Further results on the Craig-Sakamoto Equation

John Maroulas

February 2, 2008

Abstract

In this paper necessary and sufficient conditions are stated for the Craig-Sakamoto equation
\[ \det(I - sA - tB) = \det(I - sA) \det(I - tB), \]
for all scalars \( s, t \). Moreover, spectral properties for \( A \) and \( B \) are investigated.

1 Introduction

Let \( M_n(\mathbb{C}) \) be the set of \( n \times n \) matrices with elements in \( \mathbb{C} \). For \( A \) and \( B \in M_n(\mathbb{C}) \), the well known in Statistics [1] Craig-Sakamoto (CS) equation
\[ \det(I - sA - tB) = \det(I - sA) \det(I - tB) \]
for all scalars \( s, t \) has occupied several researchers. In particular, in [5] O. Trusky presented that the CS equation is equivalent to \( AB = O \), when \( A, B \) are normal and most recently in [4] Olkin and in [2] Li proved the same result in a different way. The author, together with M. Tsatsomero and P. Psarrako in [3], have investigated the CS equation involving the eigenspaces of \( A, B \) and \( sA + tB \). Being more specific, if \( \sigma(X) \) denotes the spectrum for a matrix \( X \), \( m_X(\lambda) \) the algebraic multiplicity of \( \lambda \in \sigma(X) \), and \( E_X(\lambda) \) the generalized eigenspace corresponding to \( \lambda \), we have shown in [3]:

---

1Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, GREECE. E-mail:maroulas@math.ntua.gr. This work is supported by a grant of the EPEAEK, project “Pythagoras II”.
Proposition 1  For the $n \times n$ matrices $A, B$ the following are equivalent :

I. The CS equation holds

II. for every $s, t \in \mathbb{C}$, $\sigma(sA \oplus tB) = \sigma((sa + tB) \oplus O_n)$, where $O_n$ denotes the zero matrix

III. $\sigma(sA + tB) = \{ s\mu_i + t\nu_i : \mu_i \in \sigma(A), \ \nu_i \in \sigma(B) \}$, where the pairing of eigenvalues requires either $\mu_i = 0$ or $\nu_i = 0$.

Proposition 2  Let the $n \times n$ matrices $A, B$ satisfy the CS equation. Then,

I. $m_A(0) + m_B(0) \geq n$.

II. If $A$ is nonsingular, then $B$ must be nilpotent.

III. If $\lambda = 0$ is semisimple eigenvalue of $A$ and $B$, then $\text{rank}(A) + \text{rank}(B) \leq n$.

Proposition 3  Let $\lambda = 0$ be semisimple eigenvalue of $n \times n$ matrices $A$ and $B$ such that $BE_A(0) \subset E_A(0)$. Then the following are equivalent.

I. Condition CS holds.

II. $\mathbb{C}^n = E_A(0) + E_B(0)$.

III. $AB = O$.

The remaining results in [3] are based on the basic assumption that $\lambda = 0$ is a semisimple eigenvalue of $A$ and $B$. Relaxing this restriction, we shall attempt here to look at the CS equation focused on the factorization of polynomial of two variables $f(s, t) = \text{det}(I - sA - tB)$. Also, considering the determinants in [1], new conditions necessary and sufficient on CS property are stated.

2 Spectral results

The first statement on the CS property is obtained investigating the determinantal equation through the Theory of Polynomials. By Proposition 2 II, it is clear that the CS equation is worth valuable when the $n \times n$ matrices $A$ and $B$ are singular. Especially, we define that

"$A$ and $B$ are called r-complementary, if and only if at most, $r$ rows (columns), $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ of $A$ are shifted and substituted by the corresponding $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$ rows (columns) of $B$, such that the structured matrix $N(i_1, i_2, \ldots, i_r)$ of $a$’s and $b$’s rows is nonsingular."

Note that, $n - r \leq \text{rank}(B)$. 

