CONVERGENCE OF YAMABE FLOW ON SOME COMPLETE MANIFOLDS WITH INFINITE VOLUME

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Abstract. The goal of this paper is to study a curvature normalized Yamabe flow on a non-compact complete Riemannian manifold with a fibered boundary metric $g_{\Phi}$. Examples of such spaces include special cases of non-abelian magnetic monopoles, gravitational instantons and products of asymptotically conical with closed manifolds. Assuming negative scalar curvature $\text{scal}(g_{\Phi})$, we prove long-time existence and convergence of the flow. Our work extends the results of Suárez-Serrato and Tapie to a non compact setting.

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1. Introduction and main results

The Yamabe flow equation is an evolution equation introduced by Hamilton [Ham89] as an alternative ansatz to solving the Yamabe conjecture. The conjecture, posed by Yamabe [Yam60] and proved by Trudinger [Tru68], Aubin [Aub76] and Schoen [Sch84], states that for any compact, smooth Riemannian manifold $(M, g)$ without boundary there exists a constant
scalar curvature metric, conformal to $g$. Their proofs are based on the calculus of variations and elliptic theory.

Hamilton proposed a new approach using parabolic methods. More precisely, his Yamabe flow is a parabolic evolution equation with solution given by a family of Riemannian metrics \( \{ g(t) \}_{t \in [0,T)} \) on $M$ such that

$$
\partial_t g(t) = -\text{scal}(g(t))g(t); \quad g(0) = g.
$$

On compact Riemannian manifolds such a flow shrinks the metric on regions with $\text{scal}(g(t)) > 0$. In particular a sphere collapses along the flow to a point in finite time. For this reason one introduces a volume normalized Yamabe flow, using the average scalar curvature

$$
\rho(t) = \frac{1}{\text{vol}_{g(t)}(M)} \int_M \text{scal}(g(t)) \, d\text{vol}(g(t)).
$$

The volume normalized Yamabe flow is then defined by

$$
\partial_t g(t) = (\rho(t) - \text{scal}(g(t)))g(t), \quad g(0) = g.
$$

The flow is by now well understood in the setting of compact manifolds. Hamilton [Ham89] himself proved long time existence of the volume normalized flow for any choice of initial metric. Later, Ye [Ye94] proved convergence of the flow for scalar negative, scalar flat and locally conformally flat scalar positive metrics. The case of metrics that are not conformally flat has been studied in a series of papers by Schwetlick and Struwe [ScSt03] and later by Brendle [Bre05, Bre07].

In this paper we are concerned with the Yamabe conjecture and more specifically the Yamabe flow on non-compact complete manifolds. Note that in case of infinite volume, the average scalar curvature (1.2) is ill-defined and only the unnormalized Yamabe flow has been considered in the literature references below.

The Yamabe problem in the non-compact setting has been attacked by elliptic methods by several authors. Aviles and McOwen [AvMc88] have shown that, under decay assumptions on the scalar curvature and lower bounds for the Ricci curvature, there is metric in the conformal class with constant positive scalar curvature. Grosse [Gro13] has proven that there is a metric with positive constant scalar curvature within the conformal classes of metrics with bounded geometry. Wei [Guo19] studied positive solutions of the Yamabe equations under conditions on the Yamabe invariant and volume growth on geodesic balls.

The Yamabe flow in the non-compact setting has been studied on asymptotically conical surfaces by Isenberg, Mazzeo and Sesum [IMS13], who
proved locally uniform convergence of a time-rescaled metric to a complete hyperbolic metric with finite area. The flow has also been utilized by Bing-Long Chen and Xi-Ping Zhu [BLXP02] to establish a Gap theorem for non-compact manifolds with nonnegative Ricci curvature under certain decay conditions at infinity. Ma, Cheng and Zhu [LLA12] have studied long-time existence of the Yamabe flow, under some $L^p$ conditions on the scalar curvature. Schulz [Sch19] proved global existence of the Yamabe flow on non-compact manifolds with unbounded initial curvature, provided the metric is conformally equivalent to a complete metric with bounded, non-positive scalar curvature and positive Yamabe invariant. A recent work Ma [Li21] establishes global existence of the Yamabe flow on noncompact manifolds that are asymptotically flat near infinity.

In all of these works, convergence of the flow is out of reach, since (1.2) is not defined and thus only the unnormalized Yamabe flow has been considered. In this paper we study a different type of normalization for the Yamabe flow, that allows to study convergence in the non-compact setting as well. We use the concepts of decreasing and increasing curvature-normalized flows, denoted by CYF$^{-}$ and CYF$^{+}$ respectively, as introduced by Suárez-Serrato and Tapie [SSTa11] for compact manifolds

$$
\partial_t g(t) = (\sup_M \text{scal}(g(t)) - \text{scal}(g(t)))g(t), \quad g(0) = g, \quad (\text{CYF}^{+}),
$$

$$
\partial_t g(t) = (\inf_M \text{scal}(g(t)) - \text{scal}(g(t)))g(t), \quad g(0) = g, \quad (\text{CYF}^{-}).
$$

We study such curvature normalized flows in the setting of fibered boundary manifolds, that generalize the asymptotically flat manifolds considered recently in Ma [Li21].

1.1. Yamabe flow for the conformal factor. The flow preserves the conformal class of the metric and can be written as a scalar evolution equation for the conformal factor. More precisely, assume $m := \dim M \geq 3$ and set $\eta := (m - 2)/4$. Writing $g(t) = u(t)^{1/\eta}$, the scalar curvature of $g(t)$ can be computed by ($\Delta$ is the negative Laplace Beltrami operator of $(M, g)$)

$$
\text{scal}(g(t)) = -u(t)^{-1/(1+\eta)} \left[ \frac{m-1}{\eta} \Delta u(t) - \text{scal}(g) u(t) \right].
$$

In view of this relation, the Yamabe flow (1.1) turns into

$$
\partial_t u(t)^{(m+2)/(m-2)} = \frac{m+2}{m-2} \left( (m-1) \Delta u(t) - \eta \text{scal}(g) u(t) \right)
$$

$$
\Leftrightarrow \quad \partial_t u(t) = (m-1)u(t)^{-1/\eta} \Delta u(t) - \eta \text{scal}(g) u(t)^{1-1/\eta},
$$

with the initial condition $u(t = 0) = 1$. 

1.2. Normalized Yamabe flows for the conformal factor. Similar computations as those leading to (1.6) yield the following scalar evolution equation for the conformal factor under the volume normalized Yamabe flow
\[
\partial_t u(t) - (m - 1)u(t)^{-1/\eta}\Delta u(t) = \eta \left( \rho(t) u(t) - \text{scal}(g) u(t)^{1-1/\eta} \right). \tag{1.7}
\]
The curvature normalized flows in (1.4) are similarly given by
\[
\partial_t u(t) - (m - 1)u(t)^{-1/\eta}\Delta u(t) = \eta \left( \sup_M \text{scal}(g(t)) \cdot u(t) - \text{scal}(g) u(t)^{1-1/\eta} \right), \tag{CYF⁺}
\]
\[
\partial_t u(t) - (m - 1)u(t)^{-1/\eta}\Delta u(t) = \eta \left( \inf_M \text{scal}(g(t)) \cdot u(t) - \text{scal}(g) u(t)^{1-1/\eta} \right), \tag{CYF⁻}
\]

1.3. Outline of the paper and main results. In §2, we review the geometry of fibered boundary manifolds equipped with \( \Phi \)-metrics in their open interior. We also introduce H"older spaces \( C^{k,\alpha}_\Phi(M) \), adapted to this geometry. In §3 we employ the Omori-Yau maximum principle to establish uniqueness of solutions to the Yamabe flow within \( C^{2,\alpha}_\Phi(M) \) and derive differential inequalities for the solution, that will later be used for the a priori estimates.

In §4 we study mapping properties of the heat operator with respect to these spaces. Based on that, in §5 we establish short time existence of the (unnormalized) Yamabe flow (1.1) within the class of such \( \Phi \)-manifolds, see Theorem 5.3 for the precise statement.

**Theorem 1.1.** Let \( (M, g_\Phi) \) be a \( \Phi \)-manifold of dimension \( m \geq 3 \) such that \( \text{scal}(g_\Phi) \in C^{k+1,\alpha}_\Phi(M) \), for some \( \alpha \in (0, 1) \) and any \( k \in \mathbb{N}_0 \). Then the Yamabe flow \( g(t) = u(t)^{1/\eta}g_\Phi \) with conformal factor \( u \in C^{k+2,\alpha}_\Phi(M \times [0, T]) \) solving (1.6), exists for \( T > 0 \) sufficiently small.

In §6, we turn to the increasing curvature normalized Yamabe flow (CYF⁺), introduced in (1.8), whose short-time existence follows from Theorem 1.1 by some time rescaling. The same holds also for the decreasing curvature normalized Yamabe flow CYF⁻ by a verbatim repetition of the arguments and hence we only write the proofs for CYF⁺.

In §7, we study the evolution of \( \text{scal}(g) \) along the CYF⁺. In §8 we derive a priori estimates for solutions of the increasing curvature normalized Yamabe flow. These a priori estimates allow us to apply the machinery of standard estimates of solutions to parabolic equations, which we review in §9. Subsequently, we conclude in §10 with a global existence result for CYF⁺ on \( \Phi \)-manifolds.
Theorem 1.2. Let \((M, g_\Phi)\) be a \(\Phi\)-manifold of dimension \(m \geq 3\) with \(\text{scal}(g_\Phi) \in C^{k,\alpha}_\Phi(M)\) negative, uniformly bounded away from zero and \(k \geq 4\). Then the increasing curvature normalized Yamabe flow \(CYF^+\) (see Eq. (1.8)) admits a global solution \(g = u^{4/(m-2)}g_\Phi\) for some \(u \in C^{k,\alpha}_\Phi(M \times \mathbb{R}_+)\).

Finally, in §11 we establish convergence for the \(CYF^+\) and thus settle the Yamabe conjecture on negatively curved \(\Phi\)-manifolds. Our result, see Theorem 11.3 for the precise statement, reads as follows.

Theorem 1.3. Let \((M, g_\Phi)\) be a \(\Phi\)-manifold of dimension \(m \geq 3\) such that \(\text{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M)\) is negative and uniformly bounded away from zero. Then the increasing curvature normalized Yamabe flow \(CYF^+\) converges to a Riemannian metric \(g^*\) conformal to \(g_\Phi\) with constant negative scalar curvature.

In fact, the same arguments apply in the general case of manifolds with bounded geometry, provided the flow exists at least for short time within the corresponding Hölder space. The \(\Phi\)-geometry is not essential in our arguments. One can view our contribution as an extension of Suárez-Serrato and Tapie [SST11] to a non-compact setting.

Let us conclude with a remark, that even though we only write out the proofs for \(CYF^+\), same statements hold for the decreasing curvature normalized Yamabe flow \(CYF^-\) as well.

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2. Fibered boundary manifolds and Hölder spaces

In this section we recall basic concepts of fibered boundary manifolds, insofar used in our paper. For more details see Mazzeo and Melrose [MAM98]. Furthermore, we introduce geometry-adapted Hölder spaces due to the first author’s jointly work with Gentile [CAG21]. We conclude the section with noting some properties of conformal transformations of the \(\Phi\)-metric with conformal factors in the previously defined Hölder spaces and noting a maximum principle due to Omori and Yau.

2.1. Fibered boundary manifolds and \(\Phi\)-metrics. Let \(\overline{M} = M \cup \partial M\) be a compact \(m\)-dimensional smooth manifold with boundary \(\partial M\) being the total space of a fibration over a closed manifold \(Y\) with typical fiber given...
by a closed manifold $Z$. We write $b := \dim Y$ and $f := \dim Z$ for the dimensions of the respective manifolds. We denote the fibration by

$$\phi : \partial M \to Y,$$  \hfil (2.1)

which is a smooth surjective map such that $\phi^{-1}(\{y\}) = Z_y \simeq Z$ for all $y \in Y$. Consider a collar neighborhood $U \simeq [0, 1) \times \partial M$ of the boundary $\partial M$ with a smooth boundary defining function $x : U \to [0, 1)$, i.e. $x^{-1}(\{0\}) = \partial M$ and $dx \neq 0$ at $\partial M$. We extend $x$ to a smooth nowhere vanishing function on $M$. We can now introduce a $\Phi$-metric in the open interior $M$.

**Definition 2.1.** A Riemannian metric $g_{\Phi}$ in the interior $M \subset \overline{M}$ of a fibered boundary manifold is said to be a $\Phi$-metric if in the collar $U$ it can be written as

$$g_{\Phi} = \frac{d x^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z + h := g_{\Phi,0} + h,$$ \hfil (2.2)

where $g_Y$ is a Riemannian metric on the base $Y$, $g_Z$ is a symmetric bilinear form on $\partial M$ which restricts to a Riemannian metric at each fiber $Z_y$, and the (higher order) term $h$ satisfies $|h|_{g_{\Phi,0}} = O(x)$ as $x \to 0$. We assume that $\phi : (\partial M, g_Z + \phi^* g_Y) \to (Z, g_Z)$ is a Riemannian submersion and call the Riemannian manifold $(M, g_{\Phi})$ a $\Phi$-manifold.

