Non-integrability of density perturbations in the FRW universe

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We investigate the evolution equation of linear density perturbations in the Friedmann-Robertson-Walker universe with matter, radiation and the cosmological constant. The concept of solvability by quadratures is defined and used to prove that there are no “closed form” solutions except for the known Chernin, Heath, Meszaros and simple degenerate ones. The analysis is performed applying Kovacic’s algorithm. The possibility of the existence of other, more general solutions involving special functions is also investigated.

I. INTRODUCTION

This paper is an attempt to apply the methods of differential algebra, and the differential Galois theory in particular, to a problem of cosmology. Although the considerations are mostly mathematical, the problem itself, and its solutions, are of rather practical interest. Linear perturbations of the Einstein equations in many forms are investigated due to their direct relation to such practical questions as the formation of galaxies or the CMB inhomogeneities. We, however, enter the physical domain only as the source of a theoretical problem, on which we concentrate.

The result itself is of negative nature, or, in other words, it makes any further investigation of this kind unnecessary. All the known solutions are given, together with the conditions for their validity. As no new ones can exist, this closes and completes the analysis of the given equation.

Of course, that is not to say that apart from those special cases nothing can be said about the behaviour of the solution. It is only to some extent that differential algebra can make exact the intuitive concepts of “being possible to solve” or “expressible in a closed form”. The definition of non-integrability employed here is but one of many which were born as classical mechanics evolved. The Liouville’s theorem implies that enough first integrals might yield a complete solution of a dynamical system, and accordingly many criteria regarding the existence of certain classes of first integrals were developed. The first achievements were those of Kovalevskaia and Lyapunov, greatly improved only recently by Ziglin, Morales and Ramis. The Galois theory used here, can also be applied to prove non-existence

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of meromorphic first integrals in more complex systems. Paradoxically here, we are presented with an equation simple enough not to allow for the application of those advanced methods. Thus, only a small part of the theory is put into practise and explained here. For a complete bibliography and exposition see for example [7].

The paper is organised as follows. In section 2 we derive the equation in question in a non-standard but intuitively clear way. The next two sections describe the concept of Liouvillian solutions, integrability and give the basic criteria, which we proceed to use in section 5. We also investigate the possibility of solving the problem by a combination of some special and Liouvillian functions, and give the theorem which is the main result in section 6. Finally, conclusions and finishing remarks are given in section 7.

II. DENSITY PERTURBATION EQUATION

We will be considering the Friedmann-Robertson-Walker universe given by the metric

$$ds^2 = c^2 dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right],$$

where $K$ is the curvature index, $d\Omega^2$ the distance element on a two-sphere. The universe will be filled with radiation and baryonic matter characterised by their pressures and densities $p$ and $\rho$. A non-zero cosmological constant’s effect will also be considered.

The Einstein equations for this model give

$$\begin{cases} 
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} - \frac{K}{a^2} \\
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (3p + \rho) + \frac{\Lambda}{3},
\end{cases}$$

where $G$, and $\Lambda$ are the gravitational and cosmological constants respectively. The conservation equation reduces to

$$c^2 \dot{\rho} = -3H(p + \rho e^2),$$

with the dot representing the time derivative. This can be expressed as the following transformation laws for matter and radiation respectively:

$$p_m = 0 \quad \Rightarrow \quad \frac{\rho_m}{\rho_{m0}} = \left( \frac{a_0}{a} \right)^3,$$

$$p_r = \frac{1}{3} \rho_r \quad \Rightarrow \quad \frac{\rho_r}{\rho_{r0}} = \left( \frac{a_0}{a} \right)^4.$$

$\rho_{m0}$ and $\rho_{r0}$ are the values of the densities for the moment when $a = a_0$, which can be chosen as the present day.

