THE EMBEDDING MODEL OF INDUCED GRAVITY
WITH BOSONIC SOURCES

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November 2, 2021

Abstract

We consider a theory in which spacetime is an n-dimensional surface $V_n$ embedded in an $N$-dimensional space $V_N$. In order to enable also the Kaluza-Klein approach we admit $n > 4$. The dynamics is given by the minimal surface action in a curved embedding space. The latter is taken, in our specific model, as being a conformally flat space. In the quantization of the model we start from a generalization of the Howe-Tucker action which depends on the embedding variables $\eta^a(x)$ and the (intrinsic) induced metric $g_{\mu\nu}$ on $V_n$. If in the path integral we perform only the functional integration over $\eta^a(x)$, we obtain the effective action which functionally depends on $g_{\mu\nu}$ and contains the Ricci scalar $R$ and its higher orders $R^2$ etc. But due to our special choice of the conformal factor in $V_N$ enterig our original action, it turns out that the effective action contains also the source term. The latter is in general that of a $p$-dimensional membrane ($p$-brane); in particular we consider the case of a point particle. Thus, starting from the basic fields $\eta^a(x)$, we induce not only the kinetic term for $g_{\mu\nu}$, but also the "matter" source term.

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1 Introduction

After so many years of intensive research quantization of gravity is still an unfinished project. Among many approaches followed there is the one which seems to be especially promising. This is the so-called induced gravity proposed by Sakharov [1]. His idea was to treat metrics not as a fundamental field but as one induced from more basic fields. The idea has been pursued by numerous authors [2]; especially illuminating are works by Akama, Terasawa and Naka [3]. Their basic action contains N scalar fields and it is formally just a slight generalization of the well-known Dirac-Nambu-Goto action for an n-dimensional worldsheet swept by an (n−1)-dimensional membrane.

In the present paper we are going to pursue such an approach and give a concrete physical interpretation to the N-scalar fields which we shall denote ηa(x). We shall assume that spacetime is a surface V4, called spacetime sheet, embedded in a higher dimensional space V N, and ηa(x) are the embedding functions. An embedding model has been first proposed by Regge and Teitelboim [4] and investigated by others [5]. In that model the action contains the Ricci scalar expressed in terms of the embedding functions. In our present model [6], [7] on the contrary, we start from the action which is essentially the minimal surface action weighted with a function ω(η) in V N. For a suitably chosen ω, such that it is singular (δ-function like) on certain surfaces ˆV m, also embedded in V N, we obtain on V 4 a set of worldlines. It is shown that these worldlines are geodesics of V 4 provided that V 4, described by ηa(x), is a solution to our variational procedure. I shall show that after performing functional integrations over ηa(x) we obtain two contributions to the path integral. One contribution comes from all possible ηa(x) not intersecting ˆV m and the other contribution from those ηa(x) which do intersect the surfaces ˆV m. In the effective action so obtained, the first contribution gives the Einstein-Hilbert term R plus higher order terms like R2. The second contribution can be cast into the form of the path integral over all possible worldlines Xμ(τ). We so obtain the action which contains matter sources and the kinetic term for the metric field (plus higher orders in R). So in the proposed approach both, the metric field and the matter field are induced from more basic fields ηa(x).

2 The classical model

We assume that the arena in which physics takes place is an N-dimensional space V N with N ≥ 10. Next we assume that an n-dimensional surface V n living in V n represents a possible spacetime. The parametric equation of such a “spacetime sheet” V n is given by the embedding functions ηa(xμ), a = 0, 1, 2, ..., N, where xμ, μ = 0, 1, 2, ..., n − 1 are coordinates (parameters) on V n. We suppose that the action is just the one for a minimal surface V n

\[ I[η^a(x)] = \int (\det \partial^\mu \eta^a \partial^\nu \eta^b \gamma_{ab})^{1/2} d^n x \]  