2
For example, the pair of matrices

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

is not 1 or 2–complementary, on behalf of \( \text{rank} \begin{bmatrix} A \\ B \end{bmatrix} = 3 \), but the pair

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad B = B
\]

is 1-complementary and not 2-complementary, since \( \det N(b_1, a_2, a_3) = \det \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} \neq 0 \)

and \( \det N(b_1, b_2, a_3) = \det \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} = 0. \)

**Proposition 4** Let the \( n \times n \) singular matrices \( A \) and \( B \) be \( [n - m_B(0)] \)-complementary with

\[
\theta = \sum_{i_1, \ldots, i_{n-m_B(0)}} \det N(i_1, i_2, \ldots, i_{n-m_B(0)}) \neq 0, \text{ where the sum is over all possible combinations of } i_1, \ldots, i_{n-m_B(0)} \text{ of } n - m_B(0) \text{ of the indices } 1, 2, \ldots, n. \text{ If they satisfy the CS equation, then }
\]

\[
m_A(0) + m_B(0) = n.
\]

**Proof.** Let \( \text{rank} B = b(< n) \). Then \( \lambda = 0 \) is eigenvalue of \( B \) with algebraic multiplicity \( m_B(0) = m \geq n - b \). Denoting

\[
\beta(t) = \det(tI - B) = t^n + \beta_1 t^{n-1} + \cdots + \beta_{n-m} t^m,
\]

where \( \beta_k = (-1)^k \sum B_k \) and \( B_k \) are the \( k \times k \) principal minors of \( B \), then

\[
\det(tB - I) = (-1)^n t^n \det(t^{-1}I - B) = (-1)^n (1 + \beta_1 t + \cdots + \beta_{n-m} t^{n-m}).
\]

The polynomial \( \tilde{\beta}(t) = 1 + \beta_1 t + \cdots + \beta_{n-m} t^{n-m} \) has precisely \( n - m \) nonzero roots, let \( t_1, t_2, \ldots, t_{n-m} \), since \( \tilde{\beta}(0) = 1 \neq 0 \). Moreover, we have

\[
\det(sA + tB - I) = |A|^s + f_1(t)s^{n-1} + \cdots + f_{n-1}(t)s + |tB - I|,
\]

(2)
where

\[ f_1(t) = \sum_i \det \hat{A}_i, \quad \text{with} \quad \hat{A}_i = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ tb_{i1} & \cdots & tb_{ii} - 1 & \cdots & tb_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}. \]

Note that, \( \hat{A}_i \) arises by \( A \) when the \( i \)-row of \( A \) is substituted by the \( i \)-row of \( tB - I \). Similarly,

\[ f_2(t) = \sum_{i,j} \det \hat{A}_{ij}, \quad \text{with} \quad \hat{A}_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ tb_{i1} & \cdots & tb_{ii} - 1 & \cdots & tb_{in} \\ \vdots & & \vdots \\ tb_{j1} & \cdots & tb_{jj} - 1 & \cdots & tb_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}, \]

and \( \hat{A}_{ij} \) is obtained by \( A \), substituting the \( i \) and \( j \) rows of \( A \) by the corresponding rows of \( tB - I \). The summation in \( f_2(t) \) is referred to all pairs of indices \( i, j \) by \( \{1, 2, \ldots, n\} \). Hence, by the equation (2) and the CS equation

\((-1)^n \det(sA + tB - I) = \det(sA - I) \det(tB - I), \quad \forall s, t \)

for \( t = t_1, t_2, \ldots, t_{n-m} \), we obtain

\[ |A| s^n + f_1(t_i)s^{n-1} + \cdots + f_{n-1}(t_i)s = 0, \quad \forall s \]

and consequently

\[ |A| = 0, \quad f_1(t_i) = f_2(t_i) = \cdots = f_{n-1}(t_i) = 0, \quad \text{for} \quad i = 1, 2, \ldots, n - m. \quad (3) \]

