PROBABILISTIC AVERAGES OF JACOBI OPERATORS

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Abstract. I study the Lyapunov exponent and the integrated density of states for general Jacobi operators. The main result is that questions about these, can be reduced to questions about ergodic Jacobi operators. Then, I apply this to $a(n) = 1$ and $b(n) = f(n^\rho \pmod{1})$ for $\rho > 0$ not an integer, and to obtain a probabilistic version of the Denisov–Rakhmanov–Remling Theorem.

1. Introduction

This paper is part of my effort to study the Schrödinger operator,

\begin{equation}
(Hu)(n) = u(n + 1) + u(n - 1) + f(n^\rho \pmod{1})u(n),
\end{equation}

where $u(-1) = 0$, $f : [0, 1] \to \mathbb{R}$ is a continuous function, and $\rho > 0$ is not an integer. I was intrigued by the fact that for $0 < \rho < 1$ and $f(0) \neq f(1)$, one has the absence of absolutely continuous spectrum and vanishing of the Lyapunov exponent on an interval. This is somewhat surprising since $n^\rho \pmod{1}$ has nice uniform distribution properties. We will discuss properties of these operators in Section 5. In particular, we resolve the discrepancy between the perturbative and numerical calculations of Griniasty and Fishman in [6] in Corollary 5.2 by proving an exact formula.

In order to understand the consequences and reasons for zero Lyapunov exponent, it turned out to be useful to work with general Jacobi operators, which are introduced by

\begin{equation}
J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})
\end{equation}

\begin{equation}
Ju(n) = a(n)u(n + 1) + b(n)u(n) + a(n - 1)u(n - 1),
\end{equation}

where $C_0^{-1} \leq a(n) \leq C_0$ and $-C_0 \leq b(n) \leq C_0$ for some $C_0 > 1$. We let $m_\pm(z)$ be the Weyl–Titchmarsh $m$ functions of the restrictions of $J$ to $\ell^2(\mathbb{Z}_\pm)$. $J$ is called reflectionless on $A$ if

\begin{equation}
m_+(t) = -m_-(t)
\end{equation}

for almost every $t \in A$. Denote by $L(E)$ the Lyapunov exponent of $J$, by $J^{(n)}$ the $n$-th translate of $J$, and by $\delta_J$ the Dirac measure. We have that

**Theorem 1.1.** Assume $L(E) = 0$ for almost every $E \in A$ and

\begin{equation}
\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{J^{(n)}}
\end{equation}

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in the weak * topology, then μ almost every Jacobi operator has absolutely continuous spectrum in the essential closure of A and is reflectionless there.

Remling has shown in [12] a very similar result. He has assumed that \( A \subseteq \sigma_{ac}(J) \), and concluded that every \( J \) in the \( \omega \) limit set of \( J^{(n)} \) is reflectionless on \( A \). Since
\[
\sigma_{ac}(J) \subseteq \{ E : \ L(E) = 0 \},
\]
the assumptions of the above theorem are weaker, but also the conclusion is. One can easily check that sparse potentials (as discussed in [12]) provide examples, that show that this distinction is sharp.

The above theorem will follow Theorem 4.1, which provides a formula for the Lyapunov exponent \( L(E) \) in terms of the Lyapunov exponents of the ergodic families arising in the ergodic decomposition of the limit measure \( \mu \).

Theorem 1.1 implies in particular the following result, which has to be thought of as a probabilistic analog of the Denisov–Rakhmanov–Remling theorem

**Theorem 1.2.** Let \( J \) be a Jacobi matrix with \( \sigma_{ess}(J) = [-2, 2] \) and \( L(E) = 0 \) for almost every \( E \in [-2, 2] \), then for every \( \varepsilon > 0 \)
\[
\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq n \leq N : \ |a(n) - 1| > \varepsilon \text{ or } |b(n)| > \varepsilon \} = 0.
\]

We will obtain a generalization of this theorem for finite gap operators (see Corollary 4.1) and the above claim for the Lyapunov exponent as corollaries of Theorem 4.1 in the next section.