The fluctuation is introduced by the means of the scale factor

$$a = \tilde{a}(1 + y),$$
and \( \tilde{a} \) is the solution of the original equations (2). As it is a matter perturbation only, we have

\[
\rho_r = \tilde{\rho}_r, \\
\rho_m = \tilde{\rho}_m \left( \frac{\tilde{a}}{a} \right)^{-3} \\
= \tilde{\rho}_m(1 + y)^{-3} \\
= \tilde{\rho}_m(1 - 3y).
\]

Where we use the scaling law (4) and linearise the problem. Substituting this into the second of the equations (2), we obtain

\[
\ddot{\tilde{a}}(1 + y) + 2\dot{\tilde{a}} \dot{y} = -\frac{8\pi G}{3} \tilde{a}(1 + y)\rho_r - \frac{4\pi G}{3} \tilde{a}(1 + y)\tilde{\rho}_m(1 - 3y) + \frac{1}{2} \Lambda \tilde{a}(1 + y),
\]

which, after substituting the equation satisfied by the unperturbed \( \tilde{a} \), simplifies to

\[
\ddot{y} + 2H \dot{y} - 4\pi G \tilde{\rho}_m y = 0,
\]

with the Hubble “constant”

\[
H := \frac{\dot{a}}{a}.
\]

This kind of perturbation is of the scalar type, i.e. constructed from a single function \( y \). Here, we take it to depend on time only, although in general it could also involve spatial variables. This case could then be thought of as the zeroth order coefficient in the expansion of \( y(t, r, \theta, \phi) \) in terms of eigenfunctions of the spatial Laplace operator. Another generalisation would be to consider the vector and tensor type perturbations (see [11] for details of the decomposition). As it turns out, vector perturbations also admit quite general exact solutions [10].

In order to simplify the equation (7) fully, that is, bring it to the linear form with rational coefficients, we choose new variables

\[
x := \frac{\tilde{a}}{a_0}, \quad u := H_0 t,
\]

and constant density parameters:

\[
\Omega_{r0} := \frac{8\pi G \rho_{r0}}{3H_0^2}, \quad \Omega_{m0} := \frac{8\pi G \rho_{m0}}{3H_0^2}, \quad \Omega_{K0} := -\frac{c^2 K}{\tilde{a}^2 H_0^2}, \quad \Omega_{\Lambda0} := \frac{c^2 \Lambda}{3H_0^2}
\]

which allow us to rewrite the main equation as

\[
\left( \frac{dx}{du} \right)^2 \frac{d^2 y}{dx^2} + \left[ \frac{d^2 x}{du^2} + \frac{2}{x} \frac{dx}{du} \right]^2 \frac{dy}{dx} - \frac{4}{3} \Omega_{m0} \frac{1}{x^3} y = 0,
\]

where the perturbation \( y \) is now considered as a function of \( x \). Finally, using the first of the equations (2), which in the new variables reads

\[
x^2 \left( \frac{dx}{du} \right)^2 = \Omega_{\Lambda0} x^4 + \Omega_{K0} x^2 + \Omega_{m0} x + \Omega_{r0} =: W(x),
\]
we can eliminate the derivatives with respect to \( u \), and denoting the differentiation with respect to \( x \) with a prime, we get

\[
x(\Omega_\Lambda x^4 + \Omega_K x^2 + \Omega_m x + \Omega_r)y'' + (3\Omega_\Lambda x^4 + 2\Omega_K x^2 + 3\Omega_m x + \Omega_r)y' - \frac{3}{2}\Omega_m y = 0.
\]

(14)

As follows from the definitions, \( \Omega_K \) and \( \Omega_\Lambda \) are of arbitrary signs, while \( \Omega_r \) and \( \Omega_m \) are non-negative. We take \( \Omega_m \) to be strictly positive, though, because of the nature of the examined perturbations.

We also introduce the functions \( p(x) \) and \( q(x) \) related to the above equation in the following form:

\[
y''(x) + p(x)y'(x) + q(x)y(x) = 0.
\]

(15)

They are

\[
p(x) = \frac{6\Omega_\Lambda x^4 + 4\Omega_K x^2 + 3\Omega_m x + 2\Omega_r}{2x(\Omega_\Lambda x^4 + \Omega_K x^2 + \Omega_m x + \Omega_r)} \quad \text{and} \quad q(x) = \frac{-3\Omega_m}{2x(\Omega_\Lambda x^4 + \Omega_K x^2 + \Omega_m x + \Omega_r)}.
\]

(16)

Having obtained the solutions as functions of \( x \) it is straightforward to express them as functions of the cosmological or conformal time, since \( x(u) \) satisfies equation (13), and is therefore expressible in terms of the elliptic functions. The exact formulae can be found for example in [8].