Due to the matrices \( A \) and \( B \) are \([n - m_B(0)]\)-complementary and the leading coefficient of \( f_{n-m}(t) \) is equal to the nonzero \( \theta \), then \( \deg(f_{n-m}(t)) = n - m \) and \( \deg(f_k(t)) \leq n - m \), for \( k = 1, 2, \ldots, n - m - 1 \). Moreover, by (3) we have

\[ f_1(t) = f_2(t) = \cdots = f_{n-m-1}(t) = 0, \quad \forall t \]

Reminding that \( A_\ell \) denotes the \( \ell \times \ell \) principal minor of \( A \), by \( f_1(t) = 0 \), clearly

\[ f_1(0) = \sum A_{n-1} = 0 \quad \implies \quad c_{n-1} = 0. \]

4
Similarly, by
\[ f_2(t) = 0 \implies \sum A_{n-2} = 0 \implies c_{n-2} = 0 \]
\[ \vdots \]
\[ f_{n-m-1}(t) = 0 \implies \sum A_{m+1} = 0 \implies c_{m+1} = 0, \]
and consequently
\[
\delta_A(\lambda) = |\lambda I - A| = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + (-1)^n |A| \\
= \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + (-1)^m c_m \lambda^{n-m}.
\]
In (4), \( c_m \neq 0 \), since \( (-1)^{n-m} c_m = \theta t_1 t_2 \cdots t_{n-m} \). Thus, \( \lambda = 0 \) is eigenvalue of \( A \) with algebraic multiplicity \( n - m_B(0) \), whereby we conclude
\[
m_A(0) + m_B(0) = n.
\]

Remark 1 By the proof of Proposition 4, it is evident that the equality \( m_A(0) + m_B(0) = n \) holds, when the matrices \( B \) and \( A \) are \([n - m_A(0)]\)-complementary and
\[
\theta = \sum_{j_1, \ldots, j_{n-m_A(0)}} \det N(j_1, j_2, \ldots, j_{n-m_A(0)}) \neq 0.
\]

Corollary 1 Let the \( n \times n \) singular and \([n - m_B(0)]\)-complementary matrices \( A \) and \( B \). If \( \theta \neq 0 \) and these matrices satisfy the CS equation (4), then

I. \( \lambda = 0 \) is semisimple eigenvalue of \( A \) and \( B \implies \text{rank} A + \text{rank} B = n. \)

II. \( \lambda = 0 \) is semisimple eigenvalue of \( A \implies \text{rank} A = m_B(0). \)

Proof. I. Because
\[
n - \text{rank} A \leq m_A(0) = n - m_B(0),
\]
we have \( \text{rank} A + \text{rank} B \geq m_B(0) + r \geq n \). Hence, by III, Proposition 4, we obtain the equality.

II. By the assumption and Proposition 4 we have \( \text{rank} A = n - m_A(0) = m_B(0) \).

Closing this section, we present a property of generalized eigenspaces of nonzero eigenvalues of \( A \) and \( B \).
Proposition 5. Let \( \lambda = 0 \) be semisimple eigenvalue of \( n \times n \) matrices \( A \) and \( B \) such that \( E_A(0) + E_B(0) = \mathbb{C}^n \). If for any \( \lambda \in \sigma(A) \setminus \{0\} \) (or, \( \mu \in \sigma(B) \setminus \{0\} \)), the corresponding generalized eigenspaces \( E_A(\lambda) \) (or \( E_B(\mu) \)) satisfy \( E_A(\lambda) \subseteq E_B(0) \) (or, \( E_B(\mu) \subseteq E_A(0) \)), then

I. \( A, B \) have the CS property.

II. \( E_A(\lambda) = E_{I-sA-tB}(1-s\lambda) \) and \( E_B(\mu) = E_{I-sA-tB}(1-t\mu) \).

Proof. I. Since \( E_A(\lambda) \subseteq E_B(0) \), for every \( w = w_1 + w_2 \in \mathbb{C}^n \), where \( w_1 \in \bigoplus_\lambda E_A(\lambda) \), \( w_2 \in E_A(0) \), we have \( BAw = BA(w_1 + w_2) = BAw_1 = 0 \). Thus, \( BA = O \) and consequently \( AE_B(0) \subseteq E_A(0) \). The assumption \( E_A(0) + E_B(0) = \mathbb{C}^n \), and Proposition \( \mathbb{3} \) lead to the statement I.

