On the foundations and necessity of classical gauge invariance

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Abstract. We argue that, ideally, the ways to measure magnitudes in non-quantum theories of physics (spacetime, field theory), limit drastically their possible mathematical models. In particular, gauge invariance in the Yang-Mills framework, is a necessity of our way of measuring rather than an a priori imposition on symmetry.

A general postulational basis for the geometric aspects of classical field theories is introduced, and the permitted models are studied. Some of them (for example, compatible with signature-changing metrics or variations of the speed of interactions) are new, and require a generalization of the concept of principal fiber bundle, which may be of interest both, physically and mathematically.

Keywords: Foundations of gauge theories, Yang-Mills theories, spacetimes, axiomatic approaches for field theories, Newtonian and Leibnizian structures, Galilean connection, generalized principal bundles, connections, fields of interactions, variations of the speed of light, multiverses.

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1 Introduction

There is a long philosophical tradition which claims that, in order to know the world, one has to study himself —our own structure reflects the world. In the accurate framework of the theories of physical spacetimes, Bernal, López and the author [4] developed a precise variant of this claim: our way of measuring macroscopic space and time (summarized in three minimum postulates), attaches one among four geometric structures to physical spacetime.

In the present article, our aim is to extend this viewpoint to field theory showing, in particular, that gauge invariance emerges in classical field theories as a necessity of our way of measuring —not, say, as an a priori requirement of symmetry or as an imposition of causal interactions. Certainly, this idea was already suggested by the founders of the theory some decades ago (see especially [16]). Nevertheless, we develop it in detail, introducing a postulational basis, revisiting known arguments and obtaining new possibilities. So, our results can be summarized as: (1) gauge invariance is a requirement of consistency for our measures, not an “optional” mathematically elegant assumption, (2) classical theories on spacetime, including not only General Relativity but also Galilei-Newton one, can be regarded as gauge theories in the sense of Yang-Mills, and (3) there are some natural possibilities beyond the standard framework of classical gauge theories, which might be both, physically and mathematically interesting.
Our approach is introduced in a more or less classic way. First, the minimum consensus hypotheses on measures are postulated in a mathematically rigorous way. Such postulates should be “obviously acceptable” at least as effective or consensus claims. They are even partially deducible from more elementary facts. Nevertheless, the “non-deducible” part of these postulates sounds so elementary that a universe where they did not hold would seem radically different to ours. In this sense, our conclusions can be useful for present-day proposals such as the Theories of Everything (TOEs) or parallel universes (see for example Barrow [3]): our results bound the mathematical possibilities for universes measurable in a way minimally similar to ours.

Some comments on this point are worth mentioning. To be more specific, we will consider [14], and retain part of his terminology. In this recent reference (see also [15]), Tegmark has suggested a Mathematical Universe Hypotheses (MUH). Accordingly, our Universe would be a mathematical structure, free of our particular (cultural, biological) “baggage”. The necessity to descend from the (outside, mathematical) “bird view” of the reality, to the (inside, experimentalist) “frog view” is emphasized. To get this, Tegmark suggests to analyze the symmetries of mathematical structures. In this paper, we stress the reversed perspective, going from frog postulates (expressed in a reasonably baggage-free way) to more general bird views. Our conclusions yield hints for any TOE, as it must be compatible with our conclusions (at least as a non-quantum limit). Moreover, they may suggest concrete ways to “descend” from the bird to the frog view, as suggested in [6].

This paper is organized as follows. In Section 2 we sketch the approach for spacetimes in [4]. As the study in this reference is mathematically exhaustive, here we stress the relevant physical aspects. In the first subsection §2.1 a brief account of our three postulates for spacetimes is given. In the next one §2.2 the four possible mathematical structures derived from these postulates are explained. In the last subsection §2.3 we emphasize the existence of a new element yielded by the theory. This is a function $k$ on the spacetime which controls possible changes among the four mathematical structures. Rigorously, if $k(p)$ is non-positive then $c(p) = \sqrt{-k(p)}$ is the supremum of speeds between standard observers at $p$. But $c(p)$ also admits a natural interpretation as a (possible varying) speed of propagation of interactions –light– in vacuum.

In Section 3 three postulates for the geometric contents of field theory are analyzed. The first and the third ones are easy to understand, even though they are discussed in some detail at the corresponding subsections: plainly, the first postulate (H1) states that particle fields can be described by sections in some fiber bundle space $E(M,V)$, §3.1, and the third (H3) is a technical claim on the effective (macroscopic) smoothness of the mathematical structures §3.3. As in the case of spacetimes, the second postulate (H2) is the crucial one: it states the existence of standard bases for the observation of (linear) fields at each event $p$. These bases are essential for our way of measuring and, a posteriori, they come from the automorphisms of the mathematical structures.

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1That is, they would be partially redundant, if one started at a more elementary level. For example, as explained in Section 5, the vector structure of the fiber $V$ in postulate (H1) can be regarded just as a linear approximation, or the group action on standard bases in (H2) appears necessarily because, otherwise, a group action can be attached univocally.

2Moreover, this structure would be one among others, in a ladder of multiverses with four levels of increasing generality. Many physicists may feel reluctant or clearly opposed to this type of ideas, and the public lecture by R. Penrose, “Fashion, Faith, and Fantasy in Modern Physical Theories” (delivered at 17th International Conference on General Relativity and Gravitation) may be a prominent example. Our approach is classical and, thus, independent of such controversial topics —our aim is not to dispute on them. We will regard ideas concerning parallel universes as a natural background for compelling speculative items in Physics, such as the many worlds interpretation of Quantum Mechanics or inflation.
we are trying to measure. The three postulates (H1), (H2), (H3) yield certain fibred structure, obtained from the principal fiber bundle $BE(M,V)$ of the bases of $E(M,V)$. This is somewhat more general than the usual structure in field theory. So, in §3.3 we also discuss an additional postulate (H4)* which permits to recover the familiar principle fiber bundle structure $P(M,G)$ of standard Yang-Mills theories.

In Section 4, gauge invariance in the postulated geometric framework for field theory is analyzed. Here we include the additional postulate (H4)* in order to make the approach directly applicable to the standard case—but it will be removed in Section 5. In the first subsection §4.1, gauge invariance appears naturally from our way of measuring (not as an “a priori imposition of local symmetry” from a global one), and it is independent of the causal relations on the spacetime. Careful mathematical distinctions among notions for particle fields such as gauge transformation, gauge orbit or (gauge) naturally equivalent particle fields, are introduced. These notions are equivalent in trivializable principal bundles, but conceptually different. They become natural when one considers a standard field observer as a section in the fiber bundle $P(M,G)$ of standard bases on $M$. Then, a principle of gauge invariance emerges from this framework. In the second subsection §4.2, the known consequences of this principle are revisited. So, we discuss the necessity of a connection on $P(M,G)$ and its possible interpretation as a particle field. In the last subsection §4.3 we recall how all the spacetime theories in Section 2 can be regarded also as field and gauge theories. This includes both, Galilei-Newton theory and Einstein General Relativity. In principle, they appear as gauge theories in the same footing, even though the interpretation of the canonical Levi-Civita connection for the latter might differ from the interpretation of the (non-unique) Galilean connection for the former. We stress that, here, these spacetime theories are regarded as Yang-Mills theories and, so, the structural group appears on the corresponding fiber bundle for the spacetime (the tangent bundle or associated tensor bundles); i.e., this group is not the group of diffeomorphisms of the spacetime, as frequently claimed for General Relativity (and critiqued in [17]).

In Section 5 the new possibilities which appear when postulate (H4)* is not imposed, are explored. Roughly, this means that the structural group $G$ of the classic principle bundle $P(M,G)$ may vary with the event—for spacetimes, this is equivalent to assume that function $k(p)$ is not a constant, allowing variations in the speed of interactions. In the first subsection §5.1, we explain the general mathematical structure $P(M,G_*)$ which appears. Its technical difficulties are stressed with an example, and are discussed in the second subsection §5.2. In the last subsection §5.3, we check that the principle of gauge invariance is extended naturally to this framework, and leave its possible implications for future developments.

Some conclusions are summarized in the last section.

