Efficient Estimation of Linear Functionals of Principal Components

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Abstract: We study principal component analysis (PCA) for mean zero i.i.d. Gaussian observations $X_1, \ldots, X_n$ in a separable Hilbert space $\mathbb{H}$ with unknown covariance operator $\Sigma$. The complexity of the problem is characterized by its effective rank $r(\Sigma) := \frac{\text{tr}(\Sigma)}{\|\Sigma\|}$, where $\text{tr}(\Sigma)$ denotes the trace of $\Sigma$ and $\|\Sigma\|$ denotes its operator norm. This framework includes, in particular, high-dimensional spiked covariance models as well as some models in functional PCA and kernel PCA in machine learning.

We develop a method of bias reduction in the problem of estimation of linear functionals of eigenvectors of $\Sigma$. Under the assumption that $r(\Sigma) = o(n)$, we establish the asymptotic normality and asymptotic properties of the risk of the resulting estimators and prove matching minimax lower bounds, showing their semi-parametric optimality.

Keywords and phrases: principal component analysis, spectral projections, asymptotic normality, semi-parametric efficiency.

1. Introduction

Principal Component Analysis (PCA) is commonly used as a dimension reduction technique for high-dimensional data sets. Assum ing a general framework where the data lies in a Hilbert space $\mathbb{H}$, PCA can be applied to a wide range of problems such as functional data analysis [24, 19] or machine learning [4]. The parametric setting has been well understood since the 1960's (e.g. [1] and [7]) and the asymptotic distribution of sample eigenvalues and sample eigenvectors is well known. For high-dimensional data, where the dimension $p = p(n) \rightarrow \infty$ with the sample size $n$, the spiked covariance model introduced by Johnstone in [14] has been the most common framework to study the asymptotic properties of principal components. In this model, it is assumed that the covariance matrix is given by a 'spike' and a noise part, that is

$$\Sigma = \sum_{j=1}^{l} s_j (\theta_j \otimes \theta_j) + \sigma^2 I_p,$$

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where $\sum_{j=1}^l s_i (\theta_i \otimes \theta_i) = \sigma^2 I_p$ is a low rank covariance matrix involving several orthonormal components (‘spikes’) $\theta_i$ and $\sigma^2 I_p$ is the covariance of the noise. Error bounds in this model, based on perturbation analysis, were studied in [20]. Moreover, if $\frac{p}{n} \rightarrow c \in (0, 1)$ the asymptotic distribution of sample eigenvectors was derived in [23] and in more general asymptotic regimes in [33]. Assuming sparsity of the eigenvectors (sparse PCA), inference is possible even when $\frac{p}{n} \rightarrow \infty$. This model has recently received a lot of attention, e.g. [6, 2, 31, 32, 9].

More recently, a so-called ‘effective rank’ setting for PCA has been considered, for example, in [15, 16, 17, 30, 25, 21]. In this dimension-free setting, it is assumed that the covariance $\Sigma$ is an operator acting in a Hilbert space $H$, no structural assumptions are made about $\Sigma$ and its ‘complexity’ is characterized by the effective rank $r(\Sigma) := \text{tr}(\Sigma)/\|\Sigma\|$, $\text{tr}(\Sigma)$ denoting the trace and $\|\Sigma\|$ denoting the operator (spectral) norm of $\Sigma$. In a series of papers [16, 15, 17, 18], Koltchinskii and Lounici derived sharp bounds on the spectral norm loss of estimation of $\Sigma$ by the sample covariance $\hat{\Sigma}$ that provide complete characterization of the size of $\|\hat{\Sigma} - \Sigma\|$ in terms of $\|\Sigma\|$ and $r(\Sigma)$, and obtained error bounds and limiting results for empirical spectral projection operators and eigenvectors of $\hat{\Sigma}$ under the assumption that $r(\Sigma) = o(n)$ as $n \rightarrow \infty$. In a recent paper [21], Naumov et. al. constructed bootstrap confidence sets for spectral projections in a lower dimensional regime where $r(\Sigma) = o(n^{1/3})$. In [25], Reiss and Wahl considered the reconstruction error for spectral projections.

In this paper, we further develop the results of [15] and [17] in the direction of semi-parametric statistics. In particular, we develop a bias reduction method in the problem of estimation of linear functionals of principal components (eigenvectors of $\Sigma$) and show asymptotic normality of the resulting de-biased estimators under the assumption that $r(\Sigma) = o(n)$ as $n \rightarrow \infty$. We prove a non-asymptotic risk lower bound that asymptotically exactly matches our upper bounds, thus establishing rigorously the semi-parametric optimality of our estimator in a general dimension-free setting (as long as $r(\Sigma) = o(n)$).

The problem of $\sqrt{n}$-consistent estimation of low-dimensional functionals of high-dimensional parameters has received increased attention in recent years, and in various models semi-parametric efficiency of regularisation-based estimators has been studied, see for instance [27, 12, 26, 22]. While formal calculations of the Fisher information in such models indicate optimality of these procedures, a rigorous interpretation of such efficiency claims requires some care: the standard asymptotic setting for semi-parametric efficiency [28] can not be straightforwardly applied because parameters in high-dimensional models are not fixed but vary with sample size $n$, so that establishing LAN expansions to apply Le Cam theory is not always possible or even desirable. In [12] some non-asymptotic techniques have been suggested under conditions that ensure asymptotic negligibility of the bias of candidate estimators. We take here a different approach, based on using van Trees’ inequality [10] to construct non-asymptotic lower bounds for the minimax risk in our estimation problem that match the upper bound exactly in the large sample limit.
2. Preliminaries

2.1. Some notations and conventions.

Let $\mathbb{H}$ be a separable Hilbert space. In what follows, $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{H}$ and also, with a little abuse of notation, the Hilbert–Schmidt inner product between Hilbert–Schmidt operators acting on $\mathbb{H}$. Similarly, the notation $\| \cdot \|$ is used both for the norm of vectors in $\mathbb{H}$ and for the operator (spectral) norm of bounded linear operators in $\mathbb{H}$. For a nuclear operator $A$, $\text{tr}(A)$ denotes its trace. We use the notation $\| \cdot \|_p$, $1 \leq p \leq \infty$ for the Schatten $p$-norms of operators in $\mathbb{H}$: $\|A\|_p := (\text{tr}(|A|^p))^{1/p}$, where $|A| = \sqrt{A^*A}$, $A^*$ being the adjoint operator of $A$. For $p = 1$, $\|A\|_1$ is the nuclear norm; for $p = 2$, $\|A\|_2$ is the Hilbert–Schmidt norm; for $p = \infty$, $\|A\|_\infty = \|A\|$ is the operator norm.

Given vectors $u, v \in \mathbb{H}$, $u \otimes v$ denotes the tensor product of $u$ and $v$:

$$(u \otimes v) : \mathbb{H} \mapsto \mathbb{H}, (u \otimes v)w := \langle v, w \rangle u.$$ 

Given bounded linear operators $A, B : \mathbb{H} \mapsto \mathbb{H}$, $A \otimes B$ denotes their tensor product:

$$(A \otimes B)(u \otimes v) = Au \otimes Bv, \ u, v \in \mathbb{H}.$$ 

Note that $A \otimes B$ can be extended (by linearity and continuity) to a bounded operator in the Hilbert space $\mathbb{H} \otimes \mathbb{H}$, which could be identified with the space of Hilbert–Schmidt operators in $\mathbb{H}$. It is easy to see that, for a Hilbert–Schmidt operator $C$, we have $(A \otimes B)C = ACB^*$ (in the finite-dimensional case, this defines the so called Kronecker product of matrices). On a couple of occasions, we might need to use the tensor product of Hilbert–Schmidt operators $A, B$, viewed as vectors in the space of Hilbert–Schmidt operators. For this tensor product, we use the notation $A \otimes_u B$.

Throughout the paper, the following notations will be used: for nonnegative $a, b$, $a \lesssim b$ means that there exists a numerical constant $c > 0$ such that $a \leq cb$; $a \gtrsim b$ is equivalent to $b \lesssim a$; finally, $a \asymp b$ is equivalent to $a \lesssim b$ and $b \lesssim a$. Sometimes, constant $c$ in the above relationships could depend on some parameter $\gamma$. In this case, we provide signs $\lesssim, \gtrsim$ and $\asymp$ with subscript $\gamma$. For instance, $a \lesssim_\gamma b$ means that there exists a constant $c_\gamma > 0$ such that $a \leq c_\gamma b$.

In many places in the proofs, we use exponential bounds for some random variables, say, $\xi$ of the following form: for all $t \geq 1$ with probability at least $1 - e^{-t}$, $\xi \leq Ct$. In some cases, it would follow from our arguments that the inequality holds with a slightly different probability, say, at least $1 - 3e^{-t}$. In such cases, it is easy to rewrite the bound again as $1 - e^{-t}$ by adjusting the value of constant $C$. Indeed, for $t \geq 1$ with probability at least $1 - e^{-t} = 1 - 3e^{-t - \log(3)}$, we have $\xi \leq C(t + \log(3)) \leq 2\log(3)Ct$. We will use such an adjustment of the constants in many proofs, often, without further notice.

2.2. Bounds on sample covariance.

Let $X$ be a Gaussian vector in $\mathbb{H}$ with mean $\mathbb{E}X = 0$ and covariance operator $\Sigma := \mathbb{E}(X \otimes X)$. Given i.i.d. observations $X_1, \ldots, X_n$ of $X$, let $\hat{\Sigma} = \hat{\Sigma}_n$ be the
sample (empirical) covariance operator defined as follows:

\[ \hat{\Sigma} := n^{-1} \sum_{j=1}^{n} X_j \otimes X_j. \]

**Definition 2.1.** The effective rank of the covariance operator \( \Sigma \) is defined as

\[ r(\Sigma) := \frac{\operatorname{tr}(\Sigma)}{\|\Sigma\|}. \]

The role of the effective rank as a complexity parameter in covariance estimation is clear from the following result proved in [16].

**Theorem 2.1.** Let \( X \) be a mean zero Gaussian random vector in \( \mathbb{H} \) with covariance operator \( \Sigma \) and let \( \hat{\Sigma} \) be the sample covariance based on i.i.d. observations \( X_1, \ldots, X_n \) of \( X \). Then

\[ \mathbb{E} \| \hat{\Sigma} - \Sigma \| \approx \| \Sigma \| \left( \sqrt{\frac{r(\Sigma)}{n}} \vee \frac{r(\Sigma)}{n} \right). \]  

This result shows that the size of the properly rescaled operator norm deviation of \( \hat{\Sigma} \) from \( \Sigma \), \( \mathbb{E} \| \hat{\Sigma} - \Sigma \| \| \Sigma \| \), is characterized up to numerical constants by the ratio \( \frac{r(\Sigma)}{n} \). In particular, the condition \( r(\Sigma) = o(n) \) is necessary and sufficient for operator norm consistency of \( \hat{\Sigma} \) as an estimator of \( \Sigma \). In addition to this, the following concentration inequality for \( \| \hat{\Sigma} - \Sigma \| \) around its expectation was also proved in [16].

**Theorem 2.2.** Under the conditions of the previous theorem, for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[ \| \hat{\Sigma} - \Sigma \| - \mathbb{E} \| \hat{\Sigma} - \Sigma \| \leq \| \Sigma \| \left( \sqrt{\frac{r(\Sigma)}{n}} \vee \left( \sqrt{\frac{r(\Sigma)}{n}} + \frac{t}{n} \right) \right). \]  

It immediately follows from the bounds (2.1) and (2.2) that, for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[ \| \hat{\Sigma} - \Sigma \| \leq \| \Sigma \| \left( \sqrt{\frac{r(\Sigma)}{n}} \vee \sqrt{\frac{r(\Sigma)}{n}} \vee \frac{t}{n} \right) \]  

and, for all \( p \in [1, \infty) \),

\[ \mathbb{E}^{1/p} \| \hat{\Sigma} - \Sigma \| \leq \| \Sigma \| \left( \sqrt{\frac{r(\Sigma)}{n}} \vee \sqrt{\frac{r(\Sigma)}{n}} \right). \]

**2.3. Perturbation theory and empirical spectral projections.**

The covariance operator \( \Sigma \) is self-adjoint, positively semidefinite and nuclear. It has spectral decomposition

\[ \Sigma = \sum_{r \geq 1} \mu_r P_r, \]
where \( \mu_r \) are distinct strictly positive eigenvalues of \( \Sigma \) arranged in decreasing order and \( P_r \) are the corresponding spectral projection operators. For \( r \geq 1 \), \( P_r \) is an orthogonal projection on the eigenspace of the eigenvalue \( \mu_r \). The dimension of this eigenspace is finite and will be denoted by \( n_r \). The eigenspaces corresponding to different eigenvalues \( \mu_r \) are mutually orthogonal. Denote by \( \sigma(\Sigma) \) the spectrum of operator \( \Sigma \) and let \( \lambda_j = \lambda_j(\Sigma), j \geq 1 \) be the eigenvalues of \( \Sigma \) arranged in a non-increasing order and repeated with their multiplicities. Denote \( \Delta_r := \{ j : \lambda_j = \mu_r \}, r \geq 1 \). Then \( \text{card}(\Delta_r) = m_r \). The \( r \)-th spectral gap is defined as

\[
g_r = g_r(\Sigma) := \text{dist}(\mu_r; \sigma(\Sigma) \setminus \{ \mu_r \}).
\]

Let \( \bar{g}_r = \bar{g}_r(\Sigma) := \min_{1 \leq s \leq r} g_s \).

We turn now to the definition of empirical spectral projections of sample covariance \( \hat{\Sigma} \) that could be viewed as estimators of the true spectral projections \( P_r, r \geq 1 \). In [15], the following definition was used: let \( \hat{P}_r \) be the orthogonal projection on the direct sum of eigenspaces of \( \hat{\Sigma} \) corresponding to its eigenvalues \( \{ \lambda_j(\hat{\Sigma}) : j \in \Delta_r \} \). This is not a perfect definition of a statistical estimator since the set \( \Delta_r \) is unknown and it has to be recovered from the spectrum \( \sigma(\hat{\Sigma}) \) of \( \hat{\Sigma} \).

When \( \hat{\Sigma} \) is close to \( \Sigma \) in the operator norm, the spectrum \( \sigma(\hat{\Sigma}) \) of \( \hat{\Sigma} \) is a small perturbation of the spectrum \( \sigma(\Sigma) \) of \( \Sigma \). This could be quantified by the following inequality that goes back to H. Weyl:

\[
\sup_{j \geq 1} |\lambda_j(\hat{\Sigma}) - \lambda_j(\Sigma)| \leq \|\hat{\Sigma} - \Sigma\|.
\]  

(2.5)

It easily follows from this inequality that, if \( \|\hat{\Sigma} - \Sigma\| \) is sufficiently small, then the eigenvalues \( \lambda_j(\hat{\Sigma}) \) of \( \hat{\Sigma} \) form well separated clusters around the eigenvalues \( \mu_1, \mu_2, \ldots \) of \( \Sigma \). To make the last claim more precise, consider a finite or countable bounded set \( A \subset \mathbb{R}_+ \) such that \( 0 \in A \) and 0 is the only limit point (if any) of \( A \). Given \( \delta > 0 \), define \( \lambda_{\delta}(A) := \max \{ \lambda \in A : (\lambda - \delta, \lambda) \cap A = \emptyset \} \) and let \( T_{\delta}(A) := A \setminus [0, \lambda_{\delta}(A)] \). The set \( T_\delta(A) \) will be called the top \( \delta \)-cluster of \( A \). Let \( A_1 := T_\delta(A), A_2 := T_\delta(A \setminus A_1), A_3 := T_\delta(A \setminus (A_1 \cup A_2)), \ldots \) and \( \nu = \nu_\delta := \min \{ j : A_{j+1} = \emptyset \} \). Obviously, \( \nu < \infty \). We will call the sets \( A_1, \ldots, A_\nu \) the \( \delta \)-clusters of \( A \). They provide a partition of \( A \) into sets separated by the gaps of length at least \( \delta \) and such that the gaps between the points inside each of the clusters are smaller than \( \delta \).

The next lemma easily follows from inequality (2.5).

**Lemma 2.1.** Let \( \delta > 0 \) be such that, for some \( r \geq 1 \),

\[
\|\hat{\Sigma} - \Sigma\| < \delta/2 \quad \text{and} \quad \delta < \bar{g}_r/2.
\]

Let \( \hat{A}_1^\delta, \ldots, \hat{A}_\nu^\delta \) be the \( \delta \)-clusters of the set \( \sigma(\hat{\Sigma}) \). Then \( \nu \geq r \) and, for all \( 1 \leq s \leq r \)

\[
\hat{A}_s^\delta \subset (\mu_s - \delta/2, \mu_s + \delta/2) \quad \text{and} \quad \{ j : \lambda_j(\hat{\Sigma}) \in \hat{A}_s^\delta \} = \Delta_s.
\]

Given \( \delta > 0 \) and \( \delta \)-clusters \( \hat{A}_1^\delta, \ldots, \hat{A}_\nu^\delta \) of \( \sigma(\hat{\Sigma}) \), define, for \( 1 \leq s \leq \nu \), the empirical spectral projection \( \hat{P}_s^\delta \) as the orthogonal projection on the direct sum
of eigenspaces of $\hat{\Sigma}$ corresponding to its eigenvalues from the cluster $A_\delta^s$. It immediately follows from Lemma 2.1 that, under its assumptions on $\delta$, $\hat{P}_s^\delta = \hat{P}_s$, $s = 1, \ldots, r$.

