RIEMANNIAN COUNTERPARTS TO LORENTZIAN SPACE FORMS

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Abstract. On a smooth $n$-manifold $M$ with $n \geq 3$, we study pairs $(g, T)$ consisting of a Riemannian metric $g$ and a unit length vector field $T$ with geodesic flow and integrable normal bundle. Motivated by how Ricci solitons generalize Einstein metrics via a distinguished vector field, we propose to generalize space forms by considering those pairs $(g, T)$ whose corresponding Lorentzian metric $g_L := g - 2T^\flat \otimes T^\flat$ has constant curvature. We show by examples that such pairs exist when $M$ is noncompact, and that complete metrics exist among them. When $M$ is compact, however, the situation is more rigid. In the compact setting, we prove that the only pairs $(g, T)$ whose corresponding Lorentzian metric $g$ is a space form are those where $(M, g)$ is flat and its universal covering splits isometrically as a product $\mathbb{R} \times N$. The nonexistence of compact Lorentzian spherical space forms plays a key role in our proof.

1. Introduction

In this paper we search for generalizations of space forms, taking as our motivation the way in which Ricci solitons generalize Einstein metrics — namely, via a distinguished vector field. In our case, however, the vector field arises as a “bridge” between Riemannian and Lorentzian geometry: when a nowhere vanishing vector field $T$ is present, any Riemannian metric $g$ has a Lorentzian sibling

$$g_c := g - 2T^\flat \otimes T^\flat,$$

and vice-versa: $g = g_c + 2T^{\flat L} \otimes T^{\flat L}$ (note that $T^{\flat L} = -T^\flat$, and that we are assuming for convenience here that $T$ has unit length). In this paper we therefore adopt the following strategy: we seek to classify those Riemannian manifolds whose Lorentzian sibling (1) has constant curvature. Of course, the relationship (1) is well known and has been studied extensively; see, e.g., [Ole14] for a recent analysis which includes, among other things, curvature formulae. Generally speaking, (1) can be a fruitful method by which to construct distinguished examples of Riemannian or Lorentzian manifolds. That is because in certain cases it allows properties of one metric to be more or less directly inferred from those of the other; e.g., distinguished curvature or geodesic completeness. Indeed, a well known instance of the latter is the following: if $T^\flat$ is bounded on $TM$, then $g_c$ will be complete if $g$ is (this is a consequence of [CS08, Proposition 3.4]). Of course, one expects
that the more conditions one places on $T$, the easier to read these shared properties become — the easier it becomes, so to speak, to cross the bridge “with nice properties in hand.” Therefore we have imposed, in addition to $T$ having unit length, two further properties on $T$, properties which we motivate below and which are borne out by examples. What we are then able to achieve in our Theorem is a classification when the underlying manifold is compact, a classification of those Riemannian pairs $(g, T)$ whose Lorentzian counterparts are precisely the space forms:

**Theorem.** Let $M$ be an $n$-manifold $(n \geq 3)$ and $(g, T)$ a Riemannian metric $g$ on $M$ and a unit length vector field $T$ with geodesic flow and integrable normal bundle. Then the Riemann curvature 4-tensor of $(M, g)$ is

$$R_m = \frac{1}{2} \lambda g \otimes g - 2 \lambda g \otimes (T^\flat \otimes T^\flat) - \nabla T^\flat \otimes \nabla T^\flat,$$

(2)

where $\otimes$ is the Kulkarni-Nomizu product and $\lambda$ is a constant, if and only if the corresponding Lorentzian metric $g_\cdot \cdot = g - 2 T^\flat \otimes T^\flat$ has constant curvature $\lambda$. If $M$ is compact, then $\lambda = 0$, $(M, g)$ is flat, and its universal covering splits isometrically as a product $\mathbb{R} \times N$.

