An inequality for the number of periods in a word

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Abstract

We prove an inequality for the number of periods in a word $x$ in terms of the length of $x$ and its initial critical exponent. Next, we characterize all periods of the length-$n$ prefix of a characteristic Sturmian word in terms of the lazy Ostrowski representation of $n$, and use this result to show that our inequality is tight for infinitely many words $x$. We propose two related measures of periodicity for infinite words. Finally, we also consider special cases where $x$ is overlap-free or squarefree.

1 Introduction

Let $x$ be a finite nonempty word of length $n$. We say that an integer $p$, $1 \leq p \leq n$, is a period of $x$ if $x[i] = x[i+p]$ for $1 \leq i \leq n - p$. For example, the English word alfalfa has periods 3, 6, and 7. A period $p$ is nontrivial if $p < n$; the period $n$ is trivial and is often ignored. The least period of a word is sometimes called the period and is written $\text{per}(x)$. The number of nontrivial periods of a word $x$ is written $\text{nnp}(x)$. Sometimes the prefix $x[1..p]$ is also called a period; in general, this should cause no confusion.

The exponent of a length-$n$ word $x$ is defined to be $\exp(x) = n/\text{per}(x)$. For example, the French word entente has exponent $7/3$. The initial critical exponent $\text{ice}(x)$ of a finite or infinite word $x$ is defined to be

$$\text{ice}(x) := \sup_{p \text{ a nonempty prefix of } x} \exp(p).$$

For example, $\text{ice}($phosphorus$) = 7/4$. This concept was (essentially) introduced by Berthé, Holton, and Zamboni [5].

A word $w$ is a border of $x$ if $w$ is both a prefix and a suffix of $x$. Although overlapping borders are allowed, by convention we generally rule out borders $w$ where $|w| \in \{0, |x|\}$.

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There is an obvious relationship between borders and periods: a length-$n$ word $x$ has a nontrivial period $t$ iff it has a border of length $n-t$. For example, the English word \textit{abracadabra} has periods 7, 10, and 11, and borders of length 1 and 4.

A word is \textit{unbordered} if it has no borders and \textit{bordered} otherwise. An unbordered word $x$ has only the trivial period $|x|$. On the other hand, a word of the form $a^n$, for $a$ a single letter, evidently has the largest possible number of periods; namely, $n$.

In this note we prove an inequality that gives an upper bound for $nnp(x)$, the number of nontrivial periods of (and hence, the number of borders in) a word $x$. Roughly speaking, this inequality says that, in order for a word to have many periods, it must either be very long, or have a large initial critical exponent. We also prove that our inequality is tight, up to an additive constant. To do so, in Section 3 we characterize all periods of the length-$n$ prefix of a characteristic Sturmian word in terms of the lazy Ostrowski representation of $n$. In Section 5, we propose two related measures of periodicity for infinite words, and we compute these measure for some famous words. Finally, in the last two sections, we consider the shortest binary overlap-free (resp., ternary squarefree) words having $n$ periods.

## 2 The period inequality

**Theorem 1.** Let $x$ be a bordered word of length $n \geq 1$. Let $e = \text{ice}(x)$. Then

$$nnp(x) \leq \frac{e}{2} + 1 + \frac{\ln(n/2)}{\ln(e/(e-1))}. \quad (1)$$

**Proof.** We break the bound up into two pieces, by considering the periods of size $\leq n/2$ and $> n/2$. We call these the \textit{short} and \textit{long} periods.

Let $p = \text{per}(x)$, the shortest period of $x$. If $p$ is short, then $x$ has short periods $p, 2p, 3p, \ldots, \lfloor n/(2p) \rfloor p$. Clearly $\text{ice}(x) \geq n/p$, so we get at most $e/2$ periods from this list. To see that there are no other short periods, let $q$ be some short period not on this list. Then $p < q \leq n/2$ by assumption. By the Fine-Wilf theorem [12], if a word of length $n$ has two periods $p, q$ with $n \geq p + q - \gcd(p, q)$, then it also has period $\gcd(p, q)$. Since $\gcd(p, q) \leq p$, either $\gcd(p, q) < p$, which is a contradiction, or $\gcd(p, q) = p$, which means $q$ is a multiple of $p$, another contradiction.

Next, let’s consider the long periods or, alternatively, the short borders (those of length $< n/2$). Suppose $x$ has borders $y, z$ of length $q$ and $r$ respectively, with $q < r < n/2$. Then $x = yy'y = zz'z$ for words $y'$ and $z'$. Hence $z = yt = t'y$ for some nonempty words $t$ and $t'$. Then by the Lyndon-Schützenberger theorem (see, e.g., [16]) we know there exist words $u, v$ with $u$ nonempty, and an integer $d \geq 0$, such that $t' = uv$, $t = vu$, and $y = (uv)^d u$. Hence $x$ has the prefix $z = yt = (uv)^{d+1} u$, which means $e = \text{ice}(x) \geq |z|/|uv| = r/(r-q)$.

Now the inequality $r/(r-q) \leq e$ is equivalent to $r/q \geq e/(e-1)$. Thus if $b_1 < b_2 < \cdots < b_t$ are the lengths of all the short borders of $x$, by the previous paragraph we have

$$b_1 \geq 1, \ b_2 \geq (e/(e-1))b_1 \geq e/(e-1),$$


and so forth, and hence $b_t \geq \left(\frac{e}{e-1}\right)^{t-1}$. All these borders are of length at most $n/2$, so $n/2 > b_t \geq \left(\frac{e}{e-1}\right)^{t-1}$. Hence

$$t \leq 1 + \frac{\ln(n/2)}{\ln(e/(e-1))},$$

and the result follows. \qed

It is also possible to simplify the statement of the bound (1), at the cost of being less precise.

