Parity and Ruin in a Stochastic Game

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We study an elementary two-player card game where in each round players compare cards and the holder of the smallest card wins. Using the rate equations approach, we treat the stochastic version of the game in which cards are drawn randomly. We obtain an exact solution for arbitrary initial conditions. In general, the game approaches a steady state where the card densities of the two players are proportional to each other. The leading small size behavior of the initial card densities determines the corresponding proportionality constant, while the next correction governs the asymptotic time dependence. The relaxation towards the steady state exhibits a rich behavior, e.g., it may be algebraically slow or exponentially fast. Moreover, in ruin situations where one player eventually wins all cards, the game may even end in a finite time.

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Numerous phenomena in social and economic sciences involve multiple interacting agents. The interaction between these agents often leads to exchange of quantities such as capital, goods, political opinions, etc. [1–4]. Games are widely employed in modelling collective behavior especially in the context of economics [5], with recent examples ranging from evolution of trading strategies in a stock market [6–8] to bidding in auctions [9]. Here, we investigate a stochastic null strategy card game. By considering the “thermodynamic limit” where the initial number of cards is infinite, we show that rate equations provide a natural framework for analyzing game dynamics.

Our two-player game is defined as follows. Each player starts with a certain number of cards. At each round players draw a card randomly from their deck and compare the card values. The player with the smaller card wins the round and gets both cards. This is repeated ad infinitum or until one of the players gains all cards. This stochastic adaptation of the elementary card game “war” is also motivated by a recently introduced auction bidding model where the lowest bid is rewarded [10].

Our main result is that one specific aspect of the initial card distribution, namely, the small size tail, governs the dynamics of the game. Let us denote by \( A \) and \( B \) the two players, and let their initial card densities be \( a_0(x) \) and \( b_0(x) \), respectively. In the long time limit, a steady state is approached with the card densities of both players being equal to a fraction of the total card density, \( a_\infty(x) = \alpha [a_0(x) + b_0(x)] \), with \( a_\infty(x) \) the limiting card density of player \( A \). While a family of steady state solutions characterized by the parameter \( 0 \leq \alpha \leq 1 \) is in principle possible, the leading small size behavior of the initial distributions selects a specific value \( \alpha = \lim_{x \to 0} \frac{a_0(x)}{a_0(x) + b_0(x)} \). Moreover, the next leading correction determines how the system approaches the steady state. The corresponding time dependent behavior may be algebraic or exponential. Interesting behaviors also occur when one player captures all cards. In this case, the game duration maybe finite or infinite. Additionally, using numerical simulations we show that the theoretical predictions concerning the game duration extend to deterministic realizations of the game.

Let the initial number of cards of player \( A \) and \( B \) be \( N_A \) and \( N_B \), respectively, and let the total number be \( N = N_A + N_B \). We shall take the thermodynamic limit \( N_A, N_B, N \to \infty \) such that the initial fractions \( N_A/N \) and \( N_B/N \) are fixed. Then the card exchange process can be conveniently described via rate equations for \( a(x,t) \) and \( b(x,t) \), the densities of cards with value \( x \) at time \( t \) for players \( A \) and \( B \), respectively. These densities evolve according to the nonlinear integro-differential equations

\[
\frac{\partial}{\partial t} a(x,t) = R(x,t), \quad \frac{\partial}{\partial t} b(x,t) = -R(x,t),
\]

with the gain (loss) term \( R(x,t) \) given by

\[
R = \frac{1}{A(t)B(t)} \left[ b(x,t) \int_0^x dy a(y,t) - a(x,t) \int_0^x dy b(y,t) \right].
\]

Here

\[
A(t) = \int_0^\infty dx a(x,t), \quad B(t) = \int_0^\infty dx b(x,t)
\]

are the fraction of cards possessed by players \( A \) and \( B \), respectively. Clearly,

\[
A(t) + B(t) = 1.
\]

The rate equations (1) reflect the nature of the game as the rate by which player \( A \) gains (loses) cards of value \( x \) is proportional to the fraction of his opponent’s cards which is larger (smaller) than \( x \). The overall factor \( |AB|^{-1} \) ensures that on average, every opposing pair of cards comes into play once per unit time. The minimal card value was
The second transformation essentially reduces any total density. Formed card densities are found from the relations

\[ \xi, \tau \]

\[ a(x, t) + b(x, t) = u_0(x), \]

(4)

where \( u_0(x) = a_0(x) + b_0(x) \) is the initial total density. Second, the density of the minimal card remains constant throughout the evolution: \( a(0, t) = a_0(0) \), where \( a_0(x) \equiv a(x, t = 0) \), and similarly for \( B \).

