Masses of the Goldstone modes in the CFL phase
of QCD at finite density

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Abstract

We construct the $U_L(3) \times U_R(3)$ effective lagrangian which encodes the dynamics of the low energy pseudoscalar excitations in the Color-Flavor-Locking superconducting phase of QCD at finite quark density. We include the effects of instanton-induced interactions and study the mass pattern of the pseudoscalar mesons. A tentative comparison with the analytical estimate for the gap suggests that some of these low energy momentum modes are not stable for moderate values of the quark chemical potential $\mu$. 

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I. INTRODUCTION

The fascinating possibility that the ground state of QCD at finite quark density can behave as a color superconductor has attracted much attention recently [1–16]. Of particular interest is the symmetry breaking pattern that may arise for $N_f = 3$ light quark flavors. As first shown by Alford, Rajagopal and Wilczek [4], the diquark condensates can lock the color and flavor symmetry transformations (Color-Flavor-Locking or CFL for brief). In the limit of massless quarks, these condensates spontaneously break both color and chiral symmetries. The eight gluons become massive through the Higgs mechanism while eight plus two Goldstone bosons are leftover. The two octets of states are analogous to the octets of vector and light pseudoscalar mesons in vacuum and Schäfer and Wilczek [5] have conjectured that there might exist some sort of continuity between the properties of QCD at zero density and in the CFL phase. The two other massless excitations are the Goldstone modes associated with the spontaneous breaking of the baryon number ($U(1)_B$) and axial ($U(1)_A$) symmetries. Because of the axial anomaly, the latter is not a true symmetry of QCD. However, because instantons effects are small at large densities, the associated mode can be treated as a true Goldstone mode as a first approximation. Although there are significant differences, to which we shall come back later, the situation at high density is analogous to considering the limit of large numbers of colors $N_c$ in vacuum [17], in which $U(1)_A$ breaking effects are $1/N_c$ suppressed.

For energies which are small compared to the gap of the superconducting CFL phase, the dynamics of these Goldstone modes is most conveniently described with the help of an effective lagrangian. As suggested by the symmetry breaking pattern in the CFL phase at large densities, this effective theory is analogous to Chiral Perturbation Theory ($\chi$PT) in the large $N_c$ limit of QCD in vacuum [18]. For large densities, the leading order effective lagrangian invariant under $U(3)_L \times U(3)_R$ flavor symmetry has been constructed in some recent works [19–22,24,25]. In particular, Son and Stephanov [22] have shown how the parameters of the lagrangian can be computed at large densities by matching to the underlying microscopic theory. In the present note we make a first attempt to extend these works to lower density regimes, taking into account the effects of instanton-induced interactions. By using the power counting rules of $\chi$PT we construct the effective lagrangians up to order $E^4$ in an energy expansion. We make a particular emphasis on the meson spectrum and compute their masses as function of the quark chemical potential $\mu$. For moderate densities, $\mu \lesssim 10^4$ MeV, the gauge coupling grows large and instanton effects become important. We give numerical values for the masses of the mesons simply assuming that the analytical expressions for the gap and condensates, which have been computed at weak gauge coupling, also hold at strong coupling. Our calculations suggests that some of the Goldstone modes cannot exist as low energy excitations for values of $\mu < 1000$ MeV, as their masses are very close to the unstability threshold $2\Delta$, where $\Delta$ is the gap in the CFL phase. These conclusions depend on the value of the color superconducting gap as estimated in the literature, and of a number of approximations that we have made.

In the next section, we write down the general low energy effective action for the pseudoscalar Goldstone modes up to $E^4$ in a low energy expansion and estimate the couplings that are relevant for the meson mass spectrum. In Sec. [11] we give some numbers for the
meson masses at low densities, \( \mu \lesssim 2500 \text{ MeV} \) and finally draw some conclusions.

