ASYMPTOTIC SMOOTHING EFFECT FOR WEAKLY DAMPED FORCED KORTEWEG-DE VRIES EQUATIONS

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Abstract. Weakly damped forced KdV equation provides a dissipative semigroup on $L^2$. We prove that this semigroup enjoys an asymptotic smoothing effect, i.e. that all solutions converge towards a set of smoother solutions, when time goes to infinity.

1. Introduction
We are interested in the long time behavior for solutions to weakly damped forced KdV equation that reads

$$u_t + \gamma u + uu_x + u_{xxx} = f,$$  \hspace{1cm} (1.1)

where the unknown $u$ maps $T_x \times \mathbb{R}_+^+$ into $\mathbb{R}$, and where the data are the damping parameter $\gamma > 0$, and the external force $f$, that does not depend on $t$. Throughout this article we shall assume that $f$ belongs to $\dot{L}^2(T) = \{ u \in L^2; \int_T u(x)dx = 0 \}$. Here $T$ denotes the one-dimensional torus; in other words, we are considering (1.1) on $[0,1]$ with periodic boundary conditions.

We supplement (1.1) with initial conditions at $t = 0$

$$u(0) = u_0 \in \dot{L}^2(T).$$  \hspace{1cm} (1.2)

Due to the work of Bourgain (see [2]), (1.1)-(1.2) provide a well-posed problem on $\dot{L}^2(T)$: for each initial data, we have a local in time solution to (1.1)-(1.2), that is unique in a suitable space and that depends continuously on the initial data. On the other hand, this solution extends to a global one, since we have bounds on the $L^2$ norm of the solution; for positive times, this is due to the existence of an absorbing ball in $L^2$ that captures all the trajectories after a finite transient time. In the framework of infinite-dimensional dynamical systems (see [14]), KdV equation (1.1)-(1.2) defines a dissipative nonlinear semigroup. Before the work of Bourgain, the only known global well-posedness result was for $H^2$ initial data (see [15], [1], [12]).

The history of the dynamical study of (1.1) can be overviewed as follows. When considering classical smooth solutions to (1.1), i.e. solutions that start from initial data in $H^2(T)$, Ghidaglia (see [3]) proved that the associated KdV semiflow possesses a weak global attractor, i.e. a bounded subset of $H^2$, that is invariant by...
the flow and that attracts all the trajectories when $t$ goes to $+\infty$ for the $H^2$-weak topology. Moreover, this attractor has finite $H^1$-dimension. This result was proved under the assumption that the external force $f$ belongs to $H^2$. Actually, it turns out that this weak attractor is a global attractor for the $H^2$ strong topology (see [4]). The next step is concerned with the issue of the regularity of the attractor. In [11], the authors prove that if the external force $f$ belongs to $H^3$, then the KdV equation provides a dissipative semigroup in $H^3$ that enjoys the following property: if $f \in H^{3+k}$, then the global attractor for the $H^3$ topology is a compact subset of $H^{3+k}$. This result corresponds to the so-called asymptotic smoothing effect for the KdV semigroup. We follow here the terminology of Haraux (see [9]), where the author proved a regularization at $t = +\infty$ for a dissipative wave equation. The asymptotic smoothing effect for dissipative nonlinear Schrödinger equations was first proved in [7].

In this article, we are able (due to the work of Bourgain) to consider low-regularity solutions for KdV equation and not only classical ones. We then prove using the method of [7] that the $L^2$ KdV semiflow possesses a compact global attractor, that is moreover smooth. We also remove the regularity assumption on $f$ that were needed in the articles [3], [11]. Our main result states as follows

**Theorem 1.1.** The KdV equation on $\dot{L}^2$ possesses a compact global attractor $\mathcal{A}$ in $L^2$, that is a compact subset of $H^3$.

Our result is sharp in the following sense: any stationary solution to (1.1) belongs to $\mathcal{A}$, and cannot belongs to smaller $H^3$ space than $H^3$ if the forcing term belongs to $L^2$. Moreover, as a byproduct of our theorem we have

**Corollary 1.2.** For $\beta = 1, 2, 3$ the KdV equation on $\dot{H}^\beta$ possesses a global attractor $\mathcal{A}_\beta$ that for this topology; as a set, $\mathcal{A}_\beta = \mathcal{A}$.

Actually, this means that the global attractor for KdV equation does not depend of the space chosen for the mathematical study. This corollary contains the $H^2$ case, that was left open in [11].

We would like to point out that we do not consider solutions of KdV equations below $L^2$ as in [10]; in that case the issue of the long-time behavior of the solutions of dissipative KdV is open (to the author knowledge). In fact, it is unclear that the Bourgain solutions are weak solutions to (1.1) in a classical sense, i.e. solutions to (1.1) in the distributions space.

This article is organized as follows: Section 2.1 is devoted to the definition of Bourgain function spaces. In section 2.2, we prove bilinear estimates that are slightly modified versions of those of [10] and [2]. Section 2.3 and 2.4 are concerned with the description of the Bourgain method applied to respectively the conservative and the dissipative KdV equation. In section 3.1, we introduce an auxiliary problem whose solution is a smooth approximation of the high-frequency modes of a given trajectory. Sections 3.2 and 3.3 are devoted to prove respectively the global well-posedness of this problem in respectively $L^2$ and $H^3$. Section 4 is concerned with the proof of the main result; in a first subsection, we prove that the solution of the auxiliary problem compares with the high-frequency component of the solution. Section 4.2 and 4.3 are then concerned with the proofs of respectively Theorem 1.1 and Corollary 1.2.

We complete Section 1 by introducing some notations. We denote by $\dot{H}^m_x$ (or simply $H^m_x$, since we always deal with homogeneous Sobolev spaces in the $x$ variable) the space $H^m(T) \cap \dot{L}^2(T)$. Mixed classical space-time like $L^2_T \dot{H}^m_x$ will be also used.
We denote by \( C(I; H^n_x) \) the set of continuous functions from \( I \) that take values in \( H^n_x \). For the definition of Bourgain spaces, we refer to Section 2. We set \( \partial u = u_x \) and we denote by \( D \) the square root of \(-\partial^2\). We denote by \( A \) the linear part of (1.1), i.e. \( A = \partial^3 + \gamma Id \), that is an unbounded operator on \( L^2_x \). Observe that since we are working with homogeneous Sobolev spaces \( \partial \) (or \( D \)) is an isomorphism from \( H^n_x \) into \( H^{n-1}_x \). Throughout this paper, we set \( c \) for a numerical constant and we reserve the letter \( K \) for a constant which depends on the data \( \gamma, ||f||_{L^2_x} \). These constants are allowed to vary from one line to one another.

2. The Bourgain method

The aim of this section is to introduce material and arguments that will be usefull in the sequel. We follow here the framework of [2], [5], [10] and [6].

2.1. Bourgain function spaces

Let \( X \) be the space of functions \( u \) such that

* \( u : T \times \mathbb{R} \to \mathbb{R}, u : (x, t) \to u(x, t) \).
* \( t \to u(x, t) \in S(\mathbb{R}), \forall x \in T \), where \( S(\mathbb{R}) \) stands for the Schwarz class.
* \( x \to u(x, t) \in C^\infty(T), \forall t \in \mathbb{R} \).
* \( \hat{u}(0, t) = \int_T u(x, t)dx = 0, \forall t \in \mathbb{R} \).

