On the Cauchy problem for the periodic fifth-order KP-I equation

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Abstract

The aim of this paper is to investigate the Cauchy problem for the periodic fifth order KP-I equation

\[ \partial_t u - \partial^5_\xi u - \partial^{-1}_x \partial^2_y u + u \partial_x u = 0, \quad (t, x, y) \in \mathbb{R} \times \mathbb{T}^2 \]

We prove global well-posedness for constant x mean value initial data in the space \( E = \{ u \in L^2, \partial^2_x u \in L^2, \partial^{-1}_x \partial_y u \in L^2 \} \) which is the natural energy space associated with this equation.

**Keywords**: fifth-order KP-I equation, global well-posedness.

1 Introduction

The KP equations arised in [9] as fluid mechanics models for long, weakly non-linear two-dimensional waves with a small dependence in the tranverse variable. The usual KP equations are

\[ \partial_t u + \partial^3_x u + \epsilon \partial^{-1}_x \partial_y u + u \partial_x u = 0 \tag{1.1} \]

where the coefficient \( \epsilon \) depends on the surface tension. The KP-I equation corresponds to \( \epsilon = -1 \), and the KP-II equation to \( \epsilon = 1 \). The Cauchy problem for these equations has been extensively studied in the past twenty years. The KP-II equation is known to be locally well-posed in the scale-critical space \( H^{-1/2,0}(\mathbb{R}^2) \) [5], and globally well-posed in \( L^2(\mathbb{R} \times \mathbb{T}) \) [15] and \( L^2(\mathbb{T}^2) \) [2].

As for the KP-I equation, some ill-posedness results [14, 11] have shown that this equation does not have a semilinear nature, in the sense that it cannot be treated via a perturbative method. Ionescu, Kenig and Tataru [7] thus developed the short-time Fourier restriction norm method to overcome the resonant low-high interactions responsible of the quasilinear behavior, therefore obtaining global well-posedness in the energy space on \( \mathbb{R}^2 \). The adaption [22]
in the periodic setting revealed a logarithmic divergence in the energy estimate due to a bad frequency interaction in the resonant set, establishing therefore a local well-posedness result in the Besov space $B_{2,1}^1(T^2)$ which is strictly larger than the natural energy space. To overcome this difficulty and recover a global well-posedness result in the energy space, one can look for a better dispersion effect by either removing the assumption of periodicity in one direction \[16\], or studying higher-order models.

To pursue this latter issue, we investigate the Cauchy problem for the periodic fifth-order KP-I equation

$$
\partial_t u - \partial_x^5 u - \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0, \quad (t, x, y) \in \mathbb{R} \times T^2 \tag{1.2}
$$

First, as noticed by Bourgain \[2\] in the context of the periodic KP-II equation, any (periodic in space) solution of \textbf{(1.2)} has a constant (in $y$) $x$-mean value, i.e. if $(m, n) \in \mathbb{Z}^2$ are the Fourier variables associated with $(x, y) \in T^2$, then the Fourier coefficients of $u$ with respect to $(x, y)$ satisfy the extra condition

$$
\hat{u}(t, 0, n) = 0 \text{ for } n \in \mathbb{Z} \setminus \{0\} \tag{1.3}
$$

In particular, in $t = 0$ we see that the initial data must satisfy (1.3). As in \[1, 2\], we will make the additional assumption that $\hat{u}_0(0, 0) = 0$, which is not restrictive since for data $u_0$ with non zero constant $x$ mean value, we will just have to set $v_0 := u_0 - \hat{u}_0(0, 0)$ which satisfies the above condition and the modified equation

$$
\partial_t v - \partial_x^5 v + c \partial_x v - \partial_x^{-1} \partial_y^2 v + v \partial_x v = 0
$$

with $c = \hat{u}_0(0, 0)$. Our analysis of \textbf{(1.2)} applies equally to the above modified equation, since the extra lower order term does not change the resonant function (see its definition in \textbf{(3.7)} below).

Now, to work with initial data satisfying the constraint \textbf{(1.3)}, we introduce the subspace of distributions

$$
\mathcal{D}'_0(T^2) := \{u_0 \in \mathcal{D}'(T^2), \ \hat{u}_0(0, n) = 0 \ \forall n \in \mathbb{Z}\}
$$

in which the operator $\partial_x^{-1}$ is well defined as

$$
\partial_x^{-1} u_0(x, y) := \mathcal{F}^{-1} \left\{ \frac{1}{im} \hat{u}_0(m, n) \right\}
$$

The equation \textbf{(1.2)} has another interesting feature : it possesses some conservation laws. Indeed, the mass

$$
\mathcal{M}(u_0) := \int_{T^2} u_0^2(x, y) dx dy \tag{1.4}
$$

and the energy

$$
\mathcal{E}(u_0) := \int_{T^2} \left\{ (\partial_x^2 u_0)^2(x, y) + (\partial_x^{-1} \partial_y u_0)^2(x, y) - \frac{1}{3} u_0^3(x, y) \right\} dx dy \tag{1.5}
$$
are conserved by the flow. Therefore, to obtain a global well-posedness result, it suffices to construct local solutions to (1.2) and they will be automatically extended globally in time as soon as the above quantities are bounded.

In view of the precedent remarks, we will thus work in the energy space defined as

$$\mathcal{E}(\mathbb{T}^2) := \{u_0 \in D'_0(\mathbb{T}^2) \cap L^2(\mathbb{T}^2), \partial_x^2 u_0 \in L^2(\mathbb{T}^2), \partial_x^{-1} \partial_y u_0 \in L^2(\mathbb{T}^2)\}$$

endowed with the norm

$$||u_0||_{\mathcal{E}} := \left( ||u_0||^2_{L^2} + ||\partial_x^2 u_0||^2_{L^2} + ||\partial_x^{-1} \partial_y u_0||^2_{L^2} \right)^{1/2}$$

For initial data in this space, the mass is clearly finite, and due to the anisotropic Sobolev inequality of Tom [20, Lemma 2.5] the energy is bounded as well.

The first results on the Cauchy problem for (1.2) were obtained by Iório and Nunes in the general setting of [8] where it has been shown to be locally well-posed for zero mean value initial data in the space $H^s(\mathbb{T}^2)$ for $s > 2$ by adapting the general quasi-linear theory of Kato. This model has then been studied in the work of Saut and Tzvetkov [17, 18, 19], where it has been proved that this equation is globally well-posed in the energy spaces $\mathcal{E}(\mathbb{R}^2)$ and $\mathcal{E}(\mathbb{T} \times \mathbb{R})$ by using the standard Bourgain method. Li and Xiao [13] have then pushed forward with this approach and got global well-posedness in $L^2(\mathbb{R}^2)$. However, a counter-example is built in [19] to show the failure of the bilinear estimate in the usual Bourgain spaces when $u$ is periodic in both variables, initiating thereafter a systematic study of such quasilinear behaviours in dispersive equations (see [21] for a detailed presentation of this issue). This implies that another approach is needed. Using the refined energy method of [12], Ionescu and Kenig [6] proved global well-posedness in $\mathcal{E}(\mathbb{R} \times \mathbb{T})$. Very lately, Guo, Huo and Fang [3] proved local well-posedness in $H^{s,0}(\mathbb{R}^2)$ for $s \geq -3/4$ and the initial-value problem (1.2) for periodic initial data in the energy space remained open. In this note, we prove the following.

**Theorem 1.1.** (a) For any $u_0 \in \mathcal{E}(\mathbb{T}^2)$, there exists a unique global smooth solution

$$u =: \Phi\infty(u_0) \in C(\mathbb{R}, \mathcal{E}(\mathbb{T}^2))$$

to (1.2) and moreover, for all $T > 0$ and $\sigma \geq 2$ we have

$$||\Phi\infty(u_0)||_{L^\infty_T \mathcal{E}^\sigma} \leq C(T, \sigma, ||u_0||_{\mathcal{E}^\sigma})$$

(b) Take any $u_0 \in \mathcal{E}(\mathbb{T}^2)$ and $T > 0$, then there exists a unique solution $u$ to (1.2) in the class

$$C([-T; T], \mathcal{E}) \cap \Phi(T) \cap \mathcal{B}(T)$$

This defines a continuous flow $\Phi : \mathcal{E} \to C(\mathbb{R}, \mathcal{E})$ which leaves $\mathcal{M}$ and $\mathcal{E}$ invariants.
The functions spaces $E^\infty$, $F(T)$ and $B(T)$ are defined in section 2 below.

Now, in view of the above definition of the energy space, one may be surprised by the gap in regularity between the Cauchy theory in $\mathbb{R}^2$ [3] and our well-posedness result. This is explained by the difficulty to evaluate accurately the measure of the resonant set in the periodic setting. See remark 3.7 below for more details.

To prove Theorem 1.1 we will then use the method of [7] and prove the linear, bilinear and energy estimates in the spaces $F$, $N$ and $B$.

Section 2 introduces general functions spaces and their basic properties. We prove some dyadic estimates in section 3 which we will use in sections 4 and 5 to prove energy and bilinear estimates respectively. The proof of Theorem 1.1 is finally completed in section 6.

**Notations** For positive reals $a$ and $b$, $a \lesssim b$ means that there exists a positive constant $c > 0$ (independent of the various parameters) such that $a \leq c \cdot b$.