2 Macroscopic space and time

2.1 The minimum consensus hypotheses on measures

Historically, the most relevant physical theories of spacetime are Galilei-Newton Classical Mechanics (with or without “external ether”) Special and General Relativity. As emphasized in [1], all of them share some minimum consensus hypotheses on how space and time must be measured. These hypotheses are valid not only for these theories but also, in principle, for conceivable measures of a
continuum macroscopic spacetime (including even gedanken measures). Concretely, the postulated hypotheses are:

(P1) The existence of a set $M$ of events (“here and now”) which can be suitably labelled by using four coordinates $(t,x^i), i = 1, 2, 3$, the first one (“time coordinate”) clearly distinguishable from the other three (spatial coordinates). In particular, $M$ is endowed with a structure of smooth connected manifold.

(P2) The possibility to find, at least infinitesimally, standard observers (or, more properly, standard observations realized as coordinate charts) around each event $p \in M$. These are characterized by the following minimum symmetry assumption:

Any two charts $O \equiv (t, x^i), \tilde{O} \equiv (\tilde{t}, \tilde{x}^i)$ obtained by standard observers around $p$ satisfy

$$\partial_t \tilde{t}|_p = \partial \tilde{t} t|_p, \quad \partial_{x^i} \tilde{x}^i|_p = \partial \tilde{x}^i x^j|_p, \quad \forall i, j \in \{1, 2, 3\}.$$ (2.1)

(P3) The effective smoothability of the possible geometric structures assigned to $M$ by means of the previous item (P2) —as well as by means of item (P1), but recall that smoothness was already claimed explicitly there.

These minimum consensus hypotheses are widely discussed in [4], where they are introduced rigourously. The first and last one are very easy to understand (and accept). For the key second one (P2), just recall:

• Implicitly, (P2) is assumed when one speaks on “freely falling observers” in General Relativity, or on “inertial observers” in Special Relativity and Classical Mechanics. In the latter case (P2) would hold even if a sort of ether (which selects more restrictively which observers are standard) were assumed.

• Equalities (2.1) constitute a minimum symmetry assumption which means that, given two such standard observers $O, \tilde{O}$, the comparison between their temporal (resp. spatial) coordinates at $p$ cannot privilege any of them.

In fact, $\partial_t \tilde{t}|_p = \partial \tilde{t} t|_p$ means that the $O$-time, measured with the $O$ clock, goes by as the $\tilde{O}$-time, measured with the $\tilde{O}$ clock (this is just a sensible mathematical translation of the assertion: “$O$ and $\tilde{O}$ measure at $p$ using the same unit of time”). For example, in Classical Mechanics there is a sort of “absolute time”, and standard observers measure them obtaining $\partial_t \tilde{t}|_p = 1 = \partial \tilde{t} t|_p$. Otherwise, the time observed by $\tilde{O}$ will present some “time dilation” (or contraction) $\partial_t \tilde{t}|_p$ with respect to $O$; then, we postulate that $O$-time will present an equal time dilation for $\tilde{O}$.

In a similar way, $\partial_{x^i} \tilde{x}^i|_p = \partial \tilde{x}^i x^j|_p$ means that the $i$-th spatial unit of $\tilde{O}$, measured with the $j$-th ruler of $O$, is identical to the $j$-th spatial unit of $O$, observed with the $i$-th ruler of $\tilde{O}$. Again, this is a sensible mathematical translation of the assertion: “$O$ and $\tilde{O}$ measure at $p$ using the same units of space”.

Summing up:
i) The second postulate (P2) seems to be an unavoidable symmetry in our frog perception of spacetime, where time and space are clearly different (and, consequently, the symmetry for time and space measures are a priori independent). This symmetry is expressed in a purely mathematical (“baggage free”) way and, thus, seems appropriate for searching a more general theory.

ii) Postulates (P1), (P2), (P3) have been minimum consensus hypotheses on how space and time are measured, as reflected for the fact that, historically, all the physical theories on continuum spacetime include them. But, moreover, they are so fundamental that seem unavoidable in any macroscopic theory of spacetime similar to ours.

2.2 The set of possible mathematical models

As proved in detail in [4], the careful analysis of the three postulates above, show the possibility to assign one of some (few) mathematical structures to the set of events $M$. For completeness, we sketch in the Appendix how to do this, and we only describe the final results next. Let $S^1$ be the circumference obtained from the extended real line $[-\infty, \infty]$ by collapsing $\pm \infty$ to a single point $\omega$, and let $O^k(4, \mathbb{R})$ be the groups defined around (7.2), (7.3) in the Appendix. Notice that, when $k \in (-\infty, 0)$, then $O^k(4, \mathbb{R})$ is conjugate to Lorentz group (in fact, it becomes the Lorentz group for Lorentz transformations with speed of light $c = \sqrt{-k}$), and when $k \in (0, \infty)$ then $O^k(4, \mathbb{R})$ is conjugate to the orthonormal Euclidean group. If $k = \omega$ then $O^k(4, \mathbb{R})$ is the Galilei group, and if $k = 0$ it is a mathematically dual group. Now, it is possible to prove:

(A) For each event $p \in M$, some $k(p) \in S^1$ can be assigned. Such a $k(p)$ either (i) it is univocally determined at $p$, and the group $O^{k(p)}(4, \mathbb{R})$ is assigned to $p$, or (ii) it can be chosen arbitrarily at $p$, and $\{1\} \times O(3, \mathbb{R}) \equiv \cap_{k \in S^1} O^k(4, \mathbb{R})$ is assigned to $p$. The way to assign such a $k(p)$ and the corresponding group is the following. Take the set of standard observers at $p$, and the corresponding set of bases $B_p = (\partial_t|_p, \partial_i|_p)$ of the tangent space $T_p M$ induced by these standard observers. Then, the transition matrix between two such bases belong to the group $O^{k(p)}(4, \mathbb{R})$ (see the Appendix).

(B) Now, apart from the function $k(p)$ on all $M$, we can assign to $M$:

1. In the points where $k(p) \in (-\infty, 0)$: a Lorentzian metric, as in General Relativity.

2. In the points where $k(p) = \omega$: a “Leibnizian structure”, which is a big generalization of classical Newtonian structures. Concretely, such a structure consists of a non-vanishing one form $\Omega$ and a (positive definite) Riemannian metric $h$ on its kernel (see [5] for an exhaustive study). In the classical case, there exists a “time function” $t$ and $\Omega = dt$. (Notice that, as a difference with the semi-Riemannian case, such a structure does not select any connection; nevertheless, a connection must be selected under a gauge principle as below, see Subsection 4.3)

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3 There exist a third residual possibility, namely, the existence of (at most) four values of $k(p)$ all of them positive. Even though this possibility can be handled without any problem [4 Subsect. 4.3], it disappears under very mild additional hypothesis [4 Sect. 5] and is scarcely representative (its appearance is due to the existence of “only a few” standard observers, with a special symmetry among them). So, we will not take into account this.
3. In the points where $k(p) = 0$: an “anti-Leibnizian structure”, which is a dual version of Leibnizian one.

4. In the points where $k(p) \in (0, \infty)$: a Riemannian metric.

5. In the points where $k(p)$ takes all the values of $S^1$: any of the previous structures (as well as others). Nevertheless, this case cannot hold on some parts of $M$ at the same time that any of the previous ones, if postulate (P3) is applied in a strict sense. In fact, under a smooth variation of the group assigned by (A) at each point (i.e., either one of the 6-dimensional groups $O^{k(p)}(4, \mathbb{R})$ or the 3-dimensional $O(3, \mathbb{R})$), the dimension should vary continuously, that is, it must be constant (see also Section 5). Therefore, the case $O(3, \mathbb{R})$ will be dropped, taking into account that, on one hand, we have strong experimental evidences that $k(p)$ must be non-negative in some parts of our Universe, and, on the other, the case $O(3, \mathbb{R})$ can be regarded as a special limit of the generic case $O^{k(p)}(4, \mathbb{R})$.

Summing up:

The minimum postulates (P1), (P2), (P3) about how space and time are measured (in a “macroscopic way”) imply that we can assign to spacetime either one of the four mathematical structures 1–4 above (or, eventually, the “degenerate” fifth one), or a structure on $M$ varying smoothly with the point among these four ones.

That is, only these four structures are the result of the effective symmetries perceived from our frog viewpoint. If a TOE existed or Tegmark’s MUH were true, such effective symmetries should be explained as partial symmetries or limit symmetries derived from the bird viewpoint. And, at any case, they must be taken into account in any description (even if less ambitious than a TOE) of our space-time reality.

2.3 The role of $k(p)$ and the speed of light

Some words on the role of function $k(p)$ are in order. Recall first:

(i) When $k(p) \notin (0, \infty)$ then $c(p) = \sqrt{-k(p)}$ admits the natural interpretation of supremum of relative velocities between standard observers.