## II. EFFECTIVE CHIRAL LAGRANGIAN IN THE CFL PHASE

We follow Gatto and Casalboni \[20\] and Son and Stephanov \[22\] to construct the effective lagrangian for the low energy excitations of the CLF phase of QCD. The ground state is characterized by the two diquark condensates

\[
X_{ia} \sim \epsilon^{ijk} \epsilon^{abc} \langle \psi_{bj}^{L} \psi_{ck}^{L} \rangle^{*}, \quad Y_{ia} \sim \epsilon^{ijk} \epsilon^{abc} \langle \psi_{bj}^{R} \psi_{ck}^{R} \rangle^{*},
\]

where \( a, b, c \) denote color indices, while \( i, j, k \) refer to flavor ones. Under an \( SU(3)_c \times SU(3)_L \times SU(3)_R \) the condensates transform as

\[
X \rightarrow U_L X U_c^{\dagger}, \quad Y \rightarrow U_R Y U_c^{\dagger},
\]

and as

\[
X \rightarrow e^{2i\alpha} e^{2i\beta} X, \quad Y \rightarrow e^{-2i\alpha} e^{2i\beta} Y,
\]

under \( U(1)_A \) and \( U(1)_B \) transformations defined as

\[
\psi_L \rightarrow e^{i(\alpha-\beta)} \psi_L, \quad \psi_R \rightarrow e^{i(-\alpha-\beta)} \psi_R.
\]

One can factor out the norm of the condensates and consider the unitary matrices \( X \) and \( Y \). The slow variations of the phases of these matrices then correspond to the low energy excitations. Altogether these are \( 9 + 9 = 18 \) degrees of freedom: 8 will be absorbed by the gluons through the Higgs mechanism, which leaves 10 true low energy excitations. At low energies the gluons decouple from the theory, as they are heavy degrees of freedom. It is convenient to collect all the Goldstone modes in the unitary matrix

\[
\Sigma = XY^{\dagger},
\]

which is singlet of \( SU(3)_c \) and \( U(1)_B \) and transforms as

\[
\Sigma \rightarrow e^{4i\alpha} U_L \Sigma U_R^{\dagger},
\]

under \( SU(3)_L \times SU(3)_R \times U(1)_A \). This leaves apart the low energy excitation that emerges from the spontaneous breaking of baryon number symmetry \( U(1)_B \). Because baryon number is an exact global symmetry of QCD, this Goldstone mode is always massless in the CFL phase, independent of the quark masses. As our focus here is on the effect of chiral symmetry breaking by finite quark masses, to simplify our discussion we will simply drop this degree of freedom in the sequel.

To proceed we parametrise the unitary matrix \( \Sigma \) as

\[
\Sigma = \exp \left( i \frac{\Phi}{f_{\pi}} \right),
\]

where \( f_{\pi} \) is the pion decay constant.
where $\Phi = \phi^A T^A$, $A = 1, \ldots, 9$. The $T^A$ for $A = 1, \ldots, 8$ are the Gell-Mann generators of $SU(3)$ and $T^9 = \sqrt{2/3} 1_3$, all normalized to $\text{Tr}(T^A T^B) = 2 \delta^{AB}$. We use the same nomenclature as in vacuum for the nonet of Goldstone modes,

$$
\Phi = \begin{pmatrix}
\pi_0 + \frac{1}{\sqrt{3}} \left( \eta_8 + \sqrt{2} \frac{f_\pi}{f_{\eta_0}} \eta_0 \right) \\
\sqrt{2}\pi^- \\
\sqrt{2}K^- \\
\eta_8 + \sqrt{2} \frac{f_\pi}{f_{\eta_0}} \eta_0 \\
\sqrt{2}K_0 \\
\frac{1}{\sqrt{3}} \left( -2 \eta_8 + \sqrt{2} \frac{f_\pi}{f_{\eta_0}} \eta_0 \right)
\end{pmatrix} \quad (2.8)
$$

In (2.8), $f_\pi$ and $f_{\eta_0}$ are the decay constants at finite density respectively of the octet of mesons and of the singlet $\eta^0$. A priori there is no reason for these to be equal. By matching to the microscopic theory, Son and Stephanov [22] have found

$$
f_\pi^2 = \frac{21 - 8 \log 2}{18} \frac{\mu^2}{2\pi^2}, \quad f_{\eta_0}^2 = \frac{3}{4} \frac{\mu^2}{2\pi^2},
$$

where $\mu$ is the quark chemical potential. These expressions are valid at large densities (i.e. small gauge coupling). From (2.3), the ratio $f_\pi/f_{\eta_0} \approx 1.07$ is close to one. This result is reminiscent of the OZI rule often invoked to hold in vacuum. At smaller densities, the gauge coupling grows large and the ratio could significantly depart from unity because of instanton effects.

