LINEAR ALMOST POISSON STRUCTURES AND HAMILTON-JACOBI EQUATION.
APPLICATIONS TO NONHOLONOMIC MECHANICS

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Abstract. In this paper, we study the underlying geometry in the classical Hamilton-Jacobi equation. The proposed formalism is also valid for nonholonomic systems. We first introduce the essential geometric ingredients: a vector bundle, a linear almost Poisson structure and a Hamiltonian function, both on the dual bundle (a Hamiltonian system). From them, it is possible to formulate the Hamilton-Jacobi equation, obtaining as a particular case, the classical theory. The main application in this paper is to nonholonomic mechanical systems. For it, we first construct the linear almost Poisson structure on the dual space of the vector bundle of admissible directions, and then, apply the Hamilton-Jacobi theorem. Another important fact in our paper is the use of the orbit theorem to simplify the Hamilton-Jacobi equation, the introduction of the notion of morphisms preserving the Hamiltonian system; indeed, this concept will be very useful to treat with reduction procedures for systems with symmetries. Several detailed examples are given to illustrate the utility of these new developments.

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1. Introduction

The standard Hamilton-Jacobi equation is the first-order, non-linear partial differential equation,
\[
\frac{\partial S}{\partial t} + H(q^A, \frac{\partial S}{\partial q^A}) = 0,
\]
for a function \(S(t, q^A)\) (called the principal function) and where \(H\) is the Hamiltonian function of the system. Taking \(S(t, q^A) = W(q^A) - tE\), where \(E\) is a constant, we rewrite the previous equations as
\[
H(q^A, \frac{\partial W}{\partial q^A}) = E,
\]
where \(W\) is called the characteristic function. Equations (1.1) and (1.2) are indistinctly referred as the Hamilton-Jacobi equation (see [1, 11]; see also [6] for a recent geometrical approach).

The motivation of the present paper is to extend this theory for the case of nonholonomic mechanical systems, that is, those mechanical systems subject to linear constraints on the velocities. In Remark 5.11 of our paper, we carefully summarize previous approaches to this subject. These tried to adapt the standard Hamilton-Jacobi equations for systems without constraints to the nonholonomic setting. But for nonholonomic mechanics it is necessary to take into account that the dynamics is obtained from an almost Poisson bracket, that is, a bracket not satisfying the Jacobi identity. In this direction, in a recent paper [20], the authors have developed a new approach which permits to extend the Hamilton-Jacobi equation to nonholonomic mechanical systems. However, the expression of the corresponding Hamilton-Jacobi equation is far from the standard Hamilton-Jacobi equation for unconstrained systems. This fact has motivated the present discussion since it was necessary to understand the underlying geometric structure in the proposed Hamilton-Jacobi equation for nonholonomic systems.

To go further in this direction, we need a new framework, which captures the non-Hamiltonian essence of a nonholonomic problem. Thus, we have considered a more general minimal “Hamiltonian” scenario. The starting point is a vector bundle \(\tau_D : D \rightarrow Q\) such that its dual vector bundle \(\tau_{D^*} : D^* \rightarrow Q\) is equipped with a linear almost Poisson bracket \(\{\cdot, \cdot\}_{D^*}\), that is, a linear bracket satisfying all the usual properties of a Poisson bracket except the Jacobi identity. The existence of such bracket is equivalent to the existence of an skew-symmetric algebroid structure \([\cdot, \cdot]_D, \rho_D\) on \(\tau_D : D \rightarrow Q\) (i.e. a Lie algebroid structure eliminating the integrability property), or even, the existence of an almost differential \(d^D\) on \(\tau_D : D \rightarrow Q\), that is, an operator \(d^D\) which acts on the “forms” on \(D\) and it satisfies all the properties of an standard differential except that \((d^D)^2\) is not, in general, zero. We remark that skew-symmetric algebroid structures are almost Lie structures in the terminology of [34] (see also [35]) and that the one-to-one correspondence between skew-symmetric algebroids and almost differentials was obtained in [34]. We also note that an skew-symmetric algebroid also is called a pre-Lie algebroid in the terminology introduced in some papers (see, for instance, [15] [16] [23]) and the relation between linear almost Poisson brackets and skew-symmetric algebroid structures was discussed in these papers (see also [12] [13] for some applications to Classical Mechanics).

In this framework, a Hamiltonian system is given by a Hamiltonian function \(h : D^* \rightarrow \mathbb{R}\). The dynamics is provided by the corresponding Hamiltonian vector field \(\mathcal{H}_h^{\Lambda_D^*}\) (\(\mathcal{H}_h^{\Lambda_D^*}(f) = \{f, h\}_{D^*}\), for all real function \(f\) on \(D^*\)). Here, \(\Lambda_D^*\) is the almost Poisson tensor field defined from \(\{\cdot, \cdot\}_{D^*}\). The reader can immediately recognize that we are extending the standard model, where \(D = TQ\),
$D^* = T^*Q$ and $[\cdot, \cdot]_D$ is the usual Lie bracket of vector fields which is related with the canonical Poisson bracket on $T^*Q$, so that $d\alpha$ is just, in this case, the usual exterior differential. Another important fact is the introduction of the notion of morphisms preserving the Hamiltonian system; indeed, this concept will be very useful to treat with reduction procedures for nonholonomic systems with symmetries. We remark that this type of procedures were intensively discussed in the seminal paper [3] by Bloch et al.

In the above framework we can prove the main result of our paper: Theorem 4.1 In this theorem, we obtain the Hamilton-Jacobi equation whose expression seems a natural extension of the classical Hamilton-Jacobi equation for unconstrained systems, as appears, for instance, in [1]. Moreover, our construction is preserved under the natural morphisms of the theory. This fact is proved in Theorem 4.12.

Furthermore, using the orbit theorem (see [2]), we will show that the classical form of the Hamilton-Jacobi equation: $H \circ \alpha = \text{constant}$, with $\alpha : Q \rightarrow D^*$ satisfying $d\alpha = 0$, remains valid for a special class of nonholonomic mechanical systems: those satisfying the condition of being completely nonholonomic. See Section 3 for more details and also the paper by Ohsawa and Bloch [31] for the particular case when $D$ is a distribution on $Q$.

The above theorems are applied to the theory of mechanical systems subjected to linear nonholonomic constraints on a Lie algebroid $A$. The ingredients of this theory are a Lie algebroid $\tau_A : A \rightarrow Q$ over a manifold $Q$, a Lagrangian function $L : A \rightarrow \mathbb{R}$ of mechanical type, and a vector subbundle $\tau_D : D \rightarrow Q$ of $A$. The total space $D$ of this vector subbundle is the constraint submanifold (see [7]). Then, using the corresponding linear Poisson structure on $A^*$, one may introduce a linear almost Poisson bracket on $D^*$, the so-called nonholonomic bracket. A linear almost Poisson bracket on $D$ which is isomorphic to the nonholonomic bracket was considered in [7]; however, it should be remarked that our formalism simplifies very much the procedure to obtain it. Using all these ingredients one can apply the general procedure (Theorems 4.1 and 4.12) to obtain new and interesting results. We also remark that the main part of the relevant information for developing the Hamilton-Jacobi equation for the nonholonomic system $(L, D)$ is contained in the vector subbundle $D$ or, equivalently, in its dual $D^*$ (see Theorems 4.1 and 4.12). Then, the computational cost is lower than in previous approximations to the theory.

In the particular case when $A$ is the standard Lie algebroid $\tau_Q : TQ \rightarrow Q$ then the constraint subbundle is a distribution $D$ on $Q$. The linear almost Poisson bracket on $D^*$ is provided by the classical nonholonomic bracket (which is usually induced from the canonical Poisson bracket on $T^*Q$), clarifying previous constructions [5, 19, 39]. In addition, as a consequence, we recover some of the results obtained in [20] about the Hamilton-Jacobi equation for nonholonomic mechanical systems (see Corollary 5.9). Moreover, we apply our results to an explicit example: the two-wheeled carriage. On the other hand, if our Lagrangian system on an arbitrary Lie algebroid $A$ is unconstrained (that is, the constraint subbundle $\bar{D} = A$) then, using our general theory, we recover some results on the Hamilton-Jacobi equation for Lie algebroids (see Corollary 5.1) which were proved in [25]. Furthermore, if $A$ is the standard Lie algebroid $\tau_Q : TQ \rightarrow Q$ then we directly deduce some well-known facts about the classical Hamilton-Jacobi equation (see Corollary 5.2).

Another interesting application is discussed; the particular case when the Lie algebroid is the Atiyah algebroid $\tau_A : A = TQ/G \rightarrow Q = Q/G$ associated with a principal $G$-bundle $F : Q \rightarrow \bar{Q} = Q/G$. In such a case, we have a Lagrangian function $L : A \rightarrow \mathbb{R}$ of mechanical type and a constraint subbundle $\tau_D : \bar{D} \rightarrow \bar{Q}$ of $\tau_A : \bar{A} = TQ/G \rightarrow \bar{Q}$. This nonholonomic system is precisely
the reduction, in the sense of Theorem 4.12, of a constrained system \((L, D)\) on the standard Lie algebroid \(\tau_A : A = TQ \to Q\). In fact, using Theorem 4.12 we deduce that the solutions of the Hamilton-Jacobi equations for both systems are related in a natural way by projection. We also characterize the nonholonomic bracket on \(\bar{D}^*\). All these results are applied to a very interesting example: the snakeboard. In this example, an explicit expression of the reduced nonholonomic bracket is found; moreover, the Hamilton-Jacobi equations are proposed and it is shown the utility of our framework to integrate the equations of motion.

We expect that the results of this paper will be useful for analytical integration of many difficult systems (see, as an example, the detailed study of the snakeboard in this paper and the examples in [31]) and the key for the construction of geometric integrators based on the Hamilton-Jacobi equation (see, for instance, Chapter VI in [18] and references therein for the particular case of standard nonholonomic mechanical systems).

The structure of the paper is as follows. In Section 2, the relation between linear almost Poisson structures on a vector bundle, skew-symmetric algebroids and almost differentials is obtained. In Section 3, we introduce the notion of a completely nonholonomic skew-symmetric algebroid and we prove that on an algebroid \(D\) of this kind with connected base \(Q\) the space \(H^0(dD) = \{f \in C^\infty(Q)/dDf = 0\}\) is isomorphic to \(\mathbb{R}\). We also prove that on an arbitrary skew-symmetric algebroid \(D\) the condition \(dDf = 0\) implies that \(f\) is constant on the leaves of a certain generalized foliation (see Theorem 3.4). For this purpose, we will use the orbit theorem. In Section 4, we consider Hamiltonian systems associated with a linear almost Poisson structure on the dual bundle \(D^*\) to a vector bundle and a Hamiltonian function on \(D^*\). Then, the Hamilton-Jacobi equation is proposed in this setting. Moreover, using the results of Section 3, we obtain an interesting expression of this equation. In Section 5, we apply the previous results to nonholonomic mechanical systems and, in particular, to some explicit examples. Moreover, we review in this section some previous approaches to the topic. We conclude our paper with the future lines of work and an appendix with the proof of some technical results.

2. Linear almost Poisson structures, skew-symmetric algebroids and almost differentials

Most of the results contained in this section are well-known in the literature (see [14,15,16,34,35]). However, to make the paper more self-contained, we will include their proofs.

Let \(\tau_D : D \to Q\) be a vector bundle of rank \(n\) over a manifold \(Q\) of dimension \(m\). Denote by \(D^*\) the dual vector bundle to \(D\) and by \(\tau_{D^*} : D^* \to Q\) the corresponding vector bundle projection.

**Definition 2.1.** A linear almost Poisson structure on \(D^*\) is a bracket of functions \(\{\cdot, \cdot\}_{D^*} : C^\infty(D^*) \times C^\infty(D^*) \to C^\infty(D^*)\) such that:

(i) \(\{\cdot, \cdot\}_{D^*}\) is skew-symmetric, that is,
\[
\{\varphi, \psi\}_{D^*} = -\{\psi, \varphi\}_{D^*}, \quad \text{for } \varphi, \psi \in C^\infty(D^*).
\]

(ii) \(\{\cdot, \cdot\}_{D^*}\) satisfies the Leibniz rule, that is,
\[
\{\varphi \varphi', \psi\}_{D^*} = \varphi \{\varphi', \psi\}_{D^*} + \varphi' \{\varphi, \psi\}_{D^*}, \quad \text{for } \varphi, \varphi', \psi \in C^\infty(D^*).
\]
(iii) $\{\cdot,\cdot\}_{D^*}$ is linear, that is, if $\varphi$ and $\psi$ are linear functions on $D^*$ then $\{\varphi,\psi\}_{D^*}$ is also a linear function.

If, in addition, the bracket satisfies the Jacobi identity then we have that $\{\cdot,\cdot\}_{D^*}$ is a linear Poisson structure on $D^*$.

Properties (i) and (ii) in Definition 2.1 imply that there exists a 2-vector $\Lambda_{D^*}$ on $D^*$ such that

$$\Lambda_{D^*}(d\varphi, d\psi) = \{\varphi, \psi\}_{D^*}, \quad \text{for} \quad \varphi, \psi \in C^\infty(D^*).$$

$\Lambda_{D^*}$ is called the linear almost Poisson 2-vector associated with the linear almost Poisson structure $\{\cdot,\cdot\}_{D^*}$.

Note that there exists a one-to-one correspondence between the space $\Gamma(\tau_D)$ of sections of the vector bundle $\tau_D : D \to Q$ and the space of linear functions on $D^*$. In fact, if $X \in \Gamma(\tau_D)$ then the corresponding linear function $\hat{X}$ on $D^*$ is given by

$$\hat{X}(\alpha) = \alpha(X(\tau_D(\alpha))), \quad \text{for} \quad \alpha \in D^*.$$

**Proposition 2.2.** Let $\{\cdot,\cdot\}_{D^*}$ be a linear almost Poisson structure on $D^*$.

(i) If $X$ is a section of $\tau_D : D \to Q$ and $f$ is a real $C^\infty$-function on $Q$ then the bracket $\{\hat{X}, f \circ \tau_D\}_{D^*}$ is a basic function with respect to the projection $\tau_{D^*}$.

(ii) If $f$ and $g$ are real $C^\infty$-functions on $Q$ then

$$\{f \circ \tau_{D^*}, g \circ \tau_{D^*}\}_{D^*} = 0.$$

**Proof.** Let $Y$ be an arbitrary section of $\tau_D : D \to Q$.

Using Definition 2.1, we have that

$$\{\hat{X}, (f \circ \tau_{D^*})\hat{Y}\}_{D^*} = (f \circ \tau_{D^*})\{\hat{X}, \hat{Y}\}_{D^*} + \{\hat{X}, f \circ \tau_{D^*}\}_{D^*}\hat{Y}$$

is a linear function on $D^*$. Thus, since $(f \circ \tau_{D^*})\{\hat{X}, \hat{Y}\}_{D^*}$ also is a linear function, it follows that $\{\hat{X}, f \circ \tau_{D^*}\}_{D^*}$ is a basic function with respect to $\tau_{D^*}$. This proves (i).

On the other hand, using (i) and Definition 2.1, we deduce that

$$\{(f \circ \tau_{D^*})\hat{Y}, g \circ \tau_{D^*}\}_{D^*} = (f \circ \tau_{D^*})\{(\hat{Y}, g \circ \tau_{D^*})_{D^*} + \{(f \circ \tau_{D^*}, g \circ \tau_{D^*})_{D^*}\hat{Y}

is a basic function with respect to $\tau_{D^*}$. Therefore, as $(f \circ \tau_{D^*})\{\hat{Y}, g \circ \tau_{D^*}\}_{D^*}$ also is a basic function with respect to $\tau_{D^*}$, we conclude that $\{f \circ \tau_{D^*}, g \circ \tau_{D^*}\}_{D^*} = 0$. This proves (ii).

If $(q')$ are local coordinates on an open subset $U$ of $Q$ and $\{X_\alpha\}$ is a basis of sections of the vector bundle $\tau_D^{-1}(U) \to U$ then we have the corresponding local coordinates $(q', p_\alpha)$ on $D^*$. Moreover, from Proposition 2.2 it follows that

$$\{p_\alpha, p_\beta\}_{D^*} = -C^\gamma_{\alpha\beta}p_\gamma, \quad \{q', p_\alpha\}_{D^*} = \rho^i_\alpha, \quad \{q', q'\}_{D^*} = 0,$$

with $C^\gamma_{\alpha\beta}$ and $\rho^i_\alpha$ real $C^\infty$-functions on $U$.

Consequently, the linear almost Poisson 2-vector $\Lambda_{D^*}$ has the following local expression

$$\Lambda_{D^*} = \rho^i_\alpha \frac{\partial}{\partial q'} \wedge \frac{\partial}{\partial p_\alpha} - \frac{1}{2} C^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\alpha} \wedge \frac{\partial}{\partial p_\beta}. \quad (2.1)$$
Definition 2.3. An **skew-symmetric algebroid structure** on the vector bundle $\tau_{D} : D \to Q$ is a $\mathbb{R}$-linear bracket $\{ \cdot, \cdot \}_D : \Gamma(\tau_{D}) \times \Gamma(\tau_{D}) \to \Gamma(\tau_{D})$ on the space $\Gamma(\tau_{D})$ and a vector bundle morphism $\rho_{D} : D \to TQ$, the **anchor map**, such that:

(i) $\{ \cdot, \cdot \}_D$ is skew-symmetric, that is,

$$[X,Y]_D = -[Y,X]_D, \quad \text{for } X,Y \in \Gamma(\tau_{D}).$$

(ii) If we also denote by $\rho_{D} : \Gamma(\tau_{D}) \to \mathfrak{X}(Q)$ the morphism of $C^\infty(Q)$-modules induced by the anchor map then

$$[X,fY]_D = f[X,Y]_D + \rho_{D}(X)(f)Y, \quad \text{for } X,Y \in \Gamma(D) \text{ and } f \in C^\infty(Q).$$

If the bracket $\{ \cdot, \cdot \}_D$ satisfies the Jacobi identity, we have that the pair $(\{ \cdot, \cdot \}_D, \rho_{D})$ is a **Lie algebroid structure** on the vector bundle $\tau_{D} : D \to Q$.

Remark 2.4. If $(\tau_{D}^*, \{ \cdot, \cdot \}_{D^*}, \rho_{D})$ is a Lie algebroid over $Q$, we may consider the generalized distribution $\tilde{D}$ whose characteristic space at a point $q \in Q$ is given by $\tilde{D}(q) = \rho_{D}(D_{q})$, where $D_{q}$ is the fibre of $D$ over $q$. The distribution $\tilde{D}$ is finitely generated and involutive. Thus, $\tilde{D}$ defines a generalized foliation on $Q$ in the sense of Sussmann [38]. $\tilde{D}$ is the **Lie algebroid foliation** on $Q$ associated with $D$.

Now, we will denote by $\mathcal{L}\mathcal{A}\mathcal{P}(D^*)$ (respectively, $\mathcal{L}\mathcal{P}(D^*)$) the set of linear almost Poisson structures (respectively, linear Poisson structures) on $D^*$. Denote also by $\mathcal{S}\mathcal{S}\mathcal{A}(D)$ (respectively, $\mathcal{L}\mathcal{A}(D)$) the set of skew-symmetric algebroid (respectively, Lie algebroid) structures on the vector bundle $\tau_{D} : D \to Q$. Then, we will see in the next theorem that there exists a one-to-one correspondence between $\mathcal{L}\mathcal{A}\mathcal{P}(D^*)$ (respectively, $\mathcal{L}\mathcal{P}(D^*)$) and the set of skew-symmetric algebroid (respectively, Lie algebroid) structures on $\tau_{D} : D \to Q$.

**Theorem 2.5.** There exists a one-to-one correspondence $\Psi$ between the sets $\mathcal{L}\mathcal{A}\mathcal{P}(D^*)$ and $\mathcal{S}\mathcal{S}\mathcal{A}(D)$. Under the bijection $\Psi$, the subset $\mathcal{L}\mathcal{P}(D^*)$ of $\mathcal{L}\mathcal{A}\mathcal{P}(D^*)$ corresponds with the subset $\mathcal{L}\mathcal{A}(D)$ of $\mathcal{S}\mathcal{S}\mathcal{A}(D)$. Moreover, if $\{ \cdot, \cdot \}_{D^*}$ is a linear almost Poisson structure on $D^*$ then the corresponding skew-symmetric algebroid structure $(\{ \cdot, \cdot \}_D, \rho_{D})$ on $D$ is characterized by the following conditions

$$\hat{[}X,Y\hat{]}_D = -\{ \hat{X},\hat{Y}\}_{D^*}, \quad \rho_{D}(X)(f) \circ \tau_{D^*} = \{ f \circ \tau_{D^*}, \hat{X}\}_{D^*}$$

for $X,Y \in \Gamma(\tau_{D})$ and $f \in C^\infty(Q)$.

**Proof.** Let $\{ \cdot, \cdot \}_{D^*}$ be a linear almost Poisson structure on $D^*$. Then, it is easy to prove that $\hat{[} \cdot, \cdot \hat{]}_D$ (defined as in (2.2)) is a $\mathbb{R}$-bilinear skew-symmetric bracket. Moreover, since $\{ \cdot, \cdot \}_{D^*}$ satisfies the Leibniz rule, it follows that $\rho_{D}(X)$ is a vector field on $Q$, for $X \in \Gamma(\tau_{D})$. In addition, using again that $\{ \cdot, \cdot \}_{D^*}$ satisfies the Leibniz rule and Proposition 2.2, we deduce that

$$\rho_{D}(gX) = g\rho_{D}(X), \quad \text{for } g \in C^\infty(Q) \text{ and } X \in \Gamma(\tau_{D}).$$

Thus, $\rho_{D} : \Gamma(\tau_{D}) \to \mathfrak{X}(Q)$ is a morphism of $C^\infty(Q)$-modules.

