17. An approach to higher ramification theory

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We use the notation of sections 1 and 10.

17.0. Approach of Hyodo and Fesenko

Let $K$ be an $n$-dimensional local field, $L/K$ a finite abelian extension. Define a filtration on $\text{Gal}(L/K)$ (cf. [H], [F, sect. 4]) by

$$
\text{Gal}(L/K)^i = \mathcal{Y}_{L/K}^{i-1}(U_i K_{n}^{\text{top}}(K) + N_{L/K} K_{n}^{\text{top}}(L)/N_{L/K} K_{n}^{\text{top}}(L)), \quad i \in \mathbb{Z}_n,
$$

where $U_i K_{n}^{\text{top}}(K) = \{U_i\} \cdot K_{n-1}^{\text{top}}(K)$, $U_i = 1 + P_K(i)$,

$$
\mathcal{Y}_{L/K}^{-1}: K_{n}^{\text{top}}(K) / N_{L/K} K_{n}^{\text{top}}(L) \rightarrow \text{Gal}(L/K)
$$

is the reciprocity map.

Then for a subextension $M/K$ of $L/K$

$$
\text{Gal}(M/K)^1 = \text{Gal}(L/K)^1 \text{ Gal}(L/M)/\text{ Gal}(L/M)
$$

which is a higher dimensional analogue of Herbrand’s theorem. However, if one defines a generalization of the Hasse–Herbrand function and lower ramification filtration, then for $n > 1$ the lower filtration on a subgroup does not coincide with the induced filtration in general.

Below we shall give another construction of the ramification filtration of $L/K$ in the two-dimensional case; details can be found in [Z], see also [KZ]. This construction can be considered as a development of an approach by K. Kato and T. Saito in [KS].

**Definition.** Let $K$ be a complete discrete valuation field with residue field $k_K$ of characteristic $p$. A finite extension $L/K$ is called *ferociously ramified* if $|L:K| = |k_L:k_K|_{\text{ins}}$.
In addition to the nice ramification theory for totally ramified extensions, there is a nice ramification theory for ferociously ramified extensions $L/K$ such that $k_L/k_K$ is generated by one element; the reason is that in both cases the ring extension $\mathcal{O}_L/\mathcal{O}_K$ is monogenic, i.e., generated by one element, see section 18.

17.1. Almost constant extensions

Everywhere below $K$ is a complete discrete valuation field with residue field $k_K$ of characteristic $p$ such that $|k_K : k_K^p| = p$. For instance, $K$ can be a two-dimensional local field, or $K = \mathbb{F}_q((X_1))((X_2))$ or the quotient field of the completion of $\mathbb{Z}_p[T]_{(p)}$ with respect to the $p$-adic topology.

**Definition.** For the field $K$ define a base (sub)field $B$ as

- $B = \mathbb{Q}_p \subset K$ if $\text{char}(K) = 0$,
- $B = \mathbb{F}_p(\rho) \subset K$ if $\text{char}(K) = p$, where $\rho$ is an element of $K$ with $v_K(\rho) > 0$.

Denote by $k_0$ the completion of $B(\mathbb{R}_K)$ inside $K$. Put $k = k_0^{\text{alg}} \cap K$.

The subfield $k$ is a maximal complete subfield of $K$ with perfect residue field. It is called a constant subfield of $K$. A constant subfield is defined canonically if $\text{char}(K) = 0$. Until the end of section 17 we assume that $B$ (and, therefore, $k$) is fixed.

By $v$ we denote the valuation $K^{\text{alg}}^* \to \mathbb{Q}$ normalized so that $v(B^*) = \mathbb{Z}$.

**Example.** If $K = F\{\{T\}\}$ where $F$ is a mixed characteristic complete discrete valuation field with perfect residue field, then $k = F$.

**Definition.** An extension $L/K$ is said to be constant if there is an algebraic extension $l/k$ such that $L = Kl$.

An extension $L/K$ is said to be almost constant if $L \subset L_1L_2$ for a constant extension $L_1/K$ and an unramified extension $L_2/K$.

A field $K$ is said to be standard, if $e(K|k) = 1$, and almost standard, if some finite unramified extension of $K$ is a standard field.

