

Research Article

New Results on the Geometric-Arithmetic Index

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Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_u$ denote the degree of vertex $u \in V(G)$. The geometric-arithmetic index of $G$ is defined as $\text{GA}(G) = \sum_{uv \in E(G)} \frac{2 \sqrt{d_ud_v}}{d_u + d_v}$. In this paper, we obtain some new lower and upper bounds for the geometric-arithmetic index and improve some known bounds. Moreover, we investigate the relationships between geometric-arithmetic index and several other topological indices.

1. Introduction

Let $G$ be a simple graph (i.e., graph without loops and multiple edges) with vertex set $V(G)$ and edge set $E(G)$. The integers $n = |V(G)|$ and $m = |E(G)|$ are the order and the size of the graph $G$, respectively. For $u \in V(G)$, we denote by $d_u$ the degree of vertex $u$ in $G$. The minimum and maximum degrees of a graph are denoted by $\delta$ and $\Delta$, respectively.

Graph theory has provided chemists with a variety of useful tools, such as topological indices. A topological index $\text{Top}(G)$ of a graph $G$ is a number with the property that, for every graph $H$ isomorphic to $G$, $\text{Top}(H) = \text{Top}(G)$.

Molecular descriptors play a significant role in mathematical chemistry, especially in QSPR/QSAR investigations. Among them, special place is reserved for so-called topological descriptors. A topological index is a numeric quantity from the structural graph of a molecule.

Usage of topological indices in chemistry began in 1947 when Wiener [1] developed the most widely known topological descriptor, namely, the Wiener index, and used it to determine physical properties of types of alkanes known as paraffin (see, for instance, [2, 3]). The interest of topological indices lies in the fact that they synthesize some of the properties of a molecule into a single number. With this in mind, hundreds of topological indices have been introduced and studied. Topological indices based on the vertex degree play a vital role in mathematical chemistry, and some of them are recognized as tools in chemical research.

Authors are studying various topological descriptors, such as Zagreb indices [4–6], general sum-connectivity index [7, 8], hyper-Zagreb index [9], and harmonic index [10, 11]. Besides the abovementioned ones, there are other topological descriptors based on end vertex degrees of edges of graphs that have found some applications in QSPR/QSAR research [2, 12, 13].

The geometric-arithmetic index of a graph is defined in [13] as

$$\text{GA}(G) = \sum_{uv \in E(G)} \frac{2 \sqrt{d_ud_v}}{d_u + d_v} \quad (1)$$

The geometric-arithmetic index has a number of interesting properties, e.g., see [13]. The lower and upper bounds of the geometric-arithmetic index of connected graphs and the characterizations of graphs for which these bounds are best possible can be found in [13–16].
The aim of this paper is to investigate new relationships between the geometric-arithmetic index and other topological indices. In particular, we obtain some lower and upper bounds for the geometric-arithmetic index. Moreover, we improve some known bounds.

2. Preliminaries

Let us recall some remarkable lemmas which will be used in this paper.

The first one is a very straightforward observation.

\[
\frac{n}{(1/x_1) + (1/x_2) + \cdots + (1/x_n)} \leq \sqrt[n]{\prod_{i=1}^{n} x_i} \leq \sqrt[n]{\sum_{i=1}^{n} x_i}
\]

Lemma 3 (see [19]). Let \( a = (a_i)_{i=1}^{m} \) and \( b = (b_i)_{i=1}^{m} \) be two sequences of positive numbers. For any \( r \geq 0 \),

\[
\sum_{i=1}^{m} a_i^{r+1} \leq \left( \sum_{i=1}^{m} a_i \right)^{r+1} \leq \left( \sum_{i=1}^{m} b_i \right)^{r+1}
\]

Lemma 4 (see [20]). Let \( r \leq a \leq R \) for \( 1 \leq i \leq m \) and \( r \) and \( R \) be some positive constants. Then,

\[
\sum_{i=1}^{m} a_i^n \leq \left( \frac{R}{r} \right)^{n} \left( \sum_{i=1}^{m} a_i \right)
\]

Lemma 5 (see [21]). If \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) are positive numbers, where \( m_1 \leq a_i \leq N_1 \) and \( m_2 \leq b_i \leq N_2 \) for each \( 1 \leq i \leq n \), then

\[
\sum_{i=1}^{n} a_i^2 b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} (N_1N_2 + m_1m_2).
\]

Lemma 6 (the Pólya–Szegő inequality, see p. 62 in [22]). Let \( a = (a_i)_{i=1}^{m} \) and \( b = (b_i)_{i=1}^{m} \) be two sequences of positive numbers, where \( 0 < m_1 \leq a_i \leq M_1 \) and \( 0 < m_2 \leq b_i \leq M_2 \), for \( i = 1, 2, \ldots, n \). Then,

\[
\sum_{i=1}^{m} a_i^{2} b_i^{2} - \left( \sum_{i=1}^{m} a_i b_i \right)^{2} \leq \left( \frac{M_1M_2}{m_1m_2} + \frac{m_1m_2}{M_1M_2} \right)^{2} \left( \sum_{i=1}^{m} a_i b_i \right)^{2}
\]

3. Upper Bounds for the Geometric-Arithmetic Index

In this section, we investigate the relationships between geometric-arithmetic index and some topological indices. Moreover, we obtain some upper bounds for the geometric-arithmetic index in terms of order, size, maximum degree, minimum degree, domination number, girth, number of cut edges, and number of pendant vertices.

