Algebro-geometric aspects of superintegrability: 
the degenerate Neumann system

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Abstract

In this article we use algebro-geometric tools to describe the structure 
of a superintegrable system. We study degenerate Neumann system with 
potential matrix that has some eigenvalues of multiplicity greater than 
one. We show that the degenerate Neumann system is superintegrable if 
and only if its spectral curve is reducible and that its flow can be linearized 
on the generalized Jacobians of the spectral curve. We also show that the 
generalized Jacobians of the hyperelliptic component of the spectral curves 
are models for the minimal invariant tori of the flow. Moreover the spec-
tral invariants generate local actions that span the invariant tori, while the 
moment maps for the rotational symmetries provide additional first inte-
grals of the system. Using our results we reproduce already known facts 
that the degenerate Neumann system is superintegrable, if its potencial 
matrix has eigenvalues of multiplicity greater or equal than 3.

Keywords: superintegrability – noncommutative integrability – Neu-
mann system – generalized Jacobian – algebraic geometry

1 Introduction

The Neumann system describes the motion of a particle constrained to a n-
dimensional sphere \( S^n \) in a quadratic potential. The potential is in ambient 
coordinates \( q = (q_1, \ldots, q_{n+1}) \in \mathbb{R}^{n+1} \) given as a quadratic form

\[
V(q) = \frac{1}{2} \langle Aq, q \rangle = \frac{1}{2} \sum_{i=1}^{n+1} a_i q_i^2
\]

with the potential matrix \( A = \text{diag}(a_1, \ldots, a_{n+1}) \) and \( a_i > 0 \). We study the 
degenerate case when some of the eigenvalues of \( A \) are of multiplicity greater 
than 3.

In the generic case where all the eigenvalues of the potential \( a_i \) are different, 
the Neumann system is algebraically completely integrable and its flow can be 
linearized on the Jacobian torus of an algebraic spectral curve \([1, 2, 3]\).

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In contrast to the generic Neumann system, the degenerate case has not received much attention. The Liouville integrability of degenerate Neumann case was proven by Liu in [4]. We have used the methods of algebraic geometry to study the non superintegrable case when multiplicity of the eigenvalues is less or equal 2 [5]. The reduction of the degenerate Neumann system to the Rosochatious system was recently explored by Dullin and Hansmann in [6].

In this paper we study the degenerate case with the multiplicity of some eigenvalues ≥ 3. We show that the degenerate Neumann system is superintegrable if the spectral curve is reducible and that its flow can be linearized on the generalized Jacobian of a singular hyperelliptic component of the spectral curve. We show that the generalized Jacobian of the hyperelliptic component is a model for minimal invariant tori of the flow, which are of smaller dimension in superintegrable case. Another difference between generic and superintegrable case is that the spectral invariants do not generate all of the first integrals, but only the actions that generate the invariant tori.

To linearize the flow, we will use $(n+1) \times (n+1)$ Lax equation, where the resulting spectral curve is singular in the degenerate case. Following the standard procedure and normalizing the spectral curve results in the loss of several degrees of freedom and in fact corresponds to the symplectic reduction by the additional symmetries in the degenerate case [5, 6]. In order to avoid this, we will use the generalized Jacobian of the singular spectral curve to linearize the flow as in [7, 8, 5]. The generalized Jacobian is an extension of the “ordinary” Jacobian by a commutative algebraic group (see [9] for more detailed description).

One should also mention that apart from classical case there has been a lot of interest in quantum case [10, 11, 12] for both Neumann and its reduction Rosochatius system. Our study should provide additional insight into the superintegrable nature of quantum degenerate Neumann system as well.

After the introduction, Lax representation and spectral curve of the degenerate Neumann system are discussed in section 2. In subsection 3 we study invariant manifolds of the isospectral flow. Our main result is formulated in theorem 1 and states that the minimal invariant tori of degenerate Neumann system are isomorphic to the real parts of the generalized Jacobian of the singular hyperelliptic component of the spectral curve modulo some discrete group action.