In Section 2, I collect a few results on the space of all Jacobi operators and discuss measures on that space. In Section 3 I discuss results about ergodic Schrödinger operators. The main results are stated in Section 4. The application to the potential \( V(n) = f(n^\rho \mod 1) \) is examined in Section 5. Section 6 proofs some facts about the Lyapunov exponent for general Jacobi matrices and provides the proof of Theorem 4.1 which has to be considered the main result of this paper.

2. **Probabilistic Averages of Jacobi Matrices**

Given bounded sequences \( a : \mathbb{Z} \to (0, \infty), b : \mathbb{Z} \to \mathbb{R} \), we introduce the associated Jacobi operator \( J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) by
\[
(Ju)(n) = a(n)u(n + 1) + b(n)u(n) + a(n - 1)u(n - 1).
\]
We will often identify \( J \) with \((a, b)\). Fix now \( C_0 > 1 \), and introduce \( \mathcal{J} \) as the set of all Jacobi operators, such that \((a, b)\) satisfy the inequalities
\[
\frac{1}{C_0} \leq a(n) \leq C_0, \quad -C_0 \leq b(n) \leq C_0.
\]
We endow \( \mathcal{J} \) with the strong operator topology, which just corresponds to pointwise convergence on the level of the sequences \((a, b)\). We remark that \( \mathcal{J} \) is now a compact metric space, where an explicit example of the metric is
\[
d(J, \tilde{J}) = \sum_{n \in \mathbb{Z}} \frac{1}{2|n|} (|\langle \delta_n, (J - \tilde{J})\delta_n \rangle| + |\langle \delta_n, (J - \tilde{J})\delta_{n+1} \rangle|)
\]
\[
= \sum_{n \in \mathbb{Z}} \frac{1}{2|n|} (|b(n) - \tilde{b}(n)| + |a(n) - \tilde{a}(n)|).
\]
Denote by $S$ the shift operator on $\ell^2(\mathbb{Z})$ that is
\begin{equation}
(Su)(n) = u(n + 1).
\end{equation}
Introduce $\hat{S} : \mathcal{J} \to \mathcal{J}$ by
\begin{equation}
\hat{S}J = S^* JS,
\end{equation}
and denote $J^{(n)} = \hat{S}^n J$.

We will denote by $\mathcal{M}^1$ the space of all Borel probability measures on $\mathcal{J}$. For a Jacobi matrix $J$, we introduce the corresponding Dirac measure $\delta_J$ by
\begin{equation}
\delta_J(A) = \begin{cases} 1 & J \in A \\ 0 & J \notin A. \end{cases}
\end{equation}
We will be mainly interested in the limit points of the averages of the Dirac measures of the translates of a Jacobi matrix. For this, introduce for a Jacobi matrix $J$ and an integer $N \geq 1$ the average
\begin{equation}
A_{N,J} = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{J^{(n)}},
\end{equation}
which will be a measure in $\mathcal{M}^1$.

We denote by $\omega(J)$ the (topological) $\omega$ limit set of the translates of $J$, that is
\begin{equation}
\omega(J) = \{ \tilde{J} \in \mathcal{J} : \exists n_j \to \infty : \tilde{J} = \lim_{j \to \infty} J^{(n_j)} \}.
\end{equation}
We recall that the weak * topology on $\mathcal{M}^1$ gives rise to the notion of convergence $\mu_n \to \mu$ if for every continuous function $f : \mathcal{J} \to \mathbb{R}$ we have
\begin{equation}
\lim_{n \to \infty} \int f(J) \mu_n(J) = \int f(J) \mu(J).
\end{equation}
We remark

**Lemma 2.1.** $\mathcal{M}^1$ is a compact and metrizable space in the weak * topology.

We write $\text{supp}(\mu)$ for the support of a measure $\mu$, which is the smallest closed set $A$, such that $\mu(A) = 1$. Furthermore, we call a measure $\mu \in \mathcal{M}^1$ shift invariant, if for any Borel set $A \subseteq \mathcal{J}$
\begin{equation}
\mu(A) = \mu(\hat{S}A).
\end{equation}

**Lemma 2.2.** If $\mu = \lim_{j \to \infty} A_{N_j,J}$ in the weak * topology, then
\begin{equation}
\text{supp}(\mu) \subseteq \omega(J),
\end{equation}
and $\mu$ is shift invariant.