**Corollary 2.** Let $x$ be a word of length $n \geq 1$, and let $e = \text{ice}(x)$. Then

(a) $\text{nnp}(x) \leq \frac{e}{2} + 1 + (e - \frac{1}{2}) \ln(n/2)$;

(b) $\text{nnp}(x) \leq Ce \ln n$, where $C = 3/(2 \ln 2) \doteq 2.164$.

**Proof.** (a) Start with (1). If $e > 1$, then by computing the Taylor series for $\frac{1}{\ln(e/(e-1))}$, we see that

$$\frac{1}{\ln(e/(e-1))} \leq e - \frac{1}{2}.$$

If $e = 1$, then $x$ is unbordered. The left-hand side of (a) is then 0, while the right-hand side is at least $3/2 + (1/2) \ln n/2 \geq 1$.

(b) If $n = 1$ then the desired inequality follows trivially.

Otherwise assume $n \geq 2$. It is easy to check that

$$1 + \frac{1}{2} \ln 2 = (\ln 2 - \frac{1}{2}) + \frac{1}{2} \ln 2 + (C - 1) \ln 2$$

where $C = 3/(2 \ln 2)$. Thus

$$1 + \frac{1}{2} \ln 2 \leq (\ln 2 - \frac{1}{2})e + \frac{1}{2} \ln n + (C - 1)e \ln n,$$

since $n \geq 2$ and $e \geq 1$. Now add $e \ln n$ to both sides and rearrange to get

$$\frac{e}{2} + 1 + (e - \frac{1}{2}) \ln(n/2) \leq Ce \ln n,$$

which by (a) gives the desired result. \qed

It is natural to wonder how tight the bound (1) is for a “typical” word of length $n$. The following two results imply that the expected value of the left-hand side of (1) is $O(1)$, while the expected value of the right-hand side is $\Theta(\ln n)$. Our inequality, therefore, implies nothing useful about the “typical” word.
Theorem 3. Let $k \geq 2$. Over a $k$-letter alphabet, the expected number of borders (or the number of nontrivial periods) of a length-$n$ word is $k^{-1} + k^{-2} + \cdots + k^{1-n} \leq \frac{1}{k-1}$.

Proof. By the linearity of expectation, the expected number of borders is the sum, from $i = 1$ to $n - 1$, of the expected value of the indicator random variable $B_i$ taking the value $1$ if there is a border of length $i$, and $0$ otherwise. Once the left border of length $i$ is chosen arbitrarily, the $i$ bits of the right border are fixed, and so there are $n - i$ free choices of symbols. This means that $E[B_i] = k^{n-i}/k^n = k^{-i}$. □

Theorem 4. The expected value of $\text{ice}(x)$, for finite or infinite words $x$, is $\Theta(1)$.

Proof. Let’s count the fraction $H_j$ of words having at least a $j$’th power prefix. Count the number of words having a $j$’th power prefix with period $1, 2, 3$, etc. This double counts, but shows that $H_j \leq k^{1-j} + k^{2(1-j)} + \cdots = 1/(k^{j-1} - 1)$ for $j \geq 2$. Clearly $H_1 = 1$.

Then $H_{j-1} - H_j$ is the fraction of words having a $(j-1)$th power prefix but no $j$th power prefix. These words will have an ice at most $j$. So the expected value of ice is bounded above by

$$2(H_1 - H_2) + 3(H_2 - H_3) + 4(H_3 - H_4) + \cdots = 2H_1 + H_2 + H_3 + H_4 + \cdots = 2 + \sum_{j \geq 2} 1/(k^{j-1} - 1) = 2 + \sum_{j \geq 1} 1/(k^j - 1).$$ □

3 Periods of prefixes of characteristic Sturmian words

In this section we take a brief digression to completely characterize the periods of the length-$n$ prefix of the characteristic Sturmian word with slope $\alpha$. This characterization is based on a remarkable connection between these periods and the so-called “lazy Ostrowski” representation of $n$. Theorem 6 below implies that all the periods of a length-$n$ prefix of a Sturmian characteristic word can be read off directly from the lazy Ostrowski representation of $n$.

We start by recalling the Ostrowski numeration system. Let $0 < \alpha < 1$ be an irrational real number with continued fraction expansion $[0, a_1, a_2, \ldots]$. Define $p_i/q_i$ to be the $i$’th convergent to this continued fraction, so that $[0, a_1, a_2, \ldots, a_i] = p_i/q_i$. In the (ordinary) Ostrowski numeration system, we write every positive integer in the form

$$n = \sum_{0 \leq i \leq t} d_i q_i,$$ (2)

where $d_i > 0$ and the $d_i$ have to obey three conditions:
(a) \(0 \leq d_0 < a_1\); 
(b) \(0 \leq d_i \leq a_{i+1}\) for \(i \geq 1\); 
(c) For \(i \geq 1\), if \(d_i = a_{i+1}\) then \(d_{i-1} = 0\).

See, for example, [1, §3.9]. 

The lazy Ostrowski representation is again defined through the sum (2), but with slightly different conditions:

(d) \(0 \leq d_0 < a_1\); 
(e) \(0 \leq d_i \leq a_{i+1}\) for \(i \geq 1\); 
(f) For \(i \geq 2\), if \(d_i = 0\), then \(d_{i-1} = a_i\); 
(g) If \(d_1 = 0\), then \(d_0 = a_i - 1\).

See, for example, [11, §5]. By convention, the Ostrowski representation is written as a finite word \(d_t d_{t-1} \cdots d_1 d_0\), starting with the most significant digit.

