Hom-prealternative superalgebras

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Abstract

The purpose of this paper is to introduce Hom-prealternative superalgebras
and their bimodules. Some constructions of Hom-prealternative superalgebras
and Hom-alternative superalgebras are given, and their connection with Hom-
alternative superalgebras are studied. Bimodules over Hom-prealternative su-
peralgebras are introduced, relations between bimodules over Hom-prealtern-
tive superalgebras and the bimodules of the corresponding Hom-alternative su-
peralgebras are considered, and construction of bimodules over Hom-prealter-
native superalgebras by twisting is described.

1 Introduction

Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced
first by Hartwig, Larsson and Silvestrov in [60] where a general approach to dis-
cretization of Lie algebras of vector fields using general twisted derivations ($\sigma$-
derivations) and a general method for construction of deformations of Witt and
Virasoro type algebras based on twisted derivations have been developed. The general quasi-Lie algebras, containing the quasi-Hom-Lie algebras and Hom-Lie algebras as subclasses, as well their graded color generalization, the color quasi-Lie algebras including color quasi-hom-Lie algebras, color hom-Lie algebras and their special subclasses the quasi-Hom-Lie superalgebras and hom-Lie superalgebras, have been first introduced in [60, 67–70, 98]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [78]. In particular, in [78], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras leading to Lie algebras using commutator map. Furthermore, in [78], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. Since the pioneering works [60, 67–70, 78], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. Hom-algebra structures are very useful since Hom-algebra structures of a given type include their classical counterparts and open broad possibilities for deformations, Hom-algebra extensions of homology and cohomology structures and representations, formal deformations of Hom-associative and Hom-Lie algebras, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras, [6, 33, 45, 67, 72, 79–81, 94, 104, 106]. Hom-Lie algebras, Hom-Lie superalgebras, color Hom-Lie algebras, Hom-associative color algebras, Enveloping algebras of color Hom-Lie algebras, color Hom-Leibniz algebras, omni-Hom-Lie algebras, color omni-Hom-Lie algebras, biHom-Lie algebras, biHom-associative algebras, biHom-Frobenius algebras, Hom-Ore extensions Hom-algebras, Hom-alternative algebras, Hom-center-symmetric algebras, Hom-left-symmetric color dialgebras, Hom-dendriform algebras, Rota–Baxter Hom-algebras, Hom-tridendriform color algebras, Hom-Malcev algebras, Hom-Jordan algebras, Hom-Poisson algebras, Color Hom-Poisson algebras, Hom-Akivis algebras, Hom-Lie-Yamaguti algebras, nearly Hom-associative algebras, Hom-Gerstenhaber algebras and Hom-Lie algebroids, n-Lie algebras and Hom-Nambu-Lie algebras and other n-ary Hom-algebra structures have been further investigated in various aspects for example in [1–30, 32–37, 37–44, 46–
In particular, Color Hom-Poisson algebras [24] and modules over some color Hom-algebras [27], under the name of generalized Hom-algebras, have been considered. When the grading abelian group is $\mathbb{Z}_2$, the corresponding $\mathbb{Z}_2$-graded Hom-algebras are called Hom-superalgebras. Hom-Lie superalgebra structures such as Hom-Lie superalgebras and Hom-Lie admissible superalgebras [46], Rota-Baxter operator on pre-Lie superalgebras [2], Hom-Novikov superalgebras [102] have been considered in more details. Hom-alternative superalgebras have been considered in [1] as a $\mathbb{Z}_2$-graded version of Hom-alternative algebras [76] and their relationships with Hom-Malcev superalgebras and Hom-Jordan superalgebras are established [1].

The aim of this paper is to study the $\mathbb{Z}_2$-graded version of Hom-prealternative algebras and their bimodules. In Section 2, we recall some basic notions on Hom-alternative superalgebras and their bimodules. We prove that bimodules over Hom-alternative superalgebras are closed under twisting and direct product. We show that the tensor product of super-commutative Hom-associative superalgebras and Hom-alternative superalgebras is also a Hom-alternative superalgebra. Then we recall the definition of Hom-Jordan superalgebra. Section 3 is devoted to Hom-prealternative superalgebras and Hom-alternative superalgebras and their connections. We point out that to any Hom-prealternative superalgebra one may associate a Hom-alternative superalgebra, and conversly to any Hom-alternative superalgebra it corresponds a Hom-prealternative superalgebra via an $O$-operator. Construction of Hom-prealternative superalgebras by composition is given. Bimodules over Hom-prealternative superalgebras are introduced, relations between bimodules over Hom-prealternative superalgebras and bimodules of the corresponding Hom-alternative superalgebras are considered, and a construction of bimodules over Hom-prealternative superalgebras by twisting is described.

2 Hom-prealternative algebras and bimodules

In this section, we present important basic notions and provide some construction results for Hom-alternative superalgebras.

Firstly, let us recall necessary important basic notions and notations on graded spaces and algebras. Throughout this paper, all linear spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

**Definition 2.1.** Let $G$ be an abelian group. A linear space $V$ is called $G$-graded if $V = \bigoplus_{a \in G} V_a$ for some family $(V_a)_{a \in G}$ of linear subspaces of $V$. 

(i) An element \( x \in V \) is said to be homogeneous of degree \( a \in G \) if \( x \in V_a \), and \( \mathcal{H}(V) = \bigcup_{a \in G} V_a \) denotes the set of all homogeneous elements in \( V \).

(ii) Let \( V = \bigoplus_{a \in G} V_a \) and \( V' = \bigoplus_{a \in G} V'_a \) be two \( G \)-graded linear spaces. A linear mapping \( f : V \to V' \) is said to be homogeneous of degree \( b \) if

\[
f(V_a) \subseteq V'_{a+b}, \quad \text{for all } a \in G.
\]

If, \( f \) is homogeneous of degree zero i.e. \( f(V_a) \subseteq V'_{a} \) holds for any \( a \in G \), then \( f \) is said to be even.

(iii) An algebra \((A, \cdot)\) is said to be \( G \)-graded if its underlying linear space is \( G \)-graded i.e. \( A = \bigoplus_{a \in G} A_a \), and if furthermore

\[
A_a \cdot A_b \subseteq A_{a+b}, \quad \text{for all } a, b \in G.
\]

(iv) A morphism \( f : A \to A' \) of \( G \)-graded algebras \( A \) and \( A' \) is by definition an algebra morphism from \( A \) to \( A' \), which is moreover an even mapping.

Let \( A \) be a \( \mathbb{Z}_2 \)-graded linear space with direct sum \( A = A_0 \oplus A_1 \). The elements of \( A_j \), are said to be homogeneous of degree (parity) \( j \in \mathbb{Z}_2 \). The set of all homogeneous elements of \( A \) is \( \mathcal{H}(A) = A_0 \cup A_1 \). Usually \( |x| \) denotes parity of a homogeneous element \( x \in \mathcal{H}(A) \).

**Definition 2.2.** Hom-superalgebras are triples \((A, \mu, \alpha)\) in which \( A = A_0 \oplus A_1 \) is a \( \mathbb{Z}_2 \)-graded linear space (\( \mathbb{K} \)-superspace), \( \mu : A \times A \to A \) is an even bilinear map, and \( \alpha : A \to A \) is an even linear map.

(i) Let \((A, \mu, \alpha)\) be a Hom-superalgebra. Hom-associator of \( A \) is the even trilinear map \( as_{\alpha,\mu} : A \times A \times A \to A \) given by

\[
as_{\alpha,\mu} = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu).
\]

In terms of elements, the map \( as_{\alpha,\mu} \) is given by

\[
as_{\alpha,\mu}(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)),
\]

or in usual juxtaposition notation \( xy = \mu(x, y) \),

\[
as_{\alpha,\mu}(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz).
\]

(ii) An even linear map \( f : (A, \mu, \alpha) \to (A', \mu', \alpha') \) is said to be a weak morphism of Hom-superalgebras if

\[
f \circ \mu = \mu \circ (f \otimes f),
\]

and a morphism of Hom-superalgebras if moreover \( f \circ \alpha = \alpha' \circ f \).
(iii) Hom-superalgebra \((A, \mu, \alpha)\) in which \(\alpha : A \to A\) is moreover an endomorphism of the algebra structure \(\mu\) is said to be multiplicative, and the algebra endomorphism condition
\[
\alpha \circ \mu = \mu \circ (\alpha \otimes \alpha)
\]  
(2.1)
is called the multiplicativity of \(\alpha\) with respect to \(\mu\).