On the other hand, from (2.2), we obtain that

$$\hat{[}X,fY\hat{]}_D = -\{ \hat{X},(f \circ \tau_{D^*})\hat{Y}\}_{D^*} = (\rho_{D}(X)(f) \circ \tau_{D^*})\hat{Y} - (f \circ \tau_{D^*})\{ \hat{X},\hat{Y}\}_{D^*}.$$

Therefore,

$$[X,fY]_D = f[X,Y]_D + \rho_{D}(X)(f)Y$$
and \([\cdot, \cdot]_D, \rho_D\) is a skew-symmetric algebroid structure on \(\tau_D : D \to Q\). It is clear that if \([\cdot, \cdot]_{D*}\)

is a Poisson bracket then \([\cdot, \cdot]_D\) satisfies the Jacobi identity.

Conversely, if \([\cdot, \cdot]_D, \rho_D\) is a skew-symmetric algebroid structure on \(\tau_D : D \to Q\) and \(x \in Q\)

then one may prove that there exists an open subset \(U\) of \(Q\) and a unique linear almost Poisson structure on the vector bundle \(\tau_{\tau_D^{-1}(U)*} : \tau_D^{-1}(U)^* \to U\) such that

\[
\{\check{X}, \check{Y}\}_{\tau_D^{-1}(U)*} = -[X, Y]_{\tau_D^{-1}(U)*}, \quad \{f \circ \tau_{\tau_D^{-1}(U)*}, \check{X}\}_{\tau_D^{-1}(U)*} = \rho_{\tau_D^{-1}(U)}(X)(f) \circ \tau_{\tau_D^{-1}(U)*},
\]

and

\[
\{f \circ \tau_{\tau_D^{-1}(U)*}, g \circ \tau_{\tau_D^{-1}(U)*}\}_{\tau_D^{-1}(U)*} = 0,
\]

for \(X, Y\) sections of the vector bundle \(\tau_{\tau_D^{-1}(U)} : \tau_D^{-1}(U) \to U\) and \(f, g \in C^\infty(U)\). Here, \(([\cdot, \cdot]_D^{-1}(U), \rho_{\tau_D^{-1}(U)})\) is the skew-symmetric algebroid structure on \(\tau_D^{-1}(U)\) induced, in a natural way, by the skew-symmetric algebroid structure \(([\cdot, \cdot]_D, \rho_D)\) on \(D\). In addition, we have that if \([\cdot, \cdot]_D\) satisfies the Jacobi identity then \([\cdot, \cdot]_{\tau_D^{-1}(U)*}\) is a linear Poisson bracket on \(\tau_D^{-1}(U)^*\). Thus, we deduce that there exists a unique linear almost Poisson structure \([\cdot, \cdot]_{D*}\) on \(D^*\) such that (2.2) holds.

Let \([\cdot, \cdot]_{D*}\) be a linear almost Poisson structure on \(D^*\) and \(([\cdot, \cdot]_D, \rho_D)\) be the corresponding skew-symmetric algebroid structure on \(\tau_D : D \to Q\). If \((q^i)\) are local coordinates on an open subset \(U\) of \(Q\) and \(\{X_\alpha\}\) is a basis of sections of the vector bundle \(\tau_{\tau_D^{-1}(U)} : \tau_D^{-1}(U) \to U\) such that \(\Lambda_D\) is given by (2.1) (on \(\tau_{\tau_D^{-1}(U)}\)) then

\[
[X_\alpha, X_\beta]_D = C^\gamma_{\alpha\beta} X_\gamma, \quad \rho_D(X_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i}.
\]

\(C^\gamma_{\alpha\beta}\) and \(\rho^i_\alpha\) are called the local structure functions of the skew-symmetric algebroid structure \(([\cdot, \cdot]_D, \rho_D)\) with respect to the local coordinates \((q^i)\) and the basis \(\{X_\alpha\}\).

Next, we will see that there exists a one-to-one correspondence between \(SSA(D)\) and the set of almost differentials on the vector bundle \(\tau_D : D \to M\).

**Definition 2.6.** An almost differential on the vector bundle \(\tau_D : D \to Q\) is a \(\mathbb{R}\)-linear map

\[
d^D : \Gamma(\Lambda^k\tau_{D*}) \to \Gamma(\Lambda^{k+1}\tau_{D*}), \quad k \in \{0, \ldots, n - 1\}
\]

such that

\[
d^D(\alpha \wedge \beta) = d^D \alpha \wedge \beta + (-1)^k \alpha \wedge d^D \beta, \quad \text{for } \alpha \in \Gamma(\Lambda^k\tau_{D*}) \text{ and } \beta \in \Gamma(\Lambda^r\tau_{D*}). \tag{2.3}
\]

If \((d^D)^2 = 0\) then \(d^D\) is said to be a differential on the vector bundle \(\tau_D : D \to Q\).

Denote by \(\mathcal{AD}(D)\) (respectively, \(\mathcal{D}(D)\)) the set of almost differentials (respectively, differentials) on the vector bundle \(\tau_D : D \to Q\).

**Theorem 2.7.** There exists a one-to-one correspondence \(\Phi\) between the sets \(SSA(D)\) and \(\mathcal{AD}(D)\). Under the bijection \(\Phi\) the subset \(\mathcal{LA}(D)\) of \(SSA(D)\) corresponds with the subset \(\mathcal{D}(D)\) of \(\mathcal{AD}(D)\). Moreover, we have:

(i) If \(d^D\) is an almost differential on the vector bundle \(\tau_D : D \to Q\) then the corresponding skew-symmetric algebroid structure \(([\cdot, \cdot]_D, \rho_D)\) on \(D\) is characterized by the following conditions:

\[
\alpha([X, Y]_D) = d^D(\alpha(Y))(X) - d^D(\alpha(X))(Y) - (d^D \alpha)(X, Y), \quad \rho_D(X)(f) = (d^D f)(X), \tag{2.4}
\]

for \(X, Y \in \Gamma(\tau_D), \alpha \in \Gamma(\tau_{D*})\) and \(f \in C^\infty(Q)\).
(ii) If \([\cdot, \cdot]_D, \rho_D\) is an skew-symmetric algebroid structure on the vector bundle \(\tau_D : D \to Q\) then the corresponding almost differential \(d^D\) is defined by

\[
(d^D\alpha)(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \rho_D(X_i)(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j]_D, X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)
\]

(2.5)

for \(\alpha \in \Gamma(\Lambda^k\tau_D^*)\) and \(X_0, \ldots, X_k \in \Gamma(\tau_D)\).

Proof. Let \(d^D\) be an almost differential on \(\tau_D : D \to Q\) and suppose that \(\rho_D\) and \([\cdot, \cdot]_D\) are given by (2.4). Then, using the fact that

\[
d^D(ff') = fd^Df' + f'd^Df, \quad \text{for } f, f' \in C^\infty(Q),
\]

we deduce that \([\cdot, \cdot]_D, \rho_D\) is an skew-symmetric algebroid structure on \(\tau_D : D \to Q\). Moreover, it is well-known that if \((d^D)^2 = 0\) then \([\cdot, \cdot]_D\) satisfies the Jacobi identity and the pair \(([\cdot, \cdot]_D, \rho_D)\) is a Lie algebroid structure on \(\tau_D : D \to Q\) (see, for instance, [23, 45]).

Conversely, if \([\cdot, \cdot]_D, \rho_D\) is an skew-symmetric algebroid structure on \(\tau_D : D \to Q\) and \(d^D\) is the operator defined by (2.5) then, it is clear that,

\[
(d^D\alpha)(X, Y) = \rho_D(X)(\alpha(Y)) - \rho_D(Y)(\alpha(X)) - \alpha[X, Y]_D, \quad (d^Df)(X) = \rho_D(X)(f),
\]

for \(f \in C^\infty(M)\), \(\alpha \in \Gamma(\tau_D^*)\) and \(X, Y \in \Gamma(\tau_D)\). In addition, an straightforward computation proves that

\[
d^D(\alpha \wedge \beta) = d^D\alpha \wedge \beta + (-1)^k \alpha \wedge d^D\beta, \quad \text{for } \alpha \in \Gamma(\Lambda^k\tau_D^*) \text{ and } \beta \in \Gamma(\Lambda^r\tau_D^*).
\]

Finally, it is well-known that if \([\cdot, \cdot]_D\) satisfies the Jacobi identity then \((d^D)^2 = 0\) (see, for instance, [27]).

Let \(([\cdot, \cdot]_D, \rho_D)\) be a skew-symmetric algebroid structure on the vector bundle \(\tau_D : D \to Q\) and \(d^D\) be the corresponding almost differential. If \((q^i)\) are local coordinates on an open subset \(U\) of \(Q\) and \(\{X_\alpha\}\) is a basis of sections of the vector bundle \(\tau_D^{-1}(U) \to U\) such that \(C^\gamma_{\alpha\beta}\) and \(\rho_\alpha^\beta\) are the local structure functions of the skew-symmetric algebroid structure, then

\[
d^Dx^i = \rho_\alpha^i X_\alpha, \quad d^DX^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} X_\alpha \wedge X^\beta,
\]

for all \(i\) and \(\gamma\).

From Theorems 2.5 and 2.7 we conclude the following result

**Theorem 2.8.** Let \(\tau_D : D \to Q\) be a vector bundle over a manifold \(Q\) and \(D^*\) be its dual vector bundle. Then, there exists a one-to-one correspondence between the set \(\mathcal{L}_{\text{AP}}(D^*)\) of linear almost Poisson structures on \(D^*\), the set \(\SSA(D)\) of skew-symmetric algebroid structures on \(\tau_D : D \to Q\) and the set \(\AD(D)\) of almost differentials on this vector bundle.
3. Skew-symmetric algebroids and the orbit theorem

Let \( (D, [[\cdot, \cdot]]_D, \rho_D) \) be a skew-symmetric algebroid over \( Q \) and \( d^D \) be the corresponding almost differential.

We can consider the vector space over \( \mathbb{R} \)

\[
H^0(d^D) = \{ f \in C^\infty(Q) / d^D f = 0 \}.
\]

Note that if \( D \) is a Lie algebroid we have that \( H^0(d^D) \) is the 0 Lie algebroid cohomology group associated with \( D \).

On the other hand, it is clear that if \( Q \) is connected and \( D \) is a transitive skew-symmetric algebroid, that is, \( \rho^D(D_q) = T_qQ \), for all \( q \in Q \) then

\[
H^0(d^D) \cong \mathbb{R}.
\] (3.1)

Condition (3.1) will play an important role in Section 4.1.

Next, we will see that (3.1) holds if the skew-symmetric algebroid is completely nonholonomic with connected base space.

Let \( \tilde{D} \) be the generalized distribution on \( Q \) whose characteristic space at the point \( q \in Q \) is

\[
\tilde{D}_q = \rho^D(D_q).
\]

It is clear that \( \tilde{D} \) is finitely generated. Note that the \( C^\infty \)-module \( \Gamma(D) \) is finitely generated (see [17]).

Now, denote by \( \text{Lie}^\infty(\tilde{D}) \) the smallest Lie subalgebra of \( \mathfrak{X}(Q) \) containing \( \tilde{D} \). Then \( \text{Lie}^\infty(\tilde{D}) \) is comprised of finite \( \mathbb{R} \)-linear combinations of vector fields of the form

\[
[\tilde{X}_k, [\tilde{X}_{k-1}, \ldots [\tilde{X}_2, \tilde{X}_1] \ldots]]
\]

with \( k \in \mathbb{N}, k \neq 0 \), and \( \tilde{X}_1, \ldots, \tilde{X}_k \in \mathfrak{X}(Q) \) satisfying

\[
\tilde{X}_l(q) \in \tilde{D}_q, \quad \text{for all } q \in Q
\]

(see [2]).

For each \( q \in Q \), we will consider the vector subspace \( \text{Lie}^\infty_q(\tilde{D}) \) of \( T_qQ \) given by

\[
\text{Lie}^\infty_q(\tilde{D}) = \{ \tilde{X}(q) \in T_qQ / \tilde{X} \in \text{Lie}^\infty(\tilde{D}) \}.
\]

Then, the assignment

\[
q \in Q \rightarrow \text{Lie}^\infty_q(\tilde{D}) \subseteq T_qQ
\]

defines a generalized foliation on \( Q \). The leaf \( L \) of this foliation over the point \( q_0 \in Q \) is the orbit of \( \tilde{D} \) over the point \( q_0 \), that is,

\[
L = \{ (\phi_{\tilde{X}_1} \circ \ldots \circ \phi_{\tilde{X}_l})(q_0) \in Q / \tilde{t}_l \in \mathbb{R}, \tilde{X}_l \in \mathfrak{X}(Q) \text{ and } \tilde{X}_l(q) \in \tilde{D}_q, \text{ for all } q \in Q \}.
\]

Here, \( \phi_{\tilde{X}_l}^{\tilde{t}_l} \) is the flow of the vector field \( \tilde{X}_l \) at the time \( \tilde{t}_l \) (for more details, see [2]).

**Definition 3.1.** The skew-symmetric algebroid \( (D, [[\cdot, \cdot]]_D, \rho_D) \) over \( Q \) is said to be completely nonholonomic if

\[
\text{Lie}^\infty_q(\rho_D(D)) = \text{Lie}^\infty_q(\tilde{D}) = T_qQ, \quad \text{for all } q \in Q.
\]
Thus, if $Q$ is a connected manifold, it follows that $D$ is completely nonholonomic if and only if the orbit of $\tilde{D}$ over any point $q_0 \in Q$ is $Q$.

**Remark 3.2.**

(i) Definition 3.1 may be extended for anchored vector bundles. A vector bundle $\tau_D : D \rightarrow Q$ over $Q$ is said to be **anchored** if it admits an anchor map, that is, a vector bundle morphism $\rho_D : D \rightarrow TQ$. In such a case, the vector bundle is said to be completely nonholonomic if $\text{Lie}^\infty_q(\rho_D(D)) = \text{Lie}^\infty_q(\tilde{D}) = T_q Q$, for all $q \in Q$.

(ii) If $D$ is a regular distribution on a manifold $Q$ then the inclusion map $i_D : D \rightarrow TQ$ is an anchor map for the vector bundle $\tau_D : D \rightarrow Q$. Moreover, the anchored vector bundle $\tau_D : D \rightarrow Q$ is completely nonholonomic if and only if the distribution $D$ is completely nonholonomic in the classical sense of Vershik and Gershkovich \[44\]. In this sense it is formulated in the literature the classical **Rashevsky-Chow theorem**: If $\text{Lie}^\infty_q(D) = T_q Q$, for all $q \in Q$, then each orbit is equal to the whole manifold $Q$.

Now, we deduce the following result.

**Proposition 3.3.** If the skew-symmetric algebroid $(D, [\cdot, \cdot]_D, \rho_D)$ over $Q$ is completely nonholonomic and $Q$ is connected then $H^0(d^D) \simeq \mathbb{R}$.

*Proof.* Suppose that $f \in C^\infty(Q)$ and that $d^D f = 0$. Let $q_0$ be a point of $Q$. We must prove that

$$\tilde{v}(f) = 0, \quad \text{for all } \tilde{v} \in T_{q_0} Q.$$ 

The condition $(d^D f)(q_0) = 0$ implies that $\tilde{v}(f) = 0$, for all $\tilde{v} \in \tilde{D}_{q_0}$.

Thus, first we have that

$$0 = \tilde{X}_1(\tilde{X}_2(f)) - \tilde{X}_2(\tilde{X}_1(f)) = [\tilde{X}_1, \tilde{X}_2](f),$$

for $\tilde{X}_1, \tilde{X}_2 \in \mathfrak{X}(Q)$ and $\tilde{X}_1(q), \tilde{X}_2(q) \in \tilde{D}_q$, for all $q \in Q$.

Secondly, since $D$ is completely nonholonomic then $\text{Lie}^\infty_{q_0}(\tilde{D}) = T_{q_0} Q$. Therefore, there exists a finite sequence of vector fields on $Q$, $\tilde{X}_1, \ldots, \tilde{X}_k$ such that $\tilde{X}_i(q) \in \tilde{D}_q$, for all $i \in \{1, \ldots, k\}$ and $q \in Q$,

$$\tilde{v} = [\tilde{X}_k, [\tilde{X}_{k-1}, \ldots [\tilde{X}_2, \tilde{X}_1] \ldots ]] (q_0).$$

From both considerations, we deduce the result. \qed

However, the condition $H^0(d^D) \simeq \mathbb{R}$ does not imply, in general, that the skew-symmetric algebroid $D$ is completely nonholonomic.

In fact, let $D$ be the tangent bundle to $\mathbb{R}^2$

$$\tau_{T\mathbb{R}^2} : T\mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

If $(x, y)$ are the standard coordinates on $\mathbb{R}^2$, it follows that $\{X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}\}$ is a global basis of $\Gamma(T\mathbb{R}^2) = \mathfrak{X}(\mathbb{R}^2)$. So, we can consider the skew-symmetric algebroid structure $([\cdot, \cdot]_{T\mathbb{R}^2}, \rho_{T\mathbb{R}^2})$ on $T\mathbb{R}^2$ which is characterized by the following conditions

$$[X_1, X_2]_{T\mathbb{R}^2} = 0, \quad \rho_{T\mathbb{R}^2}(X_1) = \frac{\partial}{\partial x}, \quad \rho_{T\mathbb{R}^2}(X_2) = xy \frac{\partial}{\partial y}.$$
Then, the generalized distribution $\tilde{D} = \tilde{T}\mathbb{R}^2$ on $\mathbb{R}^2$ is generated by the vector fields

$$\tilde{X}_1 = \frac{\partial}{\partial x}, \quad \tilde{X}_2 = xy \frac{\partial}{\partial y}.$$ 

Thus, the Lie subalgebra $\text{Lie}^\infty(\tilde{D})$ of $T\mathbb{R}^2$ is generated by the vector fields

$$\tilde{X}_1 = \frac{\partial}{\partial x}, \quad \tilde{X}_2 = xy \frac{\partial}{\partial y}, \quad \tilde{X}_3 = y \frac{\partial}{\partial y}.$$ 

This implies that

$$\text{Lie}^\infty_{(x_0,y_0)}(\tilde{D}) \neq T_{(x_0,y_0)}\mathbb{R}^2, \quad \text{if} \ y_0 = 0.$$ 

However, if $f \in C^\infty(\mathbb{R}^2)$ and $d^Df = 0$, we deduce that

$$\frac{\partial f}{\partial x} = 0, \quad xy \frac{\partial f}{\partial y} = 0.$$ 

Consequently, using that $f \in C^\infty(\mathbb{R}^2)$, we obtain that $f$ is constant.

Next, we will discuss the case when the generalized foliation $\text{Lie}^\infty(\tilde{D}) \neq TQ$. In fact, we will prove the following result.

**Theorem 3.4.** Let $(D, [\cdot, \cdot]_D, \rho_D)$ be a skew-symmetric algebroid over a manifold $Q$ and $f$ be a real $C^\infty$-function on $Q$ such that $d^Df = 0$. Suppose that $L$ is an orbit of $\tilde{D}$ and that $D_L$ is the vector bundle over $L$ given by $D_L = \cup_{q \in L} D_q = \tau_D^{-1}(L)$. Then:

(i) The couple $([\cdot, \cdot]_D, \rho_D)$ induces a skew-symmetric algebroid structure $([\cdot, \cdot]_{D_L}, \rho_{D_L})$ on the vector bundle $\tau_{D_L} : D_L \to L$ and the skew-symmetric algebroid $(D_L, [\cdot, \cdot]_{D_L}, \rho_{D_L})$ is completely nonholonomic.

(ii) The restriction of $f$ to $L$ is constant.

**Proof.** It is clear that

$$\rho_D(D_q) = \tilde{D}_q \subseteq \text{Lie}^\infty_q(\tilde{D}) = T_qL, \quad \text{for all} \ q \in L.$$ 

Thus, we have a vector bundle morphism

$$\rho_{D_L} : D_L \to TL.$$ 

More precisely, $\rho_{D_L} = (\rho_D)|_{D_L}$.

On the other hand, we may define a $\mathbb{R}$-bilinear skew-symmetric bracket

$$[\cdot, \cdot]_{D_L} : \Gamma(D_L) \times \Gamma(D_L) \to \Gamma(D_L).$$ 

In fact, if $X_L, Y_L \in \Gamma(D_L)$ then

$$[X_L, Y_L]_{D_L}(q) = [X, Y]_D(q), \quad \text{for all} \ q \in L$$

where $X, Y$ are sections of $\tau_D : D \to Q$ such that

$$X|_{U \cap L} = (X_L)|_{U \cap L}, \quad Y|_{U \cap L} = (Y_L)|_{U \cap L},$$

with $U$ an open subset of $Q$ and $q \in U$.

Note that the condition

$$\text{Lie}^\infty_q(\tilde{D}) = T_qL, \quad \text{for all} \ q \in L,$$

implies that

$$([X, Y]_D)|_{U \cap L} = 0.$$
for \( X, Y \in \Gamma(D) \), with \( V \) an open subset of \( Q \) and
\[
X(q) = 0, \quad \text{for all } q \in V \cap L.
\]
Therefore, the map \([\cdot, \cdot]_{D_L}\) is well-defined. We remark that if \( q \in V \cap L \) and \( \{X_\alpha\} \) is a local basis of \( \Gamma(\tau_D) \) in an open subset \( W \) of \( Q \), with \( q \in W \), such that \( X = f^\alpha X_\alpha \) in \( W \) then, using that \((f^\alpha)_{|V \cap W \cap L} = 0\) and that \( \rho_D(Y)(q) \in T_qL \), we deduce that
\[
[X, Y]_{D}(q) = f^\alpha(q)[X_\alpha, Y]_\mathcal{D}(q) - \rho_D(Y)(f^\alpha)X_\alpha(q) = 0.
\]
Moreover, the couple \((\mathcal{D}, \rho_{D_L})\) is a skew-symmetric algebroid structure on the vector bundle \( \tau_{D_L} : D_L \to L \).