(1)
where $\gamma_{ab}$ is the metric tensor of $V_N$. Dimension of a spacetime sheet $V_n$ is taken here to be arbitrary, in order to allow for the Kaluza-Klein approach. In particular we may take $n = 4$. We admit that the embedding space is curved in general. In particular let us consider the case of a conformally flat $V_N$ such that $\gamma_{ab} = \omega^{2/n} \eta_{ab}$, where $\eta_{ab}$ is the $N$-dimensional Minkowski tensor. Then Eq. (1) becomes

$$I[\eta^a(x)] = \int \omega(\eta)(\det \partial^n \eta^a \partial^n \eta_{ab})^{1/2} d^nx$$

(2)

From now on we shall forget about the origin of $\omega(\eta)$ and we shall consider it as a function of position in a flat embedding space. Indices $a, b, c$ will be raised and lowered by $\eta^{ab}$ and $\eta_{ab}$, respectively.

In principle $\omega(\eta)$ is arbitrary. But it is very instructive to choose the following function

$$\omega(\eta) = \omega_0 + \sum_i \int m_i \frac{\delta^N(\eta - \hat{\eta}_i)}{\sqrt{|\gamma|}} \sqrt{|f|}$$

(3)

where $\eta^a = \eta^a(\hat{x})$ is the parametric equation of an $m$-dimensional surface $\hat{V}_m$, called matter sheet, also embedded in $V_N$, $\hat{f}$ is the determinant of the induced metric on $\hat{V}_m$, and $\sqrt{|\gamma|}$ allows for taking curved coordinates in otherwise flat $V_N$. If we take $m = N - n + 1$, then the intersection of $V_n$ and $\hat{V}_m$ can be a (one-dimensional) line, i.e. a worldline $C_i$ on $V_n$. In general, when $m = N - n + (p + 1)$, the intersection can be a $(p + 1)$-dimensional worldsheets representing motion of a $p$-dimensional membrane (also called $p$-brane). In this paper we confine our consideration to the case $p = 0$, that is to motion of a point particle.

Inserting (3) into (2) and writing $f_{\mu\nu} \equiv \partial^\mu \eta^a \partial^\nu \eta_a$, $f \equiv \det f_{\mu\nu}$ we obtain

$$I[\eta] = \omega_0 \int d^n x \sqrt{|f|} + \int d^n x \sum_i m_i \delta^a(x - X_i)(f_{\mu\nu}\hat{X}_i^\mu \hat{X}_i^\nu)^{1/2} d\tau$$

(4)

The above result was obtained by writing $d^m \hat{x} = d^{n-1} x d^{n-1} \tau$, $\hat{f} = \hat{f}^{(m-1)}(\hat{X}_i^\mu \hat{X}_i^\mu)^{1/2}$ and taking the coordinates $\eta^a$ such that $\eta^a = (x^\mu, \eta^n, ..., \eta^{N-1})$, where $x^\mu$ are (curved) coordinates on $V_n$. The determinant of the metric of the embedding space $V_N$ in such curvilinear coordinates is then $\gamma = \det \partial^\mu \eta^a \partial^\nu \eta_a = f$.

If we vary the action (4) with respect to $\eta^a(x)$ we obtain

$$\partial_\mu \left[ \sqrt{|f|} (\omega_0 f^{\mu\nu} + T^{\mu\nu}) \partial_\nu \eta_a \right] = 0$$

(5)

where

$$T^{\mu\nu} = \frac{1}{\sqrt{|f|}} \sum_i \int d^n x m_i \delta^a(x - X_i) \frac{\hat{X}_i^\mu \hat{X}_i^\nu}{(X_i^a X_i^a)^{1/2}} d\tau$$

