On polynomial approximations to solutions of implicit differential equations

Ihor Korol
ON POLYNOMIAL APPROXIMATIONS TO SOLUTIONS OF
IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract. In this paper the possibility to present by a polynomial an independent variable
for the approximate solutions of the systems of implicit ordinary differential equations under
multi-point boundary conditions is substantiated.

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1. Introduction

There is a large number of methods which mathematicians elaborated for studying
boundary value problems (BVPs). In the papers [1], [2] the numerical-analytic method
based upon successive approximations was introduced. The polynomial version of this
method in which the successive approximations are polynomials was proposed in [1]
and then developed in [3], [4] for three- and multi-point boundary conditions. In this
paper the issue of existence and approximate construction of the solutions of multi-
point boundary conditions for the systems of implicit ordinary differential equations
of the first order are studied by using polynomial approximations.

2. Construction of successive polynomial approximations

Let us consider a system of implicit equations

\[ \frac{dx}{dt} = f(t, x, \frac{dx}{dt}), \]

with a multi-point boundary conditions

\[ A_0 x(0) + \sum_{k=1}^{q} A_k x(t_k) + A_{q+1} x(T) = d, \]

where \( x, d \in \mathbb{R}^n \), \( f : [0, T] \times D_1 \times D_2 \to \mathbb{R}^n \), \( D_1, D_2 \) are closed bounded domains in
\( \mathbb{R}^n \), \( 0 = t_0 < t_1 < \ldots < t_q < t_{q+1} = T \), \( A_k \) \( (k = 0, 1, \ldots, q + 1) \) - are \( n \times n \) matrices
so that  \( \det \left[ \sum_{k=1}^{q} A_{k} t_{k} \right] \neq 0 \).

First of all we will introduce some notations [1].

It is known that for \( f(t) \in C[0,T] \) there is a unique polynomial \( P_m(t) \) among all the polynomials \( P_m(t) \) with no more than \( m \) degree which is the best approximation for \( f(t) \):

\[
E_m(f) = \| f(t) - P_m(t) \| = \inf_{P_m(t)} \| f(t) - P_m(t) \|.
\]

Let us set in the interval \([0,T]\) the nodes

\[
t_i = \frac{T}{2} \left( \cos \frac{(2i-1)\pi}{2(p+1)} + 1 \right), \quad i = 1,2,\ldots,p+1,
\]

which are obtained by the substitution \( \tau = \frac{T}{2} (\tau' + 1) \) from the corresponding zeroes \( \tau_i' \in [-1,1] \) of the Chebyshev polynomials

\[
T_{p+1}(t) = \cos ((p+1) \arccos t).
\]

For arbitrary continuous function \( x_r(t) \) by \( f^p(t,x_r(t),y_r(t)) \) we denote the Lagrange interpolation polynomial with \( p \) degree and with respect to the points (2.3):

\[
f^p(t,x_r(t,x_0),y_r(t,x_0)) = f_1^p(t,x_r(t,x_0),y_r(t,x_0)), \ldots, f_n^p(t,x_r(t,x_0),y_r(t,x_0))
\]

where \( y_r(t) := \frac{dx_r(t)}{dt}, \quad f_j^p(t,x_r(t,x_0),y_r(t,x_0)) = a_{0j} + a_{1j} t + \ldots + a_{pj}, \quad j = 1,2,\ldots,n, \quad f_j^p(t_i,x_r(t_i),y_r(t_i)) = f_j(t_i,x_r(t_i),y_r(t_i)), \quad i = 1,2,\ldots,p+1.

Let us denote by

\[
\mathcal{T}(f,x,y,t,x_0) = f(t,x(t,x_0),y(t,x_0)) - \frac{1}{T} \int_{0}^{T} f(s,x(s,x_0),y(s,x_0)) \, ds,
\]

\[
\mathcal{L}(f,x,y,t,x_0) = \int_{0}^{t} \left( f(\tau,x(\tau,x_0),y(\tau,x_0)) - \frac{1}{T} \int_{0}^{T} f(s,x(s,x_0),y(s,x_0)) \, ds \right) d\tau.
\]

