Coefficientwise Hankel-total positivity of the row-generating polynomials for the output matrices of certain production matrices

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Abstract

Total positivity of matrices is deeply studied and plays an important role in various branches of mathematics. The aim of this paper is to study the criteria for coefficientwise Hankel-total positivity of the row-generating polynomials of generalized m-Jacobi-Rogers triangles and their applications.

Using the theory of production matrices, we present the criteria for coefficientwise Hankel-total positivity of the row-generating polynomials of the output matrices of certain production matrices. In particular, we gain a criterion for coefficientwise Hankel-total positivity of the row-generating polynomial sequence of the generalized m-Jacobi-Rogers triangle. This immediately implies that the corresponding generalized m-Jacobi-Rogers triangular convolution preserves the Stieltjes moment property of sequences and its zeroth column sequence is coefficientwise Hankel-totally positive and log-convex of higher order in all the indeterminates. In consequence, for \( m = 1 \), we immediately obtain some results on Hankel-total positivity for the Catalan-Stieltjes matrices. In particular, we in a unified manner apply our results to some combinatorial triangles or polynomials including the generalized Jacobi Stirling triangle, a generalized elliptic polynomial, a refined Stirling cycle polynomial and a refined Eulerian polynomial. For the general \( m \), combining our criterion and a function satisfying an autonomous differential equation, we present different criteria for coefficientwise Hankel-total positivity of the row-generating polynomial sequence of exponential Riordan arrays. In addition, we also derive some results for coefficientwise Hankel-total positivity in terms of compositional functions and \( m \)-branched Stieltjes-type continued fractions. Finally, we apply our criteria to: (1) rook polynomials and signless Laguerre polynomials (confirming a conjecture of Sokal on coefficientwise Hankel-total positivity of rook polynomials), (2) labeled trees and forests (proving some conjectures of Sokal on total positivity and Hankel-total positivity), (3) \( r \)-th-order Eulerian polynomials (giving a new proof for the coefficientwise Hankel-total positivity of \( r \)-th-order Eulerian polynomials, which in particular implies the conjecture of Sokal on the coefficientwise Hankel-total positivity of reversed 2-th-order Eulerian polynomials), (4) multivariate Ward polynomials,

*Supported partially by the National Natural Science Foundation of China (No. 12022105) and the Natural Science Foundation for Distinguished Young Scholars of Jiangsu Province (No. BK20200048).

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labeled series-parallel networks and nondegenerate fanout-free functions, (5) an array from the Lambert function and a generalization of Lah numbers and associated triangles, and so on.

**MSC:** 05A20; 05A15; 15B48; 30B70; 44A60; 15B05; 30E05

**Keywords:** Total positivity; Hankel-total positivity; Toeplitz-total-positivity; Stieltjes moment sequences; Pólya frequency; 3-x-log-convexity; Convolutions; Production matrices; Exponential Riordan arrays; Stieltjes-type continued fractions; Jacobi-type continued fractions; Branched continued fractions; Stieltjes-Rogers polynomials; Jacobi-Rogers polynomials; Laguerre polynomials; rth-order Eulerian polynomials; Multivariate Ward polynomials; Elliptic functions; Lambert functions

## Contents

1 **Introduction**  
1.1 Total positivity  
1.2 The $m$-Stieltjes–Rogers triangle and $m$-Jacobi–Rogers triangle  

2 **Total positivity from production matrices**  

3 **Binomial-totally positive matrices**  

4 **Row-generating polynomials of generalized $m$-Jacobi–Rogers triangles**  
4.1 Coefficientwise Hankel-total positivity of row-generating polynomials  
4.2 Combinatorial interpretations for row-generating polynomials  

5 **Row-generating polynomials of Catalan-Stieltjes matrices**  
5.1 Main results for Catalan-Stieltjes matrices  
5.2 A generalized Jacobi-Stirling triangle  
5.3 A generalized elliptic polynomial  
5.4 A refined Stirling cycle polynomial  
5.5 A refined Eulerian polynomial  

6 **Row-generating polynomials of exponential Riordan arrays**  
6.1 Main results for exponential Riordan arrays  
6.2 Rook polynomials and signless Laguerre polynomials  
6.3 Enumerative labeled trees and forests  
6.4 Stirling permutations and $r$th-order Eulerian polynomials  
6.5 Multivariate Ward polynomials  
6.6 Labeled series-parallel networks and nondegenerate fanout-free functions  
6.7 An array from the Lambert function  
6.8 A generalization of Lah numbers  

7 **Acknowledgements**
1 Introduction

Recently, in [76], based on lattice paths and branched continued fractions, Pétréolle, Sokal and Zhu developed the theory for coefficientwise Hankel-total positivity of the $m$-Stieltjes-Rogers polynomial sequence in all the indeterminates. Note that an $m$-Stieltjes-Rogers polynomial sequence coincides with the zeroth column of certain $m$-Jacobi-Rogers triangle. The purpose of this paper is to consider a more generalized problem whether the row-generating polynomial sequence of the generalized $m$-Jacobi-Rogers triangles is coefficientwise Hankel-totally positive. Since the generalized $m$-Jacobi-Rogers triangle can be viewed as the output matrix of certain lower-Hessenberg matrix, in terms of the theory of production matrices, in general, total positivity of production matrices implies coefficientwise Hankel-total positivity of the zeroth column of the output matrices, but does not deduce that of its row-generating polynomials. This motivates us to consider the stronger properties of production matrices and propose a stronger concept: binomial-total positivity. We develop some criteria for coefficientwise Hankel-total positivity of the row-generating polynomials of output matrices and then present many applications for the generalized $m$-Jacobi-Rogers triangle.

The organization of this paper is as follows. We introduce the background and definitions from total positivity in subsection 1.1 and give the definitions of the $m$-Stieltjes-Rogers triangle, the $m$-Jacobi-Rogers triangle and the generalized $m$-Jacobi-Rogers triangle in terms of lattice paths in subsection 1.2. In Section 2, we introduce the general theory for total positivity from production matrices and present our criteria to coefficientwise Hankel-total positivity of the row-generating polynomials of output matrices. We provide some kinds of binomial-totally positive matrices in Section 3 and use such matrices to give the general criteria for coefficientwise Hankel-total positivity of the row-generating polynomials of the generalized $m$-Jacobi-Rogers triangle in Section 4. We apply our criteria to a variety of combinatorial triangles from Catalan-Stieltjes matrices in Section 5 and exponential Riordan arrays in Section 6 in a unified viewpoint.

1.1 Total positivity

Total positivity of matrices has been deeply studied and plays an important role in various branches of mathematics. The literature on this fascinating subject is extensive such as classical analysis [86], representation theory [63, 64, 65, 82], network analysis [80], cluster algebra [10, 41], positive Grassmannians and integrable systems [57, 58], combinatorics [15, 45]. We also refer the reader to monographs [53, 77] for more details.

A matrix of real numbers is **totally positive** if all its minors are nonnegative and is **totally positive of order** $r$ if all minors of order $k \leq r$ are nonnegative. For a sequence $a = (a_n)_{n \geq 0}$, denote by $\Gamma(a) = [a_{i-j}]_{i,j \geq 0}$ its **Toeplitz matrix** and by $H(a) = [a_{i+j}]_{i,j \geq 0}$ its **Hankel matrix**. Their total positivity plays an important role in different fields.

Total positivity of $\Gamma(a)$ can characterize the Pólya frequency of $\mathbf{a}$. In classical analysis, the sequence $\mathbf{a}$ with $a_0 = 1$ of nonnegative real numbers is called a **Pólya frequency** sequence if its generating function has the form

$$\sum_{n \geq 0} a_n z^n = \frac{\prod_{j \geq 1} (1 + \alpha_j z)}{\prod_{j \geq 1} (1 - \beta_j z)} e^{\gamma z} \quad (1.1)$$
in some open disk centered at the origin, where \( \alpha_j, \beta_j, \gamma \geq 0 \) and \( \sum_{j \geq 1} (\alpha_j + \beta_j) < +\infty \), see Karlin [53, pp. 412] for instance. We say that \( \sum_{n \geq 0} a_n t^n \) is a **Pólya frequency ogf** in \( \mathbb{R}[[t]] \) if the sequence \( a \) is a Pólya frequency sequence. The Pólya frequency is closely related to polynomials with only real zeros. In fact, from the function representation (1.1) for the Pólya frequency, it in particular implies that \( a_0, a_1, \ldots, a_n, 0, 0, \ldots \) is a Pólya frequency sequence if and only if \( \sum_{k=0}^n a_k z^k \) has only real zeros ([53, pp. 399]). It is well known that the sequence \( a \) is a Pólya frequency sequence if and only if its infinite Toeplitz matrix \( \Gamma(a) \) is totally positive. We refer the reader to Brändén [11], Brenti [12, 13, 15] and Wang-Yeh [101] for the Pólya frequency in combinatorics.

Total positivity of \( H(a) \) is closely related to the Stieltjes moment property of \( a \). The sequence \( a \) of real numbers is a **Stieltjes moment sequence** if it has the form

\[
a_k = \int_0^{+\infty} x^k d\mu(x),
\]

where \( \mu \) is a non-negative measure on \([0, +\infty)\) (see [77, Theorem 4.4] for instance). The Stieltjes moment problem is one of classical moment problems and arises naturally in many branches of mathematics [88, 103]. Stieltjes proved that \( a \) is a Stieltjes moment sequence if and only if there exist nonnegative numbers \( \alpha_0, \alpha_1, \ldots \) such that \( \sum_{n \geq 0} a_n z^n \) has the Stieltjes-type continued fraction

\[
\sum_{n \geq 0} a_n z^n = \frac{1}{1 - \frac{\alpha_0 z}{1 - \frac{\alpha_1 z}{1 - \ldots}}}
\]

in the sense of formal power series. It is well known that an equivalent characterization for \( a \) being a Stieltjes moment sequence is that its Hankel matrix \( H(a) \) is totally positive [77]. The sequence \( a \) is **log-convex** if \( a_{k-1} a_{k+1} \geq a_k^2 \) for all \( k \geq 1 \). Clearly, a sequence of positive numbers is log-convex if and only if \( H(a) \) is totally positive of order two. Thus, log-convexity is implied by its Stieltjes moment property. In fact, log-convexity of many combinatorial sequences can be extended to Stieltjes moment property. See Liu and Wang [60] and Zhu [104] for log-convexity and Liang et al. [59], Wang and Zhu [102] and Zhu [108, 109] for Stieltjes moment property in combinatorics.

In fact, this is only the beginning of the story. Because some combinatorial objects with respect to one or more statistics often generate multivariate polynomials, our more interest is to study sequences and matrices of polynomials in one or more indeterminates \( x \). Let \( \mathbb{R} \) denote the set of all real numbers and \( x = \{x_i\}_{i \in I} \) be a set of indeterminates. A matrix \( M \) with entries in \( \mathbb{R}[x] \) is **coefficientwise totally positive** in \( x \) (we also call it \( x \)-totally positive) if all its minors are polynomials with nonnegative coefficients in the indeterminates \( x \) and is **coefficientwise totally positive of order \( r \) in \( x \) (we also call it an \( x \)-totally positive of order \( r \)) if all its minors of order \( k \leq r \) are polynomials with nonnegative coefficients in the indeterminates \( x \). A sequence \( (a_n(x))_{n \geq 0} \) in \( \mathbb{R}[x] \) is **coefficientwise Toeplitz totally positive** in \( x \) (we also call it an \( x \)-Pólya frequency sequence) if its associated infinite Toeplitz matrix is coefficientwise totally positive in \( x \). A sequence \( (a_n(x))_{n \geq 0} \) in \( \mathbb{R}[x] \) is **coefficientwise Hankel totally positive** in \( x \) (we also call it \( x \)-Stieltjes moment) if its associated infinite Hankel matrix is coefficientwise
totally positive. Similarly, we have coefficientwise Hankel totally positive of order $r$ and coefficientwise Toeplitz totally positive of order $r$ in $x$. It is $x$-log-convex if all coefficients of $\alpha_{n+1}(x)\alpha_{n-1}(x) - \alpha_n(x)^2$ are nonnegative for all $n \geq 1$. Clearly, an $x$-Stieltjes moment sequence is $x$-log-convex. Define an operator $L$ by

$$L[\alpha_i(x)] := \alpha_{i-1}(x)\alpha_{i+1}(x) - \alpha_i(x)^2$$

for $i \geq 1$. In general, we say that $(\alpha_i(x))_{i \geq 0}$ is $k$-$x$-log-convex if the coefficients of $L^m[\alpha_i(x)]$ are nonnegative for all $m \leq k$, where $L^m = L(L^{m-1})$. If $x$ contains a unique indeterminate $q$, then they reduce to be $q$-log-convex, $k$-$q$-log-convex and $q$-Stieltjes moment, respectively. More and more combinatorial polynomials were proved to have such properties. For example, the Bell polynomials, the classical Eulerian polynomials, the Narayana polynomials of type A and B, Dowling polynomials, Jacobi-Stirling polynomials, and so on, are $q$-log-convex (see Chen et al. [18, 19], Liu and Wang [60], Zhu [104, 105, 106, 107], Zhu and Sun [114] for instance), $3$-$q$-log-convex (see [108]) and $q$-Stieltjes moment (see [102, 108, 110]). We refer the reader to [75, 76, 91, 92, 111, 112, 113] for coefficientwise Hankel-total positivity in more indeterminates.

1.2 The $m$-Stieltjes–Rogers triangle and $m$-Jacobi–Rogers triangle

The $m$-Stieltjes-Rogers polynomial was the zeroth column of the $m$-Stieltjes–Rogers triangle. In [76], the $m$-Stieltjes–Rogers triangle was defined in terms of $m$-Dyck paths. Recall the definition of the $m$-Dyck path as follows [6, 17, 81]:

**Definition 1.1.** Fix an integer $m \geq 1$. A partial $m$-Dyck path is a path in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, starting on the horizontal axis but ending anywhere, using steps $(1, 1)$ ("rise") and $(1, -m)$ ("$m$-fall"). In particular, a partial $m$-Dyck path ending on the horizontal axis is called the $m$-Dyck path: see Figure 1 for an example.

![Figure 1. A 3-Dyck path of length 10.](image)

In particular, for $m = 1$, the 1-Dyck path is the well-known Dyck path. The definitions of the $m$-Stieltjes–Rogers triangle and the $m$-Stieltjes–Rogers polynomial [76] can be stated as follows:

**Definition 1.2.** Fix an integer $m \geq 1$, and let $\alpha = (\alpha_i)_{i \geq m}$ be an infinite set of indeterminates. Let $S_{n,k}^{(m)}(\alpha)$ be the generating polynomial for partial $m$-Dyck paths from $(0,0)$...
to \(((m + 1)n, (m + 1)k)\) in which each rise gets weight 1 and each \(m\)-fall from height \(i\) gets weight \(\alpha_i\). We call the infinite unit-lower-triangular array \(S^{(m)} = [S^{(m)}_{n,k}(\alpha)]_{n,k}\) the \(m\)-Stieltjes–Rogers triangle. In particular, the zeroth column \(S^{(m)}_{n,0}(\alpha)\) is called the \(m\)-Stieltjes–Rogers polynomial of order \(n\), denoted by \(S^{(m)}_n(\alpha)\).

Generally, \(S^{(m)}_n(\alpha)\) is a homogeneous polynomial of degree \(n\) with nonnegative integer coefficients. When \(\alpha_i = 1\) for all \(i \geq m\), the \(m\)-Stieltjes–Rogers polynomials \(S^{(m)}_n(1)\) reduce to the famous Fuss-Catalan numbers of order \(p = m + 1\): 

\[
\frac{1}{pm+1}(\frac{pm+1}{n})
\]

Let \(f_0(t) = \sum_{n=0}^{\infty} S^{(m)}_n(\alpha) t^n\) be the ordinary generating function for \(m\)-Dyck paths with these weights. It is known that \(f_0\) is given by the \(m\)-branched Stieltjes-type continued fraction [76]:

\[
f_0(t) = \frac{1}{1 - \alpha_m t \prod_{i_1=1}^{m} (1 - \frac{1}{1 - \alpha_{m+i_1} t \prod_{i_2=1}^{m} (1 - \frac{1}{1 - \alpha_{m+i_1+i_2} t \prod_{i_3=1}^{m} (1 - \ldots )}}),}
\]

which for \(m = 1\) in particular reduces to the classical Stieltjes-type continued fraction \((1.3)\).

The following result on Hankel-total positivity for \(m\)-Stieltjes–Rogers polynomials was one of main results in [76].

**Theorem 1.3** (Hankel-total positivity for \(m\)-Stieltjes–Rogers polynomials). For each integer \(m \geq 1\), the sequence \((S^{(m)}_n(\alpha))_{n \geq 0}\) of \(m\)-Stieltjes–Rogers polynomials is a coefficientwise Hankel-totally positive sequence in the polynomial ring \(\mathbb{Z}[\alpha]\).

From [76], we know that the \(m\)-Stieltjes–Rogers triangle can be viewed as a special \(m\)-Jacobi–Rogers triangle defined by the following well known \(m\)-Lukasiewicz path.

**Definition 1.4.** Fix \(1 \leq m \leq \infty\). A partial \(m\)-Lukasiewicz path is a path in the upper half-plane \(\mathbb{Z} \times \mathbb{N}\), starting on the horizontal axis but ending anywhere, using steps \((1, r)\) with \(-m \leq r \leq 1\): the allowed steps are thus \(r = 1\) ("rise"), \(r = 0\) ("level step"), and \(r = -\ell\) for any \(\ell > 0\) ("\(\ell\)-fall"). In particular, a partial \(m\)-Lukasiewicz path starting and ending on the horizontal axis is called the \(m\)-Lukasiewicz path. See Figure 2 for an example.

![Figure 2. A 3-Lukasiewicz path of length 12.](image)
In what follows we introduce the \( m \)-Jacobi–Rogers triangle in [76] and define a slightly generalized \( m \)-Jacobi–Rogers triangle.

**Definition 1.5.** Fix \( 1 \leq m \leq \infty \), and let \( \beta = (\beta_i^{(t)})_{-1 \leq t \leq m, i \geq t} \) be indeterminates. Let \( J_{n,k}^{(m)}(\beta) \) be the generating polynomial for partial \( m \)-Lukasiewicz paths from \( (0,0) \) to \( (n,k) \) in which each rise at height \( i \) gets weight \( \beta_i^{(-1)} \), each level step at height \( i \) gets weight \( \beta_i^{(0)} \), and each \( \ell \)-fall from height \( i \) gets weight \( \beta_i^{(t)} \). Let the infinite lower-triangular array \( J^{(m)} = (J_{n,k}^{(m)}(\beta))_{n,k \geq 0} \) be the **generalized \( m \)-Jacobi–Rogers triangle** and call its zeroth column \( J_{n,0}^{(m)}(\beta) \) the **generalized \( m \)-Jacobi–Rogers polynomial** of order \( n \), denoted by \( J_n^{(m)}(\beta) \).

For all \( \beta_i^{(-1)} = 1 \), \( J^{(m)} \) reduces to the **\( m \)-Jacobi–Rogers triangle** and the zeroth column \( J_{n,0}^{(m)}(\beta) \) reduces to the **\( m \)-Jacobi–Rogers polynomial** of order \( n \) in [76]. Though here is a slight generalization, it will provide more plentiful combinatorial triangles whose row-generating polynomials are coefficientwise Hankel-totally positive (see Section 5).

Let \( f_0(t) = \sum_{n=0}^{\infty} J_{n,0}^{(m)}(\beta) t^n \) be the ordinary generating function for the \( m \)-Jacobi–Rogers polynomials. For \( m = 1 \), we have the classical Jacobi-type continued fraction:

\[
f_0(t) = \frac{1}{1 - \beta_0^{(0)} t - \frac{\beta_0^{(-1)} \beta_1^{(1)} t^2}{1 - \beta_1^{(0)} t - \frac{\beta_1^{(-1)} \beta_2^{(1)} t^2}{1 - \beta_2^{(0)} t - \frac{\beta_2^{(-1)} \beta_3^{(1)} t^2}{1 - \ldots}}}}
\] (1.5)

which was studied by Flajolet [39] and where the \( 1 \)-Jacobi–Rogers polynomials are the classical Jacobi–Rogers polynomials [39]. For \( m > 1 \), \( f_0(t) \) can be written as an \( m \)-branched Jacobi-type continued fraction (see [99, pp. V-39–V-40], [84, pp. 22–24, 143], [98, pp. 5] for instance).

For \( 1 \leq m \leq \infty \), in terms of the weights of partial \( m \)-Lukasiewicz paths, we define a matrix \( P^{(m)}(\beta) \) as follows: \( P^{(m)}(\beta) \) is the \( m \)-banded lower-Hessenberg matrix with \( \beta_i^{(-1)} \) on the superdiagonal and \( i \geq 0 \), \( \beta_0^{(0)}, \beta_1^{(0)}, \ldots \) on the diagonal, \( \beta_1^{(1)}, \beta_2^{(1)}, \ldots \) on the first subdiagonal, \( \beta_2^{(2)}, \beta_3^{(2)}, \ldots \) on the second subdiagonal, etc., and in general

\[
P^{(m)}(\beta)_{ij} = \begin{cases} \beta_i^{(i-j)} & \text{if } i - m \leq j \leq i + 1 \\ 0 & \text{otherwise} \end{cases}
\] (1.6)

or in other words

\[
P^{(m)}(\beta) = \begin{bmatrix} \beta_0^{(0)} & \beta_0^{(-1)} \\ \beta_1^{(1)} & \beta_1^{(0)} & \beta_1^{(-1)} \\ \beta_2^{(2)} & \beta_2^{(1)} & \beta_2^{(0)} & \beta_2^{(-1)} \\ \beta_3^{(3)} & \beta_3^{(2)} & \beta_3^{(1)} & \beta_3^{(0)} & \beta_3^{(-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\] (1.7)

Obviously, \( P^{(\infty)}(\beta) \) is simply the generic lower-Hessenberg matrix.
The $m$-Jacobi–Rogers triangle $J^{(m)}$ can be viewed as the output matrix of $P^{(m)}(\beta)$ for all $\beta^{(i-1)}_1 = 1$ [76]. Then the theory of production matrices provides a powerful approach to coefficientwise Hankel-total positivity of $m$-Jacobi–Rogers polynomials $J^{(m)}_n(\beta)$.