We define the space $\mathcal{V}_{\Phi}$ of $\Phi$-vector fields as the space of smooth vector fields, that are bounded under $g_{\Phi}$. Choose local coordinates $(x, y, z)$ on $U$, where $(y)$ restrict to local coordinates on $\partial M$, lifted from $Y$; and $(z)$ restrict to local coordinates on each fibre $Z$. Then, locally, $\mathcal{V}_{\Phi}$ can be written as

$$\mathcal{V}_{\Phi} = C^\infty(\overline{M}) - \text{span}\{x^2 \partial_x, x \partial_{y_1}, ..., x \partial_{y_b}, \partial_{z_1}, ..., \partial_{z_f}\}.$$ \hfil (2.3)

The universal enveloping algebra of $\mathcal{V}_{\Phi}$ is the ring $\text{Diff}_{\Phi}^l(M)$ of differential $\Phi$-operators. We denote by $\mathcal{V}_{\Phi}^l$ a set of generators for $\text{Diff}_{\Phi}^l(M)$. By doing so, we are able to define the class of $k$-continuously $\mathbb{R}$-valued $\Phi$-differentiable functions

$$C^k_{\Phi}(M \times [0, T]) = \left\{ u \in C^0(M \times [0, T]) \left| \begin{array}{l} (V \circ \partial_t^{l_2})u \in C^0_{\Phi}(M \times [0, T]), \\
\text{for } V \in \text{Diff}_{\Phi}^{l_1}(M), \quad l_1 + l_2 \leq k \end{array} \right. \right\}.$$ \hfil (2.4)

Naturally, one can define $C^k_{\Phi}(M)$ similarly simply by taking spacial derivatives only.

Note that in contrast to the previous work [CaGe21], in this paper we do not work in function spaces that require continuity on $\overline{M}$.

### 2.2. Classical Hölder spaces.
Definition 2.2. The Hölder space $C^\alpha_0(M \times [0, T])$, for $\alpha \in (0, 1)$, is defined as the space of continuous functions $u \in C^0(M \times [0, T])$ which satisfy

$$[u]_\alpha := \sup_{M_T^\delta} \left\{ \frac{|u(p, t) - u(p', t')|}{d(p, p')^\alpha + |t - t'|^{\alpha/2}} \right\} < \infty,$$  \hspace{1cm} (2.5)

where the supremum is taken over $M_T^\delta$ with $M_T := M \times [0, T]$; the distance function $d$ is induced by the metric $g_\phi$ and is equivalently given in local coordinates $(x, y, z)$ in $U$ by the following local expression

$$d((x, y, z), (x', y', z')) = \sqrt{\left(\frac{x - x'}{(x + x')^2}\right)^2 + \left(\frac{y - y'}{(x + x')^2}\right)^2 + |z - z'|^2}.$$

The Hölder norm of any $u \in C^\alpha_0(M \times [0, T])$ is defined by

$$\|u\|_\alpha := \|u\|_{\infty} + [u]_\alpha.$$ \hspace{1cm} (2.6)

The resulting normed vector space $C^\alpha_0(M \times [0, T])$ is a Banach space. It will be useful below to note that there is an equivalent Hölder norm with spacial and time differences taken only within small local regions.

Lemma 2.3. The following defines an equivalent norm on $C^\alpha_0(M \times [0, T])$

$$\|u\|'_\alpha := \|u\|_{\infty} + [u]'_\alpha, \quad [u]'_\alpha := \sup_{M_{T, \delta}} \left\{ \frac{|u(p, t) - u(p', t')|}{d(p, p')^\alpha + |t - t'|^{\alpha/2}} \right\},$$ \hspace{1cm} (2.7)

where the supremum is taken over

$$M_{T, \delta} := \{(p, t), (p', t') \in M_T | d(p, p')^\alpha + |t - t'|^{\alpha/2} \leq \delta\}.$$

More precisely, we have the following relation between the two norms

$$\|u\|_\alpha' \leq \|u\|_\alpha \leq (1 + 2\delta^{-1})\|u\|_\alpha'.$$

Proof. It is clear that $\|u\|_\alpha' \leq \|u\|_\alpha$. To prove the second estimate, simply note for any $u \in C^\alpha_0(M \times [0, T])$ and any $(p, t), (p', t') \in M_T$ with

$$d(p, p')^\alpha + |t - t'|^{\alpha/2} \geq \delta,$$

that we can estimate the Hölder differences as follows

$$\frac{|u(p, t) - u(p', t')|}{d(p, p')^\alpha + |t - t'|^{\alpha/2}} \leq \frac{|u(p, t) - u(p', t')|}{\delta} \leq 2\delta^{-1}\|u\|_{\infty}. \hspace{1cm} \square$$

From now on we only use the Hölder norm $\|u\|'_\alpha$ defined in (2.7), which we denote from now on without the apostrophe. We also define the higher order Hölder spaces for any given $k \in \mathbb{N}$ by

$$C^k_\phi(M \times [0, T]) = \left\{ u \in C^k_\phi(M \times [0, T]) \mid (V \circ \delta^l_\phi)u \in C^\alpha_\phi(M \times [0, T]), \text{ for } V \in \text{Diff}_\phi^l(M), \ l_1 + 2l_2 \leq k \right\}.$$
which is a Banach space, cf. [BaVe14, Proposition 3.1], with the norm
\[ \|u\|_{k,\alpha} := \|u\|_{\alpha} + \sum_{1+2l\leq k} \sum_{V \in \mathcal{V}_{\Phi}^l} \|(V \circ \partial^l_t)u\|_{\alpha}. \] (2.8)

**Remark 2.4.** Sometimes, we will also use Hölder spaces for functions depending either only on spatial variables or on time variables, denoted as \( C^{k,\alpha}_{\Phi}(M) \) and \( C^{k,\alpha}_{\Phi}([\emptyset, T]) \), with Hölder brackets (for \( k = 0 \))
\[ [u]_{\alpha} = \sup \frac{|u(p) - u(p')|}{d(p, p')^{\alpha}} \] and \[ [u]_{\alpha} = \sup \frac{|u(t) - u(t')|}{|t - t'|^{\alpha/2}}, \]
respectively.

### 2.3. Conformal transformation by Hölder functions.
Since the Yamabe flow preserves the conformal class of the metric, we need to look into the effect of conformal transformation by Hölder functions. We first define the conformal class of a \( \Phi \)-metric \( g_{\Phi} \) (we tacitly assume \( m \geq 3 \))
\[ [g_{\Phi}] = \{ u^{4/(m-2)} \cdot g_{\Phi} \mid u \in C^2_{\Phi}(M), u > 0 \}. \] (2.9)
First, observe that a generic element of the conformal class \( [g_{\Phi}] \) is not a \( \Phi \)-metric in the sense of Definition 2.1, since the conformal factor \( u^{4/(m-2)} \) cannot in general be expected to admit a partial asymptotic expansion as \( x \to 0 \). However, it still has \( \mathcal{V}_{\Phi} \) as the space of bounded vector fields and thus the distance functions defined with respect to any \( g \in [g_{\Phi}] \) are equivalent. In that sense \( g \) still has the same \( \Phi \)-geometry as \( g_{\Phi} \) and we conclude

**Proposition 2.5.** The Hölder spaces defined in §2.2 do not depend on the choice of a metric \( g \in [g_{\Phi}] \).

### 2.4. Scalar curvature of \((M, g_{\Phi})\).
In this work we assume that the scalar curvature \( \text{scal}(g_{\Phi}) \) of \((M, g_{\Phi})\) is negative, bounded uniformly away from zero. In order to understand the geometric restrictions it entails, we should consider the asymptotic expansion of \( \text{scal}(g_{\Phi}) \) near the boundary \( \partial M \). Employing [O’N83, Chapter 7, Corollary 43] and the tensor properties of the scalar curvature, we obtain for the trivial fibration and vanishing higher order term \( h \)
\[ \text{scal}(g_{\Phi}) = x^2 (\text{scal}(g_Y) + b(b - 1)) + \text{scal}(g_Z), \] (2.10)
near \( x = 0 \). In the general case, additional \( O(x) \) terms (as \( x \to 0 \)) appear. Thus, negativity of \( \text{scal}(g_{\Phi}) \) (uniformly bounded away from zero) leads to a geometric restriction on the scalar curvature of the fibres \( Z \) in our work
\[ \text{scal}(g_Z) < 0. \]
3. Omori-Yau maximum principle on \((M, g_\Phi)\)

As established in the previous work [CaGe21, §2.3] by checking a volume growth condition as in [AMR16, Theorem 2.11], \(\Phi\)-manifolds are stochastically complete: i.e. the heat kernel \(H\) of the (negative) Laplace Beltrami operator \(\Delta_\Phi\), associated to \(g_\Phi\), satisfies the identity

\[
\int_M H(t, p, \tilde{p}) \, d\text{vol}_\Phi(\tilde{p}) = 1. \tag{3.1}
\]

In particular we have for any \(V \in \text{Diff}_\Phi^\ast(M)\) applied to the \(p\)-variable

\[
\int_M V H(t, p, \tilde{p}) \, d\text{vol}_\Phi(\tilde{p}) = 0. \tag{3.2}
\]

Hence, according to [AMR16, Theorem 2.8 (i) and (iii)], the Omori-Yau maximum principle does hold for \(\Phi\)-manifolds. This means that for any function \(u \in C^2_\Phi(M)\) there is a sequence \(\{p_k\}_k \subset M\) satisfying

\[
u(p_k) > \sup_M u - \frac{1}{k} \text{ and } \Delta_\Phi u(p_k) < \frac{1}{k} \tag{3.3}\]

Similarly, there exists a sequence \(\{p'_k\}_k \subset M\) such that

\[
u(p'_k) < \inf_M u + \frac{1}{k} \text{ and } \Delta_\Phi u(p'_k) > \frac{1}{k}. \tag{3.4}\]

3.1. Some enveloping theorem.

**Proposition 3.1.** Consider any \(u \in C^{2,\alpha}_\Phi(M \times [0, T]).\) Then

\[
u_{\sup}(t) := \sup_M u(\cdot, t), \quad u_{\inf}(t) := \inf_M u(\cdot, t)
\]

are differentiable almost everywhere in \((0, T)\) and at those \(t \in (0, T)\) we find in the notation of (3.3) and (3.4)

\[
\frac{\partial}{\partial t} \nu_{\sup}(t) \leq \lim_{\epsilon \to 0} \left( \limsup_{k \to \infty} \frac{\partial u}{\partial t}(p_k(t + \epsilon), t + \epsilon) \right),
\]

\[
\frac{\partial}{\partial t} \nu_{\inf}(t) \geq \lim_{\epsilon \to 0} \left( \liminf_{k \to \infty} \frac{\partial u}{\partial t}(p'_k(t + \epsilon), t + \epsilon) \right). \tag{3.5}
\]

**Proof.** Apply (3.3) to \(u(t + \epsilon)\) and find by the mean value theorem

\[
u_{\sup}(t + \epsilon) \leq \nu(p_k(t + \epsilon), t + \epsilon) + \frac{1}{k}
\]

\[
= \nu(p_k(t + \epsilon), t) + \epsilon \cdot \frac{\partial u}{\partial t}(p_k(t + \epsilon), \xi) + \frac{1}{k},
\]

for some \(\xi \in (t, t + \epsilon).\) On the other hand we can write

\[
u_{\sup}(t + \epsilon) = \nu_{\sup}(t) + \epsilon \cdot \frac{\nu_{\sup}(t + \epsilon) - \nu_{\sup}(t)}{\epsilon}
\]
≥ u(p_k(t + ε), t) + ε \cdot \frac{u_{\text{sup}}(t + ε) - u_{\text{sup}}(t)}{ε}.

Combining these two estimates leads after cancelling u(p_k(t + ε), t) to
\[
e \cdot \frac{u_{\text{sup}}(t + ε) - u_{\text{sup}}(t)}{ε} \leq e \cdot \frac{\partial u}{\partial t}(p_k(t + ε), \xi) + 1.
\]

Taking limes superior as \( k \to \infty \) on the right hand side, we obtain
\[
e \cdot \frac{u_{\text{sup}}(t + ε) - u_{\text{sup}}(t)}{ε} \leq e \cdot \limsup_{k \to \infty} \frac{\partial u}{\partial t}(p_k(t + ε), \xi).
\]

Cancelling \( ε \) on both sides, we find
\[
u_{\text{sup}}(t + ε) - u_{\text{sup}}(t) \leq \limsup_{k \to \infty} \frac{\partial u}{\partial t}(p_k(t + ε), t + ε).
\]

For any \( u \in C^2_0(M \times [0, T]) \) we can estimate
\[
\begin{align*}
\bullet \quad & \limsup_{k \to \infty} \left| \frac{\partial u}{\partial t}(p_k(t + ε), \xi) - \frac{\partial u}{\partial t}(p_k(t + ε), t + ε) \right| \leq ||u||_{2,α} ε^{α/2}, \\
\bullet \quad & \limsup_{k \to \infty} \left| \frac{\partial u}{\partial t}(p_k(t + ε), t + ε) \right| \leq ||u||_{2,α}.
\end{align*}
\]

Thus the last two summands in (3.6) are bounded uniformly in \( ε \). Repeating same arguments with the roles of \( u(t) \) replaced by \( u(t + ε) \) interchanged, we conclude that \( u_{\text{sup}} \) is locally Lipschitz and thus by theorem of Rademacher, differentiable almost everywhere. This proves the first statement.

