CLUTHING AND GLUING IN TROPICAL AND LOGARITHMIC GEOMETRY

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ABSTRACT. The classical clutching and gluing maps between the moduli stacks of stable marked algebraic curves are not logarithmic, i.e. they do not induce morphisms over the category of logarithmic schemes, since they factor through the boundary. Using insight from tropical geometry, we enrich the category of logarithmic schemes to include so-called sub-logarithmic morphisms and show that the clutching and gluing maps are naturally sub-logarithmic. Building on the recent framework developed by Cavalieri, Chan, Wise, and the third author, we further develop a stack-theoretic counterpart of these maps in the tropical world and show that the resulting maps naturally commute with the process of tropicalization.

1. INTRODUCTION

In [15], building on the classical work of Deligne and Mumford [9], Knudsen has introduced the moduli stacks $\overline{M}_{g,n}$ of stable algebraic curves of genus $g$ with $n$ marked points as well as a natural system of maps between these moduli spaces:

- the forgetful maps $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ given by forgetting the $(n+1)$-st marked point (and possibly stabilizing the resulting $n$-marked curve);
- the clutching maps $\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$ given by identifying the $(n_1+1)$-st marked point of the first curve with the $(n_2+1)$-st point of the second curve in a node; and
- the (self-)gluing maps $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ given by identifying the last two marked points of an $(n+2)$-marked curve in a node.

The clutching and gluing maps factor through the boundary of $\overline{M}_{g,n}$ and therefore they cannot induce morphisms between the moduli stacks $\overline{M}^\log_{g,n}$ of stable logarithmic curves (of genus $g$ with $n$ marked points) in the sense of [12]. The reason is that the monoidal coordinates that are normal to the boundary of $\overline{M}^\log_{g,n}$ would have to be sent to an absorbing element in the logarithmic structure of $\overline{M}^\log_{g,n}$ which is not allowed in the usual framework of logarithmic geometry (as developed in [13]).

In this note, we use insights coming from tropical geometry (see [1, Section 8]) to enrich the usual logarithmic structures in the sense of [13] to so-called pointed logarithmic structures that allow an absorbing element in the sheaf of monoidal coordinates. Using this language we may define a natural stack $\overline{M}^\log_{g,n}$ of stable logarithmic curves over the category of pointed logarithmic schemes that generalizes $\overline{M}^\log_{g,n}$ and that is representably by the algebraic moduli stack $\overline{M}_{g,n}$ with its pointed logarithmic structure coming from its boundary divisor. The stacks $\overline{M}^\log_{g,n}$ admit natural clutching and gluing maps in the category of pointed logarithmic algebraic stacks, an
important instance of what we call sub-logarithmic morphisms (see Section 3.1), whose underlying algebraic morphisms are the classical clutching and gluing maps defined in [15].

Let us now give a quick outline of the contents of this article: In Section 2 we prove an equivalence between the category of pointed toric monoids and the category of extended rational polyhedral cones, thereby providing a dictionary between the languages used in logarithmic geometry and tropical geometry respectively. In Section 3 we introduce pointed logarithmic structures and develop their basic geometric properties, expanding on [13]. In Section 4 we introduce the moduli stacks $\mathcal{M}_{g,n}^{\text{log}}$ as well as their natural clutching and gluing maps. In Section 5 we finally construct a generalization of the moduli stack $\mathcal{M}_{g,n}^{\text{trop}}$ of tropical curves, as introduced in [7], to a stack over the category of extended rational polyhedral cones as well as tropical clutching and gluing maps in this framework. Our main results, discussed in Section 4 and Section 5 can be summarized as follows (see Theorem 4.3 and Theorem 5.8 for a formal statement).

**Theorem 1.1.** The classical clutching and gluing maps between products of moduli spaces of stable curves induce sub-logarithmic morphisms and naturally commute with the process of tropicalization.

Since the $i$-th universal section of the universal family over $\mathcal{M}_{g,n}^{\text{log}}$ is itself an instance of a clutching map, it is a straightforward corollary that the tropicalization of such a universal section provides a universal section of the tropicalization of the family. We expect such tropical universal sections will be necessary for the development of tropical analogues to the standard tautological classes on these moduli spaces.

Finally, in [7, Section 6] the authors chose to use a different route to realize clutching and gluing in logarithmic geometry. The main difference here is that clutching and gluing become correspondences instead of morphisms and this approach gives rise to generalized clutching and gluing maps on the tropical side, as hinted upon in [1, Section 8.6].

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2. Pointed monoids and extended rational polyhedral cones

2.1. **Monoids and rational polyhedral cones.** A monoid $P$ is a commutative semigroup with a neutral element, usually written additively as $(P, +, 0)$. A monoid is said to be integral if the natural map $P \to P^{gp}$ into its Grothendieck group is injective. A monoid is fine if it is finitely generated and integral. An integral monoid is saturated if, whenever we have $n \cdot p \in P$ for some $n > 0$ and $p \in P^{gp}$, then $p \in P$. A fine and saturated monoid is toric if it is torsion-free. Denote by $\text{Mon}_{fs}$ the category of fine, saturated monoids and by $\text{Mon}_{tor}$ the full subcategory of toric monoids.

An ideal $I$ of $P$ is a subset such that $p + I \subseteq I$ for all $p \in P$. An ideal $p$ is called prime if, whenever $p + q \in p$ then either $p \in p$ or $q \in p$, or, equivalently, if the complement $P - p$ is a submonoid. We write $\text{Spec} P$ for the set of prime ideals of $P$. It carries a natural Zariski topology
generated by the basic open subsets \( D(f) = \{ p \in \text{Spec} P | f \not\in p \} \) for \( f \in P \) (see [14] and [21, Section 3] for details).

A \((\text{strictly convex})\) \textit{rational polyhedral cone} (cone for short) is a pair \((\sigma, N)\), consisting of a finitely generated free abelian group \( N \) and a cone \( \sigma \subseteq N_\mathbb{R} := N \otimes \mathbb{R} \), that is, a finite intersection of half-spaces \( \sigma = \cap_i H_i \) of the form

\[
H_i := \left\{ u \in N_\mathbb{R} | \langle u, m_i \rangle \geq 0 \right\},
\]

where the \( m_i \) are elements of the dual lattice \( M := \text{Hom}(N, \mathbb{Z}) \), so that \( \sigma \) does not contain any non-trivial linear subspaces. A morphism of rational polyhedral cones \((\sigma, N) \hookrightarrow (\sigma', N')\) is a morphism \( \varphi \in \text{Hom}(N, N') \) of lattices whose canonical extension to the linear map of vector spaces \( \varphi_\mathbb{R} : N_\mathbb{R} \rightarrow N'_\mathbb{R} \) maps \( \varphi(\sigma) \subset \sigma' \). We denote the category of rational polyhedral cones by \( \text{RPC} \).

The \textit{dual cone} \( \sigma^\vee \) of \( \sigma \) is given by

\[
\sigma^\vee = \left\{ v \in M_\mathbb{R} | \langle u, v \rangle \geq 0 \text{ for all } u \in \sigma \right\},
\]

It may also be described as a finite intersection of dual half-spaces built from the \( H_i \) above. A \textit{face} of a cone, i.e. an intersection \( \tau = H_m \cap \sigma \), for some \( m \in \sigma^\vee \), is written as \( \tau \leq \sigma \). The \textit{dual face} to \( \tau \) is the face of \( \sigma^\vee \) defined by \( \sigma^\vee \cap \tau^\perp \), where

\[
\tau^\perp = \left\{ u \in M_\mathbb{R} | \langle u, v \rangle = 0 \text{ for all } v \in \tau \right\}.
\]

We finish by recalling the following well-known Proposition 2.1.