For \( \rho, b \in \mathbb{R}, X^{\rho, b} \) denotes the completion of \( X \) with respect to the following norm

\[
||u||_{X^{\rho, b}} = \left( \sum_{\xi \neq 0} \int_{\mathbb{R}} (1 + |\tau - \xi|^3)^b |\xi^{2\rho}| \hat{u}(\xi, \tau)^2 d\tau \right)^{1/2}.
\]

In the sequel, we also need another space whose norm reads as follows

\[
||u||_{Y^{\rho}} = \left( \sum_{\xi \neq 0} (1 + |\tau - \xi|^3)^{-1} |\xi^{2\rho}| \hat{u}(\xi, \tau)^2 d\tau \right)^{1/2}.
\]

2.2. Bilinear estimates

We now describe estimates needed to handle the bilinear term in the KdV equation. We first recall from [10] the following statement

**Proposition 2.1.** Let \( \rho \geq -\frac{1}{2} \) be given. Then there exists a numerical constant \( c \) such that for any function \( u \) in \( X^{\rho, \frac{1}{2}} \)

\[
||D(u^2)||_{X^{\rho, -\frac{1}{2}}} \leq c ||u||_{X^{\rho, \frac{1}{2}}}^2.
\]

For later use, we need to introduce a localized version of this assertion. We now state an improved version of Proposition 2.1 as follows

**Proposition 2.2.** For \( \rho \geq -\frac{1}{2} \), there exists a numerical constant \( c \) such that for any function \( u \) in \( X^{\rho, \frac{1}{2}}, \) whose support is included in \([-2T, 2T] \times T \)

\[
||D(u^2)||_{X^{\rho, -\frac{1}{2}}} \leq c T^{1/4} ||u||_{X^{\rho, \frac{1}{2}}} ||u||_{X^{\rho, -\frac{1}{2}}}.
\]

**Proof of Proposition 2.2.** For the convenience of the reader, we indicate the complete proof of the proposition, despite the fact that one needs only minor modifications over the proof of [10].

By a duality argument, (2.4) comes from the following assertion: there exists \( c \) such that for any \( G \) that is of norm 1 in \( L^2_{\xi, \tau} \),

\[ \text{(	ext{2.4})} \]
\[ Q = \int_{\mathcal{D}} |\xi|^{1+\rho}(\xi, \tau)G(\xi, \tau)|\xi|^{\rho} d\tau < \tau - \xi^3 > 1/2 \leq c T^{1/4} ||u||_{X^{0, \frac{3}{2}}} ||u||_{X^{0, 1}}, \]  

(2.5)

where \( \mathcal{F} \) stands for the Fourier transform; here, in order to simplify the notations, we have set \( \tau := (1 + |\tau|^2)^{1/2} \), and we have written \( \int_\xi \) for the discrete sum \( \sum_\xi \) and \( d\xi \) for the counting measure on \( \mathbb{Z} \). Observe that \( Q \) reads also

\[ Q = \int_{\mathcal{D}} |\xi|^{1+\rho}\hat{u}(\xi_1, \tau_1)\hat{u}(\xi_2, \tau_2)G(\xi, \tau) < \tau - \xi^3 > 1/2 d\tau d\xi_1 d\xi_2, \]  

(2.6)

where \( \mathcal{D} = \{ \sigma = \{ \xi, \xi_1, \xi_2, \tau, \tau_1, \tau_2 \} \in \mathbb{Z}^3 \times \mathbb{R}^3; \xi_1 + \xi_2 = \xi \) and \( \tau_1 + \tau_2 = \tau \}

Observe that \( |\xi| \leq |\xi_1| + |\xi_2| \). Then \( \mathcal{D} \subset \mathcal{D}_1 \cup \mathcal{D}_2 \) where \( \mathcal{D}_1 = \{ \sigma; |\xi| \leq 2|\xi_1| \} \). Hence we will bound the integral over \( \mathcal{D}_1 \), omitting the majorization of the integral over \( \mathcal{D}_2 \) which is similar. We also have the following algebraic inequality

\[ 3|\xi_1 \xi_2| = |\tau - \xi^3| - (\tau_1 - \xi^3_1) - (\tau_2 - \xi^3_2)| \leq |\tau - \xi^3| + |\tau_1 - \xi^3_1| + |\tau_2 - \xi^3_2|. \]  

(2.7)

Therefore \( \mathcal{D}_1 \subset \mathcal{D}^0_1 \cup \mathcal{D}^1_1 \cup \mathcal{D}^2_1 \) where \( \mathcal{D}^0_1 = \{ \sigma \in \mathcal{D}_1; |\tau - \xi^3| \geq |\xi_1 \xi_2| \} \) and \( \mathcal{D}^i_1 = \{ \sigma \in \mathcal{D}_1; |\tau_i - \xi^3_i| \geq |\xi_1 \xi_2|; i = 1, 2 \} \). We now divide the majorization in three steps, according to either \( \sigma \in \mathcal{D}^0_1, \mathcal{D}^1_1 \) or \( \mathcal{D}^2_1 \). Set \( Q_i \) for the integral over \( \mathcal{D}_i \).

First case: \( \sigma \in \mathcal{D}^0_1 \). Set \( \mathcal{F}^{-1} \) for the inverse Fourier transform and \( D^\gamma u = \mathcal{F}^{-1}(|\xi|^\gamma \hat{u}) \). Observe that since \( \frac{1}{2} + \rho \geq 0 \),

\[ \frac{|\xi|^{1+\rho}}{\tau - \xi^3 > 1/2} \leq c |\xi|^{1/2} \]  

(2.8)

Therefore

\[ |Q_0| \leq \int_{\mathcal{D}^0_1} \frac{|\xi|^{1+\rho}|\hat{u}(\xi_1, \tau_1)||\hat{u}(\xi_2, \tau_2)||G(\xi, \tau)|}{\tau - \xi^3 > 1/2} d\tau_1 d\tau d\xi_1 d\xi_2 \]

\[ \leq c ||G||_{L^2_{\xi, \tau}} ||\hat{u}||_{L^2_{\xi, \tau}}^{-1/2} \leq c ||D^{-1/2}\mathcal{F}^{-1}(|\hat{u}|)||_{L^2_{\xi, \tau}} ||D^\rho \mathcal{F}^{-1}(|\hat{u}|)||_{L^2_{\xi, \tau}}. \]  

(2.9)

We now use the following assertion

**Lemma 2.3.** There exist \( c_1, c_2 \) such that for any \( u \in X^{0, \frac{3}{2}} \), supported in \([-2T, 2T] \times T \)

\[ ||u||_{L^4_{\xi, \tau}} \leq c_1 ||u||_{X^{0, \frac{3}{2}}} \leq c_2 T^{1/8} ||u||_{X^{0, \frac{3}{2}}}. \]  

(2.10)

**Proof of Lemma 2.3.** See Proposition 7.15 in [2] for the proof of the first inequality, that corresponds to the embedding \( X^{0, \frac{3}{2}} \subset L^4_{\xi, \tau} \). For the second inequality, use also the interpolation

\[ ||u||_{X^{0, \frac{3}{2}}} \leq ||u||^{2/3}_{X^{0, \frac{1}{2}}} ||u||^{1/3}_{L^4_{\xi, \tau}}, \]  

(2.11)

together with the embedding inequality

\[ ||u||_{L^2_{\xi, \tau}} \leq T^{1/4} ||u||_{L^4_{\xi, \tau}} \]  

(2.12)
that holds for \((x, t) \in [0, 1] \times [-2T, 2T]\).

Hence

\[
|Q_0| \leq c T^{1/4} \|u\|_{X^p}^{1 \over 2} \|u\|_{X^{-{1 \over 2}}}^{1 \over 2},
\]  

(2.13)

since \(\|D^\alpha F^{-1}(|\hat{u}|)\|_{X^0} = \|u\|_{X^p}^{1 \over 2}\).

Second case: \(\sigma \in D_0^1\).

