Abstract

We consider the problem of super-replication (hedging without risk) for the Arbitrage Pricing Theory. The dual characterization of super-replication cost is provided. It is shown that the reservation prices of investors converge to this cost as their respective risk-aversion tends to infinity.

Keywords: Arbitrage Pricing Theory, super-replication, large markets, risk-neutral measures

JEL classification: D4; G1

1 Introduction

Arbitrage Pricing Theory (APT) was originally introduced by Ross ([28], [29]) and later extended by [18], [9], [10] and numerous other authors. The APT assumes an approximate factor model and states that the risky asset returns in a “large” financial market are linearly dependent on a finite set of random variables, termed factors, in a way that the residuals are uncorrelated with the factors and with each other.

The APT emphasizes the role of the covariance between asset returns and those exogenous factors, while the Capital Asset Pricing Model (CAPM) of [30, 22] is based on the covariance between asset returns and the endogenous market portfolio. One of the desirable aspects of the APT is that it can be empirically tested as argued, for example, in [14]. These remarkable conclusions had a huge bearing on empirical work, see [7], [4]. Papers on the theoretical aspects of APT mainly focused on showing that the model is a good approximation in a sequence of economies when there are “sufficiently many” assets, see for example [9], [10], [18], [19], [28], [27], [1].
Ross derives the APT pricing formula under the assumption of absence of asymptotic arbitrage in the sense that a sequence of asymptotically costless and riskless finite portfolios does not asymptotically yield a positive return. Mathematical finance subsequently took up the idea of a market involving a sequence of markets with an increasing number of assets in the so-called theory of large financial markets (see, among other papers, [20, 21]) and mainly study the characterization of a notion of absence of arbitrage, using sequence of portfolio involving finitely many assets where the classical notion of no arbitrage holds true i.e. non-negative portfolios with zero cost should have zero return. For the sake of generality, continuous trading was assumed in the overwhelming majority of related papers. But these generalizations somehow overshadowed the highly original ideas suggested in [28] where a one-step model was considered. They did not answer the following natural questions either: In the APT is there a way to consider strategies involving possibly all the infinitely many assets and to exclude exact arbitrage rather than asymptotic one? A first answer was given in [2] in a measure-theoretical setup. Then [26] proposes a straightforward concept of portfolios using infinitely many assets which we will use in the present paper, see Section 2 below. This notion leads to the existence of equivalent risk-neutral probability measures (also called martingale measures or pricing measures) which are equivalent probability measures under which the asset returns has probability zero.

While questions of arbitrage for APT have been extensively studied, other crucial topics – such as utility maximization or pricing – received little attention though these are important questions in today’s markets where there is a vast array of available assets. This is particularly conspicuous in the credit market where bonds of various maturities and issuers indeed constitute an entity that may be best viewed as a large financial market, see [11]. Questions of pricing inevitably arise and current literature on APT does not provide satisfactory answers. A standard problem is calculating the superreplication cost of a claim $G$. It is the minimal amount needed for an agent selling $G$ in order to superreplicate $G$ by trading in the market. This is the hedging price with no risk and, to the best of our knowledge, it was first introduced in [5] in the context of transaction costs. In complete markets with finitely many assets the superreplication cost is just the cash flows expectation computed under the unique martingale measure. But when such markets are incomplete, there exists a so-called dual representation in terms of supremum of those expectations computed under the different risk-neutral probability measures. see e.g. [15]. Our first contribution is such a representation theorem for APT under mild conditions, see Theorem 4.2 below.

We are also able to prove the existence of optimizers for utility functions on the positive real axis, see Theorem 5.2. Such results are standard for finitely many assets, see e.g. [15]. In the context of APT, the case of
utility functions defined on the real line (i.e. admitting losses) has been considered in [26]. Finally, we establish that, when risk aversion tends to infinity, the utility indifference (or reservation) prices (see [17]) tend to the superreplication price. This links in a nice way investors’ price calculations to the preference-free cost of superhedging, see Theorem 6.2.

The model is presented in Section 2. Concepts of no arbitrage are discussed in Section 3. The dual characterization of superreplication prices in given in Section 4, the utility maximization problem is treated in Section 5. Finally, the asymptotics of reservation prices in the high risk-aversion regime is investigated in Section 6.

2 The large market model

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We consider a one step economy which contains a countable number of tradeable assets. The price of asset $i \in \mathbb{N}$ is given by $(S_i^t)_{t \in \{0,1\}}$. The returns $R_i$, $i \in \mathbb{N}$ represent the profit (or loss) created tomorrow from investing one dollar’s worth of asset $i$ today, i.e. $R_i = \frac{S_i^1}{S_i^0} - 1$. We briefly describe below our version of the Arbitrage Pricing Model, identical to that of [20, 25, 26], which is a special case of the model presented in [28, 18].

We assume that the assets’ returns are given by

$$
R_0 := r, \quad R_i := \mu_i + \bar{\beta}_i \varepsilon_i, \quad 1 \leq i \leq m;
$$

$$
R_i := \mu_i + \sum_{j=1}^{m} \beta_i^j \varepsilon_j + \bar{\beta}_i \varepsilon_i, \quad i > m,
$$

where the $\varepsilon_i$ are random variables and $r, \mu_i, \beta_i^j, \bar{\beta}_i$ are constants. Asset 0 represents a riskless investment with a constant rate of return $r \in \mathbb{R}$. For simplicity, we set $r = 0$ i.e. $S_0^0 = S_0^0$ from now on. The random variables $\varepsilon_i$, $i = 1, \ldots, m$ serve as factors which influence the return on all the assets $i \geq 1$ while $\varepsilon_i$, $i > m$ are random sources particular to the individual assets $R_i$, $i > m$.

Assumption 2.1 The $\varepsilon_i$ are square-integrable, independent random variables satisfying

$$
E(\varepsilon_i) = 0, \quad E(\varepsilon_i^2) = 1, \quad i \geq 1.
$$

Remark 2.2 If the $\varepsilon = (\varepsilon_i)_{i \geq 1}$ fail to be independent then the market will not display good pricing properties (namely, there may not exist a martingale measure having a second moment, see Proposition 4 of [25]).
We further assume $\beta_i \neq 0$, $i \geq 1$ and reparametrize the model by introducing

$$b_i := -\frac{\mu_i}{\beta_i}, \quad 1 \leq i \leq m;$$

$$b_i := -\frac{\mu_i}{\beta_i} + \sum_{j=1}^{m} \frac{\mu_j \beta_i}{\beta_j \beta_i}, \quad i > m$$

and set $b = (b_i)_{i \geq 1}$. Asset returns take then the following form

$$R_i = \bar{\beta}_i (\varepsilon_i - b_i), \quad 1 \leq i \leq m;$$

$$R_i = \sum_{j=1}^{m} \beta_j (\varepsilon_j - b_j) + \bar{\beta}_i (\varepsilon_i - b_i), \quad i > m.$$ 

For some $n \in \mathbb{N}$, a portfolio $\phi$ in the assets numbered $0, \ldots, n$ is an arbitrary sequence $\phi_i$, $0 \leq i \leq n$ of real numbers satisfying

$$\sum_{i=0}^{n} \phi_i S_0^i = x, \quad (1)$$

where $x$ is a given initial wealth. Such a portfolio will have value tomorrow given by

$$V_n^{x,\phi} := \sum_{i=0}^{n} \phi_i S_1^i = \left( x - \sum_{i=1}^{n} \phi_i S_0^i \right) \frac{S_1^0}{S_0^0} + \sum_{i=1}^{n} \phi_i S_1^i = x + \sum_{i=1}^{n} \phi_i S_0^i R_i = x + \sum_{i=1}^{n} \psi_i R_i,$$

using (1), $r = 0$ and where $\psi_i = \phi_i S_0^i$ is the amount held at time 1 in asset $i$. Using our parametrization one can easily rewrite that for some $(h_1, \ldots, h_n) \in \mathbb{R}^n$

$$V_n^{x,\phi} = x + \sum_{i=1}^{n} h_i (\varepsilon_i - b_i) =: V_n^{x,h}.$$ 

The value tomorrow that can be attained using finitely many assets is given by

$$J^x := \bigcup_{n \geq 1} \left\{ V_n^{x,h}, \ (h_1, \ldots, h_n) \in \mathbb{R}^n \right\}.$$ 