The notation $a \sim b$ stands for $a \lesssim b$ and $b \lesssim a$.

For $x \in \mathbb{R}^d$ we set $\langle x \rangle := (1 + |x|^2)^{1/2}$.

For a real $x$, we write $|x|$ to denote its integer part.

For a set $A \subset \mathbb{R}^d$, $1_A$ is the characteristic function of $A$ and if $A$ is Lebesgue-measurable, $|A|$ means its measure. When $A \subset \mathbb{Z}$ is a finite set, its cardinal is denoted $#A$.

For $M > 0$ and $s \in \mathbb{R}$, $M^s \lesssim$ means $\leq C_s M^{s-\varepsilon}$ for any choice of $\varepsilon > 0$ small enough. We define similarly $M^{s+}$.

Let $(\tau, m, n) \in \mathbb{R} \times \mathbb{Z}^2$ denote the Fourier variables of $(t, x, y) \in \mathbb{R} \times \mathbb{T}^2$. We define the unitary group

$$U(t) = e^{-i \langle \hat{\partial}_x^5 + \hat{\partial}_y^5 \rangle} = \mathcal{F}^{-1}_{xy} e^{-i \hat{\omega}(m, n) \hat{x} y}$$

where $\omega(m, n) := m^5 + \frac{n^5}{m}$.

We note $M, K \in 2\mathbb{N}$ the dyadic frequency decompositions of $|m|$ and $\langle \tau + \omega(m, n) \rangle$.

We define then $D_{M,K} := \{ (\tau, m, n) \in \mathbb{R} \times \mathbb{Z}^2, \ |m| \sim M, \langle \tau + \omega(m, n) \rangle \sim K \}$ and $D_{M,K} := \bigcup_{K' \leq K} D_{M,K'}$.

We note also $I_M := \{ 5M/8 \leq |m| \leq 8M/5 \}$ and $I_{M} := \bigcup_{M' \leq M} I_{M'}$.

We use the notations $M_1 \land M_2 := \min(M_1, M_2)$ and $M_1 \lor M_2 := \max(M_1, M_2)$.

For $M_1, M_2, M_3 \in \mathbb{R}^*_+$, $M_{\min} \leq M_{\text{med}} \leq M_{\max}$ denotes the increasing rearrangement of $M_1, M_2, M_3$, i.e

$$M_{\min} := M_1 \land M_2 \land M_3, \quad M_{\max} := M_1 \lor M_2 \lor M_3$$

and $M_{\text{med}} = M_1 + M_2 + M_3 - M_{\max} - M_{\min}$

We define now the Littlewood-Paley decomposition. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\text{supp} \chi \subset [-8/5; 8/5]$ and $\chi \equiv 1$ on $[-5/4; 5/4]$.

For $K \in 2\mathbb{N}$, we then define $\eta_1(x) := \chi(x)$ and $\eta_K(x) := \chi(x/K) - \chi(2x/K)$.
if $K > 1$, such that $\text{supp}\eta_K \subset I_K$ and $\eta_K \equiv 1$ on $\{4/5K \leq |x| \leq 5/4K\}$. Thus $\langle \tau + \omega(m, n) \rangle \in \text{supp}\eta_K \Rightarrow \langle \tau + \omega \rangle \in I_K$ and $|\tau + \omega| \sim K$ for any $K \in 2\mathbb{N}$.

When needed, we may use another decomposition $\tilde{\chi}, \tilde{\eta}$ with the same properties as $\chi, \eta$ and satisfying $\tilde{\chi} \equiv 1$ on $\text{supp}\chi$ and $\tilde{\eta} \equiv 1$ on $\text{supp}\eta$.

Finally, for $\kappa \in \mathbb{R}_+^*$, we note $\chi_{\kappa}(x) := \chi(x/\kappa)$.

We also define the Littlewood-Paley projectors associated with the sets $I_M$:

$$P_M u := \mathcal{F}^{-1}(1_{I_M}(m) \hat{u}) \quad \text{and} \quad P_{\leq M} u := \sum_{M' \leq M} P_{M'} u = \mathcal{F}^{-1}(1_{I_{\leq M}}(m) \hat{u})$$

# Functions spaces and first properties

## 2.1 Definitions

The energy space $E$ was defined in (2.4). More generally, for $\sigma \geq 2$, we define

$$E^\sigma(\mathbb{T}^2) := \{u_0 \in \mathcal{D}'(\mathbb{T}^2) \cap L^2(\mathbb{T}^2), \|u_0\|_{E^\sigma} := \|\langle m \rangle^\sigma \cdot p(m, n) \cdot \hat{u}_0\|_{L^2} < +\infty\}$$

and

$$E^\infty := \bigcap_{\sigma \geq 2} E^\sigma$$

with the weight $p$ defined as

$$p(m, n) := \langle \langle m \rangle^{-2} \frac{n}{m} \rangle, \ (m, n) \in (\mathbb{Z}^*)^2$$

so that with this definition $E = E^2$.

Let $M \in 2\mathbb{N}$. As in [7], for $b \in [0; 1/2]$ the dyadic Bourgain type space is defined as

$$X^b_M := \{f(\tau, m, n) \in L^2(\mathbb{R} \times \mathbb{Z}^2), \text{supp} f \subset \mathbb{R} \times I_M \times \mathbb{Z}, \|f\|_{X^b_M} := \sum_{K \geq 1} K^b \|\rho_K(\tau + \omega)f\|_{L^2} < +\infty\}$$

When $b = 1/2$ we simply write $X_M$.

Then, we use the $X^b_M$ structure uniformly on time intervals of size $M^{-2}$:

$$F^b_M := \{u(t, x, y) \in C(\mathbb{R}, E^\infty), \ P_M u = u, \|u\|_{F^b_M} := \sup_{t_M \in \mathbb{R}} \|p \cdot \mathcal{F}\{\chi_{M^{-2}}(t - t_M)u\}\|_{X^b_{M'}} < +\infty\}$$

and

$$N_M := \{u(t, x, y) \in L^2(\mathbb{R}, E^\infty), \ P_M u = u, \|u\|_{N_M} := \sup_{t_M \in \mathbb{R}} \left\|p \cdot |\tau + \omega + iM^2|^{-1} \mathcal{F}\{\chi_{M^{-2}}(t - t_M)u\}\right\|_{X_M} < +\infty\}$$
For a function space \( Y \hookrightarrow \mathcal{C}(\mathbb{R}, \mathbb{E}^\infty) \), we set
\[
Y(T) := \left\{ u \in \mathcal{C}([-T, T], \mathbb{E}^\infty), \ ||u||_{Y(T)} < +\infty \right\}
\]
endowed with
\[
||u||_{Y(T)} := \inf\{||\tilde{u}||_Y; \ \tilde{u} \in Y, \ \tilde{u} \equiv u \text{ on } [-T,T]\} \tag{2.1}
\]
Finally, the main function spaces are defined as
\[
F^{\sigma,b}(T) := \left\{ u \in \mathcal{C}([-T, T], \mathbb{E}^\sigma), \ ||u||_{F^{\sigma,b}(T)} := \left( \sum_{M \geq 1} M^4 ||P_M u||_{F^{\sigma,b}(T)}^2 \right)^{1/2} < +\infty \right\} \tag{2.2}
\]
and
\[
N^{\sigma}(T) := \left\{ u \in L^2([-T, T], \mathbb{E}^\sigma), \ ||u||_{N^{\sigma}(T)} := \left( \sum_{M \geq 1} M^4 ||P_M u||_{N^{\sigma}(T)}^2 \right)^{1/2} < +\infty \right\} \tag{2.3}
\]
The last space is the energy-type space which is the analogous in this context of the usual space \( L^\infty([-T; T], \mathbb{E}^\sigma) \):
\[
B^{\sigma}(T) := \left\{ u \in \mathcal{C}([-T, T], \mathbb{E}^\sigma), \ ||u||_{B^{\sigma}(T)} := \left( \sum_{M \geq 1} \sup_{t_M \in [-T,T]} ||P_M u(t_M)||_{\mathbb{E}^\sigma}^2 \right)^{1/2} < +\infty \right\} \tag{2.4}
\]
Again, for \( F^{b} \) and \( F^{\sigma,b}(T) \), if \( b = 1/2 \) we just drop it. We do the same for \( \sigma = 2 \).

For the difference equation, we use similar spaces \( \overline{F}_M \), \( \overline{N}_M \) and \( \overline{F}(T), \overline{N}(T) \) and \( \overline{B}(T) \) which are the same as the above spaces but without the weight \( p \) and at regularity \( \sigma = 0 \). Let us notice that in view of the definition of \( p \) we then have
\[
||u||_{F^{\sigma}(T)} \sim ||u||_{F^{\sigma,b}(T)} + M^{-4} \left| |\partial_x^{-1}\partial_y u|_{F^{\sigma,b}(T)} \right|
\]

### 2.2 Basic properties

We collect here some basic properties of the spaces \( X_M \), \( F(T) \) and \( N(T) \). The proof of these results can be found e.g in [7, 10, 16].

