The classical Taub–Nut system: factorization, spectrum generating algebra and solution to the equations of motion

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Abstract
The formalism of SUperSYmmetric quantum mechanics (SUSYQM) is properly modified in such a way to be suitable for the description and the solution of a classical maximally superintegrable Hamiltonian system, the so-called Taub–Nut system, associated with the Hamiltonian:

\[ H_\eta(q, p) = T_\eta(q, p) + U_\eta(q) = \sqrt{|q|^2 + \frac{1}{2m(\eta + |q|)}} - \frac{k}{\eta + |q|} \quad (k > 0, \eta > 0). \]

In full agreement with the results recently derived by Ballesteros et al for the quantum case, we show that the classical Taub–Nut system shares a number of essential features with the Kepler system, that is just its Euclidean version arising in the limit \( \eta \to 0 \), and for which a ‘SUSYQM’ approach has been recently introduced by Kuru and Negro. In particular, for positive \( \eta \) and negative energy the motion is always periodic; it turns out that the period depends upon \( \eta \) and goes to the Euclidean value as \( \eta \to 0 \). Moreover, the maximal superintegrability is preserved by the \( \eta \)-deformation, due to the existence of a larger symmetry group related to an \( \eta \)-deformed Runge–Lenz vector, which ensures that in \( \mathbb{R}^3 \) closed orbits are again ellipses. In this context, a deformed version of the third Kepler’s law is also recovered. The closing section is devoted to a discussion of the \( \eta < 0 \) case, where new and partly unexpected features arise.

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(Some figures may appear in colour only in the online journal)

1. Introduction

We consider the classical Hamiltonian in $\mathbb{R}^N$ given by:

$$
H_\eta(q, p) = T_\eta(q, p) + U_\eta(q) = \frac{|q| p^2}{2m(\eta + |q|)} - \frac{k}{\eta + |q|},
$$

(1)

where $k$ and $\eta$ are real positive parameters, $q = (q_1, ..., q_N)$, $p = (p_1, ..., p_N) \in \mathbb{R}^N$ are conjugate coordinates and momenta, and $q^2 \equiv |q|^2 = \sum_{j=1}^{N} q_j^2$. We recall that $H_\eta$ has been proven to be a maximally superintegrable Hamiltonian by making use of symmetry techniques [1]. This means that $H_\eta$ is endowed with the maximum possible number $N(2N-1)$ of functionally independent constants of motion (including $H_\eta$ itself). In fact, besides the integrals of motion provided by the $so(N)$ symmetry, $H_\eta$ is endowed with a $\eta$-deformed $ND$ Laplace–Runge–Lenz vector $R$ implying the existence of $N$ additional constants of motion coming from the components of $R$, which are given by:

$$
R_i = \frac{1}{m} \sum_{j=1}^{N} p_j (q_j p_i - q_i p_j) + \frac{q_i}{|q|} (\eta H_\eta + k), \quad i = 1, ..., N.
$$

(2)

The squared modulus of $R$ is radially symmetric, and turns out to be expressible in terms of $H_\eta$ and $L^2$:

$$
R^2 = \sum_{i=1}^{N} R_i^2 = \frac{2L^2}{m} H_\eta + (\eta H_\eta + k)^2.
$$

(3)

In turn, the Hamiltonian $H_\eta$ can be expressed in terms of hyperspherical coordinates $r, \theta_j$ and canonical momenta $p_r, p_{\theta_j}$, $(j = 1, ..., N-1)$ (see for example [2]) defined by

$$
q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k \quad (1 \leq j \leq N-1), \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k,
$$

(4)

such that

$$
r = |q|, \quad p^2 = p_r^2 + \frac{L^2}{r^2}, \quad \text{with} \quad L^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k},
$$

(5)

where the radial coordinate $r$ is canonically conjugated to the radial momentum $p_r$. Then, for a given value of $L^2 = l^2$, the Hamiltonian (1) can be written as a one-degree of freedom radial system.

