Three dimensional large $N$ monopole gas

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ABSTRACT: We study here the large $N$ limit in the presence of magnetic monopoles in the Yang-Mills/Higgs model in three dimensions. The physics in the limit depends strongly on the distribution of eigenvalues of the Higgs field in the vacuum, and we propose a particular, nondegenerate configuration. It minimizes the free energy at the moment of symmetry breaking. Given this, the magnetic monopoles show a wide hierarchy of masses, and some are vanishing as $1/N$. The dilute gas picture, then, provides an interesting framework for the large $N$ analysis.

KEYWORDS: Monopoles, Coulomb gas, Large $N$ limit.
1 Introduction

The subject of this paper is the analysis of some nonperturbative features of the Yang-Mills-Higgs (YMH) model in three dimensions for gauge group $SU(N)$ and some implications for the large $N$ limit.

Our YMH model is the direct generalization to $SU(N)$ of the Georgi-Glashow model [1]. This was one of the first models providing a spontaneous breaking of a gauge symmetry.

The vacuum expectation value of the Higgs field provides a length scale and the size for magnetic monopoles, providing their stability.

It was however argued in the known 't Hooft paper on oblique confinement [5] that such configurations should exists also in pure Yang-Mills theory of strong interactions,
and play a major role in the mechanism of quark confinement with their condensation.

Already the Georgi Glashow model in two dimensions shows analogous configurations, \textit{vortices} \cite{6}, which account for the BCS theory of superconductivity \cite{7}.

It was an idea from the early ages of QCD \cite{8, 9, 10} that quark confinement could be explained with a picture analogous to that of type II superconductors but with the role of electric and magnetic charges interchanged (and for late issues like abelian dominance in confinement see \cite{11, 12}).

For nonabelian groups $SU(N)$ the vortex solutions are not stable, mainly due to the triviality of the fundamental group, and this complicates the matter in that it requires the breaking to some subgroup with abelian factors.

This is exactly realized in the YMH model, and the infrared behavior of gauge theories with compact gauge group \cite{13}, was explicitly shown to hold in the context of the Coulomb gas approach to the ensemble of widely separated monopole solutions \cite{14}.

This treatment is based on the fact that widely separated monopoles interact via a Coulomb interaction, despite the exact form of the energy is not known explicitly for generic parameters and also the space of moduli of the generic solutions has a nontrivial geometric structure \cite{15}.

After these early years, lattice simulations have been largely employed to test the occurrence and condensation of monopoles in correspondence to the deconfinement transition, in the pure Yang-Mills theory \cite{16}.

Because of the absence of the Higgs field, monopoles are of no fixed size, and thus some mechanism of generation of mass is necessary if monopoles are to be relevant for the confinement mechanism. This is indeed found on the lattice and there are arguments in the three dimensional continuum theory \cite{17}.

Let’s recall also that the \textit{static} four dimensional theory is equivalent to the YMH model with the limit of no potential. This is one of the main reasons to study the YMH model.

Many interesting features are in the domain of the large $N$ expansion, mainly: a) the classification of Feynman graphs according to the surfaces where they can be embedded gives also the order in $1/N$, actually $1/N^2$, of the graph and this allows the contact with the string interpretation of gauge theories on one hand, and on the other with the random surface interpretation of zero dimensional matrix models; b) for suitable operators \cite{19} the correlation functions “factorize”, that is, are given by the disconnected part at leading order; c) finally $N$ is not renormalized, in that it is a dimensionless external fixed parameter.

Of course a special mention is due to the pure $N = \infty$ case, where one implicitly lets $N \rightarrow \infty$ \textit{before} removing any cutoff. This can bring the theory to have a different phase structure. There is a specific example in one dimension that is the Kazakov-Migdal phase transition.

The second property above, factorization, shows that for the operators which follow it, the large $N$ limit is a kind of semiclassical limit. Their fluctuations are suppressed,
and $1/N$ plays the role of $\hbar$ for them.

One of the recent interesting phenomena, is the emergence of a new dynamic in the $N = \infty$ limit of some theories. These include affine Toda models, principal chiral fields in one and two dimensions, two dimensional QCD, not to mention the matrix models.

In all these theories the effect of taking $N$ large is to generate an infinity of states which coalesce to form collective excitations of some other, higher dimensional, theory.

The emergence of a new dimension is the remnant of the matrix index of the diagonal fields. In these theories angular variables play an important role but are integrated, more or less explicitly, to leave an effective interaction for the eigenvalues.

An example is the theory of the principal chiral field on the line, which is equivalent, for some boundary conditions, to two dimensional QCD with its interpretation in terms of two dimensional string.

We come thus to our case of Coulomb gas of magnetic monopoles. It was constructed as a sum on the classical dilute configurations of monopoles, weighted with the determinant of gaussian fluctuations around them.

With the Coulomb gas there is, via a Sine-Gordon transform, a dual representation and the possibility to achieve an estimate about the string tension for Wilson loops of large area.

The string tension is related to the pseudo-mass of the monopoles by an exponential relation, which shows, like in the dual Landau-Ginzburg theory, that it is related to the density of monopoles.

For $SU(N)$ we have $N(N - 1)$ species of minimal monopoles, with magnetic charges in different couples of $U(1)$ sectors, and the coulomb gas can be generalized to this case, as well as the Sine-Gordon transform.

One has however to take into account the different species of monopoles that populate the model, and this was not done in. Accordingly one has to consider the determinant of quantum fluctuations around the different monopole backgrounds.

Up to this point the analysis is valid for any $N$.

A possible new behavior comes from the large $N$ limit, because there necessarily appears a distribution of masses which are present in the theory.

These masses can be of order, a priori, in the interval from 0 to $N$, and the physics at large $N$ of course has to be very different from case to case.

In this model the monopole pseudo-masses and the mass of the gauge bosons are governed by the differences of eigenvalues of the Higgs field at infinity, $\phi^\infty$, so that all the model depends on its distribution of eigenvalues in the vacuum.

In the standard picture of symmetry breaking $\phi^\infty$ can not be changed by any fluctuation, once the universe has formed. One simply fixes it. The modulus of $\phi^\infty$ gets renormalized and possibly shifted as with the arguments of the effective Higgs potential, but there is no indication on its direction in the Cartan space.
However in the course of our analysis we are lead to use the unitary gauge, because the physical degrees of freedom are explicit and the Higgs field is diagonal, and a curious phenomenon arises.

The unitary gauge is somewhat singular, because its Faddeev Popov determinant is not defined in the continuum. This turns out to be the product at each point of the Vandermonde determinant constructed with the eigenvalues of $\phi^\infty$.

This factor in the functional integral seems to provide measure zero for all configurations where some eigenvalues coincide, and to give a sort of repulsion of eigenvalues.

This immediately faces with the problem that monopole configurations, which have necessarily points where the eigenvalues coincide, have all zero weight.

Fortunately as the analysis is carried on, and still thinking that the theory is renormalizable, we can show that once the quantum fluctuations of the massive gauge fields are taken into account, the Van-der-Monde ultralocal determinant is canceled almost completely. What remains is just the Van-der-Monde determinant of the eigenvalues of $\phi^\infty$!

Still this term does not authorize us to think to a ‘quantum lifting’ of the degeneracy in Cartan directions, because again $\phi^\infty$ is fixed at the “beginning of the universe”.

However we think that at the epoch of its formation, the system is sensible to this term, and thus chooses the distribution of eigenvalues which maximizes it.

We have found this distribution, which determines back the distribution of masses of gauge bosons and monopoles in the system.

It shows a peculiar shape, and should bring peculiar consequences in the properties of the system.

It is worth noting that the same distribution of eigenvalues of ‘Higgs field’ and gauge boson masses, is found in the recent nonperturbative solution of $SU(N)$ supersymmetric Yang-Mills theory in four dimensions (the $N = 1$ case), once the complete confinement is required. It is striking that the same distribution appears in the large $N$, and as its very origin is physically not known in that case.

Coming back to our problem of the monopole gas, we have now a distribution to analyze the system and we could proceed.

Unfortunately the very difficulty to the completion of the program is still there, because the semiclassical sum needs some estimate with $N$ of the determinant in the external field of a monopole.

This problem is still unsolved despite many efforts and is nontrivial. An approach with Eguchi Kawai reduction which could provide at least the large $N$ behavior is being studied.

Let’s summarize our path in this work.

The first part of this work deals with some new large $N$ ideas for the YMH model in three dimensions. Needless to say there is a large historical and scientific background and it is of course difficult to say something really new on these subjects. Nevertheless some
latest ideas on matrix models, For example, to our knowledge, the study of Coulomb gases at large number of species is not investigated.

After one finds a reliable method to estimate the functional determinant, it will be possible to draw definite conclusions on the large \( N \) monopole gas, which seems to promise interesting features, due to the competition of factors in the functional integral which takes place in the large \( N \) limit.

2 Preliminaries

We start by considering the Yang-Mills/Higgs model built on the gauge group \( SU(N) \), so it is a 3-dimensional euclidean theory of a gauge field \( A_\mu = A_\mu^a T_a = A_\mu \cdot T \) and a matter field \( \phi = \phi^a T_a = \phi \cdot T \) both living in the adjoint representation. Both are arranged in \( N \times N \) matrix fields living in the algebra of \( SU(N) \). In all the following we will always denote with plain symbols such fields, like \( A_\mu, \phi \), and with bold symbols the vector of their components along the algebra generators, like \( A_\mu, \phi \) above. Whenever the field will be diagonalized, its vector will have only the Cartan components.

To fix the notation we take the normalization such that \( \text{tr}(T_a T_b) = \frac{1}{2} \delta_{ab} \) in the fundamental representation. We will occasionally mention also the theory with \( T_a \) in the adjoint representation, in next chapter, because they give rise to very different scenario of monopoles.

The configuration space \( \Gamma \) is the space of functions from \( \mathbb{R}^3 \) to the couple \((A, \phi)\) with finite action \( S \). In this configuration space acts a continuous \( SU(N) \) gauge group:

\[
A_\mu(x) \rightarrow \omega^{-1}(x) \partial_\mu \omega(x) + g \omega^{-1}(x) A_\mu(x) \omega(x) \\
\phi(x) \rightarrow \omega^{-1}(x) \phi(x) \omega(x)
\]

which leaves the action invariant.

Because we will focus on the large \( N \) expansion, it is necessary to adapt the parameters of the theory to this limit.

It is the standard remark \([27]\) that the large \( N \) limit is nontrivial only if the perturbative series remains finite, and this requires the coupling constants to be suppressed with powers of \( N \). Rescaling the fields one can require all terms to be of the same order \( N^2 \).