\textbf{Proposition 2.1.} \quad (i) There is a natural equivalence of categories

\[
S : \text{Mon}_{\text{tor}} \xrightarrow{\sim} \text{RPC}
\]

\[
P \mapsto \sigma_P = \text{Hom}(P, R_{\geq 0})
\]

\[
S_\sigma = \sigma^\vee \cap M \longmapsto (\sigma, N).
\]

(ii) Let \( P \) be a toric monoid and \( \sigma_P \) the associated cone. Given a face \( \tau \) of \( \sigma \), define \( p_\tau := S_\sigma - \tau^\perp \). The above equivalence induces a natural one-to-one correspondence

\[
\text{Spec} P \xrightarrow{\sim} \{ \text{faces of } \sigma \}
\]

such that \( \tau' \leq \tau \leq \sigma_P \text{ if and only if } p_\tau \text{ specializes to } p_{\tau'} .
\]

\textit{Proof.} Part (i) is well-known and we leave the details of the proof to the avid reader (also see e.g. [11, Section 1.2] or [21, Proposition 2.2]). Note, hereby, that the lattice associated to the monoid \( \sigma_P \) is given by the dual \( \text{Hom}(P^{gp}, \mathbb{Z}) \) of \( P^{gp} \).

For Part (ii), note that \( p_\tau := S_\sigma - \tau^\perp \) is a prime ideal of \( S_\sigma \) since it consists of the lattice points of the complement of a face of \( \sigma^\vee \). Given a prime ideal \( p \) of \( P \), we define the associated face \( \tau_p \) of \( \sigma_P \) to be

\[
\tau_p = \text{Cone}(P - p)^\perp \cap \sigma_P .
\]
Let \( p \) be a prime ideal of \( P \) and \( \tau = \tau_p \) the associated face of \( \sigma_p \). Since \( S_\sigma = \sigma^\vee \cap M = P \) and \( \sigma \) is a full dimensional cone inside \( \mathbb{N}_\mathbb{R} \), we have

\[
p_\tau = (\sigma^\vee - \tau^\perp) \cap M = P - (\text{Cone}(P - p)^\perp \cap \sigma)^\perp \cap M = P - (\text{Cone}(P - p) + \sigma^\perp) \cap M = P - (P - p) = p.
\]

Therefore we have \( p_\tau \cap p = p \). The converse verification that \( p_\tau \cap p = p \) is left to the reader. Finally, notice that \( \tau' \) is a proper face of \( \tau \) if and only if there is an element \( f \in S_\sigma \) such that \( f \notin p_{\tau'} \) and \( f \in p_\tau \). This implies that \( p_\tau \) is in the closure of \( p_{\tau'} \), i.e. that \( p_{\tau'} \) specializes to \( p_\tau \). \( \square \)

2.2. Pointed Monoids. A pointed monoid is a monoid \( P \) containing a non-zero absorbing element \( \infty_p \in P \) satisfying

\[
p + \infty_p = \infty_p + p = \infty_p
\]

for every element \( p \in P \). Such an absorbing element is unique when it exists. A morphism \( f: P \to Q \) between pointed monoids is a monoid morphism mapping the absorbing element of \( P \) to the absorbing element of \( Q \). Denote by \( \text{Mon}^\infty \) the category of pointed monoids. Although \( \text{Mon}^\infty \) is a subcategory of the category of monoids \( \text{Mon} \), it is not a full subcategory: a zero morphism, for instance, will never map \( \infty_p \) to \( 0_Q \). We refer the reader to [6] for further background on the theory of pointed monoids.

There is a pointification functor

\[
[\ ]^\infty: \text{Mon} \to \text{Mon}^\infty
\]

that takes a monoid \( P \) to \( P^\infty := P \cup (\infty_p) \) and a morphism \( f: P \to Q \) of monoids to the morphism \( f^\infty: P^\infty \to Q^\infty \) defined by \( f^\infty|_P = f \) and \( f^\infty(\infty_p) = \infty_Q \). Since for each pointed monoid \( P \) in \( \text{Mon}^\infty \) it is immediate that \( P^\infty = P \), we throughout this section decorate all our pointed monoids with the superscript infinity. In the other direction, a pointed monoid \( P^\infty \) has an underlying unpointed monoid \( P^0 \) obtained by removing the absorbing element; this operation does not produce a functor back to \( \text{Mon} \). Note that we may canonically identify the sets \( \text{Hom}(P^\infty, Q^\infty) \) and \( \text{Hom}(P^\infty, Q^\infty) \) in \( \text{Mon} \) since morphisms of pointed monoids must send absorbing elements to absorbing elements (this is not the case when the co-domain Q is not already pointed).

For any ideal \( I \subseteq P^0 \) there is a canonical minimal extension to an ideal \( I \subseteq P^\infty \) given by including \( \infty_p \in I \), and likewise in the other direction by removing the absorbing element. We abuse notation by using the same notation \( I \) for both ideals. For each morphism \( f: P^\infty \to Q^\infty \) of pointed monoids, the set \( f^{-1}(\infty_Q) \) is a prime ideal of \( P^\infty \).

The category of pointed toric monoids, which we denote \( \text{Mon}_{\text{tor}}^\infty \), is the full subcategory of \( \text{Mon}^\infty \) whose objects are in the image of \( \text{Mon}_{\text{tor}} \) via the above pointification functor \([\ ]^\infty \).

In the case that \( f^{-1}(\infty_Q) = (\infty_p) \) consists only of the absorbing element of \( P^\infty \), the map \( f \) is simply the image via the pointification functor of the restriction \( f|_{P^0}: P^0 \to Q^0 \); we also call such a morphism of pointed monoids toric.

Given any ideal \( I \subseteq P^\infty \), the standard construction of a quotient monoid is given by the Rees quotient [18]

\[
P^\infty/I = \{[p]|p \in P^\infty - I\} \cup \{[I]\}
\]
where the monoid operation is given by

\[
[p] + [q] \mapsto \begin{cases} 
[p + q] & \text{for } p + q \notin I \\
I & \text{otherwise.}
\end{cases}
\]

This construction works for ideals in both pointed and unpointed monoids, but always produces a pointed monoid as a quotient. The universal property of Rees quotients yields that any morphism \( f : P^\infty \to Q^\infty \) in \( \text{Mon}_{\text{tor}}^\infty \) admits a unique factorization

\[
P^\infty \xrightarrow{g} P^\infty/f^{-1}(\infty_Q) \xrightarrow{h} Q^\infty
\]

where \( g \) is the quotient map and \( h^{-1}(\infty_Q) \) consists only of the absorbing element of \( P^\infty/f^{-1}(\infty) \).

2.3. **Extended rational polyhedral cones.** Following [20] and [1, Section 2], a rational polyhedral cone \( \sigma = \text{Hom}(P^\circ, \mathbb{R}_{\geq 0}) \) has a canonical compactification given by

\[
\overline{\sigma} = \text{Hom}(P^\circ, \mathbb{R}_{\geq 0}) = \text{Hom}(P^\infty, \mathbb{R}_{\geq 0})
\]

where \( \mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\} \) is the extended non-negative real half-line, a pointed monoid under addition. We call \( \overline{\sigma} \) the associated extended rational polyhedral cone (extended cone for short).

Given a face \( \tau \) of \( \sigma \) let \( N_\tau \) denote the sublattice of \( N \) spanned by the points in \( \tau \cap N \) and set \( N(\tau) = N/N_\tau \). The cone quotient \( \sigma/\tau \) is a cone in \( N(\tau)_R \) whose faces correspond to the faces of \( \sigma \) containing \( \tau \) via the induced quotient map of vector spaces \( N_R \twoheadrightarrow N(\tau)_R \):

\[
\sigma/\tau = \{ [\tau'] \subset N(\tau)_R \mid \tau \preceq \tau' \preceq \sigma \}.
\]

We denote by \( \phi_\tau : \sigma \longrightarrow \sigma/\tau \) the corresponding quotient map of cones. Given any sequence of intermediate faces \( \tau \preceq \tau' \preceq \sigma \) the quotient map factors uniquely as a composition

\[
\phi_{\tau'} : \sigma \xrightarrow{\phi_\tau} \sigma/\tau \longrightarrow \sigma/\tau'.
\]

The extended boundary \( \overline{\sigma} - \sigma \) is a union of locally closed subsets, the faces at infinity, determined uniquely by inclusions of faces \( \tau' \preceq \tau \) in \( \sigma \) (see e.g. [16, Section 3], [17, Proposition 3.4], and [21, Section 3.4]). We phrase this observation, adapted to our situation, as follows.

