DARBOUX COORDINATES FOR SYMPLECTIC GROUPOID AND CLUSTER ALGEBRAS

LEONID O. CHEKHOV* AND MICHAEL SHAPIRO**

Dedicated to the memory of great mathematician and person Boris Dubrovin

Abstract. Using Fock–Goncharov higher Teichmüller space variables we derive Darboux coordinate representation for entries of general symplectic leaves of the $A_n$ groupoid of upper-triangular matrices and, in a more general setting, of higher-dimensional symplectic leaves for algebras governed by the reflection equation with the trigonometric $R$-matrix. The obtained results are in a perfect agreement with the previously obtained Poisson and quantum representations of groupoid variables for $A_3$ and $A_4$ in terms of geodesic functions for Riemann surfaces with holes. We represent braid-group transformations for $A_n$ via sequences of cluster mutations in the special $A_n$-quiver. We prove the groupoid relations for quantum transport matrices and, as a byproduct, obtain the Goldman bracket in the semiclassical limit.

1. Introduction

1.1. Symplectic groupoid, induced Poisson structure on the unipotent upper triangular matrices. Let $V$ denote an $n$-dimensional vector space, $\mathcal{A}$ be some subspace of bilinear forms on $V$. Fixing the basis in $V$, one can identify $\mathcal{A}$ with a subspace in the space of $n \times n$ matrices. The matrix $B$ of a change of a basis in $V$ takes a matrix of bilinear form $A \in A$ to $BAB^T$.

Below we consider an important particular case when $\mathcal{A}$ is the space of unipotent forms identified with the space of the unipotent matrices. The basis change $B$ acts on $\mathcal{A}$ only if the product $BAB^T$ be unipotent itself. We thus introduce the space of morphisms $\mathcal{M}$ identified with admissible pairs of matrices $(B, A)$ such that

$$\mathcal{M} = \{(B, A) \mid B \in GL(V), A \in A, BAB^T \in A\}.$$

We then have the standard set of maps:

- source $s: \mathcal{M} \to A \ (B, A) \to A$,
- target $t: \mathcal{M} \to A \ (B, A) \to BAB^T$,
- injection $e: A \to \mathcal{M} \ A \to (E, A)$,
- inversion $i: \mathcal{M} \to \mathcal{M} \ (B, A) \to (B^{-1}, BAB^T)$,
- multiplication $m: \mathcal{M}^{(2)} \to \mathcal{M} \ ((C, BAB^T), (B, A)) \to (CB, A)$.

*Steklov Mathematical Institute, Moscow, Russia, and Michigan State University, East Lansing, USA. Email: chekhov@msu.edu.
**Michigan State University, East Lansing, USA. Email: mshapiro@msu.edu.
such that the following diagram, where \( p_1 \) and \( p_2 \) are natural projections to the first and the second morphism in an admissible pair of morphisms, is commutative:

\[
\begin{array}{ccc}
\mathcal{M}^{(2)} & \xrightarrow{s} & \mathcal{A} \\
\downarrow{p_1} & & \downarrow{t} \\
\mathcal{M} & \xrightarrow{p_2} & \mathcal{M}
\end{array}
\]

The crucial point of the construction is the existence of a symplectic structure: a smooth groupoid endowed with a symplectic form \( \omega \in \Omega^2 \mathcal{M} \) on the morphism space \( \mathcal{M} \) that satisfies the splitting (consistency) condition [27, 42]

\[
m^* \omega = p_1^* \omega + p_2^* \omega,
\]

which implies, in particular, that the source and target maps Poisson commute being respectively an automorphism and an anti-automorphism of the initial Poisson algebra. Since \( p_1^* \omega \) and \( p_2^* \omega \) are nondegenerate, they admit a (unique) Poisson structure, and because the immersion map \( e \) is Lagrangian, this Poisson structure yields a Poisson structure on \( \mathcal{A} \)—the space of unipotent upper triangular matrices.

In 2000, Bondal [3] obtained the Poisson structure on \( \mathcal{A} \) using the algebroid construction; assuming \( B = e^g \), we obtain the Bondal algebroid using the anchor map \( D_\mathbb{A} \) to the tangent space \( T_\mathbb{A} \mathcal{A} \)

\[
D_\mathbb{A} : \mathfrak{g}_\mathbb{A} \to T_\mathbb{A} \mathcal{A}, \quad g \mapsto \mathbb{A} g + g T_\mathbb{A}
\]

where \( \mathfrak{g}_\mathbb{A} \) is the linear subspace

\[
\mathfrak{g}_\mathbb{A} := \{ g \in \mathfrak{gl}_n(\mathbb{C}), |\mathbb{A} + \mathbb{A} g + g T_\mathbb{A} \in \mathcal{A} \}
\]

of elements \( g \) leaving \( \mathbb{A} \) unipotent.

**Lemma 1.1.** [3] The map

\[
P_\mathbb{A} : T_{\mathbb{A}}^* \mathcal{A} \to \mathfrak{g}_\mathbb{A}, \quad w \mapsto P_{-1/2}(w \mathbb{A}) - P_{1/2}(w T_\mathbb{A} T),
\]

where \( P_{\pm 1/2} \) are the projection operators:

\[
P_{\pm 1/2} a_{i,j} := \frac{1 \pm \text{sign}(j - i)}{2} a_{i,j}, \quad i, j = 1, \ldots, n,
\]

and \( w \in T^* \mathcal{A} \) is a strictly lower triangular matrix, defines an isomorphism between the Lie algebroid \((\mathfrak{g}, D_\mathbb{A})\) and the Lie algebroid \((T^* \mathcal{A}, D_\mathbb{A} P_\mathbb{A})\).

The Poisson bi-vector \( \Pi \) on \( \mathcal{A} \) is then obtained by the anchor map on the Lie algebroid \((T^* \mathcal{A}, D_\mathbb{A} P_\mathbb{A})\) (see Proposition 10.1.4 in [31]) as:

\[
\Pi : T_{\mathbb{A}}^* \mathcal{A} \times T_{\mathbb{A}}^* \mathcal{A} \to \mathcal{C}^{\infty}(\mathcal{A}), \quad (\omega_1, \omega_2) \mapsto \text{Tr}(\omega_1 D_\mathbb{A} P_\mathbb{A}(\omega_2))
\]

It can be checked explicitly that the above bilinear form is in fact skew-symmetric and gives rise to the Poisson bracket

\[
\{a_{i,k}, a_{j,l}\} := \frac{\partial}{\partial da_{i,k}} \wedge \frac{\partial}{\partial da_{j,l}} \text{Tr}(da_{i,k} D_\mathbb{A} P_\mathbb{A}(da_{j,l})),
\]
having the following form in components:

\[
\{a_{ik}, a_{jl}\} = 0, \quad \text{for } i < k < j < l, \text{ and } i < j < l < k,
\]

\[
\{a_{ik}, a_{jl}\} = 2(a_{ij}a_{kl} - a_{il}a_{kj}), \quad \text{for } i < j < k < l,
\]

\[
\{a_{ik}, a_{kl}\} = a_{ik}a_{kl} - 2a_{il}, \quad \text{for } i < k < l,
\]

\[
\{a_{ik}, a_{jk}\} = -a_{ik}a_{jk} + 2a_{ij}, \quad \text{for } i < j < k,
\]

\[
\{a_{ik}, a_{il}\} = -a_{ik}a_{il} + a_{kl}, \quad \text{for } i < k < l.
\]

This bracket turned out to coincide with the bracket previously known in mathematical physics as Gavrilik–Klimyk–Nelson–Regge–Dubrovin–Ugaglia bracket [23, 33, 34, 14, 41] and it arises from skein relations satisfied by a special finite subset of geodesic functions (traces of monodromies of $SL_2$ Fuchsian systems, which are in 1-1 correspondence with closed geodesics on a Riemann surface $\Sigma_{g,s}$) described in [7]; a simple constant log-canonical (Darboux) bracket on the space of Thurston shear coordinates $z_\alpha$ on the Teichmüller space $T_{g,s}$ of Riemann surfaces $\Sigma_{g,s}$ of genus $g$ with $s = 1, 2$ holes was shown [6] to induce the above bracket on a special subset of geodesic functions identified with the matrix elements $a_{ik}$. All such geodesic functions admit an explicit combinatorial description [15], which immediately implies that they are Laurent polynomials with positive integer coefficients of $e^{z_\alpha}/2$. The Poisson bracket of $z_\alpha$ spanning the Teichmüller space $T_{g,s}$ has exactly $s$ Casimirs, which are linear combinations of shear coordinates incident to the holes, so the subspace of $z_\alpha$ orthogonal to the subspace of Casimirs parameterizes a symplectic leaf in the Teichmüller space which we call a geometric symplectic leaf.

In [7], the Poisson embedding of geometric symplectic leaf into $A_n$ was constructed. Note however that the size $n$ of matrix $A_n$ is related to the genus and the number of holes as $n = 2g + s$ (with $s$ taking only two values, 1 and 2) and that the (real) dimension of $T_{g,s}$ is $6g - 6 + 3s$ increasing linearly with $g$ whereas the total dimension of $A_n$ is obviously $n(n - 1)/2$ increasing quadratically with $n$; for $n = 3$ and $n = 4$ these two dimensions coincide and the geometric symplectic leaf having the dimension $6g - 6 + 2s$ is of maximum dimension.

For $n = 5$, the dimension of the geometric symplectic leaf has still the maximum value 8 of dimensions of symplectic leaves in $A_5$, but we have just one central element in the corresponding Teichmüller space $T_{2,1}$ and two central elements in $A_5$. For all larger $n$ the dimension of geometric symplectic leaf is strictly less than the maximal dimension of symplectic leaf in $A_n$, so the geometric systems do not describe maximal symplectic leaves in the total Poisson space of $A_n$. The Darboux coordinates in geometric situation are well known to be the above shear coordinates, but, as just mentioned, they can not help in constructing Darboux coordinates in $A_n$ for $n \geq 5$.

The first problem addressed in this publication is a construction of Darboux coordinates for a general symplectic leaf of $A_n$ and explicit expression of matrix elements $a_{ij}$ in terms of these Darboux coordinates. It was expected for long, and we show below that these Darboux coordinates are related to cluster algebras, similar to the geometric cases $n = 3, 4$.

From the integrable models standpoint, algebras (1.6) (either with a unipotent $A_n$ or with a general $A_n \in \mathfrak{gl}_n$) are known under the name of reflection equation algebras. A task closely related to the first problem is to construct a Darboux coordinate representation for a general matrix $A_n$ enjoying the reflection equation.

1.2. Standard Poisson-Lie group $G$ and its dual. Another description of the Poisson structure on the space of triangular forms $\mathcal{A}$ was obtained in [2]. Namely, the Poisson structure on triangular forms is a push-forward of the standard Poisson bracket on the dual group $G^* = SL^*_n$ to the set of fixed points of the natural involution.
A reductive complex Lie group $G$ equipped with a Poisson bracket $\{\cdot,\cdot\}$ is called a Poisson–Lie group if the multiplication map $G \times G \ni (X,Y) \mapsto XY \in G$ is Poisson. Denote by $\langle \cdot,\cdot \rangle$ an invariant nondegenerate form on the corresponding Lie algebra $\mathfrak{g} = \text{Lie}(G)$, and by $\nabla^R, \nabla^L$ the right and left gradients of functions on $G$ with respect to this form defined by

$$\langle \nabla^R f(X), \xi \rangle = \frac{d}{dt} \bigg|_{t=0} f(X e^{t\xi}), \quad \langle \nabla^L f(X), \xi \rangle = \frac{d}{dt} \bigg|_{t=0} f(e^{t\xi}X)$$

for any $\xi \in \mathfrak{g}$, $X \in G$.

Let $\pi_{>0}, \pi_{<0}$ be projections of $\mathfrak{g}$ onto subalgebras spanned by positive and negative roots, $\pi_0$ be the projection onto the Cartan subalgebra $\mathfrak{h}$, and let $R = \pi_{>0} - \pi_{<0}$. The standard Poisson-Lie bracket $\{\cdot,\cdot\}_r$ on $G$ can be written as

$$(1.7) \quad \{f_1, f_2\}_r = \frac{1}{2} \left( \langle R(\nabla^L f_1), \nabla^L f_2 \rangle - \langle R(\nabla^R f_1), \nabla^R f_2 \rangle \right).$$

The standard Poisson–Lie structure is a particular case of Poisson–Lie structures corresponding to quasitriangular Lie bialgebras. For a detailed exposition of these structures see, e. g., [41 Ch. 1], [37] and [43].

Following [37], let us recall the construction of the Drinfeld double. The double of $\mathfrak{g}$ is $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}$ equipped with an invariant nondegenerate bilinear form $\langle \langle (\xi, \eta), (\xi', \eta') \rangle \rangle = \langle \xi, \xi' \rangle - \langle \eta, \eta' \rangle$. Define subalgebras $\mathfrak{d}_\pm$ of $D(\mathfrak{g})$ by $\mathfrak{d}_+ = \{(\xi, \xi) : \xi \in \mathfrak{g}\}$ and $\mathfrak{d}_- = \{(R_+(\xi), R_-(\xi)) : \xi \in \mathfrak{g}\}$, where $R_\pm \in \text{End} \mathfrak{g}$ is given by $R_\pm = \frac{1}{2}(R \pm \text{Id})$. The operator $R_\pm = \pi_{\mathfrak{d}_\pm} - \pi_{\mathfrak{d}_\mp}$ can be used to define a Poisson–Lie structure on $D(G) = G \times G$, the double of the group $G$, via

$$(1.8) \quad \{f_1, f_2\}_D = \frac{1}{2} \left( \langle R_D(\nabla^L f_1), \nabla^L f_2 \rangle - \langle R_D(\nabla^R f_1), \nabla^R f_2 \rangle \right),$$

where $\nabla^R$ and $\nabla^L$ are right and left gradients with respect to $\langle \langle \cdot,\cdot \rangle \rangle$. The diagonal subgroup $\{(X,X) : X \in G\}$ is a Poisson–Lie subgroup of $D(G)$ (whose Lie algebra is $\mathfrak{d}_+$) naturally isomorphic to $(G, \{\cdot,\cdot\}_r)$.

The group $G^*$ whose Lie algebra is $\mathfrak{d}_-$ is a Poisson–Lie subgroup of $D(G)$ called the dual Poisson–Lie group of $G$. The Poisson bracket $\{\cdot,\cdot\}_r$ induces the Poisson bracket on $G^*$. For $G = SL_n$ the dual group $G^* = \{(X_+, Y_-) \in B_+ \times B_-\}$ satisfying the additional relation $\pi_0(X_+)\pi_0(Y_-) = \text{Id}$ where $B_+(B_-) \subset SL_n$ are Borel subgroups of nondegenerate upper (lower) triangular matrices.

The involution $\iota_{G^*} : G^* \rightarrow G^*$ takes $(X_+, Y_-)$ to $(Y_-^t, X_+^t)$.

The subgroup $U_+$ of unipotent upper triangular matrices is embedded diagonally in $G^*$. The embedding $\epsilon : U_+ \rightarrow G^*$ maps $X \in U_+$ to $(X, X)$. The image $\epsilon(U_+)$ is the set of fixed points of involution $\iota_{G^*}$.

The image $\epsilon(U_+)$ is not a Poisson subvariety of $G^*$ however the Dirac reduction induces the Poisson bi-vector $\Pi$ on $U_+$.

For sake of completeness, we remind here the definition of Dirac reduction. Let $X$ be a subvariety of a Poisson variety $(V, \{\cdot,\cdot\}_{PB})$ defined by constrains $\phi_i = \text{const}$. The second class constrains are constrains $\tilde{\phi}_a$ which have nonvanishing Poisson bracket with at least one other constraint.

Define matrix $U$ with entries $U_{ab} = \{\tilde{\phi}_a, \tilde{\phi}_b\}_{PB}$. Note that $U$ is always invertible.