The case $k(p) \in (0, \infty)$ cannot hold if we assume the Postulate of Temporal Orientation (PTO), that is, $\partial_t \tilde{t}|_p > 0$ hold for standard observers around $p$. This postulate would be clearly natural in the Universe around us.

(ii) If PTO were accepted, other assumptions might be reasonable. A, say, Postulate of Electromagnetic Interpretation (PEI) would state that $c(p)$ is equal to the speed of light at $p$ and

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4Concretely, an anti-Leibnizian structure is a non-vanishing vector field $Z$ and a Riemannian metric $h^*$ on its kernel in dual space.

5Technically, this must hold not only for the set $S_p$ of standard observers around $p$ but also for the set $S^*_p$ of observers which share the symmetries $Z$ with all the observers of $S_p$ and, thus, can be also regarded as standard, see [4, Defn. 2.1].

6The justification of this postulate (under the previously accepted (P1), (P2), (P3), (PTO)), in a reasonably general way (with “no heavy baggage”), would come from the experimental input: (1) light propagates at a finite speed in vacuum, and (2) vacuum looks like equal for all standard observers around each event $p$. As a consequence, the measured speed of light $c_p$ at each $p$ must be equal for all the standard observers. As the unique scalar quantity at each $p$ from previous postulates is $c(p)$, this leads to identify $c(p) = c_p$. 
If this were accepted then one can try to justify also the constant of the speed of light \( c(p) \equiv c \in (0, \infty) \) (independent of \( p \)) as a general fact.

Thus, one is tempted to include a fourth postulate for spacetime which, under PEI, would express the constancy of light speed:

\[
(P4)^* \quad \text{Function } k(p) \text{ is a negative constant } k = -c^2 \text{ independent of } p \text{ (and, thus, a unique group } O^k(4, \mathbb{R}) \text{ must be considered on all } M). \]

Nevertheless, some caution with \((P4)^*\) or the other “additional postulates” is needed, when one is looking for a general (bird) theory. All these postulates (as well as \((P1), (P2), (P3)\) above) collect perceptions from our frog perspective: if a TOE (or reasonably general theory) existed, it should be compatible with them. But we are not aware on how strange a TOE may seem. So, our frog postulates must be consensus hypotheses as minimal as possible. They must hold in the (relatively small) part of the Universe we can measure, but also allow generality for possible extrapolations. In this sense, it is natural to think that the inequality stated by PTO will hold in vast regions of our Universe, but not necessarily on all it. A general signature-changing metric (from Lorentzian to Riemannian, with degenerate parts which eventually may be Leibnizian or anti-Leibnizian), in the spirit of the limit of Hartle and Hawking proposal [9] (see also, for example, [8, 11] and references therein), is fully compatible with our basic three postulates and our limited experimental observations. Moreover, the evidences of, say, the constancy of the speed of light, might be non so clear (as claimed sometimes) and affect several measures — for example, the accelerated expansion of the Universe. This question might be a testable (experimentalist, frog) problem.

Summing up:

- If \((P4)^*\) is assumed (in addition to \((P1), (P2), (P3)\)) then the mathematical ambient for our description of physical spacetime is a Lorentzian 4-manifold, as in classical General Relativity.
- If \((P4)^*\) is not assumed, other possibilities, as the existence of a signature changing metric or the variation of the speed of light, appear (Subsection 5).

### 3 Field theories

Along this section we will define a general framework for field theory, based again in hypotheses on our way of measuring (close to the observer) and as “baggage free” as possible. As in the case of spacetimes, we will introduce three basic “minimum consensus hypotheses”, or postulates. Finally, we discuss a fourth one, similar to \((P4)^*\) for spacetimes, which simplifies the mathematical approach and recovers the accepted framework for classical Yang-Mills theories.

As in the case of spacetimes, we focus only in the geometric aspects on measures. That is, we will not consider essential ingredients such as energy, Lagrangians or equations of evolution.

#### 3.1 Ambient Hypotheses (H1)

Let us start finding some consensus hypotheses on our way of measuring in field theory. We will consider as a first set of hypotheses the following postulate:
(H1) (Framework) (a) The mathematical ambient for field theory is a fiber bundle $E(M,V)$, where
a set of primary physical fields on $M$ is represented by a section of the bundle $\psi : M \to E$, or particle field.

(b1). For the base $M$ (the underlying spacetime), the postulates (P1),(P2), (P3) in Section 2 hold.

(b2). The fiber $V$ (the model target space for the values of the physical magnitudes at each event) is a vector space of finite dimension $m$, and the fiber bundle $E(M,V)$ is a (real) vector bundle.

Notice that the part (a) only provides the most general ambient in Differential Geometry; its role is similar to (P1) in Section 2. The word “primary” is introduced to keep track of the fact that, by developing the theory, new physical fields may appear. Notice also that two different sections $\psi, \tilde{\psi}$ may describe the same physical fields, as discussed in Section 4.

The assertion about $M$ in (b1) only states the compatibility with our study of spacetime in previous section. The stated properties on the fiber $V$ in (b2) are a simplification, but there are reasons to assume it, at least for a “effective but general” theory:

(i) Finite dimensionality would be natural for our capability to measure only a finite number of variables, and

(ii) The linear character of the fiber $V$ is justified by the key role of linear mathematical approximations. That is, perhaps a more subtle fiber fitted better, but we can measure directly only linear approximations.

Moreover if, say, $V$ were a complex vector space, it could be also regarded as a real one, and this is a natural choice for the vector bundle structure, as the base is real.

There are two more possibilities of interest for the fiber, even accepting the arguments (i) and (ii) above: (a) to assume that $V$ has more than one connected component (say, in order to describe a discrete variable), or (b) to assume that $V$ is an affine space, with the 0 section not defined a priori (a candidate for such a section might appear a posteriori after a sort of “spontaneous symmetry breaking”). But even these cases can be handled in a similar way. Summing up, we maintain in what follows for simplicity (or, at least, as a first approximation), that $V$ is a (finite dimensional) real vector space.

3.2 Standard bases (H2)

For any vector bundle $E(M,V)$, one can construct the manifold $BE$ consisting of all the bases of the fiber $E_\cdot$ at any point $\cdot$ of $M$. This turns out a principle fiber bundle $BE(M,G_m)$, where the structural group $G_m$ is just the general linear group $G_m = \text{Gl}(m,\mathbb{R})$. $G_m$ acts on the right naturally on each fiber $BE_\cdot$: if $u = (e_1, \ldots, e_m) \in BE_\cdot$ and $g \in G_m$ then $ug$ is obtained just by multiplying formally. The following hypothesis plays a role similar to (P2) for spacetimes.

(H2) (Standard bases). For each $\cdot \in M$, our way of measuring the physical fields around $\cdot$ selects a proper subset $P_{\cdot}$ of $BE_{\cdot}$ ($P_{\cdot} \neq \emptyset, BE_{\cdot}$), whose elements will be called standard bases at $\cdot$ and satisfy:

There exists a closed subgroup $G_{\cdot} \subset G_m$ which acts freely and transitively on $P_{\cdot}$.
Notice that, chosen a bases \( u \) in \( P_p \), the claimed action allows to identify \( P_p \) and \( G_p \), but this identification depends on \( u \).

Recall that (H2) has been historically a minimum consensus hypothesis on our way of measuring, including the standard model of particle physics. For example, the bases in \( P_p \) may be orthonormal for a real or hermitian product, and \( G_p \) is then the set of orthonormal or unitary matrixes. But the \textit{a priori} reasonability of (H2) comes from the facts:

(i) \( P_p \) must be a proper subset of \( BE_p \): if no standard bases existed at \( p \) (or if all the bases were standard) there would not be ways to specify any intrinsic property of a vector \( v \) in \( E_p \), except if \( v \) is 0 or not.

(ii) A group \( G_p \) acts on each fiber \( E_p \): this is commonly assumed as a natural requirement of symmetry (even more, with \( G_p \) independent of \( p \)), but a more precise justification is the following. Assume first that \( P_p \) is any (non-empty) set of bases of \( E_p \). Choose \( u \in P_p \) and define \( G_{p,u} = \{ g \in G_m : ug \in P_p \} \). The smallest subgroup \( G_{p,u}^* \) of \( G_m \) which contains \( G_{p,u} \) is independent of \( u \). This unique subgroup \( G_{p,u}^* \) is determined from \( P_p \) and acts freely and transitively on some set of bases \( P_p^* \supset P_p \) (also univocally determined). So, one can assume that the extended set of bases \( P_p^* \) will play the role of standard bases.

