ON THE STABILIZATION FOR THE HIGH-ORDER
KADOMTSEV-PETVIASHVILI AND THE
ZAKHAROV-KUZNETSOV EQUATIONS
WITH LOCALIZED DAMPING

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Abstract. In this paper we prove the exponential decay of the energy for
the high-order Kadomtsev-Petviashvili II equation with localized damping. To
do that, we use the classical dissipation-observability method and a unique
continuation principle introduced by Bourgain in [3] here extended for the high-
order Kadomtsev-Petviashvili. A similar result is also obtained for the two-
dimensional Zakharov-Kuznetsov (ZK)equation. The method of proof works
better for the ZK equation, so we were led to make some subtle modifications
on it to include KP type equations. In fact, to reach a key estimate we use
an anisotropic Gagliardo-Nirenberg inequality to drop the y-derivative of the
norm.

1. Introduction. The main purpose of this work is to study the exponential decay
of energy for the initial value problem associated to the high order Kadomtsev-
Petviashvili equation

\[ \partial_t u + \alpha \partial_x^3 u + \beta \partial_x^5 u + \gamma \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0, \tag{1} \]

where \( u = u(x, y, t) \) (with \((x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+\)) is a real-valued function, \( \alpha, \beta \) and
\( \gamma \) are real parameters with \( \beta \neq 0 \), and the operator \( \partial_x^{-1} \) denotes the anti-derivative
defined by

\[ \partial_x^{-1} f(x, y) = \int_{-\infty}^{x} f(z, y) dz, \tag{2} \]
and via Fourier transform, by

$$\widehat{(\partial_x^{-1} f)}(\xi, \eta) = \frac{1}{i \xi} \hat{f}(\xi, \eta).$$

(3)

More precisely, we shall study the initial value problem for (1) with \((x, y)\) in appropriate bounded domain and with suitable boundary conditions in order to make the energy dissipates. Below, we will point out the details.

When \(\beta = 0\) and \(\gamma = \pm 1\), the equation (1) becomes the usual Kadomtsev-Petviashvili (KP) equation

$$\partial_t u + \partial_x^3 u \pm \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0,$$

(4)

introduced by B. B. Kadomtsev and V. I. Petviashvili in [23] in order to study the transverse stability of the solitary waves solutions of the Korteweg-de Vries (KdV) equation. In both equations (4) and (1), the number \(\gamma = -1\) corresponds to the focusing case (KP-I type), while \(\gamma = +1\) corresponds to the defocusing case (KP-II type).

Analogously to (4), the fifth order KP equation (1) is deduced by taking into account weak transverse effects in the \(y\) direction of the so-called Kawahara equation

$$\partial_t v + v \partial_x v + \varepsilon \partial_x^3 v + \partial_x^5 v = 0,$$

(5)

where \(\varepsilon \in \{-1, 0, 1\}\) instead of the KdV equation. Also known as fifth order Korteweg-de Vries (KdV) equation, (5) was deduced in [24] by Kawahara in his study of oscillatory solitary waves, which in turn occurs just when the coefficient of the fifth order derivative term dominates over that of the third order one. See also [16] and [17] for the derivation of this equation in the context of one dimensional gravity-capillary waves.

The equation (1) is an infinite-dimensional Hamiltonian system with Hamiltonian

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ -\beta |\partial_x^2 u|^2 + \alpha |\partial_x u|^2 \pm |\partial_x^{-1} \partial_y u|^2 - \frac{1}{3} u^3 \right\} dxdy,$$

(6)

where the sign + corresponds to \(\gamma = -1\) (\(KP_5 - I\)) and the sign − corresponds to \(\gamma = +1\) (\(KP_5 - II\)). The Hamiltonians are (at least formally) constant along the trajectories of (1), i.e.

$$H(u(t)) = H(u_0).$$

The \(L^2\) norm is also conserved along the flow, i.e.,

$$E(u(t)) = E(u_0), \quad \text{where} \quad E(u(t)) := \int_{\mathbb{R}^2} |u(x, y, t)|^2 dxdy.$$

(7)

The Cauchy problem for higher order KP equations has been extensively studied (see [5, 13, 15, 20, 26, 31, 32, 33, 35, 42, 43] and references therein). For the IVP associated to (4) see e.g. [2, 14, 18, 21, 22] and references therein.

Now we shall detail the main matter of this work. As pointed out in (7), the energy \(E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^2} u^2 dxdy\) is a conserved quantity. However in the bounded framework \((x, y) \in \Omega := (0, L) \times (0, L)\) the energy may be dissipated. In this sense, we consider the IVP to the high-order KP equation in a bounded domain, under
the presence of a localized non-negative function \( a(x, y) \in L^\infty(\Omega) \) as damping term

\[
\begin{aligned}
\partial_t u + \alpha \partial_x^2 u + \beta \partial_y^2 u + \gamma \partial_x^{-1} \partial_y^2 u + u \partial_x u + au &= 0, \\
(x, y) \in (0, L) \times (0, L), t \in \mathbb{R}, \\
u(0, y, t) &= u(L, y, t) = 0, \\
\partial_x u(L, y, t) &= \partial_x u(0, y, t) = 0, \\
\partial_y u(x, L, t) &= \partial_y u(0, y, t) = 0, \\
\partial_x^2 u(L, y, t) &= 0, \\
u(x, y, 0) &= u_0(x, y),
\end{aligned}
\]

(8)

where the operator \( \partial_x^{-1} \) is in this case defined by \( \partial_x^{-1} f(x, y) = \int_0^x f(z, y)dz \). Under the above boundary conditions and the restrictions \( \beta < 0 \) and \( \gamma > 0 \), the total energy associated to (8) given by

\[
E(u(t)) = \frac{1}{2} \int_0^L \int_0^L u^2(x, y, t)dx\,dy
\]

(9)
is in fact dissipated along the flow (see Proposition 2), i.e.,

\[
\frac{d}{dt} E(u(t)) \leq 0,
\]

for all \( t > 0 \). Consequently, \( E(t) \) is a non-increasing function.

The following basic question arises: Does \( E(t) \to 0 \) as \( t \to +\infty \)? We refer [39, 40, 41, 29] for studies of this problem in the context of KdV equation and some of its generalizations, and [44] for an analogous approach for the Kawahara equation. However, as far as we know the only work dealing with stabilization for two-dimensional nonlinear dispersive equations via the technique presented here on bounded domains is Gomes and Panthee [11], where the KP-II equation is studied. For some related results regarding high dimensional nonlinear dispersive equations we refer [4, 6, 7, 28] and references therein.

Here, following the strategy described in [39, 44], we use the boundary conditions from (8) together with a damping function \( a(x, y) \) and then we employ compactness arguments and the UCP property of (1) stated at Theorem 1.2 below, to prove the decay of the energy is exponential in time. We shall get the following result.