In the following sections, we will be interested in the problem of estimation of spectral projections in the case when the true covariance $\Sigma$ belongs to certain subsets of the following class of covariance operators:

$$S^{(r)}(\tau; a) := \left\{ \Sigma : \|\hat{\Sigma}\| \leq \tau, \|\Sigma\|_{\hat{g}_r(\Sigma)} \leq a \right\},$$

where $a > 1$, $r > 1$. We will allow the effective rank to be large, $r = r_n \to \infty$, but not too large such that $r_n = o(n)$ as $n \to \infty$. For $\Sigma \in S^{(r)}(\tau; a)$, we take $\delta := \tau \|\hat{\Sigma}\|$ for a sufficiently small value of the constant $\tau > 0$ in the definition of spectral projections $\hat{P}_s^\delta$.

The following lemma is an easy consequence of the exponential bound (2.3).

**Lemma 2.2.** Suppose $a > 1$ and $r_n = o(n)$ as $n \to \infty$. Take $\tau \in \left(0, \frac{4}{a^2} \wedge 2\right)$ and $\delta := \tau \|\hat{\Sigma}\|$. Then, there exists a numerical constant $\beta > 0$ such that, for all large enough $n$,

$$\sup_{\Sigma \in S^{(r)}(\tau; a)} P_\Sigma \left\{ \exists s = 1, \ldots, r : \hat{P}_s^\delta \neq \hat{P}_s \right\} \leq e^{-\beta \tau^2 n}.$$

**Proof.** By (2.3) with $t := \beta \tau^2 n$, we obtain that

$$\sup_{\Sigma \in S^{(r)}(\tau; a)} P_\Sigma \left\{ \|\hat{\Sigma} - \Sigma\| \geq C\|\Sigma\| \left( \sqrt{\frac{\tau}{n}} \vee \sqrt{\frac{\beta \tau^2 n}{n}} \right) \right\} \leq e^{-\beta \tau^2 n},$$

where $C > 0$ is a numerical constant. Take $\beta = \frac{1}{16a^2}$ and note that, for all large enough $n$, $C \sqrt{\frac{\tau}{n}} \leq \tau/4$ to obtain that

$$\sup_{\Sigma \in S^{(r)}(\tau; a)} P_\Sigma \left\{ \|\hat{\Sigma} - \Sigma\| \geq (\tau/4)\|\Sigma\| \right\} \leq e^{-\beta \tau^2 n},$$

Since $\tau/4 \leq 1/2$, we easily obtain that, for all $\Sigma \in S^{(r)}(\tau; a)$ and for all $n$ large enough with probability at least $1 - e^{-\beta \tau^2 n}$, $(1/2)\|\Sigma\| \leq \|\hat{\Sigma}\| \leq 2\|\Sigma\|$. This implies that with the same probability (and on the same event)

$$\|\hat{\Sigma} - \Sigma\| < (\tau/4)\|\Sigma\| \leq (\tau/2)\|\hat{\Sigma}\| = \delta/2.$$

On the other hand, for all $\Sigma \in S^{(r)}(\tau; a)$,

$$\delta = \tau \|\hat{\Sigma}\| \leq 2\tau \|\Sigma\| < \frac{1}{2a} \|\Sigma\| \leq \frac{\hat{g}_r(\Sigma)}{2}.$$

It remains to use Lemma 2.1 to complete the proof.

In the proofs of the main results of the paper, we deal for the most part with spectral projections $\hat{P}_r$ that were studied in detail in [15]. We use Lemma 2.2 to reduce the results for $\hat{P}_r^\delta$ to the results for $\hat{P}_r$. 


3. Main Results

Our main goal is to develop an efficient estimator of the linear functional \( \langle \theta_r, u \rangle \), where \( u \in \mathbb{H} \) is a given vector and \( \theta_r = \theta_r(\Sigma) \) is a unit eigenvector of the unknown covariance operator \( \Sigma \) corresponding to its \( r \)-th eigenvalue \( \mu_r \), which is assumed to be simple (that is, of multiplicity \( m_r = 1 \)). The corresponding spectral projection \( P_r \) is one-dimensional: \( P_r = \theta_r \otimes \theta_r \). A “naive” plug-in estimator of \( P_r \) is the empirical spectral projection \( \hat{P}_r^\delta \) with \( \delta = \tau \| \Sigma \| \) for a suitable choice of a small constant \( \tau \), as described in Lemma 2.2. According to this lemma and under its assumptions, \( \hat{P}_r^\delta \) coincides with a high probability with the one-dimensional empirical spectral projection \( \hat{P}_r := \hat{\theta}_r \otimes \hat{\theta}_r \), where \( \hat{\theta}_r \) is the corresponding unit eigenvector of \( \hat{\Sigma} \). As an estimator of \( \theta_r \), we can use an arbitrary unit vector \( \hat{\theta}_r^\delta \) from the eigenspace \( \text{Im}(\hat{P}_r^\delta) \), which with a high-probability coincides with \( \pm \hat{\theta}_r \) (under conditions of Lemma 2.2). In case \( r = 1 \), when the top eigenvalue \( \mu_1 = \| \Sigma \| \) of \( \Sigma \) is simple and the goal is to estimate a linear functional of the top principal component \( \theta_1 \), there is no need to use \( \delta \)-clusters to define an estimator of \( \theta_1 \) since \( \hat{\theta}_1 \) (a unit eigenvector in the eigenspace of the top eigenvalue \( \| \Sigma \| \) of \( \Sigma \)) is already a legitimate estimator.

Note that both \( \theta_r \) and \( -\theta_r \) are unit eigenvectors of \( \Sigma \), so, strictly speaking, \( \langle \theta_r, u \rangle \) can be estimated only up to its sign. In what follows, we assume that \( \hat{\theta}_r^\delta \) and \( \hat{\theta}_r \) (or, whenever is needed, \( \hat{\theta}_r \) and \( \theta_r \)) are properly aligned in the sense that \( \langle \hat{\theta}_r^\delta, \theta_r \rangle \geq 0 \) (which is always the case either for \( \theta_r \) or for \( -\theta_r \)). This allows us to view \( \langle \hat{\theta}_r^\delta, u \rangle \) as an estimator of \( \langle \theta_r, u \rangle \).

It was shown in [15] that “naive” plug-in estimators of the functional \( \langle \theta_r, u \rangle \), such as \( \langle \hat{\theta}_r^\delta, u \rangle \) or \( \langle \hat{\theta}_r, u \rangle \), are biased with the bias becoming substantial enough to affect the efficiency of the estimator or even its convergence rates as soon as the effective rank is large enough, namely, \( r(\Sigma) \gtrsim n^{1/2} \). Moreover, it was shown that the quantity

\[
b_r = b_r(\Sigma) := \mathbb{E}_{\Sigma} \langle \hat{\theta}_r, \theta_r \rangle^2 - 1 \in [-1, 0]
\]

plays the role of a bias parameter. In particular, the results of [15] imply that the random variable \( \langle \hat{\theta}_r, u \rangle \) concentrates around \( \sqrt{1 + b_r} \langle \theta_r, u \rangle \) (rather than around \( \langle \theta_r, u \rangle \)) with the size of the deviations of order \( O(n^{-1/2}) \) provided that \( r(\Sigma) = o(n) \) as \( n \to \infty \). Thus, the bias of \( \langle \theta_r, u \rangle \) as an estimator of \( \langle \theta_r, u \rangle \) is of the order \( (\sqrt{1 + b_r} - 1)\langle \theta_r, u \rangle \approx b_r \langle \theta_r, u \rangle \). It was shown in [15] that \( |b_r| \lesssim \frac{r(\Sigma)}{n} \) and it will be proven below in this paper that, in fact, \( |b_r| \lesssim \frac{r(\Sigma)}{n} \) (see Lemma 4.9 and bounds (4.32), (4.33)). This fact implies that, indeed, the bias of \( \langle \theta_r, u \rangle \) (and of \( \langle \hat{\theta}_r^\delta, u \rangle \)) is not negligible and affects the convergence rate as soon as \( \frac{r(\Sigma)}{n^{1/2}} \to \infty \). This resembles the situation in sparse regression (see e.g. [13, 27, 34]). If \( p \) denotes the dimension of the model and \( s \) its sparsity and if \( s \log(p) = o(n^{1/2}) \), the bias of a de-sparsified LASSO estimator for the regressor \( \beta \) is negligible, which makes it possible to prove asymptotic normality of linear forms of \( \beta \). On the other hand, if \( s \log(p) \gg n^{1/2} \), Cai and Guo [5] proved that adaptive confidence sets for linear forms do not exist. This implies that any attempt...
to further de-bias the de-sparsified LASSO or any other estimator to prove asymptotic normality is deemed to fail. Contrary to this, in our case estimation of the bias parameter $b_r$ is possible (as will be shown below).

We will state a uniform (and somewhat stronger) version of some of the results of [15] on asymptotic normality of linear forms

$$\sqrt{n}(\langle \hat{\theta}^0_r, u \rangle - \sqrt{1 + b_r}(\langle \theta_r, \Sigma \rangle, u)), u \in \mathbb{H}$$

under the assumption that $r(\Sigma) = o(n)$. To this end, define the following operator

$$C_r := \sum_{s \neq r} \frac{1}{\mu_r - \mu_s} P_s,$$

which is bounded with $\|C_r\| = \frac{1}{g_r}$. Denote

$$\sigma^2_r(\Sigma; u) := \langle \Sigma \theta_r, \theta_r \rangle \langle \Sigma C_r u, C_r u \rangle = \mu_r \langle \Sigma C_r u, C_r u \rangle.$$ 

Clearly,

$$\sigma^2_r(\Sigma; u) \leq \frac{\|\Sigma\|^2}{g_r^2} \|u\|^2. \quad (3.1)$$

Note that, if $\mathbb{H}$ is finite-dimensional (with a fixed dimension) and $\Sigma$ is non-singular, then the Fisher information for the model $X \sim N(0; \Sigma)$ is $I(\Sigma) = \frac{1}{2}(\Sigma^{-1} \otimes \Sigma^{-1})$ (see, e.g., [8]). The maximum likelihood estimator $\hat{\Sigma}$ based on $n$ i.i.d. observations of $X$ (the sample covariance) is then asymptotically normal with $\sqrt{n}$-rate and limit covariance $I(\Sigma) - \frac{1}{2}(\Sigma \otimes \Sigma)$.

An application of the Delta Method to the smooth function $g(\Sigma) := \langle \theta_r, \Sigma \rangle, u \rangle$ shows that $g(\hat{\Sigma})$ is also asymptotically normal with limiting variance $\langle (I(\Sigma)^{-1} g'(\Sigma), g'(\Sigma) \rangle$, which turns out to be equal to $\sigma^2_r(\Sigma; u)$.

For $u \in \mathbb{H}$, $r > 1$, $a > 1$ and $\sigma_0 > 0$, consider the following class of covariance operators in $\mathbb{H}$:

$$S^{(r)}(\mathbb{r}, a, \sigma_0, u) := \left\{ \Sigma : r(\Sigma) \leq r, \frac{\|\Sigma\|}{g_r(\Sigma)} \leq a, \sigma^2_r(\Sigma; u) \geq \sigma^2_0 \right\}.$$ 

We emphasize here that we regard $a$ and $\sigma_0$ as fixed constants, but $r, \|\Sigma\|$ and $\hat{g}_r$ may all possibly depend on $n$. For example, this allows that $\|\Sigma\| \to \infty$ as long as $\hat{g}_r \to \infty$ at the same rate as it is the case in factor models as considered in [33]. Note that some additional conditions on $r, a, \sigma_0, u$ are needed for the class $S^{(r)}(\mathbb{r}, a, \sigma_0, u)$ to be nonempty. Say, bound (3.1) implies that it is necessary for this that $\sigma^2_0 \leq a^2\|u\|^2$. It is also obvious that there should be $a > r$ (since $\|\Sigma\| \geq r g_r(\Sigma)$).

We will also need the following assumption on the loss function $\ell$.

Assumption 3.1. Let $\ell : \mathbb{R} \to \mathbb{R}_+$ be a loss function satisfying the following conditions: $\ell(0) = 0$, $\ell(u) = \ell(-u)$, $u \in \mathbb{R}$, $\ell$ is nondecreasing and convex on $\mathbb{R}_+$ and, for some constants $c_1, c_2 > 0$

$$\ell(u) \leq c_1 e^{c_2 u}, u \geq 0.$$
The proofs to all our theorems are in fact non-asymptotic and often can be expressed with Berry-Essen type bounds. However, for a more concise presentation we present the asymptotic statements.

In what follows, \( Z \) denotes a standard Gaussian random variable and \( \Phi \) denotes its distribution function.

**Theorem 3.1.** Let \( u \in \mathbb{H}, a > 1 \) and \( \sigma_0 > 0 \). Suppose that \( r_n > 1 \) and \( \tau_n = o(n) \) as \( n \to \infty \). Let \( \delta = \tau \| \hat{\Sigma} \| \) for some \( \tau \in \left( 0, \frac{1}{\sqrt{n}} \wedge 2 \right) \). Then

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \sup_{x \in \mathbb{R}} \mathbb{P}_{\Sigma} \left\{ \frac{\sqrt{n}(\hat{r}_n, u) - \sqrt{1 + b_r(\Sigma)(\theta_r(\Sigma), u))}}{\sigma_r(\Sigma; u)} \leq x \right\} - \Phi(x) \to 0 \text{ as } n \to \infty.
\]

Moreover, under Assumption 3.1,

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \mathbb{E}_{\Sigma} \left\{ \frac{\sqrt{n}(\hat{r}_n, u) - \sqrt{1 + b_r(\Sigma)(\theta_r(\Sigma), u))}}{\sigma_r(\Sigma; u)} - \mathbb{E}_r(Z) \right\} \to 0 \text{ as } n \to \infty.
\]

The proof of this theorem will be given in Section 4 that also includes a number of auxiliary statements used in the proofs of our main results on efficient estimation of linear functionals.

**Corollary 3.1.** Let \( u \in \mathbb{H}, a > 1 \) and \( \sigma_0 > 0 \). Suppose that \( r_n > 1 \) and \( \tau_n = o(\sqrt{n}) \) as \( n \to \infty \). Let \( \delta = \tau \| \hat{\Sigma} \| \) for some \( \tau \in \left( 0, \frac{1}{\sqrt{n}} \wedge 2 \right) \). Then

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \sup_{x \in \mathbb{R}} \mathbb{P}_{\Sigma} \left\{ \frac{\sqrt{n}(\hat{r}_n, u) - \theta_r(\Sigma, u))}{\sigma_r(\Sigma; u)} \leq x \right\} - \Phi(x) \to 0 \text{ as } n \to \infty.
\]

Moreover, under Assumption 3.1,

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \mathbb{E}_{\Sigma} \left\{ \frac{\sqrt{n}(\hat{r}_n, u) - \theta_r(\Sigma, u))}{\sigma_r(\Sigma; u)} - \mathbb{E}_r(Z) \right\} \to 0 \text{ as } n \to \infty.
\]

Our next goal is to provide a minimax lower bound on the risk of an arbitrary estimator of the linear functional \( \theta_r(\Sigma, u) \) in the case of quadratic loss \( \ell(t) = t^2, t \in \mathbb{R} \). The proof is based on van Trees' inequality and will be given in Section 7.

Let

\(\hat{S}^{(r)}(r, a, \sigma_0, u) := \left\{ \Sigma : r(\Sigma) < r, \frac{\| \Sigma \|_{\bar{g}_r(\Sigma)}}{\bar{g}_r(\Sigma)} < a, \sigma^2(\Sigma; u) > \sigma_0^2 \right\} \), \( r > 1, a > 1, \sigma_0 > 0 \).

**Theorem 3.2.** Let \( r > 1, a > 1 \) and \( \sigma_0 > 0 \). Suppose \( \hat{S}^{(r)}(r, a, \sigma_0, u) \neq \emptyset \). Then, for all statistics \( T_n(X_1, \ldots, X_n) \),

\[
\liminf_{n \to \infty} \inf_{\Sigma \in \hat{S}^{(r)}(r, a, \sigma_0, u)} \sup_n \frac{n\mathbb{E}_{\Sigma}(T_n(X_1, \ldots, X_n) - \theta_r(\Sigma, u))^2}{\sigma^2_r(\Sigma; u)} \geq 1.
\]

Moreover, for any \( \Sigma_0 \in \hat{S}^{(r)}(r, a, \sigma_0, u) \)

\[
\liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \inf_{\Sigma \in \hat{S}^{(r)}(r, a, \sigma_0, u), \| \Sigma - \Sigma_0 \|_1 \leq \varepsilon} \sup_n \frac{n\mathbb{E}_{\Sigma}(T_n(X_1, \ldots, X_n) - \theta_r(\Sigma, u))^2}{\sigma^2_r(\Sigma; u)} \geq 1.
\]
It follows from Corollary 3.1 and Theorem 3.2 that the estimator $(\hat{\theta}_r^\delta, u)$ is efficient in a semi-parametric sense for quadratic loss under the assumption that $r_n = o(n^{1/2})$. It turns out, however, that if $\frac{\delta}{n}$ converges to infinity, then not only the efficiency, but even the $\sqrt{n}$-convergence rate of this estimator fails in the class of covariance operators $\mathcal{S}(\tau)(r_n, a, \sigma_0, u)$.

