Alon-Tarsi number of signed planar graphs

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Abstract

Let \((G, \sigma)\) be any signed planar graph. We show that the Alon-
Tarsi number of \((G, \sigma)\) is at most 5, generalizing a recent result of Zhu
for unsigned case. In addition, if \((G, \sigma)\) is 2-colorable then \((G, \sigma)\) has
the Alon-Tarsi number at most 4. We also construct a signed planar
graph which is 2-colorable but not 3-choosable.

Key words. signed graph; planar graph; list coloring; Alon-Tarsi number

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1 Introduction

Let \(G\) be a simple graph with vertex set \(V(G)\) and edge set \(E(G)\). A
signed graph with underling graph \(G\) is a pair \((G, \sigma)\), where \(\sigma\) is a mapping
from \(E(G)\) to \{+1, −1\}. An edge \(e\) is positive (resp. negative) if \(\sigma(e) = +1\)
(resp. \(\sigma(e) = −1\)). In particular, we denote by \((G, +)\) (resp. \((G, −)\)) the
signed graph \((G, \sigma)\) if every edge is positive (resp. negative). We often
identify \((G, +)\) with the (unsigned) underling graph \(G\).

Recently, based on the work of Zaslavsky \cite{12}, Máčajová et al. \cite{14}
generalized the concept of chromatic number of an unsigned graph to a signed

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We say that \((G, \sigma)\) with color set \(C \subseteq \mathbb{Z}\), a proper coloring \[1\] with color set \(C\) is a mapping \(\phi: V(G) \mapsto C\) such that
\[
\phi(u) \neq \sigma(uv)\phi(v)
\]
for each edge \(uv \in E(G)\). For \(k \geq 1\), set \(M_k = \{\pm 1, \pm 2, \ldots, \pm k/2\}\) if \(k\) is even and \(M_k = \{0, \pm 1, \pm 2, \ldots, \pm (k-1)/2\}\) if \(k\) is odd. A (proper) \(k\)-coloring of a signed graph \((G, \sigma)\) is a proper coloring with color set \(M_k\). A signed graph \((G, \sigma)\) is \(k\)-colorable if it admits a \(k\)-coloring. The chromatic number of \((G, \sigma)\), denoted \(\chi(G, \sigma)\), is the minimum \(k\) for which \((G, \sigma)\) is \(k\)-colorable.

Jin et al. \[6\] and Schweser et al. \[9\] further considered the list coloring of signed graphs. For a positive integer \(k\), a \(k\)-list assignment of \((G, \sigma)\) is a mapping \(L\) which assigns to each vertex \(v\) a set \(L(v) \subseteq \mathbb{Z}\) of \(k\) permissible colors. For a \(k\)-list assignment \(L\) of \((G, \sigma)\), an \(L\)-coloring is a proper coloring \(\phi: V(G) \mapsto \bigcup_{v \in V(G)} L(v)\) such that \(\phi(v) \in L(v)\) for every vertex \(v \in V(G)\). We say that \((G, \sigma)\) is \(L\)-colorable if \(G\) has an \(L\)-coloring. A signed graph \((G, \sigma)\) is called \(k\)-choosable if \(G\) is \(L\)-colorable for any \(k\)-list assignment \(L\). The list chromatic number (or choice number) \(\chi_l(G, \sigma)\) is the minimum \(k\) for which \(G\) is \(k\)-choosable. Clearly, \(\chi_l(G, \sigma) \geq \chi(G, \sigma)\). We note that when we restrict the signed graphs \((G, \sigma)\) to \((G, +)\), both the chromatic number and list chromatic number agree with the ordinary chromatic number and list chromatic number of its underlying graph \(G\). This explains why we can identify \((G, +)\) with \(G\).

Let ‘\(<\)’ be an arbitrary fixed ordering of the vertices of \((G, \sigma)\). In view of \[1\], we define the graph polynomial of \((G, \sigma)\) as
\[
P_{G, \sigma}(\mathbf{x}) = \prod_{u \sim v, u < v} (x_u - \sigma(uv)x_v),
\]
where \(u \sim v\) means that \(u\) and \(v\) are adjacent, and \(\mathbf{x} = (x_v)_{v \in V(G)}\) is a vector of \(|V(G)|\) variables indexed by the vertices of \(G\). It is easy to see that a mapping \(\phi: V(G) \mapsto \mathbb{Z}\) is a proper coloring of \((G, \sigma)\) if and only if \(P_{G, \sigma}((\phi(v))_{v \in V(G)}) \neq 0\).

