The solvability conditions and exact solutions to some quaternion tensor systems

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Abstract: We derive necessary and sufficient conditions for the existence of the exact solution to the Sylvester-type quaternion tensor system

\[ A_i *_N X_i + Y_i *_M B_i + C_i *_N Z_i *_M D_i + F_i *_N Z_{i+1} *_M G_i = E_i, i = 1, 3 \]

using Moore-Penrose inverse, and present an expression of the general solution to the system when it is solvable. As an application of this system, we provide the solvability conditions and general solutions for the Sylvester-type quaternion tensor system

\[ A_i *_N Z_i *_M B_i + C_i *_N Z_{i+1} *_M D_i = E_i, i = 1, 4. \]

This paper can also serve as extensions to some known results.

Keywords: quaternion tensor equation; Einstein product; general solution

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1. Introduction

About two hundred years ago the skew fields of quaternions was introduced by Sir William Rowan Hamilton\[16\], which has all properties of fields except for the commutativity of multiplication. The theoretical knowledge about quaternions is a good part of algebra; see, for example, \[39\]. Dating back to 1936 \[52\], the literature on matrices over quaternions is still fragmentary. Nevertheless, renewed interest and applications have been witnessed recently, such as signal and color image procession, quaternionic quantum mechanics (qQM) and many other fields (eg. \[4\], \[7\], \[17\], \[26\], \[44\]– \[48\], \[51\], \[53\], \[54\]).

Since 1952, the first generalized Sylvester matrix equation was studied by Roth. This seems to have stimulated several authors who have discussed the generalized Sylvester matrix equations and their applications, for instance, image processing \[24\], eigenvalue assignment problems \[6\], neural network \[55\] and so on. It is known that matrix version is the special form of tensor version. Tensor theory has arisen lots of applications, such as signal processing \(32\)–\(35\), graphic analysis \[23\] and so on (eg. \[1\], \[2\], \[8\], \[10\], \[11\], \[13\]–\[15\], \[19\], \[22\], \[30\], \[36\]–\[38\], \[40\], \[41\], \[49\], \[50\]).

There are also some papers about the generalized Sylvester tensor equation (eg. \[5\], \[9\], \[19\], \[25\], \[27\], \[28\], \[43\], \[49\]). For example, Sun et al. \[43\] derived the solvability conditions and solutions for the Sylvester tensor equation

\[ A *_N X + X *_M D = C, \]

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where $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, $D \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $C \in \mathbb{R}^{J_1 \times \cdots \times J_M \times I_1 \times \cdots \times I_N}$. The solvability conditions and solutions of two-sided Sylvester tensor equation

$$A \ast_N X_1 \ast_M B + C \ast_N X_2 \ast_M D = E$$

was studied by He [19], where $A \in \mathbb{H}^{I_1 \times \cdots \times I_N \times Q_1 \times \cdots \times Q_N}$, $B \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M}$, $C \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M}$, $D \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$, $E \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$. He et al. [20] gave some solvability conditions for the Sylvester quaternion tensor system

$$A_i \ast_{N_i} X_1 + Y_1 \ast_M B_1 + C_1 \ast_N Z_1 \ast_M D_1 + F_1 \ast_N Z_2 \ast_M G_1 = E_i$$

(1.1)

where $A_i \in \mathbb{H}^{I_1 \times \cdots \times I_N \times Q_1 \times \cdots \times Q_N}$, $B_i \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M}$, $C_i \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M}$, $F_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$, $G_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$, $E_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$, $i = 1, 2, 3$.

In this article our purpose is to present a treatment of methods that are useful in the study of solving Sylvester tensor systems

$$A_i \ast_{N_i} X_1 + Y_1 \ast_M B_1 + C_1 \ast_N Z_1 \ast_M D_1 + F_1 \ast_N Z_2 \ast_M G_1 = E_i$$

(1.2)

where

$$A_i \in \mathbb{H}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}, \quad B_i \in \mathbb{H}^{O_1 \times \cdots \times O_M \times J_1 \times \cdots \times J_M},$$

$$C_i \in \mathbb{H}^{I_1 \times \cdots \times I_N \times L_1 \times \cdots \times L_N}, \quad D_i \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M},$$

$$F_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}, \quad G_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M},$$

$$E_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}, \quad i = 1, 2, 3.$$}

we establish the necessary and sufficient conditions for the existence of general solution in terms of Moore-Penrose inverse, and present general solution to System (1.2).

During investigation about (1.2), we are motivated to find that it is useful for solving the following Sylvester quaternion tensor system with five variables:

$$A_i \ast_{N_i} Z_1 \ast_M B_1 + C_1 \ast_N Z_2 \ast_M D_1 = E_i$$

(1.3)

where $A_i \in \mathbb{H}^{I_1 \times \cdots \times I_N \times Q_1 \times \cdots \times Q_N}$, $B_i \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M}$, $C_i \in \mathbb{H}^{P_1 \times \cdots \times P_M \times J_1 \times \cdots \times J_M}$, $D_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$, $E_i \in \mathbb{H}^{Q_1 \times \cdots \times Q_N \times J_1 \times \cdots \times J_M}$. Also, if $B_i, C_i$ are not all zeros, the system (1.3) can also serve as an extended form to the system (1.1).

The paper is divided, structurally, into three parts. The first part (Section 2) contains some basic notations, definitions and lemmas as tools. The second part (Section 3) give the solvability conditions and general solutions to quaternion tensor system (1.2). In the third part (Section 4), we provide the solvability conditions and general solutions to quaternion tensor system (1.3).
2. Preliminaries

A tensor $A = (a_{i_1...i_N})_{1 \leq i_j \leq I_j}$ (j = 1, ..., N) of order $N$ is a multidimensional array with $I_1I_2\cdots I_N$ entries, where $N$ is a positive integer. The sets of tensors of order $N$ with dimension $I_1 \times I_2 \times \cdots \times I_N$ over the complex field $\mathbb{C}$, the real field $\mathbb{R}$ and the real quaternion algebra $\mathbb{H}$ are represented, respectively, by $\mathbb{C}^{I_1\times I_2\times \cdots \times I_N}$, $\mathbb{R}^{I_1\times I_2\times \cdots \times I_N}$ and $\mathbb{H}^{I_1\times I_2\times \cdots \times I_N}$. There are more definitions and propositions of quaternions refer to the book [39] and the paper [54].

A tensor can be viewed as a matrix if $N = 2$. For review and convenience in reference, this section contains a summary of necessary facts about Moore-Penrose inverse, $\{i\}$-inverse of quaternion tensors over $\mathbb{H}^{I_1\times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ as well as some basic definitions related the Einstein product. Beyond this, we survey some special quaternion tensor systems and their solutions when they are solvable.

**Definition 2.1.** [12] Let $A \in \mathbb{H}^{I_1\times \cdots \times I_P \times K_1 \times \cdots \times K_N}, B \in \mathbb{H}^{K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$, their Einstein product is satisfying the associative law, which is

$$(A \ast_N B)_{i_1\cdots i_Pj_1\cdots j_M} = \sum_{k_1,\ldots,k_N=1}^{K_1,\ldots,K_N} a_{i_1\cdots i_Pk_1\cdots k_N} b_{k_1\cdots k_Nj_1\cdots j_M},$$

where $A \ast_N B \in \mathbb{H}^{I_1\times \cdots \times I_P \times J_1 \times \cdots \times J_M}$.

For simplicity we will denote the above summation by $\sum_{k_1\cdots k_N}$ or $\sum_{k_1,\ldots,k_N}$. When $N = P = M = 1$, we have that $A, B$ are quaternion matrices, and their Einstein product is the usual matrix product.

**Definition 2.2.** [19] Let $A \in \mathbb{H}^{I_1\times \cdots \times I_N \times K_1 \times \cdots \times K_N}$, the tensor $X \in \mathbb{H}^{K_1 \times \cdots \times K_N \times I_1 \times \cdots \times I_N}$ satisfying

1. $A \ast_N X \ast_N A = A$;
2. $X \ast_N A \ast_N X = X$;
3. $(A \ast_N X)^* = A \ast_N X$;
4. $(X \ast_N A)^* = X \ast_N A$,

is called the Moore-Penrose inverse of $A$, abbreviated by M-P inverse, denoted by $A^\dagger$.

