Compactness for immersions of prescribed Gaussian curvature II - geometric aspects

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Abstract: We develop a compactness result near the boundary for families of locally convex immersions. We also develop a mod 2 degree theory for immersion of constant (and prescribed) Gaussian curvature with prescribed boundary. These are then used to solve the Plateau problem for immersions of constant (and prescribed) Gaussian curvature in general Hadamard manifolds.

Key Words: Gaussian Curvature, Plateau Problem, Monge-Ampère Equation, Non-Linear Elliptic PDEs.

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Compactness for immersions of prescribed Gaussian curvature II

1 - Introduction.

In this paper, we prove a compactness result for families of locally strictly convex immersions in general Riemannian manifolds. Our motivation is the compactness result for locally convex immersions in $\mathbb{R}^{n+1}$ developed by Guan and Spruck in [6] and Trudinger and Wang in [16], and which may be expressed as follows: let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of compact, codimension 2 submanifolds of $\mathbb{R}^{n+1}$ which converges to a compact, codimension 2 submanifold, $\Gamma_0$, say. Let $(\Sigma_n)_{n \in \mathbb{N}}$ be a sequence of compact, locally convex hypersurfaces in $\mathbb{R}^{n+1}$ such that, for all $n$:

$$\partial \Sigma_n = \Gamma_n.$$  

For all $n$, let $\text{Vol}(\Sigma_n)$ and $\text{Diam}(\Sigma_n)$ be the volume and diameter of $\Sigma_n$ respectively. The reasoning of [6] and [16] readily yields:

**Theorem 1.1**

Suppose that there exists $B > 0$ such that, for all $n$:

$$\text{Vol}(\Sigma_n), \text{Diam}(\Sigma_n) \leq B.$$  

Then, there exists a $C^{0,1}$, locally convex immersion, $\Sigma_0$ such that:

(i) $\partial \Sigma_0 = \Gamma_0$; and

(ii) $(\Sigma_n)_{n \in \mathbb{N}}$ subconverges towards $\Sigma_0$.

The main interest of this result lies in its applications to the study of the Plateau problem. Indeed, after the progress made in the 80’s and early 90’s (c.f. [4], [5] and [10]), Spruck conjectured in [15] that for any $K > 0$, for any $k \in ]0, K[$, and for any smooth, compact, codimension 2 curve, $\Gamma \subseteq \mathbb{R}^{n+1}$, bounding a compact, locally strictly convex immersed hypersurface of Gaussian curvature everywhere bounded below by $K$, the Plateau problem can be solved for $\Gamma$ in the sense that there exists a smooth, locally strictly convex, immersed hypersurface, $\Sigma \subseteq \mathbb{R}^{n+1}$, whose boundary is equal to $\Gamma$ and whose Gaussian curvature is constant and equal to $k$. This conjecture was confirmed simultaneously by Guan and Spruck in [6] and Trudinger and Wang in [16]. Both papers use the Perron Method in a similar manner, and in each paper some equivalent of Theorem 1.1 plays a crucial role.

The most natural extension of the results of [6] and [16] is then to construct solutions to the Plateau problem in the case where $\mathbb{R}^{n+1}$ is replaced by any Hadamard manifold. We quickly encounter two significant obstacles to the application of the Perron Method to this more general setting. The first is that the Perron Method as described in [6] and [16] relies heavily on the existence of a totally geodesic hypersurface passing through every point and normal to any vector at that point, but this is a strong geometric property only possessed by space forms. The second obstacle, similar to the first but different, is that there exists to date no complete extension of Theorem 1.1 to general manifolds, and that its current proof is only valid in space forms, since it also relies on the existence of the same totally geodesic hypersurfaces described above.
We therefore require completely different techniques, and we proceed as follows: first, in place of the Perron Method, we develop a parametric version of the continuity method arising essentially from the marriage of the approaches of [6] and [9]. This yields a mod 2 degree theory and, used in conjunction with the appropriate generalisation of Theorem 1.1, discussed presently, and the higher order a-priori estimates already developed by the author in [11], this allows us to solve the Plateau problem for Gaussian curvature in general manifolds.

We now state the result, and for ease of presentation, we restrict attention to the case where the ambient manifold is a Hadamard manifold. Thus, let $M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. We recall that an immersed hypersurface in $M$ is a pair $\Sigma^n = (i, (S^n, \partial S^n))$ where $(S^n, \partial S^n)$ is a smooth, compact, $n$-dimensional manifold with boundary and $i : S \to M$ is a smooth immersion and $\Sigma$ is said to be locally convex if and only if its shape operator is everywhere positive definite. We say that $\Sigma$ has generic boundary if and only if for any $p \neq q \in \partial S$ such that $i(p) = i(q)$:

$$Di \cdot T_p \partial S \neq Di \cdot T_q \partial S.$$  

In other words, the two tangent spaces of $\partial \Sigma$ at these points do not coincide. Trivially, every smooth, locally convex immersion can be approximated arbitrarily closely by a smooth, locally convex immersion with generic boundary. We prove:

**Theorem 1.2**

Let $\hat{\Sigma}^n = (\hat{i}, (\hat{S}, \partial \hat{S}))$ be a locally strictly convex, immersed hypersurface in $M$ with generic boundary. Let $\phi \in C^\infty(M)$ be a smooth, positive function such that, for every $p \in \Sigma$, the Gaussian curvature of $\Sigma$ at $p$ is strictly greater than $\phi(p)$. Suppose that there exists a convex set, $K$, with smooth boundary and an open subset $\Omega \subset \partial K$ such that:

(i) $\partial \Omega$ is smooth;

(ii) $\Omega^c$ has finitely many connected components; and

(iii) $\Sigma^n$ is isotopic by locally strictly convex, immersed hypersurfaces to a finite covering of $\Omega$,

then there exists a locally strictly convex, immersed hypersurface $\Sigma^n$ in $M$ such that:

(a) $\partial \Sigma = \partial \hat{\Sigma}$;

(b) $\Sigma$ is bounded by $\hat{\Sigma}$; and

(c) for every point $p \in \Sigma$, the Gaussian curvature of $\Sigma$ at $p$ is equal to $\phi(p)$.

Remarks:

(a) the concept of being bounded (condition (b)), is described explicitly in Section 3. Heuristically, $\Sigma$ is bounded by $\hat{\Sigma}$ if and only if $\Sigma$ is, more or less, a graph over $\Sigma$;
(b) when the ambient manifold is of dimension greater than 3, immersed hypersurfaces of constant Gaussian curvature typically do not behave well under passage to the limit. We thus do not expect that an approximation argument may be used to relax the condition of genericity along the boundary;

(c) if $(\hat{S}, \partial\hat{S})$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$, then Conditions $(i)$, $(ii)$ and $(iii)$ of Theorem 1.2 are automatically satisfied;

(d) when $n = 2$, and thus when the dimension of the ambient manifold is equal to 3, Theorem 1.2 yields a stronger version of Proposition 5.0.3 of [9], which itself constitutes the analytic core of that paper; and

(e) in general manifolds (of arbitrary curvature), the situation is complicated by the possible existence of conjugate points along geodesics. However, most stages of the argument remain more or less intact, and the result may thus be adapted, albeit with stronger hypothesis, to the more general case.

The continuity method as presented here relies no less heavily on a version of Theorem 1.1 than does the Perron Method it was designed to replace. However, although the complete generalisation remains unproven, we discover that we only really require an analogous result to hold in some neighbourhood of the boundary: this suffices to obtain a-priori estimates which may then be extended across the whole hypersurface using the maximum principal. It is such a boundary compactness result that we prove and which constitutes the hardest and most technical part of this paper, but which is also of independant interest.

The result itself is rather technical, and in order to state it, we first require some notation.

Let $M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $\Gamma^{n-1} \subseteq M$ be a compact, codimension 2 immersed submanifold in $M$ which intersects itself generically, as described above. If $p$ is a point in $\Gamma$ and if $N_p$ is a unit vector normal to $\Gamma$ at $p$, then we say that $N_p$ is a strictly convex normal if and only if the shape operator of $\Gamma$ with respect to $N_p$ is strictly positive definite. We then say that $\Gamma$ is strictly convex if and only if, for every $p \in \Gamma$ there exists a strictly convex normal. For example, when $M$ is 3 dimensional, and $\Gamma$ is thus a closed curve, $\Gamma$ is convex if and only if its geodesic curvature never vanishes. In this case, the set of strictly convex normals over any point constitutes an open subinterval of the circle of unit normal vectors to $\Gamma$ at that point. If, moreover, $\Gamma$ is oriented, which is the case when $\Gamma$ bounds a locally strictly convex hypersurface, we may define a maximal strictly convex normal, $N^+$, which is a continuous vector field over $\Gamma$. Let $\Sigma$ be a locally strictly convex immersed hypersurface in $M$ such that $\partial\Sigma = \Gamma$. Suppose, moreover that the orientation of $\Sigma$ is compatible with that of $\Gamma$, and denote the outward pointing unit normal over $\Sigma$ by $N_\Sigma$. We obtain:

**Lemma 5.1**

Choose $\theta > 0$. There exists $r > 0$, which only depends on $M$, $\Gamma$ and $\theta$ such that if the angle between $N_\Sigma$ and $N^+$ is always greater than $\theta$, then, for all $p \in \Gamma$, there exists a convex subset $K \subseteq B_r(p)$ such that the connected component of $\Sigma \cap B_r(p)$ containing $p$ is embedded and is a subset of $\partial K$. 

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Remark: Details of notation and conventions are given in Section 4. Note that it is precisely at this stage that the genericity condition on the boundary is required.

This paper is structured as follows:

(i) in Sections 2 to 4 we introduce the concepts and notation used in the sequel: in Section 2, we introduce immersed hypersurfaces and describe the Banach manifold of immersed hypersurfaces modulo reparametrisation; in Section 3 we introduce locally convex immersions and describe the notion of locally convex immersions bounding other locally convex immersions; and in Section 4 we develop a higher codimension concept of convexity which is required to understand the boundary conditions used in the sequel;

(ii) in Sections 5 to 9, which together constitute the most innovative part of the paper, we determine first order a-priori bounds near the boundary for generic, locally convex immersions of prescribed curvature: in Section 5, restricting to the case where the boundary is embedded, and using the notion of “semi-convexity”, we obtain a compactness result for convex immersions which yields these a-priori bounds but requires various intuitive but technical propositions whose proofs are deferred to the subsequent two sections; in Section 6, we obtain technical results using the parabolic limit; in Section 7, we show that the limit of a sequence of semi-convex sets is also semi-convex; in Section 8, we show how, under a simple modification, the reasoning of Section 5 may be adapted to the case where the boundary is immersed and generic; and in Section 9, we obtain first order lower bounds along the boundary which are important in the sequel for the final (technical) step in obtaining second order bounds over the boundary;

(iii) in Sections 10 to 11, we recall the results of [11] to prove a conditional existence result: in Section 10, we prove compactness of families of immersions of prescribed curvature; and in Section 11, we show how Sard’s Lemma may be used along with compactness to obtain (generically) solutions which interpolate between isotopic data; and

(iv) in Section 12, we prove the existence of isotopies between the given data and other data for which solutions are known to exist, and, using the concepts of local and global rigidity, we prove Theorem 1.2.

This paper was written while the author was working at the Mathematics Department of the Universitat Autònoma de Barcelona, Bellaterra, Spain.

2 - Immersed Submanifolds and Moduli Spaces.

Let $M^{n+1}$ be a smooth Riemannian manifold. A (smooth, compact) immersed submanifold is a pair $\Sigma := (i, (S, \partial S))$ where:

(i) $(S, \partial S)$ is an oriented, compact, Riemannian manifold with boundary; and

(ii) $i : \Sigma \to M$ is a smooth immersion (i.e. $Di$ is everywhere injective).

Remark: in the sequel, all submanifolds of $M$ will be (relatively) compact. Likewise, unless stated otherwise, all submanifolds of $M$ will be smooth.
Let $\Sigma = (i, (S, \partial S))$ and $\Sigma' = (i', (S', \partial S'))$ be two immersed hypersurfaces in $M$. We say that $\Sigma$ and $\Sigma'$ are **equivalent** if and only if there exists a diffeomorphism $\phi : (S, \partial S) \to (S', \partial S')$ such that:

$$i' \circ \phi = i.$$ 

Let $\text{Exp}$ be the exponential map of $M$. Let $N_\Sigma$ be the outward pointing normal vector field over $\Sigma$. We say that $\Sigma'$ is a **graph** over $\Sigma$ if and only if there exists $f \in C^\infty_0(S)$ and a diffeomorphism $\phi : (S, \partial S) \to (S', \partial S')$ such that:

$$i' \circ \phi = \text{Exp}(fN_\Sigma).$$

In particular, $\Sigma$ and $\Sigma'$ are **equivalent** if and only if $\Sigma'$ is a trivial graph over $\Sigma$.

Let $(\Sigma_n)_{n \in \mathbb{N}} = (i_n, (S_n, \partial S_n)), \Sigma_0 = (i_0, (S_0, \partial S_0))$ be immersed submanifolds in $M$. We say that $(\Sigma_n)_{n \in \mathbb{N}}$ **converges** to $\Sigma_0$ if and only if there exists $N \geq 0$ and, for all $n \geq N$ a diffeomorphism $\phi_n : (S_0, \partial S_0) \to (S_n, \partial S_n)$ such that $(i_n \circ \phi_n)_{n \geq N}$ converges to $i_0$ in the $C^\infty$ sense.

Trivially, if $(\Sigma_n)_{n \in \mathbb{N}}$ converges to $\Sigma_0$, then there exists $N \geq 0$, and for all $n \geq N$ a vector field $X_n \in \Gamma(i_0^*TM)$ and a diffeomorphism $\phi_n : (S_0, \partial S_0) \to (S_n, \partial S_n)$ such that:

$$i_n \circ \phi_n = \text{Exp}(X_n).$$

Moreover, $(X_n)_{n \geq N}$ tends to 0 in the $C^\infty$ sense. If $\Sigma_n$ and $\Sigma_0$ have the same boundary for all $n$, then, increasing $N$ if necessary, $X_n$ may always be chosen to be normal to $\Sigma_0$ and vanishing along $\partial S_0$. In other words, $\Sigma_n$ is a graph over $\Sigma_0$ for sufficiently large $n$.

Let $(\Gamma_t)_{t \in [0, 1]} = (j_t, G_t)_{t \in [0, 1]}$ be a smooth family of (exact) immersed submanifolds without boundary in $M$. We denote by $\tilde{\mathcal{M}}$ the family of all pairs $(t, \Sigma)$ where $t \in I$ and $\Sigma$ is an immersed submanifold in $M$ such that $\partial \Sigma = \Gamma_t$. For all $t \in [0, 1]$, let $\mathcal{M}_t$ be the fibre of $\tilde{\mathcal{M}}$ over $t$. We denote by $\mathcal{M}$ the family of all pairs $(t, \Sigma)$ where $\Sigma$ denotes the equivalence class of $\Sigma$. Likewise, for all $t \in [0, 1]$, we denote by $\mathcal{M}_t$ the fibre of $\mathcal{M}$ over $t$.

For all $t$, we interpret $\mathcal{M}_t$ as a smooth Banach manifold (strictly speaking, every relatively compact open subset is an intersection of an infinite family of nested Banach manifolds).

We now briefly review the theory of Banach manifolds (see [8] for a more detailed description in the 1 dimensional case). Let $[\Sigma]$ be an element in $\mathcal{M}_t$. Let $V_{\Sigma} \subseteq \mathcal{M}_t$ be the set of those immersed hypersurfaces which are graphs over $\Sigma$. This is an open subset of $\mathcal{M}_t$, which we identify with an open subset $U_{\Sigma}$ of $C^\infty_0(S)$. Let $\Phi_{\Sigma} : U_{\Sigma} \to V_{\Sigma}$ be the canonical identification. $(U_{\Sigma}, V_{\Sigma}, \Phi_{\Sigma})$ constitutes a smooth chart of $\mathcal{M}_t$ which we call the **graph neighbourhood** of $\Sigma$.

We likewise interpret $\mathcal{M}$ also as a smooth Banach manifold. As before, let $(t, [\Sigma])$ be an element of $\mathcal{M}$, where $\Sigma = (i, (S, \partial S))$. We extend $i$ to a smooth family $(i_s)_{s \in [t-\epsilon, t+\epsilon]}$ such that, for all $s$, $(i_s, \partial S) = \Gamma_s$. Thus, if, for all $s$, we define $\Sigma_s$ by $\Sigma_s = (i_s, (S, \partial S))$, then $(s, [\Sigma_s])_{s \in [t-\epsilon, t+\epsilon]}$ is a smooth family in $\mathcal{M}$. Let $V_{\Sigma} \subseteq \mathcal{M}$ be the set of pairs $(s, [\Sigma'])$ where $\Sigma'$ is a graph over $\Sigma_s$. $V_{\Sigma}$ is an open subset of $\mathcal{M}$ which we identify with an open subset, $U_{\Sigma}$, of $[t-\epsilon, t+\epsilon] \times C^\infty_0(S)$. Let $\Phi_{\Sigma} : U_{\Sigma} \to V_{\Sigma}$ be the canonical identification. $(U_{\Sigma}, V_{\Sigma}, \Phi_{\Sigma})$
consitutes a smooth chart of \( \mathcal{M} \) which we likewise call the graph neighbourhood of \( \Sigma \). Trivially, this does not depend canonically on \( \Sigma \), but also on the choice of smooth family extending \( \Sigma \).

Let \( (t, \Sigma) \) be an element of \( \hat{\mathcal{M}} \), where \( \Sigma = (i, (S, \partial S)) \). The group of smooth diffeomorphisms of \( (S, \partial S) \) acts linearly on \( C^\infty(S) \). \( C^\infty(S) \) therefore defines a bundle \( \mathcal{E} \) over \( \hat{\mathcal{M}} \), whose fibre at \( (t, [\Sigma]) \) is \( C^\infty(S) \). Since the constant functions over \( S \) are preserved by the diffeomorphisms of \( (S, \partial S) \), these generate a subbundle of \( \mathcal{E} \) which we identify with \( \mathcal{M} \times \mathbb{R} \).

