ON SINGULAR FIBRES OF COMPLEX LAGRANGIAN FIBRATIONS

DAISUKE MATSUSHITA

ABSTRACT. We classify singular fibres over general points of the discriminant locus of projective complex Lagrangian fibrations on 4-dimensional holomorphic symplectic manifolds. The singular fibre $F$ is the following either one: $F$ is isomorphic to the product of an elliptic curve and a Kodaira singular fibre up to finite unramified covering or $F$ is a normal crossing variety consisting of several copies of a minimal elliptic ruled surface of which the dual graph is Dynkin diagram of type $A_n, \tilde{A}_n, D_n$ or $D_n$.

1. INTRODUCTION

First we define complex Lagrangian fibration.

DEFINITION 1. Let $(X, \omega)$ be a Kähler manifold with a holomorphic symplectic two form $\omega$ and $S$ a smooth manifold. A proper flat surjective morphism $f : X \to S$ is said to be a complex Lagrangian fibration if a general fibre $F$ of $f$ is a Lagrangian submanifold with respect to $\omega$, that is, the restriction of 2-form $\omega|_F$ is identically zero and $\dim F = (1/2) \dim X$.

REMARK. A general fibre $F$ of a complex Lagrangian fibration is a complex torus by Leauville’s theorem.

The plainest example of a complex Lagrangian fibration is an elliptic fibration of $K3$ surface over $\mathbb{P}^1$. In higher dimension, such a fibre space naturally occurs on a fiber space of an irreducible symplectic manifold ([7, Theorem 2] and [8, Theorem 1]). When the dimension of fibre is one, a complex Lagrangian fibration is a minimal elliptic fibration and whose singular fibre is completely classified by Kodaira [4, Theorem 6.2]. In this note, we investigate singular fibres of a projective complex Lagrangian fibration whose fibre is 2-dimensional.

THEOREM 1. Let $f : X \to S$ be a complex Lagrangian fibration on 4 dimensional symplectic manifold and $D$ the discriminant locus of $f$. Assume that $f$ is projective. For a general point $x$ of $D$, $f^{-1}(x)$ is the following one:

1991 Mathematics Subject Classification. Primary 14E35, Secondary 14D05.

*Research Fellow of the Japan Society for the Promotion of Science.
(1) There exists an étale finite covering \( f^{-1}(x) \sim \to f^{-1}(x) \) and \( f^{-1}(x) \sim \) is isomorphic to the product of an elliptic curve and a Kodaira singular fibre of type \( I_0, I_0^*, II, II^*, III, III^*, IV \) or \( IV^* \).

(2) \( f^{-1}(x) \) is isomorphic to a normal crossing variety consisting of a minimal elliptic ruled surface. The dual graph of \( f^{-1}(x) \) is the Dynkin diagram of type \( A_n, \tilde{A}_n, D_n \) or \( \tilde{D}_n \). If the dual graph is of type \( \tilde{A}_n \) or \( \tilde{D}_n \), each double curve is a section of the ruling. In the other cases, the double curve on each edge components is a bisection or a section and other double curve is a section. (See figures 1 and 2 (pp 12).)

Combining Theorem 1 with [7, Theorem 2] and [8, Theorem 1], we obtain the following corollary.

**Corollary 1.** Let \( f : X \to B \) be a fibre space of a projective irreducible symplectic manifold. Assume that \( \dim X = 4 \). Then, for a general point \( x \) of the discriminat locus of \( f \), \( f^{-1}(x) \) satisfies the properties of Theorem 1 (1) or (2).

**Remark.** Let \( S \) be a K3 surface and \( \pi : S \to \mathbb{P}^1 \) an elliptic fibration. The induced morphism \( f : \text{Hilb}^2 S \to \mathbb{P}^2 \) gives examples of singular fibres above except whose dual graphs are \( A_n \) or \( D_n \). The author does not know whether a normal crossing variety whose dual graph is \( A_n \) or \( D_n \) occur as a singular fibre of a fibre space of an irreducible symplectic manifold.

This paper is organized as follows. In section 2, we set up the proof of Theorem 1. The key proposition is stated and proved in section 3. Section 4 and 5 are devoted to the proof of Theorem 1.

**Acknowledgment.** The author express his thanks to Professors A. Fujiki, Y. Miyaoka, S. Mori and N. Nakayama for their advice and encouragement.

### 2. Preliminary

(2.1) In this section, we collect definitions and some fundamental materials which are necessary for the proof of Theorem 1.

**Definition 2.** Let \( f : X \to \Delta^1 \) be a proper surjective morphism from algebraic variety to an unit disk. \( f \) is said to be semistable degeneration if \( f \) satisfies the following two properties:

1. \( f \) is smooth over \( \Delta^1 \setminus 0 \).
2. \( f^*\{0\} \) is a reduced normal crossing divisor.
**Definition 3.** Let \( f : X \to \Delta^1 \) and \( f' : X' \to \Delta^1 \) be proper surjective morphisms from algebraic varieties to unit disks. We call \( f \) is isomorphic to \( f' \) (resp. birational) if there exists an isomorphism (resp. a birational map) \( g : X \to X' \) such that \( f' \circ g = f \).

