Stability of convergence rates: 
Kernel interpolation on non-Lipschitz domains

Tizian Wenzel \(^{1}\), Gabriele Santin \(^{1}\), and Bernard Haasdonk \(^{1}\)

\(^{1}\)Institute for Applied Analysis and Numerical Simulation, University of Stuttgart, Germany

\(^{2}\)Digital Society Center, Bruno Kessler Foundation, Trento, Italy

July 13, 2023

Abstract

Error estimates for kernel interpolation in Reproducing Kernel Hilbert Spaces (RKHS) usually assume quite restrictive properties on the shape of the domain, especially in the case of infinitely smooth kernels like the popular Gaussian kernel.

In this paper we prove that it is possible to obtain convergence results (in the number of interpolation points) for kernel interpolation for arbitrary domains \(\Omega \subset \mathbb{R}^d\), thus allowing for non-Lipschitz domains including e.g. cusps and irregular boundaries. Especially we show that, when going to a smaller domain \(\tilde{\Omega} \subset \Omega \subset \mathbb{R}^d\), the convergence rate does not deteriorate — i.e. the convergence rates are stable with respect to going to a subset. We obtain this by leveraging an analysis of greedy kernel algorithms.

The impact of this result is explained on the examples of kernels of finite as well as infinite smoothness. A comparison to approximation in Sobolev spaces is drawn, where the shape of the domain \(\Omega\) has an impact on the approximation properties. Numerical experiments illustrate and confirm the analysis.

1 Introduction

On a nonempty set \(\Omega \subset \mathbb{R}^d\) a real-valued kernel is defined as a symmetric function \(k : \Omega \times \Omega \to \mathbb{R}\). For a given set of points \(X_n := \{x_1, \ldots, x_n\} \subset \Omega\) the kernel matrix \(A \in \mathbb{R}^{n \times n}\) is defined as \(A_{ij} := k(x_i, x_j)\). If the kernel matrix is positive definite for any choice of pairwise distinct points \(X_n \subset \Omega, n \in \mathbb{N}\), then the kernel is called strictly positive definite.

In the following we focus on this class of kernels. For those kernels, there is a unique Reproducing Kernel Hilbert Space (RKHS) \(\mathcal{H}_k(\Omega)\) of functions, which satisfies

\(^{1}\)tizian.wenzel@mathematik.uni-stuttgart.de, corresponding author

\(^{1}\)gsantin@fbk.eu, orcid.org/0000-0001-6959-1070

\(^{1}\)haasdonk@mathematik.uni-stuttgart.de
1. \( k(\cdot, x) \in \mathcal{H}_k(\Omega) \quad \forall x \in \Omega, \)

2. \( f(x) = \langle f, k(\cdot, x) \rangle_{\mathcal{H}_k(\Omega)} \quad \forall x \in \Omega, f \in \mathcal{H}_k(\Omega). \)

We remark that for certain types of kernels and domains the RKHS \( \mathcal{H}_k(\Omega) \) is found to be norm equivalent to some Sobolev space with specific smoothness \( \tau > d/2 \), i.e. \( \mathcal{H}_k(\Omega) \cong H^{\tau}(\Omega) \). Examples are some radial basis function kernels like the Matérn kernels (see Eq. (20) below) on Lipschitz domains \( \Omega \).

For a function \( f \in \mathcal{H}_k(\Omega) \) and pairwise distinct points \( X_n \subset \Omega \) there is a unique minimum-norm interpolating function, which is given by the projection of \( f \) onto the subspace \( V(X_n) := \{ k(\cdot, x_i), x_i \in X_n \} \) spanned by kernel functions. It holds

\[
\Pi_{\mathcal{H}_k(\Omega), V(X_n)}(f)(x_i) = f(x_i) \quad \forall x_i \in X_n, \quad (1)
\]

where \( \Pi_{\mathcal{H}_k(\Omega), V(X_n)} \) denotes the orthogonal projector from \( \mathcal{H}_k(\Omega) \) onto the closed subspace \( V(X_n) \).

In order to quantify the worst-case (i.e. for any function \( f \in \mathcal{H}_k(\Omega) \)) pointwise error between a function and its interpolant, the so called power function \( P_{k, \Omega, X_n} \) is introduced, which is defined as

\[
P_{k, \Omega, X_n}(x) := \sup_{0 \neq f \in \mathcal{H}_k(\Omega)} \frac{|\langle f - \Pi_{\mathcal{H}_k(\Omega), V(X_n)}(f), x \rangle|}{\|f\|_{\mathcal{H}_k(\Omega)}} = \|k(\cdot, x) - \Pi_{\mathcal{H}_k(\Omega), V(X_n)}(k(\cdot, x))\|_{\mathcal{H}_k(\Omega)}. \quad (2)
\]

Note that the dependency of the power function on the kernel \( k \) and the domain \( \Omega \) was explicitly included in the subscript of the power function for later purposes.

Typical results for bounding the maximal worst case interpolation error \( \|P_{k, \Omega, X_n}\|_{L^\infty(\Omega)} \) in the number \( n \) of interpolation points \( X_n \) make use of the equivalence of the RKHS \( \mathcal{H}_k(\Omega) \) to Sobolev spaces \( H^{\tau}(\Omega) \) as mentioned earlier. These equivalence results however require restrictive assumptions on the domain \( \Omega \) such as Lipschitz boundaries or cone conditions. On the other hand, these conditions are not required for having an RKHS \( \mathcal{H}_k(\Omega) \). In this paper we close this gap by showing that the convergence rates for interpolation in RKHS are stable when going to a smaller domain — in the sense that the convergence order (in the number of interpolation points) does not deteriorate. In this way we obtain convergence rates also for non-Lipschitz domains.

The paper is organized as follows. After recalling background information on kernel based interpolation in Section 2 we review and extend an analysis of greedy algorithms in an abstract setting in Section 3. This analysis of greedy algorithms is leveraged as a tool and applied to general (non-greedy) kernel based interpolation in Section 4 where our main result is derived in Theorem 8 and discussed and applied afterwards. Section 5 illustrates the theoretical findings with numerical experiments. The final Section 6 summarizes and concludes the paper by giving an outlook to future research.

## 2 Background

We proceed by recalling background information on greedy kernel interpolation [De Marchi et al., 2005; Santin and Haasdonk, 2017; Wenzel et al., 2022a] and
on the dependency of the RKHS $\mathcal{H}_k(\Omega)$ on the domain $\Omega$ (Wendland, 2005), which is required to prove our main result.

### 2.1 Greedy kernel interpolation

A suitable choice of interpolation points $X_n \subset \Omega$ is usually unclear a priori. For this task, a popular approach is to use greedy kernel methods to pick the interpolation points step by step. One starts with an empty set $X_0 := \emptyset$ and adds points incrementally via $X_{n+1} := X_n \cup \{x_{n+1}\}$, where $x_{n+1}$ is selected according to some selection criterion. For later purposes we recall the so-called $P$-greedy selection criterion, which uses

$$x_{n+1} := \arg \max_{x \in \Omega} P_{k,\Omega,X_n}(x).$$

(3)

Since it holds $P_{k,\Omega,X_n}(x_i) = 0$ if and only if $x_i \in X_n$, the $P$-greedy selection criterion yields an infinite sequence of pairwise distinct points (given that $\Omega$ consists of infinitely many points). The convergence of $\|P_{k,\Omega,X_n}\|_{L^\infty(\Omega)}$ for $n \to \infty$ was analyzed in (De Marchi et al., 2005; Santin and Haasdonk, 2017; Wenzel et al., 2021).

We remark that there are more greedy methods available, especially the target-data dependent $f$-greedy or $f/P$-greedy (Müller, 2009; Schaback and Wendland, 2000), but also stabilization of those algorithms as analyzed in (Wenzel et al., 2021). An analysis of a unifying scale of target-data dependent algorithms including the aforementioned ones can be found in (Wenzel et al., 2022a).

### 2.2 Restrictions and extensions of domains

This section gives some background information on the dependency of the RKHS $\mathcal{H}_k(\Omega)$ on the domain $\Omega$, and we refer to (Wendland, 2005, Section 10.7) for further details. Especially, we will state our notation more precisely.

We consider two nested domains

$$\tilde{\Omega} \subset \Omega \subset \mathbb{R}^d.$$

(4)

Let $k := k_\Omega : \Omega \times \Omega \to \mathbb{R}$ be a strictly positive definite kernel defined on $\Omega$. Then we define the restricted kernel $\tilde{k}$ via

$$\tilde{k} := k_{\tilde{\Omega}} : \tilde{\Omega} \times \tilde{\Omega} \to \mathbb{R}, (x, y) \mapsto k(x, y).$$

(5)

With this notation we can distinguish between $k(\cdot, x) \in \mathcal{H}_k(\Omega)$ and $\tilde{k}(\cdot, x) \in \mathcal{H}_{\tilde{k}}(\tilde{\Omega})$ for $x \in \tilde{\Omega}$ as elements of different spaces, while both $k$ and $\tilde{k}$ refer to the same mapping.