As a matter of fact the system associated with $H_\eta$ (1) under the canonical symplectic structure can be considered as a genuine (maximally superintegrable) $\eta$-deformation of the $ND$ usual Kepler–Coulomb (KC) system, since the limit $\eta \to 0$ of $H_\eta$ (1) yields:

$$
H_0 = \frac{p^2}{2m} - \frac{k}{|q|}.
$$

(6)

Moreover $H_\eta$ can be naturally related to the Taub–Nut system [3–14] since $\mathcal{M}^N$ can be regarded as the (Riemannian) $ND$ Taub–Nut space [15]. It is also known that according to the
Perlick classification [15–18] the system (1) pertains to the class II, and thus it has to be considered as an ‘intrinsic oscillator’. In the sequel it will be clear that with respect to the Euclidean KC it plays an analogous role to the Darboux III (D-III in the following) in comparison with the standard harmonic oscillator [19].

Actually it turns out that the solution to the classical equations of motion for D-III can be found on the same footing, through the factorization of the corresponding classical Hamiltonian. The results concerning D-III will be published later in a larger more general paper [20], where the quantum cases will be also investigated, mostly through the shape invariant potentials approach.

2. Classical Taub–Nut: factorization, spectrum generating algebra and solution to the equations of motion

In the following we simplify our setting, and limit our considerations to the physical (i.e. three-dimensional) case. As mentioned in the abstract we adapt and in a sense generalize the construction and the results derived in [21]. We will study the Hamiltonian

\[ H = T(r, p) + V_{en}(r) = \frac{p^2}{2m(r + \eta)} + \frac{l^2}{2mr(r + \eta)} - \frac{k}{r + \eta} = \mathcal{K}(r)H_0, \quad (7) \]

where \( m, k \) and \( l \) are positive constants, \( \eta \) is the deformation parameter, \( p \equiv p_r \) is the radial momentum, \( H_0 \) is the ‘undeformed’ KC Hamiltonian and \( \mathcal{K}(r) \equiv \frac{l}{r + \eta} \). The main idea is to use the framework of SUSYQM in the context of classical mechanics to derive algebraically the classical trajectories (see [22]). To this end, let us multiply the Hamiltonian (7) by \( r(r + \eta) \):

\[ r(r + \eta)H = r^2 \left( \frac{p^2}{2m} + \frac{l^2}{2mr^2} - \frac{k}{r} \right) = \frac{1}{2m} \left( r^2p^2 + l^2 - 2mkr \right). \quad (8) \]

Now, as it has been done in the undeformed case by Kuru and Negro [21], at any \( r \) we can factorize (8) as follows:

\[ r^2p^2 - 2mr(k + \eta H) - 2mr^2H = A^+A^- + \gamma(H) = -l^2, \quad (9) \]

where for the time being \( A^+, A^- \) are unknown functions of \( r, p \). Paraphrasing [21] we make the following ansatz for \( A^+, A^- \):

\[ A^\pm = \left( \mp i r p + ar\sqrt{-H} + \frac{b(H)}{\sqrt{-H}} \right) e^{\mp f(r, p)}. \quad (10) \]

The ‘arbitrary function’ \( f(r, p) \) will be determined by requiring the closure of the Poisson algebra generated by \( H \) and \( A^\pm \). More precisely, we impose:

\[ \{ H, A^\pm \} = \mp i \alpha(H)A^\pm, \quad (11) \]

\[ \{ A^+, A^- \} = i\beta(H), \quad (12) \]

where the functions \( \alpha, \beta \) wait to be determined. Inserting \( A^\pm \) in (9) we get