We assume that the following conditions hold for the BVP (2.1), (2.2):

a) the vector-function \( f(t,x,y) \) is continuous in \( \Omega = [0,T] \times D_1 \times D_2 \) (and therefore it is bounded by some vector \( M \)) and Lipschitzian in \( x \) and \( y \), i.e.,

\[
|f(t,x,y)| \leq M, \quad |f(t,x,y) - f(t,x',y')| \leq K_1|x-x'| + K_2|y-y'|,
\]

where \( M \) and \( n \times n \) matrices \( K_1, K_2 \) have non-negative components. The absolute value sign and the inequalities we understand component-wise;

b) domains \( D_1 \) and \( D_2 \) satisfy the conditions

\[
D_{\beta_1} := \{ x \in \mathbb{R}^n \mid B(x,\beta_1(x)) \subset D_1 \} \neq \emptyset, \quad B(0,\beta_2(x)) \subset D_2,
\]
where \( B(x, \rho(x)) \) is the ball of radius \( \rho(x) \) with center \( x \) and

\[
\beta_1(x) = \left( \frac{T}{2} E + G \right) \cdot (M' + L_p) + T|d(x)|, \quad G = T \cdot \sum_{k=1}^{q} |HA_k| \cdot \alpha_1(t_k),
\]

\[
\beta_2(x) = 2(M + L_p) + \frac{1}{T} G (M' + L_p) + |d(x)|, \quad H = \left[ \sum_{k=1}^{q+1} \alpha_k(t_k) \right]^{-1},
\]

\[
d(x) = H \cdot \left( d - \sum_{k=0}^{q+1} A_k x \right), \quad \alpha_1(t) = 2t \left( 1 - \frac{t}{T} \right),
\]

\[
M' = \frac{1}{2} \left[ \max_{(t,x,y) \in \Omega} f(t,x,y) - \min_{(t,x,y) \in \Omega} f(t,x,y) \right],
\]

\[
L_p = (5 + lg) \max_{r} \left[ f \left( t, x_r^{p+1}(t,x_0), y_r^p(t,x_0) \right) \right] =
\]

\[
= (5 + lg) \cdot \left( \max_{r} \left[ f_1 \left( t, x_r^{p+1}(t,x_0), y_r^p(t,x_0) \right) \right], \ldotsight.
\]

\[
\left. \left. \ldots, \max_{r} \left[ f_n \left( t, x_r^{p+1}(t,x_0), y_r^p(t,x_0) \right) \right] \right) ;
\]

c) the eigenvalues \( \lambda_j(Q) \) of the matrix \( Q = K_1 \left( \frac{T}{2} E + G \right) + K_2 \left( 2E + \frac{1}{T} G \right) \) satisfy the inequalities

\[
|\lambda_j(q)| < 1, \quad j = 1, \ldots, n. \tag{2.5}
\]

Let us introduce the sequence of polynomials with \( p + 1 \) degree

\[
x_m^{p+1}(t,x_0) = x_0 + L \left( f_p, x_m^{p+1}, y_m^p, t, x_0 \right) + tHd(x_0) -
\]

\[
-tH \sum_{k=1}^{q} A_k L \left( f_p, x_m^{p+1}, y_m^p, t, x_0 \right), \quad x_0^{p+1}(t,x_0) = x_0, \quad m = 1, 2, \ldots \tag{2.6}
\]

Their derivatives look as follows:

\[
y_m^p(t,x_0) = L \left( f_p, x_m^{p+1}, y_m^p, t, x_0 \right) + Hd(x_0) -
\]

\[
-H \sum_{k=1}^{q} A_k L \left( f_p, x_m^{p+1}, y_m^p, t, x_0 \right), \quad y_0^p(t,x_0) = 0, \quad m = 1, 2, \ldots \tag{2.7}
\]

Here the above index means that this expression is a polynomials of a correspondent degree. It is easy to see that all the members of the sequence (2.6) satisfy the boundary condition (2.2) for arbitrary \( x_0 \in D_{\beta_1} \).

The next theorem establishes the convergence of the sequence (2.6) and the properties of the limit functions.