This implies the following consequence

**Lemma 2.3.** Let $\mu = \lim_{j \to \infty} A_{N_j,J}$ for some $N_j \to \infty$. Then for $\mu$ almost every $\tilde{J}$, we have that
\begin{equation}
\sigma_{\text{ess}}(\tilde{J}) \subseteq \sigma_{\text{ess}}(J).
\end{equation}

**Proof.** Follows from the fact, that the inclusion holds for every $\tilde{J} \in \omega(J)$. \hfill \Box

We will need the following result from measure theory, known as Portmanteau-Theorem (see e.g. [3] Theorem VIII.4.10.).
Theorem 2.4. \( \mu_n \to \mu \) in the weak * topology, is equivalent to that for every Borel set \( B \) with \( \mu(\partial B) = 0 \), we have that

\[
\lim_{n \to \infty} \mu_n(B) = \mu.
\]

We say that a sequence \( J^{(n)} \) converges to a set \( A \subseteq J \) along \( N_j \to \infty \) in probability if for every \( \varepsilon > 0 \)

\[
\lim_{j \to \infty} \frac{1}{N_j} \# \{ 1 \leq n \leq N_j : d(J^{(n)}, A) \geq \varepsilon \} = 0.
\]

We have the following result.

Lemma 2.5. Let

\[
\mu = \lim_{j \to \infty} A_{N_j, J},
\]

and \( S \) a support for \( \mu \). Then \( J^{(n)} \) converge to \( S \) along \( N_j \) in probability.

Proof. For \( \varepsilon > 0 \) apply the last theorem to \( B = \{ J : \ \text{dist}(S, J) \geq \varepsilon \} \) to conclude that \( A_{N_j, J}(B) \to 0 \) as \( j \to \infty \). By rewriting, one sees that this is exactly the definition of convergence in probability. \( \square \)

For \( \Lambda \subseteq \mathbb{Z} \), denote by \( J_\Lambda \) the restriction of \( J \) to \( \ell^2(\Lambda) \). We will call \( f : J \to \mathbb{R} \) compactly supported, if there is a finite set \( \Lambda \subseteq \mathbb{Z} \) such that

\[
f(J) = f(\hat{J}),
\]

whenever \( J_\Lambda = \hat{J}_\Lambda \). We have that

Lemma 2.6. For \( \mu_n, \mu \in \mathcal{M}^1 \), we have

\[
\mu_n \to \mu
\]
in the weak * topology, if and only if for every compactly supported \( f \)

\[
\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu.
\]

Proof. It clearly suffices to show that (2.18) implies weak * convergence. So given \( f : J \to \mathbb{R} \) and \( \varepsilon > 0 \), we have that show that there is an \( N \geq 1 \)

\[
\forall n \geq N : \ | \int f \, d\mu_n - \int f \, d\mu | \leq \varepsilon.
\]

Since, \( f \) is continuous, we may find for each \( J \in J \) an integer \( K \geq 1 \) such that \( |f(J) - f(\hat{J})| \leq \frac{\varepsilon}{2} \), where

\[
\hat{J} \in U_{J,K} = \{ \hat{J} : \ |J_{[-K,K]} - J_{[-K,K]}| \}.
\]

Since the \( U_{J,K} \) are open sets and \( J \) is compact, finitely many of them cover \( J \). In particular, we can find a maximal necessary \( K \). Hence, we may approximate \( f \) by \( \hat{f} \), which is supported on \( [-K,K] \) such that \( \|f - \hat{f}\|_\infty < \frac{\varepsilon}{2} \). Now the claim follows by (2.18). \( \square \)
3. Families of ergodic Schrödinger operators

In this section, we collect basic facts about ergodic Jacobi operators. For the Jacobi operator background see [3] or Section 7 of [13]. For the measure theoretic part, see [7] or [9].

Denote by $M^1_S$ the set of all shift invariant measures. One can check that $M^1_S$ will be a convex set, and in particular we will write $E$ for its extremal points.

