Some new results on the total domination polynomial of a graph

Saeid Alikhani*and Nasrin Jafari

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Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
alikhani@yazd.ac.ir

Abstract

Let $G = (V, E)$ be a simple graph of order $n$. The total dominating set of $G$ is a subset $D$ of $V$ that every vertex of $V$ is adjacent to some vertices of $D$. The total domination number of $G$ is equal to minimum cardinality of total dominating set in $G$ and is denoted by $\gamma_t(G)$. The total domination polynomial of $G$ is the polynomial $D_t(G, x) = \sum_{i=\gamma_t(G)} d_t(G, i)x^i$, where $d_t(G, i)$ is the number of total dominating sets of $G$ of size $i$. A root of $D_t(G, x)$ is called a total domination root of $G$. We denote the set of distinct total domination roots by $Z(D_t(G, x))$.

1 Introduction

Let $G = (V, E)$ be a simple graph. The order of $G$ is the number of vertices of $G$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of $S$ is the set $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$. The set $D \subset V$ is a total dominating set if every vertex of $V$ is adjacent to some vertices of $D$, or equivalently, $N(D) = V$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set in $G$. A total dominating set with cardinality $\gamma_t(G)$ is called a $\gamma_t$-set. An $i$-subset of $V$ is a subset of $V$ of cardinality $i$. Let $D_t(G, i)$ be the family of total dominating sets of $G$ which are $i$-subsets and let $d_t(G, i) = |D_t(G, i)|$. The polynomial $D_t(G; x) = \sum_{i=1}^{\gamma_t(G)} d_t(G, i)x^i$ is defined as total domination polynomial of $G$. A root of $D_t(G, x)$ is called a total domination root of $G$. We denote the set of distinct total domination roots by $Z(D_t(G, x))$.  

*Corresponding author
The corona of two graphs $G_1$ and $G_2$, as defined by Frucht and Harary in [12], is the graph $G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i$-th vertex of $G_1$ is adjacent to every vertex in the $i$-th copy of $G_2$. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added.

Recurrence relations of graph polynomials have received considerable attention in the literature. It is well-known that the independence polynomial and matching polynomial of a graph satisfies a linear recurrence relation with respect to two vertex elimination operations, the deletion of a vertex and the deletion of vertex’s closed neighborhood. Other graph polynomials in the literature satisfy similar recurrence relations with respect to vertex and edge elimination operations [13]. In contrast, it is significantly harder to find recurrence relations for the domination polynomial and the total domination polynomial. The easiest recurrence relation is to remove an edge and to compute the total domination polynomial of the graph arising instead of the one for the original graph. Indeed, for the total domination polynomial of a graph there might be such irrelevant edges, that can be deleted without changing the value of the total domination polynomial at all. An irrelevant edge is an edge $e \in E$ of $G$, such that $D_t(G, x) = D_t(G \setminus e, x)$.

The roots of graph polynomials reflect some important information about the structure of graphs. There are many papers on the location of the roots of graph polynomials such as chromatic polynomial, matching polynomial, independence polynomial, characteristic polynomial and domination polynomial. We refer the reader to [14] and its references for more information in roots of graph polynomials. In [2] we have shown that all roots of $D_t(G, x)$ lie in the circle with center $(-1, 0)$ and the radius $\sqrt[2]{2^n} - 1$, where $\delta$ is the minimum degree of $G$. Also we proved that for a graph $G$ of order $n$, if $\delta \geq \frac{2n}{3}$, then every integer root of $D_t(G, x)$ lies in the set $\{-3, -2, -1, 0\}$.

As usual we denote the complete graph, path and cycle of order $n$ by $K_n$, $P_n$ and $C_n$, respectively. Also $S_n$ is the star graph with $n$ vertices. A leaf (end-vertex) of a graph is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf.

