Alon – Tarsi numbers of direct products

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Abstract

We provide a general framework on the coefficients of the graph polynomials of graphs which are Cartesian products. As a corollary, we prove that if $G = (V, E)$ is a graph with degrees of vertices $2d(v), v \in V$, and the graph polynomial $\prod_{(i,j) \in E}(x_j - x_i)$ contains an “almost central” monomial (that means a monomial $\prod v x^c_v$, where $|c_v - d(v)| \leq 1$ for all $v \in V$), then the Cartesian product $G \square C_{2n}$ is $(d(\cdot) + 2)$-choosable.

1 Introduction

Let $F$ be a field, $x = (x_1, \ldots, x_n)$ a set of variables. For $A \subset F$ and $a \in A$ denote

$$D(A, a) := \prod_{b \in A \setminus a} (a - b).$$

For a multi-index $d = (d_1, \ldots, d_n) \in \mathbb{Z}_{>0}^n$ denote $|d| = d_1 + \ldots + d_n$, $x^d = \prod_{i=1}^n x_i^{d_i}$. For a polynomial $f \in F[x]$ denote by $[x^d]f$ the coefficient of monomial $x^d$ in polynomial $f$.

Theorem 1 (Combinatorial Nullstellensatz [1]). Choose arbitrary subsets $A_i \subset F$, $|A_i| = d_i + 1$ for $i = 1, \ldots, n$. Denote $A = A_1 \times A_2 \times \ldots \times A_n$. For any polynomial $f \in F[x]$ such that $\deg f \leq |d|$, if $[x^d]f \neq 0$, then there exists $a \in A$ for which $f(a) \neq 0$.

Alon and Tarsi [2] suggested to use Combinatorial Nullstellensatz for list graph colorings. Namely, if $G = (V, E)$ is a non-directed graph with the vertex set $V = \{v_1, \ldots, v_n\}$ and the edge set $E$, we define its graph polynomial in $n$ variables $x_1, \ldots, x_n$ as

$$F_G(x) = \prod_{(i,j) \in E} (x_j - x_i).$$

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Here each edge corresponds to one linear factor $x_j - x_i$, so the whole $F_G$ is defined up to a sign. Assume that each vertex $v_i$ has a list $A_i$ consisting of $d_i + 1$ colors, which are real numbers. A proper list coloring of $G$ subordinate to lists \( \{ A_i \} \) is a choice of colors \( a = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n = A \) for which neighboring vertices have different colors: \( a_i \neq a_j \) whenever \( (i, j) \in E \).

In other words, a proper list coloring is a choice of $a \in A$ for which $F_G(a) \neq 0$. If \( |d| = |E| \), the existence of a proper list coloring follows from \( |x^d| F_G \neq 0 \).

Define the chromatic number \( \chi(G) \) of the graph $G$ as the minimal $m$ such that there exists a proper list coloring of $G$ subordinate to equal lists of size $m$: \( A_i = \{1, \ldots, m\} \). Define the list chromatic number $\text{ch}(G)$ of the graph $G$ as the minimal $m$ such that for arbitrary lists $A_i$, \( |A_i| \geq m \), there exists a proper list coloring of $G$ subordinate to these lists. Define the Alon–Tarsi number $\text{AT}(G)$ of the graph $G$ as the minimal $k$ for which there exists a monomial \( x^d \) such that \( \max(d_1, \ldots, d_n) = k - 1 \) and \( |x^d| F_G \neq 0 \).

From above we see that the list chromatic number does not exceed the Alon–Tarsi number:

\[
\text{ch}(G) \leq \text{AT}(G).
\]

Further we consider the Alon–Tarsi numbers for the graphs which are direct products $G_1 \square G_2$ of simpler graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Recall that the vertex set of $G_1 \square G_2$ is $V_1 \times V_2$ and two pairs $(v_1, v_2)$ and $(u_1, u_2)$ are joined by an edge if and only if either $v_1 = u_1$ and $(v_2, u_2) \in E_2$ or $v_2 = u_2$ and $(v_1, u_1) \in E_1$.