2 Total positivity from production matrices

In recent years, the method of production matrices [27, 28] has become an important tool in enumerative combinatorics. It dates back to the work (the Karlin iteration) of Karlin [53, section 3.6, pp. 132-140]. It also rose in denumerable Markov chains which can only move one step to the right [56] and played an important role in continued fractions (Stieltjes [97]).

Following [76], a partially ordered commutative ring is a pair $(R, \mathcal{P})$ where $R$ is a commutative ring and $\mathcal{P}$ is a subset of $R$ satisfying the following three conditions: (1) $0,1 \in \mathcal{P}$. (2) If $a,b \in \mathcal{P}$, then $a+b \in \mathcal{P}$ and $ab \in \mathcal{P}$. (3) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. We call $\mathcal{P}$ the nonnegative elements of $R$, and we define a partial order on $R$ (compatible with the ring structure) by writing $a \leq b$ as a synonym for $b-a \in \mathcal{P}$.

Now let $(R,\mathcal{P})$ be a partially ordered commutative ring and let $\mathbf{x} = \{x_i\}_{i \in I}$ be a collection of indeterminates. In the polynomial ring $R[\mathbf{x}]$ and the formal-power-series ring $R[[\mathbf{x}]]$, let $\mathcal{P}[\mathbf{x}]$ and $\mathcal{P}[[\mathbf{x}]]$ be the subsets consisting of polynomials (resp. series) with nonnegative coefficients. Then $(R[\mathbf{x}], \mathcal{P}[\mathbf{x}])$ and $(R[[\mathbf{x}]], \mathcal{P}[[\mathbf{x}]])$ are partially ordered commutative rings; we refer to this as the coefficientwise order on $R[\mathbf{x}]$ and $R[[\mathbf{x}]]$.

A (finite or infinite) matrix with entries in a partially ordered commutative ring is called totally positive if all its minors are nonnegative; it is called totally positive of order $r$ if all its minors of size $\leq r$ are nonnegative. It follows immediately from the Cauchy–Binet formula that the product of two totally positive (resp. totally positive of order $r$) matrices is totally positive (resp. totally positive of order $r$).

We say that a sequence $\mathbf{a} = (a_n)_{n \geq 0}$ with entries in a partially ordered commutative ring is Hankel-totally positive (resp. Hankel-totally positive of order $r$) if its associated infinite Hankel matrix $H(\mathbf{a}) = [a_{i+j}]_{i,j \geq 0}$ is totally positive (resp. totally positive of order $r$). It is Toeplitz-totally positive (resp. Toeplitz-totally positive of order $r$) if its associated infinite Toeplitz matrix $\Gamma(\mathbf{a}) = [a_{i-j}]_{i,j \geq 0}$ is totally positive (resp. totally positive of order $r$). We also say that $f(t) = \sum_n f_nt^n$ is a Pólya frequency ogf (resp. Pólya frequency ogf of order $r$) in $R[[t]]$ if $(f_n)_{n \geq 0}$ is Toeplitz-totally positive (resp. Toeplitz-totally positive of order $r$).

Let $\mathbf{P} = (p_{ij})_{i,j \geq 0}$ be an infinite matrix with entries in a partially ordered commutative ring $R$. Assume that $\mathbf{P}$ is either row-finite (i.e. has only finitely many nonzero entries in each row) or column-finite. Define an infinite matrix $\mathbf{A} = (a_{nk})_{n,k \geq 0}$ by

$$a_{nk} = (\mathbf{P}^n)_{0k}$$

(2.1)

(in particular, $a_{0k} = \delta_{0k}$). We can write $a_{n,k}$ as

$$a_{nk} = \sum_{i_1,\ldots,i_{n-1}} p_{0i_1} p_{i_1i_2} p_{i_2i_3} \ldots p_{i_{n-2}i_{n-1}} p_{i_{n-1}k} .$$

(2.2)
We also have another equivalent formulation for $a_{nk}$ defined by the recurrence

$$a_{nk} = \sum_{i=0}^{\infty} a_{n-1,i} p_{ik} \quad \text{for } n \geq 1 \quad (2.3)$$

with the initial condition $a_{0k} = \delta_{0k}$. We call $P$ the **production matrix** and $A$ the **output matrix** of $P$, and we write $A = \mathcal{O}(P)$. Given a matrix $M$, define a matrix, denoted by $\overline{M}$, obtained from the matrix $M$ with the first row removed. So we have

$$\overline{\mathcal{O}(P)} = \mathcal{O}(P)P. \quad (2.4)$$

When $P$ is coefficientwise totally positive, the output matrix $\mathcal{O}(P)$ has two total-positivity properties as follows:

**Theorem 2.1** (Total positivity of the output matrix). [76, 92] Let $P$ be a row-finite or column-finite matrix with entries in a partially ordered commutative ring $R$. If $P$ is totally positive of order $r$ ($1 \leq r \leq \infty$), then

(i) the output matrix $\mathcal{O}(P)$ is totally positive of order $r$ ($1 \leq r \leq \infty$);

(ii) the zeroth column of the output matrix $\mathcal{O}(P)$ is Hankel-totally positive of order $r$.

Let $B_q = \left(\binom{n}{k}q^{n-k}\right)_{n,k}$. For a matrix $M = [M_{n,k}]_{n,k}$, define an associated matrix

$$M(q) := MB_q.$$ 

Following [91], we call the matrix $M(q)$ the **binomial row-generating matrix**. Let $M(q) = [M_{n,k}(q)]_{n,k}$. Obviously, we have

$$M_{n,k}(q) = \sum_{i} M_{n,i} \binom{i}{k} q^{i-k}. \quad (2.5)$$

Assume that $P$ is a row-finite or column-finite matrix with entries in a partially ordered commutative ring $R$ and its output matrix $\mathcal{O}(P)$ is $M$. Since the binomial matrix $B_q$ is coefficientwise totally positive in the indeterminate $q$, it follows immediately from Theorem 2.1 (i) and the Cauchy-Binet theorem that:

(i') $M(q)$ is totally positive of order $r$ in the ring $R[q]$ equipped with the coefficientwise order.

Note that the **row-generating polynomials** $M_n(q) = \sum_{k=0}^{n} M_{n,k}q^k$ of $M$ locate exactly in the zeroth column of $M(q)$. It is natural to ask whether the row-generating polynomials $M_n(q)$ are Hankel-totally positive of order $r$ or not. However, the foregoing hypotheses are not sufficient to imply that the row-generating polynomials are Hankel-totally positive in the ring $R[q]$ equipped with the coefficientwise order, or even Hankel-totally positive in the ring $R$ when $q$ is specialized to strictly positive values. The following simple example [91, Example 2.15] shows this:
Example 2.2. (Failure of Hankel-totally positive of the row-generating polynomials). Let \( P = e_{00} + \Delta \) be the upper-bidiagonal matrix with 1 on the superdiagonal and 1, 0, 0, 0, \ldots on the diagonal; it is totally positive because every bidiagonal matrix with nonnegative entries is totally positive [91, Lemma 2.1]. Then \( A = O(P) \) is the lower-triangular matrix with all entries 1, so that \( A_n(q) = \sum_{k=0}^n q^k \). Since \( A_0(q)A_2(q) - A_1(q)^2 = -q \), the sequence \( (A_n(q))_{n \geq 0} \) is not even log-convex (i.e. Hankel-totally positive of order 2) for any real number \( q > 0 \).

Thus, in order to consider the Hankel-total positivity of row-generating polynomials of \( O(P) \), we need a stronger condition on \( P \), namely:

**Definition 2.3.** Let \( P \) be a row-finite matrix with entries in a commutative ring \( R \). We say that \( P \) is **binomial-totally positive of order** \( r \) \((1 \leq r \leq \infty)\) if the matrix \( B_q^{-1}PB_q \) is totally positive of order \( r \) in the ring \( R[q] \) equipped with the coefficientwise order.

Clearly, specializing to \( q = 0 \), we see that binomial-total positivity implies total positivity; but in general it is much stronger. We will develop some ideas in [75, 91, 113] to be further. The reason for the appearance here of the combination \( B_q^{-1}PB_q \) is the following simple fact [91, Lemma 2.6]: Let \( P \) be a row-finite matrix with entries in a commutative ring \( R \), with its output matrix \( A = O(P) \) and let \( B = (b_{ij})_{i,j \geq 0} \) be a lower-triangular matrix with invertible (in \( R \)) diagonal entries. Then \( AB = b_{00}O(B^{-1}PB) \).

Applying this fact to the case \( B = B_q \), working in the ring \( R[q] \) equipped with the coefficientwise order, by Theorem 2.1, we immediately deduce:

**Theorem 2.4 (Hankel-total positivity of row-generating polynomials).** Let \( P \) be a row-finite or column-finite matrix with entries in a partially ordered commutative ring \( R \). If \( P \) is binomial-totally positive of order \( r \) \((1 \leq r \leq \infty)\), then

(ii') the sequence of row-generating polynomials of \( O(P) \) is Hankel-totally positive of order \( r \) in the polynomial ring \( R[q] \) equipped with the coefficientwise order.

Note that for any two matrices \( P_1 \) and \( P_2 \), we have

\[
B_q^{-1}P_1P_2B_q = B_q^{-1}P_1B_q^{-1}P_2B_q.
\]

So, by the Cauchy-Binet theorem, the product of two binomial-totally positive matrices is still binomial-totally positive. We immediately have:

**Theorem 2.5.** Let \( P \) be a row-finite or column-finite matrix. If \( P \) is the product of some matrices being binomial-totally positive of order \( r \) \((1 \leq r \leq \infty)\) with entries in a partially ordered commutative ring \( R \), then

(i) \( M(q) \) is totally positive of order \( r \) in the polynomial ring \( R[q] \) equipped with the coefficientwise order;

(ii) the sequence of row-generating polynomials of \( O(P) \) is Hankel-totally positive of order \( r \) in the polynomial ring \( R[q] \) equipped with the coefficientwise order.

Theorem 2.4 and Theorem 2.5 are our main tools to prove the Hankel-total positivity of row-generating polynomials.
3 Binomial-totally positive matrices

By Theorem 2.5, it is natural to study which kind of matrices are binomial-totally positive. In this section, we will provide some new kinds of matrices that are binomial-totally positive.

Define \( L(a, b, c) \) to be a lower-bidiagonal matrix with \( ka + b \) for \( k \geq 0 \) on the diagonal and \( kc \) for \( k \geq 1 \) on the subdiagonal:

\[
L(a, b, c) = \begin{bmatrix}
    b & 2a + b & & \\
    c & 2a + b & 3c & & \\
                  & 3a + b & \ddots & \ddots & \\
    \end{bmatrix}
\]  

(3.1)

For the matrix \( L(a, b, c) \), it is totally positive in the ring \( \mathbb{Z}[a, b, c] \) equipped with the coefficientwise order; this is obvious since it is a bidiagonal matrix with entries that are (coefficientwise) nonnegative. The following result shows that \( L(a, b, c) \) is also binomial-totally positive in the ring \( \mathbb{Z}[a, b, c] \) equipped with the coefficientwise order.

**Proposition 3.1.** Let \( L(a, b, c) \) be defined by (3.1). Then we have

\[
B_q^{-1}L(a, b, c)B_q = L(a, b, c + qa)
\]

(3.2)

and the matrix \( L(a, b, c) \) is binomial-totally positive in the ring \( \mathbb{Z}[a, b, c] \) equipped with the coefficientwise order.

**Proof.** By (3.2), the total positivity of \( L(a, b, c) \) in the ring \( \mathbb{Z}[a, b, c] \) equipped with the coefficientwise order immediately implies that \( L(a, b, c) \) is binomial-totally positive in the ring \( \mathbb{Z}[a, b, c] \) equipped with the coefficientwise order. So it suffices to prove (3.2). We will show that

\[
L(a, b, c)B_q = B_qL(a, b, c + qa).
\]

Since

\[
[L(a, b, c)]_{n,n} = b + na, \quad [L(a, b, c)]_{n,n-1} = nc,
\]

we have

\[
[L(a, b, c)B_q]_{n,k} = (b + na)\binom{n}{k}q^{n-k} + nc\binom{n-1}{k}q^{n-1-k}
\]

(3.4)

\[
[B_qL(a, b, c + qa)]_{n,k} = \binom{n}{k}q^{n-k}(b + ka) + \binom{n}{k+1}q^{n-1-k}(k + 1)(c + qa)
\]

(3.5)

\[
= \binom{n}{k}q^{n-k}(b + ka) + n(c + qa)\binom{n-1}{k}q^{n-1-k}.
\]

(3.6)

In consequence, we obtain

\[
[L(a, b, c)B_q]_{n,k} - [B_qL(a, b, c + qa)]_{n,k} = 0.
\]

(3.7)

This completes the proof. \( \square \)
Assume that $U(a,b,u,v)$ is a upper-bidiagonal matrix with $ka+b$ for $k \geq 0$ on the diagonal and $ku+v$ for $k \geq 1$ on the superdiagonal:

$$U(a,b,u,v) = \begin{bmatrix} b & v \\ a+b & u+v \\ 2a+b & 2u+v \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$  

(3.8)

Suppose that $J(a,b,u,v,\lambda)$ is the following tri-diagonal matrix

$$J(a,b,u,v,\lambda) = \begin{bmatrix} s_0 & r_0 \\ t_1 & s_1 & r_1 \\ t_2 & s_2 & r_2 \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$  

(3.9)

where $r_n = un+v$, $s_n = (2\lambda u+a)n+b+\lambda v$ and $t_n = \lambda(a+\lambda u)n$.

Obviously, $U(a,b,u,v) = J(a,b,u,v,0)$. Since $U$ is bidiagonal, it follows immediately that $U$ is totally positive in the ring $\mathbb{Z}[a,b,u,v]$ equipped with the coefficientwise order. But it is less obvious that $J(a,b,u,v,\lambda)$ is totally positive in the ring $\mathbb{Z}[a,b,u,v,\lambda]$ equipped with the coefficientwise order. In fact, the following result gives the stronger property that $J(a,b,u,v,\lambda)$ is binomial-totally positive in the ring $\mathbb{Z}[a,b,u,v,\lambda]$ equipped with the coefficientwise order.

**Proposition 3.2.** Let $J(a,b,u,v,\lambda)$ be defined by (3.9). Then

(i) the matrix $J(a,b,u,v,\lambda)$ is totally positive in the ring $\mathbb{Z}[a,b,u,v,\lambda]$ equipped with the coefficientwise order.

(ii) we have

$$B^{-1}_q J(a,b,u,v,\lambda) B_q = J(a,b,u,v,\lambda+q).$$  

(3.10)

Therefore, the matrix $J(a,b,u,v,\lambda)$ is binomial-totally positive in the ring $\mathbb{Z}[a,b,u,v,\lambda]$ equipped with the coefficientwise order.

**Proof.** (i) Consider the matrix

$$J^* = \begin{bmatrix} s_0 & r^*_0 \\ t^*_1 & s_1 & r^*_1 \\ t^*_2 & s_2 & r^*_2 \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$  

(3.11)

where $r^*_n = \lambda(un+v-u)$, $s_n = [2\lambda u+a]n+b+\lambda v$ and $t^*_n = [a+\lambda u]n$. It satisfies $r_it_{i+1} = r^*_it^*_i$ for all $i$, so by [111, Claim 2 in the proof of Theorem 2.1], $J(a,b,u,v,\lambda)$ is totally positive in the ring $\mathbb{Z}[a,b,u,v,\lambda]$ equipped with the coefficientwise order if and only if $J^*$ is so. On the other hand, the matrix $J^*$ is coefficientwise row-diagonally-dominant (that is, the coefficients of $s_n^*-r^*_n-t^*_n$ are nonnegative), so by [111, Proposition
3.3 (i)], it follows that $J^*$ is totally positive in the ring $\mathbb{Z}[a, b, u, v, \lambda]$ equipped with the coefficientwise order.

(ii) By (i) and $B_q^{-1}J(a, b, u, v, \lambda)B_q = J(a, b, u, v, \lambda + q)$, the matrix $J(a, b, u, v, \lambda)$ is immediately binomial-totally positive in the ring $\mathbb{Z}[a, b, u, v, \lambda]$ equipped with the coefficientwise order. So we will prove $J(a, b, u, v, 0)B_q = B_qJ(a, b, u, v, q)$. Since

$$[J(a, b, u, v, \lambda)]_{n,n+1} = un + v,$$
$$[J(a, b, u, v, \lambda)]_{n,n} = (\lambda u + a)n + b + \lambda v,$$
$$[J(a, b, u, v, \lambda)]_{n,n-1} = \lambda(a + \lambda u)n,$$

we have

$$[J(a, b, u, v, 0)B_q]_{n,k} = (un + v)\binom{n+1}{k}q^{n+1-k} + (an + b)\binom{n}{k}q^{n-k},$$

$$[B_qJ(a, b, u, v, q)]_{n,k} = \left(\begin{array}{c} n \\ k - 1 \end{array}\right)q^{n-k+1}[u(k-1) + v] + \left(\begin{array}{c} n \\ k \end{array}\right)q^{n-k}[(2qu + a)k + b + qv] +$$

$$\left(\begin{array}{c} n \\ k + 1 \end{array}\right)q^{n-k-1}q(a + qu)(k + 1)$$

$$= \left(\begin{array}{c} n \\ k - 1 \end{array}\right)q^{n-k+1}[u(k-1) + v] + \left(\begin{array}{c} n \\ k \end{array}\right)q^{n-k}q(2uk + v) +$$

$$\left(\begin{array}{c} n \\ k \end{array}\right)q^{n-k}(ak + b) + (n - k)\left(\begin{array}{c} n \\ k \end{array}\right)q^{n-k}(a + qu)(k + 1).$$

Then it is easy to check that the two right-hand sides of (3.15) and (3.16) are equal, which gives $[J(a, b, u, v, 0)B_q]_{n,k} = [B_qJ(a, b, u, v, q)]_{n,k}$. This completes the proof. □

It is easy to observe that $J(a, b, 0, 0, \lambda) = L(a, b, \lambda a)$. So if we write $\lambda = c/a$ and work in a ring of Laurent polynomials, Proposition 3.1 is a special case of Proposition 3.2. In what follows we will introduce a generalization of $L$, $U$ and $J$. Define a tridiagonal matrix

$$M(a, b, c, u, v) := \begin{bmatrix} b & v & & & \\ c & b + a & v + u & & \\ & 2c & b + 2a & v + 2u & \\ & & 3c & b + 3a & v + 3u & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$  \hspace{1cm} (3.17)

It is of the form (3.9) with $r_n = v + un$, $s_n = b + an$ and $t_n = cn$. Clearly, all of the matrices $L$, $U$ and $J$ are special cases of this general form: in particular,

$$L(a, b, c) = M(a, b, c, 0, 0),$$
$$U(a, b, u, v) = M(a, b, 0, u, v),$$
$$J(a, b, u, v, \lambda) = M(2\lambda u + a, b + \lambda u, \lambda(a + \lambda u), u, v).$$

For this matrix $M(a, b, c, u, v)$, we have the following key property.

**Proposition 3.3.** We have

$$B_q^{-1}M(a, b, c, u, v)B_q = M(2uq + a, qu + b, c + aq + uq^2, u, v).$$  \hspace{1cm} (3.21)
Proof. From definitions, we have \( M(a, b, c, u, v) = U(0, 0, u, v) + L(a, b, c) \). So
\[
B_q^{-1} M(a, b, c, u, v) B_q = B_q^{-1} U(0, 0, u, v) B_q + B_q^{-1} L(a, b, c) B_q. \tag{3.22}
\]
By Proposition 3.1 and Proposition 3.2, we get
\[
B_q^{-1} M(a, b, c, u, v) B_q = J(0, 0, u, v, q) + L(a, b, c + qa) \tag{3.23}
\]
\[
= M(2qu, qu, q^2u, v) + M(a, b, c + qu, 0, 0) \tag{3.24}
\]
\[
= M(2uq + a, qu + b, c + aq + uq^2, u, v). \tag{3.25}
\]
This completes the proof. \( \square \)

Proposition 3.3 says that the family of matrices \( M(a, b, c, u, v) \) is mapped into itself under a binomial similarity transformation. This will allow us to leverage total positivity into binomial-total positivity. The matrices \( M(a, b, c, u, v) \) are not in general totally positive; but by suitably specializing the variables \( a, b, c, u, v \), one can devise a matrix \( \widetilde{M}(a, b, c, d, e, \mu, \nu) \) in seven indeterminates \( a, b, c, d, e, \mu, \nu \) by
\[
\widetilde{M}(a, b, c, d, e, \mu, \nu) := M(c + e + (\mu + \nu)b, d + (\mu + \nu)a, (\mu + \nu)c + \nu e, b, a) \tag{3.26}
\]
or in other words (3.9) with
\[
r_n = a + bn \tag{3.27}
\]
\[
s_n = d + (\mu + \nu)a + [c + e + (\mu + \nu)b]n \tag{3.28}
\]
\[
t_n = [(\mu + \nu)c + \nu e]n. \tag{3.29}
\]
In addition, \( \widetilde{M}(a, b, c, d, e, \mu, \nu) \) satisfies the following relations
\[
\widetilde{M}(0, 0, c, d, e, \mu, \nu) = L(c + e, d, (\mu + \nu)c + \nu e)
\]
\[
\widetilde{M}(a, b, c, d, e, 0, 0) = U(c + e, d, b, a)
\]
\[
\widetilde{M}(a, b, c + \lambda b, d, e, 0, \lambda) = J(c + e, d, b, a, \lambda).
\]

For the matrix \( \widetilde{M}(a, b, c, d, e, \mu, \nu) \), the following result says that it is binomial-totally positive in seven variables \( a, b, c, d, e, \mu, \nu \).

