Transition Rate and the Photoelectric effect in the Presence of a Minimal Length

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Abstract

In this work, according to generalized uncertainty principle (GUP) and time-dependent perturbation theory, the transition rate in the present of a minimal length based on the Kempf algebra is studied. Also, we find the absorption cross section in the framework of GUP. The modified photoelectric effect is investigated and we show that the differential cross section of photoelectric effect in the framework of GUP is related to the isotropic minimal length scale. The upper bound on the isotropic minimal length is estimated.

Keywords: Phenomenology of quantum gravity; Generalized uncertainty principle; Minimal length; Absorption cross section ; Photoelectric effect

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1 Introduction

In the last decade, different theories of quantum gravity such as the string theory, loop quantum
gravity, noncommutative geometry and doubly special relativity are proposed for finding the
unification between the general theory of relativity and the standard model of particle physics
[1]. Although these theories are different in concepts, all of these studies lead to unique belief
which predicts the existence of a measurable minimal length scale. An immediate consequence
of existence of a minimal length is that the Heisenberg uncertainty principle is modified. Nowa-
days the modified uncertainty principle is called generalized uncertainty principle (GUP) [2].

The generalized uncertainty principle corresponding to the modified Heisenberg algebra can be
written as
\[
\Delta X \Delta P \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta P)^2 \right],
\]
where $\beta$ is a positive parameter [3,4] and also, Eq. (1) yields a minimal measurable length
$(\Delta X)_{\text{min}} = \hbar \sqrt{\beta}$. At this time, many studies have been done to compute the corrections of
quantum mechanics in the GUP framework. These investigations seem to modify mechanical
Hamiltonians at atomic scales [5-10]. In the recent years, a lot of papers have been devoted
to the gravity and reformulations of quantum field theory in the presence of a minimal length
scale [11-25]. Kempf and his collaborators have introduced finite resolution of length can be
obtained from the deformed Heisenberg algebra [26-28]. The Kempf algebra in a D-dimensional
space is characterized by the following deformed commutation relations
\[
[X^i, P^j] = i\hbar [(1 + \beta P^2) \delta^{ij} + \beta' P^i P^j],
\]
\[
[X^i, X^j] = i\hbar \frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2} (P^i X^j - P^j X^i),
\]
\[
[P^i, P^j] = 0,
\]
where $i, j = 1, 2, \ldots, D$ and $\beta, \beta'$ are two positive deformation parameters. In Eq. (2), $X^i$ and
$P^i$ are position and momentum operators in the GUP framework. According to Eq. (2), we
can easy to find the following an isotropic minimal length scale
\[
(\Delta X^i)_{\text{min}} = \hbar \sqrt{(D\beta + \beta')}, \quad \forall i \in \{1, 2, \ldots, D\}.
\]
It should be mentioned that in a $D + 1$-dimensional space-time the following Lorentz-covariant
deformed algebra have introduced by Quesne and Tkachuk [29,30]
\[
[X^\mu, P^\nu] = -i\hbar [(1 - \beta P_\mu P^\nu) g^{\mu\nu} - \beta' P^\mu P^\nu],
\]
\[
[X^\mu, X^\nu] = i\hbar \frac{2\beta - \beta' - (2\beta + \beta')\beta P_\mu P^\nu}{1 - \beta P_\mu P^\nu} (P^\mu X^\nu - P^\nu X^\mu),
\]
\[
[P^\mu, P^\nu] = 0,
\]
where $\mu, \nu, \rho = 0, 1, 2, \cdots, D$ and $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, \cdots, -1)$. In the present work, we study the transition rate and photoelectric effect in the presence of a minimal length. For this purpose, we consider the formalism of time-dependent perturbation theory to the interactions of atomic electron with the modification of classical radiation field. The paper is organized as follows: In Sec. 2, the Hamiltonian of atomic electron is obtained in the presence of a minimal length. From this modified Hamiltonian we find the modified transition rate. Also, we investigate the absorption cross section in the presence of a minimal length. In Sec. 3, the photoelectric effect in the presence of a minimal length is studied. According to this study we obtain the upper bound on the isotropic minimal length. Our conclusions are presented in Sec. 4. We use SI units throughout this paper.

