The Holevo bound and Landauer’s principle

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Landauer’s principle states that the erasure of information generates a corresponding amount of entropy in the environment. We show that Landauer’s principle provides an intuitive basis for Holevo bound on the classical capacity of a quantum channel.

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I. INTRODUCTION

Landauer’s principle [1] states that the erasure of classical information leads to an increase in the entropy of the environment by at least the same amount. The importance of this principle is the insight that it is not the act of obtaining information that necessarily generates heat and therefore entropy, but the erasure of information. This profound insight has led to the resolution of the problem of Maxwell’s demon by Bennett [2]. Here the missing entropy is generated when the demon’s memory is erased. Clearly Landauer’s principle provides a bridge that links classical information theory to thermodynamics. Recently, however, it has been shown that one can also use Landauer’s principle to connect the entropy of erasure and the efficiency of entanglement purification [3].

Here we demonstrate that there is also a connection between Landauer’s principle and the Holevo bound which limits the classical capacity of a quantum channel. In classical communication via quantum states, we encode classical signals in possibly mixed, non-orthogonal quantum states and send those to a receiver. The receiver then has the task to deduce from those quantum states the original classical message. The maximal information that the receiver can obtain is limited by the Holevo bound [4]. It has also been shown that this bound can be achieved asymptotically [5].

The purpose of this paper is to understand the Holevo bound in terms of Landauer’s principle. This approach is more intuitive than the unavoidably very technical, rigorous mathematical proofs. The hope is that this new approach may stimulate ideas that may lead to a better understanding of the more intricate problem of the quantum capacity of a quantum channel, a problem which is not yet fully understood (see however [6]).

This paper is organized as follows. In the next section we will present a general method for the erasure of information which was introduced by Lubkin [7] and later generalized by Vedral [8] to non-commuting variables. In section III we will first briefly introduce the Holevo bound and then proceed to show how the Holevo bound can be obtained from Landauer’s principle.

II. HOW TO ERASURE INFORMATION?

Landauer’s principle states that the erasure of a given amount of information generates at least the same amount of entropy. In the following we will present a physical scheme for the erasure of information (due to Lubkin [7]) in its form for non-commuting states as described in [3]. This scheme has the nice feature that it can easily be made optimal in the sense that the entropy of erasure can be made equal to the amount of information that has been erased.

Let us consider a measurement apparatus $\mathcal{M}$ that is initially in a pure state (e.g. the ground state of $\mathcal{M}$) and then interacts with a system. This interaction results in a measurement, i.e. after the interaction the apparatus will be in one of a set of pure states $\{|m_i\rangle\}$ with probabilities $\{p_i\}$. The amount of classical information the apparatus $\mathcal{M}$ has acquired during the measurement is given by the mutual information between system and apparatus which turns out to be equal to the von Neumann entropy $S(\rho) = \sum_i p_i S|m_i\rangle\langle m_i|$, where $S|m_i\rangle\langle m_i|$ is the average state of the apparatus [9]. If we want to reuse the apparatus $\mathcal{M}$ for another measurement, then this information has to be erased, i.e. we have to return $\mathcal{M}$ to its original pure state. This can be done with arbitrary precision by placing $\mathcal{M}$ into contact with a heat bath of temperature $T$ such that in thermal equilibrium the state of the apparatus is described by the Boltzmann distribution

$$\omega = Z^{-1} e^{-\beta H},$$

where $\beta = 1/kT$, $H$ is the Hamilton operator of the apparatus and $Z = \text{tr} e^{-\beta H}$. To erase the apparatus we place
it into contact with the heat bath. The total change of entropy of erasure is given by the sum of the changes of entropy both in the measurement apparatus as well as the heat bath. After the measurement, and prior to the erasure, the apparatus is in one of the pure states $|m_i\rangle$. Therefore it will always increase its entropy by evolving into the state $\omega$ given in Eq. (1). This change of entropy of the apparatus is given by

$$\Delta S_M = S(\omega) . \hspace{2cm} (2)$$

Now we need to evaluate the change of entropy of the heat bath $\Delta S_B$. The easiest way to do so is by determining the heat change of the bath. This is just the negative of the change of heat in the measurement apparatus $\mathcal{M}$, for which we find

$$kT\Delta S_B = tr\{H(\rho - \omega)\}$$

$$= -kT tr\{\rho - \omega\} log Z\omega$$

$$= -kT tr\{\rho - \omega\} log \omega . \hspace{2cm} (3)$$

Therefore, the total change of entropy $\Delta S_{tot}$ of measurement apparatus and heat bath together is given by

$$\Delta S_{tot} = \Delta S_M + \Delta S_B$$

$$= S(\omega) - tr\{\rho - \omega\} log \omega$$

$$= -tr\{\rho log \omega\} . \hspace{2cm} (4)$$

We can see that the amount of erased information $S(\rho)$ is never larger than the entropy of erasure, i.e.

$$S(\rho) \leq \Delta S_{tot} = -tr\{\rho log \omega\} \hspace{2cm} (5)$$

because the relative entropy $S(\rho||\omega) = tr\{\rho log \rho - \rho log \omega\}$ is always positive. If the temperature of the heat bath is chosen such that for the thermal equilibrium state of the apparatus in Eq. (1) we have $\omega = \rho$, then the entropy of erasure is exactly equal to the information that has been erased from the apparatus and the erasure is optimal.

