Charged string solutions with dilaton and modulus fields

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Abstract

We find charged, abelian, spherically symmetric solutions (in flat space-time) corresponding to the effective action of $D = 4$ heterotic string theory with the scale dependent dilaton $\phi$ and modulus $\varphi$ fields. We take into account perturbative (genus-one), moduli-dependent ‘threshold’ corrections to the coupling function $f(\phi, \varphi)$ in the gauge field kinetic term $f(\phi, \varphi)F_{\mu\nu}^2$, as well as non-perturbative scalar potential $V(\phi, \varphi)$, e.g., induced by gaugino condensation in the hidden gauge sector. Stable, finite energy, electric solutions (corresponding to on abelian subgroup of a non-abelian gauge group) have the small scale region as the weak coupling region ($\phi \rightarrow -\infty$) with the modulus $\varphi$ slowly varying towards smaller values. Stable, finite energy, abelian magnetic solutions exist only for a specific range of threshold correction parameters. At small scales they correspond to the strong coupling region ($\phi \rightarrow \infty$) and the compactification region ($\varphi \rightarrow 0$). The non-perturbative potential $V$ plays a crucial role at large scales, where it fixes the asymptotic values of $\phi$ and $\varphi$ to be at the minimum of $V$.

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1. Introduction

One of the basic features of string theory is the existence of scalar fields, such as dilaton and moduli. The latter ones are string modes in a vacuum associated with compactification of extra dimensions. In the simplest cases moduli correspond to the ‘radii’ of compact dimensions. These fields are natural partners of the metric and thus should play an important role in string gravitational physics.

Moreover, it is a generic property of Kaluza-Klein-type theories, and thus also of string theory, that scalar fields couple to the Maxwell and Yang-Mills kinetic terms of the gauge fields. In particular, in string theory, the dilaton field determines the strength of the gauge couplings at the tree level of the effective action, while string one-loop (genus-one) contributions give moduli dependent corrections to such couplings. Thus, in general, a scalar function $f$ that couples to the gauge field kinetic energy is a function of both the dilaton as well as the moduli.

The dilaton and the moduli have no potential in the effective action to all orders in string loops. To avoid a contradiction with observations they should acquire masses. Currently proposed scenarios rely on non-perturbatively induced potential $V$ due to gaugino condensation in the hidden gauge group sector\textsuperscript{1} Such a potential would generate masses for the dilaton and the moduli and at the same time provide a mechanism of supersymmetry breaking. While these scalar fields eventually get masses, it is very important to appreciate the fact that they may change with distance (or time, in the cosmological context) at small scales (or times). In other words, such fields may participate in the dynamics at small scales (or times), i.e., in the region where non-perturbative effects presumably can be neglected.

String solutions are usually discussed in perturbation theory in $\alpha'$ (string tension); occasionally it is possible to go beyond the $\alpha'$ expansion by identifying an exact conformal

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\textsuperscript{1} Other non-perturbative string effects may lead to contributions to the potential $V$ which have different functional dependencies on the dilaton $S$, e.g., $V \sim e^{-\sqrt{S}}$ \textsuperscript{2}. See also \textsuperscript{3} and references therein.
field theory which corresponds to a given tree-level solution. For example, new charged black hole string solutions have been recently obtained by taking into account the tree level coupling of the dilaton to the gauge fields (for a review see ). However, there are very few discussions in the literature where perturbative (e.g., genus-one, moduli dependent threshold corrections to the gauge couplings), or non-perturbative string effects (e.g., non-perturbatively induced potential for the dilaton and moduli) are taken into account. Such corrections may substantially modify the tree level solutions and it is thus important to include them in order to understand predictions of string theory.

The change of the gauge coupling function from the naive exponential dilaton factor in the three level action to a more realistic one, as found in string perturbation theory, may have a fundamental influence on the previously discussed solutions. Addressing the case of a general gauge coupling was one of the motivations for the present paper. One should thus consider the following improvements of the previous approaches: (i) A more general form of the dilaton potential should be taken into account. Up to now, solutions with the dilatonic mass term and/or a choice of convex potential were studied. One should analyse subtleties associated with the fact that starting from the weak coupling region the ‘gaugino condensation’ potential always has a concave region, i.e., the potential grows as the coupling increases. (ii) The dynamics of the modulus field was ignored. The modulus, however, also couples to the gauge field strength due to the one-loop threshold corrections. It also appears, along with the dilaton, in the non-perturbatively induced potential. Therefore, one may not a priori set any of the two fields equal to a constant. (iii) A possibility of additional non-perturbative dilatonic terms in was not considered.

There were discussions of cosmological solutions in the presence of a non-perturbative dilaton potential (see e.g. and refs. therein). Aspects of charged dilatonic black hole solutions with non-perturbatively induced dilaton mass included were addressed in . However, in the latter case, the potentials for the dilaton were not always taken to be ‘realistic’ or well motivated from the point of view of non-perturbative dynamics like gaugino condensation in the hidden gauge sector of the gauge group.
The present work is a step towards clarifying some of these issues. In this paper, however, we shall ignore the gravitational dynamics, thus postponing a study of black hole–type solutions for future work. We shall consider abelian electric and magnetic solutions in flat space concentrating on the role of the non-trivial functions $f$, a coupling function of scalars to the gauge field kinetic term, and $V$, a non-perturbative potential for the dilaton and the moduli.

The solutions we shall find should have generalisations to the curved space. We shall assume that they approximate the exact solutions of the whole set of equations (including the gravitational one) in the region where the curvature is small. As in the case of solitonic solutions in field theory in flat space one can ignore gravitational effects if the scale of the solutions is large compared to the gravitational scale ($\sim E/M_P^2$) (i.e. if the energy of the solutions is small enough). It is true that in the absence of a non-perturbative potential the dilaton and the metric are on an equal footing. Once the potential is generated, it introduces a new scale (different from the Planck one). This makes it possible in principle to ‘disentangle’ the metric from the dilaton and to consider dilatonic solutions in flat space with a characteristic scale being larger than the gravitational one.

In our study we shall include the dilaton $\phi$ and, for the sake of simplicity, only one modulus field $\varphi$, which is associated with an overall compactification scale. We shall look for stable spherically symmetric finite-energy solutions with a regular gauge field strength in flat $D = 4$ space-time. We shall consider a general class of functions $f$ and $V$. We will treat examples with $f$ modified by the string loop corrections, as they appear in a class of orbifold-type compactifications, and with the non-perturbative potential $V$ due to the gaugino condensation in the hidden sector of the theory, as special cases. We shall not include higher derivative terms, assuming that the fields change slowly in space.

We shall find that the abelian electric solutions are regular, have finite energy, and

\footnote{Note that regular, finite energy spherically symmetric solutions exist in the tree-level ($f = e^{-2\phi}, V = 0$) dilaton-Yang-Mills system, but are, in general, unstable (they are similar to the regular but unstable solutions of the Einstein-Yang-Mills system found in [16]). The simplest abelian solutions of this class are, in fact, stable and are the obvious limiting cases of the gravitational solutions of [4][3].}
are stable when the abelian subgroup is embedded in a non-abelian gauge group. They have the effective string coupling $e^\phi$ increasing from zero at the origin ($r = 0$) to a finite value $e^{\phi_0}$ at $r = \infty$. The asymptotic value $\phi_0$ of the dilaton corresponds to the minimum of the potential $V$. Thus the small distance region is a weak coupling region and can be studied ignoring non-perturbative corrections. The large distance region corresponds to the ‘observed’ world where the dilaton is trapped in the minimum of $V$. The modulus field $\varphi$ is slowly varying with $r$; at large scales it is fixed at the minimum of the potential $V$, while at small scales its value decreases slightly. Generic existence of such particle-like, finite energy, charged configurations may have potentially interesting applications.

Stable, regular, finite energy, abelian magnetic solutions exist for a certain range of threshold correction parameters. Here at small distances ($r \to 0$) the dilaton approaches the strong coupling region, $e^\phi \to \infty$, while the modulus goes to zero, $\varphi \to 0$, i.e., the small scale region is the compactification region. The role of the non-perturbative potential is again to fix the asymptotic values of $\phi_0$ and $\varphi_0$ to be at its minimum.

The plan of this paper is the following. In Section 2 we discuss the string effective action with the dilaton and the modulus field and its modification due to the perturbative as well as non-perturbative corrections. In Section 3 we study general properties of the solutions without specifying particular functions $f$ and $V$. In Section 4 we turn to specific examples (with only one of the two fields changing with $r$) with particular choices for the function $f$, but ignoring the potential ($V = 0$). In Section 5 we consider the case when both the dilaton and the modulus field are included in $f$, while the potential is still zero. The modifications of the solutions due to the non-perturbative potential are considered in Section 6. Section 7 contains a summary and concluding remarks.

2. Structure of low-energy string effective action

2.1. Perturbative terms in the effective action

The leading terms (in the derivative expansion) in the low-energy, $D = 4$ effective
The action of the heterotic string theory have the form

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2\partial_\mu \phi \partial^\mu \phi - 2\partial_\mu \varphi \partial^\mu \varphi - f(\phi, \varphi)F_{\mu\nu}F^{\mu\nu} - V(\phi, \varphi) + \ldots \right]. \quad (2.1) \]

Here \( g = |\text{det}g_{\mu\nu}| \), \( R \) is the scalar curvature and \( F_{\mu\nu} \) is the abelian gauge field strength. For simplicity we consider, along with the dilaton field \( \phi \), only one modulus field \( \varphi \), associated with an overall scale of compactification and ignore in the most part of the paper the axionic partners \((\alpha, \beta)\) of \( \phi \) and \( \varphi \) as well as other matter fields. The standard scalar fields \((S, T)\) of chiral multiplets of \( N = 1 \) supergravity, corresponding to the dilaton and the modulus are \([17]\)

\[ S = e^{-2\phi} + i\alpha, \quad T = e^{2\varphi/\sqrt{3}} + i\beta. \quad (2.2) \]

At the tree level

\[ f_{\text{tree}} = e^{-2\phi}, \quad V_{\text{tree}} = c e^{2\phi} = 0, \quad (2.3) \]

where \( c \) is the central charge deficit.

In the case of the supersymmetric \( D = 4 \) heterotic string \( c = 0 \) and thus \( V \) remains zero to all orders in the string perturbation theory. On the other hand, the gauge coupling function \( f \) receives a non-trivial, \( \varphi \)-dependent, string one-loop (genus-one) correction \([1]\). Thus:

\[ f_{\text{perturb}} = e^{-2\phi} + f_2(\varphi), \quad V_{\text{perturb}} = 0. \quad (2.4) \]

The modulus dependent function \( f_2 \) depends on a type of superstring vacuum one is considering. In particular, for toroidal compactifications and a class of orbifolds, it is invariant under the duality symmetry \((\varphi \rightarrow -\varphi)\) and can be schematically written in the form:

\[ f_2(\varphi) = b_0 \ln \left[(T + T^*)|\eta(T)|^4\right] + b_1, \quad T = e^{2\varphi/\sqrt{3}}. \quad (2.5) \]

Here \( \eta(T) \) is the Dedekind function (modular function of weight \(-1/2\)). It turns out that \( \ln \left[(T + T^*)|\eta(T)|^4\right] \) is always negative, has a maximum at \( \varphi = 0 \) (\( T = 1 \)) and

\[ ^4 \text{We use the space-time signature } (-+++) \text{ and the gravitational constant } G = 1. \]
approaches $-\frac{\pi}{3} T = -\frac{\pi}{3} e^{2\varphi/\sqrt{3}}$ as $\varphi \to \infty$. The important property of $f_2$ is its duality symmetry $f_2(\varphi) = f_2(-\varphi)$. The constant $b_0$ is related to the one-loop $\beta$-function coefficients associated with the $N = 2$ subsector of the massless spectrum in a symmetric orbifold compactification.\footnote{Generically $b_0 = O(1/100)$ and is negative (positive) in the case of the abelian (non-abelian) gauge group factors, although examples with reversed signs were also found \cite{19}. There is also a constant, moduli-independent and non-universal contribution \cite{20} to the gauge coupling threshold correction term, denoted as $b_1$ in eq. (2.5). According to \cite{20} $b_1 = O(1/100)$ and it is positive (negative) for the abelian (non-abelian) case. Thus, in general $f_2(\varphi)$ is positive (negative) in the abelian (non-abelian) case. In Figure 1 the function $f_2(\varphi)$ with $b_0 = -1$ and $b_1 = 0$ is plotted.}

Both $f$ and $V$ may, in principle, receive as well corrections which are non-perturbative in string coupling. Since the string coupling is related to the dilaton, both $f$ and $V$ may contain non-perturbative contributions which are non-trivial functions of $\phi$. In fact, a priori separate non-perturbative factors may appear in the kinetic term of the dilaton, in the gauge field term and in the potential so that after a field redefinition not only $V$, but also $f$ may contain non-trivial dilaton dependence. Such terms in $f$ do not actually appear in the proposed gaugino condensation scenario for supersymmetry breaking. One should, however, bear in mind, that the origin of supersymmetry breaking in string theory is not well understood. In view of that one should allow for a possibility that $f$ may contain additional, non-perturbative terms which depend on $\phi$, e.g., $\exp[-k \exp(-2\phi)]$ (implying $1/g^2 \to 1/g^2 + a e^{-k/g^2}$).

