The Nature of the Vector and Scalar Potentials and Gauge Invariance in the Context of Gauge Theory

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Abstract

Modern undergraduate textbooks in electricity and magnetism typically focus on a force representation of electrodynamics with an emphasis on Maxwell’s Equations and the Lorentz Force Law. The vector potential $\mathbf{A}$ and scalar potential $\Phi$ play a secondary role mainly as quantities used to calculate the electric and magnetic fields. However, quantum mechanics including quantum electrodynamics (QED) and other gauge theories demands a potential $(\Phi, \mathbf{A})$ oriented representation where the potentials are the more fundamental quantities. Here, we help bridge that gap by showing that the homogeneous Maxwell’s equations together with the Lorentz Force Law can be derived from assuming that the potentials represent potential energy and momentum per unit charge. Furthermore, we enumerate the additional assumptions that are needed to derive the inhomogeneous Maxwell’s equations. As part of this work we demonstrate the physical nature and importance of gauge invariance.
I. INTRODUCTION

The vector and scalar potentials have an interesting history. James Clerk Maxwell originally formulated his equations using the vector potential \( \mathbf{A} \) along with the electric \( \mathbf{E} \) and magnetic \( \mathbf{B} \) fields. In his first great paper published in 1856, Maxwell showed that Michael Faraday’s experimental work in electrodynamics could be expressed as \( \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{A} \) where \( \mathbf{E} \) is the induced electric field. Since \( \mathbf{E} \) is defined as the force per unit charge Maxwell deduced that \( \mathbf{A} \) represents a potential momentum per unit charge in the same way that the scalar potential \( \Phi \) represents a potential energy per unit charge.

In 1885, Oliver Heaviside eliminated the vector potential from Maxwell’s equations in favor of using \( \mathbf{E} \) and \( \mathbf{B} \) only. Heaviside viewed the \( \mathbf{E} \) and \( \mathbf{B} \) fields as the true physical quantities with the potentials being merely useful functions. This view was likely influenced by the gauge invariance of the potentials. Adding the gradient of an arbitrary function to \( \mathbf{A} \) left the fields the same provided you subtracted the time derivative of the same function from \( \Phi \).

Forty years later the advent of quantum mechanics began to reassert \( \mathbf{A} \) and \( \Phi \) as important quantities in their own right. In quantum mechanics, energy and canonical momentum are fundamental quantities. Force and consequently force fields have a much more limited role. Further, gauge invariance was shown to be equivalent to the phase invariance inherent in quantum mechanics\(^1\,^2\,^3\). Later still, the Aharonov Bohm effect proved that the potentials have noticeable effects in the absence of electric and magnetic fields\(^4\,^5\) and therefore established the potentials as physical quantities in their own right.

The true importance of the vector and scalar potentials became most apparent with the development of quantum electrodynamics (QED) and subsequent gauge theories. If we ask how the electromagnetic force gets from particle A to particle B then Maxwell had a simple answer. Force is carried by electric and magnetic fields through an ether similar to how transverse sound waves move through a solid. The physical nature of \( \mathbf{E} \) and \( \mathbf{B} \) makes sense in this context as does the Maxwell stress tensor. Unfortunately, ether does not exist. In QED, the force is transferred not through an elastic medium but by particles, photons. But, particles cannot carry force; particles carry (or have) energy and momentum. The \( \mathbf{A} \) and the \( \Phi \) fields represent how this momentum and energy is carried. The force type fields \( \mathbf{E} \) and \( \mathbf{B} \) are the derived quantities. (See E. J. Konopinski\(^6\) p. 502 or R. Feynman\(^11\).)
Yet almost 60 years later, the nature of the potentials is often downplayed in favor of the electric and magnetic fields. Recent work\textsuperscript{6,7,8,9} has begun to revive the potentials as representing the potential energy and momentum per unit charge. Some of this work has been incorporated into the undergraduate electrodynamics curriculum\textsuperscript{10}, but more needs to be done.\textsuperscript{11} Here we will extend that work by showing that the laws of electrodynamics result naturally from the physical meaning of the potentials in the context of the assumptions behind QED.