(6)
is the stress-energy tensor of dust. Eq. (5) can be rewritten in terms of the covariant derivative $D_\mu$ on $V_n$:

$$D_\mu [(\omega_0 f^{\mu\nu} + T^{\mu\nu}) \partial_\nu \eta_a] = 0$$

The latter equation gives

$$\partial_\nu \eta_a D_\mu T^{\mu\nu} + (\omega_0 f^{\mu\nu} + T^{\mu\nu}) D_\mu D_\nu \eta_a = 0$$

where we have taken into account that covariant derivative of metric is zero, i.e. $D_\alpha f_{\mu\nu} = 0$, and $D_\alpha f^{\mu\nu} = 0$, which implies also $\partial_\alpha \eta^c D_\mu D_\nu \eta_c = 0$, since $f_{\mu\nu} \equiv \partial_\mu \eta^a \partial_\nu \eta_a$. Contracting Eq. (8) by $\partial_\alpha \eta^a$ we have

$$D_\mu T^{\mu\nu} = 0$$

The latter are the well known equations of motion for sources. In the case of dust (8) implies that dust particles move along geodesics of spacetime $V_n$. We have thus obtained a very interesting result that the worldlines $C_i$ which are obtained as the intersections $V_n \cap \hat{V}_n^{(i)}$ are geodesics of the spacetime sheet $V_n$. The same result is obtained also directly by varying the action (4) with respect to variables $\dot{X}_\mu^i(\tau)$.

A solution to the equations of motion (5) (or (7)) gives both: a spacetime sheet $\eta^a(x)$ and worldlines $\dot{X}_\mu^i(\tau)$. Once $\eta^a(x)$ is determined, also the induced metric $g_{\mu\nu} = \partial_\mu \eta^a \partial_\nu \eta_a$ is determined. But such a metric, in general, does not satisfy Einstein’s equations. In the next section we shall see that quantum effects induce the necessary Einstein- Hilbert term $(-g)^{1/2} R$.

### 3 The quantum model

For the purpose of quantization we shall use a classical action (7) which is a generalization of the well known Howe-Tucker action (3) which is equivalent to (5):

$$I[\eta^a, g^{\mu\nu}] = \frac{1}{2} \int d^n x \sqrt{|g(\eta)|} (g^{\mu\nu} \partial_\mu \eta^a \partial_\nu \eta_a + 2 - n)$$

It is a functional of the embedding functions $\eta^a(x)$ and the Lagrange multipliers $g^{\mu\nu}$. Varying (10) with respect to $g^{\mu\nu}$ gives the constraints

$$-\frac{\omega}{4} \sqrt{|g|} g_{\alpha\beta}(g^{\mu\nu} \partial_\mu \eta^a \partial_\nu \eta_a + 2 - n) + \frac{\omega}{2} \sqrt{|g|} \partial_\alpha \eta^a \partial_\beta \eta_a = 0$$

Contracting (11) with $g^{\alpha\beta}$ we find $g^{\mu\nu} \partial_\nu \eta^a \partial_\nu \eta_a = n$, and after inserting the latter relation back into (11) we find

$$g_{\alpha\beta} = \partial_\alpha \eta^a \partial_\beta \eta_a$$

which is the expression for the induced metric on a surface $V_n$. In the following paragraphs we shall specify $n = 4$; however, whenever necessary we shall switch to a generic case of arbitrary $n$. 


In the classical theory we may say that a 4-dimensional spacetime sheet is swept by a 3-dimensional space-like hypersurface Σ which moves forward in time. The latter surface is specified by initial conditions and equations of motion then determine Σ at every value of a time-like coordinate \(x^0 = t\). Knowledge of a particular hypersurface Σ implies knowledge of the corresponding intrinsic 3-geometry given by the 3-metric \(g^{ij}\) induced on Σ \((i, j = 1, 2, 3)\). However, knowledge of data \(\eta^a(t, x^i)\) on an entire infinite Σ is just a mathematical idealization which cannot be realized in a practical situation by an observer, because of the finite speed of light.