Then, Dirac bracket of functions $f$ and $g$ on $X$ is

$$\{f, g\}_{DB} = \{f, g\}_{PB} - \sum_{a,b} \{f, \tilde{\phi}_a\}_{PB} U^{-1}_{ab} \{\tilde{\phi}_b, g\}_{PB},$$

see [26] for details.
1.3. Braid-group action on the unipotent matrices. The next important result concerning $\mathcal{A}$ is that this space admits the discrete braid-group action generated by morphisms $\beta_{i,i+1} : \mathcal{A} \to \mathcal{A}$, $i = 1, \ldots, N - 1$, such that

\begin{equation}
\beta_{i,i+1}[\mathcal{A}] = B_{i,i+1} \mathcal{A} B_{i,i+1}^T \equiv \mathcal{A} \in \mathcal{A},
\end{equation}

where the matrix $B_{i,i+1}$ has the block form

\begin{equation}
B_{i,i+1} = \begin{bmatrix}
1 & \ldots & 0 & 0 \\
\vdots & 1 & a_{i,i+1} & -1 \\
& \vdots & 1 & 0 \\
& & \ddots & \ddots \\
& & & 1
\end{bmatrix},
\end{equation}

and this action is a Poisson morphism \[3], \[11]. When acting on $\mathcal{A}_n$, $\beta_{i,i+1}$ satisfy the standard braid-group relations $\beta_{i,i+1} \beta_{i+1,i+1} \beta_{i,i+1} \mathcal{A}_n = \beta_{i+1,i+1} \beta_{i,i+1} \beta_{i,i+1} \mathcal{A}_n$ for $i = 1, \ldots, n - 2$ together with the additional relation $\beta_{n-1,n} \beta_{n-2,n-1} \cdots \beta_{2,3'} \beta_{1,2} \mathcal{A}_n = S_n \mathcal{A}_n$, where $S_n$ is an element of the group of permutations of matrix entries $a_{i,j}$ whose $n$th power is the identity transformation. Note that $\beta_{i,i+1}^2 \mathcal{A} \neq \mathcal{A}$.

In \[3], the quantum version of the above transformations was constructed for a quantum upper-triangular matrix

\begin{equation}
\mathcal{A}_n^h := \begin{bmatrix}
q^{-1/2} & a_{1,2}^h & a_{1,3}^h & \ldots & a_{1,n}^h \\
0 & q^{-1/2} & a_{2,3}^h & \ldots & a_{2,n}^h \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & q^{-1/2}
\end{bmatrix}.
\end{equation}

Here $a_{i,j}^h$ are self-adjoint ($[a_{i,j}^h]^* = a_{i,j}^h$) operators enjoying quadratic-linear algebraic relations following from the quantum reflection equation (see Theorem \[11]) and coinciding with relations obtained for quantum geodesic functions upon imposing quantum skein relations on the corresponding geodesics and $q = e^{-\hbar/2}$. The analogous quantum braid-group action is $\mathcal{A}_n^h \to B_{i,i+1}^h \mathcal{A}_n^h [B_{i,i+1}^h]^\dagger$ with

\begin{equation}
B_{i,i+1}^h = \begin{bmatrix}
1 & \ldots & 0 & 0 \\
\vdots & 1 & q^{1/2}a_{i,i+1} & -q \\
& \vdots & 1 & 0 \\
& & \ddots & \ddots \\
& & & 1
\end{bmatrix},
\end{equation}

In the geometric cases, the above braid-group morphisms are related to modular transformations generated by (classical or quantum \[28]) Dehn twists along geodesics corresponding to the geodesic functions $a_{i,i+1}$ (see \[7]). In the absence of geometric interpretation, the only possibility we may resort to is to address the second problem: to find a sequence of cluster mutations
in a quiver still to be constructed that produces the above braid-group transformation for a generic symplectic leaf of $\mathcal{A}_n$.

We solve the both formulated problems in this paper: we explicitly construct the quiver (called an $\mathcal{A}_n$-quiver) such that the entries $a_{i,j}$ of the unipotent matrix $\mathbf{A}$ are positive Laurent polynomials of the cluster quiver variables, construct a quantum version of this quiver thus realizing the representation $\{1,1\}$ and finding explicitly chains of mutations of the $\mathcal{A}_n$-quiver that produce the braid-group transformations.

1.4. Cluster algebra and cluster mutations. Cluster algebras were introduce in [21] by S.Fomin and A.Zelevinsky in 2001 in an effort to describe (dual) canonical basis of the (quantized) universal enveloping algebra. Cluster algebra is determined by combinatorial cluster structures consisting of infinite set of seeds and mutation rules. Each seed (of cluster algebra of geometric type) consists of the collection of $n$ cluster variables $X_i$, $i \in [1,n]$ (elements of quantum torus (see Section 2.2 for detailed description)) and an $n \times n$ integer skew-symmetric exchange matrix $B$. Seeds are connected by mutations. Each cluster algebra is equipped with the compatible Poisson structure [24] and allows quantization [1].

For $k \in [1,n]$ a quantum cluster mutation $\mu_k$ is defined as the following transformations of seed (see [18])

$$
\mu_k(X_j) = \begin{cases} 
X_k^{-1}, & \text{if } j = k \\
X_j \prod_{a=1}^{b_{jk}} \left( 1 + q^{2a-1} X_k^{- \text{sgn}(b_{jk})} \right)^{-b_{jk}}, & \text{otherwise}.
\end{cases}
$$

Mutation of exchange matrix is described by $\mu_k(b_{ij}) = \begin{cases} 
-b_{ij}, & \text{if } i = k \text{ or } j = k; \\
b_{ij} + \frac{b_{ik}b_{kj} + b_{ik}b_{kj}}{2}, & \text{otherwise}.
\end{cases}$

An integer skew-symmetric $n \times n$ matrix is encoded by exchange quiver with $n$ vertices labeled 1 to $n$ and multiple arrows. If $k, \ell \in [1,n]$ are indices such that $b_{k\ell} > 0$ then vertex $k$ is connected to $\ell$ by $b_{k\ell}$ arrows. Exchange matrix mutations correspond to mutations of exchange quiver.

2. $SL_n$-algebras for the triangle $\Sigma_{0,1,3}$

Let $\Sigma_{g,s,p}$ denote a topological genus $g$ surface with $s$ boundary components and $p$ marked points. In what follows, we concentrate on the case of the disk with 3 marked points on the boundary $\Sigma_{0,1,3}$. (To simplify notations, we use $\Sigma = \Sigma_{0,1,3}$.) In this section we review the definition of quantized moduli space $\mathcal{X}_{SL_n, \Sigma}$ closely related to the $SL_n$-local systems on the disk with three marked points ([16]). We call disk with three marked points $a, b, c$ triangle with vertices $a, b, c$ and use notation $S_{abc}$ (see Fig 1).

![Figure 1. Triangle $S_{1,2,3}$ with vertices 1, 2, 3.](image)

2.1. Moduli space $\mathcal{X}_{SL_n, \Sigma}$ and transport matrices. Following [16], recall that a moduli space $\mathcal{X}_{SL_n, \Sigma}$ parametrizes $SL_n$-local systems on $\Sigma$ or, equivalently, triples of flags in $\mathbb{C}^n$ in general position. Any $SL_n$-local system in the triangle $S_{123}$ determines monodromy (or transport) matrices.
Two transport matrices $M_1$ and $M_2$ correspond respectively to directed paths going from one side of $\Sigma$ to the other side as on Fig 1. If $M$ is associated to a directed path then the inverse matrix $M^{-1}$ corresponds to the same path in the opposite direction.

Consider a triple of flags in general position $(F_1)_\bullet, (F_2)_\bullet, (F_3)_\bullet$ in $\mathbb{C}^n$. Recall that a \textit{complete flag} $F_\bullet$ is a collection of consecutively embedded subspaces $\{0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \cdots \subset F_n-1 \subset F_n = \mathbb{C}^n\}$ where $F_k$ is a linear subspace of dimension $k$. Denote by $F^a = F_{n-a}$, $a = 0, 1, \ldots, n$, the vector subspace of codimension $a$.

Flag $(F_i)_\bullet$ is assigned to vertex $i$ of triangle $S_{123}$, for all $i \in [1,3]$. Consider the subtriangulation of $S_{123}$ into $\binom{n}{2}$ white upright triangles and $\binom{n-1}{2}$ upside-down black triangles (see Fig 2). Label all white upright triangles by triples $\{(a, b, c) | a, b, c \geq 0 \& a + b + c = n - 1\}$ such that subtriangle $(n - 1, 0, 0)$ is closer to vertex 1 of $S_{123}$, subtriangle $(0, n - 1, 0)$ is closer to vertex 2 and subtriangle $(0, 0, n - 1)$ is closer to vertex 3. Each white triangle $(a, b, c)$ corresponds to a line $\ell_{abc} = (F_1)^a \cap (F_2)^b \cap (F_3)^c$. Similarly, label black upside-down triangles by triples $\{(abc) | a, b, c \geq 0 \& a + b + c = n - 2\}$ where the labels satisfy similar agreement as labels of white subtriangles. Each upside-down (black) triangle $(\alpha, \beta, \gamma)$ is associated with the plane $P_{abc} = (F_1)^a \cap (F_2)^b \cap (F_3)^c$. Note that every plane $P_{abc}$ of a black triangle contains all three lines $\ell_{(a+1)bc}, \ell_{a(b+1)c}, \ell_{ab(c+1)}$ of white triangles which are neighbors of the black one. For every such plane $P_{abc}$ choose three vectors $v_{(a+1)bc} \in \ell_{(a+1)bc}, v_{a(b+1)c} \in \ell_{a(b+1)c}, v_{ab(c+1)} \in \ell_{ab(c+1)}$ such that they satisfy condition $v_{(a+1)bc} + v_{a(b+1)c} = v_{ab(c+1)}$. Hence, given a configuration of lines corresponding to triple of flags $(F_1)_\bullet, (F_2)_\bullet, (F_3)_\bullet$, the choice of one vector $v_{abc} \in \ell_{abc}$ determines uniquely all other vectors in the lines $\ell_{a'b'c'}$ for all $(a'b'c')$ (see Fig 3).

Thus, the configuration of lines $\ell_{abc}$ determines \textit{projective collection of vectors} $\{v_{abc}\}$ modulo scalar scaling. Note that exactly two vectors at vertices of any gray triangle are independent.

Define a snake as an oriented path connecting the top line $\ell_{000}$ downwards to the bottom $\ell_{0ab}$ (red path in Fig 3 is an example of a snake).

Any snake defines a projective basis $v_{\alpha_1}, \ldots, v_{\alpha_n}$ of $\mathbb{C}^n$. Note that changing orientation replaces the basis to opposite one $v_{\alpha_n}, -v_{\alpha_{n-1}}, \ldots, (-1)^{n-1}v_{\alpha_1}$. For instance, the red snake top to bottom in Fig 3 defines the basis $v_{200}^0, v_{110}, v_{011}$.

Denote by $b_p$ the basis defined by snake $p$. Let $b_{12}$ be the basis defined by the unique snake from $\ell_{000}$ to $\ell_{001}$ in the triangle $S_{123}$ and by $b_{31}$ the basis defined by the snake $\ell_{001}$ to $\ell_{000}$. The basis $b_{12}$ in Fig 3 is $b_{12} = (v_{200}, v_{110}, v_{020})$, the basis $b_{31} = (v_{002}, v_{101}, v_{200})$. Note that the snake and the basis depends on the triangle in which the snake is considered. The basis associated with the red snake in the triangle $S_{125}$ (see Fig 4) is $b_{12}' = \{(v_{200}', v_{110}', v_{020}')\}$. Note that vectors $v_{abc} \in b_{12}$ and $v_{abc}' \in b_{12}'$ are scalar multiple of each other. Similarly, we consider
Figure 3. Configuration of lines corresponding to triple of flags in $\mathbb{C}^3$. Black triangles are equipped with planes $P_{abc}$. Plane $P_{00}$ contains lines $\ell_{200}, \ell_{110}, \ell_{101}, P_{010}$ contains lines $\ell_{110}, \ell_{020}, \ell_{011}, P_{001}$ contains lines $\ell_{101}, \ell_{011}, \ell_{002}$. Vectors $v_{abc} \in \ell_{abc}$ satisfy relations $v_{101} = v_{200} + v_{110}, v_{020} = v_{101} + v_{011}, v_{011} = v_{110} + v_{002}$.

the basis $b''_{31} = (v''_{002}, v''_{101}, v''_{200})$ associated with the snake in $S_{134}$; vector $v''_{abc} \in b''_{31}$ is a scalar multiple of the vector $v_{abc} \in b_{31}$.

Figure 4. Configuration of lines corresponding to triple of flags in $\mathbb{C}^3$. Black triangles are equipped with planes $P_{abc}$. Plane $P_{100}$ contains lines $\ell_{200}, \ell_{110}, \ell_{101}, P_{010}$ contains lines $\ell_{110}, \ell_{020}, \ell_{011}, P_{001}$ contains lines $\ell_{101}, \ell_{011}, \ell_{002}$. Vectors $v_{abc} \in \ell_{abc}$ satisfy relations $v_{101} = v_{200} + v_{110}, v_{020} = v_{101} + v_{011}, v_{011} = v_{110} + v_{002}$. Lines in triangle $S_{125}$ are denoted by $\ell''_{abc}$; line in triangle $S_{135}$ are denoted by $\ell''_{abc}$.

Define $T_1 \in SL_n$ as the transformation matrix from basis $b''_{31}$ to $b'_{12}$, namely, $i$-th column of $T_1$ is $[(b''_{31})_i]_{b'_{12}}$, i.e. coordinate vector $(b''_{31})_i$ with respect to basis $b'_{12}$. Because $b''_{31}$ and $b'_{12}$ are both defined up to multiplicative scalars, $T_1$ is defined up to scalar too. Hence, condition $T_1 \in SL_n$ fixes $T_1$ uniquely.

Similarly, we define the transport matrix $T_2$ as transformation matrix from basis $b'_{12}$ to $b''_{23}$ and $T_3$ as transformation from $b''_{23}$ to $b''_{31}$, where $b''_{23}$ corresponds to the snake in the triangle sharing the common side $2-3$ with $S_{123}$.

Finally, let $M_1 = T_1, M_2 = T_2^{-1}$ (see Fig. 2).
Matrix $M_1$ is an upper-anti-diagonal matrix and $M_2$ is a lower-anti-diagonal matrix (see Example 2.2).

Fock-Goncharov coordinates $Z_\alpha$ parametrize $X_{\mathcal{SL}_n, \Sigma}$. They are associated with vertices of triangular subdivision of $\Sigma$ except vertices 1, 2, 3 of triangle and labelled using barycentric indices $(i,j,k), i + j + k = n$ denoted often below by Greek letters (see Fig. 2 for $n = 3$ and Fig. 6 for $n = 6$). The expressions for classical transport matrices were first found in [19], (see also [13] Appendix A.2).

One can factorize the transport matrix $T_1$ in a sequence of elementary basis changes corresponding to the following sequence of snake transformations.

![Figure 5. Sequence of snakes factorizing transport matrix T1](image)

Let $\chi(k)$ denote the integer step function, $\chi(k) = 0$, if $k < 0$, and 1, if $k \geq 0$. Define $n \times n$ matrices $H_k(t)$, $L_k$, $S$ as follows $H_k(t) = t^{-n-k} \text{diag}(t^\chi(1-k-1), t^\chi(2-k-1), \ldots, t^\chi(n-k-1))$, $L_k = \text{Id}_n + E_k$ for $k \in [1, n - 1]$ where $\text{Id}_n$ is the identity $n \times n$ matrix, $(E_k)_{i,j} = \delta_{k+1,i} \cdot \delta_{k,j}$ is the matrix whose only nonzero element is 1 at the position $(k + 1, k)$, $(S)_{i,j} = (-1)^{n-k} \delta_{i,n+1-j}$.

Then, $T_1 = S \left[ \prod_{j=1}^{n-1} H_{n,j}(Z,j,0,n-j) \right] \cdot L_{n-1} \prod_{p=1}^{n-2} \left[ \prod_{q=1}^{p} L_{n-q-1} H_{n,q}(Z_{n-q,p+g+1-n,n-1-p}) \right] L_{n-1}$

Here, the factor $L_{n-1} \prod_{p=1}^{n-2} \left[ \prod_{q=1}^{p} L_{n-q-1} H_{n,q}(Z_{n-q,p+g+1-n,n-1-p}) \right] L_{n-1}$ is a transformation from the basis $b_{13}$ to $b_{12}$; $\prod_{j=1}^{n-1} H_j(Z,j,0,n-j)$ is a transformation from $b_{12}$ to $b_{12}';$ finally, $S \left[ \prod_{j=1}^{n-1} H_{n,j}(Z,j,0,n-j) \right]$ is a transformation from $b_{12}'$ to $b_{13}$.

Let $\mathbb{I} = \{(a,b,c) | a, b, c \in \mathbb{Z}_n, a + b + c = n \}$ be the set of barycentric indices in the triangle with side $n$, $\tau : \mathbb{I} \rightarrow \mathbb{I}$ be the clockwise rotation by $2\pi/3$, $\tau$ acts naturally on the sequences of barycentric parameters and hence on sequences of Fock-Goncharov parameters: for $Z = (Z_{\alpha_1}, \ldots, Z_{\alpha_k})$ the sequence $\tau Z = (Z_{\tau(\alpha_1)}, \ldots, Z_{\tau(\alpha_k)})$, for any object $O(Z)$ depending on the collection $Z = (Z_{\alpha_i})_{i=1}^k$ we set $\tau O = O(\tau Z)$.