\[ a = \sqrt{2m}, \quad b(H) = -\sqrt{\frac{m}{2}}(k + \eta H), \quad \gamma(H) = \frac{m(k + \eta H)^2}{2H}, \quad (13) \]
and requiring that $A^\pm$ obey the proper Poisson brackets we arrive at
\[ f(r, p) = -i \sqrt{\frac{2}{m}} \frac{rp \sqrt{-H}}{(k - \eta H)}, \quad \alpha(H) = -\sqrt{\frac{2}{m}} \frac{2H \sqrt{-H}}{(k - \eta H)}, \quad \beta(H) = \sqrt{\frac{2m}{m}} \frac{(k + \eta H)}{\sqrt{-H}}, \] (14)
and finally:
\[ A^\pm = \left( xirp + r \sqrt{2m} H - \sqrt{\frac{m}{2}} \frac{(k + \eta H)}{\sqrt{-H}} \right) e^{\pm \sqrt{\frac{m}{2}} \frac{(k + \eta H)}{\sqrt{-H}}}, \] (15)
\[ \{H, A^\pm\} = \pm i \sqrt{\frac{2}{m}} \frac{2H \sqrt{-H}}{(k - \eta H)} A^\pm, \quad \{A^+, A^-\} = i \sqrt{\frac{2m}{m}} \frac{(k + \eta H)}{\sqrt{-H}}. \] (16)
A mandatory requirement is that in the limit $\eta \to 0$ one gets back the undeformed Poisson algebra that is in fact, for $2m = k = 1$, the result found in [21]. To make the identification even more perspicuous we can introduce $A_0 \equiv \frac{m}{\sqrt{2}} \frac{(k + \eta H)}{\sqrt{-H}}$ entailing the following $su(1, 1)$ algebra relations:
\[ \{A_0, A^\pm\} = \mp i A^\pm, \quad \{A^+, A^-\} = 2i A_0. \] (17)
Now we can define the ‘time-dependent constants of the motion’
\[ Q^\pm = A^\pm e^{\mp \alpha(H)t}, \] (18)
such that $\frac{dQ^\pm}{dt} = \{Q^\pm, H\} + \partial_t Q^\pm = 0$. Those dynamical variables take complex values admitting the polar decomposition $Q^\pm = q_0 e^{z_0 t}$ and allowing in fact to determine the motion, which turns out to be bounded for $E = -|E| < 0$. Indeed we have:
\[ \left( xirp + r \sqrt{2m} |E| - \sqrt{\frac{m}{2}} \frac{(k - \eta |E|)}{\sqrt{|E|}} \right) e^{\pm \sqrt{\frac{m}{2}} \frac{(k - \eta |E|)}{\sqrt{|E|}}} = q_0 e^{z_0 t}, \] (19)
or else
\[ \begin{cases} -irp + r \sqrt{2m} |E| - \sqrt{\frac{m}{2}} \frac{(k - \eta |E|)}{\sqrt{|E|}} = q_0 e^{\pm i \sqrt{\frac{m}{2}} \frac{(k - \eta |E|)}{\sqrt{|E|}}} + \frac{2 |E| l}{|E|} \theta_0, \\ +irp + r \sqrt{2m} |E| - \sqrt{\frac{m}{2}} \frac{(k - \eta |E|)}{\sqrt{|E|}} = q_0 e^{\mp i \sqrt{\frac{m}{2}} \frac{(k - \eta |E|)}{\sqrt{|E|}}} + \frac{2 |E| l}{|E|} \theta_0, \end{cases} \] (20)
where $\theta_0 = \sqrt{t^2 + \frac{m(k - \eta |E|)^2}{2 |E|}}$ (following from $A^+ A^- + \gamma(H) = q_0^2 + \gamma(H) = -t^2$). Summing and subtracting (20) we obtain:
\[ \begin{cases} \frac{2r \sqrt{2m} |E|}{\sqrt{|E|}} - \sqrt{2m} \frac{(k - \eta |E|)}{\sqrt{|E|}} \\ = 2q_0 \cos \left( \sqrt{\frac{2}{m}} \frac{rp \sqrt{|E|}}{(k + \eta |E|)} + \frac{2}{m} \frac{2 |E| \sqrt{|E|}}{l} \theta_0 \right), \end{cases} \] (21)
\[ \begin{cases} rp = -q_0 \sin \left( \sqrt{\frac{2}{m}} \frac{rp \sqrt{|E|}}{(k + \eta |E|)} + \frac{2}{m} \frac{2 |E| \sqrt{|E|}}{l} \theta_0 \right). \end{cases} \]
It is immediate to verify that taking the sum of the square of these two equations we obtain the equation (9) restricted to the level surface $H = -|E|$. Finally, thanks to the above relations,
we are able to obtain \( t \) as a function of \( r \):

\[
\begin{align*}
t(r) &= \frac{1}{\Omega_\eta(E)} \left[ \text{arccos} \left( -\frac{m}{2} \left( k + \eta |E| \right) - 2m |E| r \right) \right] \\
&= -\frac{2}{m \left( k + \eta |E| \right)} \sqrt{2mr(k - \eta |E|)} - 2m |E| r^2 + t^2 - \theta_0, \tag{22}
\end{align*}
\]

where \( \Omega_\eta(E) = \sqrt{\frac{2}{m \left( k + \eta |E| \right)}} \equiv \alpha(E) \) is the angular frequency of the motion. Concerning (22) it is evident that, due to the presence of the ‘inverse cosine’ function, \( t \) is a multivalued function of \( r \) defined mod \( 2\pi/\Omega \). To recover univaluedness, we have to introduce a ‘uniformization map’ which is trivially given by the periodic function \( \cos(\Omega t) \). In the limit \( \eta \to 0 \), the results for the flat KC are recovered (see [21]). We can say that the motion has been algebraically determined.

