Fermionic criticality with enlarged fluctuations in Dirac semimetals

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The fluctuations-driven continuous quantum criticality has sparked tremendous interest in condensed matter physics. It has been verified that the gapless fermions fluctuations can change the nature of phase transition at criticality. In this paper, we study the fermionic quantum criticality with enlarged Ising×Ising fluctuations in honeycomb lattice materials. The Gross-Neveu-Yukawa theory for the multicriticality between the semimetallic phase and two ordered phases that break Ising symmetry is investigated by employing perturbative renormalization group approach. We first determine the critical range in which the quantum fluctuations may render the phase transition continuous. We find that the Ising criticality is continuous only when the flavor numbers of four-component Dirac fermions \( N_f \geq 1/4 \). Using the \( \epsilon \) expansion in four space-time dimensions, we then study the Ising×Ising multicriticality stemming from the symmetry-breaking electronic instabilities. We analyze the underlying fixed-point structure and compute the critical exponents for the Ising×Ising Gross-Neveu-Yukawa universality class. Further, the correlation scaling behavior for the fermion bilinear on the honeycomb lattice at the multicritical point are also briefly discussed.

I. INTRODUCTION

The quantum phase transitions (continuous) at zero-temperature driven by non-temperature parameters are believed to be key to understand some unconventional properties of correlated many-body systems\cite{1, 2}, including strange metal phase in the high-temperature superconductors\cite{3–5}, ferromagnetic quantum criticality in the heavy fermion systems\cite{6} and deconfined quantum critical point (QCP)\cite{7, 8}. From a field-theoretical perspective, the general description for the continuous phase transition can be formulated in terms of the order parameter which acquires a nonzero value as the system is tuned across the transition. Together with the renormalization group (RG) theory\cite{9}, the Landau-Ginzburg-Wilson (LGW) paradigm provides a well understanding of the universal criticality and a effective method to calculate the critical exponents near the critical point, for example the extensively studied scalar \( O(N) \) model which captures the continuous critical behavior of a wide variety of systems\cite{2, 10, 11}.

However, the LGW paradigm has been challenged recently by various examples in which the fluctuations from emergent degrees of freedom render the transition continuous. The prime example is the transition between Neel and valence bond solid phase that separated by the deconfined QCP on the 2D square Heisenberg antiferromanets\cite{7, 8}. A large number of theoretical and numerical studies demonstrate that the deconfined QCP is continuous as a result of the emergent fractionalized “spinon” and noncompact U(1) gauge field\cite{12–16}. Since the new degrees of freedom such as spinons emerge right at the QCP, the LGW theory is fail to describe the deconfined QCP purely in terms of the space-time fluctuations of order parameter. Another example that goes beyond the LGW picture is the fermion induced quantum critical point (FIQCP) which has attracted persistent attention in Dirac fermion systems\cite{17–40}. Evidences for FIQCP have been embodied in the transition from semimetal to \( Z_3 \) Kekule valence-bond-solid (VBS) phase of (2+1)D fermions on the honeycomb lattice\cite{17, 18, 27}. The Kekule VBS pattern breaks the translational symmetry and breaks the continuous \( U(1) \) symmetry down to \( Z_3 \)\cite{27, 41, 42}, consequently, the Kekule phase transition allows a cubic term of VBS order parameter. From the view of point of LGW picture, the Kekule phase transition is expected to be first-order in the presence of cubic term of order parameter. Extensive studies, however, suggest that the presence of gapless fermion fluctuations at the critical point can dramatically change the nature of critical point and render a putatively first-order transition continuous\cite{17, 27}. Various theoretical methods have been applied to study the FIQCP, ranging from perturbative RG\cite{19, 29, 37, 38} to functional RG\cite{21, 25, 27, 43}. On the other hand, the sign-problem-free quantum Monte Carlo simulations for interacting fermions on lattice also push our understanding of FIQCP\cite{44–48}.

Building on the concept of FIQCP, the quantum criticality in the presence of external fluctuations provided by the Dirac fermions has sparked tremendous interest\cite{35, 49–51}. A representative example is the semimetal-insulator transition of interacting electrons on the honeycomb lattice\cite{49, 51}. When the interaction is sufficiently strong, the electrons may undergo continuous transition from semimetallic phase to a symmetry broken insulating phase characterized by an ordered ground state\cite{53}. A large on-site repulsive interaction, for example, gives rise to a spin-density-wave state\cite{43, 54}, whereas a nearest-neighbor interaction would induce a charge-density-wave state\cite{37}. A large number of interacting Dirac fermion model exhibit the continuous phase transition, e.g., spin