In the next section, we characterize irrelevant edges for the total domination polynomial. We consider regular graphs in Section 3 and study their total domination polynomials. Finally we study graphs with exactly two total domination roots $\{-3, 0\}$, $\{-2, 0\}$ and $\{-1, 0\}$ in Section 4.

# 2 Irrelevant edges

The easiest recurrence relation for total domination polynomial of a graph is to remove an edge and to compute the total domination polynomial of the graph arising instead of the one for the original graph. Indeed, for the total domination polynomial of a graph there might be such irrelevant edges, that can be deleted without changing the value of the total domination polynomial at all. In this section, we study these edges.

**Definition 2.1** Let $G = (V, E)$ be a graph. A vertex $v \in V$ is total domination-covered, if each total dominating set of $G \setminus v$ is a total dominating set of $G$ and the
Figure 1: The edge $e$ is not irrelevant but $u$ and $v$ are total domination covered in $G \setminus e$.

total dominating sets of $G \setminus v$ are exactly those total dominating sets of $G$ not including the vertex $v$.

The proofs of Theorems 2.2 and 2.3 follow the proofs in [13] with some minor changes:

**Theorem 2.2** Let $G = (V, E)$ be a graph. A vertex $v \in V$ is total domination-covered if there is a vertex $u \in N[v]$ such that $N[u] \subseteq N[v]$.

**Proof.** To dominate $u$, the vertex $u$ or a vertex adjacent to $u$ must be in each total dominating set of $G \setminus v$. Since every vertex adjacent to $u$ is also adjacent to $v$ in $G$, so we have the result. □

**Theorem 2.3** Let $G = (V, E)$ be a graph. If $e = \{u, v\} \in E$ is an irrelevant edge in $G$, then $u$ and $v$ are total domination-covered in $G \setminus e$.

**Proof.** By contradiction, suppose that at least one vertex (say $u$) is not total domination-covered in $G \setminus e$. Then there exist a total dominating set of $G \setminus e$ which is not a total dominating set of $G$, and this implies that $D_t(G, x) \neq D_t(G \setminus e, x)$ which is a contradiction. □

Note that the converse of Theorem 2.3 is not true. As an example for the graph in Figure 1, the vertices $u$ and $v$ are total domination-covered, while $e$ is not irrelevant edge. Because $D_t(G, x) = x^5 + 5x^4 + 8x^3 + 5x^2$ but the total domination polynomial of $G \setminus e$ is $x^5 + 5x^4 + 6x^3 + 4x^2$.

We need the following theorem to obtain more results.

**Theorem 2.4** [10] If $G = (V, E)$ is a graph and $e = \{u, v\} \in E$ with $N[v] = N[u]$, then $D_t(G, x) = D_t(G \setminus e, x) + x^2D_t(G \setminus N[u], x)$.

By Theorem 2.4, we have the following result.

**Theorem 2.5** Let $G = (V, E)$ be a graph and $e = \{u, v\} \in E$ with $N[v] = N[u]$. If there exists a support vertex $w \in N[u]$, then the edge $e$ is an irrelevant edge.

**Proof.** By theorem 2.4 we have
\[ D_t(G, x) = D_t(G \setminus e, x) + x^2 D_t(G \setminus N[u], x). \]

Note that the graph \( G \setminus N[u] \) has at least an isolated vertex, therefore \( D_t(G \setminus N[u], x) = 0 \) and we have the result. \( \square \)

**Theorem 2.6** Let \( G \) be a graph and \( e = \{u, v\} \) is an edge of \( G \). If the vertices \( u \) and \( v \) are adjacent to the support vertices, then \( e \) is an irrelevant edge.

