Analytic continuation of the resolvent of the Laplacian
and the dynamical zeta function

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Abstract

Let $s_0 < 0$ be the abscissa of absolute convergence of the dynamical zeta function $Z(s)$ for several disjoint strictly convex compact obstacles $K_i \subset \mathbb{R}^N$, $i = 1, \ldots, k_0$, $k_0 \geq 3$ and let $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1} \chi$, $\chi \in C_0^\infty(\mathbb{R}^N)$, be the cut-off resolvent of the Dirichlet Laplacian $-\Delta_D$ in $\Omega = \mathbb{R}^N \setminus \bigcup_{i=1}^{k_0} K_i$. We prove that there exists $\sigma_2 < s_0$ such that $Z(s)$ is analytic for $\Re(s) \geq \sigma_2$ and the cut-off resolvent $R_\chi(z)$ has an analytic continuation for $\Im(z) < -i\sigma_2$, $|\Re(z)| \geq C$.

Résumé

Prolongement analytique de la résolvante du Laplacien et de la fonction zeta dynamique. Soit $s_0 < 0$ l’abscisse de convergence absolue de la fonction zeta dynamique $Z(s)$ pour des obstacles compacts, disjoints et strictement convexes $K_i \subset \mathbb{R}^N$, $i = 1, \ldots, k_0$, $k_0 \geq 3$ et soit $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1} \chi$, $\chi \in C_0^\infty(\mathbb{R}^N)$, la résolvante tronquée du Laplacien de Dirichlet $-\Delta_D$ dans $\Omega = \mathbb{R}^N \setminus \bigcup_{i=1}^{k_0} K_i$. On prouve qu’il existe $\sigma_2 < s_0$ tel que $Z(s)$ est analytique pour $\Re(s) \geq \sigma_2$ et la résolvante tronquée $R_\chi(z)$ admet un prolongement analytique pour $\Im(z) < -i\sigma_2$, $|\Re(z)| \geq C$. Pour citer cet article : V. Petkov, L. Stoyanov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).

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Soit $K = K_1 \cup K_2 \cup \cdots \cup K_{k_0}$, où $K_i \subset \mathbb{R}^N$, $N \geq 2$, sont des domaines compacts, disjoints et strictement convexes ayant des frontières $\Gamma_i = \partial K_i$ et $k_0 \geq 3$. Soit $\Omega = \mathbb{R}^N \setminus K$ et $\Gamma = \partial K$. On suppose que $K$ satisfait la condition suivante : 

(H) Pour chaque couple $K_i, K_j$ de différentes composantes connexes de $K$ l’enveloppe convexe de $K_i \cup K_j$ n’a pas de points communs avec les autres composantes connexes de $K$. 

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Étant donné un rayon périodique réfléchissant $\gamma \subset \Omega$ avec $m_\gamma$ réflexions, soit $T_\gamma$ la période primitive (longueur) de $\gamma$ et soit $P_\gamma$ l’application de Poincaré linéaire associée à $\gamma$ (cf. [8]). Notons par $\lambda_i, \gamma_i$, $i = 1, \ldots, N - 1$, les valeurs propres de $P_\gamma$ telles que $|\lambda_i, \gamma| > 1$ et désignons par $\mathcal{P}$ l’ensemble de rayons primativs périodiques. Soit $\delta_{y_\gamma} = -\frac{1}{2} \log (\lambda_1, \gamma \cdots \lambda_{N-1}, \gamma)$, $r_\gamma = 0$ si $m_\gamma$ est pair et $r_\gamma = 1$ si $m_\gamma$ est impair. On considère la fonction zeta dynamique

$$Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^m r_\gamma \chi(-sT_\gamma + \delta_{y_\gamma}).$$