**Proposition 3.4.** The matrix \( \widetilde{M}(a, b, c, d, e, \mu, \nu) \) is binomial-totally positive in the ring \( \mathbb{Z}[a, b, c, d, e, \mu, \nu] \) equipped with the coefficientwise order.

**Proof.** By the definition of \( \widetilde{M}(a, b, c, d, e, \mu, \nu) \) and Proposition 3.3, we immediately have
\[
B_q^{-1} \widetilde{M}(a, b, c, d, e, \mu, \nu) B_q = \widetilde{M}(a, b, c + bq, d, e, \mu, \nu + q). \tag{3.30}
\]
So it suffices to prove that \( \widetilde{M}(a, b, c, d, e, \mu, \nu) \) is totally positive in the ring \( \mathbb{Z}[a, b, c, d, e, \mu, \nu] \) equipped with the coefficientwise order.
Consider the tridiagonal matrix

\[
M^*(a, b, c, d, e, \mu, \nu) = \begin{bmatrix}
s_0^* & r_0^* & & & \\
& s_1^* & r_1^* & & \\
& & s_2^* & r_2^* & \\
& & & \ddots & \ddots & \ddots \\
\end{bmatrix}
\] (3.31)

with

\[
\begin{align*}
r_n^* &= (a + bn)(\mu + \nu) \\
s_n^* &= d + (\mu + \nu)a + [c + e + (\mu + \nu)b]n \\
t_n^* &= (c + e)n.
\end{align*}
\] (3.32-3.34)

Obviously, the matrix \(M^*(a, b, c, d, e, \mu, \nu)\) is coefficientwise row-diagonally-dominant. So by \([111, \text{Proposition 3.3 (i)}]\), it is totally positive in the ring \(\mathbb{Z}[a, b, c, d, e, \mu, \nu]\) equipped with the coefficientwise order. Let

\[
\begin{align*}
r_n^{**} &= a + bn \\
s_n^{**} &= d + (\mu + \nu)a + [c + e + (\mu + \nu)b]n \\
t_n^{**} &= (c + e)(\mu + \nu)n.
\end{align*}
\] (3.35-3.37)

Obviously, \(r_i^{**}t_{i+1}^{**} = r_i^{**}t_{i+1}^{**}\) for all \(i\). Then by \([111, \text{Claim 2 in the proof of Theorem 2.1}]\), the tridiagonal matrix

\[
M^{**}(a, b, c, d, e, \mu, \nu) = \begin{bmatrix}
s_0^{**} & r_0^{**} & & & \\
& s_1^{**} & r_1^{**} & & \\
& & s_2^{**} & r_2^{**} & \\
& & & \ddots & \ddots & \ddots \\
\end{bmatrix}
\] (3.38)

is totally positive in the ring \(\mathbb{Z}[a, b, c, d, e, \mu, \nu]\) equipped with the coefficientwise order. For \(\tilde{M}(a, b, c, d, e, \mu, \nu)\) and \(M^{**}(a, b, c, d, e, \mu, \nu)\), we have

\[
\begin{align*}
r_n &= r_n^{**} \\
s_n &= s_n^{**} \\
t_n &= t_n^{**} - \mu en.
\end{align*}
\] (3.39-3.41)

Therefore, \(0 \leq t_n \leq t_n^{**}\) in the coefficientwise sense. In consequence, by perturbation result \([111, \text{Proposition 3.1}]\), \(\tilde{M}(a, b, c, d, e, \mu, \nu)\) is also totally positive in the ring \(\mathbb{Z}[a, b, c, d, e, \mu, \nu]\) equipped with the coefficientwise order. This completes the proof. \(\square\)

By Proposition 3.4, if we take (i) \(a \to af, b \to bf, c \to 0, \nu \to 0\) and \(\mu \to 1\), (ii) \(c \to 0, e \to c, \mu + \nu \to f\) and \(\nu \to 1\), then we have the following much simpler and cleaner way, respectively.

**Corollary 3.5.** Let \(a, b, c, d, e, f\) be elements of a partially ordered commutative ring \(R\).
(i) If \( a, b, d, e, f \geq 0 \) and \( 0 \leq c \leq e \), then the matrix \( M(e+bf, d+af, c, bf, af) \) is totally positive.

(ii) If \( a, b, d, e, f \geq 0 \) and \( 0 \leq c \leq ef \), then the matrix \( M(e+bf, d+af, c, bf, af) \) is totally positive.

Let \( f = (f_n)_n \) and \( f(t) = \sum_{n \geq 0} f_n t^n \). Define a matrix \( \Lambda := \begin{bmatrix} 0! & 1! & 2! & \cdots \end{bmatrix} \) and a matrix \( \Gamma^{(\Lambda, f(t))} := \Lambda \Gamma(f) \Lambda^{-1} \). We obtain the following result.

**Proposition 3.6.** Let \( R \) be a partially ordered commutative ring and \( f(t) \in R[[t]] \). We have

\[
B_q^{-1} \Gamma^{(\Lambda, f(t))} B_q = \Gamma^{(\Lambda, f(t))}.
\]

Therefore, if \( f(t) \) is a Pólya frequency ogf of order \( r \) in \( R[[t]] \), then the matrix \( \Gamma^{(\Lambda, f)} \) is binomial-totally positive of order \( r \) in the ring \( R \).

**Proof.** Obviously, if \( f(t) \) is a Pólya frequency ogf of order \( r \) in \( R[[t]] \), then \( \Gamma^{(\Lambda, f(t))} \) is totally positive of order \( r \) in the ring \( R \). It follows from \( B_q^{-1} \Gamma^{(\Lambda, f(t))} B_q = \Gamma^{(\Lambda, f(t))} \) that the matrix \( \Gamma^{(\Lambda, f(t))} \) is binomial-totally positive of order \( r \) in the ring \( R \). In consequence, we only need to prove

\[
B_q^{-1} \Gamma^{(\Lambda, f(t))} B_q = \Gamma^{(\Lambda, f(t))},
\]

which is immediate from the following observing that

(a) Any two Toeplitz matrices \( \Gamma(f) \) and \( \Gamma(g) \) commute; hence the matrices \( \Lambda \Gamma(f) \Lambda^{-1} \) and \( \Lambda \Gamma(g) \Lambda^{-1} \) commute.

(b) \( B_q = \Lambda \Gamma(h) \Lambda^{-1} \), where \( h_n = \frac{a_n}{c_n} \).

We complete the proof. \( \square \)

4 Row-generating polynomials of generalized \( m \)-Jacobi–Rogers triangles

In this section, we will present coefficientwise Hankel-total positivity of row-generating polynomials of generalized \( m \)-Jacobi–Rogers triangles and combinatorial interpretations for such row-generating polynomials in terms of trees or forests.
4.1 Coefficientwise Hankel-total positivity of row-generating polynomials

It follows from definitions that the production matrix of the generalized \( m \)-Jacobi-Rogers triangle \( J^{(m)} \) equals \( P^{(m)}(\beta) \). Thus, by Theorem 2.5 and results in Section 3, we can immediately state the following result concerning coefficientwise Hankel-total positivity of row-generating polynomials of the generalized \( m \)-Jacobi–Rogers triangle.

**Theorem 4.1.** Let \( R \) be a partially ordered commutative ring and \( J^{(m)} \) be the infinite generalized \( m \)-Jacobi–Rogers triangle. Assume that \( P^{(m)}(\beta) \) can be written as the product of some matrices, such as \( L(a_i, b_i, c_i) \), \( U(\hat{a}_i, \hat{b}_i, \hat{u}_i) \), \( J(\tilde{a}_i, \tilde{b}_i, \tilde{u}_i, \tilde{v}_i, \lambda_i) \), \( \tilde{M}(\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, d_i, e_i, \mu_i, \nu_i) \) and \( \Gamma^{(\lambda, f_i(t))} \) for \( i \geq 1 \), and denote by \( x \) all indeterminates in \( L \), \( U \), \( J \) and \( \tilde{M} \). If \( f_i(t) \) is a Pólya frequency ogf of order \( r \) in \( R[[t]] \) for each \( i \geq 1 \), then

(i) the binomial row-generating matrix \( J^{(m)}(q) \) is coefficientwise totally positive of order \( r \) in the ring \( R[x, q] \) equipped with the coefficientwise order;

(ii) the row-generating polynomial sequence \( (J_n(q))_{n \geq 0} \) of \( J^{(m)} \) is Hankel-totally positive of order \( r \) in the ring \( R[x, q] \) equipped with the coefficientwise order.

In particular, for the \( m \)-Stieltjes–Rogers triangle \( S^{(m)} \), we have the following result.

**Theorem 4.2.** Let \( S_n(\alpha; q) \) be the \( n \)th row-generating polynomial of the \( m \)-Stieltjes–Rogers triangle \( S^{(m)} \). If \( \alpha = (y, x_1, \ldots, x_m, y+2x_1, \ldots, 2x_m, y+2x_0, 3x_1, \ldots, 3x_m, \ldots) \), then \((S_n(\alpha; q))_{n \geq 0}\) is Hankel-totally positive in the polynomial ring \( \mathbb{Z}[x, y, q] \) equipped with the coefficientwise order.

**Proof.** Note by [76, Proposition 8.2 (b)], for

\[
(\alpha_i)_{i \geq m} = (y, x_1, \ldots, x_m, y + x_0, 2x_1, \ldots, 2x_m, y + 2x_0, 3x_1, \ldots, 3x_m, \ldots)
\]

that the production matrix of the \( m \)-Stieltjes-Rogers triangle \( S^{(m)} \) is

\[
\left[ \begin{array}{c}
\prod_{i=1}^{m} \left[ \begin{array}{cc}
1 & 1 \\
1 & 2x_i \\
1 & 3x_i \\
\vdots & \ddots
\end{array} \right]
\end{array} \right] \left[ \begin{array}{ccc}
y & 1 & 1 \\
x_0 + y & 2x_0 + y & 1 \\
3x_0 + y & \ddots & \ddots
\end{array} \right], \tag{4.1}
\]

where the product over \( i \) of these matrices is to be understood as left-to-right (i.e. \( i = 1 \) on the far left, \( i = m \) on the far right). It is easy to observe that the matrices over \( i \) in the product (4.1) are \( L(0; 1; x_i) \), and the final matrix is \( U(x_0; y; 0; 1) \). Therefore, it follows from Theorem 4.1 that \((S_n(\alpha; q))_{n \geq 0}\) is Hankel-totally positive in the polynomial ring \( \mathbb{Z}[x, y, q] \) equipped with the coefficientwise order.

The following result concerning coefficientwise Hankel-total positivity of polynomials is very useful.
Proposition 4.3. Let $A_n(q)$ be a polynomial of degree $n$ and define a new polynomial

$$
\hat{A}_n(a, b, c, d, q) = (a + bq)^n A_n \left( \frac{c + dq}{a + bq} \right)
$$

for $n \geq 0$. If $(A_n(q))_{n \geq 0}$ is coefficientwise Hankel-totally positive of order $r$ in $q$, then $(\hat{A}_n(q))_{n \geq 0}$ is coefficientwise Hankel-totally positive of order $r$ in $(a, b, c, d, q)$.

Proof. For $k \leq r$, let $A^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}(q)$ (resp. $\hat{A}^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}(a, b, c, d, q)$) be the minor of $[A_{i+j}(q)]_{i,j}$ (resp. $[\hat{A}_{i+j}(q)]_{i,j}$) by taking its rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$. In terms of the assumption that $[A_{i+j}(q)]_{i,j}$ is coefficientwise totally positive of order $r$ in $q$, $A^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}(q)$ is a polynomial with nonnegative coefficients of degree $\leq i_1 + i_2 + \cdots + i_k + j_1 + j_2 + \cdots + j_k$. Since $\hat{A}_{i+j}(q) = (a + bq)^{i+j} A_{i+j}(\frac{c + dq}{a + bq})$, taking out $(a + bq)^{i_1}, (a + bq)^{i_2}, \ldots, (a + bq)^{i_k}$ from rows and $(a + bq)^{j_1}, (a + bq)^{j_2}, \ldots, (a + bq)^{j_k}$ from columns of $\hat{A}^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}(a, b, c, d, q)$, respectively, we get

$$
\hat{A}^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}(a, b, c, d, q) = (a + bq)^{i_1+i_2+\cdots+i_k+j_1+j_2+\cdots+j_k} A^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k} \left( \frac{c + dq}{a + bq} \right). \tag{4.3}
$$

In consequence, $\hat{A}^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}(a, b, c, d, q)$ is a multivariate polynomial with nonnegative coefficients in $a, b, c, d, q$. This completes the proof. \qed

For a polynomial $A_n(q)$ with $\deg(A_n(q)) = n$, its reversed polynomial $A^*_n(q)$ is defined by $A^*_n(q) = q^n A_n(\frac{1}{q})$. The following result is immediate from Proposition 4.3.

Corollary 4.4. If $(A_n(q))_{n \geq 0}$ is coefficientwise Hankel-totally positive of order $r$ in $(x, q)$, then so is the sequence $(A^*_n(q))_{n \geq 0}$ of reversed polynomials.

This implies in particular the coefficientwise Hankel-total positivity of $(S_n^*(\alpha; q))_{n \geq 0}$ and $(J_n^*(q))_{n \geq 0}$ in Theorem 4.2 and Theorem 4.1, respectively.

Coefficientwise Hankel-total positivity of polynomials also plays an important role in triangular convolutions preserving the Stieltjes moment properties. Let $A = [a_{n,k}]_{n,k \geq 0}$ be an infinite matrix. Define the $A$-convolution

$$
z_n = \sum_{k=0}^{n} a_{nk} x_k y_{n-k} \tag{4.4}
$$

for $n \geq 0$. We say that (4.4) preserves the Stieltjes moment property: if both $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are Stieltjes moment sequences, then so is $(z_n)_{n \geq 0}$. Pólya and Szegö [79, Part VII, Theorem 42] proved for $n \geq 0$ that the binomial convolution

$$
z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k} \tag{4.5}
$$

preserves the Stieltjes moment property in $\mathbb{R}$. Recently, it was proved in [102] that the next sufficient condition for the triangular convolution preserving the Stieltjes moment property [102].
Theorem 4.5. [102] Let \( A_n(q) = \sum \limits_{k=0}^{n} a_{n,k} q^k \) be the \( n \)th row-generating function of the matrix \( A = [a_{n,k}]_{n,k} \). Assume that \((A_n(q))_{n \geq 0}\) is a Stieltjes moment sequence for any fixed \( q \geq 0 \). Then the \( A \)-convolution (4.4) preserves the Stieltjes moment property in \( \mathbb{R} \).

Clearly, if the sequence \((A_n(q))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \( q \), then \((A_n(q))_{n \geq 0}\) is a Stieltjes moment sequence for any fixed \( q \geq 0 \). In addition, it also implies \( 3-q \)-log-convexity of \((A_n(q))_{n \geq 0}\) in terms of the next result.

Proposition 4.6. [111] For a polynomial sequence \((A_n(x))_{n \geq 0}\), if the Hankel matrix \([A_{i+j}(x)]_{i,j \geq 0}\) is coefficientwise totally positive of order \( 4 \) in \( x \), then \((A_n(x))_{n \geq 0}\) is \( 3-x \)-log-convex.

In consequence, by Theorems 4.2, 4.5 and Proposition 4.6, we obtain:

Theorem 4.7. Let \( S_n(\alpha;q) \) be the \( n \)th row-generating polynomial of the \( m \)-Stieltjes-Rogers triangle \( S^{(m)} \). If \( \alpha = (y_1, x_1,\ldots, x_m; y_0, 2x_1,\ldots, 2x_m; y+2x_0, 3x_1,\ldots, 3x_m,\ldots) \), then we have

(i) \((S_n(\alpha;q))_{n \geq 0}\) is \( 3-(x,y,q) \)-log-convex;

(ii) the triangular convolution \( z_n = \sum \limits_{k=0}^{n} S^{(m)}_{n,k}(\alpha) s_k x_{n-k} \) preserves the Stieltjes moment property in \( \mathbb{R} \) for all \( x \geq 0 \) and \( y \geq 0 \).

Similarly, by Theorems 4.1, 4.5 and Proposition 4.6, we obtain:

Theorem 4.8. Let \( J^{(m)} = [J^{(m)}_{n,k}(\beta)]_{n,k} \) be the infinite generalized \( m \)-Jacobi-Rogers triangle. Assume that \( P^{(m)}(\beta) \) can be written as the product of some matrices, such as \( L(\alpha_i, b_i, c_i) \), \( U(\hat{\alpha_i}, \hat{b}_i, \hat{c}_i, \hat{v}_i) \), \( J(\alpha_i, b_i, \tilde{\alpha}_i, \tilde{b}_i, \tilde{v}_i, \lambda_i) \), \( \tilde{M}(\alpha_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i, \tilde{e}_i, \mu_i, \nu_i) \) and \( \Gamma^{(A,J,L)} \) for \( i \geq 1 \), and denote by \( x \) all indeterminates in \( L \), \( U \), \( J \) and \( \tilde{M} \). If \( f_i(t) \) is a Pólya frequency off in \( \mathbb{R}[[t]] \) for each \( i \geq 1 \), then

(i) \((J_n(q))_{n \geq 0}\) is \( 3-(x,q) \)-log-convex;

(ii) the triangular convolution \( z_n = \sum \limits_{k=0}^{n} J^{(m)}_{n,k}(\beta) x_k y_{n-k} \) preserves the Stieltjes moment property in \( \mathbb{R} \) for all indeterminates \( x \geq 0 \).

4.2 Combinatorial interpretations for row-generating polynomials

Recall [94, pp. 294–295] that an ordered tree (also called plane tree) is an (unlabeled) rooted tree in which the children of each vertex are linearly ordered. An ordered forest of ordered trees (also called plane forest) is a linearly ordered collection of ordered trees.

Using a well-known bijection from the set of ordered forests of ordered trees with \( n+1 \) total vertices and \( k+1 \) components onto the set of partial Lukasiewicz paths from \((0,0)\) to \((n,k)\) (see [95, pp. 30–36], [62, Chapter 11], [78, section 6.2] and [49, proof of Proposition 3.1] for instance), in [76], the \( m \)-Jacobi-Rogers polynomials \( J^{(m)}_{n,k}(\beta) \) are interpreted as the generating polynomials for certain classes of ordered trees, and the \( m \)-Jacobi-Rogers triangle \([J^{(m)}_{n,k}(\beta)]_{n,k}\) are interpreted as the generating polynomials for certain classes of ordered forests of ordered trees.
Proposition 4.9. [76, Proposition 6.1] Let $\beta = (\beta^{(i)}_i)_{i \geq 1}$ be indeterminates with all $\beta^{(-1)}_1 = 1$. Then the $\infty$-Jacobi–Rogers triangle $J^{(\infty)}_{n,k}(\beta)$ is the generating polynomial for ordered forests of ordered trees with $n + 1$ total vertices and $k + 1$ components in which each vertex at level $L$ with $c$ children gets weight 1 if it is a leaf ($c = 0$) and weight $\beta^{(c-1)}_{L+c-1}$ otherwise.

Remark 4.10. The $m$-Jacobi–Rogers polynomials with $m < \infty$ correspond to the forests in which each vertex has at most $m + 1$ children.

For the $m$-Stieltjes–Rogers triangle $S^m$ in Theorem 4.2 and the $m$-Jacobi–Rogers triangle $J^m$ in Theorem 4.1, obviously, their corresponding production matrix $P = \prod_{i \geq 1} P_i$ is an $m$-banded lower-Hessenberg matrix. In consequence, by Propositions 3.1, 3.2, 3.4 and 3.6, the production matrix of the binomial row-generating matrix:

$$B_q^{-1}PB_q = B_q^{-1}\left(\prod_{i \geq 1} P_i\right)B_q = \prod_{i \geq 1} B_q^{-1}P_iB_q$$

is a $r$-banded lower-Hessenberg matrix for $r = m$ or $r = m + 1$. Thus for the $m$-Stieltjes–Rogers triangle $S^m$ and the $m$-Jacobi–Rogers triangle $J^m$, their row-generating polynomials are both $r$-Jacobi–Rogers polynomials with $r = m$ or $r = m + 1$. Therefore we can interpreter them in terms of ordered forests of ordered trees.

Furthermore, for $x_0 = 0$ in Theorem 4.2, i.e.,

$$\alpha = (y, x_1, \ldots, x_m, y, 2x_1, \ldots, 2x_m, y, 3x_1, \ldots, 3x_m, \ldots),$$

then, by (4.1), we have

$$P = \left(\prod_{i = 1}^{m} \left[\begin{array}{ccc} 1 & 1 & 1 \\ x_i & 2x_i & 3x_i \\ 1 & 1 & \ddots \end{array}\right]\right)\left[\begin{array}{ccc} q + y & 1 \\ q + y & 1 \\ q + y & 1 \end{array}\right].$$

(4.7)

In consequence, by [76, Proposition 8.2 (b)], the row-generating polynomial sequence $(S_n(\alpha; q))_{n \geq 0}$ is still an $m$-Stieltjes–Rogers polynomial $S_n^{(m)}(\alpha')$ with

$$\alpha' = (q + y, x_1, \ldots, x_m, q + y, 2x_1, \ldots, 2x_m, q + y, 3x_1, \ldots, 3x_m, \ldots).$$

For $y = x_0$ in Theorem 4.2, i.e.,

$$\alpha = (x_0, x_1, \ldots, x_m, 2x_0, 2x_1, \ldots, 2x_m, 3x_0, 3x_1, \ldots, 3x_m, \ldots),$$

then the $m$-Stieltjes–Rogers polynomial $S_{n,0}^{(m)}(\alpha)$ can be interpreted in terms of increasing $(m + 1)$-ary trees and is called the **multivariate Eulerian polynomial** [76, Section 12.2].
Row-generating polynomials of Catalan-Stieltjes matrices

In this section, we will apply our results for the generalized $m$-Jacobi–Rogers triangle $J^{(m)} = (J^{(m)}_{n,k})_{n,k \geq 0}$ to tridiagonal matrices.

