Anomalous Behavior
in the Massive Schwinger Model

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Abstract

We evaluate the chiral condensate and Polyakov loop in two-dimensional QED with a fermion of an arbitrary mass ($m$). We find discontinuous $m$ dependence in the chiral condensate and anomalous temperature dependence in Polyakov loops when the vacuum angle $\theta\sim \pi$ and $m=O(\epsilon)$. These nonperturbative phenomena are due to the bifurcation process in the solutions to the vacuum eigenvalue equation.

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The Schwinger model, QED in two dimensions has been a preferred theoretical laboratory for the study of physical phenomena such as chiral symmetry, gauge symmetry, anomalies, and confinement. [1]-[7] In a nontrivial topology it allows us to inquire about finite volume and temperature effects while keeping the computations infrared safe. [8]-[24] Results at finite temperature (T) can be obtained by Wick rotating the solution on a circle $S^1$ of circumference $L$ and replacing $L$ by $T^{-1}$.

The theory is exactly solvable with massless fermions, but not with massive fermions. The effect of a small fermion mass ($m/e \ll 1$) in the one flavor case is minor other than necessitating the $\theta$ vacuum. [4, 6, 10] The opposite limit of weak coupling, or heavy fermions, has been analyzed by Coleman. [5, 6] In this work we investigate physical quantities such as chiral condensate and Polyakov loop with no restriction on values of the parameters of the system. The effect of nonvanishing fermion masses has been investigated in lattice gauge theory and light cone quantization methods as well. [23]-[27]

The Lagrangian of the system is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu (i \partial_\mu - e A_\mu) \psi - m(M + M^\dagger)$$

$$M = \bar{\psi} \frac{1}{2} (1 - \gamma^5) \psi,$$

(1)

where $\gamma^\mu = (\sigma_1, i \sigma_2)$ and $\psi^T_a = (\psi^a_+, \psi^a_-)$. We study the model on a circle of circumference $L$ and boundary conditions

$$A_\mu(t, x + L) = A_\mu(t, x)$$

$$\psi_a(t, x + L) = -\psi_a(t, x).$$

(2)

The only physical degree of freedom associated with gauge fields is the Wilson line phase $\Theta_W(t)$: [9, 11, 11]

$$e^{i\Theta_W(t)} = \exp \left\{ i e \int_0^L dx \, A_1(t, x) \right\}.$$

(3)

In the Matsubara formalism of finite temperature field theory boson and fermion fields are periodic and anti-periodic in imaginary time ($\tau$), respectively. Mathematically, the model at finite temperature $T = \beta^{-1}$ is obtained from the model defined on a circle by Wick rotation and replacement $L \rightarrow \beta$, $it \rightarrow x$ and $x \leftrightarrow \tau$. The Polyakov loop of a charge
$q$ in the finite temperature theory corresponds to the Wilson line phase:

$$P_q(x) = \exp \left\{ iq \int_0^\beta d\tau A_0(\tau, x) \right\} \Longleftrightarrow \exp \left\{ \frac{i}{e} q \Theta_W(t) \right\}. \quad (4)$$

We bosonize the fermion in the Coulomb gauge in the interaction picture defined by a massless fermion:[10, 18]

$$\psi_{\pm}(t, x) = \frac{1}{\sqrt{L}} C_{\pm} e^{\pm ip_\pm (t+q_\pm)/L} : e^{\pm i \sqrt{4\pi} \phi_{\pm}(t, x)} : ,$$

$$C_+ = 1, \quad C_- = \exp \{ i\pi (p_+ - p_-) \} ,$$

$$\phi_{\pm}(t, x) = \sum_{n=1}^\infty (4\pi n)^{-1/2} \left\{ c_{\pm, n} e^{-2\pi i n (t+q_\pm)/L} + \text{h.c.} \right\} ,$$

$$e^{2\pi i p_\pm} |\text{phys}\rangle = |\text{phys}\rangle , \quad (5)$$

where $[q_\pm, p_\pm] = i$, and $[c_{\pm, n}, c_{\pm, m}^\dagger] = \delta_{nm}$. The $:\ :$ in (5) indicates normal ordering with respect to $(c_n, c_n^\dagger)$. In physical states $p_\pm$ takes an integer eigenvalue.

