Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism

Jens Boos*

Institute for Theoretical Physics
University of Cologne, 50923 Köln, Germany

Abstract

An electromagnetic field represented by the field strength 2-form $F$ has two invariants: the scalar $B^2 - E^2$ and the pseudo-scalar $E \cdot B$. $F$ can be interpreted as curvature, in analogy to the Riemannian curvature of general relativity. The invariants then take the same form in the non-linear case of Einstein’s general relativity as applied to the exact seven parameter solution of Plebański and Demiański (PD).

The vacuum energy density $B^2 + E^2$ corresponding to an electromagnetic field can be deduced from the square of its symmetric energy momentum tensor. The square of the Bel–Robinson tensor gives the analogous expression in case of the PD solution. A general 3-form is proposed, from which the Bel–Robinson tensor can be deduced.

We also determine the Kummer tensor, a tensor cubic in curvature, for the PD solution for the first time, and calculate the pieces of its irreducible decomposition.

The calculations are carried out in two coordinate systems: in the original polynomial PD coordinates, and in a modified Boyer–Lindquist-like version introduced by Griffiths and Podolský (GP) allowing for a more straightforward physical interpretation of the free parameters. file: 05elm_inv_v5.tex, Dec 5, 2014

*E-mail: boos@thp.uni-koeln.de
From electrodynamics it is well known [17, 20] that a field configuration represented by the field strength tensor $F_{ij}$ has the two invariant quantities $I_1 := F_{ab} F^{ab}$ and $I_2 := (\ast F_{ab}) F^{ab}$, where $\ast$ denotes the left tensor dual. Thus, $I_2$ is a pseudo-scalar whereas $I_1$ is a scalar. Evaluated in a local inertial frame, these (pseudo-)invariants take the form $I_1 = 2(B^2 - E^2)$ and $I_2 = 4B \cdot E$, where $E$ and $B$ are the electric and magnetic fields, respectively.

Electrodynamics can be described as a gauge theory with the structure group $U(1)$, wherein the vector potential $A$ plays the role of the connection 1-form. Consequently, the field strength $F = dA$ is nothing but the “curvature” 2-form, and therefore the two invariants $I_1, I_2$ can be regarded as “curvature” invariants.

On the other hand, Petrov type D spacetimes can be thought of as spacetimes with Coulomb-like sources, as already pointed out by Szekeres [30]; see also Stephani et al. [29], p. 39. In this sense, they are the closest scenario to electromagnetostatics that general relativity has to offer.

The Plebański–Demiański solution [24] is of type D, has seven free parameters, and can be used to model various axisymmetric electrovacuum spacetimes. Furthermore, it contains the important Kerr solution. Various subclasses of the PD solution have been studied: Kerr–Taub–NUT–(A)deSitter by Mars and Senovilla [21]; the (rotating) $C$-metric by Hong and Teo [18, 19]; Kerr-de Sitter by Chambers [5]; and of course the Kerr metric itself, see for example Carter [8], de Felice and Bradley [7], and Cherubini et al. [6].

A direct physical interpretation of the PD coordinates has been lacking for a long time, but it is possible to transform the PD coordinates to Boyer–Lindquist-like coordinates, see Griffiths and Podolský [15].

It is therefore of interest to calculate various geometric quantities of the Plebański–Demiański solution in physically well-motivated coordinates, as well as to compare geometric quantities with their electromagnetic counterparts in this more general setting.

### 1 Plebański–Demiański solution and its parameters

The Plebański–Demiański solution of 1976 is an exact solution of the Einstein–Maxwell equations with cosmological constant [24]. Using the coordinates $\{\tau, p, q, \sigma\}$ it can be written in
terms of the pseudo-orthonormal coframe

\[
\begin{align*}
\vartheta^0 &:= \frac{1}{1-pq} \sqrt{\Phi(q)} \left(d\tau - p^2 d\sigma\right), \\
\vartheta^1 &:= \frac{1}{1-pq} \sqrt{\frac{p^2 + q^2}{\Phi(q)}} dq, \\
\vartheta^2 &:= \Theta \frac{1}{1-pq} \sqrt{\frac{p^2 + q^2}{\mathcal{P}(p)}} dp, \\
\vartheta^3 &:= \Theta \frac{1}{1-pq} \sqrt{\frac{\mathcal{P}(p)}{p^2 + q^2}} (d\tau + q^2 d\sigma).
\end{align*}
\]

The metric is

\[
g = -\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3. \Theta = \pm 1 \text{ determines the spatial orientation of}
\]

the angular parts of the coframe: for \( \Theta = +1 \), as chosen originally by PD, we would not retrieve

the flat spatial tetrad in its usual orientation, as will be shown later. At this point, we merely

include this symbol without fixing its value yet. The vector potential 1-form reads

\[
A := \frac{1-pq}{\sqrt{p^2 + q^2}} \left( \hat{\varepsilon} q \vartheta^0 + \hat{g} p \vartheta^3 \right).
\]

The related field strength is

\[
F := dA \quad \text{and the excitation (in vacuum) is given by } \mu_0 H := \star F. \quad \text{We use CGS units where } \{4\pi \varepsilon_0 = 1, \mu_0 = 4\pi\}; \quad \text{see Hehl and Obukhov [17] for a detailed introduction}
\]

to electrodynamics formulated in terms of exterior calculus. \( \mathcal{P} \) and \( \Phi \) are quartic functions

defined via

\[
\begin{align*}
\mathcal{P}(p) &:= \hat{k} + 2\hat{\eta} - \hat{\varepsilon} p^2 + 2\hat{\nu} p^3 + \left( \hat{k} - \hat{\varepsilon}^2 + \hat{g}^2 - \frac{\Lambda}{3} \right) p^4, \\
\Phi(q) &:= \hat{k} + \hat{\varepsilon}^2 + \hat{g}^2 - 2\hat{\eta} q + \hat{\varepsilon} q^2 - 2\hat{\nu} q^3 + \left( \hat{k} - \frac{\Lambda}{3} \right) q^4.
\end{align*}
\]

The identification \( \hat{k} := \hat{\gamma} - \hat{g}^2 - \Lambda/6 \) readily reproduces Eq. (3.31) given in [24]. Using computer

algebra, it is quite straightforward to show that Eqs. (1)–(3) indeed solve the Einstein–Maxwell

equations. See appendix C.1 for our check of the PD solution.

In exterior calculus (see our conventions in appendix B), the Einstein–Maxwell equations with

cosmological constant \( \Lambda \) take the form

\[
\frac{1}{2} \eta_{\mu\alpha\beta} \wedge \text{Riem}^{\alpha\beta} + \Lambda \eta_\mu = 8\pi \Sigma_\mu. \quad \text{(4)}
\]

\( \Sigma_\mu \) is the electromagnetic energy-momentum 3-form defined via

\[
\Sigma_\mu := \frac{1}{2} \left( F \wedge (e_\mu \lrcorner H) - H \wedge (e_\mu \lrcorner F) \right). \quad \text{(5)}
\]
Evaluating the left-hand side of Eq. (4) explicitly yields
\[ \frac{1}{2} \eta_{\mu\alpha\beta} \wedge \text{Riem}^{\alpha\beta} + \Lambda \eta_{\mu} = \hat{e}^2 + \hat{g}^2 \frac{(1 - pq)^4}{(p^2 + q^2)^2} \eta_{\mu}. \] (6)

The energy-momentum 3-form of the potential (2) reads
\[ \Sigma_{\mu} = \frac{(1 - pq)^4}{8\pi} \frac{\hat{e}^2 + \hat{g}^2}{(p^2 + q^2)^2} \eta_{\mu}. \] (7)

Therefore, Eqs. (1)–(3) indeed solve the Einstein–Maxwell equations. Note that the trace \( \Sigma := \vartheta_{\alpha} \wedge \Sigma^{\alpha} \) vanishes, as expected for an electromagnetic field. Furthermore, the constants \( \hat{e} \) and \( \hat{g} \) can tentatively be identified as the electric and magnetic charge, respectively.

2 New coordinates of Griffiths and Podolský

As seen above, the PD solution comes with a set of seven free parameters: \( \{ \hat{m}, \hat{n}, \hat{e}, \hat{g}, \hat{\epsilon}, \hat{k}, \Lambda \} \). Provided that at least one of the parameters \( \{ \hat{m}, \hat{n}, \hat{e}, \hat{g} \} \) does not vanish, the solution is of Petrov type D. By means of the Einstein–Maxwell equation, \( \Lambda \) can be identified straightforwardly as the cosmological constant. The parameters \( \{ \hat{e}, \hat{g} \} \) allow an interpretation as electric charge and magnetic charge, respectively, by their appearance in the energy-momentum 3-form. At any rate, the nature of the remaining four parameters remains somewhat obscure.

However, this solution contains a variety of limiting cases, such as Schwarzschild, Taub–NUT, Kerr(–Newman), de Sitter, the \( C \)-metric, and combinations thereof. It is the lack of direct physical meaning of the free parameters \( \{ \hat{m}, \hat{n}, \hat{e}, \hat{k} \} \) that makes it difficult to procure a simple limiting procedure to arrive at the aforementioned spacetimes. This has already been pointed out by Griffiths and Podolský [15]. At this point, we will briefly summarize their approach to extract a physically directly relevant coframe and vector potential.

Following Hong and Teo [18, 19], a coordinate transformation can be employed to simplify the roots of the quartics \( P, Q \) of Eq. (3). This is a promising procedure, since the quartics control the Lorentzian signature of the metric.

The transformation introduces the new coordinates \( \{ t, r, \tilde{p}, \phi \} \), as well as the new parameters
\( \alpha, \omega, a, \) and \( \ell. \) It is degenerate provided any of the new parameters (save \( \ell \)) vanish:

\[
\tau \mapsto \tau(t, \phi) := \sqrt{\frac{\omega}{\alpha}} \left( t - \frac{(\ell + a)^2}{a} \phi \right),
\]

\[
p \mapsto p(\tilde{p}) := \sqrt{\frac{\omega}{\alpha}} (\ell + a \tilde{p}),
\]

\[
q \mapsto q(r) := \sqrt{\frac{\omega}{\alpha}} r,
\]

\[
\sigma \mapsto \sigma(\phi) := -\left( \frac{\omega}{\alpha} \right)^{\frac{3}{2}} \frac{\tilde{p}}{a}.
\]

Simultaneously, the free parameters of the original PD solution are scaled according to

\[
(m, \hat{n}) \mapsto \left( \frac{\alpha}{\omega} \right)^{\frac{3}{2}} (m, n),
\]

\[
(e, \hat{g}) \mapsto \frac{\alpha}{\omega} (e, g),
\]

\[
\hat{e} \mapsto \frac{\alpha}{\omega} e,
\]

\[
\hat{k} \mapsto \alpha^2 k,
\]

\[
\Lambda \mapsto \Lambda.
\]

Note that also this scaling is degenerate in the cases of vanishing \( \omega, a \) or \( \alpha. \) With the additional parameters, there are 11 degrees of freedom. Three of these degrees of freedom can be used to adjust the roots of \( P \) to \( \tilde{p} = \pm 1, \) thereby introducing the relations

\[
\epsilon := \frac{\omega^2 k}{a^2 - \ell^2} + 4 \frac{\alpha}{\omega} \ell m - (a^2 + 3\ell^2) \left[ \frac{\alpha^2}{\omega^2} (\omega^2 k + e^2 + g^2) - \frac{1}{3} \right],
\]

\[
n := \frac{\omega^2 k \ell}{a^2 - \ell^2} - \frac{\alpha}{\omega} (a^2 - \ell^2) m + (a^2 - \ell^2) \ell \left[ \frac{\alpha^2}{\omega^2} (\omega^2 k + e^2 + g^2) - \frac{1}{3} \right],
\]

\[
k := \frac{(a^2 - \ell^2)(1 + 2\frac{\alpha}{\omega} \ell m - 3\frac{\alpha^2}{\omega^2} \ell^2 (e^2 + g^2) + \ell^2 \Lambda)}{\omega^2 + (a^2 - \ell^2) 3a^2 \ell^2}.
\]

The parameters \( \epsilon, n, k \) are now fixed by the new parameters \( \alpha, \omega, a, \) and \( \ell. \) According to [15], the remaining degree of freedom can be used to set \( \omega \) to a convenient value, if not both \( a \) and \( \ell \) vanish simultaneously. The quartics become

\[
P = \frac{\alpha^2 a^2}{\omega^2} (1 - \tilde{p}^2) (1 - \alpha_3 \tilde{p} - \alpha_4 \tilde{p}^2),
\]

\[
Q = \frac{\alpha^2}{\omega^2} \left[ \omega^2 k + e^2 + g^2 - 2mr + re^2 + 2mr^3 - (\alpha^2 k - \frac{4}{3}) r^4 \right].
\]
The constants $\alpha_3, \alpha_4$ turn out to be
\begin{align}
\alpha_3 &= 2 \frac{\alpha}{\omega} am - 4a\ell^2 \left(\omega^2 k + \epsilon^2 + g^2\right) + 4a\ell \frac{A}{3}, \\
\alpha_4 &= -\frac{\alpha^2}{\omega^2} a^2 \left(\omega^2 k + \epsilon^2 + g^2\right) + \frac{1}{3} a^2.
\end{align}

(12)

Since $-1 < \tilde{p} < 1$ due to $\mathcal{P} \geq 0$ we have a Lorentzian signature (assuming $\alpha_3$ and $\alpha_4$ are sufficiently small), a convenient parametrization is $\tilde{p} = \cos \theta$, with $\theta \in [0, \pi]$. We then arrive at the final expression for the PD coframe expressed in more familiar Boyer–Lindquist-like coordinates \(\{t, r, \theta, \phi\}\):
\begin{align}
\vartheta^0_E := \sqrt{\Delta} \frac{2}{\Omega \rho} \left[ dt - \left( a \sin^2 \theta + 4\ell \sin^2 \frac{\theta}{2} \right) d\phi \right], \\
\vartheta^1_E := \rho \frac{2}{\Omega \sqrt{\Delta}} dr, \\
\vartheta^2_E := -\rho \frac{2}{\Omega \sqrt{\chi}} d\theta, \\
\vartheta^3_E := \rho \sqrt{\chi} \frac{2}{2} \sin \theta \left( a dt - \left[r^2 + (a + \ell)^2\right] d\phi\right).
\end{align}

