Conformal totally symmetric arbitrary spin fermionic fields

R.R. Metsaev*

Department of Theoretical Physics, P.N. Lebedev Physical Institute,
Leninsky prospect 53, Moscow 119991, Russia

Abstract

Conformal totally symmetric arbitrary spin fermionic fields propagating in the flat space-time of even dimension greater than or equal to four are investigated. First-derivative metric-like formulation involving Fang-Fronsdal kinetic operator for such fields is developed. Gauge invariant Lagrangian and the corresponding gauge transformations are obtained. Gauge symmetries of the Lagrangian are realized by using auxiliary fields and the Stueckelberg fields. Realization of conformal algebra symmetries on the space of conformal gauge fermionic fields is obtained. The on-shell degrees of freedom of the fermionic arbitrary spin conformal fields are also studied.

*E-mail: metsaev@lpi.ru
1 Introduction

Although, at present time, up-to-date methods of quantization of gauge theories allow to treat higher-derivative theories and theories that do not consist higher derivatives on an equal footing, we note that the use of the famous Slavnov-Taylor identities [1] and BRST approach [2] is streamlined when the gauge fields theories do not consist higher derivatives. Note however that commonly used Lagrangian formulations of most conformal fields consist higher derivatives [3]. We recall that Lagrangian higher-derivative formulation of bosonic totally symmetric arbitrary spin conformal fields in $R^{d,1}$ space for $d = 4$ and $d \geq 4$ was obtained in the respective Ref.[3] and Ref.[4]. Alternative higher-derivative formulation of bosonic conformal fields obtained by using AdS/CFT correspondence was developed in Ref.[9]. At present time, higher-derivative Lagrangian formulation of the totally symmetric arbitrary spin conformal fermionic is known only for the case of $R^{3,1}$ space (see Ref.[3]).

The present paper is a continuation of our investigations in Refs.[10, 11]. In Ref.[10], for free bosonic and fermionic low-spin conformal fields, we developed the respective second-derivative and first-derivative Lagrangian gauge invariant metric-like formulations. The second-derivative metric-like formulation of bosonic arbitrary spin conformal fields was developed in Ref.[11]. In this paper, we develop Lagrangian gauge invariant metric-like formulation for totally symmetric fermionic arbitrary spin fields in $R^{d-1,1}$ space, $d \geq 4$.

Our approach to conformal fermionic fields is summarized as follows.

i) In addition to fields entering the higher-derivative formulation of conformal fields we introduce Stueckelberg fields and auxiliary fields.

ii) Kinetic term of our Lagrangian of conformal fermionic field does not involve higher than first order terms in the derivatives. The one-derivative contributions to the kinetic terms of Lagrangian of conformal fermionic fields are realized as the well-known Dirac, Rarita-Schwinger, and Fang-Fronsdal kinetic terms of the respective spin-$\frac{1}{2}$, spin-$\frac{3}{2}$, and spin-$(s + \frac{1}{2})$, $s > 1$, $s \in \mathbb{N}$, fermionic fields.

iii) All vector-spinor and tensor-spinor fermionic fields entering our Lagrangian are supplemented by the respective gauge symmetries. Gauge transformations of the fermionic fields do not contain higher than first order terms in derivatives. One-derivative contributions to gauge transformations of fermionic conformal fields are realized as the well-known gradient gauge transformations of the vector-spinor and tensor-spinor fermionic fields.

iv) Our first-derivative formulation is equivalent to the higher-derivative formulation. Namely, by eliminating the auxiliary fields via equations of motion and gauging away the Stueckelberg fields, we can verify that our first-derivative formulation of conformal fermionic fields amounts to the higher-derivative formulation of conformal fermionic fields.

The rest of the paper is organized as follows.

In Sec. 2, we summarize our notation. In Sec.3, we develop the first-derivative metric-like formulation for arbitrary spin conformal fermionic field. After this, in Sec.3.2, we present our result for our first-derivative gauge invariant Lagrangian and realization of gauge symmetries in our approach. In Sec.4, we discuss realization of conformal algebra symmetries on the space of gauge fields entering our approach. In Sec.5, we describe our results for number of on-shell degrees of freedom (D.o.F)
for the arbitrary spin conformal fermionic field and decomposition of those on-shell D.o.F into irreps of the $so(d - 2)$ algebra. Section 6 is devoted to the discussion of directions for future research.

2 Notation and conventions

Our conventions are as follows. $x^a$ denotes coordinates in the $R^{d-1,1}$ space, while $\partial_a$ denotes derivatives with respect to $x^a$, $\partial_a \equiv \partial/\partial x^a$. Vector indices of the Lorentz algebra $so(d - 1, 1)$ take the values $a, b, c, e = 0, 1, \ldots, d - 1$. To simplify our expressions we drop the flat metric $\eta_{ab} = (-, +, \ldots, +)$ in scalar products, i.e., we use $X^a Y^b \equiv \eta_{ab} X^a Y^b$.

A set of the creation operators $\alpha^a$, $\zeta$, $\bar{\nu}^\oplus$, $\bar{\nu}^\otimes$ and the respective set of annihilation operators $\bar{\alpha}^a$, $\bar{\zeta}$, $\nu^\oplus$, $\nu^\otimes$ will be referred to as oscillators in what follows.\(^3\) Commutation relations and the vacuum are defined as

$$[\bar{\alpha}^a, \alpha^b] = \eta^{ab}, \quad [\bar{\zeta}, \zeta] = 1, \quad [\bar{\nu}^\oplus, \nu^\oplus] = 1,$$

$$[\bar{\zeta}|0\rangle = 0, \quad [\bar{\nu}^\oplus|0\rangle = 0, \quad [\bar{\nu}^\otimes|0\rangle = 0. \quad (2.1)$$

The oscillators $\alpha^a, \bar{\alpha}^a$ and $\zeta, \bar{\zeta}, \nu^\oplus, \nu^\otimes, \bar{\nu}^\oplus, \bar{\nu}^\otimes$ transform in the respective vector and scalar representations of the Lorentz algebra $so(d - 1, 1)$. We use $2^{[d/2]} \times 2^{[d/2]}$ Dirac gamma matrices $\gamma^a, \{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and adapt the following hermitian conjugation rules for the derivatives, oscillators, and $\gamma$-matrices:

$$\bar{\partial}^\dagger = -\partial^a, \quad \gamma^a\dagger = \gamma^0 \gamma^a \gamma^0, \quad \alpha^a\dagger = \bar{\alpha}^a, \quad \zeta\dagger = \bar{\zeta}, \quad \nu^\oplus\dagger = \bar{\nu}^\oplus, \quad \nu^\otimes\dagger = \bar{\nu}^\otimes. \quad (2.3)$$