Next, we recall the definition of the characteristic Sturmian infinite word \(x_\alpha = x_1 x_2 x_3 \cdots\). It is defined by

\[x_i = \lfloor (i + 1)\alpha \rfloor - \lfloor i\alpha \rfloor\]

for \(i \geq 1\). For more about Sturmian words, see [4, 19, 3].

**Example 5.** Take \(\alpha = \sqrt{2} - 1 = [0, 2, 2, 2, \ldots]\). Then \(q_0 = 1, q_1 = 2, q_2 = 5, q_3 = 12\). The first few ordinary and lazy Ostrowski representations are given in the table below.

| \(n\) | ordinary Ostrowski | lazy Ostrowski | \(n\) | ordinary Ostrowski | lazy Ostrowski |
|-------|-------------------|----------------|-------|-------------------|----------------|
| 1     | 1                 | 1              | 15    | 1011             | 221            |
| 2     | 10                | 10             | 16    | 1020             | 1020           |
| 3     | 11                | 11             | 17    | 1100             | 1021           |
| 4     | 20                | 20             | 18    | 1101             | 1101           |
| 5     | 100               | 21             | 19    | 1110             | 1110           |
| 6     | 101               | 101            | 20    | 1111             | 1111           |
| 7     | 110               | 110            | 21    | 1120             | 1120           |
| 8     | 111               | 111            | 22    | 1200             | 1121           |
| 9     | 120               | 120            | 23    | 1201             | 1201           |
| 10    | 200               | 121            | 24    | 2000             | 1210           |
| 11    | 201               | 201            | 25    | 2001             | 1211           |
| 12    | 1000              | 210            | 26    | 2010             | 1220           |
| 13    | 1001              | 211            | 27    | 2011             | 1221           |
| 14    | 1010              | 220            | 28    | 2020             | 2020           |
In what follows, fix a suitable $\alpha$. Let $Y_n$ for $n \geq 1$ be the prefix of $x_n$ of length $n$, and define $X_n := Y_{q_n}$. Let $\text{PER}(n)$ denote the set of all periods of $Y_n$ (including the trivial period $n$). Then we have the following result, which gives a complete characterization of the periods of $Y_n$. It can be viewed as a generalization of a 2009 theorem of Currie and Saari [9, Corollary 8], which obtained the least period of $X_n$.

**Theorem 6.**

(a) The number of periods of $Y_n$ (including the trivial period $n$) is equal to the sum of the digits in the lazy Ostrowski representation of $n$.

(b) Suppose the lazy Ostrowski representation of $n$ is $\sum_{0 \leq i \leq t} d_i q_i$. Define

$$A(n) = \left\{ e q_j + \sum_{j < i \leq t} d_i q_i : 1 \leq e \leq d_j \text{ and } 0 \leq j \leq t \right\}.$$  

Then $\text{PER}(n) = A(n)$.

Part (a) follows immediately from part (b), so it suffices to prove (b) alone. We need some preliminary lemmas.

**Lemma 7.** The lazy Ostrowski representation of $n$ has length $t + 1$ if and only if

$$q_t + q_{t-1} - 1 \leq n \leq q_{t+1} + q_t - 2.$$  

**Proof.** The largest integer $N$ represented by a lazy Ostrowski representation of length $t + 1$ is the one where the coefficient of each $q_i$ takes the maximum possible values allowed by conditions (d) and (e) above, but ignoring condition (f); namely $N = a_1 - 1 + \sum_{1 \leq i \leq t} a_{i+1} q_i$. Suppose $t$ is even; an analogous proof works for the case of $t$ odd. Then

$$q_t = a_t q_{t-1} + q_{t-2}$$  

$$q_{t-1} = a_{t-1} q_{t-2} + q_{t-3}$$  

$$\vdots$$  

$$q_1 = a_1 q_0 + 0,$$

which, by telescoping cancellation, gives

$$q_{t+1} = a_{t+1} q_t + q_{t-1} + a_{t-1} q_{t-2} + \cdots + a_1 q_0. \quad (3)$$

Similarly

$$q_t = a_t q_{t-1} + q_{t-2}$$  

$$q_{t-1} = a_{t-1} q_{t-2} + q_{t-3}$$  

$$\vdots$$  

$$q_2 = a_2 q_1 + q_0,$$  

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which, by telescoping cancellation, gives
\[ q_t = a_t q_{t-1} + a_{t-2} q_{t-3} + \cdots + a_2 q_1 + q_0. \]  

Adding Eqs. (3) and (4) gives \( q_t + q_{t+1} = 1 + a_1 q_0 + \sum_{1 \leq i \leq t} a_i q_i \), and hence \( N = q_t + q_{t+1} - 2 \), as desired.

**Lemma 8.** We have \( A(n) \subseteq \text{PER}(n) \).

**Proof.** Frid [13] defined two kinds of representations in the Ostrowski system. A representation \( n = \sum_{0 \leq i \leq t} d_i q_i \) is legal if \( 0 \leq d_i \leq a_{i+1} \). A representation \( n = \sum_{0 \leq i \leq t} d_i q_i \) is valid if \( Y_n = X_t^d \cdots X_0^d \). She proved [13, Corollary 1, p. 205] that every legal representation is valid. Since the lazy Ostrowski representation is legal [11, Thm. 47], it follows that if \( n = \sum_{0 \leq i \leq t} d_i q_i \) is the lazy Ostrowski representation of \( n \), then \( Y_n = X_t^d \cdots X_0^d \).