**Epp’s theorem on elimination of wild ramification.** ([E], [KZ]) Let $L$ be a finite extension of $K$. Then there is a finite extension $k'$ of a constant subfield $k$ of $K$ such that $e(Lk'|Kk') = 1$.

**Corollary.** There exists a finite constant extension of $K$ which is a standard field.

**Proof.** See the proof of the Classification Theorem in 1.1.

**Lemma.** The class of constant (almost constant) extensions is closed with respect to taking compositums and subextensions. If $L/K$ and $M/L$ are almost constant then $M/K$ is almost constant as well.
Definition. Denote by $L_c$ the maximal almost constant subextension of $K$ in $L$.

Properties.
(1) Every tamely ramified extension is almost constant. In other words, the (first) ramification subfield in $L/K$ is a subfield of $L_c$.
(2) If $L/K$ is normal then $L_c/K$ is normal.
(3) There is an unramified extension $L'_0$ of $L_0$ such that $L_cL'_0/L_0$ is a constant extension.
(4) There is a constant extension $L'_c/L_c$ such that $LL'_c/L'_c$ is ferociously ramified and $L'_c \cap L = L_c$. This follows immediately from Epp’s theorem.

The principal idea of the proposed approach to ramification theory is to split $L/K$ into a tower of three extensions: $L_0/K$, $L_c/L_0$, $L/L_c$, where $L_0$ is the inertia subfield in $L/K$. The ramification filtration for $\text{Gal}(L_c/L_0)$ reflects that for the corresponding extensions of constants subfields. Next, to construct the ramification filtration for $\text{Gal}(L/L_c)$, one reduces to the case of ferociously ramified extensions by means of Epp’s theorem. (In the case of higher local fields one can also construct a filtration on $\text{Gal}(L_0/K)$ by lifting that for the first residue fields.)

Now we give precise definitions.

17.2. Lower and upper ramification filtrations

Keep the assumption of the previous subsection. Put
$$\mathcal{A} = \{-1, 0\} \cup \{(c, s) : 0 < s \in \mathbb{Z}\} \cup \{(i, r) : 0 < r \in \mathbb{Q}\}.$$  
This set is linearly ordered as follows:

$-1 < 0 < (c, i) < (i, j)$ for any $i, j$;
$(c, i) < (i, j)$ for any $i < j$;
$(i, i) < (i, j)$ for any $i < j$.

Definition. Let $G = \text{Gal}(L/K)$. For any $\alpha \in \mathcal{A}$ we define a subgroup $G_\alpha$ in $G$.
Put $G_{-1} = G$, and denote by $G_0$ the inertia subgroup in $G$, i.e.,
$$G_0 = \{g \in G : v(g(a) - a) > 0 \text{ for all } a \in \mathcal{O}_L\}.$$  
Let $L_c/K$ be constant, and let it contain no unramified subextensions. Then define
$$G_{c,i} = \text{pr}^{-1}(\text{Gal}(l/k)_i)$$
where $l$ and $k$ are the constant subfields in $L$ and $K$ respectively.

pr: $\text{Gal}(L/K) \to \text{Gal}(l/k) = \text{Gal}(l/k)_0$
is the natural projection and \( \text{Gal}(l/k) \) are the classical ramification subgroups. In the general case take an unramified extension \( K'/K \) such that \( K'L/K' \) is constant and contains no unramified subextensions, and put \( G_{c,i} = \text{Gal}(K'L/K')_{c,i} \).

Finally, define \( G_{i,i}, i > 0 \). Assume that \( L_c \) is standard and \( L/L_c \) is ferociously ramified. Let \( t \in \mathcal{O}_L, t \notin k_p \). Define
\[
G_{i,i} = \{ g \in G : v(g(t) - t) \geq i \}
\]
for all \( i > 0 \).

In the general case choose a finite extension \( l'/l \) such that \( l'L_c \) is standard and \( e(l|l'L_c) = 1 \). Then it is clear that \( \text{Gal}(l'/l|l'L_c) = \text{Gal}(L/L_c) \), and \( l'L/l'L_c \) is ferociously ramified. Define
\[
G_{i,i} = \text{Gal}(l'/l|l'L_c)_{i,i}
\]
for all \( i > 0 \).

**Proposition.** For a finite Galois extension \( L/K \) the lower filtration \( \{ \text{Gal}(L/K)_{\alpha} \}_{\alpha \in \mathcal{A}} \) is well defined.

