AN ADAPTIVE FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL
ELLIPTIC EQUATIONS WITH LINE DIRAC SOURCES *

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ABSTRACT. In this paper, we propose a novel adaptive finite element method for an elliptic equation with line Dirac delta functions as a source term. We first study the well-posedness and global regularity of the solution in the whole domain. Instead of regularizing the singular source term and using the classical residual-based a posteriori error estimator, we propose a novel a posteriori estimator based on an equivalent transmission problem with zero source term and nonzero flux jumps on line fractures. The transmission problem is defined in the same domain as the original problem excluding on line fractures, and the solution is therefore shown to be more regular. The estimator relies on meshes conforming to the line fractures and its edge jump residual essentially uses the flux jumps of the transmission problem on line fractures. The error estimator is proven to be both reliable and efficient, an adaptive finite element algorithm is proposed based on the error estimator and the bisection refinement method. Numerical tests show that quasi-optimal convergence rates are achieved even for high order approximations and the adaptive meshes are only locally refined at singular points.

1. INTRODUCTION

We are interested in the adaptive finite element method for the elliptic boundary value problem

$$-\Delta u = \sum_{l=1}^{N} g_l \delta_{\gamma_l} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\gamma_l, l = 1, \ldots, N$ are disjoint or intersecting line fractures strictly contained in $\Omega$, $g_l \in H^{\beta_l}(\gamma_l)$ with $\beta_l \geq 0$, and $g_l \delta_{\gamma_l}$ in source term $\sum_{l=1}^{N} g_l \delta_{\gamma_l}$ is a line Dirac measure on line fracture $\gamma_l$ satisfying

$$\langle g_l \delta_{\gamma_l}, v \rangle = \int_{\gamma_l} g_l(s)v(s)ds, \quad \forall \ v|_{\gamma_l} \in L^2(\gamma_l). \quad (1.2)$$

Although $g_l \in H^{\beta_l}(\gamma_l) \subset L^2(\gamma_l)$, the line Dirac measure $\sum_{l=1}^{N} g_l \delta_{\gamma_l} \notin L^2(\Omega)$.

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The model (1.1) has been widely used to describe monophasic flows in porous media, tissue perfusion or drug delivery by a network of blood vessels [14], and it also has applications in elliptic optimal control problems [21]. The solution of the elliptic problem (1.1) is smooth in a large part of the domain, but it becomes singular in the region close to line fractures $\gamma_l$ and the region close to the vertices of the domain [28]. The corner singularity has been well understood in the literature [3, 16, 19, 26, 27] and references therein, we shall focus on the regularity of the solution near line fractures $\gamma_l$. The smoothness of the source term can be obtained by the duality argument [29], thus the regularity of solution for problem (1.1) follows from the standard elliptic regularity theory [20, 6].

Finite element methods for the second-order elliptic equations with singular source terms date back to the 1970s, but the main focus was on point Dirac delta sources (see e.g., [7, 34, 35, 13, 4, 25]). More recently, singular sources on complex geometry [21, 22, 28, 23, 15, 14, 5], including one-dimensional (1D) fracture sources, have attracted more attention. The finite element method was studied in [22] for problems involving a $C^2$ closed fracture strictly contained in the domain, and later an adaptive finite element method was proposed to improve the convergence rate [23]. As a controlled equation in an optimal control problem, the boundary value problem (1.1) with a single $C^2$ curve fracture was solved in [21] by the linear finite element method.

Due to the lack of regularity, the finite element method for problem (1.1) has only a convergence rate $h^{1/2}$ on general quasi-uniform meshes. Later on, in order to improve the convergence rate for problem (1.1) with one line segment fracture and the coefficient function $g_l = \text{const}$, Li et al. [28] studied the regularities in both Sobolev space and weighted Sobolev space, and a finite element algorithm was proposed to approximate the singular solution at the optimal convergence rate on graded meshes, which were densely refined only at the endpoints of the line fractures. The graded finite element algorithm in [28] can be applied to problem (1.1), but the grading parameter (used to generate graded meshes) depends on the smoothness of functions $g_l$, and it could be complicated to calculate and may vary case by case for different functions $g_l$ in order to generate graded meshes on which the finite element solutions are optimal.

An alternative way to obtain optimal finite element solutions for problem (1.1) is by the adaptive finite element methods (AFEMs), which are effective numerical methods for problems with singularities. AFEMs usually consist of four steps (see e.g., [18, 33]),

SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE,

which generates a sequence of meshes, on which the finite element approximations converge to the solution of the target problem. An essential ingredient of the AFEMs is a posteriori error estimator, which is a computable quantity that depends on the finite element approximation and known data, and provides information about the size and the distribution of the error of the numerical approximation. Therefore, it can be used to guide mesh adaption and as an error estimation. For results on the a posteriori error estimations of finite element analysis for the second order elliptic problems with an $L^2$ source term can be found in [2, 39] and references therein.

Elliptic problems with point Dirac delta source term were sufficiently studied by the AFEMs, for which the residual-based a posteriori error estimators were widely employed to guide the mesh
adaptions and as the finite element solution error estimations [8, 18, 32, 36, 39]. Due to the singularity of the point Dirac delta source term, it was generally regularized to an $L^2(\Omega)$ or $L^p(\Omega)$ function with $1 < p < \infty$ by projecting the source term to a polynomial space. Therefore, the residual-based a posteriori error estimator for the Poisson problem with an $L^2$ source term [38, 2] can be applied.

Recently, the regularization techniques of projecting the source term to an $L^2(\Omega)$ or $L^p(\Omega)$ function were also applied to the elliptic problem with line Dirac delta source term [23, 31]. The resulted residual-based a posteriori error estimators were also effective in proposing adaptive finite element algorithms, which don’t rely on specific meshes. However, the associated adaptive finite element solutions involve not only the discretization error but also the regularization error [23, 31], and the error estimators might lead to over-refinement on adaptive meshes or low convergence rates for high order approximations.

Motivated by the performance of the finite element solutions on graded meshes for which the grading parameters are involved, and of AFEMs based on regularized source terms for which the meshes are generally over-refined for low order approximations, in this work we propose a novel residual-based a posteriori error estimator, which is of high order convergence rates and the adaptive meshes are only locally refined near the singularities of the solution.

Instead of regularizing the singular line Dirac source term in problem (1.1), we transfer the problem (1.1) to an equivalent interface problem with zero source term and nonzero flux jumps on line fractures $\gamma_l$. More specifically, the coefficients $g_l$ in the line Dirac source term are transferred to the flux jumps on line fractures. The new transferred problem is known as the transmission problem [27], which is defined in the same domain as the original problem excluding on line fractures. The solution of problem (1.1) excluding on the line fractures solves the transmission problem, and it is shown that the solution becomes more regular after the transmission, which implies the finite element solutions for problem (1.1) would have a higher convergence rate if the meshes conform to the line fractures. Compared with the convergence rate on general quasi-uniform meshes, the finite element method for problem (1.1) has a better convergence rate $h^{\min(\alpha,\beta+\frac{1}{2})}$ on conforming quasi-uniform meshes, where $\beta = \min_l \{\beta_l\}$ and $\alpha < \frac{\pi}{\omega}$ with $\omega$ being the largest interior angle of the polygonal domain $\Omega$.

Our residual-based a posteriori error estimator is proposed based on the transmission problem. First, we triangulate the mesh conforming to line fractures $\gamma_l$, namely, $\gamma_l$ is the union of some edges in the triangulation. Second, the error estimator consists of element residual with zero source and edge residuals involving the difference with the flux jumps $g_l$ on line fractures. We derive the reliability and efficiency of the proposed a posteriori error estimator with novel skills in handling the edge residual. Based on the derived error estimator and bisection mesh refinement method, we propose an adaptive finite element algorithm. The quasi-optimal convergence rates can be numerically achieved for finite element approximations with the adaptive meshes only locally refined at the singular points.

As far as we have known, this is the first work using the transmission problem to construct a posteriori error estimator for problems with Dirac source terms. It would be interesting to apply
the proposed AFEM to problem (1.1) with curved line segments, and to explore the applications in three-dimension, we will leave these topics to our future work. The rest of the paper is organized as follows. In Section 2, we discuss the well-posedness and global regularity of equation (1.1) in Sobolev spaces. In Section 3, we introduced the transmission problem associated with problem (1.1) and investigate its well-posedness and regularity, and also showed its relationship with problem (1.1). In Section 4, we propose a novel residual-based a posteriori error estimator, show its reliability and efficiency, and propose an adaptive finite element algorithm. In Section 5, we present various numerical test results to validate the theoretical findings.

Throughout this paper, \( C > 0 \) denotes a generic constant that may be different at different occurrences. It will depend on the computational domain, but not on the functions involved and mesh parameters.

2. WELL-POSEDNESS AND REGULARITY IN SOBOLEV SPACES

Denote by \( H^m(\Omega) \), \( m \geq 0 \), the Sobolev space that consists of functions whose \( i \)th (\( 0 \leq i \leq m \)) derivatives are square integrable. Denote by \( H^1_0(\Omega) \subset H^1(\Omega) \) the subspace consisting of functions with zero trace on the boundary \( \partial \Omega \). For \( s > 0 \), let \( s = m + t \), where \( m \in \mathbb{Z}_{\geq 0} \) and \( 0 < t < 1 \). Recall that for \( D \subseteq \mathbb{R}^d \), the fractional order Sobolev space \( H^s(D) \) consists of distributions \( v \) in \( D \) satisfying

\[
||v||_{H^s(D)}^2 := ||v||_{H^m(D)}^2 + \sum_{|\alpha|=m} \int_D \int_D \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^2}{|x-y|^{d+2t}} \, dx \, dy < \infty,
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d \) is a multi-index such that \( \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \) and \( |\alpha| = \sum_{i=1}^d \alpha_i \). We denote by \( H^s_0(D) \) the closure of \( C_0^\infty(D) \) in \( H^s(D) \), and \( H^{-s}(D) \) the dual space of \( H^s_0(D) \). Let \( \tilde{H}^s(D) \) be the space of all \( v \) defined in \( D \) such that \( \tilde{v} \in H^s(\mathbb{R}^d) \), where \( \tilde{v} \) is the extension of \( v \) by zero outside \( D \).