Lemma 1.1. \[1\] (Combinatorial Nullstellensatz) Let \(\mathbb{F}\) be an arbitrary field and let \(f = f(x_1, x_2, \ldots, x_n)\) be a polynomial in \(\mathbb{F}[x_1, x_2, \ldots, x_n]\). Suppose that the degree \(\text{deg}(f)\) of \(f\) is \(\sum_{i=1}^{n} t_i\) where each \(t_i\) is a nonnegative integer, and suppose that the coefficient of \(\prod_{i=1}^{n} x_i^{t_i}\) of \(f\) is nonzero. Then if \(S_1, S_2, \ldots, S_n\) are subsets of \(\mathbb{F}\) with \(|S_i| \geq t_i + 1\), then there are \(s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n\) so that \(f(x_1, x_2, \ldots, x_n) \neq 0\).

Note that \(P_{G, \sigma}(\mathbf{x})\) is a homogeneous polynomial. It follows from Lemma \[1\] that if there exists a monomial \(c \prod_{v \in V(G)} x_v^{l_v}\) in the expansion of \(P_{G, \sigma}(\mathbf{x})\)
such that \( c \neq 0 \) and \( t_v < k \) for all \( v \in V(G) \), then \( (G, \sigma) \) is \( k \)-choosable. Thus, the notion of Alon-Tarsi number of unsigned graphs defined by Jensen and Toft [5] can be naturally extended to signed graphs.

**Definition 1.2.** The Alon-Tarsi number of \((G, \sigma)\), denoted \(AT(G, \sigma)\), is the minimum \( k \) for which there exists a monomial \( c \prod_{v \in V(G)} x_v^{t_v} \) in the expansion of \( P_{G,\sigma}(\mathbf{x}) \) such that \( c \neq 0 \) and \( t_v < k \) for all \( v \in V(G) \).

Parallel to the unsigned case, we have

\[
AT(G, \sigma) \geq \chi_l(G, \sigma) \geq \chi(G, \sigma).
\]

For a subgraph \( H \) of \( G \), we use \((H, \sigma)\) to denote the signed subgraph of \((G, \sigma)\) restricted on \( H \), i.e., \((H, \sigma) = (H, \sigma|_{E(H)})\). Note that \( P_{H,\sigma}(\mathbf{x}) \) is a factor of \( P_{G,\sigma}(\mathbf{x}) \). From Definition 1.2, it is clear that \( AT(H, \sigma) \leq AT(G, \sigma) \).

For a vertex \( v \) in a signed graph \((G, \sigma)\), a switching at \( v \) means changing the sign of each edge incident to \( v \). For \( X \subseteq V(G) \), a switching at \( X \) means switching at every vertex in \( X \) one by one. Equivalently, a switching at \( X \) means changing the sign of every edge with exactly one end in \( X \). We denote the switched graph by \((G, \sigma^X)\). In particular, when \( X = \{v\} \) we use \((G, \sigma^v)\) to denote \((G, \sigma^{\{v\}})\). Two signed graphs \((G, \sigma)\) and \((G, \sigma')\) are switching equivalent if \( \sigma' = \sigma^X \) for some \( X \subseteq V(G) \).

It is easy to show that two switching equivalent signed graphs have the same chromatic number [7] as well as the same list chromatic number [6, 9]. For the Alon-Tarsi numbers, we have the following similar result.

**Proposition 1.3.** If two signed graphs \((G, \sigma)\) and \((G, \sigma')\) are switching equivalent then \( AT(G, \sigma) = AT(G, \sigma') \).

**Proof.** It clearly suffices to consider the case that \( \sigma' = \sigma^v \), where \( v \in V(G) \). For any edge incident with \( v \), say \( uv \), we have \( \sigma^v(\sigma uv) = -\sigma(\sigma uv) \). We use \( T(x_u, x_v) \) and \( T^v(x_u, x_v) \) to denote the factors corresponding to this edge in \( P_{G,\sigma}(\mathbf{x}) \) and \( P_{G,\sigma^v}(\mathbf{x}) \), respectively. If \( u < v \) then \( T(x_u, x_v) = x_u - \sigma(\sigma uv)x_v \), \( T^v(x_u, x_v) = x_u - \sigma^v(\sigma uv)x_v \) and hence \( T(x_u, x_v) = T^v(x_u, -x_v) \). If \( v < u \) then \( T(x_u, x_v) = x_v - \sigma(\sigma uv)x_u \) and \( T^v(x_u, x_v) = x_v - \sigma^v(\sigma uv)x_u \) and hence \( T(x_u, x_v) = -T^v(x_u, -x_v) \). In either case we have \( T(x_u, x_v) = \pm T^v(x_u, -x_v) \). Letting \( \mathbf{x}^v \) be obtained from \( \mathbf{x} \) by changing \( x_v \) to \(-x_v \), we have \( P_{G,\sigma}(\mathbf{x}) = \pm P_{G,\sigma^v}(\mathbf{x}^v) \). Therefore, for each monomial \( \prod_{v \in V(G)} x_v^{t_v} \), the coefficients of this monomial in \( P_{G,\sigma}(\mathbf{x}) \) and \( P_{G,\sigma^v}(\mathbf{x}^v) \) and hence in \( P_{G,\sigma^v}(\mathbf{x}) \) have the same absolute value. This implies that \( AT(G, \sigma) = AT(G, \sigma^v) \).
Recently, a few classical results on colorability \cite{4} and choosability \cite{6} of planar graphs were generalized to signed planar graphs. In particular, Jin et al. \cite{6} showed that every signed planar graph is 5-choosable, generalizing the well-known result of Thomassen \cite{10} which states that every (unsigned) planar graph is 5-choosable. Another generalization of Thomassen’s result was given by Zhu \cite{11}, who showed that $AT(G) \leq 5$ for any planar graph $G$, which solved an open problem proposed by Hefetz \cite{3}. Considering the above results of Jin et. al \cite{6} and Zhu \cite{11}, it is natural to ask whether the Alon-Tarsi number of each signed planar graph is at most 5. The main aim of this paper is to give an affirmative answer to this question.