The quark mass term in the microscopic lagrangian

$$
\Delta L = -\bar{\psi}_L M \psi_R + h.c. \quad (2.10)
$$

breaks chiral symmetry. Its effects can be introduced in the effective lagrangian by treating the mass matrix as an external field with vacuum expectation value

$$
M = \text{diag}(m_u, m_d, m_s), \quad (2.11)
$$

where $m_u, m_d$ and $m_s$ refer to the up, down and strange quark masses, respectively. The spurion field then transforms as

$$
M \rightarrow e^{-2i\alpha} U^\dagger_L M U_R, \quad (2.12)
$$

under $SU(3)_L \times SU(3)_R \times U(1)_A$.

To take into account the effects of the $U(1)_A$ anomaly, we also allow for the presence of the $\theta$ term in the microscopic lagrangian,

$$
\Delta L_\theta = \theta \frac{g^2}{32\pi^2} F^A_{\mu\nu} \tilde{F}^{\mu\nu,A}. \quad (2.13)
$$

As for the quark mass term, we will treat $\theta$ as an external field with vanishing expectation value which, to account for the variation of the quark measure, transforms as

$$
\theta \rightarrow \theta - 2N_f \alpha \equiv \theta - 6 \alpha \quad (2.14)
$$

under $U(1)_A$ transformations. Then any arbitrary function of the combination
\[ X = \theta - \frac{i}{2} \text{Tr} \log \Sigma \equiv \theta + \sqrt{\frac{3}{2} \frac{\eta_0}{f_{\pi}}}, \]  

(2.15)

is invariant under \( SU(3)_L \times SU(3)_R \times U(1)_A \) transformations.

With these ingredients, we are almost ready to construct the low energy effective lagrangian. One last issue is that of power counting. In QCD in vacuum, because \( M_{\pi}^2 \sim m_q \), the expansion is in powers of the pion external energy or momenta and quark masses and \( E^2 \sim p^2 \sim m_q \). Similarly, at large \( N_c \), \( U(1)_A \) breaking effects are counted as \( E^2 \sim 1/N_c \), because \( M_{\pi}^2 \sim 1/N_c \). In the CFL phase of QCD on the other hand, the power counting depends very much on the density. At finite densities, instantons effects are screened \( \propto (\Lambda/\mu)^{\alpha} \), where \( \Lambda \approx 200 \text{ MeV} \) is the scale of QCD, \( \mu \) is the quark chemical potential and \( \alpha \) is some positive constant which depends on the process under consideration. This has two consequences at large densities. The first is that \( U(1)_A \) is essentially a good symmetry and the \( \eta_0 \) is approximately massless in the chiral limit. Then, because the two condensates break chiral symmetry only through the intermediate of color transformations, there is an approximate \( Z_L^2 \times Z_R^2 \) symmetry which acts independently on the left and right-handed quarks \[ \text{[4]} \]. At high densities, this implies that the leading contribution to the mass of the Goldstone modes is \( \mathcal{O}(m_q^2) \). The natural power counting rule at high densities is then \( E^2 \sim m_q^2 \) because \( M_{\pi}^2 \sim m_q^2 \). At the moderate densities that could eventually be of interest for heavy ion collisions or for neutron stars, instantons effects are likely to be non-negligible. As \( Z_L \times Z_R \) symmetry is broken to the diagonal \( Z_2 \) by instantons, a meson mass term \( M_{\pi}^2 \sim m_q \) is allowed \[ \text{[4]} \]. If instanton effects are dominant, the power counting is that of vacuum, \( E^2 \sim m_q \). For convenience, we will adopt the counting rules of \( \chi \text{PT} \) as in vacuum and comment when necessary about the differences that arise in the CFL phase of QCD.

A final remark concerning the region of applicability of the energy expansion. In vacuum, the low energy expansion is valid for \( E^2 \lesssim f_{\pi}^2 \). In the CFL phase however, \( f_{\pi} \sim \mu \), which is independent and much larger than the gap (\( \Delta \ll \mu \)) at large densities (see (2.9)). However, the effective theory must break down for \( E \sim 2\Delta \), which is the energy necessary to excite one quasiparticle pair out of the superconducting ground state, and the low energy expansion is only valid for the more restrictive range \( E^2 \sim M_{\pi}^2 \lesssim 4\Delta^2 \) \[ \text{[4]} \].

