Eppur si espace

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March 21, 2022

Abstract

The rather wide-spread belief that cosmological expansion of a flat 3D–space (with spatial curvature $k = 0$) cannot be observationally distinguished from a kinematics of galaxies moving in a flat and non-expanding space is erroneous. We suggest that the error may have its source in a non relativistic intuition that imagines the Universe not as a spacetime but separates space from time and pictures the cosmological expansion as space evolving in time. The physical reality, however, is fundamentally different — the expanding Universe is necessarily a curved spacetime. We show here that the fact that the spacetime is curved implies that the interpretation of the observed cosmological redshift as being due to the expansion of the cosmological 3D–space is observationally verifiable. Thus it is impossible to mimic the true cosmological redshift by a Doppler effect caused by motion of galaxies in a non-expanding 3D-space, flat or curved. We summarize our points in simple spacetime diagrams that illustrate a gedanken experiment distinguishing between expansion of space and pure kinematics. We also provide all relevant mathematics. None of the previously published discussions of the issue, including a recent popular Scientific American article \cite{1}, offered a similarly clear way out of the confusion.

1 Introduction

This work was triggered by skeptical remarks made by Bohdan Paczyński and helped by several technical discussions with Michał Chodorowski. For different reasons (and using different arguments \cite{3,4}) they both advocated exploring if in the case of a flat cosmological 3D-space ($k = 0$) one needs to invoke
Figure 1: Galaxies moving in a non-expanding space
Figure 2: Galaxies moving in an expanding space.
space expansion since, they argued, the observed cosmological redshifts can be (formally) interpreted as a pure Doppler effect due to motion of galaxies in a non-expanding flat space and no observation can distinguish this from space expansion. In this paper we show that such interpretation is not possible. We are particularly interested in the case of an Universe with flat space sections i.e. with $k = 0$, but we will keep our discussion general, including also the cases $k = \pm 1$. We are not using the $c = 1$ convention here, so the light velocity $c$ appears in all formulae where it should.

1.1 Expansion or kinematics?

Figures 1 and 2 describe the source of the confusion. Let us imagine two galaxies in the expanding Universe. When the expansion progresses, the two galaxies are just further apart. The two figures seem to imply that in the case $k = 0$ one can adopt two equivalent interpretations: (a) the space is expanding, (b) galaxies move relative to each other in a non-expanding flat space. The cosmological redshift in the case $k = 0$ is therefore explained as the Doppler effect.

1.2 The metric and curvature of spacetime

Figures 3 and 4 illustrate again the same two interpretations as shown in Figures 1 and 2: galaxies in an expanding universe, and galaxies moving in a non-expanding space. This time, however, we show why the two interpretations are in reality physically very different. Indeed, the case of expanding Universe corresponds to a curved spacetime. The case of expanding galaxies corresponds to a flat (Minkowski) spacetime. It is the spacetime curvature that makes the light trajectories physically different in the two cases. This is a reflection of the fact that Newtonian and general-relativistic cosmology of an isotropic and homogenous Universe are equivalent only as long as one does not consider the propagation of light.

Figures 3 and 4 suggest also that combined measurements of redshift and distance may together reveal the physical, observable difference between the two interpretations. Note, that for these measurements it is sufficient to consider only radial motions of galaxies and light signals, and ignore angular $\theta$ and $\phi$ coordinates. This obviously means that the curvature of 3D-space is irrelevant to this problem.

The radial part of the metric of the expanding Universe may be written as,

$$ds^2 = c^2 dt^2 - R^2(t) dr^2.$$  \hspace{1cm} (1)

