Witt groups of Spinor varieties

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Abstract
We show that Witt groups of spinor varieties (aka. maximal isotropic Grassmannians) can be presented by combinatorial objects called 'even shifted young diagrams'. Our method relies on the Blow-up setup of Balmer–Calmès, and we investigate the connecting homomorphism of the localization sequence via the projective bundle formula of Walter–Nenashev, the projection formula of Calmès–Hornbostel and the excess intersection formula of Fasel.

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1 INTRODUCTION

In the 1930s, Witt [34] introduced a group structure on the set of isometry classes of quadratic forms over an arbitrary field, which is now known as the Witt group; for a nice survey article, see [1]. Witt groups give rise to a very interesting cohomology theory in algebraic geometry. Similarly to the oriented cohomology theories in the sense of Levine–Morel [24] or Panin [27] (for example, K-theory and Chow groups), Witt theory also has a localization sequence, cf. [1]. However, unlike what happens with these functors, in Witt theory pushforwards always keep track of the orientation and the relative codimension, a fact which makes computations via the localization sequence much trickier, cf. [9]. To this day, not many computations of Witt groups of elementary projective schemes have been performed: quadrics (cf. [26] and [36]), projective bundles (cf. [26] and [33], see also [20] and [30]), Grassmann varieties (cf. [4]), curves and surfaces (cf. [38]), cellular varieties over algebraically closed fields (cf. [37]), and real varieties (cf. [19] and [21]).
In [4], Balmer–Calmès adopt an innovative approach known as the ‘Blow-up setup’ from [3]. This approach is effective, because it interprets the abstract connecting homomorphism arising from the localization sequence of Witt groups in purely geometric terms. This key idea motivates the current article and, in its vein, we study Witt groups of even maximal isotropic Grassmannians $OG_+(n, E)$, maximal isotropic Grassmannians for short. Provided that the ambient space $E$ has even dimension and it comes equipped with a non-degenerate symmetric bilinear form, these spaces are defined as subschemes of the usual Grassmannians $Gr(n, E)$: maximal isotropic Grassmannians parametrise those subspaces on which the symmetric form vanishes identically.

Before stating the main result (cf. Theorem 1.1), we recall that in [4] Balmer–Calmès identified the additive generators of the Witt groups of Grassmannians and introduced a combinatorial object known as even Young diagrams, which they used as an indexing set. Roughly speaking they consider a subfamily consisting of those Young diagrams whose inner edges (that is, those which do not lie on the outer rectangular frame) have even length. In similar fashion, for maximal isotropic Grassmannians the additive generators can be described by means of even shifted Young diagrams, where shifted Young diagrams are the combinatorial object used to index the generators of the Chow ring of $OG_+(n, E)$. In this case, the outer frame consists of an upside-down staircase corresponding to the maximal shifted partition $\mu = (n-1, n-2, ..., 1)$ right-justified. As before, the even diagrams are those whose inner segments have even length and we denote by $\mathcal{G}_{n-1}$ the set of even shifted Young diagram which are contained inside $\mu$. Example 1.2 provides the full list of the even shifted Young diagrams contained in $\mu$ for $n = 7$, these are exactly those whose internal segments have even length.

Let $S$ be a scheme with $\frac{1}{2} \in \mathcal{O}_S$. Let $X$ be a scheme over $S$. Define the total Witt ring as

$$W^{\text{tot}}(X) := \bigoplus_{i \in \mathbb{Z}/4\mathbb{Z}} \bigoplus_{[L] \in \text{Pic}(S)/2} W^i(X, p^*L),$$

where $p : X \to S$ is the structure morphism. Note that this convention differs from [4], as we do not adopt the whole grading involving $\text{Pic}(X)/2$. This simplification is enough for us, because in our case the twisted Witt groups are trivial (cf. Proposition 3.8). We can now state the main theorem.

**Theorem 1.1.** Let $S$ be a regular Noetherian scheme with $\frac{1}{2} \in \mathcal{O}_S$. Let $OG_+(n, E)$ be the maximal isotropic Grassmannian of the trivial $2n$-dimensional bundle $E = \mathcal{O}_S^{2n}$ with the split hyperbolic form (In fact, we deal with a more general setting of complete flags throughout the paper). There is an isomorphism of graded $W^{\text{tot}}(S)$-modules

$$W^{\text{tot}}(OG_+(n, E)) \cong \bigoplus_{\lambda \in \mathcal{G}_{n-1}} W^{\text{tot}}(S)[-|\lambda|].$$

Here $|\lambda|$ denotes the number of boxes of the even shifted Young diagram $\lambda$ or, equivalently, the sum of all entries in the strict partition associated to $\lambda$.

**Example 1.2.** Here are the even shifted Young diagrams for $OG_+(7, E)$. 
It is worth pointing out that, although we work within the framework of the Blow-up setup of [3], our method differs from that of [4] for Type A. There the authors make use of desingularization of Schubert varieties as an intermediate step to compute the connecting homomorphism $\partial$, instead, in view of the observation that the exceptional fiber of the Blow up is a projective bundle, we are able to handle $\partial$ in Type D using the projective bundle formula and the excess intersection formula for regular schemes. This idea does not seem to have appeared in the literature so far. Since the connecting homomorphisms are not always trivial, the localization sequence cannot split in general. Note, however, that in a specific case (cf. Theorem 3.14) the non-split localization sequence can be transformed into short split exact sequences, provided that one extracts the data from the codimension two subbundle of the ambient bundle. This explains why even Young diagrams come into the picture. It would be interesting to know if this method can be used to compute the Witt groups of other homogeneous varieties. Unfortunately, we were informed by Nicolas Perrin that Type C and E do not fit in the Blow-up Setup, and therefore it can not be applied directly to these cases without modifications.

One may also try to adopt our method to study the $I$-cohomology of Type D homogeneous varieties. For $I$-cohomology of projective bundles and split quadrics using the Blow-up setup, we refer the reader to [11] and [18]. There are also recent developments on the Hermitian $K$-theory when 2 is not invertible in the base, cf. [7] and [31]. Investigating the Hermitian $K$-theory of schemes in these frameworks seems to be an interesting project.

Convention. All our schemes, if not mentioned otherwise, are assumed to be regular Noetherian with $\frac{1}{2}$ in their global sections. We refer to [1, 3, 22] and [29] for basic terminology.

2 | GEOMETRY OF SPINOR VARIETIES

2.1 | Flags

Let $(E, \beta)$ be a bilinear space (aka. a non-degenerate symmetric bilinear bundle) of rank $2n$ over a scheme $S$ with $\frac{1}{2} \in \mathcal{O}_S$. Throughout the article we assume that $S$ is connected, and the general case of our main theorem follows easily from this one.
**Definition 2.1.** The bilinear space $E = (E, \beta)$ is said to admit a complete flag if there is a filtration

$$E_\cdot := (0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E^\perp)$$

with the rank $\text{rk}(E_i) = i$ and all the inclusions are admissible (that is, their quotients are locally free).

In particular, $(E, \beta)$ is metabolic, and $E_n$ is a Lagrangian, that is, a maximal totally isotropic subbundle. The symmetric bundle $(E, \beta)$ induces a unique symmetric bundle $(E^1, \beta^1)$ with $E^1 := E_1^\perp / E_1$, see [1, Theorem 1.1.32]. Note that the complete flag $E_\cdot$ induces a complete flag

$$E^1_\cdot := (0 = E_0 \subset E_1 \subset \cdots \subset E_{n-2} \subset E_{n-1}^\perp \subset \cdots \subset E_0^\perp =: (E^1)^\perp)$$

on $(E^1, \beta^1)$, where we define $E_1^\perp := E_{i+1}/E_1$. Note that $E_{i+1}^1 / E_1$ is isomorphic to $(E_{i+1}^1 / E_i)^\perp$ in $(E^1, \beta^1)$, cf. [29, Proposition 6.5]. This procedure can be repeated and inductively one obtains complete flags $E^j_\cdot$ on $(E^j, \beta^j)$, where $E^j := ((E^j_1)^\perp / E_1^j, \beta^j_1)$ and $E^0_1 = E_1$. Each $E^j$ is a metabolic space with a Lagrangian $E^j_{n-j}$.

**Proposition 2.2.** Let $E$ be a metabolic space of rank 2. Then, $E$ has exactly two Lagrangians $N$ and $N'$. Moreover, $N' \cong N^\vee$ and $E \cong N \oplus N^\vee$.

**Proof.** This is well known if the base is smooth over a field, cf. [14, p. 77]. We include details, as we could not find a reference in the generality that we need. Assume that $N$ is a Lagrangian of $E$. Then, we have the structural exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow N^\vee \longrightarrow 0 .$$

Let $\{U_i\}$ be an affine open cover of the base scheme so that $E|_{U_i}$ is free. Since $\frac{1}{2} \in \mathcal{O}_{U_i}$, recall that the metabolic space $E|_{U_i}$ of rank 2 has exactly two Lagrangians over each affine space [22]. The first one is precisely $N|_{U_i}$ and we will denote the second one by $N'_{i}$. Note that $N_i'|_{U_i \cap U_j} = N_j|_{U_i \cap U_j}$, since both of them are different from $N|_{U_i \cap U_j}$. This shows that the different Lagrangians $N'_i$ glue together, giving rise to a new Lagrangian $N'$ of $E$. Next, take any Lagrangian $N''$ of $E$. If $N''|_{U_i} = N|_{U_i}$ for some $i$, then $N'' = N$. If not, then there is an affine open subscheme $V$ such that $N''|_V = N'|_V$. It follows that $N''|_{U_i \cap V} = N|_{U_i \cap V} = N'|_{U_i \cap V}$ which is a contradiction. This shows that $N''$ is either $N$ or $N'$. Finally, the Lagrangian $N'$ gives rise to the following diagram

$$0 \longrightarrow N \longrightarrow E \longrightarrow N^\vee \longrightarrow 0 .$$

Note that the composite $N' \longrightarrow E \longrightarrow N^\vee$ is locally an isomorphism, and hence an isomorphism.

**Remark 2.3.** Although metabolic spaces of rank 2 are split, there exist examples of metabolic spaces of rank 4 which do not split, cf. [23].
Lemma 2.4. Assume that \( V \) is totally isotropic. Take \( W \subset V^\perp / V \) to be a Lagrangian. Denote by \( L \) the pullback along the inclusion \( W \hookrightarrow V^\perp / V \) and the canonical quotient \( V^\perp \rightarrow V^\perp / V \). Then \( L \) is a Lagrangian of \( E \) such that \( V \subset L \) and \( L / V \cong W \).

\[
\begin{array}{ccc}
W & \longrightarrow & V^\perp / V \\
\uparrow & & \uparrow \\
L & \longrightarrow & V^\perp \longrightarrow E
\end{array}
\]

Proof. See [29, Proposition 6.5]. \( \square \)

Remark 2.5. Start with a complete flag \( E_\bullet \). By Proposition 2.2 and Lemma 2.4, we see that there exists a unique complete flag \( \tilde{E}_\bullet \) such that \( \tilde{E}_i = E_i \) for \( i \leq n - 1 \) and for which \( E_n / E_{n-1}, E_n / E_{n-1} \) form the two Lagrangians of \( E_{n-1} \). Therefore, this verifies that [14, p. 77] applies to our situation.