As a more abstract argument, any intrinsic property for vectors in the fiber \( E_p \) can be described by taking the group of automorphisms for this property. This group will act on \( E_p \), and a class of bases related by this action will be the set of standard bases.

It is worth comparing (H2) and (P2):

- The assumption (H2) on the action of a group between standard bases only expresses that there will be some symmetry between these bases. In (P2) we did not assume the action of any group, but we deduced its existence\(^7\) and, then, (H2) will also hold in this case.

- (H2) states that some symmetries among standard bases will exist, but we do not assume \textit{a priori} anything about this symmetry. On the contrary, in (P2) we used the concrete symmetry \((2.1)\), motivated by our familiar distinction between time and space.

### 3.3 Effective smoothness (H3), constancy of the structural group (H4)*

Now, we can state that smoothness is an effective macroscopic approximation, as claimed in (P3):

(H3) (Smoothability) All the geometric structures obtained by means of the previous item (H2) (as well as for the item (H1)) are (differentiably) smooth. In particular, \( P = \cup_{p \in M} P_p \) is a smooth submanifold of \( BE(M, G_m) \).

\(^7\)Some technical details may be taken into account. This action is obtained not for the (rather arbitrary) original set \( S_p \) of standard observers but for a natural set \( S^*_p \) constructed from the original one (see footnote\(^5\), which plays a similar role of \( P^*_p \) above. Nevertheless, the result for spacetimes is sharper, as one proves the possibility to relate the bases of \( S_p \) with elements of some of the groups \( O^k(4, \mathbb{R}) \). In all the cases, but in the residual one (footnote\(^4\) \( S^*_p \) appears naturally as \( P^*_p \) above (in fact, either \( k \) is univocally determined, and \( O^k(4, \mathbb{R}) \) acts on \( S^*_p \) or \( k \) can be chosen arbitrarily, and \( O(3, \mathbb{R}) \) acts). In the residual case, one can also define \( S^*_p \) from a minimum group \( G_p \) (as done above with \( P^*_p \)). But there exists also the possibility to determine univocally (at most) four values of \( k \). The corresponding groups \( O^k(4, \mathbb{R}) \) would be subgroups of the minimum group \( G_p \). However, this case disappear under hypotheses very scarcely restrictive \([4, \text{Section 5}]\), and the present discussion is only an academical one.
Notice that this differentiability allows $G_p$ vary from one point to another, even though this variation must be smooth. We will discuss discuss precisely what differentiability implies in Section 5 in particular, the dimension of $G_p$ will be regarded constant.

The consensus hypotheses (H1), (H2), (H3) are natural extensions of (P1), (P2) and (P3). And we can wonder, extending in Subsection 2.3 at what extent the assumption that $G_p$ is independent of $p$ is reasonable. The constancy of $G_p$ with $p$ is not as compelling as the constancy of its dimension stated above neither, in general, as (H1), (H2), (H3). With this caution, we introduce:

(H4)* There exists a closed subgroup $G \subset G_m$ such that $G_p = G$ for all $p \in M$.

We emphasize that there are as many evidences for (H4)* as for (P4)* above, that is: there is no experimental evidence against them. However, the consensus of the physical community for (H4)* might be bigger. The reason is that, in the case of the spacetime, there exist theories which admit the signature change. Moreover, under our approach the variation of $c(p)$ would admit (in principle) testable evidences. But, as far as the author knows, there is nothing analogous in the case of field theory. Nevertheless, the variation of $G_p$ may be a possibility worth of exploring theoretically, which will be discussed again in Section 5.

Recall that (H4)*, in addition to previous hypotheses, yields a structure of principal fiber bundle on $P$. Thus, summing up, we have obtained:

If (H4)* is assumed (in addition to (H1), (H2), (H3)) then the mathematical ambient for our description of field theory is a principal fiber bundle $P(M,G)$, obtained as a reduction (subbundle) of the bundle of the bases $BE(M,G_m)$; its base $M$ represents the underlying spacetime, and the fiber $G$ is a (finite dimensional) Lie group.

If (H4)* were not assumed, another possibilities of fiber bundles would appear (Subsection 5).

Two technical notes. First, notice that $E(M,V)$ can be recovered from $P(M,G)$ as an associate vector bundle (see for example [13, p. 30]). Second, once the structure of principle fiber bundle for $P(M,G)$ is accepted, one can construct an associate vector fiber bundle $E^\rho(M,V)$ for any representation $\rho : G \rightarrow \text{Aut}V$, as commonly used in particle physics for faithful representations. Even though technically useful, in principle, this does not yield more generality from the fundamental viewpoint.

4 Gauge Theory

Since the seminal paper by Yang and Mills [18], gauge principles and their interpretations in terms of connections in fiber bundles are very well-known (see for example, [7], [12]). Now, we will revisit the fundamental physical ideas. We will show that our minimum hypotheses in last section lead to a principle of gauge invariance. Thus, this principle can be also regarded as a minimum consensus hypotheses.

Next, we will assume that postulates (H1), (H2), (H3) hold. For simplicity, (H4)* will be also assumed here –as always in classical Gauge Theory--, and the possibility to remove it will be explored in the next section. So, next $P = P(M,G) \subset BE(M,G_m)$ is a principal bundle, as explained above.
4.1 Gauge invariance

Our aim is to show that, given any particle field \( \psi : M \to E \), there exists a set of particle fields, the gauge orbit of \( \psi \), \( \text{Orbit}(\psi) \), which cannot be distinguished of \( \psi \) by any experimental method. This justifies the principle of gauge invariance, which will assert that all these particle fields describe the same physical reality and, thus, will yield the same physical quantities.

We start by working in a trivializing neighbourhood \( U \subseteq M \) for \( P \), that is, \( U \) satisfies that \( P(M, G) \) admits a section on \( U \) and, thus, the restrictions \( P^U \equiv P(U, G) \) and \( E^U \equiv E(U, V) \) are trivializable bundles.

Ideally, in order to measure a particle field, an observer should take a standard basis at each event \( p \) defining some section \( \sigma : U \subseteq M \to P \) of the principal bundle \( P(M, G) \). Then, at each \( p \in U \), the basis \( \sigma(p) \) will yield some coordinates \( c(v_p) \in \mathbb{R}^m \) for any \( v_p \in E_p \). Regarding \( \sigma(p) \) as a \( m \)-uple of vectors, \( \sigma(p) = (e_1(p), \ldots, e_m(p)) \), and \( c(v_p) \) as a column vector of \( \mathbb{R}^m \), one obtains naturally a smooth map of coordinates on \( E^U \) characterized by:

\[
c : E^U \to \mathbb{R}^m, \quad v_p = \sigma(p)c(v_p).
\]

**Definition 4.1**

1. A standard (field) observer is any section \( \sigma : U \to P \).

2. Its set of associate coordinates \( c(\equiv c_\sigma) \) is the function defined by (4.1).

3. The coordinates of a particle field \( \psi : M \to E \) measured by the standard observer \( \sigma \) is the composite function \( c \circ \psi : U \to \mathbb{R}^m \).

Of course, the coordinates of a particle field change with the standard observer. If \( g_U : U \to G \) is any smooth map then the section \( \bar{\sigma} : U \subseteq M \to P, p \mapsto \sigma(p) : g_U(p) \) also defines a standard observer with coordinates

\[
c_{\bar{\sigma}} = g_U^{-1}c
\]

(notice that each \( g_U(p) \) belong to a group of matrices); one may say that the new coordinates \( c_{\bar{\sigma}} \circ \psi : U \to \mathbb{R}^m \) for the particle field \( \psi \) are obtained from coordinates \( c \circ \psi \) by means of a “passive pointwise symmetry”.

Two different standard observers \( \sigma, \bar{\sigma} \) may assign the same coordinate function to two particle fields \( \psi, \bar{\psi} \) (the particle field \( \bar{\psi} \) is obtained from \( \psi \) by means of an “active pointwise symmetry”). In this case, the fact that standard observers \( \sigma, \bar{\sigma} \) must be physically equivalent should imply that the two particle fields are physically indistinguishable on \( U \). This will be claimed below as a physical property, but let us introduce it progressively.

