Deformed black strings in five-dimensional Einstein–Yang–Mills theory

Yves Brihaye¹ and Betti Hartmann²

¹ Faculté des Sciences, Université de Mons-Hainaut, 7000 Mons, Belgium
² School of Engineering and Sciences, International University Bremen (IUB), 28725 Bremen, Germany

E-mail: yves.brihaye@umh.ac.be and b.hartmann@iu-bremen.de

Received 31 August 2005, in final form 14 October 2005
Published 22 November 2005
Online at stacks.iop.org/CQG/22/5145

Abstract
We construct the first examples of deformed non-Abelian black strings in a five-dimensional Einstein–Yang–Mills model. Assuming all fields to be independent of the extra coordinate, we construct deformed black strings, which in the four-dimensional picture correspond to axially symmetric non-Abelian black holes in gravity-dilaton theory. These solutions thus have deformed $S^2 \times \mathbb{R}$ horizon topology. We study the fundamental properties of the black strings and find that for all choices of the gravitational coupling two branches of solutions exist. The limiting behaviour of the second branch of solutions, however, depends strongly on the choice of the gravitational coupling.

PACS numbers: 04.20.Jb, 04.40.Nr, 04.50.+h, 11.10.Kk

1. Introduction

Higher-dimensional black holes have gained a lot of interest in recent years. This is mainly due to the ongoing study of theories in higher dimensions such as Kaluza–Klein theories [1, 2], (super)string theories [3] and brane world models [4]. In higher dimensions, horizon topologies other than those known in four dimensions are possible. While the first examples of higher-dimensional black holes, namely the higher-dimensional generalizations of Schwarzschild, Reissner–Nordström [5] and Kerr solutions [6], have horizon topology $S^{d-2}$ in $d$ dimensions, so-called black string solutions [7] (in the simplest version a four-dimensional Schwarzschild black hole extended into one extra dimension) with horizon topology $S^2 \times \mathbb{R}$ as well as black ring solutions [8] with horizon topology $S^2 \times S^1$ have been constructed. The black strings (in their simplest version) are—like the five-dimensional Schwarzschild black holes—static solutions of the vacuum Einstein equations. They have mainly been investigated with respect to their stability [9]. Since the black strings have entropy proportional to $M^2/\gamma_0$, where...
$M$ denotes the mass of the black string and $y_0$ is the extension in the extra dimension, the hyperspherical black holes have entropy proportional to $M^{3/2}$. Thus for a large enough $y_0$, one would expect an instability—thereafter called the ‘Gregory–Laflamme instability’—of the black string which was confirmed analytically in [9]. For recent reviews on black strings and black holes in spacetimes with compact extra dimensions, see [10, 11].

The higher dimensional Kerr solutions, which in the following we refer to as the Myers–Perry solutions, as well as the Emparan–Reall black ring solutions, are stationary solutions of the vacuum Einstein equations and thus carry angular momentum. While the Myers–Perry solutions exist only up to a maximal value of the angular momentum, black rings are balanced against gravitational collapse by rotation and thus exist only above a critical value of the angular momentum.

All existing higher-dimensional black holes have been studied intensively with respect to their uniqueness. While uniqueness theorems for a variety of static black hole solutions have been well established [12–14], stationary black holes seem to violate uniqueness. In $d = 5$, Myers–Perry solutions as well as black ring solutions exist for the same values of the mass and angular momentum and are thus not uniquely characterized by these latter parameters. Interestingly, the situation changes if five-dimensional supersymmetric black holes and black rings are studied [15, 16].

While most black hole solutions in higher dimensions have been constructed in different (dilaton-) gravity theories without additional matter fields, the first examples of black holes in a five-dimensional SO(4)-Einstein–Yang–Mills model have been constructed [17]. These black holes are hyperspherically symmetric generalizations (horizon topology $S^3$) of the coloured black hole solutions in four-dimensional SU(2) Einstein–Yang–Mills theory [18]. These solutions show that—as in four dimensions—the uniqueness theorems for higher-dimensional static black holes cannot be extended to models involving non-Abelian gauge fields.

In [19] five-dimensional black strings in an SU(2) Einstein–Yang–Mills model which has been introduced in [20] have been constructed. These are four-dimensional, spherically symmetric non-Abelian black holes extended trivially into one extra dimension and thus have horizon topology $S^2 \times \mathbb{R}$.

