FORMAL GEOMETRIC QUANTIZATION

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Abstract. Let $K$ be a compact Lie group acting in an Hamiltonian way on a symplectic manifold $(M, \Omega)$ which is prequantized by a Kostant-Souriau line bundle. We suppose here that the moment map $\Phi$ is proper so that the reduced space $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ are compact for all $\mu$. Following Weitsman [33], we define the “formal geometric quantization” of $M$ as

$$Q^{-\infty}_K(M) := \sum_{\mu \in K} Q(M_\mu) V^K_\mu.$$ 

The aim of this article is to study the functorial properties of the assignment $M \mapsto Q^{-\infty}_K(M)$.

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1. Introduction and statement of the results

Let $M$ be an Hamiltonian $K$-manifold with symplectic form $\Omega$ and moment map $\Phi : M \to \mathfrak{k}^*$. We assume the existence of a $K$-equivariant line bundle $L$ on $M$ with connection of curvature equal to $-i\omega$. In other words $M$ is pre-quantizable in the sense of [20] and we call $L$ the Kostant-Souriau line bundle.

In the process of quantization one tries to associate a unitary rep resentation of $K$ to the data $(M, \Omega, \Phi, L)$. When $M$ is compact one associates to these data a virtual representation $Q_K(M) \in R(K)$ of $K$ defined as the equivariant index of a Dolbeault-Dirac operator : $Q_K(M)$ is the geometric quantization of $M$.

This quantization process satisfies the following functorial properties :

[P1] When $N$ and $M$ are respectively pre-quantized compact Hamiltonian $K_1$ and $K_2$ manifolds, the product $M \times N$ is a pre-quantized compact Hamiltonian $K_1 \times K_2$-manifold, and we have

$$(1.1) \ Q_{K_1 \times K_2}(M \times N) = Q_{K_1}(M) \otimes Q_{K_2}(N) \quad \text{in} \quad R(K_1 \times K_2) \simeq R(K_1) \otimes R(K_2).$$

[P2] If $H \subset K$ is a connected Lie subgroup, then the restriction of $Q_K(M)$ to $H$ is equal to $Q_H(M)$.

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Note that [P1] and [P2] gives the following functorial property:

[P3] When $N$ and $M$ are pre-quantized compact Hamiltonian $K$-manifold, the product $M \times N$ is a pre-quantized compact Hamiltonian $K$-manifold, and we have $Q_K(M \times N) = Q_K(M) \cdot Q_K(N)$, where $\cdot$ denotes the product in $R(K)$.

One other fundamental property is the behaviour of the $K$-multiplicities of $Q_K(M)$ that is known as “quantization commutes with reduction”.

Let $T$ be a maximal torus of $K$, $C_K \subset \mathfrak{t}^*$ be a Weyl Chamber, $\wedge^* \subset \mathfrak{t}^*$ be the weight lattice, and $\widetilde{K} = \wedge^* \cap C_K$ be the set of dominant weights. The ring of character $R(K)$ as a $\mathbb{Z}$-basis $V^K_{\mu}, \mu \in \widetilde{K} : V^K_{\mu}$ is the irreducible $K$-representation with highest weight $\mu$.

For any $\mu \in \widetilde{K}$ which is a regular value of $\Phi$, the reduced space (or symplectic quotient) $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is an orbifold equipped with a symplectic structure $\Omega_\mu$. Moreover $L_\mu := (L|_{\Phi^{-1}(\mu) \otimes \mathbb{C} \cdot \mu})/K_\mu$ is a Kostant line orbibundle over $(M_\mu, \Omega_\mu)$. The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence $Q(M_\mu) \in \mathbb{Z}$ is defined. In [25], this is extended further to the case of singular symplectic quotients, using partial (or shift) de-singularization. So the integer $Q(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \widetilde{K}$ in particular $Q(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

The following Theorem was conjectured by Guillemin-Sternberg [16] and is known as “quantization commutes with reduction” [24, 25, 30, 28]. For complete references on the subject the reader should consult [29, 32].

**Theorem 1.1.** (Meinrenken, Meinrenken-Sjamaar). We have the following equality in $R(K)$

$$Q_K(M) = \sum_{\mu \in \widetilde{K}} Q(M_\mu) V^K_\mu.$$ 

Suppose now that $M$ is non-compact but the moment map $\Phi : M \to \mathfrak{t}^*$ is assumed to be proper (we will say simply ”$M$ is proper”). In this situation the geometric quantization of $M$ as an index of an elliptic operator is not well defined. Nevertheless the integers $Q(M_\mu), \mu \in \widetilde{K}$ are well defined since the symplectic quotient $M_\mu$ are compact.

Following Weitsman [33], we introduce the following

**Definition 1.2.** The formal quantization of $(M, \Omega, \Phi)$ is the element of $R^{-\infty}(K) := \text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$ defined by

$$Q^{-\infty}_K(M) = \sum_{\mu \in \widetilde{K}} Q(M_\mu) V^K_\mu.$$ 

A representation $E$ of $K$ is admissible if it as finite $K$-multiplicities : $\dim(\text{hom}_K(V^K_\mu, E)) < \infty$ for every $\mu \in \widetilde{K}$. Here $R^{-\infty}(K)$ is the Grothendieck group associated to the $K$-admissible representations. We have a canonical inclusion $i : R(K) \hookrightarrow R^{-\infty}(K)$: to $V \in R(K)$ we associate the map $i(V) : R(K) \to \mathbb{Z}$ defined by $W \mapsto \dim(\text{hom}_K(V, W))$. In order to simplify the notation, $i(V)$ will be written $V$. Moreover the tensor product induces a structure of $R(K)$-module on $R^{-\infty}(K)$ since $E \otimes V$ is an admissible representation when $V$ and $E$ are respectively finite dimensional and admissible representation of $K$. 

\[Q^{-\infty}_K(M) = \sum_{\mu \in \widetilde{K}} Q(M_\mu) V^K_\mu.\]
It is an easy matter to see that [P1] holds for the formal quantization process $Q^{-\infty}$. Let $N$ and $M$ be respectively pre-quantized proper Hamiltonian $K_1$ and $K_2$ manifolds: the product $M \times N$ is then a pre-quantized proper Hamiltonian $K_1 \times K_2$-manifold. For the reduced spaces we have $(M \times N)_{(\mu_1, \mu_2)} \simeq M_{\mu_1} \times N_{\mu_2}$ for all $\mu_1 \in \hat{K}_1$, $\mu_2 \in \hat{K}_2$. It follows then that
\begin{equation}
Q^{-\infty}_{K_1 \times K_2}(M \times N) = Q^{-\infty}_{K_1}(M) \hat{\otimes} Q^{-\infty}_{K_2}(N)
\end{equation}
in $R^{-\infty}(K_1 \times K_2) \simeq R^{-\infty}(K_1) \hat{\otimes} R^{-\infty}(K_2)$.

The purpose of this article is to show that the functorial property [P2] still holds for the formal quantization process $Q^{-\infty}$.

**Theorem 1.3.** Let $M$ be a pre-quantized Hamiltonian $K$-manifold which is proper. Let $H \subset K$ be a connected Lie subgroup such that $M$ is still proper as an Hamiltonian $H$-manifold. Then $Q^{-\infty}_K(M)$ is $H$-admissible and we have the following equality in $R^{-\infty}(H)$:
\begin{equation}
Q^{-\infty}_K(M)|_H = Q^{-\infty}_H(M).
\end{equation}

For $\mu \in \hat{K}$ and $\nu \in \hat{H}$ we denote $N^\mu_\nu = \dim(\text{hom}_H(V^H_\nu, V^K_\mu|_H))$ the multiplicity of $V^H_\nu$ in the restriction $V^K_\mu|_H$. In the situation of Theorem 1.3 the moment maps relative to the $K$ and $H$ actions are $\Phi_K$ and $\Phi_H = p \circ \Phi_K$, where $p : \mathfrak{t}^* \rightarrow \mathfrak{k}^*$ is the canonical projection.

**Corollary 1.4.** For every $\nu \in \hat{H}$, we have:
\begin{equation}
Q(M_{\nu,H}) = \sum_{\mu \in \hat{K}} N^\mu_\nu Q(M_{\mu,K}).
\end{equation}

where $M_{\nu,H} = \Phi^{-1}_H(H \cdot \nu)/H$ and $M_{\mu,K} = \Phi^{-1}_K(K \cdot \mu)/K$ are respectively the symplectic reductions relative to the $H$ and $K$-actions.

Since $V^K_\mu$ is equal to the $K$-quantization of $K \cdot \mu$, the “quantization commutes with reduction” Theorem tells us that $N^\mu_\nu = Q((K \cdot \mu)_{\nu,H})$ : in particular $N^\mu_\nu \neq 0$ implies that $\nu \in p(K \cdot \mu) \iff \mu \in K \cdot p^{-1}(\nu)$. Finally
\[ N^\mu_\nu Q(M_{\mu,K}) \neq 0 \iff \mu \in K \cdot p^{-1}(\lambda) \text{ and } \Phi^{-1}_K(\mu) \neq \emptyset. \]
Theses two conditions imply that we can restrict the sum of RHS of (1.4) to
\begin{equation}
\mu \in \hat{K} \cap \Phi_K(K \cdot \Phi^{-1}_H(\nu))
\end{equation}
which is finite since $\Phi_H$ is proper.

Theorem 1.3 and 1.4 gives the following extended version of [P3].

**Theorem 1.5.** Let $N$ and $M$ be two pre-quantized Hamiltonian $K$-manifold where $N$ is compact and $M$ is proper. The product $M \times N$ is then proper and we have the following equality in $R^{-\infty}(K)$:
\begin{equation}
Q^{-\infty}_K(M \times N) = Q^{-\infty}_K(M) \cdot Q_K(N)
\end{equation}

For $\mu, \lambda, \theta \in \hat{K}$ we denote $C^{\mu}_{\lambda,\theta} = \dim(\text{hom}_H(V^K_\mu, V^K_{\lambda} \otimes V^K_{\theta}))$ the multiplicity of $V^K_\mu$ in the tensor $V^K_{\lambda} \otimes V^K_{\theta}$. Since $V^K_\mu \otimes V^K_{\theta}$ is equal to the quantization of the product $K \cdot \lambda \times K \cdot \theta$, the “quantization commutes with reduction” Theorem tells us that $C^{\mu}_{\lambda,\theta} = Q((K \cdot \lambda \times K \cdot \theta)_{\mu})$ : in particular $C^{\mu}_{\lambda,\theta} \neq 0$ implies that
\[ \|\lambda\| \leq \|\theta\| + \|\mu\|. \]
Corollary 1.6. In the situation of Theorem 1.3 we have for every $\mu \in \hat{K}$:

$$Q((M \times N)_\mu) = \sum_{\lambda, \theta \in \hat{K}} C^\mu_{\lambda, \theta} Q(M_\lambda) Q(N_\theta).$$

Since $N$ is compact, $Q(N_\theta) \neq 0$ for a (**) $\theta \in \{\text{finite set}\}$. Then (*) and (**) show that the sum in the RHS of (1.7) is finite.

Weistman [33] studied the formal quantization procedure in the case of $K = U(n)$. Using a method of symplectic cutting [21, 35] he defines for two increasing sequence of positive integers $r_n, s_n$ a family of cut-spaces $M^{r_n,s_n}$ which are compact. Under the hypothesis that the cut spaces $M^{r_n,s_n}$ are smooth, he notes that $Q_K^{-\infty}(M) = \lim_{n \to \infty} Q_K(M^{r_n,s_n})$, and he was then able to show Theorem 1.5.

Our proof of Theorem 1.3 uses also a technique of symplectic cutting but which is valid for any compact Lie group actions. We have to overpass the difficulties concerning the non-smoothness of the cut-spaces. For this purpose we introduce another method of symplectic cutting that uses the wonderful compactifications of Concini-Procesi [13, 14], and we prove an extension of the “quantization commutes with reduction” Theorem to the singular setting.

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2. Quantization commutes with reduction

In this section we precise the definition of the quantization of a smooth and compact Hamiltonian manifold. We extend the definition to the case of a singular Hamiltonian manifold and we prove a “quantization commutes with reduction” Theorem in the singular setting.

In the Kostant-Souriau framework, an Hamiltonian $K$-manifold $(M, \Omega_M, \Phi_M)$ is pre-quantized if there is an equivariant Hermitian line bundle $L$ with an invariant Hermitian connection $\nabla$ such that

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi_M, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega_M,$$

for every $X \in \mathfrak{k}$. Here $X_M$ is the vector field on $M$ defined by $X_M(m) = \frac{d}{dt} e^{-tX} m|_0$. 

$(L, \nabla)$ is also called a Kostant-Souriau line bundle. Remark that the conditions (2.8) imply through the equivariant Bianchi formula the relation

$$i\langle X_M \rangle \Omega_M = -d\langle \Phi_M, X \rangle, \quad X \in \mathfrak{k}.$$

We will now recall the notions of geometric quantization.

2.1. Geometric quantization : the compact and smooth case. We suppose here that $(M, \Omega_M, \Phi_M)$ is compact and is prequantized by a Hermitian line bundle $L$. Choose a $K$-invariant almost complex structure $J$ on $M$ which is compatible with $\Omega_M$ in the sense that the symmetric bilinear form $\Omega_M(\cdot, J \cdot)$ is a Riemannian metric. Let $\overline{\mathcal{D}}_L$ be the Dolbeault operator with coefficients in $L$, and let $\overline{\mathcal{D}}'_L$ be its (formal) adjoint. The Dolbeault-Dirac operator on $M$ with coefficients in $L$ is $D_L = \overline{\mathcal{D}}_L + \overline{\mathcal{D}}'_L$, considered as an operator from $\mathcal{A}^{0,\text{even}}(M, L)$ to $\mathcal{A}^{0,\text{odd}}(M, L)$. 

Definition 2.1. The geometric quantization of $(M, \Omega, \Phi_M)$ is the element $Q_K(M) \in R(K)$ which is defined as the equivariant index of the Dolbeault-Dirac operator $D_L$.