In addition, it is clear that
\[
(\mathcal{D}_L)_q = \rho_{D_L}((D_L)_q) = \mathcal{D}_q, \quad \text{Lie}^\infty_{\mathcal{D}_q}(\mathcal{D}_L) = \text{Lie}^\infty_{\mathcal{D}_q}(\mathcal{D})
\]
for all \( q \in L \). Thus, we have that the skew-symmetric algebroid \((D_L, [\cdot, \cdot]_{D_L}, \rho_{D_L})\) is completely nonholonomic.

This proves (i).

On the other hand, it follows that the condition \( d^D f = 0 \) implies that
\[
d^D_L(f_{|L}) = 0
\]
and, since \( H^0(d^D_L) \simeq \mathbb{R} \) (as a consequence of the first part of the theorem), we conclude that \( f \) is constant on \( L \).

It will be also interesting to characterize under what conditions there exist functions \( f \in C^\infty(Q) \) such that \((d^D)^2 f = 0\). Using Equations (2.4) we easily deduce that:
\[
((d^D)^2 f)(X, Y) = ([\rho_D(X), \rho_D(Y)] - \rho_D[X, Y]_\mathcal{D}) f
\]
for all \( X, Y \in \Gamma(\tau_D) \). Now, consider the generalized distribution \( \mathcal{D} \) on \( Q \) whose characteristic space \( \mathcal{D}_q \) at the point \( q \in Q \) is:
\[
\mathcal{D}_q = \{([\rho_D(X), \rho_D(Y)] - \rho_D[X, Y]_\mathcal{D})(q) / X, Y \in \Gamma(\tau_D)\}.
\]
It is also clear that \( \mathcal{D} \) is finitely generated. Denote by \( \text{Lie}^\infty(\mathcal{D}) \) the smallest Lie subalgebra of \( \mathfrak{X}(Q) \) containing \( \mathcal{D} \). Observe that \( \text{Lie}^\infty(\mathcal{D}) \subseteq \text{Lie}^\infty(\mathcal{D}) \). We deduce that \((d^D)^2 f = 0\) if and only if \( f \) is constant on any orbit \( L \) of \( \mathcal{D} \). Of course if \( \mathcal{D} \) is completely nonholonomic then the unique functions \( f \in C^\infty(Q) \) satisfying \((d^D)^2 f = 0\) are \( f = \text{constant} \), but it has not to be always the case and it may useful to find this particular type of functions in concrete examples.

4. Linear almost Poisson structures and Hamilton-Jacobi equation

4.1. Linear almost Poisson structures and Hamiltonian systems. In this section, we will consider Hamiltonian systems associated with a linear almost Poisson structure on the dual bundle \( D^* \) to a vector bundle and with a Hamiltonian function on \( D^* \). Thus, the ingredients of our theory are:

(i) A vector bundle \( \tau_D : D \to Q \) of rank \( n \) over a manifold \( Q \) of dimension \( m \);
(ii) A linear almost Poisson structure \( \{\cdot, \cdot\}_{D^*} \) on \( D^* \) and
(iii) A Hamiltonian function \( h : D^* \to \mathbb{R} \) on \( D^* \).
The triplet \((D, \{\cdot, \cdot\}_D, h)\) is said to be a Hamiltonian system.

We will denote by \(\Lambda_{D^*}\) the linear almost Poisson 2-vector associated with \(\{\cdot, \cdot\}_{D^*}\). Then, we may introduce the vector field \(\mathcal{H}^\Lambda_{D^*}\) on \(D^*\) given by

\[
\mathcal{H}^\Lambda_{D^*} = -i(dh)\Lambda_{D^*}.
\]

\(\mathcal{H}^\Lambda_{D^*}\) is called the Hamiltonian vector field of \(h\) with respect to \(\Lambda_{D^*}\). The integral curves of \(\mathcal{H}^\Lambda_{D^*}\) are the solutions of the Hamilton vector field for \(h\).

Now, suppose that \((q^i)\) are local coordinates on an open subset \(U\) of \(Q\) and that \(\{X_\alpha\}\) is a basis of the space of sections of the vector bundle \(\tau_D^{-1}(U) \to U\). Denote by \((q^i, p_\alpha)\) the corresponding local coordinates on \(D^*\) and by \(C^\gamma_{\alpha\beta}\) and \(\rho_\alpha^i\) the local structure functions (with respect to the coordinates \((q^i)\) and to the basis \(\{X_\alpha\}\)) of the corresponding skew-symmetric algebroid structure on \(D\). Then, using (2.1), it follows that

\[
\mathcal{H}^\Lambda_{D^*} = \rho_\alpha^i \frac{\partial h}{\partial p_\alpha} \frac{\partial}{\partial p_\alpha} - (\rho_\alpha^i \frac{\partial h}{\partial q^i} + C^\gamma_{\alpha\beta} p_\gamma \frac{\partial h}{\partial p_\beta}) \frac{\partial}{\partial p_\alpha},
\]

Therefore, the Hamilton equations are

\[
\frac{dq^i}{dt} = \rho_\alpha^i \frac{\partial h}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\left(\rho_\alpha^i \frac{\partial h}{\partial q^i} + C^\gamma_{\alpha\beta} p_\gamma \frac{\partial h}{\partial p_\beta}\right).
\]

4.2. Hamiltonian systems and Hamilton-Jacobi equation. Let \((D, \{\cdot, \cdot\}_{D^*}, h)\) be a Hamiltonian system and \(\alpha : Q \to D^*\) be a section of the vector bundle \(\tau_{D^*} : D^* \to Q\).

If \(\mathcal{H}^\Lambda_{D^*}\) is the Hamiltonian vector field of \(h\) with respect to \(\{\cdot, \cdot\}_{D^*}\), we may introduce the vector field \(\mathcal{H}^\Lambda_{h,\alpha}\) on \(Q\) given by

\[
\mathcal{H}^\Lambda_{h,\alpha}(q) = (T_\alpha(q)\tau_{D^*})(\mathcal{H}^\Lambda_{D^*}(\alpha(q))), \quad \text{for } q \in Q.
\]

From (4.1), it follows that

\[
\mathcal{H}^\Lambda_{h,\alpha}(q) \in \rho_D(D_q), \quad \text{for all } q \in Q,
\]

where \((\mathcal{L}_D, \rho_D)\) is the induced skew-symmetric algebroid structure on the vector bundle \(\tau_D : D \to Q\).

Then, the aim of this section is to prove the following result.

**Theorem 4.1.** Let \((D, \{\cdot, \cdot\}_{D^*}, h)\) be a Hamiltonian system and \(\alpha : Q \to D^*\) be a section of the vector bundle \(\tau_{D^*} : D^* \to Q\) such that \(d^D\alpha = 0\). Under these hypotheses, the following conditions are equivalent:

(i) If \(c : I \to Q\) is an integral curve of the vector field \(\mathcal{H}^\Lambda_{h,\alpha}\), that is,

\[
\dot{c}(t) = (T_\alpha(c(t))\tau_{D^*})(\mathcal{H}^\Lambda_{h,\alpha}(\alpha(c(t))), \quad \text{for all } t \in I,
\]

then \(\alpha \circ c : I \to D^*\) is a solution of the Hamilton equations for \(h\).

(ii) \(\alpha\) satisfies the Hamilton-Jacobi equation

\[
d^D(h \circ \alpha) = 0.
\]
Remark 4.2. Let $\hat{D}$ be the generalized distribution on $Q$ given by $\hat{D} = \rho_D(D)$ and $\text{Lie}^\infty(\hat{D})$ be the smallest Lie subalgebra of $\mathfrak{X}(Q)$ containing $\hat{D}$. Then, using Theorem 3.4, we deduce that the Hamilton-Jacobi equation holds for the section $\alpha$ if and only if the function $h \circ \alpha : Q \to \mathbb{R}$ is constant on the leaves of the foliation $\text{Lie}^\infty(\hat{D})$.

In order to prove Theorem 4.1, we will need some previous results:

Proposition 4.3. Let $\{\cdot,\cdot\}_{D^*}$ be a linear almost Poisson structure on the dual bundle $D^*$ to a vector bundle $\tau_D : D \to Q$, $\alpha : Q \to D^*$ be a section of $\tau_{D^*} : D^* \to Q$ and $\#_{\Lambda_{D^*}} : T^*D^* \to TD^*$ be the vector bundle morphism between $T^*D^*$ and $TD^*$ induced by the linear almost Poisson 2-vector $\Lambda_{D^*}$. Then, $\alpha$ is a 1-cocycle with respect to $d^D$ (i.e., $d^D\alpha = 0$) if and only if for every point $q$ of $Q$ the subspace of $T_{\alpha(q)}D^*$

$$L_{\alpha,D}(q) = (T_q \alpha)(\rho_D(D_q))$$

is Lagrangian with respect to $\Lambda_{D^*}$, that is,

$$\#_{\Lambda_{D^*}}((L_{\alpha,D}(q))^0) = L_{\alpha,D}(q), \text{ for all } q \in Q.$$

Remark 4.4. If $D = TQ$ and $\{\cdot,\cdot\}_{T^*Q}$ is the canonical Poisson (symplectic) structure on $T^*Q$ then $\alpha$ is a 1-form on $Q$, $d^D = d^TQ$ is the standard exterior differential on $Q$ and $\rho_D = \rho_{TQ} : TQ \to TQ$ is the identity map. Thus, if we apply Proposition 4.3, we obtain that $\alpha$ is a closed 1-form if and only if $\alpha(Q)$ is a Lagrangian submanifold of $T^*Q$. This is a well-known result in the literature (see, for instance, [1]).

Proposition 4.5. Under the same hypotheses as in Proposition 4.3, if the section $\alpha$ is a 1-cocycle with respect to $d^D$, i.e., $d^D\alpha = 0$ then we have that

$$\text{Ker}(\#_{\Lambda_{D^*}}(\alpha(q))) \subseteq (L_{\alpha,D}(q))^0, \text{ for all } q \in Q.$$

The proofs of Propositions 4.3 and 4.5 may be found in the Appendix of this paper.

Proof of Theorem 4.1. It is clear that condition (i) in Theorem 4.1 is equivalent to the following fact:

(i') The vector fields $\mathfrak{H}^ {\Lambda_{D^*}}_{h,\alpha}$ and $\mathfrak{H}^ {\Lambda_{D^*}}_h$ on $Q$ and $D^*$, respectively, are $\alpha$-related, that is,

$$(T_q \alpha)(\mathfrak{H}^ {\Lambda_{D^*}}_{h,\alpha}(q)) = \mathfrak{H}^ {\Lambda_{D^*}}_h(\alpha(q)), \text{ for all } q \in Q.$$  (4.4)

Therefore, we must prove that

(i') $\iff$ (ii)

(i') $\implies$ (ii) Let $q$ be a point of $Q$. Then, using (4.2), (4.3) and (4.4), we deduce that

$$\mathfrak{H}^ {\Lambda_{D^*}}_h(\alpha(q)) \in L_{\alpha,D}(q).$$

Consequently, from Proposition 4.3, we obtain that

$$\mathfrak{H}^ {\Lambda_{D^*}}_h(\alpha(q)) = \#_{\Lambda_{D^*}}(\eta_{\alpha(q)}), \text{ for some } \eta_{\alpha(q)} \in (L_{\alpha,D}(q))^0.$$  (4.5)

Thus, since $\mathfrak{H}^ {\Lambda_{D^*}}_h(\alpha(q)) = -\#_{\Lambda_{D^*}}(dh(\alpha(q)))$, it follows that

$$\eta_{\alpha(q)} + dh(\alpha(q)) \in \text{Ker}(\#_{\Lambda_{D^*}}(\alpha(q))).$$

Now, using Proposition 4.5 and the fact that $\eta_{\alpha(q)} \in (L_{\alpha,D}(q))^0$, we conclude that

$$dh(\alpha(q)) \in (L_{\alpha,D}(q))^0.$$  (4.5)
Finally, if \( a_q \in D_q \), we have that
\[
d^D(h \circ \alpha)(q)(a_q) = dh(\alpha(q))((T_q\alpha)(\rho_D(a_q)))
\]
which implies that (see (4.3) and (4.5))
\[
d^D(h \circ \alpha)(q)(a_q) = 0.
\]

(ii) \( \implies \) (i) Let \( q \) be a point of \( Q \). Then, using that \( d^D(h \circ \alpha)(q) = 0 \), we deduce that
\[
dh(\alpha(q)) \in (\mathcal{L}_{\alpha,D}(q))^0.
\]
Therefore, it follows that
\[
\mathcal{H}_{h}^{\Lambda_{D^*}}(\alpha(q)) = -\#_{\Lambda_{D^*}}(dh(\alpha(q))) \in \#_{\Lambda_{D^*}}((\mathcal{L}_{\alpha,D}(q))^0)
\]
and, from Proposition 4.3, we obtain that there exists \( v_q \in \rho_D(D_q) \subseteq T_qQ \) such that
\[
\mathcal{H}_{h}^{\Lambda_{D^*}}(\alpha(q)) = (T_q\alpha)(v_q).
\]
This implies that
\[
\mathcal{H}_{h,\alpha}^{\Lambda_{D^*}}(q) = (T_{\alpha(q)}\tau_{D^*})(\mathcal{H}_{h}^{\Lambda_{D^*}}(\alpha(q))) = v_q
\]
and, thus,
\[
\mathcal{H}_{h}^{\Lambda_{D^*}}(\alpha(q)) = (T_q\alpha)(\mathcal{H}_{h,\alpha}^{\Lambda_{D^*}}(q)).
\]
\( \square \)

Let \((D, \{\cdot, \cdot\}_{D^*}, h)\) be a Hamiltonian system and \( \alpha : Q \to D^* \) be a section of the vector bundle \( \tau_{D^*} : D^* \to Q \).

Suppose that \((q^i)\) are local coordinates on an open subset \( U \) of \( Q \) and that \( \{X_\gamma\} \) is a basis of sections of the vector bundle \( \tau_{D^*}^{-1}(U) \to U \). Denote by \((q^i, p_\gamma)\) the corresponding local coordinates on \( D^* \) and by \( \rho^\gamma_\nu, C^\delta_\gamma_\nu \) the local structure functions of the skew-symmetric algebroid structure \((\mathcal{I}_D, \rho_D)\) with respect to the local coordinates \((q^i)\) and to the basis \( \{X_\gamma\} \). If the local expression of \( \alpha \) is
\[
\alpha(q^i) = (q^i, \alpha_\gamma(q^i))
\]
then
\[
d^D \alpha = 0 \iff C^\delta_\gamma_\nu \alpha_\delta = \rho^\gamma_\nu \frac{\partial \alpha_\nu}{\partial q^i} - \rho^\nu_\nu \frac{\partial \alpha_\gamma}{\partial q^i}, \quad \forall \gamma, \nu,
\]
and
\[
d^D(h \circ \alpha) = 0 \iff \rho^\gamma_\nu(q)\left(\frac{\partial h}{\partial q^i|_{\alpha(q)}} + \frac{\partial \alpha_\nu}{\partial p^i|_{\alpha(q)}}\right) = 0, \quad \forall \gamma, \forall q \in U.
\]

**Corollary 4.6.** Under the same hypotheses as in Theorem 4.1 if, additionally, \( H^0(d^D) \simeq \mathbb{R} \) or if the skew-symmetric algebroid \((D, [\cdot, \cdot]_D, \rho_D)\) is completely nonholonomic and \( Q \) is connected, then the following conditions are equivalent:

(i) If \( c : I \to Q \) is an integral curve of the vector field \( \mathcal{H}_{h,\alpha}^{\Lambda_{D^*}} \), that is,
\[
\dot{c}(t) = (T_{\alpha(c(t))}\tau_{D^*})(\mathcal{H}_{h}^{\Lambda_{D^*}}(\alpha(c(t)))), \quad \text{for all } t \in I,
\]
then \( \alpha \circ c : I \to D^* \) is a solution of the Hamilton equations for \( h \).

(ii) \( \alpha \) satisfies the following relation
\[
h \circ \alpha = \text{constant}.
\]
Note that if $Q$ is connected then
\[ h \circ \alpha = \text{constant} \iff \left( \frac{\partial h}{\partial q^i|_{\alpha(q)}} + \frac{\partial \alpha_i}{\partial q^i|_{q}} \frac{\partial h}{\partial p^i|_{\alpha(q)}} \right) = 0, \ \forall i \text{ and } \forall q \in U. \]

4.3. **Linear almost Poisson morphisms and Hamilton-Jacobi equation.** Suppose that $\tau_D : D \to Q$ and $\tau_{\tilde{D}} : \tilde{D} \to \tilde{Q}$ are vector bundles over $Q$ and $\tilde{Q}$, respectively, and that $\{\cdot, \cdot\}_D$ (respectively, $\{\cdot, \cdot\}_{\tilde{D}}$) is a linear almost Poisson structure on $D^*$ (respectively, $\tilde{D}^*$). Denote by $([[\cdot, \cdot]]_D, \rho_D)$ and $d^D$ (respectively, $([[\cdot, \cdot]]_{\tilde{D}}, \rho_{\tilde{D}})$ and $d^{\tilde{D}}$) the corresponding skew-symmetric algebroid structure and the almost differential on the vector bundle $\tau_D : D \to Q$ (respectively, $\tau_{\tilde{D}} : \tilde{D} \to \tilde{Q}$).

**Definition 4.7.** A vector bundle morphism $(\tilde{F}, F)$ between the vector bundles $\tau_{D^*} : D^* \to Q$ and $\tau_{\tilde{D}^*} : \tilde{D}^* \to \tilde{Q}$

\[
\begin{array}{ccc}
D^* & \xrightarrow{\tilde{F}} & \tilde{D}^* \\
\downarrow \tau_{D^*} & & \uparrow \tau_{\tilde{D}^*} \\
Q & \xrightarrow{F} & \tilde{Q}
\end{array}
\]

is said to be a **linear almost Poisson morphism** if
\[
\{\tilde{\varphi} \circ \tilde{F}, \tilde{\psi} \circ \tilde{F}\}_{D^*} = \{\tilde{\varphi}, \tilde{\psi}\}_{D^*} \circ \tilde{F}, \tag{4.6}
\]
for $\tilde{\varphi}, \tilde{\psi} \in C^\infty(\tilde{D}^*)$.

Let $(\tilde{F}, F)$ be a vector bundle morphism between the vector bundles $\tau_{D^*} : D^* \to Q$ and $\tau_{\tilde{D}^*} : \tilde{D}^* \to \tilde{Q}$. If $\tilde{X}$ is a section of $\tau_{\tilde{D}} : \tilde{D} \to \tilde{Q}$ then we may define the section $(\tilde{F}, F)^* \tilde{X}$ of $\tau_D : D \to Q$ characterized by the following condition
\[
\alpha_q((\tilde{F}, F)^* \tilde{X})(q)) = \tilde{F}(\alpha_q)(\tilde{X}(F(q))), \tag{4.7}
\]
for all $q \in Q$ and $\alpha_q \in D^*_q$.

**Theorem 4.8.** Let $(\tilde{F}, F)$ be a vector bundle morphism between the vector bundles $\tau_{D^*} : D^* \to Q$ and $\tau_{\tilde{D}^*} : \tilde{D}^* \to \tilde{Q}$. Then, $(\tilde{F}, F)$ is a linear almost Poisson morphism if and only if
\[
[[\tilde{F}, F)^* \tilde{X}, (\tilde{F}, F)^* \tilde{Y}]_D = (\tilde{F}, F)^* [[\tilde{X}, \tilde{Y}]_D], \tag{4.8}
\]
\[
(TF \circ \rho_D)((\tilde{F}, F)^* \tilde{X}) = \rho_{\tilde{D}}(\tilde{X}) \circ F, \tag{4.9}
\]
for $\tilde{X}, \tilde{Y} \in \Gamma(\tau_{\tilde{D}})$.

**Proof.** Suppose that $(\tilde{F}, F)$ is a linear almost Poisson morphism and that $\tilde{Z}$ is a section of $\tau_D : \tilde{D} \to \tilde{Q}$. From (4.7), it follows that
\[
(\tilde{F}, F)^* \tilde{Z} = \tilde{Z} \circ \tilde{F}. \tag{4.10}
\]
Now, if $\tilde{X}, \tilde{Y} \in \Gamma(\tau_{\tilde{D}})$ then, using (2.2) and (4.10), we deduce that
\[
[[\tilde{F}, F)^* \tilde{X}, (\tilde{F}, F)^* \tilde{Y}]_D = -\{\tilde{X} \circ \tilde{F}, \tilde{Y} \circ \tilde{F}\}_{D^*}. \]
Thus, from (2.2) and (4.6), we obtain that

\[ [(F, F)^*X, (F, F)^*Y]_D = (F, F)^*[X, Y]_D \]

which implies that (4.8) holds.

On the other hand, if \( \bar{f} \in C^\infty(\bar{Q}) \) then, using again (2.2) and (4.6), it follows that

\[ (\rho_D(\bar{X})(\bar{f}) \circ F) \circ \tau_{D^*} = ((\bar{f} \circ \tau_{D^*}) \circ \bar{F}, \bar{F} \circ \bar{X})_{D^*}. \]

Therefore, from (2.2) and (4.10), we have that

\[ (\rho_D(X)(\bar{f}) \circ F) \circ \tau_{D^*} = \rho_D((\bar{F}, F)^*X)((\bar{f} \circ F) \circ \tau_{D^*}) \]

and, consequently,

\[ \rho_D(X)(\bar{f}) \circ F = \rho_D((\bar{F}, F)^*X)((\bar{f} \circ F)). \]

This implies that (4.9) holds.

Conversely, assume that (4.8) and (4.9) hold.