(13)

Note that it is now convenient to set $\Theta := -1$ such that the angular part of the coframe has its standard sign for flat spacetime. The vector potential reads
\begin{align}
A := \frac{\Omega \rho}{2} \left[ \frac{2}{\sqrt{\Delta}} \sqrt{\chi} \frac{2}{2} \sin \theta \left( a dt - \left[r^2 + (a + \ell)^2\right] d\phi\right) \vartheta^3_E \right].
\end{align}

(14)

We introduced the following auxiliary functions:
\begin{align}
\Delta := \omega^2 k + \epsilon^2 + g^2 - 2mr + e^2 - 2\frac{\alpha}{\omega} mr^3 - \left(\alpha^2 \frac{e}{\omega} - \Lambda^2 r^4 = \left(\frac{\omega}{\alpha} \right)^2 \varphi, \\
\chi := 1 - \alpha_3 \cos \theta - \alpha_4 \cos^2 \theta = \frac{\omega^2}{\alpha^2 a^2 \sin^2 \theta} \mathcal{P}, \\
\Omega := 1 - \frac{\alpha}{\omega} r (\ell + a \cos \theta) = 1 - pq, \\
\rho^2 := r^2 + (\ell + a \cos \theta)^2 = \frac{\omega}{2} \left( p^2 + q^2 \right).
\end{align}

(15)

Simple computer algebra can be used to verify that Eqs. (13)–(15) solve the Einstein–Maxwell equations for any value of $\omega$. The corresponding program can be found in appendix C.2. The left-hand side of the Einstein–Maxwell equations turns out to be
\begin{align}
\frac{1}{2} \eta_{\mu \alpha \beta} \wedge \text{Riem}^{\alpha \beta} + \Lambda \eta_{\mu} = \frac{\Omega^4 (\epsilon^2 + g^2)}{\rho^4} \eta_{\mu}.
\end{align}

(16)
The energy-momentum 3-form derived from the potential \( (14) \) reads
\[
\Sigma_{\mu} = \frac{1}{8\pi} \frac{\Omega^4 (e^2 + g^2)}{\rho^4} \eta_{\mu}. \tag{17}
\]

Thus, Eqs. (13)–(12) indeed fulfill the Einstein–Maxwell equations for any \( \omega \).

Griffiths and Podolský [15] interpret the new set of free parameters to be \( \{ m, \ell, a, \alpha, e, g, \Lambda \} \), combined with a scaling degree of freedom \( \omega \). All of the parameters have a physical interpretation: mass, Taub–NUT parameter, angular momentum, acceleration parameter, electric and magnetic charge, and cosmological constant, respectively. This can be seen, for example, by determining the coframes for various choices of the parameters and comparing it to the literature.

It is noteworthy, however, that the Einstein–Maxwell equations are fulfilled for any value of \( \omega \). Therefore the interpretation of \( \omega \) as a pure scaling degree of freedom becomes questionable. Similar results have been found for the original PD solution by García and Macías [11], see also Socorro et al. [28]. How their additional parameter \( \mu \) is related to the \( \omega \) parameter of Griffiths and Podolský will be subject of a further study.

The seemingly divergent expressions \((a^2 - \ell^2)\) in Eqs. (10)1, (10)2 cancel the leading \((a^2 - \ell^2)\) of Eq. (10)3. Therefore, all limits are well-behaved, even in the case \( a = \ell = 0 \). However, in this case, the parameter \( \omega \) has to be adjusted appropriately. A compilation of spacetimes, somewhat similar as [15], can be found in Table 1.

### 3 Curvature

Let us now turn to the curvature of the PD solution. Any antisymmetric 2-form has \( \left( \frac{n}{2} \right)\left( \frac{n}{2} \right) \) independent components. The curvature components, however, are further constrained by the first Bianchi identity \( 0 = DD\vartheta^\mu = \text{Riem}^\mu_{\alpha\beta\mu\nu} \vartheta^\alpha \wedge \vartheta^\beta \). This is a vector-valued 3-form with \( n\left( \frac{n}{3} \right) \) independent components. Therefore, the curvature has \# = \( \frac{1}{12}n^2(n + 1)(n - 1) \) independent components. In four dimensions, \# = 20.

We decompose the curvature into its irreducible pieces with respect to the Lorentz group [12]:
\[
\text{Riem}_{\mu\nu} =: \text{Weyl}_{\mu\nu} + \text{Ricci}_{\mu\nu} + \text{Scalar}_{\mu\nu} \tag{18}
\]

This decomposition consists of three pieces:

- the (tracefree) Weyl curvature \( \text{Weyl}_{\mu\nu} = \frac{1}{2} \text{Weyl}_{\alpha\beta\mu\nu} \vartheta^\alpha \wedge \vartheta^\beta \)
- the tracefree Ricci part \( \text{Ricci}_{\mu\nu} := -\frac{2}{n} \frac{\partial}{\partial \mu} \mu \wedge \text{Ric}_{\nu} \) with \( \text{Ric}_{\nu} := \text{Ric}_\nu - \frac{1}{n} \text{R} \vartheta_\mu \), whereas the Ricci 1-form is given by \( \text{Ric}_\mu := e_\alpha \cdot \mu \text{Riem}_\alpha^\mu \).
| parameters | coframe |
|------------|---------|
|            | **Kerr–Newman–de Sitter** |
|            | $\vartheta^0 = \frac{\sqrt{r^2 - 2mr + a^2 + \frac{1}{3}\Lambda (r^4 + a^2r^2) + e^2}}{\sqrt{r^2 + a^2 \cos^2 \theta}} (dt - a \sin^2 \theta d\phi)$ |
|            | $\vartheta^1 = \frac{\sqrt{r^2 + a^2 \cos^2 \theta} dr}{\sqrt{r^2 - 2mr + a^2 + \frac{1}{3}\Lambda (r^4 + a^2r^2) + e^2}}$ |
| $m, a, e, \Lambda, \omega = 1$ | $\vartheta^2 = \frac{\sqrt{r^2 + a^2 \cos^2 \theta} d\theta}{\sqrt{1 - \frac{1}{3}\Lambda a^2 \cos^2 \theta}}$ |
|            | $\vartheta^3 = \frac{\sqrt{1 - \frac{1}{3}\Lambda a^2 \cos^2 \theta} \sin \theta \left[(r^2 + a^2)d\phi - adt\right]}{er \vartheta^0}$ |
|            | $A = \frac{er \vartheta^0}{\sqrt{r^2 - 2mr + a^2 + \frac{1}{3}\Lambda (r^4 + a^2r^2) + e^2}}$ |
|            | **charged Taub–NUT–de Sitter** |
|            | $\vartheta^0 = \frac{\sqrt{r^2 (1 + 2\Lambda \ell^2) + e^2 - 2mr - \ell^2 (1 - \Lambda \ell^2) + \frac{1}{3}\Lambda r^4}}{r^2 + \ell^2} [dt - 2\ell (1 - \cos \theta) d\phi]$ |
| $m, \ell, e, \Lambda, \omega = 1$ | $\vartheta^1 = \frac{\sqrt{r^2 (1 + 2\Lambda \ell^2) + e^2 - 2mr - \ell^2 (1 - \Lambda \ell^2) + \frac{1}{3}\Lambda r^4}}{\sqrt{r^2 + \ell^2} dr}$ |
|            | $\vartheta^2 = \sqrt{r^2 + \ell^2} d\theta$ |
|            | $\vartheta^3 = \sqrt{r^2 + \ell^2} \sin \theta d\phi$ |
|            | $A = \frac{er \vartheta^0}{\sqrt{r^2 (1 + 2\Lambda \ell^2) + e^2 - 2mr - \ell^2 (1 - \Lambda \ell^2) + \frac{1}{3}\Lambda r^4}}$ |
|            | **C-metric** |
| $m, \ell, e, \Lambda, \omega = a$ | $\vartheta^0 = \frac{1}{\sqrt{1 - \alpha r \cos \theta}} \sqrt{(1 - \alpha^2 r^2) \left(1 - \frac{2m}{r}\right)} dt$ |
|            | $\vartheta^1 = \frac{1}{\sqrt{1 - \alpha r \cos \theta}} \sqrt{(1 - \alpha^2 r^2) \left(1 - \frac{2m}{r}\right)}^{-1} dr$ |
|            | $\vartheta^2 = \frac{1}{\sqrt{1 - \alpha r \cos \theta}} \sqrt{1 + 2\alpha m \cos \theta} d\theta$ |
|            | $\vartheta^3 = \frac{1}{\sqrt{1 - \alpha r \cos \theta}} \sqrt{1 + 2\alpha m \cos \theta \sin \theta} d\phi$ |

Table 1: Various coframes. The metric is given by $g = -\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3$. 
the scalar part $\text{Scalar}_{\mu\nu} := -\frac{1}{n(n-1)} R_{\mu}^\rho \wedge \vartheta_{\nu}$ with $R := e_\alpha \uparrow \text{Ric}^\alpha$

Counting degrees of freedom, Eq. (18) translates into

$$20 \ (\text{Riemann}) = 10 \ (\text{Weyl}) + 9 \ (\text{tracefree Ricci}) + 1 \ (\text{Ricci scalar}) .$$

By means of the Einstein–Maxwell equations (4), we now notice that only the Weyl piece contains non-trivial information about the PD spacetime (or about any other electro-magneto vacuum spacetime with cosmological constant, for that matter):

Since $\Sigma_\mu$ is the energy-momentum 3-form of an electromagnetic field, it is traceless. Taking the trace of the dual of the Einstein equation (B.15) then implies $R = 4\Lambda$. This is equivalent to $\text{Ric}_\mu = 8\pi \Sigma_\mu$. Therefore, only the Weyl part of curvature carries truly non-trivial information.

In order to visualize the 20 independent components of $\text{Riem}_{\mu\nu}$, we use the symmetry properties of its anholonomic components $\text{Riem}_{\alpha\beta\mu\nu} := e_\beta \uparrow (e_\alpha \uparrow \text{Riem}_{\mu\nu})$, namely,

$$\text{Riem}_{\alpha\beta\mu\nu} = -\text{Riem}_{\beta\alpha\mu\nu} = -\text{Riem}_{\alpha\beta\nu\mu} \quad \text{Riem}_{\alpha\beta\mu\nu} = \text{Riem}_{\mu\nu\alpha\beta} \quad \text{Riem}_{[\alpha\beta\mu\nu]} = 0 .$$

By means of Eq. (18), this symmetry holds for all pieces of the (irreducible!) decomposition. The symmetries (20) allow us to organize all 20 components in a $6 \times 6$ matrix. It is now convenient to introduce collective anholonomic indices; we define

$$I, J \in \{ \hat{0}1, \hat{0}2, \hat{0}3, \hat{2}3, \hat{3}1, \hat{1}2 \} \mapsto \{ 1, 2, 3, 4, 5, 6 \} .$$

The components of the covariant metric on this six-dimensional space are given by the 0-(pseudo-)form $\eta^{\alpha\beta\mu\nu}$, such that in our conventions (see appendix B)

$$\eta^{I\hat{J}} = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} ,$$

with $\mathbb{I} := \text{diag}(1,1,1)$. By means of the symmetry (20)$_3$ the trace of this matrix vanishes, $\eta^{AB} \text{Riem}_{AB} = 0$, and it comprises in fact 20 independent degrees of freedom. The $6 \times 6$ curvature matrix then reads

$$\text{Riem}^{I\hat{J}} = \begin{pmatrix} -2E & 0 & 0 & 2E & 0 & 0 \\ . & E & 0 & 0 & -E & 0 \\ . & . & E & 0 & 0 & -E \\ . & . & . & 2E & 0 & 0 \\ . & . & . & . & -E & 0 \\ . & . & . & . & . & -E \end{pmatrix} + \text{diag}(Q,0,0,Q,0,0) + \frac{\Lambda}{3} \mathbb{I} .$$
The dots “.” denote matrix entries following directly from the symmetry. We defined

\[ E := -\frac{1}{2} \text{Weyl}_1 \mathbf{0}_1 \mathbf{1}, \quad B := \frac{1}{2} \text{Weyl}_1 \mathbf{0}_{\bar{2} \bar{3}}, \quad Q := -2 \ast \left( \varphi^0 \wedge \Sigma_0 \right), \]  

(24)

and \( \mathbf{1} := \text{diag} \left( 1, 1, 1, -1, -1, -1 \right) \). For PD coordinates we find

\[ E = \left( \frac{pq - 1}{p^2 + q^2} \right)^3 \left[ (3p^2 - q^2)\dot{m}q + (p^2 - 3q^2)\dot{n}p - (\dot{e}^2 + \dot{g}^2)(p^2 - q^2)(1 + pq) \right], \]  

(25)

\[ B = \left( \frac{pq - 1}{p^2 + q^2} \right)^3 \left[ (p^2 - 3q^2)\dot{m}p - (3p^2 - q^2)\dot{n}q + 2(\dot{e}^2 + \dot{g}^2)(1 + pq)q \right], \]  

(26)

\[ Q = \frac{(pq - 1)^4}{(p^2 + q^2)^2} (\dot{e}^2 + \dot{g}^2). \]  

(27)

As anholonomic components, the above expressions are coordinate independent. Therefore, we can obtain the respective expression in GP coordinates simply by replacing the coordinates and constants according to Eqs. (8), (9):

\[ E = \frac{\Omega^3}{\rho^6} \left[ (r^2 - 3(\ell + a \cos \theta)^2)\dot{m}r + (3r^2 - (\ell + a \cos \theta)^2)n(\ell + a \cos \theta) \right. \]
\[ - \left. (\dot{e}^2 + g^2)(r^2 - (\ell + a \cos \theta)^2)(1 + \alpha r(\ell + a \cos \theta)) \right], \]  

(28)

\[ B = \frac{\Omega^3}{\rho^6} \left[ (3r^2 - (\ell + a \cos \theta)^2)m(\ell + a \cos \theta) - (r^2 - 3(\ell + a \cos \theta)^2)\dot{n}r \right. \]
\[ - \left. 2(\dot{e}^2 + g^2)(1 + \alpha r(\ell + a \cos \theta))r(\ell + a \cos \theta) \right], \]  

(29)

\[ Q = \frac{\Omega^4}{\rho^7} (\dot{e}^2 + g^2). \]  

(30)

The dual of the Weyl part of the decomposition of Eq. (23) turns out to be

\[
\left( \ast \text{Weyl}_{1,I} \right) = \begin{pmatrix}
2B & 0 & 0 & 2E & 0 & 0 \\
\cdot & -B & 0 & 0 & -E & 0 \\
\cdot & \cdot & -B & 0 & 0 & -E \\
\cdot & \cdot & \cdot & -2B & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & B & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & B
\end{pmatrix}. 
\]

(31)

Conversely, the dual swaps \( E \) and \( B \), up to a sign, i.e. \( E \leftrightarrow -B \) and \( B \leftrightarrow E \). We adopt a complex null tetrad according to

\[ l := \frac{1}{\sqrt{2}} \left( \varphi^0 + \varphi^1 \right), \quad k := \frac{1}{\sqrt{2}} \left( \varphi^0 - \varphi^1 \right), \quad m := \frac{1}{\sqrt{2}} \left( \varphi^3 - i \varphi^3 \right), \quad \overline{m} := \frac{1}{\sqrt{2}} \left( \varphi^3 + i \varphi^3 \right), \]  

(32)
such that \( l^\alpha k_\alpha = -1, \ m^\alpha \bar{m}_\alpha = 1 \), with \( i^2 = -1 \). The only non-vanishing Weyl scalar is [29]
\[
\Psi_2 := \frac{1}{2} \text{Weyl}_{\alpha\beta\gamma\delta} l^\alpha k^\beta (l^\gamma k^\delta - m^\gamma \bar{m}^\delta) = -E - iB. \tag{33}
\]
The scalar \( \Psi_2 \) represents the “Coulomb” component of a given spacetime (Szekeres [30], Stephani et al. [29]). Furthermore, in the limiting Kerr case of the PD solution, we have \( E \propto m \) and \( B \propto ma \): this allows us to identify the mass \( m \) as the gravitational electric charge, and the angular momentum \( m \) as the gravitational magnetic charge. Therefore the interpretations of \( E \) and \( B \) as gravitoelectric and gravitomagnetic quantities seem reasonable.

