GEOMETRIC CHARACTERIZATIONS OF $C^1$ MANIFOLD IN EUCLIDEAN SPACES BY TANGENT CONES

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Commemorating the 150th Birthday of Giuseppe Peano (1858-1932)

Abstract. A remarkable and elementary fact that a locally compact set $F$ of Euclidean space is a smooth manifold if and only if the lower and upper paratangent cones to $F$ coincide at every point, is proved. The celebrated von Neumann’s result (1929) that a locally compact subgroup of the general linear group is a smooth manifold, is a straightforward application. A historical account on the subject is provided in order to enrich the mathematical panorama. Old characterizations of smooth manifold (by tangent cones), due to Valiron (1926, 1927) and Severi (1929, 1934) are recovered; modern characterizations, due to Gluck (1966, 1968), Tierno (1997) and Shchepin and Repovš (2000) are restated.

Introduction

A primary aim of this paper is to prove that

**Theorem (Four-cones coincidence).** A non-empty subset $F$ of $\mathbb{R}^n$ is a $C^1$-manifold if and only if $F$ is locally compact and the lower and upper paratangent cones to $F$ coincide at every point, i.e.,

$$p\text{Tan}^-(F,x) = p\text{Tan}^+(F,x) \text{ for every } x \in F.$$ 

This theorem entails numerous other existing characterizations, as well as von Neumann’s theorem (1929) that a locally compact subgroup of the general linear group is a smooth manifold.

The upper paratangent cone $p\text{Tan}^+(F,x)$ (introduced by Severi and Bouligand in 1928) and lower tangent cone $p\text{Tan}^-(F,x)$ (introduced by Clarke in 1973) are defined respectively as the upper and the lower limits of the homothetic relation

$$\frac{1}{\lambda}(F - y)$$

as $\lambda$ tends to 0 and $y$ tends to $x$ within $F$. They are closely related to the upper tangent cone $\text{Tan}^+(F,x)$ and the lower tangent cone $\text{Tan}^-(F,x)$, defined by Peano in 1887, as the upper and the lower limits of $\frac{1}{\lambda}(F - x)$ as $\lambda$ tends to 0. In general,

$$p\text{Tan}^-(F,x) \subset \text{Tan}^-(F,x) \subset \text{Tan}^+(F,x) \subset p\text{Tan}^+(F,x),$$

so that the condition of our theorem amounts to the coincidence of all these cones.

A secondary aim of this paper is to retrace historical information by direct references to mathematical papers where notions and properties first occured to the
best of our knowledge. As a consequence to this historical concern, some geometrical characterizations of $C^1$-manifold by tangent cones implement conditions and properties recovered from forgotten mathematical papers of VALIRON and SEVERI.

Section 1: Tangency and paratangency. Investigation of $C^1$-manifolds involve four tangent cones approximating, already mentioned. The upper and lower tangent cones were introduced by PEANO to ground tangency on a firm basis and to establish optimality necessary conditions; as today modern habit, PEANO defined upper and lower tangent cones as limits of sets. The upper tangent cone, which was recovered 41 years later by SEVERI and BOULIGAND in 1928, is known as BOULIGAND’s contingent cone.

Section 2: Tangency and paratangency in traditional sense compared with differentiability. Characterizations of both differentiability (called today Fréchet differentiability) and strict differentiability of functions on arbitrary sets (not necessary open, as a today habit) are stated and, with a pedagogical intent, proved. They are due essentially to GUARESCHI and SEVERI. The modern definition of differentiability of vector functions is due to GRASSMANN (1862), although there is a slighter imperfection. This imperfection was corrected by PEANO in 1887 (for scalar functions) and in 1908 (for vector functions). The notion of strict differentiability was introduced by PEANO (1892) for real functions of one real variable and by SEVERI (1934) for several variables.

Section 3: Grassmann Exterior algebra, limits of vector spaces and angles between vector spaces. Following PEANO’s Applicazioni geometriche (1887), limits of sets and exterior algebra of Grassmann are used to defined convergence of vector spaces. Exterior algebra of Grassmann is used to associate multivectors to vector spaces and, consequently, to define the notion of angle between vector spaces of same dimension and, finally, to express convergence of vector spaces by their angle. In 1888 PEANO revisited exterior algebra of Grassmann in Calcolo Geometrico secondo l’Ausdehnungslehre of Grassmann and here he introduced the terms of bi-vector, tri-vector and, more important, the modern notion of vector space.

Section 4: Four-cones Coincidence Theorem: local and global version. We state and prove our main theorem: the four-cone theorem. As its straightforward application, we show a celebrated von NEUMANN’s result (1929) that a locally compact subgroup of the general linear group is a smooth manifold. Moreover, some corollaries of the main theorem will provide efficacious test for visual reconnaissance of $C^1$-manifolds.

Appendix A: Von Neumann and alternative definitions of lower tangent cones. We comment on von Neumann’s definition of Lie algebra in his famous paper [63, (1929)] on matrix Lie groups, and present alternative definitions of the lower tangent and lower paratangent cones for encompassing von Neumann’s tangent vectors.

Appendix B: From Fréchet problem to modern characterizations of smooth manifold. In 1925 Fréchet inquires into existence of non-singular continuously differentiable parametric representations of continuous curves. This problem had been a starting, motivating and reference point for subsequent research by various mathematicians. Two basic conditions for the existence of a non-singular parametrization of a set (either curve or surface) were given by VALIRON:
in the setting of the topological manifolds, first geometrical characterizations of \( C^1 \)-manifold by tangent cones. From a historical point of view, an essential condition to a complete geometrical characterization of \( C^1 \)-manifold by tangent cones, has been a solid and univocal (but not necessary unique) definition of tangency and a \( C^1 \) version of differentiability, the so-called strict differentiability. Finally, old characterizations of smooth manifold (by tangent cones), due to Valiron (1926, 1927) and Severi (1929, 1934) are recovered; modern characterizations, due to Gluck (1966, 1968), Tierno (1997) and Shchepin and Repovš (2000) are restated.

**Remark.** In the following sections, the symbols \( \mathbb{R} \) and \( \mathbb{N} \) will denote the real and natural numbers, respectively; and \( \mathbb{N}_1 := \{m \in \mathbb{N} : n \geq 1\} \), \( \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\} \), \( \mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\} \). If not otherwise specified, any set will be a subset of some finite dimensional Euclidean space \( \mathbb{R}^n \). An open (resp. closed) ball of center \( \hat{x} \) and ray \( \varepsilon \) will be denoted by \( B_\varepsilon(x) \) (resp. \( \overline{B}_\varepsilon(x) \)). \( \mathbb{P}(\mathbb{R}^n) \) denotes the set of all subsets of \( \mathbb{R}^n \). The set of accumulation points of a given set \( A \) and its interior are denoted by \( \text{der}(A) \) and \( \text{int}(A) \), respectively.

1. Tangency and Paratangency

Let \( F \) be an arbitrary subset of Euclidean space \( \mathbb{R}^n \) and let \( x \in \mathbb{R}^n \). We will consider four types of tangent cone to \( F \) at \( x \): the lower and the upper tangent cones

\[
\tan^-(F,x), \quad \tan^+(F,x)
\]

respectively; and the lower and the upper paratangent cones

\[
p\tan^-(F,x), \quad p\tan^+(F,x)
\]

respectively. All of them are cones\(^1\) of \( \mathbb{R}^n \). They satisfy the following set inclusions:

\[
(1.1) \quad p\tan^-(F,x) \subset \tan^-(F,x) \subset \tan^+(F,x) \subset p\tan^+(F,x).
\]

The elements of \( p\tan^-(F,x) \) (resp. \( \tan^-(F,x) \), \( \tan^+(F,x) \), \( p\tan^+(F,x) \)) are referred to as lower paratangent (resp. lower tangent, upper tangent, upper paratangent) vectors to \( F \) at \( x \).

In order to define them as lower or upper limits of homothetic sets, let us introduce two types of limits of sets (the so-called Kuratowski limits). Let \( A_\lambda \) be a subset of \( \mathbb{R}^n \) for every real number \( \lambda > 0 \). The lower limit \( \text{Li}_{\lambda \to 0^+} A_\lambda \) and upper limit \( \text{Ls}_{\lambda \to 0^+} A_\lambda \) are defined by \(^2\)

\[
(1.2) \quad \text{Li}_{\lambda \to 0^+} A_\lambda := \{v \in \mathbb{R}^n : \lim_{\lambda \to 0^+} \text{dist}(v, A_\lambda) = 0\},
\]

\[
(1.3) \quad \text{Ls}_{\lambda \to 0^+} A_\lambda := \{v \in \mathbb{R}^n : \liminf_{\lambda \to 0^+} \text{dist}(v, A_\lambda) = 0\},
\]

where define \( \text{dist}(x, A) := \inf\{||x - a|| : a \in A\} \) for every \( x \in \mathbb{R}^n \) and \( A \subset \mathbb{R}^n \).

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\(^1\)In the sequel, a set \( A \subset \mathbb{R}^n \) is said to be a cone, if \( \lambda v \in A \) for every \( v \in A \) and each \( \lambda \in \mathbb{R}_+ \).

\(^2\)The limits of sets (1.2) and (1.3) were introduced by Peano: the lower limit in *Applicazioni geometriche* [41, (1887), p. 302] and the upper limit in *Lessoni di analisi infinitesimale* [44, (1893), volume 2, p. 187] (see Dolecki, Greco [14, (2007)]) for further historical details.)
Obviously $\text{Li}_{\lambda \to 0^+} A_\lambda \subset \text{Ls}_{\lambda \to 0^+} A_\lambda$. They can be characterized in terms of sequences:

$$\begin{align*}
(1.4) \quad & v \in \text{Li}_{\lambda \to 0^+} A_\lambda \iff \forall \{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \exists \{a_m\}_m \text{ with } a_m \in A_{\lambda_m} \text{ eventually, such that } \lim_m a_m = v, \\
(1.5) \quad & v \in \text{Ls}_{\lambda \to 0^+} A_\lambda \iff \exists \{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \exists \{a_m\}_m \text{ with } a_m \in A_{\lambda_m} \text{ eventually, such that } \lim_m a_m = v.
\end{align*}$$

The lower and upper tangent cones $\text{Tan}^{-}(F, x)$ and $\text{Tan}^{+}(F, x)$ are defined, respectively, by the following blow-up

$$\begin{align*}
(1.6) \quad & \text{Tan}^{-}(F, x) := \text{Li}_{\lambda \to 0^+} \frac{1}{\lambda} (F - x), \\
(1.7) \quad & \text{Tan}^{+}(F, x) := \text{Ls}_{\lambda \to 0^+} \frac{1}{\lambda} (F - x).
\end{align*}$$

Since $\text{dist}(v, \frac{1}{\lambda}(F - x)) = \frac{1}{\lambda} \text{dist}(x + \lambda v, F)$, it follows from (1.2) and (1.3) that

$$\begin{align*}
(1.8) \quad & v \in \text{Tan}^{-}(F, x) \iff \lim_{\lambda \to 0^+} \frac{1}{\lambda} \text{dist}(x + \lambda v, F) = 0, \\
(1.9) \quad & v \in \text{Tan}^{+}(F, x) \iff \liminf_{\lambda \to 0^+} \frac{1}{\lambda} \text{dist}(x + \lambda v, F) = 0.
\end{align*}$$

Therefore, in terms of sequences, from (1.4) and (1.5) 4

$$\begin{align*}
(1.10) \quad & v \in \text{Tan}^{-}(F, x) \iff \forall \{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \exists \{x_m\}_m \subset F \text{ such that } \lim_m \frac{x_m - x}{\lambda_m} = v, \\
(1.11) \quad & v \in \text{Tan}^{+}(F, x) \iff \exists \{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \exists \{x_m\}_m \subset F \text{ such that } \lim_m \frac{x_m - x}{\lambda_m} = v.
\end{align*}$$

Generally, the lower and upper tangent cones are denominated adjacent and (Bouligand) contingent cones, respectively. 5

The lower and upper paratangent cones $\text{pTan}^{-}(F, x)$ and $\text{pTan}^{+}(F, x)$ are defined, respectively, by the following blow-up

$$\begin{align*}
(1.12) \quad & \text{pTan}^{-}(F, x) := \text{Li}_{\frac{F \ni y}{\lambda \to 0^+}} \frac{1}{\lambda} (F - y), \\
(1.13) \quad & \text{pTan}^{+}(F, x) := \text{Ls}_{\frac{F \ni y}{\lambda \to 0^+}} \frac{1}{\lambda} (F - y).
\end{align*}$$

3The affine variants of the lower and upper tangent cones (1.6) and (1.7) were introduced by Peano: the lower tangent cone in Applicazioni geometriche [41, (1887), p. 305] and the upper tangent cone in Applicazioni geometriche [41, (1887) n. 11 p. 143-144] (implicitly) and in Formulario Mathematico [45, (1903) p. 296] (explicitly). See DOLECKI, GRECO [14, (2007)] for further historical details.

4A new sequential definition of the lower tangent cone was introduced in DOLECKI, GRECO [14, (2007)]; see Appendix A.

5See Aubin and Frankowska [1], Rockafellar and Wets [49], Murdokhovich [36]
According to (1.2) and (1.3), we have
\begin{align}
(1.14) \quad & v \in p\text{Tan}^-(F, x) \iff \lim_{\lambda \to 0^+} \frac{1}{\lambda} \text{dist}(y + \lambda v, F) = 0, \\
(1.15) \quad & v \in p\text{Tan}^+(F, x) \iff \liminf_{\lambda \to 0^+} \frac{1}{\lambda} \text{dist}(y + \lambda v, F) = 0.
\end{align}

Therefore, in terms of sequences, from (1.4) and (1.5)
\begin{align}
(1.16) \quad & v \in p\text{Tan}^-(F, x) \iff \left\{ \begin{array}{l}
\forall \{\lambda_m\}_m \subset \mathbb{R}^+ \text{ with } \lambda_m \to 0^+, \\
\forall \{y_m\}_m \subset F \text{ with } y_m \to x,
\end{array} \right.
\exists \{x_m\}_m \subset F \text{ such that } \lim_m \frac{y_m - y_m}{\lambda_m} = v,

(1.17) \quad & v \in p\text{Tan}^+(F, x) \iff \left\{ \begin{array}{l}
\exists \{\lambda_m\}_m \subset \mathbb{R}^+ \text{ with } \lambda_m \to 0^+, \\
\exists \{y_m\}_m \subset F \text{ with } y_m \to x,
\end{array} \right.
\exists \{x_m\}_m \subset F \text{ such that } \lim_m \frac{y_m - y_m}{\lambda_m} = v.
\end{align}

Generally, the upper and lower paratangent cone are called paratangent cone and Clarke tangent cone, respectively. 

A sequence \( \{x_m\}_m \subset \mathbb{R}^n \) converging to a point \( x \) is called a tangential sequence, if there exist an infinitesimal sequence \( \{\lambda_m\}_m \subset \mathbb{R}^+ \) and a non-null vector \( v \) such that \( \lim_{m \to \infty} \frac{x_m - x}{\lambda_m} = v \). Moreover, a couple of sequences \( \{x_m\}_m, \{y_m\}_m \subset \mathbb{R}^n \) converging to a same point \( x \) is called a paratangential couple, if there exist an infinitesimal sequence \( \{\lambda_m\}_m \subset \mathbb{R}^+ \) and a non-null vector \( v \) such that \( \lim_{m \to \infty} \frac{x_m - y_m}{\lambda_m} = v \).

