Duality in the Quantum Dissipative Villain Model and application to Mesoscopic Josephson Junction Circuits

G. Falci and U. Weiss

Abstract
We study exact self duality in the model of a Brownian particle in a washboard (WB) potential which describes a Josephson Junction (JJ) coupled to an environment, for arbitrary temperature and arbitrary form of the spectral density of the environment.

To this end we introduce the Quantum Dissipative Villain Model (QDVM), which models tunneling of a degree of freedom coupled to a linear quantum environment through an infinite set of states. We derive general exact mappings on various dual discrete representations (one-dimensional Coulomb gases or surface roughening models) which are exactly self-dual. Then we show how the QDVM maps exactly onto the WB model and use duality relations to calculate the leading terms of the total impedance of a JJ circuit, for general frequency dependence of the spectral density of the environment and arbitrary temperature.

Key words Quantum dissipative systems; Josephson tunneling; Mesoscopic systems

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1 INTRODUCTION

Tunneling in quantum dissipative systems can be described in terms of one (or few) special degree of freedom coupled to an incoherent environment of quantum oscillators [1]. If the special variable has an underlying discrete character the model may have various discrete dual representations, each corresponding to specific dressed excitations.

As an example of underlying discrete character of a continuous degree of freedom we may consider a Brownian particle in a double-well potential [1], a model which also describes a mesoscopic SQUID [2]. In some region of the parameter space the relevant physics is well represented by a two-state approximation, each state being related to a minimum of the potential.

In this paper we will focus on the WB model [3] which describes the Brownian motion of a quantum particle in a tilted washboard potential and the quantum dynamics of mesoscopic JJ circuits [3, 4], For JJs the above special variable is the phase \( \varphi \) across the junction (see fig.1) a collective degree which interacts with all the microscopic variables entering the problem (quasiparticles in the electrodes [4], excitations of the circuit [5]) modeled by an environment with given spectral density. The partition function is

\[
Z(I_x) = \int D\varphi_x \ e^{-\frac{1}{2} \int_0^\beta d\tau d\tau' \varphi(\tau) \mathcal{A}(\tau-\tau') \varphi(\tau')} + \int_0^\beta d\tau I_x(\tau) \varphi(\tau) \ e^{\int_0^\beta d\tau V \cos \varphi(\tau)}
\]

\[
\mathcal{A}(\omega) = m\omega^2 - \frac{\alpha(\omega)}{2} \iff \frac{|\omega|}{2\pi} \frac{R_Q}{Z_T(-i|\omega|)},
\]

where \( m \) is the mass and \( \alpha(\omega) \) is the damping kernel for the Brownian particle [1]. They correspond to \( C \) and \( Z(\omega) \) in the JJ circuit of fig.1, \( Z_T(\omega) \) being the impedance seen by the junction [3, 4, 5], \( Z_T(\omega) = (1/Z(\omega) + i\omega C)^{-1} \).
The WB model accounts for a variety of effects (e.g., Josephson effect, phase diffusion, macroscopic quantum phenomena, Coulomb blockade, Bloch oscillations) in different regimes, due to its “dual” structure. Indeed $\phi$ and the charge $Q$ at the junction are conjugate quantum variables, $[\hbar\phi/2e, Q] = i\hbar$ and both have an underlying discrete character (it comes from the existence of minima in the WB potential for $\phi$, whereas for $Q$ it is related to the fact that $Q$ varies in units of $2e$ due to Cooper pair tunneling). As a result pairs of “dual” effects are observed in JJ circuits, for instance the Josephson effect (a finite current flowing at zero voltage, corresponding to classical dynamics of $\phi$) and the Coulomb blockade (finite voltage drop with zero current, corresponding to classical stochastic dynamics of $Q$). It is tempting to think that “quantum duality” implies a phase transition triggered by the ratio $V/E_C$, $E_C = e^2/2C$ being the charging energy, but this is wrong: the low-frequency response of a single junction device in a circuit is determined solely by the properties of the coupling with the environment. For instance, for an ohmic environment the critical parameter is $\alpha = R_Q/Z(\omega = 0)$ ($R_Q = \hbar/4e^2 \approx 6.5 \text{k}\Omega$ is the quantum of resistance): Coulomb blockade is possible if $\alpha \ll 1$ (effectively current biased junction), whereas for $\alpha \gg 1$ (effectively voltage biased junction) the dynamics of $\phi$ is semiclassical.