The action is then:

\[
S = \int d^3x \left\{ \frac{N}{2g^2} \text{tr}(G_{\mu\nu} G_{\mu\nu}) + \frac{N}{2} \text{tr}(D_\mu \phi \cdot D_\nu \phi) + V(\phi) \right\}
\]

with \( D_\mu = \partial_\mu + g [A_\mu, \cdot] \), \( G_{\mu\nu} = [D_\mu, D_\nu] = g(\partial_\mu A_\nu - \partial_\nu A_\mu) + g^2 [A_\mu, A_\nu] \) and \( V(\phi) \) a scalar potential.

This YMH theory in three dimensions can be seen as the static version of the four dimensional minkowsky YMH, or even of self-dual pure YM theory, at finite temperature,
where the $A_0$ component assumes the part of the Higgs field (and $V = 0$). It has thus also direct phenomenological interest.

So, to make contact with the four dimensional theories, one respects four dimensional renormalizability, and takes the potential to be a quartic polynomial in the traces of the Higgs field:

$$V(\phi) = \lambda [\text{tr}(\phi^2) - \mu^2]^2$$  \hspace{0.5cm} (2.4)

From $N$ power counting $\mu$ has to be of order $N$.

Requiring the vacuum to have finite action (finite energy in four dimensions) $V(\phi)$ induces a spontaneous shift of the Higgs vacuum value from zero to some $\phi^\infty$, and the gauge group $G$ breaks down spontaneously to a subgroup $H$ which leaves $\phi^\infty$ invariant. This is called the little group.

$\Gamma$ is divided in disjoint sets classified by the winding of the two-sphere at infinity into the coset $G/H$: elements of $\pi_2(G/H)$ which is isomorphic to $\pi_1(H)$ because $\pi_1(G)$ is trivial.

In case the vacuum Higgs field has all different eigenvalues the little group $H$ is the maximal abelian subgroup $U(1)^{N-1}$, and the classification has $N-1$ topological quantum numbers: $Z^{N-1}$. Moreover thanks to the vanishing of $\pi_2(G)$, this classification is gauge invariant, because gauge transformations of $\phi^\infty$ are homotopic to the identity.

Then, in the spirit of the semiclassical approach, one considers the minimum of the action in each set $\hat{\phi}(x)$, and expands $\phi$ around it. One has a pointwise breaking of the local gauge group $G$ down to the little group which leaves $\hat{\phi}(x)$ invariant. All gauge fields not belonging to $H$ acquire a mass, while those in the little group remain massless. At points where the Higgs field has all different eigenvalues, the breaking is maximal and the little group is $U(1)^{N-1}$.

As 't Hooft shows [5], the manifold of points where two eigenvalues of $\phi$ coincide has dimension $d-3$, that is, in three dimensions, consists of isolated points. There the field configurations have the properties of magnetic monopoles, and in the next section we will review also, in the case of $SU(N)$, how the topological number which classifies the Higgs field represents the magnetic charge of the configuration under the broken symmetry group $H$. (modulo equivalence under the Weyl discrete symmetry).

The other terms in the action, namely the pure gauge and the Higgs kinetic terms, pose no obstruction, and configurations of arbitrary winding can be explicitly constructed [29, 30].

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*The choice of a function which is symmetric in the different Cartan directions is not the only one but is the more natural.

†The little group $H$ is always of the form $T' \times G'$, where $T'$ is some abelian group and $G'$ is a simple subgroup. In any case it always includes a $U(1)$ subgroup, called the electromagnetic group and generated by the Higgs field itself.

‡This happens in the gaugeless limit, where, turning off the coupling to the gauge fields, the Higgs kinetic term forces $\phi$ to have the same direction at infinity, thus giving winding zero.
Before reviewing in the next section the classification of classical solutions for SU(N), we make a remark that we will need later valid for all the configurations of nontrivial winding.

Following φ smoothly in all the space, one necessarily meets points where it has at least two coinciding eigenvalues, because otherwise the winding would have disappeared.

This statement is obviously gauge invariant so it is true even in non regular gauges like the unitary gauge.

### 3 Classical monopoles

In this section we review the classification of monopoles \[4, 31, 32, 33, 34, 35, 36, 37\] that arise in 3 dimensions for the YMH model with group SU(N), that we will need later.

The action is:

\[
S = N \int d^3x \left[ \frac{1}{2g^2} \text{tr}(G_{\mu\nu} G_{\mu\nu}) + \frac{1}{2} \text{tr}(D_\mu \phi D_\mu \phi) + \frac{\lambda}{N} (\text{tr}(\phi^2) - \mu^2)^2 \right]
\]  (3.1)

First of all one introduces what are called point monopoles, singular configurations satisfying the equations of YMH plus the requirement of minimum action:

\[
\begin{align*}
D_\mu G_{\mu\nu} &= 0 \\
\frac{\partial}{\partial \phi^a} V(\phi) &= 0 \\
D_\mu \phi^a &= 0
\end{align*}
\]  (3.2)

The last two equations impose that one gets everywhere zero contribution to the action from the Higgs sector, so that we are in what is called Higgs vacuum. Some gauge fields have to be zero and the others allow for nontrivial solutions localized near isolated points.

To see this it is useful to choose the abelian “unitary” gauge φ ∈ H (with H the Cartan subalgebra). We will write φ = φ · T, where T is the vector of (N − 1) commuting generators of H in some faithful representation and φ is a vector in R^{N−1}.

In this gauge the above equations imply that the Higgs field is constant in all space φ(x) = φ_∞ and that A_µ has nonzero components only in the algebra of the little group of φ_∞, i.e. only if [A, φ_∞] = 0.

For φ_∞ with generic eigenvalues (all different) the little group is simply U(1)^{N−1}, so that only the abelian, Cartan, gauge fields survive in the vacuum.

Here φ_∞ induces perturbatively a mass term for each gauge field that is charged with respect to it. For each root α of SU(N) the mass of the charged A_±^α fields is

\[
m_{W_\alpha} = g|\phi_\infty \cdot \alpha|.
\]

At the same time the Cartan gauge fields decouple completely from the Higgs fields.

As V(φ) is flat in all gauge directions, the V′(φ) = 0 constraint only fixes the modulus of the vacuum Higgs field: \[\text{tr}(\phi_\infty^2) = \mu^2\].

We will assume at this point a vacuum φ_∞ with all different eigenvalues. This is a gauge invariant statement and we will see that this vacuum configurations should be preferred by the system itself, in the large N limit. It could also be imposed by some gauge invariant external source.

(checked to here)
3.1 Point monopoles

The $G_{\mu\nu}$ field equation now allows for abelian $U(1)^{N-1}$ solutions with a singular Dirac string:\[4\]:

\begin{align*}
A_\mu &= T \cdot qD_\mu \\
G_{\mu\nu} &= gT \cdot q \epsilon_{\mu\nu\lambda\chi} x^\lambda |x|^3 \\
D_\mu &= (1 - z/r) \partial_\mu \tan^{-1}(y/x)
\end{align*}

$q$ is the nonabelian charge of the monopole, but as we are in the unitary gauge, it belongs to $U(1)^{N-1}$; it is for now an arbitrary vector in $\mathbb{R}^{N-1}$.

As $z \to -\infty$ we have the asymptotic form:

\begin{equation}
A_\mu = 2T \cdot q \partial_\mu \Phi \quad (0 \leq \Phi = \tan^{-1}(y/x) \leq 2\pi).
\end{equation}

Observing the phase of a loop around the string we get a realization of $\pi_1(U(1)^{N-1})$ and obtain the admissible monopoles: the generalized Dirac condition

\begin{equation}
e^{4\pi i q \cdot T} = 1.
\end{equation}

This condition restricts the possible charges $q$ to belong to a lattice in $\mathbb{R}^{N-1}$, in fact $q$ has to be reciprocal to each weight of the representation chosen for the $T$: for every weight $m_i$,

\begin{equation}
q \cdot m_i = \frac{n_i}{2} \quad n \in \mathbb{Z}.
\end{equation}

The lattice of charges depends thus on the representation chosen for the $T$’s, calling in the game also the global properties of the representation of the gauge group.

It can be more or less dense depending on the modulus of the highest weight of the representation.

One now introduces the (dual) co-roots, $\alpha_i^* = \alpha_i / \alpha_i \cdot \alpha_i$, (where $\alpha_i$ are the simple roots). For each weight they satisfy the relation $m \cdot \alpha^* = n/2$, so that they are reciprocal to the weight lattice. The coroot system defines what we have called the dual group. For $SU(N)$ the dual group is isomorphic to it, denoted $SU^*(N)$. Moreover for roots normalized to unity the coroot lattice coincides with the root lattice.

An immediate consequence of the relation with the weight lattice is that the coroots (and also their multiples) are always between the possible magnetic charges $q$. Monopoles in the adjoint representation of the dual group are thus always present.

Usually one studies the simplest cases of the fundamental and adjoint representations, for the gauge field.

For $T$ in the fundamental representation ($T = \frac{1}{2} \lambda$) the weights are the fundamental ones and the magnetic lattice is just the coroot lattice

\begin{equation}
q = \sum_{i=1,...,N-1} n_i \alpha_i^* \quad n_i \in \mathbb{Z}
\end{equation}
The monopoles of minimum charge transform in this adjoint representation.

The picture represents the coroot lattice for $SU^*(3)$, its generators as black circles. The small triangles represent the fundamental monopoles that arise for adjoint gauge generators. There is thus a nice duality: for gauge variables with fundamental (adjoint) generators, the minimum monopoles transform in the adjoint (fundamental) representation of the dual group.

A remark is due for the Weyl group which acts in the weight lattice and thus also on the magnetic charges. It is generated by the reflection with respect to the planes orthogonal to the roots, and sends every lattice that we have considered into itself. The action can be seen as a reflection also on the magnetic charges.

Seen on the Cartan generators, it simply exchanges the diagonal entries (as can be easily seen in the (overcomplete) basis $2(T_{ij})_{kl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = \begin{pmatrix} 1 & \cdots & \cdots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ -1 & \cdots & \cdots & 1 \end{pmatrix}$).

On the fields, any Weyl action is equivalent to a global gauge rotation with respect to the generator $E_\alpha + E_{-\alpha}$, where $\alpha$ is the relative root, and this shows that monopole configurations related by Weyl symmetry are gauge equivalent.

This has implications on the type of monopoles: for the usual case of gauge variables in the fundamental representation the minimum charge monopoles are all related by Weyl transformations.

