Time Variable Cosmological Constants from Cosmological Horizons

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In this paper, time variable cosmological constants, dubbed horizon cosmological constants, as analogues with de Sitter cosmological boundary against a positive cosmological constant. The horizon cosmological constants correspond to Hubble horizon, future event horizon and particle horizon are discussed respectively. When the Hubble horizon is taken as a cosmological length scale, the effective equation of state of horizon cosmological constant is quintessence like and an accelerated expansion universe is obtained. When the future event horizon and particle horizon are taken as the roles of cosmological length scales, the forms of effective equation of state of horizon cosmological constants, which are the the same as the holographic ones, are derived. But, their evolutions are so different.

I. INTRODUCTION

The observation of the Supernovae of type Ia [1, 2] provides the evidence that the universe is undergoing accelerated expansion. Jointing the observations from Cosmic Background Radiation [3, 4] and SDSS [5, 6], one concludes that the universe at present is dominated by 70% exotic component, dubbed dark energy, which has negative pressure and pushes the universe to accelerated expansion. Of course, a natural explanation to the accelerated expansion is due to a positive tiny cosmological constant. Though, it suffers the so-called fine tuning and cosmic coincidence problems. However, in 2σ confidence level, it fits the observations very well [7]. If the cosmological constant is not a real constant but is time variable, the fine tuning and cosmic coincidence problems can be removed. In fact, this possibility was considered in the past years.

In particular, the dynamic vacuum energy density based on holographic principle are investigated extensively [8, 9]. According to the holographic principle, the number of degrees of freedom in a bounded system should be finite and has relations with the area of its boundary. By applying the principle to cosmology, one can obtain the upper bound of the entropy contained in the universe. For a system with size $L$ and UV cut-off $\Lambda$ without decaying into a black hole, it is required that the total energy in a region of size $L$ should not exceed the mass of a black hole of the same size, thus $L^3\rho_\Lambda \leq LM_P^2$. The largest $L$ allowed is the one saturating this inequality, thus $\rho_\Lambda = 3c^2M_P^2L^{-2}$, where $c$ is a numerical constant and $M_P$ is the reduced Planck Mass $M_P = \frac{8\pi G}{c^2}$. It just means a duality between UV cut-off and IR cut-off. The UV cut-off is related to the vacuum energy, and IR cut-off is related to the large scale of the universe, for example Hubble horizon, future event horizon or particle horizon as discussed by [8, 9, 13]. The holographic dark energy in Brans-Dicke theory is also studied in Ref. [13, 14, 15, 16, 17, 18].

In the standard and Brans-Dicke holographic dark energy models when the Hubble horizon is taken as the role of IR cut-off, non-accelerated expansion universe can be achieved [8, 9, 13]. However, the Hubble horizon is the most natural cosmological length scale, how to realize an accelerated expansion by using it as an IR cut-off will be interesting.

As known, for any nonzero value of the cosmological constant $\Lambda$, a natural length scale and time scale

$$r_\Lambda = t_\Lambda = \sqrt{\frac{3}{|\Lambda|}}$$

can be introduced into Einstein’s theory. Reversely, a cosmological length scale and time scale may introduce a cosmological constant or vacuum energy density into Einstein’s theory. Of course, the important is how to choose a proper cosmological length scale or time scale to obtain a tiny cosmological constant or vacuum energy density. Honestly, we have not the first physical principle to determine the length or time scale. But, one can immediately relate these length or time scales to the biggest natural scales, for example Hubble horizon, particle horizon and future event horizon etc. Because, what we are talking about is the system: universe. So, for a cosmological length scale

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When the length scale is time variable, a time variable cosmological constant can be obtained. Inspired by this observation, one can consider time variable cosmological constant from this analogue and let the holographic principle alone. The important is that with this analogue an accelerated expansion will be obtained when the Hubble horizon is taken as a cosmological length scale. If the other cosmological length scales, for examples future event horizon and particle horizon, are put in, a time variable tiny cosmological constant and an accelerated expansion universe will be obtained. In this paper, we will explore these possibilities. For their explicit relations with the cosmological length scales, i.e. cosmological horizons as contrasts with the holographic cosmological constants [10, 11, 12], we dub them horizon cosmological constants.