2.2 Isotropic Grassmann bundles

Even without a symmetric structure on \( E \), the Grassmannian scheme \( Gr(d, E) \) can be defined on the functor of points as

\[
Gr(d, E)(X) = \left\{ N \subset E_X : E_X / N \text{ is a locally free } \mathcal{O}_X \text{-module of rank } 2n - d \right\}
\]

\[
= \left\{ N \subset E_X : N(x) \leftrightarrow E_X(x) \text{ is a } k(x) \text{-vector subspace of rank } d, \forall x \in X \right\},
\]

where for every given scheme \( X \) one defines \( E_X := p^* E \) with the structure morphism \( p : X \rightarrow S \) (cf. [32, p. 210–211]). Let \( L_d \) be the universal bundle of \( Gr(d, E) \). It fits into the exact sequence

\[
0 \longrightarrow L_d \longrightarrow E_{Gr(d,E)} \longrightarrow Q_d \longrightarrow 0,
\]

where \( Q_d \) is called the universal quotient bundle. The tangent bundle of \( Gr(d, E) \) can be identified with the bundle \( Hom(L_d, Q_d) \) via a second fundamental form homomorphism, cf. [13, Appendix B.5.8].

Definition 2.6. Define \( OG(n, E) \) to be the subscheme of the Grassmannian \( Gr(n, E) \) that parametrizes maximal totally isotropic subbundles in \( E \) with respect to the form \( \beta \). More concretely, the subscheme \( OG(n, E) \) is defined as the locus where the sequence

\[
0 \longrightarrow L_n \longrightarrow E_{OG(n,E)} \xrightarrow{i^* \circ \beta} L_n^\vee \longrightarrow 0
\]

on \( Gr(n, E) \) is exact. Denote the correspondent embedding by \( \kappa : OG(n, E) \hookrightarrow Gr(n, E) \).

Lemma 2.7. The relative tangent bundle \( T_{OG(n,E)/S} \) of \( OG(n, E) \) over \( S \) is isomorphic to \( \wedge^2(\kappa^* L_n^\vee) \).

Proof. Note that \( OG(n, E) \) can also be viewed as the zero locus of the regular section \( \beta|_{L_n} : L_n \otimes L_n \rightarrow \mathcal{O} \) in the symmetric power \( S^2(L_n^\vee) \). As a consequence, the normal bundle \( N_\kappa \) of \( OG(n, E) \) in \( Gr(n, E) \) is isomorphic to \( S^2(\kappa^* L_n^\vee) \). The sequence (1) becomes exact on \( OG(n, E) \), one has
\( \kappa^*Q_n \cong \kappa^*L_n^\vee \). Hence,

\[ \kappa^*T_{\text{Gr}(n,E)/S} = \kappa^*\text{Hom}(L_n, Q_n) = \text{Hom}(\kappa^*L_n, \kappa^*Q_n) = \kappa^*L_n^\vee \otimes \kappa^*L_n^\vee. \]

This implies that the exact sequence of bundles on \( OG(n, E) \) from [13, Appendix B.7]

\[
\begin{array}{cccc}
0 & \longrightarrow & T_{OG(n,E)/S} & \longrightarrow \kappa^*T_{\text{Gr}(n,E)/S} & \longrightarrow & N_x & \longrightarrow & 0 \\
\end{array}
\]

gets identified with the usual exact sequence

\[
\begin{array}{cccc}
0 & \longrightarrow & \Lambda^2(\kappa^*L_n^\vee) & \longrightarrow & \kappa^*L_n^\vee \otimes \kappa^*L_n^\vee & \longrightarrow & S^2(\kappa^*L_n^\vee) & \longrightarrow & 0, \\
\end{array}
\]

showing that \( T_{OG(n,E)/S} \cong \Lambda^2(\kappa^*L_n^\vee) \).

\[ \square \]

**Remark 2.8.** For simplicity, we will drop \( \kappa^* \) from the notation and simply write \( L_n \) instead of the more precise \( \kappa^*L_n \) to refer to the pullback bundle to \( OG(n, E) \).

If \((E, \beta)\) is metabolic, then by base-change \((E_X, \beta_X)\) is also metabolic. On the functor of points, we have

\[
OG(n, E)(X) := \left\{ N \in Gr(n, E)(X) : N^\perp = N \right\}.
\]

Elements in \( OG(n, E)(X) \) are precisely the Lagrangians of \((E_X, \beta_X)\). The following fact is well known.

**Lemma 2.9.** If \( N \) and \( N' \) are Lagrangians of \((E_X, \beta_X)\), then the function

\[ \Gamma : X \to \mathbb{Z}/2\mathbb{Z} \]

\[ x \mapsto \text{rk}(N(x) \cap N'(x)) \]

is constant on each connected component of \( X \).

**Proof.** The proof basically follows from [25, the proof of the theorem is on p. 184]. For the readers’ convenience, we provide more details. We may assume that \( X \) is connected. Let \( U_i = \text{Spec} A_i \) (\( i \in I \)) be a finite affine open cover of \( X \) on which \( N \) and \( N' \) are both trivial. One can find isometries \( \psi_i : (E_{U_i}, \beta_{U_i}) \overset{\cong}{\to} \mathbb{H}(N_{U_i}) \) and \( \phi_i : (E_{U_i}, \beta_{U_i}) \overset{\cong}{\to} \mathbb{H}(N'_{U_i}) \), where \( \mathbb{H}(N_{U_i}) \) and \( \mathbb{H}(N'_{U_i}) \) are hyperbolic spaces. Therefore, there exists an isometry \( \varphi_i : (E_{U_i}, \beta_{U_i}) \overset{\cong}{\to} (E_{U_i}, \beta_{U_i}) \) such that \( \varphi_i(N_{U_i}) = N'_{U_i} \).

Now, by the definition of isometry, we get the equality \( \beta_{U_i} = \varphi_i^* \beta_{U_i} \varphi_i \). By taking the determinant, we obtain \( (\det \varphi_i)^2 = 1 \in A_i \). Since \( U_i \) is irreducible, we see that \( \det \varphi_i = \pm 1 \).

Note that, since \( X \) is irreducible, one has \( U_i \cap U_j \neq \emptyset \). Take a point \( s \in U_i \cap U_j \). By [6, Exercise 18 (d) in Section 6], we deduce that \( \det \varphi_i(s) = \det \varphi_j(s) = (-1)^{n-q} \) over the residue field \( k(s) \), where \( q = \text{rk}(N(s) \cap N'(s)) \). Since \( 2 \) is invertible, this implies

\[ \det \varphi_i = \det \varphi_i(s) = \det \varphi_j(s) = \det \varphi_j = (-1)^{n-q}. \]

Therefore, \( \Gamma \) is well defined on each intersection \( U_i \cap U_j \) and constant on each \( U_i \). The result follows. \[ \square \]
Remark 2.10. For readers interested in more general settings, it would seem that the proof of Lemma 2.9 generalizes to integral Noetherian schemes with $\frac{1}{2}$ in their global sections.

In particular, Lemma 2.9 implies that the scheme $O(n, E)$ has two disjoint connected components $O_+(n, E)$ and $O_-(n, E)$ defined as

$$O_\pm(n, E)(X) = \{ N \in O(n, E)(X) : \text{rk}(E_n(x) \cap N(x)) \equiv n_\pm \pmod{2} \text{ for any } x \in X \},$$

where $n_+ = n$ and $n_- = n - 1$. In the sequel, we shall drop the mention of $X$ to avoid cumbersome notation. The connected component $O_+(n, E)$ is usually called the spinor variety (or maximal isotropic Grassmannian).

In view of Proposition 2.2, it is clear that any metabolic space $E$ of rank 2 admits a complete flag

$$0 = E_0 \subset E_1 = E_2 = E.$$ 

Moreover, Proposition 2.2 also implies the following result.

**Corollary 2.11.** If $E$ is metabolic of rank 2, then $O_{\pm}(1, E) = S$.

### 2.3 The closed embedding

Let $V$ be a totally isotropic subbundle of $E$. Consider the closed subscheme $i^V_\pm : O_\pm^V(n, E) \hookrightarrow O_\pm(n, E)$ given by

$$O_\pm^V(n, E) = \{ N \in O_\pm(n, E) : V \subset N \}.$$

In the special case $V = E_j$, we have the following result.

**Lemma 2.12.** The morphism

$$\Phi : O_{\pm}^{E_j}(n, E) \to O_{\pm}(n - j, E^j)$$

$$E_j \subset N \subset E \Rightarrow N/E_j \subset E^j = E_j^j/E_j$$

is an isomorphism.

**Proof.** Set $N^j := N/E_j$. By [29, Proposition 6.5], we see that $(N^j)^{1/E_j} = N^j$. To see that $\Phi$ is an isomorphism, we construct an inverse morphism $\Psi : O_{\pm}(n - j, E^j) \to O_{\pm}^{E_j}(n, E)$ by sending a Lagrangian $W \subset E^j$ to the Lagrangian $L$ of $E$ constructed as in Lemma 2.4. It is straightforward to check that $\Phi$ and $\Psi$ are inverses of each other. □

### 2.4 The Blow-up setup

In this subsection, we shall study the blow-up of the closed embedding

$$i_\pm : O_{\pm}^{E_j}(n, E) \hookrightarrow O_{\pm}(n, E),$$
where \( \iota_{\pm} := \iota_{\pm}^{E_1} \) and we use this simplification if no confusion may occur. Let \( L_{n}^{\pm} \) be the pullback of the universal bundle \( L_{n} \) on \( OG(n, E) \) along the canonical embedding \( OG_{\pm}(n, E) \hookrightarrow OG(n, E) \). Consider the universal exact sequence

\[
0 \longrightarrow L_{n}^{\pm} \longrightarrow E \longrightarrow (L_{n}^{\pm})^{\vee} \longrightarrow 0
\]

on \( OG_{\pm}(n, E) \). The subscheme \( OG_{\pm}^{E_1}(n, E) \) of \( OG_{\pm}(n, E) \) is precisely the vanishing locus of the composite

\[
\begin{array}{ccc}
E_1 & \longrightarrow & (L_{n}^{\pm})^{\vee} \longrightarrow 0 \\
\downarrow & & \\
0 & \longrightarrow & L_{n}^{\pm} \longrightarrow E
\end{array}
\]

or, equivalently, the vanishing locus of the composite

\[
\begin{array}{ccc}
0 & \longrightarrow & L_{n}^{\pm} \longrightarrow E \longrightarrow (L_{n}^{\pm})^{\vee} \longrightarrow 0 \\
& & \downarrow \\
& & (E_1)^{\vee}
\end{array}
\]

One can pullback the bundle \( L_{n}^{\pm} \) over \( OG_{\pm}(n, E) \) via the closed embedding \( \iota_{\pm} : OG_{\pm}^{E_1}(n, E) \hookrightarrow OG_{\pm}(n, E) \), which is denoted by \( \tilde{L}_{n}^{\pm} \).

