Prediction of Large Alphabet Processes and Its Application to Adaptive Source Coding *

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Abstract

The problem of predicting a sequence $x_1, x_2, \cdots$ generated by a discrete source with unknown statistics is considered. Each letter $x_{t+1}$ is predicted using information on the word $x_1 x_2 \cdots x_t$ only. In fact, this problem is a classical problem which has received much attention. Its history can be traced back to Laplace. We address the problem where each $x_i$ belongs to some large (or even infinite) alphabet. A method is presented for which the precision is greater than for known algorithms, where precision is estimated by the Kullback-Leibler divergence. The results can readily be translated to results about adaptive coding.

Keywords: Prediction of random processes, Universal coding, Kullback-Leibler divergence, Laplace problem of succession, Shannon entropy.

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1 Introduction

The problem of prediction and the closely related problem of adaptive coding of time series is well known in statistics, probability theory and information theory. The problem can be traced back to Laplace (cf. Feller [5] where the problem is referred to as the problem of succession). Presently, the problem of prediction is investigated by many researchers because of its practical applications and importance for probability theory, information theory, statistics, pattern recognition and other theoretical sciences, see [1, 7, 10, 13, 22].

Consider a source with unknown statistics which generates sequences $x_1x_2\cdots$ of letters from an alphabet $A = \{a_1, \ldots, a_n\}$. The underlying true distribution, which is unknown, is indicated by the letter $p$. Let the source generate a message $x_1\ldots x_{t-1}x_t$, $x_i \in A$ for all $i$, and let $\nu^t(a)$ denotes the count of letter $a$ occurring in the word $x_1\ldots x_{t-1}x_t$. After first $t$ letters $x_1, \ldots, x_{t-1}, x_t$ have been processed the following letter $x_{t+1}$ needs to be predicted. By definition, the prediction is the set of non-negative numbers $p^*(a_1|x_1\ldots x_t), \ldots, p^*(a_n|x_1\ldots x_t)$ which are estimates of the unknown conditional probabilities $p(a_1|x_1\ldots x_t), \ldots, p(a_n|x_1\ldots x_t)$, i.e. of the probabilities $p(x_{t+1} = a_i|x_1\ldots x_t)$; $i = 1, \ldots, n$.

Laplace suggested the following prediction $p^*_{L}(a|x_1\ldots x_t) = (\nu^t(a)+1)/(t+|A|)$, where $\nu^t(a)$ denote the count of letter $a$ occurring in the word $x_1\ldots x_{t-1}x_t$ and $|A|$ is the number of letters in the alphabet $A$. (The problem which Laplace considered was to estimate the probability that the sun will rise tomorrow, given that it has risen every day since the creation. Using our terminology, we can say that Laplace estimated $p(r)|r r \cdots r)$ and $p(\bar{r})|r r \cdots r)$, where $\{r, \bar{r}\}$ is the alphabet (“sun rises”, “sun does not rise”) and the length of $r r \cdots r$ is the number of days since the creation.)

We will estimate the precision of the prediction by the Kullback-Leibler divergence between the true distribution $p(./x_1\ldots x_t)$ and its estimation $p^*_\gamma(./x_1\ldots x_t)$, where $\gamma$ denote a prediction method. It is known that the average divergence of Laplace predictor is upper bounded by $(|A| - 1)/(t + 1)$ [20, 21], if the predictor is applied to i.i.d. source. Krivevsky [8, 9] investigated the problem of optimal minimax predictor for i.i.d. sources and showed that, loosely speaking, the precision of the optimal predictor is asymptotically equal to $(|A| - 1)/(2t) + o(1/t)$.

We can see that, on the one hand, the precision of the predictors essentially depends on the alphabet size $|A|$. On the other hand, there are many applications, where the alphabet size is unknown beforehand and can
be upper bounded only. Moreover, quite often such a bound is infinity. That is why the problems of prediction and, especially, adaptive coding for large and unbounded alphabet sources have been a subject in literature before, see [4, 14, 15, 11, 23].

In this paper we suggest a scheme of prediction for a case where a source generates letters from an alphabet with unknown (and even unbounded) size. This scheme can be applied along with Laplace, Krichevsky and any other predictors. In all cases the loss of precision is the same as if the alphabet size increases by one. In words, if the suggested scheme is applied to $s$—letter source and $s$ is unknown, the precision is the same as in case the predictor is applied to $(s + 1)$—letter source and the alphabet size $(s + 1)$ is known beforehand. It is interesting that one of such predictors was known in adaptive coding [3, 23], but its precision was not known.