If $M$ is simply connected (and noncompact), then $T$ is a gradient and $\nabla T^\flat$ is its Hessian; generally speaking, $T$ is always at least locally a gradient. Observe how (2) generalizes constant curvature, which is the first term $\frac{1}{2} \lambda g \otimes g$; Remark 1 of Section 5 will further elaborate on this. The crucial fact, however, is that the corresponding Lorentzian metric $g_\cdot \cdot$ has constant curvature $\lambda$ — and this plays the key role in our proof of the case when $\lambda > 0$. Indeed, by [CM62] and [Kli96], there are no such Lorentzian metrics when $\lambda > 0$ and $M$ is compact, a foundational result in Lorentzian geometry (see [Lun15] for a comprehensive account). As for the two properties imposed on our unit length vector field $T$ — namely, that it have geodesic flow and integrable normal bundle — they are shared across $g$ and $g_\cdot \cdot$ and ensure that the 2-tensor $\nabla T^\flat$ is symmetric; indeed, the symmetry of $\nabla T^\flat$ is equivalent to these two properties. By virtue of this symmetry, the curvature 4-tensors of $g$ and $g_\cdot \cdot$, denoted by $R_m$ and $R_{m\cdot \cdot}$, respectively, have a very simple relationship to each other:

$$R_{m\cdot \cdot} = R_m + \nabla T^\flat \otimes \nabla T^\flat.$$

(3)

This will no longer hold if either of these two conditions on $T$ is dropped. Indeed, another motivation for these properties of $T$ is that such a vector field exists prominently in de Sitter spacetime, the canonical (and noncompact) Lorentzian manifold of constant positive curvature; see Example 3 in Section 3. We close this Introduction with three more remarks. First, Riemannian manifolds satisfying (2) certainly do exist in the noncompact setting. We furnish two such examples in Section 2, both with $\lambda > 0$; note that the first of these is complete and exists in all dimensions $\geq 3$. (This makes it clear that when $\lambda > 0$, the obstruction arises from compactness, not completeness.) Second, in the proof of our Theorem, while the case $\lambda > 0$ makes direct
use of (3) and relies on [CM62] and [Kli96], the case \( \lambda \leq 0 \) relies instead on a Bochner technique; see (18) in Section 4. Finally, compact Lorentzian manifolds with constant negative curvature certainly do exist; examples can be found, e.g., in [KR85] and [Gol85]. So do flat ones: any \((S^1 \times N, -dt^2 \oplus h)\), with \((N, h)\) a compact flat Riemannian manifold, yields an example, with \(\nabla^t\) serving the role of \(T\) above.

2. The Riemannian-to-Lorentzian correspondence

The proof of our Theorem is a corollary of the following more general fact:

**Proposition.** Let \((M, g)\) be a Riemmanian \(n\)-manifold equipped with a unit length vector field \(T\) with geodesic flow and integrable normal bundle. Then the Lorentzian metric \(g_L\) defined by

\[
g_L := g - 2T^b \otimes T^b
\]

has Riemann curvature 4-tensor \(R_m\) given by

\[
R_m = R_m + \nabla T^b \otimes \nabla T^b,
\]

where \(R_m\) is the Riemann curvature 4-tensor of \(g\).

**Proof.** For \(R_m\), we adopt the sign convention

\[
R_m(a, b, c, d) = g(\nabla_a \nabla_b c, d) - g(\nabla_b \nabla_a c, d) - g(\nabla_{[a,b]} c, d),
\]

and similarly for \(R_m\). With respect to the Lorentzian metric \(g_L\), \(T\) is now unit length “timelike”: \(g_L(T, T) = -1\). Next, denoting by \(\nabla\) the Levi-Civita connection of \(g\), the Koszul formula shows that \(T\) will have \(g\)-geodesic flow, since

\[
\nabla_T T = -\nabla_T T = 0,
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). Furthermore, it’s clear from the definition of \(g\) that the \(g\)-normal bundle \(T^\perp \subseteq TM\) will remain integrable (and equal to the \(g\)-normal bundle \(T^\perp\)), so that the endomorphism

\[
D_c : T^\perp \longrightarrow T^\perp, \quad X \mapsto \nabla_X T
\]

is self-adjoint with respect to the (positive-definite) metric \(g|_{T^\perp} = g|_{T^\perp}\) induced on \(T^\perp\). Thus we have, in a neighborhood of each point of \(M\), an orthonormal basis of eigenvectors \(\{X_1, \ldots, X_{n-1}\} \subseteq T^\perp\) of \(D_c\),

\[
\nabla_{X_i} T = \lambda_i X_i, \quad i = 1, \ldots, n-1,
\]

where the eigenvalues \(\lambda_i\) are smooth functions defined on said neighborhoods. In fact these are the same eigenvalues and eigenvectors as those of the endomorphism

\[
D : T^\perp \longrightarrow T^\perp, \quad X \mapsto \nabla_X T,
\]