**Theorem 1.1.** Let \( \gamma > 0 \). Given \( R > 0 \), let \( u \) be a solution of (8) with \( \alpha > 0, \beta < 0 \) and data \( u_0 \in X^0(\Omega) \) satisfying \( \|u_0\|_{L^2(\Omega)} \leq R \), and let \( E(u(t)) \) be the energy defined by (9). Then the energy \( E(u(t)) \) decays exponentially, i.e., there exists \( \delta = \delta(R) > 0 \) such that,

\[
E(u(t)) \leq CE(u(0)) \exp(-\delta t), \quad \forall \ t \geq 0,
\]

(10)

where \( C = C(R, T) > 0 \).

According the dissipation-observability method (see Section 2) if the energy dissipates in an expected way (that is, meeting an observability condition) then it decays exponentially in time. In order to prove the observability condition for (8) we face some hardship due the order of the equation, the dimension and the presence of the term \( \partial_x^{-1} \partial_x^2 u \).

The structure of the equation (1) led us to drop the \( y \)-derivative in the norm of \( H^2(\Omega) \) (see Lemma 3.1 below), adding an extra difficulty to employ the usual a priori estimate argument. To overcome that trouble, we employ a useful anisotropic Gagliardo-Nirenberg inequality to obtain an estimate in the context of the space \( H^2_x(\Omega) \) instead of the classical one.
Our method of proof relies on a unique continuation principle (UCP) instead of the classical Holmgren’s uniqueness Theorem. More precisely, we shall prove and use the following unique continuation result.

**Theorem 1.2.** Consider the IVP
\[
\begin{cases}
\partial_t u + \alpha \partial_x^2 u + \beta \partial_y^2 u + \gamma \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0, \\
u(x, y, 0) = u_0(x, y),
\end{cases}
(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+,
\tag{11}
\]
with \(\alpha, \beta\) and \(\gamma\) real constants where \(\beta \neq 0\). Let \(u = u(x, y, t)\) be a smooth solution to (11) defined in a non degenerate interval \(I = [-T, T]\). If for some \(B > 0\),
\[
\text{supp } u(t) \subseteq [-B, B] \times [-B, B], \text{ for any } t \in I,
\]
then \(u \equiv 0\).

Inspired by the method introduced by Bourgain in [3], our approach on UCP extends the results of Esfahani and Pastor [8] and Panthee [38] to the fifth order KP equation (1) without restriction on the parameters \(\alpha, \beta, \gamma\). In the proof of Theorem 1.2 we follow closely the ideas of Esfahani and Pastor [8] to reduce the problem to a one dimensional one by choosing some key parameters appropriately. (For a thorough survey about the so-called UCP we suggest reading the introduction of Esfahani and Pastor [8] and references mentioned therein.)

For the existence of smooth solutions associated to the IVP (11), we refer the following local well-posedness result due to Iório and Nunes in [19] for initial data in \(X_s(\mathbb{R}^2)\), which denotes the set of all \(f \in H^s(\mathbb{R}^2)\) such that \((\partial_x^{-1} f)(\xi, \eta) \in H^s(\mathbb{R}^2)\), endowed with the norm \(\|f\|^2_{X_s} = \|f\|^2_{H^s} + \|\partial_x^{-1} f\|^2_{H^s}\).

**Theorem A.** Let \(s > 2\). For each \(\phi \in X_s(\mathbb{R}^2)\), there exists \(T > 0\), depending only on \(\|\phi\|_{X_s}\), and a unique solution \(u\) to (11) such that
\[
u \in C([0, T]; X_s(\mathbb{R}^2)) \cap C^1([0, T]; H^{s-5}(\mathbb{R}^2)).
\tag{12}
\]
Furthermore, the map \(\phi \mapsto u\) is continuous from \(X_s\) to the space (12). Moreover, \(T\) can be chosen independent of \(s\).

Regarding the well-posedness for the problem (8), we prove the existence and uniqueness of global solution in \(Y\) for initial data in \(u_0 \in X^0(\Omega)\) (see Appendix, Sec. 5), where
\[
Y = C([0, \infty); X^0(\Omega)) \cap L^2_{loc}((0, \infty); X^2_{x_0}(\Omega)),
\]
and \(X^2_{x_0}(\Omega)\) is the closure of \(C^\infty_0(\Omega)\) in \(X^2_\Omega(\Omega)\).

A similar approach to that one above described for the KP-5 equation (1) also works for the Zakharov-Kuznetsov (ZK) equation
\[
\partial_t u + \alpha \partial_x^2 u + \partial_x \partial_y^2 u + u \partial_x u = 0, \ (x, y, t) \in \mathbb{R}^2, \ t > 0,
\tag{13}
\]
where \(u = u(x, y, t)\) is a real-valued function and \(\alpha \neq 0\) is a real parameter. In this case, the UCP result needed (similar to the Theorem 1.2), was proved by Panthee in [37]. The existence of smooth solution is done using the classical parabolic regularization method as in Iório and Nunes [19].

The equation (13) was formally derived by Zakharov and Kuznetsov [45] as a long wave small amplitude limit of the Euler-Poisson system in the “cold plasma” approximation on the context of plasma physics (see also [27] for a rigorously justification of this formal long-wave limit). The Zakharov-Kuznetsov equation can be seen as a higher-dimensional extension of the Korteweg-de Vries model of surface
wave propagation, quite different from the KP equation which is obtained as an asymptotic model of several nonlinear dispersive systems under a different scaling.

Regarding well-posedness results for the IVP associated to (13) we refer for instance [34, 12, 9, 30, 25] and the references therein.

Similarly to (8), we consider (13) in a bounded domain \( \Omega = (0,L) \times (0,L) \), under the presence of a localized non-negative function \( a(x,y) \in L^\infty(\Omega) \) as damping term, as follows

\[
\begin{aligned}
\partial_t u + \alpha \partial_x^3 u + \partial_x \partial_y^2 u + u \partial_x^2 u + a(x,y)u &= 0, \quad (x,y) \in \Omega, \quad t \in \mathbb{R}, \\
u(0,y,t) = u(L,y,t) = 0, \quad u(x,L,t) = u(x,0,t) = 0, \\
\partial_x u(L,y,t) = 0, \quad \partial_y u(L,y,t) = 0, \\
u(x,y,0) &= u_0(x,y).
\end{aligned}
\]

(14)

The total energy associated with (14) is given by

\[
E(u(t)) = \frac{1}{2} \int_0^L \int_0^L u^2(x,y,t) \, dx \, dy.
\]

(15)

Using the same strategy and applying similar tools to those in the proof of Theorem 1.1 we get the following result.