In the adjoint representation instead monopoles are classified by the dual fundamental weights which are not all Weyl-equivalent: they are divided in classes according to the

\[ q = \sum_{i=1}^{N-1} n_i m_i^* \quad n_i \in \mathbb{Z}. \]  

(3.8)

One can say that the minimum charge monopoles transform now in the fundamental representation of the dual group. They are shown in the picture as small triangles. The monopoles that arise in this case include also the previous adjoint charges as combinations of minimal monopoles, although the lattice generated by them is not shown.

\footnote{In fact an other case mentioned in the literature \[2\] is that of gauge variables with generators in the adjoint representation. The nonzero weights are in this case the simple roots, so that the reciprocal lattice coincides with the weight lattice of the dual group $(m^* \cdot \alpha = n/2)$:}

$N - 1$ (nontrivial) elements of $\pi_1(SU(N)/Z_N)$. Weyl reflections only act within these classes.

An example for the dual $SU(3)$ is shown in the picture, the fundamental co-weights are represented by the the small triangles, the right pointing giving the $[3]$ representation, the left the $[3^*]$. The Weyl transformations are the reflections with respect to the long-dashed lines.

One sees that the fundamental monopoles come in two triplets invariant under Weyl, while the usual adjoint monopoles come in a whole sextet.

### 3.2 Regular monopoles

One notices first from (3.5) that if the $U^{N-1}(1)$ path $e^{2q \cdot T \Phi}$ ($0 < \Phi < 2\pi$) can be continuously gauged away in $SU(N)$ to the identity, then the Dirac string will decrease of intensity and disappear. The gauge transformation needed to do that is necessarily non-constant, so that one will end up with a nonconstant Higgs field. If this can be done, the point monopole is the basis for a regular one with finite energy. It is clear that there is a great freedom to construct these regular solutions of the equations of motion.

The spherically symmetric solutions have been studied in detail in [32] with a classification of all the possible magnetic charges from which one can determine a finite action configuration.

The charges which admit spherical solutions are given by $q = q' - q''$ where $q'$ and $q''$ are the roots of two embeddings of $SU(2)$ in $SU(N)$ and $q''$ must also be in the little group. One $SU(2) q'$ is needed to rotate the Higgs field to a radial gauge, and the other is a remaining freedom to define the spherical gauge configuration.

Since the factor in (3.3) is a loop in the chosen representation of $SU(N)$, the process is clearly possible for any charge if and only if the generators are in a faithful representation like the fundamental one.

The Weyl group relates monopoles of different charges by a global gauge transformation. This is a difference with the abelian case where different charges classify gauge inequivalent monopoles. This peculiarity of non-abelian theories follows mainly from the simple-connectedness of $SU(N)$, but also from the fact that colored flux lines are not gauge invariant (but covariant) and can thus be deformed or changed of color by gauge transformation.

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*In the case for example of the adjoint representation, there are $N$ inequivalent loops which join the identity to the elements of the center of $SU(N)$. So, for $N - 1$ kind of point monopoles, the string is impossible to remove and they are genuine Dirac monopoles. An other accident of this case is that the minimum charge monopoles are in this case $N(N-1)$ (the weights of all fundamental representations), while the nontrivial elements of the center $Z_N$ are $N-1$. The minimum charges can thus be divided in $(N-1)$ sets of $N$ elements, and the Weyl group acts only within each set.

All this fundamental monopoles, associated to the nontrivial paths in $SU(N)$ around the Dirac string, cannot be made regular.
Another important notice is that the charge quantization condition (3.5) does not depend on the vacuum Higgs field \( \phi_\infty \), and this is an advantage of the unitary gauge.

Regular monopoles instead do depend on the \( \phi_\infty \) boundary condition, because they are constructed transforming the Dirac string into a varying Higgs field.

As we are going to assume \( \phi_\infty \) to have all different eigenvalues, the charge \( q'' \), belonging to the little group, can only be zero, so that the magnetic charges of spherically symmetric monopoles will coincide with \( q \) roots of SU(2) embeddings\(^\|\). The figure shows them for \( SU(3) \), for \( N > 3 \) the pattern is much more complicated.

Possible spherical monopoles for \( SU(3) \)

- \( 3 \rightarrow 2+1 \) embedding   \( Q=(1/2,-1/2, 0) \)
- \( 3 \rightarrow 3 \) embedding   \( Q=(1, 0, -1) \)
- \( 3 \rightarrow 3 \) only with degenerate \( \phi \)   \( Q=(1,-1/2,-1/2) \)

Only the first two cases are possible for nondegenerate \( \phi_\infty \). For this cases the solution are well-known 't Hooft-Polyakov monopoles:

\[
\hat{A}_\mu(r, q) = T_q D_\mu + K(m_W r) \left[ E_q e^{-i\phi(i\hat{\theta} + \hat{\phi})} + E_{-q} e^{i\phi(-i\hat{\theta} + \hat{\phi})} \right]
\]

\[
\hat{\phi}(r, q) = \phi_\infty \cdot (T - qT_q) + H(m_H r)(\phi_\infty \cdot q)T_q
\]

(3.9)

\( T_q = q \cdot T \) with some \( E_q \) and \( E_{-q} \) give the SU(2) subalgebra.

Changing to a regular gauge one can “smear out” the string in \( D_\mu \) and leave an isolated singularity at the origin.

### 3.2.1 Masses

In the last expressions, \( m_W \) is the mass of the two charged gauge bosons in the chosen sector of the monopole, function of the Higgs vacuum: \( m_W = g|\phi_\infty \cdot q| = g|\phi_i^\infty - \phi_j^\infty| \); \( m_H \) is instead the mass of the Higgs field “in the sector of \( T_q \)”:

\[
m_H^2 = \frac{2\lambda}{N^2} (\phi_i^\infty - \phi_j^\infty)^2.
\]

They regulate the exponential decay of the massive field components of the monopole as well as its total classical action. We have in fact

\[
S_{cl} = N|q| \frac{m_W(q)}{g^2} C\left( \frac{\lambda}{N g^2} \right).
\]

\(^\|\)Monopoles of higher charge, empty sites in the picture, are not realizable as single three dimensional spherical configurations, although the topological argument indicates the existence of some extrema of the action. Some can be constructed as multimonompole-like configurations which possess discrete symmetry groups, with stability given by the Higgs attraction. Such configurations have been found to exist with tetrahedral, octahedral (but not icosahedral symmetry, for a late reference, see \cite{cite2} and therein).
The function $C$ is found to approach the value $4\pi$ when the argument goes to zero, which for $SU(2)$ is called BPS limit.

Considering the $N(N-1)$ unit charge monopoles, we see that, fixed $\phi^\infty$, their pseudo-mass ranges in the interval $0 \leq S_{cl} \leq 4\pi N\mu/g$.

We see that most of the properties of the various objects are ruled by the $m_W$ of the gauge bosons in the relative $SU(2)$ sector, so that all depends on the set of Higgs vacuum eigenvalues $\phi_i^\infty$.

The constraint $|\phi^\infty| = \mu \simeq O(N^{1/2})$ has important consequences on the set of masses that are present in the model in the large $N$ limit, because, as we will see, necessarily there are masses that become small at least as $O(1/N)$. We will also find masses of order $O(1/N^2)$.

All the monopoles of charge greater than 1, have higher mass proportional to their charge and they are expected to dissociate into smaller constituents. Hence their contribution to the infrared region is negligible and one can safely discard them.

We will instead use approximate solutions built by superimposing an arbitrary number of minimal monopoles at large distances.

They are constructed easily in the unitary gauge and are then regularized by means of a procedure similar to the one for the single monopole. We just need the proof of existence of the gauge transformation needed to do that, because we will work in the unitary gauge.

$$A^{(n)}_\mu(x) \equiv \sum_{a=1}^{n} \hat{A}_\mu(x - x_a, q_a)$$
$$\phi^{(n)}(x) \equiv \sum_{a=1}^{n} (\hat{\phi}(x - x_a, q_a) - \phi^\infty) + \phi^\infty$$  \hspace{1cm} (3.11)

They depend only on the parameters of the $n$ single monopoles $\{x_a, q_a\}$ and are good solution to the equations of motion for distances much bigger than the monopole sizes. This approximation is called dilute gas.

The interaction of monopoles simplifies drastically for large distances compared to the monopole size and remains function of the relative distances only.

In fact the action for such dilute configurations is found to be approximated by the self-action of each monopole plus a Coulomb-like monopole-monopole interaction term.

4 Quantum fluctuations

Coming to the much more ambitious program of quantizing the theory, the only feasible way is for now the semiclassical expansion, leaving apart the S-duality which solves the supersymmetric theories, or the first order (BF) formalism, which also falls back to semiclassical quantization.
If semiclassical it is, it should show the correct result.

However even this treatment at one loop order is a nontrivial task, because involves the calculation of functional determinants in classical backgrounds.

There are some simplifications in the BPS limit \cite{38,40} of supersymmetric theories, because the three-dimensional configurations represent four dimensional selfdual backgrounds and there are useful relations between fermion and boson determinants in this case, but we remind that the physics in this limit is drastically different. The main reason for this is that the Higgs field is massless and thus gives a further long range interaction. We thus want to consider $\lambda > 0$.

A major progress was made by Polyakov. In analogy with other simpler models he applied \cite{14} the semiclassical quantization to the $SU(2)$ monopole solutions and carried the program to evaluate the Wilson loop resumming the semiclassical expansion.

His treatment, for the compact QED, shows that the area law for the Wilson loop emerges from this semiclassical treatment precisely because of the condensation of monopoles.

Later in \cite{21} Das and Wadia have reached the same conclusion for the problem with $SU(3)$ gauge group. Recently also for the case of pure $YM_3$ they have argued that confinement arises from the gas of monopoles, using nonperturbative results from the theory of the three dimensional Coulomb gas \cite{17}.

So we have, at our disposal, the minima of the classical action that are strongly supposed to give the main contribution to the functional integral, and taking into account the gaussian fluctuations around them, one can introduce a semiclassical sum.

We will see that important nonperturbative features of the model are reproduced by this approach.

### 4.1 Semiclassical program

In \cite{14,21} the semiclassical quantization of a system with monopoles is approached through a grand canonical ensemble of magnetic particles. The sum on all configurations gives the partition function in a nonperturbative way, and after a generalized Poisson transform, a saddle point technique can be applied.

The gauge fields should be integrated perturbatively at one loop, taking into account the regular monopole backgrounds as nontrivial minima of the action.