The M-P inverse of $A$ exists and is unique. If $(i)$, $i = 1, 2, 3, 4$ of the above equation holds, then $X$ is called an $\{i\}$-inverse of $A$, denoted by $A^{(i)}$. Furthermore, $L_A$ and $R_A$ stand for the two projectors $L_A = I - A^\dagger \ast_N A$ and $R_A = I - A \ast_N A^\dagger$ induced by $A$, respectively. We say the tensor $B \in \mathbb{H}^{I_1\times \cdots \times I_N \times I_1\times \cdots \times I_N}$ is the inverse of tensor $A \in \mathbb{H}^{I_1\times \cdots \times I_N \times I_1\times \cdots \times I_N}$, if $A \ast_N B = I = B \ast_N A$, and we denote $B = A^{-1}$ [5]. Moreover, we will say that a tensor is nonsingular if it has an inverse. For an invertible tensor $A$, $A^\dagger = A^{(0)} = A^{-1}$.

Given a tensor $A = (a_{i_1\cdots i_Nj_1\cdots j_M}) \in \mathbb{H}^{I_1\times \cdots \times I_N \times J_1 \times \cdots \times J_M}$,
the tensor $B = (b_{i_1 \ldots i_M j_1 \ldots j_N}) \in \mathbb{H}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is called the quaternion transpose of $A$, and it is denoted by $A^\ast$, where $b_{i_1 \ldots i_M j_1 \ldots j_N} = \overline{a}_{j_1 \ldots j_N i_1 \ldots i_M}$. The quaternion tensor $q = q_0 + q_1 i + q_2 j + q_3 k$ stands for the conjugate of quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$. The tensor $B = (a_{j_1 \ldots j_N i_1 \ldots i_M}) \in \mathbb{H}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is called the transpose of $A$, and it is denoted by $A^T$.

We say that $D \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$ is a diagonal tensor if $z \neq i_1 \ldots i_N 1 \ldots i_N$, then $d_z = 0$. $D$ is a unit tensor, if it is diagonal and $d_{1 \ldots 1 \ldots 1} = 1$, it is denoted by $I$. We define the trace of a tensor $A = (a_{i_1 \ldots i_N j_1 \ldots j_N}) \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N} \times \cdots \times J_N}$ by $\text{tr}(A) = \Sigma_{i_1 \ldots i_N} a_{i_1 \ldots i_N i_1 \ldots i_N}$ [5]. Moreover, we say that matricization is the process of transforming a tensor into a matrix, that is a reordering of the elements of an order $N$ tensor into a matrix, this is also called unfolding or flattening. For example, a $2 \times 3 \times 4 \times 8$ tensor can be matricized into a $12 \times 16$ matrix or a $6 \times 32$ matrix, and so on [22].

The following results can be verified easily.

**Proposition 2.1.** [13] Let $A \in \mathbb{H}^{I_1 \times \cdots \times I_P \times K_1 \times \cdots \times K_N}$ and $B \in \mathbb{H}^{K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$. Then

1. $(A * N B)^{\ast} = B^\ast * N A^\ast$;
2. $I_N * N B = B$ and $B * M I_M = B$, where unit tensors $I_N \in \mathbb{H}^{K_1 \times \cdots \times K_N \times K_1 \times \cdots \times K_N}$ and $I_M \in \mathbb{H}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_M}$.

**Proposition 2.2.** [13] Let $A \in \mathbb{H}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$. Then

1. $(A^\ast)^\dagger = A$;
2. $(A^\ast)^\ast = (A^\ast)^\dagger$;
3. $(A^\ast * N A)^\dagger = A^\dagger * N (A^\ast)^\dagger$, $(A * N A^\ast)^\dagger = (A^\ast)^\dagger * N A^\dagger$;
4. $A^\dagger * N R_A = 0$ and $R_A * N A = 0$.

**Proposition 2.3.** [13] The following equalities hold:

1. \[
\begin{bmatrix}
A_1 & B_1 \\
A_2 & B_2
\end{bmatrix}^* M
\begin{bmatrix}
C \\
D
\end{bmatrix} =
\begin{bmatrix}
A_1 * M C + B_1 * M D \\
A_2 * M C + B_2 * M D
\end{bmatrix} \in \mathbb{H}^{P_1 \times \cdots \times P_N \times \alpha N};
\]
2. \[
\begin{bmatrix}
G & H
\end{bmatrix}^* N
\begin{bmatrix}
A_1 & B_1 \\
A_2 & B_2
\end{bmatrix} =
\begin{bmatrix}
G * N A_1 + H * N A_2 \\
G * N B_1 + H * N B_2
\end{bmatrix} \in \mathbb{H}^{S_1 \times \cdots \times S_N \times \beta_1 \times \cdots \times \beta_M},
\]
where $A_1 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $B_1 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_M}$, $A_2 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $B_2 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_M}$, $C \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $D \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$, $G \in \mathbb{H}^{S_1 \times \cdots \times S_N \times I_1 \times \cdots \times I_N}$, $H \in \mathbb{H}^{S_1 \times \cdots \times S_N \times L_1 \times \cdots \times L_N}$, where $\alpha = I_1 \times \cdots \times I_N$, and $\beta = J_1 + K_j, j = 1, \ldots, M$.

**Lemma 2.4.** [19] Consider tensor equation $A * N X * M B = C$. Let $A \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$, $B \in \mathbb{H}^{K_1 \times \cdots \times K_M \times L_1 \times \cdots \times L_M}$, $C \in \mathbb{H}^{I_1 \times \cdots \times I_N \times L_1 \times \cdots \times L_M}$. Then the quaternion tensor equation is consistent if and only if

$R_A * N C = 0$, $C * M L_B = 0$.

In this case, the general solution can be expressed as

$X = A^\dagger * N C * M B^\dagger + L_A * N U + V * M R_B$,

where $U$ and $V$ are arbitrary quaternion tensors with appropriate order.

**Lemma 2.5.** [19] Let $A \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$, $B \in \mathbb{H}^{K_1 \times \cdots \times K_M \times L_1 \times \cdots \times L_M}$, $C \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$, $D \in \mathbb{H}^{I_1 \times \cdots \times I_N \times L_1 \times \cdots \times L_M}$ and $E \in \mathbb{H}^{I_1 \times \cdots \times I_N \times L_1 \times \cdots \times L_M}$. Set $P = (R_A) * N C$, $Q = D * M$.
In this case, the general solution can be expressed as

$A * N X * M B + C * N Y * M D = \mathcal{E}$

(2.1)

is consistent if and only if

$\mathcal{R}_P * N \mathcal{R}_A * N \mathcal{E} = 0, \mathcal{E} * M \mathcal{L}_B * M \mathcal{L}_Q = 0, \mathcal{R}_A * N \mathcal{E} * M \mathcal{L}_D = 0, \mathcal{R}_C * N \mathcal{E} * M \mathcal{L}_B = 0$

In this case, the general solution can be expressed as

$X = A^\dagger * N \mathcal{E} * M B^\dagger - A^\dagger * N \mathcal{C} * N \mathcal{P}^\dagger * N \mathcal{E} * M B^\dagger - A^\dagger * N \mathcal{S} * N \mathcal{C}^\dagger * N \mathcal{E} * M \mathcal{Q}^\dagger * M \mathcal{D} * M B^\dagger$

$- A^\dagger * N \mathcal{S} * N \mathcal{U}_2 * M \mathcal{R}_Q * M \mathcal{D} * M B^\dagger + \mathcal{L}_A * N \mathcal{U}_4 + \mathcal{U}_5 * M \mathcal{R}_B,$

$Y = \mathcal{P}^\dagger * N \mathcal{E} * M D^\dagger + S^\dagger * N \mathcal{S} * N \mathcal{C}^\dagger * N \mathcal{E} * M \mathcal{Q}^\dagger + \mathcal{L}_P * N \mathcal{L}_S * N \mathcal{U}_1 + \mathcal{L}_P * N \mathcal{U}_2 + \mathcal{U}_3 * M \mathcal{R}_D,$

where $\mathcal{U}_i$, $i = 1, \ldots, 5$ are arbitrary quaternion tensors with appropriate order.