Let \( x \) and \( y \) be two positive numbers. Then,

\[
\frac{2xy}{x+y} \leq \sqrt{x+y} \leq \sqrt[4]{\frac{(x+y)^2}{2}} \leq \sqrt[4]{\frac{x^2+y^2}{2}}
\]

The following is the well-known inequality of arithmetic and geometric means.

Lemma 2 (inequality of arithmetic and geometric means, see [18]). Let \( x_1, \ldots, x_n \) be positive numbers. Then,

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt[2]{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}
\]

The first and second Zagreb indices are vertex-degree-based graph invariants defined as

\[
M_1(G) = \sum_{uv \in E(G)} (d_u + d_v),
\]

\[
M_2(G) = \sum_{uv \in E(G)} d_ud_v.
\]

The quantity \( M_1 \) was first considered in 1972 [6], whereas \( M_2 \) in 1975 [5]. The general Randić index is defined as follows [23]:

\[
R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha,
\]

where \( \alpha \) is a real number.

We begin with the establishment of an upper bound for the geometric-arithmetic index in terms of the first Zagreb index and the general Randić index.

Theorem 1. Let \( G \) be a graph. Then,

\[
GA(G) \leq \frac{M_1(G) + 2R_{1/2}(G)}{4}
\]

Proof. By Lemma 1, we have

\[
GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_ud_v}}{d_u + d_v}
\]

\[
\leq \sum_{uv \in E(G)} \frac{2d_ud_v}{d_u + d_v}
\]

\[
\leq \sum_{uv \in E(G)} \frac{(d_u + d_v)^2}{2} \quad \frac{\sqrt{d_ud_v}}{2}
\]

\[
= \frac{M_1(G) + 2R_{1/2}(G)}{4}
\]

as desired.
Using Lemma 1 and an argument similar to the proof of Theorem 1, we can obtain the next result.

**Corollary 1.** Let $G$ be a graph. Then,

$$ GA(G) \leq R_{1/2}(G). \quad (12) $$

From Lemma 1, we get

$$ R_{1/2}(G) = \sum_{uv \in E(G)} \sqrt{d_u d_v} \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2} = \frac{M_1(G)}{2}. \quad (13) $$

Again by Lemma 1, we have

$$ \frac{M_1(G) + 2R_{1/2}(G)}{4} = \sum_{uv \in E(G)} \frac{(d_u + d_v)/2 + \sqrt{d_u d_v}}{2} \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2} = \frac{M_1(G)}{2}. \quad (14) $$

Hence, we can see that the bounds in Theorem 1 and Corollary 1 improve the bound:

$$ GA(G) \leq \frac{M_1(G)}{2}, \quad (15) $$

established in [15].

The proof of the following result can be found in [23].

**Lemma 7 (see [23]).** Let $G$ be a graph of size $m$. Then,

$$ R_{\alpha}(G) \leq m \left( \frac{\sqrt{8m + 1} - 1}{2} \right)^{\frac{1}{\alpha}}, \quad (16) $$

for $0 < \alpha \leq 1$.

Using Corollary 1 and Lemma 7, we can drive the next result.

**Corollary 2.** Let $G$ be a graph of size $m$. Then,

$$ GA(G) \leq \frac{m (\sqrt{8m + 1} - 1)}{2}. \quad (17) $$

**Lemma 8.** Let $x$ and $y$ be two positive numbers. Then,

$$ \frac{2 \sqrt{xy}}{x + y} \leq 1, \quad (18) $$

$$ \frac{x + y}{\sqrt{xy}} \geq 2. $$

Now, we obtain an upper bound for the geometric-arithmetic index in terms of the first Zagreb index.

**Theorem 2.** Let $G$ be a graph of order $n \geq 2$, size $m$, and minimum degree $\delta$. Then,

$$ GA(G) \leq m - n + \frac{M_1(G)}{\delta^2}. \quad (19) $$

**Proof.** Notice that

$$ \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v} = \sum_{uv \in E(G)} \left( \frac{1}{d_u} + \frac{1}{d_v} \right) = n. \quad (20) $$

By Lemma 8, we have

$$ GA(G) + n = \sum_{uv \in E(G)} \left( 2 \sqrt{d_u d_v} + \frac{d_u + d_v}{d_u d_v} \right) \leq \sum_{uv \in E(G)} \left( 1 + \frac{d_u + d_v}{d_u d_v} \right) \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v} \leq m + \frac{M_1(G)}{\delta^2}, $$

and this implies the desired bound.

A dominating set of a graph is a vertex subset whose closed neighborhood includes all vertices of the graph. The domination number of a graph $G$ is the size of a minimum dominating set.

**Theorem 3 (see [24]).** Let $T$ be a tree of order $n$ with domination number $\gamma$. Then,

$$ M_1(T) \leq (n - \gamma)(n - \gamma + 1) + 4(\gamma - 1). \quad (22) $$

By Theorems 2 and 3, we have the following result for trees with the given domination number.