At order \( E^2 \), the most general lagrangian compatible with the symmetries of the condensates is

\[
\mathcal{L}_2 = V_1(X) \text{Tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \right) + \left\{ V_2(X) \text{Tr} \left( M \Sigma^\dagger \right) e^{i\theta} + h.c. \right\} \\
+ V_3(X) \left( \text{Tr} \Sigma \partial_\mu \Sigma^\dagger \right)^2 + V_4(X) \left( \text{Tr} \Sigma \partial_\mu \Sigma^\dagger \right) \partial^\mu \theta + V_5(X) \partial_\mu \theta \partial^\mu \theta .
\]

(2.16)

\[ \^ \]

\[ ^\dagger \]There is a formal analogy between this behavior and that of \( \chi \text{PT} \) in vacuum in the large \( N_c \) limit. \textit{A priori} the expansion is valid for \( E < f_{\pi} \). But because \( f_{\pi}^2 \sim N_c \Lambda^2 \) grows large as \( N_c \rightarrow \infty \) and as something is bound to happen for \( E \sim \Lambda \sim 200 \text{ MeV} \), which is the scale of quark confinement, the low energy expansion is limited to \( E \lesssim \Lambda \).
We have written (2.16) using a compact notation. Since Lorentz invariance is broken at finite density (only a $O(3)$ symmetry is preserved) all the four-vectors should be split into temporal and spatial components, and the functions $V_i$ multiplying those spatial or temporal components are not forced to be the same by symmetry arguments. For example, the first term in (2.16) should read

$$V_1(X) \text{Tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma \right) - V_{1,s}(X) \text{Tr} \left( \partial_t \Sigma \partial_\tau \Sigma \right).$$

The same situation occurs for the remaining terms and functions $V_3, V_4, V_5$ of $X$. All the couplings in (2.16) can be a priori arbitrary functions of $X$ (2.15). At large densities, $U(1)_A$ symmetry breaking effects are exponentially suppressed and the couplings only depend on the chemical potential $\mu$.

To reproduce the standard normalization of the meson kinetic terms, we impose

$$V_{1,t}(0) = \frac{f^2_\pi}{4}. \tag{2.18}$$

The ratio $V_{1,s}(0)/V_{1,t}(0) = v^2$ is the velocity squared of the Goldstone bosons. At large densities, Son and Stephanov [22] have found that $v$ is equal to the speed of sound $1/\sqrt{3}$ for all the low energy modes, including the baryon Goldstone mode.

The operators in (2.16) are not all independent. The last three terms can be transformed into each other with a field redefinition, $\eta_0/f_{\eta_0} \rightarrow \eta_0/f_{\eta_0} + \kappa \theta$. Using this freedom, we can choose to set $V_4(0)$ to zero. With this choice, the last operator becomes irrelevant for the meson spectrum and can be discarded.

The operator that is left, with coupling $V_3(X)$, contributes to the difference between $f_\pi$ and $f_{\eta_0}$. At high densities, using (2.9),

$$V_{3,t}(0) = \frac{f^2_\pi - f^2_{\eta_0}}{12} \approx 0.01 f^2_\pi, \tag{2.19}$$

which is small compared to $f^2_\pi$. At moderate densities, $V_{3,t}$ could receive large contributions from instantons as is manifest from the mixing with $V_4$ and $V_5$.

Consider now the mass term in (2.16). This term is analogous to the leading mass term in $\chi$PT in vacuum. The only difference is the occurrence of the phase $\theta$, which is absent in vacuum. This is because at zero density the condensate that breaks chiral symmetry is $\langle \bar{\psi}_L \psi_R \rangle$ which transforms like $M^\dagger$ under $SU(3)_L \times SU(3)_R \times U(1)_A$. In (2.10) the presence of the $\theta$ in the effective lagrangian is the trademark of a one instanton effect; in the CFL phase, a one instanton process can be saturated by closing its six external quark legs with the insertion of one left-handed diquark, one right-handed diquark and one chiral condensate, $\sim (\bar{\psi}_L \psi_L)(\bar{\psi}_R \bar{\psi}_R)(\bar{\psi}_R \psi_L)$. As in vacuum, a non-zero chiral condensate $\langle \bar{\psi} \psi \rangle$ leads to $M_\pi^2 \propto m_0 \langle \bar{\psi} \psi \rangle$. Instanton effects are small and can be reliably computed at high densities $\mu \gg \Lambda$. In particular, Schäfer [13] has obtained

$$\langle \bar{\psi} \psi \rangle = \langle \bar{u} u + \bar{d} d + \bar{s} s \rangle \approx -2 \left( \frac{\mu^2}{2\pi^2} \right) \frac{3\sqrt{2\pi}}{5} \frac{18}{g(\mu)} \frac{1}{\phi_A^2(\mu)} \tag{2.20}$$
where \( g(\mu) \) is the gauge coupling estimated at the scale \( \mu \) using the one-loop beta function. The factor \( G(\mu) \) is the one instanton weight integrated over all instanton sizes \( \rho \), which are peaked around \( \rho \sim \mu^{-1} \) at finite density,

\[
G(\mu) \approx 0.26 \Lambda^{-5} \left( \frac{\beta_0 \log(\mu/\Lambda)}{\mu} \right)^{2N_c} \frac{\Lambda}{\mu}^{\beta_0+5}.
\] (2.21)

