A Proof of Duality in Structure Functions Near $x = 1$

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A proof of Bloom-Gilman duality which relates an integral over the low-mass resonances in deep inelastic structure functions to an integral over the scaling region near $x = 1$ is given. It is based on general analytic properties of the corresponding virtual Compton amplitude but is insensitive to its asymptotic behaviour.

Renewed interest in the large $x(\approx 1)$ behaviour of deep inelastic structure functions has been driven in large part by new data from SLAC [1] [2]. In addition, a wealth of detailed data can be expected from CEBAF in the near future, albeit in the scaling transition region at relatively modest values of $Q^2$. Together these will refocus attention on Bloom-Gilman duality, which connects an integral over the resonant contributions below the onset of scaling to an integral over the threshold scaling region above the resonances [3]. This is closely connected to the “inclusive-exclusive” connection which relates the $x \approx 1$ behaviour of the scaling structure functions to the large $Q^2$ behaviour of the resonance form factors and which was first derived from the quark-parton model [4] [5]. As originally formulated, duality can be expressed in the following way:

$$\frac{2M}{Q^2} \int_0^\nu d\nu F_2(\nu, Q^2) = \int^{\omega'}_1 d\omega' F_2(\omega')$$

(1)

Here, $Q^2$ and $\nu$ are to be taken to be sufficiently large that $F_2(\nu, Q^2)$, the conventional structure function, scales, i.e., it becomes a function only of $\omega' \equiv (2M\nu + M_0^2)/Q^2 = 1/\nu + M_0^2/Q^2$ with $M_0$ as an “arbitrary” scale of order $M$. In the original work $F_2$ was assumed to scale exactly; below we shall discuss the inclusion of logarithmic scaling violations dictated by asymptotic freedom. The limit on the integral in (1), $\nu$, defines the transition from the non-scaling resonance region to the continuum scaling region; furthermore, $\omega' = (2M\nu + M_0^2)/Q^2$. Bloom and Gilman found that the data satisfies this sum rule rather well if $M_0$ is identified with $M$. Thus, in some average sense as specified by Eq. (1), the scaling function smoothly interpolates through the bumpy resonant region. They proposed a derivation of this finite energy sum rule using a superconvergence relation which followed from some assumptions about the high energy (i.e., large $\nu$) behaviour of $F_2$. The form of this asymptotic behaviour was guided by conventional Regge phenomenology for the corresponding $q^2 = 0$ Compton amplitude and included the presumed absence of fixed poles. If present, these would contribute only to its real part and, in their derivation, would lead to an unknown contribution to Eq. (1). In addition, the asymptotic behaviour was assumed to hold uniformly from $q^2 = 0$ into the scaling region where $q^2$ becomes large and space-like. As such, it is, in principle, sensitive to the small-$x$ behaviour of $F_2$ which is still a matter of conjecture, both theoretically and experimentally.

With this in mind, we revisit the problem and give a relatively simple proof of duality which is essentially independent of both the high energy and fixed pole behaviour of the Compton amplitude. The former enters only in as much as it determines the number of subtractions required for a fixed-$q^2$ dispersion relation in $\nu$ to converge. These play no role in the proof which is trivially amended if the number of subtractions is different from what has been perenially assumed. The full Compton amplitude, $T_2(q^2, \nu)$, whose imaginary part is proportional to $F_2(q^2, \nu)$, is, for fixed $q^2(\le 0)$, an analytic function of $\nu$ except for cuts along the real axis; (see Fig. 1).

![FIG. 1. The complex $\nu$-plane showing the contour C wrapping around the cuts beginning at the threshold, ±$\nu_0$.](image)

Using crossing symmetry this can be expressed in the form of the well-known dispersion relation:

$$T_2(q^2, \nu) = \int_0^\infty \frac{d\nu'}{\nu'^2 - \nu^2} F_2(q^2, \nu')$$

(2)