Then, if \( \bar{f}, \bar{g} \in C^\infty(\bar{Q}) \) it is clear that the real functions

\[ \bar{f} \circ \tau_{D^*} \circ \bar{F} = \bar{f} \circ F \circ \tau_{D^*}, \quad \bar{g} \circ \tau_{D^*} \circ \bar{F} = \bar{g} \circ F \circ \tau_{D^*} \]

are basic functions with respect to the projection \( \tau_{D^*} : D^* \rightarrow Q \). Therefore, we deduce that

\[ 0 = \{(\bar{f} \circ \tau_{D^*}) \circ \hat{F}, (\bar{g} \circ \tau_{D^*}) \circ \bar{F}\}_{D^*} = \{\hat{f} \circ \tau_{D^*}, \hat{g} \circ \tau_{D^*}\}_{D^*} \circ \hat{F}. \quad (4.11) \]

Now, if \( \bar{X} \in \Gamma(\tau_D) \) then, using (2.2) and (4.10), we obtain that

\[ \{\hat{f} \circ \tau_{D^*}, \hat{X} \circ \bar{F}\}_{D^*} = ((TF \circ \rho_D)((\bar{F}, F)^*\bar{X}))((\bar{f} \circ F) \circ \tau_{D^*}). \]

Consequently, from (2.2) and (4.9), it follows that

\[ \{\hat{f} \circ \tau_{D^*}, \hat{X} \circ \bar{F}\}_{D^*} = \{\hat{f} \circ \tau_{D^*}, \hat{X}\}_{D^*} \circ \hat{F}. \quad (4.12) \]

On the other hand, if \( \bar{Y} \in \Gamma(\tau_D) \) then, using (2.2), (4.8) and (4.10), we deduce that

\[ \{\hat{X} \circ \hat{F}, \bar{Y} \circ \bar{F}\}_{D^*} = \{\hat{X}, \bar{Y}\}_{D^*} \circ \hat{F}. \quad (4.13) \]

Thus, (4.11), (4.12) and (4.13) imply that \((\bar{F}, F)\) is a linear almost Poisson morphism. \(\square\)

Let \((\bar{F}, F)\) be a vector bundle morphism between the vector bundles \( \tau_{D^*} : D^* \rightarrow Q \) and \( \tau_{D^*} : \bar{D}^* \rightarrow \bar{Q} \). Denote by \( \Lambda^k \bar{F} : \Lambda^k D^* \rightarrow \Lambda^k \bar{D}^* \) the vector bundle morphism (over \( F \)) between the vector bundles \( \Lambda^k \tau_{D^*} : \Lambda^k D^* \rightarrow Q \) and \( \Lambda^k \tau_{D^*} : \Lambda^k \bar{D}^* \rightarrow \bar{Q} \) induced by \( \bar{F} \). Then, a section \( \alpha \in \Gamma(\Lambda^k \tau_{D^*}) \) is said to be \((\bar{F}, F)\)-related with a section \( \bar{\alpha} \in \Gamma(\Lambda^k \tau_{D^*}) \) if

\[ \Lambda^k \bar{F} \circ \alpha = \bar{\alpha} \circ F. \]

Now, assume that \( F \) is a surjective map and that \((\bar{F}, F)\) is a fiberwise injective vector bundle morphism, that is,

\[ \bar{F}_q = \bar{F}|_{D^*_q} : D^*_q \rightarrow \bar{D}^*_q \]

is a monomorphism of vector spaces, for all \( q \in Q \), and

\[ F(q) = F(q') \Rightarrow \bar{F}_q(D^*_q) = \bar{F}_q'(D^*_q). \]

Then, we may consider the vector subbundle \( \bar{F}(D^*) \) (over \( Q \)) of \( \tau_{D^*} : D^* \rightarrow Q \). Moreover, if \( \bar{\alpha} \) is a section of this vector subbundle we have that there exists a unique section \( \alpha \) of \( \tau_{D^*} : D^* \rightarrow Q \) such
that \( \alpha \) is \((\tilde{F}, F)\)-related with \( \bar{\alpha} \). In fact, if \( \{\bar{\alpha}_i\} \) is a local basis of sections of \( \tau_{\bar{F}(D)} : \tilde{F}(D^*) \to \tilde{Q} \), it follows that \( \{\alpha_i\} \) is a local basis of \( \Gamma(\tau_{D^*}) \).

**Theorem 4.9.** Let \((\tilde{F}, F)\) be a vector bundle morphism between the vector bundles \( \tau_{D^*} : D^* \to Q \) and \( \tau_{\bar{D}^*} : \bar{D}^* \to \bar{Q} \).

(i) If \((\tilde{F}, F)\) is a linear almost Poisson morphism then the following condition \((C)\) holds:

\( (C) \) For each \( \alpha \in \Gamma(\Lambda^k\tau_{D^*}) \) which is \((\tilde{F}, F)\)-related with \( \bar{\alpha} \in \Gamma(\Lambda^k\tau_{D^*}) \) we have that \( d^D\alpha \in \Gamma(\Lambda^{k+1}\tau_{D^*}) \) is also \((\tilde{F}, F)\)-related with \( d^D\bar{\alpha} \in \Gamma(\Lambda^{k+1}\tau_{D^*}) \).

(ii) Conversely, if condition \((C)\) holds, \( F \) is a surjective map and \((\tilde{F}, F)\) is a fiberwise injective vector bundle morphism then \((\tilde{F}, F)\) is a linear almost Poisson morphism.

**Proof.** (i) Suppose that \( \bar{X} \) and \( \bar{Y} \) are sections of \( \tau_{\tilde{D}} : \tilde{D} \to \tilde{Q} \).

Then, if \( \tilde{f} \in C^\infty(\tilde{Q}) \), using (4.9), we deduce that

\[
(d^D\tilde{f})(\bar{X}) \circ F = (\rho_D((\tilde{F}, F)^*\bar{X}))((\tilde{f} \circ F)).
\]

Thus, from (4.7), it follows that

\[
(d^D\tilde{f})(\bar{X}) \circ F =< \tilde{F}(d^D(\tilde{f} \circ F)), \bar{X} \circ F >.
\]

Therefore, we have that

\[
d^D\tilde{f} \circ F = \tilde{F} \circ d^D(\tilde{f} \circ F). \tag{4.14}
\]

Now, let \( \alpha \) be a section of \( \tau_{D^*} : D^* \to Q \) which is \((\tilde{F}, F)\)-related with \( \bar{\alpha} \in \Gamma(\tau_{D^*}) \), that is,

\[
\tilde{F} \circ \alpha = \bar{\alpha} \circ F. \tag{4.15}
\]

Then, using (4.8), (4.9) and (4.15), we obtain that

\[
(d^D\bar{\alpha})(\bar{X}, \bar{Y}) \circ F = \rho_D((\tilde{F}, F)^*\bar{X})(\alpha((\tilde{F}, F)^*\bar{Y})) - \rho_D((\tilde{F}, F)^*\bar{Y})(\alpha((\tilde{F}, F)^*\bar{X})) - \alpha[(\tilde{F}, F)^*\bar{X}, (\tilde{F}, F)^*\bar{Y}]_D
\]

which implies that

\[
(d^D\bar{\alpha})(\bar{X}, \bar{Y}) \circ F =< \Lambda^2\bar{F} \circ d^D\alpha, (\bar{X} \circ F, \bar{Y} \circ F) >.
\]

This proves that

\[
d^D\bar{\alpha} \circ F = \Lambda^2\bar{F} \circ d^D\alpha. \tag{4.16}
\]

Consequently, from (2.3), (4.14) and (4.16), we deduce the result.

(ii) If \( \tilde{X} \in \Gamma(\tau_{\bar{D}}) \) and \( \tilde{f} \in C^\infty(\tilde{Q}) \), then, using condition \((C)\), we have that

\[
(\rho_D(\tilde{X}) \circ F)(\tilde{f}) = d^D(\tilde{f} \circ F)((\tilde{F}, F)^*\tilde{X}) = (\rho_D((\tilde{F}, F)^*\tilde{X}))((\tilde{f} \circ F)).
\]

This proves that (4.9) holds.

Next, suppose that \( \bar{Y} \in \Gamma(\tau_{\bar{D}}) \) and that \( \alpha \) is a section of \( \tau_{D^*} : D^* \to Q \) which is \((\tilde{F}, F)\)-related with \( \bar{\alpha} \in \Gamma(\tau_{D^*}) \).

Then, from (4.9), it follows that

\[
\alpha[(\tilde{F}, F)^*\tilde{X}, (\tilde{F}, F)^*\tilde{Y}]_D = -(d^D\alpha)((\tilde{F}, F)^*\tilde{X}, (\tilde{F}, F)^*\tilde{Y}) + \rho_D(\bar{X})(\bar{\alpha}(\tilde{Y})) \circ F - \rho_D(\bar{Y})(\bar{\alpha}(\tilde{X})) \circ F.
\]

Thus, using condition \((C)\), we deduce that

\[
\alpha[(\tilde{F}, F)^*\tilde{X}, (\tilde{F}, F)^*\tilde{Y}]_D = -(d^D\bar{\alpha})(\bar{X}, \bar{Y}) \circ F + \rho_D(\bar{X})(\bar{\alpha}(\tilde{Y})) \circ F - \rho_D(\bar{Y})(\bar{\alpha}(\tilde{X})) \circ F.
\]
This implies that equation \[ \text{[4.8]} \]
holds. □

**Remark 4.10.** Let \((\tilde{F}, F)\) be a linear almost Poisson morphism between the vector bundles \(\tau_{D^*} : D^* \to Q\) and \(\tau_{D^*} : \tilde{D}^* \to \tilde{Q}\). Moreover, suppose that \(F\) is surjective and that \((\tilde{F}, F)\) is a fiberwise injective vector bundle morphism.

(i) From Theorem \[ \text{[4.9]} \]
we deduce that the condition \(H^0(d^D) \simeq \mathbb{R}\) implies that \(H^0(d^\tilde{D}) \simeq \mathbb{R}\). In general, the converse does not hold. However, if \(H^0(d^\tilde{D}) \simeq \mathbb{R}\) and \(f \in C^\infty(Q)\) is a \(F\)-basic function such that \(d^D f = 0\) then \(f\) is constant.

(ii) If \(F\) is a surjective submersion with connected fibers, \(V_q F \subseteq \tilde{D}_q = \rho_D(D_q)\), for all \(q \in Q\), and \(d^D f = 0\) then \(f\) is \(F\)-basic function. Here, \(V F\) is the vertical bundle to \(F\).

\[ \text{\(\Box\)} \]

Now, we will introduce the following definition.

**Definition 4.11.** Let \((D, \{\cdot, \cdot\}_D, h)\) (respectively, \((\tilde{D}, \{\cdot, \cdot\}_{\tilde{D}}, \tilde{h})\)) be a Hamiltonian system and \((\tilde{F}, F)\) be a linear almost Poisson morphism between the vector bundles \(\tau_D : D^* \to Q\) and \(\tau_{\tilde{D}} : \tilde{D}^* \to \tilde{Q}\). Then, \((\tilde{F}, F)\) is said to be **Hamiltonian** if

\[
\tilde{h} \circ \tilde{F} = h.
\]

It is clear that if \((\tilde{F}, F)\) is a Hamiltonian morphism then the Hamiltonian vector fields of \(h\) and \(\tilde{h}\), \(\mathfrak{h}^A_{\tilde{h}}\) and \(\mathfrak{h}^A_h\), are \(\tilde{F}\)-related, that is,

\[
(T\tilde{F}) \mathfrak{h}^A_{\tilde{h}} ((\tilde{F}(\beta))) = \mathfrak{h}^A_h (\tilde{F}(\beta)), \quad \text{for all} \ \beta \in D^*.
\]

This implies that if \(\mu : I \to D^*\) is a solution of the Hamilton equations for \(h\) then \(\tilde{F} \circ \mu : I \to \tilde{D}^*\) is a solution of the Hamilton equations for \(\tilde{h}\).

In addition, from Theorem \[ \text{[4.9]} \]
we deduce the following result

**Theorem 4.12.** Let \((D, \{\cdot, \cdot\}_D, h)\) (respectively, \((\tilde{D}, \{\cdot, \cdot\}_{\tilde{D}}, \tilde{h})\)) be a Hamiltonian system and \((\tilde{F}, F)\) be a Hamiltonian morphism between the vector bundles \(\tau_{D^*} : D^* \to Q\) and \(\tau_{\tilde{D}^*} : \tilde{D}^* \to \tilde{Q}\). Assume that the map \(F\) is surjective and that \((\tilde{F}, F)\) is a fiberwise injective vector bundle morphism.

(i) If \(\alpha\) is a section of the vector bundle \(\tau_{D^*} : D^* \to Q\) such that \(d^D \alpha = 0\), it satisfies the Hamilton-Jacobi equation for \(h\) (respectively, the strongest condition \(h \circ \alpha = \text{constant}\)) and it is \((\tilde{F}, F)\)-related with \(\tilde{\alpha} \in \Gamma(\tau_{\tilde{D}^*})\) then \(d^{\tilde{D}} \tilde{\alpha} = 0\) and \(\tilde{\alpha}\) satisfies the Hamilton-Jacobi equation for \(\tilde{h}\) (respectively, the strongest condition \(\tilde{h} \circ \tilde{\alpha} = \text{constant}\)).

(ii) If \(\tilde{\alpha}\) is a section of the vector subbundle \(\tilde{F}(D^*)\) of \(\tau_{\tilde{D}^*} : \tilde{D}^* \to \tilde{Q}\) such that \(d^{\tilde{D}} \tilde{\alpha} = 0\) and \(\tilde{\alpha}\) satisfies the Hamilton-Jacobi equation for \(\tilde{h}\) (respectively, the strongest condition \(\tilde{h} \circ \tilde{\alpha} = \text{constant}\)) then \(d^D \alpha = 0\) and \(\alpha\) satisfies the Hamilton-Jacobi equation for \(h\) (respectively, the strongest condition \(h \circ \alpha = \text{constant}\)), where \(\alpha\) is the section of \(\tau_{D^*} : D^* \to Q\) characterized by the condition \(\tilde{F} \circ \alpha = \tilde{\alpha} \circ F\).

### 5. Applications to nonholonomic Mechanics

#### 5.1. Unconstrained mechanical systems on a Lie algebroid

Let \(\tau_A : A \to Q\) be a Lie algebroid over a manifold \(Q\) and denote by \((\mathfrak{g}, [\cdot, \cdot]_A, \rho_A)\) the Lie algebroid structure of \(A\).
If $\mathcal{G} : A \times Q A \rightarrow \mathbb{R}$ is a bundle metric on $A$ then the **Levi-Civita connection**

$$\nabla^\mathcal{G} : \Gamma(\tau_A) \times \Gamma(\tau_A) \rightarrow \Gamma(\tau_A)$$

is determined by the formula

$$2\mathcal{G}(\nabla^\mathcal{G}_X Y, Z) = \rho_A(X)(\mathcal{G}(Y, Z)) + \rho_A(Y)(\mathcal{G}(X, Z)) - \rho_A(Z)(\mathcal{G}(X, Y))$$

$$+ \mathcal{G}(X, [Z, Y]_A) + \mathcal{G}(Y, [Z, X]_A) - \mathcal{G}(Z, [X, Y]_A)$$

for $X, Y, Z \in \Gamma(A)$. Using the covariant derivative induced by $\nabla^\mathcal{G}$, one may introduce the notion of a geodesic of $\nabla^\mathcal{G}$ as follows. A curve $\sigma : I \rightarrow A$ is **admissible** if

$$\frac{d}{dt}(\tau_A \circ \sigma) = \rho_A \circ \sigma.$$

An admissible curve $\sigma : I \rightarrow A$ is said to be a **geodesic** if $\nabla^\mathcal{G}_{\sigma(t)} \sigma(t) = 0$, for all $t \in I$.

The geodesics are the integral curves of a vector field $\xi_{\mathcal{G}}$ on $A$, the **geodesic flow** of $A$, which is locally given by

$$\xi_{\mathcal{G}} = \rho^i_B v^B \frac{\partial}{\partial q^i} - C^E_{EB} v^B v^C \frac{\partial}{\partial v^E}.$$

Here, $(q^i)$ are local coordinates on an open subset $U$ of $Q$, $\{X_B\}$ is an orthonormal basis of sections of the vector bundle $\tau^{-1}_A(U) \rightarrow U$, $(q^i, v^B)$ are the corresponding local coordinates on $A$ and $\rho^i_B, C^E_{BC}$ are the local structure functions of $A$. Note that the coefficients $\Gamma^E_{BC}$ of the connection $\nabla^\mathcal{G}$ are

$$\Gamma^E_{BC} = \frac{1}{2}(C^C_{EB} + C^B_{EC} + C^E_{BC})$$

(for more details, see [7, 9]).

The **Lagrangian function** $L : A \rightarrow \mathbb{R}$ of an (unconstrained) mechanical system on $A$ is defined by

$$L(a) = \frac{1}{2} \mathcal{G}(a, a) - V(\tau_A(a)) = \frac{1}{2} \|a\|^2_{\mathcal{G}} - V(\tau_A(a)), \quad \text{for } a \in A,$$

$V : Q \rightarrow \mathbb{R}$ being a real $C^\infty$-function on $Q$. In other words, $L$ is the kinetic energy induced by $\mathcal{G}$ minus the potential energy induced by $V$.

Note that if $\Delta$ is the Liouville vector field of $A$ then the **Lagrangian energy** $E_L = \Delta(L) - L$ is the real $C^\infty$-function on $A$ given by

$$E_L(a) = \frac{1}{2} \mathcal{G}(a, a) + V(\tau_A(a)) = \frac{1}{2} \|a\|^2_{\mathcal{G}} + V(\tau_A(a)), \quad \text{for } a \in A.$$  

On the other hand, we may consider the section $grad_\mathcal{G} V$ of $\tau_A : A \rightarrow Q$ characterized by the following condition

$$\mathcal{G}(grad_\mathcal{G} V, X) = (dA)(V)(X) = \rho_A(X)(V), \quad \forall X \in \Gamma(\tau_A).$$

Then, the solutions of the **Euler-Lagrange equations** for $L$ are the integral curves of the vector field $\xi_L$ on $A$ defined by

$$\xi_L = \xi_{\mathcal{G}} - (grad_\mathcal{G} V)^v,$$

where $(grad_\mathcal{G} V)^v \in \mathfrak{X}(A)$ is the standard vertical lift of the section $grad_\mathcal{G} V$. The local expression of the Euler-Lagrange equations is

$$\dot{q}^i = \rho^i_B v^B, \quad \dot{v}^E = -C^C_{EB} v^B v^C - \rho^E_C \frac{\partial V}{\partial q^i},$$
for all $i$ and $E$ (see [7, 9]).

Now, we will denote by $\flat$ the vector bundle isomorphism induced by $S$ and by $\flat : A^* \to A$ the inverse morphism. If $\alpha : Q \to A^*$ is a section of the vector bundle $\tau_{A^*} : A^* \to Q$ we also consider the vector field $\xi_{L,\alpha}$ on $Q$ defined by

$$\xi_{L,\alpha}(q) = (T_{\#_S(\alpha(q))} \tau_A)(\xi_L(\#_S(\alpha(q)))), \; \text{for} \; q \in Q.$$

**Corollary 5.1.** Let $\alpha : Q \to A^*$ be a 1-cocycle of the Lie algebroid $A$, that is, $d^A\alpha = 0$. Then, the following conditions are equivalent:

(i) If $c : I \to Q$ is an integral curve of the vector field $\xi_{L,\alpha}$ on $Q$ we have that $\#_S \circ \alpha \circ c : I \to A$ is a solution of the Euler-Lagrange equations for $L$.

(ii) $\alpha$ satisfies the **Hamilton-Jacobi equation**

$$d^A(E_L \circ \#_S \circ \alpha) = 0,$$

that is, the function $\frac{1}{2}\|\#_S \circ \alpha\|_{\#_S}^2 + V$ on $Q$ is constant on the leaves of the Lie algebroid foliation associated with $A$.

**Proof.** The Legendre transformation associated with the Lagrangian function $L$ is the vector bundle isomorphism $\flat : A \to A^*$ between $A$ and $A^*$ induced by the bundle metric $S$ (for the definition of the Legendre transformation associated with a Lagrangian function on a Lie algebroid, see [25]). Thus, if we denote by $S^*$ the bundle metric on $A^*$ then, the Hamiltonian function $H_L = E_L \circ \#_S$ induced by the hyperregular Lagrangian function $L$ is given by

$$H_L(\gamma) = \frac{1}{2}S^*(\gamma, \gamma) + V(\tau_{A^*}(\gamma)), \; \text{for} \; \gamma \in A^*.$$

Therefore, if $\Lambda_{A^*}$ is the corresponding linear Poisson 2-vector on $A^*$ and $\mathcal{H}^*_L\Lambda_{A^*}$ is the Hamiltonian vector field of $H_L$ with respect to $\Lambda_{A^*}$, we have that the solutions of the Hamilton equations are the integral curves of the vector field $\mathcal{H}^*_L\Lambda_{A^*}$. In fact, the vector fields $\xi_L$ and $\mathcal{H}^*_L\Lambda_{A^*}$ are $\flat$-related, that is,

$$\mathcal{T}\flat \circ \xi_L = \mathcal{H}^*_L\Lambda_{A^*} \circ \flat.$$

Consequently, if $\sigma : I \to A$ is a solution of the Euler-Lagrange equations for $L$ then $\flat \circ \sigma : I \to A^*$ is a solution of the Hamilton equations for $H_L$ and, conversely, if $\gamma : I \to A^*$ is a solution of the Hamilton equations for $H_L$ then $\#_S \circ \sigma : I \to A$ is a solution of the Euler-Lagrange equations for $L$ (for more details, see [25]).