(2.2) We refer the fundamental properties of an abelian fibration.

**Lemma 1.** Let \( f : (X, \omega) \to S \) be a projective complex Lagrangian fibration.

(1) The discriminant locus of \( f \) is pure codimension one.

(2) Let \( F \) be an irreducible component of a fibre of \( f \) and \( j : \tilde{F} \to F \) a resolution of \( F \). Then \( j^*(\omega|_F) = 0 \).

**Proof.**

(1) Let \( h : \mathcal{A} \to \Delta^2 \) be a projective flat morphism over a two dimensional polydisk. Assume that a general fibre of \( h \) is an abelian variety and \( h \) is smooth over \( \Delta^2 \setminus 0 \). For the proof of Lemma 1 (1), it is enough to prove that \( h \) is smooth morphism. Let \( \mathcal{A}^o := \mathcal{A} \setminus h^{-1}(0) \). Since \( h \) is smooth over \( \Delta^2 \setminus 0 \) and projective, there exists an \( \acute{e}tale \) finite cover \( \pi : \mathcal{A}^o \to \tilde{A}^o \) and a smooth abelian fibration \( \tilde{h}^o : \tilde{A}^o \to \Delta^2 \setminus 0 \) with a section. Since \( \Delta^2 \setminus 0 \) is simply connected, \( \tilde{A}^o \) has a level \( n \) structure. We put \( \mathcal{M}_g[n] \) the moduli space of \( g \)-dimensional abelian varieties with level \( n \)-structure. There exists a morphism from \( t : \Delta^2 \setminus 0 \to \mathcal{M}_g[n] \) and we extend \( t \) on \( \Delta^2 \) by Hartogs theorem. The moduli \( \mathcal{M}_g[n] \) has the universal family \( \mathcal{A}_g[n] \to \mathcal{M}_g[n] \). Considering the pull back of \( \mathcal{A}_g[n] \) by \( t \), we obtain a smooth abelian morphism \( \tilde{h} : \tilde{\mathcal{A}} \to \Delta^2 \) which is the extension of \( \tilde{h}^o : \tilde{\mathcal{A}}^o \to \Delta^2 \setminus 0 \). Since \( \mathcal{A}^o \to \tilde{\mathcal{A}}^o \) is a finite morphism, we extend this morphism to a finite morphism \( \nu : \mathcal{A}' \to \tilde{\mathcal{A}}^o \). The codimension of \( \mathcal{A} \setminus \tilde{\mathcal{A}}^o \) is two. By purity of branch loci, \( \nu \) is \( \acute{e}tale \). Hence \( h' : \mathcal{A}' \to \Delta^2 \) is a smooth abelian fibration. By construction, \( \mathcal{A}' \) is isomorphic to \( \mathcal{A} \) in codimension one. Let \( A \) be a \( h \)-ample divisor on \( \mathcal{A} \) and \( A' \) a proper transform on \( \mathcal{A}' \). Since every fibre of \( h' \) is an abelian variety, \( A' \) is \( h' \)-ample. Thus we obtain that \( \mathcal{A} \) is isomorphic to \( \mathcal{A}' \) and \( h \) is smooth.

(2) Let \( A \) be an \( f \)-ample divisor. We consider the following function:

\[
\lambda(s) := \int_{X_s} \omega \wedge \bar{\omega} A^{\dim S-2},
\]

where \( X_s := f^{-1}(s) \) (\( s \in S \)). Since \( f \) is flat, \( \lambda(s) \) is a continuous function on \( S \) by [3, Corollary 3.2]. Thus \( \lambda(s) \equiv 0 \) on \( S \) and

\[
\int_F \omega \wedge \bar{\omega} A^{\dim S-2} = 0.
\]

Since \( F \) and \( \tilde{F} \) is birational, \( j^*\omega = 0 \) on \( \tilde{F} \).
We review basic properties of the mixed hodge structure on a simple normal crossing variety.

**Lemma 2.** Let \( X := \sum X_i \) be a simple normal crossing variety. Then

\[
F^1H^1(X, \mathbb{C}) = \{ (\alpha_i) \in \oplus H^0(X_i, \Omega^1_{X_i}) | \alpha_i|_{X_i \cap X_j} = \alpha_j|_{X_i \cap X_j} \}.
\]

**Proof.** Let

\[
X^{[k]} := \bigcup_{i_0 < \cdots < i_k} X_{i_0} \cap \cdots \cap X_{i_k} \quad \text{(disjoint union)}.
\]

For an index set \( I = \{i_0, \cdots, i_k\} \), we define an inclusion \( \delta^I_j \)

\[
\delta^I_j : X_{i_0} \cap \cdots \cap X_{i_k} \to X_{i_1} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots \cap X_{i_k}.
\]