2 Degenerate Neumann system

The Neumann system is a Hamiltonian system on the cotangent bundle of the sphere $T^*S^n$ with the standard symplectic form. The Hamiltonian can be written in ambient coordinates $(q,p) \in T^*S^n = \{(q,p) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} ; \langle q, q \rangle = 1 \wedge \langle p, q \rangle = 0\}$ as

$$H = \frac{1}{2}(p,p) + V(q) = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + \frac{1}{2} \sum_{i=1}^{n+1} a_i q_i^2$$

and the equations of motion can be written in Hamiltonian form

$$\dot{q} = p$$

$$\dot{p} = -Aq + \varepsilon q$$

(1)
where \( \varepsilon = \|p\|^2 + c \) is chosen so that \( \|q\| = 1 \) and the particle stays on the sphere.

**Definition 1.** We will call the Neumann system degenerate if some of the eigenvalues of \( A \) are of multiplicity greater than 1.

We assume throughout this paper that \( A \) has \( k \) different eigenvalues \( \{a_1, \ldots, a_k\} \) with multiplicities \( \{m_1, \ldots, m_k\} \) and we denote by \( r \) the number of eigenvalues with multiplicity greater than one.

It is known that degenerate Neumann system is superintegrable if \( m_j \geq 3 \) for some \( j \) and is not superintegrable if all \( m_j \leq 2 \). Its structure is well described in [5] for the case of all \( m_j \leq 2 \) and in [6] for the general case.

### 2.1 Isospectral flow and spectral curve

A standard approach to study integrable systems is by writing down the system in the form of Lax equation, which describes the flow of matrices or matrix polynomials with constant eigenvalues, i.e. the isospectral flow \([2, 13]\). Eigenvalues of the isospectral flow are the first integrals of the integrable system and Lax representation maps Arnold-Liouville tori into the real part of the isospectral manifold consisting of matrices with the same spectrum. A quotient of the isospectral manifold by a suitable gauge group is in turn isomorphic to the open subset of the Jacobian of the spectral curve \([14]\).

We extend this approach to cover special properties of the superintegrable systems. The isospectral flow for Neumann system can be written in the form of a differential equation on the space of matrix polynomials of degree 2

\[
\frac{d}{dt} L(\lambda; t) = [L(\lambda; t), (\lambda^{-1} L(\lambda; t))_+]
\]

and with the initial condition of the special form

\[
L_0(\lambda) = \lambda^2 A + \lambda q \land p - q \otimes p
\]

for \( q, p \in \mathbb{R}^{n+1} \) such that \( \|q\| = 1 \) and \( \langle q, p \rangle = 0 \).

**Remark 1.** Two different Lax equations are known for a generic Neumann system with \( n \) degrees of freedom: one is using \((n+1) \times (n+1)\) matrix polynomials of degree 2 \([2] \) and the other is using \(2 \times 2\) matrix polynomials of degree \( n \) \([2, 13]\). Both of the Lax representations can be used to describe degenerate case, the \((n+1) \times (n+1)\) Lax equation make it easier to investigate the superintegrable properties of the system.

**Lemma 1.** The solution of (2) with initial condition of the form (3) is of the same form

\[
L(\lambda; t) = \lambda^2 A + \lambda q(t) \land p(t) - q(t) \otimes q(t)
\]

for all time \( t \in \mathbb{R} \). Moreover the vectors \( q(t), p(t) \in \mathbb{R}^{n+1} \) satisfy \( \|q(t)\| = 1 \) and \( \langle q(t), p(t) \rangle = 0 \) and satisfy Hamiltonian equations for Neumann system.
Proof. We will show, that the path of the form (4) can satisfy the Lax equation if \((q(t), p(t))\) are the solutions of the Neumann equations. Lemma follows from the uniqueness for the solution of the ordinary differential equations. Let’s put a curve of the form (4) into the Lax equation

\[
\frac{d}{dt}(\lambda^2 A + \lambda q \wedge p - q \otimes q) = [\lambda^2 A + \lambda q \wedge p - q \otimes q, \lambda A + q \wedge p] = 0 \cdot \lambda^3 + 0 \cdot \lambda^2 - [q \otimes q, A] \lambda - [q \otimes q, q \wedge p].
\]

and compare the terms at powers of \(\lambda\)

\[
\frac{dA}{dt} = 0
\]

\[
\frac{dq \wedge p}{dt} = -[q \otimes q, A]
\]

\[
\frac{dq \otimes q}{dt} = -[q \otimes q, q \wedge p].
\]

The leading coefficient \(A\) does not change and one can see directly that the second and third equations are satisfied if \(q(t), p(t)\) are solutions to the Neumann problem. From the uniqueness theorem for ordinary differential equations it follows that the isospectral flow \(L(\lambda; t)\) is always of the form (4).