A number of plots are reported (see figures 1 and 2), showing the behaviour of \( V_{\text{eff}}(r) \equiv \frac{\ell^2}{2m(r^2 + \eta^2)} - \frac{k}{r^2} \) as a function of \( r \), and the orbits on the phase plane \((r, p)\) for different values of the deformation parameter (for \( 2l = m = k = 1, E = -1 \), in appropriate units).
3. Explicit formula for the orbits and third Kepler’s law

As we have shown in the introduction, our system is maximally superintegrable and this maximal superintegrability is strictly related to the existence of the Runge–Lenz vector: then, as it happens for the standard KC system, we expect that this extra symmetry will play a crucial role in determining the shape of the orbits. As is well known, in the undeformed case the orbits are conic sections, namely ellipses for bounded trajectories. To identify the analytic form of the orbits when \( \eta \neq 0 \), we will first consider the simplest and more physical case, corresponding to \( \eta > 0 \). To this end, we will closely follow [23, 24]. In \( \mathbb{R}^3 \) the Runge–Lenz vector \( \mathbf{R} \), when evaluated on-shell, can be written as:

\[
\mathbf{R} = \frac{1}{m} [p(p \cdot q) - q(p \cdot p)] + \frac{q}{|q|} (k - \eta |E|). \tag{23}
\]

Again, we see that its expression is formally identical to the one holding in the flat case and is obtained by letting \( k \to k - \eta |E| = K. \) For its square we can write (again on-shell):

\[
\mathbf{R}^2 = K^2 - \frac{2l^2 |E|}{m}, \tag{24}
\]

and then

\[
-\frac{2l^2}{m} \left( |E| - \frac{K}{r} \right) = \mathbf{R}^2 + K^2 + 2K |\mathbf{R}| \cos(\theta - \theta_0). \tag{25}
\]

At this point, by elementary algebraic manipulations, it is easy to write the equation for the orbits in terms of \( r := |q| \) and \( \theta \), getting:

\[
r(\theta) = \frac{p(\eta)}{1 + \epsilon(\eta) \cos(\theta - \theta_0)}, \tag{26}
\]

\( p(\eta) \) being the parameter and \( \epsilon(\eta) \) the eccentricity of the ellipses) which is formally the same expression holding in the flat case. But now we have:

\[
\left\{
\begin{align*}
p(\eta) &\equiv p(E, \eta) = \frac{l^2}{mk}, \\
\epsilon(\eta) &\equiv \epsilon(E, \eta) = \frac{|\mathbf{R}|}{K},
\end{align*}
\right. \tag{27}
\]

so that \( \epsilon^2(\eta) = 1 - 2|E|l^2/mK^2 \). In the above expression \( \theta \) and \( \theta_0 \) are the angles that the vectors \( q \) and \( \mathbf{R} \) form with the half-line \( \theta = 0 \) (of course \( \theta_0 \) is a constant of the motion).

To check whether the third Kepler’s law holds in the deformed case as well, we have to compute the ratio \( r^2/a^3 = 4\pi^2/a^3\Omega^2 \) where

\[
\Omega(\eta)(E) = \frac{4 |E| \sqrt{|E|}}{\sqrt{2m} (k + \eta |E|)}, \tag{28}
\]

and \( a \) is the larger semi-axis defined as \( a = \frac{r_+ - r_-}{2} \). The inversion points \( r_{\pm} \) (where \( p_{\pm} = p_{r_{\pm}} = 0 \)) are obtained by taking the roots of
\[ r^2 - \frac{(k - \eta |E|)}{|E|} r + \frac{l^2}{2m |E|} = 0 \Rightarrow r_{(\eta)} = \frac{k - \eta |E|}{2 |E|} \pm \sqrt{\frac{(k - \eta |E|)^2}{4 |E|^2} - \frac{l^2}{2m |E|}}, \] (29)

entailing

\[ a_{(\eta)} = \frac{r_{(\eta)} + r_{(\eta)^-}}{2} = \frac{k - \eta |E|}{2 |E|}. \] (30)

