MIXED MULTIFRACTAL SPECTRA OF BIRKHOFF AVERAGES FOR NON-UNIFORMLY EXPANDING ONE-DIMENSIONAL MARKOV MAPS WITH COUNTABLY MANY BRANCHES

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Abstract. For a Markov map of the interval or the circle with countably many branches and finitely many neutral periodic points, we establish conditional variational formulas for the mixed multifractal spectrum of Birkhoff averages of countably many observables, in terms of the Hausdorff dimension of invariant probability measures. Using our results, we describe mixed multifractal spectra for the Rényi map generating the backward continued fraction expansion, as well as multi-cusp winding spectra for the Bowen-Series maps associated with finitely generated free Fuchsian groups with parabolic elements.

1. Introduction

Borel’s normal number theorem states that Lebesgue almost every real number has the property that the limiting frequency of each digit in the decimal expansion is $1/10$. It is therefore natural to ask for the Hausdorff dimension of the set of real numbers with a different limiting behavior. This problem was investigated by Besicovitch [6] and Eggleston [11] in the 30s and 40s, who computed the Hausdorff dimension of the set of real numbers whose digits have limiting frequency given by any prescribed probability vector.

This number-theoretic problem can be generalized as follows. Given a dynamical system on a metric space $X$, a sequence $A_1, \ldots, A_k$ of pairwise disjoint subsets of $X$, and a probability vector $\alpha = (\alpha_1, \ldots, \alpha_k)$, is it possible to describe the Hausdorff dimension of the set of initial points whose orbits visit each $A_j$ with frequency $\alpha_j$? One can also consider a frequency of visits to a countably infinite number of subsets, and even more generally, Birkhoff averages of a countably infinite number of observables. The problem is to describe \textit{mixed multifractal spectra of Birkhoff averages}, or simply \textit{mixed Birkhoff spectra} of countably many observables.

For certain maps having a Markov structure, several results are known which describe mixed Birkhoff spectra: Fan and Feng [12], and Olsen [37, 38] on finite shift spaces; Fan et al. [15] on the Hausdorff dimension of Besicovitch-Eggleston sets on infinite shift spaces; Fan et al. [13, 14] on uniformly expanding Markov interval maps with infinitely many branches. Other mixed multifractal spectra were considered in [3].

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In this paper, we describe mixed Birkhoff spectra for non-uniformly expanding one-dimensional Markov maps with a countable (finite or infinite) number of branches. The lack of uniform expansion is due to the existence of neutral periodic points. Various types of results in different settings on dimension theory for such parabolic systems (i.e., maps with neutral periodic points) are fairly in abundance. We refer to [16, 21, 27, 28, 35, 42, 51] for results on Lyapunov spectra, [31, 33, 49] on the Hausdorff dimension of limit sets, [10, 19, 26] on Birkhoff spectra of observables not necessarily related to Lyapunov exponents, and [47] on entropy spectra. However, to our knowledge, there is no known result on the mixed Birkhoff spectra of parabolic systems, in spite of its potential of broad applications.

We introduce our setting and results in more precise terms. Let

\[ M = [0,1] \text{ or } M = S^1, \]

and let \( S \) be a countable set. A Markov map \( f : \Delta \to M \) is a map such that the following holds:

- (M0) There exists a family \( \{ \Delta_a \}_{a \in S} \) of connected subsets of \( M \) with pairwise disjoint interiors such that \( \Delta = \bigcup_{a \in S} \Delta_a \) and \( f|_{\Delta_a} = f_a \) for each \( a \in S \).
- (M1) For each \( a \in S \), \( f_a \) is a \( C^1 \) diffeomorphism onto its image with appropriate one-sided derivatives.
- (M2) If \( a, b \in S \) and \( f \Delta_a \cap \Delta_b \) has non-empty interior, then \( f \Delta_a \supset \Delta_b \).

The family \( \{ \Delta_a \}_{a \in S} \) is called a Markov partition of \( f \). We say a Markov map \( f \) is fully branched if \( f \Delta_a \supset \Delta_b \) holds for all \( a \in S, b \in S \). We say \( f \) is non-uniformly expanding if \( |f'| > 1 \) everywhere except for finitely many indices \((a, x_a), a \in S\), for which \( x_a \in \Delta_a \) and \( |f'_a x_a| = 1 \). Such indices \( a \) are called neutral indices, and \( x_a \) is called a neutral periodic point if \( f^n x_a = x_a \) holds for some \( n \geq 1 \). All other indices are called expanding. Let \( \Omega \) denote the set of all neutral indices. In the case \( \Omega = \emptyset \), \( f \) is called uniformly expanding.

Remark 1.1. All neutral periodic points are topologically repelling.

Remark 1.2. Since the elements of the Markov partition may intersect each other at their boundary points, it may happen that \( a, b \in \Omega, a \neq b \) and \( x_a = x_b \).

Condition (M2) determines a transition matrix over \( S \). We will assume a strong connectivity of the associated directed graph, called finite irreducibility. Moreover, we will assume that \( f \) has uniform decay of cylinders. We refer to Section 3.1 for the definitions of these two key properties.

We aim to describe a fractal structure of the maximal invariant set

\[ J = \bigcap_{n=0}^{\infty} f^{-n} \Delta. \]

We assume \( J \neq \emptyset \), and decompose \( J \) into level sets defined as follows. Let \( \mathcal{F} \) denote the set of \( \mathbb{R} \)-valued functions on \( \Delta \) with suitable bounds on their distortion (see Section 3.3 for the precise definition). We say \( f \) has mild distortion if \( \log |f'| \in \mathcal{F} \). Let us introduce the following two infinite-dimensional vector spaces

\[ \mathcal{F}^\mathbb{N} = \{ \phi = (\phi_1, \phi_2, \ldots) : \phi_i \in \mathcal{F} \ \forall i \geq 1 \} \]

and

\[ \mathbb{R}^\mathbb{N} = \{ \alpha = (\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{R} \ \forall i \geq 1 \}. \]
For $\phi \in \mathcal{F}^N$, $\alpha \in \mathbb{R}^N$ and an integer $k \geq 1$, put $\phi_k = (\phi_1, \ldots, \phi_k)$, $\alpha_k = (\alpha_1, \ldots, \alpha_k)$ and
\[
\|\alpha_k\| = \max_{1 \leq i \leq k} |\alpha_i|.
\]
For an integer $n \geq 1$ and $x \in \bigcap_{j=0}^{n-1} f^{-j} \Delta$, we denote the multi-Birkhoff sum by
\[
S_n \phi_k(x) = (S_n \phi_1(x), \ldots, S_n \phi_k(x)),
\]
where $S_n \phi = \sum_{j=0}^{n-1} \phi \circ f^j$ for $\phi: \Delta \to \mathbb{R}$. The level sets we consider are given by
\[
B(\phi, \alpha) = \bigcap_{k=1}^{\infty} B_k(\phi, \alpha),
\]
where we have set
\[
B_k(\phi, \alpha) = \left\{ x \in J: \lim_{n \to \infty} \left\| \frac{1}{n} S_n \phi_k(x) - \alpha_k \right\| = 0 \right\}.
\]
Note that $B(\phi, \alpha)$ is the set of points in $J$ for which the Birkhoff average of each $\phi_i$ exists and is equal to $\alpha_i$. We have the multifractal decomposition
\[
J = \left( \bigcup_{\alpha \in \mathbb{R}^N} B(\phi, \alpha) \right) \cup B'(\phi),
\]
where $B'(\phi)$ denotes the set of points in $J$ for which the Birkhoff average of some $\phi_i$ does not exist. If $f|_J$ is transitive, then each non-empty level set $B(\phi, \alpha)$ is a dense subset of $J$. Our goal is to describe the mixed multifractal spectrum of Birkhoff averages
\[
b_\phi(\alpha) = \dim_H B(\phi, \alpha), \quad \alpha \in \mathbb{R}^N,
\]
where $\dim_H$ denotes the Hausdorff dimension on $M$ with respect to the standard Riemannian metric.

All measures appearing in this paper are probabilities on appropriate Borel sigma-fields. For each measure $\mu$ which is invariant under the map $f|_J: J \to J$, denote by $h(\mu) \in [0, \infty]$ the Kolmogorov-Sinai entropy of $\mu$ with respect to $f|_J$. If $f$ is differentiable at $\mu$-almost every point, then define the Lyapunov exponent of $\mu$ by $\chi(\mu) = \int \log |f'| d\mu \in [0, \infty]$. Let $\mathcal{M}(f)$ denote the set of $f|_J$-invariant measures with finite Lyapunov exponent. The dimension of a measure $\mu \in \mathcal{M}(f)$ is the number
\[
\dim(\mu) = \begin{cases} h(\mu) / \chi(\mu) & \text{if } \chi(\mu) > 0; \\ 0 & \text{if } \chi(\mu) = 0. \end{cases}
\]
A measure $\mu \in \mathcal{M}(f)$ is called expanding if $\chi(\mu) > 0$. For an ergodic expanding measure $\mu$, $\dim(\mu)$ is equal to the supremum of the Hausdorff dimensions of sets with full $\mu$-measure $[18, 20, 29, 33]$.

Our formula for $b_\phi(\alpha)$ is given in terms of the dimension of measures. Important quantities are
\[
\delta_0 = \sup \{ \dim(\mu): \mu \in \mathcal{M}(f) \}
\]
and
\[
\beta_\infty = \inf \{ \beta \in \mathbb{R}: \sup \{ h(\mu) - \beta \chi(\mu): \mu \in \mathcal{M}(f) \} < \infty \}.
\]
For $\phi \in \mathcal{F}^N$ and $k \geq 1$, we denote
\[ \int \phi_k d\mu = \left( \int \phi_1 d\mu, \ldots, \int \phi_k d\mu \right), \]
where we will always assume that $\phi_i \in L^1(\mu)$ for every $1 \leq i \leq k$, if this notation is in use. For $\alpha \in [0, \infty]$ define
\[ L(\alpha) = \left\{ x \in J : \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'x| = \alpha \right\}. \]
We are now in the position to state our main result.

**Main Theorem** (Conditional variational formulas for mixed Birkhoff spectra). Let $f : \Delta \to M$ be a finitely irreducible non-uniformly expanding Markov map which has mild distortion and uniform decay of cylinders. Let $\phi \in \mathcal{F}^N$ and $\alpha \in \mathbb{R}^N$. Then the following holds:

(a) We have $B(\phi, \alpha) \neq \emptyset$ if and only if for any integer $k \geq 1$ and any $\epsilon > 0$ there exists a measure $\mu \in \mathcal{M}(f)$ such that
\[ \left\| \int \phi_k d\mu - \alpha_k \right\| < \epsilon. \]

(b) Assume
\[ \dim_H J \leq \delta_0. \]
If $B(\phi, \alpha) \neq \emptyset$ then
\[ b_\phi(\alpha) = \lim_{k \to \infty} \dim_H B_k(\phi, \alpha) \]
\[ = \lim_{k \to \infty} \limsup_{\epsilon \to 0} \left\{ \dim(\mu) : \mu \in \mathcal{M}(f), \left\| \int \phi_k d\mu - \alpha_k \right\| < \epsilon \right\}. \]
If moreover each $\phi_i$ is bounded, then
\[ b_\phi(\alpha) = \lim_{k \to \infty} \limsup_{\epsilon \to 0} \left\{ \sup \left\{ \dim(\mu) : \mu \in \mathcal{M}(f), \left\| \int \phi_k d\mu - \alpha_k \right\| < \epsilon \right\}, \beta_\infty \right\}. \]

(c) Assume (1.2). Then
\[ \dim_H J = \delta_0 = \sup \{ \dim(\mu) : \mu \in \mathcal{M}(f) \}. \]
If moreover $f$ has a neutral periodic point, then
\[ \delta_0 = \dim_H L(0) = \lim_{\alpha \to 0} \dim_H L(\alpha). \]

**Remark 1.3.** We may replace $\mathcal{M}(f)$ by the subset $\mathcal{M}_c(f)$ of measures with compact support, in the Main Theorem as well as in all other results of this paper.

The two limits in the conditional variational formulas can be exchanged because of the monotonicity of the supremum in $\epsilon$ and $k$. The Main Theorem extends the results of Fan et al. [13, Theorem 1.1, Theorem 1.2] in which only uniformly expanding fully branched Markov interval maps with infinitely many branches were considered. The class of maps we consider is sufficiently broad, to include most of the previously studied one-dimensional Markov maps with neutral periodic points. Our class $\mathcal{F}$ of observables is strictly larger than the one considered in [13]. The
uniform decay of cylinders clearly holds for uniformly expanding maps, and for non-uniformly expanding maps we have to state it as an assumption. A number of concrete examples have this property, see \cite[Section 8]{33} and Section 5.1 for details.

The proof of the Main Theorem differs significantly from that of \cite[Theorem 1.1, Theorem 1.2]{13} in which the uniform expansion of the Markov map is essential. As stated in Main Theorem(c), the set of points with zero pointwise Lyapunov exponent has a large Hausdorff dimension, and can therefore intersect many level sets. It is convenient to restrict to level sets with positive Lyapunov exponents, but many level sets will be unaccounted for due to this restriction (see e.g. \cite[Theorem 2]{26}). To establish our Main Theorem for the complete multifractal spectrum, we take the set with zero Lyapunov exponent into a direct consideration.

What we will actually prove is the conditional variational formula for any level set whose Hausdorff dimension does not exceed $\delta_0$. The assumption (1.2) ensures that the formulas are valid for all level sets. For iterated function systems with neutral fixed points, essentially the same assumption was made in \cite[Theorem 2]{26} in order to describe Birkhoff spectra of single observables. In the setting of \cite{13}, the assumption (1.2) holds and is used implicitly in the proof. A number of concrete examples satisfy (1.2). Indeed, in Section 5.2 we show that the existence of a good induced Markov map implies (1.2).

As an immediate application of the Main Theorem, we describe the Hausdorff dimension of level sets of prescribed frequencies of visits to elements of the Markov partition. Let $f: \Delta \to M$ be a non-uniformly expanding fully branched Markov map with an infinite Markov partition $\{\Delta_i\}_{i=1}^\infty$. For each $k \geq 1$ and $\alpha \in \mathbb{R}^N$, we put

$$BE_k(\alpha) = \left\{ x \in J: \lim_{n \to \infty} \frac{1}{n} \#\{0 \leq j \leq n-1: f^jx \in \Delta_i\} = \alpha_i \quad 1 \leq i \leq k \right\}.$$

By a frequency vector we mean an element $\alpha$ of $\mathbb{R}^N$ such that $\alpha_i \geq 0$ holds for every $i \geq 1$ and $\sum_{i=1}^\infty \alpha_i \leq 1$. For each frequency vector $\alpha$ we introduce the Besicovitch-Eggleston set with frequency $\alpha$ given by

$$BE(\alpha) = \bigcap_{k=1}^\infty BE_k(\alpha).$$

**Corollary 1.4** (Dimension of Besicovitch-Eggleston sets). Let $f: \Delta \to M$ be a non-uniformly expanding fully branched Markov map with an infinite Markov partition $\{\Delta_i\}_{i=1}^\infty$ which has mild distortion and uniform decay of cylinders. Then we have $BE(\alpha) \neq \emptyset$ for each frequency vector $\alpha \in \mathbb{R}^N$. If moreover (1.2) holds, then

$$\dim_H BE(\alpha) = \lim_{k \to \infty} \dim_H BE_k(\alpha) = \lim_{k \to \infty} \lim_{\epsilon \to 0} \max \left\{ \sup \left\{ \dim(\mu): \mu \in \mathcal{M}(f), \max_{1 \leq i \leq k} |\mu(\Delta_i) - \alpha_i| < \epsilon \right\}, \beta_\infty \right\}.$$
Besicovitch-Eggleston sets were established in \cite{13,14}. See \cite{15} for analogous results on the countable Markov shift. For frequencies of visits to families of finitely many sets, a statement analogous to Corollary \cite{14} holds without the constant $\beta_\infty$.

We now investigate properties of the function $\alpha \mapsto \dim_{\text{H}} BE(\alpha)$.

**Theorem 1.5** (Properties of Besicovitch-Eggleston spectrum). Let $f : \Delta \to M$ be a non-uniformly expanding fully branched Markov map with an infinite Markov partition which has mild distortion, uniform decay of cylinders and which satisfies \cite{12}. Then the following holds:

(a) The map

$$\alpha \in \left\{ (\alpha_1, \alpha_2, \ldots) \in \mathbb{R}^\mathbb{N} : \sum_{i=1}^{\infty} \alpha_i \leq 1, \alpha_i > 0 \ \forall i \geq 1 \right\} \mapsto \dim_{\text{H}} BE(\alpha)$$

is continuous with respect to the topology of pointwise convergence;

(b) for any frequency vector $\alpha$ such that $\sum_{i=1}^{\infty} \alpha_i < 1$, we have

$$\dim_{\text{H}} BE(\alpha) = \beta_\infty.$$  

For parabolic Iterated Function Systems with finite Markov partitions, the continuity of Birkhoff spectra of a single observable was shown in \cite[Corollary 1]{26}. Theorem \cite{15}b) extends the result \cite[Theorem 1.2]{13} on uniformly expanding fully branched Markov maps.

We give two other immediate applications of our results to number-theoretic problems. Each number $x \in (0,1) \setminus \mathbb{Q}$ has the unique Backward Continued Fraction (BCF) expansion

\begin{equation}
(1.3) \quad x = 1 - \frac{1}{b_1(x) - \frac{1}{b_2(x) - \frac{1}{\ddots}}}.
\end{equation}

where each digit $b_j(x)$ is an integer greater than or equal to 2. This expansion is generated by the Rényi map (see Section \ref{section} for the definition), which is a non-uniformly expanding Markov map with a neutral fixed point. The Birkhoff spectrum of the BCF expansion digits is completely flat, in sharp contrast to the Birkhoff spectrum of the regular continued fraction expansion digits, which is a strictly increasing real-analytic function \cite{22}.