### III. LIOUVILLIAN SOLUTIONS

Equation (14) is, in general, a Fuchsian one, and the solutions can be found by means of series. In particular, in vicinities of the singular points

\[
y_\pm(x) = (x - x_0)^{\alpha_\pm} \left(1 + \sum_{n=1}^{\infty} c_n (x - x_0)^n\right),
\]

(17)

where \( \alpha_\pm \) are the characteristic exponents at the point \( x_0 \). Such solutions are local, and the area of convergence around a given point is restricted by the remaining singular points. However, there are some special cases in which the solutions can be expressed by means of known special functions, and become global. It is then much easier to investigate and understand their behaviour. Thus, we are lead to the natural question of existence of such “simple” solutions. In this section we give a short description of a class of functions, which could here be called closed-form solutions.

It is natural to start seeking for the solutions in a class of functions to which the given equation’s coefficients belong, but such a set proves to be insufficient in most cases. As we are dealing with an equation whose coefficients are rational functions, the first choice is the field \( \mathbb{C}(x) \) – rational functions with complex numbers as the field of constants. Or, in the language of differential algebra, \( (\mathbb{C}(x), \partial) \) - the above field equipped with a suitable derivation operation, which, in this case, coincides with the usual, complex one, and will be denoted by a prime throughout this section.

This field is much too small, of course, and we are soon forced to extend it, introducing new elements. Just, as \( \mathbb{C}(x) \) is an extension of \( \mathbb{C} \) obtained by adjoining an indeterminate variable \( x \), our new fields will be \( \mathbb{C}(x) \) to which
new functions are added. Naturally, the derivation on the extended field must coincide with the subfield’s derivation when restricted to it (that is essentially what a differential extension is).

To keep the new functions relatively simple, so that the usual notion of “solvability by quadratures” could still be applied, the new elements are restricted to three classes.

**Definition 1** For a differential field extension \( F \subset G \), an element \( a \in F \) is:

- **primitive** over \( G \) if \( a' \in G \),
- **exponential** over \( G \) if \( a'/a \in G \), or
- **algebraic** over \( G \) if \( P(a) = 0 \) for some \( P(x) \in G[x] \) – the ring of polynomials with coefficients in \( G \).

**Definition 2** A field extension is called Liouvillian if it is a result of a finite number of extensions, each the adjunction of an algebraic, exponential, or primitive element.

Some examples of “new” functions appearing in this process are radicals (for algebraic elements), logarithms and inverse circular function (for primitives), and trigonometric functions (for exponentials). In short, the new elements are expressible as combinations of exponentials, integrals, and algebraic and elementary functions.

Finally, we are able to formulate what we mean by integrability:

**Definition 3** A linear differential equation with coefficients in \( \mathbb{C}(x) \) is said to possess Liouvillian solutions, or be solvable by quadratures, if its solutions belong to a Liouvillian extension of \( \mathbb{C}(x) \).

### IV. MONODROMY AND GALOIS GROUPS

In order to characterise a differential equation we can introduce two groups whose invariant properties are closely connected with those of the first integrals of the given equation.

The first, called the monodromy group \( \mathcal{M} \) is associated with the continuation of the local solutions of a linear differential equation, along closed paths in the domain where the equation is defined. The group itself is an image the fundamental group of that domain, and a subgroup of \( \text{GL}(n, \mathbb{C}) \) for \( n \) the order of the equation. The only other fact, concerning \( \mathcal{M} \), that we will need here is that \( \mathcal{M} \subset \mathcal{G} \) – the differential Galois group, which we proceed to describe.

\( \mathcal{G} \) is directly connected with the extension of the base field \( \mathbb{C}(x) \) to a bigger field \( F \), which contains the solutions of the considered equation. \( \mathcal{G} \) is defined as the group of automorphisms of \( F \) that leave \( \mathbb{C}(x) \) element-wise fixed. Carrying this information regarding the equation, the Galois group enables to test for particular forms of solutions.

