COHEN-LENSTRA PROBABILITY MEASURE OVER $\mathbb{F}_q[t]$

GILYOUNG CHEONG

Abstract. A celebrated conjecture of Cohen and Lenstra says that given an odd prime $p$, a fixed finite abelian $p$-group $H$ occurs as the $p$-part of the class group of a random imaginary quadratic field over $\mathbb{Q}$ with a probability inversely proportional to $\#\text{Aut}(H)$, the size of the automorphism group of $H$. They proved that an analogous statement is true if we compute the same probability but take the $p$-part of a random finite abelian group, if each finite abelian group has probability of occurrence equal to the inverse of the size of its automorphism group. We prove a similar result where we replace finite abelian groups with finite $\mathbb{F}_q[t]$-modules and $p$-parts with $(t-a)$-parts, where $q$ is a prime power and $a$ is any element of the finite field $\mathbb{F}_q$ of size $q$. In proving our result, we describe a concrete sample space from which we pick a random $(t-a)\infty$-torsion module. When $q=p$ is a prime and $n$ goes to infinity, our distribution matches with that of Cohen and Lenstra.

Throughout the paper, we fix an arbitrary prime (number) $p$ and a prime power $q$. We do not necessarily assume that $p$ divides $q$, although we do not prohibit this case either. We denote by $\mathbb{F}_q$ the finite field of size $q$.

1. Introduction

In their influential paper [CL1983], Cohen and Lenstra conjectured that if $p$ is odd, then given a finite abelian $p$-group $H$, the probability that $H$ arises as the $p$-part of the class group of a random imaginary quadratic field over $\mathbb{Q}$ is

$$\frac{1}{\#\text{Aut}(H)} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^3} \right) \cdots,$$

where the word “random” means that we consider all the quadratic imaginary fields with a fixed upper bound for their absolute discriminants to compute the probability and then let the bound tend to infinity. The following notation is quite standard when it comes to discussing the $p$-part of a finite abelian group (or equivalently, a $\mathbb{Z}$-module):

Notation 1.1. Given any abelian group $A$, we write

$$A[p\infty] := \{ a \in A : p^N a = 0 \text{ for } N \gg 0 \},$$

and call it the $p$-part or the $p\infty$-torsion of $A$. The terminology is the most standard when $A$ is a finite abelian group. More generally, given any ideal $I$ in a commutative ring $R$ and an $R$-module $A$, we define

$$A[I\infty] := \{ a \in A : I^N a = 0 \text{ for } N \gg 0 \},$$

the $I$-part of $A$ or the $I\infty$-torsion of $A$. When $I$ is principal, say $I = (t)$, then we interchangeably write $A[t\infty]$ to mean $A[I\infty]$. We say that $A$ is an $I\infty$-torsion module if $I^N A = 0$ for $N \gg 0$.

The conjecture of Cohen and Lenstra comes from the following result of their own:

Proposition 1.2 (Example 5.9. in [CL1983], $u = 0$ and $\#\mathfrak{P}_1 = 1$). Let be $H$ a finite abelian $p$-group. Given $N \geq 1$, let $\text{Ab}_{\leq N}$ denote the set of isomorphism classes of finite abelian groups of size $\leq N$. Then
More generally, let $R$ be a Dedekind domain with finite residue fields, $p$ a nonzero prime ideal of $R$, and $H$ a finite $p^\infty$-torsion $R$-module. Given $N \geq 1$, let $\text{Mod}_{\leq N}^R$ denote the set of isomorphism classes of finite $R$-modules of size $\leq N$. Then we have

$$
\lim_{N \to \infty} \frac{\sum_{[A] \in \text{Ab}_{\leq N}, A[p^\infty] \cong H} 1/\#\text{Aut}(A)}{\sum_{[A] \in \text{Ab}_{\leq N}} 1/\#\text{Aut}(A)} = \frac{1}{\#\text{Aut}(H)} \left( 1 - \frac{1}{p^1} \right) \left( 1 - \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^3} \right) \cdots .
$$

Remark 1.3. Here, we give motivation for our work. A priori, there does not seem to be any natural reason why we are considering each isomorphism class $[A]$ with the weight $1/\#\text{Aut}(A)$. However, there is surprising evidence in numerical computations about the distribution of class groups in [CL1983], which matches the right-hand side of the above. Moreover, for some Dedekind domains $R$, the result above is no longer true when we replace $\text{Mod}_{\leq N}^R$ with $\text{Mod}_{\leq N}^R$, the set of isomorphism classes of $R$-modules with size $N$. For example, take $R = \mathbb{Z}$ and $H = \mathbb{Z}/(p)$, and we took $p = (p)$. If we take the sequence of prime numbers $p < p_1, p_2, p_3, \ldots$, there is no $p$-part of any finite abelian group of size $p_1$, so the analogous probability is 0 if $N$ only takes the subsequence $p_1, p_2, p_3, \ldots$ in the limit. However, when $N$ takes the subsequence $p_1, p_2, p_3, \ldots$ in the limit, with $H = \mathbb{Z}/(p)$, the analogous probability is 1. Hence, the limit cannot exist for $\text{Mod}_{\leq N}^R = \text{Ab}_{\leq N}$, the set of $\mathbb{Z}$-modules (or abelian groups) of size $N$, as $N \to \infty$.