We now argue that (thinking of each \( X_i \) as a single symbol) that every nonempty prefix of \( X_t^d \cdots X_0^d \) is a period of \( Y_n \). In other words,
\[
X_t, X_t^2, \ldots, X_t^d, \\
X_t^d X_{t-1}, X_t^d X_{t-1}^2, \ldots, X_t^d X_{t-1}^{d-1}, \\
\ldots, \\
X_t^d X_{t-1}^{d-1} \cdots X_1^d \ X_0, X_t^d X_{t-1}^{d-1} \cdots X_1^d X_0^2, \ldots, X_t^d X_{t-1}^{d-1} \cdots X_1^d X_0^d.
\]

are all periods of \( Y_n \).

We first handle the periods in the first line of (5), which are all powers of \( X_t \). Note that every nonempty suffix of a lazy representation is also lazy, and hence from Lemma 7 we know that \( |X_t^{d-1} \cdots X_0^d| \leq q_t + q_{t-1} - 2 = |X_t X_{t-1}| - 2 \). Furthermore every lazy representation is valid, so \( Y_n = X_t^e Z \), where \( Z = Y_{n-e_t q_t} \) is a (possibly empty) prefix of \( X_t X_{t-1} \). Then \( Y_n = X_t^e Z \) is a prefix of \( X_t^e X_t X_{t-1} \), which is a prefix of \( X_t^{e_t+2} \), which has period \( X_t^j \) for \( 0 \leq j \leq e_t \).

Next, we handle the remaining periods, if there are any. The next one in the list (5) to consider is \( X_t^d X_r \), where \( r \) is the largest index \( < t \) satisfying \( d_r > 0 \). Thus \( Y_n = X_t^d X_r Z' \), where \( Z' = Y_{n-d_r q_r} \). There are two cases to consider:

- If \( r = t - 1 \), then \( X_r Z' = X_t^{d-1} \cdots X_0^d \), and hence, as above \( |X_r Z'| \leq q_t + q_{t-1} - 2 \). It follows that \( |X_t^d X_r| = d_r q_t + q_{t-1} \geq q_t + q_{t-1} > q_t + q_{t-1} - 2 \geq |Z'| \).

- If \( r \leq t - 2 \), then
\[
|X_t^d X_r| = d_t q_t + q_r \geq q_t = a_t q_t + q_r - 2 \geq q_t + q_{t-1} - 2 \geq |X_1^{d-1} \cdots X_0^d|,
\]

where in the last step we have used Lemma 7 again.

Hence in both cases the next period in the list is of size greater than \( n/2 \), and hence so is every period following it in the list. Thus for every period \( P \) after the first line we have \( Y_n = P Z' \) where \( |P| > |Z'| \). Since \( Z' \) is also a valid Ostrowski representation of \( n - |P| \), it follows that \( Z' = Y_{n-|P|} \) is a prefix of \( P \). Thus \( Y_n \) has period \( P \), as desired. \( \square \)
Lemma 9. If \( q_t + q_{t-1} - 1 \leq n \leq q_{t+1} + q_t - 2 \) then the smallest period of \( Y_n \) is at least \( q_t \).

Proof. It suffices to prove the result for \( n = q_t + q_{t-1} - 1 \), since any period of \( Y_{n'} \), \( n' > n \), is at least as large as the smallest period of \( Y_n \). Write \( Y_{n+1} = X_t X_{t-1} \), where \( |X_t| = q_t \) and \( |X_{t-1}| = q_{t-1} \). Let \( ab \) be the last two symbols of \( X_{t-1} \). Then \( a \neq b \) and we have the well-known “almost commutative” property: \( Y_{t-1} = X_t X_{t-1}(ab)^{-1} = X_{t-1}X_t(ba)^{-1} \). Consequently, the word \( Y_{n-1} \) is a central word and has periods \( q_t \) and \( q_{t-1} \), with \( q_{t-1} \) being its smallest period [7, Proposition 1]. Since \( X_{t-1} \) is a prefix of \( X_t \), it is clear that \( Y_n \) has period \( q_t \). The word \( Y_n \) does not have period \( q_{t-1} \), since it would then be a word of length \( q_t + q_{t-1} - 1 \) with co-prime periods \( q_t \) and \( q_{t-1} \), contrary to the Fine-Wilf theorem. The word \( Y_n \) therefore does not have any period that is a multiple of \( q_{n-1} \). Furthermore, if \( Y_n \) had a period \( q \) with \( q_{t-1} < q < q_t \) and \( q \) not a multiple of \( q_{n-1} \), then the central word \( Y_{n-1} \) would have period \( q \) as well. The word \( Y_{n-1} \) would then have periods \( q \) and \( q_{t-1} \), again violating the Fine-Wilf theorem. It follows that \( Y_n \) has smallest period \( q_t \). \( \Box \)

Lemma 10. We have \( \text{PER}(n) \subseteq A(n) \).

Proof. The proof is by induction on \( n \). Certainly the result holds for \( n = 1 \). Suppose the lazy Ostrowski representation of \( n \) is \( \sum_{0 \leq i \leq d} d_i q_i \). By Lemma 7 we have \( q_t + q_{t-1} - 1 \leq n \leq q_{t+1} + q_t - 2 \). Suppose that the elements of \( A(n) \) are ordered by size and note that \( q_t \) and \( n \) are the least and greatest elements of \( A(n) \) respectively.

By Lemma 9, the minimal period of \( Y_n \) is at least \( q_t \), and clearly the maximal period of \( Y_n \) is \( n \). Consequently, if there is some \( p \in \text{PER}(n) \) such that \( p \notin A(n) \), then there are two consecutive periods \( p_1, p_2 \in A(n) \) such that \( p_1 < p < p_2 \). We find then that \( Y_{n-p_1} \) has periods \( p_2 - p_1 \) and \( p - p_1 \).

