Hydrodynamic Limit for Interacting Neurons with Spatial Dependency

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Abstract

In this paper we study the hydrodynamic limit for a stochastic process describing the time evolution of the membrane potentials of a system of neurons with spatial dependency. We do not impose on the neurons mean-field type interactions. The values of the membrane potentials evolve under the effect of chemical and electrical synapses and leak currents. The system consists of \( \varepsilon^{-2} \) neurons embedded in \([0, 1)^2\), each spiking randomly according to a point process with rate depending on both its membrane potential and position. When neuron \( i \) spikes, its membrane potential is reset to a resting value while the membrane potential of \( j \) is increased by a positive value \( \varepsilon^2 a(i, j) \), if \( i \) influences \( j \). Furthermore, between consecutive spikes, the system follows a deterministic motion due both to electrical synapses and leak currents. The electrical synapses are involved in the synchronization of neurons. For each pair of neurons \((i, j)\), we modulate this synchronizing strength by \( \varepsilon^2 b(i, j) \), where \( b(i, j) \) is a nonnegative symmetric function. On the other hand, the leak currents inhibit the activity of all neurons, attracting simultaneously their membrane potentials to the resting value.

In the main result of this paper is shown that the empirical distribution of the membrane potentials converges, as the parameter \( \varepsilon \) goes to zero, to a probability density \( \rho_t(u, r) \) which is proved to obey a non linear PDE of Hyperbolic type.

Key words: Hydrodynamic limit, Piecewise deterministic Markov process, Neuronal Systems

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1 Introduction

In this paper we present a stochastic process which describes a population of interacting neurons with spatial dependency. Our aim is to study the hydrodynamic limit of such process and characterize its limit law as well.

Neurons are electrically excitable cells whose activity consist in sudden peaks, called action potentials and often referred to as spikes. More specifically, spikes are short-lasting

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electrical pulses in the membrane potential of the cell and the higher the membrane potential the higher the probability of a spike to occur. Thus, it is quite natural to assume that the generating mechanism of spikes is given by a point process in which the spiking rate of a given neuron depends on its membrane potential. In this paper, we work under that assumption and additionally, assume that the membrane potential evolves under the effect of chemical synapses, electrical synapses and leak currents.

Electrical synapses are due to so-called gap-junction channels between neurons which induce a constant sharing of potential. The unique aspect of electrical synapses is their reciprocity. This means they are neither excitatory nor inhibitory but rather synchronizer. For each pair of neurons \((i, j)\), we modulate this synchronizing strength by \(b(i, j)\), where \((i, j) \mapsto b(i, j)\) is a nonnegative symmetric function. For instance, if \(N\) is the size of the set of neurons and \(b(i, j) = N^{-1}\), each neuron will have its membrane potential pushed to the average membrane potential of the system. In the general case, the membrane potential of each neuron is also attracted to a mean value, although this value may vary for each neuron depending on the shape of the function \(b(i, j)\).

In contrast with electrical synapses, chemical synapses are point events which can be described as follows. Each neuron \(i\) with membrane potential \(U\) spikes randomly at rate \(\varphi(U, i)\), where \(U \mapsto \varphi(U, i)\) is a non decreasing function, positive at \(U > 0\) and vanishing at 0. This last assumption implies the absence of external stimuli. When neuron \(i\) spikes, its membrane potential is immediately reset to a resting potential 0. Simultaneously, the neurons which are influenced by neuron \(i\) receive an additional positive value to their membrane potential. Specifically, the membrane potential of neuron \(j\) is increased by the value \(a(i, j)\) in each spike of \(i\), if the latter influences the former. The positiveness of the function \((i, j) \mapsto a(i, j)\) means that all chemical synapses are of the excitatory type.

Additionally to the synapses, neurons loose potential to the environment along time due to leakage channels which pushes down the membrane potential of each neuron toward the resting state. This constant outgoing flow of potential is referred to, in the neurobiological literature, as leak currents. For an account on these subjects we refer the reader to Gerstner and Küstler (2002).

Our model is inspired by the ones introduced in Galves and Löcherbach (2013), De Masi et al. (2014) and Duarte and Ost (2014). For a critical readers guide to these papers - together with the one of Fournier and Löcherbach (2014) - we refer to Galves and Löcherbach (2015). Our model is also an example of piecewise deterministic Markov processes introduced by Davis (1984). Such processes combine random jump events, in our case due to the chemical synapses, with deterministic continuous evolutions, in our case due both to electrical synapses and leak currents. The piecewise deterministic Markov processes have been used also to model neuronal systems by other authors, see for instance the papers by Riedler, Thieullen and Wainrib (2012), De Masi et al. (2014), Duarte and Ost (2014), Fournier and Löcherbach (2014) and Robert and Touboul (2014).

In the study of Hydrodynamic limits a mean-field type assumption is quite frequent. This means that \(a(i, j) = b(i, j) = N^{-1}\) for any pair of neurons \((i, j)\), with \(N\) being the size of the population of neurons. For recent neuromathematical models adopting the mean-field assumption see, among others, the models in De Masi et al. (2014) and Fournier and Löcherbach (2014). However, a more realistic description should incorporate the mutual distance among neurons. In order to achieve such accurate description, we use Kac potentials ideas and techniques developed for such potentials in statistical mechanics.
In our context, this means that the functions \( a(i,j) \) and \( b(i,j) \) considered here are quite general but are scaled by factor \( N^{-1} \), if \( N \) stands for the size of the set of neurons. For an account on hydrodynamic limits and Kac potentials we refer to De Masi and Presutti (1991) and Presutti (2008).

For each \( \varepsilon > 0 \), the set of neurons is denoted by \( \Lambda_\varepsilon = \varepsilon \mathbb{Z}^2 \cap [0,1)^2 \) and the state of our system at time \( t \geq 0 \) is specified by \( U^{(\varepsilon)}(t) = \left( U^{(\varepsilon)}_i(t), i \in \Lambda_\varepsilon \right) \), with \( U^{(\varepsilon)}_i(t) \in \mathbb{R}_+ \). For each neuron \( i \in \Lambda_\varepsilon \) and time \( t \geq 0 \), \( U^{(\varepsilon)}_i(t) \) represents the membrane potential of neuron \( i \) at time \( t \). The main result of this paper, Theorem \ref{thm:main_result}, shows that the empirical distribution of the membrane potentials converges, as \( \varepsilon \to 0 \), to a law having, at each time \( t \), \( \rho_t(u,r)dudr \) as a probability density. This means that, in the limit, for any set \( C \subset [0,1)^2 \), interval \( I \subset \mathbb{R}_+ \) and time \( t \geq 0 \), \( \int_I \int_C \rho_t(u,r)dudr \) is the limit fraction of neurons located in \( C \) whose membrane potentials are inside of \( I \) at time \( t \). The limit density \( \rho_t(u,r) \) is proved to obey a nonlinear PDE of the hyperbolic type.

In order to prove this theorem, we first approximate the true process by a handy family of processes \( Y^{(\delta,\ell,E,\tau)} \) for which the analysis of the Hydrodynamic limit is somehow easier. Once established the convergence to \( Y^{(\delta,\ell,E,\tau)} \), we obtain the result by taking \( \delta, \ell, E, \tau \to 0 \). A similar approach was used in De Masi et al. (2014), however, in the present work, we exploit the spatial structure imposed on the set of neurons to simplify the hydrodynamic limit for the auxiliary process \( Y^{(\delta,\ell,E,\tau)} \).

Our paper is organized as follows. In section \ref{sec:main_results} we introduce our model and state the main results, Theorem \ref{thm:main_result} and \ref{thm:main_result2}. In addition, at the end of the section, we argue that it is possible to work, without lost of generality, under a stronger condition on the spiking rate \( \varphi \). In section \ref{sec:tightness} we prove Theorem \ref{thm:main_result} under this stronger condition. In Section \ref{sec:auxiliary_processes} the tightness of the sequence of processes indexed by \( \varepsilon \) is proven. In section \ref{sec:auxiliary_processes} we define the family of auxiliary processes as well as the coupling algorithm for the true and auxiliary processes. Moreover, we state Theorem \ref{thm:hydrodynamic_limit} which claims that the auxiliary and true processes are close to each other. Its proof is postponed to Appendix \ref{app:proofs}. In section \ref{sec:proofs} we state the hydrodynamic limit for the auxiliary process whose proof is given in the Appendix \ref{app:proofs}. In section \ref{sec:proofs} we conclude the proofs of Theorem \ref{thm:auxiliary_processes} and Theorem \ref{thm:hydrodynamic_limit}. In the Appendix \ref{app:extensions} we extend the results to general firing rates \( \varphi \).

## 2 Model Definition and Main Results

For each \( \varepsilon > 0 \), let \( \Lambda_\varepsilon = \varepsilon \mathbb{Z}^2 \cap [0,1)^2 \) be a \( \varepsilon \)-mesh of the set \([0,1)^2 \). The set \( \Lambda_\varepsilon \) represents the set of neurons and its size is \( |\Lambda_\varepsilon| = \varepsilon^{-2} \). We consider a continuous time Markov process \((U^{(\varepsilon)}(t))_{t \geq 0}\) taking values in \( \mathbb{R}_+^{\Lambda_\varepsilon} \). For each \( t \geq 0 \) and neuron \( i \in \Lambda_\varepsilon \), \( U^{(\varepsilon)}_i(t) \) models the membrane potential of neuron \( i \) at time \( t \). The global configuration at time \( t \geq 0 \) is denoted by

\[
U^{(\varepsilon)}(t) = (U^{(\varepsilon)}_i(t), i \in \Lambda_\varepsilon).
\]

As usual in the theory of Markov processes, the dynamics of the processes is given through the infinitesimal generator \( \mathcal{L} \). We assume that the action of \( \mathcal{L} \) on any smooth test function \( f : \mathbb{R}_+^{\Lambda_\varepsilon} \to \mathbb{R} \), is given by

\[
\mathcal{L} f(u) = \sum_{i \in \Lambda_\varepsilon} \varphi(u_i,i)[f(u+\Delta_i(u)) - f(u)] - \sum_{i \in \Lambda_\varepsilon} \frac{\partial f}{\partial u_i}(u) \left[ \alpha u_i + \varepsilon^2 \sum_{j \in \Lambda_\varepsilon} b(i,j)(u_i - u_j) \right], \quad (2.1)
\]
where for all $i \in \Lambda_{\varepsilon}$, $\Delta_i : \mathbb{R}_{+}^{\Lambda_{\varepsilon}} \to \mathbb{R}_{+}^{\Lambda_{\varepsilon}}$ is defined by
\[
(\Delta_i(u))_j = \begin{cases} 
\varepsilon^2 a(i, j), & \text{if } j \neq i \\
-u_i, & \text{if } j = i 
\end{cases},
\]
with $a : [0, 1)^2 \times [0, 1)^2 \mapsto \mathbb{R}_{+}$ being a Lipschitz continuous function, $\alpha$ is a nonnegative parameter, $b : [0, 1)^2 \times [0, 1)^2 \mapsto \mathbb{R}_{+}$ is a symmetric Lipschitz continuous and

**Assumption 1.** $\varphi \in C^1(\mathbb{R}_{+} \times [0, 1)^2, \mathbb{R}_{+})$ is increasing in the first variable such that for all $r \in [0, 1)^2$, $\varphi(0, r) = 0$.

The first term in (2.1) depicts how the chemical synapses are incorporated in our model. A neuron $i$ with potential $u$ spikes at rate $\varphi(u, i)$. Intuitively this means that for any initial configuration $u \in \mathbb{R}_{+}^{\Lambda_{\varepsilon}}$ of the membrane potentials
\[
\mathbb{P}(U(t) = \Delta_i(u) \mid U(0) = u) = \varphi(u_i, i)t + o(t), \text{ as } t \to 0.
\]
Thus, the function $\varphi(\cdot, i)$ is called firing or spiking rate of the neuron $i$. Notice that under such assumption neurons may have different spike rates. The function $a(\cdot, \cdot)$, appearing in the definition of $\Delta_i(\cdot)$, mimics the chemical synapses. The value $\varepsilon^2 a(i, j)$ corresponds to the energy added to the membrane potential of neuron $j$ when neuron $i$ spikes.

The second term in (2.1) represents both electrical synapses and leak currents. They describe the deterministic time evolution of the system between two consecutive spikes. More specifically, if there is no spikes in an interval of time $[a, b]$, the membrane potential of each neuron $i \in \Lambda_{\varepsilon}$ obeys the following ordinary differential equation
\[
\frac{d}{dt}U^{(e)}_i(t) = -\alpha U^{(e)}_i(t) - \varepsilon^2 \sum_{j \in \Lambda_{\varepsilon}} b(i, j) \left[ U^{(e)}_i(t) - U^{(e)}_j(t) \right].
\]
(2.2)

The function $b(\cdot, \cdot)$ incorporates the action of the gap-junction channels. The value $\varepsilon^2 b(i, j)$ corresponds to the synchronization strength between the neurons $i, j$. Notice also that the first term of the right-hand side of (2.2) pushes neuron $i$ to the resting state 0, so that we interpret $\alpha$ as the rate in which membrane potential decreases due to leak channels.

Defining $\lambda^{(e)}(i) = \varepsilon^2 \sum_{j \in \Lambda_{\varepsilon}} b(i, j)$ and $\tilde{b}(i, j) = (\lambda^{(e)}(i))^{-1} b(i, j)$, automatically $\lambda^{(e)}(\cdot)$ and $\tilde{b}(\cdot, \cdot)$ are Lipschitz continuous, $\varepsilon^2 \sum_{j \in \Lambda_{\varepsilon}} \tilde{b}(i, j) = 1$ and we can rewrite the ODE (2.2) as
\[
\frac{d}{dt}U^{(e)}_i(t) = -\alpha U^{(e)}_i(t) - \lambda^{(e)}(i)(U^{(e)}_i(t) - \bar{U}^{(e)}_i(t)),
\]
(2.3)

where for each $t \geq 0$ and $i \in \Lambda_{\varepsilon}$,
\[
\bar{U}^{(e)}_i(t) = \sum_{j \in \Lambda_{\varepsilon}} \tilde{b}(i, j) U^{(e)}_j(t).
\]

We call $\bar{U}^{(e)}_i(t)$ the local average potential of the neuron $i$. Thus, the second term of both ODE's is, in fact, pushing with rate $\lambda^{(e)}_i$ the membrane potential of neuron $i$ to an average value which depends on $i$ itself.

We shall study a simpler situation in which all the rates $\lambda^{(e)}_i$ - and consequently the function $b(\cdot, \cdot)$ - do not change with $\varepsilon$, keeping all others properties. In this way, hereafter we shall assume that there exist functions $\lambda : [0, 1)^2 \mapsto \mathbb{R}_{+}$ and $b : [0, 1)^2 \times [0, 1)^2 \mapsto \mathbb{R}_{+}$ satisfying:
(i) $\lambda$ is Lipschitz continuous;
(ii) $b$ is Lipschitz continuous such that for each $i \in \Lambda_{\varepsilon}$, $\varepsilon^2 \sum_{j \in \Lambda_{\varepsilon}} b(i, j) = 1$;
(iii) Between consecutive spikes the membrane potential of each neuron $i \in \Lambda_{\varepsilon}$ obeys
\[
\frac{d}{dt} U_i^{(e)}(t) = -\alpha U_i^{(e)}(t) - \lambda_i(U_i^{(e)}(t) - \bar{U}_i^{(e)}(t)),
\]
where for each $t \geq 0$ and $i \in \Lambda_{\varepsilon}$, $ar{U}_i^{(e)}(t) = \sum_{j \in \Lambda_{\varepsilon}} b(i, j) U_j^{(e)}(t)$.

One can check, from our assumptions (i) and (ii), existence and uniqueness of the solution of (2.4) on any interval $[0, t]$.

Our first theorem proves the existence and uniqueness of the process described above and gives some a priori estimates on the maximal membrane potential of the system. In what follows, for any vector $u \in \mathbb{R}^{\Lambda_{\varepsilon}}$,
\[
|u| = \max_{i \in \Lambda_{\varepsilon}} \{u_i\}.
\]
Thus, the maximum membrane potential at time $t$ is $||U^{(e)}(t)||$.

**Theorem 1.** Let $\varphi$ be any function satisfying the Assumption 7.

(i) For any $\varepsilon > 0$ and any $u \in \mathbb{R}^{\Lambda_{\varepsilon}^+}$ there exists a unique strong Markov process $U^{(e)}(t)$ taking values in $\mathbb{R}^{\Lambda_{\varepsilon}^+}$ starting from $u$ whose generator is given by (2.1).

(ii) Denote by $P_u^{(e)}$ the probability law under which the process $U^{(e)}(t)$ starts from the initial configuration $U^{(e)}(0) = u \in \mathbb{R}^{\Lambda_{\varepsilon}^+}$. Then for any $R > 0$ and $T > 0$ there exists a constant $C > 0$ such that
\[
\sup_{u: ||u|| \leq R} P_u^{(e)} \left[ \sup_{t \leq T} ||U^{(e)}(t)|| < C \right] \geq 1 - c_1 e^{-c_2 \varepsilon^{-2}},
\]
where $c_1$ and $c_2$ are suitable positive constants. All the constants $C, c_1$ and $c_2$ do not depend on $\varepsilon$.

The proof of Theorem 1 is given in the Appendix. Now, we focus on the hydrodynamic limit. We suppose that for all $\varepsilon > 0$ the following assumption holds.

**Assumption 2.** There exists a smooth function $\psi_0 : \mathbb{R}_{+} \times (0, 1]^2 \to \mathbb{R}_{+}$ fulfilling the conditions:

(i) For each $r \in (0, 1]^2$, $\psi_0(\cdot, r)$ is a probability density on $\mathbb{R}_{+}$ whose support is $[0, R_0]$;
(ii) $\psi_0(\cdot, r) > 0$ on $[0, R_0]$;
(iii) $(U_i^{(e)}(0))_{i \in \Lambda_{\varepsilon}}$ is a sequence of independent random variables, $U_i^{(e)}(0)$ being distributed according to $\psi_0(u, i)du$.

**Remark 1.** The above assumption can be weakened. Indeed, all proofs work under the assumption in which items (i) and (ii) are replaced by (i') and (ii') where
(i') For each $r \in (0,1]^2$, $\psi_0(\cdot,r)$ is a probability density on $\mathbb{R}_+$ with compact support $[0,R_0(r)]; \psi_0(\cdot,r) > 0$ on $[0,R_0(r)]$.

(ii') There exits a positive parameter $R_0$ such that

$$\sup_{r \in [0,1]^2} R_0(r) \leq R_0 < \infty.$$  

Since the state space of the process changes with $\varepsilon$, it is convenient to identify our process $(U^{(\varepsilon)}(t))_{t \geq 0}$ as an element of a suitable space which is independent of $\varepsilon$. The identification is achieved through the map

$$\mathbb{R}^{\Lambda_\varepsilon} \ni U^{(\varepsilon)}(t) \mapsto \mu_{U^{(\varepsilon)}(t)} := c^2 \sum_{i \in \Lambda_\varepsilon} \delta_{(U^{(\varepsilon)}(t),i)}.$$  

In this way we identify our process with the element $t \mapsto \mu_{U^{(\varepsilon)}(t)}$ of the Skorohod space $D(\mathbb{R}_+,\mathcal{S}')$, where $\mathcal{S}'$ is the Schwartz space of all smooth functions $\phi : \mathbb{R}_+ \times [0,1]^2 \to \mathbb{R}$. The associated element $\mu_{U^{(\varepsilon)}(t)}$ has the nice biological interpretation of being the empirical distribution of the membrane potential of the neurons at time $t$.