On voit facilement qu’il existe une abscisse de convergence absolue $s_0 \in \mathbb{R}$ telle que pour $\Re(s) > s_0$ la série $Z(s)$ est absolument convergente. D’autre part, en utilisant la dynamique symbolique et les résultats de [7], on conclut que $Z(s)$ est méromorphe pour $\Re(s) > s_0 - a$, $a > 0$ (cf. [4]). En suivant les résultats récents (cf. [9] pour $N = 2$ et [10] pour $N = 3$ sous certaines conditions) on sait qu’il existe $0 < \epsilon < a$ tel que la fonction zeta dynamique $Z(s)$ admet un prolongement analytique pour $\Re(s) > s_0 - \epsilon$. On considère maintenant pour $\Im(s) < 0$ la résolvante tronquée $R_\gamma(z) = \chi(-\Delta_D - z^2)^{-1} \chi : L^2(\Omega) \to L^2(\Omega)$, où $\chi \in C^\infty_0(\mathbb{R}^N)$. $\chi = 1$ sur $K$ et $-\Delta_D$ est le Laplacien de Dirichlet dans $\Omega = \mathbb{R}^N \setminus K$. La résolvante $R_\gamma(z)$ admet un prolongement méromorphe dans $\mathbb{C}$ pour $N$ impair et dans $\mathbb{C} \setminus \{i\mathbb{R}^+\}$ pour $N$ pair avec des pôles $z_j$, $\Im(z_j) > 0$. On se propose d’étudier la liaison entre les prolongements analytiques de $Z(s)$ et $R_\gamma(z)$. Le cas $s_0 > 0$ est plus facile car on sait que pour $-i s_0 \leq \Im(z) < 0$ la résolvante tronquée $R_\gamma(z)$ est analytique [6]. Dans la suite on suppose que $s_0 < 0$. Sous l’hypothèse $s_0 < 0$, Ikawa [3] a démontré que pour tout $\epsilon > 0$ il existe $C_\epsilon > 0$ tel que $R_\gamma(z)$ est analytique pour $\Im(z) < -i(s_0 + \epsilon)$, $|\Re(z)| \geq C_\epsilon$. Un résultat similaire pour un problème du contrôle a été établi par Burq [1]. La fonction zeta dynamique $Z(s)$ est liée aux périodes des rayons périodiques et formellement $Z(s)$ ne contient pas une information sur la dynamique de rayons dans un voisinage de l’ensemble ‘non-wandering’ (capift). Dans cette Note on examine le cas $\Re(s) < s_0$ en exploitant les propriétés spectrales de l’opérateur de Ruelle $L_\gamma$ (cf. Section 2). Notre résultat principal est le suivant :

**Théorème 0.1.** Soit $s_0 > 0$. Supposons que l’opérateur de Ruelle $L_\gamma$ satisfait les estimations (6). Alors il existe $\sigma_2 < s_0$ tel que $Z(s)$ est analytique pour $\Re(s) > \sigma_2$ et la résolvante tronquée $R_\gamma(z)$ est analytique pour $\Im(z) < -i \sigma_2$, $|\Re(s)| \geq C$.

Les estimations (6) sont un analogue aux estimations de Dolgopyat [2]. Ces estimaitions ont été démontrées pour $N = 2$ dans [9] et pour $N \geq 3$ sous certaines conditions dans [10]. On espère que (6) sont valables pour $N \geq 3$ sans aucune restriction. Notons qu’il y a quelques ans, Ikawa [5] a annoncé un résultat concernant le prolongement analytique de $R_\gamma(z)$ dans un domaine $-i D_{\epsilon, \alpha}$, où

$$D_{\epsilon, \alpha} = \{ s \in \mathbb{C} : \Re(s) > s_0 - |\Im(s)|^{-\alpha}, |\Im(s)| \geq C_\epsilon, 0 < \alpha < 1 \}$$

en imposant des conditions fortes sur le comportement de la fonction propre $w$ de l’opérateur de Ruelle associée à la valeur propre maximale et un prolongement analytique de $Z(s)$ dans $D_{\epsilon, \alpha}$. De plus, il suppose qu’on ait l’estimation $|Z(s)| \leq |s|^{1-\epsilon}$, $0 < \epsilon < 1$, $s \in D_{\epsilon, \alpha}$. A notre connaissance la preuve de ce résultat n’a pas été publiée ailleurs.

1. Introduction

Let $K$ be a subset of $\mathbb{R}^N$, $N \geq 2$, of the form $K = K_1 \cup K_2 \cup \cdots \cup K_k$, where $K_i$ are compact strictly convex disjoint domains in $\mathbb{R}^N$ with $C^\infty$ boundaries $\Gamma_i = \partial K_i$ and $k_0 \geq 3$. Set $\Omega = \mathbb{R}^N \setminus K$ and $\Gamma = \partial K$. We assume that $K$ satisfies the following (no-eclipse) condition:

(H) For every pair $K_i, K_j$ of different connected components of $K$ the convex hull of $K_i \cup K_j$ has no common points with any other connected component of $K$.