Once we have an isospectral flow, its invariants can be described in terms of the characteristic polynomial and isospectral subsets of matrix polynomials. Indeed if the flow is isospectral, the characteristic polynomial

\[
P(\lambda, \mu) = \det(L(\lambda) - \mu)
\]

does not change along the flow. The characteristic polynomial is therefore a conserved object of the flow (its coefficients and also zeros are conserved quantities). The affine spectral curve of the flow (2) is defined as the set of points \((\lambda, \mu) \in \mathbb{C} \times \mathbb{C}\) that satisfy the characteristic equation

\[
\det(L(\lambda) - \mu) = 0
\]

and is also conserved along the flow. We can complete the affine spectral curve to obtain a compact Riemann surface. To obtain maximal smoothness the affine spectral curve is embedded into the surface \(O(2)\) given by two patches of \(\mathbb{C}^2\) glued together by the coordinate change

\[
(\lambda, \mu) \mapsto \left( \frac{1}{\lambda}, \frac{\mu}{\lambda^2} \right).
\]

We denote the latter coordinate by \(z = \frac{\mu}{\lambda^2}\).

Definition 2. The spectral curve of the flow (2) is the compactification of the affine spectral curve in the surface \(O(2)\). We denote it by \(C\).

There is an important relation between singularities of the spectral curve and regularity of the matrix \(L(\lambda)\).

Definition 3. We will call a matrix \(A \in \mathbb{C}^{n \times n}\) regular if all of its eigenvalues have one-dimensional eigenspace. Equivalently \(A\) is regular if all the eigenvalues have the geometric multiplicity equal to one.
It is easy to see, that if the matrix $\mathbf{L}(\lambda_0)$ is not regular, then the curve $\mathcal{C}$ has a singularity in at least one of the points of $\lambda^{-1}(\lambda_0)$. More precisely if $\mu_0$ is an eigenvalue of $\mathbf{L}(\lambda_0)$ with eigenspace of dimension grater than one, then the point $(\lambda_0, \mu_0) \in \mathcal{C}$ is singular (see [7] for proof).

### 2.2 Spectral curve for degenerate Neumann system

As the important singularities occur at $\infty$, we will calculate the affine curve in coordinates $(\lambda, z) = (\lambda, \frac{\Phi}{\lambda})$. The Lax matrix at infinity is given as a rank 2 perturbation of the matrix $\mathbf{A}$

$$\lambda^{-2} \mathbf{L}(\lambda) = \mathbf{A} + \lambda^{-1} \mathbf{q} \wedge \mathbf{p} - \lambda^{-2} \mathbf{q} \otimes \mathbf{q}.$$ 

To obtain the equation of the spectral curve, we follow [1, 4] and use Weinstei-Aronszajn determinant as follows

$$\frac{\det(z - \lambda^{-2} \mathbf{L}(\lambda))}{\det(z - \mathbf{A})} = 1 - \Phi_z(\mathbf{q}, \mathbf{p}),$$

where $\Phi_z(\mathbf{q}, \mathbf{p}) = -\lambda^{-2} \mathbf{Q}_z(\mathbf{q}) - \lambda^{-2} (\mathbf{Q}_z(\mathbf{q}) \mathbf{Q}_z(\mathbf{p}) - \mathbf{Q}_z^2(\mathbf{p}, \mathbf{q}))$ and

$$\mathbf{Q}_z(x, y) = \langle (z - \mathbf{A})^{-1} x, y \rangle$$

$$\mathbf{Q}_z(x) = \mathbf{Q}_z(x, x)$$

for any $x, y \in \mathbb{R}^{n+1}$. If we write $\Phi_z$ in co-ordinates

$$-\lambda^2 \Phi_z = \mathbf{Q}_z(\mathbf{q}) + \mathbf{Q}(\mathbf{q}) \mathbf{Q}_z(\mathbf{p}) - \mathbf{Q}_z^2(\mathbf{p}, \mathbf{q})$$

$$= \sum_{i=1}^{k} \frac{q_i^2}{z - a_i} + \sum_{i=1}^{k} \frac{q_i^2}{z - a_i} \sum_{i=1}^{k} \frac{p_i^2}{z - a_i} - \left( \sum_{i=1}^{k} \frac{p_i q_i}{z - a_i} \right)^2$$

we can write $\det(z - \lambda^{-2} \mathbf{L}(\lambda)) = \det(z - \mathbf{A})(1 - \Phi_z)$ and