The Hamiltonian in the Schrödinger picture becomes

$$H_{\text{tot}} = H_0 + H_\phi + H_{\text{mass}} - \frac{\pi}{6L}$$

$$H_0 = \frac{e^2 L}{2} P_W^2 + \frac{1}{2\pi L} \left( \Theta_W + 2\pi p \right)^2$$

$$H_\phi = \int_0^L dx \frac{1}{2} : \left[ \Pi^2 + (\phi')^2 + \frac{e^2}{\pi} \phi^2 \right] :$$

$$H_{\text{mass}} = \int_0^L dx m (M + M^\dagger) . \quad (6)$$

The conjugate pairs are $\{ p, q \} = \{ \frac{1}{2}(p_+ + p_-), q_+ + q_- \}$, $\{ \bar{p}, \bar{q} \} = \{ p_+ - p_-, \frac{1}{2}(q_+ - q_-) \}$, $\{ P_W, \Theta_W \}$, and $\{ \Pi, \phi = \phi_+ + \phi_- \}$. Note that $(\phi, \Pi)$ fields are subject to conditions $\int_0^L dx \phi(x) = 0 = \int_0^L dx \Pi(x)$. The mass operator is given by

$$M = -C_+^\dagger C_+ e^{-2\pi i \bar{q} x/L} e^{iq} \cdot L^{-1} N_0 [ e^{i\sqrt{4\pi} \phi} ] \quad (7)$$

where $N_\mu[\cdots]$ indicates that the operator inside $[ \cdots ]$ is normal-ordered with respect to a mass $\mu$.

As $[\bar{p}, H_{\text{tot}}] = 0$, we may restrict ourselves to states with $\bar{p} = 0$. $p$ takes integer eigenvalues in this subspace. The Hamiltonian (6) possesses a residual gauge symmetry
\[ \Theta_W \rightarrow \Theta_W + 2\pi \text{ and } p \rightarrow p - 1 \text{ generated by } U \text{ defined as} \]
\[ U = \exp \left( 2\pi i P_W + iq \right), \quad [U, H_{\text{tot}}] = 0. \quad (8) \]

The ground state is the \( \theta \) vacuum: \[\Phi_{\text{vac}}(\theta)\] \[ U \Delta(\theta) = e^{i\theta} \Delta(\theta). \quad (9) \]

To determine the vacuum wave function we must solve the eigenvalue equation
\[ (H_0 + H_{\text{mass}}) \Phi_{\text{vac}}(\theta) = E \Phi_{\text{vac}}(\theta). \quad (10) \]

The fermion mass term \( H_{\text{mass}} \) changes the mass of the boson field \( \phi \) from \( \mu = e/\sqrt{\pi} \) to \( \mu_1 \). The vacuum is defined with respect to the physical boson mass \( \mu_1 \). Making use of \[\text{[10, 13]}\]
\[ N_0[e^{i\sqrt{4\pi}\phi}] = B(\mu L) N_\mu[e^{i\sqrt{4\pi}\phi}] \]
\[ B(z) = \frac{z}{4\pi} \exp \left\{ \gamma + \frac{\pi}{z} - 2 \int_1^\infty \frac{du}{(e^{uz} - 1)\sqrt{u^2 - 1}} \right\}, \quad (11) \]

the mass operator in (7) is accordingly written as
\[ M = -C^+_\kappa C_\kappa \cdot e^{-2\pi i\tilde{p}_x/L} e^{iq} \cdot L^{-1} B(\mu_1 L) N_{\mu_1}[e^{i\sqrt{4\pi}\phi}]. \quad (12) \]

With this understanding we write the vacuum wave function, taking (9) into account, as
\[ \Phi_{\text{vac}}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_n \int dp_W |p_W, n\rangle e^{-in\theta + 2\pi in p_W} f(p_W) \]
\[ \int dp_W |f(p_W)|^2 = 1. \quad (13) \]

