Testing the Galactic Centre potential with S-stars

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ABSTRACT

Two groups of astronomers used the large telescopes Keck and VLT for decades to observe trajectories of bright stars near the Galactic Centre. Based on results of their observations the astronomers concluded that trajectories of the stars are roughly elliptical and foci of the orbits are approximately coincide with the Galactic Centre position. In a last few years a self-gravitating dark matter core–halo distribution was suggested by Ruffini, Argüelles, Rueda (RAR) and this model was actively used in consequent studies. In particular, recently it has been claimed that the RAR-model provides a better fit of trajectories of bright stars in comparison to the conventional model with a supermassive black hole. The dark matter distribution with a dense core having a constant density as it was suggested in the RAR-model leaves trajectories of stars elliptical like in Kepler’s two-body problem. However, in this case not the foci of the ellipses coincide with the Galactic Center but their centers while the orbital periods do not depend on semi-major axes. These properties are not consistent with the observational data for trajectories of bright stars.

Key words: Dark matter – Supermassive black holes – Sgr A* – The Galactic Centre

1 INTRODUCTION

For decades astronomers observed bright stars which are moving very closely to the Galactic Centre. Analyzing trajectories of these stars one can deduce the functional form of the gravitational potential there. Observations of Keck and GRAVITY (VLT) groups showed that in the first approximation the stars are moving along elliptical orbits and foci of these orbits are roughly coinciding with the Galactic Centre (Ghez et al. 2003, 2005, 2008; Gillessen et al. 2009a; Genzel et al. 2010; Ciurlo et al. 2020). If we apply the simplest approach based on laws of classical mechanics we could follow the Newton’s derivation of the gravitational potential from the Kepler’s laws (Kopeikin et al. 2011). For the conventional model with the supermassive black hole and for a dense core model considered below such an approach looks reasonable since for these orbits radial coordinates \( r >> r_g \) \( (r_g \) is the Schwarzschild radius for supermassive black hole) and general relativistic effects could be evaluated as perturbations of Newtonian model. Recently, Gravity Collaboration et al. (2018a, 2019a) and Keck group (Do et al. 2019) has reported discovery of the relativistic redshift of spectral lines for S2 star near its periastris passage in May 2018, the redshift was consistent with theoretical estimates done in the first post-Newtonian approximation of general relativity. These results are confirming the universality of gravity laws and did not support the alternative theories of gravity predicting different value of redshift in the weak and strong gravitational field (Capozziello et al. 2006, 2007; Capozziello & De Laurentis 2011).

In the last year Gravity Collaboration et al. (2020) reported a discovery of relativistic (Schwarzschild) precession for S2 star orbit.

If there is a supermassive black hole along with a bulk distribution of ordinary matter forming a stellar cluster and a dark matter inside the orbits of bright stars, the bulk distribution of matter causes precession of the stellar orbits in the direction which is opposite to that caused by relativistic effects of the black hole (Rubilar, Eckart 2001; Nucita, De Paolis, Ingrosso et al. 2007; Zakharov et al. 2007).

Observational data for trajectories of bright stars can be used to test predictions of general relativity and to constrain parameters of alternative theories of gravity such as \( f(R) \) (Borka et al. 2012), Yukawa potential (Borka et al. 2013), theories with massive graviton (Zakharov et al. 2016, 2018; Hees et al. 2017), black holes with a tidal charge (Zakharov et al. 2018), and others (Gravity Collaboration et al. 2019b). Constraints on variations of a fine structure constant were imposed by observations of bright stars near the supermassive black hole at the Galactic Centre as shown in (Hees et al. 2020), where other astrophysical ways to constrain the fine structure constant variations are mentioned as well.