**Proposition 2.2.** Let \( \sigma \) be a rational polyhedral cone and \( \tau \preceq \sigma \) one of its faces.

(i) Given a face \( \tau \) of \( \sigma \), we define \( i_\tau : \sigma/\tau \to \overline{\sigma} = \text{Hom}(S_\sigma, \mathbb{R}_{\geq 0}) \) by associating to an element in \( \sigma/\tau \) represented by \( \nu \in \sigma \) the unique element \( i_\tau(\nu) \in \overline{\sigma} = \text{Hom}(S_\sigma, \mathbb{R}_{\geq 0}) \) such that

\[
u \mapsto \begin{cases} 
\langle \nu, u \rangle & \text{if } u \in \tau^\perp \cap S_\sigma \\
\infty & \text{else}
\end{cases}
\]

for all \( u \in S_\sigma \). The maps \( i_\tau \) are well-defined and induce a stratification

\[
i : \bigsqcup_{\tau \preceq \sigma} \sigma/\tau \longrightarrow \overline{\sigma}
\]

by locally closed subsets.
(ii) Given a face \( \tau \) of \( \sigma \), the closure of \( i(\sigma/\tau) \) in \( \overline{\sigma} \) is equal to the canonical compactification \( \overline{\sigma/\tau} \) of \( \sigma/\tau \).

In fact, we have a commutative diagram

\[
\begin{array}{ccc}
\bigcup_{\tau \leq \tau' \leq \sigma} \sigma/\tau' & \longrightarrow & \overline{\sigma/\tau} \\
\downarrow \subseteq & & \downarrow \tau \\
\bigcup_{\tau' \leq \sigma} \sigma/\tau' & \longrightarrow & \overline{\sigma}
\end{array}
\]

Proof. Part (i) follows immediately from [17, Proposition 3.4 (i)] by noticing that \( \sigma/\tau \) is nothing but the image of \( \sigma \) in \( \text{Hom}(S_{\sigma}, \mathbb{R}) \) under the natural inclusion \( \text{Hom}(S_{\sigma}, \mathbb{R}) \to \text{Hom}(S_{\sigma}, \mathbb{R}) \). For part (ii) notice that the strata of \( \overline{\sigma/\tau} \) have to be equal to the strata \( \sigma/\tau' \) of \( \sigma \) for which \( \tau \preceq \tau' \).

We refer to \( i(\sigma/\tau) \subseteq \overline{\sigma} \) as the maximal face at infinity of \( \overline{\sigma} \) associated to \( \tau \preceq \sigma \). A morphism of extended cones is given by a continuous map of topological spaces, \( \overline{f} : \overline{\sigma} \to \overline{\sigma}' \) so that the restriction map \( f := \overline{f}|_\sigma \) is a cone morphism into some maximal face at infinity of \( \overline{\sigma}' \). The category of extended rational polyhedral cones will be denoted by \( \text{RPC}_\infty \).

Indeed, a morphism \( f : \sigma \to \sigma' \) of extended cones is completely determined by the data of a pair \( (\tau', f : \sigma \to \sigma'/\tau') \) where \( \tau' \) is a face of \( \sigma' \) and \( f \) is a morphism of cones: \( \tau' \) identifies the maximal face containing the image of \( \overline{f} \), the map \( f : \sigma \to \sigma'/\tau' \) is the underlying map of cones extending uniquely to \( \overline{f} \) by continuity, and \( \overline{f} \) is given by the composition

\[
\begin{array}{ccc}
\sigma & \xrightarrow{\overline{f}} & \sigma' \\
\downarrow \tau' & & \downarrow \tau'
\end{array}
\]

The inclusion map \( \iota_{\tau'} \) identifying the extended cone \( \overline{\sigma'/\tau'} \) with a subset of \( \text{Hom}(S_{\sigma'}, \mathbb{R} \geq 0) \) is determined by sending an element \( u \in \sigma'/\tau' \) to the homomorphism

\[
p \mapsto \begin{cases} 
\langle u, p \rangle & \text{if } p \in (\tau'^\perp \cap \sigma'\cap S_{\sigma'}) \\
\infty & \text{else}
\end{cases}
\]

and extending by continuity (see [17, Proposition 3.4] or [16, Section 3]). When \( \overline{f} = (0, f : \sigma \to \sigma') \) then \( \overline{f} \) is the canonical extension of the morphism of cones \( f \) by continuity; we call such a morphism of extended cones toric.

Given morphisms of extended cones \( \overline{f} : \overline{\sigma}_1 \to \overline{\sigma}_2 \) and \( \overline{g} : \overline{\sigma}_2 \to \overline{\sigma}_3 \), we now describe their composition. The morphisms \( \overline{f} \) and \( \overline{g} \) are determined respectively by pairs \( (\tau, f : \sigma_1 \to \sigma_2/\tau) \) and \( (\upsilon, g : \sigma_2 \to \sigma_3/\upsilon) \) with corresponding quotient maps for the cone quotients \( \phi_\tau : \sigma_2 \to \sigma_2/\tau \) and \( \phi_\upsilon : \sigma_3 \to \sigma_3/\upsilon \).

**Lemma 2.3.** Let \( \sigma \to \sigma' \) be a morphism of cones and \( \tau \preceq \sigma \). Then \( f \) factors through \( \phi_\tau : \sigma \to \sigma/\tau \) if and only if \( f(\tau) = 0 \). When such a factorization exists, it is unique.

Proof. This follows immediately from the properties of quotients of vector spaces. \( \square \)
By Lemma 2.3, there exists a unique morphism of cones \( h : \sigma_2 / \tau \to \sigma_3 / \omega \) fitting into the commutative diagram

\[
\begin{array}{ccc}
\sigma_2 & \xrightarrow{g} & \sigma_3 / \tau_3 \\
\downarrow{\phi_{\tau_2}} & & \downarrow \\
\sigma_1 & \xrightarrow{f} & \sigma_2 / \tau_2 \xrightarrow{h} \sigma_3 / \omega
\end{array}
\]

if and only if \( \omega \) is a face of \( \sigma_3 \) containing both \( \upsilon \) and \( g(\tau) \). Continuity of \( g \) ensures that the composition \( \overline{g} \circ \overline{f} := (\omega, h \circ f : \sigma_1 \to \sigma_3 / \omega) \) lands in the cone \( \sigma_3 / \omega \) for \( \omega \) the smallest face of \( \sigma_3 \) containing both \( \upsilon \) and \( g(\tau) \). This is made precise in the following lemma.

**Lemma 2.4.** Let \( \overline{f} = (0, \sigma \to \sigma') \) be a toric morphism of extended cones and let \( \sigma / \tau \) be the maximal face of \( \sigma \) at infinity corresponding to \( \tau \preceq \sigma \). Then \( \overline{f}(\sigma / \tau) \subseteq \sigma' / \omega \) where \( \omega \) is the smallest face of \( \sigma' \) containing \( \overline{f}(\tau) \).