With a change of variables this can be expressed as fol-
the origin. Now, if $Q^2$ is below gral over the resonances $\pm$ region above the resonances the right-hand-side of Eq. (5) to write
\[ \nu T_2(q^2, \omega') = (\omega' - \frac{M^2_0}{q^2}) \int_{1+\frac{M^2_0}{\omega'^2}}^\infty d\omega'' \frac{F_2(q^2, \omega'')}{(\omega'' - \omega')(\omega'' + \omega' - 2M^2_0)} \] (3)
When $Q^2 \gg M^2_0$ this reduces to
\[ \nu T_2(q^2, \omega') = \omega' \int_{1}^\infty \frac{d\omega''}{\omega'^2 - \omega''^2} F_2(q^2, \omega'') \] (4)
Following Bloom and Gilman we will assume that scaling (i.e., $F_2(q^2, \omega') \approx F_2(\omega')$) sets in when $Q^2 \approx Q^2_0 \geq M^2_0$ and $s \approx \tilde{s}$ where $\tilde{s}$ exceeds the mass-squared of the last resonance. Note that $\nu = (\tilde{s} - M^2 + Q^2)/2M$ with $Q^2 > Q^2_0$.
Consider now the contour, $C$, shown in Fig. 1 consisting of a circle of radius $\tilde{\nu}$ together with line integrals around the cuts from threshold $\nu = \pm \nu_0$ ($\equiv Q^2/2M$) to $\pm \nu$. Then, since it encloses no singularities,
\[ \int_{\nu_0}^{\tilde{\nu}} F_2(q^2, \nu) d\nu = \int_{\nu}^{\tilde{\nu}} \nu T_2(q^2, \nu) \frac{d\nu}{2\pi i} = \tilde{\nu} \int_{0}^{2\pi} \frac{d\theta}{2\pi i} \tilde{\nu} T_2(q^2, \tilde{\nu} e^{i\theta}) e^{i\theta} \] (5)
where the contour is the circle of radius $|\tilde{\nu}|$ centred at the origin. Now, if $Q^2 \geq Q^2_0$, then this relates an integral over the resonances below the scaling threshold to the full Compton amplitude evaluated at $\tilde{\nu}$ in the scaling region above the resonances. The left-hand-side is just that which occurs in the duality relationship, Eq. (6). To derive (6) we can use the representation Eq. (5) in the right-hand-side of Eq. (4) to write
\[ \int_{\nu_0}^{\tilde{\nu}} F_2(q^2, \nu) d\nu = \frac{q^2}{2M} \int_{\omega'}^{\tilde{\omega}} d\omega' \omega' \int_{1}^\infty \frac{d\omega''}{2\pi i} \frac{F_2(q^2, \omega'')}{\omega'^2 - \omega''^2} \] (6)
The contour integral on the right-hand-side is to be evaluated around a circle of radius $\tilde{\omega}'$ centred at the origin of the complex $\omega'$-plane as shown in Fig. 2.
Since the integrals are absolutely convergent the order of integration can be interchanged. Furthermore, it is easy to see that
\[ \frac{1}{2\pi i} \int_{\omega'}^{\tilde{\omega}} \frac{d\omega'}{\omega'^2 - \omega''^2} = \theta(\tilde{\omega}' - \omega'') \] (7)
Using this in Eq. (6) immediately leads to Eq. (5):
\[ \frac{2m}{q^2} \int_{0}^{\tilde{\omega}} d\nu F_2(\nu, q^2) = \int_{1}^{\tilde{\omega}'} d\omega' F_2(q^2, \omega') \] (8)
Notice that no assumption about either the high energy (i.e., large $\nu$) or the large $\omega'$ (i.e., small $x$) behaviours of $T_2$ need be made to derive this nor do we need to assume exact scaling. Furthermore, the presence of fixed poles, or unknown real parts, contributing to $T_2$ play no role since they are analytic.
\[ \omega' - \text{plane} \] (9)
\[ \frac{2M}{q^2} \int_{0}^{\tilde{\omega}} d\nu F_2(\nu, q^2) = \int_{1}^{\tilde{\omega}'} d\omega' F_2(q^2, \omega') \omega'^{2m} \]
This is valid provided $Q^2/M_0^2 \gg m \geq 0$. This condition can be relaxed by keeping explicit terms of $O(M_0^2/Q^2)$. For $m < 0$ the situation is a little more subtle since there are singularities (poles at the origin) inside $C$. Effectively the resulting sum rules turn out to be identities and contain no new information.
iii) Recall that, if the transition form factor to a given resonance is generically denoted by $G_r(q^2)$, and if, in the scaling region, $F_2(q^2, \omega') \approx A(\omega' - 1)^p$ when $\omega' \approx 1$ (assuming, for the moment, exact scaling), then saturating Eq. (8) with just the resonances leads to the relationship (valid only for large $Q^2$)
\sum_{r} G_{r}^{2}(Q^{2}) \approx \frac{A}{p+1} \left( \frac{s + M_{0}^{2} - M^{2}}{Q^{2}} \right)^{p+1} \tag{10}

The first term in the sum on the left-hand-side is just the elastic nucleon contribution proportional to its elastic form factor \( \rho \). Eq. (10) is the integral version of the local inclusive-exclusive relationship \( \rho \), a particular form of which says that the large \( Q^{2} \) behaviour of the nucleon elastic form factor is given by \( (Q^{2})^{-(p+1)/2} \) which is in reasonable agreement with data if \( p = 3 \). From Eq. (10), however, one can conclude only that at least one of the transition form factors must fall like \( (Q^{2})^{-(p+1)/2} \); equivalently, it states that the nucleon elastic form factor cannot fall slower than this power. This technique does not allow a local version of the inclusive-exclusive relationship to be derived.