2.1. Trace estimates. A sketch drawing of the domain \( \Omega \) with several line fractures is given in Figure 1(a). To obtain the trace estimates on line fractures, we first introduce the trace estimate on a general polygonal domain with no line fracture.
Lemma 2.1. [17, 30] Let $\Omega'$ be a polygonal domain with no line fracture, then the trace operator

$$\vartheta : H^s(\Omega') \rightarrow H^{s - \frac{1}{2}}(\partial \Omega')$$

is bounded for $\frac{1}{2} < s < \frac{3}{2}$.

Lemma 2.2. For the domain $\Omega$ with line segment fractures $\gamma_l$, $l = 1, \cdots, N$, it follows that the trace operator

$$\vartheta : H^s(\Omega) \rightarrow H^{s - \frac{1}{2}}(\bigcup_{l=1}^N \gamma_l)$$

is bounded for $\frac{1}{2} < s < \frac{3}{2}$.

Proof. By extending line fractures $\gamma_l$ appropriately to the boundary of the domain $\Omega$ or another line fracture and denoting the extended line fractures by $\gamma'_l$, which partition the domain $\Omega$ into $M$ polygonal subdomains $\Omega'_j$, $1 \leq j \leq M$ and $\gamma'_l$ is shared by neighboring subdomains $\Omega'_j$ (see Figure 1(b)). For any $v \in H^s(\Omega)$, it follows

$$v \in H^s(\Omega'_j), \quad j = 1, \cdots, M,$$

satisfying

$$\|v\|_{H^s(\Omega')}^2 = \sum_{j=1}^M \|v\|_{H^s(\Omega'_j)}^2 = \|v\|_{H^s(\bigcup_{j=1}^M \Omega'_j)}^2.$$  

By Lemma 2.1 if $\frac{1}{2} < s < \frac{3}{2}$, it follows for $l = 1, \cdots, N$,

$$\|v\|_{H^{s - \frac{1}{2}}(\gamma_l)} \leq \|v\|_{H^{s - \frac{1}{2}}(\gamma'_l)} \leq C \|v\|_{H^s(\bigcup_{j=1}^M \Omega'_j)} = C \|v\|_{H^s(\Omega)}.$$  

Therefore, the conclusion holds. \hfill \Box

2.2. Well-posedness and regularity. We have the following result regarding the line Dirac measure $\sum_{l=1}^N g_l \delta_{\gamma_l}$.

Lemma 2.3. For $\epsilon > 0$, the line Dirac measure $\sum_{l=1}^N g_l \delta_{\gamma_l} \in H^{-\frac{1}{2} - \epsilon}(\Omega)$ satisfying

$$\left\| \sum_{l=1}^N g_l \delta_{\gamma_l} \right\|_{H^{-\frac{1}{2} - \epsilon}(\Omega)} \leq C \sum_{l=1}^N \|g_l\|_{L^2(\gamma_l)}.$$  

Proof. The proof is based on the duality pairing (e.g., [29]). For $v \in H^{\frac{1}{2} + \epsilon}(\Omega)$, by Hölder’s inequality and Lemma 2.2, we have for $l = 1, \cdots, N$,

$$\langle g_l \delta_{\gamma_l}, v \rangle = \int_{\gamma_l} g_l(s)v(s)ds \leq C \|g_l\|_{L^2(\gamma_l)}\|v\|_{L^2(\gamma_l)} \leq C \|g_l\|_{L^2(\gamma_l)}\|v\|_{H^{\epsilon}(\gamma_l)} \leq C \|g_l\|_{L^2(\gamma_l)}\|v\|_{H^{\frac{1}{2} + \epsilon}(\Omega)}.$$  

Therefore, by definition, we have

$$\left\| \sum_{l=1}^N g_l \delta_{\gamma_l} \right\|_{H^{-\frac{1}{2} - \epsilon}(\Omega)} := \sup \left\{ \left\langle \sum_{l=1}^N g_l \delta_{\gamma_l}, v \right\rangle : \|v\|_{H^{\frac{1}{2} + \epsilon}} = 1 \right\} \leq C \sum_{l=1}^N \|g_l\|_{L^2(\gamma_l)}.$$  

\hfill \Box
The variational formulation for problem (1.1) is to find \( u \in H^1_0(\Omega) \), such that
\[
a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx = \left\langle \sum_{l=1}^N g_l \delta_{\gamma_l}, v \right\rangle, \quad \forall \, v \in H^1_0(\Omega).
\] (2.1)

By Lemma 2.3, the variational formulation (2.1) is well-posed.

Therefore, we have the following global regularity estimate.

**Lemma 2.4.** For \( \epsilon > 0 \), the elliptic boundary value problem (1.1) admits a unique solution \( u \in H^{\frac{3}{2}-\epsilon}(\Omega) \cap H^1_0(\Omega) \) satisfying
\[
\|u\|_{H^{\frac{3}{2}-\epsilon}(\Omega)} \leq C \sum_{l=1}^N \|g_l\|_{L^2(\gamma_l)}. \] (2.2)

**Proof.** The gives
\[
\|u\|_{H^{\frac{3}{2}-\epsilon}(\Omega)} \leq C \left\| \sum_{l=1}^N g_l \delta_{\gamma_l} \right\|_{H^{-\frac{1}{2}-\epsilon}(\Omega)} \leq C \sum_{l=1}^N \|g_l\|_{L^2(\gamma_l)}. \]
\]

\[\Box\]

**Remark 2.5.** Since problem (1.1) is a linear problem, so that the solution \( u \) of problem (1.1) can be obtained by summing of solutions of the following problems with one line Dirac source term for \( l = 1, \cdots, N \),
\[
- \Delta u_l = g_l \delta_{\gamma_l} \quad \text{in} \; \Omega, \quad u_l = 0 \quad \text{on} \; \partial \Omega.
\] (2.3)

By the superposition principle, one has
\[
u = \sum_{l=1}^N u_l.
\]
The estimate in Lemma 2.4 can also be obtained by first obtaining the estimates for problem (2.3), and then taking the summation of all these estimates.

Based on Lemma 2.4, we find that no matter how smooth the functions \( g_l \) are, the solution of problem (1.1) is merely in \( H^{\frac{3}{2}-\epsilon}(\Omega) \) for \( \forall \epsilon > 0 \) due to the appearance of the singular line Dirac measure \( \sum_{l=1}^N g_l \delta_{\gamma_l} \) in the source term. Then, by Lemma 2.4 and the Sobolev imbedding Theorem [30], we have the following result.

**Corollary 2.1.** For \( \epsilon > 0 \), the solution \( u \) of problem (1.1) is Hölder continuous \( u \in C^{0,1/2-\epsilon}(\Omega) \). In particular, the solution \( u \in C^0(\Omega) \).

By Corollary 2.1, we know that the solution of problem (1.1) is continuous across line fractures \( \gamma_l, \; l = 1, \cdots, N \). Next, we introduce the transmission problem of problem (1.1) to investigate the normal derivatives of \( u \) across line fractures.
3. The transmission problem

Let $n^\pm$ be the outward unit normal of the region on each side of the fracture $\gamma_l$. For a function $v$, we denote $v^\pm$ (resp. $\partial_n v^\pm$) the traces of $v$ (resp. $\nabla v$) evaluated on the fracture $\gamma_l$ from the region on each side. We define the jump of $v$ across $\gamma_l$ by $[v] = v^+ - v^-$ and the jump of its normal derivatives (or flux jumps) on $\gamma_l$ by $[\partial_n v] = n^+ \cdot \nabla v^+ + n^- \cdot \nabla v^-$. 

Based on the observation of the solution and weak solution of problem (1.1), we introduce the following interface problem,

\begin{align}
-\Delta w &= 0 \quad \text{in } \Omega \setminus \bigcup_{l=1}^N \gamma_l, \quad (3.1a) \\
[w] &= 0 \quad \text{on } \gamma_l, \ l = 1, \cdots, N, \quad (3.1b) \\
[\partial_n w] &= g_l \quad \text{on } \gamma_l, \ l = 1, \cdots, N, \quad (3.1c) \\
w &= 0 \quad \text{on } \partial \Omega. \quad (3.1d)
\end{align}

The interface problem (3.1) is known as the transmission problem of the elliptic problem (1.1) [27].

We define a space

$$V = \left\{ v \in H^1(\Omega \setminus \bigcup_{l=1}^N \gamma_l) : v|_{\partial \Omega} = 0, \ [v]|_{\gamma_l} = 0, \ l = 1, \cdots, N \right\}.$$ 

Similar to [10], multiplying a test function $v \in H^1_0(\Omega)$ on both sides of (3.1a), and applying the Green’s formula together with the interface and boundary conditions (3.1b-d), we have

$$- \int_{\Omega \setminus \bigcup_{l=1}^N \gamma_l} \Delta w v dx = \int_{\Omega \setminus \bigcup_{l=1}^N \gamma_l} \nabla w \cdot \nabla v dx - \sum_{l=1}^N \int_{\gamma_l} [\partial_n w] v ds = 0,$$

thus the variational formulation for the transmission problem (3.1) is to find $w \in V$ such that

$$\int_{\Omega \setminus \bigcup_{l=1}^N \gamma_l} \nabla w \cdot \nabla v dx = \sum_{l=1}^N \int_{\gamma_l} g_l v ds, \quad \forall v \in H^1_0(\Omega). \quad (3.2)$$

Lemma 3.1. The weak formulation (3.2) admits a unique solution $w \in V$ satisfying

$$w = u|_{\Omega \setminus \bigcup_{l=1}^N \gamma_l}, \quad (3.3)$$

where $u$ is the solution of the weak formulation (2.1).

Proof. By Lemma 2.2 and (1.2), we have for $\forall v \in H^1_0(\Omega)$,

$$\left\langle \sum_{l=1}^N g_l \delta_{\gamma_l}, v \right\rangle = \sum_{l=1}^N \int_{\gamma_l} g_l v ds.$$

Since $u$ is the solution of (2.1), so $w$ given in (3.3) satisfying $w \in V$ and solves (3.2), whose existence and uniqueness therefore follow from the well-posedness of the weak formulation (2.1). \hfill \Box

Lemma 3.1 indicates that the solution of problem (1.1) solves the transmission problem (3.1) at least in $H^{\frac{3}{2}-\epsilon}(\Omega \setminus \bigcup_{l=1}^N \gamma_l) \cap V$. 
To investigate the regularity of the transmission problem (3.1), we first consider the following interface problem,

\[ -\Delta z = 0 \quad \text{in } \Omega \setminus \Gamma_0, \]
\[ [z] = 0 \quad \text{on } \Gamma_0, \]
\[ [\partial_n z] = g \quad \text{on } \Gamma_0, \]
\[ z = 0 \quad \text{on } \partial \Omega, \]

where \( \Gamma_0 \) is a closed sufficiently smooth curve strictly contained in \( \Omega \), and \( g \in H^\beta(\Gamma_0) \) with \( \beta \geq 0 \). For problem (3.4), we recall the following result from [20, 11, 1].