**Theorem 1.4.** For any signed graph $(G, \sigma)$, if $G$ is a planar graph then $AT(G, \sigma) \leq 5$.

In \cite{2}, Alon and Tarsi showed that every bipartite planar graph is 3-choosable. The result is sharp as $K_{2,4}$ is a bipartite planar graph and $\chi_l(K_{2,4}) = 3$. The following result is a natural extension of this result for signed planar graphs.

**Theorem 1.5.** For any signed graph $(G, \sigma)$, if $G$ is planar and 2-colorable then $AT(G, \sigma) \leq 4$. Moreover, there is a signed planar graph which is 2-colorable but not 3-choosable.

## 2 Orientation and Alon-Tarsi number for signed graphs

For an unsigned graph $G$, Alon and Tarsi \cite{2} found a useful combinatorial interpretation of the coefficient for each monomial in the graph polynomial $P_G(x)$ in terms of orientations and Eulerian subgraphs. By defining hypergraph polynomial and hypergraph orientation, Ramamurthi and West \cite{8} generalized the result of Alon and Tarsi to $k$-uniform hypergraph for prime $k$. In this section we consider the signed graphs. Instead of using orientations of signed graphs, we use orientations of the underlying graphs and find that the result of Alon and Tarsi has a very natural extension for signed graphs.

Let $(G, \sigma)$ be a signed graph and ‘$<$’ be an arbitrary fixed ordering of $V(G)$. For an orientation $D$ of the underling graph $G$, we denote by $(v, u)$ the oriented edge of $D$ with direction from $v$ to $u$. We call an oriented edge $(v, u)$ $\sigma$-decreasing if $v > u$ and $\sigma(uv) = +1$, that is, $(v, u)$ is positive and oriented from the larger vertex to the smaller vertex. We note that a negative edge
will never be $\sigma$-decreasing, no matter how it is oriented. An orientation $D$ of $G$ is called $\sigma$-even if it has an even number of $\sigma$-decreasing edges and called $\sigma$-odd otherwise. For a nonnegative sequence $d = (d_v)_{v \in V(G)}$, let $\sigma EO(d)$ and $\sigma OO(d)$ denote the sets of all $\sigma$-even and $\sigma$-odd orientations of $G$ having outdegree sequence $d$, respectively.

**Lemma 2.1.** $P_{G,\sigma}(x) = \sum (|\sigma EO(d)| - |\sigma OO(d)|) \prod_{v \in V(G)} x^{d_v}$, where $d = (d_v)_{v \in V(G)}$ and the summation is taken over all $d$ such that $d_v \geq 0$ and $\sum_{v \in V(G)} d_v = |E(G)|$.

**Proof.** Let $D$ be an arbitrary orientation of $G$. For each oriented edge $e = (v, u)$, define

$$w(e) = \begin{cases} -x_v, & \text{if } e \text{ is } \sigma \text{-decreasing} \\ x_v, & \text{otherwise.} \end{cases}$$

(2)

and $w(D) = \prod_{e \in E(D)} w(e)$. Let $d_v$ be the outdegree of $v$ in $D$ for each $v \in V(G)$ and let $t$ be the number of $\sigma$-decreasing edges in $D$. It is easy to see that

$$w(D) = (-1)^t \prod_{v \in V(G)} x^{d_v}.$$  

(3)

Recall that

$$P_{G,\sigma}(x) = \prod_{u \sim v, u < v} (x_u - \sigma(uv)x_v).$$

By selecting $x_u$ or $-\sigma(uv)x_v$ from each factor $(x_u - \sigma(uv)x_v)$, we expand $P_{G,\sigma}(x)$ and obtain $2^{|E(G)|}$ monomials, each of which has coefficient $\pm 1$. For each monomial, we orient the edge $uv$ of $G$ with direction from $u$ to $v$ if, in the factor $(x_u - \sigma(uv)x_v)$, $x_u$ is selected; or from $v$ to $u$ if $-\sigma(uv)x_v$ is selected. This is clearly a bijection between the $2^{|E(G)|}$ monomials and the $2^{|E(G)|}$ orientations of $G$. Therefore,

$$P_{G,\sigma}(x) = \sum w(D),$$

(4)

where $D$ ranges over all orientations of $G$.