**Corollary 3.** Let $T$ be a tree of order $n \geq 2$ with domination number $\gamma$. Then,

$$ GA(T) \leq (n - \gamma)(n - \gamma + 1) + 4(\gamma - 1) - 1. \quad (23) $$

Since for every two real numbers $x, y$, and $xy \leq ((x + y)^2)/4$, we have the next observation.

**Lemma 9.** Let $x$ and $y$ be two real numbers, where $x + y \neq 0$. Then, $(xy/(x + y)^2) \leq (1/4)$.

Next, we establish an upper bound for the geometric-arithmetic index in terms of the second Zagreb index.

**Theorem 4.** Let $G$ be a graph of size $m$ with maximum degree $\Delta$. Then,

$$ GA(G) \leq \frac{5m}{4} - \frac{M_1(G)}{4\Delta^2}. \quad (24) $$
Proof. By Lemmas 8 and 9, we have
\[
\frac{\text{GA}(G) + \frac{M_2(G)}{4\Delta}}{\Delta} \leq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{d_u d_v}{(d_u + d_v)^2} \right)
\]
\[
\leq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{1}{4} \right)
\]
\[
\leq \sum_{uv \in E(G)} \left( 1 + \frac{1}{4} \right)
\]
\[
= \frac{5m}{4},
\]
and this implies the desired bound. \(\Box\)

In [25], it is proved that, for any tree \(T\) of order \(n\), \(M_2(T) \geq 4n - 8\). Using this and Theorem 4, we obtain the next result.

**Corollary 4.** Let \(T\) be a tree of order \(n\) with maximum degree \(\Delta\). Then,
\[
\text{GA}(T) \leq \frac{5(n - 1)}{4} - \frac{n - 2}{\Delta^2}.
\] (26)

Here, we establish an upper bound for the geometric-arithmetic index in terms of the hyper-Zagreb index.

The hyper-Zagreb index is defined as follows [9]:
\[
\text{HM}(G) = \sum_{uv \in E(G)} (d_u + d_v)^2.
\] (27)

**Theorem 5.** Let \(G\) be a graph of order \(n\), size \(m\), and minimum degree \(\delta\). Then,
\[
\text{GA}(G) \leq m - n + \frac{\text{HM}(G)}{2\delta^2}.
\] (28)

Proof. By Inequality (21), we have
\[
\text{GA}(G) + n \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v}
\]
\[
\leq \sum_{uv \in E(G)} \left( 1 + \frac{d_u + d_v}{(d_u + d_v)} \right)
\]
\[
= \sum_{uv \in E(G)} \left( \frac{d_u + d_v}{2d_u d_v} \right)^2
\]
\[
\leq \frac{\text{HM}(G)}{2\delta^2}.
\]
It leads to the desired bound. The next result is proven in [26].

**Theorem 6** (see [26]). Let \(G\) be a graph with \(n\) vertices and \(m\) edges. Then,
\[
\text{HM}(G) \leq \frac{m^3(n + 1)^6}{16n^2(n - 1)^2}.
\] (30)

Theorem 5 and 6 lead to the desired result.

**Corollary 5.** Let \(G\) be a graph of order \(n\), size \(m\), and minimum degree \(\delta\). Then,
\[
\text{GA}(G) \leq m - n + \frac{m^3(n + 1)^6}{32\delta^2n^2(n - 1)^2}.
\] (31)

The redefined third Zagreb index is defined as follows [27]:
\[
\text{ReZ}_3(G) = \sum_{uv \in E(G)} (d_u d_v)(d_u + d_v).
\] (32)

Now, we obtain an upper bound for the geometric-arithmetic index in terms of the second Zagreb index, the general Randić index, and the redefined third Zagreb index.

**Theorem 7.** Let \(G\) be a graph with maximum degree \(\Delta\) and minimum degree \(\delta\). Then,
\[
\text{GA}(G) \leq M_2(G) + \frac{R_{1/2}(G)}{\delta} - \frac{\text{ReZ}_3(G)}{2\Delta}.
\] (33)

Proof. It is easy to obtain
\[
M_2(G) - \text{GA}(G) = \sum_{uv \in E(G)} \left( d_u d_v - 2\sqrt{d_u d_v} \right)
\]
\[
= \sum_{uv \in E(G)} \left( \frac{(d_u + d_v)d_u d_v - 2\sqrt{d_u d_v}}{d_u + d_v} \right)
\]
\[
= \sum_{uv \in E(G)} \frac{(d_u + d_v)d_u d_v}{d_u + d_v} - \sum_{uv \in E(G)} 2\sqrt{d_u d_v}
\]
\[
\geq \frac{\text{ReZ}_3(G)}{2\Delta} - \frac{R_{1/2}(G)}{\delta}.
\] (34)

The desired bound follows. \(\Box\)

**Theorem 8.** Let \(G\) be a graph of order \(n\), size \(m\), maximum degree \(\Delta\), and minimum degree \(\delta\). Then,
\[
\text{GA}(G) \leq \frac{2m^2}{n} \left( 1 + \frac{1}{4} \left( 1 - \frac{1}{\sqrt{2} \Delta} \left( \frac{\Delta}{\delta} - \frac{\delta}{\Delta} \right)^2 \right) \right).
\] (35)

Proof. Now, putting \(a_{uv} = (2\sqrt{d_u d_v}/d_u + d_v)\) for each edge \(uv \in E(G), R = (\Delta/\delta), \) and \(r = (\delta/\Delta)\) in Lemma 4, we have
\[
\sum_{uv \in E(G)} \frac{2 \sqrt{d_u d_v}}{d_u + d_v} \leq \frac{m^2}{2} \left( 1 + \frac{1}{4} \left( 1 - \frac{1 - (-1)^{m+1}}{2m^2} \left( \frac{\Delta - \delta}{\Delta} \right)^2 \right) \right).
\]

On the contrary, we have
\[
\frac{n}{2} = \sum_{uv \in E(G)} \frac{d_u + d_v}{2 \sqrt{d_u d_v}} \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2 \sqrt{d_u d_v}}.
\]

Finally, we get the bound by using Inequalities (36) and (37).