\footnote{The coefficient $b_0$ contains in general a contribution due to the mixed Yang-Mills–sigma model anomaly \cite{18}. In special cases, i.e., of $Z_3$, $Z_7$ orbifolds, the total value of $b_0$ turns out to be zero. In most cases, however, e.g., $Z_4$, $Z_6$ etc. orbifolds, $f_2$ depends on the contribution of a modulus associated with one two-torus and not on a modulus associated with an overall scale of six-torus. In the following, we consider a symmetric contribution to $f_2$ which depends on an overall modulus. A study with a modulus associated with one two-torus only can be done analogously and should yield the same qualitative features.}
2.2. Non-perturbative scalar potential

We now summarize the properties of the non-perturbative potential for the dilaton and the moduli fields. A detailed structure of a non-perturbative potential \( V(\phi, \varphi) \) depends on a particular mechanism of supersymmetry breaking. We shall describe the form of \( V \) due to the gaugino condensation in the hidden sector of the gauge group [21][22][23][24][25][1][26].

The \( N = 1 \) supergravity potential can be written in terms of a K"ahler function

\[
G = K + \ln |W|^2 ,
\]

where \( K \) is the Kähler potential, and \( W \) is the superpotential. In the case of gaugino condensation \( G \) is a separable function, \( i.e., \)

\[
G = G_1(S, S^*) + G_2(T, T^*) ,
\]

where \( G_1(S, S^*) \) depends on the dilaton \( S \), and \( G_2(T, T^*) \) depends on the modulus \( T \) only. The potential is then of the form:

\[
V(S, T) = e^G \left( G_{SS^*}^{-1} |G_S|^2 + G_{TT^*}^{-1} |G_T|^2 - 3 \right) ,
\]

where \( G_S = \partial G / \partial S \), \( G = \partial^2 G / \partial S \partial S^* \), etc. In the case of symmetric orbifolds with the compactification moduli of all three two-tori equal to \( T \) the Kähler functions are of the following form [23][24][25]:

\[
G_1(S, S^*) = - \ln (S + S^*) + \ln |H(S)|^2 , \quad G_2(T, T^*) = -3 \ln (T + T^*) - 6 \ln |\eta(T)|^2 .
\]

Here

\[
H(S) = \sum_{i=1}^{J} d_i e^{-a_i S} , \quad a_i = \frac{3}{2b_{0i}} ,
\]

where \( J \) is a number of gaugino condensates and \( b_{0i} \) are the (one-loop \( N = 1 \)) \( \beta \)-functions of the gauge group factors of the hidden gauge group sector. Inserting (2.8) in the potential (2.7) one finds [23][24][25]

\[
V_{\text{non-perturb.}} \equiv V(S, T) = \frac{|H|^2}{|\eta(T)|^{12} S_R T_R^2} \left[ |S_R \partial \ln H / \partial S - 1|^2 + \frac{3}{4\pi^2} T_R^2 |\hat{G}_2(T)|^2 - 3 \right] ,
\]

\[6 \text{ For general superpotentials generating duality invariant potentials see [27].}\]
where $S_R = 2\text{Re}S$, $T_R = 2\text{Re}T$, and

$$
\hat{G}_2(T) = G_2(T) - 2\pi T_R^{-1} = -4\pi \frac{1}{\eta(T)} \frac{\partial \eta(T)}{\partial T} - 2\pi T_R^{-1}
$$

is the Eisenstein function of weight 2. It has zeros at $T = 1$ and $T = e^{i\pi/6}$, the respective $Z_2$ and $Z_3$ symmetric points of the fundamental domain of the $SL(2, \mathbb{Z})$ modular group.

This potential vanishes in the weak coupling limit $S = e^{-2\phi} \to \infty$ and has an extremum in $S$ if $S_R \frac{\partial W}{\partial S} - W = 0$. A minimum exists if $J > 1$, i.e., in cases with more than one gaugino condensate. Since the condition for a minimum in $S$ is $G_S = H^{-1} \partial H/\partial S - S_R^{-1} = 0$, the dilaton sector does not break supersymmetry. As for the extrema in $T$, $\partial V/\partial T = 0$, they are achieved at the self-dual points $T = 1$ and $T = e^{i\pi/6}$, which are saddle points of $V$, and at $T \sim 1.2$, which is the minimum of $V$. Interestingly, both $f$ and $V$ have an extremum in $T$ at the points $T = 1$ and $T = e^{i\pi/6}$ (zeros of $\hat{G}_2$), since $\partial f/\partial T = -\frac{b_0}{\pi} \hat{G}_2$ and $\partial V/\partial T \propto \hat{G}_2$ (see [28][23][24][25][27]). At these points $V$ preserves supersymmetry in the $T$ sector (note that $G_T \propto \hat{G}_2$). At a fixed, extremal value of $T$ and a fixed $\text{Im} S$,

the potential $V$ can be represented in the following form

$$
V = S_R^{-1} \sum_{i=1}^{J} e^{-a_i S_R (c_i + d_i S_R + e_i S_R^2)} , 
$$

(2.11)

where $a_i$, $c_i$, $d_i$, and $e_i$ are constants and $S_R = 2e^{-2\phi}$. For example, in the case of two gaugino condensates, $J = 2$, this potential starts from zero in the weak coupling region $\phi \to -\infty$, grows and reaches a local maximum, then decreases to a local minimum (with negative value of $V$), then has the second local maximum and finally goes to $-\infty$ at $\phi \to +\infty$. Since the potential has a local minimum in $\phi$, it may fix the value of the dilaton.

\footnote{In order to have a minimum of $V$ for a finite value of $\text{Re}S$, in such a minimum one should have, in general, $\text{Im} S \neq 0$.}

\footnote{This would give a mass to the fluctuating part of the dilaton. A generic property of this potential is that starting from a weak coupling region it first increases and has a local maximum and only then decreases to a minimum, namely, the potential is not convex everywhere. Another problem is that the value of the potential at the minimum, i.e. the effective cosmological constant, is negative in general. A local minimum with a zero cosmological constant can be achieved in the case with more than two gaugino condensates. Note, however, that in this case there are usually also other minima with negative cosmological constants (see e.g., [11]).}
3. General properties of charged solutions in flat space-time

3.1. Equations of motion

In the following we shall study charged solutions with non-trivial dilaton $\phi$ and the modulus field $\varphi$ in flat four dimensional ($D = 4$) space-time, i.e., $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_3^2$. We shall look for spherically symmetric, static solutions with non-trivial abelian gauge field strength. The same approach applies also to solutions associated with an abelian subgroup of a non-abelian gauge group. We shall refer to such solutions as abelian solutions embedded in a non-abelian theory.

The relevant part of the action (2.1) has the form

$$ S = \frac{1}{4\pi} \int d^4 x \left\{ -\frac{1}{2} (\partial \Phi_i)^2 - \frac{1}{4} f(\Phi_i) F_{\mu\nu} F^{\mu\nu} - V(\Phi_i) \right\}, \quad (3.1) $$

so that the corresponding field equations are

$$ D_\mu (f F^{\mu\nu}) = 0, \quad D_\mu F^{*\mu\nu} = 0, \quad (3.2) $$

$$ D^2 \Phi_i - \frac{1}{4} \partial_i f F_{\mu\nu} F^{\mu\nu} - \partial_i V = 0, \quad i = 1, 2, \quad (3.3) $$

where $\partial_i = \partial / \partial \Phi_i$ and $\Phi_i = (\phi, \varphi)$. It is easy to see that this system transforms into the same one under the following ‘duality transformation’ (see, e.g., [29])

$$ f \to f^{-1}, \quad F_{\mu\nu} \to f F_{\mu\nu}^*, \quad \Phi_i \to \Phi_i. $$

In particular, this implies that electric solutions for the action (3.1) with the gauge coupling function $f(\Phi_i)$ are related to the magnetic solutions of the theory with the coupling $f^{-1}(\Phi_i)$.

Eqs. (3.2–3.3) may have also standard symmetries (for fixed functions $f, V$). For example, if the modulus can be ignored, i.e., in the pure dilaton case ($\Phi_i = \phi$, $f = e^{-2\phi}$)

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9 We are making this distinction because of the opposite signs of the coefficient $b_0$ in $f$ (2.5) in the abelian and non-abelian cases.

10 For convenience we rescale the potential $V$ by the factor 4 as compared to $V$ in (2.1).
and $V = 0$), eqs. (3.2–3.3) are invariant under the usual duality transformation of $F$ combined with $\phi \rightarrow -\phi$. Such a symmetry may survive also if $V(\phi) = V(-\phi)$.

In the case when the theory is invariant under the modulus duality symmetry $\varphi \rightarrow -\varphi$ (as happens for toroidal as well as a class of orbifold compactifications) both $f$ and $V$ are invariant under this duality, which is thus a symmetry of (3.2–3.3).

Let us consider first the electric solution:

$$F_{01} = E(r) , \quad F_{0k} = F_{1k} = F_{kl} = 0 \ (k, l = 2, 3) , \quad \Phi_i = \Phi_i(r) .$$

Then eq. (3.2) reduces to

$$\frac{d}{dr}(r^2 f E) = 0 \ ,$$

which has an obvious solution

$$E = \frac{q}{r^2 f(\Phi_i(r))} .$$

Using (3.5), one simplifies the form of eqs. (3.3) to:

$$\Phi_i'' + \partial_i U - \frac{1}{x^4} \partial_i V = 0 , \quad i = 1, 2 ,$$

where

$$U \equiv -\frac{q^2}{2f} .$$

We have introduced a new coordinate $x = 1/r$ with the range $0 \leq x \leq \infty$, and $\Phi_i' \equiv d\Phi_i/dx$.

