EXPANDING TRANSLATES OF SHRINKING SUBMANIFOLDS IN HOMOGENEOUS SPACES AND DIOPHANTINE APPROXIMATION

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Abstract. On the space $\mathcal{L}_{n+1}$ of unimodular lattices in $\mathbb{R}^{n+1}$, we consider the action of $a(t) = \text{diag}(t^n, t^{-1}, \ldots, t^{-1}) \in \text{SL}(n+1, \mathbb{R})$ for $t > 1$. Let $M$ be a nondegenerate $C^{n+1}$-submanifold of an expanding horospherical leaf in $\mathcal{L}_{n+1}$. We prove that for almost every $x \in M$, the shrinking balls in $M$ of radii $t^{-1}$ around $x$ get asymptotically equidistributed in $\mathcal{L}_{n+1}$ under the action of $a(t)$ as $t \to \infty$. This result implies non-improvability of Dirichlet’s Diophantine approximation theorem for almost every point on a nondegenerate $C^{n+1}$-submanifold of $\mathbb{R}^n$, answering a question of Davenport and Schmidt (1969).

1. Introduction

After Davenport and Schmidt [5], given $0 < \lambda \leq 1$, we say that $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is DT($\lambda$) if for each sufficiently large $N \in \mathbb{N}$, there exist integers $q_1, \ldots, q_n$ and $p$ such that

$$|(q_1z_1 + \ldots + q_nz_n) - p| \leq \lambda/N^n \quad \text{and} \quad 0 < \max_{1 \leq i \leq n} |q_i| \leq \lambda N.$$ (1.1)

In a dual manner, we say that $z \in \mathbb{R}^n$ is DT$'$($\lambda$) if for each sufficiently large $N \in \mathbb{N}$ there exist integers $q$ and $p_1, \ldots, p_n$ such that

$$\max_{1 \leq i \leq n} |qz_i - p_i| \leq \lambda/N \quad \text{and} \quad 0 < |q| \leq \lambda N^n.$$ (1.2)

Dirichlet’s simultaneous approximation theorem states that every $z \in \mathbb{R}^n$ is DT(1) and DT$'$(1). Davenport and Schmidt [5] proved that for any $\lambda < 1$, almost every $z \in \mathbb{R}^n$ is not DT($\lambda$) and not DT$'$($\lambda$). In other words, Dirichlet’s theorem cannot be improved for almost all $z \in \mathbb{R}^n$. In [5] they showed that for almost every $z \in \mathbb{R}$, the vector $z = (z, z^2) \in \mathbb{R}^2$ is not DT(1/4), opening an investigation of whether almost all points on a sufficiently curved submanifold in $\mathbb{R}^n$ are not DT($\lambda$) for any $\lambda < 1$. The question was taken up in [1, 7, 2], where several non-improvability results were obtained for small $\lambda > 0$. Later Kleinbock and Weiss [11] reformulated this question in terms of dynamics on homogeneous spaces using an observation due to Dani [3] relating simultaneous Diophantine approximation to asymptotic properties.

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of individual orbits of diagonal subgroups. Using the non-divergence techniques from [10], they [11] proved that almost all points on the image of a \(l\)-nondegenerate differentiable map from an open set in \(\mathbb{R}^d\) to \(\mathbb{R}^n\) are not DT\((\lambda)\) for some very small \(\lambda > 0\), where \(l\)-nondegenerate means that at almost every point all partial derivatives of the map up to order \(l\) span \(\mathbb{R}^n\).

In [17] by proving an equidistribution result for expanding translates of analytic curve segments on the space of unimodular lattices in \(\mathbb{R}^{n+1}\), it was shown that if an analytic curve in \(\mathbb{R}^n\) is not contained in a proper affine subspace then almost all points on this curve are not DT\((\lambda)\) and not DT\('(\lambda)\) for every \(\lambda \in (0, 1)\). The analyticity is a technical assumption because of a fundamental limitation of the method of proof; namely the \((C, \alpha)\)-good property [10] of differentiable maps do not survive under composition by non-linear polynomial maps. To overcome this limitation we would require a quantitative local avoidance result, which was conjectured in [18] Section 5. In this article, we resolve this conjecture and prove a stronger equidistribution result for expanding translates of sufficiently slowly shrinking curves (cf. [16] for \(G = \text{SO}(n, 1)\)). The new equidistribution result leads to non-improvability of Dirichlet’s approximation theorem for nondegenerate manifolds as defined by Pyartli [12].

**Definition 1.1** (cf. [12, §2]). We say that a curve \(\zeta : (c, d) \to \mathbb{R}^k\) is nondegenerate at \(s \in (c, d)\), if \(\zeta^{(k-1)}(s)\) exists and the vectors \(\zeta^{(0)}(s) := \zeta(s), \zeta^{(1)}(s), \ldots, \zeta^{(k-1)}(s)\) span \(\mathbb{R}^k\).

Let \(\Omega\) be an open subset of \(\mathbb{R}^d\) and \(\phi : \Omega \to \mathbb{R}^{n+1}\) be a \(C^n\)-map. We say that \(\phi\) is nondegenerate at \(s \in \Omega\) if the following conditions are satisfied:

1. The derivative \(D\phi(s) : \mathbb{R}^d \to \mathbb{R}^{n+1}\) is injective. Let \(\mathcal{T} := D\phi(s)(\mathbb{R}^d)\).
2. There exists a subspace \(\mathcal{L}\) of \(\mathbb{R}^{n+1}\) containing \(\phi(s)\) such that \(\mathcal{T} \oplus \mathcal{L} = \mathbb{R}^{n+1}\), and there exists \(0 \neq \nu \in \mathcal{T}\) such that the map \(\rho_\nu : (-r_0, r_0) \to \mathbb{R} \nu + \mathcal{L}\) defined by \(\rho_\nu(r) = \phi(\Omega_1) \cap (r \nu + \mathcal{L})\) for all \(|r| < r_0\), for a neighborhood \(\Omega_1\) of \(s\) and some \(r_0 > 0\), is nondegenerate at 0.

We say that \(\phi\) is nondegenerate if it is nondegenerate at all \(s \in \Omega\).

**Theorem 1.2.** Let \(\psi : \Omega \to \mathbb{R}^n\) be a \((n+1)\)-times differentiable map, where \(\Omega\) is open in \(\mathbb{R}^d\). Suppose that \(\tilde{\psi} : \Omega \to \mathbb{R}^{n+1}\) given by, \(\tilde{\psi}(s) = (1, \psi(s))\) for all \(s \in \Omega\), is nondegenerate. Then given an infinite set \(\mathcal{N} \subset \mathbb{N}\), for almost every \(s \in \Omega\) and any \(\lambda \in (0, 1)\), there are no integral solutions to (1.1) and (1.2) for \(z = \psi(s)\) and infinitely many \(N \in \mathcal{N}\).

In particular, \(\psi(s)\) is not DT\((\lambda)\) or DT\('(\lambda)\) for almost any \(s \in \Omega\) and any \(\lambda \in (0, 1)\).

**Remark 1.3.** (1) The manifold \((\Omega, \psi, \mathbb{R}^n)\) is nondegenerate at \(\psi(s)\) as per Pyartli [12] if and only if the corresponding map \(\tilde{\psi}\) is nondegenerate at \(s\).