First, for any \( f_M \in X_M \), we have
\[
||f_M||_{X_M} \lesssim ||f_M||_{L^1_T}
\]
\[
||f_M||_{F^{\sigma,b}_M} \lesssim ||f_M||_{X_M} \tag{2.5}
\]
Moreover, if we take $\gamma \in L^2(\mathbb{R})$ satisfying
\[
|\hat{\gamma}(\tau)| \lesssim \langle \tau \rangle^{-4}
\] (2.6)
then for any $K_0 \geq 1$ and $t_0 \in \mathbb{R}$ we have
\[
K_0^{1/2} \left\| \chi_{K_0}(\tau + \omega) \mathcal{F} \{ \gamma(K_0(t - t_0)) \mathcal{F}^{-1} f_M \} \right\|_{L^2} + \sum_{K \geq K_0} K^{1/2} \left\| \rho_K(\tau + \omega) \mathcal{F} \{ \gamma(K_0(t - t_0)) \mathcal{F}^{-1} f_M \} \right\|_{L^2} \lesssim \| f_M \|_{X_M} \tag{2.7}
\]
and the implicit constants are independent of $M$, $K_0$ and $t_0$.

For general time multipliers $m_M \in C^4(\mathbb{R})$ bounded along with its derivatives, as in [7] we have the bounds
\[
\left\| m_M(t) f_M \right\|_{F_M} \lesssim \left( \sum_{k=0}^{4} (1 \vee M)^{-k} \left\| m_M^{(k)} \right\|_{L^\infty} \right) \| f_M \|_{F_M} \tag{2.8}
\]
and
\[
\left\| m_M(t) f_M \right\|_{N_M} \lesssim \left( \sum_{k=0}^{4} (1 \vee M)^{-k} \left\| m_M^{(k)} \right\|_{L^\infty} \right) \| f_M \|_{N_M^{\text{b},k_1}} \tag{2.9}
\]
We will also use [4, Lemma 3.4] to get a factor $T^{0+}$ in the estimates in order to avoid rescaling:

**Lemma 2.1.** Let $T \in [0; 1]$ and $0 \leq b < 1/2$. Then, for any $u \in F_M(T)$,
\[
\left\| u \right\|_{F_M^{b}(T)} \lesssim T^{(1/2-b)-} \left\| u \right\|_{F_M(T)} \tag{2.10}
\]
and the implicit constant is independent of $M$ and $T$.

The last estimate justifies the use of $\mathbf{F}(T)$ as a resolution space:

**Lemma 2.2.** Let $\sigma \geq 2$, $T \in [0; 1]$ and $u \in \mathbf{F}^\sigma(T)$. Then
\[
\left\| u \right\|_{L^\infty_T \mathbf{F}^\sigma} \lesssim \left\| u \right\|_{\mathbf{F}^\sigma(T)} \tag{2.11}
\]

### 2.3 Linear estimate
In this last subsection, we recall a linear estimate which replaces the usual estimate in the context of standard Bourgain spaces. The proof is the same as the one of [7, Proposition 3.2].
Proposition 2.3. Let $T > 0$ and $u, f \in C([-T,T], E^\infty)$ satisfying
\begin{equation}
\partial_t u - \partial_x^5 u - \partial_x^{-1} \partial_y^2 u = f
\end{equation}
on $[-T,T] \times T^2$.
Then for any $\sigma \geq 2$, we have
\begin{equation}
||u||_{F^\sigma(T)} \lesssim ||u||_{B^\sigma(T)} + ||f||_{N^\sigma(T)}
\end{equation}
and
\begin{equation}
||u||_{F(T)} \lesssim ||u||_{B(T)} + ||f||_{N(T)}
\end{equation}

3 Dyadic estimates

We prove here several estimates on the trilinear form $\int_R \sum_{Z^2} f_1 \ast f_2 \cdot f_3$ which replace [7, Corollary 5.3] in our context.

For the proof of the following easy lemmas, we refer to [16, Section 3].

Lemma 3.1. Let $f_i \in L^2(\mathbb{R} \times Z^2)$ be such that $\text{supp} f_i \subset D_{M_i, \infty K_i} \cap \mathbb{R} \times Z \times I_i$, with $M_i, K_i \in 2^N$ and $I_i \subset Z$, $i = 1, 2, 3$. Then
\begin{equation}
\int_R \sum_{Z^2} f_1 \ast f_2 \cdot f_3 \lesssim M_{\text{min}}^{1/2} K_{\text{min}}^{1/2} (\# I_{\text{min}})^{1/2} \prod_{i=1}^3 ||f_i||_{L^2}
\end{equation}

Lemma 3.2. Let $\Lambda \subset Z^2$. We assume that the projection of $\Lambda$ on the $m$ axis is contained in an interval $I \subset Z$. Moreover, we assume that the cardinal of the $n$-sections of $\Lambda$ (that is the sets $\{n \in Z, (m_0, n) \in \Lambda\}$ for a fixed $m_0$) is uniformly (in $m_0$) bounded by a constant $C$. Then we have
\begin{equation}
|\Lambda| \leq C\langle |I| \rangle
\end{equation}
Lemma 3.3. Let $I$, $J$ be two intervals in $\mathbb{R}$, and let $\varphi : I \rightarrow \mathbb{R}$ be a $C^1$ function with $\inf_{x \in J} |\varphi'(x)| > 0$. Assume that $\{n \in J \cap \mathbb{Z}, \varphi(n) \in I\} \neq \emptyset$. Then
\[
\# \{n \in J \cap \mathbb{Z}, \varphi(n) \in I\} \lesssim \langle \frac{|I|}{\inf_{x \in J} |\varphi'(x)|} \rangle
\] (3.2)

Lemma 3.4. Let $a \neq 0, b, c$ be real numbers and $I \subset \mathbb{R}$ a bounded interval. Then
\[
\# \{n \in \mathbb{Z}, an^2 + bn + c \in I\} \lesssim \langle \frac{|I|^{1/2}}{|a|^{1/2}} \rangle
\] (3.3)

The main estimates of this section are the following:

Proposition 3.5. Let $M_i, K_i \in 2^{\mathbb{N}}$, $i = 1, 2, 3$, and take $u_1, u_2 \in L^2(\mathbb{R} \times \mathbb{Z}^2)$ be such that $\text{supp}(u_i) \subset D_{M_i, \leq K_i}$. Then
\[
\left\| 1_{D_{M_3, \leq K_3}} \cdot u_1 \ast u_2 \right\|_{L^2} \lesssim \left( K_1 \wedge K_2 \right)^{1/2} M_{\text{min}}^{1/2} \cdot \left\langle \left( K_1 \vee K_2 \right)^{1/4} (M_1 \wedge M_2)^{1/4} \right\rangle \|u_1\|_{L^2} \|u_2\|_{L^2}
\] (3.4)

Moreover, if we are in the case $K_{\text{max}} \leq 10^{-10} M_1 M_2 M_3$, then
\[
\left\| 1_{D_{M_3, \leq K_3}} \cdot u_1 \ast u_2 \right\|_{L^2} \lesssim \left( K_1 \wedge K_2 \right)^{1/2} M_{\text{min}}^{1/2} \cdot \left\langle \left( K_1 \vee K_2 \right)^{1/4} (M_3 M_{\text{max}})^{-1/2} \right\rangle \|u_1\|_{L^2} \|u_2\|_{L^2}
\] (3.5)

Proof:
These estimates are the analogous of those proved in [19] Subsections 2.1 & 2.2, in the context of the bilinear estimate in standard Bourgain spaces. The proof is very similar to that one of [16] Proposition 5.5. First, we split $u_1$ and $u_2$ depending on the value of $m_i$ on an $M_3$ scale, meaning
\[
\left\| 1_{D_{M_3, \leq K_3}} \cdot u_1 \ast u_2 \right\|_{L^2} \leq \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left\| 1_{D_{M_3, \leq K_3}} \cdot u_{1,k} \ast u_{2,j} \right\|_{L^2}
\] (3.6)

with
\[u_{i,j} := 1_{[j M_3, (j+1) M_3]}(m_i) u_i\]

The conditions $|m| \sim M_3$, $m_1 \in [k M_3, (k+1) M_3]$ and $m - m_1 \in [j M_3; (j+1) M_3]$ require $j \in [-k - c; -k + c]$ for an absolute constant $c > 0$.