**Theorem 1.** Let BVP (2.1), (2.2) satisfy the conditions a)-c). Then:

1. the sequences (2.6) and (2.7) converge to the functions \( x^*(t,x_0) \) and \( y^*(t,x_0) \), respectively, as \( m \to \infty \), uniformly in \( (t,x_0) \in [0,T] \times D_{\beta_1} \):

\[
x^*(t,x_0) = \lim_{m \to \infty} x_m^{p+1}(t,x_0), \quad y^*(t,x_0) = \lim_{m \to \infty} y_m^p(t,x_0),
\]
where \(y^*(t, x_0) = \frac{dx^*(t, x_0)}{dt}\);

(2) the limit function \(x^*(t, x_0)\) satisfies the "perturbed" BVP

\[
\frac{dx}{dt} = f(t, x, \frac{dx}{dt}) + \Delta(x_0),
\]

where

\[
\Delta(x_0) = -\frac{1}{T} \int_0^T f^p(s, x^*(s, x_0), y^*(s, x_0))\, ds + Hd(x_0) -
\]

with the initial value \(x^*(0, x_0) = x_0\);

(3) the following error estimations hold:

\[
|x^*(t, x_0) - x_m^{p+1}(t, x_0)| \leq (\alpha_1(t)E + G) \cdot W_{m-1}^p, \quad (2.10)
\]

\[
|y^*(t, x_0) - y_m^p(t, x_0)| \leq \left(2E + \frac{1}{T}G\right) \cdot W_{m-1}^p, \quad (2.11)
\]

where

\[
W_{m-1}^p = \left[\sum_{k=0}^{m-1} Q^i\right] \cdot L_p + Q^{m-1}(E - Q)^{-1}.
\]

\[
\cdot \left[K_1 \left\{(\frac{T}{2}E + G) M' + T|d(x_0)|\right\} + K_2 \left\{2M + \frac{T}{2}GM' + |d(x_0)|\right\}\right].
\]

Proof. In addition to (2.6), (2.7) let us introduce the sequence of functions.

\[
x_m(t, x_0) = x_0 + \mathcal{L}(f, x_{m-1}, y_{m-1}, t, x_0) + tHd(x_0) -
\]

\[
-tH \sum_{k=1}^q A_k \mathcal{L}(f, x_{m-1}, y_{m-1}, t_k, x_0), \quad x_0(t, x_0) = x_0, \quad m = 1, 2, \ldots, \quad (2.12)
\]

\[
y_m(t, x_0) := \frac{dx_m(t, x_0)}{dt} = \mathcal{L}(f, x_{m-1}, y_{m-1}, t, x_0) + Hd(x_0) -
\]

\[
-tH \sum_{k=1}^q A_k \mathcal{L}(f, x_{m-1}, y_{m-1}, t_k, x_0), \quad y_0(t, x_0) = 0, \quad m = 1, 2, \ldots \quad (2.13)
\]

Also we introduce some notations:

\[
x_m := x_m(t, x_0), \quad x_m^{p+1} := x_m^{p+1}(t, x_0), \quad r_{m+1}(t, x_0) := |x_{m+1}(t, x_0) - x_m(t, x_0)|,
\]

\[
y_m := y_m(t, x_0), \quad y_m^p := y_m^p(t, x_0), \quad \tilde{r}_{m+1}(t, x_0) := |y_{m+1}(t, x_0) - y_m(t, x_0)|.
\]

We note [1] that

\[
|f^p(t, x_m^{p+1}, y_m^p) - f(t, x_m^{p+1}, y_m^p)| \leq L_p, \quad (2.14)
\]

and making use of (2.4) we get

\[
|f^p(t, x_m^{p+1}, y_m^p) - f(t, x_m, y_m)| \leq |f^p(t, x_m^{p+1}, y_m^p) - f(t, x_m^{p+1}, y_m^p)| +
\]

\[
+ |f(t, x_m^{p+1}, y_m^p) - f(t, x_m, y_m)| \leq L_p + K_1|x_m^{p+1} - x_m| + K_2|y_m^p - y_m|.
\]
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Using Lemma 3 of [5] we have that

\[ |\mathcal{L}(f, x, y, t, x_0)| \leq \alpha_1(t)M' \leq \frac{T}{2}M', \]  

(2.16)

\[ |TH\sum_{k=1}^{q} A_k \mathcal{L}(f, x, y, t, x_0)| \leq GM', \]  