We use operators constructed out of the derivatives, oscillators, and Dirac $\gamma$-matrices,

\[ \Box \equiv \partial^a \partial^a, \quad \not{\partial} \equiv \gamma^a \partial^a, \quad \alpha \partial \equiv \alpha^a \partial^a, \quad \bar{\alpha} \partial \equiv \bar{\alpha}^a \partial^a, \quad (2.4) \]

\[ \gamma \alpha \equiv \gamma^a \alpha^a, \quad \gamma \bar{\alpha} \equiv \gamma^a \bar{\alpha}^a, \quad \alpha^2 \equiv \alpha^a \alpha^a, \quad \bar{\alpha}^2 \equiv \bar{\alpha}^a \bar{\alpha}^a, \quad (2.5) \]

\[ N_\alpha \equiv \alpha^a \bar{\alpha}^a, \quad N_\zeta \equiv \zeta \bar{\zeta}, \quad N_{\nu^\oplus} \equiv \nu^\oplus \bar{\nu}^\oplus, \quad N_{\nu^\otimes} \equiv \nu^\otimes \bar{\nu}^\otimes, \quad (2.6) \]

\[ N_\partial \equiv N_{\nu^\oplus} + N_{\nu^\otimes}, \quad \Delta' \equiv N_{\nu^\oplus} - N_{\nu^\otimes}. \quad (2.7) \]

\[ \mathcal{A}^a \equiv \alpha^a - \gamma \alpha \gamma^a \frac{1}{2N_\alpha + d - 2} - \alpha^2 \frac{1}{2N_\alpha + d} \bar{\alpha}^a, \quad (2.8) \]

\[ \mathcal{G}^a \equiv \gamma^a - \gamma \alpha \frac{2}{2N_\alpha + d} \bar{\alpha}^a, \quad (2.9) \]

\[ \mathcal{A}_1^a \equiv \alpha^a - \gamma \alpha \gamma^a \frac{1}{2N_\alpha + d} - \alpha^2 \frac{1}{2N_\alpha + d + 2} \bar{\alpha}^a, \quad (2.10) \]

\[ \mathcal{A}_1^a \equiv \alpha^a - \gamma \alpha \gamma^a \frac{1}{2N_\alpha + d} - \alpha^2 \frac{1}{2N_\alpha + d + 2} \bar{\alpha}^a, \quad (2.11) \]

\[ \mathcal{G}_1^a \equiv \gamma^a - \gamma \alpha \frac{2}{2N_\alpha + d} \bar{\alpha}^a, \quad (2.12) \]

\[ \mathcal{G}_1^a \equiv \gamma^a - \alpha \gamma^a \frac{2}{2N_\alpha + d} \bar{\alpha}^a, \quad (2.13) \]

\(^3\)We use the oscillator to introduce generating ket-vectors and simplify our expressions (see also Refs.[14, 15, 16]).
\[
\Pi^{[1,3]} = 1 - \gamma \alpha \frac{1}{2N_\alpha + d} \gamma \bar{\alpha} - \alpha^2 \frac{1}{2(2N_\alpha + d + 2)} \bar{\alpha}^2 , \tag{2.14}
\]

\[
\Pi^{[1,2]} = 1 - \alpha^2 \frac{1}{2(2N_\alpha + d)} \bar{\alpha}^2 . \tag{2.15}
\]

The \(2 \times 2\) matrices \(\sigma_\pm, \sigma_3, \pi_\pm\), and antisymmetric products of \(\gamma\)-matrices are defined as
\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \pi_\pm = \frac{1}{2}(1 \pm \sigma_3) \tag{2.16}
\]

\[
\gamma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a) , \quad \gamma^{abc} = \frac{1}{3!}(\gamma^a \gamma^b \gamma^c \pm 5 \text{ terms}) \tag{2.17}
\]

The notation \(k' \in [n]_2\) implies that \(k' = -n, -n + 2, -n + 4, \ldots, n - 4, n - 2, n\):
\[
k' \in [n]_2 \iff k' = -n, -n + 2, -n + 4, \ldots, n - 4, n - 2, n \tag{2.18}
\]

Using the notation \(A^\dagger\) for the standard hermitian conjugated of the operator \(A\) we define operator \(A^\hat{\dagger}\) as follows
\[
A^\hat{\dagger} \equiv -\gamma^0 A^\dagger \gamma^0 . \tag{2.19}
\]

### 3 First-derivative gauge invariant Lagrangian

#### 3.1 Field content

In order to develop the first-derivative gauge invariant metric-like formulation of spin-(\(s + \frac{1}{2}\)) conformal non-chiral Dirac fermionic field in \(R^{d-1,1}\) space, \(d \geq 4\), we use the set of non-chiral spinor, vector-spinor, and tensor-spinor Dirac fields of the Lorentz algebra \(so(d - 1, 1)\) given by:

\[
\psi^{a_1 \cdots a_{s'}}_{k' \cdots k'}, \quad s' = 0, 1, \ldots, s , \quad k' \in [k s]_2 ; \tag{3.1}
\]

\[
\psi^{a_1 \cdots a_{s'}}_{k' \cdots k'}, \quad s' = \begin{cases} 0, 1, \ldots, s ; & \text{for } d \geq 6 ; \\ 1, 2, \ldots, s ; & \text{for } d = 4 ; \end{cases} \quad k' \in [k s' - 1]_2 ; \tag{3.2}
\]

where the spinor indices of the fermionic fields \(\psi^{a_1 \cdots a_{s'}}_{k' \cdots k'}\) are implicit and for some notation see (2.18). The fields \(\psi^{a_1 \cdots a_{s'}}_{k' \cdots k'}\) with \(s' = 0, s' = 1,\) and \(s' \geq 2,\) are the respective non-chiral spinor, vector-spinor, and tensor-spinor fermionic fields of the Lorentz algebra \(so(d - 1, 1)\). Chiral fermionic fields are discussed below.

For \(d \geq 6\), field content given in (3.1) can alternatively be represented as

\[
\psi^{a_1 \cdots a_{s}}_{k'} , \quad k' \in [k s]_2 ; \tag{3.3}
\]

\[
\psi^{a_1 \cdots a_{s}}_{k'}, \quad k' \in [k s - 1]_2 ; \tag{3.4}
\]

\[
\psi^{a_1 \cdots a_{s-1}}_{k'}, \quad k' \in [k s - 1]_2 ; \tag{3.5}
\]

\[
\psi^{a_1 \cdots a_{s-1}}_{k'}, \quad k' \in [k s - 1 - 1]_2 ; \tag{3.6}
\]

\[
\psi^{a_1 \cdots a_{s-1}}_{k'}, \quad k' \in [k s]_2 ; \tag{3.6}
\]
while, for the case of $d = 4$, field content (3.1) is given in (3.3)-(3.9). This is to say that fields in (3.10) enter field content only for $d \geq 6$.