Since the grading degree of Hom-associator \(\left| as_{\alpha,\mu}(x, y, z)\right| = |x| + |y| + |z|\) for \(x, y, z \in \mathcal{H}(A) = A_0 \cup A_1\) in any Hom-superalgebra \((A = A_0 \oplus A_1, \mu, \alpha)\),
\[
as_{\alpha,\mu}(A_0, A_0, A_0) \subseteq A_0,
\]  
(2.2)
\[
as_{\alpha,\mu}(A_1, A_0, A_0) \subseteq A_1,
\]  
(2.3)
\[
as_{\alpha,\mu}(A_0, A_1, A_0) \subseteq A_1,
\]  
(2.4)
\[
as_{\alpha,\mu}(A_0, A_0, A_1) \subseteq A_1,
\]  
(2.5)
\[
as_{\alpha,\mu}(A_1, A_1, A_0) \subseteq A_0,
\]  
(2.6)
\[
as_{\alpha,\mu}(A_1, A_0, A_1) \subseteq A_0,
\]  
(2.7)
\[
as_{\alpha,\mu}(A_0, A_1, A_1) \subseteq A_0,
\]  
(2.8)
\[
as_{\alpha,\mu}(A_1, A_1, A_1) \subseteq A_1.
\]  
(2.9)

**Definition 2.3.** Hom-associative superalgebras are those Hom-superalgebras \((A = A_0 \oplus A_1, \bullet, \alpha)\) obeying super \((\mathbb{Z}_2\text{-graded})\) Hom-associativity super identity,
\[
\forall x, y, z \in \mathcal{H}(A) = A_0 \cup A_1 : as_{\alpha,\bullet}(x, y, z) = 0,
\]  
(super Hom-associativity)  
(2.10)
equivalent in juxtaposition notation \(x \bullet y = \bullet(x, y)\) to
\[
(x \bullet y) \bullet \alpha(z) = \alpha(x) \bullet (y \bullet z).
\]
Hom-associativity super identity for Hom-superalgebras is equivalent to
\[
as_{\alpha,\bullet}(A_i, A_j, A_k) = \{0_A\}, \quad i, j, k \in \mathbb{Z}_2.
\]  
(2.11)

**Definition 2.4.** Left Hom-alternative superalgebras are Hom-superalgebras \((A = A_0 \oplus A_1, \bullet, \alpha)\) obeying the left Hom-alternative super identity,
\[
\forall x, y, z \in \mathcal{H}(A) = A_0 \cup A_1 : as_{\alpha,\bullet}(x, y, z) + (-1)^{|x||y|} as_{\alpha,\bullet}(y, x, z) = 0,
\]  
(2.12)
equivalent in juxtaposition notation \(x \bullet y = \bullet(x, y)\) to
\[
(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -(-1)^{|x||y|}(y \bullet x) \bullet \alpha(z) - \alpha(y) \bullet (x \bullet z).
\]
For \((x, y, z) \in A_{|x|} \times A_{|y|} \times A_{|z|}, \ |x|, |y|, |z| \in \mathbb{Z}_2\), the left super Hom-alternativity for \(|x||y| = 0\) or \(|x||y| = 1\) respectively is

\[
|x|y| = 0 : (x, y, z) \in ((A_0 \times A_0) \cup (A_1 \times A_0) \cup (A_0 \times A_1)) \times A_k, \ k \in \mathbb{Z}_2 : \\
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = -((y \cdot x) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot z)),
\]

(2.13)

\[
|x|y| = 1 : (x, y, z) \in A_1 \times A_1 \times A_k, \ k \in \mathbb{Z}_2 : \\
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = (y \cdot x) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot z).
\]

(2.14)

**Definition 2.5.** Right Hom-alternative superalgebra is a Hom-superalgebra \((A = A_0 \oplus A_1, \bullet, \alpha)\) obeying the right Hom-alternative super identity

\[
\forall x, y, z \in \mathcal{H}(A) = A_0 \cup A_1 : \\
as_{\alpha, \bullet}(x, y, z) + (-1)^{|y||z|}as_{\alpha, \bullet}(x, z, y) = 0,
\]

(2.15)

which, in juxtaposition notation \(x \cdot y = \bullet(x, y)\), is

\[
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = (-1)^{|y||z|}((x \cdot z) \cdot \alpha(y) - \alpha(x) \cdot (z \cdot y)).
\]

For \((x, y, z) \in A_{|x|} \times A_{|y|} \times A_{|z|}, \ |x|, |y|, |z| \in \mathbb{Z}_2\), the left super Hom-alternativity for \(|y||z| = 0\) or \(|y||z| = 1\) respectively is

\[
|y|z| = 0 : (x, y, z) \in A_k \times ((A_0 \times A_0) \cup (A_1 \times A_0) \cup (A_0 \times A_1)), \ k \in \mathbb{Z}_2, \\
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = -((x \cdot z) \cdot \alpha(y) - \alpha(x) \cdot (z \cdot y)),
\]

(2.16)

\[
|y|z| = 1 : (x, y, z) \in A_k \times A_1 \times A_1, \ k \in \mathbb{Z}_2, \\
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = (x \cdot z) \cdot (y \cdot z).
\]

(2.17)

**Definition 2.6.** Hom-alternative superalgebras are defined as both left and right Hom-alternative superalgebras.

**Definition 2.7.** Hom-flexible superalgebra is a Hom-superalgebra \((A, \mu, \alpha)\) obeying the Hom-flexible super-identity

\[
\forall x, y, z \in \mathcal{H}(A) = A_0 \cup A_1 : \\
as_{\alpha, \bullet}(x, y, z) + (-1)^{|x||y|+|x||z|+|y||z|}as_{\alpha, \bullet}(z, y, x) = 0,
\]

(2.18)

which, in juxtaposition notation \(x \cdot y = \bullet(x, y)\), is

\[
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = (-1)^{|x||y|+|x||z|+|y||z|}((z \cdot y) \cdot \alpha(x) - \alpha(z) \cdot (y \cdot x)).
\]

For \((x, y, z) \in A_{|x|} \times A_{|y|} \times A_{|z|}, \ |x|, |y|, |z| \in \mathbb{Z}_2\), the left super Hom-alternativity for \(|x||y| + |x||z| + |y||z| = 0\) or \(1\) respectively is

\[
|x||y| + |x||z| + |y||z| = 0 : (x, y, z) \in (A_1 \times A_0 \times A_0) \cup (A_0 \times A_1 \times A_0) \\
\cup (A_0 \times A_0 \times A_1) \cup (A_0 \times A_0 \times A_0),
\]

and

\[
|x||y| + |x||z| + |y||z| = 1 : (x, y, z) \in (A_1 \times A_0 \times A_0) \cup (A_0 \times A_1 \times A_0) \\
\cup (A_0 \times A_0 \times A_1) \cup (A_0 \times A_0 \times A_0),
\]
\[ (x \cdot y) \circ \alpha(z) - \alpha(x) \circ (y \cdot z) = -((z \cdot y) \circ \alpha(x) - \alpha(z) \circ (y \cdot x)), \quad (2.19) \]

\[ |x||y| + |x||z| + |y||z| = 1 : (x, y, z) \in (A_1 \times A_1 \times A_0) \cup (A_1 \times A_0 \times A_1) \cup (A_0 \times A_1 \times A_1), \]

\[ (x \cdot y) \circ \alpha(z) - \alpha(x) \circ (y \cdot z) = (z \cdot y) \circ \alpha(x) - \alpha(z) \circ (y \cdot x). \quad (2.20) \]