Equation (3.6) can be interpreted as corresponding to a mechanical system with the action

$$S = \int_0^\infty dx \left( \frac{1}{2} \Phi_i'^2 + \frac{q^2}{2 f} + \frac{1}{x^4} V \right) ,$$

which, at the same time, gives the energy of the field configuration as derived from the action (3.1):

$$E = S = \int_0^\infty dr \ r^2 \left[ \frac{1}{2} \left( \frac{d}{dr} \Phi_i \right)^2 + \frac{q^2}{2r^4 f} + V \right] = \int_0^\infty dx \left( \frac{1}{2} \Phi_i'^2 - U + \frac{1}{x^4} V \right) .$$

In eqs. (3.7–3.8) the summation over the index $i$ is implied. Note, that the case with $V = 0$ corresponds to the mechanical system with the conservative potential $U$, while the case with $V \neq 0$ corresponds to the mechanical system with non-conservative (time-dependent) potential

$$U(\Phi_i, x) \equiv U - \frac{1}{x^4} V .$$
3.2. Stability Constraints

Let us now discuss the stability constraints for the solutions of eqs. (3.6) under linearized perturbations. Expanding the action (3.1) around the solutions of eqs. (3.6) we find for the terms quadratic in perturbations $\eta_i$ and $B_\mu$ of the scalar fields and the vector potential, respectively:

$$S = \frac{1}{4\pi} \int d^4x \{ -\frac{1}{2}(\partial\eta_i)^2 - \frac{1}{8}\partial_i\partial_j f \eta^i \eta^j F^2_{\mu\nu}(A) + \frac{1}{2} f(\Phi_i) F^2_{01}(B) + \partial_i f \eta^i F_{01}(A) F_{01}(B)$$

$$-\frac{1}{4} f F^2_{rs}(B) - \frac{1}{2} \partial_i \partial_j V \eta^i \eta^j + \ldots \} . \quad (3.9)$$

Here $r, s = 1, 2, 3$ and the summation over $(i, j)$ is implied. Since $f$ is assumed to be positive the terms which are quadratic in $B_\mu$ give positive contributions to the energy and hence do not produce instability. Eliminating $F_{\mu\nu}$ from (3.9) and using (3.5-3.6) we get for the perturbations of the scalar fields $\eta_i = r\tilde{\eta}_i(t, r)$

$$S = \int dt \int_0^\infty dr \{ \frac{1}{2}(\partial_t \tilde{\eta}_i)^2 - \frac{1}{2}(\partial_r \tilde{\eta}_i)^2 - \frac{1}{2r^4} V_{ij} \tilde{\eta}_i \tilde{\eta}_j + \ldots \} , \quad (3.10)$$

where

$$V_{ij} = -\partial_i \partial_j U + r^4 \partial_i \partial_j V = -\partial_i \partial_j U . \quad (3.11)$$

A sufficient condition of ‘linearized’ stability of the abelian electric solutions is therefore that the matrix (3.11) should have non-negative eigenvalues for all $r$:

$$V_{ij} \geq 0 . \quad (3.12)$$

This condition has an obvious interpretation that perturbations should not decrease the energy (3.8) of the system. In the case when $V = 0$ (3.12) is satisfied if $U$ is convex within the range of variation of $\Phi_i$.

Note, however, that while the condition $V_{ij} \geq 0$ is sufficient it may not be necessary. Depending on the nature of functions $f$ and $V$ the corresponding Laplace operator may have only non-negative eigenvalues even if (3.12) is not satisfied. In other words, the
corresponding Laplace operator with the potential \( \mathcal{V} \) need not have bound states with negative energy.

The set of equations, the energy and the condition of stability for the magnetic solution

\[
F_{23} = h \sin \theta , \quad h = \text{const.}, \quad F_{01} = F_{0k} = F_{1k} = 0 , \quad k = 2, 3 , \quad \Phi_i = \Phi_i(r) ,
\]

are found by replacing \( f \) by \( f^{-1} \) and \( q \) by \( h \) in the above equations or by using

\[
U \equiv -\frac{1}{2} h^2 f .
\]

Having spelled out the basic formalism, we would now like to proceed with properties of examples.

4. Case of zero scalar potential: examples of solutions with one non-trivial scalar field

4.1. Basic relations

It is clear that the properties of the solutions are different depending on whether \( V \) is zero or not: in the former case the system is conservative, in the latter it is not. In this Section and Section 5 we shall consider the case when the effect of the potential can be ignored, \textit{i.e.}, when non-perturbative corrections are small. Thus, we shall put \( V = 0 \). In addition, in this section we address the properties of the solutions with only one non-trivial field \( \Phi_i \equiv \Phi \) for a class of functions \( f(\Phi) \).

This corresponds to the case when only one scalar field \( \Phi \) is changing with \( r \). Such a reduction to only one field could be possible if \( f \) had extrema with respect to the other field and thus the other field could be ‘frozen’, \textit{i.e.}, put to a fixed \( r \)-independent value. In principle, \( \Phi \) may be either the dilaton or the modulus, but as we shall see later it does not seem to be possible to ‘freeze out’ the \( r \) dependence of the dilaton in general.
If \( V = 0 \) eq. (3.6) has one integral of motion, \( i.e. \), the ‘energy’ of the corresponding mechanical system:

\[
\frac{1}{2} \Phi'^2 + U(\Phi) = C = \text{const.} \tag{4.1}
\]

The energy (3.8) then takes the form

\[
\mathcal{E} = \int_0^\infty dx (\Phi'^2 - C) , \tag{4.2}
\]

where \( U = -\frac{1}{2} q^2 f^{-1} \) and \( U = -\frac{1}{2} h^2 f \) for the electric and the magnetic solutions, respectively. Here \( x = 1/r \) and \( \Phi' = d\Phi/dx \), as before. In most cases, for finite energy solutions one has to set \( C = 0 \). We shall, however, consider also examples of finite energy solutions with \( C \neq 0 \).

In what follows we shall assume that \( f \) is always positive (since it is the coupling function in the gauge field kinetic term) and thus, \( U \) is always negative. Then the absolute minimum of the energy (3.8) is achieved when \( \Phi \) takes the values corresponding to the maximum of \( U \) if it exists. In looking for other, \( x \) dependent, minima of \( \mathcal{E} \) it is useful to represent (3.8) in the form

\[
\mathcal{E} = \int_0^\infty dx \frac{1}{2} \left( \Phi' \pm \sqrt{2C - 2U(\Phi)} \right)^2 \mp \int_0^{\Phi_0} d\Phi \sqrt{2C - 2U(\Phi)} - \int_0^\infty dx C , \tag{4.3}
\]

where \( \Phi_\infty \equiv \Phi(x = \infty) \) and \( \Phi_0 \equiv \Phi(x = 0) \). Note that \( x \equiv 1/r \). In the following we shall consider solutions with a regular field value, \( i.e. \), \( \Phi_0 \neq \infty \) at \( x = 0 \). However, in the core of the solution, \( i.e., \) at \( x \to \infty \) the field may or may not blow up.

The first term in (4.3) is always positive and hence the energy is minimized if

\[
\Phi' = \pm \sqrt{2C - 2U(\Phi)} \tag{4.4}
\]

is satisfied. The choice of sign is such that the second and the third term in eq. (4.3) yield a positive number. In addition, the constant \( C \) is chosen such that the second and the
third term in eq. (4.3) give a finite value of the energy. Eqs. (4.3–4.4) are analogs of the Bogomol’nyi-type equations.\footnote{Solutions satisfying eq. (4.4) extremize the energy (4.3) for any value of $C$. However, only for a special value of $C$ (in most cases $C = 0$) the solution has a finite energy. One has traded one of the two boundary conditions for $\Phi$ for the constant $C$. Note also that eqs. (4.3–4.4) bear close similarities to analogous supersymmetric systems, e.g., supersymmetric walls \cite{30}. Here the role of the superpotential is played by $W = \int_{\Phi_0}^{\Phi(x)} d\Phi \sqrt{2C - 2U(\Phi)}$. See also Section 5.2.}

Equation (4.4) implies the explicit form of solution

$$
\int_{\Phi_0}^{\Phi(x)} d\Phi \frac{1}{\sqrt{2C - 2U(\Phi)}} = \mp x .
$$

(4.5)

The upper (lower) sign solutions correspond to $\Phi$ decreasing (increasing) with increasing $x = 1/r$. Then the energy $\mathcal{E}$ of the solution is given by

$$
\mathcal{E} = \mp \int_{\Phi_0}^{\Phi_{\infty}} d\Phi \sqrt{2C - 2U(\Phi)} - \int_0^{\infty} dx C .
$$

(4.6)

For the electric solution the charge $Q$ and the scalar charge $D$ are

$$
Q = [r^2 E(r)]_{r \to \infty} = \frac{q}{f(\Phi_0)}, \quad D = -[r^2 \frac{d\Phi}{dr}]_{r \to \infty} = \mp \sqrt{2C + \frac{q^2}{f(\Phi_0)}} .
$$

(4.7)

The magnetic solution is found from the electric solution by the replacements $f \to f^{-1}$ and $q \to h$, or in other words, by taking $U = -\frac{1}{2}h^2 f$. The expression for the energy (4.6) suggests that generically for a given $f$ finite energy electric and magnetic solutions correspond to different values of $C$ as well as opposite signs in eqs. (4.5–4.7).

In order to have a finite energy, regular solution the potential $U(\Phi) = -\frac{1}{2}q^2 f^{-1}(\Phi)$ (for the electric solution) and $U(\Phi) = -\frac{1}{2}h^2 f(\Phi)$ (for the magnetic solution) should satisfy certain conditions. The nature of the solution is different in the cases with $C = 0$ or $C \neq 0$. One can show that the regular, finite energy solutions of eqs. (4.5–4.6), with the upper sign, exist only for the choice of $C = 0$. Such solutions have the property $\Phi_{\infty} \neq \infty$ and as $\Phi \to \Phi_{\infty}$, $U(\Phi)$ approaches zero faster than $(\Phi - \Phi_{\infty})^2$. We shall see that solutions of this
type correspond to the case of ‘dilatonic’-type electric solutions (Section 4.2) and a class of ‘moduli’-type magnetic solutions (Section 4.3).

Solutions with the lower sign in eqs. (4.5–4.6) exist for \( C = 0 \) or \( C \neq 0 \), depending on the nature of \( U \). If \( C = 0 \) one obtains \( \Phi_\infty = \infty \) and as \( \Phi \to \infty \), \( U \to 0 \) faster than \( \Phi^{-2} \). We shall see that such properties are found for a class of ‘moduli’-type electric solutions and a special case of the ‘dilatonic’-type magnetic solution. On the other hand, there also exist regular, finite energy solutions for a specific value of \( C = C_0 > 0 \); this is the case if \( \Phi_\infty = \infty \) and as \( \Phi \to \infty \), \( U(\Phi) - \frac{1}{2}C_0 \) approaches zero faster than \( \Phi^{-1} \). This case will turn out to correspond to an instructive example of a ‘dilatonic’-type magnetic solution.

4.2. ‘Dilatonic’ solutions

Let us consider several particular examples of \( f \). In the following we shall express explicit solutions in terms of the radial coordinate \( r \) (rather than in terms of \( x = 1/r \)). The simplest example is the one of the tree level dilaton coupling

\[
f = e^{-2\Phi} , \quad \Phi = \phi.
\]

(4.8)

The solution is given by eq. (4.5) with the upper sign and \( C = 0 \) (in order to avoid a singularity at finite \( x \)). The explicit form of the solution, its mass \( M \), charge \( Q \) and the scalar charge \( D \) (see eqs. (4.6) and (4.7)) are then:

\[
\phi = \phi_0 - \ln \left(1 + \frac{M}{r}\right) , \quad M \equiv E = |q|e^{\phi_0},
\]

(4.9)

\[
E(r) = \frac{Q}{(r + M)^2} , \quad Q = Me^{\phi_0} , \quad D = -M.
\]

(4.10)

Note that the electric field and the effective string coupling \( e^\phi \) are regular everywhere. The string coupling grows from zero at \( r = 0 \) to a finite value \( e^{\phi_0} \) at large distances. The small distance region is thus a \emph{weak coupling} region. Therefore it is consistent to ignore the non-perturbative potential \( V \) in this region. We shall see in Section 6 that once the potential \( V \)
is included, $\phi$ will be evolving to its minimum at large $r$. The solution (4.9–4.10) is stable since the condition (3.12) is satisfied for (4.8).