The upper tangent and paratangent cones share basic compactness properties which are essential to prove several propositions in next Section. These compactness properties are expressed in terms of sequences:
\begin{align}
(1.18) \quad & \text{every convergent sequence with infinitely many distinct terms, admits a tangential subsequence;} \\
(1.19) \quad & \text{every proper \footnote{We say that a couple of convergent sequences \( \{x_m\}_m, \{y_m\}_m \subset \mathbb{R}^n \) is proper if \( x_m \neq y_m \) for infinitely many \( m \).} \text{couple of convergent sequences to a same point admits a paratangential couple of subsequences.}
\end{align}

Analogous compactness properties for both lower tangent and lower paratangent cones do not hold. For example, for \( S := \{ \frac{1}{m^2} : m \in \mathbb{N}, m \geq 1 \} \) one has \( p\text{Tan}^-(S, 0) = \text{Tan}^- (S, 0) = \{0\} \) (see Example A.4); therefore the sequence \( \{\frac{1}{m^2}\}_m \) does not admit subsequences having non-null lower tangent vectors.

In the following proposition we collect well-known properties on tangent cones, which are used in subsequent proofs in Section 4.

\footnote{The upper paratangent cone was introduced as a set of straight-lines by Severi \cite{51, 1928} p. 149 and Bouligand in \cite{2, 1928} pp. 29-30; see Dolecki, Greco \cite{15, 2011} for further historical details.}

\footnote{The lower paratangent cone was introduced in 1973 by Clarke \cite{8, 1975} and redefined in terms of sequences by Thibault \cite{56, 1976, p. 1303}, Hiriart-Urruty \cite{29, 1977, p. 1381}; in Appendix A a new sequential definition is given. The upper and lower paratangent cones were expressed by blow-up in Dolecki \cite{13, 1882}.}

\footnote{Property (1.18) was first used by Cassina in \cite{6, 1930} to show the existence of a non-null upper tangent vector to a set at an accumulation point. Properties (1.18) and (1.19) were frequently and freely used by Severi and Bouligand in their works; for example, they are stated in Severi \cite{53, 1931, p. 342].}
Proposition 1.1. Let $S \subset \mathbb{R}^n$ be non-empty and $\hat{x} \in S$. Then the following properties hold.

1. (20) The upper paratangent cone is bilateral, i.e.,
   \[ p\text{Tan}^+(S, \hat{x}) = -p\text{Tan}^+(S, \hat{x}). \]

2. (21) (Bouligand [5, (1932), p. 75]) The upper paratangent cone is upper semi-continuous, i.e.,
   \[ \text{Ls}_{S \ni x \to \hat{x}} p\text{Tan}^+(S, x) \subset p\text{Tan}^+(S, \hat{x}). \]

3. (22) (Clarke [8, (1975)]) The lower paratangent cone is convex, i.e.,
   \[ p\text{Tan}^-(S, \hat{x}) \text{ is convex}. \]

4. (23) (Cornet [10, (1981)], [11, (1981)] for closed sets) If $S$ is locally compact at $\hat{x}$, then
   \[ p\text{Tan}^-(S, \hat{x}) = \text{Li}_{S \ni x \to \hat{x}} \text{Tan}^+(S, x). \]

5. (24) (Furi [20, (1995), p. 96]) $S$ is open if and only if
   \[ S \text{ is locally compact and } \text{Tan}^+(S, x) = \mathbb{R}^n \text{ for all } x \in S. \]

6. (25) (Rockafellar [47, (1979), p. 149] for closed sets) If $S$ is locally compact at $\hat{x}$, then
   \[ \hat{x} \in \text{int}(S) \iff p\text{Tan}^-(S, \hat{x}) = \mathbb{R}^n. \]

7. (26) (Cassina [6, (1930)]) The point $\hat{x}$ is an accumulation point of $S$ if and only if
   \[ \text{Tan}^+(S, \hat{x}) \text{ contains non-null vectors}. \]

8. (27) (Bouligand [2, (1928) p. 33], [5, (1932) p. 76-79]) The orthogonal projection onto the linear hull of $p\text{Tan}^+(S, \hat{x})$ is injective on a neighborhood of $\hat{x}$ in $S$. More generally, if $V$ and $W$ are vector spaces such that $V \cap p\text{Tan}^+(S, \hat{x}) = \{0\}$ and $\mathbb{R}^n = V \oplus W$, then there is $\varepsilon \in [0, +\infty)$ such that the projection along $V$ onto $W$ is injective on $S \cap B_{\varepsilon}(\hat{x})$.

Additional characterizations of open sets by tangent cones, we have that $S$ is said to be locally compact at $\hat{x}$, whenever there exists a compact neighborhood of $\hat{x}$ in $S$. If $e_n := (0, \ldots, 0, 1) \notin p\text{Tan}^+(S, \hat{x})$, then there exists an open ball $B_{\varepsilon}(\hat{x})$ such that $p$ is injective on $S \cap B_{\varepsilon}(\hat{x})$ and, moreover, defined $\varphi : p(S \cap B_{\varepsilon}(\hat{x})) \to \mathbb{R}$ by $\varphi(x_1, \ldots, x_{n-1}) := \text{the real number } x_n \text{ such that } (x_1, \ldots, x_{n-1}, x_n) \in S \cap B_{\varepsilon}(\hat{x})$ and $p(x_1, \ldots, x_{n-1}, x_n) := (x_1, \ldots, x_{n-1})$, the following property holds

\[ \varphi \text{ is Lipschitz and } \text{graph}(\varphi) = S \cap B_{\varepsilon}(\hat{x}). \]
(1.29) \( S \) is open if and only if \( S \) is locally compact and

\[
\Tan^+(S, x) = -\Tan^+(S, x) \quad \text{and} \quad \LTan^+(S, x) = \mathbb{R}^n \quad \text{for all} \quad x \in S \, .
\]

A rich and unstable terminology deals with coincidence conditions: \( \Tan^-(S, \hat{x}) = \Tan^+(S, \hat{x}) \) \(^{12}\), \( p\Tan^-(S, \hat{x}) = \Tan^+(S, \hat{x}) \) \(^{13}\), \( p\Tan^-(S, \hat{x}) = p\Tan^+(S, \hat{x}) \) \(^{14}\). Below we present unidimensional examples of all possible coincidence the various coincidence conditions.

\[
\begin{align*}
p\Tan^-(F, x) & \subset \Tan^-(F, x) \subset \Tan^+(F, x) \subset p\Tan^+(F, x) \\
p\Tan^-(F, x) & \subset \Tan^-(F, x) \subset \Tan^+(F, x) \subset p\Tan^+(F, x)
\end{align*}
\]

Example 1.2. \( F := \{0\} \cup \{1/m! : m \in \mathbb{N}_1\} \). Here \( p\Tan^-(F, 0) = \Tan^-(F, 0) = \{0\}, \Tan^+(F, 0) = \mathbb{R}_+, \ p\Tan^+(F, 0) = \mathbb{R} \).

Example 1.3. \( F := \mathbb{R}_+ \). Here \( p\Tan^-(F, 0) = \Tan^-(F, 0) = \Tan^+(F, 0) = \mathbb{R}_+, \ p\Tan^+(F, 0) = \mathbb{R} \).

Example 1.4. \( F := \{0\} \). Here \( p\Tan^-(F, 0) = \Tan^-(F, 0) = \Tan^+(F, 0) = \{0\} \).

Example 1.5. \( F := \{0\} \cup \{1/m : m \in \mathbb{N}_1\} \). Here \( p\Tan^-(F, 0) = \{0\}, \Tan^-(F, 0) = \Tan^+(F, 0) = \mathbb{R}_+, \ p\Tan^+(F, 0) = \mathbb{R} \).

Example 1.6. \( F := \{0\} \cup \{1/m : m \in \mathbb{N}_1\} \cup \{-1/m : m \in \mathbb{N}_1\} \). Here \( p\Tan^-(F, 0) = \{0\}, \Tan^-(F, 0) = \Tan^+(F, 0) = \{0\} \).

Example 1.7. \( F := \{0\} \cup \{1/m! : m \in \mathbb{N}_1\} \cup \{-1/m : m \in \mathbb{N}_1\} \). Here \( p\Tan^-(F, 0) = \{0\}, \ p\Tan^-(F, 0) = -\mathbb{R}_+, \Tan^+(F, 0) = p\Tan^+(F, 0) = \mathbb{R} \).

\(^{11}\)Here and in the sequel, \( \LTan^+(S, x) \) denotes the linear hull of tangent cone \( \Tan^+(S, x) \).

\(^{12}\)Proof of (1.29) (only sufficiency). Suppose by absurd that a point \( \hat{x} \in S \) is not interior to \( S \). Then, by local compactness, choose \( \varepsilon \in \mathbb{R}_+ \) and \( \hat{x} \not\in S \) such that \( \mathbb{B}_\varepsilon(\hat{x}) \cap S \) is closed and \( \|\hat{x} - \tilde{x}\| < \varepsilon/2 \). Now, denote by \( p(\tilde{x}) \) a projection of \( \tilde{x} \) on the closed set \( \mathbb{B}_\varepsilon(\hat{x}) \cap S \). Then \( 0 < \|\tilde{x} - p(\tilde{x})\| < \varepsilon/2, p(\tilde{x}) \in \mathbb{B}_\varepsilon(\hat{x}) \cap S \) and \( \mathbb{B}_{\|\tilde{x} - p(\tilde{x})\|/2}(\tilde{x}) \cap \mathbb{B}_\varepsilon(\hat{x}) \cap S = \emptyset \). Hence the open vector half-space \( H^+ := \{v \in \mathbb{R}^n : \langle v, \hat{x} - p(\tilde{x}) \rangle > 0\} \) has no elements in common with the upper tangent cone \( \Tan^+(S, p(\hat{x})) \). Since, by the hypothesis, \( \Tan^+(S, p(\hat{x})) = -\Tan^+(S, p(\tilde{x})) \), then even the opposite open vector half-space \( H^- := \{v \in \mathbb{R}^n : \langle v, \hat{x} - p(\tilde{x}) \rangle < 0\} \) has no elements in common with \( \Tan^+(S, p(\tilde{x})) \). Therefore the vector hyperplane \( H := \{v \in \mathbb{R}^n : \langle v, \hat{x} - p(\tilde{x}) \rangle = 0\} \) includes \( \Tan^+(S, p(\tilde{x})) \); in contradiction of the hypothesis \( \Tan^+(S, \hat{x}) = \mathbb{R}^n \).

\(^{13}\)If this equality holds, the set \( S \) is said to be “derivable at \( \hat{x} \) in AUBIN, Frankowska [1, (1990), p. 127], “geometrically derivable at \( \hat{x} \) in Rockafellar, Wets [49, (1998), p. 197-198], “tangent regular in Shchepin, Repovš [50, (2000), p. 2117]. If, in addition, \( \Tan^+(S, \hat{x}) \) is a vector space, the set \( S \) is said to be “smooth at \( \hat{x} \) in Rockafellar [48, (1985), p. 173].

\(^{14}\)If this equality holds, the set \( S \) is said to be “tangentially regular at \( \hat{x} \) in AUBIN, Frankowska [1, (1990), p. 127], “regular at \( \hat{x} \) in Rockafellar, Wets [49, (1998), p. 220], and “Clarke regular at \( \hat{x} \) in Mordukhovich [36, (2006), p. 136].
2. TANGENCY AND PARATANGENCY IN TRADITIONAL SENSE COMPARED WITH DIFFERENTIABILITY

Traditionally, intrinsic notions of tangent straight line to a curve and that of tangent plane to a surface at a point can be resumed by the following general definition.

**Definition 2.1.** Let \( \hat{x} \) be an accumulation point of a subset \( F \) of \( \mathbb{R}^n \). A vector space \( H \) of \( \mathbb{R}^n \) is said to be tangent in traditional sense to \( F \) at \( \hat{x} \) if

\[
\lim_{x \to \hat{x}} \frac{\text{dist}(x, H + \hat{x})}{\text{dist}(x, \hat{x})} = 0.
\]

Since \( \frac{\text{dist}(x, H + \hat{x})}{\text{dist}(x, \hat{x})} \) is the sinus of the angle between \( H \) and the vector \( x - \hat{x} \), the geometric meaning of (2.1) is evident: the half-line that passes through \( \hat{x} \) and \( x \in F \) and the affine space \( H + \hat{x} \) form an angle that tends to zero as \( x \) tends to \( \hat{x} \).

![Fig. 1: \( F = \{(x, y) : y^3 = x^2\}\) Fig. 2: \( F = \{(x, y) : (y - x^2)(y - 2x^2) = 0\}\)]

The sets \( F \) of fig. 1 and 2 admit everywhere tangent line in traditional sense; in both cases the tangent lines vary continuously.

Analytically Definition 2.1 becomes:

**Proposition 2.2.** A vector space \( H \) of \( \mathbb{R}^n \) is tangent in traditional sense to \( F \) at an accumulation point \( \hat{x} \) of \( F \) if and only if

\[
\text{Tan}^+ (F, \hat{x}) \subset H. \tag{2.2}
\]

According to (2.2), we assume, as a definition, that every vector space of \( \mathbb{R}^n \) is tangent in traditional sense to \( F \) at the isolated points of \( F \).

**Proof.** Necessity of (2.2). Let \( H \) be a vector space, tangent in traditional sense to \( F \) at \( \hat{x} \). Fix a non-null vector \( v \in \text{Tan}^+ (F, \hat{x}) \). There exist sequences \( \{x_m\}_m \subset F \) and \( \{\lambda_m\}_m \subset \mathbb{R}^+ \) such that \( \lim_{m} \lambda_m = 0 \), \( \lim_{m} x_m = x \) and \( \lim_{m} \frac{x_m - \hat{x}}{\lambda_m} = v \);

hence

\[
\lim_{m} \frac{x_m - \hat{x}}{\|x_m - \hat{x}\|} = \frac{v}{\|v\|}.
\]

Therefore by (2.1) we have

\[
0 = \lim_{m} \frac{\text{dist}(x_m, H + \hat{x})}{\|x_m - \hat{x}\|} = \lim_{m} \text{dist} \left( \frac{x_m - \hat{x}}{\|x_m - \hat{x}\|}, H \right) = \text{dist} \left( \frac{v}{\|v\|}, H \right).
\]

Then \( v \in H \).

Sufficiency of (2.2). Suppose that (2.1) does not hold. Then there is a sequence \( \{x_m\}_m \subset F \setminus \{\hat{x}\} \) with \( \lim_{m} x_m = \hat{x} \) and

\[
\lim_{m} \text{dist} \left( \frac{x_m - \hat{x}}{\|x_m - \hat{x}\|}, H \right) > 0. \tag{*}
\]

15Something more, the upper tangent cone \( \text{Tan}^+ (F, \hat{x}) \) is the smallest closed set \( H \) (not necessary, vector spaces) verifying (2.1).
By compactness, \( \left\{ \frac{x_m - \hat{x}}{\|x_m - \hat{x}\|} \right\}_m \) has a subsequence \( \left\{ \frac{x_{m_k} - \hat{x}}{\|x_{m_k} - \hat{x}\|} \right\}_k \) converging to a vector \( w \) of norm 1. Therefore, by (2.2), \( w \in H \); consequently \( \lim_k \text{dist}(x_{m_k} - \hat{x}, H) = 0 \), contradicting (\( \ast \)). □

Following Valiron [59, (1926)], [60, (1927), p. 47], a vector space \( H \) is said to be tangent in Valiron sense to \( F \) at \( \hat{x} \), if \( H = \text{Tan}_+^\ast(F, \hat{x}) \).

