Central Limit for the Products of Free Random Variables

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March 29, 2011

Abstract

The central limit for the products of identically-distributed positive free random variables are studied by evaluating all the moments of the limit distribution. If the logarithm of the free random variables are zero mean with unity variance, the central limit is found to confine between $\exp(\pm c_0)$ with $c_0 \approx 2.08$. This central limit distribution is not the same but very close to the log-semicircle distribution.

Let $X_i$ denote identically-distributed positive free random variables, if $\log X_i$ exist and are zero mean, the central limit distribution of $Y = \lim_{n \to \infty} \prod_{i=1}^{n} X_i^{1/\sqrt{n}}$ is bounded by $|\log Y| \leq c_0$. All the moments of $Y$ are found and high-order moments are asymptotically the same as those of log-semicircle distribution. If the variance of $\log X_i$ is small, the central limit is very close to log-semicircle distribution, showing similarity to the relationship between normal and log-normal distribution in commuting random variables. If the variance of $\log X_i$ is large, the central limit of $\log Y$ is very close to uniform distribution.

Keywords: Free central limit theorem, products of free random variables, log-semicircle distribution

2000 Mathematics Subject Classification: 46L54, 60F05, 62E17

1 Introduction

Free random variables are random infinite-dimensional linear operators that are equivalently very large random matrices. Free probability theory [Nica and Speicher (2006); Voiculescu et al. (1992)] is used to study free random variables. The theory for free random variables are applicable to study random matrices [Speicher (2003); Voiculescu et al. (1992, §4)]. The statistical properties of free random variables are equivalently that of the eigenvalues of large random matrices.

In free probability theory, the central limit theorem on the sum of independent free random variables gives semicircle distribution [Voiculescu (1986, 1991); Voiculescu et al. (1992, §3.5)]. Semicircle distribution serves the same function as the Gaussian or normal distribution for the sum of independent commuting random variables. Mathematically, if $X_1, X_2, \ldots, X_n$ are identically distributed zero mean free random variables with variance of $(R/2)^2$, the free summation or additive free convolution of

$$\frac{X_1 \boxplus X_2 \boxplus \cdots \boxplus X_n}{\sqrt{n}}$$

has the semicircle distribution of

$$p(t) = \begin{cases} \frac{1}{2\pi R^2} \sqrt{R^2 - t^2} & |t| \leq R \\ 0 & \text{otherwise} \end{cases}$$
where $R$ is the radius of the distribution.

For positive conventional commuting random variables, the product of positive random variables has a central limit as the log-normal distribution because of $x_1x_2 = \exp(\log x_1 + \log x_2)$ for positive random variables of $x_1$ and $x_2$. As the central limit for the logarithm of a random variable is normal distribution, the product of random variables has its central limit as log-normal distribution. Log-normal distribution is applicable to many physical phenomena [Limpert et al. (2001)], the shadowing of wireless channel (Rappaport, 2002, §4.9.2), and some nonlinear noise in fiber communications [Ho (2000)]. Log-normal distribution is also very close to the power law or Zipf law [Mitzenmacher (2004); Newman (2005)].

For positive free random variables, the relationship of $X_1 \boxtimes X_2$ cannot guarantee the same as $\exp(\log X_1 \boxplus \log X_2)$. Alternatively, the relationship of $\exp(X_1 \boxplus X_2)$ is not necessary equal to the free multiplication of $\exp(X_1) \boxtimes \exp(X_2)$. The log-semicircle distribution is not necessary the central limit for the product of positive free random variables.

Here, the central limit distribution is derived for the products of positive free random variables. The moments of the central limit distribution are not the same as that for log-semicircle distribution but numerically very close for the high order moments. The central limit for the sum of zero mean unity variance free random variables has a radius of 2 as from [2]. Let $X_i$ denote identically-distributed positive free random variables, if $\log X_i$ are zero mean with unity variance, the central limit of products of $X_i^{1/\sqrt{n}}$ with $n$ approaching infinity has a radius of approximately $c_0 = 2.0805$. Although very close, the central limit of the product of free random variables is not log-semicircle distribution. Both the difference and similarity are presented in later parts of this paper.

There was a long history to study the products of random matrices [Furstenberg and Kesten (1960)] to represent a physical system with the cascade of a chain of basic random elements with identical statistical properties. The results here are based on the multiplication of free random variables [Voiculescu (1987)]. Similar works of Bercovici and Pata (2000); Bercovici and Wang (2008); Chistyakov and Götze (2008); Kargin (2007b); Tucci (2010) were mainly on the behavior of the products of free random variables but none of them considered the central limit distribution. The behavior of the products of free random variables was studied in [Bercovici and Pata (2000); Bercovici and Wang (2008); Chistyakov and Götze (2008)] as infinitely divisible free random variables. In Kargin (2007b), the norm was considered for the case with unity mean finite support $X_i$. The geometric mean was found in Tucci (2010), similar to the large number theory for commutative random variables. The theory in [Bercovici and Pata (2000); Bercovici and Wang (2008); Chistyakov and Götze (2008); Kargin (2007b); Tucci (2010)] may applicable to the case that $\log X_i$ do not exist, i.e., with an atom in zero. The theory here is only applicable to the case with the existing of $\log X_i$. The results here are very similar to that of [Bercovici and Pata (2000)] but cannot derive from [Bercovici and Pata (2000)] directly.