The regular magnetic solution with the finite energy, a counterpart of the electric solution (4.9), is obtained from the lower sign solution of eq. (4.5) where $f$ in (4.8) is replaced with $f^{-1}$, and $q$ with $h$, and $C = 0$. Then

$$\phi = \phi_0 + \ln \left(1 + \frac{M}{r}\right), \quad \mathcal{E} \equiv M = |h|e^{-\phi_0}, \quad D = M.$$  \hspace{1cm} (4.11)

The fact that in the magnetic case we got a regular, finite energy solution with the same choice of $C$ as in the electric case is exceptional; it is a consequence of the property that for $f$ in (4.8) $f \to f^{-1}$ corresponds to $\phi \to -\phi$. The small distance region of the magnetic solution (4.11) is a strong coupling region and hence there non-perturbative corrections (inducing $V \neq 0$) can be significant. Like the corresponding regular electric solution, this magnetic solution is also stable.\footnote{\textsuperscript{12} Similar abelian magnetic solution for $f = e^{-2\phi}$ considered in \cite{15} was embedded in a class of non-abelian solutions and was found to be unstable. The condition of stability in the non-abelian case \cite{14,15} is different from (3.12) because of the additional non-abelian term in the potential $V$ in (3.10).}

The expressions for the dilaton in the electric (4.9–4.10) and the magnetic (4.11) solutions coincide with the expressions for the dilaton in the electric and magnetic black hole solutions of \cite{4,5} if $r$ is identified with the coordinate $\hat{r} = r - M$, where $r$ is defined outside the horizon. In terms of $r$ eqs.(4.9) and (4.11) give asymptotic large $r$ expressions for the dilaton of \cite{4,5}. For example, in the case of the electric black hole solution the metric (in the Einstein frame) takes the form (see, e.g., \cite{5})

$$ds^2 = -(1 - \frac{m}{\hat{r}})(1 + \frac{M}{\hat{r}})^{-2} dt^2 + (1 - \frac{m}{\hat{r}})^{-1} d\hat{r}^2 + \hat{r}^2 d\Omega^2,$$

while the expressions for the dilaton and the electric field coincide with (4.9) and (4.10) where $r$ is replaced by $\hat{r}$. The physical mass is $\mu = M + m$ and $Q = (M\mu)^{1/2}$. As was discussed in \cite{5} the small $\hat{r}$ region is a weak coupling region for the electric solution but a
strong coupling region for the magnetic one (which is obtained from the electric solution by the duality transformation $\phi \rightarrow -\phi$, $F_{\mu\nu} \rightarrow e^{-2\phi} F^*_{\mu\nu}$).

A simple, but important generalization of (4.8) corresponds to

$$f = e^{-2\phi} + b, \quad \Phi = \phi,$$

(4.12)

where the constant $b$ can be interpreted, e.g., as a contribution of threshold corrections to $f$ (see (2.4–2.5)) in the case when the space dependence of the modulus field can be ignored. In the case of toroidal compactifications and a class orbifolds with a fixed value of the modulus field the coefficient $b$ is generically positive (negative) for the abelian (non-abelian) gauge group factors. Assuming first that $b > 0$, we get the electric solution from eqs. (4.5–4.6) with the choice of the upper sign and $C = 0$:

$$\left[-\sqrt{e^{-2\phi} + b} + \sqrt{b}\text{Arcsinh}(\sqrt{b}e^{\phi})\right]_{\phi_0}^{\phi(r)} = -\frac{|q|}{r},$$

(4.13)

$$\mathcal{E} = \frac{|q|}{\sqrt{b}}\text{Arcsinh}(\sqrt{b}e^{\phi_0}).$$

(4.14)

Note that $\phi_\infty = -\infty$, i.e., the string coupling is increasing with $r$ from zero to a finite value $e^{\phi_0}$ and the energy is finite. The stability condition (3.12) reduces to $\frac{\partial^2}{\partial \phi^2} f^{-1} \geq 0$. It is satisfied if $\phi_0$ is such that $e^{-2\phi_0} - b \geq 0$.

In the case of $b < 0$ and the boundary condition $e^{-2\phi_0} > |b|$, eqs. (4.13–4.14) still apply with $b$ replaced by $|b|$ and $\text{Arcsinh}$ – by $\text{Arcsin}$. This solution again corresponds to the upper sign of eqs. (4.5–4.6) and $C = 0$. Since we assume that the boundary value $\phi(r = \infty) = \phi_0$ is such that $f = e^{-2\phi} - |b|$ is always positive, we again have $\phi_\infty = -\infty$. This solution is stable if the maximal value of the string coupling small enough, i.e., $e^{\phi_0} < \sqrt{|b|}$.

Interestingly, for $b < 0$ and the boundary condition $2e^{-2\phi_0} < |b|$, a regular, finite energy solution can be obtained from eqs. (4.5–4.6) with the lower sign and $C = -q^2/b > 0$. Note, however, that in this case the solution is unstable.

In contrast to the case of $f = e^{-2\phi}$ (eq.(4.8)), for $f$ in (4.12) one cannot obtain a regular, finite energy magnetic solution from a regular electric one by simply changing the
sign of \( \phi \). Recall, that the magnetic solution for (4.12) is given by eqs. (4.3-4.5) with \( f \) replaced by \( 1/f \), and \( q \) by \( h \). By simply taking \( C = 0 \) one cannot obtain a finite energy solution. It turns out that the only finite energy, regular solution exists for \( b < 0 \) and the choice of \( C = -h^2 b > 0 \). In this case:

\[
\frac{1}{\sqrt{-b}}[\text{Arcsinh}(\sqrt{-b}e^\phi)]^{\phi(r)} = \frac{|h|}{r},
\]

(4.15)

\[
\mathcal{E} = |h|(\sqrt{e^{-2\phi_0} - b} - \sqrt{-b}).
\]

(4.16)

Note that the energy (4.16) is always lower than the corresponding one (eq. (4.11)) with \( b = 0 \). For \( b > 0 \) one gets a regular solution with \( \phi_\infty = \infty \) (eqs. (4.5-4.6) with the lower sign and \( C = 0 \)). However, now the energy is infinite. It turns out that in this case there is no way of adjusting the coefficient \( C \) in order to get a finite energy solution.

Finite energy, regular ‘dilatonic’ electric and magnetic solutions with \( f \) in (4.12) are plotted in Figure 2a and Figure 2b, respectively.

4.3. ‘Modulus’ solutions

Our next example is

\[
f(\Phi) = p^2(\cosh a\Phi + s)^2.
\]

(4.17)

For large negative \( \Phi \) and \( a = 1 \) this function is the same as in (4.8). Being symmetric under the ‘duality’ symmetry \( \Phi \to -\Phi \) this \( f \) (with \( \Phi = \varphi \), \( a = 1/\sqrt{3} \)) models well the modular invariant coupling function \( f_2(\varphi) \) in (2.5).

\[\text{Note, that here } f_2(\varphi) = f_2(-\varphi), f_2(\varphi) \to -\frac{4}{3}\pi b_0 e^{\pm 2\varphi/\sqrt{3}} \text{ for } \varphi \to \pm \infty \text{ and } b_0 \text{ is generically negative (positive) in the case of the abelian (non-abelian) gauge group. A non-zero constant } s \text{ in (4.17) may be considered as accounting for a modulus independent contribution to } f.\]

\[\text{[31] In the case if string theory is invariant under the conjectured ‘dilatonic’ S-duality the coupling (4.17) (with } a = 1, s = 0 \text{) may serve also as a model of a non-perturbative duality invariant modification of the tree-level dilaton coupling function (4.8).}\]
Without loss of generality one can fix the boundary condition $\varphi_0 \equiv \varphi(1/r = 0) > 0$.
Namely, due to the duality symmetry $\varphi \to -\varphi$ solutions with the boundary condition $\varphi_0 < 0$ are related to the ones with $\varphi_0 > 0$. Then for the regular, positive energy electric solutions one finds from eqs. (4.5–4.6) (with the lower sign and $C = 0$)

$$[\sinh a\varphi + sa\varphi]^{(r)}_0 = \frac{|q|a}{|p|r}, \quad (4.18)$$

$$\mathcal{E} = \frac{2|q|}{a|p|\sqrt{1 - s^2}}[\text{Arctan}(\frac{\sqrt{1 - s^2}}{1 + s} \tanh \frac{a\varphi_0}{2})]^{(r)}_\varphi, \quad s^2 < 1, \quad (4.19)$$

$$E = \frac{q}{r^2p^2[cosh a\varphi(r) + s]^2}, \quad Q = \frac{q}{p^2(cosh a\varphi_0 + s)^2}, \quad D = \frac{q}{|p|(cosh a\varphi_0 + s)}.$$

For $s^2 > 1$ the expression for the energy is the same as the one in eq. (4.19), however, Arctan is replaced by Arctanh and $1 - s^2$ by $s^2 - 1$. For $s > -1$ the solution is regular for any $\varphi_0 > 0$. On the other hand, for $s < -1$, the regular solution exists when $\varphi_0 > 0$ satisfies $\cosh a\varphi_0 + s > 0$. The condition of stability (3.12), i.e., $\frac{d^2}{d\varphi^2} f^{-1} \geq 0$, is satisfied if $\varphi_0$ is such that $2\cosh^2 a\varphi - s \cosh a\varphi - 3 \geq 0$ for all values of $\varphi(r)$. Since there always exists a choice of $\varphi_0$ for which both of the above constraints are satisfied we get a class of stable, regular finite energy electric solutions.

In general, the electric solution (4.18–4.20) has the property that it increases from a positive finite value $\varphi_0 \equiv \varphi(1/r = 0) > 0$ to $\varphi_\infty \equiv \varphi(1/r = \infty) = +\infty$. Namely, in the core of the solution, i.e., as $r \to 0$, a decompactification ($\varphi \to \infty$) takes place. We would like to draw an analogy with the corresponding ‘dilatonic’ electric solutions with $f$ in (4.12): the role of the weak coupling at the core of the ‘dilatonic’ electric solution is now played by the decompactification at the core of the ‘modulus’ electric solution.\footnote{For a special case with $s = 0$ we have $f = p^2 \cosh^2 a\varphi$ and eqs. (4.18) and (4.19) take a more explicit form}

$$\varphi = \frac{1}{a} \text{Arcsinh}(\sinh a\varphi_0 + \frac{|q|a}{|p|r}), \quad \mathcal{E} = \frac{2|q|\pi}{a|p|} - \text{Arctan}(\tanh \frac{a\varphi_0}{2}).$$

The boundary condition $\varphi_0 > 0$ is chosen. The solution is stable for $\cosh a\varphi_0 \geq \sqrt{\frac{3}{2}}$.\footnote{For a special case with $s = 0$ we have $f = p^2 \cosh^2 a\varphi$ and eqs. (4.18) and (4.19) take a more explicit form}
We have demonstrated that stable, regular, finite energy electric solutions exist for the duality invariant function $f(\varphi)$ in (4.17). On the other hand, it turns out that there are no regular, finite energy magnetic solutions corresponding to such $f(\varphi)$, unless $s = -1$. In the latter case:

$$f = 4p^2 \sinh^4 \frac{a\varphi}{2}.$$  \hspace{1cm} (4.21)

It is crucial, that now $f(0) = 0$ and the point $\varphi = 0$ corresponds to the minimum of $f$. Here one gets stable, regular, finite energy solutions for both the electric and the magnetic cases. The explicit form of the electric solution (lower sign and $C = 0$ in eqs. (4.5–4.6)) is:

$$(\sinh a\varphi - a\varphi) - (\sinh a\varphi_0 - a\varphi_0) = \frac{|q|a}{|p|r},$$  \hspace{1cm} (4.22)

$$\mathcal{E} = \frac{|q|}{a|p|}(\coth \frac{a\varphi_0}{2} - 1).$$  \hspace{1cm} (4.23)

Without loss of generality we can choose the boundary condition $\varphi_0 > 0$ and the solution has again the property $\varphi_\infty \equiv \varphi(r = 0) = +\infty$, i.e., as $r \to 0$, a decompactification ($\varphi \to \infty$) takes place. The solution with the boundary condition $\varphi_0 < 0$ is related to the one of eqs. (4.22–4.23) by the duality symmetry $\varphi \to -\varphi$.