**Definition 2.8.** A bimodule over a Hom-alternative superalgebra \((A, \bullet, \alpha)\) consists of a \(\mathbb{Z}_2\)-graded linear space \(V\) with an even linear map \(\beta : V \to V\) and two even bilinear maps

\[
> : \ A \otimes V \to V \quad < : \ V \otimes A \to V
\]

\[
x \otimes v \mapsto x > v \quad v \otimes x \mapsto v < x
\]

such that, for any homogeneous elements \(x, y \in A\) and \(v \in V\),

\[
(v < x) < \alpha(y) + (1)^{|x||v|}(x > v) < \alpha(y) - \alpha(x) - (1)^{|x||v|} \alpha(x) > (v < y) - \beta(v) > (x \cdot y) = 0, \quad (2.21)
\]

\[
\alpha(y) > (v < x) - (y > v) < \alpha(x) - \alpha(y) > (y \cdot x) - (1)^{|x||v|} \alpha(y) > (x > v) = 0, \quad (2.22)
\]

\[
(x \cdot y) > \beta(v) + (1)^{|x||y|}(y \cdot x) > \beta(v) - \alpha(x) > (y > v) - (1)^{|x||y|} \alpha(y) > (x > v) = 0, \quad (2.23)
\]

\[
\beta(v) < (x \cdot y) + (1)^{|x||y|} \beta(v) < (y \cdot x) - (v < x) < \alpha(y) - (1)^{|x||y|} (v < y) < \alpha(x) = 0. \quad (2.24)
\]

**Remark 2.9.** The notation \(x > v\) means the left action of \(x\) on \(v\) and \(v < x\) means the right action of \(x\) on \(v\) given by the linear operators on \(V\) defined by

\[
L_{>}(x)v = x > v, \quad R_{<}(x)v = v < x.
\]

Bimodules over Hom-alternative superalgebras are closed under twisting in the sense of Theorem 2.10.

**Theorem 2.10.** Let \((V, L_{>}, R_{<}, \beta)\) be a bimodule over the multiplicative Hom-alternative superalgebra \((A, \bullet, \alpha)\). Then, \((V, L^o_{>}, R^o_{<}, \beta)\) is a bimodule over \(A\), where \(L^o_{>} = L_{>} \circ (\alpha^2 \otimes \text{Id})\) and \(R^o_{<} = R_{<} \circ (\alpha^2 \otimes \text{Id})\).

**Proof.** We only prove (2.21), as (2.22), (2.23), (2.24) are proved similarly. With

\[
x \geq v = L^o_{>}(x)v = L_{>} \circ (\alpha^2 \otimes \text{Id})(x \otimes v) = \alpha^2(x) > v,
\]

\[
v \leq x = R^o_{<}(x)v = R_{<} \circ (\text{Id} \otimes \alpha^2)(v \otimes x) = v < \alpha^2(x),
\]

for any \(x, y \in A\) and any \(v \in V\),

\[
(v \leq x) \leq \alpha(y) + (1)^{|x||v|}(x \geq v) \leq \alpha(y)
\]
by using the multiplicativity of $\alpha$ in the last term, and then (2.21) for $\alpha^2(x)$ and $\alpha^2(y)$ in $(V, L, R, \alpha, \beta)$. \hfill \Box

For two $\mathbb{Z}_2$-graded linear spaces $V = \oplus_{a \in \mathbb{Z}_2} V_a$ and $V' = \oplus_{a \in \mathbb{Z}_2} V'_a$, the tensor product $V \otimes V'$ is also a $\mathbb{Z}_2$-graded linear space such that for $a, a' \in \mathbb{Z}_2$,

$$(V \otimes V')_a = \sum_{a = a + a'} V_a \otimes V_a'.
$$

**Theorem 2.11.** Let $(A, \cdot, \alpha)$ be a super-commutative Hom-associative superalgebra and $(A', \cdot', \alpha')$ be a Hom-alternative superalgebra. Then the tensor product $(A \circ A', \ast, \alpha \circ \alpha')$ where for $x, y \in \mathcal{H}(A), a, b \in \mathcal{H}(A')$,

$$(\alpha \circ \alpha')(x \circ a) = \alpha(x) \circ \alpha'(a),$$

$$(x \circ a) \ast (y \circ b) = (-1)^{|a||b|}(x \circ y) \otimes (a \cdot b'),$$

is a Hom-alternative superalgebra

**Proof.** Let us set $X = x \circ a$, $Y = y \circ b$, $Z = z \circ c \in \mathcal{H}(A) \otimes \mathcal{H}(A')$. Then,

$$as_{\alpha \circ \alpha', \ast}(X, Y, Z) = as_{\alpha \circ \alpha', \ast}(x \circ a, y \circ b, z \circ c)$$

$$= ((x \circ a) \ast (y \circ b)) \ast ((\alpha \circ \alpha')(x \circ a) \ast ((y \circ b) \ast (z \circ c)))$$

$$= \left((x \circ a) \ast (y \circ b)\right) \ast (\alpha(z) \circ \alpha'(c)) - (\alpha(x) \circ \alpha'(a)) \ast ((y \circ b) \ast (z \circ c))$$

$$= (-1)^{|a||b|}(\alpha(x) \circ \alpha'(a)) \ast ((y \circ b) \circ (b \cdot c'))$$

$$= (-1)^{|a||b|} + |a| + |b|}(\alpha(x) \circ (y \circ z)) \ast (\alpha'(a) \circ (b \cdot c')).$$

$$as_{\alpha \circ \alpha', \ast}(X, Y, Z) + (-1)^{|X|Y|as_{\alpha \circ \alpha', \ast}(Y, X, Z)}$$

$$= as_{\alpha \circ \alpha', \ast}(x \circ a, y \circ b, z \circ c) + (-1)^{|x \circ a||y \circ b|} as_{\alpha \circ \alpha', \ast}(x \circ a, y \circ b, z \circ c)$$

$$= (-1)^{|a||b| + |a| + |b|}(\alpha(x) \circ (y \circ z)) \ast (\alpha'(a) \circ (b \cdot c'))$$

$$- (-1)^{|b||z| + |a||y||z|}(\alpha(x) \circ (y \circ z)) \ast (\alpha'(a) \circ (b \cdot c'))$$


8
and the right Hom-alternativity of $(A, \cdot, \alpha)$. Proof.

Hom-alternativity means both left and right Hom-alternativity. The left

Let $\beta: \text{averaging operator}$ and right averaging operator, meaning an even linear map $(A, \cdot, \alpha)$ be an element of the centroid, an even linear map such that for all $x, y, z \in A$

The left hand side vanishes by the left Hom-alternativity of $A$. Right averaging operator is defined as

Even linear map $\beta: \text{averaging operator}$ as

Definition 2.12 ([31]). Left averaging operator over a Hom-alternative superalgebra $(A, \cdot, \alpha)$ is an even linear map $\beta: A \to A$ satisfying

$$\alpha \circ \beta = \beta \circ \alpha,$$

$$\beta(x) \cdot \beta(y) = \beta(\beta(x) \cdot y) \quad \text{for all } x, y \in \mathcal{H}(A).$$

Right averaging operator over a Hom-alternative superalgebra $(A, \cdot, \alpha)$ is an even linear map $\beta: A \to A$ such that $\alpha \circ \beta = \beta \circ \alpha$ and

$$\beta(x) \cdot \beta(y) = \beta(x \cdot \beta(y)) \quad \text{for all } x, y \in \mathcal{H}(A).$$

Averaging operator over a Hom-alternative superalgebra $(A, \cdot, \alpha)$ is both left averaging operator and right averaging operator, meaning an even linear map $\beta: A \to A$ such that $\alpha \circ \beta = \beta \circ \alpha$ and

$$\beta(\beta(x) \cdot y) = \beta(x) \cdot \beta(y) = \beta(x \cdot \beta(y)).$$

Proposition 2.13. Let $(A, \cdot, \alpha)$ be a Hom-alternative algebra. Let $\beta: A \to A$ be an element of the centroid, an even linear map such that for all $x, y \in \mathcal{H}(A)$,

$$\beta \circ \alpha = \alpha \circ \beta,$$

$$\beta(x \cdot y) = \beta(x) \cdot y = x \cdot \beta(y).$$

Then $(A, \cdot, \beta = \beta \circ \cdot, \alpha)$ is a Hom-alternative superalgebra.