Let $d = (d_v)_{v \in V(G)}$ be the sequence of outdegrees of some orientation $D$. Clearly, $d_v \geq 0$ and $\sum_{v \in V(G)} d_v = |E(G)|$. Note that there are exactly $|\sigma EO(d)|$ (resp. $|\sigma OO(d)|$) $\sigma$-even (resp. $\sigma$-odd) orientations of $G$. It follows from (3) and (4) that the coefficient of $\prod_{v \in V(G)} x^{d_v}$ in the expansion of $P_{G,\sigma}(x)$ is $|\sigma EO(d)| - |\sigma OO(d)|$. This proves the lemma. $\square$

For an orientation $D$ of $G$, a subdigraph $H$ of $D$ is called Eulerian if $V(H) = V(D)$ and the indegree of every vertex equals its outdegree. We note
that an Eulerian subdigraph $H$ defined here is not necessarily connected. In particular, a vertex is called isolated in $H$ if it has indegree 0 (and therefore, has outdegree 0) in $H$. Further, $H$ is called $\sigma$-even (resp. $\sigma$-odd) if $H$ has an even (resp. odd) number of positive edges. Let $\sigma EE$ denote the set of all $\sigma$-even (resp. $\sigma$-odd) Eulerian subdigraphs of $D$.

**Lemma 2.2.** Let $(G, \sigma)$ be a signed graph and $D$ be an orientation of $G$ with outdegree sequence $d = (d_v)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of $P_{G, \sigma}(x)$ is equal to $\pm(|\sigma EE(D)| - |\sigma OE(D)|)$.

**Proof.** For any orientation $D' \in \sigma EO(d) \cup \sigma OO(d)$, let $D \oplus D'$ denote the set of all oriented edges of $D$ whose orientation in $D'$ is in the opposite direction. Since $D$ and $D'$ have the same outdegree sequence, $D \oplus D'$ is Eulerian. Moreover, $D \oplus D'$ contains an even number of positive edges if and only if $D$ and $D'$ are both $\sigma$-even or both $\sigma$-odd.

Now, the map $\tau : D' \mapsto D \oplus D'$ is clearly a bijection between $\sigma EO(d) \cup \sigma OO(d)$ and $\sigma EE(D) \cup \sigma OE(D)$. If $D$ is $\sigma$-even, then $\tau$ maps $\sigma EO(d)$ to $\sigma EE(D)$ and maps $\sigma OO(d)$ to $\sigma OE(D)$. In this case $|\sigma EO(d)| = |\sigma EE(D)|$ and $|\sigma OO(d)| = |\sigma OE(D)|$. Thus, $|\sigma EO(d)| - |\sigma OO(d)| = |\sigma EE(D)| - |\sigma OE(D)|$. It follows from Lemma 2.2 that the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of $P_{G, \sigma}(x)$ is equal to $|\sigma EE(D)| - |\sigma OE(D)|$. Similarly, if $D$ is $\sigma$-odd, then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of $P_{G, \sigma}(x)$ is equal to $|\sigma OE(D)| - |\sigma EE(D)|$. This proves the lemma.

By Lemma 2.2 and Definition 1.2, we have the following characterization of the Alon-Tarsi number $AT(G, \sigma)$.

**Corollary 2.3.** For any signed graph $(G, \sigma)$, $AT(G, \sigma)$ equals the minimum $k$ for which there exists an orientation $D$ of $G$ such that $|\sigma EE(D)| \neq |\sigma OE(D)|$ and every vertex has outdegree less than $k$.

## 3 Proof of Theorem 1.4

We call a plane graph (a planar graph embedded on the plane) a near triangulation if the boundary of the outer face is a cycle, called the outer facial cycle, and the boundaries of all inner faces are triangles.

**Definition 3.1.** Let $(G, \sigma)$ be a signed graph where $G$ is a near triangulation with outer facial cycle $v_1v_2 \cdots v_k$ and let $e = v_1v_2$. An orientation $D$ of $G - e$ is $\sigma$-nice for $G - e$ if the following hold:
• $|\sigma EE(D)| \neq |\sigma OE(D)|$.

• $v_1$ and $v_2$ have outdegree 0, $v_i$ has outdegree at most 2 for $i \in \{3, 4, \ldots, k\}$, and every interior vertex has outdegree at most 4.

We use the method presented in [11] to prove the following theorem.

**Theorem 3.2.** Let $(G, \sigma)$ be a signed graph where $G$ is a near triangulation with outer facial cycle $C = v_1v_2 \cdots v_k$ and let $e = v_1v_2$. Then $G - e$ has a $\sigma$-nice orientation.