With this condition, the billiard flow $\phi_\gamma$ defined on the cosphere bundle $S^*\Omega$ in the standard way is called an open billiard flow. Given a periodic reflecting ray $\gamma \subset \Omega$ with $m_\gamma$ reflections, denote by $T_\gamma$ the primitive period (length) of $\gamma$ and by $P_\gamma$ the linear Poincaré map associated to $\gamma$ (see [8]). Let $\lambda_i, \gamma, i = 1, \ldots, N - 1$, be the eigenvalues of $P_\gamma$.
with \(|\lambda_{i,j}\gamma| > 1\) and let \(\mathcal{P}\) be the set of primitive periodic rays. For \(\gamma \in \mathcal{P}\) set \(\delta_{\gamma} = -\frac{1}{2}\log(\lambda_{1,\gamma} \cdots \lambda_{N-1,\gamma})\), \(r_{\gamma} = 0\) if \(m_{\gamma}\) is even and \(r_{\gamma} = 1\) if \(m_{\gamma}\) is odd. Consider the dynamical zeta function

\[
Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{mr_{\gamma}} e^{m(-sT_{\gamma} + \delta_{\gamma})}.
\]

It is easy to show that there exists an abscissa of absolute convergence \(s_0 \in \mathbb{R}\) such that for \(\Re(s) > s_0\) the series \(Z(s)\) is absolutely convergent. On the other hand, using symbolic dynamics and the results of [7], we deduce that \(Z(s)\) is meromorphic for \(\Re(s) > s_0 - a, \, a > 0\) (see [4]). According to some recent results (see [9] for \(N = 2\) and [10] for \(N \geq 3\) under some additional conditions) there exists \(0 < \epsilon < a\) so that the dynamical zeta function \(Z(s)\) admits an analytic continuation for \(\Re(s) \geq s_0 - \epsilon\). For \(\Im(z) < 0\) consider the cut-off resolvent \(R_{\chi}(z) = \chi(\Delta_D - z^2)^{-1}\chi : L^2(\Omega) \to L^2(\Omega)\), where \(\chi \in C^\infty_0(\mathbb{R}^N)\), \(\chi = 1\) on \(K\) and \(-\Delta_D\) is the Dirichlet Laplacian in \(\Omega = \mathbb{R}^N \setminus K\). The cut-off resolvent \(R_{\chi}(z)\) has a meromorphic continuation in \(\mathbb{C}\) for \(N\) odd and in \(\mathbb{C} \setminus \{i\mathbb{R}^+\}\) for \(N\) even with poles \(z_j\) such that \(\Im(z_j) > 0\). The analytic properties and the estimates of \(R_{\chi}(z)\) play a crucial role in many problems related to the local energy decay, distribution of the resonances etc. We study the link between the analytic continuations of \(Z(s)\) and \(R_{\chi}(z)\). The case \(s_0 > 0\) is much easier, since we know that for \(-i s_0 \leq \Im(z) < 0\) the cut-off resolvent \(R_{\chi}(z)\) is analytic (see [6]).

In the following we assume that \(s_0 < 0\). The problem is to examine the link between the analyticity of \(Z(s)\) for \(\Re(s) > s_0\) and the behavior of \(R_{\chi}(z)\) for \(0 \leq \Im(z) < -i s_0\). Assuming \(s_0 < 0\), Ikawa [3] proved that for every \(\epsilon > 0\) there exists \(C_\epsilon > 0\) so that the cut-off resolvent \(R_{\chi}(z)\) is analytic for \(\Im(z) < -i(s_0 + \epsilon), \, |\Re(z)| \geq C_\epsilon\). A similar result for a control problem has been established by Burq [1]. The proofs in [3] and [1] are based on the construction of an asymptotic solution \(U_M(x, s; k)\) with boundary data \(m(x; k) = e^{ik\varphi(s)}h(x), \, k \in \mathbb{R}, \, k \geq 1\), where \(\varphi\) is a phase function \((\|\nabla \varphi\| = 1)\) and \(h \in C^\infty(\Gamma)\) has a small support. More precisely, \(U_M(\cdot, s; k)\) is \(C^\infty(\Omega)\)-valued holomorphic function in \(D_0 = \{s \in \mathbb{C} : \Re(s) > s_0\}\), and we have

\[
\langle \Delta - s^2 \rangle U_M(\cdot, s; k) = 0 \quad \text{for} \quad x \in \Omega, \quad \Re(s) > s_0, \quad (1)
\]

