ON THE RANKIN-SELBERG $L$-FACTORS FOR $\text{SO}_5 \times \text{GL}_2$

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Abstract. Let $\pi$ and $\tau$ be a smooth generic representation of $\text{SO}_5$ and $\text{GL}_2$ respectively over a non-archimedean local field. Assume that $\pi$ is irreducible and $\tau$ is irreducible or induced of Langlands’ type. We show that the $L$- and $\epsilon$-factors attached to $\pi \times \tau$ defined by the Rankin-Selberg integrals and the associated Weil-Deligne representation coincide. Similar compatibility results are also obtained for the local factors defined through the Novodvorsky’s local zeta integrals that attached to generic representations of $\text{GSp}_4 \times \text{GL}_2$.

1. Introduction

Let $F$ be a finite extension over $\mathbb{Q}_p$ and $\psi$ be a non-trivial additive character of $F$. Let $\pi$ and $\tau$ be an irreducible smooth generic representation of split $\text{SO}_{2n+1}(F)$ and $\text{GL}_r(F)$ respectively. We denote by $\phi_\pi$ the $L$-parameter attached to $\pi$ defined by Jiang-Soudry ([JS03], [JS04]) and $\phi_\tau$ the $L$-parameter attached to $\tau$ under the local Langlands correspondence for $\text{GL}_r$ ([HT01], [Hen00], [Sch13]). Then to $\pi \times \tau$ and $\psi$, one can define the (local) $L$, $\epsilon$- and $\gamma$-factors through the following approaches:

- (WD) the associated Weil-Deligne representation ([Tat79]);
- (LS) the Langlands-Shahidi method ([Sha90]);
- (RS) the Rankin-Selberg integrals ([Sou93]).

While it is natural to ask whether these approaches define the same local factors, the equalities among these local factors are actually essential for studying analytic properties of Langlands’ automorphic $L$-functions and have applications to the theory of automorphic representations. For instance, in a different setting, Yamana ([Yam14]) proved that the $L$-factors defined in [GPSR87], [PSR86] and [LR11] agree. Consequently, he was able to show that non-vanishing of global theta liftings is characterized in terms of the analytic properties of the complete $L$-functions and the occurrence in the local theta correspondence. As another example, Ikeda ([Ike92]) proved that the $L$-factors defined in [PSR87] agree with the one defined by the associated Weil-Deligne representations, provided that the representations are local components of global representations. As a result, he was able to determine the poles of the triple product $L$-functions and supply some evidents that consistent with “the Langlands’ philosophy”.

In the literature, Jiang-Soudry proved that the local factors attached to $\pi \times \tau$ and $\psi$ defined by (WD) and (LS) coincide in [JS04]. On the other hand, the $\gamma$-factors attached to $\pi \times \tau$ and $\psi$ defined by (LS) and (RS) also essentially coincide according to [Sou00] and [Kap15]. Therefore, to settle the question arising in the first paragraph, it remains to show that the $L$-factors attached to $\pi \times \tau$ given by (RS) agree with the one given by (WD) or (LS). However, this question is usually more involved, for the $L$-factors defined by (LS) are designed to have right properties with respect to the local Langlands correspondence, whereas the $L$-factors defined by (RS) are designed to control poles the local Rankin-Selberg integrals. While both approaches give essentially the same $\gamma$-factors, it is not at all clear that they give the same $L$-factors. In this note, we resolve this question when $n = r = 2$. In fact, our results allow $\tau$ to be reducible, and in addition, we also obtain similar results for the closely related local integrals attached to generic representations of $\text{GSp}_4 \times \text{GL}_2$ introduced by Novodvorsky in [Nov79].

1.1. Main Results. Let us describe our main results in this subsection.
1.1. Local factors via (WD) and (RS). Using \( \phi_\pi, \phi_\tau, \psi \) and the natural maps
\[
L^2 SO_{2n+1}(C) \times GL_r(C) = S^2 P_{2n}(C) \times GL_r(C) \hookrightarrow GL_{2n}(C) \times GL_r(C) \xrightarrow{\otimes} GL_{2n+1}(C)
\]
one can define the \( L \)-factor \( L(s, \phi_\tau \otimes \phi_\tau) \) and \( \epsilon \)-factor \( \epsilon(s, \phi_\tau \otimes \phi_\tau, \psi) \) as in [Tat79]. On the other side, one can define the (local) Rankin-Selberg \( L \)-factor \( L(s, \pi \times \tau) \) as a “g.c.d.” of poles of local Rankin-Selberg integrals (Sou93) for good sections (cf. §2.4.2). Granted the \( L \) - and \( \gamma \)-factors, the Rankin-Selberg \( \epsilon \)-factors \( \epsilon(s, \pi \times \tau, \psi) \) are defined. We indicate that these analytic local factors can be defined even when \( \tau \) is reducible.

1.1.2. Induced of Langlands’ type. As pointed out, we also consider reducible representations. To describe them, let \( \tau \) be a smooth representation of \( GL_2(F) \). We denote by \( \bar{\tau} \) the unique non-zero irreducible quotient of \( \tau \) (if exists). Certainly, \( \tau \cong \bar{\tau} \) when \( \tau \) is irreducible. On the other hand, if \( \tau \) is induced of Langlands’ type, namely, \( \tau \) is a normalized induced representation of \( GL_2(F) \) inducing from characters (not necessary unitary) \( \chi_1 \) and \( \chi_2 \) of \( F^\times \) with \( |\chi_1\chi_2^{-1}(\varpi)| \leq 1 \), then \( \bar{\tau} \) is also defined. Here \( \varpi \) is a fixed uniformizer of \( F \). Moreover, \( \tau \) is reducible precise when \( \chi_1\chi_2^{-1}(\varpi) = 1 \), where \( |\cdot|_F \) is the absolute value on \( F \) normalized so that \( |\varpi|_F = q^{-1} \) with \( q \) the size of the residue field of \( F \). In this case, \( \bar{\tau} \) is one-dimensional with \( \chi \) the character of \( F^\times \) given by \( \chi = \chi_1 \cdot \varpi^\frac{1}{2} \), and we regard it as a character of \( GL_2(F) \) via composing with the determinant map. We indicate that every irreducible induced representations of \( GL_2(F) \) can be arranged into induced of Langland’s type, and every representation that is induced of Langlands’ type admits a unique Whittaker model (JS83).

With these knowledge, our first result can be stated as follows.

**Theorem 1.1.** Let \( \pi \) be an irreducible smooth generic representation of \( SO_5(F) \) and \( \tau \) be a smooth generic representation of \( GL_2(F) \) that is irreducible or induced of Langlands’ type. Then we have
\[
L(s, \pi \times \tau) = L(s, \phi_\tau \otimes \phi_\tau) \quad \text{and} \quad \epsilon(s, \pi \times \tau, \psi) = \epsilon(s, \phi_\tau \otimes \phi_\tau, \psi).
\]

**Remark 1.2.** When \( n \leq 2 \) and \( r = 1 \), similar identities can be deduced from the results in the literature. Indeed, when \( n = 1 \), \( SO_3 \cong PGL_2 \) and the Rankin-Selberg integrals reduce to the zeta integrals studied by Jacquet-Langlands in [JL70]. In this case, the assertion is well-known. When \( n = 2 \), on the other hand, \( SO_5 \cong PGSp_4 \) and the integrals become the zeta integrals (for \( GSp_4 \)) considered by Novodvorsky in [Nov79]. In this case, the assertion follows from the results in [TB00], [Sou00], [JS04], [RS07], [Kap15] and [Tra19].

1.1.3. Local factors via Novodvorsky’s local integrals. As noticed by Soudry in [Sou93] Section 0], when \( n = r = 2 \), the Rankin-Selberg integrals are essentially equal to the (local) zeta integrals for \( GSp_4 \times GL_2 \) introduced by Novodvorsky in [Nov79]. These zeta integrals were subsequently studied by Soudry in [Sou84] (see also [LPSZ22] and [Loe20]). In particular, one can define the \( L \)-factors \( L^{Nov}(s, \pi \times \tau) \) and the \( \epsilon \)-factors \( \epsilon^{Nov}(s, \pi \times \tau, \psi) \) by using these zeta integrals (cf. §3.2.3). Here \( \pi \) (resp. \( \tau \)) is a smooth generic representation of \( GSp_4(F) \) (resp. \( GL_2(F) \)). In this note, we explicate the relation between these two integrals and obtain the following result.

**Theorem 1.3.** Let \( \pi \) be an irreducible smooth generic representation of \( GSp_4(F) \) and \( \tau \) be a smooth generic representation of \( GL_2(F) \) that is irreducible or induced of Langlands’ type. Suppose that the central character of \( \pi \) is a square. Then we have
\[
L^{Nov}(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\pi) \quad \text{and} \quad \epsilon^{Nov}(s, \pi \times \tau, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\pi, \psi).
\]

**Here \( \phi_\pi \) is the \( L \)-parameter of \( \pi \) under the local Langlands correspondence for \( GSp_4 \) established in [GT11a].**

**Remark 1.4.** When \( \pi \) is a theta lift from a generic representation of split \( GSO_4 \), same identities was obtained by Soudry in [Sou84] under a mild assumption. On the other hand, when \( \tau \) is non-supercuspidal, the equality between \( L \)-factors was proved in recent results of Loeffler-Pilloni-Skinner-Zerbes, and Loeffler in [LPSZ22] and [Loe20], respectively. We point out that in these results, there are no restriction on the central character. Here we supply some complementary results under the assumption on the central character.

1.2. Idea of the proof. In an analogous setting, Kaplan proved, under an assumption \(^1\) on intertwining operators, that the \( L \) - and \( \epsilon \)-factors attached to tempered irreducible smooth generic representations of (quasi-)split \( SO_{2n}(F) \) and \( GL_n(F) \) defined by (LS) and (RS) are the same ([Kap13]). As pointed out in op. cit.,

\(^1\)This assumption can be lifted by results in a recent preprint [Luo21]
the technique and results readily adapted to our settings, due to the similar nature and technical closeness of the constructions. In this note; however, we take a different approach\(^2\) which takes the advantage of the accidental isomorphisms $\text{PGSp}_4 \cong \text{SO}_5$ and $\mathbb{G}_m \backslash G(\text{SL}_2 \times \text{SL}_2) \cong \text{SO}_4$. These allow us to compare two local integrals and then to establish the following key identity

$$L^\text{Nov}(s, \pi \times \tau) = L(s, \pi \times \tau).$$

Once this identity is established, we can transfer the relevant results from one to another. In particular, by results in \cite{LPSZ21} and \cite{Loc20}, our task is to prove Theorem \ref{thm:main} when $\tau$ is supercuspidal. This is settled in this note.

### 1.3. Contents of the note.

We conclude this introduction with a brief description of the contents of this note. In \S\S \ref{sec:3} we will review the local Rankin-Selberg integrals for $\text{SO}_{2n+1} \times \text{GL}_r$ with $1 \leq r \leq n$ and defined the associated local factors. For this, we first introduce the good sections in local factors will be defined in \S\S \ref{sec:2}. In \S\S \ref{sec:2.4} we interpret the Rankin-Selberg integrals as formal Laurent series following the idea of Jacquet-Shalika-Piatetski-Shapiro and Kaplan. This will be used in the proof later. In \S\S \ref{sec:4} we recall the accidental isomorphisms and fit them into a commutative diagram, which is crucial when comparing two local integrals. Novodvorsky’s local integrals and the associated local factors will be introduced in \S\S \ref{sec:3.2} and the comparison between Rankin-Selberg and Novodvorsky’s local integrals will be given in \S\S \ref{sec:3.3}. Finally, the proof of Theorem \ref{thm:main} (when $\tau$ is supercuspidal) will appear in \S\S \ref{sec:4}.

### 1.4. Conventions.

Let $G$ be an $\ell$-group in the sense of \cite[Section 1]{BZ76}. In this note, by a representation of $G$ we mean a smooth representation with coefficients in $\mathbb{C}$. Let $\pi$ be a representation of $G$. We usually denote by $\mathcal{V}_\pi$ the underlying (abstract) $\mathbb{C}$-linear space and by $\omega_\pi$ the central character (if exists). When $G = \text{GL}_r(F)$ or $\text{GSp}_4(F)$ and $\mu$ is a character of $F^\times$, that is, a one-dimensional representation of $F^\times$, we often regard $\mu$ as a character of $G$ via composing with the determinant map or the similitude character accordingly. Then $\pi \otimes \mu$ will be a representation of $G$ with $\mathcal{V}_{\pi \otimes \mu} = \mathcal{V}_\pi$ and the action given by $\mu(g)\pi(g)v$ for $g \in G$ and $v \in \mathcal{V}_\pi$. If $\mu$ is a character of $F^\times$, we denote by $L(s, \mu)$, $\epsilon(s, \mu, \psi)$ and $\gamma(s, \mu, \psi)$ the $L$-, $\epsilon$- and $\gamma$-factor attached to $\mu$ and $\psi$ that appeared in the Tate thesis (\cite[Section 3.1]{Bum98}).

### 2. Rankin-Selberg Integrals and Local Factors

In this section, we review the local Rankin-Selberg integrals for $\text{SO}_{2n+1} \times \text{GL}_r$ with $1 \leq r \leq n$ developed by Soudry in \cite{Soudry93} (see also \cite{Kaplan15}) and define the associated local factors.

#### 2.1. Special orthogonal groups and their subgroups.

##### 2.1.1. Special orthogonal groups.

Define an element $j_m \in \text{GL}_m(F)$ inductively by

$$j_1 = 1 \quad \text{and} \quad j_m = \begin{pmatrix} 0 & 1 \\ J_{m-1} & 0 \end{pmatrix}.$$  

Put $S_m = j_m$ if $m$ is even, and

$$S_m = \begin{pmatrix} J_{m/2} & 2 \\ J_{m/2} & 2 \end{pmatrix}$$

if $m = 2n + 1$ is odd. Let $\text{SO}_m(F) \subset \text{SL}_m(F)$ be the special orthogonal group defined by

$$\text{SO}_m(F) = \{ g \in \text{SL}_m(F) \mid g S_m g = S_m \}$$

where $^t g$ stands for the transpose of $g$. We may regard $\text{SO}_m$ as an algebraic group defined over $\mathfrak{c}$, the valuation ring of $F$. Throughout this note, we assume $1 \leq r \leq n$. Then there is a natural embedding $\varphi : \text{SO}_{2r}(F) \hookrightarrow \text{SO}_{2n+1}(F)$ given by

$$\text{SO}_{2r}(F) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SO}_{2n+1}(F)$$

with $a, b, c, d \in \text{Mat}_{r \times r}(F)$.