It is well known (Lemma 2.6 in [9]) that \( \chi(G_1 \square G_2) = \max(\chi(G_1), \chi(G_2)) \). Much less is known about the list chromatic number (and the Alon–Tarsi number) of the Cartesian product of graphs. Borowiecki, Jendrol, Král, and Miškuf [3] gave the following bound:

**Theorem 2 ([3]).** For any two graphs $G$ and $H$,

\[
\text{ch}(G \square H) \leq \min(\text{ch}(G) + \text{col}(H), \text{col}(G) + \text{ch}(H)) - 1.
\]

Here $\text{col}(G)$ is the coloring number of $G$, i.e. the smallest integer $k$ for which there exists an ordering of vertices $v_1, \ldots, v_n$ of $G$ such that each vertex $v_i$ is adjacent to at most $k - 1$ vertices among $v_1, \ldots, v_{i-1}$.

Here we continue the previous work [6] where the toroidal grid $C_n \square C_m$ (here $C_n$ is a simple cycle with $n$ edges) was considered and it was proved that $\text{AT}(C_n \square C_{2k}) = 3$.

An explicit form of Combinatorial Nullstellensatz (the coefficient formula) was used in [6], such approach does not seem to work in the more general setting of the present paper.

We call a coefficient \( [x^d] F_G(x) \) of the graph polynomial $F_G$ central, if $\xi_i = \deg_G(v_i)/2$ for all $i$, and almost central, if $|\xi_i - \deg_G(v_i)/2| \leq 1$ for all $i$.

Our main result is the following

**Theorem 3.** Let $G$ be a graph, all vertices in which have even degree. Suppose that the graph polynomial $F_G$ has at least one non-zero almost central coefficient.
Then for $H = G \square C_{2k}$ the central coefficient is non-zero. In particular, $H$ is $(\deg H/2 + 1)$-choosable and

$$
\text{ch}(H) \leq \text{AT}(H) \leq \frac{\Delta(H)}{2} + 1 = \frac{\Delta(G)}{2} + 2.
$$

Note that Theorem 2 gives the bound $\text{ch}(H) \leq \min(\text{ch}(G) + 2, \text{col}(G) + 1)$ under the same conditions. When $\text{ch}(G)$ (or $\text{col}(G)$) is small, this bound is stronger. But it can also be weaker when $\text{ch}(G)$ and $\text{col}(G)$ are close to $\Delta(G)$. For example, if $G = C_{2l+1}$ is an odd cycle, then $F_G$ obviously has a non-zero almost central coefficient, so, by Theorem 3, $\text{ch}(C_{2l+1} \square C_{2k}) \leq 3$ (this was also proved in [6] by a different argument). On the other hand, Theorem 2 gives only $\text{ch}(C_{2l+1} \square C_{2k}) \leq 4$.

2 Coefficients as traces

Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{Z}_{\geq 0}^n$, denote $a_1 + \cdots + a_n = |a|$. Consider a polynomial

$$
P(x, y) = Q(x)R(x, y) \in F[x, y]
$$

in variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, where $Q$ is of degree at most $|a|$, $R$ is homogeneous of degree $|b|$.

Consider $nk$ variables $(x^1, x^2, \ldots, x^k)$, $x^i = (x_1^i, \ldots, x_n^i)$: for convenience, denote $x^{k+1} = x^i$. Define

$$
P_k(x^1, x^2, \ldots, x^k) = \prod_{1 \leq j \leq k} P(x^j, x^{j+1}).
$$

We are interested in the coefficient

$$
M_k := \left[ \prod_{j=1}^{k} (x^j)^{a+b} \right] P_k.
$$

It is easy to see that this coefficient is equal to

$$
\sum \prod_{j=1}^{k} \left[ (x^j)^{p^j} (x^{j+1})^{q^j} \right] R(x^j, x^{j+1}) \cdot \left[ (x^j)^{a+b-p^j-q^{j-1}} \right] Q(x^j) = \text{tr} \Phi^k,
$$