**Lemma 3.2.** Let \( z \) be the solution of the problem (3.4), then it follows \( z \in H^{\min\{1+\alpha, \beta+\frac{3}{2}\}}(\Omega \setminus \Gamma_0) \) satisfying

\[ \|z\|_{H^{\min\{1+\alpha, \beta+\frac{3}{2}\}}(\Omega \setminus \Gamma_0)} \leq C \|g\|_{H^\beta(\Gamma_0)}, \]

where \( \alpha < \frac{\pi}{\omega} \) with \( \omega \) the largest interior angle of the polygonal domain \( \Omega \).

Next, we introduce the following result from [20] Theorem 1.2.15 and Theorem 1.2.16.

**Lemma 3.3.** For a point \( x \in \gamma_l \), let \( \rho_l(x) \) be multiplication of the distances of \( x \) to the endpoints of \( \gamma_l \). Then one has \( \frac{\nu}{\rho_l} \in L^2(\gamma_l) \) for all \( v \in H^s(\gamma_l) \) when \( s \in (0, \frac{1}{2}) \), and one also has \( \frac{\partial v}{\partial \nu} \in L^2(\Omega) \) for all \( v \in H^s(\Omega) \) and \( |\nu| \leq s \) provided \( s - \frac{1}{2} \) is not an integer.

**Lemma 3.4.** For any \( \epsilon > 0 \), we have the following results,

(i) if \( g_l \in H^{\beta_l}(\gamma_l) \), it follows \( g_l \in H^{\min\{\beta_l, \frac{1}{2} - \epsilon\}}(\gamma_l) \);

(ii) if \( g_l \in H^{\beta_l}(\gamma_l) \) and \( \beta_l + \frac{1}{2} > 0 \) is not an integer, it follows \( g_l \in \tilde{H}^{\beta_l}(\gamma_l) \);

(iii) if \( g_l \in H^{\beta_l}(\gamma_l) \) and \( \beta_l + \frac{1}{2} > 0 \) is an integer, it follows \( g_l \in \tilde{H}^{\beta_l - \epsilon}(\gamma_l) \).

**Proof.** The proofs of (i) and (ii) follow from Lemma 3.3 and the definition of the fractional space \( \tilde{H}^s(D) \) defined at the beginning of Section 2. For (iii), we have \( g_l \in H^{\beta_l}(\gamma_l) \subset H^{\beta_l - \epsilon}(\gamma_l) \), then the conclusion holds by applying Lemma 3.3 and the definition of \( \tilde{H}^s(D) \).

**Theorem 3.5.** For \( \forall \epsilon > 0 \), let \( w \) be the solution of the transmission problem (3.1), if \( g_l \in H^{\beta_l}(\gamma_l), l = 1, \cdots, N \), then it follows

\[ \|w\|_{H^{\min\{\alpha+1, \beta+\frac{3}{2}, 2\epsilon\}}(\Omega \setminus \bigcup_{l=1}^N \gamma_l)} \leq C \sum_{l=1}^N \|g_l\|_{H^{\beta_l}(\gamma_l)} \cdot \]

Further, if all \( g_l \in H^{\beta_l}(\gamma_l), l = 1, \cdots, N \), it follows

\[ \|w\|_{H^{\min\{\alpha+1, \beta+\frac{3}{2}, 2\epsilon\}}(\Omega \setminus \bigcup_{l=1}^N \gamma_l)} \leq C \sum_{l=1}^N \|g_l\|_{H^{\beta_l}(\gamma_l)}, \quad \text{if } \beta + \frac{1}{2} > 0 \text{ is not an integer}; \]

\[ \|w\|_{H^{\min\{\alpha+1, \beta+\frac{3}{2}, 2\epsilon\}}(\Omega \setminus \bigcup_{l=1}^N \gamma_l)} \leq C \sum_{l=1}^N \|g_l\|_{H^{\beta_l}(\gamma_l)}, \quad \text{if } \beta + \frac{1}{2} > 0 \text{ is an integer}; \]

Here, \( \beta = \min_l\{\beta_l\} \) and \( \alpha < \frac{\pi}{\omega} \) with \( \omega \) being the largest interior angle of the polygonal domain \( \Omega \).
Proof. We first prove the case with only one line fracture $\gamma_1$ as shown in Figure 2(a). We extend $\gamma_1$ to $\gamma_1'$ which has two points of intersection with the boundary $\partial \Omega$, then $\Omega$ is partitioned into two open subdomains $\Omega^-$ and $\Omega^+$ (see Figure 2(b)). In $\Omega^+$, we extend the line fracture $\gamma_1$ to a closed $C^2$ curve $\tilde{\gamma}_1^+$ partitioning $\Omega$ into two subdomains $\Omega_0$ and $\Omega_1$ as shown in Figure 2(c), and extend $g_1$ on $\gamma_1$ to $\tilde{g}_1$ on $\tilde{\gamma}_1^+$ satisfying

$$
\tilde{g}_1 = \begin{cases} 
g_1 & \text{on } \gamma_1, \\
0 & \text{on } \tilde{\gamma}_1^+ \setminus \gamma_1. 
\end{cases}
$$

Then the transmission problem (3.1) is equivalent to the following problem

\begin{align}
-\Delta w &= 0 \quad \text{in } \Omega \setminus \tilde{\gamma}_1^+, \\
[w] &= 0 \quad \text{on } \tilde{\gamma}_1^+, \\
[\partial_n w] &= \tilde{g}_1 \quad \text{on } \tilde{\gamma}_1^+, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align}

(3.9)

Note that $\Omega^- \subset \Omega_0$, so by Lemma 3.2 and Lemma 3.4, if $g_1 \in H^{\beta_1}(\gamma_1)$, we have

$$
\|w\|_{H^{\min(1+\alpha,\beta_1+\frac{3}{2})}(\Omega^-)} \leq \|w\|_{H^{\min(1+\alpha,\beta_1+\frac{3}{2})}(\Omega_0)} \leq C\|\tilde{g}_1\|_{H^{\min(\beta_1,\frac{1}{2}-\epsilon)}(\tilde{\gamma}_1^+)} \\
= C\|g_1\|_{H^{\min(\beta_1,\frac{1}{2}-\epsilon)}(\gamma_1)} \leq C\|g_1\|_{H^{\beta_1}(\gamma_1)};
$$

(3.10)

if $g_1 \in H_0^{\beta_1}(\gamma_1)$ and $\beta_1 + \frac{1}{2} > 0$ is not an integer, we have

$$
\|w\|_{H^{\min(1+\alpha,\beta_1+\frac{3}{2})}(\Omega^-)} \leq \|w\|_{H^{\min(1+\alpha,\beta_1+\frac{3}{2})}(\Omega_0)} \leq C\|\tilde{g}_1\|_{H^{\beta_1}(\tilde{\gamma}_1^+)} = C\|g_1\|_{H^{\beta_1}(\gamma_1)},
$$

(3.11)
and if \( g_1 \in H_0^{\beta_1}(\gamma_1) \) and \( \beta_1 + \frac{1}{2} > 0 \) is an integer, we have
\[
\|w\|_{H_0^{\min(1+\alpha,\beta_1+\frac{1}{2},-\epsilon)}}(\Omega) \leq \|w\|_{H_0^{\min(1+\alpha,\beta_1+\frac{1}{2},-\epsilon)}}(\Omega_0) \leq C\|\tilde{g}_1\|_{H^{\beta_1-\epsilon}(\gamma_1^+)} \leq C\|g_1\|_{H^{\beta_1}(\gamma_1)},
\]
(3.12)

Similarly, we can also extend the line fracture \( \gamma_1 \) to a closed sufficiently smooth curve \( \tilde{\gamma}_1^- \) in \( \Omega^- \) as shown in Figure 2(d), and obtain similar estimates on \( \Omega^+ \). It can be observed that \( w \) is smooth in the neighborhood of \( \gamma_1' \setminus \gamma_1 \). Thus, it follows that if \( g_1 \in H^{\beta_1}(\gamma_1) \),
\[
\|w\|_{H_0^{\min(1+\alpha,\beta_1+\frac{1}{2},-\epsilon)}}(\Omega_0) \leq C\|g_1\|_{H^{\beta_1}(\gamma_1)};
\]
(3.13)

and if \( g_1 \in H_0^{\beta_1}(\gamma_1) \),
\[
\|w\|_{H_0^{\min(1+\alpha,\beta_1+\frac{1}{2},-\epsilon)}}(\Omega_0) \leq C\|g_1\|_{H^{\beta_1}(\gamma_1)} ;
\]
(3.14)

We can apply the regularity estimate (3.13) or (3.14) to multiple line fractures case and obtain the estimate (3.6) or (3.7) by using the superposition principle as discussed in Remark 2.5.

By Lemma 3.1 and (3.1b), we can extend the solution \( w \) of the transmission problem (3.1) from \( \Omega \setminus \bigcup_{l=1}^{N}\gamma_l \) to the whole domain \( \Omega \) by taking
\[
\bar{w} := \left\{ \begin{array}{ll}
w & \text{in } \Omega \setminus \bigcup_{l=1}^{N}\gamma_l, \\
w^+ (= w^-) & \text{on } \gamma_l, \ l = 1, \ldots, N. \end{array} \right.
\]
(3.15)

It is obvious that the extended solution
\[
\bar{w} \in C^0(\Omega) \cap H^1_0(\Omega),
\]
(3.16)

and
\[
\|\bar{w}\|_{H^1(\Omega)} = \|w\|_{V} = \|w\|_{V}.
\]
(3.17)

Therefore, (3.2) can be extended to the weak formulation
\[
a(\bar{w}, v) = \sum_{l=1}^{N} \int_{\gamma_l} g v d s, \quad v \in H^1_0(\Omega).
\]
(3.18)

**Theorem 3.6.** Let \( u \) be the solution of problem (1.1), and \( w \) be the solution of the transmission problem (3.1), then it follows
\[
u \equiv \bar{w} \quad \text{in } \Omega,
\]
where \( \bar{w} \) is the extended solution of \( w \) in \( \Omega \) by (3.15).