In this paper, we extend these latter results and discuss deformed black strings. These black strings are axially symmetric black holes in four dimensions extended trivially into one extra dimension. They thus have deformed $S^2 \times \mathbb{R}$ topology. Our solutions are translationally invariant—in contrast to the recently constructed translationally non-uniform black strings [21]. To distinguish the case studied here from the (non-) uniform black strings, we call our solutions with rotational symmetry in four dimensions ‘undeformed’ black strings and the solutions with axial symmetry in four dimensions ‘deformed’ black strings, respectively. Note also that the solutions studied here are the black hole analogues of the deformed vortex-type solutions studied in [22].

Our paper is organized as follows: in section 2, we give the model, the ansatz, the equations of motion and the boundary conditions. In section 3, we introduce the fundamental properties of black strings. In section 4, we discuss our numerical results and in section 5 we give our conclusions.

2. The model

We study the model introduced in [20]. This is an SU(2) Einstein–Yang–Mills model in five dimensions, where all fields are assumed to be independent of the extra dimension. We take the length of the extra dimension to be equal to unity.
The Einstein–Yang–Mills Lagrangian in \( d = (4 + 1) \) dimensions then reads
\[
S = \int \left( \frac{1}{16\pi G_5} R - \frac{1}{4} F_{MN}^a F^{aMN} \right) \sqrt{g} \, d^5 x
\]
(1)
with the SU(2) Yang–Mills field strengths \( F_{MN}^a = \partial_M A^a_N - \partial_N A^a_M + e \epsilon_{abc} A^b_M A^c_N \), the gauge index \( a = 1, 2, 3 \) and the spacetime index \( M = 0, \ldots, 4 \). \( G_5 \) and \( e \) denote, respectively, the five-dimensional Newton’s constant and the coupling constant of the gauge field theory. \( G_5 \) is related to the Planck mass \( M_{pl} \) by \( G_5 = M_{pl}^{-3} \) and \( e^2 \) has the dimension of [length].

Both the metric and matter fields are assumed to be independent of the extra coordinate \( y \).

The gauge fields can then be parametrized as follows \[20\]:
\[
A^a_M \, dx^M = A^a_\mu \, dx^\mu + A^a_y \, dy,
\]
(2)
while we give the generalized ansatz for the metric below.

### 2.1. The ansatz

Our aim is to construct non-Abelian black strings, which are axially symmetric in four dimensions and extended trivially into one extra dimension. We thus have three Killing vectors \( \partial_y, \partial_{\phi}, \partial_t \) associated with the black string solutions. Due to the fact that we will choose the components of \( A_y \) and \( A_\phi \) to point in the same direction of the internal space, off-diagonal components of the energy–momentum tensor appear. We thus choose the following ansatz for the metric tensor \[22\]:
\[
g^{(5)}_{MN} \, dx^M \, dx^N = e^{-\xi} \left[ -f \, dr^2 + m_\perp \, dr^2 + m_r \, d\theta^2 + l \, r^2 \sin^2 \theta (d\phi + J \, dy)^2 \right] + e^{2\xi} \, dy^2,
\]
(3)
where \( f = f(r, \theta) \), \( m = m(r, \theta) \), \( l = l(r, \theta) \), \( J = J(r, \theta) \) and \( \xi = \xi(r, \theta) \) are functions of \( r \) and \( \theta \) only. We have parametrized the metric such that the determinant of the metric \( g \) becomes independent of \( J \):
\[
\sqrt{-g} = e^{-\xi} \frac{m_r \, r^2}{f} \sqrt{l \sin \theta},
\]
(4)
Note that our parametrization here differs from that used in \[22\]; however, it can be obtained by a simple transformation of the fields.

For the gauge fields, the ansatz reads \[22\]
\[
A_\mu \, dx^\mu = \frac{1}{2er} \left[ \tau^a_\phi (H_1 \, dr + (1 - H_2) \, r \, d\theta) - n \left( \tau^a_\phi H_3 + \tau^a_0 (1 - H_3) \right) r \, d\theta \right.
+ \left( H_5 \tau^a_\phi + H_6 \tau^a_0 \right) r \, dy \]
\]
(5)
where \( H_i = H_i(r, \theta), i = 1, \ldots, 6 \) and \( \tau^a_\phi, \tau^a_0 \) and \( \tau^a_\psi \) denote the scalar product of the vector of Pauli matrices \( \vec{\tau} = (\tau_1, \tau_2, \tau_3) \) with the unit vectors \( \vec{\tau}^a_\phi = (\sin \theta \cos n\varphi, \sin \theta \sin n\varphi, \cos \theta), \vec{\tau}^a_0 = (\cos \theta \cos n\varphi, \cos \theta \sin n\varphi, -\sin \theta), \vec{\tau}^a_\psi = (-\sin n\varphi, \cos n\varphi, 0) \). \( n \) corresponds to the winding number of the configuration.