Proposition 3.1. Let $a > r$ and let $\sigma_0^2$ be sufficiently small, say,

$$\sigma_0^2 \leq \frac{1}{2} \left[ \frac{a^2}{(r-1)^2} - \frac{a}{r-1} \right].$$

Let $\tau_n = o(n)$ and $\frac{n}{\sqrt{n}} \rightarrow \infty$ as $n \rightarrow \infty$. Then, for some constant $c = c(r; a; \sigma_0) > 0$

$$\lim_{n \rightarrow \infty} \sup_{\Sigma \in \mathcal{S}(\tau)(\tau_n, a, \sigma_0, u)} \mathbb{P}_\Sigma \left\{ \left| \langle \hat{\theta}_r^\delta, u \rangle - \langle \theta_r(\Sigma), u \rangle \right| \geq c \|u\| \frac{\tau_n}{n} \right\} = 1.$$

The reason for the loss of the $\sqrt{n}$-convergence rate of plug-in estimators of linear functionals of principal components is their large bias in the case when the complexity of the problem is even moderately high (that is, $\frac{\delta}{n} \rightarrow \infty$). In [15], a method of bias reduction in this problem was suggested that led to $\sqrt{n}$-consistent estimation of linear functionals. This approach is based on concentration properties of linear functionals $(\hat{\theta}_r^\delta, u)$. To describe it, it is of importance to emphasize the dependence of the bias parameter $b_r$ on the sample size. To this end, we will write $b_r = b_r(S) = b_r(n)(\Sigma)$. The idea is to split the sample into two equal parts and to construct an estimator of the bias parameter that can be used to de-bias plug-in estimators. Assume, for simplicity, that $n$ is even and let $n' := n/2$. The sample $X_1, \ldots, X_n$ is divided into two parts, $X_1, \ldots, X_{n'}$ and $X_{n'+1}, \ldots, X_n$, of size $n'$ each. Let $\hat{\Sigma}^{(1)}, \hat{\Sigma}^{(2)}$ be the sample covariance operators based on these two samples and denote by $\hat{\theta}_r^{\delta,1}, \hat{\theta}_r^{\delta,2}$ the corresponding empirical eigenvectors (estimators of $\theta_r$). Since for any $u \in \mathbb{H}$, $(\hat{\theta}_r^{\delta,1}, u)$ and $(\hat{\theta}_r^{\delta,2}, u)$ concentrate around $\sqrt{1 + b_r^{(n')}(\Sigma)} \langle \theta_r(S), u \rangle$ and the random vectors $\hat{\theta}_r^{\delta,1}, \hat{\theta}_r^{\delta,2}$ are independent, it is natural to expect that $(\hat{\theta}_r^{\delta,1}, \hat{\theta}_r^{\delta,2})$ concentrates around

$$\langle \sqrt{1 + b_r^{(n')}(\Sigma)} \theta_r(S), \sqrt{1 + b_r^{(n')}(\Sigma)} \theta_r(S) \rangle = 1 + b_r^{(n')}(\Sigma)$$

and to use $\hat{b}_r^{(n')} := \langle \hat{\theta}_r^{\delta,1}, \hat{\theta}_r^{\delta,2} \rangle - 1$ as an estimator of $b_r^{(n')}(\Sigma)$. It was proved in [15] that, under the assumption $r(S) = o(n)$, the error $\hat{b}_r^{(n')}(\Sigma) - b_r^{(n')}(\Sigma) = o(n^{-1/2})$ which allows one to define a new estimator of the linear functional $\langle \theta_r(S), u \rangle$ with reduced bias as $\frac{1}{\sqrt{1 + \hat{b}_r^{(n')}}} \hat{\theta}_r^{(1)} + \hat{\theta}_r^{(2)}$. It was shown in [15] that this estimator is $\sqrt{n}$-consistent and asymptotically normal. It is, however, not efficient: due to a very straightforward sample split, the limiting variance of this estimator is twice as large as the optimal variance.

We describe below a more subtle construction that yields an asymptotically normal estimator of $(\theta_r(S), u)$ with optimal variance in the class of covariance
operators \(S^{(r)}(\tau_n, a, \sigma_0, u)\) with \(\tau_n = o(n)\). The idea is to use only a small portion of the data (of size \(o(n)\)) to estimate the bias parameters and to use most of the data for the estimator of the target eigenvector. The main difficulty is that the bias parameters themselves depend on the sample size.

Let \(m = m_n = o(n)\) as \(n \to \infty\). Assuming that \(m < n/3\), we split the sample \(X_1, \ldots, X_n\) into three disjoint subsamples, one of size \(n' = n'_n := n - 2m > n/3\) and two others of size \(m\) each. Clearly, \(n' = (1 + o(1))n\) as \(n \to \infty\). Denote by \(\Sigma^{(1)}, \Sigma^{(2)}, \Sigma^{(3)}\) the sample covariances based on these three subsamples and let \(\hat{\theta}_j, j = 1, 2, 3\) be the corresponding empirical eigenvectors with parameters \(\delta_j = r\|\Sigma^{(j)}\|\) for a proper choice of \(r\) (see Lemma 2.2). Let

\[
\hat{d}_r := \frac{\langle \hat{\theta}_{r,1}, \hat{\theta}_{r,2} \rangle}{\langle \hat{\theta}_{r,2}, \hat{\theta}_{r,3} \rangle^{1/2}}
\]

and

\[
\hat{\theta}_r := \frac{\hat{\theta}_{r,1}}{\hat{d}_r \vee (1/2)}.
\]

Our main goal is to prove the following result showing the efficiency of the estimator \(\langle \hat{\theta}_r, u \rangle\) of the linear functional \(\langle \theta_r(\Sigma), u \rangle\). Its proof will be given in Section 5.

**Theorem 3.3.** Let \(u \in \mathbb{H}, a > 1\) and \(\sigma_0 > 0\). Suppose that \(\tau_n > 1\) and \(\tau_n = o(n)\) as \(n \to \infty\). Take \(m = m_n\) such that \(m_n = o(n)\) and \(n\tau_n = o(m_n^2)\) as \(n \to \infty\).

Then

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \sup_{x \in \mathbb{R}} \left| \mathbb{P}_{\Sigma} \left\{ \sqrt{n} \left( \frac{\langle \hat{\theta}_r, u \rangle - \langle \theta_r(\Sigma), u \rangle}{\sigma_r(\Sigma; u)} \right) \leq x \right\} - \Phi(x) \right| \to 0 \text{ as } n \to \infty.
\]

Moreover, under Assumption 3.1 on the loss \(\ell\),

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \left| \mathbb{E}_{\Sigma} \ell \left( \frac{\langle \sqrt{n} \frac{\langle \hat{\theta}_r, u \rangle - \langle \theta_r(\Sigma), u \rangle}{\sigma_r(\Sigma; u)} \rangle}{\langle \sigma_r(\Sigma; u) \rangle} - \mathbb{E}(\ell) \right) \right| \to 0 \text{ as } n \to \infty.
\]

Finally, we show that \(\sigma_r(\Sigma; u)\) can be consistently estimated by \(\sigma_r(\hat{\Sigma}; u)\), which allows us to replace the standard deviation \(\sigma_r(\Sigma; u)\) in the normal approximation (3.2) by its empirical version. This yields the following result that can be used for constructing \(\ell_\infty\)-type confidence sets for \(\theta_r\) and for hypotheses testing of linear functionals of \(\theta_r\). See Section 6 for its proof.

**Corollary 3.2.** Under the conditions of Theorem 3.3,

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \sup_{x \in \mathbb{R}} \left| \mathbb{P}_{\Sigma} \left\{ \sqrt{n} \left( \frac{\langle \hat{\theta}_r, u \rangle - \langle \theta_r(\Sigma), u \rangle}{\sigma_r(\Sigma; u)} \right) \leq x \right\} - \Phi(x) \right| \to 0 \text{ as } n \to \infty.
\]
4. Proof of Theorem 3.1

We will prove the result for empirical eigenvectors \( \hat{\theta}_r \) rather than for \( \hat{\theta}^\delta_r \). The reduction to this case is based on Lemma 2.2 which immediately implies that

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \mathbb{P}_\Sigma \{ \hat{\theta}^\delta_r \neq \hat{\theta}_r \} \leq e^{-\beta \tau^2 n}.
\]

Therefore, denoting

\[
\xi_n(\Sigma) := \frac{\sqrt{n}(\hat{\theta}^\delta_r, u) - \sqrt{1 + b_r(\Sigma)}(\theta_r(\Sigma), u)}{\sigma_r(\Sigma; u)}
\]

and

\[
\eta_n(\Sigma) := \frac{\sqrt{n}(\hat{\theta}_r, u) - \sqrt{1 + b_r(\Sigma)}(\theta_r(\Sigma), u)}{\sigma_r(\Sigma; u)}
\]

we obtain

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} \sup_{x \in \mathbb{R}} |\mathbb{P}_{\Sigma}(\xi_n(\Sigma) \leq x) - \mathbb{P}_{\Sigma}(\eta_n(\Sigma) \leq x)| \leq e^{-\beta \tau^2 n} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Also, since \( \xi_n(\Sigma) \leq \frac{2\sqrt{n}\|u\|}{\sigma_r(\Sigma; u)} \) and \( \eta_n(\Sigma) \leq \frac{2\sqrt{n}\|u\|}{\sigma_r(\Sigma; u)} \), we obtain that

\[
\sup_{\Sigma \in S^{(r)}(\tau_n, a, \sigma_0, u)} |\mathbb{E}_\Sigma\ell(\xi_n(\Sigma)) - \mathbb{E}_\Sigma\ell(\eta_n(\Sigma))| \leq 2\ell\left(\frac{2\sqrt{n}\|u\|}{\sigma_0}\right)e^{-\beta \tau^2 n} \rightarrow 0,
\]

under Assumption 3.1.

We will prove more explicit bounds for the estimator \( \hat{\theta}_r \) stated below in Lemma 4.8 that immediately implies the result.

Our starting point is the first order perturbation expansion of the empirical spectral projection operator \( \hat{P}_r \):

\[
\hat{P}_r = P_r + L_r(E) + S_r(E)
\]

with a linear term \( L_r(E) = P_rEC_r + C_rEP_r \) and a remainder \( S_r(E) \), where \( E := \hat{\Sigma} - \Sigma \). It was proved in [15] that, under the assumption

\[
\mathbb{E}\|\hat{\Sigma} - \Sigma\| \leq \frac{(1 - \gamma)g_r}{2}
\]

for some \( \gamma \in (0, 1) \), the bilinear form of the remainder \( S_r(E) \) satisfies the following concentration inequality: for all \( u, v \in \mathbb{H} \) and for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[
\left| \langle (S_r(E) - \mathbb{E}S_r(E))u, v \rangle \right| \leq \gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\|u\|\|v\|}.
\]
Under the same assumption, it was also proved in [15] that the following representation holds for the bias \( \mathbb{E} \hat{P}_r - P_r \) of empirical spectral projections \( \hat{P}_r \):

\[
\mathbb{E} \hat{P}_r - P_r = P_r(\mathbb{E} \hat{P}_r - P_r)P_r + T_r,
\]

where the main term \( P_r(\mathbb{E} \hat{P}_r - P_r)P_r \) is aligned with the spectral projection \( P_r \) and is of order

\[
\|P_r(\mathbb{E} \hat{P}_r - P_r)P_r\| \lesssim \|\Sigma\|^2 \frac{r(\Sigma)}{n} g_r^2 r_r(\Sigma) n
\]

and the remainder \( T_r \) satisfies the bound

\[
\|T_r\| \lesssim \gamma m_r \frac{\|\Sigma\|^2}{g_r^2} \sqrt{\frac{r(\Sigma)}{n}} \sqrt{n}.
\]

Representation (4.4) is especially simple in the case when \( P_r \) is of rank 1 (\( m_r = 1 \)), which also implies that \( \hat{P}_r \) is of rank 1. In this case, \( P_r = \theta_r \otimes \theta_r, \hat{P}_r = \hat{\theta}_r \otimes \hat{\theta}_r \) for unit eigenvectors \( \theta_r, \hat{\theta}_r \) of covariance operators \( \Sigma, \hat{\Sigma} \), respectively, and

\[
P_r(\mathbb{E} \hat{P}_r - P_r)P_r = b_r P_r
\]

for a “bias parameter” \( b_r = b_r(\Sigma) : \)

\[
b_r = \mathbb{E}(\hat{\theta}_r, \theta_r)^2 - 1 \in [-1, 0].
\]

Thus, it follows from (4.4) that

\[
\mathbb{E} \hat{P}_r = (1 + b_r) P_r + T_r.
\]

We obtain from (4.1) and (4.7) that

\[
\hat{P}_r - (1 + b_r) P_r = L_r(E) + S_r(E) - \mathbb{E} S_r(E) + T_r.
\]

Denote

\[
\rho_r(u) := \langle (\hat{P}_r - (1 + b_r) P_r) \theta_r, u \rangle, u \in \mathbb{H}.
\]

As in [15], the function \( \rho_r(u), u \in \mathbb{H} \) will be used in what follows to control the linear forms \( \langle \theta_r - \sqrt{1 + b_r} \theta_r, u \rangle, u \in \mathbb{H} \). First, we need to derive some bounds on \( \rho_r(u) \).

The following lemma is an immediate consequence of (4.8), (4.3) and (4.6).

**Lemma 4.1.** Suppose condition (4.2) holds for some \( \gamma \in (0, 1) \). Then, for all \( u \in \mathbb{H} \) and for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[
|\rho_r(u) - \langle L_r(E) \theta_r, u \rangle| \lesssim \gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \|u\| \right).
\]

We will need simple concentration and normal approximation bounds for \( \langle L_r(E) \theta_r, u \rangle \) given in the next lemma.
Lemma 4.2. For all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)
\[
|\langle L_r(E)\theta_r, u \rangle| \lesssim \sigma_r(\Sigma; u) \left( \sqrt{\frac{2}{n}} \sqrt{\frac{t}{n}} \right). 
\]
(4.10)

Moreover, if \( \sigma_r(\Sigma; u) > 0 \), then
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ \frac{\sqrt{n} \langle L_r(E)\theta_r, u \rangle}{\sigma_r(\Sigma; u)} \leq x \right\} - \Phi(x) \right| \lesssim \frac{1}{\sqrt{n}},
\]
where \( \Phi \) is the distribution function of standard normal r.v.

Proof. Without loss of generality, assume that the space \( \mathbb{H} \) is finite-dimensional (the general case follows by a simple approximation argument). Since \( L_r(E) = P_rEC_r + C_rEP_r \) and \( C_r\theta_r = 0 \), we have
\[
\langle L_r(E)\theta_r, u \rangle = \langle C_rEP_r\theta_r, u \rangle = \langle E\theta_r, C_ru \rangle = \langle E, \theta_r \otimes C_ru \rangle.
\]

Since \( E \) is self-adjoint, we obtain that
\[
\langle L_r(E)\theta_r, u \rangle = \frac{1}{2} \langle E, \theta_r \otimes C_ru + C_ru \otimes \theta_r \rangle.
\]

Let \( Z, Z_1, \ldots, Z_n \) be i.i.d. standard normal vectors in \( \mathbb{H} \) such that \( X_j = \Sigma^{1/2}Z_j \).

Then
\[
E = \Sigma^{1/2}\left( n^{-1} \sum_{j=1}^n Z_j \otimes Z_j - \mathbb{E}(Z \otimes Z) \right) \Sigma^{1/2}.
\]

Defining
\[
D := \frac{1}{2} \Sigma^{1/2}(\theta_r \otimes C_ru + C_ru \otimes \theta_r)^{1/2} = \frac{1}{2} \left( \Sigma^{1/2}\theta_r \otimes \Sigma^{1/2}C_ru + \Sigma^{1/2}C_ru \otimes \Sigma^{1/2}\theta_r \right),
\]

we obtain that
\[
\langle L_r(E)\theta_r, u \rangle = \left\langle n^{-1} \sum_{j=1}^n Z_j \otimes Z_j - \mathbb{E}(Z \otimes Z), D \right\rangle
\]
\[
= n^{-1} \sum_{j=1}^n \langle (DZ_j, Z_j) - \mathbb{E}(DZ, Z) \rangle.
\]

Clearly,
\[
\langle DZ, Z \rangle \overset{d}{=} \sum_k \lambda_k g_k^2,
\]
where \( \{\lambda_k\} \) are the eigenvalues of \( D \) and \( \{g_k\} \) are i.i.d. standard normal r.v. It easily follows that
\[
\mathbb{E}(DZ, Z) = \text{tr}(D) = 0
\]
and
\[
\text{Var}(\langle DZ, Z \rangle) = 2 \sum_k \lambda_k^2 = 2\|D\|_2^2 = \sigma_r^2(\Sigma; u).
\]
We can now represent $\langle L_r(E)\theta_r, u \rangle$ as follows:

$$\langle L_r(E)\theta_r, u \rangle = n^{-1} \sum_{j=1}^{n} \sum_{k} \lambda_k (g_{k,j}^2 - 1),$$

where $\{g_{k,j}\}$ are i.i.d. standard normal r.v. Using standard exponential bounds for sums of independent $\psi_1$ r.v. (see, e.g., [30], Proposition 5.16 or Theorem 3.1.9 in [11]), we obtain that with probability at least $1 - e^{-t}$

$$\left| n^{-1} \sum_{j=1}^{n} \sum_{k} \lambda_k (g_{k,j}^2 - 1) \right| \lesssim \left( \sum_{k} \lambda_k^2 \right)^{1/2} \sqrt{\frac{t}{n}} \sqrt{\sup_k |\lambda_k|} \frac{t}{n},$$

which implies that with the same probability

$$|\langle L_r(E)\theta_r, u \rangle| \lesssim \|D\|_2 \sqrt{\frac{t}{n}} \|D\|_2 \frac{t}{n}.$$

Since $\|D\| \leq \|D\|_2 = \frac{1}{2} \sigma_r^2(\Sigma; u)$, bound (4.10) follows.