**Proposition 1.6** (Completely flat Birkhoff spectrum). For any $\alpha \in [2, \infty]$,

$$\dim_{\text{H}} \left\{ x \in (0,1) \setminus \mathbb{Q} : \lim_{n \to \infty} \frac{1}{n} (b_1(x) + \cdots + b_n(x)) = \alpha \right\} = 1.$$  

For each $n \geq 2$ consider the set

$$E(n) = \{ x \in [0,1) : b_j(x) \leq n \ \forall j \geq 1 \}.$$  

By a result of Urbański (\cite[Theorem 4.3]{49}), the Hausdorff dimension of $E(n)$ coincides with the first zero $t_0 \in [0,1]$ of the geometric pressure function associated with the Markov map generated by $n - 1$ branches of the Rényi map. Since the
Rényi map has no invariant Borel probability measure that is absolutely continuous with respect to the Lebesgue measure, we have $t_0 < 1$ by a result of Ledrappier [30]. Since any invariant measure for the Rényi map with compact support is supported on $E(n)$, for some $n \geq 2$, we obtain the following from Main Theorem(c) and Remark 1.3 answering a question of Pollicott.

**Proposition 1.7** (Limit of dimension of sets with bounded BCF digits). *We have*

$$\lim_{n \to \infty} \dim_H E(n) = 1.$$  

It follows that the set of irrational numbers in $(0, 1)$ with bounded BCF digits has Hausdorff dimension 1. For the regular continued fraction expansion, Jarník [25] proved that the set of irrational numbers in $(0, 1)$ whose digits do not exceed $n$ has Hausdorff dimension $1 - O(1/n)$. This was later significantly improved by Hensley [17] to $1 - O(1/n^2)$.

The following complementary result for the BCF digits follows from Theorem 1.5.

**Proposition 1.8** (Dimension of zero-frequency set). *We have*

$$\dim_H \left\{ x \in (0, 1) \setminus \mathbb{Q} : \lim_{n \to \infty} \frac{1}{n} \# \{1 \leq j \leq n : b_j(x) = i\} = 0 \ \forall i \geq 2 \right\} = \frac{1}{2}.$$  

In Section 2 we will prove related results for the multi-cusp winding process for the geodesic flow on hyperbolic surfaces with cusps. The rest of this paper consists of four sections. Section 3 contains preliminary results on non-uniformly expanding one-dimensional Markov maps. In Section 4 we prove the Main Theorem and Theorem 1.5. For the upper bound of the Hausdorff dimension of the level sets, we modify Bowen’s covering argument [7], and utilize the thermodynamic formalism for finite and countable Markov shifts [8, 32, 44] applied to induced systems. We will work with infinitely many induced systems which altogether exhaust the whole dynamics. For the lower bound, we construct subsets of the level sets by a Moran-like geometric construction [39], and relate the Hausdorff dimension of the level sets to the dimension of expanding measures with prescribed expected values.

In Section 5 we provide a sufficient condition for two of the assumptions in the Main Theorem: the uniform decay of cylinders, and the approximability of the Hausdorff dimension of $J$ by the dimension of expanding measures as stated in (1.2). In Section 6 we show the complete flatness of a certain mixed Birkhoff spectrum for the Rényi map in Theorem 6.1 which implies Corollary 1.6.

2. Cusp winding spectra for some hyperbolic surfaces

In this section we apply our main results to cusp winding spectra for the geodesic flow on hyperbolic surfaces modeled by a finitely generated free Fuchsian group with parabolic elements. The underlying dynamical system is derived from the Bowen-Series map associated with the action of the Fuchsian group on the boundary of hyperbolic space ([9]). This defines a uniformly expanding Markov map with infinitely many branches which are not fully branched. Other cusp winding spectra were treated in [24, 34].
2.1. Bowen-Series maps for free Fuchsian groups. We denote by \((\mathbb{D}, d)\) the Poincaré disk model of the two-dimensional hyperbolic space, where \(\mathbb{D} \subset \mathbb{R}^2\) is the open unit disk around the origin, and \(S^1 = \partial \mathbb{D}\) its boundary. We denote by 

\[ \Lambda(G) = \bigcup_{g \in G} g(0) \setminus \bigcup_{g \in G} g(0) \subset S^1 \]

the limit set of a Fuchsian group \(G\) (see [36] and also [9, Section 1] for further details), where the closure is taken with respect to the Euclidean topology on \(\mathbb{R}^2\).

We begin by recalling the definition of the Bowen-Series map from [9]. Let \(R \subset \mathbb{D}\) denote a Dirichlet fundamental domain for a finitely generated free Fuchsian group \(G\) with parabolic elements. By Poincaré’s Polyhedron Theorem, each of the finitely many sides \(s\) of \(R\) gives rise to a side-pairing transformation \(g_s \in G\), and the set 

\[ G_0 = \{ g_s : s \text{ side of } R \} \subset G \]

is a symmetric set of generators of \(G\). Moreover, each vertex \(p_j\) of \(R\) \((j = 1, \ldots, k)\) satisfies \(p_j \in S^1\) and that \(p_j\) is a parabolic fixed point of some \(g \in G\). We will assume for simplicity that for each \(p_j\) there exists \(\gamma_j \in G_0\) such that \(\gamma_j(p_j) = p_j\). We define

\[ \Gamma_0 = \{ \gamma_j^\pm \in G_0 : 1 \leq j \leq k\} \quad \text{and} \quad H_0 = G_0 \setminus \Gamma_0. \]

We will always assume that \(G\) is non-elementary, which in our setting means that \(\#G_0 \geq 4\). Each side \(s\) of \(R\) is contained in the isometric circle of \(g_s\), where we recall that the isometric circle \(C_g\) of \(g \in G\) is given by

\[ C_g = \{ z \in \mathbb{R}^2 : |g'(z)| = 1 \}. \]

Here, \(|g'(z)|\) denotes the norm of the derivative of \(g\) at \(z\) with respect to Euclidean metric on \(\mathbb{R}^2\). We then define for \(g \in G_0\),

\[ \Delta_g = S^1 \cap \{ z \in \mathbb{R}^2 : |g'(z)| \geq 1 \}. \]

Let \(\Delta = \bigcup_{g \in G_0} \Delta_g\) and define the Bowen-Series map \(f : \Delta \to \Delta\) by

\[ f|_{\Delta_g} = g|_{\Delta_g}, \quad g \in G_0. \]
That $f$ is well defined follows from the fact that, for distinct $g, h \in G_0$, we have $\Delta_g \cap \Delta_h = \emptyset$ unless $h = g^{-1} \in \Gamma_0$. In this case, we have $\Delta_g \cap \Delta_{g^{-1}} = \{ p_j \}$ where $p_j$ is the parabolic fixed point of $g$, and thus $f(p_j) = p_j$.

Note that $f$ is a Markov map with Markov partition $(\Delta_g)_{g \in G_0}$. It is not difficult to verify that $f$ is a finitely irreducible non-uniformly expanding Markov map satisfying Renyi’s condition and (M3). Hence, $f$ has uniform decay of cylinders by Lemma 5.1. For the maximal $f$-invariant set $J \subset \mathbb{S}^1$ we have

$$J = \Lambda(G).$$

There is a one-to-one correspondence between $x \in \Lambda(G)$ and symbolic sequences $\omega \in G_0^\mathbb{N}$ satisfying $\omega_i \omega_{i+1} \neq 1$ for every $i \geq 1$ (45). This correspondence is given by the coding map associated with the Markov map $f$ as defined in Section 3.1. We denote by $\Lambda_c(G) \subset \Lambda(G)$ the conical limit set of $G$ (see 30 for the definition).

By a result of Beardon and Maskit (4) we have

$$\Lambda_c(G) = \Lambda(G) \setminus \bigcup_{g \in G} \bigcup_{j=1}^k g^{-1}(p_j).$$

2.2. Multi-cusp winding process. For $x \in \Lambda_c(G)$ it follows from (2.1) that the corresponding symbolic sequence $\omega \in G_0^\mathbb{N}$ does not eventually become constant to some element of $\Gamma_0$. For $x \in \Lambda_c(G)$ we can therefore decompose its symbolic sequence $\omega$ into a sequence of blocks $(B_i(x))_{i \geq 1}$ as in [24]. Each hyperbolic generator in $\omega$ forms a block of length one. For parabolic generators in $\omega$ we build maximal blocks of consecutive appearances of the same parabolic generator. Hence, we have either $B_i(x) = h$ for some $h \in H_0$, or $B_i(x) = \gamma^n$ for some $\gamma \in \Gamma_0$ and $n \geq 1$.

Motivated by [24], we introduce the multi-cusp winding process $(a_{i,j})_{i \geq 1, 1 \leq j \leq k}$, a family of functions on $\Lambda_c(G)$ given by

$$a_{i,j}(x) = \begin{cases} n - 1 & \text{if } B_i(x) = \gamma_j^n, \quad n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

This process has an interpretation in terms of the geodesic flow on the associated hyperbolic surface $\mathbb{D}\setminus G$ (34). Namely, for a fixed initial value $o \in \mathbb{D}\setminus G$, the set of oriented geodesics $\gamma : [0, \infty) \to \mathbb{D}\setminus G$ which satisfy $\gamma(0) = o$ and which return to some compact set of $\mathbb{D}\setminus G$ infinitely often, can be identified with $\Lambda_c(G)$. This identification is obtained by lifting $\gamma$ to a geodesic $\tilde{\gamma} : [0, \infty) \to \mathbb{D}$ with $\tilde{\gamma}(0) \in R$, and identifying $\gamma$ with the endpoint $x := \lim_{t \to \infty} \tilde{\gamma}(t) \in \mathbb{S}^1$. The symbolic sequence $\omega$ is then the cutting sequence which records the sides of $R$ crossed by $\gamma(t)$ as $t \to \infty$ (45 Section 2). A block $B_i(x) = \gamma_j^n$ of length $n \geq 2$ of some parabolic generator $\gamma_j \in \Gamma_0$ then means that the geodesic $\gamma$ spirals $n - 1$ times around the cusp associated with $\gamma_j$. Here, the $i$th block refers to the $i$th return of $\gamma$ to a certain compact subset of $\mathbb{D}\setminus G$.

2.3. Mixed Birkhoff spectra for multi-cusp winding process. In order to apply our theory for mixed Birkhoff spectra to the multi-cusp winding process, we introduce the induced Markov map $\tilde{f} : K \to \mathbb{S}^1$ given by

$$\tilde{f}x = \tilde{f}|_{B_i(x) + 1}x,$$
where \(|B_1(x)|\) denotes the word length of the block \(B_1(x)\) and \(K \subset S^1\) is the countable union of pairwise disjoint partition elements
\[
K = \bigcup_{\omega \in F} \Delta_\omega.
\]
Here, \(F\) is the countable set of reduced words in \(\bigcup_{n=2}^\infty G_0^n\) given by
\[
F = \bigcup_{n=1}^\infty \{\gamma^n g: \gamma \in \Gamma_0, g \in G_0 \setminus \{\gamma^{\pm 1}\}\} \cup \{hg: h \in H_0, g \in G_0 \setminus \{h^{-1}\}\}.
\]
By (2.1), the maximal \(\tilde{\f}\)-invariant set \(\tilde{J}\) satisfies
\[
\tilde{J} = \Lambda_c(G).
\]
Let \(D_k = \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k: \alpha_j \geq 0 \quad 1 \leq \forall j \leq k\}\).

For \(\alpha \in D_k\) we define
\[
B(\alpha) = \left\{ x \in \Lambda_c(G): \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} a_{1,j}(x) + \cdots + a_{n,j}(x)) = \alpha_j \quad 1 \leq \forall j \leq k \right\}.
\]

**Proposition 2.1** (Birkhoff spectrum of multi-cusp windings).\(^\dagger\) For every \(\alpha \in D_k\) we have \(B(\alpha) \neq \emptyset\) and
\[
\dim_H B(\alpha) = \limsup_{\epsilon \to 0} \left\{ \dim(\mu): \mu \in \mathcal{M}(\tilde{f}), \left| \int a_{1,j} \, d\mu - \alpha_j \right| < \epsilon \quad 1 \leq \forall j \leq k \right\}.
\]
Moreover, \(\alpha \in D_k \mapsto \dim_H B(\alpha)\) is continuous.

**Proof.** For \(i \geq 1\) and \(1 \leq j \leq k\) we have
\[
a_{i,j} = a_{1,j} \circ \tilde{f}^{i-1}.
\]
Hence, we have that
\[
B(\alpha) = \left\{ x \in \tilde{J}: \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} a_{1,j} \circ \tilde{f}^i x = \alpha_j \quad 1 \leq \forall j \leq k \right\}.
\]

Our main task is to verify that \(\tilde{f}\) is a finitely irreducible, non-uniformly expanding Markov map with Markov partition \((\Delta_\omega)_{\omega \in F}\) which has mild distortion and uniform decay of cylinders, and satisfies (1.2). That \(\tilde{f}\) is finitely irreducible follows from the fact that \(\tilde{f}: \Delta \to \Delta\) is transitive with finite Markov partition \((\Delta_\omega)_{g \in G_0}\). Since \(f\) has uniform decay of cylinders, so has \(\tilde{f}\). Since \(B_1\) is constant on each element of the Markov partition, (1.2) holds for \(\tilde{f}\) by Proposition 5.4(b). Hence, the desired conditional variational formula follows from Main Theorem(b) for every non-empty level set \(B(\alpha)\).

To prove that \(B(\alpha) \neq \emptyset\) for every \(\alpha \in D_k\) we make use of Main Theorem(a). Namely, for \(n \geq 1\) and \(1 \leq j \leq k\) we consider the \(\tilde{f}\)-invariant probability measure on the periodic orbit \((\gamma_j^n g \gamma_j^n g \ldots)\) of \(\tilde{f}\), where \(\gamma_j \in \Gamma_0\) and \(g \in G_0 \setminus \{\gamma_j^{\pm 1}\}\). By taking suitable convex combinations of these measures and applying Main Theorem(a) we can show that \(B(\alpha) \neq \emptyset\) for every \(\alpha \in D_k\). One can prove the
continuity of the Hausdorff dimension by slightly modifying the proof of Theorem 1.5 using the uniform expansion of \( \tilde{f}^2 \) in Proposition 5.4(b), see Remark 4.13. □

Remark 2.2. In the case \( k = 1 \), our conditional variational formula for the cusp winding spectrum is analogous to the Birkhoff spectrum of the arithmetic mean of the regular continued fraction digits considered in [22, Corollary 6.6]. Note that in there the dynamical system is given by the Gauss map.

In order to investigate the frequency of multi-cusp windings, we define for \( i \geq 0 \) and \( 1 \leq j \leq k \) the sets

\[
A_{i,j} = \{ x \in \Lambda_c(G) : a_{1,j}(x) = i \}.
\]

Denote by \( D \) the set of frequency vectors \( \alpha = (\alpha_{i,j}) \in \mathbb{R}^{N \times \{1, \ldots, k\}} \). Then for every \( \alpha = (\alpha_{i,j}) \in D \) we have

\[
BE(\alpha) = \left\{ x \in \Lambda_c(G) : \lim_{n \to \infty} \frac{1}{n} \# \{ 1 \leq \ell \leq n : a_{\ell,j}(x) = i \} = \alpha_{i,j}, \quad \forall i \geq 0, \ 1 \leq \forall j \leq k \right\}.
\]

Hence, the number \( \alpha_{i,j} \) prescribes the asymptotic frequency of precisely \( i \) consecutive windings around the \( j \)th cusp.

Proposition 2.3 (Frequency of multi-cusp windings). For every frequency vector \( \alpha \in D \) we have \( BE(\alpha) \neq \emptyset \) and

\[
\dim_H BE(\alpha) = \lim_{k \to \infty} \lim_{\epsilon \to 0} \max \left\{ \sup \left\{ \dim(\mu) : \mu \in \mathcal{M}(\tilde{f}), \sup_{i \geq 0, 1 \leq j \leq k} |\mu(A_{i,j}) - \alpha_{i,j}| < \epsilon \right\}, \frac{1}{2} \right\}.
\]

Moreover, the map \( \alpha \mapsto \dim_H BE(\alpha) \) is continuous on \( D \). Furthermore, for the zero-frequency \( 0 \in D \) we have

\[
\dim_H BE(0) = \frac{1}{2}.
\]

Proof. Recall that \( \tilde{f} \) is a finitely irreducible, non-uniformly expanding Markov map with a Markov partition \( (\Delta_\omega)_{\omega \in F} \) which has mild distortion and uniform decay of cylinders, and which satisfies (1.2). Further, each \( A_{i,j} \) is a finite union of elements of \( (\Delta_\omega)_{\omega \in F} \). By slightly modifying the proofs we see that Corollary 1.4 and Theorem 1.5 also hold in this setting. By [24, Lemma 2.4] we have \( \beta_\infty = 1/2 \). Finally, that \( BE(\alpha) \neq \emptyset \) for every frequency vector \( \alpha \) can be shown as in the proof of Proposition 2.1. □

Remark 2.4. In the case \( k = 1 \), the last assertion in Proposition 2.3 is related to [34, Theorem 2] where it is shown that, for a certain Jarnık set \( \mathcal{J} \subset \Lambda_c(G) \), we have \( \dim_H \mathcal{J} = 1/2 \). Namely, by the definition of \( \mathcal{J} \), we have \( \lim_{n \to \infty} a_{n,1}(x) = \infty \) for every \( x \in \mathcal{J} \), which implies \( \mathcal{J} \subset BE(0) \).
3. Preliminaries on non-uniformly expanding Markov maps

Throughout this section, let \( f : \Delta \to M \) be a non-uniformly expanding Markov map with a Markov partition \( \{ \Delta_a \}_{a \in S} \). In Section 3.1 we introduce a symbolic coding via the Markov partition, as well as the necessary definitions and conditions appearing in our main results. In Section 3.2 we introduce a subset \( J' \) of \( J \) of full Hausdorff dimension which detects weak expansion. In Section 3.3 we define the set \( \mathcal{F} \) of admissible observables for our main results. In Section 3.4 we state two lemmas on approximations of invariant measures, which will be frequently used later on.