**Definition.** Define a generalization \( h_{L/K} : \mathcal{A} \to \mathcal{A} \) of the Hasse–Herbrand function. First, we define
\[
\Phi_{L/K} : \mathcal{A} \to \mathcal{A}
\]
as follows:
\[
\Phi_{L/K}(\alpha) = \alpha \quad \text{for } \alpha = -1, 0;
\]
\[
\Phi_{L/K}((c, i)) = \left( c, \frac{1}{e(L/K)} \int_0^i | \text{Gal}(L_c/K)_{c,t}| \, dt \right) \quad \text{for all } i > 0;
\]
\[
\Phi_{L/K}((i, i)) = \left( i, \int_0^i | \text{Gal}(L/K)_{i,t}| \, dt \right) \quad \text{for all } i > 0.
\]

It is easy to see that \( \Phi_{L/K} \) is bijective and increasing, and we introduce
\[
h_{L/K} = \Psi_{L/K} = \Phi_{L/K}^{-1}.
\]

Define the upper filtration \( \text{Gal}(L/K)^{\alpha} = \text{Gal}(L/K)_{h_{L/K}(\alpha)} \).

All standard formulas for intermediate extensions take place; in particular, for a normal subgroup \( H \) in \( G \) we have \( H_\alpha = H \cap G_\alpha \) and \( (G/H)^{\alpha} = G^\alpha H/H \). The latter relation enables one to introduce the upper filtration for an infinite Galois extension as well.

**Remark.** The filtrations do depend on the choice of a constant subfield (in characteristic \( p \)).

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Example. Let $K = \mathbb{F}_p((t))((\pi))$. Choose $k = B = \mathbb{F}_p((\pi))$ as a constant subfield. Let $L = K(b)$, $b^p - b = a \in K$. Then

- if $a = \pi^{-i}$, $i$ prime to $p$, then the ramification break of $\text{Gal}(L/K)$ is $(c, i)$;
- if $a = \pi^{-p^it}$, $i$ prime to $p$, then the ramification break of $\text{Gal}(L/K)$ is $(i, i)$;
- if $a = \pi^{-i}t$, $i$ prime to $p$, then the ramification break of $\text{Gal}(L/K)$ is $(i, i/p)$;
- if $a = \pi^{-i}p^t$, $i$ prime to $p$, then the ramification break of $\text{Gal}(L/K)$ is $(i, i/p^2)$.

Remark. A dual filtration on $K/\wp(K)$ is computed in the final version of [Z], see also [KZ].

17.3. Refinement for a two-dimensional local field

Let $K$ be a two-dimensional local field with $\text{char}(k_K) = p$, and let $k$ be the constant subfield of $K$. Denote by

$$\nu = (v_1, v_2): (K^{\text{alg}})^* \to \mathbb{Q} \times \mathbb{Q}$$

the extension of the rank 2 valuation of $K$, which is normalized so that:

- $v_2(a) = v(a)$ for all $a \in K^*$,
- $v_1(u) = w(\overline{u})$ for all $u \in K^{\text{alg}}$, where $w$ is a non-normalized extension of $v_{k_K}$ on $k_{k_K}^{\text{alg}}$, and $\overline{u}$ is the residue of $u$,
- $\nu(c) = (0, e(k|B)^{-1}v_{k_K}(c))$ for all $c \in k$.

It can be easily shown that $\nu$ is uniquely determined by these conditions, and the value group of $\nu|_{k^*}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Next, we introduce the index set

$$\mathcal{A}_2 = \mathcal{A} \cup \mathbb{Q}_+^2 = \mathcal{A} \cup \{(i_1, i_2) : i_1, i_2 \in \mathbb{Q}, i_2 > 0\}$$

and extend the ordering of $\mathcal{A}$ onto $\mathcal{A}_2$ assuming

$$(i, i_2) < (i_1, i_2) < (i'_1, i_2) < (i, i'_2)$$

for all $i_2 < i'_2$, $i_1 < i'_1$.