$$\det(\mu - \mathbf{L}) = \lambda^{2(n+1)} \det(z - \lambda^{-2} \mathbf{L}) = \lambda^{2n+2} \prod_{i=1}^{k} \left( z - a_i \right) \left( 1 + \lambda^{-2} (\mathbf{Q}_z(\mathbf{q}) - \mathbf{Q}_z(\mathbf{q}) \mathbf{Q}_z(\mathbf{p}) + \mathbf{Q}_z^2(\mathbf{p}, \mathbf{q})) \right)$$

$$= \lambda^{2n} \prod_{i=1}^{k} \left( z - a_i \right) \left( \lambda^2 + \mathbf{Q}_z(\mathbf{q}) - \mathbf{Q}_z(\mathbf{q}) \mathbf{Q}_z(\mathbf{p}) + \mathbf{Q}_z^2(\mathbf{p}, \mathbf{q}) \right).$$

Let us assume, that $\mathbf{A}$ is diagonal and has eigenvalues $a_i$, $i = 1, \ldots, k$ with multiplicities $m_i \geq 1$. Denote by $V_i \subset \{1, \ldots, n+1\}$ a subset of indexes for which the diagonal element of $\mathbf{A}$ is $a_i$. Denote by

$$||\mathbf{q}||_i^2 = \sum_{j \in V_i} q_j^2, \quad \langle \mathbf{q}, \mathbf{p} \rangle_i = \sum_{j \in V_i} q_j p_j$$

and $||\mathbf{p}||_i^2 = \sum_{j \in V_i} p_j^2$.

Then $\Phi_z$ can be written as

$$-\lambda^2 \Phi_z(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{k} \frac{||\mathbf{q}||_i^2}{z - a_i} + \sum_{i=1}^{k} \frac{||\mathbf{q}||_i^2}{z - a_i} \sum_{i=1}^{k} \frac{||\mathbf{p}||_i^2}{z - a_i} - \left( \sum_{i=1}^{k} \frac{\langle \mathbf{q}, \mathbf{p} \rangle_i}{z - a_i} \right)^2$$

$$= \sum_{i=1}^{k} \frac{F_i}{z - a_i} + \sum_{m_i > 1} \frac{K_i^2}{(z - a_i)^2}$$

$$= \frac{Q_{\mathbf{q}, \mathbf{p}}(z)}{\prod_{i,m_i=1}^{k} (z - a_i) \prod_{j,m_j>1} (z - a_j)^2}.$$
for a polynomial \(Q_{q,p}(z)\) of degree \(k - 1 + r\), where \(r\) is the number of eigenvalues with multiplicity \(> 1\). If the multiplicity of an eigenvalue \(a_i\) is 1, the terms with \((z - a_i)^2\) cancel, if on the other hand, the multiplicity is greater than one, the quadratic term remains. We have obtained a set of commuting first integrals \(F_i\) for \(i = 1, \ldots, k\) and \(K^2_j\) for \(m_j > 1\) for the degenerate Neumann system, which were first described in [4]. The integrals \(F_i\) are known as Uhlenbeck’s integrals for non-degenerate case, while the integrals \(K_j\) are total angular momenta

\[
K_j = \sum_{v,u \in V_j} (q_v p_u - q_u p_v)^2
\]

for the action of the symmetry group of rotations \(O(m_j)\) acting on the eigenspace of \(A\) for the eigenvalue \(a_j\). Moreover

\[
\sum F_i = 1
\]

and Hamiltonian \(H\) can be written as

\[
H = \sum a_i F_i + \frac{1}{2} \sum K^2_j.
\]

The number of first integrals we have obtained from the spectral curve is \(k - 1 + r < n\) and is smaller than the number of degrees of freedom. There are more integrals (arising from the actions of the orthogonal group \(O(m_j)\)), which do not appear in the algebraic picture. Taking \(\mu = \lambda^2 z\), the spectral polynomial becomes

\[
\det(\mu - L) = \lambda^{2n} \prod_{i=1}^{k} (z - a_i)^{m_i} \left( \lambda^2 + \frac{Q_{q,p}(z)}{\prod(z - a_i) \prod(z - a_j)^2} \right)
\]

\[
= \lambda^{2n} \prod_{i,m_i>1} (z - a_i)^{m_i-2} \left( \lambda^2 \prod_{m_i=1} (z - a_i) \prod_{m_i>1} (z - a_j)^2 + Q_{q,p}(z) \right). \quad (5)
\]