3. Solvable Conditions and General Solution to the System \((1,2)\)

In this section we consider the solvability conditions and the expression of the general solution to the quaternion tensor system \((1,2)\). For simplicity, put

$A_{ii} = \mathcal{R}_A * N \mathcal{C}_i, \ B_{ii} = \mathcal{D}_i * M \mathcal{L}_B_i, \ C_{ii} = \mathcal{R}_A * N \mathcal{F}_i, \ D_{ii} = \mathcal{G}_i * M \mathcal{L}_B_i,$

(3.1)

$\mathcal{E}_{ii} = \mathcal{R}_A * N \mathcal{E}_i * M \mathcal{L}_B_i, \ M_{ii} = \mathcal{R}_A * N \mathcal{C}_{ii},$

(3.2)

$N_{ii} = \mathcal{D}_i * M \mathcal{L}_B_i, \ S_{ii} = \mathcal{C}_{ii} * N \mathcal{L}_M_i, \ i = 1, 2, 3,$

(3.3)

$A_{j+1,j+1} = \begin{bmatrix} \mathcal{L}_{M,j} * N \mathcal{L}_{S,j} & -\mathcal{L}_{A,j+1,j+1} \\ \mathcal{R}_{D,j} & -\mathcal{R}_{B,j+1,j+1} \end{bmatrix}$, $B_{j+1,j+1} = \begin{bmatrix} \mathcal{R}_{D,j} \\ -\mathcal{R}_{B,j+1,j+1} \end{bmatrix},$

(3.4)

$C_{j+1,j+1} = \mathcal{L}_{M,j+1} * M \mathcal{D}_{B,j+1} = \mathcal{N}_{M,j+1},$

(3.5)

$A_{j+1,j+1} = A_{j+1,j+1} \mathcal{S}_{j+1,j+1} * N \mathcal{G}_{j+1,j+1} = \mathcal{R}_{N,j+1,j+1} * M \mathcal{D}_{j+1,j+1} * M B_{j+1,j+1}^\dagger,$

(3.6)

$\mathcal{E}_{j+1,j+1} = M_{j+1,j+1}^\dagger * N \mathcal{E}_{j,j+1} * M \mathcal{D}_{j+1,j+1} + S_{j+1,j+1}^\dagger * N \mathcal{S}_{j+1,j} * N \mathcal{C}_{j+1,j}^\dagger * N \mathcal{E}_{j+1,j+1} * M \mathcal{N}_{j+1,j+1}^\dagger - \mathcal{A}_{j+1,j+1}^\dagger * N \mathcal{E}_{j+1,j+1} * M \mathcal{N}_{j+1,j+1}^\dagger$

$* M B_{j+1,j+1}^\dagger + A_{j+1,j+1}^\dagger * M C_{j+1,j+1} * N \mathcal{M}_{j+1,j+1} + \mathcal{N}_{j+1,j+1} * N \mathcal{E}_{j+1,j+1} * M \mathcal{B}_{j+1,j+1}^\dagger + A_{j+1,j+1}^\dagger$ * M B_{j+1,j+1}^\dagger

(3.7)

$\mathcal{A}_{j+1,j+1} = \mathcal{A}_{j+1,j+1} * N \mathcal{F}_{j+1,j+1} * M \mathcal{L}_{B_{j+1}},$

(3.8)

$\mathcal{C}_{j+1,j+1} = \mathcal{R}_{A,j+1,j+1} * N \mathcal{F}_{j+1,j+1} = \mathcal{G}_{j+1,j+1} * M \mathcal{L}_{B_{j+1}},$

(3.9)

$\mathcal{E}_{j+1,j+1} = \mathcal{R}_{A,j+1,j+1} * N \mathcal{E}_{j+1,j+1} * M \mathcal{L}_{B_{j+1}}, \ \mathcal{M}_{j+1,j+1} = \mathcal{R}_{A,j+1,j+1} * N \mathcal{C}_{j+1,j+1},$

(3.10)

$\mathcal{N}_{j+1,j+1} = \mathcal{D}_{j+1,j+1} * M \mathcal{L}_{B_{j+1}} * M \mathcal{L}_{S_{j+1}}, \ \mathcal{S}_{j+1,j+1} = \mathcal{C}_{j+1,j+1} * N \mathcal{L}_{M_{j+1}}, \ j = 1, 2,$

(3.11)

$A = \begin{bmatrix} \mathcal{L}_{M,12} * N \mathcal{L}_{S,12} & -\mathcal{L}_{A,23} \\ \mathcal{R}_{D,23} & -\mathcal{R}_{B,23} \end{bmatrix}$, $B = \begin{bmatrix} \mathcal{R}_{D,23} \\ -\mathcal{R}_{B,23} \end{bmatrix},$

(3.12)

$C = \mathcal{L}_{M,12}$, $D = \mathcal{R}_{N,12},$

(3.13)

$F = \mathcal{A}_{23} * N \mathcal{S}_{23}, \ \mathcal{G} = \mathcal{R}_{N,23} * M \mathcal{D}_{23} * M \mathcal{B}_{23}^\dagger$,

(3.14)
\[ \mathcal{E} = \tilde{M}_{12}^\dagger *_N \tilde{E}_{12} *_M \tilde{D}_{12}^\dagger + \tilde{S}_{12} *_N \tilde{S}_{12} *_N \tilde{C}_{12} *_N \tilde{E}_{12} *_M \tilde{N}_{12}^\dagger - \tilde{A}_{23} *_N \tilde{E}_{23} *_M \tilde{N}_{23}^\dagger *_M \tilde{B}_{23}^\dagger + \tilde{A}_{23} *_N \tilde{C}_{23} *_N \tilde{M}_{23} *_N \tilde{E}_{23} *_M \tilde{B}_{23}^\dagger + \tilde{A}_{23} *_N \tilde{S}_{23} *_N \tilde{C}_{23}^\dagger *_N \tilde{E}_{23} \]

\( (3.15) \)

\[ \hat{A} = R_A *_N C, \hat{B} = D *_M L_B, \hat{C} = R_A *_N \mathcal{F}, \hat{D} = G_{12} *_M L_B, \]

\( (3.16) \)

\[ \hat{e} = R_A *_N \mathcal{E} *_M L_B, \tilde{M} = R_A *_N \hat{C}, \tilde{N} = \hat{D} *_M L_B, \tilde{S} = \hat{C} *_N L_{\tilde{M}}, \]

\( (3.17) \)

Before giving the basic theorems, we present a lemma which is helpful to prove the Theorem 3.2

**Lemma 3.1.** Let \( A_{11} = R_{A_{11}} *_N C_1, B_{11} = D_{11} *_M L_{B_1}, C_{11} = R_{A_{11}} *_N \mathcal{F}_1, D_{11} = G_{11} *_M L_{B_1}, E_{11} = R_{A_{11}} *_N E_1 *_M L_{B_1}, M_{11} = R_{A_{11}} *_N C_{11}, N_{11} = D_{11} *_N L_{B_{11}}, S_{11} = C_{11} *_N L_{M_{11}}. \) Consider the following tensor equation

\[ A_{11} *_N X_1 + Y_1 *_M B_1 + C_{11} *_N Z_1 *_M D_1 + F_1 *_N Z_2 *_M G_1 = \mathcal{E}_1, \]

\( (3.18) \)

which is consistent if and only if

\[ R_{M_{11}} *_N R_{A_{11}} *_N E_{11} = 0, \mathcal{E}_{11} *_M L_{B_{11}} *_M N_{11} = 0, \]

\( (3.19) \)

\[ R_{A_{11}} *_N E_{11} *_M L_{D_{11}} = 0, \mathcal{E}_{11} *_N C_{11} *_M L_{B_{11}} = 0, \]

\( (3.20) \)

In this case, the general solution can be expressed as

\[ X_1 = A_{11}^\dagger *_N (E_1 - C_1 *_N Z_1 *_M D_1 - F_1 *_N Z_2 *_M G_1) - T_1 *_M B_1 + L_{A_{11}} *_N T_2, \]

\( (3.21) \)

\[ Y_1 = R_{A_{11}} *_N (E_1 - C_1 *_N Z_1 *_M D_1 - F_1 *_N Z_2 *_M G_1) *_M B_{11}^\dagger + A_{11} *_N T_1 + T_3 *_M R_{B_{11}}, \]

\( (3.22) \)

\[ Z_1 = A_{11}^\dagger *_N E_{11} *_N M_{11}^\dagger - A_{11}^\dagger *_N C_{11} *_N M_{11}^\dagger - A_{11}^\dagger *_N S_{11} *_N C_{11}^\dagger *_N E_{11} *_M N_{11}^\dagger *_M D_{11} *_M B_{11}^\dagger - A_{11}^\dagger *_N S_{11} *_N T_4 *_M R_{A_{11}} *_M D_{11} *_M B_{11}^\dagger + L_{A_{11}} *_N T_5 + T_6 *_M R_{B_{11}}, \]

\( (3.23) \)

\[ Z_2 = M_{11}^\dagger *_N E_{11} *_M D_{11}^\dagger + S_{11} *_N S_{11} *_N C_{11}^\dagger *_N E_{11} *_M N_{11}^\dagger + L_{M_{11}} *_N L_{S_{11}} *_N T_7 + L_{M_{11}} *_N T_3 *_M R_{N_{11}} + T_8 *_M R_{D_{11}}, \]

\( (3.24) \)

where \( T_i (i = 1, \ldots, 8) \) are arbitrary quaternion tensors over \( \mathbb{H} \).