We make comments on the field content.

i) In (3.1), the fields $\psi_{k'}^a$ and $\psi_{k'}^a$ are the respective non-chiral spinor and vector-spinor fields of the Lorentz algebra, while the fields $\psi_{k'}^{a_1\ldots a'_s}$, $s' > 1$, are rank-$s'$ totally symmetric non-chiral tensor-spinor fields of the Lorentz algebra $so(d - 1, 1)$.

ii) The tensor-spinor fields $\psi_{k'}^{a_1\ldots a'_s}$ with $s' \geq 3$ satisfy the $\gamma$ triple-tracelessness constraint,

$$\gamma^a \psi_{k'}^{abba_1\ldots a'_s} = 0, \quad s' \geq 3.$$  \hspace{1cm} (3.11)

iii) The conformal dimension of the fermionic field $\psi_{k'}^{a_1\ldots a'_s}$ is given by

$$\Delta(\psi_{k'}^{a_1\ldots a'_s}) = \frac{d - 1}{2} + k'.$$  \hspace{1cm} (3.12)

In order to illustrate our field content presented in (3.1) let us use the shortcut $\psi_{k'}^{s'}$ for the field $\psi_{k'}^{a_1\ldots a'_s}$. We note then that, for $d \geq 6$ and arbitrary $s$, fields in (3.1) can be presented as in Table I.
The spinor fields \( \psi_{k'}^0 \) having \( k' \in [k_0 - 1, 2] \) do not enter the field content when \( d = 4 \). Namely, for \( d = 4 \) and arbitrary \( s \), the field content in (3.1) can be represented as in Table II.

**TABLE II.** Field content for \( d = 4 \), \( s - \text{arbitrary} \). The notation \( \psi_{k'}^s \) stands for \( \psi_{k+1}^{a_1...a_s} \).

\[
\begin{array}{cccccccc}
\psi_{-s} & \psi_{-s+2} & \cdots & \\
\psi_{-s+1} & \psi_{-s+3} & \cdots & \psi_{s-3} & \psi_{s-1} \\
\psi_{-s+1} & \psi_{-s+3} & \cdots & \psi_{s-3} & \psi_{s-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\psi_{2} & \psi_{0} & \psi_{2} \\
\psi_{1} & \psi_{1} \\
\psi_{0} \\
\end{array}
\]

We note that \( d = 6 \) is the lowest space-time dimension when the spinor fields \( \psi_{k'}^0 \) having \( k' \in [k_0 - 1, 2] \) appear in the field content. This is to say that for the case of \( d = 6 \) and arbitrary \( s \), the field content given in (3.1) can be represented as in Table III.
The notation $\psi^s_{k'}$ stands for $\psi^{a_1\ldots a_s}_{k'}$.

Field content for $d = 6$, $s$ arbitrary.

| $\psi^s_{-s-1}$ | $\psi^s_{-s+1}$ | ... | $\psi^s_{s-1}$ | $\psi^s_{s+1}$ |
|-----------------|-----------------|----|----------------|----------------|
| $\psi^s_{-s}$   | $\psi^s_{-s+2}$ | ...| $\psi^s_{s-2}$ | $\psi^s_{s}$   |
| $\psi^s_{s-1}$  | $\psi^s_{s+1}$  | ...| $\psi^s_{s-2}$ | $\psi^s_{s-1}$ |
|                 |                 | ...|                |                |

To illustrate further the field content entering our approach we note that, for the case of $d = 4$ and spin-$\frac{3}{2}$ field, the field content shown in (3.1) is given by

Field content for $d = 4$ and spin-$\frac{3}{2}$ field

\[
\begin{align*}
\psi^1_{-2} & \quad \psi^1_0 & \quad \psi^1_2 \\
\psi^1_{-1} & \quad \psi^1_1 \\
\psi^0_{-1} & \quad \psi^0_1 \\
\psi^0 &
\end{align*}
\]

(3.13)

while, for the case of $d = 6$ and spin-$\frac{3}{2}$ field, the field content in (3.1) is given by

Field content for $d = 6$ and spin-$\frac{3}{2}$ field

\[
\begin{align*}
\psi^a_{-2} & \quad \psi^a_0 & \quad \psi^a_2 \\
\psi^a_{-1} & \quad \psi^a_1 \\
\psi^a_0 &
\end{align*}
\]

(3.14)

For the case of $d = 4$ and spin-$\frac{5}{2}$ field, the field content in (3.1) is given by

Field content for $d = 4$ and spin-$\frac{5}{2}$ field
\[
\begin{aligned}
\psi_{-2}^{ab} & \quad \psi_{-1}^{ab} \\
\psi_0^{ab} & \quad \psi_1^{ab} \\
\psi_{-1}^a & \quad \psi_0^a \\
\psi_1^a & \quad \psi_0^a \\
\psi_0 & \\
\end{aligned}
\]  

(3.15)

while, for the case of \(d = 6\) and spin-\(\frac{5}{2}\) field, the field content in (3.1) is given by

Field content for \(d = 6\) and spin-\(\frac{5}{2}\) field

\[
\begin{aligned}
\psi_{-3}^{ab} & \quad \psi_{-2}^{ab} \\
\psi_{-1}^{ab} & \quad \psi_1^{ab} \\
\psi_0^{ab} & \quad \psi_3^{ab} \\
\psi_{-2}^a & \quad \psi_0^a \\
\psi_1^a & \quad \psi_2^a \\
\psi_{-1}^a & \quad \psi_1^a \\
\psi_1 & \quad \psi_0 \\
\end{aligned}
\]  

(3.16)

To simplify our presentation we use the oscillators \(\alpha^a, \zeta, \nu^\oplus, \nu^\ominus\) to collect all fields appearing in (3.1) into the ket-vector \(|\psi\rangle\) given by

\[
|\psi\rangle = \sum_{s' = 0}^{s} \frac{\zeta^{s-s'}}{(s-s')!} |\psi^{s'}\rangle, \quad |\psi^{s'}\rangle = \left( \begin{array}{c} |\psi_{s'}^u\rangle \\ |\psi_{s'}^d\rangle \end{array} \right),
\]  

(3.17)

\[
|\psi_{s'}^u\rangle \equiv \sum_{k' \in [k_s+1]_2} \frac{\nu^\oplus_{k' + k'}}{s! \left( \frac{k_s + k'}{2} \right)!} \alpha^{a_1} \ldots \alpha^{a_{s'}} \psi_{k'}^{a_1 \ldots a_{s'}} |0\rangle,
\]  

(3.18)

\[
|\psi_{s'}^d\rangle \equiv \sum_{k' \in [k_s-1]_2} \frac{\nu^\ominus_{k' + k'}}{s! \left( \frac{k_s + k'}{2} \right)!} \alpha^{a_1} \ldots \alpha^{a_{s'}} \psi_{k'}^{a_1 \ldots a_{s'}} |0\rangle,
\]  

(3.19)

where \(k_{s'}\) is given in (3.2). For \(d = 4\), we use relation (3.20) in (3.19) to respect the fact that fields appearing in (3.10) do not enter the field content when \(d = 4\). It is easy to see that ket-vector (3.17) satisfies the following algebraic constraints:

\[
(N_{\alpha} + N_{\zeta} - s)|\psi\rangle = 0,
\]  

(3.21)

\[
(N_{\zeta} + N_{\nu} - k_{s})\pi_+ |\psi\rangle = 0,
\]  

