CHAIN POLYNOMIALS OF DISTRIBUTIVE LATTICES ARE 75% UNIMODAL

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Abstract. It is shown that the numbers $c_i$ of chains of length $i$ in the proper part $L \setminus \{0, 1\}$ of a distributive lattice $L$ of length $\ell$ + 2 satisfy the inequalities

$$c_0 < \ldots < c_{\ell/2} \quad \text{and} \quad c_{3\ell/4} > \ldots > c_{\ell}.$$  

This proves 75% of the inequalities implied by the Neggers unimodality conjecture.

1. Introduction

The chain polynomial of a finite poset $P$ is defined as

$$C(P, t) = \sum_i c_i t^i,$$

where $c_i$ is the number of chains (totally ordered subsets) in $P$ of length $i$ (i.e., cardinality $i + 1$). One of the equivalent forms of a well-known poset conjecture due to Neggers [12] implies that the chain polynomial of the proper part $L \setminus \{0, 1\}$ of a distributive lattice $L$ of length $d + 1$ is unimodal, meaning that for some $k$ the coefficients of $C(L \setminus \{0, 1\}, t)$ satisfy the inequalities

$$c_0 \leq \ldots \leq c_k \geq \ldots \geq c_{d-1}.$$  

See [7] and [18] for background, references and more details concerning this unimodality conjecture. Recent progress in a special case (when the poset of join-irreducibles is graded) appears in [8], [9] and [14].

The purpose of this note is to show that the unimodality conjecture is 75% correct, in the sense that violations of unimodality can occur only for indices (roughly) between $d/2$ and $3d/4$. More precisely, we prove the following.

Theorem 1. The numbers $c_i$ of chains of length $i$ in the proper part of a distributive lattice $L$ of length $d + 1$ satisfy the inequalities

$$c_0 < \ldots < c_{(d-1)/2} \quad \text{and} \quad c_{3(d-1)/4} > \ldots > c_{d-1}.$$
The proof consists in observing that the order complex of $L \setminus \{0, 1\}$ is a nicely behaved ball, and then gathering and combining some known facts from $f$-vector theory. The pieces of the argument are stated as Propositions 2, 3, 4 and 5. Of these, only Proposition 3 seems to be new.

2. Some $f$-vector inequalities

For standard notions concerning simplicial complexes we refer to the literature, see e.g. the books \[6, 19\].

Let $\Delta$ be a $(d-1)$-dimensional simplicial complex, and let $f_i$ be the number of $i$-dimensional faces of $\Delta$. The sequence $(f_0, \ldots, f_{d-1})$ is called the $f$-vector of $\Delta$. We put $f_{-1} = 1$. The $h$-vector $(h_0, \ldots, h_d)$ of $\Delta$ is defined by the equation

$$
\sum_{i=0}^{d} f_{i-1} x^{d-i} = \sum_{i=0}^{d} h_i (x+1)^{d-i}.
$$

In the following two results we assume that $(f_0, f_1, \ldots, f_{d-1})$ is the $f$-vector of a $(d-1)$-dimensional simplicial complex $\Delta$, and that $f_0 > d$. From now on, let $d \geq 3$ and $\delta \overset{\text{def}}{=} \left\lfloor \frac{d}{2} \right\rfloor$, $\varepsilon \overset{\text{def}}{=} \left\lfloor \frac{d-1}{2} \right\rfloor$.

**Proposition 2.** Suppose that $h_i \geq 0$, for all $0 \leq i \leq d$. Then

$$
f_i < f_j, \text{ for all } i < j \text{ such that } i + j \leq d - 2.
$$

In particular, $f_0 < f_1 < \ldots < f_{\varepsilon}$.