Squaring the norm in the right-hand side of (3.6), it suffices to evaluate
\[
\int_{\mathbb{R}} \sum_{(m,n) \in \mathbb{Z}^2} \left| 1_{D_{M_3, \leq K_3}}(\tau, m, n) \right|^2 \int_{\mathbb{R}} \sum_{(m_1,n_1) \in \mathbb{Z}^2} u_{1,k}(\tau_1, m_1, n_1) \cdot u_{2,j}(\tau - \tau_1, m - m_1, n - n_1) d\tau_1^2 d\tau
\]
Using Cauchy-Schwarz inequality, the integral above is controlled by
\[ \sup_{(\tau, m, n) \in \mathcal{D}, k} |A_{\tau, m, n}| \cdot ||u_{1,k}||_L^2 \cdot ||u_{2,j}||_L^2 \]
where \( A_{\tau, m, n} \) is defined as
\[ A_{\tau, m, n} = \{ (\tau_1, m_1, n_1) \in \mathbb{R} \times \mathbb{Z}^2, m_1 \in I_k, m - m_1 \in I_j, \langle \tau_1 - \omega(m_1, n_1) \rangle \lesssim K_1, \langle \tau - \tau_1 - \omega(m - m_1, n - n_1) \rangle \lesssim K_2 \} \]
with the intervals
\[ I_k = I_{M_1} \cap [kM_3; (k + 1)M_3] \quad \text{and} \quad I_j = I_{M_2} \cap [jM_3; (j + 1)M_3] \]
Using the triangle inequality in \( \tau_1 \), we get the bound
\[ |A_{\tau, m, n}| \lesssim (K_1 \wedge K_2) |B_{\tau, m, n}| \]
where \( B_{\tau, m, n} \) is defined as
\[ B_{\tau, m, n} = \{ (m_1, n_1) \in \mathbb{Z}^2, m_1 \in I_k, m - m_1 \in I_j, \langle \tau + \omega(m, n) - \Omega(m_1, n_1, m - m_1, n - n_1) \rangle \lesssim (K_1 \vee K_2) \} \]
and the resonant function \( \Omega \) is defined as
\begin{align*}
\Omega(m_1, n_1, m_2, n_2) &= \omega(m_1, n_1) + \omega(m_2, n_2) - \omega(m_1 + m_2, n_1 + n_2) \\
&= 5m_1m_2(m_1 + m_2)\alpha(m_1, m_2) - \frac{(m_1n_2 - m_2n_1)^2}{m_1m_2(m_1 + m_2)} \\
&= \frac{5m_1m_2(m_1 + m_2)\alpha(m_1, m_2)}{m_1 + m_2} - \frac{m_1m_2}{m_1 + m_2} \left( \frac{n_1}{m_1} - \frac{n_2}{m_2} \right)^2 (3.7)
\end{align*}
with
\[ \alpha(m_1, m_2) = m_1^2 + m_1m_2 + m_2^2 \sim M_{max}^2 \]
First, in the case \( K_{max} \lesssim 10^{-10}M_1M_2M_3M_{max}^2 \), we estimate \( |B_{\tau, m, n}| \) with the help of Lemma 3.2 and 3.3. Indeed, its projection on the \( m_1 \) axis is controlled by \( |I_k| \wedge |I_j| \). Now, we compute
\[ \left| \frac{\partial \Omega}{\partial m_1} \right| = 2 \left| \frac{n_1}{m_1} - \frac{n - n_1}{m - m_1} \right| = 2 \left| \frac{m}{m_1(m - m_1)} (5m_1(m - m_1)\alpha(m_1, m - m_1) - \Omega) \right|^{1/2} \]
Thus, from the condition \( |\Omega| \lesssim K_{max} \lesssim 10^{-10}M_1M_2M_3M_{max}^2 \) we get
\[ \left| \frac{\partial \Omega}{\partial m_1} \right| \gtrsim \left| \frac{m}{m_1(m - m_1)} \cdot m_1(m - m_1)\alpha(m_1, m - m_1) \right|^{1/2} \sim M_3M_{max} \]
So we can estimate $|B_{\tau,m,n}|$ in this regime by

$$|B_{\tau,m,n}| \lesssim \langle |I_k| \wedge |I_j| \rangle \langle (K_1 \vee K_2)(M_3 M_{\max})^{-1} \rangle$$

For (3.4), note that we can neglect the localization $1_{D_{M_3,\epsilon K_3}}$, thus we can use the argument of [19, Lemma 4] and assume that $m_3 \geq 0$ on the support of $u_i$. To get a bound for $|B_{\tau,m,n}|$, we now use Lemma 3.4 instead of Lemma 3.3. Indeed, we can write

$$\tau - \omega(m,n) - \Omega(m_1, n_1, m-m_1, n-n_1)$$

which is a parabola in $n_1$ with leading coefficient

$$\left| \frac{m}{m_1(m-m_1)} \right| = \frac{1}{m_1} + \frac{1}{m-m_1} \geq \frac{1}{m_1 \wedge (m-m_1)}$$

Thus for a fixed $m_1$, the cardinal of the $n_1$-section is estimated by

$$\left( (K_1 \vee K_2)^{1/2}(M_1 \wedge M_2)^{1/2} \right)$$

thanks to (3.3). So we get the final bound

$$|B_{\tau,m,n}| \lesssim \langle |I_k| \wedge |I_j| \rangle \langle (K_1 \vee K_2)^{1/2}M_{\min}^{1/2} \rangle$$

These bounds for $|A_{\tau,m,n}|$ finally give (3.4) and (3.5) by using Cauchy-Schwarz inequality to sum over $k \in \mathbb{Z}$, since $|I_k| \lesssim M_1 \wedge M_3$ and $|I_j| \lesssim M_2 \wedge M_3$.

\[ \□ \]

**Remark 3.6.** In the context of standard Bourgain spaces, we cannot recover some derivatives in the regime $K_{\max} < M_3 M_{\max}$ since

$$\langle (K_1 \vee K_2)^{1/2}(M_3 M_{\max})^{-1/2} \rangle = 1$$

in that case. This is the main reason for the bilinear estimate to fail in [19, Section 5] and for our choice of time localization on intervals of size $M_{\text{max}}^{-2}$.

**Remark 3.7.** Estimate (3.5) may seem rough, but a more careful analysis of the dyadic bilinear estimates in the resonant case (that is, the analogous of [3, Lemma 3.1 (a)] for periodic functions) in the spirit of [22, Lemma 3.1] leads to the bound

$$(K_1 K_3)^{1/2} M_{\max}^{-1} \cdot \left\{ \left( \frac{K_2}{(M_1 \wedge M_2)M_{\max}} \right)^{1/2} \wedge \left( M_1 \wedge M_2 \frac{K_2}{(M_{\min} M_{\max})^{3/2}} \right)^{1/2} \right\}$$

showing that, in the case $K_2 = K_{\text{med}} \lesssim M_{\min} M_{\max}^3$ and $M_1 \wedge M_2 = M_{\min}$, (3.6) is actually optimal. Comparing with [3, Lemma 3.1], we see why there is such a gap in regularity between the well-posedness in $\mathbb{R}^2$ and $\mathbb{T}^2$. 

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Corollary 3.8. Assume $M_1, M_2, M_3, K_1, K_2, K_3 \in 2^b$ with $K_i \geq M_i^2$ and $f_i \in L^2(\mathbb{R} \times \mathbb{Z}^2)$ are positive functions with the support condition $\text{supp} f_i \subset D_{M_i, K_i}$, $i = 1, 2$. Then

$$\left\| \mathbb{I}_{D_{M_3, K_3}} \cdot f_1 \ast f_2 \right\|_{L^2} \lesssim M_{\min}^{1/2} M_{\max}^{-1/2} (K_{\min} K_{\max})^{1/2} K_{\med}^{1/2} \|f_1\|_{L^2} \|f_2\|_{L^2}$$

(3.8)

for any $b \in [1/4; 1/2]$, and

$$\left\| \mathbb{I}_{D_{M_3, K_3}} \cdot f_1 \ast f_2 \right\|_{L^2} \lesssim M_1^{3/2} M_{\min}^{1/2} K_{\min}^{-1/2} \|p \cdot f_1\|_{L^2} \|f_2\|_{L^2}$$

(3.9)

Proof:

(3.8) follows directly from (3.4) and (3.5) above.

For the proof of (3.9), we follow [7, Lemma 5.3]: we split

$$f_1 = \sum_{N \geq M_1^3} f_{1,N} = \mathbb{I}_{< M_1^3} (n) f_1 + \sum_{N > M_1^3} \mathbb{I}_N (n) f_1$$

such that

$$\left\| \mathbb{I}_{D_{M_3, K_3}} \cdot f_1 \ast f_2 \right\|_{L^2} \lesssim \sum_{N \geq M_1^3} N^{1/2} M_{\min}^{1/2} K_{\min}^{1/2} \|f_{1,N}\|_{L^2} \|f_2\|_{L^2}$$

after using (3.4).

Thus, using Cauchy-Schwarz inequality in $N$, we obtain

$$\left\| \mathbb{I}_{D_{M_3, K_3}} \cdot f_1 \ast f_2 \right\|_{L^2} \lesssim M_{\min}^{1/2} K_{\min}^{1/2} \|f_2\|_{L^2} \sum_{N \geq M_1^3} N^{-1/2} M_1^3 \|p \cdot f_{1,N}\|_{L^2}$$

$$\lesssim M_1^{3/2} M_{\min}^{1/2} K_{\min}^{-1/2} \|p \cdot f_1\|_{L^2} \|f_2\|_{L^2}$$

□

4 Energy estimates

In this section, we prove the energy estimates which allow to control the $B$-norm of regular solutions and the $\mathbf{B}$-norm of the difference of solutions.