**Proof.** We prove the theorem by induction on $|V(G)|$. If $|V(G)| = 3$ then $G - e$ is a path $v_2v_3v_1$. Let $D$ be the orientation of $G - e$ such that $E(D) = \{(v_3,v_2),(v_3,v_1)\}$. Clearly, $D$ is $\sigma$-nice. Now assume that $|V(G)| > 3$ and the assertion holds for graphs of order less than $|V(G)|$. We shall distinguish two cases, according to whether the outer facial cycle $C$ contains a chord incident with $v_k$.

First we consider the case that $C$ has a chord $e' = v_kv_j$ where $2 \leq j \leq k - 2$ (see Figure 1(a)). In this case $C_1 = v_1v_2 \cdots v_jv_k$ and $C_2 = v_kv_jv_{j+1} \cdots v_{k-1}$ are two cycles of $G$. For $i \in \{1,2\}$, let $G_i$ be the subgraph of $G$ formed by $C_i$ and its interior part. By the induction hypothesis, $G_i - e$ has a $\sigma$-nice orientation $D_i$, and $G_2 - e'$ has a $\sigma$-nice orientation $D_2$. We notice that $D_1$ and $D_2$ are edge disjoint. Let $D = D_1 \cup D_2$. It is clear that $D$ is an orientation of $G - e$. We will show that $D$ is $\sigma$-nice for $G - e$. It can be checked that $D$ satisfies the outdegree condition in Definition 3.1. It remains to check that $|\sigma EE(D)| \neq |\sigma OE(D)|$.

Note that both $v_k$ and $v_j$ have outdegree 0 in $D_2$. This implies that $v_k$ and $v_j$ are both isolated in any Eulerian subdigraph of $D$. Therefore, any Eulerian subdigraph $H$ of $D$ has an edge-disjoint decomposition $H = H_1 \cup H_2$, where $H_1$ and $H_2$ are Eulerian subdigraphs in $D_1$ and $D_2$, respectively. Thus, the map $\tau: H \mapsto (H_1,H_2)$ is a bijection between $\sigma EE(D) \cup \sigma OE(D)$ and $(\sigma EE(D_1) \cup \sigma OE(D_1)) \times (\sigma EE(D_2) \cup \sigma OE(D_2))$. Moreover, $H$ is $\sigma$-even if and only if both $H_1$ and $H_2$ are $\sigma$-even, or both are $\sigma$-odd. Thus, we have

\[
|\sigma EE(D)| - |\sigma OE(D)| = (|\sigma EE(D_1) \times \sigma EE(D_2)| + |\sigma OE(D_1) \times \sigma OE(D_2)|) - (|\sigma EE(D_1) \times \sigma OE(D_2)| + |\sigma OE(D_1) \times \sigma EE(D_2)|) = (|\sigma EE(D_1)| - |\sigma OE(D_1)|) \cdot (|\sigma EE(D_2)| - |\sigma OE(D_2)|) \neq 0,
\]
where the last inequality holds since $D_1$ and $D_2$ are $\sigma$-nice. This proves that 

$D$ is a $\sigma$-nice orientation of $G - e$.

Next assume that $C$ contains no chord of the form $v_kv_j$ for $j \in \{2, 3, \ldots, k-2\}$. Let $v_{k-1}, u_1, u_2, \ldots, u_s, v_1$ be the neighbors of $v_k$ and be ordered so that $v_kv_{k-1}u_1, v_ku_1u_2, \ldots, v_ku_su_1$ are inner facial cycles of $G$ (see Figure 1(b) when $k = 3$ and Figure 1(c) when $k > 3$). Let $G' = G - v_k$. It is clear that $G'$ is a near triangulation with outer facial cycle $v_1v_2 \cdots v_{k-1}u_1u_2 \cdots u_s$. Therefore, by the induction hypothesis, $G' - e$ has a $\sigma$-nice orientation $D'$.

Figure 1. Proof of Theorem 1.4.

If $k = 3$ (i.e., $C$ is a triangle), then let $D$ be the orientation of $G - e$ obtained from $D'$ by adding the vertex $v_3$ and oriented edges $(v_3, v_1)$, $(v_3, v_2)$ and $(u_i, v_3)$ for $i \in \{1, 2, \ldots, s\}$, as shown in Figure 1(b). It is easy to verify that $D$ satisfies the outdegree condition in Definition 3.1. In particular, both $v_1$ and $v_2$ have outdegree 0. Thus, $v_1$ and $v_2$ are both isolated in any Eulerian subdigraph of $D$ and therefore, by the definition of $D$, $v_3$ is also isolated in any Eulerian subdigraph of $D$. This means that each Eulerian subdigraph of $D$ is an Eulerian subdigraph of $D'$ by ignoring the isolated vertex $v_k$. Thus, $\sigma_{EE}(D) = \sigma_{EE}(D')$ and $\sigma_{OE}(D) = \sigma_{EE}(D')$. As $D'$ is $\sigma$-nice,
Case 1. Following hold:

- $v_1$ and $v_2$ have outdegree 0, $v_{k-1}$ has outdegree at most 1, each of $v_3, v_4, \ldots, v_{k-1}$ has outdegree at most 2, and each of $u_1, u_2, \ldots, u_s$ has outdegree at most 3.
- Every interior vertex has outdegree at most 4.