One of the fundamental properties is as follows:

**Lemma 1** Let \( \mathcal{G} \) be the differential Galois group of a linear differential equation \( w''(x) = r(x)w(x) \), \( r(x) \in \mathbb{C}(x) \). Then it is an algebraic subgroup of \( \text{SL}(2, \mathbb{C}) \), and one of the following cases can occur.
1. \( \mathcal{G} \) is triangulisable. There is a solution of the form \( \exp \int \omega \), where \( \omega \in \mathbb{C}(x) \).

2. \( \mathcal{G} \) is conjugate to a subgroup of

\[
D^\dagger = \left\{ \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix} \middle| c_1 \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & c_2 \\ -c_2^{-1} & 0 \end{pmatrix} \middle| c_2 \in \mathbb{C}^* \right\},
\]

where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). There is a solution of the form \( \exp \int \omega \), with \( \omega \) algebraic over \( \mathbb{C}(x) \) of degree 2.

3. \( \mathcal{G} \) is finite. All the solutions are algebraic.

4. \( \mathcal{G} = SL(2, \mathbb{C}) \). The equation has no Liouvillian solutions.

V. NON-INTEGRABILITY OF THE EQUATIONS

We now proceed to the direct analysis of the integrability of equation (14), based mainly on the Kovacic’s algorithm [1]. First we consider the equation with \( \Omega_{\Lambda_0} = 0 \), for which the calculation is not too cumbersome, and next, we outline the reasoning for the general case.

A. Models without the cosmological constant

In order to apply the algorithm, equation (15) is cast into a reduced form

\[
v''(x) = r(x)v(x),
\]

with

\[
r(x) = -\frac{1}{4x^2} - \frac{3(2\kappa x + 1)^2}{(\kappa x^2 + x + \varrho)^2} + \frac{4\kappa x + 7}{x(\kappa x^2 + x + \varrho)}.
\]

Where we have introduced new parameters

\[
\kappa := \frac{\Omega_{K_0}}{\Omega_{m0}}, \quad \varrho := \frac{\Omega_{r0}}{\Omega_{m0}}.
\]

The reduction itself is performed by means of the following change in the dependent variable

\[
y(x) = v(x) \exp \left[ -\frac{1}{2} \int_{x_0}^{x} p(s)ds \right],
\]

where the constant \( x_0 \) is arbitrary. This transformation does not change the class of solutions we are interested in, as it uses the “admissible” operations only.

In the general case, there are two distinct roots of the polynomial \( W(x) \) in equation (14) and they are both non-zero. The degenerate cases will be treated separately.
Looking for the local solutions around $x = 0$, which is a regular singular point of the equation with both characteristic exponents equal to $\frac{1}{2}$, we find

$$
v_1(x) = x^{1/2}w_1(x),
$$

$$
v_2(x) = \ln(x)v_1(x) + x^{1/2}w_2(x),
$$

and $w_1$ and $w_2$ are holomorphic at $x = 0$. Using these, it is now possible to obtain an element of the monodromy group, by continuation of the fundamental solution matrix along a small closed path around $x = 0$.

$$
\begin{pmatrix}
v_1 & v_2 \\
v'_1 & v'_2
\end{pmatrix} \rightarrow
\begin{pmatrix}
-1 & -2\pi i \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
v_1 & v_2 \\
v'_1 & v'_2
\end{pmatrix}
$$

Since $\mathcal{M} \subset \mathcal{G}$, the Galois Group cannot be $D^\dagger$, as that group does not contain non-diagonalisable matrices. Also, a triangular, non-diagonal matrix cannot generate a finite group, so that $\mathcal{G}$ itself cannot be finite. We have thus excluded cases 2 and 3 of lemma 1.