1.1. Main result. The main work of this paper is to show that the two issues in Remark 1.3 do not occur when $R = \mathbb{F}_q[t]$ and $p = (t-a)$ for any $a \in \mathbb{F}_q$. Namely, we give a concrete sample space of finite $\mathbb{F}_q[t]$-modules of fixed size that naturally gives the weighted probability inversely related to the size of the automorphism groups of the modules.

Theorem 1.4. Fix $a \in \mathbb{F}_q$. Recall that any finite $(t-a)^\infty$-torsion $\mathbb{F}_q[t]$-module is of the form

$$
H_{(t-a), \lambda} := \mathbb{F}_q[t]/(t-a)^{\lambda_1} \oplus \cdots \oplus \mathbb{F}_q[t]/(t-a)^{\lambda_l},
$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition with length $l(\lambda) = l$. We have

$$
\text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(A[(t-a)^\infty] \cong H_{(t-a), \lambda}) = \frac{\sum_{[A] \in \text{Mod}_{\geq n}^{\mathbb{F}_q[t]}, A[(t-a)^\infty] \cong H_{(t-a), \lambda}} 1/\#\text{Aut}_{\mathbb{F}_q[t]}(A)}{\sum_{[A] \in \text{Mod}_{\geq n}^{\mathbb{F}_q[t]}} 1/\#\text{Aut}_{\mathbb{F}_q[t]}(A)} = \begin{cases} 
\frac{1}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{(t-a), \lambda})} & \left( 1 - \frac{1}{q^{\lambda_1}} \right) \left( 1 - \frac{1}{q^{\lambda_2}} \right) \cdots \left( 1 - \frac{1}{q^{\lambda_l}} \right) \text{ if } n \geq |\lambda|, \\
0 & \text{if } n < |\lambda|.
\end{cases}
$$

We now discuss how Theorem 1.4 lets us avoid issues pointed out in Remark 1.3. First, the notation

$$
|\lambda| := \lambda_1 + \cdots + \lambda_l,
$$

is standard for partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$. Every finite $\mathbb{F}_q[t]$-module is a vector space over $\mathbb{F}_q$ of a finite dimension $n$, so its size must be of the form $q^n$, so considering only $\text{Mod}_{\geq n}^{\mathbb{F}_q[t]}$ in the theorem is reasonable. We denoted by $\text{Mat}_n(\mathbb{F}_q)$ the set of $n \times n$ matrices over $\mathbb{F}_q$. Each matrix $A \in \text{Mat}_n(\mathbb{F}_q)$ can be thought of as a finite $\mathbb{F}_q[t]$-module by giving $\mathbb{F}_q^n$ the $\mathbb{F}_q[t]$-action $t \cdot v := Av$ for $v \in \mathbb{F}_q^n$, so we can think of $\text{Mat}_n(\mathbb{F}_q)$ as a sample space of $\mathbb{F}_q[t]$-modules of size $q^n$ or equivalently, of dimension $n$ over $\mathbb{F}_q$, where each matrix (or module) gets an equal probability $1/q^n$ of occurrence. We explain how this sample space $\text{Mat}_n(\mathbb{F}_q)$ is related to the weight $1/\#\text{Aut}_{\mathbb{F}_q[t]}(A)$ by answering the following question:
Question 1.5. Given an \( \mathbb{F}_q[t] \)-module \( X_0 \) of dimension \( n \) over \( \mathbb{F}_q \), what is the probability that a random module (i.e., matrix) in \( \text{Mat}_n(\mathbb{F}_q) \) is isomorphic to \( X_0 \)?

We may think of \( X_0 \) as a matrix in \( \text{Mat}_n(\mathbb{F}_q) \). Consider the conjugate action of the group \( \text{GL}_n(\mathbb{F}_q) \) on \( \text{Mat}_n(\mathbb{F}_q) \), and denote by \( \text{GL}_n(\mathbb{F}_q) \cdot X_0 \) the orbit of this action containing \( X_0 \). A crucial observation is that the stabilizer subgroup of any \( A \in \text{Mat}_n(\mathbb{F}_q) \) is equal to the automorphism group of \( \mathbb{F}_q[t] \)-module (which we also use \( A \) to write) that \( A \) defines:

\[
\text{Aut}_{\mathbb{F}_q[t]}(A) = \{ g \in \text{GL}_n(\mathbb{F}_q) : g(tv) = tg(v) \text{ for all } v \in A \} = \{ g \in \text{GL}_n(\mathbb{F}_q) : g(Av) = Ag(v) \text{ for all } v \in \mathbb{F}_q^n \} = \{ g \in \text{GL}_n(\mathbb{F}_q) : gA = Ag \} = \{ g \in \text{GL}_n(\mathbb{F}_q) : gAg^{-1} = A \}.
\]

By the orbit-stabilizer theorem, we have

\[
\frac{#(\text{GL}_n(\mathbb{F}_q) \cdot X_0)}{#\text{Mat}_n(\mathbb{F}_q)} = \frac{1}{#\text{Aut}_{\mathbb{F}_q[t]}(X_0)},
\]

so the desired probability is equal to

\[
\frac{#(\text{GL}_n(\mathbb{F}_q) \cdot X_0)}{#\text{Mat}_n(\mathbb{F}_q)} = \frac{#(\text{GL}_n(\mathbb{F}_q) \cdot X_0) / #\text{GL}_n(\mathbb{F}_q)}{#\text{Mat}_n(\mathbb{F}_q) / #\text{GL}_n(\mathbb{F}_q)} = \frac{\sum_{[A] \in \text{Mod}_{\mathbb{F}_q[t]}^n} #(\text{GL}_n(\mathbb{F}_q) \cdot A) / #\text{GL}_n(\mathbb{F}_q)}{1 / #\text{Aut}_{\mathbb{F}_q[t]}(X_0)}.
\]