(2) It will be interesting to know whether the following holds: If a manifold is \(l\)-nondegenerate in the sense of Kleinbock and Margulis [10] then almost every point of the manifold is nondegenerate in the sense of Pyartli.
If $\psi$ is analytic and $\psi(\Omega)$ is not contained in a proper affine subspace of $\mathbb{R}^n$ then $\tilde{\psi}$ is nondegenerate on $\Omega \setminus Z$, where $Z$ is a proper analytic subvariety of $\Omega$ with strictly lower dimension and with zero Lebesgue measure.

As shown by Kleinbock and Weiss [11] and Shah [17, Section 2], due to the Dani’s correspondence the Theorem 1.2 can be derived as a consequence of Theorem 1.4, which is the main goal of this article.

**Notation.** Let $1 \leq d \leq n$, and $G = \text{SL}(n+1, \mathbb{R})$. For $t > 0$, let $a(t) := \text{diag}(t^n, t^{-1}, \ldots, t^{-1}) \in G$. Let $\Omega \subset \mathbb{R}^d$ be open and $\Phi : \Omega \to G$ be a $C^1$-map. Then for any $s \in \Omega$,

$$a(t)\Phi(s) = t^n I_0 \Phi(s) + t^{-1} I_n \Phi(s),$$

where $I_0 = \text{diag}(1, 0, \ldots, 0)$, $I_n = \text{diag}(0, 1, \ldots, 1) \in \text{M}(n+1, \mathbb{R})$. For any $g \in \text{M}(n+1, \mathbb{R})$, we identify $I_0 g$ with the top row of $g$ which is realized as an element of $\mathbb{R}^{n+1}$. We define $\phi : \Omega \to \mathbb{R}^{n+1}$ by

$$\phi(s) = I_0 \Phi(s) \in \mathbb{R}^{n+1}$$

for all $s \in \Omega$.

**Theorem 1.4.** Suppose that $\phi$ is a nondegenerate $(n+1)$-times differentiable map. Let $L$ be a Lie group containing $G$, $\Lambda$ a lattice in $L$, and let $x \in L/\Lambda$. Then there exists $E_x \subset \Omega$ of zero Lebesgue measure such that for every $s \in \Omega \setminus E_x$, and any bounded open convex neighborhood $C$ of $0$ in $\mathbb{R}^d$,

$$\lim_{t \to \infty} \frac{1}{\text{vol}(C)} \int_C f(a(t)\Phi(s + t^{-1}\eta)x) d\eta = \int_{Gx} f d\mu_x,$$

(1.3)

where $\text{vol}(\cdot)$ denotes the Lebesgue measure, and $\mu_x$ is the $G$-invariant probability measure on the homogeneous space $Gx$.

In particular, for any probability measure $\nu$ on $\Omega$ which is absolutely continuous with respect to the Lebesgue measure,

$$\lim_{t \to \infty} \int_{\Omega} f(a(t)\Phi(\eta)x) d\nu(\eta) = \int_{Gx} f d\mu_x.$$  

(1.4)

To derive Theorem 1.2, we need (1.4) for $\Phi(s) = \begin{pmatrix} \psi(s) \\ 0 \\ I_n \end{pmatrix}$ and a suitably chosen embedding of $G$ into $L = G \times G$ [17, §1.0.1]. But to justify (1.4) for differentiable maps, we need to prove the equidistribution of local expansion given by (1.3), which is new even for the horospherical case of $L = G$, $d = n$ and $\psi(s) = s$; cf. [8, Lemma 16] and [9, Theorem 20].

Our proof of (1.3) is quite different from the arguments of [17] for proving (1.4) for analytic maps. A new identity observed in this article allows us to describe the limiting distribution of expansion of shrinking pieces in the curve $(d = 1)$ case using equidistribution of long polynomial trajectories on homogeneous spaces [15].

This article is organized as follows. In [2] we obtain the key identity as mentioned above. In [3] we combine the result on limiting distributions of polynomial trajectories with the key identity to obtain the algebraic description of the limiting distribution of the stretching translates of the shrinking
Proof. We want to find a nilpotent matrix $B_s \in M(n+1, \mathbb{R})$ such that for any $t \neq 0$ with $s + t^{-1} \in \Omega$, we have
\[ a(|t|)\Phi(s + t^{-1}) = (I + o(t^{-1})t)\xi_s(\sigma)(I - tB_s)^{-1}, \tag{2.1} \]
where $\sigma = t/|t| = \pm 1$, $\xi_s(\pm 1) \in G$, and $o(t^{-1}) \in M_{n+1}(\mathbb{R})$ is such that $o(t^{-1})t \to 0$ as $t \to \infty$.

We note that $B_s^n \neq 0$ and $B_s^{n+1} = 0$, so
\[ P_s(t) := (I - tB_s)^{-1} = I + \sum_{k=1}^{n} t^k B_s^k \in SL(n+1, \mathbb{R}) = G. \tag{2.2} \]

We will obtain a geometric description of the set of exceptional points (Proposition 5.3) and prove that it is Lebesgue null (Proposition 5.1). We will show that in many standard examples the exceptional points are dense in $\Omega$ (Proposition 5.4).

2. Basic identity

The main new ingredient in the proof of Theorem 1.4 is the following:

**Lemma 2.1** (Basic Identity). Let $d = 1$, $\Omega \subset \mathbb{R}$ open, $\Phi : \Omega \to G$ a $C^1$-map, and $s \in \Omega$ be such that the map $\phi = I_0 \Phi : \Omega \to \mathbb{R}^{n+1}$ is $(n+1)$-times differentiable and nondegenerate at $s$. Then there exists a nilpotent matrix $B_s \in M(n+1, \mathbb{R})$ of rank $n$ such that for any $t \neq 0$ with $s + t^{-1} \in \Omega$, we have
\[ a(|t|)\Phi(s + t^{-1}) = (I + o(t^{-1})t)\xi_s(\sigma)(I - tB_s)^{-1}, \tag{2.1} \]
where $\sigma = t/|t| = \pm 1$, $\xi_s(\pm 1) \in G$, and $o(t^{-1}) \in M_{n+1}(\mathbb{R})$ is such that $o(t^{-1})t \to 0$ as $t \to \infty$.

We note that $B_s^n \neq 0$ and $B_s^{n+1} = 0$, so
\[ P_s(t) := (I - tB_s)^{-1} = I + \sum_{k=1}^{n} t^k B_s^k \in SL(n+1, \mathbb{R}) = G. \tag{2.2} \]

Proof. We want to find a nilpotent matrix $B_s \in M(n+1, \mathbb{R})$ such that
\[ \lim_{t \to \infty} a(|t|)\Phi(s + t^{-1})(I - tB_s) \in G. \]

Let $t \neq 0$ such that $s + t^{-1} \in \Omega$. In view of (1), by Taylor’s expansion,
\[ I_0 \Phi(s + t^{-1}) = \phi(s + t^{-1}) = \sum_{k=0}^{n+1} \frac{\phi^{(k)}(s)}{k!} t^{-k} + o(t^{-(n+1)}). \]