**Proposition 1.** The spectral curve \(C\) is an union of the following components

1. a point \(\lambda = 0, \mu = 0\)
2. a collection of parabolas \(\mu = a_i \lambda^2\), coming from \((z - a_i) = 0\), for \(i\) such that \(m_i > 1\).
3. a singular hyperelliptic curve \(C_h\)

\[
\lambda^2 = \frac{Q_{q,p}(z)}{\prod_{i,m_i=1}(z - a_i) \prod_{j,m_j>1}(z - a_j)^2}
\]

with the arithmetic genus \(g_a = k + r - 1\). We see that all of the first integrals \(F_i\) and \(K_j\) coming from the spectral curve are encoded in the hyperelliptic part \(C_h\). Also note, that the genus of the parabolas \(\mu = a_i \lambda^2\) is 0. This suggests that the flow of the system will be entirely described by the hyperelliptic curve \(C_h\) and its generalized Jacobian. We will also show
that the arithmetic genus \(k + r - 1\) gives the dimension of the invariant tori we can expect for the Neumann system.

The desingularisation of the hyperelliptic curve \(C_h\) is given by the equation

\[w^2 = \left( \lambda \prod_{i=1}^{k} (z - a_i) \right)^2 = Q_{q,p}(z) \prod_{i,m_i=1} \left( z - a_i \right)\]

and is of genus \(g = \left[k - \frac{1}{2}m\right] - 1\). We denote it by \(\tilde{C}_h\).

**Remark 2.** It has been shown that the normalized spectral curve \(\tilde{C}_h\) can be used to describe the dynamics of the Rosochatius system, which can be seen as a reduction of degenerate Neumann system by \(SO(m_j)\) symmetries \([5]\). We will show later in this paper that the singular hyperelliptic curve \(C_h\) describes the dynamics of the degenerate Neumann system and that the set of parabolas \(\mu = a_j \lambda^2\) accounts for the loss of dimensions in the invariant tori as a result of the superintegrability of the system.

### 3 Invariants of the flow

We have seen, that the isospectral nature of the flow \([2]\) reveals the first integrals of the system encoded as coefficients in the hyperelliptic part of the spectral curve. We have also observed, that not all of the first integrals can be expressed in terms of the spectral data. Next we would like to describe the invariant sets of the isospectral flow \([2]\) and hence the invariant tori of the Neumann system.

If an integrable system is not superintegrable, then the generic invariant sets of its flow are Arnold-Liouville tori. In the case of superintegrability, the generic invariant sets are still tori, but of lower dimension. We will show, that invariant tori of the degenerate Neumann system can be obtained naturally as real parts of the generalized Jacobian of hyperelliptic part \(C_h\) of the spectral curve.

Both the characteristic polynomial as well as the spectral curve are invariants of the flow, so the trajectories of \([2]\) lie inside the subspace of all the matrix polynomials \(L(\lambda)\) with the same spectral polynomial (or in fact the same spectral curve).

\[\mathcal{M}_C = \{ L(\lambda) = A \lambda^2 + B \lambda + C; \text{ spectral curve of } L(\lambda) \text{ is } C \}\]

The set \(\mathcal{M}_C\) is way too large to be the minimal invariant set. We therefore look for additional invariants. The flow of \([2]\) conserves the leading term \(A\), so the set

\[\mathcal{M}_{C,A} = \{ L(\lambda); \ L \in \mathcal{M}_C \text{ and } L(\infty) = A \}\]

is also an invariant of the flow. The set \(\mathcal{M}_{C,A}\) is too large to be minimal invariant torus. For the submatrices of \(L(\lambda)\), where \(A\) has a non-regular block, the second highest power of \(\lambda\) is also preserved by the flow. Let us denote by \(K_j\) the sub-matrix of \(B\) that correspond to eigenspace of \(A\) for eigenvalue \(a_j\) with multiplicity \(m_j > 1\). In case of the degenerate Neumann system the matrix \(K_j\) is an antisymmetric matrix whose entries are