**Theorem 1.3.** Given \( R > 0 \), let \( u \) be a solution of (14) with \( \alpha > 0 \) and data \( u_0 \in L^2(\Omega) \), satisfying \( \|u_0\|_{L^2(\Omega)} \leq R \), and let \( E(u(t)) \) be the energy defined by (15). Then the energy \( E(u(t)) \) decays exponentially.

The proof of Theorem 1.3 is a little simpler than the proof of Theorem 1.1 because the structure of the equation (14) enjoys good symmetry in \( L^2(\Omega) \), so that we avoid to resort to the strategy of restricting Sobolev’s norm to a single direction (\( x \)-direction) as we have done for the KP-5 equation. Furthermore, thanks to the lower order of the ZK equation, we only need to estimate \( \|u\|^2_{L^2((0,T);H^1(\Omega))} \) by \( E(u_0) \) as key one (see Lemma 4.1 below).

**Remark 1.** It worth mentioning that the technique presented seems to extend for ZK equations in higher dimension. Indeed, a careful look at Lemma 4.1 reveals that it works for ZK equations in dimension \( 2 \leq n \leq 6 \). Therefore we might obtain stabilization for ZK equations in dimension up to \( n = 6 \) whenever one can prove a UCP result similar to that for the two dimensional ZK. However, as far as we know the UCP for for higher order ZK equations is unknown.

The work is organized as follows. In Section 2 we prove the unique continuation result for the high order KP equation and exhibit a sketch of the dissipation-observability method. In Section 3 we prove the result of stabilization of the high order KP equation (Theorem 1.1) and in Section 4 we prove the result of stabilization for the ZK equation (Theorem 1.3). We finish with an appendix in which we prove the global existence of solution to (8) for initial data in \( X^0(\Omega) \) (see the definition in (17)).

**Notation.** Given any positives constants \( C, D \), by \( C \lesssim D \) we mean that there exists a constant \( c > 0 \) such that \( C \leq cD \); and, by \( C \sim D \) we mean \( C \lesssim D \) and \( D \lesssim C \).

By \( \mathcal{F}\{\phi\} \) or \( \hat{\phi} \) we denote the Fourier transform of \( \phi \), defined as

\[
\hat{\phi}(\xi,\eta) = \int_{\mathbb{R}^2} \phi(x,y)e^{-i(x\xi+y\eta)} \, dx \, dy.
\]
Given $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$, we define the space $X^k(\Omega)$ to be the Sobolev space

$$X^k(\Omega) = \{ f \in H^k(\Omega) \mid \partial_x^{-1}f(x, y) = \int_0^x f(z, y)dz \in H^k(\Omega) \} \quad (k \geq 0 \text{ integer}),$$

endowed with the norm

$$\|f\|_{X^k(\Omega)}^2 = \|f\|_{H^k(\Omega)}^2 + \|\partial_x^{-1}f\|_{H^k(\Omega)}^2.$$

We also define $H^k_x(\Omega)$ to be the normed space consisting of all functions $f$ for which $\partial_j f \in L^2(\Omega), 0 \leq j \leq k$, endowed with the norm

$$\|f\|_{H^k_x(\Omega)}^2 = \sum_{0 \leq j \leq k} \|\partial_x^j f\|_{L^2(\Omega)}^2.$$  \hfill (16)

and the space

$$X^k_x(\Omega) = \{ f \in H^k_x(\Omega) \mid \partial_x^{-1}f(x, y) = \int_0^x f(z, y)dz \in H^k_x(\Omega) \},$$

normed by

$$\|f\|_{X^k_x(\Omega)}^2 = \|f\|_{H^k_x(\Omega)}^2 + \|\partial_x^{-1}f\|_{H^k_x(\Omega)}^2.$$  \hfill (17)

$H^k_{x0}(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ in $H^k_x(\Omega)$.

2. Preliminary and basic results. Here we first establish an extension of the unique continuation result proved by Esfahani and Pastor in [8] for high-order KP equations. We finish the section with a brief outline of the dissipation-observability method which we use to conclude the proof of our main results.

2.1. Unique continuation for the high-order KP equations. Our goal here is to prove Theorem 1.2. The main idea is to use the method introduced by J. Bourgain in [3]. As in Esfahani and Pastor [8], we choose some key parameters so that the issue should be reduced to an one-dimensional problem. We make direct use the following result.

**Lemma 2.1.** Let $u = u(x, y, t)$ be a smooth solution to the IVP (11) and let $I = [-T, T]$ be a non degenerate interval. If for some $B > 0,$

$$\text{supp } u(t) \subseteq [-B, B] \times [-B, B], \text{ for any } t \in I,$$

then for all $\lambda = (\xi, \eta)$ and $\sigma = (\theta, \delta)$ in $\mathbb{R}^2,$

$$\widehat{u(t)}(\lambda + i\sigma) \lesssim e^{cB|\sigma|}.$$

**Proof.** See Lemma 2.1 in [8].

**Proof of Theorem 1.2.** We suppose by contradiction that there exists $t \in I$ such that $u(t) \neq 0.$ The integral equation associated to (11), for $t_1, t_2 \in I,$ is

$$u(t_2) = U(t_2 - t_1)u(t_1) - \frac{1}{2} \int_{t_1}^{t_2} U(t_2 - t')\partial_x(u^2(t'))dt',$$ \hfill (18)

where

$$U(t)\phi(x, y) = \int_{\mathbb{R}^2} e^{i(x\xi + y\eta + tp(\xi, \eta))}\phi(\xi, \eta)d\xi d\eta,$$

with $p(\xi, \eta) = \alpha\xi^3 - \beta\xi^5 - \gamma\eta^2.$
Now taking the Fourier transform in space variable on (18), we get
\[
\hat{u}(t_2)(\xi, \eta) = e^{i\Delta p(\xi, \eta)}(\hat{u}(t_1)(\xi, \eta) - i\xi/2 \int_0^{\Delta t} e^{i(t_1-t')p(\xi, \eta)}\hat{u}^2(t')(\xi, \eta)dt'),
\]
(19)
where without loss of generality we are assuming \(\Delta t := t_2 - t_1 > 0\).