In quantum theory a state of a surface Σ is not given by coordinates \(\eta^a(t, x^i)\), but by a wave functional \(\psi[t, \eta^a(x^i)]\). The latter represents probability amplitude that at time \(t\) an observer would get, as a result of measurement, a particular surface Σ.

The probability amplitude for the transition from a state with definite \(\Sigma_1\) at time \(t_1\) to a state \(\Sigma_2\) at time \(t_2\) is given by the Feynman path integral

\[
K(2, 1) = \langle \Sigma_2, t_2 | \Sigma_1, t_1 \rangle = \int e^{iI[\eta, g]} \mathcal{D}\eta \mathcal{D}g
\]

Now, if in Eq.(13) we perform integration only over the embedding functions \(\eta^a(x^\mu)\), then we obtain the so called effective action \(I_{\text{eff}}\)

\[
e^{iI_{\text{eff}}[g]} = \int e^{iI[\eta, g]} \mathcal{D}\eta
\]

which is functional solely of the metric \(g^{\mu\nu}\). From Eq.(14) we obtain by functional differentiation

\[
\frac{\delta I_{\text{eff}}[g]}{\delta g^{\mu\nu}} = \int \frac{\delta I[\eta, g]}{\delta g^{\mu\nu}} e^{iI[\eta, g]} \mathcal{D}\eta \equiv \left(\frac{\delta I[\eta, g]}{\delta g^{\mu\nu}}\right) = 0
\]

On the left hand side of Eq.(15) we have taken into account the constraints \(\frac{\delta I[\eta, g]}{\delta g^{\mu\nu}} = 0\) (explicitly given in Eq.(11)).

The expression \(\frac{\delta I_{\text{eff}}[\eta, g]}{\delta g^{\mu\nu}} = 0\) gives the the classical equations for the metric \(g^{\mu\nu}\), derived from the effective action.

Let us now consider a specific case in which we take for \(\omega(\eta)\) the expression (3). Then our action (10) splits into two terms

\[
I[\eta, g] = I_0[\eta, g] + I_m[\eta, g]
\]

with

\[
I_0[\eta, g] = \frac{\omega_0}{2} \int d^n x \sqrt{\left|f\right|} \left(g^{\mu\nu} \partial_{\mu} \eta^a \partial_{\nu} \eta_a + 2 - n\right)
\]

\[
I_m[\eta, g] = \frac{1}{2} \int d^n x \sqrt{|g|} \sum_i m_i \frac{\delta ^N (\eta - \hat{\eta}_i)}{\sqrt{\gamma}} \sqrt{|f|} \left(g^{\mu\nu} \partial_{\mu} \eta^a \partial_{\nu} \eta_a + 2 - n\right)
\]
The last expression can be integrated over \( m - 1 \) coordinates \( \hat{x}^\mu \), while \( \hat{x}^0 \) is chosen so to coincide with a parameter \( \tau \) of a worldline \( C_i \). We also split the metric as \( g^{\mu\nu} = n^\mu n^\nu/n^2 + \hat{g}^{\mu\nu} \), where \( n^\mu \) is a time-like vector and \( \hat{g}^{\mu\nu} \) the projection tensor, giving \( \hat{g}^{\mu\nu} \partial_\mu \eta^\alpha \partial_\nu \eta_a = n - 1 \). So we obtain

\[
I_m[\eta, g] = \frac{1}{2} \int d^n x \sqrt{|g|} \sum_i \frac{\delta^n(x - X_i(\tau))}{\sqrt{|g|}} \left( \frac{\hat{g}^{\mu\nu} \hat{X}_i^\mu \hat{X}_i^\nu}{\mu_i} + \mu_i \right) = I_m[\hat{X}_i, g]
\]

(19)

Here \( \mu_i \equiv 1/\sqrt{n^2} |C_i| \) are the Lagrange multipliers giving, after variation, the worldline constraints \( \mu_i^2 = \hat{X}_i^2 \). Equation (14) is the well known Howe-Tucker action for point particles.