**Proof.** This implication is well known. See e.g. \[5\] Proposition 7.2.5 (i). \(\square\)

**Proposition 3.** Suppose that $h_i \geq h_{d-i} \geq 0$, for all $0 \leq i \leq \delta$. Then

$$
f_{\left\lfloor 3(d-1)/4 \right\rfloor} > \ldots > f_{d-2} > f_{d-1}.
$$

**Proof.** By \[11\], the $f$-vector $f = (f_0, f_1, \ldots, f_{d-1})$ and the $h$-vector $h = (h_0, h_1, \ldots, h_d)$ satisfy

$$
f_k = \sum_{i=0}^{d} h_i \binom{d-i}{d-1-k}, \quad k = -1, \ldots, d-1.
$$

Define integer vectors $b^i$ as follows:

$$
b^i = (b_0^i, b_1^i, \ldots, b_{d-1}^i), \quad \text{where } b_k^i = \binom{i}{d-1-k}.
$$
Then, by (2),
\[ f = \sum_{i=0}^{\varepsilon} h_i b^{d-i}, \]
which we rewrite
\[ f = \sum_{i=0}^{\delta} (h_i - h_{d-i}) b^{d-i} + \sum_{i=0}^{\delta} h_{d-i} \tilde{b}^i, \]
where
\[ \tilde{b}^i \overset{\text{def}}{=} \begin{cases} b^i + b^{d-i} & \text{if } 2i \neq d \\ b^{d/2} & \text{if } 2i = d. \end{cases} \]

Let us say that a unimodal sequence
\[ a_0 \leq a_1 \leq \ldots \leq a_k \geq \ldots \geq a_n \]
peaks at \( k \) (note that this does not necessarily determine \( k \) uniquely).

It is shown in [4, Proof of Thm. 5, p. 50] that the vector \( \tilde{b}^i \) is unimodal and peaks at \( d - 1 - \left\lfloor \frac{d-i}{2} \right\rfloor \). The vector \( b^{d-i} \) is a segment of a row in Pascal’s triangle, so it is easy to see that it is unimodal and, in fact, also peaks at \( d - 1 - \left\lfloor \frac{(d-i)}{2} \right\rfloor \). One easily checks that
\[ d - 1 - \left\lfloor \frac{(d-i)}{2} \right\rfloor = \begin{cases} \left\lfloor \frac{d}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor - 1 & \text{if } d \text{ and } i \text{ are even} \\ \left\lfloor \frac{d}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor & \text{otherwise}. \end{cases} \]

Hence, both the vectors \( b^{d-i} \) (\( 0 \leq i \leq \varepsilon \)) and the vectors \( \tilde{b}^i \) (\( 0 \leq i \leq \delta \)) are unimodal and peak between \( \delta \) and \( \delta + \left\lfloor \delta/2 \right\rfloor \).

By equation (3), \( f \) is a nonnegative linear combination of the vectors \( b^{d-i} \) and \( \tilde{b}^i \). It follows from the previous paragraph that the inequalities hold for each of these vectors separately, strictly for \( b^d \), and non-strictly otherwise. For the computation of the index \( \left\lfloor 3(d-1)/4 \right\rfloor \), see again [4, pp. 50-51]. Hence, if \( h_d = 0 \) the result follows. The case when \( h_d = 1 \) requires a small extra argument to see that the inequalities are in fact strict. For this case one can proceed as in [4, Proof of Thm. 5]. \( \square \)

3. On the \( h \)-vectors of balls

We say that a simplicial complex is a polytopal \((d - 1)\)-sphere if it is combinatorially isomorphic to the boundary complex of some convex \( d \)-polytope. See Ziegler [19] for notions relating to polytopes and convex geometry.

We now review some definitions and results from the general theory of face numbers. For more about this topic, see e.g. [19] or the survey [1].
It follows from (1) that
\[ h_0 = 1, \quad h_1 = f_0 - d, \quad \text{and} \quad h_d = (-1)^{d-1} \chi(\Delta), \]
where \( \chi(\Delta) \) is the reduced Euler characteristic of \( \Delta \). In particular,

\[ h_d = \begin{cases} 
1, & \text{if } \Delta \text{ is a sphere,} \\
0, & \text{if } \Delta \text{ is a ball,} 
\end{cases} \]

where the conditions are shorthand for saying that \( \Delta \)'s geometric realization is homeomorphic to a sphere, resp. a ball.