\[
U_M(\cdot, s; k) \in L^2(\Omega) \quad \text{if} \quad \Re(s) > 0, \quad (2)
\]

\[
U_M(x, s; k) = m(x, k) + r_M(x, s; k) \quad \text{on} \quad \Gamma, \quad (3)
\]

where, for \(r_M(x, s; k)\) and \(s \in D_0, \, |s + i k| \leq 1\), we have the estimates

\[
\|r_M(\cdot, s; k)\|_{C^p(\Gamma)} \leq C_p k^{-M+p} \langle \|\nabla \varphi\|\rangle_{CM^2+2p+2(\Gamma)} + 1 \|h\|_{CM^2+2p+2(\Gamma)}^0, \quad \forall p \in \mathbb{N}. \quad (4)
\]

The function \(U_M(x, s; k)\) is given by a finite sum of series having the form

\[
\sum_{n=0}^\infty \sum_{|j|=n+2}^M \sum_{l=q=0}^M e^{-s\varphi_j(x)} \sum_{v=0}^{2q} (a_{j,q,v}(x, s; k)(s + ik)^v)(ik)^{-q}, \quad (5)
\]

where \(j = (j_1, \ldots, j_\nu)\), \(j_i \in \{1, \ldots, k_0\}\) are configurations, \(|j| = m\), \(\varphi_j(x)\) are phase functions and the amplitudes \(a_{j,q,v}(x, s; k)\) depend on \(s \in \mathbb{C}\) and \(k \in \mathbb{R}\). The main difficulty is to establish the summability of these series and to obtain for \(\Re(s) > s_0\) suitable \(C^p\) estimates of their traces on \(\Gamma\). The absolute convergence of \(Z(s)\) makes it possible to establish the absolute convergence of the series in (5) and to get crude estimates leading to (1)–(4). The dynamical zeta function \(Z(s)\) is related to the periods of periodic rays and formally from \(Z(s)\) we get no information about the dynamics of all rays in a neighborhood of the non-wandering (trapped) set. In this Note we study the case \(\Re(s) < s_0\) by means of the Ruelle operator \(L\) (see Section 2 for the definition). Our main result is the following:

**Theorem 1.1.** Let \(s_0 < 0\). Assume that for the Ruelle operator \(L_\delta\) the estimates (6) hold. Then there exists \(\sigma_2 < s_0\) such that \(Z(s)\) is analytic for \(\Re(s) > \sigma_2\) and the cut-off resolvent \(R_{\chi}(z)\) is analytic for

\[
\Im(z) < -i\sigma_2, \quad |\Re(s)| \geq C.
\]

The estimates (6) are analogous to Dolgopyat’s estimates in [2]. For open billiard flows (6) have been established in [9] for \(N = 2\) and under some conditions in [10] for \(N \geq 3\). We expect that (6) hold for \(N \geq 3\) without any restrictions.
Several years ago, Ikawa [5] announced a result concerning an analytic continuation of \( R_{\chi}(z) \) in a domain \(-iD_{\epsilon,\alpha}\), where

\[
D_{\epsilon,\alpha} = \{ s \in \mathbb{C} : \Re(s) > s_0 - |\Im(s)|^{-\alpha}, |\Im(z)| \geq C_\epsilon, \ 0 < \alpha < 1 \}
\]

assuming some strong conditions on the behavior of the eigenfunction \( \varphi \) of the corresponding Ruelle operator related to its maximal eigenvalue as well as an analytic continuation of \( Z(s) \) in \( D_{\epsilon,\alpha} \) combined with an estimate \( |Z(s)| \leq |s|^{-\epsilon}, \ 0 < \epsilon < 1, \ s \in D_{\epsilon,\alpha} \). To our best knowledge a proof of the above result of Ikawa has not been published anywhere.