**Proof.** We separate the left part of equation \( (3.18) \) into two parts \( A_{11} *_N X_1 + Y_1 *_M B_1 \) and \( C_{11} *_N Z_1 *_M D_1 + F_1 *_N Z_2 *_M G_1 \), we have

\[ A_{11} *_N X_1 + Y_1 *_M B_1 = E_1 - C_1 *_N Z_1 *_M D_1 - F_1 *_N Z_2 *_M G_1. \]

\( (3.25) \)

Applying Lemma 2.5 if \( B = D = 0 \), equation \( (3.25) \) is consistent if and only if

\[ R_{A_{11}} *_N (E_1 - C_1 *_N Z_1 *_M D_1 - F_1 *_N Z_2 *_M G_1) *_M L_{B_1} = 0, \]

\( (3.26) \)

\( \chi_1, \chi_1 \) can be written as \( (3.21), (3.22) \). That is equivalent to prove

\[ A_{11} *_N Z_1 *_M B_{11} + C_{11} *_N Z_2 *_M D_{11} = E_{11} \]

\( (3.27) \)

is consistent. Applying Lemma 2.5 that equation \( (3.27) \) is consistent if and only if \( (3.19) \) and \( (3.20) \), as well as \( Z_1, Z_2 \) can be written as \( (3.23), (3.24) \).

When the \( N = M = 1 \), the matrix form of above-mentioned Lemma was given in \( [18] \).
Theorem 3.2. Consider system (1.2). Then the following statements are equivalent:

1. System (1.2) is consistent.
2. The following equalities are satisfied:

\[
\begin{align*}
\mathcal{R}_{M_{ii}} \ast * N \mathcal{R}_{A_{ii}} \ast * N \mathcal{E}_{ii} = 0, & \quad \mathcal{E}_{ii} \ast * M \mathcal{L}_{B_{ii}} \ast * M \mathcal{L}_{\mathcal{X}_i} = 0, & \quad (i = 1, 2, 3), \\
\mathcal{R}_{A_{ii}} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{L}_{D_{ii}} = 0, & \quad \mathcal{R}_{C_{ii}} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{L}_{B_{ii}} = 0, & \quad (i = 1, 2, 3), \\
\mathcal{R}_{M_{jj+1}} \ast * N \mathcal{R}_{A_{jj+1}} \ast * N \mathcal{E}_{jj+1} = 0, & \quad \mathcal{E}_{jj+1} \ast * M \mathcal{L}_{B_{jj+1}} \ast * M \mathcal{L}_{\mathcal{Q}_{jj+1}} = 0, \\
\mathcal{R}_{A_{jj+1}} \ast * N \mathcal{E}_{jj+1} \ast * M \mathcal{L}_{B_{jj+1}} = 0, & \quad \mathcal{R}_{C_{jj+1}} \ast * N \mathcal{E}_{jj+1} \ast * M \mathcal{L}_{B_{jj+1}} = 0, & \quad (j = 1, 2), \\
\mathcal{R}_{\mathcal{M}} \ast * N \mathcal{R}_{\mathcal{A}} \ast * N \mathcal{\hat{E}} = 0, & \quad \mathcal{\hat{E}} \ast * M \mathcal{L}_{\mathcal{B}} \ast * M \mathcal{L}_{\mathcal{Q}} = 0, \\
\mathcal{R}_{\mathcal{A}} \ast * N \mathcal{\hat{E}} \ast * M \mathcal{L}_{\mathcal{B}} = 0, & \quad \mathcal{R}_{\mathcal{C}} \ast * N \mathcal{\hat{E}} \ast * M \mathcal{L}_{\mathcal{B}} = 0.
\end{align*}
\]

Furthermore, if statement (1) holds, then the general solution to (1.2) can be expressed as

\[
\begin{align*}
\mathcal{X}_i & = \mathcal{A}_{ii} \ast * N (\mathcal{E}_i - \mathcal{C}_i \ast * N \mathcal{Z}_i \ast * M \mathcal{D}_i - \mathcal{F}_i \ast * M \mathcal{Z}_{i+1} \ast * M \mathcal{G}_i) - \mathcal{T}_{i1} \ast * M \mathcal{B}_i - \mathcal{L}_{A_i} \ast * N \mathcal{T}_{i2}, \\
\mathcal{Y}_i & = \mathcal{R}_{A_i} \ast * N (\mathcal{E}_i - \mathcal{C}_i \ast * N \mathcal{Z}_i \ast * M \mathcal{D}_i - \mathcal{F}_i \ast * M \mathcal{Z}_{i+1} \ast * M \mathcal{G}_i) \ast * M \mathcal{B}_{i1} + \mathcal{A}_i \ast * N \mathcal{T}_{i1} + \mathcal{T}_{i3} \ast * M \mathcal{R}_{B_i}, \\
\mathcal{Z}_i & = \mathcal{A}_{ii} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{B}_{i1} - \mathcal{A}_{ii} \ast * N \mathcal{C}_{ii} \ast * N \mathcal{M}_{ii} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{B}_{i1} - \mathcal{A}_{ii} \ast * N \mathcal{S}_{ii} \ast * N \mathcal{C}_{ii} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{N}_{ii} \ast * M \mathcal{D}_{ii} \ast * M \mathcal{B}_{i1} - \mathcal{A}_{ii} \ast * N \mathcal{S}_{ii} \ast * N \mathcal{T}_{i4} \ast * M \mathcal{R}_{N_{ii}} \ast * M \mathcal{D}_{ii} \ast * M \mathcal{B}_{i1} + \mathcal{L}_{A_{ii}} \ast * N \mathcal{T}_{i5} + \mathcal{T}_{i6} \ast * M \mathcal{R}_{B_{ii}}, \\
\mathcal{Z}_{i+1} & = \mathcal{M}_{ii} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{D}_{ii} \ast * N \mathcal{S}_{ii} \ast * N \mathcal{C}_{ii} \ast * N \mathcal{E}_{ii} \ast * M \mathcal{N}_{ii} \ast * M \mathcal{D}_{ii} \ast * M \mathcal{B}_{i1} + \mathcal{L}_{M_{ii}} \ast * N \mathcal{L}_{S_{ii}} \ast * N \mathcal{T}_{i7} + \mathcal{L}_{M_{ii}} \ast * N \mathcal{T}_{i4} \ast * M \mathcal{R}_{N_{ii}} + \mathcal{T}_{i8} \ast * M \mathcal{R}_{D_{ii}}, \quad (i = 1, 2, 3)
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{T}_{j7} & = \begin{bmatrix} \mathcal{I}_{m_j} & 0 \end{bmatrix} \ast * N [\mathcal{A}_{jj+1} \ast * N (\mathcal{E}_{jj+1} - \mathcal{C}_{jj+1} \ast * N \mathcal{T}_{j4} \ast * M \mathcal{D}_{jj+1} - \mathcal{F}_{jj+1} \ast * N \mathcal{T}_{j8} \ast * N \mathcal{G}_{jj+1}) \\
& - \mathcal{U}_{j1} \ast * M \mathcal{B}_{jj+1} + \mathcal{L}_{A_{jj+1}} \ast * N \mathcal{U}_{j2}] \\
\mathcal{T}_{j1,5} & = \begin{bmatrix} 0 & \mathcal{I}_{m_j} \end{bmatrix} \ast * N [\mathcal{A}_{jj+1} \ast * N (\mathcal{E}_{jj+1} - \mathcal{C}_{jj+1} \ast * N \mathcal{T}_{j4} \ast * M \mathcal{D}_{jj+1} - \mathcal{F}_{jj+1} \ast * N \mathcal{T}_{j8} \ast * N \mathcal{G}_{jj+1}) \\
& - \mathcal{U}_{j1} \ast * M \mathcal{B}_{jj+1} + \mathcal{L}_{A_{jj+1}} \ast * N \mathcal{U}_{j2}], \\
\mathcal{T}_{j8} & = [\mathcal{R}_{A_{jj+1}} \ast * N (\mathcal{E}_{jj+1} - \mathcal{C}_{jj+1} \ast * N \mathcal{U}_{jj+1} \ast * M \mathcal{D}_{jj+1} - \mathcal{F}_{jj+1} \ast * N \mathcal{U}_{j2} \ast * M \mathcal{G}_{jj+1}) \ast * M \mathcal{B}_{jj+1} \\
& + \mathcal{A}_{jj+1} \ast * N \mathcal{U}_{j1} + \mathcal{U}_{j3} \ast * M \mathcal{R}_{B_{jj+1}}] \ast * M \begin{bmatrix} \mathcal{T}_{n_j} \\
0 \end{bmatrix} , \\
\mathcal{T}_{j1,6} & = [\mathcal{R}_{A_{jj+1}} \ast * N (\mathcal{E}_{jj+1} - \mathcal{C}_{jj+1} \ast * N \mathcal{U}_{jj+1} \ast * M \mathcal{D}_{jj+1} - \mathcal{F}_{jj+1} \ast * N \mathcal{U}_{j2} \ast * M \mathcal{G}_{jj+1}) \ast * M \mathcal{B}_{jj+1} \\
& + \mathcal{A}_{jj+1} \ast * N \mathcal{U}_{j1} + \mathcal{U}_{j3} \ast * M \mathcal{R}_{B_{jj+1}}] \ast * M \begin{bmatrix} 0 \\
\mathcal{T}_{n_j} \end{bmatrix} .
\end{align*}
\]
\( T_{j4} = A_{j,j+1}^\dagger *_N E_{j,j+1} *_M B_{j,j+1} - A_{j,j+1}^\dagger *_N C_{j,j+1} *_N M_{j,j+1} *_N E_{j,j+1} *_M B_{j,j+1} - A_{j,j+1}^\dagger *_N S_{j,j+1} *_N C_{j,j+1} *_N E_{j,j+1} *_N N_{j,j+1} *_M D_{j,j+1} *_M B_{j,j+1} - A_{j,j+1}^\dagger *_N S_{j,j+1} *_N U_{j4} *_M R_{N,j,j+1} *_M D_{j,j+1} *_M B_{j,j+1} + \mathcal{L}_{A_{j,j+1}} *_N U_{j5} + U_{j6} *_M R_{B,j,j+1}, \) 

