ORBIT GROUPS

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Abstract. In the paper [1] are introduced two groups generated by the orbits of an action of a group on another group by automorphisms. One is of group-theoretic nature and the other comes from homology of invariant group chains. In this note are given some properties of the first groups and is studied a natural homomorphism between these groups. More precisely, it is shown that this homomorphism is not injective nor surjective. A description of the kernel is given.

INTRODUCTION

The study of orbits of group actions on groups by group automorphism has been related to the study of finite solvable groups, in particular that of Frobenius groups (See [6], [3]). More recent work of Deaconescu and Walls is dedicated to the study of orbits and fixed points of such actions [2].

The present work has another origin coming from the study of homology theories for groups and their automorphism groups. Namely, it was found in [4] Theorem 2] a group generated by the orbits of the action of a finite group on another group. This group arrises as the first homology group of invariant group chains defined by Knudson [5] and, as expected, it is related to the generalization in this context of the abelianization of a group [1, Sec. 2].

The discovery of an abelian group generated by orbits of these actions rises the question on the existence of such an object in group theory. This is defined in [1, Sec. 2.2] together with a homomorphism to the first homology group of invariant group chains.

The first part of this work is dedicated to discuss some properties of the so-called orbit groups and the second part to study relations with the first homology group of invariant group chains of Knudson. It is shown that the homomorphism defined in [1 Sec. 2.2] is not in general surjective nor injective. Condition for elements in the kernel of this homomorphism are given.

1. Orbit groups and main properties

Consider the category having as objects groups with a group action by group automorphisms and as morphisms pairs of homomorphisms commuting with the actions. More precisely, the objects of this category are triples $(Q, G, \varphi)$ where $Q$ and $G$ are groups and $\varphi : Q \times G \to G$ is a group action by group homomorphisms, i.e.
orbits, i.e. such that 
\[ \varphi(q_1,q_2,g) = \varphi(q_1,\varphi(q_2,g)), \varphi(1,g) = g \] 
and 
\[ \varphi(q,g_1g_2) = \varphi(q,g_1)\varphi(q,g_2), \varphi(q,1) = 1, \]
or equivalently,
\[ (q_1,q_2)(g) = q_1(q_2(g)), \quad 1(g) = g, \]
and
\[ q(g_1g_2) = q(g_1)q(g_2), \quad q(1) = 1 \]
for \(q,q_1,q_2 \in Q, g,g_1,g_2 \in G,\)
if the action is avoided from the notation. The morphisms of this category are pairs \((\alpha,\beta)\) of homomorphisms \(\alpha : Q \to Q', \beta : G \to G'\) commuting with the actions, i.e. such that the diagram
\[
\begin{array}{ccc}
Q \times G & \xrightarrow{\varphi} & G \\
\alpha \times \beta \downarrow & & \downarrow \psi \\
Q' \times G' & \xrightarrow{\varphi'} & G'
\end{array}
\]
is commutative.

In this case, one has the well known semidirect product functor \(Q \rtimes \varphi G\) (or \(Q \ltimes G\)) to the category of groups. In order to keep track of the group \(G\), one can consider the corresponding functor to the category of pairs of groups. This functor, say \(L\), takes the triple \((Q,G,\varphi)\) to the pair \((G,Q \ltimes G)\) and the pair of homomorphisms \((\alpha,\beta)\) to the pair \((\beta,\alpha \times \beta)\).

**Definition 1.1** (Conf. [1]). The orbit group of the action of \(Q\) on \(G\), denoted by \(G//Q\), is a group equipped with a surjective homomorphism \(p : G \to G//Q\) and having the property that every group homomorphism \(\phi : G \to H\) identifying orbits, i.e. such that \(\phi(q(g)) = \phi(g), g \in G\), factors through \(G//Q\), i.e. there exists an unique homomorphism \(\psi : G//Q \to H\) such that the diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & H \\
p \downarrow & & \downarrow \psi \\
G//Q & \xrightarrow{\psi} & H
\end{array}
\]
is commutative.