Using the change of variable \(s = t' - t_1\) we can write (19) as
\[
\hat{u}(t_2)(\xi, \eta) = e^{i\Delta p(\xi, \eta)}(\hat{u}(t_1)(\xi, \eta) - i\xi/2 \int_0^{\Delta t} e^{-isp(\xi, \eta)}u^2(s + t_1)(\xi, \eta)ds).
\]
(20)

Since \(u(t), t \in I\), has compact support, it follows from Paley-Wiener theorem that \(\hat{u}(t_2)\) has an analytic continuation to \(\mathbb{C}^2\) as
\[
\hat{u}(t_2)(\lambda + i\sigma) = e^{i\Delta p(\lambda + i\sigma)}(\hat{u}(t_1)(\lambda + i\sigma) - \theta - i\xi/2 \int_0^{\Delta t} e^{-isp(\lambda + i\sigma)}u^2(s + t_1)(\lambda + i\sigma)ds),
\]
where \(\lambda = (\xi, \eta)\) and \(\sigma = (\theta, \delta)\), with the parameters \(\lambda\) and \(\sigma\) to be chosen later and
\[
\text{ip}(\lambda + i\sigma) = p(\xi + i\theta, \eta + i\delta)
\]

By Lemma 2.1 we have
\[
Ce^{\Delta t(\alpha(3\xi^3 \theta - \theta^3) + \gamma \frac{\eta^2 - \eta^3}{\xi^2 + \eta^2} + \beta(-5\xi^4 \theta + 10\xi^2 \theta^3 - \theta^5))} \geq |\hat{u}(t_1)(\lambda + i\sigma)| \geq |u(t_1)(\lambda + i\sigma)|
\]
(22)
\[
- \frac{\xi + i\theta}{2} \int_0^{\Delta t} e^{\frac{2\xi + i\theta}{\lambda + i\sigma}}(\alpha(3\xi^3 \theta - \theta^3) + \gamma \frac{\eta^2 - \eta^3}{\xi^2 + \eta^2} + \beta(-5\xi^4 \theta + 10\xi^2 \theta^3 - \theta^5))u^2(s + t_1)(\lambda + i\sigma)ds.
\]
We have the following two cases to consider.

1. \(\gamma = -1\). In this case, we shall prove that the equation in (11) behaves as the Kadomtsev-Petviashvili-I (KP-I) in Esfahani/Pastor [8]. In fact, taking \(\delta \sim \theta\), with \(\theta = 0, \quad \delta < 0\), and \(\xi > 0, \quad \eta > 0\) large enough such that
\[
\frac{\xi^2}{\eta|\delta|} \ll 1,
\]
(23)
we get from (22) that
\[
Ce^{-2\Delta t|\theta|/\xi} \geq |\hat{u}(t_1)(\lambda + i\sigma)| - \frac{\xi}{2} \int_0^{\Delta t} e^{-2\Delta t|\theta|/\xi}u^2(s + t_1)(\lambda + i\sigma)ds.
\]
(24)
From this is enough to follow the proof of Theorem 1.3 in [8], from inequality (4.7) onwards therein, to get a contradiction.

2. \(\gamma = 1\). In this case, the equation in (11) behaves as the Kadomtsev-Petviashvili-II (KP-II). It is enough to take \(\delta \sim \theta\), with \(\theta = 0\) and \(\delta > 0\), and \(\xi > 0, \quad \eta > 0\) satisfying (23). In fact choosing the referred parameters we arrive to the estimate
\[
Ce^{-2\Delta t|\theta|/\xi} \geq |\hat{u}(t_1)(\lambda + i\sigma)| - \frac{\xi}{2} \int_0^{\Delta t} e^{-2\Delta t|\theta|/\xi}u^2(s + t_1)(\lambda + i\sigma)|ds
\]
(25)
from which we can get a contradiction as before. We note that (25) differs from (24) by the sign of \( \delta \), but we note out that fact does not have affected the obtaining of the contradiction.

2.2. The dissipation-observability method. We present here a sketch from the dissipation-observability method which we will employ to get theorems 1.1 and 1.3. For more details see for instance [39, 46, 47].

Consider \( \Omega \subset \mathbb{R}^n \) a domain. Let \( A \) be a linear operator, and \( B \) a nonlinear operator with domain dense in \( L^2(\Omega) \). Let \( u \) be a solution to the evolution equation in \( L^2(\Omega) \),

\[
\partial_t u = Au + B(u),
\]

under suitable initial-boundary conditions. Suppose that the evolution associated to (26) satisfies a semi-group property and that the energy

\[
E(u(t)) = \int_{\Omega} u^2 dx
\]

is dissipated according to

\[
\frac{d}{dt} E(u(t)) = -Q(u(t)) \leq 0,
\]

where \( -Q(u) = \int_{\Omega} u(Au + B(u)) dx \).

**Proposition 1.** Let \( u \) be a solution of (26) satisfying (27). Suppose that

\[
\int_0^T E(u(t)) dt \leq C(T) \int_0^T Q(u(t)) dt.
\]

Then there exists \( C > 0 \) such that

\[
E(u(0)) \leq C \int_0^T Q(u(s)) ds.
\]

**Proof.** In fact, since \( u \) satisfies (27) we have (cf. Lemma 2.2 in [11])

\[
E(u_0) \leq \frac{1}{T} \int_0^T E(u(t)) dt + 2 \int_0^T Q(u(t)) dt.
\]

Thus the result follows from (30) and (28).

**Corollary 1.** Suppose that \( u \) satisfies the observability inequality (29). Then the energy \( E \) decays exponentially, i.e., there exists \( \alpha > 0 \) such that,

\[
E(u(t)) \leq CE(u(0)) \exp(-\alpha t), \quad \forall t \geq 0.
\]

**Proof.** See Theorem 2.1 in [11].

Therefore in order to prove the exponential decay of the energy \( E(u(t)) \) it is enough to prove the inequality (28).

3. Stabilization for the high-order KP-II equation. In this section we prove Theorem 1.1. In order to do that we use dissipation-observability method and follow closely the ideas posed in [39, 44].
3.1 Preliminary estimates. Consider \( a(x, y) \geq a_0 > 0 \) almost everywhere in the complement of a compact non-empty proper subset \( \Gamma \) of \( \Omega = (0, L) \times (0, L) \). Thus the damping term is acting effectively in \( \Gamma \). We assume that for some \( \delta > 0 \), \( \Gamma \subset (\delta, L - \delta) \times (\delta, L - \delta) \), so that we can apply the UCP using an extension argument.

Consider the damped high-order KP model (8). Let \( E(u(t)) \) be the energy defined in (9). We also define

\[
Q(u(t)) = \frac{1}{2} \int_0^L \left[ -\beta (\partial_x^2 u(0, y, t))^2 + \gamma (\partial_x^{-1} \partial_y u(0, y, t))^2 \right] dy + \int_0^L \int_0^L a(x, y)u^2(x, y, t) dx dy.
\] (32)

Next, we prove that the energy \( E(u) \) is a decreasing function of \( t \).