This paper is structured as follows. In Section II, we give a brief review of time variable cosmological constant. In Section III A, III B and III C, Hubble horizon, future event horizon and particle horizon will be considered as cosmological length scale respectively and their corresponding time variable cosmological constants are discussed. We will discuss their evolutions in Section III D. Conclusions are set in Section IV.

II. TIME VARIABLE COSMOLOGICAL CONSTANT

The Einstein equation with a cosmological constant is written as

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \]  

(3)

where \( T_{\mu\nu} \) is the energy-momentum tensor of ordinary matter and radiation. From the Bianchi identity, one has the conservation of the energy-momentum tensor \( \nabla^\mu T_{\mu\nu} = 0 \), it follows necessarily that \( \Lambda \) is a constant. To have a time variable cosmological constant \( \Lambda = \Lambda(t) \), one can move the cosmological constant to the right hand side of Eq. (3) and take \( \tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{\Lambda(t)}{8\pi G} g_{\mu\nu} \) as the total energy-momentum tensor. Once again to preserve the Bianchi identity or local energy-momentum conservation law, \( \nabla^\mu \tilde{T}_{\mu\nu} = 0 \), one has, in a spacially flat FRW universe,

\[ \dot{\rho}_\Lambda + 3H (1 + w_m) \rho_m = 0, \]

(4)

where \( \rho_\Lambda = M_P^2 \Lambda(t) \) is the energy density of time variable cosmological constant and its equation of state is \( w_\Lambda = -1 \), and \( w_m \) is the equation of state of ordinary matter, for dark matter \( w_m = 0 \). It is natural to consider interactions between variable cosmological constant and dark matter [11], as seen from Eq. (4). After introducing an interaction term \( Q \), one has

\[ \dot{\rho}_m + 3H (1 + w_m) \rho_m = Q, \]

(5)

\[ \dot{\rho}_\Lambda + 3H (p_\Lambda + p) = -Q, \]

(6)

and the total energy-momentum conservation equation

\[ \dot{\rho}_{tot} + 3H (\rho_{tot} + p_{tot}) = 0. \]

(7)

For a time variable cosmological constant, the equality \( \rho_\Lambda + p = 0 \) still holds. Immediately, one has the interaction term \( Q = -\dot{\rho}_\Lambda \) which is different from the interactions between dark matter and dark energy considered in the literatures [13], where a general interacting form \( Q = 3b^2H (\rho_m + \rho_\Lambda) \) is put by hand. With observation to Eq. (6), the interaction term \( Q \) can be moved to the left hand side of the equation, and one has the effective pressure of the time variable cosmological constant- dark energy

\[ \dot{\rho}_\Lambda + 3H (p_\Lambda + p^{eff}_\Lambda) = 0, \]

(8)

where \( p^{eff}_\Lambda = p_\Lambda + \frac{Q}{3H} \) is the effective dark energy pressure. Also, one can define the effective equation of state of dark energy

\[ w^{eff}_\Lambda = \frac{p^{eff}_\Lambda}{\rho_\Lambda} \]

\[ = -1 + \frac{Q}{3H \rho_\Lambda} \]

\[ = -1 - \frac{1}{3} \frac{d \ln \rho_\Lambda}{d \ln a}. \]

(9)
The Friedmann equation as usual can be written as, in a spatially flat FRW universe,

\[ H^2 = \frac{1}{3M_P^2} (\rho_m + \rho_\Lambda). \]  

(10)

### III. HORIZON COSMOLOGICAL CONSTANTS

#### A. Hubble horizon as cosmological length scale

In fact, Horvat has considered this possibility from holographic principle [11], where the Hubble horizon \( H^{-1} \) was taken as a cosmological length scale. When Hubble horizon \( H^{-1} \) is chosen, one obtains a time variable cosmological constant

\[ \Lambda(t) = 3c^2 H^2(t) \]  

(11)

which is just the one considered by Horvat [11], where \( c \) is a constant. As known, our universe is filled with dark matter and dark energy and deviates from a de Sitter one. Just to describe this gap, the constant \( c \) was introduced in. It can be seen that a \( c < 1 \) constant is expected under the consideration of the energy budget of universe. Also, one can see that, when \( c = 1 \), the de Sitter universe will be recovered. Now, the corresponding vacuum energy density can be written as

\[ \rho_\Lambda = 3c^2 M_P^2 H^2 \]  