The sigma index of \( G \) is defined in [28] as
\[
\sigma(G) = \sum_{uv \in E(G)} (d_u - d_v)^2.
\]

Here, we obtain an upper bound for the geometric-arithmetic index in terms of the first Zagreb index and the sigma index.

**Theorem 9.** Let \( G \) be a nontrivial graph with maximum degree \( \Delta \). Then,
\[
\text{GA}(G) \leq \frac{M_1(G)}{2} - \frac{\sigma(G)}{4\Delta}.
\]

**Proof.** For two real numbers \( x \) and \( y \), we have that
\[
xy = \frac{1}{4} \left( (x + y)^2 - (x - y)^2 \right).
\]

By (40), we obtain
\[
\text{GA}(G) = \sum_{uv \in E(G)} \frac{2 \sqrt{d_u d_v}}{d_u + d_v} \leq \sum_{uv \in E(G)} \frac{2d_u d_v}{d_u + d_v} \leq \frac{1}{2} \sum_{uv \in E(G)} (d_u + d_v) - \sum_{uv \in E(G)} \frac{(d_u - d_v)^2}{2(d_u + d_v)} \leq \frac{M_1(G)}{2} - \frac{\sigma(G)}{4\Delta}
\]

and this implies the desired bound.

The general first \( F \)-index of a graph \( G \) is defined in [29] as
\[
F_a(G) = \sum_{uv \in E(G)} \left( \frac{d_u + d_v}{a} \right)^a,
\]
where \( a \) is a real number. In particular, \( F_1(G) = F(G) \).

Since for every two real numbers \( x \) and \( y \), \( (x - y)^2 \geq 0 \), and we deduce that, for any graph \( G \),
\[
F(G) \geq 2M_1(G),
\]
\[
\sigma(G) = F(G) - 2M_2(G).
\]

Using these and Theorem 9, we obtain the next result.

**Corollary 6.** Let \( G \) be a nontrivial graph with maximum degree \( \Delta \). Then,
\[
\text{GA}(G) \leq \frac{M_1(G)}{2} - \frac{F(G) - 2M_2(G)}{4\Delta}.
\]

From \( F(G) \geq 2M_2(G) \), we would like to indicate that the above new bound improves the known bound:
\[
\text{GA}(G) \leq \frac{M_1(G)}{2},
\]
which was established in [15].

Now, by using the following result, we want to obtain an upper bound for trees.

**Theorem 10** (see [30]). Let \( T \) be a tree of order \( n \) with independence number \( \alpha \). Then,
\[
M_1(T) \leq \alpha^2 - 3\alpha + 4n - 4.
\]

Here, by Theorems 9 and 10, we obtain the next result.

**Corollary 7.** Let \( T \) be a tree of order \( n \) with independence number \( \alpha \) and maximum degree \( \Delta \). Then,
\[
\text{GA}(T) \leq \frac{\alpha^2 - 3\alpha + 4n - 4 - \sigma(G)}{4\Delta}.
\]

### 4. Lower Bounds for the Geometric-Arithmetic Index

In this section, we first investigate the relationships between the geometric-arithmetic index and some other topological indices, and then, we obtain some lower bounds for the geometric-arithmetic index which improve some well-known bounds.

**Theorem 11.** Let \( G \) be a graph of size \( m \) with minimum degree \( \delta \). Then,
\[
\text{GA}(G) \geq \frac{4\delta^2 m^2}{\text{HM}(G)}.
\]

**Proof.** By Lemmas 1 and 2, we have
\[
\frac{m^2}{\text{GA}(G)} = \frac{m^2}{\sum_{uv \in E(G)} (2\sqrt{d_u d_v} / (d_u + d_v))} \\
\leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2\sqrt{d_u d_v}} \\
\leq \sum_{uv \in E(G)} \frac{d_u + d_v}{4d_u d_v} \\
= \sum_{uv \in E(G)} \frac{(d_u + d_v)^2}{4d_u d_v} \\
\leq \frac{1}{4\delta^2} \sum_{uv \in E(G)} (d_u + d_v)^2 \\
= \frac{\text{HM}(G)}{4\delta^2}.
\]

The result follows. \(\square\)

Here, by Theorems 11 and 6, we have the next result.

**Corollary 8.** Let \(G\) be a graph of order \(n\) and size \(m\), with minimum degree \(\delta\). Then,

\[
\text{GA}(G) \geq \frac{64m^3 \delta^2 (n - 1)^2}{m(n + 1)^6}.
\]

Since for any real numbers \(x\) and \(y\), it holds that \(((x + y)^2/4) \leq ((x^2 + y^2)/2)\); hence, by this fact and Inequality (49), we can obtain the following result.