Observe that since \(\alpha + \rho \geq 0\),

\[
|\xi|^{1 + \rho} \over \tau - \xi^3 > 1 \over 2 \leq c \over \tau - \xi^3 > 1 \over 2 |\xi|^\rho
\]

(2.14)

Proceeding as in the first case, we thus obtain

\[
|Q_1| \leq \int_{\mathbb{D}_0^1} |\xi|^{1 + \rho} \over \tau - \xi^3 > 1 \over 2 \left|\hat{u}(\xi, \tau_1)\right| \left|\hat{u}(\xi, \tau_2)\right| \|G(\xi, \tau)\|_{L^2_{\xi_1, \tau_1}} \left|\mathcal{F}^{-1}(G)\right|_{L^2_{\xi_1, \tau_1}} \|D^{1/2} \mathcal{F}^{-1}(\hat{u})\|_{L^2_{\xi_1, \tau_1}} \, d\tau_1 d\xi_1
d\xi
\leq \left|\hat{u}(\xi, \tau_1)\right| < \tau_1 - \xi^3 > 1 \over 2 \|\xi\|_{L^2_{\xi_1, \tau_1}} \left|\mathcal{F}^{-1}(G)\right|_{L^2_{\xi_1, \tau_1}} \|D^{1/2} \mathcal{F}^{-1}(\hat{u})\|_{L^2_{\xi_1, \tau_1}},
\]

Therefore

\[
|Q_1| \leq c T^{1/4} \|u\|_{X^p}^{1 \over 2} \|u\|_{X^{-{1 \over 2}}}^{1 \over 2},
\]  

(2.15)

since due to lemma 2.2,

\[
|\mathcal{F}^{-1}(G)\|_{L^2_{\xi_1, \tau_1}} \leq c T^{1/4} \|\mathcal{F}^{-1}(G)\|_{X^0} = c T^{1/4}.
\]  

(2.16)

Third case: \(\sigma \in D_0^2\).

We just have to use

\[
|\xi|^{1 + \rho} \over \tau - \xi^3 > 1 \over 2 \leq c \over \tau - \xi^3 > 1 \over 2 |\xi|^\rho
\]

(2.17)

and

\[
|Q_2| \leq \int_{\mathbb{D}_0^2} |\xi|^{1 + \rho} \over \tau - \xi^3 > 1 \over 2 \left|\hat{u}(\xi, \tau_1)\right| \left|\hat{u}(\xi, \tau_2)\right| \|G(\xi, \tau)\|_{L^2_{\xi_2, \tau_2}} \left|\mathcal{F}^{-1}(G)\right|_{L^2_{\xi_2, \tau_2}} \|D^{1/2} \mathcal{F}^{-1}(\hat{u})\|_{L^2_{\xi_2, \tau_2}} \, d\tau_1 d\xi_1 d\xi
\]

\[
\leq \left|\hat{u}(\xi, \tau_2)\right| < \tau_2 - \xi^3 > 1 \over 2 \|\xi\|_{L^2_{\xi_2, \tau_2}} \left|\mathcal{F}^{-1}(G)\right|_{L^2_{\xi_2, \tau_2}} \|D^{1/2} \mathcal{F}^{-1}(\hat{u})\|_{L^2_{\xi_2, \tau_2}},
\]

(2.18)

and to proceed as above.

We now state a result that is similar to Proposition 2.2 for \(Y^p\) spaces. For the sake of conciseness, we omit the proof of this result, because it consists in modifying the proof of Lemma 7.42 in [2] according to the arguments given in the proof of Proposition 2.2 above. Nevertheless, we would like to point out that we do not know if Proposition 2.4 below holds true in the limiting case \(\rho = -{1 \over 2}\).

**Proposition 2.4.** Let \(\rho > -{1 \over 2}\) be given. Then for any \(\beta > -{1 \over 2}\) there exists \(c\) which depends on \(\beta, \rho\) such that for \(u\) supported in \([-2T, 2T] \times T\)
\[ ||D(u^2)||_{Y^p} \leq cT^{1/4}||u||_{X^{\rho,\frac{1}{2}}}||u||_{X^{\rho,\frac{1}{2}}} \]  

(2.20)

The introduction of \( Y^p \) is motivated by the following proposition, that will be used in the next paragraph.

**Proposition 2.5.**

\[ || \int_0^t W(t-s)D(u^2)ds ||_{X^{\rho,\frac{1}{2}}} \leq c(||D(u^2)||_{X^{\rho,\frac{1}{2}}} + ||D(u^2)||_{Y^p}), \]  

(2.21)

and

\[ \sup_t || \int_0^t W(s-t)D(u^2)(s)ds ||_{H^\rho} \leq c(||D(u^2)||_{X^{\rho,\frac{1}{2}}} + ||D(u^2)||_{Y^p}), \]  

(2.22)

**Proof of Proposition 2.5:** For the proof of (2.21), see [6] (or [2]). The proof of (2.22) is similar and we omit it. \( \square \)

### 2.3 Contraction in \( X^{\rho,b} \) spaces

Let \( W(t) \) be the free Airy group defined by \( W(t)u = \mathcal{F}^{-1}(e^{it\xi^3} \hat{u}(\xi)) \). The Bourgain method consists in performing a fixed point argument in \( X^{0,\frac{1}{2}} \) to the Duhamel form of (1.1) that reads

\[ u(t) = W(t)u_0 - \frac{1}{2} \int_0^t W(t-s)\partial(u^2)ds. \]  

(2.23)

We introduce a suitable time localization as follows: we consider a cut-off function \( \psi \in C_0^\infty(\mathbb{R}) \) that satisfies \( \psi(t) = 1 \) if \( |t| \leq 1 \) and that vanishes outside \([-2,2]\). We set \( \psi_T(t) = \psi(t/T) \). We then introduce the following semi-norm

\[ ||u||_{X^{\rho,\frac{1}{2}}_{[-T,T]}} = ||\psi_T u||_{X^{\rho,\frac{1}{2}}}. \]  

(2.24)

When needed, we also extend this definition to any interval \( I \) by setting

\[ ||u||_{X^{\rho,\frac{1}{2}}_I} = ||\psi_I u||_{X^{\rho,\frac{1}{2}}}, \]  

(2.25)

where \( \psi_I \) is a smooth cut-off function localized around \( I \). We will also use the following notation: for a family \( I \) of intervals that have the same width, the \( X^{\rho,\frac{1}{2}}_{I_{loc}} \)-norm of a function \( u \) is the supremum over the \( I \)'s of the \( X^{\rho,\frac{1}{2}}_{I_{loc}} \)-norms of \( u \).

We now proceed to the fixed point argument. Set

\[ T(u) = \psi_T(t)W(t)u_0 - \frac{1}{2} \psi_{T/2}(t) \int_0^t W(t-s)\partial(\psi_T(s)u)^2ds. \]  

(2.26)

Then, on the one hand \( ||\psi_T(t)W(t)u_0||_{X^{\rho,\frac{1}{2}}_{[-T,T]}} = c||u_0||_{L^2}, \) where \( c \) is independent of \( T \). On the other hand to proceed to the majorization of the integral term, we need the following technical argument

**Lemma 2.6.** There exists a numerical constant \( c \) such that for any \( G \) in \( H^\frac{1}{2}_t \).
for the sake of concisness, we omit the details. \(\Box\)

Proceeding as in Section 2.3, we have

\[
||\psi_T G||_{H^s_T} \leq c(\ln T)^{1/2}||G||_{H^s_T}. \tag{2.27}
\]

Proof of Lemma 2.6. Lemma 2.6 appears in [6] with a \(T^{-\varepsilon}\) factor instead of \((\ln T)^{1/2}\) in (2.27) (for any small \(\varepsilon > 0\)). A careful analysis on the proof in [6] gives (2.27); for the sake of concisness, we omit the details. \(\Box\)

At this stage, using (2.4), (2.27), Propositions 2.4 and 2.5 we have

\[
||\psi_{T/2} \int_0^t W(t-s)\partial (\psi_T u)^2 ds||_{X^{\alpha, 1/2}_{[-T/T]}} \leq c(\ln T)^{1/2}||\int_0^t W(t-s)\partial (\psi_T u)^2 ds||_{X^{\alpha, 1/2}}
\]

\[
\leq c(\ln T)^{1/2} T^{1/4}||u||^2_{X^{\alpha, 1/2}}. \tag{2.28}
\]

Hence, for \(T\) small enough, \(T\) maps a ball of radius \(2c||u_0||_{L^2}\) into itself. To prove that \(T\) is a contraction mapping is very similar. Therefore we have a fixed point \(u\) for \(T\) that is a mild solution for \(|t| \leq T/2\) of (1.1). On the other hand, applying (2.22), we prove that the integral term in (2.23) depends continuously on \(t\). Then the fixed point \(u\) belongs to \(C(-T/2, T/2; L^2)\), and is a mild solution to (1.1) in this space. Observe that since \(T\) depends only on \(||u_0||_{L^2}\), and since the \(L^2\)-norm of \(u\) is conserved along the flow, the global existence result follows promptly. We would like to point out that since the integral term in (2.23) belongs to \(C^1(-T/2, T/2; H^{-n}_x)\) for \(n\) large enough, then a Bourgain solution for (1.1)-(1.2) is also a weak solution in the classical sense.