We have used the running of \( g(\mu) \) at one-loop, \( \Lambda = 200 \text{ MeV} \) and \( \beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \equiv 9 \) is the first coefficient of the QCD beta function. Finally, \( \phi_A(\mu) \sim \langle \bar{\psi}_L \psi_L \rangle \sim \langle \bar{\psi}_R \psi_R \rangle \) is the diquark condensate in the CFL phase of QCD,

\[
\phi_A(\mu) \approx 2 \left( \frac{\mu^2}{2\pi^2} \right) \left( \frac{3\sqrt{2}\pi}{g(\mu)} \right) \Delta(\mu),
\] (2.22)

and \[11,13\]

\[
\Delta(\mu) = b'_0 512\pi^4 (2/N_f)^{5/2} g(\mu)^{-5} \mu \exp \left( -\frac{3\pi^2}{\sqrt{2}g(\mu)} \right),
\] (2.23)

is the color superconducting gap. The factor \( b'_0 \) is unknown but expected to be \( \mathcal{O}(1) \) \[13,26,27\]. At large densities \( \mu \gg \Lambda \), instantons are suppressed, \( \langle \bar{\psi}\psi \rangle \) goes to zero and the mass term of (2.16) vanishes. At the moderate densities that could be of interest for heavy ion collisions or neutron stars, this instanton effect is however not negligible. Using the Gell-Mann-Oakes-Renner relation, and \( \langle \bar{\psi}\psi \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle \equiv 3\langle \bar{u}u \rangle \), we find

\[
V_2(0) = -\frac{1}{6} \langle \bar{\psi}\psi \rangle.
\] (2.24)

At order \( E^4 \), there are a few more operators\footnote{At this order, one could also consider adding a Wess-Zumino-Witten term \[19\].}.

\[
\mathcal{L}_4 = K_1(X) \left[ \text{Tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \right) \right]^2 + K_2(X) \text{Tr} \left( \partial_\mu \Sigma \partial_\nu \Sigma^\dagger \right) \text{Tr} \left( \partial^\mu \Sigma \partial^\nu \Sigma^\dagger \right)
+ K_3(X) \text{Tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\nu \Sigma^\dagger \right)
+ \left\{ K_4(X) \text{Tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \right) \text{Tr} \left( \mathcal{M} \Sigma^\dagger \right) e^{i\theta} + h.c. \right\} + \left\{ K_5(X) \text{Tr} \left( \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \mathcal{M} \Sigma^\dagger \right) e^{i\theta} + h.c. \right\}
+ \left\{ K_6(X) \text{det}(\Sigma) \text{Tr} \left( \mathcal{M} \Sigma^\dagger \right) \text{Tr} \left( \mathcal{M} \Sigma^\dagger \right) + h.c. \right\}
+ \left\{ K_7(X) \text{det}(\Sigma) \text{Tr} \left( \mathcal{M} \Sigma^\dagger \mathcal{M} \Sigma^\dagger \right) + h.c. \right\}.
\] (2.25)

Again, we have used a compact notation for the terms in \( K_i \), \( i = 1 \ldots 5 \) which should be split into temporal and spatial components.

Like the term \( V_2 \) in (2.16), the terms \( K_4 \) and \( K_5 \) depend explicitly on \( \theta \) and so are exponentially suppressed at high densities. Similarly, at large chemical potential \( \mu \), the
remaining functions $K_i(X)$ reduce to constants $K_i(X = 0; \mu)$. In this limit, the mass pattern of the Goldstone bosons will be determined by the couplings $K_6$, $K_7$ and $K_8$ in (2.25). As first shown by Son and Stephanov [22], at large densities, these coupling constants can be computed by matching to the underlying microscopic theory.