To prove (4.11), we use the Berry-Esseen bound that implies

$$\sup_{x \in \mathbb{R}} \| \left\{ \sum_{j=1}^{n} \frac{\lambda_k (g_{k,j}^2 - 1)}{\sqrt{n} (2 \sum_{k} \lambda_k^2)^{1/2}} \right\} - \Phi(x) \| \lesssim \frac{\sum_k |\lambda_k|^3}{(\sum_k \lambda_k^2)^{3/2}} \frac{1}{\sqrt{n}},$$

and therefore

$$\sup_{x \in \mathbb{R}} \| \left\{ \frac{\sqrt{n} \langle L_r(E)\theta_r, u \rangle}{\sigma_r(\Sigma; u)} \right\} - \Phi(x) \| \lesssim \frac{\|D\|_2^3}{\|D\|_2 \sqrt{n}} \lesssim \frac{\|D\|_2}{\|D\|_2^2} \frac{1}{\sqrt{n}} \lesssim \frac{1}{\sqrt{n}}.$$

The following bounds on $\rho_r(u)$ immediately follow from (4.9) and (4.10).

**Lemma 4.3.** Suppose condition (4.2) holds for some $\gamma \in (0, 1)$. Then, for all $u \in \mathbb{H}$ and for all $t \geq 1$ with probability at least $1 - e^{-t}$

$$|\rho_r(u)| \lesssim \gamma \sigma_r(\Sigma; u) \left( \sqrt{\frac{\Sigma}{n}} \sqrt{\frac{t}{n}} \right) + \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\|u\|}.$$ (4.12)

Moreover, with the same probability

$$|\rho_r(u)| \lesssim \gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\|u\|}.$$ (4.13)

and, for $u = \theta_r$,

$$|\rho_r(\theta_r)| \lesssim \gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}.$$ (4.14)
Note that we dropped the term $\frac{1}{n}$ in some of the expressions on the right hand side of the above bounds (compare with (4.9)). This term is dominated by $\sqrt{\frac{t}{n}}$ for $t \leq n$. Moreover, it follows from the definition of $\rho_r(u)$ that it is upper bounded by $2\|u\|$. Since $\frac{\|\Sigma\|}{\sqrt{n}} \geq 1$, this easily implies that, for $t \geq n$, the right hand side of bound (4.13) (with a proper constant) is larger than $|\rho_r(u)|$.

Bound (4.14) follows from (4.9) since $\langle L_r(E)\theta_r, \theta_r \rangle = 0$.

To study concentration and normal approximation of the linear form

$$\langle \hat{\theta}_r - \sqrt{1 + b_r}\theta_r, u \rangle, \ u \in \mathbb{H},$$

it remains to prove that it can be approximated by $\langle L_r(E)\theta_r, u \rangle$.

**Lemma 4.4.** Suppose that for some $\gamma \in (0, 1)$ condition (4.2) holds and, in addition,

$$1 + b_r \geq \gamma.$$  

(4.15)

Then, for all $u \in \mathbb{H}$ and for all $t \geq 1$, with probability at least $1 - e^{-t}$

$$|\langle \hat{\theta}_r - \sqrt{1 + b_r}\theta_r, u \rangle - \langle L_r(E)\theta_r, u \rangle| \lesssim_{\gamma} \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \right).$$  

(4.16)

**Proof.** We use the following representation obtained in [15] (see (6.7) in [15]), which holds provided that $\hat{\theta}_r$ and $\theta_r$ are properly aligned so that $\langle \hat{\theta}_r, \theta_r \rangle \geq 0$:

$$\langle \hat{\theta}_r - \sqrt{1 + b_r}\theta_r, u \rangle = \frac{\rho_r(u)}{\sqrt{1 + b_r + \rho_r(\theta_r)}} - \frac{\sqrt{1 + b_r}}{\sqrt{1 + b_r + \rho_r(\theta_r)}} \rho_r(\theta_r) \langle \theta_r, u \rangle$$  

(4.17)

(it is clear from the proof given in [15] that $1 + b_r + \rho_r(\theta_r) \geq 0$). Denote

$$\nu_r := \frac{\rho_r(\theta_r)}{1 + b_r}.$$  

Then, it is easy to see that

$$\langle \hat{\theta}_r - \sqrt{1 + b_r}\theta_r, u \rangle = \rho_r(u) - \frac{b_r}{1 + \nu_r + \sqrt{1 + b_r}} \rho_r(u)$$  

(4.18)

Recall that (4.2) and (4.15) hold for some $\gamma \in (0, 1)$. If $|\nu_r| \leq 1/2$, then (4.18) easily implies that

$$|\langle \hat{\theta}_r - \sqrt{1 + b_r}\theta_r, u \rangle - \rho_r(u)| \leq \frac{1}{\gamma} (|b_r| + |\nu_r|)|\rho_r(u)| + |\nu_r| |\theta_r, u||.$$  

(4.19)

It also follows from (4.14) that, under condition (4.15),

$$|\nu_r| \lesssim_{\gamma} \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \right).$$  

(4.20)
with probability at least $1 - e^{-t}$. On the other hand, bound (4.5) implies that

$$|b_r| \lesssim \frac{||\Sigma||^2 r(\Sigma)}{g_r^2 n}. \tag{4.21}$$

It follows from (4.20) that for the condition $|\nu_r| \leq 1/2$ to hold with probability at least $1 - e^{-t}$, it is enough to have

$$\frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}} \leq c_\gamma \tag{4.22}$$

for a small enough constant $c_\gamma > 0$. Assume that (4.22) holds. Note also that it implies that $t \lesssim n$ and condition (4.2) and Theorem 2.1 imply that $\frac{||\Sigma||}{g_r} \sqrt{\frac{r(\Sigma)}{n}} \lesssim 1$. It follows from (4.19), (4.13), (4.20) and (4.21) that with probability at least $1 - 3e^{-t}$:

$$|\langle \hat{\theta}_r - \sqrt{1 + b_r \theta_r}, u \rangle - \rho_r(u)| \lesssim \gamma \left[ \left( \frac{||\Sigma||^2}{g_r^2} \right) \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) + \frac{||\Sigma||}{g_r} \sqrt{\frac{t}{n}} \right] \tag{4.23}$$

Using the facts that

$$\frac{||\Sigma||^2}{g_r^2} \frac{r(\Sigma)}{n} \lesssim \frac{||\Sigma||}{g_r} \sqrt{\frac{r(\Sigma)}{n}} \lesssim 1,$$

that

$$\frac{||\Sigma||^2}{g_r^2} \frac{t}{n} \lesssim \frac{||\Sigma||}{g_r} \sqrt{\frac{t}{n}} \lesssim 1$$

and that

$$\frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \frac{t}{n} \lesssim \frac{||\Sigma||^2}{g_r^2} \left( \frac{r(\Sigma)}{n} \right)^{1/4} \left( \frac{t}{n} \right)^{1/4} \leq \frac{||\Sigma||}{g_r} \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}}$$

(that follow from condition (4.22)), it is easy to conclude that the last term in the right hand side of bound (4.23) is dominant. Hence, with probability at least $1 - e^{-t}$

$$|\langle \hat{\theta}_r - \sqrt{1 + b_r \theta_r}, u \rangle - \rho_r(u)| \lesssim \gamma \frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \frac{||u||}{n} \tag{4.24}$$
provided that condition (4.22) holds. On the other hand, if

\[ \|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}} > c_\gamma, \]

then

\[ |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, u \rangle - \rho_r(u)| \leq |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, u \rangle| + |\rho_r(u)| \]

\[ \lesssim_{\gamma} \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|}. \]

Thus, we proved that with probability at least \(1 - e^{-t}\)

\[ |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, u \rangle - \rho_r(u)| \lesssim_{\gamma} \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|}. \] (4.25)

It remains to combine this with the bound (4.9) to complete the proof.

The following result is a slightly improved version of Theorem 6 in [15].

**Lemma 4.5.** Under conditions (4.2) and (4.15) for some \(\gamma \in (0, 1)\), the following bounds hold for all \(t \geq 1\) with probability at least \(1 - e^{-t}\):

\[ |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, u \rangle| \lesssim_{\gamma} \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|}. \] (4.26)

and

\[ |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, \theta^r \rangle| \lesssim_{\gamma} \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|}. \] (4.27)

**Proof.** Indeed, it follows from (4.16) and (4.10) that, for some constants \(C, C_\gamma > 0\) with probability at least \(1 - e^{-t}\)

\[ |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, u \rangle| \leq C \sigma_r(\Sigma; u) \left( \sqrt{\frac{t}{n}} \sqrt{\frac{t}{n}} \right) + C_\gamma \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|}. \]

Since \(\sigma_r(\Sigma; u) \lesssim \frac{\|\Sigma\|_2^2}{\|u\|}\), with the same probability

\[ |\langle \hat{\theta}^r - \sqrt{1 + b_r \theta^r}, u \rangle| \leq C \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|} + C_\gamma \frac{\|\Sigma\|_2^2 \left( \sqrt{\frac{\sigma_0}{n}} \sqrt{\frac{t}{n}} \right) \sqrt{\frac{t}{n}}}{\|u\|}. \]

We dropped the term \(\frac{t}{n}\) present in bounds (4.16) and (4.10) since for \(t \geq n\) (the only case when it is needed), the right hand side already dominates the left hand side (which is smaller than \(2\|u\|\)). Note that condition (4.2) and Theorem 2.1 imply that \(\frac{\|\Sigma\|_2^2}{\|u\|} \leq c_\gamma\) for some constant \(c_\gamma > 0\). Assuming that also
\[
\frac{\|\Sigma\|}{\sqrt{n}} \sqrt{\frac{t}{n}} \leq c_r,
\]
which implies that \( t \leq n \), we obtain that for some constant \( C_r > 0 \) with probability at least \( 1 - e^{-t} \) bound (4.26) holds. On the other hand, if \( \frac{\|\Sigma\|}{\sqrt{n}} \sqrt{\frac{t}{n}} > c_r \), then
\[
|\langle \hat{\theta}_r - \sqrt{1 + \theta_r u}, u \rangle| \leq \left( \|\hat{\theta}_r\| + \sqrt{1 + \theta_r \|\theta_r\|}\|u\| \right) \leq 2\|u\| \leq 2, \quad \text{and for all } n \text{ for } \|u\| \leq \sqrt{\frac{t}{n}} \|\Sigma\| \sqrt{\frac{t}{n}} \|u\|,
\]
implies again (4.26). For \( u = \theta_r, \langle L_r(E)\theta_r, u \rangle = 0 \) and bound (4.16) implies that with probability at least \( 1 - e^{-t} \) (4.27) holds.

The following two lemmas will be used to derive normal approximation bounds for \( \langle \hat{\theta}_r - \sqrt{1 + \theta_r u}, u \rangle \) from the corresponding bounds for \( \langle L_r(E)\theta_r, u \rangle \) as well as to control the risk for loss functions satisfying Assumption 3.1. We state them without proofs (which are elementary).

**Lemma 4.6.** For random variables \( \xi, \eta \), denote
\[
\Delta(\xi; \eta) := \sup_{x \in \mathbb{R}} |\mathbb{P}\{\xi \leq x\} - \mathbb{P}\{\eta \leq x\}|
\]
and
\[
\delta(\xi; \eta) := \inf\{\delta > 0 : \mathbb{P}\{|\xi - \eta| \geq \delta\} \leq \delta\}.
\]
Then, for a standard normal r.v. \( Z \),
\[
\Delta(\xi; Z) \leq \Delta(\eta; Z) + \delta(\xi; \eta).
\]
Under Assumption 3.1, for all \( A > 0 \)
\[
|\mathbb{E}\ell(\xi) - \mathbb{E}\ell(\eta)| \leq 4\ell(A)\Delta(\xi; \eta) + \mathbb{E}\ell(\xi)\mathbb{I}\{\xi \geq A\} + \mathbb{E}\ell(\eta)\mathbb{I}\{|\eta| \geq A\}.
\]

**Lemma 4.7.** Let \( \xi \) be a random variable such that for some \( \tau_1 \geq 0 \) and \( \tau_2 \geq 0 \) and for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)
\[
|\xi| \leq \tau_1 \sqrt{t} + \tau_2 t.
\]
Let \( \ell \) be a loss function satisfying Assumption 3.1. If \( 2c_2\tau_2 < 1 \), then
\[
\mathbb{E}\ell^2(\xi) \leq 2c_2^2 c_1^2 e^{2\tau_2^2} + \frac{\epsilon\tau_1^2}{1 - 2c_2\tau_2}. \tag{4.28}
\]

Next we prove the normal approximation bounds for linear forms \( \langle \hat{\theta}_r - \sqrt{1 + \theta_r u}, u \rangle \).

**Lemma 4.8.** Suppose that conditions (4.2) and (4.15) hold for some \( \gamma \in (0, 1) \) and also that \( n \geq 2\tau_1(S) \). Assume that, for some \( u \in \mathbb{H}, \sigma_r(S; u) > 0 \). Let \( \alpha \geq 1 \). Then the following bound holds: for some constants \( C, C_{\gamma, \alpha} > 0 \),
\[
\sup_{x \in \mathbb{R}} \mathbb{P}\left\{ \frac{\frac{n}{\sqrt{n}} \log n}{\sigma_r(S; u)} \min\left( \sqrt{\frac{\tau_1(S)}{n}} \log \left( \frac{n}{\tau_1(S)} \right), \sqrt{\frac{\tau_{\alpha}(S)}{n}} \right) \|u\| + \left( \frac{r(S)}{n} \right)^{\alpha} \right\} \leq C n^{-1/2} + \frac{C_{\gamma, \alpha}}{\sigma_r(S; u)} \|\Sigma\|_2 \left( \frac{\sqrt{\tau_1(S)}}{n} \log \left( \frac{n}{\tau_1(S)} \right) \sqrt{\frac{\log \tau_{1, S}}{n}} \right) \|u\| + \left( \frac{r(S)}{n} \right)^{\alpha}. \tag{4.29}
\]
Moreover, under Assumption 3.1 on the loss \( t \), there exist constants \( C, C_\gamma, C_{\gamma,\alpha} > 0 \) such that

\[
\| \mathbb{E} \left( \sqrt{n} (\hat{\theta}_r - \sqrt{1 + \ell, \theta_r, u}) \right) - \mathbb{E}(Z) \|
\leq c_1 e^{c_2 A} \left( C n^{-1/2} + \frac{C_\gamma,\alpha}{\sigma_r(\Sigma; u)} \right) \left( \left\| \frac{\mathbb{E}(\Sigma)}{n} \right\| \log^2 \frac{n}{\mathbb{E}(\Sigma)} \right) \left( \left\| \frac{\mathbb{E}(\Sigma)}{n} \right\| \right) + 2e^{3/2} (2\pi)^{1/4} c_1 e^{2/2} e^{-A^2/2},
\]

where

\[
\tau := C_\gamma \left\| \Sigma \right\| \left\| u \right\| / \sigma_r(\Sigma; u).
\]

**Proof.** We will use the first claim of Lemma 4.6 with

\[
\xi := \frac{\sqrt{n} (\hat{\theta}_r - \sqrt{1 + \ell, \theta_r, u})}{\sigma_r(\Sigma; u)} \quad \text{and} \quad \eta := \frac{\sqrt{n} (L_r(E) \theta_r, u)}{\sigma_r(\Sigma; u)}.
\]

It follows from bound (4.16) that, under conditions (4.2) and (4.15), for some \( C_\gamma > 0 \)

\[
\delta(\xi; \eta) \leq \inf_{t \geq 1} \left\{ \frac{C_\gamma}{\sigma_r(\Sigma; u)} \left\| \Sigma \right\| \left( \left\| \frac{\mathbb{E}(\Sigma)}{n} \right\| \log^2 \frac{n}{\mathbb{E}(\Sigma)} \right) \left( \left\| \frac{\mathbb{E}(\Sigma)}{n} \right\| \right) + c_1 e^{2} e^{-A^2/2} \right\}.
\]

Taking \( t := \alpha \log \frac{1}{\rho(\Sigma)} \) with some \( \alpha \geq 1 \) easily yields an upper bound

\[
\delta(\xi; \eta) \leq \frac{C_{\gamma,\alpha}}{\sigma_r(\Sigma; u)} \left\| \Sigma \right\| \left( \left\| \frac{\mathbb{E}(\Sigma)}{n} \right\| \log^2 \frac{n}{\mathbb{E}(\Sigma)} \right) \left( \left\| \frac{\mathbb{E}(\Sigma)}{n} \right\| \right) + \left( \frac{\mathbb{E}(\Sigma)}{n} \right) \alpha.
\]

Using bound (4.11) to control \( \Delta(\eta; Z) \), we obtain from Lemma 4.6 that bound (4.29) holds with some constants \( C, C_\gamma,\alpha > 0 \). To prove the second statement, we use the second bound of Lemma 4.6 with r.v. \( \xi := \sqrt{n} (\hat{\theta}_r - \sqrt{1 + \ell, \theta_r, u}) / \sigma_r(\Sigma; u) \) and \( \eta = Z \). The following exponential bound on \( \xi \) is an easy corollary of bound (4.26): for some constant \( C_\gamma > 0 \) and for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[
|\xi| \leq C_\gamma \frac{\left\| \Sigma \right\|}{\sigma_r(\Sigma; u)} \sqrt{t} \left\| u \right\| = \tau \sqrt{t}.
\]

Using bound (4.28) with \( \tau_1 = \tau \) and \( \tau_2 = 0 \), we obtain

\[
\mathbb{E} \ell^2(\xi) \leq 2e \sqrt{2\pi c_1^2 e^{2\Sigma \tau^2}} + c_2^2 \leq 4e \sqrt{2\pi c_1^2 e^{2\Sigma \tau^2}}.
\]

Therefore,

\[
\mathbb{E} \ell(\xi) I(|\xi| \geq A) \leq \mathbb{E}^{1/2} \ell^2(\xi) \mathbb{P}^{1/2} \{ |\xi| \geq A \} \leq 2e^{3/2} (2\pi)^{1/4} c_1 e^{2\Sigma \tau^2} e^{-A^2/2}.
\]

We also have

\[
\mathbb{E} \ell(Z) I(|Z| \geq A) \leq c_1 e^{2} e^{-A^2/4}.
\]

Using bound (4.29), we can now deduce bound (4.30) from the second statement of Lemma 4.6. \( \square \)
Lemma 4.8 immediately implies Theorem 3.1 (by passing to the limit as $n \to \infty$ in (4.29) and as $n \to \infty$ and then $A \to \infty$ in (4.30)).