### 2.4. Dissipative KdV equation

In the dissipative case, we have to take into account the damping parameter and the forcing term. We then begin with two preliminary remarks:

1. When dealing with estimates that are local in time, the damping provides \(e^{\gamma t}\)-terms in the Duhamel form of (1.1); these terms can be neglected, since they can be incorporated into the cut-off function in the Bourgain method. Of course, the damping will play a role when considering large time estimates.

2. Since the forcing term is independent of \(t\), then

\[
||A^{-1} f||_{X^{\alpha, 1/2}_{loc}} = c||f||_{H^0_{\alpha - 1/2}} \tag{2.29}
\]

is finite iff \(\beta \leq 3/2\).

We now check that the dissipative KdV equations provide a semigroup on \(L^2_x\). Proceeding as in Section 2.3, we have

**Proposition 2.7.** Assume \(u_0\) be given in \(\dot{L}^2_x\). Then there exist \(T\) which depend on \(||u_0||_{\dot{L}^2_x}, \gamma, f\) such that there exists a unique solution \(u \in C(-T/2, T/2; \dot{L}^2_x) \cap X_{loc}^{0, 1/2}\) for (1.1)-(1.2).

Actually, the solution extends also to a global solution for positive time by observing that the dissipative KdV flow has an absorbing set in \(L^2_x\). To check this point, multiply (1.1) by \(u\), integrate over \(T_x \times [0, T]\) to obtain

\[
||u(t)||^2_{L^2_x} \leq ||u_0||^2_{L^2_x} e^{-\gamma t} + \frac{||f||^2_{L^2_x}}{\gamma^2} (1 - e^{-\gamma t}) \tag{2.30}
\]
(this inequality can be rigorously proved by approximating $u_0$ and $f$ by smooth functions, by establishing (2.30) for the smooth solution $u(t)$ of (1.1) associated with this smooth data, and then by passing to the limit).

Hence there exists $t_1$ which depend on $||u_0||_{L^2_x}, \gamma, f$ such that for $t \geq t_1$

$$||u(t)||_{L^2_x}^2 \leq 2||f||_{L^2_x}^2 \frac{\gamma^2}{\gamma^2}.$$ (2.31)

As a consequence of Proposition 2.7 and of (2.31), we then have

**Corollary 2.8** Assume $u(t)$ be a solution for (1.1) in $L^2_x$, that moreover belongs to the $L^2_x$ absorbing ball for $t \geq t_1$. Then there exists $M_0, T_1$ which depend only on the data $\gamma, f$ such that for any time interval $I \subset [t_1, \infty)$ whose width is less than $T_1$

$$||u||_{\chi^1_{\gamma f}} \leq M_0.$$ (2.32)

Without loss of generality, for $M_0$ as in (2.32), we may assume

$$\varepsilon = T_1^{1/4}(\ln T_1)^{1/2}M_0 <<< 1.$$ (2.33)

In the following, we denote by $S(t)u_0 = u(t)$ the nonlinear dissipative semigroup on $L^2_x$ defined by (1.1)-(1.2).

3. **The auxiliary problem**

3.1. **Definition**

In this section, we consider a trajectory $u(t)$ in $\dot{L}^2_x$, and we set $t_1$ for his entrance time into the $\dot{L}^2_x$ absorbing ball. We are given a level set $N$, which is assumed to be large enough. We consider the orthogonal projectors $P$ and $Q$ defined as follows

$$Pu = \sum_{|\xi| \leq N} \hat{u}(\xi)e^{2i\pi \xi x},$$ (3.1)

$$Qu = \sum_{|\xi| > N} \hat{u}(\xi)e^{2i\pi \xi x}.$$ (3.2)

We plan to approximate the high-frequency part $z = Qu$ of a trajectory $u$ by $Z$ that is solution for

$$Z_t + \gamma Z + \partial^3 Z + \frac{1}{2}Q\partial(y + Z)^2 = Qf,$$ (3.3)

supplemented with initial condition at $t_1$

$$Z(t_1) = 0.$$ (3.4)

Here $y = Pu$ stands for the low-frequency part of $u$, and is a data for the equations (3.3)-(3.4). Due to standard inverse inequalities, $y$ is smooth, and then $Z$ is a classical solution for (3.3)-(3.4) in $C([t_1, t_1 + t]; QH^3_x)$. The next two sections are devoted to prove that this solution is global for positive times in respectively $L^2_x$ and $H^3_x$.

3.2. **$L^2_x$-global well-posedness for the auxiliary problem**

To begin with, we prove that (3.3)-(3.4) is globally well-posed in $QL^2_x$. 
Proposition 3.1. There exist $M, N_0$ which depend on $\gamma, f$ such that for any fixed $N \geq N_0$, for all $t \geq t_1$
\[
||Z(t)||_{L^2} \leq M. \quad (3.5)
\]

Remark Observe that if (3.5) is valid, then $Z$ is also locally (in time) bounded in $X^{0, \frac{1}{2}}$; this is the analog of Corollary 2.8.

Proof of Proposition 3.1. For the sake of simplicity, we assume that $t_1 = 0$ throughout this proof. Using the Bourgain method, we can construct, for each $Z(t_0) = Z_0$ in $QL^2$, a solution $Z$ for (3.3) that belongs to $X^{0, \frac{1}{2}}$, where $I = [t_0, t_0 + T_0]$, where $T_0$ depends on the $L^2$ norm of $Z_0$. The point is that we do not know a priori if $Z(t)$ remains bounded in $L^2$ when $t$ varies.

Fix $M$ large enough with respect to $\gamma, f$ (this point will be clarified in the sequel). Let $K$ be any other constant which depends on $\gamma, f$, and that may vary from one line to one another. Let $T_{max}$ be defined as follows
\[
T_{max} = \sup\{t \geq 0; \forall s < t, ||Z(s)||_{L^2} \leq M\}. \quad (3.6)
\]

We prove below that, if $M$ is chosen large enough, then for $N \geq N_0(M, \gamma, f)$ we have $T_{max} = +\infty$.

First step: local in time estimate

We set $W(t)$ for the free Airy group defined by $W(t)u_0 = \mathcal{F}^{-1}(e^{it\xi^2} \hat{u}_0(\xi))$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. When dealing with estimates local in time, we may assume without loss of generality that $\gamma = 0$.

Let $t_0 < T_{max}$ be given. Let $T$ be small enough as above ($T$ compares with $T_1$ defined in Corollary 2.8 above; say $T \leq 2T_1 \leq 2T$) and let $t \in [t_0, t_0 + T]$. The Duhamel’s form of (3.3) reads
\[
Z(t) = W(t - t_0)Z(t_0) - \frac{1}{2} Q \int_{t_0}^{t} W(t - s)\partial(y + Z)^2 ds + Q(\int_{t_0}^{t} W(t - s) ds) f. \quad (3.7)
\]

Set $T(Z)$ for the r.h.s of (3.7), and $I = [t_0, t_0 + T]$. $Z(t_0)$ being given, T defines a mapping from $X^{0, \frac{1}{2}}$ into itself. More precisely, due to Propositions 2.4 and 2.5, we have
\[
|| \int_{t}^{t_0} W(t - s)D(y + Z)^2 ds ||_{X^{0, \frac{1}{2}}} \leq cT^{1/4} \ln(T)^{1/2}(||y + Z||_{X^{0, \frac{1}{2}}} ||y||_{X^{0, \frac{1}{2}}} + ||Z||_{X^{-\frac{3}{2}, 0}}), \quad (3.8)
\]
where $-\frac{3}{2} > -\frac{1}{2}$ denotes a number arbitrarily close to $-\frac{1}{2}$.