Lemma 4.1. There exists $\mu_0 > 0$ small enough such that for $T \in [0; 1]$ and $u_i \in F_{M_i}(T)$, $i \in \{1, 2, 3\}$, with one of them in $F_{M_i}(T)$, then

$$\left| \int_{[0,T] \times \mathbb{T}^2} u_1 u_2 u_3 dtdx \right| \lesssim T^{\mu_0} M_{\min}^{1/2} \prod_{i=1}^3 \|u_i\|_{F_{M_i}(T)}$$

(4.1)
If moreover \( M_1 \leq M/16 \), and \( u \in \overline{F_M(T)}, \ v \in F_M(T) \), we have

\[
\left| \int_{[0,T] \times \mathbb{T}^2} P_M u \cdot P_M (P_M v \cdot \partial_x u) dt dx dy \right| 
\lesssim T^{\mu_0} M_1^{3/2} \| P_M v \|_{F_M(T)} \sum_{M_2 \sim M} \| P_M u \|^2_{F_M(T)} \tag{4.2}
\]

\textbf{Proof :}

From symmetry, we may assume \( M_1 \leq M_2 \leq M_3 \). Let \( \tilde{u}_i \in F_M \) be extensions \( u_i \) to \( \mathbb{R} \), satisfying \( \| \tilde{u}_i \|_{F_M} \leq 2 \| u_i \|_{F_M(T)} \).

Let \( \gamma \in C_c^\infty(\mathbb{R}) \) be such that \( \gamma : \mathbb{R} \to [0; 1] \) with \( \text{supp} \gamma \subset [-1; 1] \) and satisfying

\( \forall t \in \mathbb{R}, \sum_{\nu \in \mathcal{Z}} \gamma(t - \nu) = 1 \)

Then

\[
\int_{[0,T] \times \mathbb{T}^2} u_1 u_2 u_3 dt dx dy 
\lesssim \sum_{|\nu| \leq M_{\text{max}}^2} \sum_{K_1, K_2, \gamma \geq M_{\text{max}}^2} \int_{\mathbb{R} \times \mathbb{T}^2} \left( \rho K_3 (\tau + \omega) F \{ \mathbb{1}_{[0,T]} \gamma (M_{\text{max}}^2 t - \nu) \tilde{u}_3 \} \right) 
\cdot \left( \rho K_1 (\tau + \omega) F \{ \mathbb{1}_{[0,T]} \gamma (M_{\text{max}}^2 t - \nu) \tilde{u}_1 \} \right) 
\ast \left( \rho K_2 (\tau + \omega) F \{ \mathbb{1}_{[0,T]} \gamma (M_{\text{max}}^2 t - \nu) \tilde{u}_2 \} \right) d\tau
\]

where there are at most \( TM_{\text{max}}^2 \) interior terms \( \nu \) for which \( \mathbb{1}_{[0,T]} \gamma (M_{\text{max}}^2 t - \nu) = \gamma (M_{\text{max}}^2 t - \nu) \), and at most 4 remaining border terms where the integral is non-zero. The property of \( X_M \) allows us to partition the modulations at \( K_i \geq M_{\text{max}}^2 \).

Let us now observe that, using (2.7), for the interior terms we have

\[
\sup_{\nu \in \mathbb{Z}} \sum_{K_i \geq M_{\text{max}}^2} K_i^b \left\| \rho K_i(\tau + \omega) F \{ \gamma (M_{\text{max}}^2 t - \nu) \tilde{u}_i \} \right\|_{L^2} \lesssim \| \tilde{u}_i \|_{F_M(T)}
\]

Thus, since we can take \( \mu_0 = 1 \) for those terms, (4.1) follows from (3.8) with \( b = 1/2 \) and the estimate above.

For the remaining border terms, we use that

\[
\sup_{\nu} \sup_{K_i \geq M_{\text{max}}^2} K_i^{1/2} \left\| \rho K_i(\tau + \omega) \cdot \mathbb{1}_{[0,T]} \ast F \{ \gamma (M_{\text{max}}^2 t - \nu) \tilde{u}_i \} \right\|_{L^2} \lesssim \| \tilde{u}_i \|_{F_M(T)}
\]

which follows through the same argument as for the proof of (2.7) (see [16]). Thus we can use (3.8) with \( b < 1/2 \) to get (4.1).

(4.2) then follows from the one of (4.1) through the same argument as in Lemma 6.1.
We can now state our global energy estimate.

**Proposition 4.2.** Let \( T \in [0;1[ \) and \( u \in C([-T,T], E^\infty) \) be a solution of \( (1.2) \) on \([-T,T] \). Then for any \( \sigma \geq 2 \),

\[
||u||_{B^\sigma(T)}^2 \lesssim ||u_0||_{E^\sigma}^2 + T^{\mu_0} ||u||_{F(T)}^2 ||u||_{F^\sigma(T)}^2 \tag{4.3}
\]

**Proof:**

From the definition of the \( B^\sigma \) norm and the weight \( p \), we have the first estimate

\[
\sum_{M_3 > 1} \sup_{t \in [-T;T]} \left( M_3^{2\sigma} \ ||P_{M_3}u(t,M_3)||_{L^2} + M_3^{2(\sigma-2)} \ ||\partial_x^{-1}\partial_y P_{M_3}u(t,M_3)||_{L^2} \right)
\]

For the first term within the sum, using that \( u \) is a solution to \( (1.2) \), we have

\[
\sup_{t \in [-T;T]} M_3^{2\sigma} \ ||P_{M_3}u(t,M_3)||_{L^2} \lesssim M_3^{2\sigma} \ ||P_{M_3}u_0||_{L^2}
\]

\[
+ M_3^{2\sigma} \ \left| \int_{[0;T] \times T^2} P_{M_3} u \cdot P_{M_3} \left( u \partial_x u \right) dt dx dy \right|
\]

We can divide the previous integral term into

\[
\sum_{M_3 \leq M_3/16} M_3^{2\sigma} \left| \int_{[0;T] \times T^2} P_{M_3} u \cdot P_{M_3} \left( P_{M_3} u \cdot \partial_x u \right) dt dx dy \right|
\]

\[
+ \sum_{M_1, M_2, M_3 \geq 1} M_3^{2\sigma} \left| \int_{[0;T] \times T^2} P_{M_3}^2 u \cdot P_{M_1} u \cdot \partial_x (P_{M_2} u) dt dx dy \right|
\]

Using \( (4.2) \) for the first one and \( (4.1) \) for the second one, we get the bound

\[
\sum_{M_3 \leq M_3/16} M_3^{2\sigma} M_1^{3/2} \ ||P_{M_1} u||_{\overline{F_{M_1}(T)}} \ ||P_{M_3} u||_{\overline{F_{M_3}(T)}}^2
\]

\[
+ \sum_{M_1, M_3 \geq 1} M_1^{2\sigma} M_2 (M_2 \land M_3)^{1/2} \cdot ||P_{M_1} u||_{\overline{F_{M_1}(T)}} ||P_{M_2} u||_{\overline{F_{M_2}(T)}} ||P_{M_3} u||_{\overline{F_{M_3}(T)}}
\]

For the sum on the first line, we use Cauchy-Schwarz inequality to sum on \( M_1 \) (as we have \( 1/2 \) derivative to spare) and then sum on \( M_3 \) by writing \( M_2 = 2^k M_3 \) with \( k \in \mathbb{Z} \) bounded and then use of Cauchy-Schwarz inequality in \( M_3 \).

For the second line, we cut the sum into two parts \( M_2 \geq M_3 \sim M_1 \) and \( M_2 \sim M_1 \geq M_3 \), put \( 2\sigma \) derivatives on the highest frequency, and then use
Cauchy-Schwarz again to sum on the lowest frequency (we have again 1/2 extra derivative) and then the biggest. Thus the term above is bounded by the right-hand side of (4.3).

It remains to treat the sum with the antiderivative. Proceeding similarly and writing \( v := \partial_x^{-1} \partial_y u \), we get

\[
\sup_{t, M_3 \in [-T; T]} M_3^{2(\sigma - 2)} \left\| \partial_x^{-1} \partial_y P_{M_3} u(t, M_3) \right\|_{L^2} \lesssim M_3^{2(\sigma - 2)} \left\| \partial_x^{-1} \partial_y P_{M_3} u_0 \right\|_{L^2} + M_3^{2(\sigma - 2)} \left| \int_{[0; T] \times \mathbb{T}^2} P_{M_3} v \cdot P_{M_3} (u \partial_x v) \, dt \, dx \, dy \right|
\]

which is analogously dominated by

\[
\sum_{M_1 \leq M_3/16} M_3^{2(\sigma - 2)} M_1^{3/2} \left\| P_{M_1} u \right\|_{F_{M_1}(T)} \sum_{M_2 \sim M_3} \left\| P_{M_2} v \right\|_{F_{M_2}(T)}^2 + \sum_{M_1 \gtrsim M_3} \sum_{M_2 \sim 1} M_3^{2(\sigma - 2)} M_2 (M_2 \wedge M_3)^{1/2} \cdot \left\| P_{M_1} u \right\|_{F_{M_1}(T)} \left\| P_{M_2} v \right\|_{F_{M_2}(T)} \left\| P_{M_3} v \right\|_{F_{M_3}(T)}
\]

For the first line, we run the summation over \( M_1, M_2, M_3 \) as before, whereas for the second line, we split the highest frequency into \( M_2^2 (M_2 \vee M_3)^{2(\sigma - 3)} (M_2 \wedge M_3)^{3/2} \) and then perform the summation as above.

Remark 4.3. In the dyadic summations above, we see that we are 1/2-derivative below the energy space, thus a simple adaptation of our argument would actually yield local well-posedness in \( H^{s_1, s_2} (\mathbb{T}^2) \) with \( s_1 > 3/2, s_2 \geq 0 \). For our result to be more readable, we chose not to present these technical details here.