In this approach many approximations are to be taken carefully:

First, the gas of monopoles is assumed dilute, thus considering just the coulomb part of the monopole-monopole interaction and simplifying the classical and one-loop effective action.

After the generalized Poisson& Sine-Gordon transform has been used, usually one is limited to the minimum charge monopoles, assuming the higher ones dissociate rapidly**.

**One could try to consider also the higher charges. From the Wilkinson Goldhaber analysis, that spherical monopoles have limited charge (for limited $N$, $|q| \leq N - 1$). Moreover their mass grows
4.2 Grand canonical ensemble

The partition function of pure YM in 2+1 dimensions is transformed into the sum on all configurations made of any number of monopoles of charges \( \{ q_a \} \) and locations \( \{ x_a \} \). For \( SU(2) \) it is:

\[
Z = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} Q_n
\]

\[
Q_n = \sum_{\{ q_1, \ldots, q_n \}} \int \prod_{a} d^3 x_a \ e^{-\frac{2\pi}{g^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|}} \quad (q_a = \pm 1, \pm 2, \ldots)
\]

where \( \xi \) is the classical and one loop contribution to the action of the one monopole configuration [14].

For \( SU(N) \) this expression is no longer valid because the weight of each monopole depends on the charge, \( \xi = \xi(q) \). Hence one has:

\[
Z = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n \quad (4.1)
\]

\[
Q_n = \sum_{\{ q_1, \ldots, q_n \}} \int \prod_{a=1}^{n} d^3 x_a \ \xi(q_a) \cdot e^{-\frac{2\pi}{g^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|}} \quad (q_a \text{ in the root lattice}) \quad (4.2)
\]

The partition function \( Z \) can be nicely reexpressed as a functional integral using the Sine-Gordon transform [41]:

\[
Z = \int \mathcal{D} \chi e^{-\frac{\chi^2}{2\pi^2} \int d^3 x \left[ (\partial \chi)^2 - \sum_i M^2(q_i) e^{i q_i \cdot \chi} \right]}.
\]

The mass \( M^2 \) comes from the weight \( \xi \): \( M^2 = 32\pi^2 \xi / g^2 \). \( \chi \) is a \( N - 1 \) components scalar field whose propagator is just the coulomb potential, and the sum on \( i \) is the sum on all the possible magnetic charges of one monopole (the magnetic lattice). The last term in the integral is the functional generator of the multi-charge configurations. For the symmetry of the minimum charges, the roots \( q_i = -q_{-i} \), the last term becomes a cosine.

This representation of the coulomb gas has to be understood in a perturbative sense, because it is just the perturbative expansion which reproduces, diagram by diagram, the dilute gas. In this spirit the \( \chi \) field configurations have to vanish at infinity, although there is formally an infinite 'zero-point' energy of the vacuum \( \chi = 0 \), which disappears with the normalization.
4.3 Wilson loop

One can then succeed to evaluate gauge invariant operators like the Wilson loop $W_C$:

$$W_C = \frac{1}{N} \left\langle \text{Tr} e^{i \int_C A \cdot dx} \right\rangle$$  \hspace{1cm} (4.4)

for any contour $C$ in three dimensional space-time.

If we take into account the form of $A_\mu$ in the unitary gauge (3.3) for each monopole of charge $q_a$ and location $x_a$, (and after using the Stokes theorem) we can rewrite the Wilson loop as an external source for the configuration of charges $\{q_a\}$:

$$W_C = \frac{1}{N} \left\langle \text{Tr} e^{i \sum_a q_a \int_S d^2 \sigma(y) \frac{(x_a - y)_\mu}{|x_a - y|^4}} \right\rangle \eta(x) = \int d^2 \sigma(y) \frac{(x - y)_\mu}{|x - y|^3}$$  \hspace{1cm} (4.5)

Instead of the flux through the loop of $\eta(x)$, one thinks to a potential ($\eta(x)$, in the picture) which acts on the charges $q_a$ at points $x_a$, produced by a dipole layer on the surface $S$ spanning the loop.†† The problem of evaluating the Wilson loop in the functional integral is reduced to the average of the canonical ensemble under the action of this external potential.

In the limit of very large loop this potential is constant on the two sides leaving just the discontinuity, which is of crucial importance.

From the color trace the potential takes each direction along the fundamental weights $m_\alpha$:

$$W_C = \frac{1}{N} \left\langle e^{i \sum_a m_\alpha q_a \eta(x_a)} \right\rangle = \frac{1}{N} \sum_\alpha \left\langle e^{i \sum_a m_\alpha q_a \eta(x_a)} \right\rangle$$  \hspace{1cm} (4.6)

It is straightforward to evaluate this operator in the Coulomb gas ensemble, and the result is, after a shift in the $\chi$ field:

$$W_C = \frac{1}{ZN} \sum_\alpha \int D\chi e^{-\frac{a^2}{32\pi^2} \int d^3 x \left[ (\partial^2 \chi - m_\alpha \partial \eta)^2 - \sum_i M^2(q_i)e^{q_i \cdot \chi} \right]}$$  \hspace{1cm} (4.7)

4.4 Saddle point solution

The nongaussian integral above is solvable by saddle point, remembering that $M^2$ is asymptotically small.

The saddle point is given by the equations, one for each $m_\alpha$:

$$-\partial^2 \chi_\alpha + m_\alpha \partial^2 \eta = - \sum_{i \neq \alpha} M^2(q_i)q_i \sin(q_i \cdot \chi_\alpha)$$  \hspace{1cm} (4.8)

††The dipole density is unitary and in direction of $d^2 \sigma_\mu$ orthogonal to the surface. The potential is in practice the solid angle of the loop seen by the charge at $x_a$. 

††The dipole density is unitary and in direction of $d^2 \sigma_\mu$ orthogonal to the surface. The potential is in practice the solid angle of the loop seen by the charge at $x_a$. 

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(The sum on $i$ runs now only on one half of the magnetic lattice because the negative symmetric part is included in the sine).

The term $m_{\alpha} \partial^2 \eta$ imposes a discontinuity of the solution in 'internal' direction $m_{\alpha}$ and together with the constraint of perturbativeness $\chi(\pm \infty) = 0$, this leaves only the solutions of constant direction $\chi(x) = m_{\alpha} \chi_{\alpha}(x)$ (no sum).

Taking the product with $m_{\alpha}$, one gets the scalar equations

$$\left( \partial^2 \chi_{\alpha} - \partial^2 \eta \right) \frac{N-1}{2N} = \sum_{\text{half } i} M^2(q_i)m_{\alpha} \cdot q_i \sin(q_i \cdot m_{\alpha} \chi_{\alpha})$$

(4.9)

(where we used the fact that for the weights of the fundamental repr. $m_{\alpha}^2 = \frac{N-1}{2N}$).

At this point one has to specify which magnetic charges $q_i$ to use, and we see from the Sine-Gordon representation that higher charge monopoles give rise to a generalized Sine-Gordon potential which is of lower magnitude, even if it has shorter periods. This means that higher charges only perturbate the potential given by the minimal charges.

We limit then the sum over $i$ to the minimum magnetic charges, i.e. the co-roots of the first picture.

We have to evaluate the scalar product $m_{\alpha} \cdot q_i$ of a fundamental weight with the adjoint weights.

To this aim we remember that the roots $q_i$ are $N(N-1)$, and that given a fundamental weight $m_{\alpha}$, $(N-1)(N-2)$ of them are orthogonal to it, while $N-1$ have scalar product $1/2$ and the others $(N-1)$ (negative symmetric) have scalar product $-1/2$.

It is then sufficient to limit the sum to the $(N-1)$ cases all giving result $1/2$ for the scalar product:

$$\partial^2 \chi_{\alpha} = \partial^2 \eta + \frac{2N}{N-1} \frac{1}{2} \sin(\chi_{\alpha}/2)(N-1)\bar{M}_{\alpha}^2$$

(4.10)

where we have introduced the averaged $\bar{M}^2$ given by

$$\bar{M}_{\alpha}^2 = \frac{1}{(N-1)} \sum_{q_i: q_i \cdot m_{\alpha} = 1/2} M^2(q_i).$$

(4.11)

It is this quantity which carries information on the physics that we obtain in the large $N$ limit. Each $M^2(q)$ is, in Coulomb gas language, the fugacity of the monopole species $q$. $\bar{M}_{\alpha}^2$ is then directly related to the average density of monopoles, which is strongly believed to be the order parameter for confinement.

We note also that many monopoles are mutually neutral and that $\bar{M}_{\alpha}^2$ is the average in the $(N-1)$ sectors that are not orthogonal to $m_{\alpha}$. To evaluate the Wilson loop we should also average on $m_{\alpha}$, finally.

The solution of (4.10) is known explicitly:

$$\chi_{\alpha}(x) = \begin{cases} 4 \tan^{-1} e^{-M \sqrt{2x}} & x > 0 \\ -4 \tan^{-1} e^{M \sqrt{2x}} & x < 0 \end{cases}$$

(4.12)
which consists of two parts of a Sine-Gordon soliton. We remark that this soliton is permitted only because the discontinuity allows the field to vanish at infinity. Otherwise there could be an infinity of other classical solutions.

Inserting this into eq. (4.7), it gives the estimate for the Wilson loop:

\[
W_C = \frac{1}{N} \sum_\alpha \exp \left\{ -\frac{g^2}{32\pi^2} \int d^3x \left[ \frac{N-1}{2N}(\chi'_\alpha - \eta')^2 - \bar{M}^2(N-1)(\cos(\chi_\alpha/2) - 1) \right] \right\}
\]

\[
\simeq \frac{1}{N} \sum_\alpha e^{-\sigma_\alpha A} \tag{4.13}
\]

with string tensions \(\sigma_\alpha\) of

\[
\sigma_\alpha = \frac{g^2 \bar{M}}{32N\pi^2} \frac{N-1}{2} \frac{2}{N} = \frac{g}{8\pi} \frac{N-1}{N} \sqrt{\frac{\xi}{N}} \tag{4.14}
\]

This shows that confinement of quarks exists in this theory for generic values of the coupling constants and for finite \(N\), while to extract the behavior with large \(N\) it is necessary to perform the average for \(\bar{M}\) and then for the Wilson loop \(\frac{1}{N-1} \sum_\alpha e^{-\sigma_\alpha A}\).