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liquid\cite{55}, superconducting and XY phases \cite{32, 52, 56-58}, and exotic topological phases\cite{59, 60}. Moreover, the recent theoretical studies suggest the multicritical point may be achieved in three ways: (i) condensation of topological defects\cite{61}, (ii) anticommuting mass terms\cite{62}, (iii) interaction instabilities\cite{22, 29, 32}. It’s argued that the quantum criticality occurs in the vicinity of these critical points and the universality class are defined through different microscopic models. Since the presence of external fluctuations provided by gapless Dirac fermions, these critical points are not captured by the conventional $O(N)$ universality classes. Instead, they are described by the Gross-Neveu-Yukawa theory, defining qualitatively different chiral Gross-Neveu-Yukawa (GNY) universality classes\cite{63}. Indeed, the critical behavior of a large number of transitions in condensed matter systems are captured by the GNY model. To describe the critical behavior of GNY (or the gauged QED-GNY) theory, various theoretical methods and numerical methods have been applied, i.e., perturbative RG\cite{11, 18, 37, 38, 64}, nonperturbative functional RG\cite{27, 43}, large-N method\cite{54, 65} and quantum Monte Carlo simulations\cite{50, 66-68}. In view of the symmetry of the broken phase, the GNY universality class comprises chiral Ising ($\mathbb{Z}_2$) class \cite{11, 37, 38, 46, 63}, chiral XY $[O(2)]$ class\cite{17, 26, 27, 33, 50, 69} and chiral Heisenberg$[SU(2)]$ universality class\cite{11, 43, 54, 58, 65}.

Once the fermion-induced continuous criticality is established, efforts to extend the FIQCP to the multicritical point with enlarged fluctuations is significant, as this theoretical problem relates to the competing orders and multicritical behavior of correlated electrons, e.g., high-temperature superconductors\cite{3} and deconfined criticality\cite{62, 70}. Recently, the studies of multicritical point are ongoing\cite{26, 29, 36, 71}. It’s demonstrated that the multicritical point is characterized by an emergent enlarged symmetry and features a continuous transition between two ordered phases as the system is tuned through the multicritical point, for example $O(5)$ symmetry for $O(3)$ Neel order and $U(1)$ Kekule VBS. Moreover, the multicritical point exhibits an enhanced symmetry within the Yukawa sector against small perturbations that break the $O(5)$ symmetry\cite{36}. These recent progress raises two fundamental issues: (1) How the multicritical behavior is modified by the interplay between two ordered phases and the gapless Dirac fermions. (2) To what extent does the multi-criticality affect the possible competing order parameters.

Here, we solve the two issues by exploring the gapless Dirac fermions coupled to the Ising×Ising symmetry-breaking order parameters. We first formulate the theory for the multicritical point with enlarged Ising×Ising fluctuations, which we analysis using perturbative RG. By including a cubic term in the theory of Ising FIQCP, we also identify the semimetal-CDW transition is continuous for the flavors of fermions $N_f$ fulfill $N_f > 1/4$. In particular, we find that the Kekule VBS phase can be enhanced by multicritical fluctuations, the crucial ingredient for the enhancement is the anticommuting nature between the corresponding fermion bilinears and the Dirac gamma matrices in the kinetic part. Although our results are judged from the enlarged Ising×Ising criticality, the enhancement scenario can be applied to other multicritical fluctuations, such as enlarged $O(3)\times U(1)$, $O(3)\times$ Ising.

This paper is organized as follows. We formulate the critical theory with enlarged Ising×Ising fluctuations in Sec. II. After identifying the range in which the transition is continuous in Sec. III, we perform RG analyses for the multicritical point in Sec. IV. We also show the Kekule VBS are enhanced by the multicritical fluctuations in Sec. IV. Conclusions are drawn in Sec. V.

## II. SEMIMETAL TO ISING-ORDER TRANSITION

We consider the spinless Dirac fermions on a honeycomb lattice, whose low-energy effective theory in physical $2 + 1$ dimensions can be expressed as the Lagrangian density\cite{27, 29, 32}

\[ \mathcal{L}_\phi = i\bar{\Psi} \gamma^\mu \partial_\mu \Psi, \]

where the conjugate fermionic field $\bar{\Psi} = \Psi^\dagger \gamma^0$, the derivative operator reads $\partial_\mu = (\partial_0, \partial_i)$. Here the Fermi velocity $v_F = 3t/2$ was set to unity for convenience and the summation convention over repeated indices is assumed. The $\gamma^\mu$ matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \mu, \nu = 0, 1, 2,$ and $g^{0\nu} = \text{diag}(1, -1, -1)$ is a Minkowski space metric. We have defined the following $4 \times 4$ Minkowski space gamma matrices

\[ \gamma^0 = \tau^0 \otimes \sigma^3, \gamma^1 \tau^0 \otimes i\sigma^1, \gamma^2 = \tau^3 \otimes i\sigma^2, \]

where the two-component identity matrix $\tau^0$ and the standard Pauli matrices $\tau^i$ act on the valley indices ($K, -K$), the two-component Pauli matrices ($\sigma^0, \sigma^i$) act in sublattice space $(A, B)$. In the free Dirac Lagrangian, the four-component Dirac spinor is defined as $\Psi = (c_{AK}, c_{BK}, c_{A-K}, c_{B-K})^T$. In the vicinity of Dirac points, then the Bloch Hamiltonian reads $\mathcal{H} = \gamma^0 \gamma^i k_i$ with the reduced Planck constant $\hbar = 1$\cite{32}. There are two matrices anticommutate with all $\gamma^\mu$ matrices

\[ \gamma^3 = \tau^1 \otimes i\sigma^2, \gamma^5 = \tau^2 \otimes i\sigma^2. \]