Generally, tangency was regarded as an elementary, intuitive notion not needing a definition. When definitions however, were written down, they were neither precise and general, nor univocal and coherent. As a case study, the reader may consider the following definitions of Lagrange and Fréchet.

(2.3) Lagrange [32, (1813), p. 259] writes:

Ainsi, de même qu’une ligne droite peut être tangente d’une courbe, un plan peut être tangent d’une surface, et l’on déterminera le plan tangent par la condition qu’aucun autre plan ne puisse être mené par le point de contact entre celui-là et la surface.

(2.4) Fréchet [16, (1912), p. 437], [19, (1964), p. 189] writes:

Préciions d’abord que nous entendons par plan tangent à [une surface] \( S \) au point \((a, b, c)\) un plan qui soit le lieu des tangentes aux courbes situées sur \( S \) et passant par ce point (s’entendant de celles de ces courbes qui ont effectivement une tangente en ce point).

(2.5) Fréchet [19, (1964), p. 193] writes:

Un plane \( P \) passant par un point \( Q \) d’une surface \( S \) est, par définition, tangent à \( S \) en \( Q \) si,

1°) \( M \) étant un point quelconque de \( S \), distinct de \( Q \), l’angle aigu de \( M \) avec \( P \) tend vers zéro quand \( M \) tend vers \( Q \).

2°) La condition \( W \) ci-dessus [c’est-à-dire, si l’on projette \( S \) sur \( P \), il y a au moins un voisinage de \( Q \) qui appartient entièrement à cette projection] est satisfaite.

Following Guareschi [24, (1934), p. 175-176], [25, (1936), p. 131-132], the vector space generated by \( \text{Tan}_+^\ast(F, \hat{x}) \) is called linear upper tangent space of \( S \) at \( \hat{x} \); it is denoted by \( \text{LTan}_+^\ast(S, \hat{x}) \). Proposition 2.2 allows us to describe the linear upper tangent space as the smallest vector space which is tangent in traditional sense.

\[ f(x, y) = \begin{cases} x & \text{if } x \in \left\{ \frac{1}{m!} : m \in \mathbb{N} \right\} \text{ and } y = 0, \\ 0 & \text{otherwise} \end{cases}, \quad g(x, y) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{otherwise}. \end{cases} \]

Observe that the plane \( z = 0 \) is tangent to surface \( z = f(x, y) \) at \((0, 0, 0)\) in the sense of the Fréchet’s definition (2.4), but does not fulfill Fréchet’s definition (2.5). Conversely, the plane \( z = 0 \) is tangent to surface \( z = g(x, y) \) at \((0, 0, 0)\) in the sense of the Fréchet’s definition (2.5), but does not fulfill Fréchet’s definition (2.4). A critical attitude towards unstinting Fréchet’s definition of tangency is took, for example, by Valiron [59, (1926), p. 191-192] and Cinquini [12, (1955)].
In the case where \( F \) is the graph of a function \( f \), the tangency in traditional sense becomes differentiability.

**Proposition 2.3** (Guareschi [24, (1934), p. 181, 183], [25, (1936), p. 132]). If \( A \subset \mathbb{R}^d, f : A \to \mathbb{R}^n \) is a function, \( L : \mathbb{R}^d \to \mathbb{R}^n \) is a linear function and \( \hat{x} \in A \cap \text{der}(A) \), then the following three properties are equivalent:

1. \( L \) is a differential of \( f \) at \( \hat{x} \), i.e.,
   \[
   \lim_{A \ni x \to \hat{x}} \frac{f(x) - f(\hat{x}) - L(x - \hat{x})}{\|x - \hat{x}\|} = 0, \tag{2.7}
   \]
2. \( f \) is continuous at \( \hat{x} \) and \( \text{graph}(L) \) is tangent in traditional sense to \( \text{graph}(f) \) at \( (\hat{x}, f(\hat{x})) \),
3. \( f \) is continuous at \( \hat{x} \) and \( \text{Tan}^+(\text{graph}(f), (\hat{x}, f(\hat{x}))) \subset \text{graph}(L) \).

Contrary to the tangency of a straight line to a curve and that of a plane to a surface at a given point, there is no established tradition for the notion of paratangency. Nevertheless, to manifest a logical correlation between tangency and paratangency we introduce the following definition.

**Definition 2.4.** Let \( \hat{x} \) be an accumulation point of a subset \( F \) of \( \mathbb{R}^n \). A vector space \( H \) of \( \mathbb{R}^n \) is said to be paratangent in traditional sense to \( F \) at \( \hat{x} \) if

\[
\lim_{F \ni x \not\to \hat{x} \neq y} \frac{\text{dist}(x, H + y)}{\text{dist}(x, y)} = 0. \tag{2.10}
\]

Since \( \frac{\text{dist}(x, H + y)}{\text{dist}(x, y)} \) is the sinus of the angle between \( H \) and the vector \( x - y \), the geometric meaning of (2.10) is evident: the straight-lines passing through \( y \) and \( x \) in \( F \) and the affine space \( H + \hat{x} \) form an angle that tends to zero as \( x \) and \( y \) tend to \( \hat{x} \).

The concepts of tangency and paratangency were either confused or improperly identified. Surprisingly, Lebesgue used paratangency to define traditional tangency in *Du choix des définitions* [34, (1934) p. 6]:

"L’idée de tangent provient de ce jugement: tout arc suffisemment petit d’une courbe est rectiligne. […] Dire qu’un élément de courbe et un élément de droite sont indiscernables, c’est dire que la droite est pratiquement confondue avec toute corde de l’élément de courbe, d’où cette définition: a) on dit qu’une courbe \( C \) a des tangentes si, quel que soit le point \( M \) de \( C \) et de quelque manière qu’on fasse tendre les points \( M_1 \) et \( M_2 \) de \( C \) vers \( M \), la corde \( M_1 M_2 \) tend vers une position limite déterminée par la donnée de \( M \); on l’appelle tangente en \( M \)."

\[\text{Guareschi writes in [25, (1936), p. 131]}:\quad \text{"Il concetto di semiretta tangente (o semitangente) ad un insieme puntuale in suo punto d’accumulazione […] permette di strettamente collegare l’operazione analitica di differenziazione totale di una funzione di più variabili reali, in suo punto, al concetto sintetico di spazio lineare di dimensione minima contenente l’insieme tangente alla grafica della funzione stessa nel suo punto corrispondente; e tale collegamento porta a concludere che tanti sono i differenziali totali della funzione, quanti sono gli iperpiani passanti per quello spazio."}\]

\[\text{A function is said to be differentiable at a point, when it admits a differential at the point.}\]
Paratangency to curves and surfaces was used by Peano in Applicazioni geometriche [41, (1887), p. 163, 181-184] to evaluate the infinitesimal quotient of the length of an arc and its segment or its projection. Moreover, the following proposition due to Peano makes transparent the relation between paratangency and $C^1$-smoothness.

**Proposition 2.5** (Peano [41, (1887), teorema II, p. 59]). If $\gamma$ is a continuously differentiable curve and $\gamma'(t) \neq 0$, then its tangent straight line at $\gamma[t]$ is the limit of the lines passing through $\gamma[t]$ and $\gamma[u]$ when $t \neq u$ and $t, u$ tend to $t$.

Analytically we have that

**Proposition 2.6.** A vector space $H$ of $\mathbb{R}^n$ is paratangent in traditional sense to $F$ at an accumulation point $\hat{x}$ of $F$ if and only if

$$\text{pTan}^+(F, \hat{x}) \subset H. \tag{2.11}$$

**Proof.** Proceed similarly as in the proof of Proposition 2.2. $\square$

According to (2.11), we assume, as a definition, that every vector space of $\mathbb{R}^n$ is paratangent in traditional sense to $F$ at the isolated points of $F$.

As for upper tangent cones, the linear upper paratangent space of $F$ at $\hat{x}$, i.e., the linear hull of the upper paratangent cone of $F$ at $\hat{x}$, was introduced by Guareschi [26, (1941), p. 154]; it is denoted by $\text{pLTan}^+(F, \hat{x})$. By Proposition 2.6, the linear upper paratangent space is the smallest vector space which is paratangent in traditional sense.

In the case where $F$ is a graph of a function $f$, the paratangency in traditional sense becomes strict differentiability.

**Definition 2.7** (Peano in [43, (1892)] for $n = d = 1$, Severi in [54, (1934), p.185]). Let $A \subset \mathbb{R}^d$, let $f : A \to \mathbb{R}^n$ be a function and let $\hat{x} \in A \cap \text{der}(A)$. The function $f$ is said to be strictly differentiable at $\hat{x}$, if there is a linear function $L : \mathbb{R}^d \to \mathbb{R}^n$ (called strict differential of $f$ at $\hat{x}$) such that

$$\lim_{y \to \hat{x}, \ y \neq \hat{x}} \frac{f(x) - f(y) - L(x - y)}{\|x - y\|} = 0. \tag{2.20}$$

**Proposition 2.8** (Severi [54, (1934), p.189], Guareschi [26, (1941), p.161]).

If $A \subset \mathbb{R}^d$, $f : A \to \mathbb{R}^n$ is a function, $L : \mathbb{R}^d \to \mathbb{R}^n$ is a linear function and $\hat{x} \in A \cap \text{der}(A)$, then the following three properties are equivalent:

1. $L$ is a strict differential of $f$ at $\hat{x}$.
2. $f$ is continuous at $\hat{x}$ and graph$(L)$ is paratangent in traditional sense to graph$(f)$ at $(\hat{x}, f(\hat{x}))$.
3. $f$ is continuous at $\hat{x}$ and $\text{pTan}^+(\text{graph}(f), (\hat{x}, f(\hat{x}))) \subset \text{graph}(L)$.

In order to prove this Proposition 2.8 we use the following two lemmata.

**Lemma 2.9** (Cyrenian lemma for strict differentiability). Let $f$, $L$, $A$ and $\hat{x}$ be as Proposition 2.8. Then $L$ is a strict differential of $f$ at $\hat{x}$ if and only if

$\text{pTan}^+(\text{graph}(f), (\hat{x}, f(\hat{x}))) \subset \text{graph}(L)$.

$^{19}$Among the non-empty sets $H$ (not necessary, vector spaces) verifying (2.10), the smallest closed set is the upper paratangent cone.

$^{20}$As usual, “strictly differentiable function on a set $X$” stands for “strictly differentiable function at every point belonging to $X$”.
Lemma 2.10. Let $A$ be a subset of $\mathbb{R}^d$ and let $\hat{x}$. A function $f : A \to \mathbb{R}^n$ is locally Lipschitz at $\hat{x}$ if and only if $f$ is continuous at $\hat{x}$ and $\text{pTan}^+(\text{graph}(f), (\hat{x}, f(\hat{x})))$ does not contain vertical lines. \hfill \square

Proof of Proposition 2.8. (2.13) $\iff$ (2.14): It follows from Proposition 2.6.

(2.12) $\implies$ (2.14): Let $v \in \text{pTan}^+(\text{graph}(f), (\hat{x}, f(\hat{x})))$. Take $v_1 \in \mathbb{R}^d$ and $v_2 \in \mathbb{R}^n$ such that $v = (v_1, v_2)$. By definition of $\text{pTan}^+$, there are $\{\lambda_m\} \subset \mathbb{R}_{++}, \{x_m\}, \{y_m\} \subset A$ such that $\lim_{m} \lambda_m = 0$, $\lim_{m} y_m = \hat{x}$ and $\lim_{m} \frac{x_m - y_m}{\lambda_m} = v_1$ and $\lim_{m} \frac{f(x_m) - f(y_m)}{x_m - y_m} = v_2$. Hence, by Cyrenian lemma 2.9 one has $L(v_1) = v_2$, that is $v = (v_1, v_2) \in \text{graph}(L)$.

(2.14) $\implies$ (2.12). By Lemma 2.10, condition (2.14) implies the existence of $M, \varepsilon \in \mathbb{R}_{++}$ such that

\[ \|f(x) - f(y)\| \leq M\|x - y\| \text{ for every } x, y \in \mathbb{B}_\varepsilon(\hat{x}). \]

To prove (2.12), by absurd suppose that (2.12) does not hold. Then there are sequences $\{x_m\}, \{y_m\} \subset A$ with $x_m \neq y_m$ for $m \in \mathbb{N}$ such that $\lim_{m \to \infty} y_m = \hat{x}$ and

\[ \lim_{m \to \infty} \left\| \frac{f(x_m) - f(y_m)}{x_m - y_m} - L\left( \frac{x_m - y_m}{\|x_m - y_m\|} \right) \right\| > 0. \]

By compactness and (*), without loss of generality we assume that $\lim_{m} \frac{x_m - y_m}{\|x_m - y_m\|} = v_1 \in \mathbb{R}^d$ and $\lim_{m} \frac{f(x_m) - f(y_m)}{x_m - y_m} = v_2 \in \mathbb{R}^d$. Hence $(v_1, v_2) \in \text{pTan}^+(\text{graph}(f), (\hat{x}, f(\hat{x})))$.

By (2.14) we have that $(v_1, v_2) \in \text{graph}(L)$, that is $L(v_1) = v_2$. Therefore

\[ \lim_{m} \left\| \frac{f(x_m) - f(y_m)}{x_m - y_m} - L\left( \frac{x_m - y_m}{\|x_m - y_m\|} \right) \right\| = \|v_2 - L(v_1)\| = 0, \text{ contradicting (**).} \]

Corollary 2.11. Let $A$ be a subset of $\mathbb{R}^d$ and let $\hat{x} \in A \cap \text{der}(A)$. A function $\varphi : A \to \mathbb{R}^n$ is strictly differentiable at $\hat{x}$ if and only if $\varphi$ is continuous at $\hat{x}$ and $\text{pLTan}(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x})))$ does not contain vertical lines.

Proof. Necessity. Assume $\varphi$ strictly differentiable at $\hat{x}$ and denote by $L$ a strict differential of $\varphi$ at $x$. By Proposition 2.8 the function $\varphi$ is continuous at $x$ and

\[ \text{pLTan}^+(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x}))) \subset \text{graph}(L). \]

Therefore, $\text{pLTan}^+(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x})))$ does not contain vertical lines.

Sufficiency. Posit $d^\prime := \dim(\text{pLTan}^+(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x}))))$, $R_n := \{(v, w) \in \mathbb{R}^d \times \mathbb{R}^n : v = 0\}$ and $V := (R_n \oplus \text{pLTan}^+(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x}))))^\perp$ in $\mathbb{R}^d \times \mathbb{R}^n$. Since $\text{pLTan}^+(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x})))$ does not contain vertical lines, the vector space

\[ A := V \oplus \text{pLTan}^+(\text{graph}(\varphi), (\hat{x}, \varphi(\hat{x}))) \]

\[ (2.15) \lim_{m} \frac{f(x_m) - f(y_m)}{\lambda_m} = L(v) \text{ for each } v \in \mathbb{R}^d \text{ and for all sequences } \{\lambda_m\} \subset \mathbb{R}_{++}, \{x_m\}, \{y_m\} \subset A \text{ such that } \lim_{m} \lambda_m = 0, \lim_{m \to \infty} y_m = \hat{x} \text{ and } \lim_{m} \frac{x_m - y_m}{\lambda_m} = v. \]
has dimension $d$ and it does not contain verticals lines; hence, there exists a linear function $L : \mathbb{R}^d \to \mathbb{R}^n$ such that $\text{graph}(L) = \Lambda$. Therefore, being $\varphi$ continuous at $\hat{x}$, by Proposition 2.8 we have that $\varphi$ is strictly differentiable at $\hat{x}$ and $L$ is a strict differential of $\varphi$ at $\hat{x}$. ☐

3. **Grassmann Exterior Algebra, Limits of Vector Spaces and Angles Between Vector Spaces**

Let $\Lambda(\mathbb{R}^n)$ denote the graded Grassmann exterior algebra on $n$-dimensional Euclidean space $\mathbb{R}^n$:

(3.1) $\Lambda(\mathbb{R}^n) := \Lambda_0(\mathbb{R}^n) \oplus \Lambda_1(\mathbb{R}^n) \oplus \Lambda_2(\mathbb{R}^n) \oplus \Lambda_3(\mathbb{R}^n) \oplus \cdots \oplus \Lambda_n(\mathbb{R}^n)$

where $\Lambda_k(\mathbb{R}^n) := \mathbb{R}$ and $\Lambda_k(\mathbb{R}^n)$, $k = 1, \ldots, n$, is the $\binom{n}{k}$-dimensional vector space generated by linear combinations of simple $k$-vectors $\wedge_{i=1}^k v_i$, where $\{v_i\}_{i=1}^k \subset \mathbb{R}^n$.