\[k_{ik} = q_{v_j(i)} p_{v_j(k)} - q_{v_j(k)} p_{v_j(i)}\]
for indexes $v_j(i), v_j(k) \in V_j$. The entries $k_{ik}$ are angular momenta in $q_{v_j(i)}, q_{v_j(k)}$ plane and are the components of the momentum map for the $O(m_j)$ action on the eigenspace of $A$ corresponding to eigenvalue $a_j$. In fact the block $K_j \in \mathfrak{so}(m_j)$ is the value of the momentum map itself. Since the Hamiltonian $H$ is preserved by those actions, the blocks $K_j$ are also a conserved quantity. The isospectral flow (2) will therefore fix the blocks $K_j$ for all the eigenvalues $a_j$, with multiplicity greater than 1. We can then restrict ourselves to the isospectral set

$$M_{C,A,K_j} = \{L(\lambda); \quad L(\lambda) \in M_C, A, K_j \text{ are fixed}\}.$$  

By the result in [8] the isospectral manifold $M_{C,A,K_j}$ is isomorphic as an algebraic manifold to the Zariski open subset $J(C) - \Theta$ of the generalized Jacobian of the singularization of the spectral curve $C$, given by a modulus

$$m = \sum_{P \in A^{-1}(\infty)} P$$

on the normalization $\tilde{C}$ of the spectral curve. Furthermore the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & PG_{A,K_j} & \longrightarrow & M_{C,A,K_j} & \longrightarrow & M_{C,A,K_j}/PG_{A,K_j} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (C^*)^n & \longrightarrow & Jac(C) - \Theta & \longrightarrow & Jac(\tilde{C}) - \Theta & \longrightarrow & 0
\end{array}$$

where the group $PG_{A,K_j}$ is the subgroup of the projective general linear group that leaves the matrix $A$ and the sub-matrices $K_j$ invariant by conjugation. If $K_j$ are regular, the group $PG_{A,K_j}$ is isomorphic to a complex torus $(C^*)^n$, since both $A$ and $K_j$ can be diagonalized. The stabilizer $G_{A,K_j}$ is a product

$$G_{A,K_j} = T_A \times \prod_{m_j \geq 2} \mathbb{P} stab K_j$$

where $T_A$ consists of diagonal matrices $[t_{ij}]$ such that $t_{kl} = t_{kk}$ if the indexes $k$ and $l$ correspond to the same eigenvalue $a_j$ of $A$. The group of stab $K_j$ consist of all the matrices that can be diagonalized with the same coordinate change as $K_j$. The subgroup $T_j = \{exp(t(K_j); t \in C\}$ generated by $K_j$ is a subgroup of the stabilizer stab $K_j$. Denote by $T_{A,K_j}$ the group

$$T_{A,K_j} = T_A \times \prod_{m_j \geq 2} \mathbb{P} stab K_j / T_j.$$  

The orbits of the group $T_{A,K_j}$ are transversal to the isospectral flow (2), so the right space to consider as the invariant set is the quotient $M_{C,A,K_j}/T_{A,K_j}$. On the other hand we have seen from lemma [1] that the set

$$P_A = \{A\lambda^2 + \lambda p \wedge q - q \otimes q; \quad q, p \in C^{n+1} \times C^{n+1}\}$$

is also invariant with respect to the flow (2). The map $(p, q) \mapsto A\lambda^2 + \lambda p \wedge q - q \otimes q$ gives a parametrization of the complexified phase space of the Neumann system into the set of matrix polynomials. The appropriate invariant set would therefore be the intersection $P_A \cap M_{C,A,K_j}$, which we denote by $M_N$. The following theorem describes the relation among $M_N$ and $M_{C,A,K_j}/T_{A,K_j}$. 

8
Theorem 1. For any initial condition of the Neumann system such that the normalized hyperelliptic part $\tilde{C}_h$ of the spectral curve is smooth the following holds.

1. If $K_j$ is regular, the quotient of $\mathcal{M}_N$ by a discrete group is isomorphic as an algebraic variety to the open subset of the generalized Jacobian $\text{Jac}(\tilde{C}_h) - \Theta \simeq \mathcal{M}_{C,A,K_j}/\Gamma_{A,K_j}$ of the hyperelliptic spectral curve $\tilde{C}_h$.

2. If $K_j = 0$ ($K_j = 0$) the set $\mathcal{M}_N$ is isomorphic as an algebraic variety to the open subset $\text{Jac}(\tilde{C}_h)$ - $\Theta \simeq \mathcal{M}_{C,A,K_j}/\mathcal{P}G_{A,K_j}$ of the Jacobian of the desingularized hyperelliptic curve for $\tilde{C}_h$.