Case 1 can also be easily excluded applying the aforementioned algorithm. In order for the solution to be of the form

$$
P(x) \exp \left[ \int \omega(x)dx \right],
$$

with a monic $P(x) \in \mathbb{C}[x]$, and $\omega(x) \in \mathbb{C}(x)$, the following equation must be satisfied:

$$
P''(x) + 2\omega(x)P'(x) + [\omega'(x) + \omega(x)^2 - r(x)]P(x) \equiv 0,
$$

for $P(x)$ and $\omega(x)$ found according to the algorithm, as follows. First, we define a set of auxiliary quantities:

$$
\alpha^\pm_c = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4b},
$$

where $c \in \Gamma \cup \{\infty\}$ – the set of all the finite poles of $r(x)$ including infinity, and $b$ is the coefficient of $(x - c)^{-2}$ in the Laurent expansion of $r(x)$ around a given point $x = c$. Next, for all possible combinations of signs $s(\infty), s(c)$ we compute the possible degrees of $P(x)$ as

$$
d = \alpha^{s(\infty)} - \sum_{c \in \Gamma} \alpha^{s(c)},
$$

and, for the same combinations of signs $s(c)$, the respective possible $\omega$

$$
\omega(x) = \sum_{c \in \Gamma} \frac{\alpha^{s(c)}}{x - c}.
$$

Inserting the functions just obtained into equation (24) we are left with a system of linear equations determining the unknown coefficients of $P(x)$. In this particular case, the only non-negative value of $d$ is 0, so that $P(x) = 1$, and the system is reduced to

$$
\frac{3}{2x(kx^2 + x + q)} \equiv 0,
$$
which cannot hold. The first case of lemma 1 is also excluded, which means that \( G \) is SL(2, \( \mathbb{C} \)), and the equation has no Liouvillian solutions in the general case.

When we admit a double non-zero root of \( W(x) \), or in other words, when 

\[
\Omega_{m0} = 2\sqrt{\Omega_{r0}} \left( 1 - \sqrt{\Omega_{r0}} \right),
\]

we have the well known Chernin solution [2]

\[
y(x) = 2F_1 \left( -i\sqrt{3}, i\sqrt{3}, 1; \frac{x}{x + 2\Omega_{r0}} \right).
\]

In the current notation it means that \( 4\varrho\kappa = 1 \), and that the root itself is \( x_0 = -2\varrho \). As equation (15) now has three regular singular points: 0, \( x_0 \) and \( \infty \), it becomes a Riemann P-equation. The complete set of solutions is denoted by

\[
y(x) = P \left\{ \begin{array}{ccc}
0 & -2\varrho & \infty \\
0 & -i\sqrt{3} & 0 \\
0 & i\sqrt{3} & 1 \\
\end{array} \right\}.
\]

Such a P-function is not Liouvillian either, as can be immediately checked using Kimura’s Theorem [4].

Another degenerate subcase occurs when \( \varrho = 0 \), and \( \kappa \neq 0 \), so that 0 becomes a root of \( W(x) \) and there is another non-zero root \( x = -\frac{1}{\kappa} \). Like before the solution is a P-function

\[
y(x) = P \left\{ \begin{array}{ccc}
0 & -\frac{1}{\kappa} & \infty \\
-\frac{3}{2} & 0 & 0 \\
1 & \frac{1}{2} & 1 \\
\end{array} \right\}.
\]

This time however, it is a Liouvillian solution because we can express the above symbol as two base solutions:

\[
y_1(x) = x^{-3/2} \sqrt{\Omega_{m0} + \Omega_{K0}x},
\]

\[
y_2(x) = x \cdot 2F_1 \left( 1, 2; -\frac{\Omega_{K0}}{\Omega_{m0}}; x \right),
\]

where \( 2F_1 \) is the Gauss hypergeometric function.

The next subcase is \( \kappa = 0 \) and only two simple roots remain: 0 and \( -\varrho \). We accordingly get

\[
y(x) = P \left\{ \begin{array}{ccc}
0 & -\varrho & \infty \\
0 & 0 & -1 \\
0 & \frac{1}{2} & \frac{3}{2} \\
\end{array} \right\}.
\]

As before, this is a Liouvillian function, according to Kimura’s theorem, and it can be rewritten as the following two independent solutions:

\[
y_1(x) = 1 + \frac{3\Omega_{m0}}{2\Omega_{r0}} x,
\]

\[
y_2(x) = \sqrt{\Omega_{r0} + \Omega_{m0}x} \cdot 2F_1 \left( 2, -\frac{1}{2}, \frac{3}{2}; 1 + \frac{\Omega_{m0}}{\Omega_{r0}} x \right),
\]
which are the solutions discovered by Meszaros [9].