since \(\nabla_{X_i} T = \nabla_X T\) via the Koszul formula; in particular, \(\nabla_{T} X_i = \nabla_T X_i\), and the remaining covariant derivatives are, by the Koszul formula again,

\[
\nabla_{X_i} X_j = 2\lambda_i \delta_{ij} T + \nabla_{X_i} X_j.
\]
In fact (5) extends trivially to all of $TM$; furthermore, because $\nabla_T T = 0$ and $g(T, T) = 1$, this extension remains self-adjoint with respect to $g$, thus defining a symmetric 2-tensor which is precisely $\nabla T^g$ mentioned above (although we won’t need this explicitly, observe in passing that $\nabla^T T^g_L = \nabla T^g$). Now we compute the components of the curvature tensor $R_m$, with respect to the frame of eigenvectors above:

$$\{T, X_1, \ldots, X_{n-1}\}, \quad \nabla_{X_i} T = \lambda_i X_i = \nabla X_i T.$$

Note that this frame is both $g$- and $g_L$-orthonormal, but with the important difference that

$$g(T, T) = 1 = -g(T, T).$$

As mentioned in the Introduction, more general curvature formulae are provided by [Ole14], in particular of the components (7) and (8) appearing below; however, the most important curvature component in our Proposition, namely (12) below, has not (to the best of our knowledge) appeared explicitly in the literature. As it is also the most technical, we have therefore chosen to compute all the curvature components explicitly.

1. We start with the components $R_m(X_i, T, T, X_j)$:

$$R_m(X_i, T, T, X_j) = g_i(\nabla_{X_i} \nabla_T T, X_j) - g_i(\nabla_T \nabla_{X_i} T, X_j) - g_i(\nabla_{[X_i, T]} T, X_j) = -T(\lambda_i) \delta_{ij} - \lambda_i g_i(\nabla_T X_i, X_j) - \lambda_j g_i(\nabla_T X_j, X_i) - g_i(\nabla_{X_i} T, X_j) = -g_i(\nabla_T X_i, X_j)(\lambda_j - \lambda_i)$$

where the last equality is due to the fact that the same answer would have been reached starting with

$$R_m(X_i, T, T, X_j) = g(\nabla_{X_i} \nabla_T T, X_j) - g(\nabla_T \nabla_{X_i} T, X_j) - g([X_i, T] T, X_j).$$

It is worthwhile to pause here and see what would have resulted had $\nabla_T T \neq 0$. Other than the fact that the Lie brackets $[X_i, T]$ are $g$- and $g_L$-orthogonal to $T$ if and only if $\nabla_T T = 0$ (which allowed us to simplify the last term), the first term of $R_m(X_i, T, T, X_j)$ would have been

$$g_i(\nabla_{X_i} \nabla_T T, X_j) = X_i(\frac{g_i(\nabla_T T, X_j)}{-g(\nabla_T T, X_j)}) - \frac{g_i(\nabla_T T, \nabla_{X_i} T)}{-g(\nabla_T T, X_j)} + 2\lambda_i \delta_{ij} g(\nabla_T T, T)^0.$$
2. Next, the components $R_m(X_i, X_k, T, X_j)$, with $k \neq i$:

$$R_m(X_i, X_k, T, X_j) = \frac{\lambda_k X_k}{g(\nabla_{X_i} \nabla_{X_k} T, X_j)} - \frac{\lambda_i X_i}{g(\nabla_{X_i} \nabla_{X_k} T, X_j)} - \frac{\lambda_l X_l}{g(\nabla_{X_j} T, [X_i, X_k])}$$

$$= g(\nabla_{X_i} \nabla_{X_k} T, X_j) - g(\nabla_{X_k} \nabla_{X_i} T, X_j) = g(\nabla_{X_i} T, [X_i, X_k])$$

$$= R_m(X_i, X_k, T, X_j), \quad (8)$$

where in the first and second equalities we’ve used the fact that $g(\cdot, X) = g(\cdot, X)$ for any $X \in T^\perp$, which includes $[X_i, X_k]$ by integrability of $T^\perp$.