At those \( t \in (0, T) \), where \( u_{\text{sup}} \) is differentiable, we conclude from (3.6) and the first line in (3.7), taking \( ε \to 0 \)
\[
\frac{∂}{∂t} u_{\text{sup}}(t) \leq \lim_{ε \to 0} \left( \limsup_{k \to \infty} \frac{∂ u}{∂ t}(p_k(t + ε), t + ε) \right).
\]

This proves the first inequality in (3.5). The second inequality follows from the first, using (3.4), with \( u \) replaced by \( (-u) \). \( \Box \)

3.2. **Uniqueness of solutions.** We can now turn to uniqueness of solutions to the Yamabe flow. In view of Proposition 3.1, the arguments of
Proposition 3.2. Let \( a \) and \( b \) be any bounded positive functions on \( M \times [0, T] \). Let \( u \in C^2_{\alpha}(M \times [0, T]) \) be a solution to
\[
\partial_t u = a \Delta \phi u - bu,
\]
with initial value \( 0 \) at \( t = 0 \). Then \( u \equiv 0 \).

Proof. Consider first the case where \( b = 0 \). Note first by (3.3) and (3.4)
\[
\frac{\partial}{\partial t} u(p_k(t), t) \leq \frac{a(p_k(t), t)}{k}, \quad \frac{\partial}{\partial t} u(p_{k}^{\prime}(t), t) \geq -\frac{a(p_{k}^{\prime}(t), t)}{k}.
\]
Then in view of Proposition 3.1 we find almost everywhere
\[
\frac{\partial}{\partial t} u_{\sup}(t) \leq 0, \quad \frac{\partial}{\partial t} u_{\inf}(t) \geq 0.
\]
Then in view of \( u(t = 0) = 0 \), we conclude \( u \equiv 0 \). Now the general statement follows as in [CAGe21, Corollary 9.2].

Corollary 3.3. Consider the Yamabe flow equation as in Eq. (1.6)
\[
\partial_t u = (m - 1)u^{1/\eta} \Delta \phi u - \eta \text{scal}(g_\phi) u^{1-1/\eta}, \quad u_{|t=0} = u_0,
\]
for some positive initial data \( u_0 \in C^2_{\alpha}(M) \). For such a Cauchy problem, a positive solution in \( C^2_{\alpha}(M \times [0, T]) \) is unique for any given \( 0 < T < \infty \).

Proof. Suppose \( u \) and \( v \) are two positive solutions in \( C^2_{\alpha}(M \times [0, T]) \) for (3.9). Consider \( \omega = u - v \in C^2_{\alpha}(M \times [0, T]) \). Since \( u(t = 0) = v(t = 0) = u_0 \), we find \( \omega(t = 0) = 0 \). Moreover, we infer from (3.9)
\[
u^{1/\eta} \partial_t u - v^{1/\eta} \partial_t v = (m - 1) \Delta \phi \omega - \eta \text{scal}(g_\phi) \omega.
\]
From the definition of \( \omega \), we have
\[
\partial_t \omega = u^{1/\eta} \partial_t u - v^{1/\eta} \partial_t v + (v^{1/\eta} - u^{1/\eta}) \partial_t v
\]
\[
= u^{1/\eta} ((m - 1) \Delta \phi \omega - \eta \text{scal}(g_\phi) \omega + (v^{1/\eta} - u^{1/\eta}) \partial_t v)
\]
\[= -\left( \eta \text{scal}(g_\phi) u^{1-1/\eta} + \frac{\partial_t v}{\eta} \int_0^1 (sv + (1-s)u)^{1/\eta-1} dv \right) \omega
\]
\[+ (m - 1)u^{1-1/\eta} \Delta \phi \omega,
\]
where the last equality follows from Taylor’s theorem applied for the function \( f(s) := (sv + (1-s)u)^{1/\eta} \). This means that \( \omega \) is a solution of the equation
\[
\partial_t \omega = a \Delta \phi \omega + b \omega,
\]
with \( a \in C^2_{\alpha}(M \times [0, T]) \) positive and \( b \in C^0_{\alpha}(M \times [0, T]) \). Since nothing can be said about the sign of the \( b \)-term above, we consider any negative constant \( c < -\|b\|_{\infty} \) and apply an integration factor trick by writing \( \omega' = e^{ct} \omega \). We obtain an equation for \( z \)
\[
\partial_t \omega' = a \Delta \phi \omega' + (b + c) \omega',
\]
with \( \omega'|_{t=0} = \omega|_{t=0} = 0 \). Now, since \( c < -\|b\|_\infty \), we have \( (b + c) < 0 \). From Proposition 3.2, it follows that \( \omega' \equiv 0 \) and, consequently, \( \omega \equiv 0 \). \( \square \)

### 3.3. Some differential inequalities for solutions to CYF\(^+\).

As a direct consequence of Proposition 3.1 we also obtain differential inequalities for solutions to the increasing curvature normalized Yamabe flow CYF\(^+\). These will be central later in the derivation of a priori estimates.

**Corollary 3.4.** Let \( u \in C^2_0(M \times [0, T]) \) be a positive (uniformly bounded away from zero) solution to the increasing curvature normalized Yamabe flow CYF\(^+\) in (1.8). Then almost everywhere in \((0, T)\)

\[
\frac{\partial}{\partial t} u^{\sup} \leq \eta \sup_M \text{scal}(g(t)) \cdot u^{\sup} + \eta \sup_M \text{scal}(g_\Phi) \cdot u^{1-1/\eta},
\]

\[
\frac{\partial}{\partial t} u^{\inf} \geq \eta \sup_M \text{scal}(g(t)) \cdot u^{\inf} + \eta \inf_M \text{scal}(g_\Phi) \cdot u^{1-1/\eta}.
\]

**Proof.** Note first by (1.8) and (3.3)

\[
\frac{\partial}{\partial t} \left( p_k(t), t \right) \leq \frac{(m - 1)}{k} \cdot u^{-1/\eta}(p_k(t), t) + \eta \sup_M \text{scal}(g(t)) \cdot u(p_k(t), t)
\]

\[
- \eta \left[ \text{scal}(g_\Phi)(p_k(t), t) \cdot u(p_k(t), t) \right]^{1-1/\eta}.
\]

Since \( u \) is positive and uniformly bounded away from zero, we conclude

\[
\limsup_{k \to \infty} \frac{\partial u}{\partial t}(p_k(t), t) \leq \eta \sup_M \text{scal}(g(t)) \cdot u^{\sup}(t)
\]

\[
+ \eta \sup_M \text{scal}(g_\Phi) \cdot u^{\sup}(t)^{1-1/\eta}.
\]

Now the first statement follows from Proposition 3.1. The second statement follows by (3.4) along the same lines. \( \square \)

### 4. Mapping properties of the heat operator

Denote by \( \Delta_\Phi \) the (negative) Laplace Beltrami operator on a \( \Phi \)-manifold \((M, g_\Phi)\). Consider the homogeneous and inhomogeneous heat equations for some compactly supported smooth \( v \in C^\infty(M \times [0, T]) \) and \( u_0 \in C^\infty_0(M) \)

\[
(\partial_t - \Delta_\Phi)u^{\text{hom}} = 0, \quad u^{\text{hom}}(0) = u_0,
\]

\[
(\partial_t - \Delta_\Phi)u^{\text{inhom}} = v, \quad u^{\text{inhom}}(0) = 0.
\]

We denote the heat operator, corresponding to the unique self-adjoint extension of \( \Delta_\Phi \) in \( L^2 \) as well as its Schwartz kernel by the letter \( H \). Then the
solutions $u_{\text{hom}}$ and $u_{\text{inhom}}$ can be given by

$$u_{\text{hom}}(p, t) = (Hu_0)(p, t) = \int_M H(t, p, \tilde{p}) u_0(\tilde{p}) d\text{vol}_\phi(\tilde{p}),$$

$$u_{\text{inhom}}(p, t) = ( Hv)(p, t) = \int_0^t \int_M H(t - \tilde{t}, p, \tilde{p}) v(\tilde{p}, \tilde{t}) d\text{vol}_\phi(\tilde{p}) d\tilde{t},$$

where $d\text{vol}_\phi$ denotes the volume form of $g_\phi$. We do not distinguish notationally between the action of $H$ including or without convolution in time, unless necessary. However, in this section we are concerned with $H$ acting by convolution in time.

4.1. Asymptotics of the heat kernel. In order to derive mapping properties of $H$, we need to recall briefly the asymptotic structure of the heat kernel. We refer the reader to [Tave21] and [CaGe21] for further details. Specifically, the heat kernel $H$ is a smooth function in the open interior of $M^2 \times [0, \infty)$, with singular behaviour at

$$FF := \partial M \times \partial M \times [0, \infty),$$
$$FD := \{y = y'\} \times [0, \infty) \subset FF,$$
$$TD := \text{diag}(\partial M \times \partial M) \times \{t = 0\}.$$

This singular behaviour is resolved by blowing up these singular submanifolds, i.e. replacing the submanifolds by their inward pointing normal bundles, glued into $M^2 \times (0, \infty)$ in a well-defined geometric way. The inward pointing normal bundles of (the lifts of) $FF, FD, TD$ are then new boundary faces in the blowup space $M^2_b$, referred to as $ff, fd$ and $td$, respectively. The blowup space $M^2_b$ is illustrated in Figure 1.

![Figure 1. The parabolic blowup space $M^2_b$.](image)

Local coordinates on the blowup space are best understood in terms of projective coordinates, written in terms of local coordinates $(x, y, z)$ and $(\tilde{x}, \tilde{y}, \tilde{z})$ on the two copies of $M$. In the regime 1 (coordinates in the regime
2 are obtained by interchanging the roles of $x$ and $\tilde{x}$) we have the projective coordinates
\[
(x, y, z, \tilde{s}, \tilde{y}, \tilde{z}, \tau) := \left( x, y, z, \frac{\tilde{x}}{x}, \frac{\tilde{y}}{x}, \frac{\tilde{z}}{x}, \sqrt{t} \right).
\]
In these coordinates, the defining functions $\rho_{lf}, \rho_{ff}, \rho_{tb}$ of the boundary faces $lf, ff$ and $tb$, are given by $\tilde{s}, x$ and $\tau$, respectively.

In the regime 3 (coordinates in the regime 4 are obtained by interchanging the roles of $x$ and $\tilde{x}$) we have the projective coordinates
\[
(x, y, z, S', U', Z', \tau) := \left( x, y, z, \frac{S' - 1}{x}, \frac{U' - y}{x}, \frac{Z' - z}{x}, \sqrt{t} \right).
\]
In these coordinates, the defining functions $\rho_{ff}, \rho_{fd}, \rho_{tb}$ of the boundary faces $ff, fd$ and $tb$, are given by $|S'|, x$ and $\tau$, respectively.

The projective coordinates in the regime 5 are given by
\[
(x, y, z, S, U, Z, \tau) := \left( x, y, z, \frac{S'}{\tau}, \frac{U'}{\tau}, \frac{Z'}{\tau}, \sqrt{t} \right).
\]
In these coordinates, the defining functions $\rho_{fd}, \rho_{td}$ of the boundary faces $fd$ and $td$, are given by $x$ and $\tau$, respectively. And $(|S|, \|U\|, \|Z\|) \to \infty$ corresponds to $tb$. Lifting $H$ to $M^2_h$ corresponds in local coordinates simply to a change to projective coordinates (4.3), (4.4) or (4.5). Now we can state the asymptotics of the heat kernel $H$.

**Theorem 4.1.** [TaVe21, Theorem 6.1] Let $(M, g_\Phi)$ be an $m$-dimensional $\Phi$-manifold. Then the heat kernel $H$ lifts to a polyhomogeneous function $\beta^*H$ on the blowup space $M^2_h$ with the following asymptotic behavior
\[
\beta^*H \sim \rho_{lf}^\infty \rho_{ff}^\infty \rho_{tb}^\infty \rho_{fd}^0 \rho_{td}^{-m} G_0
\]
with $G_0$ being a bounded function.

**4.2. Mapping properties of the heat operator.** The mapping properties for the heat kernel $H$ proved in [Cage21, Theorem 1.1], are stronger than those presented here, since there the authors actually study Hölder spaces with respect to the distance of the incomplete metric $x^4 g_\Phi$. The mapping properties here are still sufficient for our purposes, and the proof follows along the lines of [Cage21].