To show that $G - e$ has a $\sigma$-nice orientation, we consider two cases:

**Case 1.** $G' - e$ has a special orientation $D''$ with $|\sigma EE(D'')| \neq |\sigma OE(D'')|$.

Let $D$ be the orientation of $G - e$ obtained from $D''$ by adding the vertex $v_k$ and $s + 2$ oriented edges $(v_k, v_i), (v_{k-1}, v_k)$ and $(u_i, v_k)$ for $i \in \{1, 2, \ldots, s\}$, see Figure 1(c). Then $D$ satisfies the outdegree condition of a $\sigma$-nice orientation. Since $v_1$ has outdegree 0 in $D$, by a similar discussion as above, $v_k$ is isolated in any Eulerian subdigraph of $D$. Therefore, each Eulerian subdigraph of $D$ is an Eulerian subdigraph of $D''$ by ignoring the isolated vertex $v_k$, i.e., $\sigma EE(D) = \sigma EE(D'')$ and $\sigma OE(D) = \sigma OE(D'')$. This yields that $|\sigma EE(D)| \neq |\sigma EE(D)|$ by the condition of this case. Thus, $D$ is a $\sigma$-nice orientation of $G - e$, as desired.

**Case 2.** For any special orientation $D''$ (if exists), $|\sigma EE(D'')| = |\sigma OE(D'')|$.

Recall that $D'$ is a $\sigma$-nice orientation of $G' - e$. Let $D$ be the orientation of $G - e$ obtained from $D'$ by adding the vertex $v_k$ and $s + 2$ oriented edges $(v_k, v_i), (v_{k-1}, v_k)$ and $(u_i, v_k)$ for $i \in \{1, 2, \ldots, s\}$, as shown in Figure 1(d). Clearly, $D$ satisfies the outdegree condition of a $\sigma$-nice orientation. To show that $D$ is $\sigma$-nice for $G - e$, it remains to show that $|\sigma EE(D)| \neq |\sigma OE(D)|$.

Notice that $v_1$ has outdegree 0 in $D$ and therefore, is isolated in any Eulerian subdigraph of $D$. Thus, if $H$ is an Eulerian subdigraph of $D$ and $v_k$ is non-isolated in $H$ then $H$ contains the oriented edge $(v_k, v_{k-1})$ and exactly one of the $s$ oriented edges $(u_1, v_k), (u_2, v_k), \ldots, (u_s, v_k)$. For $i \in \{1, 2, \ldots, s\}$, let

$$\sigma EE_i(D) = \{H \in \sigma EE(D): (u_i, v_k) \in H\},$$

and similarly,

$$\sigma OE_i(D) = \{H \in \sigma OE(D): (u_i, v_k) \in H\}.$$
For an Eulerian subdigraph of $D'$, we regard it as an Eulerian subdigraph of $D$ by adding $v_k$ as an isolated vertex. Then we have

$$\sigma EE(D) = \sigma EE(D') \cup \bigcup_{i=1}^{8} \sigma EE_i(D), \sigma OE(D) = \sigma OE(D') \cup \bigcup_{i=1}^{8} \sigma OE_i(D).$$

Since $D'$ is $\sigma$-nice, $|\sigma EE(D')| \neq |\sigma OE(D')|$. If we can show that $|\sigma EE_i(D)| = |\sigma OE_i(D)|$ for each $i \in \{1, 2, \ldots, 8\}$, then $|\sigma EE(D)| \neq |\sigma OE(D)|$ and we are done.

Let $i$ be any integer in $\{1, 2, \ldots, 8\}$. If $\sigma EE_i(D) \cup \sigma OE_i(D) = \emptyset$ then $|\sigma EE_i(D)| = |\sigma OE_i(D)| = 0$, as desired. Thus, we may assume that $\sigma EE_i(D) \cup \sigma OE_i(D) \neq \emptyset$. Therefore, $D$ has an Eulerian subdigraph and hence a directed cycle containing $(u_i, v_k)$. Let $C_i = u_i v_k v_{k-1} w_1 w_2 \cdots w_p$ be such a directed cycle and let $D'_i$ be the orientation of $G' - e$ obtained from $D'$ by reversing the direction of edges in the path $v_{k-1} w_1 w_2 \cdots w_p u_k$. The reversing operation decreases the outdegree of $v_{k-1}$ by 1, increases the outdegree of $u_i$ by 1, and leaves the outdegrees of other vertices in $G' - e$ unchanged. Since $D'$ is $\sigma$-nice for $G' - e$, the outdegree condition of $D'$ implies that $D'_i$ is special. Hence, $|\sigma EE(D'_i)| = |\sigma OE(D'_i)|$ by the condition of this case.