For any fixed $T > 0$, we denote the restriction of the process to $[0,T]$ by $\mu_{U^{(\varepsilon)}_{[0,T]}}$ which belongs to the space $D([0,T],\mathcal{S}')$. We write $\mathcal{P}^{(\varepsilon)}_{[0,T]}$ to denote the law on $D([0,T],\mathcal{S}')$ of the processes $\mu_{U^{(\varepsilon)}_{[0,T]}}$. Our main result shows that for any positive $T$, the sequence of laws $\mathcal{P}^{\varepsilon}_{[0,T]}$ converges, as $\varepsilon \to 0$, to a law $\mathcal{P}_{[0,T]}$ on $D([0,T],\mathcal{S}')$ which is supported by a single trajectory

$$\rho := (\rho_t(u,r)dudr)_{t \in [0,T], r \in [0,1]^2}.$$  

The function $\rho_t(u,r)$ is interpreted as the limit density function and is proved to solve the nonlinear PDE

$$\frac{\partial \rho_t(u,r)}{\partial t} + \frac{\partial}{\partial u} [V(u,r)\rho_t(u,r)] = -\varphi(u)\rho_t(u,r), \quad t > 0, u > 0 \text{ and } r \in [0,1]^2, \tag{2.6}$$  

where $V(u,r) = -\alpha u - \lambda_r(u - \bar{u}_t(r)) + \rho_t(r)$, where for each $t \geq 0$ and $r \in [0,1]^2$,

$$q_t(r) = \int_0^\infty \varphi(u)\rho_t(u,r)du, \quad p_t(r) = \int_0^{\infty} \int_{[0,1]^2} a(r',r)\varphi(u,r')\rho_t(u,s)dudr' \tag{2.7}$$  

are respectively the limit spiking rate and the limit value added to the membrane potential of the neuron at position $r$ and

$$\bar{u}_t(r) = \int_0^{\infty} \int_{[0,1]^2} ub(r,r')\rho_t(u,r)dudr' \tag{2.8}$$  

is the limit average potential of the neuron at position $r$.

The boundary conditions of (2.6) are specified by

$$\rho_0(u,r) = v_0(u,r), \quad \rho_1(0,r) = v_1(t,r). \quad \tag{2.9}$$  

$v_0(u,r)$ is determined by the problem $v_0(u,r) = \psi_0(u,r)$, while $v_1(t,r)$ has to derived together with (2.7). From our analysis we deduce that

$$v_1(t,r) = \frac{q_t(r)}{\lambda_r \bar{u}_t(r) + p_t(r)} \tag{2.10}.$$  

Since we may have $v_0(0,r) \neq v_1(0,r)$, i.e., $\psi(0,r) \neq \frac{q_0(r)}{\lambda_r v_0(r) + p_0(r)}$, the function $\rho_t(u,r)$ may not be continuous, so that we need a weak formulation of (2.6).
Definition 1. A real valued function \( \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]^2 \ni (t, r, u) \mapsto \rho_t(u, r) \) is a weak solution of (2.6)-(2.9) if for all smooth functions \( \phi(u), \mathbb{R}_+ \times [0,1]^2 \ni (t, r) \mapsto \int_0^\infty \phi(u) \rho_t(u, r) du \) is continuous in \( t \), differentiable in \( t \) and
\[
\frac{\partial}{\partial t} \int_0^\infty \phi(u) \rho_t(u, r) du - \int_0^\infty \phi'(u) V(u, r, \rho_t) \rho_t(u, r) du - \phi(0,r) V(0,r, \rho_t) v_1(t,r) = - \int_0^\infty \varphi(u,r) \phi(u) \rho_t(u, r) du,
\]
where \( V(u, r \rho_t) = -u \alpha - \lambda_r (u - \bar{u}_t(r)) + p_t(r), \) with \( \bar{u}_t(r) \) and \( p_t(r) \) as in (2.7) and (2.8).

The solution of (2.11) can be computed explicitly by the method of characteristics. Characteristics are curves along which the PDE reduces to an ODE. They are defined by the equation
\[
\frac{dx(t,r)}{dt} = V(x(t,r), r, \rho_t).
\]
The solution of (2.12) on the interval \([s, t]\), with value \( u \) at \( s \) is denoted by \( T_{s,t}(u, r), u \in \mathbb{R}_+, \) and it has the following expression:
\[
T_{s,t}(u, r) = e^{-(\alpha + \lambda_r)(t-s)} u + \lambda_r \int_s^t e^{-(\alpha + \lambda_r)(t-h)} [\bar{u}_h(r) + p_h(r)] dh.
\]

The statement of our main theorem reads as follows.

Theorem 2. Under assumptions 1 and 2, for any fixed \( T > 0, \)
\[
\mathcal{P}_{[0,T]}(\epsilon) \overset{w}{\rightarrow} \mathcal{P}_{[0,T]},
\]
(weak convergence in \( D([0,T], S') \)) as \( \epsilon \to 0, \) where \( \mathcal{P}_{[0,T]} \) is the law on \( D([0,T], S') \) supported by the distribution valued trajectory \( \omega_t \) given by
\[
\omega_t(\phi) = \int_{[0,1]^2} \int_0^\infty \phi(u,r) \rho_t(u,r) du dr, \quad t \in [0,T],
\]
for all \( \phi \in S. \) Here, \( \rho_t(u, r) \) is the unique weak solution of (2.6)-(2.9) with \( v_0 = \psi_0 \) and \( v_1 \) given by (2.10). Moreover, \( \rho_t(u, r) \) is a continuous function of \((t,u,r)\) in \( \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]^2 \backslash \{(T_{0,t}(0,r),t,r) : (t,r) \in \mathbb{R}_+ \times [0,1]^2 \} \), where it is differentiable in \( t \) and \( u \) and its derivatives satisfy (2.7). Furthermore, for any \( t \geq 0 \) and \( r \in [0,1]^2, \) \( \rho_t(u, r) \) has compact support in \( u \) and
\[
\rho_t(0,r) = \frac{q_t(r)}{\lambda_r \bar{u}_t(r) + p_t(r)} \quad \text{and} \quad \int_0^\infty \rho_t(u, r) du = 1.
\]
Its explicit expression for \( u \geq T_{0,t}(0,r) \), is:
\[
\rho_t(u, r) = \psi_0 \left( T_{0,t}^{-1}(u, r) \right) \exp \left\{ - \int_0^t \left[ \varphi \left( T_{s,t}^{-1}(u, r) \right) - \alpha - \lambda_r \right] ds \right\},
\]
and for \( u = T_{s,t}(0,r) \) for some \( 0 < s \leq t, \)
\[
\rho_t(u, r) = \frac{q_s(r)}{p_s(r) + \lambda(r) \bar{u}_s(r)} \exp \left\{ - \int_s^t \left[ \varphi \left( T_{s,h}(0,r) \right) - \alpha - \lambda_r \right] dh \right\}.
\]
Theorem 3. Assume (1) and (2). If additionally for all \( r \in [0,1)^2 \),
\[
\psi_0(0,r) = \frac{q_0(r)}{\lambda_r u_0(r) + p_0(r)}, \text{ where } q_0(r) = \int \varphi(u,r)\psi_0(u,r)du
\]
and
\[
\bar{u}_0(r) = \int ub(r,r')\psi_0(u,r)dudr', \quad p_0(r) = \int a(r',r)\varphi(u,r)\psi_0(u,r)dudr',
\]
then \( \rho_t(u,r) \) is continuous in \( \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1)^2 \).

The estimate in (2.5) provided by Theorem 1 implies that with probability going to 1 as \( \varepsilon \to 0 \) all the membrane potentials are uniformly bounded in the time interval \([0,T]\). Therefore, we are allowed to change the values of the spiking rate \( \varphi \) for those values of membrane potentials not reached by the system of neurons. In doing this we can suppose without loss of generality that the function \( \varphi \) satisfies the following stronger condition.

Assumption 3. \( \varphi \in C^1(\mathbb{R}_+ \times [0,1)^2, \mathbb{R}_+) \) is non-decreasing, Lipschitz continuous, bounded and constant for all \( u \geq u_0 \) for some \( u_0 > 0 \). We denote by \( \varphi^* = \| \varphi \|_\infty \) the sup norm of \( \varphi \).

The argument above is given precisely at the end of the Appendix 12.

3 Boundedness of the Membrane Potentials

Under the Assumption 3 we start proving a statement stronger than in Theorem 1.

Proposition 1. Let \( \varphi \) be any function satisfying the Assumption 3.

(i) For any \( \varepsilon > 0 \) and any \( u \in \mathbb{R}_+^{\Lambda_\varepsilon} \) there exists a unique strong Markov process \( U^{(\varepsilon)}(t) \) taking values in \( \mathbb{R}_+^{\Lambda_\varepsilon} \) starting from \( u \) whose generator is given by (2.1).

(ii) Calling \( N^{(\varepsilon)}(t) \) the total number of fires in the time interval \([0,t]\) we have
\[
N^{(\varepsilon)}(t) \leq \bar{N}^{(\varepsilon)}(t) \quad \text{stochastically}
\]
where \( \bar{N}^{(\varepsilon)}(t) \) is a Poisson process with intensity \( \varepsilon^{-2}\varphi^* \).

(iii) For any \( T > 0 \), \( \sup_{t \leq T} \| U^{(\varepsilon)}(t) \| \leq \| U^{(\varepsilon)}(0) \| + a^* \varepsilon^2 N(T) \), \( a^* = \| a \|_\infty \). In particular, there exist positive constants \( c_1 \) and \( c_2 \) such that for any \( \varepsilon > 0 \) and any \( U^{(\varepsilon)}(0) \):
\[
P_{U^{(\varepsilon)}(0)} \left[ \sup_{t \leq T} \| U^N(t) \| \leq \| U^N(0) \| + 2a^* \varphi^* T \right] \geq 1 - c_1 e^{-c_2 T \varepsilon^{-2}}. \tag{3.2}
\]

The constants \( c_1 \) and \( c_2 \) do not depend on \( \varepsilon \).

Proof. The items (i) and (ii) are straightforward consequences of the Assumption 3. To prove (iii), we first observe that any spike increases the membrane potential of any neuron by at most \( \varepsilon^2 a^* \). Since between spikes the largest membrane potential is attracted by its average value, it can not increase. From these two facts it follows that
\[
\sup_{t \leq T} \| U^{(\varepsilon)}(t) \| \leq \| U^{(\varepsilon)}(0) \| + a^* \varepsilon^2 N^{(\varepsilon)}(T).
\]
From this inequality and item (ii) we deduce the bound (3.2) by a large deviation estimate of the Poisson random variable \( N^{(\varepsilon)}(T) \). \( \Box \)
4 Tightness of the Sequence of Laws $P^{(\varepsilon)}_{[0,T]}$

With this section we begin the proof of Theorem 2 (under Assumption 3). We start by proving tightness of the sequence $P^{(\varepsilon)}_{[0,T]}$.

**Proposition 2.** Suppose that $U^{(\varepsilon)}(0) = u^{(\varepsilon)}$ is such that Assumption 3 is verified. Then the sequence of laws $P^{(\varepsilon)}_{[0,T]}$ of $\mu_{U^{(\varepsilon)}_{[0,T]}}$ is tight in $D([0,T], S')$.

**Proof.** For any test function $\phi \in S$ and all $t \in [0,T]$, we have

$$
\mu_{U^{(\varepsilon)}(t)}(\phi) = \varepsilon^2 \sum_{i \in \Lambda_{\varepsilon}} \phi(U_i^{(\varepsilon)}(t), i).
$$

By Mitoma (1983) it is sufficient to prove the tightness of $\mu_{U^{(\varepsilon)}(t)}(\phi), t \in [0,T] \in D([0,T], \mathbb{R})$ for any fixed $\phi \in S$. In order to do so, we shall use a tightness criterion which requires that the $L^2$-norms of the “compensators” of $\mu_{U^{(\varepsilon)}(t)}(\phi)$ are finite, see for instance Theorem 2.6.2 of De Masi and Presutti (1991). The compensators are

$$
\gamma_1^{(\varepsilon)}(t) = L[\mu_{U^{(\varepsilon)}(t)}(\phi)], \quad \gamma_2^{(\varepsilon)}(t) = L^2[\mu_{U^{(\varepsilon)}(t)}(\phi)] - 2\mu_{U^{(\varepsilon)}(t)}(\phi)L[\mu_{U^{(\varepsilon)}(t)}(\phi)];
$$

where $L$ is the generator given by (2.1). The criterion requires that there exists a positive constant $c$ so that

$$
\sup_{t \leq T} E[\gamma_1^{(\varepsilon)}(t)^2] \leq c, \quad \sup_{t \leq T} E[\gamma_2^{(\varepsilon)}(t)^2] \leq c. \tag{4.2}
$$

To prove (4.2) we start by computing $\gamma_1^{(\varepsilon)}(t)$. By its definition,

$$
\gamma_1^{(\varepsilon)}(t) = \varepsilon^2 \sum_{i \neq j} \varphi(U_j^{(\varepsilon)}(t), j) \left[ \phi \left( U_i^{(\varepsilon)}(t) + \varepsilon^2 a(j, i, i) \right) - \phi(U_i^{(\varepsilon)}(t), i) \right] + \varepsilon^2 \sum_j \varphi(U_j^{(\varepsilon)}(t), j) \left[ \phi(0, j) - \phi(U_j^{(\varepsilon)}(t), j) \right] - \alpha \varepsilon^2 \sum_j \phi'(U_j^{(\varepsilon)}(t), j) U_j^{(\varepsilon)}(t) - \varepsilon^2 \sum_j \phi'(U_j^{(\varepsilon)}(t), j) \lambda_j [U_j^{(\varepsilon)}(t) - \bar{U}_j^{(\varepsilon)}(t)].
$$

Expanding the discrete derivative, we get

$$
\gamma_1^{(\varepsilon)}(t) = \varepsilon^4 \sum_{j \neq \lambda} \varphi(U_j^{(\varepsilon)}(t), j) \phi'(U_i^{(\varepsilon)}(t), i) a(j, i) + \varepsilon^2 \sum_j \varphi(U_j^{(\varepsilon)}(t), j) \phi(0, j) - \varepsilon^2 \sum_j \varphi(U_j^{(\varepsilon)}(t), j) \phi(U_j^{(\varepsilon)}(t), j) - \alpha \varepsilon^2 \sum_j \phi'(U_j^{(\varepsilon)}(t), j) U_j^{(\varepsilon)}(t) - \varepsilon^2 \sum_j \phi'(U_j^{(\varepsilon)}(t), j) \lambda_j [U_j^{(\varepsilon)}(t) - \bar{U}_j^{(\varepsilon)}(t)] + O(\varepsilon^2),
$$

where

$$
O(\varepsilon^2) := \varepsilon^2 \sum_{j \neq \lambda} \varphi(U_j^{(\varepsilon)}(t), j) \left[ \phi \left( U_i^{(\varepsilon)}(t) + \varepsilon^2 a(j, i, i) \right) - \phi(U_i^{(\varepsilon)}(t), i) - \varepsilon^2 a(j, i) \phi'(U_i^{(\varepsilon)}(t), i) \right].
$$
Since $\phi, \phi', \phi'', a, \lambda$ are bounded as well as $\varphi$ (Assumption 3) there is a positive constant $c$ so that

$$|\gamma_1^{(\varepsilon)}(t)| \leq c \left(1 + \varepsilon^2 \sum_j U_j^{(\varepsilon)}(t) + \varepsilon^2 \sum_j \tilde{U}_j^{(\varepsilon)}(t)\right) \leq c \left(1 + 2 \sup_{t \leq T} ||U^{(\varepsilon)}(t)||\right).$$

By Proposition 1 $\sup_{t \leq T} \mathbb{E}[\gamma_1^{(\varepsilon)}(t)^2] \leq c$ for a constant $c$ not depending on $\varepsilon$.

The proof of (1.2) for $\gamma_2^{(\varepsilon)}(t)$ is simpler. We write $L = L_{\text{fire}} + L_{\text{(a+\lambda)}}$, where $L_{\text{fire}}\phi$ and $L_{\text{(a+\lambda)}}\phi$ are given respectively by the first and second terms on the right hand side of (2.4). Since $L_{\text{(a+\lambda)}}$ acts as a “derivative” we have

$$L_{\text{(a+\lambda)}}[\mu_{U^{(\varepsilon)}(t)}(\phi)] - 2\mu_{U^{(\varepsilon)}(t)}(\phi)L_{\text{(a+\lambda)}}[\mu_{U^{(\varepsilon)}(t)}(\phi)] = 0,$$

as it can be easily checked. Thus, it follows that

$$\gamma_2^{(\varepsilon)}(t) = L_{\text{fire}}[\mu_{U^{(\varepsilon)}(t)}(\phi)] - 2\mu_{U^{(\varepsilon)}(t)}(\phi)L_{\text{fire}}[\mu_{U^{(\varepsilon)}(t)}(\phi)].$$

Since $|2\mu_{U^{(\varepsilon)}(t)}(\phi)| \leq c$ and we have already proven the bound for $L_{\text{fire}}[\mu_{U^{(\varepsilon)}(t)}(\phi)]$, it remains only to bound uniformly in $t \leq T$ and in $\varepsilon$, the $L^2$-norm of $L_{\text{fire}}[\mu_{U^{(\varepsilon)}(t)}(\phi)]$. By definition,

$$L_{\text{fire}}[\mu_{U^{(\varepsilon)}(t)}(\phi)] = \varepsilon^4 \sum_j \varphi(U_j^{(\varepsilon)}(t), j) \left[\phi(U_j^{(\varepsilon)}(t) + \varepsilon^2 a(j,i), i)\phi(U_j^{(\varepsilon)}(t) + \varepsilon^2 a(j,k), k) - \phi(U_j^{(\varepsilon)}(t), i)\phi(U_j^{(\varepsilon)}(t), j)\right] + \varepsilon^4 \sum_j \varphi(U_j^{(\varepsilon)}(t), j)\left[\phi^2(0,j) - \phi^2(U_j^{(\varepsilon)}(t), j)\right]$$

$$+ 2\varepsilon^4 \sum_j \varphi(U_j^{(\varepsilon)}(t), j)\left[\phi(0,j) - \phi(U_j^{(\varepsilon)}(t), j)\right]\left[\phi(U_j^{(\varepsilon)}(t) + \varepsilon^2 a(j,i), i) - \phi(U_j^{(\varepsilon)}(t), i)\right].$$

Using the same type arguments above we can show that the $L^2$-norm of this term is bounded uniformly in $t \in [0, T]$ and in $\varepsilon$, concluding the proof. Notice that taking into account the signs we could prove that $\gamma_2^{(\varepsilon)}(t) \to 0$ as $\varepsilon \to 0$.  

5  The Auxiliary Process and the Coupling Algorithm

In this section we shall define an auxiliary process which we later shall prove that it is close to the true process uniformly in $\varepsilon$. This uniform closeness in $\varepsilon$ is the content of the Theorem 4. The proof of this result is based on coupling algorithm designed so that neurons in both processes spike together as often as possible. In the section 6 we study the hydrodynamic for the auxiliary process. Section 7 we will then conclude the proof of Theorem 2 and 3.

Throughout the section $\varepsilon$ is kept fixed so that we omit the superscript $\varepsilon$ from $U^{(\varepsilon)}(t)$ and all variables involving in the definition of the auxiliary process. Before defining the auxiliary process we shall introduce three partitions.
Definition 2 (Partition on space). Let $\ell > 0$ be a fixed parameter such that $\ell^{-1}$ is an integer number. We then partition the set $[0, 1)^2$ into small squares of side length $\ell$:

$$C_\ell = \{ C_{(m_1, m_2)} : m_1, m_2 \in \ell\mathbb{Z}^2 \cap [0, 1)^2 \}, \quad C_{(m_1, m_2)} = [m_1, m_1 + 1) \times [m_2, m_2 + 1).$$

Since we shall not use the form chosen for the elements of $C_\ell$, we take any enumeration of the set $\ell\mathbb{Z}^2 \cap [0, 1)^2$ and assume that

$$C_\ell = \{ C_m : m = 1, \ldots, \ell^{-2} \}.$$ 

For each square $C_m$ we denote by $i_m$ its center.

Definition 3 (Partition on time). Let $\delta$ and $\tau$ be positive numbers such that $\delta$ is divisible by $\tau$. We partition the interval $[0, \delta)$ into intervals of length $\tau$:

$$J_\tau = \{ J_h : h = 1, \ldots, \delta\tau^{-1} \}, \quad J_h = [\delta - h\tau, \delta - (h - 1)\tau).$$

Let us explain the role of the partitions $C_\ell$ and $J_\tau$ in the definition of the auxiliary process. The auxiliary process is denoted by $Y^{(\delta, \ell, E, \tau)}(n\delta)$ (the parameter $E$ will appear below) and is defined at discrete times $n\delta$, $n \in \mathbb{N}$. Its definition is such that neurons in the square $C_m$, having potential $U \geq 0$, spike with a constant rate $\varphi(U, i_m)$ in the time interval $[n\delta, (n + 1)\delta)$. Thus, neurons in same the square spike according the same spiking rate $u \mapsto \varphi(u, i_m)$. Moreover, in the same interval, all firing events after the first one are suppressed.

