Separation of P and NP

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Abstract. There have been many attempts to solve the P versus NP problem. However, with a new proof method, \( P \neq NP \) can be proved.
A time limit is set for an arbitrary Turing machine and an input word is rejected on a timeout. The time limit goes toward \( \infty \). Due to the halting problem, whether a word is accepted can only be determined at run time. It can be shown by Rice’s theorem, if a finite set of words are to be checked, they all have to be tested by brute force.

1 Introduction

There have been many attempts to solve the P versus NP problem. Since the Cook-Levin theorem [4], [7] in the early 70s, the problem has been the most important unsolved problem in computer science. Unfortunately, most conventional methods of proof have failed. Relativizing proof [2], natural proof [9] or algebrization [1] could not solve the problem, and it was concluded that there is no simple proof for \( P \neq NP \). However, this is a fallacy. There are many barriers to proofs of the P versus NP problem. But this does not mean that there is no simple proof.

In [10, page 24] is mentioned that the set

\[ \{(M, x, 1^t) \mid \text{NTM } M \text{ accepts } x \text{ within } t \text{ steps}\} \]

is NP-complete. A similar problem is considered in this paper. Does a deterministic TM accept any word with length \( n \) within \( t \) steps if \( n \) and \( t \) are unary coded?

\[ \{(M, 1^n, 1^t) \mid \text{TM } M \text{ accepts some } y \in \{0, 1\}^n \text{ within } t \text{ steps}\} \]

This set is NP-complete. We analyze what happens when \( t \to \infty \). With Rice’s theorem, we show that one has to test every \( y \) in the worst case. It is undecidable which \( y \) are accepted by \( M \). So it is also undecidable which \( y \) will be accepted first for \( t \to \infty \).

Although we use diagonalization indirectly via the halting problem, the proof is not relativizing. The proof is consistent with \( \mathsf{P}^\mathsf{EXP} = \mathsf{NP}^\mathsf{EXP} \). It does not violate the relativization barrier.

1.1 Procedure

In this paper, the problem of determining whether a tuple in the set is considered:

\[ S = \{(M, 1^n, 1^t) \mid \text{DTM } M \text{ accepts some } y \in \{0, 1\}^n \text{ within } t \text{ steps}\} \]
is contained. The problem is in \( \text{NP} \).
In section 3 we analyze the time complexity of the set
\[
\{(M, x, 1^t) \mid \text{TM } M \text{ accepts } x \text{ within } t \text{ steps}\}
\]
This is equivalent to the problem \((M, 1^n, 1^t) \in S\) with arbitrary but fixed \(M\).
Section 4 shows that in the worst case, all \(y \in \{0, 1\}^n\) must be checked to
decide whether \((M, 1^n, 1^t) \in S\). This means that the running time would be exponential.

2 Preliminaries

Let \(M\) be a deterministic Turing machine with input alphabet \(\Sigma = \{0, 1\}\). It
is defined, that \(L(M) := \{x \mid \text{M accepts x}\}\). In this paper, we investigate what
happens when the running time of a TM is restricted. We additionally define:
\[
L(M, t) := \{x \mid \text{M accepted x within } t \text{ steps}\}
\]
Single-tape TMs are used as Turing machines, as described in \[3\] with input
alphabet \(\Sigma = \{0, 1\}\).

3 Time Complexity of \(L(M, t)\)

In this section, the running time is considered to compute \(L(M, t)\). It is concluded
that to check whether a TM \(M\) in \(t\) steps accepts a word, \(t\) steps must be
calculated in the worst case. Unfortunately, it takes \(O(t)\) steps to find out for
fixed \(M\) and \(x\) whether \(x \in L(M, t)\) due to the linear speedup theorem. See \[8, p. 32\].

Definition 1. Let \(M\) be a TM and \(x\) be an input word. Then, \(t_M(x)\) is the
running time of \(M\) on input \(x\) if \(M\) terminates with it. \(t_M\) is a partial function.
\(t_M\) is total iff \(M\) terminates on every \(x\). And
\[
T_M(n) := \max\{t_M(x) \mid |x| \leq n \text{ and } x \in \text{dom}(t_M)\} \cup \{0\}
\]
see \[12, \text{Chapter 2.4}\].