Later parts of this paper are organized as following: Sect. 2 summarizes the main results of this paper; Sect. 3 gives the proof of the theorems related to the products of free random variables; Sect. 4 shows numerical computation of the products of free random variables, mostly to verify that the central limit distribution is very close to log-semicircle distribution if $\log X_i$ have unity variance; Sect. 5 is the conclusion of this paper.
2 Main Results

The goal here is to find out the central limit for the products of free random variables. Although we are not able to write down the limit distribution explicitly, all moments of the central limit can be derived.

Theorem 1. Let $X_i$ denote identically-distributed positive free random variables, if $\log X_i$ are zero mean and have unity variance, the limit of

$$Y = \lim_{n \to \infty} X_1^{\frac{1}{\sqrt{n}}} \otimes X_2^{\frac{1}{\sqrt{n}}} \otimes \cdots \otimes X_n^{\frac{1}{\sqrt{n}}}$$

(3)

has the following properties:

(a) The $k$-th moment of $Y$ is

$$\varphi(Y^k) = \frac{e^{k/2}}{k} L_{k-1}^{(1)}(-k).$$

(4)

(b) The asymptotic expression of $\varphi(Y^k)$ for large $k$ is

$$2\sigma_0 \frac{I_1(c_0 k)}{c_0 k}$$

(5)

with $c_0 = 2.0805$ and $\sigma_0 = 1.0034$.

(c) $Y$ is bounded by the relationship of $|\log Y| \leq c_0$.

In Theorem 1, $L_n^{(a)}(z)$ is the Laguerre polynomial Koornwinder et al. (2010) Szegö, 1975, §5.1) and $I_n(x)$ is the modified Bessel function of the first kind (Olver and Maximon, 2010, §10.25).

If $\log X_i$ do not have unity variance, we have the following corollary:

Corollary 1. If the variance of $\log X_i$ is $\sigma_v^2$, the $k$-th moment of the central limit $Y$ (3) is

$$\varphi(Y^k) = \frac{e^{\sigma_v^2 k/2}}{k} L_{k-1}^{(1)}(-\sigma_v^2 k),$$

(6)

and $|\log Y| \leq c_0$ with

$$c_0 = \sigma_v \sqrt{1 + \frac{\sigma_v^2}{4}} + 2 \log \left( \frac{\sigma_v^2}{2} + \sqrt{1 + \frac{\sigma_v^2}{4}} \right).$$

(7)

The asymptotic of bound $c_0$ is $c_0 \approx 2\sigma_v$ for small $\sigma_v$ and $c_0 \approx \sigma_v^2/2$ for large $\sigma_v$.

This study is always limited to the case with zero mean $\log X_i$. If $\log X_i$ have positive mean, $Y$ (3) grows to infinity. If $\log X_i$ have negative mean, $Y$ (3) shrinks to zero. Only the cases with zero mean $\log X_i$ is interested for the study of $Y$ (3). In practice, the mean of $\log Y$ may be studied separately as the Lyapunov exponent as in Furstenberg and Kesten (1960).

Definition 1. A positive random variable $Z$ is log-semicircle distributed with radius of $R$ if $\log Z$ has semicircle distribution with radius of $R$. 

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From Theorem 1, all moments of the central limit $Y$ is known. In principle, if all moments of a probability measure is known, the probability measure can be determined, in this case, the Hausdorff moment problems with bounded interval between $\exp(\pm c_0)$ for $Y$ (Akhiezer, 1965, §2.6.4) (Shohat and Tamarkin, 1943, §1.2.d). The probability distribution can be uniquely determined because the moments are conformed to the Carleman’s condition (Shohat and Tamarkin, 1943, §1.6). However, practical evaluation of the probability distribution is difficult. The moments of Theorem 1 can be compared with the moments of log-semicircle distribution with radius of $c_0$. When the variance of log $X_i$ is small, the distribution of $Y$ is very close to log-semicircle distribution.

The moments of log $Y$ can also be found and used to determine log $Y$ as the Hausdorff moment problems with bounded interval between $\pm c_0$. If log $X_i$ have small variance, the moments of log $Y$ are determined by Catalan number, similar to that of semicircle distribution. If log $X_i$ have large variance, the moments of log $Y$ are the same as those of zero mean uniform distribution in the interval of $\pm \sigma_v^2/2$. The distribution of log $Y$ is approximately semicircle distribution if $\sigma_v^2 \ll 16$. The distribution of log $Y$ is approximately uniform distribution if $\sigma_v^2 \gg 16$.