(2.17)

\[ \left| \mathcal{L}\left( f^{p}, x^{p+1}_{m}, y^{p}_{m}, t, x_0 \right) - \mathcal{L}\left( f^{p}, x^{p+1}_{m}, y^{p}_{m}, t, x_0 \right) \right| \leq \alpha_1(t)L_p, \]  

(2.18)

\[ \left| TH\sum_{k=1}^{q} A_k \left[ \mathcal{L}\left( f^{p}, x^{p+1}_{m}, y^{p}_{m}, t, x_0 \right) - \mathcal{L}\left( f^{p}, x^{p+1}_{m}, y^{p}_{m}, t, x_0 \right) \right] \right| \leq GL_p. \]  

(2.19)

We have to show that (2.6) is a Cauchy sequence in the space of continuous vector functions. To begin with, we establish for arbitrary \((t, x_0) \in [0, T] \times D_B\), and \(m = 0, 1, 2, \ldots\) that \(x^{p+1}_m(t, x_0) \in D_1\) and \(y^{p+1}_m(t, x_0) \in D_2\) by using (2.16)-(2.19):

\[ |x^{p+1}_1 - x_0| \leq |\mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right)| + |TH\sum_{k=1}^{q} A_k \mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right)| + + T|d(x_0)| \leq |\mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right)| + |TH\sum_{k=1}^{q} A_k \mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right)| + \]

\[ + T|d(x_0)| + \left| TH\sum_{k=1}^{q} A_k \left[ \mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right) - \mathcal{L}\left( f, x_0, y_0, t, x_0 \right) \right] \right| \leq (\alpha_1(t)E + G) (L_p + M') + T|d(x_0)| \leq \beta_1(x_0), \]

\[ |y^{p+1}_1| \leq |\mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right)| + |d(x_0)| + |H\sum_{k=1}^{q} A_k \mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right)| \leq \]

\[ \leq |\mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right) - \mathcal{L}\left( f, x_0, y_0, t, x_0 \right)| + |d(x_0)| + + |H\sum_{k=1}^{q} A_k \left[ \mathcal{L}\left( f^{p}, x^{p+1}_{0}, y^{p}_{0}, t, x_0 \right) - \mathcal{L}\left( f, x_0, y_0, t, x_0 \right) \right]| + \]

\[ + |H\sum_{k=1}^{q} A_k \mathcal{L}\left( f, x_0, y_0, t, x_0 \right)| \leq 2(M + L_p) + G (M' + L_p) + |d(x_0)| \leq \beta_2(x_0). \]

It follows that \(x^{p+1}_m(t, x_0) \in D_1, \ y^{p+1}_m(t, x_0) \in D_2\). By induction in a similar way we can establish that