We fix the residual gauge invariance by imposing the gauge condition \( r \partial_r H_1 - \partial_\theta H_2 = 0 \) \[22, 23\].

### 2.2. Equations of motion

The matter Lagrangian in terms of the field strength tensor reads
\[
\mathcal{L}_M = -\frac{1}{4} \text{trace}(F_{MN} F^{MN}) = -\frac{1}{2} \text{trace} \left[ F_{\mu \nu} e^{2\xi} \frac{f^2}{m^2 r^2} + F_{\phi \psi} \left( e^{2\xi} \frac{f^2}{m l \sin^2 \theta r^2} + e^{-\xi} \frac{J^2 f}{m} \right) \right.
\]
\[
+ F_{\theta \phi} e^{-\xi} \frac{f}{m} + F_{\theta \psi} \left( e^{2\xi} \frac{f^2}{m l \sin^2 \theta r^2} + e^{-\xi} \frac{J^2 f}{m r^2} \right) + F_{\psi \phi} e^{-\xi} \frac{f}{m r^2} \n\]
\[
+ F_{\psi \psi} \left( -\frac{e^{-\xi} f}{\sin^2 \theta r^2} + J^2 e^{-4\xi} \right) - 2 F_{\phi \psi} F_{\theta \theta} \frac{e^{-\xi} J f}{m} \n\]
\[
- 2 F_{\phi \psi} F_{\theta \psi} e^{-\xi} \frac{J f}{m r^2} - F_{\psi \phi} e^{-\xi} J^2 \right]
\]

where the non-vanishing parts of the field strength tensor are given by

\[
F_{\theta \phi} = -\frac{1}{r} \left( H_{1,0} + r H_{2,r} \right) \frac{y_n}{2},
\]

\[
F_{\phi \psi} = -n \sin \theta \left( r H_{3,\theta} - H_{1,0} \right) \frac{y_n}{2} + n \sin \theta \left( r H_{4,\theta} + H_{1,0} + \cot \theta H_{1} \right) \frac{y_n}{2},
\]

\[
F_{\theta \psi} = -n \sin \theta \left( H_{1,\theta} - 1 + H_2 H_4 + \cot \theta H_3 \right) \frac{y_n}{2} + n \sin \theta \left( H_{4,\theta} - H_2 H_3 - \cot \theta \left( H_2 - H_3 \right) \right) \frac{y_n}{2},
\]

\[
F_{\theta \phi} = \left( H_{5,r} + \frac{H_{1,0}}{r} \right) \frac{y_n}{2} + \left( \frac{H_{1,0}}{r} - H_{5,r} \right) \frac{y_n}{2},
\]

\[
F_{\phi \psi} = \left( H_{5,\theta} - H_2 H_6 \right) \frac{y_n}{2} + \left( H_{5,\theta} + H_2 H_6 \right) \frac{y_n}{2},
\]

\[
F_{\psi \psi} = n \left( H_{3, \theta} \sin \theta + H_6 \cos \theta + H_4 H_5 \sin \theta \right) \frac{y_n}{2}.
\]

The energy–momentum tensor

\[
T_{MN} = 2 \text{trace} \left( g^{AB} F_{MA} F_{NB} - \frac{1}{4} g_{MN} F_{AB} F^{AB} \right)
\]

has non-vanishing components \( T_{MM}, M = 0, \ldots, 4 \) and \( T_{\mu \nu}, T_{\psi \psi} \).

The Euler–Lagrange equations \( \nabla_M F^{MN} + i [A_M, F^{MN}] = 0 \) are obtained by varying the Lagrangian with respect to the matter fields \( H_i(r, \theta) \), while the Einstein equations read \( G_{MN} = 8 \pi G T_{MN} \). We thus obtain a system of 11 coupled partial differential equations to be solved subject to appropriate boundary conditions.

Note that in the four-dimensional picture our system corresponds to an SU(2) Einstein–Yang–Mills–Higgs dilaton system with an additional U(1) potential given in terms of \( J(r, \theta) \) [22]. The \( A_\psi \)-component of the gauge field then plays the role of a Higgs field, while \( \xi \) can be interpreted as a dilaton.