Note that $T_2 = \tau T_1$, $T_3 = \tau^2 T_1$ (see Fig. 2). The transport matrix $M_2 = (\tau M_1)^{-1}$.

Example 2.1. For $n = 3$, we have $H_1(t) = \begin{pmatrix} t^{-2/3} & 0 & 0 \\ 0 & t^{1/3} & 0 \\ 0 & 0 & t^{1/3} \end{pmatrix}$, $H_2(t) = \begin{pmatrix} t^{-1/3} & 0 & 0 \\ 0 & t^{-1/3} & 0 \\ 0 & 0 & t^{2/3} \end{pmatrix}$.

$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Transport matrices $T_1$ from side 1 to 2 to side 3 to 1, $T_2$ from side 2 to 3 to side 1 to 2 and $T_3$ from side 3 to 1 to side 2 to 3 (see Fig. 2) have the following form

$M_1 = T_1 = \text{SH}_2(Z_{201})H_1(Z_{102})L_2L_1H_2(Z_{111})L_2H_1(Z_{012})H_2(Z_{021})$

$T_2 = \text{SH}_2(Z_{120})H_1(Z_{210})L_2L_1H_2(Z_{111})L_2H_1(Z_{012})H_2(Z_{021})$

$T_3 = \text{SH}_2(Z_{021})H_1(Z_{012})L_2L_1H_2(Z_{111})L_2H_1(Z_{102})H_2(Z_{201})$. 

DARBOUX COORDINATES FOR SYMPLECTIC GROUPOID AND CLUSTER ALGEBRAS 9
Then, $M_1$ is the matrix

\[
\begin{pmatrix}
Z_{021}^{-1/3}Z_{102}^{1/3}Z_{111}^{-1/3}Z_{012}^{-2/3}Z_{201}^{2/3} & Z_{021}^{-1/3}Z_{103}^{1/3}Z_{111}^{-1/3} + Z_{111}^{1/3}Z_{111}^{2/3}Z_{201}^{2/3} & Z_{021}^{-1/3}Z_{102}^{1/3}Z_{112}^{2/3}Z_{201}^{2/3} \\
Z_{021}^{-1/3}Z_{103}^{1/3}Z_{111}^{-1/3}Z_{012}^{-2/3}Z_{201}^{-1/3} & Z_{021}^{-1/3}Z_{103}^{1/3}Z_{112}^{1/3}Z_{012}^{-2/3}Z_{201}^{-1/3} & Z_{021}^{-1/3}Z_{102}^{1/3}Z_{111}^{1/3}Z_{112}^{2/3}Z_{201}^{-1/3} \\
Z_{021}^{-1/3}Z_{102}^{1/3}Z_{111}^{-1/3}Z_{012}^{-2/3}Z_{201}^{-1/3} & 0 & 0
\end{pmatrix}.
\]

Recall, $M_2 = T_2^{-1}$. We can easily factorize $M_2$ in the product of elementary matrices noting that $S^{-1} = (-1)^{n-1}S$, $H_k(t)^{-1} = H_k(-t)^{-1} = S H_{n-k}(t) S$, $L_k^{-1} = I d_n - E_k = SL_{n-k}^I S$, where $L_j^I$ is the transpose of $L_j$. Therefore,

\[
M_2 = H_2(Z_{210})^{-1}H_1(Z_{120})^{-1}L_2^{-1}H_2(Z_{111})^{-1}L_1^{-1}L_2^{-1}H_1(Z_{201})^{-1}H_2(Z_{021})^{-1}S^{-1} \\
= SH_1(Z_{210})SSH_2(Z_{120})SSH_1(Z_{111})SSLS_1^ISSH_2(Z_{021})SSH_1(Z_{021})S(-1)^{n-1} \\
= (-1)^{n-1}SH_1(Z_{210})H_2(Z_{120})L_1H_1(Z_{111})L_2L_1H_2(Z_{021})H_1(Z_{021}).
\]

Then, $M_2$ is the following matrix

\[
\begin{pmatrix}
0 & Z_{102}^{1/3}Z_{111}^{1/3}Z_{012}^{-1/3}Z_{201}^{1/3} & 0 & Z_{102}^{1/3}Z_{111}^{1/3}Z_{012}^{-1/3}Z_{201}^{1/3} \\
0 & Z_{102}^{-1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{-1/3}Z_{021}^{1/3} & Z_{102}^{-1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{-1/3}Z_{021}^{1/3} & Z_{102}^{-1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{-1/3}Z_{021}^{1/3} \\
0 & Z_{102}^{-1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{-1/3}Z_{021}^{1/3} & Z_{102}^{-1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{-1/3}Z_{021}^{1/3} & Z_{102}^{-1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{-1/3}Z_{021}^{1/3}
\end{pmatrix}.
\]

To obtain quantum transport matrices we need to expand all entries of classical transport matrix $M_i$ in the sum of monomials $m_j$ and then replace all $m_j$ by the corresponding Weyl form $\mathfrak{m}_j$. For instance, the $(1,2)$-entry of quantum $M_1$ becomes

\[ (M_1)_{12} = Z_{021}^{-1/3}Z_{102}^{1/3}Z_{111}^{-1/3}Z_{012}^{-1/3}Z_{120}^{1/3}Z_{021}^{1/3} + Z_{102}^{1/3}Z_{111}^{1/3}Z_{012}^{1/3}Z_{120}^{1/3}Z_{021}^{1/3}Z_{120}^{1/3}Z_{021}^{1/3}.
\]

In Section 2.3 we generalize this construction to (quantum) non-normalized transport matrices that can be defined for more general class of planar quivers.

**Example 2.2.** A toy example is the one in which all $Z_\alpha$ are the units. Matrix entries then just count numbers of monomials entering the corresponding matrix elements $a_{i,j} \in (-1)^{i+1}Z_+[[Z_\alpha^{\pm 1}]]$. Then, for the $M_1$ matrix, we have the following representation:

\[ M_1 = \begin{pmatrix}
1 & 2 & 1 \\
-1 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
1 & 3 & 3 & 1 \\
-1 & -2 & -1 & 0 \\
1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \text{etc},
\]

that is, $[M_1]_{i,j} = (-1)^{i+1}\binom{n-i}{j}$ for $SL_n$. We introduce the antidiagonal unit matrix $S = \delta_{i,n+1-i}$ to distinguish it from $S = (-1)^{i+1}\delta_{i,n+1-i}$.

For $M_2$ we have

\[ M_2 = M_2^2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & -1 \\
1 & 2 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1 \\
0 & 1 & 2 & 1 \\
-1 & -3 & -3 & -1
\end{pmatrix}, \text{etc}
\]

A riddle-thirsty reader can check the following relations between these matrices:

\[ M_1^2 = M_2 = (-1)^{n+1}|S| \cdot M_1 \cdot |S|, \quad M_3 = (-1)^{n+1}|S| \cdot M_1^2 = (-1)^{n+1}I
\]

\[ M_1^T M_2 = A = \begin{pmatrix}
1 & 3 & 3 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 4 & 6 & 4 \\
0 & 1 & 4 & 6 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 1
\end{pmatrix}, \text{etc}
\]
that is $[A]_{i,j} = \left[ \begin{array}{cc} n \\ j-i \end{array} \right]$. 

2.2. Quantum torus. Let lattice $\Lambda = \mathbb{Z}^n$ be equipped with a skew-symmetric integer form $\langle \cdot, \cdot \rangle$. Introduce the $q$-multiplication operation in the vector space $\Upsilon = \text{Span}\{Z_\lambda\}_{\lambda \in \Lambda}$ as follows $Z_\lambda Z_\mu = q^{\langle \lambda, \mu \rangle}Z_{\lambda+\mu}$. The algebra $\Upsilon$ is called a quantum torus. Fix a basis $\{e_i\}$ in $\Upsilon$, we consider $\Upsilon$ as a non-commutative algebra of Laurent polynomials in variables $Z_i = Z_{e_i}$, $i \in [1, N]$. For any sequence $a = (a_1, \ldots, a_t)$, $a_i \in [1, N]$, let $\Pi_s$ denote the monomial $\Pi_s = Z_{a_1}Z_{a_2} \ldots Z_{a_t}$. Let $Z_{\lambda s} = \sum_{j=1}^t e_{a_j}$. Element $Z_{\lambda s}$ is called in physical literature the Weyl form of $\Pi_s$ and we denote it by two-sided colons $\Pi_s \cdot \cdot$ It is easy to see that $\Pi_s \cdot \cdot = Z_{\lambda s} = q^{-\sum_{j<k} \langle e_j, e_k \rangle} \Pi_s$. 

2.3. Quantum transport matrices and quantum Fock-Goncharov coordinates. Quantum transport matrices were introduced in [39]. We describe now how quantized transport matrices are expressed in terms of quantum Fock-Goncharov parameters (see also [13], [17] and [39]).

The quantum Fock-Goncharov variables form a quantum torus $\Upsilon$ with commutation relation described by the quiver shown on Fig. 6. Vertices of the quiver label quantum Fock-Goncharov coordinates $Z_\alpha$ (we use Greek letters to indicate barycentric labels) while the arrows encode commutation relations: if there are $m$ arrows from vertex $\alpha$ to $\beta$ then $Z_\beta Z_\alpha = q^{-2m}Z_\alpha Z_\beta$. Dashed arrow counts as $m = 1/2$. In particular, a solid arrow from $Z_\alpha$ to $Z_\beta$ implies $Z_\beta Z_\alpha = q^{-2}Z_\alpha Z_\beta$, a dashed arrow from $Z_\alpha$ to $Z_\beta$ implies $Z_\beta Z_\alpha = q^{-1}Z_\alpha Z_\beta$, and, for the future use, a double arrow from $Z_\alpha$ to $Z_\beta$ means $Z_\beta Z_\alpha = q^{-4}Z_\alpha Z_\beta$. Vertices not connected by an arrow commute.

![Figure 6. The quiver of Fock-Goncharov parameters for $n = 6$; note that vertices (6,0,0), (0,6,0), and (0,0,6) are missing.](image)

Consider the following planar oriented graph in the triangle $S_{123}$ dual to the quiver above. Label vertices on the left, on the right and on the bottom sides from 1 to $n$ as shown on the Figure 7. Barycentric indices label the vertices of the quiver which correspond to the faces of the dual oriented graph. Vertices of the new dual graph are colored black and white depending on whether there are inside a upright or upside-down triangle. Note that black vertex has two incoming arrows while white one has only one incoming arrow. Faces of $G$ are equipped with $q$-commuting weights $Z_\alpha$. We add also three face weights $Z_{n,0,0}$, $Z_{0,n,0}$, and $Z_{0,0,n}$ with the corresponding commutation relations (shown in red on Fig. 7).

Any maximal oriented path in the dual graph connects a vertex on the right side 1–2 of the triangle either with a vertex of the left side 1–3 or with a vertex on the bottom side 2–3. We
assign to every oriented path $P : j \leadsto i'$ from the right side to the left side or to the bottom side $P : j \leadsto i''$ the quantum weight $w(P) = \prod_{\text{face } \alpha \text{ lies to the right of the path } P} Z_{\alpha}$.

**Figure 7.** The oriented planar graph $G$ dual to the quiver of Fock-Goncharov parameters for $X_{\mathcal{S}L_6, \Sigma}$. Additional face weights $Z_{0,0,6}, Z_{0,6,0}, Z_{0,0,6}$ are painted red.

**Definition 2.3.** We define two $n \times n$ non-normalized quantum transition matrices

$$(\mathcal{M}_1)_{i,j} = \sum_{\text{directed path } P : j \leadsto i'} w(P) \quad \text{and} \quad (\mathcal{M}_2)_{i,j} = \sum_{\text{directed path } P : j \leadsto i''} w(P).$$

Note that each $\mathcal{M}_1$ is a lower-triangular matrix and $\mathcal{M}_2$ is an upper-triangular matrix.

In section 7.2 we generalize this definition. Let $\Gamma$ be a planar oriented graph in the rectangle with no sources or sinks inside (see Fig 28), $m$ univalent boundary sinks on the left labeled 1 to $m$ top to bottom and $n$ univalent boundary sources on the right labelled 1 to $n$ top to bottom. All arrows of $\Gamma$ are oriented right to left, in particular, $\Gamma$ has no oriented cycles. Note that this condition is, in particular, satisfied by the oriented planar graph $G$ (see Fig 7) considered as a graph with $n$ sources and $2n$ sinks. Indeed, we can redraw $G$ in a rectangle such that the right side of the triangle becomes the right vertical side of the rectangle while union of the left and the bottom sides becomes the left side of the triangle.

Faces of $\Gamma$ are equipped with $q$-commuting weights $Z_{\alpha}$ whose commutation relations are governed by the dual quiver (see Section 7.2 for details). We define weight of the maximal oriented path $P$ from a source $a$ to a sink $b$ as

$$(2.4) \quad w(P) = \prod_{\text{face } \alpha \text{ lies to the right of the path } P} Z_{\alpha}.$$

Then, entries of a $m \times n$ non-normalized transport matrix are $[\mathcal{M}]_{ij} = \sum_{\text{directed path } P : j \leadsto i} w(P)$.

Lemma 7.18 implies that the matrix $\mathcal{M}$ satisfies the quantum $R$-matrix relation $R_m(q)\mathcal{M} \otimes \mathcal{M} = \mathcal{M} \otimes \mathcal{M}R_n(q)$, where $R_k(q)$ is a $k^2 \times k^2$ matrix

$$(2.5) \quad R_k(q) = \sum_{1 \leq i,j \leq k} e_{ii} \otimes e_{jj} + (q-1) \sum_{1 \leq i \leq k} e_{ii} \otimes e_{ii} + (q^{-1} - 1) \sum_{1 \leq j < i \leq k} e_{ij} \otimes e_{ji}.$$
Note that $\mathcal{R}_k$ has the following properties:

\begin{equation}
\mathcal{R}_k^{-1}(q) = \mathcal{R}_k(q^{-1}), \quad \mathcal{R}_k(q) - \mathcal{R}_k^T(q^{-1}) = (q - q^{-1})P_k,
\end{equation}

where $P_k$ is the standard permutation matrix $P_k := \sum_{1 \leq i, j \leq k} \frac{1}{e_{ij} \otimes e_{ji}}$. The total transposition $\mathcal{R}_k^T(q)$ of $\mathcal{R}_k(q)$ results in interchanging the space labels $1 \leftrightarrow 2$.

In Fig. 7 we have an example of a directed network with 6 sources and 12 sinks. Adding additional face weights $Z_{000}$, $Z_{001}$, $Z_{00n}$ to the Fock-Goncharov quiver for $SL_n$ we noticed that the dual oriented planar graph $G$ can be redrawn in the rectangle with $n$ sources on the left and $2n$ sinks on the right (check Fig. 7 for $n = 6$ example); then $\mathcal{M}$ has a block matrix form $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2)$ in which we let $\mathcal{M}_1$ is the upper $n \times n$ block and $\mathcal{M}_2$ is the lower $n \times n$ block. We want to show that Lemma 7.18 implies the following commutation relations for $\mathcal{M}_1$ and $\mathcal{M}_2$:

\begin{equation}
\mathcal{R}_n(q) \mathcal{M}_i \otimes \mathcal{M}_i = \mathcal{M}_i \otimes \mathcal{M}_i \mathcal{R}_n(q), \quad i = 1, 2,
\end{equation}

and

\begin{equation}
\mathcal{M}_1 \otimes \mathcal{M}_2 = \mathcal{M}_2 \otimes \mathcal{M}_1 \mathcal{R}_n(q).
\end{equation}

We rewrite the above matrix $\mathcal{R}_{2n}(q)$ as

\[
\mathcal{R}_{2n}(q) = \sum_{1 \leq i, j \leq n} \frac{1}{e_{ii} \otimes e_{jj}} + (q - 1) \sum_{1 \leq i, j \leq n} \frac{1}{e_{ii} \otimes e_{ji}} + (q - q^{-1}) \sum_{1 \leq i, j \leq n} \frac{1}{e_{ij} \otimes e_{ji}}
\]

\[
+ \sum_{1 \leq i, j \leq n} \frac{1}{e_{n+i,n+i} \otimes e_{n+j,n+j}} + (q - 1) \sum_{1 \leq i \leq n} \frac{1}{e_{n+i,n+i} \otimes e_{n+i,n+i}}
\]

\[
+ (q - q^{-1}) \sum_{1 \leq j < i \leq n} \frac{1}{e_{n+i,n+j} \otimes e_{n+j,n+i}}
\]

In the first two lines we immediately recognize $\mathcal{R}_n(q)$ in two diagonal $n \times n$ blocks of $\mathcal{R}_{2n}(q)$: the relations for the pair of first indices $(i, j)$ and $(n + i, n + j)$ generate (2.7) for $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively; setting $(i, n + j)$, which corresponds to the third line, we obtain just a unit matrix in the left-hand side thus producing relation (2.8), whereas in the case $(n + i, j)$ (the fourth line), we have the equation

\[
\mathcal{M}_2 \otimes \mathcal{M}_1 + (q - q^{-1})P_n \mathcal{M}_1 \otimes \mathcal{M}_2 = \mathcal{M}_1 \otimes \mathcal{M}_2 \mathcal{R}_n(q).
\]

We first push the permutation operator $P_n$ through the $\mathcal{M}$-matrix product interchanging the labels of spaces in the direct product and then use the identity $(q - q^{-1})P_n = \mathcal{R}_n(q) - \mathcal{R}_n^T(q^{-1})$ obtaining

\[
\mathcal{M}_2 \otimes \mathcal{M}_1 + \mathcal{M}_1 \otimes \mathcal{M}_2(\mathcal{R}_n(q) - \mathcal{R}_n^T(q^{-1})) = \mathcal{M}_1 \otimes \mathcal{M}_2 \mathcal{R}_n(q),
\]

or

\[
\mathcal{M}_2 \otimes \mathcal{M}_1 = \mathcal{M}_1 \otimes \mathcal{M}_2 \mathcal{R}_n^T(q^{-1}),
\]

which is just another form of writing relation (2.8). This accomplishes the proof of relations 2.7 and 2.8. \qed
Remark 2.4. In [38] the cluster realization of the quantum group $U_q(SL_n)$ was constructed. Combined together with description in the form of planar networks in [39] it implies another proof of formula (2.7).