The suggested scheme can also be applied to a case of infinite alphabets. In such a case the Kullback-Leibler divergence of the suggested predictor goes to 0 if, loosely speaking, the original representation of alphabet letters has the finite average word length. It will be shown that, in fact, such a condition is necessary for existing of predictors, for which the Kullback-Leibler divergence goes to 0.

We mainly consider a case of prediction for i.i.d. sources, but all results can be easily extended to Markov sources, using well known methods, see, for example., [8, 21].

2 Definitions and Preliminaries

Consider an alphabet $A = \{a_1, \cdots, a_n\}$ with $n \geq 2$ letters and denote by $A^t$ the set of words $x_1 \cdots x_t$ of length $t$ from $A$. Let $p$ be a source which generates letters from $A$. Formally, $p$ is a probability distribution on the set of words of infinite length or, more simply, $p = (p^t)_{t \geq 1}$ is a consistent set of probabilities over the sets $A^t$; $t \geq 1$. By $M_0(A)$ we denote the set of Bernoulli (or i.i.d.) sources over $A$, and by $M_k(A)$ the set of Markov sources over $A$ of connectivity (memory) $k$; $k \geq 1$.

Denote by $D(\cdot \| \cdot)$ the Kullback-Leibler divergence and consider a source $p$ and a method $\gamma$ of prediction. The precision is characterized by the divergence

\[ r_{\gamma, p}(x_1 \cdots x_t) = D \left( p(\cdot | x_1 \cdots x_t) \| p^*_{\gamma}(\cdot | x_1 \cdots x_t) \right) \]
\[ \sum_{a \in A} p(a|x_1 \cdots x_t) \log \frac{p(a|x_1 \cdots x_t)}{p^*(a|x_1 \cdots x_t)}. \]

(Here and below \( \log \equiv \log_2 \).) As usual, high precision means divergence close to zero.

We noted in the introduction, the prediction problem is closely related to the problem of adaptive (and universal) coding and \( r_{\gamma,p} \) may also be considered as the redundancy when the prediction is used for coding. Let us comment on the relation to coding in more detail. An encoder can construct a code with codelength
\[ l^*(a|x_1 \cdots x_t) \approx -\log p^*(a|x_1 \cdots x_t) \]
for any letter \( a \in A \) (the approximation may be as accurate as you like, if the arithmetic code is used, see [16, 12]). An ideal encoder would base coding on the true distribution \( p \) and not on the prediction \( p^* \). The difference in performance (the redundancy) measured by average code length is given by
\[ \sum_{a \in A} p(a|x_1 \cdots x_t)(-\log p^*(a|x_1 \cdots x_t)) - \sum_{a \in A} p(a|x_1 \cdots x_t)(-\log p(a|x_1 \cdots x_t)) \]
\[ = \sum_{a \in A} p(a|x_1 \cdots x_t) \log \frac{p(a|x_1 \cdots x_t)}{p^*(a|x_1 \cdots x_t)} \quad (= r_{\gamma,p}). \]

So, we can see that from a mathematical point of view the prediction and adaptive coding are identical and can, therefore, be investigated together. Note that such a scheme of adaptive (or universal) coding was suggested in [3] and now is called as a PPM algorithm.

We use \( M \) to denote the model under consideration. Formally, \( M \) could be any set of sources but for this paper we mainly consider the case \( M = M_0(A) \). In fact, we shall focus on the case \( M = M_0(A) \) as results for the general Markovian case can be deduced from results for the Bernoulli case.

For fixed \( t \), \( r_{\gamma,p} \) is a random variable, because \( x_1, x_2, \ldots, x_t \) are random variables. We define the average divergence (or average redundancy) at time \( t \) by
\[ r^t(p||\gamma) = E(r_{\gamma,p}(\cdot)) = \sum_{x_1 \cdots x_t \in A^t} p(x_1 \cdots x_t) r_{\gamma,p}(x_1 \cdots x_t). \]

Related to this quantity we define the maximum average divergence (at time \( t \)) by
\[ r^t(M||\gamma) = \sup_{p \in M} r^t(p||\gamma). \]

If the Laplace predictor
\[ p^*_L(a|x_1 \cdots x_t) = \frac{\nu^t(a) + 1}{(t + |A|)} \]
is applied to i.i.d. source, its average redundancy \( r^t(M_0\|L) \) is upper bounded by \( (|A| - 1)/(t + 1) \), i.e.