Likewise, if \( (\phi_t)_{t \in [0,1]} \in C^\infty(M) \) is a smooth family of smooth functions, then it defines a section of \( \mathcal{E} \), which we also denote by \( \phi \), given by:

\[
\phi(t, [\Sigma]) = [\phi_t \circ i].
\]

For all \( t \), \( \mathcal{E} \) restricts canonically to a bundle over \( \mathcal{M}_t \), which we denote by \( \mathcal{E}_t \). Let \( (U_\Sigma, V_\Sigma, \Phi_\Sigma) \) be a graph neighbourhood of \( \mathcal{M}_t \) about \( \Sigma \). Trivially:

\[
\mathcal{E}|_{V_\Sigma} = U_\Sigma \times C^\infty(S).
\]

This yields a canonical splitting of \( T\mathcal{E}_t \) over the fibre over \( \Sigma \). Since every point in \( \mathcal{M}_t \) has a canonical graph neighbourhood, we thus obtain a canonical splitting of \( T\mathcal{E}_t \) which in turn generates a covariant derivative of \( \mathcal{E}_t \). More explicitly, for every \( \Sigma' = (i', (S', \partial S')) \in V_\Sigma \), let \( \pi_{\Sigma'} : S' \to S \) be the canonical projection. A section, \( f \), of \( \mathcal{E}_t \) is covariant constant at \( \Sigma \) if and only if there exists a function \( f_0 \in C^\infty(S) \) such that, up to second order around \( \Sigma \):

\[
f_{\Sigma'} = f_0 \circ \pi_{\Sigma'}.
\]

We advise the reader unfamiliar with the theory of Banach manifolds not to trouble himself with the details of this construction. In the sequel, it suffices to know that, locally, \( \mathcal{E}_t \) behaves like the constant bundle \( U_\Sigma \times C^\infty(S) \) and it is not really necessary to have an explicit choice of splitting of \( \mathcal{E} \).

We define the Gauss curvature mapping, \( K \), to be the mapping that associates to every element \( (t, [\Sigma]) \), where \( \Sigma = (i, (S, \partial S)) \), the function \( f \in C^\infty(S) \) whose value at the point \( p \in S \) is the Gaussian curvature of \( \Sigma \) at \( p \). \( K \) defines a smooth section of \( \mathcal{E} \) over \( \mathcal{M} \).

We determine a formula for the covariant derivative, \( \nabla K \), of \( K \) with respect to the canonical splitting of \( \mathcal{E}_t \). Let \( \Sigma = (i, (S, \partial S)) \) be an element of \( \mathcal{M}_t \). Let \( N \) be the outward pointing unit normal vector field over \( \Sigma \). Let \( R \) be the Riemann curvature tensor of \( M \). We define the operator \( W \) acting on sections of \( TS \) by:

\[
W \cdot X = R_{NX}N.
\]

**Lemma 2.1**

With respect to the canonical splitting, identifying \( T_{[\Sigma]}\mathcal{M}_t \) with \( C^\infty_0(S) \):

\[
\nabla f K = K \text{Tr}(A^{-1}(W - A^2)) f - K \text{Tr}(A^{-1} \text{Hess}(f)),
\]

where \( A \) is the shape operator of \( \Sigma \).
Proof: See Proposition 3.1.1 of [9]. □

This yields the following result, which will be of use in the sequel:

**Corollary 2.2**

\( \nabla K \) is a second order linear differential operator. Moreover:

(i) if \( \Sigma \) is strictly convex, then \( \nabla K \) is elliptic; and

(ii) when \( \text{Tr}(A^{-1}(W - A^2)) > 0 \), \( \nabla K \) has trivial kernel.

**Remark:** In particular, if the sectional curvature of \( M \) is bounded above by \(-1\) and if \( A \leqslant \text{Id} \), then \( W - A^2 \geqslant 0 \) and so, by (ii), \( \nabla K \) is invertible.

**Proof:** (i) is immediate. (ii) follows by the Maximum Principal. □

### 3 - Locally Convex Hypersurfaces.

Let \( M^{n+1} \) be a Riemannian manifold. A **locally convex hypersurface** in \( M \) is a pair \( \Sigma = (i, S^n) \) where \( S \) is an \( n \)-dimensional topological manifold and \( i : S \to M \) is a continuous map such that, for all \( p \in S \), there exists a neighbourhood, \( U \), of \( p \) in \( S \), a convex subset \( K \subseteq M \) with non-trivial interior, and an open subset \( V \subseteq \partial K \) such that \( i \) restricts to a homeomorphism from \( U \) to \( V \). We refer to such a triplet \((U, V, K)\) as a **convex chart** of \( \Sigma \). Pulling back the metric on \( M \) through \( i \) yields a natural length metric on \( \Sigma \) which we denote by \( d_\Sigma \). Let \((\Sigma_n)_{n \in \mathbb{N}} = (i_n, S_n)_{n \in \mathbb{N}} \) and \( S_0 = (i_0, S_0) \) be convex immersions. We say that \((\Sigma_n)_{n \in \mathbb{N}} \) **converges** to \( \Sigma_0 \) if and only if:

(i) \((S_n, d_{\Sigma_n})_{n \in \mathbb{N}} \) converges to \((S_0, d_{\Sigma_0})\) in the Gromov-Hausdorff sense; and

(ii) \((i_n)_{n \in \mathbb{N}} \) converges to \( i_0 \) locally uniformly.

Let \( \Sigma = (i, S) \) and \( \Sigma' = (i', S') \) be two locally convex hypersurfaces in \( M \). We say that \( \Sigma \) and \( \Sigma' \) are equivalent if and only if there exists a homeomorphism \( \phi : S \to S' \) such that:

\[ i = i' \circ \phi. \]

**Example:** Let \( K \subseteq M \) be a convex subset with non trivial interior. Then any open subset of \( \partial K \) is a locally convex hypersurface. □

**Example:** Let \( \Sigma \) be a smooth hypersurface on \( M \). \( \Sigma \) is a locally convex hypersurface if and only if its second fundamental form is everywhere non-negative definite. □

Suppose now that \( M \) is a Hadamard manifold. Let \( K \subseteq M \) be a convex set with non-trivial interior. Let \( K^o \) be the interior of \( K \). We define \( \pi_K : M \setminus K^o \to \partial K \) to be projection onto the closest point in \( \partial K \). Let \( V \subseteq \partial K \). We call the set \( \pi_K^{-1}(V) \) the **end** of \( V \), and we denote it by \( \mathcal{E}(V) \). Trivially, \( \mathcal{E}(V) \) is foliated by half geodesics leaving points in \( V \) in directions normal to \( K \). Let \( \Sigma \) be a locally convex hypersurface. Let \((U, V, K)\) and \((U', V', K')\) be convex charts of \( \Sigma \). Trivially:

\[ \pi_K^{-1}(i(U \cap U')) = \pi_K^{-1}(i(U \cap U')). \]
We thus define the end of \( \Sigma \) to be the manifold (with non-smooth, concave boundary) whose coordinate charts are the ends of the convex charts of \( \Sigma \). We denote this manifold by \( \mathcal{E}(\Sigma) \). \( \mathcal{E}(\Sigma) \) has the following properties:

(i) \( \Sigma \) naturally embeds as the boundary of \( \mathcal{E}(\Sigma) \);

(ii) in the complement of \( \Sigma \), \( \mathcal{E}(\Sigma) \) has the structure of a smooth Riemannian manifold with non-positive curvature;

(iii) \( \mathcal{E}(\Sigma) \) is foliated by half geodesics leaving points in \( \Sigma \) in directions normal to \( \Sigma \); and

(iv) there exists a natural embedding \( I : \mathcal{E}(\Sigma) \to M \) which restricts to \( i \) over \( \Sigma \) and which is a local diffeomorphism over the complement of \( \Sigma \).

Let \( K \subseteq \mathcal{E}(\Sigma) \) be a subset of the end of \( \Sigma \). Suppose moreover that \( K \) contains \( \Sigma \) and that \( K \) coincides with \( \Sigma \) outside a compact set. Let \( p \) be a point in \( \mathcal{E}(\Sigma) \setminus \Sigma \) lying on the boundary of \( K \). We say that \( K \) is boundary convex at \( p \) if and only if there exists a neighbourhood, \( U \), of \( p \) in \( \mathcal{E}(\Sigma) \), a convex subset \( K' \subseteq M \) with non trivial interior, and a neighbourhood \( V \) of \( I(p) \) in \( M \) such that \( I \) restricts to a homeomorphism from \( U \) to \( V \), and:

\[
I(K \cap U) = K' \cap V.
\]

Bearing in mind that, near any point \( p \in \Sigma \), \( \mathcal{E}(\Sigma) \) may always be extended over an open set containing \( p \), we extend this definition to also include boundary points lying in \( \Sigma \). We then say that \( K \) is boundary convex if and only if it is boundary convex at \( p \) for every \( p \in \partial K \). Importantly, the image under \( I \) of the boundary of a boundary convex set is a locally convex hypersurface.

We say that a subset \( K \subseteq \mathcal{E}(\Sigma) \) is semi-convex if and only if for every geodesic segment \( \gamma : [0, 1] \to \mathcal{E}(\Sigma) \) contained within \( \mathcal{E}(\Sigma) \), if \( \gamma(0), \gamma(1) \in K \), then the whole of \( \gamma \) is contained in \( K \).

**Proposition 3.1**

Let \( K \) be a subset of the end of \( \Sigma \) which contains \( \Sigma \) and coincides with \( \Sigma \) outside a convex set. If \( K \) is semi-convex, then \( K \) is boundary convex.

**Proof:** Let \( p \in \partial K \). If \( p \) lies in the interior of \( \mathcal{E}(\Sigma) \), then \( K \) is trivially boundary convex at \( p \). Suppose therefore that \( p \in \Sigma \). Let \((U, V, K')\) be a convex chart of \( \Sigma \) at \( p \). Let \( r > 0 \) be such that \( B_r(p) \subseteq \mathcal{E}(U) \). Consider \( X = (K' \cap B_r(p)) \cup (K \cap B_r(p)) \). Let \( \gamma : [0, 1] \to B_r(p) \) be a geodesic segment with endpoints in \( X \). Let \( \gamma' \) be a maximal subsegment of \( \gamma \) lying outside \( K' \cap B_r(p) \). Since \( \Sigma \subseteq K \), the endpoints of \( \gamma' \) are contained in \( K \cap B_r(p) \). Thus, by semi-convexity, \( \gamma' \) is contained in \( K \), and therefore also in \( X \). It follows that the whole of \( \gamma \) is contained in \( X \). Since \( \gamma \) is arbitrary, \( X \) is convex and \( K \) is therefore boundary convex at \( p \). This completes the proof. \( \square \)

Let \( K \) be a semi-convex subset of the end of \( \Sigma \) which contains \( \Sigma \) and coincides with \( \Sigma \) outside a convex set. \((\partial K, I|_{\partial K})\) defines a convex immersion in \( M \) which, by abuse of notation, we simply denote by \( \partial K \). Let \( \Sigma \) and \( \Sigma' \) be two locally convex hypersurfaces in \( M \). We say that \( \Sigma \) is bounded by \( \Sigma' \) (and \( \Sigma' \) bounds \( \Sigma \)) if and only if there exists a
semi-convex subset, \( K \subseteq E(\Sigma) \), which contains \( \Sigma \) and which coincides with \( \Sigma \) outside a convex set such that \( \Sigma' \) is equivalent to \( \partial K \). In this case, we often identify \( \Sigma' \) with \( \partial K \) and thus view it as a subset of \( E(\Sigma) \).

**Example:** Let \( K, K' \subseteq M \) be two convex sets. Then \( \partial K \) is bounded by \( \partial K' \) if and only if \( K \subseteq K' \). \( \Box \)

Let \( \Sigma = (i, S) \) be a locally convex hypersurface. For \( p \in S \), let \( N_p \subseteq UM \) be the set of supporting normals of \( \Sigma \) at \( S \). We define \( N\Sigma \) by:

\[
N\Sigma = \bigcup_{p \in S} N_p.
\]

\( N\Sigma \) defines a \( C^0 \) immersed submanifold of \( UM \) which we call the normal of \( \Sigma \).

If \( \Sigma' \) bounds \( \Sigma \), then there exists an upper semi-continuous function \( f : N\Sigma \to [0, \infty[ \) such that \( \Sigma' \) is the graph of \( f \) over \( \Sigma \). Moreover, \( f \) vanishes outside a compact set. We call \( f \) and \( \text{Supp}(f) \) respectively the graph function and graph support of \( \Sigma' \) with respect to \( \Sigma \).

The property of boundedness is preserved by passage to limits:

**Lemma 3.2**

Let \( (\Sigma_n)_{n \in \mathbb{N}}, \Sigma_0 \) and \( (\Sigma'_n)_{n \in \mathbb{N}}, \Sigma'_0 \) be convex immersions in \( M \). Suppose that, for all \( n > 0 \), \( \Sigma'_n \) bounds \( \Sigma_n \). For all \( n > 0 \), let \( f_n \) and \( X_n = \text{Supp}(f_n) \) be the graph function and graph support respectively of \( \Sigma'_n \) with respect to \( \Sigma_n \). Suppose that there exists \( R > 0 \) and that, for all \( n \), there exists a compact set \( X'_n \subseteq \Sigma_n \) such that:

(i) \( f_n \leq R \) for all \( n > 0 \);

(ii) for all \( n > 0 \), \( X_n \subseteq X'_n \); and

(iii) \( (X'_n)_{n \in \mathbb{N}} \) converges to \( X'_0 \) in the Hausdorff sense,

then \( \Sigma'_0 \) also bounds \( \Sigma_0 \).

**Proof:** For all \( n \), let \( K_n \subseteq E(\Sigma_n) \) be the semi-convex subset such that \( \partial K_n = \Sigma'_n \). The hypotheses on \( (f_n)_{n \in \mathbb{N}} \) and \( (X_n)_{n \in \mathbb{N}} \) imply that \( (K_n)_{n \in \mathbb{N}} \) is uniformly bounded. By compactness of the family of semi-convex sets, \( (K_n)_{n \in \mathbb{N}} \) subconverges in the Hausdorff sense to a semi-convex set \( K_0 \subseteq E(\Sigma_0) \), say. By Proposition 3.1, \( K_0 \) is boundary convex and so \( (I|_{\partial K_n}, \partial K_0) \) is a locally convex hypersurface. Moreover \( (I|_{\partial K_n}, \partial K_n)_{n \in \mathbb{N}} \) converges to \( (I|_{\partial K_0}, \partial K_0) \) in the sense of convex immersions. Since \( (I|_{\partial K_n}, \partial K_n) = \Sigma_n \) for all \( n \), and since \( (\Sigma'_n)_{n \in \mathbb{N}} \) converges to \( \Sigma'_0 \) in the sense of convex immersions, \( (I|_{\partial K_0}, \partial K_0) \) is equivalent to \( \Sigma'_0 \). \( \Sigma'_0 \) therefore bounds \( \Sigma_0 \), and this completes the proof. \( \Box \)

In the sequel, we require a slight variation of this definition. Let \( \Sigma = (i, (S, \partial S)) \) and \( \Sigma' = (i', (S', \partial S')) \) be (smooth) immersed hypersurfaces which are also convex. Let \( N_\Sigma \) and \( N_{\Sigma'} \) be the outward pointing normal vector fields over \( \Sigma \) and \( \Sigma' \) respectively. Let \( N_{\partial \Sigma} \) be the normal vector field over \( \partial \Sigma \) which is tangent to \( \Sigma \) and points outwards from \( \partial \Sigma \).
Suppose that \( \partial \Sigma' = \partial \Sigma =: \Gamma \). We suppose moreover that \( \Sigma' \) lies “locally strictly above” \( \Sigma \) along \( \Gamma \): i.e. for all \( p \in \Gamma \):

\[
\langle N_{\Sigma'}, N_{\partial \Sigma} \rangle > 0.
\]

Since \( \Sigma' \) is smooth, it may be extended to a (smooth) convex, immersed hypersurface \( \tilde{\Sigma}' \) strictly containing \( \partial \Sigma' \) in its interior. Let \( \Sigma'_0 \) denote the collar region of \( \tilde{\Sigma}' \) lying outside \( \Sigma' \). We define the piecewise smooth immersed hypersurface \( \tilde{\Sigma} \) by:

\[
\tilde{\Sigma} = \Sigma \cup \Sigma'_c.
\]

Since \( \tilde{\Sigma}' \) lies locally strictly above \( \Sigma \) along \( \Gamma \), \( \tilde{\Sigma} \) is also a locally convex hypersurface. We now say that \( \Sigma' \) bounds \( \Sigma \) if and only if \( \Sigma' \) bounds \( \tilde{\Sigma} \).

Suppose that \( \Sigma' \) lies locally strictly above \( \Sigma \) along \( \partial \Sigma \) and bounds \( \Sigma \). Let \( f \) be the graph function of \( \tilde{\Sigma}' \) with respect to \( \tilde{\Sigma} \). Let \( \pi : \tilde{\Sigma}' \rightarrow N\Sigma \) be the canonical projection. We say that \( \Sigma' \) strictly bounds \( \Sigma \) if and only for all \( p \in \Sigma' \setminus \partial \Sigma' \):

\[
f \circ \pi(p) > 0.
\]

In this case, the property of strict containment is preserved by small deformations:

**Lemma 3.3**

Let \( (\Sigma_n)_{n \in \mathbb{N}}, \Sigma_0 \) and \( (\Sigma'_n)_{n \in \mathbb{N}}, \Sigma'_0 \) be smooth, convex, immersed hypersurfaces. Suppose that \( \Sigma'_0 \) lies locally strictly above \( \Sigma_0 \) along \( \partial \Sigma_0 \) and strictly bounds \( \Sigma_0 \). Suppose moreover that, for all \( n \), \( \partial \Sigma_n = \partial \Sigma'_n \) and that \( (\Sigma_n)_{n \in \mathbb{N}} \) and \( (\Sigma'_n)_{n \in \mathbb{N}} \) converge to \( \Sigma_0 \) and \( \Sigma'_0 \) respectively. Then, for sufficiently large \( n \), \( \Sigma'_n \) lies locally strictly above \( \Sigma_n \) along \( \partial \Sigma_n \) and bounds \( \Sigma_n \).