**Proposition 2.** Let \( \gamma > 0 \). Suppose that \( u \) satisfies (8), and let \( E \) be given by (9). Then, for each \( \beta < 0 \), we have

\[
\frac{d}{dt} E(u(t)) = -Q(u)(t) \leq 0.
\] (33)

**Proof.** In fact,

\[
\frac{d}{dt} E(u(t)) = -\alpha \int_0^L \int_0^L u \partial_x^3 u dx dy - \beta \int_0^L \int_0^L u \partial_x^2 u dx dy
\]

\[
\quad - \gamma \int_0^L \int_0^L u \partial_x^{-1} \partial_y^2 u dx dy - \int_0^L \int_0^L a(x, y)u^2 dx dy.
\]

Now using integration by parts and the boundary conditions from (8) we get (33).

\[\square\]

From Proposition 2 we have that the energy is dissipated, i.e., \( E(u(t)) \leq E(u_0) \), for all \( t > 0 \). So, we are able to get the following crucial result to establish the proof of Theorem 1.1.

**Lemma 3.1.** Let \( u \) be a solution of (8), with \( \alpha > 0 \), \( \beta < 0 \) and \( \gamma > 0 \). Then, there exist constants \( C_1, C_2 > 0 \) depending only on \( L \) and \( T \), such that

\[
\|u\|_{L^2(0, T); H_x^2(\Omega)}^2 \leq C_1 E(u_0) + C_2 E(u_0)^{\frac{7}{2}},
\] (34)

where \( H_x^2(\Omega) \) is defined in (16), and \( \|u\|_{H_x^2(\Omega)}^2 = \|f\|_{L^2(\Omega)}^2 + \|\partial_x f\|_{L^2(\Omega)}^2 + \|\partial_x^2 f\|_{L^2(\Omega)}^2 \), with \( \Omega = (0, L) \times (0, L) \).

**Proof.** Multiplying the equation in (8) by \( xu \), integrating by parts on \( \Omega \) and then integrating on \( (0, T) \), we find

\[
\frac{1}{2} \int_0^T \int_0^L xu^2(x, y, T) dx dy - \frac{1}{2} \int_0^L \int_0^L xu_0^2(x, y) dx dy
\]

\[
+ \frac{\gamma}{2} \int_0^T \int_0^L \int_0^L (\partial_x^{-1} \partial_y u)^2 dx dy dt
\]

\[
= -\frac{3\alpha}{2} \int_0^T \int_0^L \int_0^L (\partial_x u)^2 dx dy dt + \frac{5\beta}{2} \int_0^T \int_0^L \int_0^L (\partial_x^2 u)^2 dx dy dt
\]

\[
+ \frac{1}{3} \int_0^T \int_0^L \int_0^L u^3 dx dy dt - \int_0^T \int_0^L \int_0^L x a(x, y) u^2 dx dy dt.
\] (35)
Then we have,

\[
\int_0^T \int_0^L \int_0^L \left\{ \frac{3\alpha}{2} (\partial_x u)^2 - \frac{5\beta}{2} (\partial_x^2 u)^2 \right\} dx dy dt \\
= -\frac{1}{2} \int_0^L \int_0^L x u^2(x, y, T) dx dy + \frac{1}{2} \int_0^L \int_0^L x u_0^2(x, y) dx dy \\
- \frac{\gamma}{2} \int_0^T \int_0^L \int_0^L (\partial_x^{-1} \partial_y u)^2 dx dy dt + \frac{1}{3} \int_0^T \int_0^L \int_0^L u^3 dx dy dt \\
- \int_0^T \int_0^L \int_0^L x a(x, y) u^2 dx dy dt.
\] (36)

We set \( \delta = \min\{1, \frac{3\alpha}{2}, \frac{-5\beta}{2}\} \). Therefore, since \( a(x, y) \geq 0 \), \( \alpha, \gamma > 0 \) and \( \beta < 0 \), we have from (36) that

\[
\delta \|u\|_{L^2(0, T; H^2(\Omega))}^2 \leq \frac{L}{2} \|u_0\|_{L^2(\Omega)}^2 + \delta \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt + \int_0^T \|u(t)\|_{L^2(\Omega)}^3 dt. \] (37)

In order to control the last term in (37), first we employ Cauchy-Schwarz inequality and then an anisotropic Gagliardo-Nirenberg inequality (see Theorem 15.7 in [1]) and Young inequality as follows.

\[
\|u\|_{L^4(\Omega)}^2 \leq L \|u\|_{L^4(\Omega)}^2 \leq cL \|\partial_x^2 u\|_{L^2(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{5/2} \\
\leq \frac{\delta^4}{4} \|\partial_x^2 u\|_{L^2(\Omega)}^2 + \frac{3}{4} \left( \frac{cL}{\delta} \right)^{3/2} \|u\|_{L^2(\Omega)}^{10/3}. \] (38)

We see by the Proposition 2 that \( E(u(t)) \leq E(u_0) \), where \( E(u) = \frac{1}{2} \|u\|_{L^2(\Omega)}^2 \). So, integrating (38) in the time interval we obtain

\[
\int_0^T \|u(t)\|_{L^2(\Omega)}^3 dt \leq \frac{\delta^4}{4} \int_0^T \|u(t)\|_{H^2(\Omega)}^2 dt + \frac{3}{4} \left( \frac{cL}{\delta} \right)^{3/2} T \|u_0\|_{L^2(\Omega)}^{10/3}. \] (39)

Thus, since \( \delta \leq 1 \), we have from (37) and (39) that

\[
\frac{3\delta}{4} \|u\|_{L^2(0, T; H^2(\Omega))}^2 \leq \left( \frac{L}{2} + \delta T \right) \|u_0\|_{L^2(\Omega)}^2 + \frac{3}{4} \left( \frac{cL}{\delta} \right)^{3/2} T \|u_0\|_{L^2(\Omega)}^{10/3}.
\]

Hence,

\[
\|u\|_{L^2(0, T; H^2(\Omega))}^2 \leq \frac{4}{3} \left( \frac{L}{2\delta} + T \right) E(u_0) + \frac{(cL)^{3/4}}{\delta^{3/2}} T E(u_0)^{5/3},
\]

and this concludes the proof. \( \square \)

In what follows we shall consider, without loss of generality, \( \alpha = -\beta = \gamma = 1 \).

With Lemma 3.1 and Theorem 1.2 in hand, we can prove the following result, from which, according to the Proposition 1, we get the observability inequality.

**Lemma 3.2.** Let \( Q(u) \) be defined in (32). Then, for any \( T > 0 \) and \( R > 0 \) there exist a positive constant \( C' = C'(R, T) > 0 \) such that

\[
\int_0^T E(u(t)) dt \leq C' \int_0^T Q(u(t)) dt,
\] (40)

for all solution of (8) with \( \|u_0\|_{L^2(\Omega)} \leq R \).
Proof. We prove (40) by contradiction using (34) and the unique continuation result stated in Theorem 1.2. We shall follow the ideas posed in the proof of Theorem 2.2 from [39], but with important changes in some norms and key estimates.