Many theoretical models for the Galactic Centre have been proposed in the past and some of them were rejected as being inconsistent with the analysis of observational data and at the time being the model consisting of a single supermassive black hole looks the most preferable. Nonetheless, several years ago Ruffini, Argüelles & Rueda (2015) proposed that the gravitational potential at the central part of our galaxy is better approximated by a dark matter distribution having a dense core and a diluted halo. Later, the dark matter distribution was called the RAR-model. Recently Becerra-Vergara et al. (2021) declared that this model pro-
vides a better fit of trajectories of the bright stars in comparison with the supermassive black hole model. Below we consider properties of bright star trajectories in gravitational field of a dense core in the RAR-model and conclude that these properties are not consistent with the existing observational data.

2 CENTRAL FIELD

We assume that a gravitational potential for the Galactic Centre is spherically symmetric. At this stage we use the laws of the Newtonian mechanics. This is sufficient for our goal and the general relativistic effects have a post-Newtonian order of magnitude and do not change our conclusions. Here we introduce basic notations for the central field following the textbooks by Whittaker (1917); Arnold (1989); Goldstein, Poole & Safko (2002). Let’s assume, that the potential \( U = U(r) \) does not depend on vector \( r \), it depends only on the radial distance \( r = |r| \). Then, motion of a freely-falling test particle with mass \( m \) obeys the Newtonian law of gravity

\[
m\ddot{r} = -\frac{\partial U}{\partial r},
\]

where \( e_r \) is an orthonormal vector in a radial direction. Therefore, the conservation law for angular momentum can be written in the form

\[
h = m\dot{\phi}(r)t^2 = \text{const},
\]

where \( \phi \) is the angle in the orbital plane and the dot denotes a time derivative. Introducing potential \( V(r) = U(r) + \frac{\hbar^2}{2mr^2} \), one has

\[
m\ddot{r} = -\frac{\partial V}{\partial r}.
\]

The total energy of the particle in one-dimensional case is

\[
E = \frac{m\dot{r}^2}{2} + V(r).
\]

From Eq. (4) one has

\[
r = \sqrt{\frac{2m}{E-V(r)}},
\]

therefore,

\[
t = \int \frac{dr}{\sqrt{\frac{2m}{E-V(r)}}}.
\]

Integrating Eq. (2) we get

\[
\phi = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2mE}{\hbar^2} - \frac{2mu}{\hbar^2} - u^2}}.
\]

If we introduce a new variable \( u = 1/r \), we have

\[
\phi = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\hbar^2} - \frac{2mu}{\hbar^2} - u^2}}.
\]

The corresponding motion is bounded and it is located inside a ring with an inner radius \( r_{\text{min}} \) and an outer radius \( r_{\text{max}} \).

For bounded orbits we could introduce the following angle characterizing precessional motion of the orbit, see for instance, the textbooks by Landau & Lifshitz (1993) and Arnold (1989)

\[
\Phi = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{hdr}{m^2 \sqrt{\frac{2}{m}[E - V(r)]}}.
\]

For a central potential \( U(r) \) Bertrand (1873) proved the following theorem (see also the available English translation by Santos (2011)): there are only two potentials of the central field which yield closed elliptical orbits. These potentials are the \( U_{\text{HO}} = ar^2 \) and \( U_N = -k/r \) (the harmonic oscillator and the Newtonian potentials). This result is also discussed by Landau & Lifshitz (1993) and Arnold (1989), while the book by Arnold (1990) presents a historical background leading to the discovery of Bertrand’s theorem.

3 CLOSED ORBITS IN HARMONIC OSCILLATOR POTENTIAL

As soon as a particular potential is known we can calculate the angle \( \Phi \) in Eq. (11). For instance, for the harmonic oscillator and for this case \( \Phi_{\text{HO}} = \pi/2 \) (Arnold 1989) while it is well-known that for the Newtonian potential one has \( \Phi_N = \pi \). It is known that the orbit is closed if the angle \( \Phi \) commensurable with \( 2\pi \), or in other words, if \( \Phi = 2\pi \alpha/\beta \), where \( m \) and \( n \) are integers (Arnold 1989). Thus, computation of the angle \( \Phi \) immediately tells us whether the orbit is closed or not. Clearly, they are closed in both cases of the harmonic oscillator and the Newtonian potential. What remains is to understand where the geometric center of the elliptical orbit is located with respect to the center of gravity. It is well known that for the Newtonian potential the center of gravity is not at the center of the ellipse but in its focal point. This is not the case for the harmonic oscillator potential as we demonstrate below.