Proof. Hom-alternativity means both left and right Hom-alternativity. The left and the right Hom-alternativity of $(A, \cdot, \beta = \beta \circ \cdot, \alpha)$ are proved respectively as follows. For any $x, y, z \in \mathcal{H}(A)$,

$$as_{\alpha \circ \beta}(x, y, z) = (x \cdot \beta y) \cdot \beta \alpha(z) - \alpha(x) \cdot \beta (y \cdot \beta z)$$
On the other hand, exchanging the role of $y$:

$$\partial((\beta(x) \cdot y) \cdot \alpha(z) - \beta(\alpha(x) \cdot (\beta(y) \cdot z))$$

$$= \beta((\beta(x) \cdot y) \cdot \alpha(z) - \beta(\alpha(x) \cdot (\beta(y) \cdot z))$$

$$= (\beta(x) \cdot \beta(y)) \cdot \alpha(z) - \beta(\alpha(x)) \cdot (\beta(y) \cdot z)$$

$$= (\beta(x) \cdot \beta(y)) \cdot \alpha(z) - \alpha(\beta(x)) \cdot (\beta(y) \cdot z)$$

$$= \alpha(\beta(x), \beta(y), z) \quad (2.27)$$

(Proposition 2.14. Any Hom-alternative superalgebra $(A, \cdot, \alpha)$ with an averaging operator $\partial$ is a Hom-alternative superalgebra with respect to multiplication $\ast : A \ast A \to A$ defined by $x \ast y := x \cdot \partial(y)$ and the same twisting map $\alpha$.

**Proof.** For any $x, y, z \in H(A)$,

$$(x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z) = \alpha(x) \cdot (\partial(y) \cdot \partial(z)) - \alpha(x) \cdot (\partial(y) \cdot \partial(z))$$

$$= \alpha(x) \cdot (\partial(y) \cdot \partial(z)) - \alpha(x) \cdot (\partial(y) \cdot \partial(z)) = 0.$$ 

On the one hand, exchanging the role of $x$ and $y$, yields

$$(x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z) + (-1)^{|x||y|}(y \ast x) \ast \alpha(z) - \alpha(y) \ast (x \ast z) = 0.$$ 

On the other hand, exchanging the role of $y$ and $z$, yields

$$(x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z) + (-1)^{|y||z|}(x \ast z) \ast \alpha(y) - \alpha(x) \ast (z \ast y) = 0.$$ 

□
This completes the proof.

**Definition 2.15** ([1]). A Hom-Jordan superalgebra is a Hom-superalgebra \((A, \bullet, \alpha)\) satisfying super-commutativity and Hom-Jordan super identity

\[
\forall x, y, z, t \in \mathcal{H}(A) : \\
x \bullet y = (-1)^{|x||y|} y \bullet x, \quad \text{super-commutativity} \quad (2.28) \\
\sum_{\otimes(x,y,t)} (-1)^{|t|(|x|+|z|)} a_{x,\alpha}(xy, \alpha(z), \alpha(t)) = 0, \quad \text{Hom-Jordan super identity} \quad (2.29)
\]

where \(\sum_{\otimes(a,b,c)}\) is the summation over cyclically permutated \((a, b, c)\). Hom-Jordan super identity (2.29) in juxtaposition notation \(x \bullet y = \bullet(x, y)\) is

\[
\forall x, y, z, t \in \mathcal{H}(A) : \\
\sum_{\otimes(x,y,t)} (-1)^{|t|(|x|+|z|)} ((x \bullet y) \bullet \alpha(z)) \cdot \alpha^2(t) = \\
\sum_{\otimes(x,y,t)} (-1)^{|t|(|x|+|z|)} \alpha(x \bullet y) \bullet (\alpha(z) \bullet \alpha(t)).
\]

**Remark 2.16.** If \((x, y, z, t) \in (A_0 \times A_0 \times A_0 \times A_0) \cup (A_1 \times A_1 \times A_1 \times A_1)\), then \((-1)^{|t|(|x|+|z|)} = (-1)^{|x||y|+|z|} = (-1)^{|y||t|+|z|} = 1\), and Hom-Jordan super identity is

\[
\sum_{\otimes(x,y,t)} ((x \bullet y) \bullet \alpha(z)) \cdot \alpha^2(t) = \sum_{\otimes(x,y,t)} \alpha(x \bullet y) \bullet (\alpha(z) \bullet \alpha(t)). \quad (2.30)
\]

**Theorem 2.17** ([1]). Any multiplicative Hom-alternative superalgebra is Hom-Jordan admissible, that is, for any multiplicative Hom-alternative superalgebra \((A, \bullet, \alpha)\), the Hom-superalgebra \(A^+ = (A, *, \alpha)\) is a multiplicative Hom-Jordan superalgebra, where \(x * y = xy + (-1)^{|x||y|}yx\).

### 3 Hom-prealternative and Hom-alternative superalgebras

In this section, we introduce Hom-prealternative superalgebras, give some construction theorems and study their connection with Hom-alternative superalgebras. The associated bimodules are also discussed.
3.1 Prealternative superalgebras

**Definition 3.1.** A Hom-prealternative superalgebra is a quadruple \((A, <, >, \alpha)\) in which \(A\) is a supervector space, \(<, >: A \otimes A \to A\) are even bilinear maps and \(\alpha: A \to A\) an even linear map such that, for any \(x, y, z \in H(A)\),

\[
\begin{align*}
(x \cdot x) > \alpha(y) - \alpha(x) > (x > y) &= 0, \quad (3.1) \\
(x < y) < \alpha(y) - \alpha(x) < (x \cdot y) &= 0, \quad (3.2) \\
(x > y) < \alpha(z) - \alpha(x) > (y < z) + \\
&\quad (-1)^{|x||y|}(y < x) < \alpha(z) - (-1)^{|y|} \alpha(y) < (x \cdot z) = 0, \quad (3.3) \\
(x > y) < \alpha(z) - \alpha(x) > (y < z) + \\
&\quad (-1)^{|y||z|}(x \cdot z) > \alpha(y) - (-1)^{|y||z|} \alpha(x) > (z > y) = 0, \quad (3.4)
\end{align*}
\]

where \(x \cdot y = x > y + x < y\).

**Definition 3.2.** Let \((A, <, >, \alpha)\) and \((A', <', >', \alpha')\) be two Hom-prealternative superalgebras. An even linear map \(f: A \to A'\) is said to be a morphism of Hom-prealternative superalgebras if, for any \(x, y \in H(A)\),

\[
\alpha' \circ f = f \circ \alpha, \quad f(x < y) = f(x) <' f(y) \quad \text{and} \quad f(x > y) = f(x) >' f(y).
\]

A Hom-prealternative superalgebra \((A, <, >, \alpha)\) in which \(\alpha: A \to A\) is a morphism is called a multiplicative Hom-alternative superalgebra.

**Remark 3.3.** Axioms (3.1) and (3.2) can be rewritten respectively as

\[
\begin{align*}
(x \cdot y) > \alpha(z) - \alpha(x) > (y > z) &+ \\
&\quad (-1)^{|x||y|}(y \cdot x) > \alpha(z) - (-1)^{|x||y|} \alpha(y) > (x > z) = 0, \quad (3.5) \\
(x < y) < \alpha(z) - \alpha(x) < (y \cdot z) &+ \\
&\quad (-1)^{|y||z|}(x < z) < \alpha(y) - (-1)^{|y||z|} \alpha(x) < (z \cdot y) = 0. \quad (3.6)
\end{align*}
\]

**Remark 3.4.** If \((A, <, >, \alpha)\) is a Hom-prealternative superalgebra, then so is \((A, <_\lambda = \lambda \cdot <, >_\lambda = \lambda \cdot >, \alpha)\).