**Proposition 4.2.** The heat operator $H$ acting by convolution in time, defines for any $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$ bounded linear mappings
\[
H : C^k_\Phi(M \times [0, T]) \to C^{k+2, \alpha}_\Phi(M \times [0, T]),
\]
\[
H : C^{k+1, \alpha}_\Phi(M \times [0, T]) \to \sqrt{t} C^{k+2, \alpha}_\Phi(M \times [0, T]).
\]
Proof. We first discuss the proof for $k = 0$. The statement is then equivalent to boundedness of

$$ G = V \circ H : \mathcal{C}^\infty_\Phi(M \times [0, T]) \to \mathcal{C}^\infty_\Phi(M \times [0, T]), $$

(4.8)

where $V$ is either the identity, first or second order differential $\Phi$-operator. The key is to use the (local) Hölder norm in (2.7), so that the Hölder differenced have to be estimated only for

$$ d(p, p')^\alpha + |t - t'|^{\alpha/2} \leq \delta. $$

Boundedness of $G$ is then established in the three following steps.

(i) Estimates for spatial difference: if $d(p, p')^\alpha \leq \delta$, then

$$ | Gu(p, t) - Gu(p', t) | \leq C \| u \|_\alpha d(p, p')^\alpha, $$

(ii) Estimates for time difference:

$$ | Gu(p, t) - Gu(p, t') | \leq C \| u \|_\alpha |t - t'|^{\alpha/2}, $$

(iii) Estimates for supremum norm:

$$ | Gu(p, t)| \leq C \| u \|_\alpha, $$

for some uniform constants $C > 0$ independent of $u$ and $(p, p', t, t')$. In fact, we will denote uniform positive constants always by $C$ and $c$, despite the constants possibly being different from estimate to estimate.

Proof of (i): Consider $p, p' \in M$ and write

$$ M^+ = \{ \tilde{p} \in M; d(p, \tilde{p}) \leq 3d(p, p') \}, $$

$$ M^- = \{ \tilde{p} \in M d(p, \tilde{p}) \geq 3d(p, p') \}. $$

We shall assume that $p = (x, y, z)$ and $p' = (x', y, z)$, with $x' > x$ without loss of generality. The cases where $p$ and $p'$ differ in the $(y, z)$ components, are discussed similarly. We write out the estimate in the regime 5 in Figure 1, where $fd$ meets $td$. The other regimes are simpler.

Below we use the mean value theorem with $p_L = (\xi, y, z)$ for some intermediate $\xi \in (x, x')$ and stochastic completeness (3.2) in order to replace $u(\tilde{p}, \tilde{t})$ by $(u(\tilde{p}, \tilde{t}) - u(p, \tilde{t}))$ (writing $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{z}))$

$$ Gu(p, t) - Gu(p', t) = $$

$$ (x - x') \int_0^t \int_{M^-} \partial_\xi G(t - \tilde{t}, p_L, \tilde{p}) (u(\tilde{p}, \tilde{t}) - u(p, \tilde{t})) \, d\text{vol}_\Phi(\tilde{p}) \, d\tilde{t} $$

$$ + \int_0^t \int_{M^+} G(t - \tilde{t}, p, \tilde{p}) (u(\tilde{p}, \tilde{t}) - u(p, \tilde{t})) \, d\text{vol}_\Phi(\tilde{p}) \, d\tilde{t} $$

$$ - \int_0^t \int_{M^+} G(t - \tilde{t}, p', \tilde{p}) (u(\tilde{p}, \tilde{t}) - u(p', \tilde{t})) \, d\text{vol}_\Phi(\tilde{p}) \, d\tilde{t} $$
and \( d \frac{d}{d \tau} \) variables, followed by another change of coordinates \( \tau \). Then we compute from Theorem 4.3 Estimates for \( I_1 \). First we use Hölder continuity of \( u \)

\[
|I_1| \leq |x - x'| \cdot \|u\|_{\alpha} \int_0^1 \int_{M^{-}} \partial_{\xi} G(t - \tilde{t}, p_{\xi}, \tilde{p}) \cdot d(\tilde{p}, p)^{\alpha} \, d\text{vol}_{\phi}(\tilde{p}) \, d \tilde{t}
\]

\[
\leq |x - x'| \cdot \|u\|_{\alpha} \int_0^1 \int_{M^{-}} \partial_{\xi} G(t - \tilde{t}, p_{\xi}, \tilde{p}) \cdot d(\tilde{p}, p)^{\alpha} \, d\text{vol}_{\phi}(\tilde{p}) \, d \tilde{t},
\]

where in the second estimate we used \( d(\tilde{p}, p) \leq 3d(\tilde{p}, p_{\xi}) \), obtained by exactly the same arguments as in [CAGe21, (6.1)]. Since we estimate in the regime 5 in Figure 1, where \( fd \) meets \( td \), we use the local projective coordinates \( (\tau, \xi, y, z, S', U', \tilde{Z}') \), introduced in (4.4), where

\[
S' = \frac{\tilde{x} - \xi}{\xi^2}, \quad U' = \frac{\tilde{y} - y}{\xi}, \quad \tilde{Z}' = \tilde{z} - z \quad \text{and} \quad \tau = \sqrt{t - t}. \]

Then we compute from Theorem 4.1 and \( d\text{vol}_{\phi}(\tilde{x}, \tilde{y}, \tilde{z}) \sim \tilde{x}^{-2-b} d \tilde{x} \tilde{y} d \tilde{z} \)

\[
|I_1| \leq c \cdot \frac{|x - x'|}{\xi^2} \cdot \|u\|_{\alpha} \int_0^1 \int_{M^{-}} \tau^{-m - 2} G_0 \sqrt{|S'|^2 + |U'|^2 + |Z'|^2} \times
\]

\[
d S' \, d U' \, d \tilde{Z}' \, d \tau,
\]

with \( G_0 \) being bounded and vanishing to infinite order as \( |(S, U, Z)| \to \infty \), where \( (S, U, Z) = (S'/\tau, U'/\tau, Z'/\tau) \). Let us define

\[
r(S', U', \tilde{Z}') := \sqrt{|S'|^2 + |U'|^2 + |Z'|^2}.
\]

Such a function \( r \) describes the radial distance in polar coordinates around the origin. Performing a change of coordinates, we obtain

\[
|I_1| \leq c \cdot \frac{|x - x'|}{\xi^2} \cdot \|u\|_{\alpha} \int_0^1 \int_{M^{-}} \tau^{-m - 2} r^{m-1+\alpha} G_0 \, d \tau \, d (\text{angle}) \, d \tau.
\]

Now, setting \( \sigma = r^{-1} \tau = \sqrt{|S'|^2 + |U'|^2 + |Z'|^2}^{-1} \), it follows that \( G_0 \) against any negative power of \( \sigma \) is bounded. Hence, integrating out the angular variables, followed by another change of coordinates \( \tau \mapsto \sigma \) gives

\[
|I_1| \leq c \cdot \frac{|x - x'|}{\xi^2} \cdot \|u\|_{\alpha} \int_0^1 \int_{M^{-}} \tau^{-2+\alpha} d \tau.
\]
Now, exactly as in [CAGE21, (6.3)] we find $M^- \subset \{d(p, p') \leq cr\}$ for some constant $c > 0$. Thus we can estimate even further

$$|I_1| \leq c \cdot \frac{|x - x'|}{\xi^2} \cdot \|u\|_\alpha \int_0^\infty r^{-2+\alpha} \, dr$$

$$= c \cdot \frac{|x - x'|}{\xi^2} \cdot d(p, p')^{-1+\alpha} \|u\|_\alpha. \tag{4.9}$$

In order to conclude the desired estimate of $I_1$, recall from Lemma 2.7, that we may consider only $d(p, p') \leq \delta^{1/\alpha} =: \rho$, with any positive $\rho < 1/4$. Then

$$1 - \frac{x}{x'} \leq 2\rho(x + x') \leq 4\rho.$$

Thus $x > (1-4\delta)x'$. Hence we may estimate

$$\frac{|x - x'|}{\xi^2} \leq \frac{|x - x'|}{x^2} \leq (1 - 4\rho)^{-2} \frac{|x - x'|}{x^2}$$

$$\leq 4(1 - 4\rho)^{-2} \frac{|x - x'|}{(x + x')^2} \leq 4(1 - 4\rho)^{-2}d(p, p').$$

Thus for $\delta > 0$ sufficiently small, we conclude from (4.9) and the last estimate above

$$|I_1| \leq c \cdot d(p, p')^\alpha \|u\|_\alpha.$$

4.4. Estimates for $I_2, I_3$. Similar estimates as above lead to

$$|I_2|, |I_3| \leq c \cdot \|u\|_\alpha \int_0^{cd(p, p')} r^{-1+\alpha} \, dr \leq c \|u\|_\alpha d(p, p')^\alpha,$$

from where it follows both estimates

4.5. Estimates for $I_4$. For the estimate of $I_4$, we assume again as before that the heat kernel is supported near $t^d$ meeting $t_0$, and thus work with local projective coordinates $(\tau, x', y', z', S, U, Z)$ given in (4.5), that is,

$$S = \frac{\bar{x} - x'}{x'^2}, \quad U = \frac{\bar{y} - y'}{x'^2}, \quad Z = \frac{\bar{z} - z'}{\tau} \quad \text{and} \quad \tau = \sqrt{t - t}.$$

We will obtain the estimates using integration by parts. To do so, note that one has (as the “worst case scenario” with $V \in \text{Diff}_0^2(M)$) $G = \tau^{-m-2}(X_1 X_2 H)$ with both $X_1, X_2 \in \{\partial_S, \partial_U, \partial_Z\}$. For the sake of simplicity, we shall assume $X_1 = \partial_S$. On the other hand, one has by triangle inequality

$$\partial M^+ = \left\{ d((x, y, z), (\bar{x}, \bar{x}, \bar{x})) = 3d((x, y, z), (x', y', z')) \right\}$$

$$\subseteq \left\{ 2d((x, y, z), (x', y', z')) \leq d((x', y', z'), (\bar{x}, \bar{x}, \bar{x})) \right\}. \tag{4.10}$$

Moreover we can also write for some smooth function $\ell$

$$\beta^*(d\text{vol}_\phi(P) \, dt) = \ell(x' + \tau x'^2 S, y' + \tau x' U, z' + \tau Z) \, dS \, dU \, dZ \, d\tau,$$
Since \( u(p, \tilde{t}) - u(p', \tilde{t}) =: \delta u \) is independent of \( \tilde{p} \), we can integrate by parts

\[
I_4 = \int_0^{\tau} \delta u \int_{M^+} \tau^{-1}(\partial_\zeta X_2 H) \ell \, dS \, d\mathcal{U} \, d\mathcal{Z} \, d\tau
= \int_0^{\tau} \delta u \int_{\partial M^+} \tau^{-1}(X_2 H) \ell \, d\mathcal{U} \, d\mathcal{Z} \, d\tau
- \int_0^{\tau} \delta u \int_{M^+} \tau^{-1}(X_2 H) \partial_\zeta \ell \, dS \, d\mathcal{U} \, d\mathcal{Z} \, d\tau =: I_1^1 - I_1^2.
\]

For the \( I_1^2 \)-term, note that \( \ell \) is a smooth function and therefore \( \partial_\zeta \ell = \tau x^2 \partial_x \ell \). This cancels the \( \tau^{-1} \) in the integrand and thus \( I_1^2 \) can be estimated against \( \|u\|_\alpha d(p, p')^\alpha \). For the \( I_1^1 \)-term, note by (4.10) that we can estimate

\[
|I_1^1| \leq \|u\|_\alpha d(p, p')^\alpha \int_{\partial M^+} \tau^{-1}(X_2 H) \leq \frac{1}{2} \|u\|_\alpha \int_{\partial M^+} \tau^{-1}(X_2 H) d(p', \tilde{p})^\alpha.
\]

This can now be estimated exactly as \( I_2, I_3 \), completing the proof for (i).

**Proof of (ii):** For time difference, first assume \( t' < t \) (without loss of generality) and suppose first \( t < 2t' \). Let us consider the case where \( V \) in (4.8) is a first or second order \( \Phi \)-derivative, so that we can apply stochastic completeness (3.2). Then we find by the mean value theorem for some intermediate \( \theta \in (t', t) \)

\[
G_u(p, t) - G_u(p, t') =
|t - t'| \int_{T_-} \int_M \partial_\theta G(\theta - \tilde{t}, p, \tilde{p}) (u(\tilde{p}, \tilde{t}) - u(p, \tilde{t})) \, d\mathcal{V}_p(\tilde{p}) \, d\tilde{t}
+ \int_{T_+} \int_M G(t - \tilde{t}, p, \tilde{p}) (u(\tilde{p}, \tilde{t}) - u(p, \tilde{t})) \, d\mathcal{V}_p(\tilde{p}) \, d\tilde{t}
- \int_{T'_+} \int_M G(t' - \tilde{t}, p, \tilde{p}) (u(\tilde{p}, \tilde{t}) - u(p, \tilde{t})) \, d\mathcal{V}_p(\tilde{p}) \, d\tilde{t}
=: L_1 + L_2 - L_3,
\]

with the subsets \( T_- \), \( T_+ \) and \( T'_+ \) defined as follows:

\[
T_- = [0, 2t' - t], \quad T_+ = [2t' - t, t] \quad \text{and} \quad T'_+ = [2t' - t, t'].
\]

If \( V \) is identity, the estimates follow similar to those of \( L_1 \) with \( T_- \) replaced by \([0, T]\). Using Hölder continuity of \( u \) we obtain by Theorem 4.1 in projective coordinates (4.5)

\[
|L_1| \leq C|t - t'||u|_\alpha \int_{T_-} \tau^{-3+\alpha}, \quad |L_2|, |L_3| \leq C\|u\|_\alpha \int_{T_+} \tau^{-1+\alpha},
\]
where \( \tau = \sqrt{\theta - t} \) in the first integral, and \( \tau = \sqrt{t - \tilde{t}} \) in the second.
Note that for \( \tilde{t} \in T \), we have \((0 - \tilde{t}) \geq (t - \tilde{t})\). From there we conclude immediately the statement (ii).