In general, collective excitations and duality are more involved than the Kramers-Wannier $\phi$-$Q$ duality because they depend mainly on the environment. Duality in the WB model with ohmic dissipation at $T = 0$ was studied first by Schmid who found a low-frequency approximate self-duality with $\alpha \leftrightarrow 1/\alpha$. A dual mapping onto a one-dimensional Tight-
Binding model was then introduced [9] for arbitrary temperature, which was then used by Sassetti et al. [10] to map into each other the sub-ohmic and the super-ohmic environment. Very recently it was shown that the Schmid self-duality is exact at $T = 0$ for a strictly ohmic environment [11], $m, C \to 0$ and $Z_T(\omega) = R$ limit of eq.(2), where the problem is exactly solvable [12].

In this paper we study the discrete representations and duality for the WB model. We introduce in sec.2 the QDVM and its discrete dual and self-dual representations, surface roughening and Coulomb gas models [13]. The QDVM, besides illustrating in a simple but exact mathematical framework how duality works in connection with discreteness and dissipation, can be reduced exactly to the WB model with a given environment (sec.3). thus, also the WB model has an exact self-dual structure for arbitrary environment and temperature. As an application we finally calculate the impedance of a Josephson circuit for low frequencies.

2 THE QDVM

2.1 The model

The QDVM is defined on the discretized (slice $1/\Lambda$) imaginary time axis as

$$Z(J) = \int_{-\infty}^{\infty} \left[ \prod_{\tau} d\varphi_\tau \right] e^{-\frac{1}{\Lambda^2} \sum_{\tau,\tau'} \varphi_\tau A^0_{\tau,\tau'} \varphi_{\tau'} + \frac{i}{\lambda} \sum_{\tau} J_\tau \varphi_\tau} \cdot \sum_{\{m_\tau\}} e^{-\frac{1}{\lambda} \sum_{\tau} V(\varphi_\tau - 2\pi m_\tau)^2}. \quad (3)$$

The discrete variable $m_\tau$ is coupled to $\varphi_\tau$ which undergoes gaussian fluctuations tuned by the kernel $A^0_{\tau,\tau'}$, $J_\tau$ being the source. We can interpret eq.(3) as the discretized path integral for $\varphi(\tau) - 2\pi m(\tau)$, a continuous variable with an underlying discrete character (see fig.2a) coupled to an environment.
By comparing eq.(3) with eq.(1) it is apparent that the QDVM can be obtained from the discretized version of the WB model by using the Villain approximation [14], so we can identify the kernel \( A^{0}_{\tau,\tau'} \). The Villain approximation has been successfully used to study mesoscopic JJ arrays [15], although it is not valid in the continuum limit \( V/\Lambda \rightarrow 0 \). Notice however that here we use a strategy different from ref. [15]: we first discuss the dual properties of the QDVM and then we show how to obtain exactly from it the WB model, in the limit \( V/\Lambda \rightarrow 0 \).

### 2.2 Discrete representations of the QDVM

**Surface roughening representation**  Starting from eq.(3), we can eliminate \( \{ \varphi_\tau \} \) and obtain a surface roughening model [13] on the imaginary time axis, where \( m_\tau \) is the surface height in \([\tau, \tau + 1/\Lambda]\) (see fig.2b). Each element \( m_\tau \) of the surface interacts with all the other elements.

**Coulomb gas representation I: e charges** We now introduce the charges \( e_\tau = m_{\tau + 1/\Lambda} - m_\tau \) and perform a discrete double integration of the interaction to obtain the generating functional of the QDVM in the \( e \)-representation

\[
Z(J) = Z^V(J) \cdot Z^{(e)}(J^{(e)})
\]

\[
Z^{(e)}(J^{(e)}) = \sum_{\{e_\tau\}} e^{-\frac{1}{2} \sum_{\tau,\tau'=0}^{N-1} e_{\tau'} \Delta_{\tau,\tau'} e_{\tau'} + \sum_\tau J^{(e)}_\tau e_{\tau}} . \quad (4)
\]

This is a set of interacting \( (\Delta_{\tau,\tau'}) \) integer charges \( e_\tau \in ]-\infty, \infty[ \) (see fig.2c). \( Z^V(J) \) is the generating functional for the damped harmonic oscillator [1], and the interaction is given by

\[
\Delta_{\tau} = \frac{\Lambda}{N} \sum_{|\omega|<\Lambda} \left\{ \frac{4\pi^2 m - 4\pi^2 \alpha(\omega)/2\omega^2}{m\omega^2 - \alpha(\omega)/2 + V} V e^{-i\omega\tau} \right\} .
\]
The source \( J^{(e)} \) can be obtained from the source \( J_{\tau} \) of the QDVM [16].