**Proof:** For all \( n \), let \( \Sigma_n = (i_n', (S'_n, \partial S'_n)) \). For all \( n \), \( \mathcal{E}(\tilde{\Sigma}_n) \) may be extended beyond \( \tilde{\Sigma}_n \) to contain a neighbourhood of \( \tilde{\Sigma}_n \). Let \( \mathcal{E}_{ext}(\tilde{\Sigma}_n) \) denote this extension. For sufficiently large \( N \), \( \Sigma_n \) is contained in \( \mathcal{E}_{ext}(\tilde{\Sigma}_n) \). Let \( d_n : \Sigma'_n \rightarrow \mathbb{R} \) be the signed distance in \( \mathcal{E}_{ext}(\tilde{\Sigma}_n) \) to \( \tilde{\Sigma}_n \). For sufficiently large \( n \), \( d_n \) is smooth, and \( (d_n)_{n \in \mathbb{N}} \) converges to \( d_0 \) in the \( C^\infty \) sense. However, \( d_0 > 0 \) and \( \nabla d \neq 0 \) along \( \partial \Sigma_0 \). Thus, for sufficiently large \( n \), \( d_n > 0 \) and so \( \Sigma'_n \subseteq \mathcal{E}(\tilde{\Sigma}_n) \). This completes the proof. \( \square \)

**4 - Convexity in Higher Codimension.**

Let \( M^{n+1} \) be a Riemannian manifold. Let \( \Gamma^k = (i, (G^k, \partial G^k)) \subseteq M \) be a \( k \)-dimensional immersed submanifold. Let \( N\Gamma \subseteq i^*(UM) \) be the bundle of unit normal vectors over \( \Gamma \). \( N\Gamma \) has spherical fibres of dimension \((n-k)\). For all \( N_p \in N\Gamma \), let \( A_{\Gamma}(N_p) \) be the shape operator of \( \Gamma \) with respect to \( N_p \). In other words, for all vector fields \( X \) and \( Y \) tangent to \( \Gamma \):

\[
A_{\Gamma}(N_p)(X, Y) = -\langle \nabla_X Y, N_p \rangle.
\]

For all \( p \in \Gamma \), we define define \( X_p \subseteq T_p \Gamma \) by:

\[
X_p = \{ N_p \text{ s.t. } A_{\Gamma}(N_p) > 0 \},
\]
where, for a matrix, $M$, we write $M > 0$ if and only if it is positive definite. Since the set of positive definite matrices is an open convex cone, $X_p$ is a convex subset of $N_p \Gamma$. In particular, it is contained within a hemisphere. We say that $\Gamma$ is \textbf{locally strictly convex} at $p$ if and only if $X_p$ is non-empty. We say that $\Gamma$ is \textbf{locally strictly convex} if and only if it is locally strictly convex at every point $p \in \Gamma$.

We now consider the case where $\Gamma$ is of codimension 2, in which case $N \Gamma$ is a circle bundle over $\Gamma$ and, for all $p \in \Gamma$, $X_p$ is an open interval of length at most $\pi$. We define a \textbf{convexity orientation} of $\Gamma$ to be a continuous section, $N^-\Gamma$, of $N \Gamma$ over $\Gamma$ such that, for all $p \in \Gamma$:

$$N^-(p) \in \partial X_p.$$ 

We say that $\Gamma$ carries a convexity orientation when such a section exists. A convexity orientation defines an order over $X_p$ in the following manner: we say that, given two vectors, $V_p, V'_p \in X_p$, $V_p$ lies below $V'_p$ if and only if it lies between $N^-(p)$ and $V'_p$. Given a convexity orientation, $N^-$, we define the section $N^+$ such that, for all $p$:

$$\partial X_p = \{N^+, N^-\}.$$ 

We call this vector field the \textbf{convexity coorientation} of $\Gamma$.

\textbf{Example:} If $(\hat{\Sigma}, \partial \hat{\Sigma})$ is a strictly convex immersed hypersurface in $M$, then $\Gamma := \partial \hat{\Sigma}$ is a locally strictly convex, codimension 2, immersed submanifold. Moreover, $\Gamma$ inherits a convexity orientation from $\hat{\Sigma}$ in the following manner: For $p \in \Gamma$, we identify each unit vector in $N_p \Gamma$ with the (oriented) hyperplane in $T_p M$ normal to that vector. $T_p \hat{\Sigma}$ defines a half-hyperplane with upward pointing unit normal in $X_p$. Let $H_p$ be another (oriented) hyperplane in $X_p$ that is close to $T_p \hat{\Sigma}$. We say that $H_p$ lies above (resp. below) $T_p \hat{\Sigma}$ if and only if it is a graph over (resp. beneath) $T_p \hat{\Sigma}$. We extend this to an order on $X_p$, and define $N^-(p)$ to be the end point of $X_p$ lying below $T_p \hat{\Sigma}$.

More formally, for $p \in \Gamma$, let $E = T_p M/T_p \Gamma$. $E$ is a two dimensional vector space. Moreover, $N_p \Gamma$ projects down to a circle, $S_p$, in $E$. We consider $X_p$ as a subinterval of $S_p$. Let $N_p \in X_p$ be the outward pointing exterior normal to $\hat{\Sigma}$ at $p$. $T_p \hat{\Sigma}$ defines a half-hyperplane which projects down to a half line in $E$. This half-line is parallel to the tangent line to $X_p$ at $N_p$, and thus defines an orientation on $S_p$ at $N_p$. $N^-(p)$ is then the boundary point of $S_p$ towards which $T_p \hat{\Sigma}$ points. □

Suppose that $\Gamma$ is locally strictly convex with convexity orientation, and suppose that $\partial \Sigma$ is a strictly convex immersed hypersurface such that $\partial \Sigma = \Gamma$. We say that $\Sigma$ is \textbf{compatible} with the orientation on $\Gamma$ if and only if the convexity orientation induced on $\Gamma$ by $\Sigma$ coincides with the pre-existing convexity orientation on $\Gamma$.

\section*{5 - First Order Upper Bounds.}

Let $M^{n+1}$ be an $(n + 1)$-dimensional Riemannian manifold. Let $\Gamma^{n-1} \subseteq M$ be a strictly convex, codimension 2, embedded submanifold with convexity orientation. Let $N^-$ and $N^+$
be the convexity orientation and coorientation respectively of $\Gamma$. Let $\Sigma$ be a strictly convex immersed hypersurface in $M$ such that $\partial \Sigma = \Gamma$. Suppose, moreover that $\Sigma$ is compatible with the convexity orientation on $\Gamma$. We denote by $N_\Sigma$ the outward pointing unit normal over $\Sigma$.

First order bounds near the boundary follow from the following result:

**Lemma 5.1**

Choose $\theta > 0$. There exists $r > 0$, which only depends on $M$, $\Gamma$ and $\theta$ such that if the angle between $N_\Sigma$ and $N^+\Gamma$ is always greater than $\theta$, then, for all $p \in \Gamma$, there exists a convex subset $K \subseteq B_r(p)$ such that the connected component of $\Sigma \cap B_r(p)$ containing $p$ is embedded and is a subset of $\partial K$.

**Proof:** This follows immediately from Proposition 5.2 (below). □

We establish the framework. Choose $p \in \Gamma$. Choose $r_1 > 0$, and denote the connected component of $\Gamma \cap B_{r_1}(p)$ containing $p$ by $\Gamma_0$. Reducing $r_1$ if necessary, there exists a smooth, embedded, locally strictly convex hypersurface $\hat{\Sigma} \subseteq B_{r_1}(p)$ such that $\partial \hat{\Sigma} \subseteq \partial B_{r_1}(p)$ and $\Gamma \subseteq \hat{\Sigma}$. We may suppose, moreover, that $\hat{\Sigma}$ bounds a convex set, $\hat{K}$, in $B_{r_1}(p)$. In the sequel, we will identify $M$ with $B_{r_1}(p)$, reducing $r_1$ at various stages whenever necessary. We may thus assume that $\Gamma$ divides $\hat{\Sigma}$ into two connected components: $\hat{\Sigma}^+$ and $\hat{\Sigma}^-$ which correspond to the interior and exterior respectively of $\Sigma$ with respect to $\Gamma$.

Let $N_{\hat{\Sigma}}$ be the unit normal vector field over $\hat{\Sigma}$. We may suppose that $N_{\hat{\Sigma}}$ makes an angle of less than $\theta/2$ with $N^+\Gamma$.

Since $\hat{\Sigma}$ is strictly convex, there exists $\epsilon > 0$ such that the shape operator of $\hat{\Sigma}$ is bounded below by $\epsilon \text{Id}$. Let $H$ be a strictly convex embedded hypersurface tangent to $\hat{\Sigma}$ at $p$ whose second fundamental form is strictly bounded above by $\delta \text{Id}$, for $\delta < \epsilon/2$. Let $(H_t)_{t \in [-\tau, \tau]}$ be the foliation of $M$ by hypersurfaces equidistant to $H$. We may assume that each leaf of this foliation is embedded, strictly convex and complete with second fundamental form strictly bounded above by $\delta \text{Id}$. Moreover, we may assume that $H_0 = H$ meets $\hat{\Sigma}$ at a single point, $p$. Thus, the upward pointing normal of $H_0$ coincides with that of $\hat{\Sigma}$ at this point.

Each leaf of $(H_t)_{t \in [-\tau, \tau]}$ divides $M$ into two connected components, one of which we say lies above the leaf, and the other of which we say lies below the leaf. Recalling section 3, we say that a subset $K$ of $M$ is **semi-convex** with respect to a leaf $H_t$ if and only if:

(i) it lies above that leaf; and

(ii) if $\gamma$ is a geodesic segment lying above $H_t$ whose endpoints are elements of $K$, then the whole of $\gamma$ is contained in $K$.

**Remark:** Importantly, in contrast to the situation considered in Section 3, $H_t$ is not contained in $K$. Semi-convexity is therefore no longer necessarily preserved by taking limits. This is a delicate point which will be discussed presently.

We extend $\Sigma$ to a (piecewise smooth) convex immersed hypersurface by adjoining to it $\hat{\Sigma}^-$ and denote the resulting immersed hypersurface by $\Sigma$. For all $t$, let $\Sigma_t$ be the connected component of $\Sigma$ lying above $H_t$ and containing $p$. 

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Lemma 5.1 follows immediately from the following proposition by taking intersections with a small ball about $p$:

**Proposition 5.2**

There exists $t_0 < 0$ (which only depends on $M$, $\Gamma$, $\hat{\Sigma}$, $\theta$ and $r_1$) such that $\hat{\Sigma}_{t_0}$ is embedded and (along with $H_{t_0}$) bounds a semi-convex set.

**Proof:** This follows immediately from Proposition 5.8 (below). \(\square\)

Let $T$ denote the set of all $t < 0$ such that, for all $s \in ]-t, 0[$:

(i) $\hat{\Sigma}_s$ is embedded;

(ii) $\hat{\Sigma}_s \subseteq \hat{K}$;

(iii) $\hat{\Sigma}_s$ bounds a semi-convex set above $H_s$; and

(iv) $\hat{\Sigma}_s$ intersects $H_s$ transversally along $\partial \hat{\Sigma}_s$.

**Proposition 5.3**

$T$ is non-empty.

**Proof:** Since $\hat{\Sigma}$ is a piecewise smooth, strictly convex immersion, there exists $0 < r_2 < r_1$ (which does depend on $\Sigma$) such that the connected component of the intersection of $\hat{\Sigma}$ with $B_{r_2}(p)$ containing $p$ is embedded and bounds a convex set. The portion of this convex set lying above $H_t$ for $t$ small is trivially semi-convex, and (i), (ii) and (iii) are therefore satisfied for all small $t$ less than 0, likewise so is (iv), and the result follows. \(\square\)

Let $t_0$ be the infimum of $T$. We will obtain upper bounds for $t_0$, from which Proposition 5.2 will follow. The first step involves proving that $\hat{\Sigma}_{t_0}$ is transverse to $H_{t_0}$. The main geometric obstacle is the possibility that the outward pointing normal to $\Sigma_{t_0}$ points upwards from $H_{t_0}$. This is dealt with by the following observation:

**Proposition 5.4**

For all $\theta' < \theta$, there exists $t_1 < 0$ (which only depends on $\hat{\Sigma}$, $\Gamma$, $M$, $\theta$ and $\theta'$) such that, if $d$ is the (signed) distance function in $M$ to $H = H_0$, and if $t_0 > t_1$, then, throughout $\Sigma_{t_0}$:

$$\langle N_{\Sigma}, \nabla d \rangle \leq \cos(\theta') < 1.$$  

**Proof:** Let $\nabla$ and $\nabla^\Sigma$ denote the Levi-Civita covariant derivatives over $M$ and $\Sigma$ respectively. Define the function $\phi : \Sigma \to \mathbb{R}$ by:

$$\phi = \langle N, \nabla d \rangle.$$  

Let $A$ be the shape operator of $\Sigma$. If $X$ is a vector field over $\Sigma$, then:

$$X \phi = \langle \nabla_X N_{\Sigma}, \nabla d \rangle + \langle N_{\Sigma}, \nabla_X \nabla d \rangle$$

$$= \langle A \cdot X, \nabla d \rangle + \text{Hess}(d)(N_{\Sigma}, X)$$

$$= \langle A \cdot X, \nabla^\Sigma d \rangle + \text{Hess}(d)(N_{\Sigma}, X).$$
The final line follows since the normal component of $A \cdot X$ vanishes. Now let $X = \nabla^\Sigma d/\|\nabla^\Sigma d\|^2$. Since $A$ is positive definite:

$$X \phi \geq \text{Hess}(d)(N_\Sigma, X).$$

Since $d$ is the distance to a hypersurface, $\|\nabla d\| = 1$ is constant, and so $\text{Hess}(d)(\nabla d, \cdot)$ vanishes. Thus, if $N_0$ denotes the component of $N_\Sigma$ tangent to the foliation, $(H_t)_{t \in [-\tau, \tau]}$, of level subsets of $d$, then:

$$X \phi \geq \text{Hess}(d)(N_0, X).$$

However:

$$\|N_0\|^2 = 1 - \langle N_\Sigma, \nabla d \rangle^2 = \|\nabla^\Sigma d\|^2.$$

Thus $N_0/\|\nabla^\Sigma d\|$ has norm equal to 1. Since the shape operator of $H_t$ is bounded above by $\delta I$ for all $t$, the norm of $\text{Hess}(d)$ is also bounded above by $\delta$. Thus:

$$X \phi \geq -\delta.$$

However:

$$X d = \langle X, \nabla^\Sigma d \rangle = 1.$$

Thus, if $\gamma : [0, \tau] \to \Sigma$ is an integral curve of $X$ starting at $q$, then $\gamma(s)$ meets $\Gamma$ for some $s \leq |t_0|$. There therefore exists $q' \in \Gamma$ such that:

$$\phi(q') \geq \phi(q) - \epsilon |t_0| / 2 \Rightarrow \phi(q) \leq \phi(q') + \epsilon |t_0| / 2,$$

Choosing $t_1$ sufficiently small, for all $q' \in \Gamma_{t_1}$:

$$\phi(q') + \epsilon |t_1| / 2 \leq \cos(\theta').$$

The result follows. □

**Proposition 5.5**

There exists $t_1 < 0$ (which only depends on $M$, $\hat{\Sigma}$, $\theta$ and $r_1$) such that, if $t_0 > t_1$, then $\hat{\Sigma}_{t_0}$ intersects $H_{t_0}$ transversally along $\partial \hat{\Sigma}_{t_0}$.

**Proof:** Suppose the contrary. Choose $q \in \partial \hat{\Sigma}_{t_0}$ such that $\hat{\Sigma}$ is tangent to $H_{t_0}$ at $q$. The normal to $\hat{\Sigma}$ at $q$ either points downwards into $H_{t_0}$ or upwards from $H_{t_0}$. By reducing $r_1$ if necessary, we may assume that the normal does not point upwards over $\hat{\Sigma}^{-} \setminus \{p\}$. By Proposition 5.4, for $t_1$ sufficiently small, the normal doesn’t point upwards over $\Sigma$ either, and it therefore does not point upwards anywhere over $\hat{\Sigma}_{t_0}$.

We now show that the normal cannot point downwards. Since $\hat{\Sigma}$ and $H_{t_0}$ are strictly convex with opposing normals, they meet at a single point. For $t > t_0$, let $\hat{\Sigma}'_t$ denote the connected component of $\hat{\Sigma}$ lying below $H_t$ containing $q$. Since $\hat{\Sigma}$ is piecewise smooth, for $t$ sufficiently close to $t_0$, $\hat{\Sigma}'_t$ is topologically a ball whose boundary is an embedded
topological sphere in $H_t$. Moreover, $\partial \tilde{\Sigma}'_t$ is a subset of $\partial \tilde{\Sigma}_t$. However, for all $s \in (t_0, 0]$, $\tilde{\Sigma}_s$ is transverse to $H_s$ and does not self intersect. $\partial \tilde{\Sigma}_t$ is thus also an embedded topological sphere. It follows that $\partial \tilde{\Sigma}_t$ and $\partial \tilde{\Sigma}'_t$ coincide, and $\tilde{\Sigma}$ is therefore an embedded topological sphere lying above $H_{t_0}$. $\Gamma$ is therefore not contained in $\tilde{\Sigma}$, which is absurd, and thus the normal to $\tilde{\Sigma}'_t$ does not point downwards, and this completes the proof. $\square$

The next step uses the fact that $K_{t_0}$, being the limit of a sequence of semi-convex sets, is also semi-convex. Despite being an intuitive result, its proof is rather technical, and is deferred to Section 7.

**Proposition 5.6**

There exists $t_1 < 0$ (which only depends on $M$, $\tilde{\Sigma}$, $\theta$ and $t_1$), such that, if $t_0 > t_1$, then $\partial \tilde{\Sigma}_{t_0}$ is embedded in $H_{t_0}$ and bounds an open set.

**Proof:** For $t > t_0$, let $K_t$ be the semi-convex set bounded by $\tilde{\Sigma}_t$ and $N_t$. By Proposition 7.1, $K_{t_0}$ is also semi-convex. By the preceeding proposition, $\tilde{\Sigma}_{t_0}$ is transverse to $H_{t_0}$ along $\partial \tilde{\Sigma}_{t_0}$. It follows that $\partial \tilde{\Sigma}_{t_0}$ is a (piecewise smooth) immersed submanifold of $H_{t_0}$. Suppose it is not embedded. Since $\partial \Sigma_t$ is embedded for all $t > t_0$, there exist two open subsets $\Sigma'_1$, $\Sigma'_2 \subseteq \Sigma_{t_0}$ such that:

(i) $\Sigma'_1$ and $\Sigma'_2$ are embedded; and

(ii) $\Sigma'_1 \cap H_{t_0}$ and $\Sigma'_2 \cap H_{t_0}$ meet tangentially at some point $p$.

Since $\tilde{\Sigma}_{t_0}$ bounds $K_{t_0}$, the hypersurfaces $\Sigma'_1$ and $\Sigma'_2$ divide a neighbourhood of $p$ above $H_{t_0}$ into three (roughly) wedge-shaped open sets. Consider the central one of these three wedges. It is either a subset of $K_{t_0}$ or a subset of its complement. If it is a subset of $K_{t_0}$, then we say that $\Sigma'_1 \cap H_{t_0}$ and $\Sigma'_2 \cap H_{t_0}$ lie on each others interior. Otherwise, we say that they lie on each others exterior.