This means that the distribution of \( \mathbb{F}_q[t] \)-modules in \( \text{Mat}_n(\mathbb{F}_q) \) is precisely the distribution of \( \mathbb{F}_q[t] \)-modules in \( \text{Mod}_{\mathbb{F}_q[t]}^n \), each of whose member gets the probability inversely weighted to the size of its automorphism group. That is, our observation along with Theorem 1.4 shows that for \( R = \mathbb{F}_q[t] \), the issues we have discussed in Remark 1.3 for \( R = \mathbb{Z} \) are no longer present. Notice the similarities between the numerical quantities given in Theorem 1.4 and the conclusion of Proposition 1.2. In particular, if \( q = p \), taking \( n \to \infty \) in Theorem 1.4 gives the same numerical quantity present in the first part of Proposition 1.2.

2. Related works

The work of Cohen and Lenstra [CL1983] stimulated a great deal of mathematical research. We mention a few of them here, with some relevant context. Some of the exposition here is from a lecture by Wood [Woo2016].

2.1. Cokernel of a random \( p \)-adic matrix. Given a quadratic imaginary field \( K \) of \( \mathbb{Q} \), the class group \( \text{Cl}_K \) is supported on a set \( S = \{ p_1, \ldots, p_n \} \) of finitely many nonzero primes of \( \mathcal{O}_K \), the integral closure of \( \mathbb{Z} \) in \( K \). We may construct an exact sequence

\[
\mathcal{O}_K^{S, \times} / \text{torsions} \to I_K^S \to \text{Cl}_K \to 1
\]

of abelian groups, where

\[
\mathcal{O}_K^{S, \times} := \{ f \in K^{\times} : f\mathcal{O}_K = p_1^{e_1} \cdots p_n^{e_n} \text{ for some } e_i \in \mathbb{Z} \}
\]
and $I_K^S$ is the free abelian group generated by $p_1, \ldots, p_n$ (written multiplicatively). Since $K$ is imaginary quadratic, by Dirichlet’s unit theorem, the rank of $G_K^{S, \times}$ is precisely $n = \#S$, so writing additively, our exact sequence becomes

$$Z^n \rightarrow Z^n \rightarrow \text{Cl}_K \rightarrow 0.$$ 

Since $\text{Cl}_K$ is finite, applying the right-exact functor $(-) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to the above sequence, where $\mathbb{Z}_p$ is the ring of $p$-adic integers, we get

$$Z^n \rightarrow Z^n \rightarrow \text{Cl}_K[p^\infty] \rightarrow 0,$$

an exact sequence of $\mathbb{Z}_p$-modules. Therefore, if $\text{Cl}_K$ were “uniformly distributed” (as we vary $K$ with a fixed bound for their absolute discriminants) among cokernels of $\mathbb{Z}$-linear maps $Z^n \rightarrow Z^n$, one may expect that $\text{Cl}_K[p^\infty]$ are uniformly distributed among cokernels of $\mathbb{Z}_p$-linear maps $Z^n \rightarrow Z^n$, with respect to the Haar probability measure $\mu_n$ on $\text{Mat}_n(\mathbb{Z}_p)$. The computation of such distribution is due to Friedman and Washington:

**Proposition 2.1** (Proposition 1 of Section 3 in [FW1989]). Recall that a finite abelian $p$-group is of the form $G_{p, \lambda} := \mathbb{Z}/(p)^{\lambda_1} \oplus \cdots \oplus \mathbb{Z}/(p)^{\lambda_l}$ where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of length $l = l(\lambda)$. We have

$$\mu_n(A \in \text{Mat}_n(\mathbb{Z}_p) : \text{coker}(A) \simeq G_{p, \lambda}) = \begin{cases} \frac{1}{\#\text{Aut}(G_{p, \lambda})} \prod_{i=1}^l \left(1 - \frac{1}{p^i}\right) \prod_{j=n-l(\lambda)+1}^n \left(1 - \frac{1}{p^j}\right) & \text{if } n \geq l(\lambda), \\ 0 & \text{if } n < l(\lambda), \end{cases}$$

where $\mu_n$ is the unique Haar measure on $\text{Mat}_n(\mathbb{Z}_p) = \mathbb{Z}_p^{n^2}$ with $\mu_n(\text{Mat}_n(\mathbb{Z}_p)) = 1$.

Denoting the set of isomorphism classes of finite abelian $p$-groups by $\text{Ab}_p$, Proposition 2.1 gives a measure $\text{Ab}_p$ by assigning to $[G_{p, \lambda}]$ the probability given by the right-hand side of the statement. We will use $\mu_n$ to denote this measure, although we also used it to mean the Haar measure on $\text{Mat}_n(\mathbb{Z}_p)$. The measure on $\text{Ab}_p$ given by the point-wise limit $\mu := \lim_{n \rightarrow \infty} \mu_n$ is what literature refers to as Cohen-Lenstra (probability) measure (e.g., see Section 8.1 of [EVW2016]). Fulman and Kaplan [FK2018] studies the measure $\mu_n$ (and its generalization) quite extensively. A work in progress by the author, Yifeng Huang, and Zhan Jiang studies how $\mu_n$ is related to the probability given by Theorem 1.3 Lengler [Len2010] pointed out another interesting distribution objects that follows Cohen-Lenstra measure, which is a result in Fulman’s PhD thesis:

**Proposition 2.2** (Corollary 5 in [Ful1997]). Fix any monic irreducible polynomial $P_0(t) \neq t$ in $\mathbb{F}_q[t]$ and a partition $\lambda$. Then

$$\lim_{n \rightarrow \infty} \text{Prob}_{A \in \text{GL}_n(\mathbb{F}_q)}(A[P_0^\infty] \simeq H_{P_0, \lambda}) = \frac{1}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{P_0, \lambda})} \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^2}\right) \left(1 - \frac{1}{q^3}\right) \cdots.$$

In particular, when $P_0(t) = t - 1$, the above result coincides with Theorem 1.3 when $n$ goes to infinity. Motivated by this, the author, Yifeng Huang, and Zhan Jiang investigate what happens if we replace $t - a$ with any monic irreducible $P_0(t) \in \mathbb{F}_q[t]$ for Theorem 1.3 in the same work in progress, mentioned above.

Wood [Woo2018] proves a much stronger result than Proposition 2.1, when $n$ goes to infinity, by considering a general random matrix in $\text{Mat}_n(\mathbb{Z}_p)$, whose entries are given by independent random variables in $\mathbb{Z}_p$ with a mild condition. Liu and Wood [LW2017] provides a non-abelian version of Cohen-Lenstra measure, including non-abelian groups and even infinite random groups.
3. Cycle index

To prove Theorem 1.4 we use a generating function that encodes data of similarity classes of matrices in Mat$_n(F_q)$ for various $n \geq 0$ (where $n = 0$ corresponds to the trivial vector space). In literature, such a generating function is called a “cycle index” (to be defined), and the name comes from the fact that the first generating function of this kind is given to encode information about conjugacy classes of symmetric groups $S_n$ for various $n \geq 0$, which is equivalent to information about the cycle types of permutations of $n$ letters. A more thorough explanation to the historical context is available in the first section of Fulman’s paper [Ful1999] about cycle index.

As we pointed out in the previous section, every matrix $A \in$ Mat$_n(F_q)$ gives $F_q[t]$-modules structure on $F_q^n$, and up to isomorphism, it is

$$H_{P_\nu, \lambda^{(i)} \oplus \cdots \oplus \lambda^{(r)}}.$$

In the above, we used the following notation:

$$H_{P_\nu, \lambda^{(i)} \oplus \cdots \oplus \lambda^{(r)}} = H_{(P_\nu), \lambda^{(i)}} := F_q[t]/(P_i(t))^{\lambda_i^{(i)}} \oplus \cdots \oplus F_q[t]/(P_i(t))^{\lambda_i^{(i)}}$$

where $P_i(t) \in F_q[t]$ are monic irreducible polynomials and $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_{\ell_i}^{(i)})$ are nonempty partitions, if $n \geq 1$. For $n = 0$, the only choice is the trivial module, which corresponds to the empty partition $\emptyset$. These $H_{P_\nu, \lambda^{(i)}}$ only depend on the similarity class of $A$, and they also characterize the class. Given any monic irreducible $P = P(t) \in F_q[t]$, we write $\mu_P(A)$ to mean the partition given by the $P^\infty$-torsion of $A$. For example, given the above notation, we have

$$\mu_P(A) = \lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_{\ell_i}^{(i)})$$

and $\mu_P(A) = \emptyset$, the empty partition if $P \neq P_i$ for all $i$. We say $\mu_P(A)$ is the shape of $A$ at $P$. Write $|A^1_{F_q}| = |\text{Spec}(F_q[t])|$ to mean the set of all monic irreducible polynomials in $F_q[t]$. The notation comes from the fact that it corresponds to the maximal ideals of $F_q[t]$, which are precisely the closed points of the prime spectrum Spec($F_q[t]$) of the ring $F_q[t]$. For each nonempty partition $\nu$ and $P \in |A^1_{F_q}|$, we consider a formal variable $x_{P, \nu}$. For the empty partition $\emptyset$, we put $x_{P, \emptyset} := 1$.

From the structure theorem about finitely generated modules over $F_q[t]$, which is a PID (Principal Ideal Domain), and the Chinese Remainder theorem, we note that for any two matrices $A, B \in$ Mat$_n(F_q)$, the following are equivalent:

1. $A$ and $B$ are similar;
2. $A$ and $B$ give isomorphic $F_q[t]$-module structures on $F_q^n$;
3. $A$ and $B$ have the same shape everywhere in $|A^1_{F_q}|$ (i.e., $\mu_P(A) = \mu_P(B)$ for all $P \in |A^1_{F_q}|$); and
4. $\prod_{P \in |A^1_{F_q}|} x_{P, \mu_P(A)} = \prod_{P \in |A^1_{F_q}|} x_{P, \mu_P(B)}$.