It is a known fact in ergodic theory, that $E$ are exactly the ergodic measures of the dynamical system $(\mathcal{J}, \hat{S})$, where

$$\hat{S}(J) = S^*JS.$$  (3.1)

Furthermore, it follows from Choquet’s theorem that one can write any measure $\mu \in M^1_S$ as a generalized convex combination. That is, there exists a measure $\alpha$ on $E$ such that for any $f : J \to \mathbb{R}$ continuous, one has

$$\int f d\mu = \int \left( \int f d\beta \right) d\alpha(\beta).$$  (3.2)

We call an ergodic measure $\beta \in E$ on the space of Jacobi operators $\mathcal{J}$ a family of ergodic Jacobi operators. One knows from general fact, that there is a set $\Sigma(\beta)$ such that

$$\Sigma(\beta) = \sigma(J)$$  (3.3)

for $\beta$ almost every $J$. We may define its Lyapunov exponent by

$$\gamma_\beta(z) = \lim_{N \to \infty} \frac{1}{N} \int_\mathcal{J} \log \left\| \prod_{n=N-1}^0 \frac{1}{a(n)} \left( \begin{array}{cc} z - b(n) & 1 \\ a(n)^2 & 0 \end{array} \right) \right\| d\beta(J),$$  (3.4)

and its integrated density of states by

$$k_\beta(E) = \int_\mathcal{J} \langle \delta_0, \chi_{(-\infty,E)}(J)\delta_0 \rangle d\beta(J).$$  (3.5)

We note that this quantity is equal to

$$k_\beta(E) = \lim_{N \to \infty} \frac{1}{N} \int_\mathcal{J} \text{tr}(P_{(-\infty,E)}(J_{[0,N-1]})) d\beta(J).$$  (3.6)

We will need the following result of Kotani theory.

**Theorem 3.1.** Denote by $\mathcal{Z}$ the essential closure of the set

$$\{ E \in \mathbb{R} : \gamma_\beta(E) = 0 \}$$  (3.7)

then $\beta$ almost every $J$ has purely absolutely continuous spectrum on $\mathcal{Z}$ and it is reflectionless there.

We define

$$\log(A_\beta) = \int_\mathcal{J} \log(a(0)) d\beta(J).$$  (3.8)

Furthermore $m_+(z, J) = \langle \delta_0, (J_+ - z)^{-1}\delta_0 \rangle$, where $J_+$ is the restriction of $J$ to $\ell^2(\mathbb{Z}_+)$. 

Lemma 3.2. We have that
\begin{equation}
\gamma(z) = \log(A^{-1} - 1) - \int P \log |m_+ (z, J)| d\beta(J)
\end{equation}
for every \( z \in \mathbb{C} \).

(3.10) is known as the Thouless formula. It implies that
\begin{equation}
\lim_{\varepsilon \to 0} \gamma(E + i\varepsilon) = \gamma(E)
\end{equation}
for every \( E \in \mathbb{R} \) by monotone convergence.

We now make the connection to the usual definition of ergodic Jacobi operators (see also Section 2 in \([1]\)). Let \((\Omega, T, \mu)\) be an ergodic dynamical system, and \(a: \Omega \to (0, \infty)\) and \(b: \Omega \to \mathbb{R}\) are measurable functions satisfying
\begin{equation}
\frac{1}{C_0} \leq a(\omega) \leq C_0, \quad -C_0 \leq b(\omega) \leq C_0
\end{equation}
for almost every \( \omega \). Then we can define a map
\begin{equation}
f: \Omega \to J
\end{equation}
by \( f(\omega) \) being the Jacobi operator with coefficients
\begin{equation}
\{(a(T^n \omega), b(T^n \omega))\}_{n \in \mathbb{Z}}.
\end{equation}
Introduce a measure \( \beta \) on \( J \) given by
\begin{equation}
\beta(A) = \mu(f^{-1}(A))
\end{equation}
for Borel subsets \( A \subseteq J \). Then the usual definitions of the Lyapunov exponent and the integrated density of states will just be \( \gamma_\beta \) and \( k_\beta \).