3. The flow generated by the Hamiltonian $H$ is linear on the generalized Jacobian $\text{Jac}(\tilde{C}_h)$ if $K_j$ regular or Jacobian $\text{Jac}(\tilde{C}_h)$ if $K_j = 0$.

Remark 3. The intermediate cases where $K_j \neq 0$ and $K_j$ are non-regular appears for dimensions greater than 3 and will not be covered in this article.

Proof. We can define the eigenbundle map from the isospectral set $\mathcal{M}_{C,A,K_j}$ to $\text{Jac}(\tilde{C}_h)$

$$e : \mathcal{M}_{C,A,K_j} \rightarrow \text{Jac}(\tilde{C}_h)$$

by normalizing eigenline bundles on $\tilde{C}_h$ as in [7, 8, 5]. The set $\mathcal{M}_{C,A,K_j}$ is isomorphic to an open subset of the generalized Jacobian of the singular spectral curve $C_m$, defined by the modulus

$$m = \sum_{p \in b^{-1}(\infty)} P$$

on the reducible normalized spectral curve $\tilde{C}$ [8]. We also see, that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{M}_{C,A,K_j} & \longrightarrow & \mathcal{M}_{C,A,K_j}/\Gamma_{A,K_j} \\
e & \downarrow & \downarrow e \quad \varepsilon \\
\text{Jac}(\tilde{C}_m) - \Theta_m & \longrightarrow & \text{Jac}(\tilde{C}_h) - \Theta_h
\end{array}$$

where $\Gamma_{A} = \mathcal{P}G_{A,K_j}/\prod_{m_j \geq 2} T_j$ and $T_j = \{\exp(tK_j)\}$ is a subgroup generated by the elements $K_j \in \mathfrak{so}(m_j) \subset \mathfrak{so}(n+1)$.

We have to prove, that the orbit of $\Gamma_{A} \mathcal{M}_{C,A,K_j}/\Gamma_{A}$ intersects the set $\mathcal{M}_N$ in finitely many points.

Since $K_j$ are anti-symmetric all of its eigenvalues are pure imaginary and are of multiplicity one if $K_j$ is a regular matrix, the group $\mathcal{P}G_{A,K_j}$ is isomorphic to the torus $T_c^n = (\mathbb{C}^*)^n$. We only have to prove that the set $\mathcal{M}_N$ consisting of matrices of the form (3) is a covering of the quotient $\mathcal{M}_{C,A,K_j}/\Gamma_{A,K_j}$. Or in other words (3) gives a parametrization of the quotient $\mathcal{M}_{C,A,K_j}/\Gamma_{A,K_j}$ by $(q,p) \in F^{-1}(f) \subset (T^*S^n)^C$. First note that the map

$$J^A : \mathcal{M}_N \rightarrow \mathcal{M}_{C,A,K_j}$$

defined by (3) is an immersion. We will show that any orbit of $\Gamma_{A}$ intersects the image of $J^A$ only in finite number of points. To explain how $\mathcal{P}G_{A,K_j}$ acts
on the Lax matrix $J$, note that an element $g \in \mathbb{P}GL(n+1)$ acts on a tensor product $x \otimes y$ of $x, y \in \mathbb{C}^{n+1}$

$$g : x \otimes y \mapsto (gx) \otimes (g^{-1})^Ty$$

by multiplying the first factor with $g$ and the second with $(g^{-1})^T$. The subgroup of $\mathbb{P}G_{A, K_j}$, for which the generic orbit lies in the image of $J^A$, is given by orthogonal matrices

$$D := O(n, \mathbb{C}) \cap \mathbb{P}G_{A, K_j} \simeq (\mathbb{Z}_2)^{n-r-1} \times \prod_{m_j > 2} (\mathbb{T}_j \cap O(m_j, \mathbb{C})).$$

There are special points in the image of $J^A$ that have a large isotropy group (take for example $q_i = \delta_{ij}$ and $p_i = \delta_{ik}$, $k \neq j$, where the isotropy is $(\mathbb{C}^*)^{n-2}$). But the intersection of any orbit with the image of $J^A$ coincides with the orbit of $D$. If we take the torus $\mathbb{T}_{A, K_j} < \mathbb{P}G_{A, K_j}$ the orbits of $\mathbb{T}_{A, K_j}$ will intersect image of $J^A$ only in the orbit of the finite subgroup $(\mathbb{Z}_2)^{n-r-1}$, since $\mathbb{T}_{A, K_j} \cap \mathbb{T}_j = \{Id\}$.