The last possibility is that \( \theta = \kappa = 0 \) which implies \( \Omega_{m0} = 1 \), and we simply obtain

\[
\begin{align*}
y_1(x) &= x, \\
y_2(x) &= x^{3/2}.
\end{align*}
\] (36)

Taking into account that condition (29) makes it impossible for the physical cases (\( \Omega_{m0} > 0 \)) to have zero as a triple root, we conclude that the only Liouvillian solutions of equation (14) for \( \Omega_{A0} = 0 \), appear when at least one of the parameters \( \Omega_{r0} \) or \( \Omega_{K0} \) is equal to zero.

B. Models with the cosmological constant

The reasoning for \( \Omega_{A0} \neq 0 \) is the same as in the previous section, but since the leading coefficient now contains a polynomial of the fourth degree, the calculations are somewhat more laborious.

In general, when \( W(x) \) has only simple roots, and they are all non-zero (which means \( \Omega_{r0} \neq 0 \)), we obtain, as before, a triangular element of the monodromy group \( \mathcal{M} \). This leaves us with only the first case of the Kovacic’s algorithm to check, and the physical requirement of \( \Omega_{m0} > 0 \) causes the equation to be non-integrable.

In fact, even if the roots become multiple, but still non-zero, the local solutions in the vicinity of \( x = 0 \) do not change and, again, we only need to consider the first case of the algorithm. We introduce the roots by the following formula

\[
W(x) = \Omega_{A0}(x - e_1)(x - e_2)(x - e_3)(x - e_4),
\]

Taking \( e_2 = e_4 \), we could rewrite the polynomial as \( W(x) = \Omega_{A0}(x - e_1)(x + e_1 + 2e_2)(x - e_2)^2 \), because we must have \( e_1 + e_2 + e_3 + e_4 = 0 \).

This time, the coefficients \( b \), as defined in the preceding section, is not numeric, but depends on the roots. We now have

\[
\alpha_{e_2}^\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4 \left( \frac{3}{4} + \frac{3e_1}{e_2 - e_1} - \frac{e_1}{e_1 + 3e_2} \right)} = \frac{1 \pm n}{2}. \quad (37)
\]

We see that \( n \) must be at least a positive integer, and that initially there are countably many possibilities to check.

Since the relation between the roots and the densities \( \Omega \) is known, we might obtain an important restriction of the form

\[
\frac{\Omega_0}{\Omega_{A0}} = \frac{-3(n^2 - 4)}{n^2 + 12} e_4^2, \quad (38)
\]

which means that \( \Omega_{A0} \) must be negative if \( n > 2 \). One can readily check that \( n = 1, 2 \) give no solution. Further calculations with these restrictions on \( \Omega_{A0} \) and \( n \), determining the coefficients of the appropriate polynomial \( P(x) \) require us to solve a system of \( n \) homogeneous, linear equations. Unfortunately we have been unable to do that for
general $n$, but it is easy to check for the first 50 values that the determinant of the system is not zero, with its modulus monotonically increasing with $n$. Thus we conjecture that there are no values of $n$ for which a solution exists.

When the double root becomes triple, it also becomes a pole of the third order of the function $r(x)$. This means that only the second case of Kovacic’s algorithm needs to be considered, but it yields no Liouvillian solutions.

It is impossible for $W(x)$ to have two double roots or a quadruple root, because as we noted the sum of all roots must be zero, and that would lead to $\Omega_{m0} = 0$.

Letting $\Omega_{r0} = 0$ changes the multiplicity of $x = 0$ as a singular point, and the monodromy argument no longer holds. We note also that no further increase in the multiplicity of that point is possible as $\Omega_{m0} \neq 0$.