The steady state behavior is quite generic. The cumulative card densities, \( A(x, t) = \int_0^x dy a(y, t) \) and \( B(x, t) = \int_0^x dy b(y, t) \), satisfy \( A'/A = B'/B \) in the long time limit. Thus, \( A_\infty(x) \propto B_\infty(x) \), and consequently, the limiting card densities, \( a_\infty(x) = A_\infty(x) \) and \( b_\infty(x) = B_\infty(x) \), are proportional to each other. The conservation law \( B \) implies that each of the limiting card densities equals a fraction of the overall card density

\[ a_\infty(x) = \alpha u_0(x), \quad b_\infty(x) = (1 - \alpha)u_0(x). \]

(5)

In principle, for a given total card density \( u_0(x) \), there is a family of steady state solutions characterized by \( 0 \leq \alpha \leq 1 \). Moreover, initial conditions where the densities are proportional to each other do not evolve further in time regardless of \( \alpha \). Still, for a given initial condition a specific value of \( \alpha \) is selected. This value can easily be found for a class of initial conditions with non-vanishing minimal card densities, \( u_0(0) > 0 \). Consider the density of the smallest cards \( x = 0 \). Equation (3) gives \( a_\infty(0) = \alpha u_0(0) \), while the second conservation law implies \( a_\infty(0) = a_0(0) \), and hence \( \alpha = a_0(0)/[a_0(0) + b_0(0)] \). This simple argument suggests that the smallest cards govern the outcome of the game. In the following, we solve for the full time dependent behavior and show that in general, the limiting small size tail of the two distributions dictates \( \alpha \).

To solve the time dependent behavior, we make two simplifying transformations. First, the overall rate by which the exchange occurs \( |AB|^{-1} \) can be absorbed into a modified time variable \( \tau \), defined via

\[ \tau = \int_0^t ds \ [A(s)B(s)]^{-1}. \]

(6)

The second transformation essentially reduces any total density \( u_0(x) \) to a uniform density by introducing the variable \( \xi \)

\[ \xi = \int_0^x dy u_0(y). \]

(7)

Suppressing the explicit time dependence, the transformed card densities are found from the relations \( a(\xi) \, d\xi = a(x) \, dx \) and \( b(\xi) \, d\xi = b(x) \, dx \). Clearly, these satisfy \( a(\xi) = a(x)/u_0(x) \) and \( b(\xi) = b(x)/u_0(x) \). In the following, we shall omit the bar. The conservation law \( B \) becomes

\[ a(\xi, \tau) + b(\xi, \tau) = 1, \]

(8)

i.e., the transformed total density is uniform on the interval \([0,1]\) (note that Eqs. (3) and (4) imply \( 0 \leq \xi \leq 1 \)).

The above transformations simplify the evolution equations, and given the linear dependence \( B \), it suffices to solve for \( a \)

\[ \frac{\partial}{\partial \tau} a(\xi, \tau) = b(\xi, \tau) \int_0^\xi dy a(\eta, \tau) - a(\xi, \tau) \int_0^\xi dy b(\eta, \tau). \]

Replacing \( b(\xi, \tau) \) with \( 1 - a(\xi, \tau) \) linearizes this equation

\[ \frac{\partial}{\partial \tau} a(\xi, \tau) = \int_0^\xi dy a(\eta, \tau) - \xi a(\xi, \tau), \]

and differentiating with respect to \( \xi \) yields further simplification

\[ \left( \frac{\partial}{\partial \tau} + \xi \right) \frac{\partial}{\partial \xi} a(\xi, \tau) = 0. \]

(9)

Integrating over \( \tau \) and then over \( \xi \) we arrive at our primary result, the exact time dependent solution for arbitrary initial conditions:

\[ a(\xi, \tau) = \alpha + \int_0^\xi dy a_0(\eta) e^{-\eta \tau}. \]

(10)

Hereinafter we utilize the notations \( a_0(\xi) \equiv a(\xi, \tau = 0) \), \( a_0(\xi) \equiv \frac{\partial}{\partial \xi} a_0(\xi) \), and \( \alpha = a_0(\xi = 0) \).