We consider the following spectral sequence \([3, \text{Chapter 4}]\):

\[
E^{p,q}_1 = H^q(X^{[p]}, \mathbb{C}) \implies E^{p+q}_1 = H^{p+q}(X, \mathbb{C}),
\]

where \( D_2 : E^{p,q}_1 \to E^{p+1,q}_1 \) is defined by the

\[
\bigoplus_{|I|=p} \sum_{j=0}^p (-1)^j \delta^I_j.
\]

Since this spectral sequence degenerates at \( E_2 \) level \([3, 4.8]\), we deduce

\[
\text{Gr}_1^W(H^1(X, \mathbb{C})) = \text{Ker}(\oplus_i H^1(X_i, \mathbb{C}) \xrightarrow{D_2} \oplus_{i < j} H^1(X_i \cap X_j, \mathbb{C})).
\]

Moreover \( F^1 \cap W_0 = 0 \), \( F^1H^1(X, \mathbb{C}) = F^1\text{Gr}_1^W(H^1(X, \mathbb{C})) \). Thus we obtain the assertion of Lemma 2 from the definition of \( D_2 \). \( \square \)

**Lemma 3.** Let \( f : X' \to X \) be a birational morphism between smooth algebraic varieties. Assume the following two conditions:

1. There exists a simple normal crossing divisor \( Y \) on \( X \) such that \( f \) is isomorphic on \( X \setminus Y \).
2. \( Y' := (f^*Y)|_{\text{red}} \) is a simple normal crossing divisor.

Then \( F^1H^1(Y', \mathbb{C}) \cong F^1H^1(Y, \mathbb{C}) \).

**Proof.** We consider the following exact sequence of morphisms of Mixed Hodge structures.

\[
H^0(Y', \mathbb{C}) \to H^1(X, \mathbb{C}) \to H^1(X', \mathbb{C}) \oplus H^1(Y, \mathbb{C}) \to H^1(Y', \mathbb{C}) \xrightarrow{\beta} H^2(X, \mathbb{C}) \to \]

Note that each morphism has weight \((0, 0)\). Since \( H^i(X, \mathbb{C}) \) carries pure Hodge structure of weight \( i \), \( \alpha \) and \( \beta \) are 0-map. Moreover \( F^1H^1(X, \mathbb{C}) \cong F^1H^1(X', \mathbb{C}) \). Thus we deduce that \( F^1H^1(Y', \mathbb{C}) \cong F^1H^1(Y, \mathbb{C}) \). \( \square \)
3. Kulikov model

(3.1) In this section, we prove the key proposition of the proof of Theorem 1. First we refer the following theorem due to Kulikov, Morrison [9, Classification Theorem I] and Persson [10, Proposition 3.3.1].

**Theorem 2.** Let \( g' : T' \to \Delta \) be a semistable degeneration whose general fibre is an abelian surface. Then there exists a semistable degeneration \( k : K \to \Delta \) such that \( k \) and \( g' \) is birational and \( K_k \sim_k 0 \). Moreover, exactly one of the following cases occurs:

1. \( K_0 \) is an abelian surface.
2. \( K_0 \) consists of a cycle of minimal elliptic ruled surfaces, meeting along disjoint sections. Every selfintersection number of double curve is 0.
3. \( K_0 \) consists of a collection of rational surfaces, such that the double curves on each component form a cycle of rational curves; the dual graph \( \Gamma \) of \( Y'_0 \) is a triangulation of \( S^1 \times S^1 \).

We call \( K \) a Kulikov model of type I, II or III according the case occurs (1), (2) or (3).

(3.2) We state the key proposition.

**Proposition 1.** Let \( f : (X, \omega) \to S \) be a projective complex lagrangian fibration on 4-dimensional symplectic manifold and \( D \) the discriminant locus of \( f \). If we take a general point \( x \) of \( D \) and an unit disk \( \Delta^1 \) on \( S \) such that \( \Delta^1 \) and \( D \) intersects transversally at \( x \) and \( T := X \times_S \Delta^1 \) is smooth, then

1. \( t : T \to \Delta^1 \) is birational to the quotient of Kulikov model \( K \) of Type I or Type II by a cyclic group \( G \).
2. There exists a nonzero \( G \)-equivariant element of \( F^1H^1(K, \mathbb{C}) \).

(3.3) For the proof of Proposition 1 we need the following Lemmas.

**Lemma 4.** Let \( \nu : Y \to X \) be a birational morphism such that \( (f \circ \nu)^*D \) is a simple normal crossing divisor. Then for a general point \( x \) of the discriminant locus \( D \) of \( f \),

\[ F^1H^1(Y_x, \mathbb{C}) \neq 0 \]

where \( Y_x := f^{-1}(x) \).