We can define $\gamma^{35} = i\gamma^3\gamma^5 = \tau^3 \otimes i\sigma^0$ which commutes with all $\gamma^\mu$ but anticommutates with $\gamma^\tau$ and $\gamma^5$, or explicitly, $\{\gamma^{35}, \gamma^\mu\} = 0, \{\gamma^{35}, \gamma^3\} = 0, \{\gamma^{35}, \gamma^5\} = 0$. It’s easily check that $\{\gamma^{35}, \mathcal{H}\} = 0$. The Hamiltonian possesses a symmetry implemented by $CHC^{-1} = -\mathcal{H}$, where $C$ is expressed as either $C = \sigma^0$ or $C = \gamma^0\gamma^{35}$. This symmetry is conventionally called chiral symmetry or sublattice symmetry on bipartite graphene lattice. For generality, we introduce an arbitrary number of $N_f$ fermion flavors of four-component Dirac fermions. The fermion field carries a flavor index, $\Psi = \Psi_i$ with $i = \{1, 2, .., N_f\}$, $N_f = 2$
We have set the boson and fermion velocities equally to preserve the Lorentz symmetry, $v_F = v_B = 1$, which is reasonable since the the Lorentz invariance has been argued to emerge naturally near the critical point and the velocity difference between boson and fermion is always irrelevant in Yukawa theories[30, 73]. Furthermore, for the enlarged Ising×Ising criticality, the Landau-Ginzburg action can be written as $S = \int d^D x (\mathcal{L}_{\text{GNY}} + \mathcal{L}_{\chi \phi})$, where $\mathcal{L}_{\chi \phi} = \lambda_{\chi \phi} \chi^2 \phi^2$ describes the interaction between two Ising fields with strength $\lambda_{\chi \phi}$.

### III. CDW ISING-CRITICALITY

In this section, we study the critical properties of CDW Ising criticality within field-theoretic RG in $D = 4 - \epsilon$ space-time dimensions and the modified minimal subtraction ($\overline{MS}$). Our starting point is the renormalized Lagrangian

$$\mathcal{L}_{\text{GNY}} = Z_y \bar{\Psi} i\gamma^\mu \partial_\mu \Psi + g_\chi \bar{\Psi} \chi \Psi + \mathcal{L}_\chi,$$

where the bosonic Lagrangian is given by

$$\mathcal{L}_\chi = \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} m^2 \chi^2 - \lambda \chi^4.$$ (5)

Here the parameters $m^2$ tunes the phase transition from semimetallic phase to the phase with spontaneous $Z_2$ symmetry breaking where the fermion mass are dynamically generated. To determine the nature of the transition, we introduce the cubic term in the Landau-Ginzburg Lagrangian by hand, is of the form

$$\mathcal{L}_{\text{cub.}} = b(\chi^3 + \chi^3).$$ (6)

Such kind of terms also exist in the valence-bond-solid phase as the reduction of continuous symmetry down to discrete symmetry, i.e., $Z_3$ and $Z_4$ symmetry[18, 22, 27]. According to Landau criterion, the transition should be first-order in the presence of cubic terms of order parameter in the Lagrangian[21]. In general space-time dimensions $D$, the cubic coupling have canonical dimensions $[b] = 3 - D/2$, which implies that the cubic terms is strongly relevant near upper critical dimensions $D_{\text{uc}} = 4$. By contrast, the possible $Z_4$-anisotropy $\sim \chi^4 + \chi^4$ on square lattice is marginal and can be accessible within $\epsilon$ expansion near four dimensional space-time[77]. Though the cubic term is relevant at leading order, in the following, we will show it is irrelevant in the one-loop corrections. This leaves the concept of FIQCP in the fermion systems[17, 22, 27].