Substituting \( U_{\text{HO}} = a/r^2 \) in Eq. (8), we obtain (since \( u = 1/r \))

\[
\phi = -\int_{\mu}^{\nu} \frac{du}{\sqrt{\frac{2mE}{\hbar^2} - \frac{2ma}{\hbar^2} - u^2}}.
\]

Following Whittaker (1917) we introduce notations \( \beta = \frac{2mE}{\hbar^2} \), \( \mu = 2ma \), \( \nu = u^2 \), and showed that

\[
\phi = -\frac{1}{2} \int_{\nu_0}^{\nu} \frac{dv}{\sqrt{\frac{\beta - \frac{\mu}{\hbar^2}}{\frac{\beta^2}{4} - \frac{\mu^2}{\hbar^4} - v^2}}}. \tag{13}
\]

The integral in Eq. (13) is evaluated with Euler’s substitutions and if \( \gamma \) is an integration constant, we obtain (Whittaker 1917)

\[
2(\phi - \gamma) = \arccos \left( \frac{\nu - \beta}{\frac{\beta^2}{4} - \frac{\mu^2}{\hbar^4}} \right), \tag{14}
\]

therefore, following Whittaker (1917)

\[
\frac{1}{r^2} = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \frac{\mu^2}{\hbar^4} \cos(2\phi - 2\gamma)}, \tag{15}
\]

This equation describes the ellipse whose center is at the origin of coordinates that coincides with the center of gravity. The angular frequency and the period of harmonic oscillator potential are \( \omega = \sqrt{2\beta} \) and \( T_{\text{HO}} = 2\pi/\omega \).
4 CENTRAL PART OF DARK MATTER DISTRIBUTION IN THE RAR MODEL: CORE POTENTIAL

Recently, many alternative models have been proposed for the Galactic Center in addition to the conventional model of the supermassive black hole with mass around $4 \times 10^6 M_\odot$. There are two main reasons to develop such alternative models. First, some theorists hope that alternative theories of gravity could explain dark matter phenomenon and properties of compact objects in alternative theories may be different from properties of black holes in general relativity. Second, other researchers hope that proposed dark matter distributions could provide better fits for observational data and many authors argued that instead of the supermassive black hole the Galactic Centre contains a dense core with constant density, see for instance, Chavanis & Sommeria (1998); Bilic et al. (2002a,b); Ruffini, Argüelles & Rueda (2015); Argüelles et al. (2016, 2018, 2019, 2020); Becerra-Vergara et al. (2020, 2021) and references therein. These authors considered various dark matter distributions with a low density halo and a high density core. The most popular model of this type was proposed by Ruffini, Argüelles & Rueda (2015) and it is called the RAR-model. In the RAR model the central part of the galaxy represents a ball of constant density $\rho_0$ having a radius $R$. Let us consider motion of a test particle (a star) in the potential field of the RAR model. We assume that the orbit of the particle is bounded and has a pericenter $r_p$ and an apocenter $r_a$.

In this case, one could analyze three different conditions for density and related potentials.

**Condition A.** $r_p \leq R$ and star is moving inside the ball of constant density $\rho_0$. Then we have a harmonic oscillator potential $U_{HO}(r)$ along the entire orbit, since according to the law of the Newtonian gravity the potential is formed only by the matter contained inside the radius $r$

$$4\pi G \int_0^r \rho_0 r^2 dr = U_{HO}(r) = \frac{1}{2}ar^2,$$  \hspace{1cm} (16)

where $a = 2\pi G \rho_0$, $G$ is the gravitational constant.