In order to show the existence of such an object, construct first a quotient of the semidirect product \(Q \ltimes G\) by a normal subgroup, say \(N\), such that the relation \(\overline{g} = \varphi(g), g \in G, q \in Q\) is true in \((Q \ltimes G)/N\). Such subgroup \(N\) must contain the elements \(q(g^{-1})\) for every \(g \in G, q \in Q\). In the semidirect product, the action of \(Q\) on \(G\) can be written as conjugation by the corresponding element. In this way, one has \(qg(g^{-1}) = [g,q]\) and, therefore, the commutator subgroup \([G,Q]\) must be contained in \(N\). However, the subgroup \([G,Q]\) is not necessarily normal, so, one shall take the minimal normal subgroup containing \([G,Q]\), this is \(N = [G,Q]^Q \ltimes G\). This gives the quotient \(Q \ltimes G/[G,Q]^Q \ltimes G\) together with the quotient map \(Q \ltimes G \to (Q \ltimes G)/[G,Q]^Q \ltimes G\). The map \(p : G \to G//Q\) is then obtained as the restriction of the composition
\[
G \to Q \ltimes G \to (Q \ltimes G)/[G,Q]^Q \ltimes G
\]
to its image. By the second isomorphism theorem, one obtains the formula \(G//Q = G/[G \cap [G,Q]^Q \ltimes G]\). Using the exterior definition of the semidirect product, one can show that \([G,Q]^Q \ltimes G\) is contained in \(G\) and coincides with \([G,Q]^G\). Therefore, one obtains a simpler expression \(G//Q = G/[G,Q]^G\) which is defined solely in terms of
$G$ and the action of $Q$ on $G$, because the group $[G, Q]^G$ is generated by the elements of the form $g_1 g q (g^{-1}) q_i$ for $g, g_1 \in G, q \in Q$.

The universal property of $G//Q$ means that a morphism

$$(\alpha, \beta) : (Q, G, \varphi) \rightarrow (Q', G', \varphi')$$

induces a homomorphism $\beta//\alpha : G//Q \rightarrow G'///Q'$ such that the square

$$\begin{array}{ccc}
G & \xrightarrow{\beta} & G' \\
p \downarrow & & \downarrow p' \\
G//Q & \xrightarrow{\beta//\alpha} & G'///Q',
\end{array}$$

is commutative. In other words, one has a functor from the category of groups acting on groups to the category of groups.

The orbit group functor just constructed has the following properties:

- if $i : K \rightarrow G$ is a monomorphism and $K$ is a $Q$-invariant subgroup, then $i//id : K//Q \rightarrow G//Q$ is a monomorphism;
- if $j : G \rightarrow H$ is an equivariant epimorphism, then $j//id : H//Q \rightarrow G//Q$ is an epimorphism;
- if $N \triangleleft G$ is a $Q$-invariant normal subgroup, then $N//Q \triangleleft G//Q$ is a normal subgroup and one has $(G/N)//Q \cong (G//Q)/(N//Q)$;
- there is a correspondence between subgroups of $H' < G//Q$ and $Q$-invariant subgroups $H < G$ containing $[G, Q]^G$;
- there is a correspondence between quotients of $(G//Q)/H'$ and quotients $G/H$ by $Q$-invariant subgroups such that the induced action $Q \times G/H \rightarrow G/H$ is trivial.

There is no natural transformation between the functors $Q \times G$ and $G//Q$ in the very manner they are defined. Instead, one can consider the corresponding functors $(G, Q \times G)$ and $(G//Q, (Q \times G)//[G, Q]^G)$ to the category of pairs of groups. Then, the pair of morphisms $(p, r) : (G, Q \times G) \rightarrow (G//Q, (Q \times G)//[G, Q]^G)$, where $r : Q \times G \rightarrow (Q \times G)//[G, Q]^G$ is the quotient map, is natural.

The functor $G//Q$ is a (non abelian) generalization of the abelianization of a group: take $Q = G$ acting on itself by inner automorphisms, then one has $G//G = G/[G, G] = G_{ab}$.

2. Relations between the orbit group and the first homology group of invariant group chains

Let $Q$ be a finite group, $G$ be a group, and $Q \times G \rightarrow G$ be an action of $Q$ on $G$ by group automorphisms. Denote by $H_1^Q(G, \mathbb{Z})$ the first homology group of $Q$-invariant chains on the group $G$ (see [5]). It was shown in [4] Theorem 2] the following.

$$H_1^Q(G, \mathbb{Z}) = \frac{\mathbb{Z}\left\{ \sum_{g \in [Q \times G]^G} q[g] \right| g \in G \right\}}{\mathbb{Z}\left\{ a \sum_{g \in [Q \times G]^G} q[g_2] - b \sum_{g \in [Q \times G]^G} q[g_1 g_2] + c \sum_{g \in [Q \times G]^G} q[g_1] \right\}},$$

where $a = \frac{|Q||[Q_{g_1}]||[Q_{g_2}]|}{|Q||[Q_{g_1}]||[Q_{g_2}]|}$, $b = \frac{|Q||[Q_{g_1}]||[Q_{g_2}]|}{|Q||[Q_{g_1}]||[Q_{g_2}]|}$, $c = \frac{|Q||[Q_{g_1}]||[Q_{g_2}]|}{|Q||[Q_{g_1}]||[Q_{g_2}]|}$, $g_1, g_2 \in G$. 
This result means that the abelian group $H^1_Q(G, \mathbb{Z})$ is generated by the orbits of the action of $Q$ on $G$.