Correspondingly, the semimetal-QAH quantum criticality is governed by the Lagrangian

$$\mathcal{L}_{\text{GNY}} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi + g_\chi \bar{\Psi} \chi \Psi + \mathcal{L}_\chi.$$ (7)

$$\mathcal{L}_\phi = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \lambda_\phi \phi^4.$$ (8)

We have set the boson and fermion velocities equally to preserve the Lorentz symmetry, $v_F = v_B = 1$, which is reasonable since the the Lorentz invariance has been argued to emergent naturally near the critical point and the velocity difference between boson and fermion is always irrelevant in Yukawa theories[30, 73]. Furthermore, for the enlarged Ising×Ising criticality, the Landau-Ginzburg action can be written as $S = \int d^D x (\mathcal{L}_{\text{GNY}} + \mathcal{L}_{\chi \phi})$, where $\mathcal{L}_{\chi \phi} = \lambda_{\chi \phi} \chi^2 \phi^2$ describes the interaction between two Ising fields with strength $\lambda_{\chi \phi}$.

### A. Beta functions and critical exponents

The beta function for the coupling constants are defined as the logarithmic derivatives with respect to the energy scale,

$$\beta(g_\chi) = \frac{dg_\chi}{d\ln \mu}, \beta(\lambda_\chi) = \frac{d\lambda_\chi}{d\ln \mu}.$$ (10)

In terms of the renormalization constants, the beta functions can be represented as

$$\beta(g_\chi) = (-\epsilon/2 - \gamma_\chi) g_\chi, \beta(\lambda_\chi) = (-\epsilon - \gamma_\lambda) \lambda_\chi,$$ (11)

where $\gamma_\chi$ is defined as $\gamma_\chi = d \ln Z_\chi / d \ln \mu$ for $X = g_\chi, \lambda_\chi$. We remind that the expression for the beta functions differ from the previous publications[54, 77], the difference arises from the definition of the renormalized coupling constants. Rescaling the coupling constants according to $\lambda_\chi/(8\pi^2) \rightarrow \lambda_\chi, \lambda_\chi^2/(8\pi^2) \rightarrow g_\chi^2$, the beta functions are given by

$$\beta(g_\chi^2) = -\epsilon g_\chi^2 + (2N_f + 3)g_\chi^4,$$ (12)

$$\beta(\lambda_\chi) = -\epsilon \lambda_\chi + 4N_f \lambda_\chi g_\chi^2 + 36 \lambda_\chi^2 - N_f g_\chi^4.$$ (13)
The one-loop beta functions above agree with those in the previous publications\cite{11,38}. In the limit \( g^2 = 0 \), our expressions reduce to scalar \( \phi^4 \) theory with \( Z_2 \) or Ising symmetry.

When the system is tuned to criticality with \( m^2 = 0 \), the simultaneous zeros of the set of beta functions give the fixed-point which denoted by \((g^2_{\chi^+}, \lambda_{\chi^+})\). At one-loop order, the beta functions admit four fixed-points: the Gaussian fixed-point \((0, 0)\), the bosonic Wilson-Fisher fixed-point \((0, \epsilon/36)\), and a pair of Ising GNY fixed-point

\[
(g^2_{\chi^+}, \lambda_{\chi^+})_{\pm} = \left( \frac{1}{2N_f + 3}, \frac{-2(N_f - 3) \pm W}{72(2N_f + 3)} \epsilon \right), \tag{14}
\]
defining \( W = (4N_f^2 + 132N_f + 9)^{1/2} \). Among these fixed-points, the infrared stable fixed-point is given by the positive one.

In addition to the beta function for the coupling constants, the beta function for the scalar mass squared is given by

\[
\beta(m^2_{\chi}) = \frac{dm^2_{\chi}}{d \ln \mu} = -(2 + \gamma_{m^2_{\chi}})m^2_{\chi}, \tag{15}
\]
where \( \gamma_{m^2_{\chi}} = d Z m^2_{\chi} / d \ln \mu \) is the anomalous dimension for mass squared. When the system is tuned to criticality \((m^2_{\chi} = 0)\), the inverse correlation length exponent \( \nu^{-1} \) is related to the mass squared anomalous dimension by

\[
\nu^{-1} = 2 + \gamma_{m^2_{\chi}} (g^2_{\chi^+}, \lambda_{\chi^+}). \tag{16}
\]

At one-loop order, we find \( \gamma_{m^2_{\chi}} = -12\lambda - 2N_f g^2 \), evaluating at the criticality provides \( \nu^{-1} = 2 - 0.8347\epsilon \) for \( N_f = 1 \), and \( \nu^{-1} = 2 - 0.9524\epsilon \) for \( N_f = 2 \). These results have also been calculated up to three- and four-loop in previous literatures\cite{11,38}. At the QCP, it is found empirically that the pair correlation function of the order-parameter takes the form

\[
\langle O_{\text{CDW}}(r) O_{\text{CDW}}(r') \rangle \sim \frac{1}{|r - r'|^{D - 2 + \eta_{\chi}}}, \tag{17}
\]
where \( \eta_{\chi} \) is the order-parameter anomalous dimension characterizing the long-range power-law decay of the pair correlation function. In the framework of the field theory, the order-parameter anomalous dimensions is determined by

\[
\eta_{\chi} = \frac{1}{Z_{\chi}} \frac{d Z_{\chi}}{d \ln \mu}, \tag{18}
\]
and the value is evaluated at criticality. We find the one-loop result \( \eta_{\chi} = 2N_f g^2_{\chi^+} \), the evaluation at the criticality provides \( \eta_{\chi} = 0.5714\epsilon \) for \( N_f = 2 \), which agrees exactly with Ref.\cite{11} at the corresponding order.

