ON NON-VANISHING OF COHOMOLOGIES OF GENERALIZED RAYNAUD POLARIZED SURFACES

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ABSTRACT. We consider a family of slightly extended version of the Raynaud’s surfaces $X$ over the field of positive characteristic with Mumford-Szpiro type polarizations $Z$, which have Kodaira non-vanishing $H^1(X, Z^{-1}) \neq 0$. The surfaces are at least normal but smooth under a special condition. We compute the cohomologies $H^i(X, Z^n)$ for $i, n \in \mathbb{Z}$ and study their (non-)vanishing. Finally, we give a fairly large family of non Mumford-Szpiro type polarizations $Z_{a,b}$ with Kodaira non-vanishing.

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1. Introduction

Let $X$ be a projective variety over an algebraically closed field $k$ and $Z$ an ample invertible sheaf on $X$. It is well known that Kodaira vanishing theorem does not hold if the characteristic of the field char($k$) = $p$ is positive. The first counter-example has been found by Raynaud [14]. He constructed a smooth polarized surface $(X, Z)$ with $H^1(X, Z^{-1}) \neq 0$ using Tango-structure [22]. Mukai [10] generalized Raynaud’s construction to obtain polarized smooth projective varieties $(X, Z)$ of any dimension with $H^1(X, Z^{-1}) \neq 0$. He also showed that, if a smooth projective surface $X$ is a counter-example to Kodaira vanishing, then $X$ must be either hyperelliptic with $p = 2, 3$ or of general type. The construction similar to Mukai’s has been also studied by Takeda [18, 19, 20] and Russel [15]. Mumford [13] and Szpiro [16] gave a sufficient condition for a polarized smooth projective surface to be a counter-example to Kodaira vanishing and pointed out that Raynaud’s examples are its instances. Szpiro [17] and Lauritzen-Rao [8] also gave different counter-examples to Kodaira vanishing. Mumford [11] constructed a normal polarized surface $(X, Z)$ with $H^1(X, Z^{-1}) \neq 0$ but it is not known whether desingularizations of $X$ satisfy Kodaira vanishing.

The aim of this paper is to study (non-)vanishing of $H^i(X, Z^n)$, $i, n \in \mathbb{Z}$, for a family of surfaces $X$ with Mumford-Szpiro type polarizations $Z$, which is an extension of Raynaud’s counter-examples. Recall that Raynaud’s examples are cyclic covers of ruled surfaces over smooth projective curves of genus $g \geq 2$. The degree $\ell$ of the cyclic covers is $\ell = 2$ for $p \geq 3$ and $\ell = 3$ for $p = 2$. Notice that Kodaira vanishing holds for ruled surfaces [23] [10]. The smooth curve must have a special kind of divisor called Tango-structure (Tango-Raynaud structure) and this gives a strong restriction to the genus $g$ of the curve, i.e. $p$ must divide $2g - 2$. If we consider a weaker condition called pre-Tango, which is satisfied by any smooth curves with $g \geq p$ ([21]), but in this case the obtained surface is singular.
As is implicitly described in [14], we can choose the degree $\ell$ of cyclic cover more freely. In Mukai’s construction [10], $\ell$ can be any integer $\geq 2$ with $(p, \ell) = 1$ (and a mild condition), but then we must take normalization to construct the cyclic cover. In this paper, we consider an additional condition $\ell \mid p + 1$. This assures the normality of the cyclic cover without normalization and moreover the computation of cohomologies $H^i(X, \mathcal{Z}^n)$ is much easier. Thus, we obtain a fairly large class of surfaces over the fields of positive characteristics containing many counter-examples to Kodaira vanishing, together with formulas for cohomologies $H^i(X, \mathcal{Z}^n)$. These surfaces are normal if the base curve $C$ has a pre-Tango structure and smooth if $C$ has a Tango structure.

This family would be particularly interesting in the sense that this provides a class of finitely generated graded integral algebras $(R, \mathfrak{m})$, over the fields of positive characteristics whose graded local cohomologies $H^i_m(R)$, $i < \dim R$, do not necessarily vanish at negative degrees. For a polarized variety $(X, \mathcal{L})$, we consider the section ring $(R, \mathfrak{m}) := (\bigoplus_{n\geq0} H^0(X, \mathcal{L}^n), \bigoplus_{n>0} H^0(X, \mathcal{L}^n))$, which is a finitely generated graded algebra over $\bar{k} = H^0(X, \mathcal{O}_X)$ with natural $\mathbb{N}$-grading. We have $X \cong \text{Proj}(R)$. Then by computing Čech complexes we know that $H^0_m(R) = 0$ and we have

$$0 \rightarrow R \rightarrow \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{L}^n) \rightarrow H^1_m(R) \rightarrow 0$$

and $H^{i+1}_m(R) \cong \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{L}^n)$ for $i \geq 1$. From this, we immediately know that we always have $[H^j_m(R)]_n = 0$ for $j = 0, 1$ and for all $n < 0$ and moreover, if Kodaira type vanishing holds, then we have $[H^i_m(R)]_n = 0$ for all $n < 0$ and $i < \dim R$ (see [7]). It is known that if $X$ has at most $F$-rational singularities and $X$ is obtained by generic mod $p$ reduction from a variety with at most rational singularities, then we have Kodaira type vanishing (see [5, 4, 7]). From our generalized Raynaud surfaces, we obtain examples of $R$ with $\dim R = 3$ different from this type, whose local cohomologies can be studied by analyzing cohomologies of vector bundles over smooth curves of genus $\geq 2$.

In section 2, we will present the construction of our generalized Raynaud surface $X$, which is the cyclic cover of degree $\ell$ of the ruled surface $P$ over a curve $C$ with pre-Tango structure. In section 3, we show that $K_X$ is ample if $(p, \ell) = (3, 4)$ and $p \geq 5$ (Proposition 10) and in this case we have Kodaira type vanishing $H^1(X, K_X^{-1}) = 0$ (Proposition 12). Then we apply the Mumford-Szpiro type sufficient condition for Kodaira non-vanishing to obtain the polarization $(X, \mathcal{Z})$ with Kodaira non-vanishing (Proposition 16). Then we will compute cohomologies $H^i(X, \mathcal{Z}^n)$, $i, n \in \mathbb{Z}$, in section 4 (Propositions 19, 21 and 25, Theorem 22, Corollary 4) and show some (non-)vanishing results (Corollaries 20, 24 and 26, Theorem 23). Finally, we give a class of polarizations with Kodaira non-vanishing, which are not of Mumford-Szpiro type (Theorem 28).

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2. Fibered surfaces on pre-Tango curves

In this section, we present the construction of our polarized surface, which is a cyclic cover of a ruled surface over a smooth projective curve. This is an extension of the Raynaud’s counter-example [14] allowing more variations of the degree of the cyclic cover and a weaker condition for the base curve. See [10, 18, 19, 20, 21, 24] for similar constructions and detailed description. In the following, let $k$ be an algebraically closed field of characteristic $\text{char}(k) > 0$.

2.1. pre-Tango and Tango structure. Let $C$ be a smooth projective curve over $k$ with genus $g \geq 2$. We denote by $K(C)$ the function field of $C$ and we define $K(C)^p = \{ f^p \mid f \in K(C) \}$. Then the Tango-invariant $n(C)$ is defined by

$$n(C) := \max \left\{ \deg \left[ \frac{(df)}{p} \right] \mid f \in K(C) \setminus K(C)^p \right\},$$

where $\lfloor \cdot \rfloor$ denotes round down of coefficients, see [22]. We know that $0 \leq n(C) \leq \frac{2(g - 1)}{p}$ and $C$ is called a pre-Tango curve (or a Tango curve) if $n(C) > 0$ (or $n(C) = 2g - 2$). This means the existence of an ample divisor $D$ on $C$ such that $(df) \geq p(D(>0))$ (or $(df) = pD(>0)$) with some $f \in K(C) \setminus K(C)^p$. We call the invertible sheaf $L := \mathcal{O}_C(D)$ a pre-Tango structure (or a Tango structure) of $C$.

Pre-Tango structure can be described in other way around. Consider the relative Frobenius morphism $F: C' \rightarrow C$ and let $B^1$ be the image of the push forward $F_* d : F_* \mathcal{O}_{C'} \rightarrow F_* \Omega^1_{C'}$, of the Kähler differential $d : \mathcal{O}_{C'} \rightarrow \Omega^1_{C'}$. Then we have the following short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F_* \mathcal{O}_{C'} \rightarrow B^1 \rightarrow 0.$$

Now any ample invertible subsheaf $L \subset B^1$ is a pre-Tango structure of $C$ and the existence of such subsheaves is assured if $g \geq p$ (see Cor. 1.5 [21]), namely, curves with large genus are pre-Tango.

In the rest of this section, we consider a pre-Tango structure $L = \mathcal{O}_C(D)$ of a pre-Tango curve $C$.

2.2. dividing (pre-)Tango structure. Consider the Jacobi variety $J$ which consists of all the divisors of degree 0 on $C$. It is well known that if $(e, p) = 1$, $e \in \mathbb{N}$, the map $\varphi_e : J \rightarrow J$ s.t. $\varphi_e(D_0) = eD_0$ is surjective (cf. page 42 [12]), i.e. every $D_0 \in J$ can be divided by $e$. Thus we know that, for every $\mathbb{N} \ni e \geq 2$ such that $(e, p) = 1$ and $e | \deg L$, there exists an ample invertible sheaf $N$ with $L = N^e$.