**Proof.** The image of \( \sigma / \tau \) through \( \overline{f} \) must land in some maximal face of \( \sigma' \) at infinity. Denote this face at infinity by \( \sigma' / \omega \) and assume for contradiction that \( \omega \) is not the smallest face of \( \sigma' \) containing \( \overline{f}(\tau) \). Then there is some intermediary face \( \gamma \preceq \sigma' \) with \( \gamma \prec \omega \) and \( \overline{f}(\tau) \) a subcone of \( \gamma \). We then have that \( \sigma' / \gamma \) is a maximal face of \( \sigma' / \omega \) at infinity, with \( \overline{f}(\sigma / \tau) \subseteq \sigma' / \omega \). Let \( p \in \tau \) be a vector in the maximal face of \( \tau \) and consider the limit point \( \overline{p} = \lim_{t \to \infty} tp \). In the topology of \( \sigma \), we must have \( \overline{p} \in \sigma / \tau \). Notice, however, that \( f(tp) \in \gamma \) for any \( t \in \mathbb{R}_{\geq 0} \), but \( \overline{f}(p) \in \sigma / \omega \), contradicting the continuity of \( \overline{f} \). □

### 2.4. Pointed monoids and extended cones.

In this section we are going to show that (the opposite of) the category of pointed toric monoids is equivalent to the category of extended rational polyhedral cones.

**Theorem 2.5.** There is a natural contravariant equivalence of categories

\[
\overline{S} : \text{Mon}^\infty_{\text{tor}} \to \text{RPC}^\infty
\]

\[
P^\infty \mapsto \overline{\sigma}_P = \text{Hom}(P, \mathbb{R}_{\geq 0}).
\]

**Proof.** Consider the association

\[
\overline{S} : \text{Mon}^\infty_{\text{tor}} \to \text{RPC}^\infty
\]

that takes a pointed monoid \( P^\infty \) to the extended cone

\[
\overline{S}(P^\infty) = \overline{\sigma}_P := \text{Hom}(P^\infty, \mathbb{R}_{\geq 0}),
\]

and a morphism \( f : P^\infty \to Q^\infty \) to the morphism defined by pre-composition:

\[
\overline{S}(f) : \overline{\sigma}_Q \to \overline{\sigma}_P
\]

\[
\alpha \mapsto \alpha \circ f.
\]

Pre-composition is functorial; thus we are left to check that these constructions indeed send a morphism of pointed monoids to morphisms in \( \text{RPC}^\infty \) respectively. A morphism \( f : P^\infty \to Q^\infty \) factors uniquely as

\[
P^\infty \xrightarrow{g} P^\infty / f^{-1}(\infty_Q). \xrightarrow{h} Q^\infty
\]
Since $\mathcal{S}$ restricts classically to a contravariant equivalence between $\text{Mon}_\text{tor}$ and $\text{RPC}$, it sends toric morphisms to toric morphisms. By Proposition 2.1 (ii) we know $f^{-1}(\infty_Q) = p_\tau$ for $\tau = \text{Cone}(P^0 - f^{-1}(\infty_Q))^\perp \cap \sigma_p$, so the Theorem follows from the two following Lemmas.

\textbf{Lemma 2.6.} Let $\tau \preceq \sigma_p$ be a face and set $r : P^\infty \rightarrow P^\infty/p_\tau$ to be the Rees quotient by $p_\tau$. Then $\mathcal{S}(r) = \iota_\tau$, i.e. $\mathcal{S}$ sends $r$ to the canonical inclusion of the extended face at infinity $\sigma/\tau$ in $\sigma_p$.

\textit{Proof.} By definition, $\mathcal{S}(r)$ results in the morphism
\[
\text{Hom}(P^\infty/p_\tau, \mathbb{R}_{\geq 0}) \xrightarrow{\mathcal{S}(r)} \text{Hom}(P^\infty, \mathbb{R}_{\geq 0})
\]
\[
\alpha \xrightarrow{\mathcal{S}(r)} \alpha \circ r.
\]
That this is an injective morphism of monoids is straightforward. We show that $\mathcal{S}(r)$ has image $\sigma/\tau$. Certainly, given any map $\alpha : P^\infty/p_\tau \rightarrow \mathbb{R}_{\geq 0}$ this map must send elements of $p_\tau$ to infinity, so the image of $\mathcal{S}(r)$ is contained in $\sigma/\tau$. Indeed it is onto: Let $c : P^\infty \rightarrow \mathbb{R}_{\geq 0}$ be given with $c \in \sigma/\tau$. Then
\[
c : u \mapsto \begin{cases} 
(u, p) & \text{for } p \in (\sigma^\vee \cap \tau^\perp) \cap P \\
\infty & \text{otherwise}
\end{cases}
\]
by definition and the map $P^\infty/p_\tau \rightarrow \mathbb{R}_{\geq 0}$ given by
\[
[x] \mapsto \begin{cases} 
(x, p) & \text{for } p \in (\sigma^\vee \cap \tau^\perp) \cap P \\
\infty & \text{otherwise}
\end{cases}
\]
is the preimage of $c$. \hfill \Box

\textbf{Lemma 2.7.} The functor $\mathcal{S}$ is full, faithful, and essentially surjective.

\textit{Proof.} Essentially surjective is clear since given any extended cone $\overline{\sigma}$ we have $\mathcal{S}(S^\infty_\sigma) = \overline{\sigma}$.

Let $P^\infty$ and $Q^\infty$ be given. To show $\mathcal{S}$ is full, we will find a preimage in $\text{Hom}(P^\infty, Q^\infty)$ for a choice $\overline{h} \in \text{Hom}(\overline{\sigma}_Q, \overline{\sigma}_P)$. By definition, the map $\overline{h}$ factors uniquely as
\[
\overline{\sigma}_Q \xrightarrow{\overline{h}} \overline{\sigma}_P \xrightarrow{\iota_\tau} \overline{\sigma}_P/\tau
\]
for some face $\tau \preceq \sigma_p$, that is, $\overline{h} = (\tau, h : \sigma_Q \rightarrow \sigma_P/\tau)$. By Lemma 2.6, we know that $\iota_\tau$ is the image through $\Sigma$ of the Rees quotient morphism $r : P^\infty \rightarrow P^\infty/p_\tau$, thus we may write this diagram as
\[
\begin{array}{ccc}
\text{Hom}(Q^\infty, \mathbb{R}_{\geq 0}) & \xrightarrow{\overline{h}} & \text{Hom}(P^\infty, \mathbb{R}_{\geq 0}) \\
\downarrow{\text{h}} & & \downarrow{\text{h}} \\
\text{Hom}(P^\infty/p_\tau, \mathbb{R}_{\geq 0}) & \xrightarrow{\iota_\tau} & \text{Hom}(P^\infty, \mathbb{R}_{\geq 0}) \\
\end{array}
\]
The morphism \( h \) in this factorization is toric by definition and thus induces a morphism \( h^\vee : p^\infty/p_\tau \to Q^\infty \). Our desired preimage is the composition

\[
\begin{array}{ccc}
p^\infty & \xrightarrow{f} & Q^\infty \\
\downarrow \tau & & \downarrow h^\vee \\
p^\infty/p_\tau & \xrightarrow{t_\tau} & Q^\infty 
\end{array}
\]