As $J^x$ fails to be closed in any reasonable sense, we consider strategies which can use infinitely many assets. This is more desirable from an economical
point of view, see for example [2]. We require that those strategies belong to
\[ \ell_2 := \left\{ (h_i)_{i \geq 1}, \sum_{i=1}^{\infty} h_i^2 < \infty \right\} \]
the family of square-summable sequences for integrability reasons. Recall that \( \ell_2 \) is a Hilbert space with the norm \( ||h||_{\ell_2} := \sqrt{\sum_{i=1}^{\infty} h_i^2} \). We denote by \( \Phi \) the function mapping \( \ell_2 \) to \( L^2(\Omega, \mathcal{F}, P) := \{ X : \Omega \to \mathbb{R}, E|X|^2 < \infty \} \) (shortly denoted by \( L^2(P) \) from now on) – the space of square-integrable random variables, which is again a Hilbert space with the norm \( ||X||_{L^2} := \sqrt{E(|X|^2)} \) – and which is defined by
\[ \Phi(h) := \sum_{i=1}^{\infty} h_i \varepsilon_i. \]
First the infinite sum in \( \Phi(h) \) has to be understood as the limit in \( L^2(P) \) of \( \left( \sum_{i=1}^{n} h_i \varepsilon_i \right)_{n \geq 1} \), which are Cauchy sequences. Indeed, under Assumption 2.1, for \( m > n \),
\[ E \left( \left( \sum_{i=1}^{m} h_i \varepsilon_i - \sum_{i=1}^{n} h_i \varepsilon_i \right)^2 \right) = \sum_{i=n+1}^{m} h_i^2 E(\varepsilon_i^2) + 2 \sum_{n+1 \leq i < j \leq m} h_i h_j E(\varepsilon_i \varepsilon_j) \]
\[ = \sum_{i=n+1}^{m} h_i^2 \leq \sum_{i=n+1}^{\infty} h_i^2, \]
which can be arbitrarily small for \( n \) large enough, since it is the tail sum of a converging series. Actually, under Assumption 2.1, \( \Phi \) is even an isometry by the same computation:
\[ ||\Phi(h)||_{L^2}^2 = \sum_{i=1}^{\infty} h_i^2 = ||h||_{\ell_2}^2. \]
We would like to give sense (as an \( L^2(P) \) limit of a sequence of finite sums) to the portfolio value
\[ V^{x,h} := x + \sum_{i=1}^{\infty} h_i (\varepsilon_i - b_i). \]
Using again the same kind of computation, for \( h \in \ell_2 \),
\[ E \left( \left( \sum_{i=1}^{m} h_i (\varepsilon_i - b_i) - \sum_{i=1}^{n} h_i (\varepsilon_i - b_i) \right)^2 \right) = \sum_{i=n+1}^{m} h_i^2 + \sum_{i=n+1}^{m} h_i^2 b_i^2. \]
So without any assumption on \( b \), one can not expect that the finite sums \( (\sum_{i=1}^{n} h_i (\varepsilon_i - b_i))_{n \geq 1} \) converge in \( L^2(P) \). Hence we stipulate the following as well.
Assumption 2.3 We have that $b \in \ell_2$.

Then (4) shows that $(\sum_{i=1}^{\infty} h_i (\varepsilon_i - b_i))_{n \geq 1}$ is a Cauchy-sequence in $L^2(P)$ and the infinite sum in (3) can be understood as an $L^2(P)$ limit of finite sums. Notice furthermore that

$$E \left( \sum_{i=1}^{\infty} h_i (\varepsilon_i - b_i) \right)^2 = \|h\|_{\ell_2}^2 + \|hb\|_{\ell_2}^2 \leq (1 + \|b\|_{\ell_2}^2)\|h\|_{\ell_2}^2 < \infty. \quad (5)$$

From now, we will use the notation

$$\langle h, \varepsilon - b \rangle := \sum_{i=1}^{\infty} h_i (\varepsilon_i - b_i).$$

Under Assumptions 2.1 and 2.3 the value tomorrow that can be attained using infinitely many assets with a strategy in $\ell_2$ is thus given by

$$K^x := \{V^{x,h}, h \in \ell_2\} = \{x + \langle h, \varepsilon - b \rangle, h \in \ell_2\}.$$

3 No arbitrage in large markets

In Arbitrage Pricing Theory, the classical notion of arbitrage is the asymptotic arbitrage in the sense of Ross (1976) and Huberman (1982).

Definition 3.1 There is an asymptotic arbitrage if there exists a sequence of strategies $(h(n))_{n \geq 1}$, with $h(n) = (h(n)_i)_{1 \leq i \leq n}$, such that $V^{x,h(n)}_n$ satisfies

$$E(V^{x,h(n)}_n) \xrightarrow{n \rightarrow +\infty} \infty \text{ and } \text{Var}(V^{x,h(n)}_n) \xrightarrow{n \rightarrow +\infty} 0.$$

If there exists no such sequence, then we say that there is absence of asymptotic arbitrage (AAA).

We would like to understand the link between AAA and the classical definition of no arbitrage, which says roughly that if the value of a portfolio at time 1 – with value at time zero equal to 0 – is non-negative then it should be zero. The no-arbitrage condition on a “small market” with $N$ random sources (called AOA($N$)) for some $N \geq 1$ holds true if $P \left( \sum_{i=1}^{N} h_i (\varepsilon_i - b_i) \geq 0 \right) = 1$ for $(h_1, \ldots, h_N) \in \mathbb{R}^N$ implies that $h_1 = \ldots = h_N = 0$. We prove in Lemma 3.3 that under the following assumption there is absence of arbitrage in any of the markets containing $N$ assets.

Assumption 3.2 For all $i \geq 1$,

$$P(\varepsilon_i > b_i) > 0 \text{ and } P(\varepsilon_i < b_i) > 0.$$
Lemma 3.3  Under Assumption 2.1, Assumption 3.2 implies $\text{AOA}(N)$ for any $N \geq 1$. Moreover, $\text{AOA}(N)$ implies the so called quantitative no arbitrage condition: There exists some $\alpha_N \in (0, 1)$ such that for every $(h_1, \ldots, h_N) \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} h_i^2 = 1$

$$P \left( \sum_{i=1}^{N} h_i (\varepsilon_i - b_i) < -\alpha_N \right) > \alpha_N. \quad (6)$$

Note that (6) implies that

$$P \left( \sum_{i=1}^{N} h_i (\varepsilon_i - b_i) < 0 \right) > 0. \quad (7)$$

Proof. We first prove (7). Let $(h_1, \ldots, h_N) \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} h_i^2 = 1$. Let $B_i = \{h_i (\varepsilon_i - b_i) < 0 \}$ and $I_N = \{i \in \{1, \ldots, N\}, \; h_i \neq 0\}$. Then

$$\bigcap_{i \in I_N} B_i \subset \left\{ \sum_{i=1}^{N} h_i (\varepsilon_i - b_i) < 0 \right\}.$$

As for $i \in I_N$, $P(B_i) \geq \min\{P(\{\varepsilon_i - b_i < 0\}), P(\{\varepsilon_i - b_i > 0\})\} > 0$ and the $(\varepsilon_i)_{i \geq 1}$ are independent, we get that

$$P \left( \bigcap_{i \in I_N} B_i \right) = \prod_{i \in I_N} P(B_i) > 0$$

and (7) holds true. Since this implies that the return on every non-zero portfolio is negative with positive probability, $\text{AOA}(N)$ follows for every $N \geq 1$.

We now prove (6). Introduce the following set for $n \geq 1$

$$A_n = \left\{ h \in \mathbb{R}^N : \sum_{i=1}^{N} h_i^2 = 1, \; P \left( \sum_{i=1}^{N} h_i (\varepsilon_i - b_i) < -\frac{1}{n} \right) \leq \frac{1}{n} \right\}.$$ 

Let $n_0 := \inf\{n \geq 1, A_n = \emptyset\}$. Assume that $n_0 = \infty$. For all $n \geq 1$, we thus get some $h(n) \in \mathbb{R}^N$ with $\sum_{i=1}^{N} h_i(n)^2 = 1$ and such that $P(B_n) \leq \frac{1}{n}$, with $B_n := \{\sum_{i=1}^{N} h_i(n)(\varepsilon_i - b_i) < -1/n\}$. By passing to a sub-sequence we can assume that $h(n)$ tends to some $h^* \in \mathbb{R}^N$ with $\sum_{i=1}^{N} h^*_i)^2 = 1$. Then

$$\{\sum_{i=1}^{N} h_i^*(\varepsilon_i - b_i) < 0\} \subset \lim\inf_n B_n$$

and Fatou’s Lemma implies that

$$P \left( \sum_{i=1}^{N} h^*_i (\varepsilon_i - b_i) < 0 \right) \leq \lim\inf_n P(B_n) = 0.$$
This implies that 

\[ P \left( \sum_{i=1}^{N} h_i^* (\epsilon_i - b_i) \geq 0 \right) = 1, \] 

which contradicts (7). Thus \( n_0 < \infty \), we can set \( \alpha_N = \frac{1}{n_0} \) and for every \( (h_1, \ldots, h_N) \in \mathbb{R}^N \) satisfying 

\[ \sum_{i=1}^{N} h_i^2 = 1 \] 