### 2.3. Boundary conditions

Due to the requirement of asymptotic flatness, we have for the metric functions at infinity

\[
f(r = \infty) = 1, \quad m(r = \infty) = 1, \quad l(r = \infty) = 1,
\]

\[
\xi(r = \infty) = 0, \quad J(r = \infty) = 0,
\]

while for the gauge field functions we have

\[
H_i(r = \infty) = 0, \quad i = 1, 2, 3, 4, 6, \quad H_5(r = \infty) = 1.
\]

At the regular horizon, the boundary conditions read

\[
f(r = r_h) = 0, \quad m(r = r_h) = 0, \quad l(r = r_h) = 0,
\]

\[
\partial_r \xi |_{r=r_h} = 0, \quad \partial_r J |_{r=r_h} = 0.
\]
for the metric functions and
\[ H_i(r = r_h) = 0, \quad \partial_r H_i |_{r = r_h} = 0, \quad i = 2, 3, 4, 5, 6 \] (12)
for the gauge field. Finally, the boundary conditions on the \( \rho \)- and \( z \)-axes read (due to symmetry requirements)
\[ \partial_\theta f |_{\theta = \theta_0} = \partial_\theta m |_{\theta = \theta_0} = \partial_\theta \xi |_{\theta = \theta_0} = \partial_\theta J |_{\theta = \theta_0} = 0, \quad \theta_0 = 0, \quad \frac{\pi}{2} \] (13)
for the metric fields and
\[ H_1(\theta = \theta_0) = H_2(\theta = \theta_0) = H_3(\theta = \theta_0) = \partial_\theta H_4 |_{\theta = \theta_0} = \partial_\theta H_5 |_{\theta = \theta_0} = 0, \]
\[ \theta_0 = 0, \quad \frac{\pi}{2} \] (14)
for the gauge fields.

3. Fundamental properties of the black strings

With the introduction of the new variable \( x = re \) (with \( x_h \equiv r_h e \)) the equations of motion depend only on the coupling constant
\[ \alpha^2 = 4\pi G_5. \] (15)

The entropy \( S \) of the black strings is given by
\[ S = \frac{A}{4} = \frac{1}{4} \int_0^{\pi} \int_0^{2\pi} \sqrt{g_{\theta\theta}} \sqrt{g_{\phi\phi}} \sqrt{g_{m\phi}} \sin \theta \sin \phi \, d\theta \, d\phi \, dy = \frac{y_0 \pi}{2} \left( \int_0^{\pi} d\theta \sqrt{\frac{m \sqrt{x^2}}{f}} \right) |_{x = x_h}, \]
where \( y_0 \) is the length of the extra dimension, which we set to one, \( y_0 = 1 \).

The parameter entering our boundary conditions is the horizon parameter \( x_h \). However, the interpretation of this parameter is not as straightforward as in the case of Schwarzschild-like coordinates. We thus use the area parameter \( x / \Delta \) [23] with
\[ x_\Delta = \sqrt{\frac{A}{4\pi y_0}} \] (17)

to characterize the solutions.

The mass \( M \) (per unit length of the extra dimension) is given by [23]
\[ M = \frac{1}{2\alpha^2} \lim_{x \to \infty} x^2 \partial_x f. \] (18)

Furthermore, we study the ratio of the circumference along the equator \( L_e \) and that along the poles \( L_p \) to have a measure for the deformation of the horizon of the black strings [23]:
\[ L_e = \left( \int_0^{2\pi} d\phi \sqrt{f} \sin \theta \sin \phi \right) |_{x = x_h, \theta = \pi/2}, \quad L_p = 2 \left( \int_0^{\pi} d\phi \sqrt{\frac{m \sqrt{x^2}}{f}} \right) |_{x = x_h, \phi = 0}. \] (19)

A further important property of black holes and black strings is the temperature of the solutions, which here is given by
\[ T = \frac{f_x(\theta)}{2\pi x_h \sqrt{m_2(\theta)}}, \] (20)
where we have used the expansion of the metric functions at $x_h$ [23]:

$$f(x, \theta) = f_2(\theta) \left(\frac{x - x_h}{x_h}\right)^2 + O\left(\frac{x - x_h}{x_h}\right)^3, \tag{21}$$

$$m(x, \theta) = m_2(\theta) \left(\frac{x - x_h}{x_h}\right)^2 + O\left(\frac{x - x_h}{x_h}\right)^3.$$  

The zeroth law of black hole mechanics states that the temperature is constant at the horizon, i.e. $\partial_\theta T = 0$, which requires $m_2 \partial_\theta f_2 - f_2 \partial_\theta m_2 = 0$ (this follows directly from (20)).

In the isolated horizon framework for four-dimensional black hole solutions, it has been stated [24] and in fact confirmed for four-dimensional non-Abelian black holes in SU(2) Einstein–Yang–Mills–Higgs theory [23] that a non-Abelian black hole is a bound system of a Schwarzschild black hole and the corresponding non-Abelian regular solution. To test whether this also holds true here, we have studied the binding energy $E_b$ (per unit length of the extra dimension) of the black string solutions:

$$E_b = M - M_{reg} - M_s, \quad M_s = \frac{x_\Delta}{2\alpha^2}$$  

where $M_s$ is the mass of the Schwarzschild black string (per unit length of the extra dimension) and $M_{reg}$ is the mass of the corresponding non-Abelian regular solution which has been studied in [22]. Note that for $M_{reg}$, we have used the mass of the fundamental solution, i.e. the solution on the first (1) branch of regular solutions, which we believe to be stable.