Let $C_1(G)$ be the free abelian group generated by $G$ as a set. This group has a natural action of $Q$ and the norm map $N : G \to C_1(G)^Q$ sending each element $g \in G$ to the sum of the elements on its orbit

$$\sum_{[q] \in Q} q[g] = |Q_g| \sum_{[q] \in Q/Q_g} q[g]$$

identifies $Q$-orbits and, by the universal property of the orbit group and the abelianization of a group, this induces a homomorphism (conf. [1])

$$\begin{array}{ccc}
(G//Q)_{ab} & \longrightarrow & H^1_Q(G, \mathbb{Z})
\end{array}$$

such that the diagram

$$\begin{array}{ccc}
G & \xrightarrow{p} & G//Q \\
\downarrow & & \downarrow \tilde{N} \\
(G//Q)_{ab} & \longrightarrow & H^1_Q(G, \mathbb{Z})
\end{array}$$

is commutative.

It is intuitively clear that homomorphism (2.1) may not in general be surjective, as it is induced by the non surjective norm map.

In order to have concrete examples of this, consider the case $G = \mathbb{Z}/4$, $Q = \mathbb{Z}/2 = \langle t \rangle$ acting by group inversion, i.e. $n \mapsto -n$. In this case, $(G//Q)_{ab} = G//Q = G/[G, Q]^G = G/[G, Q]$ because $G$ is abelian. The subgroup $[G, Q]$ is generated by elements of the form $n + t(-n) = 2n$, i.e. $[G, Q] = \{0, 2\}$ and $(G//Q)_{ab} \cong \mathbb{Z}/2$. On the other hand, according to [1] Theorem 3, $H^1_Q(G, \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and, therefore, homomorphism (2.1) cannot be surjective.

3. Conditions for (non) injectivity of homomorphism (2.1)

The aim of this section is to give conditions for the existence of elements in the kernel of homomorphism (2.1) and to give more detailed conditions for these elements when the order of the group $Q$ is bounded.

Assume that the element $g \in G$ is in the kernel of the map $\tilde{N} : G \to H^1_Q(G, \mathbb{Z})$. This means that, in the free group $C_1(G)^Q$, one has an equation of the form

$$\sum_{q \in Q} q(g) = 0$$

(3.1) $$\sum_{i=1}^N \left[ a_i \sum_{q \in Q/q_1} q(g_{1i}) + b_i \sum_{q \in Q/q_2} q(g_{2i}) - c_i \sum_{q \in Q/q_1, q_2} q(g_{1i}g_{2i}) \right]$$
where \( a_i = \pm \frac{|Q_{g_i}|}{|Q_{g_1} \cap Q_{g_2}|}, \) \( b_i = \pm \frac{|Q_{g_i}|}{|Q_{g_1} \cap Q_{g_2}|}, \) and \( c_i = \mp \frac{|Q_{g_i, g_2i}|}{|Q_{g_1} \cap Q_{g_2}|}. \) Equivalently,

\[
(3.2) \quad \sum_{i=1}^{N} \left[ a_i \sum_{q \in Q/Q_{g_1}} q(g_1) + b_i \sum_{q \in Q/Q_{g_2}} q(g_2) - c_i \sum_{q \in Q/Q_{g_1, g_2}} q(g_1 g_2) \right] - \sum_{q \in Q} q(g) = 0.
\]

This equation means that all terms with negative sign must coincide with some elements with positive sign.

In order to find elements in the kernel of homomorphism \( \varphi, \) one may consider special cases of equation \( (3.2) \). In the following, it will be considered that coefficients \( a_i \) and \( b_i \) are all positive and \( c_i \) are all negative.