Here \( |p_W, n\rangle \) is an eigenstate of \( P_W \) and \( p \). Since \( \langle p_W', n'|e^{\pm i q}|p_W, n\rangle = \delta(p_W'-p_W) \delta_{n',n \pm 1} \), the vacuum eigenvalue equation (10) is reduced to a Schrödinger equation
\[ \left\{ -\frac{d^2}{dp_W^2} + V(p_W) \right\} f(p_W) = \epsilon f(p_W) \quad (14) \]

where
\[ V(p_W) = \omega^2 p_W^2 - \kappa \cos(\theta - 2\pi p_W) \]
\[
\omega = \pi \mu L \quad , \quad \kappa = 4\pi mLB(\mu_1 L) \quad , \quad \mu^2 = \frac{e^2}{\pi}
\]  

and \( \epsilon = 2\pi LE_{\text{vac}} + \frac{1}{3}\pi^2 \).

To determine the boson mass \( \mu_1 \), we expand \( H_{\text{mass}} \) in (6) in power series in \( \phi \). In the vacuum

\[
H_{\text{mass}} \rightarrow \int_0^L dx \frac{2\pi m}{L} B(\mu_1 L) \langle e^{iq} + e^{-iq} \rangle_{\text{vac}} \phi^2
\]

\[
= \int_0^L dx \frac{4\pi mB(\mu_1 L)}{L} \langle \cos(\theta - 2\pi p_{W}) \rangle_f \phi^2
\]

where \( \langle F(p_{W}) \rangle_f = \int dp_{W} F(p_{W}) |f(p_{W})|^2 \). Hence

\[
\mu_1^2 = \mu^2 + \frac{8\pi mB(\mu_1 L)}{L} \langle \cos(\theta - 2\pi p_{W}) \rangle_f .
\]

(17)

(14) and (17) must be solved simultaneously. We have a Schrödinger problem in which the potential needs to be determined selfconsistently. [18] Wave functions \( f(p_{W}) \) for typical values for \( T/\mu, m/\mu, \) and \( \theta \) are displayed in fig. 1.

The chiral condensate is given by

\[
\langle \overline{\psi} \psi \rangle_\theta = \langle M + M^\dagger \rangle_\theta
\]

\[
= -2L^{-1}B(\mu_1 L) \langle \cos(\theta - 2\pi p_{W}) \rangle_f .
\]

(18)

Combining (17) and (18), one finds

\[
\mu_1^2 - \mu^2 = -4\pi m \langle \overline{\psi} \psi \rangle_\theta ,
\]

(19)

which is a PCAC relation.

The Polyakov loop is, from (4),

\[
\langle P_q \rangle_{\theta,T} = \langle e^{i(q/e)\Theta_{W}} \rangle_{\theta,L=T^{-1}}
\]

\[
= \begin{cases} 
0 & \text{for } \frac{q}{e} \neq \text{an integer} \\
\int_{-\infty}^{\infty} dp_{W} \ f(p_{W})^* f(p_{W} - \frac{q}{e}) & \text{for } \frac{q}{e} = \text{an integer}.
\end{cases}
\]

(20)

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3 We have defined the mass operator \( M \) by (7), independent of a mass \( m \). Consequently \( \langle \overline{\psi} \psi \rangle \neq 0 \) even in the limit \( e \rightarrow 0 \). It may be appropriate to define a physical \( M \) by \( M^{\text{phys}} = M - \langle M \rangle_{m,e=0} \) so that \( \langle \overline{\psi} \psi \rangle^{\text{phys}} \) vanishes in the free theory. The composite operator \( M^{\text{phys}} \) thus defined depends on \( m \).
Figure 1: Wave functions $|f(p_W)|^2$. (a) $m/\mu = 10, 1,$ and 0.1 corresponds to $\kappa = 3.61 \cdot 10^{-1}, 3.19 \cdot 10^{-2},$ and $3.18 \cdot 10^{-3}$. (b) $\theta = \pi^-$ indicates that the $\theta$ value is less than, but is very close to $\pi$. At $m/\mu = 1, \theta = 0$ and $\pi$ corresponds to $\kappa = 2.24$ and 2.19. At $m/\mu = 10$, $\kappa = 165$ for both $\theta = 0$ and $\pi$.