In the study of (non-) uniform black strings and black holes in spacetimes with extra compact dimensions, a further quantity, namely the tension along the extra dimensions, has been studied [10, 11]. The phase diagram in the mass–tension plane gives a good indication about the properties of the solutions. A detailed study of these diagrams has been done in a follow-up publication by the present authors [25].

4. Numerical results

We have solved numerically the system of partial differential equations subject to the above given boundary conditions for several values of the coupling constant $\alpha$ and of the horizon parameter $x_h$, respectively. Here, we report on our analysis of the cases $n = 1, n = 2$ and $\alpha = 0.5$. More details for other parameter values will be presented elsewhere [25].

Before we discuss the numerical results, let us recall the results for the regular case. It turns out that these are crucial for the understanding of the qualitative features of the black hole solutions. The regular case for $n = 1$ was studied in detail in [20, 26]. It has been found that several branches of solutions (which in the following we refer to as $\alpha$-branches) for varying $\alpha$ exist. $\alpha$-branch refers here to a curve giving one of the quantities of the solution (e.g. the energy) as a function of $\alpha$. Typically, several distinct curves (‘branches’) appear in an energy–$\alpha$ plot, such that for a fixed value of $\alpha$ different solutions (with different energies) exist. In [26] four branches have been constructed such that for $\alpha \in [0: 0.312], \alpha \in [0.312 : 1.268], \alpha \in [0.312 : 0.395]$ and $\alpha \in [0.395 : 0.419]$ one, two, three and four solutions, respectively, exist. It is likely that for $\alpha \in [0.395 : 0.419]$ further branches exist; however, these have not been constructed so far.

The regular $n = 2$ case was studied in [22] and only one branch of solutions in $\alpha$ has been constructed. Corresponding to the $n = 1$ case we believe, though, that further branches also exist for $n = 2$, the numerical construction of which however seems very involved.
4.1. Undeformed black strings for $n = 1$

In [19], the existence of several branches for a fixed area parameter $x_{\Delta}$ and varying $\alpha$ has been demonstrated for the $n = 1$ black strings. The behaviour of the solutions for fixed $x_{\Delta}$ and varying $\alpha$ is thus similar to that observed for regular solutions [20]. Here, we observe a new phenomenon for $\alpha$ fixed and varying $x_{\Delta}$. Since this case has not been studied in [19], we have reconsidered the case $n = 1$ here. Note that for $n = 1$, the black strings are four-dimensional spherically symmetric black holes extended trivially into the extra dimension. The solutions thus depend only on the radial coordinate $x$. For the functions we have $H_5(x) = H_4(x)$, $H_1(x) = H_2(x) = H_6(x) = J(x) = 0$.

Our numerical results for $\alpha = 0.5$ are shown in figures 1 and 2. In figure 1, we give the values of the gauge field functions $H_2(x) = H_4(x)$, $H_3(x)$ and of the metric function $\xi(x)$ at $x_{\Delta}$, $H_2(x_{\Delta})$, $H_3(x_{\Delta})$, $\xi(x_{\Delta})$, as functions of the area parameter $x_{\Delta}$. Clearly, two branches of solutions exist. The first branch (denoted by ‘1’) exists for $x_{\Delta} \in [0 : x_{\Delta}^{(\text{max})}]$ with $x_{\Delta}^{(\text{max})} \approx 0.633$. The limit $x_{\Delta} \to 0$ on this branch of solutions corresponds to the fundamental regular solution, i.e. the solution on the first $\alpha$-branch [20]. Clearly $H_2(x_{\Delta} \to 0) \to 1$ and $H_3(x_{\Delta} \to 0) \to 0$, which corresponds to the boundary conditions for the globally regular solutions at $x = 0$. At the same time $\xi(x_{\Delta} \to 0) \to 0.07216$, which is the numerically determined value for the fundamental regular solution [20]. Similarly the mass on the first (lower) branch tends to the mass of the fundamental regular solution for $x_{\Delta} \to 0$ (see figure 2).