**Proof.** We set \( \tilde{u} = u - \bar{w} \) and subtract (3.18) from (2.1), we have that
\[
a(\tilde{u}, v) = 0, \quad v \in H^1_0(\Omega).
\]

Set \( v = \tilde{u} \in H^1_0(\Omega) \), we further have
\[
C\|\tilde{u}\|^2_{H^1(\Omega)} \leq a(\tilde{u}, \tilde{u}) = 0,
\]
which gives
\[
\tilde{u} = 0, \quad \text{in } H^1_0(\Omega).
\]
Thus, by Lemma 2.4 we have

\[ \| \nabla w \|_{L^2(\Omega)} \leq C \| \bar{w} \|_{H^1(\Omega)} = C \| u \|_{H^1(\Omega)} \leq C \sum_{l=1}^{N} \| g_l \|_{L^2(\gamma_l)}. \]  

(3.20)

Next, we consider closed region \( R_\epsilon \) enclosing all line fractures such that \( \Omega = R_\epsilon \cup (\Omega \setminus R_\epsilon) \), and denote \( n_\epsilon \) the unit outward norm vector of \( \Omega \setminus R_\epsilon \) (inward for \( R_\epsilon \)) on \( \partial R_\epsilon \). For \( \forall v \in C_0^\infty(\Omega) \),

\[ -\int_\Omega \Delta \bar{w} v dx = -\lim_{R_\epsilon \to \cup_{l=1}^{N} \gamma_l} \left( \int_{\Omega \setminus R_\epsilon} \Delta \bar{w} v dx + \int_{R_\epsilon} \Delta \bar{w} v dx \right) = -\lim_{R_\epsilon \to \cup_{l=1}^{N} \gamma_l} \int_{R_\epsilon} \Delta \bar{w} v dx, \]

where we have used (3.1a) in the second equality, namely, \( \Delta \bar{w} = 0 \) in \( \Omega \setminus R_\epsilon \).

Then for \( \forall v \in C_0^\infty(\Omega) \) we have,

\[ -\int_\Omega \Delta \tilde{u} v dx = -\int_\Omega \Delta u v dx + \int_\Omega \Delta \bar{w} v dx = -\int_\Omega \Delta u v dx + \lim_{R_\epsilon \to \cup_{l=1}^{N} \gamma_l} \int_{R_\epsilon} \Delta \bar{w} v dx. \]

Applying (1.1) to the first term and Green’s formula to the second term on the right hand side of the equation above, we have

\[ -\int_\Omega \Delta \tilde{u} v dx = \sum_{l=1}^{N} \int_{\gamma_l} g_l v ds + \lim_{R_\epsilon \to \cup_{l=1}^{N} \gamma_l} \left( \int_{\partial R_\epsilon} \partial_n w v ds + \int_{R_\epsilon} \nabla w \nabla v dx \right) \]

By (3.20) and the boundedness of \( \nabla v \), we have

\[ \left| \int_{R_\epsilon} \nabla w \nabla v dx \right| \leq \| \nabla w \|_{L^2(R_\epsilon)} \| \nabla v \|_{L^2(R_\epsilon)} \leq \| \nabla w \|_{L^2(\Omega)} \| \nabla v \|_{L^\infty(\Omega)} |R_\epsilon| \to 0, \]

as \( R_\epsilon \to \cup_{l=1}^{N} \gamma_l \).

It can be observed

\[ \int_{\partial R_\epsilon} \partial_n w v ds = \sum_{l=1}^{N} \int_{\gamma_l} [\partial_n w] v ds = \sum_{l=1}^{N} \int_{\gamma_l} g_l v ds, \]

as \( R_\epsilon \to \cup_{l=1}^{N} \gamma_l \). From the discussion above, we have

\[ -\int_\Omega \Delta \tilde{u} v dx = 0, \quad v \in C_0^\infty(\Omega). \]

Since \( v \) is arbitrary, so it follows that

\[ -\Delta \tilde{u} = 0 \quad \text{in} \quad \Omega, \]

which together with the boundary condition \( \tilde{u} = u - \bar{w} = 0 \) on \( \partial \Omega \) gives \( \tilde{u} \equiv 0 \) in \( \Omega \).

\[ \square \]

Theorem 3.6 indicates that the extension \( \bar{w} \) of the solution of the transmission problem (3.1) by (3.15) solves elliptic problem (1.1).

Corollary 3.1. For \( \forall \epsilon > 0 \), we have the following results,

(i) let \( u \) be the solution of problem (1.1), then \( u|_{\Omega \cup_{l=1}^{N} \gamma_l} \) solves the transmission problem (3.1);

(ii) if \( g_l \in H^{\beta_l}(\gamma_l), \ l = 1, \cdots, N \), it follows

\[ u \in H^{\frac{3}{2} - \epsilon}(\Omega) \cap H^{\min\{\alpha + 1, \beta + 3, 2 - \epsilon\}}(\Omega \setminus \cup_{l=1}^{N} \gamma_l). \]
Further, if \( g_l \in H^\beta_0(\gamma_l), l = 1, \cdots, N, \) it follows

\[
\begin{align*}
    u &\in H^{3-\epsilon}(\Omega) \cap H^{\min\{\alpha+1,\beta+\frac{3}{2}\}}(\Omega \setminus \bigcup_{l=1}^N \gamma_l), & \text{if } \beta + \frac{1}{2} > 0 \text{ is not an integer}, \\
    u &\in H^{3-\epsilon}(\Omega) \cap H^{\min\{\alpha+1,\beta+\frac{3}{2}-\epsilon\}}(\Omega \setminus \bigcup_{l=1}^N \gamma_l), & \text{if } \beta + \frac{1}{2} > 0 \text{ is an integer},
\end{align*}
\]

where \( \alpha, \beta \) are given in Theorem 3.5.

**Proof.** The proof follows from Lemma 2.4, Lemma 3.1, Theorem 3.5 and Theorem 3.6. \( \square \)

**Remark 3.7.** The conclusions in Theorem 3.5 and Corollary 3.1 still hold, if \( \gamma_l, l = 1, \cdots, N \) are sufficiently smooth curved line segments.

4. **Adaptive finite element method**

Let \( \mathcal{T} = \{T\}_{T \in \mathcal{T}} \) be a triangulation of \( \Omega \) with triangles. Denote the set of edges of \( \mathcal{T} \) by \( \mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_B \), where \( \mathcal{E}_I \) and \( \mathcal{E}_B \) represent the set of the interior edges and the boundary edges, respectively. For any triangle \( T \in \mathcal{T} \), we denote \( h_T \) the diameter of \( T \).

The Lagrange finite element space is defined by

\[
S(T) = \{v \in C^0(\Omega) \cap H^1_0(\Omega) : v|_T \in P_k(T), \ \forall \ T \in \mathcal{T}\},
\]

where \( P_k(T) \) is the space of polynomials with degree less than or equal to \( k \) on \( T \).

4.1. **Standard finite element method.** We suppose that the mesh \( \mathcal{T} \) consists of quasi-uniform triangles with mesh size \( h := \max h_T \). Based on the variational formulation (2.1) and (3.2), the standard finite element solution for problem (1.1) is to find \( u_h \in S(T) \) such that

\[
\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx = \sum_{l=1}^N \int_{\gamma_l} g_l(s)v_h(s) \, ds, \quad \forall \ v_h \in S(T). \tag{4.1}
\]

Because of the lack of regularity in the solution \( u \in H^{3-\epsilon}(\Omega) \) for \( \forall \epsilon > 0 \) (see Lemma 2.4), the standard error estimate [12] on general quasi-uniform meshes which allow the line fractures pass through the triangles yields only a suboptimal convergence rate,

\[
\|u - u_h\|_{H^1(\Omega)} \leq C h^{\frac{3}{2} - \epsilon}. \tag{4.2}
\]

If we further assume that the quasi-uniform mesh \( \mathcal{T} \) conforms to line fractures \( \gamma_l \). Namely, \( \gamma_l \) are the union of some edges in \( \mathcal{E}_I \) and do not cross with any triangles in \( \mathcal{T} \). By Corollary 3.1, the standard error estimate of the finite element approximations on conforming quasi-uniform meshes gives a better convergence rate compared with (4.2), if all \( g_l \in H^\beta_0(\gamma_l) \), it follows

\[
\|u - u_h\|_{H^1(\Omega)} \leq C h^{\min\{\alpha,\beta+\frac{1}{2}\} - \epsilon}, \tag{4.3}
\]

and if all \( g_l \in H^\beta_0(\gamma_l) \), it follows

\[
\begin{align*}
    \|u - u_h\|_{H^1(\Omega)} &\leq C h^{\min\{k,\alpha,\beta+\frac{1}{2}\}}, & \text{if } \beta + \frac{1}{2} > 0 \text{ is not an integer}, \\
    \|u - u_h\|_{H^1(\Omega)} &\leq C h^{\min\{k,\alpha,\beta+\frac{1}{2} - \epsilon\}}, & \text{if } \beta + \frac{1}{2} > 0 \text{ is an integer},
\end{align*}
\]

where \( \alpha, \beta \) are given in Theorem 3.5.
The singularities in the solution can severely slow down the convergence of the standard finite element method associated with the quasi-uniform meshes. To improve the convergence rate, we introduce an adaptive finite element method to approximate the solution of problem (1.1).

4.2. The adaptive finite element method. In the following, we first derive a residual-based error estimator and show its reliability and efficiency. Based on the derived error estimator and bisection mesh refinement method, we then propose an adaptive finite element algorithm.

To propose an efficient and reliable residual-based error estimator, one of choices is to regularize the source term such that the regularized source term belongs to \( L^2(\Omega) \) or \( L^p(\Omega) \) with \( 1 < p < \infty \) [23, 31]. Therefore, the residual-based a posteriori error estimator for the usual Poisson equation can be applied. Let the function \( g^r \in L^2(\Omega) \) be a regularized function of the source term \( \sum_{l=1}^{N} g_l \delta_{\gamma_l} \) in (1.1), then the classical residual-based a posteriori error estimator is given by

\[
\xi = \left( \sum_{T \in T} \xi_T^2(u_h) \right)^{\frac{1}{2}},
\]

where the local indicator satisfying

\[
\xi_T(u_h)^2 = h_T^2 \| \Delta u_h + g^r \|_{L^2(T)}^2 + \frac{1}{2} \sum_{e \in \partial T \cap E} h_{T(e)} \| [\partial_n u_h] \|_{L^2(e)}^2,
\]

where \([\partial_n u_h] \) denotes the jump of the normal derivatives of \( u_h \) on the interior edges of element \( T \).