2 Transition Rate and Absorption Cross section in the Presence of a Minimal length

The purpose of this section is finding the transition rate and an absorption cross section in the presence of a minimal length based on the Kempf algebra. So that we must to introduce the representation of modified position and momentum operators which are satisfied Kempf algebra in Eq. (2). In Ref. [31], Stetsko and Tkachuk introduced the approximate representation fulfilling the Kempf algebra in the first order over the deformation parameters $\beta$ and $\beta'$

\[
X^i = x^i + \frac{2\beta - \beta'}{4}(p^2 x^i + x^i p^2),
\]

\[
P^i = p^i(1 + \frac{\beta'}{2}p^2),
\]

where the operators $x^i, p^i$ satisfy the canonical commutation relation and $p^2 = \sum_{i=1}^{D} p^i p^i$. It is interesting to note that in the special case of $\beta' = 2\beta$, the position operators commute in linear approximation over the deformation parameter $\beta$, i.e. $[X^i, X^j] = 0$. The following representations, which satisfy Kempf algebra in the special case of $\beta' = 2\beta$, was introduced by Brau [32]

\[
X^i = x^i,
\]

\[
P^i = p^i(1 + \beta p^2).
\]

2.1 Interaction of Atomic electron with the Radiation Field in the Presence of a Minimal Length

In this section, let us obtain the time-dependent perturbation theory to the interactions of atomic electron with the classical radiation field. Classical radiation field means that the
electric or magnetic field derivable from a classical radiation field \[33\]. The basic Hamiltonian, with \(A^2\) vanished, is

\[
H = \frac{p^2}{2m_e} + e\phi(x) - \frac{e}{m_ec} A \cdot p, \tag{8}
\]

where in the coulomb gauge (\(\nabla \cdot A = 0\)), we have used \(A \cdot p = p \cdot A\). We work with a monochromatic field of the plane wave for

\[
A = 2A_0 \hat{\epsilon} \cos(\frac{\omega}{c} \hat{n} \cdot x - \omega t) = A_0 \hat{\epsilon} [\exp(i\frac{\omega}{c} \hat{n} \cdot x - i\omega t) + \exp(-i\frac{\omega}{c} \hat{n} \cdot x + i\omega t)], \tag{9}
\]

where \(\hat{\epsilon}\) and \(\hat{n}\) are the polarization and propagation direction. In Eq. (9), \(\hat{\epsilon}\) is perpendicular to the propagation direction \(\hat{n}\) and from Eq. (8) treat \(-\frac{e}{m_ec} A \cdot p\) as time-dependent potential.

Now, for obtaining the Hamiltonian in the presence of a minimal length, we must replace the usual position and momentum operators with the modified position and momentum operators according to Eqs. (6) and (7), we have

\[
H_{\text{modified}} = \left( p - \frac{e}{m_ec} A \right) \left( 1 + \beta \left( p - \frac{e}{m_ec} A \right)^2 \right)^2 + e\phi(x). \tag{10}
\]

After neglecting terms of order \(\beta^2\) and higher in Eq. (10), the modified Hamiltonian can be obtained as follows

\[
H_{\text{modified}} = \frac{p^2}{2m_e} + e\phi(x) - \frac{e}{m_ec} A \cdot p + \frac{\beta}{m_e} p^4 \tag{11}
\]

\[
- 4\beta \frac{e}{m_ec} A \cdot p^2 + \frac{4\beta}{m_e} \left( \frac{e}{c} \right)^2 (A \cdot p)^2.
\]

From above equation we can consider the following modified time-dependent potential

\[
\mathcal{V} = -\frac{e}{m_ec} A \cdot p - 4\beta \frac{e}{m_ec} A \cdot p^2 + \frac{4\beta}{m_e} \left( \frac{e}{c} \right)^2 (A \cdot p)^2. \tag{12}
\]

As we know that the \(\exp(-i\omega t)\)-term is responsible for absorption, while the \(\exp(i\omega t)\)-term is responsible for stimulated emission. Now, let us treat the absorption case in the presence of a minimal length. We have

\[
\mathcal{V} = -\frac{eA_0}{m_ec} (\exp(i\frac{\omega}{c} \hat{n} \cdot x) \hat{\epsilon} \cdot p) - 4\beta \frac{eA_0}{m_ec} [\exp(i\frac{\omega}{c} \hat{n} \cdot x)(\hat{\epsilon} \cdot p)p^2] + \frac{4\beta}{m_e} \left( \frac{eA_0}{c} \right)^2 [\exp(2i\frac{\omega}{c} \hat{n} \cdot x)(\hat{\epsilon} \cdot p)^2]. \tag{13}
\]

and the modified transition rate is obtained as follows

\[
(w_{i\rightarrow n})_{\text{modified}} = (w_{i\rightarrow n})_0 + \beta^2 (w_{i\rightarrow n})_{ML}. \tag{14}
\]
where