On the other hand, we have
\[
||Q(\int_{t_0}^{t} W(s - t_0) ds) f||_{X^{0, \frac{1}{2}}} = ||Q(W(t - t_0) - Id)A^{-1} f||_{X^{0, \frac{1}{2}}}
\leq ||QA^{-1} f||_{L^2} + ||QA^{-1} f||_{X^{0, \frac{1}{2}}} \leq ||QA^{-1} f||_{H^2} \leq \frac{||f||_{L^2}}{N^{3/2}}, \quad (3.9)
\]
We now infer from (3.7)-(3.9) and from the following enhanced Poincaré inequality
\[ ||Z||_{\mathcal{X}_t^{n, 1/2}} \leq N^{-1/2} ||Z||_{\mathcal{X}_t^{n}} \] (3.10)
that
\[
||T(Z)||_{\mathcal{X}_t^{n, 1/2}} \leq c_0||Z(t_0)||_{\mathcal{L}^2} + c(M_0^2 T^{1/4} \ln T^{1/2}) + cT^{1/4} \ln T^{1/2} M_0||Z||_{\mathcal{X}_t^{n, 1/2}} +
\]
\[
+ \frac{cT^{1/4} \ln T^{1/2}}{N^{1/2}} ||Z||_{\mathcal{X}_t^{n, 1/2}}^2 + \frac{||f||_{\mathcal{L}^2}}{N^{3/2}},
\] (3.11)
where \(M_0\) is as in (2.32).

\(M_0\) being given as in Corollary 3.2 and assuming without loss of generality that \(M \geq M_0\) and \(T \leq 1\) (since \(T \leq T_1\)), then (3.11) becomes, when \(||Z||_{\mathcal{X}_t^{n, 1/2}} \leq R\),
\[
||T(Z)||_{\mathcal{X}_t^{n, 1/2}} \leq c_0 M + c \varepsilon(M + R) + R^2 \frac{N^{1/2}}{N^{3/2}} + \frac{||f||_{\mathcal{L}^2}}{N^{3/2}}.
\] (3.12)

Fix now \(R = 2c_0 M\), then if \(\varepsilon\) is small enough, and if \(N_0\) is chosen large enough with respect to \(M, ||f||_{\mathcal{L}^2}\), then for any fixed \(N \geq N_0\)
\[
||T(Z)||_{\mathcal{X}_t^{n, 1/2}} \leq R.
\] (3.13)

To prove that \(T\) is a contraction mapping on the ball of radius \(R\) in \(\mathcal{X}_t^{n, 1/2}\) is very similar, therefore we omit the details. At this stage, we have: given \(M\) large enough, there exists \(N_0\) which depends on \(M, \gamma, f\) such that for \(N \geq N_0\), as long as \(||Z(t)||_{\mathcal{L}^2} \leq M\), then \(||Z||_{\mathcal{X}_t^{n, 1/2}} \leq 2c_0 M\), where \(T\) is independent of \(M\).

Second step: global in time estimate. We prove below that \(T_{\text{max}} = +\infty\).

Multiply (3.3) by \(Z\) and integrate over \(T_x\) to obtain (some cancellations occur when integrating by parts), for \(t < T_{\text{max}} + T\,
\[
\frac{d}{dt} (||Z(t)||_{\mathcal{L}^2}^2) = e^{2\gamma t} \left( \int_{T_x} (y \partial (Z^2) - Z \partial (y^2)) + 2 \int_{T_x} f Z \right).
\] (3.14)

First case: we assume here that \(t > T\). Then there exists \(\delta \in [\frac{1}{2}, 1]\) and \(n \in \mathbb{N}\) such that \([0, t] = \bigcup_{k=0}^{n} [k \delta T, (k + 1) \delta T]\). Set \(I_k = [k \delta T, (k + 1) \delta T]\). We now integrate (3.14) over \([0, t]\), and bound the second term in the r.h.s of the resulting equation using this splitting of \([0, t]\) as follows
\[
\left| \int_{T_x [0, t]} e^{2\gamma s} Z \partial (y^2) \right| \leq \sum_{k=0}^{n} e^{2\gamma k \delta T} \left| \int_{T_x I_k} e^{2\gamma (s-\delta T)} Z \partial (y^2) \right|.
\] (3.15)

Observe that since we are dealing on estimates that are local in time, the \(e^{2\gamma (s-\delta T)}\) does not play any role when integrating over \(T \times I_k\). We now apply Proposition 2.2. and thus obtain
We now prove

**Proposition 3.2.** There exists $K, N_0$ which depend on the data $\gamma, f$ and a numerical constant $m$ such that for any fixed $N > N_0$, for $t \geq t_1$, the solution $Z$ for (3.3)-(3.4) satisfies

$$
\|Z(t)\|_{H^3} \leq KN^m. 
$$
Proof of Proposition 3.2: assume \( t_1 = 0 \) throughout this proof. We proceed in two steps.

First step: \( H^1 \)-estimate.

Multiply (3.3) by \( -\partial^2 Z - \frac{1}{2} Q(y + Z)^2 + Q\partial^{-1} f \) and integrate over \( T_x \) to obtain

\[
\frac{d}{dt} J(Z) + \gamma J(Z) + \gamma \|DZ\|_{L^2_x}^2 = \frac{\gamma}{3} \left( \int_T Z^3 - 3 \int_T y^2 Z \right) + \int_{T_x} y \partial (y^2 - (y + Z)^2), \tag{3.23}
\]

where

\[
J(Z) = \|\partial Z\|_{L^2_x}^2 + \frac{1}{3} \int_{T_x} (y^3 - (y + Z)^3) + 2 \int_{T_x} (\partial^{-1} f) Z. \tag{3.24}
\]

We first observe that

\[
|J(Z) - \|\partial Z\|_{L^2_x}^2| \leq c(\|y\|_{L^2_x}^2 + \|Z\|_{L^2_x}^2) \|Z\|_{L^2_x}^2 + \|f\|_{L^2_x} \|Z\|_{H^{-1}_x} \leq K(\|Z\|_{L^2_x}^2 + \|Z\|_{H^{-1}_x}), \tag{3.25}
\]

due to (2.31) (to bound the \( y = Pu \) term) and to (3.5) (to bound the \( Z \) term).

We complete this computation by recalling classical enhanced Poincaré inequality

\[
\|Z\|_{L^\infty_x} \leq \|Z\|_{L^2_x}^{1/2} \|\partial Z\|_{L^2_x}^{1/2} \leq N^{-1/2} \|\partial Z\|_{L^2_x}. \tag{3.26}
\]

We easily infer from (3.25)-(3.26) that to bound \( Z \) in \( H^1 \) is equivalent to bound \( J(Z) \).

We now proceed to the majorization of the r.h.s of (3.23). To begin with, we state

**Lemma 3.3.** There exists \( K \) which depends only on the data \( \gamma, f \) such that for each time interval \( I \in [t_1, +\infty) \) whose width is less than \( T_1 \), then \( u_t = a + b \) such that

\[
N^{-2} \|Pa\|_{X^{-1/2}_{\gamma, I}} + N^{-3} \|Pb\|_{L^\infty_x L^2_x} \leq K. \tag{3.27}
\]

**Proof of Lemma 3.3:** set \( b = -\frac{1}{2} \partial (u^2) \) and consider a smooth test function \( \phi = \phi(x) \). Then

\[
\left| \int_{T_x} P\partial (u^2) \phi \right| \leq \|u\|_{L^2_x}^2 \|P\partial \phi\|_{L^\infty_x} \leq N^{3/2} \|u\|_{L^2_x}^2 \|\phi\|_{L^2_x}, \tag{3.28}
\]

due to standard inverse inequalities

\[
N^{-3/2} \|P\phi\|_{L^\infty_x} + N^{-1} \|P\partial \phi\|_{L^2_x} \leq \|\phi\|_{L^2_x}. \tag{3.29}
\]

Hence, due to (2.31) and to (3.29), we obtain the boundeness of \( Pb \) as stated in (3.27).