**Proof.** Let \( E := ((f \circ \nu)^*D)_{\text{red}} \) and \( E = \sum E_i \). We take an open set \( U \) of \( S \) which satisfies the following two conditions:

1. \( D|_U \) is a smooth curve.
2. \( f \circ \nu|_U : (E|_{f^{-1}(U)})^{[k]} \to D|_U \) is a smooth morphism for every \( k \).
We consider the following exact sequences:

\[ 0 \to \mathcal{F} \to \Omega^2_{E_i} \to \Omega^2_{E_i/D} \to 0 \]

\[ 0 \to (f \circ \nu)^*\Omega^2_D \to \mathcal{F} \to (f \circ \nu)^*\Omega^1_D \otimes \Omega^1_{E_i/D} \to 0 \]

Since \( \omega \) is nondegenerate, \( \nu^*\omega \neq 0 \) on \( E_i \) on a non-\( \nu \)-exceptional divisor \( E_i \). By Lemma [4] (2), the restriction of \( \omega \) on every irreducible component of a fibre of \( f \) is zero. Thus \( \nu^*\omega = 0 \) in \( \Omega^2_{E_i/D} \). On the contrary, \((f \circ \nu)^*\Omega^2_D = 0\), we deduce

\[ \alpha(\nu^*\omega) \neq 0 \]

for non-\( \nu \)-exceptional divisor \( E_i \). Therefore, for a general point \( x \) of \( D \), \( H^0(E_{i,x}, \Omega^1_{E_{i,x}}) \neq 0 \) where \( E_{i,x} \) is the fibre of \( E_i \to D \) over \( x \). We denote by \( \alpha_i \) the restriction \( \alpha(\nu^*\omega) \) to \( E_{i,x} \). If \( E_{i,x} \cap E_{j,x} \neq \emptyset \), \( \alpha_i = \alpha_j \) on \( E_{i,x} \cap E_{j,x} \) by the construction. By Lemma [4] we deduce that \( F^1 H^1(E_x, \mathbb{C}) \neq 0 \). □

**Lemma 5.** Let \( k : \mathcal{K} \to \Delta^1 \) be a Kulikov model of type I or type II. Assume that \( k \) is birational to a projective abelian fibration \( \nu : \mathcal{T}' \to \Delta^1 \). Then

1. \( k \) is a projective morphism.
2. Every birational map \( \mathcal{K} \dashrightarrow \mathcal{K} \) is birational morphism.

**Proof.** We may assume that \( \mathcal{T}' \) is a relatively minimal model over \( \Delta^1 \). Then \( \mathcal{T}' \) and \( \mathcal{K} \) is isomorphic in codimension one, since \( \mathcal{T}' \) and \( \mathcal{K} \) have only terminal singularities and \( K_{\mathcal{T}'} \) is \( t' \)-nef. Let \( A' \) be a \( t' \)-ample divisor on \( \mathcal{T}' \) and \( A \) a proper transform on \( \mathcal{K} \). Since \( \mathcal{T}' \) and \( \mathcal{K} \) is isomorphic in codimension one, \( A \) is \( k \)-big. If \( A \) is \( k \)-nef, we conclude that \( \mathcal{T}' \) is isomorphic to \( \mathcal{K} \) by relative base point free theorem [4, Theorem 3-1-2] Then \( k \) is projective, \( \mathcal{K} \) is the unique relative minimal model and every birational map \( \mathcal{K} \dashrightarrow \mathcal{K} \) is birational morphism. Thus we will prove that \( A \) is \( k \)-nef. Since every big divisor on abelian surface is ample, \( A \) is \( k \)-nef if \( \mathcal{K} \) is of type I. In the case that \( \mathcal{K} \) is of type II, we investigate the nef cone of each component of the central fibre of \( \mathcal{K} \). Let \( V \) be a component of the central fibre. Then \( K_V \sim -2e \), where \( e \) is a double curve. Since \( e \) is a section and \( e^2 = 0 \), the nef cone of \( V \) is spanned by \( e \) and a fibre \( l \) of the ruling of \( V \). We deduce that every effective divisor on \( V \) is nef. Therefore \( A \) is \( k \)-nef in the case that \( \mathcal{K} \) is of type II. □