**Proof of (iii):** Let us consider the case where \( V \) in (4.8) is a first or second order \( \Phi \)-derivative, so that we can apply stochastic completeness (3.2). Then

\[
|G_u(p, t)| \leq \int_0^T \int_M G(t - \tilde{t}, p, \tilde{p})(u(\tilde{p}, \tilde{t}) - u(p, \tilde{t})) \, d\text{vol}_\Phi(\tilde{p}) \, d\tilde{t}
\]

Using Hölder continuity of \( u \) we obtain by Theorem 4.1 in projective coordinates (4.5)

\[
|G| \leq C\|u\|_\alpha \int \tau^{-1+\alpha} \leq C\|u\|_\alpha.
\]

If \( V \) is identity, the estimate follows along the same lines without the stochastic completeness trick. This completes the proof of the statement for \( k = 0 \). For general \( k \), in all of the above integrals we can first pass \( k \Phi \)-derivatives to the function \( u \) using integration by parts in \((S, U, Z)\) and then continue as before in case \( k = 0 \). \( \square \)

### 5. Short-time existence of the Yamabe flow

Consider a \( \Phi \)-manifold \((M, g_\Phi)\) of dimension \( m \geq 3 \) and set \( \eta := (m - 2)/4 \). We write \( \Delta_\Phi \) for the negative Laplace Beltrami operator of \((M, g_\Phi)\). In this section we construct a short-time solution to the Yamabe flow equation (1.6) of the conformal factor

\[
\partial_t u = (m - 1)u^{-1/\eta}\Delta_\Phi u - \eta \text{scal}(g_\Phi)u^{1 - 1/\eta}, \quad u|_{t=0} = 1.
\]  

(5.1)

We plan to construct a solution as a fixed point in of a contraction in \( C^{k+2,\alpha}_\Phi(M \times [0, T]) \) and some short time \( T > 0 \). We assume below that \( k = 0 \), since the general case follows the \( k = 0 \) case verbatim. We write \( u = 1 + v \) and obtain from (5.1) an equation for \( v \)

\[
\partial_t v = (m - 1)\Delta_\Phi v(1 + v)^{-1/\eta} - \eta \text{scal}(g_\Phi)(1 + v)^{1 - 1/\eta}; \quad v|_{t=0} = 0.
\]  

(5.2)

Assume \( v \in C^{2,\alpha}_\Phi(M \times [0, T]) \) with \( \|v\|_{2,\alpha} \leq \mu \) for some \( \mu < 1 \). Then the following series converge in the Banach space \( C^{2,\alpha}_\Phi(M \times [0, T]) \)

\[
(1 + v)^{-1/\eta} = \sum_{j=0}^\infty a_j v^j = 1 - \frac{v}{\eta} + \sum_{j=2}^\infty a_j v^j =: 1 - \frac{v}{\eta} + v^2 s(v)
\]

with \( \|(1 + v)^{-1/\eta}\|_{2,\alpha} \leq C_\mu \) and \( \|s(v)\|_{2,\alpha} \leq C_\mu \).
for some $C_\mu > 0$, depending only on $\mu$. Plugging the identity $(1 + v)^{-1/\eta} = 1 - v/\eta + v^2s(v)$ into (5.2) yields after rescaling the time variable by $(m - 1)$ the following flow equation

$$(\partial_t - \Delta \phi)v = -\frac{1}{\eta}v\Delta \phi v + v^2s(v)\Delta \phi v - \frac{\eta}{m - 1}\text{scal}(g_\phi) + \frac{1}{m - 1}\text{scal}(g_\phi)v
+ \frac{1}{m - 1}\text{scal}(g_\phi)v^2(1 - \eta s(v) - \eta vs(v)).$$

We will simplify the right hand side by introducing two non-linear operators, the first one containing no derivatives of $v$

$$F_1(v) := -\frac{\eta}{m - 1}\text{scal}(g_\phi) + \frac{1}{m - 1}\text{scal}(g_\phi)v
+ \frac{1}{m - 1}\text{scal}(g_\phi)v^2(1 - \eta s(v) - \eta vs(v)).$$

The second one is in a certain sense quadratic in $v$ and defined by

$$F_2(v) := -\frac{1}{\eta}v\Delta \phi v + v^2s(v)\Delta \phi v.$$ 

In this notation, (5.3) can be written as

$$\quad (\partial_t - \Delta \phi)v = (F_1 + F_2)v; \quad v|_{t=0} = 0. \quad (5.4)$$

Our intention is to prove short-time existence of solution of (5.4) by using exactly the same argument as in [CaGe21, Corollary 1.3], albeit with slightly different Hölder spaces, which gives some conditions for the contraction argument to work (the argument of [CaGe21, Corollary 1.3] in fact does not specifically depend on the Hölder spaces). To this end, we need to prove some properties of $F_1$ and $F_2$.

**Lemma 5.1.** Denote by $B$ the open ball of radius 1 in $C^{2,\alpha}_\phi(M \times [0, T])$. Then the map $F_2 : B \to C^{2,\alpha}_\phi(M \times [0, T])$ is bounded. Moreover, for any two functions $v, v' \in B \subset C^{2,\alpha}_\phi(M \times [0, T])$ satisfying

$$\|v\|_{2,\alpha}, \|v'\|_{2,\alpha} \leq \mu < 1,$$

there exists a constant $C_\mu > 0$ such that

(i) $\|F_2(v) - F_2(v')\|_{\alpha} \leq C_\mu \max(\|v\|_{2,\alpha}, \|v'\|_{2,\alpha})\|v - v'\|_{2,\alpha},$

(ii) $\|F_2(v)\|_{k,\alpha} \leq C_\mu \|v\|_{k+2,\alpha}.$

**Proof.** We shall write $\Delta = \Delta \phi$ for simplicity of notation. First, let $v \in B$ with $\|v\|_{2,\alpha} \leq \mu < 1$. Then, by the definition of $C^{2,\alpha}_\phi(M \times [0, T])$ and the
fact that $\Delta \in \text{Diff}^2_0(M)$, it follows that $\Delta v \in C^{0}_\alpha(M \times [0, T])$. We can thus estimate
\[
\|F_2(v)\|_\alpha \leq C_\mu \left( \|v\Delta v\|_\alpha + \|v^2s(v)\Delta v\|_\alpha \right) \\
\leq C_\mu \left( \|v\|_\alpha \|\Delta v\|_\alpha + \|v^2s(v)\|_\alpha \|\Delta v\|_\alpha \right) \\
\leq C_\mu \|v\|_{2,\alpha}^2,
\]
for some $C_\mu > 0$ depending only on $\mu$ and possibly changing in each estimation step. This proves the second item and in particular boundedness of $F_2 : B \to C^{0}_\alpha(M \times [0, T])$. For the first item we write for any $v, v' \in B$
\[
v^2s(v) - (v')^2s(v') =: (v - v')O_1(v, v'),
\]
where $O_1(v, v')$ is a polynomial combination in $v$ and $v'$. Eq. (5.5) implies
\[
F_2(v) - F_2(v') = -\frac{1}{\eta} \left( \Delta v(v - v') + v'\Delta(v - v') \right) \\
+ O_1(v, v') ((v - v')\Delta v + v'\Delta(v - v')).
\]
which then implies
\[
\|F_2(v) - F_2(v')\|_{1,\alpha} \leq C_\mu \left( \|\Delta v\|_\alpha \|v - v'\|_\alpha + \|v'\|_\alpha \|\Delta(v - v')\|_\alpha \right) \\
\quad + \|\Delta v\|_\alpha \|v - v'\|_\alpha + \|v'\|_\alpha \|v - v'\|_2,\alpha \\
\leq C_\mu \max[\|\Delta v\|_{2,\alpha}, \|v - v'\|_{2,\alpha}] \|v - v'\|_{2,\alpha}.
\]

\[\Box\]

**Lemma 5.2.** Assume that $\text{scal}(g_\phi) \in C^{2,\alpha}_\phi(M)$. Denote by $B$ the open ball of radius 1 in $C^{2,\alpha}_\phi(M \times [0, T])$. Then $F_1$ maps $B$ into $C^{1,\alpha}_\phi(M \times [0, T])$. Furthermore, if $v, v' \in B$ with $\|v\|_{2,\alpha} \|v'\|_{2,\alpha} \leq \mu < 1$, there exists a constant $C_\mu$ such that
(i) $\|F_1(v) - F_1(v')\|_{1,\alpha} \leq C_\mu \|v - v'\|_{2,\alpha}$
(ii) $\|F_1(v)\|_{1,\alpha} \leq C_\mu$.

**Proof.** First, consider $v \in B \subset C^{2,\alpha}_\phi(M \times [0, T])$. Since by assumption $\text{scal}(g_\phi) \in C^{2,\alpha}_\phi(M)$, we find
\[
\text{scal}(g_\phi)v^2(1 - \eta s(v) - \eta vs(v)) \in C^{1,\alpha}_\phi(M \times [0, T]).
\]
Now, assume $\|v\|_{2,\alpha} \leq \mu < 1$. We can now estimate
\[
\|F_1(v)\|_{1,\alpha} \leq C_\mu \|\text{scal}(g_\phi)\|_{1,\alpha} (1 + \|v\|_{2,\alpha} + \|v^2\|_{2,\alpha}) \leq C_\mu,
\]
for some $C_\mu > 0$ depending only on $\mu$ and possibly changing in each estimation step. This completes the proof for the second item. In particular, $F_1$ indeed maps $B$ into $C^{1,\alpha}_\phi(M \times [0, T])$. For the first item we have for any $v, v' \in B$ with $\|v\|_{2,\alpha} \|v'\|_{2,\alpha} \leq \mu < 1$
\[
\|F_1(v) - F_1(v')\|_{1,\alpha} \leq C_\mu \|\text{scal}(g_\phi)\|_{1,\alpha} \|v - v'\|_{2,\alpha} \\
+ C_\mu \|\text{scal}(g_\phi)\|_{1,\alpha} \|v^2 - (v')^2\|_{2,\alpha}.
\]
where in the final estimate we use (5.5) and its analogue for $\nu^3 s(\nu)$. This concludes the first item and, naturally, finishes the proof.

Now, exactly the same argument as in [CAGe21, Corollary 1.3] (with $\alpha = 1$) implies directly, that for $\text{scal}(g_\Phi) \in C^1(M)$, the map $H \circ (F_1 + F_2)$ is a contraction on a closed ball $\overline{B}_\mu \subset C^1(M \times [0, T])$ of radius $\mu > 0$, provided $\mu, T > 0$ are sufficiently small. Thus the flow (5.4) admits a solution $v \in \overline{B}_\mu$ as a fixed point of that contraction. Setting $u = 1 + v$, we obtain a short-time solution for the Yamabe flow (5.1) and thus to (1.1). The same argument yields a solution in $C^{k+1,2}_{\Phi}(M \times [0, T])$ for a general $k \in \mathbb{N}_0$, provided $\text{scal}(g_\Phi) \in C^{k+1,2}_{\Phi}(M)$.

**Theorem 5.3.** Consider a $\Phi$-manifold $(M, g_\Phi)$ of dimension $m \geq 3$. Assume $\text{scal}(g_\Phi) \in C^{k+1,\alpha}_{\Phi}(M)$ for some $\alpha \in (0, 1)$ and any $k \in \mathbb{N}_0$. Then the Yamabe flow (1.1) admits a unique solution $g = u^{4/(m-2)}_\Phi g_\Phi$, where $u \in C^{k+2,\alpha}_{\Phi}(M \times [0, T])$, for some $T > 0$ sufficiently small.

This proves Theorem 1.1.

6. CURVATURE-NORMALIZED YAMABE FLOW ON $\Phi$-MANIFOLDS

Consider the increasing curvature normalized Yamabe flow CYF$^+$

$$\partial_t g = (\text{scal}(g)_{\sup} - \text{scal}(g)) g,$$

where $\text{scal}(g(t))_{\sup} := \sup_M \text{scal}(g(t))$.

see (1.4), introduced by Suárez-Serrato and Tapie [SSTA11] to study entropy rigidity on the Yamabe flow in the compact setting. We are interested in the non-compact setting of a $\Phi$-manifold $(M, g_\Phi)$, which is why the usual normalization by (1.2) doesn’t work and we resort to the CYF$^+$ normalization. We can study the decreasing curvature normalized Yamabe flow CYF$^-$ with $\text{scal}(g)_{\sup}$ replaced by $\text{scal}(g)_{\inf}$ along the same lines.