Consider now \( a = f - \gamma u - \partial^1 u \). Since \( u \) is locally uniformly bounded in \( X^{0, 1/2}_{\gamma, I} \), then the bound on \( Pa \) comes from inverse inequalities on Bourgain spaces and from (2.32).

Applying (3.27), we then have
\[
\int_{T_s} \|Pb(y^2 - (y + Z)^2)\| \leq c\|Pb\|_{L^2}(\|y\|_{L^2} + \|Z\|_{L^2}) \|Z\|_{L^\infty} \leq KN^{3/2}N^{-1/2}\|\partial Z\|_{L^2}
\]
\[
\leq \frac{\gamma}{10}\|\partial Z\|_{L^2}^2 + KN^2. \tag{3.30}
\]

On the other hand, the first term in the r.h.s of (3.23) can be also bounded by
\[
\frac{\gamma}{10}\|\partial Z\|_{L^2}^2 + KN^2
\]
by the same arguments as in (3.25)-(3.26).

We now use these majorizations and integrate (3.23) over \([0, t]\) (using \(J(Z(0)) = J(0) = 0\)) to obtain
\[
J(Z(t))e^{\gamma t} \leq KN^2 e^{\gamma t} + c\int_{T_s \times [0, t]} e^{\gamma s}(Pa)(Z^2 + 2yZ). \tag{3.31}
\]

Proceeding as in (3.15)-(3.18) (considering \(t > T_1 \geq T\) and \([0, t] = \cup_{k=0}^n [k\delta t, (k+1)\delta t]\)) we obtain
\[
|\int_{T_s \times [0, t]} e^{\gamma s}\partial^{-1}(Pa)\partial(Z^2 + 2yZ)| \leq K e^{\gamma t} \sup_k \|Pa\|_{X_{\delta t}^{1/2}} \|Z\|_{X_{\delta t}^{1/2}} \|2y + Z\|_{X_{\delta t}^{1/2}} \leq KN^2 e^{\gamma t}, \tag{3.32}
\]
due to (2.32) and since \(Z\) remains also locally uniformly bounded in \(X_{\delta t}^{1/2}\), due to (3.5) (see the remark below Proposition 3.1).

We easily infer from (3.25)-(3.26), (3.31)-(3.32) that
\[
\|Z(t)\|_{H^2} \leq KN. \tag{3.33}
\]

This completes the proof of the first step.

Second step: \(L^2_t\)-estimate on \(Z_t\).

To bound the \(H^2_t\)-norm of \(Z\) is equivalent to bound the \(L^2_t\)-norm of \(Z_t\). We now proceed to this majorization. \(Z_t = w\) is solution to
\[
w_t + \gamma w + \partial^3 w + Q\partial(w(y + Z)) = -Q\partial(y_t(y + Z)). \tag{3.34}
\]

Actually \(w\) is the unique \((L^\infty \cap C)(0, t; L^2)\) solution for this equation supplemented with initial condition \(w(0) = Z_t(0)\); this holds true since \(\partial(y + Z)\) belongs to \(L^\infty_{t, x}\) \((Z\) belongs to \(L^\infty_{t, loc, H^2}\)) ans since the r.h.s. of (3.34) is smooth due to
\[
||\partial(y_t(y + Z))||_{L^2_x} \leq ||\partial y_t||_{L^2_x} ||y + Z||_{L^\infty_{x}} + ||y_t||_{L^\infty_{x}} ||\partial(y + Z)||_{L^2_x} \leq KN^4, \tag{3.35}
\]

since \(u_t\) remains bounded in \(L^\infty_{t, x} H^{-1}_{x}\).

Multiply now (3.34) by \(w\) and integrate over \(T_s \times [0, t]\) to obtain
\[
||w(t)||_{L^2_x}^2 e^{\gamma t} \leq ||w(0)||_{L^2_x}^2 + \int_{T_s \times [0, t]} e^{\gamma s}(y + Z)\partial(w^2). \tag{3.36}
\]
Proceeding as in (3.15)-(3.17) we then get

\[
\left| \int_{[0,t]} e^{\gamma s} (y + Z) \partial (w^2) \right| \leq \frac{K}{N^{1/2}} e^{\gamma t} \sup_k (||y + Z||_{X^{\alpha, \frac{1}{2}}_{t_k}} ||w||^2_{X^{\alpha, \frac{1}{2}}_{t_k}}). \tag{3.37}
\]

We now need a local estimate for the Bourgain norm of \( w \). Actually we have

\[
\sup_k ||w||_{X^{\alpha, \frac{1}{2}}_{t_k}} \leq c (||w||_{L^\infty_t L^2_x} + KN^4). \tag{3.38}
\]

We first observe that if (3.36)-(3.38) are valid, then the \( L^2_x \) bound for \( w \) follows promptly. Therefore, to complete the proof of the proposition we just have to check that (3.38) holds true.

We consider the Duhamel’s form of (3.34) that reads (setting \( \gamma = 0 \) since it does not play any role when dealing with estimates that are local in time)

\[
w(t) = W(t-t_0) w(t_0) - \int_{t_0}^{t} W(t-s) \partial (w(y + Z)) - \int_{t_0}^{t} W(t-s) \partial (y_t(y + Z)). \tag{3.39}
\]

On the one hand, due to Propositions 2.4 and 2.5, we have

\[
|| \int_{t_0}^{t} W(t-s) \partial (w(y + Z)) ||_{X^{\alpha, \frac{1}{2}}} \leq c (|t-t_0|^{1/8} ||y||_{X^{\alpha, \frac{1}{2}}} + N^{-1/2} ||Z||_{X^{\alpha, \frac{1}{2}}} ||w||_{X^{\alpha, \frac{1}{2}}}),
\]

(3.40)

that is bounded by \( \frac{1}{2} ||w||_{X^{\alpha, \frac{1}{2}}} \) if \( |t-t_0| \leq T_1 \) and \( N > N_0 \) as above.

On the other hand, using (3.35) we have

\[
|| \int_{t_0}^{t} W(t-s) \partial (y_t(y + Z)) ||_{X^{\alpha, \frac{1}{2}}} \leq ||\partial (y_t(y + Z))||_{L_t^2 L_x^2} \leq cK N^3. \tag{3.41}
\]

Hence (3.38) follows and the proof of the proposition is completed. \( \square \)

4. The main result

4.1. Comparison at \( t = +\infty \) between \( z \) and \( Z \)

This subsection is devoted to compare \( z \) and \( Z \) when times goes to \( +\infty \). We now state

**Proposition 4.1.** There exists \( K, N_0 \) which depend on \( \gamma, f \) such that for any fixed \( N \geq N_0 \)

\[
||z(t) - Z(t)||_{L^2_x} \leq Ke^{-\gamma t}. \tag{4.1}
\]

Proof of Proposition 4.1. Assume \( t_1 = 0 \) for the sake of simplicity. Set \( w = z - Z \). Then \( w \) is solution for

\[
w_t + \gamma w + \partial^3 w + Q\partial(vw) = 0, \tag{4.2}
\]

\[
w(0) = z_0 \text{ in } L^2_x, \tag{4.3}
\]
Weakerly damped KdV equations

First step: local in time estimate. Since we are dealing with estimates that are
local in time, we may assume without loss of generality that \( \gamma = 0 \). Let \( W(t) \) denote
the free Airy group. The Duhamel’s form of (4.2) reads

\[
w(t) = W(t)w_0 - Q \int_0^t W(t - s) \partial(vw) ds.
\]

(4.4)

Applying Proposition 2.2 we then obtain

\[
\|w\|_{X^0_{t, \frac{1}{2}}} \leq \|w_0\|_{L^2} + cT^{1/8} \|w\|_{X^0_{t, \frac{1}{2}}} + \|Z\|_{X^0_{t, \frac{1}{2}}} + \|Z\|_{X^0_{t, \frac{1}{2}}} + \|w\|_{X^0_{t, \frac{1}{2}}}.
\]

(4.5)