A necessary condition for equation \( (3.2) \) to hold is that the number of positive terms must coincide with the number of negative terms. In this case,

\[
(3.3) \quad \sum_{i=1}^{N} (a_i [Q : Q_{g_1}] + b_i [Q : Q_{g_2}]) = |Q| + \sum_{i=1}^{N} c_i [Q : Q_{g_1, g_2i}].
\]

Equivalently, one has

\[
(3.4) \quad \sum_{i=1}^{N} \left( \frac{|Q_{g_i}|}{|Q_{g_1} \cap Q_{g_2}|} \frac{|Q|}{|Q_{g_1}|} + \frac{|Q_{g_i}|}{|Q_{g_1} \cap Q_{g_2}|} \frac{|Q|}{|Q_{g_2}|} \right) = \sum_{i=1}^{N} \frac{|Q_{g_i, g_2i}|}{|Q_{g_1} \cap Q_{g_2}|} \frac{|Q|}{|Q_{g_1}|} + |Q|,
\]

i.e.

\[
(3.5) \quad \sum_{i=1}^{N} \left( \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} + \frac{|Q|}{|Q_{g_1}|} \frac{|Q_{g_2}|}{|Q_{g_2}|} \right) = \sum_{i=1}^{N} \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} + |Q|,
\]

that is

\[
(3.6) \quad \sum_{i=1}^{N} \left( \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} \right) = \sum_{i=1}^{N} \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} + |Q|,
\]

\[
(3.7) \quad 2 \sum_{i=1}^{N} \left( \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} \right) = \sum_{i=1}^{N} \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} + |Q|,
\]

\[
(3.8) \quad \sum_{i=1}^{N} \left( \frac{|Q|}{|Q_{g_1} \cap Q_{g_2}|} \right) = |Q|.
\]

Therefore, one has the equation

\[
(3.9) \quad \sum_{i=1}^{N} \left( \frac{1}{|Q_{g_1} \cap Q_{g_2}|} \right) = 1.
\]

That is, intersection of isotropy groups of elements in right side of equation \( (3.2) \) must form a rational partition of \( 1. \)
Equation (3.9) can be applied to bound the length \( N \) of the sum in equation (3.2). For example, if \( Q \) does not fix elements different from the identity in \( G \), then one has \( Q_{g_1} \cap Q_{g_2} = 1 \) for every pair of elements \( g_1, g_2 \in G \) that are not the neutral element. In this case one must have \( N = 1 \), i.e. equation (3.2) reduces to

\[
0 = a \sum_{q \in Q/Q_{g_1}} q(g_1) + b \sum_{q \in Q/Q_{g_2}} q(g_2) - c \sum_{q \in Q/Q_{g_1g_2}} q(g_1g_2) - \sum_{q \in Q} q(g).
\]

(3.10)

For the equation to hold, \( g_1 \) or \( g_2 \) must be in the same orbit with their product \( g_1g_2 \), for example, one has \( q_1(g_2) = g_1g_2 \), which, on the one hand, implies that \( |Q_{g_2}| = |Q_{g_1g_2}| \), i.e. \( b = c \), and on the other hand, that \( \sum_{q \in Q/Q_{g_1}} q(g_1) = \sum_{q \in Q/Q_{g_1g_2}} q(g_1g_2) \). Therefore, equation (3.10) becomes

\[
0 = a \sum_{q \in Q/Q_{g_1}} q(g_1) - \sum_{q \in Q} q(g).
\]

(3.11)

This means that \( g \) and \( g_1 \) are in the same orbit, but one has that \( g_1 = g_2^{-1} q_1(g_2) = [g_2, g_1]^{-1} \), i.e. \( \bar{g} = \bar{g}_1 \in G//Q \). And this does not give non-trivial elements in the kernel.

The length \( N \) may also be bounded in terms of the order of the group, for example, if \( |Q| = 2 \), then the only possible non-trivial length (in terms of the previous discussion) is \( N = 2 \). The only corresponding partition in this case is \( \frac{1}{2} + \frac{1}{2} \).

**Proposition 3.1.** Let \( Q \) be a group of order two. If a non-trivial element \( \bar{g} \in (G//Q)_{ab} \) satisfies equation (3.1) then it must have order two.

**Proof.** As it was shown, for length \( N = 1 \) one has \( \bar{g} = 1 \in G//Q \) which implies \( \bar{g} = 1 \in (G//Q)_{ab} \). So, the only remaining possibility is \( N = 2 \).