Let $C_i^{-1}$ be the reverse of $C_i$, i.e., $C_i^{-1} = w_p w_{p-1} \cdots w_1 v_{k-1} v_k u_i$. For each Eulerian subdigraph $H \in \sigma EE_i(D) \cup \sigma OE_i(D)$, let $H \uplus C_i^{-1}$ be the symmetry difference of the edge sets of $H$ and $C_i^{-1}$, that is, the set obtained from the edge union $H \cup C_i^{-1}$ of $H$ and $C_i^{-1}$ by deleting the directed 2-cycles. One may verify that $H \uplus C_i^{-1}$ is an Eulerian subdigraph of $D'_i$ and the map $\tau: H \mapsto H \uplus C_i^{-1}$ is a bijection between $\sigma EE_i(D) \cup \sigma OE_i(D)$ and $\sigma EE(D'_i) \cup \sigma OE(D'_i)$.

For a set $S$ of some oriented edges in an orientation of $(G, \sigma)$, we use $N^\sigma(S)$ to denote the number of positive edges in $S$. If $S$ is a directed 2-cycle, then either $N^\sigma(S) = 2$ or $N^\sigma(S) = 0$. Thus, $N^\sigma(H \uplus C_i^{-1})$ and $N^\sigma(H \cup C_i^{-1})$ have the same parity. Of course, $N^\sigma(H \cup C_i^{-1}) = N^\sigma(H) + N^\sigma(C_i^{-1}) = N^\sigma(H) + N^\sigma(C_i)$. Therefore, if $N^\sigma(C_i)$ is even, then $\tau: H \mapsto H \uplus C_i^{-1}$ maps $\sigma EE_i(D)$ to $\sigma EE(D'_i)$ and $\sigma OE_i(D)$ to $\sigma OE(D'_i)$. Similarly, if $N^\sigma(C_i)$ is odd, then it maps $\sigma EE_i(D)$ to $\sigma OE(D'_i)$ and $\sigma OE_i(D)$ to $\sigma EE(D'_i)$. Therefore, we have $|\sigma EE_i(D)| - |\sigma OE_i(D)| = \pm (|\sigma EE(D'_i)| - |\sigma OE(D'_i)|)$. Note that $D'_i$ is special. It follows from the condition of this case that $|\sigma EE_i(D)| = |\sigma OE_i(D)|$. This completes the proof of this theorem.
cycle of \( G \) and \( e = v_1v_2 \). By Theorem 3.2, \( G - e \) has a \( \sigma \)-nice orientation \( D \). Let \( D' \) be obtained from \( D \) by adding the oriented edge \( (v_1, v_2) \). Clearly, each vertex has outdegree at most 4 in \( D' \). Moreover, as \( v_2 \) has outdegree 0 in \( D' \), the orientated edge \( (v_1, v_2) \) will never appears in any Eulerian subgraph of \( D' \). Thus, \(|\sigma EE(D')| = |\sigma EE(D)|\) and \(|\sigma OE(D')| = |\sigma OE(D)|\). As \( D \) is \( \sigma \)-nice, we have \(|\sigma EE(D)| \neq |\sigma OE(D)|\). Therefore, \(|\sigma EE(D')| \neq |\sigma OE(D')|\) and hence \( AT(G, \sigma) \leq 5 \) by Corollary 2.3.

4 Proof of Theorem 1.5

For a graph \( G \), the maximum average degree of \( G \), denoted \( \text{mad}(G) \), is the maximum of \( 2|E(H)|/|V(H)| \), where \( H \) ranges over all subgraphs of \( G \). The following useful criterion on the existence of an orientation with bounded outdegree appeared in \[2\].

**Lemma 4.1.** A graph \( G \) has an orientation \( D \) such that every vertex has outdegree at most \( p \) if and only if \( \text{mad}(G) \leq 2p \).

**Corollary 4.2.** For any graph \( G \),

\[
AT(G, -) = \left\lceil \frac{\text{mad}(G)}{2} \right\rceil + 1.
\]  

**Proof.** Let \( p = \left\lfloor \frac{\text{mad}(G)}{2} \right\rfloor \). Then \( \text{mad}(G) \leq 2p \) and hence, by Lemma 4.1, \( G \) has an orientation \( \tilde{D} \) in which every outdegree is at most \( p \). As all edge in \( (G, -) \) is negative, each Eulerian subdigraph of \( D \) contains no positive edge and hence is \( \sigma \)-even. Thus \(|\sigma OE(D)| = 0\). Since the empty subgraph is a \( \sigma \)-even Eulerian subdigraph, we have \(|\sigma EE(D)| \geq 1\) and hence \(|\sigma EE(D)| \neq |\sigma OE(D)|\). Thus by Corollary 2.3, \( AT(G, -) \leq p + 1 \).