3.1. Symbolic coding. Condition (M2) determines a transition matrix \( (T_{ab}) \) over the countable alphabet \( S \) by \( T_{ab} = 1 \) if \( f \Delta_a \supset \Delta_b \) and \( T_{ab} = 0 \) otherwise. A word of length \( n \geq 1 \) is an \( n \)-string of elements of \( S \). A word \( \omega_1 \cdots \omega_{n-1} \) of length \( n \) is admissible if \( n = 1 \), or else \( n \geq 2 \) and \( T_{\omega_j \omega_{j+1}} = 1 \) holds for every \( 0 \leq j \leq n - 1 \).

Denote by \( E^n \) the set of \( n \)-admissible words and put \( E^* = \bigcup_{n=1}^{\infty} E^n \). For two words \( \omega = \omega_0 \cdots \omega_{m-1} \) and \( \eta = \eta_0 \cdots \eta_{n-1} \), denote by \( \omega \eta \) the concatenated word \( \omega_0 \cdots \omega_{m-1} \eta_0 \cdots \eta_{n-1} \). This notation extends in an obvious way to concatenations of an arbitrary finite number of words. For convenience, put \( E^0 = \{ \emptyset \} \), \( |\emptyset| = 0 \), and \( \eta \emptyset = \emptyset \eta = \emptyset \) for every \( \eta \in E^* \). We say \( f \) is finitely irreducible if there exists a finite set \( \Lambda \subset E^* \) such that for any \( \omega, \eta \in E^* \) there exists \( \lambda \in \Lambda \) such that \( \omega \lambda \eta \in E^* \). If \( f \) is fully branched, i.e., \( T_{ab} = 1 \) for all \( a, b \in S \), then it is finitely irreducible with \( \Lambda = \emptyset \).

Let \( \mathbb{N} \) denote the set of non-negative integers. The transition matrix determines a topological Markov shift

\[
X = \{ x = (x_n)_{n=0}^{\infty} : x_n \in S, \ T_{x_n x_{n+1}} = 1 \ \forall n \in \mathbb{N} \},
\]

endowed with the usual symbolic metric. The left shift acting on \( X \) is denoted by \( \sigma : (\sigma x)_n = x_{n+1} \) for every \( n \in \mathbb{N} \). For each \( \omega = \omega_0 \cdots \omega_{n-1} \in E^n \), put \( |\omega| = n \) and define the \( n \)-cylinder

\[
\Delta_\omega = \bigcap_{j=0}^{n-1} f^{-j} \Delta_{\omega_j},
\]

and put

\[
[\omega] = [\omega_0, \ldots, \omega_{n-1}] = \{ x \in X : x_j = \omega_j \ 0 \leq j \leq n - 1 \}.
\]

The Riemannian length of the cylinder \( \Delta_\omega \) is denoted by \( |\Delta_\omega| \). We say \( f \) has uniform decay of cylinders if

\[
\lim_{n \to \infty} \sup_{\omega \in E^n} |\Delta_\omega| = 0.
\]

If \( f \) has uniform decay of cylinders, then for any \( (x_n)_{n=0}^{\infty} \in X \) the set \( \bigcap_{n=0}^{\infty} f^{-n} \Delta_{x_n} \) is a singleton. The coding map \( \pi : X \to M \) given by

\[
\pi((x_n)_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} f^{-n} \Delta_{x_n}
\]

is well-defined and satisfies \( J \subset \pi(X) \). It is measurable, one-to-one except on the countable set \( \bigcup_{n=0}^{\infty} f^{-n}(\bigcup_{a \in S} \partial \Delta_a) \). Since the set \( \{ x \in X : \pi \text{ is one-to-one at } \pi(x) \} \)
is $\sigma$-invariant and $f \circ \pi = \pi \circ \sigma$ holds on this set, for any non-atomic $f$-invariant measure $\mu$ on $\pi(X)$, there exists a $\sigma$-invariant measure $\mu'$ such that $\mu = \mu' \circ \pi^{-1}$ and the entropy of $\mu'$ with respect to $\sigma$ is equal to $h(\mu)$.

3.2. Weak expansion on a set of full dimension. For a neutral index $a \in \Omega$ and an integer $n \geq 1$, let $\Delta^\chi(a)$ denote the union of all $n$-cylinders containing $x_a$ (there are at least one and at most two such cylinders, see Remark 1.2). Fix an integer $\chi \geq 1$ such that the following holds:

(i) If $a, b \in \Omega$ and $x_a \neq x_b$, then $\Delta^\chi(a) \cap \Delta^\chi(b) = \emptyset$.

(ii) For each $a \in \Omega$, there exists at most one $b \in \Omega$ such that $f(\Delta^\chi(a)) \cap \Delta^\chi(b)$ has non-empty interior.

Put

$$V = \bigcup_{a \in \Omega} \Delta^\chi(a).$$

This is a closed neighborhood of the set $\{x_a : a \in \Omega\}$. Define

$$J' = J \cap \bigcap_{n=0}^\infty \bigcap_{k=n}^\infty f^{-k}(M \setminus V).$$

Points in $J'$ can have zero pointwise Lyapunov exponent. Nevertheless, the following weak expansion estimate holds.

**Lemma 3.1.** Let $x \in J'$. For any $n' \geq 1$ there exists an integer $n \geq n'$ such that for every $\omega \in E^n$ such that $x \in \Delta_\omega$, we have $\inf_{\Delta_\omega} |(f^n)'| \geq \inf_{\Delta \setminus V} |f'| > 1$.

**Proof.** Let $n > \max\{\chi, n'\}$ be such that $f^{n-\chi}x \notin V$. For every $\omega \in E^n$ such that $x \in \Delta_\omega$, $f^{n-\chi}\Delta_\omega$ is a $\chi$-cylinder. Since cylinders of the same word lengths have disjoint interiors, $f^{n-\chi}\Delta_\omega \subset \Delta \setminus V$ holds. For every $y \in \Delta_\omega$ we have

$$|(f^n)'y| = |(f^{n-1})'(f^{n-\chi+1}y)||f'(f^{n-\chi}y)||(f^{n-\chi})'y| \geq |f'(f^{n-\chi}y)| \geq \inf_{\Delta \setminus V} |f'| > 1,$$

which yields the claim. \hfill $\square$

We restrict ourselves to the set $J'$, and use Lemma 3.1 to avoid the effect of neutral periodic points for the upper estimate of the Hausdorff dimension of the level sets in our main theorem. That this restriction carries the full Hausdorff dimension follows from the next lemma.

**Lemma 3.2.** If $f$ has uniform decay of cylinders, then $J \setminus J'$ is a countable set.

**Proof.** Condition (M2) implies that if $f(\Delta^\chi(a)) \cap \Delta^\chi(b)$ has non-empty interior then $f(\Delta^\chi(a)) \supset \Delta^\chi(b)$. From this and the choice of $\chi$, it follows that for each $a \in \Omega$, all points in $\Delta^\chi(a) \cap \bigcap_{n=1}^\infty f^{-n}V$, except possibly a countable number of points which eventually enter the boundary of some element of the Markov partition, share the same coding sequence. By the uniform decay of cylinders, there exists at most one point in $M$ for each given coding sequence. Since $\Omega$ is finite it thus follows that $\bigcap_{n=0}^\infty f^{-n}V$ is countable. Therefore, $J \setminus J' = J \cap \bigcup_{k=0}^\infty f^{-k} (\bigcap_{n=0}^\infty f^{-n}V)$ is a countable set. \hfill $\square$
3.3. Class of observables. For a function \( \phi : \Delta \to \mathbb{R} \) and an integer \( n \geq 1 \) put
\[
D_n(\phi) = \sup_{\omega \in E^n} \sup_{x,y \in \Delta_\omega} S_n \phi(x) - S_n \phi(y),
\]
and define
\[
\mathcal{F} = \{ \phi : \Delta \to \mathbb{R} : D_1(\phi) < \infty \text{ and } D_n(\phi) = o(n) \}.
\]
The first condition is non-trivial only if \( \phi \) is unbounded. We will only consider observables which belong to \( \mathcal{F} \). If \( f \) has uniform decay of cylinders, then \( \mathcal{F} \) contains the set of bounded uniformly continuous functions.

We say \( f \) satisfies Rényi’s condition if each branch \( f_a \ (a \in S) \) is \( C^2 \) and satisfies
\[
\sup_{a \in S} \sup_{\Delta_a} \frac{|f''_a|}{|f'_a|^2} < \infty.
\]

Lemma 3.3. If \( f \) has uniform decay of cylinders and satisfies Rényi’s condition, then it has mild distortion.

Proof. For every \( a \in S \) and all \( x, y \in \Delta_a \) we have
\[
\log \frac{|f'_a(x)|}{|f'_a(y)|} \leq \sup_{\Delta_a} \frac{|f''_a|}{|f'_a|^2} f(x) - f(y) \leq \sup_{\Delta_a} \frac{|f''_a|}{|f'_a|^2}.
\]
Therefore \( D_1(\log |f'|) < \infty \). Iterating this argument, for every \( n \geq 2 \), every \( \omega \in E^n \) and all \( x, y \in \Delta_\omega \) we obtain
\[
\log \frac{|(f^n)'x|}{|(f^n)'y|} \leq \sup_{a \in S} \sup_{\Delta_a} \frac{|f''_a|}{|f'_a|^2} \sum_{j=1}^{n} |f^j \Delta_\omega| \leq \sup_{a \in S} \sup_{\Delta_a} \frac{|f''_a|}{|f'_a|^2} \left( \sum_{j=1}^{n-1} \sup_{\omega \in E^{n-j}} |\Delta_\omega| + 1 \right).
\]
The uniform decay of cylinders implies that this series is \( o(n) \), and therefore \( \log |f'| \in \mathcal{F} \). \( \square \)

3.4. Approximations of expanding measures. For a subset \( F \) of \( E^* \) we use the following notation:
\[
[F] = \bigcup_{\omega \in F} [\omega], \quad |F| = \sup_{\omega \in F} |\omega|, \quad \Delta F = \bigcup_{\omega \in F} \Delta_\omega.
\]

The lemma below allows us to approximate an ergodic measure with a finite collection of cylinders under the assumption of finite irreducibility. For \( a,b \in S \) and an integer \( n \geq 1 \), let \( E^n(a,b) \) denote the set of elements of \( E^n \) whose first symbol is \( a \) and the last one is \( b \).

Lemma 3.4. Let \( f \) be finitely irreducible and have uniform decay of cylinders. Let \( \phi \in \mathcal{F}^N \), \( k \geq 1 \) and let \( \mu \in \mathcal{M}(f) \) be an ergodic expanding measure such that \( \phi_i \in L^1(\mu) \) for every \( 1 \leq i \leq k \). For any \( \epsilon > 0 \) and any integer \( n \geq 1 \), there exist \( \ell \geq n \), \( a,b \in S \) and a finite subset \( F^\ell(a,b) \) of \( E^\ell(a,b) \) such that
\[
\left| \frac{1}{\ell} \log \# F^\ell(a,b) - h(\mu) \right| < \epsilon \quad \text{and} \quad \sup_{\Delta F^\ell(a,b)} \left| \frac{1}{\ell} S_{\ell} \phi_k - \int \phi_k d\mu \right| < \epsilon.
\]

Proof. By virtue of the uniform decay of cylinders, \( f|_J \) is semi-conjugate to a countable Markov shift via the Markov partition. Then the proof of Lemma 3.4 is carried out on a symbolic level as in [46, Lemma 2.3]. \( \square \)
Since the finite irreducibility implies the transitivity, the proof of Main Theorem works verbatim to show the next lemma approximating non-ergodic measures with ergodic ones in a particular sense.

**Lemma 3.5.** Let $f$ be finitely irreducible and have uniform decay of cylinders. Let $\phi \in F^N$, $k \geq 1$ and let $\mu \in M(f)$ be a non-ergodic expanding measure such that $\phi_i \in L^1(\mu)$ for every $1 \leq i \leq k$. For any $\epsilon > 0$ there exists an ergodic expanding measure $\mu' \in M(f)$ which satisfies

$$|h(\mu) - h(\mu')| < \epsilon$$

and

$$\left\| \int \phi_k d\mu - \int \phi_k d\mu' \right\| < \epsilon.$$

### 4. On the proofs of the Main results

This section is dedicated to proofs of the Main Theorem and related results, including Theorem 1.5. In Section 4.1 we prove an upper bound (Proposition 4.1) leading to the formula in Main Theorem(b). In Section 4.2 we prove a lower bound (Proposition 4.5), and we put these together in Section 4.3 to complete the proof of the Main Theorem. In Section 4.4 we prove Theorem 1.5. A short comment on the set $B'(\phi)$ in (1.1) is given in Section 4.5.

#### 4.1. Upper bound on dimension.

The next proposition will be used to obtain upper bounds on the Hausdorff dimension of level sets.

**Proposition 4.1.** Let $f: \Delta \to M$ be a non-uniformly expanding Markov map which is finitely irreducible, has mild distortion and uniform decay of cylinders. Let $\phi \in F^N$ and $\alpha \in \mathbb{R}^N$, and let $k \geq 1$ be such that $B_k(\phi, \alpha) \neq \emptyset$. For $\epsilon > 0$ put

$$d_k(\epsilon) = \sup \left\{ \dim(\mu) : \mu \in M(f), \left\| \int \phi_k d\mu - \alpha_k \right\| < \epsilon \right\} + \epsilon,$$

and set

$$d_k = \lim_{\epsilon \to 0} d_k(\epsilon).$$

Assume

$$d_k < \delta_0.$$  \tag{4.1}

Then the following holds:

(a) If $\sup_{\Delta} \|\phi_k\| = \infty$, then

$$\dim_H B_k(\phi, \alpha) \leq d_k;$$

(b) If $\sup_{\Delta} \|\phi_k\| < \infty$, then

$$\dim_H B_k(\phi, \alpha) \leq \max\{d_k, \beta_{\infty}\}.$$

The proofs of Proposition 4.1(a) and (b) involve the same set of ideas that is outlined as follows. By Lemma 3.2 and the countable stability of Hausdorff dimension, to verify (a) it suffices to show $\dim_H (B_k(\phi, \alpha) \cap J' \cap \Delta_a) \leq d_k$ for each $a \in S$. For a subset $G$ of $E^*$ and $\beta \in \mathbb{R}$, put

$$m_\beta(G) = \sum_{\omega \in G} |\Delta_\omega|^{\beta}.$$
For sufficiently small \( \epsilon > 0 \), we will construct a family \( \{G_n\}_n \) of subsets of \( E^* \) with increasing uniform word lengths such that the corresponding family of sets of cylinders covers \( B_k(\phi, \alpha) \cap J' \cap \Delta_a \), and satisfies \( \lim_{n \to \infty} m_{d_n(\epsilon)}(G_n) = 0 \). We would like to apply the thermodynamic formalism for the induced system associated with each \( G_n \), to obtain an upper bound of \( m_{d_n(\epsilon)}(G_n) \) in terms of the Lyapunov exponent of a certain expanding measure \( \mu_n \). The problem is that \( \inf_n \chi(\mu_n) \) may be zero, due to the existence of neutral periodic points. Hence, we will fix an expanding measure \( \mu^* \) with sufficiently large dimension from the start, and use it to construct a family of expanding measures whose Lyapunov exponents are bounded away from zero.

**Proof of Proposition 4.1.** Let us start with choosing such a measure.

**Lemma 4.2.** There exists an expanding measure \( \mu^* \in \mathcal{M}(f) \) such that \( \phi_i \in L^1(\mu^*) \) for every \( 1 \leq i \leq k \) and

\[
d_k < \dim(\mu^*).
\]

**Proof.** By virtue of (4.1) and Lemma 3.5, there exists an ergodic expanding measure \( \mu \) such that \( d_k < \dim(\mu) \). If \( \phi_i \in L^1(\mu) \) for every \( 1 \leq i \leq k \), then put \( \mu^* = \mu \). Otherwise, we approximate \( \mu \) with a finite number of cylinders in the sense of Lemma 3.4 and glue them together using the finite irreducibility to construct another expanding measure \( \mu^* \) whose entropy and Lyapunov exponent are arbitrarily close to those of \( \mu \). Since the support of \( \mu^* \) is contained in the union of finitely many cylinders, \( \phi_i \in L^1(\mu^*) \) for every \( 1 \leq i \leq k \). \( \square \)

Fix an expanding measure \( \mu^* \) as in Lemma 4.2. Put

\[
A_k = \max \left\{ \left\| \int \phi_k d\mu^* \right\|, 2\|\alpha_k\|, 1 \right\} > 0.
\]

Fix \( \epsilon_0 \in (0, 7A_k) \) such that

\[
h(\mu^*) - d_k(\epsilon)\chi(\mu^*) \geq 0 \quad \text{for every } \epsilon \in (0, \epsilon_0).
\]

Put

\[
d_k(\epsilon) = \max \{d_k(\epsilon), \beta_{\infty} + \epsilon \}.
\]

**Lemma 4.3.** For any \( \epsilon \in (0, \min\{\epsilon_0, 4/3\}) \) there exists \( n(\epsilon) \geq 1 \) such that if \( n \geq n(\epsilon) \), \( a \in S \) and \( G \) is a non-empty subset of \( E^{n+1}(a, a) \) such that for every \( \omega \in G \),

\[
\sup_{\Delta_{\omega}} \left\| \frac{1}{n}S_n\phi_k - \alpha_k \right\| < \frac{\epsilon}{2}
\]

and

\[
\inf_{\Delta_{\omega}} |(f^n)'| > 1,
\]

then the following holds:

(a) If \( G \) is a finite set, then

\[
m_{d_k(\epsilon)}(G) \leq \exp \left( -\frac{\epsilon^2 \chi(\mu^*)n}{8A_k} \right).
\]
(b) If $\sup_{\Delta} \| \phi_k \| < \infty$, then
\[
m_{[n_k]}(G) \leq \exp \left( -\frac{e^2 \chi(\mu^*)n}{8A_k} \right).
\]

We finish the proof of Proposition 4.1 assuming the conclusion of Lemma 4.3. Let $\epsilon \in (0, \min\{\epsilon_0, 4/3\})$ and $n \geq n(\epsilon)$. We assume $n$ is large enough so that
\[
(4.4) \quad D_n(\log |f'|) < \epsilon^2 n \quad \text{and} \quad D_n(\phi_i) < \frac{\epsilon n}{4} \quad 1 \leq i \leq k.
\]

Let $a \in S$. Put
\[
H_n = \left\{ \omega \in \bigcup_{b \in S} E^n(a, b): \inf_{\Delta_{\omega}} (f^n)' > 1, \quad \left\| \frac{1}{n} S_n \phi_k(z) - \alpha_k \right\| < \frac{\epsilon}{4} \quad \text{for some } z \in \Delta_{\omega} \right\}.
\]

The rest of the proof of Proposition 4.1 consists of two parts. We first treat the case $\sup_{\Delta} \| \phi_k \| = \infty$. The next lemma asserts that in this case one can cover the level set with countably many subsets with ‘finite ranges’.