Suppose that (40) is not true. Then for each positive integer \( n \) there exists a solution \( u_n \) of (8) such that \( nQ(u_n) \gtrsim \|u_n\|^2_{L^2((0,T);L^2(\Omega))} \). In this case, we have a sequence \( \{u_n\} \) of solutions such that

\[
\lim_{n \to \infty} \frac{\|u_n\|^2_{L^2((0,T);L^2(\Omega))}}{Q(u_n)} = +\infty. \tag{41}
\]

Let \( \{\lambda_n\} \) and \( \{v_n\} \) be sequences defined respectively by

\[
\lambda_n = \|u_n\|_{L^2((0,T);L^2(\Omega))} \quad \text{and} \quad v_n(x,y,t) = \frac{1}{\lambda_n} u_n(x,y,t),
\]

so that

\[
\|v_n\|^2_{L^2((0,T);L^2(\Omega))} = 1. \tag{42}
\]

From (34) we have that \( \{\lambda_n\} \) is a bounded sequence, for \( \|u_n(0)\|_{L^2(\Omega)} \leq R \). Therefore, extracting a subsequence if necessary, we can assume that \( \lambda_n \to \lambda \geq 0 \).

We observe that, for all \( n \in \mathbb{N} \) the function \( v_n \) satisfies

\[
\begin{cases}
\partial_t v_n - \partial_x^2 v_n + \partial_y^2 v_n + \frac{1}{2} \partial_y^2 v_n + \lambda_n v_n \partial_x v_n + a(x,y)v_n = 0, \\
v_n(0,y,t) = 0, \quad v_n(x,L,t) = v_n(x,0,t) = 0, \\
\partial_x v_n(0,y,t) = \partial_x v_n(L,y,t) = 0, \\
\partial_x^2 v_n(L,y,t) = 0,
\end{cases} \tag{43}
\]

for \( (x,y) \in \Omega, t \in \mathbb{R} \), with initial data \( v_n(x,y,0) = \frac{1}{\lambda_n} u_n(x,y,0) \). Moreover, from (41)

\[
\int_0^T Q(v_n(t))dt \to 0, \quad \text{as} \quad n \to \infty. \tag{44}
\]

From (30) we have that \( v_n(x,y,0) \) is bounded in \( L^2(\Omega) \) yielding

\[
\|v_n(\cdot,\cdot,t)\|_{L^2(\Omega)} \leq R, \quad \forall \quad 0 \leq t \leq T. \tag{45}
\]

From (34) we see that

\[
\|v_n\|^2_{L^2((0,T);H^2(\Omega))} \leq C, \quad \forall \quad n \in \mathbb{N}. \tag{46}
\]

The identity (42) together with the inequality (46) show that

\[
\partial_t v_n = \partial_x^2 v_n - \partial_y^2 v_n - \frac{1}{2} \partial_y^2 v_n - \lambda_n v_n \partial_x v_n - a(x,y)v_n \tag{47}
\]

is bounded in \( L^2((0,T);H^{-3}(\Omega)) \). It is worth noticing that \( \lambda_n v_n \partial_x v_n \) on the RHS of (47) is bounded in \( L^2((0,T);L^p(\Omega)) \), \( 1 \leq p < 2 \), which is bounded in \( L^2((0,T);H^{-3}(\Omega)) \). Then the sequence \( v_n \) is bounded in \( L^2((0,T);H^2(\Omega)) \cap H^1((0,T);H^{-3}(\Omega)) \)

and so, there exist \( v \) such that

\[
v_{n_j} \rightharpoonup v, \quad \text{weakly in} \quad L^2((0,T);H^2(\Omega)) \cap H^1((0,T);H^{-3}(\Omega)). \tag{48}
\]

Moreover, since \( H^2(\Omega) \subset L^2(\Omega) \) and \( H^2(\Omega) \subset H^2(\Omega) \subset L^2(\Omega) \), the embedding \( H^2(\Omega) \hookrightarrow L^2(\Omega) \) is also compact. Thus, follows from the Rellich’s theorem that \( \{v_n\} \) is relatively compact in \( L^2((0,T);L^2(\Omega)) \). Then, we can assume

\[
v_{n_j} \to v, \quad \text{strongly in} \quad L^2((0,T);L^2(\Omega)). \tag{49}
\]
The fact that \( \|v_{nj}\|_{L^2((0,T);L^2(\Omega))} = 1 \) implies that
\[
\|v\|_{L^2((0,T);L^2(\Omega))} = 1.
\]  
Using the weak lower semicontinuity of convex functionals we have
\[
0 = \lim inf_{j \to \infty} \int_0^T \int_0^L Q(v_{nj}(t)) \, dt \geq \int_0^T Q(v(t)) \, dt,
\]
which implies that \( a(x,y)v \equiv 0 \) in \( \Omega \times (0,T) \). Since \( a(x,y) > 0 \) in \( \Gamma^c \) we get that \( v \equiv 0 \) in \( \Gamma^c \times (0,T) \).

We notice that the limit \( v \) satisfies
\[
\partial_t v - \partial_x^2 v + \partial_x^3 v + \partial_x^{-1} \partial_y^2 v + \lambda v \partial_x v + a(x,y)v = 0,
\]
where \( \lambda \geq 0 \) is the limit of \( \lambda_n \) as \( n \to \infty \). In either case \( \lambda = 0 \) or \( \lambda > 0 \), we shall employ the UCP provided by Theorem 1.1 to conclude that \( v \equiv 0 \) in \( \Omega \times (0,T) \). To do so, we must find an smooth extension of \( v \) in \( \mathbb{R}^2 \). Let \( \mathcal{Z} = (\delta, L-\delta) \times (\delta, L-\delta) \) and define the function
\[
w(x,y,t) = \begin{cases} 
  v(x,y,t), & (x,y,t) \in \mathcal{Z} \times (0,T) \\
  0, & (x,y,t) \in \{\mathbb{R}^2 - \mathcal{Z}\} \times (0,T).
\end{cases}
\]

Since \( \Gamma \subset (\delta, L-\delta) \times (\delta, L-\delta) \), this extension is as smooth as \( v \). Besides, \( w \) solves
\[
\begin{cases}
\partial_t w - \partial_x^2 w + \partial_x^3 w + \partial_x^{-1} \partial_y^2 w + \lambda w \partial_x w = 0, & (x,y,t) \in \mathbb{R}^2 \times (0,T) \\
w(x,y,0) = \phi(x,y), & \end{cases}
\]
where
\[
\phi(x,y) = \begin{cases} 
  v(x,y,0), & (x,y) \in \mathcal{Z} \\
  0, & (x,y) \in \mathbb{R}^2 - \mathcal{Z},
\end{cases}
\]
which is compactly supported in \( H^s(\mathbb{R}^2) \), \( s > 2 \). From Theorem A, the IVP (53) has a smooth solution \( w \). Therefore, by the unique continuation property established in Theorem 1.2, we conclude that \( w \equiv 0 \) in \( \Omega \times (0,T) \). As a result we have that \( v \equiv 0 \) in \( \Omega \times (0,T) \) which contradicts (50). Thus (40) holds. \( \square \)

3.2. Proof of Theorem 1.1. It is enough to employ Corollary 1 using Proposition 1 with the \( Q(u) \) defined in (32). In fact, from Lemma 3.2, we have
\[
\int_0^T E(u(t)) \, dt = \int_0^T \int_0^L \int_0^L u^2(x,y,t) \, dx \, dy \, dt \leq C(T) \int_0^T Q(u(t)) \, dt.
\]
This implies, from Proposition 1 that the observability inequality (29) holds. Therefore, the energy \( E \) defined in (9) above, decays exponentially. Thus the proof of Theorem 1.1 is finished.