Thus, one has the equation

\[
a_1 \sum_{q \in Q/Q_{g_1}} q(g_{11}) + b_1 \sum_{q \in Q/Q_{g_2}} q(g_{21}) - c_1 \sum_{q \in Q/Q_{g_1g_2}} q(g_{11}g_{21}) + a_2 \sum_{q \in Q/Q_{g_1}} q(g_{12}) + b_2 \sum_{q \in Q/Q_{g_2}} q(g_{22}) - c_2 \sum_{q \in Q/Q_{g_1g_2}} q(g_{12}g_{22}) - \sum_{q \in Q} q(g) = 0.
\]

(3.12)

**Case 1. A cancellation in the upper sum in equation (3.12).** As in the previous discussion, in this case one has \( b_1 = c_1 \), the corresponding terms in the upper sum cancel, \( g_{11} = 1 \in G//Q \) and one obtains the equation

\[
0 = a_1 \sum_{q \in Q/Q_{g_1}} q(g_{11}) + a_2 \sum_{q \in Q/Q_{g_2}} q(g_{12}) + b_2 \sum_{q \in Q/Q_{g_2}} q(g_{22}) - c_2 \sum_{q \in Q/Q_{g_1g_2}} q(g_{12}g_{22}) - \sum_{q \in Q} q(g).
\]

(3.13)

If the lower therm cancels with (part of) the upper remaining term, then \( g \) and \( g_{11} \) are and the same orbit and \( \bar{g} = 1 \in (G//Q)_{ab} \). Otherwise, the upper term cancels with (part of) the negative middle term which implies \( |Q_{g_1}| = |Q_{g_1g_2}| \) and, as \( |Q_{g_1} \cap Q_{g_2}| = 2 = |Q_{g_1g_2} \cap Q_{g_1g_2}| \) one has \( a_1 = c_2 \) and both remaining middle (positive) terms must cancel with the lower term. This means that both \( g_{12} \) and \( g_{22} \) are in the same orbit of \( g \) and, therefore

\[
(\bar{g})^2 = \bar{g}\bar{g} = \bar{g}_{12}\bar{g}_{22} = \bar{g}_{12}g_{22} = g_{11} = 1 \in G//Q.
\]
**Case 2.** No cancellations in the same row in equation [3.12]. One may assume that there exist elements \( q_1, q_2 \in Q \) such that \( q_1(g_{11}) = g_{12}g_{22} \) and \( q_2(g_{12}) = g_{11}g_{21} \). Then

\[
q_2q_1(g_{11}) = g_{12}(g_{12}g_{22}) = q_2(g_{12})q_2(g_{22}) = g_{11}g_{21}g_{2}(g_{22}),
\]

which implies \( [g_{11}, q_2q_1] = g_{11}^{-1}q_2q_1(g_{11}) = g_{21}g_2(g_{22}) \), i.e. \( g_{21}g_2 = 1 \in G//Q \).

As in case 1, one has \( a_1 = c_2 \) and analogously \( a_2 = c_1 \). The remaining positive terms must cancel with \( -\sum_{q \in Q} q(g) \) which implies that \( g_{21}, g_{22} \) and \( g \) are in the same orbit. So, \( g \in G//Q \) is trivial or it has order two.

Note that, if \( |Q| = 2 \), then \( |Q_{g_{11}} \cap Q_{g_{21}}| = 2 = |Q_{g_{12}} \cap Q_{g_{22}}| \) implies \( Q_{g_{12}} \cap Q_{g_{22}} = Q \) and, so, \( Q_{g_{21}} = Q_{g_{22}} = Q \). Similarly, all the other isotropy groups involved in equation [3.12] are equal to \( Q \). This means that elements in the kernel of (2.1) from equation [3.11] must come from order two elements in the fixed point subgroup \( G^Q = C_Q(G) \) in the notation used in [2] of \( G \).

As an example of this, consider \( G = \mathbb{Z}/4 \) with \( Q \) acting as inversion \( n \mapsto -n \). Then the equation,

\[
0 = ([2] + [2] - [0]) + ([0] + [2] - [2]) - 2[2]
\]

shows that the class of the (non trivial) element \( g = 2 \) in \( (G//Q)_{ab} \) is in the kernel of (2.1). This example can be generalized as follows.

**Proposition 3.2.** If \( |Q| = n \) and there is a fixed element \( g \in G \) of order \( n \), then the class of this element in \( (G//Q)_{ab} \) is in the kernel of (2.1).