### B. The nature of Ising-order transition

So far we have considered the fixed-point without the cubic term. At one-loop order, the cubic term contributes to the renormalization of bosonic self-energy and cubic vertex. If the cubic term is relevant when approaching the critical point in \( 4 - \epsilon \) dimensions, then the transition should be first-order. To confirm the nature of Ising transition on the honeycomb lattice, we write down the renormalized cubic Lagrangian

\[
L^R_{\text{cub.}} = \mu^{3-D/2} b Z_b Z_{\chi^+} \chi^3 + \chi^{3}\chi^{-3}, \tag{19}
\]
where we have introduced the renormalization constant \(Z_b\) such that \( b_0 = Z_b b \). Similarly, the beta function for \( b \) is given by

\[
\beta(b) = [-3 - D/2 - \gamma_b] b, \tag{20}
\]

with \( \gamma_b = d \ln Z_b / d \ln \mu \). Explicitly, the cubic term is relevant (infrared) for \( D \to 4 \) without loop corrections. Using the \( MS \) scheme and evaluating the one-loop correction, we find the one-loop renormalization constant

\[
Z_b = 1 + (36\lambda_{\chi} + 3N_f g^2_{\chi^+})/\epsilon, \tag{21}
\]
which implies the beta function

\[
\beta(b) = [-3 - D/2 + 36\lambda_{\chi} + 3N_f g^2_{\chi^+}] b. \tag{22}
\]
The negative slope of the beta function \( \beta(b) \) evaluated at the fixed-point determines the relevance or irrelevance of the cubic term when flowing toward the critical point, which gives

\[
\theta = (3 - D/2) - 36\lambda_{\chi^+} - 3N_f g^2_{\chi^+}. \tag{23}
\]
\( \theta > 0 \) corresponds to a relevant and \( \theta < 0 \) corresponds to an irrelevant cubic coupling. At the infrared stable Ising GNY fixed-point, we find

\[
\theta = (3 - D/2) - 4N_f + 3 + s / 2(2N_f + 3) \epsilon. \tag{24}
\]
The numerical irrelevant range is displayed in Fig. 1, the dimensions \( D \in [3, 3.34] \) allows an irrelevant cubic coupling. In the \( \theta > 0 \) range, cubic coupling is relevant, which render a second-order transition. Instead, in the \( \theta < 0 \) range, cubic coupling is irrelevant and we expect a second-order transition. We also determine the second-order range in \( D = 2 + 1 \) dimensions, the one-loop RG calculations found a critical fermion flavor number \( N_f^c = 1/4 \). Above \( N_f^c \), \( \theta \) is negative so that a continuous critical point takes place for \( N_f \geq N_f^c \).

We have also calculated the stability matrix at criticality in three dimensions space-time. From the beta functions, the stability matrix is given by the linearization of flow equations at the fixed-point on the hypersurfaces of coupling constants,

\[
\beta(X_i) = B_{i,j}(X_j - X_j^*),
\]

where \( B_{i,j} = \partial \beta_i/\partial X_j|_{X_i=X_j^*} \) and \( -B_{i,j} \) is termed stability matrix. The eigenvalues of \( -B_{i,j} \) define the critical exponents which are universal at the putative continuous critical point. More explicitly, one has

\[
\frac{dX_i}{d\ln s} = -B_{i,j}(X_j - X_j^*),
\]

the renormalization scaling factor \( s \) is accompanied by the relation \( F(X_i) \sim s^{-D} F(s^{y_i} X_i) \), where \( F \) is an universal scaling function. Choosing \( \tilde{X} = (g_\chi^2, \lambda_\chi, \lambda_\phi) \), we have three eigenvalues which are ordered as \( y_1 > y_2 > y_3 \). A second-order critical point requires \( y_1 < 0 \). The critical index \( y_1 > 0 \) implies \( X_i \) is a relevant variable, this corresponds to repelling flow. By contrast, \( y_1 < 0 \) implies \( X_i \) is an irrelevant variable, this corresponds to attractive flow. Our calculations of the critical index in \( D = 3 \) dimensions are listed in Table I, we find the critical index \( y_1 \) change sign at \( N_f^c = 0.25 \), which gives a consistent check on the critical fermion flavor number. Consequently, the two-dimensional honeycomb lattice with \( N_f = 2 \) may display a continuous phase transition with universal critical behavior.