In the limit \( \eta \to 0 \) we recover the larger semi-axis of the flat case, and then

\[ \frac{r^2}{a^3} = \frac{4\pi^2 m}{k}. \] (31)

We remind that the so-called third Kepler’s law is obtained by assuming that the ratio \( \frac{m}{M} \) between the mass of the planet and the mass of the Sun be very small, so that the reduced mass can be identified with the mass of the planet, entailing \( k = GMm \) and thus \( \frac{r^2}{a^3} = \frac{4\pi^2 m}{GM} \). In the deformed case the analogous formula reads:

\[ \frac{\tau^2}{a^3} = 4\pi^2 m (k + \eta |E|)^2 \] (32)

The Kepler’s third law is then violated as the rhs of (32), again assuming \( k = GMm \), keeps its dependence upon \( m \) and \( E \):

\[ \frac{\tau^2}{a^3} = 4\pi^2 m (k + \eta |E|)^2 \] (33)

4. Explicit evaluation of the trajectory and comparison with algebraic method

For the sake of completeness we present here the explicit derivation of the trajectory \( t(r) \) using the standard analytic method [23]. A comparison with the results obtained through the spectrum generating algebra will provide a definite proof of the correctness of the algebraic approach. The starting point is the usual Hamiltonian (7):

\[ H = \frac{r}{r + \eta} \left[ \frac{p^2}{2m} + \frac{l^2}{2mr^2} - \frac{k}{r} \right]. \] (34)

The radial momentum \( p \) is related to the radial component of the velocity through the Hamilton’s equation:

\[ \dot{r} = \partial_r H = \frac{r}{r + \eta} \frac{p}{m} \Rightarrow p = \frac{r + \eta}{r} mr. \] (35)

Inserting in the Hamiltonian the expression of \( p \) in terms of \( r \) and \( \dot{r} \) we obtain:

\[ H = \frac{r}{r + \eta} \left[ \frac{(r + \eta)^2 m}{r^2} + \frac{l^2}{2mr^2} - \frac{k}{r} \right]. \] (36)

By solving the above expression with respect to \( \dot{r}(t) \) and setting \( H = E \) we get:

\[ \dot{r}(t) = \pm \sqrt{\frac{2}{m(r + \eta)} \sqrt{E + \frac{k + \eta E}{r} - \frac{l^2}{2mr^2}}}. \] (37)
Comparing with the Euclidean case \( \eta = 0 \), besides the coupling constant metamorphosis, the essential difference consists in the presence of a nontrivial conformal factor. As a next step, we calculate \( t(r) \) by taking the positive branch of the square root:

\[
t(r) - t_0 = \frac{1}{\sqrt{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{E + k + \eta E - \frac{r^2}{2mr^2}}} = \frac{1}{\sqrt{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{E + k + \eta E - \frac{r^2}{2mr^2}}}
\]

\[
+ \frac{1}{\sqrt{2}} \eta \int_{r_0}^{r} \frac{dr}{\sqrt{E + k + \eta E - \frac{r^2}{2mr^2}}}.
\]

(38)

The two integrals involved in the above formula can be conveniently calculated by introducing the so-called eccentric anomaly \( \Psi_{(\eta)} \) through the relation \([23]\):

\[
\epsilon \Psi_{(\eta)} = - \eta \eta \eta (a). \cos \Psi_{(\eta)}, \sin \Psi_{(\eta)}.
\]

(39)

In the previous section we have already shown that the semi-major axis is given by \( a_{(\eta)} = - \eta E \) and the eccentricity reads \( \epsilon_{(\eta)} = \sqrt{1 + \frac{2E}{m(k + \eta E)^2}} \). Let us now pass to the explicit calculation of the two integrals contained in (38), setting there \( E = -|E| < 0 \). It is not too difficult to arrive at the following results:

\[
\sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{-|E| + k - \eta |E| - \frac{r^2}{2mr^2}}} = \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \int_{0}^{\Psi_{(\eta)}} d\Psi_{(\eta)} \left( 1 - \epsilon_{(\eta)} \cos \Psi_{(\eta)} \right)
\]

\[
= \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \left( \Psi_{(\eta)} - \epsilon_{(\eta)} \sin \Psi_{(\eta)} \right),
\]