\[
(w_{i \rightarrow n})_0 = \frac{2\pi}{\hbar} \left[ (\frac{eA_0}{m_e c})^2 |\langle n| \exp(i\frac{\omega}{c}\hat{n} \cdot \mathbf{x})(\hat{\epsilon} \cdot \mathbf{p})|i \rangle|^2 \right] \delta(E_n - E_i - \hbar\omega),
\]

and

\[
(w_{i \rightarrow n})_{ML} = \frac{2\pi}{\hbar} \left[ (\frac{4eA_0}{m_e c})^2 |\langle n| \exp(i\frac{\omega}{c}\hat{n} \cdot \mathbf{x})(\hat{\epsilon} \cdot \mathbf{p})|i \rangle|^2 \right] \delta(E_n - E_i - \hbar\omega) + \frac{2\pi}{\hbar} \left[ (\frac{4(eA_0)^2}{m_e c^2})^2 |\langle n| \exp(2i\frac{\omega}{c}\hat{n} \cdot \mathbf{x})(\hat{\epsilon} \cdot \mathbf{p})^2|i \rangle|^2 \right] \delta(E_n - E_i - 2\hbar\omega).
\]

2.2 Absorption Cross Section in the Presence of a Minimal Length

In this section we want to obtain the absorption cross section in the framework of GUP. An absorption cross section has a definition as follow

\[
\frac{(\text{Energy/unit time})_{\text{absorbed by the atom}}(i \rightarrow n)}{\text{Energy flux of the radiation field}},
\]

where the energy flux of the radiation field is given by

\[
I = \frac{c}{\pi} |\mathbf{E} \times \mathbf{B}|.
\]

In this case we use the following definition for the electromagnetic field

\[
\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.
\]

Let us obtain the modified energy flux of the radiation field. From Eq. (17), first we must find the classical electromagnetic field in the presence of a minimal length so that we write the electromagnetic field in Eq. (17) by using the modified position and momentum operators which are satisfied the Quesne-Tkachuk algebra, that is

\[
x^\mu \rightarrow X^\mu = x^\mu, \quad \partial^\mu \rightarrow \nabla^\mu := (1 + \beta \hbar^2 \Box) \partial^\mu,
\]

where \(\Box := \partial^\mu \partial_\mu\) is the d’Alembertian operator. If we substitute Eq. (18) into Eq. (17), we will obtain the following modified electromagnetic field

\[
\mathbf{E}_{ML}(\mathbf{x}, t) = -\frac{1}{c} (1 + \beta \hbar^2 \Box) \frac{\partial \mathbf{A}}{\partial t},
\]

\[
\mathbf{B}_{ML}(\mathbf{x}, t) = \nabla^\mu \times \mathbf{A}.
\]
After inserting Eq. (9) into Eq. (19), we have
\[ E_{ML}(x, t) = -2kA_0(1 - 2\beta\bar{\hbar}^2k^2)\cos(kx - \omega t + \theta), \]  
\[ B_{ML}(x, t) = -2(1 - 2\beta\bar{\hbar}^2k^2)k \times A_0 \hat{\epsilon}\cos(kx - \omega t + \theta), \]  
where \( k = \bar{\omega}\hat{n} \). According to Eqs. (16) and (20), we can easily obtain the energy flux in the presence of a minimal length as follows
\[ I_{\text{modified}} = \frac{c}{\pi}|E_{ML}(x, t) \times B_{ML}(x, t)| = I_0 + \beta I_{ML}, \]  
where
\[ I_0 = \frac{(\omega A_0)^2}{2\pi c}, \]
\[ I_{ML} = 4\frac{(\omega A_0)^2}{2\pi c} (\bar{\hbar}\omega c)^2. \]

Now, from Eqs. (14), (15) and (22), we can obtain the modified absorption cross section as follows
\[ (\sigma_{abs})_{\text{modified}} = (\sigma_{abs})_0 + \beta(\sigma_{abs})_{ML}, \]  
where
\[ (\sigma_{abs})_0 = \frac{(\omega_{i \rightarrow n})_0}{I_0} = \frac{4\pi^2\hbar\alpha}{m_c^2\omega} \langle n | \exp(i\bar{\omega}\hat{n} \cdot \hat{x}) \hat{\epsilon} \cdot \hat{p} | i \rangle^2 \delta(E_n - E_i - \hbar\omega), \]
\[ (\sigma_{abs})_{ML} = \frac{(\omega_{i \rightarrow n})_{ML}}{I_{ML}} = \frac{(4\pi c)^2\alpha^3}{m_c^2\omega} \hbar \langle n | \exp(i\bar{\omega}\hat{n} \cdot \hat{x})(\hat{\epsilon} \cdot \hat{p})^2 | i \rangle^2 \delta(E_n - E_i - \hbar\omega) \]
\[ + \frac{(4\pi A_0\alpha^2c^3}{m_c^2\omega} \hbar \langle n | \exp(2i\bar{\omega}\hat{n} \cdot \hat{x})(\hat{\epsilon} \cdot \hat{p})^2 | i \rangle^2 \delta(E_n - E_i - 2\hbar\omega). \]

In the above equation \( \alpha \) is \( \frac{e^2}{\hbar c} \).