\(^2\)Actually, our original approach follows \cite{Kaplan13}, but we can only obtain partial results.
2.1.2. Subgroups and non-degenerated characters. Let $U_n \subset \text{SO}_{2n+1}$, $V_r \subset \text{SO}_{2r}$ and $Z_r \subset \text{GL}_r$ be the maximal unipotent subgroups whose $F$-rational points consist upper unitriangular matrices. Define the non-degenerate characters $\psi_{U_n}: U_n(F) \to \mathbb{C}^*$ and $\tilde{\psi}_{Z_r}: Z_r(F) \to \mathbb{C}^*$ by

$$\psi_{U_n}(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n} + 2^{-1}u_{n,n+1})$$

and

$$\tilde{\psi}_{Z_r}(z) = \tilde{\psi}(z_{12} + z_{23} + \cdots + z_{r-1,r})$$

for $u = (u_{ij}) \in U_n(F)$ and $z = (z_{ij}) \in Z_r(F)$, where $\tilde{\psi} = \psi^{-1}$. Let $Q_r \subset \text{SO}_{2r}$ be the Siegel parabolic subgroup with the Levi decomposition $M_r \rtimes N_r$ with

$$M_r(F) = \left\{ m_r(a) = \begin{pmatrix} a & * \\ 0 & a^* \end{pmatrix} \mid a \in \text{GL}_r(F) \right\} \subset \text{GL}_r(F)$$

and

$$N_r(F) = \left\{ n_r(b) = \begin{pmatrix} I_r & b \\ 0 & I_r \end{pmatrix} \mid b \in \text{Mat}_{r \times r}(F) \right\}.$$ 

Here for $a \in \text{GL}_r(F)$, we denote $a^* = j_r^{-1}a^{-1}j_r$.

2.2. Induced representations. We introduce normalized induced representations that enter the definitions of local Rankin-Selberg integrals. Let $\tau$ be a representation of $\text{GL}_r(F)$, $\tau_0 \in \mathbb{C}$ and denote $\tau_{s_0} = \tau \otimes |\cdot|_{F}^{s_0 - \frac{1}{2}}$.

2.2.1. Induced representations $\rho_{\tau,s_0}$. By the decomposition $Q_r = M_r \rtimes N_r$, we can extend $\tau_{s_0}$ to a representation of $Q_r(F)$ and obtain a normalized induced representation

$$\rho_{\tau,s_0} = \text{Ind}_{Q_r}^{\text{SO}_{2r}(F)}(\tau_{s_0}) \tag{2.5}$$

of $\text{SO}_{2r}(F)$. Its underlying space $I(\tau,s_0)$ consists of smooth functions $\xi_{s_0}: \text{SO}_{2r}(F) \to \mathcal{V}_r$ satisfying

$$\xi_{s_0}(m_r(a)na) = \delta_{Q_r}^{-1}(m_r(a))\tau_{s_0}(a)\xi_{s_0}(h) \tag{2.6}$$

for $a \in \text{GL}_r(F)$, $n \in N_r(F)$ and $h \in \text{SO}_{2r}(F)$, where $\delta_{Q_r}$ is the modulus function of $Q_r(F)$. We remind that $\delta_{Q_r}(m_r(a)) = |\det(a)|_{F}^{s_0 - \frac{1}{2}}$.

2.2.2. Induced representations $\check{\rho}_{\tau^*,1-s_0}$. There are another induced representations that we need to introduce. First let $\check{Q}_r \subset \text{SO}_{2r}$ be the maximal parabolic subgroup such that $\check{Q}_r(F) = \delta_{r}^{-1}Q_r(F)\delta_{r}$ where $\delta_{r}$ is the modulus function of $Q_r(F)$ acting on $\mathcal{V}_r$ with the action $\tau^*(a) = \tau(a^*)$ for $a \in \text{GL}_r(F)$. Note that when $\tau$ is irreducible, $\tau^*$ is isomorphic to the dual $\tau^!$. Again, let $s_0$ be a complex number. Then by extending $\tau_{1-s_0}^!$ to the representation of $\check{Q}_r(F)$ on $\mathcal{V}_r$ with the action

$$\tau_{1-s_0}^!(\check{m}_r(a)\check{n}) = \tau(a^*)|\det(a)|_{F}^{\frac{1}{2}-s_0}$$

where $\check{m}_r(a) = \delta_{r}^{-1}m_r(a)\delta_{r}^r \in \check{M}_r(F)$ and $\check{n} \in \check{N}_r(F)$, we obtain another normalized induced representation

$$\check{\rho}_{\tau^*,1-s_0} = \text{Ind}_{\check{Q}_r}^{\text{SO}_{2r}(F)}(\tau_{1-s_0}^!) \tag{2.7}$$

of $\text{SO}_{2r}(F)$. Its underlying space $\check{I}(\tau^*,1-s_0)$ consists of the functions $\check{\xi}_{1-s_0}: \text{SO}_{2r}(F) \to \mathcal{V}_r$ satisfying the rule similar to that of \([2.6]\).

Remark 2.1. Observe that when $r$ is even, we have $\check{\rho}_{\tau^*,1-s_0} = \rho_{\tau^*,1-s_0}$ and $\check{I}(\tau^*,1-s_0) \simeq I(\tau^*,1-s_0)$. In general, given $\check{\xi}_{1-s_0} \in \check{I}(\tau^*,1-s_0)$, we define $\check{\xi}_{1-s_0}^!(h) = \xi_{1-s_0}(h^r\delta_r^r)$ for $h \in \text{SO}_{2r}(F)$. Then $\check{\xi}_{1-s_0}^! \in I(\tau^*,1-s_0)$ and the map $\xi_{1-s_0} \mapsto \xi_{1-s_0}^!$ gives rise to an isomorphism between linear spaces $I(\tau^*,1-s_0)$ and $I(\tau^*,1-s_0)$.

\[ \text{The } \delta_r \text{ here is slightly different from that defined in } \text{Che21b Section 4.1}. \]
2.3. Intertwining operators and good sections. As pointed out in the introduction, the definitions of the Rankin-Selberg $L$- and $\epsilon$-factors involve the so called good sections. The notion of good sections was introduced by Piatetski-Shapiro and Rallis in [PSR87], in order to define the local factors of triple product $L$-functions. This idea is adopted in [Ke92, HK96, Kap13, Yam14] and also in this note. In the followings, we use $s_0$ (resp. $s$) to denote a complex number (resp. variable) for the sake of clearness.

2.3.1. Assumptions on $\tau$. From now on, we always assume that $\tau$ is a subrepresentation of a representation (of $\text{GL}_r(F)$) parabolically induced from an irreducible supercuspidal representation (of a Levi part), and the space $\text{Hom}_{\text{Z}_r}(\tau, \psi_{Z_r}) = \mathbb{C}\Lambda_{\tau, \psi_{Z_r}}$ is one-dimensional. In particular, $\tau$ admits a central character $\omega_{\tau}$, and all irreducible generic representations of $\text{GL}_r(F)$ are included. We need these assumptions to ensure the intertwining operators that will be introduced in the next subsection enjoy certain properties (cf. Remark 2.2). Observe that $\tau'$ satisfies the same assumptions and is isomorphic to the admissible dual of $\tau$ when $\tau$ is irreducible. Moreover, the one-dimensional space $\text{Hom}_{\text{Z}_r}(\tau^*, \psi_{Z_r})$ is generated by $\Lambda_{\tau^*, \psi_{Z_r}} = \Lambda_{\tau, \psi_{Z_r}} \circ \tau(d_r)$ with

\[
\left(\begin{array}{ccc}
1 & & \\
-1 & & \\
& & \\
(1)^{r-1} & & 
\end{array}\right) \in \text{GL}_r(F).
\]

2.3.2. Standard and holomorphic sections. A function $\xi_s(h) : \mathbb{C} \times \text{SO}_{2r}(F) \to \mathcal{V}_r$ is called a section of the induced representation $\rho_{r,s_0}$ if the function $h \mapsto \xi_{s_0}(h)$ belongs to $I(\tau, s_0)$ for every $s_0 \in \mathbb{C}$. A section $\xi_s(h)$ is called standard if for every $k \in \text{SO}_{2r}(\mathfrak{o})$, the function $s \mapsto \xi_s(k)$ is a constant function. We denote by $I_{\text{std}}(\tau, s)$ the space of standard sections. Given a complex number $s_0$ and an element $\xi_{s_0} \in I(\tau, s_0)$, it is easy to see that there is a unique $\xi_s \in I_{\text{std}}(\tau, s)$ so that $\xi_{s_0} = \xi_s_{s_0}$.

2.3.3. Intertwining operators. To define good sections, we first introduce the intertwining operator $M(\tau, s)$ and its normalization $M^1(\tau, s)$ between the induced representations defined in 2.2. Denote $\omega_r \in \text{SO}_{2r}(F)$ by

\[
\omega_r = \left\{ \begin{array}{ll}
I_r & \text{if } r \text{ is even}, \\
I_r \begin{pmatrix} I_{r} & 0 \\
0 & 1 \\
\end{pmatrix} I_{2r-2} & \text{if } r \text{ is odd}.
\end{array} \right.
\]

Then for $\Theta(s_0) \gg 0$, the intertwining operator $M(\tau, s_0) : I(\tau, s_0) \to I(\tau^*, 1 - s_0)$ is defined by the following convergent integral

\[
M(\tau, s_0)\xi_{s_0}(h) = \int_{\tilde{N}_r(F)} \xi_{s_0}(\omega_r^{-1} \tilde{n} h) d\tilde{n}.
\]

For the choice of the Haar measure on $\tilde{N}_r(F)$, see [Kap15, Page 398]. In particular, when $r = 2$ so that $\tilde{N}_2(F) = N_2(F) \cong F$, $d\tilde{n}$ is simply the Haar measure on $F$ that is self-dual with respect to $\psi$. To define $M(\tau, s_0)\xi_{s_0}(h)$ for arbitrary $s_0$, one has to apply the meromorphic continuation. More precisely, given $\xi_{s_0} \in I(\tau, s_0)$, let $\xi_s \in I_{\text{std}}(\tau, s)$ be the standard section such that $\xi_{s_0} = \xi_s_{s_0}$. We define $M(\tau, s)\xi_s(h)$ by the same integral (2.10). Then in addition to absolutely convergent for $\Theta(s) \gg 0$, the integral also admits meromorphic continuation to whole complex plane. Now $M(\tau, s_0)\xi_{s_0}^{\circ}(h)$ is understood as $M(\tau, s)\xi_s(h)|_{s = s_0}$. One can define the intertwining operator $M(\tau^*, s_0)$ from $I(\tau^*, s_0)$ to $I(\tau, 1 - s_0)$ in a similar fashion.
The normalized intertwining operator $M^I_\psi(\tau, s)$ can be defined as follows. We first apply our second assumption on $\tau$ to define Shahidi’s local coefficient $\mathbf{1}_\gamma(s, \tau, \Lambda^2, \psi)$ as in [Sou00, Page 521] (see also [Kap13, Pages 398-399]). Then $\gamma(s, \tau, \Lambda^2, \psi)$ lies in $\mathbb{C}(q^{-s})$ by our first assumption on $\tau$ together with [CSS80 Prop. 2.1] and the following rationality results

\[(2.11) \quad P_0(q^{-2s})M(\tau, s)\xi_s \in \tilde{I}^{\text{hol}}(\tau^*, 1 - s) \quad \text{and} \quad P_0^*(q^{-2s})\tilde{M}(\tau^*, s)\tilde{\xi}_s \in I^{\text{hol}}(\tau, 1 - s)\]

for all $\xi_s \in I^{\text{std}}(\tau, s)$ and $\tilde{\xi}_s \in \tilde{I}^{\text{std}}(\tau^*, s)$, where $P_0(X)$ and $P_0^*(X)$ are some non-zero polynomials. Now the normalized intertwining operator $M^I_\psi(\tau, s)$ is defined to be

\[(2.12) \quad M^I_\psi(\tau, s) = \gamma(2s - 1, \tau, \Lambda^2, \psi)M(\tau, s).\]

One can define $\tilde{M}(\tau^*, s)$ in a similar way. Then we have

\[(2.13) \quad M^I_\psi(\tau, s) \circ M^I_\psi(\tau^*, 1 - s) = 1 \quad \text{and} \quad M^I_\psi(\tau^*, 1 - s) \circ M^I_\psi(\tau, s) = 1.\]

**Remark 2.2.** There are two places that we need the assumptions on $\tau$ made in (2.3.1). The first is the rationality property of the intertwining operators in (2.11). The second is the identities in (2.13). When $\tau$ is irreducible, these follow from well-known results of Shahidi in [Sha81 Section 2.3]. One of the key steps to get these results is to realize $\tau$ as a subrepresentation of a representation that is parabolically induced from an irreducible supercuspidal representation. When $\tau$ is irreducible, this follows from Jacquet’s subrepresentation theorem. Here we make it as an assumption and then apply (various of) the proofs in [Sha81] to obtain (2.11) and (2.13).