(40)

\[
\sqrt{\frac{m}{2}} \eta \int_{r_0}^{r} \frac{dr}{\sqrt{-|E| + k - \eta |E| - \frac{r^2}{2mr^2}}} = \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \eta \int_{0}^{\Psi_{(\eta)}} d\Psi_{(\eta)}
\]

\[
= \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \eta \Psi_{(\eta)}.
\]

(41)

Hence, dividing and multiplying the output of the second integral by the same quantity \( a_{(\eta)} \) and rearranging the two integrals in a single expression, we get the trajectory (with the initial condition \( t_0 = 0 \)):

\[
t(r) = \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \left( \frac{\eta + a_{(\eta)}}{a_{(\eta)}} \right) \Psi_{(\eta)} - \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \epsilon_{(\eta)} \sin \Psi_{(\eta)},
\]

(42)

namely:

\[
t(r) = \sqrt{\frac{ma_{(\eta)}^3}{k - \eta |E|}} \left( \frac{\eta + a_{(\eta)}}{a_{(\eta)}} \right) \left[ \Psi_{(\eta)} - \frac{a_{(\eta)}}{\eta + a_{(\eta)}} \epsilon_{(\eta)} \sin \Psi_{(\eta)} \right]
\]

\[
= \frac{1}{\Omega_{(\eta)}(E)} \left[ \Psi_{(\eta)} - \frac{a_{(\eta)}}{\eta + a_{(\eta)}} \epsilon_{(\eta)} \sin \Psi_{(\eta)} \right],
\]

(43)
which is the deformed Kepler’s equation:

\[
\Omega_{(q)}(E) t(r) = \Psi_{(q)} - \frac{a_{(q)}}{\eta + a_{(q)}} \epsilon_{(q)} \sin \Psi_{(q)}. \tag{44}
\]

The frequency of the motion is given by

\[
\Omega_{(q)}(E) = \sqrt{\frac{k - \eta |E|}{m a_{(q)}^3}} \frac{a_{(q)}}{\eta + a_{(q)}} = \sqrt{\frac{2}{m} \frac{|E| \sqrt{|E|}}{k + \eta |E|}}, \tag{45}
\]

which is nothing but the same frequency obtained through the spectrum generating algebra. Now we have just to plug in the equation (44) the explicit form of \(\Psi_{(q)}\) and check whether it coincides with the one derived via the algebraic method. By solving for \(\Psi_{(q)}\) one gets

\[
\Psi_{(q)} = \arccos \left[ \frac{1}{\epsilon_{(q)}} \left( 1 - \frac{r}{a_{(q)}} \right) \right], \tag{46}
\]

whence, owing to the well known relation \(\sin(\arccos(x)) = \sqrt{1 - x^2}\), it follows

\[
\Omega_{(q)}(E) t(r) = \arccos \left[ \frac{1}{\epsilon_{(q)}} \left( 1 - \frac{r}{a_{(q)}} \right) \right] - \frac{a_{(q)}}{\eta + a_{(q)}} \sqrt{\epsilon_{(q)}^2 - \left( 1 - \frac{r}{a_{(q)}} \right)^2}. \tag{47}
\]

Equation (47) represents the trajectory calculated through the standard analytic method.

On the other hand, the equation (22) for the trajectory derived by means of the algebraic method yields (in the case \(\theta_0 = 0\)):

\[
\Omega_{(q)}(E) t(r) = \arccos \left( -\frac{\sqrt{\frac{2}{m} \frac{|E| \sqrt{|E|}}{k + \eta |E|}}}{q_0 \sqrt{|E|}} \right) \left( \sqrt{2} \frac{(k - \eta |E|) - 2 |E| r}{q_0 \sqrt{|E|}} \right)
\]

\[
- \frac{2}{m (k + \eta |E|)} \sqrt{2mr(k - \eta |E|) - 2m |E| r^2 - l^2}, \tag{48}
\]

where \(q_0 = \sqrt{-l^2 + \frac{m(k - \eta |E|)^2}{2 |E|}}\). After easy algebraic manipulations equation (48) acquires the form:

\[
\Omega_{(q)}(E) t(r) = \arccos \left[ \frac{1}{\epsilon_{(q)}} \left( 1 - \frac{r}{a_{(q)}} \right) \right] - \frac{a_{(q)}}{\eta + a_{(q)}} \sqrt{\epsilon_{(q)}^2 - \left( 1 - \frac{r}{a_{(q)}} \right)^2}. \tag{49}
\]

In other words, by the algebraic method we get \(t(r)\) evaluated for \(-\epsilon_{(q)}\). As we expected, this expression is just the \(\eta\)-deformation of the Kuru–Negro result [21].