Now we can define $G_{i_1, i_2}$, where $G$ is the Galois group of a given finite Galois extension $L/K$. Assume first that $L_c$ is standard and $L/L_c$ is ferociously ramified. Let $t \in \mathcal{O}_L$, $\tilde{t} \notin k_L^{\text{reg}}$ (e.g., a first local parameter of $L$). We define

$$G_{i_1, i_2} = \left\{ g \in G : \nu(t^{-1}g(t) - 1) \geq (i_1, i_2) \right\}$$

for $i_1, i_2 \in \mathbb{Q}$, $i_2 > 0$. In the general case we choose $l'/l$ ($l$ is the constant subfield of both $L$ and $L_c$) such that $l'L_c$ is standard and $l'L/l'L_c$ is ferociously ramified and put

$$G_{i_1, i_2} = \text{Gal}(l'L/l'L_c)^{i_1, i_2}.$$
In a similar way to 17.2, one constructs the Hasse–Herbrand functions
\( \Phi_{2, L/K} : A_2 \to A_2 \) and \( \Psi_{2, L/K} = \Phi_{2, L/K}^{-1} \) which extend \( \Phi \) and \( \Psi \) respectively. Namely,
\[
\Phi_{2, L/K}(i_1, i_2) = \int_{(0,0)}^{(i_1, i_2)} |\text{Gal}(L/K)|_t |dt.
\]

These functions have usual properties of the Hasse–Herbrand functions \( \varphi \) and \( h = \psi \), and one can introduce an \( A_2 \)-indexed upper filtration on any finite or infinite Galois group \( G \).

### 17.4. Filtration on \( K^{\text{top}}(K) \)

In the case of a two-dimensional local field \( K \) the upper ramification filtration for \( K_{ab}^\text{op}/K \) determines a compatible filtration on \( K_{2}^{\text{top}}(K) \). In the case where \( \text{char}(K) = p \) this filtration has an explicit description given below.

From now on, let \( K \) be a two-dimensional local field of prime characteristic \( p \) over a quasi-finite field, and \( k \) the constant subfield of \( K \). Introduce \( v \) as in 17.3. Let \( \pi_k \) be a prime of \( k \).

For all \( \alpha \in \mathbb{Q}_2^+ \) introduce subgroups
\[
Q_\alpha = \{ \pi_k, u : u \in K, v(u - 1) \geq \alpha \} \subset VK_2^{\text{top}}(K);
\]
\[
Q^{(n)}_\alpha = \{ a \in K_2^{\text{top}}(K) : p^n a \in Q_\alpha \};
\]
\[
S_\alpha = \text{Cl} \bigcup_{n \geq 0} Q^{(n)}_\alpha.
\]

For a subgroup \( A \), \( \text{Cl} A \) denotes the intersection of all open subgroups containing \( A \).

The subgroups \( S_\alpha \) constitute the heart of the ramification filtration on \( K_2^{\text{top}}(K) \). Their most important property is that they have nice behaviour in unramified, constant and ferociously ramified extensions.

**Proposition 1.** Suppose that \( K \) satisfies the following property.

(*) The extension of constant subfields in any finite unramified extension of \( K \) is also unramified.

Let \( L/K \) be either an unramified or a constant totally ramified extension, \( \alpha \in \mathbb{Q}_2^+ \). Then we have \( N_{L/K} S_\alpha, L = S_\alpha, K \).

**Proposition 2.** Let \( K \) be standard, \( L/K \) a cyclic ferociously ramified extension of degree \( p \) with the ramification jump \( h \) in lower numbering, \( \alpha \in \mathbb{Q}_2^+ \). Then:

1. \( N_{L/K} S_\alpha, L = S_{\alpha + (p - 1)h}, K \), if \( \alpha > h \);
2. \( N_{L/K} S_\alpha, L \) is a subgroup in \( S_{p\alpha, K} \) of index \( p \), if \( \alpha \leq h \).
Now we have ingredients to define a decreasing filtration $\{\text{fil}_\alpha K_2^{\text{top}}(K)\}_{\alpha \in \mathcal{A}_2}$ on $K_2^{\text{top}}(K)$. Assume first that $\overline{K}$ satisfies the condition (*). It follows from [KZ, Th. 3.4.3] that for some purely inseparable constant extension $K'/K$ the field $K'$ is almost standard. Since $K'$ satisfies (*), and is almost standard, it is in fact standard.