2.3.4. **Good sections.** We now define good sections for $\rho_{\tau, s_0}$. By (2.11) and the fact that $\gamma(s, \tau^*, \Lambda^2, \psi) \in \mathbb{C}(q^{-s})$, we have $M^I_\psi(\tau^*, 1 - s)\tilde{I}^{\text{hol}}(\tau^*, 1 - s) \subset \mathbb{C}(q^{-s}) \otimes C I^{\text{hol}}(\tau, s)$. Thus we can define the space $I^{\text{std}}(\tau, s)$ of good sections of $\rho_{\tau, s_0}$ to be the $\mathbb{C}[q^{-s}, q^s]$-submodule of $\mathbb{C}(q^{-s}) \otimes C I^{\text{hol}}(\tau, s)$ generated by

\[(2.14) \quad I^{\text{hol}}(\tau, s) \cup \tilde{M}^I_\psi(\tau^*, 1 - s)\tilde{I}^{\text{hol}}(\tau^*, 1 - s).\]

Similarly, we can define the space $\tilde{I}^{\text{std}}(\tau^*, 1 - s)$ of good sections of $\tilde{\rho}_{\tau^*, 1 - s_0}$. These definitions are independent of $\psi$ by [Kap15 (5.6)]. Also by (2.13), we have

\[(2.15) \quad M^I_\psi(\tau, s)I^{\text{std}}(\tau, s) = \tilde{I}^{\text{std}}(\tau^*, 1 - s) \quad \text{and} \quad \tilde{M}^I_\psi(\tau^*, 1 - s)I^{\text{std}}(\tau^*, 1 - s) = I^{\text{std}}(\tau, s).\]

We point out that these identities are crucial in defining the Rankin-Selberg $L$- and $\epsilon$-factors attached to $\pi \times \tau$. Especially we need them to guarantee the (Rankin-Selberg) $\epsilon$-factors are units in $\mathbb{C}[q^{-s}, q^s]$. For more reasons of using good sections instead of using merely holomorphic sections, we recommend the readers to consult [Kap15 pages 589-590].

2.4. **Integrals and local factors.** Let $\pi$ be an irreducible generic representation of $\text{SO}_{2n+1}(F)$ and fix a non-zero element $\Lambda_{\pi, \psi_{U_n}} \in \text{Hom}_{U_n(F)}(\pi, \psi_{U_n})$. Let $\tau$ be a representation of $\text{GL}_r(F)$ as in (2.3.1).

2.4.1. **Rankin-Selberg integrals.** To define the local Rankin-Selberg integrals for $\pi \times \tau$, let us put

\[(X_{n, r}(F) = \begin{cases}
I_r & \quad x \in \text{Mat}_{(n-r)xr}(F), \quad x' = -j_r x j_{n-r} \\
1 & \quad x' = I_r
\end{cases}) \subset \text{SO}_{2n+1}(F).\]

Then the integrals are given by

\[(2.16) \quad \Psi(v \otimes \xi_s) = \int_{V_r(F) \backslash \text{SO}_{2r}(F)} \int_{X_{n, r}(F)} W_v(g(\bar{x}h))f_{\xi_s}(h) d\bar{x} dh\]

where $W_v(g) = \Lambda_{\pi, \psi_{U_n}}(\pi(g)v)$ is the Whittaker function associated to $v \in V_r$ for $g \in \text{SO}_{2n+1}(F)$ and $f_{\xi_s}(h) = \Lambda_{\tau, \psi_{2r}}(\xi_s(h))$ for $\xi_s \in I^{\text{std}}(\tau, s)$ and $h \in \text{SO}_{2r}(F)$ (this is NOT the Whittaker function associated to $\xi_s$).

\[\text{If } \tau \text{ is irreducible, then by [Sha90 Thm. 3.5] and a recent result of Cogdell-Shahidi-Tsai in [CST17], the local coefficient } \gamma(s, \tau, \Lambda^2, \psi) \text{ and the } \gamma\text{-factor } \gamma(s, \phi, \Lambda^2, \psi) \text{ are equal up to a unit in } \mathbb{C}[q^{-s}, q^s].\]
the other hand, by using the normalized intertwining operator $M^\dagger_{\psi}(\tau, s)$, one obtains another integral $\hat{\Psi}(v \otimes \xi_s)$ attached to $v$ and $\xi_s$. More precisely, let

$$
\delta_{n, \tau} = \begin{pmatrix} I_{r-1} & 1 \\
-1 & I_{2(n-r)+1} \\
1 & I_{r-1} 
\end{pmatrix} \in SO_{2n+1}(F).
$$

Then we define

$$
(2.17) \quad \hat{\Psi}(v \otimes \xi_s) = \int_{V_r(F) \backslash SO_{2r}(F)} \int_{X_{n, r}(F)} W_v(q(\bar{x}h)) \delta_{n, r}^s f^s_{M^\dagger_{\psi}(\tau, s)}(\delta_h \eta s \xi_s(h)) d\bar{x} dh
$$

where $f^s_{M^\dagger_{\psi}(\tau, s)}(h) = \Lambda_{\tau, \bar{\psi}_\tau}(M^\dagger_{\psi}(\tau, s) \xi_s(h))$.

The integrals (2.16) and (2.17), which are originally absolute convergence in some half-planes that depend only upon (the classes of) $\pi$ and $\tau$, have meromorphic continuations to whole complex plane, and give rise to rational functions in $q^{-s}$. Furthermore, there exist $v \in V_\pi$ and $\xi_s \in I^{\text{std}}(\tau, s)$ such that $\hat{\Psi}(v \otimes \xi_s) = 1$ according to [Sou93 Section 6]. Consequently, there exists a non-zero rational function $\gamma(s, \pi \times \tau, \psi) \in \mathbb{C}(q^{-s})$, depending only on $\psi$ and (the classes of) $\pi$ and $\tau$ such that

$$
(2.18) \quad \hat{\Psi}(v \otimes \xi_s) = \gamma(s, \pi \times \tau, \psi) \hat{\Psi}(v \otimes \xi_s)
$$

for all $v \in V_\pi$ and $\xi_s \in \mathbb{C}(q^{-s}) \otimes_{\mathbb{C}} I^{\text{std}}(\tau, s)$. Concerning the compatibility for the $\gamma$-factors defined by different methods, we have the following result.

**Theorem 2.3** ([LS94], [Sou00], [Kap15]). Let $\pi$ be an irreducible generic representation of $SO_{2n+1}(F)$ and $\tau$ be an irreducible generic representation of $GL_r(F)$. Then we have $\omega_r(-1)^n \gamma(s, \pi \times \tau, \psi) = \gamma(s, \phi_{\pi} \otimes \phi_{\tau}, \psi)$.

The following lemma, which is standard, is needed in formulating the Rankin-Selberg $L$- and $\epsilon$-factors.

**Lemma 2.4.** Let $v \in V_\pi$ and $\xi_s \in I^{\text{std}}(\tau, s)$. Then we have $\hat{\Psi}(v \otimes \xi_s) \in \mathbb{C}(q^{-s})$ as a meromorphic function. Moreover, there exists a non-zero polynomial $P(X) \in \mathbb{C}[X]$ depending at most upon $\psi$ and (the classes of) $\pi$ and $\tau$ such that $P(q^{-s}) \hat{\Psi}(v \otimes \xi_s) \in \mathbb{C}(q^{-s}, q^s)$.

**Proof.** If $\xi_s \in I^{\text{std}}(\tau, s)$, then the assertions follows from results in [Sou93 Section 4.3], namely, $\hat{\Psi}(v \otimes \xi_s)$ gives an element in $\mathbb{C}(q^{-s})$ and there exists $P_1(X) \in \mathbb{C}[X]$ such that $P_1(q^{-s}) \hat{\Psi}(v \otimes \xi_s)$ is contained in $\mathbb{C}(q^{-s}, q^s)$. Since $\psi^{\text{hol}}(\tau, s) = \mathbb{C}(q^{-s}, q^s) \otimes_{\mathbb{C}} I^{\text{std}}(\tau, s)$, same assertions hold when $\xi_s \in I^{\text{hol}}(\tau, s)$. To complete the proof, we may assume $\xi_s = M^\dagger_{\psi}(\tau^*, 1 - s) \xi_{1-s}$ for some $\xi_{1-s} \in I^{\text{std}}(\tau^*, 1 - s)$ by (2.14). Then by (2.11) and the fact that $\gamma(s, \tau^*, \Lambda^2, \psi)$ is a non-zero element in $\mathbb{C}(q^{-s})$, there exists a non-zero $P_2(X) \in \mathbb{C}[X]$ so that

$$
P_2(q^{-s}) \xi_s = P_2(q^{-s}) M^\dagger_{\psi}(\tau^*, 1 - s) \xi_{1-s} \in I^{\text{hol}}(\tau, s).
$$

Now the proof follows as if we set $P(X) = P_1(X) P_2(X)$.

2.4.2. Rankin-Selberg local factors. We are ready to define the (local) Rankin-Selberg $L$- and $\epsilon$-factors attached to $\pi \times \tau$. Let $I_{\pi \times \tau}(s) \subset \mathbb{C}(q^{-s}, q^s)$ be the $\mathbb{C}(q^{-s}, q^s)$-submodule spanned by the integrals $\hat{\Psi}(v \otimes \xi_s)$ for $v \in V_\pi$ and $\xi_s \in I^{\text{std}}(\tau, s)$. The submodule $I_{\pi \times \tau}(s)$ is independent of $\psi$ by [Sou93 Proposition 10.2] (see also [Kap15 Section 5.1]). Moreover, Lemma 2.4 implies that $I_{\pi \times \tau}(s)$ is in fact a fractional ideal. Now since $1 \in I_{\pi \times \tau}(s)$ by [Sou93 Chapter 6], the ideal $I_{\pi \times \tau}(s)$ admits a generator $P_{\pi \times \tau}(q^{-s})^{-1}$ for some $P_{\pi \times \tau}(X) \in \mathbb{C}[X]$ with $P_{\pi \times \tau}(0) = 1$. We then define

$$
L(s, \pi \times \tau) = P_{\pi \times \tau}(q^{-s})^{-1} \quad \text{and} \quad \epsilon(s, \pi \times \tau, \psi) = \omega_r(-1)^n \gamma(s, \pi \times \tau, \psi) \frac{L(s, \pi \times \tau)}{L(1 - s, \pi \times \tau^*)}.
$$

Then $I_{\pi \times \tau}(s) = L(s, \pi \times \tau) \mathbb{C}(q^{-s}, q^s)$ according to the definition. On the other hand, a standard argument implies that $\epsilon(s, \pi \times \tau, \psi)$ is a unit in $\mathbb{C}(q^{-s}, q^s)$ ([Kap13 Claim 3.2]).

---

5By combining with the multiplicity one property of certain Hom spaces ([Sou93 Chapter 8]).
Remark 2.5. The definition of \( L(s, \pi \times \tau) \) seems to depend on the integrals when \( r \) is odd, that is, one can define the \( L \)-factor by using either \( \Psi(v \otimes \xi_s) \) or \( \tilde{\Psi}(v \otimes \xi_s) \), while the induced representations involved are different. However, it does not since in (2.17), the section is twisted by \( \delta_r \), and the isomorphism in Remark (2.1) extends to an isomorphism between the spaces of relevant standard sections due to the fact \( \delta^{-1}_r \mathrm{SO}_{2r}(\mathfrak{o}) \delta_r = \mathrm{SO}_{2r}(\mathfrak{o}) \).

2.5. Integrals and formal Laurent series. In this subsection, we associate to each integral \( \Psi(v \otimes \xi_s) \) (with \( v \in \mathcal{V}_r \) and \( \xi_s \in I^\text{hol}(\tau, s) \)) a formal Laurent series \( \Psi_{v \otimes \xi_s}(X) \) such that \( \Psi_{v \otimes \xi_s}(q^{-m}) = \Psi(v \otimes \xi_{s_0}) \) for all \( s_0 \) with \( R(s_0) = 0 \). These formal Laurent series will be used in our proofs. Let \( \mathbb{C}[X, X^{-1}] \) be the space of the formal sums \( f(X) = \sum_{m \in \mathbb{Z}} c_m X^m \) with \( c_m \in \mathbb{C} \). Following [IPSS83], elements in \( \mathbb{C}[X, X^{-1}] \) are called formal Laurent series. Note that \( \mathbb{C}[X, X^{-1}] \) can be regard (in a natural way) as a \( \mathbb{C}[X, X^{-1}] \)-module with torsion. The idea of interpreting integrals as formal Laurent series first appeared in [IPSS83], which allows them to prove the multiplicativity of their \( \gamma \)-factors, and as a byproduct, the upper bounds of poles of their integrals. In [Kap13] section 8, Kaplan adopted the same idea to bound the poles for his integrals. He also provided some background definitions and results which explicate the connection between integrals and series in [Kap13] Section 7. In this note, however, we don’t need such a generality.

Let \( v \in \mathcal{V}_r \) and \( \xi_s \in I^\text{hol}(\tau, s) \). Then the integral \( \Psi(v \otimes \xi_s) \) can be formally written as
\[
\Psi(v \otimes \xi_s) = \sum_{m \in \mathbb{Z}} q^{-m} \Psi^m(v \otimes \xi_s)
\]
with
\[
\Psi^m(v \otimes \xi_s) = \int_{Z_r(F) \backslash \mathrm{GL}_r(F)} \int_{\mathrm{SO}_s(\mathfrak{o})} W_r(\xi_s)(a) \det(a)^{\gamma} d\mathfrak{x} d\mathfrak{y}
\]
by the Iwasawa decomposition \( \mathrm{SO}_{2r}(F) = Q_r(F) \mathrm{SO}_r(\mathfrak{o}) \) and the identification \( V_r(F) \backslash Q_r(F) \cong Z_r(F) \backslash \mathrm{GL}_r(F) \).

Here we denote
\[
W_{\xi_s}(a)(k) = \Lambda_r(\psi_{\xi_s}(\tau(a) \xi_s(k)))
\]
and
\[
\mathrm{GL}_r^m(F) = \{ a \in \mathrm{GL}_r(F) \mid |\det(a)|_F = q^{-m} \}
\]
for \( m \in \mathbb{Z} \). Suppose that \( \xi_s \) is a standard section. Then the integrals \( \Psi^m(v \otimes \xi_s) \) are actually independent of \( s \). We show that they converge absolutely and vanishing for all \( m < 0 \). Indeed, it follows from the [Sou93] Proposition 4.2] that the integral \( \Psi^m(v \otimes \xi_s) \) is bounded by a finite sum of integrals of the form
\[
\int f(\alpha_1, \alpha_2, \ldots, \alpha_r) \chi_1(\chi_{\alpha_1}) \ldots \chi_r(\alpha_r) |\xi_1|^{\text{R}(s_0)} d^s \alpha_1 d^s \alpha_2 \ldots d^s \alpha_r
\]
with the domain of integration
\[
\{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in F^r \mid |\alpha_1 \alpha_2 \ldots \alpha_r|_F = q^{-m} \}
\].