The second branch of solutions (denoted by ‘2’ in figure 1) similarly exists for $x_{\Delta} \in [0 : x_{\Delta}^{(\text{max})}]$. Clearly, the solutions on this second branch are distinct from those on the first branch having higher mass (see figure 2) and different values of $H_2(x_{\Delta})$, $H_3(x_{\Delta})$ and $\xi(x_{\Delta})$ (see figure 1). In the limit $x_{\Delta} \to 0$, we find again that $H_2(x_{\Delta} \to 0) \to 1$ and $H_3(x_{\Delta} \to 0) \to 0$, while $\xi(x_{\Delta} \to 0) \to -1.262$ tends to the value of the corresponding solution of the second $\alpha$-branch of regular solutions. Strong evidence for this is also given by an inspection of the mass curve in figure 2, where in the limit $x_{\Delta} \to 0$, the mass tends to that of the regular solution of the second $\alpha$-branch.

Figure 1. The values of the gauge field functions $H_2 = H_4$, $H_5$ and of the metric function $\xi$ at $x_{\Delta}$, $H_2(x_{\Delta})$, $H_3(x_{\Delta})$, $\xi(x_{\Delta})$, on the first (1) and second branch (2) of solutions are shown as a function of the area parameter $x_{\Delta}$ for the $n = 1$ black strings with $\alpha = 0.5$. 
The binding energy $E_b$ (see figure 2) is negative on the first branch, indicating indeed that non-Abelian black strings are bound systems of a Schwarzschild black string and the corresponding regular non-Abelian vortex solution. On the second branch, the binding energy becomes positive, indicating an instability of the solutions. Note that for both branches, we have used the mass of the regular solution on the second $\alpha$-branch to obtain the binding energy for our black string solutions on the second branch, the binding energy would have been negative.

Results for different values of $\alpha$ will be given elsewhere [25]. However, we believe that for all values of $\alpha$ two branches of black string solutions will exist. The critical behaviour, however, will strongly depend on the value of $\alpha$. For $\alpha = 0.5 \in [0.419 : 1.268]$ two regular solutions exist and the second branch terminates in the corresponding regular solution of the second $\alpha$-branch. We believe that if $\alpha \in [\alpha_1^{(n)} : \alpha_2^{(n)}]$, where $n$ indicates the number of regular solutions available for this range of $\alpha$, i.e. the number of branches, the black string solutions on the second branch will tend to the $n$th regular solution, i.e. to the solution on the $n$th branch, for $x_\Delta \rightarrow 0$. For example, the black strings on the second branch of solutions for $\alpha = 0.35 \in [0.312 : 0.395]$ would tend to the third solution, i.e. the solution on the third $\alpha$-branch of regular solutions. For $\alpha \in [0 : 0.312]$ for which only the fundamental regular solution exists, we expect that the second branch terminates at some finite $x_\Delta > 0$. At this point, the solution bifurcates with the branch of the Einstein–Maxwell-dilaton solution with $H_2(x) = H_3(x) \equiv 0$, $H_5(x) \equiv 1$ and $\xi(x)$ given by the corresponding function of the Einstein–Maxwell-dilaton solution.

4.2. Deformed black strings for $n = 2$

Our results for the deformed black string solutions ($n = 2$) are given in figures 2–4.

The first branch exists for $x_\Delta \in [0 : x_{\Delta,\text{max}}]$ with $x_{\Delta,\text{max}} \approx 1.31$. In the limit $x_\Delta \rightarrow 0$, the solution approaches the corresponding regular solution which has been constructed in [22].

![Figure 2](image-url)
When increasing $x_\Delta$, our results show that both the mass $M$ (see figure 2) and the corresponding value of the horizon parameter $x_h$ increase, while the temperature $T$ decreases (see figure 3). We have confirmed numerically that the temperature on the horizon is constant and our solutions thus fulfill the zeroth law of black hole mechanics. The deformation parameter $L_\sigma/L_p$ decreases, but stays very close to one indicating that the horizon is only deformed slightly. On this branch, the absolute value of the new function $|J(x, \theta)|$ stays small. In figure 4 we show the values of $J$ at the horizon, $J(x_\Delta, \theta = 0)$, together with the values of $\xi(x_\Delta, \theta = 0)$. These latter values are positive on the first branch of solutions. The
values $H_2(x_\Delta, \theta)$ and $H_4(x_\Delta, \theta)$ decrease from one, while $H_5(x_\Delta, \theta)$ increases from zero for increasing $x_\Delta$. We demonstrate the $x_\Delta$-dependence for $H_2(x_\Delta, \theta = 0)$ and $H_5(x_\Delta, \theta = 0)$ in figure 4.

As in the $n = 1$ case, the limit $x_\Delta \to 0$ corresponds to the fundamental regular deformed vortex solution [22].