II. Let \( \lambda \in \sigma(A) \setminus \{0\} \), and \( x_k \in E_A(\lambda) \) be generalized eigenvector of \( A \) of order \( k \). By the assumption, \( x_k \in E_B(0) \), and yields

\[
(I - sA - tB)x_k = (I - sA)x_k = x_k - s(\lambda x_k + x_{k-1}) = (1 - s\lambda)x_k - sx_{k-1}.
\]

Thus, for all chain \( x_1, \ldots, x_k, \ldots, x_\tau \) of \( \lambda \), we have

\[
(I - sA - tB) \begin{bmatrix} x_1 & \cdots & x_\tau \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_\tau \end{bmatrix} \begin{bmatrix} 1 - s\lambda & \cdots & -s \\ 0 & 1 - s\lambda & \cdots & O \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 1 - s\lambda & -s \\ 0 & \cdots & 0 & 1 - s\lambda \end{bmatrix}_{\tau \times \tau}.
\]

Moreover, by the statement III in Proposition II, \( s\lambda \) and \( t\mu \in \sigma(sA + tB) \). The equivalence of CS equation and \( \mathbb{C}^n = E_A(0) + E_B(0) \) in Proposition \( \mathbb{3} \) and the assumption \( E_A(\lambda) \subseteq E_B(0) \), lead to \( E_B(\mu) \subseteq E_A(0) \). Similarly, if \( y_\ell \in E_B(\mu) \) is generalized eigenvector of order \( \ell \), then \( y_\ell \in E_A(0) \) and

\[
(I - sA - tB)y_\ell = (I - tB)y_\ell = y_\ell - t(\mu y_\ell + y_{\ell-1}) = (1 - t\mu)y_\ell - ty_{\ell-1},
\]

and for all chain \( y_1, \ldots, y_\ell, \ldots, y_\sigma \) we obtain

\[
(I - sA - tB) \begin{bmatrix} y_1 & \cdots & y_\sigma \end{bmatrix} = \begin{bmatrix} y_1 & \cdots & y_\sigma \end{bmatrix} \begin{bmatrix} 1 - t\mu & \cdots & -t \\ 0 & 1 - t\mu & \cdots & O \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 1 - t\mu & -t \\ 0 & \cdots & 0 & 1 - t\mu \end{bmatrix}_{\sigma \times \sigma}.
\]

Clearly, by (5) and (6) are implied the equations in II, for any \( s, t \). □
Remark 2  For $z \in E_A(0) \cap E_B(0)$ obviously $(I - sA - tB)z = z, \ \forall \ s, \ t$. Therefore by the above proposition the Jordan canonical form of $I - sA - tB$, and the matrix

$$F = I_\nu \bigoplus_{\lambda \neq 0} \begin{pmatrix} 1 - s\lambda_A & -s & 0 \\ & 1 - s\lambda_A & \ddots \\ & & & -s \\ 0 & & & 1 - s\lambda_A \end{pmatrix} \bigoplus_{\mu \neq 0} \begin{pmatrix} 1 - t\mu_B & -t & 0 \\ & 1 - t\mu_B & \ddots \\ & & & -t \\ 0 & & & 1 - t\mu_B \end{pmatrix},$$

are similar.