Euclidean inner product of $\mathbb{R}^n$ induces an inner product on $\Lambda(\mathbb{R}^n)$; on simple $k$-vectors it is described by

(3.2) $\langle \wedge_{i=1}^k v_i, \wedge_{i=1}^k w_i \rangle := \det(\langle v_i, w_j \rangle)_{ij}$.

With respect to the associated norm on $\Lambda(\mathbb{R}^n)$, a simple $k$-vectors $\wedge_{i=1}^k v_i$ has a non-null norm if and only if the $k$ vectors $\{v_i\}_{i=1}^k$ are linearly independent.

Let $\mathcal{G}(\mathbb{R}^n, d)$ denote the set of all $d$-dimensional vector (sub)spaces of $\mathbb{R}^n$ and define $\mathcal{G}(\mathbb{R}^n) := \bigcup_{0 \leq d \leq n} \mathcal{G}(\mathbb{R}^n, d)$. If $d \geq 1$, the *angle* between two (non oriented) $d$-dimensional vector spaces $V$ and $W$ is a real number denoted by $\text{ang}(V, W)$ and well defined by

(3.3) $\text{ang}(V, W) := \arccos \left( \frac{\| \wedge_{i=1}^d v_i \wedge_{j=1}^d w_j \|}{\| \wedge_{i=1}^d v_i \| \| \wedge_{i=1}^d w_i \|} \right)$,

where $\{v_i\}_{i=1}^d$ and $\{w_i\}_{i=1}^d$ are arbitrary bases of $V$ and $W$, respectively.

For every basis $\{v_i\}_{i=1}^d$ of a $d$-dimensional vector space $V$, one has:

(3.4) the norm $\| \wedge_{i=1}^d v_i \|$ is the $d$-dimensional elementary measure of $d$-parallelepiped $P(\{v_i\}_{i=1}^d) := \{\sum_{i=1}^d \alpha_i v_i : \sum_{i=1}^d \alpha_i = 1 \text{ and } 0 \leq \alpha_i \leq 1 \text{ for } 1 \leq i \leq d\}$;

(3.5) $\text{dist}(x, V) = \frac{\| \wedge_{i=1}^d v_i \wedge_{j=1}^d w_j \|}{\| \wedge_{i=1}^d w_j \|}$ for every $x \in \mathbb{R}^n$;

(3.6) if $v_i'$ is the orthogonal projection of vector $v_i$ on a $d$-dimensional vector space $W$, then

$$
\| \wedge_{i=1}^d v_i' \| = \| \wedge_{i=1}^d v_i \| \cos(\text{ang}(V, W)),
$$

i.e., $\cos(\text{ang}(V, W))$ is the reduction factor for $d$-dimensional measure under orthogonal projection of $V$ on $W$.\(^{23}\) Hence $\text{ang}(V, W) = \frac{\pi}{2}$ if and only if $V \cap W^\perp \neq \{0\}$.

**Proposition 3.1** (see Peano [41, (1887) thm. 5 p. 35, thm. 7 p. 39] and [41, (1887) thm. 2 p. 34, thm. 2 p. 36]). If $V_m, V \in \mathcal{G}(\mathbb{R}^n, d)$ and $d \geq 1$, then following properties are equivalent:

(3.7) $V \subset \text{Li}_{m \to \infty} V_m$,

(3.8) $\lim_{m \to \infty} \text{ang}(V_m, V) = 0$.

\(^{23}\) To verify (3.6), fix an orthonormal basis $\{w_i\}_{i=1}^d$ of $W$. Then observe that $v_i' = \sum_{k=1}^d \langle v_i, w_k \rangle w_k$ for $1 \leq i \leq d$, and $\cos(\text{ang}(V, W)) = \frac{\| \wedge_{i=1}^d v_i \wedge_{j=1}^d w_j \|}{\| \wedge_{i=1}^d v_i \| \| \wedge_{j=1}^d w_j \|}$. Hence

$$
\| \wedge_{i=1}^d v_i' \|^2 = \det(\sum_{i=1}^d \langle v_i, w_i \rangle w_i, \sum_{i=1}^d \langle v_i, w_i \rangle w_i)_{ij} = \det(\sum_{i=1}^d \langle v_i, w_i \rangle v_i, \sum_{i=1}^d \langle v_i, w_i \rangle w_i)_{ij} = \det^2(\langle v_i, w_j \rangle)_{ij} = \langle \wedge_{i=1}^d v_i, \wedge_{i=1}^d w_i \rangle^2 = \| \wedge_{i=1}^d v_i \|^2 \cos^2(\text{ang}(V, W)).
$$
(3.9) there are bases \(\{v_i\}_{i=1}^d\) and \(\{v_i^{(m)}\}_{i=1}^d\) of \(V\) and \(V_m\) respectively, such that \(v_i = \lim_{m \to \infty} v_i^{(m)}\) for \(1 \leq i \leq d\).

**Proof.** (3.7) \(\Rightarrow\) (3.9). Let \(\{v_i\}_{i=1}^d\) be a basis of \(V\). By (3.7), for every \(1 \leq i \leq d\) there is a sequence \(\{v_i^{(m)}\}_m\) such that \(v_i^{(m)} \in V_m\) and \(\lim_m v_i^{(m)} = v_i\). Hence, by continuity of exterior product, one obtains that

(3.9) \(\Rightarrow\) (3.8). Let \(\{v_i\}_{i=1}^d\) and \(\{v_i^{(m)}\}_{i=1}^d\) as in (3.9). By continuity of exterior product we have (3.7). Therefore, from Definition (3.3) the required (3.8) follows.

(3.8) \(\Rightarrow\) (3.7). Let \(\{v_i\}_{i=1}^d\) be an orthonormal basis of \(V\); moreover, for every \(m \in \mathbb{N}\), let \(\{v_i^{(m)}\}_{i=1}^d\) be an orthonormal basis of \(V_m\) such that \(\langle \wedge_{i=1}^d v_i^{(m)}, \wedge_{i=1}^d v_i \rangle \geq 0\). From orthonormality it follows that \(\| \wedge_{i=1}^d v_i \| = \| \wedge_{i=1}^d v_i^{(m)} \| = 1\); hence, by condition (3.8), we have that \(\lim_m \wedge_{i=1}^d v_i^{(m)} = \wedge_{i=1}^d v_i\). On the other hand, by continuity of exterior product, one obtains that

\[
\lim_m \text{dist}(x, V_m) = \lim_m \| \wedge_{i=1}^d v_i^{(m)} \wedge x \| = \| \wedge_{i=1}^d v_i \wedge x \| = \text{dist}(x, V)
\]

for every \(x \in \mathbb{R}^n\). Therefore, for every \(x \in V\), we have \(\lim_m \text{dist}(x, V_m) = 0\), i.e., \(x \in \text{Li}_m V_m\).

**Corollary 3.2.** Let \(V_m, V\) be as in Proposition 3.1. The following equivalence holds:

(3.10) \(V \subset \text{Li}_m V_m \iff \text{dist}(x, V_m) = \lim_{m \to \infty} \text{dist}(x, V_m)\) for all \(x \in \mathbb{R}^n\).

**Proof.** \(\iff\) By the definition of the lower limit \(\text{Li}_m\), it is obvious. \(\Rightarrow\) Let \(\{v_i\}_{i=1}^d\) be a basis of \(V\). By the definition of the lower limit of sets, \(V \subset \text{Li}_m V_m\) implies that for every \(1 \leq i \leq d\) there is a sequence \(\{v_i^{(m)}\}_m\) such that \(v_i^{(m)} \in V_m\) and \(\lim_m v_i^{(m)} = v_i\). Hence, by the continuity of the exterior product and \(\| \wedge_{i=1}^d v_i \| \neq 0\), one has that

\[
\lim_m \text{dist}(x, V_m) = \lim_m \frac{\| \wedge_{i=1}^d v_i^{(m)} \wedge x \|}{\| \wedge_{i=1}^d v_i^{(m)} \|} = \frac{\| \wedge_{i=1}^d v_i \wedge x \|}{\| \wedge_{i=1}^d v_i \|} = \text{dist}(x, V)
\]

for every \(x \in \mathbb{R}^n\).

**Corollary 3.3.** Let \(V_m, V\) be as in Proposition 3.1. The following equivalence holds:

(3.11) \(V \subset \text{Li}_m V_m \iff \text{Ls}_m V_m \subset V\).

**Proof.** \(\subseteq\) : Being \(V \subset \text{Li}_m V_m\), by (3.10) we have that

\[
\lim_m \text{dist}(x, V_m) = \text{dist}(x, V)
\]

for every \(x \in \mathbb{R}^n\). Now, for \(x \in \text{Ls}_m V_m\), Definition of upper limit of sets entails \(\liminf_m \text{dist}(x, V_m) = 0\); therefore, by (3.7) one obtains \(\text{dist}(x, V) = 0\), i.e., \(x \in V\).

(3.11) \(\supseteq\) : Fix \(\hat{x} \in V\) and choose an orthonormal basis \(\{v_i^{(m)}\}_{i=1}^d\) of \(V_m\). By absurd, suppose \(\hat{x} \not\in \text{Li}_m V_m\), i.e., \(\limsup_m \text{dist}(\hat{x}, V_m) = \alpha > 0\). Then there exist
\{w_i\}_{i=1}^d \subset \mathbb{R}^n \text{ and an infinite subset } N \text{ of } \mathbb{N} \text{ such that } \lim_{N \ni m \to \infty} \text{dist}(\hat{x}, V_m) = \alpha \text{ and } \lim_{N \ni m \to \infty} v_i^m = w_i \text{ for } 1 \leq i \leq d. \text{ Now, let } W \text{ denote the } d\text{-dimensional vector space generated by the orthonormal basis } \{w_i\}_{i=1}^d. \text{ First, by the continuity of exterior product we have }

\lim_{N \ni m \to \infty} \text{dist}(\hat{x}, V_m) = \lim_{N \ni m \to \infty} \| (\wedge_{i=1}^d v_i^m) \wedge \hat{x} \| = \| (\wedge_{i=1}^d w_i) \wedge \hat{x} \| = \text{dist}(\hat{x}, W);

\text{hence } \hat{x} \notin W, \text{ since } \lim_{N \ni m \to \infty} \text{dist}(\hat{x}, V_m) = \alpha > 0. \text{ On the other hand, from } \text{‘(3.9) } \Rightarrow \text{ (3.7)’ of Proposition 3.1 it follows that } W \subset \text{Li}_{N \ni m \to \infty} V_m; \text{ hence } W \subset \text{Li}_{N \ni m \to \infty} V_m \subset \text{Ls}_{m \to \infty} V_m \subset V; \text{ therefore, being both } W \text{ and } V \text{ d-dimensional vector spaces, we have } W = V \text{ in conflict with the fact that } \hat{x} \in V \setminus W. \square

Let us define continuity of set-valued functions. Let \( A \subset \mathbb{R}^n \) and \( \hat{x} \in A; \) as usual, a set-valued function \( \varphi : A \to \mathcal{P}(\mathbb{R}^n) \) is said to be lower (resp. upper) semicontinuous at \( \hat{x} \), whenever \( \varphi(\hat{x}) \subset \text{Li}_{A_{\hat{x}}} \varphi(x) \) \(^{24}\) (resp. \( \text{Ls}_{A_{\hat{x}}} \varphi(x) \subset \varphi(\hat{x}) \) \(^{25}\)); moreover \( \varphi \) is said to be continuous at \( \hat{x} \), if \( \varphi \) is both lower and upper semicontinuous. \(^{26}\) The following well known elementary properties are useful:

(3.12) \( \varphi \) is lower semicontinuous at \( \hat{x} \) if and only if

\[ \varphi(\hat{x}) \subset \text{Li}_{m \to \infty} \varphi(x_m) \text{ for every sequence } \{x_m\}_m \subset A \text{ converging to } \hat{x}; \]

(3.13) \( \varphi \) is upper semicontinuous at \( \hat{x} \) if and only if

\[ \text{Ls}_{m \to \infty} \varphi(x_m) \subset \varphi(\hat{x}) \text{ for every sequence } \{x_m\}_m \subset A \text{ converging to } \hat{x}; \]

(3.14) \( \varphi \) is continuous at \( \hat{x} \) if and only if

\[ \text{Ls}_{m \to \infty} \varphi(x_m) \subset \varphi(\hat{x}) \subset \text{Li}_{m \to \infty} \varphi(x_m) \text{ for every } \{x_m\}_m \subset A \text{ converging to } \hat{x}. \]

By (3.11)-(3.13), Proposition 3.1 and Corollary 3.3 can be restated in terms of continuity of vector-space-valued functions.

**Theorem 3.4.** Let \( A \) be a subset of \( \mathbb{R}^n \) and let \( \tau : A \to \mathcal{G}(\mathbb{R}^n, d) \) be a vector-space-valued function with \( d \geq 1 \). For every \( x \in A \), the following properties are equivalent:

(3.15) \( \tau \) is lower semicontinuous at \( x \),

(3.16) \( \tau \) is upper semicontinuous at \( x \),

(3.17) \( \tau \) is continuous at \( x \),

(3.18) \( \lim_{A_{\hat{x}} \to \hat{x}} \text{ang}(\tau(y), \tau(x)) = 0. \) \( \square \)

4. **Four-cones Coincidence Theorem: Local and Global Version**

The geometry of manifolds was originated by Riemann’s Habilitationsschrift (1854) Über die Hypothesen welche der Geometrie zu Grunde liegen \(^{[46]}\) (1878) p. 272-287. Today axiomatic formulation of manifolds by coordinate systems and regular atlas was presented by Veblen and Whitehead in \([61, 62, (1931)]\) and was elaborated in a series of celebrated works by Whitney in the 1930’s. There are various kinds of finite dimensional manifolds; all are topological, i.e., they are locally homeomorphic to Euclidean spaces. In the sequel we will consider (sub)manifolds of Euclidean spaces that are \( C^1 \) smooth.

\(^{24}\) i.e., the set \( \{x \in A : B \cap \varphi(x) \neq \emptyset\} \) is open in \( A \), for every open ball \( B \).

\(^{25}\) i.e., the set \( \{x \in A : \overline{B} \cap \varphi(x) \neq \emptyset\} \) is closed in \( A \), for every closed ball \( \overline{B} \).