4. Statement of the results

For a Jacobi operator \( J \in J \) and a sequence \( N_j \to \infty \), we introduce the Lyapunov exponent by
\begin{equation}
\mathcal{L}(z, J, \{N_j\}) = \limsup_{j \to \infty} \frac{1}{N_j} \log \left\| \prod_{n=0}^{N_j-1} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & 1 \\ a(n)^2 & 0 \end{pmatrix} \right\|
\end{equation}
where \( J = (a, b) \), and the integrated density of states by
\begin{equation}
k(E, J, \{N_j\}) = \lim_{j \to \infty} \frac{1}{N_j} \text{tr}(P(-\infty, E)(J_{[0, N_j-1]})),
\end{equation}
where the limit is assumed to exist.

Theorem 4.1. Assume that we have a measure \( \alpha \) on \( \mathcal{E} \) such that for \( N_j \to \infty \)
\begin{equation}
\lim_{j \to \infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} \delta_{\mathcal{E}(n)} = \int_\mathcal{E} \beta d\alpha(\beta)
\end{equation}
then
\begin{equation}
\mathcal{L}(z, J, \{N_j\}) = \int_\mathcal{E} \gamma_\beta(z) d\alpha(\beta)
\end{equation}
for $\text{Im}(z) > 0$ and almost every $z \in \mathbb{R}$, and

\begin{equation}
(4.5) \quad k(E, J, \{N_j\}) = \int_E k_\beta(E)d\alpha(\beta).
\end{equation}

The statement is actually stronger, since one may replace the almost every by quasi-every (in the sense of potential theory). Theorem 1.1 is now an easy corollary.

**Corollary 4.2.** Let $A \subseteq \mathbb{R}$ be a set of positive measure. Assume that

\begin{equation}
(4.6) \quad \mathcal{T}(E, \{N_j\}, J) = 0
\end{equation}

for almost every $E \in A$. Then $\mu$ almost every $J$ is reflectionless on $A$.

**Proof.** Since $\gamma_\beta(E) \geq 0$ for every $\beta$, we can conclude by (4.4) that for $\alpha$ almost every $\beta$, we have

\begin{equation}
\gamma_\beta(E) = 0
\end{equation}

for almost every $E \in A$. The result now follows by Theorem 3.1. \qed

We also obtain the following probabilistic version of the Denisov–Rakhmanov–Remling theorem (see [4], [11], [12]). For this recall, that a set $\epsilon$ is called a finite gap set, if

\begin{equation}
(4.7) \quad \epsilon = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g+1}, E_{2g+2}].
\end{equation}

Furthermore, one has a finite dimensional torus $T(\epsilon)$ of reflectionless Jacobi-operators which have spectrum $\epsilon$. We have that

**Corollary 4.3.** Let $\epsilon$ be a finite gap set, and assume that

\begin{equation}
\sigma_{\text{ess}}(J) = \epsilon
\end{equation}

and $\mathcal{T}(E, \{N_j\}, J) = 0$ for almost every $E \in \epsilon$, then

\begin{equation}
J^{(n)} \rightarrow T(\epsilon)
\end{equation}

in probability along $N_j$, where $T(\epsilon)$ denotes the isospectral torus.

**Proof.** Assume there is a subsequence of $N_j$ such that convergence in probability does not hold. By passing to a further subsequence of $N_j$, we may assume that (4.3) holds. Since $\gamma_\beta(z) \geq 0$ everywhere, it follows from our result that for $\alpha$ almost every $\beta$, we have that

\begin{equation}
\Sigma(\beta) = \epsilon
\end{equation}

and that $\gamma_\beta = 0$ on $\epsilon$. Hence, Kotani’s theory implies that these operators are reflectionless on $\epsilon$, which implies in turn that $\beta$ is supported on $T(\epsilon)$ (see e.g. Section 8 in [17]). This is a contradiction by Lemma 2.5. \qed

5. **The family of potentials**

In this section, we examine the family of Schrödinger operators given by (1.1) in some detail. Introduce for a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ and $r < \rho < r + 1$, where $r$ is a nonnegative integer, the sequences

\begin{equation}
(5.1) \quad a(n) = 1, \quad b(n) = f(n^\rho \pmod{1}).
\end{equation}