2.3. Construction of the ruled surface $P$ and the divisor $E + C''$. Tensoring (1) by $L^{-1}$ to take the global sections, we have

$$0 \rightarrow H^0(C, B^1 \otimes L^{-1}) \rightarrow H^1(C, L^{-1}) \xrightarrow{F_*} H^1(C, L^{-p}).$$

On the other hand, we have the short exact sequence

$$0 \rightarrow B^1 \rightarrow F_* \Omega^1_{C'} \rightarrow \Omega^1_C \rightarrow 0$$
where \( c \) is the Cartier operator \([2]\). By tensoring \([2]\) by \( \mathcal{L}^{-1} \) to take the global sections, we have

\[
0 \longrightarrow H^0(C, \mathcal{B}^1 \otimes \mathcal{L}^{-1}) \longrightarrow H^0(C, F_{*}(\Omega_C^{\ell}(-pD))) \xrightarrow{c_{(-D)}} H^0(C, \Omega_C^1 \otimes \mathcal{L}^{-1}).
\]

Then we know \( \text{Ker} F_{*} \cong H^0(C, \mathcal{B}^1 \otimes \mathcal{L}^{-1}) = \text{Ker} c_{(-D)} \cong \{ df \mid f \in K(C), (df) \geq pD \} \), which is non-trivial since \( C \) is pre-Tango (cf. Lemma 12 \([22]\)).

Now take any \( 0 \neq df_0 \in H^0(C, \mathcal{B}^1 \otimes \mathcal{L}^{-1}) \). Then \( \xi := \eta(df_0) \) is a non-trivial element in \( H^1(C, \mathcal{L}^{-1}) \cong \text{Ext}^{1}_{\mathcal{O}_C}(\mathcal{L}, \mathcal{O}_C) \), so that we have a non-splitting extension

\[
0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0
\]

where \( \mathcal{E} \) is a locally free sheaf of rank 2.

Moreover, we have \( 0 = F_{*}^{\ast} \xi \in H^1(C, \mathcal{L}^{-p}) \cong \text{Ext}^{1}_{\mathcal{O}_C}(\mathcal{L}^p, \mathcal{O}_C) \) and the corresponding split extension

\[
0 \longrightarrow \mathcal{O}_C \longrightarrow F_{*}^{\ast} \mathcal{E} \longrightarrow \mathcal{L}^p \longrightarrow 0
\]

is just the Frobenius pullback of the sequence \([3]\). Using the splitting maps and tensoring by \( \mathcal{L}^{-1} \) we obtain another exact sequence

\[
0 \longrightarrow \mathcal{O}_C \longrightarrow F_{*}^{\ast} \mathcal{E} \otimes \mathcal{L}^{-p} \longrightarrow \mathcal{L}^{-p} \longrightarrow 0.
\]

Now from the sequences \([3]\) and \([4]\) we obtain two ruled surfaces and their canonical cross sections \( \sigma \) and \( \tau \). Namely,

\[
\pi : P = \mathbb{P}(\mathcal{E}) \longrightarrow C, \quad E := \sigma(C) \subset P
\]

where \( E \) is determined, as a Cartier divisor, by the global section \( s \) that is the image of 1 by the inclusion \( H^0(C, \mathcal{O}_C) \hookrightarrow H^0(C, \mathcal{E}) = H^0(P, \mathcal{O}_P(1)) \) induced from \([3]\) and

\[
\pi' : P' = \mathbb{P}(F_{*}^{\ast} \mathcal{E} \otimes \mathcal{L}^{-p}) \cong \mathbb{P}(F_{*}^{\ast} \mathcal{E}) \longrightarrow C, \quad \bar{C}' := \tau(C)
\]

where \( \bar{C}' \) is determined, as a Cartier divisor, by the global section \( t' \) that is the image of 1 by the inclusion \( H^0(C, \mathcal{O}_C) \hookrightarrow H^0(C', F_{*}^{\ast} \mathcal{E} \otimes \mathcal{L}^{-p}) \cong H^0(P', \mathcal{O}_{P'}(1)) \) induced from \([4]\). Now we define the morphism \( \varphi : P \longrightarrow P' \) over \( C \) by taking the \( p \)-th power of the coordinates of \( \pi^{-1}(x) \cong \mathbb{P}^1_k(\subset P) \) to obtain the coordinates of \( (\pi')^{-1}(x) \cong \mathbb{P}^1_k(\subset P') \) for every \( x \in C \). Then we set \( C'' = \varphi^{-1}(\bar{C}') \). By construction, we have \( \mathcal{O}_{P'}(C'') \cong \mathcal{O}_P(p) \otimes \pi^{'*} \mathcal{L}^{-p} \) and \( C'' \) is a degree \( p \) curve in \( P \). We know that \( E \cap C'' = \emptyset \). \( E \) is smooth since \( E \cong C \) via \( \sigma \).

2.4. Purely inseparable cover \( \pi \mid_{C''} : C'' \longrightarrow C \). Now as a Cartier divisor we write \( D = \{(U_i, g_i)\}_i \), where \( C = \bigcup_i U_i \) is an open covering, \( g_i \in K(C) \) is the local equation of \( D \). By taking a finer covering, we can assume that \( \mathcal{E} \mid_{U_i} \) are the free \( \mathcal{O}_{U_i} \)-modules. Then we can describe \((df) \geq pD \) by \( f = \{(U_i, g^p_i c_i)\}_i \in K(C) \) with \( K(C)^p \not\ni c_i \in \mathcal{O}_{U_i} \), i.e. \((df) \mid_{U_i} = (g^p_i dc_i) \). Then,

**Proposition 1.** We have

\[
C'' \mid_{U_i} = \text{Proj} \mathcal{O}_{U_i}[x, y]/(c_i x^p + y^p).
\]

In particular, \( C'' \) is an purely inseparable covering of \( C \).
Proof. The sequence (3) is locally as follows:

\[ 0 \rightarrow \mathcal{O}_{U_i} \rightarrow \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i} g_i^{-1} \rightarrow \mathcal{O}_{U_i} g_i^{-1} \rightarrow 0 \]

so that we have \( P |_{U_i} = \text{Proj}(\mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i} g_i^{-1})) = \text{Proj} \mathcal{O}_{U_i}[x, y] \), where the indeterminates \( x \) and \( y \) represent the free basis \( 1 \) and \( g_i^{-1} \). On the other hand, we know that the sequence (4) is locally as follows:

\[ 0 \rightarrow \mathcal{O}_{U_i} \overset{i}{\rightarrow} (\mathcal{O}_{U_i} \otimes \mathcal{O}_{U_i} g_i^{-p}) \otimes \mathcal{O}_{U_i} g_i^p \overset{j}{\rightarrow} \mathcal{O}_{U_i} g_i^p \rightarrow 0 \]

where we view

\[ (\mathcal{O}_{U_i} \otimes \mathcal{O}_{U_i} \cdot g_i^{-p}) \otimes \mathcal{O}_{U_i} \cdot g_i^p \cong \mathcal{O}_{U_i} \cdot g_i^p \oplus \mathcal{O}_{U_i} \cdot 1 \]

\[ \cong \mathcal{O}_{U_i}(c_i g_i^p, 1) \oplus \mathcal{O}_{U_i}(g_i^p, 0) \]

and we define \( i(a) = a(c_i g_i^p, 1) \) and \( j(a(c_i g_i^p, 1) + b(g_i^p, 0)) = b g_i^p \) for \( a, b \in \mathcal{O}_{U_i} \). Thus \( P' |_{U_i} = \text{Proj}(\mathcal{O}_{U_i} g_i^p \oplus \mathcal{O}_{U_i} g_i^{-1})) \cong \text{Proj} \mathcal{O}_{U_i}[x', y'] \) where the indeterminates \( x' \) and \( y' \) represents the free basis \( g_i^p \) and \( 1 \). Also \( C'' = \tau(C) \) is locally the zero locus of \( i'' |_{U_i} = c_i g_i^p + 1 \), so that we have

\[ C'' |_{U_i} = \text{Proj} \mathcal{O}_{U_i}[x', y']/(c_i x' + y') \]

Since \( \varphi : P = \text{Proj} \mathcal{O}_{U_i}[x, y] \rightarrow P' = \text{Proj} \mathcal{O}_{U_i}[x', y'] \) is induced by the Frobenius \( \mathcal{O}_{U_i}[x', y'] \ni x', y' \mapsto x^p, y^p \in \mathcal{O}_{U_i}[x, y] \), we have

\[ C'' |_{U_i} = \text{Proj} \mathcal{O}_{U_i}[x, y]/(c_i x^p + y^p) \].

\[ \square \]

Remark 1. By a similar discussion to the proof of Proposition \( \exists \) we can show

\[ E |_{U_i} = \text{Proj}(\mathcal{O}_{U_i}[x, y]/(x)) \cong \text{Spec} \mathcal{O}_{U_i}[y]. \]

Later we will construct cyclic covers of \( P \) ramified at \( E + C'' \) and the smoothness of the cyclic covers depends on the smoothness of \( E \) and \( C'' \). Since \( E \cong C \) is smooth by definition, we have to see if \( C'' \) is smooth. To this end, we must prepare the following lemma.