By the definition of \( A(n) \), the period \( p_1 \) has the form

\[
p_1 = d_t q_t + d_{t-1} q_{t-1} + \cdots + d_{j+1} q_{j+1} + a q_j
\]

for some \( a \leq d_j \). Hence \( n - p_1 \) has lazy representation (possibly including some leading 0’s) \( (d_j - a)d_{j-1} \cdots d_0 \). By the induction hypothesis, we have \( \text{PER}(n - p_1) \subseteq A(n - p_1) \).

However, since \( p_2 \) and \( p_1 \) are consecutive periods of \( Y_n \), we have \( p_2 - p_1 = q_j \) if \( a < d_j \) or \( p_2 - p_1 = q_{j'} \), where \( j' \) is the largest index \( < j \) such that \( d_{j'} > 0 \), if \( a = d_j \). By the definition of \( A(n - p_1) \), the least element of \( A(n - p_1) \) is \( q_j \) if \( a < d_j \) or \( q_{j'} \) if \( a = d_j \). It follows that \( p_2 - p_1 \) is the least element of \( A(n - p_1) \). However, \( p - p_1 \) is smaller than \( p_2 - p_1 \), so we have \( p - p_1 \in \text{PER}(n - p_1) \) but \( p - p_1 \notin A(n - p_1) \) which is a contradiction. \( \Box \)

Theorem 6 now follows from Lemmas 8 and 10.

Let us now apply these results to the infinite Fibonacci word \( f = 01001010 \cdots \), which equals the Sturmian characteristic word \( x_\alpha \) for \( \alpha = (3 - \sqrt{5})/2 = [0, 2, 1, 1, 1, \ldots] \). Recall that the \( n \)th Fibonacci number is defined by \( F_0 = 0, F_1 = 1, \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \). An easy induction shows that \( q_i = F_{i+2} \) for \( i \geq 0 \). Here the ordinary Ostrowski representation corresponds to the familiar and well-studied Fibonacci (or Zeckendorf) representation [15, 24] as a sum of distinct Fibonacci numbers. The lazy Ostrowski representation, on the other
hand, corresponds to the so-called “lazy Fibonacci representation”, as studied by Brown [6]. This representation has the property that it contains no two consecutive 0’s.

Theorem 6 now has the following implications for the Fibonacci word.

Corollary 11.

(a) If the lazy Fibonacci representation of \( n \) is \( n = F_{t_1} + F_{t_2} + \cdots + F_{t_r}, \) for \( t_1 < t_2 < \cdots < t_r, \) then the periods of the length-\( n \) prefix of the Fibonacci word are

\[
F_{t_r}, \ F_{t_r} + F_{t_{r-1}}, \ F_{t_r} + F_{t_{r-1}} + F_{t_{r-2}}, \ \ldots, \ F_{t_r} + F_{t_{r-1}} + \cdots + F_{t_1}.
\]

(b) The shortest prefix of \( f \) having exactly \( n \) periods (including the trivial period) is of length \( F_{n+3} - 2, \) for \( n \geq 1. \)

(c) The longest prefix of \( f \) having exactly \( n \) periods (including the trivial period) is of length \( F_{2n+2} - 1, \) for \( n \geq 1. \)

(d) The least period of \( f[0..m-1] \) is \( F_n \) for \( F_{n+1} - 1 \leq m \leq F_{n+2} - 2 \) and \( n \geq 2. \)

Proof.

(a) This is just a restatement of Theorem 6 for the special case \( \alpha = (3 - \sqrt{5})/2. \)

(b) This corresponds to the lazy Fibonacci representation \( 1^n \), which equals the sum \( F_2 + F_3 + \cdots + F_{n+1}, \) for which a classical Fibonacci identity gives \( F_{n+3} - 2. \)

(c) This corresponds to the lazy Fibonacci representation \( (10)_n, \) which equals the sum \( F_3 + F_5 + \cdots + F_{2n+1}, \) for which a classical Fibonacci identity gives \( F_{2n+2} - 1. \)

(d) Theorem 6 implies that the least period of every \( n \) with Ostrowski representation of length \( t \) is \( F_{t+1}. \) Lemma 7 implies that \( q_{t-1} + q_{t-2} - 1 \leq n \leq q_t + q_{t-1} - 2; \) in other words, \( F_{t+1} + F_t - 1 \leq n \leq F_{t+2} + F_{t+1} - 2, \) or \( F_{t+2} - 1 \leq n \leq F_{t+3} - 2. \)

\( \square \)

For another connection between Ostrowski numeration and periods of Sturmian words, see [21]. Saari [20] determined the least period of every factor of the Fibonacci word, not just the prefixes; also see [18, Thm. 3.15].
4 Tightness of the period inequality

Returning to our period inequality, it is natural to wonder if the bound (1) is tight. We exhibit a class of binary words for which it is.

Let \( g_s \), for \( s \geq 1 \), be the prefix of length \( F_s + 2 - 2 \) of \( f \). Thus, for example, \( g_1 = \epsilon, g_2 = 0, g_3 = 010, g_4 = 010010 \), and so forth. We now show that the bound (1) is tight, up to an additive factor, for the words \( g_s \). Let \( \tau = (1 + \sqrt{5})/2 \), the golden ratio.

**Theorem 12.** Take \( x = g_s \) for \( s \geq 4 \). Then the left-hand side of (1) is \( s - 2 \), while the right-hand side is asymptotically \( s + c \) for \( c = 3 + \tau^2/2 - (\ln 2/\sqrt{5})/\ln \tau = 1.19632 \).

**Proof.** Take \( x = g_s \). By definition we have \( n = |x| = F_{s+2} - 2 \). By Corollary 11 (b) we know that \( g_s \) has \( s - 1 \) periods, and hence \( s - 2 \) nontrivial periods. Thus \( \text{nnp}(x) = s - 2 \).