**Definition 4.2** Let \( \psi, \bar{\psi} : M \to E \) be two particle fields which satisfy: for each standard observer \( \sigma : U \to P \) there exists another standard observer \( \bar{\sigma} : U \to P \) such that the corresponding coordinate functions of the particle fields coincide, i.e., \( c_{\bar{\sigma}} \circ \psi = c_\sigma \circ \bar{\psi} \).

Then, \( \psi \) and \( \bar{\psi} \) are called naturally indistinguishable.

**Remark 4.3** Trivially the binary relation “to be naturally indistinguishable” in the set of all the particle fields is transitive (and a relation of equivalence).

On a trivializing open subset \( U \) for \( P \), we can characterize easily all the particle fields which are naturally indistinguishable to a prescribed one \( \psi_U : U \to E \) as follows. Fix a standard observer \( \sigma_0 : U \to P \). Associated to \( \sigma_0 \) one has an action \( \cdot \) of \( G \) on \( E^U \) defined as:

\[
g \cdot v_p = \sigma_0(p)g c_0(v_p), \quad \forall g \in G, \forall v_p \in E^U,
\]

(4.2)
is well defined on all \( M \).

**Definition 4.4**

1. A gauge transformation of the particle field \( \psi : M \to E \) is any particle field \( \tilde{\psi} \) obtained from (4.4) for some trivializing \( U \) and some function \( g_U : U \to G \). Consider functions \( g_U : U \to G \) such that \( g_U \cdot \psi \) is identically equal to the identity matrix \( I_m \in G \) on a neighborhood of the boundary \( \partial U \) of \( U \) in \( M \).

2. The gauge orbit \( \text{Orb}(\tilde{\psi}) \) of \( \tilde{\psi} \) is the smallest set of particle fields which satisfies:
   - (i) \( \tilde{\psi} \in \text{Orb}(\tilde{\psi}) \) and,
   - (ii) if \( \tilde{\psi} \in \text{Orb}(\tilde{\psi}) \) then any gauge transformation of \( \tilde{\psi} \) belongs to \( \text{Orb}(\tilde{\psi}) \).

**Remark 4.5**

1. If the principle fiber bundle \( P(M,G) \) is trivializable, then \( \text{Orb}(\tilde{\psi}) \) is just the set of all the gauge transformations of \( \tilde{\psi} \) at some points: as no global section exists, there exists some \( p_0 \in M \) (any point not included in the domain of the section) and a neighborhood \( U_0 \ni p_0 \) such that \( \tilde{\psi}(p) = \psi(p) \) for all \( p \in U_0 \). So, in the non-trivializable case, the binary relation \( \tilde{\psi} \) is related by means of a gauge transformation with \( \psi \) on the set of all particle fields, is not transitive (compare with Remark 4.3). Nevertheless, to belong to the same orbit is obviously a relation of equivalence. Moreover, \( \text{Orb}(\tilde{\psi}) \) can be constructed as follows. Consider the smallest relation of equivalence which contains the binary relation \( \text{To be gauge related} \) (i.e., the intersection of all the relations of equivalence which contain the binary relation induced by Defn. 4.4(1)). Then, \( \text{Orb}(\tilde{\psi}) \) is the class of equivalence of \( \tilde{\psi} \).

2. Anyway, the smallest set which defines \( \text{Orb}(\tilde{\psi}) \) can be constructed explicitly as the union \( \bigcup_{n=0}^{\infty} \text{Orb}_n(\tilde{\psi}) \), where each \( \text{Orb}_n(\tilde{\psi}) \) is defined recursively as follows:
   - (i) \( \text{Orb}_0(\tilde{\psi}) = \{ \tilde{\psi} \} \),
   - (ii) \( \text{Orb}_n(\tilde{\psi}) \) is the set of all the particle fields obtained as a gauge transformation of some particle field in \( \text{Orb}_{n-1}(\tilde{\psi}) \).

Thus, if \( \tilde{\psi}, \tilde{\psi}' \) belong to the same orbit, there is a finite chain of gauge transformations which sends the first field into the second one.

---

8The usual name is “local” gauge transformation, but we will not use it, in order to avoid confusions with local properties such as the trivialization of fiber bundles.
Theorem 4.6  If two particle fields $\psi, \bar{\psi} : M \rightarrow E$ belong to the same gauge orbit ($\text{Orbit}(\psi) = \text{Orbit}(\bar{\psi})$) then they are naturally indistinguishable.

Proof. From Remark 4.5(2) there exists a chain of particle fields $\psi_0, \psi_1, \ldots, \psi_k = \bar{\psi}$, such that each two consecutive fields are related by a gauge transformation and, thus, are naturally indistinguishable. As this relation is transitive (Remark 4.3) the result follows. □

Remark 4.7 Along this section, we have considered three conditions on particle fields $\psi, \bar{\psi}$:

(a) To be related by the a gauge transformation (Defn. 4.4(1)).
(b) To lie in the same gauge orbit (Defn. 4.4(2)).
(c) To be naturally indistinguishable (Defn. 4.2).

If $P(M, G)$ is trivializable (as happens necessarily if, for example, $M$ is contractible), these three conditions coincide. But in general one only has (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

In principle, this distinction might be regarded as a mathematical subtlety which does not affect the essence of our physical discussion. Nevertheless, we will take it into account below not only in order to be totally accurate but also to bear in mind that we are dealing with three different concepts — apart from the fact that non-local experimental effects type Aharonov-Bohm may stress its physical importance.

Up to now, a name such as “naturally indistinguishable” has been introduced just as a mathematical definition, being the name only suggested by other more elementary ones such as “standard observer”. But in order to make a physical theory we must postulate in a precise way at what extent this mathematical definition corresponds with reality. Taking into account the discussion above Defn. 4.2 if $\psi$ and $\bar{\psi}$ are naturally indistinguishable then no direct measure by standard observers can distinguish between them. Nevertheless, (in the non-trivializable case) no domain $U$ of any standard observer covers all of $M$. Thus, in principle, the possibility that an indirect measure of some global property on $M$ distinguished them, must be taken into account. Of course, this would not be the case if $\psi$ and $\bar{\psi}$ are related by a gauge transformation. In this case, the possible differences between the two fields would be measurable in the region where $\psi \neq \bar{\psi}$. But the equivalence between standard observers must imply that both particle fields represent the same reality. That is, the physical fields described by them must generate the same measurable magnitudes and cannot be distinguished by any experimental method. Nevertheless, this is not sufficient yet. It is obvious that the relation “to represent the same physical fields” is a relation of equivalence, but to be gauge related is not, recall Remark 4.5. Thus, particle fields in the same gauge orbit must also represent the same physical fields. So, we arrive naturally to the following consensus hypotheses about our measures:

(PGI)* (Principle of Gauge Invariance). Under postulates (H1), (H2), (H3), (H4)*, all the particle fields in the same gauge orbit $\text{Orb}(\psi)$ are physically identical, that is, all of them describe the same set of (primary) physical fields.

Remark 4.8 Notice that we have “deduced” (PGI)* from simple interpretations about standard field observers. In order to avoid a formalization of these interpretations, we state (PGI)* as a new postulate. But we emphasize that (PGI)* emerges as a requirement of our way of measuring, not as an a priori assumption on symmetry.
4.2 Necessity of fiber connections and gauge fields

Now, we revisit the classical implications of (PGI)* on the necessity of gauge fields. Under the classical viewpoint on field theory, one assumes the existence of a Lagrangian density \( L \) defined on the space of 1-jets \( J^1(E) \) (roughly, each element of this space gives a point \( p \) of \( M \), an element \( v_p \) of the fiber \( E_p \), and the differential \( \theta \) of some section which sends \( p \) to \( v_p \)). Given a particle field \( \psi : M \to E \) which describes the primary physical fields, \( L \) can be applied to the the section \( d\psi \) of 1-jets induced from \( \psi \), defining so a function \( L(d\psi) : M \to \mathbb{R} \).

Fix a standard observer \( \sigma_0 : U \to P \) and the associate action (4.2) on \( E^U \). This action also induces a natural action on \( J^1(E^U) \). As \( G \) is related to symmetries of the physical system, the Lagrangian density is assumed to be invariant by the action, i.e.,

\[
L(g_0d\psi) = L(d\psi) \quad \forall g_0 \in G,
\]

For jets \( d(g_0s) = g_0ds \) and, thus:

\[
L(d(g_0s)) = L(ds).
\]

i.e., \( L \) is invariant under a “global gauge transformations on \( U \)”. Nevertheless, if we consider a particle field related by a gauge transformation \( \tilde{\psi} = g_U\psi \), with non-constant \( g_U : U \to G \), then the action does not commute with the operation of taking the induced jets, that is:

\[
d(g_U\tilde{\psi}) \neq g_Ud\psi.
\]

Thus, if \( L \) does not depend trivially on the \( \theta \) part of the jet,

\[
L(d\tilde{\psi}) \neq L(d\psi),
\]

i.e., \textit{in general, \( L \) is not invariant on the gauge orbit of \( \psi \)}.