Using the following notations [85], [59]:

\[
\begin{align*}
(x, y, z)_1 &= (x \cdot y) > \alpha(z) - \alpha(x) > (y > z), \quad (3.7) \\
(x, y, z)_2 &= (x > y) < \alpha(z) - \alpha(x) > (y < z), \quad (3.8) \\
(x, y, z)_3 &= (x < y) < \alpha(z) - \alpha(x) < (y \cdot z), \quad (3.9)
\end{align*}
\]

the axioms in Definition 3.1 of Hom-prealternative superalgebras can be rewritten for any \(x, y, z \in H(A)\) as

\[
(x, x, z)_1 = (y, x, x)_3 = 0 \quad (3.10)
\]
\[(x, y, z)_2 + (-1)^{|y|} (y, x, z)_3 = 0, \quad (3.11)\]
\[(x, y, z)_2 + (-1)^{|z|} (x, z, y)_1 = 0. \quad (3.12)\]

The following definition is motivated by [59, Definition 17, Definition 18].

**Definition 3.5.** A Hom-prealternative superalgebra \((A, \prec, \succ, \alpha)\) is said to be left Hom-alternative if
\[(x, y, z)_i + (-1)^{|x||y|} (y, x, z)_i = 0, \quad i = 1, 2, 3. \quad (3.13)\]
and right Hom-alternative if
\[(x, y, z)_i + (-1)^{|y||z|} (x, z, y)_i = 0, \quad i = 1, 2, 3. \quad (3.14)\]

**Definition 3.6.** A Hom-prealternative superalgebra algebra \((A, \prec, \succ, \alpha)\) is said to be flexible if
\[(x, y, x)_i = 0, \quad i = 1, 2, 3. \quad (3.15)\]

**Theorem 3.7.** If \((A, \prec, \succ, \alpha)\) is a left Hom-prealternative superalgebra, then \(\text{Alt}(A) = (A, \bullet, \alpha)\) is a left Hom-alternative superalgebra. If \((A, \prec, \succ, \alpha)\) is a right Hom-prealternative superalgebra, then \(\text{Alt}(A) = (A, \bullet, \alpha)\) is a right Hom-alternative superalgebra.

**Proof.** For any \(x, y, z \in \mathcal{H}(A)\),
\[
as_\bullet(z, x, y) = (z \bullet x) \bullet \alpha(y) - \alpha(z) \bullet (x \bullet y)
= (z \prec x + z \succ x) \prec \alpha(y) + (z \bullet x) \succ \alpha(y) -
\alpha(z) \prec (x \bullet y) - \alpha(z) \succ (x \bullet y)
= ((z \prec x) \prec \alpha(y) - \alpha(z) \prec (x \bullet y)) + ((z \succ x) \prec \alpha(y) - \alpha(z) \succ (x \succ y)) +
((z \bullet x) \succ \alpha(y) - \alpha(z) \succ (x \succ y))
= (z, x, y)_3 + (z, x, y)_2 + (z, x, y)_1
= -(-1)^{|x||y|} ((z, y, x)_3 + (z, y, x)_2 + (z, y, x)_1)
= -(-1)^{|x||y|} as_\bullet(z, y, x).
\]
The left alternatively is proved analogously. \(\Box\)

Note that the left and right Hom-alternativity for dialgebras is not defined in the same way that the one of algebras with one operation; so the two terminologies must not be confused.
Proposition 3.8. Let \((A, \prec, \succ, \alpha)\) be a flexible \(\text{Hom-prealternative superalgebra.} \) Then \((A, \bullet, \alpha)\) is a flexible \(\text{Hom-alternative superalgebra.} \)

Theorem 3.9. Let \((A, \prec, \succ, \alpha)\) be a \(\text{Hom-prealternative superalgebra.} \) Then \(A' = (A, \prec', \succ', \alpha)\) is also a \(\text{Hom-prealternative superalgebra with} \)
\[
x \prec' y = (-1)^{|x||y|} y \succ x,
\]
\[
x \succ' y = (-1)^{|x||y|} y \prec x.
\]

Proof. We prove only (3.3), as (3.1), (3.2) and (3.4) are proved similarly. For any \(x, y, z \in \mathcal{H}(A), \)
\[
(x \succ' y) \prec' \alpha(z) - \alpha(x) \succ' (y \prec z) +
(-1)^{|x||y|} (y \prec' x) \prec' \alpha(z) - (-1)^{|x||y|} \alpha(y) \prec' (x \bullet z)
\]
\[
= (-1)^{|x||y|} (y \prec x) \prec' \alpha(z) - (z \prec y) +
(-1)^{|x||y|} \alpha(z) \succ' (z \succ y) +
\]
\[
= (-1)^{|x||y| + |x||y| + |z|} \alpha(z) \succ (y \prec x) - (1)^{|x||y| + |x||y| + |z|} (z \succ y) \prec \alpha(x) +
\]
\[
= (-1)^{|x||y| + |x||y| + |z|} \alpha(z) \succ (z \bullet x) \succ \alpha(y) = 0
\]
by axiom (3.4) for \((A, \prec, \succ, \alpha). \)

Note that \(\text{Alt}(A') = \text{Alt}(A)^\text{op}, \) that is, \(x \bullet' y = (-1)^{|x||y|} y \bullet x, \) for \(x, y \in \mathcal{H}(A). \)

Theorem 3.10. Let \((A, \prec, \succ, \alpha)\) be a \(\text{Hom-prealternative superalgebra.} \) Let us define the operation \(x \bullet y = x \prec y + x \succ y \) for any homogeneous elements \(x, y \) in \(A. \) Then \(\text{Alt}(A) = (A, \bullet, \alpha)\) is a \(\text{Hom-alternative superalgebra.} \)

Proof. Let us prove the left alternativity. For any homogeneous \(x, y, z \in A, \)
\[
as_\bullet(x, y, z) + (-1)^{|x||y|} as_\bullet(y, x, z) =
\]
\[
(x \prec y) \prec \alpha(z) + (x \succ y) \prec \alpha(z) + (x \bullet y) \succ \alpha(z) - \alpha(x) \prec (y \bullet z)
\]
\[
- \alpha(x) \succ (y \succ z) - \alpha(x) \succ (y \prec z) + (-1)^{|x||y|} (y \prec x) \prec \alpha(z)
\]
\[
+ (-1)^{|x||y|} (y \succ x) \prec \alpha(z) + (-1)^{|x||y|} (y \bullet x) \succ \alpha(z) - (-1)^{|x||y|} \alpha(y) \prec (x \bullet z)
\]
\[
- (-1)^{|x||y|} \alpha(y) \succ (x \prec z) - (-1)^{|x||y|} \alpha(y) \succ (x \succ z).
\]
The left hand side vanishes by using one axiom (3.3) and twice axiom (3.6). \(\square\)
The Hom-alternative superalgebra \( \text{Alt}(A) = (A, \bullet, \alpha) \) in Theorem 3.10 is called the associated Hom-alternative superalgebra of \((A, \prec, \succ, \alpha)\). We call \((A, \prec, \succ, \alpha)\) a compatible Hom-prealternative superalgebra structure on the Hom-alternative superalgebra \( \text{Alt}(A) \).

Theorem 2.17, Theorem 3.9 and Theorem 3.10 yield the following corollary.

**Corollary 3.11.** Let \((A, \prec, \succ, \alpha)\) be a multiplicative Hom-prealternative superalgebra. Then \((A, *, \alpha)\) is a multiplicative Hom-Jordan superalgebra with

\[
 x * y = x \prec y + y \succ x + (-1)^{|x||y|} y \prec (-1)^{|x||y|} y \succ x.
\]

Let us define the notion of \(O\)-operator for Hom-alternative superalgebras.

