_R)^{3/4}} \exp[-2\sqrt{V_0/E_R}]$ with $V_0$ the optical-lattice depth and $E_R$ the recoil energy [35, 36], we find

$$\frac{t_2}{t_1} = \sqrt{2} \frac{U_{am}^2}{\mu_m^2} \exp \left[ -2 \sqrt{\frac{V_0}{E_R}} \right].$$

This expression shows that $t_2$ has the same sign as $t_1$ and can vary at fixed $t_1$ (or fixed lattice geometry) through the tuning of $U_{am}$ and $\mu_m$. We also see that even if the molecular tunneling is smaller than the atomic tunneling ($t_m < t_1$), $t_2$ can be comparable

\(^1\) In combination with any bare interatomic interaction present, one would obtain an effective interatomic interaction as discussed in Ref. [32].
tering length and elongated tube limit, with

\[ \mu_m = \frac{\delta \mu}{m} \]

denoting tube indexes in the plane perpendicular to \( r \). The one-particle Hamiltonian \( H^0_\sigma = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + m(\omega_r^2 r^2 + \omega_z^2)\) includes the kinetic energy in the \( z \) direction, the global trapping potential, and the spin-dependent chemical potentials. The on-site coupling constant (taken positive for attractive interaction) is given as \( g = -2\hbar^2 a_s / m \ell^2 (1 - 1.033a_s / \ell) \) in the highly elongated tube limit, with \( a_s \) the two-body s-wave scattering length and \( \ell \) the oscillator length of the transverse confinement in a tube [39]. In the tube array of lattice spacing \( d \), \( \ell \sim (V_0 / E_R)^{-1/3} d / \pi \). The ST (PT) of strength \( t_1 \) (\( t_2 \)) takes place between nearest-neighbor tubes (\( r' \)).

Applying the BdG mean-field theory [40] (which has successfully described tube lattices without PT [25] and a variety of tube confinements [12–15, 21, 23]), we construct a mean-field Hamiltonian \( H_M \) by correspondingly replacing the quartic operators in Eq. [4] with quadratic ones coupled to three different mean fields,

\[
H_M = \int_z \sum_r \left[ \sum_\sigma \psi_\sigma^\dagger (H^0_\sigma + U_\sigma) \psi_\sigma + (\Delta_r \psi_{\sigma r}^\dagger \psi_{\sigma r} + H.c.) \right] + \int_{(rr')} \sum_\sigma T_{\sigma rr'} \psi_{\sigma r'}^\dagger \psi_{\sigma r}. \tag{8}
\]

Here we introduce a tunneling field \( T_{\sigma rr'}(z) \) as a new ingredient to describe the effective tunneling under the influence of both \( t_1 \) and \( t_2 \). We rotate \( H_M \) into the quasiparticle basis \( \gamma_n \) through a Bogoliubov transformation \( \psi_\sigma = \sum_n (|u_{n\sigma} r\rangle \gamma_n + \sigma \sigma' \bar{\gamma}_{n\sigma} \langle \sigma'|) \) \( \sigma = \sigma - \sigma' \) and derive extended BdG equations for the quasiparticle wave functions \( u_{n\sigma} \) and \( v_{\sigma r} \) as well as the corresponding energies \( \epsilon_{n\sigma} \).

In Sec. IV we present the results for a spherically trapped system (\( \omega_r = \omega \)).
IV. RESULTS

From now on, we take a realistic setup $d = -a_s = 0.5 \mu m$ for $^6$Li systems in the Feshbach resonant regime and use the binding energy $\epsilon_b = mg^2/4\hbar^2$ as the energy unit for the following results. We look at the influence of $t_2$ at fixed $t_1 = 0.014\epsilon_b$, the latter corresponding to a typical lattice depth of $7E_R$ and thus into the dimensional crossover regime. In Fig. [3] we plot the axial profiles of $\rho$, $s$, and the average of $|\Delta|$ by tracing out the $r$ degree of freedom. The first and second columns correspond to a lower global polarization of $P = 25\%$ (LP) and a higher one of $P = 50\%$ (HP), respectively. From top row to bottom, $t_2$ is chosen to be either zero, comparable to $t_1$, or larger than $t_1$, respectively. We see that in the LP case at $t_2 = 0$, the axial profile exhibits (i) an FFLO center with oscillatory $\Delta$, (ii) a BCS-like shoulder with non oscillatory $\Delta$, and (iii) a normal tail having zero $\Delta$. At the intermediate $t_2$ value, this trilayered structure remains. However, the FFLO center shrinks, the BCS-like region extends toward the center accompanied by a drop in imbalance, and the normal tail grows. This indicates a transfer of unpaired majority atoms from the center to the tail, implying an enhancement of a Meissner-like effect in the central region. We notice that the gap profile develops small ripples between the BCS-like shoulder [(ii)] and the normal tail [(iii)], suggesting the incipience