The potential, $V(p_W)$, in Eq. (14) consists of two terms; harmonic oscillator and cosine potentials. The strength of the cosine term, $\kappa$, is

$$\kappa = \begin{cases} e^{\gamma}m\mu_1L^2 & \text{for } \mu_1L \gg 1 \\ 4\pi mL & \text{for } \mu_1L \ll 1. \end{cases}$$

Depending on relative strength, the behavior of the ground state wave function is quite different. If $\omega^2 \gg \kappa$, the potential is approximated by the harmonic term, i.e. $f(p_W) = (\omega/\pi)^{1/4}e^{-\omega p_W^2/2}$. The condition is satisfied if $m/\mu \ll 1 \ll \mu L$ or if $m/\mu \ll \mu L \ll 1$. In this regime $\langle \cos(\theta - 2\pi p_W) \rangle_f = e^{-\pi/\mu L} \cos \theta$ so that

$$\mu_1 = \sqrt{\mu^2 + (me^\gamma \cos \theta)^2 + me^\gamma \cos \theta}$$
\[ \langle \bar{\psi} \psi \rangle_\theta = -\frac{e^\gamma}{2\pi} \mu_1 \cos \theta \]

\[ \langle P_e \rangle_\theta = e^{-\pi \mu/4T} \sim 0 \quad \text{for} \quad \frac{m}{\mu} \ll 1 \ll \mu L = \frac{\mu}{T} \]  \hspace{1cm} (22)

and

\[ \mu_1^2 = \mu^2 + \frac{8\pi m}{L} e^{-\pi/\mu L} \cos \theta \]

\[ \langle \bar{\psi} \psi \rangle_\theta = -\frac{2}{L} e^{-\pi/\mu L} \cos \theta \]

\[ \langle P_e \rangle_\theta = e^{-\pi \mu/4T} \sim 1 \quad \text{for} \quad \frac{m}{\mu} \ll \mu L = \frac{\mu}{T} \ll 1 \]  \hspace{1cm} (23)

In the opposite limit \( \kappa \gg \omega^2 \), the cosine term dominates. However, the harmonic potential cannot be ignored as it lifts the degeneracy of the cosine potential. In the large volume limit the harmonic potential selects one of the minima of the cosine potential at \( p_W = \overline{\theta}/2\pi \) where \( \overline{\theta} = \theta - 2\pi[\theta + \pi)/2\pi]. \) Hence \( V \sim \frac{1}{2} e^\gamma m \mu_1 L^2 (2\pi p_W - \overline{\theta})^2 \) so that \( f(p_W) = (\mu L)^{1/4} e^{-\mu L (2\pi p_W - \overline{\theta})^2}/8\pi \) where \( \mu^2 = 2m e^\gamma. \) [For \( \theta \sim \pi (\mod 2\pi), \) \( f \) has two peaks at \( p_W \sim \pm \frac{1}{2}. \)] This leads to \( \langle \cos(\theta - 2\pi p_W) \rangle_f = e^{-\pi \mu L} \sim 1. \) Consequently

\[ \mu_1 = \tilde{\mu} = 2e\gamma m \]

\[ \langle \bar{\psi} \psi \rangle_\theta = -\frac{e^{2\gamma}}{\pi} m \]

\[ \langle P_e \rangle_\theta = e^{-\pi e\gamma m/2T} \quad \text{for} \quad m \gg \mu, \mu L = \frac{\mu}{T} \gg 1. \]  \hspace{1cm} (24)

Notice that the chiral condensate increases linearly with \( m. \) However, there is no \( \theta \) dependence to the leading order. (See the previous footnote, too.)