Theorem 10.46 in (Wendland, 2005) states that each function $\tilde{f} \in \mathcal{H}_{\tilde{k}}(\tilde{\Omega})$ has a natural extension to a function $E \tilde{f} \in \mathcal{H}_k(\Omega)$ and it holds $\|E \tilde{f}\|_{\mathcal{H}_k(\Omega)} = \|\tilde{f}\|_{\mathcal{H}_{\tilde{k}}(\tilde{\Omega})}$. The operator $E : \mathcal{H}_{\tilde{k}}(\tilde{\Omega}) \to \mathcal{H}_k(\Omega)$ will be called the extension operator. Especially it holds $Ek(\cdot, x) = k(\cdot, x)$ for all $x \in \tilde{\Omega}$.

For $X_n \subset \tilde{\Omega}$ we define

$$\tilde{V}(X_n) := \text{span}\{\tilde{k}(\cdot, x_i), x_i \in X_n\} \subset \mathcal{H}_{\tilde{k}}(\tilde{\Omega}),$$

$$V(X_n) := \text{span}\{k(\cdot, x_i), x_i \in X_n\} \subset \mathcal{H}_k(\Omega),$$

(6)

(7)
where we recalled the same definition of $V(X_n)$ given above.

We have the following result, which states that these projections and extensions commute.

**Lemma 1.** For $\tilde{f} \in \mathcal{H}_k(\tilde{\Omega})$ and $X_n \subset \tilde{\Omega}$ it holds

$$E\Pi_{\mathcal{H}_k(\tilde{\Omega}),V(X_n)}(\tilde{f}) = \Pi_{\mathcal{H}_k(\Omega),V(X_n)}(E\tilde{f}).$$

**Proof.** Let

$$\Pi_{\mathcal{H}_k(\tilde{\Omega}),V(X_n)}(\tilde{f}) = \sum_{j=1}^{n} \alpha_j^{(n)} \tilde{k}(\cdot, x_j),$$

with the kernel expansion coefficients $\{\alpha_j^{(n)}\}_{j=1}^{n} \subset \mathbb{R}$ being determined by the interpolation conditions (see Eq. (1)), i.e. $\tilde{f}(x_i) = \sum_{j=1}^{n} \alpha_j^{(n)} \tilde{k}(x_i, x_j)$ for $i = 1, \ldots, n$. Based on the linearity of the extension operator $E$ and the identity $E\tilde{k}(\cdot, x) = k(\cdot, x)$ it holds

$$E\Pi_{\mathcal{H}_k(\tilde{\Omega}),V(X_n)}(\tilde{f}) = E \sum_{j=1}^{n} \alpha_j^{(n)} \tilde{k}(\cdot, x_j) = \sum_{j=1}^{n} \alpha_j^{(n)} E\tilde{k}(\cdot, x_j)$$

$$= \sum_{j=1}^{n} \alpha_j^{(n)} k(\cdot, x_j) = \Pi_{\mathcal{H}_k(\Omega),V(X_n)}(E\tilde{f}),$$

where in the last step we used $E\tilde{f}|_{\tilde{\Omega}} = \tilde{f}$ and $X_n \subset \tilde{\Omega}$, so that the interpolation coefficients $\{\alpha_j^{(n)}\}_{j=1}^{n}$ are the same both in $\mathcal{H}_k(\tilde{\Omega})$ and $\mathcal{H}_k(\Omega)$.

The following lemma shows that the power functions $P_{k,\Omega,X_n}$ and $P_{k,\tilde{\Omega},X_n}$ coincide on $\tilde{\Omega}$, if $X_n \subset \tilde{\Omega}$:

**Lemma 2.** For $X_n \subset \tilde{\Omega}$ it holds

$$P_{k,\Omega,X_n}(x) = P_{k,\tilde{\Omega},X_n}(x) \quad \forall x \in \tilde{\Omega}.$$

**Proof.** From the definition of the power function in Eq. (2), by using the fact that $E\tilde{k}(\cdot, x) = k(\cdot, x)$ for all $x \in \tilde{\Omega}$, and by Lemma 1 we can calculate

$$P_{k,\Omega,X_n}(x) = \|k(\cdot, x) - \Pi_{\mathcal{H}_k(\Omega),V(X_n)}(k(\cdot, x))\|_{\mathcal{H}_k(\Omega)}$$

$$\equiv \|E\tilde{k}(\cdot, x) - \Pi_{\mathcal{H}_k(\Omega),V(X_n)}(E\tilde{k}(\cdot, x))\|_{\mathcal{H}_k(\Omega)}$$

$$= \|E\tilde{k}(\cdot, x) - E\Pi_{\mathcal{H}_k(\tilde{\Omega}),V(X_n)}(\tilde{k}(\cdot, x))\|_{\mathcal{H}_k(\tilde{\Omega})}$$

$$= \|\tilde{k}(\cdot, x) - \Pi_{\mathcal{H}_k(\tilde{\Omega}),V(X_n)}(\tilde{k}(\cdot, x))\|_{\mathcal{H}_k(\tilde{\Omega})}$$

$$\equiv P_{k,\tilde{\Omega},X_n}(x),$$

which holds for all $x \in \tilde{\Omega}$.  

We remark that Lemma 1 and Lemma 2 can be formulated for general closed subspaces $V_n \subset \mathcal{H}_k(\tilde{\Omega})$, $EV_n =: V_n \subset \mathcal{H}_k(\Omega)$ instead of the particular kernel-based subspaces $\tilde{V}(X_n), V(X_n)$ considered here.
2.3 Related work

From a high level point of view, our main result (in Theorem 8 below) allows us to derive (worst-case) error estimates (in the number of interpolation points) for kernel-based interpolation for arbitrary (infinite) sets \( \tilde{\Omega} \subset \mathbb{R}^d \), e.g. non-Lipschitz domains. This principle holds for general continuous kernels \( k \) with \( \sup_{x \in \Omega} k(x, x) < \infty \). Both the error estimates for non-Lipschitz domains as well as for general continuous bounded kernels was not available so far. Thus here we briefly comment on related work on kernel-based interpolation, with a particular focus on the dependency on the underlying domain.

First we note that most worst case error estimates are formulated in terms of the fill distance, which is usually defined as

\[
h_{X_n, \Omega} := \sup_{x \in \Omega} \min_{x_j \in X_n} \| x - x_j \|_2.
\]

For bounded domains and uniformly distributed points there exist constants \( c, C > 0 \) such that

\[
 cn^{-1/d} \leq h_{X_n, \Omega} \leq Cn^{-1/d}.
\]

Using this relationship between the number of points \( n \) and the fill distance \( h_{X_n, \Omega} \), any error estimate expressed in the fill distance can be turned into an error estimate based on the number of points and vice versa.

A first line of research works via the power function of Eq. (2), and (Schaback, 1995, Section 4) provides a good overview of the used techniques: (Madych and Nelson, 1988, 1992) consider kernels \( k(x, y) := \Phi(\| x - y \|_2) \) (radial basis function kernels) and derive error estimates by investigating the polynomial expansion of the radial basis function \( \Phi \) around zero. Those references assume \( \Omega \) to have a “sufficiently smooth boundary”. The article (Madych and Nelson, 1990) (for infinitely smooth kernels) as well as (Wu and Schaback, 1993) (for finitely smooth kernels) consider kernels \( k(x, y) := \Phi(x - y) \) (translational invariant kernels) and derive error estimates with Fourier transform techniques, as this is the standard tool for translational invariant problems. The paper (Madych and Nelson, 1990) assumes \( \Omega \) to satisfy a cone condition, while (Wu and Schaback, 1993) does not state to require a cone condition. They work with a local fill distance, which is defined as

\[
h_{\rho, X_n}(x) := \max_{y \in B_\rho(x)} \min_{x_j \in X_n} \| y - x_j \|_2,
\]

(using \( B_\rho(x) = \{ y \in \mathbb{R}^d \mid \| x - y \|_2 < \rho \} \)) and is assumed to be locally small enough, i.e. “\( h_{\rho, X_n} \leq h_0 \)”.

However, a close inspection of the interplay of \( h_{\rho, X_n}, h_0 \) and \( \rho \) (see (Wu and Schaback, 1993, Eq. (5.5))) reveals that plenty of interpolation points \( X_n \) are required in the neighborhood \( B_\rho(x) \), thus requiring many interpolation points in possible cusps of the domain \( \Omega \). This then breaks the coupling of the local fill distance to the number of interpolation points of Eq. (6). Therefore, if one wants to obtain convergence rates in terms of interpolation points with the results from (Wu and Schaback, 1993), a cone condition is required.

This is one on the contributions of our work over previous works (Madych and Nelson, 1988, 1990, 1992; Wu and Schaback, 1993): We obtain convergence rates in terms of the number of points for domains without such cone conditions and...
also for more general kernels than radial or translational invariant kernels. This is achieved by a considerably new proof technique.

A second line of research deals with sampling inequalities (Narcowich et al., 2005, 2006; Wendland and Rieger, 2005). In contrast to the power function analysis before, this approach also allows for approximation statements on functions outside the RKHS of the used kernel. By applying the sampling inequalities to the residual $f - s_n$ and leveraging norm equivalences, namely between the RKHS and corresponding Sobolev spaces, it is possible to derive error estimates (in terms of the fill distance) for kernel-based interpolation. In all of those references, the underlying assumption is that $\Omega$ is a compact domain with Lipschitz boundary, which is required e.g. to ensure a continuous extension operator to $\mathbb{R}^d$.