4. Stabilization for the ZK equation: Proof of Theorem 1.3. The proof of Theorem 1.3 is quite similar, but a little simpler, than that of Theorem 1.1. Indeed, as we explained at the introduction, unlike the KP-5 equation, the linear part of (14) enjoys better symmetry in \( L^2(\Omega) \) and the equations is of third order, so we will not need to resort to the strategy of restricting Sobolev’s norm to a single direction (\( x \) direction) as we have done for the KP-5 equation. Hence, we give only the main steps here.

Consider the damped ZK model (14). Let \( E(u(t)) \) be the energy defined in (15). We also define
\[
Q(u(t)) = \frac{\alpha}{2} \int_0^L (\partial_x u(0,y,t))^2 \, dy + \frac{1}{2} \int_0^L (\partial_y u(0,y,t))^2 \, dy + \int_0^L \int_0^L a(x,y) u^2 \, dx \, dy.
\]
A straightforward computation shows that the energy is a decreasing function of $t$. In fact, using integration by parts and boundary conditions from (14), we have

$$\frac{d}{dt} E(u(t)) = -Q(u(t)) \leq 0.$$  \hfill (57)

**Lemma 4.1.** Let $u$ be a solution of (14), with $\alpha > 0$. Then, there exist constants $C_1, C_2 > 0$ depending on $\alpha, L$ and $T$, such that

$$\|u\|_{L^2((0,T),H^1(\Omega))}^2 \leq C_1 E(u_0) + C_2 E(u_0)^2.$$  \hfill (58)

where $E(\cdot)$ is the functional defined in (15).

**Proof.** Multiplying the equation in (14) by $xu$ and integrating by parts on $(0, L) \times (0, L)$ and then integrating on $(0, T)$ we get

$$\frac{3\alpha}{2} \int_0^T \int_0^L \int_0^L (\partial_x u)^2 dxdy + \frac{1}{2} \int_0^T \int_0^L (\partial_y u)^2 dxdy$$

$$= -\frac{1}{2} \int_0^T \int_0^L x u^2(\delta px, T) dxdy + \frac{1}{2} \int_0^T \int_0^L x u_0^2(\delta px, T) dxdy$$

$$+ \frac{1}{3} \int_0^T \int_0^L \int_0^L w^3 dxdy - \int_0^T \int_0^L \alpha(x,y) u dxdy.$$  \hfill (59)

Thus, since $a(x, y) \geq 0$, $\alpha > 0$ we have from (59) and (57) that

$$\delta \|u\|_{L^2((0,T),H^1(\Omega))}^2 \leq (L + \delta T)E(u_0) + \frac{2}{3} \int_0^T \|u\|_{L^2(\Omega)}^3 dt,$$  \hfill (60)

where $\delta = \min\{3\alpha, 1\}$. Now from the Gagliardo-Nirenberg posed in Friedman [10] (Theorem 10.1, p. 27), we have

$$\|u\|_{L^3(\Omega)} \leq C\|u\|_{H^1(\Omega)}^\frac{1}{2} \|u\|_{L^2(\Omega)}^\frac{3}{2}.$$  \hfill (61)

Thus, employing (61) at (60) joint with (15) and (57), we find

$$\delta \|u\|_{L^2((0,T),H^1(\Omega))}^2 \leq (L + \delta T)E(u_0) + \frac{2C}{3} E(u_0) \int_0^T \|u\|_{H^1(\Omega)}^3 dt$$

$$\leq (L + \delta T)E(u_0) + \frac{2CT^\frac{1}{2}}{3} E(u_0) \|u\|_{L^2((0,T),H^1(\Omega))}.$$  \hfill (62)

Hence, by Young inequality we get

$$\delta \|u\|_{L^2((0,T),H^1(\Omega))}^2 \leq (L + \delta T)E(u_0) + \frac{2CT}{9\delta} E(u_0)^2 + \delta \|u\|_{L^2((0,T),H^1(\Omega))}^2.$$  \hfill (62)

Therefore,

$$\|u\|_{L^2((0,T),H^1(\Omega))}^2 \leq \frac{2L + \delta T}{\delta} E(u_0) + \frac{4CT}{9\delta^2} E(u_0)^2,$$  \hfill (63)

which concludes the proof. $\square$

From now on, the analysis is very close to that one employed to obtain Lemma 3.2. Indeed, we have:

**Lemma 4.2.** Let $Q(u)$ be defined in (56). Then, for any $T > 0$ and $R > 0$ there exists a positive constant $C' = C'(R, T) > 0$ such that

$$\int_0^T E(u(t))^2 dt \leq C' \int_0^T Q(u(t)) dt,$$  \hfill (64)

for all solution of (14) with $\|u_0\|_{L^2(\Omega)} \leq R.$
Finally, an argument similar to that one used in (55) provides the conclusion of our analysis. This completes the proof of Theorem 1.3.

5. Appendix. The argument of proof of the global well-posedness of problem (8) and (14) are quite similar. So for simplicity, we will address only (8). We consider Appendix.

Proof. From Lemma 3.1 it follows that \( A^* = \partial_x^3 v - \partial_y^3 v + \partial^{-1}_x \partial_y^2 v \) on

\[
D(A^*) = \{ v \in X^5(\Omega) : v(0, y) = v(L, y) = 0, v(x, 0) = v(x, L) = 0, \\
\partial_x v(0, y) = \partial_x v(L, y) = \partial_y^2 v(L, y) = 0 \}.
\]

and its adjoint \( A^* \), defined by

\[
D(A) = \{ v \in X^5(\Omega) : v(0, y) = v(L, y) = 0, v(x, 0) = v(x, L) = 0, \\
\partial_x v(0, y) = \partial_x v(L, y) = \partial_y^2 v(L, y) = 0 \}.
\]

Lemma 5.1. A and its adjoint \( A^* \) are dissipative operators.