\textbf{Remark 4.9} Under our approach, the equality in (4.7) is not necessary by reasons of mathematical elegance (or by any causality condition). As \( L(d\psi) \) is assumed to be a quantity physically meaningful, the equality in (4.7) is a requirement of (PGI)*.

The known answer to this drawback is that one is forced to reconsider the definition of \( L(d\psi) \), and regard the primary vector fields as a part of the set of all the physical fields, introducing some new physical fields, to restore (PGI)*. Remarkably, such fields determine a connection in the principle bundle \( P(M,G) \) (and, thus, in \( E(M,V) \)). This allows to define a covariant derivative \( D\psi \) which involves the derivatives of \( \psi \) but is naturally \( G \)-invariant. That is, the connection yields a way to generate 1-jets from \( \psi \) which is free of the problem (4.6) and, thus:

\[
L(D\tilde{\psi})(= L(D(g_U\psi)) = L(g_UD\psi)) = L(D\psi),
\]

in agreement with (PGI)*.

We can reformulate this physical input as follows:

(1) The operation of taking jets from a particle field yields different measures by different standard observers. Thus, according to (PGI)*, such jets cannot generate by themselves physically meaningful quantities (as the Lagrangian density function \( L(d\psi) \)).
(2) The underlying reason is the following. If there is no a connection in $P(M,G)$, a standard observer $\sigma: U \rightarrow P$ does not have any way to compare the base he chooses at some point $\sigma(p) \in P_p$ with the bases he chooses at neighboring points. The only known mathematical element which allows such a comparison is a connection.

(3) Nevertheless, in general, a particle fibre bundle has no any canonical connection. So, the introduction of a connection in $P(M,G)$ is interpreted as the existence of a new physical field.

In fact, the introduced connection or gauge field, must be combined with the differentiation of the section representing the particle field, in order to yield a well defined operation for standard observers (covariant derivative). So, the gauge fields have a interpretation as fields of interactions with particle fields. This interpretation (which looks rather “baggage dependent”), may have further consequences for the theory.\footnote{Say, the Lagrangian density not only is defined by using covariant derivatives, but also must include a new term which represents the free Lagrangian density for the gauge fields. (As these fields can be also measured by using standard observers, the new Lagrangian term must depend on the curvature of the connection –Utiyama’s theorem– in order to preserve (PGI)$^*$.}

In their own right, these items are compelling for the general existence of gauge fields. Nevertheless, one can wonder at what extent they are unavoidable, and additional arguments can be provided for each item:

1.- The bad transformation of measures when taking jets, relies on the computation of the left hand side of (4.6), which involves derivatives of $g_U$. Nevertheless, even under (4.6) some Lagrangian densities could be defined in such a way that the equality in (4.7) holds. However, such counterexamples are in some sense degenerate and non-generic\footnote{As a simple example let $M$ be Lorentz-Minkowski spacetime, $V = \mathbb{R}^2$, $E$ diffeomorphic to $M \times V$ and the principle bundle $P(M,G)$ be obtained as a reduction of the bundle of the bases $BE(M,G_2)$ with structural group: $G = \left\{ \left( \begin{array}{cc} \lambda & 0 \\
0 & 1 \end{array} \right) : \lambda > 0 \right\}$. Taking a global section $\sigma: M \rightarrow P$, we can identify $E \equiv M \times \mathbb{R}^2$, $P \equiv M \times G$. The gauge orbit of a particle field $\psi = (\psi^1, \psi^2)$ is the set of all the sections $x \rightarrow (\lambda M(x)\psi^1(x), \psi^2(x))$, $x \in M$ constructed for any positive function $\lambda_M$ on $M$. Notice that the directions of the components $\psi^1, \psi^2$ are gauge independent. Then, a Lagrangian density which does not depend on $\psi^1$ will be gauge invariant, even if it depends on the derivatives of $\psi^2$.}. And the usage of a potentially big set of scalar quantities involving derivatives of $\psi$, as arbitrary Lagrangian densities, seems a basic requirement. So, item (1) is supported.

2.- The only differential operators defined in arbitrary manifolds with no additional structure deals with the differential or the Lie bracket. The latter has the same problems of gauge invariance than the former (as well as other problems: the Lie bracket must be applied to pairs of sections, it is defined only for the tangent bundle and associate tensor bundles, etc.) So, in order to ensure gauge invariance, an additional structure must be defined. Connections in principle fiber bundles (concretely, in reductions of a fiber bundle of bases) are designed specifically to compare bases at different points, supporting item (2).

3.- A well-known exception to the inexistence of a unique canonical connection is the fiber bundle of orthonormal bases for a semi-Riemannian metric. In fact, one has a controlled set of possible connections, and only one torsion free. This (Levi-Civita) connection is the canonical choice. Then, one can interpret the Levi-Civita connection as: (i) a gauge field, which appears in a rather rigid way, or (ii) a pure consequence of a different structure, the metric. In principle, the
difference between these two “baggage” interpretations might yield consequences for quantities as the Lagrangian term associated to the connection (footnote 9). At any case, Levi-Civita connection affects only to spacetime (to be discussed in Subsection 4.3). If this case (and eventually other exceptions of principal fiber bundles with a canonical connection) is excluded, the interpretation (i) of the connection as a field of interactions appears clearly, in agreement with item (3).

Summing up, with the above precisions:

Assuming (PGI)*, if $G$ is not a discrete group and enough physical quantities depends on the derivatives of a particle field $\psi : M \to E$, then a connection on $P(M,G)$ must be included in the geometry of $P$; eventually, such a connection will be regarded as a physical field in its own right.

4.3 Spacetime theories as Yang-Mills theories

Postulates on spacetimes were introduced under a viewpoints somewhat different to gauge theory. But, as stressed in Section 3.1, consensus hypotheses (H1), (H2), (H3), (H4)*, can be regarded as extensions of (P1), (P2), (P3), (P4)*. In particular, the principle of gauge invariance (PGI)* is also a consensus hypotheses for spacetimes. We revisit the conclusions for this case.

(P1), (P2), (P3) were introduced to measure spacetime $M$ itself, not any field on it. Nevertheless, the tangent vector bundle is naturally associated to $M$ and, finally, standard observers led to a submanifold $P$ of the principle bundle of the bases $BTM(M,G)$. Connections in $BTM(M,G)$ can be expressed easily as a (Koszul) differential operator $\nabla$ on $TM$, as we will do here.

Under (P4)* ($k \equiv -c^2$), the submanifold $P$ is the principle bundle of frames $F(M,O_k(4,\mathbb{R}))$ (orthonormal bases up to the normalization of the first vector to $c$), for a Lorentzian metric $g$. Without loss of generality, we can take the fiber bundle of orthonormal frames $F(M,O_1(4,\mathbb{R}))$, $O_1(4,\mathbb{R}) = O_{k=-1}(4,\mathbb{R})$.

Let us review briefly this well-known case. The Levi-Civita connection $\nabla^g$ is automatically selected in $F(M,O_1(4,\mathbb{R}))$, and any other Koszul connection $\nabla$ on $M$ can be written as $\nabla_X Y = \nabla^g_X Y + T_S(X,Y) + T_A(X,Y)$ where $T_S, T_A$ are vector valued bilinear maps, $T_S$ symmetric and $T_A$ skew-symmetric. Notice that $T_S, T_A$ are tensor fields canonically associated to $\nabla$ through $g$.

Accepting that some physical fields can be described by means of a section on $TM$ or its associated tensor bundles, (PGI)* applies, and implies the existence of a connection in $F(M,O_1(4,\mathbb{R}))$. Necessarily this connection is type $\nabla = \nabla^g + T_A$, where $T_A$ is (up to a factor) the torsion of $\nabla^g$. Then, it is natural to consider $\nabla^g$ as the gauge field, and $T_A$ as a new possible particle field (some authors have tried to use the torsion to describe different physical effects, see for example [1]). An arbitrary connection on $M$, which does not come a priori from a connection on $F(M,O_1(4,\mathbb{R}))$ (say, as a priori in Palatini method for Einstein equations) would lie out of the scope of (PGI)*, but would be also equivalent to consider a particle field $T_S$, in addition to $\nabla^g, T_A$.