Some more details on convergence rates for kernel interpolation using the Gaussian kernel are discussed in Section 4.3.2.

Both the power function analysis as well as the sampling inequality approach are quite constructive. This is in contrast to our approach, which mostly makes use of abstract approximation theory and leverages greedy kernel analysis to apply this to possibly non-greedy kernel-based approximation.

3 Analysis of greedy algorithms in an abstract setting

We start in Section 3.1 by recalling the abstract analysis of greedy algorithms in Hilbert spaces of (Binev et al., 2011), which was later refined in (DeVore et al., 2013). This review will serve as background for the refined analysis in Section 3.2, where we both lift unnecessary restrictive assumptions and extend the results.

3.1 Review of abstract setting

We start by reviewing the framework from (Binev et al., 2011; DeVore et al., 2013): For this let $\mathcal{H}$ be a Hilbert space with norm $\| \cdot \| = \| \cdot \|_\mathcal{H}$. Furthermore let $\mathcal{F} \subset \mathcal{H}$ be a compact (and infinite) subset and assume, for notational convenience only, that it holds $\| f \|_\mathcal{H} \leq 1$ for all $f \in \mathcal{F}$. Now a weak greedy algorithm with constant $\gamma \in (0, 1]$ is considered that selects a sequence $\{f_0, f_1, \ldots \} \subset \mathcal{F}$ of elements. It starts by picking any element $f_0 \in \mathcal{F}$ that satisfies

$$
\| f_0 \| \geq \gamma \cdot \max_{f \in \mathcal{F}} \| f \|,
$$

and proceeds by defining $V_n := \text{span}\{f_0, \ldots, f_{n-1}\}$ and selecting the next elements according to

$$
\text{dist}(f_n, V_n)_\mathcal{H} \geq \gamma \cdot \sup_{f \in \mathcal{F}} \text{dist}(f, V_n)_\mathcal{H}, \quad (7)
$$

The resulting sequence $\{f_0, f_1, \ldots \} \subset \mathcal{F}$ will not be unique, but the subsequent analysis holds for any sequence that is selected according to this weak selection criterion.
We recall that two important quantities $d_n(F)_H$ and $\sigma_n(F)_H$ considered in the analysis, that are defined as
\begin{equation}
\begin{aligned}
d_n(F)_H &:= \inf_{Y_n \subset H} \sup_{f \in F} \text{dist}(f, Y_n)_H, \\
\sigma_n(F)_H &:= \sup_{f \in F} \text{dist}(f, V_n)_H,
\end{aligned}
\end{equation}
where $d_n(F)_H$ is the Kolmogorov $n$-width of $F$ in $H$. For the proof of the main result of (DeVore et al., 2013), a lemma is required which is recalled here:

**Lemma 3.** (DeVore et al., 2013, Lemma 2.1) Let $G = (g_{i,j})_{i,j=1}^K$ be a $K \times K$ lower triangular matrix with rows $g_1, \ldots, g_K$, $W$ be any $m$-dimensional subspace of $\mathbb{R}^K$, and $P$ be the orthogonal projection of $\mathbb{R}^K$ onto $W$. Then
\[ \prod_{i=1}^K g_{i,i}^2 \leq \left\{ \frac{1}{m} \sum_{i=1}^K \|P g_i\|_2^2 \right\}^m \left\{ \frac{1}{K-m} \sum_{i=1}^K \|g_i - P g_i\|_2^2 \right\}^{K-m} \]
where $\| \cdot \|_2$ is the Euclidean norm of a vector in $\mathbb{R}^K$.

Then, the main statement (DeVore et al., 2013, Theorem 3.2) puts the quantities $\sigma_n(F)_H$ and $d_n(F)_H$ from Eq. (8) into relation:

**Theorem 4.** (DeVore et al., 2013, Theorem 3.2) For the weak greedy algorithm with constant $\gamma$ in a Hilbert space $H$ and for any compact set $F$, we have the following inequalities between $\sigma_n := \sigma_n(F)_H$ and $d_n := d_n(F)_H$, for any $N \geq 0, K \geq 1$, and $1 \leq m < K$:
\[ \prod_{i=1}^K \sigma_{N+i}^2 \leq \gamma^{-2K} \left( \frac{K}{m} \right)^m \left( \frac{K}{K-m} \right)^{K-m} \sigma_N^{2m} d_m^{2K-2m}. \]

### 3.2 Generalization of abstract setting

This subsection generalizes and extends the results recalled in Section 3.1, especially Theorem 3.2. In particular:

1. **We lift the unnecessarily restrictive assumption on $F \subset H$ being compact and replace it by $F$ being precompact.**

2. **We extend the result by introducing an additional subset $\tilde{F} \subset F$ and analyzing its effect.**

Both these points are addressed in our main abstract statement in Theorem 5 below.

In order to understand why $F$ being precompact is sufficient, we recall that for the Kolmogorov $n$-widths of a set $F \subset H$ it holds
\[ d_n(F)_H = d_n(F)_H, \]
i.e. the Kolmogorov widths for a set and for its closure coincide (Pinkus, 2012, Theorem 1.1 (page 10)). This can also be seen easily from the definition of the Kolmogorov $n$-width in Eq. (8), as it already incorporates the supremum
Moreover, the weak selection criteria, i.e. $\gamma \in (0,1)$, do not require that the supremum $\sup_{f \in \mathcal{F}} f$ actually needs to be attained within $\mathcal{F}$. Beyond these points, the compactness (in conjunction with its boundedness) of $\mathcal{F}$ is only used to have $d_n(\mathcal{F})_\mathcal{H} \to 0$ (see [Pinkus 2012, Prop. 1.2 (page 10)]), for which, however, the precompactness of $\mathcal{F}$ is sufficient.

Regarding the second point, we additionally consider another set $\tilde{\mathcal{F}} \subset \mathcal{F}$, i.e. $\tilde{\mathcal{F}} \subset \mathcal{F} \subset \mathcal{H}$.

Now we consider running a greedy algorithm that picks subsequently elements $f_0, f_1, \ldots$ from $\tilde{\mathcal{F}}$ according to

\[ f_n \in \tilde{\mathcal{F}} \text{ such that } \text{dist}(f_n, V_n)_\mathcal{H} \geq \gamma \cdot \sup_{f \in \tilde{\mathcal{F}}} \text{dist}(f, V_n)_\mathcal{H}. \tag{9} \]

It is crucial to point out that the elements are selected from $\tilde{\mathcal{F}} \subset \mathcal{F}$ according to the largest error (measured as distance in $\mathcal{H}$) which is attained for the elements in $\tilde{\mathcal{F}}$.

Like this we are ready to formulate our main abstract result in Theorem 5, which is a slight modification of [DeVore et al. 2013, Theorem 3.2]. A full proof is provided.

**Theorem 5.** Consider a precompact set $\mathcal{F}$ in a separable Hilbert space $\mathcal{H}$ and a subset $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ that contains infinitely many elements. Consider a greedy algorithm that selects pairwise different elements $\{f_0, f_1, \ldots\}$ from $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ according to Eq. (9). We have the following inequalities between $\sigma_n(\tilde{\mathcal{F}})$ and $d_n(\mathcal{F})$ for any $N \geq 0, K \geq 1, 1 \leq m < K$:

\[
\prod_{i=1}^{K} \sigma_{N+i}(\tilde{\mathcal{F}})_\mathcal{H}^2 \leq \gamma^{-2K} \left(\frac{K}{m}\right)^m \left(\frac{K}{K-m}\right)^{K-m} \sigma_{N+1}(\tilde{\mathcal{F}})_\mathcal{H}^{2m} d_m(\mathcal{F})_\mathcal{H}^{2K-2m}. \tag{10}
\]

**Proof.** We proceed similarly to the proof of [DeVore et al. 2013, Theorem 3.2]: As $\mathcal{H}$ is separable, we assume without loss of generality that $\mathcal{H}$ is $\ell^2(\mathbb{N} \cup \{0\})$. For the infinite sequence $(f_n)_{n \geq 0} \subseteq \tilde{\mathcal{F}} \subset \mathcal{H}$ consisting of pairwise different elements, we consider the orthonormal system $(f^*_n)_{n \geq 0}$ obtained by Gram-Schmidt orthogonalization. For the orthogonal projector $P_n : \mathcal{H} \to V_n \equiv \text{span}\{f_0, \ldots, f_{n-1}\}$ it then holds

\[ P_n f = \sum_{i=0}^{n-1} (f, f^*_i) f^*_i. \]

Especially $f_n$ can be expressed in this orthogonal basis, and we collect the coefficients in an (infinite dimensional) lower triangular matrix $A$:

\[ A := (a_{i,j})_{i,j=0}^\infty, \quad a_{i,j} := (f_i, f^*_j)_\mathcal{H}. \]