Lemma 2. \( \Omega_{C''/C} \cong \pi^* \mathcal{O}_C(D) \).

Proof. By Proposition \( \exists \) we have \( \mathcal{O}_{C''} |_{U_i} = \mathcal{O}_{U_i}[x_i, y_i]/(c_i x_i^p + y_i^p) \) and \( (df) \geq pD \) with \( f = \{(U_i, g_i^p c_i)\} \ni K(C) \). Thus on \( U_i \cap U_j \) we have \( g_i^p c_i = g_j^p c_j \) so that

\[ (g_i^p c_i)(g_i^{-1} x_i)^p + y_i^p = c_i x_i^p + y_i^p = c_j x_j^p + y_j^p = (g_j^p c_j)(g_j^{-1} x_j)^p + y_j^p \]

Thus we have \( g_i^{-1} x_i = g_j^{-1} x_j \) and \( y_i = y_j \), and then

\[ g_i^{-1} dx_i = d(g_i^{-1} x_i) = d(g_j^{-1} x_i) = g_j^{-1} dx_j \]

for \( d := d_{C''/C} \). Now on \( \bar{U_i} = U_i \cap \{ y_i \neq 0 \} \), we have \( \mathcal{O}_{C''} |_{\bar{U_i}} = \mathcal{O}_{\bar{U}_i}[X_i]/(c_i X_i^p + 1) \) with \( X_i := x_i/y_i \) and then

\[ \Omega_{C''/C} |_{\bar{U}_i} = \mathcal{O}_{C''} |_{\bar{U}_i} \cdot g_i^{-1} dX_i = \pi^* \mathcal{O}_C(D) |_{\bar{U}_i} \]

Notice that we have \( g_i^{-1} dX_i = g_j^{-1} dX_j \) on \( \bar{U}_i \cap \bar{U}_j \).
On the other hand, on \( \hat{U}_i = U_i \cap \{ x_i \neq 0 \} \) we can write
\[
\mathcal{O}_{C''} |_{\hat{U}_i} = \mathcal{O}_C[Y_i]/(c_i + Y_i^p) \quad \text{with } Y_i = x_i^{-1}
\]
and then
\[
\Omega_{C''/C} |_{\hat{U}_i} = \Omega_{C''} |_{\hat{U}_i} \cdot dY_i = \Omega_{C''} |_{\hat{U}_i} \cdot x_i^{-2} dx_i.
\]
By setting \( t = g_i^{-1}x_i = g_j^{-1}x_j \) on \( U_i \cap U_j \), we have
\[
x_i^{-2} dx_i = x_i^{-2} g_i dt = g_i^{-1} t^{-2} dt
\]
and so that we have \( \Omega_{C''/C} |_{\hat{U}_i} \cong \pi^* \mathcal{O}_C(D) |_{\hat{U}_i} \). Consequently, we have \( \Omega_{C''/C} \cong \pi^* \mathcal{O}_C(D) \), locally free of rank 1, as required.

The following result first appeared in Mukai’s paper in Japanese \([10]\) Prop. 5) with a brief outline of the proof and his result is for varieties of arbitrary dimensions. We give here a detailed proof in the case of curves for the readers convenience.

**Theorem 3.** Let \( C \) be a pre-Tango curve. Then \( C'' \) is smooth if and only if \( C \) is Tango.

**Proof.** In the following we will denote the restriction of \( \pi : P \rightarrow C \) to \( C'' \subset P \) also by \( \pi \). Now we consider the sequence
\[
0 \rightarrow \pi^* \mathcal{O}_C(pD) \xrightarrow{df} \pi^* \Omega_C \xrightarrow{\psi} \Omega_{C''} \xrightarrow{\rho} \Omega_{C''/C} \rightarrow 0
\]
where \( df \) is the multiplication by \( df = \{(g_i^p dc_i)\}_i \). The exactness of \( \pi^* \Omega_C \rightarrow \Omega_{C''} \rightarrow \Omega_{C''/C} \rightarrow 0 \) is well known. The multiplication by \( df \) is injective since \( dc_i \neq 0 \) and \( C \) is smooth. Moreover we have \( \ker \psi \supset \text{Im } df \). To see this we have only to show that \( \psi(dc_i) \), which is by definition the Kähler differential of the image of \( c_i \) by \( \pi^* : \mathcal{O}_C \rightarrow \mathcal{O}_{C''} \), is trivial. But this is immediate since, by Proposition \([1]\) \( \pi^* : \mathcal{O}_C \rightarrow \mathcal{O}_{C''} \) is locally the canonical inclusion \( \mathcal{O}_C \hookrightarrow \mathcal{O}_C[x,y]/(c_i x^p + y^p) \). Thus \( [5] \) is exact if and only if Ker \( \psi \subset \text{Im } df \).

Now \( \Omega_C \) is locally free of rank 1 since \( C \) is smooth. Then we know by NAK that \( C \) being Tango, i.e. \( (dc_i) = 0 \) is equivalent with \( \ker \rho = 0 \). This implies that \([5] \) is exact, and then we have \( \Omega_{C''} \cong \Omega_{C''/C} \), which is locally free of rank 1 by Lemma \([2] \) and \( C'' \) is smooth. Conversely, assume that \( C'' \) is smooth. Then since \( \Omega_{C''/C} \) and \( \Omega_{C''} \) are locally free module of rank 1, we must have \( \ker \rho = \text{Im } \psi = 0 \) so that we have \( \ker \rho = 0 \), i.e., \( C \) is Tango.

### 2.5. Construction of cyclic cover of \( P \) ramified at \( E + C'' \). In this section, we will construct a cyclic cover \( X \) of \( P \) of suitable degree ramified at \( E + C'' \). We choose \( \ell \geq 2 \) such that \( \ell \mid p + 1 \) and \( \ell \mid e \), and set
\[
\mathcal{M} := \mathcal{O}_P \left( -\frac{p+1}{\ell} \right) \otimes \pi^* \mathcal{N}^{\ell/2}.
\]
Then we have \( \mathcal{M}^{-\ell} = \mathcal{O}_P(E + C'') \). Now we define an \( \mathcal{O}_P \)-algebra structure in \( \bigoplus_{i=0}^{\ell-1} \mathcal{M}^i \) with the multiplication defined by
\[
\mathcal{M}^i \times \mathcal{M}^j \rightarrow \mathcal{M}^{i+j} \quad (a, b) \mapsto a \otimes b
\]
if \( i + j \leq \ell - 1 \) and
\[
\mathcal{M}^i \times \mathcal{M}^j \longrightarrow \mathcal{M}^{i+j-\ell} \\
(a, b) \mapsto a \otimes b \otimes \zeta
\]
if \( i + j > \ell \), with \( \zeta = s \otimes t'' \) where \( s \) and \( t'' \) are the global sections defining \( E \) and \( C'' \). Then we obtain
\[
\psi : X := \text{Spec} \left( \bigoplus_{i=0}^{\ell-1} \mathcal{M}^i \right) \longrightarrow P,
\]
which is the cyclic cover of the ruled surface \( P \) ramified at \( E + C'' \) of degree \( \ell \), where \( \text{Spec} \) denotes the affine morphism. Now we will define \( \phi = \pi \circ \psi : X \to C \).

**Remark 2.** \( \phi : X \to C \) is an extension of Raynaud’s original counter-example to Kodaira vanishing. Namely, let \( C \) be a Tango curve and let \( e = \ell = 3 \) if \( p = 2 \) and \( e = \ell = 2 \) if \( p \geq 3 \), then we obtain the example as given in [14].

We define \( \tilde{E} = \psi^{-1}(E) \) and \( \tilde{C}'' = \psi^{-1}(C'') \), then we have
\[
(6) \quad \ell \tilde{E} = \psi^* E \quad \text{and} \quad \ell \tilde{C}'' = \psi^* C''
\]
and
\[
(7) \quad \psi_* \mathcal{O}_X = \bigoplus_{i=0}^{\ell-1} \mathcal{M}^i.
\]
Moreover, we have the following, which will be used later.

**Lemma 4.** For \( k \geq 1 \), we have \( \psi_* \mathcal{O}_X(-k \tilde{E}) \cong \mathcal{O}_P(-kE) \oplus \bigoplus_{i=1}^{\ell-1} \mathcal{M}^i \).

**Proof.** From the exact sequence
\[
0 \longrightarrow \mathcal{O}_X(-k \tilde{E}) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{k \tilde{E}} \longrightarrow 0,
\]
we obtain by (7)
\[
0 \longrightarrow \psi_* \mathcal{O}_X(-k \tilde{E}) \longrightarrow \bigoplus_{i=0}^{\ell-1} \mathcal{M}^i \longrightarrow \psi_* \mathcal{O}_{k \tilde{E}} \longrightarrow R^1 \psi_* \mathcal{O}_X(-k \tilde{E}),
\]
where \( R^1 \psi_* \mathcal{O}_X(-\tilde{E}) = 0 \) since \( \psi : X \to P \) is an affine morphism. Also since \( \psi : \tilde{E} \cong E \), we have \( \psi_* \mathcal{O}_{k \tilde{E}} \cong \mathcal{O}_{kE} = \mathcal{O}_P/\mathcal{O}_P(-kE) \). Then we obtain the following diagram:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \psi_* \mathcal{O}_X(-k \tilde{E}) & \longrightarrow & \mathcal{O}_P \oplus \bigoplus_{i=1}^{\ell-1} \mathcal{M}^i & \longrightarrow & \psi_* \mathcal{O}_{k \tilde{E}} & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{O}_P(-kE) \oplus \bigoplus_{i=1}^{\ell-1} \mathcal{M}^i & \longrightarrow & \mathcal{O}_P \oplus \bigoplus_{i=1}^{\ell-1} \mathcal{M}^i & \longrightarrow & \mathcal{O}_P/\mathcal{O}_P(-kE) & \longrightarrow & 0
\end{array}
\]
from which we have \( \psi_* \mathcal{O}_X(-k \tilde{E}) \cong \mathcal{O}_P(-kE) \oplus \bigoplus_{i=1}^{\ell-1} \mathcal{M}^i \) by 5-lemma.