3. Finally, the components $R_m(X_k, X_i, X_j, X_l)$, with $k \neq i, l \neq j$:

$$R_m(X_k, X_i, X_j, X_l) = \frac{g(\nabla_{X_k} \nabla_{X_i} X_j, X_l)}{g(\nabla_{X_k} \nabla_{X_i} X_j, X_l)} - \frac{g(\nabla_{X_l} \nabla_{X_i} X_j, X_k)}{g(\nabla_{X_l} \nabla_{X_i} X_j, X_k)}, \quad (a)$$

will be computed term-by-term, using the Koszul formula, beginning with $(a)$:

$$g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) = g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) + 2\lambda_i \delta_{ij} g(\nabla_{X_k} T, X_l)$$

$$= \frac{1}{2} \left[ X_k(g(\cdot, Z) + Z(g(\cdot, X_i)) - X_i(g(\cdot, X_k)) - g(\cdot, [X_i, X_k]) + g(\cdot, [X_i, Z]) \right]$$

$$+ 2\lambda_i \lambda_k \delta_{ij} \delta_{kl},$$

Every “$g(\cdot, \cdot)$” here contains at least one $X \in T^\perp$, so that it equals $g(\cdot, \cdot)$. We thus have that

$$g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) = \frac{1}{2} \left[ X_k(g(\cdot, Z) + Z(g(\cdot, X_i)) - X_i(g(\cdot, X_k)) - g(\cdot, [X_i, X_k]) + g(\cdot, [X_i, Z]) \right]$$

$$+ 2\lambda_i \lambda_k \delta_{ij} \delta_{kl}$$

$$= g(\nabla_{X_k} Z, X_l) + 2\lambda_i \lambda_k \delta_{ij} \delta_{kl}$$

$$= g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) + 2\lambda_i \lambda_k \delta_{ij} \delta_{kl}. \quad (9)$$
Similarly with \((b)\),

\[
g_c(\nabla_{X_i} \nabla_{X_k} X_j, X_l) = g_c(\nabla_{X_i} \nabla_{X_k} X_j, X_l) + 2\lambda_k \delta_{kj} g_c(\nabla_{X_i} T, X_l)
\]

\[
= \frac{1}{2} \left[ X_l(\nabla_i (g_c(Z, X_l))) + Z(\nabla_i (g_c(X_l, X_i))) - X_l(\nabla_i (g_c(X_l, Z)))
\right.
\]

\[
- g_c(Z, [X_l, X_i]) - g_c([X_l, Z], X_i) + g_c([X_l, [X_l, Z]] + 2\lambda_k \lambda_i \delta_{kl} \delta_{il}.
\]

For \((c)\), we observe that because \([X_k, X_i] \in T^\perp\), it is of the form \([X_k, X_i] = \sum_{r=1}^{n-1} a_r X_r\) with the \(a_r\)'s smooth locally defined functions, which in turn implies that \(\nabla_{[X_k, X_i]} X_j = \nabla_{[X_k, X_i]} X_j + 2a_j \lambda_j T\). Hence

\[
g_c(\nabla_{[X_k, X_i]} X_j, X_l) = g(\nabla_{[X_k, X_i]} X_j, X_l).
\]

Taken together, \((9)\), \((10)\), and \((11)\) yield

\[
\text{Rm}_c(X_k, X_i, X_j, X_l) = \text{Rm}(X_k, X_i, X_j, X_l) + 2\lambda_i \lambda_k \delta_{ij} \delta_{il} - \delta_{ij} \delta_{kl}
\]

\[
= \text{Rm}(X_k, X_i, X_j, X_l) + (\nabla T^\circ \nabla T^\circ)(X_k, X_i, X_j, X_l).
\]

(12)

Given \((7)\), \((8)\), and \((12)\), and the fact that \((\nabla T^\circ \nabla T^\circ)(T, \cdot, \cdot, \cdot) = 0\), the proof is complete. \(\square\)

3. Examples

We provide three examples illustrating our Proposition, all in the noncompact setting, beginning with the canonical example:

**Example 1.** Consider \(n\)-dimensional de Sitter spacetime \((\mathbb{S}^n_1, g_c)\) \((n \geq 3)\), which is the warped product \((\mathbb{R} \times \mathbb{S}^{n-1}, -dt^2 + f(t) \hat{g})\), where \(\hat{g}\) is the standard (round) Riemannian metric on \(\mathbb{S}^{n-1}\) and

\[
f(t) = r^2 \cosh^2 \left( \frac{t}{r} \right),
\]

with \(r\) the radius of \(\mathbb{S}^{n-1}\); consult, e.g., [Har17] and [BEE96, p. 183ff.]. This is a geodesically complete Lorentzian manifold with constant positive curvature \((\lambda = 1/r^2)\); furthermore,

\[
T := \nabla^t t = -\partial_t
\]

is a unit timelike gradient, hence has geodesic flow and integrable normal bundle. The corresponding Riemannian metric

\[
g := g_c + 2T^\circ L \otimes T^\circ L = dt^2 + f(t) \hat{g}
\]

is also geodesically complete (see [O'N83, Lemma 40, p. 209]). In fact the eigenvalues \(\lambda_i\) of \(D_t\) are easily determined here; we shall compute them in
Section 5. Finally, note that on any warped product \((\mathbb{R} \times N, -dt^2 + f(t)h)\), the gradient \(T := \nabla t = -\partial_t\) is unit timelike.