Now, in order to obtain the commutation relations for normalized transport matrices $M_1$ and $M_2$ we have to eliminate extra variables $Z_{n,0,0}$, $Z_{0,n,0}$ and $Z_{0,0,n}$. Henceforth, we normalize the matrices $M_i$, $i = 1, 2$ by multiplying them by corresponding monomials. More exactly, we multiply the matrix $M_1$ by $D_1^{-1}$, where $D_1$ is the function \( \alpha \) commuting with all elements of $M_1$. Clearly, $D_1^{-1}M_1$ will be independent of $Z_{n00}$. Similarly, we multiply $M_2$ by $\dot{D}_1^{-1}D_2^{-1}\cdot$\( \alpha \) where $D_2 = \tau^2D_1$ is the similar monomial that starts with the variable $Z_{00n}$. These multiplications preserve the form of relations (2.7) and their only effect on (2.7) is the appearance of the constant factor in the $R$-matrix in the right-hand side (this is because $D_1$ commutes with $\dot{D}_1^{-1}D_2^{-1}\cdot$\( M_2 \), and $\dot{D}_1^{-1}D_2^{-1}\cdot$ commutes with $D_1^{-1}M_2$; only $D_1$ and $D_2$ do not commute.

We now define the normalized quantum monodromy or normalized quantum transport matrices of the standard $SL_n$ quiver:

Definition 2.5. Normalized quantum transport matrices are defined as follows

$$M_1 = QS\mathcal{M}_1D_1^{-1}$$ and $M_2 = QS\mathcal{M}_2D_1^{-1}D_2^{-1}$

where $Q = \text{diag}\{(-q)^{n-1}\}$.

Note that

$$Q \otimes Q\mathcal{R}_n(q) = \mathcal{R}_n(q)Q \otimes Q$$

for any diagonal matrix $Q$ and

$$S \otimes S\mathcal{R}_n(q) = \mathcal{R}_n^T(q)S \otimes S$$

for any antidiagonal matrix $S$.

We have therefore proved the following theorem.

Theorem 2.6. The above $M_1$ and $M_2$ satisfy the relations

$$\mathcal{R}_n^T(q)M_i \otimes M_i = 2M_i \otimes M_i\mathcal{R}_n(q)$$

where

$$M_1 \otimes M_2 = 2M_2 \otimes M_1\mathcal{R}_n(q)$$

(2.9) $R_n(q) = q^{-1/n}\left[\sum_{i,j}^1e_{ii} \otimes e_{jj} + (q - 1)\sum_i^1e_{ii} \otimes e_{ii} + (q - q^{-1})\sum_{i>j}^1e_{ij} \otimes e_{ji}\right]$ is the quantum trigonometric $R$-matrix

Remark 2.7. G.Schrader and A.Shapiro obtained independently a proof of Theorem 2.6 using the cluster representation of the mapping class group action that will be published elsewhere [10].

Theorem 2.8. The quantum transport matrices $T_i$ satisfy the quantum groupoid relation

$$T_1T_2T_3 = \text{Id}.$$
We now consider the pattern in the figure below. We do not indicate arrows on edges recalling that all paths in $T_1$ go from right to left and from top to bottom whereas all paths in $T_3$ go from left to right and from top to bottom. Two paths: $j \to k$ from $T_1$ and $k \to i$ from $T_3$ share the common horizontal leg; if we remove this leg then the remaining part of the union of $j \to k$ and $k \to i$ is a path that first goes from right to left, then (in a general Case I) has the leftmost vertical edge, then goes from left to right. In a very special Case II, the path does not have the last part; this happens only for $k = n$ and only if the last horizontal part of the path $j \to k = n$ is strictly longer than the shared horizontal leg

![Diagram](image)

Case I

In Case I, given a path $j \to k$ encompassing the region $A$ and a path $k \to i$ encompassing the regions $B$ and $C$ we have the corresponding path $j \to k + 1$ encompassing the regions $A$ and $B$ and the path $k + 1 \to i$ encompassing the region $C$. These pairs of paths are in bijection being the only two possible combinations of paths having the same union of domains $A \cup B \cup C$. Contributions from these two pairs of paths have opposite signs and since $\mathcal{B} \mathcal{C} \mathcal{A} = q \mathcal{B} \mathcal{C} \mathcal{A}$ they are mutually canceled in the sum (2.10).

The only pairs of paths ($j \to k$, $k \to i$) that do not have counterparts are those for which the region $C$ is absent (Case II):

![Diagram](image)

Case II

In this case, $\mathcal{B} \mathcal{A} = \mathcal{B} \mathcal{A}$, $k$ is necessarily equal $n$, and after removing the common leg, all these pairs of paths are in bijection with single paths going from right to left and encompassing the regions $A$ and $B$; note that these paths are exactly paths constituting the matrix $M_2$! So the sum in (2.10) just gives the matrix $M_2$ (up to the factor $QS$, which we can now reconstruct). We have therefore proved that $T_3T_1 = M_2$
In Section 7.2 we present another proof of the groupoid property using quantum Grassmannian.

**Remark 2.10.** The semiclassical limit of Theorem 2.6 statement reads
\[
\{ M_1 \otimes M_2 \} = 2 \otimes M_1 - \frac{1}{n} I \otimes \hat{I} + r_n
\]
where
\[
r_n = \sum_i \frac{1}{2} e_{ii} \otimes e_{ii} + 2 \sum_{i > j} \frac{1}{2} e_{ij} \otimes e_{ji}
\]
is the semiclassical $r$-matrix. Equivalently,
\[
\{ [M_1]_{ab}, [M_2]_{cd} \} = -\frac{1}{n} [M_1]_{ab} [M_2]_{cd} + [M_1]_{ad} [M_2]_{cb} \theta (b - d), \quad \theta (x) = \begin{cases} 2, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases}
\]

**Remark 2.11.** Note that for the trigonometric $R$-matrix (2.9) the quantum relation
\[
R_n^T(q) M_i \otimes M_i = 2 \otimes M_i = 1 \otimes M_i R_n(q)
\]
has an equivalent form of writing
\[
M_i \otimes M_i R_n^T(q) = R_n(q) M_i \otimes M_i \quad \text{for } i = 1, 2.
\]
Both these relations generate the same quantum algebra on elements of the matrices $M_1$ and $M_2$ and have the same semiclassical limit
\[
\{ [M_i]_{ab}, [M_i]_{cd} \} = [M_i]_{ad} [M_i]_{cb} \theta (b - d) - \theta (a - c)) \quad i = 1, 2, \quad \theta (x) = \begin{cases} 2, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases}
\]

**Example 2.12.** For $n = 3$ direct computations show
\[
\frac{1}{2} M_1 \otimes M_2 = 2 \otimes \frac{1}{2} M_1 R_3(q),
\]
where
\[
R_3(q) = q^{-1/3}
\]
\[
\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-1} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\
\end{pmatrix}
\]
Similarly, for both $i = 1, 2$, we have
\[
R_n^T(q) M_i \otimes M_i = 2 \otimes M_i \otimes M_i R_3(q)
\]
Furthermore, direct computation shows $T_2 T_3 T_1 = \text{Id.}$
3. Goldman brackets and commutation relations between transport matrices

To obtain a full-dimensional (without zero entries) form of transport matrices we define transport matrices along more general paths.

Namely, let $D = (D, M)$ be a disk $D$ with four marked boundary points $M = \{A, B, D, C\}$ in clockwise order, $(\Delta ABC, \Delta BCD)$ be a triangulation of $D$ and we also assume clockwise orientation of all triangle sides inside every triangle. The space $\mathcal{X}_{SL_n, D}$ coincides with the space of quadruple of flags; one flag assigned to each marked point. Each oriented side of triangulation gives a projective basis in $\mathbb{C}^N$. This allows to define transport matrix for any pair of oriented sides as transition matrix for the pair of corresponding bases; the matrix $S$ acts by changing the orientation of the corresponding side. Let $T_{BC \leftarrow AB}$ be a transport in $\Delta ABC$ from side $AB$ to $BC$. Pay attention that the sides are oriented and the order of endpoints in side notation matters. Similarly, $T_{CB \leftarrow DC}$ is a transport matrix in $\Delta BCD$ from $DC$ to $CB$. Note that $T_{DC \leftarrow CB} = T_{CB \leftarrow DC}^{-1}$. Then, we define a transport $T_{DC \leftarrow AB}$ from $AB$ to $DC$ as $T_{DC \leftarrow AB} = T_{DC \leftarrow CB} S T_{BC \leftarrow AB} = T_{CB \leftarrow DC} S T_{BC \leftarrow AB}$. Crossing side $CD$ between two triangles add an extra factor $S = T_{CB \leftarrow BC}$ because by our agreement all triangles are oriented counterclockwise that implies the opposite orientation of the common side considered in two neighboring triangles that contain the side. Similarly, $T_{BD \leftarrow AB} = T_{BD \leftarrow CB} S T_{BC \leftarrow AB}$.

\[ \begin{array}{c}
\begin{array}{c}
T_{BC \leftarrow BA} \\
A \\
C
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
B \\
D
\end{array} \\
T_{BD \leftarrow CB}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
T_{CB \leftarrow DC}
\end{array}
\end{array} \]

Figure 8. Disk with four marked points.

In the quantum case, the cluster variables of triangle $\Delta ABC$ commute with those of triangle $\Delta BCD$, so the product of two Weyl-ordered monomials is itself a Weyl-ordered monomial, in which we perform an amalgamation of variables on the side $BC$. Due to the double action of the matrix $S$, the amalgamation of boundary (frozen) variables in neighbour triangles respects the surface orientation, so, we amalgamate pairwise variables on the sides $BC$ of the two triangles ordered in the same direction, from $B$ to $C$. After the amalgamation, we unfreeze the obtained new variables. Therefore, for any path along triangles that does not go more than once through any given triangle, the ordering is the Weyl ordering. It was explicitly demonstrated in [39] that the Weyl ordering is preserved by quantum mutations which now include mutations of amalgamated variables as well as mutations of variables in the interior of triangles.

We now show that the commutation relations from Theorem 2.6 together with the groupoid condition (Theorem 2.8) imply the commutativity relations and (semiclassical) Goldman brackets.

We begin with the pattern in the figure below.
We use the identities
\[ M_i^{-1} \otimes M_k^{-1} = R_n(q)M_k^{-1} \otimes M_i^{-1} \]
\[ M_j \otimes M_i = M_i \otimes M_j R^{-T}(q) \]
\[ R_n(q)S \otimes 2 = S \otimes 2 R_n(q)^T \]
for any antidiagonal matrix \( S \)

We then have
\[ [M_jSM_i^{-1}] \otimes [M_iSM_k^{-1}] = M_i \otimes M_j R_n^{-T}(q)S \otimes 2 R_n(q)M_k^{-1} \otimes M_i^{-1} \]
\[ = M_i \otimes M_j S \otimes 2 R_n^{-1}(q)R_n(q)M_k^{-1} \otimes M_i^{-1} = [M_iSM_k^{-1}] \otimes [M_jSM_i^{-1}], \]
so these two matrix products commute. This is consistent with the quantum mapping class group transformations: flipping \( BC \) edge separates the paths \( AB \rightarrow BD \) and \( AC \rightarrow CD \) into two adjacent triangles.

Consider now the case of two intersecting paths (we consider a single intersection inside a quadrangle).

We then have
\[ [M_jSM_i^{-1}] \otimes [M_iSM_k^{-1}] = M_i \otimes M_j R_n^{-T}(q)S \otimes 2 R_n^{-T}(q)M_i^{-1} \otimes M_k^{-1} \]
\[ = M_i \otimes M_j S \otimes 2 R_n^{-1}(q)R_n(q)M_k^{-1} \otimes M_i^{-1}. \]

Note that the combination
\[ R_n^{-1}(q)R_n^{-T}(q) = q^{2n} \left[ \sum_{i,j} \frac{1}{n} \frac{1}{2} \frac{1}{2} e_{i,j} e_{j,i} + (q^{-2} - 1) \sum_{i} \frac{1}{n} \frac{1}{2} \frac{1}{2} e_{i,i} e_{i,i} + \sum_{i>j} \frac{1}{n} \frac{1}{2} \frac{1}{2} e_{i,i} e_{j,j} \right] \]
does not admit a nice decomposition itself. However, in the semiclassical limit \( q \rightarrow 1 \) with \( q = e^{\hbar/2} \), it becomes
\[ R_n^{-1}(q)R_n^{-T}(q) = I \otimes I + \hbar \left[ \frac{1}{n} I \otimes I - P_n \right] + O(\hbar^2), \]
with $P_n$ the permutation matrix, so the term linear in $\hbar$ therefore gives rise to the Goldman bracket for $SL_n$ [25].

Let a polygon be triangulated into collection of triangles containing triangle $\Delta ABC$, let $EF$ be another side of triangulation different from sides of $\Delta ABC$, let $\gamma$ be a path connecting $EF$ to $AB$ crossing any side of triangulation at most once and crossing neither $AC$ nor $BC$ (see Figure 9). Denote by $T_\gamma = T_{BA\leftarrow EF}$ the composition of transport matrices along the path $\gamma$. Figure 9). Denote by $T_{\gamma}$, define transport matrices $M_1 = T_{AB\leftarrow CA}^{-1} T_\gamma$, $M_2 = T_{BC\leftarrow AB} T_\gamma$.

![Figure 9. $M_1 = T_{CA\leftarrow AB} ST_{BA\leftarrow EF}$, $M_2 = T_{BC\leftarrow AB} ST_{BA\leftarrow EF}$.](image)

**Theorem 3.1.** The transport matrices $M_1$ and $M_2$ in Fig. 9 satisfy the commutation relations

$$M_1 \otimes M_2 = M_2 \otimes M_1 R_n(q),$$

$$R_n^T(q) M_i \otimes M_i = M_i \otimes M_i R_n(q) \quad \text{for } i = 1, 2.$$  

and

$$M_2^{-1} M_{BC\leftarrow CA} M_1 = 1.$$  

**Proof.** Consider a transport matrix corresponding to a path not passing twice through the same triangle. It is given by the matrix product $M_{i_{m-1}}^{(m-1)} S \cdots S M_{i_2}^{(2)} S M_{i_1}^{(1)} = T_{BA\leftarrow EF}$ where $i_k = 1, 2$ and variables of $M_{i_k}^{(k)}$ and $M_{i_p}^{(p)}$ commute for distinct $k$ and $p$. Every such product satisfies the relation $R_n^T(q) T \otimes T = T \otimes TR_n(q)$. Then,

$$M_1^{(m)} S T \otimes M_2^{(m)} S T = M_1^{(m)} \otimes M_2^{(m)} S \otimes ST \otimes T = M_2^{(m)} \otimes M_1^{(m)} R_n(q) S \otimes ST \otimes T = M_2^{(m)} S \otimes M_1^{(m)} S R_n^T(q) T \otimes T.$$

General algebras of transport matrices in an ideal triangle decomposition of $\Sigma_{g,s,p}$—a genus $g$ Riemann surface with $s$ holes and $p > 0$ marked points on the hole boundaries (note that an external edge of an ideal triangle decomposition corresponds to a starting/terminating edge of a path, so we allow only paths starting and terminating at these edges). Transition matrices in this pattern enjoy Fock–Rosly Poisson algebra [20] also considered in [11].