Lemma 1. For an arbitrary multi-taped TM \(M\) with input \(x\), it takes \(\Omega(t/\log^2(t))\)
computation time in the worst case to find out whether \(x\) is accepted by \(M\) within \(t\) steps, i.e., whether \(x \in L(M, t)\).

Proof. Let \(T\) be a fully time-constructible function. According to the time hierarchy theorem\[11\], there exists a TM \(M\) with \(L(M) \in \text{DTIME}(T)\), but \(L(M) \not\in \text{DTIME}(T')\) if \(T' \log(T') \in o(T)\) \[6, p. 112\]. Let \(T' = T/\log^2(T)\). Then \(T' \log(T') = T/\log^2(T) \log(T/\log^2(T)) \leq T/\log(T) \in o(T)\).
Let \(x_n\) with \(n \in \mathbb{N}\) be a sequence with \(|x_n| \leq n\) and \(T_M(n) = t_M(x_n)\). To check,
if \(x_n \in L(M, T_M(n))\) one need \(\Omega(T_M/\log^2(T_M))\) computation time.

Remark 1. Lemma \[1\] can also be applied to single taped TMs as a special case
of multi-taped TMs.
4 Proof of \( P \neq \text{NP} \)

**Lemma 2.** Let \( W = \{w_1, \ldots, w_m\} \) be a finite set of words and \( M \) be a TM. To find out if every \( w \in W \) is not accepted by \( M \) within \( t \) steps, i.e. \( W \cap L(M, t) = \emptyset \), one needs a running time of \( \Omega(m \cdot t \cdot \log^2(t)) \) in worst case.

**Proof.** Proof via induction over \( |W| \). For \( W = \{w_1\} \) it has been proven in Lemma 1. In the inductive step let \( M \) be an arbitrary deterministic TM and \( W = \{w_1, \ldots, w_m\} \) be a finite set of input words.

According to Rice’s theorem, it is undecidable whether \( W \cap L(M) = \emptyset \) or \( W \cap L(M) = \{w_m\} \) or \( W \cap L(M) \) is something else. \( W \) is finite, so there exists a TM \( T_W \) for \( W \) and a \( t \) such that \( \exists T_W \forall t \geq T_W \exists w \in L(M) \iff w \in L(M, t) \).

If one wants to check if \( W \cap L(M, t) = \emptyset \), one has to check whether \( \{w_1, \ldots, w_{m-1}\} \cap L(M, t) = \emptyset \) and \( w_m \notin L(M, t) \). This requires by induction hypothesis \( \Omega((m-1) \cdot t \cdot \log^2(t)) + \Omega(t \cdot \log^2(t)) \) computation time in the worst case.

If \( t < T_W \) should be, then this property is undecidable. So it is valid too that one needs \( \Omega(m \cdot t \cdot \log^2(t)) \) computation time.

**Remark 2.** That each \( x \in W \) must be tested independently of the other words from \( W \), can be seen especially if \( M \) is a UTM and each \( x \in W \) is the code of a TM.

Consider the following set from Section 1.1:

\[
S = \{(M, 1^n, 1^t) \mid \text{DTM } M \text{ and } \exists y \in \{0,1\}^n \text{ with } y \in L(M, t)\}
\]

The problem whether \((M, 1^n, 1^t)\) is in \( S \) is \( \text{NP} \)-complete for an arbitrary \( M \). Moreover, it holds:

\[
(M, 1^n, 1^t) \in S \iff \exists y \in \{0,1\}^n : y \in L(M, t) \iff \{0,1\}^n \cap L(M, t) \neq \emptyset
\]

The set \( \{0,1\}^n \) has the cardinality \( 2^n \). Due to Lemma 2, one needs a computation time of \( \Omega(2^n \cdot t \cdot \log^2(t)) \) to check whether a tuple is in \( S \) in the worst case.

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