Writing the sequence of formal variables $x_{P, \nu}$ as $\mathbf{x} = (x_{P, \nu})$, we define the cycle index of $\text{GL}_\bullet(F_q) = \{\text{GL}_n(F_q)\}_{n \in \mathbb{Z}_{\geq 0}}$ to be

$$\sum_{n=0}^\infty \mathcal{Z}(\text{GL}_n(F_q), \mathbf{x})u^n \in \mathbb{Q}[\mathbf{x}][[u]],$$

where

$$\mathcal{Z}(\text{GL}_n(F_q), \mathbf{x}) := \frac{1}{\#\text{GL}_n(F_q)} \sum_{A \in \text{GL}_n(F_q)} \prod_{P \in |A^1_{F_q}|} x_{P, \mu_P(A)}.$$
Note that $GL_0(F_q) = \text{Mat}_0(F_q)$ is the singleton of the identity $F_q$-linear automorphism of the trivial vector space. Writing $\text{Mat}_n(F_q) = \{\text{Mat}_n(F_q)\}_{n \in \mathbb{Z}_{\geq 0}}$, the relative cycle index of $(\text{Mat}_\bullet(F_q), GL_\bullet(F_q))$ is defined to be

$$\sum_{n=0}^{\infty} Z_{GL_n(F_q)}(\text{Mat}_n(F_q), x) u^n \in \mathbb{Q}[x][[u]],$$

where

$$Z_{GL_n(F_q)}(\text{Mat}_n(F_q), x) := \frac{1}{\#GL_n(F_q)} \sum_{A \in \text{Mat}_n(F_q)} \prod_{P \in |A|_{F_q}} x_{P, \mu(P)}(A).$$

Note that the relative cycle index is defined with a sum over $\text{Mat}_n(F_q)$, whereas the cycle index is defined with a sum over a smaller set $GL_n(F_q)$. The key result we use for proving Theorem 1.4 is due to Stong, who introduced the relative cycle index:

**Lemma 3.1** (Lemma 1 in [Sto1988]). We have

$$\sum_{n=0}^{\infty} Z_{GL_n(F_q)}(\text{Mat}_n(F_q), x) u^n = \sum_{n=0}^{\infty} \sum_{A \in \text{Mat}_n(F_q)} \left( \prod_{P \in |A|_{F_q}} x_{P, \mu(P)}(A) \right) u^n = \prod_{P \in |A|_{F_q}} \sum_{\nu \in \#\text{Aut}_{F_q}[t]} x_{P, \nu}(H_{P, \nu}),$$

where $\nu$ runs over all the partitions (including the empty one).

In proving Theorem 1.4 we will also use the following corollary to the above identity, originally due to Kung, who introduced the cycle index for $GL_\bullet(F_q)$:

**Lemma 3.2** (Lemma 1 in [Kun1981]). We have

$$\sum_{n=0}^{\infty} Z(GL_n(F_q), x) u^n = \sum_{n=0}^{\infty} \sum_{A \in GL_n(F_q)} \left( \prod_{P \in |A|_{F_q}} x_{P, \mu(P)}(A) \right) u^n = \prod_{P \in |A|_{F_q}} \sum_{\nu \in \#\text{Aut}_{F_q}[t]} x_{P, \nu}(H_{P, \nu}),$$

where $\nu$ runs over all the partitions (including the empty one).

**Proof of Lemma 3.2** If we take $x_{t, \nu} = 0$ for every nonempty partition $\nu$ in the expression

$$Z_{GL_n(F_q)}(\text{Mat}_n(F_q), x) = \frac{1}{\#GL_n(F_q)} \sum_{A \in \text{Mat}_n(F_q)} \prod_{P \in |A|_{F_q}} x_{P, \mu(P)}(A),$$

we get $Z(GL_n(F_q), x)$ because any square matrix is invertible if and only if it does not have 0 eigenvalue (or equivalently, if it does not have any invariant factor divisible by $t$). Hence, applying the same evaluation in the right-hand side of the identity in Lemma 3.1 we obtain the result. □
3.1. Proof of Lemma 3.1. We also provide a proof of Lemma 3.1 which is from Lemma 1 in Stol1988. To our best knowledge, this argument is essentially due to Kung (Lemma 1 in Kun1981), which is used to prove Lemma 3.2. We excerpt this proof from the mentioned references for the sake of completeness. Furthermore, the reader can observe that the proof of Lemma 3.1 also explicitly shows where the weight inverse to the size of automorphism groups come into play, which also appeared when we answered Question 1.5.

Proof of Lemma 3.1. Fix \( n \geq 0 \), monic irreducibles \( P_1, \ldots, P_r \in \mathbb{A}_q^1 \), and partitions \( \nu^{(1)}, \ldots, \nu^{(r)} \) such that \( |\nu^{(1)}| \deg(P_1) + \cdots + |\nu^{(r)}| \deg(P_r) = n \). We compute the \( \mathbb{Q} \)-coefficient of \( x_{P_1,\nu^{(1)}} \cdots x_{P_r,\nu^{(r)}} u^n \) from each side of the identity we want to establish in the statement of Lemma 3.1. When \( n = 0 \), both sides gives 1, so let us assume that \( n \geq 1 \). For the left-hand side, we count the monomials that are equal to \( x_{P_1,\nu^{(1)}} \cdots x_{P_r,\nu^{(r)}} u^n \) by considering how many ways to put an \( F_q[t] \)-module structure on \( \mathbb{F}_q^n \) so that it is isomorphic to \( H_{P_1,\nu^{(1)}} \oplus \cdots \oplus H_{P_r,\nu^{(r)}} \), where we used the notation defined in Section 3. We first count the number of ways to decompose \( \mathbb{F}_q^n \) into \( V_1 \oplus \cdots \oplus V_r \), where \( V_i \) is an \( F_q \)-vector space of dimension \( |\nu^{(i)}| \deg(P_i) \) and then we count the number of \( F_q[t] \)-module structures on each \( V_i \) isomorphic to \( H_{P_i,\nu^{(i)}} \). The first count is