4. **Reflection equation and groupoid of upper triangular matrices**

4.1. **Factorization of an upper-triangular bilinear form $A$.** We assume again that both $M_1$ and $M_2$ are triangular matrices.
The main theorem concerns a special combination of \( M_1 \) and \( M_2 \):

\[
\mathbb{A} := M_1^T M_2.
\]

Note that the transposition in the quantum case is formal: the quantum ordering is preserved, only matrix elements are permuted. Also, since \( M_1 \) and \( M_1^T \) are upper-anti-diagonal matrices and \( M_2 \) is a lower-anti-diagonal matrix, the matrix \( \mathbb{A} \) is automatically upper-triangular.

**Theorem 4.1.** The matrix \( \mathbb{A} = M_1^T M_2 \) satisfies the quantum reflection equation

\[
R_n(q) \frac{1}{2} R_n^T(q) R_n(q) A = \frac{1}{2} A R_n^T(q) \frac{1}{2} R_n(q) A,
\]

with the trigonometric R-matrix (2.9), where \( R_n^T(q) \) is a partially transposed (with respect to the first space) R-matrix.

**Proof.** The proof is a short direct calculation that uses only R-matrix relations. Note that transposing with respect to the first space the second relation in Theorem 2.6 we obtain

\[
M_1^T \otimes M_2 = M_2 R_n^T(q) M_1^T
\]

and the total transposition of the first relation gives

\[
R_n(q) M_1^T \otimes M_2^T = M_2^T \otimes M_1^T R_n^T(q),
\]

so

\[
R_n(q) M_1^T M_2^T R_n^T(q) M_1^T M_2 = R_n(q) M_1^T M_2^T R_n^T(q) M_2^T M_1 = M_1^T R_n^T(q) M_2^T M_2
\]

\[
= M_1^T M_2^T R_n(q) M_1^T M_2 R_n(q) = M_1^T M_2^T R_n^T(q) M_1^T M_2 R_n(q),
\]

which completes the proof. \( \square \)

We thus conclude that we have a Darboux coordinate representation for operators satisfying the reflection equation. Moreover all matrix elements of \( \mathbb{A} \) are Laurent polynomials with positive coefficients of \( Z_\alpha \) and \( q \). In particular, positive integers in equation (2.3) count numbers of monomials in the corresponding Laurent polynomials.

By construction of quantum transport matrices in Sec. 2.3 all matrix elements of \( M_1 \) and \( M_2 \) are Weyl-ordered. For \( [\mathbb{A}]_{i,j} = \sum_{k,i} [M_1]_{k,i} [M_2]_{k,j} \) we obtain that for \( i < j \), \([M_1]_{k,i}\) commutes with \([M_2]_{k,j}\) (all paths contributing to \([M_1]_{k,i}\) are disjoint from all paths contributing to \([M_2]_{k,j}\) for \( i < j \), so the corresponding products are also Weyl-ordered, \( [\mathbb{A}]_{i,j} = \sum_{k,i} [M_1]_{k,i} [M_2]_{k,j} \).

For \( i = j \), the corresponding two paths share the common starting edge, and then \( [\mathbb{A}]_{i,i} = q^{-1/2} [M_1]_{i,i} [M_2]_{i,i} \). This explains the appearance of \( q^{-1/2} \) factors on the diagonal of the quantum matrix \( \mathbb{A}_\alpha \) (see (1.11), [8]). Note that all Weyl-ordered products of \( Z_\alpha \) are self-adjoint and we assume that all Casimirs are also self-adjoint.

To obtain a full-dimensional (not upper-triangular) form of the matrix \( \mathbb{A} \) let us consider adjoint action by any transport matrix:

**Theorem 4.2.** Any matrix \( \mathbb{A}' := M_1^T S T \mathbb{A} S M_1 \) where \( M_1 \) is a (transport) matrix satisfying commutation relations of Theorem 2.6 and commuting with \( \mathbb{A} = M_1^T M_2 \) satisfies the quantum reflection equation of Theorem 4.1.

The proof is again a direct computation; note also that if we represent \( \mathbb{A} = M_1^T M_2 \), then \( M_1' = M_1 S M_1 \) and \( M_2' = M_2 S M_1 \) satisfy commutation relations from Theorem 3.1 and \( \mathbb{A}' := M_1^T M_2' \) then satisfy the quantum reflection equation.
5. The quiver for an upper-triangular $A_n$ and the braid-group action

The goal in this section is, first, to construct the quiver corresponding to the Darboux coordinates of the upper-triangular matrix $A_n$ and, second, to present the braid-group action on elements of $A_n$ via chains of mutations in the newly constructed quivers.

5.1. $A_n$-quiver. We begin by considering in more details the structure of matrix entries of the product $M^T_1 M_2$. Up to inessential numerical factor, we observe that $(M^T_1 M_2)_{i,j} \sim \sum_k (M_1)_{k,i} (M_2)_{k,j}$ for non-normalized transport matrices (see Definition 7.4), and matrix entries $(M_1)_{k,i}$ and $(M_2)_{k,j}$ have coinciding monoidal structures in the respective boundary (frozen) variables $Z_{(k,0,n-k)}$ and $Z_{(n-k,k,0)}$ (see Fig. 6). In the matrix product $M^T_1 M_2$ we then have the dependence only on the pairwise products of these variables, which we therefore amalgamate to obtain a single new variable $\tilde{Z}_k := Z_{(k,0,n-k)} Z_{(n-k,k,0)}$. This results in a “twisted” pattern shown in Fig. 14. Another outcome of this amalgamation procedure is that the resulting quiver admits $n$ new Casimirs depicted in Fig. 15: exactly one such Casimir per every remaining frozen variable $Z_{(0,i,n-i)}$. We can then replace $Z_{(0,i,n-i)}$ by the corresponding commuting product of elements,

$$Z_{(0,i,n-i)} \rightarrow Z_k^2 \prod_{j=1}^{i-1} Z_{j,i-j,n-i} \prod_{j=1}^{n-i-1} Z_{j,i,n-i-j}.$$  

Because this product is a Casimir, it becomes an isolated vertex in the obtained quiver. We eventually unfreeze the variables $Z_k$, and then the connected part of the resulting quiver, which we call the $A_n$ quiver, contains only unfrozen variables. A convenient, and the most symmetric, way of drawing this quiver is to “cut out” the half-sized triangle located in the left-lower corner, then reflect this small triangle through the diagonal passing through its left-lower corner preserving the incidence relations for arrows in the both parts of the quiver, then glue pairwise the amalgamated variables $Z_{(k,0,n-k)}$ and $Z_{(n-k,k,0)}$. Amalgamation operations become planar in this procedure schematically depicted below for $A_5$.

Note that $[n/2]$ original Casimirs give rise to $[n/2]$ Casimir elements

$$C_k = \prod_{i=1}^{n-k-1} Z_{(k,i,n-k-i)} \prod_{j=1}^{k-1} Z_{(n-k,i,j)}.$$  

of the $A_n$ quiver (vertices of the same color contribute to the same quiver).

We present the resulting quivers for $n = 3, 4, 5, 6$ (they look aesthetically nicer for odd $n$). In the figure, vertices of the same color enter the same Casimir element and we have $[n/2]$
independent Casimir elements depicted in Fig. 16 in every quiver.

Since all Casimirs of $A_n$ are generated by $\lambda$-power expansion terms for $\det(A_n + \lambda A_n^T)$, we automatically obtain the following lemma

**Lemma 5.1.** $\det(A_n + \lambda A_n^T) = P(C_1, \ldots, C_{[n/2]})$, where $C_i$ are Casimirs of the $A_n$-quiver.

**Figure 10.** Fat graphs $\Gamma_{1,1}$ and $\Gamma_{1,2}$ corresponding to the respective quivers for $A_3$ and $A_4$. We indicate closed paths that yield elements $a_{1,2}$ of $A_3$ and $a_{1,2}$ and $a_{1,3}$ of $A_4$. For example, setting all $Z_\alpha = 1$, using formulas from [6] for the corresponding geodesic functions we obtain $\text{tr}(LR) = 3$ for $a_{1,2}$ in $A_3$ and $\text{tr}(LLR) = 4$ and $\text{tr}(LLRR) = 6$ for the respective $a_{1,2}$ and $a_{1,3}$ in $A_4$, where $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are matrices of respective left and right turns undergoing by paths at three-valent vertices of a spine. It is easy to see that the above values of $a_{i,j}$ coincide with those in Example 2.2.

**Remark 5.2.** In the cases $n = 3$ and $n = 4$, the constructed quivers are those of geometric systems: these cases admit three-valent fat-graph representations in which $Y$-cluster variables are identified with (exponentiated) Thurston shear coordinates $z_\alpha$ enumerated by edges of the corresponding graphs, and nontrivial commutation relations are between variables on adjacent edges; for $n = 3$ and $n = 4$ these graphs are the relative spines of Riemann surfaces $\Sigma_{1,1}$ and $\Sigma_{1,2}$ depicted in Fig. 10, the Laurent polynomials for entries of $A_n$ coincide up to a linear change of log-canonical variables with the expressions obtained by identifying these entries with geodesic functions corresponding to closed paths on these graphs; for more details and for the explicit construction of geodesic functions, see [6], [8]. Note here that, likewise all $a_{i,j}$ constructed in this paper, all geodesic functions for all surfaces $\Sigma_{g,s}$ are positive Laurent polynomials of $e^{z_\alpha/2}$. 

5.2. **Braid-group action through mutations.** Our second major goal in this paper is to find a representation of the braid-group action from Sec. 1.3 in terms of cluster mutations of the $A_n$-quiver. It is well-known that if we identify $a_{i,j}$ with the geodesics functions, these braid-group transformations correspond to Dehn twists along geodesics corresponding to geodesic functions $a_{i,i+1}$ on $\Sigma_{g,s}$. We know that every Dehn twist on a Riemann surface $\Sigma_{g,s}$ can be presented as a chain of mutations of shear variables on edges of the corresponding spine $\Gamma_{g,s}$.

Whereas, for $n = 3$ and $n = 4$, $A_n$ quivers are geometric, and the corresponding mutation sequences are identical, for larger $n$ quivers become essentially different and we have to start from the scratch. However, knowing the answer for $n = 3$ and 4 helps in guessing the answer for a general $n$. To confirm our result we used computer experiments. We begin with the example of $A_5$-quiver:

![Diagram of A_5-quiver]

Then the following chain of standard mutations preserves the form of the original quiver with the interchanged vertices 3 and 4: $\beta_{3,4} = \mu_1\mu_2\mu_3\mu_2\mu_1$; here, $\mu_i$ is the quiver mutation at vertex $i$.

Possibly a most explicit way to represent a set of elementary braid-group transformations for a general $A_n$ quiver is as follows: take another copy of the triangle representing the quiver, reflect it and glue the resulting triangle to the original one along the bottom side of the latter in a way that amalgamated variables on the sides of two triangles match and the colored vertices representing Casimir elements are stretched along straight lines, see Fig. 11. The sequences of mutations corresponding to elementary generating elements $\beta_{i,i+1}$ of the braid groups are indicated in the figure.

6. **Casimirs**

In this section, we derive complete sets of Casimirs for all relevant (sub)varieties of cluster variables related to regular quivers associated to $SL_n$ systems. All proofs are direct calculations and will be omitted.

6.1. **The full-rank $SL_n$ quiver.**

**Lemma 6.1.** The complete set of Casimir operators for the full-rank $SL_n$ quiver consists of $\left[\frac{n}{2}\right]$ monomials of Fock-Goncharov parameters $Z_{abc}$ depicted in the figure below for the example of $SL_6$: numbers at vertices indicate the power with which the corresponding variable comes into the product; all nonnumbered variables have power zero. All Casimirs correspond to closed broken-line paths in the $SL_n$ quiver with reflections at the boundaries (the “frozen” variables at boundaries enter the product with powers two, powers of non-frozen variables can be 0, 1, 2, and 3, and they count how many times the path goes through the corresponding variable. The total Poisson dimension of the full-rank quiver is therefore $\frac{(n+2)(n+1)}{2} - 3 - \left[\frac{n}{2}\right]$. 
Figure 11. Visualising the braid-group action on the union of two triangles, each of which represents the corresponding quiver, for $A_5$ and $A_6$. (Note that we can continue this unfolding periodically along the stripes of the same color with the period $n$.) The dotted line indicates the line of the triangle gluing. Dashed blocks encompass sets of cluster variables sequences of mutations at which produce elementary braid-group transformation $\beta_{i,i+1}$: every such sequence commences with mutating the lowest element inside a box (corresponding to a six-valent vertex), then its upper-right neighbor and so on until we reach the upper element, mutate it, and repeat mutations at all inner elements in the reverse order. So, every such braid-group transformation is produced by a sequence of $2n - 5$ mutations in the quiver corresponding to $A_n$.

Figure 12. Three central elements of the full-rank quiver for $SL_6$.

Remark 6.2. All Casimir operators from Lemma 6.1 remain Casimirs for the extended $SL_n$ quiver obtained by adding three more variables $Z_{n00}, Z_{0n0}$ and $Z_{00n}$ at the vertices of the triangle (the variables $Z_{6,0,0}, Z_{0,6,0}$, and $Z_{0,0,6}$ in Fig. 17). For the extended $SL_n$-quiver we have to add one more Casimir operator which is just the product of all frozen variables along all three boundaries of the obtained triangle-shaped quiver.
Figure 13. Two central elements of the reduced quiver for $SL_6$. Every Casimir of the complete quiver in Fig. 12 has its counterpart in the reduced quiver except the third element.

Note that transport matrix $M_1$ does not depend on the last row of parameters $Z_{00c}, b + c = n$, while transport matrix $M_2$ does not depend on the left side $Z_{a0c}, a + c = n$. For completeness, we also present Casimirs for a reduced quiver in which we eliminate one of the three sets of frozen variables. In this case, every Casimir of the full-rank quiver has its counterpart in the reduced quiver except the element that is represented by a triangle-shaped path in the full-rank quiver (such an element exists only for even $n$).

Lemma 6.3. The complete set of Casimir operators for the reduced $SL_n$ quiver consists of \( \left\lfloor \frac{n-1}{2} \right\rfloor \) monomials of Fock-Goncharov variables depicted in Fig. 13 for the example of $SL_6$: numbers at vertices indicate the power with which the corresponding variable comes into the product; all nonnumbered variables have power zero. All Casimirs correspond, as in Fig. 12, to closed broken-line paths in the corresponding full-rank quiver with reflections at the boundaries (the “frozen” variables at boundaries enter the product with powers two), but now the path is split into two parts separated by two reflections at the side of the triangle that corresponds to the erased frozen variables; these two parts enter with opposite signs; the corresponding Casimir therefore contains cluster variables both in positive and in negative powers. As in the case of full-rank quiver, these powers count (with signs) how many times the path goes through the corresponding variable). The total Poisson dimension of the reduced $SL_n$ quiver is therefore $n(n+1)/2 - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor$.

In our construction below, an important role is played by the additional Casimir for the reduced quiver that appears if we add the variable $Z_{n00}$ at the summit of the corresponding triangle. In this case, besides the Casimirs in Lemma 6.3, we have one more central element $D_1$:

Lemma 6.4. The complete set of Casimir operators for the reduced $SL_n$ quiver with the ("frozen") cluster variable $(0,0,n)$ added comprises all Casimirs described in Lemma 6.3 plus the element $D_1$ given by the following formula. The element

\[
D_1 = \prod_{k=1}^{n} \left[ \prod_{i+j=N-k} [Z_{kij}]^{k/n} \right];
\]

is central for the subset of $Z_{kij}$ with $k > 0$. Moreover, the only elements do not commute with $D_1$ are $Z_{00n}$ and $Z_{0n0}$. More accurately, $Z_{00n}D_1 = q^{1/n}D_1Z_{00n}$ and $Z_{0n0}D_1 = q^{-1/n}D_1Z_{0n0}$.

6.2. Casimirs for the upper-triangular matrices. Entries of the matrix $A := M_1^T M_2$ depend on all variables of the $SL_n$ quiver, but due to the transposition, two sets of the frozen variables become amalgamated, that is, only their products appear in the entries of the matrix $A$. We explicitly show this amalgamation in Fig. 14.
Figure 14. The amalgamation of the $SL_n$-quiver corresponding to the triangle $\Sigma_{0,1,3}$. (The example in the figure corresponds to $SL_6$.)