**Lemma 4.4.** If $\sup_{\Delta} \| \phi_k \| = \infty$, then there exists a countable family $\{B_{k,p}(\phi, \alpha)\}_{p=1}^\infty$ of subsets of $B_k(\phi, \alpha)$ such that
\[
B_k(\phi, \alpha) = \bigcup_{p=1}^\infty B_{k,p}(\phi, \alpha),
\]

and for each $p \geq 1$ there exists a sequence $\{C_{n,p}\}_{n=1}^\infty$ of finite unions of 1-cylinders such that $f^n(B_{k,p}(\phi, \alpha)) \subset C_{n,p}$ for every $n \geq 1$.

**Proof.** Let $1 \leq i \leq k$ be such that $\sup_{\Delta} |\phi_i| = \infty$. For each integer $t \geq 1$ put
\[
B_k(\phi, \alpha, t) = \left\{ x \in B_k(\phi, \alpha): \left| \frac{1}{n} S_n \phi_i(x) - \alpha_i \right| \leq 1 \quad \forall n \geq t \right\}.
\]

Note that $B_k(\phi, \alpha) = \bigcup_{t=1}^\infty B_k(\phi, \alpha, t)$. The unboundedness of $\phi_i$ implies that if $n \geq t$ then $f^n(B_k(\phi, \alpha, t))$ is contained in a finite union of 1-cylinders, the number of which may depend on $n$. The set $I = \bigcup_{t=1}^\infty \{t\} \times E^t$ is a countably infinite set. Write $I = \{a_p\}_{p=1}^\infty$ and define $B_{k,p}(\phi, \alpha) = B_k(\phi, \alpha, t) \cap \Delta_{\omega}$ where $(t, \omega) = a_p$. Then $\{B_{k,p}(\phi, \alpha)\}_{p=1}^\infty$ has the desired properties. \(\square\)

By Lemma 3.1 and Lemma 4.4, for each $p \geq 1$ with $B_{k,p}(\phi, \alpha) \neq \emptyset$ there exists a finite set $H_p^n \subset H^n$ such that
\[
B_{k,p}(\phi, \alpha) \cap J' \cap \Delta_a \subset \bigcup_{\omega \in H_p^n} \Delta_{\omega}: \omega \in \bigcup_{n=N}^\infty H_p^n \right\}.
\]

For each $\omega \in H_p^n$ fix $\lambda(\omega) \in \Lambda$ and $z(\omega) \in \Delta_{\omega}$ satisfying
\[
(4.5) \quad \omega \lambda(\omega)a \in E^* \quad \text{and} \quad \left\| \frac{1}{n} S_n \phi_k(z(\omega)) - \alpha_k \right\| < \frac{\epsilon}{4}.
\]
Using (4.4) and (4.5) we have
\[
\sup_{\Delta, \lambda(\omega)} \|S_{n+\lambda(\omega)}(\phi_k - (n + |\lambda(\omega)|)\alpha_k)\| \leq \sup_{\Delta, \lambda(\omega)} \|S_n\phi_k - S_n\phi_k(z(\omega))\|
\]
\[
+ \|S_n\phi_k(z(\omega)) - n\alpha_k\|
\]
\[
+ \sup_{\Delta, \lambda(\omega)} \|S_{|\lambda(\omega)|}\phi_k \circ f^n - |\lambda(\omega)|\alpha_k\|
\]
\[
< \varepsilon + \frac{\varepsilon n}{2} + \sup_{\Delta, \lambda(\omega)} \|S_{|\lambda(\omega)|}\phi_k\| + |\lambda(\omega)| \cdot \|\alpha_k\|
\]
< \varepsilon(n + |\lambda(\omega)|),
\]
where the last inequality holds for sufficiently large \(n\) since \(\Lambda\) is a finite set by the finite irreducibility. For \(0 \leq q \leq |\Lambda|\) put
\[
H^n_p(q) = \{\omega \in H^n_p : |\lambda(\omega)| = q\}.
\]
Pick \(q_0 \in \{0, \ldots, |\Lambda|\}\) which maximizes the quantity \(m_{dk(\varepsilon)}(\{\omega\lambda(\omega)a : \omega \in H^n_p(\cdot)\})\).
Put
\[
G_n = \{\omega\lambda(\omega)a : \omega \in H^n_p(q_0)\}.
\]
Put \(\rho = \min_{\Lambda} |\Delta_{\lambda_a}| > 0\). For each \(\omega \in H^n_p\) we have
\[
\left| \frac{\Delta_{\lambda(\omega)a}}{\Delta_{\omega}} \right| \geq e^{-\varepsilon n} \frac{|f^n \Delta_{\lambda(\omega)a}|}{|f^n \Delta_{\omega}|} \geq e^{-\varepsilon n} |\Delta_{\lambda(\omega)a}| \geq e^{-2\varepsilon n} \rho,
\]
where the first inequality is from (4.4). Hence
\[
m_{dk(\varepsilon)}(H^n_p) \leq C_{\varepsilon, \rho, k, n} m_{dk(\varepsilon)}(\{\omega\lambda(\omega)a : \omega \in H^n_p\})
\]
\[
= C_{\varepsilon, \rho, k, n} \sum_{q=0}^{\lfloor \frac{|\Lambda|}{2} \rfloor} m_{dk(\varepsilon)}(\{\omega\lambda(\omega)a : \omega \in H^n_p(q)\})
\]
\[
\leq C_{\varepsilon, \rho, k, n} (|\Lambda| + 1) m_{dk(\varepsilon)}(G_n),
\]
where \(C_{\varepsilon, \rho, k, n} = (\rho^{-1} e^{2\varepsilon n})^{dk(\varepsilon)}\). Lemma 4.3(a) applied to the finite set \(G_n\) gives
\[
\sum_{n=N}^{\infty} m_{dk(\varepsilon)}(H^n_p) \leq \rho^{-dk(\varepsilon)}(|\Lambda| + 1) \sum_{n=N}^{\infty} \exp \left( \frac{2\varepsilon^d_k(\varepsilon) - \varepsilon^2(\lambda(\mu)^*)}{8A_k} \right) n.
\]
The last series is summable for \(\varepsilon\) small enough. From the uniform decay of cylinders, the Hausdorff \(d_k(\varepsilon)\)-measure of the set \(B_{k,p}(\phi, \alpha) \cap J' \cap \Delta_a\) is finite. Letting \(\varepsilon \to 0\) we obtain \(\dim_H(B_{k,p}(\phi, \alpha) \cap J' \cap \Delta_a) \leq d_k\). Since \(p \geq 1\) is arbitrary, we obtain \(\dim_H(B_k(\phi, \alpha) \cap J' \cap \Delta_a) \leq d_k\).

In the case \(\sup_\Delta \|\phi_k\| < \infty\), Lemma 3.1 implies that for each integer \(N \geq 1\),
\[
B_k(\phi, \alpha) \cap J' \cap \Delta_a \subset \bigcup_{n=N}^{\infty} \left\{ \Delta_\omega : \omega \in \bigcup_{n=N}^{\infty} H^n \right\}.
\]
The rest of the proof is a repetition of the previous argument. We make use of Lemma 4.3(b) instead of Lemma 4.3(a), and replace \(B_{k,p}(\phi, \alpha)\) by \(B_k(\phi, \alpha)\), \(H^n_p\) by \(H^n\), and \(d_k(\varepsilon)\) by \(\delta_k(\varepsilon)\). This completes the proof of Proposition 4.1. \(\square\)
Proof of Lemma 4.3. Let $\epsilon \in (0, \min \{\epsilon_0, 4/3\})$ and $a \in S$. Let $n \geq 1$ and let $G$ be the non-empty subset of $E^n(a, a)$ in the statement of Lemma 4.3. Put $\tilde{X} = \bigcap_{m=0}^{\infty}(\sigma^n)^{-m}[G] \subset X$, $\tilde{\sigma} = \sigma^n|_{\tilde{X}}$ and $K = \pi \tilde{X}$, where $\pi$ is the coding map defined in Section 3.1. Then $\tilde{X}$ is topologically conjugate to the full shift on $\#G$-symbols. The induced map $f^n|_K: K \to K$ is topologically semi-conjugate to this full shift, namely $\pi|_{\tilde{X}}$ is one-to-one except on countable number of points where it is two-to-one, and the following diagram commutes:

$$\begin{array}{c}
\tilde{X} \xrightarrow{\tilde{\sigma}} \tilde{X} \\
\pi|_{\tilde{X}} \downarrow \quad \downarrow \pi|_{\tilde{X}} \\
K \xrightarrow{f^n|_K} K.
\end{array}$$

Define the induced potential $\tilde{\psi}: \tilde{X} \to \mathbb{R}$ given by

$$\tilde{\psi} = -\log |(f^n)' \circ \pi|.$$

Fix $z \in K$. For $\beta \in \mathbb{R}$ we have

$$\sum_{x \in \tilde{\sigma}^{-m}z} \exp \left( \beta \sum_{j=0}^{m-1} \tilde{\psi}(\tilde{\sigma}^j x) \right) \geq \left( \inf_{x \in K} \sum_{j=0}^{m-1} e^{\beta \tilde{\psi}(x)} \right)^m \geq \left( e^{-\beta D_n(\log |f'|) \frac{m}{\beta}} \right)^m.$$

Taking logarithms, dividing by $m$ and then letting $m \to \infty$, we have

$$\liminf_{m \to \infty} \frac{1}{m} \log \sum_{x \in \tilde{\sigma}^{-m}z} \exp \left( \beta \sum_{j=0}^{m-1} \tilde{\psi}(\tilde{\sigma}^j x) \right) \geq \log m \beta(G) - D_n(\log |f'|).$$

That the lower-limit is actually a limit follows from sub-additivity. We show that this limit is equal to the pressure $P(\beta \tilde{\psi})$ of the potential $\beta \tilde{\psi}$ with respect to $\tilde{\sigma}$ as defined in [32]. For every $m \geq 1$ we have

$$D_m(\log |(f^n)'|_K) \leq D_m(\log |f'|) = o(mn).$$

Therefore, the Markov map $f^n|_K$ with Markov partition $\{\Delta_\omega\}_{\omega \in G}$ has mild distortion. By a slight modification of the proof of [32] Theorem 1.2 we obtain

$$(4.6) \quad P(\beta \tilde{\psi}) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{x \in \tilde{\sigma}^{-m}z} \exp \left( \beta \sum_{j=0}^{m-1} \tilde{\psi}(\tilde{\sigma}^j x) \right).$$

Moreover, the pressure satisfies the variational principle

$$(4.7) \quad P(\beta \tilde{\psi}) = \sup_{\tilde{\mu} \in \mathcal{M}(\tilde{\sigma})} \left\{ \tilde{h}(\tilde{\mu}) + \beta \int \tilde{\psi} d\tilde{\mu}: \int \tilde{\psi} d\tilde{\mu} > -\infty \right\},$$

where $\mathcal{M}(\tilde{\sigma})$ denotes the space of $\tilde{\sigma}$-invariant measures endowed with the weak* topology, and $\tilde{h}(\tilde{\mu})$ the entropy of $\tilde{\mu} \in \mathcal{M}(\tilde{\sigma})$ with respect to $\tilde{\sigma}$.

The rest of the proof consists of two parts. We first prove Lemma 4.3(a). Assume $G$ is a finite set. For each $m \geq 1$ and $x \in \tilde{\sigma}^{-m}z$, we have $\#(\tilde{\sigma}^{-m}z) = (\#G)^m$ and
\[ \left| \sum_{j=0}^{m-1} \psi(\bar{s}^j x) \right| \leq m \sup_{\Delta G} \log |(f^n)'|. \] Hence for every \( \beta \in \mathbb{R} \),

\[ P(\beta \tilde{\psi}) \leq \sup_{m \geq 1} \frac{1}{m} \log \sum_{x \in \bar{\sigma}^{-m} z} \exp \left( \beta \sum_{j=0}^{m-1} \psi(\bar{s}^j x) \right) \leq \log \#G + \sup_{\Delta G} |(f^n)'|^{\beta} < \infty. \]

Substituting \( \beta = d_k(\epsilon) \) into (4.7) gives

\[ \infty > P(d_k(\epsilon) \tilde{\psi}) \geq \log m_{d_k(\epsilon)}(G) - d_k(\epsilon) D_n(\log |f'|). \]

Pick \( \tilde{\mu}_n \in \mathcal{M}(\bar{\sigma}) \) such that \( \int \tilde{\psi} d\tilde{\mu}_n > -\infty \), and

\[ \tilde{h}(\tilde{\mu}_n) + d_k(\epsilon) \int \tilde{\psi} d\tilde{\mu}_n \geq \log m_{d_k(\epsilon)}(G) - \epsilon - d_k(\epsilon) D_n(\log |f'|). \]

The measure \( \mu_n = (1/n) \sum_{j=0}^{n-1} f_*(\pi|_{\bar{\sigma}})_* \tilde{\mu}_n \) is in \( \mathcal{M}(f) \). Since \( \sup_{G} \tilde{\psi} < 0 \) from Lemma 4.3(ii), \( \int \tilde{\psi} d\tilde{\mu}_n < 0 \) and so \( \chi(\mu_n) > 0 \) holds. Lemma 4.3(i) gives

\[ \text{(4.8)} \quad \left\| \int \phi_k d\mu_n - \alpha_k \right\| \leq \frac{\epsilon}{2}. \]

Hence, by the definition of \( d_k(\epsilon) \) we have

\[ h(\mu_n) - d_k(\epsilon) \chi(\mu_n) \leq -\epsilon \chi(\mu_n) < 0. \]

This yields

\[ 0 > (h(\mu_n) - d_k(\epsilon) \chi(\mu_n)) n = \tilde{h}(\tilde{\mu}_n) + d_k(\epsilon) \int \tilde{\psi} d\tilde{\mu}_n \]

\[ \geq \log m_{d_k(\epsilon)}(G) - \epsilon - d_k(\epsilon) D_n(\log |f'|). \]

The Lyapunov exponent of \( \mu_n \) may become arbitrarily close to 0 as \( n \) increases. To circumvent this problem, we introduce another expanding measure

\[ \nu_n = \left( 1 - \frac{\epsilon}{7A_k} \right) \mu_n + \frac{\epsilon}{7A_k} \mu^*, \]

which is in \( \mathcal{M}(f) \) and satisfies

\[ \left\| \int \phi_k d\nu_n - \alpha_k \right\| \leq \left( 1 - \frac{\epsilon}{7A_k} \right) \left\| \int \phi_k d\mu_n - \alpha_k \right\| + \frac{\epsilon}{7A_k} \left\| \int \phi_k d\mu_n \right\| \]

\[ + \frac{\epsilon}{7A_k} \left\| \int \phi_k d\mu^* \right\| \]

\[ \leq \frac{\epsilon}{2} + \frac{\epsilon}{7A_k} \left\| \int \phi_k d\mu_n \right\| + \frac{\epsilon}{3} \quad \text{by (4.2) and (4.8)} \]

\[ < \frac{5\epsilon}{6} + \frac{\epsilon}{7A_k} \left( \| \alpha_k \| + \frac{\epsilon}{2} \right) \quad \text{by (4.8)} \]

\[ < \frac{5\epsilon}{6} + \frac{\epsilon}{14} + \frac{\epsilon^2}{14} \quad \text{by (4.2)} \]

\[ < \epsilon. \]
The last inequality holds if \( \epsilon < 4/3 \). Therefore, the definition of \( d_k(\epsilon) \) gives \( d_k(\epsilon) \geq 0 \) and

\[
h(\nu_n) - d_k(\epsilon)\chi(\nu_n) \leq -\epsilon\chi(\nu_n) \leq -\frac{\epsilon^2}{7A_k}\chi(\mu^*).
\]

On the other hand, from (4.3) and (4.9),

\[
(h(\nu_n) - d_k(\epsilon)\chi(\nu_n)) n = \left(1 - \frac{\epsilon}{7A_k}\right)(h(\mu_n) - d_k(\epsilon)\chi(\mu_n)) n
\]

\[
+ \frac{\epsilon}{7A_k}(h(\mu^*) - d_k(\epsilon)\chi(\mu^*)) n
\]

\[
\geq \log m_{d_k(\epsilon)}(G) - \epsilon - d_k(\epsilon)D_n(\log |f'|),
\]

and therefore

\[
\log m_{d_k(\epsilon)}(G) \leq (h(\nu_n) - d_k(\epsilon)\chi(\nu_n)) n + \epsilon + d_k(\epsilon)D_n(\log |f'|)
\]

\[
\leq -\frac{\epsilon^2}{7A_k}\chi(\mu^*) n + \epsilon + d_k(\epsilon)D_n(\log |f'|)
\]

\[
\leq -\frac{\epsilon^2}{8A_k}\chi(\mu^*) n.
\]

The last inequality holds for \( n \) large enough. This completes the proof of Lemma 4.3(a).