**Proof.** Suppose that the vertices \( u, v \) are adjacent to the support vertices \( w \) and \( z \), respectively. Then, every total dominating set of \( G \) include support vertices \( w \) and \( z \). So the vertices on edge \( e \), under any total dominating set of \( G \) are dominated and adjacency between them is ineffective. Therefore \( D_t(G, x) = D_t(G \setminus e, x) \) and \( e \) is an irrelevant edge. \( \square \)

Let to compute the total domination polynomial of a family of graphs which has shown in Figure 2 using the irrelevant edges. An \((n, k)-firecracker\) \( F(n, k) \) is a graph obtained by the concatenation of \( n \), \( k \)-stars \( S_k \) by linking one leaf from each. See figure 2. The following easy theorem gives the total domination number of this kind of graphs:

**Theorem 2.7** For every natural numbers \( n \) and \( k \), we have \( \gamma_t(F(n, k)) = 2^n \).

**Proof.** Let \( D \) be a minimum total dominating set of \( F(n, k) \). Then \( \{v_1, v_2, \ldots, v_n\} \subseteq D \) and for every \( v_i \), the set \( D \) contains exactly one vertex that is adjacent to \( v_i \). So \( \gamma_t(F(n, k)) = 2^n \). \( \square \)

**Theorem 2.8** For every natural numbers \( n \) and \( k \geq 3 \),

\[ D_t(F(n, k), x) = (x(x + 1)^{(k-1)} - x)^n. \]

**Proof.** By Theorem 2.6 every edge that linking \( k \)-stars together is an irrelevant edge. Therefore \( D_t(F(n, k), x) = (D_t(S_k, x))^n \) and we have the result. \( \square \)

Now, we generalize the definition of firecracker graphs. An \((k_1, k_2, \ldots, k_n)-firecracker\) \( F(k_1, \ldots, k_n) \) is a a graph obtained by the concatenation of \( k_i \)-stars \( S_{k_i} \) by linking one leaf from each (see Figure 3). If \( k_i \geq 3 \) \( (1 \leq i \leq n) \), then every edge
that linking $k_i$-stars’s together, is an irrelevant edge. Therefore $D_t(F(k_1, \ldots, k_n)) = \prod_{i=1}^{n}(x(x+1)^{(k_i-1)} - x)$.

Here, we are interested to examine the effect on the total domination polynomial of a graph when we remove a vertex. Recall that a vertex $v \in G$ is called essential vertex, if $D_t(G \setminus v, x) = 0$ ([10]).

Lemma 2.9 Let $G = (V, E)$ be a graph. The vertex $v \in V(G)$ is an essential vertex if and only if $v$ is a support vertex of $G$.

Proof. Since $D_t(G, x) = 0$ if and only if $G$ has an isolated vertex, so we have the result. □

Theorem 2.10 [10] If $G = (V, E)$ is a graph and $v \in V(G)$, then

$$D_t(G, x) = D_t(G \setminus v, x) + D_t(G \odot v, x) - D_t(G \oslash v, x).$$

where $G \odot v$ is the graph obtained from $G$ by removing all edges between vertices of $N(v)$ and $G \oslash v = G \odot v \setminus v$.

Lemma 2.11 Let $G = (V, E)$ be a graph and $v$ is a support vertex of $G$. Then $D_t(G, x) = D_t(G \odot v, x)$.

Proof. Since the vertex $v$ is a support vertex, so $D_t(G \setminus v, x) = D_t(G \odot v, x) = 0$. Therefore, by Theorem 2.10 we have the result. □

3 Total domination polynomial of regular graphs

In this section, we study some coefficients of the total domination polynomial of regular graphs and then compute the total domination polynomial of cubic graphs of order 10.

We denote the family of all total dominating sets of $G$ with cardinality $i$ and contain a vertex $v$ by $D^v_t(G, i)$ and $d^v_t(G, i) = |D^v_t(G, i)|$. Two graphs $G$ and $H$ are said to be total dominating equivalent or simply $D_t$-equivalent, if $D_t(G, x) = D_t(H, x)$ and written $G \sim H$. It is evident that the relation $\sim$ of $D_t$-equivalent is an equivalence relation on the family $G$ of graphs, and thus $G$ is partitioned into equivalence classes, called the $D_t$-equivalence classes. Given $G \in G$, let
\[ [G] = \{ H \in \mathcal{G} : H \sim G \}. \]