The configuration of $Y^{(\delta, \ell, E, \tau)}$ is updated at every time interval $[n\delta, (n + 1)\delta)$. Neurons in a common square have the same updating rule, so that we need to specify it in each square for a single neuron. For that sake, denote by $\bar{Y}^{(\delta, \ell, E, \tau)}_{i_m}(n\delta)$ the average potential of neuron $i_m$ in the auxiliary process at time $n\delta$ and take $i \in C_m$. Conditionally on $\bar{Y}^{(\delta, \ell, E, \tau)}_{i_m}(n\delta) = \bar{y}(m)$, suppose first that $i$ have not spiked during the interval $[n\delta, (n + 1)\delta)$. Then the value of its membrane potential at time $(n + 1)\delta$ is obtained by first letting the value of its current potential evolve, for a time $\delta$, under the attraction of $\bar{y}(m)$ and then taking into account the effect of the spikes in the interval $[0, \delta]$. If, on the other hand, $i$ have spiked in the interval $J_h$, its potential is updated by first setting its current potential to 0 and then applying the earlier updating rule during the interval $[\delta - (h - 1)\tau, \delta)$. This means that the potential of $i$ is then attracted for a time $(h - 1)\tau$ by $\bar{y}_{i_m}$ and next the effect of the spikes during $[\delta - (h - 1)\tau, \delta)$ is taken into account. Before giving the precise definition of the auxiliary process, we need introduce a third partition.

Definition 4 (Partition on the membrane potential at time 0). Let $E$ be a positive real number which divides $R_0$. We then partition the interval $[0, R_0]$ into subintervals

$$I_E = \{ I_k : k = 1, \ldots, R_0E^{-1} \}, \quad I_k = [(k-1)E, kE).$$

For each $I_k$ we denote its center by $E_k$.

For each neuron $i \in \Lambda_\ell$, the value $Y^{(\delta, \ell, E, \tau)}(0)$ will be defined by first picking a point in $[0, R_0]$ according to the probability density $\psi_0(u, i)du$ and then redefining it as $E_k$ if the chosen value belongs to $I_k$. The precise definition of the auxiliary process is given now.

Initially, we consider the map $[0, R_0] \ni u \mapsto \Phi_0(u)$ which assigns $\Phi_0(u) = E_k$ if $u \in I_k$ and we then put

$$Y^{(\delta, \ell, E, \tau)}(0) = \Phi_0(U_i(0)), \quad \text{for each } i \in \Lambda_\ell.$$
The definition of the process is done by induction. Now, suppose that the configuration $Y^{(\delta,\ell,E,\tau)}(n\delta) = y = (y_i, i \in \Lambda_e)$ is given and consider the sequence of independent exponential random variables $(\xi_i)_{i \in \Lambda_e}$ which are independent of anything else, having rates $\varphi(y_i, i_m)$ when $i \in C_m$. Notice that we keep constant the spiking intensity of the neurons. We write $N(m, h)$ to denote the number of neurons in $C_m$ spiking in the interval $J_h \in \mathcal{J}_\tau$,

$$N(m, h) = \sum_{i \in C_m} \mathbb{1}_{\{\xi_i \in J_h\}}, \ J_h = [\delta - h\tau, \delta - (h - 1)\tau), \quad (5.3)$$

while the contribution, due to spikes of other neurons, to the membrane potential of those neurons in $C_m$ which spike in $J_h$ is given by

$$S(m, h) = \varepsilon^2 \sum_{m' = 1}^{\ell - 1} a(i_{m'}, i_m)N(m', s), \ h = 2, \ldots, \delta\tau^{-1}, \quad (5.4)$$

and for $h = 1$, we set $S(m, 1) = 0$. Neurons which do not spike in $[0, \delta)$ will have their membrane potentials increased by

$$S(m, \delta) = \varepsilon^2 \sum_{m' = 1}^{\ell - 1} a(i_{m'}, i_m)N(m', h). \quad (5.5)$$

The average potential of neuron $i_m$ (at time $n\delta$) is defined by

$$\bar{y}(m) = \varepsilon^2 \sum_{m' = 1}^{\ell - 1} \sum_{i \in C_{m'}} b(i_{m'}, i_m)y_i. \quad (5.6)$$

Notice that the electrical synaptic strength is constant in each square $C_{m'}$. Setting for simplicity $\lambda_m = \lambda_{m'}$, we write,

$$\Phi_t,\bar{y}(m)(y_i) = e^{-t(\alpha + \lambda_m)}y_i + \frac{\lambda_m}{\alpha + \lambda_m}(1 - e^{-t(\alpha + \lambda_m)})\bar{y}(m), \ 0 \leq t \leq \delta, \ i \in C_m, \quad (5.7)$$

for deterministic flow attracting the value $y_i$ to $\bar{y}(m)$, and set

$$Y_i^{(\delta,\ell,E,\tau)}((n + 1)\delta) = \Phi_{(h - 1)\tau,\bar{y}(m)}(y_i) + S(m, \delta), \ i \in C_m, \ \text{if} \ \xi_i > \delta. \quad (5.8)$$

Thus neurons which do not spike follow the deterministic flow for a time $\delta$. Afterwards, we add to their membrane potentials the value $S(m, \delta)$, generated by the spiking of other neurons, only at the end of the interval $[n\delta, (n + 1)\delta)$.

For those neurons which spike in the interval $J_h$, we set

$$Y_i^{(\delta)}((n + 1)\delta) = \Phi_{(h - 1)\tau,\bar{y}(m)}(0) + S(m, h), \ i \in C_m, \ \text{if} \ \xi_i \in J_h. \quad (5.9)$$

This is the value of the membrane potential of a neuron initially having potential 0, following the deterministic flow for the remaining time $(h - 1)\tau$ and receiving an additional potential $S(m, h)$, due to spikes of other neurons in the time interval $[\delta - (h - 1)\tau, \delta)$.

**Remark 2.** Notice that all variables $N(m, h)$, $S(m, h)$, $S(m, \delta)$ and $\bar{y}(m)$ depend on also on $n$. We shall stress this dependency in the analysis of the hydrodynamic limits for $Y^{(\delta,\ell,E,\tau)}$, section 6.

**Remark 3.** The definition of the auxiliary process $Y^{(\delta,\ell,E,\tau)}$ has to be such that $Y^{(\delta,\ell,E,\tau)}$ is close to the true process. Thus, we have some freedom in choosing the distribution of the spiking neurons in the auxiliary process and the above formulas (5.8) and (5.9) could be modified. The choice we have made is convenient, specially in the analysis of the hydrodynamic limit for $Y^{(\delta,\ell,E,\tau)}$. 

5.1 Coupling the Auxiliary and True Processes

In this section we present a coupling algorithm for the two processes \((U(n\delta))_{n \geq 1}\) and \((Y^{(\delta,E,\tau)}(n\delta))_{n \geq 1}\). The algorithm is designed so that neurons in both processes spike together as often as possible.

At time 0, it is set, for each \(i \in \Lambda_\varepsilon\), \(Y_i^{(\delta,E,\tau)}(0) = \Phi_0(U_i(0))\). Then, for \(n = 0, 1, \ldots\) the input of the algorithm is the configuration \((U(n\delta), Y^{(\delta)}(n\delta))\) and its output is the new configuration \((U((n+1)\delta), Y^{(\delta)}((n+1)\delta))\). The following auxiliary variables are used in the algorithm.

- \((u, y) \in \mathbb{R}^\Lambda_+ \times \mathbb{R}^\Lambda_+\) representing the configuration of membrane potentials in the two processes and \(\bar{y}(m) = \varepsilon^2 \sum_{m' \in C_m} b(i_{m'}, i_m) y_i\) representing the average membrane potential of the neuron \(i_m\).
- Independent random times \(\xi_i^1, \xi_i^2, \xi_i \in (0, \infty), i \in \Lambda_\varepsilon\), indicating possible times of updates.
- \(q = (q_i, i \in \Lambda_\varepsilon) \in \{0, 1\}^{\Lambda_\varepsilon}\). The variable \(q_i\) indicates the occurrence of spike of the neuron \(i\) in the auxiliary process.
- \(\beta = (\bar{\beta}_i, i \in \Lambda_\varepsilon) \in \{0, 1, \ldots, \delta^{-1}\}^{\Lambda_\varepsilon}\). The variable \(\beta_i\) indicates in which subinterval of length \(\tau\) the neuron \(i\) has spiked in the auxiliary process. The condition \(\beta_i = 0\) means the neuron \(i\) has not spiked.
- \(L \in [0, \delta]\) indicates the remaining time after every update of the system.

The deterministic flows follow by the processes \(U\) and \(Y^{(\delta,E,\tau)}\) make part of the coupling algorithm. Recall that the deterministic flow of the process \(Y^{(\delta,E,\tau)}\), given in \([5.7]\), is defined by

\[
\Phi_t \bar{y}(m)(y_i) = e^{-t(\alpha + \lambda_m)} y_i + \frac{\lambda_m}{\alpha + \lambda_m} \left(1 - e^{-t(\alpha + \lambda_m)}\right) \bar{y}(m), \quad 0 \leq t \leq \delta, \ i \in C_m,
\]

where \(\bar{y}(m) = \varepsilon^2 \sum_{m' \in C_m} b(i_{m'}, i_m) y_i\) and \(\lambda_m = \lambda_{i_m}\).

The deterministic flow of the \(U\) at time \(t\), denoted by \(\Psi_t^1(u)\), is the solution on the interval \([0, t]\), with value \(u \in \mathbb{R}_+\) at 0, of the ordinary differential equation

\[
\frac{d}{dt} U_i(t) = -\alpha U_i(t) - \lambda_i(U_i(t) - \bar{U}_i^{(\varepsilon)}(t)), \quad \bar{U}_i(t) = \varepsilon^2 \sum_{j \in \Lambda_\varepsilon} b(i, j) U_j(t).
\]

The coupling can be described as follows. Conditionally on \((U(n\delta), Y^{(\delta,E,\tau)}(n\delta)) = (u, y)\), we attach to each neuron \(i\) two independent random clocks \(\xi_i^1\) and \(\xi_i^2\). For \(i \in C_m\), \(\xi_i^1\) has intensity \(\varphi(\Psi_t^1(u))\), \(\varphi(y_i, i_m)\), while \(\xi_i^2\) intensity \(\varphi(|\Psi_t^1(u)|, \tau)\). Random clocks associated to different neurons are independent. If \(\xi_i^1\) rings first, then the neuron \(i\) spikes in both process and the coupling is successful. On the other hand, if \(\xi_i^2\) rings first, then the neuron \(i\) fires only in the process whose the membrane potential of \(i\) at time \(\xi_i^2\) is the largest. Whenever the neuron \(i\) fires in the interval \(J_h\), in the auxiliary process, we set \(q_i = 1\) and \(\beta_i = h\) and disregard other spikes of \(i\) in the auxiliary process. Thus, all others possible spikes of \(i\) will be considered in the true process \(U_i\). For this reason we also consider a random clock \(\xi_i\) with intensity \(\varphi(\Psi_t^1(u), i)\) whose rings will indicate the
next spikes of $i$ in the true process. All the random clocks are taken into account only if they happen in the interval of time $[0, \delta]$.

Our algorithm is given below.

\textbf{Algorithm 1} Coupling algorithm

1: \textbf{Input:} $(U^{(\epsilon)}(n\delta), Y^{(\delta)}(n\delta))$
2: \textbf{Output:} $(U^{(\epsilon)}((n+1)\delta), Y^{(\delta)}((n+1)\delta))$
3: \textbf{Initial values:} $(u, y) \leftarrow (U(n\delta), Y(n\delta))$, $q_i \leftarrow 0$ and $\beta_i \leftarrow 0$, for all $i \in \Lambda_\varepsilon$, $L \leftarrow \delta$
4: \textbf{while} $L > 0$ \textbf{do}
5: \hspace{0.5cm} For each $i \in \Lambda_\varepsilon$, choose independent random times
6: \hspace{1cm} \textbullet \hspace{0.5cm} $\xi_i^1$ with intensity $\varphi(\Psi_i^1(u_i), i) \land \varphi(y_i, i_m)$ for all neurons in $C_m$
7: \hspace{1cm} \textbullet \hspace{0.5cm} $\xi_i^2$ with intensity $|\varphi(\Psi_i^1(u_i), i) - \varphi(y_i, i_m)|$ for all neurons in $C_m$
8: \hspace{1cm} \textbullet \hspace{0.5cm} $\xi_i$ with intensity $\varphi(\Psi_i^1(u_i))$
9: \hspace{1cm} $R = \inf_{i \in \Lambda_\varepsilon: q_i = 0} (\xi_i^1 \land \xi_i^2) \land \inf_{i \in \Lambda_\varepsilon: q_i = 1} \xi_i$
10: \textbf{if} $R \geq L$ then
11: \hspace{1cm} \textbf{Stop situation:}
12: \hspace{1.5cm} $y_i \leftarrow \Phi_{\delta,y_m}(y_i) + S(m, \delta)$, for all $i \in \Lambda_\varepsilon \cap C_m$ such that $q_i = 0$
13: \hspace{1.5cm} $y_i \leftarrow \Phi_{(\beta_i-1)\tau,\delta}(0) + S(m, \beta_i)$, for all $i \in \Lambda_\varepsilon \cap C_m$ such that $q_i = 1$
14: \hspace{1.5cm} $u_i \leftarrow \Psi_{L}^i(u_i)$, $L \leftarrow 0$
15: \textbf{else if} $R = \xi_i^1 < L$ then
16: \hspace{1.5cm} $L \leftarrow L - R$, $q_i \leftarrow 1$, $\beta_i \leftarrow \delta \tau^{-1} - (\lceil \frac{R}{\tau} \rceil - 1)$
17: \hspace{1.5cm} $u_i \leftarrow 0$
18: \hspace{1.5cm} $u_j \leftarrow \Psi_{R}^j(u_j) + \varepsilon^2 a(i, j)$ for all $j \neq i$
19: \textbf{else if} $R = \xi_i^2 < L$ then
20: \hspace{1.5cm} if $\varphi(\Psi_{R}^i(u_i), i) > \varphi(y_i, i_m)$ then
21: \hspace{1.5cm} $L \leftarrow L - R$, $u_i \leftarrow 0$
22: \hspace{1.5cm} $u_j \leftarrow \Psi_{R}^j(u) + \varepsilon^2 a(i, j)$ for all $j \neq i$
23: \hspace{1.5cm} end if
24: \hspace{1.5cm} if $\varphi(\Psi_{R}^i(u), i) \leq \varphi(y_i, i_m)$ then
25: \hspace{1.5cm} $L \leftarrow L - R$, $q_i \leftarrow 1$, $\beta_i \leftarrow \delta \tau^{-1} - (\lceil \frac{R}{\tau} \rceil - 1)$, $u_i \leftarrow \Psi_{R}^i(u)$ for all $i \in \Lambda_\varepsilon$
26: \hspace{1.5cm} end if
27: \textbf{else if} $R = \xi_i < L$ then
28: \hspace{1.5cm} $L \leftarrow L - R$, $u_i \leftarrow 0$
29: \hspace{1.5cm} $u_j \leftarrow \Psi_{R}^j(u) + \varepsilon^2 a(i, j)$ for all $j \neq i$
30: \hspace{1.5cm} end if
31: \textbf{end if}
32: \textbf{end while}
33: $(U((n+1)\delta)), Y((n+1)\delta) \leftarrow (u, y)$
34: \textbf{Return} $(U((n+1)\delta), Y((n+1)\delta))$

\section{5.2 Consequences of the Coupling Algorithm}

Theorem 4 is the main result of this section. It states that typically the difference of the potentials $\Delta_i(n) = |U_i(n\delta) - Y_i^{(\delta, \ell, E, \tau)}(n\delta)|$ is small (proportionally to $\delta$). In addition, it claims that the proportion of neurons having large values of $\Delta_i(n)$ is also small (again proportional to $\delta$).
Definition 5. A label $i \in \Lambda$ is called “good at time $k\delta$” if for all $n = 1, \ldots, k$ the following is true:

(i) Either $\xi_1^i$ rings first and $\xi_i$ does not ring on interval $[(n - 1)\delta, n\delta]$;
(ii) or neither $\xi_1^i$ nor $\xi_2^i$ have rang on the interval $[(n - 1)\delta, n\delta]$.

We call $G_n$ the set of good labels at time $n\delta$ and $B_n = \Lambda \setminus G_n$ the set of bad labels. If $i \in G_n$ we call $\Delta_i(n) := |U_i(n\delta) - Y_i^{(\delta, t, E, \tau)}(n\delta)|$. Finally, we set

$$\theta_n = \max\{\Delta_i(k), i \in G_n, k \leq n\}.$$

We now enunciate the Theorem 4. Its proof is postponed to the Appendix 9.

Theorem 4. Under Assumption 3, for any fixed $T > 0$, there exist $\delta_0 > 0$ and a constant $C$ depending on $\|\varphi\|_{\infty}$ and on $T$ such that for all $\delta \leq \delta_0$,

$$\theta_n \leq C\delta \quad \text{and} \quad \varepsilon^2|B_n| \leq C\delta \quad \text{for all } n \text{ such that } n\delta \leq T,$$

with probability $\geq 1 - c_1 \delta^{-1} e^{-c_2 \delta^{-2} \delta^4}$. The constants $c_1$ and $c_2$ do not depend on $\varepsilon$ and $\delta$.

As a by product, we obtain an upper bound for the $L_1$- distance between the variables $\mu_{U_{(n\delta)}}(\phi)$ and $\mu_{Y^{(\delta, t, E, \tau)}}(\phi)$, for each test function $\phi \in S$. This result will be used, in section 7 in the analysis of the Hydrodynamic for $U$. Let

$$T = \left\{t \in [0, T] : t = n2^{-q}T, n, q \in \mathbb{N}\right\}. \quad (5.10)$$

Recall that $P_u^{(c)}$ denotes the law under which $U(\cdot)$ starts from $U(0) = u$. Denote by $\tilde{P}_u^{(c)}$ the law under which the process $Y^{(\delta, t, E, \tau)}(\cdot)$ starts from $\Phi_0(u) = (\Phi_0(u_i), i \in \Lambda_\varepsilon)$ and write $Q_u^{(c)}$ for the probability law governing the coupled process defined above. By abuse of notation, we shall denote the associated expectations also by $P_u^{(c)}$, $\tilde{P}_u^{(c)}$ and $Q_u^{(c)}$.

Proposition 3. Let $t \in T$, $\delta \in \{2^{-q}T, q \in \mathbb{N}\}$ such that $t = \delta n$ for some positive integer $n$ and $\phi \in S$. Then, there exists a constant $C$, not dependent on $\delta$, such that

$$Q_u^{(c)} \left[ \varepsilon^2 \sum_{i \in C_m} \phi(U_i(t), i) - \varepsilon^2 \sum_{i \in C_m} \phi(Y_i(t), i_m) \right] \leq C\|\varphi\|_{\text{Lip}} \ell^3 \frac{e^{-C\varepsilon^{-2} \delta^4}}{\delta} + C\delta \ell^2. \quad (5.11)$$

In particular,

$$Q_u^{(c)} \left[ |\mu_{U(t)}(\phi) - \mu_{Y^{(\delta, t, E, \tau)}(n\delta)}(\phi)| \right] \leq C\|\varphi\|_{\text{Lip}} \ell^3 \frac{e^{-C\varepsilon^{-2} \delta^4}}{\delta} + C\delta. \quad (5.12)$$

The proof is given in Appendix 10. Next, in section 6 we study the hydrodynamic limit for the approximating process and, in section 7 we conclude the proof of Theorems 2 and 3.
6 Hydrodynamic Limit for the Auxiliary Process

In this section, we initially describe the random evolution of the membrane potentials in the auxiliary process. Next, we define a deterministic version of this evolution taking into account the average behaviour of the auxiliary process in each time interval $[n\delta,(n+1)\delta)$. Beside, we also consider the random variables which compute the number of neurons of the auxiliary process in a given square with a given potential and, from the dynamics of these variables, we define a second deterministic evolution. The main theorem of this section, Theorem 5, states that both the random potentials and the counting variables becomes deterministic as $\varepsilon \to 0$ and they are described respectively by the first and second deterministic evolutions.