4 Curvature invariants

The Kretschmann invariant \( K \) and the Chern–Pontryagin pseudo-invariant \( P \) can be defined as squares of the Riemannian curvature:
\[
K := \frac{1}{2} \text{Riem}_{\alpha\beta\gamma\delta} \text{Riem}^{\alpha\beta\gamma\delta} = -\ast \left[ \text{Riem}_{\alpha\beta} \wedge (\ast \text{Riem}^{\alpha\beta}) \right], \tag{34}
\]
\[
P := \frac{1}{2} (\ast \text{Riem}_{\alpha\beta\gamma\delta}) \text{Riem}^{\alpha\beta\gamma\delta} = \ast \left( \text{Riem}_{\alpha\beta} \wedge \text{Riem}^{\alpha\beta} \right). \tag{35}
\]
\( \ast \) denotes the Hodge dual acting on forms, and \( \ast \) denotes the left tensor dual acting on the left pair of antisymmetric indices. We can now use the decomposition (18). Since it is irreducible, the individual parts are orthogonal with respect to each other, such that upon squaring there appear no cross terms, see García et al. [12]:
\[
K =: K^{\text{Weyl}} + K^{\text{Ric}} + K^R, \tag{36}
\]
\[
P =: P^{\text{Weyl}} + P^{\text{Ric}} + P^R. \tag{37}
\]
We obtain:
\[
K^{\text{Weyl}} = -24 \left( B^2 - E^2 \right), \quad K^{\text{Ric}} = 4Q^2, \quad K^R = \frac{4}{3} \Lambda^2, \tag{38}
\]
\[
P^{\text{Weyl}} = -48EB, \quad P^{\text{Ric}} = 0, \quad P^R = 0. \tag{39}
\]
This is a remarkably simple structure, and coincides with the case of electrodynamics.

For the Kretschmann scalar \( K \) of the Kerr spacetime, this result is well-known, see O’Neill [23], theorem 2.7.2 (for the definition of the two functions \( I \) and \( J \), here referred to as \( E \) and \( B \), respectively) and corollary 2.7.5 for the form of the Kretschmann scalar (the relative factor 2 arises due to our definition of Kretschmann, see Eq. (34)).
5 Bel and Bel–Robinson tensors

With the curvature invariants taking a form so closely related to the invariants of vacuum electrodynamics, we will now try to establish further analogies between the energy momentum of an electric field and its gravitational almost-counterpart, the Bel and Bel–Robinson tensors.

The Bel tensor can be defined via the tensor dual, see Senovilla [26]:

\[ 2B_{\mu\nu\rho\sigma} := \text{Riem}_{\mu\alpha\beta\rho} \text{Riem}_{\nu}^{\alpha\beta\sigma} + (\star \text{Riem}_{\mu\alpha\beta\rho}) (\star \text{Riem}_{\nu}^{\alpha\beta\sigma}) \]

\[ + (\star \text{Riem}_{\mu\alpha\beta\rho}) (\star \text{Riem}_{\nu}^{\alpha\beta\sigma}) (\star \text{Riem}_{\nu}^{\alpha\beta\sigma}) \]  

(40)

It has the following symmetries:

\[ B_{[\mu\nu]}^{\rho\sigma} = B_{\mu\nu}^{[\rho\sigma]} = 0, \quad B_{\mu\nu\rho\sigma} = B_{\rho\sigma\mu\nu}, \quad B^{\alpha}_{\alpha\rho\sigma} = 0 \]  

(41)

Note that \( B^{\alpha}_{\mu\alpha\sigma} \neq 0 \). The Bel–Robinson tensor can be defined as (Senovilla [26])

\[ \tilde{B}_{\mu\nu\rho\sigma} := \text{Weyl}_{\mu\alpha\beta\rho} \text{Weyl}_{\nu}^{\alpha\beta\sigma} + (\star \text{Weyl}_{\mu\alpha\beta\rho}) (\star \text{Weyl}_{\nu}^{\alpha\beta\sigma}) \]  

(42)

It is completely symmetric in all its indices and completely tracefree [25, 26, 27]:

\[ \tilde{B}_{\mu\nu\rho\sigma} = \tilde{B}_{(\mu\nu\rho\sigma)}, \quad \tilde{B}^{\alpha}_{\nu\alpha\sigma} = 0 \]  

(43)

There are several other definitions in the literature for the Bel and Bel–Robinson tensors, for a review see Douglas [9]. In the following we will use the definitions above.

The Bel and Bel–Robinson tensors are interesting, since they are the closest tensorial objects available to describe gravitational energy momentum (see e.g. Garecki [13, 14], Mashhoon [22], and the references above). However, Eqs. (40) and (42) do not allow such a conclusion yet. Therefore we will motivate this interpretation briefly by employing analogies from electrodynamics:

Expressed in terms of components \( F = \frac{1}{2} F_{\alpha\beta} \partial^\alpha \wedge \partial^\beta \), the symmetric tracefree electromagnetic energy momentum \( \left( \begin{smallmatrix} 0 & 1 \end{smallmatrix} \right) \) tensor defined via \( T_{\mu\nu} := e_{\mu} \downarrow \Sigma_{\nu} \) can be written as

\[ T_{\mu\nu} = \frac{1}{2} \left[ F_{\mu\alpha} F^\alpha_{\nu} + (\star F_{\mu\alpha}) (\star F^\alpha_{\nu}) \right] \]  

(44)

This form is quite similar to Eq. (40). Furthermore, inserting the Riemannian curvature 2-form
into Eq. (5) and contracting over both indices yields

$$B_\mu := \frac{1}{2} \left[ \text{Riem}_{\alpha\beta} \wedge (e_\mu \hook\star \text{Riem}^{\alpha\beta}) - (\star \text{Riem}_{\alpha\beta}) \wedge (e_\mu \hook \text{Riem}^{\alpha\beta}) \right]$$

$$= \frac{1}{4} \text{Riem}_{\alpha\beta\gamma\delta} \text{Riem}^{\alpha\beta\gamma\delta} \eta_\mu - \text{Riem}_{\mu\beta\gamma\delta} \text{Riem}^{\alpha\beta\gamma\delta} \eta_\alpha.$$  

(45)

On the other hand, the electromagnetic energy momentum 3-form turns out to be

$$\Sigma_\mu = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \eta_\mu - F_{\mu\alpha} F^{\beta\alpha} \eta_\beta.$$  

(46)

The similarity between Eqs. (45) and (46) is obvious. Is it also possible to find a 3-form $\Sigma_{\nu\rho\sigma}$, such that $B_{\mu\nu\rho\sigma} = e_\mu \hook \Sigma_{\nu\rho\sigma}$? For the Bel–Robinson tensor the answer is affirmative:

$$\tilde{B}_{\mu\nu\rho\sigma} = e_\mu \hook \left[ \text{Weyl}_{\rho\alpha} \wedge (e_\nu \hook \star \text{Weyl}^{\alpha}) - (\star \text{Weyl}_{\rho\alpha}) \wedge (e_\nu \hook \text{Weyl}^{\alpha}) \right] =: e_\mu \hook \tilde{\Sigma}_{\nu\rho\sigma}.$$  

(47)

The 3-form $\tilde{\Sigma}_{\nu\rho\sigma}$ is (up to a factor of 2) the energy momentum of Eq. (5) where we replaced the 2-form $F$ with the Riemannian curvature 2-form. The only modification arises due to the tensorial indices of the Riemann curvature 2-form. After performing the only possible non-trivial trace (summation over $\alpha$, unique up to a sign) we end up with the correct energy momentum $\left(\frac{3}{2}\right)$-valued 3-form.

Forming the Hodge dual $\star$ and subsequently building an interior product $\hook$ then yields the completely symmetric, tracefree energy-momentum in complete analogy to the case of vacuum electrodynamics. See the proof of Eq. (47) in appendix A.

For the Bel tensor this procedure is not straightforward, because the symmetry $B_{\mu\nu\rho\sigma} = B_{\rho\sigma\mu\nu}$ has to be put in by hand, and a part of its trace has to be subtracted as well:

$$B_{\mu\nu\rho\sigma} = e_\mu \hook \tilde{\Sigma}_{\nu\rho\sigma} + e_\rho \hook \tilde{\Sigma}_{\sigma\mu\nu} - \frac{1}{2} \left( g_{\mu\nu} \text{Tr}_{\rho\sigma} + g_{\rho\sigma} \text{Tr}_{\mu\nu} \right) + \frac{1}{8} g_{\mu\nu} g_{\rho\sigma} \text{Tr}_{\alpha} \text{Tr}_{\alpha},$$

$$\text{Tr}_{\mu\nu} := \frac{1}{2} \left( e_\mu \hook \tilde{\Sigma}_{\alpha\sigma\nu} + e_\nu \hook \tilde{\Sigma}_{\alpha\sigma\mu} \right)$$  

(48)

These complications are rooted in the following property of the Weyl tensor, that is not valid for the Riemann tensor (only in vacuum, where they coincide) $^{[10,31]}$:

$$\text{Weyl}_{\mu\alpha\beta\gamma} \text{Weyl}_{\nu}^{\alpha\beta\gamma} = \frac{1}{4} g_{\mu\nu} \text{Weyl}_{\alpha\beta\gamma\delta} \text{Weyl}_{\nu}^{\alpha\beta\gamma\delta}$$  

(50)

This relation, see the references above, is also inherited by the Bel–Robinson tensor. A similar relation holds for the electromagnetic energy momentum, that is, $T_{\mu\alpha} T_{\nu}^{\alpha} = \frac{1}{4} g_{\mu\nu} T_{\alpha\beta} T_{\alpha\beta}$.

Therefore, the Bel–Robinson tensor seems to be of greater physical interest in non-vacuum spacetimes. Furthermore, it seems to be the direct analogon of the energy momentum tensor.
of the electromagnetic field. We introduce collective anholonomic indices

$$I, J \in \{\hat{0}0, \hat{0}1, \hat{0}2, \hat{0}3, \hat{1}1, \hat{1}2, \hat{1}3, \hat{2}2, \hat{2}3, \hat{3}3\} \mapsto \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$ (51)

and find for the collective components of Bel–Robinson

$$\left( \tilde{B}_{IJ} \right) = (E^2 + B^2) \begin{pmatrix}
6 & 0 & 0 & 0 & -2 & 0 & 0 & 4 & 0 & 4 \\
. & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & 6 & 0 & 0 & -4 & 0 & -4 \\
. & . & . & . & . & -4 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & -4 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & 6 & 0 & 2 \\
. & . & . & . & . & . & . & . & 2 & 0 \\
. & . & . & . & . & . & . & . & . & 6
\end{pmatrix}. \quad (52)$$

The quantity $E^2 + B^2$ nicely resembles the (positive definite) vacuum energy density of an electromagnetic field (see e.g. Hehl and Obukhov [17], Eq. (E.1.34)). Algebraically, it turns out to be the magnitude squared of the Weyl scalar, $\Psi_2 \bar{\Psi}_2 = E^2 + B^2$. On the other hand, for Petrov type D spacetimes, the Newman Penrose formalism relates the Kretschmann and Pontryagin (pseudo-)invariants of the Weyl tensor to the Weyl scalar $\Psi_2$ as follows:

$$K^{\text{Weyl}} - i\mathcal{P}^{\text{Weyl}} = 24 (\Psi_2)^2 \Rightarrow 24 \Psi_2 \bar{\Psi}_2 = \sqrt{(K^{\text{Weyl}})^2 + (\mathcal{P}^{\text{Weyl}})^2}. \quad (53)$$

Accordingly, the energy density quantity should be expressible as an invariant, since the square of the Pontryagin pseudo-invariant is again an invariant. We find for the PD solution

$$24 (E^2 + B^2) = \sqrt{\tilde{B}_{\alpha\beta\gamma\delta} \tilde{B}^{\alpha\beta\gamma\delta}}. \quad (54)$$

The analogon of Eq. (54) within vacuum electrodynamics for a field configuration $(\mathbf{E}, \mathbf{B})$ is

$$\sqrt{T_{\alpha\beta} T^{\alpha\beta}} = \mathbf{E}^2 + \mathbf{B}^2. \quad (55)$$

Bonilla and Senovilla [4] interpret the Bel–Robinson tensor as an energy-squared expression. By expanding it in terms of the complex null tetrad $\{l, k, m, \overline{m}\}$, see Eq. (32), they define an effective square root for a completely symmetric, tracefree rank 4 tensor. According to Eq. (16) [4], the symmetric, tracefree square root $t_{\alpha\beta}$ then reads (for any type D spacetime)

$$t_{\alpha\beta} = \epsilon 6 \sqrt{\Psi_2 \bar{\Psi}_2} \left( m_{(\alpha} \overline{m}_{\beta)} + l_{(\alpha} k_{\beta)} \right). \quad (56)$$
\( \epsilon = \pm 1 \) can be chosen freely. For the PD solution we find:

\[
( t_{\alpha \beta} ) = \epsilon 3 \sqrt{E^2 + B^2} \text{diag}(1,-1,1,1) \tag{57}
\]

This is clearly symmetric and tracefree.