For any $B_s \in M(n+1, \mathbb{R})$ and $\sigma = t/|t| = \pm 1$, we have
\[ a(|t|)I_0 \Phi(s + t^{-1})(I - tB_s) = |t|^n \phi(s + t^{-1})(I - tB_s) \]
\[ = \sigma^n \left( \sum_{k=0}^{n+1} \frac{\phi^{(k)}(s)}{k!} t^{n-k} + o(t^{-1}) \right) (I - tB_s) \]
\[ = \sigma^n \left( -\phi(s)B_s t^{n+1} + \sum_{k=1}^{n} \frac{\phi^{(k-1)}(s)}{(k-1)!} - \frac{\phi^{(k)}(s)}{k!} t^{n-k+1} B_s \right) \]
\[ + \sigma^n \xi_{s,1} + o(t^{-1}) t, \tag{2.3} \]
where
\[ \xi_{s,1} = \frac{\phi^{(n)}(s)}{n!} - \frac{\phi^{(n+1)}(s)}{(n+1)!} B_s. \quad (2.4) \]

We want to choose \( B_s \) such that all the coefficients of positive powers of \( t \) vanish in (2.3); in other words, we want
\[ \phi(s) B_s = 0 \quad \text{and} \quad \frac{\phi^{(k)}(s)}{k!} B_s = \frac{\phi^{(k-1)}(s)}{(k-1)!} \] for \( 1 \leq k \leq n. \quad (2.5) \]

By our assumption, \( \{ \phi^{(k)}(s)/k! : 0 \leq k \leq n \} \) is a basis of \( \mathbb{R}^{n+1} \). Therefore there exists a unique matrix \( B_s \) such that (2.5) holds. Moreover, \( B_s \) is nilpotent matrix of rank \( n \). In particular, \( \det(I - tB_s) = 1 \) for all \( t \in \mathbb{R} \).

Now by (2.3) and (2.5), we have the following key identity:
\[ a(|t|) I_n \Phi(s + t^{-1})(I - tB_s) = \sigma^n \xi_{s,1} + o(t^{-1})t. \quad (2.6) \]

Also, since \( \Phi \) is differentiable at \( s \),
\[ a(|t|) I_n \Phi(s + t^{-1})(I - tB_s) = |t|^{-1}(I_n \Phi(s) + O(t^{-1}))(I - tB_s) \]
\[ = \sigma \xi_{s,2} + O(t^{-1}), \quad (2.7) \]
where
\[ \xi_{s,2} = -I_n \Phi(s) B_s. \quad (2.8) \]

In view of (1), combining (2.6) and (2.7):
\[ a(|t|) \Phi(s + t^{-1})(I - tB_s) = \xi_s(\sigma) + o(t^{-1})t, \quad (2.9) \]
where in view of (2.4) and (2.8), \( \sigma = t/|t| = \pm 1 \) and
\[ \xi_s(\sigma) = \sigma^n \xi_{s,1} + \sigma \xi_{s,2}. \quad (2.10) \]

Now (2.1) follows from (2.9). Since the left hand side of (2.9) belongs to \( G \) for all \( t \), by taking \( t \to \pm \infty \), we get \( \xi_s(\pm 1) \in G \).

Though it is straightforward to verify the basic identity, the path that lead us to conceive the identity involved an intricate study of interactions of linear dynamics of intertwining copies of \( SL(2, \mathbb{R}) \) in \( G \) using Weyl group actions using [19, Lemma 4.1].

3. Limiting distribution of polynomial trajectories and stretching translates of shrinking curves

Our proof of Theorem 1.4 for \( d = 1 \) is based on Lemma 2.1 and the following result on limiting distribution of polynomial trajectories on homogeneous spaces which was proved using Ratner’s description [13] of ergodic invariant measures for unipotent flows.
3.0.1. Notation. Let $L$ be a Lie group containing $G$ and $\Lambda$ be a lattice in $G$. Let $x \in L/\Lambda$. Let $\mathcal{H}_x$ denote the collection of all connected Lie subgroups $H$ of $L$ such that $Hx$ is closed and admits an $H$-invariant probability measure, say $\mu_H$, which is ergodic with respect to an Ad$_L$-unipotent one-parameter subgroup of $L$. Then $\mathcal{H}_x$ is countable [13, §2].

If $H_1, H_2 \in \mathcal{H}_x$ then there exists $H \in \mathcal{H}_x$ such that $H \subset H_1 \cap H_2$ and $H$ contains all Ad$_L$-unipotent one-parameter subgroups of $H_1 \cap H_2$ [14, §2].

**Theorem 3.1** (Shah [15]). Let $Q : \mathbb{R} \to G = \text{SL}(n+1, \mathbb{R})$ be a map whose each coordinate is a polynomial and the identity element $I \in Q(\mathbb{R})$. Let $H$ be the smallest Lie subgroup of $L$ containing $Q(\mathbb{R})$ such that $Hx$ is closed. Then $H \in \mathcal{H}_x$, and for any $f \in C_c(L/\Lambda)$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(Q(t)x) \, dt = \int_{Hx} f \, d\mu_H.$$  

The following is its straightforward reformulation via change of variable.

**Corollary 3.2.** Let the notation be as in Theorem 3.1. Then for any $f \in C_c(L/\Lambda)$ and $c < d$,

$$\lim_{T \to \infty} \frac{1}{d-c} \int_c^d f(Q(Ts)x) \, ds = \int_{Hx} f \, d\mu_H.$$  

From this result we can deduce its following variation.

**Corollary 3.3.** Let the notation be as in Theorem 3.1. Let $\rho : \mathbb{R} \to G$ be a measurable map and $\nu$ be a finite measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Then for any $f \in C_c(L/\Lambda)$,

$$\int_{\mathbb{R}} f(\rho(\eta)Q(T\eta)x) \, d\nu(\eta) \xrightarrow{T \to \infty} \int_{\mathbb{R}} \left[ \int_{Hx} f(\rho(\eta)y) \, d\mu_H(y) \right] d\nu(\eta). \tag{3.1}$$  

**Proof.** We can assume that $|f| \leq 1$. And since $\nu$ is finite, due to Lusin’s theorem, we can replace $\rho$ and $d\nu(\eta)/d\eta$ by continuous functions with compact support. Let $s \in \mathbb{R}$. Given $\epsilon > 0$, there exists $\delta_s > 0$ such that for all $\eta \in (s - \delta_s/2, s + \delta_s/2)$ and $y \in L/\Lambda$,

$$|(d\nu/d\eta)(\eta) - (d\nu/d\eta)(s)| \leq \epsilon \quad \text{and} \quad |f(\rho(\eta)y) - f(\rho(s)y)| \leq \epsilon.$$  

Using these approximations, by Corollary 3.2 for any $0 < \delta < \delta_s$ there exists $T_{s,\delta} \geq 1$ such that

$$\left| \int_{s-\delta/2}^{s+\delta/2} f(\rho(\eta)Q(T\eta)x) \, d\nu(\eta) - \delta \cdot (d\nu/d\eta)(s) \cdot \int_{Hx} f(\rho(s)y) \, d\mu_H(y) \right| \leq 2\epsilon \delta,$$

for all $T \geq T_{s,\delta}$. We use convergence in measure to complete the proof.  

**Theorem 3.4.** Let $d = 1$ and the notation be as in Theorem 1.4 and Notation 7.0.4. Let $s \in \Omega$. Then there exists $H_s \in \mathcal{H}_x$ such that the following
holds: Let $\nu$ be a finite measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Then for any $f \in C_c(L/\Lambda)$,

$$
\lim_{t \to \infty} \int_{\mathbb{R}} f(a(t)\Phi(s + \eta t^{-1})x) \, d\nu(\eta)
= \int_{\mathbb{R}} \left[ \int_{H_s x} f(a(|\eta|)\xi_s(\text{sign}(\eta))y) \, d\mu_{H_s}(y) \right] \, d\nu(\eta),
$$

where $\text{sign}(\eta) = \eta/|\eta| = \pm 1$ and $\xi_s(\pm 1) \in G$ are given by (2.10).