The remaining of the section these deterministic evolutions will be used to define the hydrodynamic evolution for the auxiliary processes. When necessary we shall make explicit the dependence both on $\varepsilon$ and $n$ writing $Y^{(\varepsilon,\delta,\ell,E,\tau)}$, $N^{(\varepsilon)}_{n,m}(m,h)$, $Y^{(\varepsilon)}_{n,m}(m)$, $S^{(\varepsilon)}_{n,m}(m,h)$ and $S^{(\varepsilon)}_{n,m}(m,\delta)$. 

6.1 Hydrodynamic Evolution of the Auxiliary Process

Throughout the subsection the parameters $\delta, \ell, E, \tau$ are kept fixed, so that we omit the superscript in all variables considered below. In what follows we work in $C_m$ and doing so we drop also the dependency on $m$ unless some confusion may arise.

We denote by $\mathcal{E}^{(\varepsilon)}_{n}$ the random set of potentials which the auxiliary process (restricted to $C_m$) assume at time $n\delta$. By (5.2), we have $\mathcal{E}^{(\varepsilon)}_{0} = \{E^{(\varepsilon)}_{0,k} : k = 1, \ldots, R_0E^{-1}\}$ where we set $E^{(\varepsilon)}_{0,k} = E_k$. At time $\delta$, the potential of neurons which spike in the $J_h = [\delta - h\tau, \delta - (h-1)\tau)$, independently of their initial membrane potentials, will be a value $E^{(\varepsilon)}_{1,h} \in \mathcal{E}^{(\varepsilon)}_{1}$. By (5.9), we immediately see that

$$E^{(\varepsilon)}_{1,h} = \Phi_{(h-1)\tau,y^{(\varepsilon)}_{0}(m)}(0) + S^{(\varepsilon)}_{1}(m,h), \ h = 1, \ldots, \delta^{-1}. \quad (6.1)$$

On the other hand, at time $\delta$, the membrane potential of those neurons which initially had potential $E^{(\varepsilon)}_{0,k}$ and do not spike will be a value $E^{(\varepsilon)}_{1,k+\delta^{-1}} \in \mathcal{E}^{(\varepsilon)}_{1}$. Recalling (5.8), it is readily verified that

$$E^{(\varepsilon)}_{1,k+\delta^{-1}} = \Phi_{\delta,y^{(\varepsilon)}_{n,k-1}}(m)\left(E^{(\varepsilon)}_{0,k}\right) + S^{(\varepsilon)}_{1}(m,\delta), \ k = 1, \ldots, |\mathcal{E}_{0}^{(\varepsilon)}| \quad (6.2)$$

where $E^{(\varepsilon)}_{0,k} \in \mathcal{E}_{0}$. Thus, we may split the elements of the finite set $\mathcal{E}^{(\varepsilon)}_{1}$ into two groups. The first group consists of those potentials satisfying (5.2), reached for only by neurons which do not spike in $[0,\delta)$. On the other hand, due to spikes of neurons in the interval $[0,\delta)$ some potentials are “created” at time $\delta$. This leads to the second group of potentials, those satisfying (6.1). Moreover, the following chain of inequalities holds

$$0 = E^{(\varepsilon)}_{1,1} < \ldots < E^{(\varepsilon)}_{1,\delta^{-1}} < E^{(\varepsilon)}_{1,1+\delta^{-1}} < \ldots < E^{(\varepsilon)}_{1,R_0E^{-1}+\delta^{-1}}.$$ 

Iterating the argument above, for each $n\delta \leq T$, we may also split the elements of the set $\mathcal{E}^{(\varepsilon)}_{n}$ into two groups. Those potentials belonging to the first group satisfy

$$E^{(\varepsilon)}_{n,k+\delta^{-1}} = \Phi_{\delta,y^{(\varepsilon)}_{n-1,k}}(m)\left(E^{(\varepsilon)}_{n-1,k}\right) + S^{(\varepsilon)}_{n}(m,\delta), \ k = 1, \ldots, |\mathcal{E}^{(\varepsilon)}_{n-1}|, \quad (6.3)$$
where \( E_{n-1,k}^{(e)} \in \mathcal{E}_{n-1}^{(e)} \), while the potentials of the second group satisfy
\[
E_{n,h}^{(e)} = \Phi_{(h-1)\tau}^{(e)}(0) + S_{n}^{(e)}(m, h), \ h = 1, \ldots, \delta\tau^{-1}.
\] (6.4)

From our definitions, we have also that
\[
0 = E_{n,1}^{(e)} < \ldots < E_{n,\delta\tau^{-1}}^{(e)} < E_{n,1+\delta\tau^{-1}}^{(e)} < \ldots < E_{n,R_0 E^{-1}+n\delta\tau^{-1}}^{(e)}.
\]

Now, writing
\[
e_{n}^{(e)}(m) = \bar{\Phi}_{Y^{(e,\delta,\ell,E,\tau)}}^{(e)}(0) \left[ S_{n}^{(e)}(m, \delta) \right], \ 1 \leq m \leq \ell^2, \ n\delta \leq T,
\] (6.5)

to denote the expected value of the local average membrane potential \( Y_{n}^{(e,\delta,\ell,E,\tau)} \) at time \( n\delta \), we set \( D_0^{(e)} = \mathcal{E}_0^{(e)} \) and then recursively define for \( k = 1, \ldots, |D_{n-1}^{(e)}|, \)
\[
D_{n,k+\delta\tau^{-1}}^{(e)} := \Phi_{(h-1)\tau}^{(e)}(m) \left( D_{n-1,k}^{(e)} \right) + \bar{\Phi}_{Y^{(e,\delta,\ell,E,\tau)}}^{(e)}(0) \left[ S_{n}^{(e)}(m, \delta) \right], \ \text{with} \ D_{n-1,k}^{(e)} \in D_{n-1}^{(e)},
\] (6.6)
\[
D_{n,h}^{(e)} := \Phi_{(h-1)\tau}^{(e)}(m) (0) + \bar{\Phi}_{Y^{(e,\delta,\ell,E,\tau)}}^{(e)}(0) \left[ S_{n}^{(e)}(m, h) \right], \ h = 1, \ldots, \delta\tau^{-1}.
\] (6.7)

Notice that, by an abuse of notation, we keep denoting the associated expectation of a probability measure \( P \) also by \( P \).

Given \( E_{n,k}^{(e)} \in \mathcal{E}_{n}^{(e)} \), we write \( \eta_{n}^{(e)}(m, k) \) to denote the number of neurons of \( Y^{(e,\delta,\ell,E,\tau)} \), in \( C_m \), with membrane potential \( E_{n,k}^{(e)} \) at time \( n\delta \). Finally, we write
\[
\zeta_{0}^{(e)}(m, k) = \bar{\Phi}_{Y^{(e,\delta,\ell,E,\tau)}}^{(e)}(0) \left[ \eta_{0}^{(e)}(m, k) \right],
\]
to denote the expected number of neurons of \( Y^{(e,\delta,\ell,E,\tau)} \) in the square \( C_m \) whose potential at time 0 is \( E_{0,k}^{(e)} \), and iteratively we set
\[
\zeta_{n}^{(e)}(m, k + \delta\tau^{-1}) = \zeta_{n-1}^{(e)}(m, k) e^{-\delta\phi(D_{n-1,k+i,m}^{(e)})}, \ D_{n-1,k}^{(e)} \in D_{n-1}^{(e)},
\] (6.8)

and for \( h = 1, \ldots, \delta\tau^{-1}, \)
\[
\zeta_{n}^{(e)}(m, h) = \sum_{k} \zeta_{n-1}^{(e)}(m, k) \left( e^{-\delta(h-1)\tau} \varphi(D_{n-1,k+i,m}^{(e)}) - e^{-\delta(h-1)\tau} \varphi(D_{n-1,k+i,m}^{(e)}) \right).
\] (6.9)

Suppose we have computed the number neurons of \( Y^{(e,\delta,\ell,E,\tau)} \), in \( C_m \), with a given potential \( E_{n-1,k}^{(e)} \). Then, the probability of a neuron with such potential does not spike in the interval \([0, \delta]\) is exactly \( e^{-\delta\phi(E_{n-1,k+i,m}^{(e)})} \). Thus, we expect that the number of neurons having potential \( E_{n,k}^{(e)} \) at the next step satisfies
\[
\eta_{n}^{(e)}(m, k + \delta\tau^{-1}) \approx \eta_{n-1}^{(e)}(m, k) e^{-\delta\phi(E_{n-1,k+i,m}^{(e)})}.
\]

This relation explains (6.8). Similarly, we notice that the expected fraction of those neurons having potential \( E_{n-1,k}^{(e)} \), which spike in the interval \( J_h = [\delta - h\tau, \delta - (h-1)\tau) \) is precisely
\[
\eta_{n}(m, k) \left( e^{-\delta(h-1)\tau} \varphi(E_{n-1,k+i,m}^{(e)}) - e^{-\delta(h-1)\tau} \varphi(E_{n-1,k+i,m}^{(e)}) \right).
\]
Then, summing over \( k \) we get the random version of (6.9).

We shall show that the random membrane potentials \( E_{n,k}^{(\varepsilon)} \) are close (proportionally to \( \varepsilon^{1/2} \)) to the deterministic values \( D_{n,k}^{(\varepsilon)} \) defined above. Furthermore, it will be shown that the collection of counting variables \( \eta_n^{(\varepsilon)}(m,k) \) are close to the values \( \zeta_n^{(\varepsilon)}(m,k) \). Here, close means again to be proportional to \( \varepsilon^{1/2} \).

**Theorem 5.** There exist positive constants \( C, c_1 \) and \( c_2, \) not depending on \( \varepsilon \) such that for all \( n \) with \( 0 \leq n \delta \leq T \), \( E_{n,k}^{(\varepsilon)} \in \mathcal{E}_n^{(\varepsilon)} \) and \( D_{n,k}^{(\varepsilon)} \in \mathcal{D}_n^{(\varepsilon)} \),

\[
|E_{n,k}^{(\varepsilon)} - D_{n,k}^{(\varepsilon)}| \leq C\varepsilon^{1/2}, \quad \varepsilon^2|\eta_n^{(\varepsilon)}(m,k + \delta\tau^{-1}) - \zeta_n^{(\varepsilon)}(m,k + \delta\tau^{-1})| \leq E\ell^2\varepsilon^{1/2}
\]

for \( k = 1, \ldots, |\mathcal{C}_n^{(\varepsilon)}| \) and

\[
\varepsilon^2|\eta_n^{(\varepsilon)}(m,h) - \zeta_n^{(\varepsilon)}(m,h)| \leq \tau\ell^2\varepsilon^{1/2}, \quad h = 1, \ldots, \delta\tau^{-1},
\]

with probability \( \geq 1 - c_1\varepsilon^{2}\varepsilon^{-1} \).

The proof is given in the Appendix 11.

**Remark 4.** The constant \( C, c_1 \) and \( c_2, \) which does depend on \( \varepsilon \), turns out to have a bad dependency on the parameters \( \delta, \ell, E \) and \( \tau \). However all these parameters are fixed in this section, so that the Theorem 5 implies that both \( E_{n,k}^{(\varepsilon)} \) and \( D_{n,k}^{(\varepsilon)} \), as well as \( \varepsilon^2\eta_n(m,k) \) and \( \varepsilon^2\zeta_n(m,k) \) are close to each other as \( \varepsilon \to 0 \) (keeping \( \delta, \ell, E, \tau \) fixed).

### 6.2 The Limit Trajectory of the Auxiliary Process

As a consequence of the Theorem 5 we shall prove that the law of \( \mu_{Y^{(\varepsilon)}}(n\delta) \) converges in the Hydrodynamic limit to a limit law denote by \( \rho^{(\delta,\ell,E,\tau)}_{0}(u,r) \) to be defined below. The limit as \( \varepsilon \to 0 \) of \( D_{n,k}^{(\varepsilon)} \) and \( \zeta_{n,m}(D_{n,k}^{(\varepsilon)}) \) will appear in its definition. In what follows we make explicit the dependence on \( \delta, \ell, E, \tau \) writing \( D_{n,k}^{(\varepsilon,\delta,\ell,E,\tau)}, \zeta_{n,m}^{(\varepsilon,\delta,\ell,E,\tau)}(m,k), e_{n}^{(\varepsilon,\delta,\ell,E,\tau)}(m), S_{n}^{(\varepsilon,\delta,\ell,E,\tau)}(m,\delta) \) and \( S_{n}^{(\varepsilon,\delta,\ell,E,\tau)}(m,\delta) \) and \( S_{n}^{(\varepsilon,\delta,\ell,E,\tau)}(m,\delta) \) and \( S_{n}^{(\varepsilon,\delta,\ell,E,\tau)}(m,\delta) \)

We set for each \( 1 \leq k \leq R_0E^{-1} \) and \( 1 \leq m \leq \ell^{-2} \),

\[
\zeta_{0}^{(\delta,\ell,E,\tau)}(m,k) := \lim_{\varepsilon \to 0} \varepsilon^2\zeta_{0}^{(\varepsilon,\delta,\ell,E,\tau)}(m,k), \quad I_{0,k} = I_k \in I_E.
\]

By Assumption 2 this limit exits and it is equal to \( \ell^2 \int_{I_k} \psi_0(u, i_m)du \). The value \( \zeta_{0}^{(\delta,\ell,E,\tau)}(m,k) \) has the nice probabilistic meaning of being the limit fraction of neurons, inside \( C_m \), whose membrane potential is \( D_{0,k}^{(\delta,\ell,E,\tau)} = D_{0,k}^{(\delta,\ell,E,\tau)} = E_{k} \).

The function \( \rho_{0}^{(\delta,\ell,E,\tau)}(u,r) \) is then obtained by distributing the number \( \zeta_{0,m}(D_{0,k}^{(\delta,\ell,E,\tau)}) \) uniformly over the rectangle \( I_{0,k} \times C_m \):

\[
\rho_{0}^{(\delta,\ell,E,\tau)}(u,r) := \frac{\zeta_{0}^{(\delta,\ell,E,\tau)}(m,k)}{E\ell^2}, \quad (u,r) \in I_{0,k} \times C_m.
\]
We now give its definition at a general step \( n\delta \). We first compute the limit potentials \( D_{1,h}^{(\delta,\ell,E,\tau)} \) and \( D_{n,k+\delta\tau-1}^{(\delta,\ell,E,\tau)} \). Taking the limit as \( \varepsilon \to 0 \) of in the expressions (6.6) and (6.7), it follows that

\[
D_{n,k+\delta\tau-1}^{(\delta,\ell,E,\tau)} = \Phi_{\delta,\ell,E,\tau}(m) \left( D_{n-1,k}^{(\delta,\ell,E,\tau)} \right) + s_n^{(\delta,\ell,E,\tau)}(m), \quad (6.11)
\]

where the functions \( c_n^{(\delta,\ell,E,\tau)}(m) \), \( s_n^{(\delta,\ell,E,\tau)}(m) \) and \( s_n^{(\delta,\ell,E,\tau,h)}(m) \) are obtained by letting \( \varepsilon \to 0 \):

\[
e_n^{(\delta,\ell,E,\tau)}(m) = \lim_{\varepsilon \to 0} e_n^{(\varepsilon,\delta,\ell,E,\tau)}(m),
\]

\[
s_n^{(\delta,\ell,E,\tau)}(m) = \lim_{\varepsilon \to 0} P_\varepsilon(x,\delta,\ell,E,\tau)(0) \left[ S_n^{(\varepsilon,\delta,\ell,E,\tau)}(m,\delta) \right],
\]

\[
s_n^{(\delta,\ell,E,\tau,h)}(m) = \lim_{\varepsilon \to 0} P_\varepsilon(x,\delta,\ell,E,\tau)(0) \left[ S_n^{(\varepsilon,\delta,\ell,E,\tau)}(m,h) \right].
\]

We need also to compute the limit as \( \varepsilon \to 0 \) of the numbers \( \zeta_n^{(\varepsilon,\delta,\ell,E,\tau)}(m,k) \). By letting \( \varepsilon \to 0 \) in (6.8), it is clear that

\[
\zeta_n^{(\delta,\ell,E,\tau)}(m,k+\delta\tau^{-1}) = \zeta_{n-1}^{(\delta,\ell,E,\tau)}(m,k)e^{-\delta\Phi(D_{n-1,k}^{(\delta,\ell,E,\tau)}i_m)}.
\]

Similarly, sending \( \varepsilon \to 0 \) in (6.9), we have that

\[
\zeta_n^{(\delta,\ell,E,\tau)}(m,h) = \sum_k \zeta_{n-1}^{(\delta,\ell,E,\tau)}(m,k) \left( e^{-(h-1)\tau\Phi(D_{n-1,k}^{(\delta,\ell,E,\tau)}i_m)} - e^{-h\tau\Phi(D_{n-1,k}^{(\delta,\ell,E,\tau)}i_m)} \right). \tag{6.17}
\]

Now, consider the set of intervals \( \mathcal{I}_{n,k}^{(\delta,\ell,E,\tau)} = \left\{ I_{n,k}^{(\delta,\ell,E,\tau)} \right\} \) for \( h = 1, \ldots, \delta\tau^{-1} \), the intervals are of the form

\[
I_{n,k}^{(\delta,\ell,E,\tau)} = \left[ D_{n,k}^{(\delta,\ell,E,\tau)}, D_{n,k+\delta\tau^{-1}}^{(\delta,\ell,E,\tau)} \right], \tag{6.18}
\]

while for \( k = 1, \ldots, |D_{n-1}^{(\delta,\ell,E,\tau)}| \), \( I_{n,k+\delta\tau^{-1}}^{(\delta,\ell,E,\tau)} \) is the interval having center in the value \( D_{n,k+\delta\tau^{-1}}^{(\delta,\ell,E,\tau)} \) whose length satisfies

\[
|I_{n,k+\delta\tau^{-1}}^{(\delta,\ell,E,\tau)}| = e^{-(\alpha+\lambda_m)\delta} |I_{n,k}^{(\delta,\ell,E,\tau)}| \tag{6.19}
\]

Finally, we set

\[
\rho_{n\delta}^{(\delta,\ell,E,\tau)}(u,r) = \frac{\zeta_n^{(\delta,\ell,E,\tau)}(m,k)}{|I_{n,k}^{(\delta,\ell,E,\tau)}|^{1/2}}, \quad (u,r) \in I_{n,k}^{(\delta,\ell,E,\tau)} \times C_m. \tag{6.20}
\]

Notice that \( \rho_{n\delta}^{(\delta,\ell,E,\tau)}(u,r) \) is obtained by distributing the number \( \zeta_n^{(\delta,\ell,E,\tau)}(m,k) \) uniformly over the rectangle \( I_{n,k} \times C_m \). Furthermore, for all \( r \in [0,1)^2 \), the function \( \rho_{n\delta}^{(\delta,\ell,E,\tau)}(u,r) \) is a probability density on \( \mathbb{R}_+ \), i.e,

\[
1 = \int_0^\infty \rho_{n\delta}^{(\delta,\ell,E,\tau)}(u,r)du. \tag{6.21}
\]

As an immediate consequence of the definition of \( \rho_{n\delta}^{(\delta,\ell,E,\tau)} \) and of Theorem 5,

**Corollary 1** (Hydrodynamic limit for the auxiliary process). Let \( t \in \mathcal{T} \), \( \delta \in \{2^{-q}T, q \in \mathbb{N}\} \) such that \( t = \delta n \) for some positive integer \( n \) and \( \phi \in \mathcal{S} \). Then, almost surely as \( \varepsilon \to 0 \),

\[
\mu_{Y^{(\varepsilon,\delta,\ell,E,\tau)}}(\phi) \xrightarrow{w} \int_{[0,1)^2} \int_0^\infty \phi(u,r)\rho_t^{(\delta,\ell,E,\tau)}(u,r)dudr. \tag{6.22}
\]
6.3 Convergence of $P_{n\delta}^{(\delta, \ell, E, \tau)}$ as $\ell, E, \tau \to 0$ and its Consequences

We shall next prove that the limit evolution $\rho_{n\delta}^{(\delta, \ell, E, \tau)}(u, r)$ converges as $\ell, E, \tau \to 0$ to a function denoted by $P_{n\delta}^{(\delta)}(u, r)$. Its explicit expression will be given in the Proposition [4] below. Before going to this proposition, we shall make some considerations which motivate the definitions of all ingredients involved in the definition of $P_{n\delta}^{(\delta)}(u, r)$.