Now let us substitute our specific action (16)-(19) into the expression (14) for the effective action. The functional integration now runs over two distinct classes of spacetime sheets \( V_n \) [represented by \( \eta^a(x) \)]:

(a) those \( V_n \) which either do not intersect the matter sheets \( \hat{V}_m^{(i)} \) [represented by \( \eta^a_i(\hat{x}) \)], or if they do, the intersections are just single points, and

(b) those \( V_n \) which do intersect \( \hat{V}_m^{(i)} \), the intersections being worldlines \( C_i \).

The sheets \( V_n \) which correspond to the case (b) have two distinct classes of points (events):

(b1) the points outside the intersection, i.e., outside the worldlines \( C_i \),
(b2) the points on the intersection, i.e., the events belonging to \( C_i \).

The measure \( D\eta^a(x) \) can be factorized into the contribution which corresponds to the case (a) or (b1) \( (x \notin C_i) \), and into the contribution which corresponds to the case (b2) \( (x \in C_i) \):

\[
D\eta = \prod_{a,x} (|g(x)|)^{1/4} d\eta^a(x)
\]

\[
= \prod_{a,x \notin C_i} (|g(x)|)^{1/4} d\eta^a(x) \prod_{a,x \in C_i} (|g(x)|)^{1/4} d\eta^a(x) \equiv D_0\eta D_m\eta
\]

(20)

The additional factor \( (|g(x)|)^{1/4} \) comes from the requirement that the measure be invariant under reparametrizations of \( x^\mu \) (see Ref. [4] for details). From the very definition of \( \prod_{a,x \in C_i} (|g(x)|)^{1/4} d\eta^a(x) \) as the measure of the set of points on the worldlines \( C_i \) [each \( C_i \) being represented by an equation \( x = \hat{X}_i^\mu(\tau) \)] we conclude that

\[
D_m\eta^a(x) = D\hat{X}_i^\mu(\tau)
\]

(21)

The effective action then satisfies [using (16)-(21)]:

\[
e^{iI_{eff}[g]} = \int e^{iI_0[\eta,g]} D_0\eta e^{iI_m[\hat{X}_i,g]} D\hat{X}_i \equiv e^{iW_0} e^{iW_m}
\]

(22)

\[
I_{eff} = W_0 + W_m
\]

(23)

The measure \( D_0\eta \) includes all those sheets \( V_n \) that do not intersect a matter sheet [case (a)], and also all those which do intersect [case (b1)], apart from the points on \( \hat{V}_m^{(i)} \) [case (a)].
The first factor in the product \( 22 \) contains the action \( 10 \). The latter has the same form as the action for \( N \) scalar fields in a curved background spacetime with the metric \( g_{\mu \nu} \). The corresponding effective action has been studied and derived Refs. \[ 10 \]. Using the same procedure and taking our specific constants \( \omega_0 / 2 \) and \( (n - 1) \) occurring in Eq.\( \[ 10 \] \) we find for the effective Lagrangian the following expression:

\[
L_{\text{eff}} = n \omega_0^{-1} (4 \pi)^{-n/2} \sum_{j=0}^{\infty} (n - 2)^{n/2-j} a_j(x) \Gamma(j - \frac{n}{2}) \tag{24}
\]

with

\[
a_0(x) = 1 \tag{25}
\]
\[
a_1(x) = \frac{R}{6} \tag{26}
\]
\[
a_2(x) = \frac{1}{12} R^2 + \frac{1}{180} (R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} - R_{\alpha \beta} R^{\alpha \beta}) - \frac{1}{20} D_\mu D^{\mu} R \tag{27}
\]