Now we consider the $K \times K$ matrix $G = (g_{i,j})$ which is formed by the rows and columns of $A$ with indices from $\{N+1, \ldots, N+K\}$. Each row $g_i$ is the restriction of $A_{N+i}$ to the coordinates $N+1, \ldots, N+K$. Let $\mathcal{H}_m$ be the $m$-dimensional so-called Kolmogorov subspace of $\mathcal{H}$ for which $\text{dist}(\mathcal{F}, \mathcal{H}_m) = d_m(\mathcal{F})$. Then, $\text{dist}(f_{N+i}, \mathcal{H}_m) \leq d_m(\mathcal{F}), i = 1, \ldots, K$. Let $\tilde{W}$ be the linear space which is the
Corollary 6. This is summarized in the following corollary. Finally we obtain Theorem 3 can be leveraged:

\[ \| P g_i \|_{L^2}^2 \leq \| g_i \|_{L^2}^2 = \| f_{N+i} - \Pi_{V_{N+i}}(f_{N+i}) \|_{H}^2 \]

\[ \leq \sup_{f \in F} \| f - \Pi_{V_{N+i}}(f) \|_{H}^2 = \sigma_{N+i}(\tilde{F})^2_H \quad \forall m \geq n. \]

Furthermore,

\[ \| g_i - P g_i \|_{L^2} \leq \| g_i - \tilde{P} g_i \|_{L^2} = \text{dist}(g_i, \tilde{W}) \]

\[ \leq \text{dist}(f_{N+i}, H_m) \leq d_m(F), \quad i = 1, \ldots, K. \]

Finally we obtain

\[ g_{i,i} \equiv a_{N+i,N+i} = \| f_{N+i} - \Pi_{V_{N+i}}(f_{N+i}) \|_H = \text{dist}(f_{N+i}, V_{N+i})_H \]

\[ \geq \gamma \cdot \sigma_{N+i}(\tilde{F})_H. \]

Now Theorem \[5\] can be leveraged:

\[ \prod_{i=1}^{K} g_{i,i}^2 \leq \left\{ \frac{1}{m} \sum_{i=1}^{K} \| P g_i \|_{L^2}^2 \right\}^m \left\{ \frac{1}{K-m} \sum_{i=1}^{K} \| g_i - P g_i \|_{L^2}^2 \right\}^{K-m} \]

\[ \Rightarrow \prod_{i=1}^{K} \gamma^2 \sigma_{N+i}(\tilde{F})^2 \leq \left\{ \frac{1}{m} \sum_{i=1}^{K} \sigma_{N+i}(\tilde{F})^2_H \right\}^m \left\{ \frac{1}{K-m} \sum_{i=1}^{K} d_m(F)^2 \right\}^{K-m} \]

\[ \Rightarrow \prod_{i=1}^{K} \sigma_{N+i}(\tilde{F})^2 \leq \gamma^{-2K} \left( \frac{K}{m} \right)^m \left( \frac{K}{K-m} \right)^{K-m} \sigma_{N+i}(\tilde{F})^{2m} d_m(F)^{2K-2m}. \]

Similarly to \[\text{ DeVore et al., 2013, Corollary 3.3}\], we can derive decay statements for \( \sigma_n(\tilde{F})_H \) based on decay statements for \( d_n(F)_H \) due to Theorem \[5\].

This is summarized in the following corollary.

Corollary 6. For the greedy algorithm of Eq. \[9\] we have the following.

i) If \( d_n(F)_H \leq C_n n^{-\alpha}, n = 1, 2, \ldots \), then it holds

\[ \sigma_n(\tilde{F})_H \leq C_1 n^{-\alpha}, \quad (11) \]

for \( n = 1, 2, \ldots \) with \( C_1 := 2^{5\alpha+1} C_0 \gamma^{-2} \).

ii) If \( d_n(F)_H \leq C_0 e^{-c_0 n^\alpha}, n = 1, 2, \ldots \), then it holds

\[ \sigma_n(\tilde{F})_H \leq \sqrt{2C_0 \gamma^{-1} e^{-c_1 n^\alpha}} \]

\[ \quad (12) \]

for \( n = 2, 3, \ldots \) with \( \tilde{C}_0 := \max\{1, C_0\} \) and \( c_1 = 2^{-(2+\alpha)}c_0 < c_0 \).
Proof. For the first statement i), we can directly reuse (DeVore et al., 2013, Proof of Corollary 3.3). The only difference is that we have $\sigma_n(\tilde{F})_H$ instead of $\sigma_n(F)_H$, which does not matter since the proof of (DeVore et al., 2013, Corollary 3.3) is purely algebraical.

For the second statement ii), we proceed similarly, however we note that (DeVore et al., 2013, Proof of Corollary 3.3) is slightly inaccurate, because its Eq. (3.9) uses
\[
\sigma_{2n+1} \leq \sqrt{2C_0} \gamma^{-1} e^{-c_0 2^{-1-\alpha} (2n)^\alpha},
\]
however
\[
\sigma_{2n+1} \leq \sqrt{2C_0} \gamma^{-1} e^{-c_0 2^{-1-\alpha} (2n+1)^\alpha},
\]
would be required. Therefore we include the full proof: We leverage Theorem 5 for $N = 0, K = n$, thus Eq. (10) turns into
\[
\prod_{i=1}^n \sigma_i(\tilde{F})_H^2 \leq \gamma^{-2n} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sigma_1(\tilde{F})_H^2 d_m(F)^{2n-2m}.
\]
For $0 < x := m/n < 1$ it holds $x^{-x}(1-x)^{x-1} \leq x$, thus we have
\[
\left( \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \right)^{1/n} = x^{-x}(1-x)^{x-1} \leq 2.
\]
Using $\sigma_1(\tilde{F})_H \leq 1$ due to $\|f\|_H \leq 1$ for all $f \in F$ we then obtain
\[
\left( \prod_{i=1}^n \sigma_i(\tilde{F})_H \right)^{1/n} \leq \sqrt{2} \gamma^{-1} d_m(F)^{(n-m)/n}.
\]
Finally using the monotonicity of $\sigma_i(\tilde{F})_H$ and picking $m = \lfloor n/2 \rfloor$ gives
\[
\sigma_n(\tilde{F})_H \leq \sqrt{2} \gamma^{-1} \cdot d_m(F)^{(n-m)/n} = \sqrt{2} \cdot d_{\lfloor n/2 \rfloor}(F)^{(n-\lfloor n/2 \rfloor)/n} \quad (13)
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha} (n-\lfloor n/2 \rfloor)n}\]
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha} n}\]
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha} \lfloor n/2 \rfloor}\]
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha} n}\]
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha} \lfloor n/2 \rfloor}.
\]

4 Kernel interpolation on arbitrary domains

We start in Section 4.1 by recalling a convenient connection of the abstract analysis of Section 3 to greedy kernel interpolation which was introduced in (Santin and Haasdonk, 2017) and subsequently leveraged as well in (Wenzel et al., 2021, 2022a, b). By doing so, we are able to derive our main result in Theorem 8 in Section 4.2. In Section 4.3 several applications of that main result are given.
4.1 Convenient connection

We make use of the convenient connection between the abstract setting and the kernel setting as introduced in (Santin and Haasdonk, 2017), and extend it to the consideration of two sets $\tilde{\Omega} \subset \Omega \subset \mathbb{R}^d$ and choose 

$$H := H_k(\Omega),$$

$$\tilde{F} := \{ k(\cdot, x), x \in \tilde{\Omega} \} \subset H,$$

$$F := \{ k(\cdot, x), x \in \Omega \} \subset H,$$

(14)

for some continuous kernel $k : \Omega \times \Omega \to \mathbb{R}$ such that $k(x, x) \leq 1$ (in order to satisfy the convenience assumption $\|f\|_H \leq 1 \ \forall f \in F$). As long as $\sup_{x \in \Omega} k(x, x) < \infty$, this can always be enforced by normalizing the kernel. Thus we have

$$V_n = \text{span}\{ f_0, ..., f_{n-1} \} = \text{span}\{ k(\cdot, x_1), ..., k(\cdot, x_n) \} = \text{span}\{ k(\cdot, x_i), x_i \in X_n \} = V(X_n),$$

for $X_n \subset \tilde{\Omega}$. With these choices we can relate the quantities from Eq. (8) from the abstract setting to the kernel setting as done in (Santin and Haasdonk, 2017):

$$\sigma_n(F)_H \equiv \sup_{f \in F} \text{dist}(f, V_n)_H = \sup_{f \in F} \| f - \Pi_{V_n}(f) \|_H$$

$$= \sup_{x \in \tilde{\Omega}} \| k(\cdot, x) - \Pi_{H_k(\tilde{\Omega}), V(X_n)}(k(\cdot, x)) \|_{H_k(\tilde{\Omega})} = \| P_{k, \tilde{\Omega}, X_n} \|_{L^\infty(\tilde{\Omega})},$$

(15)

$$d_n(F)_H \equiv \inf_{Y_n \subset H} \sup_{f \in F} \text{dist}(f, Y_n)_H = \inf_{Y_n \subset H} \sup_{f \in F} \| f - \Pi_{Y_n}(f) \|_H$$

$$\leq \inf_{Y_n \subset H} \sup_{f \in F} \| f - \Pi_{Y_n}(f) \|_{H_k(\tilde{\Omega})} = \inf_{X_n \subset \Omega} \| P_{k, \tilde{\Omega}, X_n} \|_{L^\infty(\tilde{\Omega})}.$$  

(16)

For brevity, in Eq. (16) and in the following we suppress the dimension and size constraints $\dim(Y_n) = n$ and $|X_n| = n$. The precompactness of $F$ is equivalent to $d_n(F)_H \xrightarrow{n \to \infty} 0$ (see e.g. (Pinkus, 2012, Prop. 1.2 (page 10)) together with the comments on (Pinkus, 2012, Th. 1.1 i) (page 10)) and thus, due to Eq. (16), $\inf_{X_n \subset \Omega} \| P_{k, \tilde{\Omega}, X_n} \|_{L^\infty(\tilde{\Omega})} \xrightarrow{n \to \infty} 0$ implies precompactness of $F$.