\( T_{j+1,4} = M_{j,j+1} *_N E_{j,j+1} *_N D_{j,j+1} + S_{j,j+1} *_N S_{j,j+1} *_N C_{j,j+1} *_N E_{j,j+1} *_N N_{j,j+1} *_M N_{j,j+1} + \mathcal{L}_{M_{j,j+1}} *_N \mathcal{L}_{S_{j,j+1}} *_N U_{j2} + \mathcal{L}_{M_{j,j+1}} *_N U_{j4} *_M R_{N,j,j+1} + U_{j8} *_M R_{D,j,j+1}, (j = 1, 2) \)

\( U_{17} = \begin{bmatrix} I_s & 0 \end{bmatrix} *_N [A^\dagger *_N (E - C *_N U_{14} *_M D - F *_N U_{24} *_M G) - V_1 *_M B + L_A *_N V_2], \)

\( U_{25} = \begin{bmatrix} 0 & I_s \end{bmatrix} *_N [A^\dagger *_N (E - C *_N U_{14} *_M D - F *_N U_{24} *_M G) - V_1 *_M B + L_A *_N V_3], \)

\( U_{18} = [R_A *_N (E - C *_N U_{14} *_M D - F *_N U_{24} *_M G) *_M B^\dagger + A *_N U_{j1} + U_{j3} *_M R_B] *_M \begin{bmatrix} I_t & 0 \end{bmatrix}, \)

\( U_{26} = [R_A *_N (E - C *_N U_{14} *_M D - F *_N U_{24} *_M G) *_M B^\dagger + A *_N U_{j1} + U_{j3} *_M R_B] *_M \begin{bmatrix} 0 & I_{t} \end{bmatrix}, \)

\( U_{14} = \hat{A}^\dagger *_N \hat{E} *_M \hat{B}^\dagger - \hat{A}^\dagger *_N \hat{C} *_N \hat{M}^\dagger *_N \hat{E} *_M \hat{B}^\dagger - \hat{A}^\dagger *_N \hat{S} *_N \hat{V}_4 *_M \hat{R}_{N} *_M \hat{D} *_M \hat{B}^\dagger + \mathcal{L}_{\hat{A}} *_N \hat{V}_5 + \hat{V}_6 *_M \hat{R}_{\hat{B}}, \)

\( U_{24} = \hat{M}^\dagger *_N \hat{E} *_N \hat{D}^\dagger + \hat{S}^\dagger *_N \hat{S} *_N \hat{C} *_N \hat{E} *_M \hat{N}^\dagger + \mathcal{L}_{\hat{M}} *_N \hat{L}_S *_N \hat{V}_7 + \mathcal{L}_{\hat{M}} *_N \hat{V}_4 *_M \hat{R}_{\hat{N}} + \hat{V}_8 *_M \hat{R}_{\hat{B}}, \)

and \( T_{11}, T_{12}, T_{13}, U_{j1}, U_{j2}, U_{j3}, T_{14}, T_{15}, T_{16}, T_{37}, T_{38}, U_{15}, U_{16}, U_{27}, U_{28}, V_t, \) are arbitrary tensors with appropriate sizes over \( \mathbb{H} \), \( m, j, s \) is the same as the column block of \( F_j, F_2 \), respectively, \( n, j, t \) is the same as the row block of \( G_j, G_2 \), respectively \( (i=1, 2, 3; j=1, 2; t=1, \ldots, 8) \).

**Proof.** \( \text{[1]} \iff \text{[2]} \).

We separate the tensor system into three groups

\( A_1 *_N X_1 + Y_1 *_M B_1 + C_1 *_N Z_1 *_M D_1 + F_1 *_N Z_2 *_M G_1 = E_1, \)

\( A_2 *_N X_2 + Y_2 *_M B_2 + C_2 *_N Z_2 *_M D_2 + F_2 *_N Z_3 *_M G_2 = E_2, \)

and

\( A_3 *_N X_3 + Y_3 *_M B_3 + C_3 *_N Z_3 *_M D_3 + F_3 *_N Z_4 *_M G_3 = E_3. \)

It follows from the Lemma 3.1 that (3.50), (3.51) and (3.52) are consistent, respectively, if and only if (3.28) and (3.29) hold. The general solution of (3.50), (3.51), (3.52) can be expressed
as (3.34), (3.35), (3.36), (3.37), where $T_{i1} \cdots T_{i8}$, $i = 1, 2, 3$ are arbitrary appropriate size tensor over $\mathbb{H}$.

Let $Z_2$ in (3.37) (when $i = 1$) be equal to $Z_2$ in (3.36) (when $i = 2$) and $Z_3$ in (3.37) (when $i = 2$) be equal to $Z_3$ in (3.36) (when $i = 3$). Note

- When $i = 1$,

$$Z_{i+1}(Z_2) = M_{11}^\dagger * N E_{11} * M D_{11}^\dagger + S_{11}^\dagger * N S_{11} * N C_{11}^\dagger * N E_{11} * M N_{11}^\dagger + L_{M_{11}} * N L_{S_{11}} * N T_{27}$$

$$+ L_{M_{11}} * N T_{24} * M R_{N_{11}} + T_{28} * M R_{D_{11}},$$

$$Z_i(Z_2) = \bigg[ A_{12}^\dagger * N E_{22} * M B_{12}^\dagger - A_{12}^\dagger * N C_{22} * N M_{12}^\dagger * N E_{22} * M B_{12}^\dagger - A_{12}^\dagger * N S_{22} * N C_{12}^\dagger * N E_{22} * M N_{12}^\dagger + M_{N_{12}} * N D_{22} * M B_{12}^\dagger + L_{A_{22}} * N T_{25} + T_{26} * M R_{B_{22}},$$

$$Z_{i+1}(Z_3) = M_{12}^\dagger * N E_{22} * M D_{22}^\dagger + S_{22}^\dagger * N S_{22} * N C_{22}^\dagger * N E_{22} * M N_{22}^\dagger + L_{M_{22}} * N L_{S_{22}} * N T_{37}$$

$$+ L_{M_{22}} * N T_{34} * M R_{N_{22}} + T_{38} * M R_{D_{22}},$$

$$Z_i(Z_3) = A_{33}^\dagger * N E_{33} * M B_{33}^\dagger - A_{33}^\dagger * N C_{33} * N M_{33}^\dagger * N E_{33} * M B_{33}^\dagger - A_{33}^\dagger * N S_{33} * N C_{33}^\dagger * N E_{33} * M N_{33}^\dagger + M_{N_{33}} * N D_{33} * M B_{33}^\dagger + L_{A_{33}} * N T_{35} + T_{36} * M R_{B_{33}},$$

Then equating (3.33), (3.54) and (3.55), (3.56), respectively. We have the following

$$A_{12} * N \begin{bmatrix} T_{17} \\ T_{25} \end{bmatrix} + \begin{bmatrix} T_{18} \\ T_{26} \end{bmatrix} * M B_{12} + C_{12} * N T_{14} * M D_{12} + F_{12} * N T_{24} * M \mathcal{H}_{12} = \mathcal{E}_{12},$$

and

$$A_{23} * N \begin{bmatrix} T_{27} \\ T_{35} \end{bmatrix} + \begin{bmatrix} T_{28} \\ T_{36} \end{bmatrix} * M B_{23} + C_{23} * N T_{24} * M D_{23} + F_{23} * N T_{34} * M \mathcal{H}_{23} = \mathcal{E}_{23},$$

where $A_{j,j+1}, B_{j,j+1}, C_{j,j+1}, D_{j,j+1}, F_{j,j+1}, E_{j,j+1}$ are given in (3.4) – (3.7) ($j = 1, 2$).