To show faithful, let \( f, f' \in \text{Hom}(p^\infty, Q^\infty) \) with \( \overline{\mathfrak{S}}(f) = \overline{\mathfrak{S}}(f') \). They both admit unique factorizations

\[
\begin{array}{ccc}
p^\infty & \xrightarrow{f} & Q^\infty \\
\downarrow \tau & & \downarrow t_\tau \\
p^\infty/p_\tau & \xrightarrow{t_\tau} & Q^\infty \\
p^\infty & \xrightarrow{f'} & Q^\infty \\
\downarrow \tau' & & \downarrow t'_{\tau'} \\
p^\infty/p_{\tau'} & \xrightarrow{t'_{\tau'}} & Q^\infty 
\end{array}
\]

for some faces \( \tau \) and \( \tau' \) of \( \sigma_p \). Since \( \overline{\mathfrak{S}}(f) = \overline{\mathfrak{S}}(f') \), the images of these factorizations through \( \overline{\mathfrak{S}} \) both result in the same diagram

\[
\begin{array}{ccc}
\overline{\mathfrak{S}}_Q & \xrightarrow{\overline{\mathfrak{S}}(f) = \overline{\mathfrak{S}}(f')} & \overline{\mathfrak{S}}_p \\
\downarrow \overline{\mathfrak{S}}(\tau) = \overline{\mathfrak{S}}(\tau') & & \downarrow \overline{\mathfrak{S}}(\tau) = \overline{\mathfrak{S}}(\tau') \\
\overline{\mathfrak{S}}(\tau)/\overline{\mathfrak{S}}(\tau) & \xrightarrow{\iota_\tau = \iota_{\tau'}} & \overline{\mathfrak{S}}(\tau)/\overline{\mathfrak{S}}(\tau')
\end{array}
\]

Thus \( \tau = \tau' \), which forces \( t_\tau = t'_{\tau'} \). Since this functor is already faithful for toric morphisms, we have \( f = f' \). \( \square \)

**Remark 2.8.** One may continue by extending the category of affine normal toric varieties (with toric morphisms) to include also sub-toric morphisms i.e. toric morphisms to a closed torus orbit. The above techniques go through to provide an extension of the foundational equivalence of categories between toric semigroups, convex rational polyhedral cones and normal affine toric varieties, interpreting the data of sub-toric morphisms combinatorially as the extra morphisms coming from canonically extending cones or equipping totally absorbing elements to toric monoids. A standard gluing argument lifts this to the toric case, providing equivalences between the following three categories:

- normal toric varieties with sub-toric morphisms;
- extended rational polyhedral fans; and
- pointed toric \( \mathbb{P}_1 \)-schemes (i.e. monoidal schemes, see [8, Sections 2-4]).

In particular, this completes a result of Cortinas-Haesemeyer-Walker-Weibel [8, Theorem 4.4] by identifying the necessary structure missing from the categories of normal toric varieties and rational polyhedral fans.

### 3. Pointed logarithmic structures

Let \( X \) be a scheme. Following K. Kato (see [13, (1.1), (1.2)]), a *pre-logarithmic structure on \( X \) is a pair \((M_X, \alpha_X)\) consisting of a sheaf of monoids \( M_X \) on the étale site of \( X_{\text{et}} \) together with a morphism \( \alpha_X : M_X \to O_X \). A pre-logarithmic structure \((M_X, \alpha_X)\) on \( X \) is said to be a *logarithmic structure*, if \( \alpha_X \) induces a natural isomorphism

\[
\alpha^* O_X^+ \simeq O_X^+.
\]
In the following we are going to refer to the triple \((X, M_X, \alpha_X)\) simply as a \textit{logarithmic scheme} and denote it by \(X\).

A \textit{sheaf of pointed monoids} is a sheaf of monoids whose values lie in the category \(\text{PMon}\). Note that the structure sheaf \(O_X\) of \(X\) is a sheaf of pointed monoids on \(X\) with absorbing element \(0\).

**Definition 3.1.** A logarithmic structure \((M_X, \alpha_X)\) on \(X\) is a \textit{pointed logarithmic structure} if \(M_X\) is a sheaf of pointed monoids and \(\alpha_X\) is a morphism of sheaves of pointed monoids.

A triple \((X, M_X, \alpha_X)\) consisting of a scheme \(X\) together with a pointed logarithmic structure \((M_X, \alpha_X)\) will be referred to as a \textit{pointed logarithmic scheme}. Logarithmic schemes form a category \(\text{LSch}\) (see [13]). A morphism \(f : X \rightarrow Y\) of logarithmic schemes consists of a morphism \(f : X \rightarrow Y\) of the underlying schemes together with a morphism \(f^*M_Y \rightarrow M_X\) of monoid sheaves on \(X\) that makes the natural diagram commute. If \(f = \text{id}_X\) we refer to \(f^\circ\) as a \textit{morphism of logarithmic structures} on \(X\).

**Definition 3.2.** A morphism \(f : X \rightarrow Y\) of pointed logarithmic schemes is a morphism of logarithmic schemes such that \(f^\circ : f^*M_Y \rightarrow M_X\) is a morphism of sheaves of pointed monoids.

The category of pointed logarithmic schemes is a faithful subcategory of the category of logarithmic schemes and will be denoted by \(\text{PLSch}\).

3.1. **Pointification.** There is a natural \textit{pointification functor}

\[ [.]^\circ : \text{LSch} \rightarrow \text{PLSch} \]

that sends a logarithmic scheme \(X\) to a pointed logarithmic scheme \(X^\circ\) by applying the functor \([.]^\circ\) from Section 2.2 on local sections; i.e. for an étale open \(U\) on \(X\) we have \(M^\circ(U) = M(U)^\circ = M(U) \sqcup \{\infty\}\) and \(\alpha^\circ\) is given by sending \(\infty\) to \(0 \in O_X\). To avoid confusion, hereafter we restrict the use of this notation to identify pointed logarithmic schemes or stacks. We will note when an underlying monoid is pointed if it is not clear from context.

We refer to a morphism \(f : X^\circ \rightarrow Y^\circ\) of pointed logarithmic schemes as a \textit{sub-logarithmic morphism}. If \(f = g^\circ\) for a morphism \(g : X \rightarrow Y\) of the underlying unpointed logarithmic schemes, i.e. if \(f^\circ(m) = \infty_Y \in M_Y\) if and only if \(m = \infty_X \in M_X\), we say that \(f\) is a \textit{purely logarithmic morphism}.

A pointed logarithmic scheme is said to be \textit{coherent} (or \textit{fine}, \textit{separated}) if it is of the form \(X^\circ\) for an (unpointed) logarithmic scheme \(X\) that is \textit{coherent} (or \textit{fine}, \textit{separated}, respectively).

Fix an algebraically closed base field \(k\). In the following the term \textit{pointed logarithmic scheme} will refer to a fine and saturated pointed logarithmic scheme that is locally of finite type over the base field \(k\). The fiber product in this category is given by

\[ X^\circ \times_Z Y^\circ = (X \times_Z Y)^\circ, \]

where \(X \times_Z Y\) denotes the fiber product in the category of fine and saturated logarithmic schemes.
3.2. **Geometry of sub-logarithmic morphisms.** We say that a morphism \( f : X \rightarrow Y \) of pointed logarithmic schemes is a **closed logarithmic immersion** if the underlying morphism \( f : X \rightarrow Y \) is a closed immersion and \( f^\flat \) is surjective.

**Proposition 3.3.** Let \( Y \) be a pointed logarithmic scheme and \( I \subseteq M_Y \) be an ideal sheaf. Then there is a unique logarithmic scheme \( Y/I \) together with a closed logarithmic immersion \( i_I : Y/I \rightarrow Y \) such that, every sub-logarithmic morphism \( f : X \rightarrow Y \) with \( f^\flat(I) = \infty_X \) uniquely factors as

\[
X \longrightarrow Y/I \longrightarrow Y
\]
in \( \text{PLSch} \).

In analogy with Section 2.2 above we refer to the closed logarithmic subscheme \( Y/I \) as the **Rees quotient** of \( Y \) by the ideal sheaf \( I \).

**Proof of Proposition 3.3.** Denote by \( J \) the ideal sheaf in \( \mathcal{O}_Y \) that is generated by the non-unit elements in \( I \). The underlying scheme of \( Y/I \) is given as \( \text{Spec} \mathcal{O}_Y/J \) or, in other words, as the closed subscheme of \( Y \) defined by \( J \). There is a unique factorization

\[
X \longrightarrow Y/I \longrightarrow Y
\]
on the level of the underlying schemes. The closed subscheme is endowed with the pointed logarithmic structure \( M/I \) that is given by the Rees quotients \( M(U)/I(U) \) on étale opens \( U \) of \( Y \). The universal property of the Rees quotient on local sections then yields the existence and uniqueness of the factorization

\[
X \longrightarrow Y/I \longrightarrow Y
\]
in \( \text{PLSch} \).