For simplicity, we will describe the dimensionless cosmological scale factor by a power-law function,

$$R(\tau) = \tau^n; \quad \tau \equiv t/t_*,$$  \hspace{1cm} (2)

where $t_* = \text{const}$ is an arbitrary scale; for example it could be $t_* = 100 \, \text{Mpc}/c$. 
The only relevant Riemann curvature tensor component, $R_{trtr}$, of the two-dimensional metric (1) with the scale factor (2) obeys

$$R_{trtr} \sim \frac{n(n-1)}{(ct)^2}, \quad (3)$$

i.e. vanishes for $n = 0$ and $n = 1$. In both cases, the spacetime (1) must be therefore flat i.e. described by the two-dimensional Minkowski metric. Indeed, $n = 0$ corresponds directly to the Minkowski metric,

$$ds^2 = c^2 dt^2 - dr^2, \quad (4)$$

while $n = 1$ gives the Milne metric,

$$ds^2 = c^2 dt^2 - \left(\frac{t}{t_\ast}\right)^2 dr^2, \quad (5)$$

which, after the coordinate transformation

$$T = t \cosh(r/ct_\ast), \quad X = ct \sinh(r/ct_\ast), \quad (6)$$

takes the form $ds^2 = c^2 dT^2 - dX^2$, identical with the Minkowski metric (4). Milne $n = 1$ metric (5) and Minkowski $n = 0$ metric (4) are thus identical. However, from the $t = \text{const.}$ 3D–space point of view one can conclude that space is expanding. Of course since in this case the 4D space-time is flat this is just an illusion. In the following we will show that this illusion is the source of confusion about the expansion of cosmological 3D–space. In a curved spacetime expansion is not illusory as it corresponds to physical observables.

## 2 A gedanken experiment

Figures 3 and 4 show the history of a radar measurement of the distance between two galaxies. At the spacetime event $[t_1, 0]$ a light signal is sent by an observer in the galaxy located at $r=0$. At the spacetime event $[t_0, r_0]$ this signal is reflected from a distant galaxy located in $r=r_0$ and returns to the observer at the spacetime event $[t, 0]$. The $[t, r]$ coordinates in the metric (1) are comoving with the galaxies, so that $r=0$ for the first galaxy and $r=r_0$ for the second hold at all times.

The observer continuously measures the pair of moments $t_1$ (at which the signal is emitted) and $t$ (at which the signal returns) constructing a function $t_1(t)$. He also continuously records the redshift $z(t)$ of the other galaxy, and defines and continuously records the “radar distance”

$$D_{\text{RAD}}(t) = c \frac{t - t_1}{2}. \quad (7)$$

In the remaining part of this article we will discuss these measurements and show, that from the knowledge of the three functions, $t_1(t)$, $D_{\text{RAD}}(t)$, and $z(t)$, the observer may definitely decide whether or not the space is expanding. We stress that the conclusion is based on the results of his gedanken experiment only, and is independent of the definition of 3D–space that he, or his colleagues, might adopt.
Figure 3: Same as Figure 1 but with added radar measurement of distance. The light trajectories reveal different spacetime structure from that in Fig. 4 (next page). Obviously this difference will show up in simultaneous measurements of the redshift and distance. Thus, measuring the redshift and distance, one can learn whether or not space is really expanding.
Figure 4: Same as Figure 2 but with added radar measurement of distance. The light trajectories reveal \textit{different} spacetime structure from that in Fig. \ref{fig:3} (previous page).
2.1 The radar distance in the expanding Universe

Along the trajectory of a light signal $ds = 0$, and therefore one has,

$$c \int_{t_1}^{t_0} \frac{dt}{R(t)} = r_0 = c \int_{t_0}^{t} \frac{dt}{R(t)}$$

(8)

or, using (2),

$$\tau_1 = \tau \left[ 1 - 2\epsilon(1 - n) \tau^{-(1-n)} \right]^{1/(1-n)},$$

(9)

$$\tau_0 = \tau \left[ 1 - \epsilon(1 - n) \tau^{-(1-n)} \right]^{1/(1-n)},$$

(10)

where we introduced the notation

$$\epsilon \equiv \frac{r_0}{ct} = \text{const.}$$

(11)

We do not assume that $\epsilon \ll 1$.