Remark 2.2. • We can define the Dolbeault-Dirac operator $D^+_L$ for any invariant almost complex structure $J$. If $J_0$ and $J_1$ are equivariantly homotopic the indices of $D^+_L$ and $D^+_L$ coincides (see [28]).

• Since the set of compatible invariant almost complex structure on $M$ is path-connected, the element $Q_K(M) \in R(K)$ does not depend of the choice of $J$.

2.2. Geometric quantization: the compact and singular case. We are interested to defined the geometric quantization of singular compact Hamiltonian manifolds: here "singular" means that the manifold is obtain by symplectic reduction.

Let $(N, \Omega_N)$ be a smooth symplectic manifold equipped with an Hamiltonian action of $K \times H$: we denote $(\Phi_K, \Phi_H): N \to \mathfrak{t}^* \oplus \mathfrak{h}^*$ the corresponding moment map. We assume that $N$ is pre-quantized by a $K \times H$-equivariant line bundle $L$ and we suppose that the map $\Phi_H$ is proper. One wants to define the quantization of the (compact) symplectic quotient

$$N/\!\!/\theta H := \Phi_H^{-1}(0)/H.$$ 

When $0$ is a regular value of $\Phi_H$, $N/\!\!/\theta H$ is a compact symplectic orbifold equipped with an Hamiltonian action of $K$: the corresponding moment map is induced by the restriction of $\Phi_K$ to $\Phi_H^{-1}(0)$. The symplectic quotient $N/\!\!/\theta H$ is pre-quantized by the line orbibundle

$$L_0 := \left( L|_{\Phi_H^{-1}(0)} \right)/H.$$ 

Definition 2.1 extends to the orbifold case, so one can still defined the quantization of $N/\!\!/\theta H$ as an element $Q_K(N/\!\!/\theta H) \in R(K)$.

Suppose now that $0$ is not a regular value of $\Phi_H$. Let $T_H$ be a maximal torus of $H$, and let $C_H \subset \mathfrak{t}_H^*$ be a weyl chamber. Since $\Phi_H$ is proper, the convexity theorem says that the image of $\Phi_H$ intersects $C_H$ in a closed locally polyhedral convex set, that we denoted $\Delta_H(M)$ [22].

We consider an element $a \in \Delta_H(M)$ which is generic and sufficiently closed to $0 \in \Delta_H(M)$: we denote $H_a$ the subgroup of $H$ which stabilizes $a$. When $a \in \Delta_H(M)$ is generic, one can shows (see [25]) that

$$N/\!\!/\theta a H := \Phi_H^{-1}(a)/H_a$$ 

is a compact $K$-Hamiltonian orbifold, and that

$$L_a := \left( L|_{\Phi_H^{-1}(a)} \right)/H_a.$$ 

is a $K$-equivariant line orbibundle over $N/\!\!/\theta a H$: we can then define like in Definition 2.1 the element $Q_K(N/\!\!/\theta a H) \in R(K)$ as the equivariant index of the Dolbeault-Dirac operator on $N/\!\!/\theta a H$.

Proposition-Definition 2.3. The elements $Q_K(N/\!\!/\theta a H) \in R(K)$ do not depend of the choice of the generic element $a \in \Delta_H(M)$, when $a$ is sufficiently closed to $0$. The common value will be taken as the geometric quantization of $N/\!\!/\theta H$, and still denoted $Q_K(N/\!\!/\theta H)$.

Proof. When $N$ is compact and $K = \{e\}$, the proof can be founded in [25] and in [28]. The $K$-theoretic proof of [28] extends naturally to our case. □
2.3. Quantization commutes with reduction: the singular case. In section 2.2 we have defined the geometric quantization $Q_K(N/\mathcal{H}) \in R(K)$ of a compact symplectic reduced space $N/\mathcal{H}$. We will compute its $K$-multiplicities like in Theorem 2.1.

For every $\mu \in \hat{K}$, we consider the coadjoint orbit $K \cdot \mu \simeq K/K_\mu$ which is pre-quantized by the line bundle $\mathbb{C}_{[\mu]} \simeq K \times_{K_\mu} \mathbb{C}_\mu$. We consider the product $N \times K \cdot \mu$ which is an Hamiltonian $K \times H$ manifold which is pre-quantized by $K \times H$-equivariant line bundle $L \otimes \mathbb{C}_{[\mu]}^{-1}$. The moment map $N \times K \cdot \mu \to \mathfrak{k}^* \times \mathfrak{h}^*, (n, \xi) \to (\Phi_K(n) - \xi, \Phi_H(n))$ is proper, so the reduced space

$$(N/\mathcal{H})_\mu := (N \times K \cdot \mu)/(0,0)K \times H$$

is compact. Following Proposition 2.3 we can then define its quantization $Q((N/\mathcal{H})_\mu) \in \mathbb{Z}$. The main result of this section is the

**Theorem 2.4.** We have the following equality in $R(K)$:

$$(2.10) \quad Q_K(N/\mathcal{H}) = \sum_{\mu \in \hat{K}} Q((N/\mathcal{H})_\mu) V^K_\mu.$$

**Proof.** The proof will occupied the remaining of this section. The starting point is to state another definition of the geometric quantization of a symplectic reduced space which uses the Atiyah-Singer’s theory of transversally elliptic operators.

2.3.1. Transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [8, 9] and for a short introduction see [28].

Let $\mathcal{X}$ be a compact $K_1 \times K_2$-manifold. Let $p: T\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-,-)_X$ be a $K_1 \times K_2$-invariant Riemannian metric. If $E^0, E^1$ are $K_1 \times K_2$-equivariant vector bundles over $X$, a $K_1 \times K_2$-equivariant morphism $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$ is called a symbol. The subset of all $(x, v) \in T\mathcal{X}$ where $\sigma(x, v): E^0_x \to E^1_v$ is not invertible is called the characteristic set of $\sigma$, and is denoted by $\text{Char}(\sigma)$.

Let $T_{K_2}\mathcal{X}$ be the following subset of $T\mathcal{X}$:

$$T_{K_2}\mathcal{X} = \{(x, v) \in TM, (v, X_M(x))_M = 0 \text{ for all } X \in \mathfrak{g}_2\}.$$

A symbol $\sigma$ is elliptic if $\sigma$ is invertible outside a compact subset of $T\mathcal{X}$ (i.e. $\text{Char}(\sigma)$ is compact), and is $K_2$-transversally elliptic if the restriction of $\sigma$ to $T_{K_2}\mathcal{X}$ is invertible outside a compact subset of $T_{K_2}\mathcal{X}$ (i.e. $\text{Char}(\sigma) \cap T_{K_2}\mathcal{X}$ is compact).

An elliptic symbol $\sigma$ defines an element in the equivariant $K$-theory of $T\mathcal{X}$ with compact support, which is denoted by $K_{K_1 \times K_2}(T\mathcal{X})$, and the index of $\sigma$ is a virtual finite dimensional representation of $K_1 \times K_2$ [3, 4, 7, 6].

A $K_2$-transversally elliptic symbol $\sigma$ defines an element of $K_{K_1 \times K_2}(T_{K_2}M)$, and the index of $\sigma$ is defined as a trace class virtual representation of $K_1 \times K_2$ (see [1] for the analytic index and [4, 8] for the cohomological one): in fact $\text{Index}^\mathcal{X}(\sigma)$ belongs to the tensor product $R(K_1) \otimes R^{-\infty}(K_2)$.

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1$K \cdot \mu$ denotes the coadjoint orbit with the opposite symplectic form.
Remark that any elliptic symbol of $T\mathcal{X}$ is $K_2$-transversally elliptic, hence we have a restriction map $K_{K_1 \times K_2}(T\mathcal{X}) \to K_{K_1 \times K_2}(T\mathcal{K}_2\mathcal{X})$, and a commutative diagram

\[
\begin{array}{ccc}
K_{K_1 \times K_2}(T\mathcal{X}) & \xrightarrow{\text{Index}^\mathcal{X}} & K_{K_1 \times K_2}(T\mathcal{K}_2\mathcal{X}) \\
\downarrow & & \downarrow \\
R(K_1) \otimes R(K_2) & \xrightarrow{\text{Index}^\mathcal{X}} & R(K_1) \otimes R^{-\infty}(K_2)
\end{array}
\]

Using the excision property, one can easily show that the index map $\text{Index}^\mathcal{U} : K_{K_1 \times K_2}(T\mathcal{K}_2\mathcal{U}) \to R(K_1) \otimes R^{-\infty}(K_2)$ is still defined when $\mathcal{U}$ is a $K_1 \times K_2$-invariant relatively compact open subset of a $K_1 \times K_2$-manifold (see [23][section 3.1]).

2.3.2. Quantization of singular space : second definition. Let $(\mathcal{X}, \Omega_\mathcal{X})$ be an Hamiltonian $K_1 \times K_2$-manifold pre-quantized by a $K_1 \times K_2$-equivariant line bundle $L$. The moment map $\Phi_2 : \mathcal{X} \to \mathfrak{t}_2^*$ relative to the $K_2$-action is supposed to be proper. Take a compatible $K_1 \times K_2$-invariant almost complex structure on $\mathcal{X}$. We choose a $K_1 \times K_2$-invariant Hermitian metric $\|v\|^2$ on the tangent bundle $T\mathcal{X}$, and we identify the cotangent bundle $T^*\mathcal{X}$ with $T\mathcal{X}$. For $(x, v) \in T\mathcal{X}$, the principal symbol of the Dolbeault-Dirac operator $\overline{\partial}_L + \partial^*_L$ is the Clifford multiplication $c_x(v)$ on the complex vector bundle $\Lambda^*_{x\mathcal{X}} \otimes L_x$. It is invertible for $v \neq 0$, since $c_x(v)^2 = -\|v\|^2$.

When $\mathcal{X}$ is compact, the symbol $c_x$ is elliptic and then defines an element of the equivariant $K$-group of $T\mathcal{X}$. The topological index of $c_x \in K_{K_1 \times K_2}(T\mathcal{X})$ is equal to the analytical index of the Dolbeault-Dirac operator $\overline{\partial}_L + \partial^*_L$:

\[
(2.12) \quad \mathcal{Q}_{K_1 \times K_2}(\mathcal{X}) = \text{Index}^\mathcal{X}(c_x) \quad \text{in} \quad R(K_1) \otimes R(K_2).
\]

When $\mathcal{X}$ is not compact the topological index of $c_x$ is not defined. In order to give a topological definition of $\mathcal{Q}_{K_1}(\mathcal{X}/0K_2)$, we will deform the symbol $c_x$ as follows. Consider the identification $\mathfrak{t}_2^* \simeq \mathfrak{t}_2$ defined by a $K_2$-equivariant scalar product on the Lie algebra $\mathfrak{t}_2$. From now on the moment map $\Phi_2$ will take values in $\mathfrak{t}_2$, and we define the vectors field on $\mathcal{X}$

\[
(2.13) \quad \kappa_x = (\Phi_2(x))_{\mathcal{M}}(x), \quad x \in \mathcal{X}.
\]

We consider now the symbol

$$c^{\kappa_x}_x(v) = c(v - \kappa_x), \quad v \in T_x\mathcal{X}.$$  

Note that $c^{\kappa_x}_x(v)$ is invertible except if $v = \kappa_x$. If furthermore $v$ belongs to the subset $T_{K_2}M$ of tangent vectors orthogonal to the $K_2$-orbits, then $v = 0$ and $\kappa_x = 0$. Indeed $\kappa_x$ is tangent to $K_2 \cdot x$ while $v$ is orthogonal.

Since $\kappa$ is the Hamiltonian vector field of the function $\|\Phi_2\|^2$, the set of zeros of $\kappa$ coincides with the set $\text{Cr}([\|\Phi_2\|^2])$ of critical points of $\|\Phi_2\|^2$.

Let $\mathcal{U} \subset \mathcal{X}$ be a $K_1 \times K_2$-invariant open subset which is relatively compact. If the border $\partial \mathcal{U}$ does not intersect $\text{Cr}([\|\Phi_2\|^2])$, then the restriction $c^{\kappa_x}_{\mathcal{U}}|_{\mathcal{U}}$ defines a class in $K_{K_1 \times K_2}(T\mathcal{K}_2\mathcal{U})$ since

$$\text{Char}(c^{\kappa_x}_{\mathcal{U}}|_{\mathcal{U}}) \cap T\mathcal{K}_2\mathcal{U} \simeq \text{Cr}([\|\Phi_2\|^2]) \cap \mathcal{U}$$

is compact. In this situation the index of $c_{\mathcal{U}}$ is defined as an element $\text{Index}^\mathcal{U}(c_{\mathcal{U}}|_{\mathcal{U}}) \in R(K_1) \otimes R^{-\infty}(K_2)$.

**Theorem 2.5.** The $K_2$-invariant part of $\text{Index}^\mathcal{U}(c^{\kappa_x}_{\mathcal{U}}|_{\mathcal{U}})$ is equal to:

- $\mathcal{Q}_{K_1}(\mathcal{X}/0K_2)$ when $\Phi_2^{-1}(0) \subset \mathcal{U}$,
• 0 in the other case.

Proof. When \( K_1 = \{e\} \), the proof is done in [28] (see section 7). This proof works equally well in the general case.

Remark 2.6. If \( \mathcal{X} \) is compact we can take \( \mathcal{U} = \mathcal{X} \) in the last Theorem. In this case the symbols \( c^*_\mathcal{X} \) and \( c_\mathcal{X} \) defines the same class in \( K_{K_1 \times K_2}(T\mathcal{X}) \) so they have the same index. Theorem 2.5 corresponds then to the traditional “quantization commutes with reduction” phenomenon: \( \{ Q_{K_1 \times K_2}(\mathcal{X}) \}^{K_2} = Q_{K_1}(\mathcal{X} \parallel_0 K_2) \).