Here, some comments are in order, which seem to imply a sort of difference between the classical and the quantum case. Namely, according to the results obtained in [2] in the quantum case, for \(\eta > 0\), one has a very simple coupling constant metamorphosis, amounting just to replace in the Euclidean system \(k\) by \(k + \eta E\) both in the expressions for the discrete spectrum and for the corresponding eigenfunctions. However, in the classical case, just looking at formula (37), it turns out that we have not just coupling constant metamorphosis. This is because, besides the term \(k + \eta E\), there is an extra \(\eta\) dependence in the overall conformal factor. An analogous behaviour is exhibited by the classical D-III [20].
5. The case $\eta < 0$: new features

This section is devoted to a terse investigation of the main features arising in the case $k > 0$ for negative values of the deformation parameter $\eta$. In this case the conformal factor $\frac{r}{r + \eta}$ can be more conveniently written as $\frac{r}{r - |\eta|}$ which emphasizes the singularity at $r = |\eta|$. One relevant question is whether the singularity can be overcome or not. In the first case there might be trajectories intersecting the line $r = |\eta|$. In the second case the phase plane $(r, \dot{r})$ will consist of two non-overlapping domains. In particular, for closed orbits one may ask under what conditions the following (mutually excluding) inequalities for the inversion points hold:

$$r_{1} > |\eta|, \quad r_{2} < |\eta|.$$  \hfill (50)

A careful analysis of (50) shows that to characterize the corresponding regions of this plane one has to look at both parameters $\eta$ and $\lambda$, a characteristic length scale defined \footnote{We point out that in the case $l = 0$ the motion takes place on straight lines implying that no two-dimensional closed orbits are allowed.} as

$$\lambda := \frac{l^2}{2mr} > 0,$$

or better at their ratio $\alpha := \frac{\eta}{\lambda}$, and at the behaviour of the effective potential

$$V_{\text{eff}}(r) = \frac{l^2}{2mr(\sqrt{r^2 - |\eta|^2})} - \frac{k}{r - |\eta|} = -\frac{k}{r - |\eta|} \left[ r - \frac{\lambda}{r - \alpha \eta} \right].$$

5.1. Case $\alpha < 1$

The most interesting situation occurs in the case $\alpha < 1$, where one has indeed two non-overlapping regions separated by the straight-line $r = |\eta|$ (see figure 3).

- In the right domain $r > |\eta|$ the conformal factor is positive. We have a Riemannian manifold with non-constant curvature and there will be closed trajectories whenever the energy belongs to the (negative) open interval $(0, V_{\text{eff}}(r_c))$, where $V_{\text{eff}}(r_c) = -\frac{k}{\lambda(1 + \sqrt{1 - \alpha^2})}$ is the value of the effective potential at the critical point $r_c = \lambda (1 + \sqrt{1 - \alpha})$.
- In the left domain $r < |\eta|$ the conformal factor is negative entailing that the kinetic energy is also negative. In order to get a physically significant system we are naturally led to define in this region a new Hamiltonian $\tilde{H} := -\frac{l^2}{2mr(|\eta| - r)} - \frac{k}{|\eta| - r}$, namely to look at the system obtained by time-reversal...
As it is clearly shown by figure 4, after that transformation in the region \( \eta < |r_0| \), the effective potential acquires a typical 'confining' shape. There will be closed orbits for any positive energy higher than \( \sim -V_{\text{eff}}(r) \), where \( \lambda_\alpha = -\sqrt{1 - \alpha} \). We point out that the minimum of the potential is a monotonically decreasing function of \( \eta \), so that it goes to infinity as \( \eta \) goes to zero.

In both regions, the SGA approach allows us to find the solution \( \hat{t}(r) \). In the domain \( r > |\eta| \), we have only to change the sign of the parameter \( \eta \) in the formula (22). Conversely, in the domain \( 0 < r < |\eta| \), we have to factorize the new Hamiltonian \( \tilde{H} \) following the same steps as in the case \( \eta > 0 \).

5.2. Case \( \alpha > 1 \)

Here the dynamics is certainly less interesting because no closed orbits will come out (see figure 5).