In order to make use of the abstract result of Theorem 5 and the resulting decay statements of Corollary 6 we need to understand the quantity $\sigma_n(\tilde{F})_H$ which was defined in Eq. (8). We have the following lemma:

**Lemma 7.** It holds

$$\sigma_n(\tilde{F})_H = \| P_{k, \tilde{\Omega}, X_n} \|_{L^\infty(\tilde{\Omega})}.$$

**Proof.** By using the definition of $\sigma_n(\tilde{F})_H$ from Eq. (8), our choice of $\tilde{F}$ and $H$
in Eq. (14), and Lemma 2, we have
\[
\sigma_n(\tilde{F})_H = \sup_{f \in \tilde{F}} \text{dist}(f, V_n)_H
= \sup_{x \in \tilde{F}} \|k(\cdot, x) - \Pi_{\mathcal{H}_k(\tilde{\Omega})}(k(\cdot, x))\|_{\mathcal{H}_k(\tilde{\Omega})}
= \sup_{x \in \tilde{\Omega}} \|P_{k,\tilde{\Omega},X_n}(x)\|_{\mathcal{H}_k(\tilde{\Omega})}
= \sup_{x \in \tilde{\Omega}} \|P_{k,\tilde{\Omega},X_n}(\cdot, x)\|_{\mathcal{H}_k(\tilde{\Omega})}
= \sup_{x \in \tilde{\Omega}} \|P_{k,\tilde{\Omega},X_n}(\cdot, x)\|_{\mathcal{H}_k(\tilde{\Omega})}
\]
which concludes the proof.

Furthermore the weak greedy selection criterion from Eq. (9) results in
\[
x_{n+1} \in \tilde{\Omega} \text{ such that } P_{k,\tilde{\Omega},X_n}(x_{n+1}) \geq \gamma \cdot \|P_{k,\tilde{\Omega},X_n}\|_{\mathcal{H}_k(\tilde{\Omega})},
\]
i.e. a weak P-greedy algorithm on \(\tilde{\Omega}\).

4.2 Main result

Corollary 6 in conjunction with Lemma 7 now immediately yields the following theorem.

**Theorem 8.** Let \(\tilde{\Omega} \subset \mathbb{R}^d\) be an arbitrary infinite set. If there exists a superset \(\Omega \supset \tilde{\Omega}\) and a sequence of (non-necessarily nested) sets of points \((X_n)_{n \in \mathbb{N}} \subset \Omega\) such that for some \(\alpha > 0\) it holds
\[
1. \text{(Algebraic decay case)}
\|f - \Pi_{\mathcal{H}_k(\tilde{\Omega})}(f)\|_{L^\infty(\tilde{\Omega})} \leq C_0 n^{-\alpha} \cdot \|f\|_{\mathcal{H}_k(\tilde{\Omega})} \quad \forall f \in \mathcal{H}_k(\tilde{\Omega}),
\]
then the weak P-greedy algorithm with parameter \(\gamma \in (0, 1]\) using \(k\) applied to \(\tilde{\Omega}\) (see Eq. (17)) gives a nested sequence of sets of points \((\tilde{X}_n)_{n \in \mathbb{N}} \subset \tilde{\Omega}\) such that
\[
\|f - \Pi_{\mathcal{H}_k(\tilde{\Omega})}(f)\|_{L^\infty(\tilde{\Omega})} \leq C_1 n^{-\alpha} \cdot \|f\|_{\mathcal{H}_k(\tilde{\Omega})} \quad \forall f \in \mathcal{H}_k(\tilde{\Omega})
\]
for \(n = 1, 2, \ldots\) with \(C_1 := 2^{5\alpha+1} C_0 \gamma^{-2}\).

2. \text{(Exponential decay case)}
\[
\|f - \Pi_{\mathcal{H}_k(\tilde{\Omega})}(f)\|_{L^\infty(\tilde{\Omega})} \leq C_0 e^{-c_0 n^{\alpha}} \cdot \|f\|_{\mathcal{H}_k(\tilde{\Omega})} \quad \forall f \in \mathcal{H}_k(\tilde{\Omega}),
\]
then the weak P-greedy algorithm with parameter \(\gamma \in (0, 1]\) using \(k\) applied to \(\tilde{\Omega}\) (see Eq. (17)) gives a nested sequence of sets of points \((\tilde{X}_n)_{n \in \mathbb{N}} \subset \tilde{\Omega}\) such that
\[
\|f - \Pi_{\mathcal{H}_k(\tilde{\Omega})}(f)\|_{L^\infty(\tilde{\Omega})} \leq \sqrt{2} C_0 \gamma^{-1} e^{-c_1 n^{\alpha}} \cdot \|f\|_{\mathcal{H}_k(\tilde{\Omega})} \quad \forall f \in \mathcal{H}_k(\tilde{\Omega})
\]
for \(n = 2, 3, \ldots\) with \(C_0 = \max\{1, C_1\}\) and \(c_1 = 2^{-(2+\alpha)} c_0 < c_0\).
Proof. We make use of the choices from Eq. (14) to connect the abstract setting with the kernel setting: The prerequisites from Eq. (18) and (19) are equivalent to bounds on \(\inf_{X_n \subset \Omega} \| P_{k,\Omega,X_n} \|_{L^\infty(\Omega)}\) (see Eq. (2)). Thus, due to Eq. (16) we obtain bounds on \(d_n(\mathcal{F})_H\) as

\[
d_n(\mathcal{F})_H \leq \inf_{X_n \subset \Omega} \| P_{k,\Omega,X_n} \|_{L^\infty(\Omega)} \leq \begin{cases} C_0 n^{-\alpha} & \text{algebraic decay case} \\ C_0 e^{-c_0 n^\alpha} & \text{exponential decay case} \end{cases}
\]

for \(n = 1, 2, \ldots\). This ensures also the precompactness of the set \(\mathcal{F} = \{ k(\cdot, x), x \in \Omega \}\) in \(H = H_k(\Omega)\).

From Lemma 7 we recall \(\sigma_n(\tilde{\mathcal{F}})_H = \| P_{k,\tilde{\Omega},X_n} \|_{L^\infty(\tilde{\Omega})}\), and thus an application of Corollary 6 by using the bounds on \(d_n(\mathcal{F})_H\) gives

\[
\| P_{k,\tilde{\Omega},X_n} \|_{L^\infty(\tilde{\Omega})} = \sigma_n(\tilde{\mathcal{F}})_H \leq \begin{cases} C_1 n^{-\alpha} & \text{algebraic decay case} \\ \sqrt{2} \tilde{C}_0 \gamma^{-1} e^{-c_1 n^\alpha} & \text{exponential decay case} \end{cases}
\]

for \(n = 1, 2, \ldots\) respective \(n = 2, 3, \ldots\) with \(C_1, \tilde{C}_0, c_1\) as specified in the statement of the theorem. Applying the \(\| \cdot \|_{L^\infty(\tilde{\Omega})}\)-norm to Eq. (2) finally gives

\[
\sup_{0 \neq f \in H_k(\tilde{\Omega})} \frac{\| f - \Pi_{H_k(\tilde{\Omega}),\tilde{\mathcal{V}}(X_n)}(f) \|_{L^\infty(\tilde{\Omega})}}{\| f \|_{H_k(\tilde{\Omega})}} = \frac{\| P_{k,\tilde{\Omega},X_n} \|_{L^\infty(\tilde{\Omega})}}{\| f \|_{H_k(\tilde{\Omega})}} \leq \begin{cases} C_1 n^{-\alpha} & \text{algebraic decay case} \\ \sqrt{2} \tilde{C}_0 \gamma^{-1} e^{-c_1 n^\alpha} & \text{exponential decay case} \end{cases}.
\]

Rearranging gives the final result

\[
\| f - \Pi_{H_k(\tilde{\Omega}),\tilde{\mathcal{V}}(X_n)}(f) \|_{L^\infty(\tilde{\Omega})} \leq C_1 n^{-\alpha} \frac{\sqrt{2} \tilde{C}_0 \gamma^{-1} e^{-c_1 n^\alpha}}{\sqrt{2} \tilde{C}_0 \gamma^{-1} e^{-c_1 n^\alpha}}
\]

for all \(f \in H_k(\tilde{\Omega})\).