\(^{26}\) Since \( \text{Li}_{A_{\hat{x}}} \varphi(x) \subset \text{Ls}_{A_{\hat{x}}} \varphi(x) \), the continuity of \( \varphi \) at \( \hat{x} \) amounts to the equalities:

\[ \varphi(\hat{x}) = \text{Li}_{A_{\hat{x}}} \varphi(x) = \text{Ls}_{A_{\hat{x}}} \varphi(x). \]
Definition 4.1. A non-empty subset $S$ of $\mathbb{R}^n$ is said to be a $C^1$-manifold of $\mathbb{R}^n$ at a point $x \in S$, if there are an open neighborhood $\Omega$ of $x$ in $\mathbb{R}^n$, an affine subspace $H$ of $\mathbb{R}^n$ and a $C^1$-diffeomorphism from $\Omega$ onto another open set of $\mathbb{R}^n$ such that

\begin{equation}
\xi(S \cap \Omega) = \xi(\Omega) \cap H.
\end{equation}

The dimension of $H$ is said to be the dimension of $S$ at $x$ and it is denoted by $\text{dim}(S, x)$.

A set $S$ is said to be a $C^1$-manifold of $\mathbb{R}^n$, if it is a $C^1$-manifold of $\mathbb{R}^n$ at every point. If the dimension $d = \text{dim}(S, x)$ does not depend on $x \in S$, then $S$ is said to be a \textit{d-dimensional $C^1$-manifold}.  \footnote{If $S$ is a $C^1$-manifold of $\mathbb{R}^n$, then $S_i := \{x \in S : \text{dim}(S, x) = i\}$ is an $i$-dimensional $C^1$-manifold (if non-empty), and, for $0 \leq i \neq j \leq n$, $S_i$ and $S_j$ are separated (i.e. $S_i \cap S_j = \emptyset$).}

Elementary and well known facts on arbitrary $C^1$-manifolds are resumed in the following proposition.

Proposition 4.2. Let $S$ be a $C^1$-manifold of $\mathbb{R}^n$ and let $x$ be an arbitrary point of $S$. Then the following four properties hold:

\begin{enumerate}
  \item $S$ is a topological manifold (hence, a locally compact set);
  \item $\text{dim}(S, x)$ is a vector space and $d\xi(x)(\text{Tan}^+(S, x)) = H - \xi(x)$ and $\text{dim}(S, x) = \text{dim}(\text{Tan}^+(S, x))$ where $\xi$ and $H$ are as in Definition 4.1 and $d\xi(x)$ denotes the differential of $\xi$ at $x$;
  \item all four tangent cones coincide, i.e. $\text{pTan}^+(S, x) = \text{Tan}^+(S, x) = \text{Tan}^-(S, x) = \text{pTan}^-(S, x)$; \footnote{The relevant equality $\text{tan}^+(S, x) = \text{pTan}^-(S, x)$ was proved by Clarke \cite{Clarke75}, pp. 254-256} for every $x \in S$ where $\text{pTan}^+$, $\text{pTan}^-$, $\text{tan}^+$, and $\text{tan}^-$ denote any one of the four tangent cones $\text{tan}^+$, $\text{tan}^-$, $\text{pTan}^+$ and $\text{pTan}^-$, then
  \item $\text{Tan}(S \cap \Omega, x) = \text{tan}(S, x)$ for every neighborhood $\Omega$ of $x$ in $\mathbb{R}^n$;
  \item they are \textit{stable by diffeomorphisms}, i.e., $\text{Tan}(\xi(S \cap \Omega), \xi(x)) = d\xi(x)(\text{tan}(S \cap \Omega, x))$ for every open neighborhood $\Omega$ of $x$ and for every $C^1$-diffeomorphism from $\Omega$ to another open set of $\mathbb{R}^n$. \footnote{The equality in (4.7) is a consequence of the description of $\text{tan}$ in terms of sequences (see (1.11), (1.10), (1.17) and (1.16)), since $\xi$ is $C^1$-diffeomorphism and, consequently, by Cyrenian lemma 2.9 one has}
\end{enumerate}

\begin{equation}
\lim_{m \to +\infty} \frac{x_m - y_m}{\lambda_m} = v \iff \lim_{m \to +\infty} \frac{\xi(x_m) - \xi(y_m)}{\lambda_m} = d\xi(x)(v)
\end{equation}

for $\lambda_m \to 0^+$, $v \in \mathbb{R}^n$, $\{y_m\}_m, \{x_m\}_m \subset S \cap \Omega$ with $y_m \to x$.\footnote{The equality in (4.7) is a consequence of the description of $\text{tan}$ in terms of sequences (see (1.11), (1.10), (1.17) and (1.16)), since $\xi$ is $C^1$-diffeomorphism and, consequently, by Cyrenian lemma 2.9 one has}
they fix vector spaces, i.e.,
\[ \text{Tan}(V, 0) = V \]
for every vector subspace \( V \) of \( \mathbb{R}^n \).

To prove both (4.3) and (4.4), fix \( x \in S \) and let \( (\Omega, H, \xi) \) be as in Definition 4.1. The following equalities hold:
\[ (*) \quad \text{Tan}(\xi(S \cap \Omega), \xi(x)) = d\xi(x)(\text{Tan}(S \cap \Omega, x)) = d\xi(x)(\text{Tan}(S, x)). \]
The first and the second equality of (\( (*) \)) are due to (4.7) and (4.6), respectively. On the other hand
\[ (**) \quad \text{Tan}(\xi(\Omega) \cap H, \xi(x)) = \text{Tan}(H, \xi(x)) = \text{Tan}(H - \xi(x), 0) = H - \xi(x), \]
where the equalities are due to (4.6), (4.7) and (4.8), respectively. Finally, combining (\( *) \) and (\( ** \)) with (4.1), we have (4.3) and (4.4). Besides, the tangent cone \( \text{Tan}^+(S, x) \) is a vector subspace of \( \mathbb{R}^n \) having the same dimension of \( S \) at \( x \), because it is the preimage of the vector space \( H - \xi(x) \) under the linear isomorphism \( d\xi(x) \).

To verify (4.5), fix \( x \in S \) and let \( (\Omega, H, \xi) \) be as in Definition 4.1. From (4.3) it follows that
\[ (***) \quad d\xi(y)(\text{Tan}^+(S, y)) = H - \xi(y) = H - \xi(x) \quad \text{for every} \quad y \in S \cap \Omega. \]

1\textsuperscript{st} \textit{case}: \( \dim(S, x) = 0 \). Obviously (4.5) holds, since \( x \) is an isolated point of \( S \). 2\textsuperscript{nd} \textit{case}: \( d := \dim(S, x) \geq 1 \). Choose a basis \( \{w_i\}_{i=1}^d \) of the vector space \( W := H - \xi(x) \) and a sequence \( \{y_m\}_m \subset S \) converging to \( x \). Since \( \Omega \) is an open neighborhood of \( x \), there is a natural number \( \bar{m} \) such that \( y_m \in S \cap \Omega \) for every natural number \( m \geq \bar{m} \). For every \( y_m \) with \( m \geq \bar{m} \), by (\( *** \)) define a basis \( \{v_i^{(m)}\}_{i=1}^d \) of the \( d \)-dimensional vector space \( \text{Tan}^+(S, y_m) \) by
\[ v_i^{(m)} := (d\xi(y_m))^{-1}(w_i) \quad \text{for} \quad 1 \leq i \leq d \]
and, analogously, define a basis \( \{v_i\}_{i=1}^d \) of the \( d \)-dimensional vector space \( \text{Tan}^+(S, x) \) by
\[ v_i := (d\xi(x))^{-1}(w_i) \quad \text{for} \quad 1 \leq i \leq d \]
Since \( \xi \) is \( C^1 \)-diffeomorphism, the map \( y \mapsto (d\xi(y))^{-1} \) is continuous. Hence
\[ \lim_{m \to \infty} v_i^{(m)} = v_i \quad \text{for} \quad 1 \leq i \leq d. \]
Therefore, by Proposition 3.1, \( \text{Tan}^+(S, x) \subset \lim_{m \to \infty} \text{Tan}^+(S, y_m) \); consequently, by Corollary 3.3, \( \lim_{m \to \infty} \text{Tan}^+(S, y_m) \subset \text{Tan}^+(S, x) \). Hence Theorem 3.4 entails (4.5). \( \square \)

**Lemma 4.3.** Let \( F \) be a subset of \( \mathbb{R}^n \) such that \( F \subset \text{der}(F) \) and
\[ (4.9) \quad c_n \notin \text{pLTan}^+(F, x) \quad \text{for every} \quad x \in F. \]
Let \( \hat{x} \in F \). Then there exist \( \varepsilon \in \mathbb{R}_{++}, \ A \subset \mathbb{R}^{n-1} \) with \( A \subset \text{der}(A) \) and there exists a function \( \varphi : A \to \mathbb{R} \) strictly differentiable on \( A \) such that
\[ (4.10) \quad \text{graph} (\varphi) = F \cap B_\varepsilon(\hat{x}). \]
Moreover, if \( F \) is a \( d \)-dimensional topological manifold (resp. locally compact at \( \hat{x} \)), then \( A \) is a \( d \)-dimensional topological manifold (resp. locally compact at \( \hat{t} \), where \( \hat{t} \) is the element of \( A \) such \( \hat{x} = (\hat{t}, \varphi(\hat{t})) \)).
\textbf{Proof.} By (1.28) there are \( \varepsilon \in \mathbb{R}_{++}, A \subset \mathbb{R}^{n-1} \) and \( \varphi : A \to \mathbb{R} \) such that
\[(*)1 \quad \varphi \text{ is continuous and } \text{graph}(\varphi) = F \cap B_{\varepsilon}(\hat{x}).\]
Since \( B_{\varepsilon}(\hat{x}) \) is an open set, we have
\[(*)2 \quad \text{pLTan}^+(\text{graph}(\varphi), (t, \varphi(t))) = \text{pLTan}^+(F, (t, \varphi(t))) \text{ for every } t \in A;\]
therefore, by (4.9)
\[(*)3 \quad \text{pLTan}^+(\text{graph}(\varphi), (t, \varphi(t))) \text{ does not contain vertical lines.}\]
for every point \( t \in A' := \{ t \in A : e_n \notin \text{pLTan}^+(F, (t, \varphi(t))) \}. \) On the other hand, being \( F \subset \text{der}(F), \) graph(\varphi) has no isolated point; therefore \( A \subset \text{der}(A). \) Now, by Corollary 2.11 we have that \( \varphi \) is strictly differentiable on \( A', \) as required. Finally, if \( F \) is locally compact at \( \hat{x} \) (resp. a \( d \)-dimensional topological manifold), then the set \( A \) is locally compact at \( \hat{t} \) (resp. a \( d \)-dimensional topological manifold), since it is homeomorphic to \( \text{graph}(\varphi) = F \cap B_{\varepsilon}(\hat{x}). \)
\[\Box\]

\textbf{Lemma 4.4.} Let \( A \subset \mathbb{R}^d \) with \( A \subset \text{der}(A) \) and let \( \varphi : A \to \mathbb{R}^n \) be strictly differentiable. For every \( x \in A, \) the following properties hold:
\[(4.11) \quad \text{pTan}^+(\text{graph}(\varphi), (x, \varphi(x))) = \{(v, L(v)) : v \in \text{pTan}^+(A, x)\},\]
\[(4.12) \quad \text{pTan}^-(\text{graph}(\varphi), (x, \varphi(x))) = \{(v, L(v)) : v \in \text{pTan}^-(A, x)\},\]
where \( L \) denote a strict differential of \( \varphi \) at \( x. \)

\textbf{Proof.} Fix \( x \in A. \) We will prove only (4.11); a similar proof of the (4.12) is left to the reader. 1\textsuperscript{st} claim: \( \text{pTan}^+(\text{graph}(\varphi), (x, \varphi(x))) \subset \{v, L(v) : v \in \text{pTan}^+(A, x)\}. \) Let \( v \in \mathbb{R}^d \) and \( r \in \mathbb{R}^n \) such that \( (v, r) \in \text{pTan}^+(\text{graph}(\varphi), (x, \varphi(x))) \). Then there exist sequences \( \{\lambda_m\}_m \subset \mathbb{R}_{++} \) and \( \{x_m\}_m, \{y_m\}_m \subset A \) such that \( \lim_m \lambda_m = 0, \) \( \lim_m (x_m, \varphi(x_m)) = (x, \varphi(x)) \) and
\[(*)1 \quad \lim_{m \to \infty} \frac{(x_m - y_m, \varphi(x_m) - \varphi(y_m))}{\lambda_m} = (v, r).\]
By Cyrenian lemma 2.9, we have \( \lim_m \frac{\varphi(x_m) - \varphi(y_m)}{\lambda_m} = L(v), \) since \( \lim_m \frac{x_m - y_m}{\lambda_m} = v. \)
Hence, \( v \in \text{pTan}^+(A, x) \) and \( (v, r) = (v, L(v)) \), as required.

2\textsuperscript{nd} claim: \( \{v, L(v) : v \in \text{pTan}^+(A, x)\} \subset \text{pTan}^+(\text{graph}(\varphi), (x, \varphi(x))). \) Let \( v \in \text{pTan}^+(A, x). \) Then there exist sequences \( \{\lambda_m\}_m \subset \mathbb{R}_{++}, \{x_m\}_m, \{y_m\}_m \subset A \) such that \( \lim_m \lambda_m = 0, \) \( \lim_{m \to \infty} x_m = x \) and
\[(*)2 \quad \lim_{m \to \infty} \frac{x_m - y_m}{\lambda_m} = v.\]
Being \( \varphi \) strictly differentiable, property (*2) and Cyrenian lemma 2.9 imply
\[(*)3 \quad \lim_{m \to \infty} \frac{\varphi(x_m) - \varphi(y_m)}{\lambda_m} = L(v).\]
Since \( \lim_m (x_m, \varphi(x_m)) = (x, \varphi(x)) \) and \( \lim_m \lambda_m = 0, \) from (*2) and (*3) follows that \( (v, L(v)) \in \text{pTan}^+(\text{graph}(\varphi), (x, \varphi(x))) \), as required.
\[\Box\]

\textbf{Lemma 4.5.} Let \( A \subset \mathbb{R}^n \) with \( A \subset \text{der}(A) \) and let \( \varphi : A \to \mathbb{R} \) be strictly differentiable. If \( A \) is a \( C^1 \)-manifold of \( \mathbb{R}^n \) at a point \( i \in A, \) then \( \text{graph}(\varphi) \) is a \( C^1 \)-manifold of \( \mathbb{R}^{n+1} \) at \( (i, \varphi(i)). \)
**Proof.** 1st case: A is a non-empty open subset of some vector subspace $V$ of $\mathbb{R}^n$. Let $V^\perp$ denote the orthogonal complement of $V$. Moreover, let $\nu: (A + V^\perp) \times \mathbb{R} \to (A + V^\perp) \times \mathbb{R}$ the function such that $\nu(t + y, z) := (t + y, z - \varphi(t))$ for every $(t, y, z) \in A \times V^\perp \times \mathbb{R}$. Since both domain and codomain of $\nu$ coincide with the open set $\Omega := (A + V^\perp) \times \mathbb{R}$ of $\mathbb{R}^{n+1}$ and, besides, $\nu$ is bjective and strictly differentiable, we have that $\nu$ is a $C^1$-diffeomorphism from $\Omega$ onto $\Omega$. On the other hand we have

\begin{equation}
(\ast 1) \quad \nu(\text{graph}(\varphi) \cap \Omega) = \nu(\text{graph}(\varphi)) = A \times \{0\} = \Omega \cap (V \times \{0\}) = \nu(\Omega) \cap (V \times \{0\}).
\end{equation}

Therefore, by Definition 4.1, graph($\varphi$) is a $C^1$-manifold of $\mathbb{R}^{n+1}$.