From now on we will work with this topological definiton for the geometric quantization of the reduced \( K_1 \)-Hamiltonian manifold \( \mathcal{X} \parallel_0 K_2 \) (which is possibly singular): \( Q_{K_1}(\mathcal{X} \parallel_0 K_2) = \{ \text{Index}^\mathcal{U}(c^*_\mathcal{X}|_\mathcal{U}) \}^{K_2} \) where \( \mathcal{U} \) is any relatively compact neighborhood of \( \Phi_2^{-1}(0) \) such that \( \partial \mathcal{U} \cap \text{Cr}(\|\Phi_2\|^2) = \emptyset \). The functorial properties still holds in this singular setting. In particular:

[2] If \( H \subset K_1 \) is a connected lie subgroup, then the restriction of \( Q_{K_1}(\mathcal{X} \parallel_0 K_2) \) to \( H \) is equal to \( Q_H(\mathcal{X} \parallel_0 K_2) \).

2.3.3. Proof of theorem 2.4. We come back in the situation of sections 2.2 and 2.3.

First we apply Theorem 2.5 to \( \mathcal{X} = N, K_1 = K \) and \( K_2 = H \). (2.10) is trivially true when \( 0 \notin \text{Image}(\Phi_H) \). So we suppose now that \( 0 \in \text{Image}(\Phi_H) \), and we consider an \( K \times H \)-invariant open subset \( \mathcal{U} \subset N \) which is relatively compact and such that \( \Phi_H^{-1}(0) \subset \mathcal{U} \) and \( \partial \mathcal{U} \cap \text{Cr}(\|\Phi_H\|^2) = \emptyset \).

We have \( Q_K(N \parallel_0 H) = \{ \text{Index}^\mathcal{U}(c^*_N|_\mathcal{U}) \}^H \) and one want to compute its \( K \)-multiplicities \( m_\mu, \mu \in \hat{K} \). Here \( \kappa_\mu \) is the vectors field on \( N \) associated to the moment map \( \Phi_H \) (see (2.13)).

Take \( \mu \in \hat{K} \). We denote \( c_{-\mu} \) the principal symbol of the Dolbeaut-Dirac operator on \( \overline{K \cdot \mu} \) with values in the line bundle \( \mathcal{C}_{-\mu} \) : we have \( \text{Index}^{K \cdot \mu}(c_{-\mu}) = (V^K_\mu)^* \).

We know then that the multiplicity of \( \{ \text{Index}^\mathcal{U}(c^*_N|_\mathcal{U}) \}^H \) relatively to \( V^K_\mu \) is equal to

\[
m_\mu := \left[ \text{Index}^\mathcal{V}(c^*_N|_\mathcal{U} \circ c_{-\mu}) \right]^{K \times H}
\]

with \( \mathcal{V} = \mathcal{U} \times K \cdot \mu \). This identity is due to the fact that we have a ”multiplication”

\[
K_{K \times H}(T\mathcal{U}) \times K_K(T(K \cdot \mu)) \longrightarrow K_{K \times H}(T_{K \times H}(\mathcal{U} \times K \cdot \mu))
\]

\[
(\sigma_1, \sigma_2) \mapsto \sigma_1 \circ \sigma_2.
\]

so that \( \text{Index}^{K \times K \cdot \mu}(\sigma_1 \circ \sigma_2) = \text{Index}^H(\sigma_1) \cdot \text{Index}^{K \cdot \mu}(\sigma_2) \) in \( R^{-\infty}(K \times H) \). See [1].

Consider now the case where \( \mathcal{X} = N \times K \cdot \mu, K_1 = \{e\} \) and \( K_2 = K \times H \). After Theorem 2.5 we know that

\[
Q((N \parallel_0 H)_\mu) = \left[ \text{Index}^\mathcal{V}(c^*_\mathcal{X}|_\mathcal{V}) \right]^{K \times H},
\]

where \( \kappa \) is the vector field on \( N \times \overline{K \cdot \mu} \) associated to the moment map

\[
\Phi : N \times \overline{K \cdot \mu} \longrightarrow \mathfrak{k}^* \times \mathfrak{h}^*
\]

\[
(x, \xi) \longmapsto (\Phi_K(x) - \xi, \Phi_H(n))
\]

Note that \( \mathcal{V} = \mathcal{U} \times K \cdot \mu \) is a neighborhood of \( \Phi^{-1}(0) \subset (\Phi_H)^{-1}(0) \).
Our aim now is to prove that the quantities (2.14) and (2.15) are equal.

Since the definition of $\kappa$ needs the choice of an invariant scalar product on the Lie algebra $\mathfrak{k} \times \mathfrak{h}$, we will precise its definition. Let $\|\cdot\|_K$ and $\|\cdot\|_H$ be two invariant Euclidean norm respectively on $\mathfrak{k}$ and $\mathfrak{h}$. For any $r > 0$ we consider on $\mathfrak{k} \times \mathfrak{h}$ the invariant Euclidean norm $\|(X, Y)\|_r^2 = r^2\|X\|_K^2 + \|Y\|_H^2$.

Let $\kappa^K$ be the vector field on $N \times \overline{K \cdot \mu}$ associated to the map $N \times \overline{K \cdot \mu} \to \mathfrak{k}^*, (x, \xi) \mapsto \Phi_K(x) - \xi$, and where the identification $\mathfrak{k} \simeq \mathfrak{k}^*$ is made through the Euclidean norm $\|\cdot\|_K$ (see (2.13)). For $(x, \xi) \in N \times \overline{K \cdot \mu}$, we have the decomposition

$$\kappa^K(x, \xi) = (\kappa_1(x, \xi), \kappa_2(x, \xi)) \in T_xN \times T_xK \cdot \mu.$$

Let $\kappa^H$ be the vector field on $N \times \overline{K \cdot \mu}$ associated to the map $N \times \overline{K \cdot \mu} \to \mathfrak{h}^*, (x, \xi) \mapsto \Phi_H(x)$, and where the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ is made through the Euclidean norm $\|\cdot\|_H$. For $(x, \xi) \in N \times \overline{K \cdot \mu}$, we have the decomposition

$$\kappa^H(x, \xi) = (\kappa^H(x), 0) \in T_xN \times T_xK \cdot \mu.$$

For any $r > 0$, we denote by $\kappa_r$ the vector field on $N \times \overline{K \cdot \mu}$ associated to the map $\Phi$, and where the identification $\mathfrak{k} \times \mathfrak{h} \simeq \mathfrak{k}^* \times \mathfrak{h}^*$ is made through the Euclidean norm $\|\cdot\|_r$. We have then

$$\kappa_r = \kappa^H + r \kappa^K$$

$$= (\kappa^H + r \kappa_1, r \kappa_2)$$

Now we can precise (2.15). Take an invariant relatively compact neighborhood $\mathcal{U}$ of $\Phi^H_\mu(0)$ such that $\partial \mathcal{U} \cap \{\text{zeros of } \kappa^H\} = \emptyset$. With the help of a invariant Riemannian metric on $X$ we define

$$\varepsilon_H = \inf_{x \in \partial \mathcal{U}} \|\kappa^H(x)\| > 0 \text{ and } \varepsilon_K = \sup_{(x, \xi) \in \partial \mathcal{U} \times K \cdot \mu} \|\kappa_1(x, \xi)\|.$$

Note that for any $0 \leq r < \frac{\varepsilon_H}{\varepsilon_K}$, we have $\partial \mathcal{U} \times K \cdot \mu \cap \{\text{zeros of } \kappa^H + r \kappa_1\} = \emptyset$, and then $\partial \mathcal{V} \cap \{\text{zeros of } \kappa_r\} = \emptyset$ for the neighborhood $\mathcal{V} := \mathcal{U} \times K \cdot \mu$ of $\Phi^{-1}(0)$. We can then use Theorem 2.9: for $0 < r < \frac{\varepsilon_H}{\varepsilon_K}$ we have

$$Q((N \parallel_0 H)_\mu) = \left[\text{Index}^\mathcal{V}(c^r_X|_\mathcal{V})\right]^{K \times H}.$$

We are now close to the end of the proof. Let us compare the symbols $c^r_N|_\mathcal{V}$ and $c_N^H|_\mathcal{U} \circ c_{-\mu}$ in $K_{K \times H}(T_{K \times H}(\mathcal{U} \times K \cdot \mu))$. First one sees that the symbols $c_X^r$ is equal to the product $c_N \circ c_{-\mu}$ hence the symbols $c_N^H|_\mathcal{U} \circ c_{-\mu}$ is equal to $c_X^r|_\mathcal{V}$ when $r = 0$. Since for $r < \frac{\varepsilon_H}{\varepsilon_K}$, the path $s \in [0, r] \to c_X^r|_\mathcal{V}$ defines an homotopy of $K \times H$-transversally elliptic symbols on $\mathcal{V}$, we get

$$\text{Index}^\mathcal{V}(c_X^r|_\mathcal{V}) = \text{Index}^\mathcal{V}(c_N^H|_\mathcal{U} \circ c_{-\mu})$$

and then $m_\mu = Q((N \parallel_0 H)_\mu)$. □

3. Wonderful compactifications and symplectic cutting

In this section we use the wonderful compactifications of Concini-Procesi [13, 14] to perform symplectic cutting.
3.1. Wonderful compactifications: definitions. We study here the wonderfull compactifications from the Hamiltonian point of view.

We consider a compact connected Lie group $K$ and its complexification $K_C$. Let $T$ be a maximal torus of $K$, and let $W := N(T)/T$ be the Weyl group. Consider $t^*$, the dual of the Lie algebra of $T$, with the lattice $\wedge^*$ of real weights. Let $C_K \subset t^*$ be a Weyl chamber and let $\hat{K} := \wedge^* \cap C_K$ be the set dominants weights.

**Definition 3.1.** A polytope $P$ in $t^*$ is $K$-adapted if:

i) the vertices of $P$ are regular elements of $\wedge^*$,

ii) $P$ is $W$-invariant,

iii) $P$ is Delzant.

**Example:** When $K$ as a trivial center, the convex hull of $W \cdot \mu$ is a $K$-adapted polytope for any regular dominant weight $\mu$.

**Proposition 3.2.** There exists $K$-adapted polytopes in $t^*$.

**Proof.** Let us use the dictionary between polytopes and projective fan \cite{27}. Conditions ii) and iii) of Definition 3.1 means that we are looking after a smooth projective $W$-invariant fan $\mathcal{F}$ in $t$. Condition i) means that each cone of $\mathcal{F}$ of maximal dimension should not be fixed by any element of $W \setminus \{\text{Id}\}$. For a proof of the existence of search fan, see \cite{11, 12}. In particular condition $(\ast)$ in Proposition 2 of \cite{12} implies i). □

In the rest of this section, we consider a $K$-adapted polytope $P$. Let $[P]_1$ be the union of all the closed facet of dimension 1: we label the elements of $[P]_1 \cap \hat{K}$ by $\{\lambda_1, \cdots, \lambda_N\}$. Since $P$ is $K$-adapted, when $\lambda_i$ is a vertex of $P$ there exits $\alpha_{j_1}, \cdots, \alpha_{j_r}$ belonging to $[P]_1 \cap \wedge^* = W \cdot \{\lambda_1, \cdots, \lambda_N\}$ such that $\alpha_{j_1} - \lambda_i, \cdots, \alpha_{j_r} - \lambda_i$ is a basis of the lattice $\wedge^*$.

Let $V_{\lambda_i}$ be an irreducible representation of $K$ with highest weight $\lambda_i$: these representations extend canonically to the complexification $K_C$. We denote $\rho : K_C \rightarrow \Pi_{i=1}^N GL(V_{\lambda_i})$ the representation of $K_C$ on $V := \oplus_{i=1}^N V_{\lambda_i}$. We consider the vector space

$$E = \oplus_{i=1}^N \text{End}(V_{\lambda_i})$$

equipped with the action of $K_C \times K_C$ given by: $(g_1, g_2) \cdot f = \rho(g_1) \circ f \circ \rho(g_2)^{-1}$. Let $\mathbb{P}(E)$ the projective space associated to $E$: it is equipped with an algebraic action of the reductive group $K_C \times K_C$. We consider the map $g \mapsto [\rho(g)]$ from $K_C$ into $\mathbb{P}(E)$, that we denote $\bar{\rho}$.

**Lemma 3.3.** The map $\bar{\rho} : K_C \rightarrow \mathbb{P}(E)$ is an embedding.

**Proof.** Let $g \in K_C$ such that $\bar{\rho}(g) = [\text{Id}]$: there exist $a \in \mathbb{C}^*$ such that $\rho(g) = a \text{Id}$. The Cartan decomposition gives

$$\rho(k) = \frac{a}{|a|} \text{Id} \quad \text{and} \quad \rho(e^{iX}) = |a| \text{Id}$$

for $g = ke^{iX}$ with $k \in K$ and $X \in \mathfrak{k}$. Since there exist $Y, Y' \in \mathfrak{t}$ and $u, u' \in K$ such that $k = ue^Y u^{-1}$ and $X = u' \cdot Y'$, \eqref{3.16} gives

$$\rho(e^Y) = \frac{a}{|a|} \text{Id} \quad \text{and} \quad \rho(e^{iY'}) = |a| \text{Id}. \tag{3.17}$$
Since each element of \([P_1] \cap \wedge^* = W \cdot \{\lambda_1, \cdots, \lambda_N\}\) is a weight for the action of \(T_C\) on \(\oplus_{i=1}^N V_{\lambda_i}\), (3.17) implies that for every \(\alpha \in [P_1] \cap \wedge^*\) we have

\[
e^{i(\alpha,Y)} = \frac{a}{|a|} \quad \text{and} \quad e^{-i(\alpha,Y')} = |a|.
\]

Since there exists \(\alpha_{j_0}, \cdots, \alpha_{j_r} \in [P_1] \cap \wedge^*\) such that \(\alpha_{j_1} - \alpha_{j_0}, \cdots, \alpha_{j_r} - \alpha_{j_0}\) is a basis of the lattice \(\wedge^*\), (3.18) implies that \(Y' = 0\) and that \(Y \in \ker(Z \subset t \to e^Z)\).

We have proved that \(g = e\). \(\square\)

Let \(T_C \subset K_C\) the complexification of the (compact) torus \(T \subset K\).