\[
r^t(M_0\|L) \leq (|A| - 1)/(t + 1),
\]

see \[20\, 21\]. (Here \( e = 2.718... \) is the Euler number and, as before, \( \nu^t(a) \) denote the count of letter \( a \) occurring in the word \( x_1 \ldots x_{t-1}x_t \).) In \[9\] Krichevsky investigated the problem of optimal predictor for the set of i.i.d. sources and showed that for any predictor \( \pi \) the maximal redundancy is asymptotically lower bounded by \( (|A| - 1) \log e/2t \):

\[
\lim_{t \to \infty} \sup (2t \ r^t(M_0\|\pi)) \geq (|A| - 1) \log e.
\]

He has also suggested the predictor

\[
p^*_K_1(a|x_1 \ldots x_t) = (\nu^t(a) + 0.5)/(t + |A|)
\]

\[5\]

and shown that it is asymptotically optimal:

\[
\lim_{t \to \infty} \sup (2t \ r^t(M_0\|K_1)) = (|A| - 1) \log e.
\]

Predictors for i.i.d. sources can be easily extended to Markov sources (see, for example, \[8\, 21\]) and to the general stationary and ergodic sources, as it was suggested in \[18\, 19\]. But, it is worth noting that, as it is shown in \[19\], there exist such stationary and ergodic sources that their divergence does not go to 0. More precisely, for any predictor \( \gamma \) there exists such a stationary and ergodic source \( \tilde{p} \), that

\[
\lim_{t \to \infty} \sup (r^t(\tilde{p}\|\gamma)) \geq \log |A|,
\]

(see \[19\, 1\, 10\]). But, on the other hand, it is shown in \[18\, 19\] that there exists such a predictor \( \rho \), that the following average

\[
R^t(p\|\rho) = t^{-1} \sum_{i=1}^{t} r^i(p\|\rho)
\]

(6)

goes to 0 for any stationary and ergodic source \( p \), where \( t \) goes to infinity. That is why we will focus our attention on \( R^t(p\|\gamma) \) and the following supremum:

\[
R^t(M\|\gamma) = \sup_{p \in M} (t^{-1} \sum_{i=1}^{t} r^i(p\|\gamma)).
\]

\[7\]
From this definition and estimation for Laplace predictor \(4\) we can easily obtain that
\[
R^t(M_0 \parallel L) = (|A| - 1) \log t/t + O(1/t).
\] (8)

Krichevsky \[8\] showed that for any predictor \(\gamma\) and the set of i.i.d. sources \(M_0\), the following lower bound is valid.
\[
R^t(M_0 \parallel \gamma) \geq (|A| - 1) \log t/(2t) + O(1/t).
\] (9)

He also suggested the predictor
\[
p^*_K(a|x_1 \cdots x_t) = (\nu^t(a) + 1)/(|A|/2)
\] (10)
and showed that
\[
R^t(M_0 \parallel K_2) = (|A| - 1) \log t/(2t) + O(1/t).
\] (11)

Having taken into account (9) and (11), we can see that the predictor (10) is asymptotically optimal.

3 The new scheme

Let, as before, \(p\) be an i.i.d. source generated letters from the alphabet \(A\). The probability distribution \(p(a), a \in A\) is unknown and each letter \(x_{t+1}\) should be predicted (or encoded) using information on the word \(x_1x_2 \cdots x_t\) only.

The suggested scheme can be applied to any predictor (or letterwise coding), but we will use the Laplace predictor as the main example. We start the description using a simple example. Let \(A = \{a_0, a_1, a_2, \ldots\}\) and \(t = 4, x_1x_2x_3x_4 = a_0a_2a_0a_0\). The “common” Laplace predictor is as follows:
\[
p^*_{L}(x_5 = a_0) = (3 + 1)/(4 + 3) = 4/7, p^*_{L}(x_5 = a_1) = (0 + 1)/(4 + 3) = 1/7,
\]
\[
p^*_{L}(x_5 = a_0) = (1 + 1)/(4 + 3) = 2/7,
\]
see \(3\). In this example we suggest to group letters into two subsets \(A_0 = \{a_0, a_1\}, A_1 = \{a_3\}\) and carry out the prediction into two steps. First the original sequence \(a_0a_2a_0a_0\) is represented as \(A_0A_1A_0A_0\) and belonging to the subsets is predicted as follows
\[
p^*_{L}(x_5 \in A_0) = (3 + 1)/(4 + 2) = 2/3, p^*_{L}(x_5 \in A_1) = (1 + 1)/(4 + 2) = 1/3.
\]
We know that $A_1$ contains one letter $(a_2)$, hence, $p^*_L(x_5 = a_2) = 1/3$. The sequence $A_0A_1A_0A_0$, which contains three letters $A_0$, is used for predicting conditional probabilities $p(x_5 = a_i/x_5 \in A_0)$, $i = 0, 1$. Again, we apply the Laplace predictor \( \text{L} \) to the sequence $A_0A_0A_0 = a_0a_0a_0$ and obtain the following prediction: $p(x_5 = a_0/x_5 \in A_0) = (3 + 1)/(3 + 2) = 4/5$, $p(x_5 = a_1/x_5 \in A_0) = (0 + 1)/(3 + 2) = 1/5$. So, combining all predictions, we obtain $p^*_L(x_5 = a_0) = (2/3)(4/5) = 8/15$, $p^*_L(x_5 = a_1) = (2/3)(1/5) = 2/15$, $p^*_L(x_5 = a_0) = 1/3$. We can see that this prediction and "common" one are different.