\[ |x^{p+1}_m - x_0| \leq \beta_1(x_0), \ |y^{p+1}_m| \leq \beta_2(x_0). \]
Now we consider the differences $x_m - x_m^{p+1}$ and $y_m - y_m^p$. For $m = 1$ we have
\[
| x_1 - x_1^{p+1} | \leq | \mathcal{L}(f, x_0, y_0, t, x_0) - \mathcal{L}(f^p, x_0^{p+1}, y_0^p, t, x_0) | + \\
+ TH \sum_{k=1}^q A_k \left| \mathcal{L}(f, x_0, y_0, t, x_0) - \mathcal{L}(f^p, x_0^{p+1}, y_0^p, t, x_0) \right| \leq \alpha_1(t)E + G \|L_p\| \tag{2.20}
\]
\[
| y_1 - y_1^p | \leq | \mathcal{Z}(f, x_0, y_0, t, x_0) - \mathcal{Z}(f^p, x_0^{p+1}, y_0^p, t, x_0) | + \\
+ H \sum_{k=1}^q A_k \left| \mathcal{L}(f, x_0, y_0, t, x_0) - \mathcal{L}(f^p, x_0^{p+1}, y_0^p, t, x_0) \right| \leq \alpha_1(t)E + G \|L_p\| \tag{2.21}
\]
Using (2.14)-(2.21) and Lemma 4 of [5] we get
\[
| x_2 - x_2^{p+1} | \leq | \mathcal{L}(f, x_1, y_1, t, x_0) - \mathcal{L}(f^p, x_1^{p+1}, y_1^p, t, x_0) | + \\
+ TH \sum_{k=1}^q A_k \left| \mathcal{L}(f, x_1, y_1, t, x_0) - \mathcal{L}(f^p, x_1^{p+1}, y_1^p, t, x_0) \right| \leq \alpha_1(t)E + K_1 (\alpha_2(t)E + \alpha_1(t)G) + \alpha_1(t)K_2 (2E + \frac{1}{T}G) \|L_p\| + \\
+ TH \sum_{k=1}^q A_k [\alpha_1(t_k)E + K_1 (\alpha_2(t_k)E + \alpha_1(t_k)G) + \alpha_1(t_k)K_2 (2E + \frac{1}{T}G)] \|L_p\| \leq \alpha_1(t)E + G \|E + K_1 (\frac{1}{T}E + G) + K_2 (2E + \frac{1}{T}G) \|L_p\| \leq \alpha_1(t)E + G \|E + Q \|L_p\| , \tag{2.22}
\]
We can obtain by induction that
\[
| x_m(t, x_0) - x_m^{p+1}(t, x_0) | \leq (\alpha_1(t)E + G) \left[ \sum_{i=1}^{m-1} Q^i \right] \|L_p\| , \tag{2.22}
\]
\[
| y_m(t, x_0) - y_m^p(t, x_0) | \leq (2E + \frac{1}{T}G) \left[ \sum_{i=1}^{m-1} Q^i \right] \|L_p\| . \tag{2.23}
\]
Now we have to estimate \( r_{m+1}(t, x_0) \) and \( \hat{r}_{m+1}(t, x_0) \) for every \( m = 0, 1, 2, \ldots \) by using Lemmas 3 and 4 of [5]:

\[
\begin{align*}
\quad r_1(t, x_0) & \leq |\mathcal{L}(f, x_0, y_0, t, x_0)| + T|d(x_0)| + \\
+ TH \sum_{k=1}^{q} A_k \mathcal{L}(f, x_0, y_0, t_k, x_0) & \leq \left( \frac{T}{2} E + G \right) M' + T|d(x_0)| = \gamma_1(x_0),
\end{align*}
\]

\[
\begin{align*}
\hat{r}_1(t, x_0) & \leq |\mathcal{Z}(f, x_0, y_0, t, x_0)| + |d(x_0)| + \left| H \sum_{k=1}^{q} A_k \mathcal{L}(f, x_0, y_0, t_k, x_0) \right| \\
& \leq 2M + |d(x_0)| + \frac{1}{2}GM' = \gamma_2(x_0),
\end{align*}
\]

\[
\begin{align*}
\quad r_2(t, x_0) & \leq |\mathcal{L}(f, x_1, y_1, t, x_0) - \mathcal{L}(f, x_0, y_0, t, x_0)| + \\
+ TH \sum_{k=1}^{q} A_k \left[ \mathcal{L}(f, x_1, y_1, t_k, x_0) - \mathcal{L}(f, x_0, y_0, t_k, x_0) \right] & \leq 2 \max_{t \in [0, T]} |K_1 r_1(\tau, x_0) + K_2 \hat{r}_1(\tau, x_0)| + \\
+ \left( 1 - \frac{T}{t} \right) \int_{0}^{t} \left[ K_1 r_1(\tau, x_0) + K_2 \hat{r}_1(\tau, x_0) \right] d\tau & < (a_1(t)E + G) \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)],
\end{align*}
\]

\[
\begin{align*}
\hat{r}_2(t, x_0) & \leq |\mathcal{Z}(f, x_1, y_1, t, x_0) - \mathcal{Z}(f, x_0, y_0, t, x_0)| + \\
+ TH \sum_{k=1}^{q} A_k \left[ \mathcal{L}(f, x_1, y_1, t_k, x_0) - \mathcal{L}(f, x_0, y_0, t_k, x_0) \right] & \leq 2 \max_{t \in [0, T]} |K_1 r_1(\tau, x_0) + K_2 \hat{r}_1(\tau, x_0)| + \\
+ \left( 1 - \frac{T}{t} \right) \int_{0}^{t} \left[ K_1 r_1(\tau, x_0) + K_2 \hat{r}_1(\tau, x_0) \right] d\tau & < (a_1(t)E + G) \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)].
\end{align*}
\]