where \( R \), \( R_{\alpha \beta} \) and \( R_{\alpha \beta \gamma \delta} \) are the Ricci scalar, the Ricci tensor and the Riemann tensor, respectively. The function \( \Gamma(y) = \int_0^\infty e^{-t} t^{y-1} dt \); it is divergent at negative integers \( y \) and finite at \( y = \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \). The effective Lagrangian (24) is thus divergent in even dimensional spaces \( V_n \). For instance, when \( n = 4 \), the argument in Eq.\( \[ 24 \] \) is \( j - 2 \) which, for \( j = 0, 1, 2, ... \), is indeed a negative integer. Therefore, in order to obtain a finite effective action in 4-dimensions one needs to introduce a suitable cut off parameter, so that \( L_{\text{eff}} \) depends on that parameter. On the contrary, in an odd dimensional space \( L_{\text{eff}} \) is finite and has the form

\[
L_{\text{eff}} = \lambda_0 + \lambda_1 R + \lambda_2 R^2 + \lambda_3 (R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} - R_{\alpha \beta} R^{\alpha \beta}) + \lambda_4 D_\mu D^{\mu} R \tag{28}
\]

where

\[
\lambda_0 = K \Gamma(-\frac{n}{2}) , \quad \lambda_1 = \frac{K}{6(n-2)} \Gamma(1 - \frac{n}{2}) , \quad \lambda_2 = \frac{K}{12(n-2)^2} \Gamma(2 - \frac{n}{2})
\]
\[
\lambda_3 = \frac{K}{180(n-2)^3} \Gamma(3 - \frac{n}{2}) , \quad \lambda_4 = -\frac{K}{20(n-2)^4} \Gamma(4 - \frac{n}{2}) \tag{29}
\]

and \( K \equiv \omega_0^{-1} n(n-2)^{n/2} (4 \pi)^{-n/2} \). For instance, when \( n = 5 \), we have \( K = \omega_0^{-1} 3 \left( \frac{3}{\pi^2} \right)^{5/2} \), and

\[
\lambda_0 = -\frac{\sqrt{3}}{\pi^2 \omega_0} , \quad \lambda_1 = -\frac{5}{36} \lambda_0 , \quad \lambda_2 = \frac{5}{144} \lambda_0 , \quad \lambda_3 = \frac{5}{2160} \lambda_0 , \quad \lambda_4 = -\frac{5}{240} \lambda_0 \tag{30}
\]

Here \( \lambda_0 \) is the cosmological constant, while \( \lambda_1 \) is related to the gravitational constant \( G \) in \( n \) dimensions according to \( \lambda_1 \equiv (16 \pi G)^{-1} \). In 5 dimensions we have from (30) that \( (16 \pi G)^{-1} = \frac{\sqrt{3}}{\pi^2 \omega_0} \frac{5}{36} \). This last relation shows how
the induced (5-dimensional) gravitational constant is calculated in terms of \( \omega_0 \) which is a free parameter of our embedding model.

So, if take seriously the Kaluza-Klein theories in which spacetime has more than 4 dimensions, then it makes sense to consider a spacetime sheet \( V_n \) of an odd dimension \( n = 5, 7 \) or 9, etc., which leads straightforwardly to a finite effective action, without need to introduce a cut off parameter. Such a higher dimensional effective action can then be reduced to 4 dimensions by taking the extra dimensions compactified on a very small length (e.g. Plank length).

In the above calculation of the effective action we have considered all functions \( \eta^a(x) \) entering the path integral (14) as representing distinct spacetime sheets \( V_n \). However, because of the reparametrization invariance there exist equivalence classes of functions representing the same \( V_n \). This complication must be taken into account, and the conventional approach is to introduce ghost fields which cancel the unphysical degrees of freedom. An alternative approach, explored in Ref. [11] is to assume that all possible embedding functions \( \eta^a(x) \) can be nevertheless interpreted to describe physically distinct spacetime sheets \( V_n \). This is possible if we the extra degrees of freedom in \( \eta^a(x) \) describe deformations of \( V_n \). Such a deformable surface \( V_n \) is then a different concept than a non-deformable surface \( V_n \). The path integral can be straightforwardly performed in the case of \( V_n \), as we did it in arriving at the result (24).