It will be convenient to describe the general case using the notation of a tree. Let $\Upsilon$ be a rooted tree, which contains $|A|$ leaves, and let each leaf be marked by one letter from $A$ in such a way that different leaves are marked by different letters. We will mark each vertex $\mu \in \Upsilon$ by a subset of $A_\mu$ as follows. We consider the subtree $\Upsilon_\mu$ whose root is the vertex $\mu$ and define the subset $A_\mu$ by the set of all letters, which mark the leaves of the subtree $\Upsilon_\mu$. Note that $\Upsilon_{\text{root}} = A$. Let us consider an example.

**EXAMPLE**

Let us proceed with the description of the prediction scheme. Let there be an alphabet $A$ and a rooted tree $\Upsilon$, which leaves are marked by letters from $A$. We denote vertexes of the depth one by $\alpha_i, i = 1, ..., k$, the vertexes of the depth 2 by $\alpha_{i,j}, i = 1, ..., k, j = 1, ..., k_i$, etc., where $k$ is the number of the depth 1 vertexes, $k_i$ is the number of the sons of the vertex $i$, etc. The prediction is carried out as follows. First a generated sequence $x_1 \ldots x_t$ is represented as the sequence $A_{\alpha_1}A_{\alpha_2} \ldots A_{\alpha_t}$, where $\alpha_{i,j}$ is such a vertex of the first level that the letter $x_j$ belongs to the subset $A_{\alpha_{i,j}}$. Then, $A_{\alpha_1}A_{\alpha_2} \ldots A_{\alpha_t}$ is considered as the sequence generated by an i.i.d. source with the alphabet $\{A_{\alpha_i}, i = 1, ..., k\}$ and the next letter $A_{\alpha_t+1}$ is predicted (say, by Laplace predictor). In fact, $p^*_L(A_{\alpha_t})$ is the estimation of $Pr(x_{t+1} \in A_{\alpha_t})$. Then, for each vertex $\alpha_{i,j}, i = 1, ..., k, j = 1, ..., k_i$ of the depth 2, which is not a leaf, we estimated the probability $Pr(x_{t+1} \in A_{\alpha_{i,j}}/x_{t+1} \in A_{\alpha_t})$. For this purpose for each $i$ we consider all letters $A_{\alpha_i}$ in the sequence $A_{\alpha_{i,1}}A_{\alpha_{i,2}} \ldots A_{\alpha_{i,t}}$ and organize the following sequence of their sons $A_{\alpha_{i,1}} \ldots A_{\alpha_{i,s}}$, whose length $s$ equals to the count of occurrences $A_{\alpha_i}$ in the sequence $A_{\alpha_{i,1}}A_{\alpha_{i,2}} \ldots A_{\alpha_{i,t}}$. This sequence is considered as a generated by an i.i.d. source and the next letter $A_{\alpha_{i,s}}$ is predicted. As a result, we obtain $p^*_L(A_{\alpha_{i,j}})$, which are, in fact, the estimations of the probabilities $Pr(x_{t+1} \in A_{\alpha_{i,j}}/x_{t+1} \in A_{\alpha_t})$. And so on.
Then we calculate the predictor $P^*_{\Psi}(x_{t+1} = a)$ as a product $Pr(x_{t+1} \in A_{\alpha_i})$
$Pr(x_{t+1} \in A_{\alpha_{i,j}}/x_{t+1} \in A_{\alpha_i}) Pr(x_{t+1} \in A_{\alpha_{i,j,m}}/x_{t+1} \in A_{\alpha_{i,j}})$ ..., where $a \in A_{\alpha_i}$ $a \in A_{\alpha_{i,j}}$ $a \in A_{\alpha_{i,j,m}}$, ... The following example is to illustrate all steps.