Figure 15. Five new central elements of the main quiver for $SL_6$ due to amalgamation. (These central elements are used to set all diagonal elements of the upper-triangular matrix $A = M_1^T M_2$ to be the unities.)
Figure 16. Three remaining central elements of the full-rank quiver for $SL_6$ after amalgamation and setting the diagonal elements of $A$ equal to unities.

It is easy to see that all Casimirs from Lemma 6.1 remain Casimirs in the amalgamated quiver. We just keep in mind that either four or two frozen variables (depending on the Casimir element) become pairwise amalgamated. More, this amalgamation results in the appearance of $n-1$ new Casimirs; it is technically more convenient to consider these new Casimirs and some products of them with old Casimirs to obtain a complete set of Casimir operators for the amalgamated quiver.

We use these new Casimirs to eliminate the dependence of $A$ on remaining $n-1$ frozen variables: diagonal entries of $A = M_1^T M_2$ are particular products of these Casimirs, and we adjust the values of these Casimir operators to make all diagonal elements of $A$ equal to the unities in the classical case and $q^{-1/2}$ in the quantum case (the latter can be justified by assuming the condition that all Casimirs are self-adjoint operators).

Lemma 6.5. The complete set of central elements for the amalgamated quiver in Fig. 14 comprises $n-1$ new Casimirs depicted in Fig. 15 for the case of $SL_6$ and $\left[\frac{n}{2}\right]$ central elements (products of old Casimirs with the new ones) depicted in Fig. 16.

7. Proof of Theorem 2.6

In this section we accomplish the proof of Theorem 2.6.

7.1. Normalized and non-normalized transport matrices.

Remark 7.1. An equivalent of the semiclassical limit of Theorem 2.6 (see Remark 2.10) was proved in [24].

To prove the theorem we consider extended Fock-Goncharov quiver with additional vertices labelled $(n00), (0n0)$ and $(00n)$ and construct the oriented dual planar bicolored graph $G$ (Figure 7). Vertices of $G$ are colored into black and white color as follows: a black vertex has two incoming arrows and one outgoing, while a white vertex has two outgoing and one incoming arrows.

Then, we define non-normalized quantum transport matrices $M_1$ and $M_2$ as quantization of boundary measurement matrices of graph $G$ introduced by Postnikov in [35].

Namely, we assign to every maximal oriented path $P$ connecting a source of $G$ to a sink a quantum weight $w(P)$ in the quantum torus $\Upsilon$ defined by formula 2.4 (see Fig 17).

We define the boundary measurement between source $p$ and sink $q$ as $M_{pq} = \sum_{P:p\to q} w(P)$.

Noting that $G$ has $n$ sources and $2n$ sinks, we organize boundary measurements $M_{pq}$ into $2n \times n$
matrix that consists of two \( n \times n \) blocks: the top \( n \) rows \( M_1 = M_{[1,n]} \) and the bottom \( n \) rows \( M_2 = M_{[n+1,2n]} \).

**Example 7.2.** Consider the triangular network of \( SL_3 \) (Fig. 18)

Quantum transport matrices have the following form:

\[
M_1 = \begin{pmatrix}
Z_{120} & Z_{111} & Z_{210} & Z_{102} & Z_{210} & Z_{102} & Z_{210} & Z_{102} \\
Z_{111} & Z_{210} & Z_{102} & Z_{210} & Z_{102} & Z_{210} & Z_{102} & Z_{210} \\
Z_{210} & Z_{102} & Z_{210} & Z_{102} & Z_{210} & Z_{102} & Z_{210} & Z_{102} \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

Then, \( M_1 = \text{QSD}_{1}^{-1} M_1 \), where \( D_1 = Z_{120} Z_{210} Z_{111} \) and

\[
M_1 = \begin{pmatrix}
Z_{120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Z_{210} & Z_{111} & Z_{300} Z_{210} Z_{102} & Z_{210} Z_{111} Z_{300} Z_{210} Z_{102} & Z_{120} Z_{210} Z_{111} Z_{300} Z_{210} Z_{102} & Z_{210} Z_{111} Z_{300} Z_{210} Z_{102} & Z_{120} Z_{210} Z_{111} Z_{300} Z_{210} Z_{102}
\end{pmatrix}.
\]
Similarly, 

\[
M_2 = \begin{pmatrix}
0 & 0 & Z_{012}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-2/3}Z_{120}^{-2/3} \\
0 & 0 & Z_{102}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-2/3}Z_{120}^{-2/3} \\
Z_{012}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-2/3}Z_{111}^{-2/3}Z_{210}^{-1/3}Z_{120}^{-1/3} \\
Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} \\
Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3} & Z_{012}^{-1/3}Z_{111}^{-1/3} Z_{210}^{-1/3}Z_{120}^{-1/3}
\end{pmatrix}
\]

Hence, \(M_2 = QS\cdot D^{-1}D^{-1}\cdot \mathcal{M}_2\), where \(D_2 = \begin{pmatrix} Z_{003}^{-2}Z_{102}^{-2}Z_{201}^{-2}Z_{012}^{-1}Z_{111}^{-1}Z_{210}^{-1}Z_{120}^{-1} \end{pmatrix}\) and

\[
\mathcal{M}_2 = \begin{pmatrix}
Z_{300}Z_{003}Z_{102}Z_{201} & Z_{300}Z_{003}(1 + Z_{111})Z_{210}Z_{102}Z_{201} & Z_{300}Z_{003}Z_{120}Z_{111}Z_{210}Z_{102}Z_{201} & Z_{300}Z_{003}Z_{120}Z_{111}Z_{210}Z_{102}Z_{201}
\end{pmatrix}
\]

Notice that both \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are non-normalized quantum transport matrices of network shown on Figure 18.

7.2. Quantum Grassmannian and measurement map. Following Postnikov we call a planar network an oriented graph embedded in the disk whose faces are equipped with (quantum) weights. Let \(\mathcal{N}\) be a network in the disk with neither sources nor sinks inside and non-interlacing sources and sinks on the boundary. One can always draw such network in the rectangle \(R\) with sources on the right side, sinks on the left side, and no boundary vertices on either bottom or top side. An example of such network \(\mathcal{N}\) with sources labelled 1 and 2 on the right and sinks 3, 4, 5 on the left is drawn on Figure 19.

**Remark 7.3.** We don’t assume any additional conditions on the network. In particular, a network can contain oriented cycles etc.

![Figure 19. Network \(\mathcal{N}\) in rectangle \(R\).](image)

Denote by \(\text{Faces}(\mathcal{N})\) the set of all faces of network \(\mathcal{N}\). In the example on Figure 19, \(\text{Faces}(\mathcal{N}) = \{\alpha, \beta, \gamma, \delta, \epsilon\}\). Consider the vector space \(\tilde{V}\) with the basis \(\{\omega | \omega \in \text{Faces}(\mathcal{N})\}\) formed by all the faces of the network. \(\tilde{\Lambda}\) is the integer lattice in \(\tilde{V}\) \(\mathbb{Z}\)-spanned by this basis. We equip it with the integer skew-symmetric form \(\Omega\) as follows.

**Definition 7.4.** [35] A planar bicolored graph, or simply a plabic graph is a planar (undirected) graph \(G\), without orientations of edges, such that each boundary vertex \(b_i\) is incident to a single edge, together with a coloring of internal edges into two colors: black and white. A perfect orientation of a plabic graph is a choice of orientation of its edges such that each black internal vertex \(v\) is incident to exactly one edge directed away from \(v\); and each white \(v\) is incident to exactly one edge directed towards \(v\). A plabic graph is called perfectly orientable if it has a perfect orientation.

Let us transform the oriented graph \(G\) of network into plabic graph \(G^{pl}\) by coloring inner vertices of \(G\) into black and white colors according to the rule: black vertex has two incoming arcs and one outgoing; white vertex has one incoming and two outgoing. Then, we forget the orientations of arcs of \(G\) and keep only the color of vertices. We remove boundary sources and sinks so that any arcs connecting inner vertex to the boundary one becomes a half-arc (see
For a plabic graph $G^{pl}$ we define an oriented dual graph $(G^{pl})^*$ as follows. Vertices of $(G^{pl})^*$ are faces of $G^{pl}$. For every black and white vertex $x$ of $G^{pl}$ we define 3 arcs of $(G^{pl})^*$ that cross half-edges attached to $x$ in counterclockwise direction if $x$ is black and clockwise direction if $x$ is white (see Fig. 20).

For $\theta, \phi \in \text{Faces}(\mathcal{N})$ let $\#(\theta \rightarrow \phi)$ denote the number of arrows from $\theta$ to $\phi$ in $(G^{pl})^*$. Define the form $\Omega \in (\tilde{V})^* \wedge (\tilde{V})^*$ by the formula

$$\Omega(\theta, \phi) = \frac{1}{2} (#(\theta \rightarrow \phi) - #(\phi \rightarrow \theta)).$$

Example 7.5. The plabic graph $G^{pl}$ and its dual for the network Fig. 19 are shown on the Fig. 21.

Let $V$ be the quotient space $V = \tilde{V} / \sum_{\theta \in \text{Faces}(\mathcal{N})} \theta$ and $\Lambda$ be the integer lattice in $V$ induced by the lattice $\tilde{\Lambda}$ in $\tilde{V}$. Note that $\sum_{\theta \in \text{Faces}(\mathcal{N})} \theta \in \ker(\Omega)$ and the push forward of $\Omega$ to $V$ is well-defined.

Let $a \in \text{Faces}(\mathcal{N})$, $Z_a$ denote its weight. The quantum torus $\mathcal{Y}_N$ is generated by weights $Z_a$, $a \in \text{Faces}(\mathcal{N})$ satisfying commutation relations $Z_a Z_b = q^{\Omega(a,b)} Z_b Z_a$. Note that $q^{\Omega(a,b)} Z_a Z_b = q^{\Omega(b,a)} Z_b Z_a$. The Weyl ordering of a monomials in $Z_a$ is defined by $Z_a Z_b = q^{\Omega(a,b)} Z_b Z_a$. Evidently, $Z_a Z_b = Z_b Z_a$. We define weight of any vector in $\Lambda \otimes \mathbb{Q}$ by $Z_{a+b} = Z_a Z_b = Z_b Z_a$ and $Z_{a\lambda} = Z_a Z_{\lambda}$ for any $\lambda \in \mathbb{Q}$. Since every $Z_a$ is a positively defined self-adjoint operator in $\ell^2(\mathbb{R})$ having a continuous spectrum $(0, \infty)$, any rational power of it is itself a positively defined self-adjoint operator. Weights satisfy relation $\prod_{a \in \text{Faces}(\mathcal{N})} Z_a = 1$, or, equivalently,

$$\sum_{a \in \text{Faces}(\mathcal{N})} a = 0 \in V.$$

Let $p$ be the maximal oriented path from a source $i$ on the right to the sink $j$ on the left of a network. Complete the path to the loop $\tilde{p} = g \circ p$ where $g$ is the path around the boundary.
of the rectangle going in the clockwise direction from \( j \) to \( i \). The oriented loop \( \tilde{p} \) defines a covector \( \partial \in \Lambda^* \).

For any \( \alpha \in \text{Faces}(N) \) we compute \( \tilde{p}(\alpha) \) as follows. Let \( r \) be a half infinite ray with starting point inside face \( \alpha \) and directed towards infinity and \( s \) be an intersection point of \( r \) and path \( \tilde{p} \), \( T_s r \) be the unit tangent vector to \( r \) at \( s \) directed in the same direction as \( r \), \( T_s \tilde{p} \) is the unit tangent vector to \( \tilde{p} \) at \( s \). We assume that \( r \) is chosen generic, in the sense that for any intersection point \( s \) vectors \( T_s r \) and \( T_s \tilde{p} \) are linearly independent.

We define the signed intersection number \( \text{int}_s(\tilde{p}, r) \) of \( \tilde{p} \) and \( r \) at \( s \) to be 1 if orientation of basis \( (T_s \tilde{p}, T_s r) \) coincides with counterclockwise orientation of the plane and \(-1\) otherwise. For any point \( b \) in rectangle and any ray \( r \) starting from \( b \) we define \( \tilde{p}(\alpha) = \text{ind}(\tilde{p}, r) = \sum_{s_i \in \tilde{p} \cap r} \text{int}_{s_i}(\tilde{p}, r) \)
where the sum is taken over all intersection points \( s_i \) of \( \tilde{p} \) and \( r \). Note that \( \tilde{p}(\alpha) \) does not depend on the exact position of starting point while the starting point varies inside the same connected component of complement to \( \tilde{p} \). Since any face \( \alpha \) lies entirely in some connected component of \( \tilde{p} \) we conclude that \( \tilde{p}(\alpha) \) is well defined.

Hence, \( \partial \in \hat{\Lambda}^* \).

Then, we assign to any path \( p \) a vector \( v_p = \sum_{\alpha \in \text{Faces}(N)} \tilde{p}(\alpha) = \Lambda. \) Note that the trivial path \( \phi \) has the vector \( v_{\phi} = \sum_{\alpha \in \text{Faces}(N)} \alpha = 0 \in V \). In the example in Section 2.3 where any maximal oriented path \( p \) is non-selfintersecting the vector \( v_p \) is the sum of all faces to the right from the path.

Assign to path \( p \) its weight \( w_p := Z_{v_p} \).

Let \( S \) be the set of all sources of net, \( F \) be the set of all sinks of the net. Following construction of Postnikov [35], we define for any source \( a \in S \) and sink \( b \in F \) a quantum boundary measurement \( \text{Meas}^h(a, b) = \sum_{p:a \rightarrow b} (-1)^{\text{cross}(p)} w_p \), where the sum is taken over all oriented paths \( p \) from \( a \) to \( b \) where the crossing index \( \text{cross}(p) \) is the number of self-crossings of the path \( p \). For the examples of networks in Section 2.3 \( \text{Meas}^h(a, b) = \sum_{p:a \rightarrow b} w_p \).

Let \( n = |S| \), \( m = |F| \). Define an \( m \times n \) matrix \( \text{Meas}^h \) by \( \text{Meas}^h = \text{Meas}^h(a, b) \) \( a \in S, b \in F \). Superindex \( h \) indicates the quantum version of boundary measurements in contrast to the classical measurements defined in [35].

In Example 19 the matrix \( \text{Meas}^h = \begin{pmatrix} Z_{\alpha+\beta} & Z_{\alpha} \\ Z_{\alpha+\beta+\gamma} & Z_{\alpha+\gamma} \\ Z_{\alpha+\beta+\gamma+\delta} & 0 \end{pmatrix} \).

Define \( (m+n) \times n \) extended boundary measurement matrix \( \widetilde{\text{Meas}}^h \) of network \( N \). Columns of \( \widetilde{\text{Meas}}^h \) are labelled by boundary sources of network; rows are labelled by all boundary vertices.

To describe matrix elements of \( \widetilde{\text{Meas}}^h \) we introduce the order \( \text{ord}_N(b) \) of boundary vertex \( b \). Namely, let \( b \in [1, m+n] \) be the index of boundary vertex in the chosen counterclockwise labelling. Let \( p \) be the number of sources among boundary vertices with indices from 1 to \( b-1 \). The order is defined by the formula \( \text{ord}_N(b) = \begin{cases} p, & \text{if } b \text{ is not a source;} \\ p + \frac{1}{2}, & \text{if } b \text{ is a source.} \end{cases} \)

Let \( \mathbb{I}(i) \in [1, m+n] \) be the index of \( i \)th source, \( i \in [1, n] ; \mathbb{J} : [1, n] \rightarrow [1, m+n] \) is an increasing function.

We define \( \langle \widetilde{\text{Meas}}^h \rangle_{ji} = \begin{cases} (-1)^{\mathbb{I}(i) + \text{ord}_N(j)} q^{\text{ord}_N(j)} \text{Meas}^h(i, j), & \text{if } j \text{ is not a source;} \\ q^{-\text{ord}_N(j)} \delta_{\mathbb{I}(i), j}, & \text{otherwise.} \end{cases} \)
Example 7.6. In Example [19], the matrix $\widehat{\text{Meas}}^\hbar = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{3/2} \\ -q^2 Z_{\alpha+\beta} & q^2 Z_{\alpha+\gamma} \\ -q^2 Z_{\alpha+\beta+\gamma} & q^2 Z_{\alpha+\gamma} \\ -q^2 Z_{\alpha+\beta+\gamma+\delta} & 0 \end{pmatrix}$.