4.3 Applications of the main result

There are several implications of Theorem 8 and we discuss them separately in the next subsections. Under the notion non-Lipschitz we address in the following quite arbitrary domains \(\tilde{\Omega}\), emphasizing that they can e.g. have irregular boundaries. We only require convergence rates on a larger domain \(\Omega\), which are frequently available under some conditions on \(\Omega\), e.g. having a Lipschitz boundary.

The general overall idea is always to leverage Theorem 8 to carry over convergence rates (in the number of interpolation points) from a larger domain \(\Omega\) to a smaller domain \(\tilde{\Omega} \subset \Omega\).

4.3.1 Error estimates for optimally chosen points

Theorem 8 states \(\| \cdot \|_{L^\infty(\tilde{\Omega})}\) convergence rates for interpolation on \(\tilde{\Omega}\) via \(P\)-greedily chosen interpolation points \(X_n\), as soon as one knows convergence rates
for interpolation on a larger domain $\Omega \supset \tilde{\Omega}$. However, these greedily chosen points do not need to be optimal. For optimal points we have the trivial estimate

$$\inf_{X_n \subset \tilde{\Omega}} \| P_{k,\tilde{\Omega},X_n} \|_{L^\infty(\tilde{\Omega})} \leq \| P_{k,\tilde{\Omega},\hat{X}_n} \|_{L^\infty(\tilde{\Omega})} \quad \forall X_n \subset \tilde{\Omega},$$

where the right hand side can be further upper bounded with help of Theorem 8. This implies that the estimates of Theorem 8 are not limited to greedily selected points: It is also possible to obtain error estimates for optimally chosen interpolation points on $\tilde{\Omega}$.

4.3.2 Stability of the convergence rates with respect to the domain

Theorem 8 implies the stability of the convergence rate with respect to the domain: Given a kernel $k$ and two nested domains $\tilde{\Omega} \subset \Omega$, the convergence rate on $\tilde{\Omega}$ (in terms of polynomial convergence $n^{-\alpha}$ or exponential convergence $\exp(-cn^\alpha)$) is at least as fast as the convergence on the larger domain $\Omega$. The case of an even faster convergence rate is exemplified in a numerical example in Subsection 5.1.

An important application of this fact are kernels of infinite smoothness such as the Gaussian kernel: Sampling inequalities for those kernels (Lee and Yoon, 2014; Rieger and Zwicknagl, 2014, 2010) usually assume quite restrictive assumptions on the domain. Here, $C$ is a generic constant.

1. Theorem 3.5 in (Rieger and Zwicknagl, 2010) gives a rate of decay $\exp(-C \log(n)n^{1/d})$ for bounded Lipschitz domains satisfying an interior cone condition, while (Rieger and Zwicknagl, 2014, Theorem 4.5) gives a decay of $\exp(-C \log(n)n^{1/d})$ for compact cubes $\Omega$.

2. Theorem 3.5 in (Rieger and Zwicknagl, 2014) gives an improved rate of $\exp(-C \log(Cn)n^{1/d})$ for bounded domains that are star-shaped with respect to a ball, if oversampling near the boundary is used.

3. Similar estimates for a broader class of kernels are given in (Lee and Yoon, 2014). However, they are restricted to sets which are given as the union of cubes of the same side length.

Using our approach we can directly conclude the best available convergence rate for any bounded domain $\tilde{\Omega}$. To this end we simply consider a compact cube $\Omega$ such that $\tilde{\Omega} \subset \Omega$. Then the application of Theorem 8 in conjunction with (Rieger and Zwicknagl, 2014, Theorem 3.5) directly gives the same convergence rate for $\tilde{\Omega}$. We summarize this statement in the following corollary.

Corollary 9. Consider the Gaussian kernel $k : \tilde{\Omega} \times \tilde{\Omega} \to \mathbb{R}$ and a bounded domain $\tilde{\Omega} \subset \mathbb{R}^d$. Then there exists a nested sequence of point sets $X_n \subset \tilde{\Omega}$ such that it holds

$$\| f - \Pi_{\mathcal{H}_{\tilde{\Omega}}(\hat{\Omega})}(f) \|_{L^\infty(\tilde{\Omega})} \leq C' \cdot e^{-C \log(n)n^{1/d}} \cdot \| f \|_{\mathcal{H}_{\tilde{\Omega}}(\hat{\Omega})},$$

These sampling inequalities are usually formulated in terms of the fill distance $h_n \equiv h_{X_n,\tilde{\Omega}}$. For asymptotically uniformly distributed points it holds $h_n \asymp n^{-1/d}$. Thus we directly state these results in terms of the number of interpolation points.
for all $f \in \mathcal{H}_k(\Omega)$ and $n \in \mathbb{N}$.

**Proof.** The proof is very similar to the proof of Theorem 8 for the exponential decay case, we only need to address the additional $\log(n)$ term in the exponent: Since $\Omega$ is bounded, we can consider a compact cube $\Omega$ such that $\tilde{\Omega} \subset \Omega$. Due to [Rieger and Zwicknagl, 2010, Theorem 4.5] we have the existence of (non-necessarily nested) set of points $(X_n)_{n \in \mathbb{N}} \subset \Omega$ such that for $n \in \mathbb{N}, f \in \mathcal{H}_k(\Omega)$ it holds

$$\|f - \Pi_{\mathcal{H}_k(\tilde{\Omega}),\tilde{V}(X_n)}(f)\|_{L^\infty(\Omega)} \leq C_0 \exp(-c_0 \log(n)^{\frac{1}{d}}).$$

Therefore it follows that $d_n(f) \leq C_0 \exp(-c_0 \log(n)^{\frac{1}{d}})$. Leveraging Corollary 11 from Appendix A we obtain

$$\sigma_n(f)_{\mathcal{H}} \leq \sqrt{2C_0 e^{-c_1 \log(n)}}$$

for $n = 4, 5, \ldots$ with $c_1 = 2^{-(3+\alpha)}c_0 < C_0$. Now we conclude as in the proof of Theorem 8:

$$\sup_{0 \neq f \in \mathcal{H}_k(\tilde{\Omega})} \frac{\|f - \Pi_{\mathcal{H}_k(\tilde{\Omega}),\tilde{V}(X_n)}(f)\|_{L^\infty(\tilde{\Omega})}}{\|f\|_{\mathcal{H}_k(\tilde{\Omega})}} = \|P_{\tilde{\Omega},X_n} f\|_{L^\infty(\tilde{\Omega})}$$

$$= \sigma_n(f)_{\mathcal{H}} \leq \sqrt{2C_0 e^{-c_1 \log(n)}}$$

Rearranging the last equation (and possibly an adjustment of the constants to include $n \in \{1, 2, 3\}$) gives the final result.

We remark that the use of Theorem 8 may possibly lead to non-optimal convergence rates, see e.g. Subsection 5.1. Another possible application of Theorem 8 is given in the case when $\tilde{\Omega}$ is a manifold embedded in a larger ambient domain $\Omega \subset \mathbb{R}^d$. The same machinery works also in this case, but due to the smaller intrinsic dimension of the manifold it is sometimes possible to obtain a faster convergence by working directly in $\tilde{\Omega}$.

### 4.3.3 Convergence rates for non-Lipschitz domains

Theorem 8 can be used to deduce convergence rates for interpolation on non-Lipschitz domains and thus allows a comparison to Sobolev spaces: We consider a kernel $k$ such that $\mathcal{H}_k(\Omega) \asymp H^\tau(\Omega)$ for $\tau > d/2$ for a well-shaped domain $\Omega$ (e.g. satisfying a Lipschitz boundary), see also Section 4. In this case, decay statements as in Eq. (18) are available, e.g. via sampling inequalities [Narcowich et al, 2003, 2006; Wendland and Rieger, 2005]. Then, Theorem 8 states that the same convergence rates (with adjusted constants) also hold in $\mathcal{H}_k(\tilde{\Omega})$, i.e. in the RKHS over the smaller domains $\tilde{\Omega} \subset \Omega$. On top, there is an algorithm to obtain such interpolation points, namely the (weak) $P$-greedy algorithm applied to $\tilde{\Omega}$. This is exemplified in a numerical example in Subsection 5.2.

It is crucial to point out that the decay property of the power function, which is independent of the shape of the domain $\Omega$, is also a striking difference compared to Sobolev spaces $H^\tau(\Omega)$ on $\Omega$ if defined in their standard way via the...
existence of integrable weak derivates. Using that definition of Sobolev spaces, the RKHS $\mathcal{H}_k(\bar{\Omega})$ of the considered kernel can be smaller than the corresponding Sobolev space. The reason for this is that the RKHS $\mathcal{H}_k(\bar{\Omega})$ always allows for an extension from $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^d$ (see Section 2.2), while for Sobolev spaces specific assumptions on the boundary are required in order to have an extension operator (Adams and Fournier, 2003; Agranovich, 2015). However, if one defines Sobolev spaces via the restriction of the global Sobolev spaces to $\Omega \subset \mathbb{R}^d$, see e.g. (Agranovich, 2015, Theorem 5.1.1) or (Novak and Triebel, 2006, Section 2.4), the spaces will coincide with $\mathcal{H}_k(\Omega)$.