4.1. Proof of Proposition 3.1

Denote

$$A_r(\Sigma) := 2 \text{tr}(P_r \Sigma P_r) \text{tr}(C_r \Sigma C_r) = 2 \sum_{s \neq r} \frac{\mu_r \mu_s m_s}{(\mu_r - \mu_s)^2}.$$  

It was shown in [17] that

$$E \left\| L_r(E) \right\|^2 = \frac{A_r(\Sigma)}{n},$$

where $E = \hat{\Sigma} - \Sigma$. Note that

$$\frac{A_r(\Sigma)}{2} \leq \frac{\mu_r}{g_r}(\text{tr}(\Sigma) - \mu_r) \leq \frac{\|\Sigma\|^2}{g_r^2} r(\Sigma) \tag{4.32}$$

and

$$\frac{A_r(\Sigma)}{2} \geq \frac{\mu_1 \mu_r}{(\mu_1 - \mu_r)^2 \vee \mu_r^2}(r(\Sigma) - 1). \tag{4.33}$$

Lemma 4.9. The following representation holds:

$$b_r(\Sigma) = -\frac{1}{2} \frac{A_r(\Sigma)}{n} + \beta_r,$$

where

$$|\beta_r| \lesssim \frac{\|\Sigma\|^3}{g_r^3} \left( \frac{\text{tr}(\Sigma)}{n} \sqrt{\frac{r(\Sigma)}{n}} \right)^3.$$

Proof. Recall representation (4.4) and bound (4.6). Note that

$$b_r = \text{tr}(P_r(\hat{E}P_r - P_r)P_r)$$

and

$$\hat{E}P_r - P_r = ES_r(E).$$

We will use the following representation for $S_r(E)$ (based on perturbation series for $P_r$) that easily follows from Lemma 4 in [18]:

$$S_r(E) = P_r\Sigma C_r E + C_rE P_rC_r E + C_r E P_r C_r E - P_r E C_r E P_r - C_r E P_r C_r E + S_r^{(3)}(E),$$

where

$$\|S_r^{(3)}(E)\| \lesssim \frac{\|E\|^3}{g_r^3}.$$  

Since $P_r C_r = C_r P_r = 0$ this implies

$$P_r S_r(E) P_r = -P_r E C_r E P_r + P_r S_r^{(3)}(E) P_r."
Therefore we obtain
\[ b_r = \mathbb{E} \text{tr}(P_r S_r(E) P_r) = -\mathbb{E} \text{tr}(P_r E C_r^2 E P_r) + \mathbb{E} \text{tr}(P_r S_r^{(3)}(E) P_r) \]
\[ = -\mathbb{E} \|P_r E C_r\|_2^2 + \mathbb{E} \text{tr}(P_r S_r^{(3)}(E) P_r) = -\frac{1}{2} \mathbb{E} \|P_r E C_r + C_r E P_r\|_2^2 + \mathbb{E} \text{tr}(P_r S_r^{(3)}(E) P_r) \]
\[ -\frac{1}{2} \mathbb{E} \|L_r(E)\|_2^2 + \mathbb{E} \text{tr}(P_r S_r^{(3)}(E) P_r) = -\frac{1}{2} A_r(\Sigma) \frac{1}{n} + \mathbb{E} \text{tr}(P_r S_r^{(3)}(E) P_r). \]
Thus, \( \beta_r = \mathbb{E} \text{tr}(P_r S_r^{(3)}(E) P_r) \) and, using bound (2.4), we get
\[ |\beta_r| \leq \mathbb{E} \|S_r^{(3)}(E)\|_1 \leq \mathbb{E} \|S_r^{(3)}(E)\| \lesssim \frac{\mathbb{E}\|E\|^3}{g_r^3} \]
\[ \lesssim \frac{\|\Sigma\|^3}{g_r^3} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{r(\Sigma)}{n}} \right)^3, \]
which completes the proof.

It follows from the lower bound (4.33) on \( \frac{A_r(\Sigma)}{2} \) and the bound of Lemma 4.9 that, under the assumption \( r(\Sigma) \leq n \), with some constant \( C > 0 \)
\[ |b_r| \geq \frac{\mu_1 \mu_r}{(\mu_1 - \mu_r)^2} \frac{r(\Sigma) - 1}{n} - C \frac{\|\Sigma\|^3}{g_r^3} \left( \frac{r(\Sigma)}{n} \right)^{3/2}. \] (4.34)

Next note that
\[ |\langle \hat{\theta}_r - \theta_r, u \rangle| \geq |\sqrt{1 + b_r} - 1| |\langle \hat{\theta}_r, u \rangle| - |\langle \hat{\theta}_r - \sqrt{1 + b_r} \theta_r, u \rangle| \]
\[ \geq \frac{|b_r|}{1 + \sqrt{1 + b_r}} |\langle \theta_r, u \rangle| - |\langle \hat{\theta}_r - \sqrt{1 + b_r} \theta_r, u \rangle| \]
\[ \geq \frac{|b_r|}{2} |\langle \theta_r, u \rangle| - |\langle \hat{\theta}_r - \sqrt{1 + b_r} \theta_r, u \rangle|. \]
Using bounds (4.26) and (4.34), we obtain that for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)
\[ |\langle \hat{\theta}_r - \theta_r, u \rangle| \geq \frac{1}{2} |\langle \hat{\theta}_r, u \rangle| \left( \frac{\mu_1 \mu_r}{(\mu_1 - \mu_r)^2} \frac{r(\Sigma) - 1}{n} - C \frac{\|\Sigma\|^3}{g_r^3} \left( \frac{r(\Sigma)}{n} \right)^{3/2} - C_\gamma \frac{\|u\|}{g_r} \sqrt{\frac{t}{n}} \right). \] (4.35)
We will show that there exists a covariance \( \Sigma_0 \in S^{(r)}(r_n, a, \sigma_0, u) \) such that
\[ |\langle \theta_r(\Sigma_0), u \rangle| \geq \frac{\|u\|}{2}, \]
\[ \frac{\mu_1(\Sigma_0) \mu_r(\Sigma_0)}{\left( \mu_1(\Sigma_0) - \mu_r(\Sigma_0) \right)^2} \geq c_1 \]
for some constant \( c_1 > 0 \) that might depend on \( r, a, \sigma_0 \) and \( r(\Sigma_0) - 1 \geq r_n/2 \).
Assuming that such a \( \Sigma_0 \) exists, we choose \( t_n \to \infty \), \( t_n = o\left( \frac{r_n^2}{n} \right) \) and applying bound (4.35) to \( \Sigma = \Sigma_0 \), we immediately obtain that
\[ \sup_{\Sigma \in S^{(r)}(r_n, a, \sigma_0, u)} \mathbb{P}_\Sigma \left\{ |\langle \hat{\theta}_r - \theta_r(\Sigma), u \rangle| \geq \left( \frac{c_1 r_n}{8 n} - \frac{C_\gamma a}{4} \left( \frac{r_n}{n} \right)^{3/2} \right) \right\} \]
Since
\[ \left( c_1 \frac{\tau_n}{n} - \frac{C}{4} a^3 \left( \frac{\tau_n}{n} \right)^{3/2} - C_\gamma a \sqrt{\frac{\tau_n}{n}} \right) \| u \| = \left( \frac{c_1}{8} + o(1) \right) \frac{\tau_n}{n} \| u \| , \]
this implies the claim of Proposition 3.1.

It remains to define a $\Sigma_0$ with the desired properties. Let
\[ \Sigma_0 = \sum_{s=1}^{r+1} \mu_s P_s , \]
where $P_s = \theta_s \otimes \theta_s, s = 1, \ldots, r, \theta_1, \ldots, \theta_r$ being arbitrary orthonormal vectors in $\mathbb{H}$ and $P_{r+1}$ is an orthogonal projection on a $d$-dimensional subspace of $\mathbb{H}$ orthogonal to $\theta_1, \ldots, \theta_r$. Let $\mu_s := \mu_1 \left( 1 - \frac{s-1}{a} \right), s = 1, \ldots, r+1$. Then $\bar{g}_r(\Sigma_0) = \frac{\mu_r}{\mu_s}$ and the condition $\frac{\| \Sigma_0 \|}{g_r(\Sigma_0)} \leq a$ is satisfied. For simplicity, assume that $\| u \| = 1$. Moreover, since $\theta_1, \ldots, \theta_r$ are arbitrary orthonormal vectors, we can assume without loss of generality that, for $r > 1$, $u := \frac{1}{\sqrt{r}} \theta_1 + \frac{1}{\sqrt{r}} \theta_r$. Then $\langle \theta_r(\Sigma_0), u \rangle = \frac{1}{\sqrt{r}} > \frac{1}{2} \| u \|$ and, by a simple computation,
\[ \sigma_r^2(\Sigma_0; u) = \sum_{s \neq r} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} \| P_s u \|^2 = \frac{1}{2} \frac{\mu_1 \mu_r}{(\mu_1 - \mu_r)^2} = \frac{1}{2} \left[ \frac{a^2}{(r-1)^2} - \frac{a}{r-1} \right] . \]

Assuming that $\sigma_0^2 \leq \frac{a}{2} \left[ \frac{a^2}{(r-1)^2} - \frac{a}{r-1} \right]$, we conclude that the condition $\sigma_r^2(\Sigma_0; u) \geq \sigma_0^2$ is satisfied. For $r = 1$, we can assume that $u := \frac{1}{\sqrt{a}} \theta_1 + \frac{1}{\sqrt{a}} \theta_2$ with a slight modification of the argument. Finally, we take dimension $d = d_n$ so that
\[ r(\Sigma_0) = \sum_{s=1}^{r} \frac{\mu_s}{\mu_1} + \frac{\mu_{r+1}}{\mu_1} d_n = \sum_{s=1}^{r} \left( 1 - \frac{s-1}{a} \right) + \left( 1 - \frac{r}{a} \right) d_n \in (\tau_n/2 + 1, \tau_n] , \]
Then $\Sigma_0 \in \mathcal{S}^{(r)}(\tau_n, a, \sigma_0, u)$. This completes the proof.

5. Proof of Theorem 3.3
Recall that the estimator $\hat{\theta}_r$ is based on empirical eigenvectors $\hat{\theta}_r^{\delta_j,j}, j = 1, 2, 3$ with parameters $\delta_j = r \| \Sigma(j) \|$ and with a proper choice of $r$ (as in Lemma 2.2). These eigenvectors are in turn defined in terms of empirical spectral projections $\bar{P}_r^{\delta_j,j}$ of sample covariances $\hat{\Sigma}(j)$ (based on $\delta_j$-clusters of its spectrum $\sigma(\Sigma(j))$).

We will, however, replace $\hat{\theta}_r$ by the estimator $\hat{\theta}_r$ defined in terms of empirical spectral projections $\bar{P}_r^{\delta_j,j}, j = 1, 2, 3, \bar{P}_r^{\delta_j,j}$ being the orthogonal projection onto direct sum of eigenspaces of $\Sigma(j)$ corresponding to its eigenvalues $\lambda_k(\Sigma(j)), k \in \Delta_r$. Since $\text{card}(\Delta_r) = m_r = 1, \bar{P}_r^{(j)} = \hat{\theta}_r^{(j)} \otimes \hat{\theta}_r^{(j)}$ and we can define
\[ \hat{d}_r := \frac{(\hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)})}{(\hat{\theta}_r^{(2)}, \hat{\theta}_r^{(3)})^{1/2}} \]
and

\[ \hat{\theta}_r := \frac{\hat{\theta}_r^{(1)}}{d_r \lor (1/2)}. \]

The reduction to this case is based on Lemma 2.2 (implying that \( \hat{P}_r^{(j)} = \hat{P}_r^{(j)} \)
with a high probability) and is straightforward (as in the proof of Theorem 3.1).

The rest of the proof is based on several lemmas stated and proved below.

**Lemma 5.1.** Suppose that for some \( \gamma \in (0, 1) \) condition (4.2)
holds for the sample covariance \( \hat{\Sigma}^{(2)} \) based on \( m \) observations:

\[ \mathbb{E} \| \hat{\Sigma}^{(2)} - \Sigma \| \leq \frac{(1 - \gamma)g_r}{2} \]

Then, for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[ \left| \langle \hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)} \rangle - \sqrt{1 + b_r^{(n')} \sqrt{1 + b_r^{(m)}}} \right| \lesssim \gamma \frac{\| \Sigma \|^2}{g_r} \left( \sqrt{\frac{r(\Sigma)}{n'}} \lor \sqrt{\frac{r(\Sigma)}{m}} \right) \sqrt{\frac{t}{n'}}. \]  

(5.2)

and with the same probability

\[ \left| \langle \hat{\theta}_r^{(2)}, \hat{\theta}_r^{(3)} \rangle - (1 + b_r^{(m)}) \right| \lesssim \gamma \frac{\| \Sigma \|^2}{g_r} \left( \sqrt{\frac{r(\Sigma)}{m}} \lor \sqrt{\frac{r(\Sigma)}{m}} \right) \sqrt{\frac{t}{m}}. \]  

(5.3)

**Proof.** Obviously, condition (5.1) holds also for the sample covariance \( \hat{\Sigma}^{(2)} \)
(which is based on a sample of the same size \( m \)). Moreover, it also holds for the sample covariance \( \hat{\Sigma}^{(1)} \) based on \( n' \geq m \) observations since the sequence \( n \mapsto \mathbb{E} \| \hat{\Sigma}_n - \Sigma \| \) is non-increasing (see, e.g., Lemma 2.4.5 in [29]).