On the other hand, by Corollary 2.3, \( G \) has an orientation \( D \) such that each outdegree is at most \( AT(G, -) - 1 \). Thus, by Lemma 4.1, \( \text{mad}(G) \leq 2(\text{AT}(G, -) - 1) \), i.e., \( AT(G, -) \geq \frac{\text{mad}(G)}{2} + 1 \). Therefore, \( AT(G, -) \geq p + 1 \) since \( AT(G, -) \) is an integer. This proves the corollary.

**Proof of Theorem 1.5.** For a signed graph \((G, \sigma)\), Schweser and Stiebitz \[9\] showed that \( \chi(G, \sigma) \leq 2 \) if and only if \((G, \sigma)\) is switching equivalent to \((G, -)\). Thus, by Proposition 1.3, it suffices to consider the case when \((G, \sigma) = (G, -)\), i.e., \( \sigma(uv) = -1 \) for each \( uv \in E(G) \). Let \( H \) be any subgraph of a planar graph \( G \). Then by Euler’s formula for planar graph we
have $2|E(H)|/|V(H)| \leq 6$, i.e., $\mad(G) \leq 6$. By Corollary 4.2, $AT(G, -) \leq 4$. This proves the first part of Theorem 1.5.

Let $(G, -)$ be the negative planar graph as shown in Figure 2. We show that $(G, -)$ is not 3-choosable.

![Figure 2. A non-3-choosable negative planar graph $(G, -)$.](image)

Define a 3-list assignment $L$ as follows:

- $L(a) = L(a') = \{0, -1, -2\}$.
- $L(b) = L(b') = \{0, -1, 2\}$.
- $L(c) = L(c') = \{0, 1, -2\}$.
- $L(d) = L(d') = \{0, 1, 2\}$.

It suffices to show that $(G, -)$ is not $L$-colorable. Suppose to the contrary that $\phi$ is an $L$-coloring of $(G, -)$. Let $V = \{a, b, c, d\}$.

**Claim 1**: There exists some $x \in V$ such that $\phi(x) = 0$.

Suppose to the contrary that $\phi(x) \neq 0$ for each $x \in V$. Then $\phi(a) \in \{-1, -2\}$, $\phi(b) \in \{-1, 2\}$, $\phi(c) \in \{1, -2\}$ and $\phi(d) \in \{1, 2\}$. Note that $(G[V], -)$ is a negative complete graph. Thus $\phi(x) \neq -\phi(y)$ for two distinct $x$, $y$ in $V$. If $\phi(a) = -1$ then $\phi(c) = -2$ and $\phi(d) = 2$. Now, $\phi(c) = -\phi(d)$, a contradiction. Similarly, if $\phi(a) = -2$ then $\phi(b) = -1$ and $\phi(d) = 1$ and hence $\phi(b) = -\phi(d)$. This is also a contradiction. Thus, Claim 1 follows.

**Claim 2**: Let $x \in V$. If $\phi(x) = 0$ then $\phi(N(x')) = -L(x')$.

We only prove the case that $x = a$ and the other three cases can be settled in the same way. Since $\phi(a) = 0$, we have $\phi(b) \in \{-1, 2\}$, $\phi(c) \in \{1, -2\}$ and $\phi(d) \in \{1, 2\}$. If $\phi(b) = -1$ then $\phi(c) = -2$ and $\phi(d) = 2$. Thus,
\[ \phi(c) = -\phi(d), \text{ a contradiction. Therefore, } \phi(b) = 2. \] Similarly, if \( \phi(c) = -2 \) then \( \phi(b) = -1 \) and \( \phi(d) = 1 \). We also have a contradiction as \( \phi(b) = -\phi(d) \). Therefore, \( \phi(c) = 1 \). Finally, as \( N(a') = \{a, b, c\} \) and \( L(a') = \{0, -1, -2\} \), we have \( \phi(N(a')) = \{\phi(a), \phi(b), \phi(c)\} = \{0, 2, 1\} = -L(a') \). This proves Claim 2.

Now, by Claim 1, let \( x \in V \) satisfy \( \phi(x) = 0 \). Then, \( \phi(N(x')) = -L(x') \) by Claim 2. As \( \phi(x') \in L(x') \) we have \( -\phi(x') \in \phi(N(x')) \), that is, \( -\phi(x') = \phi(y) \) for some \( y \in N(x') \). Thus, \( \phi \) is not proper since \( xy \) is a negative edge. This is a contradiction and hence completes the proof of Theorem 1.5.

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