Let us now return to Eq.(22). In the second factor of Eq.(22) the functional integration runs over all possible worldlines \( X^\mu(\tau) \). Though they are obtained as intersections of various \( V_n \) with \( \hat{V}_m^{(i)} \), we may consider all those worldlines as lying in the same effective spacetime \( V_n^{(eff)} \) with the intrinsic metric \( g_{\mu\nu} \). In other words, in the effective theory, we identify all those various \( V_n \)'s, having the same induced (intrinsic) metric \( g_{\mu\nu} \), to be one and the same spacetime. If one considers the embedding space \( V_N \) of sufficiently high dimension \( N \), then there is enough freedom to obtain as the intersection any possible worldline in the effective spacetime \( V_n^{(eff)} \).

When the condition for the classical approximation is satisfied, i.e., when \( I_m >> h = 1 \), then only those trajectories \( \hat{X}_i^a \) which are close to the classically allowed ones effectively contribute:

\[
e^{iW_m} = e^{iI_m[X_i,g]} , \quad W_m = I_m
\]

The effective action is then the sum of the gravitational field kinetic term \( W_0 \) given in Eq.(28) and the source term \( I_m \) given in (13). Variation of \( I_{eff} \) with respect to \( g^{\mu\nu} \) then gives the equations of the gravitational field in the presence of point-particle sources with the stress energy tensor \( T^{\mu\nu} \) as given in Eq.(6):

\[
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} + \lambda_0 g^{\mu\nu} + \text{(higher order terms)} = -8\pi G T^{\mu\nu}
\]

However, in general the classical approximation is not satisfied, and in the evaluation of the matter part \( W_m \) of the effective action one must take into
account the contribution of all possible paths \( \hat{X}^\mu_i(\tau) \). So we have (confining to the case of only one particle, omitting the subscript \( i \), and taking \( \mu = 1 \))

\[
e^{iW_m} = \int_{x_0}^{x_b} dX \exp \left( \frac{i}{2} \int_{\tau_a}^{\tau_b} d\tau \, m \left( g_{\mu\nu} \hat{X}^\mu_i(\tau) \hat{X}^\nu_i(\tau) + 1 \right) \right) = K(x_b, \tau_b; x_a, \tau_a) \equiv K(b,a)
\]

which is the propagator or the Green function satisfying (for \( \tau_b \geq \tau_a \))

\[
\left( i \frac{\partial}{\partial \tau_b} - H \right) K(x_b, \tau_b; x_a, \tau_a) = -\frac{1}{\sqrt{|g|}} \delta^n(x_b - x_a) \delta(\tau_b - \tau_a) \quad (34)
\]

where \( H = (|g|)^{-1/2} \partial_e (|g|)^{1/2} \partial^e \). From (34) we have

\[
K(b,a) = -i \left[ i \frac{\partial}{\partial \tau_b} - H \right]^{-1}_{x_b,\tau_b;x_a,\tau_a} \quad (35)
\]

where the inverse Green function is treated as a matrix in \((x, \tau)\) space.

Using the following relation \([12]\) for Gaussian integration

\[
\int y_m y_n \prod_{i=1}^N dy_i dy_j \, e^{-y_i A_{ij} y_j} \propto \frac{(A^{-1})_{mn}}{(\det A_{ij})^{1/2}}
\]

we can rewrite the Green function in terms of the second quantized field

\[
K(a,b) = \int \psi^*(x_b, \tau_b) \psi(x_b, \tau_b) \mathcal{D}\psi^* \mathcal{D}\psi \exp \left[ -i \int d\tau \, d^a x \, \sqrt{|g|} \psi^*(i\partial_{\tau} - H)\psi \right]
\]

(37)