The following representation is obvious:

\[ \langle \hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)} \rangle = \sqrt{1 + b_r^{(n')} \sqrt{1 + b_r^{(m)}}} \langle \theta_r, \theta_r \rangle \]

\[ + \sqrt{1 + b_r^{(m)}} \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')} \theta_r, \theta_r} \rangle \]

\[ + \sqrt{1 + b_r^{(n')}} \langle \hat{\theta}_r^{(2)} - \sqrt{1 + b_r^{(m)} \theta_r, \theta_r} \rangle \]

\[ \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')} \theta_r, \theta_r}, \hat{\theta}_r^{(2)} - \sqrt{1 + b_r^{(m)} \theta_r, \theta_r} \rangle. \]

(5.4)

By bound (4.27), with probability at least \( 1 - e^{-t} \)

\[ \left| \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')} \theta_r, \theta_r} \rangle \right| \lesssim \gamma \frac{\| \Sigma \|^2}{g_r} \left( \sqrt{\frac{r(\Sigma)}{n'}} \lor \sqrt{\frac{r(\Sigma)}{m}} \right) \sqrt{\frac{t}{n'}}. \]  

(5.5)

Similarly, with probability at least \( 1 - e^{-t} \)

\[ \left| \langle \hat{\theta}_r^{(2)} - \sqrt{1 + b_r^{(m)} \theta_r, \theta_r} \rangle \right| \lesssim \gamma \frac{\| \Sigma \|^2}{g_r} \left( \sqrt{\frac{r(\Sigma)}{m}} \lor \sqrt{\frac{r(\Sigma)}{m}} \right) \sqrt{\frac{t}{m}}. \]  

(5.6)
To bound the last term in the right hand side of (5.4), we apply bound (4.26) to \( \hat{\theta}^{(1)}_r \) conditionally on the second sample (similarly to the proof of Theorem 6 in [15]). This yields that with probability at least \( 1 - e^{-t} \)

\[
|\langle \hat{\theta}^{(1)}_r - \sqrt{1 + b^{(m)}_r} \theta_r \rangle - \langle \hat{\theta}^{(2)}_r - \sqrt{1 + b^{(m)}_r} \theta_r \rangle| \lesssim \gamma \frac{\|\Sigma\|}{g_r} \sqrt{\frac{t}{m}} \|\hat{\theta}^{(2)}_r - \sqrt{1 + b^{(m)}_r} \theta_r\|.
\]

(5.7)

On the other hand, under the assumption that \( \langle \hat{\theta}^r, \theta_r \rangle \geq 0 \),

\[
\|\hat{\theta}^{(2)}_r - \sqrt{1 + b^{(m)}_r} \theta_r\| \leq \|\hat{\theta}^{(2)}_r - \theta_r\| + \left| \sqrt{1 + b^{(m)}_r} - 1 \right|
\]

\[
= \sqrt{2 - 2\langle \hat{\theta}^{(2)}_r, \theta_r \rangle} + \frac{|b^{(m)}_r|}{\sqrt{1 + b^{(m)}_r} + 1} \leq \sqrt{2 - 2\langle \hat{\theta}^{(2)}_r, \theta_r \rangle^2} + |b^{(m)}_r| \leq \sqrt{2} \|\hat{\theta}^{(2)}_r - P_r\| + |b^{(m)}_r|.
\]

By a standard perturbation bound (see, e.g., [15]),

\[
\|\hat{\theta}^{(2)}_r - P_r\| \leq 4 \frac{\|\Sigma^{(2)} - \Sigma\|}{g_r}.
\]

Thus,

\[
\|\hat{\theta}^{(2)}_r - \sqrt{1 + b^{(m)}_r} \theta_r\| \leq 4 \frac{\|\Sigma^{(2)} - \Sigma\|}{g_r} + |b^{(m)}_r|.
\]

(5.8)

Using the exponential bound (2.3) on \( \|\Sigma^{(2)} - \Sigma\| \) and bound (4.21), we obtain that with probability at least \( 1 - e^{-t} \)

\[
\|\hat{\theta}^{(2)}_r - \sqrt{1 + b^{(m)}_r} \theta_r\| \lesssim \frac{\|\Sigma\|}{g_r} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) + \frac{\|\Sigma\|^2 r(\Sigma)}{g^2_r} \frac{t}{m}.
\]

(5.9)

Under assumption (5.1), we have \( \frac{\|\Sigma\|}{g_r} \sqrt{\frac{r(\Sigma)}{m}} \lesssim 1 \), which implies \( \frac{\|\Sigma\|^2 r(\Sigma)}{g^2_r} \frac{t}{m} \lesssim \frac{\|\Sigma\|}{g_r} \sqrt{\frac{r(\Sigma)}{m}} \). Thus, the first term in the right hand side of bound (5.9) is dominant. Moreover, we can drop the term \( \frac{\|\Sigma\|}{g_r} \sqrt{\frac{r(\Sigma)}{m}} \) and, for \( t \leq m \), we can also drop the term \( \frac{\|\Sigma\|^2 r(\Sigma)}{g^2_r} \frac{t}{m} \) in the right hand side. Since the left hand side of (5.9) is not larger than 2, for \( t > m \), the term \( \frac{\|\Sigma\|}{g_r} \sqrt{\frac{r(\Sigma)}{m}} \) is larger (up to a constant) than the left hand side. Thus, the term \( \frac{\|\Sigma\|^2 r(\Sigma)}{g^2_r} \frac{t}{m} \) can be dropped for all the values of \( t \) and the bound (5.9) simplifies as follows

\[
\|\hat{\theta}^{(2)}_r - \sqrt{1 + b^{(m)}_r} \theta_r\| \lesssim \frac{\|\Sigma\|}{g_r} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right).
\]

(5.10)
and it still holds with probability at least $1 - e^{-t}$. It follows from bound (5.7) and (5.10) that for all $t \geq 1$ with probability at least $1 - 2e^{-t}$

$$
|\langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(1')} \theta_r}, \hat{\theta}_r^{(2)} - \sqrt{1 + b_r^{(1)}} \theta_r \rangle| \lesssim \gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{n'}}.
$$

(5.11)

Taking into account that $n' \geq m$, it easily follows from representation (5.4) and bounds (5.5), (5.6) and (5.11) that with probability at least $1 - e^{-t}$

$$
|\langle \hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)} \rangle - \sqrt{1 + b_r^{(1'')} \sqrt{1 + b_r^{(1)'}} \theta_r | \theta_r \rangle} - \sqrt{1 + b_r^{(1)'} \sqrt{1 + b_r^{(1)}} \theta_r | \theta_r \rangle} \lesssim \gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}},
$$

which proves (5.2). The proof of bound (5.3) is similar.

Define

$$
\Delta_1 := \frac{\langle \hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)} \rangle}{\sqrt{1 + b_r^{(1'')}}} - 1
$$

and

$$
\Delta_2 := \frac{\langle \hat{\theta}_r^{(2)}, \hat{\theta}_r^{(3)} \rangle}{1 + b_r^{(1)}} - 1.
$$

Assuming that

$$
1 + b_r^{(1'')} \geq (3/4)^2 \quad \text{and} \quad 1 + b_r^{(1)} \geq (3/4)^2,
$$

(5.12)

we obtain that, for some constant $C_\gamma > 0$ and for $t \geq 1$ on an event $E$ of probability at least $1 - e^{-t}$

$$
|\Delta_1| \lor |\Delta_2| \leq C_\gamma \frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}}.
$$

(5.13)

Next we have

$$
\hat{d}_r = \frac{\langle \hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)} \rangle}{\langle \hat{\theta}_r^{(2)}, \hat{\theta}_r^{(3)} \rangle} = \frac{\langle \hat{\theta}_r^{(1)}, \hat{\theta}_r^{(2)} \rangle / ((1 + b_r^{(1'')} \sqrt{1 + b_r^{(1)'}}) 1/2) (1 + b_r^{(1)}) 1/2)}{\langle \hat{\theta}_r^{(2)}, \hat{\theta}_r^{(3)} \rangle / (1 + b_r^{(1)}) 1/2} \sqrt{1 + b_r^{(1)}}
$$

$$
= \frac{1 + \Delta_1}{\sqrt{1 + \Delta_2}} \sqrt{1 + b_r^{(1)'}} = \sqrt{1 + b_r^{(1)'}} + \frac{1 + \Delta_1}{\sqrt{1 + \Delta_2}} \sqrt{1 + b_r^{(1)'}}
$$

which implies

$$
|\hat{d}_r - \sqrt{1 + b_r^{(1)'}}| \leq \sqrt{1 + b_r^{(1)'}} \left| \frac{(1 + \Delta_1)^2 - (1 + \Delta_2)}{\sqrt{1 + \Delta_2(1 + \Delta_1 + \sqrt{1 + \Delta_2})}} \right| \leq \frac{2|\Delta_1| + \Delta_1^2 + |\Delta_2|}{\sqrt{1 + \Delta_2(1 + \Delta_1 + \sqrt{1 + \Delta_2})}}.
$$

(5.14)

Under the assumption that

$$
\frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \leq c_\gamma
$$

(5.15)
for a sufficiently small constant $c_\gamma > 0$, bounds \((5.14)\) and \((5.13)\) imply that on the event $E$

$$\left| \frac{\hat{d}_r}{\sqrt{1 + b_r^{(n')}}} - 1 \right| \lesssim_\gamma \frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \tag{5.16}$$

Moreover, on the same event $E$,

$$\hat{d}_r \geq \sqrt{1 + b_r^{(n')}} - \frac{2|\Delta_1| + |\Delta_2|}{\sqrt{1 + \Delta_2(1 + \Delta_1 + \sqrt{1 + \Delta_2})}}$$

$$\geq \frac{3}{4} - \frac{2|\Delta_1| + |\Delta_2|}{\sqrt{1 + \Delta_2(1 + \Delta_1 + \sqrt{1 + \Delta_2})}} \geq \frac{1}{2}, \tag{5.17}$$

$$\left| \frac{\sqrt{1 + b_r^{(n')}}}{\hat{d}_r} - 1 \right| \lesssim_\gamma \frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \tag{5.18}$$

and also, using bound \((4.21)\), we obtain that

$$|\hat{d}_r - 1| \leq |\sqrt{1 + b_r^{(n')}} - 1| + \frac{2|\Delta_1| + |\Delta_2|}{\sqrt{1 + \Delta_2(1 + \Delta_1 + \sqrt{1 + \Delta_2})}}$$

$$\leq |b_r^{(n')}}| + \frac{2|\Delta_1| + |\Delta_2|}{\sqrt{1 + \Delta_2(1 + \Delta_1 + \sqrt{1 + \Delta_2})}}$$

$$\lesssim_\gamma \frac{||\Sigma||^2}{g_r^2} \frac{r(\Sigma)}{n'} + \frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \tag{5.19}$$

and

$$\left| \frac{1}{\hat{d}_r} - 1 \right| \lesssim_\gamma \frac{||\Sigma||^2}{g_r^2} \frac{r(\Sigma)}{n'} + \frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \tag{5.20}$$

The key ingredient of the proof of Theorem \ref{thm:efficient_estimation} is the following lemma.

**Lemma 5.2.** Suppose that, for some $\gamma \in (0, 1)$, conditions \((5.1)\) and \((5.12)\) hold. Then, for all $t \geq 1$ with probability at least $1 - e^{-t}$

$$\left| \langle \tilde{\theta}_r - \theta_r, u \rangle - \langle L_{\epsilon}(\tilde{\Sigma}(1) - \Sigma)\theta_r, u \rangle \right| \lesssim_\gamma \frac{||\Sigma||^2}{g_r^2} \left( \sqrt{\frac{r(\Sigma)}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \|u\|. \tag{5.21}$$

**Proof.** We use the following simple representation:

$$\langle \tilde{\theta}_r - \theta_r, u \rangle = \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')}}\theta_r, u \rangle$$

$$+ \left( \frac{1}{\hat{d}_r} - 1 \right) \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')}}\theta_r, u \rangle + \left( \frac{\sqrt{1 + b_r^{(n')}}}{\hat{d}_r} - 1 \right) \langle \theta_r, u \rangle \tag{5.22}$$

that holds on the event $E$ (where $\hat{d}_r \geq 1/2$). Using bounds \((5.18)\) and \((5.20)\) that both hold under assumption \((5.15)\) on the event $E$ as well as bound \((4.26)\)
(applied to \( \hat{\theta}_r^{(1)} \) with \( n = n' \)), we obtain that with probability at least \( 1 - 2e^{-t} \)

\[
\left| \langle \hat{\theta}_r - \theta_r, u \rangle - \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')} \theta_r}, u \rangle \right| \\
\lesssim \gamma \frac{\|\Sigma\|}{g_r^2} \sqrt{\frac{r}{m}} \left( \sqrt{\frac{r}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \|u\| 
\]

It is easy to check that the last term in the right hand side is dominant yielding the simpler bound

\[
\left| \langle \hat{\theta}_r - \theta_r, u \rangle - \langle \hat{\theta}_r^{(1)} - \sqrt{1 + b_r^{(n')} \theta_r}, u \rangle \right| \\
\lesssim \gamma \frac{\|\Sigma\|}{g_r^2} \sqrt{\frac{r}{m}} \left( \sqrt{\frac{r}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \|u\|.
\]

(5.23)

that holds under assumption (5.15) with probability at least \( 1 - e^{-t} \). Since the left hand side is bounded by \( 5\|u\| \), bound (5.23) also holds trivially when

\[
\frac{\|\Sigma\|^2}{g_r^2} \left( \sqrt{\frac{r}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \|u\| > c_\gamma.
\]

It remains to combine (5.23) with the bound (4.16) (applied to \( \hat{\theta}_r^{(1)} \)) to complete the proof.

\[
\square
\]

The following statement is an immediate consequence of Lemma 5.2 and Lemma 4.2. As always, we dropped the terms \( \frac{t}{n'}, \frac{1}{m} \) from the bounds since the left-hand side is smaller than \( 3\|u\| \) and, for \( t \geq n' \) or \( t \geq m \) (the only cases when these terms might be needed), it is dominated by the expression with \( \sqrt{\frac{t}{n'}}, \sqrt{\frac{t}{m}} \) only.

**Corollary 5.1.** Suppose that, for some \( \gamma \in (0, 1) \), conditions (5.1) and (5.12) hold. Then, for all \( t \geq 1 \) with probability at least \( 1 - e^{-t} \)

\[
\left| \langle \hat{\theta}_r - \theta_r, u \rangle \right| \lesssim \gamma \frac{\|\Sigma\|}{g_r} \sqrt{\frac{t}{m}} \|u\| + \frac{\|\Sigma\|^2}{g_r} \left( \sqrt{\frac{r}{m}} \sqrt{\frac{t}{m}} \right) \sqrt{\frac{t}{m}} \|u\|.
\]

Lemma 5.2 implies the following statement. This, in turn, implies Theorem 3.3.

**Lemma 5.3.** Suppose that \( m^2 \geq 2nr(\Sigma) \) and conditions (5.1) and (5.12) hold for some \( \gamma \in (0, 1) \). For a given \( u \in H \), suppose that \( \sigma_r(\Sigma; u) > 0 \). Let \( \alpha \geq 1 \).
Then the following bounds holds: for some constants $C, C_{\gamma, \alpha} > 0$,
\[
\sup_{x \in \mathbb{R}} \left\{ \frac{\sqrt{n} (\hat{\theta}_r - \theta_r, u)}{\sigma_r (\Sigma; u)} \leq x \right\} - \Phi(x) 
\leq C (n')^{-1/2} + \frac{C_{\gamma, \alpha}}{\sigma_r (\Sigma; u)} \| \Sigma \|^2 \left( \sqrt{\frac{nr(\Sigma)}{m^2}} \log \frac{m^2}{nr(\Sigma)} \right) \sqrt{n \log^2 \frac{m^2}{nr(\Sigma)}} \| u \| + \left( \frac{nr(\Sigma)}{m^2} \right)^{\alpha}.
\] (5.24)

Moreover, denote
\[
\tau_1 := C_\gamma \left( \frac{\| \Sigma \|}{g_r} \sqrt{\frac{\| \Sigma \|^2}{g_r^2} \sqrt{\frac{nr(\Sigma)}{m^2}}} \| u \| \right)
\]
and
\[
\tau_2 := C_\gamma \frac{\| \Sigma \|^2}{g_r^2} \sqrt{\frac{n}{m^2}} \| u \|.
\]

Suppose that Assumptions 3.1 on the loss $\ell$ holds and $c_2 \tau_2 \leq 1/4$. There exist constants $C, C_\gamma, C_{\gamma, \alpha} > 0$ such that
\[
\| \mathbb{E} \left( \sqrt{n} (\hat{\theta}_r - \theta_r, u) \right) \| - \mathbb{E} \| Z \| 
\leq c_1 e^{c_2 A} \left( C (n')^{-1/2} + \frac{C_{\gamma, \alpha}}{\sigma_r (\Sigma; u)} \| \Sigma \|^2 \left( \sqrt{\frac{nr(\Sigma)}{m^2}} \log \frac{m^2}{nr(\Sigma)} \right) \sqrt{n \log^2 \frac{m^2}{nr(\Sigma)}} \| u \| + \left( \frac{nr(\Sigma)}{m^2} \right)^{\alpha} + 2e^{3/2} (2\pi)^{1/4} e^{c_2^2 r_4^2} (e^{-A^2/2r_4^2} \lor e^{-A/2r_2}) + c_1 e^{c_2^2} e^{-A^2/4}. \right)
\] (5.25)

Proof. The proof is similar to that of Lemma 4.8. To prove (5.24), we apply the first bound of Lemma 4.6 to the random variables
\[
\xi := \frac{\sqrt{n} (\hat{\theta}_r - \theta_r, u)}{\sigma_r (\Sigma; u)}, \quad \eta := \frac{\langle L_r (\hat{\Sigma}(1) - \Sigma) \theta_r, u \rangle}{\sigma_r (\Sigma; u)}.
\]
and use the bound of Lemma 5.2 with $t = \alpha \log \left( \frac{m^2}{nr(\Sigma)} \right)$ to control $\delta(\xi, \eta)$.

To prove the bound (5.25), observe that, by bound (5.24), for all $t \geq 1$ with probability at least $1 - e^{-t}$
\[
|\xi| \leq \tau_1 \sqrt{t} \lor \tau_2 t.
\]
Under assumption $c_2 \tau_2 \leq 1/4$, bound (4.28) implies that
\[
\text{Eff}^2(\xi) \leq 2e \sqrt{2\pi c_1} e^{2\gamma^2 t} + \frac{6c_1^2}{1 - 2c_2 \tau_2} \leq 4e \sqrt{2\pi c_1} e^{2\gamma^2 t}.
\]
Therefore,
\[
\text{Eff}(\xi) I(|\xi| \geq A) \leq \text{Eff}^{1/2}(\xi) \text{Eff}^{1/2}(|\xi| \geq A) \leq 2e^{3/2} (2\pi)^{1/4} c_1 e^{c_2^2 r_4^2} (e^{-A^2/2r_4^2} \lor e^{-A/2r_2}).
\]
It remains to repeat the rest of the proof of the second statement of Lemma 4.8.  \( \square \)
6. Proof of Corollary 3.2

The proof is based on a deterministic bound on $|\sigma_r^2(\Sigma; u) - \sigma_r^2(\Sigma; u)|$ for a small perturbation $\Sigma$ of $\Sigma$ provided by the following lemma.