The \(N\) dependence of \(\bar{M}\), that is of \(\xi\), is not known explicitly. As in [14], \(\xi\) is the one loop partition function in a single monopole background, before the integration of the zero mode translation coordinate:

\[
\xi(\mathbf{q}) = N^{9/2}m_W^3 \left( \frac{m_W^3(q)}{g^3} \right) A(\lambda/g^2)e^{-N \frac{m_W(q)}{g^2} C(\lambda/N g^2)} \tag{4.15}
\]

The condition of validity of the saddle point approximation, which represents the low Debye density, is \(\bar{\xi} \ll 1\). It can be seen to hold for finite \(N\) from the above expression, where the exponential vanishes asymptotically. In the limit of \(N\) large one needs a more precise estimate on the whole average \(\bar{\xi}\). This we will attempt in the next sections.

4.5 Determinant

In the previous sections, \(\xi(\mathbf{q})\) plays the part of the statistical-quantum weight of a particular background configuration, so that the higher it is, the higher is the importance of that particular configuration, although it may have large action.

Up to now we assumed \(\xi\) to be some fixed quantity. Now, in order to draw some conclusion about the string tension \(\sigma\), we need to find something more precise on it.

The evaluation of \(\xi\) is the problem of calculating the functional determinant of the fluctuating fields around the one monopole solution. It would be a hopeless problem to calculate it exactly in an arbitrary external field, as it is equivalent to the solution of a Schroedinger or Dirac equation in an external potential*.

We will try to extract some information from the high \(N\) analysis of the problem.

*In [23] the expression has calculated numerically with the heat kernel method for \(SU(2)\), but the estimate of the behavior with \(N\) has yet to be done.
The idea comes from the fact that after gauge fixing, and better in the unitary gauge, the Higgs field has only $N$ components, while its effective action, upon integration of the gauge sector, is of order $N^2$. Hence the saddle point should be applicable, and the Higgs field is a semiclassical quantity with respect to $1/N$, which acts like $\hbar \to 0$ to suppress its fluctuations.

We will treat the fields in one loop approximation around the one monopole configuration $\hat{A}_\mu, \hat{\phi}$ adding the fluctuating fields $a_\mu, \varphi$, so that: $A_\mu = \hat{A}_\mu + a_\mu, \phi = \hat{\phi} + \varphi$.

In doing so, we are faced with the problem of gauge fixing, because there are zero modes of the action. The gauge invariance involves the total field $(\hat{A}_\mu + a_\mu, \hat{\phi} + \varphi)$, and one can split the gauge variation between the fluctuating and the background fields in an arbitrary manner.

Among the possible (infinite) choices, one can assign the whole field variation either to the fluctuations or to the background. The latter choice is of little or no utility, the first, instead, is quite convenient in that it keeps away the gauge invariance problem from the background fields.

So we will keep the background fields in some fixed gauge, and consider the gauge group as acting on the sole fluctuations:

\[
\delta_{\text{gauge}}(\hat{A}_\mu, \hat{\phi}) = 0
\]
\[
\delta_{\text{gauge}}(a_\mu, \varphi) = \delta_{\text{(A,\phi)}}(a_\mu, \varphi) = (D_\mu \alpha, -ig[\phi, \alpha]).
\]

The gauge for the classical fields is left for now unspecified, even though the radial gauge satisfies automatically the background gauge. In the next section instead we will choose for all fields the unitary gauge.

The partition function in a one-monopole background, $\xi$, is a gauge invariant object but not gauge independent, at least in principle. Every gauge fixing could provide different physical insight, as happens already with spontaneous symmetry breaking.

According to 't Hooft and Polyakov $\xi$ it is better calculated in the so called background, "natural", gauge, for two reasons: one is the presence of zero modes, to appear in a moment, for the action of fluctuations, and they are best treated in the background gauge; the second is that candidates with opposite features, the unitary gauges, usually addressed to be non-renormalizable, can be cured only introducing a nonpolynomial term in the action.

This can be seen from the Vandermonde determinant appearing in the measure of $\varphi$ after elimination of the $(SU(N))$ 'angular' part of $\varphi$: $D\varphi \to D\varphi \prod_x \Delta^2(\varphi(x))$.

There are two ways to deal with this $\Delta^2$; it can be reabsorbed in the measure by a (nonlinear but nonsingular) change of $\varphi_i$ variables, but it yields a nonlinear model on a singular curved target space of the kind (for $SU(2)$ for example):

\[
L = \varphi^{-4/3}(\partial_\mu \varphi)^2
\]

Alternatively, thinking to a lattice, one can exponentiate the determinant and convert it in the divergent logarithmic potential: $\delta(0) \sum_{i \neq j} \int \log |\phi_i - \phi_j| d^3x$, but the continuum...
limit seems problematic due to the ultralocal nature of $\Delta^2$. This procedure can be justified with care starting from the $R_\xi$ kind of gauges [tH,V] or with other kind of regular gauge fixing.

I can note however, that this divergent extra potential, being exponentiated without the help of ghost fields, carries an $\hbar$ factor, and thus already is a quantum correction to the action.

As the theory is renormalizable for every gauge $R_\xi$, in the $\phi$ effective action there should be, then, a divergent term from the massive gauge fields which cancels it. In effect it is what we find after the one loop analysis.

Hence for the large $N$ saddle point analysis on the effective potential, there should be no serious problem from this term.

We will adopt in the following paragraph the background gauge and in the next the unitary one.

### 4.5.1 Background Gauge

Here we take, as gauge fixing:

$$F_\kappa = \hat{D}_\mu a_\mu - i\kappa g[\hat{\phi}, \varphi] = B(x)$$

In the limit $\kappa \to \infty$ the second term reproduces the unitary gauge. This is a variant of the pure background gauge (that has $B = 0$, $\kappa = 1$), already considered by [3] or of the "natural gauge" that appears in [14]. It has a long history in the literature, due to its double features of renormalizability together with massive ghost fields.

The usual gaussian averaging of the gauge fixing $\delta(F_k - B)$, to get a quadratic term in the action, is in some contrast with the high $N$ analysis because removes the gauge fixing and leaves $N^2$ degrees of freedom. On the other hand fixing strictly the gauge in the case $k \neq 1$ presents some subtleties.

We will denote together the fluctuating fields $a_\mu(x)$ and $\varphi(x)$ with $\Phi = (a_\mu \varphi)$ and the full quadratic action will be, in matrix form, $S_{quadr} = \langle \Phi M \Phi \rangle$. Scalar product $\Phi \cdot \Phi$ will imply integration on space-time.

About the action we just need to say, for now\(^1\), that it is annihilated by the following zero modes:

$$\Phi_0(\alpha) = (a_\mu(\alpha) \varphi(\alpha)) = \left(\begin{array}{c} \hat{D}_\mu \alpha \\ -ig[\hat{\phi}, \alpha] \end{array}\right)$$

\(^1\)Explicitly it is:

$$S_{quadr} = N \text{tr} \int \left[ ([\hat{D}_\mu, a_\nu] - [\hat{D}_\nu, a_\mu])^2 + \frac{g^2}{2} G_{\mu\nu}[a_\mu, a_\nu] + (g[a_\mu, \hat{\phi} + \varphi] + [\hat{D}_\mu, \varphi])^2 + \varphi V''(\varphi) \varphi \right]$$

Together with the proper mass term for gauge fields, $\frac{1}{2}Ng^2[\hat{\phi}, a_\mu]^2$, there is also the bilinear mixing $N([a_\mu, \varphi][\hat{D}_\mu, \hat{\phi}] + [a_\mu, \hat{\phi}][\hat{D}_\mu, \varphi])$. 

The complete field $\Phi$ can thus be decomposed in zero modes plus nonzero ones, $\Phi_n$, eigenfunctions of the action:

$$\Phi = R_i \Phi^{(i)}_0 + \Phi_0(\alpha) + \xi_n \Phi_n$$

The translational modes (4.20) are written in a gauge so that they satisfy the gauge-fixing (4.18) with $\kappa = 1$, $B = 0$. This is useful because they are orthogonal to the gauge modes. They are normalized to

$$|\Phi^{(i)}_0|^2 = \mathcal{N}_i = N \text{tr} \int \frac{1}{g^2} \hat{G}_{\mu i}^2 + (\hat{D}_i \hat{\phi})^2. \quad (4.21)$$

All these zero modes are treated with the standard Faddeev-Popov method to extract the integration on collective coordinates $R_i$, $\alpha(x)$:

$$1 = \text{Det}_{ij} \left[ \frac{\partial}{\partial R_j} \Phi \cdot \Phi^{(i)}_0 \right] \text{Det} \left[ \frac{\delta}{\delta \alpha(x)} F_k(\Phi(y)) \right] \cdot \int d^3R \text{d}D \alpha \delta(F_k - B) \prod_{(i)} \delta \left( \Phi \cdot \Phi^{(i)}_0 \right).$$

Insertion of this unity in the functional integral replaces the flat fluctuations with the right variables $R_i$, $\alpha(x)$ and gives the two determinants. The first Det gives $\prod_i \mathcal{N}_i^{1/2}$, the second is the Faddeev-Popov determinant

$$\text{Det} \left[ \frac{\delta}{\delta \alpha(x)} F_k(\Phi(y)) \right] = \text{Det}[\mathcal{M}_{FP}] = \text{Det} \left[ \hat{D}_\mu D_\mu - \kappa g^2 [\hat{\phi}, [\hat{\phi}, \cdot]] \right] \quad (4.22)$$

After elimination of the first delta, the remaining fluctuations represent the functional integral restricted the non translational modes:

$$Z = e^{-S_{cl}} \int d^3R \prod_i \mathcal{N}_i^{1/2} \int \mathcal{D}\Phi \delta(F_k(\Phi) - B) \text{Det}[\mathcal{M}_{FP}] e^{-\int i\Phi, \mathcal{M}\Phi} \quad (4.23)$$

The Faddeev-Popov determinant can be evaluated, in one loop approximation, in the sole classical background fields.

Then, as seen from (4.21), each factor in $\prod_i \mathcal{N}_i^{1/2}$ is the action in a space-time direction without the potential. Because we consider spherical monopoles only, all $\mathcal{N}_i$ are equal and

$$\mathcal{N}_i = \frac{N}{3} \text{tr} \int \frac{1}{g^2} \hat{G}^2 + (\hat{D}\hat{\phi})^2 \quad (4.24)$$

They coincide with the action in the BPS limit, hence, after the discussion of section 3.2.1 in the large $N$ limit we also take $\mathcal{N} = N \frac{4\pi mW}{3g^2}$. 