The regular positive energy magnetic solution is obtained from (4.5–4.6) with the upper sign and $C = 0$:

$$\coth \frac{a\varphi(r)}{2} - \coth \frac{a\varphi_0}{2} = \frac{|p||h|a}{r},$$  \hspace{1cm} (4.24)

$$\mathcal{E} = \frac{|h||p|}{a}(\sinh a\varphi_0 - a\varphi_0).$$  \hspace{1cm} (4.25)

We chose again the boundary condition $\varphi_0 > 0$. Now, $\varphi_\infty = 0$, i.e., the magnetic solution corresponds to the compactification at the self-dual point $\varphi = 0$. Interestingly, for the duality invariant $f$ with $f(0) = 0$ (e.g., given by (4.21)) the electric (4.22–4.23) and the magnetic (4.24–4.25) solutions have complementary features, similar to the ones of the
dilatonic solution with $f$ in (4.8). Now, however, the role of the strong-weak coupling regions is played by the compactification - decompactification regions.\footnote{15}

One can convince oneself that the finite energy electric and magnetic solutions with qualitatively the same behaviour exist for a general positive definite, duality invariant function $f$ with the following properties: $f(\varphi)$ has the minimum at $\varphi = 0$, $f(0) = 0$, and as $\varphi \to \infty$, $f$ grows faster than $\varphi^2$. In Figure 3 we show the explicit numerical solutions for the ‘realistic’ example of $f$ corresponding to the toroidal and a class of orbifold compactifications (see eqs. (2.4–2.5))

$$f = b_0 \ln \left[ \frac{(T + T^*)|\eta(T)|^4}{2|\eta(1)|^4} \right], \quad b_0 = O(1/100), \quad T = e^{2\phi/\sqrt{3}}. \quad (4.26)$$

5. **Case of zero scalar potential: solutions with two non-trivial scalar fields**

5.1. **General remarks and magnetic solution**

In this section we shall study the solutions in a more ‘realistic’ case when both the dilaton and the modulus field can change in space. We shall choose the coupling function in the following form

$$f(\phi, \varphi) = f_1(\phi) + f_2(\varphi), \quad (5.1)$$

\footnote{Analogous stable, regular, finite energy electric and magnetic solutions exist also for another example of duality invariant $f$ with the property $f(0) = 0$, namely $f = p^2\sinh^2 a\varphi$. In this case the electric solution (lower sign and $C = 0$ in eqs. (4.5–4.6)) is given by

$$\cosh(a\varphi) - \cosh(a\varphi_0) = \frac{|q|a}{|p|r}, \quad E = -\frac{|q|}{|p|a} \ln \tanh\left( \frac{a\varphi_0}{2} \right),$$

while the magnetic solution (upper sign and $C = 0$ in eqs. (4.5–4.6)) is

$$\ln \left[ \frac{\tanh\left( \frac{a\varphi_0}{2} \right)}{\tanh\left( \frac{a\varphi}{2} \right)} \right] = -\frac{|p||h|a}{r}, \quad E = \frac{|h||p|}{a} (\cosh a\varphi_0 - 1).$$

The qualitative behaviour of the above solutions is the same as of (4.22–4.25). Namely, as $r \to 0$ the electric and magnetic solutions correspond to decompactification ($\varphi \to \infty$) and compactification at the self-dual point ($\varphi \to 0$), respectively.}
which is a generalization of the perturbative expressions (2.4–2.5) where \( f_1 = e^{-2\phi} \) and \( f_2 = b_0 \ln [(T + T^*)|\eta(T)|^4] + b_1 \), with \( T = e^{2\varphi/\sqrt{3}} \). The presence of a non-perturbative potential will be ignored. The system of equations for the two scalars \( \Phi_i = (\phi, \varphi) \) in the case of the electric solution takes the form (eq. (3.6) with the \( U = -\frac{1}{2}q^2(f_1 + f_2)^{-1} \) and \( V = 0 \)):

\[
\Phi_i'' + \frac{q^2}{2(f_1 + f_2)^2} \frac{df_i}{d\Phi_i} = 0 , \quad i = 1, 2 .
\] (5.2)

The first integral of this system

\[
\frac{1}{2}\Phi_1'^2 + \frac{1}{2}\Phi_2'^2 - \frac{q^2}{2(f_1 + f_2)} = C = \text{const.}
\] (5.3)

remains quite complicated.

The system of eqs. (5.2) reduces to the one-scalar case considered in the previous section if one of the scalars \( \Phi_i \) is fixed to be at the extremum of the corresponding function \( f_i \). While the tree-level dilatonic coupling \( f_1 \) in (5.1) does not have a local extremum (and thus the dilaton cannot be ‘frozen’ at a constant value) the modulus coupling \( f_2 \) does have an extremum at \( \varphi = 0 \). If \( \varphi = 0 \) then (5.2) reduces to the case of the dilatonic coupling (4.12) discussed in Section 4.2.

In the next subsection we shall find electric solutions of eq. (5.2) using a perturbative approach, i.e., by assuming that \( f_1 \gg f_2 \). This assumption is satisfied, in fact, in the case of the threshold correction in (2.5). We shall see that in such a case one can reduce the system of the two second-order coupled differential equations (5.2) to a set of two first-order coupled differential equations.

Let us first consider, however, a simpler case of magnetic solutions. The corresponding equations (3.6) for the magnetic case with \( f \) given by eq. (5.1) take the form of the two decoupled second order differential equations:

\[
\Phi_i'' - \frac{1}{2}h^2 \frac{df_i}{d\Phi_i} = 0 , \quad i = 1, 2 .
\] (5.4)
Since the total energy (3.8) for the magnetic case is now the sum of separate contributions for each $\Phi_i$, it is minimized if one combines the minimal energy magnetic solutions for each $\Phi_i$ independently. Such solutions were already discussed in Section 4. Let us represent $f$ in (5.1) in the form

$$f = \tilde{f}_1 + \tilde{f}_2, \quad \tilde{f}_1 = e^{-2\varphi} + f_2(0), \quad \tilde{f}_2 = f_2(\varphi) - f_2(0).$$  (5.5)

With this choice the duality invariant function $\tilde{f}_2(\varphi) = \tilde{f}_2(-\varphi)$ has the minimum at $\varphi = 0$ with $\tilde{f}_2(0) = 0$ and thus corresponds to the finite energy, magnetic ‘modulus’ solutions as discussed in Section 4.3. For example, for the ‘realistic’ choice (2.4–2.5), $\tilde{f}_2(\varphi) = b_0 \ln [(T+T^*)|\eta(T)|^4/(2|\eta(1)|^4]$ (eq. (4.26)), the corresponding solution is depicted in Figure 3 (dashed line). On the other hand, we saw in Section 4.2 that in the case of $\tilde{f}_1$ (eq. (4.12) with $b = f_2(0) \neq 0$) the ‘dilatonic’ magnetic solution for $\phi$ has a regular, finite energy solution (4.15–4.16) (Figure 2b) only when $b = f_2(0) < 0$. For the case of toroidal compactifications and a class of orbifold compactifications (with $f$ defined in eqs. (2.4–2.5)) the constraints on $\tilde{f}_1$ and $\tilde{f}_2$ can be satisfied for an abelian gauge group ($b_0 < 0$ in eq. (2.5)) if the modulus-independent threshold correction $b_1$ in (2.5) is such that $f_2(0) < 0$. Note, however, that although not implausible, the constraints for existence of a finite energy ‘dilatonic’ magnetic solution might be difficult to satisfy.

5.2. Electric solutions: perturbative approach

To find electric solutions with both fields being non-trivial fields it turns out to be useful to draw an analogy with the scalar field solutions in the supersymmetric theories. Namely, due to the existence of the first integral (5.3) one can represent the energy of the system in a simplified form assuming there exists a function $W(\phi, \varphi)$ (‘superpotential’) which is related to the potential $U = -\frac{1}{2}q^2(f_1 + f_2)^{-1}$ in the following way:

$$2C - 2U = \left(\frac{\partial W}{\partial \phi}\right)^2 + \left(\frac{\partial W}{\partial \varphi}\right)^2.$$  (5.6)
In this case the energy of the system can be rewritten as:

\[ E = \frac{1}{2} \int_0^\infty dx \left[ \left( \phi' \pm \frac{\partial W}{\partial \phi} \right)^2 + \left( \varphi' \pm \frac{\partial W}{\partial \varphi} \right)^2 \right] \mp (W_{x=\infty} - W_{x=0}) - \int_0^\infty dx \ C \ , \quad (5.7) \]

where \( W_{x=\infty} \) and \( W_{x=0} \) denote the values of the superpotential at \( \phi(x = \infty) \), \( \varphi(x = \infty) \) and \( \phi(x = 0) \), \( \varphi(x = 0) \), respectively.

The energy (5.7) is extremized if:

\[ \phi' = \mp \frac{\partial W}{\partial \phi} \ , \ \varphi' = \mp \frac{\partial W}{\partial \varphi} \ ; \quad (5.8) \]

then

\[ E = \mp [W_{x=\infty} - W_{x=0}] - \int_0^\infty dx \ C \ . \quad (5.9) \]

Thus, one has reduced the system of two coupled second order differential equations (5.2) to a more tractable system of two coupled first order differential equations (5.8). Equations of motion (5.8) are analogs of the Bogomol'nyi equations and eq. (5.9) is an analog of the Bogomol'nyi bound.\(^{16}\)

Such a significant simplification can take place only if it is possible to find a superpotential \( W \) satisfying eq. (5.6). We will now determine the explicit form of \( W \) using the approximation \( f_1 \gg f_2 \).\(^{17}\) Expanding \( W \) in powers of \( f_2(\varphi) \), \( W = W_0(\phi) + W_1(\phi, \varphi) + \ldots \), it is easy to show that

\[ W(\phi, \varphi) = \int_{\phi_a}^{\phi} d\phi \sqrt{\frac{q^2}{f_1(\phi)}} + 2C - \frac{1}{2} q^2 f_2(\varphi) \int_{\phi_b}^{\phi} \frac{d\phi}{f_1^{2}(\phi) \sqrt{\frac{q^2}{f_1(\phi)}} + 2C} + \ldots . \quad (5.10) \]

The equations of motion (5.8) then take the form

\[ \phi' = \mp \sqrt{\frac{q^2}{f_1(\phi)}} + 2C + \ldots , \quad (5.11) \]

\[ \varphi' = \pm q^2 \frac{\partial f_2(\varphi)}{\partial \varphi} \int_{\phi_b}^{\phi} \frac{d\phi}{f_1^{2}(\phi) \sqrt{\frac{q^2}{f_1(\phi)}} + 2C} + \ldots . \]

\(^{16}\) Note a clear similarity with supersymmetric configurations, e.g., global supersymmetric domain walls.\(^{30}\)

\(^{17}\) This will turn out to be the case for (2.4–2.5) as long as the boundary condition \( |f_2(\varphi_0)|e^{2\phi_0} \ll 1 \) is satisfied.
Note that the lower boundary value \( \phi_a \) in the first integral in eq. (5.10) is arbitrary and the equations of motion (5.11) as well as the energy (5.9) do not depend on it. However, the solution does depend on the lower boundary value \( \phi_b \) in the second integral in eq. (5.10). It should be chosen so that to minimize the energy (5.9) and yield a regular solution. Thus, we have traded the two of the four boundary conditions on \((\phi, \varphi)\) for the two constants \(C\) and \(\phi_b\).