**Example 2.** On \(\mathbb{R}^3 = \{ (x^1, x^2, x^3) \}\), consider a Lorentzian metric \(g\), whose components \((g_{ij})\) with respect to the coordinate basis \(\{\partial_1, \partial_2, \partial_3\}\) take the form

\[
(g_{ij}) = \begin{pmatrix}
0 & 1 & 0 \\
1 & H(x^1) & 0 \\
0 & 0 & H(x^1)/2
\end{pmatrix},
\]

where \(H(x^1)\) is a smooth function, to be determined below, such that \(g\) will have constant positive curvature on an open subset of \(\mathbb{R}^3\). In dimension 3, this is equivalent to being an Einstein metric with positive Einstein constant, which we can take to be 1:

\[
\text{Ric}_L = g_l.
\] (13)

Though we forego the computations here, it is straightforward to show that (13) will hold if and only if

\[
H(x^1) = (x^1 + a)^2/2, \quad a \in \mathbb{R}.
\]

For example, if we take \(a = 2\), then \(g\) will be a Lorentzian metric in the open subset \(\{(x^1, x^2, x^3) : x^1 > -2\}\). Finally, let \(f(x^1)\) be a smooth function and consider its \(g\)-gradient:

\[
\nabla^g f = -\frac{f'}{2} (x^1 + 2)^2 \partial_1 + f' \partial_2.
\]

Another computation shows that \(g((\nabla^g f, \nabla^g f)) = -1\) if and only if

\[
(f')^2 = \frac{2}{(x^1 + 2)^2}.
\]

Taking the smooth solution \(f(x^1) = \sqrt{2} \ln \left(\frac{x^1 + 2}{2}\right)\), we thus have that the pair \((g, \nabla^g f)\), when restricted to \(\{(x^1, x^2, x^3) : x^1 > -2\}\), yields a Lorentzian manifold with constant positive curvature and a unit timelike vector field with geodesic flow and integrable normal bundle. The corresponding Riemannian metric \(g\) is then

\[
g := g_l + 2df \otimes df.
\]

With respect to \(g\), \(\nabla^g f\) will also be a unit length vector field with geodesic flow and integrable normal bundle.

**Example 3.** Our last example is not one of constant curvature, but it does involve an important class of Lorentzian manifolds. On \(\mathbb{R}^4 = \{(v, u, x, y)\}\), consider the Lorentzian \(pp\)-wave metrics

\[
g_l = H(u, x, y) du \otimes du + dv \otimes dv + dv \otimes du + dx \otimes dx + dy \otimes dy,
\] (14)