(3.4) **Proof of Proposition [4]**. Let \( \nu : Y \to X \) be a birational morphism such that \((f \circ \nu)^*D \) is a simple normal crossing divisor. We define \( T'' := \nu^*T \). If we choose \( x \) generally, \( F^1 H^1(T''_x, \mathbb{C}) \neq 0 \) by Lemma [4]. By Semistable reduction theorem [3, Theorem 11'], there exists a generically finite surjective morphism \( \eta : \mathcal{T}' \to T'' \) such that \( \eta : T' \to \Delta^1 \) is a semistable degeneration. By Theorem [3], there exists the Kulikov model \( k : \mathcal{K} \to \Delta^1 \) which is birational to \( T' \). We denote by \( \mathcal{K}_0 \) the central fibre of \( \mathcal{K} \). Then \( F^1 H^1(\mathcal{K}_0, \mathbb{C}) \neq 0 \) since \( F^1 H^1(\mathcal{K}_0, \mathbb{C}) \cong F^1 H^1(T'_0, \mathbb{C}) \).
due to Lemma 3 and $F^1 H^1(T'_0, \mathbb{C}) \neq 0$. Thus $\mathcal{K}$ is of type I or type II. Let $G$ be the galois group of a cyclic extension $K(T')/K(T)$ and $g$ a generator of $G$. Since $k$ is birational to $t'$, there exists a birational map $\Phi_g : \mathcal{K} \dashrightarrow \mathcal{K}$ corresponding to $g$. By Lemma 3, $\Phi_g$ is a birational morphism and $G$ acts on $\mathcal{K}$ holomorphically. Therefore $T$ is birational to the quotient $\mathcal{K}/G$. We claim that $F^1 H^1(\mathcal{K}_0, \mathbb{C})$ carries a nonzero $G$-equivariant element. Let $Z$ be a $G$-equivariant resolution of indeterminancy of $T' \dashrightarrow \mathcal{K}$. Then $F^1 H^1(T'_0, \mathbb{C}) \cong F^1 H^1(Z_0, \mathbb{C}) \cong F^1 H^1(\mathcal{K}_0, \mathbb{C})$ by Lemma 3. Let $\alpha$ be a nonzero $G$-equivariant element of $F^1 H^1(T'_0, \mathbb{C})$. The pull back of $\alpha$ in $F^1 H^1(Z_0, \mathbb{C})$ is nonzero $G$-equivariant element. Thus there exists a nonzero $G$-equivariant element in $F^1 H^1(\mathcal{K}_0, \mathbb{C})$. 

\begin{enumerate}
\item[4.1] In this section, we prove the following proposition.
\end{enumerate}

**Proposition 2.** Let $t : T \to \Delta^1$ be an abelian fibration which is birational to the quotient of a Kulikov model $\mathcal{K}$ of type I by a cyclic group $G$. Assume that

\begin{enumerate}
\item[(1)] $T$ is smooth.
\item[(2)] $K_T \sim \mathcal{O}$.
\item[(3)] There exists a nonzero $G$-equivariant element of $F^1 H^1(\mathcal{K}_0, \mathbb{C})$.
\end{enumerate}

Then the representation $\rho : G \to \text{Aut} H^1(\mathcal{K}_0, \mathbb{C})$ is faithful and the central fibre $T_0$ of $T$ satisfies the properties of Theorem 1

\begin{enumerate}
\item[4.2] We need the following lemma to prove Proposition 2.
\end{enumerate}

**Lemma 6.** The central fibre $\mathcal{K}_0$ admits an $G$-equivariant elliptic fibration over an elliptic curve.

**Proof.** Since $\mathcal{K}_0$ is an abelian surface, it is enough to prove that $\mathcal{K}_0$ admits a $G$-equivariant fibration. Let $g$ be a generator of $G$. We consider the following morphism:

$$H^1(\mathcal{K}_0, \mathbb{C}) \oplus H^1(\mathcal{K}_0, \mathbb{C}) \xrightarrow{(\text{id} - g^*)} H^1(\mathcal{K}_0, \mathbb{C}).$$

Since $G$ is a cyclic group, the kernel of $\text{id} - g^*$ is $G$-invariant. Moreover this kernel is nonzero by Proposition 1. Therefore $H^1(\mathcal{K}_0, \mathbb{C})$ has a $G$-equivariant sub Hodge structure and we conclude that $\mathcal{K}_0$ admits a $G$-equivariant fibration. 