Now, assume only (H4)* instead of (P4)*. Then $k$ is also constant but non-negative values are permitted. In particular, when $k = \omega$, the associated Leibnizian structure $\Omega, h$ (the former a non-vanishing one form and the latter a Riemannian metric on its kernel) is equivalent to a reduction

$$GM(M,O^\omega(4,\mathbb{R})) \subset BTM(M,G_4),$$

where the structural group $O^\omega(4,\mathbb{R})$ is, the (matrix) Galilean group.
(PG1)* implies the existence of a connection in the fiber bundle $GM(M, O^2(4, \mathbb{R}))$, or Galilean connection. Such a connection $\nabla$ parallelizes the Leibnizian structure ($\nabla \Omega = 0, \nabla h = 0$). Galilean connections always exist, and, in fact, they have as many degrees of freedom as the connections (with and without torsion) which parallelize a semi-Riemannian metric. Nevertheless, symmetric Galilean connections exist if and only if $\Omega$ is closed (for a complete study of all these questions, see [3]).

Galilean connections can be reconstructed by means of a Koszul-type formula from a “gravitational vector field” and a “vorticity field” (plus the torsion, if non-symmetric ones are also considered), [5, formula (13)]. But none of these tensor fields are selected by our postulates and, thus, neither a Galilean connection.

So, classical Galilei-Newton theory can be regarded as a (proper) gauge theory, where the connection is not univocally determined a priori, but appears from the necessity of gauge invariance. In this sense, the interpretation of the Galilean connection as a physical gauge field becomes more clear than in the case of General Relativity.

The cases $k = 0, k \in (0, \infty)$ are analogous to the previous ones—even though they do not have analogs in classical theories.

5 A further possibility

As we showed in Section 2 if one drops (P4)* among consensus hypotheses, new possibilities appear, some of them with interesting interpretations, as the possible variations of the speed of light. This also suggests the possibility to remove (H4)* in general, and explore the physical and mathematical consequences.

5.1 Framework for pointwise structural groups

Recall that, under (H1), (H2), (H3), one has the vector fiber bundle $E(M, V)$, the principle fiber bundle of all the bases $BE(M, G_m)$ and the subbundle $P$ of $BE(M, G_m)$ containing the standard bases. As justified in (H2), some (closed) Lie group $G_p$ depending on $p$ must act freely and transitively on each fiber of $P_p$. For simplicity, all the groups will be considered as connected in what follows.

This space will be denoted $P(M, G_*)$. Nevertheless, its structure is not totally well defined yet, as a precise notion of smoothability (required by (H3)) must be provided. Clearly, this notion must include these two items:

(i) $P$ is a smooth submanifold embedded in $BE(M, G_m)$ as a closed subset.

(ii) Both, $G_p$ an its action on $P_p$ varies smoothly with $p$.

The first item is defined accurately, but the second is not, and will be analyzed in the next subsection. Now, we emphasize that (i) is not sufficient. In fact, any sensible definition of (ii) would imply that $\dim G_p$ is independent of $p$, but this is not implied by (i):

11In fact, famous Leibniz’s objection to Newton’s inertial observers, can be seen as a way to claim for gauge invariance, and Newton’s answer on the spinning water-bucket might be regarded as the way to postulate the existence of a connection—a gauge field for classical space and time.
Example 5.1 Put \( M = (-\infty, 1), V = \mathbb{R}^2 \) and \( E = M \times V \), and construct \( P \) as follows. For any \( v \in V \setminus \{0\} \), let \( B_v \) the (ordered) base \( B_v = (v, w) \) univocally determined by: \( v \cdot w = 0, \| v \| = \| w \| \), and \( B_v \) is positively oriented (the usual scalar product \( \cdot \), norm \( \| \cdot \| \) and orientation are considered on \( V = \mathbb{R}^2 \)). Fix \( v_0 \in V \) with \( \| v_0 \| > 1 \). For each \( p \in M \) put:

\[
P_p = \begin{cases} 
B_v : \| v - v_0 \|^2 = p & 0 \leq p < 1 \\
B_v : \| v \|^2 = 1/|p| & -\infty < p < 0
\end{cases}
\]

Notice that \( P = \bigcup_{p \in M} P_p \) is a closed embedded submanifold of dimension 2 in \( BE(M, G_2) \). Clearly, \( G_p \) is 1-dimensional (in fact, a circle) at any \( p \neq 0 \), but it is the (0-dimensional) trivial group at \( p = 0 \). Notice also that \( P \) does not admit any (continuous) local section in a neighborhood of \( p = 0 \).

5.2 Problem on smoothness and structure of \( P(M, G_*) \)

In order to ensure the smoothness according to (ii) above, one must ensure that \( \dim G_p \) is constant, but this condition seems too weak by itself. Notice that \( \{ G_p : p \in M \} \) is just a set of subgroups of \( G_m \), with no further structure. A possibility would be to add an assumption such as “all the groups \( G_p \) are diffeomorphic”. In fact, this happened in the case of spacetimes when \( k \in (-\infty, 0) \), and this was enough to model variations of speed of light. Nevertheless, there are reasons to avoid such an assumption: (a) it is not justified by first principles, (b) technical problems would not be solved in general, as the diffeomorphisms between the groups \( G_p \) may be non-canonical, and (c) it is a relatively strong hypothesis and may forbid some interesting possibilities.

In fact, objection (c) would happen if that assumption is imposed to spacetimes. Notice that the groups \( O^{(p)}(4, \mathbb{R}) \) are all conjugate (and thus, diffeomorphic) if either \( k(p) \in (-\infty, 0) \) or \( k(p) \in (0, \infty) \). But the first case corresponds to the Lorentz group, and the second one to the orthonormal group, which are topologically very different. Nevertheless, this change of topology may be interesting, and can occur in a smooth way:

Example 5.2 Let \( M = \mathbb{R}^4 \), and then \( TM = M \times \mathbb{R}^4, BTM = M \times G_4 \). Let \( k : \mathbb{R}^4 \rightarrow \mathbb{R} \) be any smooth function with non-constant sign. Let \( \mathcal{M}_4(\mathbb{R}) \) be the set of square matrixes \( 4 \times 4 \) and consider the map \( F : M \times G_4(\equiv BTM) \rightarrow M \times \mathcal{M}_4(\mathbb{R}) \times \mathbb{R} \) defined as:

\[
F(p, A) = (p, A^t I_{4}^{k(p)} A, \det A).
\]

Recall that \( N = \{(p, I_{4}^{k(p)}, 1) : p \in M \} \subset M \times \mathcal{M}_4(\mathbb{R}) \times \mathbb{R} \) is a closed embedded submanifold of the codomain. Now, choose \( P = F^{-1}(N) \). \( P \) has a structure of fiber bundle \( P(M, G_*) \) with pointwise fiber \( G_p = O^{(p)}(4, \mathbb{R}) \) for all \( p \). As \( F \) has constant rank on \( P \), \( P \) becomes a smooth submanifold of \( BTM \), even at the changes of sign of \( k(p) \) (where the topology of \( G_p \) changes).

One can explore some alternatives for the meaning of smoothability hypothesis (ii). At any case, from any reasonable definition of (ii) one would have:

(i)\(_1\) Constancy of the dimension: the dimension of \( G_p \) is independent of \( p \).

(i)\(_2\) Existence of local sections (or standard observers): let \( \pi : P \rightarrow M \) is the natural projection, for any \( p \in M \) there exists a neighborhood \( U \subset M \) and a map \( \sigma : U \rightarrow P \), such that, \( \pi \circ \sigma \) is the identity of \( U \).
(ii) Compatibility of sections and actions: given two sections on \( \sigma_1, \sigma_2 : U \subset M \to P \), define \( g_U(p) \in G_p \subset G_m \) by means of \( \sigma_2(p) = \sigma_1(p)g_U(p) \), for all \( p \in U \); then both, the map \( g_U : U \to G_m \) and
\[
\pi^{-1}(U) \subset P \to \pi^{-1}(U), \quad u_p(\in P_p) \mapsto u_pg_U(p)
\]
are smooth.

These items can be imposed as a (provisional) definition. Thus, summing up,

For any field theory, (H1), (H2), (H3) imply the fiber bundle space \( P(M,G_*) \).