Firstly, we consider the solvability conditions and general solution to the equation (3.57) and (3.58), respectively.

- When $j = 1,$

$$T_{j+1,4}(T_{24}) = \bigg[ \mathcal{M}_{12}^\dagger * N \mathcal{E}_{12} * N \mathcal{D}_{12}^\dagger + \mathcal{S}_{12}^\dagger * N \mathcal{S}_{12} * N \mathcal{C}_{12} * N \mathcal{E}_{12} * M \mathcal{N}_{12}^\dagger + \mathcal{L}_{\mathcal{M}_{12}} * N \mathcal{L}_{\mathcal{S}_{12}} * N \mathcal{U}_{17}$$

$$+ \mathcal{L}_{\mathcal{M}_{12}} * N \mathcal{U}_{14} * M \mathcal{R}_{\mathcal{N}_{12}} + \mathcal{U}_{18} * M R_{\mathcal{D}_{12}},$$

(3.59)
• When \( j = 2 \),

\[
T_{j,4}(T_{24}) = \bar{A}_{23}^\dagger *_{M} \bar{E}_{23} *_{N} \bar{B}_{23} \bar{F}_{23} - \bar{A}_{23}^\dagger *_{N} \bar{C}_{23} *_{N} \bar{M}_{23}^\dagger *_{N} \bar{E}_{23} *_{M} \bar{B}_{23} \bar{F}_{23} - \bar{A}_{23}^\dagger *_{N} \bar{S}_{23} *_{N} \bar{C}_{23}^\dagger *_{N} \bar{E}_{23} *_{M} \bar{B}_{23} \bar{F}_{23} + \bar{L}_{\bar{A}_{23}} *_{N} \bar{U}_{25} + \bar{U}_{26} *_{M} \bar{R}_{\bar{B}_{23}}.
\]

Equating (3.59) and (3.60), we have

\[
A *_{N} \begin{bmatrix} \bar{U}_{17} \\ \bar{U}_{25} \end{bmatrix} + \begin{bmatrix} \bar{U}_{18} & \bar{U}_{26} \end{bmatrix} *_{M} B + C *_{N} U_{14} *_{M} D + F *_{N} U_{24} *_{M} G = \mathcal{E},
\]

where \( A, B, C, D, F, G, \mathcal{F} \) are given by (3.12) – (3.15). Then we consider the solvability condition and general solution to equation (3.61).

Using Lemma (3.1) over again, equation (3.61) is consistent if and only if (3.32) and (3.33) hold. And \( U_{17}, U_{25}, U_{18}, U_{26}, U_{14}, U_{24} \) can be expressed as (3.34) – (3.39), where \( V_{t}, t = 1, \ldots, 8 \) are arbitrary quaternion tensors.

\[
\square
\]

4. Solvable conditions and general solution to the System (1.3)

In this section, an general solution to System (1.3) using Moore-Penrose is given and the necessary and sufficient conditions are investigated. For simplicity, put

\[
M_{k} = \mathcal{R}_{A_{k}} *_{N} C_{k}, \quad N_{k} = D_{k} *_{M} L_{B_{k}}, \quad S_{k} = C_{k} *_{N} L_{M_{k}}, \quad k = 1, \ldots, 4,
\]

\[
\bar{A}_{i} = \begin{bmatrix} L_{M_{i}} & *_{N} L_{s_{i}} \\ -L_{A_{i+1}} \end{bmatrix}, \quad \bar{B}_{i} = \begin{bmatrix} \mathcal{R}_{D_{i}} \\ -\mathcal{R}_{B_{i+1}} \end{bmatrix}, \quad \bar{C}_{i} = L_{M_{i}}, \quad \bar{D}_{i} = \mathcal{R}_{N_{i}}, \quad i = 1, 2, 3, \quad (4.1)
\]

\[
\bar{F}_{i} = A_{i+1}^\dagger *_{N} S_{i+1}, \quad \bar{G}_{i} = \mathcal{R}_{N_{i+1}} *_{M} D_{i+1} *_{M} B_{i+1}^\dagger, \quad i = 1, 2, 3, \quad (4.2)
\]

\[
\bar{E}_{i} = A_{i+1}^\dagger *_{N} E_{i} *_{M} D_{i}^\dagger + S_{i}^\dagger *_{N} S_{i} *_{N} C_{i}^\dagger *_{N} E_{i} *_{M} N_{i}^\dagger - A_{i+1}^\dagger *_{N} E_{i+1} *_{M} B_{i+1}^\dagger + A_{i+1}^\dagger *_{N} C_{i+1} *_{N} M_{i+1}^\dagger *_{N} E_{i+1} *_{M} B_{i+1}^\dagger + A_{i+1}^\dagger *_{N} S_{i+1} *_{N} C_{i+1}^\dagger *_{N} E_{i+1} *_{M} N_{i+1}^\dagger *_{M} D_{i+1} *_{M} B_{i+1}^\dagger, \quad i = 1, 2, 3, \quad (4.3)
\]

\[
A_{i} = \mathcal{R}_{A_{i}} *_{N} \bar{C}_{i}, \quad B_{i} = \bar{D}_{i} *_{M} L_{B_{i}}, \quad C_{ii} = \mathcal{R}_{\bar{A}_{i}} *_{N} \bar{F}_{i}, \quad D_{ii} = \bar{G}_{i} *_{M} L_{B_{i}}, \quad \mathcal{E}_{ii} = \mathcal{R}_{\bar{A}_{i}} *_{N} \bar{E}_{i} *_{M} L_{B_{i}}, \quad M_{ii} = \mathcal{R}_{A_{i}} *_{N} C_{ii}, \quad i = 1, 2, 3, \quad (4.4)
\]

\[
N_{ii} = D_{ii} *_{M} L_{B_{ii}}, \quad S_{ii} = C_{ii} *_{N} L_{M_{ii}}, \quad i = 1, 2, 3, \quad (4.5)
\]

\[
A_{j,j+1} = \begin{bmatrix} L_{M_{j}} & *_{N} L_{S_{j}} \\ -L_{A_{j+1,j+1}} \end{bmatrix}, \quad B_{j,j+1} = \begin{bmatrix} \mathcal{R}_{D_{j}} \\ -\mathcal{R}_{B_{j+1,j+1}} \end{bmatrix}, \quad (4.6)
\]

\[
C_{j,j+1} = L_{M_{j}}, \quad D_{j,j+1} = \mathcal{R}_{N_{j}}, \quad (4.7)
\]

\[
F_{j,j+1} = A_{j+1,j+1}^\dagger *_{N} S_{j+1,j+1}, \quad G_{j,j+1} = \mathcal{R}_{N_{j+1,j+1}} *_{M} D_{j+1,j+1} *_{M} B_{j+1,j+1}^\dagger, \quad (4.8)
\]
According the proof of Theorem 3.2, we get the following theorem:

**Theorem 4.1.** Consider system (3.32). Then the following statements are equivalent:

1. System (3.32) is consistent.
2. The following equalities are satisfied:

\[
\begin{align*}
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} * N \mathcal{E}_i & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, * M \mathcal{L}_N_i = 0, \quad (4.22) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, = 0, \quad (4.23) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} * N \mathcal{E}_i & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, * M \mathcal{L}_N_i = 0, \quad (4.24) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, = 0, \quad (i = 1, 2, 3), \quad (4.25) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} * N \mathcal{E}_i & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, * M \mathcal{L}_N_i = 0, \quad (4.26) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, = 0, \quad (i = 1, 2, 3), \quad (4.27) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} * N \mathcal{E}_i & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, * M \mathcal{L}_Q = 0, \quad (4.28) \\
\mathcal{R}_{M_i} * N \mathcal{R}_{A_i} & = 0, \quad \mathcal{E}_i * M \mathcal{L}_B, = 0, \quad (i = 1, 2, 3), \quad (4.29)
\end{align*}
\]
Furthermore, if (1) holds, then the general solution to System (1.3) can be expressed as

\[
\begin{align*}
Z_k &= A^1_i * N \mathcal{E}_i * M B^i_k - A^1_k * N C_k * N \mathcal{M}_i * N \mathcal{E}_k * M B^i_k - A^1_k * N \mathcal{S}_k * N C^i_k * N \mathcal{E}_k * M \mathcal{N}_k * M \\
&\quad + D_{i,M} B^i_k - A^1_k * N S_k * N W_{k1} * M \mathcal{R}_N * M D_{i,M} B^i_k + \mathcal{L}_{A_k} * M W_{k4} + W_{k5} * M \mathcal{R}_B, \\
Z_{k+1} &= \mathcal{M}_i * N \mathcal{E}_i * M D^i_k + S^i_k * N S_k * N C^i_k * N \mathcal{E}_i * M \mathcal{N}_i * M \mathcal{L}_{M_k} * N L_{S_k} * N W_{k3} + \mathcal{M}_{M_k} * N W_{k2} \\
&\quad * M \mathcal{R}_N + W_{k3} * M \mathcal{R}_D, \quad (k = 1, 2, 3, 4)
\end{align*}
\]  

