A simple BRST system with quadratically constrained ghosts

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Abstract

I present a toy model for the Berkovits pure spinor superparticle. It is a $D = 1$, $N = 2$ superparticle with no physical degrees of freedom. We study the cohomology in various ways, in particular finding an explicit expression for the ‘b’-field. Finally, we construct the topological string B-model from a straightforward generalization of the system.

Keywords: BRST, Superstring, Pure spinors

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I. INTRODUCTION

The pure spinor string \[1\], introduced by Berkovits in the year 2000, is the first string model to be quantized covariantly with respect to the full super-Poincaré symmetry group of flat (super)space-time. See \[2\] for an early review article. It is also arguably the most promising model for calculating higher loop amplitudes \[3, 4, 5, 6\], and for covariant quantization in Ramond-Ramond backgrounds \[7\].

Characterized by a free field world-sheet action and a BRST charge, it has one unconventional feature: its bosonic ghost fields obey certain quadratic constraints. This has interesting implications for the BRST cohomology, which defines physical states and operators. It also leads to some difficulties. Firstly, a clean expression for the ‘b’-field, which is important in amplitude computations, is not known in the ‘minimal’ formalism \[3\]. Secondly, a covariant BRST treatment of the quadratic constraints requires infinitely many ghosts-for-ghosts \[8, 9\]. These issues apply equally to the more straightforward superparticle, which we will consider from now on.

To begin with, let us compare the pure spinor superparticle to a similar but ordinary Hamiltonian BRST system, without the constrained ghosts. See \[10\] for a review of such systems.

Consider a generic abelian gauge theory, whose operators act on some superspace with co-ordinates \(X, \theta\). Suppose that it has independent fermionic gauge generators, or first class constraints \(\{g_\alpha\}, \alpha = 1, \ldots, N\). We introduce bosonic ghost variables \(l^\alpha\), such that the local gauge symmetry is replaced with a global symmetry defined by fermionic, nilpotent BRST charge

\[
\Omega = l^\alpha g_\alpha, \quad \Omega^2 = 0.
\]  

A physical state \(\psi\) is in the cohomology of \(\Omega\) at ghost-number 0

\[
g_\alpha \psi(X, \theta) = 0.
\]

We see that \(\psi\) is simply a physical wavefunction in the Dirac quantization of the gauge theory. At higher ghost numbers, the cohomology vanishes (at least locally), as can be shown by making a (local) canonical change of basis such that \(g_\alpha\) are the first \(N\) momenta.

Now let us consider the \(\mathcal{N} = 1\) Berkovits superparticle \[11\]. Its gauge theory origin is
currently unclear, but the BRST charge is defined to be
\[ \Omega = \lambda^\alpha D_\alpha, \quad \Omega^2 = -i \lambda^\alpha \Gamma_{\alpha\beta} \lambda^\beta \frac{\partial}{\partial X_a}, \] (3)
where \( D_\alpha \) is a fermionic covariant derivative which acts on a \( D = 10, \mathcal{N} = 1 \) superspace with co-ordinates \( X^a, \theta^\alpha \), where \( a = 0, ..., 9 \), and \( \alpha = 1, ..., 16 \). The variables \( \lambda^\alpha \) are bosonic ghosts, which are made to obey the so-called pure spinor constraints
\[ \lambda^\alpha \Gamma_{\alpha\beta} \lambda^\beta = 0, \] (4)
in order that \( \Omega \) be nilpotent. See \[11, 12\] for more details. We note that the Berkovits BRST charge resembles that of the earlier abelian gauge theory. However, constraints \( D_\alpha \) are second class and not abelian, which is compensated for by the quadratic pure spinor constraints. Also, the state cohomology turns out to be non-zero at ghost numbers zero to three.

In this article, we build and analyze a toy model, which has analogous features to the pure spinor superparticle, but which is simpler to handle. The aim is to help better understand the harder properties of the string, by constructing a clean system, where everything relating to the ghost constraints is easy to solve. In particular, there are only finitely many ghosts and we are able to find an explicit expression for the ‘b’-field, without use of picture changing operators \[3\], or adding non-minimal fields in the style of \[13\].

The paper is structured as follows: First we construct the simple system and calculate its spectrum. Next we calculate the operator cohomology, which includes finding the ‘b’-field, and we make a BRST implementation of the ghost constraint. We then calculate a partition function, and show equivalence to a simple bosonic model. Finally, as an exercise, we use our toy system to construct the topological string B-model.