For $P^{(m)}(\beta)$ with $m = 1$, we use the notation $\beta_i = \beta_i^{(1)}$, $\gamma_i = \beta_i^{(0)}$ and $\delta_i = \beta_i^{(-1)}$ for $i \geq 0$ and also write $J_{n,k}^{(1)}(\beta)$ as $J_{n,k}$ for short. So $P^{(m)}(\beta)$ reduces to a tridiagonal matrix denoted by

$$P = \begin{bmatrix}
\gamma_0 & \delta_0 & & \\
\beta_1 & \gamma_1 & \delta_1 & \\
& \beta_2 & \gamma_2 & \delta_2 & \\
& & \beta_3 & \gamma_3 & \delta_3 & \\
& & & \ddots & \ddots & \ddots \\
\end{bmatrix}. \quad (5.1)
$$

Meanwhile we observe that the output matrix $O(P) = J^{(1)}$ satisfies the recurrence relation:

$$J_{n,k} = \delta_{k-1}J_{n-1,k-1} + \gamma_kJ_{n-1,k} + \beta_{k+1}J_{n-1,k+1}, \quad (5.2)$$

with initial conditions $J_{n,k} = 0$ unless $0 \leq k \leq n$ and $J_{0,0} = 1$. The generalized 1-Jacobi–Rogers triangle $J^{(1)}$ is often called the Catalan-Stieltjes triangle [1, 72]. In particular, for all $\delta_i = 1$, $J^{(1)}$ is called the Stieltjes matrix [39]. The $x$-log-convexity, $x$-log-convexity of higher order and coefficientwise Hankel-total positivity of $(J_{n,k})_{n \geq 0}$ have been studied, see [21, 71, 72, 92, 102, 104, 106, 108, 111] for instance. But the following question is open.

**Question 5.1.** For the Catalan-Stieltjes matrix defined by (5.2), let $J_n(q) = \sum_k J_{n,k}q^k$ be its $n$th row-generating polynomial. Find conditions ensuring coefficientwise Hankel-total positivity of the sequence $(J_{n,k}(q))_{n \geq 0}$.

In general, for Question 5.1, there exists the cases such that $J_n(q)$ is not coefficientwise Hankel-totally positive: see e.g. Example 2.2. For the Catalan-Stieltjes triangle defined by (5.2), let all $\delta_k$, $\gamma_k$ and $\beta_k$ be linear in $k$. In such case, if the production matrix $P$ equals one of $L(a, b, c)$, $U(\tilde{a}, \tilde{b}, \tilde{u}, \tilde{v})$, $J(\tilde{a}, \tilde{b}, \tilde{u}, \tilde{v}, \lambda)$, and $\widetilde{M}(\tilde{a}, \tilde{b}, \tilde{c}, d, e, \mu, \nu)$, then, by Theorem 4.1, the row-generating polynomial sequence $(J_n(q))_{n \geq 0}$ is Hankel-totally positive in the polynomial ring $\mathbb{Z}[a, b, c, d, e, \mu, \nu, q]$ equipped with the coefficientwise order.

5.1 Main results for Catalan-Stieltjes matrices

In this subsection, we consider nonlinear cases for $\delta_k$, $\gamma_k$ and $\beta_k$, i.e., $\delta_k$, $\gamma_k$ and $\beta_k$ are nonlinear functions in $k$. We will give the first result for Question 5.1 as follows.

**Theorem 5.2.** Let $J = [J_{n,k}]_{n,k}$ be a Catalan-Stieltjes triangle defined by (5.2) with $\delta_k = (ak+b)(uk+v)$, $\gamma_k = [(ax+cu)k^2+(ay+bx+cv-cu)k+by]$ and $\beta_{k+1} = c(y+kx)(k+1)$. Then we have

(i) the binomial row-generating matrix $J(q)$ is totally positive in the polynomial ring $\mathbb{Z}[a, b, c, u, v, x, y, q]$ equipped with the coefficientwise order;
(ii) both \( (J_n(q))_{n \geq 0} \) and its reversed polynomial sequence \( (J_n^*(q))_{n \geq 0} \) are Hankel-totally positive in the polynomial ring \( \mathbb{Z}[a, b, c, u, v, x, y, q] \) equipped with the coefficientwise order and \( 3\cdot(a, b, c, u, v, x, y, q) \)-log-convex;

(iii) the sequence \( (J_n(q))_{n \geq 0} \) is Hankel-totally positive in the ring \( \mathbb{Z}[a, b, c, u, v, x, y, q] \) equipped with the coefficientwise order and \( 3\cdot(a, b, c, u, v, x, y, q) \)-log-convex;

(iv) the convolution \( r_n = \sum_{k \geq 0} J_{n,k}s_k t_{n-k} \) preserves the Stieltjes moment property in \( \mathbb{R} \) for all \( a, b, c, u, v, x, y \geq 0 \).

**Proof.** For this Catalan-Stieltjes triangle with \( \delta_k = (ak + b)(ak + v) \), \( \gamma_k = [(ax + cu)k^2 + (ay + bx + cv - cu)k + by] \) and \( \beta_{k+1} = c(y + kx)(k + 1) \), we have its production matrix

\[
P = \begin{bmatrix}
\gamma_0 & \delta_0 \\
\beta_1 & \gamma_1 & \delta_1 \\
\beta_2 & \gamma_2 & \delta_2 \\
\beta_3 & \gamma_3 & \delta_3 \\
\vdots & \ddots & \ddots 
\end{bmatrix} = L(a, b, c) U(x, y, u, v). \tag{5.3}
\]

Hence, (i), (ii) and (iv) are immediate from Theorem 4.1, Corollary 4.4 and Theorem 4.8, respectively. For \( q = 0 \), (ii) implies (iii) because \( J_n(0) = J_{n,0} \). \( \square \)

If we change the order of \( L(a, b, c) \) and \( U(x, y, u, v) \) in (5.3), we immediately obtain the following result for Question 5.1.

**Theorem 5.3.** Let \( \mathbf{J} = [J_{n,k}]_{n,k} \) be a Catalan-Stieltjes triangle defined by (5.2) with \( \delta_k = (ak + a + b)(ku + v) \), \( \gamma_k = (ax + cu)k^2 + (ay + bx + cv + cu)k + by + cv \) and \( \beta_{k+1} = c[y + x(k + 1)](k + 1) \). Then we have conclusions (i)-(iv) in Theorem 5.2.

**Remark 5.4.** We want to point out that the 1-Jacobi–Rogers triangle has the restricted condition: all \( \beta_i^{(-1)} = 1 \). Then such condition enforces \( a = u = 0 \) for the Catalan-Stieltjes triangle in Theorems 5.2 and 5.3. Meanwhile, \( \gamma_k \) must be a linear function in \( k \). Such examples for the Catalan-Stieltjes triangles are restricted. So we see the good for the generalized \( m \)-Jacobi–Rogers triangle \( \mathbf{J}^{(m)} \) without the restriction \( \beta_i^{(-1)} = 1 \).

For the Catalan-Stieltjes triangle \( \mathbf{J} = [J_{n,k}]_{n,k} \) defined by (5.2), let \( \tilde{J}_{n,k} := J_{n,k}! \). Then by (5.2), the associated triangle \( [\tilde{J}_{n,k}]_{n,k \geq 0} \) satisfies the recurrence relation:

\[
\tilde{J}_{n,k} = k\delta_{k-1}\tilde{J}_{n-1,k-1} + \gamma_k\tilde{J}_{n-1,k} + \frac{\beta_{k+1}}{k+1}\tilde{J}_{n-1,k+1}, \tag{5.4}
\]

with initial conditions \( \tilde{J}_{n,k} = 0 \) unless \( 0 \leq k \leq n \) and \( \tilde{J}_{0,0} = 1 \). From the view point of production matrices, that is to say \( \tilde{J} = \mathbf{J} \Lambda = \mathcal{O}(\Lambda^{-1}\mathbf{P}\Lambda) \), where \( \Lambda = \text{diag}(k!) \). Thus, some Catalan-Stieltjes triangles \( \mathbf{J} \) may not satisfy the condition of Theorems 5.2 or 5.3, but their associated matrices \( \tilde{\mathbf{J}} \) do. So we can still obtain some positivity results from \( \tilde{\mathbf{J}} \).

For the Catalan-Stieltjes triangle in the following result, we have the same conclusions in (i) and (ii) of Theorem 5.2, but we want to list and emphasize the similar results (ii) and (iv).
Proposition 5.5. Let $J = [J_{n,k}]_{n,k}$ be a Catalan-Stieltjes triangle defined by (5.2) with \( \delta_k = (ak + b)u, \gamma_k = [(ax + cu)k^2 + (ay + bx)k + by] \) and \( \beta_{k+1} = c(y + kx)(k + 1)^2 \). Let \( \tilde{J}_n(q) = \sum_{k=0}^n J_{n,k}k!q^k \). Then we have

(ii) both \((\tilde{J}_n(q))_{n \geq 0}\) and \((\tilde{J}_n^*(q))_{n \geq 0}\) are Hankel-totally positive in the polynomial ring \( \mathbb{Z}[a, b, c, u, x, y, q] \) equipped with the coefficientwise order and \(-3-(a, b, c, u, x, y, q)\)-log-convex;

(iv) the convolution \( r_n = \sum_{k \geq 0} J_{n,k}k!s_k t_{n-k} \) preserves the Stieltjes moment property in \( \mathbb{R} \) for all \( a, b, c, u, x, y \geq 0 \).

Similar to Proposition 5.5, we also obtain:

Proposition 5.6. Let $J = [J_{n,k}]_{n,k}$ be a Catalan-Stieltjes triangle defined by (5.2) with \( \delta_k = (ak + a + b)u, \gamma_k = [(ax + cu)k^2 + (ay + bx + 2cu)k + by + cu] \) and \( \beta_{k+1} = c(y + x(k + 1))(k + 1)^2 \). Then we have the same conclusions (i)-(iv) in Proposition 5.5.

5.2 A generalized Jacobi-Stirling triangle

The Jacobi-Stirling numbers \( JS_n^k(z) \) of the second kind satisfy the following recurrence relation:

\[
\begin{cases}
JS_0^0(z) = 1, & JS_n^0(z) = 0, \quad \text{if } k \not\in \{1, \ldots, n\}, \\
JS_n^k(z) = JS_{n-1}^{k-1}(z) + k(k+z)JS_n^{k-1}(z), & n, k \geq 1,
\end{cases}
\]

where \( z = \alpha + \beta + 1 \) and \( \alpha, \beta \geq -1 \). They were introduced in [37] and are the coefficients of the integral composite powers of the Jacobi differential operator

\[
\ell_{\alpha, \beta}[y](t) = \frac{1}{(1-t)^{\alpha}(1+t)^\beta} \left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)'.
\]

For \( \alpha = \beta = 0 \), the Jacobi-Stirling numbers reduce to the Legendre-Stirling numbers of the second kind. It was found that the Jacobi-Stirling numbers and Legendre-Stirling numbers have many similar properties with the classical Stirling numbers. In fact, there has been an extensive literature in recent years on Jacobi-Stirling numbers and the Legendre-Stirling numbers, see Andrews et al. [3, 4, 5], Egge [34], Everitt et al. [36, 37], Gelineau and Zeng [43], Mongelli [70] and Zhu [105, 107] for details.

Let \( U(n,k) \) be the central factorial numbers [89, A036969], which first occurred in MacMahon [67, pp. 106] and were also defined in Riordan’s book [83, pp. 213-217] by

\[
x^{2n} = \sum_{k=0}^{2n} U(n,k) x \prod_{i=1}^{k-1} (x + k - i).
\]

They satisfy the following recurrence

\[
U(n,k) = U(n-1,k-1) + k^2 U(n-1,k)
\]

with \( U(0,0) = 1 \). Obviously, the central factorial numbers are simply the Jacobi-Stirling numbers specialized to \( z = 0 \). In addition, the central factorial numbers have a close connection with the famous Genocchi numbers \( G_n \) by

\[
G_{2n+2} = \sum_{k=0}^{n} (-1)^{k+1} U(n,k)(k!)^2
\]
and also play an important role in the proof of one inequality concerning rank and crank moments related to Andrews’ spt-function \([42]\).

Motivated by Jacobi-Stirling numbers, Legendre-Stirling numbers, and central factorial numbers, we introduce a \textbf{generalized Jacobi-Stirling triangle} \([\text{JS}_{n,k}]_{0 \leq k \leq n}\) defined by

\[
\text{JS}_{n,k} = \text{JS}_{n-1,k-1} + (xk^2 + yk)\text{JS}_{n-1,k}
\]

where \(\text{JS}_{n,k} = 0\) unless \(0 \leq k \leq n\) and \(\text{JS}_{0,0} = 1\). Obviously, the generalized Jacobi-Stirling number \(\text{JS}_{n,k}\) is a common generalization of the Jacobi-Stirling number, the Legendre-Stirling number, and the central factorial number. From \([105, 107]\), we know that the row-generating polynomials of \([\text{JS}_{n,k}]_{0 \leq k \leq n}\) are \(q\)-log-convex. In 2018, the author conjectured that the row-generating polynomials of \([\text{JS}_{n,k}]_{0 \leq k \leq n}\) are Hankel-totally positive in the polynomial ring \(\mathbb{Z}[x, y, q]\) equipped with the coefficientwise order. Now, we will obtain a weaker result in the following. Let \(\tilde{\text{JS}}_{n,k} = \text{JS}_{n,k}k!\) for \(n \geq k \geq 0\). It follows from (5.10) that the associated triangle \(\tilde{\text{JS}}\) satisfies the recurrence relation

\[
\tilde{\text{JS}}_{n,k} = k\tilde{\text{JS}}_{n-1,k-1} + (xk^2 + yk)\tilde{\text{JS}}_{n-1,k}
\]

with the initial condition \(\tilde{\text{JS}}_{0,0} = 1\). Obviously, the associated triangle \(\tilde{\text{JS}}\) is the Catalan-Stieltjes triangle in Theorem 5.3 specialized to \(a = v = 1\) and \(b = c = u = 0\). Therefore, applying Theorem 5.3 to the associated triangle \(\tilde{\text{JS}}\), we immediately obtain:

\textbf{Proposition 5.7.} Let \([\text{JS}_{n,k}]_{0 \leq k \leq n}\) be the generalized Jacobi-Stirling triangle defined by (5.10) and \(\text{JS}(q) = \sum_{k=0}^{n} \text{JS}_{n,k}k!q^k\). Then we have

(i) the binomial row-generating matrices \(\text{JS}(q)\) and \(\tilde{\text{JS}}(q)\) are totally positive in the polynomial ring \(\mathbb{Z}[x, y, q]\) equipped with the coefficientwise order;

(ii) both \((\tilde{\text{JS}}(q))_{n \geq 0}\) and \((\tilde{\text{JS}}^*_n(q))_{n \geq 0}\) are Hankel-totally positive in the polynomial ring \(\mathbb{Z}[x, y, q]\) equipped with the coefficientwise order and \(3-(x, y, q)\)-log-convex;

(iii) the convolution \(r_n = \sum_{k=0}^{n} \text{JS}_{n,k}k!s_k t_{n-k}\) preserves the Stieltjes moment property in \(\mathbb{R}\) for both \(x, y \geq 0\).

\textbf{5.3 A generalized elliptic polynomial}

The Jacobi elliptic functions \(sn(u, \alpha), cn(u, \alpha)\) and \(dn(u, \alpha)\) appear in different fields (i.e., real and complex analysis, number theory, combinatorics, physics and so on) and have been extensively studied (see \([25, 33, 40, 85]\) for instance).

For a fixed modulus \(\alpha\), \(sn(u, \alpha)\) is defined as the inverse of an elliptic integral:

\[
\text{sn}(u, \alpha) = y \iff u = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-\alpha^2\sin^2 t)}}.
\]

The elliptic functions \(cn(u, \alpha)\) and \(dn(u, \alpha)\) are defined by

\[
\text{cn}(u, \alpha) = \cos \text{am}(u, \alpha); \quad \text{dn}(u, \alpha) = \sqrt{1 - \alpha^2 \sin^2 \text{am}(u, \alpha)},
\]
where $am(u, \alpha)$ is the inverse of an elliptic integral: by definition

$$am(u, \alpha) = \phi \quad \text{iff} \quad u = \int_0^\phi \frac{dt}{\sqrt{1 - \alpha^2 \sin^2 t}}.$$  

(5.14)

The functions $cn(u, \alpha)$ and $dn(u, \alpha)$ have power series expansions:

$$cn(u, \alpha) = \sum_{n \geq 0} (-1)^{n-1} c_n(\alpha^2) \frac{u^{2n}}{(2n)!};$$  

(5.15)

$$dn(u, \alpha) = \sum_{n \geq 0} (-1)^{n-1} d_n(\alpha^2) \frac{u^{2n}}{(2n)!},$$  

(5.16)

where $c_n(\lambda)$ and $d_n(\lambda)$ are polynomials of degree $n-1$ and $d_n(\lambda) = \lambda^{n-1} c_n(\frac{1}{\lambda})$. We call $c_n(\lambda)$ and $d_n(\lambda)$ the elliptic polynomials. Flajolet [39, Theorem 4] proved that the coefficient $c_{n,k}$ of the polynomial $c_n(\lambda)$ counts the alternating permutations over $[2n]$ having $k$ minima of even value and

$$\sum_{n=0}^{\infty} c_n(\lambda)t^n = \frac{1}{1 - s_0 t - \frac{r_1 t^2}{1 - s_1 t - \frac{r_2 t^2}{1 - s_2 t - \frac{r_3 t^2}{1 - s_3 t - \cdots}}}},$$  

(5.17)

where $s_n = 4(1 + \lambda)n^2 + 4n + 1$ and $r_{n+1} = (2n + 1)^2 (2n + 2)^2 \lambda$ for $n \geq 0$. Obviously, the polynomial $c_n(\lambda)$ is the zeroth column of the Catalan-Stieltjes triangle $C = [C_{n,k}]_{n,k}$, which satisfies the recurrence

$$C_{n,k} = (2k - 1)kC_{n-1,k-1} + [4(\lambda + 1)k^2 + 4k + 1]C_{n-1,k} + 4\lambda(1 + 2k)(k + 1)C_{n-1,k+1}$$  

(5.18)

with $C_{0,0} = 1$. Let its row-generating polynomial $C_n(\lambda, q) = \sum_{k=0}^n C_{n,k}q^k$. Obviously, $C_n(\lambda, 0) = c_n(\lambda)$. Therefore we call $C_n(\lambda, q)$ the generalized elliptic polynomial. By Theorem 5.2 with $a = x = 2$, $b = u = v = y = 1$ and $c = 4\lambda$, we immediately obtain:

**Proposition 5.8.** Let the Catalan-Stieltjes triangle $C$ be defined by (5.18). Then we have

(i) the binomial row-generating matrix $C(\lambda, q)$ is totally positive in the polynomial ring $\mathbb{Z}[\lambda, q]$ equipped with the coefficientwise order;

(ii) $(C_n(q))_{n \geq 0}$ is Hankel-totally positive in the polynomial ring $\mathbb{Z}[\lambda, q]$ equipped with the coefficientwise order and 3-$(\lambda, q)$-log-convex;

(iii) $(c_n(\lambda))_{n \geq 0}$ and $(d_n(\lambda))_{n \geq 0}$ are Hankel-totally positive in the polynomial ring $\mathbb{Z}[\lambda]$ equipped with the coefficientwise order and 3-$\lambda$-log-convex;

(iv) the convolution $r_n = \sum_{k \geq 0} C_{n,k}s_k t_{n-k}$ preserves the Stieltjes moment property in $\mathbb{R}$ for $\lambda \geq 0$. 

25
5.4 A refined Stirling cycle polynomial

Let \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) be the signless Stirling number. It counts the number of permutations of \( n \) elements which is the product of \( k \) disjoint cycles. It is well-known that it satisfies the following recurrence

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = (n - 1) \left[ \begin{array}{c} n - 1 \\ k \end{array} \right] + \left[ \begin{array}{c} n - 1 \\ k - 1 \end{array} \right].
\]

The row-generating polynomial \( s_n(\lambda) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \lambda^k \) is called the Stirling cycle polynomial, which has many nice properties [24]. For example, its explicit formula is given as \( s_n(\lambda) = \lambda(\lambda + 1) \cdots (\lambda + n - 1) \). Define a generalized cycle-triangle \( [S_{n,k}]_{n,k} \) satisfying the following recurrence

\[
S_{n,k} = [b_0(n - 1) + b_1]S_{n-1,k-1} + [a_0(n - 1) + a_1]S_{n-1,k}
\]

(5.19)

with \( S_{0,0} = 1 \). From this recurrence relation, we derive the row generating polynomial

\[
S_n(a_0, a_1, b_0, b_1, \lambda) := \sum_k S_{n,k} \lambda^k = \prod_{k=0}^{n-1} [(a_0 + b_0 \lambda)k + a_1 + b_1 \lambda]
\]

(5.20)

for \( n \geq 1 \). Obviously, \( S_n(a_0, a_1, b_0, b_1, \lambda) \) is a polynomial in five indeterminates \( a_0, a_1, b_0, b_1 \) and \( \lambda \). In addition, \( S_n(1, 0, 0, 1, \lambda) = s_n(\lambda) \), \( S_n(1, 0, 0, 1, 1) = n! \) and \( S_n(1, 1, 1, 0, 1) = (2n - 1)!! \). By [111, (4.3)], we have

\[
\sum_{n=0}^{\infty} S_n(a_0, a_1, b_0, b_1, \lambda)t^n = \frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \gamma_1 t - \frac{\beta_2 t^2}{1 - \gamma_2 t - \frac{\beta_3 t^2}{1 - \gamma_3 t - \cdots}}}}.
\]

(5.21)

where \( \beta_{k+1} = [(a_0 + b_0 \lambda)k + b_1 \lambda + a_1](a_0 + b_0 \lambda)(k + 1) \) and \( \gamma_k = [2(a_0 + b_0 \lambda)k + a_1 + b_1 \lambda] \).