**Definition 2.13.** Define \( Bl_{\pm}(n, E) \) to be the Grassmannian scheme \( Gr(n - 1, L_{n}^{\pm}) \) over \( OG_{\pm}^{E_1}(n, E) \), and denote by \( \tilde{\alpha}_{\pm} : Bl_{\pm}(n, E) \to OG_{\pm}^{E_1}(n, E) \) the projection.

Let \( P_{n-1}^{\pm} \) be the universal bundle of \( Bl_{\pm}(n, E) := Gr(n - 1, L_{n}^{\pm}) \), and let \( P_{n-1}^{\pm} \hookrightarrow L_{n}^{\pm} \) be the canonical inclusion. Note that one has \( E_1 \subset L_{n}^{\pm} \) and \( P_{n-1}^{\pm} \subset L_{n}^{\pm} \subset E_{\perp} \). Consider the filtration

\[
P_{n-1}^{\pm} \subset L_{n}^{\pm} \subset (P_{n-1}^{\pm})^{\perp} \subset E
\]

over \( Bl_{\pm}(n, E) \). By taking the quotient, we obtain a Lagrangian \( T^{\pm} := L_{n}^{\pm}/P_{n-1}^{\pm} \) inside the metabolic space \( (P_{n-1}^{\pm})^{\perp}/P_{n-1}^{\pm} \) of rank 2. By Proposition 2.2, we see that inside this metabolic space there is a unique Lagrangian \( T^{\pm} \), which is different from \( T^{\mp} \) and is in the other component. By Lemma 2.4, the bundle \( T^{\pm} \) can be lifted to a Lagrangian \( \pi L_{n}^{\pm} \subset E \) such that

\[
P_{n-1}^{\pm} \subset \pi L_{n}^{\pm} \subset (P_{n-1}^{\pm})^{\perp} \subset E
\]

By the universal property of \( OG_{\pm}(n, E) \), we get a morphism

\[
\pi_{\pm} : Bl_{\pm}(n, E) \to OG_{\pm}(n, E)
\]

from the Lagrangian \( \pi L_{n}^{\pm} \subset E \) over \( Bl_{\pm}(n, E) \) such that \( \pi L_{n}^{\pm} = \pi^{*} L_{n}^{\pm} \). Thus, we just adopt a simplified notation and write \( L_{n}^{\pm} \) to refer to the pullback \( \pi L_{n}^{\pm} \) over \( Bl_{\pm}(n, E) \).
Definition 2.14. Define $E_\pm(n, E)$ to be the vanishing locus of the composite

\[
P_{n-1}^\pm \xrightarrow{\pi} \tilde{L}_n^\pm \rightarrow L_n^\pm / P_{n-1}^\pm
\]

over $B_l^\pm(n, E)$. Define $i_\pm : E_\pm(n, E) \hookrightarrow B_l^\pm(n, E)$ to be the closed embedding, and $\tilde{v}_\pm : U_\pm(n, E) \hookrightarrow B_l^\pm(n, E)$ to be its open complement. Define $v_\pm := \pi_\pm \circ \tilde{v}_\pm$ and $\alpha_\pm := \tilde{\alpha}_\pm \circ \tilde{v}_\pm$.

Note that $P_{n-1}^\pm = L_n^\pm \cap L_n^\pm$, and $E_1 \subset P_{n-1}^\pm$ if and only if $E_1 \subset L_n^\pm$. Therefore, $E_\pm(n, E)$ can be identified with the fiber product $\pi^{-1}_\pm OG_{E_1}^E(n, E)$, where the pullback morphism

\[
\tilde{\pi}_\pm : E_\pm(n, E) \to OG_{E_1}^E(n, E)
\]
classifies $E_1 \subset P_{n-1}^\pm \subset L_n^\pm$ over $E_\pm(n, E)$ by the universal property of $OG_{E_1}^E(n, E)$. Note also that $E_\pm(n, E) = E_\pm(n, E)$ by construction, and $\tilde{\alpha}_\pm \circ \tilde{i}_\pm = \tilde{\pi}_\pm$. Therefore, we only write $E(n, E)$ to denote $E_\pm(n, E)$. The following result is proved in [28] when the base is $C$.

Theorem 2.15. The scheme $B_l^\pm(n, E)$ is the blow-up of $OG_{E_1}^E(n, E)$ along $OG_{E_1}^E(n, E)$ with the exceptional fiber $E(n, E)$, and $v_\pm : U_\pm(n, E) \to OG_{E_1}^E(n, E)$ is the open complement of $OG_{E_1}^E(n, E)$ inside $OG_{E_1}^E(n, E)$. Moreover, the morphism $\alpha_\pm : U_\pm(n, E) \to OG_\pm^E(n, E)$ is an affine bundle.

The situation is depicted in the following diagram

\[
\begin{align*}
OG_{E_1}^E(n, E) & \xrightarrow{i_\pm} OG_{E_1}^E(n, E) \xleftarrow{v_\pm} U_\pm(n, E) \\
E(n, E) & \xrightarrow{i_\pm} B_l^\pm(n, E) \xleftarrow{\alpha_\pm} OG_{E_1}^E(n, E)
\end{align*}
\]

and the Blow-up setup (cf. [3, Setup 1.1]) is therefore satisfied by Theorem 2.15. When viewed in terms of functors of points, (2) has the following interpretation:

\[
\begin{align*}
\left\{ \{L_n^\pm \mid L_n^\pm \supset E_1\} \right\} & \xrightarrow{\left\{ \{L_n^\pm \mid L_n^\pm \not\supset E_1\} \right\}} \left\{ \{L_n^\pm \mid E \supset L_n^\pm\} \right\} \\
\left\{ \{P_{n-1}^\pm, L_n^\pm, L_n^\pm\} \mid P_{n-1}^\pm \supset E_1\right\} & \xrightarrow{\left\{ \{P_{n-1}^\pm, L_n^\pm, L_n^\pm\} \mid L_n^\pm \not\supset E_1\right\}} \left\{ \{L_n^\pm \mid L_n^\pm \supset E_1\} \right\}.
\end{align*}
\]

Proof. By the compatibility of blow-ups and pullbacks, one can reduce to the case when $S$ is affine, and so we can take $E$ to be free and $E_1 = \mathcal{O}$. We need to check that $B := B_l^\pm(n, E)$ has the universal property of the blow-up, that is, it is final in the category of schemes over $X := OG_{E_1}^E(n, E)$ for
which the preimage of $Z := \text{OG}_E^\pm(n, E)$ is an effective Cartier divisor. Let us check that $B$ is an object of this category. There is an exact sequence
\[
0 \longrightarrow E_1 \longrightarrow \tilde{L}_n^\mp \longrightarrow L_n^\mp / E_1 \longrightarrow 0
\]
of vector bundles over $Y := \text{OG}_E^\pm(n, E)$. By dualizing it, we note that $B \cong \mathbb{P}_Y((\tilde{L}_n^\mp)^\vee)$ (respectively, $E(n, E) \cong \mathbb{P}_Y((L_n^\mp/E_1)^\vee)$) is a projective bundle of relative dimension $n-1$ (respectively, $n-2$) over $Y$, and $E(n, E)$ is an effective Cartier divisor of $B$.

Suppose that $f : W \to X$ is a morphism for which $f^{-1}Z$ is an effective Cartier divisor. On $X$, consider the morphism of bundles $s : L_n^\pm \to E_1^\vee \cong \mathcal{O}_X$ given by the composition $L_n^\pm \to E \to E_1^\vee \cong \mathcal{O}_X$. By dualizing, we note that $E \cong \mathbb{P}_Y((\tilde{L}_n^\mp/E_1)^\vee)$ is a projective bundle of relative dimension $n-1$ (respectively, $n-2$) over $Y$. We claim that $(E + E_1) / K$ is a Lagrangian of $E^\perp_1 / E_1$. This is because
\[
(K + E_1) / K \subset (K^\perp \cap E_1^\perp) / K = ((K + E_1) / K)^\perp
\]
(cf. [29, Section 6]), and $(K + E_1) / K$ is a rank one bundle. If not $K + E_1 = K$, then $E_1 \subset K \subset L_n^\pm$ and $L_n^\pm / K \subset E_1^\perp$, which contradicts the subjectivity of $L_n^\pm \to I \subset E / E_1^\perp$ and $I$ is invertible. Since $K^\perp / K$ is of rank two, we see that $(K + E_1) / K$ is either $L_n^\pm / K$ or $L_n^\pm / K$. If $(K + E_1) / K = L_n^\pm / K$, then $E_1 \subset L_n^\pm$ which is a contradiction, as we have already seen. Thus, $E_1 \subset L_n^\pm$, which shows that $W$ is a scheme over $Y$. By the universal property of the Grassmannian bundle $B := \text{Gr}_Y(n-1, L_n^\perp)$, the inclusion $K \subset L_n^\perp$ defines the wanted morphism $g : W \to B$ such that $\pi \circ g = f$.

For the uniqueness, if $g' : W \to B$ is a map such that $\pi \circ g' = f$, then $g'^* (L_n^\perp) = L_n^\perp$ and $g'^* P_{n-1} \subset L_n^\perp$. Note that $g'^* P_{n-1} \subset E_1^\perp$ on $W$, which forces $g'^* P_{n-1} \subset K$ and hence to be equal by dimension counting. Thus, $g'^* \tilde{L}_n^\mp = \tilde{L}_{n, W}^\mp$ is the unique Lagrangian in the other component containing $K = g'^* P_{n-1}$, which implies $g = g'$.

Finally, we see that $U_\beta = B - E(n, E) \cong \mathbb{P}_Y((\tilde{L}_n^\mp)^\vee) - \mathbb{P}_Y((\tilde{L}_n^\mp/E_1)^\vee) \cong X - Z$. The scheme $\mathbb{P}_Y((\tilde{L}_n^\mp)^\vee) - \mathbb{P}_Y((\tilde{L}_n^\mp/E_1)^\vee)$ is an affine bundle over $Y$, but need not to be a vector bundle in general.

Consider the case of $\text{OG}_E^\pm(2, E)$. Suppose that $(E, \beta)$ admits a complete flag
\[
0 \subset E_1 \subset E_2 = E_2^\perp \subset E^\perp_1 \subset E.
\]
By Remark 2.5, there is a Lagrangian $E_2$ in the other component such that the filtration
\[
0 \subset E_1 \subset E_2 = E_2^\perp \subset E^\perp_1 \subset E
\]
is a complete flag.