Moreover if $H_s \supset G$, then $Gx = H_s x$, $\mu_x = \mu_{H_s}$, and

$$
\lim_{t \to \infty} \int_{\mathbb{R}} f(a(t)\Phi(s + \eta t^{-1})x) \, d\nu(\eta) = |\nu| \cdot \int_{Gx} f \, d\mu_x.
$$

Proof. Let $\eta \neq 0$. For $t \gg 1$, writing $h = \eta^{-1}t$, by (2.1) and (2.2),

$$
a(t)\Phi(s + \eta t^{-1})x = a(|\eta|)a(|h|)\Phi(s + h^{-1})x
= a(|\eta|)(I + o(h^{-1})h)\xi_s(\text{sign}(\eta))P_s(h)x
= (I + |\eta|^{-1}o(h^{-1})h)a(|\eta|)\xi_s(\text{sign}(\eta))P_s(h)x
= (I + |\eta|^{-1}o(t^{-1})t)a(|\eta|)\xi_s(\text{sign}(\eta))P_s(t\eta^{-1})x.
$$

Since $f$ is bounded, we can ignore the integration over a small neighborhood of 0, outside which $|\eta|^{-1}o(t^{-1})t$ is close to 0 uniformly for all large $t$. So by uniform continuity of $f$ we can ignore the factor $(I + |\eta|^{-1}o(t^{-1})t)$, and hence

$$
\lim_{t \to \infty} \int_{\mathbb{R}} f(a(t)\Phi(s + \eta t^{-1})x) \, d\nu(\eta)
= \lim_{t \to \infty} \int_{\mathbb{R}} f(a(|\eta|)\xi_s(\text{sign}(\eta))P_s(t\eta^{-1})x) \, d\nu(\eta).
$$

By (2.2), $P_s(0) = I$. Let $H_s \subset H_x$ be the smallest subgroup containing $P_s(\mathbb{R})$. Applying Corollary (3.3) to the image of $\nu$ on $\mathbb{R}$ under the map $\eta \mapsto \eta^{-1}$, from (3.3) we obtain (3.2). 

\[\square\]

4. Stretching Translates of Shrinking Submanifolds

In this section we will obtain the analogue of Theorem 3.4 for $d \geq 2$.

Notation. Let $d \geq 2$ and $n \geq 2$. Let $\Phi : \Omega \subset \mathbb{R}^d \to G = \text{SL}(n + 1, \mathbb{R})$ be a $C^1$-map. Fix $s \in \Omega$, and suppose that $\phi = I_0\Phi : \Omega \to \mathbb{R}^{n+1}$ is $(n + 1)$-differentiable and nondegenerate at $s$. So the derivative $D\phi(s) : \mathbb{R}^d \to \mathbb{R}^{n+1}$ of $\phi$ as $s$ is injective. Let $\text{SO}(d)$ be the special orthogonal group acting on $T := D\phi(s)(\mathbb{R}^d)$. Since $\phi(s) \neq 0$, by Definition (1.1(2)), $d \leq n$.

Theorem 4.1. There exists a rational function $\xi_s : \text{SO}(d) \to G$ such that the following holds. Let $L$ be a Lie group containing $G$, $\Lambda$ be a lattice in $L$, and $x \in L/\Lambda$. Then there exists a closed subgroup $H_s$ of $L$ such that $H_s x$
is closed and admits an $H_s$-invariant probability measure, say $\mu_{H_s}$, and for any open bounded convex neighbourhood $C$ of 0 in $\mathbb{R}^d$ and any $f \in C_c(L/\Lambda)$,

$$\lim_{t \to \infty} \frac{1}{\text{vol}(C)} \int_C f(a(t)\Phi(s + t^{-1}\eta)x) \, d\eta = \int_{g \in \text{SO}(d)} \int_0^1 \left[ \int_{Hx} f(a(r)\xi_s(g)y) \, d\mu_{H_s}(y) \right] r^{d-1} \, dr \, dg. \quad (4.1)$$

Here we fix a unit vector $e_1 \in \mathcal{T}$, and let $r_g = \sup \{ r \geq 0 : rge_1 \in D\phi(C) \}$, and $f \cdot dg$ denotes a suitably normalized Haar integral on $\text{SO}(d)$.

Moreover if $H_s \supset G$, then $\overline{Gx} = H_s x$, $\mu_x = \mu_{H_s}$, and

$$\lim_{t \to \infty} \frac{1}{\text{vol}(C)} \int_C f(a(t)\Phi(s + t^{-1}\eta)x) \, d\eta = \int_{\overline{Gx}} f \, d\mu_x. \quad (4.2)$$

**Realizing the manifold as a graph over a tangent.** Let $\mathcal{L}$ be a subspace of $\mathbb{R}^{n+1}$ containing $\phi(s)$ as in Definition 1.1. Since $D\phi(s)$ is an injection, and $\mathcal{T} \oplus \mathcal{L} = \mathbb{R}^{n+1}$, by the implicit function theorem, there exist open neighborhoods $\Delta$ of 0 in $\mathcal{T}$ and $\Omega_1$ of $s$ in $\mathbb{R}^d$, and a $C^{n+1}$-diffeomorphism $\Psi : \Delta \to \Omega_1$ and a $C^{n+1}$ map $F : \Delta \to \mathcal{L}$ such that $\Psi(0) = s$ and

$$\phi(\Psi(\eta)) = \phi(s) + \eta + F(\eta), \quad \forall \eta \in \Delta. \quad (4.3)$$

In particular, $DF(0) = 0$ and $D\Psi(0) = D\phi(s)^{-1}$.

Fix an open bounded convex neighborhood $C$ of 0 in $\mathbb{R}^d$. Let $C_1 = D\phi(0)(C) \subset \mathcal{T}$. Then for any $f \in C_c(L/\Lambda)$,

$$\lim_{t \to \infty} \frac{1}{\text{vol}(C)} \int_C f(a(t)\Phi(s + t^{-1}\eta)x) \, d\eta,$$

changing the variable $\kappa$ to $\eta$ such that $s + t^{-1}\kappa = \Psi(t^{-1}\eta)$,

$$= \lim_{t \to \infty} \frac{1}{\text{vol}(C)} \int_{t\Psi^{-1} \circ \Lambda} f(a(t)\Phi(t^{-1}\eta)x) \cdot |\det(D\Psi(t^{-1}\eta))| \, d\eta$$

$$= \lim_{t \to \infty} \frac{1}{\text{vol}(C_1)} \int_{C_1} f(a(t)\Phi(t^{-1}\eta)x) \, d\eta, \quad (4.4)$$

if any of the limits exists. Because since

$$\eta = t\Psi^{-1}(s + t^{-1}\kappa) = D\Psi(0)^{-1}(\kappa) + O(t^{-2})t = D\phi(s)(\kappa) + O(t^{-1}),$$

$$\lim_{t \to \infty} \text{vol}(t\Psi^{-1}(s + t^{-1}\kappa)\Delta C_1) = 0,$$

and $\text{vol}(C) = |\det(D\Psi(0))| \text{vol}(C_1)$.