The regularization technique introduced above is an effective approach to propose adaptive finite element algorithm. However, the corresponding adaptive finite element solution involves not only the discretization error but also the regularization error. When it applied to problem (1.1), it may lead to over-refinements on the meshes or low convergence rates for high order approximations.

For analysis convenience, we extend \( g_l \) from \( \gamma_l \) to \( E_l \) by defining

\[
f = \begin{cases} 
g_l, & e \in \gamma_l, \quad l = 1, \ldots, N, \\
0, & e \in E_l \setminus \bigcup_{l=1}^{N} \gamma_l. 
\end{cases}
\]

Let \( n \) be the outward unit normal derivative of triangle \( T \in \mathcal{T} \). By Corollary 3.1 we have \([\partial_n u] = g_l = f \) for \( e \in \gamma_l \), and \([\partial_n u] = 0 = f \) for \( e \in E_l \setminus \bigcup_{l=1}^{N} \gamma_l \), so \([\partial_n u] \) is also extended to \( E_l \) in the sense

\[
[\partial_n u]_e = f|_e, \quad e \in E_l.
\]

Motivated by the equivalence of the elliptic problem (1.1) and the transmission problem (3.1) in the domain excluding the line fractures, we propose the following residual-based a posteriori error estimator,

\[
\eta = \left( \sum_{T \in \mathcal{T}} \eta_T^2(u_h) \right)^{\frac{1}{2}},
\]

where the local indicator on \( T \in \mathcal{T} \) is defined by,

\[
\eta_T(u_h)^2 = h_T^2 \| \Delta u_h \|_{L^2(T)}^2 + \frac{1}{2} \sum_{e \in \partial T \cap E} h_{T(e)} \| f - [\partial_n u_h] \|_{L^2(e)}^2.
\]
Remark 4.1. If the equation in (1.1) has an additional source term \( q(x) \in L^2(\Omega) \), namely,
\[
-\Delta u = \sum_{l=1}^{N} g_l \delta_{\gamma_l} + q(x) \quad \text{in } \Omega,
\]
then the transmission problem (3.1a) will be modified as
\[
-\Delta w = q(x) \quad \text{in } \Omega \setminus \bigcup_{l=1}^{N} \gamma_l,
\]
and the local indicator (4.10) is therefore given by
\[
\eta_T(u_h)^2 = h_T^2 \| \Delta u_h + q(x) \|^2_{L^2(T)} + \frac{1}{2} \sum_{e \in \partial T \cap E_l} h_T \| f - [\partial_n u_h] \|^2_{L^2(e)}.
\]

Before we present the efficiency and reliability of the proposed a posteriori error estimator (4.9), we first prepare some necessary inequalities and estimates.

**Lemma 4.2 (Trace inequality [9]).** For any element \( T \in T \), \( \forall e \subset \partial T \), we have
\[
\| v \|_{L^2(e)} \leq Ch_T^{-1/2} (\| v \|_{L^2(T)} + h_T \| \nabla v \|_{L^2(T)}), \quad \forall v \in H^1(T).
\]

**Lemma 4.3 (Inverse inequality [9]).** For any element \( T \in T \) and \( v \in P_k(T) \), \( \forall e \subset \partial T \), we have
\[
\| \nabla^j v \|_{L^2(T)} \leq Ch_T^{-j} \| v \|_{L^2(T)}, \quad \forall 0 \leq j \leq k.
\]

**Lemma 4.4 (Interpolant error estimate [39]).** For any \( v \in H^l(T) \), \( l \geq 1 \), it follows
\[
\| v - \pi v \|_{H^m(T)} \leq Ch^{l-m} \| v \|_{H^l(T)},
\]
where \( m = 0, 1 \) and \( \pi v \in S(T) \) represents the nodal interpolant of \( v \).

In the following analysis, we make use of the equivalence of problem (1.1) to the transmission problem (3.1) as discussed in Section 3, and we pay special attention to handle the flux jumps (4.8) on line fractures \( \gamma_l \) in the following reliability analysis.

**Theorem 4.5 (Reliability).** Assume that \( u \) and \( u_h \) are the solution of (1.1) and (4.1), respectively. Then the residual-based a posteriori error estimator \( \eta \) satisfies the global bound,
\[
\| \nabla (u - u_h) \|_{L^2(\Omega)} \leq C \eta(u_h).
\]

**Proof.** Let \( e_u = u - u_h \), we have
\[
\| \nabla e_u \|_{L^2(\Omega)}^2 = \int_{\Omega} \nabla e_u \cdot \nabla e_u \, dx = \int_{\Omega} \nabla e_u \cdot \nabla (e_u - \pi e_u) \, dx,
\]
where we have used the Galerkin orthogonality to subtract an interpolant \( \pi e_u \in S(T) \) to \( e_u \). Note that by Corollary 3.1 we have
\[
\Delta e_u = \Delta u - \Delta u_h = -\Delta u_h, \quad \text{in } \Omega \setminus \bigcup_{l=1}^{N} \gamma_l.
\]
Thus splitting (4.13) into a sum over the elements and using Green’s formula, we have
\[ \sum_{T \in T} \int_T \nabla e_u \cdot \nabla (e_u - \pi e_u) \, dx = \sum_{T \in T} \int_T -\Delta e_u (e_u - \pi e_u) \, dx + \int_{\partial T} \mathbf{n} \cdot \nabla e_u (e_u - \pi e_u) \, ds \]
\[ = \sum_{T \in T} \left( \int_T \Delta u_h (e_u - \pi e_u) \, dx + \int_{\partial T \cap \mathcal{E}_i} \mathbf{n} \cdot \nabla e_u (e_u - \pi e_u) \, ds \right), \]
where we have used \( \pi e_u = e_u = 0 \) on \( \partial \Omega \). Note that \( e_u \) is continuous by Corollary 2.1 and the continuity of the finite element solution, so we have \( (e_u^+ - \pi e_u^+) |_e = (e_u^- - \pi e_u^-) |_e \) for any \( e = \partial T_+ \cap \partial T_- \in \mathcal{E}_i \). Thus, it follows
\[ \int_{e \cap \partial T^+} \mathbf{n} \cdot \nabla e_u (e_u - \pi e_u) \, ds + \int_{e \cap \partial T^-} \mathbf{n} \cdot \nabla e_u (e_u - \pi e_u) \, ds \]
\[ = \int_e \mathbf{n}^+ \cdot \nabla e_u^+ (e_u^+ - \pi e_u^+) + \mathbf{n}^- \cdot \nabla e_u^- (e_u^- - \pi e_u^-) \, ds \]
\[ = \int_e \left( (\mathbf{n}^+ \cdot \nabla u^+ + \mathbf{n}^- \cdot \nabla u^-) - (\mathbf{n}^+ \cdot \nabla u_h^+ + \mathbf{n}^- \cdot \nabla u_h^-) \right) (e_u - \pi e_u) \, ds \]
\[ = \int_e [\partial_n u_h] (e_u - \pi e_u) \, ds - \int_e [\partial_n u_h] (e_u - \pi e_u) \, ds. \]

This, together with (4.8), implies that
\[ \sum_{T \in T} \int_{\partial T \cap \mathcal{E}_i} \mathbf{n} \cdot \nabla e_u (e_u - \pi e_u) \, ds = \sum_{e \in \mathcal{E}_i} \left( \int_e [\partial_n u](e_u - \pi e_u) \, ds - \int_e [\partial_n u_h](e_u - \pi e_u) \, ds \right) \]
\[ = \sum_{e \in \mathcal{E}_i} \int_e (f - [\partial_n u_h]) (e_u - \pi e_u) \, ds. \]

Returning to the sum over the elements with simply distributing half of \( f - [\partial_n u_h] \) on \( T_+ \) and the remaining half on \( T_- \), we have
\[ \| \nabla e_u \|_{L^2(\Omega)}^2 = \sum_{T \in T} \left( \int_T \Delta u_h (e_u - \pi e_u) \, dx + \frac{1}{2} \sum_{e \in \partial T \cap \mathcal{E}_i} \int_e (f - [\partial_n u_h]) (e_u - \pi e_u) \, ds \right). \] (4.15)

Next, we estimate the terms on the right hand side of (4.15) one by one.

Using Cauchy-Schwarz inequality and Lemma 4.4, we have
\[ \int_T \Delta u_h (e_u - \pi e_u) \, dx \leq \| \Delta u_h \|_{L^2(T)} \| e_u - \pi e_u \|_{L^2(T)} \leq C h_T \| \Delta u_h \|_{L^2(T)} \| \nabla e_u \|_{L^2(T)}. \] (4.16)

Then, using Cauchy-Schwarz inequality, Lemma 4.2 and Lemma 4.4, we have
\[ \int_e (f - [\partial_n u_h]) (e_u - \pi e_u) \, ds \leq \| f - [\partial_n u_h] \|_{L^2(e)} \| e_u - \pi e_u \|_{L^2(e)} \]
\[ \leq C \left( h_T^{-1} \| e_u - \pi e_u \|_{L^2(T)}^2 + h_T \| \nabla (e_u - \pi e_u) \|_{L^2(T)}^2 \right)^{1/2} \| f - [\partial_n u_h] \|_{L^2(e)} \]
\[ \leq C h_T^{1/2} \| f - [\partial_n u_h] \|_{L^2(e)} \| \nabla e_u \|_{L^2(T)} \] (4.17)

The estimate (4.12) now follows from (4.15)-(4.17). \( \square \)
Let \( \overline{f} \in P_k(e) \) be the \( L^2 \)-projection of \( f \). We define the oscillation on \( e \in E_T \) by
\[
osc(e)^2 = h_e \| f - \overline{f} \|^2_{L^2(e)},
\]
where \( h_e \) is the length of \( e \). Let \( e = \partial T_+ \cap \partial T_- \) with \( T_+ \) and \( T_- \) being two adjacent triangles, and we set \( \omega_e = T_+ \cup T_- \), then for any \( T \in \omega_e \) there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 h_T \leq h_e \leq C_2 h_T.
\]

For a triangle \( T \in T \) with vertices \( x_1, x_2, x_3 \), we denote \( (\lambda_{x_1}, \lambda_{x_2}, \lambda_{x_3}) \) the barycentric coordinates on \( T \). We define a bubble function \( b_T \) in \( T \) by
\[
b_T = 27\lambda_{x_1}\lambda_{x_2}\lambda_{x_3}.
\]
For an edge \( e = x_i x_j \in \partial T \subset E_T \), we define an edge bubble function \( b_e \) in \( e \) by
\[
b_e = 4\lambda_{x_i}\lambda_{x_j}.
\]

For the bubble functions \( b_T \) and \( b_e \), we have the following results.