If we classify the configurations in eq.(4) according to the number of charges of each species we obtain the Coulomb gas representation

\[
Z^{(e)}(J^{(e)}) = \sum_{\{\bar{M}_e\}} \left[ \prod_{\bar{e} = -\infty}^{\infty} (\bar{M}_e!)^{-1} \right] \sum_{\{\tau_l\}} e^{-\frac{i}{2} \sum_{l,l' = 1}^{M_T} e_l \Delta_{\tau_l \tau_{l'}} e_{l'} + \sum_{l} J^{(e)}_{\tau_l} e_l} .
\]

Here each term of the summation is labeled by a set \( \{\bar{M}_e\} \) where \( \bar{M}_e \) is the number of charges \( \bar{e} \neq 0 \) present. The charges are numbered with the index \( l = 1, \ldots, M_T \), where \( M_T = \sum_{\bar{e} = -\infty}^{\infty} \bar{M}_e \). For each set \( \{\bar{M}_e\} \) only a special configuration enters eq.(4). The properties of the kernel enforce the charge neutrality condition \( \sum_{\bar{e} = -\infty}^{\infty} \bar{M}_e \bar{e} = 0 \).

**Coulomb gas representation II: n charges** A different Coulomb gas representation can be obtained if we apply the Poisson transformation to eq.(4) (or to eq.(3)). We obtain the \( n \)-representation of the QDVM

\[
Z(J) = Z^0(J) \cdot Z(n)(J^{(n)})
\]

\[
Z(n)(J^{(n)}) = \sum_{\{n_{\tau}\}} e^{-\frac{i}{2} \sum_{\tau, \tau' = 0}^{N-1} n_{\tau} D_{\tau, \tau'} n_{\tau'} + \sum_{\tau} J^{(n)}_{\tau} n_{\tau}} ,
\]

which is again a gas of interacting integer charges \( n_{\tau} \in ] - \infty, \infty[ \). The partition function \( Z^0(J) \) of the free Brownian particle [1] appears and again \( J^{(n)}_{\tau} \) can be obtained explicitly from \( J_{\tau} \) [16]. The generating functional \( Z(n)(J^{(n)}) \) has the same form of eq.(4) with a different interaction given by

\[
D_{\tau} = \Lambda \frac{V}{N} \delta_{\tau, \tau'} + \Lambda^2 G^0_{\tau} = \Lambda \frac{1}{N} \sum_{|\omega| < \Lambda} \left\{ \frac{1}{V} + \frac{1}{m \omega^2 - \alpha(\omega)/2} \right\} e^{-i \omega \tau} .
\]

where \( G^0_{\tau} \) is the discrete Green’s function of the free Brownian particle [1]. By proceeding as before, a Coulomb gas form for \( Z(n)(J^{(n)}) \) is obtained, structurally identical to eq.(4).
2.3 Self-duality

The fact that $Z^{(e)}(J^{(e)})$ and $Z^{(n)}(J^{(n)})$ are structurally identical indicates that the QDVM possesses a self-dual structure. The relation between the interactions $\Delta_\tau$ and $D_\tau$ assumes a simple form if we introduce the function $h^0(w) := Z_T(-iw)/R_Q$. Then $(1/\Lambda) D_\omega = 2\pi h^0(|\omega|)/|\omega| + 1/V =: (2\pi/|\omega|) h(|\omega|)$ and $(1/\Lambda) \Delta_\omega = (2\pi/|\omega|) [1/h(|\omega|)]$. So the exact self-duality in the QDVM is given by $h(\omega) \leftrightarrow 1/h(\omega)$, which is a sort of a Norton transformation.

For low frequencies we obtain the $\alpha \leftrightarrow 1/\alpha$ and the sub-ohmic $\leftrightarrow$ super-ohmic dualities analogous to the ones possessed by the WB model [8, 10].

Self-duality is a powerful tool for obtaining nonperturbative results. For instance from our low-energy results we can infer the existence of the critical line $\alpha = 1$ for ohmic dissipation in the QDVM, analogously to the WB model [8]. For any dual mapping, perturbative results in a certain representation may correspond to nonperturbative results in a different representation (see the next section). An appealing feature of self-duality is that one can write exact relations (as the eqs. (8) and (10) we will use below) and exact equations for the correlation functions [16].