**Definition 3.12.** Let \((V, L, R, \beta)\) be a bimodule of the Hom-alternative superalgebra \((A, \bullet, \alpha)\). An even linear map \(T : V \to A\) is called an \(O\)-operator associated to \((V, L, R, \beta)\) if for any \(u, v \in V\),

\[
 T(u) \bullet T(v) = T(L(T(u))v + R(T(v))u), \quad (3.16)
\]

\[
 T \circ \beta = \alpha \circ T. \quad (3.17)
\]

**Theorem 3.13.** Let \(T : V \to A\) be an \(O\)-operator of the Hom-alternative superalgebra \((A, \bullet, \alpha)\) associated to the bimodule \((V, L, R, \beta)\). Then \((V, \prec, \succ, \beta)\) is a Hom-prealternative superalgebra structure, where for all \(u, v \in V\),

\[
 u \prec v = R(T(v))u \quad \text{and} \quad u \succ v = L(T(u))v.
\]

Therefore, \((V, \bullet = \prec + \succ, \beta)\) is the associated Hom-alternative superalgebra of this Hom-prealternative superalgebra, and \(T\) is a homomorphism of Hom-alternative superalgebras. Furthermore, \(T(V) = \{T(v), v \in V\} \subseteq A\) is a Hom-alternative subalgebra of \((A, \bullet, \alpha)\), and \((T(V), \prec, \succ, \alpha)\) is a Hom-prealternative superalgebra, where for all \(u, v \in V\),

\[
 T(u) \prec T(v) = T(u \prec v) \quad \text{and} \quad T(u) \succ T(v) = T(u \succ v).
\]

The associated Hom-alternative superalgebra \((T(V), \bullet = \prec + \succ, \alpha)\) is just the Hom-alternative subalgebra structure of \((A, \bullet, \alpha)\), and \(T\) is a homomorphism of Hom-prealternative superalgebras.

**Proof.** For any homogeneous elements \(u, w, w \in V\),

\[
 (u \succ v) \prec \beta(w) - \beta(u) \succ (v \prec w) + (-1)^{|u||w|}(v \prec u) - \beta(w) =
\]

\[
 (-1)^{|u||w|}\beta(v) \prec (u \bullet w) = (T(u)v)T\beta(w) - T\beta(u)(vT(w)) + (-1)^{|u||w|}(vT(u))T\beta(w).
\]
\[
(-1)^{|u||v|}\beta(v)T(uT(w) + T(u)w) = (T(u)v)\alpha(T(w)) - \alpha(T(u))(vT(w)) + \\
(-1)^{|u||v|}(vT(u))\alpha(T(w)) - (-1)^{|u||v|}\beta(v)(T(u)T(w)) = 0. \quad \text{(by (2.21))}
\]

The other identities are checked similarly, and the rest of the proof is easy. \(\square\)

**Definition 3.14.** A Hom-alternative Rota-Baxter superalgebra of weight \(\lambda\) is a Hom-alternative superalgebra \((A, \cdot, \alpha)\) together with an even linear self-map \(R : A \to A\) such that \(R \circ \alpha = \alpha \circ R\) and
\[
R(x) \cdot R(y) = R \left( R(x) \cdot y + x \cdot R(y) + \lambda x \cdot y \right). \quad \text{(3.18)}
\]

**Corollary 3.15.** Let \((A, \cdot, \alpha)\) be a Hom-alternative superalgebra and \(R : A \to A\) a Rota-Baxter operator of weight 0 on \(A\). Then

(i) \(A_R = (A, <, >, \alpha)\) is a Hom-prealternative superalgebra, where
\[
x < y = x \cdot R(y) \quad \text{and} \quad x > y = R(x) \cdot y,
\]
for any homogeneous elements \(x, y \in \mathcal{H}(A)\).

(ii) \((A, \bullet, \alpha)\) is also a Hom-alternative superalgebra with
\[
x \bullet y = R(x) \cdot y + x \cdot R(y).
\]

**Proposition 3.16.** Let \((V, <, >, \beta)\) be a bimodule over the Hom-alternative superalgebra \((A, \cdot, \alpha)\) and \(R : A \to A\) be a Rota-Baxter operator of weight 0 on \(A\). Then, \((V, \triangleleft, \triangleright, \beta)\), where
\[
v \triangleleft x = v \cdot R(x) \quad \text{and} \quad x \triangleright y = R(x) \triangleright v,
\]
is a bimodule over \((A, \bullet, \alpha)\).

**Proof.** For any homogeneous elements \(x, y \in A\) and \(v \in V\),
\[
(v \triangleleft x) \triangleleft \alpha(y) - \beta(v) \triangleleft (x \bullet y) = (v \triangleleft R(x)) \triangleleft \alpha(y) - \beta(v) \triangleleft (R(x) \cdot y + x \cdot R(y)) = (v \triangleleft R(x)) \triangleleft \alpha(y) - \beta(v) \triangleleft (R(x) \cdot R(y)) = (-1)^{|x||y|} \left( (R(x) \triangleright v) \triangleleft \alpha(y) - \alpha(x) \triangleright (v \triangleleft y) \right) \equiv (-1)^{|x||y|} \left( (x \triangleright v) \triangleleft \alpha(y) - \alpha(x) \triangleright (v \triangleleft y) \right). \quad \text{(2.21)}
\]

The other identities are proved similarly. \(\square\)
Theorem 3.17. Let \((A, <, \succ, \alpha)\) be a Hom-prealternative superalgebra, and let \(\beta : A \to A\) be an even Hom-prealternative superalgebra endomorphism. Then \(A_\beta = (A, <_\beta = \beta \circ <, \succ_\beta = \beta \circ \succ, \beta \alpha)\) is a Hom-prealternative superalgebra. Moreover, suppose that \((A', <', \succ')\) is another prealternative superalgebra and \(\alpha' : A \to A'\) is a prealternative superalgebra endomorphism that satisfies \(f \circ \beta = \alpha' \circ f\), then

\[ f : (A, <_\beta = \beta \circ <, \succ_\beta = \beta \circ \succ, \beta \alpha) \to (A', <'_{\alpha'}, \succ'_{\alpha'}, \alpha' \circ <', \alpha') \]

is a morphism of Hom-prealternative superalgebras.

Proof. For all \(x, y, z \in \mathcal{H}(A)\),

\[
(x \succ_\beta y) <_\beta \beta \alpha(z) - \beta \alpha(x) \succ_\beta (y <_\beta z) \\
= \beta((\beta(x) \succ \beta(y))) < \beta(\alpha(z)) - \beta(\alpha(x)) \succ \beta((\beta(y) < \beta(z))) \\
= \beta^2((x \succ y) < \alpha(z) - \alpha(x) \succ (y < z)) \\
= (-1)^{|x||y|}\beta^2(\alpha(y) < (x \bullet z) - (-1)^{|x||y|}(y < x) < \alpha(z)) \\
= (-1)^{|x||y|}\beta(\beta(\alpha(y)) < \beta(x \bullet z) - (-1)^{|x||y|}\beta(y < x) < \beta(\alpha(z))) \\
= (-1)^{|x||y|}\beta(\beta(\alpha(y)) < \beta(x \bullet z) - (-1)^{|x||y|}\beta(y < x) < \beta^2\alpha(z)) \\
= (-1)^{|x||y|}\beta(\beta(\alpha(y)) < \beta(x \bullet z) - (-1)^{|x||y|}\beta(y < x) < \beta(\alpha(z)))
\]

The other axioms are proved similarly. For the second part,

\[ f \circ <_{\alpha} = f \circ \alpha \circ < = \alpha' \circ f \circ < = \alpha' \circ < \circ (f \otimes f) = <_{\alpha'} \circ (f \otimes f). \]

Analogue equalities hold for \(>_{\alpha}\) and \(>_{\alpha'}\).

Taking \(\beta = \alpha^{2^{n-1}}\) yields the following result.

Corollary 3.18. Let \((A, <, \succ, \alpha)\) be a multiplicative Hom-prealternative superalgebra. Then,

(i) For \(n \geq 0\), \(A^n = (A, <^{(n)} = \alpha^{2^{n-1}} \circ <, \succ^{(n)} = \alpha^{2^{n-1}} \circ \succ, \alpha^{2^n})\) is a multiplicative Hom-prealternative superalgebra, called the \(n\)th derived multiplicative Hom-prealternative superalgebra.