(6) holds true. \( \blacksquare \)

It is well-known that absence of arbitrage in markets with finitely many assets is equivalent to the existence of an equivalent martingale measure, see e.g. [13, 15]. In the present setting with infinitely many assets we need to use a concept that is somewhat more technical. We say that EMM2 (equivalent martingale measure with density in \( L^2 \)) holds true if the set of martingale measures having a finite moment of order 2 is not empty i.e.

\[ \mathcal{M}_2 := \left\{ Q \sim P, \frac{dQ}{dP} \in L^2(P), E_Q(\epsilon_i) = b_i, \forall i \geq 1 \right\} \neq \emptyset. \]

**Remark 3.4** If \( Q \in \mathcal{M}_2 \) and if Assumptions 2.1 and 2.3 hold true then \( E_Q(V^{0,h}) = 0 \) for all \( h \in \ell_2 \). Indeed, \( (\sum_{i=1}^{n} h_i(\epsilon_i - b_i))_{n \geq 1} \) converges in \( L^2 \) to \( V^{0,h} \) and for all \( n \geq 1, \)

\[ E_Q\left( \sum_{i=1}^{n} h_i(\epsilon_i - b_i) \right) = \sum_{i=1}^{n} h_i E_Q(\epsilon_i - b_i) = 0. \]

Using the Cauchy-Schwarz inequality, we get that

\[
\left| E_Q\left( \sum_{i=1}^{n} h_i(\epsilon_i - b_i) \right) - E_Q(V^{0,h}) \right| \\
\leq \left( E\left( \frac{dQ}{dP} \right)^2 \right)^{1/2} \left( E\left( \sum_{i=1}^{n} h_i(\epsilon_i - b_i) - V^{0,h} \right)^2 \right)^{1/2} \rightarrow 0,
\]

as \( n \rightarrow \infty \). Thus \( E_Q(V^{0,h}) = \lim_{n \rightarrow \infty} E_Q\left( \sum_{i=1}^{n} h_i(\epsilon_i - b_i) \right) = 0, \) see also Lemma 3.4 of Rásonyi (2016).

Unfortunately Assumptions 2.1, 2.3 and 3.2 are not sufficient to ensure that EMM2 holds true (see Proposition 4 of [24]). So we also postulate the following.

**Assumption 3.5** We have that

\[ \sup_{i \geq 1} E \left[ |\epsilon_i|^3 \right] < \infty. \]

In Corollary 1 of [25] it is showed that under Assumptions 2.1, 3.2 and 3.5

\[ \text{AAA} \iff \text{Assumption 2.3} \iff \text{EMM2}. \quad (8) \]

With this in hand, one can show that AAA implies the classical no arbitrage condition stated with infinitely many assets.
Lemma 3.6 Assume that Assumptions 2.1, 3.2 and 3.5 holds true. Assume that AAA holds true. Then, if for some $h \in \ell_2$ one has that $\langle h, \varepsilon - b \rangle \geq 0$ a.s. then $\langle h, \varepsilon - b \rangle = 0$ a.s.

Proof. Under AAA, we have that both Assumption 2.3 and EMM2 holds true (see (5)). Let $h \in \ell_2$. Then $\langle h, \varepsilon - b \rangle$ is well-defined. Assume that $\langle h, \varepsilon - b \rangle \geq 0$. Fix some $Q \in \mathcal{M}_2$, then $E_Q(\langle h, \varepsilon - b \rangle) = 0$ (see Remark 3.4). Thus $\langle h, \varepsilon - b \rangle = 0$ $Q$-a.s. and also $P$-a.s. since $P$ and $Q$ are equivalent. $\blacksquare$

Lemma 3.7 Assume that Assumptions 2.1 and 2.3 hold true and that $\sup_{i \geq 1} E|\varepsilon_i|^\gamma \leq C_1 < \infty$ for some $\gamma \geq 2$. Then there is a constant $C_\gamma$ such that, for all $h \in \ell_2$

$$E|\langle h, \varepsilon - b \rangle|^\gamma \leq C_\gamma \|h\|_{\ell_2}^\gamma \left(1 + \|b\|_{\ell_2}^\gamma\right).$$

Remark 3.8 Note that if $\{X_n|\gamma, n \in \mathbb{N}\}$ is uniformly integrable then for any $\beta \leq \gamma$, $\{X_n|\beta, n \in \mathbb{N}\}$ is also uniformly integrable. Indeed, trivially $|X_n|^{\beta} \leq |X_n|^\gamma + 1$ and then

$$\lim_{N \to \infty} \sup_n E[|X_n|^{\beta}1_{\{|X_n|^{\beta} \geq N\}}] \leq \lim_{N \to \infty} \sup_n E[(|X_n|^\gamma + 1)1_{\{|X_n|^\gamma \geq N-1\}}] \leq \lim_{N \to \infty} \sup_n E[2|X_n|^\gamma1_{\{|X_n|^\gamma \geq N-1\}}] = 0$$

by the definition of uniform integrability. Here the second inequality holds since, for $N \geq 2$ on the set $\{|X_n|^\gamma \geq N-1\}$ one has $1 \leq |X_n|^\gamma$.

Thus, under Assumptions 2.1, 2.3 and 5.5 for any $c > 0$, $\{(V^x, h)^2, h \in \ell_2, \|h\|_{\ell_2} \leq c\}$ and also $\{|V^x, h|, h \in \ell_2, \|h\|_{\ell_2} \leq c\}$ are uniformly integrable.

Proof. Let $h(n) := (h_1, \ldots, h_n, 0, 0, \ldots)$ and $b(n) := (b_1, \ldots, b_n, 0, 0, \ldots)$, $n \geq 1$. The Marcinkiewicz-Zygmund and triangle inequalities imply for
some $C_2 > 0$ that
\[
E|\langle h(n), \varepsilon - b \rangle|^\gamma = E \left| \sum_{i=1}^{n} h_i(\varepsilon_i - b_i) \right|^\gamma \\
\leq C_2 E \left( \sum_{i=1}^{n} h_i^2(\varepsilon_i - b_i)^2 \right)^{\gamma/2} = C_2 \left\| \sum_{i=1}^{n} h_i^2(\varepsilon_i - b_i)^2 \right\|_{L^{\gamma/2}}^{\gamma/2} \\
\leq C_2 \left( \sum_{i=1}^{n} \| h_i^2(\varepsilon_i - b_i)^2 \|_{L^{\gamma/2}} \right)^{\gamma/2} = C_2 \left( \sum_{i=1}^{n} |h_i|^2 \| \varepsilon_i - b_i \|^2_{L^\gamma} \right)^{\gamma/2} \\
\leq C_2^{2\gamma/2} \left( \sup_{i \geq 1} \| \varepsilon_i \|_{L^\gamma}^2 + \sum_{i=1}^{n} |h_i|^2 \| b_i \|^2 \right)^{\gamma/2} \\
\leq C_2^{2\gamma-1} \left( \sup_{i \geq 1} E|\varepsilon_i|^{\gamma} \| h(n) \|_{\ell_2} \| h(n) \|_{\ell_2} \| b(n) \|_{\ell_2} \right) \\
\leq C_2^{2\gamma-1} \| h(n) \|_{\ell_2} \left( C_1 + \| b(n) \|_{\ell_2} \right) \\
\leq C_2 \| h(n) \|_{\ell_2} \left( 1 + \| b(n) \|_{\ell_2} \right)
\]
and Fatou’s lemma finishes the proof. \qed

For all $x \geq 0$, we introduce the set of attainable wealth at time 1, allowing the possibility of throwing away money:
\[
C^x := K^x - L^+_2.
\]

**Proposition 3.9** Assume that Assumptions 2.7, 2.8, 3.2 and 3.3 hold true. Fix some $z \in \mathbb{R}$ and let $B \in L^2(P)$ such that $B \notin C^x$. Then there exists some $\eta > 0$ such that
\[
\inf_{h \in \ell_2} P(z + \langle h, \varepsilon - b \rangle < B - \eta) > \eta.
\]

**Proof.** Assume that (10) is not true. Then, for all $n \geq 1$, there exists some $h(n) \in \ell_2$ such that $P(V_n < B - \frac{\eta}{n}) \leq \frac{1}{n}$, where $V_n := z + \langle h(n), \varepsilon - b \rangle$. Set $\kappa_n := (V_n - (B - \frac{\eta}{n})) 1_{\{V_n \geq B - \frac{\eta}{n}\}}$. Then $P(|V_n - \kappa_n - B| > \frac{\eta}{n}) = P(V_n < B - \frac{\eta}{n}) \leq \frac{1}{n}$ and thus $(V_n - \kappa_n)_{n \geq 1}$ converges to $B$ in probability.