Let us again consider the steady state. In the long time limit \( \tau \to \infty \), the integral in (10) vanishes and the densities become uniform \( a(\xi, \tau) \to \alpha \) and \( b(\xi, \tau) \to 1 - \alpha \). Hence in terms of the original variable \( x \), both densities are proportional to \( u_0(x) \) according to Eq. (3), with \( \alpha = a_0(\xi = 0) = a_0(x = 0)/u_0(x = 0) \). Even when \( u_0(x) \) vanishes or diverges as \( x \to 0 \), the parameter \( \alpha \) is well-defined and using l’Hospital rule, it is given by

\[ \alpha = \lim_{x \to 0} \frac{a_0(x)}{a_0(x) + b_0(x)}. \]

(11)

Thus, if the two distributions exhibit different leading behaviors, say \( \lim_{x \to 0} b_0(x)/a_0(x) = 0 \), then player A eventually ruins player B. Hence, the small card tail \( x \to 0 \) provides the necessary selection criteria determining which of the family of solutions \( B \) is eventually selected by the dynamics.

We now study the approach to the steady state. For example, the density \( A(\tau) = \int_0^1 d\xi a(\xi, \tau) \) is given by

\[ A(\tau) = \alpha + \int_0^1 d\xi (1 - \xi) a_0(\xi) e^{-\xi \tau}. \]

(12)

While the steady state behavior is determined by the leading small argument behavior of \( a_0(\xi) \), the relaxation towards the final state is governed by the correction to
the leading behavior. Let us consider the following small argument behavior

\[ a_0(\xi) \approx \alpha + \gamma \xi^\delta \quad \xi \to 0, \quad (13) \]

with \( \delta > 0 \). Then, one has \( A(\tau) - \alpha \approx \gamma \Gamma(\delta + 1) \tau^{-\delta} \). However, in terms of the actual time variable \( t \), a richer variety of behaviors is exhibited.

First, suppose that the system approaches an active steady state, i.e., \( 0 < \alpha < 1 \). Then \( dt/d\tau \sim B(\tau) \sim \tau^{-\delta} \) and consequently, \( t \sim \tau^{1-\delta} \). Therefore, for \( \delta \leq 1 \) representing weak initial advantage of the eventual winner, the game duration is infinite:

\[
1 - A(t) \sim \begin{cases} 
\frac{t^{1+\delta}}{e^{const \times t}} & \delta < 1; \\
1 & \delta = 1.
\end{cases}
\quad (15)
\]

In the complementary situation of strong initial advantage for the eventual winner, \( \delta > 1 \), the game duration is finite:

\[
A(t_f) = 1. \quad (16)
\]

The terminal time can be determined from the integral \( t_f = \int_0^\infty d\tau A(\tau) \) \([1 - A(\tau)]\). Using Eq. (14) and recalling that \( \alpha = 1 \) yields this time as an explicit function of the initial conditions

\[
t_f = - \int_0^1 dx \frac{1 - \xi}{\xi} a'_0(\xi) + \int_0^1 dx_1 dx_2 \frac{(1 - \xi_1)(1 - \xi_2)}{\xi_1 + \xi_2} a'_0(\xi_1)a'_0(\xi_2). \quad (17)
\]

For example, the initial density \( a_0(\xi) = 1 - \xi^2 \) yields \( t_f = \frac{2}{15} + \frac{16}{15} \ln 2 \approx 0.87269 \). Additionally, the time dependent approach towards the final state is algebraic,

\[
1 - A(t) \sim (t_f - t)^{1+\delta}, \quad (18)
\]

sufficiently close to the terminal time \( t \to t_f \). As expected, the density decreases linearly with time when the disparity between the two players becomes very large in the limit \( \delta \to \infty \).

Thus if the system approaches a trivial steady state with one player winning all cards, the temporal behavior can be algebraically slow or exponentially fast. Moreover, every positive power can be realized. Remarkably, if the initial disparity between the two players is sufficiently large, the game ends in a finite time. Interestingly, such disparity is expressed only in terms of the density of the smallest cards, with the larger cards practically irrelevant to the game outcome.

Next, we analyze the time dependent evolution of the entire card density, not simply the overall number density. Evaluating the leading behavior of the density (14) in the long time limit, we find that the density exhibits a boundary layer structure

\[
a(\xi, \tau) - \alpha \approx \begin{cases} 
\gamma \xi^\delta \tau^{-\delta} & \xi \ll \tau^{-1} ; \\
\gamma \Gamma(\delta + 1) \tau^{-\delta} & \xi \gg \tau^{-1}.
\end{cases} \quad (19)
\]

The scale \( \xi_0 \sim \tau^{-1} \) underlies the distribution. Cards larger than this scale have already relaxed to the limiting distribution, while cards smaller than this scale have yet to exchange hands and hence, are still dominated by the initial distribution. In other words, smaller cards are slower to equilibrate, consistent with the fact that the small size tail dominates the asymptotic behavior.