(ii) For \(n \geq 0\), \(A^n = (A, <^{(n)} = \alpha^{2^{n-1}} \circ (< + \succ), \alpha^{2^n})\) is a multiplicative Hom-alternative superalgebra, called the \(n\)th derived multiplicative Hom-alternative superalgebra.
3.2 Bimodules of Hom-prealternative superalgebras

**Definition 3.19.** Let \((A, \prec, \succ, \alpha)\) be a Hom-prealternative superalgebra. An \(A\)-bimodule is a supervector space \(V\) with an even linear map \(\beta: V \to V\) and four even linear maps

\[
\begin{align*}
L_\succ &: A \to \text{gl}(V) & L_\prec &: A \to \text{gl}(V) \\
x \mapsto L_\succ(x)(v) = x \succ v, & x \mapsto L_\prec(x)(v) = x \prec v, \\
R_\succ &: A \to \text{gl}(V) & R_\prec &: A \to \text{gl}(V) \\
x \mapsto R_\succ(x)(v) = v \succ x, & x \mapsto R_\prec(x)(v) = v \prec x,
\end{align*}
\]

satisfying the following relations:

\[
\begin{align*}
L_\succ(x \cdot y + (-1)^{|x||y|} y \cdot x)\beta(v) &= L_\succ(\alpha(x))L_\succ(y) + (-1)^{|x||y|} L_\succ(\alpha(y))L_\succ(x), \quad (3.19) \\
R_\succ(\alpha(y))(L_\bullet(x) + (-1)^{|x||v|} R_\bullet(x))v &= L_\succ(\alpha(x))R_\succ(y)v + (-1)^{|x||v|} R_\succ(x \succ y)\beta(v), \quad (3.20) \\
R_\prec(\alpha(y))L_\succ(x) + (-1)^{|x||v|} R_\prec(\alpha(y))R_\prec(x) &= L_\prec(\alpha(x))R_\prec(y) + (-1)^{|x||v|} R_\prec(x \circ y)\beta(v), \quad (3.21) \\
R_\prec(\alpha(y))R_\succ(x)v + (-1)^{|x||v|} R_\prec(\alpha(y))L_\prec(x)v &= L_\prec(\alpha(x))R_\prec(y)v + (-1)^{|x||v|} R_\prec(x \bullet y)\beta(v), \quad (3.22) \\
L_\prec(y \prec x)\beta(v) + (-1)^{|x||y|} L_\prec(x \succ y)\beta(v) &= L_\prec(\alpha(y))L_\bullet(x)v + (-1)^{|x||y|} L_\prec(\alpha(y))L_\succ(x)v, \quad (3.23) \\
R_\prec(\alpha(x))L_\succ(y) + (-1)^{|x||v|} L_\prec(\alpha(y) \succ x)\beta(v) &= L_\prec(\alpha(y))R_\succ(x)v + (-1)^{|x||y|} L_\prec(\alpha(y))L_\succ(x)v, \quad (3.24) \\
R_\prec(\alpha(x))R_\succ(y)v + (-1)^{|x||v|} R_\prec(\alpha(y))R_\bullet(x)v &= R_\succ(y \prec x)\beta(v) + (-1)^{|x||y|} R_\prec(x \succ y)\beta(v), \quad (3.25) \\
L_\prec(y \succ x)\beta(v) + (-1)^{|x||y|} R_\prec(\alpha(x))L_\bullet(y)v &= L_\prec(\alpha(y))L_\succ(x)v + (-1)^{|x||y|} L_\prec(\alpha(y))R_\succ(y)v, \quad (3.26) \\
R_\prec(\alpha(x))R_\prec(y)v + (-1)^{|x||v|} R_\prec(\alpha(y))R_\prec(x)v &= R_\prec(x \bullet y + (-1)^{|x||y|} y \cdot x)\beta(v), \quad (3.27) \\
R_\prec(\alpha(y))L_\prec(x) + (-1)^{|x||v|} L_\prec(x \prec y)\beta(v) &= L_\prec(\alpha(x))(R_\bullet(y) + (-1)^{|x||y|} L_\bullet(y))v, \quad (3.28)
\end{align*}
\]

where \(\text{gl}(V)\) is the set of even linear maps of \(V\) onto \(V\), \(\bullet = \prec + \succ\) and

\[
x \cdot y = x \prec y + x \succ y, \quad L_\bullet = L_\succ + L_\prec, \quad R_\bullet = R_\prec + R_\succ
\]

for any homogeneous \(x, y, v\).
Remark 3.20. Axioms (3.19)-(3.28) are respectively equivalent to

$$(x \bullet y + (-1)^{|y|}y \bullet x) \succ v \beta(v) = \alpha(x) \succ (y \succ v) + (-1)^{|y|}\alpha(y) \succ (x \succ v),$$

$$(3.29)$$

$$(x \bullet y + (-1)^{|y|}y \bullet x) \succ \alpha(y) = \alpha(x) \succ (v \succ y) - (-1)^{|y|}\alpha(x) \succ (v \succ y),$$

$$(3.30)$$

$$(v \prec x) < \alpha(y) + (-1)^{|y|}|x|v(x \succ y) < \alpha(y) = \beta(v) < (x \bullet y) + (-1)^{|y|}\alpha(x) \succ (v \prec y),$$

$$(3.31)$$

$$(v \prec x) < \alpha(y) + (-1)^{|y|}|x|v(x \succ y) < \alpha(y) = \alpha(x) \prec (v \bullet y) + (-1)^{|y|}\beta(v) \succ (x \bullet y),$$

$$(3.32)$$

$$(y \succ x) < \beta(v) + (-1)^{|y|}|y|y \succ x \prec \beta(v) = \alpha(y) \prec (x \bullet y) + (-1)^{|y|}|y|\alpha(x) \succ (y \succ x),$$

$$(3.33)$$

$$(y \succ x) < \beta(v) + (-1)^{|y|}|y|y \succ x \prec \beta(v) = \alpha(y) \succ (v \prec x) + (-1)^{|y|}|y|\alpha(y) \prec (x \succ y),$$

$$(3.34)$$

$$(v \succ y) < \alpha(x) + (-1)^{|y|}|y|y \bullet x \succ \alpha(y) = \beta(v) < (y \prec x) + (-1)^{|y|}|y|\beta(v) \succ (x \succ y),$$

$$(3.35)$$

$$(y \prec x) < \beta(v) + (-1)^{|y|}|y|y \bullet y \prec \alpha(x) = \alpha(y) \succ (x \prec y) + (-1)^{|y|}|y|\alpha(y) \prec (v \succ x),$$

$$(3.36)$$

$$(y \prec x) < \beta(v) + (-1)^{|y|}|y|y \bullet y \prec \alpha(x) = \beta(v) < (x \bullet y) + (-1)^{|y|}|y|\alpha(y) \succ (y \bullet x),$$

$$(3.37)$$

$$(x \prec y) < \alpha(y) + (-1)^{|y|}|y|y \bullet y \prec \beta(v) = \alpha(x) \prec (v \bullet y) + (-1)^{|y|}|y|\alpha(y) \succ (y \bullet y),$$

$$(3.38)$$

Proposition 3.21. Let $$(A, \prec, \succ, \alpha)$$ be a Hom-prealternative superalgebra.
Then $$(A, l_\prec, r_\succ, \alpha)$$ is a bimodule of the associated Hom-alternative superalgebra $${\text{Alt}}(A) = (A, \bullet, \alpha)$$.

Proposition 3.22. Let $$(V, \prec, \succ, \beta)$$ be a bimodule over the Hom-alternative superalgebra $$(A, \bullet, \alpha)$$ and $${\mathcal{R}} : A \rightarrow A$$ be a Rota-Baxter operator on $A$. Then $$(V, 0, \triangleright, 0, \triangleleft, \beta)$$, with $x \triangleright v = {\mathcal{R}}(x) \succ v$ and $v \triangleleft x = v \prec {\mathcal{R}}(x)$, is a bimodule over the Hom-prealternative superalgebra $A_{\mathcal{R}} = (A, \prec, \succ, \alpha)$.