**Corollary 9.** Let \(G\) be a graph of size \(m\) with minimum degree \(\delta\). Then,

\[
\text{GA}(G) \geq \frac{2\delta^2 m^2}{F(G)}.
\]

We start with a lower bound for the geometric-arithmetic index in terms of the general \(F\)-index.

**Theorem 12.** Let \(G\) be a nontrivial graph of size \(m\) with minimum degree \(\delta\). Then,

\[
\text{GA}(G) \geq \frac{\sqrt{2\delta m^2}}{F_{1/2}^2(G)}
\]

**Proof.** Set \(r = 1, a_{uv} = \sqrt{2d_ud_v}\), and \(b_{uv} = \sqrt{d_u^2 + d_v^2}\) for each \(uv \in E(G)\). By Lemmas 1 and 3, we have

\[
\text{GA}(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\
\leq \sum_{uv \in E(G)} \frac{\sqrt{2d_u d_v}}{d_u + d_v} \\
= \sum_{uv \in E(G)} \frac{(\sqrt{2d_u d_v})^2}{\sqrt{d_u^2 + d_v^2}} \\
\geq \frac{\left(\sum_{uv \in E(G)} \sqrt{2d_u d_v}\right)^2}{\sum_{uv \in E(G)} (d_u^2 + d_v^2)} \\
\geq \frac{\sqrt{2\delta m^2}}{F_{1/2}^2(G)}.
\]

The result follows. \(\square\)

The harmonic index is defined as follows [11]:

\[
H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}.
\]

**Theorem 13.** Let \(G\) be a nontrivial graph of order \(n\), size \(m\), and minimum degree \(\delta\). Then,

\[
\text{GA}(G) \geq \delta (H(G) + n) - 2m.
\]

**Proof.** Notice that

\[
\text{GA}(G) + 2m = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{uv \in V(G)} d_u \\
\leq \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{uv \in V(G)} \delta \\
= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + n \delta \\
\leq \delta H(G) + n \delta.
\]

The result follows. \(\square\)

**Corollary 10.** Let \(G\) be a nontrivial graph of order \(n\), size \(m\), and minimum degree \(\delta\). Then,
Corollary 11. Let $G$ be a nontrivial graph of order $n$, size $m$, and minimum degree $\delta$. Then,
\[ \text{GA}(G) \geq \frac{R_{1/2}^\prime(G)}{\Delta} + \delta n - 2m. \] (57)

Theorem 14 (see [31]). Let $G$ be a connected graph of order $n \geq 3$. Then,
\[ H(G) \geq \frac{2(n-1)}{n}. \] (59)

A cut edge of a graph is an edge whose removal increases the number of connected components of the graph.

Lemma 10 (see [32]). Let $G$ be a connected graph of order $n$ and $k'$ cut edges. Then,
\[ m \leq \frac{(n-k')(n-k'-1)}{2} + k'. \] (60)

Now, by Theorems 13 and 14, and Lemma 10, we can obtain the next result.

Corollary 12. Let $G$ be a connected graph of order $n$, $k'$ cut edges, and minimum degree $\delta$. Then,
\[ \text{GA}(G) \geq \delta \left( \frac{2(n-1)}{n} + n \right) - 2 \left( \frac{(n-k')(n-k'-1)}{2} + k' \right). \] (61)

Here, we will use the following particular case of Jensen’s inequality.

Lemma 11. Let $f(x)$ be a convex function defined in $x > 0$. For $x_1, x_2, \ldots, x_m > 0$,
\[ f\left( \frac{x_1 + x_2 + \cdots + x_m}{m} \right) \leq \frac{1}{m} \left( f(x_1) + f(x_2) + \cdots + f(x_m) \right). \] (62)

The general sum-connectivity index is defined as follows [8]:
\[ \chi_a(G) = \sum_{uv \in E(G)} (d_u + d_v)^a. \] (63)

Now, we obtain a lower bound for the geometric-arithmetic index in terms of the general sum connectivity index.

Theorem 15. Let $G$ be a graph of size $m$ and minimum degree $\delta$. Then,
\[ \text{GA}(G) \geq \frac{4\sqrt{3} \sqrt{m^3}}{\sqrt{\chi_a(G)}}. \] (64)

Proof. Since $f(x) = (1/x^2)$ is a convex function for $x > 0$, from Lemmas 1 and 11, we have
\[
\left( \frac{m}{\text{GA}(G)} \right)^2 \leq \frac{1}{m} \sum_{uv \in E(G)} \left( \frac{d_u + d_v}{2\sqrt{d_u d_v}} \right)^2 \leq \frac{1}{m} \sum_{uv \in E(G)} \left( \frac{d_u + d_v}{4d_u d_v (d_u + d_v)} \right)^2 \leq \frac{1}{16m} \sum_{uv \in E(G)} (d_u + d_v)^4 \leq \frac{\chi_a(G)}{16m^4},
\]
as desired. \qed

Now, we obtain an upper bound for the geometric-arithmetic index in terms of the sigma index.