\begin{enumerate}
\item[4.3] Proof of Proposition 3. We will construct a suitable resolution $Z$ of $\mathcal{K}/G$ and the unique relative minimal model $W$ of $Z$ over $\Delta^1$. By Lemma 3, there exists a $G$-equivariant elliptic fibration on the central fibre $\mathcal{K}_0$ of $\mathcal{K}$. We denote this fibration by $\pi : \mathcal{K}_0 \to C$. By construction, the action of $G$ on $C$ is translation. Let $H$ be the kernel of the representation $\rho : G \to \text{Aut} H^1(\mathcal{K}_0, \mathbb{C})$. Since the action of $H$ on $\mathcal{K}_0$ is a translation, $\mathcal{K}/H$ is smooth. It is enough to consider the action of $G/H$ on $\mathcal{K}/H$ for the investigation of the singularities of $\mathcal{K}/G$. If the action of
Let \( \eta \) be a morphism \( K \) by \( F \). We claim that the representation \( \rho : G \to \text{Aut} H^1(K_0, \mathbb{C}) \) is faithful. If \( H \) is not trivial, then \( \mathcal{K} \) is not \( G \)-invariant and \( K_{\mathcal{K}/G} \not\sim 0 \). Since \( \mathcal{K}/G \cong T \) and \( K_T \sim 0, H \) is trivial. By adjunction formula, \( K \) is smooth. Moreover \( \mathcal{K}/G \) is the unique relative minimal model over \( \Delta^1 \) since it has no rational curves. On the contrary, \( T \) is a relative minimal model over \( \Delta^1 \), \( T \cong \mathcal{K}/G \). By construction, the central fibre of the quotient \( \mathcal{K}/G \) is an hyperelliptic surface. Since every hyperelliptic surface is the étale quotient of the product of elliptic curves, \( T_0 \cong (\mathcal{K}/G)_0 \) satisfies the property of Theorem [1]. We claim that the representation \( \rho : G \to \text{Aut} H^1(K_0, \mathbb{C}) \) is faithful. If \( H \) is not trivial, then \( \mathcal{K} \) is not \( G \)-invariant and \( K_{\mathcal{K}/G} \not\sim 0 \). Since \( \mathcal{K}/G \cong T \) and \( K_T \sim 0, H \) is trivial. In the following, we assume that the action of \( G/H \) on \( C/H \) is trivial. Since \( \pi : \mathcal{K}_0 \to C/H \) is \( G/H \)-equivariant, the singularities of \( \mathcal{K}/G \) consists of several copies the product of a surface quotient singularity and an elliptic curve. The list of surface quotient singularities which occur above is found in [1, Table 5 (p157)]. We construct the relative minimal resolution \( Z \) of \( \mathcal{K}/G \) by the minimal resolution of surface quotient singularities. If the singularities of \( \mathcal{K}/G \) consists of the product of Du Val singularities and an elliptic curve only, then \( Z \) is a relative minimal model over \( \Delta^1 \) and we put \( W = Z \). In other cases, we obtain a relative minimal model \( W \) after birational contractions of \( Z \) (cf. [1, pp 156–158]). In both cases, \( W \) is the unique minimal model by the similar argument in Lemma [1]. Since \( W \) is birational to \( T \) and \( T \) is a relative minimal model over \( \Delta^1 \), \( T \cong W \). By construction, the central fibres \( W_0 \) admit a fibration over \( C/H \). Note that the fibre of \( W_0 \to C/H \) is a Kodaira singular fiber of type \( I_0^*, II, II^*, III, III^*, IV \) or \( IV^* \). Since \( \text{Sing} W_0 \) forms multi sections of \( W_0 \to C/H \), there exists an étale finite cover \( \tilde{C} \to C/H \) and the base change \( W_0 \times_{C/H} \tilde{C} \) is isomorphic to the product of a Kodaira singular fibre and an elliptic curve. Finally, we prove that the representation \( \rho : G \to \text{Aut} H^1(K_0, \mathbb{C}) \) is faithful. We derive a contradiction assuming that \( H \) is not trivial. If \( W = Z \), then there exists a morphism \( \eta : W \to \mathcal{K}/G \) such that \( \eta^* K_{\mathcal{K}/G} \sim K_W \). Since the action of \( H \) on \( K_0 \) is translation, \( K_K \) is not \( G \)-equivariant and \( K_{\mathcal{K}/G} \not\sim 0 \). However, \( K_W \sim \eta^* K_{\mathcal{K}/G} \) and \( K_W \sim 0 \), that is a contradiction. If \( W \not\cong Z \) we consider the base change \( Z_1 := Z \times_{\mathcal{K}/G} \mathcal{K}/H \). Since \( Z \) is obtained by blowing up along singular locus of \( \mathcal{K}/G \) (cf. [1, pp 158]) and the singular locus of \( \mathcal{K}/G \) consists of elliptic curves, \( Z_1 \) is smooth. Let \( \eta_1 : Z_1 \to W, \eta_2 : Z_1 \to \mathcal{K}/H \) and \( k/H : \mathcal{K}/H \to \Delta^1 \). We denote by \( F \) the central fibre of \( k/H \) with reduced structure. Note that \( (k/H)^*(0) = mF \), where \( m \) is the order of \( H \). Then

\[ \eta_1^* K_W \sim \eta_2^* K_{\mathcal{K}/H} - \eta_2^* F. \]

By adjunction formula, \( K_{\mathcal{K}/H} \sim (m - 1)F \). Thus \( K_W \not\sim 0 \). However, this is a contradiction because \( K_W \sim 0 \). □
(5.1) In this section, we prove the following proposition and Theorem 3.

**Proposition 3.** Let \( t : T \rightarrow \Delta^1 \) be an abelian fibration which is birational to the quotient of a Kulikov model \( \mathcal{K} \) of type II by a cyclic group \( G \). Assume that

1. \( T \) is smooth.
2. \( K_T \sim t_0 \).
3. There exists a nonzero \( G \)-equivariant element of \( F^1 H^1(\mathcal{K}_0, \mathbb{C}) \).

Then the representation \( \rho : G \rightarrow \text{Aut} H^1(\mathcal{K}_0, \mathbb{C}) \) is faithful and the central fibre \( T_0 \) of \( T \) satisfies the properties of Theorem 1 (2).

(5.2) For the proof of Proposition 3, we investigate the action of \( G \) on the central fibre of \( \mathcal{K} \).

**Lemma 7.** Let \( g \) be a generator of \( G \) and \( m \) the smallest positive integer such that every component is stable under the action of \( g^m \). We denote by \( H \) the subgroup of \( G \) generated by \( g^m \). Then

1. The representation \( \sigma : G \rightarrow \text{Aut} F^1 H^1(\mathcal{K}_0, \mathbb{C}) \) is trivial.
2. The action of \( H \) is free and the central fibre of the quotient \( \mathcal{K}/H \) is a cycle of minimal elliptic ruled surfaces.