Next let’s compute \( \text{ice}(g_s) \). Corollary 11 (d) states that the least period of the prefix \( f[0..m-1] \) equals \( F_s \) for \( F_{s+1} - 1 \leq m \leq F_{s+2} - 2, s \geq 2 \). It follows that the exponent of the prefix \( f[0..m-1] \) is \( m/F_s \) for \( F_{s+1} - 1 \leq m \leq F_{s+2} - 2, s \geq 2 \). For fixed \( s \), the quantity \( m/F_s \) is maximized at \( m = F_{s+2} - 2 \), which gives an exponent of \( (F_{s+2} - 2)/F_s \). It remains to see that the sequence \( ((F_{s+2} - 2)/F_s)_{s \geq 2} \) is strictly increasing. For this it suffices to show that \( (F_{s+2} - 2)/F_s < (F_{s+3} - 2)/F_{s+1} \) for \( s \geq 2 \), or, equivalently,

\[
F_{s+2}F_{s+1} - F_sF_{s+3} < 2F_{s+1} - 2F_s. \tag{6}
\]

But an easy induction shows that the left-hand side of (6) is \( (-1)^s \), while the right-hand side is \( 2F_{s-1} \geq 2 \). Thus we see \( e = \text{ice}(g_s) = (F_{s+2} - 2)/F_s \).

Hence the right-hand side of (1) is

\[
\frac{F_{s+2} - 2}{2F_s} + 1 + \frac{\ln((F_{s+2} - 2)/2)}{\ln(F_{s+2} - 2)}. \tag{7}
\]

Now use the Binet formula for Fibonacci numbers, which implies that \( F_s \sim \tau^s/\sqrt{5} \), and the fact that \( \lim_{s \to \infty} F_s/F_{s-1} = \tau \), to obtain that the right-hand side of (1) is asymptotically

\[
\frac{\tau^2}{2} + 1 + (s + 2) - (\ln 2/\sqrt{5})/\ln \tau.
\]

This gives the desired result. \( \square \)

5 Two measures of periodicity

Corollary 2 suggests that the quantity

\[
M(x) := \frac{\text{nnp}(x)}{\text{ice}(x) \ln |x|}
\]

is a good measure of periodicity.
is a measure of periodicity for finite words $x$. It also suggests studying the following measures of periodicity for infinite words $x$. For $n \geq 2$ let $Y_n$ be the prefix of length $n$ of $x$. Then define

$$P(x) := \limsup_{n \to \infty} M(Y_n)$$
$$p(x) := \liminf_{n \to \infty} M(Y_n)$$

From Theorem 4, we know that for the “typical” infinite word $x$ we have $P(x) = p(x) = 0$. Thus it is of interest to find words $x$ where $P(x)$ and $p(x)$ are large. In this section we compute these measures for several infinite words.

**Theorem 13.** Let $f$ denote the Fibonacci infinite word. Then $P(f) = 1/(\tau^2 \ln \tau) \approx 0.79375857$ and $p(f) = 1/(2\tau^2 \ln \tau) \approx 0.396879286$.  

*Proof.* This follows immediately from Corollary 11, together with the calculation of ice given in the proof of Theorem 12. \qed

The *period-doubling word* $d$ is defined to be the fixed point of the morphism sending $1 \to 10$ and $0 \to 11$; see [10].

**Theorem 14.** $P(d) = \frac{1}{2\ln 2} \approx 0.7213$ and $p(d) = \frac{1}{4\ln 2} \approx 0.36067$.

*Proof.* Since $d$ is not a Sturmian word, or even closely related to one, we need to use different techniques from those we used previously.

Let $r(n)$ denote the number of periods (including the trivial period) in the length-$n$ prefix of $d$. We use $(n)_2$ to denote the canonical base-2 representation of $n$, and $(n,p)_2$ to denote the base-2 representation of $n$ and $p$ as a sequence of pairs of bits (where the shorter representation is padded with leading zeros, if necessary).

We can use the theorem-proving software Walnut to calculate the periods of prefixes of $d$. (For more about Walnut, see [17].) We sketch the ideas briefly.

We can write a first-order logical formula pdp($m,p$) stating that the prefix of length $m \geq 1$ of $d$ has period $p$, $1 \leq p \leq m$:

$$\text{pdp}(m,p) := (1 \leq p \leq m) \land d[0..m - p - 1] = d[p..m - 1]$$
$$= (1 \leq p \leq m) \land \forall t (0 \leq t < m - p) \implies d[t] = d[t + p].$$

Such a formula can be automatically translated, using Walnut, to an automaton that recognizes the language

$$\{(n,p)_2 : \text{the length-}n \text{ prefix of } d \text{ has period } p\}.$$

We depict it below.
Such an automaton can be automatically converted by \texttt{Walnut} to a linear representation for $r(n)$, as discussed in [8]. This is a triple $(v, \rho, w)$ where $v, w$ are vectors, and $\rho$ is a matrix-valued morphism, such that $r(n) = v \cdot \rho((n)_2) \cdot w$. The values are given below:

$$v = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho(1) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$  

From this, using the technique described in [14], we can easily compute the relations

$$r(0) = 0$$
$$r(2n + 1) = r(n) + 1, \quad n \geq 0$$
$$r(4n) = r(n) + 1, \quad n \geq 1$$
$$r(4n + 2) = r(n) + 1, \quad n \geq 0.$$  

Reinterpreting this definition for $r$, we see that $r(n)$ is equal to the length of the (unique) factorization of $(n)_2$ into the factors 1, 00, and 10. It now follows that

(a) The smallest $m$ such that $r(m) = n$ is $m = 2^n - 1$;

(b) The largest $m$ such that $r(m) = n$ is $m = \lfloor 2^{2n+1}/3 \rfloor$, with $(m)_2 = (10)^n$.