3 Photoelectric Effect in the Presence of a Minimal Length Scale

The photoelectric effect means that the ejection of an electron when an atom is placed in the radiation field. The process of photoelectric effect is considered to be the transition from an atomic state to a continuum state [33]. According to previous section, \(| i \rangle\) is the ket for an
atomic state, while $|n\rangle$ can be taken to be a plane-wave state $|k_f\rangle$. Now, we want to study the modified differential cross section for the photoelectric effect by using our earlier formula for modified absorption cross section. To find the number of states it is convenient to use the box normalization convention for plane-wave states. Assuming a plane-wave state normalized that means if we integrate the square modulus of its wave function for a cubic box of side $L$, we obtain unity. Also, the state is considered to satisfy the periodic boundary condition with periodicity of the side of the box. In the limit $L \rightarrow \infty$, the number of states is reduced to the number of dots in three-dimensional lattice space. If we consider the energy of the final-state plane wave ($E = \frac{\hbar k_f^2}{2m_e}$) and the periodic boundary condition, we can easily find the following number of states in the interval between $E$ and $E + dE$

$$\rho(E) = \left(\frac{L}{2\pi}\right)^3 \frac{m_e k_f}{\hbar^2} dEd\Omega,$$

where $d\Omega$ is the solid angle element. According to Eqs. (24) and (25) the modified differential cross section for the photoelectric effect is obtained as follows

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{modified}} = \left(\frac{d\sigma}{d\Omega}\right)_0 + \beta \left(\frac{d\sigma}{d\Omega}\right)_{ML}, \quad (26)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{4\pi^2 \hbar \alpha}{m_e^2 \omega}\right)^{\frac{3}{2}} |\langle k_f | \exp(i\frac{\omega}{c} \hat{n} \cdot \mathbf{x}) \hat{\epsilon} \cdot \mathbf{p} | i \rangle|^2 \left(\frac{L}{2\pi}\right)^3 \frac{m_e k_f}{\hbar^2}, \quad (27)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{ML} = \frac{(4\pi c)^2 \alpha}{m_e^2 \omega^3 \hbar} |\langle k_f | \exp(i\frac{\omega}{c} \hat{n} \cdot \mathbf{x})(\hat{\epsilon} \cdot \mathbf{p})^2 | i \rangle|^2 \left(\frac{L}{2\pi}\right)^3 \frac{m_e k_f}{\hbar^2}, \quad (28)$$

By considering the initial-state wave function is the ground-state hydrogen atom wave function, Eqs. (27) and (28) become

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{4\pi^2 \hbar \alpha}{m_e^2 \omega}\right)^{\frac{3}{2}} \hat{\epsilon} \cdot \int \frac{d^3 x}{L^3} \frac{\exp(-i\mathbf{k}_f \cdot \mathbf{x})}{L^3} \exp(i\frac{\omega}{c} \hat{n} \cdot \mathbf{x})(-i\hbar \nabla)[\exp(-Zr/a_0)(Z/a_0)^2]$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{ML} = \left(\frac{4\pi c \alpha^2}{m_e^2 \omega^3 \hbar}\right)^{\frac{3}{2}} \hat{\epsilon} \cdot \int \frac{d^3 x}{L^3} \exp(-i\mathbf{k}_f \cdot \mathbf{x}) \exp[i\frac{\omega}{c} \hat{n} \cdot \mathbf{x})(-i\hbar \nabla)(-\hbar^2 \nabla^2)[\exp(-Zr/a_0)]$$

$$\times \left(\frac{Z}{a_0}\right)^2 \left[\left(\frac{4\pi^2 \alpha A_0}{m_e^2 \omega^3 \hbar}\right) \int \frac{d^3 x}{L^3} \exp(-i\mathbf{k}_f \cdot \mathbf{x}) \exp(2i\omega c \hat{n} \cdot \mathbf{x})(\hat{\epsilon} \cdot (-i\hbar \nabla))^2 \exp(-Zr/a_0)(Z/a_0)^2\right], \quad (29) \quad (30)
where $a_0$ is the Bohr radius. After using the integrating by parts and the perpendicular $\hat{n}$ to $\hat{\epsilon}$, we will obtain the modified differential cross section for the photoelectric effect as follows