**Proof.**

(1) By Proposition 3, there exists a \( G \)-equivariant element in \( F^1 H^1(\mathcal{K}_0, \mathbb{C}) \). Since \( \dim F^1 H^1(\mathcal{K}_0, \mathbb{C}) = 1 \), every element of \( F^1 H^1(\mathcal{K}_0, \mathbb{C}) \) is \( G \)-invariant.

(2) From the assumption there exists an action of \( g^m \) on each component of the central fibre of \( \mathcal{K} \). Let \( V \) be a component of the central fibre and \( \pi : V \rightarrow C \) a ruling. Since every fibre of \( \pi \) is \( \mathbb{P}^1 \) and \( C \) is an elliptic curve, \( g^m \) maps a fibre of \( \pi \) to a fibre of \( \pi \), that is, \( \pi \) is \( g^m \)-equivariant. From Lemma 2 (1) and Lemma 2, holomorphic one forms on \( V \) is invariant under the action of \( g^m \). Thus, the action of \( g^m \) on \( C \) is translation. Therefore \( V/H \) is a minimal elliptic ruled surface. From the assumption that each component is stable under the action of \( g^m \), the central fibre of the quotient \( \mathcal{K}/H \) is a cycle of minimal elliptic ruled surfaces.

(5.3) **Proof of Proposition 3.** From Lemma 4, \( \mathcal{K}/H \) is smooth and the central fibre of \( \mathcal{K}/H \) is a cycle of minimal elliptic ruled surfaces. Let \( \Gamma \) be the dual graph of the central fibre of \( \mathcal{K} \) and \( g \) a generator of \( G \). Considering \( \mathcal{K}/H \) in stead of \( \mathcal{K} \), we may assume that the action of \( g^m \) is trivial if the action of \( g^m \) on \( \Gamma \) is trivial.

(5.3.1) If the action of \( G \) is free, \( \mathcal{K}/G \) is smooth and this is a relative minimal model over \( \Delta^1 \). Every component of the central fibre of \( \mathcal{K}/G \) is the minimal elliptic ruled
surface $V$ which has a section $e$ such that $K_V \sim -2e$ and $e^2 = 0$. Thus we show that $T \cong \mathcal{K}/G$ by similar argument as in the proof of Lemma 5. We claim that the representation $\rho : G \rightarrow \operatorname{Aut}H^1(\mathcal{K}_0, \mathbb{C})$ is faithful. If $\ker \rho$ is not trivial, then $K_\mathcal{K}$ is not $G$-equivariant and $K_{\mathcal{K}/G} \not\sim 0$. However $K_T \sim 0$, we obtain $\ker \rho$ is trivial.

Since $\Gamma$ is a Dynkin diagram of type $\tilde{A}_n$ and $G$ is a cyclic group, the action of $G$ on $\Gamma$ is either rotation or reflection.

1. If the action of $G$ on $\Gamma$ is rotation, the central fibre of $\mathcal{K}/G$ is a cycle of minimal elliptic ruled surfaces. Each double curve is a section of a minimal elliptic ruled surface.

2. If the action of $G$ on $\Gamma$ is reflection, the central fibre $(\mathcal{K}/G)_0$ of $\mathcal{K}/G$ is a chain of minimal elliptic ruled surfaces. Let $V$ be the edge component of the central fibre and $V'$ the component such that $V \cap V' \neq \emptyset$. Note that $(\mathcal{K}/G)_0 = 2V + 2V' + (\text{other components})$. By adjunction formula,

$$K_V \equiv -V'|_V.$$

Therefore, the double curve $V \cap V'$ is a bisection of the ruling of $V$. Every other double curve is a section.

(5.3.2) If the action of $g$ is not free, we need the following lemma.

**Lemma 8.** If the action of $G$ has fixed points, then the action of $G$ on $\Gamma$ is reflection and it preserves two vertices. The fixed locus of the central fibre consists of sections or bisections of the ruling of components corresponding to the fixed vertices.

Assuming this Lemma, the central fibre of the quotient $\mathcal{K}/G$ is a chain of minimal elliptic ruled surfaces. The singularities of $\mathcal{K}/G$ consists of several copies of the product of $A_1$ singularity and an elliptic curve. Thus the unique relative minimal model $W$ over $\Delta^1$ is obtained by blowing up along singular locus. Since $T$ is a relative minimal model over $\Delta^1$, $W \cong T$. The dual graph of the central fibre of $W$ is $A_n$, $D_n$ or $\tilde{D}_n$. The double curve on the edge component is a bisection or a section. Every other double curve is section. We claim that the representation $\rho : G \rightarrow \operatorname{Aut}H^1(\mathcal{K}_0, \mathbb{C})$ is faithful. If $H$ is not trivial, $K_\mathcal{K}$ is not $G$-equivariant. Since $K_W$ is the pull back of $K_{\mathcal{K}/G}$ is crepant, $K_W \not\sim 0$ if $H$ is not trivial. However, $K_T \sim 0$, that is a contradiction.