**Lemma 6.1.** Let $m_r = 1$. Denote $E := \Sigma - \Sigma$ and suppose that $\|E\| \leq g_r/4$. Then

$$|\sigma_r^2(\Sigma; u) - \sigma_r^2(\Sigma; u)| \lesssim \frac{\|E\|^2}{g_r^2} \|u\|^2.$$  \hspace{1cm} (6.1)

and

$$\left| \frac{\sigma_r(\Sigma; u)}{\sigma_r(\Sigma; u)} - 1 \right| \lesssim \frac{1}{\sigma_r^2(\Sigma; u)} \frac{\|E\|^2}{g_r^2} \|u\|^2.$$  \hspace{1cm} (6.2)

**Proof.** We use the Riesz representation of the spectral projector $P_r(\Sigma)$

$$P_r(\Sigma) = \frac{1}{2\pi i} \oint_{\gamma_r} R_\Sigma(\eta) d\eta,$$

where $R_B(\eta) = (B - \eta I)^{-1}$ denotes the resolvent of operator $B$ and $\gamma_r$ is the circle in $\mathbb{C}$ with center $\mu_r$ and radius $g_r/2$ (and with counterclockwise orientation). Since $\|E\| \leq \frac{g_r}{4}$ and $m_r = 1$, it is easy to see that there is only one eigenvalue $\lambda_r(\Sigma)$ of $\Sigma$ inside $\gamma_r$ and that $\text{dist}(\eta; \lambda_r(\Sigma)) \geq \frac{g_r}{4}, \eta \in \gamma_r$. Note also that, for all $\eta \in \gamma_r$,

$$\|R_\Sigma(\eta)\| \leq \frac{2}{g_r}, \quad \|R_\Sigma(\eta)\| \leq \frac{4}{g_r}$$  \hspace{1cm} (6.3)

and

$$R_\Sigma(\eta) - R_\Sigma(\eta) = (\Sigma - \eta I + E)^{-1} - (\Sigma - \eta I)^{-1}$$

$$= \left[ (I + R_\Sigma(\eta)E)^{-1} - I \right] R_\Sigma(\eta).$$  \hspace{1cm} (6.4)

It follows that, for all $\eta \in \gamma_r$,

$$\|R_\Sigma(\eta) - R_\Sigma(\eta)\| \leq \frac{2}{g_r} \|I + R_\Sigma(\eta)E\|^{-1} \|E\| \|R_\Sigma(\eta)\|^k \lesssim \frac{8\|E\|^2}{g_r^2}.$$  \hspace{1cm} (6.5)

Denote $A(\Sigma) := \theta_r(\Sigma) \otimes u + u \otimes \theta_r(\Sigma)$, $B(\Sigma) := P_r(\Sigma) \otimes C_r(\Sigma) + C_r(\Sigma) \otimes P_r(\Sigma)$ and

$$D(\Sigma) := B(\Sigma) A(\Sigma) = \theta_r(\Sigma) \otimes C_r(\Sigma) u + C_r(\Sigma) u \otimes \theta_r(\Sigma).$$

We have

$$\frac{1}{2\pi i} \oint_{\gamma_r} R_\Sigma(\eta) \otimes R_\Sigma(\eta) d\eta = \sum_{s, s'=1}^{\infty} \frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{(\mu_s - \eta)(\mu_{s'} - \eta)} P_s \otimes P_{s'}$$

$$= \sum_{s \neq r} \frac{1}{\mu_r - \mu_s} (P_r \otimes P_s + P_s \otimes P_r)$$

$$= P_r(\Sigma) \otimes C_r(\Sigma) + C_r(\Sigma) \otimes P_r(\Sigma) = B(\Sigma).$$  \hspace{1cm} (6.6)
Hence, using (6.3) and (6.5), we derive the following bound for any bounded operator $H$:

$$
\left\| (B(\tilde{\Sigma}) - B(\Sigma))H \right\|
\leq \frac{g_r}{2} \|E\| \left( \frac{4}{g_r} + \frac{2}{g_r} \right) \leq \frac{24\|E\|\|H\|}{g_r^2}.
$$

(6.7)

Note also that

$$
\|A(\tilde{\Sigma})\| \leq 2\|u\|,
$$

(6.8)

and, using the bound $\|C_r(\Sigma)\| \leq \frac{1}{g_r}$,

$$
\|B(\Sigma)H\| \leq \|P_r(\Sigma)HC_r(\Sigma)\| + \|C_r(\Sigma)HP_r(\Sigma)\| \leq \frac{2}{g_r}\|H\|.
$$

(6.9)

Finally, observe that, by standard perturbation bounds,

$$
\|A(\tilde{\Sigma}) - A(\Sigma)\| \leq 2\|\theta_r(\tilde{\Sigma}) - \theta_r(\Sigma)\|\|u\|
\leq 2\|P_r(\tilde{\Sigma}) - P_r(\Sigma)\|_2\|u\| \leq 2\sqrt{2}\|P_r(\tilde{\Sigma}) - P_r(\Sigma)\|_2\|u\|
\leq \frac{8\sqrt{2}\|E\|\|u\|}{g_r}.
$$

(6.10)

It follows from bounds (6.7), (6.8), (6.9) and (6.10) that

$$
\|D(\tilde{\Sigma}) - D(\Sigma)\| \leq \|(B(\tilde{\Sigma}) - B(\Sigma))A(\tilde{\Sigma})\| + \|B(\Sigma)(A(\tilde{\Sigma}) - A(\Sigma))\|
\leq \frac{24\|E\|\|A(\tilde{\Sigma})\|}{g_r^2} + \frac{2}{g_r}\|A(\tilde{\Sigma}) - A(\Sigma)\| \leq \frac{48\|E\|\|u\|}{g_r^2} + \frac{2}{g_r}\|A(\tilde{\Sigma}) - A(\Sigma)\|
\leq \frac{80\|E\|\|u\|}{g_r^2}.
$$

(6.11)

Now, recall that

$$
\sigma^2_r(\Sigma; u) = \langle \Sigma \theta_r(\Sigma), \theta_r(\Sigma) \rangle \langle \Sigma C_r(\Sigma)u, C_r(\Sigma)u \rangle
= \frac{1}{2} \left\| \Sigma^{1/2} (\theta_r(\Sigma) \otimes C_r(\Sigma)u + C_r(\Sigma)u \otimes \theta_r(\Sigma) \Sigma^{1/2} ) \right\|^2_2
= \frac{1}{2} \left\| \Sigma^{1/2} D(\Sigma) \Sigma^{1/2} \right\|^2_2 = \frac{1}{2} \text{tr}(\Sigma^{1/2} D(\Sigma) \Sigma^{1/2} \Sigma^{1/2} D(\Sigma) \Sigma^{1/2})
= \frac{1}{2} \text{tr}(\Sigma D(\Sigma) \Sigma D(\Sigma)).
$$

(6.12)
Hence, by the duality between operator and nuclear norms and since rank$(D(\Sigma)) \leq 2$, rank$(D(\tilde{\Sigma})) \leq 2$, we have that

$$|\sigma^2_r(\tilde{\Sigma}; u) - \sigma^2_r(\Sigma; u)| = \frac{1}{2} \left| \text{tr}(\tilde{\Sigma} D(\tilde{\Sigma}) \tilde{\Sigma} D(\tilde{\Sigma})) - \text{tr}(\Sigma D(\Sigma) \Sigma D(\Sigma)) \right|$$

$$= \frac{1}{2} \left| \text{tr}((\tilde{\Sigma} - \Sigma) D(\tilde{\Sigma}) \tilde{\Sigma} D(\tilde{\Sigma})) + \text{tr}(\Sigma D(\Sigma)(\tilde{\Sigma} - \Sigma) D(\tilde{\Sigma})) \right|$$

$$\leq \frac{1}{2} \|	ilde{\Sigma} - \Sigma\| \left( \|D(\tilde{\Sigma})\Sigma D(\Sigma)\|_1 + \|D(\tilde{\Sigma})\Sigma D(\Sigma)\|_1 \right)$$

$$+ \frac{1}{2} \|	ilde{\Sigma} - \Sigma\| \left( \|D(\tilde{\Sigma})\Sigma D(\Sigma)\| + \|D(\tilde{\Sigma})\Sigma D(\Sigma)\| \right)$$

$$+ \|D(\tilde{\Sigma}) - D(\Sigma)\left( \|\tilde{\Sigma} D(\Sigma)\Sigma D(\Sigma)\| \right) + \|\Sigma D(\Sigma)\Sigma D(\Sigma)\| \right).$$  \hspace{1cm} (6.13)

It remains to observe that \( \|C_r(\Sigma)\| \leq \frac{1}{g_r}, \|C_r(\tilde{\Sigma})\| \leq \frac{2}{g_r} \) and that

\[
\|D(\Sigma)\| \leq 2\|\theta_r(\Sigma) \otimes C_r(\Sigma)u\| \leq 2\|C_r(\Sigma)\|\|u\| \leq \frac{2\|u\|}{g_r},
\]

\[
\|D(\tilde{\Sigma})\| \leq 2\|\theta_r(\tilde{\Sigma}) \otimes C_r(\tilde{\Sigma})u\| \leq 2\|C_r(\tilde{\Sigma})\|\|u\| \leq \frac{4\|u\|}{g_r}
\]

and

\[
\|\tilde{\Sigma}\| \leq \|\Sigma\| + \|E\| \leq \|\Sigma\| + \frac{g_r}{4} \leq 2\|\Sigma\|
\]

implying the bounds

\[
\|D(\tilde{\Sigma})\Sigma D(\Sigma)\| \leq \frac{32\|\Sigma\|\|u\|^2}{g_r^2}, \quad \|D(\tilde{\Sigma})\Sigma D(\Sigma)\| \leq \frac{8\|\Sigma\|\|u\|^2}{g_r^2},
\]

\[
\|\tilde{\Sigma} D(\Sigma)\Sigma\| \leq \frac{8\|\Sigma\|\|u\|}{g_r} \quad \text{and} \quad \|\Sigma D(\Sigma)\Sigma\| \leq \frac{8\|\Sigma\|\|u\|}{g_r}.
\]  \hspace{1cm} (6.14)

Bound (6.1) now follows from (6.13), (6.11) and (6.14). Bound (6.2) follows from (6.1).

It remains to apply this lemma to \( \tilde{\Sigma} = \hat{\Sigma} \) and to use standard bounds on \( \|\tilde{\Sigma} - \Sigma\| \) to obtain the following inequalities.

**Proposition 6.1.** Suppose that condition (4.2) holds for some \( \gamma \in (0,1) \). Then, there exists a constant \( c_\gamma > 0 \) such that for all \( t \in [1,c_\gamma n] \) with probability at least \( 1 - e^{-t} \)

\[
|\sigma^2_r(\hat{\Sigma}; u) - \sigma^2_r(\Sigma; u)| \leq \frac{\|\Sigma\|^3}{g_r^3} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \right) \|u\|^2
\]  \hspace{1cm} (6.15)
and
\[
\left| \frac{\sigma_r(\hat{\Sigma}; u)}{\sigma_r(\Sigma; u)} - 1 \right| \lesssim \frac{1}{\sigma_r^2(\Sigma; u)} \frac{\|\Sigma\|^3}{g_r^3} \left( \sqrt{\frac{r(\Sigma)}{n}} \sqrt{\frac{t}{n}} \|u\|^2 \right).
\] (6.16)

The consistency of estimator \( \sigma_r(\hat{\Sigma}; u) \) immediately follows:

**Proposition 6.2.** Suppose \( \tau_n > 1, \tau_n = o(n) \) as \( n \to \infty \). For any sequence \( \delta_n \to 0 \) such that \( \frac{\tau_n}{n} = o(\delta_n^2) \) as \( n \to \infty \),

\[
\sup_{\Sigma \in \mathcal{S}(r)(\tau_n, a, \sigma_0, u)} \mathbb{P}_\Sigma \left\{ \left| \frac{\sigma_r(\hat{\Sigma}; u)}{\sigma_r(\Sigma; u)} - 1 \right| \geq \delta_n \right\} \to 0 \quad \text{as} \quad n \to \infty.
\]

Corollary 3.2 can be easily proved using the first statement of Theorem 3.3, Proposition 6.2 and Lemma 4.6.

7. Proof of Theorem 3.2

Note that the set \( \mathcal{S}(r)(\tau, a, \sigma_0, u) \) is open in nuclear norm topology. This easily follows from the continuity of functions \( \Sigma \mapsto \|\Sigma\|, \Sigma \mapsto \bar{g}_r(\Sigma) \) and \( \Sigma \mapsto \sigma_r^2(\Sigma; u) \) with respect to the operator norm (for the last function, see Lemma 6.1) and, as a consequence, with respect to the nuclear norm, and of the functions \( \Sigma \mapsto \text{tr}(\Sigma) \) and \( \Sigma \mapsto r(\Sigma) \) with respect to the nuclear norm.

Let \( \Sigma = \sum_{s=1}^{\infty} \mu_s P_s \in \mathcal{S}(r)(\tau, a, \sigma_0, u) \). Without loss of generality, assume that \( \Sigma \) is of finite rank. Otherwise, consider \( \Sigma_N := \sum_{s=1}^{N} \mu_s P_s \). Clearly,

\[
r(\Sigma_N) \leq r(\Sigma) < \tau
\]

and, for all \( N > r \),

\[
\frac{\|\Sigma_N\|}{\bar{g}_r(\Sigma_N)} = \frac{\|\Sigma\|}{\bar{g}_r(\Sigma)} < a.
\]

Moreover, since \( \|\Sigma_N - \Sigma\| \to 0 \) as \( N \to \infty \), we also have that \( \sigma_r^2(\Sigma_N; u) \to \sigma_r^2(\Sigma; u) \) as \( N \to \infty \), implying that \( \sigma_r^2(\Sigma_N; u) > \sigma_0^2 \) for all large enough \( N \). Thus, \( \Sigma_N \in \mathcal{S}(r)(\tau, a, \sigma_0, u) \) for a sufficiently large \( N \) and we can replace \( \Sigma \) by \( \Sigma_N \). Assuming that rank(\( \Sigma \)) < \( \infty \), let \( L := \text{Im}(\Sigma) \). We can now restrict \( \Sigma \) to an operator acting from \( L \) to \( L \), which is non-singular. In what follows, all the covariance operators from the class \( \mathcal{S}(r)(\tau, a, \sigma_0, u) \) that are of interest to us will have \( L \) as an image and could be viewed as operators from \( L \) to \( L \). For simplicity, we just assume that \( H = L \) is a finite-dimensional space. For a fixed \( \Sigma \), consider the following parametric family of perturbations of \( \Sigma \):

\[
\Sigma_t := \Sigma + \frac{tH}{\sqrt{n}}, |t| \leq c,
\]

where \( H \) is a self-adjoint operator and \( c > 0 \) is a constant. Denote

\[
\mathcal{S}_{\Sigma, c} := \{ \Sigma_t : t \in [-c, c] \}.
\]
Since the set $\hat{S}^{(r)}(r, a, \sigma_0, u)$ is open in nuclear norm topology, there exists $\delta > 0$ such that the condition
\begin{equation}
\frac{c\|H\|_1}{\sqrt{n}} < \delta,
\end{equation}
implies that $S_{\Sigma, c} \subset \hat{S}^{(r)}(r, a, \sigma_0, u)$. Moreover, we will assume that
\begin{equation}
\delta < \|\Sigma^{-1}\|^{-1}
\end{equation}
and
\begin{equation}
\delta < \frac{1}{4} \bar{g}_r(\Sigma).
\end{equation}
Under these assumptions and condition (7.1), $\Sigma_t$ is a small enough perturbation of $\Sigma$ so that $\Sigma_t$ is non-singular and we can define in a standard way the one-dimensional spectral projection operator $P_t := P_r(\Sigma_t) = \theta_t \otimes \theta_t$, where $\theta_t = \theta_r(\Sigma_t)$ is the corresponding unit eigenvector as well as operators $C_t := C_r(\Sigma_t)$ and
\begin{equation}
L_t(H) := L_r(\Sigma_t)(H) = P_t H C_t + C_t H P_t.
\end{equation}
It is easy to see that (for a given $c > 0$ and large enough $n$ so that the perturbation is small) one can choose $t \mapsto \theta_t$ in such a way that $\langle \theta_t, \theta_t' \rangle \geq 0$, $t, t' \in [-c, c]$.

Based on these definitions, we also define the functions $g(t) := \langle \theta_t, u \rangle$ and $\sigma^2(t) := \sigma_r^2(\Sigma_t; u)$. Concerning the function $g$, we need the following lemma.

**Lemma 7.1.** The function $g$ is continuously differentiable in the interval $[-c, c]$ and the following statements hold:

1) $g'(t) = \frac{1}{\sqrt{n}} \langle L_t(H) \theta_t, u \rangle$, $t \in [-c, c]$.

2) $|g'(t) - g'(0)| \lesssim \frac{\|H\|^2}{g^2 r} \|u\|$, $t \in [-c, c]$.