**Proposition 2.16.** There are isomorphisms of schemes $\text{OG}_+(2, E) \cong \mathbb{P}(E_2^\perp)$ and $\text{OG}_-(2, E) \cong \mathbb{P}(E_2^\perp)$. 

Proof. Recall that $\iota_+: OG^E_\pm(2,E) \cong S$. Note that the pullback $L_2^+$ of the universal bundle $L_2^+$ on $OG^E_+(2,E)$ via the morphism $\iota_+: OG^E_+(2,E) \hookrightarrow OG^E_+(2,E)$ is isomorphic to $E_2$, and $\tilde{L}_2^-$ is isomorphic to $\tilde{E}_2$. Note that $OG^E_+(2,E)$ is a closed subscheme of $OG_+(2,E)$ of codimension one. Therefore, the morphism $\pi_+: Bl_+(2,E) \cong \mathbb{P}(\tilde{E}^\vee_2) \rightarrow OG^E_+(2,E)$ is an isomorphism of schemes, and similarly $OG^E_-(2,E) \cong \mathbb{P}(E^\vee_2)$.

2.5 | Picard groups

In this subsection, we study Picard groups of spinor varieties.

**Proposition 2.17** [3, Theorem 1.3]. Let $H^*$ be a homotopy invariant cohomology theory for regular schemes. Assume that $H^*$ is oriented, so that it admits push-forward maps satisfying flat-base change, which are defined for all proper morphisms (examples of such theories are $K$-theory and Chow groups). Then, we have the following equivalence

$$H^*(OG^E_{\pm}(n,E)) \cong \bigoplus_{i=1}^{2^{n-1}} H^*(S).$$

Proof. First of all let us observe that if $\text{rk}(E^{n-1}) = 2$, we can make the identification $S = OG^E_+(1,E^{n-1})$. Finally, we inductively apply Lemma 2.12, homotopy invariance and [3, Theorem 1.3].

**Proposition 2.18.** $\text{Pic}(OG^E_{\pm}(n,E)) \cong \mathbb{Z} \oplus \text{Pic}(S)$ and $\text{Pic}(Bl_{\pm}(n,E)) \cong \mathbb{Z}[\mathcal{O}(E(n,E))] \oplus \text{Pic}(OG^E_{\pm}(n,E))$.

Proof. Let us recall that for $X$ regular, one can identify $\text{Pic}(X) = CH^1(X)$. The first formula is then obtained by Proposition 2.17. Note that we have the following isomorphisms induced by pullbacks

$$CH^1(OG^E_{\pm}(n,E)) \cong CH^1(OG^E_{+}(n,E)) \cong \cdots \cong CH^1(OG^E_{+}(n-2,E^n-2))$$

The last group is isomorphic to $CH^1(\mathbb{P}(E^{n-2}_2)^\vee)$, which is itself equal to $\mathbb{Z} \oplus \text{Pic}(S)$. The second formula is obtained by [3, Proposition A.6 (i)]. A similar computation applies to $OG^E_-(n,E)$.

**Remark 2.19.** Note that in the isomorphism $\text{Pic}(OG^E_+(n,E)) \cong \mathbb{Z} \oplus \text{Pic}(S)$, the group $\mathbb{Z}$ is generated by $\mathcal{O}(1)$ coming from $OG^E_+(2,E^{n-2}) \cong \mathbb{P}((E^{n-2}_2)^\vee)$ which is known as the Pfaffian line bundle, and $\text{Pic}(S)$ comes from pullback from $S$.

2.6 | The canonical bundle

**Lemma 2.20.** $\det L^E_\pm \cong \det \tilde{E}_n \otimes \mathcal{O}(-2)$.

Proof. Consider the case when $n = 2$. By Lemma 2.7, we see that $\det L^E_2 \cong T^\vee_{\mathbb{P}(E^\vee_2)/S}$, where the last bundle is isomorphic to $T^\vee_{\mathbb{P}(E^\vee_2)/S}$. Now, the exact sequence (cf. [13, Appendix B5.8])

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E^\vee_2)/S} \longrightarrow \tilde{E}_2 \otimes \mathcal{O}(1) \longrightarrow T_{\mathbb{P}(E^\vee_2)/S} \longrightarrow 0$$

shows that $\det T^\vee_{\mathbb{P}(E^\vee_2)/S} \cong \det(\tilde{E}_2 \otimes \mathcal{O}(1)) \cong \mathcal{O}(2) \otimes \det E_2$. 

The general case can be reduced to \( n = 2 \). There are exact sequences

\[
0 \rightarrow E_{n-2} \rightarrow L_n^+ \rightarrow L_n^+ / E_{n-2} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow E_{n-2} \rightarrow E_n \rightarrow E_n / E_{n-2} \rightarrow 0
\]
on \( OG_+(2, E^{n-2}) \). Thus, we conclude that \( \det L_n^+ \cong \det E_n \otimes \mathcal{O}(-2) \).

Let \( N_{\pm} \) be the normal bundle of the closed embedding \( \iota : OG_{\pm}^+(n, E) \hookrightarrow OG_{\pm}(n, E) \). We want to compute the class \( \omega_{\pm} := \det N_{\pm} \) in \( \text{Pic}(OG_{\pm}^+(n, E)) \) as it is important for Witt groups.

**Proposition 2.21.** \( \omega_{+} \cong \mathcal{O}(-2) \otimes \det E_n \otimes E_1^{\otimes(n-2)} \).

**Proof.** On the one hand, the exact sequence

\[
0 \longrightarrow T_{OG_+^+(n,E)/S} \longrightarrow T_{OG_+^+(n,E)/S} \longrightarrow N_{\pm} \longrightarrow 0
\]

shows that

\[
\det N_{\pm} \cong \det T_{OG_+^+(n,E)/S} \otimes \det T_{OG_+^+(n,E)/S}.
\]

Let \( U \) be the cokernel of the embedding \( E_1 \rightarrow L_n^+ \) on \( OG_+^+(n, E) \). Since \( OG_+^+(n, E) \cong OG_+(n-1, E^1) \), the bundle \( U \) can be identified with the universal bundle on \( OG_+(n-1, E^1) \). The tangent bundle of \( OG_+^+(n, E) \) is isomorphic to \( \Lambda^2 U^\vee \). Therefore, \( \det N_{\pm} \cong \det \Lambda^2(L_n^+)^\vee \otimes \det \Lambda^2 U \). On the other hand, note that the exact sequence

\[
0 \rightarrow U^\vee \rightarrow (L_n^+)^\vee \rightarrow E_1^\vee \rightarrow 0
\]
on \( OG_+^+(n, E) \) induces the following exact sequence

\[
0 \longrightarrow \Lambda^2 U^\vee \longrightarrow \Lambda^2(L_n^+)^\vee \longrightarrow U^\vee \otimes E_1^\vee \longrightarrow 0,
\]

which shows that

\[
\det N_{\pm} \cong \det \Lambda^2(L_n^+)^\vee \otimes \det \Lambda^2 U \cong \det(U^\vee \otimes E_1^\vee)
\]

\[
\cong \det U^\vee \otimes (E_1^\vee)^\otimes(n-1) \cong \det(L_n^+)^\vee \otimes (E_1^\vee)^\otimes(n-2).
\]

Now, by Lemma 2.20, we see that

\[
\det N_{\pm} \cong \det (L_n^+)^\vee \otimes (E_1^\vee)^\otimes(n-2) \cong \mathcal{O}(-2) \otimes \det E_n \otimes E_1^{\otimes(n-2)}. \]

**Remark 2.22.** A similar argument shows that \( \omega_{-} \cong \mathcal{O}(-2) \otimes \det E_n \otimes E_1^{\otimes(n-2)} \).

3 | WITT GROUPS OF MAXIMAL ISOTROPIC GRASSMANNIAN

3.1 | Some preliminary results

The following theorems are useful for our computations.

**Theorem 3.1** (Localization). Let \( X \) be a scheme with \( \frac{1}{2} \in \mathcal{O}_X \). Let \( Z \) be a closed subset and \( U \) be its open complement. Let \( \mathcal{L} \) be a line bundle on \( X \). Then, there is a 12-term periodic long exact sequence

\[
\cdots \rightarrow W^{i-1}(U, \mathcal{L}_U) \rightarrow W^i(Z, \mathcal{L}_U) \rightarrow W^i(X, \mathcal{L}_U) \rightarrow W^i(U, \mathcal{L}_U) \rightarrow W^{i+1}(Z, \mathcal{L}_U) \rightarrow \cdots.
\]
For the proof, we refer the reader to Balmer [1, Theorem 1.5.5].

**Theorem 3.2** (Dévissage). Let \( \iota : Z \hookrightarrow X \) be a regular embedding of codimension \( d \), where \( Z \) and \( X \) are both regular separated schemes of finite Krull dimension with \( \frac{1}{2} \) in their global sections. Let \( \omega_\iota \) be dual of the determinant of the normal bundle \( N_\iota \) of the embedding \( \iota \). Then, there is an isomorphism

\[
\iota_* : W^{i-d}(Z, \omega_\iota \otimes \mathcal{L}_Z) \rightarrow W^i(X, \mathcal{L}).
\]

The above theorem is proved by Gille in [16] (see also [36, Corollary 6.2]). Combining the two theorems together, one obtains the following corollary.

**Corollary 3.3.** Let \( \iota : Z \hookrightarrow X \) be a regular embedding of codimension \( d \), where \( Z \) and \( X \) are both regular separated schemes of finite Krull dimension with \( \frac{1}{2} \) in their global sections. Let \( \omega_\iota \) be dual of the determinant of the normal bundle \( N_\iota \) of the embedding \( \iota \). Let \( U := X - Z \). Then, one has the following 12-term periodic long exact sequence

\[
\cdots \rightarrow W^{i-1}(U, \mathcal{L}_U) \xrightarrow{\partial} W^{i-d}(Z, \omega_\iota \otimes \mathcal{L}_Z) \xrightarrow{\iota_*} W^i(X, \mathcal{L}) \rightarrow W^i(U, \mathcal{L}_U) \xrightarrow{\partial} W^{i-d+1}(Z, \omega_\iota \otimes \mathcal{L}_Z) \rightarrow \cdots
\]

**Theorem 3.4** (Homotopy invariance). Let \( X \) be a regular scheme with \( \frac{1}{2} \in \mathcal{O}_X \). Assume that \( p : E \rightarrow X \) is an affine bundle and let \( \mathcal{L} \) be a line bundle on \( X \). Then the pullback induces an isomorphism

\[
p^* : W^i(X, \mathcal{L}) \cong W^i(E, p^* \mathcal{L}).
\]

A proof can be found either in [2] or in [15]. For the singular case, see instead [20].