In addition, since $\tau_{A^*} \circ \flat = \tau_A$, it follows that

$$\xi_{L,\alpha}(q) = (T_{\alpha(\gamma)}\tau_{A^*})(\mathcal{H}^*_L\Lambda_{A^*}(\alpha(\gamma))) = \mathcal{H}^*_L\Lambda_{A^*}(q), \; \text{for} \; q \in Q,$$

i.e., $\xi_{L,\alpha} = \mathcal{H}^*_L\Lambda_{A^*}$.

Thus, using Theorem 4.1 (or, alternatively, using Theorem 3.16 in [25]), we deduce the result. \(\square\)

Next, we will apply Corollary 5.1 to the particular case when $A$ is the standard Lie algebroid $TQ$ and $\alpha$ is a 1-coboundary, that is, $\alpha = dS$ with $S : Q \to \mathbb{R}$ a real $C^\infty$-function on $Q$. Note that, in this case, the bundle metric $S$ on $TQ$ is a Riemannian metric $g$ on $Q$ and that $\#_S \circ \alpha = \#_g \circ dS$ is just the gradient vector field of $S$, $grad_g S$, with respect to $g$. 


Corollary 5.2. Let $S : Q \to \mathbb{R}$ be a real $C^\infty$-function on $Q$. Then, the following conditions are equivalent:

(i) If $c : I \to Q$ is an integral curve of the vector field $\xi_{L,\mathcal{A}}$ on $Q$ we have that $\nabla_q S \circ c : I \to A$ is a solution of the Euler-Lagrange equations for $L$.

(ii) $S$ satisfies the **Hamilton-Jacobi equation**

\[ d(E_L \circ \nabla_q S) = 0, \]

that is, the function $\frac{1}{2}\|\nabla_q S\|^2 + V$ on $Q$ is constant.

Remark 5.3. Corollary 5.2 is a consequence of a well-known result (see Theorem 5.2.4 in [1]).

Now, let $L : A \to \mathbb{R}$ (respectively, $\bar{L} : \bar{A} \to \mathbb{R}$) be the Lagrangian function of an unconstrained mechanical system on a Lie algebroid $\tau_A : A \to Q$ (respectively, $\tau_{\bar{A}} : \bar{A} \to \bar{Q}$) and $(\bar{F}, F)$ be a linear Poisson morphism between the Poisson manifolds $(A^*, \{\cdot, \cdot\}_A)$ and $(\bar{A}^*, \{\cdot, \cdot\}_{\bar{A}})$ such that:

(i) $F : Q \to \bar{Q}$ is a surjective map.

(ii) For each $q \in Q$, the linear map $\bar{F}_q = \bar{F}|_{A_q^*} : A_q^* \to \bar{A}_{\bar{F}(q)}^*$ satisfies the following conditions

\[ \bar{S}^*(\bar{F}_q(\beta), \bar{F}_q(\beta')) = S^*(\beta, \beta'), \quad \text{for} \quad \beta, \beta' \in A_q^*, \]

\[ F(q) = F(q') \implies \bar{F}_q(A_q^*) = \bar{F}_{q'}(A_{q'}^*), \]

where $S^*$ (respectively, $\bar{S}^*$) is the bundle metric on $A^*$ (respectively, $\bar{A}^*$). Note that the first condition implies that $\bar{F}_q$ is injective and an isometry.

(iii) If $V : Q \to \mathbb{R}$ (respectively, $\bar{V} : \bar{Q} \to \mathbb{R}$) is the potential energy of the mechanical system on $A$ (respectively, $\bar{A}$) we have that $\bar{V} \circ F = V$.

Then, we deduce that $(\bar{F}, F)$ is a Hamiltonian morphism between the Hamiltonian systems $(A, \{\cdot, \cdot\}_A, H_L)$ and $(\bar{A}, \{\cdot, \cdot\}_{\bar{A}}, H_{\bar{L}})$, where $H_L$ (respectively, $H_{\bar{L}}$) is the Hamiltonian function on $A^*$ (respectively, $\bar{A}^*$) associated with the Lagrangian function $L$ (respectively, $\bar{L}$).

Moreover, using Theorem 4.12 we conclude that

Corollary 5.4. (i) If $\alpha : Q \to A^*$ is a 1-cocycle for the Lie algebroid $A$ ($d^A\alpha = 0$), it satisfies the **Hamilton-Jacobi equation**

\[ d^A(E_L \circ \#_S \circ \alpha) = 0 \]  

(5.1)

and it is $(\bar{F}, F)$-related with $\bar{\alpha} \in \Gamma(\tau_{\bar{A}^*})$ then $d^A\bar{\alpha} = 0$ and $\bar{\alpha}$ is a solution of the Hamilton-Jacobi equation

\[ d^A(E_L \circ \#_{\bar{S}} \circ \bar{\alpha}) = 0. \]  

(5.2)

(ii) If $\bar{\alpha} : \bar{Q} \to \bar{A}^*$ is a 1-cocycle for the Lie algebroid $\bar{A}$ ($d^{\bar{A}}\bar{\alpha} = 0$) and it satisfies the Hamilton-Jacobi equation (5.2) then $d^A\alpha = 0$ and $\alpha$ is a solution of the Hamilton Jacobi equation (5.1). Here, $\alpha : Q \to A^*$ is the section of $\tau_{A^*} : A^* \to Q$ characterized by the condition $\bar{F} \circ \alpha = \bar{\alpha} \circ F$.

A particular example of the above general construction is the following one.

Let $F : Q \to \bar{Q} = Q/G$ be a principal $G$-bundle. Denote by $\phi : G \times Q \to Q$ the free action of $G$ on $Q$ and by $T\phi : G \times TQ \to TQ$ the tangent lift of $\phi$. $T\phi$ is a free action of $G$ on $TQ$. Then, we may consider the quotient vector bundle $\tau_{\bar{A}} = \tau_{TQ/G} : \bar{A} = TQ/G \to \bar{Q} = Q/G$. The sections of this
vector bundle may be identified with the vector fields on $Q$ which are $G$-invariant. Thus, using that a $G$-invariant vector field is $F$-projectable and that the standard Lie bracket of two $G$-invariant vector fields is also a $G$-invariant vector field, we can define a Lie algebroid structure $([\cdot, \cdot]_A, \rho_A)$ on the quotient vector bundle $\tau_A = \tau_{TQ/G} : \tilde{A} = TQ/G \to Q = Q/G$. The resultant Lie algebroid is called the \textit{Atiyah (gauge) algebroid associated with the principal bundle} $F : Q \to \tilde{Q} = Q/G$ (see [25, 27]).

On the other hand, denote by $T^*\phi : G \times T^*Q \to T^*Q$ the cotangent lift of the action $\phi$. Then, the space of orbits of $T^*\phi$, $T^*Q/G$, may be identified with the dual bundle $\tilde{A}^*$ to $\tilde{A}$. Under this identification, the linear Poisson structure on $\tilde{A}^*$ is characterized by the following condition: the canonical projection $\tilde{F} : A^* = T^*Q \to T^*Q/G \simeq \tilde{A}^*$ is a Poisson morphism, when on $A^* = T^*Q$ we consider the linear Poisson structure induced by the standard Lie algebroid $\tau_A = \tau_{TQ} : A = TQ \to Q$, that is, the Poisson structure induced by the canonical symplectic structure of $T^*Q$ (an explicit description of the linear Poisson structure on $A^* \simeq T^*Q/G$ may be found in [32]).

Thus, $(\tilde{F}, F)$ is a linear Poisson morphism between $A^* = T^*Q$ and $\tilde{A}^* \simeq T^*Q/G$ and, in addition, $\tilde{F}$ is a fiberwise bijective vector bundle morphism.

Now, suppose that $G = g$ is a $G$-invariant Riemannian metric on $Q$ and that $V : Q \to \mathbb{R}$ is a $G$-invariant function on $Q$. Then, we may consider the corresponding mechanical Lagrangian function $L : A = TQ \to \mathbb{R}$. Moreover, it is clear that $g$ and $V$ induce a bundle metric $\bar{G}$ on $\tilde{A} = TQ/G$ and a real function $\tilde{V} : \tilde{Q} \to \mathbb{R}$ and, therefore, a mechanical Lagrangian function $\tilde{L} : \tilde{A} = TQ/G \to \mathbb{R}$.

On the other hand, we have that for each $q \in Q$ the map $\tilde{F}_q : A_q^* = T_q^*Q \to \tilde{A}_{F(q)}^* \simeq (T^*Q/G)_{F(q)}$ is a linear isometry. Consequently, using Corollary 5.4, we deduce the following result

**Corollary 5.5.** \textit{There exists a one-to-one correspondence between the 1-cocycles of the Atiyah algebroid $\tau_A = \tau_{TQ/G} : \tilde{A} = TQ/G \to \tilde{Q} = Q/G$ which are solutions of the Hamilton-Jacobi equation for the mechanical Lagrangian function $\tilde{L} : \tilde{A} = TQ/G \to \mathbb{R}$ and the $G$-invariant closed 1-forms $\alpha$ on $Q$ such that the function $\frac{1}{2}\|\#_g \circ \alpha\|_g^2 + V$ is constant.}

**An explicit example:** \textit{The Elroy’s Beanie.} This system is probably the most simple example of a dynamical system with a non-Abelian Lie group of symmetries. It consists in two planar rigid bodies attached at their centers of mass, moving freely in the plane (see [29]). So, the configuration space is $Q = SE(2) \times S^1$ with coordinates $q = (x, y, \theta, \psi)$, where the three first coordinates describe the position and orientation of the center of mass of the first body and the last one the relative orientation between both bodies. The Lagrangian $L : TQ \to \mathbb{R}$ is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1 \dot{\theta}^2 + \frac{1}{2}I_2 (\dot{\theta} + \dot{\psi})^2 - V(\psi)$$

where $m$ denotes the mass of the system and $I_1$ and $I_2$ are the inertias of the first and the second body, respectively; additionally, we also consider a potential function of the form $V(\psi)$. The kinetic energy is associated with the Riemannian metric $\bar{G}$ on $Q$ given by

$$\bar{G} = m(dx^2 + dy^2) + (I_1 + I_2)d\theta^2 + I_2 d\psi \otimes d\psi + I_2 d\psi \otimes d\theta + I_2 d\psi^2.$$

The system is $SE(2)$-invariant for the action

$$\Phi_g(q) = (z_1 + x \cos \alpha - y \sin \alpha, z_2 + x \sin \alpha + y \cos \alpha, \alpha + \theta, \psi)$$

where $g = (z_1, z_2, \alpha)$. 
Let \( \{\xi_1, \xi_2, \xi_3\} \) be the standard basis of \( \mathfrak{se}(2) \),
\[
[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = -\xi_2 \quad \text{and} \quad [\xi_2, \xi_3] = \xi_1.
\]
The quotient space \( \bar{Q} = Q/SE(2) = (SE(2) \times S^1)/SE(2) \simeq S^1 \) is naturally parameterized by the coordinate \( \psi \). The Atiyah algebroid \( TQ/SE(2) \to Q \) is identified with the vector bundle: \( \tau_A : \bar{A} = TS^1 \times \mathfrak{se}(2) \to S^1 \). The canonical basis of sections of \( \tau_A \) is:
\[
\left\{ \frac{\partial}{\partial \psi}, \xi_1, \xi_2, \xi_3 \right\}.
\]
Since the metric \( \bar{G} \) is also \( SE(2) \)-invariant we obtain a bundle metric \( \bar{\bar{G}} \) and a \( \bar{\bar{G}} \)-orthonormal basis of sections:
\[
\left\{ X_1 = \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \left( \frac{\partial}{\partial \psi} - \frac{I_2}{I_1 + I_2} \xi_3 \right), \quad X_2 = \frac{1}{\sqrt{m}} \xi_1, \quad X_3 = \frac{1}{\sqrt{m}} \xi_2, \quad X_4 = \frac{1}{\sqrt{I_1 + I_2}} \xi_3 \right\}
\]
In the coordinates \( (\psi, v^1, v^2, v^3, v^4) \) induced by the orthonormal basis of sections, the reduced Lagrangian is
\[
\bar{L} = \frac{1}{2} \left( (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2 \right) - V(\psi).
\]
Additionally, we deduce that
\[
\begin{align*}
[X_1, X_2]_A &= -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}} X_3, \quad [X_1, X_3]_A = \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} X_2, \\
[X_1, X_4]_A &= 0, \quad [X_2, X_3]_A = 0, \\
[X_2, X_4]_A &= -\frac{1}{\sqrt{I_1 + I_2}} X_3, \quad [X_3, X_4]_A = \frac{1}{\sqrt{I_1 + I_2}} X_2.
\end{align*}
\]
Therefore, the non-vanishing structure functions are
\[
C_{12}^3 = -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}}, \quad C_{13}^2 = \sqrt{\frac{I_2}{I_1(I_1 + I_2)}}, \quad C_{24}^3 = -\frac{1}{\sqrt{I_1 + I_2}}, \quad C_{34}^2 = \frac{1}{\sqrt{I_1 + I_2}}.
\]
Moreover,
\[
\rho_A(X_1) = \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial}{\partial \psi}, \quad \rho_A(X_2) = 0, \quad \rho_A(X_3) = 0, \quad \rho_A(X_4) = 0.
\]
The local expression of the Euler-Lagrange equations for the reduced Lagrangian system \( \bar{L} : \bar{A} \to \mathbb{R} \) is:
\[
\begin{align*}
\dot{\psi} &= \sqrt{\frac{I_1 + I_2}{I_1 I_2}} v^1, \\
\dot{v}^1 &= -\sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi}, \\
\dot{v}^2 &= -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^3 + \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\
\dot{v}^3 &= \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^2 - \frac{1}{\sqrt{I_1 + I_2}} v^2 v^4, \\
\dot{v}^4 &= 0.
\end{align*}
\]
From the two first equations we obtain the equation:
\[
\ddot{\psi} = -\frac{I_1 + I_2 \frac{\partial V}{\partial \psi}}{I_1 I_2}.
\]

A section \( \alpha : S^1 \to \tilde{A}^* \), \( \alpha(\psi) = (\psi, \alpha_1(\psi), \alpha_2(\psi), \alpha_3(\psi), \alpha_4(\psi)) \), is a 1-cocycle, i.e. \( d^A \alpha = 0 \), if and only if \( \alpha_2(\psi) = 0 \), \( \alpha_3(\psi) = 0 \) and \( \frac{\partial \alpha_4}{\partial \psi} = 0 \). Therefore, the Hamilton-Jacobi equation \( d^A(E_L \circ \# \circ \alpha) = 0 \) is
\[
\frac{\partial V}{\partial \psi} + \frac{\partial \alpha_1}{\partial \psi} \alpha_1 = 0.
\]
Thus, integrating we obtain
\[
2V(\psi) + (\alpha_1(\psi))^2 = k_1
\]
with \( k_1 \) constant. Therefore,
\[
\alpha_1(\psi) = \sqrt{k_1 - 2V(\psi)}
\]
and all the solutions of the Hamilton-Jacobi equation are of the form
\[
\alpha(\psi) = (\psi; \sqrt{k_1 - 2V(\psi)}, 0, 0, k_2).
\]
with \( k_2 \) constant.

5.2. Mechanical systems subjected to linear nonholonomic constraints on a Lie algebroid.

Let \( \tau_A : A \to Q \) be a Lie algebroid over a manifold \( Q \) and denote by \((\llbracket \cdot, \cdot \rrbracket_A, \rho_A)\) the Lie algebroid structure on \( A \).

A mechanical system subjected to linear nonholonomic constraints on \( A \) is a pair \((L, D)\), where:

(i) \( L : A \to \mathbb{R} \) is a Lagrangian function of mechanical type, that is,
\[
L(a) = \frac{1}{2} g(a, a) - V(\tau_A(a)), \quad \text{for } a \in A,
\]
and
(ii) \( D \) is the total space of a vector subbundle \( \tau_D : D \to Q \) of \( A \). The vector subbundle \( D \) is said to be the constraint subbundle.

This kind of systems were considered in [7, 9, 14].

We will denote by \( i_D : D \to A \) the canonical inclusion. We also consider the orthogonal decomposition \( A = D \oplus D^\perp \) and the associated orthogonal projectors \( P : A \to D \) and \( Q : A \to D^\perp \). Then, the solutions of the dynamical equations for the nonholonomic (constrained) system \((L, D)\) are just the integral curves of the vector field \( \xi_{(L,D)} \) on \( D \) defined by
\[
\xi_{(L,D)} = TP \circ \xi_L \circ i_D,
\]
where \( \xi_L \) is the solution of the free dynamics (see Section [5.1]) and \( TP : TA \to TD \) is the tangent map to the projector \( P \).

In fact, suppose that \((q^i)\) are local coordinates on an open subset \( U \) of \( Q \) and that \( \{X_B\} = \{X_\gamma, X_b\} \) is a basis of sections of the vector bundle \( \tau_A^{-1}(U) \to U \) such that \( \{X_\gamma\} \) (respectively, \( \{X_b\} \)) is an orthonormal basis of sections of the vector subbundle \( \tau_D^{-1}(U) \to U \) (respectively, \( \tau_D^{-1}(U) \to U \)).
We will denote by \((q^i, v^B) = (q^i, v^\gamma, v^b)\) the corresponding local coordinates on \(A\). Then, the local equations defining the vector subbundle \(D\) are
\[
v^b = 0, \quad \text{for all} \ b.
\]
Moreover, if \(\rho^i_B\) and \(C^E_{BC}\) are the local structure functions of \(A\), we have that the local expression of the vector field \(\xi_{(L,D)}\) is
\[
\xi_{(L,D)} = \rho^i_B v^\gamma \frac{\partial}{\partial q^i} - (C^\nu_B v^\gamma v^\nu + \rho^i_B \frac{\partial V}{\partial q^i}) \frac{\partial}{\partial v^\nu}.
\]
Thus, the dynamical equations for the constrained system \((L, D)\) are
\[
\dot{q}^i = \rho^i_B v^\gamma, \quad \dot{v}^\nu = -C^\nu_B v^\gamma v^\nu - \rho^i_B \frac{\partial V}{\partial q^i}, \quad v^b = 0.
\]
On the other hand, the \textit{constrained connection} \(\bar{\nabla} : \Gamma(\tau_A) \times \Gamma(\tau_A) \to \Gamma(\tau_A)\) associated with the system \((L, D)\) is given by
\[
\bar{\nabla}_X Y = P(\nabla^A_X Y) + \nabla^A_X Q, \quad \text{for} \ X, Y \in \Gamma(\tau_A).
\]
Therefore, if \(\bar{\Gamma}^E_{BC}\) are the coefficients of \(\bar{\nabla}\), we have that
\[
\bar{\Gamma}^i_{\gamma\nu} = \Gamma^i_{\gamma\nu} = \frac{1}{2} (C^\nu_B + C^\gamma_B + C^\delta_B), \quad \Gamma^a_{\gamma\nu} = 0.
\]
Consequently, Eqs. (5.4) are just the \textit{Lagrange-D'Alembert equations} for the system \((L, D)\) considered in [7] (see also [9, 14]).

Next, we will introduce a linear almost Poisson structure \(\{\cdot, \cdot\}_D^*\) on \(D^*\).

Denote by \(\{\cdot, \cdot\}_A^*\) the linear Poisson bracket on \(A^*\) induced by the Lie algebroid structure on \(A\). Then,
\[
\{\varphi, \psi\}_D^* = \{\varphi \circ i_D^*, \psi \circ i_D^*\}_A^* \circ P^*,
\]
for \(\varphi, \psi \in C^\infty(D^*)\), where \(i_D^* : A^* \to D^*\) and \(P^* : D^* \to A^*\) are the dual maps of the monomorphism \(i_D : D \to A\) and the projector \(P : D \to A\), respectively.

It is easy to prove that \(\{\cdot, \cdot\}_D^*\) is a linear almost Poisson bracket on \(D^*\). Moreover, if \((q^i, p_B) = (q^i, p_\gamma, p_b)\) are the dual coordinates of \((q^i, v^B) = (q^i, v^\gamma, v^b)\) on \(A^*\) then it is clear that \((q^i, p_\gamma)\) are local coordinates on \(D^*\) and, in addition, the local expressions of \(i_D^*\) and \(P^*\) are
\[
i_D^*(q^i, p_\gamma, p_b) = (q^i, p_\gamma), \quad P^*(q^i, p_\gamma) = (q^i, p_\gamma, 0).
\]
Thus, from (2.1), (5.5) and (5.6), we have that
\[
\{\varphi, \psi\}_D^* = \rho^i_B \left( \frac{\partial \varphi}{\partial q^i} \frac{\partial \psi}{\partial p_\gamma} - \frac{\partial \varphi}{\partial p_\gamma} \frac{\partial \psi}{\partial q^i} \right) - C^\gamma_B p^\gamma \frac{\partial \varphi}{\partial p_\gamma} \frac{\partial \psi}{\partial p_b},
\]
for \(\varphi, \psi \in C^\infty(D^*)\).

On the other hand, one may introduce a linear Poisson bracket \(\{\cdot, \cdot\}_A\) on \(A\) in such a way that the vector bundle map \(b_\beta : A \to A^*\) is a Poisson isomorphism, when on \(A^*\) we consider the linear Poisson structure \(\{\cdot, \cdot\}_A^*\). Since the local expression of \(b_\beta\) is
\[
b_\beta(q^i, v^B) = (q^i, v^B)
\]
we deduce that the local expression of the linear Poisson bracket \(\{\cdot, \cdot\}_A\) is
\[
\{\bar{\varphi}, \bar{\psi}\}_A = \rho^i_B \left( \frac{\partial \bar{\varphi}}{\partial q^i} \frac{\partial \bar{\psi}}{\partial v^B} - \frac{\partial \bar{\varphi}}{\partial v^B} \frac{\partial \bar{\psi}}{\partial q^i} \right) - C^E_{BC} v^E \frac{\partial \bar{\varphi}}{\partial v^B} \frac{\partial \bar{\psi}}{\partial v^C},
\]
for $\tilde{\varphi}, \tilde{\psi} \in C^\infty(A)$.