At high temperatures \( (T = L^{-1} \gg \mu), \omega^2/\kappa = \pi \mu^2/4mT. \) So long as \( m \neq 0, \) eventually the cosine term dominates. However, the both terms in the potential become small in this limit. A good estimate is obtained by treating the cosine term as a perturbation. Numerical evaluation supports the result. (See fig. 3 below.)

Write Eq. (14) as \( (H_0 + V')f = \epsilon f \) where \( V' \) is the cosine term. Eigenstates of \( H_0 \) are denoted by \( \{|n\rangle \} (n = 0, 1, 2, \cdots). \) Then

\[ \langle n | \cos(\theta - 2\pi p_W) | 0 \rangle = \frac{(e^{i\theta} + (-1)^n e^{-i\theta})}{2} e^{-\pi \mu L} \frac{1}{\sqrt{n!}} \left( \frac{2\pi}{\mu L} \right)^{1/2}. \]  \hspace{1cm} (25)
Figure 2: $m/\mu$ dependence of chiral condensates at given $T/\mu$ and $\theta$.

$f$ is given by $|f⟩ = |0⟩ - \sum_{n=1} |n⟩⟨n|V'|0⟩/2\omega n$ so that

$$\langle \cos(\theta - 2\pi p_W)⟩_f = e^{-\pi^2/\omega} \cos \theta$$

$$+ \frac{\kappa}{2\omega} e^{-2\pi^2/\omega} \left\{ \int_0^{2\pi^2/\omega} dz \frac{e^z - 1}{z} + \cos 2\theta \int_0^{2\pi^2/\omega} dz \frac{e^{-z} - 1}{z} \right\}.$$  \hspace{1cm} (26)

For $T = L^{-1} \ll \mu$ ($\omega \gg 1$) the first term in (26) dominates over the rest to reproduce (22). For $T \gg \mu$ ($\omega \ll 1$), $\int_0^{2\pi^2/\omega} dz (e^z - 1)/z \sim (\omega/2\pi^2) e^{2\pi^2/\omega}$ so that $\langle \cos(\theta - 2\pi p_W)⟩_f \sim m/\pi T$. We obtain

$$\mu_1^2 = \mu^2 + 8m^2$$

$$\langle \overline{\psi} \psi \rangle_\theta = -\frac{2m}{\pi}$$

$$\langle P_e \rangle_\theta = e^{-\pi\mu/4T} \left( 1 - \frac{6.60 m}{\mu} e^{-\pi T/\mu} \cos \theta \right) \text{ for } T \gg \mu, m.$$  \hspace{1cm} (27)

In the intermediate range of parameter values Eqs. (14) and (17) must be solved numerically. The computational algorithm is the following. With given $(\mu L, m/\mu, \theta)$, we first assign an input value for $\kappa = \kappa_{in}$. The potential in (14) is specified with $(\mu L, \kappa_{in}, \theta)$. Eq. (14) determines $f(p_W)$, from which one can determine $\mu_1$, solving (17). With this
new $\mu_1$ one recomputes $\kappa = \kappa_{\text{out}}$ by (15). Schematically

$$
\kappa_{\text{in}} \rightarrow V(p_W) \rightarrow f(p_W) \rightarrow \mu_1 \rightarrow \kappa_{\text{out}}.
$$

(28)

$\kappa_{\text{out}}$ must coincides $\kappa_{\text{in}}$. This gives a consistency condition to determine $\kappa$ with given $(\mu L, m/\mu, \theta)$.

In fig. 2 chiral condensates are plotted as functions of $m/\mu$. Asymptotically ($m \gg \mu$) the behavior is given by (24). $T$ dependence of chiral condensates is displayed in fig. 3. [24] Low and high temperature limits agree with (22) and (27). The $T$ dependence of chiral condensates with given $m$ and $\theta$ is smooth. This is consistent with the Mermin-Wagner theorem that in a one-dimensional system there is no phase transition at finite temperature. [28]