## 6 Kummer–Weyl tensor

With the quadratic expression given in such a concise form, we may proceed to cubic quantities. A candidate is the Kummer tensor:

\[
K^{\mu \nu \rho \sigma}[T] := T^{\alpha \beta \mu \nu} * T^*_{\alpha \gamma \beta \delta} T_{\gamma \rho \delta \sigma}. \tag{58}
\]

The Kummer tensor can be defined for any tensor \( T \) of rank \( \binom{n}{2} \) which is antisymmetric according to \( T_{(\mu\nu)\alpha\beta} = T_{\mu\nu(\alpha\beta)} = 0 \). \( *T^*_{\alpha\beta\gamma\delta} \) denotes the double tensor dual. Without taking into account further symmetries that \( T \) might have, the Kummer tensor satisfies

\[
K^{\alpha \beta \mu \nu} = K^{\mu \nu \alpha \beta}. \tag{59}
\]

Therefore, in \( n = 4 \) dimensions, the Kummer tensor can be thought of as a symmetric \( 16 \times 16 \) matrix with 136 independent components. See the recent article by Baekler et al. [1] for an extensive and systematic introduction of the Kummer tensor.

They decompose the Kummer tensor into six pieces \((1)\) \( K \), with \( I = 1, \ldots, 6 \). In terms of degrees of freedom, \( 136 = 35 + 45 + 20 + 20 + 15 + 1 \). The pieces read (see [1], Eqs. (90)–(94), (99), and (100)):

\[
\begin{align*}
(1) K^{\alpha \beta \mu \nu} &:= K^{(\alpha \beta \mu \nu)}, \\
(2) K^{\alpha \beta \mu \nu} &:= \frac{1}{2} \left( K^{(\alpha \beta \mu \nu)} - K^{(\beta \alpha \mu \nu)} \right), \\
(3) K^{\alpha \beta \mu \nu} &:= \frac{1}{3} \left( K^{(\alpha \beta \mu \nu)} - K^{(\alpha (\nu \mu) \beta)} + K^{(\beta \alpha (\mu \nu))} - K^{(\beta (\nu \mu) \alpha)} \right), \\
(4) K^{\alpha \beta \mu \nu} &:= \frac{1}{3} \left( K^{(\alpha \beta [\mu \nu])} + K^{(\beta \alpha [\nu \mu])} + K^{(\beta \mu [\nu \alpha])} \right), \\
(5) K^{\alpha \beta \mu \nu} &:= \frac{1}{2} \left( K^{(\alpha \beta [\mu \nu])} - K^{(\beta [\alpha \mu \nu])} \right), \\
(6) K^{\alpha \beta \mu \nu} &:= K^{(\alpha \beta [\mu \nu])}. 
\end{align*} \tag{60}
\]

It is useful to introduce the two cubic invariants

\[
S := (1) K^{\alpha \beta \mu \nu}, \tag{61}
\]
\[
\mathcal{A} := \eta_{\alpha \beta \gamma \delta} (6) K^{\alpha \beta \gamma \delta}. \tag{62}
\]
$S$ may be called the Kummer scalar, and $A$ the (axial) Kummer pseudo-scalar.

We now turn back to general relativity: The Riemann curvature tensor fulfills the required symmetries, see Eq. (20), and so does the Weyl tensor. Due to the pair commutation symmetry of Riemann and Weyl, both Kummer–Riemann and Kummer–Weyl fulfill the additional symmetry

$$K_{\mu\nu\alpha\beta}[\text{Weyl} / \text{Riem}] = K_{\nu\mu\beta\alpha}[\text{Weyl} / \text{Riem}].$$

(63)

As shown in Sec. 3, the Weyl part is the only non-trivial vacuum contribution to curvature. Therefore, in the following we will evaluate the Kummer–Weyl tensor $K[\text{Weyl}]$.

The irreducible parts can be represented as matrices. (1)$K$ is completely symmetric, and can therefore be — somewhat redundantly — visualized as a symmetric $10 \times 10$ matrix. For the PD solution, (2)$K = (5)K = 0$. (3)$K$ is symmetric in its first two indices (and by Eq. (59) also in its second two), that is, (3)$K_{\mu\nu\alpha\beta} = (3)K_{\nu\mu\alpha\beta}$. This also allows for a $10 \times 10$ representation. (4)$K$ does not exhibit any obvious symmetry, therefore it has to be represented as a $16 \times 16$ matrix. Finally, (6)$K$ is completely antisymmetric and must therefore be proportional to the $\eta$ metric of Eq. (22).

We define the following abbreviations:

$$P_0 := -9B^2 - 5E^2,$$
$$P_1 := \frac{8}{3}E\left(-3B^2 + 7E^2\right),$$
$$P_2 := \frac{2}{3}E\left(15B^2 - 17E^2\right),$$
$$P_3 := 3B\left(5B^2 - 3E^2\right),$$
$$P_4 := \frac{8}{3}E\left(3B^2 - E^2\right),$$
$$P_5 := (P_4)^{-1}P_6, \quad P_6 := 3B\left(B^2 - 3E^2\right).$$

(64) - (67)
The completely symmetric and antisymmetric pieces turn out to be

\[
\left( ^{(1)}K[W]_{IJ} \right) = \mathbf{E} \begin{pmatrix}
-12E^2 & 0 & 0 & 0 & 0 & 0 & P_1 & 0 & 0 \\
. & 4E^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & 0 & 0 & 0 & 0 & 0 & 0 & P_0 \\
. & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & -12E^2 & 0 & 0 & -P_0 & 0 \\
. & . & . & . & . & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & -P_0 & 0 & 0 \\
. & . & . & . & . & . & . & -4E^2 & 0 \\
. & . & . & . & . & . & . & . & -12E^2 \\
\end{pmatrix}, \quad (68)
\]

\[
\left( ^{(6)}K[W]_{IJ} \right) = \frac{1}{3} P_6 \left( \eta_{IJ} \right). \quad (69)
\]

The invariants read

\[
S[\text{Weyl}] = 24E \left( 3B^2 - E^2 \right), \quad (70)
\]
\[
A[\text{Weyl}] = 24B \left( 3E^2 - B^2 \right). \quad (71)
\]

All components may be written in terms of simple expressions \( E \left( \alpha B^2 - \beta E^2 \right) \) or \( B \left( \gamma E^2 - \delta B^2 \right) \). The symmetric part \( ^{(1)}K \) is proportional to the electric part \( E \), whereas the antisymmetric part \( ^{(6)}K \) is proportional to the magnetic part \( B \). The same holds for their invariants \( S \) and \( A \). The results agree with the respective expressions for the Kerr metric, first obtained by Baekler [2].

The \( ^{(3)}K \) piece is neither proportional to \( E \) or \( B \):

\[
\left( ^{(3)}K[W]_{IJ} \right) = \mathbf{E} \begin{pmatrix}
0 & 0 & 0 & 0 & P_1 & 0 & 0 & P_2 & 0 \\
. & -\frac{1}{2}P_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & -\frac{1}{2}P_2 & 0 & 0 & 0 & P_3 & 0 & 0 \\
. & . & . & -\frac{1}{2}P_2 & 0 & -P_3 & 0 & 0 & 0 \\
. & . & . & . & 0 & 0 & 0 & -P_2 & 0 \\
. & . & . & . & . & \frac{1}{2}P_2 & 0 & 0 & 0 \\
. & . & . & . & . & . & \frac{1}{2}P_2 & 0 & 0 \\
. & . & . & . & . & . & . & 0 & 0 \\
. & . & . & . & . & . & . & \frac{1}{2}P_1 & 0 \\
. & . & . & . & . & . & . & . & 0 \\
\end{pmatrix}, \quad (72)
\]
The piece \((^4K)\) reads

\[
\left( ^4K[W]_{IJ} \right) = P_4
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{1}{8} \\
-2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & P_5 & 0 & 0 & -P_5 & 0 \\
-\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & -P_5 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & P_5 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & -P_5 & 0 & 0 & P_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & P_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & -P_5 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(73)

The irreducible decomposition (60) holds for any tensor \(T\) fed into the Kummer machine (58). It is expected, however, that this irreducible decomposition will simplify when the Kummer machine is applied to a tensor of higher symmetry than \(T\), say, the Weyl tensor.

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The Bel–Robinson 3-form, see Eq. (47), is given by

$$\tilde{\Sigma}_{\nu\rho\sigma} := \text{Weyl}_{\rho a} \wedge (e_{\nu} \lrcorner \star \text{Weyl}_{a}^{\alpha} \sigma) - (\star \text{Weyl}_{\rho a}) \wedge (e_{\nu} \lrcorner \text{Weyl}_{a}^{\alpha} \sigma). \quad (A.1)$$

We expand the Weyl 2-forms in components, $\text{Weyl}_{\mu\nu} = \frac{1}{2} \text{Weyl}_{\alpha\beta\mu\nu} \vartheta^{\alpha} \wedge \vartheta^{\beta}$ and find

$$\tilde{\Sigma}_{\nu\rho\sigma} = \frac{1}{4} \text{Weyl}_{\omega\tau\rho\alpha} \text{Weyl}_{\omega\tau}^{\rho\sigma} \eta_{\nu} \eta_{\omega} - \frac{1}{2} (\text{Weyl}_{\nu\tau\rho\alpha} \text{Weyl}_{\omega\tau}^{\rho\sigma} + \text{Weyl}_{\nu\tau\sigma\alpha} \text{Weyl}_{\omega\tau}^{\rho\sigma}) \eta_{\nu}. \quad (A.2)$$

We now evaluate the dual of this expression. The Hodge star only acts on the $(n-1)$-forms $\eta_{\mu}$ according to $\star \eta_{\mu} \equiv \star \vartheta_{\mu} = (-1)^{p(n-p)+1} \vartheta_{\mu}$; see [17], Eq. (C.2.90). Since the coframe $\vartheta_{\mu}$ is a 1-form, we have $p = 1$, and we are in four dimension, that is, $n = 4$. Therefore, $\star \vartheta_{\mu} = \vartheta_{\mu}$. Applying the interior product to this 1-form then yields the metric tensor, $e_{\mu} \lrcorner \vartheta_{\nu} = g_{\mu\nu}$, because frame $e_{\mu}$ and coframe $\vartheta_{\nu}$ are dual to each other. This yields

$$e_{\mu} \lrcorner \star \tilde{\Sigma}_{\nu\rho\sigma} = \frac{1}{2} \text{Weyl}_{\nu\tau\rho\alpha} \text{Weyl}_{\nu}^{\tau\sigma} + \frac{1}{2} \text{Weyl}_{\nu\tau\sigma\alpha} \text{Weyl}_{\nu}^{\tau\rho} - \frac{1}{4} g_{\mu\nu} \text{Weyl}_{\omega\tau\alpha\rho} \text{Weyl}_{\omega\tau}^{\rho\sigma}. \quad (A.3)$$

The definition of the Bel tensor, see Eq. (40), is equivalent to

$$B_{\mu\nu\rho\sigma} = \text{Riem}_{\mu\alpha\beta\rho} \text{Riem}_{\nu}^{\alpha\beta} \sigma + \text{Riem}_{\mu\alpha\beta\sigma} \text{Riem}_{\nu}^{\alpha\beta} \rho + \frac{1}{8} g_{\mu\rho} g_{\nu\sigma} \text{Riem}_{\alpha\beta\gamma\delta} \text{Riem}_{\alpha\beta\gamma\delta} - \frac{1}{2} g_{\mu\rho} \text{Riem}_{\alpha\beta\gamma\rho} \text{Riem}_{\alpha\beta\gamma} \sigma - \frac{1}{2} g_{\rho\sigma} \text{Riem}_{\alpha\beta\gamma\mu} \text{Riem}_{\alpha\beta\gamma} \nu. \quad (A.4)$$

It coincides with the definition of the Bel–Robinson tensor (42) when substituting the Weyl tensor for the Riemann tensor (in fact, it gives twice the Bel–Robinson tensor, rooted in these properties of the Weyl tensor: its left and right dual coincide, and its double dual is again the Weyl tensor, up to a sign).

Inserting the vacuum relation (50) into the last summand of Eq. (A.4) then yields

$$\tilde{B}_{\mu\nu\rho\sigma} = \frac{1}{2} \text{Weyl}_{\mu\alpha\beta\rho} \text{Weyl}_{\nu}^{\alpha\beta} \sigma + \frac{1}{2} \text{Weyl}_{\mu\alpha\beta\sigma} \text{Weyl}_{\nu}^{\alpha\beta} \rho - \frac{1}{4} g_{\mu\nu} \text{Weyl}_{\alpha\beta\gamma\rho} \text{Weyl}_{\alpha\beta\gamma} \sigma. \quad (A.5)$$

The relation (A.5) is also found in the literature, see Bel [3], Eq. (15), Robinson [25], Eq. (3.1), Garecki [14], Eq. (1), Douglas [9], Eq. (19), and So [27], Eq. (1). It coincides with Eq. (A.3)
and hence the proof is concluded: The Bel–Robinson tensor can indeed be expressed as

\[ \tilde{B}_{\mu\nu\rho\sigma} = e_\mu \cdot \tilde{\Sigma}_{\nu\rho\sigma}. \] (A.6)

### B Exterior calculus

The following is a brief outline of our notation in exterior calculus.

For a Riemannian spacetime, the anholonomic coframe is given by \( \partial^\mu = e_\mu^a dx^a \) in terms of the holonomic coordinate cobasis \( dx^i \). Similarly, the anholonomic frame is \( e_\mu = e^a_\mu \partial_a \), where \( \partial_i \) is the holonomic coordinate basis. The expansion coefficients \( e_\mu^a \) are called the tetrad. Frame and coframe are dual to each other, that is, \( e_\nu \cdot \partial^\mu = \delta^\mu_\nu \), where \( \cdot \) denotes the interior product. We use Greek indices for anholonomic frame components and Latin indices for holonomic coordinate components.