Similarly, we can use \texttt{Walnut} to determine the smallest period $p$ of every length-$n$ prefix of $d$. We use the predicate

$$\text{pdlp}(n, p) := \text{pdp}(n, p) \land \forall q \ (1 \leq q < p) \implies \text{pdp}(n, q).$$

This gives the automaton
Inspection of this automaton shows that least period of the prefix of length $n$ is, for $s \geq 2$, equal to $3 \cdot 2^{s-2}$ for $2^s \leq n < 5 \cdot 2^{s-2}$ and $2^s$ for $5 \cdot 2^{s-2} \leq n < 2^{s+1}$. It follows that the initial critical exponent of every prefix of $d$ of length $n$, for $2^t - 1 \leq n \leq 2^{t+1} - 2$, is $2 - 2^{1-t}$.

The result now follows. \hfill \square

**Theorem 15.** Let $t = t_0t_1t_2 \cdots = 01101001 \cdots$ be the Thue-Morse word, the fixed point of the morphism $\mu$ described above. Then $P(t) = 3/(10 \ln 2) \approx 0.4328$ and $p(t) = 0$.

**Proof.** We have $\text{ice}(x) = 5/3$ for every prefix $x$ of $t$ of length $\geq 5$, a claim that can easily be verified with Walnut.

For the value of $p(t)$, it suffices to observe that $\text{mnp}(x) = 1$ if $x$ is a prefix of $t$ of length $3 \cdot 2^n + 1$ for $n \geq 0$, which can also be verified with Walnut.

For $P(t)$ it suffices to show that the shortest prefix of $t$ having $n$ nontrivial periods is of length $2^{2n-1} + 2$. For this we can use Walnut, but the analysis is somewhat complicated. Letting $v(n)$ denote the number of nontrivial periods of the length-$n$ prefix of $t$, we can mimic what we did for the period-doubling word, obtaining the matrices and the following relations for $n \geq 0$:

\[
\begin{align*}
v(4n) &= v(n) + [n \neq 0] \\
v(4n + 3) &= v(4n + 1) \\
v(8n + 1) &= v(2n + 1) + t_n \\
v(8n + 2) &= v(2n + 1) + t_n \\
v(8n + 6) &= v(4n + 1) + 1 - t_n \\
v(16n + 5) &= v(2n + 1) + 1 \\
v(16n + 13) &= v(4n + 1) + 1.
\end{align*}
\]

Here $[n \neq 0]$ is the Iverson bracket, which evaluates to 1 if the condition holds and 0 otherwise.
Now a tedious induction on $m$, which we omit, shows that

- $m$ is even and $v(m) \geq n \implies m \geq 2^{2n-3} + 2$;
- $m$ is odd and $v(m) \geq n \implies m \geq 2^{2n-2} + 1$,

and furthermore $v(2^{2n-3} + 2) = n$ for $n \geq 2$. It follows that the shortest prefix of $t$ having $n$ nontrivial periods is of length $2^{2n-1} + 2$ for $n \geq 2$, from which the desired result follows. □

**Remark 16.** The Walnut commands for the last two results are available on the third author’s web page, at

https://cs.uwaterloo.ca/~shallit/papers.html.

Walnut itself is available at

https://github.com/hamousavi/Walnut.

**Remark 17.** It would be interesting to compute the values of

$$D_1 := \inf_{n \geq 1} \sup_{x \in \{0,1\}^n} M(x),$$

$$D_2 := \liminf_{n \to \infty} \sup_{x \in \{0,1\}^n} M(x).$$

Theorem 13 shows that $D_2 \geq 1/(2\tau^2 \ln \tau) = 0.396879286$. Thus, for example, for every sufficiently large $n$ there is a length-$n$ binary string $x$ with $M(x) \geq 0.396$.

### 6 Shortest overlap-free binary word with $p$ periods

In this section and the following one, we consider how quickly the number of periods can grow if we enforce an upper bound on the exponent of repetitions occurring in the word.

Recall that an **overlap** is a word of the form $axaxa$, where $a$ is a single letter and $x$ is a (possibly empty) word. An example in English is the word alfalfa. We say a word is **overlap-free** if no finite factor is an overlap.

Define $f(p)$ to be the length of the shortest binary overlap-free word having $p$ nontrivial periods. Recall that we call a border $w$ of $x$ **short** if $|w| < |x|/2$.

Define the morphism $\mu$ by $\mu(0) = 01$ and $\mu(1) = 10$. If $w = axa$ for a single letter $a$ and (possibly empty) word $x$, define $\gamma(w) = a^{-1} \mu^2(w) a^{-1}$, or, in other words, the word $\mu^2(w)$ with an $a$ removed from the front and back.

**Lemma 18.** Define a sequence of words $(A_n)_{n \geq 3}$ as follows:

$$A_n = \begin{cases} 
001001100100, & \text{if } n = 3; \\
\gamma(A_{n-1}), & \text{if } n \geq 4.
\end{cases}$$

Then $A_n$ is a palindrome with $n$ short palindromic borders for $n \geq 3$. 


Proof. Observe that if \( w \) is a palindrome, then so is \( \gamma(w) \). Write \( \overline{a} = 1 - a \) for \( a \in \{0, 1\} \).

We now prove the claim by induction on \( n \). It is true for \( n = 3 \), since the borders are 0, 00, and 00100.