Suppose that $\Sigma'_1 \cap H_{t_0}$ and $\Sigma'_2 \cap H_{t_0}$ lie on each others interior. Let $P_1$ and $P_2$ be the respective tangent hyperplanes of $\Sigma'_1$ and $\Sigma'_2$ at $p$. We identify these with their images under the exponential map. $P_1$ and $P_2$ do not coincide. Indeed, suppose the contrary. By strict convexity, the interiors of $\Sigma'_1$ and $\Sigma'_2$ coincide in a single point. This point is contained in $K_{t_0}$. However, $K_{t_0}$ is connected and also contains $p$, which is absurd and the assertion follows.

By convexity, near $p$, $\Sigma'_1$ lies above $P_1$ and $\Sigma'_2$ lies above $P_2$. However, the region lying above both $P_1$ and $P_2$ forms a wedge making an angle at $p$ strictly greater than 0 and strictly less than $\pi$. In particular, $\Sigma'_1$ and $\Sigma'_2$ intersect transversally at $p$. They therefore also intersect over a hypersurface contained inside this wedge. However, since $H_{t_0}$ is strictly convex, this wedge lies strictly above $H_{t_0}$, and therefore $\Sigma'_1$ and $\Sigma'_2$ also meet at some point above $H_{t_0}$. This contradicts the hypothesis that $\partial \tilde{\Sigma}_t$ is an embedded submanifold of $H_t$ for all $t > t_0$. It follows that these two submanifolds do not lie on each others interior.

Suppose that $\Sigma'_1 \cap H_{t_0}$ and $\Sigma'_2 \cap H_{t_0}$ lie on each others exterior. Let $\gamma$ be a geodesic arc, tangent to $H_{t_0}$ at $p$ and normal to the common tangent space of $\Sigma'_1 \cap H_{t_0}$ and $\Sigma'_2 \cap H_{t_0}$. Near $p$, $\gamma$ lies above $H_{t_0}$ and has endpoints inside $K_{t_0}$. Moving $\gamma$ upwards slightly yields a geodesic arc lying above $H_{t_0}$, having endpoints inside $K_{t_0}$ whilst itself not being contained
within $K_{t_0}$. This contradicts semi-convexity. It follows that these two submanifolds do not lie on each others exterior, and this completes the proof. □

**Proposition 5.7**

There exists $t_1 < 0$ (which only depends on $M$, $\hat{\Sigma}$, $\theta$ and $r_1$) such that, if $t_0 > t_1$, then $\hat{\Sigma}_t \subseteq \hat{K}$.

**Proof:** By Proposition 5.6, for $t$ sufficiently small, $\Sigma_t$ does not intersect $\hat{\Sigma}_t^-$. By Proposition 6.2, semi-convexity and the hypotheses on $\Sigma$ along the boundary, for all sufficiently small $t$, $\Sigma_t$ does not intersect $\hat{\Sigma}_t^+$. It is therefore contained within the set bounded by $\hat{\Sigma}_t$ and $H_t$, and the result follows. □

**Proposition 5.8**

There exists $t_1 < 0$ (which only depends on $M$, $\hat{\Sigma}$, $\theta$ and $r_1$) such that, if $t_0 > t_1$, then $t_0$ cannot be the infimum of $T$.

**Proof:** Let $t_1$ be as in Propositions 5.4, 5.5, 5.6 and 5.7 and suppose that $t_0 > t_1$. $\partial \hat{\Sigma}_{t_0}$ is embedded, is transverse to $H_{t_0}$, and is bounded away from $B_{r_1}(p)$. Thus, for all $t < t_0$ sufficiently close to $t_0$, $\hat{\Sigma}_t$ is embedded, is contained in $\hat{K}$, meets $H_t$ transversally, and, along with $H_t$, bounds a subset of $B_{r_1}(p)$. For all $t$, let $K_t$ be the closure of this subset. It thus remains to show that $K_t$ is semi-convex for all $t$ sufficiently close to $t_0$.

Suppose that there exists a sequence $(t_n)_{n \in \mathbb{N}} < t_0$ converging to $t_0$ such that, for all $n$, $K_{t_n}$ is not semi-convex. Then, for all $n$, there exists $p_n, q_n \in K_n := K_{t_n}$ and a geodesic arc $\gamma_n$ such that:

(i) $p_n$ and $q_n$ are the endpoints of $\gamma_n$;

(ii) $\gamma_n$ lies above $H_{t_n}$; and

(iii) there exists a point $r_n \in \gamma_n$ which lies inside $K_n$.

Without loss of generality, $(p_n)_{n \in \mathbb{N}}$, $(q_n)_{n \in \mathbb{N}}$, $(\gamma_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ converge to $p_0$, $q_0$, $\gamma_0$ and $r_0$ respectively. Trivially, $\gamma_0$ lies above $H_{t_0}$. Suppose first that $p_0 \neq q_0$. Suppose that $r_0$ does not coincide with either of the endpoints. Since $K_{t_0}$ is semi-convex, $\gamma_0$ lies inside $K_{t_0}$. $r_0$ therefore lies on the boundary of $K_{t_0}$, and $\gamma_0$ is therefore an interior tangent to $\hat{\Sigma}_{t_0}$ at this point, which contradicts local strict convexity. Likewise, if $r_0$ coincides with an end point, $p_0$, say, then $\gamma_0$ is contained inside $K_{t_0}$ and points outwards (or is tangent) to $\hat{\Sigma}_{t_0}$ at $p_0$, which also contradicts local strict convexity and semi-convexity. It follows that $p_0$ and $q_0$ coincide.

If $p_0 = q_0$ is an interior point of $K_{t_0}$, then $\gamma_n$ trivially lies inside $K_{t_n}$ for all sufficiently large $n$. Suppose therefore that $p := p_0 = q_0$ is a boundary point of $\hat{\Sigma}_{t_0}$. By local strict convexity, there exists a neighbourhood of $\hat{\Sigma}$ about $p$ which lies on the boundary of a convex set, $X$. For all $n$, the intersection of $X$ with the region lying above $H_{t_n}$ is a subset of $K_n$. However, for sufficiently large $n$, $p_n$ and $q_n$ both lie in $X$. For all such $n$, $\gamma_n$ is contained within $X$ and therefore within $K_n$, which is absurd.

There therefore exists $\epsilon$ such that, for all $t > t_0 - \epsilon$, $\hat{\Sigma}_t$ satisfies the hypotheses defining $T$, and therefore $t \in T$. This is absurd and the result follows. □
6 - Parabolic Limits.

Let $M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $\hat{\Sigma}^n$ be a locally strictly convex immersed hypersurface in $M$. Let $\Gamma \subseteq \hat{\Sigma}$ be an embedded hypersurface. Let $\epsilon > 0$ be such that the shape operator of $\hat{\Sigma}$ is everywhere bounded below by $\epsilon$. Choose $p \in \Gamma$. Let $H^n$ be a strictly convex embedded hypersurface of $M$ which is an exterior tangent to $\hat{\Sigma}$ at $p$. Let $\delta > 0$ be such that the shape operator of $H$ is everywhere bounded above by $\delta$ and suppose that $\delta < \epsilon/2$. For simplicity, we assume throughout the rest of this section that the shape operator of $H$ at $p$ is equal to $\delta I_d$. The general case is similar.

Let $d$ be the signed distance function in $M$ to $H$. In particular, for $q \in \hat{\Sigma}$ near $p$, $d(q) \leq 0$. For all $t$, let $H_t$ be the level hypersurface at distance $t$ from $H$. For small $t > 0$, let $\hat{\Sigma}_t$ and $\Gamma_t$ be the connected components of $\hat{\Sigma}$ and $\Gamma$ respectively lying above $H_t$ and containing $p$, and let $\hat{K}_t$ denote the compact set bounded by $\hat{\Sigma}_t$ and $H_t$. For small $t$, $\Gamma_t$ divides $\hat{\Sigma}_t$ into two components, which we denote by $\hat{\Sigma}_t^+$ and $\hat{\Sigma}_t^-$.  

Choose $t_0 < 0$. Let $(p_n)_{n \in \mathbb{N}} \in \hat{K}_{t_0}$ be a sequence converging to $p$. We consider a geodesic chart for $H$ about $p$, and thus identify a neighbourhood of $p$ in $H$ with a neighbourhood of 0 in $T_pH$. Let $(e_1, ..., e_n)$ be an orthonormal basis for $T_pH$. There exists $r > 0$ such that $\hat{\Sigma}$ is the graph of a function, $f$ over $B_r(p)$. By Taylor’s Theorem, with respect to $(e_1, ..., e_n)$:

$$f(x) = -\langle x|A|x \rangle + O(||x||^3),$$

where $A$ is a positive definite matrix. With respect to these coordinates, for all $n$, $p_n = (q_n, s_n)$, where $q_n \in T_pH$ and $s_n < 0$. For all $n$, we define $\hat{f}_n : B_r/\sqrt{|s_n|}(p) \to ]-\infty, 0]$ and $\hat{\gamma}_n \in T_{pH}$ by:

$$\hat{f}_n(x) = f(\sqrt{|s_n|}x)/|s_n|; \quad \hat{\gamma}_n = q_n/\sqrt{|s_n|}.$$

Trivially, $(\hat{f}_n)_{n \in \mathbb{N}}$ converges in the $C^\infty_{loc}$ sense over $T_pH$ to $\hat{f}_0$, where:

$$\hat{f}_0(x) = -\langle x|A|x \rangle.$$

Moreover, for all $n$, since $p_n \in \hat{K}_{s_n}$:

$$|f(q_n)| \leq |s_n| \Rightarrow \text{LimSup}_{n \to \infty} \frac{\epsilon}{2} \|\hat{\gamma}_n\|^2 \leq \text{LimSup}_{n \to \infty} \frac{1}{|s_n|} |f(q_n)| \leq 1.$$

There thus exists $\hat{\gamma}_0 \in H$ towards which $(\hat{\gamma}_n)_{n \in \mathbb{N}}$ subconverges. In particular, $\hat{f}_0(q_0) \geq -1$. We call $(\hat{f}_0, \hat{\gamma}_0)$ a parabolic limit of $(\hat{\Sigma}, q_n)_{n \in \mathbb{N}}$.

Likewise, if we suppose that $(e_1, ..., e_{n-1})$ is tangent to $\Gamma$ at $p$, then, reducing $r$ if necessary, the projection of $\Gamma$ onto $H$ is the graph of some function $g$, over the space spanned by $(e_1, ..., e_{n-1})$. For all $n$, we define $\hat{g}_n : B_r/\sqrt{|s_n|}(p) \to \mathbb{R}$ by:

$$\hat{g}_n(x') = g(\sqrt{|s_n|}x')/\sqrt{|s_n|}.$$
Trivially, \((\hat{g}_n)_{n \in \mathbb{N}}\) subconverges in the \(C^\infty_{\text{loc}}\) sense over the space spanned by \((e_1, \ldots, e_{n-1})\) to \(\hat{g}_0 := 0\). It follows that the parabolic limit of \(\Gamma_{s_n}\) is the intersection of the graph of \(\hat{f}_0\) with a vertical hyperplane in \(\mathbb{R}^n \times \mathbb{R}\).

For \(p \in M\), we call a **geodesic hyperplane** at \(p\) an immersed hypersurface consisting of geodesics passing through \(p\). Explicitly, \(P \subseteq M\) is a geodesic hyperplane if and only if there exists a hyperplane \(H \subseteq T_pM\) such that:

\[
P = \{\text{Exp}(V_p) \text{ s.t. } V_p \in H\}.
\]

For all \(n\), let \(P_n\) be the geodesic hyperplane tangent to \(H_{s_n}\) at \(p_n\). Reducing \(r\) if necessary, \(P_n\) is the graph of the function \(\phi_n : B_r(q_n) \to \mathbb{R}\), where, by convexity:

\[
-s_n \leq \phi_n(x) \leq -s_n + \langle x - q_n | B_n | x - q_n \rangle + O(\|x\|^3),
\]

where \((B_n)_{n \in \mathbb{N}}\) converges to \(\delta \text{Id}\). For all \(n\), we define \(\hat{\phi}_n\) by:

\[
\hat{\phi}_n(x) = \phi_n(\sqrt{|s_n|}x)/|s_n|.
\]

\((\hat{\phi}_n)_{n \in \mathbb{N}}\) converges in the \(C^\infty_{\text{loc}}\) sense over \(T_pH\) to \(\hat{\phi}_0\) where:

\[
\hat{\phi}_0(x) = \delta \|x - \hat{q}_0\|^2 - 1.
\]

Thus, the parabolic limit of the geodesic hyperplanes tangent to \(H_{s_n}\) at \(p_n\) is a paraboloid on \((\hat{q}_0, -1)\). Finally, in like manner, the parabolic limit of a sequence of geodesics tangent to \(H_{s_n}\) at \(p_n\) is the intersection of this paraboloid with a vertical plane in \(\mathbb{R}^n \times \mathbb{R}\).

Parabolic limits are of use in obtaining technical results concerning \(\Sigma\).

**Proposition 6.1**

For \(\delta\) sufficiently small, there exists \(t_0 < 0\) (which only depends on \(\hat{\Sigma}, \Gamma, H\) and \(M\)) such that, for all \(q \in K_{t_0} \setminus \{p\}\), if \(t > t_0\) is such that \(q \in H_t\), and if \(P\) is the geodesic hyperplane tangent to \(H_t\) at \(q\), then:

(i) \(P\) intersects \(\hat{\Sigma}_t\) transversally; and

(ii) \(P\) intersects \(\Gamma_t\) transversally.

**Proof:** (i) Suppose the contrary. Let \((p_n)_{n \in \mathbb{N}} \in \hat{K}_{t_0}\) be a sequence converging to \(p\). For all \(n\), let \(s_n < 0\) be such that \(p_n \in H_{s_n}\) and let \(P_n\) be the geodesic hyperplane tangent to \(H_{s_n}\) at \(p_n\). Trivially, \(P_n\) intersects \(\hat{\Sigma}_n\) non-trivially for all \(n\). Suppose that, for all \(n\), \(P_n\) is tangent to \(\hat{\Sigma}_n\) at some point. It follows that the parabolic limit of \((P_n)_{n \in \mathbb{N}}\) is tangent to the parabolic limit of \((\hat{\Sigma}_n)_{n \in \mathbb{N}}\) at some point. This is absurd, and the first assertion follows.

(ii) Suppose the contrary. Let \((p_n)_{n \in \mathbb{N}} \in \hat{\Sigma}\) be a sequence converging towards \(p\). For all \(n\), let \(s_n < 0\) be such that \(p_n \in H_{s_n}\), let \(\Gamma_n = \Gamma_{s_n}\) and let \(P_n\) be the geodesic hyperplane tangent to \(H_{s_n}\) at \(p_n\). We suppose that, for all \(n\):

\[
P_n \cap \Gamma_n = \emptyset.
\]

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The parabolic limit of $P_n$ intersects the parabolic limit of $\Gamma_n$ transversally. Thus, for sufficiently large $n$, $P_n \cap \Gamma_n \neq \emptyset$, which is absurd. It follows that, for $t_0$ sufficiently small, $P$ intersects $\Gamma_{t_0}$. Transversality follows as in the proof of part $(i)$, and this completes the proof. □

**Proposition 6.2**

Choose $\theta \in ]0, \pi/2[$. For $\delta$ sufficiently small, there exists $t_0 < 0$ (which only depends on $\hat{\Sigma}$, $\Gamma$, $H$, $M$ and $\theta$) such that for $t > t_0$ and for all $q \in \hat{\Sigma}_t \cap H_t$, there exists a geodesic segment, $\gamma$, joining $q$ to $\Gamma$ such that the hyperplane spanned by $\partial_t \gamma$ and $T\gamma$ at the point of intersection of $\gamma$ with $\Gamma$ makes an angle strictly less than $\theta$ with $T\hat{\Sigma}$.

**Proof:** Suppose the contrary. Let $(p_n)_{n \in \mathbb{N}} \in \hat{\Sigma}$ be a sequence converging to $p$, and let $\hat{p}_0$ be its parabolic limit. For all $n$, let $s_n < 0$ be such that $p_n \in H_{s_n}$ and let $\gamma_n$ be a geodesic segment tangent to $H_{s_n}$ at $p$ and terminating in $\Gamma_{s_n}$. Suppose that, for all $n$, the hyperplane spanned by $\partial_t \gamma_n$ and $T\gamma_n$ at the point of intersection of $\gamma_n$ with $\Gamma_{s_n}$ makes an angle of at least $\theta$ with $T\hat{\Sigma}$. Let $\hat{\gamma}_0$ and $\hat{\Gamma}_0$ be the parabolic limits of $(\gamma_n)_{n \in \mathbb{N}}$ and $(\Gamma_{s_n})_{n \in \mathbb{N}}$ respectively. Then, at its point of intersection with $\hat{\Gamma}_0$, $\hat{\gamma}_0$ is tangent to the vertical hyperplane containing $\hat{\Gamma}_0$. $\hat{\gamma}_0$ is thus entirely contained in this vertical hyperplane. It follows that every parabolic limit of every sequence of geodesic segments joining $(p_n)_{n \in \mathbb{N}}$ to $\Gamma$ is contained in the vertical hyperplane containing $\hat{\Gamma}_0$. When $\hat{p}_0 \notin \hat{\Gamma}_0$, this is trivially absurd. When $\hat{p}_0 \in \hat{\Gamma}_0$, there exists a parabolic limit of such geodesic segments which is normal to the hyperplane containing $\hat{\Gamma}_0$, which is also absurd. The result follows. □

**Proposition 6.3**

For $\delta$ sufficiently small, there exists $t_0 < 0$ (which only depends on $\hat{\Sigma}$, $\Gamma$, $H$ and $M$) such that, for $t > t_0$, if $\gamma$ is a geodesic segment lying in $K_{t_0}$ such that:

(i) $\gamma$ is tangent to $H_t$; and

(ii) the endpoints of $\gamma$ both lie in $\Gamma$,

then there exists a sequence of geodesic segments $(\gamma_n)_{n \in \mathbb{N}}$ converging to $\gamma$ such that, for all $n$:

(i) $\gamma_n$ is tangent to $H_t$; and

(ii) the end points of $\gamma_n$ lie in $\hat{\Sigma}_{t_0}^-$.