In principle, a systematic and non-ambiguous strategy to compute the coefficients of the effective theory is to use the background field technique [28], introducing external sources and symmetry breaking order parameters and integrating out the quark fields. At high densities, gluon exchange is suppressed and this amounts to a quark one-loop calculation, which would fix the coefficients of all the operators to arbitrary order in the meson fields. A slightly different strategy has been advocated in [22]: set the meson fields to zero, $\Sigma = \frac{1}{3}$ in the operators of (2.25), and compare the shift in ground state energy induced by non-zero quark masses in both the effective and microscopic theories. This approach is a priori ambiguous, as different operators could contribute to the shift in ground state energy. In practice, to order $\epsilon^4$, enough constraints can be derived to completely fix the couplings $K_6$, $K_7$ and $K_8$. The diagrams in the full microscopic theory are those of Fig.1. The rhs diagram can only contribute to $K_6$ and $K_8$. The lhs diagram can contribute to $K_7$. These couplings can be fixed by considering two different quark mass patterns,

$$\mathcal{M}_1 = m_1 \mathbf{1}_3 \quad \text{and} \quad \mathcal{M}_2 = \text{diag}(0, 0, m_s) = -\frac{m}{\sqrt{3}} \lambda^8 + \frac{m}{\sqrt{6}} \lambda^9.$$ (2.26)

In the effective theory, these respective choices lead to the following shifts in ground state energy density

$$\Delta \varepsilon_1 = - \left\{ (9m^2K_6 + \text{h.c.}) + 9m^2K_7 + (3m^2K_8 + \text{h.c.}) \right\},$$ (2.27)

$$\Delta \varepsilon_2 = - \left\{ (m_s^2K_6 + \text{h.c.}) + m_s^2K_7 + (m_s^2K_8 + \text{h.c.}) \right\}. $$ (2.28)

A special feature of a quark mass term is that it couples fermions and antifermions. At finite density, quark mass effects are then suppressed by the necessity to excite antiparticles with characteristic momentum $\sim 2\mu$ [24,25]. A simple way to see this is to consider the effective theory for the fermion excitations near the surface of the Fermi sea. In momentum space, the free fermion lagrangian is

$$\mathcal{L} = \bar{\psi}(\gamma^\mu p_\mu + \mu \gamma^0)\psi - m\bar{\psi}\psi.$$ (2.29)

If we consider the fermion mass term as a perturbation and decompose the fermions field $\psi$ into positive and negative energy components $\psi_+$ and $\psi_-$, using the projectors

$$\Lambda^\pm = \frac{1}{2}(1 \pm \alpha \cdot \hat{q}),$$ (2.30)

and $\Lambda^\pm \psi^\pm = \psi^\pm$, the lagrangian becomes

$$\mathcal{L} = \bar{\psi}_+(q_0 - |\vec{q}| + \mu)\psi_+ + \bar{\psi}_-(q_0 + |\vec{q}| + \mu)\psi_- - m(\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+).$$ (2.31)

This shows that the fermion mass term couples particles and antiparticles. If we integrate out the antifermions, we get at leading order
\[ \mathcal{L} \approx \psi_+ (q_0 - q_\parallel) \psi_+ - \frac{m^2}{\mu} \psi_+ \psi_+ \]  

(2.32)

where \( q_\parallel \approx |\vec{q}| - \mu \). This shows that quark mass effects are suppressed at high densities, as one could have expected. Note that the modified mass term \( \propto m^2/\mu \) has the same structure as a chemical potential term. This is a relevant operator which modifies the shape of the Fermi surface \([29]\).

The calculation of the \( l.h.s. \) diagram of Fig. 1 gives

\[ \frac{\Delta \varepsilon_1}{\Delta \varepsilon_2} = \frac{3 m^2}{m_s^2} , \]  

(2.33)

to \( \mathcal{O}(\frac{m^2 \Delta^4}{\mu^4}) \), where \( \Delta \) stands for generic gap and/or antigaps. This diagram simply gives a trivial shift of the vacuum energy, which in the effective theory corresponds to a constant (independent of \( \Sigma \)) invariant operator

\[ \Delta \mathcal{L} \propto \text{Tr}(M^\dagger M) , \]  

(2.34)

and, consequently, \( K_7 = 0 \).

Estimates of the \( r.h.s. \) diagram of Fig. 1 indicate that \([24]\)

\[ K_{6,8} \sim \frac{\Delta \bar{\Delta}}{\mu^2} \log(\Delta/\mu) , \]  

(2.35)

where \( \Delta \) is a generic expression for a quark gap and \( \bar{\Delta} \) is an antigap. The dependence on antigaps arises from the structure of the \( r.h.s. \) diagram with both particles and antiparticles propagators. Unfortunately, the expression of the antigap is not known. Preliminary estimates have revealed that it is gauge-dependent \([12]\). It is however a physical quantity (pole of the antiquark quasi-particles and holes), which should be gauge-invariant on quasi-particles mass-shell. At weak coupling, the antigap is presumably much smaller than the gap. Here, we will assume that \( K_{6,8} \approx 0 \) at large and moderate densities.