(12)

which takes the same form as the so-called holographic dark energy based on holographic principle. With this vacuum energy, the Friedmann equation (10) can be rewritten as

\[ \rho_m = 3(1 - c^2) M_P^2 H^2. \]  

(13)

To protect a positive dark matter energy density, a constraint

\[ c^2 > 1 \]  

(14)

is required. Immediately, a scaling solution is obtained

\[ \frac{\rho_m}{\rho_\Lambda} = \frac{1 - c^2}{c^2}. \]  

(15)

Substituting Eq. (15) into Eq. (14), one has

\[ \rho_\Lambda = \frac{c^2}{1 - c^2} \rho_m \sim a^{-3(1-c^2)}. \]  

(16)

Here, one can see a rather different result on \( \rho_m \) from the standard evolution \( a^{-3} \). In this case, the deceleration parameter becomes

\[ q = -\frac{\ddot{a}}{a^2} = -\frac{\dot{H} + H^2}{H^2} = \frac{1}{2} - \frac{3}{2} c^2. \]  

(17)

To obtain a current accelerated expansion universe, i.e. \( q < 0 \), and to protect positivity of dark matter energy density, one obtains a constraint to the constant \( c \)

\[ \frac{1}{3} < c^2 < 1. \]  

(18)

The effective equation of state of vacuum energy density is

\[ w_\Lambda^{\text{eff}} = -1 - \frac{1}{3} \frac{d \ln \rho_\Lambda}{d \ln a} = -c^2. \]  

(19)

Under the constraint Eq.(18), one can see that a quintessence like dark energy is obtained. This is tremendous different from holographic dark energy model where non-accelerated expansion universe can be achieved when the Hubble horizon taken as the IR cut-off [8, 9, 12]. Also, it is easily that the de Sitter universe will be recovered when \( c = 1 \). Once the constant \( c \) deviates from \( c = 1 \), a scaling solution will be obtained.
B. Future event horizon as a cosmological length scale

The future event horizon is defined as

$$R_e = a \int_0^\infty \frac{dt}{a} = a \int_a^\infty \frac{da'}{Ha'^2}$$

(20)

which is the boundary of the volume a fixed observer may eventually observe. Taking it as the role of cosmological length scale, one has the vacuum energy density

$$\rho_\Lambda = 3c^2 M_p^2 / R_e^2.$$  

(21)

Defining the dimensionless energy densities $$\Omega_m = \rho_m / (3M_p^2 H^2)$$ and $$\Omega_\Lambda = \rho_\Lambda / (3M_p^2 H^2),$$ the Friedmann equation is rewritten as

$$\Omega_m + \Omega_\Lambda = 1.$$  

(22)

The energy conservation equation (4) can be rewritten as

$$\frac{d}{dx} \ln H + \frac{3}{2} (1 - \Omega_\Lambda) = 0,$$  

(23)

where $$x = \ln a.$$ Combining Eq. (20), Eq. (21) and the definition of the dimensionless energy density $$\Omega_\Lambda,$$ one has

$$\int_a^\infty \frac{d \ln a'}{Ha'} = \frac{c}{aH} \sqrt{\Omega_\Lambda}.$$  

(24)

Taking the derivative with respect to $$x = \ln a$$ from the both sides of the above equation (24), one has the differential equation

$$\frac{d}{dx} \ln H + \frac{1}{2} \frac{d}{dx} \ln \Omega_\Lambda = \frac{\sqrt{\Omega_\Lambda}}{c} - 1.$$  

(25)

Substituting Eq. (23) into above differential equation, one obtains the differential equation of $$\Omega_\Lambda$$

$$\Omega'_\Lambda = \Omega_\Lambda \left( 1 - 3\Omega_\Lambda + \frac{2}{c} \sqrt{\Omega_\Lambda} \right),$$  

(26)

where ' denotes the derivative with respect to $$x = \ln a.$$ This equation describes the evolution of the dimensionless energy density of dark energy. Clearly, one can see that this equation is different from the corresponding one derived in [9] for its different form. Given the initial conditions $$\Omega_{\Lambda 0},$$ the evolution of the horizontal cosmological constant will be obtained. Once the values of $$c$$ and $$\Omega_{\Lambda 0}$$ are fixed, $$\Omega_\Lambda,$$ as a function of $$\ln a$$ or redshift $$z,$$ describes the evolution of the dark energy completely. From Eq. (9), it is easy to obtain the effective equation of state of dark energy

$$w_\Lambda^{eff} = -1 - \frac{1}{3} \frac{d \ln \rho_\Lambda}{d \ln a}$$

$$= -1 - \frac{1}{3} \left( 1 + 2 \frac{c}{\sqrt{\Omega_\Lambda}} \right).$$  

(27)