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = 32e^2 k_f \frac{(\hat{\epsilon} \cdot \mathbf{k}_f)^2}{m_e c \omega} \left(\frac{Z}{a_0}\right)^5 \frac{1}{\left[(\frac{Z}{a_0})^2 + q^2\right]^4},$$

(31)

$$\left(\frac{d\sigma}{d\Omega}\right)_{ML} = (128\pi)e^2 k_f \frac{(\hbar^2 \hat{\epsilon} \cdot \mathbf{k}_f)^2}{m_e c^3 \omega^3} \left(\frac{Z}{a_0}\right)^5 \frac{1}{\left[(\frac{Z}{a_0})^2 + q^2\right]^4} \left[\frac{(k_f c)^4}{a_0 m_e} + \frac{(\hat{\epsilon} \cdot \mathbf{k}_f)^2 (cA_0)^2}{a_0 m_e}\right],$$

(32)

where $q = (\mathbf{k}_f - \frac{\omega}{c} \hat{n})$ and $(\hat{\epsilon} \cdot \mathbf{k}_f)^2 = k_f^2 \sin^2(\theta) \cos^2(\varphi)$. If we consider the first term $\left(\frac{d\sigma}{d\Omega}\right)_0$, the usual differential cross section and the second term is $\left(\frac{d\sigma}{d\Omega}\right)_{modified}$ the relative modification of differential cross section can be obtained as follows

$$\Delta\left(\frac{d\sigma}{d\Omega}\right)_{ML} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{8\pi}{c^2 \omega^2} \left[\frac{(k_f c)^4}{a_0 m_e} + \frac{(\hat{\epsilon} \cdot \mathbf{k}_f)^2 (cA_0)^2}{a_0 m_e}\right].$$

(33)

Another hand, if we substitute $\beta' = 2\beta$ into Eq. (3), we will obtain the following isotropic minimal length

$$(\Delta X_i)_{min} = \sqrt{\frac{D + 2}{2}} \frac{\hbar}{\sqrt{2\beta}}, \quad \forall i \in \{1, 2, \cdots, D\}. \quad (34)$$

The isotropic minimal length in three spatial dimensions is given by

$$(\Delta X^i)_{min} = \hbar \sqrt{5\beta}. \quad (35)$$

Hence, by inserting Eq. (35) into Eq. (33), we will obtain the relative differential cross section as follows

$$\Delta\left(\frac{d\sigma}{d\Omega}\right)_{ML} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{8\pi}{c^2 \omega^2} \left[\frac{(k_f c)^4}{a_0 m_e} + \frac{(\hat{\epsilon} \cdot \mathbf{k}_f)^2 (cA_0)^2}{a_0 m_e}\right].$$

(36)

Now we can estimate the upper bound on the isotropic minimal length in modified photoelectric effect. If we consider the value of differential cross section is about $10^{-28} m^2$ and also considering $\omega \approx 10^{14} Hz, a_0 \approx 10^{-11} m, K_f \approx 10^{18} m^{-1}$. Here, according to Eq. (36), we can estimate the following upper bound on the the isotropic minimal length

$$10^{-28} \approx (\Delta X^i)_{min}^2 10^{32}, \quad (37)$$

$$\Delta X^i_{min} \leq 10^{-30} m.$$
4 Conclusions

Heisenberg believed that every theory of elementary particles contain a minimal length scale [34,35]. Today we know that every theory of quantum gravity predicts a minimal observable distance. An immediate consequence of GUP is a modification of position and momentum operators. This study has found the transition rate and photoelectric effect in the presence of a minimal length. First, we have considered the time- dependent perturbation theory and then the modified Hamiltonian of atomic electron was obtained up to the first order over the deformation parameter $\beta$. According to the modified Hamiltonian the transition rate in the framework of GUP was investigated. We have seen that the modified transition rate was included in two terms, one term was usual transition rate and the second term was its correction due to the considered minimal length effect. Hence, we have assumed the modified cross section in two terms and then two terms of modified cross section have been obtained. It is necessary to note that, in the limit $\beta \rightarrow 0$, the modified cross section become the usual cross section. Also, the photoelectric effect in the presence of a minimal length was investigated. We have shown that the relative differential cross section was related to the isotropic minimal length. The upper bound on the isotropic minimal length scale has been estimated. It is interesting to note that the upper bound on the isotropic minimal length was close to the Planck length scale.

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