We managed to construct the second branch of solutions for $x_\Delta \in [x_{\Delta, \text{end}} : x_{\Delta, \text{max}}]$ with $x_{\Delta, \text{end}} \approx 0.2$. At $x_\Delta = x_{\Delta, \text{max}}$ the branches merge into a single solution. The mass of the solutions on the second branch is higher than that of the corresponding solution on the first branch for the same value of $x_\Delta$ and thus the same entropy (see figure 2).

The other features of the second branch are that, when $x_\Delta$ decreases from $x_{\Delta, \text{max}}$, the parameter $x_h$ and the mass decrease, while the temperature $T$ increases and stays higher than the temperature of the corresponding solution on the first branch. As compared to the solutions on the first branch, the solutions on the second branch have a much stronger deformed horizon. This can be noted by observing the curve $L_e/L_p$ in figure 3 and the data plotted in figure 4. We also note that the value of the metric function $\xi$ at the horizon becomes negative on the second branch and that the value of $J$ at the horizon now deviates significantly from zero.

Finally, let us mention that the values of $H_2(x_\Delta, \theta)$ and $H_4(x_\Delta, \theta)$ start to increase again, while $H_3(x_\Delta, \theta)$ decreases for decreasing $x_\Delta$. The detailed curves for $H_2(x_\Delta, \theta = 0)$ and $H_5(x_\Delta, \theta = 0)$ are shown in figure 4.

We strongly believe that this second branch extends all the way back to $x_\Delta = 0$ similar to the $n = 1$ case. Then $H_2(x_\Delta \to 0, \theta) \to 1$, $H_4(x_\Delta \to 0, \theta) \to 0$, $\xi(x_\Delta \to 0, \theta) \to \xi_0 < 0$, $J(x_\Delta \to 0, \theta) \to J_0 < 0$, where $\xi_0$ and $J_0$ have not been determined so far. Correspondingly, $x_h \to 0$ in this limit, $L_e/L_p \to 1$ (as for the first branch) and $T \to \infty$, since $T$ is not defined for regular solutions.

In figure 2, we also show (the negative of) the binding energy per winding number $-E_b/n$ for $\alpha = 0.5$. Clearly, the binding energy on the first branch of solutions (note the inversion of the branches with respect to the plot of the mass) is negative and the non-Abelian black strings on the first branch are thus bound systems of the corresponding regular non-Abelian vortex solutions of [22] and the Schwarzschild black string. On the second branch, the situation changes. The binding energy becomes positive. This signals that the non-Abelian black strings are unstable to decay into the non-Abelian, globally regular vortex solutions and a Schwarzschild black string on this second branch of solutions.

Comparing the $n = 1$ and $n = 2$ solutions for the same value of $\alpha$, we find that the extension of the branches in $x_\Delta$ is bigger for $n = 2$ as compared to $n = 1$. Furthermore, when comparing the respective first and second branches for $n = 1$ and $n = 2$, the mass (per winding number) of the $n = 2$ solution is always lower than that of the $n = 1$ solution.

Results for different values of $\alpha$ will be given elsewhere. However, we believe that the scenario is similar to the $n = 1$ case, such that the second branch terminates in the corresponding regular solution of the $n$th $\alpha$-branch for $x_\Delta \to 0$ if more than one $\alpha$-branch exist, or in an Einstein–Maxwell-dilaton solution for finite $x_\Delta$ with $H_2(x, \theta) = H_4(x, \theta) = H_5(x, \theta) = H_6(x, \theta) = 0$, $H_2(x, \theta) \equiv 1$, $J(x, \theta) = 0$ and $\xi(x, \theta) = \xi(x)$ given by the corresponding function of the Einstein–Maxwell-dilaton solution if only one $\alpha$-branch exists for the regular solutions.

5. Conclusions

In this paper, we have constructed black strings as solutions of a five-dimensional Einstein–Yang–Mills model. We have presented our results for $n = 1, n = 2$ and fixed $\alpha = 0.5$. We find that two branches of solutions exist in both cases, which in the limit of $x_\Delta \to 0$ tend
Deformed black strings in five-dimensional Einstein–Yang–Mills theory

to the corresponding regular solutions of the first, respectively second α-branch. We believe that this is a generic feature of the system. For other values of α, the second branch of black string solutions will terminate in the nth available regular solution. This means that in specific parameter ranges, the black string solutions will start to show oscillatory behaviour in the gauge field functions. For those values of α, for which only one globally regular solution exists, we believe that the second branch will terminate at a finite value of the area parameter in an Einstein–Maxwell-dilaton solution. More details will be given elsewhere [25].