### 3 Products of Free Random Variables

Instead of consider the product of positive identically distributed free random variables $X_i$, we assume that $v_i = \log X_i$ exist such that $\varphi(v_i) = E\{v_i\} = 0$ and $\varphi(v_i^2) = E\{v_i^2\} = 1$, $i = 1, \ldots, n$. The free random variables of $\exp(v_i)$, $i = 1, \ldots, n$, are always positive. Define

$$Y_n = \exp\left(\frac{v_1}{\sqrt{n}}\right) \otimes \exp\left(\frac{v_2}{\sqrt{n}}\right) \otimes \cdots \otimes \exp\left(\frac{v_n}{\sqrt{n}}\right),$$

and

$$Y = \lim_{n \to \infty} Y_n.$$  

Same as (8), the free random variable $Y$ is considered here.

**Lemma 1. Voiculescu (1987)** For free random variable $A$ with probability measure $\mu_A(t)$, define its moment function as

$$\psi_A(z) = \int \frac{zt}{1-zt} d\mu_A(t)$$

and its S-transform as

$$S_A(z) = \frac{1 + z}{z} \chi_A(z),$$

where $\chi_A(z)$ is the inverse function by composition of the moment function $\psi_A(z)$. If $A$ and $B$ are positive free random variables,

$$S_{A \otimes B}(z) = S_A(z)S_B(z).$$

Using Lemma 1 for $Y_n$ (9), $S_{Y_n}(z) = S_{v_i/\sqrt{n}}(z)^n$ and $S_Y(z) = \lim_{n \to \infty} S_{Y_n}(z)$. Although Lemma 1 may extend to free random variables that are not positive (Raj Rao and Speicher, 2007), free random variables $\exp(v_i/\sqrt{n})$ here are always positive.
Lemma 2. The $S$-transform of $Y$ \((2)\) is

$$S_Y(z) = \exp \left(-z - \frac{1}{2}\right). \tag{13}$$

Proof. We first derive $S_{e^{\psi_i/\pi}}(z)$. The $S$-transform of $e^{(\psi_i/\sqrt{n})}$ can be found as a power series of $1/\sqrt{n}$, in essence the perturbation analysis of the $S$-transform. Assume that $\psi_i(z) = z^2 + \sum_{k=3}^{\infty} \varphi(v_i^k) z^k$ as $\psi_i$ are zero mean, unity variance free random variables, where $\varphi(v_i^k) = E\{v_i^k\}$ is the $k$th moments of $u_i$. Note that the moments of $\varphi(v_i^k)$, $k \geq 3$, is not very important to arrive with the results but includes here for completeness.

The moment function of $\exp(\psi_i)$ is

$$\psi_{e^{\psi_i}}(z) = \frac{z}{1-z} + \frac{z(1+z)}{2(1-z)^3} + \sum_{k \geq 3} \varphi(v_i^k) \frac{\partial^k}{\partial t^k} \left[ \frac{z}{e^{-t} - z} \right]_{t=0}, \tag{14}$$

where $\frac{\partial^k}{\partial t^k} \left[ \frac{z}{e^{-t} - z} \right]_{t=0} = \sum_{l \geq 1} t^l z^l$ is the coefficient corresponding to $\varphi(v_i^k)$ for the moment function $\psi_{e^{\psi_i}}(z)$. The summation of $\sum_{l \geq 1} t^l z^l$ may be obtained by expanding $e^{\psi_i}$ in $\sum_{k \geq 0} \varphi(e^{kv_i}) z^k$.

The moment function of $\exp(\psi_i/\sqrt{n})$ as a power series of $1/\sqrt{n}$ is

$$\psi_{e^{\psi_i/\sqrt{n}}}(z) = \frac{z}{1-z} \left[ 1 + \frac{1+z}{2n(1-z)^2} + \varphi(v_i^3) \frac{g_3(z)}{6n^2} - \varphi(v_i^4) \frac{g_4(z)}{24n^2} + O(n^{-\frac{5}{2}}) \right], \tag{15}$$

where

$$g_3(z) = \frac{1+4z+z^2}{(1-z)^3},$$

$$g_4(z) = \frac{z^3 + 11z^2 + 11z + 1}{(1-z)^4}.$$

The inverse function of \((15)\) can also be expressed as a power series of $1/\sqrt{n}$ to

$$\chi_{e^{\psi_i/\sqrt{n}}}(z) = \frac{z}{1+z} \left[ 1 - \frac{1}{n} \left( z + \frac{1}{2} \right) - \varphi(v_i^3) \frac{h_3(z)}{n^2} + \varphi(v_i^4) \frac{h_4(z)}{n^2} + O(n^{-\frac{5}{2}}) \right], \tag{16}$$

where

$$h_3(z) = z^2 + z + \frac{1}{6},$$

$$h_4(z) = z^3 + 2z^2 + \frac{7}{6} z + \frac{5}{24}.$$

The $S$-transform of $e^{\psi_i/\sqrt{n}}$ is

$$S_{e^{\psi_i/\sqrt{n}}}(z) = 1 - \frac{1}{n} \left( z + \frac{1}{2} \right) - \varphi(v_i^3) \frac{h_3(z)}{n^2} + \varphi(v_i^4) \frac{h_4(z)}{n^2} + O(n^{-\frac{5}{2}}). \tag{17}$$