**Lemma 4.6**. For the element bubble function \( b_T \) in \( (4.18) \), it has the following properties,
\[
0 \leq b_T(x) \leq 1, \quad \forall x \in T, \quad \text{and} \quad b_T(x) = 0, \quad \forall x \in \partial T,
\]
Moreover, for any \( v \in P_k \), it follows
\[
\| v \|_{L^2(T)} \leq C \| b_T^{1/2} v \|_{L^2(T)}.
\]

**Lemma 4.7**. For \( e = \partial T_+ \cap \partial T_- \), the edge bubble function \( b_e \) defined by \( (4.19) \) has the following properties,
\[
0 \leq b_e(x) \leq 1, \quad \forall x \in \omega_e, \quad \text{and} \quad b_e(x) = 0, \quad \forall x \in \partial \omega_e \setminus e,
\]
where \( \partial \omega_e = \partial T_+ \cup \partial T_- \). Moreover, for any \( v \in P_k \), it follows
\[
\| v \|_{L^2(e)} \leq C \| b_e^{1/2} v \|_{L^2(e)},
\]
\[
\| \nabla (b_e v) \|_{L^2(\omega_e)} \leq C h_e^{-1/2} \| v \|_{L^2(e)},
\]
\[
\| b_e v \|_{L^2(\omega_e)} \leq C h_e^{1/2} \| v \|_{L^2(e)}.
\]

**Theorem 4.8** (Efficiency). For the local indicator \( \eta_T \) defined in \( (4.10) \), it follows
\[
\eta_T(u_h) \leq C \left( \| \nabla e_u \|_{L^2(\omega_T)} + osc(\partial T) \right), \quad \forall T \in T,
\]
where \( w_T = \bigcup_{e \in \partial T} w_e \), and
\[
osc(\partial T)^2 = \sum_{e \in \partial T} osc(e)^2.
\]

**Proof.** Using Green’s formula, \( (4.14) \) and \( (4.20) \), we have
\[
\int_T \nabla e_u \cdot \nabla (\Delta u_h b_T) \, dx = -\int_T \Delta e_u \Delta u_h b_T \, dx + \int_{\partial T} \nabla e_u \cdot \mathbf{n} \Delta u_h b_T \, ds = \int_T \Delta u_h \Delta u_h b_T \, dx,
\]
\[
(4.27)
\]
Since \( \Delta u_h \) is a piecewise polynomial over \( T \), according to \( (4.21) \) we have
\[
\| \Delta u_h \|^2_{L^2(T)} \leq C \| \Delta u_h b_T^{1/2} \|^2_{L^2(T)}.
\]
Using the Cauchy-Schwarz inequality, Lemma 4.3 and (4.20), it follows that
\[
\|\Delta u_h\|_{L^2(T)} \leq C \int_T \nabla e_u \nabla (\Delta u_h b_T) \, dx \leq C \|\nabla e_u\|_{L^2(T)} \|\nabla (\Delta u_h b_T)\|_{L^2(T)}
\]
\[
\leq Ch_T^{-1} \|\nabla e_u\|_{L^2(T)} \|\Delta u_h b_T\|_{L^2(T)} \leq Ch_T^{-1} \|\nabla e_u\|_{L^2(T)} \|\Delta u_h\|_{L^2(T)},
\]
which gives
\[
h_T \|\Delta u_h\|_{L^2(T)} \leq C \|\nabla e_u\|_{L^2(T)}. \tag{4.28}
\]

We now extend \( \overline{f} - [\partial_n u_h] \) from edge \( e \) to \( w_e \) by taking constants along the normal on \( e \). The resulting extension \( E(\overline{f} - [\partial_n u_h]) \) is a piecewise polynomial in \( \omega_e \), then according to (4.24)-(4.25), we have
\[
\|\nabla (E(\overline{f} - [\partial_n u_h]) b_e)\|_{L^2(\omega_e)} \leq C h_e^{-\frac{1}{2}} \|\overline{f} - [\partial_n u_h]\|_{L^2(e)}, \tag{4.29}
\]
\[
\|E(\overline{f} - [\partial_n u_h]) b_e\|_{L^2(\omega_e)} \leq C h_e^{\frac{1}{2}} \|\overline{f} - [\partial_n u_h]\|_{L^2(e)}. \tag{4.30}
\]

Using arguments similar to those leading to (4.27), it follows
\[
\int_{\omega_e} \nabla e_u \nabla (E(\overline{f} - [\partial_n u_h]) b_e) \, dx = \sum_{T \in \omega_e} \int_T \nabla e_u \nabla (E(\overline{f} - [\partial_n u_h]) b_e) \, dx
\]
\[
= \sum_{T \in \omega_e} \left( \int_T -\Delta e_u E(\overline{f} - [\partial_n u_h]) b_e \, dx + \int_{\partial T} \nabla e_u \cdot n \, E(\overline{f} - [\partial_n u_h]) b_e \, ds \right)
\]
\[
= \sum_{T \in \omega_e} \left( \int_T \Delta u_h E(\overline{f} - [\partial_n u_h]) b_e \, dx + \int_{\partial T} \nabla e_u \cdot n \, E(\overline{f} - [\partial_n u_h]) b_e \, ds \right).
\]

Note that \( \overline{f} - [\partial_n u_h] \) and \( b_e \) are continuous on \( e \in \mathcal{E}_T \), and \( b_e = 0 \) on \( (\cup_{T \in \omega_e} \partial T) \setminus e \), so we have
\[
\sum_{T \in \omega_e} \int_{\partial T} \nabla e_u \cdot n \, E(\overline{f} - [\partial_n u_h]) b_e \, ds
\]
\[
= \int_e \left( (n^+ \cdot \nabla u^+ + n^- \cdot \nabla u^-) - (n^+ \cdot \nabla u_h^+ + n^- \cdot \nabla u_h^-) \right) (\overline{f} - [\partial_n u_h]) b_e \, ds
\]
\[
= \int_e [\partial_n u] (\overline{f} - [\partial_n u_h]) b_e \, ds - \int_e [\partial_n u_h] (\overline{f} - [\partial_n u_h]) b_e \, ds
\]
\[
= \int_e (f - [\partial_n u_h]) (\overline{f} - [\partial_n u_h]) b_e \, ds,
\]
where we used (4.8) in the last equality. Therefore, we get
\[
\int_{\omega_e} \nabla e_u \nabla (E(\overline{f} - [\partial_n u_h]) b_e) \, dx = \int_{\omega_e} \Delta u_h E(\overline{f} - [\partial_n u_h]) b_e \, dx
\]
\[
+ \int_e (f - [\partial_n u_h]) (\overline{f} - [\partial_n u_h]) b_e \, ds
\]
\[
= \int_{\omega_e} \Delta u_h E(\overline{f} - [\partial_n u_h]) b_e \, dx + \int_e (\overline{f} - [\partial_n u_h])^2 b_e \, ds
\]
\[
+ \int_e (f - \overline{f}) (\overline{f} - [\partial_n u_h]) b_e \, ds.
\]
It follows from (4.23), we obtain
\[ \| \bar{f} - [\partial_n u_h] \|_{L^2(e)}^2 \leq C \| (\bar{f} - [\partial_n u_h])h^{1/2}_e \|_{L^2(e)}^2. \]

Using Cauchy-Schwarz inequality and (4.29)-(4.30), (4.22), we have
\[ \| \bar{f} - [\partial_n u_h] \|_{L^2(e)}^2 \leq C \left( \int_{\omega_e} \nabla e_u \nabla(E(\bar{f} - [\partial_n u_h])b_e) \,dx - \int_{\omega_e} \Delta u_h E(\bar{f} - [\partial_n u_h])b_e \,dx \right. \]
\[ \left. - \int_{\partial e} (f - \bar{f})(\bar{f} - [\partial_n u_h])b_e \,ds \right) \]
\[ \leq C \left( \| \nabla e_u \|_{L^2(\omega_e)} \| \nabla(E(\bar{f} - [\partial_n u_h])b_e) \|_{L^2(\omega_e)} \right. \]
\[ + \| \Delta u_h \|_{L^2(\omega_e)} \| E([\bar{f} - [\partial_n u_h])b_e \|_{L^2(\omega_e)} \right. \]
\[ + \| (f - \bar{f})b_e \|_{L^2(\omega)} \| \bar{f} - [\partial_n u_h] \|_{L^2(e)} \]
\[ \leq C \left( h^{-1/2} \| \nabla e_u \|_{L^2(\omega_e)} \| \bar{f} - [\partial_n u_h] \|_{L^2(e)} \right. \]
\[ + h^{1/2} \| \Delta u_h \|_{L^2(\omega_e)} \| \bar{f} - [\partial_n u_h] \|_{L^2(e)} \right. \]
\[ + \| (f - \bar{f})b_e \|_{L^2(\omega)} \| \bar{f} - [\partial_n u_h] \|_{L^2(e)} \]
\[ \leq C h^{-1/2} \| \bar{f} - [\partial_n u_h] \|_{L^2(e)} \left( \| \nabla e_u \|_{L^2(\omega_e)} + h \| \Delta u_h \|_{L^2(\omega_e)} + osc(e) \right), \]
which gives
\[ h^{1/2}_e \| \bar{f} - [\partial_n u_h] \|_{L^2(e)} \leq C \left( \| \nabla e_u \|_{L^2(\omega_e)} + h \| \Delta u_h \|_{L^2(\omega_e)} + osc(e) \right). \] (4.31)

Together with the triangle inequality, (4.28) and (4.31), we obtain the estimation
\[ h^{1/2}_e \| f - [\partial_n u_h] \|_{L^2(e)} \leq C \left( \| \nabla e_u \|_{L^2(\omega_e)} + osc(e) \right). \] (4.32)

The required estimation now follows form (4.28) and (4.32). □

The corresponding algorithm is summarized as follows.

**Algorithm 1** Adaptive finite element algorithm for the elliptic equation.

1: Input: an initial mesh $T^0$; a constant $0 < \theta < 1$; the maximum number of mesh refinements $n$.