2. Ruelle operator

Introduce the spaces

\[
\Sigma_A = \{ \ldots, -m, \ldots, -1, \eta_0, \eta_1, \ldots, \eta_m, \ldots \}: 1 \leq \eta_j \leq \kappa_0, \ \eta_j \in \mathbb{N}, \ \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z} \},
\]

\[
\Sigma_A^+ = \{ (\eta_0, \eta_1, \ldots, \eta_m, \ldots) : 1 \leq \eta_j \leq \kappa_0, \ \eta_j \in \mathbb{N}, \ \eta_j \neq \eta_{j+1} \text{ for all } j \geq 1 \}.
\]

We define the operator \( \sigma : \Sigma_A^+ \to \Sigma_A^+ \) by \( (\sigma \xi)_i = \xi_{i+1} \), \( i \in \mathbb{N} \). Given \( \xi \in \Sigma_A \), let

\[
\ldots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi), P_1(\xi), P_2(\xi), \ldots
\]

be the successive reflection points of the unique billiard trajectory in the exterior of \( K \) such that \( P_j(\xi) \in K_{\xi_j} \) for all \( j \in \mathbb{Z} \). Set \( f(\xi) = \| P_0(\xi) - P_1(\xi) \| \). Following [4], one constructs a sequence \( \{ \varphi_{\xi,j} \}_{j=-\infty}^{\infty} \) of phase functions such that for each \( j, \varphi_{\xi,j} \) is defined and smooth in a neighborhood \( U_{\xi,j} \) of the segment \( [P_j(\xi), P_{j+1}(\xi)] \) in \( \Omega \) and

(i) \( \nabla \varphi_{\xi,j}(P_j(\xi)) = \frac{P_{j+1}(\xi) - P_j(\xi)}{\| P_{j+1}(\xi) - P_j(\xi) \|} \),

(ii) \( \varphi_{\xi,j} = \varphi_{\xi,j+1} \) on \( \partial U_{\xi,j} \cap U_{\xi,j+1} \),

(iii) for each \( x \in U_{\xi,j} \) the surface \( C_{\xi,j}(x) = \{ \gamma \in U_{\xi,j} : \varphi_{\xi,j}(\gamma) = \varphi_{\xi,j}(x) \} \) is strictly convex with respect to its normal field \( \nabla \varphi_{\xi,j} \). For any \( y \in U_{\xi,j} \) denote by \( G_{\xi,j}(y) \) the Gauss curvature of \( C_{\xi,j}(x) \) at \( y \).

Now define \( g : \Sigma_A \to \mathbb{R} \) by

\[
g(\xi) = \frac{1}{N - 1} \ln \frac{G_{\xi,0}(P_1(\xi))}{G_{\xi,0}(P_0(\xi))}.
\]

By Sinai’s Lemma, there exist \( \tilde{f}, \tilde{g} \) depending on future coordinates only and \( \chi_1, \chi_2 \) such that

\[
f(\xi) = \tilde{f}(\xi) + \chi_1(\sigma \xi), \quad g(\xi) = \tilde{g}(\xi) + \chi_2(\sigma \xi), \quad \xi \in \Sigma_A.
\]

Setting \( \tilde{r}(\xi, s) = -s \tilde{f}(\xi) + \tilde{g}(\xi) + i \pi \), we define the Ruelle transfer operator \( L_s : C(\Sigma_A^+) \to C(\Sigma_A^+) \) by \( L_s u(\xi) = \sum_{\sigma \eta = \xi} e^{\tilde{r}(\xi, \eta)} u(\eta) \) for any continuous (complex-valued) function \( u \) on \( \Sigma_A^+ \) and any \( \xi \in \Sigma_A^+ \). For our analysis the Dolgopyat type estimates [2] for the norms of \( L_s^n \) play a crucial role. Following the results in [9,10] there exist constants \( C > 0, \sigma_0 < s_0 \) and \( 0 < \rho < 1 \) so that for \( s = \tau + it \) with \( \tau \geq \sigma_0 \) and \( n = p[\log |t|] + l, \ p \in \mathbb{N}, \ 0 \leq l \leq [\log |t|] - 1 \), we have

\[
\| L_s^n \|_{L^\infty} \leq C \rho^{|\log |t||} e^{Pr(\tau - \tilde{f} + \tilde{g})}, \quad |t| \geq t_0.
\]

Pr(\( G \)) being the topological pressure of the function \( G \) (see [7]).