The order $\nu$ of submatrix $I_\nu$ of $F$ declares the number of linear independent eigenvectors which correspond to the eigenvalue $\lambda = 1$ of $I - sA - tB$. Clearly, these eigenvectors belong to $E_B(0) \backslash E_A(\lambda)$, $E_A(0) \backslash E_B(\mu)$, and $E_A(0) \cap E_B(0)$, and $\nu$ is equal to

$$\nu = n - (\text{rank}A + \text{rank}B) = n - \left(\text{dim} \bigcup_{\lambda \neq 0} E_A(\lambda) + \text{dim} \bigcup_{\mu \neq 0} E_B(\mu)\right).$$

3 Criteria for CS equation

Let

$$f(s, t) = \det(I - sA - tB) = \sum_{p+q=n} m_{pq}s^pt^q, \quad p + q \leq n. \quad (7)$$

Denoting by $x = \begin{bmatrix} 1 & s & s^2 & \cdots & s^n \end{bmatrix}^T$, $y = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix}^T$, then (7) is written obviously

$$f(s, t) = x^TMy,$$

where $M = [m_{pq}]_{p,q=0}^n$, with $m_{00} = 1$.

Proposition 6 Let $A, B \in M_n(\mathbb{C})$. The CS equation holds for the pair of matrices $A$ and $B$ if and only if $\text{rank}M = 1$.

Proof. Let $A$ and $B$ managed by the CS property. Then the equation (1) is formulated as

$$x^TMy = x^Ta^Tb^Ty, \quad (8)$$

where

$$a = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_0 \end{bmatrix}^T, \quad b = \begin{bmatrix} 1 & b_{n-1} & \cdots & b_0 \end{bmatrix}^T,$$
and \( a_i, b_i \) are the coefficients of characteristic polynomials

\[
det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0, \quad det(\lambda I - B) = \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_0.
\]

Hence, by (8) for any \( s_1 \neq s_2 \neq \cdots \neq s_{n+1} \) and \( t_1 \neq t_2 \neq \cdots \neq t_{n+1} \) we have

\[
V^T(M - ab^T)W = O, \tag{9}
\]

where

\[
V = \begin{bmatrix} 1 & \cdots & 1 \\ s_1 & \cdots & s_{n+1} \\ \vdots & \ddots & \vdots \\ s_1^n & \cdots & s_{n+1}^n \end{bmatrix}, \quad W = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_{n+1} \\ \vdots & \ddots & \vdots \\ t_1^n & \cdots & t_{n+1}^n \end{bmatrix}.
\]

Clearly, by (9), we recognize that \( M = ab^T \), i.e., \( \text{rank } M = 1 \).

Conversely, if \( \text{rank } M = 1 \), then \( M = k\ell^T \), where the vectors \( k, \ell \in \mathbb{C}^{n+1} \). Therefore,

\[
f(s, t) = x^TMy = x^T k \ell^T y = k(s)\ell(t),
\]

where \( k(s) \) and \( \ell(t) \) are polynomials. Since, \( f(0, 0) = 1 = k(0)\ell(0) \), and

\[
\begin{align*}
det(I - sA) &= f(s, 0) = k(s)\ell(0), \\
det(I - tB) &= f(0, t) = k(0)\ell(t)
\end{align*}
\]

clearly,

\[
f(s, t) = k(s)\ell(0)k(0)\ell(t) = det(I - sA) det(I - tB).
\]

\[\square\]

**Example 1** Let the matrices

\[
A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - \gamma & 1 \\ 0 & 0 & 1 - \gamma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \gamma & 0 \\ 1/\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

We have

\[
f(s, t) = det(I - sA - tB) = 1 + 2(\gamma - 1)s + (\gamma - 1)^2s^2 - t^2 + (1 - \gamma)t^2s
\]

\[
= x^T \begin{bmatrix} 1 & 0 & 1 \\ 2(\gamma - 1) & 1 - \gamma & 0 \\ \gamma - 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y
\]

8
and

$$det(I - sA) = (1 + (\gamma - 1)s)^2, \quad det(I - tB) = 1 - t^2.$$ 

By the criterion (Proposition 3) easily we recognize that $A$, $B$ have the CS property only for $\gamma = 1$.