The convergence of $\rho_{n\delta}^{(\delta, \ell, E, \tau)}(u, r)$ is direct. Indeed, by (6.10), $\rho_{n\delta}^{(\delta, \ell, E, \tau)}(u, r) = \rho_{0}^{(E)}(u, r)$ and

$$\lim_{E \to 0} \rho_{0}^{(E)}(u, r) = \rho_{0}^{(\delta)}(u, r) = \rho_{0}^{(\delta)}(u, r), \quad (u, r) \in \mathbb{R}_{+} \times [0, 1)^2.$$  

(6.23)

Now, we set $u_{0}^{(\delta)}(r) := \lim_{\ell, E, \tau \to 0} u_{0}^{(\delta, \ell, E, \tau)}(m)$ and $\delta p_{0}^{(\delta)}(r) := \lim_{\ell, E, \tau \to 0} s_{1}^{(\delta, \ell, E, \tau)}(m)$ where the index $m = m(r, \ell)$ is such that for each $\ell, r \in C_{m}$. Let us compute their explicit expressions. By equality (6.5)

$$e_{0}^{(\varepsilon, \delta, \ell, E, \tau)}(m) = \varepsilon^{2} \sum_{m' = 1}^{\ell - 2} \sum_{k = 1}^{R_{0}E^{-1}} b(i_{m'}, i_{m})E_{0,k}^{(E)}(\varepsilon, \delta, \ell, E, \tau)(m, k)$$

so that taking the limit as $\varepsilon^{2} \to 0$, we get from (6.10) and (6.23) that

$$e_{0}^{(\delta, \ell, E, \tau)}(m) = E\varepsilon^{2} \sum_{m', k} b(i_{m'}, i_{m})E_{0,k}^{(E)}(\varepsilon, \delta, \ell, E, \tau)(E_{0,k}^{(E)}(m', k)).$$

From this last expression and using once more (6.10), we immediately have

$$\tilde{u}_{0}^{(\delta)}(r) = \int_{[0,1)^2} \int_{0}^{\infty} ub(r', r)\rho_{0}^{(\delta)}(u, r')dudr'.$$

We now derive the expression of $\delta p_{0}^{(\delta)}(r)$. Since, by definition - see (5.5) -

$$\tilde{P}^{(\varepsilon)}_{Y(\varepsilon, \delta, \ell, E, \tau)}[S_{1}^{(\varepsilon, \delta, \ell, E, \tau)}(m, \delta)] = \varepsilon^{2} \sum_{m' = 1}^{\ell - 2} \sum_{k = 1}^{R_{0}E^{-1}} a(i_{m'}, i_{m})S_{0}^{(\varepsilon, \delta, \ell, E, \tau)}(m, k)(1 - e^{-\delta \varphi(E_{0,k}^{(E)}(\delta, i_{m'}))}).$$

It follows as before that

$$s_{1}^{(\delta, \ell, E, \tau)}(m) = E\varepsilon^{2} \sum_{m' = 1}^{\ell - 2} \sum_{k = 1}^{R_{0}E^{-1}} a(i_{m'}, i_{m})P_{0}^{(\delta, \ell, E, \tau)}(E_{0,k}^{(E)}(m', k))(1 - e^{-\delta \varphi(E_{0,k}^{(E)}(\delta, i_{m'}))}).$$

Therefore, using again (6.10) and then taking $\ell, E \to 0$ in the above expression, we deduce that

$$\delta p_{0}^{(\delta)}(r) := \lim_{\ell, E, \tau \to 0} s_{1, m}^{(\delta, \ell, E, \tau)} = \int_{[0,1)^2} \int_{0}^{\infty} a(r', r)\rho_{0}^{(\delta)}(u, r')(1 - e^{-\delta \varphi(u, r')})dudr',$$

where

$$\rho_{0}^{(\delta)}(r) = \int_{[0,1)^2} \int_{0}^{\infty} a(r', r)\rho_{0}^{(\delta)}(u, r')(1 - e^{-\delta \varphi(u, r')})dudr'.$$
Notice that by equations (6.16) and (6.20), for \( (u, r) \in I_{1,k+b\tau-1}^{(\delta, \ell, E, \tau)} \times C_m, \)

\[
\rho_0^{(\delta, \ell, E, \tau)} (u, r) = \rho_0 \left( \Phi_{\delta, \ell, E, \tau}^{-1} \left( \Phi_{\delta, \ell, E, \tau} (u) - e^{\delta (\alpha + \lambda_m)} s_{1,m} \right)(1, m) - \alpha - \lambda_m \right),
\]

where for each \( E, E_0^{(\bar{E})} \) is such that \( \Phi_{\delta, \ell, E, \tau}^{-1} (u) - e^{\delta (\alpha + \lambda_m)} s_{\delta, \ell, E, \tau} \) \( \in I_{1,k}^{(\delta, \ell, E, \tau)} \). From this and the above equalities, it follows that

\[
E_0^{(\bar{E})} \to \Phi_{\delta, \ell, E, \tau}^{-1} (u) - \delta \rho_0^{(\delta)} (r), \text{ as } E, \ell \to 0,
\]

so that

\[
\rho_0^{(\delta)} (u, r) = \rho_0 \left( \Phi_{\delta, \ell, E, \tau}^{-1} (u) - e^{\delta (\alpha + \lambda_r)} \rho_0^{(\delta)} (r) \right) e^{-\delta [\varphi_{\delta, \ell, E, \tau}^{-1} (u) - e^{\delta (\alpha + \lambda_r)} \rho_0^{(\delta)} (r) \delta_r - \alpha - \lambda_r]},
\]

for \( u \geq x_0 (r) = \frac{\lambda_r}{\alpha + \lambda_r} (1 - e^{-\delta (\alpha + \lambda_r)}) u_0^{(\delta)} (r) + \delta p_0^{(\delta)} (r) \). This formula express the flow of potentials of those neurons which do not spike in the interval \( [0, \delta] \).

Now, take a pair \( (u, r) \in I_{1,h}^{(\delta, \ell, E, \tau)} \times C_m \). In this case, by (6.12), there exists a sequence \( h = h(u, r, \tau) \) such that

\[
(1 - e^{-(h-1)\tau (\alpha + \lambda_m)}) e_{1,m}^{(\delta, \ell, E, \tau)} + s_{n,m}^{(\delta, \ell, E, \tau, h)} = D_{1,h}^{(\delta, \ell, E, \tau)} \to u, \text{ as } \tau, \ell \to 0,
\]

and this implies that there exits a time \( 0 \leq t_0 (u, r) \leq \delta \) such that

\[
u = (1 - e^{-t_0 (u, r) (\alpha + \lambda_r)}) u_0^{(\delta)} (r) + \int_{[0,1]} \int_0^\infty a(r', r) \rho_0^{(\delta)} (r, v, r') \left[ e^{-(\delta - t_0 (v, r)) \varphi(v, r') - e^{-\delta \varphi(v, r')}} \right] dv dr'.
\]

The time in which a neuron at position \( r \) has to spike in order to accumulate up to time \( \delta \) a potential \( u \) is exactly \( \delta - t_0 (u, r) \).

Similarly notice that,

\[
\frac{D_{1,h+1}^{(\delta, \ell, E, \tau)} - D_{1,h}^{(\delta, \ell, E, \tau)}}{\tau} \to \lambda_r u_0^{(\delta)} (r) e^{-(\delta - t_0 (u, r)) (\alpha + \lambda_r)} + \bar{p}_0^{(\delta)} (r), \text{ as } \tau, \ell \to 0,
\]

where the function \( \bar{p}_0^{(\delta)} (r) \) is given by

\[
\bar{p}_0^{(\delta)} (r) = \int_{[0,1]} \int_0^\infty a(r', r) \varphi(u', r') \rho_0^{(\delta)} (u', r') e^{-t_0 (u, r) \varphi(u', r')} du' dr'.
\]

Thus, letting in \( E, \ell, \tau \to 0 \) in (6.17) we deduce that

\[
\rho_0^{(\delta)} (u, r) = \frac{q_0^{(\delta)} (r)}{\lambda_r u_0^{(\delta)} (r) e^{-(\delta - t_0 (u, r)) (\alpha + \lambda_r)} + \bar{p}_0^{(\delta)} (r)},
\]

where \( u_0^{(\delta)} (r), p_0^{(\delta)} (r) \) and \( t_0 (u, r) \) as above and

\[
q_0^{(\delta)} (r) = \int_0^\infty \rho_0^{(\delta)} (v, r) \varphi(v, r) e^{-t_0 (u, r) \varphi(v, r)} dv.
\]
To conclude, we notice that the convergence in (6.24) holds if and only if
\[
    u < \frac{\lambda_r}{\alpha + \lambda_r} (1 - e^{-\delta(\alpha + \lambda_r)}) \bar{u}_0 (r) + \delta p_0^\delta (r) = x_0 (r).
\]

All considerations we have done above, in fact, may be extended directly to any \(n\delta\). Thus we have

**Proposition 4.** For all \(n\delta \leq T\), there exists the limit of \(\rho_{n\delta}^{(\delta,\ell,E,\tau)} (u,r)\) as \(\ell, E, \tau \to 0\). Moreover, let \(\rho_{n\delta}^\delta, \bar{u}_{n\delta}^\delta (r)\) and \(p_{n\delta}^\delta (r)\) be functions defined by
\[
    \rho_{n\delta}^\delta (u,r) = \lim_{\ell,E,\tau \to 0} \rho_{n\delta}^{(\delta,\ell,E,\tau)} (u,r), \quad (u,r) \in \mathbb{R}_+ \times [0,1]^2,
\]
\[
    \bar{u}_{n\delta}^\delta (r) = \int_{[0,1]^2} \int_0^{\infty} u(r',r) \rho_{n\delta}^\delta (u,r') dudr', \quad (6.26)
\]
\[
    p_{n\delta}^\delta (r) = \int_{[0,1]^2} \int_0^{\infty} a(r',r) \left(1 - e^{-\delta \varphi(u,r')} \right) \frac{1}{\delta} \rho_{n\delta}^\delta (u,r') dudr', \quad (6.27)
\]
and then set
\[
    x_n (r) = \frac{\lambda_r}{\alpha + \lambda_r} (1 - e^{-\delta(\alpha + \lambda_r)}) \bar{u}_{n\delta}^\delta (r) + \delta p_{n\delta}^\delta (r), \quad r \in [0,1)^2. \quad (6.28)
\]

Then for all pairs \((u,r)\) satisfying \(u \geq x_n (r)\),
\[
    \rho_{(n+1)\delta}^\delta (u,r) = \rho_{n\delta}^\delta \left( \Phi^{-1}_{\delta,\bar{u}_{n\delta}^\delta (r)} (u) - e^{\delta(\alpha + \lambda_r)} p_{n\delta}^\delta (r) \delta, r \right)
\]
\[
    \times \exp \left\{ - \delta \left[ \varphi \left( \Phi^{-1}_{\delta,\bar{u}_{n\delta}^\delta (r)} (u) - e^{\delta(\alpha + \lambda_r)} p_{n\delta}^\delta (r) \delta, r \right) - \alpha - \lambda_r \right] \right\}. \quad (6.29)
\]

Now, set for any pair \((u,r)\) such that \(u < x_n (r)\),
\[
    \rho_{n\delta}^\delta (u,r) = \int_{[0,1]^2} \int_0^{\infty} a(r',r) \varphi(v,r') \rho_{n\delta}^\delta (v,r') e^{-t_{n\delta}(u,r) \varphi(v,r')} dvdr', \quad (6.30)
\]
\[
    q_{n\delta}^\delta (u,r) = \int_0^{\infty} \rho_{n\delta}^\delta (v,r) \varphi(v,r) e^{-t_{n\delta}(u,r) \varphi(v,r')} dv, \quad (6.31)
\]

where the function \(t_{n\delta}(u,r)\) appearing in the definition of \(\rho_{n\delta}^\delta (u,r)\) and \(q_{n\delta}^\delta (u,r)\) is defined through the relation
\[
    u = \left( 1 - e^{-t_{n\delta}(u,r) (\alpha + \lambda_r)} \right) \bar{u}_{n\delta}^\delta (r)
\]
\[
    + \int_{[0,1]^2} \int_0^{\infty} a(r',r) \rho_{n\delta}^\delta \left[ e^{-\delta - t_{n\delta}(u,r)} \varphi(v,r') - e^{-\delta \varphi(v,r')} \right] dvdr'.
\]

Then it follows that for all pairs \((u,r)\) with \(u < x_n (r)\),
\[
    \rho_{(n+1)\delta}^\delta (u,r) = \frac{q_{n\delta}^\delta (u,r)}{\lambda_r \rho_{n\delta}^\delta (r) e^{-\delta - t_{n\delta}(u,r)(\alpha + \lambda_r)} + \rho_{n\delta}^\delta (u,r)}. \quad (6.32)
\]

Furthermore, in either cases, for each \(r \in [0,1)\) and \(n\delta \leq T\),
\[
    \int_0^{\infty} \rho_{n\delta}^\delta (u,r) du = 1. \quad (6.33)
\]
Hence the supports of $\rho^{(\delta)}_{n+1}$ have support $[0, R_{n+1}(r)]$, where

$$R_{n+1}(r) = e^{-\delta(\alpha+\lambda_r)} R_n(r) + \frac{\lambda_r}{\alpha + \lambda_r} (1 - e^{-\delta(\alpha+\lambda_r)}) \tilde{u}^{(\delta)}_{n\delta}(r) + \tilde{\delta} p^{(\delta)}_{n\delta}(r). \quad (6.34)$$

Since $p^{(\delta)}_{n\delta}(r) \leq \varphi^* a^*$ and $\tilde{u}^{(\delta)}_{n\delta}(r) \leq R_n(r)$, it follows that for all $n$ such that $n \delta \leq T$,

$$R_n(r) \leq R_{n-1}(r) + \varphi^* a^* \delta \leq R_0 + n \delta \varphi^* a^* \leq R_0 + T \varphi^* a^*. \quad (6.35)$$

Hence the supports of $\rho^{(\delta)}_{n\delta}$ are uniformly bounded. By iterating (6.29) and by using the explicit form of the inverse flow $\Phi^{-1}_{\delta, \tilde{u}^{(\delta)}_{n\delta}(r)}(u)$, we get

$$\rho^{(\delta)}_{n+1}(u) = e^{\delta(n+1)(\alpha+\lambda_r)} \psi_0 \left( e^{\lambda(n+1)\delta} u - \sum_{s=0}^{n} e^{\lambda(s+1)\delta(\alpha+\lambda_r)} [x_s(r) - 2p^{(\delta)}_{s\delta}(r) \delta], r \right)$$

$$\times \exp \left\{ - \sum_{s=0}^{n} \delta \varphi \left( e^{\lambda(n+1-s)\delta} u - \sum_{h=s}^{n} e^{(h-s)\delta(\alpha+\lambda_r)} [x_h(r) - 2p^{(\delta)}_{h\delta}(r) \delta], r \right) \right\} \quad (6.36)$$

for all

$$u \geq u^{*}_{n+1}(r) = \sum_{s=0}^{n} e^{-\delta(n-s)(\alpha+\lambda_r)} \left[ \frac{\lambda_r}{\alpha + \lambda_r} (1 - e^{-\delta(\alpha+\lambda_r)}) \tilde{u}^{(\delta)}_{n\delta}(r) + \tilde{\delta} p^{(\delta)}_{n\delta}(r) \right], \quad (6.37)$$

where $\psi_0$ is the initial density and and $x_n(r)$ is defined in (6.28).

The following results will be used in the analysis of the hydrodynamic of the true process. We first collect some properties of the functions $\tilde{u}^{(\delta)}_{n\delta}(r)$ and $p^{(\delta)}_{n\delta}(r)$.

**Proposition 5.** There exist $\delta_0$ and a positive constant $C$ depending on $\varphi^*, a^*, T, R_0, \lambda^*$ and $\alpha$ such that for all $\delta \leq \delta_0$ and all $n$ satisfying $n \delta \leq T$,

(i) $|\tilde{u}^{(\delta)}_{n(n+1)\delta}(r) - \tilde{u}^{(\delta)}_{n\delta}(r)| + |p^{(\delta)}_{n(n+1)\delta}(r) - p^{(\delta)}_{n\delta}(r)| \leq C \delta$,

(ii) $|\tilde{u}^{(\delta)}_{n\delta}(r) - \tilde{u}^{(\delta)}_{n\delta}(r')| + |p^{(\delta)}_{n\delta}(r) - p^{(\delta)}_{n\delta}(r')| \leq C |r - r'|$.

**Proof.** We will show only that $|\tilde{u}^{(\delta)}_{n(n+1)\delta}(r) - \tilde{u}^{(\delta)}_{n\delta}(r)| < C \delta$, since all other bounds are likewise obtained. By definition,

$$\tilde{u}^{(\delta)}_{n(n+1)\delta}(r) = \int_{[0,1]^2} \int_{0}^{x_n(r')} ub(r', r) \rho^{(\delta)}_{n(n+1)\delta}(u, r') du dr' + \int_{[0,1]^2} \int_{0}^{\infty} ub(r', r) \rho^{(\delta)}_{n(n+1)\delta}(u, r') du dr'. \quad (6.38)$$

Using (6.29) in the second term in (6.38) and then making change of variables

$$v = \Phi^{-1}_{\delta, \tilde{u}^{(\delta)}_{n\delta}(r)}(u) - \delta e^{\delta(\alpha+\lambda_r)} p^{(\delta)}_{n\delta}(r), \quad \frac{dv}{du} = e^{\delta(\alpha+\lambda_r)},$$

the second term in (6.38) becomes

$$\int_{[0,1]^2} \int_{0}^{\infty} b(r', r) \left[ \Phi^{-1}_{\delta, \tilde{u}^{(\delta)}_{n\delta}(r)}(u) + \delta p^{(\delta)}_{n\delta}(r) \right] \rho^{(\delta)}_{n\delta}(u, r') e^{-\delta\varphi(u, r')} du dr'.$$
Since \( p^{(\delta)}_{n\delta}(r) \leq \varphi^* a^* \) and \( \tilde{u}^{(\delta)}_{n\delta}(r) \leq R_n(r) \leq R_0 + T \varphi^* a^* \), we deduce from (6.38) and the integral above that there exists a positive constant depending on \( \varphi^*, a^*, T, R_0, \lambda^* \) and \( \alpha \) such that

\[
|\tilde{u}^{(\delta)}_{(n+1)\delta}(r) - \tilde{u}^{(\delta)}_{n\delta}(r)| \leq \int_{[0,1)^2} \int_0^{x_n(r')} ub(r', r') \rho^{(\delta)}_{(n+1)\delta}(u, r') dudr' + C\delta. \quad (6.39)
\]

Thus, it suffices to show that the integral on the right-hand side of (6.39) is \( \leq C\delta \). For that sake, we first notice that by (6.32), for any pair \((u, r)\) with \( u < x_n(r) \)

\[
\rho^{(\delta)}_{(n+1)\delta}(u, r) \leq \frac{\varphi^*}{\min\{e^{-\delta(\alpha+\lambda r)}, e^{-\delta\varphi^*}\}}(\lambda_r \tilde{u}^{(\delta)}_{n\delta}(r) + p^{(\delta)}_{n\delta}(r)) := C_1(\delta, n, r, \varphi^*). \quad (6.40)
\]

Then, we upper bound the integral in (6.39) by

\[
\int_{[0,1)^2} \frac{C_1(\delta, n, r', \varphi^*)}{2} x_n^2(r') b(r, r') dudr'. \quad (6.41)
\]

Since \( C_1(\delta, n, r, \varphi^*) x_n(r) \to 1 \) as \( \delta \to 0 \) uniformly in \( r \) and \( n \), and \( x_n(r) \leq C\delta \), we get the result from (6.41). \( \square \)

Finally, we prove equicontinuity of the function \( \rho^{(\delta)}_{n\delta} \). The proof is an immediate consequence of the definition of \( \rho^{(\delta)}_{n\delta} \) and the Proposition 5.