\[
\frac{\#GL_n(\mathbb{F}_q)}{\#GL_{|\nu^{(1)}| \deg(P_1)}(\mathbb{F}_q) \cdots \#GL_{|\nu^{(r)}| \deg(P_r)}(\mathbb{F}_q)}
\]

and the second count is

\[
\frac{\#(GL_{|\nu^{(1)}| \deg(P_1)}(\mathbb{F}_q) \cdot A_{P_1,\nu^{(1)}}) \cdots \#(GL_{|\nu^{(r)}| \deg(P_r)}(\mathbb{F}_q) \cdot A_{P_r,\nu^{(r)}})}{\#GL_{|\nu^{(1)}| \deg(P_1)}(\mathbb{F}_q) \cdots \#GL_{|\nu^{(r)}| \deg(P_r)}(\mathbb{F}_q)} = \frac{\#GL_{|\nu^{(1)}| \deg(P_1)}(\mathbb{F}_q) \cdots \#GL_{|\nu^{(r)}| \deg(P_r)}(\mathbb{F}_q)}{\#Aut_{F_q[t]}(H_{P_1,\nu^{(1)}}) \cdots \#Aut_{F_q[t]}(H_{P_r,\nu^{(r)}})}
\]

where \( A_{P,\nu} \) is any matrix in Mat_{|\nu| \deg(P)}(\mathbb{F}_q) that gives \( \mathbb{F}_q[\nu \deg(P)] \mathbb{F}_q \) the \( F_q[t] \)-module structure isomorphic to \( H_{P,\nu} \). We used the notation from our answer to Question 1.5 and the orbit-stabilizer theorem.

Therefore, the coefficient of \( x_{P_1,\nu^{(1)}} \cdots x_{P_r,\nu^{(r)}} u^n \) on the left-hand side is equal to

\[
1 \cdot 1 \cdot \frac{\#GL_n(\mathbb{F}_q)}{\#GL_{|\nu^{(1)}| \deg(P_1)}(\mathbb{F}_q) \cdots \#GL_{|\nu^{(r)}| \deg(P_r)}(\mathbb{F}_q)} = \frac{\#GL_{|\nu^{(1)}| \deg(P_1)}(\mathbb{F}_q) \cdots \#GL_{|\nu^{(r)}| \deg(P_r)}(\mathbb{F}_q)}{\#Aut_{F_q[t]}(H_{P_1,\nu^{(1)}}) \cdots \#Aut_{F_q[t]}(H_{P_r,\nu^{(r)}})}
\]

which is the coefficient of \( x_{P_1,\nu^{(1)}} \cdots x_{P_r,\nu^{(r)}} u^n \) on the right-hand side. This finishes the proof.  

4. Proof of Theorem 1.4

In this section, we provide a proof of Theorem 1.4 using Lemma 3.1 and Lemma 3.2.

Proof of Theorem 1.4. Without loss of generality, we assume that the partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is nonempty (i.e., \( l > 0 \)). First, notice that for any monic irreducible polynomial \( P_0 = P_0(t) \in \mathbb{F}_q[t] \), we have

\[
\sum_{\{A\} \in \text{Mod}_{\mathbb{F}_q[t]}^{[n], \lambda}, A[P_0] \simeq H_{P_0,\lambda}} 1/\#Aut_{\mathbb{F}_q[t]}(A) = \frac{\sum_{\{A\} \in \text{Mod}_{\mathbb{F}_q[t]}^{[n], \lambda}, A[P_0] \simeq H_{P_0,\lambda}} \#(GL_n(\mathbb{F}_q) \cdot A)/\#GL_n(\mathbb{F}_q)}{\sum_{\{A\} \in \text{Mod}_{\mathbb{F}_q[t]}^{[n], \lambda}} \#(GL_n(\mathbb{F}_q) \cdot A)/\#GL_n(\mathbb{F}_q)} = \frac{\sum_{\{A\} \in \text{Mod}_{\mathbb{F}_q[t]}^{[n], \lambda}, A[P_0] \simeq H_{P_0,\lambda}} \#(GL_n(\mathbb{F}_q) \cdot A)}{\#Mat_n(\mathbb{F}_q)} = \frac{\text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(A[P_0] \simeq H_{P_0,\lambda})}{\#Mat_n(\mathbb{F}_q)}.
\]
so the only nontrivial part of Theorem 1.4 is to prove the second equality.

Next, we reduce the statement to the case when \( a = 0 \). Fix any monic polynomial \( f(t) \in \mathbb{F}_q[t] \) with \( \deg(f) \leq n \). For any matrix \( A \in \text{Mat}_n(\mathbb{F}_q) \) and a subspace \( V \subset \mathbb{F}_q^n \) of dimension equal to \( \deg(f) \), we have \( f(A|_V) = 0 \) if and only if \( f((A + aI)|_V) = 0 \), where \( I \) is the \( n \times n \) identity matrix. This implies that \( f(t) \) is an invariant factor of \( A \) if and only if \( f(t - a) \) is an invariant factor of \( A + aI \). Considering the case where \( f(t) \) is a power of \( t \), we have

\[
\text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(A[(t - a)\infty]) \simeq H_{(t-a),\lambda} = \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}((A + aI)((t - a)\infty]) \simeq H_{(t-a),\lambda} = \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(A[t\infty] \simeq H_{(t),\lambda}).
\]

We have (for example, from (1.6) on p.181 of [Mac1995]) that

\[
\#\text{Aut}_{\mathbb{F}_q[t]}(H_{(t),\lambda}) = \#\text{Aut}_{\mathbb{F}_q[t]}(H_{(t-a),\lambda}),
\]

so we may apply the statement of Theorem 1.4 with \( a = 0 \) to obtain the general statement for any \( a \in \mathbb{F}_q \). Hence, we can now assume that \( a = 0 \).