Assuming first that there are no multiple roots of $W(x)$, the algorithm immediately yields the following fundamental solutions

$$y_1(x) = x^{-3/2} \sqrt{\Omega_{0}x^3 + \Omega_{K0}x + \Omega_{m0}},$$

$$y_2(x) = y_1(x) \int \left( \frac{x}{\Omega_{0}x^3 + \Omega_{K0}x + \Omega_{m0}} \right)^{3/2} dx,$$

which holds even if one of the roots becomes double. This solution is also known, and was found by Heath [3].

As above, a triple root would mean $\Omega_{m0} = 0$ and is thus not physical.

VI. ON MORE GENERAL SOLUTIONS

Although the concept of Liouvillian solutions gives us a simple and applicable formulation of solvability by quadratures, it is easily seen to be insufficient on itself as a mean to discard an equation as insolvable in general. The fact that the Bessel or hypergeometric functions (except for some special cases) are not Liouvillian is the best example of that.

Of course, there are no algorithms for finding more complex solutions, but it is possible to extend the considered class somewhat. Following the paper of Bronstein [6], we try to find solutions in the form

$$y_1(x) = m(x)F_1[\xi(x)],$$

$$y_2(x) = m(x)F_2[\xi(x)],$$

where $F_1(\xi)$ and $F_2(\xi)$ are fundamental solutions of a given target equation

$$y''(\xi) = u(\xi)y(\xi),$$

and $m(x)$ and $\xi(x)$ are Liouvillian over $\mathbb{C}(x)$. The quoted paper presents an algorithmic approach for target equations which have an irregular singularity at infinity, and $\xi(x) \in \mathbb{C}(x)$. It also offers a check for a certain class of algebraic $\xi(x)$.

Upon substituting the form of solutions (40) into equation (18), and using the target equation (41), we are left with an expression containing only $F(\xi)$ and $F'(\xi)$. Making the coefficients equal to zero gives the fundamental equations
to be solved
\[ 3\xi''(x)^2 - 2\xi''(x)\xi'(x) + 4u[x(x)]\xi'(x)^4 - 4r(x)\xi'(x)^2 = 0, \tag{42} \]
\[ m(x) = \frac{1}{\sqrt[\xi(x)]}. \tag{43} \]

The main theorem of [6] makes it possible to find all rational solutions of equation (42), by bounding the degrees of the polynomials involved.

**Theorem 1** Let \( \prod_i Q_i \) be the square-free decomposition of the denominator of \( r \in \mathbb{C}(x) \). If the order of \( u(x) \) at infinity \( \nu_{\infty}(u) < 2 \), then any solution \( \xi \in \mathbb{C}(x) \) of (42) can be written as \( \xi = P/Q \) where

\[ \xi = \prod_i Q_i^{2-\nu_{\infty}(u)} \in \mathbb{C}[x], \tag{44} \]

and \( P \in \mathbb{C}[x] \) is such that either \( \deg(P) \leq \deg(Q) + 1 \) or

\[ \deg(P) = \deg(Q) + \frac{2 - \nu_{\infty}(r)}{2 - \nu_{\infty}(u)}. \tag{45} \]

For the algebraic case, they also provide a possible (but not exhaustive) ansatz of the form

\[ \xi(x) = P \left( x^{1/(2-\nu_{\infty}(u))} \right) \prod_{i>2} Q_i^{(r-2)(2-\nu_{\infty}(u))}, \tag{46} \]

and a bound for \( \deg(P) \) of either

\[ \deg(P) < (2 - \nu_{\infty}(u))(\deg(Q) + 2) \]

or

\[ \deg(P) = (2 - \nu_{\infty}(u))\deg(Q) + 2 - \nu_{\infty}(r). \]

Here, we choose to investigate the Bessel and Kummer functions, as the solutions of the target equation. The other classical classes of \( _0F_1 \) and \( _1F_1 \) functions, to which theorem 1 is applicable, are equivalent to these two classes, rationally or algebraically.