We have proved that the level set of Lax matrices $\mathcal{M}_N = \{L(\lambda) = J^A(q, p)\}$ is an immersed sub-manifold of $\mathcal{M}_{C, A, K_j}$ that intersects the orbits of torus $\mathbb{T}_{A, K_j} \simeq (\mathbb{C}^*)^{n-1}$ in only finite number of points and is thus a covering of the quotient $\mathcal{M}_{C, A, K_j}/\mathbb{T}_{A, K_j}$.

To prove the assertion $[3]$, note that the matrix polynomial $M(\lambda)$ in the Lax equation (2) is given as a polynomial part $R(\lambda, L(\lambda))_s$ for a polynomial $R(z, w) = zw$ and such isospectral flows are mapped by $e$ to linear flows on the Jacobian $\text{Jac}(\mathcal{C})$ (see [13] for reference).

Taking into account the real structure on $\mathcal{C}$, the invariant tori on the Hamiltonian flow generated by $H$ can be described as a real part of the generalized Jacobian.

**Theorem 2.** For any initial condition of the Neumann system such that $K_j$ are regular and $\tilde{C}_h$ smooth the following holds.

1. the invariant tori of the flow of $H$ are $(\mathbb{Z}_2)^{n-r-1}$ coverings of the real part of the generalized Jacobian.

2. The rotations generated by total angular momenta $K_j^2$ are precisely the rotations of the factors $S^1$ in the fiber $\mathbb{T}^r$ in the fibration $\mathbb{T}^r \to \text{Jac}(\mathcal{C})^S \to \text{Jac}(\mathcal{C})^S$, which is the real part of the fibration $(\mathbb{C}^*)^r \to \text{Jac}(C_m) \to \text{Jac}(\mathcal{C})$.

**Proof.** The groups $\mathbb{T}_j$ defined in the proof of theorem 1 are the complexification of the group of rotations in the eigenspace of $A$ for the eigenvalue $a_j$ that are generated by the Hamiltonian vector fields of $K_j^2$. The flows of the total angular momenta $K_j$ therefore generate the fibers $\mathbb{C}^+$ in the extension $(\mathbb{C}^*)^r \to \text{Jac}(\mathcal{C}_h) \to \text{Jac}(\mathcal{C}_{\tilde{h}})$.

To prove the claim about the real part, let us denote by $S_j$ singular points of $\mathcal{C}_h$ corresponding to the eigenvalue $a_j$ of the potential $A$ with multiplicity $> 1$. Since we assumed $K_j$ is regular, two points $P_j^+$ and $P_j^-$ on the smooth curve $\tilde{C}_h$ map to $S_j$. Since the eigenvalues of $K_j$ are pure imaginary, the value of $\mu$ at the points $P_j^\pm$ is also pure imaginary. On $\mathcal{C}_h$ and $\tilde{C}_h$ there is a natural real structure $J$ induced by the conjugation of $(\lambda, \mu) \in \mathbb{C}^2$. The points $P_j^\pm$ that are glued in the singular point $S_j$ form a conjugate pair $P_j^+ = JP_j^-$. If we follow
the argument in [15] we can find that the real structure of the fiber \((\mathbb{C}^*)^r\) in the extension \((\mathbb{C}^*)^r \to \text{Jac}(\mathcal{G}_h) \to \text{Jac}(\tilde{\mathcal{G}}_h)\) is given by the maps

\[ z_j \mapsto \frac{1}{\bar{z}_j} \]

and that the real part of \((\mathbb{C}^*)^r\) is the torus \((S^1)^r\) given by \(z_j\bar{z}_j = 1\).

The above theorems can be used to describe the structure of the degenerate Neumann system as a superintegrable system. The set of local actions that generates the invariant tori of the Neumann system depends only on the spectral invariants \(F_1, \ldots, F_{k-r^2-1}, K^2_1, \ldots, K^2_r\), that come from the hyperelliptic part of the spectral curve. Apart from the spectral invariants, there is another set of first integrals given by the functions of matrices \(K_j \in \mathfrak{so}(m_j)\) that are not the functions of the total angular momenta \(K^2_j\). The latter first integrals correspond to \(O(m_j)\) symmetries of the degenerate Neumann system and their Hamiltonian flows are transversal to the invariant tori.

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