where \(H(u, x, y)\) is a smooth function independent of \(v\). For the role such metrics play in modeling gravitational waves, as well as their geometric
properties more generally, consult [SHN+17] and [FS06]. Define the vector field
\[
T := \frac{1}{2} (H + f^2 + h^2 + 1) \partial_v - \partial_u + f \partial_x + h \partial_y,
\]
with \(f(u, x, y)\) and \(h(u, x, y)\) smooth functions, to be determined below, such that \(T\) will have geodesic flow and integrable normal bundle (note that \(g_\varepsilon(T, T) = -1\) for any \(f, h\)). We now sketch the conditions on \(f, h,\) and \(H\) needed for these two properties to hold. To begin with, the normal bundle \(T^\perp\) is spanned by the orthonormal frame \(\{X, Y, Z\}\) given by
\[
X = f \partial_v + \partial_x, \quad Y = h \partial_v + \partial_y, \quad Z = \frac{1}{2} (H + f^2 + h^2 - 1) \partial_v - \partial_u + f \partial_x + h \partial_y.
\]
(In general, these vector fields are not eigenvectors of \(D_\varepsilon\) in (4); e.g., \(\nabla_\varepsilon T = f_x X + h_x Y\) and \(\nabla_\varepsilon Y = f_y X + h_y Y\).) To ensure integrability of \(T^\perp\) via Frobenius’s Theorem, we seek to satisfy the Lie bracket conditions
\[
g_\varepsilon(T, [X, Y]) = g_\varepsilon(T, [X, Z]) = g_\varepsilon(T, [Y, Z]) = 0.
\]
Computing these yields (consult, e.g., [BEE96, Chapter 13] for all relevant covariant derivatives), respectively, the following necessary and sufficient conditions on \(f, h,\) and \(H\):
\[
h_x = f_y, \quad ff_x + fh_y - f_u = \frac{H_x}{2}, \quad hh_y + fh_x - h_u = \frac{H_y}{2}. \quad (15)
\]
Incidentally, the latter two equations also ensure that \(T\) has geodesic flow: \(\nabla_\varepsilon T = 0\). Many functions \(f, h,\) and \(H\) exist satisfying (15); e.g., if \(f, h,\) and \(H\) are functions of \(x\) and \(y\) only, then we may take any \(f, h\) satisfying \(h_x = f_y\) and set \(H = f^2 + h^2\). One particularly interesting case—a so called plane wave—is given by the choices
\[
f(x, y) = y, \quad h(x, y) = x, \quad H(x, y) = x^2 + y^2,
\]
because, not only will \(T\) have geodesic flow and integrable normal bundle, but for this choice of \(H\) the metric \(g_\varepsilon\) is in fact geodesically complete and has vanishing \(N\)-Bakry-Émery tensor \(\text{Ric}_u^{7/2}\) (i.e., with “synthetic dimension” \(N = 7/2\)):
\[
\text{Ric}_u^{7/2} := \text{Ric}_\varepsilon + \text{Hess} u - \frac{du \otimes du}{7/2 - 4} = 0.
\]
(For the definition and properties of such tensors, see [WW16]. In fact \(\nabla_\varepsilon u = \partial_v\) is a parallel “lightlike” vector field for all pp-waves (14), so that its Hessian, \(\text{Hess} u\), vanishes.) The corresponding Riemannian metric
\[
g := g_\varepsilon + 2T^\flat \otimes T^\flat
\]
satisfies \(\text{Ric} = \text{Ric}_\varepsilon\), though the Riemannian gradient \(\nabla u\) is not parallel.
4. Proof of Theorem

Proof of Theorem. Suppose a pair \((g, T)\) satisfying (2) exists on a compact manifold \(M\) of dimension \(\geq 3\). We first consider the case \(\lambda \leq 0\); it turns out that the obstruction in this case occurs at the level of the Ricci tensor, via a well known equation ((18) below) and Riccati analysis, as follows. Set 

g_* := g - 2T^\flat \otimes T^\flat.

Then

\[
\text{Ric}(T, T) = \sum_{i=1}^{n-1} \text{Rm}(X_i, T, T, X_i)
\]

so that if \((M, g_*)\) has constant curvature \(\lambda \leq 0\) — more generally, if \((M, g_*)\) is Einstein with nonpositive Einstein constant — then

\[
\text{Ric}(T, T) = \text{Ric}_L(T, T) = (n - 1) \lambda g_*(T, T) \geq 0.
\]

(17)

Next, setting \(i = j\) in (6) and summing over \(i = 1, \ldots, n - 1\) yields the following Bochner-type equation,

\[
T(\text{div } T) = -\text{Ric}(T, T) - \sum_{i=1}^{n-1} \lambda_i^2,
\]

where we’ve used the fact that

\[
\text{div } T = \sum_{i=1}^{n-1} \lambda_i = \text{tr}_g \nabla T^\flat.
\]

Now, via the Schwarz inequality

\[
\sum_{i=1}^{n-1} \lambda_i^2 \geq \frac{1}{n - 1} (\lambda_1 + \cdots + \lambda_{n-1})^2,
\]

(18) reduces to

\[
T(\text{div } T) \leq -\text{Ric}(T, T) - \frac{(\text{div } T)^2}{n - 1},
\]

(19)

which permits, in turn, the following well known Riccati analysis: since \(T\) is complete (\(M\) being compact),

\[
\text{Ric}(T, T) \geq 0 \quad \Rightarrow \quad \text{div } T = 0 \quad \Rightarrow \quad \text{Ric}(T, T) = \lambda_i = 0.
\]