**Proof:** Suppose the contrary. Let $(p_n)_{n \in \mathbb{N}} \in \hat{K}_{t_0}$ be a sequence converging to $p$. For all $n$, let $s_n < 0$ be such that $p_n \in H_{s_n}$ and let $\gamma_n$ be a geodesic segment tangent to $H_{s_n}$ at $p_n$ with both end points in $\Gamma_{s_n}$. We suppose that, for all $n$, there exists $\epsilon_n > 0$ such that if $q_n \in H_{s_n}$ is such that $d(q_n, p_n) < \epsilon_n$, then no geodesic segment tangent to $H_{s_n}$ at $q_n$ has both endpoints in $\hat{\Sigma}_{s_n}^-$. Let $\hat{\gamma}_0$, $\hat{p}_0$, $\hat{\Gamma}_0$ and $\hat{\Sigma}_{s_n}^-$ be the parabolic limits of $(\gamma_n)_{n \in \mathbb{N}}$, $(p_n)_{n \in \mathbb{N}}$, $(\Gamma_n)_{n \in \mathbb{N}}$ and $(\hat{\Sigma}_{s_n}^-)_{n \in \mathbb{N}}$ respectively. Let $V_0$ be the horizontal unit vector at $(p_0, -1)$ normal to the vertical hyperplane containing $\hat{\Gamma}_0$ and pointing towards $\hat{\Sigma}_{0}^-$. For all $n$, let $V_n$ be a
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unit vector tangent to $H_{s_n}$ at $p_n$ and suppose that $\hat{V}_0$ is the parabolic limit of $(V_n)_{n\in\mathbb{N}}$. For all $n$, let $\eta_n : \mathbb{R} \to H_{s_n}$ be the geodesic in $H_{s_n}$ such that:

$$\partial_t \eta_n(0) = V_n,$$

and let $X_n$ be the parallel transport of $\partial_t \gamma_n(0)$ along $\eta_n$ (with respect to the Levi-Civita covariant derivative of $H_{s_n}$). Let $\text{Exp}$ be the exponential map of $M$ and for all $n$ define:

$$\phi_{n,t}(s) = \text{Exp}(sX_n(t)).$$

If $\hat{X}_0$ is the unit tangent vector to $\hat{\gamma}_0$ at $\hat{p}_0$, then the parabolic limit of $(\phi_n)_{n\in\mathbb{N}}$ is $\hat{\phi}_{0,t}(s)$, where:

$$\hat{\phi}_{0,t}(s) = (s\hat{X}_0 + t\hat{V}_0, \delta s^2 - 1).$$

The intersection of this family with $\hat{\Sigma}_0^-$ is transverse to $\hat{\Gamma}_0$ at the intersection of $\hat{\gamma}_0$ with $\hat{\Gamma}_0$. Thus, for sufficiently large $n$, and sufficiently small $t$, the two endpoints of the geodesic segment $s \mapsto \phi_{n,t}(s)$ both lie in $\hat{\Sigma}_q^-$. This is absurd, and the result follows. □

7 - Semi-Convexity.

In this section we show that the property of being semi-convex is preserved after taking limits. Using the same notation as in the Section 5, we show:

**Proposition 7.1**

There exists $t_1 < 0$ (which only depends on $M$, $\hat{\Sigma}$, $\theta$ and $r_1$) such that, if $t_0 \geq t_1$, then $\hat{\Sigma}_t$ bounds a semi-convex set above $H_{t_0}$.

For $p \in M$, if $P$ is a geodesic hyperplane at $p$ (see Section 6), then we say that two points $q_1, q_2 \in P$ are **coaxial** if and only if they both lie on the same radial geodesic on opposite sides of $p$. We require the following technical result:

**Lemma 7.2**

Choose $\varphi > 0$. Let $K \subseteq M$ be compact. There exists $r > 0$ (which only depends on $\varphi$ and $K$) such that, if $P$ is a geodesic hyperplane at $p \in K$, if $q_1, q_2 \in P$ are coaxial points and if $X$ is a Jacobi field over the geodesic joining $q_1$ to $q_2$ such that:

(i) $d(q_1, p), d(q_2, p) < r$;

(ii) $\|X(q_0)\| \leq 1$ and $X$ lies strictly above $TP$ at $q_0$; and

(iii) $\|X(q_1)\| = 1$ and $X$ lies strictly above $TP$ at $q_1$, making an angle of at least $\varphi$ with $TP$ at that point.

Then $X$ lies strictly above $TP$ at every point of the geodesic joining $q_0$ to $q_1$. 20
Proof: Assume the contrary. Let \((r_n)_{n \in \mathbb{N}}\) be a sequence converging to 0. For all \(n\), let \(p_n \in K\) be a point, \(P_n\) a geodesic hyperplane at \(p_n\), \(q_{1,n}, q_{2,n}\) two coaxial points in \(P_n\) and \(X_n\) a Jacobi field over the geodesic joining \(q_{1,n}\) to \(q_{2,n}\) such that:

(i) \(\max(d(q_{1,n}, p_n), d(q_{2,n}, p_n)) = r_n\);
(ii) \(\|X_n(q_{1,n})\| \leq 1\) and \(X_n\) lies strictly above \(TP_n\) at \(q_{1,n}\); and
(iii) \(\|X_n(q_{2,n})\| = 1\) and \(X_n\) lies strictly above \(TP_n\) at \(q_{2,n}\), making an angle of at least \(\varphi\) with \(TP_n\) at this point.

Suppose, moreover, that, for all \(n\), \(X_n\) is tangent to \(TP_n\) at some point lying between \(q_{1,n}\) and \(q_{2,n}\), \(x_n\), say. By compactness, there exists \(p_0 \in K\) towards which \((p_n)_{n \in \mathbb{N}}\) subconverges. Let \(g\) be the Riemannian metric of \(M\). For all \(n\), define \(g_n = r_n^{-2}g\).

The sequence of pointed manifolds \((M, g_n, p_n)_{n \in \mathbb{N}}\) converges towards \((\mathbb{R}^{n+1}, g_{\text{Euc}}, 0)\) in the \(C^\infty\) Cheeger/Gromov sense, where \(g_{\text{Euc}}\) is the Euclidean metric over \(\mathbb{R}^{n+1}\). For all \(n\), \(P_n\) is also a geodesic hyperplane of \((M, g_n)\) and so \((P_n, p_n)_{n \in \mathbb{N}}\) subconverges in the \(C^\infty\) Cheeger/Gromov sense for pointed, immersed submanifolds to a pointed, affine hyperplane \((P_0, 0)\). Likewise, there exist coaxial points \(q_{1,0}, q_{2,0} \in P_0\), a Jacobi field \(X_0\), and a point \(x_0\) lying between \(q_{1,0}\) and \(q_{2,0}\) towards which \((q_{1,n})_{n \in \mathbb{N}}, (q_{2,n})_{n \in \mathbb{N}}, (r_n X_N)_{n \in \mathbb{N}}\) and \(x_0\) subconverge respectively. Moreover:

(i) \(\max(d(q_{1,0}, 0), d(q_{2,0}, 0)) = 1\);
(ii) \(\|X_0(q_{1,0})\| \leq 1\) and \(X_0\) lies (not necessarily strictly) above \(TP\) at \(q_{1,0}\); and
(iii) \(\|X_0(q_{2,0})\| = 1\) and \(X_0\) lies strictly above \(TP\) at \(q_{2,0}\).

It follows that \(X_0\) is not tangent to \(P\) at any point along the closed geodesic joining \(q_{1,0}\) to \(q_{2,0}\), except possibly at \(q_{1,0}\). Moreover, if \(X_0\) is tangent to \(P\) at \(q_{1,0}\), then its derivative in the direction normal to \(P\) at this point is non vanishing. However, \(X_0\) is tangent to \(TP\) at \(x_0\). It follows from the first assertion that \(x_0 = q_{1,0}\), but then the derivative of \(X_0\) in the direction normal to \(P\) at \(q_{1,0}\) vanishes, and this contradicts the second assertion. This is absurd and the result follows. \(\square\)

This lemma allows us to prove Proposition 7.1:

Proof of Proposition 7.1: Let \(K_{t_0}\) be the set bounded by \(\hat{\Sigma}_{t_0}\) and \(H_{t_0}\). Let \(\gamma : [0, 1] \to M\) be a geodesic above \(H_{t_0}\) with endpoints in \(K_{t_0}\). We aim to show that the whole of \(\gamma\) is contained in \(K_{t_0}\). It suffices to consider the case where both endpoints of \(\gamma\) lie in \(\hat{\Sigma}_{t_0}\). The remaining cases are similar and much simpler. Recall that \(\hat{\Sigma}_{t_0}\) divides into two components, \(\hat{\Sigma}_{t_0}^-\) and \(\Sigma_{t_0}\). These components have different properties and we thus consider the various resulting cases separately. Let \(\hat{K}_{t_0}\) be the set bounded by \(\hat{\Sigma}_{t_0}\) and \(H_{t_0}\). We may assume that \(\hat{K}_{t_0}\) is semi-convex. Since the endpoints of \(\gamma\) lie in \(\hat{\Sigma}_{t_0} \subseteq K_{t_0}\), the whole of \(\gamma\) therefore lies in \(K_{t_0}\). Thus, by choosing \(t_1\) sufficiently small, we may assume that \(\gamma\) is sufficiently short to satisfy the hypotheses of Proposition 7.1 with \(\varphi = \theta/2\).

Suppose that \(\gamma\) lies strictly above \(H_{t_0}\). Then there exists \(\epsilon > 0\) such that \(\gamma\) lies above \(H_{t_0 + \epsilon}\). Since \(\hat{\Sigma}_{t_0 + \epsilon}\) is semi-convex, \(\gamma\) lies in \(K_{t_0 + \epsilon} \subseteq K_{t_0}\) and the result follows in this case. We thus assume that \(\gamma\) meets \(H_{t_0}\) at some point, \(s \in [0, 1]\).
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Suppose that \( \gamma \) is transverse to \( H_{t_0} \) at \( s \). Then, \( s \) is an endpoint of \([0,1]\) and, without loss of generality, \( s = 0 \). By strict convexity of \( H_{t_0} \), \( \gamma([0,1]) \) lies strictly above \( H_{t_0} \). Suppose that \( \gamma(0) \) lies in \( \hat{\Sigma}^-_{t_0} \). By Proposition 6.1, both \( \Sigma \) and \( \Gamma \) are transverse to \( H_{t_0} \) at this point. There thus exists a smooth curve \( \eta : [0, \epsilon] \to M \) such that:

(i) \( \eta(0) = \gamma(0) \);
(ii) \( \partial_t \eta(0) \) is transverse to \( TH_{t_0} \);
(iii) for \( s > 0 \), \( \eta(s) \) lies strictly above \( H_{t_0} \); and
(iv) for all \( s \), \( \eta(s) \) lies in \( \hat{\Sigma}^- \).

For all \( s \in [0, \epsilon[ \), let \( \gamma_s \) be the unique geodesic joining \( \eta(s) \) to \( \gamma(1) \). For sufficiently small \( s \), \( \gamma_s \) lies strictly above \( H_{t_0} \). Since, for all \( t > t_0 \), \( \hat{\Sigma}_t \) is semi-convex, for all sufficiently small \( s \), \( \gamma_s \) is contained in \( K_{t_0} \). The result follows in this case by taking limits.

Suppose that \( \gamma(0) \) lies in \( \Sigma \setminus \Gamma \). By Proposition 5.4, after reducing \( t_1 \) if necessary, the outward pointing normal to \( \Sigma \) makes an angle of at least \( \theta/2 \) with \( H_{t_0} \) at \( \gamma(0) \). There therefore exists a smooth curve \( \eta : [0, \epsilon[ \to M \) such that:

(i) \( \eta(0) = \gamma(0) \);
(ii) \( \partial_t \eta(0) \) is transverse to \( TH_{t_0} \);
(iii) for \( s > 0 \), \( \eta(s) \) lies strictly above \( H_{t_0} \); and
(iv) for all \( s \), \( \eta(s) \) lies in \( K_{t_0} \).

For all \( s \in [0, \epsilon[, \) let \( \gamma_s \) be the unique geodesic joining \( \eta(s) \) to \( \gamma(1) \). For sufficiently small \( s \), \( \gamma_s \) lies strictly above \( H_{t_0} \), and the result follows in this case as before. This completes the case where \( \gamma \) is transverse to \( H_{t_0} \) at \( s \), and we thus suppose that \( \gamma \) is tangent to \( H_{t_0} \) at \( s \).

Let \( P \) be the geodesic hyperplane tangent to \( H_{t_0} \) at \( \gamma(s) \). Suppose that \( \gamma(0) \) and \( \gamma(1) \) both lie in \( \hat{\Sigma}^-_{t_0} \setminus \Gamma \). Since \( \Sigma \) bounds a strictly convex set, \( K \), \( \gamma \) is transverse to \( \hat{\Sigma}^- \) at \( \gamma(0) \) and \( \gamma(1) \) (for otherwise, by strict convexity, it could only intersect \( \hat{\Sigma}^- \) at one point, which is absurd). Let \( X \) be a Jacobi field over the geodesic joining \( \gamma(0) \) and \( \gamma(1) \) such that \( X \) equals the unit upward pointing normal to \( P \) at both endpoints. By Lemma 7.2, \( X \) lies everywhere above \( TP \). Thus, if \( \gamma_t \) is a geodesic variation of \( \gamma \) with Jacobi field \( X \), then, for sufficiently small \( t \), \( \gamma_t \) lies strictly above \( P \) and therefore also above \( H_{t_0} \). Moreover, by transversality, for sufficiently small \( t \), \( \gamma_t \) intersects \( \hat{\Sigma}^-_{t_0} \) at two points near \( \gamma(0) \) and \( \gamma(1) \). We thus obtain a family of geodesic segments lying strictly above \( H_{t_0} \) with endpoints in \( \hat{\Sigma}_{t_0} \) converging towards \( \gamma \). By semi-convexity, all these geodesic segments are contained within \( K_{t_0} \), and thus, taking limits, \( \gamma \) is contained within \( K_{t_0} \). This proves the result in this case.

Suppose that \( \gamma(0) \) lies in \( \hat{\Sigma}^-_{t_0} \setminus \Gamma \) and \( \gamma(1) \) lies in \( \Sigma \setminus \Gamma \). As before, \( \gamma \) is transverse to \( \hat{\Sigma} \) at \( \gamma(0) \). By Proposition 5.4, after reducing \( t_1 \) if necessary, the outward pointing normal to \( \Sigma \) makes an angle of at least \( \theta/2 \) with \( TP \) at \( \gamma(1) \). Let \( X \) be a Jacobi field over \( \gamma \) such that \( X(0) \) is the upward pointing normal vector over \( P \) at \( \gamma(0) \) and \( X(1) \) points into \( K_{t_0} \) making
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an angle of at least $\theta/2$ with $TP$ at $\gamma(1)$. By Lemma 7.2, $X$ lies everywhere above $TP$. Thus, if $\gamma_t$ is a geodesic variation of $\gamma$ with Jacobi field $X$, then, for sufficiently small $t$, $\gamma_t$ lies strictly above $P$ and therefore also above $H_{t_0}$. Moreover, for small $t$, $\gamma_t(1)$ lies inside $K_{t_0}$, and, by transversality, $\gamma_t$ intersects $\tilde{\Sigma}_{t_0}$ at some point near $\gamma(0)$. We thus obtain a family of geodesic segments lying strictly above $H_{t_0}$ with endpoints in $K_{t_0}$ converging towards $\gamma$. By semi-convexity, all these geodesic segments are contained within $K_{t_0}$, and thus, taking limits, $\gamma$ is contained within $K_{t_0}$. This proves the result in this case.

Suppose that both $\gamma(0)$ and $\gamma(1)$ lie in $\Sigma \setminus \Gamma$. By Proposition 5.4, after reducing $t_1$ if necessary, the outward pointing normal to $\Sigma$ makes an angle of at least $\theta/2$ with $P$ at both these points. Let $X$ be a Jacobi field over $\gamma$ such that both $X(0)$ and $X(1)$ point into $K_{t_0}$ at $\gamma(0)$ and $\gamma(1)$ respectively, making an angle of at least $\theta/2$ with $TP$ at these points. By Lemma 7.2, $X$ lies everywhere above $TP$, and the result follows in this case as before.

We now consider the case where at least one end point of $\gamma$ lies on $\Gamma$. Suppose that $\gamma(0)$ lies on $\Gamma$ but $\gamma(1)$ doesn’t. By Proposition 6.1, $\Gamma$ is transverse to $P$ at $\gamma(0)$. Let $X$ be a Jacobi field over $\gamma$ such that $X(0)$ is tangent to $\Gamma$ and points strictly upwards from $P$ at $\gamma(0)$. If $\gamma(1)$ lies in $\tilde{\Sigma}_{t_0}$, then we suppose that $X(1)$ is the upward pointing unit normal over $P$ at $\gamma(1)$. If $\gamma(1)$ lies in $\Sigma_{t_0}$, then we assume that $X(1)$ points into $K_{t_0}$ at $\gamma(1)$, making an angle of at least $\theta/2$ with $TP$ at this point. By Lemma 7.2, $X$ lies everywhere above $TP$, and the result follows in this case as before.

Finally suppose that both $\gamma(0)$ and $\gamma(1)$ lie on $\Gamma$. It follows by Proposition 6.3 that, after increasing $t_1$ if necessary, there exists a small deformation of $\gamma$ whose end points both lie on $\tilde{\Sigma}_{t_0} \setminus \Gamma$. We thus reduce this case to an earlier case, and this completes the proof. □

8 - Immersed Boundaries.

Let $M^{n+1}$ be an $(n + 1)$-dimensional manifold. We recall that the reasoning of Section 5 is only valid when the boundary is embedded. We now show how this reasoning may be adapted by a simple modification to also treat the case where the boundary is permitted to have self intersections.

Let $\Gamma^{n-1} = (i, (G^{n-1}, \partial G^{n-1}))$ be a compact, codimension 2, immersed submanifold in $M$. We say that $\Gamma$ is generic if and only if, for all $p \ne q$ such that $i(p) = i(q)$:

$$T_p\Gamma \ne T_q\Gamma.$$ 

This definition is motivated by the following elementary result:

**Proposition 8.1**

(i) Let $\Gamma \subseteq M$ be a compact, codimension 2, immersed submanifold. There exists a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of generic, compact, codimension 2, immersed submanifolds which converges to $\Gamma$ in the $C^\infty$ sense.

(ii) Let $(\Gamma_t)_{t \in [0,1]} \subseteq M$ be a smooth family of compact, codimension 2, immersed submanifolds such that $\Gamma_0$ and $\Gamma_1$ are generic. There exists a sequence $(\Gamma_{n,t})_{n \in \mathbb{N}}$...
of smooth families of generic, compact, codimension 2, immersed submanifolds such that:

(a) for all \(n\), \(\Gamma_{n,0} = \Gamma_0\) and \(\Gamma_{n,1} = \Gamma_1\); and

(b) \((\Gamma_{n,t})_{n \in \mathbb{N}}\) converges to \((\Gamma_t)\) in the \(C^\infty\) sense.