The following are the Dehn-Sommerville relations:

(4) \text{If } \Delta \text{ is a sphere then } h_i = h_{d-i}, \text{ for all } 0 \leq i \leq d.

Hence, for spheres all \( f \)-vector information is encoded in the shorter \( g \)-vector
\[ g_i = h_i - h_{i-1}, \]
where the conditions are shorthand for saying that \( \Delta \)'s geometric realization is homeomorphic to a sphere, resp. a ball.

The following are the Dehn-Sommerville relations:

(4) \text{If } \Delta \text{ is a sphere then } h_i = h_{d-i}, \text{ for all } 0 \leq i \leq d.

(5) \text{If } \Delta \text{ is a polytopal sphere, then } g_i \geq 0 \text{ for all } i \geq 0.

If \( \Delta \) is a \((d - 1)\)-ball, its boundary complex \( \partial \Delta \) is a \((d - 2)\)-sphere.
Furthermore, \( \partial \Delta \)'s \( f \)-vector is determined by that of \( \Delta \), as shown by the following consequence of the Dehn-Sommerville relations, due to McMullen and Walkup [11], see also [2, Coroll. 3.9]:

(6) \text{If } \Delta \text{ is a ball with boundary } \partial \Delta, \text{ then } h_i^\Delta - h_{d-i}^\Delta = g_i^{\partial \Delta}.

Say that a \((d - 1)\)-ball \( \Delta \) admits a polytopal embedding if \( \Delta \) is isomorphic to a subcomplex of the boundary complex of some simplicial \( d \)-polytope. The following was shown by Kalai [10, §8] and Stanley [17, Coroll. 2.4]:

(7) \text{If } \Delta \text{ admits a polytopal embedding, then } g_i^{\partial \Delta} \geq 0 \text{ for all } i \geq 0.

Combining (5), (6) and (7), we deduce the following result.

**Proposition 4.** If \( \Delta \) is a \((d - 1)\)-ball, such that either the boundary sphere \( \partial \Delta \) is polytopal or \( \Delta \) admits a polytopal embedding, then
\[ h_i \geq h_{d-i} \geq 0, \text{ for all } 0 \leq i \leq \delta. \]

\[ \square \]

4. **Proof of Theorem 1**

We refer to [16, Ch. 3] for basic facts and notation concerning distributive lattices.

Let \( L \) be a distributive lattice of length \( d + 1 \), and let \( \Delta_L = \Delta(L \setminus \{0, 1\}) \) be the order complex of its proper part. Thus, \( \Delta_L \) is a pure simplicial complex of dimension \( d - 1 \).
Proposition 5. Suppose that $L$ is not Boolean. Then the complex $\Delta_L$ is a $(d - 1)$-ball satisfying

(i) $\Delta_L$ admits a polytopal embedding,
(ii) $\partial \Delta_L$ is polytopal.

Proof. By Birkhoff’s representation theorem (see [16, Ch. 3]) we have that $L = J(P)$, where $J(P)$ is the family of order ideals of some poset $P$ ordered by inclusion. Let $B$ denote the Boolean lattice of all subsets of $P$. Then $\Delta_B = \Delta(B \setminus \{0, 1\})$ is a polytope boundary (the barycentric subdivision of the boundary of a $d$-simplex). Furthermore, $\Delta_L$ is embedded in $\Delta_B$ as a full-dimensional subcomplex. Finally, $\Delta_L$ is a shellable ball [3, 13]. Thus, part (i) is proved.