**Corollary 10.** Consider a kernel $k : \Omega \times \Omega \to \mathbb{R}$ and a domain $\Omega \subset \mathbb{R}^d$ such that $\mathcal{H}_k(\Omega) \cong \mathcal{H}_\tau(\Omega)$ for some $\tau > d/2$. Furthermore let $\bar{\Omega} \subset \Omega$ be a measurable, non-Lipschitz domain such that $\mathcal{H}_\tau(\bar{\Omega}) \not\cong \mathcal{H}_\tau(\Omega)$, i.e. there is no continuous embedding of $\mathcal{H}_\tau(\bar{\Omega})$ into $\mathcal{H}_\tau(\Omega)$. Then the following inclusions of the function spaces hold:

$$\mathcal{H}_\tau(\bar{\Omega}) \hookrightarrow \mathcal{H}_\tau(\Omega)$$

$i.e.$ the RKHS $\mathcal{H}_k(\bar{\Omega})$ over $\bar{\Omega}$ is only a subset of the corresponding Sobolev space $\mathcal{H}_\tau(\bar{\Omega})$ over $\bar{\Omega}$.

**Proof.** The equivalence $\mathcal{H}_k(\Omega) \cong \mathcal{H}_\tau(\Omega)$ was stated as an assumption, and holds often in practice, see e.g. (Wendland, 2005, Corollary 10.48). The embedding $\mathcal{H}_k(\bar{\Omega}) \hookrightarrow \mathcal{H}_k(\Omega)$ follows from (Wendland, 2005, Theorem 10.46). For the subset-relation $\mathcal{H}_k(\bar{\Omega}) \subset \mathcal{H}_\tau(\bar{\Omega})$ we have the following reasoning. Let $E$ be the extension operator $E : \mathcal{H}_k(\bar{\Omega}) \to \mathcal{H}_k(\Omega)$, so that

$$f \in \mathcal{H}_k(\bar{\Omega}) \Rightarrow Ef \in \mathcal{H}_k(\Omega)$$

$$\Rightarrow Ef \in \mathcal{H}_\tau(\Omega)$$

$$\Rightarrow f = Ef|_{\bar{\Omega}} \in \mathcal{H}_\tau(\bar{\Omega}).$$

For the last step we simply used that in the computation of the Sobolev norm, we are just integrating over a smaller domain $\bar{\Omega} \subset \Omega$. It holds $\mathcal{H}_\tau(\bar{\Omega}) \neq \mathcal{H}_k(\bar{\Omega})$ because otherwise this would contradict the assumption $\mathcal{H}_\tau(\bar{\Omega}) \not\cong \mathcal{H}_\tau(\Omega)$.

**5 Numerical experiments**

In the following numerical experiments we use the basic, the linear and the quadratic Matérn kernels, which are defined as

$$k_{\text{basic}}(x, y) = e^{-||x-y||^2},$$
$$k_{\text{lin}}(x, y) = e^{-||x-y||^2} \cdot (1 - ||x-y||^2),$$
$$k_{\text{quadr}}(x, y) = e^{-||x-y||^2} \cdot \frac{1}{3} (3 + 3||x-y||^2 + ||x-y||^4).$$

(20)

For Lipschitz domains $\Omega$, the RKHS $\mathcal{H}_k(\Omega)$ of these kernels is norm equivalent to certain Sobolev spaces. Further details are provided in the following.
5.1 Improvement of the convergence rates on subdomains

The first numerical experiment will illustrate that it is also possible to obtain a faster convergence order, i.e. the proven stability of the convergence order also allows to have an improved convergence order. For this we consider the Lipschitz domains

$$\Omega = B_1(0),$$
$$\tilde{\Omega} = \Omega \setminus B_{1/2}(0) \subset \Omega,$$

with $B_r(0) := \{x \in \mathbb{R}^2 \mid \|x\| < r\}$. We use the kernel

$$k(x, y) := k_{\text{quadr}}(x, y) + \chi(x)\chi(y) \cdot k_{\text{lin}}(x, y)$$

where $k_{\text{quadr}}$ is the quadratic Matérn kernel and $k_{\text{lin}}$ is the linear Matérn kernel as defined in Eq. (20) and $\chi$ is the indicator function of $B_{1/2}(0)$. For Lipschitz domains $\Omega'$ such as $\Omega$ or $\tilde{\Omega}$, the quadratic Matérn kernel satisfies $H_{k_{\text{quadr}}}(\Omega') \approx H_{\tau_1}(\Omega')$ with $\tau_1 = (d + 5)/2 \approx 7/2$, while the linear Matérn kernel satisfies $H_{k_{\text{lin}}}(\Omega') \approx H_{\tau_2}(\Omega')$ with $\tau_2 = (d + 3)/2 \approx 5/2$. Therefore, according to [Santin and Haasdonk, 2017] the expected convergence rate of $P$-greedy using $k_{\text{quadr}}$ or $k_{\text{lin}}$ on either $\Omega$ or $\tilde{\Omega}$ is

$$\frac{1}{2} - \frac{\tau}{d} = \begin{cases} -0.75 & \text{linear Matérn kernel } k_{\text{lin}}, \\ -1.25 & \text{quadratic Matérn kernel } k_{\text{quadr}}. \end{cases} \quad (21)$$

The numerically observed convergence using $k$ is visualized in Figure 1. One can observe that the decay of $P$-greedy using $k$ on the smaller domain $\tilde{\Omega} \subset \Omega$ is faster compared to the decay on the larger domain $\Omega$. The decay on $\tilde{\Omega}$ follows an asymptotic of $n^{-1.25}$ because it holds $k_{\tilde{\Omega} \times \tilde{\Omega}} = k_{\text{quadr}}$ in accordance with Eq. (21). The decay on $\Omega$ is slower and seems to follow $n^{-0.75}$ which is motivated by the contribution of the linear Matérn kernel $k_{\text{lin}}$.

We point out that the application of Theorem 8 provides only the same convergence rate for the smaller domain $\tilde{\Omega}$ in this example, while we can observe an even faster convergence. So our notion of “stability” does not exclude such better rates, but rather should express that it prevents “worse” rates.

5.2 Convergence rates for non-Lipschitz domains

In this subsection we consider the application of the $P$-greedy algorithm to the two domains

$$\Omega := [0, 1]^2,$$
$$\tilde{\Omega} := \{x \in \Omega \mid \|x\| > 1\} \subset \Omega,$$

so that the domain $\tilde{\Omega}$ has two cusps at $(0, 1)^T, (1, 0)^T \in \mathbb{R}^2$, i.e. it does not have a Lipschitz boundary. For this example we use in Subsection 5.2.1 the basic Matérn kernel $k_{\text{basic}}$ and the linear Matérn kernel $k_{\text{lin}}$, and in Subsection 5.2.1 the Gaussian kernel.

\footnote{We also considered “monotonicity” as possible notion instead of “stability”, but this would be misleading as monotonicity in convergence behaviour might be misunderstood as the (trivial) decay of convergence rates over increasing dimension.}
Figure 1: Top: Visualization of the selected points for $P$-greedy applied to $\tilde{\Omega}$ (left) and $\Omega$ (right). Bottom: Visualization of the decay $\|P_{k,\tilde{\Omega},X_n}\|_{L^\infty(\tilde{\Omega})}$ and $\|P_{k,\Omega,X_n}\|_{L^\infty(\Omega)}$ (y-axis) in the number $n$ of selected interpolation points (x-axis). The convergence rate on the smaller domain $\tilde{\Omega}$ is faster compared to the larger domain $\Omega$.

5.2.1 Matérn kernels

Although $\tilde{\Omega}$ does not have a Lipschitz boundary, the decay of the power function follows the expected decay for Lipschitz domains like $\Omega$. Especially – in accordance with Theorem 8 – we can observe a convergence rate as for the Matérn kernels

$$\|P_{k,\tilde{\Omega},X_n}\|_{L^\infty(\tilde{\Omega})} \asymp n^{-\frac{1}{2} - \frac{\tau}{d}}$$

\[
= \begin{cases} 
  n^{-1/4} & \text{basic Matérn kernel} \\
  n^{-3/4} & \text{linear Matérn kernel} 
\end{cases}
\]

as visualized in Figure 2. Furthermore, the points chosen by $P$-greedy seem to be asymptotically uniformly distributed in $\tilde{\Omega}$ and in particular no clustering next to the cusps is observed. Moreover it can be observed that, despite having the same
rate, the blue $\|P_{\tilde{k},\tilde{\Omega},X_n}\|_{L^\infty(\tilde{\Omega})}$ curve is lower than the magenta $\|P_{k,\Omega,X_n}\|_{L^\infty(\Omega)}$ curve, i.e., there is a smaller multiplicative constant in front of the rates. This makes actually sense, since 500 points were selected in $\tilde{\Omega}$, which results in a higher density compared to 500 points within $\Omega$.

Figure 2: Visualization of the $P$-greedy algorithm applied to $\Omega$ and $\tilde{\Omega} \subset \Omega$ as defined in Eq. (22): Left: Basic Matérn kernel, right: Linear Matérn kernel. On top the decay of $\|P_{k,\Omega,X_n}\|_{L^\infty(\Omega)}$ respectively $\|P_{\tilde{k},\tilde{\Omega},X_n}\|_{L^\infty(\tilde{\Omega})}$ is displayed. The $P$-greedy selected points are visualized in the middle row (for $\tilde{\Omega}$) and bottom row (for $\Omega$).