**Proof:** This follows from Sard’s Lemma in the usual manner. Explicitly, a generic codimension 2 immersion self-intersects over a submanifold of codimension 4, from which (i) follows, and every immersion in a generic isotopy of codimension 2 immersions self-intersects over a submanifold of codimension 3, from which (ii) follows. See [7] for details. \(\square\)

Let \((\Gamma_n)_{n \in \mathbb{N}}\) be a sequence of strictly convex, codimension 2, immersed submanifolds with convexity orientation. For all \(n \in \mathbb{N} \cup \{0\}\), let \(N_n^+\) be the convexity coorientation of \(\Gamma_n\). Suppose that \((\Gamma_n)_{n \in \mathbb{N}}\) converges in the \(C^\infty\) sense to a strictly convex, codimension 2, immersed submanifold, \(\Gamma_0\) and suppose, moreover, that \(\Gamma_0\) is generic. In particular, by taking a subsequence, we may suppose that \(\Gamma_n\) is also generic for all \(n\).

**Lemma 8.2**

Choose \(\theta > 0\). There exists \(r > 0\) such that if \((\Sigma_n)_{n \in \mathbb{N}}\) is a sequence of strictly convex, immersed hypersurfaces such that, for all \(n\):

(i) \(\partial \Sigma_n = \Gamma_n\); and

(ii) the outward pointing unit normal over \(\Sigma_n\) makes an angle of at least \(\theta\) with \(N_n^+\) along \(\Gamma_n\),

then, for all \(n\), and for all \(p \in \Gamma_n\):

(i) the connected component of \(\Sigma_n \cap B_r(p)\) is embedded and lies on the boundary of a convex subset of \(B_r(p)\); and

(ii) this connected component only meets one connected component of \(\Gamma_n \cap B_r(p)\).

**Remark:** Using this result in conjunction with the compactness of the family of bounded convex sets, we obtain \(C^{0,\alpha}\) compactness near the boundary for families of locally convex immersed hypersurfaces. In particular, this result may be used to extend the conclusions of [12] to the case of compact hypersurfaces with non-trivial boundary (see [14]).

**Proof:** For all \(n \in \mathbb{N} \cup \{0\}\), choose \(p_n \in \Gamma_n\) and suppose that \((p_n)_{n \in \mathbb{N}}\) converges to \(p_0\). For all \(n \in \mathbb{N} \cup \{0\}\), let \(q_n \in M\) be the image of \(p_n\). Choose \(r > 0\) such that, for all \(n \in \mathbb{N} \cup \{0\}\), the connected component of \(\Gamma_n \cap B_r(q_n)\) containing \(p_n\) is embedded, and denote this component by \(\Gamma_{n,0}\). For all \(n\), we identify \(M\) with \(B_r(p_n)\), reducing \(r\) whenever necessary.

As in Section 5, for all \(n \in \mathbb{N} \cup \{0\}\), let \(H_n\) be a strictly convex, embedded hypersurface tangent to \(\Gamma_n\) at \(p_n\) such that:

(i) the outward pointing normal to \(H_n\) at \(p_n\) makes an angle of no more than \(\theta/2\) with \(N_n^+\) at \(p_n\); and
(ii) the shape operator of $H_n$ is everywhere strictly bounded above by $\delta \text{Id}$, where $\delta$ is small.

We suppose, moreover, that $(H_n)_{n \in \mathbb{N}}$ converges to $H_0$ in the $C^\infty$ sense. Likewise, as in Section 5, for all $n \in \mathbb{N}$, we extend $H_n$ to a foliation $(H_{n,t})_{t \in \mathbb{R}}$.

Since $\Gamma_0$ is generic, we may suppose that $H_0$ is transverse at $q_0$ to every connected component of $\Gamma_0 \cap B_r(q_0)$ not equal to $\Gamma_{0,0}$. Thus, reducing $r$ if necessary, for all $n$, if $\Gamma'_{n,0}$ is a connected component of $\Gamma_n \cap B_r(q_n)$ which is different from $\Gamma_{n,0}$, then $\Gamma'_{n,0}$ is transverse to $H_{n,t}$, for all $t$.

Let $t_0 < 0$ be as in Section 5, and, for all $t \in ]t_0, 0[$, let $\Sigma_{n,t}$ be the connected component of $\Sigma_n$ containing $p_n$ which lies above $H_{n,t}$. Define $T$ to be the set of all $t \in ]-t_0, 0[$ such that $\Gamma_{n,0}$ is the only connected component of $\Gamma_n \cap B_r(q_n)$ which intersects $\Sigma_{n,t}$. Trivially, $T$ is non-empty. Let $t_1 = \inf T$ and suppose that $t_1 > t_0$. Let $\Gamma'_{n,0} \neq \Gamma_{n,0}$ be the connected component of $\Gamma_n \cap B_r(q_n)$ which intersects $\Sigma_{n,t_1}$. For $t > t_1$, the reasoning of Section 5 proceeds as in the case where the boundary is embedded, and it follows that $\Sigma_{n,t_1}$ is embedded, is transverse to $H_{t_1}$ and bounds a semi-convex set above $H_{t_1}$. $\Gamma'_{n,0}$ is therefore tangent to $H_{n,t_1}$ at the point of intersection, since, otherwise $\Gamma'_{n,0}$ would intersect $\Sigma_{n,t_1}$ non-trivially at some point lying above $H_{n,t_1}$, which is absurd. However, this contradicts the definition of $r$. It follows that $t_1 = t_0$, and the result now follows as in the case of Lemma 5.1 by taking intersections with a ball of radius less than $t_0$. □

9 - First Order Lower Bounds.

Let $M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $\Gamma^{n-1} \subseteq M$ be a generic, strictly convex, codimension 2, immersed submanifold with convexity orientation. Let $A_\Gamma$ be the shape operator of $\Gamma$ and let $N^-$ and $N^+$ be the convexity orientation and coorientation respectively of $\Gamma$. As in [2], second order bounds require uniform lower bounds on the angle between $N^-$ and the normal to any hypersurface of constant Gaussian curvature with boundary equal to $\Gamma$. This is guaranteed by the following result:

**Proposition 9.1**

For all $k > 0$, there exists $\phi > 0$ (which only depends on $M$, $\Gamma$ and $\theta$) such that if $(\Sigma^n, \partial \Sigma^n)$ is a smooth, convex immersed hypersurface such that:

(i) $\partial \Sigma = \Gamma$;

(ii) the Gaussian curvature of $\Sigma$ is at least $k$; and

(iii) the outward pointing normal to $\Sigma$ over $\Gamma$ makes an angle of at least $\theta$ with $N^+(p)$,

then the outward pointing normal to $\Sigma$ over $\Gamma$ also makes an angle of at least $\phi$ with $N^-(p)$. 

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Let $r > 0$ and let $\Sigma$ be a $C^{0,1}$ locally convex hypersurface in $M$ such that:

(i) $\partial \Sigma \subseteq \Gamma \cup B_r(p)$;

(ii) $\Sigma$ is compatible with the orientation on $\Gamma$;

(iii) the outward pointing normal to $\Sigma$ along $\Gamma$ always makes an angle of at least $\theta$ with $N^+$; and

(iv) the outward pointing normal to $\Sigma$ at $p$ coincides with $N^-(p)$.

Let $\text{Symm}(\mathbb{R}^n)$ denote the set of positive definite, symmetric matrices over $\mathbb{R}^n$. For $t > 0$, we define $F_t \subseteq \text{Symm}(\mathbb{R}^n)$ by:

$$F_t = \{ A \in \text{Symm}(\mathbb{R}^n) \text{ s.t. } A \succeq 0 & \det(A) \geq t \}.$$

Observe that if $A \in F_t$ and if $M \succeq 0$, then $A + M \in F_t$. In the language of [1], this implies that $F_t$ is a Dirichlet set. In particular, if $A \notin F_t$ and $M \succeq 0$, then $A - M \notin F_t$.

**Proposition 9.1** is proven using barriers, which are constructed using the following result:

**Proposition 9.2**

Choose $\delta > 0$. There exists a neighbourhood $U$ of $p$ and a smooth function $f : U \to \mathbb{R}$ such that:

(i) $f \geq 0$ along $\partial(U \cap \Sigma)$;

(ii) there exists $q \in U \cap \Sigma$ such that $f(q) < 0$; and

(iii) for all $q \in B_r(p)$, the shape operator of the level subset of $f$ passing through $q$ with respect to $\nabla f$ is conjugate to an element of $F_{\delta}^c$.

Let $S$ be a smooth, immersed hypersurface in $M$ such that:

(i) $\partial S = \partial \Gamma$;

(ii) the upward pointing normal to $S$ at $p$ is equal to $N^-(p)$; and

(iii) the shape operator of $S$ at $p$ is supported along the subspace $T_p \Gamma$.

Let $H$ be a strictly concave immersed hypersurface in $M$ such that:

(i) the downward pointing normal to $H$ at $p$ lies in $X_p$ and makes an angle of at most $\theta/2$ with $N^+(p)$; and

(ii) $\Gamma$, $\Sigma$ and $S$ locally lie strictly above $H$.

Let $d_p$, $d_S$ and $d_H$ denote the (signed) distance in $M$ to $p$, $S$ and $H$ respectively. Observe that $(\nabla d_S, \nabla d_H)$ is a linearly independant pair which spans the space of normal vectors to $\Gamma$ at $p$. For any two functions, $f$ and $g$, we define the $(n - 2)$-dimensional distribution, $E(f, g)$, near $p$ by:

$$E(f, g) = \langle \nabla f, \nabla g \rangle^\perp,$$

where $\langle U, V \rangle$ here represents the subspace spanned by the vectors $U$ and $V$. Let $e_1, \ldots, e_{n-1}$ be an orthonormal basis for $T_p \Gamma$ with respect to which $A_{\Gamma}(N^-)$ is diagonal. Let $\lambda_1, \ldots, \lambda_{n-1}$
be the corresponding eigenvalues. We may suppose that $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$. We extend $(e_1, \ldots, e_{n-1})$ to a local frame in $TM$ such that, for all vectors, $X$, at $p$:

$$\langle \nabla_X e_i, \nabla dS \rangle = -\text{Hess}(dS)(X, e_i),$$

$$\langle \nabla_X e_i, \nabla dH \rangle = -\text{Hess}(dH)(X, e_i).$$

Define the distribution $E$ near $P$ to be the span of $e_1, \ldots, e_{n-1}$.

**Proposition 9.3**

If $D$ represents the Grassmannian distance between two $(n-1)$-dimensional subspaces, then:

$$D(E, E(dS, dH)) = O(d_p^2).$$

**Proof:** By definition of $e_i$, for all vectors $X$ at $p$:

$$X \langle e_i, \nabla dS \rangle = X \langle e_i, \nabla dH \rangle = 0.$$  

The result follows. $\Box$

For any smooth function, $f$, we define $D(f, E)$ by:

$$D(f, E) = \text{Det}(\text{Hess}(f)|_E),$$

where $\text{Hess}(f)|_E$ is the restriction of the Hessian of $f$ to $E$.

**Proposition 9.4**

Let $f$ be such that $f(p), \nabla f(p) = 0$ and the restriction of $\text{Hess}(f)$ to $H$ at $p$ is positive definite. There exists a function $x$ such that $x(p), \text{Hess}(x)(p) = 0$ and:

$$D(dS + x(dH - f), E) = O(d_p^2).$$

**Proof:** The Hessian of $xf$ vanishes at $p$. Likewise, the Hessian of the second order term $xd_H$ vanishes over $(\nabla d_H)^\perp$ and therefore over $E$ at $p$. It follows that the term $x(dH - f)$ does not affect the restriction of the Hessian of the function to $E$ at $p$. Thus:

$$\nabla D = \text{Tr}(\text{Adj}(\text{Hess}(dS)|_E) \nabla(\text{Hess}(dS + x(dH - f))(e_i, e_j))),$$

where $\text{Adj}(\text{Hess}(dS)|_E)$ is the adjugate matrix of $\text{Hess}(dS)|_E$. If more than one of the eigenvalues of $\text{Hess}(dS)|_E$ vanishes, then $\text{Adj}(\text{Hess}(dS)|_E)$ also vanishes, and the result follows trivially by taking $x = 0$. Suppose therefore that only one eigenvalue of $\text{Hess}(dS)|_E$ vanishes. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of the adjugate matrix, then $\mu_1 = \lambda_2 \ldots \lambda_{n-1}$ and $\mu_2 = \ldots = \mu_{n-1} = 0$. Define the vectors $U$ and $V$ at $p$ by:

$$U = \nabla D(dS, E),$$

$$V = \nabla D(dS + x(dH - f), E).$$
Denote $P = x(d_H - f)$. At $p$:

$$
\text{Hess}(P) = \nabla x \otimes \nabla d_H + \nabla d_H \otimes \nabla x.
$$

At $p$, for all $i$, by definition, $\langle e_i, \nabla d_H \rangle = 0$. Thus, recalling the formula for $\nabla e_i$:

$$
\begin{align*}
X\text{Hess}(P)(e_i, e_j) &= (\nabla_X \text{Hess}(P))(e_i, e_j) + \text{Hess}(P)(\nabla_X e_i, e_j) + \text{Hess}(P)(e_i, \nabla_X e_j) \\
&= (\nabla_X \text{Hess}(P))(e_i, e_j) + \langle \nabla x, e_j \rangle \langle \nabla_X e_i, \nabla d_H \rangle + \langle \nabla x, e_i \rangle \langle \nabla_X e_j, \nabla d_H \rangle \\
&= (\nabla_X \text{Hess}(P))(e_i, e_j) - \text{Hess}(d_H)(X, e_i)x_{;j} - \text{Hess}(d_H)(X, e_j)x_{;i}.
\end{align*}
$$

We extend $(e_i)_{1 \leq i \leq n-1}$ to an orthonormal basis $(e_i)_{0 \leq i \leq n}$ for $T_pM$. With respect to this basis, for all $k$:

$$
\frac{1}{\mu_1} \langle V - U, e_k \rangle = (d_{H;11} - f_{;11})x_{;k} - 2f_{;1k}x_{;1}.
$$

Consider the linear map, $M$, given by:

$$
(M\xi)_k = (d_{H;11} - f_{;11})\xi_k - 2f_{;1k}\xi_1.
$$

Suppose that $M\xi = 0$. Then, in particular, bearing in mind that $d_{H;11} \leq 0$ and $f_{;11} \geq 0$:

$$
\begin{align*}
(d_{H;11} - 3f_{;11})\xi_1 &= 0 \\
\Rightarrow \xi_1 &= 0 \\
\Rightarrow \xi &= 0.
\end{align*}
$$

$M$ is therefore invertible, and there exists $\xi$ such that:

$$
M\xi = -U.
$$

If we define $x$ such that:

$$
\begin{align*}
x(p) &= 0, \quad \nabla x(p) = \xi, \quad \text{Hess}(x)(p) = 0,
\end{align*}
$$

then:

$$
\nabla D(d_S + x(d_H - f), E) = 0.
$$

This completes the proof. □

Define $\Phi_0$ by:

$$
\Phi_0 = d_S + x(d_H - f).
$$

For $M > 0$, define $\Phi$ by:

$$
\Phi = d_S + x(d_H - f) + Md_H^2.
$$

**Proposition 9.5**

If $D$ represents the Grassmannian distance between two $(n-2)$-dimensional subspaces then:

$$
D(E(d_S, d_H), E(\Phi, d_H)) = O(d_p^2) + O(d_H).
$$

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Proof: Since \( xf \) is of order 3 at \( p \):
\[
\nabla \Phi = \nabla d_S + (x + 2Md_H)\nabla d_H + O(d_p^2) + O(d_H).
\]
Thus:
\[
\langle \nabla \Phi, \nabla d_H \rangle = \langle \nabla d_S + O(d_p^2) + O(d_H), \nabla d_H \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) here represents the subspace generated by two vectors. The result follows. \( \Box \)

Corollary 9.6

If \( D \) represents the Grassmannian distance between two \( (n-2) \)-dimensional subspaces, then:
\[
D(E, E(\Phi, d_H)) = O(d_p^2) + O(d_H).
\]

Proof: This follows from the triangle inequality and Proposition 9.3. \( \Box \)

Finally, we recall the following technical property of convex sets. Let \( UM \) be the bundle of unit spheres in \( TM \). Let \( K \subset M \) be a compact, convex set with non-trivial interior. For all \( q \in \partial K \), let \( N(q) \subseteq U_q M \) be the set of supporting normals to \( K \) at \( q \). This set is a closed, convex subset subset of \( U_q M \). Moreover, we have the following continuity result:

Proposition 9.7

Let \( q_0, (q_n)_{n \in \mathbb{N}} \in \partial K \) be such that \( (q_n)_{n \in \mathbb{N}} \) converges to \( q_0 \). For all \( n \), let \( N_n \) be an element of \( N(q_n) \). If \( d \) denotes the distance in \( UM \), then \( (d(N_n, N(q_0)))_{n \in \mathbb{N}} \) converges to 0.

We now prove Proposition 9.2:

Proof of Proposition 9.2: For \( \epsilon > 0 \), define the open set \( U_\epsilon \subseteq M \) by:
\[
U_\epsilon = \{ p \in M \text{ s.t. } d_p(x) < \epsilon \text{ and } d_H(x) < \epsilon^2 \}.
\]
\( \partial(\Sigma \cap U_\epsilon) \) consists of two components: \( \partial \Sigma \cap U_\epsilon = \Gamma \cap U_\epsilon \) and \( \partial U_\epsilon \cap \Sigma \). We first obtain lower estimates for \( \Phi_0 \) along these two components.

We choose \( f \) such that, along \( \Gamma \), \( (f - d_H) = O(d_p^2) \). Consequently, \( x(f - d_H) = O(d_p^2) \) along \( \Gamma \). Thus, since \( O(d_p^2) = O(d_H) \) along \( \Gamma \) and since \( d_S \) vanishes along \( \Gamma \), there exists \( K_1 > 0 \) such that, along \( \Gamma \):
\[
|d_S + x(d_H - f)| \leq K_1 d_H^2.
\]
This yields lower bounds for \( \Phi_0 \) along \( \partial \Sigma \cap U_\epsilon \).