One can perform a functional integration over \( B \) (with \( e^{-\frac{3}{2} \int B^2} \)) to remove the \( \delta(F^k - B) \):

\[
Z = \int d^3 R e^{-S_{cl}} N^{3/2} \det \left[ \hat{D}_\mu \hat{D}_\mu - \kappa g^2 [\hat{\phi}, [\hat{\phi}, \cdot]] \right] \int \tilde{D} \Phi e^{-\int \Phi (M + \frac{\lambda}{2} \mathcal{M}_{gf}) \Phi}
\]

\[
= \int d^3 R e^{-S_{cl}} N^{3/2} \det \left[ \hat{D}_\mu \hat{D}_\mu - \kappa g^2 [\hat{\phi}, [\hat{\phi}, \cdot]] \right] \tilde{D}^{-1/2} \left[ \mathcal{M} + \frac{\lambda}{2} \mathcal{M}_{gf} \right] (4.25)
\]

Here with \( \tilde{\Phi} \mathcal{M}_{gf} \Phi = F^2_k(\Phi) \) we denote the standard gauge fixing term arising from \( F_\kappa \), while with \( \tilde{D} \Phi = \tilde{D} a \tilde{D} \varphi \), as with the determinant \( \tilde{\det} \), we integrate on translation-fixed fluctuations.

This is analogous to the formula of Polyakov [14], but does not show the explicit dependence on \( \kappa \); one has to go through the loop expansion in gauge bosons and ghosts in the external field, and proceed to resum all contributions of generic order in \( g \) if one wants to control the limit \( \kappa \to \infty \). Actually in this limit the ghost masses become large so that they decouple, but at the same time the coupling with gauge and Higgs also diverges, so that the resumming is nontrivial. We will see the first order in an other way in the next section.

Now we proceed in a different direction, we eliminate the \( \delta(F^k - B) \) by direct integration of the gauge modes.

This point of view has the advantage of showing the physical degrees of freedom \( \Phi_n \), together with the explicit dependence on \( \kappa \) that we want to compare with the unitary gauge.

Briefly, instead of \( \tilde{\det} \left[ \mathcal{M} + \frac{\lambda}{2} \mathcal{M}_{gf} \right] \) we get the decoupled product of two determinants \( \tilde{\det}_{gf} [\mathcal{M}] \det^2 \left[ \sqrt{\frac{\lambda}{2}} \mathcal{M}_{FP} \right] \) plus a measure Jacobian dependent on \( \kappa \). The second determinant cancels formally the FP determinant, and \( \lambda/2 \) disappears in the normalization. \( \kappa \) of course remains in the measure, but the limit is nonsingular.

Let's expand as before the fields as \( \Phi = R,i \Phi_0^{(i)} + \Phi_0(\alpha) + \xi_n \Phi_n \). We can eliminate the zero modes, taking into account the Jacobian from \( \Phi \) to \( (\alpha, R_i) \).

We get (still ignoring contributions at more than one loop):

\[
Z = \int d^3 R e^{-S_{cl}} \prod_i \mathcal{N}_i^{1/2} \sqrt{\det \left[ \hat{D}_\mu \hat{D}_\mu - g^2 [\hat{\phi}, [\hat{\phi}, \cdot]] \right]} \int \mathcal{D}(\xi_n \Phi_n) e^{-\xi_n^2} \int \Phi_n \mathcal{M} \Phi_n (4.26)
\]

The dependence on \( \kappa \) and \( B \) seems disappeared, but the eigenfunctions \( \Phi_n \) have to satisfy the gauge fixing so that they are sensitive to \( \kappa \) and \( B \).

Remembering that \( \Phi_n \) was satisfying the natural gauge (4.18) \( B = 1, \kappa = 0 \), to pass to a choice with \( B \neq 0, \kappa \neq 1 \) we have to perform a gauge transformation of the \( \Phi_n \), so that they satisfy the new gauge fixing:

\[
\Phi_n \to \Phi_n + \Phi_0(\alpha_n). (4.27)
\]
This gauge transformation is:

\[
\alpha_n(B, \kappa - 1) = \left[ \hat{D}_\mu \hat{D}_\nu - g^2 \hat{\phi} \hat{\phi} \right]^{-1} \left( ig(\kappa - 1) \hat{\phi} + B \right). \tag{4.28}
\]

Notice that \( \alpha_n(B, \kappa - 1) \to \infty \) when \( \kappa \to \infty \).

Because we are just mixing with components along the gauge zero modes, the quadratic form in the action is unaffected.

The new basis \( \Phi_n \) is still orthogonal to the translational modes. For this reason we can retain the same measure of translations \( \mathcal{N}^{3/2} \).

However we have now eigenfunctions which are non normalized and not even mutually orthogonal, and we have to re-normalize the measure.

\[
\mathcal{D}_n \mathcal{D}_\phi = \mathcal{D}[\xi_n \Phi_n] = \prod_n d\xi_n \to \mathcal{D}[\xi_n (\Phi_n + \Phi_0(\alpha_n))] = \prod_n d\xi_n \cdot J
\]

\[
J = \sqrt{\det_{mn} [1_{mn} + \Phi_0(\alpha_m) \Phi_0(\alpha_n)]}
\]

\[
e^{-\int \phi \mathcal{M} \phi} = e^{-\mathcal{M}_{nn} \xi_n^2} \to e^{-\int (\phi + \Phi_0(\alpha)) \mathcal{M} (\phi + \Phi_0(\alpha))} = e^{-\mathcal{M}_{nn} \xi_n^2}
\]

The new jacobian \( J \) carries the information about the gauge dependence on \( \kappa \) of the functional integral. \( J \), although a formal expression, is explicitly function of the gauge parameter. The partition function is:

\[
\xi = \int d^3 R e^{-S_{cl} \mathcal{N}^{3/2}} \sqrt{\text{Det} \left[ \hat{D}_\mu \hat{D}_\nu - g^2 \hat{\phi} \hat{\phi} \right]} \cdot J \cdot \prod_n d\xi_n e^{-\mathcal{M}_{nn} \xi_n^2} \tag{4.30}
\]

For example in the case \( \kappa = 1 \), \( B \neq 0 \) we have \( \alpha_n(B, \kappa - 1) = \alpha(B) \) and \( J = \sqrt{\det_{mn} [1_{mn} + |\phi_0(\alpha(B))|]} \) It does not depend explicitly on eigenfunctions, but when \( \kappa \neq 1 \) we have:

\[
\alpha_n(B, \kappa - 1) = \left[ \hat{D}_\mu \hat{D}_\nu - g^2 \hat{\phi} \hat{\phi} \right]^{-1} \left( ig(\kappa - 1) \hat{\phi} + \varphi_n + B \right). \tag{4.32}
\]

We can rewrite \( J \) explicitly as a function of \( \kappa \) as:

\[
J = \sqrt{\det \left[ 1_{mn} + (\kappa - 1)^2 \hat{D}_\mu a^{(m)}_{n} \hat{D}_\mu a^{(n)}_{\mu} \right]}
\]

and after rescaling \( (\kappa - 1)^2 \), we can write:

\[
J = \sqrt{\det \left[ \hat{D}_\mu a^{(m)}_{n} \hat{D}_\mu a^{(n)}_{\mu} \right]}
\]

\[
\cdot \sqrt{\det \left[ 1_{mn} + \frac{1}{(\kappa - 1)^2} \left( \hat{D}_\mu a^{(m)}_{n} \hat{D}_\mu a^{(n)}_{\mu} \right) \right]^{-1}} \tag{4.34}
\]
The conclusion that we would like to draw from the present calculation is that in the limit \( \kappa \to \infty \) the singular behavior of the FP determinant in (4.25) has been canceled by the gauge bosons.

We pass directly to the unitary gauge, then, and we extract some information about the large \( N \) limit.

4.5.2 Unitary gauge

In this section we fix the gauge by requiring the Higgs field to be diagonal.

Choosing the unitary gauge eliminates all the local gauge invariance apart from the little group and the aforementioned discrete Weyl group.

In the case that we consider, we assume the background Higgs field to have all different eigenvalues and the fluctuation to be small with respect to it, so this is precisely the case. We will verify a posteriori whether this vacuum configuration is preferred.

The residual \( U^{N-1}(1) \) abelian gauge invariance has to be cured in a second moment by means of a further gauge fixing.

In this unitary gauge the Goldstone bosons are explicitly "eaten up" with the standard mechanism by the relative gauge fields which acquire one polarization more together with the mass.

At the same time two things happen: first the gauge fixing requires, through proper handling of the integration measure, the introduction of a Faddeev-Popov determinant; second the massive gauge fields have a Proca propagator, which carries bad behavior at large momentum.

For this last peculiarity the unitary gauge is usually addressed as non renormalizable, because the gauge fields produce, even at one loop, a new set of counterterms not present in the original lagrangian.

There is a lot of literature on the self-canceling of some of these non-renormalizable divergences, starting from [47], the remaining divergences are found (see [48] and therein) to vanish on the equations of motions, so that on-shell amplitudes do not suffer of this problem.

We will see that the Faddeev-Popov determinant participates exactly to render the theory manifestly renormalizable.

This can already be inferred from the \( \xi \to \infty \) limit of the \( R_\xi \) gauge (for the charged gauge fields): for any value of \( \xi \), the theory is renormalizable [27, 13, 14, 15, 16], so that the only counterterms needed are of the same form of the lagrangian. The limit \( \xi \to \infty \) is well defined for the massive gauge propagator, so that assuming some suitable regularization (Dimensional regularization is not suitable for this scope because the divergences we have are of the kind \( \delta(0) \) which vanish identically: \( \int d^4p = 0 \)) the cubic divergences cancel order by order between ghosts and massive gauge fields (or, alternatively, between the Faddeev-Popov determinant and the gauge fields). While the charged ghosts acquire an infinite mass, thus decoupling, also their coupling to the Higgs becomes large, leaving a
correct counterterm to the gauge divergences. It follows that in the careful limit we do not expect any new effective interactions.

A cancelation of this kind has been proven to happen in an abelian gauge model by Appelquist and Quinn [49].

Explicitly we impose the unitary gauge by the constraint

\[ F(\varphi) \equiv \varphi^{\text{charged}} = 0 \]  

(4.35)

It has no derivatives and acts on a field which transforms locally under the gauge group, hence it requires the pointwise Faddeev-Popov jacobian:

\[
\frac{\delta}{\delta \alpha(y)} F(\delta \alpha \varphi(x)) \Big|_{F=0} = [\phi, \cdot] \delta(x-y) = [\hat{\phi} + \varphi, \cdot] \delta(x-y) \]  

(4.36)

that gives the following functional determinant:

\[
\text{Det}[\mathcal{M}_{FP}] = \prod_x \prod_{i<j} (\hat{\phi}_i + \varphi_i - \hat{\phi}_j - \varphi_j)^2 = \prod_x \Delta^2(\phi(x)) \]  

(4.37)

where \( \Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j) \). Here a regularization has to be implicit to make sense of the infinite product.