**Corollary 3.4.** Let \( f : X \rightarrow Y \) a sub-logarithmic morphism. There is a unique factorization

\[
X \longrightarrow Y/I \longrightarrow Y
\]
where \( \tilde{f} : X \rightarrow Y/I \) is a purely logarithmic morphism and \( i_I : Y/I \rightarrow Y \) is the Rees quotient of \( Y \) by the ideal sheaf \( I = (f^\flat)^{-1}(\infty_Y) \subseteq M_X \).

**Proof.** The preimage \( I = (f^\flat)^{-1}(\infty_Y) \) is an ideal sheaf in \( M_X \), thus the factorization coming from Proposition 3.3 yields the claim. Notice that by construction the logarithmic structure on \( Y/I \) is given by identifying all elements in \( (f^\flat)^{-1}(\infty_Y) \) with \( \infty_{Y/I} \) and therefore the morphism \( \tilde{f} : X \rightarrow Y/I \) is purely logarithmic.

4. **Clutching and gluing in logarithmic geometry**

Let \( S \) be a logarithmic scheme. In [12] F. Kato has defined a logarithmic curve over \( S \) as a logarithmically smooth integral morphism \( X \rightarrow S \) (later also assumed to be proper) such that each geometric fiber is a reduced and connected curve. In particular, in [12, Section 1] Kato gives a combinatorial characterization of the étale local structure of such curves. Rather than developing a fully fledged theory of logarithmically smooth morphisms in the category of pointed logarithmic schemes, we directly generalize F. Kato’s characterization to our setting (using the notation from [7, Section 6.3]).
Definition 4.1. Let $S$ be a pointed logarithmic scheme. A logarithmic curve is a proper and flat sublogarithmic morphism $\pi: X \to S$, each of whose fibers is a reduced and connected curve, such that every geometric point $x$ of $X$ has an étale neighborhood $U$ that admits a strict étale morphism to a logarithmic scheme $V$ over $S$ that is of one of the following three types:

(i) $V = \text{Spec} \mathcal{O}_S[u]$, with $M_V = \pi^*M_S$;
(ii) $V = \text{Spec} \mathcal{O}_S[u]$, with $M_V = \pi^*M_S \oplus \mathbb{N}v$ with $\pi^*(v) = u$; or
(iii) $V = \text{Spec} \mathcal{O}_S[x, y]/(xy - t)$ for some $t \in O_S$, and

$$M_V = \pi^*M_S \oplus \mathcal{O}_S^\alpha \oplus \mathcal{O}_S^\beta/(\alpha + \beta = \delta)$$

for some $\delta \in M_S$ and $\pi^*(\alpha) = x, \pi^*(\beta) = y$, and $\pi^*(\delta) = t$.

The main difference from the situation in [12] is that in the situation of Part (iii) above we now allow $\delta = \infty$ in $M_V$. Denote by $\overline{M}_{g,n}^{\log}$ the stack over $(\text{PLSch}, \tau_{\text{str.et.}})$ whose fiber over a pointed logarithmic scheme is the groupoid of stable logarithmic curves over $S$ of genus $g$ with $n$ marked sections. The following Proposition 4.2 is an immediate generalization of [12, Theorem 4.1].

Proposition 4.2. The stack $\overline{M}_{g,n}^{\log}$ over $(\text{PLSch}, \tau_{\text{str.et.}})$ is representable by the algebraic stack $\overline{M}_{g,n}^{\log}$ with the pointed logarithmic structure induced from its boundary.

Proof. The functor $[\cdot]_{\infty}$ from Section 3 above defines a morphism of sites

$$p: (\text{PLSch}, \tau_{\text{str.et.}}) \to (\text{LSch}, \tau_{\text{str.et.}}).$$

We are going to show that the pullback $p^*\mathcal{M}_{g,n}^{\log}$ to $\text{PLSch}$ is equivalent to $\overline{M}_{g,n}^{\log}$. Then, since by [12, Theorem 4.1] we know that $\mathcal{M}_{g,n}^{\log}$ is representable by the algebraic stack $\mathcal{M}_{g,n}$ with the logarithmic structure coming from its boundary, the stack $\overline{M}_{g,n}^{\log}$ is representable by $\mathcal{M}_{g,n}$ with its pointed logarithmic structure coming from the boundary.

Given a pointed logarithmic scheme $S^{\infty}$, the fiber $p^*\mathcal{M}_{g,n}^{\log}(S^{\infty})$ over $S^{\infty}$ is the colimit

$$\lim_{\substack{T \to S^{\infty} \to \text{colim}}} \mathcal{M}_{g,n}^{\log}(T)$$

in the category of groupoids, where $T^{\infty}$ is in $\text{PLSch}/S^{\infty}$. By Corollary 3.4 above, we may restrict $T^{\infty}$ to the collection of all Rees quotients $Y/V(I)$ for ideal sheaves $I$ in $M_X$. In other words $p^*\mathcal{M}_{g,n}^{\log}(S)$ is the groupoid of proper and flat sub-logarithmic morphisms $X \to S$, each of whose fibers is a reduced and connected curve, such that the restriction to every Rees quotient $S/I$ is an object of $\mathcal{M}_{g,n}^{\log}(S/I)$, i.e. a logarithmic curve in the classical sense of F. Kato. This datum is equivalent to the datum of a logarithmic curve over $S^{\infty}$ in the sense of Definition 4.1 above and therefore we have an equivalence between the groupoids $\overline{M}_{g,n}^{\log}(S^{\infty})$ and $p^*\mathcal{M}_{g,n}^{\log}(S^{\infty})$. \hfill \Box

Theorem 4.3. The classical clutching and gluing maps induce natural sub-logarithmic clutching maps

$$\overline{M}_{g,n+1}^{\log} \times \overline{M}_{g',n'+1}^{\log} \to \overline{M}_{g+g',n+n'}^{\log}$$

$$([C, s_1, \ldots, s_n, \bullet], [C', s_1', \ldots, s_n', \bullet]) \mapsto [C \sqcup_{s_{1+}} C', s_1, \ldots, s_n, s_1', \ldots, s_n']$$

and (self-)gluing maps

$$\overline{M}_{g-1,n+2}^{\log} \to \overline{M}_{g,n}^{\log}$$

$$[C, s_1, \ldots, s_n, \bullet, \bullet] \mapsto [C/_{s_{n+}}, s_1, \ldots, s_n].$$
Étale locally around the new node $\star \sim \bullet$ the logarithmic structure (in the notation of Definition 4.1 (iii) above) is given by

$$M_V = \pi^* M_S \oplus O_S^\times \alpha \oplus O_S^\times \beta / (\alpha + \beta = \delta)$$

where $\delta = \infty \in M_S$.

Proof. The two formulas above define a functor between the fibered categories over $\text{PLSch}$ whose restriction to the category of small pointed logarithmic scheme (in the sense of [12]) are exactly the classical clutching and gluing morphisms for the algebraic moduli stacks $\overline{M}_{g,n}$. This observation, together with Proposition 5.6, implies the claim. □

Remark 4.4. Let $S = (\text{Spec } k, k^* \oplus P^\circ)$ be a standard logarithmic point, where $P^\circ$ is a sharp (unpointed) monoid. In [10, Theorem 4.4] we have seen that the groupoid $M^{\text{log}}_{g,n}(S)$ is equivalent to the groupoid of metrized curve complexes (in the sense of [5]) with edge lengths in $P^\circ$. If we take $S^\infty = (\text{Spec } k, k^* \oplus P^\infty)$, this equivalence extends to an equivalence between $\overline{M}^{\text{log}}_{g,n}(S^\infty)$ and the groupoid of metrized curve complexes with edge lengths in $P^\infty$. From this point of view, the clutching map is given by connecting the two marked points $\star$ and $\bullet$ by an edge of length $\infty$.