Since \( \Sigma \) is a convex immersion, and since \( \partial \Sigma = \Gamma \) is smooth, \( \Sigma \) has a unique supporting normal at \( p \), which coincides with \( \nabla d_S \). Now let \( V \) be a field of unit vectors defined near \( p \) such that \( V(p) \) makes an angle of exactly \( \theta \) with \( N^+(p) \). For \( q \in M \), let \( U_q M \) be the unit sphere in \( T_q M \). Let \( D_q \) be the distance in \( U_q M \) and let \( C_q \) be the shortest geodesic in \( U_q M \) joining \( V(q) \) to \( \nabla d_S(q) \). Near \( p \), \( V(p), \nabla d_S \) and \( -\nabla d_H \) are configured as shown in Figure 1:
However, by definition, $\nabla d_S(p) = N_p$. By Property (iii) of $\Sigma$, we can extend $\Sigma$ slightly beyond $\Gamma$ to a $C^{0,1}$ locally strictly convex hypersurface whose set of supporting normals at $p$ is contained within $C_p$. Thus, by Proposition 9.7, there exists a continuous function $\delta : [0, \infty] \to [0, \infty]$ such that $\delta(0) = 0$ and, for all $q \in \Sigma$, if $N_q$ is a supporting normal to $\Sigma$ at $q$, then:

$$D_q(N_q, C_q) \leq \delta(d_p(q)).$$

Thus, if, for all $q \in \Sigma$, $\pi_q$ is the orthogonal projection onto a supporting hyperplane of $\Sigma$ at $q$, then:

(i) there exists $c > 0$ such that, for all $q$ sufficiently close to $p$:

$$\|\pi_1(\nabla d_H)\| \geq c;$$

and

(ii) for all $q$ sufficiently close to $p$:

$$\langle \pi_q(\nabla d_S), \pi_q(\nabla d_H) \rangle \geq -\delta(d_p(q)).$$

Now consider $q_0 \in \Sigma \cap \partial U$. Let $\gamma : I \to \Sigma$ be an integral curve of $\pi_q(\nabla d_H)$ such that $\gamma(0) \in \partial \Sigma$ and $\gamma(1) = q_0$ (which is defined by approximating $\Sigma$ by smooth hypersurfaces). Bearing in mind that $d_H \geq 0$ along $\partial \Sigma$ and $d_S$ vanishes along $\partial \Sigma$:

$$d_H(q_0) \leq \epsilon^2$$

$$\Rightarrow \quad \text{Length}(\gamma) \leq \epsilon^2 e^{-1}$$

$$\Rightarrow \quad (d_S \circ \gamma)(1) \geq -\delta(\epsilon) \epsilon^2 e^{-1}.$$ 

Thus:

$$[d_S + x(d_H - f)](q_0) \geq -\delta(\epsilon) O(\epsilon^2),$$

for all appropriate functions $f$ and $x$. There thus exists $\delta_1 > 0$ such that, along $\Sigma \cap \partial U_\epsilon$:

$$\Phi_0 > -\delta_1 d_H.$$

Moreover, $\delta_1$ tends to 0 as $\epsilon$ tends to 0.
Thus, if we choose $M = \text{Max}(\delta_1 \epsilon^{-2}, K_1)$, then $\Phi \geq 0$ along $\partial(\Sigma \cap U)$. Since $\text{Hess}(\Phi_0)$ is bounded, by Proposition 9.4 and Corollary 9.6:

$$D(\Phi_0, E(\Phi, d_H)) = O(\epsilon^2).$$

However:

$$\text{Hess}(\Phi) = \text{Hess}(\Phi_0) + 2M \nabla d_H \otimes \nabla d_H + 2Md_H \text{Hess}(d_H).$$

Denote:

$$A = \frac{1}{\|\nabla \Phi\|} \text{Hess}(\Phi)|_{\nabla \Phi \perp}.$$ 

$A$ is the shape operator of the level sets of $\Phi$. If $A$ is not non-negative definite, then it trivially lies in $F_\delta$. Suppose, therefore, that $A$ is non-negative definite. Let $0 \leq \lambda_1 \leq ... \leq \lambda_n$ be the eigenvalues of $A$, and let $0 \leq \lambda'_1 \leq ... \leq \lambda'_{n-1}$ be the eigenvalues of the restriction of $A$ to $E(\Phi, d_H)$. Observe that $\nabla d_H \otimes \nabla d_H$ vanishes on $E(\Phi, d_H)$. Moreover, since $H$ is concave, $2Md_H \text{Hess}(d_H)$ is negative definite. It follows that the eigenvalues of the restriction of $A$ to $E(\Phi, d_H)$ are less than the eigenvalues of the restriction of $\|\nabla \Phi\|^{-1} \text{Hess}(\Phi_0)$ to this subspace. Thus, since $\|\nabla \Phi\|$ also remains uniformly bounded away from 0, by the preceding calculations:

$$\lambda'_1 \cdot ... \cdot \lambda'_{n-1} = O(\epsilon^2).$$

However, by the minimax principal, for $1 \leq i \leq (n-1)$:

$$0 \leq \lambda_i \leq \lambda'_i.$$ 

Thus:

$$\lambda_1 \cdot ... \cdot \lambda_{n-1} = O(\epsilon^2).$$

Consequently, since $\lambda_n = O(M) = O(\delta_1 \epsilon^{-2})$, for $\epsilon$ sufficiently small:

$$\text{Det}(A) < \delta.$$ 

Thus $A \in F_\delta$, and property $(iii)$ now follows. Since $f$ is non-negative over $\partial(\Sigma \cap U_\epsilon)$, property $(i)$ also follows. Since $f(p) = 0$ and $(\nabla f)(p) = N^+(p)$, deforming $f$ slightly yields a function which still satisfies conditions $(i)$ and $(iii)$ but also satisfies condition $(ii)$. This completes the proof. □

We now obtain Proposition 9.1:

**Proof of Proposition 9.1:** Assume the contrary. Let $(\Sigma_n, \partial \Sigma_n)_{n \in \mathbb{N}}$ be a sequence of convex immersed hypersurfaces such that:

(i) $\partial \Sigma_n = \Gamma$; and

(ii) the Gaussian curvature of $\Sigma$ is at least $k$.

Suppose, moreover, that there exists $(p_n)_{n \in \mathbb{N}}, p_0 \in \Gamma$ such that $(p_n)_{n \in \mathbb{N}}$ converges to $p_0$ and the angle that the exterior normal of $\Sigma_n$ makes with $N^-(p_n)$ at $p_n$ tends to 0.
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By Lemma 8.2, there exists \( r > 0 \) such that, for all \( n \), the connected component of \( \Sigma_n \cap B_r(p_n) \) containing \( p_n \) is embedded and bounds a convex set. For all \( n \), we denote this connected component by \( \Sigma_{n,0} \). By compactness of the family of convex sets, there exists a convex immersion \( \Sigma_0 \) to which \( (\Sigma_{n,0})_{n \in \mathbb{N}} \) converges in the \( C^{0,\alpha} \) sense for all \( \alpha \). Let \( f \) be as in Proposition 9.2 with \( \delta < k \). For sufficiently large \( n \), \( f \) achieves a strict local minimum at some interior point \( q_n \in \Sigma_n \).

Let \( \text{Hess}^0(f) \) be the Hessian of \( f \) over \( M \), and, for all \( n \), let \( \text{Hess}^n(f) \) be the Hessian of the restriction of \( f \) to \( \Sigma_n \). At \( q_n \):

\[
\text{Hess}^n(f) = \text{Hess}^0(f)|_{\nabla f \perp} - \| \nabla f \| A_{n,0},
\]

where \( A_{n,0} \) is the shape operator of \( \Sigma_{n,0} \) at \( q_n \). By the Maximum Principal, at \( q_n \):

\[
\Rightarrow \text{Hess}^0(f)|_{\nabla f \perp} - \| \nabla f \| A_{n,0} \geq 0
\]

\[
\Rightarrow \frac{1}{\| \nabla f \|} \text{Hess}^0(f)|_{\nabla f \perp} \geq \| \nabla f \| A_{n,0}
\]

This is absurd by definition of \( f \), and the result follows. \( \Box \)

10 - Compactness.

Let \( M^{n+1} \) be a Hadamard manifold. Let \( (\Gamma_{m-1})_{m \in \mathbb{N}}, \Gamma_0^{n-1} \subseteq M \) be generic, locally strictly convex, codimension 2, immersed submanifolds with convexity orientation such that \( (\Gamma_m)_{m \in \mathbb{N}} \) converges to \( \Gamma_0 \). For all \( m \), let \( N_m^- \) and \( N_m^+ \) be the convexity orientation and coorientation respectively of \( \Gamma_m \). Let \( (\phi_m)_{m \in \mathbb{N}}, \phi_0 : M \to ]0, \infty[ \) be smooth, positive functions such that \( (\phi_m)_{m \in \mathbb{N}} \) converges to \( \phi_0 \) in the \( C^\infty_{\text{loc}} \) sense. Let \( (\Sigma_m^n)_{m \in \mathbb{N}} \subseteq M \) be smooth, immersed, strictly convex, compact hypersurfaces such that, for all \( m \):

(i) \( \partial \Sigma_m = \Gamma_m \);

(ii) \( \Sigma_m \) is compatible with the orientation of \( \Gamma_m \); and

(iii) the Gaussian curvature of \( \Sigma_m \) at any point \( p \in \Sigma_m \) is equal to \( \phi_m(p) \).

We obtain the following precompactness result:

**Lemma 10.1**

Let \( \theta \in ]0, \pi[ \) be an angle and let \( D > 0 \) be a positive real number. Suppose that, for all \( m \):

(i) the outward pointing normal to \( \Sigma_m \) makes an angle of at least \( \theta \) with \( N_m^\perp \) at every point of \( \Gamma_m \); and

(ii) the diameter of \( \Sigma_m \) is no greater than \( D \).

Then there exists a strictly convex, smooth immersed hypersurface, \( (\Sigma_0, \partial \Sigma_0) \subseteq M \) towards which \( (\Sigma_m)_{m \in \mathbb{N}} \) subconverges. Moreover:

(i) \( \partial \Sigma_0 = \Gamma_0 \); and

(ii) the Gaussian curvature of \( \Sigma_0 \) at any point \( p \in \Sigma_0 \) is equal to \( \phi_0(p) \).
Proof: By the Arzela-Ascoli Theorem of [13], it suffices to obtain a-priori bounds for all the derivatives of the shape operators of the hypersurfaces \((\Sigma_m)_{m \in \mathbb{N}}\). For all \(m\), let \(A_m\) be the shape operator of \(\Sigma_m\). Let \((p_m)_{m \in \mathbb{N}}, p_0\) be points such that:

(i) for all \(m\), \(p_m \in \Gamma_m\); and
(ii) \((p_m)_{m \in \mathbb{N}}\) converges to \(p_0\).

Choose \(\epsilon > 0\). There exists \(r_1 > 0\) and, for all \(m\), a smooth, embedded, strictly locally convex hypersurface \(\hat{\Sigma}_m\) such that:

(i) \(p_m \in \hat{\Sigma}_m\);
(ii) \(\hat{\Sigma}_m\) is complete with respect to \(B_{r_1}(p_m)\), and along with \(\partial B_{r_1}(p_m)\) bounds a convex set, \(K_m\);
(iii) the connected component of \(\Gamma_m \cap B_{r_1}(p_m)\) containing \(p_m\), which we denote by \(\Gamma_{m,0}\), is itself contained in \(\hat{\Sigma}_m\);
(iv) the outward pointing normal over \(\hat{\Sigma}_m\) makes an angle of no more than \(\theta/2\) with \(N_m^+\) along \(\Gamma_{m,0}\); and
(v) the Gaussian curvature of \(\hat{\Sigma}_m\) at the point \(q\) is at least \(\phi_m(q) + \epsilon\).

Moreover, we may assume that \((\hat{\Sigma}_m)_{m \in \mathbb{N}}\) converges towards \(\hat{\Sigma}_0\).

By Lemma 8.2, reducing \(r_1\) if necessary we may assume that, for all \(m\), the connected component of the intersection of \(\Sigma_m\) with \(B_{r_1}(p_m)\) containing \(p_m\), which we denote by \(\Sigma_{m,0}\), is embedded, lies on the boundary of a convex set, \(K_m\) such that \(K_m \subseteq K_m\). By compactness of the family of compact sets, there exists a convex set \(K_0\) to which \((K_m)_{m \in \mathbb{N}}\) converges in the Haussdorf sense. The angle that the normal to \(K_0\) makes with \(T\hat{\Sigma}_0\) at \(p_0\) is strictly less than \(\pi\). Thus, for all \(m\), \(\Sigma_m\) is a graph over some (almost) fixed hypersurface over a uniform radius about \(p\): formally, reducing \(r_1\) further if necessary, for all \(m\), there exists a smooth embedded hypersurface \(S_m \subseteq M\) and an open subset \(\Omega_m \subseteq S_m\) with smooth boundary such that:

(i) \(p_m \in S_m\) and \(S_m\) is complete with respect to \(B_{r_1}(p_m)\);
(ii) \(\Gamma_m\) is a graph over \(\partial \Omega_m\); and
(iii) \(\Sigma_{m,0}\) and \(\hat{\Sigma}_m\) are graphs of functions \(f_m\) and \(\hat{f}_m\) respectively over \(\Omega_m\) such that \(\hat{f}_m \geq f_m\).

Moreover, we may suppose that \((S_m)_{m \in \mathbb{N}}\) converges to \(S_0\) and that \((\hat{f}_m)_{m \in \mathbb{N}}\) converges to \(\hat{f}_0\) in the \(C^\infty_{\text{loc}}\) sense. Using this construction in conjunction with Proposition 5.1 of [11] and Proposition 9.1, we obtain \(K_1 > 0\) such that, for all \(m\) and for all \(p \in \Gamma_m\):

\[\|A_m(p)\| \leq K_1.\]
Since the diameter of $\Sigma_m$ is uniformly bounded above, by Proposition 6.1 of [11], we obtain $K_2 > 0$ such that, for all $m$, and for all $p \in \Sigma_m$:

$$\|A_m(p)\| \leq K_2.$$ 

Again, using the above construction along with Theorem 1 of [3], we show that there exists $\epsilon > 0$ and uniform $C^{0,\alpha}$ bounds for $(A_m)_{m \in \mathbb{N}}$. The Schauder estimates then yield uniform $C^k$ bounds for $(A_m)_{m \in \mathbb{N}}$ for all $k$. The result now follows by the Arzela-Ascoli Theorem of [13]. □

Let $(\hat{\Sigma}_m)_{m \in \mathbb{N}}, \Sigma_0 \subseteq M$ be locally strictly convex, immersed hypersurfaces in $M$ with generic boundaries such that $(\hat{\Sigma}_m)_{m \in \mathbb{N}}$ converges to $\Sigma_0$. Let $(\phi_m)_{m \in \mathbb{N}}, \phi_0 : M \to ]0, \infty[$ be smooth, positive functions such that $(\phi_m)_{m \in \mathbb{N}}$ converges to $\phi_0$ in the $C^\infty_{\text{loc}}$ sense.

Lemma 10.1 can be refined to the following result:

**Lemma 10.2**

Let $(\Sigma_m)_{m \in \mathbb{N}}$ be strictly convex smooth immersed hypersurfaces in $M$ such that, for all $m$:

(i) $\Sigma_m$ is bounded by $\hat{\Sigma}_m$; and

(ii) for all $p \in \Sigma_m$, the Gaussian curvature of $\Sigma_m$ at $p$ is equal to $\phi_m(p)$.

There exists a strictly convex smooth immersed hypersurface, $\Sigma_0$ in $M$ to which $(\Sigma_m)_{m \in \mathbb{N}}$ subconverges. Moreover:

(i) $\Sigma_0$ is bounded by $\hat{\Sigma}_0$; and

(ii) for all $p \in \Sigma_0$, the Gaussian curvature of $\Sigma_0$ at $p$ is equal to $\phi_0(p)$.

**Proof:** Since $(\hat{\Sigma}_m)_{m \in \mathbb{N}}$ converges to $\hat{\Sigma}_0$, there exists $D > 0$ such that, for all $m$, the diameter of $\hat{\Sigma}_m$ is bounded above by $D$. Likewise, for all $m$, $\Gamma_m := \partial \hat{\Sigma}_m$ is locally strictly convex and, if $N_m^-$ and $N_m^+$ denote the convexity orientation and coorientation respectively of $\Gamma_m$, then there exists $\theta > 0$ such that the angle that the outward pointing unit normal to $\hat{\Sigma}_m$ makes with $N_m^+$ along $\Gamma_m$ is everywhere bounded below by $\theta$.

For all $m$, let $\pi_m : \hat{\Sigma}_m \to \Sigma_m$ be the canonical projection. Since $M$ has non-positive curvature, for all $m$, $\pi_m$ is distance decreasing, and the diameter of $\Sigma_m$ is thus bounded above by $D$. Moreover, for all $m$, since $\hat{\Sigma}_m$ bounds $\Sigma_m$, the angle that the outward pointing unit normal to $\Sigma_m$ makes with $N_m^+$ along $\Gamma_m$ is everywhere bounded below by $\theta$. It follows by Lemma 10.1 that there exists a strictly convex immersed hypersurface, $\Sigma_0$ towards which $(\Sigma_m)_{m \in \mathbb{N}}$ subconverges such that, for all $p \in \Sigma_0$, the Gaussian curvature of $\Sigma_0$ at $p$ is equal to $\phi_0(p)$. By Lemma 3.2, $\hat{\Sigma}_0$ bounds $\Sigma_0$ and this completes the proof. □
11 - Local Deformation.

Let $M^{n+1}$ be a Hadamard manifold. Let $(\hat{\Sigma}_t)_{t \in [0,1]}$ be a smooth family of locally convex immersed hypersurfaces in $M$ with generic boundary. For all $t$, denote $\Gamma_t = \partial \hat{\Sigma}_t$. Let $\epsilon > 0$ and let $(\phi_t)_{t \in [0,1]} \in C^\infty(M,]0,\infty[)$ be a smooth family such that, for all $t$, the Gaussian curvature of $\hat{\Sigma}_t$ is everywhere greater than $\phi_t + \epsilon$.

For all $t \in [0,1]$ let $\mathcal{M}_t$ be as in Section 2 and let $\mathcal{N}_t$ be the family of (equivalence classes) of convex immersed hypersurfaces, $[\Sigma]$ in $M$ such that $\partial [\Sigma] = \partial \hat{\Sigma}_t$ and $\Sigma$ is strictly bounded by $\hat{\Sigma}_t$. By Lemma 3.3, $\mathcal{N}_t$ is an open subset of $\mathcal{M}_t$ and is therefore interpreted as a smooth Banach manifold. Let $\mathcal{M}$ be as in Section 2 and let $\mathcal{N}$ be the family of all pairs $(t,[\Sigma])$ where $t \in [0,1]$ and $[\Sigma] \in \mathcal{N}_t$. $\mathcal{N}$ is likewise an open subset of $\mathcal{M}$.