It is in the range of $$-(1 + \frac{2}{3})/3 < w_\Lambda^{eff} < -1/3,$$ when one notices the dark energy density ratio $$0 \leq \Omega_\Lambda \leq 1.$$ The form of the effective equation of state of the horizon cosmological constant is the same as the one of holographic dark energy. However, for the differences of the evolution of $$\Omega_\Lambda,$$ its behaviors will be different from the one of holographic dark energy. One can also easily have the deceleration parameter

$$q = -\frac{\dot{H} + H^2}{H^2}$$

$$= -1 - \frac{d \ln H}{d \ln a}$$

$$= \frac{1}{2} - \frac{3}{2} \Omega_\Lambda.$$  

(28)

To have an current accelerated expansion of the universe, $$\Omega_{\Lambda 0} > 1/3$$ is required.
C. Particle horizon as cosmological length scale

The particle horizon is defined as

\[ R_p = a(t) \int_0^t \frac{dt'}{a} = a \int_0^a \frac{da'}{Ha'^2} \]  

(29)

which is the length scale a particle can pass from the beginning of the universe. In this case, the vacuum energy density is given as

\[ \rho_\Lambda = 3c^2 \frac{M_p^2}{R_p^2}. \]  

(30)

Repeating the analysis and calculations as done in III B, one has the differential equation of \( \Omega_\Lambda \)

\[ \Omega'_\Lambda = \Omega_\Lambda \left( 1 - 3 \Omega_\Lambda - \frac{2}{c} \sqrt{\Omega_\Lambda} \right), \]  

(31)

where \( ' \) denotes the derivative with respect to \( x = \ln a \). The effective equation of state and deceleration parameter, the same as that of future event horizon case, are given as

\[ w^\text{eff}_\Lambda = -\frac{1}{3} \left( 1 + \frac{2}{c} \sqrt{\Omega_\Lambda} \right), \]  

(32)

\[ q = \frac{1}{2} - \frac{3}{2} \Omega_\Lambda. \]  

(33)

Though they have the same forms, the evolutions of them are different from that of the future event horizon case for their differences between the evolution equations of \( \Omega_\Lambda \).

D. Evolution curves of horizon cosmological constants

Now, it is proper to discuss the evolutions of horizon cosmological constants. Here, we just discuss the evolutions of effective equation of state and dimensionless energy density. In the holographic dark energy model, the value of the parameter \( c \) determines the property of holographic dark energy. When \( c > 1 \), \( c = 1 \) and \( c < 1 \) when the event horizon taken as the role of IR cut-off, the holographic dark energy behaves like quintessence, cosmological constant and phantom, respectively. In our cases, there are some differences and common grounds. In the Hubble horizon cosmological constant case, the effective equation of state of time variable cosmological constant is always quintessence like or true constant when the value of \( c \) is \( c < 1 \) and \( c = 1 \) respectively. However, in the corresponding holographic case, non-accelerated expansion universe can be obtained when Hubble horizon taken as an IR cut-off. In the future event horizon and particle horizon cases, the behaviors of effective equation of state determined by \( c \) are the same as that of the holographic ones. We have plotted the evolutions of equation of state and dimensionless energy density with respect to the redshift \( z \) in Fig. 1.