**Definition 3.4.** Let \(X_P\) be the Zariski closure of \(\bar{\rho}(K_C)\) in \(\mathbb{P}(E)\) and let \(\mathcal{Y}_P \subset X_P\) be the Zariski closure of \(\bar{\rho}(T_C)\) in \(\mathbb{P}(E)\).

Since \(\bar{\rho}(K_C) = K_C \times K_C \cdot [Id]\) and \(\bar{\rho}(T_C) = T_C \times T_C \cdot [Id]\) are orbits of algebraic group actions their Zariski closures coincide with their closures for the Euclidean topology.

**Theorem 3.5.** The varieties \(X_P\) and \(\mathcal{Y}_P\) are smooth.

The proof will be done in the next section.

3.2. **Smoothness of \(X_P\) and \(\mathcal{Y}_P\).**

Let \(E\) be a complex vector space equipped with a linear action of a reductive group \(G\). Let \(Z \subset \mathbb{P}(E)\) be a projective variety which is \(G\)-stable. We have the classical fact

**Lemma 3.6.**
- \(Z\) possess closed \(G\)-orbits.
- \(Z\) is smooth if \(Z\) is smooth near its closed \(G\)-orbits.

**Proof.** Let \(z_0 \in Z\) and consider the Zariski closure \(\overline{G \cdot z_0} \subset Z\). If \(G \cdot z_0\) is not closed, we take \(z_1 \in \overline{G \cdot z_0} \setminus G \cdot z_0\): we have \(\dim G \cdot z_1 < \dim G \cdot z_0\). By induction we find a sequence \(z_1, \cdots, z_p\) with \(z_{k+1} \in \overline{G \cdot z_k} \setminus G \cdot z_k\) for \(k < p\) and \(G \cdot z_p\) closed.

For the second point, we have just to note that if \(Z\) is singular, the subvariety \(Z^{sing} \subset Z\) of singular points is \(G\)-stable and then contains a closed \(G\)-orbits. \(\square\)

We are interested here respectively in
- the \(K_C \times K_C\)-variety \(X_P \subset \mathbb{P}(E) \subset \mathbb{P}(\text{End}(V))\)
- the \(T_C \times T_C\)-variety \(\mathcal{Y}_P \subset \mathbb{P}(E)\).

Since the diagonal \(Z_C = \{(t,t)|t \in T_C\}\) stabilizes \([Id]\), its action on \(\mathcal{Y}_P\) is trivial. Hence we will restrict ourself to the action of \(T_C \times T_C/Z_C \cong T_C\) on \(\mathcal{Y}_P\) : for \(t \in T_C\) and \([y] \in \mathcal{Y}_P\) we take \(t \cdot [y] = [\rho(t) \circ y]\).

3.2.1. **The case of \(\mathcal{Y}_P\).**

We apply Lemma 3.6 to the \(T_C\)-variety \(\mathcal{Y}_P = \overline{T_C \cdot [Id]}\) in \(\mathbb{P}(E)\). Let \(\{\alpha_j, j \in J\}\) be the \(T_C\) weights on \((\oplus_{i=1}^N V_{\lambda_i}, \rho)\) counted with their multiplicity. Their exists an orthonormal basis \(\{v_j, j \in J\}\) of \(\oplus_{i=1}^N V_{\lambda_i}\) such that \(Id = \sum_{j \in J} v_j \otimes v_j^*\) and

\[
\rho(e^Z) = \sum_{j \in J} e^{i(\alpha_j,Z)} v_j \otimes v_j^*, \quad Z \in tC.
\]

So the action of \(e^Z \in T_C\) on \([Id] \in \mathbb{P}(E)\) is \(e^Z \cdot [Id] = \left[\sum_{j \in J} e^{i(\alpha_j,Z)} v_j \otimes v_j^*\right]\). We introduce a subset \(J'\) of \(J\) such that for every \(j \in J\) there exists a unique \(j' \in J'\) such that \(\alpha_j = \alpha_{j'}\). So the variety \(\mathcal{Y}_P\) belongs to \(\mathbb{P}(E')\) where \(E' = \oplus_{j' \in J'} \mathbb{C} m_{j'}\) with \(m_{j'} = \sum_{j, \alpha_j = \alpha_{j'}} v_j \otimes v_j^*\). The closed \(T_C\) orbits in \(\mathbb{P}(E')\) are \([m_{j'}], j' \in J\).
Lemma 3.7. $[m_{j_o}] \in \mathcal{Y}_P$ if and only if $\alpha_{j_o}$ is a vertex of the polytope $P$.

Proof. If $\alpha_{j_o}$ is a vertex of $P$, there exists $X \in \mathfrak{t}$ such that $\langle \alpha_{j_o}, X \rangle > 0$ whenever $\alpha_{j_o} \neq \alpha_j$. Hence $e^{-i s X \cdot \alpha_{j_o}} \cdot [\text{Id}]$ tends to $[m_{j_o}]$ when $s \to +\infty$. If $\alpha_{j_o}$ is not a vertex of $P$, there exist $L \subset J \setminus \{j_o\}$ such that $\alpha_{j_o} = \sum_{l \in L} a_l \alpha_l$ with $0 < a_l < 1$ and $\sum a_l = 1$. So $\mathcal{Y}_P$ belongs to the closed subset defined by
\[
\left[ \sum_{j \in J'} \delta_{j} m_{j} \right] \in \mathbb{P}(E') \quad \text{such that} \quad \Pi_{l \in L} |\delta_l| = |\delta_{j_o}|.
\]
Hence $[m_{j_o}] \notin \mathcal{Y}_P$. □

Remark 3.8. When $\alpha_j$ is a vertex of the polytope $P$, the multiplicity of $\alpha_j$ in $\bigoplus_{i=1}^N V_{\lambda_i}$ is equal to one, so $m_j = v_j \otimes v_j^*$.

Consider now a vertex $\alpha_{j_o}$ of $P$ (for $j_o \in J'$). We consider the open subset $\mathcal{V} \subset \mathbb{P}(E')$ defined by $\left[ \sum_{j' \in J'} \delta_{j'} m_{j'} \right] \in \mathcal{V} \Leftrightarrow \delta_{j_o} \neq 0$, and the diffeomorphism $\psi : \mathcal{V} \to \mathbb{C}^{J' \setminus \{j_o\}}$, $\left[ \sum_{j' \in J'} \delta_{j'} m_{j'} \right] \mapsto (\frac{\delta_{j'}}{\delta_{j_o}})_{j' \neq j_o}$. The map $\psi$ realizes an algebraic diffeomorphism between $\mathcal{Y}_P \cap \mathcal{V}$ and the affine subvariety
\[
\mathcal{Z} := \{(t_{j'} - \alpha_{j_o})_{j' \neq j_o} \mid t \in T_{\mathbb{C}}\} \subset \mathbb{C}^{J' \setminus \{j_o\}}.
\]

The set of weights $\alpha_j$, $j \in J$ contains all the lattice points that belongs to the one dimensional faces of $P$. Since the polytope $P$ is $K$-adapted, there exists a subset $L_{j_o} \subset J'$ such that $\alpha_l - \alpha_{j_o}$, $l \in L_{j_o}$ is a $\mathbb{Z}$-basis of the group of weights $\Lambda^*$. And for every $j' \neq j_o$ we have
\[
(3.20) \quad \alpha_{j'} - \alpha_{j_o} = \sum_{l \in L_{j_o}} n_{j'}^l (\alpha_l - \alpha_{j_o}) \quad \text{with} \quad n_{j'}^l \in \mathbb{N}.
\]

We define on $\mathbb{C}^{L_{j_o}}$ the monomials $P_{j'}(Z) = \Pi_{l \in L_{j_o}} (Z_l)^{n_{j'}^l}$. Note that $P_{j'}(Z) = Z_l$ when $j' = l \in L_{j_o}$. Now it is not difficult to see that the map
\[
\begin{array}{ccc}
\mathbb{C}^{L_{j_o}} & \to & \mathbb{C}^{J' \setminus \{j_o\}} \\
Z & \mapsto & (P_{j'}(Z))_{j' \neq j_o}
\end{array}
\]
realizes an algebraic diffeomorphism between $\mathcal{C}^{L_{j_o}}$ and $\mathcal{Z}$.

Finally we have shown that $\mathcal{Y}_P$ is smooth near $[m_{j_o}]$; hence $\mathcal{Y}_P$ is a smooth subvariety of $\mathbb{P}(E)$. Since $T_{\mathbb{C}}$ acts on $\mathcal{Y}_P$ with a dense orbit, $\mathcal{Y}_P$ is a smooth projective toric variety.

3.2.2. The case of $\mathcal{X}_P$. Let $E := \bigotimes_{i=1}^N \text{End}(V_{\lambda_i})$. The closed $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbit in $\mathbb{P}(E)$ are those passing through $[\lambda_i \otimes \lambda^*_i]$ where $\lambda_i \in V_{\lambda_i}$ is a highest weight vector (that we take of norm 1 for a $K$-invariant hermitian structure).

Lemma 3.9. $[\lambda_i \otimes \lambda^*_i] \in \mathcal{X}_P$ if and only if $\lambda_i$ is a vertex of the polytope $P$.

Proof. If $\lambda_i$ is a vertex of $P$, we have proved in Lemma 3.7 that $[\lambda_i \otimes \lambda^*_i]$ belongs to $\mathcal{Y}_P$ and so belongs to $\mathcal{X}_P$. We prove the converse in Corollary 3.11. □

For the remaining of this section we consider a vertex $\lambda_i \in \hat{K}$ of the polytope $P$. Let $B^+, B^-$ the Borel subgroups fixing respectively the elements $[\lambda_i \otimes \lambda^*_i] \in \mathbb{P}(V_{\lambda_i})$ and $[\lambda^*_i] \in \mathbb{P}(V_{\lambda_i}^*)$. Consider also the unipotent subgroups $N^\pm \subset B^\pm$ fixing respectively the elements $v_{\lambda_i} \in V_{\lambda_i}$ and $v^*_i \in V_{\lambda_i}^*$. 
We consider the open subset $\mathcal{V}_{\text{End}} \subset \mathbb{P}(E)$ of elements $[f]$ such that $\langle \nu_{\lambda_o}^*, f(\nu_{\lambda_o}) \rangle \neq 0$: $\mathcal{V}_{\text{End}}$ is $B^- \times B^+$ stable. Consider the open subset $\mathcal{V} \subset \mathbb{P}(V_{\lambda_o})$ and $\mathcal{V}^* \subset \mathbb{P}(V_{\lambda_o}^*)$ defined by:

- $[v] \in \mathcal{V} \iff \langle \nu_{\lambda_o}^*, v \rangle \neq 0 : \mathcal{V}$ is $B^-$ stable,
- $[\xi] \in \mathcal{V}^* \iff \langle \xi, \nu_{\lambda_o} \rangle \neq 0 : \mathcal{V}^*$ is $B^+$ stable.

We consider now the rational maps $l : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_o})$, $f \mapsto f(\nu_{\lambda_o})$ and $r : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_o}^*)$, $f \mapsto \nu_{\lambda_o}^* \circ f$. The map $l$ and $r$ are defined on $\mathcal{V}_{\text{End}}$ : they defined respectively $B^-$-equivariant map from $\mathcal{V}_{\text{End}}$ into $\mathcal{V}$, and $B^+$-equivariant map from $\mathcal{V}_{\text{End}}$ into $\mathcal{V}^*$.

The orbits $K_{\mathcal{C}} \cdot \nu_{\lambda_o} \subset \mathbb{P}(V_{\lambda_o})$ and $K_{\mathcal{C}} \cdot \nu_{\lambda_o}^* \subset \mathbb{P}(V_{\lambda_o}^*)$ are closed and we have

$$K_{\mathcal{C}} \cdot \nu_{\lambda_o} \cap \mathcal{V} = N^- \cdot \nu_{\lambda_o} \simeq N^-$$

$$K_{\mathcal{C}} \cdot \nu_{\lambda_o}^* \cap \mathcal{V}^* = N^+ \cdot \nu_{\lambda_o}^* \simeq N^+.$$ 

The rational map $(l, r) : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_o}) \times \mathbb{P}(V_{\lambda_o}^*)$ induced then a map $q : \mathcal{V}_{\text{End}} \cap \mathcal{X}_P \to N^- \times N^+$ which is $N^- \times N^+$-equivariant:

$$q \left( (n^-, n^+) \cdot x \right) = (n^-, n^+) \times q(x)$$

for $x \in \mathcal{V}_{\text{End}} \cap \mathcal{X}_P$, and $n^\pm \in N^\pm$.

We can now finish the arguments. The set $N^- T_{\mathcal{C}} N^+ \subset K_{\mathcal{C}}$ is dense in $K_{\mathcal{C}}$, so it is now easy to see that the map

$$N^- \times N^+ \times \mathcal{Y}_P \cap \mathcal{V}_{\text{End}} \to \mathcal{X}_P \cap \mathcal{V}_{\text{End}}$$

$$(n^-, n^+, y) \mapsto (n^-, n^+) \cdot y$$

is a diffeomorphism. We have proved previously that $\mathcal{Y}_P \cap \mathcal{V}_{\text{End}}$ is a smooth affine variety, hence $\mathcal{X}_P$ is smooth near the closed orbit $K_{\mathcal{C}} \times K_{\mathcal{C}} : [\nu_{\lambda_o} \otimes \nu_{\lambda_o}^*] \subset \mathcal{X} \cap \mathcal{V}_{\text{End}}$. Lemma 3.6 tells us then that $\mathcal{X}_P$ is smooth.

3.3. Hamiltonian actions. First consider an Hermitian vector space $V$. The Hermitian structure on $\text{End}(V)$ is $(A, B) := \text{Tr}(AB^*)$, hence the associated symplectic structure on $\text{End}(V)$ is defined by the relation $\Omega_{\text{End}}(A, B) := -\text{Im}(\text{Tr}(AB^*))$.