**Proposition 6.** There exists a constant \( C \) such that for all \( \delta \) sufficiently small, for any \( n \) and \( m \), with \( n\delta \leq T, \, r \in [0,1)^2 \),

\[
|\rho^{(\delta)}_{n\delta}(u, r) - \rho^{(\delta)}_{n\delta}(v, r)| \leq C \max\{|u - v|, \delta\}, \text{ for } u, v \in [0, u^*_n(r)) \quad (6.42)
\]

and

\[
|\rho^{(\delta)}_{n\delta}(u, r) - \rho^{(\delta)}_{n\delta}(v, r)| \leq C|u - v|, \text{ for } u, v \in [u^*_n(r), \infty). \quad (6.43)
\]

Moreover, for all \( n\delta \leq T \) and all \( r, r' \in [0,1)^2 \),

\[
|\rho^{(\delta)}_{n\delta}(u, r) - \rho^{(\delta)}_{n\delta}(u, r')| \leq C|r - r'|, \text{ for } u \in [0, u^*_n(r) \land u^*_n(r')] \cup \left[u^*_n(r) \lor u^*_n(r'), \infty\right), \quad (6.44)
\]

and for all \( n\delta \leq T, \, m\delta \leq T \),

\[
|\rho^{(\delta)}_{n\delta}(u, r) - \rho^{(\delta)}_{m\delta}(u, r)| \leq C|n - m|\delta, \text{ for } u \in [0, u^*_n(r) \land u^*_m(r)] \cup \left[u^*_n(r) \lor u^*_m(r), \infty\right). \quad (6.45)
\]

Finally, if \( \psi_0 \) satisfies the conditions in Theorem 3, then also

\[
|\rho^{(\delta)}_{n\delta}(u^*_n(r)_+, r) - \rho^{(\delta)}_{n\delta}(u^*_n(r)_-, r)| \leq C\delta
\]

and

\[
|\rho^{(\delta)}_{n\delta}(u, r) - \rho^{(\delta)}_{n\delta}(u, r')| \leq C|r - r'|, \quad |\rho^{(\delta)}_{n\delta}(u, r) - \rho^{(\delta)}_{m\delta}(u, r)| \leq C|n - m|\delta.
\]
7 Hydrodynamic limit for the True Process

We can now conclude the proof of Theorem 7. Given any positive real number $T$, recall $\mathcal{T} = \{ t \in [0,T] : t = n2^{-q}T, q, n \in \mathbb{N} \}$. For each $\delta = 2^{-q}T$, $q \geq 1$, we consider the following function defined on $[0,T] \times [0,1)^2$:

$$F_t^{(\delta)}(r) = p_n^{(\delta)}(r) + \frac{(P_n^{(\delta)}(r) - P_{n+1}^{(\delta)}(r))(t-n\delta)}{\delta}, \text{ for } n\delta \leq t < (n+1)\delta, \ r \in [0,1)^2.$$  

By Proposition 6 there exists a constant $C > 0$ not depending on $\delta$ such that

$$|F_t^{(\delta)}(r) - F_s^{(\delta)}(r')| \leq C(|t-s| + |r-r'|), \text{ for } (t,r) \in [0,T] \times [0,1)^2. \quad (7.1)$$  

An analogous argument implies that there exists also a continuous function $\bar{F}_n^{(\delta)}(r), t \in [0,T], r \in [0,1]^2$. In particular, it follows

$$\lim_{\delta \to 0} \sup_{r \in [0,1]^2: n\delta \leq T} \sup_{t \in [n\delta,(n+1)\delta]} |p_t^{(0)}(r) - \bar{F}_n^{(\delta)}(r)| = 0. \quad (7.2)$$  

An analogous argument implies that there exists also a continuous function $\tilde{u}_t^{(0)}(r), t \in [0,T], r \in [0,1]^2$ in which

$$\lim_{\delta \to 0} \sup_{r \in [0,1]^2: n\delta \leq T} \sup_{t \in [n\delta,(n+1)\delta]} |\tilde{u}_t^{(0)}(r) - \bar{u}_n^{(\delta)}(r)| = 0. \quad (7.3)$$  

Defining for each $t \in [0,T], r \in [0,1]^2$:

$$u_t^{*,0}(r) = e^{-(\alpha + \lambda_r)t} \left( \int_0^t \lambda_r \bar{u}_s^{(0)}(r)e^{(\alpha + \lambda_r)s}ds + \int_0^t p_s^{(0)}(r)e^{(\alpha + \lambda_r)s}ds \right), \quad (7.4)$$  

it follows from (7.2) and (7.3) that

$$\lim_{\delta \to 0} \sup_{r \in [0,1]^2: n\delta \leq T} \sup_{t \in [n\delta,(n+1)\delta]} |u_t^{*,0}(r) - u_n^{*,\delta}(r)| = 0. \quad (7.5)$$  

Denoting below by $\zeta$ the elements of the form $\zeta = 2^{-q}$, with $q \in \mathbb{N}$. By (7.5), for each $\zeta$ there exists $\delta_\zeta$ such that for all $\delta < \delta_\zeta$ the following holds. For all $r \in [0,1)^2$ and $n$ such that $n\delta \leq T$, if $|u_n^{*,0}(r) - u| \geq \zeta$ then $u_n^{*,0}(r) - u$ has the same sign as $u_n^{*,\delta}(r) - u$. We can then use the Proposition 6 and a Arzelà-Ascoli type of argument to deduce that the function $\rho_t^{(\delta)}(u,r)$ converges uniformly to a continuous function $\rho_t(r,u), t \in \mathcal{T}, r \in [0,1)^2, |u - u_t^{*,0}(r)| \geq \zeta$ with compact support. By continuity $\rho_t(u,r)$ can extended to all $t \in [0,T], r \in [0,1)^2$ and $|u - u_t^{*,0}(r)| \geq \zeta$. By a diagonalization procedure we extend the above to all $t,r$ and $u$ with $u \neq u_t^{*,0}(r)$. Then by (6.24), (6.25) and the Lebesgue Dominated Convergence Theorem, for all $t \in \mathcal{T}$,

$$1 = \int_0^\infty \rho_t(u,r)du, \quad p_t^{(0)}(r) = \int_{[0,1]^2} \int_0^\infty a(r',r)\varphi(u,r')p_t(u,r')dudr'.$$
and
\[ \bar{u}^{(0)}_t(r) = \int_{[0,1]^2} \int_0^\infty u b(r', r) \rho_t(u, r') du dr'. \]

By continuity, all these equalities hold for all \( t \in [0, T] \). Thus, \( p_t^{(0)}(r) \) and \( \bar{u}_t^{(0)}(r) \) coincide with \( p_t(r) \) and \( \bar{u}_t(r) \) given by (2.17) and (2.18) and therefore from now on we omit the superscript 0. Finally sending \( \delta \to 0 \) in (6.36) and (6.32) we show that \( p_t(u, r) \) satisfies (2.16)-(2.17).

We claim that \( p_t(u, r) \) is a weak solution for (2.6)-(2.9) with \( v_0 = \psi_0 \) and \( v_1 \) as in (2.10). This will be a direct consequence of the

**Lemma 1.** If \( p_t(r, u) \) is given by (2.16)-(2.17), then for any real valued test function \( \phi \) on \( \mathbb{R}_+ \times [0,1]^2 \),
\[
\int_{[0,1]^2} \int_0^\infty \phi(u, r)p_t(u, r) du dr = \int_{[0,1]^2} \int_0^t \phi(T_{s,t}(0), r)q_s(r) \exp \left\{ -\int_s^t \varphi(T_{s,h}(0), r) dh \right\} ds dr
+ \int_{[0,1]^2} \int_0^\infty \phi(T_{0,t}(u), r)\psi_0(u, r) \exp \left\{ -\int_0^t \varphi(T_{0,s}(u), r) ds \right\} du dr. \tag{7.6}
\]

**Proof.** Noticing that \( u_t^{(r)}(r) = T_{0,t}(0, r) \), we start writing
\[
\int_{[0,1]^2} \int_0^\infty \phi(u)\rho_t(u, r) du dr = \int_{[0,1]^2} \int_0^t \phi(u)\rho_t(u, r) du dr
+ \int_{[0,1]^2} \int_0^\infty \phi(u)\rho_t(u, r) du dr. \tag{7.7}
\]

Now, using (2.17) and making the change of variables \( v = T_{s,t}(0) \) in the first integral of the right-hand side of (7.7), we obtain the first integral of the right-hand side of (7.6).

To complete the proof we use (2.16) and make the change of variable \( v = T_{0,t}^{-1}(u, r) \) in the second integral of the right-hand side of (7.7). \( \square \)

Immediately from (7.6) follows that for any test function \( \phi, \int_0^\infty \phi(u)\rho_t(u, r) du \) is differentiable in \( t \) and that its derivative satisfies (2.11). Furthermore, taking \( \phi(u, r') = a(r', r) \varphi(u, r'), \varphi(u, r') = b(r', r)u \) and \( \varphi(u) = \varphi(u, r) \), we conclude that the functions \( \bar{u}_t(r), p_t(r) \) and \( q_t(r) \) are differentiable in \( t \) and from this that \( p_t(u, r) \) is differentiable in \( t \) and \( u \) in the open set \( \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]^2 \ \setminus \{(t, u, r) : u = T_{0,t}(0, r\} \). Hence, by (2.11) \( p_t(u, r) \) satisfies (2.6) is this set and the boundary conditions (2.9) with \( v_0 = \psi_0 \) and \( v_1 \) as in (2.10).

We shall next prove uniqueness for (2.11), so that limit \( p_t(u, r) \) does not depend on the converging subsequence, implying full convergence. For all smooth functions \( \phi \), setting \( g(t, r, du) = \rho_t(u, r) du \), we rewrite (2.11) in the following way,
\[
\partial_t \int_0^\infty \phi(u)g(t, r, du) = \int_0^\infty \phi'(u)[-\alpha u - \lambda_r(u - \bar{u}_t(r))] + p_t(r)]g(t, r, du)
+ \int_0^\infty \varphi(u, r)[\phi(0) - \phi(u)]g(t, r, du), \tag{7.8}
\]
where \( g(0, r, du) = \psi_0(u, r) du \) and
\[
\bar{u}_t(r) = \int \int ub(r', r)g(t, r', du) dr', \quad p_t(r) = \int \int a(r', r)\varphi(u, r')g(t, r', du) dr'.
\]
Now consider the function $\mathcal{L}_{\bar{u}_t(r), p_t(r)}$ defined on $C^1(\mathbb{R}_+, \mathbb{R})$, by

$$
\mathcal{L}_{\bar{u}_t(r), p_t(r)} \phi(u) = \varphi(u, r)[\phi(0) - \phi(r)] + \phi'(u)[-\alpha u - \lambda_r(u - \bar{u}_t(r)) + p_t(r)],
$$

(7.9)

and then define a stochastic process $(U(t))_{t \geq 0}$,

$$
U(t) = (U_r(t), r \in [0,1)^2),
$$

in which, for any $r_1, \ldots, r_n \in [0,1)^2$,

(i) the collection of stochastic processes $(U_{r_1}(t))_{t \geq 0}, \ldots, (U_{r_n}(t))_{t \geq 0}$ are independent and

(ii) for all $r \in [0,1)^2$, the function $\mathcal{L}_{\bar{u}_t(r), p_t(r)}$ is the time-dependent generator of the process $(U_r(t))_{t \geq 0}$.

We then deduce from equations (7.8) and (7.9) that for all $r$ and $t$, $g(t, r, du)$ is the law of $U_r(t)$. Notice that, by independence, the law of $(U(t))_{t \geq 0}$ is determined by the collection of laws

$$
\{g(t, r, du) : t \geq 0, r \in [0,1)^2\}.
$$

On the other hand, given a stochastic processes $(U(t))_{t \geq 0}$, $U(t) = (U_r(t), r \in [0,1)^2)$, fulfilling item (i) above and whose time-dependent generator $\mathcal{L}_{k_t(r), h_t(r)}$ and law $p(t, r, du)$ of $U_r(t)$ are such that:

(i) for any $r \in [0,1)^2$, $\mathbb{R}_+ \ni t \mapsto p(t, r, du)$ is a continuous function;

(ii) for any $t \geq 0$, $[0,1)^2 \ni r \mapsto p(t, r, du)$ is a measurable function;

(iii) for all $r$ and $t \leq T$, the support of $p(t, r, du)$ is contained in $[0, C_T]$.

(iv) $k_t(r) = \int \int ub(r', r)p(t, r', du)dr'$. $h_t(r) = \int a(r', r)\varphi(u, r')p(t, r', du)dr'$;

(v) $\mathcal{L}_{k_t(r), h_t(r)} \phi(u) = \varphi(u, r)[\phi(0) - \phi(r)] + \phi'(u)[-\alpha u - \lambda_r(u - k_t(r)) + h_t(r)], \phi \in C^1(\mathbb{R}_+, \mathbb{R})$,

then it is easy to check that the law $p(t, r, du)$ satisfies (7.8) replacing the functions $\bar{u}_t(r)$ and $p_t(r)$ respectively by $k_t(r)$ and $h_t(r)$.

Therefore, from these considerations it follows that the uniqueness problem of (7.8) reduces to prove

**Proposition 7.** Let $(U_r(t))_{t \geq 0}$ and $(V_r(t))_{t \geq 0}$ be two stochastic processes having generators $\mathcal{L}_{k_t(r), h_t(r)}$ and $\mathcal{L}_{k'_t(r), h'_t(r)}$ and laws $p(t, r, du)$ and $q(t, r, du)$ satisfying conditions (i) – (v). If $U(0) = V(0)$, then for any $T > 0$, $U_r(t) = V_r(t), 0 \leq t \leq T$ almost surely.

**Proof.** Fix $T > 0$. Notice that by assumptions $U(0) = V(0)$ and (iii) – (iv) there exists a constant $C > 0$ such that $(k_t(r) + h_t(r)) \wedge (k'_t(r) + h'_t(r)) \leq C$, so that $t \leq T$,

$$
|U_r(t) - V_r(t)| \leq \int_0^t (\lambda_r k_s(r) + h_s(r))ds + \int_0^t (\lambda_r k'_s(r) + h'_s(r))ds \leq 2TC.
$$

(7.10)
Coupling $U_r$ and $V_r$ such that they have the most common jumps possible, we obtain using conditions (i) and (ii) that

\[
\frac{d}{dt} \mathbb{E} \left[ |U_r(t) - V_r(t)| \right] \leq \mathbb{E} \left[ |\varphi(U_r(t), r) - \varphi(V_r(t), r)| \right] \left( U_r(t) \wedge V_r(t) - |U_r(t) - V_r(t)| \right) \\
- \mathbb{E} \left[ \varphi(U_r(t), r) \wedge \varphi(V_r(t), r) U_r(t) - V_r(t) \right] - (\alpha + \lambda_r) \mathbb{E} \left[ U_r(t) - V_r(t) \right] \\
+ \lambda_r |k_r(t) - k'_r(t)| + |h_r(t) - h'_r(t)|.
\]

Dropping the negative terms on the right-hand side of the inequality above, using the Lipschitz property of $\varphi$, (7.10), condition (iv), and writing $\gamma_r(t) = \mathbb{E} \left[ |U_r(t) - V_r(t)| \right]$, we obtain for all $t \leq T$,

\[
\frac{d}{dt} \gamma_r(t) \leq ||\varphi||_{\text{Lip}} 2TC \gamma_r(t) + \int_{[0,1]^2} \left[ \lambda^* b(r, r') \gamma_r(t) + \varphi^* a(r, r') \gamma_r(t) \right] dr',
\]

where $\varphi^* = ||\varphi||_{\infty}$ and $\lambda^* = ||\lambda||_{\infty}$. From the inequality above, we conclude that for $t \leq T$

\[
\gamma_r(t) \leq \tilde{C} \left( \int_0^t \gamma_r(s) ds + \int_0^t \int_{[0,1]^2} b(r, r') \gamma_r(s) dr' ds + \int_0^t \int_{[0,1]^2} a(r, r') \gamma_r(s) dr' ds \right),
\]

(7.11)

where $\tilde{C} = \max \{ ||\varphi||_{\text{Lip}2TC}, \lambda^*, \varphi^* \}$. Iterating $n$ times the inequality in (7.11), we get for all $t \leq T$

\[
\gamma_r(t) \leq \tilde{C} \int_0^t \gamma_r(s) ds + \frac{(\tilde{C}t)^n}{n!} \leq \tilde{C} \int_0^t \gamma_r(s) ds + \frac{(\tilde{C}T)^n}{n!}
\]

(7.12)

Since $n$ is arbitrary, we get the theorem result by first letting $n \to \infty$ and then applying Gronwall’s lemma.

We shall now prove that the true process converges to $\rho_t(u, r) du dr$ in the hydrodynamic limit. Recall that $\mathcal{P}_{[0,T]}^{(c)}$ is the law on $D([0, T], \mathcal{S}')$ of the processes $\mu_{\mathcal{U}_{[0,T]}^{(c)}}$. By the tightness proved in Proposition 2, we have convergence by subsequences $\mathcal{P}_{[0,T]}^{(c)}$ to a measure valued process $\mathcal{P}_{[0,T]}$. We will show that any such limit measure $\mathcal{P}_{[0,T]}$ is given by supported by the single deterministic trajectory $\rho_t(u, r) du dr, t \in [0, T], r \in [0,1]^2$ where $\rho_t(u, r)$ is the limit of $\rho_t^{(\delta)}(u, r)$ found above.

The following support property will be used in the sequel.

**Proposition 8.** Any weak limit $\mathcal{P}_{[0,T]}$ of $\mathcal{P}_{[0,T]}^{(c)}$ satisfies

\[
\mathcal{P}_{[0,T]}(C([0, T], \mathcal{S}')) = 1,
\]

where $C([0, T], \mathcal{S}')$ is the space of all continuous trajectories $[0, T] \to \mathcal{S}'$.

**Proof.** For each $\phi \in \mathcal{S}'$, consider the function on $D([0, T], \mathcal{S}')$ given by

\[
\Delta_\phi(x) = \sup_{t \in [0, T]} |x_t(\phi) - x_{t-}(\phi)|.
\]

(7.13)
It is not difficult to prove that the function $\Delta_\phi$ is continuous in the Skorohod norm (See for instance De Masi and Presutti (1991), section 2.7). Then for any $\zeta > 0$, by Chebyshev’s inequality and the weak convergence

$$\mathcal{P}\{x : \Delta_\phi(x) > \zeta\} \leq \zeta^{-1} \lim_{\epsilon \to 0} \mathcal{P}^{(\epsilon)}_{[0,T]}[\Delta_\phi(\mu_{\phi}^{(\epsilon)})].$$

If there are no spikes at $t$, then it is clear that

$$|\mu_{U_\phi(t)}(\phi) - \mu_{U_\phi(t_\epsilon)}(\phi)| = 0.$$

On the other hand, if $j$ spikes at $t$, then

$$|\mu_{U_\phi(t)}(\phi) - \mu_{U_\phi(t_\epsilon)}(\phi)| = \epsilon^2 \phi(U_{i_\epsilon}(t_\epsilon)) + \epsilon^2 \sum_{i \neq j} |\phi(U_{i_\epsilon}(t_\epsilon) + \epsilon^2 a(j, i)) - \phi(U_{i_\epsilon}(t_\epsilon))| \leq \epsilon^2 \phi^* + \epsilon^2 \phi^* a^*,$$

where in the above inequality we have used the smoothness of $\phi$. Thus, it follows that

$$\lim_{\epsilon \to 0} \mathcal{P}^{(\epsilon)}_{[0,T]}[\Delta_\phi(\mu_{\phi}^{(\epsilon)})] = 0,$$

so that $\mathcal{P}\{x : \Delta_\phi(x) > \zeta\} = 0$. By the arbitrariness of $\zeta$, we deduce that $\mathcal{P}\{x : \Delta_\phi(x) = 0\} = 1$. Now by the arbitrariness of $\phi$, we conclude the proof of the theorem.

Let us denote the elements of $C([0,T], \mathcal{S}')$ by $\omega = (\omega_t, t \in [0,T])$ and let $t \in [0,T]$. Suppose $\mathcal{P}_{[0,T]}$ is the weak limit of $\mathcal{P}^{(\epsilon)}_{[0,T]}$. We shall prove that $\mathcal{P}_{[0,T]}$ is supported by $\{\omega_t = \rho_t(\phi) d\mu dr\}$. Thus $\mathcal{P}_{[0,T]}$ coincides with $\rho_t(\phi) d\mu dr$ on the rationals of $[0,T]$ and by continuity on all $t \in [0,T]$ and therefore any weak limit of $\mathcal{P}^{(\epsilon)}_{[0,T]}$ is supported by $\rho_t(\phi) d\mu dr$.