To be complete we should also include the contribution of instantons to the mass of the singlet meson \( \eta_0 \), which enters the effective lagrangian through a function of \( X \) alone,

\[ \mathcal{L}_0 = -V_0(X) , \]  

(2.36)

This piece is \( E^0 \) as it involves no derivatives of the meson fields. Expanding to second order in \( X \) gives

\[ M_{\eta_0}^2 = \frac{3}{2} \frac{V_0''(0)}{f_{\eta_0}^2} . \]  

(2.37)

The constant \( V_0'' \) (and more generally the whole function \( V(X) \)) could in principle be computed at large densities using instanton calculus. Unfortunately, the result is not known. However, at large to moderate densities we might expect this contribution to be suppressed.
For three flavors, the two remaining legs of a one instanton contribution to \( M_{\eta_0}^2 \) can only be closed with the insertion of either a quark mass term or a chiral condensate \( \langle \bar{\psi} \psi \rangle \). In the chiral limit, the latter only arises through another instanton process. On dimensional grounds we would expect

\[
M_{\eta_0}^2 \propto \mu^4 G(\mu) |\langle \bar{\psi} \psi \rangle| \leftarrow (2.38)
\]

in the chiral limit, but from such a rough estimate, it is not reasonable to infer whether \( M_{\eta_0} \) ever grows large at low densities.

### III. MESON MASS PATTERN

In this section we give numerical estimates of the pseudoscalar mesons as function of the quark chemical potential.

At very large densities, instanton effects are suppressed: meson masses are linear in the quark masses and presumably very small. At lower densities, a non-zero \( \langle \bar{\psi} \psi \rangle \) condensate induced by instantons introduce contributions to the meson masses which are proportional to \( m_q^{1/2} \). Here, we will estimate this last effect as it is dominant in the low density regime of the theory. We will use the expression of the chiral condensate and the gap as computed at weak coupling \((2.20)\) and we will assume that the corrections are small even at moderate densities \( \mu \sim 600 \text{ MeV} \).

The masses of the charged pions and kaons deduced from \((2.19)\) read

\[
M_{\pi^\pm}^2 = \frac{2V_2}{f^2} (m_u + m_d) \leftarrow , \quad M_{K^\pm}^2 = \frac{2V_2}{f^2} (m_u + m_s) \leftarrow , \quad M_{K_0^-,\bar{K}_0}^2 = \frac{2V_2}{f^2} (m_d + m_s) \leftarrow ,
\]

\((3.1)\)

In the limit of no \( U(1)_A \) breaking effects, the neutral mesons \( \pi^0, \eta_8 \) and \( \eta_0 \) are strongly mixed. Their mass matrix reads

\[
\frac{2V_2}{f^2} \left( \begin{array}{ccc}
m_u + m_d & \frac{m_u - m_d}{\sqrt{3}} & \frac{f_\eta}{f_{\eta_0}} \frac{\sqrt{2}(m_u - m_d)}{\sqrt{3}} \\
\frac{2(m_u - m_d)}{\sqrt{3}} & m_u + m_d - 4m_s & \frac{f_\pi}{f_{\eta_0}} \frac{\sqrt{2}(m_u + m_d - 2m_s)}{\sqrt{3}} \\
\frac{f_\pi}{f_{\eta_0}} \frac{\sqrt{2}(m_u + m_d - 2m_s)}{\sqrt{3}} & \frac{f_\eta}{f_{\eta_0}} \frac{\sqrt{2}(m_u + m_d)}{\sqrt{3}} & \frac{2(m_u + m_d + m_s)}{3}\end{array} \right) .
\]

\((3.2)\)

We have assumed that the contribution from the quark masses and the chiral condensate is parametrically larger than the two-instantons contribution to the mass of \( \eta_0 \). As the latter effect, if important, would increase the mass of \( \eta_0 \), our numerical estimates should be considered as lower bounds on the mass of \( \eta' \). Neglecting the difference between \( f_\pi \) and \( f_{\eta_0} \), we get the mass eigenvalues characteristic of ideal mixing,

\[
M_{uu}^2 = \frac{2V_2}{f^2} 2m_u \leftarrow , \quad M_{dd}^2 = \frac{2V_2}{f^2} 2m_d \leftarrow , \quad M_{ss}^2 = \frac{2V_2}{f^2} 2m_s .
\]