**Nondegenerate curves on the manifold via Pyartli’s twisting.** Let $r_0 > 0$ be such that for any $0 \neq w \in \mathcal{T}$, the curve $\rho_w : (-r_0, r_0) \to \mathbb{R}w + \mathcal{L} \cong \mathbb{R}^{1+(n+1)-d}$ given by

$$\rho_w(r) = \phi(\Psi(rw)) = \phi(s) + rw + F(rw),$$

parametrizes the one-dimensional submanifold $\phi(\Omega_1) \cap (\mathbb{R}w + \mathcal{L})$. By Definition 1.1 we pick $0 \neq v \in \mathcal{T}$ such that $\rho_v$ is nondegenerate at 0.
Remark 4.2. Fix a basis of $\mathcal{T}$. Let $w = (w_1, \ldots, w_d) \in \mathcal{T}$. Then $\rho_w(0) = \phi(s) \in \mathcal{L}$, $\rho_w^{(1)}(0) = w$, and for $2 \leq i \leq \dim \mathcal{L}$,

$$\rho_w^{(i)}(0) = \sum_{i_j \geq 0, i_1 + \ldots + i_d = i} \frac{i!}{i_1! \cdots i_d!} \cdot \partial_{i_1} \cdots \partial_{i_d} F(0) \cdot w_{i_1} \cdots w_{i_d} \in \mathcal{L}. \quad (4.5)$$

Therefore $\rho_w$ is nondegenerate at 0, if and only if the determinant of the matrix whose 1-st row is $\phi(s)$ and the $i$-th row is $\rho_w^{(i)}(0)$ for $2 \leq i \leq \dim \mathcal{L}$ with respect to a fixed basis in $\mathcal{L}$ is nonzero. Since $\rho_v$ is nondegenerate, $\rho_w$ is nondegenerate for all $w \in \mathcal{T}$ outside an $\mathbb{R}$-invariant algebraic subvariety of strictly lower dimension.

Choose an orthonormal basis $\{e_i : 1 \leq i \leq d\}$ for $\mathcal{T}$. Let $\gamma : \mathbb{R} \to \mathcal{T}$ be the curve given by

$$\gamma(r) = re_1 + \sum_{i=2}^{d} r^{n-d+i} e_i \in \mathcal{T}, \quad \forall r \in \mathbb{R}.$$

For $g \in \text{SO}(d)$, let $\zeta_{g\gamma} : (-r_0, r_0) \to \phi(\Omega_1)$ be the curve given by

$$\zeta_{g\gamma}(r) = \phi(\Psi(g\gamma(r))) = \phi(s) + g\gamma(r) + F(g\gamma(r)). \quad (4.6)$$

Lemma 4.3 ([12 Lemma 5]). Let $g \in \text{SO}(d)$ be such that the curve $\rho_{ge_1}$ is nondegenerate at 0 in $\mathcal{L} + \mathbb{R}ge_1$. Then $\zeta_{g\gamma}$ is nondegenerate at 0 in $\mathbb{R}^{n+1}$.

Proof. We observe that

$$\zeta_{g\gamma}^{(k)}(0) = \rho_{ge_1}^{(k)}(0) \text{ for } 0 \leq k \leq (n-d+1) = \dim(\mathcal{L} + \mathbb{R}ge_1) - 1,$$

$$\zeta_{g\gamma}^{(n-d+i)}(0) = ge_i \text{ modulo } \mathcal{L} + \mathbb{R}ge_1, \quad \text{for } 2 \leq i \leq d. \quad (4.7)$$

So $\{\zeta_{g\gamma}^{(k)}(0) : 0 \leq k \leq n\}$ spans $\mathbb{R}^{n+1}$. \qed

Polar fibering. For $t \geq 1$, let $T_t : \text{SO}(d) \times [0, \infty) \to \mathbb{R}^d$ be given by

$$T_t(g, r) = tg\gamma(t^{-1}r) = g \cdot t\gamma(t^{-1}r) = g \cdot (re_1 + \sum_{i=2}^{d} t^{-(n-d+i-1)} r^{n-d+i} e_i). \quad (4.8)$$

We recall that $2 \leq d \leq n$. Let $dg$ denote a Haar integral on $\text{SO}(d)$. For a fixed $r > 0$, under $T_t(\cdot, r)$, the Haar measure on $\text{SO}(d)$ projects to a rotation invariant measure on the sphere of radius $|t\gamma(t^{-1}r)|$ in $\mathbb{R}^d$ centered at 0. Then the image of the integral $dg \times |t\gamma(t^{-1}r)|^{d-1} |t\gamma(t^{-1}r)|$ under the map $T_t$ corresponds to a multiple of the Lebesgue integral on $\mathbb{R}^d$.

Let $r_{g,t} = \sup\{r \geq 0 : T_t(g, r) \in C_1\}$. Now $T_t(g, r) = rge_1 + O(t^{-1})$ uniformly in $g$ and bounded $r$. Therefore $r_{g,t} = r_g + O(t^{-1})$, where

$$r_g = \sup\{r \geq 0 : gge_1 \in C_1\}.$$
By (4.8), \(\left|t \gamma(t^{-1} r) \right|^{d-1} \frac{d}{dt} \left|t \gamma(t^{-1} r) \right| = 1 + O(t^{-1} r)^2\). Therefore continuing (4.9), by the change of variable \(\eta = T_t(g, r)\),

\[
\lim_{t \to \infty} \frac{1}{\text{vol}(C_1)} \int_{C_1} f(a(t) \Phi(\Psi(\eta \gamma) x)) d\eta
\]

\[
= \lim_{t \to \infty} \int_{g \in SO(d)} \left[ \int_0^{r \gamma(t)} f((a(t) \Phi(\Psi(t^{-1} T_t(g, r)) \gamma(t^{-1} r) \right] d\gamma)
\]

\[
= \lim_{t \to \infty} \int_{g \in SO(d)} \left[ \int_0^{r \gamma(t)} f(a(t) \Phi(\Psi(g \gamma(t^{-1} r)) \gamma(t^{-1} r) \right] d\gamma)
\]

where for each \(t\) the Haar integral \(dg\) is normalized such that the integral of the expression equals 1 for the constant function \(f \equiv 1\).

**Proof of Theorem 4.1.** In view of (4.6) and (4.9),

\[
I_0 \Phi(\Psi(g \gamma(r))) = \zeta_{g \gamma}(r) \in \mathbb{R}^{n+1}. \tag{4.10}
\]

Let \(\{\hat{e}_k : 0 \leq k \leq n\}\) denote the standard basis of \(\mathbb{R}^{n+1}\) consisting of row vectors. For \(g \in SO(d)\), let \(M(g) \in M(n+1, \mathbb{R})\) be such that with respect to the right action \(\mathbb{R}^{n+1}\),

\[
\hat{e}_k M(g) = \zeta_{g \gamma}^{(k)}(0) / k!, \quad \forall 0 \leq k \leq n.
\]

Now \(\zeta_{g \gamma}\) is nondegenerate at 0 if and only if \(\det(M(g)) \neq 0\). By (4.5) and (4.7), \(\det(M(g))\) is a polynomial in coordinates of \(g\). Therefore the set \(Z_s = \{g \in SO(d) : \det(M(g)) = 0\}\) is an affine subvariety of \(SO(d)\). Since \(\phi\) is nondegenerate at \(s\), there exists \(g \in SO(d)\) such that \(ge_1 = v\) and \(\rho_v\) is nondegenerate at 0. Therefore by Lemma 4.3 we have that \(g \not\in Z_s\). Therefore \(Z_s\) is a strictly lower dimensional subvariety of \(SO(d)\), where \(d \geq 2\). Hence \(Z_s\) is null with respect to \(dg\).