III. THE HOLEVO BOUND

A. Classical Information via a quantum channel

The transmission of classical information via a quantum channel proceeds in the following way. Initially the sender, Alice, holds a long classical message. She encodes letter $i$ (which appears with probability $p_i$) of this message into a possibly mixed quantum state $\rho_i$. These quantum states are handed over to the receiver, Bob, who then has the task to infer Alice’s classical message from these quantum states. The upper bound for the capacity for such a transmission, i.e. the information $I$ that Bob can obtain about Alice’s message per sent quantum state, is given by the Holevo bound

$$I \leq I_H = S(\rho) - \sum_i p_i S(\rho_i) . \hspace{2cm} (6)$$

In fact, equality can be achieved for large message-blocksizes as has been proven by Holevo. The aim of the next section is to show how one can justify Holevo's bound from the assumption of the validity of Landauer’s principle.

B. Holevo’s bound from Landauer’s principle

The idea behind the derivation of the Holevo bound from Landauer’s principle is to determine an upper bound on the entropy that is generated when Bob erases the information that the message system carries in its state $\rho_i$. In this way we directly obtain an upper bound on the information received by Bob. To this end we consider different ways for erasing the information that Alice has originally encoded. The two methods of erasure are schematically presented in Fig. 1. Step 2 of procedure (2) corresponds to the erasure of Bob’s information. In the following the accompanying entropy of erasure will be computed as the difference of the entropies of erasure in procedure (1) and the first step of procedure (2).

![FIG. 1. A letter of a classical message is encoded with probability $r^i_\alpha$ into a pure quantum state $|\phi^i_\alpha\rangle$. The information about the original message contained in this encoding can be deleted in two ways. (1) directly by placing the systems into contact with a heat bath in state $\rho$, or (2) by first using heat baths in state $\rho_i$ and then a heat bath in state $\rho$.](image-url)
1. Direct erasure

A message letter $i$ which appears with probability $p_i$, is encoded by Alice with probability $r_{i\alpha}$ in state $|\phi_{i\alpha}\rangle$. We will now delete the information encoded in these pure state by bringing them into contact with a heat bath. We chose the temperature of this heat bath such that the thermal equilibrium state of the message system is $\rho$. This ensures that the erasure is optimal, in the sense that it produces the smallest possible amount of heat. This procedure is illustrated as part (1) in Fig. 1. Following the analysis of Lubkin’s erasure in section II, the entropy of erasure is given by

$$\Delta S^{(2)}_{er} = - \sum_i p_i tr\{\rho_i \log \rho\} = S(\rho) .$$

(7)

Note that all information has been deleted because now every quantum system is in the same state $\rho$ so that there is no correlation between the original letter $i$ and the encoded quantum state left after the erasure!

2. Two step erasure

First step: We begin by performing a partial erasure on the encoded quantum systems (see step 1 of procedure (2) in Fig 1). For a fixed $i$ which appears with probability $p_i$, we place the encoded pure states into contact with a heat bath. The temperature $T$ of the heat bath is chosen such that the thermal equilibrium state of the message system is $\rho_i$. Again this choice ensures that the erasure is optimal. According to our analysis of the Lubkin erasure in section II, the entropy of erasure is then found to be

$$\Delta S^{(1)}_{er} = - \sum_i p_i \sum_\alpha tr\{r_{i\alpha}^i |\phi_{i\alpha}^i\rangle \langle \phi_{i\alpha}^i| \log \rho_i\}$$

$$= - \sum_i p_i tr\{\rho_i \log \rho_i\}$$

$$= \sum_i p_i S(\rho_i) .$$

(8)

After this first step in the erasure procedure there is still some information left in the physical systems. The letter $i$ of the classical message is correlated with the state $\rho_i$ of the quantum system. In fact, this is exactly the situation in which Bob is after he received a message which is encoded as in described in subsection III A. To obtain a bound on the information that Bob is now holding, we need to find a bound on the entropy of erasure of his quantum systems.

Second step: In order to carry out step 2 of procedure (2) we place each of Bob’s systems, which is in one of the states $\rho_i$ with probability $p_i$, into contact with a heat bath such that the thermal equilibrium state of the message system is $\rho$. As the average state of the systems is $\rho = \sum_i p_i \rho_i$, we expect the erasure to be optimal again. We can see easily (Fig. 1) that this second step of erasure, just generates an amount of entropy that is the difference between the entropy of erasure of the first procedure and that of the first step of the second procedure. Therefore the entropy of erasure of Bob’s systems which are in one of the states $\rho_i$’s is

$$\Delta S_{er} = \Delta S^{(2)}_{er} - \Delta S^{(1)}_{er}$$

$$= S(\rho) - \sum_i p_i S(\rho_i) .$$

(9)

As the largest possible amount of information available to the receiver Bob is bounded by his entropy of erasure we have

$$I \leq \Delta S_{er} = S(\rho) - \sum_i p_i S(\rho_i) = I_H .$$

(10)

Therefore we have obtained the Holevo bound on the information in the states $\rho_i$ which appear with probabilities $p_i$. While this derivation only establishes the Holevo bound as an upper bound, one may argue that if Bob’s quantum states contain less than $I_H$, then we would expect to be able to find an even lower entropy of erasure for his message. Of course such an argument for the achievability of the Holevo bound cannot replace a full analytical proof, but merely forms the basis for a conjecture which is likely to be true.

IV. CONCLUSIONS

In this paper we have shown a different way of understanding the origin of the Holevo bound on the classical information capacity of an encoding using mixed quantum states. While the resulting bound is of course not new in itself we hope that this approach to the Holevo bound may help to stimulate new insights into the more difficult and yet to be solved question of the quantum capacity of a quantum channel.

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