Theorem 16. Let $G$ be a simple connected graph of size $m$ with maximum degree $\Delta$, $p$ pendent vertices, and minimum nonpendent vertex degree $\delta_i$. Then,
\[ \text{GA}(G) \geq \frac{2p\sqrt{\Delta}}{1 + \Delta} + \sqrt{\frac{4(m-p)^2 - (m-p/\delta_i)^2(\sigma(G) - p(\delta_i - 1)^2)}{\Delta + \delta_i/2\sqrt{\Delta\delta_i}}} + \sqrt{2(2\sqrt{\Delta\delta_i}/\Delta + \delta_i)} \] (66)

Proof. We partition all the edges into two parts: pendent edges and nonpendent edges, so
\[ \text{GA}(G) = \sum_{uv \in E(G)} d_u + d_v + \sum_{d_u,d_v \neq 1} 2\sqrt{d_u d_v} \geq \frac{2\sqrt{d_u d_v}}{1 + \Delta}. \] (67)

On one hand, for the pendent edges, it is not hard to check that $(2\sqrt{d_u d_v}/1 + d_v)$ decreases in $2 \leq d_v \leq \Delta$; thus,
\[ \sum_{d_u,d_v \neq 1} 2\sqrt{d_u d_v} \geq 2p\sqrt{\Delta}. \] (68)

Now, we consider the nonpendent edges. It is easy to see that the function $x + (1/x)$ gets its maximum value when $x$ attains the maximum or minimum value. From $(\Delta/\delta) \geq (d_u/d_v) \geq (\delta/\Delta)$ for all $u$ and $v \in V(G)$, we have
\[ \sqrt{\frac{d_u}{d_v}} + \sqrt{\frac{d_v}{d_u}} \leq \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \] (69)
which is equivalent to
\[
\frac{2\sqrt{\Delta \delta_1}}{\Delta + \delta_1} \leq \frac{2\sqrt{d_u d_v}}{d_u + d_v} \leq 1.
\] (70)

Set \(a_{uv} = 1\) and \(b_{uv} = (2\sqrt{d_u d_v}/d_u + d_v)\) for each edge \(uv \in E(G)\), \(M_1 = m_1 = M_2 = 1\), and \(m_2 = (2\sqrt{\Delta \delta_1}/\Delta + \delta_1)\) in Lemma 6, and we have

\[
\sum_{uv \in E(G)d_u, d_v \neq 1} 1^2 \sum_{uv \in E(G)d_u, d_v \neq 1} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v}\right)^2
= (m - p) \sum_{uv \in E(G)d_u, d_v \neq 1} \left(1 - \left(\frac{d_u - d_v}{d_u + d_v}\right)^2\right)
\leq \frac{1}{4} \left(\frac{1}{(2\sqrt{\Delta \delta_1}/\Delta + \delta_1)} + \frac{2\sqrt{\Delta \delta_1}}{\Delta + \delta_1}\right)^2 \left(\sum_{uv \in E(G)d_u, d_v \neq 1} \frac{2\sqrt{d_u d_v}}{d_u + d_v}\right)^2,
\] (71)

which implies that

\[
\sum_{uv \in E(G)d_u, d_v \neq 1} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \geq \frac{\sqrt{4(m - p)\sum_{uv \in E(G)d_u, d_v \neq 1} \left(1 - \left(\frac{d_u - d_v}{d_u + d_v}\right)^2\right)}}{\sqrt{\Delta + \delta_1/2\sqrt{\Delta \delta_1}} + \sqrt{2\sqrt{\Delta \delta_1}/\Delta + \delta_1}}
\geq \frac{\sqrt{4(m - p)^2 - (m - p/\delta_1^2)\sum_{uv \in E(G)d_u, d_v \neq 1} (d_u - d_v)^2}}{\sqrt{\Delta + \delta_1/2\sqrt{\Delta \delta_1}} + \sqrt{2\sqrt{\Delta \delta_1}/\Delta + \delta_1}}
\geq \frac{\sqrt{4(m - p)^2 - (m - p/\delta_1^2)(\sigma(G) - \sum_{uv \in E(G)d_u = 1} (d_v - 1)^2)}}{\sqrt{\Delta + \delta_1/2\sqrt{\Delta \delta_1}} + \sqrt{2\sqrt{\Delta \delta_1}/\Delta + \delta_1}}
\geq \frac{\sqrt{4(m - p)^2 - (m - p/\delta_1^2)(\sigma(G) - p(\delta_1 - 1)^2)}}{\sqrt{\Delta + \delta_1/2\sqrt{\Delta \delta_1}} + \sqrt{2\sqrt{\Delta \delta_1}/\Delta + \delta_1}}
\] (72)

Finally, the result follows from (67), (68), and (72).

Next, results are immediate consequences of Theorem 16 with the setting \(p = 0\).