Part (ii) requires a small convexity argument. Alternatively, it follows from Provan’s result [13] that $\Delta_L$ can be obtained from a simplex via repeated stellar subdivisions. Since this part is not needed for the main result of this paper, details of the proof are left out. □

We now have all the pieces needed to prove Theorem 1. We may assume that $L$ is not Boolean, since in that case $\Delta_L$ is a sphere and Theorem 1 is a special case of [11 Thm. 5]. Then, by Propositions 4 and 5 we have that

$$h_i \geq h_{d-i} \geq 0, \text{ for all } 0 \leq i \leq \delta.$$ 

Furthermore, by Propositions 2 and 3 it follows that the $f$-vector of $\Delta_L$ satisfies

$$f_0 < \ldots < f_{\lfloor(d-1)/2\rfloor} \text{ and } f_{\lceil3(d-1)/4\rceil} > \ldots > f_{d-1}.$$ 

Since $f_i = c_i$ for all $i$, the proof of Theorem 1 is complete.

References

[1] L. J. Billera and A. Björner, *Face numbers of polytopes and complexes*, in “Handbook of Discrete and Computational Geometry, 2nd Ed.” (ed. J. E. Goodman and J. O’Rourke), CRC Press, Boca Raton, FL, 2004, pp. 407–430.

[2] L. J. Billera and C. W. Lee, *The numbers of faces of polytope pairs and unbounded polyhedra*, European J. of Combinatorics **2** (1981), 307 – 322.

[3] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), 159–183.

[4] A. Björner, *Partial unimodality for $f$-vectors of simplicial polytopes and spheres*, in “Jerusalem Combinatorics ’93” (eds. H. Barcelo and G. Kalai), Contemporary Math. Series, Vol. 178, Amer. Math. Soc., 1994, pp. 45–54.
[5] A. Björner, *The homology and shellability of matroids and geometric lattices*, in “Matroid Applications” (ed. N. White), Cambridge Univ. Press, 1992, pp. 226–283.

[6] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, 139, Springer-Verlag, New York-Heidelberg-Berlin, 1993.

[7] F. Brenti, *Unimodal, Log-Concave and Polya Frequency Sequences in Combinatorics*, Memoirs Amer. Math. Soc. 413, Amer. Math. Soc., 1989.

[8] P. Brändén, *Sign-graded posets. unimodality of W-polynomials and the Charney-Davis conjecture*, Electronic Journal of Combinatorics 11 (2) (2004), # R9.

[9] J. D. Farley, *Linear extensions of ranked posets, enumerated by descents. A problem of Stanley from the 1981 Banff Conference on Ordered Sets*, Advances in Appl. Math. (to appear), preprint, 2003.

[10] G. Kalai, *The diameter of graphs of convex polytopes and f-vector theory*, in “Applied geometry and discrete mathematics, The Victor Klee Festschrift”, DIMACS Series in Discrete Math. and Theor. Computer Sci., Vol. 4, Amer. Math. Soc., Providence, R.I., 1991, pp. 387–411.

[11] P. McMullen and D. W. Walkup, *A generalized lower bound conjecture for simplicial polytopes*, Mathematika 18 (1971), 264 – 273.

[12] J. Neggers, *Representations of finite partially ordered sets*, J. Comb. Inf. Syst. Sci. 3 (1978), 113–133.

[13] J. S. Provan, *Decompositions, shellings, and diameters of simplicial complexes and convex polyhedra*, Ph.D. Thesis, Cornell Univ., 1977.

[14] V. Reiner and V. Welker, *On the Charney-Davis and the Neggers-Stanley conjectures*, J. Combinat. Theory, Series A, to appear.

[15] R. P. Stanley, *The number of faces of simplicial convex polytopes*, Advances in Math. 35 (1980), 236 – 238.

[16] R. P. Stanley, *Enumerative Combinatorics*, Vol 1, Cambridge Univ. Press, 1997.

[17] R. P. Stanley, *A monotonicity property of h-vectors and h*-vectors*, Europ. J. Combinatorics 14 (1993), 251 – 258.

[18] R. P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, in “Mathematics: frontiers and perspectives”, Amer. Math. Soc., Providence, R.I., 2000.

[19] G. M. Ziegler, *Lectures on Polytopes*, GTM-series, Springer-Verlag, Berlin, 1995.

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