It follows that if \(\text{Ric}(T, T) = 0\), then \(T\) must be parallel, and that the case \(\text{Ric}(T, T) > 0\) cannot occur. The latter implies that \(\lambda < 0\) cannot occur; the former, that if \(\lambda = 0\), then \(\nabla T^\flat\) is zero and (2) vanishes, so that \((M, g)\) is flat. Furthermore, its universal covering splits isometrically as a product \(\mathbb{R} \times N\), by the de Rham Decomposition Theorem (see [Pet16, p. 384]). This settles the case \(\lambda \leq 0\). For the case \(\lambda > 0\), we will employ a different strategy;
indeed, since $\text{Ric}(T, T) < 0$ when $\lambda > 0$ (via (17)), the Riccati analysis applied to (19) is unavailable here. Instead, we substitute $g = g_L + 2T^b \otimes T^b$ into our Proposition (note that $T^b \otimes T^b = T^b \otimes T^b$), to obtain

\[ R_{m_i} = R_m + \nabla T^b \otimes \nabla T^b \]

\[ = (2) \frac{1}{2} \lambda g \otimes g - 2\lambda g \otimes (T^b \otimes T^b) \]

\[ = \frac{1}{2} \lambda g_i \otimes g_i + 2\lambda g_i \otimes (T^b \otimes T^b) - 2\lambda g_i \otimes (T^b \otimes T^b) \]

\[ = \frac{1}{2} \lambda g_i \otimes g_i, \]

where we’ve used the fact that $(T^b \otimes T^b) \otimes (T^b \otimes T^b) = 0$. But as mentioned in the Introduction, such a (compact) Lorentzian manifold is impossible when $\lambda > 0$, by [CM62] and [Kli96].

5. Concluding Remarks

Remark 1. Observe that for a curvature 4-tensor of the form (2), the term $g \otimes (T^b \otimes T^b)$ is zero on the frame $\{T, X_1, \ldots, X_{n-1}\}$ except for the component

$$g \otimes (T^b \otimes T^b)(X_i, T, T, X_i) = 1.$$ 

In particular, the sectional curvature of any 2-plane containing $T$ is

$$\lambda - 2\lambda = -\lambda,$$

as opposed to $\lambda$, which would have been the case with constant curvature $R_m = \frac{1}{2} \lambda g \otimes g$. On the other hand, the term $\nabla T^b \otimes \nabla T^b$ is zero except for the component

$$(\nabla T^b \otimes \nabla T^b)(X_i, X_j, X_j, X_i) = 2\lambda_i \lambda_j \quad , \quad i \neq j,$$

so that 2-planes spanned by $\{X_i, X_j\}$ now have sectional curvature

$$\lambda - 2\lambda_i \lambda_j.$$ 

For this reason, we may regard (2) as a curvature tensor for which constant curvature has been “broken by $T$.”

Remark 2. Our Theorem is uninteresting in dimension 2. Indeed, although any Riemannian 2-manifold satisfies $R_m = \frac{1}{2} Kg \otimes g$ with $K$ the Gaussian curvature (see, e.g., [Lee18, p. 250]), and although $K$ is neither constant nor signed in general, nevertheless $K = -\lambda$ if (2) is satisfied, because in dimension 2,

$$\nabla T^b \otimes \nabla T^b = 0 \quad , \quad g \otimes (T^b \otimes T^b) = \frac{1}{2} g \otimes g.$$ 

But by the Gauss-Bonnet Theorem, when the Euler characteristic is zero (i.e., the manifold is a 2-torus or a Klein bottle), the only compact Riemannian 2-manifold with constant curvature is the flat one.
Remark 3. Via (3), the Ricci tensor of $g$ is related to that of $g_L$ as follows:

$$\text{Ric} = \text{Ric}_L + 2\text{Rm}_L(T, \cdot , \cdot , T) - \text{tr}_g(\nabla T^\flat \otimes \nabla T^\flat).$$  \hfill (20)

It is not signed in general. Indeed, in the specific case of (2) with $\lambda > 0$, note that while $\text{Ric}(T, T) < 0$ by (17) (because now $\lambda > 0$), $\text{Ric}(X_i, X_i)$ is given by

$$\text{Ric}(X_i, X_i) = \sum_{k\neq i} \text{Rm}(X_k, X_i, X_i, X_k) + 2\lambda_i \sum_{k\neq i} \lambda_k,$$

where we observe that, because $g_L(T, T) = -1$, $\text{Rm}_L(T, X_i, X_i, T) = 1/2 \lambda (g^\flat \otimes g)(T, X_i, X_i, T) = -\lambda$.