Here \( f \geq 0 \) is a Bruhat-Schwartz function on \( F^r \) and \( \chi_1, \chi_2, \ldots, \chi_r \in \mathcal{X}_{\pi, \tau} \) where \( \mathcal{X}_{\pi, \tau} \) is a finite set of positive characters of \( F^r \) that depends only on (the classes of) \( \pi \) and \( \tau \). The assertions then follow immediately.

Now we define the associated formal Laurent series by
\[
\Psi_{v \otimes \xi_s}(X) = \sum_{m \in \mathbb{Z}} X^m \Psi^m(v \otimes \xi_s)
\]
for \( v \in \mathcal{V}_r \) and \( \xi_s \in I^\text{std}(\tau, s) \). In general, if \( \xi_s \) is a holomorphic section, then \( \xi_s = \sum_j P_j(q^{-s}) \xi_s^j \) is a finite sum of standard sections \( \xi_s^j \) with coefficients \( P_j(q^{-s}) \in \mathbb{C}[q^{-s}, q^s] \). We then define
\[
\Psi_{v \otimes \xi_s}(X) = \sum_j P_j(X) \Psi_{v \otimes \xi_s}(X).
\]
Evidently, these formal Laurent series have only finitely many negative terms and \( \Psi(v \otimes \xi_s)(q^{-s_0}) = \Psi(v \otimes \xi_{s_0}) \) for all \( s_0 \) in some right half-plane plane.

\footnote{For this only, one can also apply the Funibi’s theorem.}
3. Novodvorsky v.s. Soudry

Let $\pi$ be an irreducible generic representation of $SO_5(F)$ and $\tau$ be a generic representation of $GL_2(F)$ that is irreducible or induced of Langlands’ type (cf. §1.1.2). In particular, $\tau$ satisfies the assumptions in §2.3.1. On the other hand, via the accidental isomorphism between $PGSp_4$ and $SO_5$, $\pi$ can also be regarded as an irreducible smooth generic representation of $GSp_4(F)$ with trivial central character. In [Nov79], Novodvorsky suggested a constriction of $L$- and $\epsilon$-functions attached to generic automorphic representations of $GSp_4 \times GL_2$ over global fields, whose corresponding local theory was later studied by Soudry in [Sou94]. To $v \in \mathcal{V}_x$, $u \in \mathcal{V}_y$, and a Bruhat-Schwartz function $\phi$ on $F^2$, one can attach the Novodvorsky’s local integral $Z^\text{Nov}(s, v \otimes u, \phi)$. Then as indicated in [Sou93, Section 0], the Rankin-Selberg integrals introduced in §2.4.1 essentially equal to Novodvorsky’s integrals when $n = r = 2$. The goal of this section is to explicate their relation, and as a consequence, to show that the $L$-factors defined by these two integrals coincide.

3.1. Accidental isomorphisms. In this subsection only, let $\mathbb{F}$ be an arbitrary field with the characteristic different from 2. We describe the accidental isomorphisms $\tilde{\vartheta} : PGSp_4 \cong SO_5$ and $\tilde{\vartheta}' : GSp(\mathbb{GL}_2 \times \mathbb{GL}_2) \cong SO_4$, and then fit them and the embedding $\varrho$ in (2.2) (with $n = r = 2$) into a commutative diagram.

3.1.1. The accidental isomorphism $\tilde{\vartheta}$. Let $(W, (\cdot, \cdot))$ be the 4-dimensional symplectic space over $\mathbb{F}$ and $GSp(W) = \{g \in GL(W) \mid (gw, gw') = v(g)(w, w')\text{ for all }w, w' \in W\}$ be the similitude symplectic group with $v : GSp(W) \to \mathbb{F}^\times$ the similitude character. We fix an ordered basis $\{w_1, w_2, w_2^*, w_1^*\}$ of $W$ so that the associated Gram matrix is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Let $(\tilde{V}, (\cdot, \cdot))$ be the 6-dimensional quadratic space over $\mathbb{F}$ with $\tilde{V} = \Lambda^2 W$ and the symmetric bilinear form $(\cdot, \cdot)$ defined by

$$v_1 \Lambda v_2 = (v_1, v_2)(w_1 \Lambda w_2 \Lambda w_1^* \Lambda w_2^*)$$

for $v_1, v_2 \in \tilde{V}$. Let $\tilde{\vartheta} = v_1 \Lambda w_2^* + w_2 \Lambda w_1^* \in \tilde{V}$ and put $V = \{v \in \tilde{V} \mid (v, \tilde{\vartheta}) = 0\}$. Denote also by $(\cdot, \cdot)$ the restriction of the symmetric bilinear form to $V$, so that $(V, (\cdot, \cdot))$ becomes a 5-dimensional quadratic space over $\mathbb{F}$. Let $\tilde{\vartheta} : GSp(W) \to SO(V)$ be the homomorphism defined by $\tilde{\vartheta}(g) = v(g)^{-1} \Lambda^2 g$ for $g \in GSp(W)$. Then because $\tilde{\vartheta}(h) \tilde{\vartheta} = \tilde{\vartheta}$, the homomorphism induces an exact sequence

$$1 \to \mathbb{F}^\times \overset{\iota}{\to} GSp(W) \overset{\tilde{\vartheta}}{\to} SO(V) \to 1$$

where $\iota(a) = aI_W$ with $I_W : W \to W$ the identity map. Now $\tilde{\vartheta}$ is the one induced from this exact sequence.

3.1.2. The accidental isomorphism $\tilde{\vartheta}'$. Let $(V', (\cdot, \cdot'))$ be the 4-dimensional quadratic space over $\mathbb{F}$ with $V' = M_2(\mathbb{F})$ the space of two-by-two matrices with entries in $\mathbb{F}$, and the symmetric bilinear form $(\cdot, \cdot')$ given by

$$(v_1', v_2') = \det(v_1' + v_2') - \det(v_1') - \det(v_2')$$

for $v_1', v_2' \in V'$. Here $\det$ stands for the determinant map from $V'$ onto $\mathbb{F}$. Let $GSO(V') = \{h \in GL(V') \mid (hv, hv') = \nu'(h)(v, v')\text{ for all }v, v' \in V'\}$ be the similitude special orthogonal group with $\nu' : GSO(V') \to \mathbb{F}^\times$ the similitude character. Note that $SO(V') = \text{ker}(\nu')$. Define a surjective homomorphism $\tilde{\vartheta}' : GL_2(\mathbb{F}) \times GL_2(\mathbb{F}) \to GSO(V')$ by $\tilde{\vartheta}'(a_1, a_2)v' = a_1v'a_2^{-1}$ for $a_1, a_2 \in GL_2(\mathbb{F})$ and $v' \in V'$. Then we have an exact sequence

$$1 \to \mathbb{F}^\times \overset{\iota'}{\to} GL_2(\mathbb{F}) \times GL_2(\mathbb{F}) \overset{\tilde{\vartheta}'}{\to} GSO(V') \to 1$$

with $\iota'(\alpha) = (\alpha I_2, \alpha I_2)$ for $\alpha \in \mathbb{F}^\times$. Since $\nu'(\tilde{\vartheta}'(a_1, a_2)) = \text{det}(a_1)\text{det}(a_2)^{-1}$, the restriction gives rise to an exact sequence

$$1 \to \mathbb{F}^\times \overset{\iota'}{\to} G(\mathbb{SL}_2 \times \mathbb{SL}_2)(\mathbb{F}) \overset{\tilde{\vartheta}'}{\to} SO(V') \to 1$$

where

$$G(\mathbb{SL}_2 \times \mathbb{SL}_2)(\mathbb{F}) = \{(a_1, a_2) \in GL_2(\mathbb{F}) \times GL_2(\mathbb{F}) \mid \text{det}(a_1) = \text{det}(a_2)\}.$$ 

Now $\tilde{\vartheta}'$ is the one induced from this exact sequence.
3.1.3. A commutative diagram. We fit ϑ, π′ and ϑ into a commutative diagram. For this, let
e_1 = w_1 \wedge w_2, \ e_2 = w_1 \wedge w_2', \ v_0 = w_1 \wedge w_1' - w_2 \wedge w_2', \ f_2 = w_2 \wedge w_1', \ f_1 = w_1' \wedge w_2'
be an ordered basis of V so that its Gram matrix is the matrix S_2 in (2.1). On the other hand, let
\[
e_1' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \ e_2' = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}; \ f_2' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}; \ f_1' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\]
be an ordered basis of V′. Then the associated Gram matrix is J_4. Using these bases, we can identify SO(V) with SO_2(F) and GSp_4(W) with GSp_4(F). Next, let \( \varphi' : G(SL_2 \times SL_2)(F) \rightarrow GSp_4(F) \) be the embedding given by
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta' \\ \gamma' & \delta' \end{pmatrix}.
\]
We then have the following commutative diagram
\[
\begin{array}{ccc}
1 & \longrightarrow & F^* \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Ind}_{B_2(F)}^{GL_2(F)}(\chi)
\end{array}
\]
where \( \text{Ind}_{B_2(F)}^{GL_2(F)}(\chi) \) is the induced representation of \( GL_2(F) \) by \( \chi \).

3.2. Novodvorsky’s zeta integrals and local factors. We introduce Novodvorsky’s zeta integrals for generic representations of GSp_4 \times GL_2 and then define the associated local factors in this subsection. Let \( \pi \) be an irreducible generic representation of GSp_4(F) (see [RS07, Page 34]) and \( \tau \) be a generic representation of GL_2(F) that is irreducible or induced of Langlands’ type and put \( \omega = \omega_0 \omega_\tau \). Given \( v \in \mathcal{V}_\pi \) (resp. \( u \in \mathcal{V}_\tau \)), we let \( W_v \) (resp. \( W'_u \)) be the associated Whittaker function with respect to \( \psi \) (resp. \( \psi' \)).

For convenience, let us introduce the following notations
\[
t(\alpha, \beta) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \quad z(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}; \quad w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
for \( \alpha, \beta \in F^* \) and \( y \in F \).

3.2.1. Induced representations for GL_2. Let \( B_2 \subset GL_2 \) be the standard upper triangular Borel subgroup. For a pair \( \chi = (\chi_1, \chi_2) \) of characters of \( F^* \). Then \( \chi \) induces a character on \( B_2(F) \) by \( \chi(t(\alpha, \beta)z(y)) = \chi_1(\alpha)\chi_2(\beta) \).

Let \( \tau_\chi = \text{Ind}_{B_2(F)}^{GL_2(F)}(\chi) \) to a the normalized induced representation of GL_2(F). Its underlying space, which denote by \( \mathcal{B}(\chi) \), consists of smooth functions \( f : GL_2(F) \rightarrow \mathbb{C} \) satisfying \( f(bg) = \chi(b)\delta_{B_2}^{-1}(b)f(g) \) for \( b \in B_2(F) \) and \( g \in GL_2(F) \), where \( \delta_{B_2} \) is the modulus function of \( B_2 \). More generally, for a complex number \( s_0 \), we denote \( \chi^{s_0} = (\chi_1)^{s_0} |\alpha_{\chi_2}|^{\frac{1}{2} - s_0} \).

Then we have the induced representation \( \tau_{\chi^{s_0}} \) with the underlying space \( \mathcal{B}(\chi^{s_0}) = \mathcal{B}(\chi^{s_0}) \). The spaces \( \mathcal{B}_{\text{std}}(\chi, s) \) and \( \mathcal{B}_{\text{hol}}(\chi, s) \) of standard and holomorphic sections of \( \tau_{\chi^{s_0}} \) can be defined in similar ways as in 2.3.3.

Let \( \chi' = (\chi_2, \chi_1) \). Then \( M(\chi, s_0) : \mathcal{B}(\chi, s_0) \rightarrow \mathcal{B}(\chi', 1-s_0) \) the intertwining map can be defined as follows. For \( s_0 \) with \( \Re(s_0) \gg 0 \) (depending only on \( \chi \)), it’s given by the convergent integral
\[
M(\chi, s_0)f_{s_0}(a) = \int_F f_{s_0}(wz(y)a)dy.
\]
Here \( dy \) is the Haar measure on \( F \) that is self-dual with respect to \( \psi \). For an arbitrary \( s_0 \), \( M(\chi, s_0)f_{s_0} \) can be defined as in 2.3.3.
3.2.2. Godement sections. Let $\mathcal{S}(F^2)$ be the space of Bruhat-Schwartz functions on $F^2$. For $\phi \in \mathcal{S}(F^2)$, we define the Godement section

$$f^\phi_s(a; \chi) = \chi_1(\det(a))|\det(a)|^s \int_{F^s} \phi((0, \alpha)a) \chi_1 \chi_2^{-1}|\alpha|^2 d^\star \alpha$$

for $a \in GL_2(F)$. Here $d^\star \alpha$ is any Haar measure on $F^\times$. Let us review some of its properties. First note that the integral is essentially a local Tate integral associated to the character $\chi_1 \chi_2^{-1}$ (for every fixed $a$). Consequently, this integral converges absolutely in some right half-plane and admits a meromorphic continuation to whole complex plane. Moreover, as a meromorphic function, it becomes an element in $\mathbb{C}(q^{-s})$, and

$$L(2s, \chi_1 \chi_2^{-1})^{-1} f^\phi_s(a; \chi) \in \mathbb{C}[q^{-s}, q^s]$$

for every $a \in GL_2(F)$ and $\phi \in \mathcal{S}(F^2)$. By changing variables, one sees immediately that (as a function on $GL_2(F)$)

$$f^\phi_s(-; \chi) \in \mathcal{B}(\chi, s_0)$$

for every $s_0$ at which it is defined. Furthermore, if we define the (symplectic) Fourier transform $\hat{\phi}$ of $\phi$ by

$$\hat{\phi}(x, y) = \int_F \int_F \phi(x', y') \psi(x'y - xy') dx' dy'$$

with $dx', dy'$ the Haar measures on $F$ that is self-dual with respect to $\psi$, then as meromorphic functions

$$(3.5) \quad \chi_1(-1) \gamma(2s - 1, \chi_1 \chi_2^{-1}, \psi) M(\chi, s) f^\phi_s(a; \chi) = f^\phi_{s'}(a; \chi')$$

by [GJ79 Section 4B]. Finally, we have the following lemma, whose proof is borrowed from that of [JL70 Proposition 3.2].