In the sequel, $t \in T$ and $\delta \in \{2^{-n}T, n \geq 1\}$. Fix $\zeta > 0$. Since, by Proposition 9, the support of $\mathcal{P}$ is contained in $C([0,T], \mathcal{S}')$ and the projection is a continuous map in $C([0,T], \mathcal{S}')$, we can use the Converging Map Theorem to deduce that

$$\mathcal{P}\{w : |w_t(\phi) - \int \phi \rho_t d\mu dr| > \zeta\} = \lim_{\epsilon \to 0} \mathcal{P}^{(\epsilon)}_{[0,T]}(|\mu_{\phi}^{(\epsilon)}(\phi)| - \int \phi \rho_t d\mu dr) > \zeta).$$

(7.14)

Moreover, given any $\eta > 0$, for any $\delta$ fixed and $\ell, E, \tau$ sufficiently small we have by the Dominated Convergence Theorem that

$$\left| \int \phi \rho_t^{(\delta \ell, \ell, E, \tau)} d\mu dr - \int \phi \rho_t d\mu dr \right| < \eta. \quad (7.15)$$

For the same reason, for all $\delta$ small enough

$$\left| \int \phi \rho_t d\mu dr - \int \phi \rho_t^{(\delta \ell, \ell, E, \tau)} d\mu dr \right| < \eta. \quad (7.16)$$

Next, we fix $(\delta, \ell, E, \tau)$ such that (7.15) and (7.16) hold and then apply Corollary 11 for $\epsilon$ small enough to get that

$$\left| S^{(\epsilon)}_{\phi} - \int \phi \rho_t^{(\delta \ell, \ell, E, \tau)} d\mu dr \right| < \eta.$$
Furthermore, by (5.12) for all \( \varepsilon \) sufficiently small,
\[
Q_u^{(\varepsilon)} \left[ \mu_{U(\varepsilon)}(\phi) - \mu_{Y(\varepsilon, s, t, E, r)}(\phi) \right] < \eta.
\]
Collecting the above estimates and by the arbitrariness of \( \eta \), we then get
\[
\mu_{U(\varepsilon)}(\phi) \xrightarrow{\varepsilon \to 0} \int \phi \rho_t du dr
\]
and, therefore,
\[
\lim_{\varepsilon \to 0} P^{(\varepsilon)}_{[0, T]} \left( |\mu_{U(\varepsilon)}(\phi) - \int \phi \rho_t du dr| > \zeta \right) = 0
\]
From (7.14), it follows that
\[
P \left( w : |w_t(\phi) - \int \phi \rho_t du dr| > \zeta \right) = 0.
\]
Since \( \zeta \) is arbitrary we conclude the proof of Theorem 2.

Finally, to prove Theorem 3 we have to show that
\[
\lim_{u \to u^*_t(r)} \rho_t(u, r) = \psi_0 \left( T_{0, t}^{-1}(u^*_t(r), r), r \right) \exp \left\{ - \int_0^t \left[ \varphi \left( T_{s,t}^{-1}(u^*_t(r), r), r \right) - \alpha - \lambda(r) \right] ds \right\},
\]
where \( u^*_t(r) = T_{0, t}(0, r) \). For \( u < u^*_t(r) \) we recall that,
\[
\rho_t(u, r) = \frac{q_s(r)}{p_s(r) + \lambda r \bar{u}_s(r)} \exp \left\{ - \int_s^t \varphi \left( T_{s,h}(0, r), r \right) - \alpha - \lambda(r) dh \right\}
\]
s being such that \( u=T_{s,t}(0, r) \). By continuity
\[
\lim_{s \to 0} T_{s,h}(0, r) = T_{0,h}(0, r) = T_{h,t}^{-1}(u^*_t(r), r).
\]
Since we have proved earlier the continuity of \( q_s(r), p_s(r) \) and \( \bar{u}_s(r) \),
\[
\lim_{s \to 0} \frac{q_s(r)}{p_s(r) + \lambda r \bar{u}_s(r)} = \frac{q_0(r)}{p_0(r) + \lambda r \bar{u}_0(r)} = \psi_0(0, r) = \psi_0(T_{0,t}^{-1}(u^*_t(r), r), r).
\]
Taking the limit as \( s \to 0 \) in (7.18) we finish the Theorem.

8 Acknowledges

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Appendix 1: proof of Theorem 1

Proof of (i). For any $\varepsilon > 0$ and $u \in \mathbb{R}_{+}^{\Lambda_{c}}$, let $N_{i}^{(\varepsilon)}(t)$, $t \geq 0$, be the simple point process on $R_{+}$ which counts the jump events of neuron $i \in \mathbb{R}_{+}^{\Lambda_{c}}$ up to time $t$, when the initial configuration is $u$. Define $a^{i} = \sum_{j \in \Lambda_{c}} a(i,j)$ and $a^{*} = \max\{a^{i}, i \in \Lambda_{c}\}$ and, following De Masi et al. (2014), consider the following random variable, for all $t > 0$,

$$K(\varepsilon)(t) = \sum_{i \in \Lambda_{c}} \int_{0}^{t} \mathbb{I}_{\{U_{i}(s) \leq 2a^{*}\}} dN_{i}^{(\varepsilon)}(s).$$

The random variable $K(\varepsilon)(t)$ counts the number of spikes of neurons whose the potential is at most $2a^{*}$.

Suppose $U_{i}$ fires at time $t$, in this case

$$\tilde{U}(t) = \varepsilon^{2} \sum_{j \in \Lambda_{c}} \varepsilon^{2} \sum_{k \neq i} b(j,k) [U_{k}(t_{-}) + \varepsilon^{2} a(i, k)] = \tilde{U}(t_{-}) - \varepsilon^{2} U_{i}(t_{-}) + \varepsilon^{2} a^{i}.$$

In the other hand, by the definition of $a^{*}$ and $N^{(\varepsilon)}(t)$, for all $t > 0$,

$$||U(t)|| \leq ||U(0)|| + N^{(\varepsilon)}(t) \varepsilon^{2} a^{*}. \quad (8.1)$$

Now, using the expression of $\tilde{U}(t)$ above and adapting the proof of Theorem 1 of De Masi et al (2014), we have the following inequality, for all $t > 0$,

$$N^{(\varepsilon)}(t) \varepsilon^{2} a \leq \tilde{U}(0) + 2a^{*} \varepsilon^{2} K^{(\varepsilon)}(t). \quad (8.2)$$

Since we can bound $K^{(\varepsilon)}(t)$ by a Poisson process of intensity $\varepsilon^{-2} \varphi(2a^{*})$, applying the inequality $[8.2]$ in $[8.1]$, we get that the number of jumps of the process is finite almost surely on any finite time interval, for each fix $\varepsilon > 0$. To conclude the proof just notice that the construction of the process can be achieved by gluing together trajectories given by the deterministic flow between successive jump times. This procedure is feasible since the number of jumps of the process is finite on any finite interval.

Proof of (ii). Notice that by $[8.1]$ and $[8.2]$, we have for all $t \leq T$,

$$||U(t)|| \leq 2||U(0)|| + 2a^{*} \varepsilon^{2} K^{(\varepsilon)}(T).$$

Since $K^{(\varepsilon)}(T)$ is stochastically bounded by a Poisson random variable of rate $\varepsilon^{-2} \varphi(2a^{*})T$, it follows that

$$\mathbb{P}(K^{(\varepsilon)}(T) > 2\varepsilon^{-2} \varphi(2a^{*})T) \leq e^{-\varepsilon^{-2} \varphi(2a^{*})T(3-\varepsilon)}.$$

Then, taking $C = 2R + 4a^{*} \varphi(2a^{*})T$ the result follows where $c_{1} = e^{6}$ and $c_{2} = 3 \varphi(2a^{*})T$. \qed

9 Appendix 2: proof of Theorem 4

The proof follows the same the steps of the proof of Theorem 4 of De Masi (2014). We start providing an estimate of the the total number of spikes for both processes $U^{(\varepsilon)}$ and $Y^{(\delta)}$ in the interval $[0, T]$. Recall that $Q_{u}^{(\varepsilon)}$ is the probability law governing the coupled process in which $U^{(\varepsilon)}(0) = u$ and $Y^{(\delta)}_{i}(0) = \Phi_{0}(u_{i})$ for all $i \in \Lambda_{c}$.
**Proposition 9.** Let $A_{[0,T]}$ be the event when either $U^{(e)}$ or $Y^{(δ)}$ have more than $2\varphi^*\varepsilon^{-2}δ$ spikes in an interval $[(k - 1)δ, kδ)$, for $k = 1, \ldots, Tδ^{-1}$. Then, under Assumption 3

$$Q_u^{(e)}(A_{[0,T]}) \leq 2Tδ^{-1}e^{-\varphi^*δ\varepsilon^{-2}(3-\varepsilon)},$$

for any initial configuration $u \in \mathbb{R}_{+}^{\Lambda_ε}$.

**Proof.** Fix $k \in \{1, \ldots, Tδ^{-1}\}$ and let $N([(k - 1)δ, kδ))$ denote the number of spikes of the $U^{(e)}$ process in the interval $[(k - 1)δ, kδ)$. Then, under Assumption 3 $N([(k - 1)δ, kδ))$ is stochastically bounded by

$$Z := \sum_{j \in \Lambda_ε} N_j^*([(k - 1)δ, kδ))$$

where $(N_j^*)_{j \in \Lambda_ε}$ are iid Poisson processes with intensity $\varphi^*$. Since $Z$ is distributed as a Poisson random variable with rate $\varepsilon^{-2}δ\varphi^*$, it follows that

$$Q_u^{(e)}(N([(k - 1)δ, kδ))) \geq 2\varphi^*δ\varepsilon^{-2} \leq \mathbb{P}(Z \geq 2\varphi^*δ\varepsilon^{-2}) \leq e^{-2\varphi^*δ\varepsilon^{-2}(3-\varepsilon)}.$$

Bounding in the same manner the number of spikes of the $Y^{(δ)}$ process in the interval $[(k - 1)δ, kδ)$ and then summing over $k$ we complete the proof.

From now on, we suppose that, in both processes $U^{(e)}$ and $Y^{(δ)}$, the spiking rate of each neuron is $\leq \varphi^*$ and the number of spikes of all neurons in any step $[(k - 1)δ, kδ]$ is $\leq 2\varphi^*δ\varepsilon^{-2}$. Moreover, writing $B^* = C + R_0 + 2a^*\varphi^*T$, then we also assume that for all $t \leq T$ and $kδ \leq T$,

$$||U(t)|| \leq B^*, \quad ||\tilde{U}(t)|| \leq b^*B^*, \quad ||Y^{(δ)}(kδ)|| \leq B^*,$$ (9.1)

where $\tilde{U}(t) = (\tilde{U}_i(t), i \in \Lambda_ε)$.

By Assumption 2 (5.2) and Proposition 9 such assumptions provide small error probability.

In what follows, $C$ is a constant which may change from one appearance to another.

**Bounds on the increments of $B_n$:**

We start noticing that

$$|B_k| \leq |B_{k-1}| + |A_k^1 \cap G_{k-1}| + |A_k^2 \cap G_{k-1}| \leq |B_{k-1}| + |A_k^1| + |A_k^2 \cap G_{k-1}|,$$

where $G_{k-1}$ is the set of good labels at time $kδ$ (recall Definition 5) and

- $A_k^1$ is the set of all labels $i$ for which the clocks $ξ_i^1$ and $ξ_i$ associated to label $i$ ring during $[(k - 1)δ, kδ]$,
- $A_k^2$ is the set of all labels $i$ for which a clock $ξ_i^2$ associated to label $i$ rings during $[(k - 1)δ, kδ]$.

Recall the definitions of the random clocks $ξ_i^1$, $ξ_i^2$ and $ξ_i$ appearing in the coupling algorithm given in Subsection 5.1. We shall prove that

$$P\left[|A_k^1| > \varepsilon^{-2}(\delta\varphi^*)^2\right] \leq e^{-C\varepsilon^{-2}δ^4},$$ (9.2)

$$P\left[|A_k^2 \cap G_{k-1}| > 2C\varepsilon^{-2}δ[θ_{k-1} + δ + ℓ]\right] \leq e^{-C\varepsilon^{-2}δ^4},$$ (9.3)
where the constant $C$ appearing in (9.2) and (9.3) may be different.

Then, from (9.2) and (9.3), we deduce that with probability $\geq 1 - 2e^{-Ce^{-2}\delta^4}$,

$$|B_k| \leq |B_{k-1}| + \epsilon^{-2}(\delta \varphi^*)^2 + 2Ce^{-2}\delta [\theta_{k-1} + \delta] \leq |B_{k-1}| + Ce^{-2}\delta [\theta_{k-1} + \delta].$$

(9.4)

Iterating the upper bound and using that $k \leq T\delta^{-1}$, we will then conclude that with probability $\geq 1 - 2ke^{-Ce^{-2}\delta^4} \geq 1 - \delta^{-1}Ce^{-Ce^{-2}\delta}$,

$$\epsilon^2|B_k| \leq \epsilon^2|B_1| + C\delta \sum_{h=1}^{k-1}(\theta_h + \delta),$$

(9.5)

where $C$ depends only on $T$. Since by definition $\theta_k \leq \theta_{k+1}$, we may bound the right-hand of (9.5) by $C(\theta_{k-1} + \delta)$, implying that with probability $\geq 1 - \delta^{-1}Ce^{-Ce^{-2}\delta}$,

$$\epsilon^2|B_k| \leq C(\theta_{k-1} + \delta),$$

(9.6)

for each $k \leq T\delta^{-1}$.

Proof of (9.2). $|A^1_k|$ is stochastically upper bounded by $Z^* := \sum_{i\in \Lambda_{k}} 1\{Z_i^* \geq 2\}$, where $Z_1^*, \ldots, Z_N^*$ are independent Poisson variables of parameter $\varphi^* \delta$. Thus, writing $p^* = P(N^*_1 \geq 2)$, we have

$$e^{-\varphi^*\frac{1}{2}\delta^2(\varphi^*)} \leq p^* \leq \frac{1}{2}(\delta \varphi^*)^2, \quad p^* \approx \frac{1}{2}(\delta \varphi^*)^2$$

as $\delta \to 0$.

Therefore, $Z^*$ is the sum of $\epsilon^{-2}$ Bernoulli random variables, each having mean value $p^*$. Invoking the Hoeffding’s inequality, we get (9.2).

Proof of (9.3). We shall prove that the random variable $|A^2_k \cap G_{k-1}|$ is stochastically dominated by

$$\tilde{Z} := \sum_{i\in \Lambda_{k}} 1\{\tilde{Z}_i \geq 1\},$$

(9.7)

where $\tilde{Z}_i, i \in \Lambda_{k}$, are independent Poisson variables of parameter $C(\theta_{k-1} + \delta + \ell)\delta$. Once (9.7) is established, (9.3) will then follow straightly.

Noticing that, since,

$$|A^2_k \cap G_{k-1}| \leq \sum_{i\in \Lambda_{k}} 1\{\xi_i^2 < \delta, i \in G_{k-1}\},$$

it suffices to show the intensity of each random clock $\xi_i^2, i \in G_{k-1}$, is $\leq C(\theta_{k-1} + \delta + \ell)\delta$.

For that sake, we shall write

$$y := Y((k - 1)\delta), \quad u := U((k - 1)\delta) \quad \text{and} \quad u(t) := U((k - 1)\delta + t), t \in [0, \delta].$$

Now, for any $i \in G_{k-1} \cap C_m$, the intensity of $\xi^2_i$ is

$$|\varphi(u_i(t), i) - \varphi(y_i, i_m)| \leq \|\varphi\|_{Lip} \{u_i(t) - y_i + \ell\},$$

where $\|\varphi\|_{Lip}$ is the Lipschitz constant of the function $\varphi$. Denoting the number of spikes of $U_j$ in interval $[s, t]$ by $N_j([s, t])$, we have

$$|u_i(t) - y_i| \leq |u_i - y_i|e^{-(\alpha + \lambda_i)\delta} + y_i \left(1 - e^{-(\alpha + \lambda_i)\delta}\right) + \lambda_i \int_0^\ell u_i(s)e^{-(\alpha + \lambda_i)(t - s)}ds$$

$$+ \epsilon^2 \sum_{j \in \Lambda_{k}} a(j, i)N_j([(k - 1)\delta, (k - 1)\delta + t]).$$

(9.8)
Since for all $i \in \Lambda_{\epsilon}$, $y_i, \bar{u}_i(s) \leq B^*$ and $\sum_{j \in \Lambda_{\epsilon}} a(j, i) N_{j}( [(k-1)\delta, (k-1)\delta+\ell] ) \leq 2(a \varphi)^* \epsilon^{-2}\delta$, then if additionally $i \in G_{k-1}$, it follows that

$$|u_i(t) - y_i| \leq \theta_{k-1} + (\alpha + \lambda_i) \delta + \lambda_i \delta + 2a^* \varphi^* \delta,$$

and therefore

$$|\varphi(u_i(t)) - f(y_i)| \leq \|\varphi\|_{Lip} \left( \theta_{n-1} + 2(\alpha + \sup_i \lambda_i) \delta + 2a^* \varphi^* + \ell \right) \leq C(\theta_{k-1} + \delta + \ell),$$

so that

$$|A_k \cap G_{k-1}| \leq \sum_{i \in \Lambda_{\epsilon}} \mathbb{1}_{\{\bar{Z}_i < \delta\}} \text{ stochastically},$$

where the $\bar{Z}_i$ are independent Poisson random variables of intensity $C(\theta_{k-1} + \delta + \ell)\delta$.

**Bounds on $\theta_k$:**

The final bound on $\theta_k$ is reported in (9.16) at the end of this subsection. We start by characterizing the elements $i \in G_k$ as $i \in G_{k-1} \cap (C_k \cup F_k)$ where:

(i) $C_k$ is the set of all labels $i$ for which a clock associated to label $i$ rings only once during $[(k-1)\delta, k\delta]$, and it is a clock $\tau^1_i$.

(ii) $F_k$ is the set of all labels $i$ which do not have any spike during $[(k-1)\delta, k\delta]$.

This means we study labels $i$ which are good at time $(k-1)\delta$ and which stay good at time $k\delta$ as well. We shall use in the proofs the following formula for the potential $U_i(t)$ of a neuron which does not fire in the interval $[s, t]$:

$$U_i(t) = e^{-(\alpha+\lambda_i)(t-s)} U_i(s) + \lambda_i \int_s^t e^{-(\alpha+\lambda_i)(t-h)} \left\{ \bar{U}_i(h) dh + \frac{\epsilon^2}{\lambda_i} \sum_{j \in \Lambda_{\epsilon}} a(j, i) dN_{j}(h) \right\}, \quad (9.9)$$

$N_{j}(t)$ denoting the total number of fires in the processes $U$ of neuron $j$ till time $t$. For the $Y^\delta$ process we shall instead use (5.7) and the expressions thereafter.

- Labels $i \in C_k \cap G_{k-1}$. For such labels $i$ there is a random time $t \in [(k-1)\delta, k\delta]$ at which a $\tau^1_i$ event happens. By (9.9),

$$U_i(k\delta) = \lambda_i \int_0^\delta e^{-(\alpha+\lambda_i)(\delta-s)} \bar{U}_i(s) ds + e^{-(\alpha+\lambda_i)\delta} \epsilon^2 \sum_{j \in \Lambda_{\epsilon}} e^{(\alpha+\lambda_j)\delta} dN_j(s),$$

because $U_i(t_+) = 0$. Since $\|\bar{U}(t)\| \leq B^*$ and $N([(k-1)\delta, k\delta]) \leq 2a^* \epsilon \delta^{-2}$ we immediately see that $U_i(k\delta) \leq C\delta$. In the same way, $Y^{\delta}_i(k\delta) \leq C\delta$, and therefore,

$$D_i(k) \leq C\delta. \quad (9.10)$$

Notice that the bound does not depend on $D_i(k-1)$.

- Labels $i \in F_k \cap G_{k-1}$. 