The $S$-transform of $Y_n$ \((5)\) becomes a power series of $1/\sqrt{n}$ using the relationship of $S_Y(z) = [S_{e^{\psi_i/\sqrt{n}}}(z)]^n$ as

$$S_{Y_n}(z) = \exp \left\{ -z - \frac{1}{2} \frac{\varphi(v_i^3)}{\sqrt{n}} h_3(z) + \frac{1}{n} \left[ \varphi(v_i^4) h_4(z) - \frac{1}{2} \left( z + \frac{1}{2} \right) \right] + O(n^{-\frac{3}{2}}) \right\} \tag{18}$$
and
\[ S_Y(z) = \lim_{n \to \infty} S_{Y_n}(z) = \exp\left(-z - \frac{1}{2}\right). \] (19)

The S-transform of (19) is infinitely divisible from the theory of Bercovici and Pata (2000); Bercovici and Wang (2008); Chistyakov and Götzé (2008). In practice, all terms higher than the second-order is not required in (15), (16) and (18). The S-transform of (18) shows the convergent properties with the increase of \( n \), similar to the analysis of Kargin (2007a).

For non-symmetric free random variable \( v_i \) with non-zero skewness proportional to \( \phi(v_i^3) \), the convergence is far slower than the symmetric free random variable with zero skewness.

With the S-transform of \( Y \), the moment function of \( Y \) can be founded.

**Lemma 3** (Lagrange-Bürmann formula). *If the dependence between \( w \) and \( z \) is given by \( w/\phi(w) = z \) and \( \phi(0) \neq 0 \), the inversion series for \( w = g(z) \) is given by

\[
g(z) = \sum_{k>0} \frac{z^k}{k!} \lim_{w \to 0} \frac{d^{k-1}}{dw^{k-1}} \phi(w)^k. \] (20)

This lemma also called Lagrange inversion formula and was given in (Whittaker and Watson, 1927, §7.32). A variation of this lemma (20) was used in Kargin (2007a,b).

**Part a of Theorem 2** The inversion of the moment function \( \psi_Y(z) \) is given by

\[
\chi_Y(z) = \frac{z}{1 + z} \exp\left(-z - \frac{1}{2}\right). \] (21)

Compared with the Lagrange-Bürmann formula of (20), \( \phi(w) = (1 + w) \exp(w + \frac{1}{2}) \).

From the definition of moment function in Lemma 1 the \( k \)-th moment of \( Y \) is given by

\[
\varphi(Y^k) = \frac{e^{k/2}}{k!} \frac{d^{k-1}}{dw^{k-1}} (1 + w)^k e^{kw} \bigg|_{w=0} \] (22)

or

\[
\varphi(Y^k) = \frac{1}{k} \exp\left(\frac{k}{2} \sum_{m=0}^{k-1} \frac{k^m}{m!(m+1)} \right), \] (23)

by expanding \( e^{kw} \) and collecting all terms with \( w^{k-1} \) in \((1 + w)^k e^{kw}\).

Using the generalized Laguerre polynomial \( L_n^{(a)}(z) \) Koornwinder et al. (2010) (Szegő, 1975, §5.1), the moment (23) can be expressed as

\[
\varphi(Y^k) = \frac{e^{k/2}}{k} L_{k-1}^{(1)}(-k). \] (24)

Confluent hypergeometric function \( _1F_1(a; b; z) \) (Szegő, 1975, §5.3) can also be used instead of the Laguerre polynomial. The moment (24) becomes

\[
\varphi(Y^k) = e^{k/2} _1F_1(-k + 1; 2; -k) = M_{k, \frac{1}{2}}(-k)/k, \] (25)
where $M_{\kappa, \mu}(z)$ is the Whittaker function (Whittaker and Watson, 1927, §16.1).

The asymptotic expression for large $k$ for (24) determines the upper bound of $Y$. The Laguerre polynomial in (24) has different simple asymptotic expressions (Szegö, 1975, §8.22) that is for $L_n^{(\alpha)}(-x)$, $0 < x \ll n$ as from Borwein et al. (2008). The expression in (Szegö, 1975, Theorem 8.22.4) may extend to negative number to give rough asymptotic of $e^{x/2}L_{n-1}^{(1)}(-x) \sim \sqrt{n/x}I_1(2\sqrt{nx})$, where $I_1(x)$ is the modified Bessel function of the first kind (Olver and Maximon, 2010, §10.25). Ignore the smooth term, the rough asymptotic gives $\phi(Y_k) \sim 1/2e^{2k/\sqrt{\pi k}}$ by the asymptotic of $I_1(x) \sim e^x/\sqrt{2\pi x}$ (Olver and Maximon, 2010, §10.40.1). Although for $0 < x \ll n$, the simple asymptotic gives very close result for the case $n \approx x$.