Let \(g \in SO(d) \setminus Z_s\). Let \(B(g) = M(g)^{-1} BM(g)\), where \(B\) is the lower triangular matrix such that \(\hat{e}_0 B = 0\) and \(\hat{e}_k B = \hat{e}_{k-1}\) for \(1 \leq i \leq n\). Then

\[
\zeta_{g \gamma} \gamma(0) B(g) = 0 \quad \text{and} \quad (\zeta_{g \gamma}^{(k)}(0) / k!) B(g) = \zeta_{g \gamma}^{(k-1)}(0) / (k - 1)!, \quad \forall 1 \leq k \leq n,
\]

as in (4.5). In view of (4.10), let

\[
\xi_s(g) = I_0 \left( \frac{\zeta_{g \gamma}^{(n)}(0)}{n!} - \frac{\zeta_{g \gamma}^{(n+1)}(0)}{(n + 1)!} \cdot B(g) \right) - I_0 \Phi(s) B(g).
\]

Then by (4.10) and (2.9),

\[
a(t) \Phi(\Psi(g \gamma(t^{-1}))) = (I + o(t^{-1}) t) \xi_s(g)(I - tB(g))^{-1}. \tag{4.11}
\]

In particular, \(\xi_s(g) \in G\). As in (2.2),

\[
(I - tB(g))^{-1} = \sum_{k=0}^n (tB(g))^k, \quad \forall t \in \mathbb{R}. \tag{4.12}
\]

Let \(f(g)\) be the \(\mathbb{R}\)-span of \(\{B^k(g) : 0 \leq k \leq n\}\). Then one has

\[
f(g) = M(g)^{-1} f M(g), \tag{4.13}
\]

where \(f\) is the \(\mathbb{R}\)-span of \(\{B^k : 0 \leq k \leq n\}\).
We fix \( x \in L / \Lambda \). Let \( H(g) \in H_x \) be the smallest Lie subgroup such that its Lie algebra contains \( f(g) \). By Theorem 3.3 in view of (4.9) and (4.11),
\[
\lim_{t \to \infty} \int_0^T f(a(t)\Phi(\Psi(g^\gamma(t^{-1}r)))) r^{d-1} dr = \int_0^T \left[ \int_{H(g)x} f(a(r)\xi_s(g)y) d\mu_{H(g)}(y) \right] r^{d-1} dr. \tag{4.14}
\]

Claim 1. There exists \( H_s \in H_x \) such that \( H(g) = H_s, \forall g \in SO(d) \setminus (Z_s \cup Z_{x,s}) \), where \( Z_{x,s} \) is a Haar-null subset of \( SO(d) \).

To prove this, let \( H \in H_x \). For \( g \in SO(d) \setminus Z_s \), we have \( H(g) \in H_x \) and
\[
H(g) \subset H \iff f(g) \subset \text{Lie}(H) \iff M(g)^{-1}fM(g) \subset \text{Lie}(H). \tag{4.15}
\]
So define
\[
Z_s(H) = \{ g \in SO(d) : fM(g) \subset M(g) \cdot \text{Lie}(H) \}.
\]
Then \( Z_s(H) \) is an affine subvariety of \( SO(d) \). So \( SO(d) \setminus Z_s \) is locally compact, and hence of Baire second category. For every \( g \in SO(d) \setminus Z_s \),
\[
Z_s(H(g)) \supset \{ g' \in SO(d) \setminus Z_s : H(g') \subset H(g) \} \ni g. \tag{4.16}
\]
Since \( H_x \) is countable, \( SO(d) \setminus Z_s \) is covered by a countable union of closed sets \( Z_s(H(g)) \), where \( g \in SO(d) \setminus Z_s \). So there exists \( g_0 \in SO(d) \) such that \( Z_s(H(g_0)) \) contains a non-empty open subset of \( SO(d) \). Since \( d \geq 2 \), any non-empty open subset of \( SO(d) \) is Zariski dense in \( SO(d) \). Therefore \( Z_s(H(g_0)) = SO(d) \). So \( H(g) \subset H(g_0) \) for all \( g \in SO(d) \setminus Z_s \). Define
\[
Z_{x,s} = \bigcup \{ g \in SO(d) \setminus Z_s : H_s(g_0) \not\subset H(g) \}.
\]
Let \( g \in Z_{x,s} \). By (4.16), \( g_0 \not\in Z_s(H(g)) \). So \( Z_s(H(g)) \) is a proper affine subvariety of \( SO(d) \) of strictly lower dimension, and it is Haar-null on \( SO(d) \).

Also \( g \in Z_s(H(g)) \) and \( H(g) \in H_x \). Therefore, since \( H_x \) is countable, \( Z_{x,s} \) is Haar-null on \( SO(d) \). Put \( H_s = H(g_0) \). Then \( H(g) = H_s \) for all \( g \in SO(d) \setminus (Z_s \cup Z_{x,s}) \). So the Claim 1 holds.

Continuing (4.17) using (4.14), by Claim 1, since \( Z_s \cup Z_{x,s} \) is Haar-null,
\[
\lim_{t \to \infty} \int_{g \in SO(d)} \int_0^T f((a(t)\Phi(\Psi(g^\gamma(t^{-1}r))))x) r^{d-1} dr \] \[= \int_{g \in SO(d) \setminus (Z_s \cup Z_{x,s})} \int_0^T \left[ \int_{gH_s} f(a(r)\xi_s(g)y) d\mu_{H_s}(y) \right] r^{d-1} dr dg. \tag{4.17}
\]
This completes the proof of Theorem 4.1.

5. Equidistribution of Translates of Nondegenerate Manifolds

Let the notation be as in the statement of Theorem 1.4. In view of (4.2) in Theorem 4.1, let
\[
E_x = \{ s \in \Omega : G \not\subset H_s \}. \tag{5.1}
\]
To derive Theorem 1.4 from Theorem 4.1 we will show that $E_x$ is a countable union of sets of the form $\phi^{-1}(W)$, where $W$ is a proper subspace of $\mathbb{R}^{n+1}$ (Proposition 5.3), and the following holds:

**Proposition 5.1.** Let $\Omega \subset \mathbb{R}^d$ be an open set, and $\phi : \Omega \rightarrow \mathbb{R}^{n+1}$ be a $n$-times differentiable nondegenerate map. Then for any nonzero linear functional $\ell : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, the set $\{s \in \Omega : \ell(\phi(s)) = 0\}$ has zero Lebesgue measure. In fact, if $d = 1$, then $\phi^{-1}(\ker \ell)$ is discrete in $\Omega$.