**Lemma 3.5** (Two periodicity on the twists). Let \( \mathcal{L} \) and \( \mathcal{M} \) be line bundles on \( X \). Then we have an isomorphism

\[
\text{per}_{\mathcal{M}} : W^i(X, \mathcal{L}) \rightarrow W^i(X, \mathcal{L} \otimes \mathcal{M}^{\otimes 2}),
\]

which, by definition, is the product \((\mathcal{M}, \varphi) \otimes -\), where \( \varphi : \mathcal{M} \rightarrow \text{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{M}^{\otimes 2}) \) is the canonical form in \( W^0(X, \mathcal{M}^{\otimes 2}) \).

See [1] for a proof.

**Theorem 3.6** (Projective bundles). Let \( E \) be a vector bundle over \( S \) with odd rank \( r \) such that \( E \) fits into an exact sequence \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \). Let \( q : \mathbb{P}(E) \rightarrow S \) be the projection morphism. Suppose that \( \text{rk}(E') - 1 \equiv \text{rk}(E'') - 1 \equiv 0 \mod 2 \). Then, for any line bundle \( L \) over \( S \), there are isomorphisms

\[
W^i(\mathbb{P}(E), q^* L) \cong W^{i-r}(S, L \otimes \det E) \oplus W^i(S, L) \quad \text{and} \quad W^i(\mathbb{P}(E), q^* L \otimes \mathcal{O}(1)) = 0
\]

for any \( i \in \mathbb{Z} \).

See Nenashev [26] and Walter [33].
Remark 3.7. It is worth pointing out that, since for regular schemes Witt groups on vector bundles and those on coherent sheaves are isomorphic, in this paper we do not strictly distinguish between the two.

3.2 Two simple cases

Let \( \mathcal{L} \) be a line bundle on \( OG_{\pm}(n, E) \). By homotopy invariance, dévissage theorem and Proposition 2.21, the 12-term periodic long exact sequence takes the form

\[
W^{i-n+1}(OG_{\pm}^{E_1}(n, E), L \otimes \text{det} \tilde{E}_n \otimes E_1^{\otimes (n-2)}) \xrightarrow{(\iota)_*} W^i(OG_{\pm}(n, E), L) \xrightarrow{\nu^*} W^i(U_{\pm}(n, E), L)
\]

Here, we suppressed even twists and we made a choice of a trivialization. A similar exact sequence can be written on \( OG_{\pm}(n, E) \) replacing + with − and \( \tilde{E}_n \) with \( E_n \). The following result is proved in [8] when the base is a field of characteristic \( \neq 2 \).

Proposition 3.8. For any line bundle \( L \) over \( S \), one has \( W^i(OG_{\pm}(n, E), L \otimes \mathcal{O}(1)) = 0 \).

Proof. Recall that both \( OG_{\pm}(2, E) \) and \( OG_{\pm}(2, E) \) are projective bundles satisfying the condition of Theorem 3.6. It follows that \( W^i(OG_{\pm}(2, E), L \otimes \mathcal{O}(1)) = 0 \) for any line bundle \( L \) over \( S \) and any \( i \in \mathbb{Z} \). By inductively applying the localization sequence (3), the result follows from Theorem 2.15, homotopy invariance and Lemma 2.12.

Proposition 3.9. Let \( L \) be a line bundle over \( S \) and assume \( n \) to be even. Then, the sequence

\[
0 \longrightarrow W^{(i-n-1)}(OG_{\pm}^{E_1}(n, E), L \otimes \text{det} \tilde{E}_n) \xrightarrow{(\iota)_*} W^i(OG_{\pm}(n, E), L) \xrightarrow{\alpha^*} W^i(U_{\pm}(n, E), L) \longrightarrow 0
\]

is split exact. Hence, we have the following isomorphism

\[
W^i(OG_{\pm}(n, E), L) \cong W^{(i-n-1)}(OG_{\pm}^{E_1}(n, E), L \otimes \text{det} \tilde{E}_n) \oplus W^i(U_{\pm}(n, E), L).
\]

Proof. As \( n \) is even, [3, Theorem 1.4 (A)] applies here and, as a consequence, the localization sequence breaks down into short split exact sequences. Finally, the last statement follows directly from homotopy invariance.

3.3 Preparation for the case of \( n \) odd

It now remains to compute \( W^i(OG_{\pm}(n, E), L) \) for \( n \) odd. This case is more complicated and forms the technical heart of the paper. We only deal with one component, the other one can be obtained \textit{mutatis mutandis}. Let us begin by observing that the proof of [3, Proposition 3.3] provides more information, which will be useful for the proof. Rewrite Diagram (2) starting with the canonical...
closed embedding \( \iota_{1+} : OG_+^{E_2}(n, E) \to OG_+^{E_1}(n, E) \) as follows:

\[
\begin{align*}
OG_+^{E_2}(n, E) & \xrightarrow{\iota_{1+}} OG_+^{E_1}(n, E) \xleftarrow{\nu_{1+}} U_+^{E_1}(n, E) \\
E^{E_1}(n, E) & \xrightarrow{\iota_{1+}} B_+^{E_1}(n, E) \xleftarrow{\delta_{1+}} OG_+^{E_2}(n, E)
\end{align*}
\]

\((4)\)

At the level of functor of points, the previous diagram can be described by

\[
\begin{array}{ccc}
\{L_+^+, L_n^+ \supset E_2\} & \xrightarrow{\gamma} & \{L_+^+, L_n^+ \supset E_1\} \\
\{P_{n-1}, L_n^+, L_n^-\} & \xrightarrow{\alpha} & \{P_{n-1}, L_n^+, L_n^-\}
\end{array}
\]

Note that, in Diagram (4), \( \bar{\iota}_{1+} \) is a regular embedding of codimension one, while \( \iota_{1+} \) is a regular embedding of codimension \( n - 2 \). Suppose that \( n \) is odd. As in Proposition 3.9 and [3, Theorem 1.4 (A)], the short exact sequence in the following diagram

\[
0 \to W^{i-(n-2)}(OG_+^{E_2}(n, E), \omega_{n+}) \xrightarrow{(\iota_{1+})_*} W^i(OG_+^{E_1}(n, E)) \xrightarrow{\nu_{1+}^*} W^i(U_+^{E_1}(n, E)) \to 0
\]

\[
W^i(B_+^{E_1}(n, E)) \xleftarrow{\delta_{1+}^*} W^i(OG_+^{E_2}(n, E))
\]

has a right splitting:

\[
\beta := (\pi_{1+})_* \circ \bar{\alpha}_{1-}^n \circ (\alpha_{1-})^{-1}.
\]

**Remark 3.10.** By Proposition 2.21 and Lemma 2.12, we see that \( \omega_{n+} \cong \mathcal{O}(-2) \otimes E_n \otimes E^{(n-4)}_1 \).

**Lemma 3.11.** Assume that \( n \) is an odd number. The maps

\[
(\pi_{1+})_* : W^i(B_+^{E_1}(n, E)) \cong W^i(OG_+^{E_1}(n, E)) : \pi_{1+}^*
\]

are isomorphisms and inverse to each other.

**Proof.** The excess normal bundle of the left Cartesian square in Diagram (4) is precisely the universal quotient bundle \( Q \) of the projective bundle \( \pi_{1+} : E_{E_1}^{E_1}(n, E) \to OG_+^{E_1}(n, E) \), which is the cokernel of the map \( \mathcal{O}_{E_1^{E_1}(n,E)}(-1) \to \bar{\pi}_{1+}^* \mathcal{N}_{1+}^* \).
By Lemma A.2, it is enough to prove that $\pi_{1+}^n$ is an isomorphism. Consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W^{i-1}(E^B_{+}(n, E), \mathcal{O}(-1)) & \longrightarrow & W^i(B^B_{+}(n, E)) & \longrightarrow & W^i(U^B_{+}(n, E)) & \longrightarrow & 0 \\
& & \uparrow & \uparrow \pi_{1+} & & \uparrow & \uparrow 0 & & \\
0 & \longrightarrow & W^{i-(n-2)}(OG^B_{+}(n, E), \det \tilde{E}_n \otimes E_1) & \longrightarrow & W^i(OG^B_{+}(n, E)) & \longrightarrow & W^i(U^B_{+}(n, E)) & \longrightarrow & 0,
\end{array}
$$

where commutativity of the left square is obtained by using the excess intersection formula (cf. [10]) applied to the left Cartesian square of Diagram (4). The left vertical arrow of the previous diagram is an isomorphism by Theorem A.4, because $\tilde{\pi}_{1+}$ is a projective bundle of relative dimension $n - 3$, which is even by hypothesis.

**Lemma 3.12.** Assume $n$ to be an odd number. The short exact sequence

$$
0 \longrightarrow W^{i-(n-2)}(OG^B_{+}(n, E), \det \tilde{E}_n \otimes E_1) \longrightarrow W^i(OG^B_{+}(n, E)) \longrightarrow W^i(U^B_{+}(n, E)) \longrightarrow 0
$$

has a left splitting $(\tilde{\pi}_{1+})^* \circ (\tilde{\pi}_{1-})^{-1} \circ (\tilde{\pi}_{1-})_* \circ \pi_{1+}^n$.

**Proof.** By the projective bundle formula (cf. Theorem A.4), we see that $(\tilde{\pi}_{1-})_*$ and $(\tilde{\pi}_{1+})_*$ are isomorphisms. Consider the following diagram

$$
\begin{array}{ccccccccc}
W^{i-1}(E^B_{+}(n, E), \mathcal{O}(-1)) & \stackrel{(\tilde{\pi}_{1-})_*}{\longrightarrow} & W^{i-(n-2)}(OG^B_{+}(n, E), \det \tilde{E}_n \otimes E_1) & \stackrel{(\tilde{\pi}_{1+})_*}{\longrightarrow} & W^i(B^B_{+}(n, E)) & \stackrel{(\tilde{\pi}_{1+})_*}{\longrightarrow} & W^i(U^B_{+}(n, E)) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
W^{i-1}(E^B_{+}(n, E), \mathcal{O}(-1)) & \stackrel{(\tilde{\pi}_{1+})_*}{\longrightarrow} & W^i(OG^B_{+}(n, E)) & \stackrel{(\tilde{\pi}_{1+})_*}{\longrightarrow} & W^i(U^B_{+}(n, E)) & \longrightarrow & 0
\end{array}
$$

The result follows by its commutativity.

**Lemma 3.13.** The sequence

$$
0 \longrightarrow W^{i-(n-2)}(OG^B_{+}(n, E), \det \tilde{E}_n \otimes E_1) \longrightarrow W^i(OG^B_{+}(n, E)) \longrightarrow W^i(U^B_{+}(n, E)) \longrightarrow 0
$$

is split exact. Moreover, $W^i(OG^E_{+}(n, E)) \cong W^i(OG^E_{-}(n, E))$.