**Lemma 4** (Asymptotic of Laguerre polynomial). For large $k$ and $\beta > 0$,

$$e^{\beta k/2}L_{k-1}^{(1)}(-\beta k) = \frac{\zeta^{3/2}}{\beta^{4/3}(4 + \beta)^{1/3}} I_1(\zeta k) [1 + O(k^{-1})],$$

where

$$\zeta = \sqrt{\beta \left(1 + \frac{\beta}{4}\right) + 2 \log \left(\frac{\sqrt{\beta}}{2} + \sqrt{1 + \frac{\beta}{4}}\right)}.$$  

The asymptotic approximation (26) is a special case that can be derived directly from (Frenzen and Wong, 1988, Eq. 4.7) with higher-order terms there. The same asymptotic approximation can also be found in (Koornwinder et al., 2010, §18.15.19) Temme (1990). Using the asymptotic approximation of Whittaker function (Olde Daalhuis, 2010, §13.21.6), the asymptotic expression (26) can also be derived.

**Part b of Theorem 1**. Using Lemma 4 for (24), part b of Theorem 1 is with $\beta = 1$ in (26), we obtain

$$\phi(Y^k) = 2\alpha_0 I_1(c_0 k) / c_0 k [1 + O(k^{-1})],$$

with $c_0 = \sqrt{5}/2 + 2 \log (1/2 + \sqrt{5}/2) = 2.0805$ and $\alpha_0 = c_0^{3/2} \times 5^{-1/2}/2 = 1.0034$.

Note that the above evaluation is very close to that using (Szegö, 1975, Theorem 8.22.4) that gives $c_0 = 2$.

**Lemma 5** (Markov inequality). Let $X$ be positive random variable and $h$ as a non-negative real function, then

$$\Pr(h(X) \geq a) \leq \frac{E\{h(X)\}}{a}$$

for all $a > 0$.

The format of Markov inequality like Lemma 5 is the same as that in (Grimmett and Stirzaker, 1992, §7.3.1).

**Proof of part c of Theorem 1**. For positive $Y$ and $r$, and $h(Y) = Y^k$, Markov inequality may be written as

$$\Pr(Y^k \geq r^k) = \Pr(Y \geq r) \leq \frac{\phi(Y^k)}{r^k}.$$
With \( k \to \infty \), using the asymptotic of \( I_1(z) \sim e^z/\sqrt{2\pi z} \) [Olver and Maximon, 2010, §10.40.1] for (28), we obtain
\[
\text{Pr} (Y \geq r) = 0 \quad \text{if} \quad r > \exp(c_0) = \lim_{k \to \infty} \phi(Y_k)^{1/k},
\]
where \( c_0 \) is the same as that in (28).

With
\[
Y^{-1} = \lim_{n \to \infty} X_n^{-\frac{1}{2\sqrt{n}}} \otimes X_{n-1}^{-\frac{1}{2\sqrt{n}}} \otimes \cdots \otimes X_1^{-\frac{1}{2\sqrt{n}}}
\]
and following the procedure from (14) to (31), we may obtain
\[
\text{Pr} (Y^{-1} \geq r) = 0 \quad \text{if} \quad r > \exp(c_0).
\]

The only difference between \( Y \) and \( Y^{-1} \) is that sign of all odd-order terms in (14) is negative. All the odd-order terms do not change the results to (31).

With both (31) and (33), \( Y \) is bounded between \( \exp(\pm c_0) \). Equivalently, \( |\log Y| \leq c_0 \).

With all the moments of the central limit of \( Y \) from (24), the probability measure is determined as a Hausdorff moment problem. The Hausdorff moment problem has a solution if \( \phi(Y_k) \exp(-c_0 k) \) is completely monotonic [Akhiezer, 1965, §2.6.4]. We numerically verify that \( \phi(Y_k) \exp(-c_0 k) \) is completely monotonic up to \( k = 200 \).

**Remark 2** (Saddle point approximation for parts b and c of Theorem 1). The asymptotic expression of (28) may be found using saddle point approximation to sum each term in (23). For large \( k \), binomial coefficient is
\[
\binom{k}{m+1} \sim \frac{2^k}{\sqrt{\pi k/2}} \exp\left(-\frac{(m+1-k/2)^2}{k/2}\right),
\]
that is based on the normal approximation of binomial distribution by de Moivre-Laplace limit theorem [Grimmett and Stirzaker, 1992, §4.4.4]. The factorial of \( m \approx \sqrt{2\pi mm^m e^{-m}} \) from the Stirling formula [Diaconis and Freedman, 1986], [Whittaker and Watson, 1927, §12.33].

For large \( k \), the moments (23) can be rewritten as
\[
\phi(Y^k) \sim \frac{2^k e^{k/2}}{k^{\sqrt{n}/k/2}} \sum_{m=0}^{k-1} \frac{1}{\sqrt{2\pi m}} \exp(-f(m))
\]
with
\[
f(m) = -\log(k) \ast m + \frac{(m+1-k/2)^2}{k/2} + m\log(m) - m.
\]

The saddle point is located at \( m_0 = \kappa k \) with
\[
\kappa = \frac{1}{4} W_0 \left[ 4 \exp\left(2 - \frac{4}{k}\right) \right] \sim \frac{1}{4} W_0 (4e^2) \approx 0.62
\]
and \( f''(m_0) = 4/k + 1/m_0 \), where \( W_0 \) is the Lambert W function at the main branch [Corless et al. 1996].
We may obtain
\[ \varphi(Y^k) \sim \frac{1}{k \sqrt{2\pi(\kappa + \frac{1}{4})}} \exp(c_0 k), \] (38)
where
\[ c_0 = \log 2 + \kappa + 2\kappa^2 \approx 2.0807. \] (39)

The bound of \( c_0 \) in (39) has almost no difference with that obtain based on Lemma 4 although the approximation (34) is valid only when the saddle point \( m_0 \) is in the middle and close to \( k/2 \).