(3.22)

\[
(N_{\zeta} + N_{\nu} - k_{s} + 1)\pi_- |\psi\rangle = 0.
\]  

(3.23)
From (3.21) we learn that the ket-vector $|\psi\rangle$ (3.17) is degree-$s$ homogeneous polynomial in the oscillators $\alpha^a$. Using (3.18), (3.19), we introduce the following ket-vectors

$$|\psi_u\rangle = \sum_{s'=0}^{s} \frac{\zeta^{s-s'}}{\sqrt{(s-s')!}} |\psi_{s'}\rangle,$$

$$|\psi_u\rangle = \sum_{s'=0}^{s} \frac{\zeta^{s-s'}}{\sqrt{(s-s')!}} |\psi_{s'}\rangle.$$  

(3.24)

From (3.22), we learn that the ket-vector $|\psi_u\rangle$ (3.24) is degree-$k_s$ homogeneous polynomial in the oscillators $\zeta$, $\upsilon^0$, $\upsilon^\alpha$, while, from (3.23), we learn that ket-vector $|\psi_u\rangle$ (3.24) is degree-$(k_s - 1)$ homogeneous polynomial in the oscillators $\zeta$, $\upsilon^0$, $\upsilon^\alpha$. In terms of the ket-vector $|\psi\rangle$, the $\gamma$ triple-tracelessness constraint (3.11) takes the form

$$\gamma \bar{\alpha} \alpha^2 |\psi\rangle = 0.$$  

(3.25)

### 3.2 Lagrangian and gauge symmetries

We are now ready to discuss gauge invariant Lagrangian. Lagrangian of conformal spin-$(s + \frac{1}{2})$ fermionic field we found takes the form

$$i\mathcal{L} = \langle \psi | E | \psi \rangle,$$

$$E \equiv E_{(1)} + E_{(0)};$$

$$E_{(1)} \equiv \frac{\partial}{\partial \zeta} - \alpha \alpha \bar{\alpha} \bar{\alpha} + \gamma \alpha \bar{\alpha} \bar{\alpha} + \frac{1}{2} \gamma \alpha \bar{\alpha} \bar{\alpha} + \frac{1}{2} \alpha^2 \gamma \bar{\alpha} \bar{\alpha} \bar{\alpha} - \frac{1}{4} \alpha^2 \bar{\alpha} \bar{\alpha}^2 - (3.27)$$

$$E_{(0)} = (1 - \gamma \alpha \gamma \bar{\alpha} - \frac{1}{4} \alpha^2 \bar{\alpha}^2)g^\gamma + (\gamma \alpha - \frac{1}{2} \alpha^2 \gamma \bar{\alpha})\bar{g} + (\gamma \bar{\alpha} - \frac{1}{2} \alpha \gamma \bar{\alpha}^2)g;$$

$$g^\gamma = g^\gamma_{\zeta}(\upsilon^0 \sigma_+ + \bar{\upsilon}^0 \sigma_-),$$

$$g = \zeta \bar{\upsilon}^0 g_{\zeta}, \quad \bar{g} = -g_{\zeta} \upsilon^0 \bar{\zeta},$$

$$g^\zeta_{\zeta} \equiv \frac{2s + d - 2}{2s + d - 2 - 2N_\zeta}, \quad g_{\zeta} \equiv \left(\frac{2s + d - 3 - N_\zeta}{2s + d - 4 - 2N_\zeta}\right)^{1/2}.$$  

(3.28) (3.29) (3.30) (3.31)

Note that the $E_{(1)}$ (3.27) is the well known first-order Fang-Fronsdal operator represented in terms of the oscillators. The bra-vector $\langle \psi |$ is defined according the rule $\langle \psi | \equiv (|\psi\rangle)^\dagger \gamma^0$.

We now proceed with the discussion of gauge symmetries of Lagrangian (3.26). First, we introduce gauge transformation parameters given by:

$$\xi_{k' - 1}^{a_1 \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s - 1, \quad k' \in [k_s + 1];$$

$$\xi_{k' - 1}^{a_1 \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s - 1, \quad k' \in [k_s + 1];$$

(3.32) (3.33)

where the spinor indices of the fields $\xi_{k' - 1}^{a_1 \cdots a_{s'}}$ are implicit. The parameters $\xi_{k' - 1}^{a_1 \cdots a_{s'}}$ are non-chiral, spinor, vector-spinor, and tensor-spinor fields of the Lorentz algebra $so(d - 1, 1)$.

Gauge transformation parameters appearing in (3.32), (3.33) can alternatively be represented as

$$\xi_{k' - 1}^{a_1 \cdots a_{s - 1}}; \quad k' \in [k_s];$$

$$\xi_{k' - 1}^{a_1 \cdots a_{s - 1}}; \quad k' \in [k_s - 1];$$

(3.34) (3.35)

4In this paper, we adapt the formulation in terms of the $\gamma$ triple-traceless fermionic fields in Ref.[17]. For the use of unconstrained gauge fields see Refs.[18, 19].
\begin{align}
\xi_{k'_{-1}}^{a_{1\ldots a_{s'}}}, & \quad k' \in [k_{s-1}]_2; \quad (3.36) \\
\xi_{k'_{-1}}^{a_{1\ldots a_{s'}}}, & \quad k' \in [k_{s-1} - 1]_2; \quad (3.37) \\
\cdots & \quad \cdots \\
\xi_{k'_{-1}}^{a_{1\ldots a_{s'}}}, & \quad k' \in [k_2]_2; \quad (3.38) \\
\xi_{k'_{-1}}^{a_{1\ldots a_{s'}}}, & \quad k' \in [k_2 - 1]_2; \quad (3.39) \\
\xi_{k'_{-1}}^{a_{1\ldots a_{s'}}}, & \quad k' \in [k_1]_2; \quad (3.40) \\
\xi_{k'_{-1}}^{a_{1\ldots a_{s'}}}, & \quad k' \in [k_1 - 1]_2; \quad (3.41)
\end{align}

We note that

i) In (3.32),(3.33) the fields \( \xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}} \) and \( \xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}} \) are the respective non-chiral spinor and vector-spinor fields of the Lorentz algebra, while the fields \( \xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}}, s' > 1 \), are rank-\( s' \) totally symmetric non-chiral tensor-spinor fields of the Lorentz algebra \( so(d - 1, 1) \).

ii) The vector-spinor fields \( \xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}} \) and tensor-spinor fields \( \xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}} \) with \( s' \geq 1 \) satisfy the \( \gamma \)-tracelessness constraint

\[
\gamma^{a_{1\ldots a_{s'}}} = 0, \quad s' \geq 1.
\]

iii) The conformal dimension of the gauge transformation parameter \( \xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}} \) is given by

\[
\Delta(\xi_{k'_{-1}}^{\alpha_{1\ldots a_{s'}}}) = \frac{d - 1}{2} + k' - 1.
\]