**Remark 3** In equation (5), if $b^T a = 0$ then $M^2 = 0$, and $M\left(\frac{1}{\|b\|^2}b\right) = a$. Therefore,

$$M = P \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} P^{-1}$$

where $P = \begin{bmatrix} a & p_2 & \cdots & p_{n-1} \\ \frac{1}{\|b\|^2}b \end{bmatrix}$ and $p_k, \ldots, p_{n-1}$ is an orthonormal basis of $\text{span}\{a, b\}$. Then $P^{-1} = \begin{bmatrix} \frac{1}{\|a\|^2}a & p_2 & \cdots & p_{n-1} \end{bmatrix}^T$.

Following we note by $M\left(\frac{1}{\|a\|^2}a, \frac{1}{\|b\|^2}b\right)$ the leading principal minor of order $p + q (\leq n)$, which is defined by the $i_1, \ldots, i_p$ rows of $A$ and $j_1, \ldots, j_q$ rows of $B$, i.e.,

$$M\left(\begin{smallmatrix} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{smallmatrix}\right) = \begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_q} & \cdots & a_{i_1j_p} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_q} & \cdots & a_{i_2j_p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_qj_1} & a_{i_qj_2} & \cdots & a_{i_qj_q} & \cdots & a_{i_qj_p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_pj_1} & a_{i_pj_2} & \cdots & a_{i_pj_q} & \cdots & a_{i_pj_p} \end{vmatrix}$$

for $i_1 < i_2 < j_1 < i_3 < \cdots < j_q < \cdots < i_p$. Thus, we clarify a determinental expression of coefficients $m_{pq}$ in (7):

$$m_{pq} = (-1)^{p+q} \sum_{1 \leq i_1 < j_1 < \cdots < j_q < i_p \leq n} M\left(\begin{smallmatrix} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{smallmatrix}\right), \quad m_{00} = 1. \quad (10)$$

For example, for $n \times n$ matrices $A$ and $B$ the coefficients of $t, st, s^2$ and $s^2t$ are respectively equal to

$$m_{01} = - \sum_{1 \leq j \leq n} M(b_j) = - (b_{11} + b_{22} + \cdots + b_{nn}) = -trB$$
Hence, for the matrix $M$

$$m_{11} = \sum_{1 \leq i < j \leq n} M\begin{pmatrix} a_i \\ b_j \end{pmatrix} = \sum_{i, j = 1}^{n} \left( \begin{vmatrix} a_{ii} & a_{ij} \\ b_{ji} & b_{jj} \end{vmatrix} + \begin{vmatrix} b_{ii} & b_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \right)$$

$$m_{20} = \sum_{1 \leq i, j \leq n} M(a_{ij}) = \sum_{i, j = 1}^{n} \left| a_{ii} \right| \left| a_{ij} \right|$$

and

$$m_{21} = -\sum_{1 \leq i \leq j \leq k \leq n} M\begin{pmatrix} a_{i,j} \\ b_k \end{pmatrix} = -\sum_{i, j \leq k \leq n} \left( \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ b_{ki} & b_{kj} & b_{kk} \end{vmatrix} + \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ b_{ki} & b_{kj} & a_{kk} \end{vmatrix} + \begin{vmatrix} b_{ii} & b_{ij} & b_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \right)$$

Hence, for the matrix $M$ in (7) we have:

$$M = \begin{bmatrix}
1 & -\sum M(b_j) & \sum M(b_{i,j}) & \cdots & (-1)^{n-1} \sum M(b_{j_1,\ldots,j_{n-1}}) & (-1)^n |B| \\
-\sum M(a_i) & \sum M\begin{pmatrix} a_i \\ b_j \end{pmatrix} & -\sum M\begin{pmatrix} a_i \\ b_{j_1,j_2} \end{pmatrix} & \cdots & (-1)^n \sum M\begin{pmatrix} a_i \\ b_{j_1,\ldots,j_{n-1}} \end{pmatrix} & 0 \\
\sum M(a_{i_1,i_2}) & -\sum M\begin{pmatrix} a_{i_1,i_2} \\ b_{ji} \end{pmatrix} & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & (-1)^n \sum M\begin{pmatrix} a_{i_1,\ldots,i_{n-1}} \\ b_j \end{pmatrix} & 0 & \cdots & 0 & 0 \\
(-1)^n |A| & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}$$

The zeros in $M$ correspond to the coefficients of monomials of $f(s,t)$ with degree $\geq n + 1$. These terms are not presented in $\det(I - sA - tB)$, since by (10) the order of principal minors is greater than $n$. Moreover, the dimension of $M$ in (7) should be less than $n + 1$, since the CS equation make sense for singular matrices.