After submitting this article to arXiv.org, P.A Grassi pointed out to me some overlap with earlier work \[14, 15, 16, 17, 18\], in which he and collaborators had made use of a similar but not identical toy model. However, the main result of this paper, which is the expression for the ‘b’-field is not part of the overlap.

II. THE TOY MODEL

The toy model is a supersymmetric quantum particle with target space given by flat \( D = 1, \mathcal{N} = 2 \) superspace co-ordinates \( X \) and \( \theta^a \), supplemented by bosonic ghost-number
one variables \( u^a \), where \( a = 1, 2 \). Variables \( \theta^a \) and \( u^a \), are vectors of an \( \text{SO}(2) \) R symmetry. Indices are raised and lowered with a kronecker delta.

The BRST charge is defined as

\[
\Omega = u^a D_a, \tag{5}
\]

where

\[
D_a = \frac{\partial}{\partial \theta^a} - i \theta_a \frac{\partial}{\partial X} \tag{6}
\]

are the fermionic covariant derivatives, and the ghosts also obey a quadratic constraint

\[
u^a u_a = 0, \tag{7}\]

which describes a one-dimensional complex surface. By construction, the quadratic constraint is the minimum required such that \( \Omega \) be nilpotent

\[
\Omega^2 = -i u^a u_a \frac{\partial}{\partial X}. \tag{8}\]

The above combination of quadratic constraint and superspace describes the simplest non-trivial such system.

There is a gauge symmetry associated with quadratic constraint:

\[
\delta_\varepsilon \frac{\partial}{\partial u^a} = [\varepsilon(\tau) \frac{\partial}{\partial u^a}, u^b u_b] = 2 \varepsilon(\tau) u_a, \tag{9}\]

for small local parameter \( \varepsilon \) on the world-line \( \tau \). It is this which kills one degree of freedom of the \( u^a \) momenta.

We define a BRST and gauge invariant Hamiltonian

\[
H = \frac{1}{2} \left( \frac{1}{i} \frac{\partial}{\partial X} \right)^2, \tag{10}\]

which is shown later to be BRST-exact, as is typical of particle theories which come from string theories.

The global Poincaré symmetry generators

\[
\frac{\partial}{\partial X}, \quad Q_a = \frac{\partial}{\partial \theta^a} + i \theta_a \frac{\partial}{\partial X} \tag{11}\]

(anti-)commute with the BRST charge and Hamiltonian.
III. STATE COHOMOLOGY

A. Zero momentum cohomology

It is instructive to calculate the zero-momentum cohomology. At zero momentum the BRST charge becomes

$$\Omega_0 = u^a \frac{\partial}{\partial \theta_a}, \quad \Omega_0^2 = 0 \tag{12}$$

which is nilpotent even for unconstrained $u^a$. Based on a technique by Berkovits [2], we can relate the cohomology of $\Omega_0$ with constrained ghosts $H(\Omega_0 | u^a u_a = 0)$ to that with unconstrained ghosts $H(\Omega_0)$.

A BRST-closed wavefunction $F_0(u, \theta)$ obeys

$$\Omega_0 F_0 = u^a u_a F_1, \quad \Omega_0 F_1 = 0, \tag{13}$$

for some function $F_1(u, \theta)$. The $F$’s also obey transformations

$$\delta F_p = \Omega_0 G_p, \quad \delta F_0 = u^a u_a G_1, \tag{14}$$

for $p = 0, 1$. If $F_1$ is zero, then $F_0$ belongs to the unconstrained cohomology $H(\Omega_0)$, which is known to be simply a constant. Otherwise, $F_1$ belongs to $H(\Omega_0)$. So the $F_p$’s give the physical content of the spectrum. Note that we are just interested in the cohomology holomorphic in $u^a$. The corresponding wavefunction is

$$\psi = C + u^a \theta_a \tilde{C} \tag{15}$$

where $C$ and $\tilde{C}$ are the constants corresponding to $F_0$ and $F_1$.

In general, the above is a series of recursion relations. For the Berkovits superparticle, the first two steps are $\Omega_0 F_0 = \lambda^\alpha \Gamma_{\alpha \beta}^a \lambda^\beta F_1$ and $\Omega_0 F_1 = \lambda^\alpha \Gamma_{\alpha \beta} F_2$, where $F_0$, $F_1$, $F_2^a$ correspond to the ghost, gluon, and gluino of super Yang-Mills.

So we see that in effect the quadratic constraints determine the spin content of the physical spectrum.