**Proof.** Let $\delta \in (-1, 1)$. Similarly to (4.17) (see also (6.6) in [15]),
\begin{equation}
g(t + \delta) - g(t) = \langle (P_{t+\delta} - P_t) \theta_t, u \rangle - \langle (P_{t+\delta} - P_t) \theta_t, \theta_t \rangle \rangle 1 \langle (P_{t+\delta} - P_t) \theta_t, \theta_t \rangle
\end{equation}

Applying the first order perturbation expansion (similar to (4.1)) to the spectral projections $P_t, P_{t+\delta}$, we obtain that
\begin{equation}
P_{t+\delta} - P_t = L_t(\delta H/\sqrt{n}) + S_t(\delta H/\sqrt{n})
\end{equation}
with the remainder term satisfying the bound
\begin{equation}
\|S_t(\delta H/\sqrt{n})\| \lesssim \frac{\delta^2 \|H\|^2}{g^2 r} = O(\delta^2).
\end{equation}
Moreover, since $C_t \theta_t = 0$,
\begin{equation}
\langle L_t(\delta H/\sqrt{n}) \theta_t, \theta_t \rangle = \frac{1}{\sqrt{n}} \langle (P_t H C_t + C_t H P_t) \theta_t, \theta_t \rangle = 0
\end{equation}
and therefore we have that
\[ |\langle (P_{t} + \delta - P_{t})\theta_{t}, \theta_{t} \rangle| \lesssim \frac{\delta^{2}\|H\|^{2}}{g_{r}^{2}n} = O(\delta^{2}). \quad (7.8) \]

Hence, using again (7.4), (7.6) and (7.8), we have that
\[ \frac{g(t + \delta) - g(t)}{\delta} = \frac{1}{\sqrt{n}} \frac{\langle L_{t}(H)\theta_{t}, u \rangle}{1 + O(\delta)} + O(\delta). \quad (7.9) \]

Passing to the limit as \( \delta \to 0 \) implies the first assertion.

We now prove the second claim. First note that
\[ |g'(t) - g'(0)| = |\langle L_{t}(H/\sqrt{n})\theta_{t} - L_{0}(H/\sqrt{n})\theta_{0}, u \rangle| \leq |\langle L_{0}(H/\sqrt{n})\theta_{t} - L_{0}(H/\sqrt{n})\theta_{0}, u \rangle| + |\langle L_{0}(H/\sqrt{n})\theta_{t} - \theta_{0}, u \rangle| \leq \|L_{t}(H/\sqrt{n}) - L_{0}(H/\sqrt{n})\| \|u\| + \|L_{0}(H/\sqrt{n})\| \|\theta_{t} - \theta_{0}\| \|u\|. \quad (7.10) \]

Also,
\[ L_{t}(H/\sqrt{n}) = -\frac{1}{2\pi i} \oint_{\gamma_{r}} R_{\Sigma_{t}}(\eta) \frac{H}{\sqrt{n}} R_{\Sigma_{t}}(\eta) d\eta, \quad (7.11) \]
where \( \gamma_{r} \) is the circle of radius \( g_{r}/2 \) with the center at \( \mu_{r} \) and with counterclockwise orientation. Therefore, by a standard argument already used in the proof of Lemma 6.1,
\[ \|L_{t}(H/\sqrt{n}) - L_{0}(H/\sqrt{n})\| \leq g_{r} \frac{1}{2} \sup_{\eta \in \gamma_{r}} \|R_{\Sigma_{t}}(\eta) - R_{\Sigma}(\eta)\| \|R_{\Sigma}(\eta)\| + \|R_{\Sigma_{t}}(\eta)\| \|H\| \frac{1}{\sqrt{n}}. \quad (7.12) \]

By (6.3) and (6.5), we have
\[ \|R_{\Sigma}(\eta)\| \leq \frac{2}{g_{r}}, \quad \|R_{\Sigma_{t}}(\eta)\| \leq \frac{4}{g_{r}} \]
and
\[ \|R_{\Sigma_{t}}(\eta) - R_{\Sigma}(\eta)\| \leq \frac{8}{g_{r}^{2}} \frac{|t|\|H\|}{\sqrt{n}}. \]

Therefore, it follows from (7.12) that
\[ \|L_{t}(H/\sqrt{n}) - L_{0}(H/\sqrt{n})\| \leq \frac{24|t|\|H\|^{2}}{g_{r}^{2}n}. \quad (7.13) \]

It remains to observe that
\[ \|L_{0}(H)\| = \|P_{r}HC_{r} + C_{r}HP_{r}\| \leq \frac{2\|H\|}{g_{r}} \]
and
\[ \| \theta_t - \theta_0 \| \leq \| P_t - P_0 \|_2 \leq \frac{4\sqrt{2}|t|\| H \|}{g_\epsilon \sqrt{n}} \]
(where we also used the fact that \( \text{rank}(P_t - P_0) \leq 2 \) and \( \| P_t - P_0 \|_2 \leq \sqrt{2}\| P_t - P_0 \| \). This implies the bound
\[ \| L_0(H/\sqrt{n}) \| \| \theta_t - \theta_0 \| \| u \| \leq \frac{8\sqrt{2}|t|\| H \|^2}{g_\epsilon^2 n} \| u \|. \quad (7.14) \]

The second assertion follows from the bounds \( (7.10), (7.13) \) and \( (7.14) \).

The continuity of the derivative \( g'(t) \) easily follows from the continuity of the functions \( t \mapsto \theta_t \) and \( t \mapsto L_t(H/\sqrt{n}) \) (which could be proved using representation \( (7.11) \)).

We will study the following estimation problem. Let \( \Sigma \) be fixed and let \( X_1, \ldots, X_n \) be i.i.d. random variables in \( \mathbb{H} \) sampled from \( N(0; \Sigma), |t| \leq c, \) \( t \) being an unknown parameter. The goal is to estimate the function \( g(t) \) based on the observations \( X_1, \ldots, X_n \). We will use van Trees inequality to obtain a minimax lower bound on the risk of estimation of \( g(t) \) with respect to quadratic loss. To this end, let \( \pi \) be a smooth probability density on \([-1, 1]\), satisfying the boundary conditions \( \pi(-1) = \pi(1) = 0 \) as well the condition \( J_\pi := \int_{-1}^{1} \frac{\pi'(s)^2}{\pi(s)} ds < \infty \).

Let \( \pi_c(t) := \frac{1}{c} \pi \left( \frac{t}{c} \right) \), \( t \in [-c, c] \) be a prior on \([-c, c]\). Then (see e.g. [10]), for any estimator \( T_n = T_n(X_1, \ldots, X_n) \) of \( g(t) \) the following bound holds
\[ \sup_{|t| \leq c} n\mathbb{E}_t(T_n - g(t))^2 \geq n \int_{-c}^{c} \mathbb{E}_t(T_n - g(t))^2 \pi_c(t) dt \]
\[ \geq \frac{n \left( \int_{-c}^{c} g'(t) \pi_c(t) dt \right)^2}{\int_{-c}^{c} \mathbb{I}_n(t) \pi_c(t) dt + J_\pi}, \quad (7.15) \]

where \( \mathbb{I}_n(t) = n\mathbb{I}(t) \) denotes the Fisher information for the model
\[ X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(0, \Sigma), \]
\( t \in [-c, c] \). Let \( \mathbb{I}(t) := \mathbb{I}_1(t) \). It is well known that the Fisher information for the model \( X \sim N(0; \Sigma) \) with non-singular covariance matrix \( \Sigma \) is \( \mathbb{I}(\Sigma) = \frac{1}{2}(\Sigma^{-1} \otimes \Sigma^{-1}) \) (see, e.g., [8]). Thus,
\[ \mathbb{I}_n(t) = n\mathbb{I}(t) = n \left\langle \mathbb{I}(\Sigma) \frac{d\Sigma_t}{dt}, \frac{d\Sigma_t}{dt} \right\rangle = \frac{n}{2} \left\langle (\Sigma_t^{-1} \otimes \Sigma_t^{-1}) \frac{H}{\sqrt{n}}, \frac{H}{\sqrt{n}} \right\rangle \]
\[ = \frac{1}{2} \langle \Sigma_t^{-1} H \Sigma_t^{-1}, H \rangle = \frac{1}{2} \text{tr}(\Sigma_t^{-1} H \Sigma_t^{-1} H). \]

We will now bound the numerator of the expression in the right hand side of inequality \((7.15)\) from below and its denominator from above.
Bound on the numerator. We use Lemma 7.1 to obtain that for some constant $B_1 > 0$

$$\left( \int_{-c}^{c} g'(t) \pi_c(t) dt \right)^2 = \left( \int_{-c}^{c} [g'(0) + (g'(t) - g'(0))] \pi(t/c) dt/c \right)^2$$

$$\geq g'(0)^2 + 2g'(0) \int_{-c}^{c} (g'(t) - g'(0)) \pi(t/c) dt/c$$

$$\geq g'(0)^2 - 2g'(0) \int_{-c}^{c} |g'(t) - g'(0)| \pi(t/c) dt/c$$

$$\geq g'(0)^2 - B_1 c g'(0) \int_{-c}^{c} |t| \pi(t) dt \|H\|_2^2 g_{\pi n}^2 \|u\|$$

$$= \frac{\langle L_r(H) \theta_r, u \rangle^2}{n} - \left| \frac{\langle L_r(H) \theta_r, u \rangle}{g_{\pi n}^2} \right| \|H\|_2^2 g_{\pi n}^2 \|u\|. \quad (7.16)$$

Bound on the denominator. First note that, by a simple computation,

$$J_{\pi c} = J_\pi / c^2. \quad (7.17)$$

Then, we need to bound $I_n(t) = \frac{1}{2} \text{tr}(\Sigma t^{-1} H \Sigma^{-1} H)$ in terms of $I_n(0) = \frac{1}{2} \text{tr}(\Sigma^{-1} H \Sigma^{-1} H)$.

Assume that

$$c \frac{\|\Sigma^{-1} H\|}{\sqrt{n}} \leq \frac{1}{2}. \quad (7.18)$$

Arguing as in the proof of Lemma 6.1, we easily get that

$$\Sigma t^{-1} = \Sigma^{-1} + \left[ I + t \frac{\Sigma^{-1} H}{\sqrt{n}} \right]^{-1} \Sigma^{-1}, \quad (7.19)$$

where

$$\|D\| \leq 2|t| \frac{\|\Sigma^{-1} H\|}{\sqrt{n}} \leq 1.$$

Furthermore, note that

$$\text{tr} (\Sigma t^{-1} H \Sigma^{-1} H) = \text{tr}(\Sigma^{-1} H \Sigma^{-1} H) + 2 \text{tr}(D \Sigma^{-1} H \Sigma^{-1} H) + \text{tr}(D \Sigma^{-1} H D \Sigma^{-1} H).$$

and thus we have that

$$I_n(t) \leq I_n(0) + \|D\| \|\Sigma^{-1} H \Sigma^{-1} H\|_1 + \frac{\|D \Sigma^{-1} H \Sigma^{-1} D\|_2}{2}$$

$$\leq I_n(0) + \left( \|D\| + \frac{\|D\|^2}{2} \right) \|\Sigma^{-1} H\|_2^2 \leq I_n(0) + 3 \frac{|t| \|\Sigma^{-1} H\|_2^3}{\sqrt{n}}. \quad (7.20)$$
Using (7.20), we obtain the following bound:
\[
\int_{-c}^{c} \mathbb{I}_n(t) \pi_c(t) dt \leq \mathbb{I}_n(0) + 3 \frac{\|\Sigma^{-1} H\|_2}{\sqrt{n}} \int_{-c}^{c} |t| \pi(t/c) dt / c \leq \mathbb{I}_n(0) + \frac{3c \|\Sigma^{-1} H\|_2^3}{\sqrt{n}}.
\] (7.21)

Substituting (7.16), (7.21) and (7.17) into van Trees inequality (7.15) and taking into account that
\[
\mathbb{I}_n(0) = \frac{1}{2} \text{tr}(\Sigma^{-1} H \Sigma^{-1} H) = \frac{1}{2} \|\Sigma^{-1/2} H \Sigma^{-1/2}\|_2
\]
and
\[
\langle L_r(H) \theta, u \rangle = \langle (P_r H C_r + C_r H P_r) \theta, u \rangle = \langle H \theta, C_r u \rangle = \frac{1}{2} \langle H, \theta_r \otimes C_r u + C_r u \otimes \theta_r \rangle = \langle \Sigma^{-1/2} H \Sigma^{-1/2}, \Sigma^{-1/2} B \Sigma^{-1/2} \rangle,
\]
where
\[
B := \frac{1}{2} (\Sigma \theta_r \otimes C_r u + C_r u \otimes \Sigma \theta_r),
\]
we obtain that
\[
\sup_{|t| \leq c} n \mathbb{E}_t (T_n - g(t))^2 \geq \frac{\langle \Sigma^{-1/2} H \Sigma^{-1/2}, \Sigma^{-1/2} B \Sigma^{-1/2}\rangle^2 - |\langle \Sigma^{-1/2} H \Sigma^{-1/2}, \Sigma^{-1/2} B \Sigma^{-1/2}\rangle|}{\frac{1}{2} \|\Sigma^{-1/2} H \Sigma^{-1/2}\|_2^2 + \frac{3c \|\Sigma^{-1} H\|_2^3}{\sqrt{n}} + J_\pi / c^2}
\] (7.22)

In what follows, we set \( H := B \). Note that with this choice of \( H \)
\[
2 \|\Sigma^{-1/2} B \Sigma^{-1/2}\|_2^2 = \frac{1}{2} \|\Sigma^{1/2} \theta_r \otimes \Sigma^{1/2} C_r u + \Sigma^{1/2} C_r u \otimes \Sigma^{1/2} \theta_r\|_2^2
\]
\[
= \frac{1}{2} \left( \|\Sigma^{1/2} \theta_r \otimes \Sigma^{1/2} C_r u\|_2^2 + \|\Sigma^{1/2} C_r u \otimes \Sigma^{1/2} \theta_r\|_2^2 \right) = \|\Sigma^{1/2} \theta_r\|_2^2 \|\Sigma^{1/2} C_r u\|_2^2 = \sigma_r^2(\Sigma; u).
\]

Also, by a simple computation (using that \( \text{rank}(B) = 2 \)), we have that
\[
\|B\| \leq \|B\|_2 \leq \frac{1}{\sqrt{2}} \frac{\|\Sigma\|_2}{g_r} \|u\|, \quad \|B\|_1 \leq \frac{\|\Sigma\|_2}{g_r} \|u\|
\] (7.23)
and that
\[
\|\Sigma^{-1} B\|_2 \leq \frac{1}{\sqrt{2}} \frac{\|\Sigma\|}{g_r} \|u\|.
\] (7.24)

These bounds imply that, for any given \( c > 0 \) and for all \( n \) large enough, \( H = B \) satisfies condition (7.1) for a small enough \( \delta \) such that \( S_{\Sigma; c} \subset \mathcal{S}_{(\tau, a, \sigma_0, u)} \) and conditions (7.2), (7.3) hold. Also, \( H = B \) satisfies condition (7.18) (for any given \( c > 0 \) and all large enough \( n \)).
For $H = B$, inequality (7.22) becomes
\[
\sup_{|t| \leq \varepsilon} n E_\varepsilon (T_n - g(t))^2 \\
\geq \left( 1 - \frac{B_1 c \|B\|^2}{2g_1 \sqrt{n}} + \frac{3c \|\Sigma^{-1} B\|^3}{\sqrt{n}} + J_\pi / c^2 \right) \left( 1 + \frac{D_1 \|\Sigma\| c \|B\|}{g_1^3 \sqrt{n}} \right) \sup_{|t| \leq \varepsilon} \sigma^2(t) \left( 1 \right).
\]

It remains to replace $\sigma^2(\Sigma; u)$ with $\sigma^2(t) = \sigma^2(\Sigma; u)$. To this end, we use the bound (6.1) to obtain that for some constant $D_1 > 0$
\[
\sup_{t \in [-c, c]} \frac{\sigma^2(t)}{\sigma^2(\Sigma; u)} \leq 1 + \frac{D_1}{\sigma^2(\Sigma; u)} \sup_{t \in [-c, c]} \frac{\|\Sigma\|^2 c \|B\|}{g_1^3 \sqrt{n}} \|u\|^3.
\]

It follows from (7.25) that
\[
\sup_{t \in [-c, c]} \frac{\sigma^2(t)}{\sigma^2(\Sigma; u)} \leq 1 + \frac{D_1}{\sigma^2(\Sigma; u)} \sup_{t \in [-c, c]} \frac{\|\Sigma\|^4 c}{g_1^3 \sqrt{n}} \|u\|^3 \leq 1,
\]

which holds for any given $c > 0$ and all large enough $n$ and which, in view of bounds (7.23), implies that
\[
\sup_{t \in [-c, c]} \frac{\|\Sigma\|^4 c}{g_1^3 \sqrt{n}} \|u\|^3 \leq 1.
\]

Under condition (7.28), bounds (7.27) and (7.26) (and also bounds (7.23) and (7.24)) imply that
\[
\frac{D_1}{\sigma^2(\Sigma; u)} \frac{\|\Sigma\|^2 c \|B\|}{g_1^3 \sqrt{n}} \leq 1.
\]

It remains to pass to the limit in inequality (7.29) first as $n \to \infty$ and then as $c \to \infty$ to complete the proof.

A local version of the theorem easily follows from the above arguments since, for all $\varepsilon > 0$, $c > 0$ and for all large enough $n$, $S_{\Sigma_0, c} \subset \{ \Sigma : \|\Sigma - \Sigma_0\|_1 \leq \varepsilon \}$. 

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