First we prove that $\sup_n \| h(n) \|_{\ell_2} < \infty$. Indeed assume that $\sup_n \| h(n) \|_{\ell_2} = \infty$. By extracting a subsequence (which we continue to denote by $n$) we may and will assume $\| h(n) \|_{\ell_2} \to \infty$, $n \to \infty$. Define $h_i(n) := h_i(n)/\| h(n) \|_{\ell_2}$
for all $n, i$. Clearly, $\hat{h}(n) \in \ell_2$ with $\|\hat{h}(n)\|_{\ell_2} = 1$ and $V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}}$ goes to 0 in probability and also a.s. Now under Assumptions 2.1, 2.3, 3.2 and 3.5 there exists some equivalent martingale probability measure $Q \in \mathcal{M}_2$ (see (8)). If

$$E_Q \left( V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}} \right) \to 0,$$  \hspace{1cm} (11)

as $E_Q V^{0,\hat{h}(n)} = 0$ (see Remark 3.4) we deduce that $\frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}}$ goes to zero in $L^1(Q)$ and also $Q$-a.s. (along a subsequence) and, as $Q$ is equivalent to $P$, $P$-a.s. This implies that $V^{0,\hat{h}(n)}$ goes to 0 a.s. As the family $\{V^{0,\hat{h}(n)}\}, n \geq 1$ for $\|\hat{h}(n)\|_{\ell_2} \leq 1$ is uniformly integrable (see Assumption 3.5 and Remark 3.8), we get that $V^{0,\hat{h}(n)} \to 0$ in $L^2(P)$ as well. But this is absurd since using the isometry property (see (10)), we get that

$$\|V^{0,\hat{h}(n)}\|_{L^2}^2 = \|\hat{h}(n)\|_{\ell_2}^2 + \|\hat{h}(n)b\|_{\ell_2}^2 \geq 1$$

for all $n \geq 1$. This contradiction shows that necessarily $\sup_n \|\hat{h}(n)\|_{\ell_2} < \infty$. Now we prove that (11) holds true. Since $V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}}$ goes to 0 $P$-a.s. and by the Cauchy-Schwarz inequality,

$$E_Q \left( V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}} \right) \leq \sqrt{E \left( \frac{dQ}{dP} \right)^2} \sqrt{E \left( V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}} \right)^2}.$$

It remains to show the uniform integrability of $\left( V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}} \right), n \in \mathbb{N}$ under $P$. Notice that

$$\left| V^{0,\hat{h}(n)} - \frac{\kappa_n}{\|\hat{h}(n)\|_{\ell_2}} \right|^2 = \frac{|B - z - \frac{1}{N}\sum_{i=1}^{\infty} V^{0,\hat{h}(n)}\{\hat{h}(n)\geq \frac{n - i}{|\max(n)|_{\ell_2}}\} + |V^{0,\hat{h}(n)}|}{\|\hat{h}(n)\|_{\ell_2}^2} \leq \frac{|B|^2 + |z|^2 + \frac{1}{N}}{\|\hat{h}(n)\|_{\ell_2}^2} + |V^{0,\hat{h}(n)}| \leq c|B|^2 + |V^{0,\hat{h}(n)}|^2,$$

for $n$ big enough, with some constant $c$. Since we know the uniform integrability under $P$ of $|V^{0,\hat{h}(n)}|^2, n \in \mathbb{N}$ and that $B^2$ is also integrable, (11) is proved.

We have concluded that $\sup_n \|\hat{h}(n)\|_{\ell_2} < \infty$. Then the Banach-Saks Property (recall that $\ell_2$ has the Banach-Saks Property), there exists a subsequence $(n_k)_{k \geq 1}$ and some $h^* \in \ell_2$ such that for $\hat{h}(N) := \frac{1}{N} \sum_{k=1}^{N} h(n_k)$

$$\|\hat{h}(N) - h^*\|_{\ell_2}^2 \to 0, \hspace{1cm} N \to \infty.$$

Hence, using (5),

$$E \left( V^{z,\hat{h}(N)} - V^{z,h^*} \right)^2 \leq E \left( \|\hat{h}(N) - h^*\|_{\ell_2}^2 \right) \leq (1 + \|b\|_{\ell_2}^2) \|\hat{h}(N) - h^*\|_{\ell_2}^2 \to 0, \hspace{1cm} N \to \infty.$$
So $V^{z,\hat{h}(N)} \to V^{z,h^*}$ a.s. as well. Then

$$V^{z,\hat{h}(N)} = z + \hat{h}(N), \varepsilon - b) = \frac{1}{N} \sum_{k=1}^{N} (z + \langle h(n_k), \varepsilon - b \rangle) = \frac{1}{N} \sum_{k=1}^{N} V^{z,h(n_k)}$$

and $V^{z,\hat{h}(N)} - \frac{1}{N} \sum_{k=1}^{N} \kappa_{n_k}$ converges to $B$ in probability and also a.s. for a subsequence for which we keep the same notation. This implies that $\frac{1}{N} \sum_{k=1}^{N} \kappa_{n_k}$ converges a.s. and thus $B \in C^z$, a contradiction. □

**Corollary 3.10** Assume that Assumptions 2.1, 2.3, 3.2 and 3.5 hold true and fix some $z \in \mathbb{R}$. Then $C^z$ is closed in probability.

**Proof.** We prove that there is an equivalence between (i) $C^z$ is closed in probability and (ii) $B \in L^2(P) \setminus C^z$ implies that there exists some $\eta > 0$ such that

$$\inf_{h \in \ell_2} P(z + \langle h, \varepsilon - b \rangle < B - \eta) > \eta. \quad (12)$$

Assume that $C^z$ is closed in probability and let $B \in L^2(P) \setminus C^z$. If (12) does not hold true, going through the first paragraph of the proof of Proposition 3.9 one gets that $B \in C^z$ and the contradiction is immediate. Conversely, assume that $C^z$ is not closed in probability. Then one can find some $h(n) \in \ell_2$ and $\kappa_n \in L^+_2$ such that $\theta_n = z + \langle h(n), \varepsilon - b \rangle - \kappa_n \in C^z$ converges in probability to some $\theta^* \notin C^z$. Then

$$\inf_{h \in \ell_2} P(z + \langle h, \varepsilon - b \rangle < \theta^* - \eta) \leq P(z + \langle h(n), \varepsilon - b \rangle - \kappa_n < \theta^* - \eta) \to 0,$$

which contradicts (12). So (i) and (ii) are equivalent. But Proposition 3.9 shows that (ii) holds true under Assumptions 2.1, 2.3, 3.2 and 3.5. We conclude that (i) holds true: $C^z$ is closed in probability. □

We now provide a quantitative version of the NA condition (see Assumption 3.1) in the spirit of (6).

**Proposition 3.11** Assume that Assumptions 2.1, 2.3, 3.2 and 3.5 hold true. Then there exists some $\alpha > 0$, such that for all $h \in \ell_2$ satisfying $\|h\|_{\ell_2} = 1$

$$P(\langle h, \varepsilon \rangle < -\alpha) > \alpha.$$ 

Note that changing $h$ by $-h$, we find that $P(\langle h, \varepsilon \rangle > \alpha) > \alpha$ and thus

$$P(|\langle h, \varepsilon \rangle| > \alpha) = P(\langle h, \varepsilon \rangle < -\alpha) + P(\langle h, \varepsilon \rangle > \alpha) > 2\alpha.$$

**Proof.** We argue by contradiction. Assume that for $n \geq 1$, there exists $h(n)$ with $\|h(n)\|_{\ell_2} = 1$ and

$$P(< h(n), \varepsilon > < -\frac{1}{n}) \leq \frac{1}{n}.$$
Using Hölder’s inequality and the fact that \( \{ \langle h(n), \varepsilon \rangle \geq -\frac{1}{n} \} \subset \{ | \langle h(n), \varepsilon \rangle | \leq \frac{1}{n} \} \), we get that

\[
E(\langle h(n), \varepsilon \rangle^2) = E(\langle h(n), \varepsilon \rangle^2 1_{\{\langle h(n), \varepsilon \rangle \leq -\frac{1}{n} \}}) + E(\langle h(n), \varepsilon \rangle^2 1_{\{\langle h(n), \varepsilon \rangle \geq -\frac{1}{n} \}})
\]

\[
\leq \left( E(\langle h(n), \varepsilon \rangle^2) \right)^{\frac{3}{2}} \left( E(1_{\{\langle h(n), \varepsilon \rangle \leq -\frac{1}{n} \}}) \right)^{\frac{1}{2}} + \frac{1}{n^2}
\]

\[
\leq \left( E(\langle h(n), \varepsilon \rangle^2) \right)^{\frac{3}{2}} \left( \frac{1}{n} \right)^{\frac{1}{2}} + \frac{1}{n^2}
\]

\[
\leq C_3^{\frac{2}{3}} \left( \frac{1}{n} \right)^{\frac{1}{2}} + \frac{1}{n^2}.
\]

using Lemma 3.7. When \( n \to \infty \), this contradicts (recall \((2)\))

\[
E(\langle h(n), \varepsilon \rangle^2) = \| h(n) \|^2_{\ell^2} = 1.
\]

\[\square\]

The following lemma proves that under the NA condition (see Assumption 3.2) any admissible strategy is bounded.