On each fiber \( P_p \), of \( P \), a Lie group \( G_p \) acts freely and transitively, and such actions satisfies the requirements of smoothability (i), (ii) above, the latter implying (ii)\(_1\) (ii)\(_2\) (ii)\(_3\).

In particular, \( P(M,G_*) \) admits standard observers.

5.3 Gauge invariance

Notice that, fixing a standard observer \( \sigma_0 : U \to P \) one has associate coordinates as in (4.1) and an action of each \( G_p \) on \( E_p \) as in (4.2). Thus, the notions of gauge transformation and gauge orbit makes sense and, reasoning as in Subsection 4.1, one arrives at:

(PGI) (Generalized Principle of Gauge Invariance). Under hypotheses (H1), (H2), (H3), all the particle fields in the same orbit \( \text{Orb}(\psi) \) are physically identical.

Now, reasons as those in Subsection 4.2 show the necessity of a geometrical object to compare different bases by a standard observer. That is, we must extend the notion of connection to \( P(M,G_*) \).

Recall that there are different well-known ways to define a connection in a principle bundle. The definition as a distribution admits an obvious extension for \( P(M,G_*) \). Say, a connection on \( P(M,G_*) \) is a distribution \( H \) (horizontal distribution) on \( P \) such that: (i) at each \( u_p \in P_p \) the subspace \( H_{u_p} \subset T_{u_p}P \) is complementary to the vertical subspace determined by vectors tangent to the fiber \( P_p \), and (ii) the distribution is invariant by the action of \( G_p \) at each \( p \) \((R_{g_p}^*H_{u_p} = H_{u_{g_p}})\).

As far as we know, such fibered spaces and connections have not been studied systematically, even though the possibility to extend the formalism of principle fiber bundles is well-known [10]. So, we stop here. We emphasize that the mathematical study of such connections and the possible associated physical phenomenons in \( P(M,G_*) \) appears as questions worth to be studied.

6 Conclusions

Finally, some of the points along this work are emphasized:

1. Our postulates for both, spacetimes and field theories are truly “minimum consensus hypotheses on our way of measuring”:
   - they are not only shared by the standard theories but also they are apparently unavoidable for physical theories on a Universe minimally similar to ours, and
• they are based on minimal symmetries from the experimental viewpoint, expressed in a fundamental (reasonably baggage free) sense.

2. Gauge invariance is not an a priori imposition for mathematical beauty (i.e. to impose that a global gauge invariance must be also a pointwise-local one) but a necessity for the validity of our way of measuring.

Gauge invariance is also independent of causal relations in the spacetime. As suggested in [16], if a standard observer \( O \) “has chosen” two standard bases \( B_p \) and \( B_q \) at two causally related events \( p,q \in M \), and one “changes” the chosen basis at \( q \), there is no experimental way at \( p \) to distinguish if this change has been carried out—except when a connection exists.

3. Connections are geometric elements to ensure gauge invariance. They are required under minimum hypotheses on the physical magnitudes in field theories (such as the existence of enough Lagrangians). In most fiber bundles no canonical connection exist. So, a connection must be introduced under the (baggage) interpretation of a interaction physical field.

4. General Relativity as well as Galilei-Newton theory can be regarded as gauge theories, in the same sense than Yang-Mills field theories. That is, a connection is required on a vector fiber bundle (in this case, the tangent bundle or one of its associated tensor bundles), and this connection is necessary to preserve gauge invariance under a pointwise gauge transformation (for a finite-dimensional gauge subgroup of \( G_4 \)). In both, Galilei-Newton and General Relativity, there are the same degrees of freedom for the connections compatible with the underlying spacetime structure—i.e, the underlying Lorentzian metric or “Leibnizian structure”. The difference between them (which might affect the interpretation of the connection as a physical field) comes from the fact that, for the case of Galilei-Newton, there is no a preferred connection obtained by imposing the vanishing of the torsion.

5. The possibility to vary the structural group \( G_p \subset G_m \) with \( p \in M \) in the bundle structure \( P(M,G_4) \) (in particular, the variation of \( k(p) \) in spacetimes) is allowed by the theory, and it seems an interesting possibility, as shown in spacetimes:

- “Mild” variations of \( G_p \) (as variations as a conjugate group, which would be equivalent to have a fixed subgroup of \( G_m \) and a different action on each fiber) may model effects such as the variation in the speed of propagation of interactions.
- “Strong” variations of \( G_p \) (including changes of topology, which can be carried out even in a smooth way) may describe more drastic effects such as changes of signature in a metric of the fiber.

Even more, new problems appear from the purely mathematical viewpoint (existence of connections on \( P(M,G_4) \), associate operators, etc.)

6. The strong mathematical conclusions obtained for both the spacetime and the field theories can be interpreted from different viewpoints as:

- A positivist simplification of our models of Universe.
- A set of limit requirements of compatibility for any Theory of Everything.
• A bound for the number of possible parallel Universes minimally similar to ours (even selected by the antropic principle).

• Additionally, for adherents to the Mathematical Universe Hypothesis [14] (or related ones, see [2]), a help to descend from the bird to the frog views. This was explained in [6] for the case of spacetimes, and is suggested now for gauge invariance: among the mathematical structures compatible with the fiber bundle structure $P(M, G_*)$, only those with a gauge symmetry can contain observers similar to ours.

7 Appendix: a sketch of the mathematical development for spacetime postulates

The mathematical translation of the second postulate (P2) in Section 2 is the following. Let $B_p = (\partial t_p, \partial x^1_p, \partial x^2_p, \partial x^3_p)$, $\tilde{B}_p = (\partial \tilde{t}_p, \partial \tilde{x}^1_p, \partial \tilde{x}^2_p, \partial \tilde{x}^3_p)$ be the bases of the tangent space obtained by two standard observers around $p$, with transition matrix:

$$A = \left( \begin{array}{cc} \frac{\partial \tilde{t}}{\partial \tilde{t}}|_p & \frac{\partial \tilde{x}^1}{\partial \tilde{t}}|_p \\ \frac{\partial \tilde{x}^1}{\partial \tilde{t}}|_p & \frac{\partial \tilde{x}^2}{\partial \tilde{t}}|_p \end{array} \right).$$

Rewriting $A$, equation (2.1) is:

$$A = \left( \begin{array}{cc} a_{00} & a_h \\ a_v & \hat{A} \end{array} \right) \quad \Rightarrow \quad A^{-1} = \left( \begin{array}{cc} a_{00} & \tilde{a}_h \\ a_v & \tilde{A} \end{array} \right), \quad (7.1)$$

where $a_{00} \in \mathbb{R}$, $\hat{A}$ is a submatrix $3 \times 3$ with transpose $\hat{A}^t$, and $a_h, a_v, \tilde{a}_h, \tilde{a}_v$ are four three-uples of real numbers —the Postulate of Time Orientation would yield $a_{00} > 0$.

The matrices satisfying (7.1) can be computed directly from the elementary algorithm to calculate the inverse matrix. In order to describe the results, define $O(k)(4, \mathbb{R})$, $k \in \mathbb{R}, k \neq 0$, as the group of the real matrices $4 \times 4$ which preserve the matrix

$$I^{|k|}_4 = \left( \begin{array}{cccc} k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

by congruence, that is:

$$A^t I^{|k|}_4 A = I^{|k|}_4 \quad (7.2)$$

or, equally, by taking inverses:

$$(A^t)^{-1} I^{|1/k|}_4 A^{-1} = I^{1/k}_4. \quad (7.3)$$

Definition (7.2) (resp. (7.3)) is extended naturally to the case $k = 0$ (resp $k = \omega$) just by assuming additionally $\det^2 A = 1$. Then, it is not difficult to check the existence of some $k(p)$ as described in the point (A) of Subsection 2.2. Moreover, it is easy to identify the four geometrical structures in $T_pM$ preserved by $O(k(p))(4, \mathbb{R})$. These structures are: (1) a Lorentzian scalar product if $k(p) \in (-\infty, 0)$, (2) a non-zero 1-form $\Omega_p$ and a Euclidean (positive definite) scalar product $h_p$ in the
kernel of $\Omega_p$ if $k(p) = \omega$, (3) a non-zero vector $Z_p$ and a (positive definite) scalar product $h_p^*$ in the kernel of $Z_p$ in dual space $T_p^*M$ if $k(p) \in 0$, and (4) an Euclidean scalar product if $k(p) \in (0, \infty)$. These structures, varying smoothly with $p$, yield the four structures 1–4 described in the point (B) of Subsection 2.2.

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