Second, we collect the gauge transformation parameters into a ket-vector \( |\xi> \) given by

\[
|\xi> = \sum_{s'=0}^{s-1} \sqrt{(s - 1 - s')!} |\xi_{s'}^{s'}>, \quad |\xi_{s'}^{s'}> = \left( \begin{array}{c}
|\xi_{u}^{s'}> \\
|\xi_{d}^{s'}>
\end{array} \right),
\]

\[
|\xi_{u}^{s'}> = \sum_{k'=|k_{s+1}|} \frac{(t^\otimes)^{k_{s+1}+1}}{s! (k_{s+1}+1)!} \alpha^a \cdots \alpha^{a_{s'}} \xi_{k'_{-1}}^{\alpha_{a_{1\ldots a_{s'}}}} |0>,
\]

\[
|\xi_{d}^{s'}> = \sum_{k'=|k_{s+1}|} \frac{(t^\otimes)^{k_{s+1}+1}}{s! (k_{s+1}+1)!} \alpha^a \cdots \alpha^{a_{s'}} \xi_{k'_{-1}}^{\alpha_{a_{1\ldots a_{s'}}}} |0>.
\]

We note that that ket-vector appearing in (3.44) satisfies algebraic constraints given by

\[
(N_{a} + N_{\zeta} - s + 1)|\xi> = 0, \quad (3.47)
\]

\[
(N_{\zeta} + N_{\nu} - k_{s}) \pi_{+} |\xi> = 0, \quad (3.48)
\]

\[
(N_{\zeta} + N_{\nu} - k_{s} + 1) \pi_{-} |\xi> = 0, \quad (3.49)
\]

where \( k_{s'} \) is defined in (3.2). Algebraic constraints (3.47) imply that ket-vector \( |\xi> \) (3.44) is degree-\( s - 1 \) homogeneous polynomial in the oscillators \( \alpha^a, \zeta \). Introducing the notation

\[
|\xi_{u}^{s'}> = \sum_{s'=0}^{s-1} \sqrt{(s - 1 - s')!} |\xi_{u}^{s'}>, \quad |\xi_{d}^{s'}> = \sum_{s'=0}^{s-1} \sqrt{(s - 1 - s')!} |\xi_{d}^{s'}>,
\]

\[
|\xi_{u}> = \sum_{s'=0}^{s-1} \sqrt{(s - 1 - s')!} |\xi_{u}^{s'}>, \quad |\xi_{d}> = \sum_{s'=0}^{s-1} \sqrt{(s - 1 - s')!} |\xi_{d}^{s'}>.
\]
we note that algebraic constraint (3.48) implies that ket-vector $|\xi_u\rangle$ (3.50) is degree-$k_s$ homogeneous polynomial in the oscillators $\zeta, v^\ominus, v^\oplus$, while, from algebraic constraint (3.49), we learn that ket-vector $|\xi_d\rangle$ (3.50) is degree-$(k_s - 1)$ homogeneous polynomial in the oscillators $\zeta, v^\ominus, v^\oplus$. In terms of the ket-vector $|\xi\rangle$, $\gamma$-tracelessness constraint (3.42) takes the form

$$\gamma \bar{\alpha} |\xi\rangle = 0 .$$

(3.51)

Third, we note that gauge transformations can be presented in terms of $|\psi\rangle$ and $|\xi\rangle$. Namely, the gauge transformations take the form

$$\delta |\psi\rangle = G |\xi\rangle , \quad G \equiv \alpha \partial - g + \gamma \alpha \frac{1}{2N_{\alpha} + d - 2} g^\Gamma - \alpha^2 \frac{1}{2N_{\alpha} + d} \bar{g} ,$$

(3.52)

where operators $g^\Gamma, g, \bar{g}$ are defined in (3.29)-(3.31).

**Chiral conformal fermionic fields.** In the above discussion, we considered conformal non-chiral Dirac fermionic fields (3.17). Extension of our discussion to the case of conformal chiral fermionic fields is straightforward. To this end we introduce matrix $\Gamma_*$ defined as

$$\Gamma_* \equiv \gamma_* \sigma_3 , \quad \Gamma_*^2 = 1 , \quad \Gamma_*^\dagger = \Gamma_* ,$$

$$\gamma_* \equiv \epsilon \gamma^0 \gamma^1 \ldots \gamma^{d-1} , \quad \gamma_*^2 = 1 , \quad \gamma_*^\dagger = \gamma_* , \quad \epsilon^2 = (-)^{(d-2)/2} ,$$

(3.53)

(3.54)

where $\sigma_3$ is defined in (2.16). Now we introduce chiral ket-vectors $|\psi_\pm\rangle$ defined as

$$|\psi_\pm\rangle = \Pi_\pm |\psi\rangle , \quad \Pi_\pm \equiv \frac{1}{2} (1 \pm \Gamma_*) .$$

(3.55)

We verify that the matrix $\Gamma_*$ anticommutes with operator $E$ (3.26) entering Lagrangian (3.26),

$$\{\Gamma_*, E\} = 0 .$$

(3.56)

Using (3.56) and the relations $\{\Gamma_*, \gamma^0\} = 0, \Pi_\pm^\dagger = \Pi_\pm$, we see that Lagrangian (3.26) is decomposed as

$$\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_- \quad \text{i} \mathcal{L}_\pm = \langle \psi_\pm | E | \psi_\pm \rangle .$$

(3.57)

The Lagrangian $\mathcal{L}_+$ describes positive chirality conformal fermionic field $|\psi_+\rangle$, while the Lagrangian $\mathcal{L}_-$ describes negative chirality conformal fermionic field $|\psi_-\rangle$. Note also that the projectors $\Pi_\pm$ are commuting with operator $G$ (3.52), $[\Pi_\pm, G] = 0$. Taking this into account, we see that the Lagrangians $\mathcal{L}_+$ and $\mathcal{L}_-$ are invariant under the respective gauge transformations

$$\delta |\psi_+\rangle = G |\xi_+\rangle , \quad \delta |\psi_-\rangle = G |\xi_-\rangle , \quad |\xi_\pm\rangle \equiv \Pi_\pm |\xi\rangle .$$

(3.58)

## 4 Realization of conformal symmetries

In order to complete our formulation of spin-$(s + \frac{1}{2})$ conformal fermionic field we should describe a realization of symmetries of the conformal algebra on a space of the ket-vector $|\psi\rangle$ (3.17). We now present the realization we obtained.