**Lemma 3.12** Assume that Assumptions 2.1, 2.3, 3.2 and 3.5 hold true. Let \( y \geq 0 \) and \( h \in \ell^2 \) such that \( y + \langle h, \varepsilon - b \rangle \geq 0 \). Then there exists some \( \hat{\alpha} \) such that

\[
\| h \|_{\ell^2} \leq \frac{y}{\hat{\alpha}}.
\]

**Proof.** Recall \( \alpha > 0 \) from Proposition 3.11. As \( b \in \ell^2 \), there exists some \( M_\alpha \geq 1 \) such that \( \left( \sum_{i \geq M_\alpha} b_i^2 \right)^{1/2} \leq \alpha/2 \). By the admissibility condition, we get that a.s

\[
y + \sum_{i=1}^{M_\alpha-1} h_i(\varepsilon_i - b_i) + \sum_{i \geq M_\alpha} h_i(\varepsilon_i - b_i) \geq 0. \tag{13}
\]

If all the \( h_i \) are zero then there is nothing to prove. Assume that \( h_i \neq 0 \) for some \( i \in \{1, \ldots, M_\alpha - 1\} \). Let

\[
A = \left\{ \sum_{i=1}^{M_\alpha-1} h_i(\varepsilon_i - b_i) < 0 \right\} \quad \text{and} \quad B = \left\{ y + \sum_{i \geq M_\alpha} h_i(\varepsilon_i - b_i) \leq 0 \right\}.
\]

From the no arbitrage condition in the market with \( M_\alpha - 1 \) assets (see Assumption 3.2 and \((7)\) in Lemma 3.3) we get that \( P(A) > 0 \). Assume that \( P(B) > 0 \). As the \( (\varepsilon_i)_{i \geq 1} \) are independent, we get that \( P(A \cap B) = P(A)P(B) > 0 \) which contradicts the admissibility of \( h \) (see \((13)\)). Thus a.s.

\[
y + \sum_{i \geq M_\alpha} h_i(\varepsilon_i - b_i) \geq 0. \tag{14}
\]
Let $\hat{h} = (0, \ldots, 0, h_{M\alpha}, h_{M\alpha+1}, \ldots)$ and $\hat{b} = (0, \ldots, 0, b_{M\alpha}, b_{M\alpha+1}, \ldots)$. Recall that $\|\hat{b}\|_{\ell_2} \leq \alpha/2$. Proposition 3.11 implies that

$$P\left( \frac{\langle \hat{h}, \varepsilon \rangle}{\|\hat{h}\|_{\ell_2}} < -\alpha \right) > \alpha.$$ 

Then on the set $\{ \frac{\langle \hat{h}, \varepsilon \rangle}{\|\hat{h}\|_{\ell_2}} < -\alpha \}$, using (14), we get that

$$0 \leq y + \langle \hat{h}, \varepsilon \rangle - \langle \hat{h}, \hat{b} \rangle \leq y - \alpha \|\hat{h}\|_{\ell_2} + \|\hat{b}\|_{\ell_2} \|\hat{h}\|_{\ell_2}$$

Thus $\|\hat{h}\|_{\ell_2} \leq \frac{2y}{\alpha}$.

Now take $i \in \{1, \ldots, M\alpha - 1\}$ such that $h_i \neq 0$ and let

$$C = \left\{ \sum_{i \geq M\alpha} h_i (\varepsilon_i - b_i) \leq 0 \right\} \quad \text{and} \quad D = \left\{ y + \sum_{i=1}^{M\alpha-1} h_i (\varepsilon_i - b_i) < 0 \right\}.$$

We have that $E_Q(\sum_{i \geq M\alpha} h_i (\varepsilon_i - b_i)) = 0$ (recall Remark 3.4). Thus $Q(C) > 0$. As $Q$ and $P$ are equivalent, we get that $P(C) > 0$. Assume that $P(D) > 0$. As the $(\varepsilon_i)_{i \geq 1}$ are independent, we get that $P(C \cap D) = P(C) P(D) > 0$ which contradicts the admissibility of $h$ (see (13)). Thus a.s.

$$y + \sum_{i=1}^{M\alpha-1} h_i (\varepsilon_i - b_i) \geq 0. \quad (15)$$

Let $\tilde{h} = (h_1, \ldots, h_{M\alpha-1}), \tilde{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{M\alpha-1})$ and $\tilde{b} = (b_1, \ldots, b_{M\alpha-1})$. From the AOA$(M\alpha-1)$ (see Assumption 3.2 and (6) in Lemma 3.3), there exists some $\tilde{\alpha}$ such that

$$P\left( \frac{\langle \tilde{h}, \tilde{\varepsilon} - \tilde{b} \rangle}{\|\tilde{h}\|_{\ell_2}} < -\tilde{\alpha} \right) > \tilde{\alpha}.$$ 

Then on the set $\{ \frac{\langle \tilde{h}, \tilde{\varepsilon} - \tilde{b} \rangle}{\|\tilde{h}\|_{\ell_2}} < -\tilde{\alpha} \}$, we get from (15) that

$$0 \leq y + \langle \tilde{h}, \tilde{\varepsilon} - \tilde{b} \rangle \leq y - \tilde{\alpha} \|\tilde{h}\|_{\ell_2}.$$ 

Thus $\|\tilde{h}\|_{\ell_2} \leq \frac{2y}{\alpha}$ and

$$\|\tilde{h}\|_{\ell_2} = \left( \frac{\|\tilde{h}\|_{\ell_2}^2}{\alpha} + \frac{\|\tilde{h}\|_{\ell_2}^2}{\alpha^2} \right)^{\frac{1}{2}} \leq \sqrt{\frac{4y^2}{\alpha^2} + \frac{y^2}{\alpha^2}} = \frac{y}{\alpha}.$$ 

\[\square\]
4 Superreplication price

Let $G \in L^0$ be a random variable which will be interpreted as the payoff of some derivative security at time $T$. The superreplication price is the minimal initial wealth needed for hedging $G$ without risk

$$\pi(G) := \inf \{ z \in \mathbb{R} : V^{z,h} \geq G \text{ for some } h \in \ell_2 \}$$

where $\pi(G) = +\infty$ is the set is empty. We refer to [15] for more information about this preference-free price.

**Lemma 4.1** Assume that Assumptions 2.1, 2.3, 3.2 and 3.5 holds true. Then $\pi(G) > -\infty$ and there exists $h \in \ell_2$ such that $\pi(G) + \langle h, \varepsilon - b \rangle \geq G$ a.s.

**Proof.** Assume that $\pi(G) = -\infty$. Then for all $n \geq 1$, there exists $h_n \in \ell_2$ such that $-n + \langle h_n, \varepsilon - b \rangle \geq G$ a.s. Thus, $\langle h_n, \varepsilon - b \rangle \geq G + n \geq (G + n) \land 1$ a.s. It follows that $(G + n) \land 1 \in C^0$ (see [21]), which is closed in probability (see Corollary 3.10). Thus $1 \in C^0$, i.e. $\langle h, \varepsilon - b \rangle \geq 1$ a.s., which contradicts AAA (or Assumption 2.3), see Lemma 3.6. So $\pi(G) > -\infty$.

If $\pi(G) = +\infty$, the second claim is trivial. So we can assume that, $\pi(G)$ is finite. Then for all $n \geq 1$, there exists $h_n \in \ell_2$ such that $\pi(G) + \frac{1}{n} + \langle h_n, \varepsilon - b \rangle \geq G$ a.s. It follows that $G - \pi(G) - \frac{1}{n} \in C^0$ and thus as $C^0$ is closed $G - \pi(G) \in C^0$, i.e. there exists $h \in \ell_2$ such that $\pi(G) + \langle h, \varepsilon - b \rangle \geq G$ a.s. □

We are now in position to prove our duality result.