Remark 4.4. Even with the local well-posedness result mentioned above, our result is in sharp contrast with the local well-posedness of [3] in the case of \( \mathbb{R}^2 \). This highlights the quasilinear behaviour of equation (4.3) in the periodic setting. From the technical point of view, the \( X_M \) structure is used in [3] on time intervals on size \( M^{-1} \), whereas in our case, the use of the counting measure instead of the Lebesgue measure in the localized bilinear Strichartz estimates requires us to work on time intervals of size \( M^{-2} \) which explains the gap in regularity between these results.

To deal with the difference of solutions, we also prove the following proposition.

Proposition 4.5. Assume \( T \in [0; 1] \) and \( u, v \in \mathcal{C}([-T, T], E^\infty) \) are solutions to (4.3) on \([-T, T]\) with initial data \( u_0, v_0 \in E^\infty \). Then

\[
||u - v||_{B(T)}^2 \lesssim ||u_0 - v_0||_{L^2}^2 + T^{\mu_0} ||u + v||_{F(T)} ||u - v||_{F(T)}^2 \tag{4.4}
\]

and

\[
||u - v||_{B(T)}^2 \lesssim ||u_0 - v_0||_{E}^2 + T^{\mu_0} ||v||_{F^3(T)} ||u - v||_{F^3(T)}^2 \tag{4.5}
\]
Proof:

We proceed as in the previous proposition, except that now \( w := u - v \) solves the equation

\[
\begin{aligned}
&\partial_t w - \partial_x^2 w - \partial_x^{-1}\partial_y^2 w + \partial_x \left( w \frac{u + v}{2} \right) = 0 \\
&w(t = 0) = u_0 - v_0
\end{aligned}
\] (4.6)

For (4.4), we write

\[
||u - v||_B^2(T) = \sum_{M_3 \geq 1, \mu_3 \in \mathbb{R}} \sup_{t \in [0,T]} ||P_{M_3}(u - v)(t,M_3)||_{L^2}^2 \lesssim \sum_{M_3 \geq 1} \left\{ ||P_{M_3}(u_0 - v_0)||_{L^2}^2 \right\}
\]

\[
+ \int_{[0,T] \times T^2} P_{M_3} w \cdot P_{M_3} \left( w \partial_x w + w \partial_x v + v \partial_x w \right) \, dt \, dx \, dy
\]

The first integral term with \( w \partial_x w \) can be estimated by \( ||w||_{B(T)} ||w||_F^2 \) the exact same way as the first term in the previous proposition with \( \sigma = 0 \).

As in (4.2), for the other two terms, we use again (4.1) and (4.2) to bound them with

\[
T^{\nu_0} \left\{ \sum_{M_3 \geq 1} \sum_{M_1 \lesssim M_3} \left( M_2 M_1^{1/2} \Pi_1 + M_1^{3/2} \Pi_2 \right) \\
+ \sum_{M_3 \geq 1} \sum_{M_1 \lesssim M_3} \sum_{M_2 \sim M_1} M_2 M_1^{3/2} (\Pi_1 + \Pi_2) \\
+ \sum_{M_3 \geq 1} \sum_{M_1 \sim M_3} \sum_{M_2 \sim M_3} M_2^{3/2} (\Pi_1 + \Pi_2) \right\}
\]

where we have noted

\[
\Pi_1 := ||w||_{F_{M_3}} ||w||_{F_{M_1}} ||v||_{F_{M_2}} \quad \text{and} \quad \Pi_2 := ||w||_{F_{M_3}} ||v||_{F_{M_1}} ||w||_{F_{M_2}}
\]

Observe that, as for Proposition (4.2) above, we have 1/2 derivative to spare. Moreover, using the relation between the \( M_i \)'s, we can always place all \((3/2)\)derivatives on the term containing \( v \), thus we can sum by using Cauchy-Schwarz to bound all these terms with the right-hand side of (4.4).

By the same token as for (4.3), we can estimate the left-hand side of (4.5) by \( ||u_0 - v_0||_E^2 \) plus two integral terms

\[
\sum_{M_3 \geq 1} M_3^4 \left| \int_{[0,T] \times T^2} P_{M_3} w \cdot P_{M_3} \left( w \partial_x w + w \partial_x v + v \partial_x w \right) \, dt \, dx \, dy \right| \quad (4.7)
\]
and

\[
\sum_{M_3 \neq 1} \left| \int_{[0;T] \times T^2} P_{M_3} W \cdot P_{M_3} \left( w \partial_x W + w \partial_y V + v \partial_x W \right) \ dt \ dx \ dy \right| \tag{4.8}
\]

where in the latter \( W := \partial_x^{-1} \partial_y w \) and \( V := \partial_x^{-1} \partial_y v \).

For (4.7), we proceed exactly as previously. Again, the first integral term has already been treated in the proof of (4.3). For the other terms, now we have 11/2 derivatives to distribute, and using again the relation between the \( M_i \)'s we can place 2 derivatives on each \( w \) and the remaining ones on \( v \) and then run the summations, the worst case being the term \( P_{M_1} w \cdot \partial_x P_{M_2} v \) in the regime \( M_1 \ll M_2 \sim M_3 \) since there are 5 highest derivatives, thus we need to put 3 on \( v \).

It remains to treat (4.8). Once again, the first term within the integral appeared in the proof of the previous proposition, thus we only need to deal with the last two terms. We proceed the same way as above, since there are 3/2 derivatives to share, and \( w \) can absorb 2, \( V \) can absorb 1 and \( v \) can absorb 3. \( \square \)

5 Short-time bilinear estimates

The aim of this section is to prove the bilinear estimates for both the equation and the difference equation. We mainly adapt [7].

**Proposition 5.1.** There exists \( \mu_1 > 0 \) small enough such that for any \( T \in [0; 1] \) and \( \sigma \geq 2 \) and \( u, v \in F^\sigma(T) \),

\[
||\partial_x(uv)||_{N^\sigma(T)} \lesssim T^{\mu_1} \left\{ ||u||_{F^\sigma(T)} \ ||v||_{F(T)} + ||u||_{F(T)} \ ||v||_{F^\sigma(T)} \right\} \tag{5.1}
\]

**Proof:**
Using the definition of \( F^\sigma(T) \) (2.2) and \( N^\sigma(T) \) (2.3), the left-hand side of (5.1) is bounded by

\[
\sum_{M_1, M_2, M_3} M_3^2 \ ||P_{M_1} \partial_x (P_{M_1} u \cdot P_{M_2} v)||_{N_{M_3}(T)}
\]

For \( M_1, M_2 \in 2^{\mathbb{N}} \), let us choose extensions \( u_{M_1} \) and \( v_{M_2} \) of \( P_{M_1} u \) and \( P_{M_2} v \) to \( \mathbb{R} \) satisfying \( ||u_{M_1}||_{F_{M_1}} \leq 2 ||P_{M_1} u||_{F_{M_1}(T)} \) and similarly for \( v_{M_2} \). Since the previous term is symmetrical with respect to \( u \) and \( v \), we can assume \( M_1 \leq M_2 \).

To treat the term above, from the definition of the \( F_M^b \) and \( N_M \) norms, the property of the space \( X_M \) (2.7) and the use of Lemma 2.1 it suffices to show that there exists \( b \in [0; 1/2) \) such that for all \( K_i \geq M_i^2, \ i = 1, 2, 3 \) and
Indeed, for a smooth partition of unity $\gamma : \mathbb{R} \to [0;1]$ satisfying $\text{supp} \gamma \subset [-1;1]$ and for all $t \in \mathbb{R}$

$$\sum_{\nu \in \mathbb{L}} \gamma(t - \nu)^2 = 1$$

then define for $|\nu| \lesssim M_2 \max M_3^{-2}$

$$f_{1,\nu}^{K_i} := \rho_{K_1}(r + \omega) \cdot \mathcal{F} \left\{ \gamma (M_2 \max M_3^{-2} t - \nu) u_{M_1} \right\}$$

with $\rho_{K_1}$ a non-homogeneous dyadic decomposition of unity partitioned at $K_1 = M_2$, and similarly for $f_{2,\nu}^{K_i}$. Then the norm within the sum is bounded by the left-hand side of (5.2) (after taking the supremum over $\nu$), whereas summing on $K_i \geq M_2^2$, $i = 1, 2$, using (2.7) and Lemma 2.1 and summing on $M_1, M_2$ then the right-hand side of (5.2) is controlled by the right-hand side of (5.1) (see e.g. [10] for the full details).

We then separate two cases depending on the relation between the $M_i$’s.