II. THE LORENTZ FORCE LAW

We start by deriving the Lorentz force on a particle with charge $q$ and moving with a speed $v$. We assume that the net force on the particle due to all the other charges in the universe can be described entirely in terms of a potential energy per unit charge $\Phi$ and potential momentum per unit charge $A$. This assumption is known as minimal coupling. Minimal coupling is a reasonable assumption based on the fact that electromagnetic force is mediated by photons—which as particles carry energy and momentum. With minimal coupling, the total energy $H$ and momentum $P$ of our charged particle moving at a speed $v$ are:

\begin{equation}
H = E + q\Phi, \\
P = p + qA,
\end{equation}

where $E$ and $p$ represent the relativistic energy and momentum of the charged particle.

In special relativity the quantity

\begin{equation}
\left(\frac{E}{c}\right)^2 - p \cdot p = (mc)^2
\end{equation}

is invariant for all reference frames, where $m$ is the rest mass of the particle and $c$ is the speed of light in a vacuum. Inserting Eq. (1) into Eq. (2) and solving for $H$ in terms of $P$ we obtain the Hamiltonian for the system:\textsuperscript{12}

\begin{equation}
H = c\sqrt{(P_q)^2 + (mc)^2} + q\Phi.
\end{equation}

The force $F$ on the particle is equal to the time derivative of its momentum $F = \dot{p} = \dot{P} - q\dot{A}$. We can therefore use Hamilton’s equations of motion ($\dot{x} = \frac{\partial H}{\partial P_x}$ and $\dot{P}_x = -\frac{\partial H}{\partial x}$) to determine
F. Assuming \( A \) and \( \Phi \) have no explicit dependence on \( P \) and using \( \dot{x} = v_x = \frac{\partial H}{\partial P_x} = -\frac{1}{q} \frac{\partial H}{\partial A_x} \)

we obtain that

\[
F_x = (\dot{p}_q)_x = -q \dot{A}_x - \frac{\partial}{\partial x} (q \Phi - q v \cdot A). \tag{4}
\]

Here we have also used \( v \cdot \frac{\partial A}{\partial x} = \frac{\partial}{\partial x} (v \cdot A) \) resulting from \( \frac{\partial v}{\partial x} = 0 \). Expanding Eq. (4) in terms of partial derivatives and rearranging gives

\[
F_x = q \left( -\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} \right) + qv_y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - qv_z \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right). \tag{5}
\]

This is clearly the x-component of the Lorentz force law with the quantities in parenthesis being \( E_x, B_z, \) and \( B_y \), respectively.

Equation (5) can be expressed somewhat simpler in relativistic 4-vector or tensor notation. Four-vectors have one temporal and three spatial components and are extremely useful for how they transform under a Lorentz transformation. For example the position 4-vector \((ct, x, y, z) = (ct, x)\) will transform to \((ct', x')\), yet the scalar product \((ct)^2 - x \cdot x = (ct')^2 - x' \cdot x'\)

remains the same for any Lorentz transformation to any inertial coordinate system. This invariant and its relationship to the Lorentz transformation is similar to the dot product and its relationship to rotation.

The negative sign in the scalar product is dealt with by introducing two types of vectors that are related by flipping the sign of the spatial components. The components of a contravariant 4-vector are represented by superscripts, e.g. \( x^\mu \) where \((x^0, x^1, x^2, x^3) = (ct, x, y, z)\). The components of a covariant 4-vector are represented by subscripts, e.g. \( x_\mu \) represents \((ct, -x, -y, -z)\). The scalar product then is the product of one covariant and one contravariant vector (the order is immaterial) and is represented as \( x^\mu y_\mu \). (Here, and for the rest of the paper, we use the summation notation where two repeated indexes in a product—one covariant and one contravariant—implies a sum over the indices.) The usefulness of the scalar product is that if \( x^\mu \) and \( y^\mu \) are 4-vectors (in other words \( x^\mu x_\mu \) and \( y^\mu y_\mu \) are invariant) then the scalar product \( x^\mu y_\mu \) is also invariant.