From (7) and (9) it follows that

$$D_{\text{RAD}}(\tau) = \frac{c t_\tau \tau}{2} \left[ 1 - \left( 1 - \frac{2\epsilon(1 - n)}{\tau^{1-n}} \right)^{1/(1-n)} \right].$$

(12)

2.2 The redshift measurement in expanding universe

The cosmological redshift measured in time $t$ equals (cf. Figures 3 and 4 and equation (10)),

$$1 + z(\tau) = \frac{R(\tau)}{R(\tau_0)} = \left[ 1 - \epsilon(1 - n) \right]^{-n/(1-n)} \equiv \mathcal{X}.$$  

(13)

Let us define $\delta \equiv 1 - n$. Milne’s universe correspond to $\delta = 0$. In this limit, one has

$$\mathcal{X}(\delta \to 0) = e^\epsilon$$

(14)

3 Is space expanding?

In this Section we calculate the increasing “redshift distance” to a distant galaxy by supposing that the measured redshift may be interpreted as a pure Doppler effect in a non-expanding space. If space is not expanding, the redshift distance should be equal to the measured radar distance (12). If the two distances are not equal one concludes that the space must be expanding and that this expansion is not equivalent to recession of galaxies in a static space.

The Doppler interpretation of the redshift assumes that,

$$1 + z(\tau) = \left( \frac{1 + v/c}{1 - v/c} \right)^{1/2}.$$  

(15)
Equating $1 + z(\tau)$ in (13) and (15) one calculates the velocity of the galaxy $v(\tau)$,
\[
v(\tau) = c \frac{\lambda^2 - 1}{\lambda^2 + 1}.
\] (16)

From (14) and (16) it follows that in Milne’s universe, i.e. for $\delta \to 0$, one has
\[
v(\delta \to 0) = c \frac{\epsilon^2 - 1}{\epsilon^2 + 1} = c \tanh(\epsilon)
\] (17)

The distance to the galaxy, according to the supposed interpretation of the redshift as a pure Doppler effect is,
\[
D_{\text{RED}}(\tau) = \int_0^{\tau'} v(\tau)d\tau, \quad \tau' \equiv \frac{\tau + \tau_1}{2}.
\] (18)

For $\delta \ll 1$ one has from (12) and from (18),
\[
D_{\text{RAD}}(\tau; \delta \ll 1) = \frac{ct_{\tau} \tau}{2} \left[1 - e^{-2\epsilon}\right]
- \frac{ct_{\tau} \tau}{2} \left[2 e^{-2\epsilon} (\ln \tau - \epsilon) \right] \epsilon \delta,
D_{\text{RED}}(\tau; \delta \ll 1) = \frac{ct_{\tau} \tau}{2} \left[1 - e^{-2\epsilon}\right]
+ \frac{ct_{\tau} \tau}{2} \left[\ln \frac{\tau}{2} + \epsilon(2 \ln \tau - \ln \frac{\tau}{2} - 2) \right] \epsilon \delta.
\] (19)

From the above formula one sees that
\[
D_{\text{RAD}}(\tau) = D_{\text{RED}}(\tau) \quad \text{for} \quad \delta = 0
D_{\text{RAD}}(\tau) \neq D_{\text{RED}}(\tau) \quad \text{for} \quad \delta \neq 0 \quad (\delta \ll 1).
\] (20)

4 The Milne Universe

In this Section we derive the form of the metric of Milne’s Universe in a way that may be novel to some readers.

The Hubble law says that in the local Universe the small redshift $z \ll 1$ of a nearby galaxy is proportional to the galaxy distance, $cz = H\tilde{r}$. For $z \ll 1$, one may use the non-relativistic formula $z = v/c$ and rewrite the Hubble law as $v = H\tilde{r}$. Let a galaxy located at a distance $\tilde{r}$ has its velocity $v$. What would be the velocity of a galaxy at the distance $\tilde{r} + d\tilde{r}$? The spacetime of the Universe is described locally by the Minkowski metric, where the answer is given by the special relativistic formula for adding velocities,
\[
v + dv = \frac{v + Hd\tilde{r}}{1 + vHd\tilde{r}/c^2} = v + Hd\tilde{r} \left(1 - v^2/c^2\right).
\] (21)