In the general case of \( \Omega_\Lambda \neq 0 \), and no multiple roots of \( W(x) \), the denominator of \( r(x) \) is \( 16x^2W(x)^2 \), and hence its square-free decomposition has only one term \( Q_2 = xW(x) \). We take first the Bessel equation,

\[ F''(\xi) = \left( \frac{4n^2 - 1}{4\xi^2} - \epsilon \right) F(\xi), \tag{47} \]

where \( \epsilon \) is 1 for the Bessel and -1 for modified Bessel functions. We have \( \nu_{\infty}(u) = 0 \), and \( \nu_{\infty}(r) = 2 \), so

\[ Q = \prod_i Q_{2i+2}^2 = 1, \tag{48} \]

and \( \deg(P) \leq 1 \). Finally, substituting \( \xi(x) = c_1 x + c_0 \) into equation (42) yields no non-constant solutions.

Proceeding in the same way, for Kummer functions, algebraic forms of \( \xi(x) \), and possible cases of the roots of \( W(x) \), we found no new solutions of this more general class.

Thus, we can finally formulate the main results as the following theorem:
Theorem 2 If $\Omega_{m0} > 0$, $\Omega_{r0} \geq 0$, $\Omega_{K0} \in \mathbb{R}$, and $\Omega_{\Lambda0} \in \mathbb{R}$ (unless $W(x)$ admits a double root, which requires the assumption of $\Omega_{\Lambda0} \geq 0$), then the matter density perturbation equation (14) does not possess any solutions Liouvillian over $\mathbb{C}(x)$, or, in other words, is not solvable by quadratures, except for the following three cases:

1. Heath’s solution ($\Omega_{r0} = 0$). It is given by (39), for $\Omega_{\Lambda0} \neq 0$, and by (33), when $\Omega_{\Lambda0} = 0$.

2. Meszaros’s solution ($\Omega_{K0} = 0$, $\Omega_{\Lambda0} = 0$). Given by (35).

3. Matter only ($\Omega_{\Lambda0} = \Omega_{r0} = \Omega_{K0} = 0$, $\Omega_{m0} = 1$). Given by (36).

Moreover, there exist no solutions of the class $m(x)F[\xi(x)]$, where $F(\xi)$ is a classical special function of the type $\,_{0}F_{1}$ or $\,_{1}F_{1}$, $m(x)$ is Liouvillian over $\mathbb{C}(x)$, and $\xi(x) \in \mathbb{C}(x)$ or is algebraic of the form (46).

However, when $\Omega_{\Lambda0} = 0$ and $W(x)$ admits a double root, the non-Liouvillian solution of Chernin’s, given by (30), applies.

VII. CONCLUSIONS

The problem studied in this paper is a good example of a possible application of the differential Galois theory in practise. The known solution of the perturbation equation were originally discovered in different contexts and over the span of a few years, whereas here, one theory allows for full analysis.

Furthermore, the existence of other closed-form solutions has been ruled out. The one exception is the case of a double root of $W(x)$, when the cosmological constant is negative. Although $\Lambda$ is usually assumed to be positive now, it still remains a viable mathematical possibility. However, we expect that that case will not yield any new solutions, as the “manual” check for the first, simplest candidates failed.

Despite the fact that the analysis of perturbations can be performed numerically, without the need of explicit solutions, we feel that a more exact approach is always valuable – providing a better understanding of the complexity of the problem, revealing hidden, special solutions, or simply providing a more exact formula for numerical work. It also reflects, in some sense, the fact that general relativity, as the next step in the theory of gravitation, introduces non-integrability into the basic equations. This can be seen clearly in a more advanced employment of the Galois theory applied to cosmological models themselves, for example in [7] and [5].

From the physical point of view, this model is a relatively simple one, and it is already known that the solutions found are not on themselves strong enough to explain the density fluctuations levels today. The main problem is the fact that they would need to be exponential, whereas they grow at most like a fixed power of the scale factor, and there is not enough time for the initial perturbations to increase sufficiently. One has also to be aware that linear instability might be dumped when considered in the quadratic regime, so the least to be required of the linear equation’s solutions is appropriate amplification. Nevertheless, these basic effects, being the first approximation, have physical meaning and should be taken into account when constructing more complicated models or explaining observational data.
We hope to investigate this field further in the future, as it does not limit the results to negative statements on integrability only, as we tried to demonstrate in this paper.

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