$T$ dependence of Polyakov loops is displayed with various values of $m/\mu$ in fig. 4. Notice that at $\theta = 0$ the curve smoothly changes as $m/\mu$. However, around $\theta = \pi$ nontrivial $T$ dependence is observed for $m \geq \mu$. The origin of this behavior is traced back to the wave function $f(p_W)$ in the corresponding problem on $S^1$ with $L = T^{-1}$. $f(p_W)$ at moderately low temperature has two dominant peaks at $p_W = \pm (1 - \epsilon)$ around $\theta = \pi$, whereas it has only one dominant peak at $p_W = 0$ for $\theta = 0$. As (20) shows, the Polyakov loop is determined by the overlap of $f(p_W)^*$ and $f(p_W - 1)$. Hence it vanishes quickly for $\theta = 0,$

Figure 3: $T$ dependence of chiral condensates at fixed $m/\mu$ and $\theta$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3.png}
\caption{$T$ dependence of chiral condensates at fixed $m/\mu$ and $\theta$.}
\end{figure}
Figure 4: $T/\mu$ dependence of Polyakov loops with given $m/\mu$ and $\theta$. Two-step behavior is observed at $\theta \sim \pi$.

but approaches $\sim .5$ for $\theta \sim \pi$ at low, but moderate $T$. As $T$ further gets lowered, the two peaks become narrower and sharper. When the width of the peaks becomes smaller than $\epsilon$, the overlap of the wave functions and Polyakov loop vanish. If $\theta$ is not exactly $\pi$, but is very close to $\pi$, the asymmetry in the potential, enhanced by the factor $m/T^2$, becomes important at sufficiently low $T$ and the wave function has a sharp peak around one of the minima. See the wave functions displayed in fig. 1. In the numerical evaluation presented in fig. 4, the computer picks a value for $\theta$ which is not exactly $\pi$. The transition from the plateau ($\sim .5$) to zero at $\theta \sim \pi$ is caused by this change of the wave function. This explains the two-step behavior observed in fig. 4.

In fig. 3 we observe that the pattern of the $T$ dependence of $\langle \bar{\psi}\psi \rangle$ changes as $m/\mu$ is increased. At $\theta = \pi$, $\langle \bar{\psi}\psi \rangle/\mu$ decreases (increases) as $T/\mu$ increases for $m/\mu = .1$ ($m/\mu = 1$).

Indeed there arises a discontinuity in the $m$ dependence of chiral condensates at low temperature at $\theta = \pi$, at least in our approximation scheme. We have displayed it at $T/\mu = 0.03$ in fig. 5. We observe that $\langle \bar{\psi}\psi \rangle/\mu$ discontinuously changes at $m/\mu = .437$.

To understand the origin of the discontinuity, we consider a solution $\kappa_{\text{in}} = \kappa_{\text{out}}$ in (28). Write $\kappa_{\text{out}} = g(\kappa_{\text{in}})$. We are looking for a solution to $g(\kappa) = \kappa$, or a fixed point
Figure 5: A discontinuity in chiral condensates is observed at the $m/\mu \sim 0.437$ when $T/\mu = 0.03$ and $\theta = \pi$.

$k_{\text{fix}}$ of $g(\kappa)$. As $m/\mu$ varies with given $T/\mu$ and $\theta$, $g(\kappa)$, and therefore $k_{\text{fix}}$ change. At $\theta = \pi$, there is a critical value for $m/\mu$ at which the fixed point bifurcates. In a certain range of $m/\mu$, there appear three fixed points, two stable and one unstable. Among the two stable fixed points, one of them has a larger chiral condensate and therefore a lower energy density, corresponding to the vacuum. This bifurcation induces a discontinuous change in chiral condensates as $m/\mu$ varies. We have displayed the mapping $k_{\text{out}} = g(k_{\text{in}})$ in the critical region in fig. 6 (a). We observe that saddle node bifurcation takes place at $m/\mu = 0.4368$.