### Table I: Numerical values of critical index for different \( N_f \) in three dimensions space-time (\( D = 3 \)), these indices determine critical behavior for \( D < 3.34 \).

| \( N_f \) | \( y_1 \) | \( y_2 \) | \( y_3 \) |
|--------|--------|--------|--------|
| 0.1    | 0.2319 | -1     | -1.4737 |
| 0.2    | 0.0642 | -1     | -1.7539 |
| 0.25   | 1.4 × 10^{-16} | -1 | -1.8571 |
| 0.26   | -0.0117 | -1 | -1.8757 |
| 0.3    | -0.0552 | -1 | -1.9437 |
| 1      | -0.4042 | -1 | -1.9437 |
| 10     | -0.3837 | -1 | -1.8079 |
| 100    | -0.0607 | -1 | -1.1363 |
| 10^5   | -0.67 × 10^{-4} | -1 | -1.0000 |

### IV. Ising×Ising Criticality

The previous section focus only on the Ising criticality, we now turn to the critical behavior with enlarged Ising×Ising fluctuations. The model under consideration is given by \( L_{11} = L_{\text{CDW-QAH}} + L_{\psi} \), and the corresponding renormalized Lagrangian is given by

\[
L_{11}^R = Z_\psi \tilde{\Psi} i\gamma^\mu \partial_\mu \tilde{\Psi} + \mu^2 Z_{\lambda_{\phi}} \phi + \frac{1}{2} Z_{\phi} (\partial_\mu \phi)^2 - \frac{1}{2} Z_{\phi} m^2 \phi^2 - \mu^2 Z_{\lambda_{\phi}} \phi \chi^4 + \mu^2 Z_{\phi} \phi \chi^4 + \mu^2 Z_{\phi} \chi^4,
\]

Similar models have been used to discuss the coexisting orders and Mott multicriticality in Dirac systms, see Refs. [29, 36]. As defined in Sec. III, the \( Z_i \) renormalization factors. We have also introduced the interaction between the CDW dynamical fluctuating and QAH dynamical fluctuating, which is given by \( L_{\chi_{\phi}} \). Evaluating these renormalization factors at one-loop order, the beta functions of the rescaled coupling constants are given by the following differential equations

\[
\beta(g_\phi^2) = -\epsilon g_\phi^2 + (2 N_f + 3) g_\phi^4 + 3 g_\phi^2 g_\psi^2, \quad \beta(g_\psi^2) = -\epsilon g_\psi^2 + (2 N_f + 3) g_\psi^4 + 3 g_\psi^2 g_\phi^2, \quad \beta(g_{\lambda_{\phi}}^2) = -\epsilon g_{\lambda_{\phi}}^2 + (2 N_f + 3) g_{\lambda_{\phi}}^4 + 3 g_{\lambda_{\phi}}^2 g_{\chi_{\phi}}^2, \quad \beta(g_{\chi_{\phi}}^2) = -\epsilon g_{\chi_{\phi}}^2 + (2 N_f + 3) g_{\chi_{\phi}}^4 + 3 g_{\chi_{\phi}}^2 g_{\psi_{\chi_{\phi}}}^2, \quad \beta(g_{\psi_{\chi_{\phi}}}^2) = -\epsilon g_{\psi_{\chi_{\phi}}}^2 + (2 N_f + 3) g_{\psi_{\chi_{\phi}}}^4 + 3 g_{\psi_{\chi_{\phi}}}^2 g_{\psi_{\chi_{\psi}}}^2 + 8 g_{\psi_{\chi_{\psi}}}^2 \lambda_{\phi}^2,
\]

To confirm the fixed-point on the critical hypersurface denoted by \( X_i^* = (g_\phi^2, g_\psi^2, \chi_{\phi}, \lambda_{\phi}, \lambda_{\chi_{\phi}}) \), we look for the solution for the simultaneous zero of these beta functions, \( \beta(X_i^*) = 0 \). Eqs.(28) and (29) admit four solutions, \( A_1: (g_\phi^2, g_\psi^2) = (0, 0), A_2: (\epsilon/(2 N_f + 3), 0), A_3: [0, \epsilon/(2 N_f + 3)] \) and \( A_4: g_\chi^2 = \frac{\epsilon}{2 N_f + 6}, g_\phi^2 = \frac{\epsilon}{2 N_f + 6} \).