We now briefly discuss the case where the number of card flavors is finite, or in other words, discrete card distributions

\[
a(x, t) = \sum_{n=1}^k a_n(t) \delta(x - x_n), \quad (20)
\]

\[
b(x, t) = \sum_{n=1}^k b_n(t) \delta(x - x_n),
\]

with \( x_1 = 0 \) and \( x_n < x_{n+1} \). The discrete version of the rate equations can be written and solved directly using a series of transformation which mimics the ones used above. Instead, we shall insert the initial conditions (20) in the general continuous case solution (14).

Denote again \( u_n(t) = u_n(0) + b_n(0) \) the total card concentration. The variable \( \xi_n = \sum_{m=1}^{n-1} u_m(0) \) plays the role of \( \xi \) and the time variable \( \tau \) remains as in Eq. (14). The solution (14) reads

\[
a_n(\tau) = \frac{a_1(0)}{u_1(0)} + \sum_{m=2}^n \left( \frac{a_m(0)}{u_m(0)} - \frac{a_{m-1}(0)}{u_{m-1}(0)} \right) e^{-\xi_m \tau}. \quad (21)
\]

Since all terms in the summation eventually vanish, the two players approach a limiting distribution which is proportional to the initial distribution \( a_n(\infty) = \alpha u_n(0) \) with \( \alpha = a_1(0)/u_1(0) \), in accordance with Eq. (14). In general, the approach to the steady state is exponential. We first discuss the case \( 0 < \alpha < 1 \). Since \( A_\infty = \alpha \), one has \( t \to \alpha(1 - \alpha) \tau \) asymptotically. Hence, the relaxation towards the steady state is exponential

\[
A(t) - \alpha \sim e^{-const \times t}. \quad (22)
\]

In the complementary case when one player wins all cards, \( \alpha = 1 \), the approach is dominated by the first non-vanishing term in the summation, namely, the first non-vanishing \( b_n(0) \). In this case, \( dt/d\tau \propto \exp(-const \times \tau) \), and consequently, the game duration is finite as in
In summary, the behavior in the discrete case is different from the continuous case in that the time-dependent behavior is generally exponential. An additional difference is that when one player captures all cards, the game duration is always finite.

In the above, we discussed games with an infinite number of cards. Nevertheless, one can apply the above results to realistic situations when both players start with a finite number, say $N$, cards. We note that the time unit used above is of the order $N^2$ rounds in an actual game. For the case $\delta > 1$ one therefore predicts a duration

$$T_f \sim N^2,$$

with $T_f$ the number of rounds. The duration in the marginal $\delta = 1$ case can be estimated using the average time it takes for player $B$ to get down to one card $B(t) = N^{-1}$. Utilizing the exponential decay of $B(t)$, we find that there is an additional logarithmic dependence, $T_f \sim N^2 \ln N$, in this case.

Monte Carlo simulations are consistent with these predictions. In the simulations, each player starts with $N$ cards drawn from a uniform distribution in the range $0 < x < 1$. Eventually, the player holding the smallest card wins. Our theory describes the stochastic realization of the game where cards are drawn randomly from the deck. We also examined the deterministic case where the card order is fixed throughout the game. In this version, the winner of a round places both cards in the bottom of the deck. In both cases, we find diffusive terminal times as in Eq. (23). Nevertheless, the two cases differ with the stochastic game ending faster than the deterministic one (see Fig. 1). Additionally, we find that this diffusive time scale characterizes the entire distribution of terminal times, as fluctuations in the terminal time are proportional to the mean $(T_f^2 - \langle T_f \rangle^2) \propto (T_f)^2$.

In closing, we studied a stochastic two-player card game using the rate equations approach. We found that extremal characteristics of the initial conditions select a particular steady state out of a family of possible solutions. Eventually, the card densities of the players become proportional to each other. However, the players generally possess different overall number of cards and it is even possible that one player gains all cards. The approach towards the steady state exhibits rich behavior. Large cards tend to equilibrate faster than small cards, and the distribution develops a boundary layer structure. The time dependent behavior of the overall density is algebraic in cases where a steady state is approached. In the complementary case where one player gains all cards, the game may end in a finite or an infinite time. The relative initial advantage of the winner, characterized by the correction to the leading extremal behavior, determines the game duration in this case.

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