**Proof.** As before, one has the length \( n \) equation

\[
0 = ([g] + [g] - [g^2]) + ([g^2] + [g] - [g^3]) + \cdots + ([g^{n-1}] + [g] - [1]) + ([1] + [g] - [g]) - n[g].
\]

Before returning to the general case of equation [3.11], note that the class \([1] \) of the neutral element in \( H^1_Q(G, \mathbb{Z}) \) must be zero. This means that one may consider equation [3.11] in the quotient \( C_Q^*(G)/[Z[1]] \) instead.

Considering this, if some \( g_{11} = 1 \), then \( Q_{g_{11}} = Q \) and \( Q_{g_{11}} \cap Q_{g_{21}} = Q_{g_{21}} \) giving \( b_i = c_i = 1 \). So, the corresponding term of the sum is zero:

\[
a_i \sum_{q \in Q/Q_{g_{11}}} q(g_{1i}) + b_i \sum_{q \in Q/Q_{g_{21}}} q(g_{2i}) - c_i \sum_{q \in Q/Q_{g_{11},g_{21}}} q(g_{1i}g_{2i}) = 0,
\]

and one may assume \( g_{1i} \neq 1 \neq g_{2i} \).

Also, if \( g_{2i} = g_{1i}^{-1} \) then these elements have the same isotropy group, so \( a_i = b_i = 1 \) and the corresponding term of the sum is just \( [g_{1i}] + [g_{1i}^{-1}] \).
Now, in equation (3.14) one can rearrange the sum in the right side to have

\[
\sum_{q \in Q} q(g) = \sum_{i=1}^{N} a_i \sum_{q \in Q} q(g_{1i}) + \sum_{i=1}^{M} b_i \sum_{q \in Q} q(g_{2i}) - \sum_{i=1}^{N} c_i \sum_{q \in Q} q(g_{1i}g_{2i})
\]

Counting the number of elements in the sum, one obtains the equation

\[
1 = \sum_{i=1}^{N} \left( \frac{1}{|Q_{g_{2i}}|} \right) - \sum_{j=1}^{M} \left( \frac{1}{|Q_{h_{1j}}|} \right)
\]

where all elements \(g_{1i}, g_{2i}, h_{1j}, h_{2j} \in G\) are different from the neutral element. For free actions, for example, one obtains the equation \(1 = N - M\).

Consider the set-theoretic quotient map \(p : G \to G/Q\), where the elements of this set are represented by sums of the form \(x = \sum_{[q] \in Q/Q_g} q(g)\), then all elements in the same inverse image of \(p\) can be summed up. This gives an equation of the form

\[
\sum_{i, p(g_{1i}) = p(g)} a_i + \sum_{i, p(g_{2i}) = p(g)} b_i - \sum_{i, p(g_{1i}g_{2i}) = p(g)} c_i - \sum_{i, p(h_{1j}) = p(g)} d_i - \sum_{i, p(h_{2j}) = p(g)} e_i + \sum_{i, p(h_{1j}h_{2j}) = p(g)} f_i = 0
\]

As the generators of the free group \(C_1^Q(G)\) are independent, this equation can only be true if the coefficients are all zero, giving the equations:

\[
\sum_{i, p(g_{1i}) = p(g)} a_i + \sum_{i, p(g_{2i}) = p(g)} b_i - \sum_{i, p(g_{1i}g_{2i}) = p(g)} c_i - \sum_{i, p(h_{1j}) = p(g)} d_i - \sum_{i, p(h_{2j}) = p(g)} e_i + \sum_{i, p(h_{1j}h_{2j}) = p(g)} f_i = 0
\]

where all elements \(g_{1i}, g_{2i}, h_{1j}, h_{2j} \in G\) are different from the neutral element. These equations, together with condition (3.14) give a description of the kernel of (2.1).
References

[1] Aquino, C., Jiménez, R., Mijangos, M., Morales Meléndez, Q.: On Invariant (co)homology of a group, preprint

[2] Deaconescu, M., Walls, G.L.: Groups acting on groups, Algebra Logic 52(5), 387–391 (2013)

[3] Feit, W.: On the structure of frobenius groups, Canad. J. Math. 9, 587–596 (1957)

[4] Jimenez, R., López Madrigal, A., Morales Meléndez, Q.: A spectral sequence for homology of invariant group chains, Mosc. Math. J. 18(1), 149–162 (2018)

[5] Kevin, P. K.: Homology of invariant group chains, J. Algebra 29815–33 (2006)

[6] Thompson, J.G.: Normal p-complements for finite groups, Math. Z. 72, 332–354 (1960)

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