If the variance of \( \log X_i \) is not equal to unity, Corollary 1 may be used.

**Corollary 1.** With \( \varphi(v^2) = \sigma_v^2 \), following the procedure from (14) to (19), we get
\[ S_Y(z) = \exp \left[ -\sigma_v^2 \left( z + \frac{1}{2} \right) \right]. \] (40)

Following the procedure from (21) to (24), we obtain \( \varphi(Y^k) = e^{\sigma_v^2 v k/2} k L_1^{(1)}(-\sigma_v^2 k). \) (41)

The asymptotic of Laguerre polynomial of Lemma 4 has \( \beta = \sigma_v^2 \) in (28) with \( c_0 \) given by (27) and
\[ \alpha_0 = \frac{c_0^3}{2\sigma_v^2 (4 + \sigma_v^2)^{1/4}}. \] (42)

Following the procedure from (30) to (33), we obtain
\[ c_0 = \sigma_v \sqrt{1 + \frac{\sigma_v^2}{4}} + 2\log \left( \frac{\sigma_v}{2} + \sqrt{1 + \frac{\sigma_v^2}{4}} \right). \] (43)

The relationship of \( \alpha_0 \) versus \( \sigma_v^2 \) is helpful to understand the asymptotic (41) as compared with that with \( \sigma_v^2 = 1 \). For small \( \sigma_v^2 \ll 1 \), \( \alpha_0 \to 1 \) and similar to the case of \( \sigma_v^2 = 1 \). For large \( \sigma_v^2 \gg 1 \), \( \alpha_0 \sim \sqrt{2\sigma_v}/8 \) proportional to \( \sigma_v \) but increase far slower than \( \sigma_v^2 \). Except if \( \sigma_v^2 \gg 32 \), we may assume that \( \alpha_0 \) is close to unity.

The 2k-th moments of semicircle distribution is given by the k-th Catalan number \( C_k = \frac{1}{2^{k+1}} \binom{2k}{k} \) as \((\text{Bressoud, 2010}, \S 26.5)\) \((\text{Nica and Speicher, 2006}, \S 2)\)
\[ \left( \frac{R}{2} \right)^{2k} C_k. \] (44)

**Lemma 6** (Moments for log-semicircle distribution). Let \( Z \) as positive random variable having log-semicircle distribution with radius of \( R \), the k-th moment of \( Z \) is
\[ \varphi(Z^k) = 2 \frac{I_1(kR)}{kR}. \] (45)
Proof. From definition and using (2), the $k$-th moment of $Z$ is
\[
\varphi(Z^k) = \frac{2}{\pi R^2} \int_{-R}^{+R} e^{kx} \sqrt{R^2 - x^2} \, dx. \tag{46}
\]
With $x = R \sin \theta$, the integral becomes
\[
\varphi(Z^k) = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} e^{kR \sin \theta \cos^2 \theta} \, d\theta. \tag{47}
\]
First move the integration interval to 0 to $\pi$, the integration becomes $I_0(kR) - I_2(kR)$ by using (Olver and Maximon, 2010, §10.32.3). Using (Olver and Maximon, 2010, §10.29.1),
\[
I_0(kR) - I_2(kR) = \frac{I_1(kR)}{kR} \quad \text{and} \quad \varphi(Z^k) = 2 \frac{I_1(kR)}{kR}. \tag{48}
\]
Comparing (45) of log-semicircle distribution with the asymptotic of (28) for $\varphi(Y^k)$, the high-order moments of $Y$ are very close to those of log-semicircle distribution when log $X$ have unity or less variance. The central limit for the product of free random variables $Y$ has similar high-order moments as those for log-semicircle distribution with $R = c_0$ in (45).