Thus \( S_n(a_0, a_1, b_0, b_1, \lambda) \) is exactly the zeroth column of the Catalan-Stieltjes triangle \( S = [S_{n,k}]_{n,k} \) satisfying the recurrence

\[
S_{n,k} = S_{n-1,k-1} + [2(a_0 + b_0 \lambda)k + a_1 + b_1 \lambda]S_{n-1,k} + [(a_0 + b_0 \lambda)k + b_1 \lambda + a_1](a_0 + b_0 \lambda)(k + 1)S_{n-1,k+1}
\]

(5.22)

with \( S_{0,0} = 1 \). Let its row-generating polynomial \( S_n(q) = \sum_{k=0}^{n} S_{n,k}q^k \). Obviously, \( S_n(0) = S_n(a_0, a_1, b_0, b_1, \lambda) \). In [111, Theorem 1.3 (i)], we proved that \( (S_n(a_0, a_1, b_0, b_1, \lambda))_n \) is coefficientwise Hankel-totally positive in \((a_0, a_1, b_0, b_1, \lambda)\) and 3-(\(a_0, a_1, b_0, b_1, \lambda\))-log-convex, which can be extended to those of \( S_n(q) \). Further results for total positivity are stated as follows, which is obvious by Theorem 5.2 with \( a = u = 0, b = v = 1, c = x = a_0 + b_0 \lambda \) and \( y = a_1 + b_1 \lambda \).

**Proposition 5.9.** Let \( S = [S_{n,k}]_{n,k} \) be the Catalan-Stieltjes triangle defined by (5.22). Then we have

(i) the binomial row-generating matrix \( S(q) \) is totally positive in the polynomial ring \( \mathbb{Z}[a_0, a_1, b_0, b_1, \lambda, q] \) equipped with the coefficientwise order;
(ii) \((S_n(q))_{n \geq 0}\) and \((S^*_n(q))_{n \geq 0}\) are Hankel-totally positive in the ring \(\mathbb{Z}[a_0, a_1, b_0, b_1, \lambda, q]\) equipped with the coefficientwise order and \(3\)-\((a_0, a_1, b_0, b_1, \lambda, q)\)-log-convex;

(iii) the convolution \(r_n = \sum_{k \geq 0} S_{n,k} s_{n-k}\) preserves the Stieltjes moment property in \(\mathbb{R}\) for all \(a_0, a_1, b_0, b_1, \lambda \geq 0\).

### 5.5 A refined Eulerian polynomial

Let \([A_{n,k}]_{n,k}\) be a triangular array satisfying the following recurrence

\[
A_{n,k} = (b_0 n - b_0 k + b_2)A_{n-1,k-1} + (a_1 k + a_2)A_{n-1,k}
\]

(5.23)

with \(A_{0,0} = 1\). The triangle \([A_{n,k}]_{n,k}\) has been studied in different papers. It was conjectured that \([A_{n,k}]_{n,k}\) is coefficientwise totally positive in the indeterminates \(a_1, a_2, b_0, b_2\) in [20]. Let \(A_n(a_1, a_2, b_0, b_2, \lambda) = \sum_k A_{n,k} \lambda^k\). Obviously, \(A_n(a_1, a_2, b_0, b_2, \lambda)\) can be regarded as a polynomial in five indeterminates \(a_1, a_2, b_0, b_2\) and \(\lambda\). In fact, this polynomial \(A_n(a_1, a_2, b_0, b_2, \lambda)\) can be viewed as a common generalization of many famous combinatorial polynomials. For example:

- Let \(\binom{n}{k}\) be the classical Eulerian number counting the number of permutations of \(n\) elements having \(k - 1\) descents [24]. Then \(A_n(1, 0, 1, 1, \lambda)\) is the classical Eulerian polynomial \(E_n(\lambda) = \sum_k \binom{n}{k} \lambda^k\) and \(A_n(1, 1, 1, 1, \lambda) = E_{n+1}(\lambda)/\lambda\);

- Let \(\binom{n}{k}^B\) be the Eulerian number of type B counting the elements of signed group \(B_n\) with \(k\) \(B\)-descents [14]. Then the Eulerian polynomial of type \(B\) equals \(A_n(2, 1, 2, 1, \lambda)\).

By [111, (4.6)], we know

\[
\sum_{n=0}^{\infty} A_n(a_1, a_2, b_0, b_2, \lambda)t^n = \frac{1}{1 - \gamma_0 t - \beta_1 t^2},
\]

(5.24)

where \(\beta_{k+1} = \lambda(a_1 b_0 k + a_2 b_0 + a_1 b_2)(k + 1)\) and \(\gamma_k = (a_1 + b_0 \lambda)k + a_2 + b_2\lambda\). It gives that \(A_n(a_1, a_2, b_0, b_2, \lambda)\) is exactly the zeroth column of the next Catalan-Stieltjes triangle \(\mathcal{A} = [\mathcal{A}_{n,k}]_{n,k}\) satisfying the recurrence

\[
\mathcal{A}_{n,k} = \mathcal{A}_{n-1,k-1} + [(a_1 + b_0 \lambda)k + a_2 + b_2\lambda]\mathcal{A}_{n-1,k} + \lambda(a_1 b_0 k + a_2 b_0 + a_1 b_2)(k + 1)\mathcal{A}_{n-1,k+1}
\]

(5.25)

with \(\mathcal{A}_{0,0} = 1\). Denote its row-generating polynomial \(\mathcal{A}_n(q) = \sum_{k=0}^{n} \mathcal{A}_{n,k} q^k\). Obviously, \(\mathcal{A}_n(0) = A_n(a_1, a_2, b_0, b_2, \lambda)\). In [111, Theorem 1.3 (v)], we proved for \(0 \in \{a_2, b_2, a_1 - a_2, b_0 - b_2\}\) that \((A_n(a_1, a_2, b_0, b_2, \lambda))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \((x, \lambda)\)
and 3-\((x, \lambda)\)-log-convex,\(^1\) which can be extended to those of \(\mathcal{A}_n(q)\). In addition, we get more results for total positivity as follows.

**Proposition 5.10.** Let \([\mathcal{A}_{n,k}]_{n,k}\) be defined by (5.25). If \(0 \in \{a_2, b_2, a_1 - a_2, b_0 - b_2\}\), then

(i) the binomial row-generating matrix \(\mathcal{A}(q)\) is totally positive in the polynomial ring \(\mathbb{Z}[x, \lambda, q]\) equipped with the coefficientwise order;

(ii) \((\mathcal{A}_n(q))_{n \geq 0}\) and \((\mathcal{A}^*_n(q))_{n \geq 0}\) are Hankel-totally positive in the ring \(\mathbb{Z}[x, \lambda, q]\) equipped with the coefficientwise order and 3-\((x, \lambda, q)\)-log-convex;

(iii) the convolution \(r_n = \sum_{k \geq 0} \mathcal{A}_{n,k} s_k t_{n-k}\) preserves the Stieltjes moment property in \(\mathbb{R}\) for \(x \geq 0\) and \(\lambda \geq 0\).

**Proof.** (1) If \(a_2 = 0\), then the desired results in (i)-(iii) follow from Theorem 5.2 by taking \(a = u = 0, b = v = 1, c = a_1, x = b_0 \lambda\) and \(y = b_2 \lambda\).

(2) If \(b_2 = 0\), then the desired results in (i)-(iii) follow from Theorem 5.2 by taking \(a = u = 0, b = v = 1, c = b_0 \lambda, x = a_1\) and \(y = a_2\).

(3) If \(a_1 = a_2\), then the desired results in (i)-(iii) follow from Theorem 5.3 by taking \(a = u = 0, b = v = 1, c = a_1, x = b_0 \lambda\) and \(y = b_2 \lambda\).

(4) If \(b_0 = b_2\), then the desired results in (i)-(iii) follow from Theorem 5.3 by taking \(a = u = 0, b = v = 1, c = b_0 \lambda, x = a_1\) and \(y = a_2\). \(\square\)

### 6 Row-generating polynomials of exponential Riordan arrays

The Riordan array is vast and still growing and the applications cover a wide range of subjects, such as enumerative combinatorics, combinatorial sums, recurrence relations and computer science, among other topics \([8, 21, 28, 29, 68, 87, 93]\). The **exponential Riordan array** \([8, 28, 29]\), denoted by \(\mathbf{R} = [\mathbf{R}_{n,k}]_{n,k} = (g(t), f(t))\), is an infinite lower triangular matrix whose exponential generating function of the \(k\)th column is \(g(t)f^k(t)/k!\) for \(k \geq 0\), where \(g(0)f'(0) \neq 0\) and \(f(0) = 0\). In other words, for \(n, k \geq 0\),

\[
\mathbf{R}_{n,k} = \frac{n!}{k!} [t^n] g(t)f^k(t).
\]  

(6.1)

Let \(R_n(q) = \sum_{k=0}^n \mathbf{R}_{n,k} q^k\) be the row-generating polynomial of \(\mathbf{R}\). Then we have

\[
\sum_{n \geq 0} R_n(q) \frac{t^n}{n!} = g(t) \exp(qf(t)).
\]  

(6.2)

The group law is given by

\[
(g, f) * (h, \ell) = (g \times h(f), \ell(f)).
\]  

(6.3)

The identity for this law is \(I = (1, t)\) and the inverse of \((g, f)\) is \((g, f)^{-1} = (1/(g(\overline{f})), \overline{f})\), where \(\overline{f}\) is the compositional inverse of \(f\), i.e., \(\overline{f}(f(t)) = f(\overline{f}(t)) = t\).

---

\(^1\)Here \(x\) means the remaining variables of \(\{a_1, a_2, b_0, b_2\}\) excluding one being 0 in the hypothesis.
In [22, 23], Chen et al. gave two criteria for total positivity of ordinary Riordan arrays. In [113], we proved: Given a Pólya frequency ogf $g(t)$ of order $r$, if one of $f(t)$ and $1/f'(t)$ is a Pólya frequency ogf of order $r$, then exponential Riordan arrays $(g(t), f(t))$ and $(g(f(t)), f(t))$ are totally positive of order $r$. The following question concerning coefficientwise Hankel-total positivity of the row-generating polynomials is open.

**Question 6.1.** What conditions can ensure coefficientwise Hankel-total positivity of the sequence of row-generating polynomials for the exponential Riordan array $(g(t), f(t))$?

Generally speaking, if one of $f(t)$ or $g(t)$ is known, then it is a difficult problem to choose another proper function to form an exponential Riordan array $(g(t), f(t))$ with such desired properties. In what follows our results will give the partial answers.

### 6.1 Main results for exponential Riordan arrays

In [113], we obtained the following criterion for coefficientwise total positivity.

**Theorem 6.2.** [113] If $1/f'(t)$ is a Pólya frequency ogf of order $r$, then

(i) $(e^{\lambda f(t)}, f(t))$ is coefficientwise totally positive of order $r$ in $\lambda$;

(ii) $(f'(x), f(x))$ is totally positive of order $r$;

(iii) the sequence of row-generating polynomials of $(e^{\lambda f(t)}, f(t))$ is coefficientwise Hankel-totally positive of order $r$ in $(q, \lambda)$;

(iv) the sequence of row-generating polynomials of $(f'(x), f(x))$ is coefficientwise Hankel-totally positive of order $r$ in $q$;

(v) the sequence $(f_{n+1})_{n \geq 0}$ is Hankel-totally positive of order $r$, where $f(t) = \sum_{n \geq 1} f_n \frac{t^n}{n!}$.

Let $f(t) = \sum_{n \geq 1} f_n \frac{t^n}{n!}$. For the exponential Riordan array $(1, f(t))$, its entry $R_{n,k}$ is the generating polynomial for unordered forests of increasing ordered trees on $n+1$ total vertices with $k$ components in which each vertex with $i$ children gets a weight $x_i$. In particular, $f_n = R_{n,1}$ is the generating polynomial for increasing ordered trees $n+1$ vertices in which each vertex with $i$ children gets a weight $x_i$. We refer the reader to [9, 75].

In what follows we will extend Theorem 6.2 to the following generalized result.

**Theorem 6.3.** Let $R$ be a partially ordered commutative ring and $1/f'(t) = \varphi(t)\phi(t)$. Assume that $[R_{n,k}]_{n,k} = (\varphi(f(t)) \exp(\lambda f(t)), f(t))$ and $R_n(q) = \sum_k R_{n,k} q^k$. If both $\varphi$ and $\phi$ are Pólya frequency ogf of order $r$ in $R[[t]]$, then

(i) $(\varphi(f(t)) \exp(\lambda f(t)), f(t))$ is totally positive of order $r$ in the polynomial ring $R[\lambda]$ equipped with the coefficientwise order;

(ii) both $(R_n(q))_{n \geq 0}$ and $(R'_n(q))_{n \geq 0}$ are Hankel-totally positive of order $r$ in the polynomial ring $R[\lambda, q]$ equipped with the coefficientwise order;
(iii) both \( (T_n(\lambda))_{n \geq 0} \) and \( (T_n^*(\lambda))_{n \geq 0} \) are Hankel-totally positive of order \( r \) in the polynomial ring \( R[\lambda] \) equipped with the coefficientwise order, where \( \varphi(f(t)) \exp(\lambda f(t)) = \sum_n T_n(\lambda) \frac{t^n}{n!} \).

(iv) \( (f_{n+1})_{n \geq 0} \) is Hankel-totally positive of order \( r \) in \( R \), where \( f(t) = \sum_{n \geq 1} f_n \frac{t^n}{n!} \).

**Remark 6.4.**  (1) Note that \( \frac{1}{f'(t)} = \varphi(t) \phi(t) \) implies \( f'(t) = \varphi(f(t)) \phi(f(t)) \). This type equation often is called the autonomous differential equation [9]. It plays an important role in the flow function of Copeland [89, A145271], which is also closely related to formal group laws for elliptic curves, the Abel equation, the Schröder’s functional equation, Koenigs functions for compositional iterates, the renormalization group equation and Hopf algebra.

(2) By taking \( \lambda = 0 \) and \( \phi = 1 \), (i) and (ii) of Theorem 6.3 implies Theorem 6.2 (ii) and (iv), respectively;

(3) By taking \( \varphi = 1 \), (i) and (ii) of Theorem 6.3 implies Theorem 6.2 (i) and (iii), respectively.

In order to prove Theorem 6.3, we need the following equivalent characterization for the exponential Riordan arrays.

**Proposition 6.5.** [28] Let \( [R_{n,k}]_{n,k \geq 0} = (g(t), f(t)) \) be an exponential Riordan array. Then there exist two sequences \( (z_n)_{n \geq 0} \) and \( (a_n)_{n \geq 0} \) such that

\[
R_{0,0} = 1, \quad R_{n,0} = \sum_{i \geq 0} i! z_i R_{n-1,i}, \quad R_{n,k} = \frac{1}{k!} \sum_{i \geq k-1} i! (z_{i-k} + ka_{i-k+1}) R_{n-1,i}\]

for \( n, k \geq 1 \). In particular,

\[
Z(t) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))}, \quad A(t) = f'(\bar{f}(t)),
\]

where \( Z(t) = \sum_{n \geq 0} z_n t^n \) and \( A(t) = \sum_{n \geq 0} a_n t^n \).

Associated to each exponential Riordan array \( R = (g(t), f(t)) \), there is a production matrix \( P \) such that

\[
\overline{R} = RP,
\]

where \( \overline{R} \) is obtained from \( R \) with the first row removed. Assume that \( z_{-1} = 0 \). Deutsch et al. [28] showed the production matrix

\[
P = [p_{i,j}]_{i,j \geq 0} = \begin{bmatrix}
  z_0 & a_0 \\
  \frac{1!}{1!}(z_0 + a_1) & \frac{1!}{2!}(z_0 + 2a_1) \\
  \frac{2!}{2!}(z_2 + a_2) & \frac{2!}{3!}(z_0 + 3a_1) \\
  \frac{3!}{3!}(z_2 + a_3) & \frac{3!}{4!}(z_0 + 4a_1) \\
  \vdots & \vdots & \ddots & \ddots
\end{bmatrix}.
\]
where the elements
\[ p_{i,j} = \frac{i!}{j!} (z_{i-j} + ja_{i-j+1}) \quad \text{for} \quad i, j \geq 0. \] (6.8)

**Proof of Theorem 6.3:** Let \( g(t) = \varphi(f(t)) \exp(\lambda f(t)) \). We derive
\[ g'(t) = \frac{\varphi'(f(t)) f'(t) + \varphi(f(t)) \lambda f'(t)}{\varphi(f(t))}. \] (6.9)

In addition, \( f(f(t)) = t \) also implies \( f'(f(t)) f'(t) = 1 \). Then for the exponential Riordan array \((\varphi(f(t)) \exp(\lambda f(t)), f(t))\), by Proposition 6.5, we get its two formal power series
\[ Z(t) = \frac{\varphi'(t)}{\varphi(t)f(t)} + \frac{\lambda}{f(t)} = \phi(t)(\varphi'(t) + \lambda \varphi(t)), \quad A(t) = \phi(t)\varphi(t). \] (6.10)

It follows from (6.7) that its production matrix
\[ P = \Lambda \Gamma(Z) \Lambda^{-1} + \Lambda \Gamma(A) \Theta \Lambda^{-1} \]
\[ = \Lambda \Gamma(\phi(t)) \left( \lambda \Gamma(\varphi(t)) + \Gamma(\varphi'(t)) + \Gamma(\varphi(t)) \begin{pmatrix} 0 & 1 \\ 2 \\ \vdots \end{pmatrix} \right) \Lambda^{-1} \]
\[ = \Lambda \Gamma(\phi(t)) \left( \lambda \Gamma(\varphi(t)) + \begin{pmatrix} 0 & 1 \\ 2 \\ \vdots \end{pmatrix} \Gamma(\varphi(t)) \right) \Lambda^{-1} \]
\[ = \Lambda \Gamma(\phi(t)) \begin{pmatrix} \lambda & 1 \\ \lambda & 2 \\ \lambda & 3 \\ \vdots \end{pmatrix} \Gamma(\varphi(t)) \Lambda^{-1} \]
\[ = \underbrace{\Lambda \Gamma(\phi(t)) \Lambda^{-1}}_{\text{R}(\lambda, \varphi)} \underbrace{\begin{pmatrix} \lambda & 1 \\ \lambda & 1 \\ \vdots & \vdots \end{pmatrix}}_{\text{R}(\lambda, \varphi)} \Lambda^{-1} \] (6.11)

Thus (i) follows from Theorem 4.1 (i) with \( q = 0 \) and (ii) is immediate from Theorem 4.1 (ii) and Corollary 4.4. By \( R_n(0) = R_{n,0} = T_n(\lambda) \), (ii) implies (iii). Note \( A(t) = \phi(t)\varphi(t) \) implies \( f'(t) = \varphi(f(t))\phi(f(t)) \). Taking \( \phi(t) = 1 \) and \( \lambda = 0 \), we have
\[ (\varphi(f(t)) \exp(\lambda f(t)), f(t)) = (f'(t), f(t)). \]

Thus combining (iii) and \( T_n(0) = f_{n+1} \) gives (iv). \( \square \)

Taking \( r \to \infty \) in Theorem 6.3, we get the following result, whose (v) is by Theorem 4.5.
Theorem 6.6. Let $R$ be a partially ordered commutative ring. If there exist two Pólya frequency ogfs $\varphi$ and $\phi$ in $R[[t]]$ such that $1/f(t) = \varphi(t)\phi(t)$, then

(i) the exponential Riordan array $(\varphi(f(t)) \exp(\lambda f(t)), f(t))$ is totally positive in the polynomial ring $R[\lambda]$ equipped with the coefficientwise order;

(ii) both $(R_n(q))_{n \geq 0}$ and $(R'_n(q))_{n \geq 0}$ are Hankel-totally positive in the polynomial ring $R[\lambda, q]$ equipped with the coefficientwise order, where $R_n(q)$ is the row-generating polynomial of $(\varphi(f(t)) \exp(\lambda f(t)), f(t));$

(iii) both $(T_n(\lambda))_{n \geq 0}$ and $(T'_n(\lambda))_{n \geq 0}$ are Hankel-totally positive in the polynomial ring $R[\lambda]$ equipped with the coefficientwise order, where $\varphi(f(t)) \exp(\lambda f(t)) = \sum_n T_n(\lambda) t^n$;

(iv) the sequence $(f_{n+1})_{n \geq 0}$ is Hankel-totally positive in $R$, where $f(t) = \sum_{n \geq 1} f_n t^n$;

(v) the convolution $z_n = \sum_{k=0}^n R_{n,k} x_k y_{n-k}$ preserves the Stieltjes moment property in $\mathbb{R}$ for $\lambda \geq 0$ and $R = \mathbb{R}$, where $[R_{n,k}]_{n,k} = (\varphi(f(t)) \exp(\lambda f(t)), f(t)).$

For Theorem 6.6, if $\phi(t) = 1$ and $\varphi(t) = \prod_{i=0}^{m} \frac{1}{1-x_i t}$, then by (6.11), the production matrix of the exponential Riordan array $(f'(t), f(t))$ is

$$P = 
\begin{pmatrix}
\lambda & 1 \\
\lambda & 1 \\
\lambda & 1 \\
\vdots & \vdots \\
\end{pmatrix} \Lambda(\varphi(t)) \Lambda^{-1}. \quad (6.12)
$$

By (6.12) with $q = 0$ and [76, Proposition 12.20: b], $f_{n+1}$ equals the multivariate Eulerian polynomial of negative type $Q_n^{(m)-}(x)$ in [76, Section 12.3.1], which was defined in terms of increasing multi-$m$-ary trees on $n + 1$ vertices in each $i$-edge gets a weights $x_i$ and can be expressed in terms of differential operators [76, Section 12.3.1]. Meanwhile, for the exponential Riordan array $(f'(t), f(t))$, its entry $R_{n,k}$ is the generating polynomial for unordered forests of increasing multi-$m$-ary trees on $n + 1$ total vertices with $k + 1$ components in each $i$-edge gets a weight $x_i$. When $m \to \infty$, the multivariate Eulerian polynomial of negative type gives the Eulerian symmetric function of negative type $Q_n^{(\infty)-}(x)$ [76, Section 12.3.1]. Note $T_n(0) = f_{n+1}$ and $R_n(q) = T_n(\lambda)$. Thus, both $R_n(q)$ and $T_n(\lambda)$ are the generalizations of $Q_n^{(m)-}(x)$. In addition, when $m \to \infty$, both $R_n(q)$ and $T_n(\lambda)$ are also extended to the generalizations of $Q_n^{(\infty)-}(x)$.