3 THE WB MODEL

3.1 Mapping of the QDVM onto the WB model

Now we show that if we take the continuum limit in a special way the QDVM on a lattice reduces exactly to the WB model. Indeed from the $n$-representation we obtain the exact perturbation series in $V$ as given in ref. [8].
The proof goes as follows. First one considers the Coulomb gas representation for \( n \)-charges of the QDVM with the substitution \( V \rightarrow \tilde{V} = \Lambda/2 \ln(2\Lambda/V) \).

Then one shows that the configurations where only the charges \( n_l = \pm 1 \) appear give the exact perturbation expansion in \( V \) for the WB potential \[8\]. Finally one shows that all the other charge configurations carry an additional factor \( \Lambda \) with a negative power and give a vanishing contribution to the partition function as \( \Lambda \rightarrow \infty \).

The consequence of this mapping is that the WB model possesses an exact self duality for arbitrary \( T \) and arbitrary spectral densities which generalizes the approximate Schmid self-duality \[8\].

Notice that the exact self-dual mapping is not evident if one starts directly from the WB model. In fact the perturbation expansion is recasted in a Coulomb gas with only \( n_l = \pm 1 \) charges, whereas in the instanton expansion (the dual Coulomb gas) all kind of charges \( e_l \in ] -\infty, \infty[ \) may appear. Self-duality can be exploited by carrying over the exact results for the correlation functions of the QDVM to the WB model.

### 3.2 Impedance of a JJ circuit

As an application we calculate the leading terms of the retarded \( \langle \varphi \varphi \rangle \) correlator which is related to the impedance \( Z(\Omega) \) seen by the source of the circuit in fig.1 Starting from the thermal \( \langle \varphi \varphi \rangle \) correlator of the Villain model we perform an analytic continuation, \( \omega_n \rightarrow -i\omega \), and then take the continuum limit with the prescription of sec.3.1. It would be desirable to perform the analysis in real-time \[1\], but at present we content ourself to check the QDVM method against available exact results \[12\]. We we start with the...
exact expression for the correlator \((1/\Lambda) \langle \varphi \varphi \rangle_{\omega_n} \) vs. \(\langle nn \rangle_{\omega_n}\)

\[
\Lambda \mathcal{G}_\omega^0 - \left( \Lambda \mathcal{G}_\omega^0 \right)^2 \Lambda \langle nn \rangle_{\omega_n} \longrightarrow x^0_R(\omega) - \left[ x^0_R(\omega) \right]^2 \left[ X^0_R(0) - X^0_R(\omega) \right],
\]

(8)

where \(x^0_R\) is the retarded correlator of the free Brownian particle. and \(X^0_R\) can be evaluated by analytic continuation of the first term of the Coulomb gas expansion:

\[
2\Lambda \sum_\sigma e^{D_\sigma - D_0} e^{i\omega_n \sigma} \xrightarrow{\omega_n \to -i\omega} -4\Lambda^2 e^{-\Lambda/\tilde{V}} \int_0^\infty ds e^{i\omega s} \Im e^{F_0^>(s)}
\]

Here \(F_0^>(s)\) accounts for free circuit fluctuations [3, 7, 17, 1].

\[
F_0^>(s) = \int_0^\infty d\omega \frac{2 \Re Z_T(\omega)}{R_Q} \left[ \cosh \frac{\beta \omega}{2} \left( \cos \omega s - 1 \right) - i \sin \omega s \right]
\]

In particular \(\Im X_R^{(n)}(\omega) = (V/2)^2 \left[ P_0(\omega) - P_0(-\omega) \right]\), and its Hilbert transform gives \(\Re X_R\). We introduced \(P_0(\omega) = \int_{-\infty}^\infty ds \exp\{i\omega s\} \exp\{F_0^>(s)\}\) which can be calculated numerically for almost arbitrary environment at all \(T\) [3, 7, 1].