Using the bracket $\{\cdot,\cdot\}_A$, one may define a linear almost Poisson bracket $\{\cdot,\cdot\}_{nh}$ on $D$ as follows. If $\tilde{\varphi}$ and $\tilde{\psi}$ are real $C^\infty$-functions on $D$ then

$$\{\tilde{\varphi}, \tilde{\psi}\}_{nh} = \{\tilde{\varphi} \circ P, \tilde{\psi} \circ P\}_A \circ i_D.$$

We have that

$$\{\tilde{\varphi}, \tilde{\psi}\}_{nh} = \rho^\sharp_i \left( \frac{\partial \tilde{\varphi}}{\partial q^i} \frac{\partial \tilde{\psi}}{\partial v^\gamma} - \frac{\partial \tilde{\varphi}}{\partial v^\gamma} \frac{\partial \tilde{\psi}}{\partial q^i} \right) - C^\gamma_{\beta\delta} v^\gamma \frac{\partial \tilde{\varphi}}{\partial v^\beta} \frac{\partial \tilde{\psi}}{\partial v^\delta}.
$$

Thus, a direct computation proves that $\{\cdot,\cdot\}_{nh}$ is just the **nonholonomic bracket** introduced in [7]. Note that, using (5.3) and (5.8), we obtain that $\xi_{(L,D)}$ is the Hamiltonian vector field of the function $(E_L)|_D$ with respect to the nonholonomic bracket $\{\cdot,\cdot\}_{nh}$, i.e.,

$$\tilde{\varphi} = \xi_{(L,D)}(\tilde{\varphi}) = \{\tilde{\varphi}, (E_L)|_D\}_{nh},$$

for $\tilde{\varphi} \in C^\infty(D)$ (see also [7]).

Moreover, if $g_D$ is the restriction of the bundle metric $g$ to $D$ and $b_{g_D} : D \to D^*$ is the corresponding vector bundle isomorphism then, from (5.7) and (5.8), we deduce that

$$\{\varphi \circ g_D, \psi \circ g_D\}_{nh} = \{\varphi, \psi\}_{D^*} \circ g_D,$$

for $\varphi, \psi \in C^\infty(D^*)$.

For this reason, $\{\cdot,\cdot\}_{D^*}$ will also be called the **nonholonomic bracket associated with the constrained system** $(L, D)$.

We will denote by $([\cdot,\cdot], \rho_D)$ (respectively, $d^D$) the corresponding skew-symmetric algebroid structure (respectively, almost differential) on the vector bundle $\tau_D : D \to Q$ and by $\#_{g_D} : D^* \to D$ the inverse morphism of $b_{g_D} : D \to D^*$.

Then, from (2.2) and (5.5), it follows that

$$[X,Y]_D = P[i_D \circ X, i_D \circ Y]_A,$$

for $X, Y \in \Gamma(\tau_D)$. Therefore, using (2.5), we have that

$$d^D \alpha = \Lambda^k i^*_D (d^A(P^* \circ \alpha)), \quad \text{for } \alpha \in \Gamma(\Lambda^k \tau_D^*). \tag{5.10}$$

On the other hand, if $\alpha : Q \to D^*$ is a section of the vector bundle $\tau_{D^*} : D^* \to Q$ one may consider the vector field $\xi_{(L,D)\alpha}$ on $Q$ given by

$$\xi_{(L,D)\alpha}(q) = (T_{\#_{g_D}(\alpha(q))} \tau_D)(\xi_{(L,D)}(\#_{g_D}(\alpha(q)))), \quad \text{for } q \in Q. \tag{5.11}$$

**Corollary 5.6.** Let $\alpha : Q \to D^*$ be a 1-cocycle of the skew-symmetric algebroid $(D, [,\cdot], \rho_D)$, that is, $d^D \alpha = 0$. Then, the following conditions are equivalent:

(i) If $c : I \to Q$ is an integral curve of the vector field $\xi_{(L,D)\alpha}$ on $Q$ we have that $\#_{g_D} \circ \alpha \circ c : I \to D$ is a solution of the Lagrange-D'Alembert equations for the constrained system $(L, D)$.

(ii) $\alpha$ satisfies the nonholonomic Hamilton-Jacobi equation

$$d^D((E_L)|_D \circ \#_{g_D} \circ \alpha) = 0.$$

If, additionally, $H^0(d^D) \simeq \mathbb{R}$ (or the skew-symmetric algebroid $(D, [,\cdot], \rho_D)$ is completely nonholonomic and $Q$ is connected) then conditions (i) and (ii) are equivalent to

(iii) $(E_L)|_D \circ \#_{g_D} \circ \alpha = \text{constant}$. 

Proof. Denote by \( h_{(L,D)} : D^* \to \mathbb{R} \) the Hamiltonian function on \( D^* \) given by \( h_{(L,D)} = (E_L)_D \circ \#_D^* \), by \( \Lambda_{D^*} \) the linear almost Poisson 2-vector on \( D^* \) and by \( \mathcal{H}_{h_{(L,D)}}^{\Lambda_{D^*}} \) the Hamiltonian vector field of \( h_{(L,D)} \) with respect to \( \Lambda_{D^*} \). Then, the vector fields \( \xi_{(L,D)} \) and \( \mathcal{H}_{h_{(L,D)}}^{\Lambda_{D^*}} \) on \( D \) and \( D^* \), respectively, are \( \varphi_{D^*} \)-related. Thus, from (5.11) and since \( \tau_{D^*} \circ \varphi_{D^*} = \tau_D \), it follows that

\[
\mathcal{H}_{h_{(L,D)}}^{\Lambda_{D^*}}(q) = (T_{\alpha(q)}\tau_{D^*})(\mathcal{H}_{h_{(L,D)}}^{\Lambda_{D^*}}(\alpha(q))) = \xi_{(L,D)\alpha}(q),
\]

that is, the vector fields \( \mathcal{H}_{h_{(L,D)}}^{\Lambda_{D^*}} \) and \( \xi_{(L,D)\alpha} \) are equal.

Moreover, if \( \sigma : I \to D \) is a curve on \( D \), we have that \( \sigma \) is a solution of the Lagrange-D'Alembert equations for the constrained system \((L,D)\) if and only if \( \varphi_{D^*} \circ \sigma : I \to D^* \) is a solution of the Hamilton equations for \( h_{(L,D)} \).

Therefore, using Theorem 4.1, we deduce that conditions (i) and (ii) are equivalent.

In addition, if \( H^0(d^D) \simeq \mathbb{R} \) (or if \((D,\{\cdot,\cdot\}_D,\rho_D)\) is completely nonholonomic and \( Q \) is connected) then, from Corollary 4.6 it follows that conditions (i), (ii) and (iii) are equivalent. \( \square \)

**Remark 5.7.** Let \( D^0 \) be the annihilator of \( D \) and \( \mathcal{I}(D^0) \) be the algebraic ideal generated by \( D^0 \). Thus, a section \( \nu \) of the vector bundle \( \Lambda^k A^* \to Q \) belongs to \( \mathcal{I}(D^0) \) if

\[
\nu(q)(v_1, \ldots, v_k) = 0, \quad \text{for all } q \in Q \text{ and } v_1, \ldots, v_k \in D_q.
\]

Now, let \( \mathcal{Z}(\tau_{D^*}) \) be the set defined by

\[
\mathcal{Z}(\tau_{D^*}) = \{ \alpha \in \Gamma(\tau_{D^*})/d^D \alpha = 0 \}
\]

and \( \mathcal{Z}(\tau_{(D^\perp)^0}) \) be the set given by

\[
\mathcal{Z}(\tau_{(D^\perp)^0}) = \{ \hat{\alpha} \in \Gamma(\tau_{(D^\perp)^0})/d^A \hat{\alpha} \in \mathcal{I}(D^0) \}
\]

where \((D^\perp)^0\) is the annihilator of the orthogonal complement \( D^\perp \) of \( D \) and \( \tau_{(D^\perp)^0} : (D^\perp)^0 \to Q \) is the corresponding vector bundle projection. Then, using (5.10), we deduce that the map

\[
\mathcal{Z}(\tau_{D^*}) \to \mathcal{Z}(\tau_{(D^\perp)^0}), \quad \alpha \to P^* \circ \alpha
\]

defines a bijection from \( \mathcal{Z}(\tau_{D^*}) \) on \( \mathcal{Z}(\tau_{(D^\perp)^0}) \). In fact, the inverse map is given by

\[
\mathcal{Z}(\tau_{(D^\perp)^0}) \to \mathcal{Z}(\tau_{D^*}), \quad \hat{\alpha} \to i_D^* \circ \hat{\alpha}.
\]

On the other hand, if \( f \) is a real \( C^\infty \)-function on \( Q \) then

\[
d^D f = 0 \iff (d^A f)(Q) \subseteq D^0.
\]

\( \diamond \)

Let \((L,D)\) (respectively, \((\bar{L},\bar{D})\)) be a nonholonomic system on a Lie algebroid \( \tau_A : A \to Q \) (respectively, \( \tau_A : \bar{A} \to \bar{Q} \)) and \((\bar{F},\tilde{F})\) be a linear almost Poisson morphism between the almost Poisson manifolds \((D^*,\{\cdot,\cdot\}_{D^*})\) and \((\bar{D}^*,\{\cdot,\cdot\}_{\bar{D}^*})\) such that:

(i) \( F : Q \to \bar{Q} \) is a surjective map.

(ii) For each \( q \in Q \), the linear map \( \bar{F}_q = \bar{F}|_{D_q^*} : D_q^* \to \bar{D}_{F(q)}^* \) satisfies the following conditions

\[
S_{D^*}(\bar{F}_q(\beta),\bar{F}_q(\beta')) = S_{D^*}(\beta,\beta'), \quad \text{for } \beta,\beta' \in D_q^*,
\]

\[
F(q) = F(q') \implies \bar{F}_q(D_q^*) = \bar{F}_{q'}(D_{q'}^*),
\]

where \( S_{D^*} \) (respectively, \( S_{D^*} \)) is the bundle metric on \( D^* \) (respectively, \( \bar{D}^* \)).
(iii) If \( V : Q \to \mathbb{R} \) (respectively, \( \tilde{V} : \tilde{Q} \to \mathbb{R} \)) is the potential energy for the nonholonomic system on \( A \) (respectively, \( \tilde{A} \)) we have that \( \tilde{V} \circ F = V \).

Then, we deduce that \((\bar{F}, F)\) is a Hamiltonian morphism between the Hamiltonian systems \((D, \{\cdot, \cdot\}_D^*, h_{(L,D)})\) and \((\tilde{D}, \{\cdot, \cdot\}_{\tilde{D}}^*, h_{(L,\tilde{D})})\), where \( h_{(L,D)} \) (respectively, \( h_{(L,\tilde{D})} \)) is the constrained Hamiltonian function on \( D^* \) (respectively, \( \tilde{D}^* \)) associated with the nonholonomic system \((L, D)\) (respectively, \((L, \tilde{D})\)).

Moreover, using Theorem \[4.12\] we conclude that

**Corollary 5.8.**

(i) If \( \alpha : Q \to D^* \) is a 1-cocycle for the skew-symmetric algebroid \( D \) (\( d^D\alpha = 0 \)), it satisfies the Hamilton-Jacobi equation

\[
d^D((E_L)|_D \circ \#_{\tilde{D}} \circ \alpha) = 0 \tag{5.12}
\]

(respectively, the strongest condition \((E_L)|_D \circ \#_{\tilde{D}} \circ \alpha = \) constant ) and it is \((\bar{F}, F)\)-related with \( \tilde{\alpha} \in \Gamma(\tau_{D^*}) \) then \( d^D\tilde{\alpha} = 0 \) and \( \tilde{\alpha} \) is a solution of the Hamilton-Jacobi equation

\[
d^D((E_L)|_D \circ \#_{\tilde{D}} \circ \tilde{\alpha}) = 0 \tag{5.13}
\]

(respectively, \( \tilde{\alpha} \) satisfies the strongest condition \((E_L)|_D \circ \#_{\tilde{D}} \circ \tilde{\alpha} = \) constant ).

(ii) If \( \tilde{\alpha} : \tilde{Q} \to \tilde{D}^* \) is a 1-cocycle for the skew-symmetric algebroid \( \tilde{D} \) (\( d^{\tilde{D}}\tilde{\alpha} = 0 \)) and it satisfies the Hamilton-Jacobi equation \[5.13\] (respectively, the strongest condition \((E_L)|_{\tilde{D}} \circ \#_{\tilde{D}} \circ \tilde{\alpha} = \) constant ) then \( d^D\tilde{\alpha} = 0 \) and \( \tilde{\alpha} \) is a solution of the Hamilton Jacobi equation \[5.12\] (respectively, \( \alpha \) satisfies the strongest condition \((E_L)|_D \circ \#_{\tilde{D}} \circ \alpha = \) constant ). Here, \( \alpha : Q \to D^* \) is the section of \( \tau_{D^*} : D^* \to Q \) characterized by the condition \( \tilde{F} \circ \alpha = \tilde{\alpha} \circ F \).

5.2.1. The particular case \( A = TQ \). Let \( L : TQ \to \mathbb{R} \) be a Lagrangian function of mechanical type on the standard Lie algebroid \( \tau_{TQ} : TQ \to Q \), that is,

\[
L(v) = \frac{1}{2}g(v, v) - V(\tau_Q(v)), \quad \text{for } v \in TQ,
\]

where \( g \) is a Riemannian metric on \( Q \) and \( V : Q \to \mathbb{R} \) is a real \( C^\infty \)-function on \( Q \). Suppose also that \( D \) is a distribution on \( Q \). Then, the pair \((L, D)\) is a mechanical system subjected to linear nonholonomic constraints on the standard Lie algebroid \( \tau_{TQ} : TQ \to Q \).

Note that, in this case, the linear Poisson structure on \( A^* = T^*Q \) is induced by the canonical symplectic structure on \( T^*Q \). Moreover, the corresponding nonholonomic bracket \( \{\cdot, \cdot\}_D \) on \( D^* \) was considered by several authors or, alternatively, other almost Poisson structures (on \( D \) or on \( \#_{\tilde{g}}(D) \subseteq A^* = T^*Q \) which are isomorphic to \( \{\cdot, \cdot\}_{\tilde{D}^*} \) also were obtained by several authors (see \[2, 19, 22, 33\]).

Now, denote by \( \#_g : T^*Q \to TQ \) (respectively, \( \#_{\tilde{g}} : D^* \to D \)) the inverse morphism of the musical isomorphism \( g : TQ \to T^*Q \) (respectively, \( \tilde{g} : D \to D^* \)) induced by the Riemannian metric \( g \) (respectively, by the restriction \( g_D \) of \( g \) to \( D \)), by \( d \) the standard exterior differential on \( Q \) (that is, \( d = d^{TQ} \) is the differential of the Lie algebroid \( \tau_{TQ} : TQ \to Q \)), by \( \xi_{(L,D)} \in \mathfrak{X}(D) \) the solution of the nonholonomic dynamics and by \( \xi_{(L,D)\alpha} \in \mathfrak{X}(Q) \) its projection on \( Q \), \( \alpha \) being a section of the vector bundle \( \tau_{D^*} : D^* \to Q \) (see \[5.11\]). Using this notation, Corollary \[5.6\] and Remark \[5.7\] we deduce the following result

**Corollary 5.9.** Let \( \alpha : Q \to D^* \) be a section of the vector bundle \( \tau_{D^*} : D^* \to Q \) such that \( d(P^* \circ \alpha) \in \mathcal{J}(D^0) \). Then, the following conditions are equivalent:
(i) If \( c : I \to Q \) is an integral curve of the vector field \( \xi_{(L,D)} \) on \( Q \) we have that \#_{g_D} \circ \alpha \circ c : I \to D \) is a solution of the Lagrange-D'Alembert equations for the constrained system \((L, D)\).

(ii) \( d((E_L)_{|D} \circ \#_{g_D} \circ \alpha)(Q) \subseteq D^0 \).

Remark 5.10. As we know, the Legendre transformation associated with the Lagrangian function \( L : TQ \to \mathbb{R} \) is the musical isomorphism \( \flat g \): \( TQ \to T^*Q \). Moreover, it is clear that \( X(Q) \subseteq D^0 \), where \( X \) is the vector field on \( Q \) given by \( X = \#_{g_D} \circ \alpha \). Thus, Corollary 5.9 is a consequence of some results which were proved in [20] (see Theorem 4.3 in [20]). On the other hand, if \( H_0^{(dD)} \cong \mathbb{R} \) (or if \( Q \) is connected and the distribution \( D \) is completely nonholonomic in the sense of Vershik and Gershkovich [44]) then (i) and (ii) in Corollary 5.9 are equivalent to the condition

\[
(E_L)_{|D} \circ \#_{g_D} \circ \alpha = \text{constant}.
\]

A Hamiltonian version of this last result was proved by Ohsawa and Bloch [31] (see Theorem 3.1 in [31]).

Remark 5.11. Previous approaches. There exists some different attempts in the literature of extending the classical Hamilton-Jacobi equation for the case of nonholonomic constraints [10, 33, 36, 40, 41, 42, 43]. These attempts were non-effective or very restrictive (and even erroneous), because, in many of them, they try to adapt the standard proof of the Hamilton-Jacobi equations for systems without constraints, using Hamilton’s principle. See [37] for a detailed discussion on the topic.

To fix ideas, consider a lagrangian system \( L : TQ \to \mathbb{R} \) of mechanical type, that is, 

\[
L(v_q, q) = \frac{1}{2} g_q(v_q, v_q) - V(q), \quad \text{for } v_q \in T_qQ,
\]

and nonholonomic constraints determined by a distribution \( D \) of \( Q \), whose annihilator is \( D^0 = \text{span}\{\mu_i dq^i\} \).

The idea of many of these previous approaches consist in looking for a function \( S : Q \to \mathbb{R} \) called the characteristic function which permits characterize the solutions of the nonholonomic problem. For it, define first the generalized momenta

\[
p_i = \frac{\partial S}{\partial q^i} + \lambda_b \mu_i^b,
\]

which satisfy the constraint equations \( S^{ij} \mu_i^a = 0 \). These last conditions univocally determine \( \lambda_b \) as functions of \( q \) and \( \partial S/\partial q \) and therefore we find the momenta as functions

\[
p_i = p_i(q^i, \frac{\partial S}{\partial q^i}).
\]  

(5.14)

By inserting these expressions for the generalized momenta in the Hamiltonian of the system, we obtain a version of the Hamilton-Jacobi equation (in its time-independent version):

\[
H(q^i, p_i) = \tilde{H}(q^i, \frac{\partial S}{\partial q^i}) = \text{constant}.
\]  

(5.15)

However, if we start with a curve \( c : I \to Q \) satisfying the differential equations

\[
\dot{c}^i(t) = \frac{\partial H}{\partial p_i}(c^j(t), \frac{\partial S}{\partial q^j}(c(t)) + \lambda_b \mu_j^b)
\]  

(5.16)

in general, it is not true that the curve \( \gamma(t) = (\dot{c}(t), p_i(t)) \) is a solution of the nonholonomic equations. This is trivially checked since from Equation (5.15) we deduce that:

\[
0 = \frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j} \left[ \frac{\partial S}{\partial q^j} \frac{\partial S}{\partial q^i} + \frac{\partial \lambda_b}{\partial q^j} \mu_j^b + \lambda_b \frac{\partial \mu_j^b}{\partial q^i} \right]
\]  

(5.17)
but, on the other hand,

\[ \dot{p}_i = \frac{d}{dt} \left[ \frac{\partial S}{\partial q_i^b} + \lambda_b \mu_i^b \right], \]

\[ = \dot{q}^j \left[ \frac{\partial^2 S}{\partial q_i^b \partial q_j^b} + \frac{\partial \lambda_b}{\partial q_j^b} \mu_i^b + \lambda_b \frac{\partial \mu_i^b}{\partial q_j^b} \right]. \quad (5.18) \]

Substituting Equation (5.17) in Equation (5.18), a curve \( \gamma(t) = (c^i(t), p_i(t)) \) satisfying (5.16) is solution of the nonholonomic equations (that is, \( \dot{p}_i = -\frac{\partial H}{\partial q_i} + \Lambda_b \mu_i^b \)) if it verifies the following condition:

\[ \lambda_b \left( \frac{\partial \mu_i^b}{\partial q_j^b} \dot{q}^j - \frac{\partial \mu_i^b}{\partial q_j^b} \dot{q}^j \right) \delta q^i = 0, \quad \delta q \in D_q. \quad (5.19) \]

It is well-known (see [36, 37]) that condition (5.19) takes place when the solutions of the nonholonomic problem are also of variational type. However, nonholonomic dynamics is not, in general, of variational kind (see [8, 24, 26]). Indeed, a relevant difference with the unconstrained mechanical systems is that a nonholonomic system is not Hamiltonian in the standard sense since the dynamics is obtained from an almost Poisson bracket, that is, a bracket not satisfying the Jacobi identity (see [5, 19, 22, 39]).

An explicit example: The two-wheeled carriage (see [30]). The system has configuration space \( Q = SE(2) \times T^2 \), where \( SE(2) \) represents the rigid motions in the plane and \( T^2 \) the angles of rotation of the left and right wheels. We use standard coordinates \( (x, y, \theta, \psi_1, \psi_2) \in SE(2) \times T^2 \). Imposing the constraints of no lateral sliding and no sliding on both wheels, one gets the following nonholonomic constraints:

\[ \dot{x} \sin \theta - \dot{y} \cos \theta = 0, \]
\[ \dot{x} \cos \theta + \dot{y} \sin \theta + r \dot{\theta} + a \dot{\psi}_1 = 0, \]
\[ \dot{x} \cos \theta + \dot{y} \sin \theta - r \dot{\theta} + a \dot{\psi}_2 = 0, \]

where \( a \) is the radius of the wheels and \( r \) is the half the length of the axle.