**Case A : Low $\times$ High $\to$ High.**

We assume $M_1 \lesssim M_2 \sim M_3$. In that case, for (5.2) it is sufficient to prove

$$M_3 \sum_{K_3 \geq M_3^2} K_3^{-1/2} \left\| \mathbb{1}_{D_{M_3 \leq K_3}} \cdot p \cdot f_{1}^{K_1} \ast f_{2}^{K_2} \right\|_{L^2}$$

$$\lesssim \ln(M_{\min}) M_2^{1/2} M_3^{-2b} (K_1 \land K_2)^b (K_1 \lor K_2)^{1/2} \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2}$$

(5.3)

for a $b \in \left[1/4; 1/2\right)$. Since $K_i \geq M_2^2$, $i = 1, 2$, then for the large modulations, we combine (5.3) for both $f_1$ and $f_2$ with the obvious bound

$$p(m_1 + m_2, n_1 + n_2) \lesssim M_3^3 M_3^{-3} p(m_1, n_1) + p(m_2, n_2)$$

(5.4)

to bound the sum for $K_3 \geq M_2^2 M_3^2$ by

$$M_1^{1/2} M_3^{-2b} (K_1 \land K_2)^{1/2} (K_1 \lor K_2)^b \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2}$$

for any $b \in \left[0; 1/2\right]$. For the small modulations $M_2^2 \leq K_3 \leq M_3^2 M_3^2$, the sum runs over about $\ln(M_1)$ dyadic integers. Moreover, using the definition of $\Omega$ (3.7), we can replace (5.4) with

$$p(m_1 + m_2, n_1 + n_2) \lesssim p(m_2, n_2) + M_3^{1/2} M_3^{-3} K_3^{1/2}$$

(5.5)
Indeed, this follows from the definition of $\Omega$ which implies

$$\frac{|n_1|}{|m|} \lesssim \frac{|n_2|}{|m_2|} + \left( \frac{|m_1|}{|m_2|} \Omega + 5m_2^2 \alpha(m, m_2) \right)^{1/2}$$

In the case $K_{\text{max}} \leq 10^{-10} M_1 M_2 M_3 M_{\text{max}}^2$, we then use (5.5), use the bound on $K_{\text{max}}$ and then use (3.5) to get the estimate

$$\left\| (D_{\mathcal{L}, < K_3} \cdot p \cdot f_{1}^{K_1} \ast f_{2}^{K_2}) \right\|_{L^2} \lesssim \left\| (D_{\mathcal{L}, < K_3} \cdot (p \cdot f_{1}^{K_1}) \ast (p \cdot f_{2}^{K_2})) \right\|_{L^2} \lesssim (K_{\text{min}} K_{\text{max}})^{1/2} K_{\text{med}} M_{\text{min}}^{1/2} M_{\text{max}}^{1-2b} \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2}$$

for any $b \in [0; 1/2]$. Thus the sum in this regime is estimated with

$$\ln(M_{1}) M_{\text{min}}^{1/2} M_{\text{max}}^{-2b} (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2}$$

which suffices for (5.5).

In the regime $K_{\text{max}} \geq M_1 M_2 M_3 M_{\text{max}}^2$, we apply again (5.5), lose a factor $K_{\text{max}}^{1/2}$ in the first term, and then use (3.4) instead of (3.5) to obtain

$$\left\| (D_{\mathcal{L}, < K_3} \cdot p \cdot f_{1}^{K_1} \ast f_{2}^{K_2}) \right\|_{L^2} \lesssim K_{\text{max}}^{1/2} M_{\text{min}}^{-1/2} M_{\text{max}} \left\| (D_{\mathcal{L}, < K_3} \cdot (p \cdot f_{1}^{K_1}) \ast (p \cdot f_{2}^{K_2})) \right\|_{L^2} \lesssim (K_{\text{min}} K_{\text{max}})^{1/2} K_{\text{med}} M_{\text{max}}^{-5/4} M_{\text{min}}^{-2b} \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2}$$

for any $b \in [1/4; 1/2]$ which is controled by the estimate in the previous regime.

**Case B : High × High → Low.**

We assume now $M_1 \sim M_2 \gtrsim M_3$. (5.5) becomes in this case

$$M_{\text{max}}^2 M_3 \sum_{K_3 \gtrsim M_3^{1/2}} K_{\text{max}}^{-1/2} \left\| (D_{\mathcal{L}, < K_3} \cdot p \cdot f_{1}^{K_1} \ast f_{2}^{K_2}) \right\|_{L^2} \lesssim \ln(M_{\text{max}}) M_{\text{min}}^{-2b} M_{\text{max}}^{-1/2} (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2} \quad (5.6)$$

For the high modulations $K_3 \geq M_{\text{min}}^{-2} M_{\text{max}}^6$, we use (5.4) along with the obvious bound

$$p(m_1 + m_2, n_1 + n_2) \lesssim M_{\text{max}}^{3} M_{\text{min}} (p(m_1, n_1) + p(m_2, n_2)) \quad (5.7)$$

to estimate the left-hand side of (5.6) with

$$M_{\text{max}}^{-1/2} (K_1 \wedge K_2)^{1/2} (K_1 \vee K_2)^b \left\| p \cdot f_{1}^{K_1} \right\|_{L^2} \left\| p \cdot f_{2}^{K_2} \right\|_{L^2}$$

for any $b \in [3/8; 1/2]$. 19
In the regime $M_3^2 \leq K_3 \leq M_3^{-2} M_2^6$ we replace (5.7) with

$$p(m_1 + m_2, n_1 + n_2) \lesssim M_{\min}^{-2} M_{\max}^2 p(m_1, n_1) + M_{\min}^{-5/2} K_{\max}^{1/2}$$

(5.8)

Indeed, this follows from the same argument as for (5.5). Proceeding then as in the previous case, we infer the final bound

$$\ln(M_{\max}) M_{\max}^{3 - 2b} M_{\min}^{-1/2} (K_1 \land K_2)^{1/2} (K_1 \lor K_2)^b \left\| p \cdot f_1^{K_1} \right\|_{L^2} \left\| p \cdot f_2^{K_2} \right\|_{L^2}$$

The end of this section is devoted to the short-time bilinear estimate for the difference equation.

**Proposition 5.2.** There exists $\mu_2 > 0$ small enough such that for any $T \in ]0; 1]$ and $u \in \mathcal{F}(T)$, $v \in \mathcal{F}(T)$,

$$\left\| \partial_x (uv) \right\|_{\mathcal{N}(T)} \lesssim T^{\mu_2} \left\| u \right\|_{\mathcal{F}(T)} \left\| v \right\|_{\mathcal{F}(T)}$$

(5.9)

**Proof:**

Similarly to (5.2), now it suffices to prove

$$M_{\max}^{-2} M_{\min}^{-1} \left\| 1_{D_{M_3, K_3}} \ast f_1^{K_1} \ast f_2^{K_2} \right\|_{L^2}$$

$$\lesssim M_2^3 (K_1 \land K_2)^b (K_1 \lor K_2)^{1/2} \left\| f_1^{K_1} \right\|_{L^2} \left\| p \cdot f_2^{K_2} \right\|_{L^2}$$

(5.10)

We proceed as above, except that now $u$ and $v$ do not play a symmetric role anymore, thus we have to separate three cases. (5.10) then follows directly from (3.8) in the cases $High \times High \to Low$ and $Low \times High \to High$ and from (3.9) in the case $High \times Low \to High$.

□

6 Proof of Theorem 1.1

We finally turn to the proof of our main result.

The starting point is the local well-posedness result for smooth data of Iòrio and Nunes.

**Theorem 6.1 (E).** Assume $u_0 \in E^\infty$. Then there exists $T = T(||u_0||_{E^\infty}) \in ]0; 1]$ and a unique solution $u \in C([-T; T], E^\infty)$ of (1.2) on $[-T; T] \times \mathbb{T}^2$. 

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6.1 Global well-posedness for smooth data

In view of this result and of the conservation of the energy, for Theorem 1.1 (a) it remains to prove (1.7), which will follow from the following proposition along with (2.11).

Proposition 6.2. Let \( \sigma \geq 2 \). For any \( R > 0 \), there exists a positive \( T = T(R) \sim R^{-1/(\mu_0 + \mu_1)} \) such that for any \( u_0 \in E^\infty \) with \( \|u_0\|_E \leq R \), the corresponding solution \( u \in \mathcal{C}([-T; T], E^\infty) \) satisfies

\[
\|u\|_{F^\sigma(T)} \leq C_\sigma \|u_0\|_E
\]

Proof: We fix \( \sigma \geq 2 \) and \( R > 0 \) and take \( u_0 \) as in the proposition.

Let \( T = T([\|u_0\|_{E^1}) \in [0; 1] \) and \( u \in \mathcal{C}([-T; T], E^\infty) \) be the solution to (1.2) given by Theorem 6.1. Then, for \( T' \in [0; T] \), we define

\[
\mathcal{X}_\sigma(T') := \|u\|_{B^\sigma(T')} + \|u\|_{N^\nu(T')} + \|u\|_{F^\sigma(T')}
\]

In order to perform our continuity argument, we will use the following lemma, whose proof is a straightforward adaptation of [16, Lemma 8.3].