where the sum is over all $(p^1, \ldots, p^k)$, $(q^1, \ldots, q^k)$ such that

$$
p^j = (p_1^j, \ldots, p_n^j), q^j = (q_1^j, \ldots, q_n^j) \in \mathbb{Z}_{\geq 0}^n, \quad |p^j| + |q^j| = |b|;
$$

for $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2); \alpha^i, \beta^i \in \mathbb{Z}_{\geq 0}^n; |\alpha^1| + |\alpha^2| = |\beta^1| + |\beta^2| = |b|,$

$$
\Phi(\alpha, \beta) = \left[ (x)^{\alpha^1} (y)^{\alpha^2} \right] R(x, y) \cdot \left[ (x)^{a+b-\alpha^1-\beta^2} \right] Q(x).
$$

If $Q$ is homogeneous, then $\Phi(\alpha, \beta) \neq 0$ only if $|\alpha^1| + |\beta^2| = |b|$, i.e. if $|\alpha^1| = |\beta^1|, |\alpha^2| = |\beta^2|$; the same is true for $\Phi(\beta, \alpha).$
3 Cartesian product of a graph and an even cycle

Let $G$ be a graph, all vertices in which have even degree. Denoting $V(G) = \{v_1, \ldots, v_n\}$, we take $a_i = \deg(v_i)/2$, $Q(x) = F_G(x)$, $R(x, y) = \prod_j (y_j - x_j)$, $b_i = 1$, and let $k$ be even.

Then $\Phi(\alpha, \beta) \neq 0$ only if $\alpha_j^1 = 1 - \alpha_j^2 \leq 1, \beta_j^1 = 1 - \beta_j^2 \leq 1$ for all $j$, $|\alpha^1| = |\beta^1|; \text{if this is the case, then}$

$$
\Phi(\alpha, \beta) = (-1)^{|\alpha^1|} \left[ (x)^{a+b-a^1-(b-\beta^1)} \right] F_G(x) = (-1)^{|\alpha^1|} \left[ (x)^{a+\beta^1-a^1} \right] F_G(x).
$$

Note that

$$
\left[ x^{a+\beta^1-a^1} \right] F_G(x) = (-1)^{|E(G)|} \left[ x^{a+\beta^1-a^1} \right] F_G(x), \quad (2)
$$

since simultaneously changing the choice in each linear factor $x_i - x_j$ of $F_G$ we get one of these monomials from the other. Thus, matrix $\Phi$ is (skew-)symmetric; therefore all eigenvalues of the matrix are real (or all are imaginary). Then the $k$-th powers of all non-zero eigenvalues are real and have the same sign. It follows that $\text{tr } \Phi^k \neq 0$ if at least one of the coefficients of the form

$$
\left[ x^{a+\beta^1-a^1} \right] F_G(x)
$$

is non-zero; in other words, if there is at least one non-zero coefficient $[x^\xi] F_G(x)$ with $|\xi_i - \deg(v_i)/2| \leq 1$ for all $i$. Theorem 3 is proved.

**Remark 1.** Define a generalized graph polynomial $Q$ for a graph or multigraph $G = (V, E)$ with even degrees, $V = \{v_1, \ldots, v_n\}$, as a product of factors $x_i \pm x_j$ for all edges $v_i v_j \in E$ (one multiple for each edge). Note that it satisfies the symmetry or antisymmetry property (2), with some multiple $\pm 1$ on the place of $(-1)^{|E(G)|}$. Therefore the same argument shows that if $Q$ has a non-zero almost central coefficient, then the polynomial

$$
\prod_{i=1}^{2k} Q(x^i, x^{i+1}) \prod_{i=1}^n \prod_{j=1}^n (x_j^i - x_j^{i+1}), \quad \text{where } x^{2k+1} \equiv x^1,
$$

has a non-zero central coefficient (that is, a coefficient of $\prod_{i,j} (x_j^i)^{\deg(v_i)/2+1}$).