Among these solutions, only \( A_4 \) corresponds to a stable fixed-point[11]. The equations for \( \lambda_{\phi} \) and \( \lambda_{\chi_{\phi}} \) are symmetric, so they enjoy the same value at criticality. We have solved the fixed-point numerically, for instance for \( N_f = 1 \), Eqs.(28) to (32) admit an infrared stable fixed point:

\[
X^* = (0.125 \epsilon, 0.0281 \epsilon, 0.0281 \epsilon, 0.0346 \epsilon, 0.0346 \epsilon),
\]

at which the critical behavior is universal. The stable infrared fixed-point and the RG flow spanned by \( g_\phi^2, \lambda_{\chi} \) and \( \lambda_{\chi_{\phi}} \) are illustrated in Fig. 2. In the presence of enlarged Ising×Ising fluctuations, we observe that the enlarged fluctuation brings the system to a dual GNY fixed-point denoted by \( S \).
which yields the fermion anomalous dimensions

\[ \eta_\chi, \eta_\phi, \text{and fermions anomalous dimension } \eta_\psi. \]

We provide Pade estimate for the correction length exponents with [0/1] extrapolation.

### Table II: Critical exponents for the chiral Ising\times Ising universality class in \( D = 3 \) for varying flavors of Dirac fermion \( N_f \):

| \( N_f \) | \( \nu_{0/1} \) | \( \eta = \eta_\chi = \eta_\phi \) | \( \eta_\psi \) |
|----------|-----------------|-----------------|-----------------|
| 1/4      | 1.6318          | 0.0769          | 0.1538          |
| 1/2      | 1.5863          | 0.1478          | 0.1428          |
| 1        | 1.5254          | 0.25            | 0.125           |
| 2        | 1.4589          | 0.4             | 0.1             |

### A. Critical exponents

We now turn to the computation of critical exponents. At one-loop order, the fermion field renormalization has additional contribution compared with the Ising criticality, it is easily calculated the field renormalization coefficient (see details in Ref. \([74]\)):

\[ Z_\psi = 1 - (g_\chi^2 + g_\phi^2)/(2\epsilon), \]

which yields the fermion anomalous dimensions \( \eta_\psi = (g_\chi^2 + g_\phi^2)/2 \) such that \( 2\Delta_\psi = D - 1 + \eta_\psi \), where \( \Delta_\psi \) is the scaling dimensions for Dirac fermions. As stated in Sec. IIIA, the boson anomalous dimensions are given by \( \eta_\chi = 2N_f g_\chi^2 \) and \( \eta_\phi = 2N_f g_\phi^2 \) respectively. The inverse correlation length exponent characterizes the divergence of correlation length as the mass squared is tuned to zero or the transition is approached. At the critical point, we find the inverse correlation length exponent, at one-loop order, is given by

\[ 1/\nu = 2 - 12\lambda_\chi^* - 2N_f g_\chi^2 - \lambda_\chi^*, \]

Interestingly, the GNY model with Ising\times Ising criticality also supports the emergent supersymmetry (SUSY) scenario \([79, 80]\). For \( N_f = 1/2 \), the quantitative estimates of the critical exponents finds

\[ 1/\nu = 2 - 0.5217\epsilon, \quad \eta = \epsilon/7, \]

with \( \eta = \eta_\chi = \eta_\phi \). We point also that, owing to the existence of strongly relevant mixed term between two different Ising fluctuating fields, the supersymmetry scaling relation \( 1/\nu = (D - \eta)/2 \) in the chiral Ising GNY model not holds exactly at criticality\([11]\). In general case for \( N_f \geq 1/4 \), the critical exponents define a new universality class termed chiral Ising\times Ising universality class. To obtain the estimate for the critical exponents at the physical dimensions \( \epsilon = 1 \), we employ simple Pade approximant\([37]\). At one-loop order, the Pade approximant provides only [0/1] extrapolation for the exponents, our estimates for different \( N_f \) are listed in the Table. II. For \( N_f = 1 \), the theory describes the quantum criticality of spinless electrons from semimetallic state to insulating state which break sublattice or \( Z_2 \) symmetry, i.e., CDW phase. The \( N_f = 2 \) GNY model describes the similar transition of spinful fermions on the honeycomb lattice.

### B. Scaling dimensions of fermion bilinears

Apart from the order-parameter anomalous dimensions, the pair correlation function fermion bilinear also develops universal long-range power-law decay at criticality. It’s very interesting to ask for the behavior of the bilinear correlation at criticality. In general, the microscopic order-parameter on an underlying lattice model can be identified with the bilinears obtained in the continuum limit, i.e., valence-bond-solid order or Neel order\([75, 76]\). So, the observable value of fermion bilinears is accessible to quantum Monte Carlo simulations\([68]\). The fermion bilinears are gauge-invariant while the fermion fields anomalous dimensions are not in a gauge theory such as QED theory, which allows us to calculate the scaling dimensions of the fermion bilinear.