Short time existence of CYF$^+$ (as well as CYF$^-$) follows by a simple time rescaling. Indeed, let $g(t) = u(t)^{1/n} g_\Phi$ be family of Riemannian metrics satisfying the (un normalized) Yamabe flow (1.1) with $u \in C^{2,\alpha}(M \times [0, T])$. Consider the functions

$$f(t) = \exp \left( \int_0^t \eta \text{scal}(g(\theta))_{\sup} d \theta \right),$$

$$F(t) = \int_0^t f(\theta)^n d \theta - f(\theta)^n.$$

(6.1)
Note that $f$ is positive and $F$ is a primitive for $f$ satisfying $F(0) = 0$. Moreover, since $dF/dt > 0$, it follows that $F^{-1}$ is well-defined. Thus, we can define a 1-parameter family of Riemannian metrics by

$$\tilde{g}(\tau) := \tilde{u}(\tau)^{1/\eta}g_\Phi, \quad \text{where} \quad \tilde{u}(\tau) := (fu)(F^{-1}(\tau)) \quad (6.2)$$

One can easily check from $u \in C^2_\alpha(M \times [0, \bar{T}])$ that $\tilde{u} \in C^2_\alpha(M \times [0, \bar{T}])$ with $\bar{T} = \max F$. Moreover, it follows from direct computations that

$$\partial_\tau \tilde{g} = (\text{scal}(\tilde{g})_{\text{sup}} - \text{scal}(\tilde{g})) \tilde{g}, \quad \tilde{g}(0) = g_\Phi. \quad (6.3)$$

It is also possible to invert the process and obtain a solution of the standard Yamabe flow, proving said relation. This proves short time existence of the increasing curvature normalized Yamabe flow $\text{CYF}^+$ (and similarly $\text{CYF}^-$).

7. Evolution of the scalar curvature along $\text{CYF}^+$

We begin with an easy observation.

**Lemma 7.1.** Let $M$ be any $m$-dimensional smooth manifold. Given any two Riemannian metrics $g$ and $\tilde{g}$ on $M$ related by a conformal transformation $\tilde{g} = u^{1/\eta} \cdot g$ for some positive $u \in C^2(M)$, then for any $f \in C^2(M)$

$$\Delta_{\tilde{g}}f = \frac{1}{u^{1/\eta} \cdot \Delta_g f + \frac{2}{u^{1+1/\eta}} \cdot g(\nabla f, \nabla u).}$$

**Proof.** First, note that $\tilde{g}^{-1} = u^{-1/\eta} \cdot g^{-1}$ and $\sqrt{|\det \tilde{g}|} = u^{m/2\eta} \cdot \sqrt{|\det g|}$. Thus, we compute in local coordinates

$$\Delta_{\tilde{g}}f = \frac{1}{\sqrt{|\det \tilde{g}|}} \sum_j \partial_j \left( \sqrt{|\det \tilde{g}|} \cdot \sum_i \tilde{g}^{ij} \cdot \partial_i f \right)$$

$$= \frac{1}{u^{m/2\eta} \cdot \sqrt{|\det g|}} \sum_j \partial_j \left( u^{(m-2)/2\eta} \cdot \sqrt{|\det g|} \cdot \sum_i g^{ij} \cdot \partial_i f \right)$$

$$= \frac{1}{u^{1/\eta} \cdot \sqrt{|\det g|}} \sum_j \partial_j \left( \sqrt{|\det g|} \cdot \sum_i g^{ij} \cdot \partial_i f \right)$$

$$+ \frac{2}{u^{1+1/\eta}} \cdot \sum_{i,j} g^{ij} \cdot \partial_i u \cdot \partial_i f = \frac{1}{u^{1/\eta}} \cdot \Delta_g f + \frac{2}{u^{1+1/\eta}} \cdot g(\nabla f, \nabla u). \quad \square$$

Note that the increasing curvature normalized Yamabe flow (1.4) can be rewritten as (recall $\text{scal}(g)_{\text{sup}}$ denotes the supremum of $\text{scal}(g)$)

$$\frac{1}{\eta} \partial_\tau u = \left( \text{scal}(g)_{\text{sup}} - \text{scal}(g) \right) u. \quad (7.1)$$
From here we conclude immediately
\[
\frac{1}{\eta} \partial_t (u^{-1} \Delta \phi u) = -\frac{1}{\eta} u^{-2} \partial_t u \cdot \Delta \phi u + \frac{1}{\eta} u^{-1} \Delta \phi (\partial_t u)
\]
\[
= -u^{-1} (\text{scal}(g)_{\sup} - \text{scal}(g)) \cdot \Delta \phi u
\]
\[
+ u^{-1} \Delta \phi \left( (\text{scal}(g)_{\sup} - \text{scal}(g)) u \right)
\]
\[
= u^{-1} \left( \text{scal}(g) \cdot \Delta \phi u - \Delta \phi (\text{scal}(g) u) \right).
\]

Moreover, from Lemma 7.1 we obtain
\[
u^{-1} \Delta \phi (\text{scal}(g) u) = u^{-1} \text{scal}(g) \Delta \phi u + \Delta \phi \text{scal}(g) + 2u^{-1} \text{scal}(g) (\nabla u, \nabla \text{scal}(g))
\]
\[
= u^{-1} \text{scal}(g) \Delta \phi u + u^{1/\eta} \Delta \phi \text{scal}(g),
\]
where \( \Delta \phi \) is the negative Laplace Beltrami operator of the conformally transformed metric \( g = u^{1/\eta} \cdot g \phi \). Combined with (7.2) this gives
\[
\frac{1}{\eta} \partial_t (u^{-1} \Delta \phi u) = -u^{1/\eta} \Delta \phi \text{scal}(g).
\]

On the other hand, from (1.4) is also straightforward that
\[
\partial_t u^{-1/\eta} = u^{-1/\eta} (\text{scal}(g) - \text{scal}(g)_{\sup}).
\]

Finally, combining (7.2) and (7.4) with the transformation formula for the scalar curvature, cf. (1.5)
\[
\text{scal}(g(t)) = \text{scal}(u^{1/\eta} g \phi) = -u^{-1/\eta} \left[ \frac{m-1}{\eta} u^{-1} \Delta \phi u - \text{scal}(g \phi) \right].
\]

provides us the expression
\[
\partial_t \text{scal}(g) = (m-1) \Delta \phi \text{scal}(g) + \text{scal}(g) (\text{scal}(g) - \text{scal}(g)_{\sup}).
\]

Based on (7.6), we can now prove the following

**Lemma 7.2.** Suppose \( \text{scal}(g \phi) \in C^4_\phi(M) \) is negative and bounded away from zero\(^1\), that is, there are constants \( a_1, a_2 > 0 \) such that
\[
-\infty < -a_1 \leq \text{scal}(g \phi) \leq -a_2 < 0.
\]
Then along CYF\( ^+ \) with positive solution \( u \in C^4_\phi(M \times [0, T]) \), supremum of the the scalar \( \text{scal}(g(t))_{\sup} = \sup_M \text{scal}(g(t)) \) is non-increasing.

**Proof.** By Theorem 5.3, CYF\( ^+ \) exists for short time in \( C^4_\phi(M \times [0, T]) \). From the transformation rule of the scalar curvature (7.5), it follows that \( \text{scal}(g) \in C^2_\phi(M \times [0, T]) \). Applying the arguments of §3 to \( \text{scal}(g) \), we

\(^1\)In fact, boundedness away from zero for the scalar curvature will only become important in the next section, but we list it here as a condition for consistency.
conclude from (7.6) by Proposition 3.1, similar to Corollary 3.4, for almost all \( t \in [0, T] \)

\[
\partial_t \text{scal}(g(t))_{\sup} \leq \text{scal}(g(t))_{\sup}(\text{scal}(g(t))_{\sup} - \text{scal}(g(t))_{\sup}) = 0. \quad (7.8)
\]

This implies directly that \( \text{scal}(g)_{\sup} \) is non-increasing along CYF\(^+\).

Knowing that the supremum of the scalar curvature is non-increasing in time, the next results shows that the scalar curvature approaches its supremum at an exponential rate.

**Lemma 7.3.** Suppose \( \text{scal}(g_\Phi) \in C^{4,\alpha}_0(M) \) is negative, bounded away from zero as in Lemma 7.2. Then along CYF\(^+\) with positive solution \( u \in C^{4,\alpha}_0(M \times [0, T]) \)

\[
\| \text{scal}(g(t))_{\inf} - \text{scal}(g(t))_{\sup} \| \leq Ce^{\text{scal}(g_\Phi)_{\sup}t},
\]

with \( C > 0 \) a constant independent of \( T \), where \( \text{scal}(g(t))_{\inf} := \inf_M \text{scal}(g(t)) \).

**Proof.** Applying the arguments of §3 to \( \text{scal}(g) \), we conclude from (7.6) by Proposition 3.1, similar to Corollary 3.4, for almost all \( t \in [0, T] \)

\[
\partial_t \text{scal}(g)_{\inf} \geq \text{scal}(g)_{\inf}(\text{scal}(g)_{\inf} - \text{scal}(g)_{\sup}). \quad (7.9)
\]

From here it follows that \( \text{scal}(g)_{\inf} \) is non-decreasing along the CYF\(^+\). Combining (7.9) with (7.8), we find

\[
\partial_t (\text{scal}(g)_{\sup} - \text{scal}(g)_{\inf}) \leq -\text{scal}(g)_{\inf}(\text{scal}(g)_{\inf} - \text{scal}(g)_{\sup})
\]

\[
= \text{scal}(g)_{\inf}(\text{scal}(g)_{\sup} - \text{scal}(g)_{\inf})
\]

\[
\leq \text{scal}(g_\Phi)_{\sup}(\text{scal}(g)_{\sup} - \text{scal}(g)_{\inf}).
\]

Integrating both sides of the last inequality gives

\[
(\text{scal}(g)_{\sup} - \text{scal}(g)_{\inf})(t) \leq Ce^{\text{scal}(g_\Phi)_{\sup}t}, \quad (7.10)
\]

where \( C \) depends only on the initial data. This means that the difference between the supremum and the infimum of the scalar curvature decreases exponentially along the flow. Consequently, the scalar curvature approaches \( \text{scal}(g)_{\sup} \) at an exponential rate too, therefore implying the desired outcome.

□

8. **Uniform estimates along CYF\(^+\) Yamabe flow**

We start immediately with the central result of the section. If we assume \( \text{scal}(g_\Phi) \in C^{4,\alpha}_0(M) \), then the solution \( u \in C^{4,\alpha}_0(M \times [0, T']) \) of CYF\(^+\) exists by Theorem 5.3 for \( T' > 0 \) sufficiently small. Assume \( u \) in fact exists in \( C^{4,\alpha}_0 \) on a larger time interval \([0, T)\) with maximal time \( T \geq T' \). Then even in the maximal time interval \([0, T)\) we obtain \( T \)-independent a priori estimates.
Theorem 8.1. Assume $\text{scal}(g_\Phi) \in C^1_{\Phi}(M)$ is negative and bounded away from zero as in Lemma 7.3. Let $u \in C^1_{\Phi}(M \times [0, T))$ to be the solution of CYF\(^+\) extended to a maximal time interval $[0, T)$. Then there exist constants $c_1, c_2 > 0$, depending on $u(0), \sup |\text{scal}(g_\Phi)|$ and $\inf |\text{scal}(g_\Phi)|$, and independent of $T$, such that

$$0 < c_1 \leq u(p, t) \leq c_2, \text{ for all } (p, t) \in M \times [0, T).$$

Proof. First, we consider the flow for a short time interval $[0, T']$, where $u$ is guaranteed to be positive. The estimates below will show that $u$ stays positive, bounded away from zero uniformly on $[0, T']$ and thus all of the arguments hold on the maximal interval $[0, T)$. By the differential inequalities in Proposition 3.4 we have (a priori almost everywhere on $[0, T']$, however as just explained a posteriori almost everywhere on the full time interval)

$$\frac{\partial}{\partial t} u_{\inf} \geq \eta \sup_M \text{scal}(g(t)) \cdot u_{\inf} + \eta \inf_M |\text{scal}(g_\Phi)| \cdot u_{\inf}^{1-1/\eta},$$

$$\frac{\partial}{\partial t} u_{\sup} \leq \eta \sup_M \text{scal}(g(t)) \cdot u_{\sup} + \eta \sup_M |\text{scal}(g_\Phi)| \cdot u_{\sup}^{1-1/\eta}. \tag{8.1}$$

Multiplying both sides of the first inequality by $\frac{1}{\eta} u_{\inf}^{1/\eta-1}$, and of the second inequality by $\frac{1}{\eta} u_{\sup}^{1/\eta-1}$, we obtain

$$\frac{\partial}{\partial t} u_{\inf}^{1/\eta} \geq \sup_M \text{scal}(g(t)) \cdot u_{\inf}^{1/\eta} + \inf_M |\text{scal}(g_\Phi)|,$$

$$\frac{\partial}{\partial t} u_{\sup}^{1/\eta} \leq \sup_M \text{scal}(g(t)) \cdot u_{\sup}^{1/\eta} + \inf_M |\text{scal}(g_\Phi)|. \tag{8.2}$$

Write $\omega_1 := u_{\inf}^{1/\eta}$ and $\omega_2 := u_{\sup}^{1/\eta}$. We obtain from (8.2)

$$\frac{\partial}{\partial t} \omega_1 \geq \inf_M \text{scal}(g_\Phi) \cdot \omega_1 + \inf_M |\text{scal}(g_\Phi)| =: b \omega_1 + a,$$

$$\frac{\partial}{\partial t} \omega_2 \leq \sup_M \text{scal}(g_\Phi) \cdot \omega_2 + \sup_M |\text{scal}(g_\Phi)| =: B \omega_2 + A, \tag{8.3}$$

where in the first inequality we used the fact that by (7.9) $\text{scal}(g)_{\inf}$ is non-decreasing in time, while the second inequality from Lemma 7.2, since $\text{scal}(g)_{\sup}$ is non-increasing in time.