The metric \( g \) is introduced as the symmetric tensor field \( g = g_{ab} dx^a \otimes dx^b = g_{\alpha\beta} \theta^\alpha \otimes \theta^\beta \). \( g_{ij} \) is used to raise and lower coordinate indices, and \( g_{\mu\nu} \) applies to anholonomic indices. We use the degree of freedom granted by the tetrad to set \( g_{\mu\nu} = \text{diag}(-1,1,1,1) \), thereby enforcing a pseudo-orthonormal coframe \( \theta^\mu \).

After the appearance of the metric, the Hodge dual \( * \) can be introduced, mapping \( p \)-forms to \( (n-p) \)-forms. We introduce the \( \eta \)-basis:

\[
\begin{align*}
\eta & := *1 & \text{4-form} \\
\eta_\mu & := e_\mu \cdot \eta & = * (\partial^\mu) & \text{3-form} \\
\eta_{\mu\nu} & := e_\nu \cdot \eta_\mu & = * (\partial^\mu \wedge \partial^\nu) & \text{2-form} \\
\eta_{\mu\nu\rho} & := e_\rho \cdot \eta_{\mu\nu} & = * (\partial^\mu \wedge \partial^\nu \wedge \partial^\rho) & \text{1-form} \\
\eta_{\mu\nu\rho\sigma} & := e_\sigma \cdot \eta_{\mu\nu\rho} & = * (\partial^\mu \wedge \partial^\nu \wedge \partial^\rho \wedge \partial^\sigma) & \text{0-form}
\end{align*}
\]

\( \wedge \) denotes the exterior product of forms and \( \partial^\mu = g_{\mu\alpha} \theta^\alpha \). The Hodge dual acts on a \( p \)-form \( \omega \) as follows, mapping it to an \( (n-p) \)-form:

\[
* \omega = * \left( \frac{1}{p!} \omega^{a_1 \ldots a_p} \theta_{a_1} \wedge \cdots \wedge \theta_{a_p} \right) := \frac{1}{p!(n-p)!} \omega^{a_1 \ldots a_p} \eta_{a_1 \ldots a_p}^{\alpha_1 \ldots \alpha_p} \quad (B.7)
\]

\( \eta_{\mu\nu\rho\sigma} \) is the totally antisymmetric unit tensor. It can be used to define a tensor dual acting on \( p \in [0,n] \) antisymmetric indices. For a tensor \( T_{\mu\nu\alpha\beta} \) satisfying \( T_{(\mu\nu)\alpha\beta} = T_{\mu\nu(\alpha\beta)} = 0 \), we define
the left, right, and double tensor dual according to

\[ T^\kappa\lambda\alpha\beta := \frac{1}{2} \eta_{\kappa\lambda}^{\mu\nu} T_{\mu\nu\alpha\beta}, \]
\[ T^*_{\mu\nu\rho\sigma} := \frac{1}{2} T_{\mu\nu\alpha\beta} \eta^{\alpha\beta}_{\rho\sigma}, \]
\[ *T^*_{\kappa\lambda\rho\sigma} := \frac{1}{4} \eta_{\kappa\lambda}^{\mu\nu} T_{\mu\nu\alpha\beta} \eta^{\alpha\beta}_{\rho\sigma}. \]

(B.9)

In a pseudo-orthonormal coframe, the metric compatible, torsion free Levi-Civita connection 1-form \( \Gamma^\mu_{\nu\rho} \) is antisymmetric \( \Gamma^\mu_{\nu\rho} = -\Gamma^\mu_{\rho\nu} \). In Riemannian geometry, it is completely determined by the coframe:

\[ \Gamma^\mu_{\nu\rho} = \frac{1}{2} \left( e^\mu_{\{\nu} - e^\nu_{\rho\} \Omega^{\alpha}_{\nu} \right) \varphi^\alpha - e_{\{\nu} g_{\rho\alpha\}} \varphi^\alpha = \frac{1}{2} \left( \partial_{\{\nu} g_{\rho\alpha\}} \right) \varphi^\alpha \]

\[ = \frac{1}{2} \left( \partial_{\{\nu} g_{\rho\alpha\}} + \partial_{\rho} g_{\nu\alpha\} - \partial_{\alpha} g_{\nu\rho\} \right) dx^\alpha = \Gamma_{\nu\rho} dx^\alpha \]

(B.10)

The antisymmetric curvature 2-form is given by

\[ R^\mu_{\nu\alpha\beta} := d\Gamma^\mu_{\nu\alpha\beta} = \frac{1}{2} Riem^\mu_{\alpha\beta\nu} \varphi^\alpha \wedge \varphi^\beta. \]

Its contraction, the Ricci 1-form, reads

\[ \text{Ric}^\mu_{\nu} := e^\mu_{\{\alpha} \text{Riem}_{\alpha\beta\nu} = \text{Ric}_{\alpha\mu} \varphi^\alpha. \]

The Ricci scalar is then

\[ g^\mu_{\nu} = 8 \pi \text{Ric}^\mu_{\nu}. \]

(B.14)
Using the dual of the Einstein 3-form, we may write the Einstein equations as

\[ \text{Ric}_\mu - \frac{1}{2} R\theta_\mu + \Lambda \theta_\mu = 8\pi \Sigma_\mu. \]  

(B.15)

### C Computer algebra code

The computer algebra system Reduce [16] was employed to carry out the calculations, supplemented by the package Excalc providing an efficient framework for exterior calculus. The source codes are listed below. For a review of the computer algebra system Reduce with Excalc applied to general relativity and beyond, see e.g. Socorro et al. [28].