2nd case: $A$ is an arbitrary non-empty subset of $\mathbb{R}^n$. Since $A$ is a $C^1$-manifold at $\hat{t}$, there exist an open neighborhood $\Omega$ of $\hat{t}$ in $\mathbb{R}^n$, a vector subspace $V$ of $\mathbb{R}^n$ and a $C^1$-diffeomorphism $\xi$ from $\Omega$ to another open subset of $\mathbb{R}^n$ such that

\begin{equation}
(\ast 2) \quad \xi(A \cap \Omega) = \xi(\Omega) \cap V.
\end{equation}

Now, let us define the strict differentiable function $\psi : \xi(\Omega) \cap V \to \mathbb{R}$ by $\psi(y) := \varphi(\xi^{-1}(y))$. Since the domain of $\psi$ is an open subset of the vector space $V$, by the first case we have that graph($\psi$) is a $C^1$-manifold of $\mathbb{R}^{n+1}$. On the other hand, let us define the $C^1$-diffeomorphism $\mu : \Omega \times \mathbb{R} \to \xi(\Omega) \times \mathbb{R}$ by $\mu(t, r) := (\xi(t), r)$. Clearly,

\begin{equation}
(\ast 3) \quad \mu^{-1}(\text{graph}(\psi)) = \text{graph}(\varphi) \cap ((A \cap \Omega) \times \mathbb{R}) = \text{graph}(\varphi) \cap (\Omega \times \mathbb{R}).
\end{equation}

Hence,

\begin{equation}
(\ast 4) \quad \text{graph}(\varphi) \cap (\Omega \times \mathbb{R}) \text{ is a } C^1\text{-manifold of } \mathbb{R}^{n+1},
\end{equation}

since it is the image of the $C^1$-manifold graph($\psi$) by the $C^1$-diffeomorphism $\mu^{-1}$. Therefore graph($\varphi$) is a $C^1$-manifold at $(\hat{t}, \varphi(\hat{t}))$, because $\Omega \times \mathbb{R}$ is an open neighborhood of $(\hat{t}, \varphi(\hat{t}))$. \hfill \Box

**Theorem 4.6 (Four-cones coincidence theorem: local version).** Let $F \subseteq \mathbb{R}^n$ and let $\hat{x} \in F$. Then $F$ is a $C^1$-manifold at $\hat{x}$ if and only if the following three properties hold:

(4.13) $F$ is locally compact at $\hat{x}$,

(4.14) $\text{pTan}^- (F, \hat{x}) = \text{pTan}^+ (F, \hat{x})$,

(4.15) there exists an open ball $B_\delta(\hat{x})$ centered at $\hat{x}$ such that

$$\text{pTan}^+(F, x) = \text{pLTan}^+(F, x)$$

for every $x \in F \cap B_\delta(\hat{x})$.

**Proof.** Necessity. Let $F$ be a $C^1$-manifold of $\mathbb{R}^n$ at $\hat{x}$. Clearly, $F$ is locally compact at $\hat{x}$. On the other hand, Property (4.4) of Proposition 4.2 implies the coincidence of the lower and upper paratangent cones, as required. Sufficiency. Assume (4.13), (4.14) and (4.15) are true.

1st case: $\dim(\text{pLTan}^+(F, \hat{x})) = 0$. Since $\text{Tan}^+(F, \hat{x}) \subseteq \text{pLTan}^+(F, \hat{x})$, we have $\text{Tan}^+(F, \hat{x}) = \{0\}$. Hence, by (1.26), $\hat{x}$ is an isolated point of $F$. Therefore $F$ is a $C^1$-manifold of dimension zero at $\hat{x}$.

2nd case: $\dim(\text{pLTan}^+(F, \hat{x})) = n$. By (4.14) we have $\text{pTan}^- (F, \hat{x}) = \mathbb{R}^n$. Hence, being $F$ locally compact at $\hat{x}$, property (1.25) implies $\hat{x} \in \text{int}(F)$. Therefore $F$ is a $C^1$-manifold of dimension $n$ at $\hat{x}$, since $\hat{x}$ is an interior point of $F$. 

3rd case: $0 < \dim(pLTan^+(F, \hat{x})) < n$. Choose two non null vectors $v_0, v_n \in \mathbb{R}^n$ such that $v_0 \in pLTan^+(F, \hat{x})$ and $v_n \notin pLTan^+(F, \hat{x})$. Without loss of generality, assume that $w = e_n$. By (1.23) we have
\[ pTan^-(F, \hat{x}) = \left( \left. \frac{d}{dt} \right|_{t=0} \phi(t) \right) \cap pLTan^+(F, \hat{x}). \]
Moreover, by (4.14) and (4.15), the equality $pTan^-(F, \hat{x}) = pLTan^+(F, \hat{x})$ holds; hence (1) implies $v_0 \in Li_{F, \hat{x}, \rightarrow} \left( \frac{d}{dt} \right|_{t=0} \phi(t) \cap pLTan^+(F, \hat{x})$; consequently, there exists a positive real number $\delta_1 \leq \delta$ such that the non null vector $v_0 \in F \cap (B_\delta(\hat{x}))$. Therefore, by (1.26),
\[ F \cap B_{\delta_1}(\hat{x}) \subset \text{der}(F \cap B_{\delta_1}(\hat{x})). \]
Now, by (1.21) and (4.15) we have
\[ Ls_{F, \hat{x}, \rightarrow} pLTan^+(F, x) \subset pLTan^+(F, \hat{x}); \]
consequently, $e_n \notin Ls_{F, \hat{x}, \rightarrow} pLTan^+(F, x)$. Therefore, there exists a positive real number $\delta_2 \leq \delta_1$ such that
\[ e_n \notin pLTan^+(F, x) \quad \text{for every} \quad x \in F \cap B_{\delta_1}(\hat{x}). \]
From (2), (4) and Lemma 4.3 there are a positive real number $\varepsilon \leq \delta_2$, a subset $A$ of $\mathbb{R}^{n-1}$ with $A \subset \text{der}(A)$ and there exists a function $\varphi : A \rightarrow \mathbb{R}$ strictly differentiable such that
\[ \text{graph}(\varphi) = F \cap B_{\varepsilon}(\hat{x}). \]
and
\[ A \text{ is locally compact at } \hat{t} \]
where $\hat{t} \in A$ and $(\hat{t}, \varphi(\hat{t})) = \hat{x}$. Being $B_\varepsilon(\hat{x})$ an open set, by (5) and (5.6) we have
\[ pTan^-(\text{graph}(\varphi), (t, \varphi(t))) = pTan^-(F, (t, \varphi(t))) \quad \text{and} \quad pTan^+(\text{graph}(\varphi), (t, \varphi(t))) = pTan^+(F, (t, \varphi(t))) \quad \text{for every} \quad t \in A. \]
Hence, by (4.14) and (4.15) we obtain
\[ pTan^-(\text{graph}(\varphi), (\hat{t}, \varphi(\hat{t}))) = pTan^+(\text{graph}(\varphi), (\hat{t}, \varphi(\hat{t}))) \]
and
\[ pTan^+(\text{graph}(\varphi), (t, \varphi(t))) = pLTan^+(\text{graph}(\varphi), (t, \varphi(t))) \quad \text{for every} \quad t \in A. \]
Therefore, from Lemma 4.4 it follows that
\[ pTan^-(A, \hat{t}) = pTan^+(A, \hat{t}) \quad \text{and} \]
\[ pTan^+(A, t) = pLTan^+(A, t) \quad \text{for every} \quad t \in A. \]
Now, by induction, assume that this theorem 4.6 holds for subsets of $\mathbb{R}^{n-1}$. Then, by (6), (7) and (8) , we have that the subset $A$ of $\mathbb{R}^{n-1}$ is a $C^1$-manifold at $\hat{t}$. Therefore, by Lemma 4.5, graph($\varphi$) (i.e. $F \cap B_{\varepsilon}(\hat{x})$) is a $C^1$-manifold of $\mathbb{R}^n$ at $\hat{x}$, as required.

\textbf{Theorem 4.7 (Four-cones coincidence theorem: global version).} A non-empty subset $F$ of $\mathbb{R}^n$ is a $C^1$-manifold if and only if $F$ is locally compact and the lower and upper paratangent cones to $F$ coincide at every point, i.e.,
\[ pTan^-(F, x) = pTan^+(F, x) \quad \text{for every} \quad x \in F. \]
It is certainly worthwhile to remark that, by (1.1) and (1.23), condition (4.16) amounts to the set inclusion
\[(4.17) \quad \text{pTan}^+(F, x) \subset \underset{F \ni y \to x}{\overset{\text{Li}}{\text{Li}}} \text{Tan}^+(F, y) \text{ for every } x \in F.\]

**Proof.** *Necessity.* Let \( F \) be a \( C^1 \)-manifold of \( \mathbb{R}^n \). Clearly, \( F \) is locally compact. On the other hand, Property (4.4) of Proposition 4.2 implies the coincidence of the lower and upper paratangent cones, as required. *Sufficiency.* Let \( F \) be locally compact and let both lower and upper paratangent cones coincide at every point. Then (1.20) and (1.22) imply that the upper paratangent cones to \( F \) are vector space, i.e.,
\[(*) \quad \text{pTan}^+(F, x) = \text{pLTan}^+(F, x) \]
for every \( x \in F \). Hence, Theorem 4.6 implies that \( F \) is \( C^1 \)-manifold of \( \mathbb{R}^n \) at its every point, as required.

Let us denote with \( \text{GL}_n(\mathbb{R}) \) the general linear group, i.e. the multiplicative group of the \( n \times n \) invertible matrices with real entries. We denote with \( E \) the unit of \( \text{GL}_n(\mathbb{R}) \). Let \( \text{M}_n(\mathbb{R}) \) be the algebra of \( n \times n \) matrices, endowed with Euclidean topology. Clearly every subgroup of \( \text{GL}_n(\mathbb{R}) \) which is a \( C^1 \) manifold of \( \text{M}_n(\mathbb{R}) \), is necessarily a locally compact set with respect Euclidean topology. Conversely, we will apply main Theorem 4.7 to prove that

**Corollary 4.8 (von Neumann, [63, (1929)]).** A locally compact subgroup \( \mathcal{G} \) of \( \text{GL}_n(\mathbb{R}) \) is a \( C^1 \)-manifold of \( \text{M}_n(\mathbb{R}) \).

**Proof.** By Theorem 4.7, it is enough to prove (4.17), that is, we must prove that
\[(*) \quad \text{pTan}^+(\mathcal{G}, A) \subset \underset{H \to A}{\overset{\text{Li}}{\text{Li}}} \text{Tan}^+(\mathcal{G}, H) \]
for every \( A \in \mathcal{G} \). 1st case: Let \( A \) be the unit \( E \). Let \( V \in \text{pTan}^+(\mathcal{G}, E) \); by definition there exist three sequences \( \{\lambda_m\}_m \subset \mathbb{R}_{++}, \{A_m\}_m \subset \mathcal{G}, \{B_m\}_m \subset \mathcal{G} \) such that
\[(*) \quad \lim_{m} \lambda_m = 0^+, \quad \lim_{m} A_m = E, \quad \lim_{m} \frac{A_m - B_m}{\lambda_m} = V. \]

In order to show that \( V \in \underset{H \to E}{\overset{\text{Li}}{\text{Li}}} \text{Tan}^+(\mathcal{G}, H) \), let \( \{H_k\}_k \subset \mathcal{G} \) be such that \( \lim_k H_k = E \). Define sequences of matrices \( \{H_{k,m}\}_m \) and \( \{V_k\}_k \) by
\[(*) \quad H_{k,m} := H_k \cdot A_m \cdot B_m^{-1}, \quad V_k := H_k \cdot V. \]
Observe that
\[(*) \quad V_k \in \text{Tan}^+(\mathcal{G}, H_k) \text{ for every } k \in \mathbb{N}, \]
since \( \{H_{k,m}\}_m \subset \mathcal{G}, \lim_m H_{k,m} = H_k \) and \( \lim_m \frac{H_{k,m} - H_k}{\lambda_m} = \lim_m \frac{H_k \cdot A_m \cdot B_m^{-1} - H_k}{\lambda_m} = \lim_k H_k \cdot (\frac{A_m - B_m}{\lambda_m}) \cdot B_m^{-1} = H_k \cdot V \cdot E = V_k \). On the other hand, \( \lim_k V_k = V \). Hence, because \( \{H_k\}_k \subset \mathcal{G} \) is an arbitrary sequence converging to \( E \), from the definition of the lower limit of sets it follows that \( V \in \underset{H \to E}{\overset{\text{Li}}{\text{Li}}} \text{Tan}^+(\mathcal{G}, H) \).

2nd case: \( A \) is an arbitrary element of \( \mathcal{G} \). Since
\[\frac{A_m - B_m}{\lambda_m} = A \frac{A^{-1} A_m - A^{-1} B_m}{\lambda_m} \]
for every \( A_m, B_m, \in \mathcal{G} \) and \( \lambda_m \in \mathbb{R}_{++} \), by the definitions of tangent and upper paratangent cones we have
\[(*) \quad \text{Tan}^+(\mathcal{G}, A) = A \cdot \text{Tan}^+(\mathcal{G}, E) \text{ and } \text{pTan}^+(\mathcal{G}, A) = A \cdot \text{pTan}^+(\mathcal{G}, E). \]
Therefore, from the 1st case it follows that $p\Tan^+(G, A) = A \cdot p\Tan^+(G, E) \subset A \cdot \Li_{H \to E} \Tan^+(G, H) = \Li_{H \to A} \Tan^+(G, H)$, as required. □

Continuous variability of tangent spaces (in traditional sense) does not assure that a set is a $C^1$-manifold (for example, see fig. 1 and 2 of Section 2). In order to characterize $C^1$-manifolds, in the following two corollaries simple conditions are added to the continuous variability of tangent spaces.

**Corollary 4.9.** A non-empty subset $F$ of $\mathbb{R}^n$ is a $C^1$-manifold if and only if $F$ is locally compact and the following two properties hold.

(4.18) the map $x \mapsto \Tan^+(F, x)$ is lower semicontinuous on $F$,

(4.19) $\lim_{x, y \to \hat{x}} \frac{\text{dist}(y, x + \Tan^+(F, \hat{x}))}{\|y - x\|} = 0$ for every $\hat{x} \in \text{der}(F)$.

In the case where $\Tan^+(F, \hat{x})$ is a vector space, condition (4.19) means that the angle between the straight-line passing through two distinct points $y$ and $x$ of $F$ and the vector space $\Tan^+(F, \hat{x})$ tangent to $F$ at $\hat{x}$, tends to zero as $x$ and $y$ tend to $\hat{x}$ (see fig. 3 above).

**Proof.** By (1.23) the lower semicontinuity of $x \mapsto \Tan^+(F, x)$ amounts to

(*) $\Tan^+(F, x) = p\Tan^-(F, x)$ for every $x \in F$

On the other hand, by footnote 19, condition (4.19) becomes

(**) $p\Tan^+(F, \hat{x}) \subset \Tan^+(F)$ for every $\hat{x} \in F$.