34
This means that \( i \) is good at time \( (k - 1)\delta \) and does not jump, neither in the \( U \) nor in the \( Y^{(\delta)} \) process. Let \( U((k - 1)\delta) = x \) and \( Y^{(\delta)}((k - 1)\delta) = y \). By (9.9) and (5.7) \( |U_i(k\delta) - Y^{(\delta)}_i(k\delta)| = D_i(k), \ i \in C_m, \) is bounded by

\[
D_i(k) \leq \left| e^{-\delta(\alpha + \lambda_i)} u_i - e^{-\delta(\alpha + \lambda_m)} y_i \right| + \int_{(k-1)\delta}^{k\delta} \lambda_i e^{-(\alpha + \lambda_i)(k\delta - t)} U_i(t) \, dt - \lambda_m \int_{(k-1)\delta}^{k\delta} y_i e^{-(\alpha + \lambda_m)(k\delta - t)} \, dt \right| + \varepsilon^2 \sum_{j \in \Lambda_c} a_{i,j} \int_{(k-1)\delta}^{k\delta} e^{-(\alpha + \lambda_j)(k\delta - t)} dN_j(t) - \varepsilon^2 \sum_{m'} a(i_{m'}, i_m) \tilde{N}((k - 1)\delta, k\delta). \tag{9.11}
\]

Thus, it suffices to bound each term on the right hand side of (9.11). We start bounding the first one:

\[
|e^{-\delta(\alpha + \lambda_i)} u_i - e^{-\delta(\alpha + \lambda_m)} y_i| \leq B^* \delta |\lambda_i - \lambda_m| + e^{-\delta(\alpha + \lambda_m)} |u_i - y_i|.
\]

Since, \( |\lambda_i - \lambda_m| \leq ||\lambda||_{\text{Lip}} \ell \), and supposing \( \ell \leq \delta \), we can bound the last sum by \( C\delta^2 + \theta_{k-1} \).

Now let’s bound the second term on the right-hand side of (9.11). It is easy to see that it is bounded by

\[
||\lambda||_{\text{Lip}} B^* \ell \delta (1 + \lambda_m) + \lambda_m \delta |\bar{y}_m - \bar{u}_m| + \lambda_m \int_{(k-1)\delta}^{k\delta} \left[ |\bar{U}_i(t) - \bar{u}_i| + |\bar{U}_m(t) - \bar{u}_m| \right] dt.
\]

To control the second and third terms we notice that for any \( i \in \Lambda_{\varepsilon} \), \( |\bar{U}_i(t) - \bar{u}_i| \leq C\delta \) and \( |\bar{U}_i(t) - \bar{y}_i| \leq C\delta \). In addition, for any \( i \in C_m, m = 1, \ldots, \ell^2, |\bar{U}_i(t) - \bar{u}_m| \leq C\ell \). Requiring that \( \ell \leq \delta \), from these three inequalities we can bound the sum above by \( C\delta^2(\delta + \theta_{k-1}) \).

The argument to bound the third term on (9.11) is a bit more tricky. First we bound this term by

\[
\varepsilon^2 \sum_{j} a_{i,j} \int_{(k-1)\delta}^{k\delta} (k\delta - t)(\alpha + \lambda_i) dN_j(t) + \varepsilon^2 \sum_{m'} \sum_{j \in C_m} |a(j, i) - a(i_{m'}, i_m)| N_j(\delta(k - 1), k\delta) + \varepsilon^2 \sum_{m'} \sum_{i,m} a(i_{m'}, i_m) \left| N_{C_{m'}}(\delta(k - 1), k\delta) \right| - \tilde{N}_{C_{m'}}(\delta(k - 1), k\delta).
\]

The first two terms above are easily bounded. One can check that the sum of the two can be bounded by \( C\delta^2 \). To control the third term, we shall show that

\[
\left| N_{C_{m'}}(\delta(k - 1), k\delta) \right| \leq 4(\varphi^* \delta)^2 \varepsilon^2 \ell^2 \tag{9.12}
\]

Indeed, its difference is smaller or equal to

\[
\sum_{j \in C_{m'} \cap A_i^2} N_j(\delta(k - 1), k\delta) + |C_{m'} \cap A_i^2|,
\]

so that it suffices to control this two terms. We start with the second one. We know that with probability \( \geq 1 - e^{-C\varepsilon^2 \delta^4} \),

\[
|C_{m'} \cap A_i^2| = |C_{m'} \cap A_i^2 \cap \mathcal{G}_{k-1}| + |C_{m'} \cap A_i^2 \cap \mathcal{B}_{k-1}| \leq 2C\ell^2 \varepsilon^2 \delta(\theta_{k-1} \delta) + C\delta^2 |\mathcal{B}_{k-1}|,
\]

35
where we used (9.3) and that the number of neurons among those in $B_{k-1} \cap C_{m'}$ which spiked in a time $\delta$ is bounded by a Poisson random variable of intensity $\varphi^\star \delta |B_{k-1} \cap C_{m'}|$. Thus, it remains only to bound the first term on. We start noticing that

$$P\left[ \sum_{j \in A_k \cap C_{m'}} N_j((k-1)\delta, k\delta) \geq 4(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \right] \leq P\left[ \sum_{j \in A_k \cap C_{m'}} N_j((n-1)\delta, n\delta) \geq 4(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \right]$$

where we used (9.3) and that the number of neurons among those in $A_k \cap C_{m'}$ is bounded by a Poisson random variable of intensity $(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2$.

The last term is bounded using (9.2). Let $A \subset C_{m'}$, $|A| \leq (\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2$, then

$$P\left[ \sum_{j \in A_k \cap C_{m'}} N_j((k-1)\delta, k\delta) \geq 4(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \mid A_k \cap C_{m'} = A \right] \leq P^*[\sum_{j \in A} (N_j^* - 2) \geq 2(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2],$$

where $P^*$ is the law of independent Poisson variables $N_j^*$, $j \in A$, each one of parameter $\varphi^\star \delta$ and conditioned on being $N_j^* \geq 2$. Thus the probability that $N_j^* - 2 = k$ is

$$P^*[N_j^* - 2 = k] = Z_{\xi}^{-1} \frac{\xi^k}{(k + 2)!}, \quad Z_{\xi} = \xi^{-2} (e^\xi - 1 - \xi), \quad \xi = \varphi^\star \delta.$$

Denoting by $X_j$ independent Poisson variables of parameter $\xi$ we have that $N_j^* - 2 \leq X_j$ stochastically for $\xi$ small enough, hence for $\delta$ small enough. Indeed for any integer $k$ we have

$$P^*[N_j^* - 2 \geq k] \leq P[X_j \geq k]$$

because for $k \geq 1$,

$$P^*[N_j^* - 2 \geq k] \leq \frac{2\xi^k}{(k + 2)!}, \quad P[X_j \geq k] \geq e^{-\xi} \frac{\xi^k}{k!},$$

hence when $3e^{-\xi} \geq 2$.

Since $X = \sum_{j \in A} X_j$ is a Poisson variable of parameter $|A|\xi \leq (\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \varphi^\star \delta$ we have

$$P^*\left[ \sum_{j \in A} (N_j^* - 2) \geq 2(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \right] \leq P^*\left[ X \geq 2(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \right],$$

where the expectation $E^*(X)$ of $X$ is smaller (for $\delta$ small) than $(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2$. As a consequence,

$$P^*\left[ \sum_{j \in A} (N_j^* - 2) \geq 2(\varphi^\star \delta)^2 \varepsilon^{-2} \ell^2 \right] \leq e^{-C \varepsilon^{-2} \delta^2 \ell^2}.$$

In conclusion for $i \in F_k \cap G_{k-1}$:

$$D_i(n) \leq \theta_{n-1}(1 + C\delta) + C\delta |B_{k-1}| \varepsilon^2 + C\delta^2$$

with probability $\geq 1 - e^{-C \varepsilon^{-2} \delta^2 \ell^2}$. Together with (9.10) this proves that with probability $\geq 1 - e^{-C \varepsilon^{-2} \delta^2 \ell^2}$,

$$\theta_k \leq \max\{C\delta; \theta_{k-1}(1 + C\delta) + C\delta |B_{k-1}| \varepsilon^2 + C\delta^2\}.$$
Iteration of the inequalities:

By (9.16), \( \varepsilon^2 |B_k| \leq C(\theta_{k-1} + \delta) \) for all \( k\delta \leq T \) with probability \( 1 - \delta^{-1}Ce^{-C\varepsilon^2\delta^4} \). By (9.16), with probability \( 1 - \delta^{-1}Ce^{-C\varepsilon^2\delta^4} \), we have

\[
\theta_k \leq \max \left( C\delta, [1 + C\delta] \theta_{k-1} + C\delta^2 \right).
\]

Iterating this inequality we obtain

\[
\theta_k \leq C \sum_{s=0}^{k-1} [1 + C\delta]^s \delta^2 + (1 + C\delta)^k C\delta,
\]

and since,

\[
C \sum_{s=0}^{k-1} [1 + C\delta]^s \delta^2 + (1 + C\delta)^k C\delta = C\delta[(1 + C\delta)^k - 1] + (1 + C\delta)^k C\delta \leq Ce^{CT\delta}
\]

where we have used once more that \( k\delta \leq T \), we conclude that

\[
\theta_k \leq C\delta
\]

for all \( \delta \leq \delta_0 \), with probability \( \geq 1 - \delta^{-1}Ce^{-C\varepsilon^2\delta^4} \). This finishes the proof of Theorem 4.

10 Appendix 3: proof of proposition 3

Proof. Fix \( \phi \in \mathcal{S} \). By (9.11), the left-hand side of (5.11) is not changed if replace \( U(t) \) and \( Y(t) \) by \( U^*(t) = \min\{U(t), B^*\} \) and \( Y^*(t) = \min\{Y(t), B^*\} \). Now, by the smoothness of \( \phi \),

\[
Q_u^{(e)} \left[ \varepsilon^2 \sum_{i \in C_m} \phi(U_i(t), i) - \varepsilon^2 \sum_{i \in C_m} \phi(Y_i(t), i_m) \right] \leq ||\varphi||_{\text{Lip}} (\varepsilon^2 \sum_{i \in C_m} |U^*(t) - Y^*(t)| + \ell^2).
\]

Applying the Theorem 4 and using that \( |U^*(t) - Y^*(t)| \leq B^* \), we get the upper bound in (5.11). \( \square \)

11 Appendix 4: proof of Theorem 5

Proof. Let \( \mathcal{F}_n \) be the sigma-algebra generated by the variables \( \xi_i = \xi_i(k), k \leq n-1, i \in \Lambda_{\varepsilon} \) appearing in (5.3). Observe that all variables \( Y^{(e,\delta,t,E)}(n\delta), e_n^{(e)}(m), S_n^{(e)}(m,h) \) and \( S_n^{(e)}(m,\delta) \) are \( \mathcal{F}_n \) - measurable. In what follows, the constants \( C, c_1 \) and \( c_2 \) may change from appearance to another. We also will write for simplicity \( \tilde{F}^{(e)} = \tilde{F}^{(e)}_{Y^{(e,\delta,t,E)}(0)} \)

The proof is made by induction. For \( n = 0 \), (i) holds because \( E^{(e)}_{0,k} = D^{(e)}_{0,k} \). For (ii), we start noticing that

\[
\zeta_{0,m}(D^{(e)}_{0,k}) = P^{(e)}_{U^{(e)}(0)}[\eta_{0,m}(E^{(e)}_{0,k})] = \ell^2 \varepsilon^{-2} \int_{I_k} \psi_0(u, i_m) du
\]
and that \( \eta_{0,m}(E_{0,k}^{(e)}) \) is a sum of \( \ell^2 \varepsilon^{-2} \) i.i.d Bernoulli random variables having expected value \( \int_{I_k} \psi_0(u, i_m) du \). By Hoeffding inequality we deduce that

\[
\varepsilon^2 |\eta_{0,m}(E_{0,k}) - \zeta_{0,m}(D_{0,k})| > E\ell^2 \varepsilon^{1/2} = C
\]

with probability \( \leq 2e^{-c_2 \varepsilon^{-1}} \) where \( c_2 = 2E^2 \ell^2 \). Therefore, it follows, for \( n = 0 \), that the inequality above holds for all \( k \) and \( m \) with probability larger or equal to

\[
1 - c_1 e^{-c_2 \varepsilon^{-1}},
\]

establishing the Theorem in the case \( n = 0 \). We now suppose that the result holds for \( k \leq n \). Introduce the set \( G_n \) in which for all \( k \leq n \) such that \( n\delta \leq T \)

- \( |E_{n,k}^{(e)} - D_{n,k}^{(e)}| \leq C\varepsilon^{1/2}, \)
- \( \varepsilon^2 |\eta_{n,m}(E_{n,k+\delta\tau-1}^{(e)}) - \zeta_{n,m}(D_{n,k+\delta\tau-1}^{(e)})| \leq E\ell^2 \varepsilon^{1/2}, \quad k = 1, \ldots, |E_n^{(e)}|, \) and
- \( \varepsilon^2 |\eta_{n,m}(E_{n,h}^{(e)}) - \zeta_{n,m}(D_{n,h}^{(e)})| \leq \tau\ell^2 \varepsilon^{1/2}, \quad h = 1, \ldots, \delta\tau^{-1}. \)

By the inductive hypothesis, \( \tilde{P}^{(e)}(G_n) \geq 1 - c_1 e^{-c_2 \varepsilon^{-1/2}} \).

Since,

\[
|E_{n+1,k+\delta\tau-1}^{(e)} - D_{n+1,k+\delta\tau-1}^{(e)}| \leq |E_{n,k}^{(e)} - D_{n,k}^{(e)}| + \lambda_m \delta \left| \tilde{y}_n^{(e)}(m) - \tilde{e}_n^{(e)}(m) \right| + \left| \tilde{S}_{n+1}(m, \delta) - \tilde{P}^{(e)}_{Y(y, \delta, \ell, E, r)}(0) \left[ \tilde{S}_{n+1}^{(e)}(m, \delta) \right] \right|
\]

we have that on \( G_n \),

\[
|E_{n+1,k+\delta\tau-1}^{(e)} - D_{n+1,k+\delta\tau-1}^{(e)}| \leq C\varepsilon^{1/2} + \left| \tilde{S}_{n+1}(m, \delta) - \tilde{P}^{(e)}_{Y(y, \delta, \ell, E, r)}(0) \left[ \tilde{S}_{n+1}^{(e)}(m, \delta) \right] \right|
\]

We shall show that there exist positive constants \( c, c_1 \) and \( c_2 \) not depending on \( \varepsilon \) such that

\[
\left| \tilde{S}_{n+1}(m, \delta) - \tilde{P}^{(e)}_{Y(y, \delta, \ell, E, r)}(0) \left[ \tilde{S}_{n+1}^{(e)}(m, \delta) \right] \right| \leq c \varepsilon^{1/2}
\]  
(11.1)

with probability \( \geq 1 - c_1 e^{-c_2 \varepsilon^{-1}} \). For that sake, we first write

\[
N_{n+1}(m, k, \delta) = \sum_{i \in C_m} \mathbb{1}_{\{\xi_i < \delta\}}, \quad \xi_i \sim \exp(\varphi(E_{n,k}^{(e)}, i_m))
\]

and then by the conditional version of Hoeffding’s inequality we deduce that

\[
\tilde{P}^{(e)}(\varepsilon^2 |N_{n+1}(m, k, \delta) - \eta_{n,m}(E_{n,k}^{(e)})(1 - e^{-\varphi(E_{n,k}^{(e)}, i_m)})| > E\ell^2 \varepsilon^{1/2} |\mathcal{F}_n| \leq c_1 e^{-c_2 \varepsilon^{-1}} \]  
(11.2)

Since on \( G_n \)

\[
|\varphi(E_{n,k}^{(e)}, i_m) - \varphi(D_{n,k}^{(e)}, i_m)| \leq C\varepsilon^{1/2},
\]

noticing that \( N_{n+1}(m, \delta) = \sum_k N_{n+1}(m, k, \delta) \) and \( \varepsilon^2 \zeta(D_{n,k}^{(e)}) \leq 1 \), then it follows together with (11.2) that there exist constants \( C, c_1 \) and \( c_2 \) such that
\[ \tilde{P}^{(e)} \left( G_n, \varepsilon^2 \right) \left| S_n^{(e)}(m, \delta) - \tilde{P}^{(e)} \left[ S_n^{(e)}(m, \delta) \right] \right| > C \varepsilon^{1/2} \left| F_n \right| \leq c_1 e^{-c_2 \varepsilon^{-1}}, \]

proving (11.1). Therefore,
\[ \tilde{P}^{(e)} \left( G_n, \varepsilon \right) \left| E_n^{(e)}(m, \delta) - D_n^{(e)} \left[ m, \delta \right] \right| > C \varepsilon^{1/2} \left| F_n \right| \leq c_1 e^{-c_2 \varepsilon^{-1}}. \]

A similar argument may be used to prove that we may replace in the probability above \( E_n^{(e)}(m, \delta) \) and \( D_n^{(e)} \left[ m, \delta \right] \) respectively by \( E_n^{(e)}(m, k) \) and \( D_n^{(e)} \left[ m, k \right] \). Thus, summing over all \( k, h \) and \( m \) we prove (i) of Theorem 5 for \( n + 1 \).

Now, we noticing that \( \eta_n+1(m, k + \delta \tau^{-1}) = \eta_n(m, k) - N_n+1(m, k, \delta) \) and remembering that by (6.8), \( \zeta_n+1(m, k + \delta \tau^{-1}) = \zeta_n+1(m, k)e^{-\delta \varphi(D_n^{(e)},m)} \), we easily see, together with (11.2), that
\[ \tilde{P}^{(e)} \left( G_n, \varepsilon \right) \left| \eta_n+1(m, k + \delta \tau^{-1}) - \zeta_n+1(m, k + \delta \tau^{-1}) \right| > C \varepsilon^{1/2} \left| F_n \right| \leq c_1 e^{-c_2 \varepsilon^{-1}}, \]

for some suitable constants not depending on \( \varepsilon \). A similar argument shows that the same type of bound for \( \varepsilon^2 |\eta_n+1(m, h) - \zeta_n+1(m, h)| \) it also holds, finishing the proof of Theorem 5.

\[ \square \]

12 Appendix: proof of Theorem 2 for general firing rates

Let \( \varphi, R, T \) and \( C \) as in Theorem 1. Let \( \phi \) be any bounded continuous functions on \( D([0, T], S^*) \). We need to prove that
\[ \lim_{\varepsilon \to 0} \mathcal{P}^{(e)}_{[0,T]}(\phi) = \phi(\rho), \]
where \( \lim_{\varepsilon \to 0} \mathcal{P}^{(e)}_{[0,T]}(\phi) \) is the expected value of \( \phi \) under the law of \( \mu_{U^{(e)}} \) and \( \phi(\rho) \) is the value of \( \phi \) on the element \( \rho := (\rho_t, dudr) \) of \( D([0, T], S^*) \).

Let \( A \) be the set \( A = \{ ||U^{(e)}(t)|| \leq C, t \in [0, T] \} \), then by Theorem 1
\[ \lim_{\varepsilon \to 0} \left| \mathcal{P}^{(e)}_{[0,T]}(\phi) - \mathcal{P}^{(e)}_{[0,T]}(\phi_{1A}) \right| = 0. \] (12.1)

Now, let us call \( \mathcal{P}^{(e)}_{[0,T]} \) the law of the process with a spiking rate \( \varphi * (\cdot, \cdot) \) which satisfies Assumption 3 and coincides with \( \varphi \) for \( u \leq C \). Thus,
\[ \mathcal{P}^{(e)}_{[0,T]}(\phi_{1A}) = \mathcal{P}^{(e)}_{[0,T]}(\phi_{1A}). \] (12.2)

Notice that we have proved Theorem 2 under Assumption 3 we have convergence for the process with rate \( \varphi^* \) to a limit density that we call \( \rho^* = (\rho^*_t, dudr)_{t \in [0, T]} \). It follows then, from (12.1) and (12.2), that
\[ \lim_{\varepsilon \to 0} \left| \mathcal{P}^{(e)}_{[0,T]}(\phi) \right| = \psi(\rho^* 1_A). \]

We claim that \( \rho^* = \rho^* 1_A \). Indeed, taking \( \phi(w) = \sup \{ w_1(1), t \leq T \} \wedge 1 \), we immediately see that \( 1 = \lim_{\varepsilon \to 0} \mathcal{P}^{(e)}_{[0,T]}(\phi) = \phi(\rho^* 1_A) \). This last equality implies that \( \rho^* \) have support in \( [0, C] \). As a consequence,
\[ \lim_{\varepsilon \to 0} \mathcal{P}^{(e)}_{[0,T]}(\phi) = \phi(\rho^* 1_A) = \phi(\rho^*), \]
which concludes the proof of the Theorem.
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