Assuming, for simplicity, that the center of mass of the carriage is situated on the center of the axle the Lagrangian is given by:

\[ L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} C \dot{\psi}_1^2 + \frac{1}{2} C \dot{\psi}_2^2, \]

where \( m \) is the mass of the system, \( J \) the moment of inertia when it rotates as a whole about the vertical axis passing through the point \( (x, y) \) and \( C \) the axial moment of inertia. Note that \( L \) is the kinetic energy associated with the Riemannian metric \( g \) on \( Q \) given by

\[ g = m(dx^2 + dy^2) + Jd\theta^2 + C d\psi_1^2 + C d\psi_2^2. \]
The constraints induce the distribution $D$ locally spanned by the following $g$-orthonormal vector fields

$$
X_1 = \frac{1}{\Lambda_1} \left( 2r \frac{\partial}{\partial \psi_1} - a \frac{\partial}{\partial \theta} - ar \cos \theta \frac{\partial}{\partial x} - ar \sin \theta \frac{\partial}{\partial y} \right),
$$

$$
X_2 = \frac{1}{\Lambda_2} \left( a^2(J - m_1 r^2) \frac{\partial}{\partial \psi_1} + (a^2J + 4Cr^2 + a^2m_1 r^2) \frac{\partial}{\partial \psi_2} + ar(2C + m_1 a^2) \frac{\partial}{\partial \theta} 
- a(a^2J + 2Cr^2) \cos \theta \frac{\partial}{\partial x} - a(a^2J + 2Cr^2) \sin \theta \frac{\partial}{\partial y} \right),
$$

where

$$
\Lambda_1 = \sqrt{4Cr^2 + 2a^2J + am_1 r^2},
\Lambda_2 = \sqrt{(a^2J + 2Cr^2)(2C + m_1 a^2)(a^2J + 4Cr^2 + a^2r^2m_1)}
$$

We will denote by $(x, y, \theta, \psi_1, \psi_2, v^1, v^2)$ the local coordinates on $D$ induced by the basis $\{X_1, X_2\}$.

In these coordinates, the restriction, $L_D : D \rightarrow \mathbb{R}$, of $L$ to $D$ is:

$$
L_{|D} = \frac{1}{2}((v^1)^2 + (v^2)^2).
$$

The distribution $D^\perp$ orthogonal to $D$ is generated by

$$
D^\perp = \{X_3 = \tan \theta \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, X_4 = \frac{J \sec(\theta)}{rm_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial \theta} + \frac{aJ}{Cr \psi_1} \frac{\partial}{\partial \psi_1}, X_5 = \frac{2C \sec(\theta)}{am_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \}
$$

Moreover, since the standard Lie bracket $[X_1, X_2]$ of the vector fields $X_1$ and $X_2$ is orthogonal to $D$, it follows that (see [5.9])

$$
[X_1, X_2]_{|D} = 0, \quad \rho_D(X_1) = X_1, \quad \rho_D(X_2) = X_2,
$$

where $([\cdot, \cdot]_{|D}, \rho_D)$ is the skew-symmetric algebroid structure on the vector bundle $\tau_D : D \rightarrow Q$.

The local expression of the vector field $\xi_{(L,D)}$ is:

$$
\xi_{(L,D)} = \left( \frac{2rv^1}{\Lambda_1} + \frac{a^2(J - m_1 r^2)v^2}{\Lambda_2} \right) \frac{\partial}{\partial \psi_1} + \left( \frac{a^2J + 2Cr^2 + a^2m_1 r^2}{\Lambda_2} \right) \frac{v^2 \partial}{\partial \psi_2} 
+ \left( \frac{ar(2C + m_1 a^2)}{\Lambda_2} \right) \frac{\partial}{\partial \theta} - \left( \frac{arv^1 \cos \theta}{\Lambda_1} + \frac{a(a^2J + 2Cr^2)v^2 \cos \theta}{\Lambda_2} \right) \frac{\partial}{\partial x} 
- \left( \frac{arv^1 \sin \theta}{\Lambda_1} + \frac{a(a^2J + 2Cr^2)v^2 \sin \theta}{\Lambda_2} \right) \frac{\partial}{\partial y}
$$

Furthermore, if $\{X^1, X^2\}$ is the dual basis of $\{X_1, X_2\}$ and $\alpha : Q \rightarrow D^\ast$ is a section of the vector bundle $\tau_{D^\ast} : D^\ast \rightarrow Q$

$$
\alpha = \alpha_1 X^1 + \alpha_2 X^2, \quad \text{with } \alpha_1, \alpha_2 \in C^\infty(Q)
$$

then

$$
\alpha \text{ is a 1-cocycle } \iff X_1(\alpha_2) - X_2(\alpha_1) = 0.
$$

In particular, taking

$$
\alpha = K_1 X^1 + K_2 X^2, \quad \text{with } K_1, K_2 \in \mathbb{R}
$$

trivially is satisfied the 1-cocycle condition.
In addition, since \( E_L = L \), we deduce that
\[
(E_L)|_D = \frac{1}{2}((v^1)^2 + (v^2)^2)
\]
which implies that
\[
(E_L)|_D \circ \#_{g_D} \circ \alpha = K_1^2 + K_2^2 = \text{constant}.
\]
Thus, using Corollary \[5.6\] we conclude that to integrate the nonholonomic mechanical system \((L, D)\) is equivalent to find the integral curves of the vector field on \(Q = S^1 \times S^1 \times \mathbb{R}^2\) given by
\[
\xi_{(L,D),\alpha} = K_1X_1 + K_2X_2.
\]
which are easily obtained.

It is also interesting to observe that, in this particular example,
\[
\mathrm{Lie}^\infty(D) = \{ \text{ad} \psi^1 - \text{ad} \psi^2 + 2r \theta \}^0
\]
and, thus, \(D\) is not completely nonholonomic. From Theorem \[3.4\] it is necessary to restrict the initial nonholonomic system to the orbits of \(D\), that in this case are
\[
L_k = \{(x,y,\theta,\psi_1,\psi_2) \in SE(2) \times \mathbb{T}^2 \mid a(\psi_1 - \psi_2) + 2r \theta = k, \text{ with } k \in \mathbb{R}\}
\]
to obtain a completely nonholonomic skew-symmetric algebroid structure on the vector bundle \(\tau_{D_{L_k}} : D_{L_k} \to L_k\). Note that on \(L_k\) we can use, for instance, coordinates \((x,y,\psi_1,\psi_2)\).

5.2.2. The particular case \(\tilde{A} = TQ/G\). Let \(F : Q \to \tilde{Q} = Q/G\) be a principal \(G\)-bundle and \(\tau_{\tilde{A}} = \tau_{TQ/G} : \tilde{A} = TQ/G \to \tilde{Q} = Q/G\) be the Atiyah algebroid associated with the principal bundle (see Section \[5.1\]).

Suppose that \(g\) is a \(G\)-invariant Riemannian metric on \(Q\), that \(V : Q \to \mathbb{R}\) is a \(G\)-invariant real \(C^\infty\)-function and that \(D\) is a \(G\)-invariant distribution on \(Q\). Then, we may consider the corresponding nonholonomic mechanical system \((L, D)\) on the standard Lie algebroid \(\tau_A = \tau_{TQ} : A = TQ \to Q\).

Denote by \(\xi_{(L,D)} \in \mathfrak{X}(D)\) the nonholonomic dynamics for the system \((L, D)\) and by \(\{\cdot, \cdot\}_D\), the nonholonomic bracket on \(D^*\).

The Riemannian metric \(g\) and the function \(V : Q \to \mathbb{R}\) induce a bundle metric \(\tilde{g}\) on the Atiyah algebroid \(\tau_{\tilde{A}} = \tau_{TQ/G} : \tilde{A} = TQ/G \to \tilde{Q} = Q/G\) and a real \(C^\infty\)-function \(\tilde{V} : \tilde{Q} \to \mathbb{R}\) on \(\tilde{Q}\) such that \(\tilde{V} \circ F = V\), where \(F : Q \to \tilde{Q} = Q/G\) is the canonical projection. Moreover, the space of orbits \(\tilde{D}\) of the action of \(G\) on \(D\) is a vector subbundle of the Atiyah algebroid \(\tau_{\tilde{A}} = \tau_{TQ/G} : \tilde{A} = TQ/G \to \tilde{Q} = Q/G\). Thus, we may consider the corresponding nonholonomic mechanical system \((\tilde{L}, \tilde{D})\) on \(\tilde{A} = TQ/G\).

Let \(\tilde{F} : A = TQ \to \tilde{A} = TQ/G\) be the canonical projection. Then, \((\tilde{F}, F)\) is a fiberwise bijective morphism of Lie algebroids and \(\tilde{F}(D) = \tilde{D}\). Therefore, using some results in \[7\] (see Theorem \[4.6\] in \[7\]) we deduce that the vector field \(\xi_{(L,D)}\) is \(\tilde{F}_D\)-projectable on the nonholonomic dynamics \(\tilde{\xi}_{(L,D)} \in \mathfrak{X}(\tilde{D})\) of the system \((\tilde{L}, \tilde{D})\). Here, \(\tilde{F}_D : D \to \tilde{D} = D/G\) is the canonical projection.

On the other hand, if \(P : A = TQ \to D\) and \(\tilde{P} : \tilde{A} = TQ/G \to \tilde{D} = D/G\) are the orthogonal projectors then it is clear that
\[
\tilde{F}_D \circ P = \tilde{P} \circ F
\]
which implies that
\[
\tilde{F} \circ P^* = \tilde{F}_D \circ \tilde{P}^*,
\]
(5.21)
where \( \tilde{F} : A^* \rightarrow T^*Q \rightarrow \tilde{A}^* \simeq T^*Q/G \) and \( \tilde{F}_D : D^* \rightarrow \tilde{D}^* \simeq D^*/G \) are the canonical projections. Moreover, if on \( \tilde{A}^* \) we consider the linear Poisson structure induced by the Atiyah algebroid \( \tau_{\tilde{A}} = \tau_{TQ/G} : \tilde{A} = TQ/G \rightarrow \tilde{Q} = Q/G \) then, as we know, \( \tilde{F} : A^* = T^*Q \rightarrow \tilde{A}^* \simeq T^*Q/G \) is a Poisson morphism. Thus, using this fact, (5.5) and (5.21), we deduce the following result

**Proposition 5.12.** The pair \((\tilde{F}_D, F)\) is a linear almost Poisson morphism, when on \(D^*\) and \(\tilde{D}^*\) we consider the almost Poisson structures induced by the nonholonomic brackets \(\{\cdot, \cdot\}_{D^*}\) and \(\{\cdot, \cdot\}_{\tilde{D}^*}\), respectively.

Note that Proposition 5.12 characterizes the nonholonomic bracket \(\{\cdot, \cdot\}_{\tilde{D}^*}\).

We also note that the linear map \((\tilde{F}_D)_{q} = (\tilde{F}_D)_{|D^*} : D^*_q \rightarrow \tilde{D}^*_q \simeq (D^*/G)_{F(q)}\) is a linear isometry, for all \(q \in Q\). Therefore, from Remark 5.7 and Corollary 5.8, it follows

**Corollary 5.13.** Let \(S\) the set of the \(1\)-cocycles \(\tilde{\alpha}\) of the skew-symmetric algebroid \(\tau_{\tilde{D}} = \tau_{D/G} : \tilde{D} = D/G \rightarrow \tilde{Q} = Q/G\) which are solution of the nonholonomic Hamilton-Jacobi equation

\[
d^\mathcal{D}((E_L)_{\mathcal{D}} \circ \#_{\mathcal{D}} \circ \tilde{\alpha}) = 0
\]

(respectively, which satisfy the strongest condition \((E_L)_{\mathcal{D}} \circ \#_{\mathcal{D}} \circ \tilde{\alpha} = \text{constant}\). Then, there exists a one-to-one correspondence between \(S\) and the following sets:

(i) The set of the \(G\)-invariant \(1\)-cocycles \(\alpha\) of the skew-symmetric algebroid \(\tau_{D} : D \rightarrow Q\) which are solutions of the nonholonomic Hamilton-Jacobi equation

\[
d^\mathcal{D}((E_L)_{\mathcal{D}} \circ \#_{\mathcal{D}} \circ \alpha) = 0
\]

(respectively, which satisfy the strongest condition \((E_L)_{\mathcal{D}} \circ \#_{\mathcal{D}} \circ \alpha = \text{constant}\).

(ii) The set of the \(G\)-invariant \(1\)-forms \(\gamma : Q \rightarrow (D^\perp)^0 \subseteq T^*Q\) on \(Q\) which satisfy the following conditions

\[
d\gamma \in \mathcal{I}(D^0) \quad \text{and} \quad d(E_L \circ \#_g \circ \gamma)(Q) \subseteq D^0
\]

(respectively, which satisfy the strongest conditions \(d\gamma \in \mathcal{I}(D^0)\) and \(E_L \circ \#_g \circ \gamma = \text{constant}\).

**An explicit example: The snakeboard.**

The snakeboard is a modified version of the traditional skateboard, where the rider uses his own momentum, coupled with the constraints, to generate forward motion. The configuration manifold is \(Q = SE(2) \times T^2\) with coordinates \((x, y, \theta, \psi, \phi)\) (see [1, 21]).

The system is described by a Lagrangian

\[
L(q, \dot{q}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} (J + 2J_1)\dot{\theta}^2 + \frac{1}{2} J_0(\dot{\psi} + \dot{\theta})^2 + J_1\dot{\phi}^2
\]
where \( m \) is the total mass of the board, \( J > 0 \) is the moment of inertia of the board, \( J_0 > 0 \) is the moment of inertia of the rotor of the snakeboard mounted on the body’s center of mass and \( J_1 > 0 \) is the moment of inertia of each wheel axles. The distance between the center of the board and the wheels is denoted by \( r \). For simplicity, as in [21], we assume that \( m + J_0 + 2J_1 = mr^2 \).

The inertia matrix representing the kinetic energy of the metric \( g \) on \( Q \) defined by the snakeboard is

\[
g = m dx^2 + mdy^2 + mr^2 d\theta^2 + J_0 d\theta \otimes d\psi + J_0 d\psi \otimes d\theta + J_1 d\psi^2 + 2J_1 d\phi^2.
\]

Since the wheels are not allowed to slide in the sideways direction, we impose the constraints

\[
-\dot{x} \sin(\theta + \phi) + \dot{y} \cos(\theta + \phi) - r \dot{\theta} \cos \phi = 0 \\
-\dot{x} \sin(\theta - \phi) + \dot{y} \cos(\theta - \phi) + r \dot{\theta} \cos \phi = 0.
\]

To avoid singularities of the distribution defined by the previous constraints we will assume, in the sequel, that \( \phi \neq \pm \pi/2 \).

Define the functions

\[
a = -r(\cos \phi \cos(\theta - \phi) + \cos \phi \cos(\theta + \phi)) = -2r \cos^2 \phi \cos \theta \\
b = -r(\cos \phi \sin(\theta - \phi) + \cos \phi \sin(\theta + \phi)) = -2r \cos^2 \phi \sin \theta \\
c = \sin(2\phi).
\]

The constraint subbundle \( \tau_D : D \hookrightarrow Q \) is

\[
D = \text{span} \left\{ \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi}, a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial \theta} \right\}.
\]

The Lagrangian function and the constraint subbundle are left-invariant under the \( SE(2) \) action:

\[
\Phi_g(q) = (\alpha + x \cos \gamma - y \sin \gamma, \beta + x \sin \gamma + y \cos \gamma, \gamma + \theta, \psi, \phi)
\]

where \( g = (\alpha, \beta, \gamma) \in SE(2) \).

We have a principal bundle structure \( F : Q \rightarrow \bar{Q} \) where \( \bar{Q} = (SE(2) \times \mathbb{T}^2)/SE(2) \simeq \mathbb{T}^2 \), being its vertical bundle \( VF = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta} \right\} \). We have that

\[
S = D \cap VF = \text{span} \left\{ Y_3 = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial \theta} \right\}
\]

and therefore,

\[
S^\perp \cap D = \text{span} \left\{ Y_1 = \frac{\partial}{\partial \phi}, Y_2 = \frac{\partial}{\partial \psi} - \frac{J_0 c}{k} Y_3 \right\} = \text{span} \left\{ Y_1 = \frac{\partial}{\partial \phi}, Y_2 = \frac{\partial}{\partial \psi} - \frac{J_0}{2mr^2} (\tan \phi) Y_3 \right\}
\]

where \( k = m(a^2 + b^2 + c^2 r^2) = 4mr^2 (\cos^2 \phi) \) (away form \( \phi = \pm \pi/2 \)).

Note that if \( \{\xi_1, \xi_2, \xi_3\} \) is the canonical basis of \( \mathfrak{se}(2) \) then

\[
Y_2 = \frac{\partial}{\partial \psi} - \frac{J_0 \sin \phi}{mr^2} \left[ -r(\cos \phi) \xi_1 + (\sin \phi) \xi_3 \right] \\
Y_3 = -2r(\cos^2 \phi) \xi_1 + (\sin 2\phi) \xi_3
\]
where $\xi_i$ $(i = 1, 2, 3)$ is the left-invariant vector field of $SE(2)$ such that $\xi_i(e) = \xi_i$, $e$ being the identity element of $SE(2)$.

Next, we will denote by $\{X_1, X_2, X_3\}$ the $g$-orthonormal basis of $D$ given by

\[
X_1 = \frac{1}{\sqrt{2J_1}} \frac{\partial}{\partial \phi},
\]

\[
X_2 = \frac{1}{\sqrt{f(\phi)}} \left( \frac{\partial}{\partial \psi} - \frac{J_0 \sin \phi}{m} \left[ -r \cos(\phi) \xi_1 + \left( \frac{\sin \phi}{r} \right) \xi_3 \right] \right)
\]

\[
X_3 = \frac{1}{\sqrt{m}} \left[ -(\cos \phi) \xi_1 + \left( \frac{\sin \phi}{r} \right) \xi_3 \right],
\]

where $f(\phi) = J_0 - \frac{J_0^2 \sin^2 \phi}{m}$. The vector fields $\{X_1, X_2\}$ describe changes in the internal angles $\phi$ and $\psi$, while $X_3$ represents the instantaneous rotation when the internal angles are fixed.

Consider now the corresponding Atiyah algebroid

$$TQ/SE(2) \simeq (TT^2 \times TSE(2))/SE(2) \longrightarrow \tilde{Q} = T^2.$$ 

Using the left translations on $SE(2)$, we have that the tangent bundle of $SE(2)$ may be identified with the product manifold $SE(2) \times se(2)$ and therefore the Atiyah algebroid is identified with the vector bundle $\tau_2 = \tau_A : A = TT^2 \times se(2) \longrightarrow T^2$. The canonical basis of $\tau_A : TT^2 \times se(2) \longrightarrow T^2$ is

$$\left\{ \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi}, \xi_1, \xi_2, \xi_3 \right\}.$$ 

The anchor map and the linear bracket of the Lie algebroid $\tau_A : TT^2 \times se(2) \longrightarrow T^2$ is given by

$$\rho_A(\frac{\partial}{\partial \psi}) = \frac{\partial}{\partial \psi}, \quad \rho_A(\frac{\partial}{\partial \phi}) = \frac{\partial}{\partial \phi}, \quad \rho_A(\xi_i) = 0, \quad i = 1, 2, 3
\]

$$[\xi_1, \xi_2]_\tilde{A} = -\xi_2, \quad [\xi_2, \xi_3]_\tilde{A} = \xi_1,$$ 

being equal to zero the rest of the fundamental Lie brackets.

We select the orthonormal basis of sections, $\{X'_1, X'_2, X'_3, X'_4, X'_5\}$, where

\[
X'_1 = \frac{1}{\sqrt{2J_1}} \frac{\partial}{\partial \phi},
\]

\[
X'_2 = \frac{1}{\sqrt{f(\phi)}} \left( \frac{\partial}{\partial \psi} - \frac{J_0 \sin \phi}{m} \left[ -r \cos(\phi) \xi_1 + \left( \frac{\sin \phi}{r} \right) \xi_3 \right] \right)
\]

\[
X'_3 = \frac{1}{\sqrt{m}} \left[ -(\cos \phi) \xi_1 + \left( \frac{\sin \phi}{r} \right) \xi_3 \right],
\]

and $\{X'_4, X'_5\}$ is an orthonormal basis of sections of the orthogonal complement to $\tilde{D}$, $\tilde{D}^\perp$, with respect to the induced bundle metric $\mathcal{G}_A$.

Taking the induced coordinates $(\psi, \phi, v^1, v^2, v^3, v^4, v^5)$ on $TT^2 \times se(2)$ by this basis of sections, we deduce that the space of orbits $D$ of the action of $SE(2)$ on $D$ has as local equations, $v^4 = 0$ and $v^5 = 0$, being a basis of sections of $D$, $\{X'_1, X'_2, X'_3\}$. Moreover, in these coordinates the reduced
Lagrangian $\bar{L} : T\mathbb{T}^2 \times \mathfrak{se}(2) \longrightarrow \mathbb{R}$ is

$$
\bar{L} = \frac{1}{2} \left( (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2 + (v^5)^2 \right).
$$

Now, we consider the reduced nonholonomic mechanical system $(\bar{L}, \bar{D})$.

After some straightforward computations we deduce that

$$
[X'_1, X'_2]_D = -\frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} X'_3, \quad [X'_1, X'_3]_D = \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} X'_2, \quad [X'_2, X'_3]_D = 0.
$$

Therefore, the non-vanishing structure functions are:

$$
C^3_{12} = -C^3_{21} = -\frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}}, \quad C^2_{13} = -C^3_{31} = \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}}.
$$

Moreover,

$$
\rho_{\bar{D}}(X'_1) = \frac{1}{\sqrt{2J_1}} \frac{\partial}{\partial \phi}, \quad \rho_{\bar{D}}(X'_2) = \frac{1}{\sqrt{f(\phi)}} \frac{\partial}{\partial \psi}, \quad \rho_{\bar{D}}(X'_3) = 0.
$$

This shows that $\rho_{\bar{D}}(\bar{D}) = T_q\mathbb{T}^2$ and then the skew-symmetric algebroid $\bar{D} \longrightarrow \mathbb{T}^2$ is completely nonholonomic.