Hence, from conditions (4.18) and (4.19) it follows that $p\Tan^+(F, \hat{x}) \subset p\Tan^-(F, x)$; and conversely. Therefore Theorem 4.7 entails both necessity and sufficiency of the conditions (4.18) and (4.19). □

**Corollary 4.10.** A non-empty subset $F$ of $\mathbb{R}^n$ is a $C^1$-manifold if and only if $F$ is locally compact and the following two properties hold:

(4.20) the map $x \mapsto \Tan^+(F, x)$ is continuous on $F$,

(4.21) $\lim_{x, y \to \hat{x}} \frac{\text{dist}(y, x + \Tan^+(F, \hat{x}))}{\|y - x\|} = 0$ for every $\hat{x} \in \text{der}(F)$.

In the case where the upper tangent cones $\Tan^+(F, x)$ are vector spaces, condition (4.21) means that the angle between the straight-line passing through two distinct points $y$ and $x$ of $F$ and the tangent vector space to $F$ at $\hat{x}$ tends to zero as $x$ and $y$ tend to $\hat{x}$ (see fig. 4 above).

**Proof.** Necessity: it is known. Sufficiency. We must prove (4.17). Hence, it is enough to show that, for every $\hat{x} \in \text{der}(F)$, the following set inclusion holds

(4.22) $p\Tan^+(F, \hat{x}) \cap \{v \in \mathbb{R}^n : \|v\| = 1\} \subset \Li_{F \ni y \to \hat{x}} \Tan^+(F, y)$.
To prove (4.22) fix \( \hat{x} \in \text{der}(F) \) and \( v \in p\text{Tan}^+(F, \hat{x}) \) with \( \|v\| = 1 \). By (1.17) there exist sequences \( \{x_m\}, \{y_m\} \subset F \) converging to \( \hat{x} \) such that \( \lim_{m \to \infty} \frac{y_m - x_m}{\|y_m - x_m\|} = v \).

By (4.21) and the following triangular inequality
\[
\text{dist}(v, \text{Tan}^+(F, x_m)) \leq \frac{\|y_m - x_m + \text{Tan}^+(F, x_m)\|}{\|y_m - x_m\|} + \|v - \frac{y_m - x_m}{\|y_m - x_m\|}\|
\]
we have that \( \lim_{m \to \infty} \text{dist}(v, \text{Tan}^+(F, x_m)) = 0 \); hence, \( v \in \text{Ls}_{F, \hat{x} \to \hat{x}} \text{Tan}^+(F, F) \).

Thus, from (4.20) it follows that \( v \in \text{Li}_{F, \hat{x} \to \hat{x}} \text{Tan}^+(F, y) \), as (4.22) requires. \( \square \)

Both old and modern characterization of \( C^1 \)-manifolds can be deduced from four-cones coincidence theorem 4.7; as example, we state and prove the following theorem, due to Tierno (see [57, (1997)], [58, (2000)]).

**Theorem 4.11 (Tierno’s theorem).** A non-empty set \( F \) of \( \mathbb{R}^n \) is a \( d \)-dimensional \( C^1 \)-manifold if and only if \( F \) is locally compact and the upper tangent and upper paratangent cones to \( F \) coincide and are \( d \)-dimensional vector spaces \( 31 \) at every point, i.e.,
\[
(4.23) \quad \text{Tan}^+(F, x) = p\text{LTan}^+(F, x) \quad \text{and} \quad \dim(\text{LTan}^+(F, x)) = d \quad \text{for every} \quad x \in F.
\]

This theorem provides an efficacious test for visual geometrical reconnaissance of \( C^1 \)-manifolds. In fact, it follows that \( F \) is a \( d \)-dimensional \( C^1 \)-manifold if and only if
\[
(4.24) \quad \text{at every point of} \ F, \text{the upper tangent vectors to} \ F \text{form a} \ d \text{-dimensional vector space which is paratangent in traditional sense to} \ F.
\]

In symbols, by Proposition 2.6 this condition becomes
\[
(4.25) \quad \text{Tan}^+(F, x) = \text{LTan}^+(F, x), \quad \dim(\text{LTan}^+(F, x)) = d \quad \text{and} \quad p\text{LTan}^+(F, x) \subset \text{LTan}^+(F, x) \quad \text{for every} \quad x \in F.
\]

**Proof.** *Necessity.* By Proposition 4.2, it is obvious. *Sufficiency.* By Theorem 4.7 it is enough to show that \( p\text{Tan}^-(F, x) = p\text{Tan}^+(F, x) \) for every \( x \in F \). The first equality in (4.23) means:
\[
(*1) \quad \text{Tan}^+(F, x) = \text{LTan}^+(F, x) = p\text{Tan}^+(F, x) = p\text{LTan}^+(F, x)
\]
for every \( x \in F \). Hence, by (1.21) and (1.23) we have
\[
(*2) \quad p\text{LTan}^+(F, x) \supset \text{Ls}_{F, y \to x} p\text{LTan}^+(F, y)
\]
and
\[
(*3) \quad p\text{Tan}^-(F, x) = \text{Li}_{F, y \to x} p\text{LTan}^+(F, y),
\]
for every \( x \in F \). By (4.23) the vector spaces \( p\text{LTan}^+(F, x) \) and \( p\text{LTan}^+(F, y) \) have the same dimension; hence, from (*2) and Corollary 3.3 it follows that
\[
(*4) \quad p\text{LTan}^+(F, x) = \text{Li}_{F, y \to x} p\text{LTan}^+(F, y).
\]

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30 In the proof of Corollary we have show that condition (4.21) imply the following set inclusion:
\( p\text{Tan}^+(F, \hat{x}) \subset \text{Ls}_{F, y \to x} \text{Tan}^+(F, x) \) for every \( x \in F \). The converse also holds.

31 The \( d \)-dimensionality condition cannot be dropped in (4.23). In fact, define \( F := \{ x \in \mathbb{R}^n : \|x\| = 0 \text{ or } \frac{x}{\|x\|} \in \mathbb{N} \} \). Then \( \text{Tan}^+(F, 0) = p\text{LTan}^+(F, 0) = \mathbb{R}^n \); moreover, for every \( x \in F \) with \( \|x\| \neq 0 \), one has \( \text{Tan}^+(F, x) = p\text{LTan}^+(F, x) = \{ v \in \mathbb{R}^n : (v, x) = 0 \} \) and \( \dim(p\text{LTan}^+(F, x)) = n - 1 \). Notice that \( F \) is a \( C^1 \)-manifold only at every \( x \neq 0 \).
Therefore, by (∗3) and (∗4) we obtain that \( \text{pTan}^-(F, x) = \text{pTan}^+(F, x) \) for every \( x \in F \), as required.

\[ \square \]

Appendix A. Von Neumann and alternative definitions of lower tangent cones

Five years before the rediscovery of the upper tangent cone by Bouligand and Severi, in [63, (1929)] Von Neumann \(^{32}\) showed that a closed matrix group \( G \) is a Lie group by describing the associated Lie algebra (called \( \text{Infinitesimalgruppe} \)) as the set of all upper tangent vector at unit \( E \) to the group \( G \). The elements of \( G \) are non-singular real matrices \( n \times n \); hence, being \( G \) a subset of Euclidean space \( M_n(\mathbb{R}) \) of all real matrices \( n \times n \), the upper tangent vectors are elements of \( M_n(\mathbb{R}) \).

More explicitly and clearly, Von Neumann define an \( \text{Infinitesimalgruppe} J \) of \( G \) as the set of all matrices \( V \in M_n(\mathbb{R}) \) such that there exist an infinitesimal sequence \( \{\varepsilon_m\}_{m} \subset \mathbb{R}^{++} \) and a sequence \( \{A_m\}_{m} \in \mathbb{N} \subset G \) such that

\[
(A.1) \quad \lim_{m \to \infty} \frac{A_m - E}{\varepsilon_m} = V.
\]

Moreover, to show that the \( \text{Infinitesimalgruppe} J \) is a Lie algebra, von Neumann proved that, for every matrix \( V \) belonging to the \( \text{Infinitesimalgruppe} J \), there exists a family of matrices \( \{B_\lambda\}_{\lambda \in (0,1]} \subset M_n(\mathbb{R}) \) such that

\[
(A.2) \quad \lim_{\lambda \to 0^+} \frac{B_\lambda - E}{\lambda} = V.
\]

It is well known that vectors \( V \) verifying (A.2), constitute the lower tangent cone \( \text{Tan}^-(G, E) \). Therefore, the definition (A.1) and property (A.2) can be resumed by

\[
(A.3) \quad J := \text{Tan}^+(G, E) \quad \text{and} \quad \text{Tan}^+(G, E) = \text{Tan}^-(G, E).
\]

Crucial properties of general Lie groups (as “the infinitesimal group \( J \) is mapped into \( G \) by exp” or “some neighborhood of \( E \) in \( G \) is mapped into the infinitesimal group \( J \) by log”) are verified by von Neumann by the following immediate consequence of (A.2): for every \( V \in J \), there exists a sequence \( \{A_m\}_{m} \in \mathbb{N} \) such that

\[
(A.4) \quad \lim_{m \to \infty} m(A_m - E) = V.
\]

Tangent vectors in this sense are lower tangent, and conversely. In fact

**Proposition A.1.** Let \( F \) be a subset of \( \mathbb{R}^n \) and let \( x \in F \). Then

\[
(A.5) \quad \text{Tan}^-(F, x) = \underset{\text{Li} \ M_n \to \infty}{\text{Li} \ m(F - x)}.
\]

In terms of sequences, \( v \in \text{Tan}^-(F, x) \) if and only if there exists a sequence \( \{x_m\}_{m} \subset F \) (converging to \( x \)) such that

\[
(A.6) \quad \lim_{m \to \infty} m(x_m - x) = v.
\]

Analogously, with respect the lower paratangent cones we have

**Proposition A.2.** Let \( F \) be a subset of \( \mathbb{R}^n \) and let \( x \in F \). Then

\[
(A.7) \quad \text{pTan}^-(F, x) = \underset{\text{Li} \ M_n \to \infty}{\text{Li} \ m(F - y)}.
\]

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\(^{32}\)Von Neumann’s manuscript was received by Mathematische Zeitschrift February 2, 1927.
In terms of sequences, \( v \in \text{pTan}^-(F, x) \) if and only if, for every sequence \( \{y_m\}_m \subset F \) converging to \( x \), there exists a sequence \( \{x_m\}_m \subset F \) (converging to \( x \)) such that
\[
\lim_{m \to \infty} m(y_m - x_m) = v.
\]

The proof of Propositions A.1 and A.2 is an immediate consequence of the following lemma.

**Lemma A.3.** Let \( F \subset \mathbb{R}^n \), \( x \in F \) and \( \Phi : F \to \mathcal{P}(\mathbb{R}^n) \). Then
\[
\text{Li}_{x \in \Phi(\mathcal{N})} \lambda \Phi(y) = \text{Li}_{x \in \Phi(\mathcal{N})} m \Phi(y)
\]

**Proof.** The set inclusion \( \subseteq \) is obvious. For proving the converse set inclusion, choose an arbitrary element \( v \in \text{Li}_{x \in \Phi(\mathcal{N})} m \Phi(y) \). By the definition of lower limit we have that
\[
\lim_{x \in \Phi(\mathcal{N})} \text{dist}(v, m \Phi(y)) = 0
\]

As usual, for every real number \( \lambda \) let \( \lfloor \lambda \rfloor \) denote the least integer number greater than or equal to \( \lambda \). Observe that \( \lim_{\lambda \to \infty} \frac{1}{\lfloor \lambda \rfloor} = 1 \), because \( \lfloor \lambda \rfloor - \lambda \leq 1 \). Therefore, by (A.9) and the following triangular inequality
\[
\text{dist}(v, \lambda \Phi(y)) = \lambda \text{dist}\left(\frac{v}{\lambda}, \Phi(y)\right) \leq \lambda \left(\left\| \frac{v}{\lambda} - \frac{v}{\lfloor \lambda \rfloor} \right\| + \text{dist}\left(\frac{v}{\lfloor \lambda \rfloor}, \Phi(y)\right)\right)
\]
\[
\leq \|v\| \left(1 - \frac{1}{\lfloor \lambda \rfloor}\right) + \frac{1}{\lfloor \lambda \rfloor} \text{dist}(v, \lfloor \lambda \rfloor \Phi(y)),
\]
we have
\[
\lim_{x \in \Phi(\mathcal{N})} \text{dist}(v, \lambda \Phi(y)) = 0,
\]
that is \( v \in \text{Li}_{x \in \Phi(\mathcal{N})} \lambda \Phi(y) \), as required. \( \square \)

**Example A.4.** Let \( \{x_m\}_m \subset \mathbb{R}_{++} \) be an infinitesimal decreasing sequence. Posit \( S := \{x_m : m \in \mathbb{N}\} \). Then
\[
1 \in \text{Tan}^-(S, 0) \iff \exists \{m_k\}_k \subset \mathbb{N} \text{ such that } \lim_{k \to \infty} \frac{x_{m_k}}{1/k} = 1;
\]

\[
1 \in \text{Tan}^-(S, 0) \iff \lim_{m \to \infty} \frac{x_{m+1}}{x_m} = 1.33
\]

In particular, for \( x_m := m! \), \( \text{Tan}^-(S, 0) = \{0\} \) (see Example 2.1 in Dolecki, Greco [15, (2011), p. 305]). \( \square \)

The following two examples were commented in Dolecki, Greco [15, (2011), p. 305].

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33To prove (A.13) observe that \( 1 \in \text{Tan}^-(S, 0) \) amounts to \( \lim_{\lambda \to 0^+} \frac{\text{dist}(\lambda, S)}{\lambda} = 0 \). Hence, for \( \lambda_m := \frac{x_m + x_{m+1}}{2} \), \( \lim_{m \to \infty} \frac{\text{dist}(\lambda_m, S)}{\lambda_m} = 0 \). Since \( \frac{\text{dist}(\lambda_m, S)}{\lambda_m} = \frac{x_m - x_{m+1}}{x_m + x_{m+1}} \) and \( 0 \leq \frac{1}{2} \left(1 - \frac{x_{m+1}}{x_m}\right) \leq \frac{x_{m+1}}{x_m} \), we have \( \lim_{m \to \infty} \frac{x_{m+1}}{x_m} = 1 \), as required. Conversely, assume \( \lim_{m \to \infty} \frac{x_{m+1}}{x_m} = 1 \). For every \( k \in \mathbb{N} \), define \( m_k := \max\{m \in \mathbb{N} : \frac{1}{k} \leq x_m\} \). Then \( \lim_{k \to \infty} \frac{x_{m_k}}{1/k} = 1 \). Hence (A.12) entails \( 1 \in \text{Tan}^-(S, 0) \).
Example A.5. Let $A$ be denote the set $\{(t,t \sin(1/t)) : t \in \mathbb{R} \setminus \{0\}\}$. Then \(\text{Tan}^{-1}(A, (0, 0)) = \{(h,k) \in \mathbb{R}^2 : |k| \leq |h|\}\). \(\Box\)

Example A.6. Let $B$ be denote the set $\{(t,-t) : t \in \mathbb{R}, t < 0\} \cup \{(1/m,1/m) : m \in \mathbb{N}_1\}$. Then \(\text{Tan}^{-1}(B, (0, 0)) = \{t(1,1) : t \in \mathbb{R}_+\} \cup \{t(-1,1) : t \in \mathbb{R}_+\}\). \(\Box\)

APPENDIX B. FROM FRÉCHET PROBLEM TO MODERN CHARACTERIZATIONS OF SMOOTH MANIFOLD

In [31, (1887), vol. III p. 587] JORDAN defines a curve as a continuous image of an interval. By means of a notion of rectifiability, JORDAN gives mathematical concreteness and coherence to the usage of the term “length” and, moreover, by parametrization of sets he provides fresh impetus to the study of local and global properties of sets.