Similarly,
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From (2.24) and assumption c) we obtain the inequality

\[
\begin{align*}
\frac{d}{dt} r_3(t, x_0) & \leq \left\{ \left(1 - \frac{1}{2} \right) \int_0^t \left[ K_1 (\alpha_1(\tau)E + G) + K_2 \left(2E + \frac{1}{T}G\right) \right] d\tau + \\
& + \frac{1}{T} \int_0^T \left[ K_1 (\alpha_1(\tau)E + G) + K_2 \left(2E + \frac{1}{T}G\right) \right] d\tau + \\
& + \frac{1}{T} \sum_{k=1}^q A_k \left(1 - \frac{1}{2} \right) \int_0^{t_k} \left[ K_1 (\alpha_1(\tau)E + G) + K_2 \left(2E + \frac{1}{T}G\right) \right] d\tau + \\
& + \frac{1}{T} \sum_{k=1}^q A_k \left[ K_1 \left(2E + \frac{1}{T}G\right) \right] \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)] \leq \right. \\
\end{align*}
\]

\[
\leq (\alpha_1(t)E + G) \cdot Q \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)],
\]

We can show by induction that for arbitrary \( m = 0, 1, 2, \ldots \)

\[
\begin{align*}
r_{m+1}(t, x_0) & \leq (\alpha_1(t)E + G) \cdot Q^{m-1} \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)] , \quad (2.24) \\
\hat{r}_{m+1}(t, x_0) & \leq \left(2E + \frac{1}{T}G\right) \cdot Q^{m-1} \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)] . \quad (2.25)
\end{align*}
\]

From (2.24) and assumption c) we obtain the inequality

\[
\begin{align*}
|\gamma_{m+j}(t, x_0) - \gamma_m(t, x_0)| & \leq \sum_{i=0}^{j} |\gamma_{m+i+1}(t, x_0) - \gamma_{m+i}(t, x_0)| \leq \\
& \leq \sum_{i=0}^{j} r_{m+i+1}(t, x_0) \leq \sum_{i=0}^{j} (\alpha_1(t)E + G) Q^{m+i-1} [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)] \leq \\
\leq (\alpha_1(t)E + G) \cdot Q^{m-1} (E - Q)^{-1} \cdot [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)]. 
\end{align*}
\]

For the derivatives \( y_m(t, x_0) \) from (2.25) in a similar way we have:

\[
\begin{align*}
|y_{m+j}(t, x_0) - y_m(t, x_0)| & \leq \\
& \leq (2E + \frac{1}{T}G) Q^{m-1} (E - Q)^{-1} [K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)]. \quad (2.27)
\end{align*}
\]
It follows that (2.12) and (2.13) are uniformly convergent sequences:
\[ \lim_{m \to \infty} x_m(t, x_0) = x^*(t, x_0), \lim_{m \to \infty} y_m(t, x_0) = y^*(t, x_0). \]
Taking the limit as \( j \to \infty \) in (2.26) and (2.27) we get the error estimates
\[ \left|x^*(t, x_0) - x_m(t, x_0)\right| \leq (\alpha_1(t)E + G) \cdot Q^{m-1} (E - Q)^{-1} \cdot \left[K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)\right], \]
\[ \left|y^*(t, x_0) - y_m(t, x_0)\right| \leq (2E + \frac{1}{2}G) \cdot Q^{m-1} (E - Q)^{-1} \cdot \left[K_1 \gamma_1(x_0) + K_2 \gamma_2(x_0)\right]. \]
Combining the last two inequalities with (2.22) and (2.23), we get the error estimates (2.10) and (2.11). Passing to the limit as \( m \to \infty \) in (2.6) we obtain that \( x^*(t, x_0) \) satisfies the integral equation
\[ x(t) = x_0 + L(f, x, y, t, x_0) + tHd(x_0) - tH \sum_{k=1}^{q} A_k L(f, x, y, t^k, x_0). \]
While differentiating it, we get that \( x^*(t, x_0) \) is a solution of the perturbed BVP (2.8)-(2.9).

The following statement gives necessary and sufficient conditions for the existence of a solution of the BVP (2.1)-(2.2).