Denote
$$\text{fil}_{\alpha_1, \alpha_2} K_2^{\text{top}}(K) = S_{\alpha_1, \alpha_2};$$
$$\text{fil}_{\alpha_2} K_2^{\text{top}}(K) = \text{Cl} \bigcup_{\alpha_1 \in \mathbb{Q}} \text{fil}_{\alpha_1, \alpha_2} K_2^{\text{top}}(K) \text{ for } \alpha_2 \in \mathbb{Q}_+;$$
$$T_K = \text{Cl} \bigcup_{\alpha \in \mathbb{Q}_+^2} \text{fil}_\alpha K_2^{\text{top}}(K);$$
$$\text{fil}_{\epsilon, i} K_2^{\text{top}}(K) = T_K + \{\{t, u\} : u \in k, v_k(u - 1) \geq i\} \text{ for all } i \in \mathbb{Q}_+,$$
if $K = k\{\{t\}\}$ is standard;
$$\text{fil}_{\epsilon, i} K_2^{\text{top}}(K) = N_{K'/K} \text{fil}_{\epsilon, i} K_2^{\text{top}}(K'), \text{ where } K'/K \text{ is as above;}$$
$$\text{fil}_0 K_2^{\text{top}}(K) = U(1)K_2^{\text{top}}(K) + \{t, \mathfrak{r}_K\}, \text{ where } U(1)K_2^{\text{top}}(K) = \{1 + P_K(1), K^*\},$$
$t$ is the first local parameter;
$$\text{fil}_{-1} K_2^{\text{top}}(K) = K_2^{\text{top}}(K).$$

It is easy to see that for some unramified extension $\overline{K}/K$ the field $\overline{K}$ satisfies the condition (*), and we define $\text{fil}_\alpha K_2^{\text{top}}(K)$ as $N_{\overline{K}/K} \text{fil}_\alpha K_2^{\text{top}}(\overline{K})$ for all $\alpha \geq 0$, and $\text{fil}_{-1} K_2^{\text{top}}(K)$ as $K_2^{\text{top}}(K)$. It can be shown that the filtration $\{\text{fil}_\alpha K_2^{\text{top}}(K)\}_{\alpha \in \mathcal{A}_2}$ is well defined.

**Theorem 1.** Let $L/K$ be a finite abelian extension, $\alpha \in \mathcal{A}_2$. Then $N_{L/K} \text{fil}_\alpha K_2^{\text{top}}(L)$ is a subgroup in $\text{fil}_{\Phi_{2,K}(\alpha)} K_2^{\text{top}}(K)$ of index $|\text{Gal}(L/K)\alpha|$. Furthermore,
$$\text{fil}_{\Phi_{2,K}(\alpha)} K_2^{\text{top}}(K) \cap N_{L/K} K_2^{\text{top}}(L) = N_{L/K} \text{fil}_\alpha K_2^{\text{top}}(L).$$

**Theorem 2.** Let $L/K$ be a finite abelian extension, and let
$$\Upsilon_{L/K}^{-1} : K_2^{\text{top}}(K)/N_{L/K} K_2^{\text{top}}(L) \to \text{Gal}(L/K)$$
be the reciprocity map. Then
$$\Upsilon_{L/K}^{-1}(\text{fil}_\alpha K_2^{\text{top}}(K) \mod N_{L/K} K_2^{\text{top}}(L)) = \text{Gal}(L/K)^\alpha$$
for any $\alpha \in \mathcal{A}_2$.

**Remarks.** 1. The ramification filtration, constructed in 17.2, does not give information about the classical ramification invariants in general. Therefore, this construction can be considered only as a provisional one.

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2. The filtration on $K_2^{\text{top}}(K)$ constructed in 17.4 behaves with respect to the norm map much better than the usual filtration $\{U_i K_2^{\text{top}}(K)\}_{i \in \mathbb{Z}_+}$. We hope that this filtration can be useful in the study of the structure of $K^{\text{top}}$-groups.

3. In the mixed characteristic case the description of “ramification” filtration on $K_2^{\text{top}}(K)$ is not very nice. However, it would be interesting to try to modify the ramification filtration on $\text{Gal}(L/K)$ in order to get the filtration on $K_2^{\text{top}}(K)$ similar to that described in 17.4.

4. It would be interesting to compute ramification of the extensions constructed in sections 13 and 14.

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