4 Applications

4.1 Cartesian product of several cycles

Consider a Cartesian product of odd cycles $G = C_{2k_1+1} \square \ldots \square C_{2k_n+1}$, such that

$$
\frac{1}{k_1} + \ldots + \frac{1}{k_n} \leq 1.
$$

Our goal is to show that the graph polynomial $F_G$ has a non-zero almost central coefficient. We employ the Alon-Tarsi method:
Theorem 4 (see Corollary 1.2, Corollary 2.3 in [2]). Let $G$ be a non-directed graph on vertices $v_1, \ldots, v_n$. Suppose $G$ has an orientation $D$ with outdegrees $d_{out}(v_i) = d_i$, and there are no odd directed cycles in $D$. Then the coefficient $[x^d]F_G$ (where $d = (d_1, \ldots, d_n)$) is non-zero.

We are going to build an orientation of $G$, such that the outdegree of any vertex lies in $\{n - 1, n, n + 1\}$, and there are no odd directed cycles. We are going to denote vertices of $G$ by $v = (v_1, \ldots, v_n)$, $0 \leq v_i \leq 2k_i$.

We divide $G$ into $2^n$ boxes $H_i$, $0 \leq i < 2^n$: if the binary notation of $i$ is $b_i, 1, \ldots, b_i, n$, then

$$H_i = \{(v_1, \ldots, v_n) \mid 0 \leq v_j \leq k, \text{ if } b_{i,j} = 0; \ k < v_j \leq 2k \text{ otherwise}\}.$$ 

These boxes may be colored alternately white and black (in a chess board). We direct all edges sticking out of black boxes outward, and from white boxes inward. Note that any directed cycle is contained in some box $H_i$ and has therefore even length.

The remaining task is to obtain the orientation of the box of dimension $n$ with outdegree of any vertex lying in $\{n - 1, n\}$. We will use this orientation for all white boxes $H_i$’s, for the black boxes use the reversed orientation. This guarantees that in white boxes all outdegrees are in $\{n - 1, n\}$; in black boxes the indegrees are in $\{n - 1, n\}$, therefore the outdegrees are in $\{n, n + 1\}$.

To prove the existence of such orientation we are going to use the following theorem (see, for example, Theorem 3 in [4]):

**Theorem 5.** There exists an orientation of $G = (V, E)$ with $l_v \leq d_{out}(v) \leq u_v$ for any $v$, if and only if for any $W \subset V$ the following two conditions hold:

1. $|E(W)| \leq \sum_{v \in W} u_v$;
2. $|\overline{E}(W)| \geq \sum_{v \in W} l_v$,

where $\overline{E}(W) = E(V) \setminus E(V \setminus W)$ is the set of edges incident to at least one vertex of $W$.

**Proposition 6.** Let $H = P_{k_1} \square \ldots \square P_{k_n}$ ($P_i$ is a path of length $i$). There exists an orientation of $H$ with outdegrees of all vertices lying in $\{n - 1, n\}$ if and only if

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} \leq 1. \quad (4)$$

**Proof.** First of all, note that the condition (4) is necessary: the sum of outdegrees of all vertices does not exceed the number of edges in a graph, so

$$(n - 1) \prod_{i=1}^{n} k_i \leq \sum_{i=1}^{n} (k_i - 1) \prod_{1 \leq j \leq n, j \neq i}^{n} k_i = \prod_{i=1}^{n} k_i \cdot \sum_{i=1}^{n} \left(1 - \frac{1}{k_i}\right), \quad (5)$$
which is equivalent to (4). To show that the condition (4) is sufficient, we are going to verify two conditions from Theorem 5 with \( l(v) = n - 1, u(v) = n \) for each \( v \). The first condition holds:

\[
|E(W)| \leq \frac{1}{2} \sum_{v \in W} d(v) \leq n|W|.
\]