\((3.3)\)

To give an idea of the numerical values of the meson masses at \( \mu \sim 600 \text{ MeV} \), we take \( m_u = 4 \text{ MeV}, m_d = 7 \text{ MeV} \) and \( m_s = 150\) and evaluate the value of the chiral condensate for \( \Lambda = 200 \text{ MeV} \) and \( b'_0 = 1 \), to find
\[ M_{\pi^\pm}^2 \approx 30 \text{ MeV}, \quad M_{K^\pm}^2 \approx 112 \text{ MeV}, \quad M_{K^0,\bar{K}^0}^2 \approx 113 \text{ MeV}, \quad (3.4) \]
\[ M_{\bar{u}u}^2 \approx 26 \text{ MeV}, \quad M_{\bar{d}d}^2 \approx 34 \text{ MeV}, \quad M_{\bar{s}s}^2 \approx 71 \text{ MeV}. \quad (3.5) \]

A more accurate analysis of the meson masses would require to take into account terms which are proportional to \( m_q^2 \). However, in the low density regime, and due to the smallness of \( K_6 \) and \( K_8 \), these represent negligible corrections to the above estimates.

Because the expression for the chiral condensate involves the square of the gap, which is only known up to a factor \( b'_0 \) assumed to be \( \mathcal{O}(1) \) \[11,5,13,26,27\], there is some theoretical uncertainty in the meson masses at low densities and the values quoted are at most indicative. At this order, this uncertainty can be cancelled by considering the mass over gap ratios. These ratios are independent of the gap and, moreover, tell us whether the mesons can exist as low energy excitations in the CFL phase at chemical potential \( \mu \). The ratios of the charged and neutral meson masses over two times the gap are shown in Figs. 4 and 5. For the strange particles, the ratio is larger than one or very close to the instability threshold unless \( \mu > 1 \text{ GeV} \), which suggests that these particles do not exist as stable low energy excitations at low densities.

**IV. CONCLUSIONS**

We have considered the effective lagrangian for the Goldstone modes in the Color-Flavor-Locking phase of QCD at high densities, up to \( E^4 \) in a low energy expansion \( E \ll 2\Delta \). A tentative analysis of the meson mass pattern that emerges from this lagrangian, including instanton effects, suggests that the kaons and the pure strange neutral meson may be absent from the spectrum for moderate densities. Their masses are significantly larger than two times the gap for \( \mu \lesssim 1 \text{ GeV} \). For the pions, the situation is much better as their masses are significantly smaller than the instability threshold for all densities.

Obviously, there is room for improvements. Our approach has been far from systematic and we have made some ad hoc assumptions. Although the effects are likely to be small at moderate densities, it is of much interest to gain a better understanding of the quadratic quark mass effects \[24,25\]. A calculation of the contribution of instantons to the mass of \( \eta_0 \) in the chiral limit would also prove to be most useful. If a spontaneously broken, approximate \( U(2)_L \times U(2)_R \) flavor symmetry survives in the CFL phase of QCD at moderate densities, it could also be interesting to determine the potential for the \( \eta_0, V_0(X) \). The effective theory for the low energy excitations should be very much analogous to the large \( N_c \) lagrangian of Di Vecchia and Veneziano \[31\] and Witten \[32\].

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FIG. 1. Quark loops in the microscopic theory. $\mathcal{M}$ ($\mathcal{M}^\dagger$) stands for one insertion of the (conjugate) quark mass matrix. The $lhs$ diagram is like in vacuum. The $rhs$ diagram arises only in presence of a diquark condensate.

FIG. 2. Color gap (in MeV) as a function of the quark chemical potential $\mu$ (in MeV) for $b_0' = 1$. 
FIG. 3. Plot of $(-\langle \bar{\psi}\psi \rangle)^{1/3}$ (in MeV) as function of the quark chemical potential $\mu$ (in MeV), where $\langle \bar{\psi}\psi \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle$ is the chiral quark condensate induced by instantons.

FIG. 4. Ratios of charged pions (lower curve) and kaons (two upper curves) masses over two times the gap as function of the quark chemical potential (in MeV).
FIG. 5. Ratios of neutral meson masses \((M_{\bar{u}u} < M_{\bar{d}d} < M_{\bar{s}s})\) over two times the gap as functions of the quark chemical potential (in MeV).