In [76], the $m$-branched Stieltjes-type continued fraction (1.4) plays a heart role in coefficientwise Hankel-total positivity. In the following, we also present the $m$-branched Stieltjes-type continued fraction for the ordinary generating function of the row-generating polynomials for exponential Riordan arrays.

Theorem 6.7. Assume that $1/f(t) = \phi(t)\varphi(t) = \prod_{i=0}^{m} (1 + x_i t)$. Let $R_n(q)$ and $T_n(\lambda)$ be defined in Theorem 6.6.

(i) If $\phi(t) = 1$, then we have the $(m+1)$-branched Stieltjes-type continued fraction

$$1 + \sum_{n \geq 0} (q + \lambda) R_n(q) t^n = \frac{1}{1 - \alpha_{m+1} t \prod_{i=1}^{m+1} \frac{1}{1 - \alpha_{m+1+i} t \prod_{i=2}^{m+1} \frac{1}{1 - \cdots}}}$$

32
with coefficients
\[(\alpha_i)_{i \geq m+1} = (\lambda, q, x_0, \ldots, x_m, \lambda + q, 2x_0, \ldots, 2x_m, \lambda + q, 3x_0, \ldots, 3x_m, \ldots);\]

In particular, we have
\[1 + \sum_{n \geq 0} \lambda T_n(\lambda) t^n = \frac{1}{1 - \alpha_{m+1} t \prod_{i_1=1}^{m+1} \frac{1}{1 - \alpha_{m+1+i_1} t \prod_{i_2=1}^{m+1} \frac{1}{1 - \cdots}}};\]

with coefficients \[(\alpha_i)_{i \geq m+1} = (\lambda, x_0, \ldots, x_m, \lambda, 2x_0, \ldots, 2x_m, \lambda, 3x_0, \ldots, 3x_m, \ldots).\]

(ii) If \(\varphi(t) = 1\), then the \((m+1)\)-branched Stieltjes-type continued fraction
\[\sum_{n \geq 0} R_n(q) t^n = \frac{1}{1 - \alpha_{m+1} t \prod_{i_1=1}^{m+1} \frac{1}{1 - \alpha_{m+1+i_1} t \prod_{i_2=1}^{m+1} \frac{1}{1 - \cdots}}};\]

with coefficients \[(\alpha_i)_{i \geq m+1} = (\lambda, q, x_0, \ldots, x_m, \lambda + q, 2x_0, \ldots, 2x_m, \lambda + q, 3x_0, \ldots, 3x_m, \ldots);\]

In particular, we have
\[\sum_{n \geq 0} T_n(\lambda) t^n = \frac{1}{1 - \alpha_{m+1} t \prod_{i_1=1}^{m+1} \frac{1}{1 - \alpha_{m+1+i_1} t \prod_{i_2=1}^{m+1} \frac{1}{1 - \cdots}}};\]

with coefficients \[(\alpha_i)_{i \geq m+1} = (\lambda, x_0, \ldots, x_m, \lambda + q, 2x_0, \ldots, 2x_m, \lambda, 3x_0, \ldots, 3x_m, \ldots).\]

(iii) We have the \(m\)-branched Stieltjes-type continued fraction
\[1 + \sum_{n \geq 0} x_0 f_{n+1} t^{n+1} = \frac{1}{1 - \alpha_m t \prod_{i_1=1}^{m} \frac{1}{1 - \alpha_{m+i_1} t \prod_{i_2=1}^{m} \frac{1}{1 - \cdots}}};\]

with coefficients \[(\alpha_i)_{i \geq m} = (x_0, \ldots, x_m, 2x_0, \ldots, 2x_m, 3x_0, \ldots, 3x_m, \ldots).\]
Proof. (i) If \( \phi(t) = 1 \) and \( \varphi(t) = \prod_{i=0}^{m}(1 + x_i t) \), then by (6.11), we have

\[
\begin{pmatrix}
\lambda & 1 \\
\lambda & 1 \\
\lambda & 1 \\
\vdots & \ddots & \ddots \\
\lambda & 1 \\
\end{pmatrix}
\Lambda \Gamma(\varphi(t)) \Lambda^{-1}
\]

\[
= \begin{pmatrix}
\lambda & 1 \\
\lambda & 1 \\
\lambda & 1 \\
\vdots & \ddots & \ddots \\
\lambda & 1 \\
\end{pmatrix}
\Lambda \left( \prod_{i=0}^{m} \Gamma(1 + x_i t) \right) \Lambda^{-1}
\]

\[
= \begin{pmatrix}
\lambda & 1 \\
\lambda & 1 \\
\lambda & 1 \\
\vdots & \ddots & \ddots \\
\lambda & 1 \\
\end{pmatrix}
\prod_{i=0}^{m} \Lambda \Gamma(1 + x_i t) \Lambda^{-1}
\]

\[
= \begin{pmatrix}
\lambda & 1 \\
\lambda & 1 \\
\lambda & 1 \\
\vdots & \ddots & \ddots \\
\lambda & 1 \\
\end{pmatrix}
\prod_{i=0}^{m} \begin{pmatrix}
1 & 1 & 1 \\
2x_i & 1 & 1 \\
3x_i & 1 & 1 \\
\vdots & \ddots & \ddots \\
\end{pmatrix}
\]

(6.13)

In consequence, the production matrix of \( (\varphi(f(t)) \exp(\lambda f(t)), f(t)) B_q \) equals

\[
B_q^{-1} PB_q = \begin{pmatrix}
q + \lambda & 1 \\
q + \lambda & 1 \\
q + \lambda & 1 \\
\vdots & \ddots & \ddots \\
q + \lambda & 1 \\
\end{pmatrix}
\prod_{i=0}^{m} \begin{pmatrix}
1 & 1 & 1 \\
2x_i & 1 & 1 \\
3x_i & 1 & 1 \\
\vdots & \ddots & \ddots \\
\end{pmatrix}
\]

which in terms of Proposition 7.6 (Odd contraction formula for m-Stieltjes-Rogers polynomials) and Proposition 8.2 (b) in [76] is exactly the production matrix for the \((m + 1)\)-branched Stieltjes-type continued fraction

\[
1 + \sum_{n \geq 0} (q + \lambda) R_n(q) t^n = \frac{1}{1 - \alpha_m t \prod_{i_1=1}^{m} \frac{1}{1 - \alpha_{m+i_1} t \prod_{i_2=1}^{m} \frac{1}{1 - \alpha_{m+i_1+i_2} t \prod_{i_3=1}^{m} \frac{1}{1 - \cdots}}}}
\]

(6.14)

with coefficients

\[
(\alpha_i)_{i \geq m+1} = (\lambda + q, x_0, \ldots, x_m, \lambda + q, 2x_0, \ldots, 2x_m, \lambda + q, 3x_0, \ldots, 3x_m, \ldots).
\]
(ii) Similarly, if \( \varphi(t) = 1 \) and \( \phi(t) = \prod_{i=0}^{m}(1 + x_i t) \), then

\[
B_q^{-1}PB_q = \prod_{i=0}^{m} \begin{bmatrix}
1 & 1 & 1 & \cdots \\
x_i & 2x_i & 3x_i & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} \begin{pmatrix}
q + \lambda & 1 \\
q & q + \lambda & 1 \\
q & q & q + \lambda & 1 \\
1 & \cdots & \cdots & \cdots
\end{pmatrix},
\]

which in terms of [76, Proposition 7.2 and Proposition 8.2 (b)] implies the desired \((m+1)\)-branched Stieltjes-type continued fraction

\[
\sum_{n\geq0} R_n(q) t^n = \frac{1}{1 - \alpha_{m} t \prod_{i_1=1}^{m}1 - \alpha_{m+i_1} t \prod_{i_2=1}^{m}1 - \alpha_{m+i_1+i_2} t \prod_{i_3=1}^{m}1 - \cdots}
\]

with coefficients

\[
(\alpha_i)_{i \geq m+1} = (\lambda + q, x_0, \ldots, x_m, \lambda + q, 2x_0, \ldots, 2x_m, \lambda + q, 3x_0, \ldots, 3x_m, \ldots).
\]

(iii) For \( \phi(t) = 1 \) and \( \lambda = 0 \), we have \( T_n(0) = f_{n+1} \) and the production matrix

\[
P = \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots \\
0 & 1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix} \prod_{i=0}^{m} \begin{bmatrix}
1 & 1 & 1 & \cdots \\
x_i & 2x_i & 3x_i & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix},
\]

which in terms of Proposition 7.6 (Odd contraction formula for \( m \)-Stieltjes-Rogers polynomials) and Proposition 8.2 (b) in [76] is exactly the production matrix for the \( m \)-branched Stieltjes-type continued fraction

\[
1 + \sum_{n\geq0} x_0 f_{n+1} t^{n+1} = \frac{1}{1 - \alpha_{m} t \prod_{i_1=1}^{m}1 - \alpha_{m+i_1} t \prod_{i_2=1}^{m}1 - \alpha_{m+i_1+i_2} t \prod_{i_3=1}^{m}1 - \cdots}
\]

with coefficients \((\alpha_i)_{i \geq m} = (x_0, \ldots, x_m, 2x_0, \ldots, 2x_m, 3x_0, \ldots, 3x_m, \ldots)\). \( \square \)
For Theorem 6.7, it follows from [76, Section 12] that the \( f_{n+1} \) equals the \textbf{multivariate Eulerian polynomial} \( Q_n^{(m)}(x) \) in [76, Section 12.2], which was defined in terms of increasing \((m+1)\)-ary trees on \( n + 1 \) total vertices in which each \( i \)-edge gets a weight \( x_i \) and has differential expressions [76, Section 12.2]. Meanwhile, for the exponential Riordan array \( (f'(t), f(t)) \), its entry \( R_{n,k} \) is the generating polynomial for unordered forests of increasing \((m+1)\)-ary trees on \( n + 1 \) total vertices with \( k + 1 \) components in which each \( i \)-edge gets a weight \( x_i \). In particular, \( R_{n,1} = f_{n+1} \). When \( m \to \infty \), the multivariate Eulerian polynomial \( Q_n^{(m)}(x) \) gives the \textbf{Eulerian symmetric function} \( Q_n^{(\infty)}(x) \) [76, Section 12.2]. Note \( T_n(0) = f_{n+1} \) and \( R_n(q) = T_n(\lambda) \). Thus, both \( R_n(q) \) and \( T_n(\lambda) \) are the generalizations of the multivariate Eulerian polynomial \( Q_n^{(m)}(x) \). In addition, when \( m \to \infty \), both \( R_n(q) \) and \( T_n(\lambda) \) also give the generalizations of the Eulerian symmetric function \( Q_n^{(\infty)}(x) \). Copeland also call the multivariate Eulerian polynomial as the \textbf{refined Eulerian polynomial} [89, A145271].

Finally, based on the exponential Riordan arrays, we also consider an associated array as follows.

\textbf{Theorem 6.8.} Let \( f(t) \in \mathbb{R}[[t]] \). Define \( [F_{n,k}]_{n,k} := (1, f(t)) \) and

\[
\left( \frac{1}{1 - q f(t)} \right)^y = \sum_{n \geq 0} F_n^\circ(q, y) \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{k=0}^{n} F_{n,k}^\circ(q) y^k \frac{t^n}{n!}. \tag{6.18}
\]

If \( 1/f(t) \) is a Pólya frequency ogf in \( \mathbb{R}[[t]] \), then

(i) \( [F_{n,k}^\circ(q)]_{n,k} \) is coefficientwise totally positive in \( q \);

(ii) \( (F_n^\circ(q, y))_{n \geq 0} \) is a Stieltjes moment sequence (of real numbers) for \( q \geq 0 \) and \( y \geq 0 \);

(iii) \( t_n = \sum_{k=0}^{n} F_n^\circ(q) r_k s_{n-k} \) preserves the Stieltjes moment property in \( \mathbb{R} \) for \( q \geq 0 \);

(iv) if \( (F_{n,k})_{k=0}^{n} \) is a Pólya frequency sequence, then so is \( (F_{n,k}^\circ(q))_{k=0}^{n} \) for \( q \geq 0 \).

\textbf{Proof.} (i) Let \( \langle y \rangle_k = (y + 1) \cdots (y + k - 1) \). By [24, Theorem B, p.141], we have

\[
F_n^\circ(q, y) = \sum_{k} F_{n,k} q^k \langle y \rangle_k \tag{6.19}
\]

\[
= \sum_{k} F_{n,k} q^k \sum_{i} \binom{k}{i} y^i \tag{6.20}
\]

\[
= \sum_{i} \left( \sum_{k} F_{n,k} q^k \binom{k}{i} \right) y^i, \tag{6.21}
\]

where \( \binom{k}{i} \) is the signless Stirling number of the first kind. This implies that

\[
F_n^\circ(q, i) = \sum_{k} F_{n,k} q^k \binom{k}{i}. \tag{6.22}
\]
So we have the decomposition

\[
[F_{n,k}^\circ(q)]_{n,k} = [F_{n,k}q^k]_{n,k} \left[ \begin{array}{c} n \\ k \end{array} \right]_{n,k} = [F_{n,k}]_{n,k} \begin{pmatrix} 1 & q & q^2 & \cdots \\ \end{pmatrix} \left[ \begin{array}{c} n \\ k \end{array} \right]_{n,k}.
\]

(6.23)

Note by Theorem 6.6 (i) that \([F_{n,k}]_{n,k}\) is totally positive in \(\mathbb{R}\). In addition, it is known that \(\left[ \begin{array}{c} n \\ k \end{array} \right]_{n,k}\) is totally positive (see [15] for instance), which also follows from Theorem 6.6 (i) because \(\left[ \begin{array}{c} n \\ k \end{array} \right]_{n,k} = (1, -\log(1 - t))\) in terms of

\[
\sum_{n,k} \left[ \begin{array}{c} n \\ k \end{array} \right]_{n,k} x^k t^n = (1 - t)^{-x} = e^{x \log \frac{1}{1-t}}.
\]

(6.24)

Thus, applying the classical Cauchy-Binet formula to the decomposition above (6.23), we have \([F_{n,k}^\circ(q)]_{n,k}\) is coefficientwise totally positive in \(q\).

(ii) and (iii) In terms of the proof in (i), we have \(F_{n}^\circ(q,y) = \sum_{k} F_{n,k}q^k(y)_k\). Note that \((q^n(y)_n)_{n\geq0}\) is a Stieltjes moment sequence for \(q > 0\) and \(y > 0\) because

\[
\sum_{n\geq0} q^n(y)_nt^n = \frac{1}{1 - (1 - 2qt) / (1 - (y + 1)qt / (1 - \cdots))}.
\]

(6.25)

by [38, section 26]. Thus by Theorem 6.6 (v), we have \((F_{n}^\circ(q,y))_{n\geq0}\) is a Stieltjes moment sequence (of real numbers) for \(q \geq 0\) and \(y \geq 0\). This combines Theorem 4.5 to give (iii).

Finally, for (iv), it follows from the Brenti’s result [12, Theorem 2.4.3]: if a polynomial \(\sum_{k=0}^n a_n,k y^k\) has only real zeros, then so does \(\sum_{k=0}^n a_n,k (y)_k\). This completes the proof.

\[\square\]

### 6.2 Rook polynomials and signless Laguerre polynomials

Let \(\mathfrak{S}_n(q)\) denote the rook polynomial of a square of side \(n\), which is given by

\[
\mathfrak{S}_n(q) = \sum_{k=0}^n \binom{n}{k}^2 k! q^k
\]

(see [83, Chapter 3. Problems 18] for instance). It coincides with the matching polynomial of the complete bipartite graph \(K_{n,n}\). It has only real zeros in terms of the rook theory or matching polynomials and is strongly \(q\)-log-convex [114]. Sokal conjectured the following stronger property.
**Conjecture 6.9.** [90] The sequence $(\mathcal{S}_n(q))_{n \geq 0}$ is coefficientwise Hankel-totally positive in $q$.

For the Laguerre polynomial $L_n^{(\alpha)}(q)$ with $\alpha \geq -1$ (see [2] for instance), its exponential generating function is

$$\sum_{n \geq 0} L_n^{(\alpha)}(q) \frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{qt}{t-1}\right)$$  \hspace{1cm} (6.26)

and its explicit formula is

$$L_n^{(\alpha)}(q) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{n!}{k!} (-q)^k$$  \hspace{1cm} (6.27)

for $n \geq 0$. Let $\widetilde{L}_n^{(\alpha)}(q) = L_n^{(\alpha)}(-q)$ for $n \geq 0$. It is obvious for $\alpha = 0$ that

$$\mathcal{S}_n(q) = q^n \widetilde{L}_n^{(0)}(1/q).$$  \hspace{1cm} (6.28)

For signless Laguerre polynomials $\widetilde{L}_n^{(\alpha)}(q)$, the following result was recently proved in [113], whose (ii) also implies Conjecture 6.9. Now we present a different proof.

**Proposition 6.10.** Let $\alpha \geq -1$ and $\widetilde{L}_n^{(\alpha)*}(q) = q^n \widetilde{L}_n^{(\alpha)}(1/q)$. Then we have

(i) the triangular matrix $[\binom{n+\alpha}{n-k} \frac{n!}{k!}]_{n,k}$ is totally positive;

(ii) both $(\widetilde{L}_n^{(\alpha)}(q))_{n \geq 0}$ and $(\widetilde{L}_n^{(\alpha)*}(q))_{n \geq 0}$ are coefficientwise Hankel-totally positive in $q$ and $3$-$q$-log-convex;

(iii) the convolution

$$z_n = \sum_{k \geq 0} \binom{n+\alpha}{n-k} \frac{n!}{k!} x_k y_{n-k}$$  \hspace{1cm} (6.29)

preserves the Stieltjes moment property in $\mathbb{R}$.

**Proof.** Let

$$g(t) = \frac{1}{(1-t)^{\alpha+1}}, \quad f(t) = \frac{t}{1-t}.\hspace{1cm} (6.30)$$

Rewrite (6.26) as

$$\sum_{n \geq 0} \widetilde{L}_n^{(\alpha)}(q) \frac{t^n}{n!} = g(t) \exp(qf(t)).$$  \hspace{1cm} (6.31)

Obviously, $\widetilde{L}_n^{(\alpha)}(q)$ is the row-generating polynomial of $(g(t), f(t)) := [\widetilde{L}_n^{(\alpha)}]_{n,k}$. By (6.7), we have two formal power series

$$Z(t) = (\alpha + 1)(1 + t), \quad A(t) = (1 + t)^2.$$  \hspace{1cm} (6.32)
Hence we get the production matrix of \((g(t), f(t))\) is

\[
\begin{bmatrix}
\alpha + 1 & 1 \\
\alpha + 1 & \alpha + 3 & 1 \\
2(\alpha + 2) & \alpha + 5 & 1 \\
3(\alpha + 3) & \alpha + 7 & 1 \\
\ldots & \ldots & \ldots
\end{bmatrix}.
\]

Thus, the exponential Riordan array \([\tilde{L}_{n,k}]_{n,k}\) satisfies the recurrence

\[
\tilde{L}_{n,k}^{(\alpha)} = \tilde{L}_{n-1,k-1}^{(\alpha)} + (\alpha + 1 + 2k)\tilde{L}_{n-1,k}^{(\alpha)} + (\alpha + k + 1)(k + 1)\tilde{L}_{n-1,k+1}^{(\alpha)}
\]

(6.33)

with \(\tilde{L}_{0,0}^{(\alpha)} = 1\). It follows from Theorem 5.2 by taking \(a = u = 0\), \(b = c = x = v = 1\) and \(y = \alpha + 1\) that we get the desired results in (i)-(iii).

\[\square\]

### 6.3 Enumerative labeled trees and forests

Let \(f_{n,k}\) denote the number of forests of rooted trees on \(n\) labeled vertices having \(k\) components (i.e. \(k\) trees) and it was proved that

\[
f_{n,k} = \binom{n-1}{k-1} n^{n-k},
\]

(6.34)

which also counts the number of rooted labeled trees on \(n + 1\) vertices with a root degree \(k\), see [89, A137452] for instance. In particular, \(f_{n,1} = n^{n-1}\) is exactly the number of rooted trees on \(n\) labeled vertices and \(\sum_k f_{n,k} = (n + 1)^{n-1}\) is the number of forests of rooted trees on \(n\) labeled vertices. Denote by \(T(t) = \sum_{n \geq 1} n^{n-1} t^n / n!\) the tree function, which is closely related to the famous Lambert function \(W(t) = -T(-t)\). Then one has the exponential generating function

\[
\sum_{n \geq 0} \sum_{k=1}^{n} f_{n,k} q^k t^n / n! = \exp(qT(t)).
\]

(6.35)

In [113], we proved that the matrix \([f_{n,k}]_{n,k}\) is totally positive and its row-generating polynomial is coefficientwise Hankel-totally positive. In addition, \((n^{n-1})_{n \geq 1}\) and \(((n + 1)^{n-1})_{n \geq 1}\) are Stieltjes moment sequences (i.e., their Hankel matrices are totally positive, respectively). More generally, the convolution \(z_n = \sum_{k \geq 0} f_{n+1,k+1} x_k y_{n-k}\) preserves Stieltjes moment property of sequences. Some of results above were also independently proved in [91]. Note that

\[
T(t) = te^{-t}, \quad \text{and} \quad \frac{1}{T(t)} = \frac{e^t}{1 - t}
\]

(6.36)

is a Pólya frequency ogf. Thus by Theorem 6.6, we have a more generalized result as follows.
Proposition 6.11. Let $F^{(i,j)} = [F^{(i,j)}_{n,k}]_{n,k}$ be the exponential Riordan array $(e^{rT(t)}, T(t))$ for $i \in \{0, 1\}$ and $j \in \{0, 1\}$ and its row-generating polynomial $F^{(i,j)}_n(q) = \sum_k F^{(i,j)}_{n,k} q^k$. Then we have

(i) the lower-triangle $F^{(i,j)}$ is totally positive;

(ii) both the row-generating polynomial sequence $(F^{(i,j)}_n(q))_n$ and its reversed polynomial sequence $(F^{(i,j)*}_n(q))_n$ are coefficientwise Hankel-totally positive in $q$ and $3$-q-log-convex;

(iii) the convolution $z_n = \sum_{k=0}^n F^{(i,j)}_{n,k} x_k y_{n-k}$ preserves the Stieltjes moment property in $\mathbb{R}$.