As a check of the theory we calculated analytically the low-frequency behavior of \(Z(\Omega)\) at \(T = 0\), using the toy ohmic environment \(\Re Z_T(\omega)/R_Q = \alpha^{-1} \exp\{-|\omega|/\tilde{\omega}\}\). For \(\alpha < 1\) we find

\[
\frac{\Re Z(\Omega)}{R_Q} = \frac{\Omega}{2\pi} \Im x_R(-\Omega) \xrightarrow{\Omega \to 0^+} \frac{\Re Z_T(\Omega)}{R_Q} \left\{ 1 - A \Omega^{-2+2/\alpha} - B\Omega^2 \right\}
\]

(9)

which reproduces known exact results [12], i.e. the leading behavior of the correlation function is \(\sim \Omega^2\) for \(0 < \alpha < 1/2\) and \(\sim \Omega^{-2+2/\alpha}\) for \(1/2 < \alpha < 1\). It corresponds to a vanishing junction admittance, which describes the Coulomb blockade. The coefficients in (9) depend on the details of the environment at frequencies of the order of the cutoff \(\tilde{\omega}\) and can be evaluated numerically.
Then we start with the exact expression of the $\langle \varphi \varphi \rangle$ correlation function in the $e$-representation and then perform the analytic continuation

$$\frac{1}{\Lambda} \langle \varphi \varphi \rangle_{\omega_n} = \Lambda G_{\omega_n}^{\tilde{V}} - \left( \frac{2\pi \Lambda \tilde{G}_{\omega_n}^{\tilde{V}}}{\omega_n} \right)^2 \Lambda \langle ee \rangle_{\omega_n} \xrightarrow{\omega_n \to -i\omega}$$

$$x_{R}^{\tilde{V}}(\omega) - \left[ \frac{2\pi \Lambda \tilde{G}_{\omega_n}^{\tilde{V}} x_{R}^{\tilde{V}}(\omega)}{\omega} \right]^2 \left[ X_{R}^{(e)}(0) - X_{R}^{(e)}(\omega) \right]$$

(10)

where $G_{\omega_n}^{\tilde{V}}$ is the discrete Green’s function and $x_{R}^{\tilde{V}}$ is the retarded correlation function for the damped oscillator. As before we consider the first contribution in the Coulomb gas expansion eq. (5) for $\Lambda \langle ee \rangle_{\omega_n}$

$$2\Lambda \sum_{\sigma} e^{i\omega_n \sigma} e^{i\omega_n \sigma} \xrightarrow{\omega_n \to -i\omega} -4\Lambda^2 \int_0^{\infty} ds e^{i\omega s} \Im e^{\Delta^{(e)}(s)}$$

(11)

A different correlator for the fluctuations appears, $F^{>}(s)$, coming from the analytic continuation of $\Delta_{\tau}$, from which we can define a $P(\omega)$ function which enters the expression for $Z(\Omega)$. To evaluate $Z(\Omega)$ at all frequencies we need a regularization scheme, which is not needed for checking analytically the power-low behavior at low frequencies. For $\alpha > 1$ we find

$$\frac{\Re Z(\Omega)}{R_Q} \xrightarrow{\Omega \to 0^+} A \Omega^{2\alpha - 2} + B \Omega^2$$

which in this case gives also the junction impedance. Again known exact results [12] are reproduced.

4 CONCLUSIONS

In summary we have exploited the Schmid [9] self-duality in the WB model. Using the QDVM we have shown that it is exact, that it holds for finite temperatures and that it can be extended to an environment with general
frequency dependence of the spectral density. Exact low-frequency results for the strictly ohmic environment are reproduced using this method.

The study of a general environment is important for design and performances of the devices, for instance for an optimal control of the noise. In the Coulomb blockade regime it is possible to treat a general electromagnetic environment in realistic situations using the $P$-function theory \cite{5, 7, 17}. The QDVM theory can be combined with the $P$-function theory to calculate quantities of direct experimental interest different regimes.

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Figure captions

**Figure 1**: The JJ circuit described by the WB model: the special variable is the phase across the junction $\varphi$ and the impedance $Z(\omega)$ is modeled by an environment of quantum oscillators. Classical and quantum fluctuations of $\varphi$ are modulated by the fluctuations of the environment.

**Figure 2**: (a) A typical trajectory in imaginary time for the coordinate of the QDVM, $\varphi(\tau) - 2\pi m(\tau)$. (b) A typical configuration after the elimination of $\{\varphi_\tau\}$: the set $\{m_\tau\}$ is associated to a one-dimensional interface; each elementary $(1/\Lambda)$ piece of surface interacts with all the other elements. (c) The corresponding charge configuration in the Coulomb gas representation eq.(5): in general charges may take any integer value in $e_t \in ] - \infty, \infty[.$
Falci and Weiss - Figure 1

\[ V \]

\[ \phi \]

\[ Z(\omega) \]

\[ I_x \]
Falci and Weiss - Figure 2

\[ \phi_\tau + 2\pi n_\tau \]

(a)

(b)

(c)