3. Idea of the proof of Theorem 1.1

Fix \( l \in \{ 1, \ldots, \kappa_0 \} \). Given a phase function \( \varphi(x) \) and an amplitude \( h(x) \in C^\infty(\Gamma) \), we wish to construct an asymptotic solution \( U_M(x, s; k) \) which is a holomorphic function for \( \Re(s) > \sigma_2 \) and \( U_M \) has properties similar to (1)–(4). The first approximation of \( U_M \) is an infinite sum

\[
u_{0,l}(x, -iv) = \sum_{n=0}^{\infty} \sum_{|j| = n+2, j_l = l} u_j(x, -iv),
\]
where \( u_j(x, -is) = (-1)^{m-1}e^{-xy_j(x)}a_j(x) \) is related to a configuration \( j = (j_1, \ldots, j_m) \) by using successive phase functions \( y_j(x) \) and amplitudes \( a_j(x) \) determined by the transport equation (see [5]). To justify the convergence of this series, we need to compare the general term with a suitable composition of operators related to the dynamics. Let \( \mu = (\mu_0 = 1, \mu_1, \ldots) \in \Sigma^+_A \). It follows from [3] that there exists a unique point \( y(\mu) \in K_1 \) such that the ray \( y(\mu) \) issued from a point \( y(\mu) \) in direction \( \nabla \varphi(y(\mu)) \) follows the configuration \( \mu \). Let \( Q_0(\mu) = y(\mu), Q_1(\mu), \ldots, \) be the consecutive reflection points of this ray. Define \( f_j^+(\mu) = \| Q_j(\mu) - Q_{j+1}(\mu) \| \), and

\[
 g_j^+(\mu) = \frac{1}{N-1} \ln \frac{G_{\mu, j}(Q_{j+1}(\mu))}{G_{\mu, j}(Q_j(\mu))} < 0,
\]

where \( G_{\mu, j}(y) \) denotes the Gauss curvature of the surface \( C_{\mu, j}(x) = \{z: \varphi(\mu_0, \mu_1, \ldots, \mu_j)(z) = \varphi(\mu_0, \mu_1, \ldots, \mu_j)(x)\} \) at \( y \).

We define an extension \( e: \Sigma^+_A \to \Sigma_A \). For \( s \in \mathbb{C} \) and \( \xi \in \Sigma^+_A \) with \( \xi_0 = 1 \), following [5], set

\[
 \phi^+(\xi, s) = \sum_{n=0}^\infty (-s)\left[ f_n^+(\sigma(\xi)) - f_n^+(\xi) \right] + \left[ g_n^+(\sigma(\xi)) - g_n^+(\xi) \right].
\]

Formally, we define \( \phi^+(\xi, s) = 0 \) when \( \xi_0 \neq 1 \), thus obtaining a function \( \phi^+: \Sigma^+_A \times \mathbb{C} \to \mathbb{C} \). Set \( \chi(\sigma(\xi)) = -s \chi_1(\xi) + \chi_2(\xi) \) and for any \( s \in \mathbb{C} \) define the operator \( G_s: C(\Sigma^+_A) \to C(\Sigma_A^+) \) by

\[
 G_s v(\xi) = \sum_{\sigma \in \mathbb{C}, \eta_0 = 1} e^{-\phi^+(\eta, s) + \chi(\sigma(\eta)) - s f(\sigma(\eta)) + \tilde{g}(\eta)} v(\eta), \quad v \in C(\Sigma_A^+), \xi \in \Sigma_A^+.
\]

Fix an arbitrary \( l \in \{1, \ldots, k_0 \} \) and an arbitrary point \( x_0 \in I_l \). Define the function \( \phi^-: (x_0; \cdot, \cdot): \Sigma_A \times \mathbb{R} \to \mathbb{R} \) (depending on \( l \) as well) as follows. First, set \( \phi^-: (x_0; \eta, s) = 0 \) if \( \eta_0 \neq l \). Next, assume that \( \eta \in \Sigma_A \) satisfies \( \eta_0 = l \).

There exists a unique billiard trajectory in \( \Omega \) with successive reflection points \( \hat{P}_j(x_0; \eta) \in \partial K_{\eta_j} (\infty < j < 0) \) such that \( x_0 = \hat{P}_{-1}(x_0; \eta) + t \nabla \varphi_{\eta_0} (\hat{P}_{-1}(x_0; \eta)) \) for some \( t > 0 \). In general the segment \([\hat{P}_{-1}(x_0; \eta), x_0] \) may intersect the interior of \( K_l \). If this is the case, set again \( \phi^-: (x_0; \eta, s) = 0 \). Otherwise, denote \( \hat{P}_0(x_0; \eta) = x_0 \) and for any \( j < 0 \) set

\[
 f_j^-(x_0; \eta) = \| \hat{P}_{j+1}(x_0; \eta) - \hat{P}_j(x_0; \eta) \|, \quad g_j^-(x_0; \eta) = \frac{1}{N-1} \ln \frac{G_{\eta, j}(\hat{P}_{j+1}(x_0; \eta))}{G_{\eta, j}(\hat{P}_j(x_0; \eta))},
\]

and define \( \phi^-: (x_0; \eta, s) = -s \sum_{j=-\infty}^{-1} f_j^-(\sigma(\eta)) - f_j^-(x_0; \eta) ] + \sum_{j=-\infty}^{-1} g_j^-(\sigma(\eta)) - g_j^-(x_0; \eta) \).