**Theorem 4.2** Assume that Assumptions 2.1, 2.3, 3.2 and 3.5 hold true and let $G \in L^2(P)$. Then

$$\pi(G) = \sup_{Q \in \mathcal{M}_2} E_Q(G).$$

**Proof.** Let $x$ be such that there exists $h \in \ell_2$ such that $x + \langle h, \varepsilon - b \rangle \geq G$ a.s. Fix $Q \in \mathcal{M}_2$ (which is non-empty by Corollary 1 of [25], see [8]). As $G \in L^2(P)$, $E_Q(G)$ is well-defined by the Cauchy-Schwarz inequality. Using Remark 3.4, $E_Q(x + \langle h, \varepsilon - b \rangle) = x$. Thus $x \geq E_Q(G)$ and as this holds true for all $Q \in \mathcal{M}_2$ and for all $x$ such that there exists $h \in \ell_2$ such that $x + \langle h, \varepsilon - b \rangle \geq G$ a.s., it follows that $\pi(G) \geq \sup_{Q \in \mathcal{M}_2} E_Q(G)$.

For the other inequality, it is enough to prove that $G - \sup_{Q \in \mathcal{M}_2} E_Q(G) \in C^0$. Indeed, this will imply that there exists $h \in \ell_2$ such that $\sup_{Q \in \mathcal{M}_2} E_Q(G) + \langle h, \varepsilon - b \rangle \geq G$ a.s. which shows, by definition of $\pi(G)$, that $\sup_{Q \in \mathcal{M}_2} E_Q(G) \geq \pi(G)$.

Assume this is not true. Then $\{G - \sup_{Q \in \mathcal{M}_2} E_Q(G) \} \cap (C^0 \cap L^2(P)) = \emptyset$. As $C^0$ is closed in probability (see Corollary 3.10), we can apply the Hahn-Banach theorem and there exists $Z \in L^2(P) \setminus \{0\}$ and $\theta, \gamma \in \mathbb{R}$ such that

$$E(ZX) \leq \gamma < \theta \leq E(Z(G - \sup_{Q \in \mathcal{M}_2} E_Q(G))).$$
Choose \( \tilde{E} \) in the small market with \( \tilde{Q} \) that \( \tilde{Q} \) proof. ✷

One may wonder whether Remark 4.3

\[
\text{Choose some } h^i \text{ to be the infinite sequence which is zero everywhere except in the } i\text{th coordinate: } h^i \in \ell_2. \text{ Then } <\pm h^i, \varepsilon - b> = \pm (\varepsilon_i - b_i) \in C^0 \cap L^2(P). \text{ Thus } E(Z(\varepsilon_i - b_i)) = 0 \text{ for all } i \geq 1.
\]

Define \( Q \) by \( \frac{dQ}{dP} = \frac{Z}{E(Z)}. \) Then \( \frac{dQ}{dP} \in L^2, E_Q(\varepsilon_i - b_i) = 0 \text{ for all } i \geq 1 \text{ and } E_Q(G) > \sup_{Q \in \mathcal{M}_2} E_Q(G). \)

Choose some \( Q \in \mathcal{M}_2 \) and let \( \eta \in (0,1). \) Set \( Q_\eta = (1 - \eta)Q + \eta\bar{Q}. \) It is clear that \( Q_\eta \in L^2 \) and that \( Q_\eta \sim P. \) It is also clear that \( E_{Q_\eta}(\varepsilon_i - b_i) = 0 \) for all \( i \geq 1 \text{ and } Q_\eta \in \mathcal{M}_2 \) follows. Moreover, one can always find some \( \eta \in (0,1), \) such that \( E_{Q_\eta}(G) > \sup_{Q \in \mathcal{M}_2} E_Q(G). \) This contradiction completes the proof.

\textbf{Remark 4.3} One may wonder whether \( \pi_n(G) \) the superreplication price of \( G \) in the small market with \( n \) random sources \( (\varepsilon_i)_{1 \leq i \leq n} \) converges to \( \pi(G) \) the superreplication price of \( G \) in the large market. The answer is \textit{no} in general: If \( G \) needs to be hedge with all the \( (\varepsilon_i)_{i \geq 1} \), then \( \pi_n(G) = +\infty. \) Let us sketch a concrete example: Let \( \varepsilon_i, i \in \mathbb{N} \) be standard Gaussian, let \( b_i = 0 \) for all \( i \in \mathbb{N} \) and define \( G := \sum_{i=1}^{\infty} i^{-1}\varepsilon_i. \) There exists no \( x, h_1, \ldots, h_n \) with

\[
x + \sum_{j=1}^{n} h_j\varepsilon_j \geq G
\]

since this would mean

\[
\sum_{j=1}^{n} (h_j - j^{-1})\varepsilon_j + \sum_{j \geq n+1} j^{-1}\varepsilon_j \geq -x
\]

where the left-hand side is a Gaussian random variable with non-zero variance. It follows that \( \pi_n(G) = \infty \) while \( \pi(G) = 0, \) trivially.

\section{Utility maximisation}

The idea of modeling preferences of agents by utility functions goes back to \[6. \] This approach was revived after the appearance of [23] and led to the axiomatic treatment [33]. We follow this traditional viewpoint and model economic agents’ preferences by concave increasing utility functions \( U \), where concavity of \( U \) is related to risk aversion.

So let us suppose that \( U : (0,\infty) \rightarrow \mathbb{R} \) is a concave strictly increasing differentiable function. Note that we extend \( U \) to \([0,\infty)\) by (right)-continuity \( (U(0) \text{ may be } -\infty). \) We also set \( U(x) = -\infty \text{ for } x \in (-\infty,0). \) For a contingent claim \( G \in L^0 \) and \( x \in \mathbb{R}, \) we define

\[
\Phi(U,G,x) := \{ h \in \ell_2, \; EU^+(V^{x,h} - G) < +\infty \}.
\]
We also introduce the set of strategies which dominate $G$ a.s. starting from a given wealth $x \in \mathbb{R}$

$$\mathcal{A}(G, x) := \left\{ h \in \ell_2, \ V^{x,h} \geq G \text{ a.s.} \right\}.$$ 

Finally, we set

$$\mathcal{A}(U, G, x) := \Phi(U, G, x) \cap \mathcal{A}(G, x).$$

Note that even for $x \geq \pi(G)$, $\mathcal{A}(U, G, x)$ might be empty. Indeed, from Lemma 4.1, we know that there exists some $h \in \mathcal{A}(G, x)$, but $h$ might not belong to $\Phi(U, G, x)$. But this holds true under appropriate assumptions, as proved in the lemma below.

**Lemma 5.1** Let Assumptions 2.1, 2.3, 3.2 and 3.5 hold true. Assume that $G \geq 0$ a.s. and $U(x_0) = 0$, $U'(x_0) = 1$, for some $x_0 \geq 0$. Then $\mathcal{A}(G, x) = \mathcal{A}(U, G, x)$ for all $x \in \mathbb{R}$.

**Proof.** As $U$ is increasing, concave and differentiable with $U(x_0) = 0$ and $U'(x_0) = 1$, for all $x \in \mathbb{R}$,

$$U(x) \leq U(\max(x_0, x)) \leq U(x_0) + \max(x - x_0, 0)U'(x_0) \leq \max(x - x_0, 0) \leq |x - x_0|,$$

since $x_0 \geq 0$. If $x < \pi(G)$ then $\mathcal{A}(G, x) = \emptyset$ and $\mathcal{A}(G, x) = \mathcal{A}(U, G, x) = \emptyset$. Let $x \geq \pi(G)$. Then by Lemma 4.1, $\mathcal{A}(G, x) \neq \emptyset$. Let $h \in \mathcal{A}(G, x)$. Then $V^{x,h} \geq G \geq 0$ a.s. and $h \in \mathcal{A}(0, x)$. Hence, we get that

$$U^+(x + \langle h, \varepsilon - b \rangle - G) \leq U^+(x + \langle h, \varepsilon - b \rangle) \leq U^+(x + \langle h, \varepsilon - b \rangle)1_{\{x + \langle h, \varepsilon - b \rangle \geq x_0\}} + U^+(x_0)1_{\{x + \langle h, \varepsilon - b \rangle < x_0\}} = U(x + \langle h, \varepsilon - b \rangle)1_{\{x + \langle h, \varepsilon - b \rangle \geq x_0\}} \leq x + \langle h, \varepsilon - b \rangle, \quad (16)$$

since $h \in \mathcal{A}(0, x)$. Using (5), we get that

$$E\langle h, \varepsilon - b \rangle^2 \leq \|h\|_{\ell_2}^2 (1 + \|b\|_{\ell_2}^2).$$

Cauchy-Schwarz inequality and Lemma 3.2 imply that

$$EU^+(x + \langle h, \varepsilon - b \rangle - G) \leq x + \sqrt{E(\langle h, \varepsilon - b \rangle^2)} \leq x + \|h\|_{\ell_2} \sqrt{1 + \|b\|_{\ell_2}^2} \leq x + \frac{x}{\alpha} \sqrt{1 + \|b\|_{\ell_2}^2} < +\infty. \quad (17)$$

\[\square\]
We now define the supremum of the expected utility at the terminal date when delivering claim $G$, starting from initial wealth $x \in \mathbb{R}$:

$$u(G, x) := \sup_{h \in \mathcal{A}(U, G, x)} \mathbb{E} U(V^x_h - G),$$

where $u(G, x) = -\infty$ if $\mathcal{A}(U, G, x) = \emptyset$. The following result establishes that there exists an optimal investment for the investor we are considering.