We are left to prove Lemma 4.3(b). If \( G \) is a finite set, then the desired inequality is a consequence of Lemma 4.3(a) and \( d_k(\epsilon) \geq d_k(\epsilon) \). If \( G \) is a countably infinite set, then the variational principle does not preclude the case where both numbers in (4.7) are \( \infty \). We claim this is not the case for \( \beta = d_k(\epsilon) \). Indeed, for any measure \( \tilde{\mu} \in M(\tilde{\sigma}) \) with \( \int \tilde{\psi}d\tilde{\mu} > -\infty \), the measure \( \mu = (1/n) \sum_{j=0}^{n-1} f_j^*(\pi|\tilde{X}) \tilde{\mu} \) is in \( \mathcal{M}(f) \) and satisfies

\[
(h(\mu) - d_k(\epsilon)\chi(\mu)) n = \tilde{h}(\tilde{\mu}) + d_k(\epsilon) \int \tilde{\psi}d\tilde{\mu},
\]

by Abramov-Kac’s formula (\cite{10 Theorem 2.3} [\cite{52 Theorem 5.1}]). This implies

\[
P(\overline{d_k(\epsilon)}(\tilde{\psi}) \leq n \cdot \sup \{h(\mu) - d_k(\epsilon)\chi(\mu) : \mu \in \mathcal{M}(f)\}.
\]

Since \( \overline{d_k(\epsilon)} > \beta_\infty \) by definition, we have that the right-hand side is finite. We repeat the same argument as before, replacing \( d_k(\epsilon) \) by \( \overline{d_k(\epsilon)} \). This completes the proof of Lemma 4.3.

4.2. Lower bound on dimension. The next proposition will be used to obtain lower bounds on the Hausdorff dimension of level sets.

**Proposition 4.5.** Let \( f : \Delta \to M \) be a non-uniformly expanding Markov map which is finitely irreducible and has uniform decay of cylinders. Let \( \phi \in \mathcal{F}_\infty^N \), \( \alpha \in \mathbb{R}^N \) and let \( \{\mu_i\}_{i=1}^\infty \) be a sequence of ergodic expanding measures such that

\[
\lim_{i \to \infty} \left\| \int \phi_i d\mu_i - \alpha_i \right\| = 0.
\]

Then \( B(\phi, \alpha) \neq \emptyset \) and

\[
b_\phi(\alpha) \geq \lim \sup_{i \to \infty} \dim(\mu_i).
\]
Proof of Proposition 4.3. The proof consists of two steps. We first approximate each measure $\mu_i$ with a finite collection of cylinders in the sense of Lemma 3.4 and use the finite irreducibility to glue orbits together to construct a Borel subset $\Gamma$ such that $\dim_H \Gamma \geq \limsup_{i \to \infty} \dim(\mu_i)$ in Step 1. Next we show in Step 2 that $\Gamma$ is contained in $B(\phi, \alpha)$.

Step 1: Construction of a Borel set $\Gamma$. We begin by claiming that we may assume $h(\mu_i) > 0$ for every $i \geq 1$ without any loss of generality. To see this, fix an expanding measure $\nu$ and define a sequence $\{\nu_i\}_{i=1}^{\infty}$ of expanding measures as follows: if $h(\mu_i) > 0$ then $\nu_i = \mu_i$. If $h(\mu_i) = 0$, then take $\delta_i \in (0,1)$ and define $\nu_i = (1 - \delta_i)\mu_i + \delta_i \nu$ so that $\lim_{i \to \infty} \int \phi_i d\nu_i - \alpha_i = 0$ and $\limsup_{i \to \infty} \dim(\mu_i) = \limsup_{i \to \infty} \dim(\nu_i)$. The claim follows from Lemma 3.5.

Choose a subsequence of $\{\mu_i\}_{i=1}^{\infty}$ so that $\{\dim(\mu_i)\}_{i=1}^{\infty}$ converges to the maximal possible limit. Let $\Lambda$ be the finite subset of $E^*$ given by the finite irreducibility of $f$. By taking a further subsequence and relabeling the indices if necessary, we may assume the following holds for every $i \geq 1$:

(4.10) $h(\mu_i) > \frac{1}{i}$;

(4.11) $\left\| \int \phi_i d\mu_i - \alpha_i \right\| < \frac{1}{i}$.

Put $c_i = \sup_{\lambda \in \Lambda} \sup_{\Delta \Lambda} \left\| S_\lambda | \phi_i \right\|$.

Since $\phi \in \mathcal{F}^\Delta$ and $\Lambda$ is finite, $c_i < \infty$ holds for every $i \geq 1$. Clearly, $c_i$ is monotone increasing in $i$. In view of Lemma 3.4, fix an integer $\ell_i \geq 1$, symbols $a_i, b_i \in S$ and a finite subset $F^{\ell_i} = F^{\ell_i}(a_i, b_i)$ of $E^\ell_i(a_i, b_i)$ for which the following holds for every $i \geq 1$:

(4.12) $\ell_i \geq i$;

(4.13) $\frac{\ell_i}{i} \geq \sup_{\lambda \in \Lambda} \sup_{\Delta \Lambda} \log |(f^{[\lambda]}'|);$

(4.14) $ic_i + i|\Lambda| \cdot \|\alpha_i\| \leq \frac{\ell_i}{i};$

(4.15) $\left| \frac{1}{\ell_i} \log \# F^{\ell_i} - h(\mu_i) \right| \leq \frac{1}{i};$

(4.16) $\sup_{\Delta F^{\ell_i}} \left\| \frac{1}{\ell_i} S_{\ell_i} \phi_i - \int \phi_i d\mu_i \right\| \leq \frac{1}{i};$

(4.17) $\sup_{\Delta F^{\ell_i}} \left| \frac{1}{\ell_i} \log |(f^{\ell_i}')| - \chi(\mu_i) \right| \leq \frac{1}{i}$.

For each $(b_i, a_i)$ fix $\lambda_{b_i a_i} \in \Lambda$ such that $b_i \lambda_{b_i a_i} a_i \in E^*$, and for each $(b_i, a_{i+1})$ fix $\lambda_{b_i a_{i+1}} \in \Lambda$ such that $b_i \lambda_{b_i a_{i+1}} a_{i+1} \in E^*$. For each integer $t \geq 0$ we define a finite
subset \( F_{t_1}^t \) of \( E^* \) as follows. Put \( F_{t_1}^{t_0} = E^0 \) and \( F_{t_1}^{t_1} = F_{t_1}^t \). For \( t \geq 2 \), \( F_{t_1}^{t,t} \) is the set of elements of \( E^* \) of the form

\[
\omega_1 \lambda_{b_1 a_1} \omega_2 \lambda_{b_2 a_2} \cdots \omega_t \lambda_{b_t a_t}
\]

with \( \omega_1, \ldots, \omega_t \in F_{t_1}^t \). Clearly we have

\[
(4.18) \quad \# F_{t_1}^{t,t} = (\# F_{t_1}^t)^t.
\]

Let \( \{t_i\}_{i=1}^\infty \) be a sequence of positive integers to be determined later. For integers \( N \geq 2 \) and \( s \geq 0 \), denote by \( E_{t_1,\ldots,t_{N-1}}^{t,s} \) the set of elements of \( E^* \) of the form

\[
\eta_1 \lambda_{b_1 a_2} \eta_2 \lambda_{b_2 a_3} \cdots \eta_{N-1} \lambda_{b_{N-1} a_N} \eta_N
\]

with \( \eta_i \in F_{t_1}^{t_i} \) for \( i = 1, \ldots, N-1 \) and \( \eta_N \in F_{t_1}^{t,N,s} \). Clearly, all words in \( E_{t_1,\ldots,t_{N-1}}^{t,s} \) have the same length and

\[
(4.19) \quad \# E_{t_1,\ldots,t_{N-1}}^{t,s} = \left( \prod_{i=1}^{N-1} \# F_{t_1}^{t_i,t_i} \right) \max\{\# F_{t_1}^{t,N,s},1\}.
\]

For each integer \( T \geq t_1 \) there exists a unique \( N = N(T) \) and \( s = s(T) \) such that

\[
(4.20) \quad T = t_1 + \cdots + t_{N-1} + s, \quad 0 \leq s \leq t_N - 1.
\]

Set

\[
\Gamma = \bigcap_{T=t_1}^\infty \Delta E_{t_1,\ldots,t_{N(T)-1},s(T)}.
\]

This is an intersection of decreasing closed sets, and so a compact set. For each \( \omega \in E_{t_1,\ldots,t_{N-1},s} \) fix a point \( x(\omega) \in \Delta_\omega \cap \Gamma \). Let \( \nu_T \) denote the uniform distribution on the finite set \( \bigcup \{x(\omega)\colon \omega \in E_{t_1,\ldots,t_{N-1},s}\} \). Let \( \nu \) be an accumulation point of the sequence \( \{\nu_T\}_{T=t_1}^\infty \) in the weak*-topology on the space of measures on \( M \). We show that

\[
(4.21) \quad \liminf_{r \to 0} \frac{\log \nu(B(x,r) \cap \Delta)}{\log r} \geq \lim_{i \to \infty} \operatorname{dim}(\mu_i) \quad \text{for every } x \in \Gamma,
\]

where \( B(x,r) \) denotes the connected open subset of \( M \) of Riemannian length \( 2r \) centered at \( x \). From the mass distribution principle and the compactness of \( \Gamma \) it follows that

\[
(4.22) \quad \operatorname{dim}_H \Gamma \geq \liminf_{i \to \infty} \operatorname{dim}(\mu_i).
\]

In order to show \((4.21)\), for each integer \( T \geq t_1 \) in \((4.20)\) define

\[
r_T = \exp \left( -t_{N-1} \ell_{N-1} \left( \chi(\mu_{N-1}) + \frac{3}{N-1} \right) - s \ell_N \left( \chi(\mu_N) + \frac{2}{N} \right) \right).
\]

**Lemma 4.6.** One can choose \( \{t_i\}_{i=1}^\infty \) inductively so that \( r_T \searrow 0 \) as \( T \nearrow \infty \).

**Proof.** If \( s < t_N - 1 \) then \( N(T+1) = N(T) \) and \( s(T+1) = s(T) + 1 \), and so

\[
\frac{r_{T+1}}{r_T} = \exp \left( -\ell_N \left( \chi(\mu_N) + \frac{2}{N} \right) \right) < 1.
\]
If \( s = t_N - 1 \), then \( N(T + 1) = N(T) + 1 \) and \( s(T + 1) = 0 \), and so

\[
r_T = \exp \left( -t_{N-1} \ell_{N-1} \left( \chi(\mu_{N-1}) + \frac{3}{N-1} \right) - (t_N - 1) \ell_N \left( \chi(\mu_N) + \frac{2}{N} \right) \right),
\]

and

\[
r_{T+1} = \exp \left( -t_N \ell_N \left( \chi(\mu_N) + \frac{3}{N} \right) \right).
\]

Therefore, by choosing \( \{t_i\}_{i=1}^\infty \) inductively such that \( t_i \) is large enough compared to \( t_{i-1} \) one can make sure that \( r_{T+1}/r_T \leq 1/2 \) for every \( T \geq t_1 \).

Let \( r \in (0, r_{t_1}] \). Then there exists \( T \geq t_1 \) such that

\[
(4.23) \quad r_{T+1} < r \leq r_T.
\]

For each \( \omega \in E^{t_1, \ldots, t_{N-1}, \Delta} \) we have

\[
(4.24) \quad \sup_{\Delta_\omega} \log |(f[\omega]|'\right) \leq \sum_{i=1}^{N-1} t_i \ell_i \left( \chi(\mu_i) + \frac{2}{i} \right) + s \ell_N \left( \chi(\mu_N) + \frac{2}{N} \right)
\]

\[
\leq t_{N-1} \ell_{N-1} \left( \chi(\mu_{N-1}) + \frac{3}{N-1} \right) + s \ell_N \left( \chi(\mu_N) + \frac{2}{N} \right),
\]

where the last inequality holds provided each \( t_i \) is chosen large enough compared to \( t_1, \ldots, t_{i-1} \).

**Lemma 4.7.** We have

\[
\kappa := \inf_{a \in S} |f \Delta_a| = \inf_{\omega \in E^*} |f[\omega]| \Delta_\omega| > 0.
\]

**Proof.** Let \( b \in E^* \). For each \( a \in S \) there exists \( \lambda \in \Lambda \) such that \( a \lambda b \in E^* \). Take one connected component of \( (f^{[\lambda]}|_{\Delta_b}^{-1}) \Delta_b \) and denote it by \( A \). Since \( f^{[\lambda]}(fA) = \Delta_b \), the mean value theorem implies \( \sup_{[\lambda]} |(f^{[\lambda]})|'||fA| \geq |\Delta_b| \). Since \( \Lambda \) is a finite set, \( |f \Delta_a| \geq |fA| \geq |\Delta_b|/\sup_{\lambda \in \Lambda} \sup_{[\lambda]} |(f^{[\lambda]})|' \geq 0 \) as required. \( \square \)

By the mean value theorem, Lemma 4.7 and (1.24),

\[
|\Delta_\omega| \geq \frac{|f[\omega]| \Delta_\omega|}{\sup_{\Delta_\omega} |(f[\omega]|')|}
\]

\[
\geq \kappa \exp \left( -t_{N-1} \ell_{N-1} \left( \chi(\mu_{N-1}) + \frac{3}{N-1} \right) - s \ell_N \left( \chi(\mu_N) + \frac{2}{N} \right) \right)
\]

\[
= \kappa r_T.
\]

Hence, for any \( x \in \Gamma \) we have

\[
\# \{ \omega \in E^{t_1, \ldots, t_{N-1}, \Delta}: B(x, r) \cap \Delta_\omega \neq \emptyset \} \leq \frac{2r_T}{\inf_{\omega \in E^{t_1, \ldots, t_{N-1}, \Delta}} |\Delta_\omega|} + 2 \leq 2(\kappa^{-1} + 1).
\]

**Lemma 4.8.** \( \nu(\partial \Delta_\omega) = 0 \).
Proof. By construction, each set at the $T$-th level contains a finite number of sets at the $(T + 1)$-th level, and this number of branches is independent of the sets at the $T$-th level. From the positivity of entropy (4.10) and (4.15) with $i = N - 1$ we have $\#F_{t_{N - 1}} > 1$. Hence, the number of branches at the $T$-th level in the case $s < t_{N - 1}$ is $\#F_{t_{N - i}t_{N - 1}} > 1$. A similar argument shows that the number of branches in the case $s = t_{N - 1}$ is $\#F_{t_{N}t_{N - 1}} > 1$. For each $\omega \in E_{t_{1}, \ldots, t_{N - 1}, s}$ and for every $p \geq T$,

$$\nu_{p}(\Delta \omega) = \frac{1}{\#(F_{t_{1}})^{t_{1}} \cdots \#(F_{t_{N - 1}})^{t_{N - 1}} \#(F_{t_{N}})^{s}}.$$ 

Since the number of branches is strictly bigger than 1 and cylinders of fixed length have pairwise disjoint interiors, the claim follows. \qed

From Lemma 4.8 the weak*-convergence of measures gives $\nu(\Delta \omega) = \lim_{p \to \infty} \nu_{p}(\Delta \omega)$. By (4.2a) we then have

$$\nu(B(x, r) \cap \Gamma) \leq \frac{2(\kappa^{-1} + 1)}{\#(F_{t_{N - 1}})^{t_{N - 1}} \#(F_{t_{N}})^{s}} \leq 2(\kappa^{-1} + 1) \times$$

$$\exp \left(-t_{N - 1} \ell_{N - 1} \left(h(\mu_{N - 1}) - \frac{1}{N - 1}\right) - s \ell_{N} \left(h(\mu_{N}) - \frac{1}{N}\right)\right).$$

By the definition of $r_{T}$ and (4.23) this yields

$$\log \frac{\nu(B(x, r) \cap \Gamma)}{\log r} \geq \frac{\log \nu(B(x, r) \cap \Gamma)}{\log r_{T}} \geq \frac{t_{N - 1} \ell_{N - 1} (h(\mu_{N - 1}) - 1/(N - 1)) + s \ell_{N} (h(\mu_{N}) - 1/N)}{t_{N - 1} \ell_{N - 1} (\chi(\mu_{N - 1}) + 3/(N - 1)) + s \ell_{N} (\chi(\mu_{N}) + 2/N)} + \frac{\log(2(\kappa^{-1} + 1))}{\log r_{T}}.$$

We have $T \to \infty$ as $r \to 0$, and so $N \to \infty$ by (4.20). Hence (4.21) follows.