Next, similarly to [5], introduce the operator \( \mathcal{M}_{n,s}(x_0): C(\Sigma_A^+) \to C(\Sigma_A^+) \) by

\[
 (\mathcal{M}_{n,s}(x_0) v)(\xi) = \sum_{\sigma \in \mathbb{C}} e^{-\phi^-: (x_0; a^{n+1}(\gamma), s) - \chi(\sigma a^{n+1}(\gamma)) - s f(\sigma(\gamma)) + \tilde{g}(\gamma)} v(\gamma), \quad v \in C(\Sigma_A^+), \xi \in \Sigma_A^+.
\]

Introduce the function \( \psi_s(\xi) = e^{-x \psi(\xi)} h(\varphi(\xi)) \) if \( \xi_0 = 1 \) and \( \psi_s(\xi) = 0 \) otherwise and define the norms

\[
 \| f \|_{r, p} = \max_{x \in \Gamma} \max_{a^{(1)}, \ldots, a^{(p)} \in T_x \Gamma} \| (D_{a^{(1)}} \cdots D_{a^{(p)}} f)(x) \|, \quad \| f \|_{r, (p)} = \max_{0 \leq j \leq p} \| f \|_{r, j}
\]

where \( \| a^{(j)} \| = 1 \) for all \( j = 1, \ldots, p \).

**Theorem 3.1.** There exist global constants \( C > 0, c > 0, a \in (0, 1) \) and \( \theta \in (0, 1) \) depending only on \( K \) such that for any choice of \( l \in \{1, \ldots, k_0 \} \) the following holds: For any integers \( p \geq 1 \) and \( n \geq 1 \), any \( \xi \in \Sigma_A^+ \) with \( \xi_0 = l \) and any \( s \in \mathbb{C} \) with \( \Re(s) > s_0 - a \) we have

\[
 \left| \left( L_s^{\mu} \mathcal{M}_{n,s}(\cdot) \tilde{G}_s \tilde{v}_s \right)(\xi) - \sum_{l = n+2, j_{l+2} = l} u_j(\cdot, -is) \right|_{r, p} \leq C(\theta + ca)^n e^{C(\|s\|_{\Re(s)} + 1 + \| \psi \|_{r, 0} + \| \nabla \psi \|_{r, (1)})^{1/2}} \sum_{j=0}^p \left( \| \nabla \psi \|_{r, j} + \| \nabla \psi \|_{r, j+1} \right)^{j+1} \| h \|_{r, p-j}.
\]
A similar estimate holds for \( p = 0 \); in this case the sum in the right-hand side of (7) has to be replaced by \( [(|s| + \|\nabla \varphi\|_{\Gamma,(1)})\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)}] \). Applying the case \( p = 0 \), we reduce the convergence of \( w_{0,l} \) to the summability of the series \( \sum_{n=0}^{\infty} L_n^m \). On the other hand, for \( \tau \geq \sigma_0, |t| \geq 2 \) the estimates (6) yield
\[
\sum_{n=0}^{\infty} \| L_n^m \|_{\infty} \leq \frac{C}{1 - \rho^{[\log |t|]-1}} \sum_{j=0}^{[\log |t|]} e^{j Pr(-\tau \tilde{f} + \tilde{g})} \leq C_1 \max \{ \log |t|, |t|^{Pr(-\tau \tilde{f} + \tilde{g})} \}.
\]
Moreover, for \( \sigma_0 \) sufficiently close to \( \sigma_0 \) there exists \( 0 < \beta < 1 \) such that \( \| L_n^m M_{n,s} G \|_{\Gamma,0} \leq C |t|^{1+\beta} \) and we conclude that \( \| w_{0,l}(x, -it + t) \|_{\Gamma,0} \leq B |t|^{1+\beta} \). Exploiting the case \( p \geq 1 \), we obtain similar estimates for \( \| w_{0,l}(x, -it + t) \|_{\Gamma,p} \). Repeating this procedure, we complete the construction of \( U_M \).

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