**Theorem 5.2** Let Assumptions 2.1, 2.3, 3.2 and 3.5 hold true. Let $G \geq 0$ and $x \in [\pi(G), \infty)$. There exists $h^* \in \mathcal{A}(U, G, x)$ such that

$$u(G, x) = \mathbb{E} U(V^x_{h^*} - G).$$

**Proof.** If $U$ is constant there is nothing to prove. Else there exists $x_0 > 0$ such that $U'(x_0) > 0$. Replacing $U$ by $U - \frac{U(x_0)}{U'(x_0)}$, we may and will suppose that $U(x_0) = 0$ and $U'(x_0) = 1$. Let $x \geq \pi(G)$ and let $h_n \in \mathcal{A}(G, x) = \mathcal{A}(U, G, x)$ (see Lemmata 4.1 and 5.1) be a sequence such that

$$\mathbb{E} U(V^x_{h_n} - G) \uparrow u(G, x), \ n \to \infty.$$

By Lemma 3.12

$$\sup_{n \in \mathbb{N}} \|h_n\|_{\ell_2} \leq \frac{x}{\alpha} < \infty.$$

Hence as $\ell_2$ has the Banach-Saks Property, there exists a subsequence $(n_k)_{k \geq 1}$ and some $h^* \in \ell_2$ such that for $\tilde{h}_n := \frac{1}{n} \sum_{k=1}^{n} h_{n_k}$

$$\|\tilde{h}_n - h^*\|_{\ell_2} \to 0, \ n \to \infty$$

for some $h^* \in \ell_2$. Note that $\sup_{n \in \mathbb{N}} \|\tilde{h}_n\|_{\ell_2} \leq \frac{x}{\alpha} < \infty$. Using (19), we get that

$$E(\tilde{h}_n - h^*, \epsilon - b)^2 \leq \|\tilde{h}_n - h^*\|_{\ell_2}^2 (1 + \|b\|_{\ell_2}^2) \to 0,$$

when $n \to \infty$. In particular, $(\tilde{h}_n - h^*, \epsilon - b) \to 0, \ n \to \infty$ in probability. Hence also $U(V^x_{\tilde{h}_n} - G) \to U(V^x_{h^*} - G)$ in probability by continuity (right continuity in 0) of $U$ on $[0, \infty)$. We claim that the family $U^+(V^x_{\tilde{h}_n} - G), \ n \in \mathbb{N}$ is uniformly integrable. Indeed using (16), we have that

$$U^+(V^x_{\tilde{h}_n} - G) \leq x + \langle \tilde{h}_n, \epsilon - b \rangle.$$

Thus, from Assumption 3.5 and Remark 3.8 we get that $\{U^+(V^x_{\tilde{h}_n} - G), \ h_n \in \ell_2, \|h_n\|_{\ell_2} \leq \frac{x}{\alpha}\}$ is uniformly integrable and thus

$$\lim_{n \to \infty} E\left(U^+(V^x_{\tilde{h}_n} - G)\right) = E\left(U^+(V^x_{h^*} - G)\right).$$
Fatou’s lemma used for $-U^{-}$ implies that

$$
E \left( -U^{-}(V^{x,h^*} - G) \right) \geq \lim_{n \to \infty} \sup E \left( -U^{-}(V^{x,h_n} - G) \right),
$$

By concavity of $U$,

$$
U(V^{x,h_n} - G) = U \left( \frac{1}{n} \sum_{k=1}^{n} (V^{x,h_{nk}} - G) \right) \geq \frac{1}{n} \sum_{k=1}^{n} U \left( V^{x,h_{nk}} - G \right)
$$

and we get that

$$
EU(V^{x,h^*} - G) \geq \lim_{n \to \infty} \sup EU(V^{x,h_n} - G) \geq u(G, x),
$$

and the proof is finished. $\square$

# 6 Convergence of reservation price to the super-replication price

We go on incorporating a sequence of agents in our model. A measure of risk aversion have been introduced by [3] and [24] with the “absolute risk-aversion” functions $r_n$ defined by

$$
r_n(x) := -\frac{U''_n(x)}{U'_n(x)} \to \infty, \ n \to \infty.
$$

Keeping these preliminary considerations in mind, Assumption 6.1 below says that the sequence of agents we consider have asymptotically infinite aversion towards risk.

**Assumption 6.1** Suppose that $U_n : (0, \infty) \to \mathbb{R}$, $n \in \mathbb{N}$ is a sequence of concave strictly increasing twice continuously differentiable functions such that

$$
\forall x \in (0, \infty) \quad r_n(x) := -\frac{U''_n(x)}{U'_n(x)} \to \infty, \ n \to \infty.
$$

Note that we extend each $U_n$ to $[0, \infty)$ by (right)-continuity ($U_n(0)$ may be $-\infty$). We also set $U_n(x) = -\infty$ for $x \in (-\infty, 0)$.

We introduce the value functions for our sequence of utility functions $(U_n)_{n \geq 1}$. For $x \in \mathbb{R}$ we define

$$
u_n(G, x) := \sup_{h \in \mathcal{A}(U_n, G, x)} EU_n(V^{x,h} - G)
$$

where $u_n(G, x) = -\infty$ if $\mathcal{A}(U_n, G, x) = \emptyset$. The utility indifference (or reservation) price $p_n(G, x)$, introduced by [17], is defined as

$$
p_n(G, x) = \inf \{ z \in \mathbb{R} : u_n(G, x + z) \geq u_n(0, x) \}.
$$
Intuitively, it seems reasonable that under Assumption 6.1 the utility prices \( p_n(G, x) \) tend to \( \pi(G) \). This is proved below under a suitable set of assumptions, relying on the results of Sections 4 and 5.

**Theorem 6.2** Assume that Assumptions 2.2, 2.3, 3.2 and 3.5 holds true. Suppose that \( x > 0 \) and \( G \in L^0 \). Then the utility indifference prices \( p_n(G, x) \) are well-defined and converge to \( \pi(G) \) as \( n \to \infty \).

**Proof.** Fix some \( x > 0 \). Notice that Assumption 6.1 remains true if we replace each \( U_n \) by \( \alpha_n U_n + \beta_n \) for some \( \alpha_n > 0, \beta_n \in \mathbb{R} \). Also, the utility indifference prices defined by these new functions are the same as the ones defined by the original \( U_n \). Hence by choosing \( \alpha_n := 1/U''_n(x) \) and \( \beta_n := -U'_n(x)/U''_n(x) \), we may and will suppose that for all \( n \in \mathbb{N} \), \( U_n(x) = 0 \), and \( U''_n(x) = 1 \).

If \( \pi(G) = +\infty \). By definition for all \( z \in \mathbb{R} \), \( n \geq 1 \), \( \emptyset = \mathcal{A}(G, z) = \mathcal{A}(U_n, G, z) \) and \( u_n(G, x + z) = -\infty \). But \( u_n(0, x) \geq EU_n(x) = 0 \). Thus \( p_n(G, x) = +\infty \) for all \( n \geq 1 \) and the claim is proved.