If \([G] = \{G\}\), then \(G\) is said to be total dominating unique or simply \(\mathcal{D}_t\)-unique. In this section, similar to \([4]\), we determine the \(\mathcal{D}_t\)-equivalence classes for cubic graphs of order 10. The proofs of Theorems 3.1 and 3.2 follow the proofs in \([4]\) with some minor changes:

\[
\begin{align*}
G_1 & \quad G_{11} \\
G_2 & \quad G_{12} \\
G_3 & \quad G_{13} \\
G_4 & \quad G_{14} \\
G_5 & \quad G_{15} \\
G_6 & \quad G_{16} \\
G_7 & \quad G_{17} \\
G_8 & \quad G_{18} \\
G_9 & \quad G_{19} \\
G_{10} & \quad G_{20} \\
G_{11} & \quad G_{21} \\
G_{12} & \quad G_{22} \\
G_{13} & \quad G_{23} \\
G_{14} & \quad G_{24} \\
G_{15} & \quad G_{25} \\
G_{16} & \quad G_{26} \\
G_{17} & \quad G_{27} \\
G_{18} & \quad G_{28} \\
G_{19} & \quad G_{29} \\
G_{20} & \quad G_{30} \\
G_{21} & \quad G_{31} \\
G_{22} & \quad G_{32} \\
G_{23} & \quad G_{33} \\
G_{24} & \quad G_{34} \\
G_{25} & \quad G_{35} \\
G_{26} & \quad G_{36} \\
G_{27} & \quad G_{37} \\
G_{28} & \quad G_{38} \\
G_{29} & \quad G_{39} \\
G_{30} & \quad G_{40} \\
G_{31} & \quad G_{41} \\
G_{32} & \quad G_{42} \\
G_{33} & \quad G_{43} \\
G_{34} & \quad G_{44} \\
G_{35} & \quad G_{45} \\
G_{36} & \quad G_{46} \\
G_{37} & \quad G_{47} \\
G_{38} & \quad G_{48} \\
G_{39} & \quad G_{49} \\
G_{40} & \quad G_{50} \\
G_{41} & \quad G_{51} \\
G_{42} & \quad G_{52} \\
G_{43} & \quad G_{53} \\
G_{44} & \quad G_{54} \\
G_{45} & \quad G_{55} \\
G_{46} & \quad G_{56} \\
G_{47} & \quad G_{57} \\
G_{48} & \quad G_{58} \\
G_{49} & \quad G_{59} \\
G_{50} & \quad G_{60}
\end{align*}
\]

Figure 4: Cubic graphs of order 10.

\[ \text{Lemma 3.1} \quad \text{Let} \ G = (V, E) \ \text{be a vertex transitive graph of order} \ n \ \text{and} \ v \in V. \ \text{For} \ \text{any} \ 1 \leq i \leq n, \ \text{we have} \ d_t(G, i) = \frac{n}{i}d_t(G, i). \]
Theorem 3.3 Domination polynomial of the Petersen graph.

(i) The Petersen graph is a vertex transitive graph, we calculate \( d(G) \) is a vertex transitive graph, so for every vertices \( v \) and \( u \), \( d^i_v(G,u) = d^i_v(G,u) \). If \( D \) is a total dominating set of size \( i \), then there are exactly \( i \) vertices \( v_j, v_{j2}, \ldots, v_{ji} \) such that \( D \) counted in \( d^i_{k} (G,i) \), for any \( 1 \leq k \leq i \). Therefore \( d_i(G,i) = 2d^i_v(G,i) \) and the proof is complete. □

Lemma 3.2 Let \( G = (V,E) \) be \( k \)-regular graph of order \( n \). Then \( d_i(G,i) = \binom{n}{i} \) for all \( i > n - k \).