Step 2: Verification of $\Gamma \subset B(\phi, \alpha)$. It remains to fix $\{t_{i}\}_{i=1}^{\infty}$ so that $\Gamma \subset B(\phi, \alpha)$ holds, namely for all $x \in \Gamma$,

$$\lim_{n \to \infty} \frac{1}{n} S_{n} \phi_{k}(x) - \alpha_{k} = 0 \text{ for every } k \geq 1.$$

For each $i \geq 2$ we choose $t_{i}$ large enough compared to $t_{1}, \ldots, t_{i - 1}$ so that the following holds in addition to all the previously used inequalities:

(4.26) $$\sum_{j=1}^{i} \frac{3t_{j} \ell_{j}}{j} \leq 4t_{i} \ell_{i};$$

(4.27) $$c_{i - 1} + \sup_{1 \leq p \leq t_{i}} \sup_{\Delta F_{t_{i}}} \|S_{p} \phi_{i}\| + (|\Lambda| + \ell_{i}) \|\alpha_{i}\| \leq t_{i - 1}.$$ 

Let $k \geq 1$. Let $T \geq t_{1}$ be such that $k \leq N - 1$. 
Lemma 4.9. Let $i \geq 1$, $t \geq 1$ be integers such that $\eta_i \in F_{\ell, i}^t$. Then

$$\sup_{\Delta_{n, b_{a+1}}} \| S_{|\eta|, \lambda_{b_{a+1}}^i} |\phi_k - |\eta|, \lambda_{b_{a+1}}^i \| \alpha_k \| \leq \frac{2t\ell_i}{t} + t\epsilon + |\Lambda| \cdot \| \alpha_k \|.$$  

Proof. The definition of $c_k$ gives

$$\sup_{\Delta_{n, b_{a+1}}} \| S_{|\eta|, \lambda_{b_{a+1}}^i} |\phi_k - |\eta|, \lambda_{b_{a+1}}^i \| \alpha_k \| < c_k + |\Lambda| \cdot \| \alpha_k \|.$$  

The desired inequality for $t = 1$ is a consequence of (4.11), (4.16) and (4.28). In the case $t \geq 2$, we have $\eta_i = \omega_1 \lambda_{b_{a_i}} \omega_2 \cdots \lambda_{b_{a_i}} \omega_t$ with $\omega_1, \ldots, \omega_t \in F_{\ell, i}$ and $\lambda_{b_{a_i}} \in \Lambda$. By (4.11) and (4.16), for every $1 \leq j \leq t - 1$ we have

$$\sup_{\Delta_{n, b_{a+1}}} \| S_{|\omega_j|, \lambda_{b_{a+1}}^i} |\phi_k - |\omega_j|, \lambda_{b_{a+1}}^i \| \alpha_k \| \leq \sup_{\Delta_{n, j}} \| S_{|\ell|, \phi_k - \ell_i \alpha_k} \| + \sup_{\Delta_{n, j}} \| S_{|\alpha|, \phi_k - |\lambda_{b_{a+1}}^i| \alpha_k} \|$ 

$$\leq \sup_{\Delta_{n, j}} \left( \| S_{|\ell|, \phi_k - \ell_i \int \phi_k d\mu_i} \| + \| \ell_i \int \phi_k d\mu_i - \ell_i \alpha_k \| \right) 
\leq \frac{2\ell_i}{t} + c_k + |\Lambda| \cdot \| \alpha_k \|.$$

Summing this inequality over all $1 \leq j \leq t$ yields the desired inequality. The case $t = 1$ is covered by the same argument. \hfill \Box

Recall that each $\omega \in E_{\ell_1, \ldots, \ell_{N-1}}^t$ has the form

$$\omega = \eta_1 \lambda_{b_{a_1}} \cdots \eta_{k-1} \lambda_{b_{a_{k-1}} a_k} \cdots \eta_{N-1} \lambda_{b_{N-1} a_N} \eta_N,$$  

with $\eta_i \in F_{\ell, i}^t$, $\lambda_{b_{a_i+1}} \in \Lambda$ for $1 \leq i \leq N - 1$ and $\eta_N \in F_{\ell N}^s$. Use the triangle inequality and then split

$$\sup_{\Delta_\omega} \| S_{|\omega|, \phi_k} - |\omega| \alpha_k \| \leq I + II,$$  

where

$$I = \sup_{\Delta_{\omega I}} \| S_{|\omega_I|, \phi_k} - |\omega_I| \alpha_k \| \text{ and } II = \sup_{\Delta_{\omega II}} \| S_{|\omega_{II}|, \phi_k} - |\omega_{II}| \alpha_k \|,$$  

and $\omega_I$, $\omega_{II} \in E^s$ are given by

$$\omega_I = \eta_1 \lambda_{b_{a_1}} \cdots \eta_{k-1} \lambda_{b_{a_{k-1}} a_k} \text{ and } \omega_{II} = \eta_k \lambda_{b_{a_{k+1}} a_k} \cdots \eta_{N-1} \lambda_{b_{N-1} a_N} \eta_N.$$
Applying Lemma 4.9 to $\eta_i \lambda_b a_{i+1}$ for $k \leq i \leq N - 1$ in $\omega$ and using once more Lemma 4.9 to deal with the last word $\eta_N$ yields

\[(4.29) \quad II \leq \sum_{i=k}^{N-1} \left( \frac{2t_i}{i} + \frac{t_i}{c_i} + t_i |\Lambda| \cdot \|\alpha_k\| \right) + s \left( \frac{2\ell_N}{N} + c_N + |\Lambda| \cdot \|\alpha_k\| \right)
\]

\[
= \sum_{i=k}^{N-1} 2\frac{t_i}{i} + \sum_{i=k}^{N-1} t_i \left(c_i + |\Lambda| \cdot \|\alpha_k\| \right) + s \left( \frac{2\ell_N}{N} + c_N + |\Lambda| \cdot \|\alpha_k\| \right)
\]

\[
\leq \sum_{i=k}^{N-1} 2\frac{t_i}{i} + t_{N-1} (N - 1) \left( c_{N-1} + |\Lambda| \cdot \|\alpha_k\| \right) + s \left( \frac{2\ell_N}{N} + c_N + |\Lambda| \cdot \|\alpha_k\| \right)
\]

\[
\leq \sum_{i=k}^{N-1} 2\frac{t_i}{i} + \frac{t_{N-1} \ell_{N-1}}{N - 1} + \frac{3s\ell_N}{N} \quad \text{by (4.14)}
\]

\[
\leq \sum_{i=k}^{N-1} \frac{3t_i}{i} + \frac{3s\ell_N}{N}.
\]

The same argument gives

\[ I \leq \sum_{i=1}^{k-1} \frac{3t_i}{i}. \]

Now, let $x \in \Gamma$. For each integer $n \geq t_1 \ell_1$ there exists an integer $T \geq t_1$ such that if $\omega \in E^{t_1,...,t_{N-1},s}$ and $x \in \Delta_\omega$, then the following holds:

\[(4.30) \quad 0 \leq n - |\omega| \leq |\Lambda| + \ell_N; \]

\[(4.31) \quad \|S_n |\omega| \phi_k(\sigma^{\omega}|x)\| \leq c_{N-1} + \sup_{1 \leq p \leq \ell_N} \sup_{\Delta F^\ell_N} \|S_p \phi_k\|. \]

From (4.29), (4.30), (4.31) we have

\[
\|S_n \phi_k(x) - n\alpha_k\| \leq \|S_n |\omega| \phi_k(x) - |\omega| \alpha_k\| + \|S_n - |\omega| \phi_k(\sigma^{\omega}|x) - (n - |\omega|)\alpha_k\|
\]

\[
\leq \sum_{i=1}^{N-1} \frac{3t_i}{i} + \frac{3s\ell_N}{N} + c_{N-1}
\]

\[
+ \sup_{1 \leq p \leq \ell_N} \sup_{\Delta F^\ell_N} \|S_p \phi_k\| + (|\Lambda| + \ell_N)\|\alpha_k\|.
\]

On the right-hand side, we use (4.26) to bound the first two terms, and (4.27) to bound the remaining terms. We have

\[
\|S_n \phi_k(x) - n\alpha_k\| \leq \frac{4t_{N-1} \ell_{N-1}}{N - 1} + \frac{3s\ell_N}{N} + t_{N-1}
\]

\[
\leq \frac{4n}{N - 1} + \frac{t_{N-1} \ell_{N-1}}{N - 1} \quad \text{by (4.12), (4.30)}
\]

\[
\leq \frac{5n}{N - 1} \quad \text{by (4.30)}.
\]
Since $N$ increases as $n$ does, it follows that $\lim_{n \to \infty} \| (1/n) S_n \phi_k (x) - \alpha_k \| = 0$. Since $k \geq 1$ is arbitrary, this completes the proof of Proposition 4.5. \hfill \Box

4.3. **Proof of the Main Theorem.** Let $f : \Delta \to M$ be a non-uniformly expanding Markov map which is finitely irreducible, has mild distortion and uniform decay of cylinders. Let $\phi \in \mathcal{F}_N$ and $\alpha \in \mathbb{R}_N$.

**Proof of Main Theorem (a).** Assume that for any $\epsilon > 0$ and any integer $k \geq 1$ there exists a measure $\mu \in \mathcal{M}(f)$ such that $\| \int \phi_k d\mu - \alpha_k \| < \epsilon$. Then there exists an expanding measure $\mu' \in \mathcal{M}(f)$ such that $\| \int \phi_k d\mu' - \alpha_k \| < \epsilon$. By Lemma 3.5, there exists an ergodic expanding measure $\mu''$ such that $\| \int \phi_k d\mu'' - \alpha_k \| < \epsilon$.

Proposition 4.5 shows $B(\phi, \alpha) \neq \emptyset$. Conversely, if $x \in B(\phi, \alpha)$ then using the finite irreducibility and $D_1(\phi_i) < \infty$ for each $i \geq 1$, one can find for any $\epsilon > 0$ and any integer $k \geq 1$ a measure $\mu \in \mathcal{M}(f)$ supported on a single periodic orbit such that $\| \int \phi_k d\mu - \alpha_k \| < \epsilon$. \hfill \Box

**Proof of Main Theorem (b).** Assume (1.2) and $B(\phi, \alpha) \neq \emptyset$. Put $d_{\infty} = \lim_{k \to \infty} d_k$, where $d_k$ is the constant defined in Proposition 4.4. Below we will verify $b_{\phi}(\alpha) = d_{\infty}$ by showing that the upper bound obtained in Section 4.1 and the lower bound in Section 4.2 coincide. We then show that if all $\phi_i$ are bounded then the constant $\beta_\infty$ is a lower bound for $b_{\phi}(\alpha)$. Lastly we show the remaining assertions.

We begin by claiming that

\begin{equation}
(4.32) \quad b_{\phi}(\alpha) \geq d_{\infty}.
\end{equation}

Indeed, by the definition of $d_{\infty}$, for every $c > 0$ and every $k \geq 1$ there exists $\mu_k \in \mathcal{M}(f)$ such that $\dim(\mu_k) > d_{\infty} - c$ and $\| \int \phi_k d\mu_k - \alpha_k \| < 1/k$. We may assume that each $\mu_k$ is expanding and, by Lemma 3.5, also ergodic. Then Proposition 4.5 yields $b_{\phi}(\alpha) \geq d_{\infty} - c$. Since $c > 0$ was arbitrary, we have thus shown $b_{\phi}(\alpha) \geq d_{\infty}$.

To prove the reverse inequality we distinguish two cases. First, we assume $\sup_{\Delta} \| \phi_k \| = \infty$ for some $k_0 \geq 1$. Condition (1.2) implies $b_{\phi}(\alpha) \leq \delta_0$. If this inequality is strict, then we have $d_{\infty} \leq b_{\phi}(\alpha) < \delta_0$ from (1.32), and Proposition 4.1(a) gives $b_{\phi}(\alpha) \leq d_{\infty}$. We claim that if $b_{\phi}(\alpha) = \delta_0$ then $d_{\infty} = \delta_0$ holds, for otherwise $d_{\infty} < \delta_0$ and so there exists $\epsilon > 0$ such that $d_{\infty} + \epsilon < \delta_0$. Hence, we have $d_k < d_{\infty} + \epsilon < \delta_0$ for $k \geq k_0$ large enough and Proposition 4.1(a) gives the desired contradiction $b_{\phi}(\alpha) \leq d_{\infty} + \epsilon < \delta_0$.

Next we assume $\sup_{\Delta} \| \phi_k \| < \infty$ for every $k \geq 1$. A slight modification of the argument in the previous case with Proposition 4.1(b) shows $b_{\phi}(\alpha) \leq \max\{d_{\infty}, \beta_\infty\}$. The reverse inequality and the equalities $b_{\phi}(\alpha) = \max\{d_{\infty}, \beta_\infty\} = d_{\infty}$ are consequences of (1.32) and the next lemma which is an adaptation of [13, Lemma 5.1] to our setting.

**Lemma 4.10.** Let $k \geq 1$ be an integer, $\epsilon > 0$ and $\mu \in \mathcal{M}(f)$ be an expanding measure be such that

\[ \left\| \int \phi_k d\mu - \alpha_k \right\| < \epsilon. \]
If \( \sup_{\Delta} \| \phi_k \| < \infty \), then there exists an expanding measure \( \nu \in \mathcal{M}(f) \) such that

\[
\dim(\nu) > \beta_{\infty} - \epsilon \quad \text{and} \quad \left\| \int \phi_k d\nu - \alpha_k \right\| < 2\epsilon.
\]

**Proof.** One can adapt [13, Lemma 2.5] to our setting to show that there exists a sequence \( \{ \mu_n \}_{n=1}^{\infty} \) of expanding measures which are supported on a finite union of cylinders and such that \( \lim_{n \to \infty} \dim(\mu_n) = \beta_{\infty} \) and \( \lim_{n \to \infty} \chi(\mu_n) = \infty \). Using this and arguing along the line of the proof of [13, Lemma 5.1] proves the assertion of the lemma. \( \square \)

We are almost ready to complete the proof of the Main Theorem(b). For the proof of the remaining assertions note that, for each fixed \( k \geq 1 \) we have

\[
\dim_H B(\phi, \alpha_k) = \lim_{\epsilon \to 0} \sup \left\{ \dim(\mu) : \mu \in \mathcal{M}(f), \left\| \int \phi_k d\mu - \alpha_k \right\| < \epsilon \right\}.
\]

Therefore \( b_{\phi}(\alpha) = \dim_H B(\phi, \alpha) = \lim_{k \to \infty} \dim_H B(\phi, \alpha_k) \) holds. The fact that one may restrict the supremum in the conditional variational formula to compactly supported measures follows from Lemma 3.4 and Lemma 3.5. \( \square \)

**Proof of Main Theorem(c).** Assume (1.2). The first assertion follows from Main Theorem(b). Now assume in addition that \( f \) has a neutral periodic point. Condition (1.2) means that there is an expanding measure with dimension arbitrarily close to \( \delta_0 \). Considering convex combinations of such measures and \( f \)-invariant measures supported on single neutral periodic orbits, we obtain an expanding measure with dimension arbitrarily close to \( \delta_0 \) and Lyapunov exponent arbitrarily close to 0. From this and (b), the second assertion in (c) follows. This completes the proof of the Main Theorem. \( \square \)

4.4. **Proof of Theorem 1.5.** Let \( f : \Delta \to M \) be a non-uniformly expanding fully branched Markov map which has mild distortion and uniform decay of cylinders, with an infinite Markov partition \( \{ \Delta_i \}_{i=1}^{\infty} \). Assume (1.2).

**Lemma 4.11.** We have

\[
\limsup_{\alpha \to \infty} \sup_{\alpha} \{ \dim(\mu) : \mu \in \mathcal{M}(f), \chi(\mu) = \alpha \} \leq \beta_{\infty}.
\]

**Proof.** For any \( \beta > \beta_{\infty} \) we have \( \sup \{ h(\mu) - \beta \chi(\mu) : \mu \in \mathcal{M}(f) \} < \infty \) and

\[
\alpha \cdot \sup \{ \dim(\mu) : \mu \in \mathcal{M}(f), \chi(\mu) = \alpha \} \leq \sup \{ h(\mu) - \beta \chi(\mu) : \mu \in \mathcal{M}(f), \chi(\mu) = \alpha \} + \alpha \beta \leq \sup \{ h(\mu) - \beta \chi(\mu) : \mu \in \mathcal{M}(f) \} + \alpha \beta.
\]

Dividing both sides by \( \alpha \) and letting \( \alpha \to \infty \), we have

\[
\limsup_{\alpha \to \infty} \sup_{\alpha} \{ \dim(\mu) : \mu \in \mathcal{M}(f), \chi(\mu) = \alpha \} \leq \beta.
\]

Since \( \beta > \beta_{\infty} \) is arbitrary, the desired inequality holds. \( \square \)
Proof of Theorem 1.5(a). Let

\[ D = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{\infty} \alpha_i \leq 1, \quad \alpha_i > 0 \quad \forall i \geq 1 \right\}. \]

Since \( \lim_{k \to \infty} \dim_H BE_k(\alpha) = \dim_H BE(\alpha) \) by Main Theorem(b) and since this convergence is monotone, it suffices to show that for sufficiently large \( k \geq 1 \), \( \dim_H BE_k(\cdot) \) is continuous on \( D \). Since the upper semi-continuity is a consequence of the formula in Main Theorem(b), it suffices to show the lower semi-continuity.

Let \( \alpha \in D \). Since the dimension does not become lower than \( \beta_\infty \) by Corollary 1.4, if \( \dim_H BE_k(\alpha) = \beta_\infty \) then there is nothing to prove. Assume \( \dim_H BE_k(\alpha) > \beta_\infty \). It suffices to show that for every \( 1 \leq j \leq k \) and every strictly monotone sequence \( \{\theta_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) converging to 0 as \( n \to \infty \),

\[ (4.33) \quad \dim_H BE_k(\alpha) \leq \liminf_{n \to \infty} \dim_H BE_k(\alpha + \theta_n e_j), \]

where \( e_j = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{R}^N \). By symmetry we only have to prove (4.33) for \( j = 1 \). We may assume \( \theta_n > 0 \) for every \( n \geq 1 \) for simplicity.