At very low $T$ the critical mass $m_c/\mu$ can be determined analytically. Suppose that $m = O(\mu)$. For $\mu L = \mu/T \gg 1$, $\kappa = e^\gamma m \mu_1 L^2$. At $\theta = \pi$,  

$$V = (\pi \mu L p_W)^2 + \kappa \cos 2\pi p_W.$$  

(29)

There is always a solution which satisfies $(\mu L)^2 > 2\kappa$ or $\mu^2 > 2e^\gamma m\mu_1$. In this case $f(p_W)$ is sharply localized around $p_W = 0$. This yields  

$$\mu_1 = \sqrt{\mu^2 + (m e^\gamma)^2 - m e^\gamma}.$$  

(30)

If $(\mu L)^2 < 2\kappa$ or $\mu^2 < 2e^\gamma m\mu_1$, $f(p_W)$ is localized around $\pm \bar{p}_W \neq 0$ where  

$$2\pi \bar{p}_W = \frac{2m\mu_1 e^\gamma}{\mu^2} \sin 2\pi \bar{p}_W.$$
Figure 6: $\kappa_{\text{in}} - \kappa_{\text{out}}$ plots. (a) At $T/\mu = 0.03$ and $\theta = \pi$. As $m/\mu$ changes, the fixed point bifurcates at $m/\mu = 0.4368$. (b) With a fixed $m/\mu = 0.4365$, $T/\mu$ is varied. The curves remain almost critical.

$$\mu_1 = \sqrt{\mu^2 + (me^\gamma \cos 2\pi \bar{p}_W)^2 - me^\gamma \cos 2\pi \bar{p}_W}$$

(31)

A solution to (31) exists only for $m/\mu > 0.435$. This solution corresponds to a bigger $\mu_1$ or $-\langle \bar{\psi} \psi \rangle$, and therefore to a lower energy density. In other words $m_c/\mu = 0.435$ at $T = 0$. Numerically we have found $m_c/\mu = 0.435, 0.437,$ and $0.454$ at $T/\mu = 0.01, 0.03$ and $0.07$, respectively. Above $T/\mu = 0.12$ the function $g(x)$ has only one fixed point for all values of $m/\mu$ so that the discontinuity disappears.

It is not clear, however, if the discontinuity discovered above is real in the full theory, or just an artifact of the approximation in use. In determining the boson mass $\mu_1$ in (16) and (17) we have ignored nonlinear terms in the $\phi$ field in $H_{\text{mass}}$, retaining only the $\phi^2$ term. Those nonlinear terms are expected to affect the boson mass.

Furthermore, the Mermin-Wagner theorem ensures that there should be no discontinuity in $\langle \bar{\psi} \psi \rangle$ when $T/\mu$ varies with $m/\mu$ kept fixed. The discontinuity in the $m/\mu$ dependence is consistent with the Mermin-Wagner theorem only if $m_c/\mu$ is universal, being independent of $T/\mu$. Numerically we have found that $m_c/\mu$ is almost universal, although there appears tiny dependence on $T/\mu$. In fig. 6 (b) we have plotted $\kappa_{\text{in}} - \kappa_{\text{out}}$ with a fixed $m/\mu = 0.4365$ at various $T/\mu$. The curve is critical at $T/\mu \sim 0.03$, while it is slightly off at higher or lower temperature despite it may be very hard to see visually.
It is striking that the critical value $m_c/\mu$ is almost insensitive to $T/\mu$ to such a degree of accuracy.

There are two possible scenarios when the whole interactions are taken into account. It may turn out that the discontinuity in $m/\mu$ disappears, being replaced by a crossover transition. Or the discontinuity is real, taking place at a universal mass value $m_c/\mu$. Further investigation is necessary to determine which picture is right.

In this paper we have evaluated $T$- and $m$-dependence of chiral condensates and Polyakov loops. We have demonstrated that at $\theta \sim \pi$ there appears anomalous behavior when $m/\mu$ is $0.4 \sim 0.5$. Mathematically these anomalous phenomena are related to the bifurcation process in the solutions of the vacuum equation. Electromagnetic interactions and fermion mass collaborate to induce anomalous behavior of the vacuum solutions. This is reminiscent of chaotic dynamics in nonlinear systems. More detailed analysis will be reported separately.

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