Let us illustrate the above approach for \(f\) defined in eq. (5.1) with \(f_1 = e^{-2\phi}\) and \(f_2\) such that the approximation \(f_1 \gg f_2\) is valid. In this case (5.10) gives

\[
W(\phi, \varphi) = |q|(e^\phi - e^{\phi_a}) - \frac{1}{6}|q|f_2(\varphi)(e^{3\phi} - e^{3\phi_b}) . \tag{5.12}
\]

While the first equation of motion (5.11) (upper sign and \(C = 0\)) corresponds to the standard ‘dilatonic’ electric solution (4.8) with the property \(\phi_\infty = -\infty\), the second equation in (5.10) takes the form:

\[
\varphi' = \frac{1}{6}|q|\frac{\partial f_2(\varphi)}{\partial \varphi}(e^{3\phi} - e^{3\phi_b}) . \tag{5.13}
\]

In the following we choose \(\phi_b = -\infty\) which turns out to correspond to a regular solution with the energy:

\[
\mathcal{E} = |q|e^{\phi_0} \left(1 - \frac{1}{6}f_2(\varphi_0)e^{2\phi_0}\right) . \tag{5.14}
\]

One can then rewrite eq. (5.13) as:

\[
(\varphi')^2 = \frac{1}{6}|q|e^{3\phi_0}\frac{df_2}{dx} , \tag{5.15}
\]

where \(\frac{df_2}{dx} = \frac{\partial f_2(\varphi)}{\partial \varphi} \varphi'\).

Since the left hand side of eq. (5.15) is positive, in order for the right hand side to be positive, \(\frac{df_2}{dx}\) has to be positive everywhere. In the abelian case (see eq. (2.5)), \(f_2 > 0\) and it grows with \(\varphi\). The constraint \(\frac{df_2}{dx} > 0\) implies that \(\varphi\) grows as \(x = 1/r \to \infty\). In other words, near the core, the solution tends toward a larger compactification radius. On the other hand, in the non-abelian case, \textit{i.e.}, for abelian electric solutions embedded in the
non-abelian gauge sector, \( f_2 < 0 \) and decreases as \( \varphi \) increases. In this case \( \frac{df_2}{dx} > 0 \) implies that as \( x \to \infty \), \( \varphi \) approaches smaller values. Namely, near the core the solution tends toward a smaller compactification radius.

If the same constant \( \tilde{f} = \mathcal{O}[f_2(\varphi_0)] \) is added to \( f_1 = e^{-2\phi} \) and subtracted from function \( f_2 \) (c.f. eq. (5.5)), the solution should remain the same. This is indeed the case. Note, that the solution (5.11) with \( \tilde{f}_1 = e^{-2\phi} + \tilde{f} \) yields the ‘dilatonic’ solution (4.13) with \( \phi_\infty = -\infty \) and the energy (4.14) which, in the limit \( e^{-2\phi} \gg |\tilde{b}| \), is given by \( \mathcal{E}_0 = |q|e^{\phi_0}(1 - \frac{1}{6}\tilde{f}e^{2\phi_0}) \). On the other hand, a solution for \( \varphi \) depends on \( \frac{\partial \tilde{f}_2}{\partial \varphi} \) (see eq. (5.15)), and is thus independent of an additional constant in \( \tilde{f}_2 = f_2(\varphi) - \tilde{f} \). This part of the solution gives the following contribution to the energy: \( \mathcal{E}_1 = -\frac{1}{6}|q|e^{3\phi_0}(f_2(\varphi_0) - \tilde{f}) \). The total energy \( \mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 \) is therefore unchanged (see eq. (5.14)).

In order to find an explicit expression for \( \varphi \) let us first choose a simple duality invariant function \( f_2 = p^2 \sinh^2 a \varphi \) which models well (up to an irrelevant constant) the duality invariant threshold correction \( f_2 \) in (2.5). We will allow \( p^2 \) to be negative as well (this will model the case of a non-abelian embedding). Using the explicit solution for \( \phi \) in eq. (5.15) one finds

\[
\ln \left[ \frac{\tanh a\varphi(r)}{\tanh a\varphi_0} \right] = \frac{1}{6}a^2p^2e^{2\phi_0} \left[ 1 - \frac{1}{(1 + |q|e^{\phi_0}r)^2} \right].
\]

Clearly, as \( r \to 0 \), \( \varphi \) increases (decreases) for \( p^2 > 0 \) (\( p^2 < 0 \)).

In order to confirm this general behaviour for duality invariant functions we have also evaluated numerically the solution for \( \varphi \) with \( f_2 \) in (4.26), i.e., in the ‘realistic’ example corresponding to toroidal and a class of orbifold compactifications. In this case the results are given in Figure 4.

We shall now show that the above solutions with \( f_2 > 0 \) (abelian case) are unstable, while those with \( f_2 < 0 \) (embedding in a non-abelian gauge group) are stable. This instability is a generic property of the electric two-scalar solutions with \textit{positive} and convex function \( f_2(\varphi) \). In fact, for an arbitrary \( \varphi \) the matrix of second derivatives of \( f^{-1} \) in (3.11) is of the form

\[
\partial_i \partial_j [e^{-2\phi} + f_2(\varphi)]^{-1} =
\]
\[
\frac{1}{(e^{-2\phi} + f_2)^3} \left( 4e^{-2\phi}(e^{-2\phi} - f_2) - 4f_2'e^{-2\phi} \right) - 4f_2'e^{-2\phi} f_2^3(1/f_2)'' - f_2''e^{-2\phi} \right), \tag{5.17}
\]
where \( f'_2 = \frac{\partial f_2}{\partial \phi} \) and \( f''_2 = \frac{\partial^2 f_2}{\partial \phi^2} \). For \( f_2 > 0 \) and \( f''_2 > 0 \) there is a region where \( (f_2^{-1})'' < 0 \) and (5.17) is not positive definite. This is obvious, in particular, if \( f_2 \) is small compared to \( e^{-2\phi} \). This implies that perturbative electric solutions corresponding to the case of the abelian group \( (f_2(\phi) > 0) \) are unstable. Since the tree-level dilaton solution is stable, it is clear that the instability in this case is due to the modulus sector.

Note, however, that the abelian electric solutions corresponding to the case of embedding into a non-abelian gauge group, \textit{i.e.}, to the case of \( f_2(\phi) < 0 \), are \textit{stable}. In fact, in this case
\[
f^3_2(f_2^{-1})'' - f''_2e^{-2\phi} = |f''_2|e^{-2\phi} + f^3_2(f_2^{-1})''
\]

is positive if \( e^{-2\phi} \gg |f_2| \).

The perturbative approach to obtaining electric solutions is valid for a large range of boundary conditions. In particular, as long as \( |f_2(\phi_0)|e^{2\phi_0} \ll 1 \) the solutions have the features described above. One should, however, keep in mind that for boundary conditions \( \phi_0 \gg 0 \) the approximation \( f_1 \gg f_2 \) is not valid anymore. In this case one can find special solutions if \( f_1(\phi) \) and \( f_2(\phi) \) have similar functional dependence. Then \( \phi \) and \( \phi \) are related as well. For \( \phi \gg 0 \) (a large compactification radius \( T \)) and \( f_2 \) given in (2.5) this is indeed the case. In the region of large \( \phi \), \( f_2 \) can be approximated as
\[
f_2(\phi) \simeq \tilde{b}T = \tilde{b}e^{2a\phi}, \tag{5.18}
\]
where \( a = 1/\sqrt{3}, \tilde{b} = -\frac{\pi}{3}b_0 \) (\( b_0 \) is the constant in eq. (2.5)). For a solution to exist we have to assume that the constant \( \tilde{b} \) is positive, \textit{i.e.}, the special solution below exists only in the abelian case. Eqs. (5.2) are then satisfied if \( \phi \) is related to \( \phi \) by
\[
\phi = -a\phi + k, \quad k = -\frac{1}{2} \ln (\tilde{b}a^2), \tag{5.19}
\]
and \( \phi \) is the solution (4.9) for the standard ‘dilatonic’ case with \( f_1 \) given by (4.8), \textit{i.e.},
\[
\phi = \phi_0 - \ln \left( 1 + \frac{M}{r} \right),
\]
with the rescaled ‘charge’ $q'$

$$|q'| = \frac{|q|a^2}{1 + a^2} < |q|. \quad (5.20)$$

Note, that this special solution has the property that in the region $r \to 0$ the string coupling approaches zero and the compactification radius approaches infinity.\textsuperscript{18}

Using the same arguments as above one can show that this particular solution is unstable, as is the case in general for any electric solution with two scalars and positive and convex $f_2(\phi)$, \textit{i.e.}, the abelian two-scalar electric solutions.

6. Solutions in the case with non-perturbative scalar potential

Let us now return to the analysis of the magnetic and electric solutions for the system (3.6) with a non-perturbative scalar potential $V$ included. Note that in this case the system of equations (3.6) does not correspond to a conservative mechanical system any more, and thus has no obvious integrals of motion.

We shall mainly consider the non-perturbative potential (eq. (2.10)) due to gaugino condensation in the hidden sector of the gauge group. In this case $V$ depends on both the dilaton, $S = e^{-2\phi} + i\alpha$, and the modulus, $T = e^{2\phi/\sqrt{3}} + i\beta$. In the following we shall take the imaginary part of $S$ to be constant and the imaginary part of $T$ to be zero. In general, a non-perturbative potential should vanish in the limit of small string coupling $e^\phi \to 0$; \textit{e.g.}, for $V$ in eq. (2.10), $V \to \exp[-a_0 \exp(-2\phi)] \to 0$. In addition, $V$ in (2.10) has a minimum at $T_0 \sim 1.2$ and $S_0 = e^{-2\phi_0} + \alpha_0$. As discussed in section 2.2, the potential is not convex everywhere; in addition to the minimum, it also has saddle points and local maxima.

The system of equations (3.6) is of the form:

$$\phi'' + \frac{\partial U}{\partial \phi} - \frac{1}{x^4} \frac{\partial V}{\partial \phi} = 0,$$

\textsuperscript{18} The gravitational analog of this special solution provides a particular generalization of the solution of [4][5] to the case with an additional exponential coupling of the modulus to the gauge field term. Such a solution was recently discussed in [32].
\[ \varphi'' + \frac{\partial U}{\partial \varphi} - \frac{1}{x^4} \frac{\partial V}{\partial \varphi} = 0 , \quad (6.2) \]

where \( U = -\frac{1}{2} q^2 f^{-1} \) and \( U = -\frac{1}{2} h^2 f \) for the electric and magnetic cases, respectively. Here again \( x = 1/r \) and \( \phi' = d\phi/dx, \varphi' = d\varphi/dx \). The solutions of eqs. \((6.1-6.2)\) have to satisfy the stability constraints \((3.12)\). We shall consider a class of functions \( f = f_1(\phi) + f_2(\varphi) \) (eqs. \((2.4-2.5)\)). We will find that for the minimal energy configurations both the dilaton and the modulus will asymptotically \((at \ r \to \infty)\) approach the values corresponding to the minimum of the potential, \(i.e.,\) the main role of the non-perturbative potential is to fix the asymptotic values of the dilaton and the modulus to be at its minimum.

6.1. ‘Dilatonic’ case

For the electric solutions in the absence of a non-perturbative potential (Section 5) and a class of functions \( f \) in \((2.4-2.5)\), the modulus field \( \varphi \) varies only slightly. From its boundary value \( \varphi = \varphi_0 \) at \( r = \infty \) it increases (abelian case) or decreases (non-abelian embedding case) as one approaches the core \( r \to 0 \) (see Figure 4).\(^{19}\) In the magnetic case, \( \varphi \) also varies slowly from \( \varphi_\infty = 0 \) at the origin to the asymptotic value \( \varphi_0 \) at \( r \to \infty \) (dashed line in Figure 3). We saw that in the case without a non-perturbative potential (Section 5) the modulus contribution did not affect the evolution of the dilaton \( \phi \) significantly.\(^{20}\)

In this section we shall therefore neglect a radial variation of the modulus, \(i.e.,\), we shall simplify the problem by ‘freezing out’ the modulus and study only eq. \((6.1)\) with

\[ f = e^{-2\phi} + b , \quad V = V(\phi) . \quad (6.3) \]

\(^{19}\) Recall, that only the solution corresponding to the case of embedding in the non-abelian theory is stable (see Section 5.2).