Denote by $J$ the associated Jacobi operator. For $\alpha \in [0, 1] \setminus \mathbb{Q}$, introduce the skew-shift $T_\alpha : [0, 1]^r \rightarrow [0, 1]^r$ by

\begin{equation}
(T_\alpha \omega)_k = \begin{cases}
\omega_0 + \alpha & \text{if } k = 0 \\
\omega_k + \omega_{k-1} & \text{if } 1 \leq k \leq r - 1.
\end{cases}
\end{equation}
Similarly as in the last part of Section 3 we let $\beta_\alpha$ be the measure on $\mathcal{J}$ given by the pushforward of the Lebesgue measure on $[0,1]^r$ under

$$0,1]^r \ni \omega \mapsto \{1, f((T^n\omega)_r)\}_{n \in \mathbb{Z}} \in \mathcal{J}.$$  

Lemma 5.1. We have that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{J(n)} = \int_0^1 \beta_\alpha d\alpha$$

in the weak* topology.

Proof. By Lemma 2.6 it suffices to check convergence for compactly supported functions $g$. Since, every such $g$ will be a continuous function of $\{b(n)\}_{n=1}^K$ for some $K \geq 1$ it suffices to check that these converge. Since, $f$ is also continuous. It suffices to show that $\{(n+j)^\rho\}_{j=-K}^K$ has the same distribution in $[0,1]^{2K+1}$ as the orbits of the skew-shifts would as $n \to \infty$.

Furthermore, by Lemma 2.1 in [8] and an easy argument we see that both $(n+j)^\rho$ and $(T^{n+j}\omega)_r$ are essentially given by degree $r$ polynomials, and thus uniquely determined by $\{(n+j)^\rho\}_{j=0}^r$ and $\{(T^{n+j}\omega)_r\}_{j=0}^r$.

Now Lemma 2.3 in [8] implies that the coefficients of the first polynomial are uniformly distributed, and a quick computation shows the same for the skew-shift, finishing the proof.

This lemma combined with Theorem 4.1 implies

Corollary 5.2. For almost every $E$, we have that

$$L(E) = \int_0^1 \gamma_{\beta_\alpha}(E) d\alpha.$$  

This corollary resolves the discrepancy between the numerical and perturbation theoretical computations in [13] and shows in particular that the Lyapunov exponent only depends on the integer part of $\rho$.

In particular, in the case of $r = 0$, that is $0 < \rho < 1$, one can compute that $\gamma_{\beta_\alpha}(E) = 0$ for exactly

$$E \in [-2 + f(\alpha), 2 + f(\alpha)].$$

Hence, we see that $L(E) = 0$ for

$$E \in [-2 + \max(f), 2 + \min(f)],$$

which was first observed by Simon and Zhu in [14] for continuum Schrödinger operators. We observe further spectral properties in the following result.

Theorem 5.3. We have that

(i) Stolz: If $f$ extends to a smooth function on the circle, then $\mathcal{J}$ has purely absolutely continuous spectrum in $[-2 + \max(f), 2 + \min(f)]$.

(ii) Remling’s Oracle: If $f(0) \neq f(1)$, then the absolutely continuous spectrum of $\mathcal{J}$ is empty.

Proof. Part (i) is [15]. Part (ii) follows from Remling’s Oracle Theorem, which is found in [12].
6. THE INTEGRATED DENSITY OF STATES AND THE LYAPUNOV EXPONENT

In this section, we will prove Theorem 4.1. For this, we have to discuss some further properties of the Lyapunov exponent.