**Corollary 13.** For a graph \(G\) of size \(m\) with maximum degree \(\Delta\) and minimum degree \(\delta \geq 2\),

\[
\text{GA}(G) \geq \frac{\sqrt{4m^2 - (m/\delta^2)\sigma(G)}}{\sqrt{\Delta + \delta_1/2\sqrt{\Delta \delta_1}} + \sqrt{2\sqrt{\Delta \delta_1}/\Delta + \delta}}
\] (73)

**Theorem 17.** Let \(G\) be a graph of size \(m\), maximum degree \(\Delta\), and minimum degree \(\delta\). Then,

\[
\text{GA}(G) \geq \frac{\sqrt{4M_2(G)\chi_2(G) - \frac{m^2}{4} \Delta_2^2 + \delta^2}}{\Delta \delta}
\] (74)

Now, we obtain a lower bound for the geometric-arithmetic index in terms of the second Zagreb index and the general sum connectivity index.
The eccentricity $\varepsilon(v)$ of $v$ is defined as

$$\varepsilon(v) = \max\{d(v, w) : w \in V(G)\},$$

where $d(v, w)$ is the length of a shortest path connecting $v$ and $w$. The radius $r$ and diameter $D$ are defined as the minimum and maximum values among $\varepsilon(v)$ over all vertices $v \in V(G)$, respectively.

Xu [34] showed that, for any nontrivial connected graph $G$ of order $n$, size $m$, and radius $r$, $H(G) \geq (m/n - r)$. Using this and Theorem 18, we obtain the next result.

**Corollary 15.** Let $G$ be a nontrivial connected graph of order $n$, size $m$, and radius $r$. Then,

$$GA(G) \geq \frac{\sqrt{2} m}{n - r}.$$  

**Theorem 21.** Let $G$ be a nontrivial connected graph of size $m$ and radius $r$. Then,

$$GA(G) \geq \frac{R_{1/2}(G)}{n - r}.$$  

**Proof.** Note that, for each vertex $u \in V(G)$, we have $d_u \leq n - \varepsilon(u)$. Thus, for each edge $uv \in E(G)$,

$$GA(G) = \sum_{u \in V(G)} \frac{2 \sqrt{d_u d_v}}{d_u + d_v} \geq \sum_{u \in V(G)} \frac{2 \sqrt{d_u d_v}}{2n - 2r} = \frac{R_{1/2}(G)}{n - r},$$

as desired.

The proof of next results can be found in [33].

**Theorem 19 (see [33]).** Let $G$ be a triangle-free graph of order $n$ and the minimum degree $\delta \geq k (k \leq (n/2))$. Then,

$$H(G) \geq \frac{2k(n-k)}{n}.$$  

**Theorem 20 (see [33]).** Let $G$ be a triangle-free graph of order $n$ and size $m$. Then,

$$H(G) \geq \frac{2m}{n}.$$  

Applying Theorems 18–20, it leads to the next results.

**Corollary 14.** Let $G$ be a triangle-free graph of order $n$ without isolated edges, and the minimum degree $\delta \geq k (k \leq (n/2))$. Then,

$$GA(G) \geq \frac{\sqrt{2}k(n-k)}{n},$$

$$GA(G) \geq \frac{\sqrt{2}m}{n}.$$  

We can see that Inequality (82) improves the next well-known result for triangle-free graphs [13]. Let $G$ be a graph of order $n$ and size $m$ without isolated vertex. Then,
\[
\frac{1}{\sqrt{d_u d_v}} \geq \frac{1}{n - 1 - (p/2)} \quad (90)
\]

Thus,
\[
GA(G) \geq \frac{p}{\sqrt{n-1}} + \frac{m-p}{n-1-(p/2)} \quad (91)
\]

The desired result follows.

In [35], Kulli et al. defined the first and second generalized multiplicative Zagreb indices:
\[
MZ_1^a(G) = \prod_{uv \in G} (d_u + d_v)^a, \\
MZ_2^a(G) = \prod_{uv \in G} (d_u d_v)^a. \quad (92)
\]

Here, we obtain a lower bound in terms of the first and second generalized multiplicative Zagreb indices. □

**Theorem 23.** Let G be a nontrivial graph of size m. Then,
\[
GA(G) \geq 2m \left\lceil \frac{MZ_2^{1/2}(G)}{MZ_1^{1/2}(G)} \right\rceil \quad (93)
\]

**Proof.** By Lemma 2, we obtain
\[
\frac{GA(G)}{2m} = \frac{1}{m} \sum_{uv \in G} \frac{\sqrt{d_u d_v}}{d_u + d_v} \geq \sqrt{\prod_{uv \in G} \frac{\sqrt{d_u d_v}}{d_u + d_v}} \quad (94)
\]

\[
= \sqrt{\prod_{uv \in G} \sqrt{d_u d_v}/(d_u + d_v)} = \sqrt{\frac{MZ_2^{1/2}(G)}{MZ_1^{1/2}(G)}}
\]

as desired. □

**Theorem 24.** Let G be a graph of size m and minimum degree δ. Then,
\[
GA(G) \geq \frac{4\delta^2 m^2}{HM(G)} \quad (95)
\]

**Proof.** By Lemma 1, we get
\[
\frac{GA(G)}{2m} = \frac{1}{m} \sum_{uv \in E(G)} \frac{\sqrt{d_u d_v}}{d_u + d_v} \geq \frac{1}{m} \sum_{uv \in E(G)} \frac{2d_u d_v}{(d_u + d_v)^2} \geq \frac{m}{\sum_{uv \in E(G)} (d_u + d_v)^2/2d_u d_v} \geq \frac{m}{(1/2\delta^2) \sum_{uv \in E(G)} (d_u + d_v)^2} = \frac{2\delta^2 m}{HM(G)},
\]

as desired.