**Lemma 5.** For \( k \geq 0 \) and \( 1 \leq r \leq \ell - 1 \), we have
\[(i) \quad \psi_* \mathcal{O}_X(k \ell \tilde{E}) = \bigoplus_{i=0}^{\ell-1} \mathcal{M}^i(kE),
\]
We define $H$ \textup{(}cf. Claim 3.12 of \cite{3}\textup{)}

Proposition 6

Proof. Using \eqref{eq:6} and \eqref{eq:7}, we have

Lemma 4

Thus $S$ finite and

Corollary 7.

Then, we have

M

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result by Esnault-Viehweg. Let

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Remark 3.

The cyclic cover constructed by Mukai \cite{10} is more general than ours.

from Theorem 3. □

Proof. We apply Prop. 6 in the case of $(X, \mathcal{H}, \ell, E) := (P, \mathcal{M}^{-1}, \ell, E + C'')$. Since $C''$ and $E$ are reduced curves, we have $\mathcal{O}_P([\frac{1}{\ell}(E + C'')]) = \mathcal{O}_P$ for $0 \leq i \leq \ell - 1$. Thus $\text{Spec}(A)$ is nothing but our surface $X$ and normal. Since $\dim X = 2$, we know that $X$ is Cohen-Macaulay by Serre’s $(S_2)$ condition. The final statement follows from Theorem 3. □

Remark 3. The cyclic cover constructed by Mukai \cite{10} is more general than ours. Let $L, D, N, e$ be as in \cite{2.5}. Choose $\ell \geq 2$ such that $\ell \mid e$ and $(\ell, p) = 1$. Notice that the last condition is weaker than our condition $\ell \mid (p + 1)$. Mukai’s construction is as follows. For any $\alpha \in \mathbb{N}$ such that $\ell \mid (p + \alpha)$ we write

\[ 0 \sim C'' - pE + p\pi^*(D) = C'' + \alpha E + \ell K \text{ with } K := -\frac{p + \alpha}{\ell}E + \frac{p}{\ell}\pi^*D \]

and set

\[ \mathcal{M}_\alpha := \mathcal{O}_P(K) = \mathcal{O}_P \left( -\frac{p + \alpha}{\ell}E \right) \otimes \pi^*N^{p/\ell}. \]

Then, we have $\mathcal{M}_{\alpha}^{-\ell} = \mathcal{O}_P(C'' + \alpha E)$. Now we consider

\[ \psi' : X' := \text{Spec} \bigoplus_{i=0}^{\ell-1} \mathcal{M}_\alpha^i \longrightarrow P, \]
which is normal if and only if \( \alpha = 1 \). Thus we take the normalization \( X \) of \( X' \) to obtain the cyclic cover \( \phi : X \to P \). Corollary \[\text{[7]}\] also holds for this construction. But the normalization \( \tilde{f} : X \to X' \) makes it difficult to compute the cohomologies \( H^i(X, \mathcal{Z}^n) = H^i(X', f_*\mathcal{Z}^n) \) for a polarization \((X, \mathcal{Z})\).

3. Basic properties of the surfaces

We will show some basic properties of our surface \( X \). Also we will define the Mumford-Szpiro type polarization \((X, \mathcal{Z})\) in the end of this section.

The cross section \( \tilde{E} \subset X \) has a positive self-intersection number.

**Proposition 8.** The self-intersection number of \( \tilde{E} \) is \( \tilde{E}^2 = \frac{1}{\ell} \cdot \deg D (> 0) \).

**Proof.** We compute \( \tilde{E}^2 = \left( \frac{\psi^*(E)}{\ell}, \frac{\psi^*(E)}{\ell} \right) = \frac{\deg \psi}{\ell^2} \cdot E^2 = \frac{1}{\ell} \cdot E^2 = \frac{1}{\ell} \cdot \deg D \). \( \square \)

Now we consider the canonical divisor \( K_X \).

**Proposition 9.** \( K_X \sim \phi^*(K_C - \frac{p\ell - p - \ell}{\ell} \cdot D) + (p\ell - p - \ell - 1)\tilde{E} \).

**Proof.** We have \( K_X \sim \psi^*K_P + (\ell - 1)\tilde{E} + (\ell - 1)\tilde{C}'' \) by the branch formula. Applying \( \psi^* \) to \( C'' \sim pE - p\pi^*D \) to obtain \( \tilde{C}'' \sim p\tilde{E} - \frac{p}{\ell} \cdot \phi^*(D) \). Then a direct computation, together with the well known formula \( K_P = -2E + \pi^*K_C + \pi^*(D) \), shows the required result. \( \square \)

**Proposition 10.** \( K_X \) is ample if \( (p, \ell) = (3, 4) \) or \( p \geq 5 \).

**Proof.** We have \( K_X = \phi^*A + B \), where \( A = K_C - (p\ell - p - \ell)D/\ell, B = (p\ell - p - \ell - 1)\tilde{E} \) by Proposition \[\text{[9]}\]. Since \( 0 < \deg D \leq \frac{2(g-1)}{p} \) and \( g \geq 2 \), we see \( \deg A > 0 \). Also we see that \( \deg B \leq 0 \) if and only if \( (p, \ell) = (2, 2), (2, 3), (3, 2) \). Thus, since \( \ell \mid (p + 1) \), we have \( \deg B > 0 \) if and only if \( (p, \ell) = (3, 4) \) and \( p \geq 5 \). In these cases, we have \( K_X^2 > 0 \) and \( K_X.H > 0 \) for every irreducible curve \( H \in \text{Pic}(P) \) in \( P \) by Proposition \[\text{[8]}\] (cf. Prop. V.2.3 \[\text{[3]}\]). Thus \( K_X \) is ample by Nakai-Moishezon criterion. \( \square \)

Now we are interested in whether \( H^1(X, K_X^{-1}) = 0 \) holds if \( K_X \) is ample.

**Lemma 11.** We have

\[
H^1(X, K_X^{-1}) = \bigoplus_{i=0}^{\ell-1} H^1(P, K_P^{-1} - \frac{(p + 1)(\ell - 1 + i)}{\ell}E + \frac{p(\ell - 1 + i)}{\ell} \pi^*D)
\]

**Proof.** Since \( \psi : X \to P \) is an affine morphism, we have \( H^1(X, K_X^{-1}) = H^1(P, \psi_*(K_X^{-1})) \). By branch formula and \( C'' \sim pE - p\pi^*D \), we have

\[
K_X^{-1} = \psi^*(K_P^{-1} - \frac{(p + 1)(\ell - 1)}{\ell}E + \frac{p(\ell - 1)}{\ell} \pi^*D)
\]

so that

\[
\psi_*(\mathcal{O}_X(K_X^{-1})) = \psi_*\mathcal{O}_X \otimes \mathcal{O}_P \left( K_P^{-1} - \frac{(p + 1)(\ell - 1)}{\ell}E + \frac{p(\ell - 1)}{\ell} \pi^*D \right)
\]
where \( \psi_* \mathcal{O}_X = \bigoplus_{i=0}^{\ell-1} \mathcal{M}^i \). Then we obtain the above stated result. \( \square \)

Since Kodaira vanishing holds for \( P \) (see [10, 23]), to show that \( H^1(X, K_X^{-1}) = 0 \) we have only to show that

\[
L_i := K_P + \frac{(p+1)(\ell - 1 + i)}{\ell} E - \frac{p(\ell - 1 + i)}{\ell} \pi^* D
\]

are ample for \( i = 0, \ldots, \ell - 1 \).

**Proposition 12.** \( H^1(X, K_X^{-1}) = 0 \) holds for \( p \geq 5 \) or \( p = 3 \) and \( \ell = e = 4 \).

**Proof.** Let \( f \) be any fiber of \( \pi : P \to C \). Then we have \( \pi^* D = \deg D \cdot f \) and the numerical equivalence \( K_P = -2E + 2(g - 1) \cdot f + \deg D \cdot f \). Thus we have \( L_i = u_i \cdot E + v_i \cdot f \) where

\[
u_i := \frac{(p+1)(\ell - 1 + i)}{\ell} - 2, \quad v_i := 2g - 2 - \frac{p\ell - p - \ell + pi}{\ell} \cdot \deg D.
\]

Then, using the condition \( \ell \mid (p + 1) \), we can show that \( L_i, E > 0 \) and \( L_i, f > 0 \) if \( p \geq 5 \) or \( p = 3 \) and \( \ell = e = 4 \). Also a straightforward computation shows that \( L_i^2 > 0 \). Then by Nakai-Moishezon’s criteria, \( L_i, i = 0, \ldots, \ell - 1 \), are ample. \( \square \)

Next we consider the fibers \( X_y := \phi^{-1}(y) (\subset X) \) for \( y \in C \).