Relativistic equations are expressed in their simplest form in terms of 4-vectors (and their scalar and tensor counterparts). Important examples of contravariant 4-vectors include proper velocity \( \eta^\mu = \gamma(c, v) \) where \( \gamma \) is the relativity factor \( \gamma = (1 - (v/c)^2)^{-1/2} \), 4-momentum \( p^\mu = m\eta^\mu \) where \( m \) is the rest mass, the 4-potential \( A^\mu = (\Phi/c, A) \), and the 4-current density \( J^\mu = (c\rho, J) \), where \( \rho \) and \( J \) are the charge and current densities. Covariant and contravariant 4-vector derivatives have the negative sign reversed for the spatial
part. The covariant derivative \( \partial_\mu \) is \( \left( \frac{1}{c} \frac{\partial}{\partial t}, \nabla \right) \) and the contravariant derivative \( \partial^\mu \) is \( \left( \frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) \) where \( \nabla \) is the gradient in Cartesian coordinates.

The Lorentz force equation (5) becomes in the tensor notation (after multiplying by \( \gamma \)):

\[
K^\mu = q \eta_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu),
\]

(6)

where \( K^\mu \) is the Minkowski force whose spatial part \( \mathbf{K} \) is \( \gamma \mathbf{F} \). (The temporal portion of the Minkowski force is \( \gamma \frac{dW}{dt} \), where \( \frac{dW}{dt} \) is the applied power.) This can be simplified further by defining the anti-symmetric field 4-tensor \( F^{\mu \nu} \equiv (\partial^\mu A^\nu - \partial^\nu A^\mu) \). Evaluating \( F^{\mu \nu} \) we see that its 6 unique values are the x, y, and z components of \( \pm E/c \) and \( \pm B \).

### III. GAUGE INVARIANCE

Equation (6) introduces a difficulty with \( A^\mu \). If we let \( A^\mu \to \tilde{A}^\mu = A^\mu + \partial^\mu f \) where \( f \) is any arbitrary differentiable function it will lead to the exact same field tensor \( F^{\mu \nu} \) as the original \( A^\mu \). This property is known as gauge invariance. And, at first glance, it seems to be a major problem in interpreting \( A^\mu \) as a physical quantity since \( A^\mu \) is arbitrary to whole classes of functions!

It is important to note that this affects not just electrodynamics but all representations of energy and momentum because the derivation of Eq. (6) is more general than just electrodynamics. If we absorb \( q \) into \( A^\mu \) then Eq. (6) applies to all systems that have minimal coupling. This includes conservative fields as a special case where the spatial components of \( A^\mu \) are zero.

It is well understood—though not well known outside gauge theory—that gauge invariance is a general property of classical mechanics.\(^{13}\) Consider the well-known transformation of a Lagrangian \( L \to L' = L - \frac{df}{dt} \), where \( f \) is any function of \( x \) and \( t \). This transformed Lagrangian \( L' \) produces the same equations of motion as the original \( L \). (Recall that the Euler-Lagrange equations come from finding the path that minimizes the action \( S = \int L dt \). Therefore, by the fundamental law of calculus, the difference between the \( S \) and \( S' \) is a constant and cannot affect the path of least action.)

But, transforming \( L \) changes the momentum \( \mathbf{p} \) and energy \( H \) of the system. Using the appropriate chain rule for the full derivative, \( \frac{df}{dt} = \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} \), where the sum is over the three spatial components (x,y,z), we calculate \( \mathbf{p}' \) and \( H' \) in the usual way:
\[ p'_i = \frac{\partial L'}{\partial \dot{x}_i} = p_i - \frac{\partial f}{\partial x_i} \]

\[ H' = \sum p'_i \dot{x}_i - L' = H + \sum \left( -\frac{\partial f}{\partial x_i} \right) \dot{x}_i - \left( -\frac{df}{dt} \right) = H + \frac{\partial f}{\partial t}, \] (7)

which are the equations for gauge invariance. In other words the new \((H', p')\) have the same equations of motion as the original \((H, p)\) for gauge transformations.