Proof. Let \( G \) be a \( k \)-regular graph of order \( n \) with vertex set \( V \). Each vertex of \( G \) is\( \gamma \)-adjacent to \( k \) vertices. Let \( S \) be a \((k-1)\)-subsets of \( V \), and \( V' = V \setminus S \). Then \( V' \) is a total dominating set for \( G \) of size \( n - k + 1 \) and the number of total dominating set of size \( n - k + 1 \) for \( G \) is equal to number of ways of choosing \( k - 1 \) vertex of \( V \). Therefore \( d_i(G,n-k+1) = \binom{n}{k-1} = \binom{n-k+1}{k-1} \). Similarly for each \( 2 \leq i \leq k - 1 \), we have \( d_i(G,n-i) = \binom{n}{n-i} \). So for any \( i > n - k \), \( d_i(G,i) = \binom{n}{i} \) and the proof is complete. □

In the study of the total domination polynomial of regular graphs, it is natural to ask about the total domination polynomial of Petersen graph and its \( D_t \)-equivalence class. To answer to this question, we consider exactly 21 cubic graphs of order 10 given in Figure 4 (see [4]). There are just two non-connected cubic graphs of order 10. Note that the graph \( G_{17} \) is the Petersen graph. The following theorem gives the total domination polynomial of the Petersen graph.

Theorem 3.3 The total domination polynomial of Petersen graph \( P \) is

\[
D_t(P,x) = x^{10} + 10x^9 + 45x^8 + 110x^7 + 140x^6 + 72x^5 + 10x^4.
\]

Proof. We have \( \gamma_t(P) = 4 \). Since the Petersen graph \( P \) is a 3-regular graph of order 10, by Lemma 3.2, we have \( d_t(P,i) = \binom{10}{i} \), for \( i = 8,9,10 \). On the other hand, since \( P \) is a vertex transitive graph, we calculate \( d_t(P,i) \), for \( i = 4,5,6 \) using Lemma 3.1. So we have the result.

Using Maple we computed the total domination polynomial of cubic graphs of order 10. As some consequences we stat the following results for graphs in Figure 4.

Theorem 3.4 (i) The Petersen graph \( P \) is not \( D_t \)-unique. More precisely, the three graphs \( G_{12}, G_{14} \) and \( P \cong G_{17} \) are \( D_t \)-equivalent.

(ii) The three graphs \( G_1, G_8 \) and \( G_9 \) are \( D_t \)-equivalent.

(iii) The graphs \( G_2, G_3, G_4, G_5, G_6, G_7, G_{10}, G_{11}, G_{13}, G_{15}, G_{16}, G_{18}, G_{20}, G_{21} \) are \( D_t \)-unique.
4 On the graphs with exactly two total domination roots

Graphs whose certain polynomials have few roots can sometimes give interesting information about the structure of graph. The characterization of graphs with few distinct roots of characteristic polynomials (i.e., graphs with few distinct eigenvalues) have been the subject of many researchers \[6, 7, 8, 9\]. Also the first authors has studied graphs with few domination roots in [4]. In [2] we have shown that all roots of \(D_i(G, x)\) lie in the circle with center \((-1, 0)\) and the radius \(\sqrt{2^n - 1}\), where \(\delta\) is the minimum degree of \(G\). Also we proved that for a graph \(G\) of order \(n\), if \(\delta \geq \frac{2n}{3}\), then every integer root of \(D_i(G, x)\) lies in the set \([-3, -2, -1, 0]\). Motivated by these integer roots, and a conjecture in [2] which states that for every integer root \(r\) of \(D_i(G, x)\), \(r \in \{-3, -2, -1, 0\}\), we study graphs with exactly two total domination roots \([-1, 0]\), \((-2, 0)\) and \((-3, 0)\), in this section.

4.1 Graphs with exactly two total domination roots \([-1, 0]\)

In this subsection, first we state and prove the following theorem to present a necessary condition for a graph to have exactly two total domination roots \(-1\) and \(0\).