We will now assume that we are given a fixed Jacobi operator \( J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \). We denote by \( J_\Lambda \) its restriction to \( \ell^2(\Lambda+) \), \( \mathbb{Z} = \{0, 1, 2, 3, \ldots\} \), and by \( J_\Lambda \) its restriction to \( \ell^2(\Lambda) \) for \( \Lambda \subseteq \mathbb{Z} \) an interval. Denote by \( E_j(\Lambda) \) an increasing enumeration of the eigenvalues of \( J_\Lambda \), introduce the density of states measure \( \nu_n \) of \( J_{[0,n-1]} \) by

\[
\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{E_j([0,n-1])}.
\]

For \( \text{Im}(z) > 0 \), introduce the Weyl–Titchmarsh \( m \) function

\[
m_+(z, J) = \langle \delta_0, (J_+ - z)^{-1} \delta_0 \rangle.
\]

We will show the following theorem, which is essential in the proof of Theorem 4.1. Similar results can be found in Poltoratski–Remling [10].

**Theorem 6.1.** Given a sequence \( N_j \to \infty \), assume that

\[
\lim_{j \to \infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} \delta_{J(n)} = \mu.
\]

in the weak * topology on \( \mathcal{M}^1 \). Then

(i) **The integrated density of states measures converge in the weak * topology**

\[
\lim_{j \to \infty} \nu_{N_j} = \nu.
\]

(ii) **For \( \text{Im}(z) > 0 \) and \( A \) as defined in (6.7), we have that**

\[
\mathcal{T}(z, \{N_j\}) = \log(A^{-1}) - \int J \log|m_+(z, J)|d\mu(J).
\]

(iii) **For almost every \( E \in \mathbb{R} \), we have that**

\[
\mathcal{T}(E, \{N_j\}) = \lim_{\varepsilon \to 0} \mathcal{T}(E + i\varepsilon, \{N_j\}).
\]

Since the map \((a, b) \mapsto \log(a(0))\) is continuous, we see that (6.3) implies

\[
A := \exp \left( \int J \log(a(0))d\mu(J) \right) = \lim_{j \to \infty} \exp \left( \frac{1}{N_j} \sum_{n=0}^{N_j+1} \log(a(n)) \right),
\]

where \( A \) is the constant in (6.5). We are now ready for

**Proof of the Theorem 4.1.** We first observe that

\[
\log(A^{-1}) = \int E \log(A^{-1}_\beta) d\alpha(\beta).
\]
We may compute for almost every $E$, that
\[
L(E) = \lim_{\varepsilon \to 0} L(E + i\varepsilon)
\]
by (6.6)
\[
= \lim_{\varepsilon \to 0} \left( \log(A^{-1}) - \int_{\mathcal{J}} \log |m_+ (E + i\varepsilon, J)| d\mu(J) \right)
\]
by (6.3)
\[
= \lim_{\varepsilon \to 0} \left( \int_{\varepsilon} \log(A^{-1}) - \left( \int_{\mathcal{J}} \log |m_+ (E + i\varepsilon, J)| d\beta(J) \right) d\alpha(\beta) \right)
\]
by (3.9)
\[
= \int_{\varepsilon} \gamma_\beta(E) d\alpha(\beta)
\]
by (3.11).
This implies the first claim. The second claim follows by Thouless’ formula.

We now proceed to prove Theorem 6.1. Introduce by $s$ and $c$ the sine and cosine solution of $J$ (as a formal difference equation), satisfying the initial conditions
\[
\begin{pmatrix} c(z, 0) \\ s(z, 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(6.8)
We observe that, we have that
\[
\begin{pmatrix} c(z, n) \\ s(z, n) \end{pmatrix} = \prod_{j=n-1}^0 \frac{1}{a(j)} \begin{pmatrix} z - b(j) & 1 \\ a(j)^2 & 0 \end{pmatrix}.
\]
(6.9)
We note that
\[
c(z, n) = \frac{\det(z - J_{[0, n-1]})}{\prod_{j=0}^{n-1} a(j)}, \quad s(z, n) = \frac{\det(z - J_{[1, n-1]})}{\prod_{j=0}^{n-1} a(j)}.
\]
(6.10)
For $\text{Im}(z) > 0$, we denote by $u_+(z, n)$ the solution of
\[
Hu_+ = zu_+, \quad u_+ \in \ell^2(Z_+), \quad u_+(z, -1) = 1.
\]
(6.11)
We then have that $u_+(z, 0) = -a(0)m_+ (z, J)$. We obtain that
\[
u_+(z, N) = (-1)^N \prod_{n=0}^N a(n)m_+ (z, J^{(n)}).
\]
(6.12)
Hence, we obtain that