B. The Full Cohomology

An attractive property of this toy system is that the full cohomology can be directly solved for all ghost numbers at once. It is convenient to change to U(1) covariant co-
ordinates \( u = u^1 + iu^2, \tilde{u} = u^1 - iu^2 \). Recall that \( u^a \)'s are complex so that \( \tilde{u} \) is not the complex conjugate of \( u \). The BRST charge and constraint become

\[
\Omega = uD + \tilde{u}\bar{D}, \quad u\tilde{u} = 0 \tag{16}
\]

where

\[
D = \frac{\partial}{\partial \theta} - \frac{i}{2}\bar{\theta}\partial X, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{i}{2}\theta\partial X. \tag{17}
\]

On the quadratic constraint surface, a general wavefunction has the form

\[
\psi = f_0(X, \theta) + \sum_{p=1}^{\infty} \left( u^p f_p(X, \theta) + \tilde{u}^p \tilde{f}_p(X, \theta) \right). \tag{18}
\]

The cohomology \( H^p(\Omega) \) splits into three distinct cases. When \( p \geq 2 \), then

\[
f_p \in H(D) = 0, \quad \tilde{f}_p \in H(\bar{D}) = 0, \tag{19}
\]

where note that \( D \) and \( \bar{D} \) are nilpotent operators. For \( p = 1 \), the difference to the above case, is that the variations of \( f_1 \) and \( \tilde{f}_1 \) are related

\[
\delta f_1 = Dg_0, \quad \delta \tilde{f}_1 = \bar{D}g_0, \tag{20}
\]

for some function \( g_0(x, \theta) \). The wavefunction takes the form

\[
\psi_1 = \frac{1}{2}(u\bar{\theta} + \bar{u}\theta)\tilde{C}(X), \quad \delta \tilde{C}(X) = \partial_X G(X), \tag{21}
\]

for some function \( G(X) \), where \( \tilde{C} \) has no equation of motion. When \( p = 0 \), \( f_0 \) is a constant. In other words, a physical wavefunction takes the form

\[
\psi = C(X) + u^a \theta_a \tilde{C}(X), \tag{22}
\]

where

\[
\partial_X C = 0, \quad \delta \tilde{C}(X) = \partial_X G(X). \tag{23}
\]

\( C \) and \( \tilde{C} \) resemble the ghost and anti-ghost of a Batalin-Vilkovisky theory.
IV. OPERATOR COHOMOLOGY

It’s straightforward to construct BRST-closed ghost number 0 operators. However, it is harder to find those which are BRST exact. Specifically, given a BRST-exact, ghost number 0 operator $F = [G, \Omega]$, it is a non-trivial problem to find $G$, because of the gauge symmetry associated with the quadratic constraint. The natural ghost number $-1$ operator $\partial/\partial u^a$ is not gauge invariant but transforms as in eqn (9). So it is not obvious how to construct negative ghost number operators. This is the essence of the ‘b’-field problem of the Berkovits superparticle. The expression for b, which is defined by $\{b, \Omega\} = P^a P_a$, where $b$ is a regular and nilpotent operator, is not known.

Let us find a basis for the BRST-exact operators. One can think of these operators as effective constraints of the model, in the sense that they map physical states to BRST-exact states. They must be BRST-closed, which leads to the candidates $-i \partial/\partial X$ and $Q_a$. Using equation (22), we find that they annihilate physical states in the following sense

$$-i \frac{\partial}{\partial X} \psi_{\text{phys}} \sim 0, \quad Q_a \psi_{\text{phys}} \sim 0,$$

(24)

where $\psi_1 \sim \psi_2$ implies that $\psi_1 = \psi_2 + \Omega \phi$ for some state $\phi$. Thus, both candidates are BRST-exact. We define the corresponding $b$ and $h_a$ ghosts by

$$\{b, \Omega\} = -i \frac{\partial}{\partial X}, \quad [h_a, \Omega] = Q_a.$$

(25)

We can get a handle on $b$ and $h$ from seeing how they act on physical states. For $b$, acting on wavefunction eqn (18)

$$\Omega_b u^p f_p = \Omega u^{p-1} D f_p, \quad \Omega b \bar{u}^p \tilde{f}_p = \Omega \bar{u}^{p-1} D \tilde{f}_p$$

(26)

for $p \geq 1$, and $bf_0 = 0$. So we see that $b$ acts differently on the $u = 0$, $\bar{u} = 0$ and $u = \bar{u} = 0$ sectors. Thus we define projection operator

$$P = \int d\bar{u} \delta(\bar{u}) - \int du \int d\bar{u} \delta(u) \delta(\bar{u}),$$

(27)

such that

$$P \psi = \sum_{p=1}^{\infty} u^p f_p(X, \theta),$$

(28)
and an analogous operator $\tilde{P}$. Based on the above observations, we make an educated guess

$$b = u^{-1}P\bar{D} + \bar{u}^{-1}\tilde{P}D.$$  

(29)

Indeed $[b, u\bar{u}] \approx 0$, and

$$\{b, \Omega\} = \{D, \bar{D}\} \left( \int du\delta(u) + \int d\bar{u}\delta(\bar{u}) - \int dud\bar{u}\delta(u)\delta(\bar{u}) \right) \approx -i\frac{\partial}{\partial X}. \quad (30)$$

Note that despite the presence of $u^{-1}$ and $\bar{u}^{-1}$ terms, $b$ is not a singular operator.