Proceeding as in the proof of Proposition 4.1, for \( T \leq T_1 \) and \( N \geq N_0 \) we have
that for any \( t_0 \geq 0 \), for \( I = [t_0, t_0 + T] \), then

\[
\|w\|_{X^0_{t, \frac{1}{2}}} \leq 2\|w(t_0)\|_{L^2}.
\]

(4.6)

Second step: global in time estimate. Multiply (4.2) by \( w \), and integrate over \( T_x \)
to obtain

\[
\frac{d}{dt}(\|w(t)\|_{L^2}^2 e^{2\gamma t}) = \int_T e^{2\gamma t} v\partial(w^2)
\]

(4.7)

Integrate now over \([0, t]\). We then have

\[
\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 e^{-2\gamma t} + \int_{T \times [0, t]} e^{2\gamma (s-t)} v\partial(w^2).
\]

(4.8)

For \( t \in [0, T] \), we have, due to Proposition 2.2

\[
\int_{T \times [0, T]} e^{2\gamma(s-t)} v\partial(w^2) \leq c\|v\|_{X_{[0, T]}^{0, \frac{1}{2}}} \|w\|_{X_{[0, T]}^0} \|w\|_{X_{[0, T]}^0}.
\]

(4.9)

We now apply the following enhanced Poincaré inequality

\[
\|w\|_{X_{[0, T]}^0} \leq \frac{1}{N^{1/2}} \|w\|_{X_{[0, T]}^{0, \frac{1}{2}}},
\]

(4.10)

and the fact that \( v \) is bounded in \( X_{loc}^{0, \frac{1}{2}} \) (see (2.32) and (3.5)) to obtain

\[
\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 (e^{-2\gamma t} + \frac{K}{N^{1/2}}).
\]

(4.11)

We now choose \( t = T \) in (4.11), and then consider \( N \) large enough such that

\[
\delta = e^{-2\gamma T} + \frac{K}{N^{1/2}} < e^{-\gamma T}.
\]

(4.12)

It is now easy to prove inductively that for any integer \( n \), for any \( t \in [nT, (n+1)T] \)

\[
\|w(t)\|_{L^2}^2 \leq 2\|w(nT)\|_{L^2}^2 \leq 2\|w_0\|_{L^2}^2 \delta^n
\]

(4.13)
actually the first inequality in (4.13) comes from (4.6). To complete the proof of the proposition is then straightforward.

4.2 Proof of Theorem 1.1

To begin with, we consider the following decomposition of the semigroup defined by

\[ S(t)u_0 = S_1(t)u_0 + S_2(t)u_0 = (y(t) + Z(t)) + (z(t) - Z(t)); \]

(4.14)

this is well defined for \( t \geq t_1 \), where \( t_1 \) is the entrance time of the trajectory issued from \( u_0 \) in the \( L^2_x \) absorbing ball.

On the one hand, due to Proposition 4.1, the part \( S_2(t) \) converges uniformly to 0 in \( L^2_x \). On the other hand, due to Proposition 4.2 (and due to classical inverse inequalities for the \( y \) term) the part \( S_1(t) \) is a compact mapping for \( t \) large enough. Therefore the semigroup \( S(t) \) is asymptotically compact (see [8]), Theorem 1.1.1 in [14] applies, and the existence of a global attractor \( \mathcal{A} \) for the semigroup is established. Moreover \( \mathcal{A} \) is a bounded subset of \( \dot{H}^3_x \).

The remaining part of the proof is then devoted to establish the compactness of \( \mathcal{A} \). For this purpose, we use a suitable version of a classical argument due to J. Ball.

Let \( u(t) \) be a complete trajectory in \( \mathcal{A} \). Then \( u(t) \) belongs to a bounded set of \( H^3_x \). Hence \( v = u_t \) is the unique solution in \( C(\mathbb{R}; L^2_x) \cap X^0_{\text{loc}} \) to

\[ v_t + \gamma v + \partial^3 v + \partial(uv) = 0, \]

supplemented with \( v(0) = u_t(0) \), and that satisfies the following energy equality

\[ e^{2\gamma t}||v(t)||^2_{L^2} = ||v(0)||^2_{L^2} - \int_{[0,t] \times T_x} e^{2\gamma s}(\partial u)^2 \]

(To prove this equality, first regularize \( u_0, f \), and then establish (4.16) for this smooth approximation. Then pass to the limit since this smooth approximation converges towards \( v \) in \( X^0_{\text{loc}} \cap L^\infty_t L^2_x \)).

Consider now a sequence \( \phi_n \) that belongs to \( \mathcal{A} \), and that is weakly convergent in \( H^3_x \) towards \( \phi \). Going back to (1.1), we see that this sequence is strongly convergent is equivalent to prove that \( \frac{d}{dt} S(t)\phi_n|_{t=0} \) strongly converges in \( L^2_x \) towards \( \frac{d}{dt} S(t)\phi|_{t=0} \).

Consider now \( \theta > 0 \). Set \( u_n(t) = S(t-\theta)\phi_n \), and \( v_n = \frac{d}{dt} u_n \). Define \( u(t) = S(t-\theta)\phi \) and \( v = u_t \).

We now plan to pass to the limit in the integral term in (4.16). Since \( \phi_n \) strongly converges (up to a subsequence extraction) towards \( \phi \) in \( H^{3-\delta} \), for \( \delta > 0 \), then \( u_n \) converges strongly (locally in time) towards \( u \) in \( X^\beta_{\text{loc}} \), at least for \( \beta \leq 3/2 \), i.e. when Bourgain solutions are defined. Hence in order to pass to the limit in the integral term in (4.16), we just have to prove that

\[ \int_{[0,t] \times T_x} e^{2\gamma s}(\partial u)(v_n^2 - v^2) \to 0, \]

when \( n \to 0 \). For this purpose, we use Proposition 2.2 that leads to the following inequality: for \( \delta \in (0, \frac{1}{2}) \), there exists \( C(\delta, t, \gamma) \) which depends on \( \delta, t, \gamma \) such that
\[ | \int_{[0,t] \times T_x} e^{2\gamma s} u \partial (v_n^2 - v^2) | \leq C(\delta, t, \gamma) \| u \|_{X_{\text{loc}}^{\delta, \frac{1}{2}}} \| v - v^n \|_{X_{\text{loc}}^{-\delta, \frac{1}{2}}} \| v + v^n \|_{X_{\text{loc}}^{-\delta, \frac{1}{2}}}. \tag{4.18} \]

Since \( v^n(0) \) strongly converges towards \( v(0) \) in \( H^{-\delta}_x \), then \( v^n \) strongly converges (locally in time) towards \( v \) in \( X^{-\delta, \frac{1}{2}}_{\text{loc}} \); this last assertion can be proved on the Duhamel’s form of (4.15) proceeding as above. This allows us to pass to the limit in (4.16) specified with \( u_0 = \phi_n \) and \( t = \theta \). We thus obtain

\[ \limsup \| v_n(0) \|_{L_x^2}^2 \leq \limsup \| v_n(-\theta) \|_{L_x^2}^2 e^{-2\gamma \theta} - \int_{[0,\theta] \times T_x} e^{2\gamma(s-\theta)} (\partial u) v^2 \] \tag{4.17} \]

On the other hand, applying once again (4.16) with \( u_0 = \phi \), we infer from (4.17)

\[ \limsup \| v_n(0) \|_{L_x^2}^2 \leq \limsup \| v_n(-\theta) \|_{L_x^2}^2 e^{-2\gamma \theta} + \| v(-\theta) \|_{L_x^2}^2 e^{-2\gamma \theta} + \| v(0) \|_{L_x^2}^2. \tag{4.18} \]

Since \( v_n \) remains bounded in \( L^\infty(\mathbb{R}; L_x^2) \), then when \( \theta \to +\infty \), (4.18) becomes

\[ \limsup \| v_n(0) \|_{L_x^2}^2 \leq \| v(0) \|_{L_x^2}^2. \tag{4.19} \]

Therefore \( v_n(0) \) converges strongly in \( L^2_x \), and the proof of the theorem is completed. \( \square \)

### 4.3 Proof of Corollary 1.2

If we prove that for \( \beta \in \{1, 2, 3\} \), the KdV equation provides a dissipative semigroup on \( H^\beta_x \), then it is standard to prove that it possesses a global attractor \( A_\beta \) that satisfies

\[ A_3 \subset A_2 \subset A_1 \subset A. \tag{4.20} \]

Due to the regularity result of Theorem 1.1, then all these inclusions become equalities. We then proceed to prove the existence of absorbing balls for each \( \beta \in \{1, 2, 3\} \). The proof relies on a priori estimates for regularized solutions, and on a limiting argument. We just indicate below the a priori estimates, since the limiting arguments are standard.