**Lemma 3.1.** For every $f_s \in \mathcal{B}^\std(\chi, s)$, there exists $\phi \in \mathcal{S}(F^2)$ such that $f^\phi_s(-; \chi) = f_s$. On the other hand, we have $L(2s, \chi_1 \chi_2^{-1})^{-1} f^\phi_s(-; \chi) \in \mathcal{B}^{hol}(\chi, s)$ for every $\phi \in \mathcal{S}(F^2)$.

**Proof.** Let $f_s \in \mathcal{B}^\std(\chi, s)$ be given. We define $\phi(x, y) \in \mathcal{S}(F^2)$ as follows. If $(x, y) \notin (0, 1)GL_2(\mathfrak{o})$, then $\phi(x, y) = 0$, while if $(x, y) = (0, 1)a$ for some $a \in GL_2(F)$, then we set $\phi(x, y) = c^{-1} \chi_1(\det(a))^{-1} f_s(a)$, where $c = \vol(\mathfrak{o}, d^\star \alpha)$. Note that $\phi(x, y)$ is independent of $s$ since $f_s$ is a standard section. Moreover, it’s clear from the definition that $\phi \in \mathcal{S}(F^2)$. We show that $f^\phi_s(a; \chi)$ is absolute convergence for all $a$ and $f^\phi_s(-; \chi) = f_s$. For this, it suffices to check that $f^\phi_s(a; \chi) = f_s(a)$ for $a \in GL_2(\mathfrak{o})$. But by the definition of $\phi$, we have

$$f^\phi_s(a; \chi) = \chi_1(\det(a)) \int_{F^s} \phi((0, \alpha)a) \chi_1 \chi_2^{-1}|\alpha|^2 d^\star \alpha$$

$$= \chi_1(\det(a)) \int_{F^s} \phi((0, 1)t(1, \alpha)a) \chi_1 \chi_2^{-1}(\alpha) d^\star \alpha$$

$$= f_s(a).$$

This proves the first assertion.

To prove the second assertion, we note that by definition, $L(2s, \chi_1 \chi_2^{-1})^{-1} f^\phi_s(-; \chi)$ is right $K_0$-invariant and $L(2s, \chi_1 \chi_2^{-1})^{-1} f^\phi_s(a; \chi) \in \mathbb{C}[q^{-s}, q^s]$ for all $a \in GL_2(F)$, where $K_0 \subset GL_2(\mathfrak{o})$ is some open compact subgroup that only depends on $\phi$. So our task is to show that if $f_s$ is a non-zero section of $\tau^\std_{\chi_0}$ that is right $K_0$-invariant for some open compact subgroup $K_0 \subset GL_2(\mathfrak{o})$ that is independent of $s$, and $f_s(a) \in \mathbb{C}[q^{-s}, q^s]$ for all $a \in GL_2(F)$, then $f_s \in \mathcal{B}^{hol}(\chi, s)$. Let $k_1, \ldots, k_m$ be representatives of the double cosets $B_2(F)\backslash GL_2(F)/K_0$. Let $r = m$ if $f_s(k_j) \neq 0$ for all $1 \leq j \leq m$. On the other hand, if $f_s(k_j) = 0$ for some $1 \leq j \leq m$, then after reindex, we may assume $f_s(k_{j'}) \neq 0$ for $1 \leq j \leq r$ and $f_s(k_{j}) = 0$ for $r < j \leq m$. For each $1 \leq j \leq r$, define $f_s^{(j)}(a) = \chi_1(b) \delta_{B_2}(b)^{1/2}$ if $a = bk_j k$ for some $b \in B_2(F)$ and $k \in K_0$, and $f_s^{(j)}(a) = 0$ if $a \notin B_2(F)k_j K_0$. Then $f_s^{(j)} \in \mathcal{B}^\std(\chi, s)$ are well-defined and we have

$$f_s = f_s(k_1)f_s^{(1)} + \cdots + f_s(k_r)f_s^{(r)} \in \mathcal{B}^{hol}(\chi, s).$$

This completes the proof. □
3.2.3. Zeta integrals and the associated local factors. Now we are in the position to introduce Novodvorsky’s zeta integrals for $\pi \times \tau$ and define the associated local factors. For the ease of notation, let us denote $f^\phi_s(\cdot;\mu)$ to be $f^\phi_s(\cdot;\langle 1,\mu \rangle)$ for every character $\mu$ of $F^\times$. Using the embedding $g'$ (cf. (4.11)), we form the integral

$$Z(s, v \otimes u, \phi) = \int_{F^\times \mathbb{G}(F) \backslash G(\mathbb{S}_L \times \mathbb{S}_L)} W_v(g'(a_1, a_2)) f^\phi_s(a_1; \omega^{-1}) W_u(a_2) d^*(a_1, a_2)$$

for $v \in \mathcal{V}_\pi$, $u \in \mathcal{V}_\tau$, and $\phi \in \mathcal{S}(F^2)$, where

$$Z^\phi_s(F) = \{ (z(x), z(y)) \in \mathbb{G}(\mathbb{S}_L \times \mathbb{S}_L)(F) \mid x, y \in F \}$$

and $F^\times$ is viewed as a subgroup of $G(\mathbb{S}_L \times \mathbb{S}_L)(F)$ via the embedding $i'$. Moreover, we have put $W'_u(a) = \Lambda_{\tau, \psi_2}(\tau(a) u)$ for $u \in \mathcal{V}_\tau$ and $a \in G\mathbb{L}_2(F)$. As expected, this integral converges absolutely in some right half-plane, admits meromorphic continuation to whole complex plane, and becomes an element in $\mathbb{C}(q^{-s})$. Moreover, if we put

$$I_{\pi \times \tau}^{\text{Nov}}(s) = \{ Z(s, v \otimes u, \phi) \mid v \in \mathcal{V}_\pi, u \in \mathcal{V}_\tau, \phi \in \mathcal{S}(F^2) \} \subset \mathbb{C}(q^{-s})$$

then it forms a fractional ideal that contains $1$. As a result, the ideal $I_{\pi \times \tau}^{\text{Nov}}(s)$ has the generator $P_{\pi \times \tau}^{\text{Nov}}(q^{-s})^{-1}$ for some $P_{\pi \times \tau}(X) \in \mathbb{C}[X]$ with $P_{\pi \times \tau}^{\text{Nov}}(0) = 1$. We define

$$L_{\pi \times \tau}^{\text{Nov}}(s) = P_{\pi \times \tau}^{\text{Nov}}(q^{-s})^{-1}$$

to be the Novodvorsky’s local $L$-factor attached to $\pi \times \tau$.

Next, let us describe the function equation and then define the Novodvorsky’s $\epsilon$- and $\gamma$-factor attached to $\pi \times \tau$ and $\psi$. Given $v \in \mathcal{V}_\pi$ and $u \in \mathcal{V}_\tau$, we put $\tilde{W}_v(g) = W_v(g) \omega^{-1}(\psi')$, and $W'_u(a) = W_u(a) \omega^{-1}(\text{det}(a))$ for $g \in G\mathbb{S}_p_4(F)$ and $a \in G\mathbb{L}_2(F)$. Then $\tilde{W}_v$ (resp. $\tilde{W}'_u$) defines an element in the Whittaker model of $\pi^\vee$ (resp. $\pi'^\vee$) with respect to $\psi$ (resp. $\psi'$) with $\pi^\vee$ stands for the admissible dual of $\pi$. Note that if $\pi$ is induced of Langlands’ type, then $\pi'^\vee$ is also. Now we define

$$\tilde{Z}(s, v \otimes u, \phi) = \int_{F^\times \mathbb{G}(F) \backslash G(\mathbb{S}_L \times \mathbb{S}_L)} \tilde{W}_v(g'(a_1, a_2)) f^\phi_s(a_1; \omega) \tilde{W}'_u(a_2) d^*(a_1, a_2).$$

Then the functional equation reads

$$\tilde{Z}(1-s, v \otimes u, \phi) = \gamma_{\pi \times \tau}(s, \pi \times \tau, \psi) Z(s, v \otimes u, \phi)$$

every $v$, $u$, and $\phi$, where $\gamma_{\pi \times \tau}(s, \pi \times \tau, \psi) \in \mathbb{C}(q^{-s})$ denotes the Novodvorsky’s $\gamma$-factor. The Novodvorsky’s $\epsilon$-factor is defined in a similar way, namely,

$$\epsilon_{\pi \times \tau}(s, \pi \times \tau, \psi) = \frac{L_{\pi \times \tau}^{\text{Nov}}(s, \pi \times \tau)}{L_{\pi \times \tau}^{\text{Nov}}(1-s, \pi'^\vee \times \tau'^\vee)}.$$

Applying the identity $\tilde{\phi} = \phi$ and the facts $(\pi^\vee)' \cong \pi, (\pi'^\vee)' \cong \tau$, one deduces easily that $\epsilon_{\pi \times \tau}(s, \pi \times \tau, \psi)$ is a unit in $\mathbb{C}[q^{-s}, q^s]$.

Now we record results in the literature on compatibility between the local factors defined by Novodvorsky’s zeta integrals and the associated Weil-Deligne representations. For $G\mathbb{S}_p_4(F)$, we use the local Langlands correspondence established by Gan-Takeda in [GT11]. We remark here that when $\pi$ has trivial central character so that it can also be regarded as a representation of $\text{SO}_5(F)$, the associated $L$-parameters defined by [JS04], [RS07] and [GT11] are all the same.

**Theorem 3.2** ([Sou84], [GT11]). Let $\pi$ be an irreducible generic representation of $G\mathbb{S}_p_4(F)$ and $\tau$ be an irreducible representation of $G\mathbb{L}_2(F)$. Suppose that $\phi_\pi = \phi_{\tau_1} \oplus \phi_{\tau_2}$ for some irreducible generic representations $\tau_1$ and $\tau_2$ of $G\mathbb{L}_2(F)$ with $\omega_{\tau_1} = \omega_{\tau_2}$, and if $\tau$ is supercuspidal, then $\tau$ is not isomorphic to any representations of the form $\tau_j^s \otimes | \cdot |^{s_j}$ for some $s_j \in \mathbb{C}$ with $j = 1, 2$. Then we have

$$L_{\pi \times \tau}^{\text{Nov}}(s) = L(s, \phi_\pi \otimes \phi_\tau) \quad \text{and} \quad \epsilon_{\pi \times \tau}(s, \pi \times \tau, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\tau, \psi).$$

More recently, we have the following results on the compatibility between $L$-factors.
Theorem 3.3 (LPSZ21, Loc20). Let π be an irreducible generic representation of $\text{GSp}_4(F)$ and τ be a generic representation of $\text{GL}_2(F)$ that is irreducible or induced of Langlands’ type. Suppose that τ is non-supercuspidal. Then we have
\[ L^{\text{Nov}}(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\tau). \]

Remark 3.4. Strictly speaking, results in the literature deal with the case where τ is irreducible. However, same results carry over to the case when τ is reducible. In particular, [Son84, Theorem 2.1] and [LPSZ21, Theorem 8.9(i)] are applicable to the case when τ is induced of Langlands’ type.

3.3. A comparison between two integrals. The aim of this subsection is to prove the following proposition, which is the core of this note.

Proposition 3.5. Let π be an irreducible generic representation of $\text{SO}_5(F)$ and τ be a generic representation of $\text{GL}_2(F)$ that is irreducible or induced of Langlands’ type. Then we have
\[ L(s, \pi \times \tau) = L^{\text{Nov}}(s, \pi \times \tau). \]
Here we also regard π as a representation of $\text{GSp}_4(F)$ with trivial central character.

To prove proposition Proposition 3.5 some preparations are needed.

3.3.1. Induced representations of $\text{GSO}_4(F)$. Using the basis $\{e'_1, e'_2, f'_2, f'_1\}$ in (3.1.3) we have the following commutative diagram
\[
\begin{array}{cccc}
1 & \rightarrow & F^\times & \rightarrow \text{GL}(\text{SL}_2 \times \text{SL}_2)(F) & \rightarrow \text{SO}_4(F) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & F^\times & \rightarrow \text{GL}_2(F) \times \text{GL}_2(F) & \rightarrow \text{GSO}_4(F) & \rightarrow & 1
\end{array}
\]
where the second and the third vertical lines are natural inclusions. Now if $\tau_1$ and $\tau_2$ are representations of $\text{GL}_2(F)$ with $\omega_{\tau_1}, \omega_{\tau_2} = 1$, then $\tau_1 \boxtimes \tau_2$ can be viewed as a representation of $\text{GSO}_4(F)$ via the bottom exact sequence. On the other hand, let $Q'_2 \subset \text{GSO}_4$ be the Siegel parabolic subgroup whose intersection with $\text{SO}_4$ is $Q_2$. It admits a Levi decomposition $Q'_2 = M'_2 \times N_2$ with
\[ M'_2(F) = \left\{ m_{2}(a, \beta) = \begin{pmatrix} a & \beta a^* \\ \beta^* & a^* \end{pmatrix} \mid a \in \text{GL}_2(F) \text{ and } \beta \in F^\times \right\} \cong \text{GL}_2(F) \times \text{GL}_1(F). \]

Given a representation τ of $\text{GL}_2(F)$ and character μ of $F^\times$, we obtain a representation of $Q'_2(F)$ on $\mathcal{V}_\tau$ with the action $(\tau \boxtimes \mu)(m_{2}(a, \beta)n)u = \mu(\beta)\tau(a)u$ for $u \in \mathcal{V}_\tau$ and $n \in N_2(F)$. Let
\[ \rho'_{\tau \boxtimes \mu} = \text{Ind}^{\text{GSO}_4(F)}_{Q'_2(F)}(\tau \boxtimes \mu) \]
be a normalized induced representation of $\text{GSO}_4(F)$ whose underlying space $I'(\tau \boxtimes \mu)$ consisting smooth functions $\xi' : \text{GSO}_4(F) \rightarrow \mathcal{V}_\tau$ satisfying
\[ \xi'(m_{2}(a, \beta)nh) = \delta_{Q'_2}^{\mathfrak{m}}(m_{2}(a, \beta))\mu(\beta)\tau(a)\xi'(h) \]
for $n \in N_2(F)$ and $h \in \text{GSO}_4(F)$, where $\delta_{Q'_2}$ is the modulus function of $Q'_2$ and is given by $\delta_{Q'_2}(m_{2}(a, \beta)) = |\det(a)|_F |\beta||_{F^\times}^{-1}$. More generally, for $s_0 \in \mathbb{C}$, let $\rho'_{\tau \boxtimes \mu, s_0}$ be the representation of $\text{GSO}_4(F)$ inducing from the datum $(\tau \boxtimes \mu, s_0)$. Its underlying space is also denoted by $I'(\tau \boxtimes \mu, s_0)$. The spaces $I'_{\text{std}}(\tau \boxtimes \mu, s)$ and $I'_{\text{hol}}(\tau \boxtimes \mu, s)$ of standard and holomorphic sections of $\rho'_{\tau \boxtimes \mu, s_0}$ can be defined in similar ways as in (2.3.2).