Now assume the result is true for \( n \); we prove it for \( n + 1 \). Suppose \( n \) short palindromic borders of \( A_n \) are \( w_1, w_2, \ldots, w_n \), and each starts with the letter \( a \). From the observation above, we know that \( A_{n+1} = \gamma(A_n) \) is a palindrome. We claim that \( \overline{a}, \gamma(w_1), \gamma(w_2), \ldots, \gamma(w_n) \) are short palindromic borders of \( \gamma(A_n) \).

To see that \( \overline{a} \) is a border of \( A_{n+1} \), note that \( A_n = awa \) for some \( w \), so \( \gamma(A_n) = \overline{aaa} \mu^2(w) \overline{aaa} \).

Otherwise, let \( w_i \) be a palindromic border of \( A_n \). Since it is short, we have \( A_n = w_i y w_i \) for some \( y \). Then \( \gamma(w_i) \) is both a prefix and suffix of \( \gamma(A_n) \) and hence is a palindromic border of \( A_{n+1} \). The claim about the length of the borders is trivial.

Thus \( A_{n+1} \) has at least \( n + 1 \) palindromic short borders. \( \square \)

**Corollary 19.** We have \( f(1) = 2, f(2) = 5, \) and \( f(p) \leq (17/6)^{p-2} + 2/3 \) for \( p \geq 3 \).

**Proof.** For \( p = 1 \), the shortest binary overlap-free word with 1 nontrivial period is 00. For \( p = 2 \) it is 00100.

Next we argue, by induction on \( p \), that that each \( A_p \), for \( p \geq 3 \), is overlap-free. The base case is \( p = 3 \), and is easy to check. Otherwise assume the result is true for \( A_p \). We now use a classical result that if a word \( x \) is overlap-free, then so is \( \mu(x) \) [23]. Applying this twice, we see that \( \mu^2(A_p) \) is overlap-free. Then \( A_{p+1} = \gamma(A_p) \) is overlap-free, since it is a factor of \( \mu^2(A_p) \).

As we have seen above, \( A_p \) has \( p \) borders and hence \( p \) nontrivial periods. The only thing left to verify is that \( |A_p| = (17/6)^{p-2} + 2/3 \) for \( p \geq 3 \). This is an easy induction, and is left to the reader. \( \square \)

**Remark 20.** One can go from \( A_p \) to \( A_{p+1} \), for \( p \geq 3 \), via the following procedure, which we state without proof. Write \( A_p \) in terms of its run-length encoding, that is, \( A_p = a^{e_1} b^{e_2} a^{e_3} b^{e_4} \ldots \), where \( a \neq b \) and all the \( e_i \) are positive. Then, considering \( c^e \) as the pair \((c, e)\), apply the following morphism:

\[
\begin{align*}
(0, 1) & \rightarrow 1101 \\
(1, 1) & \rightarrow 0010 \\
(0, 2) & \rightarrow 11001101 \\
(1, 1) & \rightarrow 00110010
\end{align*}
\]

Finally, drop the last two symbols.

**Remark 21.** We conjecture that the words \( A_p \) constructed above are actually the shortest overlap-free binary words with \( p \) periods with \( p \geq 3 \), but we do not currently have a proof of this claim in general. The sequence \((f(p))\) is sequence \( A334811 \) in the On-Line Encyclopedia of Integer Sequences [22].
7 Shortest squarefree ternary word with $p$ periods

Recall that a square is a nonempty word of the form $xx$, such as the English word murmur. A word is squarefree if no finite factor is a square.

Let $g(p)$ be the length of the shortest ternary squarefree word having $p$ nontrivial periods. Here are the first few values of $g$, computed through exhaustive search.

| $p$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $g(p)$ | 1 | 3 | 7 | 23 | 59 |

Theorem 22. For $p \geq 3$ we have $g(p) \leq \frac{17}{12}4^{p-1} + 1/3$.

Proof. Consider the words $A_p$ defined above. Suppose $A_p$ starts and ends with the letter $a$. Let $B_p$ be the word whose $i$’th letter is the number of occurrences of $\bar{a}$ between the $i$’th and the $(i+1)$’th occurrence of $a$. For example, we have

$$B_3 = 0102010$$
$$B_4 = 02012012021020121020$$
$$B_5 = 02012012021020121012021020120120120210201201201202102012012012021012021020120120120210201201201202102012012012021020120120120210120210$$

Then each $B_p$ is squarefree. For if $B_p$ had a square, say $c_1c_2 \cdots c_tc_1c_2 \cdots c_t$, then $A_p$ has the overlap

$$ab^{c_1}ab^{c_2} \cdots ab^{c_1}ab^{c_2} \cdots ab^{c_t}a,$$

where $b = \bar{a}$, a contradiction.

Furthermore, each border of $A_p$, except the border of length 1, corresponds via this map to a border of $B_p$. So $\text{mp}(B_p) = p - 1$. By induction we can show $|A_p| = |B_p|/2 = (17/12)4^{p-2} + 1/3$ for $p \geq 4$. It follows that $g(p) \leq (17/12)4^{p-1} + 1/3$.

Remark 23. Our bound is clearly not optimal. It would be interesting to obtain better bounds for $g(p)$. The sequence $(g(p))$ is sequence A332866 in the On-Line Encyclopedia of Integer Sequences [22].

Remark 24. One can go from $B_p$ to $B_{p+1}$, for $p \geq 4$, using the following procedure, which we state without proof. Take $B_p$ and replace every other 1 in it with 3. Then apply the following morphism:

$$0 \rightarrow 0201$$
$$1 \rightarrow 2101$$
$$2 \rightarrow 2021$$
$$3 \rightarrow 0121.$$

Finally, drop the last letter.
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