In additional to compare the moments of (24) to (45), the moments of log $Y$ may compare with those of a semicircle distribution given by (44). Using (25) for arbitrary $\sigma^2_v$, the exponential moment generating function of log $Y$ is
\[
\varphi(e^{s \log Y}) = \varphi(Y^s) = e^{\sigma^2_v s/2} F_1 \left(1 - s; 2; -\sigma^2_v s\right). \tag{49}
\]
Using the Kummer transformation (Olde Daalhuis, 2010, §13.2.39) with
\[
e^{\sigma^2_v s/2} F_1 \left(1 - s; 2; -\sigma^2_v s\right) = e^{-\sigma^2_v s/2} F_1 \left(1 + s; 2; \sigma^2_v s\right), \tag{50}
\]
the moment generating function (49) is an even function with $\varphi(Y^s) = \varphi(Y^{-s})$. All the odd moments of log $Y$ are equal to zero. For any positive integer $k$, the $2k$-th moments of log $Y$ is given by
\[
\varphi(\log^{2k} Y) = \lim_{s \to 0} \frac{d^{2k}}{ds^{2k}} e^{\sigma^2_v s/2} F_1 \left(1 - s; 2; -\sigma^2_v s\right). \tag{51}
\]
The expansion of
\[
F_1(1 - s; 2; -\sigma^2_v s) = 1 + \sum_{k=1}^{\infty} \sigma^2_v (s - 1)(s - 2)\cdots(s - k)s^k \quad \frac{k!(k + 1)!}{k! \cdots k!} = 1 + \sum_{k=1}^{\infty} f_k s^k \tag{52}
\]
gives the coefficient $f_k$ as
\[
f_0 = 1, \quad f_k = \sum_{i=\lceil k/2 \rceil}^{k} \frac{s(i + 1, k + 1 - i)\sigma^2_i}{i!(i + 1)!}, \quad k \geq 1. \tag{53}
\]
where \( s(n, k) \) is the Stirling number of the first kind (Bressoud, 2010, §26.8), and \([k/2]\) is the smallest integer larger than or equal to \( k/2 \). The moments becomes

\[
\varphi(\log^{2k} Y) = (2k)! \sum_{i=0}^{2k} \frac{f_i \sigma_{4k-2i}^k}{2^{2k-i}(2k-i)!}.
\]

(54)

For some low-order moments, we have:

**Corollary 2.** If the variance of \( \log X_i \) is \( \sigma_v^2 \), we have:

\[
\begin{align*}
\varphi(\log^2 Y) &= \sigma_v^2 + \frac{\sigma_v^4}{12}, \\
\varphi(\log^4 Y) &= 2\sigma_v^4 + \frac{\sigma_v^6}{3} + \frac{\sigma_v^8}{80}, \\
\varphi(\log^6 Y) &= 5\sigma_v^6 + \frac{5\sigma_v^8}{4} + \frac{23\sigma_v^{10}}{240} + \frac{\sigma_v^{12}}{448}.
\end{align*}
\]

(55) \( \text{(56) (57)} \)

The 2\( k \)-th moment of \( \varphi(\log^{2k} Y) \) is always a linear combination of \( \sigma_v^{2k} \) to \( \sigma_v^{4k} \). The \( \sigma_v^{2k} \) term dominates for small \( \sigma_v^2 \) but the \( \sigma_v^{4k} \) term dominates for large \( \sigma_v^2 \).

The coefficient of \( \sigma_v^{2k} \) for \( \varphi(\log^{2k} Y) \) is the Catalan number \( C_k \). The Catalan number is given by \( i = 2k \) in (54) using \( f_{2k} \) that further uses the lowest order term of \( i = k \) in (54) with \( s(k+1, k+1) = 1 \) (Bressoud, 2010, §26.8.1).

The coefficient of \( \sigma_v^{4k} \) for \( \varphi(\log^{2k} Y) \) is given by

\[
C_k = \frac{1}{(2k+1)2^{2k}}
\]

(58)

using \( s(i+1, 1) = (-1)^i i! \) (Bressoud, 2010, §26.8.14) and \( \sum_{i=0}^{2k} (-2)^i \binom{2k}{i+1} = 1 \).

If \( \sigma_v^2 \) is small, the first term of \( \varphi(\log^{2k} Y) \) dominates, we get \( \varphi(\log^2 Y) \approx C_k \sigma_v^{2k} \). Compared with (44), \( \log Y \) has semicircle distribution with radius of \( R = 2\sigma_v \).

If \( \sigma_v^2 \) is very big, the last term of \( \varphi(\log^{2k} Y) \) dominates, we get

\[
\varphi(\log^{2k} Y) \approx \frac{1}{2k+1} \left( \frac{\sigma_v^2}{2} \right)^{2k},
\]

(59)

that is the 2\( k \)-th moment of a uniform distribution between \( \pm \sigma_v^2/2 \).

Because moments of (59) must be far larger than \( C_k \sigma_v^{2k} \), with \( C_k \approx 4^k/\sqrt{\pi} k^3 \) (Bressoud, 2010, §26.5.6), the variance \( \sigma_v^2 \) needs to be far larger than \( 2^4 \) for (59) to dominate. Conversely, \( \sigma_v^2 \) needs to be far smaller than \( 2^4 \) for (59) to be negligible.

**Corollary 3.** If the variance of \( \log X_i \) is \( \sigma_v^2 \) and \( \sigma_v^2 \ll 16 \), the distribution of \( \log Y \) with \( Y \) given by (3) is approximately semicircle distribution with radius of \( 2\sigma_v \).

**Corollary 4.** If the variance of \( \log X_i \) is \( \sigma_v^2 \) and \( \sigma_v^2 \gg 16 \), the distribution of \( \log Y \) with \( Y \) given by (3) is approximately uniform distribution between \( \pm \sigma_v^2/2 \).

Using large matrices for free random variables, Corollary 4 is for matrices \( X_i \) and \( Y \) having a very large condition number.