Let $U(V)$ be the unitary group. Let $u(V)$ be the Lie algebra of $U(V)$. We will use the identification $\epsilon : u(V) \simeq u(V)^*$, $X \mapsto \epsilon_X$ where $\epsilon_X(Y) = -\text{Tr}(XY)$. The action $U(V) \times U(V)$ on $\text{End}(V)$ is $(g, h) \cdot A = gAh^{-1}$. The moment map relative to this action is

$$\text{End}(V) \to u(V)^* \times u(V)^*$$

$$A \mapsto -\frac{1}{2} (iAA^*, -iA^*A).$$

We consider now the projective space $\mathbb{P}(\text{End}(V))$ equipped with the Fubini-Study symplectic form $\Omega_\text{FS}$. Here the action of $U(V) \times U(V)$ on $\mathbb{P}(\text{End}(V))$ is hamiltonian with moment map

$$\mathbb{P}(\text{End}(V)) \to u(V)^* \times u(V)^*$$

$$[A] \mapsto \left( \frac{iAA^*}{\|A\|^2}, \frac{-iA^*A}{\|A\|^2} \right).$$

where $\|A\|^2 = \text{Tr}(A^*A)$ (see 26[Section 7]). If $\rho : K \to U(V)$ is a connected Lie subgroup, we can consider the action of $K \times K$ on $\mathbb{P}(\text{End}(V))$. Let $\pi_K : u(V)^* \to \mathfrak{k}^*$
be the projection which is dual to the inclusion \( \rho : \mathfrak{t} \hookrightarrow u(V) \). The moment map for the action of \( K \times K \) on \( (\mathbb{P}(\text{End}(V)), \Omega_{FS}) \) is then

\[
\mathbb{P}(\text{End}(V)) \longrightarrow \mathfrak{t}^* \times \mathfrak{t}^* \quad [A] \longmapsto \frac{1}{\|A\|^2}(\pi_K(iAA^*), -\pi_K(iA^*A)).
\]

We are interested here respectively in

- the projective variety \( X_p \subset \mathbb{P}(\text{End}(V)) \) with the action of \( K \times K \),
- the projective variety \( Y_p \subset \mathbb{P}(\text{End}(V)) \) with the action of \( T \times T \),

where \( V = \bigoplus_{i=1}^N V_{\lambda_i} \). The Fubini-Study two-form restrict into symplectic forms on \( X_p \) and \( Y_p \). The action of \( K \times K \) on \( X_p \) is Hamiltonian with moment map

\[
\Phi_{K×K} : X_p \longrightarrow \mathfrak{t}^* \times \mathfrak{t}^* \quad [x] \longmapsto \frac{1}{\|x\|^2}(\pi_K(ixx^*), -\pi_K(ix^*x)).
\]

Since the diagonal \( Z = \{(t, t)|t \in T\} \) acts trivially on \( Y_p \) we restrict ourself to the action of \( T \times T/Z \simeq T \) on \( Y_p \). Let us compute the moment map \( \Phi_T : Y_p \rightarrow \mathfrak{t}^* \) associated to this action. First we have

\[
\Phi_T([y]) = \frac{\pi_T(iyy^*)}{\|y\|^2} = \frac{\pi_T(iyy^*)}{\|y\|^2}
\]

where \( \pi_T : u(V)^* \rightarrow \mathfrak{t}^* \) is the projection which is dual to \( \rho : \mathfrak{t} \rightarrow u(V) \). Since \( \rho(X) = i \sum_{j \in J} \alpha_j(X)v_j \otimes v_j^* \), a small computation shows that for \( B \in u(V) \simeq u(V)^* \) we have \( \pi_T(B) = -i \sum_{j \in J} (Bv_j, v_j)\alpha_j \). Finally for any \([y] \in Y_p\) we get

\[
\Phi_T([y]) = \sum_{j \in J} \frac{\|yv_j\|^2}{\|y\|^2} \alpha_j.
\]

Together with the action on \( T \), we have also an action of the Weyl group \( W = N(T)/T \) on \( Y_p \) : for \( \bar{w} \in W \) we take

\[
\bar{w} \cdot [y] = [\rho(w) \circ y \circ \rho(w)^{-1}] \quad [y] \in Y_p.
\]

This action is well defined since the diagonal \( Z \subset T \times T \) acts trivially on \( Y_p \). The set of weights \( \{\alpha_j|j \in J\} \) is stable under the action of \( W \), hence it is an easy fact to verify that the map \( \Phi_T \) is \( W \)-equivariant.

A dense part of \( Y_p \) is formed by the elements \( e^Z \cdot [Id] = [\rho(e^Z)] \). Take \( Z = X + iY \in \mathfrak{t}_C \). We have \( \Phi_T(e^Z \cdot [Id]) = \psi_T(Y) \in \mathfrak{t}^* \) with

\[
\psi_T(Y) = \frac{1}{\sum_{j \in J} e^{-2(\alpha_j, Y)}} \sum_{j \in J} e^{-2(\alpha_j, Y)} \alpha_j.
\]

Hence the image of the moment map \( \Phi_T : Y_p \rightarrow \mathfrak{t}^* \) is equal to the closure of the image of the map \( \psi_T : \mathfrak{t} \rightarrow \mathfrak{t}^* \).

**Proposition 3.10.** The map \( \psi_T \) realises a diffeomorphism between \( \mathfrak{t} \) and the interior of the polytope \( P \subset \mathfrak{t}^* \).

**Proof.** Consider the function \( F_T : \mathfrak{t} \rightarrow \mathbb{R}, F_T(Y) = \ln \left( \sum_{j \in J} e^{(\alpha_j, Y)} \right) \), and let \( L_T : \mathfrak{t} \rightarrow \mathfrak{t}^* \) be its Legendre transform : \( L_T(X) = dF_T|_X \). Note that we have \( L_T(-2Y) = \psi_T(Y) \).
We see that $F_T$ is strictly convex. So, it is a classical fact that $L_T$ realizes a
diffeomorphism of $t$ onto its image, and for $\xi \in t^*$ we have

$$\xi \in \text{Image}(L_T) \iff \lim_{Y \to \infty} F_T(Y) - \langle \xi, Y \rangle = \infty$$

$$\iff \lim_{Y \to \infty} \sum_{j \in J} e^{(\alpha_j, -\xi, Y)} = \infty.$$

In order to conclude we need the following

**Lemma 3.11.** Let $\{\beta_j, j \in J\}$ be a sequence of elements of $t^*$, and let $Q$ be its
convex hull. We have

$$\lim_{Y \to \infty} \sum_{j \in J} e^{(\beta_j, Y)} = \infty \iff 0 \in \text{Interior}(Q)$$

**Proof.** First we see that $0 \not\in \text{Interior}(Q)$ if and only there exists $v \in t - \{0\}$
such that $\langle \beta_j, v \rangle \leq 0$ for all $j$: for such vector $v$, the map $t \to \sum_{j \in J} e^{(\beta_j, v)}$
is bounded. Suppose now that $\lim_{Y \to \infty} \sum_{j \in J} e^{(\beta_j, Y)} \neq \infty$. Then there exists a
sequence $(X_k)_k \in t$ such that $\lim_k |X_k| = \infty$ and for all $j$ the sequence $(\langle \beta_j, X_k \rangle)_k$
remains bounded. If $v$ is a limit of a subsequence of $(\frac{\beta_j}{|X_k|})_k$ we have then $\langle \beta_j, v \rangle \leq 0$
for all $j$. \hfill $\square$

**Lemma 3.12.** For $[y] \in \mathcal{Y}_P$ we have $\Phi_{K \times K}([y]) = (\Phi_T([y]), -\Phi_T([y])).$

**Proof.** It’s sufficient to consider the case $y = \rho(e^Z) = \sum_{j \in J} e^{(\alpha_j, Z)} v_j \otimes v_j^*$, for
$Z = X + iY \in t_C$. Then $yy^* = y^*y = \sum_j e^{-2(\alpha_j, Y)} v_j \otimes v_j^* = \rho(e^{2iY})$. So it remains
to prove that $\pi_K(iyy^*) = \pi_T(iyy^*)$. We have to check that $\langle \pi_K(iyy^*), [U, V] \rangle = 0$
for $U \in t$ and $V \in t$. We have

$$\langle \pi_K(iyy^*), [U, V] \rangle = -i \text{ Tr} \left( yy^* \rho([U, V]) \right)$$

$$= -i \text{ Tr} \left( \rho(e^{2iY})[\rho(U), \rho(V)] \right)$$

$$= -i \text{ Tr} \left( [\rho(e^{2iY}), \rho(U)]\rho(V) \right) = 0.$$

$\square$

**Theorem 3.13.** We have

- $\text{Image}(\Phi_T) = P$,
- $\text{Image}(\Phi_{K \times K}) = \{(k_1 \cdot \xi, -k_2 \cdot \xi) \mid \xi \in P \text{ and } k_1, k_2 \in K\}$,
- $\mathcal{Y}_P \subset \Phi_{K \times K}^{-1}(t^* \times t^*)$,
- $\Phi_{K \times K}^{-1}(\text{Interior}(C)) \subset \mathcal{Y}_P$, where $C = C_K \times -C_K$.

**Proof.** The first point follows from Proposition 3.10. Since the map $(k_1, t, k_2) \mapsto k_1 k_2$ from $K \times T_C \times K$ into $K_C$ is onto, we have

$$\mathcal{X}_P = (K \times K) \cdot \mathcal{Y}_P.$$

So if $[x] \in \mathcal{X}_P$, there exist $[y] \in \mathcal{Y}$ and $k_1, k_2 \in K$ such that $[x] = (k_1, k_2) \cdot [y]$, hence

$$\Phi_{K \times K}([x]) = (k_1, k_2) \cdot \Phi_{K \times K}([y])$$

(3.27) $\Phi_{K \times K}([x]) = (k_1 \cdot \Phi_T([y]), -k_2 \cdot \Phi_T([y]))$

The second point is proved. The third point follows also from the identity (3.27)
when $k_1 = k_2 = e$. Consider now $[x] = (k_1, k_2) \cdot [y]$ such that $\Phi_{K \times K}([x])$ belongs to
the interior of the cone $C_K \times -C_K$. Then $k_1 \cdot \Phi_T([y])$ and $k_2 \cdot \Phi_T([y])$ are regular points of $C_K$. This implies that $k_1, k_2 \in N(T)$ and $k_2 k_1^{-1} \in T$. So

\[
[x] = (k_1, k_2) \cdot [y] = (e, k_2 k_1^{-1}) \cdot ((k_1, k_1) \cdot [y]) \in \mathcal{Y}_P
\]

since $\mathcal{Y}_P$ is stable under the actions of $T \times T$ and $W$. \qed

**Corollary 3.14.** If $[v_{\lambda} \otimes v_{\lambda}^*] \in \mathcal{X}_P$ then $\lambda_i$ is a vertex of the polytope $P$.

**Proof.** Let $x = v_{\lambda} \otimes v_{\lambda}^*$, and suppose that $[x]$ belongs to $\mathcal{X}_P$. In order to show that $[x] \in \mathcal{Y}_P$, we compute $\Phi_{K \times K}([x])$. We see that $xx^* = x^*x = x$ and $\|x\| = 1$ so $\Phi_{K \times K}([x]) = (\pi_K(ix), -\pi_K(ix))$. For $X \in \mathfrak{t}$ we have

\[
\langle \pi_K(ix), X \rangle = -i \, \text{Tr} \left( v_{\lambda} \otimes v_{\lambda}^* \rho(X) \right)
\]

\[
= -i \langle \rho(X)v_{\lambda}, v_{\lambda} \rangle
\]

\[
= \langle \lambda, X \rangle.
\]

We have then $\Phi_{K \times K}([x]) = (\lambda_i, -\lambda_i)$ with $\lambda_i$ being a regular point of $C_K$: hence $[x] \in \mathcal{Y}_P$. Now we can conclude with the help of Lemma 3.7. Since $[v_{\lambda} \otimes v_{\lambda}^*]$ belongs to $\mathcal{Y}_P$, the weight $\lambda_i$ is a vertex of the polytope $P$. \qed

**Remark 3.15.** In this section, Theorem 3.13 was obtained without using the fact that the varieties $\mathcal{X}_P$ and $\mathcal{Y}_P$ are smooth. Hence Corollary 3.14 can be used to prove the smoothness of $\mathcal{X}_P$.

### 3.4. Symplectic cutting.

Let $(M, \Omega_M, \Phi_M)$ be an Hamiltonian $K$-manifold. At this stage the moment map $\Phi_M$ is not assumed to be proper. We consider also the Hamiltonian $K \times K$-manifold $\mathcal{X}_P$ associated to a $K$-adapted polytope $P$.

The purpose of this section is to define a symplectic cutting of $M$ which uses $\mathcal{X}_P$. The notion of symplectic cutting was introduced by Lerman in [21] in the case of a torus action. Later Woodward [35] extends this procedure to the case of a non-abelian group action (see also [24]). The method of symplectic cutting that we define in this section is different from the one of Woodward.

We have two actions of $K$ on $\mathcal{X}_P$ : the action from the left (resp. right), denoted $\cdot_1$ (resp. $\cdot_r$), with moment map $\Phi_1 : \mathcal{X}_P \rightarrow \mathfrak{t}^*$ (resp. $\Phi_r$). We consider now the product of $M \times \mathcal{X}_P$ with

- the action $k \cdot_1 (m, x) = (k \cdot m, k \cdot_1 x) :$ the corresponding moment map is $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot_2 x) :$ the corresponding moment map is $\Phi_2(m, x) = \Phi_r(x)$.

**Definition 3.16.** We denote $M_P$ the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action $\cdot_1 : M_P := (\Phi_1)^{-1}(0)/K$.

Note that $M_P$ is compact when $\Phi_M$ is proper. The action $\cdot_2$ on $M \times \mathcal{X}_P$ induces an action of $K$ on $M_P$. The moment map $\Phi_2$ induces an equivariant map $\Phi_{M_P} : M_P \rightarrow \mathfrak{t}^*$. Let $Z \subset (\Phi_1)^{-1}(0)$ be the set of points where $(K, \cdot_1)$ as a trivial stabilizer.