Conformal symmetries for $R^{d-1,1}$ space are described by the $so(d, 2)$ algebra. This algebra consists of translation generators $P^a$, Lorentz rotation generators $J^{ab}$ which span $so(d - 1, 1)$ algebra,
dilatation generator $D$, and conformal boost generators $K^a$. We use the following commutators of the conformal algebra:

\[
[D,P^a] = -P^a, \quad [P^a,J^{bc}] = \eta^{ab}P^c - \eta^{ac}P^b, \quad (4.1)
\]

\[
[D,K^a] = K^a, \quad [K^a,J^{bc}] = \eta^{ab}K^c - \eta^{ac}K^b, \quad (4.2)
\]

\[
[P^a,K^b] = \eta^{ab}D - J^{ab}, \quad [J^{ab},J^{ce}] = \eta^{bc}J^{ae} + 3 \text{ terms}. \quad (4.3)
\]

Let $|\psi\rangle$ stands for fermionic field which propagate in $R^{d-1,1}$. Action for the free field $|\psi\rangle$ should be invariant under the transformations

\[
\delta_G|\psi\rangle = G\text{diff}|\psi\rangle, \quad (4.4)
\]

where the realization of the generators of the conformal algebra $so(d,2)$ in terms of differential operators $G\text{diff}$ acting on the ket-vector $|\psi\rangle$ is given by

\[
P^a = \partial^a, \quad J^{ab} = x^a\partial^b - x^b\partial^a + M^{ab}, \quad (4.5)
\]

\[
D = x\partial + \Delta, \quad K^a = K^a_{\Delta,M} + R^a, \quad (4.6)
\]

\[
K^a_{\Delta,M} \equiv -\frac{1}{2}x^2\partial^a + x^aD + M^{ab}x^b. \quad (4.7)
\]

In expressions (4.5)-(4.7), the quantities $\Delta$ and $M^{ab}$ stand for the respective operator of conformal dimension and operator of the Lorentz algebra spin. An operator $R^a$ appearing in (4.6) depends on derivatives with respect to space-time coordinates and does not depend on space-time coordinates $x^a$, $[P^a,R^b] = 0$.\(^6\) The operator $M^{ab}$ is known for arbitrary spin conformal fermionic field. We note that, in higher-derivative approach, the operator $R^a$ is trivial, while, in our first-derivative formulation, the operator, in general, $R^a$ is non-trivial. This is to say that, in our first-derivative approach, the complete formulation of conformal fields requires finding, among other things, the operator $R^a$.

Thus, all that remains, is to find the operators $M^{ab}$, $\Delta$, and $R^a$. For the case of totally symmetric arbitrary spin-$(s + \frac{1}{2})$ conformal fermionic field, the spin operator $M^{ab}$ and the conformal dimension operator $\Delta$ are given by

\[
M^{ab} = \alpha^a\bar{\alpha}^b - \alpha^b\bar{\alpha}^a + \frac{1}{2}\gamma^{ab}, \quad (4.8)
\]

\[
\Delta = \frac{d-1}{2} + \Delta', \quad \Delta' \equiv N_{\nu\bar{\nu}} - N_{\nu\bar{\nu}}. \quad (4.9)
\]

Obviously, expression for the conformal dimension operator $\Delta$ given in (4.9) can be read from the relations given in (3.12). Realization of the operator $R^a$ on fermionic ket-vector $|\psi\rangle$ we found takes the form

\[
R^a = r^a_{(0)} + r^a_{(1)} + R^a_G + R^a_E, \quad (4.10)
\]

\[
r^a_{(0)} = r^a_{(0,1)}G^a + r^a_{(0,1)}\bar{\alpha}^a + \bar{r}_{(0,1)}A^a, \quad r^a_{(1)} = r_{1,1}^a\partial^a, \quad (4.11)
\]

\(^5\)Note that in our approach, only $so(d-1,1)$ symmetries are realized manifestly. The $so(d,2)$ symmetries of conformal fields could be realized manifestly by using ambient space approach (see, e.g., Refs.[20]-[23].)

\(^6\)For conformal currents and shadow fields considered in Refs.[24, 25, 26], the operator $R^a$ is independent of the derivatives.
\[ R_G^a = G r_G^a, \]  
\[ R_E^a = r_E^a E, \]  
\[ r_{0,1}^e = g_{\xi}^e (\bar{v}^0 \sigma_+ - v^0 \sigma_+), \]  
\[ r_{0,1} = 2 \zeta g_{\xi}^e \bar{v}^0, \]  
\[ r_{1,1} = -2 v^0 g_{\xi}^e, \]  
\[ \bar{r}_{0,1} = -2 v^0 \bar{v}^0, \]  
\[ r_G^a = r_{G,1} G_1^a \Pi^{[1,3]} + r_{G,2} G_1^a \Pi^{[1,3]} \gamma \bar{\alpha} + r_{G,3} G_1^a \alpha^2 \]  
\[ + r_{G,4} G_1^a \Pi^{[1,3]} \alpha^2 + r_{G,5} G_1^a \Pi^{[1,3]} \gamma \bar{\alpha} \alpha^2 + r_{G,6} G_1^a \alpha^2, \]  
\[ r_E^a = r_{E,1} G_1^a \Pi^{[1,3]} + r_{E,2} G_1^a \Pi^{[1,3]} \gamma \bar{\alpha} + r_{E,3} G_1^a \alpha^2 \]  
\[ + \left( r_{E,4} G_1^a \Pi^{[1,3]} + r_{E,5} G_1^a \Pi^{[1,3]} \gamma \bar{\alpha} + r_{E,6} G_1^a \alpha^2 \right), \]  
\[ r_{G,n} = v^0 \bar{r}_{G,n} \bar{v}^0 \sigma_+ + v^0 \bar{r}_{G,n} \bar{v}^0 \sigma_-, \quad n = 2, 4, 9; \]  
\[ r_{G,n} = (v^0 \bar{v}^0 \bar{r}_{G,n} + v^0 \bar{r}_{G,n} \bar{v}^0 \sigma_+ - \zeta), \quad n = 1, 8; \]  
\[ r_{G,n} = \zeta (v^0 \bar{v}^0 \bar{r}_{G,n} \sigma_+ + v^0 \bar{r}_{G,n} \bar{v}^0 \sigma_+), \quad n = 3, 5; \]  
\[ r_{G,n} = (v^0 \bar{v}^0 \bar{r}_{G,n} \sigma_+ + v^0 \bar{v}^0 \bar{r}_{G,n} \sigma_-), \quad n = 7; \]  
\[ r_{G,n} = \zeta^2 (v^0 \bar{r}_{G,n} \bar{v}^0 \sigma_+ + v^0 \bar{v}^0 \bar{r}_{G,n} \sigma_+), \quad n = 6; \]  
\[ r_{E,n} = v^0 \bar{r}_{E,n} \bar{v}^0 \sigma_+ + v^0 \bar{r}_{E,n} \bar{v}^0 \sigma_-, \quad n = 1, 2, 3, 7, 8; \]  
\[ r_{E,n} = (v^0 \bar{v}^0 \bar{r}_{E,n} \sigma_+ + v^0 \bar{r}_{E,n} \bar{v}^0 \sigma_+ - \zeta), \quad n = 4, 5, 9, 10, 11, 12; \]  
\[ r_{E,n} = (v^0 \bar{v}^0 \bar{r}_{E,n} \sigma_+ + v^0 \bar{v}^0 \bar{r}_{E,n} \sigma_-), \quad n = 6, 13, 14; \]  
\[ r_{E,n} = (v^0 \bar{v}^0 \bar{r}_{E,n} \sigma_+ + v^0 \bar{v}^0 \bar{r}_{E,n} \sigma_-), \quad n = 15; \]  
\[ r_{E,n}^r = r_{E,n}, \quad \text{for } n = 1, 2, 3, \]  
where the quantities \( G, E \) (4.12),(4.13) are defined in (3.52),(3.26), while the quantities \( g_\xi^e, g_\zeta \) (4.14), (4.15) are defined in (3.31). In (4.18)-(4.27), the quantities \( \bar{r}_{G,n} \pm \) and \( \bar{r}_{E,n} \pm \) remain to be arbitrary functions of the operators \( N_\xi, \Delta' \).