The $b$-field is perhaps more suggestively written as

$$b = \int_C du^a D_a \delta(u^b u_b) - \int d^2 u \frac{u^a}{u^b u_b} D_a \delta(u^1)\delta(u^2), \quad (31)$$

where $C$ is a sum of paths orthogonal to $u = 0$, and $\bar{u} = 0$ respectively, and passing through the point $(u, \bar{u})$. One can think of

$$\hat{v}_a = \int_C du_a \delta(u^b u_b) - \int d^2 u \frac{u^a}{u^b u_b} \delta(u^1)\delta(u^2) \quad (32)$$

as a gauge-invariant, ghost number -1 operator replacement for ghost momenta $-i\partial/\partial u^a$.

Our expression is similar to that of Oda and Tonin’s pure spinor b-field [18, 19]. To transfer their approach to our toy model we simply replace $\hat{v}_a$ with $Y_a = (k_a/k.u)$, where $k_a$ is some arbitrary constant. However, a problem is that $Y_a$ is not strictly allowed as an operator since it is singular. One also has to find an interpretation for $k^a$, since $b$ should be unique.

Using that $Q = D + i\bar{\theta}\partial_X$, a similar analysis for $h$ yields

$$h = -\bar{\theta}b + \frac{\partial}{\partial u} P, \quad (33)$$

$$[h, \Omega] = Q \left( \int du\delta(u) + \int d\bar{u}\delta(\bar{u}) - \int dud\bar{u}\delta(u)\delta(\bar{u}) \right) \approx Q. \quad (34)$$

It will be useful to find the corresponding $b$ field expression for the pure spinor case. Note that the $b$ and $h$ operators aren’t hermitian. We cure this in the following section by giving a BRST treatment of the quadratic constraint.

V. A BRST IMPLEMENTATION OF THE GHOST CONSTRAINT

When performing a path integral over the ghosts $u$ and $\bar{u}$, rather than doing a patch-wise integration over the constraint surface, it makes sense to use BRST methods. It also turns
out to be useful in defining a hermitian ‘b’-field later on. Following my earlier work \[8\] for the Berkovits superparticle, we implement the constraint with new BRST charge

$$\Delta = \rho u \tilde{u}, \quad \Delta^2 = 0,$$

(35)

where \(\rho\) is a fermionic ghost. \(\Omega\) is nilpotent and maps between cohomology classes of \(\Delta\) in the sense that

$$\Omega^2 = \{-i\partial_X \frac{\partial}{\partial \rho}, \Delta\} \simeq 0, \quad \{\Omega, \Delta\} = 0,$$

(36)

where \(A \simeq B\) means that \(A\) and \(B\) are in the same \(\Delta\)-equivalence class. A physical operator \(F\) is BRST-closed if

$$[F, \Delta] = 0, \quad [F, \Omega] \simeq 0,$$

(37)

and is defined up to a variation

$$\delta F \simeq [G, \Omega], \quad [G, \Delta] = 0$$

(38)

Similarly, a physical state \(\psi\) obeys

$$\Delta \psi = 0, \quad \Omega \psi \simeq 0,$$

(39)

and is defined up to variation

$$\delta \psi \simeq \Omega \phi, \quad \Delta \phi = 0.$$

(40)

There is one ghost number operator for each BRST charge

$$G_\Delta = \rho \frac{\partial}{\partial \rho}, \quad G_\Omega = u^a \frac{\partial}{\partial u^a} - 2\rho \frac{\partial}{\partial \rho},$$

(41)

where \(G_\Omega\) must be \(\Delta\)-closed, hence the \(\rho\) dependence. The physical wavefunction in eqn \(22\) now appears at \(G_\Delta\)-ghost number one. A dual wavefunction appears at ghost number zero.