**Definition 3.17.** We denote $M_P'$ the quotient $Z/K \subset M_P$. 

$M'_p$ is an open subset of smooth points of $M_p$ which is invariant under the $K$-action. The symplectic structure of $M \times \mathcal{X}_p$ induces a canonical symplectic structure on $M'_p$ that we denote $\Omega_{M'_p}$. The action of $K$ on $(M'_p, \Omega_{M'_p})$ is Hamiltonian with moment map equal to the restriction of $\Phi_{M_p} : M_p \to \mathfrak{k}^*$ to $M'_p$.

We start with the easy

**Lemma 3.18.** The image of $\Phi_{M_p} : M_p \to \mathfrak{k}^*$ is equal to the intersection of the image of $\Phi_M : M \to \mathfrak{k}^*$ with $K \cdot P$.

Let $\mathcal{U}_p = K \cdot \text{Interior}(P) \subset K \cdot P$. We will show now that the open and dense subset $(\Phi_{M_p})^{-1}(\mathcal{U}_p)$ of $M_p$ belongs to $M'_p$. Afterwards we will prove that $\Phi_{M_p}^{-1}(\mathcal{U}_p)$ is quasi-symplectomorphic to the open subset $\Phi_M^{-1}(\mathcal{U}_p)$ of $M$.

We consider the open and dense subset of $\mathcal{X}_p$ which is equal to the open orbit $\bar{\rho}(K_C)$. From Lemma 3.13 we know that

\begin{equation}
\Theta : K \times \mathfrak{k} \rightarrow \bar{\rho}(K_C) \quad (k, X) \mapsto [\rho(ke^{iX})]
\end{equation}

is a diffeomorphism. Through $\Theta$, the action of $K \times K$ on $K \times \mathfrak{k}$ is $k \cdot (a, X) = (ka, X)$ for the action "from the left" and $k \cdot (a, X) = (ak^{-1}, k \cdot X)$ for the action "from the right".

We consider now the map $\psi_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$ defined by $\psi_K(X) = \Phi_I([\rho(e^{iX})])$. In other words,

$$\psi_K(X) = \pi_K(i\rho(e^{i2X})) \frac{\operatorname{Tr}(\rho(e^{iX}))}{\operatorname{Tr}(\rho(e^{i2X}))}.$$

Consider the function $F_K : \mathfrak{k} \rightarrow \mathbb{R}$, $F_K(X) = \ln(\operatorname{Tr}(\rho(e^{-iX})))$. Let $L_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$ be its Legendre transform.

**Proposition 3.19.**

- We have $\psi_K(X) = L_K(-2X)$, for $X \in \mathfrak{k}$,
- The function $F_K$ is strictly convex,
- The map $\psi_K$ realizes an equivariant diffeomorphism between $\mathfrak{k}$ and $\mathcal{U}_p$,
- The image of $\Phi_I : \mathcal{X}_p \rightarrow \mathfrak{k}^*$ is equal to the closure of $\mathcal{U}_p$,
- $\Phi_I^{-1}(\mathcal{U}_p) = \bar{\rho}(K_C)$.

**Proof.** For $X, Y \in \mathfrak{k}$ we consider the function $\tau(s) = F_K(X + sY)$. Since $F_K$ is $K$-invariant we can restrict our computation to $X \in \mathfrak{k}$. We will use the decomposition of $Y \in \mathfrak{k}$ relatively to the $T$-weights on $\mathfrak{t}_C : Y = \sum_\alpha Y_\alpha$ where $\text{ad}(Z)Y_\alpha = \alpha(Z)Y_\alpha$ for any $Z \in \mathfrak{k}$, and $Y_0 \in \mathfrak{k}$. We have

$$\tau'(s) = \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} \left( \rho(e^{-iX_s}) \rho \left( \frac{e^{i \text{ad}(X_s)}}{i \text{ad}(X_s)} - 1 \right) Y \right)$$

$$= \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} \left( \rho(e^{-iX_s}) \rho(Y) \right)$$

$$= \frac{1}{\text{Tr}(\rho(e^{-iX_s}))} \langle \pi_K(i\rho(e^{-iX_s})), Y \rangle$$

where $X_s = X + sY$. Since by definition $\tau'(0) = \langle L_K(X), Y \rangle$, the first point is proved. For the second derivative we have

$$\tau''(0) = -\left( \frac{\text{Tr}(\rho(e^{-iX})\rho(Y))}{\text{Tr}(\rho(e^{-iX}))} \right)^2 + \frac{\text{Tr} \left( \rho(e^{-iX}) \rho \left( \frac{e^{i \text{ad}(X)}}{i \text{ad}(X)} - 1 \right) Y \right)}{\text{Tr}(\rho(e^{-iX}))}$$

$$= R_1 + R_2$$
where
\[
R_1 = \frac{\text{Tr}(\rho(e^{-iX})\rho(iY_0)\rho(iY_0))}{\text{Tr}(\rho(e^{-iX}))} - \left(\frac{\text{Tr}(\rho(e^{-iX})\rho(iY_0))}{\text{Tr}(\rho(e^{-iX}))}\right)^2
\]
\[
= \sum_j e^{-\langle \alpha_j, X \rangle} (\alpha_j, Y_0)^2 - \left(\frac{\sum_j e^{-\langle \alpha_j, X \rangle} (\alpha_j, Y_0)^2}{\sum_j e^{-\langle \alpha_j, X \rangle}}\right)^2
\]
and
\[
R_2 = \frac{1}{\text{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, \beta \neq 0} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} \text{Tr}(\rho(e^{-iX})\rho(iY_\alpha)\rho(iY_\beta))
\]
\[
= \frac{1}{\text{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, \beta \neq 0} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} e^{-\langle \alpha_j, X \rangle} \|\rho(\alpha)\varepsilon_j\|^2.
\]

It is now easy to see that $R_1$ and $R_2$ are positive and that $R_1 + R_2 > 0$ if $Y \neq 0$. We have proved that $F_K$ is strictly convex, So, its Legendre transform $L_K$ realizes a diffeomorphism of $\mathfrak{k}$ onto its image. Using the first point we know that $\psi_K$ realizes a diffeomorphism of $\mathfrak{k}$ onto its image. The map $\psi_K$ is equivariant and coincides with $\psi_T$ on $\mathfrak{k}$. We have proved in Proposition 3.19 that the image of $\psi_T$ is equal to the interior of $P$, hence the image of $\psi_K$ is $\mathcal{U}_P$.

For the last two points we first remark that
\[(3.29) \quad \Phi_l([\rho(ke^{iX})]) = k \cdot \psi_K(X)\]
hence the image of $\Phi_l$ is the closure of $\mathcal{U}_P$. If we use the fact that $\psi_K$ is a diffeomorphism from $\mathfrak{k}$ onto $\mathcal{U}_P$, (3.29) shows that $\Phi_l^{-1}(K \cdot \xi) \cap \bar{\rho}(K_C)$ is a non empty and closed subset of $\Phi_l^{-1}(K \cdot \xi)$ for any $\xi \in \mathcal{U}_P$ (in fact it is a $K \times K$-orbit). On the other hand $\Phi_l^{-1}(K \cdot \xi) \cap (\mathcal{U}_P \setminus \bar{\rho}(K_C))$ is also a closed subset of $\Phi_l^{-1}(K \cdot \xi)$ since $\bar{\rho}(K_C)$ is open in $\mathcal{U}_P$. Since $\Phi_l^{-1}(K \cdot \xi)$ is connected the second subset is empty : in other words $\Phi_l^{-1}(K \cdot \xi) \subset \bar{\rho}(K_C)$.

We introduce now the equivariant diffeomorphism
\[(3.30) \quad \Upsilon : K \times \mathcal{U}_P \rightarrow \bar{\rho}(K_C)
\]
\[(k, \xi) \mapsto \Theta(k, \psi^{-1}_K(\xi)).\]

We now look at $K \times \mathcal{U}_P$ equipped with the symplectic structure $\Upsilon^*(\Omega_{X_P})$, and the Hamiltonian action of $K \times K$ : the moment maps satisfy
\[(3.31) \quad \Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi \quad \text{and} \quad \Upsilon^*(\Phi_r)(k, \xi) = -\xi.\]

**Proposition 3.20.** We have
\[
\Upsilon^*(\Omega_{X_P}) = d\lambda + d\eta
\]
where $\lambda$ is the Liouville 1-form on $K \times T^* \simeq T^*K$ and $\eta$ is an invariant 1-form on $\mathcal{U}_P \subset T^*$ which is killed by the vectors tangent to the $K$-orbits.

**Proof.** Let $E_1, \ldots, E_r$ be a basis of $\mathfrak{k}$, with dual basis $\xi^1, \ldots, \xi^r$. Let $\omega^i$ the 1-form on $K$, invariant by left translation and equal to $\xi^i$ at the identity. The Liouville 1-form is $\lambda = -\sum_i \omega^i \otimes E_i$. For $X \in \mathfrak{k}$ we denote $X_l(k, \xi) = \frac{d}{dt} e^{-itX} \cdot l(k, \xi)$ and $X_r(k, \xi) = \frac{d}{dt} e^{-itY} \cdot r(k, \xi)$ the vectors fields generated by the action of $K \times K$. Since $\iota(X_l)d\lambda = -d\Phi_l(X)$ and $\iota(X_r)d\lambda = -d\Phi_r(X)$, the closed invariant 2-form $\beta = \Upsilon^*(\Omega_{X_P}) - d\lambda$ is $K \times K$ invariant and is killed by the vectors tangent to the
We have proved that $\Psi$ is onto. So to the $K$-orbits we see that $(0, m, x)$, we denot $\phi = \eta$ where $\eta$ is an invariant 1-form on $U_P$ which is killed by the vectors tangent to the $K$-orbits. Hence $\Psi : M \times X_P \mapsto \Lambda M \times X_P$.

**Theorem 3.21.** $\Phi_{\Phi}^{-1}(U_P)$ is an open and dense subset of smooth points in $M_P$. There exist an equivariant diffeomorphism $\Psi : \Phi_{\Phi}^{-1}(U_P) \to \Phi_{\Phi}^{-1}(U_P)$ such that

$$\Psi^*(\Omega_{M_P}) = \Omega_M + d\Phi_{\Phi}^*\eta.$$ 

Here $\eta$ is an invariant 1-form on $U_P$ which is killed by the vectors tangent to the $K$-orbits. Moreover the map $\Omega' = \Omega_M + t d\Phi_{\Phi}^*\eta$, defines an homotopy of symplectic 2-forms between $\Omega_M$ and $\Omega_{M_P}$.

**Remark 3.22.** The map $\Psi$ will be called a quasi-symplectomorphism.

**Proof.** Consider the immersion

$$\Psi : \Phi_{\Phi}^{-1}(U_P) \mapsto M \times X_P$$

$$m \mapsto (m, \Upsilon(e, \Phi_M(m))).$$

We have $\Phi_1(\psi(m)) = \Phi_M(m) + \Upsilon^*\Phi_1(e, \Phi_M(m)) = 0$, and $\Phi_2(\psi(m)) = \Upsilon^*\Phi_1(e, \Phi_M(m)) = \Phi_M(m) \in U_P$ (see (2)). Hence for all $m \in \Phi_{\Phi}^{-1}(U_P)$, we have $\psi(m) \in \Phi_1^{-1}(U_P)$, and its class $[\psi(m)] \in M_P$ belongs to $\Phi_{\Phi}^{-1}(U_P)$.

We denote $\Psi : \Phi_{\Phi}^{-1}(U_P) \to \Phi_{\Phi}^{-1}(U_P)$ the map $m \mapsto [\psi(m)]$. Let us show that it defines a diffeomorphism. If $\Psi(m) = \Psi(m')$, there exists $k \in K$ such that

$$(m, \Upsilon(e, \Phi_M(m))) = k \cdot (m', \Upsilon(e, \Phi_M(m')))$$

$$= (k \cdot m', k \cdot \Upsilon(e, \Phi_M(m')))$$

$$(k \cdot m', \Upsilon(k^{-1}, k \cdot \Phi_M(m'))) \in \Phi_{\Phi}^{-1}(U_P).$$

Since $\Upsilon$ is a diffeomorphism, we must have $k = e$ and $m = m'$: the map $\Psi$ is one to one. Consider now $(m, x) \in \Phi_1^{-1}(0)$ such that $\Phi_M((m, x)) = \Phi_1(x) \in U_P$ : then $x \in \Phi_1^{-1}(U_P) = \overline{\rho(K_C)} = \text{Image}(\Upsilon)$. We have $x = \Upsilon(k, \xi)$ where $\xi = -\Phi_r(x) = \Phi_M(m)$. Finally

$$x = (m, \Upsilon(k, \Phi_M(m)))$$

$$= k^{-1} \cdot (k \cdot m, \Upsilon(k, k \cdot \Phi_M(m)))$$

$$= k^{-1} \cdot \psi(k \cdot m).$$

We have proved that $\Psi$ is onto.
In order to show that \( \Psi \) is a submersion we must show that for \( m \in \Phi^{-1}_M(\mathcal{U}_P) \)
\[
\text{Image}(T_m\psi) \oplus T_{\psi(m)}(K \cdot 1 \psi(m)) = T_{\psi(m)}\Phi^{-1}_1(0).
\]
Here \( T_m\psi : T_mM \to T_{\psi(m)}(M \times X_P) \) is the tangent map, and \( T_{\psi(m)}(K \cdot 1 \psi(m)) \) denotes the tangent space at \( \psi(m) \) of the \((K, \cdot 1)\)-orbit. We have \( \dim(\text{Image}(T_m\psi)) + \dim(T_{\psi(m)}(K \cdot 1 \psi(m))) = \dim(T_{\psi(m)}\Phi^{-1}_1(0)) \) so it is sufficient to prove that
\[
\text{Image}(T_m\psi) \cap T_{\psi(m)}(K \cdot 1 \psi(m)) = \{0\}.
\]
Consider \( (v, w) \in \text{Image}(T_m\psi) \cap T_{\psi(m)}(K \cdot 1 \psi(m)) \). There exists \( X \in \mathfrak{t} \) such \( (v, w) = \frac{d}{dt}|_0 e^{tx} \cdot 1 \psi(m) \):
\[
v = \frac{d}{dt}|_0 e^{tx} \cdot m \quad \text{and} \quad w = \frac{d}{dt}|_0 e^{tx} \cdot \Upsilon(e, \Phi_M(m))
\]
In the other hand since \( (v, w) \in \text{Image}(T_m\psi) \), we have
\[
w = \frac{d}{dt}|_0 \Upsilon(e, \Phi_M(e^{tx} \cdot m))
\]
Since \( e^{tx} \cdot \Upsilon(e, \Phi_M(m)) = \Upsilon(e^{-tx}, \Phi_M(e^{tx} \cdot m)) \) we obtain that
\[
\frac{d}{dt}|_0 \Upsilon(e^{-tx}, \Phi_M(e^{tx} \cdot m)) = \frac{d}{dt}|_0 \Upsilon(e, \Phi_M(e^{tx} \cdot m))
\]
or in other words \( \frac{d}{dt}|_0 \Upsilon(e^{-tx}, \Phi_M(m)) = 0 \). Since \( \Upsilon \) is a diffeomorphism we have \( X = 0 \), and then \( (v, w) = 0 \).