#### C.1 Plebański–Demiański coframe and vector potential

The listing below first defines the Plebański–Demiański coframe and its accompanying vector potential. Then a variety of standard programs is included (see appendix C.3) for decomposing the curvature, calculating invariants, and calculating various tensorial quantities. Thereby the validity of the solution can easily be checked. This is not only reasonable for itself, it also serves the purpose to exclude any inconsistencies in our own notation. In a third step, the coordinate transformations and rescalings introduced by Griffiths and Podolský [15] are checked for consistency.

```
1 % %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
2 %
3 % REDUCE file checking the Plebanski-Demianski
4 % solution, and calculating second order
5 % curvature invariants within exterior calculus
6 %
7 % last edited by J. Boos, Dec 1, 2014
8 %
9 % file: plebanski_demianski_v5.rei
10 %
11 % conventions: 05_elm_inv_v5.pdf
12 %
13 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
14
15 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
16 % load the package excalc for exterior calculus
17 %
18 ```
% and adjust the line length of the output
%

load_package excalc $
linelength(200) $

% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% define the frame
%
% constants: c_m ......... mass - like parameter
% c_n ......... NUT - like parameter
% c_epsilon.... ?
% c_gamma...... ?
% c_e ......... electric charge
% c_g ......... (hypothetical) magnetic charge
% c_lambda..... cosmological constant
%
%
% auxiliary functions

clear {c_qq, c_pp, c_delta, c_hh, sqrt_c_qq, sqrt_c_pp} $
pform {c_qq, c_pp, c_delta, c_hh, sqrt_c_qq, sqrt_c_pp} = 0 $

% set the domains

fdomain c_qq = c_qq(q), sqrt_c_qq = sqrt_c_qq(q),
    c_pp = c_pp(p), sqrt_c_pp = sqrt_c_pp(p),
    c_delta = c_delta(p, q), c_hh = c_hh(p, q) $

% Plebanski - Demianski coframe

coframe o(0) = 1/c_hh*sqrt_c_qq/sqrt(c_delta)*(d tau - p**2*d sigma ) ,
    o(1) = 1/c_hh*sqrt(c_delta)/sqrt_c_qq*d q ,
    o(2) = cv_sign/c_hh*sqrt(c_delta)/sqrt_c_pp*d p ,
    o(3) = cv_sign/c_hh*sqrt_c_pp/sqrt(c_delta)*(d tau + q**2*d sigma )

with metric g = -o(0)*o(0) + o(1)*o(1) + o(2)*o(2) + o(3)*o(3) $

% conventional sign, see 05(elm_inv_v5).pdf for details

cv_sign := -1 $

25
% specify the functions explicitly, except for the quartics \( c_{qq}, c_{pp} \)
\[
c_{\text{delta}} := p^2 + q^2
\]
\[
c_{\text{hh}} := 1 - p*q
\]
% specify the chain rule for the square root expressions
\[
\begin{align*}
\n@({\sqrt c_{qq}, q}) & := \frac{\@c_{qq, q}}{(2*{\sqrt c_{qq}})} \\
\n@({\sqrt c_{pp}, p}) & := \frac{\@c_{pp, p}}{(2*{\sqrt c_{pp}})} \\

\sqrt c_{qq}^2 & := c_{qq} \\
\sqrt c_{pp}^2 & := c_{pp}
\end{align*}
\]
% denote the frame by \( e \)
\[
\text{frame } e
\]
% show the frame as a test
\[
\text{displayframe}
\]
% vector potential
\[
\begin{align*}
\text{clear } a1 & $ \\
\text{pform } a1 = 1 & $ \\
a1 & := \frac{c_{\text{hh}}}{\sqrt{c_{\text{delta}}}}*(\frac{c_{e}q}{\sqrt{c_{qq}}*o(0)} + \frac{c_{g}p}{\sqrt{c_{pp}}*o(3)})$
\end{align*}
\]
% calculate quantities
% in "einstein_maxwell_v1.rei" $ 
% in "curvature_v1.rei" $ 
% in "invariants_v1.rei" $ 
% in "weyl_def_v1.rei" $ 
% in "newman_penrose_v1.rei" $ 
% in "bel_v1.rei" $ 
% in "bel_robinson_v1.rei" $ 
% in "kummer_v1.rei" $ 
% % final substitution of the quartics,
% various checks

% quartic functions

c_pp := c_k + 2*c_n*p - c_epsilon*p**2 + 2*c_m*p**3 - (c_k + c_e**2 + c_g**2 - c_lambda/3)*p**4$

c_qq := c_k + c_e**2 + c_g**2 - 2*c_m*q + c_epsilon*q**2 - 2*c_n*q**3 - (c_k - c_lambda/3)*q**4$

sqrt_c_pp := \sqrt{c_pp}$

sqrt_c_qq := \sqrt{c_qq}$

% solution of Einstein-Maxwell equations?
% (this expression should vanish identically)
write emtest3(a) := emtest3(a)$

% solution of Maxwell equations?
write maxhom3 := maxhom3$
write maxinhom3 := maxinhom3$

% define shorthands as they appear in the curvature
ee := -1/2*weyl0(-0,-1,-0,-1)$
b := 1/2*weyl0(-0,-1,-2,-3)$
qq := -2*# (o(0) ^ sigma3(-0))$

% find greatest common divisor
on gcd$

% Kretschmann invariants, also check decomposition
write kretschmannw0 / (bb**2 - ee**2)$
write kretschmannr0 / q**2$
write kretschmanns0 / c_lambda**2$
write kretschmann0 - kretschmannw0 - kretschmannr0 - kretschmanns0$

% Pontryagin pseudo-invariants, also check decomposition
write pontryaginw0 / ee / bb$
write pontryaginr0$
write pontryagins0$
write pontryagin0 - pontryaginw0 - pontryaginr0 - pontryagins0$

% greatest common divisor not always needed in the following
off gcd$
% check the expressions for ee and bb

\[ ee_{\text{test}} := \frac{(p \cdot q - q^2)}{(p^2 + q^2)} \times 3((3p^2 - q^2)c_m q + (p^2 - 3q^2)c_n p - (c_e^2 + c_g^2)(1 + p \cdot q)(p^2 - q^2)) \]

\[ bb_{\text{test}} := \frac{(p \cdot q - q^2)}{(p^2 + q^2)} \times 3((p^2 - 3q^2)c_m p - (3p^2 - q^2)c_n q + 2(c_e^2 + c_g^2)(1 + p \cdot q)p \cdot q) \]

write \( ee - ee_{\text{test}} \)

write \( bb - bb_{\text{test}} \)

% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% introduce coordinates of Griffiths & Podolsky
%
%
% transform coordinates

\[ q := \sqrt{\frac{c_\alpha}{c_w}} r \]

\[ p := \sqrt{\frac{c_\alpha}{c_w}}(c_l + c_a \cdot \cos(\theta)) \]

\[ c_m := \left(\frac{c_\alpha}{c_w}\right)^{3/2} c_{\text{gp}_m} \]

\[ c_n := \left(\frac{c_\alpha}{c_w}\right)^{3/2} c_{\text{gp}_n} \]

\[ c_e := \frac{c_\alpha}{c_w} c_{\text{gp}_e} \]

\[ c_g := \frac{c_\alpha}{c_w} c_{\text{gp}_g} \]

\[ c_{\text{epsilon}} := \frac{c_\alpha}{c_w} c_{\text{gp}_{\text{epsilon}}} \]

\[ c_K := c_\alpha^2 c_{\text{gp}_K} \]

\[ c_{\lambda} := c_{\text{gp}_{\lambda}} \]

% constrain old, free parameters in terms of the new parameters

\[ c_{\text{gp}_{\text{epsilon}}} := \frac{c_w^2 c_{\text{gp}_{K2}} + 4c_\alpha/c_w c_l c_{\text{gp}_m} - (c_a^2 + 3c_l^2)(c_\alpha^2/2 c_w^2 c_{\text{gp}_K} + c_{\text{gp}_e}^2 + c_{\text{gp}_g}^2) - c_{\text{gp}_{\lambda}}/3}{c_{\text{gp}_{\lambda}}/3} \]

\[ c_{\text{gp}_n} := c_w^2 c_{\text{gp}_k2} c_l - c_\alpha^2 + 3c_l^2)(c_\alpha^2/2 c_w^2 c_{\text{gp}_K} + c_{\text{gp}_e}^2 + c_{\text{gp}_g}^2) - c_{\text{gp}_{\lambda}}/3 \]

\[ c_{\text{gp}_k} := (c_a^2 - c_l^2)(1 + 2c_\alpha/c_w c_l c_{\text{gp}_m}/c_w - 3c_\alpha^2/c_w^2 c_{\text{gp}_K} + c_{\text{gp}_e}^2 + c_{\text{gp}_g}^2) c_{\lambda} \]

\[ c_{\text{gp}_k2} := 1 + 2c_\alpha/c_w c_l c_{\text{gp}_m}/c_w - 3c_\alpha^2/c_w^2 c_{\text{gp}_K} + c_{\text{gp}_e}^2 + c_{\text{gp}_g}^2) c_{\lambda} \]

% auxiliary functions

\[ \text{aux}_a3 := 2c_\alpha/c_w c_a/c_{\text{gp}_m}/c_w - 4c_\alpha^2/c_w c_a/c_l/c_w^2/(c_w^2 c_{\text{gp}_K} + c_{\text{gp}_e}^2 + c_{\text{gp}_g}^2) + 4/3 c_{\text{gp}_{\lambda}}c_a/c_l \]
aux_a4 := -c_alpha**2*c_a**2/c_w**2*(c_w**2*c_gp_k + c_gp_e**2 + c_gp_g**2) + c_gp_lambda/3*c_a**2 $  

% check the transformations of the abbreviations 
% (all of these expressions yield zero) 

c_gp_delta := c_w**2*c_gp_k + c_gp_e**2 + c_gp_g**2 - 2*c_gp_m*r + c_gp_epsilon*r**2 - 2*c_alpha/c_w*c_gp_n*r**3 - (c_alpha**2*c_gp_k - c_gp_lambda/3)*r**4$  
write c_gp_delta - c_w**2/c_alpha**2*c_qq $  

c_gp_ch := 1 - aux_a3*cos(theta) - aux_a4*cos(theta)**2 $  
write c_gp_ch - c_pp*c_w**2/c_alpha**2/c_a**2/(1-cos(theta)**2) $  

c_gp_rhoe := sqrt(r**2 + (c_l + c_a*cos(theta))**2) $  
write c_gp_rhoe**2 - c_w/c_alpha*(p**2 + q**2) $  

c_omeg := 1 - c_alpha/c_w*r*(c_l + c_a*cos(theta)) $  
write c_omeg - (1 - p*q) $  
end $  

file: plebanski_demianski_v5.rei 

C.2 Griffiths–Podolský coframe and vector potential

For the sake of completeness, we also checked the expressions for the components of the Riemann and Weyl tensors as well as the invariants directly in the Griffiths–Podolský coframe. These calculations are quite time-consuming, even after several optimizations. We assume that this is due to the extensive redefinitions of the original constants and the non-polynomial structure of the coframe in the GP coordinates.

The Bel, Bel–Robinson, and Kummer tensors are not evaluated again, since their structure follows algebraically from the confirmed structure of the Riemann and Weyl tensor.
% file: griffiths_podolsky_v4.rei
% conventions: 05_elm_inv_v5.pdf
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% load the package excalc for exterior calculus
% and adjust the line length of the output
%
load_package excalc $
linthlength(200) $

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% define the frame and vector potential
%
% constants: c_m ........ mass
% c_a ........ angular momentum per mass
% c_alpha...... acceleration parameter
% c_lambda .... cosmological constant
% c_l .......... Taub-NUT parameter
% c_e .......... electric charge
% c_g .......... (hypothetical) magnetic charge
% c_w.......... scaling degree of freedom,
% here set to unity (see below)
%
% auxiliary functions
clear c_delta, c_rho, c_ch, c_omega $
pform {c_delta, c_rho, c_ch, c_omega} = 0 $
%
% auxiliary functions for some square root expressions
clear c_sqrt_delta, c_sqrt_ch $
pform {c_sqrt_delta, c_sqrt_ch} = 0 $
50  % set their domain
51  fdomain  c_omega = c_omega(r, theta),
52     c_delta = c_delta(r),
53     c_rho = c_rho(r, theta),
54     c_chl = c_chl(theta),
55     c_sqrt_delta = c_sqrt_delta(r),
56     c_sqrt_chl = c_sqrt_chl(theta) $
57
58  % specify chain rule implementation for the square roots
59  @( c_sqrt_delta, r) := @( c_delta, r)/(2* c_sqrt_delta) $
60  @( c_sqrt_chl, theta) := @(c_chl, theta)/(2* c_sqrt_chl) $
61  c_sqrt_delta**2 := c_delta $
62  c_sqrt_chl**2 := c_chl $
63  @(c_rho, r) := r/c_rho $
64  @(c_rho, theta) := -c_a*\sin(\theta)*(c_l + c_a*\cos(\theta))/c_rho $
65
66  % coframe ( employed half angle formula for \sin**2(\theta/2) )
67  coframe  o(0) = 1/c_omega*c_sqrt_delta/c_rho*(d t - (c_a*\sin(\theta) **2 - 2*c_l*\cos(\theta) + 2*c_l)*d phi ),
68     o(1) = 1/c_omega*c_rho/c_sqrt_delta* d r ,
69     o(2) = -cv_sign/c_omega*c_rho/c_sqrt_chl*d theta ,
70     o(3) = cv_sign/c_omega*\sin(\theta)*c_sqrt_chl/c_rho*(c_a*d t - (r**2 + (c_a+c_l)**2)*d phi )
71  with metric  g = -o(0)*o(0) + o(1)*o(1) + o(2)*o(2) + o(3)*o(3) $
72
73  % conventional sign,
74  % see 05_elm_inv_v3.pdf for details
75  cv_sign := -1 $
76  % cv_sign**2 := 1 $
77
78  % set scaling dof to 1
79  % c_w := 1 $
80
81  % denote the frame by e
82  frame e $
83
84  % show the frame as a test
85  displayframe $
86
87  % vector potential
88  clear a1 $
89  pform a1 = 1 $
\[ a_1 := c_e \cdot c_\omega / c_r \cdot r / c_\sqrt{\delta} \cdot o(0) + c_g \cdot c_\omega / c_r \cdot (c_l / c_a + \cos(\theta)) / c_\sqrt{\chi} / \sin(\theta) \cdot o(3) \]

% express all occurring sines in terms of cosines
for all x let \( \sin(x)^2 = 1 - \cos(x)^2 \)

% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% calculate quantities
% %

in "einstein_maxwell_v1.rei"
in "curvature_v1.rei"
in "invariants_v1.rei"

% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% introduce the constants and auxiliary
% functions at the end, this reduces computing time
%
% define a procedure to update all important entities
% (run this after each substitution, otherwise it takes much longer)
procedure update $ begin$
  emtest3(a) := emtest3(a) $
sigma3(a) := sigma3(a) $
kretschmann0 := kretschmann0 $
kretschmannw0 := kretschmannw0 $
kretschmannr0 := kretschmannr0 $
kretschmanns0 := kretschmanns0 $
pontryagin0 := pontryagin0 $
pontryaginw0 := pontryaginw0 $
pontryaginr0 := pontryaginr0 $
pontryagins0 := pontryagins0 $
end$

% auxiliary functions

c_\delta := c_w^2 \cdot aux_k + c_e^2 + c_g^2 - 2 \cdot c_m \cdot r + aux_\epsilon_\text{r} \cdot r
**2 - 2*c_alpha/c_w*aux_n*r**3 - (c_alpha**2*aux_k - c_lambda/3)*r
**4 $ update () $
c_ch := 1 - aux_a3*cos(theta) - aux_a4*cos(theta)**2 $ update () $
c_rho := sqrt(r**2 + (c_l + c_a*cos(theta))**2) $ update () $
c_omega := 1 - c_alpha/c_w*r*(c_l + c_a*cos(theta)) $ update () $
c_sqrt_delta := sqrt(c_delta) $ update () $
c_sqrt_chi := sqrt(c_ch) $ update () $
c_sqrt := sqrt(c_sqrt)

% constants
aux_a3 := 2*c_alpha*c_a*c_m/c_w - 4*c_alpha**2*c_a*c_l/c_w**2*(c_w**2*aux_k + c_e**2 + c_g**2) + 4/3*c_lambda*c_a*c_l $ update () $
aux_a4 := -c_alpha**2*c_a**2/c_w**2*(c_w**2*aux_k + c_e**2 + c_g**2)

aux_epsilon := c_w**2*aux_k2 + 4*c_alpha/c_w*c_l*c_m - (c_a**2 + 3*c_l**2)*(c_alpha**2/c_w**2*(c_w**2*aux_k + c_e**2 + c_g**2) - c_lambda/3) $ update () $
aux_n := c_w**2*aux_k2*c_l - c_alpha*(c_a**2 - c_l**2)*c_m/c_w + (c_a**2 - c_l**2)*c_l*(c_alpha**2/c_w**2*(c_w**2*aux_k2*(c_a**2 - c_l**2) + c_e**2 + c_g**2) - c_lambda/3) $ update () $
aux_k := (1 + 2*c_alpha*c_l*c_m/c_w - 3*c_alpha**2*c_l**2/c_w**2*(c_e**2 + c_g**2) + c_l**2*c_lambda)/(c_w**2/(c_a**2 - c_l**2) + 3*c_alpha**2*c_l**2)
aux_k2 := (1 + 2*c_alpha*c_l*c_m/c_w - 3*c_alpha**2*c_l**2/c_w**2*(c_e**2 + c_g**2) + c_l**2*c_lambda)/(c_w**2 + 3*c_alpha**2*c_l**2)

% solution of Einstein-Maxwell equations?
% (this expression should vanish identically)
write emtest3(a) := emtest3(a) $

% solution of Maxwell equations?
write maxhom3 := maxhom3 $
write maxinhom3 := maxinhom3 $

% define shorthands as they appear in the curvature
ee := -1/2*weyl0(-0,-1,-0,-1) $
bb := 1/2*weyl0(-0,-1,-2,-3) $
qq := -2*(o(0) ^ sigma3(-0)) $

% find greatest common divisor
on gcd $

% Kretschmann invariants, also check decomposition
write kretschmannw0 / (bb**2 - ee**2) $
write kretschmannr0 / qq**2 $
write kretschmans0 / c_lambda**2 $
write kretschmann0 - kretschmannw0 - kretschmannr0 - kretschmans0 $

% Pontryagin pseudo-invariants, also check decomposition
write pontryaginw0 / ee / bb $
write pontryaginr0 $
write pontryagins0 $
write pontryagin0 - pontryaginw0 - pontryaginr0 - pontryagins0 $

% greatest common divisor not always needed in the following
off gcd $

% check the expressions for ee and bb
ee_test := c_omega**3/c_rho**6*((q**2 - 3*p**2)*c_m_hat*q + (3*q**2 - p**2)*c_n_hat*p - (c_e_hat**2 + c_g_hat**2)*(q**2 - p**2)*(1 + c_alpha*p*q)) $
bb_test := c_omega**3/c_rho**6*((3*q**2 - p**2)*c_m_hat*p - (q**2 - 3*p**2)*c_n_hat*q - 2*(c_e_hat**2 + c_g_hat**2)*(1 + c_alpha*p*q)*p *q) $
c_m_hat := c_m $
c_n_hat := aux_n $
c_e_hat := c_e $
c_g_hat := c_g $
p := c_1 + c_a*cos(theta) $
q := r $
write ee - ee_test $
write bb - bb_test $

end $

file: griffiths_podolsky_v4.rei
C.3 Universal programs for curvature decomposition, invariants, and other geometric objects

The source codes below are a collection of standard code snippets that can be included in a Reduce program one after the other, once a coframe has been defined. For a suitable application, see the code above in appendix C.1.

C.3.1 Check of Einstein–Maxwell equations