**Proof.** Consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W^{i-1}(E^B_{+}(n, E), \mathcal{O}(-1)) & \longrightarrow & W^i(B^B_{+}(n, E)) & \longrightarrow & W^i(U^B_{+}(n, E)) & \longrightarrow & 0 \\
& & \uparrow & \uparrow \pi_{1+} & & \uparrow & \uparrow 0 & & \\
0 & \longrightarrow & W^{i-(n-2)}(OG^B_{+}(n, E), \det \tilde{E}_n \otimes E_1) & \longrightarrow & W^i(OG^B_{+}(n, E)) & \longrightarrow & W^i(U^B_{+}(n, E)) & \longrightarrow & 0.
\end{array}
$$
All vertical maps are isomorphisms. The commutativity of the left square diagram (respectively, the right square) follows from the excess intersection formula (respectively, functoriality of pullback). The upper sequence is split exact by Lemma A.5. Note that $\text{Bl}_+^{E_1}(n, E)$ (respectively, $E_{E_1}(n, E)$) is a projective bundle over $O_G^{E_2}(n, E)$ of relative dimension $n - 2$ (respectively, $n - 3$), and the condition of Lemma A.5 is satisfied (cf. Proof of Theorem 2.15). This proves the first statement. The last statement follows from the first one in combination with homotopy invariance and Lemma 3.12.

3.4 The main theorem

We can now state our main result.

**Theorem 3.14.** There is an isomorphism of graded $W^{\text{tot}}(S)$-modules

\[
W^{\text{tot}}(O_G^+(n, E)) \cong W^{\text{tot}}(O_G^{E_2^+}(n, E))[-(2n - 3)] \oplus W^{\text{tot}}(O_G^{E_2^+}(n, E))
\]

if $n \geq 3$ is odd.

**Proof.** By [3, Theorem 1.4 (B)], the connecting homomorphism $\delta$ in the localization sequence (3) can be fitted into the commutative diagram

\[
\begin{array}{ccc}
W^i(U^+(n, E)) & \xrightarrow{\delta} & W^{i-(n-2)}(O_G^{E_1^+}(n, E), \det E_n) \\
\alpha_i^+ & \cong & \langle \pi_i^+ \rangle \\
W^i(O_G^{E_1^+}(n, E)) & \xrightarrow{\pi_i^+} & W^i(E(n, E)).
\end{array}
\]

From now on, for simplicity, we shall suppress all the twists, since they can be recovered if necessary. Our strategy is to further decompose the $n - 1$ cases (that is, when one has $O_G^{E_1^+}(n, E)$) into the $n - 2$ cases and analyze the map $(\pi^+)_* \circ \bar{\pi}^-_*$. Look at the following picture

\[
\begin{array}{ccc}
W^i(U_+^1(n, E)) & \xrightarrow{\alpha_i^+} & W^i(U_{E_1}^+(n, E)) \\
\beta & \cong & \langle \eta_{1-} \rangle \\
W^i(B_+^{E_1^+}(n, E)) & \xrightarrow{(\pi_1^+)_*} & W^i(O_G^{E_1^+}(n, E)) \\
\langle \eta_{1-} \rangle & \xrightarrow{(\pi_1^+)_* \circ \bar{\pi}_*} & W^i(O_G^{E_1^+}(n, E)) \\
W^{i-2(n-2)}(O_G^{E_1^+}(n, E)) & \xrightarrow{\epsilon \cong} & W^{i-2(n-2)}(O_G^{E_1^+}(n, E)) \\
\gamma & \cong & \langle \eta_{1-} \rangle \\
W^{i-2(n-2)}(O_G^{E_1^+}(n, E)) & \xrightarrow{(\pi_1^+)_* \circ \bar{\pi}_*} & W^{i-2(n-2)}(B_+^{E_1^+}(n, E)).
\end{array}
\]

where $\beta$ and $\gamma$ are defined as the depicted composition, and $\epsilon := (\pi_1^+)_* \circ (\pi_1^-)_*^{-1}$. □

**Sublemma 3.15.** The composition $\nu^+_1 \circ (\pi_1^+)_* \circ \bar{\pi}^-_* \circ (\eta_{1-})_*$ is an isomorphism.
Proof of Sublemma 3.15. Consider the following diagram

\[
\begin{array}{c}
U^B_+(n, E) \xrightarrow{\psi_1} O^B_+(n, E) \xrightarrow{i_1} O^+_+(n, E) \xleftarrow{\psi_1} U_+(n, E) \\
E(n, E) \xrightarrow{i_1} B^+(n, E) \xrightarrow{\alpha} O^B_-(n, E) \\
U^B_+(n, E) \xrightarrow{\psi_1} B^+(n, E) \xrightarrow{\alpha} O^B_+(n, E),
\end{array}
\]

where we make the following interpretation on functors of points:

\[
\begin{align*}
\left\{ L^n_+ \mid L^n_+ \supset B_1 \right\} & \rightarrow \left\{ L^n_+ \mid L^n_+ \supset B_1 \right\} \rightarrow \left\{ L^n_+ \mid E \supset L^n_+ \right\} \leftarrow \left\{ L^n_+ \mid L^n_+ \not\supset E_1 \right\} \\
\left\{ (P_{n-1}, L^n_+ + L^n_-) \mid L^n_+ \supset P_{n-1} \right\} & \rightarrow \left\{ (P_{n-1}, L^n_+ + L^n_-) \right\} \rightarrow \left\{ L^n_+ \mid E \supset L^n_+ \right\} \\
\left\{ L^n_+ \mid L^n_+ \supset B_1 \right\} & \rightarrow \left\{ (P_{n-1}, L^n_+ + L^n_-) \right\} \rightarrow \left\{ L^n_+ \mid L^n_+ \not\supset E_1 \right\}.
\end{align*}
\]

Note that the lower right rectangle is Cartesian and tor-independent. This follows from \( \tilde{\pi}_- \) being flat. By the base change formula, we have

\[\psi^*_1 \circ (\tilde{\pi}_+)^* \circ (\tilde{\pi}_-)^* \circ (\tau_1^-)^* = \alpha^*_{1-}\]

and the claim follows, since \( \alpha^*_{1-} \) is an affine bundle. \( \square \)

Sublemma 3.16. The composition \((\psi^*_{1+})^* \circ (\tilde{\pi}_+)^* \circ (\tilde{\pi}_-)^* \circ (\tau_1^-)^* \) is zero.

Proof of Sublemma 3.16. By Lemma 3.12 and 3.13, we see that \( \ker(\psi^*_{1+}) = \ker(\psi^*_{1+}) = \im(\psi^*_{1+}) \). Therefore, it is enough to show that

\[\tau^*_{1+} \circ (\tilde{\pi}_+)^* \circ (\tilde{\pi}_-)^* \circ (\tau_1^-)^* = 0.\]

Note that we have the following diagram

\[
\begin{array}{c}
O^B_+(n, E) \xrightarrow{i^*} O^B_+(n, E) \\
\xleftarrow{\psi_{1-}} E(n, E) \xrightarrow{\pi_-} O^B_-(n, E) \\
B^B_+(n, E) \xrightarrow{\pi_1^-} E(n, E),
\end{array}
\]
where the left square is cartesian and tor-independent. Observe that, in view of the functoriality of pullbacks and the base change formula [9, Theorem 5.5], one has

$$t_{1+}^* \circ (\hat{\pi}_+)^* \circ \hat{\pi}^* \circ \beta = (\bar{\alpha}_{1+})_* \circ \pi_{1+}^* \circ \pi_{1-}^* \circ \bar{\alpha}_{1-}^* \circ \omega$$.

By Lemma 3.11, we see that the composition $\pi_{1+}^* \circ \pi_{1-}^*$ is the identity map. Since $\bar{\alpha}_{1-} : Bl_{E}^E(n, E) \to OG_{E}^E(n, E)$ is a projective bundle of odd relative dimension, Theorem A.7 implies that $(\bar{\alpha}_{1+})_* \circ \bar{\alpha}_{1-}^*$ vanishes.

**Sublemma 3.17.** The composition $\gamma \circ (\hat{\pi}_+)^* \circ \hat{\pi}^* \circ (\psi_{1-})_*$ vanishes.

**Proof.** Consider the following diagram

$$\begin{array}{c}
OG^n_+(n, E) \xleftarrow{\pi_{1+}} E(n, E) \xrightarrow{\pi_{1-}} OG^n_+(n, E) \\
\xrightarrow{\psi_{1-}} B^n_+(n, E) \xrightarrow{\alpha_{1-}} OG^n_+(n, E),
\end{array}$$

where the right square diagram is cartesian and tor-independent. Note that

$$\gamma \circ (\hat{\pi}_+)^* \circ \hat{\pi}^* \circ (\psi_{1-})_* = (\bar{\alpha}_{1-})_* \circ \pi_{1+}^* \circ (\psi_{1+})_* \circ \bar{\alpha}_{1-}^*.$$

By Lemma 3.11, we see that the composition $\pi_{1+}^* \circ (\psi_{1-})_*$ is the identity map. Since $\bar{\alpha}_{1-} : Bl_{E}^E(n, E) \to OG_{E}^E(n, E)$ is a projective bundle of odd relative dimension, Theorem A.7 implies that $(\bar{\alpha}_{1-})_* \circ \bar{\alpha}_{1-}^*$ vanishes.

**Sublemma 3.18.** The composition $\gamma \circ (\hat{\pi}_+)^* \circ \hat{\pi}^* \circ \beta$ vanishes.

**Proof of Sublemma 3.18.** Form the following diagram:

$$\begin{array}{c}
OG^n_-(n, E) \xleftarrow{\psi_{1-}} Bl^n_+(n, E) \xrightarrow{\pi_{1+}} OG^n_+(n, E) \\
\xrightarrow{\varphi_{1+}} H_+(n, E) \xrightarrow{\varphi_{1-}} E(n, E) \xrightarrow{\pi_{1-}} OG^n_+(n, E) \\
\xrightarrow{\psi_{1+}} T(n, E) \xrightarrow{\varphi_{1-}} H_-(n, E) \xrightarrow{\varphi_{1-}} B^n_+(n, E) \\
\xrightarrow{\psi_{1-}} OG^n_-(n, E) \xrightarrow{\varphi_{1-}} OG^n_+(n, E),
\end{array}$$

where all the square diagrams labeled with $\square$ are defined by fiber products, and we define $\Psi_{1-} := \bar{\alpha}_{1+} \circ \varphi_{1-} \circ \bar{\varphi}_{1-}^*$ as depicted in the picture. Next, we draw the following picture with diagrams labeled
with □ being fiber products

\[
\begin{array}{c}
\Psi_-
\end{array}
\]

where we define \(\Psi'_- := \tilde{\alpha}'_1 - o q'_+ o \varphi'_+\) and \(\Phi_+ := \tilde{\pi}_1 + o \Psi'_+.\) Let us observe the following facts.