Integrating (21) we get,
\[
H \int_0^{\tilde{r}} d\tilde{r} = c \int_0^{v} \frac{d(v/c)}{1 - v^2/c^2},
\] (22)
and finally,

\[ v = c \tanh \left( \frac{H \tilde{r}}{c} \right) \]  

(23)

The Hubble constant \( H \) may be only a function of time, \( H = H(t) \). If the velocity is constant in time, the distance \( \tilde{r} \) must obey \( \tilde{r} \sim 1/H(t) \). On the other hand, one knows that \( \tilde{r}(t) = R(t) r \), with \( r = \text{const} \) being the comoving coordinate, and \( H(t) = \dot{R}(t)/R(t) \). From these conditions, one concludes that \( \dot{R}/R \sim 1/R \), or \( R(t) \sim t \). Therefore, in this case, the metric takes the form

\[ ds^2 = c^2 dt^2 - \left( \frac{t}{t_*} \right)^2 dr^2. \]  

(24)

We have arrived again at the Milne metric. Note that the velocity expressed in terms of the comoving coordinate takes the form that explicitly shows \( v = \text{const} \),

\[ v = c \tanh \left( \frac{r}{ct_*} \right) = c \tanh(\epsilon), \]  

(25)

which agrees with (17).

The Milne spacetime is therefore the result of a particular globalization of the local Minkowskian kinematics.

One can also deduce (25) from the (already mentioned) transformation between the Minkowski metric expressed in terms of \( T, X \) and the equivalent Milne’s metric expressed in terms of \( t, \epsilon \),

\[ T = t \cosh(\epsilon), \quad X = c t \sinh(\epsilon). \]  

(26)

Indeed, from the above equation, it follows directly that,

\[ v = \frac{X}{T} = c \tanh(\epsilon). \]  

(27)

The full, four dimensional version of the Milne metric that follows from the full four dimensional Minkowski \( ds^2 = c^2 dt^2 - dX^2 - X^2 [d\theta^2 + \sin^2 \theta d\phi^2] \) after the coordinate transformation (6, 26) and \( \theta = \theta, \phi = \phi \), has the form

\[ ds^2 = c^2 dt^2 - \left( \frac{t}{t_*} \right)^2 \left[ dr^2 + \sinh^2 \left( \frac{r}{ct_*} \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]. \]  

(28)

The Milne space \( t = \text{const}, dt = 0 \) has the metric

\[ dt^2 = dr^2 + \sinh^2 \left( \frac{r}{ct_*} \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

(29)

and therefore corresponds to the case \( k = -1 \).

The Milne and Minkowski metrics describe the same non-curved spacetime of special relativity, however they differently divide the spacetime into space and time. In the Minkowski’s interpretation in the \( [X, T] \) coordinates, galaxies are moving with constant velocities \( X/T = \tanh(\epsilon) \) in space that is not expanding. In the Milne’s interpretation, in the comoving coordinates \( [\tau, \epsilon] \) the space is expanding at the rate \( R(\tau) = \tau \), and galaxies have fixed positions \( \epsilon = \text{const} \) in an expanding space. Our condition that equality of radar and redshift distance measurements indicates a non-expanding space picks up Minkowski’s interpretation as true — Milne’s interpretation is an illusion, the space of a (4–D) flat Universe is not expanding.
5 Conclusions

The concept of a global 3-space is not fundamental in Einstein’s relativity, and no universal, invariant, definition of global 3-space is possible. An observer may invariantly define only his local 3-D space.

In this article we have not attempted to define the global space. Instead, we showed that independent of the definition, in a non-expanding space the radar and redshift distances should be the same. When they are not, one must conclude that space is expanding. We have shown that if the cosmological spacetime is curved the two measurements cannot agree. Thus, the expansion of space relates to the curvature of spacetime. The curvature of the 3D-space has nothing to do with it.

M.A.A. and St.B. acknowledge the support from Polish State Committee for Scientific Research, grants 1P03D 012 26 and N203 009 31/1466, and A.M. acknowledges the support from the French-Polish LEA-Astro-PF.

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