**Proposition 13.** Every \( X_y \) has a singularity at the intersection with the curve \( \tilde{C}'' \), which is the cusp of the form \( Z^\ell = W^p \).

**Proof.** The fiber \( X_y \) may have singularities at the intersection with \( \tilde{E} + \tilde{C}'' \), which are the inverse image of \( \mathbb{P}^1 \cap (E + C'') (\subset P) \) by \( \psi \). By Proposition [1] \( \tilde{C}'' \subset P \) is locally defined by the equations \( c_iX^p + Y^p \in \mathcal{O}_{U_i}[X,Y] \) with \( y \in U_i \). Thus \( Z = (c_iX^p + Y^p)^{1/\ell} \) is a local coordinate of \( \phi^{-1}(U_i) \subset X \). Setting the new coordinate \( W = c_i^{1/p}X + Y \) we have \( Z^\ell = W^p \) as required. Moreover, a similar argument shows that \( \phi^{-1} \cap E \) is not singular (cf. Remark [1]). \( \square \)

Although \( X_y \) is birational with \( \mathbb{P}^1 \), it has a positive geometric genus.

**Proposition 14.** The geometric genus of \( X_y \) is \( \frac{(\ell - 1)(p - 1)}{2} (> 0) \).

**Proof.** By normalization, we can assume that \( X_y \) is smooth and \( \psi_y = \psi \mid_{X_y} : \phi^{-1}(y) \to \mathbb{P}^1 \cong \pi^{-1}(y), y \in C \), is a finite separated morphism. By taking the normalization, we can assume from the beginning that \( X_y := \phi^{-1}(y) \) is a smooth curve and the degree \( \deg \psi_y (= \ell) \) is preserved. Also the ramification divisor for \( \psi_y \) is \( (\ell - 1)(\tilde{E} \cap X_y) + (\ell - 1)(\tilde{C}'' \cap X_y) \), whose degree is \( (\ell - 1)(p + 1) \). Thus by Hurwitz formula we obtain the required result. \( \square \)

Mumford and Szpiro generalized Raynaud’s examples and obtained the following result.

**Theorem 15** (Mumford and Szpiro [13, 16]). Let \( \phi : X \to C \) be a fibration from a smooth projective surface to a smooth projective curve and assume that each fiber is reduced and irreducible with positive geometric genus. Then if there exists a
cross section $Γ ⊂ X$ of $φ$ with positive self intersection number, we have (i) $Z = \mathcal{O}_X(Γ) \otimes φ^*(\phi_0\mathcal{O}_X(Γ)|_Γ)$ is ample, and (ii) $H^1(X, Z^{-1}) ≠ 0$.

By Proposition 8 and Proposition 14, we know that our surface $X$ is an instance of this theorem when $Γ = \tilde{E}$. Moreover,

**Proposition 16.** In this case, we have $Z = \mathcal{O}_X(\tilde{E}) \otimes φ^*\mathcal{N}^e/ℓ = \mathcal{O}_X(\tilde{D})$ where $\tilde{D} = ψ^{-1}(E) + φ^{-1}D'$ with $D' = 1/\ell D$.

**Proof.** We have $φ_0\mathcal{O}_X(\tilde{E}) |_E = φ_0(\mathcal{O}_X(\tilde{E}) \otimes \mathcal{O}_E) = (π_0 \circ ψ_0)(ψ_0^*\mathcal{O}_P(1/\ell E) \otimes \mathcal{O}_E) = ψ_0(ψ_0^*\mathcal{O}_E \otimes \mathcal{O}_P(1/\ell E))$. Now from the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-\tilde{E}) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0$$

we obtain,

$$0 \rightarrow ψ_0\mathcal{O}_X(-\tilde{E}) \rightarrow ψ_0\mathcal{O}_X \rightarrow ψ_0\mathcal{O}_E \rightarrow R^1ψ_0\mathcal{O}_X(-\tilde{E})$$

and $R^1ψ_0\mathcal{O}_X(-\tilde{E}) = 0$ since $ψ$ is an affine morphism. Thus by Lemma 4 we have

$$ψ_0\mathcal{O}_E \cong ψ_0\mathcal{O}_X/ψ_0\mathcal{O}_X(-\tilde{E}) \cong \bigoplus_{i=0}^{ℓ-1} \mathcal{M}_i \mathcal{O}_P(-E) \oplus \bigoplus_{i=-1}^{ℓ-1} \mathcal{M}_i \mathcal{O}_E \cong \mathcal{O}_E,$$

and then

$$φ_0\mathcal{O}_X(\tilde{E}) |_E = π_0(\mathcal{O}_E \otimes \mathcal{O}_P(1/\ell E)) = \mathcal{O}_C(1/\ell D)(= \mathcal{N}^e/ℓ).$$

since $E$ is the canonical section of $π : P \rightarrow C$. □

Notice that if $e = ℓ = 2$ when $\text{char } k ≥ 3$ and $e = ℓ = 3$ when $\text{char } k = 2$ then $Z$ in Proposition 16 is the same as the ample invertible sheaf of Raynaud’s counter-example to Kodaira vanishing.

4. Cohomologies for Mumford-Szpiro type polarization

In this section, we will compute the cohomologies $H^i(X, Z^n)$ for the Mumford-Szpiro type polarization $(X, Z)$ given in Proposition 16 namely $Z = \mathcal{O}_X(\tilde{E}) \otimes φ^*\mathcal{N}_ℓ$ where we set $\mathcal{N}_ℓ = N^\dagger$.

First of all, we summarize the well-known facts about ruled surfaces, which are necessary in our computation of cohomologies.

**Lemma 17.** For the ruled surface $π : P = \mathbb{P}(\mathcal{E}) \rightarrow C$, we have

(i) $π_0\mathcal{O}_P(k) = S^k(\mathcal{E})$, which is the $k$th component of the symmetric algebra $S(\mathcal{E})$.

We will understand $S^k(\mathcal{E}) = 0$ for $k < 0$.

(ii) for a locally free sheaf $\mathcal{F}$ on $P$, $H^i(P, \mathcal{O}_P(n) \otimes π^*\mathcal{F}) \cong H^i(C, S^n(\mathcal{E}) \otimes \mathcal{F})$ for $n ≥ 0$ and $i \in \mathbb{Z}$.

(iii)

$$R^1π_0\mathcal{O}_P(n) = \begin{cases} 0 & \text{if } n ≥ -1 \\ S^{-n-2}(\mathcal{E}) \otimes \mathcal{L}^\dagger & \text{if } n ≤ -2 \end{cases}$$

For an extension of (ii) with $\mathcal{F}$ any coherent sheaf, see Proposition 7.10.13 [1].
Proof. (i) is Proposition II.7.11(a) [6]. Now we have $R^1\pi_*\mathcal{O}_P(n) = \pi_*\mathcal{O}_P(-(n + 2))\wedge L^v$ by Exer. III.8.4(c) [9]. Then applying (i) we obtain (iii), cf. Appendix A [9]. Finally, we have $S^n(\mathcal{E}) \otimes F \cong \pi_*\mathcal{O}_P(n) \otimes \pi^*F$ by (i) and $R^i\pi_*\mathcal{O}_P(n) \otimes \pi^*F = R^i\pi_*\mathcal{O}_P(n) \otimes F = 0$ for $n \geq 0$ and $i \geq 0$ by (iii). Thus we obtain (ii) by Leray spectral sequence.

The following vanishing result will also be used.

**Proposition 18.** For any $1 \leq m$ and $k < e$, we have $H^0(C, S^m(\mathcal{E})^v \otimes \mathcal{N}^k) = 0$.

**Proof.** Since rank $\mathcal{E} = 2$ and $\mathcal{L}$ is the surjective image of $\mathcal{E}$, we have rank $S^m(\mathcal{E}) = m + 1$ and there exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m \subset \mathcal{F}_{m+1} = S^m(\mathcal{E})$$

such that $\mathcal{F}_j$ is a locally free sheaf of rank $\mathcal{F}_j = j$ and

$$0 \to \mathcal{F}_{j-1} \to \mathcal{F}_j \to \mathcal{L}^j \to 0$$

for $j = 1, \ldots, m + 1$. Now taking the dual, tensoring by $\mathcal{N}^k$ and taking the global sections, we have for $j = 2, \ldots, m + 1$

$$0 \to H^0(C, \mathcal{L}^{-j} \otimes \mathcal{N}^k) \to H^0(C, \mathcal{F}_j^v \otimes \mathcal{N}^k) \overset{\psi_j}{\to} H^0(C, \mathcal{F}_{j-1}^v \otimes \mathcal{N}^k).$$

If $\deg \mathcal{L}^j \otimes \mathcal{N}^{-k} = (ej - k) \cdot \deg \mathcal{N} > 0$, i.e. $ej > k$, we have $H^0(C, \mathcal{L}^{-j} \otimes \mathcal{N}^k) = 0$ so that $\psi_j$ is an inclusion. Thus we have

$$\psi_2 \circ \cdots \circ \psi_{m+1} : H^0(C, S^m(\mathcal{E})^v \otimes \mathcal{N}^k) \subset H^0(C, \mathcal{F}_1^v \otimes \mathcal{N}^k) = H^0(C, \mathcal{L}^{-1} \otimes \mathcal{N}^k)$$

if $ej > k$ for all $j = 2, \ldots, m + 1$, i.e. if $2e > k$. Now $H^0(C, \mathcal{L}^{-1} \otimes \mathcal{N}^k) = 0$ and thus $H^0(C, S^m(\mathcal{E})^v \otimes \mathcal{N}^k) = 0$, if $\deg \mathcal{L}^{-1} \otimes \mathcal{N}^k = (k - e) \cdot \deg \mathcal{N} < 0$, i.e. if $e > k$. □

4.1. **Computation of $H^2(X, \mathbb{Z}^n)$**. Now we compute $H^2(X, \mathbb{Z}^n)$.