However \( \text{Det}[\mathcal{M}_{FP}] \) can be exponentiated without the help of ghost fields thanks to the relation \( \text{Det}A = e^{\text{Tr} \log A} \) to yield an effective potential for the Higgs field:

\[
e^{\sum_x \sum_{i<j} \log(\phi_i - \phi_j)} = e^{\delta(0) \int d^3x \log \Delta^2(\phi(x))}.
\]  

(4.38)

Then, because the action is multiplied by a \(-1/\hbar\) factor, we need to multiply by a \(-\hbar\) factor, so that it ends up describing a one loop correction to the bare action (in the form of a repulsion of the \( \phi \) eigenvalues, as in matrix models).

In three dimensions this correction has a cubic divergent coupling constant and a non polynomial structure.

Together with this one loop correction, we have then to consider the other contributions from the propagating fluctuations, namely the one loop diagrams of gauge fields \( a_\mu \) and diagonal Higgs field \( \varphi \), to calculate, at one loop:

\[
\xi = N^{3/2} \int d^3R \int \mathcal{D}\varphi \int \mathcal{D}a_\mu e^{-\frac{1}{\hbar}\left[ S_{\text{cl}} + S_{\text{quadr}} - \hbar \delta(0) \int d^3x \log \Delta^2(\phi(x)) \right]}. \]  

(4.39)

\[\S\]

\(\S\) It is the same thing to consider the \( \xi \to \infty \) limit of the \( R_\xi \) background gauges of the last section, namely \( [\phi^0, \phi] = 0 \) or \( [\hat{\phi}, \varphi] = 0 \).

\(\S\) Now we would like, later, to exploit the fact that in the unitary gauge the Higgs field has only \( N \) components, while its effective action is still of order \( N^2 \). In the large \( N \) limit this implies that the fluctuations \( \varphi \) are suppressed, and \( \phi \) is in all respect a classical field.

In particular it will be possible to apply the saddle point method on its effective action:

\[
e^{-\Gamma[\varphi]} = \int \mathcal{D}a_\mu e^{-\frac{1}{\hbar}\left[ S_{\text{cl}} + S_{\text{quadr}} - \hbar \delta(0) \int d^3x \log \Delta^2(\phi(x)) \right]}. \]  

(4.40)

Of course this calculation is still not simple, because it depends on the classical field \( \hat{A} \).
To this aim, we recall the action of fluctuations

\[ S_{\text{quadr}} = N \text{tr} \int \left[ (\hat{D}_\mu, a_\nu) - (\hat{D}_\nu, a_\mu) \right]^2 + \frac{g}{2} \hat{G}_{\mu\nu}[a_\mu, a_\nu] + (g[a_\mu, \hat{\phi} + \varphi] + [\hat{D}_\mu, \varphi])^2 + \right. \\
\left. + i \varphi V''(\hat{\phi}) \varphi \right] \quad (4.41) \]

understanding that \( \varphi \) is diagonal.

All the effect that we want to discuss arises from the massive gauge fields circulating in a loop, so we calculate the divergent part of it.

Because the gauge propagator is constant at large momentum, the loop contributes with an arbitrary number of insertions of interaction terms.

The cubic interactions in the gauge or Higgs fields do not enter at one loop, and simultaneous interactions of two \( a_\mu \) with a Higgs field plus a background gauge give perturbative corrections to what we need, so we leave them apart.

From the above action (4.41) we take the relevant quadratic interaction terms of the fluctuating \( a^{ij}_\mu \) with the external fields:

\[ g^2 a^{ij}_\mu a^{ji}_\mu (2\hat{\phi}^\infty_i - 2\hat{\phi}^\infty_j + \varphi'_i - \varphi'_j)(\varphi'_i - \varphi'_j) \quad (4.42) \]

where the Higgs field \( \phi = \hat{\phi} + \varphi \) has been decomposed in a different way: the asymptotic constant field \( \hat{\phi}^\infty \) which regulates the gauge bosons mass, plus the remaining classical nonuniform background and fluctuating fields which have to be treated as a total external field \( \varphi' \): \( \phi = \hat{\phi}^\infty + \varphi' \).

The loop consists thus of the same charged \( a^{ij}_\mu \) field running along the loop with propagator

\[ \frac{-g_{\mu\nu}}{p^2 - m_{ij}^2} + \frac{1}{m_{ij}^2} \frac{p_\mu p_\nu}{p^2 - m_{ij}^2} \quad (4.43) \]

(with \( m_{ij} = g|\hat{\phi}^\infty_i - \hat{\phi}^\infty_j| \)) and an arbitrary number of insertions of \( v_{ij} = g^2(2\hat{\phi}^\infty_i - 2\hat{\phi}^\infty_j + \varphi'_i - \varphi'_j)(\varphi'_i - \varphi'_j) \).

We insert this \( v_{ij} \) at zero external momentum, because we are dealing with the divergent part. We have thus, for \( n \) insertions,

\[ \sum_{i < j} v_{ij}^n \int d^3p \text{tr} \left( \frac{-1}{p^2 - m_{ij}^2} + \frac{1}{m_{ij}^2} \frac{p \times p}{p^2 - m_{ij}^2} \right)^n = \delta(0) \sum_{i < j} \left( \frac{v_{ij}}{m_{ij}^2} \right)^n \quad (4.44) \]

There is a combinatorial factor which comes from the \((n-1)!\) ways to insert the interactions compared with the \( n! \) ways to attach the resulting counterterm. Hence the factor is \( 1/n \).

Summing up all these divergent contributions we reconstruct the logarithmic potential:

\[ \h \delta(0) \sum_{i \neq j} \sum_n \frac{1}{n} \left( \frac{g^2(2\hat{\phi}^\infty_i - 2\hat{\phi}^\infty_j + \varphi'_i - \varphi'_j)(\varphi'_i - \varphi'_j)}{m_{ij}^2} \right)^n = \]

\[ \h \delta(0) \sum_{i \neq j} \left( \log |\phi_i - \phi_j| - \log |\hat{\phi}^\infty_i - \hat{\phi}^\infty_j| \right) \quad (4.45) \]
It clearly cancels only part of the Faddeev-Popov Vandermonde determinant above in (4.39), and leaves the second term function of the Higgs vacuum only:

\[-\hbar \delta(0) \int \log \Delta^2(\hat{\phi}^\infty)\] (4.46)

Now, on one hand it is just a constant which is the same for all the topological sectors and can be absorbed in the normalization for what regards \(\xi\) and all fluctuations, on the other it can be thought as a potential correcting the vacuum constant value \(\hat{\phi}^\infty\).

This result needs some discussion. A similar mechanism is well known from the study of the perturbative corrections to the Higgs potential \([12]\): the quantum corrections keep a nonzero Higgs v.e.v. also in the limit of no bare breaking \(\mu \to 0\).

From this point of view it provides a repulsive potential for the eigenvalues of the \(\phi^\infty\) v.e.v., justifying the assumption of \(\hat{\phi}_i \neq \hat{\phi}_j\) for the vacuum value.

It may appear strange that we have obtained an effective potential with a divergent \(\delta(0) \simeq \Lambda^3\) constant, because it seems to be stronger than any of the renormalized other terms in the action, in the continuum limit.

Nonetheless the fact that it depends only on \(\hat{\phi}^\infty\) does not allows us to treat it like the other terms in the effective action. The scalar potential \(V(\phi)\) is of a very different nature because it depends on the fluctuations also. It can lead the field to attain its minimum, which has to be the vacuum for the system, in order to be stable against local perturbations.

The effective potential (4.46) is expected to play a role let’s say, just at the moment of symmetry breaking, when the sources should decide which direction in the Cartan space to choose. It is at this stage that the Vandermonde potential is important and the quantum theory requires one unique direction with nondegenerate Higgs field.

Eventually one must leave to external sources just the discrete choice among the \(N(N-1)\) possible vacuums related by Weyl.

All this analysis is independent of the large \(N\) limit, but for our dilute gas of monopoles, we must would like to know \(\phi^\infty\). In the next paragraph we will find it according to the above discussion, for large \(N\).

### 4.6 The Higgs vacuum

The vacuum field \(\phi^\infty\) plays an important role in the dilute gas picture, because it decides if the monopoles are relevant to confinement.

According to the discussion about the unitary gauge in last section, the vector of eigenvalues \(\phi^\infty \in \mathbb{R}^{N-1}\) is defined by the minimum of

\[-\sum_{i<j} \log |\phi_i^\infty - \phi_j^\infty|\] (4.47)
with the constraint

$$\sum_i (\phi_i^\infty)^2 = \mu^2$$

(4.48)

and we recall that \(\mu^2\) is of order \(N\), so that the components of \(\phi^\infty\) are of order 1.

The solution for finite \(N\), although existing, is not illuminating. We instead turn to the large \(N\) limit and introduce the (non standard) density of eigenvalues:

$$\rho(\phi^\infty) = \left( N \frac{d\phi_i^\infty}{di} \right)^{-1}$$

(4.49)

It is of order 1, as a consequence of last constraint.

We want to solve for it to obtain the distribution of eigenvalues of the vacuum Higgs field. The equation for \(\rho(x)\) is, from (4.47), (4.48):

$$N \cdot P \int_{-a}^{a} \rho(x) \int_{-y}^{y} dx = 2\lambda x$$

(4.50)

where the right hand side is the Lagrange multiplier for the constraint (4.48) which is equivalent to \(\int x^2 \rho(x) dx = \mu\).

In the large \(N\) limit the Lagrange multiplier is negligible (we do not expect an attraction of eigenvalues coming from this constraint, because \(\mu^2 \sim N\)) and we have to solve:

$$P \int_{-a}^{a} \rho(x) \int_{-y}^{y} dx = 0$$

(4.51)

The by now standard method [50] is to introduce the resolvent of \(\rho\) as

$$F(x) = P \int_{-a}^{a} \rho(y) \int_{-y}^{y} dx$$

(4.52)

which has the following properties: It is analytic out of the cut \([-a, a]\) on the real axis; it goes to zero at infinity as \(\frac{1}{|x|}\); It is real on the real axis \([-a, a]\) excluded; near the cut it has zero real part and a discontinuity in the imaginary part given by the unknown \(\pi \rho(x)\).