(4.30)

where

\[
\begin{align*}
W_{i1} &= \begin{bmatrix} I_{p_i} & 0 \end{bmatrix} * N \begin{bmatrix} A^1_i & * N (\hat{E}_i - \hat{C}_i * N W_{i2} * M \hat{D}_i - \hat{F}_i * N W_{i+1,2} * M \hat{G}_i) - \mathcal{T}_i & * M \hat{B}_i + \mathcal{L}_{\hat{A}_i} * N \mathcal{T}_i \end{bmatrix}, \\
W_{i1,4} &= \begin{bmatrix} 0 & I_{p_i} \end{bmatrix} * N \begin{bmatrix} A^1_i & * N (\hat{E}_i - \hat{C}_i * N W_{i2} * M \hat{D}_i - \hat{F}_i * N W_{i+1,2} * M \hat{G}_i) - \mathcal{T}_i & * M \hat{B}_i + \mathcal{L}_{\hat{A}_i} * N \mathcal{T}_i \end{bmatrix}, \\
W_{i3} &= [\mathcal{R}_{\hat{A}_i} * N (\hat{E}_i - \hat{C}_i * N W_{i2} * M \hat{D}_i - \hat{F}_i * N W_{i+1,2} * M \hat{G}_i) * M \hat{B}_i & \hat{A}_i * N \mathcal{T}_i] \\
&\quad + T_{i3} * M \mathcal{R}_{\hat{B}_i} | * M \begin{bmatrix} I_{p_i} & 0 \end{bmatrix}, \\
W_{i+1,5} &= [\mathcal{R}_{\hat{A}_i} * N (\hat{E}_i - \hat{C}_i * N W_{i2} * M \hat{D}_i - \hat{F}_i * N W_{i+1,2} * M \hat{G}_i) * M \hat{B}_i & \hat{A}_i * N \mathcal{T}_i] \\
&\quad + T_{i3} * M \mathcal{R}_{\hat{B}_i} | * M \begin{bmatrix} 0 & I_{p_i} \end{bmatrix}, \\
W_{i2} &= A^1_i * N \mathcal{E}_{i,M} B^i_{i,M} - A^1_i * N C_{ii} * N \mathcal{M}_{i,M} * N \mathcal{E}_{ii} * M B^i_{ii} - A^1_i * N \mathcal{S}_{ii} * N C^i_{ii} * N \mathcal{E}_{ii} * M \mathcal{N}^i_{ii} * M \\
&\quad * M \mathcal{D}_{ii} * M B^i_{ii} - A^1_i * N \mathcal{S}_{ii} * N T_{i4} * M \mathcal{R}_{N_{ii}} * M \mathcal{D}_{ii} * M B^i_{ii} & + \mathcal{L}_{A_{ii}} * N \mathcal{T}_{i5} + T_{i6} * M \mathcal{R}_{B_{ii}}, \\
W_{i+1,2} &= \mathcal{M}_{i,M} * N \mathcal{E}_{ii} * M D^i_{i,M} + S^i_{i,M} * N S_{ii} * N C^i_{ii} * N \mathcal{E}_{ii} * M \mathcal{N}^i_{ii} * L_{N_{ii}} * N \mathcal{T}_{i7} + \mathcal{L}_{M_{ii}} * N L_{S_{ii}} * N \mathcal{T}_{i4} \\
&\quad * M \mathcal{R}_{N_{ii}} + T_{i8} * M \mathcal{R}_{D_{ii}}, \quad (i = 1, 2, 3) \\
T_{i7} &= \begin{bmatrix} I_{m_{i,j+1}} & 0 \end{bmatrix} * N \begin{bmatrix} A^1_{i,j+1} & * N (\mathcal{E}_{j,j+1} - \mathcal{C}_{j,j+1} * N T_{j4} * M \mathcal{D}_{j,j+1} - \mathcal{F}_{j,j+1} * N T_{j+1,4} * M \mathcal{G}_{j,j+1}) \\
&\quad - \mathcal{U}_{i1} * M B_{j,j+1} + \mathcal{L}_{A_{j,j+1}} * N \mathcal{U}_{j2} \end{bmatrix}, \\
T_{j+1,5} &= \begin{bmatrix} 0 & I_{m_{i,j+1}} \end{bmatrix} * N \begin{bmatrix} A^1_{i,j+1} & * N (\mathcal{E}_{j,j+1} - \mathcal{C}_{j,j+1} * N T_{j4} * M \mathcal{D}_{j,j+1} - \mathcal{F}_{j,j+1} * N T_{j+1,4} * M \mathcal{G}_{j,j+1}) \\
&\quad - \mathcal{U}_{i1} * M B_{j,j+1} + \mathcal{L}_{A_{j,j+1}} * N \mathcal{U}_{j2} \end{bmatrix}, \\
T_{j8} &= [\mathcal{R}_{A_{j,j+1}} * N (\mathcal{E}_{j,j+1} - \mathcal{C}_{j,j+1} * N \mathcal{U}_{i,j+1} * M \mathcal{D}_{j,j+1} - \mathcal{F}_{j,j+1} * N \mathcal{U}_{i2} * M \mathcal{G}_{j,j+1}) * M B^i_{j,j+1} \\
&\quad + A_{j,j+1} * N \mathcal{U}_{j1} + U_{j3} * M \mathcal{R}_{B_{i,j+1}} ] * M \begin{bmatrix} I_{m_{i,j+1}} & 0 \end{bmatrix},
\end{align*}
\]  

(4.32) (4.33) (4.34) (4.35) (4.36) (4.37) (4.38) (4.39) (4.40) (4.41) (4.42)
\[ T_{j+1,6} = [\mathcal{R}_j \mathbb{A}_{j,j+1} * N (\mathcal{E}_{j,j+1} - \mathbb{C}_{j,j+1} * N \mathcal{U}_{12} * M \mathcal{D}_{j,j+1} - \mathcal{F}_{j,j+1} * N \mathcal{U}_{22} * M \mathcal{G}_{j,j+1}) * M \mathbb{B}_{j,j+1}^\dagger + \mathbb{A}_{j,j+1} * N \mathcal{U}_{j1} + \mathcal{U}_{j3} * M \mathcal{R}_{\mathbb{B}_{j,j+1}} * M \begin{bmatrix} 0 & I_n \end{bmatrix}], \]

\[ T_{j4} = \mathbb{A}_{j,j+1} * N \mathcal{E}_{j,j+1} * M \mathbb{B}_{j,j+1}^\dagger - \mathbb{A}_{j,j+1} * N \mathcal{C}_{j,j+1} * N \mathcal{M}_{j,j+1} * N \mathcal{E}_{j,j+1} * M \mathbb{B}_{j,j+1}^\dagger - \mathbb{A}_{j,j+1} * N \mathcal{S}_{j,j+1} * N \mathcal{C}_{j,j+1} * N \mathcal{E}_{j,j+1} * M \mathcal{N}_{j,j+1}^\dagger * M \mathcal{D}_{j,j+1} * M \mathbb{B}_{j,j+1}^\dagger - \mathbb{A}_{j,j+1} * N \mathcal{S}_{j,j+1} * M \mathcal{R}_{\mathcal{N}_{j,j+1}} * M \mathcal{D}_{j,j+1} * M \mathbb{B}_{j,j+1}^\dagger + \mathcal{L}_{\mathbb{A}_{j,j+1}} * N \mathcal{U}_{j5} + \mathcal{U}_{j6} * M \mathcal{R}_{\mathbb{B}_{j,j+1}}, \]

\[ T_{j+1,4} = \mathcal{M}_{j,j+1}^\dagger * N \mathcal{E}_{j,j+1} * N \mathcal{D}_{j,j+1}^\dagger + \mathcal{S}_{j,j+1}^\dagger * N \mathcal{S}_{j,j+1} * N \mathcal{C}_{j,j+1} * N \mathcal{E}_{j,j+1} * M \mathcal{N}_{j,j+1}^\dagger + \mathcal{L}_{\mathcal{M}_{j,j+1} * N \mathcal{S}_{j,j+1} * N \mathcal{U}_{j7} + \mathcal{L}_{\mathcal{M}_{j,j+1}} * N \mathcal{U}_{j4} * M \mathcal{R}_{\mathcal{N}_{j,j+1}} * M \mathcal{U}_{j8} + \mathcal{U}_{j6} * M \mathcal{R}_{\mathbb{B}_{j,j+1}}, \]