**Proposition 19.** For $k \geq 0$ and $1 \leq r \leq \ell - 1$,

$$H^2(X, \mathbb{Z}^n) = \begin{cases} \bigoplus_{i=\lceil \frac{2}{\ell} + 1 \rceil}^{\ell-1} H^2(P, \mathcal{O}_P) \left( -\frac{i(p+1)}{\ell} + k \right) \otimes \pi^*\mathcal{N}_\ell^{ip+n} & \text{if } n = k\ell \geq 0 \\ H^2(P, \mathcal{O}_P(r+1+k-\ell)) \otimes \pi^*\mathcal{N}_\ell^n & \text{if } n = k\ell + r > 0 \\ H^2(P, \mathcal{O}_P(n) \otimes f^*\mathcal{N}_\ell^n) & \text{if } n < 0. \end{cases}$$
Proof. Since $\psi : X \to P$ is an affine morphism, the Leray spectral sequence degenerates so that we have $H^2(X, \mathcal{Z}^n) = H^2(P, \psi_* (\mathcal{O}_X(n\mathcal{E})) \otimes (\psi^* \circ \pi^*) \mathcal{N}_\ell^n)) = H^2(P, \psi_* \mathcal{O}_X(n\mathcal{E}) \otimes \pi^* \mathcal{N}_\ell^n)$. Then by Lemma 17 and Lemma 18 we compute

$$H^2(X, \mathcal{Z}^n) = \begin{cases} 
H^2(P, \mathcal{O}_P(k) \otimes \pi^* \mathcal{N}_\ell^n) 
\oplus \bigoplus_{i=1}^{\ell-1} H^2(P, \mathcal{O}_P\left(-\frac{i(p+1)}{\ell} + k\right) \otimes \pi^* \mathcal{N}_\ell^{ip+n}) & \text{if } n = k\ell \geq 0 \\
H^2(P, \mathcal{O}_P(r + 1 + k - \ell) \otimes \pi^* \mathcal{N}_\ell^n) 
\oplus \bigoplus_{i=1}^{\ell-1} H^2(P, \mathcal{O}_P\left(-\frac{i(p+1)}{\ell} + k + 1\right) \otimes \pi^* \mathcal{N}_\ell^{ip+n}) & \text{if } n = k\ell + r > 0 \\
H^2(P, \mathcal{O}_P(n) \otimes \pi^* \mathcal{N}_\ell^n) 
\oplus \bigoplus_{i=1}^{\ell-1} H^2(P, \mathcal{O}_P\left(-\frac{i(p+1)}{\ell}\right) \otimes \pi^* \mathcal{N}_\ell^{ip+n}) & \text{if } n < 0.
\end{cases}$$

Moreover, in the case of $n = k\ell \geq 0$, we have $-\frac{i(p+1)}{\ell} + k \geq 0$ if $i \leq \frac{n}{p+1}$. Also in the case of $n = k\ell + r > 0$, we have $-\frac{i(p+1)}{\ell} + k + 1 \geq 0$ if $i \leq \frac{n+\ell-r}{p+1}$. Now by Lemma 17(ii), we have $H^2(P, \mathcal{O}_P(j) \otimes \pi^* \mathcal{N}^n) = H^2(C, S^j(\mathcal{E}) \otimes \mathcal{N}^n) = 0$ for $j \geq 0$. Thus we do not have to consider the direct summands with the indices $i$ in the above specified ranges. \(\square\)

As an immediate consequence, we have the following vanishing result.

**Corollary 20.** We have $H^2(X, \mathcal{Z}^n) = 0$ (i) if $\ell \mid n$ and $n \geq (\ell - 1)(p+1)$, in particular $n \geq p(p+1)$, or (ii) if $\ell \not\mid n$ and $n \geq p(p+1) - 1$. In particular, $H^2(X, \mathcal{Z}^n) = 0$ for all $n \geq p(p+1)$.

**Proof.** By Proposition 19 and Lemma 17(ii), we know that, for $n \geq 0$, $H^2(X, \mathcal{Z}^n) = 0$ if (i) $n = k\ell \geq 0$ and $\ell - 1 < \left\lfloor \frac{n}{p+1} + 1 \right\rfloor$, or (ii) $n = k\ell + r > 0$, $\ell - 1 < \left\lfloor \frac{n+\ell-r}{p+1} + 1 \right\rfloor$ and $r+1+k-\ell \geq 0$. The second condition of (i) is equivalent to $\frac{n}{p+1} + 1 - (\ell - 1) \geq 1$, i.e., $n \geq (p+1)(\ell - 1)$. Since $\ell \mid (p+1)$, we have $\ell - 1 \leq p$ so that $n \geq p(p+1)$ implies in particular $n \geq (p+1)(\ell - 1)$. Similarly, the second condition of (ii) is equivalent to $n \geq (p+1)(\ell - 1) - (\ell - r)$. Since $\ell \mid (p+1)$ and $1 \leq r \leq \ell - 1$, we have $(p+1)(\ell - 1) - (\ell - r) \leq p(p+1) - 1$. Thus in particular the second condition of (ii) is satisfied for $n \geq p(p+1) - 1$. Moreover, by the first and the third condition of (ii), we have $n \geq (\ell - r)(\ell - 1)$. Since $1 \leq r$ and $\ell \mid (p+1)$, we have $(\ell - r)(\ell - 1) \leq (\ell - 1)^2 \leq p^2 < p(p+1) - 1$. Thus, in this case, $n \geq p(p+1) - 1$ suffices for the vanishing. \(\square\)

4.2. **Computation of $H^1(X, \mathcal{Z}^n)$**. The case of $n \geq 0$ can be computed by the same method as in Proposition 19 except the difference in the dimension of cohomologies.

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Proposition 21. Let $n \geq 0$. For $k \geq 0$, $1 \leq r \leq \ell - 1$, we have

$$H^1(X, \mathcal{Z}^n) = \left\{ \begin{array}{ll}
H^1(C, S^k(\mathcal{E}) \otimes \mathcal{N}_\ell^n) \\
\oplus \bigoplus_{i=1}^{\ell-1} H^1(C, S^{-\frac{(p+1)k}{\ell}} \otimes \mathcal{N}_\ell^{i^p+n}) \\
\oplus \bigoplus_{i=1}^{\ell-1} H^1(P, \mathcal{O}_P \left(\frac{-i(p+1)}{\ell} + k\right) \otimes \pi^* \mathcal{N}_\ell^{i^p+n}) & \text{if } n = k\ell \geq 0
\end{array} \right.
$$

Moreover, by Lemma [17(ii)], the first term in the case of $n = k\ell + r > 0$ is

$$H^1(P, \mathcal{O}_P(r + 1 + k - \ell)) \otimes \pi^* \mathcal{N}_\ell^n \cong H^1(C, S^{r+1-k-\ell}(\mathcal{E}) \otimes \mathcal{N}_\ell^n)$$

if $r + k \geq \ell - 1$.

Now we consider the case of $n < 0$.

Theorem 22. For $n < 0$, we have

$$H^1(X, \mathcal{Z}^n) = \bigoplus_{i=1}^{\ell-1} H^0(C, S^{\frac{(p+1)k}{\ell}} - 2(\mathcal{E}) \otimes \mathcal{N}_\ell^{i^p-\ell+n}).$$

Proof. Consider a part of the five-term exact sequence

$$0 \rightarrow H^1(C, \phi_* \mathcal{Z}^n) \rightarrow H^1(X, \mathcal{Z}^n) \rightarrow H^0(C, R^1 \phi_* \mathcal{Z}^n) \rightarrow H^2(C, \phi_* \mathcal{Z}^n)$$

for the Leray spectral sequence $E_2^{pq} = H^p(C, R^q \phi_* \mathcal{Z}^n) \Rightarrow H^{p+q}(X, \mathcal{Z}^n)$. We have $H^2(C, \phi_* \mathcal{Z}^n) = 0$ since $\dim C = 1$, and moreover an easy calculation using Lemma [4] shows

$$H^1(C, \phi_* \mathcal{Z}^n) = H^1(C, \pi_* \mathcal{O}_P(n) \otimes \mathcal{N}_\ell^n) \oplus \bigoplus_{i=1}^{\ell-1} H^1(C, \pi_* \mathcal{O}_P\left(-\frac{i(p+1)}{\ell}\right) \otimes \mathcal{N}_\ell^{i^p+n})$$

and this is $= 0$ by Lemma [17(i)]. Thus, we have

$$H^1(X, \mathcal{Z}^n) = H^0(C, R^1 \phi_* \mathcal{Z}^n) \quad (n < 0).$$

On the other hand, in a part of the five-term exact sequence

$$0 \rightarrow R^1 \pi_* (\psi_* \mathcal{Z}^n) \rightarrow R^1 \phi_* \mathcal{Z}^n \rightarrow \pi_* (R^1 \psi_* \mathcal{Z}^n)$$

for $\psi : X \rightarrow P$ and $\pi : P \rightarrow C$, we have $R^1 \psi_* \mathcal{Z}^n = 0$ since $\psi$ is an affine morphism. Thus we have

$$R^1 \phi_* \mathcal{Z}^n = R^1 \pi_* (\psi_* \mathcal{Z}^n) \quad (n \in \mathbb{Z}).$$