The question whether the non-Abelian black strings are unstable to decay to hyperspherically symmetric non-Abelian black holes, i.e., would have an instability corresponding to the ‘Gregory-Laflamme instability’ for Schwarzschild black strings, is beyond the scope of this paper. A first step to answer this question would be the numerical construction of hyperspherically symmetric non-Abelian black holes of the SU(2) Einstein–Yang–Mills system in five dimensions.

We remark that next to the ‘fundamental’ solutions constructed here excited solutions could also exist—similar to what is known for four-dimensional spherically symmetric black hole solutions in SU(2) Einstein–Yang–Mills–Higgs theory [27]. These solutions would have a number m, m ∈ ℤ, of zeros of the gauge field functions. The construction of these solutions has not been done so far—neither in the case of globally regular vortices nor in the case of black strings—and is left to a future publication.

Acknowledgment

We thank the Belgian FNRS for financial support.

References

[1] Kaluza T 1921 Sitzungsber Preuss. Akad. Wiss. Berlin (Math. Phys.) 966–72
[2] Klein O 1926 Z. Phys. 37 895
[3] See, e.g., Polchinski J 1998 String Theory (Cambridge: Cambridge University Press)
[4] Akama K 1983 Pregeometry Gauge Theory and Gravitation, Proc., Nara, 1982 (Lecture Notes in Physics vol 176) ed K Kikkawa, N Nakano and H Nariai (Berlin: Springer) pp 267–71 (Preprint hep-th/0001113)
  Rubakov V A and Shapiro M E 1983 Phys. Lett. B 125 136
  Rubakov V A and Shapiro M E 1983 Phys. Lett. B 125 139
  Davi G and Shifman M 1997 Phys. Lett. B 396 64
  Davi G and Shifman M 1997 Phys. Lett. B 407 452
  Antoniadis I 1990 Phys. Lett. B 246 377
  Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 429 263
  Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 436 257
  Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370
  Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690
  Cremades D, Ibanez L E and Marchesano F 2002 Nucl. Phys. B 643 93
  Kokorelis C 2004 Nucl. Phys. B 677 115
[5] Tangherlini F R 1963 Nuovo Cimento 27 636
[6] Myers R C and Perry M J 1986 Ann. Phys., NY 172 304
[7] Horowitz G T and Strominger A 1991 Nucl. Phys. B 360 197
[8] Emparan R and Reall H 2002 Phys. Rev. Lett. 88 101101
[9] Gregory R and Laflamme R 1988 Phys. Rev. D 37 305
[10] Kol B 2004 The phase transition between caged black holes and black strings—a review Preprint hep-th/0411240
[11] Harmark T and Obers N A 2005 Phases of Kaluza–Klein black holes—a brief review Preprint hep-th/0503020
[12] Gibbons G W, Ida D and Shiromizu T 2002 Phys. Rev. Lett. 89 041101
  Gibbons G W, Ida D and Shiromizu T 2002 Phys. Rev. D 66 044010
  Gibbons G W, Ida D and Shiromizu T 2003 Prog. Theor. Phys. Suppl. 148 284
[13] Rogatko M 2003 Phys. Rev. D 67 084025
Rogatko M 2004 Phys. Rev. D 70 044023

[14] Kodama H 2004 Prog. Theor. Phys. 112 249

[15] Gutowski J B 2004 J. High Energy Phys. JHEP08(2004)049

[16] Elvang H, Emparan R, Mateos D and Reall H S 2004 Phys. Rev. Lett. 93 211302

[17] Brihaye Y, Chakrabarti A, Hartmann B and Tchrakian D H 2003 Phys. Lett. B 561 161

[18] Volkov M S and Galtsov D V 1989 JETP Lett. 50 346
Bizon P 1990 Phys. Rev. Lett. 64 2844

[19] Hartmann B 2004 Phys. Lett. B 602 231

[20] Volkov M S 2002 Phys. Lett. B 524 369

[21] Wiseman T 2003 Class. Quantum Grav. 20 1117

[22] Brihaye Y, Hartmann B and Radu E 2005 Phys. Rev. D 71 085002

[23] Hartmann B, Kleihaus B and Kunz J 2002 Phys. Rev. D 65 024027

[24] Ashtekar A, Corichi A and Sudarsky D 2001 Class. Quantum Grav. 18 919

[25] Brihaye Y, Hartmann B and Radu E 2005 Black strings in (4 + 1)-dimensional Einstein–Yang–Mills theory
Preprint hep-th/0508028 (Phys. Rev. D at press)

[26] Brihaye Y and Hartmann B 2002 Phys. Lett. B 534 137

[27] Breitenlohner P, Forgacs P and Maison D 1992 Nucl. Phys. B 383 357
Breitenlohner P, Forgacs P and Maison D 1995 Nucl. Phys. B 442 126