In particular, (21) need not be negative in general, so that the Ricci tensor need not be signed. In fact this is evident in Example 1 of Section 3 above, namely, de Sitter spacetime $(\mathbb{R} \times S^{n-1}, -dt^2 + f(t)\hat{g})$, here with radius $r = 1$ so that $\lambda = 1$. By standard formulae for covariant derivatives on warped products (see, e.g., [O’N83, p. 206]), for any $X \in T^{\perp L}$,

$$\nabla_X T = \frac{T(f)}{f} X = -2 \tanh(t) X,$$

so that each $\lambda_i = -2 \tanh(t)$, and hence

$$\text{Ric}(X_i, X_i) = (n - 3) - 8(n - 2) \tanh^2(t).$$

For $n \geq 4$, this can be positive, e.g., at $t = 0$. In general, when (2) holds, (20) takes the form

$$\text{Ric} = (n - 1)\lambda g + \lambda (g \otimes g)(T, \cdot , \cdot , T) - \text{tr}_g(\nabla T^\flat \otimes \nabla T^\flat).$$

Remark 4. Regarding (19), note that if $\lambda > 0$, then even in the case of equality,

$$T(\text{div} T) = (n - 1)\lambda - \frac{(\text{div} T)^2}{n - 1},$$

nontrivial complete solutions exist. Indeed, if $s$ is an affine parameter along an integral curve of $T$, then

$$(n - 1)\sqrt{\lambda} \tanh(\sqrt{\lambda} s + c), \quad \pm(n - 1)\sqrt{\lambda}$$

are complete solutions, so that the Riccati analysis above is unavailable here.
References

[BEE96] John K. Beem, Paul E. Ehrlich, and Kevin L. Easley. *Global Lorentzian Geometry*. Marcel Dekker, Inc., 2nd edition, 1996.

[CM62] Eugenio Calabi and Lawrence Markus. Relativistic space forms. *Annals of Mathematics*, pages 63–76, 1962.

[CS08] Anna Maria Candela and Miguel Sánchez. *Geodesics in semi-Riemannian manifolds: geometric properties and variational tools*, volume 4. European Mathematical Society Zürich, 2008.

[FS06] José L. Flores and Miguel Sánchez. On the geometry of pp-wave type spacetimes. In *Analytical and Numerical Approaches to Mathematical Relativity*, pages 79–98. Springer, 2006.

[Gol85] William M. Goldman. Nonstandard Lorentz space forms. *Journal of Differential Geometry*, 21(2):301–308, 1985.

[Har17] Thomas Hartman. Lecture Notes on Classical de Sitter Space. *arXiv:1205.3855*, 2017.

[Kli96] Bruno Klingler. Complétude des variétés lorentziennes à courbure constante. *Mathematische Annalen*, 306(2):353–370, 1996.

[KR85] Ravi S. Kulkarni and Frank Raymond. 3-dimensional Lorentz space-forms and Seifert fiber spaces. *Journal of Differential Geometry*, 21(2):231–268, 1985.

[Lee18] John M. Lee. *Introduction to Riemannian Manifolds*, volume 176. Springer, 2nd edition, 2018.

[Lun15] David Lundberg. On the non-existence of compact Lorentzian manifolds with constant positive curvature. *Master’s Thesis, Lund University*, 2015.

[Ole14] Benjamín Olea. Canonical variation of a Lorentzian metric. *Journal of Mathematical Analysis and Applications*, 419(1):156–171, 2014.

[O’N83] Barrett O’Neill. *Semi-Riemannian Geometry with Applications to Relativity*, volume 103. Academic press, 1983.

[Pet16] Peter Petersen. *Riemannian Geometry*, volume 171. Springer, 3rd edition, 2016.

[SHN+17] Christina Sorroni, Denson C. Hill, Pavel Nurowski, Lydia Bieri, David Garfinkle, and Nicolás Yunes. The Mathematics of Gravitational Waves: A Two-Part Feature. *Notices of the AMS*, 64(7):684–707, 2017.

[WW16] Eric Woolgar and William Wylie. Cosmological singularity theorems and splitting theorems for N-Bakry-Émery spacetimes. *Journal of Mathematical Physics*, 57(2):022504, 2016.

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