2: Output: the numerical solution $u_h^n$; a new refined mesh $T^n$.

3: for $i = 0$ to $n$ do
  Solve the discrete equation (4.1) for the finite element solution $u_h^i$ on $T^i$;
  Computing the local error estimation $\eta_T^i(u_h^i)$ and the total error estimation $\eta^i(u_h^i)$ by (4.10) and (4.9);
  if $i < n$ then
    Mark a subset $\tilde{T}^i \subset T^i$ of elements to refined such that,
    \[ (\sum_{T \in \tilde{T}^i} \eta_T^i(u_h^i)^2)^{1/2} \geq \theta \eta^i(u_h^i); \]
    Refine each element $T \in \tilde{T}^i$ by longest edge bisection to obtain a new mesh $T^{i+1}$.
  end if
end for
Table 1. Case 1-6 in Example 5.1

| Case number | \( r_0 \) | \( r_1 \) | \( g_1 \in \) | \( u \in \) |
|-------------|-----------|-----------|-------------|-------------|
| Case 1      | \(-\frac{1}{4} + 10^{-3}\) | 1         | \( H^\frac{1}{2}(\gamma_1) \) | \( H^\frac{3}{2} - \epsilon(\Omega) \cap H^\frac{3}{2}(\Omega \setminus \gamma_1) \) |
| Case 2      | \( \frac{1}{4} + 10^{-3}\)  | 1         | \( H^\frac{1}{2}(\gamma_1) \) | \( H^\frac{3}{2} - \epsilon(\Omega) \cap H^\frac{3}{2}(\Omega \setminus \gamma_1) \) |
| Case 3      | 0         | 1         | \( C^\infty(\gamma_1) \)  | \( H^\frac{3}{2} - \epsilon(\Omega) \cap H^\frac{3}{2}(\Omega \setminus \gamma_1) \) |
| Case 4      | \( \frac{1}{4} + 10^{-3}\)  | 0         | \( H^\frac{1}{2}(\gamma_1) \) | \( H^\frac{3}{2} - \epsilon(\Omega) \cap H^\frac{3}{2}(\Omega \setminus \gamma_1) \) |
| Case 5      | \( \frac{1}{2} + 10^{-3}\)  | 0         | \( H^0(\gamma_1) \)  | \( H^\frac{3}{2} - \epsilon(\Omega) \cap H^\frac{3}{2}(\Omega \setminus \gamma_1) \) |
| Case 6      | 1 + 10^{-3}| 0         | \( H^0(\gamma_1) \)  | \( H^\frac{3}{2} - \epsilon(\Omega) \cap H^\frac{3}{2}(\Omega \setminus \gamma_1) \) |

Figure 3. Example 5.1: the initial meshes.

5. Numerical examples

5.1. The standard finite element method. In this subsection, we present numerical examples to verify the convergence rate of the standard finite element method solving equation (1.1). The quasi-uniform meshes are considered in this subsection, that is, each triangle is divided into four equal triangles in each mesh refinement. Since the solution \( u \) is unknown, we use the following numerical convergence rate

\[
\mathcal{R} = \log_2 \frac{|u_h^j - u_h^{j-1}|_{H^1(\Omega)}}{|u_h^{j+1} - u_h^j|_{H^1(\Omega)}},
\]

(5.1)

where \( u_h^j \) is the finite element solution on the mesh \( T^j \) obtained after \( j \) refinements of the initial triangulation \( T^0 \).

Example 5.1. In this example, we test the convergence rates of the finite element solutions on quasi-uniform meshes. We consider problem (1.1) in a square domain \( \Omega = (0,1)^2 \) with one line fracture \( \gamma_1 = Q_1Q_2 \) for \( Q_1 = (0.25,0.5) \) and \( Q_2 = (0.75,0.5) \). We take the function \( g_1 = ((x - 0.25)(0.75 - x))^{r_0} + r_1 \) on \( \gamma_1 \). For different parameters \( r_0, r_1 \) in Case 1-6 listed in Table 1, we show the smoothness of the corresponding function \( g_1 \), and the regularity for the solution \( u \) of problem (1.1) followed by Corollary 3.1.
Test 1. We take the initial mesh as the Union-Jack mesh and the line fracture $\gamma_1$ pass through the triangles in the mesh as shown in Figure 3(a). The convergence rates of the finite element solutions based on $P_1$, $P_2$ polynomials are shown in Table 2 and we find that suboptimal convergence rates $R \approx 0.5$ are obtained for Case 1–6, which is due to $u \in H^{2,\epsilon}(\Omega)$, $\forall \epsilon > 0$ regardless of the smoothness of $g_1$ as indicated by Lemma 2.4. The contours of the finite element solution for Case 1–6 are shown in Figure 4.

Test 2. We take the initial mesh as Figure 3(b), whose elements conforming to the line fracture $\gamma_1$. The convergence rates of the finite element solutions based on $P_1$, $P_2$ polynomials are shown in Table 3. From the results, we can find that the convergence rates $0.5 < R < 2$ depends on the smoothness of the function $g_1$ and the degree of the polynomials. The results in Table 3 satisfy the theoretical expectations shown in Corollary 3.1.

From the two tests above, we confirm that the finite element solution on the meshes conforming to the line fracture shows better convergence rates than that on meshes with the line fracture passing through the triangles. So we will always consider the initial meshes that conform to line fractures for the remaining examples.

5.2. Adaptive finite element method. The parameter $\theta$ in Algorithm 1 is taken as $\theta = 0.25$ in following examples. On adaptive meshes, the convergence rate of the a posteriori error estimator
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Table 2. $H^1$ convergence history of finite element solutions in Example 5.1 Test 1 on Union-Jack meshes.

|     | $P_1$       | $P_2$       |
|-----|-------------|-------------|
|     | 6 7 8 9     | 4 5 6 7     |
| Case1 | 0.477 0.485 0.490 0.493 | 0.484 0.489 0.493 0.495 |
| Case2 | 0.475 0.486 0.492 0.496 | 0.493 0.497 0.498 0.499 |
| Case3 | 0.485 0.491 0.495 0.497 | 0.495 0.498 0.499 0.499 |
| Case4 | 0.476 0.487 0.493 0.496 | 0.499 0.500 0.500 0.500 |
| Case5 | 0.476 0.487 0.493 0.497 | 0.503 0.501 0.500 0.500 |
| Case6 | 0.474 0.487 0.493 0.497 | 0.505 0.501 0.500 0.500 |

Table 3. $H^1$ convergence history of finite element solutions in Example 5.1 Test 2 on conforming quasi-uniform meshes.

|     | $P_1$       | $P_2$       |
|-----|-------------|-------------|
|     | 6 7 8 9     | 5 6 7 8     |
| Case1 | 0.786 0.786 0.785 0.783 | 0.792 0.786 0.781 0.777 |
| Case2 | 0.927 0.937 0.945 0.951 | 1.045 1.039 1.033 1.028 |
| Case3 | 0.905 0.916 0.925 0.932 | 1.000 1.000 1.000 1.000 |
| Case4 | 0.969 0.979 0.986 0.990 | 1.253 1.252 1.251 1.251 |
| Case5 | 0.988 0.994 0.997 0.999 | 1.500 1.501 1.501 1.501 |
| Case6 | 0.996 0.999 1.000 1.000 | 1.865 1.886 1.902 1.914 |

ξ in (4.5) or η in (4.9) for $P_k$ polynomials is called quasi-optimal if

$$\xi \approx N^{-0.5k}, \quad \text{or} \quad \eta \approx N^{-0.5k}.$$

Here and in what follows, we abuse the notation $N$ to represent the total number of degrees of freedom.

**Example 5.2.** We apply the AFEM to the Example 5.1 to test the performance of the proposed a posteriori error estimator (4.9) and the corresponding Algorithm 1. We take the mesh in Figure 3(b) as the initial mesh. The convergence rates of the error estimator η based on $P_1$ and $P_2$ polynomials are shown Figure 6. From the results, we find that the convergence rates of η are quasi-optimal. The contours of the AFEM approximations for different cases are shown in Figure 5 from which we can find that these solutions are almost identical to these in Example 5.1 Test 1.

For Case 1–6, the function $g_1$ is sufficiently smooth on $\gamma_1$ except near the endpoints $Q_1$ and $Q_2$ of the line fracture $\gamma_1$, so the solution is more singular near these two endpoints compared with any other regions in the domain. Figure 7 and Figure 8 show the adaptive meshes of $P_1$, $P_2$ approximations, respectively. We can see clearly that the error estimator guide the mesh refinements densely around the endpoints $Q_1$ and $Q_2$. We also find that the more regular the solution is, the less dense the mesh concentrates at the endpoints $Q_1$ and $Q_2$. Here, Case 3 is an example in [28].
solved by the graded finite element method, which showed optimal convergence rates with mesh refinements concentrating at the singular points \( Q_1 \) and \( Q_2 \) as well.

**Example 5.3.** We take this example from [23]. More specifically, we consider problem (1.1) on an L-shaped domain \( \Omega = (-1, 1)^2 \setminus [0,1)^2 \) and take the line fractures \( \bigcup_{l=1}^6 \gamma_l = \partial \Omega_1 \) with \( \Omega_1 = (-0.8,-0.2)^2 \setminus [-0.5,-0.2]^2 \) as shown in Figure 9(a). The function \( g_l = 5 \) on \( \gamma_l, l = 1, \cdots, 6 \). We apply the AFEMs based on the residual-based a posteriori error estimators \( \xi \) in (4.5) and \( \eta \) in
Figure 7. Example 5.2: adaptive meshes based on $P_1$ polynomials.

(4.9) to solve this problem, respectively. Both AFEMs take the mesh in Figure 8(a) as their initial mesh.