Using the criterion in Proposition (6) in the above formulation of $M$, it is clear the next necessary and sufficient conditions.
Proposition 7  The \( n \times n \) matrices \( A \) and \( B \) have the CS property if and only if

\[
\sum M(a_{i_1, \ldots, i_p}) \sum M(b_{j_1, \ldots, j_q}) = \sum M\left(\begin{array}{c} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{array}\right), \quad \text{for } p + q \leq n,
\]

and

\[
\sum M(a_{i_1, \ldots, i_p}) \sum M(b_{j_1, \ldots, j_q}) = 0, \quad \text{for } p + q > n.
\]

(11)

Example 2 In (1) let \( A \) be a nilpotent matrix. Then,

\[
\sum M(a_i) = \sum M(a_{i,j}) = \cdots = |A| = 0,
\]

and by Proposition 7 clearly

\[
\sum M\left(\begin{array}{c} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{array}\right) = 0 \quad ; \quad p, q = 1, 2, \ldots, n - 1.
\]

In this case, \( M = \begin{bmatrix} 1 & 0 \\ 0 & b_{n-1} & \cdots & b_1 & b_0 \end{bmatrix} \).

The equations (11) give also an answer to the problem "For the \( n \times n \) matrix \( A \), clarify the set \( CS(A) = \{ B : A \) and \( B \) follow the CS property \}.

If \( a(s) = det(I - sA) \) and \( b(t) = det(I - tB) \), easily we turn out the \( \mu \)-th order derivative of polynomials at the origin

\[
\frac{1}{p!} a^{(p)}(0) = \sum M(a_{i_1, \ldots, i_p}), \quad \frac{1}{q!} b^{(q)}(0) = \sum M(b_{j_1, \ldots, j_q}),
\]

and even

\[
\frac{1}{p!q!} \frac{\partial^{p+q} f(0,0)}{\partial s^{p} \partial t^{q}} = \sum M\left(\begin{array}{c} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{array}\right).
\]

Thus, if we use the Taylor’s expansion of polynomials in (1), by the relationships

\[
a^{(p)}(0) b^{(q)}(0) = \frac{\partial^{p+q} f(0,0)}{\partial s^{p} \partial t^{q}}, \quad \text{for } p + q \leq n,
\]

\[
a^{(p)}(0) b^{(q)}(0) = 0, \quad \text{for } p + q > n,
\]

the equations (11) arise again.
References

[1] M. Dumais ans G.P. Styan, A bibliography on the distribution of quadratic forms in normal variables, with special emphasis on the Craig-Sakamoto theorem and on Cochran’s theorem, In George Styan ed., Three Bibliographies and a Guide, Seventh International Workshop on Matrices and Statistics, Fort Lauderdale, 1-9, 1988.

[2] C-K. Li, A simple proof of the Craig-Sakamoto Theorem, Linear Algebra and Its Applications, 321, (2000), 281-283.

[3] J. Maroulas, P. Psarrakos and M. Tsatsomeros, Separable characteristic polynomials of pencils and property L, Electronic Journal of Linear Algebra, 7, (2000), 182-190.

[4] I. Olkin, A determinantal proof of the Craig-Sakamoto Theorem, Linear Algebra and Its Applications, 264, (1997), 217-223.

[5] O. Trussky, On a matrix theorem of A.T. Craig and H. Hotelling, Indagationes Mathematicae, 20, (1958), 139-141.