**Theorem 4.1** If \(G = (V, E)\) is a graph of order \(n\) with \(r\) support vertices, then \(d_t(G, n - 1) = n - r\).

**Proof.** Let \(A \subseteq V\) be the set of all support vertices of \(G\). For every vertex \(v \in V \setminus A\), the set \(V \setminus \{v\}\) is a total dominating set of \(G\). So \(d_t(G, n - 1) = n - r\). \(\Box\)

**Theorem 4.2** Let \(G = (V, E)\) be a graph of order \(n\). If \(Z(D_t(G, x)) = \{-1, 0\}\), then \(G\) has at least two support vertices.

**Proof.** Let \(G\) be a graph of order \(n\) and \(D_t(G, x) = x^a(x + 1)^b\), such that \(a + b = n\) and \(a = \gamma_t(G) \geq 2\). By Theorem 4.1, \(d_t(G, n - 1) = n - r\), where \(r\) is the number of support vertices. So \(n - r = b\) and \(a = r\). Therefore we have result. \(\Box\)

4.2 Graphs with exactly two total domination roots \([-2, 0]\)

In this subsection, we present a necessary condition for graphs two total domination roots \([-2, 0]\). We recall that a vertex cut of a graph \(G\) is a subset \(V'\) of \(V(G)\) such that \(G - V'\) is not connected and a \(k\)-vertex cut of \(G\) is a vertex cut of \(k\) vertices. The connectivity, \(\kappa(G)\) of a connected graph \(G\) (which contains no complete graph factor) is the smallest integer \(k\) for which \(G\) has a \(k\)-vertex cut. To obtain our result, we introduce graphs in Figure 5 which denoted by \(\mathcal{H}\).

Infinite family \(\mathcal{H}\) are connected cubic graphs. For \(k \geq 2\), let \(H_k\) be the graph constructed as follows. Consider two copies of the path \(P_{2k}\) with respective vertex sequences \(a_1b_1a_2b_2 \ldots a_kb_k\) and \(c_1d_1c_2d_2 \ldots c_kd_k\). Let \(A = \{a_1, a_2, \ldots, a_k\}\), \(B = \{b_1, b_2, \ldots, b_k\}\), \(C = \{c_1, c_2, \ldots, c_k\}\) and \(D = \{d_1, d_2, \ldots, d_k\}\). For each \(i \in \{1, 2, \ldots, k\}\), join \(a_i\) to \(d_i\) and \(b_i\) to \(c_i\). To complete the construction of the graph \(H_k\), join \(a_1\) to \(b_k\) and \(c_1\) to \(d_k\). We note that \(H_k\) are cubic graphs of order \(4k\).
Theorem 4.3 [6] If $G$ is a 3-connected graph of order $n$, then $\gamma_t(G) \leq \frac{n}{2}$ with equality if and only if $G = K_4$ or $G \in \mathcal{H}$ or $G$ is the generalized Petersen graph $GP$ of order 16 shown in Figure 6.

![Figure 5: The graphs $\mathcal{H}$.](image)

![Figure 6: The generalized Petersen graph of order 16.](image)

Theorem 4.4 Let $G$ be a simple graph of order $n$. If $Z(D_t(G, x)) = \{-2, 0\}$, then $\kappa(G) \leq 2$.

Proof. Let $G$ be a 3-connected graph and $D_t(G, x) = x^{\gamma_t}(x + 2)^{\beta}$, where $\gamma_t + \beta = n$. So by Theorem 4.1, we have $d_t(G, n - 1) = n = 2\beta$, and so $\beta = \frac{n}{2}$, $\gamma_t(G) = \frac{n}{2}$. By Theorem 1.3, $G = K_4$ or $G \in \mathcal{H}$ or $G$ is the generalized Petersen graph, $GP$, of order 16. But $D_t(K_4, x) = x^4 + 4x^3 + 6x^2$, $D_t(GP, x) = x^8(x^4 + 8x^3 + 28x^2 + 48x + 30)^2$, and so $G \neq K_4, GP$. On the other hand, for every $k = 2, \ldots, m$, $H_k \in \mathcal{H}$ is a 3-regular graph of order $4k$ with $\gamma_t(H_k) = 2k$. So by Lemma 3.2, we have $d_t(H_k, 4k - 2) = 2k(4k - 1)$. By the assumption, we shall have

$$D_t(G, x) = x^{2k}(x + 2)^{2k} = \sum_{i=\gamma_t(G)}^{2k} \binom{2k}{i} x^{2k-i} x^{2k+i}.$$ 