The local expression of the vector field $\xi_{(L, \bar{D})}$ is

$$
\xi_{(L, \bar{D})} = \frac{v^1}{\sqrt{2J_1}} \frac{\partial}{\partial \phi} + \frac{v^2}{\sqrt{f(\phi)}} \frac{\partial}{\partial \psi} - \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} v^1 \frac{\partial}{\partial v^2} + \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} v^2 \frac{\partial}{\partial v^3}.
$$

Let $\{(X')^1, (X')^2, (X')^3\}$ be the dual basis of $\bar{D}^*$. It induces a local coordinate system: $(\phi, \psi, p_1, p_2, p_3)$ on $D^*$ and, therefore, the non-vanishing terms of the nonholonomic bracket are:

$$
\{\phi, p_1\}_{\bar{D}^*} = \frac{1}{\sqrt{2J_1}}, \quad \{\psi, p_2\}_{\bar{D}^*} = \frac{1}{\sqrt{f(\phi)}}, \quad \{p_1, p_2\}_{\bar{D}^*} = \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} p_3, \quad \{p_1, p_3\}_{\bar{D}^*} = -\frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} p_2.
$$

Now, we study the Hamilton-Jacobi equations for the snakeboard system. A section $\alpha : \mathbb{T}^2 \longrightarrow \bar{D}^*$, $\alpha = \alpha_1(\phi, \psi)(X')^1 + \alpha_2(\phi, \psi)(X')^2 + \alpha_3(\phi, \psi)(X')^3$, is a 1-cocycle $(d^D \alpha = 0)$ if and only if:

$$
0 = \frac{1}{\sqrt{2J_1}} \frac{\partial \alpha_2}{\partial \phi} - \frac{1}{\sqrt{f(\phi)}} \frac{\partial \alpha_1}{\partial \psi} + \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} \alpha_3 \quad (5.22)
$$

$$
0 = \frac{1}{\sqrt{2J_1}} \frac{\partial \alpha_3}{\partial \phi} - \frac{J_0 \cos \phi}{r \sqrt{2J_1 m f(\phi)}} \alpha_2 \quad (5.23)
$$

$$
0 = \frac{1}{\sqrt{f(\phi)}} \frac{\partial \alpha_3}{\partial \psi}. \quad (5.24)
$$
Finally, since the skew-symmetric algebroid is completely nonholonomic, the Hamilton-Jacobi equation is rewritten as

\[(\alpha_1(\phi, \psi))^2 + (\alpha_2(\phi, \psi))^2 + (\alpha_3(\phi, \psi))^2 = \text{constant}\] (5.25)

Now, we will use this equation for studying explicit solutions for the snakeboard, showing the availability of our methods for obtaining new insights in nonholonomic dynamics.

From Equation (5.24) we obtain that \[\alpha_3 = \alpha_3(\phi)\]. Then it is clear that also \[\alpha_2 = \alpha_2(\phi)\]. Assume that \[\alpha_1 = \text{constant}\]. Therefore, Equations (5.22) and (5.23) are now, in this case, a system of ordinary differential equations:

\[
0 = \frac{d\alpha_2}{d\phi} + \frac{J_0 \cos \phi}{r \sqrt{mf(\phi)}} \alpha_3
\]

\[
0 = \frac{d\alpha_3}{d\phi} - \frac{J_0 \cos \phi}{r \sqrt{mf(\phi)}} \alpha_2.
\]

Moreover, observe that all the solutions of these equations automatically satisfy Equation (5.25) since

\[
\alpha_2 \frac{d\alpha_2}{d\phi} + \alpha_3 \frac{d\alpha_3}{d\phi} = 0 \quad \text{and} \quad \alpha_1 = \text{constant}
\]

Solving explicitly the system of equations (5.26) and (5.27) we obtain that

\[
\alpha_2(\phi) = C_1 \sqrt{f(\phi)} + \frac{J_0 C_2}{r \sqrt{m}} \sin \phi
\]

\[
\alpha_3(\phi) = \frac{J_0 C_1}{r \sqrt{m}} \sin \phi - C_2 \sqrt{f(\phi)}
\]

with \(C_1, C_2\) arbitrary constants. Therefore,

\[
\alpha(\phi, \psi) = (\phi, \psi; \sqrt{2} J_1 C_0, C_1 \sqrt{f(\phi)} + \frac{J_0 C_2}{r \sqrt{m}} \sin \phi, \frac{J_0 C_1}{r \sqrt{m}} \sin \phi - C_2 \sqrt{f(\phi)})
\]

is an 1-cocycle of the skew-symmetric algebroid \(\bar{D} \rightarrow \mathbb{T}^2\), for all \((C_0, C_1, C_2) \in \mathbb{R}^3\) and moreover it satisfies Equations (5.25). Hence, we can use Corollary 5.6 to obtain solutions of the reduced snakeboard problem. First, we calculate the integral curves of the vector field \(\xi_{(\bar{L}, \bar{D})\alpha}\):

\[
\dot{\phi}(t) = C_0
\]

\[
\dot{\psi}(t) = \sqrt{f(\phi(t))} \left( C_1 \sqrt{f(\phi(t))} + \frac{J_0 C_2}{r \sqrt{m}} \sin \phi(t) \right)
\]

\[= C_1 + \frac{J_0 C_2}{r \sqrt{mf(\phi(t))}} \sin \phi(t)\]

whose solutions are:

\[
\dot{\phi}(t) = C_0 t + C_3
\]

\[
\dot{\psi}(t) = C_1 t - \frac{C_2}{C_0} \log \left[ \sqrt{2} \left( \sqrt{J_0 \cos(C_0 t + C_3)} + \sqrt{mr^2 - J_0 \sin^2(C_0 t + C_3)} \right) \right] + C_4 \quad \text{(if } C_0 \neq 0)\]

\[
\dot{\psi}(t) = C_1 t + \frac{\sqrt{J_0 C_2 t \sin(C_3)}}{\sqrt{mr^2 - J_0 \sin^2(C_3)}} + C_4 \quad \text{(if } C_0 = 0)\]
for all constants $C_i \in \mathbb{R}, 1 \leq i \leq 5$. Now, by a direct application of the nonholonomic equation we obtain that

\[
\begin{align*}
v^1(t) &= \sqrt{2J_1} C_0 \\
v^2(t) &= C_1 \sqrt{f(C_0 t + C_3)} + \frac{J_0 C_2}{r \sqrt{m}} \sin(C_0 t + C_3) \\
v^3(t) &= \frac{J_0 C_1}{r \sqrt{m}} \sin(C_0 t + C_3) - C_2 \sqrt{f(C_0 t + C_3)}
\end{align*}
\]

are solutions of the reduced nonholonomic problem.

6. Conclusions and Future Work

In this paper we have elucidated the geometrical framework for the Hamilton-Jacobi equation. Our formalism is valid for nonholonomic mechanical systems. The basic geometric ingredients are a vector bundle, a linear almost Poisson bracket and a Hamiltonian function both on the dual bundle. We also have discussed the behavior of the theory under Hamiltonian morphisms and its applicability to reduction theory. Some examples are studied in detail and, as a consequence, it is shown the utility of our framework to integrate the dynamical equations. However, in this direction more work must be done.

In particular, as a future research, we will study new particular examples, testing candidates for solutions of the nonholonomic Hamilton-Jacobi equation of the form $\alpha = dD f$, for some $f \in C^\infty(Q)$ (if there exists) and moreover we will study the complete solutions for the Hamilton-Jacobi equation using the groupoid theory. In this line, we will study the construction of numerical integrators via Hamilton-Jacobi theory [18]. We will also discuss the extension of our formalism to time-dependent Lagrangian systems subjected to affine constraints in the velocities. It would be interesting to describe the Hamilton-Jacobi theory for variational constrained problems, giving a geometric interpretation of the Hamilton-Jacobi-Bellman equation for optimal control systems. Finally, extensions to classical field theories in the present context could be developed.

Appendix

Let $\{\cdot, \cdot\}_D$ be a linear almost Poisson structure on a vector bundle $\tau_D : D \to Q$, $([\cdot, \cdot]_D, \rho_D)$ be the corresponding skew-symmetric Lie algebroid structure on $D$ and $\alpha : Q \to D^*$ be a section of $\tau_{D^*} : D^* \to Q$. If $q \in Q$ then we may choose local coordinates $(q^U) = (q^i, q^\gamma)$ on an open subset $U$ of $Q$, $q \in U$, and a basis of sections $\{X_A\} = \{X_i, X_\gamma\}$ of the vector bundle $\tau_{D^{-1}}(U) \to U$ such that

\[
\rho_D(X_i)(q) = \frac{\partial}{\partial q^i}|_q, \quad \rho_D(X_\gamma)(q) = 0. \tag{A.1}
\]

Suppose that

\[
\rho_D(X_A) = \rho_A^U \frac{\partial}{\partial q^U}, \quad [X_A, X_B]_D = C^{C_C}_{AB} X_C \tag{A.2}
\]

and that the local expression of $\alpha$ in $U$ is

\[
\alpha(q^U) = (q^U, \alpha_A(q^U)). \tag{A.3}
\]
Denote by $\Lambda_{D^*}$ the linear almost Poisson 2-vector on $D^*$ and by $(q^I, p_A) = (q^i, q^a, p_i, p_a)$ the corresponding local coordinates on $D^*$. Then, from (2.1), it follows that
\begin{equation}
\Lambda_{D^*}(\alpha(q)) = \frac{\partial}{\partial q^i|_{\alpha(q)}} \wedge \frac{\partial}{\partial p_i|_{\alpha(q)}} - \frac{1}{2} C_{AB}^C(q) \frac{\partial}{\partial p_A|_{\alpha(q)}} \wedge \frac{\partial}{\partial p_B|_{\alpha(q)}}. \tag{A.4}
\end{equation}

Moreover, using (A.1), (A.2) and (A.3), we obtain that
\begin{align*}
(dD^A)(X_i(q), X_j(q)) &= \frac{\partial \alpha_j}{\partial q^i|_q} - \frac{\partial \alpha_i}{\partial q^j|_q} - C_{ij}^A(q) \alpha_A(q), \\
(dD^A)(X_i(q), X_\gamma(q)) &= \frac{\partial \alpha_\gamma}{\partial q^i|_q} - C_{i\gamma}^A(q) \alpha_A(q), \\
(dD^A)(X_\gamma(q), X_\nu(q)) &= -C_{\gamma\nu}^A(q) \alpha_A(q).
\end{align*}
\tag{A.5}

On the other hand, let $L_{\alpha,D}(q)$ be the subspace of $T_{\alpha(q)} D^*$ defined by (4.3). Then, from (A.1) and (A.3), we deduce that
\begin{equation}
L_{\alpha,D}(q) = \langle \{ \frac{\partial}{\partial q^i|_{\alpha(q)}} + \frac{\partial \alpha_A}{\partial q^i|_q} \frac{\partial}{\partial p_A|_{\alpha(q)}} \} \rangle
\end{equation}
\tag{A.6}
which implies that
\begin{equation}
(L_{\alpha,D}(q))^0 = \{ dq^a(\alpha(q)), dp_j(\alpha(q)) - \frac{\partial \alpha_j}{\partial q^i|_q} dq^i(\alpha(q)), dp_\gamma(\alpha(q)) - \frac{\partial \alpha_\gamma}{\partial q^i|_q} dq^i(\alpha(q)) \}. \tag{A.7}
\end{equation}

In addition, using (A.4), one may prove that
\begin{align*}
\#_{\Lambda_{D^*}}(dq^a(\alpha(q))) &= 0, \\
\#_{\Lambda_{D^*}}(dp_j(\alpha(q)) - \frac{\partial \alpha_j}{\partial q^i|_q} dq^i(\alpha(q))) &= -\frac{\partial}{\partial q^j|_{\alpha(q)}} - \left( \frac{\partial \alpha_j}{\partial q^i|_q} - C_{ij}^A(q) \alpha_A(q) \right) \frac{\partial}{\partial p_i|_{\alpha(q)}} - C_{ij}^A(q) \alpha_A(q) \frac{\partial}{\partial p_j|_{\alpha(q)}}, \\
\#_{\Lambda_{D^*}}(dp_\gamma(\alpha(q)) - \frac{\partial \alpha_\gamma}{\partial q^i|_q} dq^i(\alpha(q))) &= -\left( \frac{\partial \alpha_\gamma}{\partial q^i|_q} - C_{i\gamma}^A(q) \alpha_A(q) \right) \frac{\partial}{\partial p_i|_{\alpha(q)}} - C_{i\gamma}^A(q) \alpha_A(q) \frac{\partial}{\partial p_\gamma|_{\alpha(q)}}. \tag{A.8}
\end{align*}

**Proof of Proposition 4.3.** From (A.5), (A.6), (A.7) and (A.8), we deduce the result. \(\square\)

**Proof of Proposition 4.5.** Suppose that
\begin{equation}
\beta_{\alpha(q)} = \lambda_{Ud} dq^U(\alpha(q)) + \mu^A dp_A(\alpha(q)) \in T^*_{\alpha(q)} D^*. \tag{A.9}
\end{equation}
Then, using (A.4) and (A.5), it follows that
\begin{equation}
\beta_{\alpha(q)} \in Ker \#_{\Lambda_{D^*}}(\alpha(q)) \iff \mu^i = 0, \quad \lambda_i = -\frac{\partial \alpha_\gamma}{\partial q^i|_q} \mu^\gamma, \quad \text{for all } i.
\end{equation}

Thus, from (A.7), we conclude that
\begin{equation}
\beta_{\alpha(q)} \in (L_{\alpha,D}(q))^0. \tag{A.10}
\end{equation}
\(\square\)
REFERENCES

[1] Abraham R, Marsden JE, Foundations of Mechanics, Second Edition, Benjamin, New York, 1978.
[2] Agrachev AA, Sachkov Y, Control theory from the Geometric viewpoint, volume 87 of Encyclopedia of Mathematical Sciences, Springer-Verlag, New York-Heidelberg-Berlin, 2004.
[3] Bloch AM, Krishnaprasad PS, Marsden JE, Murray RM, Nonholonomic Mechanical Systems with Symmetry, Arch. Rational Mech. Anal. 136 (1996), 21-99.
[4] Bullo F, Żebrowski M, On mechanical control systems with nonholonomic constraints and symmetries, Systems & Control Letters 45 (2) (2002), 133-143.
[5] Cantilini F, de León M, Martín de Diego D, On almost-Poisson structures in nonholonomic mechanics, Nonlinearity 12 (1999), 721–737.
[6] Carrión E, Gracia X, Marín-G, Martín E, Muñoz-M, Román-R, Geometric Hamilton-Jacobi theory, Int. J. Geom. Meth. Mod. Phys. 3 (7) (2006), 1417–1458.
[7] Cortés J, de León M, Marrero JC, Martín E, Nonholonomic Lagrangian systems on Lie algebroids, Discrete and Continuous Dynamical Systems: Series A, 24 (2) (2009), 213-271.
[8] Cortés J, de León M, Martín de Diego D, Martín S, Geometric Description of Vakonomic and Nonholonomic Dynamics. Comparison of solutions, SIAM Journal on Control and Optimization 41 (5) (2003), 1389-1412.
[9] Cortés J and Martínez E, Mechanical control systems on Lie algebroids, IMA J. Math. Control Inform. 21 (2004), 457–492.
[10] Edén RJ, The Hamiltonian dynamics of non-holonomic systems, Proc. Roy. Soc. London. Ser. A. 205 (1951), 564–583.
[11] Godbillon Ch., Géométrie Différentielle et Mécanique Analytique, Hermann, Paris, 1969.
[12] Grabowska K, Grabowski J, Variational calculus with constraints on general algebroids, J. Phys. A: Math. Theor. 41 (2008), 175204 (25 pp).
[13] Grabowska K, Urbański P, Grabowski J, Geometrical mechanics on algebroids, Int. J. Geom. Methods Mod. Phys. 3 (3) (2006), 559–575.
[14] Grabowski J, de León M, Marrero JC, Martín de Diego D, Nonholonomic constraints: a new viewpoint, J. Math. Phys. 50 (1) (2009), 013520.
[15] Grabowski J, Urbański P, Lie algebroids and Poisson-Nijenhuis structures, Rep. Math. Phys. 40 (1997), 195–208.
[16] Grabowski J, Urbański P, Algebroids – general differential calculi on vector bundles, J. Geom. Phys. 31 (1999), 111–141.
[17] Greub W, Halperin S, Vanstone R, Connections, Curvature and Cohomology. Vol I, Academic Press, New York, 1972.
[18] Hairer E, Lübbert C and Wanner G, Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations, Springer Series in Computational Mathematics, 31 (2002), Springer-Verlag Berlin.
[19] Ibort A, de León M, Marrero JC, Martín de Diego D, Dirac brackets in constrained dynamics, Fortschr. Phys. 47 (1999), 459–492.
[20] Iglesias D, de León M, Martín de Diego D, Towards a Hamilton-Jacobi Theory for Nonholonomic Mechanical Systems, J. Phys. A: Math. Theor. 41 (2008) 015205.
[21] Koon WS, Marsden JE, Optimal Control for Holonomic and Nonholonomic Mechanical Systems with Symmetry and Lagrangian Reduction, SIAM Journal on Control and Optimization 35 (1997) 901–929.
[22] Koon WS, Marsden JE, Poisson reduction of nonholonomic mechanical systems with symmetry, Rep. Math. Phys. 42 (1/2) (1998), 101–134.
[23] Kosmann-Schwarzbach Y, Magri F, Poisson-Nijenhuis structures, Ann. Inst. H. Poincaré Phys. Théor. 53 (1990), 35–81.
[24] de León M, Marrero JC, Martín de Diego D, Vakonomic mechanics versus non-holonomic mechanics: A unified geometrical approach, J. Geom. Phys. 35 (2000) 126–144.
[25] de León M, Marrero JC, Martín de Diego D, Lagrangian submanifolds and dynamics on Lie algebroids, J. Phys. A: Math. Gen. 38 (2005), R241–R308.
[26] Lewis AD, Murray RM, Variational principles for constrained systems: theory and experiment, International Journal of Nonlinear Mechanics 30 (6) (1995), 793–815.
[27] Mackenzie K, General Theory of Lie Groupoids and Lie Algebroids, London Mathematical Society Lecture Note Series: 213, Cambridge University Press, 2005.
[28] Marrero JC, Martín de Diego D, Martín E, Discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids, Nonlinearity 19 (2006), no. 6, 1313–1348. Corrigendum: Nonlinearity 19 (2006), no. 12, 3003–3004.
[29] Marsden JE, Montgomery R, Ratiu T, Reduction, symmetry, and phases in mechanics, Mem. Amer. Math. Soc. 88 (1990), no. 436.
[30] Neimark J, Fufaev N, Dynamics of Nonholonomic Systems, Translations of Mathematical Monographs Vol. 33 Providence: Am. Math. Soc., 1972.
[31] Ohsawa T, Bloch AM, Nonholonomic Hamilton-Jacobi Equation and Integrability, Preprint arXiv:0906.3357.
[32] Ohtsuka JP, Ratiu TS, Momentum maps and Hamiltonian reduction, Progress in Mathematics, 222, Birkhäuser Boston, Inc., Boston, MA, 2004.
[33] Pavon M, Hamilton-Jacobi equations for nonholonomic dynamics, J. Math. Phys. 42 (2005), 032902.
[34] Popescu P, Almost Lie structures, derivations and R-curvature on relative tangent spaces, Rev. Roum. Mat. Pures Appl. 37 (8) (1992), 779–789.
[35] Popescu M, Popescu P, Geometric objects defined by almost Lie structures, Proc. Workshop on Lie algebroids and related topics in Differential Geometry (Warsaw) 54 (Warsaw: Banach Center Publications) (2001), 217–233.
[36] Rumyantsev VV, Forms of Hamilton’s Principle for Nonholonomic Systems, Mechanics, Automatic Control and Robotics 2 (10) (2002), 1035–1048.
[37] Sumbatov AS, Nonholonomic Systems, Regular and Chaotic Dynamics 7 (2) (2002), 221–238.
[38] Sussmann HJ, Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. 180 (1973), 171–188.
[39] Van der Schaft AJ, Maschke BM, On the Hamiltonian formulation of non-holonomic mechanical systems, Rep. Math. Phys. 34 (1994) 225–233.
[40] Van Dooren R, The generalized Hamilton-Jacobi method for non-holonomic dynamical systems of Chetaev’s type, Zeitschrift Fur Angewandte Mathematik Und Mechanik 55 (1975) 407–411.
[41] Van Dooren R, Motion of a rolling disc by a new generalized Hamilton-Jacobi method, Journal of Applied Mathematics and Physics 27 (1976) 501–505.
[42] Van Dooren R, Second form of the generalized Hamilton–Jacobi method for nonholonomic dynamical systems, Journal of Applied Mathematics and Physics 29 (1978) 828–834.
[43] Van Dooren R, On the generalized Hamilton-Jacobi method for nonholonomic dynamical systems, Dienst Analytische Mechanica, Tw, Vub (1979) 1–6.
[44] Vershik AM, Gershkovich VYa, Nonholonomic Dynamical Systems, Geometry of Distributions and Variational Problems, in V.I. Arnold, S.P. Novikov (eds.), Dynamical Systems VII, Encyclopaedia of Mathematical Sciences, 16 Springer-Verlag, Berlin Heidelberg 1994.
[45] Xu P, Gerstenhaber algebras and BV-algebras in Poisson geometry, Comm. Math. Phys. 200 (1999), 545–560.