**Proof.** Let $d = 1$. Suppose $s \in \Omega \subset \mathbb{R}$ and a sequence $\{s_i\} \subset \Omega \setminus \{s\}$ are such that $\ell(\phi(s_i)) = 0$ and $s_i \rightarrow s$ as $i \rightarrow \infty$. By Taylor’s expansion,

$$0 = \ell(\phi(s_i)) = \sum_{k=0}^{n} \frac{\ell(\phi^{(k)}(s))}{k!} (s_i - s)^k + o(s - s_i)^n$$

for all $i$. Let $0 \leq m \leq n$ be such that $\ell(\phi^{(k)}(s)) = 0$ for $0 \leq k < m$. Then dividing both sides by $(s_i - s)^m$, and letting $i \rightarrow \infty$, $\ell(\phi^{(m)}(s)) = 0$. By induction $\ell(\phi^{(k)}(s)) = 0$ for all $0 \leq k \leq n$. But this contradicts our assumption that $\phi^{(k)}(s)$ for $k = 0, \ldots, n$ are linearly independent in $\mathbb{R}^{n+1}$. Therefore $\phi^{-1}(\ker \ell)$ is discrete in $\Omega$ if $d = 1$.

Now we consider the case of $d \geq 2$. Suppose on the contrary that $\phi^{-1}(\ker \ell) \subset \Omega$ has strictly positive Lebesgue measure in $\mathbb{R}^d$. Let $s \in \Omega$ be a Lebesgue density point of $\phi^{-1}(\ker \ell)$. Then for a unit ball $C$ about $0$ in $\mathbb{R}^d$, by (4.4), (4.9) and (4.6),

$$1 = \lim_{t \rightarrow \infty} \frac{1}{\text{vol}(C)} \int_{\eta \in C} \chi_{\ker \ell}(\phi(s + t^{-1}\eta)) \, d\eta$$

$$= \lim_{t \rightarrow \infty} \int_{g \in \text{SO}(d)} \int_{0}^{r_g} \chi_{\ker \ell}(\phi(g\gamma(t^{-1}r)))r^{d-1} \, dr \, dg$$

$$= \lim_{t \rightarrow \infty} \int_{g \in \text{SO}(d)} \int_{0}^{r_g} \chi_{\ker \ell}(\zeta_{g\gamma}(t^{-1}r))r^{d-1} \, dr = 0,$$

because for every $g \in \text{SO}(d) \setminus Z_s$, there exists $t_g > 0$, such that $r \mapsto \zeta(g\gamma)(r)$ is nondegenerate for all $|r| < t_g^{-1}r_g$, and hence by the case of $d = 1$, $\{r \in [0, t_g^{-1}r_g] : \ell(\zeta_{g\gamma}(r)) = 0\} = \zeta_{g\gamma}^{-1}(\ker \ell) \cap (0, t_g^{-1}r_g]$ is a countable set. \hfill \Box

In order to describe the exceptional set $E_x$, we will use a crucial result from [17], which is generalized in [20] for arbitrary $G$. We begin with some some notation and observations. Let

$$P^- = \{g \in G : \{a(t)ga(t)^{-1} : t \geq 1\} \text{ is compact}\}$$

$$U = \{u(z) := \left(\frac{1}{2} I_n , \frac{1}{z} \right) : z \in \mathbb{R}^n\}.$$

Let $\{\hat{e}_k : 0 \leq k \leq n\}$ denote the standard basis of $\mathbb{R}^{n+1}$, which is treated as the space of top rows of matrices in $M(n + 1, \mathbb{R})$. We identify $\mathbb{R}^n$ with $\text{span}\{\hat{e}_k : 1 \leq k \leq n\}$. Then $P^-$ is the stabilizer of the line $\mathbb{R} \cdot \hat{e}_0$ for the right action of $G$ on $\mathbb{R}^{n+1}$ and $\tilde{e}_0 u(z) = \tilde{e}_0 + z \in \mathbb{R}^{n+1}$, $\forall z \in \mathbb{R}^n$. Therefore

$$P^- U = \{g \in G : g_{00} := \langle \tilde{e}_0 g, \tilde{e}_0 \rangle \neq 0\} = \{g \in G : I_0 g \notin \{0\} \times \mathbb{R}^n\}, \quad (5.2)$$
and it is a Zariski open dense neighborhood of the identity in $G$.

For a finite dimensional representation $V$ of $G$, define

$$V^+ = \{ v \in V : \lim_{t \to \infty} a(t)^{-1}v = 0 \}, \quad V^- = \{ v \in V : \lim_{t \to \infty} a(t)v = 0 \},$$

$$V^0 = \{ v \in V : a(t)v = v, \forall t > 0 \},$$

and let $\pi_+$, $\pi_0$, and $\pi_-$ denote the natural projections onto $V^+$, $V^0$ and $V^-$, respectively, with respect to the decomposition $V = V^+ \oplus V^0 \oplus V^-$. 

**Proposition 5.3.** Let $E$ be a proper subspace of $\mathbb{R}^{n+1}$. Then for any finite dimensional representation $V$ of $G$ and a nonzero $v \in V$, if

$$gv \in V^0 + V^-,$$ 

then $\pi_0(gv) \neq 0$ for all $g \in E$ and $Z_G(\{a(t) : t > 0\})$ fixes $\pi_0(gv)$. 

**Proof.** For every $g \in P^-U$, there exists a unique $\bar{g} \in \mathbb{R}^n$ such that $P^-g = P^-u(\bar{g})$, and $I_0g = g_00(\bar{e}_0 + \bar{g})$. Since $P^-$ stabilizes $V^0 + V^-$, by (5.3)

$$u(\bar{g})v \subset V^0 + V^-,$$ 

for all $g \in E$. 

**Claim 1.** Let $h \in E$. Then for any proper subspaces $W_k$ of $\mathbb{R}^n$ for $1 \leq k \leq n$,

$$\{ \bar{g} - \bar{h} : g \in E \} \not\subset \bigcup_{k=1}^n W_k.$$

On the contrary, suppose $\{ \bar{g} - \bar{h} : g \in E \} \subset \bigcup_{k=1}^n W_k$. For every $g \in P^-U$, $\bar{g} - \bar{h} = g_00^{-1}I_0g - h_00^{-1}I_0h$. Therefore $I_0E \subset \bigcup_{k=1}^n (W_k \oplus I_0h)$. Since $W_k \oplus I_0h$ is a proper subspace of $\mathbb{R}^{n+1}$, this leads to a contradiction, so Claim 1 holds.

According to [17, Corollary 4.4], if [5,3] and Claim 1 hold, $\pi_0(u(h)v) \neq 0$ and it is fixed by $Z_G(\{a(t) : t > 0\})$. For any $b \in P^-$ and $w \in V$, $\pi_0(bw) = \lambda\pi_0(w)$ for some $\lambda \neq 0$. Therefore we conclude that $\pi_0(hw) \neq 0$ and it is fixed by $Z_G(\{a(t) : t > 0\})$. 

**Proposition 5.3.** Let $H \in \mathcal{H}_K$ be such that $g \not\subset H$. Let

$$E_H = \{ s \in \Omega : \Phi(s) \in P^-U \text{ and } H_s \subset H \}. \quad (5.5)$$

For any $s \in E_H$, there there exists a neighborhood $\Omega_s$ of $s$ such that $\Phi(\Omega_s \cap E_H)$ is contained in the union of at most $n$ proper subspaces of $\mathbb{R}^{n+1}$. 