We can now compute the pull-back by \( \Psi \) of the symplectic form \( \Omega_{M_L} \). We have
\[
\Psi^*(\Omega_{M_L}) = \psi^*(\Omega_M + \Omega_{X_P})
= \Omega_M + \Phi_M^*\Upsilon^*(\Omega_{X_P})
= \Omega_M + d\Phi_M^*\eta.
\]

It remains to prove that for every \( t \in [0, 1] \), the 2-form \( \Omega^t = \Omega_M + td\Phi_M^*\eta \) is non-degenerate. Take \( t \neq 0 \), \( m \in \Phi^{-1}_M(\mathcal{U}_P) \) and suppose that the contraction of \( \Omega^t|_m \) by \( v \in T_mM \) is equal to 0. For every \( X \in \mathfrak{t} \) we have
\[
0 = \Omega^t(X_M(m), v)
= -\iota(v)d(\Phi_M, X)|_m + t\iota(v)\iota(X_M)d\Phi_M^*\eta|_m
= -\iota(v)d(\Phi_M, X)|_m
\]
since \( \iota(X_M)d\Phi_M^*\eta = d\Phi_M^*(\iota(X_M)\eta) = 0 \). Thus we have \( T_m\Phi_M(v) = 0 \), and then \( \iota(v)d\Phi_M^*\eta = 0 \). Finally we have that \( 0 = \iota(v)\Omega^t|_m = \iota(v)\Omega_M|_m \). But \( \Omega_M \) is non-degenerate, so \( v = 0 \). \( \Box \).

3.5. Formal quantization: second definition. We suppose here that the Hamiltonian \( K \)-manifold \( (M, \Omega_M, \Phi_M) \) is proper and admits a Kostant-Souriau line bundle \( L \). Now we consider the complex \( K \times K \)-submanifold \( X_P \) of \( P(E) \). Since \( \mathcal{O}(-1) \) is a \( K \times K \)-equivariant Kostant-Souriau line bundle on the projective space \( \mathbb{P}(E) \) the restriction
\[
L_P = \mathcal{O}(-1)|_{X_P}
\]
is a Kostant-Souriau line bundle on \( X_P \). Hence \( L \boxtimes L_P \) is a Kostant line bundle on the product \( M \times X_P \). In section 2.2 we have have defined the quantization \( Q_K(M_P) \) of the (singular) reduced space \( M_P := (M \times X_P)\#_0(K, \cdot 1) \).
Notation: $O_K(r)$ will be any element $\sum_{\nu \in \hat{K}} m_{\nu} V_{\mu}^{K}$ of $R^{-\infty}(K)$ where $m_{\mu} = 0$ if $\|\mu\| < r$. The limit $\lim_{r \to +\infty} O_K(r) = 0$ defines the notion of convergence in $R^{-\infty}(K)$.

**Proposition 3.23.** Let $\varepsilon_P > 0$ be the radius of the biggest ball center at $0 \in \mathfrak{t}^*$ which is contains in the polytope $P$. We have

$$
Q_K(M_P) = \sum_{\|\mu\| < \varepsilon_P} Q((M_P)_{\mu}) V_{\mu}^{K} + O_K(\varepsilon_P).
$$

**Proof.** Theorem 2.4 - “Quantization commutes with reduction in the singular setting” - tells us that $Q_K(M_P) = \sum_{\mu \in \hat{K}} Q((M_P)_{\mu}) V_{\mu}^{K}$ where $(M_P)_{\mu}$ is the symplectic reduction

$$(M_P \times K_2 : \mu) / \partial K_2 \cong (M \times X_P \times K_2 : \mu) / \partial_0 K_2 \times K_1.$$

Recall what the $K_1$, $K_2$-action are: $k_1 (m,x,\xi) = (km,kx,\xi)$ and $k_2 (m,x,\xi) = (m,kx,\xi)$ for $(m,x,\xi) \in M \times X_P \times K_2 : \mu$ and $k \in K$.

Since the image of $\Phi_{M_P}$ is equal to the intersection of $K : P = \mathcal{U_P}$ with the image of $\Phi_M$, we have

$$Q((M_P)_{\mu}) = 0 \quad \text{if} \quad \mu \notin P \cap \text{Image}(\Phi_M).$$

We will now exploit Theorem 3.21 to show that $Q((M_P)_{\mu}) = Q((M_{\nu})_{\mu})$ if $\mu$ belongs to the interior of $P$.

There exists a quasi-symplectomorphism $\Psi$ between the open subset $\Phi_{M_P}^{-1}(\mathcal{U_P})$ of $M$ and the open and dense subset $\Phi_{M_P}^{-1}(\mathcal{U_P})$ of $M_P$. Moreover one can see easily that the restriction of the Kostant line bundle $L_P \to X_P$ to the open subset $\mathcal{U_P}$ is trivial. If $L_{M_P}$ is the Kostant line bundle on $M_P$ induced by $L \boxtimes L_P$, we have that the pull-back of the restriction $L_{M_P}|_{\Phi_{M_P}^{-1}(\mathcal{U_P})}$ by $\Psi$ is equivariantly diffeomorphic to the restriction of $L$ to $\Phi_{M_P}^{-1}(\mathcal{U_P})$.

Take now $\mu \in \hat{K}$ that belongs to the interior of the polytope $P$. The element $Q((M_P)_{\mu}) \in \mathbb{Z}$ is given by the index of a transversally elliptic symbol defined in a (small) neighborhood of $\Phi_{M_P}^{-1}(\mathcal{U_P}) \subset M_P$. This symbol is defined through two auxiliary data: the Kostant line bundle $L_{M_P}$ and a compatible almost complex structure $J$ which defined in a neighborhood of $\Phi_{M_P}^{-1}(\mu)$. If we pull back everything by $\Psi$, we get a transversally elliptic symbol living in a (small) neighborhood of $\Phi_{M_P}^{-1}(\mu) \subset M$ which is defined by the Kostant line bundle $L$ and an almost complex structure $J_1$ compatible with the symplectic structure $\Omega_1 := \Omega_M + d\Phi_{M_P}^* \eta$. But since $\Omega_1 = \Omega_M + td\Phi_{M_P}^* \eta$ defines an homotopy of symplectic structures, any almost complex structure compatible with $\Omega_M$ is homotopic to $J_1$. We have then shown that $Q((M_{\nu})_{\mu}) = Q((M_P)_{\mu})$ for any $\mu$ belonging to the interior of $P$. So we have

$$Q_K(M_P) = \sum_{\mu \in \text{Interior}(P)} Q(M_{\mu}) V_{\mu}^{K} + \sum_{\nu \in \partial P} Q((M_P)_{\nu}) V_{\nu}^{K}.$$ 

Since for $\nu \in \partial P$ we have $\|\nu\| \geq \varepsilon_P$, the last equality proves (3.33). □

We work now with the dilated polytope $P$, for any integer $n \geq 1$. The polytope $nP$ is still $K$-adapted, so one can consider the reduced spaced $M_{nP}$ and Proposition 3.23 gives that

$$Q_K(M_{nP}) = \sum_{\|\mu\| < n\varepsilon_P} Q(M_{\mu}) V_{\mu}^{K} + O_K(n\varepsilon_P).$$
for any integer $n \geq 1$. We can summarize the result of this section in the following

**Proposition 3.24.** Let $(M, \Omega_M)$ be a pre-quantized Hamiltonian $K$-manifold, with a proper moment map $\Phi_M$.

- For any integer $n \geq 1$, the (singular) compact Hamiltonian manifold $M_{nP}$ contains as an open and dense subset, the open subset $\Phi_M^{-1}(nP)$ of $M$.
- We have $Q_K^{-\infty}(M) = \lim_{n \to \infty} Q_K(M_{nP})$.

4. **Functorial properties : Proof of Theorem 1.3**

This section is devoted to the proof of Theorem 1.3. We will use in a crucial way the characterisation of $Q_K^{-\infty}$ given in Proposition 3.24.

Let $H \subset K$ be a connected Lie subgroup. Here we consider a pre-quantized Hamiltonian $K$-manifold $M$ which is proper as an Hamiltonian $H$-manifold. We want to compare $Q_K^{-\infty}(M)$ and $Q_H^{-\infty}(M)$. For $\mu \in \tilde{K}$ and $\nu \in \tilde{H}$ we denote $N^\mu_\nu$ the multiplicity of $V^H_\nu$ in the restriction $V^K_\mu|_H$. We have seen in the introduction that $N^\mu_\nu Q(M_{\mu,K}) \neq 0$ only for the $\mu$ belonging to finite subset $\tilde{K} \cap \Phi_K (K \cdot \Phi_H^{-1}(\nu))$. Then $Q_K^{-\infty}(M)$ is $H$-admissible and we have the following equality in $R^{-\infty}(H)$:

\[
Q_K^{-\infty}(M)|_H = \sum_{\nu \in H} m_\nu V^H_\nu
\]

with $m_\nu = \sum_{\mu} N^\mu_\nu Q(M_{\mu,K})$. We will now prove that $Q_K^{-\infty}(M)|_H = Q_H^{-\infty}(M)$.

**Lemma 4.1.** The restriction $Q_K^{-\infty}(M)|_H$ is equal to $\lim_{n \to \infty} Q_K(M_{nP})|_H$.

**Proof.** Let us denote $P^o$ and $\partial P$ respectively the interior and the border of the $K$-adapted polytope $P$. We write

\[
Q_K^{-\infty}(M) = \sum_{\mu \in nP^o} Q(M_{\mu,K})V^K_\mu + \sum_{\mu \notin nP^o} Q(M_{\mu,K})V^K_\mu.
\]

On the other side

\[
Q_K(M_{nP}) = \sum_{\mu \in nP^o} Q(M_{\mu,K})V^K_\mu + \sum_{\mu \notin nP^o} Q((M_{nP})_{\mu,K})V^K_\mu.
\]

So the difference $D(n) = Q_K^{-\infty}(M) - Q_K(M_{nP})$ is equal to

\[
D(n) = -\sum_{\mu \notin nP^o} Q((M_{nP})_{\mu,K})V^K_\mu + \sum_{\mu \notin nP^o} Q(M_{\mu,K})V^K_\mu.
\]

We show now that the restriction $D(n)|_H$ tends to 0 in $R^{-\infty}(H)$ as $n$ goes to infinity. For this purpose, we will prove that for any $c > 0$ there exist $n_c \in \mathbb{N}$ such that $D(n)|_H = O_H(c)$ for any $n \geq n_c$.

For $c > 0$ we consider the compact subset of $\mathfrak{k}^*$ defined by

\[
K_c = \Phi_K (K \cdot \Phi_H^{-1}(\xi \in \mathfrak{h}^*, ||\xi|| \leq c)) .
\]

Let $n_c \in \mathbb{N}$ such that $K_c$ is included in $K \cdot (n_cP^o)$ : hence $K_c \subset K \cdot (nP^o)$ for any $n \geq n_c$. We know that for $\mu \in \tilde{K}$, we have $N^\mu_\nu Q(M_{\mu,K}) \neq 0$ only for $\mu \in \Phi_K (K \cdot \Phi_H^{-1}(\nu))$, and for $\mu' \in \tilde{K}$, we have $N^\mu_{\nu'} Q((M_{nP})_{\mu',K}) \neq 0$ only for $\mu' \in nP \cap \Phi_K (K \cdot \Phi_H^{-1}(\nu))$. 
Then if \( n \geq n_c \), we have
\[
N_{\mu}^\nu Q(M_{\mu,K}) = N_{\mu'}^\nu Q((M_{n,P})_{\mu',K}) = 0
\]
for any \( \nu \in \hat{H} \cap \{ \xi \in \mathfrak{h}^* \mid \| \xi \| \leq c \} \), \( \mu \neq nP^o \) and \( \mu' \in n\partial P \). It means that
\[
D(n)|_H = O_H(c)
\]
for any \( n \geq n_c \). \( \square \)

Since \( Q_K(M_{n,P})|_H = Q_H(M_{n,P}) \), we are no led to the

**Lemma 4.2.** The limit \( \lim_{n \to \infty} Q_H(M_{n,P}) = Q_H^\infty(M) \).

**Proof.** Theorem 2.4 - “Quantization commutes with reduction in the singular setting” tells us that \( Q_H(M_{n,P}) = \sum_{\nu \in \hat{H}} Q((M_{n,P})_{\nu,H})V_{\nu}^H \) where \( (M_{n,P})_{\nu,H} \) is the symplectic reduction
\[
(M_{n,P} \times H \cdot \nu)_{\Omega} \cong (M \times X_{n,P} \times H \cdot \mu)_{\Omega} \cong H \times K.
\]

For \( c > 0 \) we consider the compact subset of \( K_c \) defined in (4.37). Let \( n_c \in \mathbb{N} \) such that \( K_c \subset K \cdot (nP^o) \) for any \( n \geq n_c \). It implies that
\[
\Phi^{-1}_K(\xi \in \mathfrak{h}^*, \| \xi \| \leq c) \subset \Phi^{-1}_K(K \cdot (nP^o))
\]
for \( n \geq n_c \). Since \( M_{n,P} \) “contains” as an open subset \( \Phi^{-1}_K(K \cdot (nP^o)) \), the arguments similar to those used in the proof of Proposition 3.23 show that \( Q((M_{n,P})_{\nu,H}) = \tilde{Q}(M_{\nu,H}) \) for \( \| \nu \| \leq c \) and \( n \geq n_c \). It means that
\[
Q_H(M_{n,P}) = \sum_{\| \nu \| \leq c} Q(M_{\nu,H})V_{\nu}^H + O_H(c) \quad \text{when} \quad n \geq n_c.
\]

It follows that \( \lim_{n \to \infty} Q_H(M_{n,P}) = \sum_{\nu \in \hat{H}} Q(M_{\nu,H})V_{\nu}^H = Q_H^\infty(M) \). \( \square \)

5. The case of an Hermitian space

Let \( (E, h) \) be an Hermitian vector space of dimension \( n \).