\[ U_{17} = \begin{bmatrix} I_s & 0 \end{bmatrix} * N [\mathcal{A}^\dagger * N (\mathcal{E} - \mathbb{C} * N \mathcal{U}_{14} * M \mathcal{D} - \mathcal{F} * N \mathcal{U}_{24} * M \mathcal{G}) - \mathcal{V}_1 * M \mathcal{B} + \mathcal{L}_A * N \mathcal{V}_2], \]

\[ U_{25} = \begin{bmatrix} 0 & I_s \end{bmatrix} * N [\mathcal{A}^\dagger * N (\mathcal{E} - \mathbb{C} * N \mathcal{U}_{14} * M \mathcal{D} - \mathcal{F} * N \mathcal{U}_{24} * M \mathcal{G}) - \mathcal{V}_1 * M \mathcal{B} + \mathcal{L}_A * N \mathcal{V}_3], \]

\[ U_{18} = [\mathcal{R}_A * N (\mathcal{E} - \mathbb{C} * N \mathcal{U}_{14} * M \mathcal{D} - \mathcal{F} * N \mathcal{U}_{24} * M \mathcal{G}) * M \mathbb{B}^\dagger + \mathcal{A} * N \mathcal{U}_{j1} + \mathcal{U}_{j3} * M \mathcal{R}_B] * M \begin{bmatrix} I_t & 0 \end{bmatrix}, \]

\[ U_{26} = [\mathcal{R}_A * N (\mathcal{E} - \mathbb{C} * N \mathcal{U}_{14} * M \mathcal{D} - \mathcal{F} * N \mathcal{U}_{24} * M \mathcal{G}) * M \mathbb{B}^\dagger + \mathcal{A} * N \mathcal{U}_{j1} + \mathcal{U}_{j3} * M \mathcal{R}_B] * M \begin{bmatrix} 0 & I_t \end{bmatrix}, \]

\[ U_{14} = \mathcal{A}^\dagger * N \widehat{\mathcal{E}} * M \mathbb{B}^\dagger - \mathcal{A}^\dagger * N \widehat{\mathcal{C}} * N \mathcal{M}^\dagger * N \widehat{\mathcal{E}} * M \mathbb{B}^\dagger - \mathcal{A}^\dagger * N \widehat{\mathcal{S}} * N \widehat{\mathcal{C}}^\dagger * N \widehat{\mathcal{E}} * M \mathcal{N}^\dagger * M \widehat{\mathcal{D}} * M \mathbb{B}^\dagger - \mathcal{A}^\dagger * N \widehat{\mathcal{S}} * N \mathcal{V}_4 * M \mathcal{R}_{\mathcal{N}} * M \widehat{\mathcal{D}} * M \mathbb{B}^\dagger + \mathcal{L}_{\mathcal{A}^\dagger} * N \mathcal{V}_5 + \mathcal{V}_6 * M \mathcal{R}_B, \]

\[ U_{24} = \mathcal{M}^\dagger * N \widehat{\mathcal{E}} * M \mathbb{B}^\dagger + \mathcal{S}^\dagger * N \widehat{\mathcal{S}} * N \widehat{\mathcal{C}} * N \widehat{\mathcal{E}} * M \mathcal{N}^\dagger + \mathcal{L}_{\mathcal{M}^\dagger} * N \mathcal{L}_{\mathcal{S}} * N \mathcal{V}_7 + \mathcal{L}_{\mathcal{M}^\dagger} * N \mathcal{V}_4 * M \mathcal{R}_{\mathcal{N}} + \mathcal{V}_8 * M \mathcal{R}_B, \]

and \( T_{11}, T_{12}, T_{13}, U_{11}, U_{12}, U_{13}, T_{14}, T_{15}, T_{16}, T_{37}, T_{38}, U_{15}, U_{16}, U_{27}, U_{28}, V_t, t = 1, \ldots, 8 \) are arbitrary tensors with appropriate sizes over \( \mathbb{H} \). \( \pi, m_j, s \) is the same as the row block of \( \mathcal{C}_i, \mathcal{C}_{j+1}, \mathcal{C}_3 \), respectively. \( q_i, n_j, t \) is the same as the column block of \( \mathcal{D}_i, \mathcal{D}_{j+1}, \mathcal{D}_2, (i = 1, 2, 3; \ j = 1, 2) \).

**Proof.** (1) \( \iff (2) \): We divide the proof into two parts:

- Firstly, we separate the system (1.3) into four equations as following:

\[ A_1 * M \mathbb{Z}_1 * N \mathbb{B}_1 + C_1 * M \mathbb{Z}_2 * N \mathbb{D}_1 = \mathbb{E}_1, \]

\[ A_2 * M \mathbb{Z}_2 * N \mathbb{B}_2 + C_2 * M \mathbb{Z}_3 * N \mathbb{D}_2 = \mathbb{E}_2, \]
\[ \mathcal{A}_3 \ast_M \mathcal{Z}_3 \ast_N \mathcal{B}_3 + \mathcal{C}_3 \ast_M \mathcal{Z}_4 \ast_N \mathcal{D}_3 = \mathcal{E}_3, \quad (4.54) \]
\[ \mathcal{A}_4 \ast_M \mathcal{Z}_4 \ast_N \mathcal{B}_4 + \mathcal{C}_4 \ast_M \mathcal{Z}_5 \ast_N \mathcal{D}_4 = \mathcal{E}_4. \quad (4.55) \]

Using four of Lemma 2.5, it can be easily proved that the solvable conditions (1.22), (1.23) hold when the system is consistent, and the expression of general solution are given by (1.30) and (1.31), (k=1,2,3,4).

Let \( Z_2 \) in Equation (4.52) be equal to \( Z_2 \) in Equation (1.53), \( Z_3 \) in Equation (1.53) be equal to \( Z_3 \) in Equation (4.54) and \( Z_4 \) in Equation (4.54) be equal to \( Z_4 \) in Equation (4.54). Then we obtain a new tensor system as following

\[
\begin{align*}
\tilde{A}_1 \ast_N \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} + \begin{bmatrix} W_{13} & W_{25} \end{bmatrix} \ast_M \mathcal{B}_1 + \mathcal{C}_1 \ast_N \mathcal{W}_{12} \ast_M \mathcal{D}_1 + \mathcal{F}_1 \ast_N \mathcal{W}_{22} \ast_M \mathcal{G}_1 &= \mathcal{E}_1, \\
\tilde{A}_2 \ast_N \begin{bmatrix} W_{21} \\ W_{31} \end{bmatrix} + \begin{bmatrix} W_{23} & W_{35} \end{bmatrix} \ast_M \mathcal{B}_2 + \mathcal{C}_2 \ast_N \mathcal{W}_{22} \ast_M \mathcal{D}_2 + \mathcal{F}_2 \ast_N \mathcal{W}_{32} \ast_M \mathcal{G}_2 &= \mathcal{E}_2, \\
\tilde{A}_3 \ast_N \begin{bmatrix} W_{31} \\ W_{41} \end{bmatrix} + \begin{bmatrix} W_{33} & W_{45} \end{bmatrix} \ast_M \mathcal{B}_3 + \mathcal{C}_3 \ast_N \mathcal{W}_{32} \ast_M \mathcal{D}_3 + \mathcal{F}_3 \ast_N \mathcal{W}_{42} \ast_M \mathcal{G}_3 &= \mathcal{E}_3.
\end{align*}
\]

(4.56)

Obviously, the necessary and sufficient conditions and general solution can be given by Theorem 3.2. Therefore, more details are omitted here.

\[ \square \]

5. Conclusion and Further Work

We have obtained a necessary and sufficient condition for the existence of the general solution to (1.2) via Einstein product by using M-P inverse in Theorem 3.2. We also have presented an expression of the general solution to (1.2) when it is solvable. Moreover, the general solution to tensor equation (1.3) has been considered. Some known results can be viewed as special cases of the one obtained in this paper.

Based on the above conclusion, we next consider the generalization of tensor system:

\[ \mathcal{A}_i \ast_N \mathcal{X}_i + \mathcal{Y}_i \ast_M \mathcal{B}_i + \mathcal{C}_i \ast_M \mathcal{Z}_i \ast_N \mathcal{D}_i + \mathcal{F}_i \ast_N \mathcal{Z}_{i+1} \ast_M \mathcal{G}_i = \mathcal{E}_i, i = 1, 2, \ldots, n, \]

giving the solvable conditions and presenting the general solution. This work is more meaningful, but we need some new methods to solve it.

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