```reduce
% REDUCE file
% purpose: checks if Einstein-Maxwell equations are fulfilled for coframe and vector potential
% last edited by J. Boos, Dec 3, 2014
% file: einstein_maxwell_v1.rei
% conventions: 05_elm_inv_v5.pdf
%
% Levi-Civita connection for Riemannian geometry
%
write " connection...."$
clear conx1$
pform conx1(a, b) = 1$
riemannconx conx1$
write " done."$
%
% eta basis
%
write " eta basis...."$
clear eta0, eta1, eta2, eta3, eta4$
pform eta0(a, b, c, d) = 0,
    eta1(a, b, c) = 1,
    eta2(a, b) = 2,
    eta3(a) = 3,
    eta4 = 4$
eta4 := # 1$
```
eta3(a) := e(a) _| eta4 $
eta2(a, b) := e(b) _| eta3(a) $
eta1(a, b, c) := e(c) _| eta2(a, b) $
eta0(a, b, c, d) := e(d) _| eta1(a, b, c) $
write " done." $

% Riemann curvature 2-form
write " Riemann curvature..." $
% curvature is an antisymmetric 2-form
clear riem2 $
pform riem2(a, b) = 2 $
antisymmetric riem2 $
riem2(a, -b) := d conx1(a, -b) + conx1(a, -c) ^ conx1(c, -b) $

% Einstein 3- form
write " Einstein 3- form..." $
% Einstein tensor components
clear einstein3 $
pform einstein3(a) = 3 $
einstein3(-a) := 1/2 * eta1(-a, -b, -c) ^ riem2(b, c) $

% electrodynamics
write " electrodynamics..." $
% field strength
clear f2 $
pform f2 = 2 $
f2 := d a1 $
% excitation
clear h2 $
pform h2 = 2 $ 

h2 := # f2 $ 

% check of the homogeneous Maxwell equation:  
% this term should vanish 

clear maxhom3 $ 

pform mmaxhom3 = 3 $ 

maxhom3 := d f2 $ 

% check of the inhomogeneous Maxwell equation:  
% this term should vanish 

clear maxinhom3 $ 

pform mtaxinhom3 = 3 $ 

maxinhom3 := d h2 $ 

% electromagnetic energy-momentum 3-form  

clear sigma3 $ 

pform sigma3(a) = 3 $ 

sigma3(a) := 1/2 * ( f2 ^ (e(a) _| h2) - h2 ^ (e(a) _| f2) ) $ 

% trace  

clear trace4 $ 

pform trace4 = 4 $ 

trace4 := o(-a) ^ sigma3(a) $ 

write " done." $ 

% check of Einstein-Maxwell equations  
% Einstein-Maxwell equations  

clear emtest3 $ 

pform emtest3(a) = 3 $ 

emtest3(a) := einstein3(a) + c_lambda * eta3(a) - 2* sigma3(a) $ 

end $ 

file: einstein_maxwell_v1.rei 

C.3.2 Decomposition of curvature 

/**/ Decomposition of curvature 

% % purpose: calculates components of curvature,  
% its decomposition, and duals 
% last edited by J. Boos, Dec 3, 2014 
% file: curvature_v1.rei
% conventions: 05_elm_inv_v5.pdf

% Riemann curvature

\begin{verbatim}
write " Riemann curvature..." $

% tensor components
\begin{verbatim}
clear riem0 $
pform riem0(a, b, c, d) = 0 $
riem0(c, d, a, b) := e(d) _\| (e(c) _\| riem2(a, b)) $
\end{verbatim}

% left, right and double dual of Riemann
\begin{verbatim}
clear riemld0, riemrd0, riemdd0 $
pform \{ riemld0(a,b,c,d), riemrd0(a,b,c,d), riemdd0(a,b,c,d) \} = 0 $
riemld0(a,b,c,d) := 1/2*eta0(a,b,i,j)*riem0(-i,-j,c,d) $
riemrd0(a,b,c,d) := 1/2*riem0(a,b,-i,-j)*eta0(i,j,c,d) $
riemdd0(a,b,c,d) := 1/4*eta0(a,b,i,j)*riem0(-i,-j,-k,-l)*eta0(k,l,c,d) $
\end{verbatim}

write " done."
\end{verbatim}

% Ricci curvature

\begin{verbatim}
write " Ricci curvature..." $

% Ricci 1-form
\begin{verbatim}
clear ricci1 $
pform ricci1(a) = 1 $
ricci1(a) := e(-b) _\| riem2(a, b) $
\end{verbatim}

% Ricci 0-form
\begin{verbatim}
clear ricci0 $
pform ricci0 = 0 $
ricci0 := e(-a) _\| ricci1(a) $
\end{verbatim}

% traceless Ricci 1-form
\begin{verbatim}
clear tracelessricci1 $
pform tracelessricci1(a) = 1 $
tracelessricci1(a) := ricci1(a) - 1/4*o(a)*ricci0 $
\end{verbatim}

write " done."
\end{verbatim}

\end{verbatim}
write " irredicible decomposition of curvature..." 

% Ricci part of curvature

clear riccipart2 
pform riccipart2(a, b) = 2 
riccipart2(a, b) := -1/2*( o(a) ^ tracelessricci1(b) - o(b) ^ tracelessricci1(a)) 
clear riccipart0 
pform riccipart0(a, b, c, d) = 0 
riccipart0(c, d, a, b) := e(d) _| ( e(c) _| riccipart2(a, b))

% scalar part of curvature

clear scalarpart2 
pform scalarpart2(a, b) = 2 
scalarpart2(a, b) := -1/12*ricci0 *(o(a) ^ o(b)) 
clear scalarpart0 
pform scalarpart0(a, b, c, d) = 0 
scalarpart0(c, d, a, b) := e(d) _| ( e(c) _| scalarpart2(a, b))

% Weyl 2-form

clear weyl2 
pform weyl2(a, b) = 2 
weyl2(a, b) := riem2(a, b) - riccipart2(a, b) - scalarpart2(a, b)

% Weyl anholonomic components

clear weyl0 
pform weyl0(a, b, c, d) = 0 
weyl0(c, d, a, b) := e(d) _| ( e(c) _| weyl2(a, b))

% Weyl dual anholonomic components

clear weyl0d 
pform weyl0d(a, b, c, d) = 0 
weyl0d(c, d, a, b) := e(d) _| ( e(c) _| ( # weyl2(a, b)))

% left, right and double dual of Weyl

clear weyl0ld, weyl0rd, weyl0dd 
pform {weyl0ld(a,b,c,d), weyl0rd(a,b,c,d), weyl0dd(a,b,c,d)} = 0 
weyl0ld(a,b,c,d) := 1/2*eta0(a,b,i,j)*weyl10(-i,-j,c,d) 
weyl0rd(a,b,c,d) := 1/2*weyl10(a,b,-i,-j)*eta0(i,j,c,d) 
weyl0dd(a,b,c,d) := 1/4*eta0(a,b,i,j)*weyl10(-i,-j,-k,-l)*eta0(k,l,c,d)

write " done." 

done $
C.3.3 Kretschmann and Pontryagin invariants

\verbatim
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% REDUCE file
% purpose: calculates curvature invariants
% (Kretschmann and Pontryagin)
% last edited by J. Boos, Dec 3, 2014
% file: invariants_v1.rei
% conventions: 05_elm_inv_v5.pdf
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 
% quadratic (pseudo-) invariants
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\write " Kretschmann invariant..." $ 
\% Weyl part of Kretschmann 0-form
\clear kretschmannw0 $ 
\pform kretschmannw0 = 0 $ 
kretschmannw0 := - # ( weyl2(-a, -b) ^ (# weyl2(a, b))) $ 
\% traceless Ricci part of Kretschmann 0-form
\clear kretschmannr0 $ 
\pform kretschmannr0 = 0 $ 
kretschmannr0 := - # ( riccipart2(-a, -b) ^ (# riccipart2(a, b))) $ 
\% scalar part of Kretschmann 0-form
\clear kretschmanns0 $ 
\pform kretschmanns0 = 0 $ 
kretschmanns0 := - # ( (# scalarpart2(a, b)) ^ scalarpart2(-a, -b)) $ 
\% Kretschmann 0-form
\clear kretschmann0 $ 
\pform kretschmann0 = 0 $ 
kretschmann0 := - # ( riem2(-a, -b) ^ (# riem2(a, b))) $ 
\write " done." $ 
\write " Pontryagin pseudo-invariant..." $ 
% Weyl part of Pontryagin 0-pseudo-form
\clear pontryaginw0 $ 
\pform pontryaginw0 = 0 $ 

40
pontryagin0 := # ( weyl2(-a, -b) ^ weyl2(a, b)) $

% traceless Ricci part of Pontryagin 0-pseudo-form
clear pontryaginr0 $
pform pontryaginr0 = 0 $
pontryaginr0 := # ( riccipart2(-a, -b) ^ riccipart2(a, b)) $

% scalar part of Pontryagin 0-pseudo-form
clear pontryagins0 $
pform pontryagins0 = 0 $
pontryagins0 := # ( scalarpart2(-a, -b) ^ scalarpart2(a, b)) $

% Pontryagin 0-pseudo-form
clear pontryagin0 $ 
pform pontryagin0 = 0 $
pontryagin0 := # ( weyl2(-a, -b) ^ weyl2(a, b)) $

write " done." $
end $

file: invariants_v1.rei

C.3.4 Definition of the Weyl tensor components

write " symbolic Weyl tensor..." $

% set entries of symbolic Weyl tensor
% (calculational trick; oly possible if we already know how the Weyl tensor looks like)
clear symbweyl0 $ 
pform symbweyl0(a, b, c, d) = 0 $
for i := 0:3 do for j := 0:3 do for k := 0:3 do for l := 0:3 do
  symbweyl0(-i, -j, -k, -l) := 0
  symbweyl0(-0, -1, -0, -1) := -2 * symb_ee
  symbweyl0(-0, -1, -1, -0) := - symbweyl0(-0, -1, -0, -1)
  symbweyl0(-1, -0, -0, -1) := - symbweyl0(-0, -1, -0, -1)
  symbweyl0(-1, -0, -1, -0) := symbweyl0(-0, -1, -0, -1)
  symbweyl0(-0, -1, -2, -3) := 2 * symb_bb
  symbweyl0(-1, -0, -2, -3) := - symbweyl0(-0, -1, -2, -3)
  symbweyl0(-0, -1, -3, -2) := - symbweyl0(-0, -1, -2, -3)
  symbweyl0(-1, -0, -3, -2) := symbweyl0(-0, -1, -2, -3)
  symbweyl0(-2, -3, -0, -1) := symbweyl0(-0, -1, -2, -3)
  symbweyl0(-3, -2, -0, -1) := - symbweyl0(-0, -1, -2, -3)
  symbweyl0(-2, -3, -1, -0) := - symbweyl0(-0, -1, -2, -3)
  symbweyl0(-3, -2, -1, -0) := symbweyl0(-0, -1, -2, -3)
  symbweyl0(-0, -2, -0, -2) := symb_ee
  symbweyl0(-2, -0, -0, -2) := - symbweyl0(-0, -2, -0, -2)
  symbweyl0(-0, -2, -2, -0) := - symbweyl0(-0, -2, -0, -2)
  symbweyl0(-2, -0, -2, -0) := symbweyl0(-0, -2, -0, -2)
  symbweyl0(-0, -2, -3, -1) := - symb_bb
  symbweyl0(-3, -0, -3, -1) := - symbweyl0(-0, -3, -0, -3)
  symbweyl0(-0, -3, -3, -0) := - symbweyl0(-0, -3, -0, -3)
  symbweyl0(-0, -3, -1, -2) := - symb_bb
  symbweyl0(-3, -0, -1, -2) := - symbweyl0(-0, -3, -1, -2)
  symbweyl0(-0, -3, -2, -1) := - symbweyl0(-0, -3, -1, -2)
  symbweyl0(-1, -2, -0, -3) := symbweyl0(-0, -3, -1, -2)
  symbweyl0(-2, -1, -0, -3) := symbweyl0(-0, -3, -1, -2)
  symbweyl0(-2, -3, -2, -3) := 2 * symb_ee
  symbweyl0(-3, -2, -3, -2) := - symbweyl0(-2, -3, -2, -3)
  symbweyl0(-2, -3, -3, -2) := - symbweyl0(-2, -3, -2, -3)
  symbweyl0(-3, -2, -3, -2) := symbweyl0(-2, -3, -2, -3)
  symbweyl0(-3, -1, -3, -1) := - symb_ee
symbweyl0(-1, -3, -3, -1) := -symbweyl0(-3, -1, -3, -1)

symbweyl0(-3, -1, -1, -3) := -symbweyl0(-3, -1, -3, -1)

symbweyl0(-1, -3, -1, -3) := symbweyl0(-3, -1, -3, -1)

symbweyl0(-1, -2, -1, -2) := -symbweyl0(-1, -2, -1, -2)

symbweyl0(-2, -1, -1, -2) := -symbweyl0(-1, -2, -1, -2)

symbweyl0(-1, -2, -2, -1) := -symbweyl0(-1, -2, -1, -2)

symbweyl0(-2, -1, -2, -1) := symbweyl0(-1, -2, -1, -2)

write " done." $

end $

file: weyl_def_v1.rei

C.3.5 Newman–Penrose formalism

% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% % REDUCE file
% %
% % purpose: defines a complex null tetrad and
% % calculates the complex Weyl scalars
% %
% % last edited by J. Boos, Dec 3, 2014
% %
% % file: newman_penrose_v1.rei
% %
% % conventions: 05_elm_inv_v5.pdf
% %
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Newman Penrose formalism

write " setting up complex null tetrad..." $

% define complex null tetrad
% (conventions Senovilla GRG 1997)

% as 1-forms
clear {np_m1, np_mbar1, np_k1, np_l1} $
pform np_m1 = 1, np_mbar1 = 1, np_k1 = 1, np_l1 = 1 $
np_m1 := 1/sqrt(2)*(o(2) - i*o(3)) $
np_mbar1 := 1/sqrt(2)*(o(2) + i*o(3)) $
np_k1 := 1/sqrt(2)*(o(0) - o(1)) $
np_l1 := 1/sqrt(2)*(o(0) + o(1)) $

% components
clear {np_m0, np_mbar0, np_k0, np_l0} $
pform np_m0(a) = 0, np_mbar0(a) = 0, np_k0(a) = 0, np_l0(a) = 0 $
% Newman-Penrose coefficients for Weyl tensor
clear \{ np_psi_0 , np_psi_1 , np_psi_2 , np_psi_3 , np_psi_4 \} $
\text{pform} \{ np_psi_0 , np_psi_1 , np_psi_2 , np_psi_3 , np_psi_4 \} = 0 $

% transverse wave component in \( k \) direction
np_psi_0 := \text{symbweyl0}(-a,-b,-c,-d)*np_10(a)*np_m0(b)*np_10(c)*np_m0(d) $

% longitudinal wave component in \( k \) direction
np_psi_1 := \text{symbweyl0}(-a,-b,-c,-d)*np_10(a)*np_k0(b)*np_10(c)*np_m0(d) $

% "Coulomb" component
np_psi_2 := \frac{1}{2}\text{symbweyl0}(-a,-b,-c,-d)*np_10(a)*np_k0(b)*(np_10(c)*np_k0(d) - np_m0(c)*np_mbar0(d)) $

% longitudinal wave component in \( l \) direction
np_psi_3 := \frac{1}{2}\text{symbweyl0}(-a,-b,-c,-d)*np_k0(a)*np_10(b)*np_k0(c)*np_m0(d) $

% transverse wave component in \( l \) direction