So $d_t(G, 4k - 2) = 4k(2k - 1)$, which is a contradiction. Therefore $G \notin \mathcal{H}$ and we have result. \qed
As an example of family of graph $G$ with $Z(D_t(G, x)) = \{-2, 0\}$, let $H$ be an arbitrary graph of order $n$ and consider $n$ copies of graph $P_3$. By definition, the graph $H(3)$ is obtained by identifying each vertex of $H$ with an end vertex of a $P_3$ (see Figure 7). By Theorem 2.6 we compute the total domination polynomial of $H(3)$ (see also [3]).

**Theorem 4.5** For any graph $H$ of order $n$, we have $D_t(H(3), x) = x^{2n}(x + 2)^n$.

**Proof.** Let $D$ be a total dominating set of $H(3)$ of size $k \geq n$ in Figure 7. Obviously $\{v_1, v_2, \ldots, v_n\} \subset D$ and for any $v_i$, the set $D$ contains exactly one vertex that is adjacent to $v_i$, so $\gamma_t(H(3)) = 2n$. On the other hand, for all $i, j$, $1 \leq i \neq j \leq n$ and $e = \{u_i, u_j\}$, the vertices $u_i, u_j$ are adjacent to support vertices $v_i, v_j$. Therefore each edge in $H$ is an irrelevant edge, and so we have

$$D_t(H(3), x) = (D_t(P_3, x))^n = x^{2n}(x + 2)^n.$$  

□

### 4.3 Graphs with exactly two total domination roots $\{-3, 0\}$

In this subsection, we shall characterize graphs whose total domination polynomial have exactly two roots $-3$ and 0. To do this, we need the following result.

**Theorem 4.6** [6] Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = \frac{2n}{3}$ if and only if $G$ is $C_3$, $C_6$ or $H(3)$ (in Figure 7) for some connected graph $H$.

The next theorem classifies all connected graphs without end-vertices, whose total domination polynomial have just two roots $\{-3, 0\}$.

**Theorem 4.7** Let $G = (V, E)$ be a graph of order $n$ with $\delta(G) \geq 2$. Then $Z(D_t(G, x)) = \{-3, 0\}$ if and only if $G$ is $C_3$ or $C_6$.

**Proof.** First note that $D_t(C_3, x) = x^2(x+3)$ and $D_t(C_6, x) = x^4(x+3)^2$. Let $D_t(G, x) = x^\alpha(x+3)^\beta$. Therefore $\alpha + \beta = n$ and $\alpha = \gamma_t(G)$. By Theorem 4.4 we have $d_t(G, n-1) = \ldots$
\( n - r \) where \( r \) is the number of support vertices of \( G \). Therefore \( d_t(G, n - 1) = n \). On the other hand

\[
D_t(G, x) = x^\alpha(x + 3)^\beta = \sum_{i=0}^{\beta} \binom{\beta}{i} x^{\alpha+i\beta-i}.
\]

Therefore we have \( d_t(G, n - 1) = n = 3\beta \) and so \( \gamma_t(G) = \frac{2\beta}{3} \). By Theorem 4.6, \( G \) is \( C_3 \), \( C_6 \) or \( H(3) \) for some connected graph \( H \). Since \( \delta(G) \geq 2 \) and by Theorem 4.5, \( G \) is \( C_3 \) or \( C_6 \). \( \square \)

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