Surprisingly for JORDAN’s epoch, continuous curves did not fit to common intuition on 1-dimensionality and null area of their loci. In fact, PEANO in [42, 1890] constructed a continuous curve filling a square. Clearly, PEANO’s curve is not simple. An example of a simple continuous curve of non-null area was given by LEBESGUE [33, (1903)] and by OSGOOD [39, (1903)]. NALLI [37, 38, (1911)] characterized the locus of simple continuous plane curves by means of local connectedness (a new notion, introduced by NALLI). Three years later, MAZURKIEWICZ [35, (1914)] and HAHN [28, (1914)] proved the celebrated theorem: “A set of Euclidean space is a continuous image of a compact interval if and only if it is a locally connected continuum”.

In absence of differentiable properties, the continuity alone does not capture intuitive curve aspects. Aware of this lack, to recover geometric properties of the locus of a continuous curve, FRÉCHET (see [17, (1925), p. 292-3] and [18, (1928), p. 152-154]) proposed the following problem: Find a non-singular parametric representation\(^{36}\) of the locus of a continuous curve having tangent straight-line at every point. Let’s quote FRÉCHET from the first reference:

On sait qu’une courbe continue sans point multiple et ayant une tangente déterminée en chaque point peut avoir une représentation paramétrique constituée de fonctions dérivables $x(t), y(t), z(t)$, mais dont les dérivées peuvent exceptionnellement s’annuler à la fois […]

\(^{34}\)Fix $|\alpha| \leq 1$ and let $\theta$ be $\arcsin|\alpha|$. Then $(x_m, y_m) := (\frac{1}{\pi+2m\pi}, \frac{1}{\pi+2m\pi} \sin(\frac{1}{\pi+2m\pi} \alpha)) = (\frac{1}{\pi+2m\pi}, \frac{1}{\pi+2m\pi} \theta \alpha) \in A$ for every natural number $m$. Since $\lim_{m \to \infty} m(x_m, y_m) = (\frac{1}{2\pi}, \frac{1}{2\pi} \alpha)$, by Proposition A.1 we have

\(\ast\) 

$$(1, \alpha) \in \text{Tan}^{-1}(A, (0, 0)).$$

Analogously, $(x_m, y_m) := (-\frac{1}{\pi+2m\pi}, -\frac{1}{\pi+2m\pi} \sin(-\frac{1}{\pi+2m\pi} \alpha)) = (-\frac{1}{\pi+2m\pi}, -\frac{1}{\pi+2m\pi} \theta \alpha) \in A$ for every natural number $m$. Since $\lim_{m \to \infty} m(x_m, y_m) = (-\frac{1}{2\pi}, -\frac{1}{2\pi} \alpha)$ by Proposition A.1 we have

\(\ast\ast\) 

$$(1, \alpha) \in \text{Tan}^{-1}(A, (0, 0)).$$

Now, from \(\ast\) and \(\ast\ast\) if follows that $\text{Tan}^{-1}(A, (0, 0)) \supset \{(h,k) \in \mathbb{R}^2 : |k| \leq |h|\}$; the opposite inclusion is due to the fact that $\text{Tan}^{-1}(A, (0, 0)) \subset \text{Tan}^{-1}(A, (0, 0)) = \{(h,k) \in \mathbb{R}^2 : |k| \leq |h|\}$.

\(^{35}\)By applying Proposition A.1 to two sequences $\{\frac{1}{m}, \frac{1}{m}\} \cup \{-\frac{1}{m}, -\frac{1}{m}\}$ we have that $(\pm, 1) \in \text{Tan}^{-1}(B, (0, 0))$. Hence $\text{Tan}^{-1}(B, (0, 0)) = \{t(1,1) : t \in \mathbb{R}_+\} \cup \{t(-1,1) : t \in \mathbb{R}_+\}$, since $\text{Tan}^{-1}(B, (0, 0)) \subset \text{Tan}^{-1}(B, (0, 0)) = \{t(1,1) : t \in \mathbb{R}_+\} \cup \{t(-1,1) : t \in \mathbb{R}_+\}$.

\(^{36}\)Here and in the sequel, “non-singular parametric representation” stands for “differentiable parametric representation with everywhere non-null derivative”.

\(\Box\)
Ce qui précède nous encourage à proposer la question suivante, dont la solution à première vue ne paraît pas douteuse:

*S'il une courbe continue est douée partout (ou en un point) d'une tangente, peut-on la représenter paramétriquement par des fonctions dérivables partout (ou au point correspondant)?* Bien entendu, dans cet énoncé, la tangente est définie géométriquement, c’est-à-dire comme limite d’une corde.

Fréchet’s confidence about a solution to his problem was deluded in 1926 by Valiron [60, (1927)]. After making precise and explicit the meaning of both *tangent half-straight-line* and *tangent straight-line*, Valiron gives the following proposition.

**Theorem B.1** (Valiron [60, (1927), p. 47]). *If a continuous curve admits a continuously variable oriented tangent straight-line*\(^{37}\) *at its points, then it has a non-singular continuously differentiable parametric representation.*

Valiron [59, (1926)] provides an analogous proposition for surfaces of ordinary 3-dimensional space. To attain this aim, he introduces the concept of *oriented tangent plane* to a surface \(F\) \(^{38}\), takes into account continuously turning oriented tangent plane and, in addition, adopts the following condition at every point \(x \in F\):

\[
(B.1) \quad \text{(Valiron [59, (1926), p. 190]) The orthogonal projection on the oriented tangent plane to } F \text{ at } x \text{ is injective on an open neighborhood of } x \text{ in } F.
\]

**Theorem B.2** (Valiron [59, (1926)]). *Let } F \subset \mathbb{R}^3 \text{ be homeomorphic to a 2-dimensional open connected set. If } F \text{ admits a continuously variable oriented tangent plane and the condition } (B.1) \text{ holds, then } F \text{ locally coincides with the graph of a continuously differentiable function.}*

Following Pauc’s counterexample [40, (1940), p. 96] to Fréchet problem, Choquet, in his thesis [7, (1948), p. 170], provides necessary and sufficient conditions for the Fréchet supposition to hold. As Valiron, Pauc and Choquet make precise and explicit the meaning of tangent straight-line. Besides, Choquet considers a more general problem: *If a variety admits a linear tangent variety at every point (or has certain regularity), is there a regular parametrization (or a parametrization having an analogous degree of regularity)?* In this spirit, Zahorski and Choquet proves the following two propositions.

**Proposition B.3** (Zahorski, see Choquet [7, (1947), p. 173-174]). *If a continuous arc admits a tangent straight-line at all but (possibly) countably many points, then it has a differentiable parametric representation.*

\(^{37}\)Let us express definitions given by Valiron by means of vectors. Let \(\gamma : I \to \mathbb{R}^n\) be a continuous curve on an open interval \(I\) of real numbers. For a given parameter \(t \in I\), a half straight-line issued from \(\gamma(t)\) along an unit vector \(v(t)\), is said to be an oriented tangent half-straight-line, if \(v(t) = \lim_{h \to 0^+} \frac{\gamma(t+h) - \gamma(t)}{\|\gamma(t+h) - \gamma(t)\|} \). Moreover, if \(v(t) = \lim_{h \to 0^+} \frac{\gamma(t+h) - \gamma(t)}{\|\gamma(t+h) - \gamma(t)\|} = \lim_{h \to 0^+} \frac{\gamma(t+h) - \gamma(t)}{\|\gamma(t+h) - \gamma(t)\|}\), the straight-line through \(\gamma(t)\) along the unit vector \(v(t)\) is said to be an oriented tangent straight-line. If \(\gamma\) admits an oriented tangent straight-line at every point and the map \(t \to v(t)\) is continuous, then Valiron says that \(\gamma\) admits a continuously variable oriented tangent straight-line.

\(^{38}\)Using terminology of Section 2, a 2-dimensional vector space \(H\) is an oriented tangent plane to a set \(F\) at a point \(x\), if \(H\) is equal to the upper tangent cone to \(F\) at \(x\). In other words, \(F\) admits an oriented tangent plane at a point \(x\) if and only if the upper tangent cone at \(x\) is a 2-dimensional vector space.
Proposition B.4 (Choquet [7, (1947), p.174]). A continuous image of a compact interval is a rectifiable curve if and only if it admits a Lipschitzian differentiable parametric representation.

Invoking seminal papers of Fréchet [17, (1925)], and Valiron [59, 60, (1926, 1927)], Severi looks for non-singular continuously differentiable parametric representations of a curve (resp. surface). Main ingredients of the solutions of Severi are strict differentiability and paratangency. Strict differentiability ensures that curves (resp. surfaces) have a continuously turning tangent straight line (resp. plane); it is geometrically characterized in terms of paratangency (see Proposition 2.8). On the other hand, aware of the need of Valiron’s condition (B.1), Severi assumes the following simplicity condition and, consequently, ensures Valiron’s condition by replacing Valiron’s oriented tangent plane by a paratangent plane.

Definition B.5 (Severi [55, (1929), p.194], [52, (1930), p.216], [53, (1931), p.341], [54, (1934), p.194]). A $d$-dimensional topological manifold $F$ of $R^n$ satisfies the Severi simplicity condition, if the dimension of the linear hull of the upper paratangent cone to $F$ at every point is at most $d$.

Theorem B.6 (Severi [54, (1934), p.194, 196]). If $F$ is a topological manifold of dimension one (resp. two) satisfying Severi simplicity condition, then the upper paratangent cone at every point is a one (resp. two) dimensional vector space which varies continuously.

According to (B.1) we consider the

(B.2) Valiron condition for a set $F \subset R^n$: For every point $x \in F$, the orthogonal projection on the linear upper tangent space $LTan^+(F,x)$ is injective on an open neighborhood of $x$ in $F$.

The following theorem extends Valiron’s Theorem B.2.

Theorem B.7 (Valiron theorem). A non-empty subset $F$ of $R^n$ is a $d$-dimensional $C^1$-manifold if and only if $F$ is a $d$-dimensional topological manifold, Valiron condition (B.2) is satisfied and

(B.3) $x \mapsto Tan^+(F,x)$ is a continuous map from $F$ to $\mathbb{G}(R^n,d)$, i.e.,

$$pTan^-(F,x) = LTan^+(F,x) \text{ and } \dim(LTan^+(F,x)) = d \text{ for every } x \in F.$$ 39 \hfill □

The following theorem extends Severi’s Theorem B.6, involving a simplicity condition. Severi simplicity condition B.5 can be restated as

(B.4) $\dim(pLTan^+(F,x)) \leq d$ for every $x \in F$;

or, equivalently, (B.5) at every $x \in F$ there exists a $d$-dimensional vector space which is paratangent in traditional sense to $F$.

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39The $d$-dimensionality condition cannot be dropped in (B.3), as it is shown by the following example. Consider the set $F := \{ (x,y,z) \in R^3 : (x^2 + y^2 + z^2)^2 = 4(x^2 + y^2) \}$. $F$ is a torus generated by turning the circle $S := \{ (0,y,z) : (y - 1)^2 + z^2 = 1 \}$ about the $z$-axis. Since the circle $S$ is tangent to $z$-axis at $(0,0,0)$, the set $F$ is a 2-dimensional $C^1$-manifold at every point different from $(0,0,0)$; while $pTan^-(F,(0,0,0)) = Tan^+(F,(0,0,0)) = LTan^+(F,(0,0,0)) = R e_3$ and, consequently, $\dim(LTan^+(F,(0,0,0))) = 1$. 

Theorem B.8 (Severi theorem). A non-empty subset $F$ of $\mathbb{R}^n$ is a $d$-dimensional $C^1$-manifold if and only if $F$ is a $d$-dimensional topological manifold and Severi simplicity condition holds.

The following corollary of Theorem B.7 (Valiron theorem) is due to Gluck.

Corollary B.9 (Gluck [21, (1966), p. 199, 202] and [22, (1968), p. 45]). A non-empty set $F \subset \mathbb{R}^n$ is a $d$-dimensional $C^1$-manifold if and only if $F$ is a $d$-dimensional topological manifold and there exists a continuous map $L\text{Tan} : F \to \mathcal{G}(\mathbb{R}^n, d)$ such that, for every $x \in F$,

(B.6) $L\text{Tan}(x)$ is tangent in traditional sense to $F$ at $x$,

(B.7) the orthogonal projection of $F$ on $L\text{Tan}(x)$ is injective on an open neighborhood of $x$ in $F$.

Proof. Necessity. By Proposition 4.2 it is obvious. Sufficiency. Let $\hat{x} \in F$. By (B.6) and Proposition 2.2,

\begin{equation}
\text{(*)1) } \text{Tan}^+(F, \hat{x}) \subset L\text{Tan}(\hat{x}).
\end{equation}

On the other hand, being both $F$ and $L\text{Tan}(F, \hat{x})$ $d$-dimensional topological manifolds, by Brouwer domain invariance theorem and (B.7) the orthogonal projection of $F$ into $\hat{x} + L\text{Tan}(\hat{x})$ map $\Omega$ onto an open neighborhood of $\hat{x}$ in $\hat{x} + L\text{Tan}(\hat{x})$; hence,

\begin{equation}
\text{(*)2) } \text{Tan}^+(F, \hat{x}) = L\text{Tan}(\hat{x}).
\end{equation}

Hence, being $\hat{x}$ an arbitrary point of $F$, the maps $x \to \text{Tan}^+(F, x)$ and $x \to L\text{Tan}(F, x)$ are equal. Therefore, applying Theorem B.7, we have that $F$ is a $C^1$-manifold, as required.

In [21] and [22] Gluck shows a very elaborated and stimulating characterization of $C^1$-manifold which is based on “secant map” and “shape function”. Gluck’s characterization can be proved by Theorem B.8 (Severi theorem); however, we will prove only its unidimensional instance, since introducing the “shape function” is not an immediate task.

Corollary B.10 (Gluck [21, (1966), p. 200] and [22, (1968), p. 33]). Let $F$ be a one-dimensional topological manifold of $\mathbb{R}^n$. Then $F$ is a $C^1$-manifold if and only if the function $\Sigma$ (called secant map) from $(F \times F) \setminus \{(x, x) : x \in F\}$ to $\mathcal{G}(\mathbb{R}^n, 1)$ which assigns to each pair $x, y$ of distinct points of $F$ the unidimensional vector space generated by $x - y$, admits a continuous extension over all $F \times F$.

Proof. Necessity. Assume $F$ is a $C^1$-manifold of $\mathbb{R}^n$. In order to have the required extension, it is enough to assign $\text{Tan}^+(F, x)$ to every pair $(x, x)$. Sufficiency. Let $pL\text{Tan}(F, x)$ denote the value of the extension of $\Sigma$ at $(x, x)$. Clearly $pL\text{Tan}(F, x)$ is a one-dimensional vector space which is paratangent in traditional sense at $x$. Therefore, by Theorem B.8 $F$ is a one-dimensional $C^1$-manifold.

Another modern characterization of $C^1$-manifold is due to Shchepin and Repovš (2000); by four-cones theorem it can be rigorously proved.

Corollary B.11 (Shchepin and Repovš [50, (2000), p. 2717]). A non-empty subset $F$ of $\mathbb{R}^n$ is a $d$-dimensional $C^1$-manifold if and only if $F$ is locally compact.
and the upper-tangent and upper-paratangent cones to $F$ coincide and their linear hull is a $d$-dimensional vector space at every point, i.e.,

$$\text{(B.8) } \text{Tan}^+(F,x) = \text{pTan}^+(F,x) \text{ and dim(LTan}^+(F,x)) = d \text{ for every } x \in F. \quad \square$$

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