**Condition B.** $r_p \leq R < r_a$ and the particle moves both inside and outside of the dense core.

**Condition B1.** If we assume that the mass of the dark matter contained inside of the pericenter of the particle’s orbit is equal to the mass of the presumed BH

$$4\int_0^{r_a} \rho_0 r^2 dr \approx M_{BH}$$  \hspace{1cm} (17)

and the mass of the dark matter outside of the ball of radius $r_p$ is much smaller than the mass of the BH

$$4\pi \int_{r_p}^{r_a} \rho(r)r^2 dr \ll M_{BH},$$  \hspace{1cm} (18)

then this case is very similar to the standard approach consisting of a supermassive black hole and perhaps an extended matter distribution with a negligibly small mass inside a quasi-elliptical orbit of a star. As we noted earlier for our needs the Newtonian approximation looks reasonable since currently the observed bright stars move very far away from the black hole horizon.

**Condition B2.** If

$$4\pi G \int_0^r \rho(r)r^2 dr = U_{HO}(r) \approx U_{HO}(r) = \frac{1}{2}ar^2, \quad \forall r \in (R, r_a),$$  \hspace{1cm} (19)

then one could use $U_{HO}(r)$ as a suitable approximation for the gravitational potential. Eq. (19) is valid for very small density gradients for all $r$ in the interval $R < r < r_a$ (or in other words, for these $r$ we have $\rho(r) \approx \rho_0$). This case is similar to the case with Condition A, where $U_{HO}(r)$ determines the exact expression for the gravitational potential, however, trajectories in the harmonic oscillator potential are not exact solutions but suitable approximations for potential $U_{HO}(r)$.

Condition B3. If conditions B1 and B2 are not valid, then the theoretical model for gravitational potential can not be approximated with $U_{HO}(r)$ and $U_N(r)$. Therefore, in principle there is an opportunity that there are bounded orbits which are not elliptical and not closed. One should note that observed bounded orbits at the Galactic Center are closed and quasi-elliptical in the first approximation. This case deserves a separate detailed investigation but we do not pursue it in this Letter.

**Condition C.** In this case $R < r_p$ and relations (17,18) are valid then $U_N(r)$ could be chosen as a good approximation for a gravitational potential. Otherwise, similarly to case B3, gravitational potential can not be approximated with $U_{HO}(r)$ and $U_N(r)$.

5 CONCLUSIONS

High-precision astrometric monitoring of quasi-elliptical trajectories of stars with high eccentricities is instrumental for distinguishing $U_{HO}(r)$ and $U_N(r)$ potentials since the centers of the elliptic orbits of the stars should coincide with the Galactic Center in the case of the RAR potentials for a dense core while in the case of the Newtonian potential star foci of the ellipses coincide with the Center. As it was noted in Section 3 the orbital periods of stars moving in the field of harmonic oscillator potential are constant and they do not depend on semi-major axes. Assume that we have to choose a suitable potential for the Galactic Centre from two options $U_{HO}(r)$ and $U_N(r)$. An inspection of a set of trajectories with high eccentricities (see, for instance, Fig. 16 in paper by Gillessen et al. (2009a), Fig. 16 (right panel) from paper by Genzel et al. (2010) and Fig.8 in paper by Gillessen et al. (2017)) clearly showed that stars are moving around the joint focus but not the centre.

Therefore, currently, we can state with a rather good confidence that all bounded observed orbits of bright stars near the Galactic Centre are elliptical and their foci practically coincide with the Galactic Centre. Therefore, we have to conclude that the central potential inside the spherical shell where these observed bounded orbits are located must be the Newtonian $U_N(r)$. In Section 3 it was shown that in the case of a harmonic oscillator potential $U_{HO}(r)$ which corresponds to the RAR dark matter model with a dense core inside a ball with a constant density and ellipse centers for different orbits should coincide with the Galactic Centre. Periods of trajectories in $U_{HO}(r)$ potential do not depend on semi-major axis. These properties are not supported by observational data reported by Keck and GRAVITY (VLT) teams which means that the conventional model with a supermassive black hole generating approximately the Newtonian potential in a spherical shell where bright star orbits are located, is a preferable model for physical interpretation of the observed trajectories.

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DATA AVAILABILITY

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