1. \(\pi_1^\pm, \pi'_1^\pm, \varphi_1^\pm, \varphi'_1^\pm, \varphi_1^\pm, \varphi'_1^\pm, \varphi_1^\pm, \varphi'_1^\pm\) are birational and locally complete intersection.
2. \(\tilde{\pi}_1^\pm, q_1^\pm, q'_1^\pm, \tilde{\alpha}_1^\pm, \tilde{\alpha}'_1^\pm\) are projective bundles of relative dimension \(n - 2\) (an odd number).
3. \(\tilde{\pi}_1^\pm\) are projective bundles of relative dimension \(n - 3\) (an even number).

All these claims follow from Theorem 2.15 and the properties of fiber products.

Note that all square diagrams labeled with □ are tor-independent. Only the square labeled with □ in the left lower corner of the first displayed diagram requires an explanation. Since \(\pi'_1^\pm, \varphi'_1^\pm\) are birational and locally complete intersection, we are reduced to show that a fiber square of four regular immersions of the same codimension is tor-independent. The proof for this fact follows from the argument of [10, Lemma 22]. By repeatedly applying the base change formula, we see that

\[
\gamma o(\tilde{\pi}_+)^- o \tilde{\pi}_+^- o \beta = (\Psi^-)_+ o \Psi_+^-.
\]

Since \(\tilde{\pi}_1^-\) is an isomorphism (cf. Theorem A.4), it is enough to show that \(\tilde{\pi}_1^- o (\Psi^-)_+ o \Psi_+^- = 0\).

Note that

\[
(\Psi^-)_+ = (\tilde{\pi}_1^-)^+ o (\varphi'_+)^+ o (q'_+)^+ o (\varphi'_+)^+ o (\varphi'_+)^+ o (\varphi'_+)^+\]

and

\[
(\Psi'_+)_+ = (\tilde{\pi}_1^-)^+ o (\varphi'_+)^+ o (q'_+)^+ o (\varphi'_+)^+ o (\varphi'_+)^+ o (\varphi'_+)^+.
\]

By Lemma A.2, we see \((\varphi'_+)^+ o (\varphi'_+)^+ = \text{id}\), and the result follows from \((q'_+)^+ o q'_+ = 0\) (cf. Theorem A.7).

\[\Box\]

**Proof of Theorem 3.14 (continue).** Let \(\vartheta_+ : OG_{+}^E(n, E) \to OG_{+}^E(n, E)\) denote the composition \(t_+ o \pi_1^+.\) By the above sublemmas, we have constructed the following short exact sequence

\[
0 \longrightarrow W' + (2n - 1)(OG_{+}^E(n, E), L \otimes \omega_{\vartheta_+}) \longrightarrow W'(OG_{+}^E(n, E), L) \longrightarrow (\Omega_+)\circ(W'(OG_{+}^E(n, E), L) \longrightarrow 0,
\]

where \(L\) is a line bundle over the base \(S,\) and \(\Omega_+ := (\varpi^+)^- o (\varphi^+)^- o (\varphi^+)^- o (\varphi^+)^-\). Note here that \([\omega_{\vartheta_+}] = 1 \in \text{Pic}(OG_{+}^E(n, E))/2\). This formula follows from Proposition 2.21 and Remark 3.10.
It is clear that $\Omega_+$ is a morphism of graded $W^\text{tot}(S)$-modules, because it is the composition of morphisms of graded $W^\text{tot}(S)$-modules. Using the projection formula, we obtain the following short exact sequence of graded $W^\text{tot}(S)$-modules

$$0 \longrightarrow W^\text{tot}(OG^E_+(n,E))[-2n+3] \xrightarrow{\partial_*} W^\text{tot}(OG_+(n,E)) \xrightarrow{\Omega_+} W^\text{tot}(OG^E_+(n,E)) \longrightarrow 0.$$ 

Since $OG^E_+(n,E) \cong OG_+(n-2,E^2)$, by induction on $n$ we can assume that $W^\text{tot}(OG^E_+(n,E))$ is a graded projective $W^\text{tot}(S)$-module. In fact, we can reduce to the $n=1$ case and $OG_+(1,E) \cong S$, therefore this exact sequence splits.

\[ \square \]

3.5 \quad \textbf{Proof of Theorem 1.1}

Note that trivial hyperbolic bundles always admit canonical complete flags. The statement of Theorem 1.1 for $n$ odd follows inductively from Theorem 3.14 and results in Section 2.3. For $n$ even, in view of Proposition 3.9 and Lemma 3.13, one has the following isomorphism of graded $W^\text{tot}(S)$-modules:

$$W^\text{tot}(OG_+(n,E)) \cong W^\text{tot}(OG_+(E^1)(n,E))[-n-1, \det(\bar{E}_n)] \oplus W^\text{tot}(OG^E_+(n,E)),$$

which allows us to reduce to the odd case. By the assumption, $\det(\bar{E}_n)$ is trivial, and the result follows.

\textit{Remark 3.19.} For odd $n$, one can obtain a visual interpretation of Theorem 3.14. In fact, in this case one has that even shifted Young diagrams must either have the first two rows of maximal length (as in those appearing in the first line of Example 1.2) or two empty columns on the right (in the second line). The number of boxes in the first two rows is precisely $2n-3$, which, not by accident, also appears as the shift in the statement of Theorem 3.14.

For $n$ even there are two classes of even shifted Young diagrams: those whose first row is full and those whose last column is empty. Note that this fact can be viewed as a diagrammatic counterpart to the recursive description given in Formula (7).

\section*{APPENDIX: EULER CLASSES AND PROJECTIVE BUNDLES BY HENG XIE}

The projective bundle theorem for Witt groups was independently proved by Walter [33] and Nenashev [26]. In the main body of the paper, we need to apply two theorems (Theorem A.4 and A.7 in this appendix), which were proved as Theorem 1.2 and Theorem 1.4 in the unpublished [33]. Although the approach by Nenashev is closer to our needs, his description of the underlying maps appearing in the projective bundle theorem is not very explicit. The aim of this appendix is to highlight some maps that in [26] are sort of hidden and simultaneously solve the lack of a published reference for the theorems that we need. It is worth mentioning that Fasel [11] has written down explicitly the underlying maps for the projective bundle theorem of $I^j$-cohomology, which can be compared with our Theorem A.4 and A.7.

\subsection*{A.1 \quad Projective bundles}

Let $p: \mathcal{E} \to X$ be a vector bundle of rank $r+1$ over a regular scheme $X$ with $\frac{1}{2} \in \mathcal{O}_X$. Let $s: X \to \mathcal{E}$ be the zero section of the bundle $\mathcal{E}$. Let $q: \mathbb{P}(\mathcal{E}) \to X$ be the projective bundle associated to the
vector bundle $\mathcal{E}$. Then, there is an exact sequence of vector bundles on $\mathbb{P}(\mathcal{E})$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \longrightarrow q^* \mathcal{E} \longrightarrow Q \longrightarrow 0,$$

where $Q$ is the universal quotient bundle. Consider the projective bundle $q' : \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \rightarrow X$.

The canonical split exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \oplus \mathcal{O} \rightarrow \mathcal{E} \rightarrow 0$ induces two closed embeddings $\iota : \mathbb{P}(\mathcal{O}) = X \rightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ and $\iota' : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$. Let $\nu : U \hookrightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ be the associated open complement of $\iota$.

There is a canonical map $\alpha : U \rightarrow \mathbb{P}(\mathcal{E})$, which is a vector bundle.

Let $\mathcal{W}_{\mathbb{P}(\mathcal{E})}$ denote the vector bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ over $\mathbb{P}(\mathcal{E})$. Consider the projective bundle $\mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})})$ over $\mathbb{P}(\mathcal{E})$ equipped with the projection $\tilde{\alpha} : \mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})}) \rightarrow \mathbb{P}(\mathcal{E})$. By the universal property of projective bundles, there is a morphism $\pi : \mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})}) \rightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ defined by the rank one subbundle $\mathcal{O}_{\mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})})}(-1) \subset \tilde{\alpha}^* \mathcal{W}_{\mathbb{P}(\mathcal{E})} \subset \mathcal{E}_{\mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})})} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})})}$. One can form the following Cartesian diagram

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{O}) & \longrightarrow & \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \\
\downarrow & & \downarrow \\
\mathbb{P}(\mathcal{O}) & \longrightarrow & \mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})}).
\end{array}
$$

\textbf{Proposition A.1.} The projective bundle $\mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})})$ is the blow-up of $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ along $\mathbb{P}(\mathcal{O})$ with exceptional fiber $\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})})$. Moreover, $U \cong \mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})}) - \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})})$ is the total space of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ over $\mathbb{P}(\mathcal{E})$.

The whole setting is depicted in the following picture

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{O}) & \longrightarrow & \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \leftarrow U \\
\downarrow & & \downarrow \\
\mathbb{P}(\mathcal{O}) & \longrightarrow & \mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})}).
\end{array}
$$

where $\alpha := \tilde{\alpha} \circ \nu$. Set $B := \mathbb{P}(\mathcal{W}_{\mathbb{P}(\mathcal{E})})$ and $E := \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})})$.

\textit{Proof.} By the compatibility of blow-ups and pullbacks, one can reduce to the case when $X$ is affine, and suppose that $\mathcal{E}$ is free. This case is done in [17, Chapter V, Example 2.11.4] $\square$

\section{A.2 \hspace{1em} Pushforward and pullback}

Let $f : X \rightarrow Y$ be a proper morphism, and let $\mathcal{L}$ be a line bundle on $Y$. Recall two functorial maps from [9]: the pullback

$$f^* : W^i(Y, \mathcal{L}) \rightarrow W^i(X, f^* \mathcal{L})$$

and the pushforward

$$f_* : W^{i+\dim X}(X, f^* \mathcal{L} \otimes \omega_{X/Y}) \rightarrow W^{i+\dim Y}(Y, \mathcal{L}).$$

For properties of pushforward and pullback, we refer to [9]. The following lemma is useful throughout several arguments in our paper: it is an analog of [13, Proposition 6.7 (b)].
Lemma A.2. Let $\pi : B \to Y$ be a proper birational morphism. Suppose that $[\omega_\pi] = 1$ in $\text{Pic}(B)/2$, and that $\pi_*(1_B) = 1_Y$ in Witt groups. Then, $\pi_* \circ \text{per} \circ \pi^*(y) = y$ for any $y \in W^i(Y, \mathcal{L})$.

Proof. By the projection formula [9, Theorem 6.5], we see that $\pi_* \circ \text{per} \circ \pi^*(y) = \pi_* \circ \text{per} \circ (1_B \cdot \pi^*(y)) = \pi_*(1_B) \cdot y = y$. □

Remark A.3. If $B$ is the blow up of $Y$ along a regular closed subscheme $Z$, then the condition $\pi_*(1_B) = 1_Y$ is satisfied, cf. [4, Proposition 3.15].