Proof. For any homogeneous elements $x, y \in A$ and $v \in V$,

$$(v \triangleleft x) \triangleleft \alpha(y) + (x \triangleright v) \triangleleft \alpha(y)$$

$$= (v \triangleleft {\mathcal{R}}(x)) \triangleleft {\mathcal{R}}(\alpha(y)) + ({\mathcal{R}}(x) \triangleright v) \triangleleft {\mathcal{R}}(\alpha(y))$$

$$= (v \triangleleft {\mathcal{R}}(x)) \triangleleft \alpha({\mathcal{R}}(y)) + ({\mathcal{R}}(x) \triangleright v) \triangleleft \alpha({\mathcal{R}}(y))$$

$$= {\mathcal{B}}(v) \triangleleft ({\mathcal{R}}(x) \cdot {\mathcal{R}}(y)) + \alpha({\mathcal{R}}(x)) \triangleleft (v \triangleleft {\mathcal{R}}(y))$$

$$= (v \triangleleft {\mathcal{R}}(x)) \triangleleft {\mathcal{R}}(\alpha(y)) + ({\mathcal{R}}(x) \triangleright v) \triangleleft {\mathcal{R}}(\alpha(y))$$

$$= {\mathcal{B}}(v) \triangleleft ({\mathcal{R}}(x) \cdot {\mathcal{R}}(y)) + \alpha({\mathcal{R}}(x)) \triangleleft (v \triangleleft {\mathcal{R}}(y))$$

$$= (v \triangleleft {\mathcal{R}}(x)) \triangleleft \alpha({\mathcal{R}}(y)) + ({\mathcal{R}}(x) \triangleright v) \triangleleft \alpha({\mathcal{R}}(y))$$

$$= {\mathcal{B}}(v) \triangleleft ({\mathcal{R}}(x) \cdot {\mathcal{R}}(y)) + \alpha({\mathcal{R}}(x)) \triangleleft (v \triangleleft {\mathcal{R}}(y))$$
Proof. For any homogeneous elements \(x, y \in A\) and \(v \in V\),

i) The statement (i) follows from axioms (3.29), (3.31), (3.34) and (3.37).

ii) For (ii), the axiom (2.23) is verified as follows,

\[
(x \cdot y) \cdot \beta(v) + (-1)^{|y||x|} (y \cdot x) \cdot \beta(v) - \alpha(x) \cdot (y \cdot v) - (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v)
= (x \cdot y) \cdot \beta(v) + (x \cdot y) \cdot \beta(v) + (x \cdot y) \cdot \beta(v)
+ (-1)^{|y||x|} (y \cdot x) \cdot \beta(v) + (-1)^{|x||y|} (y \cdot x) \cdot \beta(v)
+ (-1)^{|x||y|} (x \cdot x) \cdot \beta(v)
- \alpha(x) \cdot (y \cdot v) - \alpha(x) \cdot (y \cdot v) - \alpha(x) \cdot (y \cdot v)
- (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v) - (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v)
- (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v).
\]

The left hand side vanishes by axioms (3.29) and (3.33). The other axioms are verified analogously: axiom (2.24) comes from axioms (3.35) and (3.37); axiom (2.22) comes from axioms (3.34), (3.36) and (3.38); axiom (2.21) comes from axioms (3.30), (3.31) and (3.32).

iii) It suffices to take \(R_{\succ} = 0\) and \(L_{\prec} = 0\).

\[\square\]

Theorem 3.23. Let \((V, L_{\prec}, R_{\prec}, L_{\succ}, R_{\succ}, \beta)\) be a bimodule over the Hom-prealternative superalgebra \((A, \prec, \succ, \alpha)\) and \(\text{Alt}(A) = (A, \bullet, \alpha)\) the associated Hom-alternative superalgebra. Then

(i) \((V, L_{\succ}, R_{\succ}, \beta)\) is a bimodule over \(\text{Alt}(A)\).

(ii) \((V, L_{\bullet} = L_{\prec} + L_{\succ}, R_{\bullet} = R_{\prec} + R_{\succ}, \beta)\) is a bimodule over \(\text{Alt}(A)\).

(iii) If \((V, L, R, \beta)\) is a bimodule of \(\text{Alt}(A)\), then \((V, 0, R, L, 0, \beta)\) is a bimodule over \((A, \prec, \succ, \alpha)\).

Proof. For any homogeneous elements \(x, y \in A\) and \(v \in V\),

i) The statement (i) follows from axioms (3.29), (3.31), (3.34) and (3.37).

ii) For (ii), the axiom (2.23) is verified as follows,

\[
(x \cdot y) \cdot \beta(v) + (-1)^{|x||y|} (y \cdot x) \cdot \beta(v)
- \alpha(x) \cdot (y \cdot v) - (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v)
= (x \cdot y) \cdot \beta(v) + (x \cdot y) \cdot \beta(v) + (x \cdot y) \cdot \beta(v)
+ (-1)^{|y||x|} (y \cdot x) \cdot \beta(v) + (-1)^{|x||y|} (y \cdot x) \cdot \beta(v)
+ (-1)^{|x||y|} (x \cdot x) \cdot \beta(v)
- \alpha(x) \cdot (y \cdot v) - \alpha(x) \cdot (y \cdot v) - \alpha(x) \cdot (y \cdot v)
- (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v) - (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v)
- (-1)^{|x||y|} \alpha(y) \cdot (x \cdot v).
\]

The left hand side vanishes by axioms (3.29) and (3.33). The other axioms are verified analogously: axiom (2.24) comes from axioms (3.35) and (3.37); axiom (2.22) comes from axioms (3.34), (3.36) and (3.38); axiom (2.21) comes from axioms (3.30), (3.31) and (3.32).

iii) It suffices to take \(R_{\succ} = 0\) and \(L_{\prec} = 0\).

\[\square\]

Theorem 3.24. Let \((V, L_{\prec}, R_{\prec}, L_{\succ}, R_{\succ}, \beta)\) be a bimodule over the multiplicative Hom-prealternative superalgebra \((A, \prec, \succ, \alpha)\), and let \(\text{Alt}(A) = (A, \bullet, \alpha)\)
be the associated Hom-alternative superalgebra. Then both $(V, L_\prec, R_\prec, \beta)$ and $(V, L_\succ = L_\prec + L_\succ, R_\succ = R_\prec + R_\succ, \beta)$ are bimodules over $\text{Alt}(A)$, where

\begin{align*}
L_\alpha^\alpha &= L_\prec \circ (\alpha^2 \otimes \text{Id}), & L_\alpha^\alpha &= L_\succ \circ (\alpha^2 \otimes \text{Id}), \\
R_\alpha^\alpha &= R_\prec \circ (\alpha^2 \otimes \text{Id}), & R_\alpha^\alpha &= R_\succ \circ (\alpha^2 \otimes \text{Id}).
\end{align*}

Proof. We only prove (2.23) in detail, as the other axioms are verified similarly. Putting $\succ_\alpha = L_\alpha^\alpha$, for any homogeneous elements $x, y \in A$ and $v \in V$,

\begin{align*}
(x \circ y + (-1)^{|x||y|} y \circ x) \succ_\alpha \beta(v) &= \alpha^2(x \circ y + (-1)^{|x||y|} y \circ x) \succ \beta(v) \\
&= (\alpha^2(x) \circ \alpha^2(y) + (-1)^{|x||y|} \alpha^2(y) \circ \alpha^2(x)) \succ \beta(v) \\
&= \alpha^3(x) \succ (\alpha^2(y) \succ v) + (-1)^{|x||y|} \alpha^3(y) \succ (\alpha^2(x) \succ v) \\
&= \alpha(x) \succ (y \succ v) + (-1)^{|x||y|} \alpha(y) \succ (x \succ v). \square
\end{align*}

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