**Lemma 6.2.** Assume (6.3), then for $\text{Im}(z) > 0$
\[
\mathcal{L}(z, \{N_j\}) = -\lim_{j \to \infty} \frac{1}{N_j} \log |u_+(z, N_j)|
\]
by (6.3)
\[
= -\lim_{j \to \infty} \frac{1}{N_j} \log \left( |u_+(z, N_j) + |u_+(z, N_j + 1)| \right)^2
\]
\[
\quad \quad = \log(A^{-1}) - \int_{\mathcal{J}} \log |m_+(z, (a, b))| d\mu(a, b).
\]
(6.13)
**Proof.** Define $L_+(z, \{N_j\})$ as the limit
\[
L_+(z, \{N_j\}) = -\lim_{j \to \infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} \log |a(n)m_+ (z, J^{(n)})|,
\]
Lemma 6.3. We have that shows that (6.15) \( c \) (6.13). In order to see the remaining inequality, observe that \( m \) (6.14) \( |L| \) we see that \( c \) which shows the claim. \( \square \)

As in the proof of Lemma 6.2, one can now show that

\[
\lim_{j \to \infty} \frac{1}{N_j} \log |c(z, N_j)| = \lim_{j \to \infty} \frac{1}{N_j} \log \sqrt{|c(z, N_j)|^2 + |c(z, N_j)|^2},
\]

which implies for \( \text{Im}(z) > 0 \) by the Ruelle–Osceledec theorem

\[
\lim_{j \to \infty} \frac{1}{N_j} \log |u_+(z, N_j)| = \lim_{j \to \infty} \frac{1}{N_j} \log |c(z, N_j)|,
\]

since the cosine solution \( c \) can never decay, since \( J \) is self-adjoint. In particular, the limit on the right hand side of (6.16) exists for every \( z \) with \( \text{Im}(z) > 0 \).

Lemma 6.4. We have that

\[
\frac{1}{n} \log |c(z, n)| = \int \log |z - t| dv_n - \frac{1}{n} \sum_{j=0}^{n-1} \log |a(j)|.
\]

Proof. This is a consequence of (6.10) and (6.11). \( \square \)
Lemma 6.5. Assume \((6.3)\), then
\begin{equation}
\nu = \lim_{j \to \infty} \nu_{N_j}
\end{equation}
exists, and for \(\text{Im}(z) > 0\)
\begin{equation}
\lim_{j \to \infty} \frac{1}{N_j} \log |c(z, N_j)| = \log(A^{-1}) + \int \log |t - z| dv.
\end{equation}
Furthermore, \((6.19)\) even holds for almost every \(z \in \mathbb{R}\).

Proof. \((6.18)\) follows from \((6.17)\) and the fact that the family of functions \(t \mapsto \log |t - z|\) for \(\text{Im}(z) > 0\) separates points on the real axis. For the last statement, observe that \((6.17)\) remains valid for \(z \in \mathbb{R}\), and then use Theorem A.7. in [13]. \(\square\)

This shows (i) of Theorem 6.1. Next, we observe that

Lemma 6.6. For every \(E \in \mathbb{R}\), we have that
\begin{equation}
\mathcal{T}(E, \{N_j\}) \leq \log(A^{-1}) + \int \log |t - E| dv
\end{equation}

Proof. First observe that \(\mathcal{T}(z, \{N_j\})\) is a submean function of \(z\), and \(z \mapsto \log(A^{-1}) + \int \log |t - z| dv\) is subharmonic. This implies the claim by Theorem 1.1. in [2]. \(\square\)

We now come to

Proof of Theorem 6.1 (iii). This is a consequence of the last lemma, and the fact that
\begin{equation}
|c(E, N_j)| \leq \left\| \prod_{n=N_j-1}^{0} \frac{1}{a(n)} \begin{pmatrix} z - b(n) & 1 \\ a(n)^2 & 0 \end{pmatrix} \right\|
\end{equation}
by \((6.3)\). \(\square\)

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