Remark 7.7. In [35] a boundary measurement map is defined as a map $\text{Meas}$ from the space $\text{Net}_{m \times n}$ of networks with $n$ sources and $m$ sinks to $Gr(n, m + n)$. For each $\mathcal{N} \in \text{Net}_{m \times n}$, Postnikov organized boundary measurements $\text{Meas}(i, j)$ from source $i$ to sink $j$, $i \in [1, n]$, $j \in [1, m]$ into an $(m + n) \times n$ matrix $\widehat{\text{Meas}}$ which represents $\text{Meas}(\mathcal{N})$. Let’s define the diagonal $(m + n) \times (m + n)$ matrix $D^\hbar = \text{diag}(q^{\text{ord}(j)})_{j=1,m+n}$. Then, $\widehat{\text{Meas}}^\hbar = D^\hbar \widehat{\text{Meas}}$. We denote by $\text{Mat}_{(m+n)\times n}(\Upsilon_N)$ the space of $(m + n) \times n$ matrices with elements in $\Upsilon_N$. The group $GL_n(\Upsilon_N)$ of invertible $n \times n$ matrices with entries from $\Upsilon_N$ acts on $\text{Mat}_{(m+n)\times n}(\Upsilon_N)$ by the right multiplication. We define the homogeneous space $Gr^\hbar(n, m+n)$ as the right quotient $Gr^\hbar(n, m+n) = \text{Mat}_{(m+n)\times n}(\Upsilon_N)/GL_n(\Upsilon_N)$. We define the map $\text{Meas}^\hbar : \text{Net}_{m \times n} \to Gr^\hbar(m, m+n)$ as the composition $\text{Net}_{m \times n}^\hbar \to \text{Mat}_{(m+n)\times n}(\Upsilon_N) \to Gr^\hbar(n, m+n)$ which is a quantization of Postnikov measurement map $\text{Meas}$.

Definition 7.8. Two quantum networks $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent if $\text{Meas}^\hbar(\mathcal{N}_1) = \text{Meas}^\hbar(\mathcal{N}_2)$.

Simple equivalence relations (M1-M3,R1-R3) on the space of networks ( [35]) are simple local network transformations preserving boundary measurements. Please note that in the figures below we draw the plabic graph assuming that it is equipped with a perfect orientation. Different choices of compatible perfect orientation give the same result.

The following claims generalize similar statements for commuting weights (cf. [35]).

Let $\alpha_0$ be the label of one face of oriented planar quiver $\Gamma$ in the disk attached to the boundary of the disk, $\text{Faces}_0 = \text{Faces} \setminus \{\alpha_0\} = \{\alpha_1, \ldots, \alpha_k\}$, $k = |\text{Faces}_0|$. Let $\{\alpha_i| i \in [1, k]\}$ denote the standard basis in the lattice $\Lambda \simeq \mathbb{Z}^k$; for $\mathbf{v}, \mathbf{w} \in \Lambda$ let $\mathbf{v} \cdot \mathbf{w}$ denote the standard dot product $\alpha_i \cdot \alpha_j = \delta_{ij}$, $Z_i = Z_{\alpha_i}$ be the generators of a quantum torus $\Upsilon$. We say that an infinite linear combination $\sum_{\lambda \in \Lambda} \alpha_{\lambda} Z_{\lambda}$ is a Laurent series if there exist integers $b_1, \ldots, b_n \in \mathbb{Z}$ such that $\alpha_{\lambda} = 0$ unless $\lambda \cdot \alpha_j \geq b_j \forall j \in [1, n]$. Any Laurent series $u \in \hat{L}$ where $\hat{L} = \mathbb{Q}[q, q^{-1}][Z_1^{-1}, \ldots, Z_k^{-1}][[Z_1, \ldots, Z_k]]$. Let $\mathfrak{m} \subset \mathbb{Q}[q, q^{-1}[Z_1, \ldots, Z_k]$ be the maximal ideal generated by $Z_i$. Note that for any $x \in \mathfrak{m}$ the expression $(1 + x)^{-1} \in \hat{L}$ and $(1 + x^{-1})^{-1} = x(1 + x)^{-1} \in \hat{L}$.

We say that $x, y \in \hat{L}$ $q$-commute if $xy = q^r yx$ for some $r \in \mathbb{Q}$.

We call an oriented planar network whose faces are equipped with $q$-commuting face weights in $\hat{L}$ a quantum network.

Define 6 elementary moves (M1-M3), (R1-R3) as shown below.

![Figure 22](image-url)
Figure 23. Elementary move M2.

\[ Z_{\alpha} Z_{\gamma} \quad \rightarrow \quad Z_{\alpha} Z_{\gamma} \]

Figure 24. Elementary move M3.

\[ Z_{\alpha} \quad \rightarrow \quad Z_{\alpha} \]

Figure 25. Elementary move R1.

\[ \sum_{j=1}^{\infty} (-1)^{j-1} Z_{\alpha+j} \]

Figure 26. Elementary move R2.

\[ Z_{\epsilon} \quad \rightarrow \quad Z_{\epsilon} \]

Figure 27. Elementary move R3.
Definition 7.9. Two networks are *move equivalent* if they are connected by a sequence of elementary moves.

The corresponding (move) equivalence is called *quantum* (move) equivalence.

The following result extends the results of [35] to quantum networks.

Lemma 7.10. Two quantum move equivalent networks are quantum equivalent.

Proof. The proof follows [35]. Compared to the commutative case we just need to check one additional condition that new parameters form q-commutative family. The cases R2, R3, M2, and M3 are evident. Let’s consider M1 and R1. The case M1 is proved in [17]. We give the proof here for completeness.

We want to show that \( \alpha'Z', \beta'Z', \gamma'Z', \delta'Z' \text{ and } \epsilon'Z' \text{ q-commute}. \) Indeed,

\[
Z_{\alpha}'Z'_{\beta} = (Z_{\alpha} + Z_{\alpha+\epsilon}) \sum_{j=1}^{\infty} (-1)^{j-1} Z_{\beta+j\epsilon} = q^{-\Omega(\alpha,\beta+\epsilon)}Z_{\alpha+\beta+\epsilon};
\]

more, \( Z'_{\beta}Z'_{\alpha} = q^{\Omega(\alpha,\beta+\epsilon)}Z_{\alpha+\beta+\epsilon}. \) Therefore, \( Z'_{\beta}Z'_{\alpha} = q^{2\Omega(\alpha,\beta+\epsilon)}Z_{\alpha}'Z_{\beta}'. \) All other pairs of parameters can be checked similarly.

Different perfect orientations are in one-to-one correspondence with the *almost perfect matchings* (see [36]). Up to evident symmetries, there are two essentially different perfect orientations. The straightforward computation shows that the elementary move M1 does not change measurements for any choice of perfect orientation.

The case R1 is similar. \( \square \)

Definition 7.11. ([35]) We say that a plabic network (or graph) is *reduced* if it has no isolated connected components and there is no network/graph in its move-equivalence class to which we can apply a reduction (R1) or (R2). A leafless reduced network/graph is a reduced network/graph without non-boundary leaves.

The following statements are proved in [35].

Lemma 7.12. [35] Any network is move equivalent to a reduced network.

Lemma 7.13. [35] Two reduced equivalent networks are (M1-M3)-move equivalent.

Definition 7.14. We call a maximal simple oriented path \( P = (p_0, p_1, \ldots, p_h) \) ( \( p_i \neq p_j \) for all \( i \neq j \) ) in a oriented planar network \( \mathcal{N} \) *unequivocal* if there is no oriented path \( (p_k, q_1, \ldots, q_t, p_e) \) such that \( q_s \neq p_r \) for all \( 1 \leq s \leq t \) and \( 0 \leq r \leq h \).

Lemma 7.15. Let \( P \) be an unequivocal path in a oriented planar network \( \mathcal{N} \in \text{Net}_{m,n} \). Reversing the orientation of \( P \) makes the new oriented planar network \( \mathcal{N}' \) which has the same quantum Grassmann measurement \( \text{Meas}^h(\mathcal{N}) = \text{Meas}^h(\mathcal{N}') \).

Figure 28. Change of orientation of the path \( P: 2 \sim 4 \)
Example 7.16. The corresponding extended quantum measurement matrices are

\[
\tilde{\text{Meas}}^h = \begin{pmatrix}
q^{1/2} & 0 \\
0 & q^{3/2} \\
-q^2Z_{\alpha+\beta} & q^2Z_{\alpha} \\
-q^2Z_{\alpha+\beta+\gamma} & q^2Z_{\alpha+\gamma} \\
-q^2Z_{\alpha+\beta+\gamma+\delta} & 0
\end{pmatrix}
\quad \text{and} \quad
(\tilde{\text{Meas}}^h)' = \begin{pmatrix}
q^{1/2} & 0 & 0 \\
qZ_{\beta} & qZ_{\delta+\epsilon+\beta} & 0 \\
0 & 0 & q^{3/2} \\
-q^2Z_{\alpha+\beta+\gamma+\delta} & q^2Z_{\alpha+\gamma} & 0
\end{pmatrix}.
\]

Note that \(\tilde{\text{Meas}}^h C = (\tilde{\text{Meas}}^h)'\), where \(C = \begin{pmatrix} 1 & 0 \\ q^{-1/2}Z_{\beta} & q^{-1/2}Z_{\delta+\epsilon+\beta} \end{pmatrix}\).

Indeed, \((\tilde{\text{Meas}}^h)'_{31} = (\tilde{\text{Meas}}^h)_{31} + q^{-1/2}(\tilde{\text{Meas}}^h)_{32}Z_{\beta} = -q^2Z_{\alpha+\beta} + q^{3/2}Z_{\alpha}Z_{\beta}\). Recall that \(Z_{\alpha}Z_{\beta} = q^{1/2}Z_{\alpha+\beta}\). Therefore, \((\tilde{\text{Meas}}^h)'_{31} = 0 = ((\tilde{\text{Meas}}^h)')_{31}\). Similarly, we can prove equalities for all the entries of these 5 × 2 matrices and we observe that \(\text{Meas}^h(N) = \text{Meas}^h(N') \in \text{Gr}^h(2, 5)\).

Proof. Let the simple unequivocal path \(P \in N\) goes from a boundary vertex \(a\) to a boundary vertex \(b\). We will denote by \(P^{-1}\) the path in \(N'\) obtained from \(P\) by orientation reversing. We assume first that the boundary vertices are labelled so that \(1 < a < b \leq m + n\). Since path \(P\) is unequivocal there is only one path \(P\) from \(a\) to \(b\), and \(\text{Meas}^h(a, b) = w_P\). Moreover, any other path \(P_t\) from \(s\) to \(t\) where both \(s\) and \(t\) are distinct from \(a\) and \(b\) has at most one common interval \([q_1, q_2]\) with path \(P\). The first point \(q_1\) (counting from \(s\)) where two paths meet has two incoming arrows and one outgoing and, hence, is colored black, the point \(q_2\) where two paths part is white (see Figure 29). Similarly, any path from \(a\) to a sink different from \(b\) separates from \(P\) at a white point; any path from a sink different from \(a\) to \(b\) joins path \(P\) at a black point.

Recall that \(\tilde{\text{Meas}}^h\) is the extended bounded measurement matrix of the network \(N\). \((\tilde{\text{Meas}}^h)\) is the extended boundary measurement matrix of \(N'\). Let \(F_P \subseteq \text{Faces}\) denote the subset of all faces to the right of the path \(P\), \(v_P = \sum_{\alpha \in F_P} \alpha\). Then, \(w_P = Z_{v_P}, F_{P^{-1}} = \text{Faces} \setminus F_P, v_{P^{-1}} = -v_P, w_{P^{-1}} = Z_{v_{P^{-1}}} = Z_{-v_P} = (w_P)^{-1}\).

Consider first the case \(1 \leq a < b\).

Let \(s < a < t < b\) in the cyclic order of the boundary vertices (see Figure 29). Observe, \(w_{b \to q_1 \to q_2 \to t} = Z_{\alpha}Z_{\delta+\gamma}Z_{\alpha}\) is a directed path in \(b\) to \(t\) and \((\tilde{\text{Meas}}^h)_{ta} = (Q_{q_2}Q_{q_1})_{ta}\). Since the equality holds for any directed path from \(b\) to \(t\), and \((\tilde{\text{Meas}}^h)_{ta} = (Q_{q_2}Q_{q_1})_{ta}\), we conclude that \((\tilde{\text{Meas}}^h)_{tb} = (\tilde{\text{Meas}}^h)'_{tb}\).

Consider the case \(a < s < t < b\). Similarly, \((\tilde{\text{Meas}}^h)'_{ts} = (\tilde{\text{Meas}}^h)'_{ts} + (\tilde{\text{Meas}}^h)'_{bs}(\tilde{\text{Meas}}^h)'_{tb}\). Therefore, \((\tilde{\text{Meas}}^h)'_{ts} = (\tilde{\text{Meas}}^h)'_{ts} + (\tilde{\text{Meas}}^h)'_{bs}(\tilde{\text{Meas}}^h)'_{tb} = 0\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Change of the orientation of the path \(P : a \to b, s < a < t < b\).}
\end{figure}
In the same way we investigate all the remaining mutual positions of $s$, $t$ and $a, b$ which leads to the following matrix identity. Let $a_0 = \mathcal{J}^{-1}(a)$, $f_0$ be the index of source of $\mathcal{N}$ such that $b$ lies between the source $f_0$ and $f_0 + 1$ (equivalently, $\mathcal{J}(f_0) < b < \mathcal{J}(f_0 + 1)$. Note that $a_0 \leq f_0$ since $a < b$. Define $n \times n$ matrix $C$ for $1 < a < b$ as follows

$$C_{ij} = \begin{cases} 
\delta_{ij}, & \text{if } i < a_0 \text{ or } i > f_0; \\
q^{-1}\delta_{i-1,j}, & \text{if } a_0 < i \leq f_0; \\
(-1)^{\lfloor j-a_0+1/2 \rfloor} q^{-1/2}(\text{Meas}^h)_{ja} (\text{Meas}^h)_{ba}^{-1} & \text{if } i = a_0.
\end{cases}$$

Then, $(\text{Meas}^h)' = \overline{\text{Meas}^h} C$.\[\]

Note that $(\text{Meas}^h)' = \overline{\text{Meas}^h} C$ implies $\overline{\text{Meas}^h} = (\text{Meas}^h)' C^{-1}$ where $C^{-1}$ is obtained from $C$ by changing the signs of the off-diagonal elements and adjusting the powers of $q$. More exactly, define $n \times n$ matrix $\tilde{C}$ for $1 < b < a$ as follows

$$\tilde{C}_{ij} = \begin{cases} 
\delta_{ij}, & \text{if } i \leq f_0 \text{ or } i > a_0; \\
q\delta_{i+1,j}, & \text{if } f_0 < i < a_0; \\
(-1)^{\lfloor j-a_0+1/2 \rfloor} q^{1/2}(\text{Meas}^h)_{ja} (\text{Meas}^h)_{ba}^{-1} & \text{if } i = a_0.
\end{cases}$$

The formulas above induce the same matrix identity $(\text{Meas}^h)' = \overline{\text{Meas}^h} C$. These matrix identities prove the statement.\[\]

**Example 7.17.** Consider the networks in Figure 30.

![Figure 30](image)

**Figure 30.** Change of the orientation of the unequivocal path $P : 5 \rightarrow 13, 1 < 5 < 13$.

Matrices $\overline{\text{Meas}^h}$ and $(\text{Meas}^h)'$ have the following form.

$$\overline{\text{Meas}^h} = \begin{pmatrix}
q^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & qZ^{-a_2} & 0 & qZ^{-\sum_{j=2}^4 a_j} & 0 & 0 & 0 & 0 \\
0 & q^{3/2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & qZ^{-a_4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{3/2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^2Z_{a_6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{7/2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{1/2} & 0 & 0 \\
q^5Z_{\sum_{j=1}^2 a_j} & -q^5Z_{\sum_{j=3}^6 a_j} & -q^5Z_{\sum_{j=5}^7 a_j} & -q^5Z_{\sum_{j=7}^9 a_j} & q^5Z_{a_9} & 0 & 0 & 0 \\
0 & q^5Z_{\sum_{j=3}^6 a_j} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^5Z_{\sum_{j=5}^7 a_j} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^5Z_{a_9} & 0 & 0 & 0 & 0 \\
-\frac{q^6}{11}Z_{\sum_{j=1}^2 a_j} & q^6Z_{\sum_{j=6}^8 a_j} & -q^6Z_{\sum_{j=5}^7 a_j} & -q^6Z_{\sum_{j=9}^a_7 a_j} & q^6Z_{a_{11}} & 0 & 0 & 0 \\
0 & q^6Z_{\sum_{j=6}^8 a_j} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{q^6}{17}Z_{\sum_{j=12}^3 a_j} & q^6Z_{\sum_{j=13}^9 a_j} & -q^6Z_{\sum_{j=12}^3 a_j} & -q^6Z_{\sum_{j=9}^a_7 a_j} & q^6Z_{a_{11}} & 0 & 0 & 0 \\
0 & q^6Z_{\sum_{j=13}^9 a_j} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$
The boundary measurement matrix has size $3n \times n$. It consists of three $n \times n$ blocks on top of each other. The top $n \times n$ block $(\text{Meas}^h)_{[1,n]}$ is the diagonal matrix with $j$th diagonal elements $q^{j-1/2}$. The middle block $(\text{Meas}^h)_{[n+1,2n]}$ is a scalar multiple of the transport matrix $q^n T_2^{-1} \Xi$ and the bottom block $(\text{Meas}^h)_{[2n+1,3n]}$ is a scalar multiple of the transport matrix $q^n T_1 \Xi$ for the network.