In Kargin (2007b), \( X_i \) has unity mean with \( \varphi(X_i) = 1 \). Using Jensen inequality (Grimmett and Stirzaker, 1992, §5.6.15) for convex function \( -\log(x) \), \( \varphi(\log X_i) \leq \log \varphi(X_i) \). If \( X_i \) has unity mean, the mean of \( \log X_i \) is less or equal to zero and the products of \( X_i \) may shrink to zero.
Figure 1: Simulated central limit distribution of log $Y$ for $\sigma_v^2 = 1$ and $\sigma_v^2 = 16$ and compared with semicircle distribution. The x-axis is normalized by $c_0$ [13].

4 Numerical Comparison

From the above results, it may be very certain that

$$\exp \left( \frac{v_1 \oplus v_2 \oplus \cdots \oplus v_n}{\sqrt{n}} \right) \neq \exp \left( \frac{v_1}{\sqrt{n}} \right) \otimes \exp \left( \frac{v_2}{\sqrt{n}} \right) \otimes \cdots \otimes \exp \left( \frac{v_n}{\sqrt{n}} \right). \quad (60)$$

In the limit of $n \to \infty$, the left hand side of (60) is a log-semicircle distribution with radius of $2\sigma_v$. Although the logarithm of right hand side of (60) or log $Y$ has an upper bounded of $c_0$, this upper bound is determined by (43) that is typically larger than $2\sigma_v$. Of course, the distribution of $Y$ is numerically very close to log-semicircle distribution when $\sigma_v^2 \leq 16$.

The simulated distribution of log $Y$ is shown in Fig. 1 when the variance of log $X_i$ is either $\sigma_v^2 = 1$ or 16. Each free random variables of $v_i$ (or log $X_i$) are approximated by $512 \times 512$ positively defined complex random matrices with eigenvalues determined by unbounded normal distribution with the variance of either $\sigma_v^2 = 1$ or 16. The distribution of $X_i$ is log-normal distribution that does not have an unique solution from its moments [Heyde (1963)]. The products of $Y_n$ is for $n = 256$ random matrices. The multiplication of $X_1X_2$, for example, is given by $X_{1}^{1/2}X_2X_{1}^{1/2}$. The probability density of Fig. 1 is given by the histogram of 51,200 eigenvalues.

From Fig. 1 the eigenvalue of log $Y_{256}$ is very close to semicircle distribution if log $X_i$ have unity variance of $\sigma_v^2 = 1$. The discrepancy to semicircle distribution is very difficult to observe. With large variance of $\sigma_v^2 = 16$, log $Y_{256}$ has observable difference with respect to semicircle distribution but still very close. With $\sigma_v^2 = 16$, the distribution of log $Y_{256}$ is almost the same as semicircle distribution near $\pm c_0$. The probability density has the trend to be more uniformly distributed and is larger than semicircle distribution in the third quarter point of $\pm0.75c_0$ but smaller than the semicircle distribution around zero.

If the variance of log $X_i$ is larger than 16, numerical simulation for the central limit is difficult. The central limit $Y$ has a big conditional number and all small eigenvalues have
big numerical errors. In the further numerical results for $\sigma_v^2 = 64$, big eigenvalues of both $Y$ and $Y^{-1}$ larger than unity are numerically very uniformly distributed.

There are many methods to compare the moments of (45) with that of (24) to get a sense on the similarity between log-semicircle distribution with that of $Y$. One of the method is to find the implied or corresponding radius for each moment by numerically solving

$$2 I_1(kr_k) \cdot kr_k = e^{k/2} I^{(1)}_{k-1}(-k)$$

as a function of $k$ and shown in Fig. 2. The radius $r_k$ are reduced from $r_1 = 2.0816$ to $r_{100} = 2.0805$ for large $k$ with a difference of about $11 \times 10^{-4}$. The difference of $r_{100} - c_0$ is about $3 \times 10^{-5}$.

The corresponding radius can also be found by

$$C_{2k} \left( \frac{r_{2k}}{2} \right)^{2k} = \varphi(\log^{2k} Y),$$

that is also shown in Fig. 2. For log $Y$, the radius $r_k$ are reduced from $r_1 = 2.0817$ to $r_{100} = 2.0805$ for large $k$ with a difference of about $12 \times 10^{-4}$. The difference of $r_{100} - c_0$ is about $6 \times 10^{-5}$.

The radius in Fig. 2 shows that log $Y$ can be well-approximated by semicircle distribution with radius of $R = c_0$ for $\sigma_v^2 = 1$.

5 Conclusion

If $X_i$ are positive free random variables with zero mean log $X_i$, central limit for the product of $X_i^{1/\sqrt{n}}$ are bounded between $\exp(\pm c_0)$. If log $X_i$ has unity variance, $c_0 \approx 2.08$ as a
special case. If the variance of \( \log X_i \) is far smaller than 16, the central limit distribution is very close to log-semicircle distribution. If the variance of \( \log X_i \) is far larger than 16, the logarithm of the central limit is very close to uniform distribution.

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