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2 The global polarization $P$ is defined as the ratio of total imbalance to the total number of particles. The LP case we focus on herein is somewhat higher than the critical polarization of 15\%–18\% verified in experiments.
of an FFLO layer [(i)] here. At the large $t_2$ value, the FFLO center is completely conquered by the BCS-like state and disappears, leaving a large fully polarized tail and a thin FFLO layer in between them. Because the FFLO and BCS centers are distinctive of one- and three-dimensional trapped systems, respectively, this result shows the evolution of the system from 1D toward 3D, driven by $t_2$ (compared with increasing $t_1$).

In the HP case, the system always has a center with oscillatory $\Delta$ and a fully polarized normal tail. In the oscillatory-pairing region, the imbalance profile exhibits characteristic out-of-phase oscillations, with the concurrence of local minima (maxima) of $s$ and local maxima (minima) of $|\Delta|$. This behavior is due to the competition between superfluid and ferromagnetic orders. An increase in $t_2$ enhances this competition, augmenting the magnitude of the out-of-phase oscillations and repelling a portion of the unpaired majority to the normal tail region. At large $t_2$ ($= 0.05\epsilon_b$), the oscillations are large enough that the minima of $s$ are almost zero. Such a case is less like an FFLO state (oscillatory pairing accompanied by finite polarization), but more like spatial alternation of fully paired superfluid and highly polarized normal gas. This phenomenon, which is analogous to the phase separation in the LP case, is taken as a signature of the dimensional crossover between 1D and 3D at higher polarizations. We also notice that the structure of the profiles is reminiscent of that of a system with vortex cores embedded in a superfluid bulk.

We find that PT affects the pairing order not only along but also across the tubes. Figure 4 shows the value of the gap function at the center of each tube ($z = 0$) for various $t_2$ and $P$ (convention as presented in Fig. 3). Here we show data for $5 \times 5$ tubes in the fourth quadrant of the $10 \times 10$ tube array, in which the top left entry of each panel corresponds to the most central tube. The other quadrants are similar due to fourfold rotational symmetry.

![FIG. 4: (Color online) Value of the gap function (in units of $\hbar \omega$) at the center of each tube ($z = 0$) for various $t_2$ and $P$ (convention as presented in Fig. 3). Here we show data for $5 \times 5$ tubes in the fourth quadrant of the $10 \times 10$ tube array, in which the top left entry of each panel corresponds to the most central tube. The other quadrants are similar due to fourfold rotational symmetry.](image-url)
tice in Figs. 3 and 4 that \( t_2 \) enhances the maximum magnitude of the gap function, as expected from Eq. (10). This enhancement raises the critical temperature above which the pairing order vanishes and hence agrees with the increase of the superfluid transition temperature in quasi-one-dimensional systems [22].

Finally, we look at the phase separation of fully paired and fully polarized regions as a function of \( t_2 \). We consider the combined fraction of particles in the highly paired (\( s/\rho < 5\% \)) and highly polarized (\( s/\rho > 95\% \)) regions of the axial profiles; \( \gamma \equiv \int_0^\gamma \rho(\theta(0.05 - s/\rho) + \theta(s/\rho - 0.95))/\int_0^\pi \rho, \) where \( \theta \) is the step function. The larger \( \gamma \) is, the stronger phase separation the system shows. Figure 5(a) shows that \( \gamma \) monotonically increases with \( t_2 \) at three various polarizations when \( t_1 \) is fixed. In the cases of \( P = 12.5\% \) and \( 25\% \) the sudden changes indicate the occurrence of the BCS-like center replacing the FFLO center. For comparison we plot also \( \gamma \) vs \( t_2 \) at fixed \( t_2 \) in Fig. 5(b) and observe that \( \gamma \) shows almost no change at the three polarizations. This result highlights that it is \( t_2 \), rather than \( t_1 \), that accounts for the phase separation and hence is essential for the correct model describing the physics at the incipience of the dimensional crossover regime.

V. CONCLUSION

By considering the microscopic physics of cold atomic systems, we have incorporated both ST and PT processes and enhanced the maximum magnitude of the gap function, as expected from Eq. (10). This enhancement raises the critical temperature above which the pairing order vanishes and hence agrees with the increase of the superfluid transition temperature in quasi-one-dimensional systems [22].

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