**Proof.** Let $F$ be the closure of the subgroup of $G$ generated by all unipotent elements of $G$ contained in $H$. Then $F \neq G$. Since $H$ is a connected Lie group, $F$ is a real algebraic subgroup of $G$. Since $F$ admits no nontrivial characters, we choose a finite dimensional representation $V$ of $G$ with a vector $p_F \in V$ such that $F$ fixes $p_F$ and $V$ has no nonzero $G$-fixed vector.
Claim 2. For any \( s \in E_H \), \( \Phi(s)p_F \in V^0 + V^- \).

To see this, for any \( g \in \text{SO}(d) \), since \( \Psi(0) = s \), we have
\[
\lim_{t \to \infty} \pi_+(\Phi(\Psi(g\gamma(t^{-1})))p_F) = \pi_+(\Phi(s)p_F).
\]
(5.6)

Now let \( s \in E_H \) and \( g \in \text{SO}(d) \setminus (Z_s \cup Z_{x,s}) \). By (4.11),
\[
a(t)\Phi(\Psi(g\gamma(t^{-1})))p_F = (I + o(t^{-1})t)\xi_s(g)(I - tB_s(g))^{-1}p_F
\]
\[
= (I + o(t^{-1})t)\xi_s(g)p_F.
\]
(5.7)
because by (4.12), (4.13) and (4.15), \( (I - tB_s(g))^{-1} \) is a unipotent element of \( G \) contained in \( H \), so it fixes \( p_F \). Therefore from (5.6) we conclude that \( \pi_+(\Phi(s)p_F) = 0 \), otherwise (5.7) diverges as \( t \to \infty \). So Claim 2 holds.

Now suppose \( s \in E_H \) is such that for any neighborhood \( \Omega_s \) of \( s \), \( \phi(\Omega_s \cap E_H) \) is not contained in the union of any \( n \) proper subspaces of \( \mathbb{R}^{n+1} \).

Therefore in view of Claim 2, by Proposition 5.2 applied to \( \mathcal{E} = \Phi(E_H) \),
\[
\pi_0(\Phi(s)p_F) \neq 0, \text{ and it is fixed by } Z_G(\{a(t) : t > 0\}). \tag{5.8}
\]
Claim 3. \( \pi_0(\Phi(s)p_F) \) is fixed by \( u(z) \in U \) for some \( z \in \mathbb{R}^n \setminus \{0\} \).

To prove this, by our assumption we pick a sequence \( (s_i) \subset E_H \) such that \( s_i \to s \) and \( \phi(s_i) \notin \text{R} \phi(s) \), \( \forall i \). Since \( \phi(s_i) = I_0\Phi(s_i) \) and \( I_0P^-\Phi(s) \subset \text{R} \phi(s) \), we have \( \Phi(s_i) \notin P^-\Phi(s) \) for all \( i \). Therefore \( \Phi(s_i)\Phi(s)^{-1} = b_iu(z_i) \), where \( b_i \to I \) in \( P^- \) and \( 0 \neq z_i \to 0 \) as \( i \to \infty \). Let \( t_i = |z_i|^{-1/(n+1)} \). After passing to a subsequence, there exist \( 0 \neq z \in \mathbb{R}^n \) such that as \( i \to \infty \),
\[
a(t_i)\Phi(s_i)\Phi(s)^{-1}a(t_i)^{-1} = a(t_i)b_i(t_i^{-1}) \cdot u(z_i/|z_i|) \to u(z). \tag{5.9}
\]

Now since \( \Phi(s_i)p_F \in V^0 + V^- \),
\[
\lim_{i \to \infty} a(t_i)\Phi(s_i)p_F = \lim_{i \to \infty} \pi_0(\Phi(s_i)p_F) = \pi_0(\Phi(s)p_F).
\]
On the other hand by (5.9), as \( i \to \infty \),
\[
a(t_i)\Phi(s_i)p_F = a(t_i)\Phi(s_i)\Phi(s)^{-1}a(t_i)^{-1} \cdot a(t_i)\Phi(s)p_F \to bu(z) \cdot \pi_0(\Phi(s)p_F).
\]
Therefore \( \pi_0(\Phi(s)p_F) \) is fixed by \( u(z) \). This proves Claim 3.

By Claim 3 and (5.8), \( \pi_0(\Phi(s)p_F) \neq 0 \) and is fixed by the subgroup generated by \( u(z) \) and \( Z_G(\{a(t) : t > 0\}) \). Since every nontrivial element of \( U \) is conjugated to \( u(z) \) by an element of \( Z_G(\{a(t) : t > 0\}) \), we have that \( \pi_0(\Phi(s)p_F) \) is fixed by \( Z_G(\{a(t) : t > 0\})U \), which is a parabolic subgroup of \( G \). So \( \pi_0(\Phi(s)p_F) \neq 0 \) is fixed by \( G \), a contradiction to our choice of \( V \). \( \square \)

Proof of Theorem 4.4. By (5.1), (5.2), and (5.5),
\[
E_x = \phi^{-1}(\{0\} \times \mathbb{R}^n) \bigcup \{E_H : H \in \mathcal{H}_x \text{ and } H \not\subset G\}.
\]
Since \( \mathcal{H}_x \) is countable, by Proposition 5.3 \( E_x \) is a countable union of sets of the form \( \phi^{-1}(W) \), where \( W \) is a proper subspace of \( \mathbb{R}^{n+1} \). Therefore by Proposition 5.1, the Lebesgue measure of \( E_x \) is zero. Let \( s \in \Omega \setminus E_x \). Then \( H_s \supset G \). Therefore by Theorem 4.1, we get (4.2), which is same as
(1.3). Now (1.4) can deduced from (1.3) using the Lebesgue points of $\nu$ and convergence in measure. □

Next we show that the exceptional set is dense in many examples.

**Proposition 5.4.** Let $L = G = \text{SL}(n + 1, \mathbb{R})$ and $\Lambda = \text{SL}(n + 1, \mathbb{Z})$. Let $\phi : \mathbb{R}^d \to \mathbb{R}^{n+1}$ be a polynomial map with coefficients in $\mathbb{Q}$ and that its image is not contained in a proper subspace of $\mathbb{R}^{n+1}$, in particular it is nondegenerate on a Zariski open dense set $\Omega \subset \mathbb{R}^d$. Let $\Phi : \Omega \subset \mathbb{R}^d \to G$ be a map such that $I_0 \Phi(s) = \phi(s)$ for all $s \in \Omega$. Let $x \in \text{SL}(n+1, \mathbb{Q})/\Lambda \subset L/\Lambda$. Then $E_x \supset \Omega \cap \mathbb{Q}^d$.

**Proof.** Let $s \in \Omega \cap \mathbb{Q}^d$. In the notation of §4, $T$ and $\text{SO}(d)$ are defined over $\mathbb{Q}$. We choose an orthonormal basis $\{e_i : 1 \leq i \leq d\}$ of $T$ to be defined over $\mathbb{Q}$. Let $g \in \text{SO}(d)(\mathbb{Q})$. Then $M(g) \in \text{GL}(n+1, \mathbb{Q})$, and hence $f(g)$ is defined over $\mathbb{Q}$, and it is an abelian subalgebra consisting of nilpotent matrices. Therefore $H(g)$ is an abelian unipotent group defined over $\mathbb{Q}$. Since $\text{SO}(d)(\mathbb{Q})$ is Zariski dense in $\text{SO}(d)$, $H_s$ is a unipotent group defined over $\mathbb{Q}$, so it does not contain $G$. So $s \in E_x$. □

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