Assume now that \( \pi(G) < \infty \). First remark that

\[
\pi_n(G, x) \leq \pi(G). \tag{18}
\]

Indeed, we may take a strategy \( \hat{h} \in \mathcal{A}(G, \pi(G)) \) (which is non-empty, see Lemma 4.1). Then \( V^{\pi(G)}, \hat{h} \geq G \) a.s. and as \( U_n \) is non-decreasing,

\[
u_n(0, x) \leq \sup_{h \in \mathcal{A}(U_n, 0, x)} EU_n(V^{x + \pi(G), h + \hat{h}} - G) \leq \sup_{h \in \mathcal{A}(U_n, G, x + \pi(G))} EU_n(V^{x + \pi(G), h} - G) = u_n(G, x + \pi(G)),
\]

where the second inequality follows for the fact that if \( h \in \mathcal{A}(U_n, 0, x) \subset \mathcal{A}(0, x) \) then \( h + \hat{h} \in \mathcal{A}(G, x + \pi(G)) = \mathcal{A}(U_n, G, x + \pi(G)) \) (see Lemma 5.1).

So by definition of the utility indifference price \( (18) \) follows and we have that \( p_n(G, x) \leq \pi(G) < \infty \) for all \( n \geq 1 \). Thus, to prove that \( \lim_{n \to \infty} p_n(G, x) = \pi(G) \) it is enough to show that \( \liminf_n p_n(G, x) \geq \pi(G) \). Assume that this is not the case. Hence we can find a subsequence \( (n_k)_{k \geq 1} \) and some \( \eta > 0 \) such that \( p_{n_k}(G, x) \leq \pi(G) - \eta \) for all \( k \geq 1 \). We may and will assume that \( x \geq \eta \). By definition of \( p_{n_k}(G, x) \) we have that

\[
u_{n_k}(G, x + \pi(G) - \eta) \geq \nu_{n_k}(0, x).
\]

Assume that \( \lim_{k \to +\infty} \nu_{n_k}(G, x + \pi(G) - \eta) = -\infty \) is proved. Then

\[
\liminf_{k \to +\infty} \nu_{n_k}(0, x) = -\infty.
\]

But

\[
\liminf_{k \to +\infty} \nu_{n_k}(0, x) \geq \liminf_{k \to +\infty} EU_{n_k}(x) = 0,
\]


a contradiction.

It remains to prove that \( \lim_{k \to +\infty} u_n(G, y) = -\infty \) with \( y = x + \pi(G) - \eta < x + \pi(G) \). For ease of notation, we will prove that \( \lim_{n \to +\infty} u_n(G, y) = -\infty \).

First we show that \( x + G \notin C_{y} \). Indeed if this is not the case, there exists some \( X \in L^0_+ \) and \( h \in \ell_2 \) such that \( x + G = V_{y,h} - X \) a.s., hence \( G \leq V^{y-x,h}_T \) a.s. Therefore we must have \( y - x \geq \pi(G) \): A contradiction.

Applying Proposition 3.9 we get some \( \gamma > 0 \) such that \( \inf_{h \in \ell_2} P(A_h) > 0 \), where

\[
A_h := \{ y + \langle h, \varepsilon - b \rangle < x + G - \gamma \}.
\]

Note that we can always assume that \( x \geq \gamma \). As \( y \geq \pi(G) \), Lemma 4.1 implies that \( A(U_n, G, y) \neq \emptyset \). Hence for all \( h \in A(U_n, G, y) \), we get that

\[
EU_n(y + \langle h, \varepsilon - b \rangle - G) \leq E1_{A_h}U_n(x - \gamma) + E1_{\Omega \setminus A_h}U_n^+(y + h(\varepsilon - b)) \\
\leq \gamma U_n(x - \gamma) + EU_n^+(y + \langle h, \varepsilon - b \rangle).
\]

Here we used the fact that \( U_n(x - \gamma) \leq U_n(x) = 0 \). Using (17), we get that

\[
EU_n^+(y + \langle h, \varepsilon - b \rangle) \leq y + \frac{y}{\alpha} \sqrt{1 + \|b\|_{l_2}^2}.
\]

Thus,

\[
u_n(G, y) \leq \gamma U_n(x - \gamma) + y + \frac{y}{\alpha} \sqrt{1 + \|b\|_{l_2}^2} \to -\infty,
\]

when \( n \) goes to infinity, by Lemma 4 of [8].

\[\square\]

Acknowledgments

M.R. was supported by the National Research, Development and Innovation Office, Hungary [Grant KH 126505] and by the “Lendület” programme of the Hungarian Academy of Sciences [Grant LP 2015-6].

References

[1] M. Ali Khan and Y. Sun. Asymptotic Arbitrage and the APT with or without Measure-Theoretic Structures. Journal of Economic Theory, 101:222–251, 2001.

[2] M. Ali Khan and Y. Sun. Exact arbitrage, well-diversified portfolios and asset pricing in large markets. Journal of Economic Theory, 110:337–373, 2003.

[3] K. Arrow. Essays in the Theory of Risk-Bearing. North-Holland, Amsterdam, 1965.
[4] E. Barucci and C. Fontana C. *Factor Asset Pricing Models: CAPM and APT*. In: *Financial Markets Theory*, Springer, London, 2017.

[5] B. Bensaid, J.-Ph. Lesne, H. Pagés and J. Scheinkman. Derivative asset pricing with transaction costs. *Mathematical Finance*, 1992, 2(2):63–86, 1992.

[6] D. Bernoulli. Theoriae Novae de Mensura Sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae. Volume V.*, 1738. Translated by L. Sommer as “Exposition of a new theory on the measurement of risk”, *Econometrica*, 1954, 22:23–36.

[7] S. J. Brown and M. I. Weinstein. A new approach to testing asset pricing models: The bilinear paradigm. *Journal of Finance*, 38(3):711–743, 1983.

[8] L. Carassus, M. Rásonyi. Convergence of utility indifference prices to the superreplication price. *Math. Methods Oper. Res.*, 64:145–154, 2006.

[9] G. Chamberlain. Funds, factors, and diversification in arbitrage pricing models. *Econometrica*, 51:1305–1323, 1983.

[10] G. Chamberlain and M. Rothschild. Arbitrage, factor structure, and mean-variance analysis on large asset markets. *Econometrica*, 51:1281–1304, 1983.

[11] C. Cuchiero, I. Klein and J. Teichmann. A new perspective on the fundamental theorem of asset pricing for large financial markets. *Theory Probab. Appl.*, 60:561–579, 2016.

[12] J. Cvitanić and I. Karatzas. Hedging and portfolio optimization under transaction costs: a martingale approach. *Mathematical Finance*, 6:133–165, 1996.

[13] R.C. Dalang, A. Morton, and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastics Rep.*, 29:185–201, 1990.

[14] Ph. H. Dybvig and S. A. Ross. Yes, The APT Is Testable. *Journal of Finance*, 40:1173–1188, 1985.

[15] H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter & Co., Berlin, 2002.

[16] Ch.-F. Huang and R. H. Litzenberger. *Foundations for financial economics*. Elsevier, 1988.

[17] R. Hodges and K. Neuberger. Optimal replication of contingent claims under transaction costs. *Rev. Futures Mkts.*, 8:222-239, 1989.
[18] G. Huberman. A simple approach to arbitrage pricing theory. *J. Econom. Theory*, 28:289–297, 1982.

[19] J. E. Ingersoll. Some results in the theory of arbitrage pricing. *J. Finance*, 39:1021–1039, 1984.

[20] Yu. M. Kabanov and D. O. Kramkov. Asymptotic arbitrage in large financial markets. *Finance Stoch.*, 2:143–172, 1998.

[21] I. Klein. A fundamental theorem of asset pricing for large financial markets. *Math. Finance*, 10:443–458, 2000.

[22] J. Lintner. The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. *Rev. Econom. Statist.*, 47:13–37, 1965.

[23] K. Menger. Das Unsicherheitsmoment in der Wertlehre. *Zeitschrift für Nationalökonomie*, 5:459–485, 1934.

[24] J. Pratt. Risk aversion in the small and in the large. *Econometrica*, 32:122–136, 1964.

[25] M. Rásonyi. Arbitrage pricing theory and risk-neutral measures. *Decis. Econ. Finance*, 27:109–123, 2004.

[26] M. Rásonyi. On optimal strategies for utility maximizers in the Arbitrage Pricing Model. *Int. J. Theor. Appl. Finan.* vol. 19, paper no. 1650047, 2016.

[27] H. Reisman. A general approach to the arbitrage pricing theory (APT). *Econometrica*, 56:473–476. 1988.

[28] S. A. Ross. The arbitrage theory of capital asset pricing. *J. Econom. Theory*, 13:341–360, 1976.

[29] S. A. Ross. Return, risk and arbitrage. In: *“Risk and Return in Finance”*, I. Friend and J. L. Bicksler, Eds. Balinger, Cambridge, MA, 1977.

[30] W. Sharpe. Capital asset prices: a theory of market equilibrium under conditions of risk. *J. Finance*, 33:885–901, 1964.

[31] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.