In Lemma 3.3, take \( x_{P,\nu} = 1 \) on both sides for all \( P(t) \neq t \) to get

\[
\sum_{n=0}^{\infty} \sum_{A \in \text{Mat}_n(\mathbb{F}_q)} \frac{x_{\mu(A)(t)}}{\#\text{GL}_n(\mathbb{F}_q)} u^n = \left( \sum_{\nu} \frac{x_{\mu,\nu}[\nu]}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{(t),\lambda})} \right) \left( \prod_{P(t) \neq t} \sum_{\nu} \frac{u[\nu]\deg(P)}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{P,\nu})} \right).
\]

Now, take \( x_{t,\nu} = 0 \) for all nonempty \( \nu \neq \lambda \) and \( x_{t,\lambda} = 1 \). Then

\[
1 + \sum_{n=1}^{\infty} \left( \frac{\#\{A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \text{ or empty} \}}{\#\text{GL}_n(\mathbb{F}_q)} \right) u^n = \left( 1 + \frac{u[\lambda]}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{(t),\lambda})} \right) \left( \prod_{P(t) \neq t} \sum_{\nu} \frac{u[\nu]\deg(P)}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{P,\nu})} \right) = \left( 1 + \frac{u[\lambda]}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{(t),\lambda})} \right) (1 + u + u^2 + \cdots).
\]

For the last equality above, we used Lemma 3.2,

\[
\sum_{n=0}^{\infty} \sum_{A \in \text{GL}_n(\mathbb{F}_q)} \frac{\prod_{P \in [\mathbb{F}_q]} x_{P,\mu(A)(P)} u^n}{\#\text{GL}_n(\mathbb{F}_q)} = \prod_{P(t) \neq t} \sum_{\nu} \frac{u[\nu]\deg(P)}{\#\text{Aut}_{\mathbb{F}_q[t]}(H_{P,\nu})},
\]

where we took all \( x_{P,\nu} = 1 \). Hence, reading the coefficients of \( u^n \) in the previous identity, we have

\[
\frac{\#\{A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \}}{\#\text{GL}_n(\mathbb{F}_q)} + 1 = \frac{\#\{A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \}}{\#\text{GL}_n(\mathbb{F}_q)},
\]

if \( n \geq |\lambda| \). We have also used the following observation: \( \mu_A(t) \) is empty (i.e., \( A \) does not have 0 has an eigenvalue) precisely when \( A \) is invertible. In addition, if \( n < |\lambda| \), then reading the coefficients of \( u^n \) gives

\[
\frac{\#\{A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \}}{\#\text{GL}_n(\mathbb{F}_q)} = 1,
\]

which just means \( \#\{A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \} = 0 \).
Thus, if \( n \geq |\lambda| \), then

\[
\frac{\# \{ A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \}}{\# \text{Mat}_n(\mathbb{F}_q)} = \frac{1}{\# \text{Aut}_{\mathbb{F}_q[t]}(H(t),\lambda)} \frac{1}{\# \text{GL}_n(\mathbb{F}_q)} = \frac{1}{\# \text{Aut}_{\mathbb{F}_q[t]}(H(t),\lambda)} \left( 1 - \frac{1}{q} \right) \left( 1 - \frac{1}{q^2} \right) \cdots \left( 1 - \frac{1}{q^n} \right),
\]

whereas if \( n < |\lambda| \), then

\[
\frac{\# \{ A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \}}{\# \text{Mat}_n(\mathbb{F}_q)} = 0.
\]

Since

\[
\text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(A[t^n] \simeq H(t),\lambda) = \frac{\# \{ A \in \text{Mat}_n(\mathbb{F}_q) : \mu_A(t) = \lambda \}}{\# \text{Mat}_n(\mathbb{F}_q)},
\]

this finishes the proof. \( \square \)

5. FURTHER DIRECTIONS

Lastly, we address some further works in progress mentioned in Section 2 and possible directions in the near future. Another way to look at Theorem 1.4 is as follows: given \( P_0 \in |\mathbb{A}_q^3| \) with \( \deg(P_0) = 1 \) and any \( P_0 \)-torsion \( \mathbb{F}_q[t] \)-module \( H_{P_0,\lambda} = \mathbb{F}_q[t]/(P_0(t))^{\lambda_1} \oplus \cdots \oplus \mathbb{F}_q[t]/(P_0(t))^{\lambda_l} \), we have

\[
\text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(A[P_0^n] \simeq H_{P_0,\lambda}) = \frac{\sum_{[A] \in \text{Mod}_{\mathbb{F}_q[t]}^{P_0^n} : A[P_0^n] \simeq H_{P_0,\lambda}} 1/\# \text{Aut}_{\mathbb{F}_q[t]}(A)}{\sum_{[A] \in \text{Mod}_{\mathbb{F}_q[t]}^{P_0^n}} 1/\# \text{Aut}_{\mathbb{F}_q[t]}(A)} = \begin{cases} 
\frac{1}{\# \text{Aut}_{\mathbb{F}_q[t]}(H_{P_0,\lambda})} \left( 1 - \frac{1}{q} \right) \left( 1 - \frac{1}{q^2} \right) \cdots \left( 1 - \frac{1}{q^n} \right) & \text{if } n \geq |\lambda| \deg(P_0) \\
0 & \text{if } n < |\lambda| \deg(P_0).
\end{cases}
\]