3.3.2. A key lemma. The following lemma, whose proof is easy, plays an important role in the sequel.

Lemma 3.6. Let $\chi = (\chi_1, \chi_2)$ be a pair of characters of $F^\times$ and τ be a representation of $\text{GL}_2(F)$ with $\omega_{\chi_1} \chi_2 = 1$. Then under the identification $(\text{GL}_2(F) \times \text{GL}_2(F))/F^\times \cong \text{GSO}_4(F)$, we have $\tau_\chi \boxtimes \tau \cong \rho'_{(\tau \boxtimes \chi_1) \boxtimes \chi_2}$.

Proof. Let $f \in \mathcal{B}(\chi)$ and $u \in \mathcal{V}_\tau$. Define a function $\xi'_{f \boxtimes u} : \text{GSO}_4(F) \rightarrow \mathcal{V}_\tau$ by
\[ \xi'_{f \boxtimes u}(\tilde{\beta}(a_1, a_2)) = f(a_1)\tau(a_2)u \]
for $a_1, a_2 \in \text{GL}_2(F)$. Then $\xi'_{f \boxtimes u} \in I'(\tau \boxtimes \chi_1) \boxtimes \chi_2)$ and the map $f \boxtimes u \mapsto \xi'_{f \boxtimes u}$ from $\mathcal{B}(\chi) \boxtimes \mathcal{V}_\tau$ onto $I'(\tau \boxtimes \chi_1) \boxtimes \chi_2)$ meets the requirement. □
Lemma 3.6 has an application to compute the local coefficients $\gamma(s, \tau, \Lambda^2, \psi)$. For this we note that the restriction induces a surjection from $\mathcal{I}_s^{\text{std}}(\tau \boxtimes \mu, s)$ onto $\mathcal{I}_s^{\text{std}}(\tau, s)$.

**Corollary 3.7.** Let $\tau$ be a generic representation of $\text{GL}_2(F)$ that is irreducible or induced of Langlands’ type. We have $\gamma(s, \tau, \Lambda^2, \psi) = \gamma(s, \omega_\tau, \psi)$.

**Proof.** When $\tau$ is irreducible, the assertion would follow from the results in [Sha90] and [CST17], but since $\tau$ can be reducible, we shall provide a proof here. Let $\xi_s \in \mathcal{I}_s^{\text{std}}(\tau, s)$. Then by definition of the local coefficient,

$$\int_F \xi_s(\omega_2 n_2(y)) \psi(y) dy = \gamma(2s - 1, \tau, \Lambda^2, \psi) \int_F M(\tau, s) \xi_s(\omega_2 n_2(y)) \psi(y) dy$$

where $dy$ is an arbitrary Haar measure on $F$, and we have slightly abuse the notation (cf. (2.4)) to denote

$$n_2(y) = \begin{pmatrix} 1 & -y \\ 1 & y \end{pmatrix} \in \text{SO}_4(F)$$

for $y \in F$. To compute $\gamma(s, \tau, \Lambda^2, \psi)$, the idea is find a proper $\chi = (\chi_1, \chi_2)$ and then use Lemma 3.6. More precisely, let $\chi_1 = 1$, $\chi_2 = \omega_\tau^{-1}$ and consider the induced representation $\rho'_\tau \boxtimes \omega_\tau^{-1}, s_0$ of $\text{GSO}_4(F)$. On one hand, we have

$$\tau_{\chi, u} \boxtimes \tau \equiv \rho'_\tau \boxtimes \omega_\tau^{-1}, s_0$$

by Lemma 3.6 and the isomorphism (cf. 3.6)

$$\mathcal{B}_s^{\text{std}}(\chi, s) \otimes_{\mathbb{C}} \mathcal{V}_r \cong \mathcal{I}_s^{\text{std}}(\tau \boxtimes \omega_\tau^{-1}, s).$$

On the other hand, we have a surjection from $\mathcal{I}_s^{\text{std}}(\tau \boxtimes \omega_\tau^{-1}, s)$ onto $\mathcal{I}_s^{\text{std}}(\tau, s)$ given by the restriction. In view of these, we may assume that $\xi_s$ in (3.7) is a restriction of $\xi'_{f, \psi}$ for some $f_s \in \mathcal{B}_s^{\text{std}}(\chi, s)$ and $u \in \mathcal{V}_r$. Here $\xi'_{f, \psi}$ is the $\mathcal{V}_r$-valued function on $\text{GSO}_4(F)$ defined by (3.6) with $f$ replaced by $f_s$.

Now simple computations show that $\vartheta'(w, d_2) = \omega_2$ and $\vartheta'(z(y), I_2) = n_2(y)$ for $y \in F$. We remind here that $d_2$ is given by (2.8). From these, we get that

$$\xi_s(\omega_2 n_2(y)) = \xi'_{f, \psi}(\vartheta'(wz(y), d_2)) = f_s(wz(y)) \tau(d_2) u$$

and

$$M(\tau, s) \xi_s(\vartheta'(a_1, a_2)) = \int_F \xi'_{f, \psi}(\vartheta'(wz(y) a_1, d_2 a_2)) dy = M(\chi, s) f_s(a_1) \tau(d_2) a_2 u$$

for $(a_1, a_2) \in G(\text{SL}_2 \times \text{SL}_2)(F)$ by (3.4). It follows that the LHS of (3.7) becomes

$$\tau(d_2) u \int_F f_s(wz(y)) \psi(y) dy$$

while the RHS of (3.7) can be written as

$$\gamma(2s - 1, \tau, \Lambda^2, \psi) \tau(d_2) u \int_F M(\chi, s) f_s(wz(y)) \psi(y) dy.$$

Combining these with [Bum98, Proposition 4.5.9], the proof follows.

### 3.3.3. Proof of Proposition 3.7

We first relate the Rankin-Selberg integrals and Novodvorsky’s zeta integrals. The idea of which is similar to that of the proof of Corollary 3.7. Let $\xi_s \in \mathcal{I}_s^{\text{std}}(\tau, s)$ and $v \in \mathcal{V}_r$. Recall that the Rankin-Selberg integral $\Psi(v, s)$ is defined by

$$\Psi(v, s) = \int_{\mathcal{G}_2(F) \setminus \text{SO}_4(F)} W_v(\varrho(h)) f_{\xi_s}(h) dh$$

with $f_{\xi_s}(h) = \Lambda_r \varphi_{\xi_s}(\xi_s(h))$ for $h \in \text{SO}_4(F)$.

Let $\chi = (1, \omega_\tau^{-1})$ be a pair of characters of $F^\times$. Then we have the isomorphism (cf. (3.6))

$$\mathcal{B}_s^{\text{std}}(\chi, s) \otimes_{\mathbb{C}} \mathcal{V}_r \cong \mathcal{I}_s^{\text{std}}(\tau \boxtimes \omega_\tau^{-1}, s).$$
between representations of GSO₄(F) and the surjection from \( I_{\text{std}}(\tau \otimes \omega^{-1}, s) \) onto \( I_{\text{std}}(\tau, s) \) given by the restriction. To connect the integrals, assume that \( \xi_s \) is a restriction of \( \xi_{f_s, \otimes u} \) for some \( f_s \in B_{\text{std}}(\chi, s) \) and \( u \in \mathcal{V}_\tau \). Here \( \xi_{f_s, \otimes u} \) is the \( \mathcal{V}_\tau \)-valued function on GSO₄(F) defined by (3.6) with \( f \) replaced by \( f_s \). By Lemma 3.1 we may further assume that \( f_s = f_s^0(-; \omega^{-1}) \) for some \( \phi \in S(F^2) \). Recall that we also denote \( f_s^0(-; \mu) \) to be \( f_s^0(-; (1, \mu)) \) for every character \( \mu \) of \( F^\times \). Then for \( (a_1, a_2) \in G(\text{SL}_2 \times \text{SL}_2)(F) \), we have

\[
\int_{\mathcal{V}_\tau} G_{\phi_1} (\tau \otimes \omega^{-1}) = f_s(a_1) W'_u(a_2) = f_s^0(a_1; \omega^{-1}) W'_u(a_2).
\]

Now since \( \vartheta' \) maps \( Z_{\text{std}}^0(F) \) onto \( V_2(F) \), one sees that

\[
\Psi(v \otimes \xi_s) = \int_{\mathcal{V}_\tau} G_{\phi_1} (\tau \otimes \omega^{-1}) \Psi_{\tau, s}(\vartheta'(a_1, a_2)) f_s^0(a_1; \omega^{-1}) W'_u(a_2) d\vartheta'(a_1, a_2) = Z(s, v \otimes u, \phi)
\]

by the commutative diagram (3.2).

We also need to connect \( \Psi_{\tau, s} \tau, \phi \) with \( \tilde{Z}(1 - s, v \otimes u, \phi) \). For this we first recall that

\[
\Psi(v \otimes \xi_s) = \int_{\mathcal{V}_\tau} G_{\phi_1} (\tau \otimes \omega^{-1}) \Psi_{\tau, s}(\vartheta'(a_1, a_2)) f_s^0(a_1; \omega^{-1}) W'_u(a_2) d\vartheta'(a_1, a_2) = Z(s, v \otimes u, \phi)
\]

by (3.2).

The identities just derived have two consequences. The first consequence is the following identities

\[
\gamma^{\text{Nov}}(s, \pi \times \tau, \psi) \gamma(s, \pi \times \tau, \psi) = \gamma(s, \phi_{\tau} \otimes \phi_{\tau}, \psi).
\]

Indeed, if \( \tau \) is irreducible, then it comes from (3.9), (3.10) and Theorem 2.3. Suppose that \( \tau \) is reducible so that \( \tau = \tau_{a_2} \) for a pair of characters \( \mu = (\mu_1, \mu_2) \) of \( F^\times \) with \( \mu_1 \mu_2^{-1} = |\cdot| \cdot F \). Note that \( \phi_{\tau} = \phi_{\mu_1} \otimes \phi_{\mu_2} \). In this case, the first identity remains valid, so it suffices to establish the second identity. But this follows at once from the multiplicativity of the Rankin-Selberg \( \gamma \)-factors proved in [Sou00] and Theorem 2.3. As another consequence, we show that

\[
I^{\text{Nov}}_{\pi \times \tau}(s) \subseteq I^{\text{Nov}}_{\pi \times \tau}(s) \subseteq I(2s, \omega_{\tau}) I_{\pi \times \tau}(s).
\]

To verify the first containment, it is enough to show that \( \Psi(v \otimes \xi_s) \in I^{\text{Nov}}_{\pi \times \tau}(s) \) for \( v \in \mathcal{V}_s \) and \( \xi_s \in I^{\text{std}}(\tau, s) \) or \( \xi_s = M^{\phi}_{\tau}(\tau_1, 1 - s) \kappa^{\phi}_{1 - s} \) for some \( \xi_{1 - s} \in I^{\text{std}}(\tau_1, 1 - s) \). If \( \xi_s \in I^{\text{std}}(\tau, s) \), then the assertion follows from (3.9). Suppose that \( \xi_s = M^{\phi}_{\tau}(\tau_1, s) \kappa^{\phi}_{1 - s} \) for some \( \xi_{1 - s} \in I^{\text{std}}(\tau_1, 1 - s) \). We may assume that \( \xi_{1 - s} \) is the restriction of \( \xi_{1 - s, \otimes u} \) with \( f_{1 - s} = f^0_{1 - s}(-; \omega_{\tau}) \in B_{\text{std}}((1, \omega_{\tau}), 1 - s) \) for some \( u \in \mathcal{V}_s \) and \( \phi \in S(F^2) \). Then

\[
\Psi(v \otimes \xi_s) = Z(s, v \otimes u', \phi)
\]

where \( u' \) is the unique element in \( \mathcal{V}_s \) such that \( \tilde{W}_u'(a) = W'_u(a_2 d_{a_*} \omega_{\tau}^{-1}(a) \) for \( a \in \text{GL}_2(F) \). This can be derived in a similar way as we obtained (3.10). This verifies the first containment, while the second containment follows from Lemma 3.1 and (3.9).
Now we are ready to prove the identity between the $L$-factors. First note that if $\tau = \tau_0 \otimes | \cdot |^{s_0}_F$ for some representation $\tau_0$ of $\text{GL}_2(F)$ and $s_0 \in \mathbb{C}$, then

$$L^{Nov}(s, \pi \times \tau) = L^{Nov}(s + s_0, \pi \times \tau_0) \quad \text{and} \quad L(s, \pi \times \tau) = L(s + s_0, \pi \times \tau_0).$$

So we may assume without loss of generality that $\omega_\tau$ is unitary. The containments in (3.12) imply

$$L(s, \pi \times \tau) = L^{Nov}(s, \pi \times \tau)P(q^{-s})$$

where $P(q^{-s}) \in \mathbb{C}[q^{-s}, q^s]$ is a factor of $L(2s, \omega_\tau)^{-1}$. Replacing $\tau$ with $\tau^s$ and $s$ with $1 - s$, we also have

$$L(1 - s, \pi \times \tau) = L^{Nov}(1 - s, \pi \times \tau^s)P^*(q^{1-s})$$

with $P^*(q^{1-s}) \in \mathbb{C}[q^{-s}, q^s]$ a factor of $L(2 - 2s, \omega_{\tau^s})^{-1}$. Since $\gamma_{Nov}(s, \pi \times \tau, \psi) = \gamma(s, \pi \times \tau, \psi)$ and the $L$-factors $L(2s, \omega_\tau)$ and $L(2 - 2s, \omega_{\tau^s})$ have no common poles, we conclude that $L^{Nov}(s, \pi \times \tau) = L(s, \pi \times \tau)$ as desired. □

3.3.4. Two corollaries. Proposition 3.5 has two corollaries.

**Corollary 3.8.** Theorem 1.3 holds when $\tau$ is non-supercuspidal.

**Proof.** This follows immediately from Theorem 2.3, Theorem 3.3 and Proposition 3.5.

**Corollary 3.9.** Theorem 1.3 holds when $\tau$ is non-supercuspidal.

**Proof.** To prove this corollary, we simply note that

$$L^{Nov}(s, (\pi \otimes \mu) \times \tau) = L^{Nov}(s, \pi \times (\tau \otimes \mu))$$

for every character $\mu$ of $F^*$. Now if $\omega_\pi = \mu^2$ for some character $\mu$, then $\pi_0 := \pi \otimes \mu^{-1}$ has trivial central character, and hence

$$L^{Nov}(s, \pi \times \tau) = L^{Nov}(s, (\pi_0 \otimes \mu) \times \tau) = L^{Nov}(s, \pi_0 \times (\tau \otimes \mu)) = L(s, \pi_0 \times (\tau \otimes \mu)) = L(s, \phi_{\pi_0} \otimes \phi_{\tau \otimes \mu})$$

by Proposition 3.3 and Corollary 3.8. On the other hand, since

$$\phi_{\pi_0} \otimes \phi_{\tau \otimes \mu} \cong \phi_{\pi_0} \otimes (\phi_\tau \otimes \phi_\mu),$$

as 8-dimensional representations by [CT11a, Main Theorem (iv)], we get that $L^{Nov}(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\tau)$. Finally, the result for the $c$-factors follows from this and (3.11).