**Theorem 2.** Under the conditions of Theorem 1, the limit function \( x^*(t, x_0) \) is a solution of the BVP (2.1)-(2.2) if and only if \( x_0^* \) verifies the determining equation
\[ \Delta(x_0) = -\frac{1}{T} \int_{0}^{T} f(s, x^*(s, x_0), y^*(s, x_0)) ds + Hd(x_0) + H \sum_{k=1}^{q} A_k L(f, x^*, y^*, t_k, x_0) = 0. \]

**Proof.** The proof can be carried out in the same way as for the corresponding statements from [2] (Theorem 2.3).\( \Box \)

### 3. Sufficient existence conditions

Consider the \( m \)-th approximation to the determining equation (2.28)
\[ \Delta_m^p(x_0) = -\frac{1}{T} \int_{0}^{T} f^p(s, x_m^{p+1}(s, x_0), y_m^p(s, x_0)) ds + Hd(x_0) + H \sum_{k=1}^{q} A_k L(f^p, x_m^{p+1}, y_m^p, t_k, x_0) = 0. \]

**Theorem 3.** Suppose that the conditions of Theorem 1 hold. Furthermore, assume that
\[ \text{d) there exists a closed, convex subset } D' = D'_1 \times D'_2 \subset D_1 \times D_2 \text{ so that for arbitrary } m \text{ and fixed } p \text{ the approximate determining equation (3.1) has only one solution } x_0 = x_{0m}^p \text{ with non-zero topological index;} \]
e) on the boundary $\partial D$ of the subset $D$ the inequality
\[
\inf_{x_0 \in \partial D} |\Delta_{m_0} (x_0)| > \left( E + \frac{1}{T} G \right) W_m
\]
holds.

Then there exists a solution $x = x^*(t)$ to the BVP (2.1)-(2.2) with the initial value $x^*(0) = x_0^0$, where $x_0^0 \in D_1'$. 

Proof. Similarly to (2.15) and making use of (2.10) and (2.11), we get
\[
|f(t,x^*,y^*) - f^p(t,x_{m_0}^{p+1},y_{m_0}^{p})| \leq \left[ K_1 (\alpha_1 (t) E + G) + K_2 \left( 2E + \frac{1}{T} G \right) \right] W_{m-1}^p + L_{p}.
\]

For the deviation of the exact and approximate determining functions we have that
\[
|\Delta(x_0) - \Delta^p_{m_0}(x_0)| \leq \frac{1}{T} \int_0^T |f^p(s,x^*(s,x_0),y^*(s,x_0)) - f^p(s,x_{m_0}^{p+1}(s,x_0),y_{m_0}^p(s,x_0))| + H \sum_{k=1}^q A_k |\mathcal{L}(f^p,x^*,y^*,t_k,x_0)|
\]
\[-L(f^p,x_{m_0}^{p+1},y_{m_0}^p,t_k,x_0)| \leq (E + \frac{1}{T} G) (QW_{m-1}^p + L_{p}) \leq (E + \frac{1}{T} G) W_m^p.
\]

Similarly to Theorem 3.1 of [2], one can prove that the vector fields $\Delta(x_0)$ and $\Delta^p_{m_0}(x_0)$ are homotopic, which completes the proof of Theorem 3. 

\[\square\]

REFERENCES

[1] Samoilenko, A. M. and Ronto, N. I.: Numerical-Analytic Methods of Investigating Solutions of Boundary Value Problems, Naukova Dumka, Kiev, 1985 (in Russian).

[2] Samoilenko, A. M. and Ronto, N. I.: Numerical-Analytic Methods in the Theory of Boundary Value Problems, Naukova Dumka, Kiev, 1992 (in Russian).

[3] Ronto, M. and Samoilenko, A. M.: Numerical-Analytic Methods in the Theory of Boundary Value Problems, World Scientific, Singapore, 2000.

[4] Korol, I. I. and Korol, I. Yu.: Using of polynomial approximation method for solving of multi-point BVPs, Naukovij Visnik Uzhgorods’koho Universitetu, Matematika, 4, (1999), 71-78.

[5] Ronto, M. and Mészáros, J.: Some remarks on the convergence analysis of the numerical-analytic method based upon successive approximations, Ukrainskij Matematicheskij Zhurnal, 48(1), (1996), 90-95.