IV. CONCLUSIONS

In this paper, time variable cosmological constants, dubbed horizon cosmological constants, as analogues with de Sitter cosmological boundary against a positive cosmological constant are explored. The horizon cosmological constants correspond to Hubble horizon, future event horizon and particle horizon are discussed respectively. When the Hubble horizon is taken as a cosmological length scale, the effective equation of state of horizon cosmological constant \( w^\text{eff}_\Lambda = -c^2 \) is quintessence like. In this case, the deceleration parameter \( q = 1/2 - 3c^2/2 \) is negative when \( 1/3 < c^2 < 1 \), i.e. an accelerated expansion universe is obtained. That is tremendous different from the holographic dark energy model in Einstein’ gravity theory and Brans-Dicke theory where non-accelerated expansion universe can exist when the Hubble horizon taken as an IR cut-off. It also can be seen that, once the constant \( c = 1 \), the de Sitter universe will be recovered. When future event horizon and particle horizon are taken as the cosmological length scales, the forms of effective equation of state of horizon cosmological constants are the same as the holographic ones. But, their evolutions and behaviors are different from the holographic one for their differences between the evolution equations, see Eq. (26) and Eq. (31). These also can be seen from the evolution curves in Fig. 1.
FIG. 1: The evolutions of effective equation of state \( w_\Lambda(z) \) and dimensionless density parameter \( \Omega_\Lambda(z) \) of horizon cosmological constants with respect to the redshift \( z \), where the values \( \Omega_\Lambda,0 = 0.70, \, c = 0.5 \) (upper panel), \( c = 0.9 \) (lower panel) are adopted. Here, the solid lines and dashed lines denote the future event horizon and particle horizon respectively.

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[1] A.G. Riess, et al., Astron. J. 116 1009(1998) [astro-ph/9805201].
[2] S. Perlmutter, et al., Astrophys. J. 517 565(1999) [astro-ph/9812133].
[3] D.N. Spergel et al., Astrophys. J. Suppl. 148 175(2003) [astro-ph/0302209].
[4] D.N. Spergel et al., Astrophys. J. Suppl.170 377(2007) [astro-ph/0603449].
[5] M. Tegmark et al., Phys. Rev. D 69 (2004) 103501 [astro-ph/0310723].
[6] M. Tegmark et al., Astrophys. J. 606 (2004) 702 [astro-ph/0310725].
[7] E. Komatsu et al., Astrophys. J. Suppl. 180, 330 (2009) [arXiv:0803.0547].
[8] S.D.H. Hsu, Phys. Lett. B594 13(2004) [arXiv:hep-th/0403052].
[9] M. Li, Phys. Lett. B603 1(2004) [hep-th/0403127]; Q.G. Huang and M. Li, JCAP 0503, 001 (2005) [hep-th/0410095]; Q.G. Huang and M. Li, JCAP 0408, 013 (2004) [astro-ph/0404229]; B. Chen, M. Li and Y. Wang, Nucl. Phys. B 774, 256 (2007) [astro-ph/0611623]; J.F. Zhang, X. Zhang and H.Y. Liu, Eur. Phys. J.C 52 693(2007) [arXiv:0708.3121].
[10] P. Horava, D. Minic, Phys. Rev. Lett. 85 1610(2000).
[11] R. Horvat, Phys. Rev. D70 087301(2004).
[12] C.J. Feng, Phys. Lett. B 663 367(2008).
[13] L.X. Xu, W.B. Li, J.B. Lu, Eur. Phys. J. C 60 135 (2009).
[14] Y. Gong, Phys. Rev. D 61 (2000) 043505.
[15] Y. Gong, Phys. Rev. D 70 (2004) 064029.
[16] M. R. Setare, Phys. Lett. B644 99(2007).
[17] N. Banerjee, D. Pavon, Phys. Lett. B 647 447(2007).
[18] B. Nayak, L. P. Singh, [arXiv:0803.2930].
[19] B. Wang, C.Y. Lin, E.O Abdalla, Phys. Lett. B 637 357(2006); H. Kim, H.W. Lee, Y.S. Myung, Phys. Lett. B 632 605(2006); B. Hu, Y. Jing, Phys. Rev. D 73 123510(2006); W. Zimdahl, D. Pavon, [arXiv:astro-ph/0606555]; H.M. Sadjadi, JCAP0702 026(2007); M.R. Setare, E.C. Vaginas, Int. J. Mod. Phys. D 18 147(2009); Q. Wu, Y. Gong, A. Wang, J.S. Alcaniz, [arXiv:0705.1006]; J.F. Zhang, X. Zhang, H.Y. Liu, Phys. Lett. B 659 26(2008); C. Feng, B. Wang, Y. Gong, R.K. Su, [arXiv:0706.4043]; S.F. Wu, P.M. Zhang, G.H. Yang, Class. Quan. Grav. 26 055020(2009); M. A. Rashid, M. U. Farooq, M. Jamil, [arXiv:0901.3724].