The first inequality is equivalent to $(e^{-bt} \omega_1)' \geq a e^{-bt}$. Hence, integration on both sides over $[0, t]$ gives the following estimate

$$\omega_1(t) \geq e^{bt} \omega_1(0) + \frac{a}{b} (e^{bt} - 1)$$

$$\iff u_{\inf}^{1/\eta}(t) \geq u_{\inf}^{1/\eta}(0) e^{\inf_M \text{scal}(g_\Phi) \cdot t} + \frac{\inf_M |\text{scal}(g_\Phi)|}{\inf_M \text{scal}(g_\Phi)} (e^{\inf_M \text{scal}(g_\Phi) \cdot t} - 1)$$
were converted into Hölder regularity by mapping and Theorem C concluding the proof.

**Proposition 8.2.** Assume \( \text{scal}(g_\Phi) \in C^{4,\alpha}_\Phi(M) \) is negative and bounded away from zero as in Lemma 7.3. Let \( u \in C^{4,\alpha}_\Phi(M \times [0, T)) \) to be the solution of CYF\(^+\) extended to a maximal time interval \([0, T)\). Then there exists a constant \( C > 0 \), depending on \( u(0), \sup |\text{scal}(g_\Phi)| \) and \( \inf |\text{scal}(g_\Phi)| \), and independent of \( T \), such that

\[
\| \partial_t u \|_{\infty} \leq C e^{\text{sup}_M |\text{scal}(g_\Phi)|} t .
\]

**Proof.** The CYF\(^+\) flow (1.4) can be rewritten as (cf. (7.1))

\[
\frac{1}{\eta} \partial_t u = (\text{scal}(g)_{\sup} - \text{scal}(g)) u .
\]

Then, employing Lemma 7.3 and Theorem 8.1, it follows directly that

\[
\| \partial_t u \|_{\infty} \leq |\eta| \| \text{scal}(g)_{\sup} - \text{scal}(g) \|_{\infty} \| u \|_{\infty}
\]

\[
\leq C e^{\text{sup}_M |\text{scal}(g_\Phi)|} t .
\]

\[\blacksquare\]

### 9. Parabolic Schauder estimates on \( \Phi \)-manifolds

In the previous work [BaVe19], a priori estimates like in Theorem 8.1 and Proposition 8.2 were converted into Hölder regularity by mapping properties of some heat parametrix for \( (\partial_t + u^{-1/\eta} \Delta_\Phi) \). This approach fails here since the parametrix construction in [CAGE21, Theorem 10.1] does not work for \( u \in C^4_\Phi(M \times [0, T)) \). Therefore, we argue here by reducing to classical parabolic Schauder estimates. The statements below are folklore on manifolds with bounded geometry, which includes \( \Phi \)-manifolds.

Consider for any fixed \( \delta > 0 \) a countable family of points \( \{p_i\} \in M \), such that the \( \delta \)-balls \( B_\delta(p_i) \) around these points (with distance measured
with respect to \( g_\phi \) cover \( M \). Let \( \delta > 0 \) be sufficiently small, such that the \( \delta \)-balls stay inside local coordinate neighborhoods. Obviously, we are interested only in those \( p_i = (x_i, y_i, z_i) \in \mathcal{U} \) in the collar neighborhood of the boundary \( \partial M \). Writing \( B_\delta(0) \in \mathbb{R}^m \) for an open ball of radius \( \delta \) around the origin, we define
\[
\Psi_i : B_\delta(0) \times [0, \delta^2] =: Q_\delta \rightarrow B_\delta(p_i) \times [0, \delta^2],
\]
\[
(S, U, Z, t) \mapsto \left( x_i + x_i^2 S, y_i + x_i U, z_i + Z, t \right)
\]
(9.1)

Away from the collar \( \mathcal{U} \) of the boundary, we may define \( \Psi_i \) as usual local coordinate parametrizations. Clearly, the choice of \( \Psi_i \) and the notation \( (S, U, Z) \) is motivated by the projective coordinates (4.4). We compute the action of \( \Phi \)-derivatives under the pullback by the transformation \( \Psi_i \)
\[
\begin{align*}
\Psi_i^\ast (x^2 \partial_x f)(S) &= (x^2 \partial_x f)(x_i + x_i^2 S) = (1 + x_i S)^2 \partial S \Psi_i^\ast f, \\
\Psi_i^\ast (x \partial_x f)(U) &= (x \partial_x f)(y_i + x_i U) = (1 + x_i S) \partial U \Psi_i^\ast f, \\
\Psi_i^\ast (\partial_z f)(S) &= (\partial_z f)(z_i + Z) = \partial Z \Psi_i^\ast f.
\end{align*}
\]
(9.2)

Hence we obtain for the heat equation
\[
\Psi_i^\ast \left( (\partial_t - \Delta_\phi) f \right) = \left( \partial_t - \widetilde{\Delta}_\phi \right) \Psi_i^\ast f,
\]
(9.3)

where \( \widetilde{\Delta}_\phi = \partial_x^2 + \partial_u^2 + \partial_z^2 \) plus first order derivatives in \( (S, U, Z) \), up to coefficients that are bounded in \( Q_\delta \), uniformly in \( i \). Moreover, we observe the following

**Lemma 9.1.** Consider the classical Hölder space \( C^{k,\alpha}(Q_\delta) \) with Hölder norm denoted by \( \| \cdot \|_{k,\alpha,Q_\delta} \). Then the Hölder norm \( \| \cdot \|_{k,\alpha} \) on \( C^{k,\alpha}_\Phi(M \times [0, \delta^2]) \) defined in terms of (2.7), is equivalent to
\[
\sup_i \| \Psi_i^\ast u \|_{k,\alpha,Q_\delta}.
\]

**Proof.** The statement follows from (9.2) and the fact that, taking the local expression of \( d \) in Definition 2.2, we find in the collar \( \mathcal{U} \) (we denote the transformation (9.1) without the time variable, again by \( \Psi_i \))
\[
d \left( \Psi_i(S, U, Z), \Psi_i(S', U', Z') \right) \sim \| (S - S', U - U', Z - Z') \|.
\]

\[ \square \]

Now we are ready to convert the a priori estimates in Theorem 8.1 into uniform Hölder regularity on \([0, T]\), where \([0, T]\) is the maximal time interval, where the CYF\(^+\) flow solution \( u \) exists in \( C^{d,\alpha}_\Phi(M \times [0, T]) \). We use the classical Krylov-Safonov estimates, see [KrSa80] and the nice exposition in [Pic19, Theorem 12].
Proposition 9.2. Assume $\text{scal}(g_\Phi) \in C^{0,\alpha}_b(M)$ is negative and bounded away from zero. Let $u \in C^{1,\alpha}_b(M \times [0, T))$ be the solution of CYF$^+$ extended to a maximal time interval $[0, T)$. Then $u \in C^{3,\alpha}_b(M \times [0, T])$ with $T$-independent Hölder norm.

Proof. Consider the CYF$^+$ flow equation in (1.8)

$$\partial_t u(t) - (m-1)u(t)^{-1/\eta} \Delta u(t) = \eta \left( \sup_M \text{scal}(g(t)) \cdot u(t) - \text{scal}(g_\Phi)u(t)^{1-1/\eta} \right) =: f$$

Pulling back under $\Psi_i$ we obtain with $a := (m-1)\Psi_i^* u^{-1/\eta}$

$$\left( \partial_t - a \cdot \tilde{\Delta}_\Phi \right) \Psi_i^* u = \Psi_i^* f.$$

From Theorem 8.1 we infer that $\Psi_i^* f$ and $u, u^{-1}$ (and hence also $a, a^{-1}$) are bounded in $Q_\delta$, uniformly in $i$. Thus by the Krylov-Safonov estimate, see [KrSa80] and cf. [Pic96, Theorem 12], we find for some uniform constant $C > 0$, depending only on $\delta, \|u\|_{\infty}$ and $\|u^{-1}\|_{\infty}$

$$\|\Psi_i^* u\|_{a, Q_\delta} \leq C \left( \|\Psi_i^* u\|_{\infty, Q_\delta} + \|\Psi_i^* f\|_{\infty, Q_\delta} \right)$$

$$\leq C \left( \|u\|_{\infty} + \|f\|_{\infty} \right).$$

Thus $\Psi_i^* u \in C^\alpha(Q_\delta)$. By Lemma 9.1 we conclude $u \in C^\alpha_b(M \times [0, \delta^2])$. We extend the regularity statement to the whole time interval $[0, T)$ (with constants independent of $T$) iteratively, by setting $t = \delta^2 + t'$ and obtaining by the argument above $u \in C^\alpha_b(M \times [\delta^2, 2\delta^2])$, and repeating the iteration, till we reach $T$. \qed

This first gain in Hölder regularity can now be converted into higher order regularity by standard parabolic Schauder estimates, see [Kry96] and the exposition in [Pic96, Theorem 6].

Proposition 9.3. Assume $\text{scal}(g_\Phi) \in C^{0,\alpha}_b(M)$ is negative and bounded away from zero. Let $u \in C^{1,\alpha}_b(M \times [0, T))$ be the solution of CYF$^+$ extended to a maximal time interval $[0, T)$. Then $u \in C^{3,\alpha}_b(M \times [0, T])$ with $T$-independent Hölder norm.

Proof. Consider (9.4). Standard parabolic Schauder estimates, see [Kry96] and cf. [Pic96, Theorem 6], assert that for any $\alpha \in C^{k,\alpha}(Q_\delta)$ positive, uniformly bounded from below away from zero, and for any $\Psi_i^* f \in C^{k,\alpha}(Q_\delta)$, a uniformly bounded solution $\Psi_i^* u$ satisfies

$$\|\Psi_i^* u\|_{k+2,\alpha, Q_\delta} \leq C \left( \|\Psi_i^* u\|_{\infty, Q_\delta} + \|\Psi_i^* f\|_{k,\alpha, Q_\delta} \right)$$

$$\leq C \left( \|u\|_{\infty} + \|f\|_{k,\alpha} \right).$$
By Lemma 9.2, \( a = (m-1)\Psi_i u^{-1/n} \in C^\alpha(Q_\delta) \) and \( \Psi_i f \in C^\alpha(Q_\delta) \) uniformly in \( i \). Thus we may apply (9.5) with \( k = 0 \) and conclude that \( \Psi_i u \in C^{2,\alpha}(Q_\delta) \), uniformly in \( i \). By Lemma 9.1 we conclude \( u \in C^\alpha(M \times [0, \delta^2]) \). As before, we may extend the regularity statement to the whole time interval \([0, T]\) (with constants independent of \( T \)) by setting \( t = \delta^2 + t' \), concluding \( u \in C^\alpha(M \times [\delta^2, 2\delta^2]) \), and repeating the iteration, till we reach \( T \).

Since \( \text{scal}(g_\phi) \in C^4(M) \), this implies that \( a \in C^{2,\alpha}(Q_\delta) \) and \( \Psi_i f \in C^{2,\alpha}(Q_\delta) \) uniformly in \( i \). Applying now (9.5) with \( k = 2 \), we conclude exactly as above \( u \in C^\alpha(M \times [0, T]) \). In fact, in case \( \text{scal}(g_\phi) \in C^\alpha(M) \), we can iterate the arguments till \( u \in C^{4,\alpha}(M \times [0, T]) \). This proves the statement. \( \square \)

**Remark 9.4.** The arguments above show that in fact, if \( \text{scal}(g_\phi) \in C^4(M) \) with \( k \geq 4 \) is negative and bounded away from zero, the CYF\(^+\) flow solution \( u \in C^\alpha(M \times [0, T]) \) on any time interval \([0, T]\) is in fact in \( C^\alpha(M \times [0, T]) \).

### 10. Global existence of the CYF\(^+\) on \( \Phi\)-manifolds

We prove global existence of the flow, i.e. \( u \in C^\alpha(M \times [0, \infty)) \) by a contradiction. Assume the maximal time \( T > 0 \) is finite. In that case we will now restart the flow at \( t = T \), which contradicts maximality of \( T \). Restarting the flow at \( t = T \) means constructing a solution \( u' \) to the (unnormalized) Yamabe flow equation (1.6) with initial condition \( u'(0) = u(T) \). A rescaling of the time function, as in §6, yields short time existence of the curvature normalized Yamabe flow.