The second condition looks like

\[
|E(V)| - |E(V \setminus W)| \geq (n - 1)|W|.
\]

Denoting \( U = V \setminus W \), it is equivalent to

\[
|E(U)| - (n - 1)|U| \leq |E(V)| - (n - 1)|V|
\]

for each \( U \subset V \). Thus, to prove that the second condition holds, it is sufficient to show that the function

\[
f(U) = |E(U)| - (n - 1)|U|
\]

reaches its maximum value at \( U = V \).

For \( 1 \leq i \leq n \), \( p = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n) \), \( 1 \leq p_j \leq k_j \), denote

\[
U(i, p) = \{ v \in U \mid v_j = p_j \text{ for any } j \neq i \}.
\]

Then

\[
|E(U)| \leq n|U| - \sum_{i,p} \chi(|U(i, p)| > 0).
\]

Denote \( l = 1 - \sum_{i=1}^{n} \frac{1}{k_i} \geq 0; \) then

\[
|U| = \left( \sum_{i=1}^{n} \frac{1}{k_i} + l \right) |U| = l|U| + \sum_{i,p} \frac{|U(i, p)|}{k_i}.
\]

It follows that

\[
f(U) \leq l|U| + \sum_{i,p} g(U, i, p),
\]

where

\[
g(U, i, p) = \begin{cases} 
0, & \text{if } |U(i, p)| = 0, \\
\frac{|U(i, p)|}{k_i} - 1, & \text{otherwise}.
\end{cases}
\]

In conclusion, note that

\[
f(U) \leq l|U| + \sum_{i,p} g(U, i, p) \leq l|U| \leq l|V| = f(V).
\]

\( \square \)
Corollary 7. Let \( G = C_{2k_1+1} \square \ldots \square C_{2k_m+1} \square C_{2k_{m+1}} \square \ldots \square C_{2k_n} \), \( 0 \leq m < n \), and, additionally, 
\[
\frac{1}{k_1} + \cdots + \frac{1}{k_m} \leq 1.
\]
Then \( \ch(G) \leq \AT(G) = n + 1 \).

Proof. The upper bound follows from the construction described above; the lower bound is obvious: in each monomial of the graph polynomial \( F_G \) there is a variable of degree at least \( n \).

Note that even though the result is sharp for the Alon–Tarsi number, it is far from sharp for the list chromatic number when \( n \) is large enough. For example, if \( k_i \geq 2 \) for all \( i = 1, \ldots, m \) (it is so for sure if \( m \geq 2 \) and \( \sum 1/k_i \leq 1 \)), then graph \( G \) is triangle free and has maximum degree \( 2n \), which yields \( \ch(G) \leq (2 + o(1)) \frac{n}{\log n} \) by a result of Molloy [7], a recent improvement of the \( O\left(\frac{n}{\log n}\right) \) bound first given by Johansson [5].

4.2 Powers of cycles

Proposition 8. Let \( C_n^p \) be the \( p \)-th power of a cycle \( C_n \), i.e. a graph on the vertex set \( \{v_1, \ldots, v_n\} \), in which \( v_i \) and \( v_j \) are adjacent if and only if \( j \in \{i-p, \ldots, i-1, i+1, \ldots, i+p\} \) (the indices are modulo \( n \)). Suppose \( p+1 \) divides \( n \) or \( n \geq p(p+1) \). Then
\[
\ch(C_n^p \square C_{2k}) \leq \AT(C_n^p \square C_{2k}) \leq p + 2.
\]

Proof. In [8] the Alon–Tarsi number \( \AT \) for powers of cycles is estimated. If \( p+1 \) divides \( n \), it is shown that the central coefficient of \( F_{C_n^p} \) is non-zero; if \( p+1 \) does not divide \( n \), but \( n \geq p(p+1) \), then it is shown that for a graph \( H_n^p \), obtained by adding some matching to the graph \( C_n^p \), there is a non-zero coefficient of \( F_{H_n^p} \) with degree of each variable in \( \{p, p+1\} \). This coefficient is a linear combination of almost central coefficients of \( F_{C_n^p} \), so at least one of them is also non-zero.