The physical meaning of gauge invariance can be made clearer with an example reminiscent of the equivalence principle. Imagine a rocket which when observed from an inertial reference frame has a constant acceleration \(a\). An observer inside the rocket will clearly feel as if he is continually pulled downward. Observing a ball falling from someplace near to the nose of the rocket, he would conclude that there is a potential energy \(V = may\) where \(m\) is the mass of the ball and \(y\) is its height. The observer in the inertial reference frame, though, would insist that there was no downward force at all and definitely no potential energy. She would see the floor accelerating into the ball, not the other way around. She would claim that the observer inside the rocket sees in his reference frame a ball that is gaining momentum \(= -mat\). Defining a potential momentum \(A\) in the same manner as potential energy is defined she could just as easily say that \(A = +mat\) to keep momentum conserved.

We could do the same in a different accelerating reference system in which case there would be another set of \((V, A)\) that are valid for a person on a rocket and produce the correct equations of motion in his reference system. In principle, the same should work for any reference system, even one that varies with both position and time. An observer in this frame will note that the person on the rocket sees a different set of \((V, A)\) that are now space and time dependent.

Again, it is important to note, that this it not a change of reference system. All of these sets of \((V, A)\) are valid for the rocket’s reference frame. All of these observers agree that the observer on the rocket sees a ball that has a Hamiltonian of the form of Eq. (3) and has a force law of the form of Eq. (6). What they don’t agree on is how the rocket observer should interpret \(A^\mu\). Gauge invariance reflects the fact that the motion of an object (for a set reference frame) should not depend on how you observe it. Gauge invariance is a necessary part of all of physics and is one of the cornerstones of all advanced theories of motion including classical mechanics, quantum mechanics, gauge theory, and general relativity.
IV. MAXWELL’S EQUATIONS

With the derivation of the Lorentz force complete, we turn our attention to Maxwell’s equations. There are many excellent derivations of Maxwell’s equations starting from a variety of different assumptions. See references \(^\text{15, 16, 17, 18, 19}\) for a small sampling. Frisch and Wilets\(^\text{18}\) do a particularly good job of not only deriving the equations but listing the important ingredients that are necessary for its derivation.

Here we will attempt to do the same but from the perspective of the potentials. We will show that there are five relatively independent and necessary conditions underlying electrodynamics:

1. **Minimal Coupling**: \( p \rightarrow p - qA \),

2. **Gauge Invariance**: \( \partial_\mu A^\mu = 0 \),

3. **The 4-potential is carried by massless particles**: \( \partial_\nu \partial_\nu A^\mu = 0 \) in the absence of charge,

4. **The 4-potential is directly proportional to 4-current density that created it**: \( A^\mu \propto J^\mu \)

5. **Conservation of Charge**: \( \partial_\mu J^\mu = 0 \).

We have already used the first postulate to derive the Lorentz force law and have shown the necessity of gauge invariance (the second postulate). The third postulate is due to massless photons mediating the electromagnetic force. The fourth postulate is necessary to derive the inhomogeneous Maxwell’s equations and is supported by its simplicity and that it produces the correct field equations. The final postulate is local charge conservation.

These postulates conveniently separate the electromagnetic interaction into three separate processes. The fourth postulate represents how charged particles generates a 4-potential. The third postulate dictates how the 4-potential traverses from the source. (In this case it is carried by massless “non-interacting” photons.) The first postulate represents how a charged particle reacts to the 4-potential it receives. Finally, the second and the fifth postulates represent important additional restrictions.

These are the assumptions that are needed to derive all of electrodynamics. Furthermore, it is straightforward to vary these postulates to model other forces. For example, giving the photon a mass will alter the third postulate and lead to the Yukawa potential.
A. The Homogeneous Maxwell’s Equations

The four homogeneous Maxwell equations are due to the minimal coupling condition although they are hidden in the definition of $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$. When written out in terms of spatial and time derivatives $F^{\mu\nu}$ includes gradients as well as curls and time derivatives of the components of $A^\mu$ in a simple form. Therefore, we might expect there to be relationships in the derivatives of $F^{\mu\nu}$ similar to

\[ \nabla \cdot (\nabla \times A) = 0 = \frac{\partial}{\partial x_i} \left[ \epsilon_{ijk} \left( \frac{\partial}{\partial x_j} A_k \right) \right] \]

\[ \nabla \times (\nabla \Phi) = 0 = \frac{\partial}{\partial x_i} \left[ \epsilon_{ijk} \left( \frac{\partial}{\partial x_j} \Phi \right) \right]. \tag{8} \]