In the sequel, we obtain a lower bound in terms of the first Zagreb index. □

**Theorem 25.** Let G be a graph of size m, maximum degree Δ, and minimum degree δ. Then,
\[
GA(G) \geq \frac{\delta m}{\Delta} + 2m - \frac{M_1(G)}{\delta} \quad (96)
\]

**Proof.** By Lemma 8, we have
\[
\frac{GA(G)}{2m} \geq \frac{M_1(G)}{\delta} \geq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{d_u + d_v}{\sqrt{d_u d_v}} \right) \geq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_u d_v}}{d_u + d_v} + 2 \right) \quad (98)
\]

\[
= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{uv \in E(G)} 2 \geq \delta m / \Delta + 2m,
\]

and this implies the desired bound. □

Zhou [36] proved that, for any triangle-free graph of order n and size m, $M_1(G) \leq mn$. Together with Theorem 25, we get the next result.

**Corollary 16.** Let G be a triangle-free graph of order n, size m, maximum degree Δ, and minimum degree δ. Then,
\[
GA(G) \geq m \left( \frac{\delta}{\Delta} + 2 - \frac{n}{\delta} \right) \quad (99)
\]

Inequality (98) leads to the following results.
**Corollary 17.** Let $G$ be a graph of size $m$, maximum degree $\Delta$, and minimum degree $\delta$. Then,

$$GA(G) \geq \delta H(G) + 2m - \frac{M_1(G)}{\delta},$$

$$GA(G) \geq \frac{R_{1/2}(G)}{\Delta} + 2m - \frac{M_1(G)}{\delta}.$$  \hspace{1cm} (100)

Note that, for every two real numbers $x$ and $y$, $(x + y)^2/xy \geq 4$. Applying this, we obtain a lower bound for the geometric-arithmetic index in terms of the hyper-Zagreb index.

**Theorem 26.** Let $G$ be a graph of size $m$, maximum degree $\Delta$, and minimum degree $\delta$. Then,

$$GA(G) \geq \frac{\delta m}{\Delta} + 4m - \frac{HM(G)}{\delta^2}.$$ \hspace{1cm} (101)

**Proof.** From the above inequality, we have

$$GA(G) + \frac{HM(G)}{\delta^2} \geq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_ud_v}}{d_u + d_v} + \frac{(d_u + d_v)^2}{d_ud_v} \right) \geq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_ud_v}}{d_u + d_v} + 4 \right) = \sum_{uv \in E(G)} \frac{2\sqrt{d_ud_v}}{d_u + d_v} + \sum_{uv \in E(G)} 4 \geq \frac{\delta m}{\Delta} + 4m,$$  \hspace{1cm} (102)

and this implies the desired bound.

Here, we obtain a lower bound for the geometric-arithmetic index in terms of the first Zagreb index. \hfill \square

**Theorem 27.** Let $G$ be a graph of size $m$ and minimum degree $\delta$. Then,

$$GA(G) \geq 2m - \frac{M_1(G)}{2\delta}.$$ \hspace{1cm} (103)

**Proof.** From the fact that $x + (1/x) \geq 2$ for any $x > 0$, we have

$$GA(G) + \frac{M_1(G)}{2\delta} \geq \sum_{uv \in E(G)} \left( \frac{2\sqrt{d_ud_v}}{d_u + d_v} + \frac{d_u + d_v}{2\sqrt{d_ud_v}} \right) \geq \sum_{uv \in E(G)} 2 = 2m,$$ \hspace{1cm} (104)

and this implies the desired bound. \hfill \square

**Theorem 28** (see [37]). Let $G$ be a graph of size $m$ and diameter $D > 1$. Then,

$$M_1(G) \leq m^2 - m(D - 3) + (D - 2).$$ \hspace{1cm} (105)

Now, by Theorems 27 and 28, we have the following result.

**Corollary 18.** Let $G$ be a graph of size $m$, minimum degree $\delta$, and diameter $D > 1$. Then,

$$GA(G) \geq 2m - \frac{m^2 - m(D - 3) + (D - 2)}{2\delta}.$$ \hspace{1cm} (106)

**Theorem 29** (see [38]). Let $G$ be a graph of size $m$, with $t$ triangles and pendent vertex $p$. Then,

$$M_1(G) \leq m(p + 2) + 3t.$$ \hspace{1cm} (107)

Again, by Theorems 27 and 29, we have the following result.

**Corollary 19.** Let $G$ be a graph of size $m$, with $t$ triangles, leaf number $L$, and minimum degree $\delta$. Then,

$$GA(G) \geq 2m - \frac{m(p + 2) + 3t}{2\delta}.$$ \hspace{1cm} (108)

**Theorem 30** (see [39]). Let $G$ be a triangle- and quadrangle-free graph with $n > 1$ vertices. Then,

$$M_1(G) \leq n(n - 1).$$ \hspace{1cm} (109)

Also, by Theorems 27 and 30, we have the following result.

**Corollary 20.** Let $G$ be a triangle- and quadrangle-free graph of order $n$, size $m$, and minimum degree $\delta$. Then,

$$GA(G) \geq 2m - \frac{n(n - 1)}{2\delta}.$$ \hspace{1cm} (110)

**Data Availability**

The data used to support the findings of the study are provided within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interests.

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