Test 1. We first consider the AFEM based on the residual-based a posteriori error estimator $\xi$ in (4.5). For simplicity of presentation, we denote $\gamma = \bigcup_{l=1}^{6} \gamma_l$, and $g|_{\gamma_l} = g_l, l = 1, \cdots, 6$. Instead of directly discretizing (1.1), one discretize its regularized problem, which is to replace the line Dirac source term $\sum_{l=1}^{N} g_l \delta_{\gamma_l}$ by its regularized data [23],

$$g^r(x) = \int_{\gamma} g(y) \delta^r(y-x) \, dy \in L^2(\Omega).$$

Here, the line Dirac approximation $\delta^r$ of the 2-dimensional Dirac distribution $\delta$ is defined by

$$\delta^r(x) = \frac{1}{r^2} \psi \left( \frac{x}{r} \right),$$

satisfying

$$\lim_{r \to 0} \delta^r(x) = \lim_{r \to 0} \frac{1}{r^2} \psi \left( \frac{x}{r} \right) = \delta(x),$$
where $r$ is the regularization parameter depending on the local mesh size, and $\psi(x)$ is the Dirac approximation [24, 37, 23]. Here, we take $r = 0.05$, and 

$$
\psi(x) = \frac{1}{4} \prod_{i=1}^{2} \chi_{[-1,1]}(x_i),
$$

in which $\chi_{[-1,1]}(x_i)$ is the characteristic function. The contour of the finite element solution based on $P_1$ polynomials is shown in Figure 9(b).

**Test 2.** We then consider the AFEM based on the residual-based a posteriori error estimator $\eta$ in (4.9), namely, the Algorithm 1 for problem (1.1). The contour of the finite element solution based on $P_1$ polynomials is shown in Figure 9(c), which is comparable to the contour in Test 1 as shown in Figure 9(b).

Since $g_l \in C^\infty$ are sufficiently smooth on line fractures $\gamma_l$, so the solution is more singular at the endpoints of line fractures $\gamma_l$ and the reentrant corner of the domain. The adaptive meshes from Test 1 and Test 2 based on $P_1$ polynomials are shown in Figure 10(a) and Figure 10(b), respectively. From the results, we find that both meshes are densely refined at the endpoints of the line fractures $\gamma_l$ and the reentrant corner of the domain, but the mesh from Test 1 is also densely refined on the whole line fractures $\gamma_l$, $l = 1, \cdots, 6$. Similar adaptive meshes can also be found for Test 1 and Test 2 based on $P_2$ polynomials as shown in Figure 11(a)-(b). These results imply that the error
estimator $\eta$ in (4.9) guides the mesh refinements effectively by only densely refining the triangles around the endpoints of the line fractures, where the solution is more singular.

The convergence rates of the error estimator $\xi$ and $\eta$ based on $P_1$ polynomials are shown in Figure 10(c). We can find that the error estimators from both Test 1 and Test 2 are quasi-optimal with $\xi \approx N^{-0.5}$ and $\eta \approx N^{-0.5}$. The convergence rates based on $P_2$ polynomials are shown in Figure 11(c). From the results, we can find that the error estimator $\eta \approx N^{-1}$ for Test 2 is quasi-optimal, but the error estimator $\xi(\approx N^{-0.5})$ for Test 1 does not achieve the quasi-optimal rate even with more dense refined meshes.

**Example 5.4.** In this example, we first introduce four intersecting line fractures $\gamma_l = QQ_l$, $l = 1, \cdots, 4$, where $Q(0.5,0.5)$, $Q_1(0.25,0.5)$, $Q_2(0.75,0.5)$, $Q_3(0.5,0.25)$ and $Q_4(0.5,0.75)$. Here, we consider three types of geometries of $\Omega$. Geometry 1 consists of two line fractures $\gamma_2$ and $\gamma_4$; Geometry 2 consists of three line fractures $\gamma_2$, $\gamma_3$ and $\gamma_4$; Geometry 3 consists of all line fractures $\gamma_l$, $l = 1, \cdots, 4$. The initial meshes of Geometry 1–3 are shown in Figure 13. The functions $g_l$ on
(a) Test 1: adaptive mesh based on regularization
(b) Test 2: adaptive mesh based on Algorithm 1
(c) Error estimators

Figure 11. Example 5.3: Adaptive meshes and error estimators based on $P_2$ polynomials.

Figure 12. Example 5.4: AFEM solutions based on $P_1$ polynomials.

Each line fracture $\gamma_l$ are taken as the following,

$$g_1 = -g_2 = -g_3 = g_4 = -1.$$

The history of the error estimators are reported in Figure 14 which shows that the convergence rates of the error estimators are quasi-optimal for all the three cases. Figure 15-16 and Figure 12 show the corresponding adaptive mesh refinements and the numerical solutions, respectively. We can see clearly that the error estimator successfully guide the mesh refinement around the singular points $Q_i$, where the solution shows singularity.

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Figure 13. Example 5.4 initial meshes.

Figure 14. Example 5.4 error estimators.

Figure 15. Example 5.4 adaptive meshes based on \( P_1 \) polynomials.

References

[1] S. Adjerid, I. Babuska, R. Guo and T. Lin. An enriched immersed finite element method for interface problems with nonhomogeneous jump conditions. arXiv preprint, [arXiv:2008.11877] 2020.
Figure 16. Example 5.4 adaptive meshes based on $P_2$ polynomials.

[2] M. Ainsworth and J.T. Oden. A posteriori error estimation in finite element analysis. *Wiley Interscience, New York*, 2000.

[3] T. Apel. Anisotropic finite elements: local estimates and applications. *Advances in Numerical Mathematics*. B. G. Teubner, Stuttgart, 1999.

[4] R. Araya, E. Behrens and R. Rodríguez. A posteriori error estimates for elliptic problems with Dirac delta source terms. *Numerische Mathematik*, 105:193–216, 2006.

[5] S. Ariche, C. De Coster and S. Nicaise. Regularity of solutions of elliptic or parabolic problems with Dirac measures as data. *SeMA Journal*, 73:379–426, 2016.

[6] S. Alinhac, P. Gérard, S. S. Wilson. Pseudo-differential operators and the Nash-Moser theorem. *Stud. Math.*, 82, AMS, Providence, RI, 2007.

[7] I. Babuška. Error-bounds for finite element method. *Numerische Mathematik*, 16:322–333, 1971.

[8] P. Binev, W. Dahmen and R. DeVore. Adaptive Finite Element Methods with convergence rates. *Numerische Mathematik*, 97:219-268, 2004.

[9] S. Brenner and L. Scott. The mathematical theory of finite element methods. *Volume 15 of Texts in Applied Mathematics*, 3rd edn. Springer, New York, 2008.

[10] J. H. Bramble and J. T. King. A finite element method for interface problems in domains with smooth boundaries and interfaces. *Advances in Computational Mathematics*, 6(1):109–138, 1996.

[11] Z. Chen and J. Zou. Finite element methods and their convergence for elliptic and parabolic interface problems. *Numerische Mathematik*, 79(2):175–202, 1998.

[12] Philippe G. Ciarlet. The Finite Element Method for Elliptic Problems. *Université Pierre et Marie Curie, Paris, France*, 1974.

[13] E. Casas. $L^2$ estimates for the finite element method for the Dirichlet problem with singular data. *Numerische Mathematik*, 47:627–632, 1985.

[14] C. D’Angelo. Finite element approximation of elliptic problems with Dirac measure terms in weighted spaces: applications to one- and three-dimensional coupled problems. *SIAM J. Numer. Anal.*, 50(1):194–215, 2012.

[15] C. D’Angelo and A. Quarteroni. On the coupling of 1D and 3D diffusion-reaction equations. Application to tissue perfusion problems. *Mathematical Models and Methods in Applied Sciences*, 18(8):1481–1504, 2008.

[16] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[17] Z. Ding. A proof of the trace theorem of Sobolev spaces on Lipschitz domains. *Proceedings of the American Mathematical Society*, 124(2):591–600, 1996.

[18] W. Dörfler. A convergent adaptive algorithm for Poisson equation. *SIAM Journal on Numerical Analysis*, 33(3):1106-1124, 1996.
[19] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[20] P. Grisvard. *Singularities in Boundary Value Problems*, volume 22 of *Research Notes in Applied Mathematics*. Springer-Verlag, New York, 1992.

[21] W. Gong, G. Wang and N. Yan. Approximations of elliptic optimal control problems with controls acting on a lower dimensional manifold. *SIAM J. Control Optim.*, 52(3):2008–2035, 2014.

[22] L. Heltai and N. Rotundo. Error estimates in weighted Sobolev norms for finite element immersed interface methods. *Computers and Mathematics with Applications*, 78(11):3586–3604, 2019.

[23] L. Heltai and W. Lei. Adaptive finite element approximations for elliptic problems using regularized forcing data. *arXiv preprint*, arXiv:2110.15029, 2021.

[24] B. Hosseini, N. Nigam, and J. M. Stockie On regularizations of the Dirac delta distribution. *Journal of Computational Physics*, 305:423–447, 2016.

[25] P. Houston, and T. P. Wihler. Discontinuous Galerkin methods for problems with Dirac delta source. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(6):1467–1483, 2012.

[26] V. Kondrat’ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obšč.*, 16:209–292, 1967.

[27] H. Li, A. Mazzucato and V. Nistor. Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains. *Electron. Trans. Numer. Anal.*, 37:41–69, 2010.

[28] H. Li, X. Wan, P. Yin and L. Zhao. Regularity and finite element approximation for two-dimensional elliptic equations with line Dirac sources. *Journal of Computational and Applied Mathematics*, 393:113518, 2021.

[29] J. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. 1*. Springer-Verlag, 1972.

[30] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000.

[31] F. Millar, I. Muga, and S. Rojas. Projection in negative norms and the regularization of rough linear functionals. *Numerische Mathematik*, 150:1087–1121, 2022.

[32] P. Morin, R. Nochetto, and K. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM Journal on Numerical Analysis*, 38(2):466-488, 2000.

[33] P. Morin, R. Nochetto, and K. Siebert. Convergence of adaptive finite element methods. *SIAM Review*, 44:631-658, 2002.

[34] R. Scott. Finite element convergence for singular data. *Numerische Mathematik*, 21:317–327, 1973.

[35] R. Scott. Optimal $L^\infty$ estimates for the finite element method on irregular meshes. *Math. Comp.*, 30:681–697, 1976.

[36] R. Stevenson. Optimality of a standard adaptive finite element method. *Foundations of Computational Mathematics volume*, 7(2):245–269, 2007.

[37] A. K. Tornberg. Multi-dimensional quadrature of singular and discontinuous functions. *BIT Numerical Mathematics volume*, 42:644-699,2002.

[38] R. Verfürth. A posteriori error estimation and adaptive mesh-refinement techniques. *Journal of Computational and Applied Mathematics*, 50(1-3):67–83,1994.

[39] R. Verfürth. A review of a posteriori error estimation and adaptive mesh-refinement techniques. *Wiley-Teubner, Chichester*, 1996.

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