Let $X_0 \subseteq \mathcal{N}$ be the set of all pairs $(t,[\Sigma])$ in $\mathcal{N}$ such that the Gaussian curvature of $\Sigma$ is equal to $\phi_t$. By Lemma 10.2 and the Geometric Maximum Principal, $X_0$ is compact. Let $P = (t_0,[\Sigma])$ be a point in $X_0$, where $\Sigma = (i,(S,\partial S))$. Let $(i_t)_{t \in [t_0-\epsilon,t_0+\epsilon]}$ be a smooth family of immersions such that $i_0 = i$ and, for all $t$, $\Gamma_t = (i_t,\partial S)$. We define the family $(\Sigma_t)_{t \in [t_0-\epsilon,t_0+\epsilon]}$ by:

$\Sigma_s = (i_s,(S,\partial S)).$

Let $(U_P,V_P,\Phi_P)$ be the resulting graph neighbourhood of $\mathcal{N}$ about $\Sigma$.

Consider the Gauss curvature mapping $K$. This is a smooth section of $\mathcal{E}$. If we identify $T_P \mathcal{N}_t$ with $C_0^\infty(S)$, then its covariant derivative, $\nabla K$, defines a mapping from $C_0^\infty(S)$ to $C^\infty(S)$. By Corollary 2.2, $\nabla K$ is a second order elliptic linear differential operator. It is therefore Fredholm. Since it maps from $C_0^\infty(S)$ to $C^\infty(S)$, it is of index 0. There therefore exists a finite dimensional vector subspace $E \subseteq C^\infty(S)$ such that if $M$ is defined by:

$M : E \oplus C_0^\infty(S) \to C^\infty(S); (f,\phi) \mapsto \nabla K \cdot \phi + f,$

then $M$ is surjective. Since $M$ differs from $\nabla K$ by a compact (in fact, finite rank) operator, it is Fredholm of index $m$, where $m$ is the dimension of $E$. Let $f_1,\ldots,f_n$ be a basis of $E$. For $Q := (t_Q,\Sigma_Q) \in U_P$, where $\Sigma_Q = (i_Q,(S_Q,\partial S_Q))$, let $\pi_Q : (S_Q,\partial S_Q) \to (S,\partial S)$ be the canonical projection (recall that $\Sigma_Q$ is a graph over $\Sigma_{t_Q}$). For all $i$, we define $f_{i,Q} \in C^\infty(S_Q)$ by:

$f_{i,Q} = f_i \circ \pi_Q.$

For all $i$, $Q \mapsto f_{i,Q}$ defines a section of $\mathcal{E}|_{U_P}$, which we denote by $F_i$. We now define $\hat{K}_P : \mathbb{R}^m \times U_P \to \mathcal{E}|_{U_P}$ by:

$\hat{K}_P(\sum_{i=1}^n \lambda_i e_i, (t,[\Sigma])) = K(\Sigma) + \sum_{i=1}^n \lambda_i F_i(t,[\Sigma]).$

By reducing $U_P$ if necessary, we may assume that $\nabla \hat{K}_P$ is Fredholm and surjective at every point of $\mathbb{R}^m \times U_P$. Since $\hat{K}_P$ is now a function over an open subset of $\mathcal{M}$ (as opposed to $\mathcal{M}_t$), it’s derivative has index $(m+1)$. 

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More generally, let $\psi : U_P \to [0, \infty]$ be a smooth function such that:

(i) $\psi = 1$ near $(t_0, [\Sigma])$; and

(ii) the support of $\psi$ is contained in $U_P$.

Let $U_P' \subseteq U_P$ be a neighbourhood of $(t_0, [\Sigma])$ such that $\psi = 1$ over $U_P'$. We define $\Psi_P : \mathbb{R}^m \to \Gamma(E)$ by:

$$\Psi_P(\sum_{i=1}^{n} \lambda_i e_i) = \sum_{i=1}^{n} \lambda_i \psi F_i.$$ 

By compactness of $X_0$, there exist finitely many points $P_1, ..., P_n \in X_0$ such that:

$$X_0 \subseteq \bigcup_{i=1}^{n} U'_{P_i} =: \Omega.$$ 

Denote $m = m_1 + ... + m_n$ and define $\Psi : \mathbb{R}^m \to \Gamma(E)$ by:

$$\Psi = \Psi_{P_1} \oplus ... \oplus \Psi_{P_n}.$$ 

Define $\hat{K} : \mathbb{R}^m \times \mathcal{N} \to \mathcal{E}$ by:

$$\hat{K}(v, (t, [\Sigma])) = K([\Sigma]) + \Psi(v).$$ 

For $v \in \mathbb{R}^m$, define $X_v$ by:

$$X_v = \left\{ (t, [\Sigma]) \in \mathcal{N} \text{ s.t. } \hat{K}(v, (t, [\Sigma])) = \phi_t \right\}.$$ 

**Proposition 11.1**

There exists $r > 0$ such that:

(i) for $\|v\| < r$, $X_v$ is compact; and

(ii) for $\|v\| < r$, $X_v \subseteq \Omega$.

**Proof:** (i) Let $(t_m, [\Sigma_m])_{m \in \mathbb{N}}$ be a sequence in $X_v$. Let $(\Sigma'_m)_{m \in \mathbb{N}}$ be a sequence of smooth, immersed, compact hypersurfaces in $M$ such that, for all $m$, $\Sigma_m$ is a graph over $\Sigma'_m$. Suppose, moreover, that $(\Sigma'_m)_{m \in \mathbb{N}}$ converges to $\Sigma'_0$. For all $m \in \mathbb{N} \cup \{0\}$, choose $f_m \in C^\infty(\Sigma'_m)$ and suppose that $(f_m)_{m \in \mathbb{N}}$ converges in the $C^\infty$ sense to $f_0$. For all $m$, let $\pi_m$ be the canonical projection onto $\Sigma'_m$. With small modifications, Lemma 10.2 adapts to the case where $\phi_m = f_m \circ \pi_m$ for all $m$, and likewise to the case where $\phi_m$ is a finite linear combination of such functions. It follows that the closure of $X_v$ in $\mathcal{M}$ is relatively compact.

Let $(t, [\Sigma])$ be a limit point of $X_v$. By Lemma 3.3, $\Sigma$ is bounded by $\hat{\Sigma}_t$. Suppose that $\Sigma \notin \mathcal{N}_t$. Then $\hat{\Sigma}_t$ does not strictly bound $\Sigma$, and $\Sigma$ is thus an interior tangent to $\hat{\Sigma}_t$ at some point, $p$, say (possibly in $\partial \hat{\Sigma}$). However, for $v$ sufficiently small, $\|\Psi(v)\| \leq \epsilon$ and so the Gaussian curvature of $\hat{\Sigma}_t$ at $p$ is strictly greater than that of $\Sigma$ at $p$. This contradicts
the Geometric Maximum Principal (see, for example, [11]). There thus exists \( r > 0 \) such that for \( \|v\| < r \), the closure of \( X_v \) is contained in \( \mathcal{N} \) and so \( X_v \) is compact. \((i)\) follows.

\((ii)\) Suppose the contrary. There exists \( (v_n)_{n\in\mathbb{N}} \) which converges to 0 and \( (t_n, [\Sigma_n])_{n\in\mathbb{N}} \) such that, for all \( n \):

\[
(t_n, [\Sigma_n]) \in X_{v_n}, \quad (t_n, [\Sigma_n]) \notin \Omega.
\]

As in the previous paragraph, by Lemma 10.2, \( (t_n, [\Sigma_n])_{n\in\mathbb{N}} \) subconverges to \( (t_0, [\Sigma_0]) \in X_0 \). Thus, for sufficiently large \( n \), \( (t_n, (\Sigma_n))_{n\in\mathbb{N}} \in \Omega \), which is absurd. \((ii)\) follows, and this completes the proof. \(\square\)

Define \( X \subseteq \mathbb{R}^m \times \Omega \) by:

\[
X = \left\{ (v, (t, [\Sigma])) \in \mathbb{R}^m \times \Omega \text{ s.t. } \hat{K}(v, (t, [\Sigma])) = \phi_t \right\}.
\]

**Proposition 11.2**

\( X \) is an \((m + 1)\)-dimensional smooth, embedded submanifold of \( \mathbb{R}^m \times \Omega \).

**Proof:** By construction, \( \hat{K} \) is everywhere Fredholm of index \((m + 1)\) and surjective. The result now follows by the Implicit Function Theorem for Banach manifolds. \(\square\)

**Proposition 11.3**

There exists \( (v_n)_{n\in\mathbb{N}} \in \mathbb{R}^m \) such that:

\( (i) \) \( (v_n)_{n\in\mathbb{N}} \) converges to 0;

\( (ii) \) for all \( n \), \( X_{v_n} \) is a (potentially empty) 1-dimensional, smooth, compact, embedded submanifold of \( \Omega \); and

\( (iii) \) \( \partial X_{v_n} \subseteq \mathcal{N}_0 \cup \mathcal{N}_1 \).

**Proof:** Let \( \pi : \mathbb{R}^m \times \Omega \to \mathbb{R}^m \) be projection onto the first factor. Let \( \pi_X \) be the restriction of \( \pi \) to \( X \). By Sard’s Lemma, the set of critical values of \( \pi_X \) has Lebesgue measure 0. Let \( (v_n)_{n\in\mathbb{N}} \in \mathbb{R}^m \) be a sequence of non-critical values of \( \pi_X \) converging to 0. By the Submersion Theorem, for all \( n \), \( X_{v_n} \) is a 1-dimensional, smooth, embedded submanifold of \( X \) and therefore of \( \Omega \). By Proposition 11.1 we may suppose moreover that, for all \( n \), \( X_{v_n} \) is compact. \( (i) \) and \( (ii) \) follow. For all \( n \), the end points of \( X_{v_n} \) lie in the (manifold) boundary of \( X \). Since this is contained in \( \mathcal{N}_0 \cup \mathcal{N}_1 \), \( (iii) \) follows. This completes the proof. \(\square\)

**12 - Local and Global Rigidity.**

Let \( M^{n+1} \) be an \((n + 1)\)-dimensional Hadamard manifold. Let \( \hat{\Sigma} \subseteq M \) be a convex immersed hypersurface. Choose \( \phi \in C^\infty(M) \). Let \( \Sigma = (i, (S, \partial S)) \) be another convex immersed hypersurface. We say that \( \Sigma \) is a solution to the problem \((\hat{\Sigma}, \phi)\) if and only if:

\( (i) \) \( \partial \Sigma = \partial \hat{\Sigma}; \)

\( (ii) \) \( \Sigma \) is bounded by \( \hat{\Sigma} \); and
(iii) for all $p \in S$, the Gaussian curvature of $\Sigma$ at $p$ is equal to $(\phi \circ i)(p)$.

**Definition 12.1**

(i) We say that $(\hat{\Sigma}, \phi)$ is locally rigid if and only if, for all solutions, $\Sigma$ to $(\hat{\Sigma}, \phi)$, the linearisation, $DK$, of the Gauss Curvature Operator, $K$, over $\Sigma$ is invertible.

(ii) We say that $(\hat{\Sigma}, \phi)$ is globally rigid if and only if there exists at most one solution, $\Sigma$ to $(\hat{\Sigma}, \phi)$.

We recall the following properties of local and global rigidity:

**Proposition 12.2**

(i) If $(\hat{\Sigma}, \phi)$ is locally rigid, then $(\hat{\Sigma}, \phi')$ is also locally rigid for all $\phi'$ sufficiently close to $\phi$.

(ii) If $(\hat{\Sigma}, \phi)$ is locally and globally rigid, then $(\hat{\Sigma}, \phi')$ is globally rigid for all $\phi'$ sufficiently close to $\phi$.

**Proof:** See [11]. □

Now let $(\hat{\Sigma}_t)_{t \in [0,1]}$ be a smooth family of locally strictly convex, immersed hypersurfaces in $M$ with generic boundaries. Let $\epsilon > 0$ and let $(\phi_t)_{t \in [0,1]} \in C^\infty(M, [0,1])$ be a smooth family of smooth, positive functions such that, for all $t$, the Gaussian curvature of $\Sigma_t$ at any point $p$ is no less than $\phi_t(p) + \epsilon$. Using local and global rigidity, we obtain existence:

**Lemma 12.3**

Suppose that $(\hat{\Sigma}_0, \phi_0)$ is both locally and globally rigid. If there exists a solution $\Sigma_0$ to $(\hat{\Sigma}_0, \phi_0)$, then there exists a solution to $(\hat{\Sigma}_1, \phi_1)$.

**Remark:** It follows that proving existence of solutions for a given problem reduces to showing the existence of a smooth isotopy by locally strictly convex immersions to a locally and globally rigid problem for which solutions are known to exist.

**Proof:** Let $\mathcal{N}$, $m \in \mathbb{N}$ and $\Psi : \mathbb{R}^m \to \Gamma(\mathcal{E})$ be as in Section 11 and, for all $v \in \mathbb{R}^m$, define $X_v \subseteq \mathcal{N}$ by:

$$X_v = \{(t, [\Sigma]) \in \mathcal{N} \text{ s.t. } K([\Sigma]) + \Psi(v) = \psi_t\}.$$ 

Let $(v_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^m$ be as in Proposition 11.3. Since $(\hat{\Sigma}_0, \phi_0)$ is locally rigid, there exists $N > 0$ such that, for all $n \geq N$, $X_{v_n} \cap \mathcal{N}_0$ is non-empty, and thus, in particular, $X_{v_n}$ is non-empty. Since $(\hat{\Sigma}_0, \phi_0)$ is also globally rigid, it follows by Proposition 12.2 that, for sufficiently large $n$, $\Psi(v_n) + \phi_0$ is too, and therefore that $X_{v_n} \cap \mathcal{N}_0$ consists of a single point.

Let $\pi : \mathcal{N} \to [0,1]$ be the canonical projection. For all $n \geq N$, $X_{v_n}$ is a smooth, embedded, compact, 1-dimensional submanifold of $\mathcal{N}$. It is thus homeomorphic, either to a compact interval or to a circle. By local and global rigidity, the restriction of $\pi$ to $X_{v_n}$ is a local diffeomorphism near the unique point lying in $\pi^{-1}(\{0\})$. It follows that $X_{v_n}$ has non-trivial (manifold) boundary, and is therefore not a circle. It is thus a compact interval.
By Proposition 11.3, the endpoints of $X_{v_n}$ lie in $N_0 \cup N_1$. By global rigidity, only one endpoint of $X_{v_n}$ lies in $N_0$, and the other therefore lies in $N_1$.

For all $n$, let $\Sigma_n$ be such that $(1, [\Sigma_n])$ is the unique endpoint of $X_{v_n}$ in $N_1$. By Lemma 10.2, there exists $\Sigma_0$ to which $(\Sigma_n)_{n \in \mathbb{N}}$ subconverges and $\Sigma_0$ is a solution of $(\Sigma_1, \psi_1)$. This completes the proof. □

Lemma 12.3 may be easily adapted to treat the case where the metric of the underlying manifold also varies, and we obtain Theorem 1.2:

**Proof of Theorem 1.2:** Let $(\hat{\Sigma}_t)_{t \in [0,1]}$ be an isotopy by convex, immersed hypersurfaces such that $\hat{\Sigma}_0 = \hat{\Sigma}$ and $\hat{\Sigma}_1$ is a finite covering of $\Omega$. For ease of presentation, we will assume that the covering is of order one: the general case is almost identical. Let $p \in K$ be an interior point. Let $d_0, d_1 : M \to \mathbb{R}$ be given by:

$$d_0(x) = d(x, K), \quad d_1(x) = d(x, p).$$

Both $d_0$ and $d_1$ are smooth outside $K$. For $t \in [0,1]$, define $d_t$ by:

$$d_t = td_1 + (1 - t)d_0.$$

Trivially, $\partial K$ is isotopic by smooth convex immersions to $d_0^{-1}([r])$ for all $r \geq 0$. Choose $r_0$ such that $K \subseteq B_{r_0}(p)$. For all $t$, $d_t^{-1}(\{r_0\})$ is a convex, embedded hypersurface and we thus obtain an isotopy by smooth convex immersions between $d_0^{-1}(\{r_0\})$ and $d_1^{-1}(\{r_0\})$.

We may thus define $(\hat{\Sigma}_t)_{t \in [1,2]}$ such that $\hat{\Sigma}_2$ is a geodesic sphere with a finite number of open sets removed. Let $g$ be the Riemannian metric on $M$. Define $(g_t)_{t \in [0,2]}$ such that $g_2 = g$ for all $t$.

We may assume that $\hat{\Sigma}_2$ is as small as we wish. Define $(\hat{\Sigma}_t)_{t \in [2,3]}$ and $(g_t)_{t \in [2,3]}$ such that:

(i) $g_2 = g$;

(ii) $g_3$ is complete with constant curvature equal to 1;

(iii) for all $t$, $\hat{\Sigma}_t$ is a geodesic sphere with respect to $g_t$ with a finite number of open sets removed.

Define $(\hat{\Sigma}_t)_{t \in [3,4]}$ and $(g_t)_{t \in [3,4]}$ such that:

(i) for all $t$, $g_t = g_3$ is the complete hyperbolic metric;

(ii) for all $t$, $(\hat{\Sigma}_t)$ is a geodesic sphere with a finite number of open sets removed; and

(iii) $\hat{\Sigma}_4$ is a horosphere with a finite number of open sets (including a neighbourhood of the infinite point) removed.

Let $(\psi_t)_{t \in [0,4]} \in C^\infty(M)$ be a smooth family of smooth, positive valued functions such that:

(i) $\psi_0 = \psi$;

(ii) for all $t$ and for all $p \in \hat{\Sigma}_t$, the Gaussian curvature of $\hat{\Sigma}_t$ at $p$ is greater than $\psi_t(p)$; and
(iii) $\psi_4$ is constant and equal to $1 - \delta$ for some $\delta < 1$.

The problem $(\Sigma_4, \psi_4)$ in $(M, g_4) = \mathbb{H}^{n+1}$ is locally and globally rigid and has a non-trivial solution (see [11]). By Proposition 8.1, this isotopy by locally strictly convex, immersed hypersurfaces may be deformed to an isotopy by locally strictly convex, immersed hypersurfaces whose boundaries are generic. Existence therefore follows by (an appropriately modified version of) Lemma 12.3, and this completes the proof. □

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