\(^{20}\) From the discussion in Section 2.2 it follows that \( f \) in \((2.4-2.5)\) and \( V \) in \((2.10)\) are both extremal at \( \varphi = 0 \) (the self-dual point \( T = 1 \)). Therefore, the eqs. \((6.1-6.2)\) admit a constant solution \( \varphi = 0 \). Note, however, that at \( \varphi = 0 \), \( V \) in \((2.10)\) has a local maximum in the \( \varphi \) direction.\(^{23}\) \(^{27}\). That means this solution is \textit{unstable}. Namely, in the region \( r \to \infty \) the corresponding Laplace operator for the linearized perturbations (Section 3.2) has a potential \((3.11)\) which is unbounded from below and thus, should have negative eigenvalues.
Keeping in mind that the constant $b \ll 1$, we can ignore its presence, except in the strong coupling region.

For the electric solution ($U = -\frac{1}{2}q^2 f^{-1}$) without the non-perturbative potential, the small $r$ region corresponds to a weak coupling region, $\phi \to -\infty$ as $r \to 0$ (see Figure 2a and eqs. (4.13–4.14)). From eq. (6.1) one then concludes that in this region ($x \to \infty$) one can neglect the contribution of the non-perturbative potential $V$. This is also consistent with the fact that $V$ is expected to be negligible in the weak coupling limit. Thus, the small radius ($x = 1/r \to \infty$) behaviour corresponds to a solution satisfying $\phi' = \sqrt{q^2 e^{2\phi} + c^2}$ and it is of the form:

$$
\phi = -c|q|(x - x_0) + \log (2c) + \ldots , \quad c \neq 0 \quad (6.4)
$$

$$
\phi = -\log |q|(x - x_0) + \ldots , \quad c = 0 \quad (6.5)
$$

where, $c$ and $x_0$ are constants determined by the properties of the potential. Note, that the special form of the solution with $c = 0$ which yields a finite contribution to the energy of the system can be obtained only for a special choice of the parameters of the potential.

On the other hand, the asymptotic form of $\phi$ at large distances ($x = 1/r \to 0$) can be found by expanding

$$
\phi = \phi_0 + k_1 x + l_1 x^2 + m_1 x^3 + n_1 x^4 + \ldots .
$$

Eq. (6.1) is then satisfied to the leading order in $x$ if

$$
k_1 = l_1 = m_1 = 0, \quad n_1 \neq 0, \quad (\frac{\partial V}{\partial \phi})_{\phi_0} = 0 \quad (6.6)
$$

where $n_1$ is determined by

$$
\frac{q^2}{2} \left( \frac{1}{f^2} \frac{\partial f}{\partial \phi} \right)_{\phi_0} - n_1 \left( \frac{\partial^2 V}{\partial \phi^2} \right)_{\phi_0} = 0 .
$$

Thus,

$$
\phi(r \to \infty) = \phi_0 + \frac{n_1}{r^4} + \mathcal{O}(\frac{1}{r^5}) ,
$$

30
and the asymptotic value of \( \phi \) should correspond to an extremum of \( V \). Moreover, \( f \) in (6.3) is positive and \( \frac{\partial f}{\partial \phi} \) is negative. Making the assumption that \( \phi \) should be growing monotonically with \( r \), i.e., \( n_1 < 0 \), we conclude from (6.7) that \( \frac{\partial^2 V}{\partial \phi^2} \phi_0 \) should be positive, i.e., that \( \phi_0 \) should correspond to a minimum of \( V \). This implies that even though the potential \( V \) may have local maxima on the way from the weak coupling region to a non-perturbative minimum (as it is actually the case for the gaugino condensation potentials) the final point \( \phi_0 \) of the ‘evolution’ of \( \phi \) from \( -\infty \) is at the minimum of \( V \). Note a crucial dependence of this conclusion on the properties of the function \( f \).

Similar analysis of eq. (6.1) can be carried out in the magnetic case \( (U = -\frac{1}{2}h^2f) \). As we have found in the \( V = 0 \) case, the small radius region \( (r \to 0) \) corresponds to a strong coupling region \( (\phi \to \infty) \) (see Figure 2b and eqs. (4.15–4.16)). That means that the presence of the potential in eq. (6.1) cannot be neglected anymore. In the strong coupling region the non-perturbative potential (2.10) becomes

\[
V(\phi \to \infty) \to V_0e^{2\phi} .
\]

(6.9)

Then (6.1) takes the following form in the \( r \to 0 \) region

\[
\phi'' + h^2e^{-2\phi} - \frac{2}{x^4} V_0 e^{2\phi} = 0 .
\]

(6.10)

This equation is solved by a particular solution

\[
\phi(r \to 0) = \ln \frac{k}{r} , \quad 2V_0k^4 + k^2 = h^2 .
\]

(6.11)

---

\(^{21}\) It is interesting to compare this conclusion to the one in the case of time-dependent cosmological solutions. There the dilaton \( \phi \) is changing with time and not with space and the gauge field is assumed to be constant so that there is no \( f \)-dependent contribution to the equation of motion (see [3] and references therein). The existence of a local maximum in \( V \) seems to preclude a natural evolution of the dilaton from a weak coupling region where \( V = 0 \) to a ‘non-perturbative’ minimum at late times unless one fine tunes the initial value of the dilaton ‘velocity’ so that it can pass over the maximum to be trapped in the minimum.

\(^{22}\) Note that this limiting form of \( V \) is the same as in the case of the potential corresponding to a ‘central charge deficit’ \( c \neq 0 \), i.e., \( V = ce^{2\phi} \) (eq. (2.3)).

\(^{23}\) In the case of the non-perturbative potential (2.10) \( V_0 < 0 \) and then the additional constraint \( 1 + 8h^2V_0 > 0 \) has to be satisfied in order to have a positive value of \( k \). Detailed properties of solutions for other forms of the potential in the strong coupling region will be be discussed elsewhere [33].
The solution (6.11) is the same (up to a constant) as the asymptotic form of the solution with $V = 0$ (eq.(4.15)) in the region $r \to 0$, i.e., $\phi(r \to 0) = \ln \frac{|h|}{r}$.

As for the large $r$ asymptotics, here we shall again assume that the regular solutions satisfy $\phi(r \to \infty) = \phi_0$. Then using the expansion (6.5) we find the relations similar to (6.6–6.7):

$$k_1 = l_1 = m_1 = 0, \quad n_1 \neq 0, \quad \left(\frac{\partial V}{\partial \phi}\right)_{\phi_0} = 0,$$

$$
\frac{h^2}{2} \left(\frac{\partial f}{\partial \phi}\right)_{\phi_0} + n_1 \left(\frac{\partial^2 V}{\partial \phi^2}\right)_{\phi_0} = 0,
$$

and

$$\phi(r \to \infty) = \phi_0 + \frac{n_1}{r^4} + O\left(\frac{1}{r^5}\right).$$

The assumption that $\phi$ is decreasing monotonically as $r \to \infty$ implies that now $n_1$ should be positive. Since $\frac{\partial f}{\partial \phi}$ is negative, we find from (6.13) that $\left(\frac{\partial^2 V}{\partial \phi^2}\right)_{\phi_0}$ should be positive, so that, as in the electric case, $\phi_0$ should correspond to the minimum of $V$.

We thus conclude that the important role of the potential is to fix the large distance asymptotic value of the dilaton to be at the minimum of $V$. The energy (3.8) of these solutions can be finite for a specific type of the potential. In addition, the value of the potential at the minimum should be zero, i.e., $V(\phi_0) = 0$. If $V(\phi_0) \neq 0$ (which is, unfortunately, a generic property of the gaugino condensation potentials) it is easy to see that the integral in (3.8) is divergent at $r = \infty$. This divergence is simply due to the presence of a constant vacuum energy density proportional to the cosmological constant $V(\phi_0)$ and thus can be ignored in the present context.

6.2. Solutions with both dilaton and modulus

We will now study the general case when both the dilaton and the modulus in (6.1–6.2) are changing with $r$. Let us consider first the electric solutions. At small radius $r \to 0$

24 Similar asymptotic behaviour of the dilaton was found in the massive dilaton case in [12] [13].
the electric solutions correspond to the weak coupling region, $\phi \to -\infty$, and thus, the potential $V$ term can be neglected in both eq. (6.1) and eq. (6.2). Thus, in the $r \to 0$ region the electric solution can be discussed along the lines in Section 5.2.

Let us now determine the large $r$ asymptotic behaviour of both $\phi$ and $\varphi$. For the electric solution we shall again assume that in the large distance region $\phi$ and $\varphi$ approach constant values $\phi_0$ and $\varphi_0$. Using the expansion

$$\Phi_i = \Phi_{0i} + k_ix + l_ix^2 + m_ix^3 + n_ix^4 + \ldots , \quad \Phi_i = (\phi, \varphi) ,$$

(6.15)

one finds that eqs. (6.1–6.2) are satisfied to the leading order in $x$ if

$$k_i = l_i = m_i = 0, \quad n_i \neq 0 , \quad (\frac{\partial V}{\partial \Phi_i})_{\Phi_0} = 0 , \quad (6.16)$$

$$q^2 \left( \frac{1}{f^2} \frac{\partial f}{\partial \phi} \right) \phi_0 - \frac{2}{\sum_{j=1}^2 n_j \left( \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \right) \phi_0} = 0 , \quad i = (1, 2) . \quad (6.17)$$

As a result,

$$\Phi_i(r \to \infty) = \Phi_{0i} + \frac{n_i}{r^4} + \mathcal{O}(\frac{1}{r^5}) , \quad (6.18)$$

where the asymptotic value $\Phi_{0i} = (\phi_0, \varphi_0)$ corresponds to an extremum of $V(\Phi)$. As follows from (2.5), $f$ is positive, $\frac{\partial f}{\partial \phi}$ is negative and $\frac{\partial f}{\partial \varphi}$ is positive (negative) for the abelian (non-abelian embedding) case. For consistency with the behaviour of the solutions at small $r$ (Section 5.2) we are to assume that as $r$ increases $\phi$ should be growing and $\varphi$ should be falling (growing) in the abelian (non-abelian embedding) case, namely, $n_1 < 0$, and $n_2 > 0 \ (n_2 < 0)$. These constraints in turn imply that in the case when the mixed second derivative term $\left( \frac{\partial^2 V}{\partial \phi \partial \varphi} \right)_{\Phi_0}$ is small, the matrix $\left( \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \right)_{\Phi_0}$ is positive definite, i.e., both $\phi_0$ and $\varphi_0$ correspond to the minimum of $V$.

We will now show that the potential $V$ in (2.10) has actually the mixed second derivative equal to zero, $\left( \frac{\partial^2 V}{\partial \phi \partial \varphi} \right)_{\Phi_0} = 0$. In fact, $V$ belongs to a class of potential which are determined in terms of a separable Kähler function $G = G_1(S, S^*) + G_2(T, T^*)$ (see eqs. (2.6–2.9)). For such a potential one can show that its mixed second derivatives are always
proportional to \(|G_S G_T|\), where \(G_S\) and \(G_T\) are derivatives of \(G\) with respect to \(S\) and \(T\).

Using the form of the \(N = 1\) supergravity potential \(V\) in eq. (2.7) one obtains:

\[
\frac{\partial^2 V}{\partial S \partial T} = G_S G_T \left\{ V + e^\phi \left[ G_T^{-1} (G_T^{-1})_T + 1 \right] \right\} + G_S G_T^{-1} \cdot e^\phi \cdot G_T^{-1} \cdot G_{TT}.
\]  

(6.19)

Thus, if either \(G_S = 0\) and/or \(G_T = 0\) (either one or both \(S\) and \(T\) sectors preserve supersymmetry), then the mixed second derivatives are zero. In the case of the potential (2.10) the supersymmetry is preserved at the minimum in the dilaton direction. Therefore:\[25\],

\[
\left( \frac{\partial^2 V}{\partial S \partial \phi} \right)_{\phi_0} = 0.26\]  

This implies that the modulus field \(\phi\) should also be asymptotically fixed at the minimum of \(V\).