Let us simplify notation by writing \( u_0 = u(T) \) and \( \Delta = \Delta_\phi \). We linearize (1.6) by setting \( u' = u_0 + v \) for its solution with initial condition \( u'(0) = u_0 \). We obtain from the second equation in (1.6)

\[
(\partial_t - (m-1)u_0^{-1/n} \Delta) \; v = F_1(v) + F_2(v); \quad v|_{t=0} = 0,
\]

(10.1)

where we have abbreviated

\[
F_1(v) = Q_2(v), \quad F_2(v) = (m-1)u_0^{-1/n}\Delta u_0 - \text{scal}(g_\phi)u_0^{-1/n} + Q_1(v),
\]

The terms \( Q_1(v) \) include linear combinations of \( v \) with coefficients given in terms of \( u_0 \) and \( \Delta u_0 \). The terms \( Q_2(v) \) include quadratic combinations of \( v \) and \( \Delta v \) with coefficients given again in terms of \( u_0 \) and \( \Delta u_0 \).

Note that by Proposition 9.3, \( u_0 \in C^\alpha(M) \). Thus, \( F_1 \) contains quadratic combinations of \( v \) and \( \Delta v \), and \( F_2 \) — linear combinations of \( v \), with coefficients being in both cases elements of \( C^\alpha(M \times [0, T']) \).
Before we can establish short time existence of \( v \) by setting up a fixed point as in §5, we note a general result from parabolic Schauder theory. This is basically a non-constructive analogue of [CaGe21, Theorem 10.1].

**Proposition 10.1.** Consider \( a \in C^k_{\Phi}(M) \) positive, uniformly bounded away from zero. Then the inhomogeneous heat equation \((\partial_t - a \cdot \Delta_\Phi) v = f, \) with \( v(t = 0) = 0 \) and \( f \in C^k_{\Phi}(M \times [0, T']) \), has a parametrix \( Q \) acting as a bounded linear map

\[
Q : C^k_{\Phi}(M \times [0, T']) \to C^{k+2,\alpha}_{\Phi}(M \times [0, T']),
\]

\[
Q : C^{k+2,\alpha}_{\Phi}(M \times [0, T']) \to t C^{k+2,\alpha}_{\Phi}(M \times [0, T']).
\]

**Proof.** Consider the inhomogeneous heat equation with \( f \in C^k_{\Phi}(M \times [0, T']) \) and initial value \( v_0 \in C^{k+2,\alpha}_{\Phi}(M) \)

\[
(\partial_t - a \cdot \Delta_\Phi) v = f, \quad v(t = 0) = v_0.
\]

Then, by reducing the argument to local \( \delta \)-balls as in §9, we can follow the proof of [LSU67, Theorem 5.1 on p.320], and conclude for some uniform constant \( C > 0 \) existence of a unique solution \( v \in C^{k+2,\alpha}_{\Phi}(M \times [0, T']) \) with

\[
\|v\|_{k+2,\alpha} \leq C \left( \|f\|_{k,\alpha} + \|v_0\|_{k+2,\alpha} \right)
\]

This proves the first mapping property in (10.2) by setting \( v_0 = 0 \). For the second mapping property in (10.2), set \( f = 0 \) and obtain a solution \( v = Rv_0 \) with the solution operator \( R \) acting as a bounded linear map

\[
R : C^{k+2,\alpha}_{\Phi}(M) \to C^{k+2,\alpha}_{\Phi}(M \times [0, T']).
\]

The solution operator \( Q \) of the inhomogeneous problem is then given by

\[
Qf(p, t) = t \int_0^t \left( Rf(\tilde{t}) \right) (p, t - \tilde{t}) d\tilde{t}.
\]

Indeed, a direct computation shows

\[
\left( \partial_t - a \cdot \Delta_\Phi \right) Qf(p, t) = f(p, t) + t \int_0^t \left( \partial_t - a \cdot \Delta_\Phi \right) \left( Rf(\tilde{t}) \right) (p, t - \tilde{t}) d\tilde{t}
\]

This implies directly the second mapping property in (10.2) and completes the proof. \( \square \)

We can now conclude with the proof of Theorem 1.2.

**Corollary 10.2.** Assume \( \text{scal}(g_\Phi) \in C^k_{\Phi}(M) \) is negative and bounded away from zero with \( k \geq 4 \). Then the increasing curvature normalized Yamabe flow CYF \(^+\) exists for all times with conformal factor \( u \in C^k_{\Phi}(M \times [0, \infty)) \).
Proof. Using Proposition 10.1, we can construct a solution \( v \in C^{k,\alpha}_\Phi(M \times [0, T']) \) to (10.1) for some \( T' > 0 \) sufficiently small, as a fixed point of
\[
Q \circ (F_1 + F_2) : C^{k,\alpha}_\Phi(M \times [0, T']) \to C^{k,\alpha}_\Phi(M \times [0, T']),
\]
in the same way as in §5. Rescaling time as in §6, we obtain a solution \( u \in C^{k,\alpha}_\Phi(M \times [0, T + \varepsilon]) \) to CYF\(^+\), with \( \varepsilon > 0 \) sufficiently small. Finally, the arguments of Proposition 9.3, cf. Remark 9.4 imply that \( u \in C^{k,\alpha}_\Phi(M \times [0, T + \varepsilon]) \) with \( T \)-independent Hölder norm. This contradicts maximality of \( T > 0 \) and hence the flow exists for all times. \( \square \)

11. Convergence of the CYF\(^+\) on \( \Phi \)-manifolds

This last section presents the convergence of the CYF\(^+\). The argument uses a compact embedding of (weighted) Hölder spaces, where the weight is defined in terms of the boundary defining function \( x \) is extended to a smooth nowhere vanishing function on \( M \).

Definition 11.1. The weighted Hölder space \( x^\gamma C^{k,\alpha}_\Phi(M) \) is defined as the space of functions \( u = x^\gamma v \) with \( v \in C^{k,\alpha}_\Phi(M) \) and the norm \( \|u\|_{k,\alpha,\gamma} := \|v\|_{k,\alpha} \).

We now obtain the following folklore compactness result.

Proposition 11.2. Consider any \( 0 < \beta < \alpha < 1 \) and \( \gamma > 0 \). Then the following inclusion is compact
\[
t : C^{k,\alpha}_\Phi(M) \hookrightarrow x^{-\gamma} C^{k,\beta}_\Phi(M).
\]

Proof. Let \( \{u_n\}_n \) be a bounded sequence of functions in \( C^{k,\alpha}_\Phi(M) \) and, for any \( \delta > 0 \), let \( M_\delta \) be the compact submanifold given by
\[
M_\delta = M \setminus \{p \in M \mid x(p) < \delta\}.
\]
We know that \( C^{k,\alpha}_\Phi(M_\delta) \hookrightarrow C^{k,\beta}_\Phi(M_\delta) \) compactly for any \( \delta > 0 \). Therefore, \( \{u_n|_{M_\delta}\}_n \) admits a subsequence \( \{u_{n_j}|_{M_\delta}\}_j \) which converges in \( C^{k,\beta}_\Phi(M_\delta) \). Now consider a sequence \( \delta_i := 1/i \) for \( i \in \mathbb{N} \). We define convergent subsequences in \( C^{k,\beta}_\Phi(M_{\delta_i}) \) for any \( i \) by an iterative procedure: given a convergent subsequence \( \{u_{n_j}|_{M_{\delta_i}}\}_j \subset C^{k,\beta}_\Phi(M_{\delta_i}) \), we choose a convergent subsequence \( \{u_{n_j}|_{M_{\delta_{i+1}}}\}_j \subset C^{k,\beta}_\Phi(M_{\delta_{i+1}}) \) from \( \{u_{n_j}|_{M_{\delta_{i+1}}}\}_j \). Define the diagonal sequence by
\[
\{v_j := u_{n_j}|_{M_{\delta_j}}\}_j.
\]
We claim that \( \{v_j\}_j \) is a Cauchy sequence in \( x^{-\gamma} C^{k,\beta}_\Phi(M) \). In fact
\[
\|v_j\|_{x^{-\gamma} C^{k,\beta}_\Phi(M\setminus M_{\delta_j})} = \|x^\gamma v_j\|_{C^{k,\beta}_\Phi(M\setminus M_{\delta_j})} \leq C\delta_j^\gamma,
\]
where \( C > 0 \) is an upper bound for the norms of \( \{u_n\}_n \subset C^{k,\alpha}_\Phi(M) \). Now, let \( \varepsilon > 0 \) and choose \( \delta_0 \in \mathbb{N} \) sufficient large such that \( C\delta_j^\gamma \leq \varepsilon/4 \). The sequence \( \{v_j|_{M_{\delta_j}}\}_j \subset C^{k,\beta}_\Phi(M_{\delta_j}) \) converges by construction and thus converges also in
\( x^{-\gamma}C_{\Phi}^{k,\beta}(M; \xi_{0}) \). Hence, there exists some \( N_0 \in \mathbb{N} \) sufficiently large, such that for every \( j, j' \geq N_0 \)

\[
\|v_j - v_{j'}\|_{x^{-\gamma}C_{\Phi}^{k,\beta}(M; \xi_{0})} \leq \varepsilon/2,
\]

(11.4)

Hence for \( J_0 = \max(j_0, N_0) \), we have for any \( j, j' \geq J_0 \)

\[
\|v_j - v_s\|_{x^{-\gamma}C_{\Phi}^{k,\beta}(M)} \leq \|v_j - v_s\|_{x^{-\gamma}C_{\Phi}^{k,\beta}(M; \xi_{0})} + \|v_j - v_{s}\|_{x^{-\gamma}C_{\Phi}^{k,\beta}(M; \xi_{0})} < \varepsilon/2 + 2\varepsilon/2 = \varepsilon.
\]

Hence, \( \{v_j\} \) is a Cauchy sequence in \( x^{-\gamma}C_{\Phi}^{k,\beta}(M) \) and by completeness, it admits a convergent subsequence. This proves the statement. \( \square \)

We can finally prove convergence of the CYF\(^+\) flow, i.e. Theorem 1.3.

**Theorem 11.3.** Let \((M, g_\Phi)\) be a \( \Phi \)-manifold such that \( \text{scal}(g_\Phi) \in C_{\Phi}^{4,\alpha}(M) \) is negative and bounded away from zero. Consider the global solution \( u \in C_{\Phi}^{4,\alpha}(M \times \mathbb{R}_+) \) of CYF\(^+\). Then the family of metrics \( \{g(t) = u(t)^{1/\eta}g_\Phi\}_{t \geq 0} \) converges to a metric \( g^* = (u^*)^{1/\eta}g_\Phi \) with constant negative scalar curvature.

**Proof.** By Proposition 8.2, \( \|\partial_1 u(t)\|_\infty \) decreases exponentially. From there it is easy to check that \( u(t) \in L^\infty(M) \) is a Cauchy sequence and hence admits a well-defined limit \( u^* \in L^\infty(M) \). By Proposition 11.2, \( u(t) \in C_{\Phi}^{4,\alpha}(M) \) admits a convergent subsequence in \( x^{-\gamma}C_{\Phi}^{4,\beta}(M) \) for any \( \beta < \alpha \) and \( \gamma > 0 \). Hence \( u^* \in x^{-\gamma}C_{\Phi}^{4,\beta}(M) \) with scalar curvature \( \text{scal}^* \in x^{-\gamma}C_{\Phi}^{4,\beta}(M) \) such that for some divergent sequence \( (t_n) \in \mathbb{R}_+ \)

\[
\|\text{scal}(g(t_n)) - \text{scal}^*\|_{x^{-\gamma}C_{\Phi}^{2,\beta}(M)} \to 0 \quad \text{for} \quad n \to \infty,
\]

(11.5)

In particular, \( \text{scal}(g(t_n)) \) converges pointwise to \( \text{scal}^* \). Note that by Lemma 7.2 the supremum \( \sup_M \text{scal}(g(t)) \) is non-increasing and by (7.9) the infimum \( \inf_M \text{scal}(g(t)) \) is non-decreasing. Thus \( \sup_M \text{scal}(g(t)) \) and \( \inf_M \text{scal}(g(t)) \) are bounded from below and above, respectively, and thus both convergent as \( t \to \infty \). By Lemma 7.3

\[
\limsup_{t \to \infty} \text{scal}(g(t)) = \lim_{t \to \infty} \inf \text{scal}(g(t)) =: \text{const}.
\]

We compute from pointwise convergence of \( \text{scal}(g(t)) \) to \( \text{scal}^* \) at any \( p \in M \)

\[
\text{scal}^*(p) = \lim_{n \to \infty} \text{scal}(g(t_n))(p) \leq \limsup_{n \to \infty} \text{scal}(g(t_n)) \leq \sup_{M} \text{scal}^* \leq \text{const}.
\]

(11.6)

Similar argument applied to the infimum of \( \text{scal}^* \) yields

\[
\text{scal}^*(p) = \lim_{n \to \infty} \text{scal}(g(t_n))(p) \geq \liminf_{n \to \infty} \text{scal}(g(t_n)) \geq \inf_{M} \text{scal}^* \geq \text{const}.
\]

(11.7)
Combining (11.6) and (11.7), proves the statement. □

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