A.3 | Pushforward and projective bundles

For the case of the projective bundle $q : \mathbb{P}(\mathcal{E}) \to X$, the exact sequence

$$
0 \longrightarrow \Omega_{\mathbb{P}(\mathcal{E})/X} \longrightarrow q^* \mathcal{E}^\vee \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow 0
$$

yields a canonical isomorphism $\omega_{\mathbb{P}(\mathcal{E})/X} := \det(\Omega_{\mathbb{P}(\mathcal{E})/X}) \cong q^* \mathcal{E}^\vee \otimes \mathcal{O}(-r - 1)$. Therefore, we are able to rewrite the pushforward as

$$
q_* : W^i(\mathbb{P}(\mathcal{E}), q^* \mathcal{L} \otimes \det \mathcal{E}^\vee \otimes \mathcal{O}(-r - 1)) \to W^{i-r}(X, \mathcal{L}).
$$

A.4 | Euler class

Let $\mathcal{L}$ be a line bundle on $X$. Let $\rho : \mathcal{V} \to X$ be a vector bundle of rank $d$ and let $v : X \to \mathcal{V}$ be its zero section. The Euler class of $\mathcal{V}$ is the following composition of maps

$$
W^i(X, \mathcal{L}) \xrightarrow{\rho_*} W^{i+d}(\mathcal{V}, \rho^*(\det \mathcal{V}^\vee \otimes \mathcal{L})) \xrightarrow{(\rho^*)^{-1}} W^{i+d}(X, \det \mathcal{V}^\vee \otimes \mathcal{L}),
$$

which shall be denoted by $e(\mathcal{V})$ (cf. [10] or [11, Section 3] for details). For the relations between pushforwards, pullbacks and Euler classes, we refer to [10] (see also [11]), and the excess intersection formula plays an important role in this paper.

A.5 | Witt groups of projective bundles

We use notations as in Section A.1.

Theorem A.4 Theorem 1.2 [33]. Assume that $r$ is an even number. The maps of groups

$$
q_* \circ \text{per} : W^i(\mathbb{P}(\mathcal{E}), \mathcal{O}(-1)) \xrightarrow{\epsilon} W^{i-r}(X, \det \mathcal{E}) : e(\mathcal{V}) \circ q^*
$$

are inverse isomorphisms.

Proof. By Proposition A.1, [3, Setup 1.1 and Hypothesis 1.2] is satisfied. By [9], we have a pushforward map $\pi_* : W^i(B, \omega_\pi) \to W^i(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}))$. Note that by [3, Proposition 2.1] one has $\omega_\pi = (0, r) \in \text{Pic}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O})) \oplus \mathbb{Z} \cong \text{Pic}(B)$. Since $r$ is even, we have a periodicity isomorphism $\text{per} : W^i(B) \to W^i(B, \omega_\pi)$. Form the following morphisms of short exact sequences coming from...
Here all the squares are commutative and in particular the commutativity of the lower left diagram follows by the excess intersection formula (cf. [10]). (Note here that \( \pi \) is of finite Tor-dimension, since \( \pi \) is the composition \( B \hookrightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \times \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \), where the first morphism is a regular immersion and the second morphism is the projection.) All the short exact sequences are split, by the computation of Witt groups of projective bundles (cf. [26]). Recall the isomorphism \( \omega \cong \mathcal{E}(-1) \) (cf. [3, Appendix A]). The excess normal bundle of the left Cartesian square of Diagram A.2 is the universal quotient bundle \( Q \) on \( \mathbb{P}(\mathcal{E}) \cong \mathcal{E} \). Theorem A.4 will follow if we can prove that \( \pi^* \circ \pi^* \circ \text{per} = \text{id} \) and \( \pi^* \circ \text{per} \circ \pi^* = \text{id} \).

Noting that \( \pi^* \circ \text{per} \circ \pi^* = \text{id} \) by Lemma A.2. It is therefore enough to prove that \( \pi^* \) is an isomorphism. Form the Cartesian diagram

\[
\begin{array}{ccc}
\mathbb{P}(\mathcal{E}) & \overset{t'}{\longrightarrow} & \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \\
g \downarrow \cong & & \pi \downarrow \\
\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)) & \overset{t'}{\longrightarrow} & B,
\end{array}
\]

where \( g \) is an isomorphism since \( t' \) factors through \( U \). By applying the localization theorem to \( t' \) and \( t' \), we obtain the following morphism of split exact sequences of Witt groups (cf. Lemma A.5)

\[
0 \longrightarrow W^{i-1}(\mathbb{P}(\mathcal{E}), \omega_{\mathcal{E}}) \overset{(t')_{i}}{\longrightarrow} W^{i}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O})) \overset{t'^*}{\longrightarrow} W^{i}(\mathbb{P}(\mathcal{E})) \longrightarrow 0
\]

\[
0 \longrightarrow W^{i-1}(\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)), \omega_{\mathcal{E}}) \overset{(t')_{i}}{\longrightarrow} W^{i}(\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))) \overset{t'^*}{\longrightarrow} W^{i}(\mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))) \longrightarrow 0.
\]

The left square is commutative by [10] (recall that \( \pi \) is of finite Tor-dimension and note that the excess normal bundle is trivial). This allows us to conclude that the map \( \pi^* \) is an isomorphism.

**Lemma A.5.** Suppose that \( 0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{L} \to 0 \) is an exact sequence with \( \mathcal{L} \) a line bundle on \( X \). Let \( \iota : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}') \) be the induced closed embedding. Then, the sequence

\[
0 \longrightarrow W^{i-1}(\mathbb{P}(\mathcal{E}), \omega_{\mathcal{E}}) \overset{\iota^*}{\longrightarrow} W^{i}(\mathbb{P}(\mathcal{E}')) \overset{\iota^*}{\longrightarrow} W^{i}(\mathbb{P}(\mathcal{E})) \longrightarrow 0
\]

is split exact.
Proof. Let \( q : \mathbb{P}(\mathcal{E}) \to X \) and \( q' : \mathbb{P}(\mathcal{E}') \to X \) be the projections. Let \( U := \mathbb{P}(\mathcal{E}') - \mathbb{P}(\mathcal{E}) \). Note that \( U \) is an affine bundle over \( X \), and let \( p : U \to X \) be the projection. By [26, Section 5], the exact sequence

\[
0 \longrightarrow W^{i-1}(\mathbb{P}(\mathcal{E}), \omega_i) \xrightarrow{i^*} W^{i}(\mathbb{P}(\mathcal{E}')) \xrightarrow{v^*} W^{i}(U) \longrightarrow 0
\]

splits on the right by the map \( q'^* \circ (p^*)^{-1} \). By [26, Theorem 3.6], we also know that \( q^* : W^i(X) \to W^i(\mathbb{P}(\mathcal{E})) \) is an isomorphism. Therefore, one has \( \tau^* \circ q'^* \circ (q^*)^{-1} = \text{id} \) and \( \tau^* \) is split surjective.

It is enough to show that \( \tau^* \circ t_{\ast} = 0 \). By the self-intersection formula [10, Theorem 33], we see that \( \tau^* \circ t_{\ast} = e(\omega_i) \). Note that the Euler class \( e(\omega_i) \) is hyperbolic [12, Proposition 14], since \( \omega_i \) is a line bundle. Therefore, \( \tau^* \circ t_{\ast} = 0 \). The result follows. \( \square \)

Remark A.6. In [26, Theorem 5.1] Nenashev is able to show that \( W^{i-r}(X, \det \mathcal{E}) \) and \( W^i(\mathbb{P}(\mathcal{E}), \mathcal{O}(1)) \) are isomorphic, however, it is more subtle to gain an understanding of the intermediate maps whose composition gives rise to this isomorphism, and this is the precisely the purpose of Theorem A.4.

**Theorem A.7** Theorem 1.4 [33]. Assume that \( r \) is an odd number. The sequence of maps of groups

\[
\cdots \longrightarrow W^i(X) \xrightarrow{q^*} W^i(\mathbb{P}(\mathcal{E})) \xrightarrow{q, \text{oper}} W^{i-r}(X, \det \mathcal{E}) \xrightarrow{e(\mathcal{E})} W^{i+1}(X) \longrightarrow \cdots
\]

is exact.

Proof. Let \( p : \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \to X \) be the canonical projection. Consider the exact sequence

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E} \oplus \mathcal{O})}(-1) \longrightarrow p^*(\mathcal{E} \oplus \mathcal{O}) \longrightarrow \mathcal{G} \longrightarrow 0 \tag{A.5}
\]

with \( \mathcal{G} \) the universal quotient bundle and form the following ladder diagram.

\[
\begin{array}{ccccccccc}
\cdots & \longrightarrow & W^i(U') & \xrightarrow{\delta} & W^{i+1}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}), \mathcal{O}(-1)) & \longrightarrow & W^{i+1}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}), \mathcal{O}(-1)) & \xrightarrow{v^*} & W^{i+1}(U') & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & W^i(X) & \xrightarrow{q^*} & W^i(\mathbb{P}(\mathcal{E})) & \xrightarrow{q, \text{oper}} & W^{i-r}(X, \det \mathcal{E}) & \xrightarrow{e(\mathcal{E})} & W^{i+1}(X) & \longrightarrow & \cdots \\
\end{array}
\tag{A.6}
\]

Note that all the vertical maps in (A.6) are isomorphisms and that the upper sequence is exact. It is now enough to prove that all squares in Diagram A.6 are commutative.

By [3, Lemma 4.2 (B)], we have the following commutative diagram

\[
\begin{array}{ccccccccc}
W^i(U') & \xrightarrow{\delta} & W^{i+1}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}), \mathcal{O}(-1)) & \xrightarrow{v^*} & W^{i+1}(U') \\
\uparrow & & \uparrow & & \uparrow \\
W^i(\mathbb{P}(\mathcal{E} \oplus \mathcal{O})) & \xrightarrow{t^*} & W^i(\mathbb{P}(\mathcal{E}))
\end{array}
\]

which explains the commutativity of the left square in (A.6). In order to conclude that the middle square is commutative, we use that, by Theorem A.4, the map \( p_* \text{oper} \) is the inverse of \( e(\mathcal{G}) \circ p^* \), so that the commutativity follows from \( p_* \circ t'_{\ast} = q_{\ast} \). Finally, in order to see that the right square is commutative, we observe that \( \alpha^* \circ t_{\ast} = v^* \) on the level of Picard groups. By applying the pullback
$\iota^*$ to (A.5), we see that $\iota^* G \cong E$ and therefore $\iota^* v = \alpha^* E$. Now one has

$$v^* e(G) p^* = e(v^* G) v^* p^* = e(v^* G) \alpha^* = e(\alpha^* E) \alpha^* = \alpha^* e(E).$$

\square

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