If the conditions for ”classical” approximation are satisfied, such that the phase in (37) is much greater than \( \hbar = 1 \), then only those paths \( \psi(\tau, x) \), \( \psi^*(\tau, x) \) which are close to the extremal path, along which the phase is zero, effectively contribute to \( K(a, b) \). Then the propagator is simply

\[
K(b, a) \propto \exp \left[ -i \int d\tau \, d^a x \, \sqrt{|g|} \psi^*(i\partial_{\tau} - H)\psi \right]
\]

(38)

The effective, one-particle, ”matter” action \( W_m \) is then

\[
W_m = -\int d\tau \, d^a x \, \sqrt{|g|} \psi^*(\tau, x) (i\partial_{\tau} - H)\psi(\tau, x)
\]

(39)

If we assume that \( \tau \)-dependence of the field \( \psi(\tau, x) \) is given by \( \psi(\tau, x) = e^{im\tau} \phi(x) \), then Eq.(39) simplifies to the usual well known expression for a scalar field:

\[
W_m = \int d^a x \, \sqrt{|g|} \phi^*(x) \left( \frac{1}{\sqrt{|g|}} \partial_{\mu} (\sqrt{|g|} \partial^\mu) + m^2 \right) \phi(x)
\]

\[
= -\frac{1}{2} \int d^a x \, \sqrt{|g|} \left( g^{\mu\nu} \partial_{\mu} \phi^* \partial_{\nu} \phi - m^2 \right)
\]

(40)

where the surface term has been omitted.

Starting from our basic fields \( \eta^a(x) \) which are the embedding functions for a spacetime sheet \( \eta^a(x) \), we arrived at the effective action \( I_{eff} \) which contains the kinetic term \( W_0 \) for the metric field \( g^{\mu\nu} \) (see Eq.(28)) and the source term \( W_m \) (see Eq.(32) and \([13]\), or \([40]\)). Both, the metric field \( g^{\mu\nu} \) and the bosonic matter field \( \phi \) are induced from the basic fields \( \eta^a(x) \).
4 Conclusion

We have investigated a model which seems to be very promising in attempts to find a consistent relation between quantum theory and gravity. Our model exploits the approach of induced gravity and the concept of embedding spacetime in a higher dimensional space and has the following interesting property: what appears as worldlines in e.g. 4-dimensional spacetime are just the intersections of a spacetime sheet $V_4$ with "matter" sheets $\hat{V}_m(i)$. Various choices of spacetime sheets then give various configurations of worldlines. Instead of $V_4$ it is convenient to consider a spacetime sheet $V_n$ of arbitrary dimension $n$. When passing to the quantized theory, a spacetime sheet is no longer definite. All possible alternative spacetime sheets are taken into account in an expression for a wave functional or a Feynman path integral. The points of intersection of a $V_n$ with a matter sheet $\hat{V}_m(i)$ are treated specially, and it is found that their contribution to a path integral is identical to the contribution of a point-particle path. We have paid special attention to the effective action which results after having functionally integrated out all possible embeddings which give the same induced metric tensor. We have found that the effective action, besides the Einstein-Hilbert term and corresponding higher-order terms, contains also the source term. The expression for the latter is equal to that of a classical (when such an approximation can be used) or quantum point-particle source described by a scalar (bosonic) field.

In other words, we have found that ($n$-dimensional) Einstein’s equations (including $R^2$ and higher derivative terms) with classical or quantum point-particle sources are effective equations resulting after having performed the quantum average over all possible embeddings of spacetime. Gravity - as described by Einstein’s general relativity - is thus considered not as fundamental phenomenon, but as being induced quantum mechanically from more fundamental phenomena.

In our embedding model of gravity with bosonic sources, new and interesting possibilities are open. For instance, instead of a 4-dimensional spacetime sheet we can consider a sheet which possesses additional dimensions, parametrized either with the usual or the Grassmann coordinates. In such a way we expect to include, on the one hand, via the Kaluza-Klein mechanism, also other interactions besides the gravitational one, and on the other hand, the fermionic sources.
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