5.1. The quantization of \( E \). Let \( U := U(E) \) be the unitary group with Lie algebra \( u \). We use the isomorphism \( \epsilon : u \to u^* \) defined by \( \langle \epsilon(X), Y \rangle = -\text{Tr}(XY) \in \mathbb{R} \). For \( v, w \in E \), let \( v \otimes w^* : E \to E \) be the linear map \( x \mapsto h(x, w)v \).

Let \( E_\mathbb{R} \) be the space \( E \) viewed as a real vector space. Let \( \Omega \) be the imaginary part of \(-h\), and let \( J \) the complex structure on \( E_\mathbb{R} \). Then on \( E_\mathbb{R} \), \( \Omega \) is a (constant) symplectic structure and \( \Omega(\cdot, J\cdot) \) defines a scalar product. The action of \( U \) on \((E_\mathbb{R}, \Omega)\) is Hamiltonian with moment map \( \Phi : E \to u^* \) defined by \( \langle \Phi(v), X \rangle = \frac{1}{2i}\Omega(Xv, v) \). Through \( \epsilon \), the moment map \( \Phi \) is defined by
\[
(5.38) \quad \Phi(v) = \frac{1}{2i}v \otimes v^*.
\]

The pre-quantization data \((L, \langle \cdot, \cdot, \cdot \rangle, \nabla)\) on the Hamiltonian U-manifold \((E_\mathbb{R}, \Omega, \Phi)\) is a trivial line bundle \( L \) with a trivial action of \( U \) equipped with the Hermitian structure \((s, s')_v = e^{-\frac{h(v, v)}{2}}ss'\) and the Hermitian connexion \( \nabla = d - i\theta \) where \( \theta \) is the 1-form on \( E \) defined by \( \theta = \frac{1}{2}i\Omega(v, dv) \).

The traditional quantization of the Hamiltonian \( U \)-manifold \((E_\mathbb{R}, \Omega, \Phi)\), that we denote \( Q_U^L(E) \), is the Bargmann space of entire holomorphic functions on \( E \) which are \( L^2 \) integrable with respect to the Gaussian measure \( e^{-h(v,v)}\Omega^n \). The representation \( Q_U^{L,\nu}(E) \) of \( U \) is admissible. The irreducible representations of \( U \)
that occur in $Q^2_U(E)$ are the vector subspaces $S^j(E^*)$ formed by the homogeneous polynomial on $E$ of degree $j \geq 0$.

On the other hand, the moment map $\Phi$ is proper (see [5.38]). Hence we can consider the formal quantization $Q^{-\infty}_U(E) \in R^{-\infty}(U)$ of the $U$-action on $E$.

**Lemma 5.1.** The two quantizations of $(E, \Omega, \Phi)$, $Q^2_U(E)\text{ and } Q^{-\infty}_U(E)$ coincide in $R^{-\infty}(U)$. In other words, we have

\[Q^{-\infty}_U(E) = S^\bullet(E^*) := \sum_{j \geq 0} S^j(E^*) \text{ in } R^{-\infty}(U).\]

**Proof.** Let $T \subset U$ be a maximal torus with Lie algebra $t \subset u$. There exists an orthonormal basis $(e_k)_{k=1,\ldots,n}$ of $E$ and characters $(\chi_k)_{k=1,\ldots,n}$ of $T$ such that $t \cdot e_k = \chi_k(t)e_k$ for all $k$. The family $(ie_k \otimes e^*_k)_{k=1,\ldots,n}$ is then a basis of $t$ such that $id\chi_l(ie_k \otimes e^*_k) = \delta_{l,k}$. The set $\hat{U} \subset t^* \subset u^*$ of dominants weights is composed, through $\epsilon$, by the elements $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a decreasing sequence of integer.

The formal quantization $Q^{-\infty}_U(E) \in R^{-\infty}(U)$ is defined by

\[Q^{-\infty}_U(E) = \sum_{\lambda_1 \geq \cdots \geq \lambda_n} Q(E_\lambda)V_\lambda\]

where $E_\lambda = \Phi^{-1}(U \cdot \lambda)/U$ is the reduced space and $V_\lambda$ is the irreducible representation of $U$ with highest weight $\lambda$.

It is now easy to check that

\[E_\lambda = \begin{cases} \{\text{pt}\} & \text{if } \lambda = (0, \ldots, 0, -j) \text{ with } j \geq 0, \\ \emptyset & \text{in the other cases,} \end{cases}\]

and then

\[Q(E_\lambda) = \begin{cases} 1 & \text{if } \lambda = (0, \ldots, 0, -j) \text{ with } j \geq 0, \\ 0 & \text{in the other cases.} \end{cases}\]

Finally $[5.39]$ follows from the fact that $V(0, \ldots, 0, -j) = S^j(E^*)$. $\square$

**5.2. The quantization of $E$ restricted to a subgroup of $U$.** Let $K \subset U$ be a connected Lie subgroup with Lie algebra $k^*$. Let $K_C \subset GL(E)$ be its complexification. The moment map relative to the $K$-action on $(E_R, \Omega)$ is the map

$\Phi_K : E \to k^*$

equal to the composition of $\Phi$ with the projection $u^* \to k^*$.

**Lemma 5.2.** The following conditions are equivalent:

(a) the map $\Phi_K$ is proper,
(b) $\Phi_K^{-1}(\{0\}) = \{0\}$,
(c) $\{0\}$ is the only closed $K_C$-orbit in $E$,
(d) for every $v \in E$ we have $0 \in K_C \cdot v$,
(e) $S^\bullet(E^*)$ is an admissible representation of $K$,
(f) the $K$-invariant polynomials on $E$ are the constant polynomials.
PROOF. The equivalence $(a) \iff (b)$ is due to the fact that $\Phi_K$ is quadratic.

Let $\mathcal{O}$ be a $K_e$-orbit in $E$. Classical results of Geometric Invariant Theory \cite{11,18} assert that $\overline{\mathcal{O}} \cap \Phi^{-1}_K(0) \neq \emptyset$ and that $\mathcal{O}$ is closed if and only if $\mathcal{O} \cap \Phi^{-1}_K(0) \neq \emptyset$. Hence $(b) \iff (c) \iff (d)$.

After Lemma 8.1 we know that $Q_u^{-\infty}(E) = S^\bullet(E^*)$. Since $Q_u^{-\infty}(E)$ is $K$-admissible when $\Phi_K$ is proper (see Section 11), we have $(a) \implies (e)$.

For every $\mu \in \widehat{K}$, the $\mu$-isotropic component $[S^\bullet(E^*)]_\mu$ is a module over $[S^\bullet(E^*)]_0 = [S^\bullet(E^*)]^K$. Hence $\dim [S^\bullet(E^*)]_\mu < \infty$ implies that $[S^\bullet(E^*)]^K = \mathbb{C}$. We have $(e) \implies (f)$.

Finally $(f) \implies (d)$ follows from the following fundamental fact. For any $v, w \in E$ we have $\overline{K_C \cdot v} \cap \overline{K_C \cdot w} \neq \emptyset$ if and only if $P(v) = P(w)$ for all $P \in [S^\bullet(E^*)]^K$. \(\Box\)

Theorem 13 implies the following

**Proposition 5.3.** Let $K \subset U(E)$ be a closed connected subgroup such that $S^\bullet(E^*)$ is an admissible representation of $K$. For every $\mu \in \widehat{K}$, we have

$$\dim ([S^\bullet(E^*)]_\mu) = Q(E_{\mu,K})$$

where $[S^\bullet(E^*)]_\mu$ is the $\mu$-isotropic component of $S^\bullet(E^*)$ and $E_{\mu,K}$ is the reduced space $\Phi^{-1}_K(K \cdot \mu)/K$.

In the following examples the condition $\Phi^{-1}_K(0) = \{0\}$ is easy to check.

1. the subgroup $K \subset U(E)$ contains the center of $U(E)$,
2. $E = \Lambda^2 \mathbb{C}^n$ or $E = S^2(\mathbb{C}^n)$ and $K = U(n) \subset U(E)$,
3. $E = M_{n,k}$ is the vector space of $n \times k$-matrices and $K = U(n) \times U(k) \subset U(E)$.

**References**

[1] M. F. Atiyah, Elliptic operators and compact groups, Springer, 1974. Lecture notes in Mathematics, 401.
[2] M. F. Atiyah, Convexity and commuting hamiltonians, Bull. London Math. Soc., 14, 1982, p. 1-15.
[3] M. F. Atiyah, G. B. Segal, The index of elliptic operators II, Ann. Math. 87, 1968, p. 531-545.
[4] M. F. Atiyah, I. M. Singer, The index of elliptic operators I, Ann. Math. 87, 1968, p. 484-530.
[5] M. F. Atiyah, I. M. Singer, The index of elliptic operators III, Ann. Math. 87, 1968, p. 546-604.
[6] M. F. Atiyah, I. M. Singer, The index of elliptic operators IV, Ann. Math. 93, 1971, p. 139-141.
[7] N. Berline, E. Getzler and M. Vergne, Heat kernels and Dirac operators, Grundlehren, vol. 298, Springer, Berlin, 1991.
[8] N. Berline and M. Vergne, The Chern character of a transversally elliptic symbol and the equivariant index, Invent. Math., 124, 1996, p. 11-49.
[9] N. Berline and M. Vergne, L’indice équivariant des opérateurs transversalement elliptiques, Invent. Math., 124, 1996, p. 51-101.
[10] M. Brion, Variétés Sphériques. Notes de la session de S.M.F. “Opérations hamiltoniennes et opération de groupes algébriques” (Grenoble), p. 1-60, 1997.
[11] J.-L. Brylinski, Decomposition simpliciale d’un réseau, invariente par un groupe fini d’automorphismes, C. R. Acad. Sci. Paris, 288, 1979, no 2, A137-A139.
[12] J. L. Colliot-Thélène, D. Harari and A. N. Skorobogatov, Compactification équivariante d’un tore (d’après Brylinski et Kunnemann), Expositiones Mathematicae 23, 2005, p. 161-170.
C. De Concini, C. Procesi, Complete symmetric varieties. In Invariant Theory (Montecatini, 1982), p. 1-44. Springer, Berlin, 1983.

C. De Concini, C. Procesi, Complete symmetric varieties II, Intersection theory. In Algebraic groups and related topics (Kyoto/Nagoya, 1983), p. 481-513. North-Holland, Amsterdam, 1985.

J. J. Duistermaat, The heat equation and the Lefschetz fixed point formula for the Spin\(^c\)-Dirac operator, Progress in Nonlinear Differential Equation and Their Applications, vol. 18, Birkhauser, Boston, 1996.

V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math., 67, 1982, p. 515-538.

L. Jeffrey and F. Kirwan, Localization and quantization conjecture, Topology, 36, 1997, p. 647-693.

G. Kempf and L. Ness, The length of vectors in representation spaces. Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), p. 233-243, Lecture Notes in Math., 732, Springer, Berlin, 1979.

F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Princeton Univ. Press, Princeton, 1984.

B. Kostant, Quantization and unitary representations, in Modern Analysis and Applications, Lecture Notes in Math., Vol. 170, Springer-Verlag, 1970, p. 87-207.

E. Lerman, Symplectic cut, Math Res. Lett. 2, 1995, p. 247-258.

E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, Non-Abelian convexity by symplectic cuts, Topology, 37, 1998, p. 245-259.

E. Meinrenken, On Riemann-Roch formulas for multiplicities, J. Amer. Math. Soc., 9, 1996, p. 373-389.

E. Meinrenken, Symplectic surgery and the Spin\(^c\)-Dirac operator, Advances in Math., 134, 1998, p. 240-277.

E. Meinrenken, R. Sjamaar, Singular reduction and quantization, Topology, 38, 1999, p. 699-762.

D. Mumford, J. Fogarty and F. Kirwan, Geometric Invariant Theory, 3rd Edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1992.

T. Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1988.

P.-E. Paradan, Localization of the Riemann-Roch character, J. Funct. Anal. 187, 2001, p. 442-509.

R. Sjamaar, Symplectic reduction and Riemann-Roch formulas for multiplicities, Bull. Amer. Math. Soc. 33, 1996, p. 327-338.

Y. Tian, W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, Invent. Math. 132, 1998, p. 229-259.

M. Vergne, Multiplicity formula for geometric quantization, Part I, Part II, and Part III, Duke Math. Journal 82, 1996, p. 143-179, p. 181-194, p. 637-652.

M. Vergne, Quantification géométrique et réduction symplectique, Séminaire Bourbaki 888, 2001.

J. Weitsman, Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds. EuroConférence Moshé Flato 2000, Part I (Dijon). Lett. Math. Phys., 56, 2001, no. 1, p. 31-40.

E. Witten, Two dimensional gauge theories revisited, J. Geom. Phys. 9, 1992, p. 303-368.

C. Woodward, The classification of transversal multiplicity-free group actions, Ann. Global Anal. Geom. 14, 1996, no. 1, p. 3-42.

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