It is natural to ask what happens when \( \deg(P_0) > 1 \). Of course, the case \( n < |\lambda| \deg(P_0) \) evidently gets the same answer as above because \( \dim_{\mathbb{F}_q}(H_{P_0,\lambda}) = |\lambda| \deg(P_0) \). A simple application of the orbit-stabilizer theorem shows that we get the same answer for the case \( n = |\lambda| \deg(P_0) \) regardless of \( \deg(P_0) \). However, for any \( P_0 \in |\mathbb{A}_q^3| \) with \( \deg(P_0) > 1 \), it turns out that the resulting distribution is not the same as above, although we get a striking similarity to the above one as \( n \to \infty \). This is a joint work in progress by the author, Yifeng Huang, and Zhan Jiang, as mentioned in Section 2.

We also recall Proposition 2.2 which is a statement about a distribution of \( \mathbb{F}_q[t] \)-modules given by a random matrix in \( \text{GL}_n(\mathbb{F}_q) \) instead of \( \text{Mat}_n(\mathbb{F}_q) \). To our best knowledge, it is still not known whether there is any reasonable connection between Proposition 2.2 and Theorem 1.4 except the numerical matching when \( n \to \infty \). One possible approach is to observe that

\[
\text{Prob}_{A \in \text{GL}_n(\mathbb{F}_q)}(A[0^n] \simeq H_{(t-1),\lambda}) = \text{Prob}_{A \in \text{GL}_n(\mathbb{F}_q) - I}(A[t^n] \simeq H(t),\lambda),
\]

where \( I \) is the \( n \times n \) identity matrix and \( \text{GL}_n(\mathbb{F}_q) - I := \{ A - I : A \in \text{GL}_n(\mathbb{F}_q) \} \). When one tries to replace \( \text{Mat}_n(\mathbb{F}_q) \) with \( \text{GL}_n(\mathbb{F}_q) - I \) in the proof of Theorem 1.4 in Section 4, there are many places that the same proof does not go through, but based on numerical evidence, errors must sum up to a quantity that goes to 0 as \( n \to \infty \). Therefore, it would be interesting to see any explicit computation for the left-hand side of Proposition 2.2 when \( P_0(t) = t - 1 \) for various fixed \( n \geq 1 \).
6. Acknowledgments

This work is initiated as a part of the author’s dissertation, which is partly supported by Rackham one-term dissertation fellowship. The sincere gratitude goes to Karen Smith, Michael Zieve, and Rackham Graduate School at the University of Michigan, who recommended and granted the fellowship. The author also thanks Brad Rodgers and Ofir Gorodetsky for introducing cycle index, Yifeng Huang, Harry Richman, and Zhan Zhang for helpful discussions, Nathan Kaplan and Yuan Liu for thoughtful comments on the older version of this paper, and Cat Tompkins for proofreading.

References

[CL1983] Henri Cohen and Hendrik W. Lenstra, Jr., Heuristics on class groups of number fields, in Number theory, Noordwijk 1983 (Noordwijk, 1983), Lecture Notes in Math. 1068, Springer-Verlag, New York, 1984, pp. 33-62.

[FW1989] Eduardo Friedman and Lawrence C. Washington, Divisor class groups of curves over a finite field, in Théorie des Nombres (Quebec, PQ, 1987), de Gruyter, Berlin, 1989, pp. 227-239.

[Ful1997] Jason Fulman, Probability in the classical groups over finite fields: symmetric functions, stochastic algorithms and cycle indices, PhD thesis, Harvard University, 1997.

[Ful1999] Jason Fulman, Cycle indices for the finite classical groups, in Journal of Group Theory 2 (1999), 251-289.

[FK2018] Jason Fulman and Nathan Kaplan, Random partitions and Cohen-Lenstra heuristics, arXiv preprint: 1803.03722

[Kun1981] Joseph Kung, The cycle structure of a linear transformation over a finite field, Linear Algebra Appl. 36 (1981), 141-155.

[LW2017] Yuan Liu and Melanie Matchett Wood, The free group on n generators modulo n + u random relations as n goes to infinity, arXiv preprint: 1708.08509

[Len2010] Johannes Lengler, The Cohen-Lenstra heuristic: Methodology and results, Journal of Algebra, 323 (2010) 2960-2976.

[Mac1995] Ian G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., The Clarendon Press, Oxford, 1995.

[Neu1999] Jürgen Neukirch, Algebraic Number Theory, Springer, 1999

[Sto1988] Richard Stong, Some asymptotic results on finite vector spaces, Advances in Applied Mathematics 9, 167-199 (1988).

[Woo2015] Melanie Matchett Wood, Random integral matrices and the Cohen Lenstra Heuristics, arXiv preprint: 1504.04391

[Woo2016] Melanie Matchett Wood, Nonabelian Cohen-Lenstra heuristics and function field theorems, Seminar talk: Joint IAS/Princeton University Number Theory Seminar, available at: https://video.ias.edu/puias/2016/1117-MelanieWood

[EVW2016] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland, Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields, Annals of Mathematics, 183 (2016), 729-786

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109-1043, USA

E-mail address: gcheong@umich.edu