Obviously, $F^{(0,0)} = [f_{n,k}]_{n,k}$ and $F^{(1,1)} = [f_{n+1,k+1}]_{n,k}$, which have different combinatorial interpretations. It is natural to ask the following question.

Question 6.12. Find the combinatorial interpretation for $F^{(0,1)}$ and $F^{(1,0)}$, respectively.

An ordered forest of rooted trees is simply a forest of rooted trees in which we have specified a linear ordering of the trees. Let $f_{n,k}^{ord}$ be the number of ordered forests of rooted trees on the vertex set $[n]$ with $k$ components and it is obvious that $f_{n,k}^{ord} = f_{n,k} k!$. It is known that

$$\sum_{n,k=0}^n f_{n,k}^{ord} q^k \frac{k^n}{n!} = \frac{1}{1 - q T(t)}. \tag{6.37}$$

Let $F_n^{ord}(q) = \sum_{k=0}^n f_{n,k}^{ord} q^k$. Sokal [91, Conjectures 6.7 and 6.8] made the following two conjectures.

Conjecture 6.13. The polynomial sequence $(F_n^{ord}(q))_{n \geq 0}$ is coefficientwise Hankel-totally positive in $q$.

Conjecture 6.14. The polynomial sequence $(F_n^{ord}(q)/n!)_{n \geq 0}$ is coefficientwise Hankel-totally positive in $q$.

The following result confirms Conjecture 6.14 and gives a stronger support for Conjecture 6.13.

Proposition 6.15. Let $F_n^{ord}(q)$ and $f_{n,k}^{ord}$ be defined above. Then we have

(i) $(F_n^{ord}(q)/n!)_{n \geq 0}$ is coefficientwise Hankel-totally positive in $q$;

(ii) the convolution $z_n = \sum_{k \geq 0} \frac{1}{n!} f_{n,k}^{ord} x_k y_{n-k}$ preserves the Stieltjes moment property in $\mathbb{R}$.

(iii) $(F_n^{ord}(q))_{n \geq 0}$ is a Stieltjes moment sequence (of real numbers) for any fixed $q \geq 0$.

Proof. Note that the sequence $(n^n/n!)_{n \geq 0}$ is a Stieltjes moment sequence because

$$\frac{n^n}{n!} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left( \frac{\sin x}{x} e^{x \cot v} \right)^n dx \tag{6.38}$$
(see [55, Corollary 2.4] for instance) and \((1/n)_{n \geq 1}\) is also a Stieltjes moment sequence. Thus their product sequence \((n^{n-1}/n!)_{n \geq 1}\) is Stieltjes moment. In terms of Stieltjes’s continued fraction criterion for Stieltjes moment sequences, there exist nonnegative real numbers \(\alpha_0, \alpha_1, \alpha_2, \ldots\) such that

\[
\mathcal{T}(t) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} t^n = \frac{t}{1 - \frac{\alpha_0 t}{1 - \frac{\alpha_1 t}{1 - \ldots}}}.
\]

(6.39)

Thus we have

\[
\sum_{n \geq 0} F_{ord}(q) \frac{t^n}{n!} = \frac{1}{1 - q \mathcal{T}(t)} = \frac{1}{1 - \frac{qt}{1 - \frac{\alpha_0 t}{1 - \frac{\alpha_1 t}{1 - \ldots}}}}.
\]

(6.40)

By (1.4) and Theorem 1.3, we immediately get that \((F_{ord}(q)/n!)_{n \geq 0}\) is a 1-Stieltjes-Rogers polynomial sequence with coefficientwise Hankel-total positivity in \(q\). This also implies that (ii) holds in terms of Theorem 4.5. For (iii), note that \((n!)_n\) is a Stieltjes moment sequence and the product of two Stieltjes moment sequences is still a Stieltjes moment sequence. Thus (iii) is immediate from (i). □

In fact, the ordered forest number \(f_{ord, n,k}\) also counts the number of functional digraphs on the vertex set \([n]\) with \(k\) cyclic vertices. Recall that a functional digraph is a directed graph \(G = (V, E)\) in which every vertex has out-degree 1 and a vertex of a functional digraph is cyclic if it lies on one of the cycles (or equivalently, is the root of one of the underlying trees). Let \(\psi_{n,k}\) be the number of functional digraphs on the vertex set \([n]\) with \(k\) (weakly connected) components, whose bivariate exponential generating function is

\[
\sum_{n \geq 0} \sum_{k=0}^n \psi_{n,k} y^k \frac{t^n}{n!} = \frac{1}{(1 - \mathcal{T}(t))^y}.
\]

(6.41)

Let \(\psi_n(y) = \sum_{k=0}^n \psi_{n,k} y^k\). Sokal [91, Conjecture 6.9] made the following conjecture.

**Conjecture 6.16.** Let \(\psi_{n,k}\) and \(\psi_n(y)\) be defined above.

(i) The lower-triangular matrix \([\psi_{n,k}]_{n,k}\) is totally positive.

(ii) The polynomial sequence \((\psi_n(y))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \(y\).

(iii) The sequence \((\psi_{n+1,1})_{n \geq 0}\) is Hankel-totally positive (i.e. is a Stieltjes moment sequence).

Let \(\psi_n(q, y)\) be the generating polynomial for functional digraphs on the vertex set \([n]\) with a weight \(q\) for each cyclic vertex and a weight \(y\) for each component. Obviously,
\(\psi_n(q, 1) = F_{n}^{ord}(q)\) and \(\psi_n(1, y) = \psi_n(y)\). In addition, one has the bivariate exponential generating function

\[
\sum_{n \geq 0} \psi_n(q, y) \frac{t^n}{n!} = \frac{1}{(1 - qT(t))^{y}}.
\]

(6.42)

Let \(\psi_{n,k}^{Y}(q) = [y^{k}]\psi_{n}(q, y)\). Sokal [91, Conjecture 6.10] also conjectured the following, which in particular implies Conjecture 6.16.

**Conjecture 6.17.** Let \(\psi_{n}(q, y)\) and \(\psi_{n,k}^{Y}(q)\) be defined above.

(i) The lower-triangular matrix \([\psi_{n,k}^{Y}(q)]_{n,k}\) is coefficientwise totally positive in \(q\).

(ii) The polynomial sequence \((\psi_{n}(q, y))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \((q, y)\).

(iii) The polynomial sequence \((\psi_{n+1,1}(q))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \(q\).

From [91], we know that \(\psi_{n+1,1}^{Y} = \sum_{k=1}^{n} (k - 1)! f_{n,k}q^{k}\), which enumerates connected functional digraphs on the vertex set \([n]\) with a weight \(q\) for each cyclic vertex; equivalently, they enumerate cyclically ordered forests of rooted trees on the vertex set \([n]\) with a weight \(q\) for each tree. Its exponential generating function is \(-\log(1 - qT(t))\).

For Conjectures 6.16 and 6.17, we obtain the next result, whose (i) confirms Conjecture 6.17 (i) and implies Conjecture 6.16 (i), (ii) gives a stronger support for Conjecture 6.17 (ii), Conjecture 6.16 (ii) and Conjecture 6.13, and (iii) provide a stronger support for Conjecture 6.17 (iii) and confirms Conjecture 6.16 (iii).

**Proposition 6.18.** (i) The lower-triangular matrix \([\psi_{n,k}^{Y}(q)]_{n,k}\) is coefficientwise totally positive in \(q\).

(ii) For any fixed \(y > 0\) and \(q > 0\), \((\psi_{n}(q, y))_{n \geq 0}\) is a Stieltjes moment sequence (of real numbers).

(iii) For any fixed \(q \geq 0\), \((\psi_{n+1,1}^{Y}(q))_{n \geq 0}\) is a Stieltjes moment sequence (of real numbers).

(iv) The convolution \(z_n = \sum_{k \geq 0} \psi_{n,k}^{Y}(q) x_k y_{n-k}\) preserves the Stieltjes moment property in \(\mathbb{R}\) for \(q \geq 0\).

(v) The polynomial \(\psi_{n}(q, y)\) in \(y\) has only real zeros for \(q > 0\).

**Proof.** For (i) and (ii), they follow from Theorem 6.8 (i) and (ii), respectively. (iv) is immediate from (ii). Finally, for (iii), by Proposition 6.11 (iii) that the convolution \(z_n = \sum_{k \geq 0} f_{n,k} x_k y_{n-k}\) preserves the Stieltjes moment property in \(\mathbb{R}\), for any fixed \(q > 0\), \(\psi_{n+1,1}^{Y} = \sum_{k=1}^{n} (k - 1)! f_{n,k}q^{k}\) forms a Stieltjes moment sequence for \(n \geq 0\).

(v) It is obviously that \(\sum_{k} f_{n,k}q^{k} = \sum_{k} (n-1)^{n-k} x^{k} = (n+x)^{n-1}\) has only real zeros. In consequence, \((f_{n,k})_{k}\) is a Pólya frequency sequence, which by Theorem 6.8 (iv) implies that \((\psi_{n,k}^{Y}(q))_{k}\) is a Pólya frequency sequence for \(q > 0\). So the polynomial \(\psi_{n}(q, y)\) in \(y\) has only real zeros for \(q > 0\). \(\Box\)
6.4 Stirling permutations and rth-order Eulerian polynomials

A few years ago, Sokal ever conjectured that the sequence of reversed 2th-order Eulerian polynomials is coefficientwise Hankel-totally positive. In [76], authors gave a combinatorial proof of coefficientwise Hankel-total positivity for rth-order Eulerian polynomials. Now we will present an algebraic proof.

Let \( r = (r_1, \ldots, r_n) \) and \( |r| = \sum_i r_i \) for \( r_i \in \mathbb{N} \), and define the multiset \( M_r = \{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\} \) consisting of \( r_i \) copies of the letter \( i \). A permutation of \( M_r \) is a word \( w_1 \cdots w_{|r|} \) containing \( r_i \) copies of the letter \( i \), for each \( i \in [n] \); it is called a Stirling permutation of \( M_r \) when the word \( w = w_1 \cdots w_L \) satisfies the condition: if \( i < j < k \) and \( w_i = w_k \) imply \( w_j \geq w_i \). Stirling permutations were introduced by Gessel and Stanley [46] for the case \( r_1 = \ldots = r_n = 2 \); this was generalized to \( r_1 = \ldots = r_n = r \) (which we denote by the shorthand \( r = (r^n) \)) by Gessel [44] and Park [73, 74], and to general multisets \( M_r \) by Brenti [12, 16] and others [51, 32]. We refer to Stirling permutations of \( M_v \) as \( r \)-Stirling permutations of order \( n \).

For any word \( w = w_1 \cdots w_L \) on a totally ordered alphabet \( A \), a pair \((i, i+1)\) with \( 1 \leq i \leq L - 1 \) is called a descent if \( w_i > w_{i+1} \). Let \( \langle n \rangle^{(r)} \) denote the number of \( r \)-Stirling permutations with \( k \) descents and call them the rth-order Eulerian numbers.\(^2\) The rth-order Eulerian polynomial is defined to be

\[
A_n^{(r)}(x) = \sum_{k=0}^{n} \frac{\langle n \rangle^{(r)}}{\langle k \rangle^{(r)}} x^k.
\]

(6.43)

The rth-order Eulerian numbers satisfy the recurrence

\[
\langle n \rangle^{(r)} = [rn - (r-1) - k] \langle n-1 \rangle^{(r)} + (k+1) \langle n-1 \rangle^{(r)}
\]

for \( n \geq 1 \) \hspace{1cm} (6.44)

with initial condition \( \langle 0 \rangle^{(r)} = \delta_{0k} \).

Note that a bijection from \( r \)-Stirling permutations of order \( n \) to increasing \((r+1)\)-ary trees with \( n \) vertices was found by Gessel [44] (see [73]) and independently by Janson et al. [51]. In [76], by using this bijection to map statistics between Stirling permutations and trees, authors proved that the rth-order Eulerian polynomial is a special case of the multivariate Eulerian polynomial on increasing ary trees and obtained:

**Corollary 6.19.** [76] For any integer \( r \geq 1 \), the rth-order Eulerian polynomial \( A_n^{(r)}(x) \) defined in (6.43) equals the \( r \)-Stieltjes–Rogers polynomial \( S_n^{(r)}(\alpha) \) and has the \( r \)-branched Stieltjes-type continued fraction

\[
\sum_{n \geq 0} A_n^{(r)}(x)t^n = \frac{1}{1 - \alpha_x t \prod_{i_1=1}^{r} \frac{1}{1 - \alpha_{r+i_1} t \prod_{i_2=1}^{r} \frac{1}{1 - \alpha_{r+i_1+i_2} t \prod_{i_3=1}^{r} \frac{1}{1 - \cdots}}}}.
\]

(6.45)

\(^2\)Here we follow the convention of Graham et al. [47] that (when \( n \geq 1 \)) \( \langle n \rangle^{(r)} \) is nonzero for \( 0 \leq k \leq n - 1 \).
with coefficients \((\alpha_i)_{i \geq r} = (1, x, \ldots, x, 2, 2x, 2x, \ldots, 2x, 3, 3x, 3x, \ldots, 3x, \ldots).\) Therefore the sequence \((A_n^{(r)}(x))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \(x\).

A slightly different definition of the \(r\)th-order Eulerian polynomials was used in [35]:

\[
E_0^{[r]}(x) = 1 \quad (6.46)
\]

\[
E_n^{[r]}(x) = xA_n^{(r)}(x) = \sum_{k=0}^{n-1} \binom{n}{k}^{(r)} x^{k+1} \quad \text{for } n \geq 1 \quad (6.47)
\]

Then the reversed \(r\)th-order Eulerian polynomials are defined by \(E_n^{[r]*}(x) = x^n E_n^{[r]}(1/x)\). A few years ago, Sokal ever conjectured:

**Conjecture 6.20.** The sequence \((E_n^{[2]*}(x))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \(x\).

Using the branched continued fraction for \(A_n^{(r)}(x)\), in [76] authors also obtained:

**Corollary 6.21.** [76] For any integer \(r \geq 1\), the reversed \(r\)th-order Eulerian polynomial \(E_n^{[r]*}(x)\) equals the \(r\)-Stieltjes–Rogers polynomial \(S_n^{(r)}(\alpha)\) with \((\alpha_i)_{i \geq r} = (1, \ldots, 1, x, 2, \ldots, 2, x, 3, \ldots, 3, x, \ldots).\) Therefore the sequence \((E_n^{[r]*}(x))_{n \geq 0}\) is coefficientwise Hankel-totally positive in \(x\).

For the coefficientwise Hankel-total positivity of \((A_n^{(r)}(x))_{n \geq 0}\) and \((E_n^{[r]*}(x))_{n \geq 0}\), by Theorem 6.6, we will give an algebraic proof from the exponential Riordan array.

Let \(F(t) = \sum_n A_n^{(r)}(x) t^n / n!\). In [7], it was proved

\[
F(t) = \frac{h \left( e^{(1-x)t} \overline{h}(x) \right)}{x \left( 1 - h \left( e^{(1-x)t} \overline{h}(x) \right) \right)},
\]

where the compositional inverse of the function \(h(z)\) is given by

\[
\overline{h}(z) = z \exp \left( \sum_{k=1}^{r-1} \binom{r-1}{k} (-z)^k \right). \quad (6.48)
\]

Let

\[
f(t) := F(t) - 1 = \frac{h \left( e^{(1-x)t} \overline{h}(x) \right) - x}{x \left( 1 - h \left( e^{(1-x)t} \overline{h}(x) \right) \right)}. \quad (6.49)
\]

We have

\[
h \left( e^{(1-x)t} \overline{h}(x) \right) = \frac{x(1 + f(t))}{1 + xf(t)}, \quad (6.50)
\]
which implies
\[
x e^{\overline{f}(1-x)^{r}} \exp \left( \sum_{k=1}^{r-1} \frac{(r-1)(-x)^k}{k} \right) = \frac{x(1+t)}{1+xt} \exp \left( \sum_{k=1}^{r-1} \frac{1}{k} \left( \frac{r-1}{k} \right) \left( -\frac{x(1+t)}{1+xt} \right)^k \right).
\]

(6.51)

Then we derive
\[
\frac{1}{\overline{f}(t)} = (1+xt)^r(1+t),
\]
whose coefficient sequence is coefficientwise Toeplitz-totally positive in \( x \). Then by Theorem 6.6 and Theorem 6.7, we conclude more results for total positivity related to \( A_n^{(r)}(x) \) and \( E_n^{(r)}(x) \) as follows.

**Proposition 6.22.** Let \( f(t) \) be defined by \((6.49)\) and \( \left\langle \binom{n}{k}^{(i,j)} \right\rangle_{n,k} \) be the exponential Riordan array \( ((1+xf(t))^i(1+f(t))^j, f(t)) \) for \( i \in \{0,1,\ldots,r\} \) and \( j \in \{0,1\} \). Then

(i) the triangle \( \left\langle \binom{n}{k}^{(i,j)} \right\rangle_{n,k} \) is coefficientwise totally positive in \( x \);

(ii) the row-generating polynomial \( (A_n^{(i,j)}(q))_n \) of \( \left\langle \binom{n}{k}^{(i,j)} \right\rangle_{n,k} \) and its reversed polynomial \( (A_n^{(i,j)*}(q))_n \) are coefficientwise Hankel-totally positive in \( (x,q) \);

(iii) the \( r \)th-order Eulerian polynomial \( (A_n^{(r)}(x))_{n \geq 1} \) is coefficientwise Hankel-totally positive in \( x \);

(iv) the sequence \( (T_n^{(i,j)}(x))_{n \geq 0} \) is coefficientwise Hankel-totally positive in \( x \), where \( (1+xf(t))^i(1+f(t))^j = \sum_{n \geq 0} T_n^{(i,j)}(x) \frac{t^n}{n!} \);

(v) the convolution \( z_n = \sum_{k=0}^{n} \binom{n}{k}^{(i,j)} x_k y_{n-k} \) preserves the Stieltjes moment property in \( \mathbb{R} \) for \( x \geq 1 \);

(vi) we have the \( r \)-branched Stieltjes-type continued fraction

\[
1 + \sum_{n \geq 1} A_n^{(r)}(x) t^n = \frac{1}{1 - \alpha_r t \prod_{i_1=1}^{r} \frac{1}{1 - \alpha_{r+i_1} t \prod_{i_2=1}^{r} \frac{1}{1 - \alpha_{r+i_1+i_2} t \prod_{i_3=1}^{r} \frac{1}{1 - \alpha_{r+i_1+i_2+i_3} t \prod_{i_4=1}^{r} \frac{1}{1 - \cdots}}}}.
\]

(6.53)

with coefficients \( (\alpha_i)_{i \geq r} = (1,x,x,\ldots,x,2,2x,2x,\ldots,2x,3,3x,3x,\ldots,3x,\ldots) \).

Obviously, \( A_n^{(i,j)}(0) = T_n^{(i,j)}(x) \) and \( A_n^{(r,1)}(0) = A_n^{(r)}(x) \). Therefore, we ask the following question.