Now an easy calculation using Lemma [4] and Lemma [17(iii)] shows

$$R^1 \pi_* (\psi_* \mathcal{Z}^n) = \left\{ \begin{array}{ll}
\bigoplus_{i=1}^{\ell-1} R^1 \pi_* \mathcal{M}^i \otimes \mathcal{N}_\ell^{-1} \\
S^{-n-2}(\mathcal{E}) \otimes \mathcal{N}_\ell^{n-\ell} \oplus \bigoplus_{i=1}^{\ell-1} R^1 \pi_* \mathcal{M}^i \otimes \mathcal{N}_\ell^n & \text{if } n = -1
\end{array} \right. \quad \text{if } n \leq -2$$

But by Proposition [18] and $\deg \mathcal{N}_\ell > 0$, we have

$$H^0(C, S^{-n-2}(\mathcal{E}) \otimes \mathcal{N}_\ell^{n-\ell}) = 0 \quad \text{for } n \leq -2.$$
Thus by (8) and (9) we have

\[ H^1(X, \mathcal{Z}^n) = \bigoplus_{i=1}^{\ell-1} H^0(C, R^1\pi_*\mathcal{M}^i \otimes \mathcal{N}^n_{\ell}). \]

By relative Serre duality, the well-known formula \( \omega_{P/C} = \mathcal{O}_P(-2) \otimes \pi^*\mathcal{L} \) and Lemma 17(i), we compute

\[ R^1\pi_*\mathcal{M}^i \cong \pi_*((\mathcal{M}^i \otimes \omega_{P/C})^\vee) = S^{(i(p+1)/\ell - 2)}(\mathcal{E})^\vee \otimes \mathcal{N}_{\ell}^{ip-\ell}. \]

and we obtain the above stated result. \( \square \)

We give here some specific instances of Theorem 22.

**Example 4.** Let \( n < 0 \). Then,

- if \( \ell = p + 1 \):
  
  \[ H^1(X, \mathcal{Z}^n) = H^0(C, \mathcal{N}_{\ell}^{p-1+n}) \oplus \bigoplus_{i=3}^{p} H^0(C, S^{i-2}(\mathcal{E})^\vee \otimes \mathcal{N}_{\ell}^{ip-1-n}) \]

  where the first term vanishes for \( n < -(p - 1) \). In particular, if \( p = 2 \) (and then \( \ell = 3 \)), we have \( H^1(X, \mathcal{Z}^n) = 0 \) for \( n \leq -2 \) and moreover \( H^1(X, \mathcal{Z}^{-1}) \neq 0 \) since this is exactly the Raynaud’s counter-example.

- if \( 2\ell = p + 1 \):
  
  \[ H^1(X, \mathcal{Z}^n) = H^0(C, \mathcal{N}_{\ell}^{\ell-1+n}) \oplus \bigoplus_{i=2}^{\frac{p-1}{2}} H^0(C, S^{2i-2}(\mathcal{E})^\vee \otimes \mathcal{N}_{\ell}^{ip-\frac{p+1}{2}+n}) \]

  where the first term vanishes for \( n < -\frac{p-1}{2} \). In particular, if \( p = 3 \) (and then \( \ell = 2 \)), we have \( H^1(X, \mathcal{Z}^n) = 0 \) for \( n \leq -2 \) and moreover \( H^1(X, \mathcal{Z}^{-1}) \neq 0 \) since this is exactly the Raynaud’s counter-example.

Now we show some non-vanishing results.

**Theorem 23.** \( H^1(X, \mathcal{Z}^n) \neq 0 \) for every \( n \) such that \( -(\ell - \lceil \frac{2\ell}{p+1} \rceil) \leq n \leq -1 \), where \( \lceil \cdots \rceil \) denotes the round up.

**Proof.** Since \( \mathcal{L} = \mathcal{N}^{e} = \mathcal{N}_{\ell}^{e} \) is the surjective image of \( \mathcal{E} \) (cf. (3)), we have the short exact sequence

\[ S^k_\ell(\mathcal{E}) \rightarrow \mathcal{N}_{\ell}^k \rightarrow 0 \]

for any \( k \in \mathbb{N} \) such that \( \ell \mid k \). Taking the dual and tensoring by \( \mathcal{N}_{\ell}^k \), we obtain

\[ 0 \rightarrow \mathcal{O}_C \rightarrow S^k_\ell(\mathcal{E})^\vee \otimes \mathcal{N}_{\ell}^k. \]

Then we have

\[ k = H^0(C, \mathcal{O}_C) \subset H^0(C, S^k_\ell(\mathcal{E})^\vee \otimes \mathcal{N}_{\ell}^k). \]

Applying this result, we know that the term \( H^0(C, S^{(i\ell + 1)/\ell - 2}(\mathcal{E})^\vee \otimes \mathcal{N}_{\ell}^{ip-\ell+n}), 1 \leq i \leq \ell - 1 \), in Theorem 22 is non-trivial if

\[ \frac{i(p + 1)}{\ell} - 2 \geq 0 \quad \text{and} \quad \ell \left( \frac{i(p + 1)}{\ell} - 2 \right) = ip - \ell + n, \]
namely $n = -(\ell - i)$, with $\left\lceil \frac{2\ell}{p+1} \right\rceil \leq i \leq \ell - 1$. \hfill \Box$

For small characteristics, the evaluation of the non-vanishing degrees in Theorem 22 is best possible.

**Corollary 24.** If $p = 2$ or 3, we have $H^1(X, \mathcal{Z}^n) = 0$ for every $n < -(\ell - \left\lceil \frac{2\ell}{p+1} \right\rceil)$.

**Proof.** Since $\ell | p + 1$, we have only to consider the cases $(p, \ell) = (2, 3), (3, 2), (3, 4)$ and

$$- \left( \ell - \left\lceil \frac{2\ell}{p+1} \right\rceil \right) = \begin{cases} -1 & \text{if } (p, \ell) = (2, 3) \\ -1 & \text{if } (p, \ell) = (3, 2) \\ -2 & \text{if } (p, \ell) = (3, 4). \end{cases}$$

The first two cases are already shown in Example 4. Then we assume $(p, \ell) = (3, 4)$ in the following. We have

$$H^1(X, \mathcal{Z}^n) = \bigoplus_{i=0}^{3} H^0(C, S^{i-2}(\mathcal{E})^\vee \otimes \mathcal{N}_t^{3i-4+n})$$

by Theorem 22. Since $\deg \mathcal{N}_t > 0$, we have $H^0(X, \mathcal{N}_t^{2+n}) = 0$ for $n < -2$. Moreover, since $\mathcal{N}_t^{5+n} = \mathcal{N}^{(5+n)}_{\psi}$ by definition and since $\frac{e}{4} < e$ for $n < -2$, we have $H^0(C, S^1(\mathcal{E})^\vee \otimes \mathcal{N}_t^{5+n}) = 0$ for $n < -2$ by Proposition 18. Thus, in this case we have $H^1(X, \mathcal{Z}^n) = 0$ for $n < -2$. \hfill \Box

4.3. Computation of $H^0(X, \mathcal{Z}^n)$. Since $\mathcal{Z}$ is ample, we have $H^0(X, \mathcal{Z}^n) = 0$ for $n < 0$. For $n \geq 0$, we have

**Proposition 25.** For $n, k \geq 0$ and $1 \leq r \leq \ell - 1$, we have

$$H^0(X, \mathcal{Z}^n) = \begin{cases} \bigoplus_{i=0}^{\ell-1} H^0(C, S^{-(\ell+1)+i}(\mathcal{E}) \otimes \mathcal{N}_t^{ip+n}) & \text{if } n = k\ell \geq 0 \\ H^0(C, S^{r+1-k-\ell}(\mathcal{E}) \otimes \mathcal{N}_t^{n}) \\ \bigoplus_{i=1}^{\ell-1} H^0(C, S^{-i(p+1)+k+1}(\mathcal{E}) \otimes \mathcal{N}_t^{ip+n}) & \text{if } n = k\ell + r > 0. \end{cases}$$

**Proof.** By Lemma 5 we compute

$$H^0(X, \mathcal{Z}^n) = H^0(P, \psi_* \mathcal{O}_X(nE) \otimes \mathcal{N}_t^n)$$

$$= \begin{cases} \bigoplus_{i=0}^{\ell-1} H^0(C, \pi_* \mathcal{M}(kE) \otimes \mathcal{N}_t^n) & \text{if } n = k\ell \geq 0 \\ H^0(C, \pi_* \mathcal{O}_X(r + 1 + k - \ell) \otimes \mathcal{N}_t^n) \\ \bigoplus_{i=1}^{\ell-1} H^0(C, \pi_* \mathcal{M}(k+1)E) \otimes \mathcal{N}_t^n) & \text{if } n = k\ell + r > 0. \end{cases}$$