example

**Theorem 1.** Let $x_1 \ldots x_t$ be a sequence generated by an i.i.d. source from an alphabet $A$. If the letter $x_{t+1}$ is predicted by a suggested scheme according to a tree $\Psi$ with the set of vertex $\Lambda$, then the following upper bound for the precision (or redundancy) (11) is valid:

$$r^t(p|L_\Psi) \leq \sum_{\lambda \in \Lambda} \min\{(|\lambda| - 1)/(t + 1), p(\lambda)\}, \tag{12}$$

where $p(\lambda) = \sum_{a \in \lambda} p(a)$, $L_\Psi$ is a notation of the predictor.

*Proof* is in Appendix.

First, we note that

4 Infinite and unbounded alphabets
References

[1] Algoet P., *Universal Schemes for Learning the Best Nonlinear Predictor Given the Infinite Past and Side Information*, IEEE Trans. Inform. Theory, v. 45, pp. 1165-1185, 1999.

[2] P. Billingsley, *Ergodic theory and information*, John Wiley & Sons (1965).

[3] Cleary J., Witten I. Data compression using adaptive coding and partial string matching. IEEE Transactions on Communications, v.32, n. 4, 1984, pp.396-402.

[4] Elias P. Universal codeword sets and representations of integers. IEEE Trans. Inform. Theory, v.21, N 2, 191975, pp. 194-203.

[5] Feller W., *An Introduction to Probability Theory and Its Applications*, vol.1. John Wiley & Sons, New York, 1970.

[6] Gallager R.G. *Information Theory and Reliable Communication*. Wiley, New York,1968.

[7] Kieffer J. *Prediction and Information Theory*, Preprint, 1998. (available at ftp://oz.ee.umn.edu/users/kieffer/papers/prediction.pdf/)

[8] Krichevsky R., *Universal Compression and Retrieval*. Kluver Academic Publishers, Dordrecht, 1994.

[9] Krichevskii R., *Laplace’s Law of Succession and Universal Encoding*, IEEE Trans. Inform. Theory, v. 44, pp. 296-303, 1998.

[10] Morvai G., Yakowitz S. J., Algoet P.H. *Weakly convergent nonparametric forecasting of stationary time series*. IEEE Trans. Inform. Theory, v.43, 1997, pp. 483 - 498.

[11] A.Moffat. Implementing the PPM data compression scheme. IEEE Transactions on Communications, v.38, n.11, 1990. pp. 1917- 1921

[12] A.Moffat,R.Neal,I.Witten. *Arithmetic Coding Revisited*, ACM Transactions on Information Systems,16:3(1998), pp.256-294.

[13] Nobel A.B. On optimal sequential prediction. IEEE Trans. Inform. Theory, v.49, 2003, n.1. pp.83-98.
[14] Orlitsky A., Santhanam N.P., Zhang J. Bounds on compression of unknown alphabets. In: Proceedings of 2003 IEEE International Symposium on Information Theory, Yokohama, p.111.

[15] Orlitsky A., Santhanam N.P. Performance of universal codes over infinite alphabets. Proceedings of Data Compression Conference (DCC), 2003.

[16] J.Rissanen, Generalized Kraft inequality and arithmetic coding, IBM J. Res. Dev., 20:5(1976), pp. 198-203.

[17] J.Rissanen. Universal coding, information, prediction, and estimation. IEEE Trans. Inform. Theory, v.30, n. 4, 1984, pp.629-636.

[18] Ryabko B.Ya.Twice-universal coding.Problems of Information Transmission.1984,n3, pp.173-177.

[19] Ryabko, B. Ya. Prediction of random sequences and universal coding. Problems of Inform. Transmission 24 (1988), no. 2, 87–96.

[20] Ryabko, B. Ya. A fast adaptive coding algorithm. Problems of Inform. Transmission 26 (1990), no. 4, 305–317.

[21] B. Ryabko, F. Topsoe. On Asymptotically Optimal Methods of Prediction and Adaptive Coding for Markov Sources. Journal of Complexity, Vol. 18, No. 1, Mar 2002, pp. 224-241.

[22] Szpankowski W. Average case analysis of algorithms on sequences. John Wiley and Sons, New York, 2001.

[23] Witten I., Bell T. The zero-frequency problem: Estimating the probabilities of novel events in adaptive text compression. IEEE Trans. Inform. Theory, v. 37, n.4, pp. 1085- 1094, 1991.