5. Clutching and gluing in tropical geometry

In order to find tropical analogues for the clutching and gluing maps, as originally introduced in [15], it has been realized in [1, Section 8] that, instead of working with morphisms of generalized cone complexes, one has to consider morphisms between their canonical compactifications. In this section we apply the same principle to the tropical moduli stacks introduced in [7] and construct tropical clutching and gluing maps that naturally commute with tropicalization.

5.1. Extended cone stacks. A (rational polyhedral) cone complex $\Sigma$ is a topological space $|\Sigma|$ that arises as a colimit of a diagram consisting of sharp cones and proper face morphisms such that the induced maps $\sigma \to |\Sigma|$ (called the faces of $\Sigma$) are injective. We refer to the reader to [7, Definition 2.1] for a more axiomatic definition of this notion. The category of cone complexes will be denoted by $\text{RPCC}$; its morphisms are continuous maps $\Sigma \to \Sigma'$ that restrict to morphisms of cones on the faces of $\Sigma$.

The category $\text{RPCC}$ naturally carries a Grothendieck topology, the so-called face topology, given by face embeddings $\sigma \hookrightarrow \Sigma$. A morphism $\Sigma \to \Sigma'$ in $\text{RPCC}$ is said to be strict if its restriction to a face of $\Sigma$ induces an isomorphism onto a face of $\Sigma'$. The tuple $(\text{RPCC}, \tau_{\text{face}}, P_{\text{strict}})$ therefore defines a geometric context in the sense of [7, Section 1] and we may define cone spaces and cone stacks as geometric spaces and geometric stacks in this context respectively.

Given a cone complex $\Sigma$, we may define its canonical extension $\overline{\Sigma}$ by replacing every cone in $\Sigma$ by its canonical extension and gluing along the face incidences of $\Sigma$.

Proposition 5.1. Let $\Sigma$ be a cone complex.

(i) Given a face $\tau$ of $\Sigma$ there is a unique cone complex $\Sigma/\tau$ together with a morphism $\Sigma \to \Sigma/\tau$ such that, whenever $\Sigma \to \Sigma'$ is morphism of cone complexes that sends $\tau$ to the origin, there is a unique morphism $\Sigma/\tau \to \Sigma'$ that makes the natural diagram

$$
\begin{array}{ccc}
\Sigma & \rightarrow & \Sigma' \\
\downarrow & & \downarrow \\
\Sigma/\tau
\end{array}
$$

valid.
(ii) For every face \( \tau \) of \( \Sigma \) there is a unique continuous injection \( \Sigma/\tau \to \Sigma \) that is induced by the maps 
\[ \sigma/\tau \to \sigma \]
from Proposition 2.2 above for faces \( \sigma \) of \( \Sigma \) that contain \( \tau \).

(iii) The injections \( \Sigma/\sigma \to \Sigma \) induce a stratification
\[ \bigsqcup_{\sigma \in \Sigma} \Sigma/\sigma \cong \Sigma \]
into locally closed subsets.

**Proof.** The quotient \( \Sigma/\tau \) is the colimit of the diagram of sharp cones \( \sigma/\tau \), where \( \sigma \) is a face of \( \Sigma \) containing \( \tau \). The universal property then follows from the fact that the \( \sigma/\tau \) are quotients (see Lemma 2.3). This proves part (i). Parts (ii) and (iii) now follow immediately from the analogous results for extended cones in Proposition 2.2 above. \( \square \)

Define a category \( \text{RPCC}^\infty \) of extended (rational polyhedral) cone complexes consisting of canonical extensions \( \Sigma \) of cone complexes \( \Sigma \). Its morphisms are continuous maps \( \Sigma \to \Sigma' \) that restrict to a morphism of cone complexes \( \Sigma \to \Sigma'/\sigma' \) for some cone \( \sigma' \) in \( \Sigma' \). The arguments in Section 2.3 for extended cones show that the composition of two such maps is again of this form.

The natural functor
\[ (.)^\infty : \text{RPCC} \to \text{RPCC}^\infty \]
\[ \Sigma \mapsto \Sigma \]
is faithful and essentially surjective. Using this functor we may endow the category \( \text{RPCC}^\infty \) with a natural face topology that is minimal so that the functor \( (.)^\infty \) induces a continuous morphism of sites
\[ e : (\text{RPCC}^\infty, \tau_{\text{face}}) \to (\text{RPCC}, \tau_{\text{face}}) \].

We define the class of strict morphisms in \( \text{RPCC}^\infty \) to be the image of the class of strict morphisms in \( \text{RPCC} \) under \( (.)^\infty \). This makes the tuple \( (\text{RPCC}^\infty, \tau_{\text{face}}, P_{\text{strict}}) \) into a geometric context in the sense of [7, Section 2] so that the morphism \( e \) is a morphism of geometric contexts.

**Definition 5.2.** A geometric space in the context \( (\text{RPCC}^\infty, \tau_{\text{face}}, P_{\text{strict}}) \) is called an extended cone space and a geometric stack an extended cone stack.

Given a cone stack \( \mathcal{C} \), by [7, Proposition 1.10], the pullback \( e^* \mathcal{C} \) is already extended cone stack. The association \( \mathcal{C} \mapsto e^* \mathcal{C} \) defines a faithful and essentially surjective functor from the 2-category of cone stacks to the 2-category of extended cone stacks.

5.2. The extended moduli stack \( \overline{\mathcal{M}}_{g,n}^{\text{trop}} \). In [7, Section 3] moduli stacks \( \mathcal{M}_{g,n}^{\text{trop}} \) of tropical curves are defined as cone stacks whose fiber over a sharp cone \( \sigma \) is the groupoid of stable tropical curves of genus \( g \) with \( n \) marked legs and edge lengths in the dual monoid \( S_{\sigma} \). We are now going to extend this construction to tropical curves with edge lengths in pointed monoids.

**Definition 5.3.** Let \( \overline{\sigma} \) be an extended cone. An extended tropical curve \( \Gamma \) over \( \overline{\sigma} \) is a finite graph \( G \) (possibly with legs) together with a non-negative vertex weight \( h : V(G) \to \mathbb{Z}_{\geq 0} \) and an edge length function \( d : E(G) \to S_{\sigma}^\infty \setminus \{0\} \).
Recall that the genus of $\Gamma$ is defined to be the number
\[
g(\Gamma) = b_1(G) + \sum_{v \in V(G)} h(v)
\]
and that $\Gamma$ is said to be stable if for all vertices $v \in V(G)$ the inequality $2h(v) - 2 + |v| > 0$ holds.

The argument in [7, Proposition 2.3] immediately shows that there is a unique stack $M_{g,n}^{\text{trop}}$ over the site $(\text{RPCC}^{\infty}, \tau_{\text{face}})$ (up to equivalence) whose fiber over an extended cone $\sigma$ is the groupoid of stable extended tropical curves $\Gamma$ over $\sigma$ of genus $g$ with $n$ marked legs.

**Proposition 5.4.** The moduli stack $M_{g,n}^{\text{trop}}$ is an extended cone stack.

**Proof.** We show that $M_{g,n}^{\text{trop}}$ is the pullback $\epsilon^* M_{g,n}^{\text{trop}}$ of $M_{g,n}^{\text{trop}}$ along $\epsilon$. Since by [7, Theorem 1] the moduli stack $M_{g,n}^{\text{trop}}$ is a cone stack, the extended moduli stack $M_{g,n}^{\text{trop}}$ is an extended cone stack.