iv) Since, on the right-hand-side of Eq. (10), \( T_{2} \) is to be evaluated in the scaling region, an alternative approach might be to use the estimate from asymptotic freedom directly. Unfortunately the canonical light-cone, operator-product machinery \( 11 \) only gives an asymptotic estimate for \( T_{2} \) in the unphysical region where \( x > 1 \). Since \( \nu T_{2} \) is a purely analytic function there it must be expandable in powers of \( x^{-1} \), or in powers of \( \omega' \):

\[
\nu T_{2}(q^{2}, \omega') \approx \sum_{n=0}^{\infty} c_{n}(q^{2})\omega'^{n} \quad (\omega' < 1) \tag{11}
\]

with \( c_{n}(q^{2}) \approx (\ln -q^{2})^{(n-2)/2}b_{1} \). Here, \( b_{1} \) is the coefficient of the leading term in the \( \beta \)-function and \( a_{n} \) the anomalous dimensions of appropriate operators occurring in the expansion of \( T_{2} \). This representation cannot be used in Eq. (11) since that requires \( \omega' > 1 \). On the other hand, \( c_{n}(q^{2}) \) can be related to the full moments of the structure functions by expanding the dispersion relation, Eq. (11), in powers of \( \omega' \) and comparing coefficients; this is, of course, why the predictions of asymptotic freedom are expressed in terms of these moments \( 11 \). By inverting them, the \( q^{2} \) evolution of \( F_{2} \) can then be determined:

\[
F_{2}(q^{2}, \omega') = \int_{t}^{\infty} \frac{dn}{2\pi t} \omega'^{n} c(n, q^{2}) \tag{12}
\]

Here \( c(n, q^{2}) \) is the analytic continuation of \( c_{n}(q^{2}) \) to complex values of \( m \) and the integration is along a line parallel to the imaginary axis standing to the right of all singularities, Eq. (12) can now be used in the right-hand-side of Eq. (10) to determine the \( q^{2} \) evolution of the duality relationship. Such a procedure leads to \( A \propto (\ln Q^{2})^{a} \) and \( p \approx p_{0} + p' \ln \ln Q^{2} \), where both \( a \) and \( p' \) are known, calculable constants in QCD \( 10 \). So, as far as the leading \( Q^{2} \) behaviour is concerned, Eq. (10) reads:

\[
\sum_{r} G_{r}^{2}(Q^{2}) \approx \frac{(\ln Q^{2})^{a}}{(Q^{2})^{p+1+p' \ln \ln Q^{2}}} \tag{13}
\]

A cautionary note must be added, however: since \( \omega'^{n} = e^{n \ln \omega'} \), the \( \omega' \rightarrow 1 \) behaviour of \( F_{2} \) and, consequently, the derivation of Eq. (13), is sensitive to the large moments \( (|n| \rightarrow \infty) \). There is, therefore, a delicate question of the uniformity of the expansions which cast possible doubt on the extrapolation to such small values of \( \omega' \) close to 1. Nevertheless, this behaviour is an intriguing possibility which should be tested phenomenologically. On the other hand, even if Eq. (13) is, in fact, valid it is unlikely that it can be naively extrapolated to its local form and the QCD corrections to the elastic form factor thereby inferred \( 11 \).

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[1] Throughout this paper we use the standard notation in which the scaling variable \( x \equiv Q^{2}/2M\nu; q \) is the four-momentum transferred to the target, which will normally be taken to be a nucleon with mass \( M \). For convenience we define \( Q^{2} = -q^{2} \). In the target rest frame the energy transferred is \( q \cdot M = q_{0} \equiv \nu \) where \( p \) is the target four-momentum. We also use \( s \equiv (p + q)^{2} = M^{2} + 2\nu - Q^{2} \).

[2] J. Arrington et al., Phys. Rev. C53, 2248 (1996).

[3] E. Bloom and F. Gilman, Phys. Rev. D4, 2901 (1971).

[4] S. D. Drell and T. M. Yan, Phys. Rev. Letters 24, 181 (1977); G. B. West, Phys. Rev. Letters 24, 1206 (1977).

[5] See also, e.g., S. J. Brodsky and G. R. Farrar, Phys. Rev. D11, 1309 (1975). For more recent related work, see C. E. Carlson and N. C. Mukhopadhyay, Phys. Rev. D41, 2343 (1993); 47, R1737 (1993) and W. Melnitchouk and A. W. Thomas, Phys. Letters, B377, 11 (1996).

[6] If neglected terms of \( O(M_{0}^{2}/Q^{2}) \) are kept, they simply change the lower limit on the integral on the right-hand-side from 1 to \( 1 + M_{0}^{2}/Q^{2} \).

[7] For the nucleon \( G_{1}(q^{2}) \) is related to the usual Dirac and Pauli elastic form factors, \( F_{1,2} \), by \( G_{1}^{2}(q^{2}) = F_{1}^{2}(q^{2}) + (\mu^{2}q^{2}/4M^{2})F_{2}^{2}(q^{2}) \), where \( \mu \) is the nucleon magnetic moment.

[8] For a review of the application of the operator-product expansion to structure functions, see, e.g., T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford (1984).

[9] Working with these moments, A. De Rujula, H. Georgi and H. D. Politzer, Ann. Phys. (N. Y.) 103, 315 (1977), tried to show how duality and precocious scaling were consequences of the \( m \)-behaviour of the moments as given by QCD.

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[11] As was done by A. De Rujula et al., Phys. Rev. D10, 1649 (1974).