The unique function with these requirements is

$$F(x) = C \frac{1}{\sqrt{x^2 - a^2}}$$

(4.53)

from which we finally read (and normalize) the distribution \(\rho(x)\):

$$\rho(x) = \frac{1}{\pi} \frac{1}{\sqrt{a^2 - x^2}}$$

(4.54)

The result is thus an inverted semicircle law.
Its domain is defined by the constraint (4.48) (we have introduced the fixed scale \( \tilde{\mu}^2 = \mu^2 / N \)):

\[
N \int_{-a}^{a} x^2 \rho(x) dx = N \tilde{\mu}^2
\]

which gives:

\[
a^2 = \frac{2}{\pi} \tilde{\mu}^2 \quad (4.56)
\]

\( \rho(x) \) of (4.54) represents the following Higgs configuration in the large \( N \) limit:

\[
\phi_i^\infty = \sqrt{\frac{2}{\pi}} \tilde{\mu} \cos\left(\pi \frac{i}{N}\right).
\]

Of course there are \( N! \) equivalent configurations related by Weyl permutations. We show the ordered ones in the picture.

Let’s make a few comments on this result:

- First, in the large \( N \) limit we find that the Higgs eigenvalues remain of order one, while in the usual picture there is no indication and one could also assign all the vacuum expectation value \( \mu \) to a single eigenvalue.

- Second, the Weyl degeneracy is completely broken by \( \phi^\infty \), and no symmetry remains apart from the abelian \( U(1)^{N-1} \).

- Third, and more important, our vacuum value (4.57) shows a peculiar characteristic, namely that, near the edges of the distribution, differences of eigenvalues are of order \( \frac{1}{N^2} \). As a consequence the masses of gauge bosons in the \( SU(2) \) sectors near the two ends are vanishing as \( 1/N^2 \). In the middle one has the normally expected \( 1/N \) masses.
This facts means that there will be an infinite number of very light monopoles, with masses vanishing as \(1/N\).

The implications on the semiclassical picture of monopole gas are interesting, and we will deal with them in the next section.

On the other hand, independently of the monopole gas, the presence of an infinity of massless modes can lead to a new phase of the model, of course present only in the \(N = \infty\) sector.

Let us mention also a nice correspondence with the solvable cases of \(\mathcal{N} = 2 \rightarrow \mathcal{N} = 1\) supersymmetric gauge theories in four dimensions.

In the \(SU(N)\) generalization of the \((\mathcal{N} = 1)SYM_4\), as analyzed by Douglas and Shenker [22], the \(\phi^\infty\) field is represented by the points in the moduli space of the \(\mathcal{N} = 2\) theory which become the vacua of the theory broken to \(\mathcal{N} = 1\).

The exact solution shows that those points are exactly of the form of our \(\phi^\infty\). In particular the Weyl degeneracy is completely broken, and moreover the different charged sectors will have masses of gauge bosons and of monopoles given again by \(\cos(\pi \frac{1}{N}) - \cos(\pi \frac{1}{N})\), for example \(\simeq 1/N^2\) near the edges of the distribution.

We remark that in the supersymmetric theory the ground state appears in a full nonperturbative and geometric way, while in our case it must be stressed that we are not dealing exactly with quantum corrections, but with a global factor in the Higgs measure, expected to be important just during the formation of the system.

A main feature, the hierarchy of masses, is the same, and this is at least curious.

**4.7 The hierarchy of masses and monopole degeneracies**

Here we come to the observation that in the large \(N\) limit there is a hierarchy of masses which arises from the model, once given the distribution of Higgs eigenvalues. As \(N\) gets large, many of the objects become very light, potentially bringing in a different physics, like happens in other cases in this limit.

We have the gauge bosons and monopole pseudo-masses

\[
m_W(q) = g|\phi^\infty \cdot q|
\]

\[
S_{cl}(q) = 4\pi N \frac{|\phi^\infty \cdot q|}{g}
\]

which are function of the \(SU(2)\) root \(q\) which specifies the color sector. Explicitly as a function of the Higgs eigenvalues

\[
|\phi^\infty \cdot q| = g \frac{1}{\sqrt{2}} |\phi_\alpha^\infty - \phi_\beta^\infty| \tag{4.58}
\]

when \(q\) is the charge in the sector \((ij)\).

The possible values are \(N(N - 1)\) and we see that according to the distribution of eigenvalues in the Higgs field, they range in the interval \((\mu/N^2, 1)\).
Moreover at the ends of the Higgs domain, ±µ, where the eigenvalues concentrate, we have masses of order \(1/N^2\), while from the standard differences we get \(\sim N^2\) masses of order \(1/N\) and more.

We can look at the distribution of differences:

\[
\sigma(d) = \int \int dx dy \rho(x) \rho(y) \delta(|x-y| - d)
\]

\[
= 4 \int_{-a}^{d/2} dt \frac{1}{\sqrt{(a^2 - t^2)(a^2 - (t+d)^2)}}
\]

which is logarithmic singular for \(d \to 0\) showing the phenomenon:

\[
\sigma(d) \simeq \frac{2}{\pi^2} \frac{1}{a} \log a/d.
\]

because masses are proportional to \(d\) this is also the distribution of masses in the model near the edges.

The monopole pseudo-mass \(S_{cl}\) has a \(N\) factor which compensates this vanishing in the middle part of the distribution, but still many monopoles tend to have zero action in the limit.

4.8 Dilute gas in the large \(N\) limit

We will use these informations in the dilute gas ensemble of section 4.2.

We have to perform the average of \(\xi(q)\) in the subspace of the simple roots which are not orthogonal to a given weight \(m_\alpha\). These are of the form: \(q = m_\alpha \pm m_\beta (\beta \neq \alpha)\). Then

\[
\bar{\xi}(m_\alpha) = \frac{1}{(N-1)} \sum_{\beta \neq \alpha} \xi(m_\alpha - m_\beta)
\]

For large \(N\) the average \(\bar{\xi}\) has the form:

\[
\bar{\xi} \sim N^{9/2} \int dx \rho(x) \frac{m_W^{9/2}}{g^3} A e^{-Nm_W^2} \]

which leads to the following average

\[
N^{9/2} A \int_{-a}^{a} dx \frac{1}{\sqrt{a^2 - x^2}} \frac{((x-b)\bar{\mu})^{9/2}}{g^3} e^{-N(x-b)\bar{\mu}}
\]

where the large \(N\) limit has to be performed.

Here the first observation is that the exponential in the large \(N\) limit becomes a delta function (times a \(1/N\) factor).

But then the \((x-b)^{9/2}\) prefactor drives the integral to zero. In fact the integral is expanded and factors of \(1/N\) appear leaving derivatives of the delta function.
However we know that also the factor $A$ depends on $N$, being currently an open problem.

In case $A$ will turn out not to grow faster than $N^2$, the string tension will be suppressed and vanish in the large $N$ limit. This could be explained noticing that also in the large $N$ limit, as in the BPS case, all the Higgs fields become massless. This important peculiarity distinguishes between the large $N$ YM $H$ and pure YM theory at $\lambda > 0$, whereas for $SU(2)$ the Higgs field just provides an explicit scale for the monopole stability. A complete discussion will be available by the time the nontrivial quantity $A$ in the one monopole determinant will be studied.

## A A remark on the Gribov problem

Here we want to make some remarks on the Gribov ambiguity in the present case where the unitary gauge is chosen.

It turns out that no Gribov problem is present, and the unitary gauge is thus just a feasible gauge with some unusual properties. Moreover the absence of unphysical fields, and the fact that the Higgs field is already diagonal, makes it a good tool to investigate the quantum theories.

The main point observation in the Gribov problem is that the Faddeev-Popov determinant carries the information on how good is our choice of gauge. It represents the local jacobian for the change of variables from the connection space to the quotient by the gauge group action where we have our physical theory defined.

At points in functional space where the FP determinant vanishes, it means that we are not taking a complete quotient, in fiber bundle language the section we have chosen becomes tangent to the fiber along some gauge direction.

These points usually mark what is called the Gribov horizon, the name because it is the boundary of the maximal region where a section can be continued.

Here, to be concrete, we have in three dimensions a gauge connection and a matter field so the space of fields $\Gamma$ is that of couples $(A_\mu, \phi)$, functions from $\mathbb{R}^3$ to the gauge algebra. As we have seen $\Gamma$ is disconnected in components according to the total magnetic charge, as described in section 2.

In every component acts the gauge group $\mathcal{G} = \{ g(x) : \mathbb{R}_3 \to SU(N) \}$, and no boundary conditions have to be imposed at infinity because by homotopic arguments gauge transformations connected to the identity do not change component.

This is best and more appropriately seen in the regular gauges, while in the unitary gauge we know that $A_\mu$ becomes singular along some Dirac string (still with $\mathcal{S}_{cl} < \infty$). Nevertheless the unitary gauge has nice properties, namely the FP determinant and the gauge fixing only depend on the Higgs field, so that one can draw some conclusion.

The unitary gauge introduced in section 4.5.2, has a Faddeev-Popov determinant
which turns out to be
\[ \prod_x \prod_{ij} (\phi_i(x) - \phi_j(x))^2. \tag{A.1} \]

Now we recall, as remarked at the end of in section 2, that in nontrivial sectors of the Higgs winding at infinity, \( \phi \) has necessarily some coinciding eigenvalues at some point.

This is proven in any regular gauge but is valid also in other gauges because it’s a gauge invariant statement.

In the semiclassical picture of monopole gas, at each monopole location two (or more) eigenvalues coincide.

Hence we find that in the unitary gauge the FP determinant above seems to vanish identically for any nontrivial configuration. More suggestively one can think that the Gribov horizon is made of monopole configurations. The same happens at each field configuration throughout the whole nontrivial sectors.

This would mean that the unitary gauge is an ill defined section, in that it does not fix the gauge. Explicitly it would leaves intact some gauge subgroup at the points in space where we have a monopole.

However there are two points which solve this seemingly bad problem.

- The determinant in the form above is ill defined. It requires us to live in a distribution space, whereas we usually consider smooth functions. After this remark it appears evident that no gauge invariance remains unfixed, because the smoothness constrains the gauge variation at the "origin" to follow that in the neighbor. So there is no such thing as the gauge variation at a single point, even if the theory has local invariance.

- In subsection 4.5.2 we proved that the Faddeev-Popov determinant and its vanishing is canceled by the gauge loops, so that if the jacobian vanishes it is just because the change of variables is singular, and in fact at the same time the integral takes care of this and diverges by the same amount so to correct the measure.

This is different from the standard Gribov phenomenon where the vanishing of the FP determinant is a unavoidable problem in the context of a renormalizable theory.
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