(5.4) **Proof of Lemma 7.** If the action of $G$ on $\Gamma$ is rotation, there exists no fixed points. Thus the action of $G$ on $\Gamma$ is reflection. We derive the contradiction assuming that $G$ fixes one of edges of $\Gamma$. Let $C$ be the elliptic curve corresponding to the edge which is fixed by $G$. From Lemma 2 and Lemma 4, the action of $G$ on $C$ preserves holomorphic one form on $C$. Therefore $C$ is fixed locus of the action of $G$. The singularities of the quotient $\mathcal{K}/G$ consist of several copies of the
product of $A_1$ singularity and an elliptic curve. Let $w : W \to \mathcal{K}/G$ be the blowing up along $C$. We denote by $V_i$ each components of $W_0$. Let $V_0$ be the exceptional divisor coming from the blowing up along $C$. Since the central fibre $W_0$ of $W$ is a chain of minimal elliptic ruled surfaces, there exists components $V_1$ and $V_2$ such that $V_0 \cap V_1 \neq \emptyset$, $V_2 \cap V_1 \neq \emptyset$ and $V_i \cap V_j = \emptyset$ ($i = 0, 1, 2, j \neq 0, 1, 2$). Then $W_0 = V_0 + 2V_1 + 2V_2 + (\text{Other component}).$ Since $W$ is smooth along $V_1,$

$$K_{V_1} \equiv K_W + V_1|_{V_1} \equiv (\frac{-1}{2}V_0 - V_2)|_{V_1}.$$ by adjunction formula. Let $l$ be a fibre of ruling of $V_1$. Then

$$K_l \equiv K_{V_1} + l|_l \equiv (\frac{-1}{2}V_0 - V_2).l.$$ Since every double curve of $W_0$ is a section, $\deg K_l = -3/2$. However this is a contradiction because $l \cong \mathbb{P}^1$. Therefore $G$ fixes two vertices. Let $V$ be one of the component corresponding to the fixed vertices and $\pi : V \to C$ the ruling of $V$. Since $C$ is an elliptic curve and every fibre $\pi$ is $\mathbb{P}^1$, $\pi$ is $G$-equivariant. By Lemma 7 (1), the action of $G$ on $V$ preserves one form on $V$. Since the action of $G$ is not free, $G$ acts on $C$ trivially. There exist two fixed points on each fibre of the ruling of $V$. Thus we obtain the rest of assertion of Lemma 7. □

The proof of Proposition 3 is completed. □

(5.5) Proof of Theorem 1. We take a general point $x$ of the discriminant locus of $f$ and an unit disk $x \in \Delta^1$ such that $T := X \times_S \Delta^1$ is smooth. By Proposition 2, the abelian fibration $T \to \Delta^1$ satisfies assumptions of Proposition 2 or 3. Then $T_0$ satisfies the assertions of Theorem 1 by Proposition 2 and 3. □
The direction of ruling

Figure 1. Figures of $\tilde{A}_n$ and $\tilde{D}_n$ case.

The direction of ruling

Figure 2. Figures of $A_n$, $A_n$ and $D_n$ case. Bold line represents bisection and line represents section.

References

[1] W. Barth, C. Peters and van de Ven, Compact complex surface, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4, Springer Verlag, Berlin-New York, (1984). x+304 pp.

[2] A. Fujiki, Closedness of the Douady spaces of complex Kähler spaces, Publ. Res. Inst. Math. Sci., 14, 1978, 1–52.

[3] F. Griffith and W. Schmid, Recent developments in Hodge theory: a discussion of techniques and results, in Discrete Subgroups of Lie Groups, Bombay, Oxford University Press, (1973).

[4] K. Kodaira, On compact complex analytic surfaces II, Ann. Math., 77 (1963), 563–626.
[5] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in *Algebraic geometry, Sendai, 1985*, Adv. Stud. Pure Math. 10, T. Oda ed., Kinokuniya and North-Holland (1987), 283–360.

[6] F. Kempf, G. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal embeddings I*, Lecture Notes in Mathematics, 339, Springer Verlag, Berlin-New York, (1973). viii+209 pp.

[7] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology, 38 (1999), 79-81.

[8] D. Matsushita, *Addendum to: On fibre space structures of a projective irreducible symplectic manifold*, to appear Topology.

[9] D. Morrison, *The birational geometry of degenerations: an overview*, in *The birational geometry of degenerations*, Robert Friedman and David R. Morrison ed., Progress in Mathematics, 29 (1981), 1–32.

[10] U. Persson, *On degenerations of algebraic surfaces*, Mem. Amer. Math. Soc. 11 (1977), no. 189.

Research Institute for Mathematical Sciences, Kyoto University, Oiwake-Cyo Kitashirakawa, Sakyo-Ku Kyoto 606-8052 Japan

E-mail address: tyler@kurims.kyoto-u.ac.jp