Let $\sigma$ be a rational polyhedral cone. The fiber $\epsilon^* M_{g,n}^{\text{trop}}(\sigma)$ is the colimit
\[
\lim_{\tau \to \sigma} M_{g,n}^{\text{trop}}(\tau)
\]
in the category of groupoids, taken over all morphisms $\tau \to \sigma$ of extended cones (see [7, Section 5] and, more generally, [19, Tag 04WA]). Since morphisms of extended cones factor through the embeddings of maximal faces at infinity, we may restrict to the system of morphisms $\tilde{\tau}: \sigma/\tau \to \sigma$, where $\tau$ runs through all faces of $\sigma$, without changing the colimit. Denote by $q_\tau: S_\sigma \subseteq S_{\sigma/\tau} \to S_{\sigma/\tau}$ the induced map on the level of dual monoids. Associating to $(\Gamma, \tilde{\tau})$ the tropical curve $\Gamma_{\tilde{\tau}}$ whose underlying graph is the one of $\Gamma$ and whose edge length is given by
\[
d_{\tilde{\tau}}: d \to S_\sigma \xrightarrow{q_\tau} S_{\sigma/\tau} \subseteq S_{\sigma/\tau}
\]
then defines an equivalence between $\epsilon^* M_{g,n}^{\text{trop}}(\sigma)$ and $M_{g,n}^{\text{trop}}(\sigma)$. □

5.3. **Clutching and gluing.** In [7, Section 6] the authors have introduced a new incarnation of the tropicalization map for the moduli space of curves as a smooth and surjective logarithmic morphism
\[
trop_{g,n}: M_{g,n}^{\log} \longrightarrow M_{g,n}^{\text{trop}}.
\]
For this to make sense one has to use the theory of Artin fans (see [2–4, 22]) in order to lift the moduli stack $M_{g,n}^{\text{trop}}$ to a stack over the category of logarithmic schemes (in a slight abuse of notation also denoted by $M_{g,n}^{\text{trop}}$) that is representable by algebraic stack with a logarithmic structure.

**Definition 5.5.** Let $S$ be a pointed logarithmic scheme. A family of extended tropical curves over $S$ is a collection $\Gamma_q$ of tropical curves, indexed by the geometric points $q$ of $S$, with edge lengths in $M_{S,q}$ such that, whenever $t$ is a geometric point of $S$ that specializes to $q$, then the tropical curve $\Gamma_t$ is obtained from $\Gamma_q$ by endowing the underlying graph $G_q$ of $\Gamma_q$ with the edge length
\[
d: E(G_q) \longrightarrow \overline{M}_{S,q} \longrightarrow \overline{M}_{S,t}
\]
and contracting all edges for which this edge length is zero.
Denote by $\mathcal{M}_{g,n}^{trop}$ the fibered category over $\text{PLSch}$ whose fiber over pointed logarithmic scheme $S$ is the groupoid of families of stable tropical curves over $S$ of genus $g$ with $n$ marked legs.

**Proposition 5.6.** The fibered category $\mathcal{M}_{g,n}^{trop}$ is representable by an algebraic stack with a pointed logarithmic structure that is logarithmically étale over $k$.

**Proof.** Let $p$ be the morphism of sites induced from the functor $[.]^\infty$ in Section 3 above. The fibered category $p^*\mathcal{M}_{g,n}^{trop}$ is equivalent to $\mathcal{M}_{g,n}^{trop}$ on the category of pointed logarithmic schemes, as we may apply Proposition 5.4 for every geometric point of a pointed logarithmic scheme. Then, since $\mathcal{M}_{g,n}^{trop}$ is an algebraic stack with a logarithmic structure that is logarithmically étale over $k$ by [7, Theorem 3], the claim follows. $\square$

**Definition 5.7.** Let $S$ be a pointed logarithmic scheme whose underlying scheme is a point. Given a logarithmic curve $X$ over $S$, the dual tropical curve $\Gamma_X$ is the extended tropical curve consisting of:

(i) one vertex $v$ for each irreducible component $X_v$ of $X$, with vertex weighted $h(v)$ the genus of the normalization of $X$;

(ii) a leg $l_i$ incident to the vertex $v$ for each marked point $x_i$ on $X_v$; and

(iii) for each node $x_e$ of $X$ connecting two component $X_v$ and $X_v'$ with logarithmic equation $\alpha + \beta = \delta_e$ (see Definition 4.1 above) an edge $e$ connecting the two vertices $v, v'$ of length $d(e) = \delta_e \in M_S$.

In particular, we have a natural strict, smooth, and surjective tropicalization morphism

$$\text{trop}_{g,n} : \mathcal{M}^\log_{g,n} \longrightarrow \mathcal{M}^{trop}_{g,n}$$

that is given by associating to a family of logarithmic curves over a pointed logarithmic scheme $S$ the family of dual tropical curves over $S$.

**Theorem 5.8.** There are natural clutching maps

$$\mathcal{M}^{trop}_{g,n+1} \times \mathcal{M}^{trop}_{g',n'+1} \longrightarrow \mathcal{M}^{trop}_{g+g',n+n'}$$

and gluing maps

$$\mathcal{M}^{trop}_{g-1,n+2} \longrightarrow \mathcal{M}^{trop}_{g,n}$$

that make the induced diagrams

$$\begin{array}{ccc}
\mathcal{M}^\log_{g,n+1} \times \mathcal{M}^\log_{g',n'+1} & \xrightarrow{\text{trop}} & \mathcal{M}^{trop}_{g,n+1} \times \mathcal{M}^{trop}_{g',n'+1} \\
\downarrow & & \downarrow \\
\mathcal{M}^\log_{g+g',n+n'} & \xrightarrow{\text{trop}} & \mathcal{M}^{trop}_{g+g',n+n'} \\
\end{array}$$

$$\begin{array}{ccc}
\mathcal{M}^\log_{g-1,n+2} & \xrightarrow{\text{trop}} & \mathcal{M}^{trop}_{g-1,n+2} \\
\downarrow & & \downarrow \\
\mathcal{M}^\log_{g,n} & \xrightarrow{\text{trop}} & \mathcal{M}^{trop}_{g,n} \\
\end{array}$$

commute.

**Proof.** Define the clutching map

$$\mathcal{M}^{trop}_{g,n+1} \times \mathcal{M}^{trop}_{g',n'} \longrightarrow \mathcal{M}^{trop}_{g+g',n+n'}$$

as the unique map whose restriction to an extended rational polyhedral cone $\sigma$ is given by the association

$$((\Gamma, l_1, \ldots, l_n, \bullet), (\Gamma', l'_1, \ldots, l'_{n'}, \bullet)) \longmapsto [\Gamma \sqcup_{\bullet} \Gamma'].$$
The tropical curve $\Gamma \sqcup_{\star \bullet} \Gamma'$ is defined by taking the amalgamated sum of the underlying graphs of $\Gamma$ and $\Gamma'$ over the legs $\star$ and $\bullet$ and endowing the resulting graph with the generalized edge length

$$d(e) = \begin{cases} 
  d_\Gamma(e) & \text{if } e \in E(\Gamma) \\
  d_{\Gamma'}(e) & \text{if } e \in E(\Gamma') \\
  \infty & \text{if } e = \{\star \sim \bullet\}
\end{cases}$$

with values in $S_0^\infty$. From the explicit description of the pointed logarithmic structures in Theorem 4.3 we obtain that this map commutes with tropicalization.

A completely analogous construction also gives a (self)-gluing map

$$\mathcal{M}_{g-1,n+2}^{\text{trop}} \to \mathcal{M}_{g,n}$$

$$[\Gamma, l_1, \ldots, l_n, \star, \bullet] \mapsto [\Gamma/\star \bullet]$$

that naturally commutes with tropicalization, again by Theorem 4.3 above. \qed

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