Suppose by contradiction that (4.33) does not hold for \( j = 1 \). Taking a subsequence of \( \{\theta_n\}_{n=1}^{\infty} \) if necessary we may assume there exists \( \epsilon > 0 \) such that

\[ (4.34) \quad \dim_H BE_k(\alpha) - \epsilon > \beta_\infty \]

and

\[ (4.35) \quad \sup_{n \geq 1} \dim_H BE_k(\alpha + \theta_n e_1) < \dim_H BE_k(\alpha) - 4\epsilon. \]

Lemma 4.12. For each \( n \geq 1 \) there exist \( \epsilon_n > 0 \) such that for any expanding measure \( \mu \) satisfying

\[ \mu(\Delta_1) = \alpha_1 + \theta_n \quad \text{and} \quad |\mu(\Delta_j) - \alpha_j| < \epsilon_n \quad 2 \leq \forall j \leq k, \]

we have

\[ \dim_H BE_k(\alpha + \theta_n e_1) > \dim(\mu) - \epsilon. \]

Proof. Follows from Corollary 1.4. \( \square \)

By Corollary 1.4 there exists a sequence \( \{\mu_p\}_{p=1}^{\infty} \) of expanding measures such that

\[ (4.36) \quad \dim(\mu_p) > \dim_H BE_k(\alpha) - \epsilon \quad \forall p \geq 1, \]

and

\[ (4.37) \quad \lim_{p \to \infty} \sup_{1 \leq j \leq k} |\mu_p(\Delta_j) - \alpha_j| = 0. \]

By Lemma 4.7 and \( D_1(\log |f'|) < \infty \) we have \( \inf_{\Delta_j} |f'| > 2 \) for all \( j \) but finitely many \( j \geq 1 \). Therefore, if \( k \geq 1 \) is sufficiently large, then the assumption \( \alpha \in D \) and (4.37) together imply

\[ (4.38) \quad \inf_{p \geq 1} \chi(\mu_p) \geq \left( 1 - \sum_{j=1}^{k} \alpha_j \right) \inf_{\Delta \in \bigcup_{j=1}^{k} \Delta_j} \log |f'| > 0. \]
From Lemma 4.11 and (4.34), (4.36) it follows that
\[(4.39) \quad \sup_{p \geq 1} h(\mu_p) \leq \sup_{p \geq 1} \chi(\mu_p) < \infty.\]
Choose a subsequence \(\{\mu_{p_n}\}_{n=1}^{\infty}\) of \(\{\mu_p\}_{p=1}^{\infty}\) such that
\[(4.40) \quad \alpha_1 + \theta_n < \mu_{p_n}(\Delta_1) \quad \forall n \geq 1 \quad \text{or} \quad \alpha_1 \geq \mu_{p_n}(\Delta_1) \quad \forall n \geq 1,
\]
and
\[(4.41) \quad |\mu_{p_n}(\Delta_j) - \alpha_j| < \epsilon_n \quad 2 \leq \forall j \leq k, \forall n \geq 1.
\]
We first assume the first alternative in (4.40). Since \(\alpha \in D\) we have \(\alpha_1 > 0\). For each \(n \geq 1\) fix a measure \(\zeta_n \in \mathcal{M}(f)\) such that
\[(4.42) \quad \sup_{n \geq 1} \chi(\zeta_n) < \infty
\]
and
\[(4.43) \quad \zeta_n(\Delta_1) < \frac{\alpha_1}{2} \quad \text{and} \quad \zeta_n(\Delta_j) = \alpha_j \quad 2 \leq \forall j \leq k.
\]
For each \(n \geq 1\) define \(t_n \in (0, 1)\) implicitly by
\[(1 - t_n)\zeta_n(\Delta_1) + t_n \mu_{p_n}(\Delta_1) = \alpha_1 + \theta_n.
\]
Note that \(t_n \to 1\), as \(n \to \infty\), and define
\[
\nu_n = (1 - t_n)\zeta_n + t_n \mu_{p_n}.
\]
Then \(\nu_n(\Delta_1) = \alpha_1 + \theta_n\), and from (4.41) and (4.43) we have \(|\nu_n(\Delta_j) - \alpha_j| < \epsilon_n\) for every \(2 \leq j \leq k\). Then we have for \(n\) large enough,
\[
\dim_H BE_k(\alpha + \theta_n e_1) > \dim(\nu_n) - \epsilon \quad \text{by Lemma 4.12}
\]
\[
> \dim(\mu_{p_n}) - 2\epsilon \quad \text{by (4.38), (4.39), (4.42)}
\]
\[
> \dim_H BE_k(\alpha) - 3\epsilon \quad \text{by (4.36)},
\]
where for the proof of the second inequality, a direct computation gives
\[
\dim(\nu_n) - \dim(\mu_{p_n}) = \frac{(1 - t_n)h(\zeta_n)\chi(\mu_{p_n}) - (1 - t_n)\chi(\zeta_n)h(\mu_{p_n})}{((1 - t_n)\chi(\zeta_n) + t_n\chi(\mu_{p_n}))\chi(\mu_{p_n})} \to 0, \text{ as } n \to \infty.
\]
Here, we used the uniform bounds in (4.38), (4.39), (4.42) and the fact that \(t_n \to 1\) as \(n \to \infty\). This yields a contradiction to (4.35) and completes the proof in the first case.

In the case where the second alternative in (4.40) holds for the subsequence \(\{\mu_{p_n}\}_{n=1}^{\infty}\), we slightly modify the previous argument. Since \(\alpha \in D\) we have \(\alpha_1 < 1\). For each \(n \geq 1\) fix a measure \(\zeta_n \in \mathcal{M}(f)\) such that \(\sup_{n \geq 1} \chi(\zeta_n) < \infty\), \(\zeta_n(\Delta_1) > \frac{\alpha_1 + 1}{2}\) and \(\zeta_n(\Delta_j) = \alpha_j\) for every \(2 \leq j \leq k\). Define \(t_n \in (0, 1)\) implicitly by
\[(1 - t_n)\zeta_n(\Delta_1) + t_n \mu_{p_n}(\Delta_1) = \alpha_1 + \theta_n\]
and define a measure \(\nu_n = (1 - t_n)\zeta_n + t_n \mu_{p_n}\).
Proceeding in the same way as before we obtain a contradiction to (4.35). This completes the proof of Theorem 1.5(a).
\[\square\]
Proof of Theorem 1.5(b). Changing the indices of the Markov partition if necessary, we may assume that \(\inf_{\Delta_i} |f'|\) is monotone increasing in \(i\). Since \(f\) has mild distortion, we have \(D_1(\log |f'|) < \infty\) and thus

\[
\lim_{i \to \infty} \inf_{\Delta_i} |f'| = \infty.
\]

Let \(\alpha \in \mathbb{R}^N\) be a frequency vector such that \(\sum_{i=1}^{\infty} \alpha_i < 1\). In view of Corollary 1.4 it suffices to show that

\[
\lim_{k \to \infty} \lim_{\epsilon \to 0} \sup \left\{ \dim(\mu) : \mu \in M(f), \max_{1 \leq i \leq k} |\mu(\Delta_i) - \alpha_i| < \epsilon \right\} \leq \beta_\infty.
\]

Fix \(\epsilon_0 \in (0, 1 - \sum_{i=1}^{\infty} \alpha_i)\) and put \(c = 1 - \sum_{i=1}^{\infty} \alpha_i - \epsilon_0 > 0\). For each integer \(k \geq 1\) take \(\epsilon(k) \in (0, \epsilon_0/k)\). Then

\[
\lim_{\epsilon \to 0} \sup \left\{ \dim(\mu) : \mu \in M(f), \max_{1 \leq i \leq k} |\mu(\Delta_i) - \alpha_i| < \epsilon(k) \right\} \leq \sup \left\{ \dim(\mu) : \mu \in M(f), \sum_{i=k+1}^{\infty} \mu(\Delta_i) > c \right\},
\]

where the last inequality follows from

\[
1 - \sum_{i=1}^{k} \alpha_i - k\epsilon(k) > 1 - \sum_{i=1}^{k} \alpha_i - \epsilon_0 > c > 0.
\]

The assertion in (4.44) implies

\[
\lim_{k \to \infty} \inf \left\{ \chi(\mu) : \mu \in M(f), \sum_{i=k+1}^{\infty} \mu(\Delta_i) > c \right\} = \infty.
\]

By (4.47) and Lemma 4.11, letting \(k \to \infty\) in (4.46) proves the desired estimate in (4.45). This completes the proof of Theorem 1.5(b). \(\square\)

Remark 4.13. If the map \(f\) in Theorem 1.5 is uniformly expanding, then \(\dim_{H} B_{\epsilon}(\alpha)\) is continuous on the set of all frequency vectors. This follows from a slight modification of the proof of Theorem 1.5(a).

4.5. Irregular sets with no Birkhoff average. Recall that \(B'(\phi)\) in (1.1) is the set of points in \(J\) for which the Birkhoff average of some \(\phi_i\) does not exist. If \(B'(\phi) \neq \emptyset\) and (1.2) holds, then using arguments in [1] one can show that \(\dim_{H} B'(\phi) = \dim_{H} J\).

5. Verifications of the assumptions

This section is geared to applications of the Main Theorem to concrete examples in the next section. Sufficient conditions for the uniform decay of cylinders and the approximability by dimensions of expanding measures (1.2) are given in Section 5.1 and Section 5.2 respectively.
5.1. **Uniform decay of cylinders.** Let \( f : \Delta \to M \) be a non-uniformly expanding Markov map. Inspired by [33, Section 8], we introduce the following condition:

(M3) there exists \( s > 1 \) such that if \( a_1, a_2 \in S \) are such that \( f \Delta_{a_1} \supset \Delta_{a_2} \) and \( a_2 \) is an expanding index, or else \( a_1 \neq a_2 \), then \( \inf_{\Delta_{a_1a_2}} |(f^2)'| \geq s \).

Condition (M3) implies that there is no neutral periodic point which is not a fixed point.

**Lemma 5.1.** Let \( f : \Delta \to M \) be a non-uniformly expanding Markov map. If (M3) holds, then \( f \) has uniform decay of cylinders.

For a proof of Lemma 5.1 we need the next lemma. For \( a \in S \) such that \( aa \) is admissible and \( n \geq 1 \), let \( a^n \in E^n \) denote the \( n \)-string of \( a \).

**Lemma 5.2.** Let \( a \in \Omega \) be such that \( aa \) is admissible. Then \( |\Delta_{a^n}| \to 0 \) as \( n \to \infty \).

**Proof.** Put \( I = \bigcap_{n=1}^{\infty} \Delta_{a^n} \) and assume \( |I| > 0 \) by contradiction. Since \( f(I) = I \) and \( |f'| \geq 1 \) everywhere on \( I \), \( |f'| \equiv 1 \) on \( I \). Since there are only finitely many neutral points, we obtain a contradiction. \( \Box \)

**Proof of Lemma 5.1.** We slightly modify the proof of [33, Lemma 8.1.2]. Let

\[
g(n) = \max\{|\Delta_{a^n}| : a \in \Omega, \ aa \text{ is admissible}\}.
\]

The finiteness of \( \Omega \) and Lemma 5.2 give \( \lim_{n \to \infty} g(n) = 0 \). Let \( \omega \in E^n \). Look at the longest block of the same index in \( \Omega \) appearing in \( \omega \). If the length of this block exceeds \( \sqrt{n} \) then we have \( |\Delta_{\omega}| \leq g(\lfloor \sqrt{n} \rfloor) \) by (M2). Otherwise, we have \( |\Delta_{\omega}| \leq s^{(\lfloor \sqrt{n} \rfloor-1)/2} \) by (M3). Hence, \( f \) has uniform decay of cylinders. \( \Box \)

**Remark 5.3.** Condition (M3) is not a necessary condition for the uniform decay of cylinders. The Farey map \( f : [0, 1) \to [0, 1) \) defined by

\[
f(x) = \begin{cases} 
\frac{x}{1-x} & \text{for } 0 \leq x < \frac{1}{2}, \\
\frac{1-x}{x} & \text{for } \frac{1}{2} \leq x \leq 1,
\end{cases}
\]

is a non-uniformly expanding Markov map with a Markov partition \( \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\} \). Although (M3) does not hold, \( f \) has uniform decay of cylinders. This follows, for instance, from the estimates in [50, Example 4].

5.2. **Approximability by expanding dimension.** One sufficient condition for (1.2) is the existence of an induced Markov map which captures sets of full Hausdorff dimension, some power of which is uniformly expanding, finitely irreducible and satisfies Rényi’s condition. Then, the thermodynamic formalism for the induced map and a finite approximation technique provide us with measures of dimension arbitrarily close to \( \dim_H J \). However, to verify Rényi’s condition for induced maps usually requires explicit formulas of individual maps (see e.g., [9, Lemma 2.7 and Lemma 2.8], [28, Lemma 3.4]). Below we provide a convenient sufficient condition for (1.2), replacing Rényi’s condition by conditions in terms of the Markov partition of the original map.
Proposition 5.4. Let $f : \Delta \to M$ be a non-uniformly expanding Markov map with $C^2$ branches satisfying Rényi’s condition and (M3). Assume there exist a subset $F$ of $E^*$ and a function $\tau : \{\Delta_\omega\}_{\omega \in F} \to \mathbb{N} \setminus \{0\}$ such that the following holds:

(i) the cylinders $\{\Delta_\omega\}_{\omega \in F}$ have disjoint interiors, and for each $\omega \in F$ and $0 \leq n \leq \tau(\Delta_\omega) - 1$, $f^n \Delta_\omega$ does not contain a neutral fixed point;

(ii) the induced map $\tilde{f} : x \in K \mapsto f^{\tau}(\Delta(x))x \in M$

where $K = \bigcup_{\omega \in \tilde{F}} \Delta_\omega$ and $\Delta(x)$ is an element containing $x$, is a finitely irreducible non-uniformly expanding Markov map with a Markov partition $\{\Delta_\omega\}_{\omega \in \tilde{F}}$;

(iii) one has

$$\dim_H J = \dim_H \left( \bigcap_{n=0}^\infty \tilde{f}^{-n} K \right);$$

(iv) if $a \in \Omega$, $aa$ is admissible and $\Delta_a$ contains no neutral fixed point, then $\Delta_a$ is closed.

Then the following holds:

(a) there exists an ergodic expanding measure in $M(f)$ with dimension arbitrarily close to $\dim_H J$. In particular, (1.2) holds for $f$;

(b) $\tilde{f}$ has mild distortion and satisfies (1.2), and $\tilde{f}^2$ is uniformly expanding.

Proof. By Lemma 5.1, $f$ has uniform decay of cylinders. Let $\tilde{X} \subset F^\mathbb{N}$ ($\tilde{X} \subset X$) denote the topological Markov shift determined by $\tilde{f}$ with alphabet $F$, and let $\tilde{\pi} : \tilde{X} \to \bigcap_{n=0}^\infty \tilde{f}^{-n} K \subset K$ denote the coding map defined as in Section 3.1, namely

$$\tilde{\pi}(\omega) \in \bigcap_{n=0}^\infty \tilde{f}^{-n} \Delta_{\omega_n}$$

for $\omega = (\tilde{\omega}_n)_{n=0}^\infty \in \tilde{X}$. Define the induced potential $\tilde{\psi} : \tilde{X} \to \mathbb{R}$ given by

$$\tilde{\psi} = - \log |D\tilde{f} \circ \tilde{\pi}|.$$

By Lemma 3.3 below, $\tilde{\psi}$ is Hölder continuous. For each $\beta \in \mathbb{R}$, denote by $P(\beta \tilde{\psi})$ the pressure of the potential $\beta \tilde{\psi}$ (see (4.6) or [33] Section 2.1 for the definition). The critical exponent is given by

$$\delta := \inf \{ \beta \geq 0 : P(\beta \tilde{\psi}) \leq 0 \}.$$

We first show $\dim_H J \leq \delta$. Let $\epsilon > 0$. Since $(\tilde{f})^2$ is uniformly expanding by Lemma 5.6 below, it follows that $\beta \mapsto P(\beta \tilde{\psi})$ is strictly decreasing. Hence, $P((\delta + \epsilon) \tilde{\psi}) < 0$. By a standard covering argument we verify that the Hausdorff $(\delta + \epsilon)$-measure of $\tilde{\pi}(\tilde{X})$ is finite. Namely, for sufficiently large $n \in \mathbb{N}$, we cover $\tilde{\pi}(\tilde{X})$ by cylinders $\omega \in F^n$. Since $\tilde{\psi}$ is Hölder continuous and $(\tilde{f})^2$ is uniformly expanding, $\tilde{\psi}$ has the bounded distortion property (cf. [33] Lemma 2.3.1)). In particular,

$$\sup_{\omega \in F^n} \frac{|\tilde{\pi}(\omega)|^{\delta + \epsilon}}{ \exp \left( (\delta + \epsilon) \sup |\tilde{\psi}| \right)} < \infty.$$
Combining this with \( P((\delta + \epsilon)\tilde{\psi}) < 0 \), it follows that the Hausdorff \((\delta + \epsilon)\)-measure of \( \tilde{\pi}(X) \) is finite. Therefore

\[
\dim_H J = \dim_H \left( \bigcap_{n=0}^{\infty} \hat{f}^{-n} K \right) \leq \dim_H \tilde{\pi}(X) \leq \delta.
\]

Next, we verify that \( \delta_0 \geq \delta \). To prove this we first make use of the approximation property of the pressure function \( \beta \mapsto P(\beta \tilde{\psi}) \) by compact invariant subsystems. Let \( \tilde{X}_n \) denote the finite subshift of \( \tilde{X} \) given by all sequences containing exclusively the first \( n \) symbols of the alphabet of \( \tilde{X} \). Denote the corresponding pressure for the potential \( \beta \tilde{\psi}|_{\tilde{X}_n} \) by \( P_n(\beta \tilde{\psi}) \). By \cite[Theorem 2.1.6]{33}, \( P(\beta \tilde{\psi}) = \lim_{n \to \infty} P_n(\beta \tilde{\psi}) \) holds. Denote by \( \delta_n \) the unique solution of the equation \( P_n(\beta \tilde{\psi}) = 0 \). It is then straightforward to verify that \( \delta = \lim_{n \to \infty} \delta_n \) (see \cite[Theorem 1.7]{23}). Let \( \tilde{\mu}_n \) denote the equilibrium state for the potential \( \delta_n \tilde{\psi}|_{\tilde{X}_n} \). Since \( \tau \) is integrable against \( \tilde{\mu}_n \), the measure

\[
\mu_n = \left( \sum_{\omega \in F} \tau(\Delta_\omega) \tilde{\mu}_n(\Delta_\omega) \right)^{-1} \sum_{\omega \in F} \sum_{j=0}^{\tau(\Delta_\omega)-1} f_j^* \tilde{\mu}_n|_{\Delta_\omega}
\]

is in \( \mathcal{M}(f) \), ergodic, expanding and satisfies \( \dim(\mu_n) = \delta_n \) by Abramov-Kac’s formula. This proves \( \delta_0 \geq \delta \).