np_psi_4 := \frac{1}{2}\text{symbweyl0}(-a,-b,-c,-d)*np_k0(a)*np_mbar0(b)*np_k0(c)*np_mbar0(d) $

\text{write} "\quad\quad\text{done.}\quad$ 
\text{end} $

file: newman_penrose_v1.rei

C.3.6 Bel tensor

%/%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% % REDUCE file
% % purpose: calculates the Bel tensor
% % last edited by J. Boos, Dec 3, 2014
% % file: bel_v1.rei
% % conventions: 05_elm_inv_v5.pdf
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Bel tensor
% definition with duals
\[ \text{robinson0}(a,b,c,d) = 0 \]
\[ \text{robinson0}(a,b,c,d) := \frac{1}{2}(\text{riem0}(a,i,j,c)\text{riem0}(b,-i,-j,d) + \text{riemld0}(a,i,j,c)\text{riemld0}(b,-i,-j,d) + \text{riemrd0}(a,i,j,c)\text{riemrd0}(b,-i,-j,d) + \text{riemdd0}(a,i,j,c)\text{riemdd0}(b,-i,-j,d)) \]
% definition with duals carried out already
\[ \text{robinson20}(a,b,c,d) = 0 \]
\[ \text{robinson20}(a,b,c,d) := \text{riem0}(a,i,j,c)\text{riem0}(b,-i,-j,d) + \text{riem0}(a,i,j,d)\text{riem0}(b,-i,-j,c) - \frac{1}{2}(\text{g}(a,b)\text{riem0}(i,j,k,c)\text{riem0}(-i,-j,-k,d) + \text{g}(c,d)\text{riem0}(i,j,k,a)\text{riem0}(-i,-j,-k,b)) + \frac{1}{8}\text{g}(a,b)\text{g}(c,d)\text{riem0}(i,j,k,l)\text{riem0}(-i,-j,-k,-l) \]
% this tensor vanishes, since the above two methods are equivalent
% clear robinson0 $ 
% clear robinson20 $ 
% pform robinson0(a,b,c,d) = 0 $ 
% robinson0(a,b,c,d) := robinson20(a,b,c,d) - robinson20(a,b,c,d) 
% introduce most general form of energy-momentum like term without fixed summation indices
\[ \text{gen3}(a,b,c,d,e) = 3 \]
\[ \text{gen3}(k,a,b,c,d) := \frac{1}{2}((\text{riem2}(a,b) \ ^ (e(k) \ _| \ (\# \text{riem2}(c,d))) - (\# \text{riem2}(a,b)) \ ^ (e(k) \ _| \ \text{riem2}(c,d))) \)
% test symmetries of this 3-form
% yields zero: symmetric in b,c 
% robinson0(k,a,b,c) := e(k) \ _| \ (\# \text{gen3}(a,b,-j,j,c)) - e(k) \ _| \ (\# \text{gen3}(a,c,-j,j,b)) 
% yields zero: symmetric in a,k
% robinson0(k,a,b,c) := e(k) \ _| \ (\# \text{gen3}(a,b,-j,j,c)) - e(a) \ _| \ (\# \text{gen3}(k,b,-j,j,c)) 
% hypothetical Robinson 3-form
% does not yield zero: symmetry in ka <-> bc needs to be put in by hand!
\[ \text{robinsonhyp0}(a,b,c,d) = 0 \]
\[ \text{robinsonhyp0}(k,a,b,c) := e(k) \ _| \ (\# \text{gen3}(a,b,-j,j,c)) - e(b) \ _| \ (\# \text{gen3}(c,k,-j,j,a)) \]
% build up hypothetical Robinson tensor by means of 3-form above
clear robinsontilde0$

pform robinsontilde0(k,a,b,c) = 0$

robinsontilde0(k,a,b,c) := 1/2 * ( e(k) _| (# gen3(a,b,-j,j,c)) + e(b) _| (# gen3(c,k,-j,j,a)) )$

% define trace of the above

clear robinsontrace0$

pform robinsontrace0(a,b) = 0$

robinsontrace0(a,b) := robinsontilde0(j,-j,a,b)$

% subtract the trace manually

clear robinson2tilde0$

pform robinson2tilde0(k,a,b,c) = 0$

robinson2tilde0(k,a,b,c) := 2*( robinsontilde0(k,a,b,c) - 1/4*( g(k,a)*robinsontrace0(b,c) + g(b,c)*robinsontrace0(k,a) ) + 1/16*g(k,a)*g(b,c)*robinsontrace0(i,-i) )$

% yields zero: Bel tensor is now indeed traceless in its first and second pair of indices

clear tracetest0$

pform tracetest0(a,b) = 0$

tracetest0(a,b) := robinson2tilde0(-i,i,a,b)$

% yields zero: Bel tensor can be written as 3-form, with symmetries put in by hand and traces subtracted by hand as well

clear robinson0$

pform robinson0(a,b,c,d) = 0$

robinson0(a,b,c,d) := robinson0(a,b,c,d) - robinson2tilde0(a,b,c,d)$

write " done."

end$

file: bel_v1.rei

C.3.7 Bel–Robinson tensor
Bel–Robinson tensor

write " Bel–Robinson tensor"$

% definition with duals
\set\clear\belrobinson0
\pform\belrobinson0(a,b,c,d) = 0$
\belrobinson0(a,b,c,d) := \weyl0(a,i,j,c) \ast \weyl0(b,-i,-j,d) + \weylld0(a,i,j,c) \ast \weylld0(b,-i,-j,d)$

% definition with duals carried out already
\set\clear\belrobinson20
\pform\belrobinson20(a,b,c,d) = 0
\belrobinson20(a,b,c,d) := \weyl0(a,i,j,c) \ast \weyl0(b,-i,-j,d) + \weyl0(a,i,j,d) \ast \weyl0(b,-i,-j,c) - 1/2 \ast (g(a,b) \ast \weyl0(i,j,k,c) \ast \weyl0(-i,-j,-k,d) + g(c,d) \ast \weyl0(i,j,k,a) \ast \weyl0(-i,-j,-k,b)) + 1/8 * g(a,b) * g(c,d) \ast \weyl0(i,j,k,l) \ast \weyl0(-i,-j,-k,-l)$

% this tensor vanishes, since the above two methods are equivalent
\set\clear\belrobinsontest0
\pform\belrobinsontest0(a,b,c,d) = 0
\belrobinsontest0(a,b,c,d) := \belrobinson0(a,b,c,d) - \belrobinson20(a,b,c,d)$

% introduce most general form of energy–momentum like term without fixed summation indices
\set\clear\gen23
\pform\gen23(a,b,c,d,e) = 3
\gen23(a,b,c,d) := \weyl2(a,b) ^ (e(k) _\mid (\# \weyl2(c,d))) - (\# \weyl2(a,b)) ^ (e(k) _\mid \weyl2(c,d))$

% the Bel–Robinson 3-form is the following trace of the above
\set\clear\belrobinson3
\pform\belrobinson3(a,b,c) = 3
\belrobinson3(a,b,c) := \gen23(a,b,-j,j,c)$

% build up Bel–Robinson tensor by means of 3-form above
% (this time, no traces need to be subtracted since it is tracefree by design)
\set\clear\belrobinsontilde0
\pform\belrobinsontilde0(k,a,b,c) = 0
\belrobinsontilde0(k,a,b,c) := e(k) _\mid (\# \belrobinson3(a,b,c))$

% check if it is indeed traceless
\set\clear\beltracetest0
\pform\beltracetest0(a,b) = 0
\beltracetest0(a,b) := \belrobinson0(-i,a,i,b)$

% this tensor vanishes if the Bel–Robinson tensor can indeed be written in terms of a 3-form
clear belrobinontest0 

pform belrobinontest0(a,b,c,d) = 0 
belrobinontest0(a,b,c,d) := belrobin0(a,b,c,d) - 
    belrobinontilde0(a,b,c,d) 

% express Bel-Robinson tensor in terms of the complex null tetrad

clear np_belrobinson0 
pform np_belrobinson0(a,b,c,d) = 0

np_belrobinson0(a,b,l,m) := 4*(ee**2+bb**2)*(
    np_m0(a)*np_mbar0 (b) + np_m0(b)*np_mbar0 (a)
    + np_l0(a)*np_k0(b) + np_l0(b)*np_k0(a)
    + np_m0(l)*np_mbar0 (m) + np_m0(m)*np_mbar0 (l)
    + np_l0(l)*np_k0(m) + np_l0(m)*np_k0(l)
    + ( np_l0(a)*np_m0(b) + np_l0(b)*np_m0 (a) )
    * ( np_k0(l)*np_mbar0 (m) + np_k0(m)*np_mbar0 (l) )
    + ( np_k0(a)*np_m0(b) + np_k0(b)*np_m0 (a) )
    * ( np_l0(l)*np_mbar0 (m) + np_l0(m)*np_mbar0 (l) )
    + np_l0(a)*np_l0 (b)*np_k0(l)*np_k0 (m)
    + np_k0(a)*np_k0 (b)*np_l0(l)*np_l0 (m)
    + np_m0(a)*np_m0 (b)*np_mbar0 (l)*np_mbar0 (m)
    + np_mbar0 (a)*np_mbar0 (b)*np_m0(l)*np_m0(m)

) 

% this tensor vanishes if the above expression is indeed the Bel-Robinson tensor

clear np_test 
pform np_test (a,b,c,d) = 0

np_test (a,b,c,d) := belrobinson0(a,b,c,d) - np_belrobinson0(a,b,c,d) 

% define Bonilla’s and Senovilla’s square root of the Bel-Robinson tensor

clear sqrt_belrobinson0 
pform sqrt_belrobinson0(a,b) = 0

sqrt_belrobinson0(a,b) := 3*sqrt(ee**2+bb**2)*(
    np_m0(a)*np_mbar0 (b) + np_m0(b)*np_mbar0 (a)
    + np_m0(b)*np_mbar0 (a) + np_l0(a)*np_k0(b) + np_l0(b)*np_k0(a) ) 

write " done."

end 

file: bel_robinson_v1.rei

C.3.8 Kummer tensor
% REDUCE file
% purpose: calculates the Kummer tensor
% last edited by J. Boos, Dec 3, 2014
% file: kummer_v1.rei
% conventions: 05_elm_inv_v5.pdf

% Kummer - Weyl tensor (calculate with symbolic Weyl
tensor to save computation time)

write "  Kummer - Weyl tensor"$

% (symbolic) Kummer Weyl tensor in anholonomic components

\( \text{clear symbkummerw0} \) 

\( \text{pform symbkummerw0(a, b, c, d) = 0} \) 

\( \text{symbkummerw0(i, j, k, l) := symbweyl0(a, i, b, j) * symbweyl0(-a, -c, -b, -d) * symbweyl0(c, k, d, l)} \) 

write "  Kummer-Weyl tensor symbolic irreducible decomposition..."

\( \text{clear symbkummer1w0} \) 

\( \text{pform symbkummer1w0(a, b, c, d) = 0} \) 

\( \text{symbkummer1w0(a, b, c, d) := ( symbkummerw0(a, b, c, d) + symbkummerw0(a, b, d, c) + symbkummerw0(a, d, b, c) + symbkummerw0(a, d, c, b) + symbkummerw0(a, c, d, b) + symbkummerw0(a, c, b, d) + symbkummerw0(c, a, b, d) + symbkummerw0(c, a, d, b) + symbkummerw0(c, d, a, b) + symbkummerw0(c, d, b, a) + symbkummerw0(c, b, a, d) + symbkummerw0(c, b, a, d) + symbkummerw0(b, c, a, d) + symbkummerw0(b, c, d, a) + symbkummerw0(b, d, c, a) + symbkummerw0(b, d, a, c) + symbkummerw0(b, a, c, d) + symbkummerw0(b, a, c, d) + symbkummerw0(d, b, a, c) + symbkummerw0(d, b, c, a) + symbkummerw0(d, c, a, b) + symbkummerw0(d, c, a, b) + symbkummerw0(d, a, c, b) + symbkummerw0(d, a, c, b) ) / 24} \) 

write "  2 / 6..."

\( \text{clear symbkummer2w0} \) 

\( \text{pform symbkummer2w0(a, b, c, d) = 0} \) 

\( \text{symbkummer2w0(a, b, c, d) := ( symbkummerw0(a, b, c, d) + symbkummerw0(c, b, a, d) - symbkummerw0(b, a, d, c) - symbkummerw0(} \)
d, a, b, c ) / 4 $

write " 3 / 6..." $

clear symbkummer3w0 $
pform symbkummer3w0(a, b, c, d) = 0 $
symbkummer3w0(a, b, c, d) := ( symbkummer0(a, b, c, d) + symbkummer0(a, b, d, c) - symbkummer0(a, d, c, b) - symbkummer0(a, c, d, b) + symbkummer0(b, a, c, d) + symbkummer0(b, a, d, c) - symbkummer0(b, d, c, a) - symbkummer0(b, c, d, a) ) / 6 $

write " 4 / 6..." $

clear symbkummer4w0 $
pform symbkummer4w0(a, b, c, d) = 0 $
symbkummer4w0(a, b, c, d) := ( symbkummer0(a, b, c, d) - symbkummer0(a, b, d, c) + symbkummer0(c, b, a, d) - symbkummer0(c, b, d, a) + symbkummer0(b, a, d, c) - symbkummer0(b, a, c, d) + symbkummer0(b, c, d, a) - symbkummer0(b, c, a, d) ) / 6 $

write " 5 / 6..." $

clear symbkummer5w0 $
pform symbkummer5w0(a, b, c, d) = 0 $
symbkummer5w0(a, b, c, d) := ( symbkummer0(a, b, c, d) - symbkummer0(c, b, a, d) - symbkummer0(b, a, d, c) + symbkummer0(d, b, a, c) ) / 4 $

write " 6 / 6..." $

clear symbkummer6w0 $
pform symbkummer6w0(a, b, c, d) = 0 $
symbkummer6w0(a, b, c, d) := ( symbkummer0(a, b, c, d) - symbkummer0(a, b, d, c) + symbkummer0(a, d, c, b) + symbkummer0(a, c, d, b) + symbkummer0(c, a, b, d) - symbkummer0(c, a, d, b) + symbkummer0(c, d, a, b) - symbkummer0(c, d, b, a) + symbkummer0(c, b, d, a) - symbkummer0(c, b, a, d) + symbkummer0(b, c, a, d) - symbkummer0(b, c, d, a) + symbkummer0(b, a, c, d) - symbkummer0(b, a, d, c) + symbkummer0(b, d, c, a) - symbkummer0(b, d, a, c) + symbkummer0(d, b, a, c) - symbkummer0(d, b, c, a) + symbkummer0(d, c, b, a) - symbkummer0(d, c, a, b) + symbkummer0(d, a, c, b) - symbkummer0(d, a, b, c) ) / 24 $

write " done. "$

write " Kummer-Weyl scalar..." $

clear symbkws0 $
pform symbkws0 = 0 $
symbkws0 := symbkummer1w0(a, -a, b, -b) $
write " done. "$

write " Kummer-Weyl axial scalar..." $
clear symbkwas0 $
$pform$ symbkwas0 = 0 $
symbkwas0 := \text{eta0}(-a, -b, -c, -d) \times \text{symbkummer6w0}(a, b, c, d)$ $
write " done." $

% gives zero: decomposition is indeed correct
% clear symbkummerdectest0 $
% pform symbkummerdectest0(a,b,c,d) = 0 $
% symbkummerdectest0(a,b,c,d) := symbkummerw0(a,b,c,d) - symbkummer1w0(a,b,c,d) - symbkummer2w0(a,b,c,d) - symbkummer3w0(a,b,c,d) - symbkummer4w0(a,b,c,d) - symbkummer5w0(a,b,c,d) - symbkummer6w0(a,b,c,d)$

end $

file: kummer_v1.rei