Then apply Lemma 17(i). \hfill \Box

**Remark 5.** According to Proposition 25, we know that the lower bound $B$ such that $H^0(X, \mathcal{Z}^n) = 0$ for $n \geq B$, depends on the vanishing of cohomologies of type $H^0(C, S^m(\mathcal{E}) \otimes \mathcal{N}_t^n)$ for $m, n > 0$. Hence it seems to be difficult to give a general estimation of $B$. 16
By the similar argument as in the proofs of Proposition 19 and 21, we know that we have only to consider fewer direct summands than Proposition 25 in some cases. Namely,

- if \( n = k\ell \) and \( 0 \leq n < (p + 1)(\ell - 1) \), we have
  \[
  H^0(X, Z^n) = \bigoplus_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} H^0(C, S^{-\frac{i(p+1)}{\ell}+k}\mathcal{E}) \otimes \mathcal{N}_t^{ip+n} \]

- if \( n = k\ell + r \), \( 0 < r \leq \ell - 1 \), and \( 0 < n < p(\ell - 1) + r - 1 \), we have
  \[
  H^0(X, Z^n) = H^0(C, S^{r+\ell-k}\mathcal{E}) \otimes \mathcal{N}_t^{n} \bigoplus_{i=1}^{\lfloor \frac{n+\ell-r}{p+1} \rfloor} H^0(C, S^{-\frac{i(p+1)}{\ell}+k+1}\mathcal{E}) \otimes \mathcal{N}_t^{ip+n} \]

Then we have

**Corollary 26.** \( H^0(X, Z^n) = 0 \) if \( n = k\ell + r > 0 \) with \( 0 < r \leq \ell - 1 \) and \( 0 \leq k \leq \min\{\ell - r - 2, \frac{p+1}{\ell} - 2\} \).

**Proof.** By the above formula for \( n = k\ell + r \), \( 0 < r \leq \ell - 1 \), we know that \( H^0(X, Z^n) = 0 \) if \( r + 1 + k - \ell < 0 \) and \( n + \ell - r < p + 1 \). From the latter inequality, we have

\[
 n = k\ell + r < \min\{p + 1 - \ell + r, p(\ell - 1) + r - 1\}
\]

and then together with the former inequality we have

\[
k < \min\left\{\ell - r - 1, \frac{p+1}{\ell} - 1, p - \frac{p+1}{\ell}\right\} = \min\{\ell - r - 1, \frac{p+1}{\ell} - 1\}
\]

where the last equation is by \( \ell \geq 2 \).

\[\square\]

5. **Families of non-vanishing polarizations**

We have considered the Mumford-Szpiro type polarization given in Proposition 16. Raynaud’s example is also of this kind. We show that much more varieties of polarizations can serve as counter-examples to Kodaira vanishing. We first consider

\[
 Z_{a,b} := \mathcal{O}_X(a\tilde{E}) \otimes \phi^*\mathcal{N}^b \quad (a, b \geq 1).
\]

**Proposition 27.** \( Z_{a,b} \) is ample.

**Proof.** We have \( E^2 = \deg D > 0 \) and also \( E, C > 0 \) for every irreducible curve \( C \in P \) (see Prop. V.2.3 [6]). Thus \( \mathcal{O}_P(nE), n > 0 \), is ample by Nakai-Moishezon criteria and, since \( \psi : X \to P \) is a finite morphism, \( \psi^*\mathcal{O}_P(nE) = \mathcal{O}_X(n\tilde{E}), n > 0 \), is also ample. In particular, \( \mathcal{O}_X(a\tilde{E}), a \geq 1, \) is ample. On the other hand, \( \mathcal{N}^b \) is ample so that in particular \( \pi^*\mathcal{N}^b \) is semi-ample (i.e., its sufficiently large powers are generated by global sections). Consequently, \( \mathcal{O}_X(a\tilde{E}) \otimes \phi^*\mathcal{N}^b \) is ample. \[\square\]
Then, by carrying out a similar argument as the proofs of Theorem 22 and Theorem 23, we have

**Theorem 28.** $H^1(X, Z_{a,b}^{-1}) \neq 0$ for all $a \geq 1$ and $1 \leq b \leq \ell - 1$.

**Proof.** Let $Q$ be an invertible sheaf on $C$ and set $Z = \mathcal{O}_X(-a\tilde{E}) \otimes \phi^*Q$. Now consider the following Leray spectral sequence of $\phi : X \rightarrow C$

$$E_2^{p,q} = H^p(C, R^q\phi_*\mathcal{Z}^{-1}) \Rightarrow H^{p+q}(X, \mathcal{Z}^{-1}) \quad (p \geq 0).$$

We have $E_2^{0,0} = 0$ since dim $C = 1$. Then by the 5-term exact sequence we have

$$H^1(X, \mathcal{Z}^{-1}) \rightarrow H^0(C, R^1\phi_*\mathcal{Z}^{-1}) \rightarrow 0.$$

Thus we have only to show $H^0(C, R^1\phi_*\mathcal{Z}^{-1}) = H^0(C, R^1\phi_*\mathcal{O}_X(-a\tilde{E}) \otimes \mathcal{Q}^{-1}) \neq 0$.

Considering the 5-term exact sequence

$$0 \rightarrow R^1\pi_*(\psi_*\mathcal{O}_X(-a\tilde{E})) \rightarrow R^1(\pi \circ \psi)_*\mathcal{O}_X(-a\tilde{E}) \rightarrow \pi^*(R^1\psi_*\mathcal{O}_X(-a\tilde{E})),$$

where $R^1\psi_*\mathcal{O}_X(-a\tilde{E}) = 0$ since $\psi : X \rightarrow P$ is an affine morphism, we have

$$R^1\phi_*\mathcal{O}_X(-a\tilde{E}) = R^1(\pi \circ \psi)_*\mathcal{O}_X(-a\tilde{E}) \cong R^1\pi_*(\psi_*\mathcal{O}_X(-a\tilde{E})).$$

Thus by Lemma 4 we obtain

$$R^1\phi_*\mathcal{O}_X(-a\tilde{E}) = R^1\pi_*\mathcal{O}_P(-aE) \oplus \bigoplus_{i=1}^{\ell-1} (R^1\pi_*\mathcal{M}^i)$$

$$= R^1\pi_*\mathcal{O}_P(-aE) \oplus \bigoplus_{i=1}^{\ell-1} (S^{(ip+i-2\ell)/\ell}(\mathcal{E}) \otimes \mathcal{N}^{(\ell-ip)e/\ell})^\vee.$$

We note that the last equation is shown in the end of the proof of Theorem 22. Thus we have

$$H^0(C, R^1\phi_*\mathcal{O}_X(-a\tilde{E}) \otimes \mathcal{Q}^{-1}) \supset \bigoplus_{i=1}^{\ell-1} H^0(C, S^{(ip+i-2\ell)/\ell}(\mathcal{E}) \otimes \mathcal{N}^{(\ell-ip)e/\ell})^\vee \otimes \mathcal{Q}^{-1}.$$

(Actually we can show that this inclusion is really an equation.)

On the other hand, from $E \rightarrow L \rightarrow 0$ we have

$$S^{(ip+i-2\ell)/\ell}(\mathcal{E}) \otimes \mathcal{N}^{(\ell-ip)e/\ell} \rightarrow L^{(ip+i-2\ell)/\ell} \otimes \mathcal{N}^{(\ell-ip)e/\ell} \rightarrow 0.$$

Taking the dual and tensoring by $\mathcal{Q}^{-1}$, we have

$$0 \rightarrow (L^{(ip+i-2\ell)/\ell} \otimes \mathcal{N}^{(\ell-ip)e/\ell})^\vee \otimes \mathcal{Q}^{-1} \rightarrow (S^{(ip+i-2\ell)/\ell}(\mathcal{E}) \otimes \mathcal{N}^{(\ell-ip)e/\ell})^\vee \otimes \mathcal{Q}^{-1}$$

and

$$(L^{(ip+i-2\ell)/\ell} \otimes \mathcal{N}^{(\ell-ip)e/\ell})^\vee = (\mathcal{N}^{(ip+i-2\ell)e/\ell} \otimes \mathcal{N}^{(\ell-ip)e/\ell})^\vee = \mathcal{N}^{(\ell-i)e/\ell}.$$

Thus we have for $i = 1, \ldots, \ell - 1$

$$H^0(C, \mathcal{N}^{(\ell-i)e/\ell} \otimes \mathcal{Q}^{-1}) \subset H^0(C, R^1\phi_*\mathcal{O}_X(-a\tilde{E}) \otimes \mathcal{Q}^{-1}).$$

In particular, taking $Q = \mathcal{N}^{(\ell-i)e/\ell} = \mathcal{N}^{(\ell-i)e/\ell}$ with $i = 1, \ldots, \ell - 1$, we have $k \subset H^0(C, R^1\phi_*\mathcal{O}_X(-a\tilde{E}) \otimes \mathcal{Q}^{-1})$ as required and in this case $\mathcal{Z}$ is exactly what we defined. □
The cohomologies $H^i(X, \mathcal{Z}_{a,b}^n)$, $i, n \in \mathbb{Z}$, can also be computed by a similar method to what we have described.

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