5.2.2 Gaussian kernel

Also for the Gaussian kernel the decay of the power function for $P$-greedy on the non-Lipschitz domain $\tilde{\Omega}$ follows at least the same decay as for Lipschitz domains like $\Omega$ as visualized in Figure 3.
This especially illustrates the results of Subsection 4.3.2 about the Gaussian kernel, where it was elaborated that there are no convergence rates known so far for interpolation with the Gaussian kernel on non-Lipschitz domains. The visualization of the chosen interpolation points in Figure 3 shows that the points cluster next to the boundary, which can be expected for analytic kernels in accordance with results in (Rieger and Zwicknagl, 2014). This clustering is especially visible in the sharp corners of the domain $\tilde{\Omega}$.

![Figure 3: Top: Visualization of the selected points for $P$-greedy applied to $\tilde{\Omega}$ (left) and $\Omega$ (right). Bottom: Visualization of the decay $\|P_{k,\tilde{\Omega},X_n}\|_{L^\infty(\tilde{\Omega})}$ and $\|P_{k,\Omega,X_n}\|_{L^\infty(\Omega)}$ (y-axis) in the number $n$ of selected interpolation points (x-axis) for the Gaussian kernel. The convergence rate on the smaller domain $\tilde{\Omega}$ is clearly at least as fast as on the larger domain.](image)

### 6 Conclusion

We extended an abstract analysis of greedy algorithms in Hilbert spaces and proved that we obtain at least the same convergence rates, if we approximate
only a smaller set of elements. By using a convenient link between this abstract setting, greedy kernel approximation and restriction and extension properties of Reproducing Kernel Hilbert Spaces, we transferred these results to potentially non-greedy kernel interpolation: In this case, the results imply a stability of the convergence rates with respect to the size of the domain. Especially we were able to obtain approximation rates for kernel interpolation on general even non-Lipschitz domains, including cases with cusps and irregular boundaries or even manifolds. A comparison with Sobolev spaces was drawn. Comments on the implications of these results were provided.

Future work will address the distribution of the $P$-greedy selected points on non-Lipschitz domains. For Lipschitz domains the point distribution for kernels of finite smoothness was already analyzed in [Wenzel et al., 2021], however its analysis made use of Sobolev space arguments, which are not necessarily available for non-Lipschitz domains.

Acknowledgements: The authors acknowledge the funding of the project by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy - EXC 2075 - 390740016 and funding by the BMBF under contract 05M20VSA.

References

Robert A Adams and John JF Fournier. Sobolev spaces. Elsevier, 2003.

Mikhail S Agranovich. Sobolev spaces, their generalizations and elliptic problems in smooth and Lipschitz domains. Springer, 2015.

Peter Binev, Albert Cohen, Wolfgang Dahmen, Ronald DeVore, Guergana Petrova, and Przemyslaw Wojtaszczyk. Convergence rates for greedy algorithms in reduced basis methods. SIAM J. Math. Anal., 43(3):1457-1472, 2011. ISSN 0036-1410. doi: 10.1137/100795772. URL http://dx.doi.org/10.1137/100795772

Stefano De Marchi, Robert Schaback, and Holger Wendland. Near-optimal data-independent point locations for radial basis function interpolation. Advances in Computational Mathematics, 23(3):317–330, 2005. ISSN 1019-7168. doi: 10.1007/s10444-004-1829-1. URL http://dx.doi.org/10.1007/s10444-004-1829-1

Ronald DeVore, Guergana Petrova, and Przemyslaw Wojtaszczyk. Greedy algorithms for reduced bases in Banach spaces. Constructive Approximation, 37(3):455–466, 2013. ISSN 0176-4276. doi: 10.1007/s00365-013-9186-2. URL http://dx.doi.org/10.1007/s00365-013-9186-2

Mun Bae Lee and Jungho Yoon. Sampling inequalities for infinitely smooth radial basis functions and its application to error estimates. Applied Mathematics Letters, 36:40–45, 2014. ISSN 0893-9659. doi: https://doi.org/10.1016/j.aml.2014.05.001. URL https://www.sciencedirect.com/science/article/pii/S0893965914001268
WR Madych and SA Nelson. Multivariate interpolation and conditionally positive definite functions. *Approx Theory and its Applications*, 4:77–89, 1988.

WR Madych and SA Nelson. Multivariate interpolation and conditionally positive definite functions. II. *Mathematics of Computation*, 54(189):211–230, 1990.

WR Madych and SA Nelson. Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation. *Journal of Approximation Theory*, 70(1):94–114, 1992.

Stefan Müller. *Komplexität und Stabilität von kernbasierten Rekonstruktionsmethoden* (Complexity and Stability of Kernel-based Reconstructions). PhD thesis, Fakultät für Mathematik und Informatik, Georg-August-Universität Göttingen, 2009. URL [https://ediss.uni-goettingen.de/handle/11858/00-1735-0000-0006-B3BA-E](https://ediss.uni-goettingen.de/handle/11858/00-1735-0000-0006-B3BA-E).

Francis Narcowich, Joseph Ward, and Holger Wendland. Sobolev bounds on functions with scattered zeros, with applications to radial basis function surface fitting. *Mathematics of Computation*, 74(250):743–763, 2005.

Francis J. Narcowich, Joseph D. Ward, and Holger Wendland. Sobolev Error Estimates and a Bernstein Inequality for Scattered Data Interpolation via Radial Basis Functions. *Constructive Approximation*, 24(2):175–186, 2006. ISSN 1432-0940. doi: 10.1007/s00365-005-0624-7. URL [https://doi.org/10.1007/s00365-005-0624-7](https://doi.org/10.1007/s00365-005-0624-7).

Erich Novak and Hans Triebel. Function Spaces in Lipschitz Domains and Optimal Rates of Convergence for Sampling. *Constructive approximation*, 23 (3), 2006.

A. Pinkus. *n*-Widths in Approximation Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2012. ISBN 9783642698941. URL [https://books.google.de/books?id=a6boCAAAQBAJ](https://books.google.de/books?id=a6boCAAAQBAJ).

C. Rieger and B. Zwicknagl. Improved exponential convergence rates by oversampling near the boundary. *Constructive Approximation*, 39(2):323–341, 2014. doi: 10.1007/s00365-013-9211-5.

Christian Rieger and Barbara Zwicknagl. Sampling inequalities for infinitely smooth functions, with applications to interpolation and machine learning. *Advances in Computational Mathematics*, 32(1):103, 2010.

G. Santin and B. Haasdonk. Convergence rate of the data-independent P-greedy algorithm in kernel-based approximation. *Dolomites Research Notes on Approximation*, 10:68–78, 2017. URL [www.emis.de/journals/DRNA/9-2.html](www.emis.de/journals/DRNA/9-2.html).

Robert Schaback. Multivariate interpolation and approximation by translates of a basis function. *Series In Approximations and Decompositions*, 6:491–514, 1995.
In the following we have an extension of Theorem 6 that additionally includes a logarithmic term.

Corollary 11. For the greedy algorithm of Eq. \( \text{(9)} \) we have the following

\[ \text{i)} \text{ If } d_n(F)_H \leq C_0 e^{-c_0 \log(n)n^\alpha}, n = 1, 2, \ldots, \text{ then it holds} \]

\[ \sigma_n(\tilde{F})_H \leq \sqrt{2\tilde{C}_0 \gamma^{-1} e^{-\tilde{c}_1 \log(n)n^\alpha}} \]

for \( n = 4, 5, \ldots \) with \( \tilde{C}_0 := \max\{1, C_0\} \) and \( \tilde{c}_1 = 2^{-(3+\alpha)}C_0 < c_0 \).

Proof. The proof is a modification of the proof of Theorem 6. We continue at...
Eq. (13):

\[
\sigma_n(\tilde{F})_H \leq \sqrt{2} \gamma^{-1} \cdot d_m(\mathcal{F})^{(n-m)/n} = \sqrt{2} \cdot d_{\lfloor n/2 \rfloor}(\mathcal{F})^{(n-\lfloor n/2 \rfloor)/n} \\
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 \lfloor n/2 \rfloor^\alpha \log(\lfloor n/2 \rfloor) \log(n-\lfloor n/2 \rfloor)/n} \\
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha \lfloor n/2 \rfloor} \log(n/2) \log(n-\lfloor n/2 \rfloor)/n} \\
\leq \sqrt{2} \gamma^{-1} \cdot \tilde{C}_0^{1/2} e^{-c_0 2^{-\alpha \lfloor n/2 \rfloor} \log(n/2)}. \\
\]

Now using \log(n/2) = \log(n) - \log(2) \geq \frac{1}{2} \log(n) for \ n \geq 4 we obtain

\[
\sigma_n(\tilde{F})_H \overset{n \geq 4}{\leq} \sqrt{2} \tilde{C}_0 \gamma^{-1} e^{-c_0 2^{-\alpha \lfloor n/2 \rfloor} \log(n) n^\alpha}.
\]

\[\square\]