With the full BRST approach we can now write a hermitian b field, given by

$$b = \rho \frac{\partial}{\partial \rho} \left(1/u \hat{D} P + 1/\tilde{u} \hat{D} \tilde{P}\right) + h.c.$$

(42)

where

$$\{b, \Omega\} \simeq -i \frac{\partial}{\partial X}, \quad \{b, \Delta\} = 0.$$

(43)

The idea is that the old b-field expression acts on the \(G_\Delta\)-ghost number one states, and its hermitian conjugate acts on the dual ghost number 0 state. There is a similar modification for \(h\).
VI. A PARTITION FUNCTION

Another way to find the spectrum is through a zero-momentum partition function, following the approach of Berkovits and Nekrasov\(^9\) for the superparticle. The Lefschetz trace formula shows how the graded trace over physical states of a BRST invariant operator is equal to that over all states, as explained in chapter 14 of [10]. By a suitable choice of operator, we can get information about the spectrum without calculating it directly. This reads as

\[
\chi(t) = \text{Tr}_{\text{Phys}}(-)^F t^K = \text{Tr}_{\text{All}}(-)^F t^K
\]

(44)

where \(t\) is a free parameter, \(F\) is the fermion number and \(K\) is a BRST invariant number operator

\[
F = \theta^a \frac{\partial}{\partial \theta^a} - \frac{\partial}{\partial \rho} \rho, \quad K = \theta^a \frac{\partial}{\partial \theta^a} + u^a \frac{\partial}{\partial u^a} + 2 \frac{\partial}{\partial \rho} \rho.
\]

(45)

The trace over all states splits up into a product of separate traces over \(\theta\), \(u\) and \(\rho\) respectively

\[
\chi(t) = (1 - t)^2(1 - t)^{-2}(-t^2(1 - t^{-2})) = 1 - t^2
\]

(46)

The 1 corresponds to a bosonic scalar with K-number 0, and the \(-t^2\) refers to a fermionic scalar with K-number 2. These are the \(C\) and \(\tilde{C}\) respectively in equation (22).

VII. EQUIVALENCE TO A SIMPLE BOSONIC MODEL

Consider the following system with nilpotent BRST charge

\[
\Omega = \eta \frac{\partial}{\partial X},
\]

(47)

where \(\eta\) is a fermionic ghost, and \(X\) is as before. This happens to be the simplest non-trivial BRST system. A general wavefunction

\[
\psi = C(X) + \eta \tilde{C}(X)
\]

(48)

in the cohomology \(H(\Omega)\) obeys

\[
\partial_X C(X) = 0, \quad \delta \tilde{C} = \partial_X G(X),
\]

(49)

for some function \(G(X)\). Since this matches the state cohomology of our model as seen in section [III.B] the two systems are equivalent. To summarize, our supersymmetric system with bosonic ghosts is exactly equivalent to a bosonic one with a fermionic ghost.
VIII. THE TOPOLOGICAL STRING

If we take a direct sum of three copies of the above bosonic model labelled by \( i = 1, 2, 3 \), and complexify the \( X^i \)'s, we end up with the particle version of the Topological string B-model. The BRST charge is \( \Omega = \eta^i \partial X^i \) which corresponds to the exterior derivative. Doing the same for our equivalent toy model, the BRST charge is given by \( \Omega = u^{ai} D_{ai} \), where each variable now has an \( i \) index. We can move from world-line to world-sheet to get the full string theory, in the manner of the pure spinor string.

As an exercise, we construct the topological B-model open superstring with a flat background metric. The action

\[
S = \int d^2z \left( \frac{1}{2} \bar{\partial} X^i \partial X^j \eta_{ij} + \frac{1}{2} \bar{\partial} X^i \partial X^j \eta_{ij} + \bar{\partial} \theta^a \pi_{zai} + \bar{\partial} u^a v_{zai} + \bar{\partial} \xi^i \rho_z^i \text{ (right movers)} \right)
\]

(50)
is in the style of the pure spinor string, where \( \pi_z, v_z \) and \( \xi \) are left-moving conjugate momenta to \( \theta, u \) and \( \rho \) respectively. The BRST charges are now

\[
\Omega = \oint dz u^{ai} d_{zai}, \quad \Delta = \sum_i \oint dz \rho_z^i u^{ai} u_{ai},
\]

(51)

where \( d_{zai} = \pi_{zai} - i(\partial X_i - \frac{1}{2} \theta_a i \partial \theta^a) \theta_{ai} \) is the stringified covariant derivative, with no summation implied over \( i \).

The respective central charge contributions from \( X^i, \theta^a, u^a \) and \( \rho_z^i \) are 6, -12, +12 and -6. It is gratifying that they sum to zero as expected.

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