It is straightforward to check that $(\text{Meas}^h)' = \text{Meas}^h C$.

Consider the network for $SL_n$ shown for $n = 6$ in Figure 31. Let $\Xi = \text{diag}(-1^{n-1}, (-1)^{n-2}, \ldots, 1)$. The boundary measurement matrix $\widetilde{\text{Meas}}^h$ has size $3n \times n$. It consists of three $n \times n$ blocks on top of each other. The top $n \times n$ block $(\text{Meas}^h)_{[1,n]}$ is the diagonal matrix with $j$th diagonal elements $q^{j-1/2}$. The middle block $(\text{Meas}^h)_{[n+1,2n]}$ is a scalar multiple of the transport matrix $q^n T_2^{-1} \Xi$ and the bottom block $(\text{Meas}^h)_{[2n+1,3n]}$ is a scalar multiple of the transport matrix $q^n T_1 \Xi$ for the network 17.

**Figure 31.** Changing orientation of the bold paths. The big arrow shows the direction of transport matrix $T_3$.

The orientation change of all blue paths (see Figure 31) leads to the new Grassmann measurement $(\widetilde{\text{Meas}}^h)'$. The submatrix $(\text{Meas}^h)'_{[n+1,2n]}$ is a scalar multiple of $T_3 \Xi$. Comparing
blocks \((\text{Meas}^h)_{[n+1,2n]}\) and \((\text{Meas}^h)_{[n+1,2n]}\) we obtain \(T_2^{-1}T_1^{-1} = T_3\). This is clearly equivalent to the second part of Theorem 3.1: \(T_1T_2T_3 = 1\).

Commutation relations between face weights induce \(R\)-matrix commutation relations between entries of \(\text{Meas}^h\).

Namely, the next lemma describes the commutation relation between elements of \(\text{Meas}^h\).

**Lemma 7.18.** \(R_m\text{Meas}^h \otimes \text{Meas}^h = \text{Meas}^h \otimes \text{Meas}^hR_n\), where \(R_m, R_n\) are given by formula 2.5.

**Proof.** We will prove this statement using factorization of matrix \(\text{Meas}^h\) into a product of elementary matrices. Let \(\mathcal{N}\) be a network in rectangle with \(m\) sinks on the left and \(n\) sources on the right all whose arrows are directed right to left. Then, clearly, \(\mathcal{N}\) can be presented as a concatenation of elementary networks of two special kinds (see Fig. 32 and 33).

**Figure 32.** Elementary forks (decreasing on the left and increasing on the right)

Figure 33 shows concatenation of a fork.

**Figure 33.** Adding fork to the second sink. Note, \(\bullet Z_{f_2}Z_{f_1}Z_{f_2} \ldots Z_{f_4} = 1\).

Using the construction above, we write that, \(\text{Meas}^h = \prod_i X_i\), where \(X_i = L_i\) or \(X_i = U_i\) is an \(m_i \times m_{i+1}\) matrix where \((m_{i+1} = m_i - 1\) if we add the decreasing fork or \(m_{i+1} = m_i + 1\) if increasing.)

\[
L_i = \begin{pmatrix}
  t_1 & 0 & \ldots & \ldots & 0 & 0 & \ldots \\
  0 & t_2 & \ldots & \ldots & 0 & 0 & \ldots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \ldots & t_i & 0 & 0 & \ldots \\
  0 & \ldots & Z_{f_1}Z_{a_1} & 0 & 0 & \ldots \\
  0 & \ldots & 0 & t_{i+1} & 0 & \ldots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, \quad t_1 = \bullet Z_{f_1}, t_2 = \bullet Z_{f_1}Z_{f_2}, \text{etc}
\]
and

\[
U_i = \begin{pmatrix}
  t_1 & 0 & \ldots & \ldots & 0 & 0 & \ldots \\
  0 & t_2 & \ldots & \ldots & 0 & 0 & \ldots \\
  \vdots & \vdots & \ddots & 0 & \ldots & \ldots & \ldots \\
  0 & \ldots & \ldots & \ldots & 0 & 0 & \ldots \\
  0 & 0 & \ldots & \ldots & t_{i+1} & \ldots & \ldots 
\end{pmatrix}, \quad t_1 = Z_{f_1}, t_2 = Z_{f_1}Z_{f_2}, \text{ etc.}
\]

Variables \( t_j \) commute with each other and commute with \( Z_{a_i} \) unless \( j = i \). Otherwise, \( Z_{a_i}t_j = qt_jZ_{a_i}, Z_{a_i}t_j = q^{-1/2}Z_{a_i}t_j \). All entries of \( X_i \) commute with all entries of \( X_j \) for \( i \neq j \).

Clearly, it is enough to check the relations for \( 2 \times 1 \) and \( 1 \times 2 \) matrices.

For \( m_i = 1 \), we have \( R_m^1 \L_i \otimes \L_i = \L_i \otimes L_i R_{m+1}, R_m^1 U_i \otimes U_i = U_i \otimes U_i R_{m+1} \). Indeed,

\[
\begin{pmatrix}
  q & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & q^{-1} & 1 & 0 \\
  0 & 0 & 0 & q
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2 \\
  \frac{a}{b} \\
  \frac{ab}{b^2}
\end{pmatrix}
= \begin{pmatrix}
  q \frac{a^2}{b^2} \\
  q \frac{ab}{b^2} \\
  qba \\
  qab
\end{pmatrix} = \begin{pmatrix}
  2 \\
  1 \\
  \frac{a}{b} \\
  \frac{b}{a}
\end{pmatrix} \cdot (q)
\]

Then, \( R_m^1 \Meas^h \otimes \Meas^h = R_m^1 \prod_{i=1}^n X_i \otimes \prod_{i=1}^n X_i = R_m^1 X_1 \prod_{i=2}^n X_i \otimes X_1 \prod_{i=2}^n X_i = X_1 \otimes X_1 R_m^1 \prod_{i=2}^n X_i \otimes \prod_{i=2}^n X_i = \cdots = \prod_{i=1}^n X_i \otimes \prod_{i=1}^n X_i R_m^1 = \Meas^h \otimes \Meas^h R_m^1 \).

8. Concluding remarks

In this paper, we have found the Darboux coordinate representation for matrices \( \mathbb{A} \) enjoying the quantum reflection equation. We have also solved the problem of representing the braid-group action for the upper-triangular \( \mathbb{A} \) in terms of mutations of cluster variables associated with the corresponding quiver.

In conclusion, we indicate some directions of development. The first interesting problem is to construct mutation realizations for braid-group and Serre element actions that are Poisson automorphisms in the case of block-upper triangular matrices \( \mathbb{A} \) (the corresponding action in terms of entries of a block-upper-triangular \( \mathbb{A} \) was constructed in \([9]\)). It is not difficult to construct planar networks producing block-triangular transport matrices \( M_1 \) and \( M_2 \) enjoying the standard Lie Poisson algebra, then \( \mathbb{A} = M_1^T M_2 \) will satisfy the semiclassical reflection equation.

The second direction of development is based on the semiclassical groupoid construction; explicit calculations in Sec.12 of \([10]\) show that, given that \( B \) is a general \( SL_n \)-matrix endowed with the standard semiclassical Lie Poisson bracket, solving the matrix equation \( BAB^T = A' \), where \( \mathbb{A} \) and \( \mathbb{A}' \) are unipotent upper-triangular matrices, we obtain that entries of \( \mathbb{A} \) are uniquely determined (provided all upper-right and lower-left minors of \( B \) are nonzero); the Lie Poisson bracket on \( B \) produces the reflection equation bracket on \( \mathbb{A} \), the mapping \( \mathbb{A} \rightarrow \mathbb{A}' \) is a Poisson anti-automorphism, and finally, entries of \( \mathbb{A} \) and \( \mathbb{A}' \) are mutually Poisson commute. Therefore, having a set \( X_n \) of Darboux coordinates for entries of \( B \) we automatically obtain that \( a_{i,j} \in \mathbb{Z}[e^{X_a}] \) (the determinant of the linear system determining \( a_{i,j} \) is the product of central...
elements of the Lie Poisson algebra for $B$). We therefore obtain a hypothetically alternative Darboux coordinate representation, this time for admissible pairs $(B, \bar{A})$.

ACKNOWLEDGEMENTS

The authors are grateful to Alexander Shapiro for many useful discussion. Our research was supported RFBR Grant No. 18-01-00273 (L.Ch.) and NSF research grant DMS #1702115 (M. S.). While working on this project, we benefited from support of the following institutions and programs: Research Institute for Mathematical Sciences, Kyoto (M. S., Spring 2019), Research in Pairs Program at the Mathematisches Forschungsinstitut Oberwolfach (M. S., Summer 2019), and Mathematical Science Research Institute, Berkeley (M. S., Fall 2019). We are grateful to all these institutions for their hospitality and outstanding working conditions they provided.

REFERENCES

[1] A. Berenstein and A. Zelevinsky, Quantum cluster algebras Advances Math. 195 (2005) 405–455; math/0404446.
[2] P. P. Boalch, Stokes Matrices, Poisson Lie Groups and Frobenius Manifolds, Invent. Math. 146 (3) (2001) 479–506. doi:10.1007/s002220100170.
[3] A. Bondal, A symplectic groupoid of triangular bilinear forms and the braid groups, preprint IHES/M/00/02 (Jan. 2000).
[4] V. Chari and A. Pressley, A Guide to Quantum Groups. Cambridge University Press, 1994.
[5] Chekhov L., Fock V., A quantum Techm¨ uller space, Theor. Math. Phys. 120 (1999), 1245–1259, math.QA/9908165.
[6] Chekhov L., Fock V., Quantum mapping class group, pentagon relation, and geodesics, Proc. Steklov Math. Inst. 226 (1999), 149–163.
[7] Chekhov L., Fock V., Observables in 3d gravity and geodesic algebras, Czech. J. Phys. 50 (2000) 1201–1208.
[8] Chekhov L.O., Mazzocco M., Isomonodromic deformations and twisted Yangians arising in Teichmüller theory, Advances Math. 226(6) (2011) 4731-4775, arXiv:0909.5350.
[9] L. Chekhov and M. Mazzocco, Block triangular bilinear forms and braid group action, Comm. Math. Phys. 322 (2013) 49–71.
[10] L. Chekhov and M. Mazzocco, On a Poisson space of bilinear forms with a Poisson Lie action, Russ. Math. Surveys 72(6) (2017) 1109–1156; arXiv:1404.0988v2.
[11] Chekhov L.O., Mazzocco M. and Rubtsov V., Algebras of quantum monodromy data and decorated character varieties, arXiv:1705.01447 (2017).
[12] Chekhov, L. and Shapiro,M. Teichm¨ uller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables, Intl. Math. Res. Notices 2013; doi: 10.1093/imrn/rnt016. (arXiv:1111.3963, 20pp)
[13] Coman, I., Gabella M, and Teschner J. 2015. Line Operators in Theories of Class S, Quantized Moduli Space of Flat Connections, and Toda Field Theory. Journal of High Energy Physics 2015 (10). Springer Verlag. doi:10.1007/JHEP10(2015)143. (ArXiv: 1505.05898)
[14] Dubrovin B., Geometry of 2D topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., 1620, Springer, Berlin, (1996) 120–348.
[15] Fock V.V., Dual Teichmüller spaces arXiv:dg-ga/9702018v3, (1997).
[16] V. V. Fock and A. B. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, Publ. Math. Inst. Hautes Études Sci. 103 (2006), 1-211. math.AG/0311149 v4.
[17] V. Fock, A. Goncharov. Cluster X-varieties, amalgamation, and Poisson-Lie groups. Algebraic geometry and number theory. (2006): 65-76. math.AG/0702397 v6.
[18] V. V. Fock and A. B. Goncharov. Cluster ensembles, quantization and the dilogarithm II: The intertwiner, in: Progress in Mathematics. Springer Basel, (2009) pp. 655-673. doi:10.1007/978-0-8176-4745-2-15
[19] V. Fock, A. Goncharov. The quantum dilogarithm and representations of quantum cluster varieties, Inventiones Math. 175(2) (2008) 223–286. math.AG/0702397 v6.
[20] V. V. Fock, A. B. Goncharov. Cluster ensembles, quantization and the dilogarithm II: The intertwiner, in: Progress in Mathematics. Springer Basel, (2009) pp. 655-673. doi:10.1007/978-0-8176-4745-2-15
[21] S. Fomin and A. Zelevinsky Cluster algebras I: Foundations, J. Amer. Math. Soc. 15 (2002), 497-529
[22] Fomin S., Shapiro M., and Thurston D., Cluster algebras and triangulated surfaces. Part I: Cluster complexes, Acta Math. 201 (2008), no. 1, 83–146.

[23] A. M. Gavrilik and A. U. Klimyk, q-Deformed orthogonal and pseudo-orthogonal algebras and their representations Lett. Math. Phys., 21 (1991) 215–220.

[24] M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebra and Poisson geometry, Moscow Math. J. 3(3) (2003) 899-934.

[25] Goldman W.M., Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. 85 (1986) 263–302.

[26] Henneaux M. and Teitelboim Cl., Quantization of Gauge Systems, Princeton University Press, 1992.

[27] M. V. Karasev, Analogues of objects of Lie group theory by nonlinear Poisson brackets, Math. USSR Izvestia 28 (1987) 497–527.

[28] R. M. Kashaev, On the spectrum of Dehn twists in quantum Teichmüller theory, in: Physics and Combinatorics, (Nagoya 2000). River Edge, NJ, World Sci. Publ., 2001, 63–81; math.QA/0008148.

[29] R. M. Kaufmann and R. C. Penner, Closed/open string diagrammatics, Nucl. Phys. B748 (2006) 335–379.

[30] Korotkin D. and Samtleben H., Quantization of coset space σ-models coupled to two-dimensional gravity, Comm. Math. Phys. 190(2) (1997) 411–457.

[31] Kirill Mackenzie General Theory of Lie Groupoids and Lie Algebroids, LMS Lect. Note Series 213 (2005).

[32] A. Molev, E. Ragoucy, P. Sorba, Coideal subalgebras in quantum affine algebras, Rev. Math. Phys., 15 (2003) 789–822.

[33] Nelson J.E., Regge T., Homotopy groups and (2+1)-dimensional quantum gravity, Nucl. Phys. B 328 (1989), 190–199.

[34] Nelson J.E., Regge T., Zertuche F., Homotopy groups and (2 + 1)-dimensional quantum de Sitter gravity, Nucl. Phys. B 339 (1990), 516–532.

[35] Postnikov A., Total positivity, Grassmannians, and networks. arXiv:math/0609764

[36] Postnikov, A., Speyer, D., Williams, L., Matching polytopes, toric geometry, and the non-negative part of the Grassmannian, Journal of Algebraic Combinatorics 30 (2009), no. 2, 173-191.

[37] A. Reyman and M. Semenov-Tian-Shansky, Group-theoretical methods in the theory of finite-dimensional integrable systems. Encyclopaedia of Mathematical Sciences, vol.16, Springer, Berlin, 1994 pp.116–225.

[38] Schrader, G., Shapiro, A. A cluster realization of U_q(sl_n) from quantum character varieties. Invent. math. 216, 799-846 (2019). https://doi.org/10.1007/s00222-019-00857-6

[39] G. Schrader and A. Shapiro, Continuous tensor categories from quantum groups I: algebraic aspects, arXiv:1708.08107

[40] G. Schrader and A. Shapiro, Private communication.

[41] Ugaglia M., On a Poisson structure on the space of Stokes matrices, Int. Math. Res. Not. 1999 (1999), no. 9, 473–493, http://arxiv.org/abs/math.AG/9902045math.AG/9902045.

[42] A. Weinstein, Coisotropic calculus and Poisson groupoids, J. Math. Soc. Japan 40(4) (1988) 705–727.

[43] M. Yakimov, Symplectic leaves of complex reductive Poisson–Lie groups, Duke Math. J. 112 (2002) 453–509.