The following remarks are in order.

i) The operators \( r_{G,0}^a \) and \( r_{E,1}^a \) (4.11) entering the operator \( R_G^a \) are fixed unambiguously, while the operators \( R_G^a, R_E^a \) remain to be arbitrary because of the \( \bar{r}_{G,n}, \bar{r}_{E,n} \) are arbitrary functions of the \( N_\xi, \Delta' \). The reason for arbitrariness governed by the operator \( R_G^a \) is clear: the \( so(d, 2) \) algebra transformations of all gauge fields are defined by module of gauge transformations. As the operator \( R_G^a \) (4.12) is proportional to the operator of gauge transformation \( G \), the action of the operator \( R_G^a \)
on ket-vector $|\psi\rangle$ takes the form of gauge transformation with a gauge transformation parameter equal to $r^a_G|\psi\rangle$. The reason for arbitrariness governed by the operator $R_E^a$ is also clear: the $so(d,2)$ algebra transformations of fermionic fields are defined by module of $\tau E|\psi\rangle$, where $\tau$ is an arbitrary operator that satisfies the condition $\tau^\dagger = \gamma^0 r^\dagger \gamma^0$. It is easy to see that operator $r^a_E$ given in (4.17) satisfies this condition.

ii) Considering $R_E^a = 0$, $R_E^{\alpha} = 0$, we verify the commutator $[K^a, K^b] = 0$.

iii) Considering $R_E^a \neq 0$, $R_E^{\alpha} \neq 0$, we find the following expression for commutator $[K^a, K^b]$:

$$[K^a, K^b] = W^{ab},$$  \hspace{1cm} (4.28)

From (4.28), we learn that the commutator $[K^a, K^b]$ is proportional to the operator of gauge transformations $G$ and to operator $E$ as it should be for fermionic gauge fields.

iv) It is easy to see that the operator $R^a$ (4.10) is commuting with the projectors $\Pi_\pm$ (3.55) which enter positive and negative chirality conformal fields $|\psi\rangle$,

$$[\Pi_\pm, R^a] = 0.$$  \hspace{1cm} (4.33)

Using (4.33), we make sure that the projectors $\Pi_\pm$ are commuting with all generators of conformal algebra (4.5),(4.6). This implies that Lagrangians $L_+$ and $L_-$ (3.57) for the respective positive and negative chirality conformal fields are invariant under conformal algebra transformations.

To summarize we note that the Lagrangian, gauge transformations, and the operator $R^a$ of the conformal fermionic fields are fixed unambiguously by the following three requirements.

i) Lagrangian and gauge transformations of fermionic conformal fields should not consist higher than first order terms in derivatives.

ii) the operator $R^a$ entering conformal boost generators should not consist higher than first order terms in derivatives;

iii) Action of the fermionic gauge field should be invariant under conformal algebra transformations and gauge transformations.

These three requirements allow us to fix the Lagrangian and gauge transformations unambiguously. The operator $R^a$ is also fixed unambiguously by module of the gauge transformation operator $G$ and the operator $E$ (as it should be for fermionic gauge fields).

5 On-shell degrees of freedom of conformal field

For $d = 4$ and $s \geq 1$, on-shell D.o.F of the spin-$(s + \frac{1}{2})$ conformal fermionic field were found in Ref.[3]. Decomposition of on-shell D.o.F into irreps of the $so(2)$ algebra was considered only for the case of spin-$\frac{3}{2}$ conformal field in Ref.[28]. For arbitrary values $s$, $s \geq 1$, and arbitrary dimension of space, $d > 4$, on-shell D.o.F of the spin-$(s + \frac{1}{2})$ conformal fermionic field have not been discussed so far in the literature. In this section, first, we present our result for on-shell D.o.F

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7Discussion of uniqueness of the higher-derivative formulation for interacting spin-2 conformal field theory may be found in Ref.[27].
of the totally symmetric arbitrary spin-\((s + \frac{1}{2})\) conformal fermionic field in \(R^{d-1,1}\) space. Second, we present our results for the decomposition of the on-shell D.o.F into irreps of the \(so(d - 2)\) algebra for the case of arbitrary values of \(s\) and \(d\).

In order to find on-shell D.o.F of the conformal fermionic field we use the light-cone gauge. This is to say that we use fields that transform as non-chiral representations of the \(so(d - 2)\) algebra and decompose the on-shell D.o.F of conformal fermionic field into such non-chiral representations.\(^8\) We find that complex-valued on-shell D.o.F of the totally symmetric spin-\((s + \frac{1}{2})\) conformal non-chiral fermionic field in \(d\)-dimensional space, \(d \geq 4\), are described by the following set of non-chiral half-integer fields of the \(so(d - 2)\) algebra:

\[
\begin{align*}
\psi_{k'}^{i_1 \ldots i_{s'}}^{}, & \quad s' = 0, 1, \ldots, s, \quad k' \in [k, 1]; \tag{5.1} \\
\psi_{k'}^{i_1 \ldots i_{s'}}^{}, & \quad s' = \begin{cases} 
0, 1, \ldots, s; & \text{for } d \geq 6; \\
1, 2, \ldots, s; & \text{for } d = 4;
\end{cases} \quad k' \in [k, s - 1];
\end{align*}
\]

where \(k, s'\) is defined in (3.2) and vector indices of the \(so(d - 2)\) algebra take the following values \(i = 1, 2, \ldots, d - 2\). In (5.1), the fields \(\psi_{k'}^i\) and \(\psi_{k'}^{i_1 \ldots i_{s'}}\) are the respective non-chiral spinor and vector-spinor fields of the \(so(d - 2)\) algebra, while the fields \(\psi_{k'}^{i_1 \ldots i_{s'}}^{}, s' \geq 2\), are rank-\(s'\) totally symmetric tensor-spinor fields of the \(so(d - 2)\) algebra. Fields \(\psi_{k'}^{i_1 \ldots i_{s'}}\) with \(s' \geq 1\) are \(\gamma\)-traceless,

\[
\gamma^i \psi_{k'}^{i_2 \ldots i_{s'}}^{} = 0, \quad s' \geq 1. \tag{5.2}
\]

In view of (5.2), the vector-spinor and tensor-spinor fields \(\psi_{k'}^{i_1 \ldots i_{s'}}\) transform as non-chiral irreps of the \(so(d - 2)\) algebra. Obviously, our conformal fermionic field is related to non-unitary representation of the conformal algebra \(so(d, 2)\).\(^9\)