In the region \( r > |\eta| \) the effective potential will be proportional to \( -(r - |\eta|)^{-1} \) while in the bounded region \( 0 < r < |\eta| \) its image is the full real line and furthermore it exhibits an inflection point for the value \( \hat{r} = \hat{\lambda}(1 - (\alpha - 1)^{1/3} + (\alpha - 1)^{2/3}) \).

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**Figure 4.** Potential \( V_{\text{eff}}(r) \) after the time-reversal transformation, i.e. \( \tilde{V}_{\text{eff}}(r) = -V_{\text{eff}}(r) \) calculated for \( \alpha = \frac{4}{3} \). The latter is contained into the segment \( 0 < r < |\eta| \).

**Figure 5.** Potential \( V_{\text{eff}}(r) \) calculated for \( \alpha = 2 \).

(see [25] for an analogous discussion about the D-III system). As it is clearly shown by figure 4, after that transformation in the region \( 0 < r < |\eta| \), the effective potential acquires a typical 'confining' shape. There will be closed orbits for any positive energy higher than \( \tilde{V}_{\text{eff}}(r_-) \), where \( r_- = \lambda(1 - \sqrt{1 - \alpha}) \). We point out that the minimum of the potential is a monotonically decreasing function of \( |\eta| \), so that it goes to infinity as \( |\eta| \) goes to zero.

In both regions, the SGA approach allows us to find the solution \( \hat{t}(r) \). In the domain \( r > |\eta| \), we have only to change the sign of the parameter \( \eta \) in the formula (22). Conversely, in the domain \( 0 < r < |\eta| \), we have to factorize the new Hamiltonian \( \tilde{H} \) following the same steps as in the case \( \eta > 0 \).
5.3. Case $\alpha = 1$

When the parameter $\alpha = 1$ the situation is totally different from the previous cases and things are definitely less clear. It looks like that the singularity could be overcome, in the sense that it does not affect the potential term. By the way, a plot of the effective potential for this particular value of the parameter $\alpha$, i.e. $V_{\text{eff}}(r) = -\frac{k}{r}$, shows that the distinction between the two regions ($r > |\eta|$, $r < |\eta|$) seems to vanish, since we have a single continuous line with a monotonically increasing behaviour (see figure 6). However, this is not the case because the singularity still appears in the kinetic term, that turns out to be divergent on the boundary $r = |\eta|$. In any case, the dynamics is not interesting since no closed orbits are allowed.

6. Concluding remarks and open problems

One of the main results obtained in our paper is the constructive proof that the spectrum generating algebra technique can be successfully employed to attack and solve maximally superintegrable systems on spaces with variable curvature. As already mentioned throughout the article, in the next future we will provide the analogous results for the classical D-III system. Moreover, for positive values of the deformation parameters, we will exhibit the exact solution to the corresponding quantum problems based on the shape invariant potentials techniques [20], making a comparison with different approaches proposed in the literature [2, 26, 27]. As a further interesting result, we have shown that, for negative values of the deformation parameter, in the case $\alpha < 1$, it is possible to introduce a new Hamiltonian, defined in the punctured open ball $0 < r < |\eta|$, that is characterized by confining properties (closed orbits/bounded motion). The latter is related to the first Hamiltonian by means of the transformation $H \rightarrow \tilde{H} = -H$. As a matter of fact, the behaviour of the classical effective potential strongly suggests that in the quantum case there will be bound states also in that region, while in the limit $|\eta| \rightarrow 0$ the minimum of the potential will go to infinity. With the proper changes, we expect that similar features will hold for D-III as well.

Actually, in this context our final aim is twofold:

1. We will focus future investigations on the quantum systems exactly on the case where the deformation parameters take negative values. There, due to the confining nature of the potentials, we expect the most interesting results from a physical point of view [25].

![Figure 6. Potential $V_{\text{eff}}(r)$ calculated for $\alpha = 1$. The centrifugal and gravitational contributions add up to give the behaviour $-kr^{-1}$. In this case the singularity in the effective potential disappears.](image-url)
(2) We will try to solve all the classical problems belonging to the Perlick’s families I and II [16] by means of the spectrum generating algebra approach. We are encouraged to proceed further in this direction inasmuch as we have seen that in the classical deformed versions of Taub–Nut and D-III the coupling constant metamorphosis, hardly applicable to the full Perlick’s families, does not seem to be the essential feature.

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