Following Ref. \([54]\), we add an infinitesimally weak fermion bilinear \( m_0 \Psi_0 M \Psi_0 \) in the bare Lagrangian, then the renormalized quantity is given by \( Z_m \Psi m_R \Psi M \Psi \). The beta function for the weak mass reads

\[ \beta(m) = \frac{dm}{d\ln \mu} = -(1 + \gamma_m)m, \]

where \( \gamma_m = d\ln Z_m/d\ln \mu \) is the anomalous dimensions for \( m \). The scaling dimensions of the bilinear is then given by

\[ \Delta_{\langle \Psi M \Psi \rangle} = D - 1 - \gamma_m(g_\chi^2, \lambda_\chi^*). \]

It’s straightforward to calculate the scaling dimensions of QAH fermion bilinear in the Ising criticality, we find

\[ \Delta_{\langle \Psi^{35} \Psi \rangle} = 3 - \frac{4N_f + 3}{4N_f + 6}\epsilon + O(\epsilon^2). \]
this result is also concide with the scaling dimensions of the flavor singlet and adoint fermion bilinears calculated in the chiral Ising GNY model[77]. Since the anticommutating relation $[\gamma^3, \gamma^\mu] = 0$, the CDW and QAH fermion bilinear are expected to have the same scaling dimensions at one-loop order $\Delta_{(\bar{\psi}_i \gamma^3 \psi_j)} = \Delta_{(\bar{\psi}_\mu \psi)}$. Similarly, the CDW and QAH bilinear at criticality show power-law decay as Eq. (17), and the scaling dimensions at the Ising$\times$Ising criticality is given by

$$\Delta_{(\bar{\psi}_i \gamma^3 \psi_j)} = \Delta_{(\bar{\psi}_\mu \psi)} = 3 - \frac{2N_f + 3}{2N_f + 6} \epsilon + O(\epsilon^2), \quad (42)$$

which is apparently larger than those in the Ising criticality for relatively small $N_f$. Therefore, the QAH bilinear correlations at the Ising$\times$Ising criticality decay faster than it at the Ising criticality.

Further, the bilinears $\langle \bar{\Psi}_3 \gamma^3 \Psi \rangle$ and $\langle \bar{\Psi}_5 \gamma^5 \Psi \rangle$ are of interests, they correspond to Kekule VBS order parameter on the honeycomb lattice[42, 78]. The corresponding scaling dimensions in the Ising and Ising$\times$Ising criticality can be calculated respectively as

$$\Delta_{\text{KVBS}}^{\text{Ising}} = 3 - \frac{4N_f + 15}{4N_f + 12} \epsilon + O(\epsilon^2), \quad (43)$$

$$\Delta_{\text{KVBS}}^{\text{Ising}} = 3 - \frac{2N_f + 9}{2N_f + 6} \epsilon + O(\epsilon^2), \quad (44)$$

with $\Delta_{\text{KVBS}} = \Delta_{(\bar{\psi}_i \gamma^3 \psi_j)}$. An important observation from this result is that the Kekule VBS correlation has been enhanced tremendously by the Ising$\times$Ising fluctuations as the bilinear scaling dimensions is decreased.

By now, we have concentrated on the Ising$\times$Ising criticality of spinless electrons. The spinful electrons on the graphene lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons. The spinful electrons on the honeycomb lattice are believed to undergo a metallicity of spinless electrons.

V. CONCLUSIONS

In this paper, with the help of one-loop perturbative RG analysis in $d = 4 - \epsilon$, we have studied the fermionic quantum criticality with enlarged Ising$\times$Ising fluctuation in the two dimensional honeycomb materials. To get an understanding of whether the semimetal-insulator transition is weak first-order or second-order transition, we include a cubic term of the order-parameter in the theory and study its fate as the infrared stable fixed-point is approached. The semimetal-CDW transition is modeled in terms of an Ising GNY theory with a generalized flavors of Dirac fermions $N_f$, we find the cubic term is always irrelevant if $N_f$ fulfills $N_f \geq 1/4$ in three space-time dimensions. The irrelevance implies that the extra fluctuations from fermions change the nature of transition and render it continuous[18, 21, 27]. We also calculate the complete second-order regime for varying space-time and $N_f$, as shown in Fig. 1.

Moreover, the tricritical point for the semimetal-transition that breaks Ising$\times$Ising symmetry is investigated. Using $\epsilon$ expansion, we have calculated the critical exponents for the Ising$\times$Ising universality class, including inverse correlation length exponent $1/\nu$, boson anomalous dimension $\eta_\Phi$, fermions anomalous dimension $\eta_\Phi$. The exponents for different value of $N_f$ are shown in Table II. Further, the $\epsilon$ expansion has been used to calculated the scaling dimensions for the fermion bilinear on the honeycomb lattice. In particular, we observe that the scaling dimensions for the Kekule valence-bond-solid at Ising$\times$Ising criticality is smaller than the value at Ising criticality. This means that the Kekule valence-bond-solid is enhanced tremendously by the enlarged Ising$\times$Ising fluctuations. The crucial ingredient for the enhancement is the anticommuting nature between the corresponding fermion bilinear matrix and the Dirac gamma matrices.

Acknowledgments

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