**Question 6.23.** Let \( i \in \{0,1,\ldots,r\} \) and \( j \in \{0,1\} \). Find the combinatorial interpretation of the triangular array \( \left\langle \binom{n}{k}^{(i,j)} \right\rangle_{n,k} \) and polynomials \( A_n^{(i,j)}(q) \) and \( T_n^{(i,j)}(x) \), respectively.
6.5 Multivariate Ward polynomials

The Ward numbers $W_{n,k}$ were studied in [100], which satisfy the recurrence

$$W_{n,k} = (n + k - 1)W_{n-1,k-1} + kW_{n-1,k}$$

(6.54)

for $n \geq k \geq 0$ with $W_{0,0} = 1$. The row-generating polynomial of the Ward triangle $W_n(x) = \sum_{k=0}^{n} W_{n,k} x^k$ is called the Ward polynomial. It is known that $W_n(x) = (1 + x)^n A_n^{(2)} \left( \frac{1 + x}{1 + x} \right)$, which combines Propositions 4.3 and 6.2 to give that both $W_n(x)$ and $W_n^*(x)$ are coefficientwise Hankel-totally positive in $x$, see [35] for a different proof from the Thon-continued fraction. In fact, Elvey Price and Sokal [35] obtained a Thron-type continued fraction for the ordinary generating function of a five-variable Ward polynomial. See [89, A134991/A181996/A269939] for further information on the Ward numbers and Ward polynomials. There are different combinatorial interpretations for the Ward polynomials. For example, the Ward polynomial $W_n(x)$ is the generating polynomial for phylogenetic trees on $n + 1$ labeled leaves in which each internal vertex gets a weight $x$. More generally, a multivariate Ward polynomial $W_n(x_1, x_2, \ldots)$ is the generating polynomial for phylogenetic trees on $n + 1$ labeled leaves in which each internal vertex with $i \geq 2$ children gets a weight $x_{i-1}$. We list the first few $W_n(x_1, x_2, \ldots)$ as follows:

$$W_0(x_1, x_2, \ldots) = 1, \quad W_1(x_1, x_2, \ldots) = x_1, \quad W_2(x_1, x_2, \ldots) = 3x_1^2 + x_2.\quad (6.55)$$

Let

$$\\mathcal{W}(t, x) = \sum_{n \geq 0} W_n(x_1, x_2, \ldots) \frac{t^{n+1}}{(n + 1)!} = t + \sum_{n \geq 2} W_{n-1}(x_1, x_2, \ldots) \frac{t^n}{n!}.$$  

(6.56)

It is known that the compositional inverse of the generic power series is

$$\\mathcal{F}(t, x) = t - \sum_{n \geq 2} x_{n-1} \frac{t^n}{n!}.$$  

(6.57)

That is to say

$$\\mathcal{W}(t, x) = t + \sum_{n \geq 2} x_{n-1} \frac{\\mathcal{W}(t, x)^n}{n!}.$$  

(6.58)

We refer the reader to [24, pp. 151], [48, 35], [83, pp. 181] for more properties of $W_n(x_1, x_2, \ldots)$. By (6.58), we derive an autonomous differential equation

$$\\mathcal{W}'(t, x) = \frac{1}{1 - \sum_{n \geq 1} x_{n-1} \frac{\\mathcal{W}(t, x)^n}{n!}}.$$  

(6.59)

Hence, by Theorem 6.3 (iv), we have:

**Proposition 6.24.** Let $x_1, x_2, \ldots$ be elements of a partially ordered commutative ring $R$ such that $\frac{1}{1 - \sum_{n \geq 1} x_{n-1} \frac{\\mathcal{W}(t, x)^n}{n!}}$ is a Pólya frequency ogf of order $r$ in the ring $R$, then the sequence $(W_n(x_1, x_2, \ldots))_{n \geq 1}$ is Hankel-totally positive of order $r$ in the ring $R$. 

46
6.6 Labeled series-parallel networks and Nondegenerate fanout-free functions

Let \( s_n \) be the number of labeled series-parallel networks with \( n \) edges. It dates from MacMahon studying Yoke-trains and multipartite compositions [66]. It also counts the number of plane increasing trees on \( n \) vertices, where each vertex of outdegree \( k \geq 1 \) can be in one of 2 colors. For \( n \geq 1 \), the first few are 1, 2, 8, 52, 472, . . . . Gutkovskiy [89, A006351] noted that it satisfies the recurrence

\[
\begin{align*}
  s_1 &= 1, \\
  s_n &= s_{n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k} s_k s_{n-k}.
\end{align*}
\]  

(6.60)

Let \( s(t) = \sum_{n \geq 1} s_n \frac{t^n}{n!} \). Bala [89, A006351] gave the autonomous differential equation

\[
s'(t) = \frac{1 + s(t)}{1 - s(t)}.
\]

(6.61)

We refer the reader to Lomnicki [61], Moon [69] and [89, A006351] for more information on \( s_n \).

Let \( p_n \) be 1, 4, 32, 416, 7552, 176128, . . . for \( n \geq 1 \), see [89, A005172]. It counts the number of nondegenerate fanout-free functions of \( n \) variables with the basic logical operation “AND” rank 1 [50] and also is the number of plane increasing trees on \( n \) vertices, where each vertex of outdegree \( k \geq 1 \) can be colored in \( 2^{k+1} \) ways (see Bala’s remarks in [89, A005172]). Kruchinin [89, A005172] obtained for \( n > 1 \) that

\[
p_n = \sum_{k=1}^{n-1} \sum_{j=1}^{k} \sum_{i=0}^{j} \frac{2^{n-i+j-1}(n+k-1)!}{(k-j)!(n-i+j-1)!j!} \left[ \binom{n-i+j-1}{j-i} \right].
\]

(6.62)

Luschny [89, A005172] also gave \( p_n = 2p_{n-1} + \sum_{j=1}^{n-1} \binom{n}{j} p_j p_{n-j} \) for \( n > 1 \). Let the exponential generating function \( p(t) = \sum_{n \geq 1} p_n \frac{t^n}{n!} \). It satisfies the autonomous differential equation

\[
p'(t) = \frac{1 + 2p(t)}{1 - 2p(t)}
\]

(6.63)

with \( p(0) = 0 \). We refer the reader to [89, A042977] for more properties on \( p_n \).

In order state some properties related to \( s_n \) and \( p_n \) for \( n \geq 1 \), we will consider a generalization of \( s(t) \) and \( p(t) \). Let \( a \) and \( b \) be indeterminates. Define \( S(t) = \sum_{n \geq 1} S_n(a, b) \frac{t^n}{n!} \) by

\[
S'(t) = \frac{1 + a S(t)}{1 - b S(t)}.
\]

(6.64)

By (6.64), we have

\[
\frac{1}{S(t)} = \frac{1 + a t}{1 - b t}.
\]

(6.65)

Obviously, the coefficient sequence of \( \frac{1 + a t}{1 - b t} \) is coefficientwise Toeplitz-totally positive in \( a \) and \( b \). It follows from Theorem 6.6 that we immediately obtain:
Theorem 6.25. Let $a$ and $b$ be indeterminates and $S(t) = \sum_{n \geq 1} S_n(a,b) \frac{t^n}{n!}$ satisfy the equation (6.64). Define an exponential Riordan array $[S_{n,k}^{(i,j)}(a,b)]_{n,k}$ := $\left( \frac{(1+a S(t))^i}{(1-b S(t))^j}, S(t) \right)$ for $i \in \{0,1\}$ and $j \in \{0,1\}$. Then we have

(i) the triangle $[S_{n,k}^{(i,j)}(a,b)]_{n,k}$ is coefficientwise totally positive in $(a,b)$;

(ii) the row-generating polynomial sequence $(S_{n}^{(i,j)}(q))_n$ and its reversed polynomial sequence $(S_{n}^{(i,j)}(q))_n$ are coefficientwise Hankel-totally positive in $(a,b,q)$ and 3-$(a,b,q)$-log-convex;

(iii) the zeroth column sequence $(S_{n}^{(i,j)}(a,b))_n$ is coefficientwise Hankel-totally positive in $(a,b)$ and 3-$(a,b)$-log-convex;

(iv) the sequence $(S_n(a,b))_{n \geq 1}$ is coefficientwise Hankel-totally positive in $(a,b)$ and 3-$(a,b)$-log-convex;

(v) the sequence $(S_n^\circ(b))_{n \geq 0}$ is coefficientwise Hankel-totally positive in $b$ and 3-$b$-log-convex, where $\frac{1}{1-b S(t)} = \sum_{n \geq 0} S_n^\circ(b) \frac{t^n}{n!}$;

(vi) the convolution $z_n = \sum_{k=0}^{n} S_{n,k}^{(i,j)}(a,b) x_k y_{n-k}$ preserves the Stieltjes moment property in $\mathbb{R}$ for $a \geq 0$ and $b \geq 0$.

In particular, taking $a = b = 1$ in (6.64), then $S(t)$ reduces to $s(t)$ and taking $a = b = 2$ in (6.64) yields $p(t)$. So the following is immediate from Theorem 6.25.

Proposition 6.26. We have

(i) the sequence $(s_{n+1})_{n \geq 0}$ is Stieltjes moment and 3-log-convex;

(ii) the sequence $(s_n^\circ)_{n \geq 0}$ is a Stieltjes moment sequence (of real numbers) and 3-log-convex, where $\frac{1}{1-s(t)} = \sum_{n \geq 0} s_n^\circ \frac{t^n}{n!}$;

(iii) the sequence $(p_{n+1})_{n \geq 0}$ is a Stieltjes moment sequence (of real numbers) and 3-log-convex;

(iv) the sequence $(p_n^\circ)_{n \geq 0}$ is a Stieltjes moment sequence (of real numbers) and 3-log-convex, where $\frac{1}{1-2 p(t)} = \sum_{n \geq 0} p_n^\circ \frac{t^n}{n!}$.

It is very interesting to find an answer to the following question.

Question 6.27. Find the combinatorial interpretation of the triangular array $[S_{n,k}^{(i,j)}(a,b)]_{n,k}$ and its row-generating polynomial $S_n^\circ(q)$, respectively.
6.7 An array from the Lambert function

The Lambert $W$ function was defined in [26] by

$$W e^W = x.$$  

The $n$-th derivative of $W$ is given implicitly by

$$\frac{d^n W(x)}{dx^n} = \frac{e^{-nW(x)} \beta_n(W(x))}{(1 + W(x))^{2n-1}}$$  \hspace{1cm} (6.66)

for $n \geq 1$, where $\beta_n(x) = (-1)^{n-1} \sum_{k=0}^{n-1} \beta_{n,k} x^k$ are polynomials satisfying the recurrence relation

$$\beta_{n+1}(x) = -(nx + 3n - 1)\beta_n(x) + (1 + x)\beta'_n(x)$$  \hspace{1cm} (6.67)

for $n \geq 1$, and the array $[\beta_{n,k}]_{n \geq 1, k \geq 0}$ satisfies the recurrence relation

$$\beta_{n+1,k} = (3n - k - 1)\beta_{n,k} + n\beta_{n,k-1} - (k + 1)\beta_{n,k+1}$$  \hspace{1cm} (6.68)

for $n,k \geq 0$ with $\beta_{1,0} = 1$. Kalugin and Jeffrey [54] proved that each polynomial $(-1)^{n-1}\beta_n(x)$ has all positive coefficients and $\frac{dW(x)}{dx}$ is a completely monotonic function.

An explicit expression for the coefficients $\beta_{n,k}$ is

$$\beta_{n,k} = \sum_{m=0}^{k} \frac{1}{m!} \binom{2n-1}{k-m} \sum_{i=0}^{m} \binom{m}{i} (-1)^{i} (i + n)^{m+n-1}.$$  \hspace{1cm} (6.69)

See [89, A042977] for more properties of $\beta_{n,k}$. Jovovic [89, A042977] gave

$$\sum_n (-1)^{n-1} \beta_n(x) \frac{t^n}{n!} = W(e^x(x - t(1 + x)^2)) + x.$$  \hspace{1cm} (6.70)

Let

$$\beta(t) = \frac{-W(e^x(x - t(1 + x)^2)) + x}{1 + x}.$$  \hspace{1cm} (6.71)

We derive that

$$\overline{\beta}(t) = \frac{x + [(1 + x)t - x]e^{-(1+x)t}}{(1 + x)^2}, \quad \frac{1}{\overline{\beta}(t)'} = \frac{e^{(1+x)t}}{1 - t}.$$  \hspace{1cm} (6.72)

It is not hard to prove that $e^{(1+x)t}/(1 - t)$ is a Pólya frequency function in $x$. Therefore, by Theorem 6.6, we obtain:

**Proposition 6.28.** Let $\beta(t)$ be defined by (6.71) and $\beta^{(i,j)} = [\beta^{(i,j)}_{n,k}]_{n,k}$ be the exponential Riordan array $\left(\frac{e^{(1+x)t}}{(1-\overline{\beta}(t))'}, \beta(t)\right)$ for $i \in \{0, 1\}$ and $j \in \{0, 1\}$. Then

(i) the triangle $\beta^{(i,j)}$ is coefficientwise totally positive in $x$.
6.8 A generalization of Lah numbers

In [113], we studied a generalized Lah triangle \( \mathcal{L} = [\mathcal{L}_{n,k}]_{n,k \geq 0} \) satisfying the following recurrence relation

\[
\mathcal{L}_{n,k} = c \mathcal{L}_{n-1,k-1} + [ab(n-1) + bk + abd + c\lambda] \mathcal{L}_{n-1,k} + b\lambda(k+1) \mathcal{L}_{n-1,k+1}
\]

where \( \mathcal{L}_{0,0} = 1 \). For \( a = b = c = 1 \) and \( d = \lambda = 0 \), \( \mathcal{L} \) reduces to the well-known signless Lah triangle \( \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{i=0}^{k} \frac{x_i}{i!} \) which counts the number of partitions of \( [n] \) into \( k \) lists, where a list means an ordered subset [89, A008297] and satisfies

\[
\exp \left( \frac{qt}{1-t} \right) = \sum_{n \geq 0} \mathcal{L}_{n,k} \frac{t^n}{n!} \quad (6.74)
\]

In addition, the row-generating function \( \sum_{k=0}^{\infty} \frac{(n-1)_k}{k!} q^k \) is called the Lah polynomial. For \( 2 \leq a \leq 4 \), \( b = c = 1 \) and \( d = \lambda = 0 \), \([\mathcal{L}_{n,k}]_{n,k} \) reduces to the triangle [89, A035342, A035469, A049029] enumerating unordered \( n \)-vertex \( k \)-forests composed of \( k \) plane increasing quartic (a-ary) trees. In [113], we showed some results concerning total positivity for the triangle \( \mathcal{L} \) under some special conditions. As an application of Theorem 6.6, we have the following generalized result.

**Proposition 6.30.** Let \( \mathcal{L} \) be the generalized Lah triangle defined by (6.73) and \( \mathcal{L}_{n}(q) = \sum_{k \geq 0} \mathcal{L}_{n,k} q^k \). Assume that \( \{a, ad\} \in \mathbb{N} \) and \( c > 0 \). If \( 0 \leq ad \leq a + 1 = m \), then we have

(i) the generalized Lah triangle \( \mathcal{L} \) is coefficientwise totally positive in \((b, \lambda)\);

(ii) the polynomial sequence \( \{\mathcal{L}_{n}(q)\}_{n \geq 0} \) and its reversed polynomial sequence \( \{\mathcal{L}^*_{n}(q)\}_{n \geq 0} \) are coefficientwise Hankel-totally positive in \((b, \lambda, q)\) and \(3-(b, \lambda, q)\)-log-convex;
(iii) \((L_{n,k})_{n \geq 0}\) is coefficientwise Hankel-totally positive in \((b, \lambda)\) and \(3-(b, \lambda)\)-log-convex;

(iv) the convolution \(z_n = \sum_{k \geq 0} L_{n,k} x_k y_{n-k}\) preserves Stieltjes moment property in \(\mathbb{R}\) for \(b \geq 0\) and \(\lambda \geq 0\);

(v) for \(d = 0\), the \(m\)-branched Stieltjes-type continued fraction expression

\[
\sum_{n \geq 0} L_n(q)t^n = \frac{1}{1 - \alpha_m t \prod_{i_1=1}^{m} \frac{1}{1 - \alpha_{m+i_1} t \prod_{i_2=1}^{m} \frac{1}{1 - \alpha_{m+i_1+i_2} t \prod_{i_3=1}^{m} \frac{1}{\ldots}}}}
\]

(6.75)

with coefficients

\[
(\alpha_i)_{i \geq m} = (c(q + \lambda), b, \ldots, b, c(q + \lambda), 2b, \ldots, 2b, c(q + \lambda), 3b, \ldots, 3b, \ldots).
\]

(vi) the \((m-1)\)-branched Stieltjes-type continued fraction expression

\[
1 + \sum_{n \geq 1} b f_n t^n = \frac{1}{1 - \alpha_{m-1} t \prod_{i_1=1}^{m-1} \frac{1}{1 - \alpha_{m-1+i_1} t \prod_{i_2=1}^{m-1} \frac{1}{1 - \alpha_{m-1+i_1+i_2} t \prod_{i_3=1}^{m-1} \frac{1}{\ldots}}}}
\]

(6.76)

with \((\alpha_i)_{i \geq m-1} = (b, \ldots, b, 2b, \ldots, 2b, 3b, \ldots, 3b, \ldots)\), where \(\sum_{n \geq 1} f_n t^n = \frac{(1-\lambda t)^{-1} - 1}{b}\).

Proof. It follows from [113, Proposition 4.7] that \(L\) is the exponential Riordan array \((g(t)e^{\lambda f(t)}, f(t))\), where

\[
g(t) = (1 - abt)^{-d}, \quad f(t) = c \left[ (1 - abt)^{-\frac{1}{a}} - 1 \right].
\]

(6.77)

Then

\[
\tilde{f}(t) = \frac{1 - (1 + \frac{bt}{c})^{-a}}{ab}, \quad 1/\tilde{f}'(t) = c(1 + \frac{bt}{c})^{a+1}.
\]

(6.78)

Obviously, the coefficient sequence of \(c(1 + \frac{bt}{c})^{a+1}\) is coefficientwise Toeplitz-totally positive in \(b\) and

\[
(1 + \frac{b}{c} f(t))^{ad} = \left[ (1 - abt)^{-\frac{1}{a}} \right]^{ad} = (1 - abt)^{-d} = g(t).
\]

(6.79)

Hence, by Theorem 6.6, we have (i)-(iv)
For (v), by taking $q \to c(q + \lambda)/b$ and $t \to bt$, it suffices to prove for the reduced case $b = c = 1$ and $\lambda = 0$ that

$$
\sum_{n \geq 0} \mathcal{L}_n(q)t^n = \frac{1}{1 - \alpha_m t \prod_{i=1}^m \frac{1}{1 - \alpha_{m+i_1} t \prod_{i_2=1}^m \frac{1}{1 - \ldots}}}
$$

with coefficients

$$(\alpha_i)_{i \geq m} = (q,1,\ldots,1,q,2,\ldots,2,q,3,\ldots,3,\ldots).$$

For the reduced case, we have

$$\frac{1}{f'(t)} = (1 + t)^m.$$

Hence we immediately get the continued fraction expansion for the reduced case by Theorem 6.7. This completes the proof. \(\square\)

We propose the following interesting question.

**Question 6.31.** What is the combinatorial interpretation of the triangular array $[\mathcal{L}_{n,k}]_{n,k \geq 0}$ and the polynomial $\mathcal{L}_n(q)$, respectively?

For the reversed generalized Lah polynomial $\mathcal{L}^*_n(q) = q^n \mathcal{L}_n(1/q)$ and the reversed generalized Lah numbers $\mathcal{L}^*_{n,k} = \mathcal{L}_{n,n-k}$, the following is immediate from Proposition 6.30.

**Proposition 6.32.** Assume that $\{a,ad\} \in \mathbb{N}$ and $c > 0$. If $0 \leq ad \leq a + 1 = m$, then we have the following results.

(i) The reversed generalized Lah triangle $[\mathcal{L}^*_{n,k}]_{n,k \geq 0}$ satisfies the next recurrence

$$
\mathcal{L}^*_{n,k} = c\mathcal{L}^*_{n-1,k} + [ab(n-1) + b(n-k) + abd + c\lambda] \mathcal{L}^*_{n-1,k-1} + b\lambda(n-k+1)\mathcal{L}^*_{n-1,k-2}
$$

for $n, k \geq 1$, where $\mathcal{L}^*_{0,0} = 1$.

(ii) The exponential generating function of $\mathcal{L}^*_n(q)$ can be written as

$$
\sum_{n \geq 0} \mathcal{L}^*_n(q)\frac{t^n}{n!} = (1 - abqt)^{-d} \exp \left( \frac{(q\lambda + 1)c}{q} \left[ \frac{(1 - abqt)^{-\frac{1}{b}} - 1}{b} \right] \right). \quad (6.81)
$$

(iii) The sequence $(\mathcal{L}^*_n(q))_{n \geq 0}$ is coefficientwise Hankel-totally positive in $(b,\lambda,q)$ and 3-$(b,\lambda,q)$-log-convex. In particular, $(\mathcal{L}^*_n,q)_{n \geq 0}$ is coefficientwise Hankel-totally positive in $(b,\lambda)$ and 3-$(b,\lambda)$-log-convex.

(iv) The convolution $z_n = \sum_{k \geq 0} \mathcal{L}^*_{n,k}xky_{n-k}$ preserves Stieltjes moment property of sequences for $b \geq 0$ and $\lambda \geq 0$;
(v) For \( d = 0 \), we have the \( m \)-branched Stieltjes-type continued fraction expansion

\[
\sum_{n\geq 0} \mathcal{L}_n^\ast(q) t^n = \frac{1}{1 - \alpha_m t \prod_{i_1=1}^m \frac{1}{1 - \alpha_{m+i_1} t \prod_{i_2=1}^m \frac{1}{1 - \alpha_{m+i_1+i_2} t \prod_{i_3=1}^m \frac{1}{1 - \ldots}}} \tag{6.82}
\]

with coefficients

\[
(\alpha_i)_{i\geq m} = (c(1 + q\lambda), bq, \ldots, bq, c(1 + q\lambda), 2bq, \ldots, 2bq, c(1 + q\lambda), 3bq, \ldots, 3bq, \ldots).
\]

7 Acknowledgements

The author wants to thank the anonymous referees for many valuable remarks and suggestions to improve the original manuscript and is very grateful to one of referees for suggesting some results for the matrices \( M \) and \( \tilde{M} \) in Section 3.

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