4. PROOF OF THE MAIN RESULTS

We prove Theorem 1.1 and Theorem 1.3 in this section. By Corollary 3.8 and Corollary 3.9, it remains to prove these theorems when $\tau$ is supercuspidal. So let us assume that $\tau$ is supercuspidal from now on. In view of (3.13), we may further assume that $\omega_\tau$ is unitary. Moreover, once Theorem 1.1 is verified, one can apply the similar argument in Corollary 3.9 to prove Corollary 1.3. Therefore, our focus will be on proving Theorem 1.1.

By Theorem 2.3 it suffices to establish the identity between the $L$-factors. To establish such an identity, we often use the following principle: If the $L$-factors $L(s, \pi \times \tau)$ and $L(1 - s, \pi \times \tau^s)$ have no common poles and the same holds for the $L$-factors $L(s, \phi_\pi \otimes \phi_\tau)$ and $L(1 - s, \phi_\pi \otimes \phi_\tau^s)$, then $L(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\tau)$. This is again a simple consequence of Theorem 2.3.

We begin with a lemma. To state it; however, we need to introduce some notations. Given $\alpha, \beta \in F^*$, let us put $i(\alpha, \beta) = g(\text{id}_2(t(\lambda, \beta)))$. This is an element in the diagonal torus of $\text{SO}_5(F)$ (cf. (2.2), (2.3) and (3.3)). According to [Sou93, Proposition 2.2], there exists a finite set $\mathcal{X}_v$ of finite functions on $F^*$, which depends only on (the class of) $\pi$, such that for $v \in \mathcal{V}_v$

$$W_v(i(\alpha, \beta)) = \sum_{\eta \in \mathcal{X}_v} \eta(\alpha, \beta) f_{v, \eta}(\alpha, \beta)$$

for some $f_{v, \eta} \in S(F^2)$ ([Sou93, Proposition 2.2]). Since $\eta$ are finite functions, they can be further written as

$$\eta(\alpha, \beta) = \eta_1(\alpha) \eta_2(\beta)|\alpha|^a_F |\beta|^2_F (\log q_{-1})^n (|\alpha|^a_F)^n (|\beta|^2_F)^{-n}$$

for some characters $\eta_1, \eta_2$ of $F^*$ and non-negative integers $n_{\eta_1}, n_{\eta_2}$ by [LL70, Lemma 8.1].

---

7Strictly speaking, we shall also replace $\pi$ with $\pi^s$ in $L^{Nov}(s, \pi \times \tau)$, but since $\pi$ has trivial central character (as a representation of $\text{GSp}_4(F)$), it is actually self-dual by [IIB00, Proposition 2.3].

8This is in fact a special case of more general results the asymptotic expansions of Whittaker functions ([CS80], [LM09]).
Lemma 4.1. Let $\pi$ be an irreducible generic representation of $SO_5(F)$. Then we have
\[ L(s, \pi \times \tau) \prod_{\eta \in \mathfrak{X}_\pi} L(2s, \eta_2 \omega_\tau) M \mathbb{C}[q^{-s}, q^s] \]
where $M = \max \{2^{n_{\eta^2}} | \eta \in \mathfrak{X}_\tau \}$.

Proof. This is equivalent to show that
\[ \Psi(v \otimes \xi) \in L(2s, \omega_\tau) \prod_{\eta \in \mathfrak{X}_\pi} L(2s, \eta_2 \omega_\tau) M \mathbb{C}[q^{-s}, q^s] \]
for every $v \in V_\pi$ and $\xi \in I^{sd}(\tau, s)$. To prove this, we first analyze the poles of $\Psi(v \otimes \xi)$ when $\xi \in I^{sd}(\tau, s)$. Using the Iwasawa decomposition $SO_4(F) = V_2(F)SO_4(\mathfrak{o})$, the right $SO_4(\mathfrak{o})$-finiteness and the fact that $\xi$ is a standard section, $\Psi(v \otimes \xi)$ can be written as a finite sum of the integrals of the form
\[ \int_{F^*} \int_{F^*} W_\nu(t(\alpha \beta, \beta)) W_\nu(t(\alpha, 1)) \omega_\tau(\beta) |\alpha|_{F^{s,1}}^{-1} ||\beta|_{F^{s,2}}^{n_1} d^s \alpha d^s \beta \]
for some $v' \in V_\pi$ and $\nu \in V_\tau$. Since $\tau$ is supercuspidal, $W_\nu(t(\alpha, 1))$ is a Bruhat-Schwartz function on $F^\times$ (Jac07 Lemma 14.3). This, together with the asymptotic expansion of $W_\nu(t(\alpha \beta, \beta))$ and the fact that $S(F^2) \cong S(F) \otimes_{\mathbb{C}} S(F)$, the integral can be further written as a finite sum of the integrals of the form
\[ \int_{F^*} \int_{F^*} f_1(\alpha) f_2(\beta) |\alpha|_{F^{s,1}}^{-1} |\beta|_{F^{s,2}} d^s \alpha \]
for some $f_1 \in S(F^*)$ and $f_2 \in S(F)$, where $S(F^*)$ and $S(F)$ stands for the space of Bruhat-Schwartz functions on $F^*$ and $F$ respectively. Clearly, the integral involving $f_1$ is always absolutely convergent (and hence has no poles). On the other hand, poles of the integral involving $f_2$ are contained in that of $\prod_{\eta \in \mathfrak{X}_\pi} L(2s, \eta_2 \omega_\tau) M$. Consequently, we find that
\[ \Psi(v \otimes \xi) \in \prod_{\eta \in \mathfrak{X}_\pi} L(2s, \eta_2 \omega_\tau) M \mathbb{C}[q^{-s}, q^s] \]
for every $v \in V_\pi$ and $\xi \in I^{sd}(\tau, s)$. From the definition of $I^{\text{hol}}(\tau, s)$, it is clear that some assertion holds if we replace $\xi$ with any holomorphic section.

We also need to analyze the poles of $\Psi(v \otimes \xi)$ when $\xi = M_1^0(\tau^*, 1 - s) \xi_1^{* - s}$ for some $\xi_1^{* - s} \in I^{\text{hol}}(\tau^*, 1 - s)$. For this, we use [CS98 Theorem 5.1] and Corollary 3.7 to deduce that
\[ L(2s, \omega_\tau)^{-1} M_1^0(\tau^*, 1 - s) \xi_1^{* - s} \in I^{\text{hol}}(\tau^*, 1 - s) \]
for every $\xi_1^{* - s} \in I^{\text{hol}}(\tau^*, 1 - s)$. It follows that
\[ \Psi(v \otimes M_1^0(\tau^*, 1 - s) \xi_1^{* - s}) \in L(2s, \omega_\tau) \prod_{\eta \in \mathfrak{X}_\pi} L(2s, \eta_2 \omega_\tau) M \mathbb{C}[q^{-s}, q^s] \]
for every $\xi_1^{* - s} \in I^{\text{hol}}(\tau^*, 1 - s)$. This completes the proof. \(\square\)

Remark 4.2. The characters $\eta_1, \eta_2$ appearing in the asymptotic expansions of the Whittaker functions of $\pi$ can be explicitly described by a result in the recent preprint of Chen in [Che21a].

Lemma 4.1 has the following consequence.

Corollary 4.3. Theorem 1.1 holds when $\pi$ is tempered.

Proof. Since $\pi$ is tempered and $\omega_\tau$ is unitary, the poles of $L(2s, \eta_2 \omega_\tau)$ are contained in the left half-plane $\Re(s) \leq 0$ for every $\eta \in \mathfrak{X}_\tau$. This follows from the fact that the exponents of a tempered representations are non-negative ([Wal03 Proposition III.3.2]). By Lemma 4.1 we see that the $L$-factors $L(s, \pi \times \tau)$ and $L(1 - s, \pi \times \tau^*)$ have no common poles. On the other hand, since the associated $L$-parameters are also tempered, same assertions hold for $L(s, \phi_\pi \otimes \phi_\tau)$ and $L(1 - s, \phi_\pi \otimes \phi_\tau^*)$. Now the corollary follows from the principle mentioned in the beginning of this section. \(\square\)
4.1. **Proof of Theorem 1.1** Now we are ready to prove Theorem 1.1. In view of Theorem 3.2 it is nature to separate the proof into two cases, depending on whether \( \sigma \) is endoscopy or not. Here we follow the terminologies in [GT11a] Section 6. In the proofs, we often regard \( \pi \) as a representation of \( \text{GSp}_4(F) \) with trivial central character.

4.1.1. **Endoscopy case.** In this case, \( \phi_\pi = \phi_{\tau_1} \oplus \phi_{\tau_2} \) for some irreducible generic representations \( \tau_1, \tau_2 \) of \( \text{GL}_2(F) \) with \( \omega_{\tau_1} = \omega_{\tau_2} = 1 \). According to [GT11a] Section 7, \( \pi \) is the theta lift of the representation \( \tau_1 \otimes \tau_2 \) of \( \text{GSO}_4(F) \). On the other hand, by Theorem 3.2 it remains to prove the case when \( \tau \) is isomorphic to an unramified twisted of \( \tau_1^\# \) or \( \tau_2^\# \). Assume without loss of generality that \( \tau \) is isomorphic to an unramified twisted of \( \tau_1^\# \), so that \( \tau_1 \) is also supercuspidal. If \( \tau_2 \) is an essentially discrete series representation and \( \tau_1 \) and \( \tau_2 \) are not isomorphic, then \( \pi \) is a discrete series representation by [GT11a] Theorem 5.6 (ii). In this case, Theorem 1.1 follows from Corollary 4.3. If \( \tau_1 \) and \( \tau_2 \) are isomorphic, then \( \pi \) is a tempered representation by [GT11a] Lemma 5.1 (a), Theorem 8.2 (i) and hence in this case, Theorem 1.1 again follows from Corollary 4.3.

Suppose that \( \tau_2 \) is an induced representation. Then since \( \omega_{\tau_2} = 1 \), \( \tau_2 \) is inducing from the characters \( \chi \) and \( \chi^{-1} \) of \( F^\times \). We may assume \( |\chi(z)| \geq 1 \). Then since \( \pi \) is generic, we must have \( \chi \not\equiv |z|_F^\pm \) and \( \pi \cong (\tau_1 \otimes \chi^{-1}) \rtimes \chi \) by [GT11b] Lemma 5.2 (a), Theorem 8.2 (v)). Here we follow the notations in [RS07] Page 35, meaning that \( \pi \) is a normalized induced representation of \( \text{GSp}_4(F) \) inducing from its Siegel parabolic subgroup with the data \( (\tau_1 \otimes \chi^{-1}) \rtimes \chi \). Now by Lemma 5.1 and [Che21a] Lemma 5.2, Type (X), we find that

\[
L(s, \pi \times \tau) \in L(2s, \omega_\tau)M+1C[q^s, q^s].
\]

Since \( \omega_\tau \) is assumed to be unitary, the \( L \)-factors \( L(s, \pi \times \tau) \) and \( L(1-s, \pi \times \tau^*) \) have no common poles. On the other hand, since

\[
\phi_\sigma \otimes \phi_{\tau_1} \cong (\phi_{\tau_1} \otimes \phi_{\tau_2}) \otimes \phi_{\tau_2} \cong (\phi_{\tau_1} \otimes \phi_{\tau_2}) \otimes (\phi_{\tau_1} \otimes \phi_{\tau_2}^{-1})
\]

as 8-dimensional representations, we see that the poles of \( L(s, \phi_\sigma \otimes \phi_{\tau_1}) \) are also lie in the vertical line \( \Re(s) = 0 \). Consequently, the \( L \)-factors \( L(s, \phi_\sigma \otimes \phi_{\tau_1}) \) and \( L(1-s, \phi_\sigma \otimes \phi_{\tau_1}) \) also have no common poles. This proves the Theorem 1.1 when \( \phi_\sigma \) is endoscopy.

4.1.2. **Non-endoscopy case.** In this case, the \( \pi \) has a non-zero theta lift to split \( \text{GSO}_6(F) \), and the description of \( \phi_\sigma \) can be found in [GT11a] Lemma 6.2, Theorem 5.6 (iii)]. More precisely, \( \phi_\sigma \) is either an irreducible 4-dimensional representation or \( \phi_\sigma = \phi \oplus \phi^\circ \) for some irreducible 2-dimensional representation \( \phi \), whose dual \( \phi^\circ \) is not isomorphic \( \phi \). In this first situation, \( \pi \) is a discrete series representation and hence we can apply Corollary 4.3 to conclude the proof.