For \(d \geq 6\), field content (5.1) can alternatively be represented as

\[
\begin{align*}
\psi_{k'}^{i_1 \ldots i_{s'}}, & \quad k' \in [k, s]; \tag{5.3} \\
\psi_{k'}^{i_1 \ldots i_{s'}}, & \quad k' \in [k, s - 1]; \tag{5.4} \\
\psi_{k'}^{i_1 \ldots i_{s - 1}}, & \quad k' \in [k, s - 1]; \tag{5.5} \\
\psi_{k'}^{i_1 \ldots i_{s - 1}}, & \quad k' \in [k, s - 1]; \tag{5.6} \\
\ldots & \quad \ldots \\
\psi_{k'}^i, & \quad k' \in [k, 1]; \tag{5.7} \\
\psi_{k'}^i, & \quad k' \in [k, 1]; \tag{5.8} \\
\psi_{k'}^k, & \quad k' \in [k, 1]; \tag{5.9} \\
\psi_{k'}^k, & \quad k' \in [k, 1]; \tag{5.10}
\end{align*}
\]

For \(d = 4\), field content (5.1) is given in (5.3)-(5.9). This is to say that spinor fields in (5.10) enter field content only for \(d \geq 6\).

Number of the complex-valued on-shell D.o.F of non-chiral fermionic fields shown in (5.1) is given by

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\(^8\)Alternative methods for counting on-shell D.o.F were discussed in Refs.[28, 29].

\(^9\)Detailed study of representations of (super)conformal algebras may be found, e.g., in Refs.[30]-[34].
\[ n = 2^{d-2} (d-3)(2s + d - 2) \frac{(s + d - 3)!}{s!(d-2)!}. \]

For \( d = 4 \), relation (5.11) gives the following expressions for \( n \):

\[ n \big|_{s \text{-arbitrary}; \ d=4} = 2(s + 1)^2. \]

Result for \( n \) in (5.12) was obtained in Ref.[3]. Thus, for \( d = 4 \) and arbitrary \( s \), our result (5.11) agrees with result in Ref.[3] and provides \( n \) (5.11) for the case of arbitrary \( s \) and \( d \).

To summarize, our result presented in relation (5.1) provides the decomposition of the complex-valued on-shell D.o.F into non-chiral half-integer representations of \( so(d-2) \) algebra, while our expression for \( n \) (5.11) gives the complex-valued number of on-shell D.o.F of non-chiral fermionic conformal fields appearing in (5.1).

Expression for \( n \) in (5.11) describes number of the complex-valued on-shell D.o.F of non-chiral conformal field. Numbers of complex-valued on-shell D.o.F of chiral conformal fermionic fields \( |\psi_{\pm}\rangle \) (3.55), which we denote by \( n_{\pm} \), are obtained in a obvious way

\[ n_{\pm} = \frac{1}{2} n. \]

Finally, let us explain how \( n \) (5.11) is obtained from decomposition in (5.1). By definition, \( n \) (5.11) is a sum of tensor-spinorial components of fermionic fields (5.1) that subject to algebraic \( \gamma \)-tracelessness constraint (5.2). Namely, the \( n \) can be represented as

\[ n = \sum_{s'=0}^{s} n^{s'}, \quad n^{s'} = \sum_{k' \in [k'_s]} n(\psi^{s'}_{k'}), \quad \sum_{k' \in [k'_s-1]} n(\psi^{s'}_{k'}), \]

\[ n(\psi^{s'}_{k'}) = 2^{d-2} \frac{(s'+d-4)!}{s!(d-4)!}, \]

where \( n(\psi^{s'}_{k'}) \) is a number of D.o.F of rank-\( s' \) \( \gamma \)-traceless non-chiral tensor-spinor field \( \psi^{i_1\ldots i_{s'}}_{k'} \). In other words, \( n(\psi^{s'}_{k'}) \) is a dimension of the rank-\( s' \) \( \gamma \)-traceless non-chiral tensor-spinor field of the \( so(d-2) \) algebra. Using expression given in (5.15), we find that relation for \( n^{s'} \) (5.14) leads to following expression:

\[ n^{s'} = 2^{d-2} \frac{(s'+d-4)!}{s!(d-4)!} (2k_{s'} + 1) \]

\[ = 2^{d-2} \frac{(s'+d-4)!}{s!(d-4)!} (2s' + d - 3). \]

We now note that expression for \( n \) given in (5.11) is obtained by substitution of \( n^{s'} \) (5.16) into \( n \) in (5.14) and using the following relation,

\[ \sum_{s'=0}^{s} \frac{(s'+t)!}{s'!} = \frac{(s + t + 1)!}{(t + 1)s!}. \]
6 Conclusions

In this paper, we generalized results in Refs.[10, 11] to the case of arbitrary spin conformal fermionic fields. The results presented in this paper might have the following further interesting applications and generalizations.

i) BRST formulation of conformal fermionic fields. BRST approach have fruitfully been used for the investigation of gauge invariant formulation of massive fields (see, e.g., Refs.[35]). We noted in Ref.[11], that gauge invariant formulation of massive fields and our formulation of conformal fields have many common features. Recent discussion of BRST method for higher-spin massive fermionic fields may be found in Ref.[36]. Therefore we think that first-derivative formulation of conformal fermionic fields in the framework of BRST method can straightforwardly be reached. The second-derivative BRST formulation of bosonic conformal fields was obtained in Ref.[37].

ii) Interacting conformal fields theories. We note that most of approaches to theories of interacting higher-spin fields in Refs.[38]-[48] can straightforwardly be adapted for the case of conformal fields. In our approach to conformal fields, use of Stueckelberg fields is similar to the use of Stueckelberg fields in gauge invariant approach to massive fields. As is well known, the Stueckelberg fields provide systematical setup for the investigation of interacting massive gauge fermionic fields (see, e.g., Ref.[49]). We think therefore that application of our approach to theory of interacting conformal fermionic fields might lead to new interesting development. It is worthwhile to mention the BRST approach also involves Stueckelberg fields and this approach turns out to be fruitful for the investigation of interacting higher-spin field theories (see, e.g., Refs.[50]-[55]).

iii) Fermionic unconstrained conformal gauge fields. Formulations of various higher-spin field theories in terms of unconstrained gauge fields were studied in Refs.[18, 19]. We think that adaptation of those formulations for conformal fields might be useful for better understanding the first-derivative conformal fermionic fields.

iv) Mixed-symmetry conformal fermionic fields. In the recent time, mixed-symmetry fields have extensively been studied in the literature (see, e.g., Ref.[56]). We think that, in view of potentially interesting application to string theory, study of the first-derivative formulation of mixed-symmetry conformal fermionic fields is important. Discussion of higher-derivative mixed-symmetry bosonic conformal fields may be found in Ref.[5], while the discussion of self-dual conformal fields in the framework of the second-derivative approach may be found in Ref.[57]. Needless to say that the study of conformal fermionic fields along the lines in Ref.[58] could be also of some interest.

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