Proof. In fact, integration by parts gives

\[
(Av, v) = -\frac{1}{2} \int_0^L (\partial_y^2 v(0, y))^2 dy \leq 0 \quad \text{and} \quad (A^* v, v) = -\frac{1}{2} \int_0^L (\partial_y^2 v(0, y))^2 dy \leq 0,
\]

so the proof is concluded. \( \square \)

Consider the linear problem

\[
\begin{aligned}
\partial_t u + \alpha \partial_x^2 u + \beta \partial_y^2 u + \gamma \partial_x^{-1} \partial_y^2 u &= f(x, y, t), \quad (x, y) \in (0, L) \times (0, L), t \geq 0, \\
u(0, y, t) &= u(L, y, t) = 0, \quad u(x, 0, t) = u(x, L, t) = 0, \\
\partial_x u(L, y, t) &= \partial_x u(0, y, t) = 0, \\
\partial_x^2 u(L, y, t) &= \partial_x^2 u(0, y, t) = 0, \\
u(x, y, 0) &= u_0(x, y).
\end{aligned}
\]

The solution of (65) can be written as

\[
u(t) = S(t)u_0 + I(f)(t),
\]

where \( I(f)(t) := \int_0^t S(t-s)f(\cdot, s) ds \).

From Lemma 5.1 it follows that \( A \) generates a strongly continuous semigroup of contractions \( \{S(t)\}_{t \geq 0} \) in \( L^2(\Omega) \) (see Corollary 1.4.4 in Pazy [36]). Thus the mild solution \( S(t)u_0, t \geq 0 \), to the linear problem (65) is such that

\[
S(t)u_0 \in C([0, \infty), L^2(\Omega)) \quad \text{and} \quad \|S(t)u_0\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}.
\]

Furthermore, by Lemma 3.1 we see that \( S(t)u_0 \in C([0, \infty), H^2_{x_0}(\Omega)) \), i.e.,

\[
\|S(t)u_0\|_{L^2((0, T); H^2_{x_0}(\Omega))} \lesssim \|u_0\|_{L^2(\Omega)}.
\]

Remark 2. Estimates (67) and (68) hold with \( \partial_x^{-1}u_0 \) in place of \( u_0 \).

For each \( 0 < T \leq \infty \), we defined

\[
Y_T = C([0, T); L^2(\Omega)) \cap L^2((0, T); X^2_{x_0}(\Omega)),
\]

where by \( X^k_{x_0}(\Omega) \) we mean the space

\[
X^k_{x_0}(\Omega) = \{ f \in H^k_{x_0}(\Omega) : \partial_x^{-1} f(x, y) = \int_0^x f(z, y) dz \in H^k_{x_0}(\Omega), \quad k \geq 0 \}
\]

For simplicity, the space \( Y_{\infty} \) will be denoted by \( Y \).
Lemma 5.2. Let $T > 0$. Then the map $I(\cdot)$ is continuous from $L^1((0,T); H^1_{x_0}(\Omega))$ to $Y_T$.

Proof. Multiplying the equation in (65) by $xu$, integrating by parts on $(0,L) \times (0,L)$ and integrate on $(0,T)$ as in (35), we find

$$0 \leq \frac{1}{2} \int_0^T \int_0^L xu^2(x,y,T)dxdy + \frac{3\alpha}{2} \int_0^T \int_0^L \int_0^L (\partial_x u)^2 dxdydt - \frac{5\beta}{2} \int_0^T \int_0^L \int_0^L (\partial_x^2 u)^2 dxdydt + \frac{\gamma}{2} \int_0^T \int_0^L \int_0^L (\partial_x^{-1} \partial_y u)^2 dxdydt = \int_0^T \int_0^L \int_0^L xu f dxdydt,$$

(69)

Since $I(u)$ is a mild solution of (65) with $u_0 \equiv 0$, we get

$$\|u\|_{L^2(\Omega)} = \|I(u)\|_{L^2(\Omega)} \leq \|f\|_{L^1(0,T; H^1_{x_0}(\Omega))},$$

(70)

and thus,

$$\int_0^T \int_0^L \int_0^L xu f dxdydt \leq L\|u\|_{C(0,T; L^2(\Omega))} \|f\|_{L^1(0,T; H^1_{x_0}(\Omega))} \leq L\|f\|_{L^1(0,T; H^1_{x_0}(\Omega))}.\quad (71)$$

Now gathering (69), (70) and (71), we arrive to the result. \qed

Theorem 5.3. Given a $u_0 \in X^0(\Omega)$, there exists a unique solution $u$ to the problem (8) in $Y$.

Proof. Note that, by the identity $(\partial_x u)^2 = \frac{1}{2} \partial_x^2 (u^2) - u \partial_x^2 u$ and the Sobolev embedding $H^2_{x_0}(\Omega) \hookrightarrow L^\infty(\Omega)$, we have

$$\|u \partial_x u\|^2_{H^1_{x_0}(\Omega)} \lesssim \|u\|^2_{L^\infty(\Omega)} \|\partial_x u\|^2_{L^2(\Omega)} + \|\partial_x^2 (u^2)\|^2_{L^2(\Omega)} + \|u \partial_x^2 u\|^2_{L^2(\Omega)} \lesssim \|u\|^2_{L^\infty(\Omega)} \|\partial_x u\|^2_{H^1_{x_0}(\Omega)} + \|u\|^2_{H^2_{x_0}(\Omega)} + \|u\|^2_{L^\infty(\Omega)} \|\partial_x^2 u\|^2_{H^1_{x_0}(\Omega)},$$

(72)

and therefore

$$\int_0^T \|u \partial_x u\|^2_{H^1_{x_0}(\Omega)} dt \lesssim \int_0^T \|u\|^2_{H^2_{x_0}(\Omega)} dt.\quad (73)$$

A straightforward calculation shows that (73) also holds with $\partial_x^{-1} (u \partial_x u)$ in place of $u \partial_x u$.

Furthermore, the map $u \mapsto u \partial_x u$ is locally Lipschitz from $L^2((0,T); H^2_{x_0}(\Omega))$ to $L^1((0,T); H^1_{x_0}(\Omega))$. In fact, let $\Phi$ be the referred map. Then, for $u,v \in L^2((0,T); H^2_{x_0}(\Omega))$ we have from (72) and (73) that

$$\|\Phi(u) - \Phi(v)\|_{L^1((0,T); H^1_{x_0}(\Omega))} \lesssim \{\|u\|_{L^2((0,T); H^2_{x_0}(\Omega))} + \|v\|_{L^2((0,T); H^2_{x_0}(\Omega))}\} \|u - v\|_{L^2((0,T); H^2_{x_0}(\Omega))}.\quad (74)$$

Finally, the result follows from (67), (68), Lemma 5.2 and Proposition 2. \qed
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