To finish the proof of Proposition \ref{5.2}(a), it remains to verify the Hölder continuity of \( \tilde{\psi} \). It suffices to show that there exists \( C > 0 \) such that for every \( \omega \in F \) and all \( x, y \in \Delta_\omega \),

\[
(5.1) \quad \log \frac{(\tilde{f})'x}{(\tilde{f})'y} \leq C|\tilde{f}x - \tilde{f}y|.
\]

This does not follow from the mild distortion of \( f \), and we need an elaborate argument based on decompositions of each branch \( \tilde{f}|_{\Delta_\omega} \) into finitely many blocks of iterates of \( f \). In order to treat neutral indices we need a couple of lemmas. In the next paragraph, we prove a general lemma on a control of distortions of each branch whose domain contains a neutral fixed point.

Let \( 0 < c < 1 \) and let \( g : [0, c] \to [0, 1] \) be a \( C^2 \) map such that \( g0 = 0, g'0 = 1 \) and \( g' > 1 \) on \( (0, c] \). Define a sequence \( \{p_n\}_{n=0}^\infty \) in \( [0, c] \) inductively by \( p_0 = c \) and \( gp_n = p_{n-1} \) for \( n \geq 1 \). Note that \( p_n \) decreases monotonically to 0 as \( n \to \infty \). Put \( I_n = (p_n, p_{n-1}] \).

**Lemma 5.5.** There exists a constant \( C_0 > 0 \) such that for any \( n \geq 1 \) and all \( x, y \in I_n \),

\[
\log \frac{(g^n)'x}{(g^n)'y} \leq C_0|g^n x - g^n y|.
\]

**Proof.** Put \( C = \sup_{[0,c]} |g''|. \) For \( x, y \in I_n \) and \( 0 \leq k \leq n - 1 \) we have

\[
\log \frac{g'(g^k x)}{g'(g^k y)} \leq |g'(g^k x) - g'(g^k y)| \leq C|g^k x - g^k y| \leq C|I_{n-k}|.
\]
Therefore
\[
\sup_{x,y \in I_n} \log \frac{(g^{n-k})'(g^k x)}{(g^{n-k})'(g^k y)} \leq C,
\]
and
\[
\log \frac{g'(g^k x)}{g'(g^k y)} \leq C \frac{|g^k x - g^k y|}{|I_{n-k}|} \leq C e^{2C} \frac{|g^k x - g^k y|}{|g I_1|} |I_{n-k}|.
\]
Put \(C_0 = C e^{2C}/|g I_1|\). Summing this over all \(0 \leq k \leq n - 1\) yields the desired inequality. This completes the proof of Lemma 5.5. \(\square\)

The next lemma will be used to treat the occurrence of the same neutral index consecutively which does not correspond to a neutral fixed point.

**Lemma 5.6.** There exists \(\rho > 1\) such that if \(a \in \Omega\) is such that \(aa\) is admissible and \(\Delta_a\) does not contain a neutral fixed point, then
\[
\inf_{\Delta_{aa}} |(f^2)'| \geq \rho.
\]
Moreover, \((\tilde{f})^2\) is uniformly expanding.

**Proof.** Since \(\Delta_a\) is closed by the assumption (iv) in Proposition 5.4 and \(|f'| \geq 1\), there is a fixed point in \(\Delta_a\). By the assumption of Lemma 5.6 this fixed point is not neutral. Further, since \(|f'| \geq 1\), there is no other fixed point in \(\Delta_a\). It follows that \(|(f^2)'|\) is bounded away from 1 on \(\Delta_{aa}\) for otherwise, since \(\Delta_a\) is closed, there would exist a neutral periodic point in \(\Delta_a\) of period two. The uniform expansion of \((\tilde{f})^2\) follows from (M3). \(\square\)

We now prove (5.1). Let \(\omega \in E^*\). We decompose \(\omega\) into a concatenation of admissible words \(\omega = \eta_0 \cdots \eta_k\) with the following properties:

- each \(\eta_i \in E^* (0 \leq i \leq k)\) consists of: (a) only expanding indices; (b) a single neutral index \(a\) for which \(\Delta_a\) contains a neutral fixed point; (c) a single neutral index \(a\) for which \(\Delta_a\) does not contain a neutral fixed point;
- if \(k' \leq k\) and \(\omega = \eta_1' \cdots \eta_{k'}'\) is another decomposition with the first property, then \(k = k'\) and \(\eta_i = \eta_i'\) for every \(0 \leq i \leq k\).

We claim that there exists a constant \(C_0 > 0\) which is independent of \(\omega\) such that the following holds for every \(0 \leq i \leq k\) and all \(x, y \in \Delta_{\eta_i}\):

\[(5.2) \log \frac{(f^{\eta_i})'x}{(f^{\eta_i})'y} \leq C_0 |f^{\eta_i}x - f^{\eta_i}y|.
\]

Indeed, an inequality of the form (5.2) in the case (a) applies to \(\eta_i\) follows from Rényi’s condition and (M3). For those \(i\) such that (b) applies to \(\eta_i\), we use Lemma 5.5. For those \(i\) such that (c) applies to \(\eta_i\), we use Rényi’s condition, Lemma 5.6 and the finiteness of the number of neutral indices.

Assume \(k \geq 1\). Condition (M3) implies

\[(5.3) \inf_{\Delta_{\eta_i \eta_{i+1}}} |(f^{\eta_i \eta_{i+1}})'| \geq s \quad 0 \leq \forall i \leq k - 1.
\]
Now, let $x, y \in \Delta_\omega$. The chain rule gives
\[
\log \left( \frac{(\tilde{f})'x}{(f)'y} \right) = \log \left( \frac{(f|\eta_k)'x}{(f|\eta_k)'y} \right) + \sum_{i=1}^{k} \log \left( \frac{(f|\eta_{i+1}-\eta_i)'x}{(f|\eta_{i+1}-\eta_i)'y} \right).
\]
Applying (5.2) to the right-hand side, we have
\[
(5.4) \quad \log \left( \frac{(\tilde{f})'x}{(f)'y} \right) \leq C_0 \sum_{i=1}^{k+1} |f|_{\eta_{i+1}-\eta_i} |x - f|_{\eta_{i+1}-\eta_i} y |.
\]
We apply the mean value theorem to each term in the series and use the uniform expansion inf $|f|_{\eta_{i+1}-\eta_i} |x - f|_{\eta_{i+1}-\eta_i} y | \geq s^{k-i+1}$ from (5.3) to deduce that
\[
\log \left( \frac{(\tilde{f})'x}{(f)'y} \right) \leq C_0 \sum_{i=1}^{k} \frac{2s-1}{s-1} |\tilde{f}x - \tilde{f}y|.
\]
Put $C = C_0 \frac{2s-1}{s-1}$. Note that (5.2) covers the case $k = 1$. This completes the proof of (5.1) and hence that of Proposition 5.4(a).

By Lemma 5.6, $\tilde{f}^2$ is uniformly expanding. Then (5.1) implies that $\tilde{f}$ has mild distortion. The above approximation by finite subsystems implies (1.2) for $\tilde{f}$. This completes the proof of Proposition 5.4(b). □

Remark 5.7. The assumption on the smoothness of $f$ in Proposition 5.4 can be relaxed, so as to accommodate the Manneville-Pomeau type neutral fixed points. It is enough to assume that a branch containing a neutral fixed point $p$ in its domain is $C^{1+\theta}$, $\theta \in (0, 1)$, concave near $p$, and that there exist $A \neq 0$, $\beta \in (0, \frac{\theta}{1-\theta})$ such that $f'(x) = 1 + A(x - p)^\beta + o((x - p)^\beta)$ ($x \to p$). A control of distortion as in Lemma 5.5 holds with exponent $\theta$, which one can prove using Thaler’s asymptotics [48, Lemma 2, Corollary] on $p_n$ as $n \to \infty$. For details, see [35, 49] for example.

6. Mixed Birkhoff spectrum for the BCF expansion

For uniformly expanding Markov maps with finitely many branches, the Birkhoff spectra of Hölder continuous observables which are not cohomologous to a constant are non-constant real-analytic functions [39, 41]. For uniformly expanding Markov maps with infinitely many branches, Fan et al. [13, Theorem 7.2] constructed an example of a bounded continuous observable with a locally flat spectrum. In their example, the spectrum on the flat part is constantly equal to a constant in $(0, 1)$. Using the Main Theorem, we show that a certain mixed Birkhoff spectrum naturally associated with the Rényi map is constantly equal to the full dimension 1.

The digits in the BCF expansion (1.3) are generated by iterating the Rényi map
\[
f: x \in [0, 1) \mapsto \frac{1}{1 - x} \left[ \frac{1}{1 - x} \right] \in [0, 1).
\]
This means that for all \( x \in (0, 1) \setminus \mathbb{Q} \),
\[
b_j(x) = \left\lfloor \frac{1}{1 - f_j^{-1}(x)} \right\rfloor + 1 \quad \forall j \geq 1.
\]
The graph of \( f \) can be obtained from that of the Gauss map by reflecting the latter in the line \( x = 1/2 \). For this reason, (1.3) is called the Backward Continued Fraction (BCF) expansion of the irrational number \( x \). The map \( f \) is a fully branched non-uniformly expanding Markov map having \( x = 0 \) as a unique neutral fixed point.

The behavior of the arithmetic mean of digits is very much peculiar. Aaronson [1] proved that the arithmetic mean converges to 3 in measure as \( n \to \infty \).Aaronson and Nakada [2] proved that
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} b_j(x) = 2 \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} b_j(x) = \infty
\]
for Lebesgue a.e. \( x \in (0, 1) \setminus \mathbb{Q} \). As in Corollary 1.6, we have found that the multifractal spectrum of the arithmetic mean of digits is completely flat. In fact, a stronger statement holds.

**Theorem 6.1** (Completely flat mixed Birkhoff spectrum). For any \( \alpha \in [2, \infty) \) and any \( \phi = (\phi_i)_{i=1}^{\infty} \in \mathcal{F}^{\mathbb{N}} \) such that \( \phi_i \) is bounded for every \( i \geq 1 \), we have
\[
\dim_{\text{H}} \left\{ x \in (0, 1) \setminus \mathbb{Q}: \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} b_j(x) = \phi_i(0) \quad \forall i \geq 1, \right. 
\left. \lim_{n \to \infty} \frac{1}{n} \log |f^n(x)| = 0, \right.
\left. \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} b_j(x) = \alpha \right\} = 1.
\]

To prove Theorem 6.1 we need a conditional variational formula for the Lyapunov spectrum of the Rényi map.

**Lemma 6.2.** There exists an ergodic expanding measure with dimension arbitrarily close to 1. Moreover, for every \( \alpha \in (0, \infty) \),
\[
\dim_{\text{H}} L(\alpha) = \limsup_{\epsilon \to 0} \{ \dim(\mu) : \mu \in \mathcal{M}(f), \ |\chi(\mu) - \alpha| < \epsilon \}.
\]

**Proof.** Clearly, \( f \) satisfies (M3). A direct calculation shows that \( f \) satisfies Rényi’s condition, and so \( \log |f'| \in \mathcal{F} \) by Lemma 3.3. It is easy to see that the first return map to the interval (1/2, 1) is a fully branched Markov map which satisfies all the assumptions in Proposition 5.4. Hence, the first assertion of Lemma 6.2 holds. The second one is a consequence of Main Theorem(b). \( \square \)

**Proof of Theorem 6.1** We start with the following lemma.

**Lemma 6.3.** There exists a sequence \( \{\mu_p\}_{p=1}^{\infty} \) of expanding measures such that
\[
\lim_{p \to \infty} \dim(\mu_p) = 1 \quad \text{and} \quad \lim_{p \to \infty} \chi(\mu_p) = 0,
\]
\[
\int b_1d\mu_p < \infty \quad \text{for every} \quad p \geq 1 \quad \text{and} \quad \lim_{p \to \infty} \int b_1d\mu_p = \infty.
\]
Proof. By Lemma 6.2 there exists an ergodic expanding measure with dimension arbitrarily close to 1. Since such measures are approximated by a finite collection of cylinders in the sense of Lemma 4, it is possible to take a sequence \( \{\xi_p\}_{p=1}^{\infty} \) of expanding measures such that \( \chi(\xi_p) \geq p^{-1/4} \) and \( \int b_1 d\xi_p < \infty \) hold for every \( p \geq 1 \) and \( \lim_{p \to \infty} \dim(\xi_p) = 1 \). Let \( \delta_p \) denote the unit point mass at the fixed point of \( f \) in \( [1 - \frac{1}{p}, 1 - \frac{1}{p+1}] \), and put
\[
\mu_p = \left(1 - \frac{1}{\sqrt{p}}\right) \xi_p + \frac{1}{\sqrt{p}} \delta_p.
\]
Then we have \( h(\delta_p) = 0, \chi(\delta_p) \leq 2 \log(p + 1) \) and
\[
\int b_1 d\mu_p \geq \frac{1}{\sqrt{p}} \int b_1 d\delta_p = \frac{p+1}{\sqrt{p}} \to \infty, \quad p \to \infty.
\]
Moreover, for any \( c > 1 \) there exists \( p_0 \geq 1 \) such that \( \chi(\mu_p) \leq c \chi(\xi_p) \), for every \( p \geq p_0 \). Hence
\[
\dim(\mu_p) \geq \frac{1 - \frac{1}{\sqrt{p}}}{c \chi(\xi_p)} h(\xi_p),
\]
which implies \( \lim\inf_{p \to \infty} \dim(\mu_p) \geq 1/c \). Decreasing \( c \) to 1 yields \( \lim_{p \to \infty} \dim(\mu_p) = 1 \).

If \( \lim \sup_{p \to \infty} \chi(\mu_p) > 0 \), then from Main Theorem(b) it follows that \( \dim_H L(\alpha) = 1 \) for \( \alpha = \lim \sup_{p \to \infty} \chi(\mu_p) \). This contradicts the fact that the Lyapunov spectrum \( \alpha \in [0, \infty) \mapsto \dim_H L(\alpha) \) is strictly monotone decreasing [21, Theorem 4.2].

Let \( \{\mu_p\}_{p=1}^{\infty} \) be a sequence of expanding measures as in Lemma 6.3. Let \( \alpha \in [2, \infty) \). For each integer \( j \geq 1 \) fix \( p(j) \geq 1 \) such that
\[
\int b_1 d\mu_{p(j)} > \alpha + j.
\]
Define \( t_j \in (0, 1] \) implicitly by
\[
t_j \int b_1 d\mu_{p(j)} + (1-t_j) \int b_1 d\delta_1 = \alpha + \frac{1}{j} \in (2, \alpha + j],
\]
and put
\[
\nu_j = t_j \mu_{p(j)} + (1-t_j) \delta_1.
\]
Then \( \nu_j \) is an expanding measure and satisfies \( \int b_1 d\nu_j = \alpha + 1/j \). Note that we have \( t_j \to 0 \) as \( j \to \infty \).

Let \( \phi = (\phi_i)_{i=1}^{\infty} \in \mathcal{F}^N \) be such that \( \phi_i \) is bounded (not necessarily continuous) for every \( i \geq 1 \). Put \( \alpha = (\phi_i(0))_{i=1}^{\infty} \in \mathbb{R}^N \). By a diagonal argument, one can choose a subsequence \( \{\nu_{j_k}\}_{k=1}^{\infty} \) of \( \{\nu_j\}_{j=1}^{\infty} \) such that \( \lim_{k \to \infty} \| \int \phi_k d\nu_{j_k} - \phi_k(0) \| = 0 \) for each \( k \geq 1 \). Since \( h(\delta_1) = 0 = \chi(\delta_1), \dim(\nu_j) = \dim(\mu_{p(j)}) \) holds. For each \( k \geq 1 \), Proposition 1.5 yields
\[
\limsup_{\epsilon \to 0} \left\{ \dim(\mu) : \mu \in \mathcal{M}(f), \left| \int \phi_k d\mu - \alpha_k \right| < \epsilon, \left| \int b_1 d\mu - \alpha \right| < \epsilon \right\} \\
\geq \lim_{t \to \infty} \dim(\nu_{j_t}) = \lim_{t \to \infty} \dim(\mu_{p(j_t)}) = 1.
\]
Since \( k \geq 1 \) is arbitrary, the Hausdorff dimension of the level set is bounded from below by 1, and so it is 1. In the case \( \alpha = \infty \), for each \( p \geq 2 \) define an expanding measure \( \nu_p = (1 - \frac{1}{p})\mu_p + \frac{1}{p}\delta_1 \). From Lemma \ref{lem:expansion_measure} \( \lim_{p \to \infty} \int b_1 d\nu_p = \infty \). A slight modification of the proof of Proposition \ref{prop:dimension} and the same reasoning in the case where \( \alpha \) is finite shows that the Hausdorff dimension of the level set is bounded from below by \( \lim_{p \to \infty} \dim(\nu_p) = \lim_{p \to \infty} \dim(\mu_p) = 1 \). This completes the proof of Theorem \ref{thm:dimension}. \( \square \)

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