Suppose that we are in the second situation. Then \( \phi \) corresponds to an irreducible supercuspidal representation \( \sigma \) of \( \text{GL}_2(F) \), and hence \( \phi_\sigma = \phi_\sigma \oplus \phi_\sigma^\circ \). Comparing this with [RS07] Table A.7 (VII), we have

\[
\pi \cong \omega_{\sigma}^{-1} \rtimes \sigma
\]

with \( \omega_{\sigma}^{-1} \neq 1 \) and \( \omega_{\sigma}^{-1} \neq 1 \), where \( \eta \) is a quadratic character of \( F^\times \) such that \( \sigma \cong \sigma \rtimes \eta \). The notation means that \( \pi \) is a normalized induced representation of \( \text{GSp}_4(F) \) inducing from its the Klingen subgroup with the data \( \sigma \rtimes \omega_{\sigma}^{-1} \). We indicate that \( \pi \) is not a discrete series representation, and is tempered if and only if \( \omega_{\sigma} \) is unitary. The rest of the proof is devoted to verify Theorem 1.1 for this case. First note that by Lemma 5.1 and [Che21a] Lemma 5.2, (VII)], we have

\[
L(s, \pi \times \tau) \in L(s, \pi \times \tau) \mathbb{C}[q^s, q^s]
\]

where

\[
L(s, \pi \times \tau) = L(2s, \omega_\tau)L(2s, \omega_\tau \omega_{\sigma})L(2s, \omega_\tau \omega_{\sigma}^{-1})M.
\]

On the other hand, since

\[
\phi_\sigma \otimes \phi_{\tau_1} \cong (\phi_\sigma \otimes \phi_{\tau_1}) \otimes (\phi_\sigma \otimes \phi_{\tau_1})
\]

as 8-dimensional representations, we have \( L(s, \phi_\sigma \otimes \phi_{\tau_1}) = L(s, \phi_\sigma \otimes \phi_{\tau_1})L(s, \phi_\sigma \otimes \phi_{\tau_1}) \). The proof of the identity \( L(s, \pi \times \tau) = L(s, \phi_\sigma \otimes \phi_{\tau_1}) \) consists of three steps.
The first step is to show
\[(4.1) \quad L(s, \phi_\tau \otimes \phi_\tau) \in \mathcal{L}(s, \pi \times \tau) \mathbb{C}[q^{-s}, q^s].\]
For this we first recall that by [GJ78 Proposition 1.2], \(L(s, \phi_\tau \otimes \phi_\tau)\) has a pole at \(s_0 \in \mathbb{C}\) if and only if \(\sigma \equiv \tau^\nu \otimes |_{\mathcal{F}}^{-s_0}\), in which case the pole is simple. The assertion being trivial when \(L(s, \phi_\tau \otimes \phi_\tau) = 1\). So let us assume that \(L(s, \phi_\tau \otimes \phi_\tau)\) has poles. Then we have
\[\sigma \equiv \tau^\nu \otimes |_{\mathcal{F}}^{-s_0} \quad \text{or} \quad \sigma^\nu \equiv \tau^\nu \otimes |_{\mathcal{F}}^{-s_0}\]
for some complex numbers \(s_0\) and \(s_0^\nu\). In the first case, since \(\omega_\nu \omega_\tau = |_{\mathcal{F}}^{-2s_0}\) by considering the central characters on both sides. We find that
\[L(s, \phi_\tau \otimes \phi_\tau) \in L(2s, \omega_\tau \omega_\sigma)^M \mathbb{C}[q^{-s}, q^s].\]
Similar argument shows
\[L(s, \phi_\tau \otimes \phi_\tau) \in L(2s, \omega_\tau \omega_\sigma^{-1})^M \mathbb{C}[q^{-s}, q^s]\]
and the assertion follows.

Our second step is to show
\[(4.2) \quad L(s, \pi \times \tau) \in L(s, \phi_\tau \otimes \phi_\tau) \mathbb{C}[q^{-s}, q^s].\]
Note that by [441], if \(L(s, \pi \times \tau)\) and \(L(1-s, \pi \times \tau^\nu)\) have no common poles, then \(L(s, \pi \times \tau) = L(s, \phi_\tau \otimes \phi_\tau)\). Note also that \(L(s, \pi \times \tau)\) and \(L(1-s, \pi \times \tau^\nu)\) have common poles would imply either \(\omega_\sigma = |_{\mathcal{F}}|^{-2}\) or \(\omega_\sigma^\nu = |_{\mathcal{F}}|^{-2}\). With these in mind, we now prove the second assertion. Certainly, it suffices to verify the assertion when \(L(s, \pi \times \tau)\) and \(L(1-s, \pi \times \tau^\nu)\) have common poles. For this we introduce an auxiliary \(\lambda \in \mathbb{R}\) and consider the representation
\[\pi_\lambda \equiv (\omega_\sigma^{-1} \cdot |_{\mathcal{F}}^{-2\lambda}) \times (\sigma \otimes |_{\mathcal{F}}^\lambda).\]
Note that the central character of \(\pi_\lambda\) is again trivial. Let \(\epsilon > 0\) be small so that \(\pi_\lambda\) remains irreducible for \(|\lambda| < \epsilon\). Then for \(0 < |\lambda| < \epsilon\), we have \(L(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\pi)\) by the observations just mentioned. So for such \(\lambda\), one has
\[(4.3) \quad \Psi(v_\lambda \otimes \xi_\lambda) \in L(s, \phi_\lambda \otimes \phi_\lambda) \mathbb{C}[q^{-s}, q^s]\]
for every \(v_\lambda \in \mathcal{V}_{\pi, \lambda}\) and \(\xi_\lambda \in \mathcal{P}_{\mathfrak{hol}}(\tau, s)\). We would like to show this also holds when \(\lambda = 0\). To do so, we use the holomorphicity of Whittaker functionals and the formal Laurent series. More precisely, by [CS80 Section 2] and [Sha78 Section 3], for a given \(v \in \mathcal{V}_\pi\), there exists for each \(\lambda\) a function \(W_\lambda\) in the Whittaker model of \(\pi_\lambda\) such that
- \(W_0 = W_0^\text{def}\);
- there is an open compact subgroup \(K\) such that \(W_\lambda(gk) = W_\lambda(g)\) for all \(g \in \text{SO}_5(F), k \in K\) and \(\lambda\);
- for each \(g \in \text{SO}_5(F)\), we have \(W_\lambda(g) \in \mathbb{C}[q^{-\lambda}, q^\lambda]\) as a function of \(\lambda\).

Let \(v_\lambda \in \mathcal{V}_{\pi, \lambda}\) be such that its associated Whittaker function is \(W_\lambda\). Then the second property implies that there exists \(N > 0\), which is independent of \(\lambda\) such that
\[(4.4) \quad W_{v_\lambda}(f_{(\alpha\beta, \beta)}) = 0\]
if \(\alpha\) or \(\beta\) does not contain in \(\varpi^{-N} \mathfrak{o}\). For \(\xi_\lambda \in \mathcal{P}^{\text{std}}(\tau, s)\), the formal Laurent series \(\Psi_{v_\lambda \otimes \xi_\lambda}(X)\) can be written as
\[\Psi_{v_\lambda \otimes \xi_\lambda}(X) = \sum_{m \in \mathbb{Z}} X^m \Psi^m(v_\lambda \otimes \xi_\lambda)\]
with \(\Psi^m(v_\lambda \otimes \xi_\lambda)\) given by (2.20). Form this discussions in (2.20) we know that \(\Psi^m(v_\lambda \otimes \xi_\lambda)\) is independent of \(s\). Moreover, by (3.3), one sees that it can be written as a finite sum, each of which gives an element in \(\mathbb{C}[q^{-\lambda}, q^\lambda]\) as a function of \(\lambda\). It follows that \(\Psi^m(v_\lambda \otimes \xi_\lambda) \in \mathbb{C}[q^{-\lambda}, q^\lambda]\) as a function of \(\lambda\) for all \(m\).

To proceed, observe that \(L(s, \phi_\pi, \phi_\pi) = L(s + \lambda, \phi_\pi, \phi_\pi)\) for \(|\lambda| < \epsilon\). Thus if \(P(X) \in \mathbb{C}[X]\) is such that \(P(0) = 1\) and \(L(s, \phi_\pi, \phi_\pi) = P(q^{-s})^{-1}\), then \(L(s, \phi_\pi, \phi_\pi) = P(q^{-s})^{-1}\). Now let us write
\[(4.5) \quad P_{\pi, \tau}(q^{-\lambda}X)\Psi_{v_\lambda \otimes \xi_\lambda}(X) = \sum_{m \in \mathbb{Z}} c_m(\lambda)X^m.\]
Then we have $c_m(\lambda) \in \mathbb{C}[q^{-\lambda}, q^{\lambda}]$ as a function of $\lambda$. As explained in [IPSSS2] (4.3) and [Kap13, Section 8.2], (4.3) implies that for each $\ell \geq 0$, there is $N_\ell > 0$ such that $c_m(\lambda) = 0$ whenever $|m| > N_\ell$. We claim that there exists $N > 0$ such that $c_m(\lambda) = 0$ for all $|\lambda| < \varepsilon$ and $|m| > N$. Indeed, since each non-zero $c_m(\lambda)$ has finitely many roots, while there are uncountable many $0 < |\lambda| < \varepsilon$, there must exists $N > 0$ such that $c_m(\lambda) = 0$ for all $0 < |\lambda| < \varepsilon$ and $|m| > N$. Since $c_m(\lambda)$ are continuous, the claim follows. The implies that (4.3) is a finite sum. In particular, by substituting $\lambda = 0$ and $X = q^{-s}$ in (4.3), we get that

$$
(4.6) \quad \Psi(v \otimes \xi_s) \in L(s, \phi_\pi \otimes \phi_\tau)\mathbb{C}[q^{-s}, q^s]
$$

for all $v \in \mathcal{V}_\pi$ and $\xi_s \in \mathcal{I}^{\mathrm{std}}(\tau, s)$, and hence for all $\xi_s \in L(1, \phi_\pi \otimes \phi_\tau)$ by the definition of holomorphic sections.

To verify (4.2), it remains to show that (4.10) still valid if $\xi_s = M^a(1/(\tau, s)\xi^*_{1-s})$ for some $\xi^*_{1-s} \in \mathcal{I}^{\mathrm{hol}}(\tau^*, 1-s)$. But this follows immediately from the identity $\gamma(s, \pi \times \tau^*, \psi) = \gamma(s, \phi_\pi \otimes \phi_{\tau^*}, \psi)$ and (4.6) since

$$
\frac{\Psi(v \otimes M^a(1/(\tau^*, 1-s)\xi^*_{1-s}))}{L(s, \phi_\pi \otimes \phi_{\tau^*})} = \frac{\Psi(v \otimes \xi^*_{1-s})}{L(1-s, \phi_\pi \otimes \phi_{\tau^*})} \in \mathbb{C}[q^{-s}, q^s].
$$

This finishes the proof of our second step.

The last step is to show that $L(s, \pi \times \tau^*) = L(s, \phi_\pi \otimes \phi_{\tau^*}) = L(s, \phi_\pi \otimes \phi_{\tau^*})L(s, \phi_{\tau^*} \otimes \phi_\tau)$. We divide the proof into two conditions depending on whether $\tau^*$ is isomorphic to an unramified twist of $\tau$. Suppose first that $\tau^*$ is not isomorphic to an unramified twist of $\tau$. In this case, we can not have both $\sigma$ and $\sigma^\vee$ are isomorphic to unramified twists of $\tau^*$. Consequently, the $L$-factors $L(s, \phi_\pi \otimes \phi_\tau)$ and $L(1-s, \phi_\pi \otimes \phi_{\tau^*})$ have no common poles, and hence $L(s, \pi \times \tau^*) = L(s, \phi_\pi \otimes \phi_\tau)$ by (4.2).

Suppose that $\tau^*$ is isomorphic to an unramified twist of $\tau$. Then by (3.13) and because the similar identity holds for $L(s, \phi_\pi \otimes \phi_{\tau^*})$, we may further assume that $\sigma$ is self-dual, i.e. $\tau^* \cong \tau$. Now if $\sigma$ is not isomorphic to unramified twists of $\tau^*$, then we have $L(s, \phi_\pi \otimes \phi_{\tau^*}) = 1$ and the assertion follows again from (4.2). So let us assume that $\tau^* \cong \tau \otimes |\cdot|_{F^\nu}^{s_0}$ for some $s_0 \in \mathbb{C}$. Then we have $\sigma^\vee \cong \tau \otimes |\cdot|_{F^\nu}^{s_0} \cong \tau^* \otimes |\cdot|_{F^\nu}^{s_0}$ and hence by [GJ98, Corollary 1.3]

$$
L(s, \phi_\pi \otimes \phi_\tau) = (1 - q^{-\kappa(s-s_0)})^{-1} (1 - q^{-\kappa(s+s_0)})^{-1}
$$

with $\kappa = 2$ if $\tau \cong (\tau \otimes \eta)$ and $\kappa = 1$ otherwise, where $\eta$ is the unramified quadratic character of $F^\times$. Similarly, we have

$$
L(1-s, \phi_\pi \otimes \phi_{\tau^*}) = (1 - q^{-\kappa(1-s-s_0)})^{-1} (1 - q^{-\kappa(1-s+s_0)})^{-1}.
$$

We are done if we can show that these $L$-factors have no common poles. Suppose in contrary that they have common poles. Then from the shapes of the $L$-factors, we must have

$$
\frac{2\pi n \sqrt{-1}}{\kappa \ln q}
$$

for some $n \in \mathbb{Z}$. It follows that $\omega^{-1}_{\tau^*} = \omega^{-1}_{\tau} \cdot |\cdot|_{F^\nu}^{s_0}$ by the assumption $\sigma^\vee \cong \tau^* \otimes |\cdot|_{F^\nu}^{s_0}$. But this contradicts to the assumption on $\omega^{-1}_{\tau^*}$ since $\tau \otimes \omega^{-1}_{\tau^*} \cong \tau^* \cong \tau$. This concludes the proof of the non-endoscopy case, and hence the proof of Theorem [1.1].

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