Lemma 6.3. Let \( u \in \mathcal{C}([-T; T], E^\infty) \), \( \sigma \geq 2 \) and \( T \in [0; 1] \). Then, \( \mathcal{X}_\sigma : [0; T] \to \mathbb{R} \) defined above is continuous and nondecreasing, and furthermore

\[
\lim_{T \to 0} \mathcal{X}_\sigma(T') \leq \|u_0\|_E
\]

Recalling (2.13), (4.3), (5.1) for \( \sigma \geq 2 \), we then get

\[
\begin{aligned}
\|u\|_{F^\sigma(T)} &\lesssim \|u\|_{B^\sigma(T)} + \|u\|_{N^\nu(T)} + \|u\|_{F^\sigma(T)} \\
\|u\|_{B^\sigma(T)} &\lesssim \|u_0\|_{E} + T^{\mu_0} \|u\|_{F(T)} + \|u\|_{F^\sigma(T)} \\
\|\partial_x(uv)\|_{N^\nu(T)} &\lesssim T^{\nu_1} \left( \|u\|_{F^\sigma(T)} \|v\|_{F(T)} + \|u\|_{F^\sigma(T)} \|v\|_{F^\sigma(T)} \right)
\end{aligned}
\]

Thus, combining those estimates first with \( \sigma = 2 \), we deduce that

\[
\mathcal{X}_2(T)^2 \leq c_1 \|u_0\|_E^2 + c_2 T^{\mu_0} \mathcal{X}_2(T)^3 + c_3 T^{2\mu_1} \mathcal{X}_2(T)^4
\]

for \( T \in [0; 1] \). Let us set \( R := c_1^{1/2} \|u_0\|_E \). Then we choose \( T_0 = T_0(R) \in [0; 1] \) small enough such that

\[
c_2 T_0^{\mu_0} (2R) + c_3 T_0^{2\mu_1} (2R)^2 < 1/2
\]

Thus, using Lemma 6.3 above and a continuity argument, we get that

\[
\mathcal{X}_2(T) \leq 2R \text{ for } T \leq T_0
\]

Using then (2.13), we deduce that

\[
\|u\|_{F(T)} \leq \|u_0\|_E
\]
for $T \leq T_0$.

Using again (2.13)-(4.3)-(5.1) for $\sigma \geq 3$ along with (6.5), we then obtain

$$
\begin{align*}
\|u\|_{B^r(T)} & \lesssim \|u\|_{B^r(T)} + \|f\|_{N^r(T)}, \\
\|u\|_{B^r(T)} & \lesssim \|u_0\|_{E^r} + T^{\mu_0} \|u_0\|_E \|u\|_{F^r(T)}, \\
\|\partial_x (u^2)\|_{N^r(T)} & \lesssim T^{\mu_1} \|u_0\|_E \|u\|_{F^r(T)}
\end{align*}
$$

We thus infer

$$
\mathcal{X}(T)^2 \lesssim \tilde{c}_1 \|u_0\|_{E^r}^2 + \tilde{c}_2 T^{\mu_0} R \mathcal{X}(T)^2 + \tilde{c}_3 T^{2\mu_1} R^2 \mathcal{X}(T)^2
$$

So, up to choosing $T_0$ even smaller, such that

$$
\tilde{c}_2 T^{\mu_0} R + \tilde{c}_3 T^{2\mu_1} R^2 < 1/2
$$

we finally obtain (6.1).

\[\square\]

### 6.2 Uniqueness

Let $u, v$ be two global solutions of (1.2) with data $u_0, v_0 \in E(\mathbb{T}^2)$, and fix $T_* > 0$.

Using now (2.14)-(4.4)-(5.9), we get that for $T \in [0; T_*]$,

$$
\begin{align*}
\|u - v\|_{F(T)} & \lesssim \|u - v\|_{H(T)} + \left\| \partial_x \left( (u - v) \frac{u + v}{2} \right) \right\|_{N(T)} \\
\|u - v\|_{H(T)} & \lesssim \|u_0 - v_0\|_{L^2} + T^{\mu_0} \|u + v\|_{F(T)} \|u - v\|_{F(T)} \\
\left\| \partial_x \left( (u - v) \frac{u + v}{2} \right) \right\|_{N(T)} & \lesssim T^{\mu_1} \|u + v\|_{F(T)} \|u - v\|_{F(T)}
\end{align*}
$$

(6.6)

Hence we infer

$$
\mathcal{X}(T)^2 \lesssim \|u_0 - v_0\|_{L^2}^2 + T^{\mu_0} \|u + v\|_{F(T_*)} \mathcal{X}(T)^2 + T^{2\mu_1} \|u + v\|_{F(T_*)} \mathcal{X}(T)^2
$$

where, as in the previous subsection,

$$
\mathcal{X}(T) := \|u - v\|_{H(T)} + \left\| \partial_x \left( (u - v) \frac{u + v}{2} \right) \right\|_{N(T)}
$$

Thus, taking $T_0 \in [0; 1]$ small enough such that

$$
T^{\mu_0} \|u + v\|_{F(T_*)} + T^{2\mu_1} \|u + v\|_{F(T_*)} < 1/2
$$

we deduce that $\mathcal{X}(T) \lesssim \|u_0 - v_0\|_{L^2}$ on $[0; T_0]$ which yields $u \equiv v$ on $[-T_0; T_0]$ provided $u_0 = v_0$. Since $T_0$ only depends on $\|u + v\|_{F(T_*)}$, we can then repeat this argument a finite number of time to reach $T_*$.  

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6.3 Existence

Again, in view of the conservation of mass, momentum and energy, it suffices to construct local in time solutions. To this aim we proceed as in [7, Section 4].

Take $R > 0$, and let $u_0 \in \mathbf{E}$ with $\|u_0\|_{\mathbf{E}} \leq R$ and take $(u_{0,j}) \in (\mathbf{E}^\infty)^N$ with $\|u_{0,j}\|_{\mathbf{E}} \leq R$, such that $(u_{0,j})$ converges to $u_0$ in $\mathbf{E}$. Using the same argument as for Theorem [14(a)], it suffices to prove that there exists $T = T(R) > 0$ such that $(\Phi^\infty(u_{0,j}))$ converges to a Cauchy sequence in $C([-T;T], \mathbf{E})$. Indeed, this provides the conservation of the mass, momentum energy for the corresponding limit, which allows us to extend the result to any time $T > 0$.

Let $T = T(R)$ given by Proposition [5.2]. For a fixed $M > 1$ and $k, j \in \mathbb{N}$, we can split

$$\|\Phi^\infty(u_{0,k}) - \Phi^\infty(u_{0,j})\|_{L^\infty_T \mathbf{E}} \leq \|\Phi^\infty(u_{0,k}) - \Phi^\infty(P_{< M} u_{0,k})\|_{L^\infty_T \mathbf{E}} + \|\Phi^\infty(P_{< M} u_{0,k}) - \Phi^\infty(P_{< M} u_{0,j})\|_{L^\infty_T \mathbf{E}} + \|\Phi^\infty(P_{< M} u_{0,j}) - \Phi^\infty(u_{0,j})\|_{L^\infty_T \mathbf{E}}$$

The middle term is controlled with the standard energy estimate

$$\|u - v\|^2_{L^\infty_T \mathbf{E}} \leq \|u_0 - v_0\|^2_{\mathbf{E}} + \left(\|u + v\|_{L^1_T L^\infty} + \|\partial_x(u + v)\|_{L^1_T L^\infty}\right) \|u - v\|^2_{L^\infty_T \mathbf{E}} \leq \|u_0 - v_0\|^2_{\mathbf{E}} + T \|u + v\|_{L^\infty_T \mathbf{E}} \|u - v\|^2_{L^\infty_T \mathbf{E}}$$

where the second line follows from a Sobolev inequality. Since

$$\|\Phi^\infty(P_{< M} u_{0,j})\|_{L^\infty_T \mathbf{E}} \leq C_\sigma \|P_{< M} u_{0,j}\|_{\mathbf{E}^\sigma}$$

thanks to (1.7), we deduce that

$$\|\Phi^\infty(P_{< M} u_{0,k}) - \Phi^\infty(P_{< M} u_{0,j})\|_{L^\infty_T \mathbf{E}} \leq C(M) \|u_{0,k} - u_{0,j}\|_{\mathbf{E}}$$

Therefore it remains to treat the first and last terms. Writing $u := \Phi^\infty(u_{0,k})$, $v := \Phi^\infty(P_{< M} u_{0,k})$ and $w := u - v$, a use of (2.11) provides

$$\|\Phi^\infty(u_{0,k}) - \Phi^\infty(P_{< M} u_{0,k})\|_{L^\infty_T \mathbf{E}} \lesssim \|w\|_{\mathbf{F}(T)}$$

As before, defining now $\tilde{\mathbf{\mathcal{V}}}(T') := \|w\|_{\mathbf{B}(T')} + \|w\|_{\mathbf{N}(T')}$, we get from (2.13), (4.5) and (5.1) the bound

$$\tilde{\mathbf{\mathcal{V}}}(T')^2 \lesssim \|u_0 - v_0\|^2_{\mathbf{E}} + T^{\mu_0} \|w\|_{\mathbf{F}^3(T)} \tilde{\mathbf{\mathcal{V}}}(T')^2 + T^{2\mu_1} \|w\|^2_{\mathbf{F}(T')} \tilde{\mathbf{\mathcal{V}}}(T')^2$$

From (6.1), we can bound

$$\|w\|_{\mathbf{F}^3(T)} \lesssim \|P_{< M} u_{0,k}\|_{\mathbf{E}^3} \lesssim C(M)$$

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and
\[ ||u + v||_{F(T)}^2 \lesssim R^2 \]

Thus, taking \( M \) large enough and \( T < T(R) \) small enough such that
\[ T^{\mu_0} C(M) + T^{2\mu_1} R^2 < 1/2 \]

concludes the proof.

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