Let us finally consider the magnetic solutions. At small radius, \(r \to 0\), the magnetic solutions have the strong coupling region, \(\phi \to +\infty\). The potential \(V\) term in eq. (6.1), however, does not qualitatively modify the particular dilaton solution in this region (see eq.(6.11)). In addition, the potential term can be neglected in eq. (6.2) since it is proportional to \(e^{-2\phi} x^{-4} \propto r^2\). Therefore, as \(r \to 0\), solutions for the modulus can be discussed along the lines in Section 5.1. The large radius behaviour can be studied in the same way as for the electric solutions with similar conclusions. Thus, in both the electric and the magnetic, two–field cases the presence of the potential \(V\) term in the equations of motion (6.1–6.2) fixes the asymptotic values of \(\phi_0\) and \(\varphi_0\) to be at the minimum of \(V\).

7. Concluding remarks

In this paper we have studied electric and magnetic configurations with nontrivial dilaton and modulus fields that are extremas of the \(D = 4\) string effective action with

\[25\] Recall that \(\phi\) and \(\varphi\) are related to \(S\) and \(T\) according to eq. (2.2).

\[26\] Note that even if this were not the case, the asymptotic value of the dilaton would still be fixed at the minimum. Since \(f_1(\phi) \gg f_2(\varphi)\), eq. (6.17) implies that \(|n_1| \gg |n_2|\). Then in the first equation in (6.17) the contribution of the term with the mixed second derivative could be neglected. Thus the conclusion of Section 6.1 that \(\phi_0\) corresponds to the minimum of the potential along the \(\phi\) direction, is still valid. Note, however, that if the term with the mixed second derivative is not negligible in the second equation in (6.17), then \(\varphi_0\) need not correspond to the minimum of the potential in the \(\varphi\) direction.
loop (higher genus) and non-perturbative string corrections included. We have confined our analysis to the case of the abelian charged solutions associated with an abelian gauge group as well as abelian charged solutions that can be embedded into a non-abelian gauge theory.

We have found regular, stable, finite energy solutions with both the dilaton $\phi$ and the modulus $\varphi$ varying with the radius $r$. In the electric solution the dilaton $\phi$ always approaches the weak coupling region in the core, $\phi \to -\infty$ as $r \to 0$ (Figure 2a). The modulus $\varphi$ changes only slightly from its asymptotic value $\varphi_0$ at $r = \infty$ (see Figure 4). For the solution in the case of an abelian gauge group $\varphi$ increases, i.e., the compactification radius grows as one approaches the core region ($r \to 0$). In other words, in the core region there is a tendency towards decompactification. On the other hand, for the abelian electric solution embedded into a non-abelian gauge theory $\varphi$ slightly decreases at small scales, i.e., in the core region there is a tendency towards compactification. Only the latter solutions turn out to be stable. The main role of the non-perturbative potential $V(\phi, \varphi)$ is to fix the asymptotic values of $\phi$ and $\varphi$ at $r = \infty$ to be at the minimum of $V$. This happens in spite of the fact that $V$ may have other extrema, corresponding to saddle points or maxima. These charged, finite energy, stable solutions can be interpreted as interesting particle-like field configurations.

Regular, finite energy magnetic solutions exist only in the abelian case and for a specific range of the moduli-independent part of the threshold corrections in the gauge coupling function $f$ (see Section 5.1). Here, in the core ($r \to 0$), the solution approaches the the strong coupling region, $\phi \to \infty$ (Figure 2b). As $r \to 0$ the modulus tends towards the region of compactification at the self-dual point, $\varphi \to 0$ (see Figure 3, dashed line). The presence of the non-perturbative potential again fixes the large $r$ asymptotic values of the two fields $\phi_0$ and $\varphi_0$ to be at the minimum of the potential.

We have also considered solutions corresponding to the models in which $f$ depends only on $\varphi$ and is invariant under the modulus duality symmetry, $f(\varphi) = f(-\varphi)$, or what
we called the ‘modulus’ solutions. In the model with \( f(0) = 0 \) and \( f \) is growing faster than \( \varphi^2 \) we have found stable, regular, finite energy ‘modulus’ solutions for both the electric and magnetic cases. Such electric and magnetic solutions (Figure 3) exhibit analogous complementary behaviour as the corresponding ‘dilatonic’ solutions (Figure 2), with the weak coupling \( (\phi \to -\infty) \) – strong coupling \( (\phi \to \infty) \) regions replaced by the compactification \( (\varphi \to 0) \) – decompactification \( (\varphi \to \infty) \) regions.

The threshold corrections to \( f \) and the non-perturbative contributions to \( V \) respect the modulus duality symmetry. That means the dyonic modulus solutions with a nontrivial axion field of the complex modulus field \( T \) can be obtained by employing the \( SL(2, \mathbb{Z}) \) \( T \)-duality symmetry of the full string effective action.

As for the duality in the dilatonic sector, it is present in the tree level string effective action but appears to be violated by the perturbative as well as non-perturbative string corrections (unless it is somehow restored in some particular cases in the full string theory as was suggested in [31]). At the tree level, the electric and magnetic solutions are related by the transformation \( \phi \to -\phi \) (Figure 2, \( b = 0 \) case) what makes it possible to concentrate only on the pure electric or pure magnetic ‘dilatonic’ solutions. Tree-level dyonic solutions with non-trivial axion of the complex dilaton field \( S \) can be found by the \( SL(2, \mathbb{R}) \) \( S \)-duality transformations as in [32][33]. Since the dilatonic \( S \)-duality is, however, broken by both perturbative and non-perturbative string corrections discussed in this paper, one can no longer relate the electric and magnetic ‘dilatonic’ solutions (see Figure 2, \( b \neq 0 \) case).

In our study we have assumed the metric to be flat. This is consistent provided the energy of the solutions is small enough. It is important of course to extend the analysis of this paper to the gravitational case, i.e., to find the the corresponding black hole -type solutions (generalizing those of [1][5][12][13]) of the string effective action with the threshold corrected coupling \( f \) as well as the non-perturbative potential \( V \) included.

Another direction is to generalize our solutions to a non-abelian case. The non-abelian dilatonic black hole solutions, which are the analogs of the ‘ordinary’ non-abelian black
hole solutions in the case with the three level dilaton coupling \( f = e^{-2\phi} \) were recently discussed in [36]. These non-abelian solutions have finite energy, but are unstable. It may be interesting to study their modifications in the presence of a more realistic coupling \( f \) and potential \( V \).

At the tree-level of toroidally compactified heterotic string theory there exist monopole solutions discussed in [37]. Since the gauge field is non-trivial the dilaton is changing in space and grows at small distances. One could try to understand how such solutions are changed by perturbative as well as non-perturbative corrections to the string effective action.

It would be important to find cosmological analogs of the solutions presented in this paper. For a recent discussion of cosmological solutions of the superstring effective action with the one-loop generated modulus coupling to the \( R^2 \) term see [38].

Another open problem is how to generalize, to all orders in \( \alpha' \), the solutions of the string effective action with perturbative as well as non-perturbative corrections included. At the tree level a solution, which is exact to all orders in \( \alpha' \), can, in principle, be obtained by identifying the corresponding 2\( d \) conformal field theory. However, once perturbative as well as non-perturbative contributions to the effective action are taken into account, the equivalence between extrema of the string effective action and 2\( d \) conformal field theories is not established and is unlikely to exist.

To conclude, we would like to emphasize again the generic features of our stable, finite energy electric solution. It is a particle-like configuration with the weak coupling region inside the core. It seems that the proper dilaton boundary condition for analogous solutions in string theory should be \( \phi \to -\infty \) at \( r \to 0 \), i.e., small string coupling at small scales. This corresponds to an appealing ‘asymptotic freedom’ scenario in which both perturbative and non-perturbative string corrections are negligible in the small distance.

\[^{27}\] As was discussed above, a stable, finite energy magnetic solution exists only for a certain range of modulus independent threshold corrections.
region. Thus, in this region the tree-level string theory applies, supersymmetry and other symmetries are unbroken.\footnote{In the cosmological context the small distance region corresponds to an early time era.} On the other hand, the growth of the dilaton with $r$ implies that at large distances, or in ‘our world’, the string coupling becomes relatively strong, so that non-perturbative corrections can no longer be ignored. In this region supersymmetry is spontaneously broken and both the dilaton and the modulus fields are stabilized at the minimum of the non-perturbative potential.

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**Figure Captions**

**Figure 1:** Duality invariant function $f = b_0 \ln \left[ (T + T^*) |\eta(T)|^4 \right]$ (eqs. (2.5), (4.36)) versus $\varphi$ with $T = \exp(2\varphi/\sqrt{3})$ for $b_0 = -1$.

**Figure 2:** Electric (Figure 2a) and magnetic (Figure 2b) ‘dilaton’–type solutions $\phi(r)$ (eqs. (4.13) and (4.15)) for $f = e^{-2\phi} + b$ (eq. (4.12)) with different values of the parameter $b$ and $\phi(r = \infty) = \phi_0 = 0$. In Figure 2a the electric solutions with $b = 0$ and $b = 1/100$ are indistinguishable. Stable, regular, finite energy electric solutions exist for $b > -e^{-\phi_0}$ while the corresponding magnetic solutions exist only for $b < 0$. For the boundary condition $\phi_0 \neq 0$ the solutions are numerically different but the qualitative behaviour is not changed. The energy of the electric and magnetic solutions increase with increasing $\phi_0$.

**Figure 3:** Electric (solid line) and magnetic (dashed line) ‘modulus’–type solutions for $\varphi(r)$ in the case of $f = b_0 \ln \left[ (T + T^*) |\eta(T)|^4/(2|\eta(1)|^4) \right]$, $T = \exp(2\varphi/\sqrt{3})$ (eq. (4.26)). The $r = \infty$ boundary condition with $T_0 = 1.2$ or $\varphi_0 = 0.158$, is used. We have chosen $b_0 = -1$, the electric charge $|q| = 1$, and the magnetic charge $|h| = 1$. The energy of the electric solution is $E_0 = 3.065$ and of the magnetic solution $E_0 = 0.0386$. For the electric solutions with $b_0 \neq -1$, $|q| \neq 1$ and the magnetic solutions with $b_0 \neq -1$, $|h| \neq 1$ the radius $r$ is rescaled by $\sqrt{-b_0/|q|}$ and $|h|/\sqrt{-b_0}$, respectively, and the energy by $|q|/\sqrt{-b_0}$ and $\sqrt{-b_0}/|h|$, respectively. For the boundary condition $T_0 \neq 1.2$ the solutions are numerically different but the qualitative behaviour is not changed. For larger (smaller) $T_0$ the electric energy decreases (increases), while the magnetic one increases (decreases).

**Figure 4:** The plot of the modulus field $\varphi(r)$ in the case of the two-scalar electric solution with $f = f_1(\varphi) + f_2(\varphi)$ (5.1), $f_1 = e^{-2\phi} + b$ (4.12) and $f_2 = b_0 \ln \left[ (T + T^*) |\eta(T)|^4/(2|\eta(1)|^4) \right]$ (4.26) with $b = O(b_0)$. The electric charge is $|q| = 1$ and the $r = \infty$ boundary condition for $\varphi$ is $\phi_0 = 0$. The boundary condition for $T$ is $T_0 = 1.2$ or $\varphi_0 = 0.158$. The solution for $\phi(r)$ is the same as shown in Figure 2a. It turns out (see discussion in Section 5.2) that only the solutions with $b_0 > 0$ (non-abelian embedding case) are stable.
Fig. 4

For $b_0 = 1/10$, $b_0 = 1/100$, $b_0 = -1/100$, and $b_0 = -1/10$, the graphs show how the function $r$ changes with different values of $b_0$. The curves are as follows:

- $b_0 = 1/10$: A dashed line.
- $b_0 = 1/100$: A solid line.
- $b_0 = -1/100$: A dashed-dotted line.
- $b_0 = -1/10$: A dotted line.

The $x$-axis represents the variable $r$, while the $y$-axis represents the function values.