4.3 Multigraphs

Note that Theorem 3 can be applied to multigraphs. In particular, non-trivial bounds can be obtained for graphs with large choice number by adding multiple edges to them. To give an example, we prove the following proposition:

Proposition 9. Let \( G \) be a graph, all vertices of maximum degree in which may be covered by some vertex-disjoint cycles. Then
\[
\AT(G \square C_{2k}) \leq \Delta(G) + 1 = \Delta(G \square C_{2k}) - 1.
\]

Proof. Denote the set of edges contained in these cycles as \( F \). Consider a graph \( G' \), which can be obtained from \( G \) by adding a multiple edge to every edge from
the set $E(G) \setminus F$. Obviously, $\AT(G \square C_{2k}) \leq \AT(G' \square C_{2k})$. If we show that $F_{G'}$ has a non-zero almost central coefficient, then

$$\AT(G' \square C_{2k}) \leq \frac{\Delta(G')}{2} + 2 = \Delta(G) + 1.$$ 

Consider another graph $G''$, which can be obtained from $G$ by adding a multiple edge to every edge. Note that the central coefficient of $F_{G''} = F_G^2$ is non-zero: the central coefficient of $F_{G''}$ is the sum of products of “opposite” coefficients of $F_G$, each summand in this sum has the same sign (which depends on the parity of the number of edges in $G$, cf. (2)). But the central coefficient of $F_{G''}$ is a linear combination of almost central coefficients of $F_{G'}$; it follows that at least one of them is also non-zero.

**Corollary 10.**

$$\AT(K_n \square C_{2k}) = \ch(K_n \square C_{2k}) = n.$$ 

**Proof.** We have

$$n \geq \AT(K_n \square C_{2k}) \geq \ch(K_n \square C_{2k}) \geq \ch(K_n) \geq n,$$

where the first inequality follows from Proposition 9, the second from (1), the third and fourth are clear. So all inequalities turn into equalities. □

Next proposition is a generalization of Theorem 3 for arbitrary graphs (not necessarily with even degrees, not necessarily with a non-zero almost central coefficient.) Roughly speaking, it bounds the choosability in dependence on how not-so-far-from-central coefficient does the graph polynomial have. It also gives Corollary 10 (we skip the details).

**Proposition 11.** Let $G = (V,E)$ be a graph; denote $V = \{v_1, \ldots, v_n\}$. For any $\eta = (\eta_1, \ldots, \eta_n)$ denote $l_G(\eta, i) = |\eta_i - \deg_G(v_i)/2|$. Consider a non-zero coefficient $[x^l]F_G(x)$ of the graph polynomial $F_G$. Partition the set $\{1, \ldots, n\}$ onto sets

$$N = \{i : \tau_i = \deg_G(v_i)/2\},$$

$$A_1 = \{i : \tau_i \leq \deg_G(v_i)/2 - 1\},$$

$$A_2 = \{i : \tau_i = \deg_G(v_i)/2 - 1/2\},$$

$$B_1 = \{i : \tau_i \geq \deg_G(v_i)/2 + 1\},$$

$$B_2 = \{i : \tau_i = \deg_G(v_i)/2 + 1/2\}.$$

Additionally, let $A_3$ be an arbitrary subset of $A_1$ of size $\max(0, |A_1| - |B_1|)$; let $B_3$ be an arbitrary subset of $B_1$ of size $\max(0, |B_1| - |A_1|)$. Then $G \square C_{2k}$ is $f$-choosable, where

$$f(v_i) = \begin{cases} 
\deg_G(v_i)/2 + 2, & \text{if } i \in N, \\
\deg_G(v_i)/2 + l_G(\tau, i) + 1, & \text{if } i \in A_1 \cup B_1 \setminus A_3 \setminus B_3, \\
\deg_G(v_i)/2 + l_G(\tau, i) + 2, & \text{if } i \in A_2 \cup B_2 \cup A_3 \cup B_3.
\end{cases}$$
Proof. Define a multiset $A$ as follows:

- each $i \in A_1 \setminus A_3$ occurs $2(l_G(\tau, i) - 1)$ times in $A$;
- each $i \in A_2 \cup A_3$ occurs $2l_G(\tau, i)$ times in $A$.

Multiset $B$ is defined similarly. Note that

$$|A| = \sum_{i \in A_1 \cup A_2} 2l_G(\tau, i) - 2 \min(|A_1|, |B_1|).$$

Similarly,

$$|B| = \sum_{i \in B_1 \cup B_2} 2l_G(\tau, i) - 2 \min(|A_1|, |B_1|).$$

It follows that $|A| = |B| =: m$. Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$.

Consider $2^m$ polynomials

$$Q_\varepsilon(x) = F_G(x) \cdot \prod_{j=1}^{m} (x_{a_j} \pm x_{b_j})$$

indexed by the choice $\varepsilon$ of $m$ signs. Using the relation $x_{a_j} = \frac{1}{2}((x_{a_j} + x_{b_j}) + (x_{a_j} - x_{b_j}))$ we see that the polynomial $Q := F_G(x) \cdot \prod x_{a_i}$ is a linear combination of $Q_\varepsilon$’s. Note that

$$\left[ x^\tau \cdot \prod x_{a_i} \right] Q = [x^\tau] F_G \neq 0,$$

therefore there exists $\varepsilon$ such that

$$\left[ x^\tau \cdot \prod x_{a_i} \right] Q_\varepsilon \neq 0.$$

Note that $Q_\varepsilon$ is a generalized graph polynomial of a certain multigraph on the ground set $V$ with degree function $2f(v_i) - 4$. The coefficient of $x^\tau \cdot \prod x_{a_i}$ is almost central for $Q_\varepsilon$.

Now it follows from the Remark 1 that the polynomial (3) (for $Q = Q_\varepsilon$) has a non-zero central coefficient.

Finally, since the graph polynomial $F_G \square C_{2k}$ divides this polynomial, graph $G \square C_{2k}$ is $f$-choosable. \hfill $\Box$

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References

[1] N. Alon. Combinatorial Nullstellensatz. *Combinatorics, Probability and Computing*, 8(1-2):7–29, 1999.
[2] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.

[3] M. Borowiecki, S. Jendrol, D. Král, and J. Miškuf. List coloring of Cartesian products of graphs. *Discrete Mathematics*, 306(16):1955–1958, 2006.

[4] A.U. Frank and A. Gyárfás. How to orient the edges of a graph? *Colloq. Math. Soc. János Bolyai*, 18:353–364, 1976.

[5] A. Johansson. Asymptotic choice number for triangle free graphs. *DIMACS Technical Report 91–5*, 1996.

[6] Z. Li, Z. Shao, F. Petrov, and A. Gordeev. The Alon–Tarsi Number of A Toroidal Grid. *arXiv preprint arXiv:1912.12466*, 2019.

[7] M. Molloy. The list chromatic number of graphs with small clique number. *Journal of Combinatorial Theory, Series B*, 134:264–284, 2019.

[8] A. Prowse and D.R. Woodall. Choosability of Powers of Circuits. *Graphs and Combinatorics*, 19:137–144, 2003.

[9] G. Sabidussi. Graphs with given group and given graph-theoretical properties. *Canadian Journal of Mathematics*, 9:515–525, 1957.