First case: \( \beta = 1 \).

For a solution \( u \) of (1.1) issued from \( u_0 \) in \( H^1 \), multiply (1.1) by \( -\partial^2 u - u^2/2 + \partial^{-1} f \), and integrate over \( T_x \) to obtain

\[ \frac{d}{dt} J_1(u) + \gamma J_1(u) + \gamma \| \partial u \|_{L_x^2}^2 = \frac{2\gamma}{3} \int_T u^3, \tag{4.21} \]

where

\[ J_1(u) = \| \partial u \|_{L_x^2}^2 - \frac{1}{3} \int_T u^3 + 2 \int_T (\partial^{-1} f) u. \tag{4.22} \]

Without loss of generality, we may assume that \( u \) belongs to the \( L^2 \) absorbing set for \( t \geq 0 \). Then, proceeding as in (3.25)-(3.26), we have
\[ |J_1(u) - ||\partial u||_{L^2_t}^2| \leq K + \frac{1}{4}||\partial u||_{L^2_t}^2. \] (4.23)

Since the r.h.s. of (4.21) is bounded by a quantity similar to the r.h.s. of (4.23), then the existence of an \( H^1 \) absorbing ball follows promptly.

Second case: \( \beta = 2 \).

Consider \( u \) that is solution to (1.1) supplemented with \( u_0 \) in \( H^2 \). Due to the first case, this solution is global in \( H^1 \) and we may assume without loss of generality that \( u \) belongs to the \( H^1 \) absorbing set for \( t \geq 0 \). Multiply (1.1) by \( \partial^2 u + \partial(u\partial u) - \partial f \) and integrate over \( T \) to obtain

\[ \frac{d}{dt} J_2(u) + 2\gamma J_2(u) = - \int_T u_t(\partial u)^2 + \gamma \int_T f \partial u, \] (4.24)

where

\[ J_2(u) = ||\partial^2 u||_{L^2_t}^2 - \int_T u(\partial u)^2 + 2 \int_T f \partial u. \] (4.25)

We easily observe that

\[ |J_2(u) - ||\partial^2 u||_{L^2_t}^2| \leq K + \frac{1}{4}||\partial^2 u||_{L^2_t}^2, \] (4.26)

Therefore, we just have to bound the r.h.s of (4.24). For this purpose, we proceed as in Lemma 3.3, and we split \( u_t = a + b = (u_t + \frac{1}{2}\partial(u^2)) + (-\frac{1}{2}\partial(u^2)) \); we then observe that \( a \) remains (locally in time) bounded in \( X_{loc}^{-\frac{1}{2}} \) and \( b \) in \( L^\infty_t L^2_x \). The bound of \( b \) is valid since \( u \) remains in the \( H^1 \) absorbing set, and the bound on \( a \) comes from (4.29) below.

On the one hand, due to Agmon inequality as in (3.26)

\[ \left| \int_T b(\partial u)^2 \right| \leq ||b||_{L^2_t} ||u||_{H^1_t}^{3/2} ||u||_{H^2_t}^{1/2} \leq K ||u||_{H^1_t}^{1/2}. \] (4.27)

On the other hand, we split \([0, t] = \bigcup_{k=1}^{m} I_k\), where the \( I_k \) are disjoint intervals of constant size \( T \) and thus obtain, proceeding as in (3.15)-(3.16)

\[ \left| \int_{[0, t] \times T} e^{2\gamma s} a(\partial u)^2 \right| \leq ce^{2\gamma t} \sup_k ||\partial^{-1} a||_{X_{I_k}^{-\frac{1}{2}}} ||\partial u||_{X_{I_k}^{-\frac{1}{2}}} ||\partial u||_{X_{I_k}^{-\frac{1}{2}}} + cT_{1/4}(\ln T_1)^{1/2} ||u||_{X_{I_k}^{0}} ||u||_{X_{I_k}^{3/2}}. \] (4.28)

Assume that the size of each \( I_k \) compares with \( T_1 \), \( T_1 \) being as in Corollary 2.8. Then applying Proposition 2.2 to the Duhamel's form of (1.1) leads to (using also (2.29))

\[ ||u||_{X_{I_k}^{3/2}} \leq c ||u_0||_{H^{3/2}} + c ||f||_{L^2_t} + cT_{1/4}(\ln T_1)^{1/2} ||u||_{X_{I_k}^{0}} ||u||_{X_{I_k}^{3/2}}. \] (4.29)

Due to (2.33), we then have

\[ ||u||_{X_{I_k}^{3/2}} \leq c(||u_0||_{H^{3/2}} + ||f||_{L^2_t}) \leq K(1 + ||u||^{1/2}_{L_t^\infty H^2}). \] (4.30)
Using then that \( u \) is bounded in \( X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}} \), for \( \delta \leq 1 \), due to the analog of (4.29)-(4.30) in \( X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}} \) and since \( u \) belongs to the \( H^1 \) absorbing ball, we infer from (4.28)-(4.30) that

\[
\left| \int_{[0,t] \times I} e^{2\gamma s} a(\partial u)^2 \right| \leq K (1 + \|u\|^{1/2}_{L^\infty L^2})^3. \tag{4.31}
\]

To complete the proof of the existence of the \( H^2 \) absorbing ball is then straightforward.

Third case: \( \beta = 3 \).

In this case we observe that \( v = e^{\gamma t} u_t \) is solution to

\[
v_t + v_{xxx} + (uv)_x = 0. \tag{4.32}
\]

We then use a pacing argument similar of those used in [13]. For \( |t| \leq T_1 \), \( T_1 \) being as in Corollary 2.8, Proposition 2.2 leads to

\[
\|v(t)\|_{L^2_x}^2 \leq \|v_0\|_{L^2_x}^2 + T_1^{1/4}\|u\|_{X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}}} \|v\|_{X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}}} \|v\|_{X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}}} \|v\|_{X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}}}. \tag{4.33}
\]

When dealing on the Duhamel’s form of (4.32), we may prove (due to Proposition 2.4 and 2.5 as above) that for \( \delta \in (0, \frac{1}{2}) \)

\[
\|v\|_{X_{0, \text{loc}}^{\frac{3}{2}, \frac{1}{2}}} \leq c \|v_0\|_{H^{\frac{3}{2}, \frac{1}{2}}}. \tag{4.34}
\]

We then infer from (4.32)-(4.34) that for \( |t| \leq T_1 \),

\[
\|v(t)\|_{L^2_x}^2 \leq \|v_0\|_{L^2_x}^2 + K \|v_0\|_{L^2_x}^2, \tag{4.35}
\]

where \( \frac{3}{2} \) denotes any arbitrarily number close to \( \frac{3}{2} \).

We now pass to a global in time estimate using a pacing argument and the dissipation constant \( \gamma \). We infer from (4.35) that for any \( t \geq 0 \),

\[
\|u_t(t + T_1)\|_{L^\infty l^2}^2 e^{2\gamma T_1} \leq \|u_t(t)\|_{L^\infty l^2}^2 + K \|u_t(t)\|_{L^\infty l^2}^2, \tag{4.36}
\]

since \( u_t \) remains bounded in \( L^\infty (\mathbb{R}_+; H^{-1}_x) \) (assuming without loss of generality that \( u \) belongs to the \( H^2 \) absorbing ball). The existence of an \( L^2 \) absorbing set for \( u_t \) follows promptly. Then we have the existence of the \( H^3 \) absorbing set for \( u \), and the proof of Corollary 1.2 is completed \( \Box \).

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