Here, the Levi-Civita symbol is defined by $\epsilon_{ijk} = 0$ if any two of $i,j,k$ are the same, $\epsilon_{ijk} = 1$ for all even permutations of 123, and $\epsilon_{ijk} = -1$ for all odd permutations of 123. It is straight-forward to show that there is a relationship for $F^{\mu\nu}$ similar to Eq. (8),

\[ \partial_\mu \frac{1}{2} \epsilon^{\mu\rho\sigma} F_{\rho\sigma} = 0, \tag{9} \]

that is valid as long as $A^\mu$ is differentiable. (This is essentially due to the fact that partial derivatives commute for differentiable functions.) Here, the doubly covariant $F_{\rho\sigma}$ is obtained by changing the signs of the components of $F^{\mu\nu}$ that have both a spatial and temporal part, for example $F^{01}$ or $F^{30}$ but not $F^{31}$. We also extend the Levi-Civita symbol to four dimensions in the expected way. Evaluating Eq. (9) leads to the four homogeneous Maxwell’s equations. This demonstrates that the homogeneous equations are valid for any system that acts on a particle with a Hamiltonian of the form of Eq. (3).

B. The Inhomogeneous Maxwell’s Equations

In order to simplify the more complicated derivation of the inhomogeneous Maxwell’s equations we split the derivation into two parts. First we derive the form of the equation for regions where there are neither charges nor currents, $J^\mu = 0$ using the third postulate. Then we add a term proportional to $J^\mu$ in accordance with postulate 4 to account for the sources creating the 4-potentials.

We start by examining the particles (photons) that are assumed to be mediating the electromagnetic force. The relation between the relativistic energy and momentum of a
A particle having mass $m$ is given by Eq. (2). For a free particle (such as a photon at a location at which there is no 4-current) the total momentum and energy $(H, P)$ equals the relativistic energy and momentum $(E, p)$, respectively. Substituting the appropriate quantum mechanical operators for $p$ and $H$ we see that a particle with mass $m$ must have a wave function $\psi$ that satisfies the differential equation

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2 \psi = 0,$$

or in covariant notation

$$\partial_\lambda \partial^\lambda A^\mu = 0,$$  

valid for regions where $J^\mu = 0$. Note that extending this model to give the photon a small mass is straightforward. This equation represents a wave equation for the $A^\mu$. It should also be noted that in choosing our operators for $H$ and $p$ we have chosen a particular gauge. We will have to enforce gauge invariance later.

We incorporate the source term $J^\mu = (c\rho, J)$ by noting the 4-vector nature of $J^\mu$ and that the fields are linear in $J^\mu$ by postulate 4. The simplest 4-vector equation (linear in $J^\mu$) that reduces to Eq. (10) for $J^\mu = 0$ is

$$\partial_\lambda \partial^\lambda \tilde{A}^\mu = \mu_0 J^\mu,$$  

where $\mu_0$ is a constant and we have marked $\tilde{A}^\mu$ with a tilde to remind us that this equation is for a particular gauge.

To proceed we need to determine the particular gauge of $\tilde{A}^\mu$ such that Eq. (11) leads to the conservation of charge (postulate 5),

$$\partial_\mu J^\mu = 0.$$  

The solution to Eq. (11) for the boundary condition that $A^\mu = 0$ at infinity is well known,

$$\tilde{A}^\mu = \frac{\mu_0}{c} \int \frac{J^\mu(ct', x', y', z') \delta(R - c(t - t'))}{R} d(ct') dx' dy' dz',$$  

where $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ and $\delta$ is the Dirac delta function and the integral is over all space. To determine the gauge of $\tilde{A}^\mu$ we need to determine the value of

$$\partial_\mu \tilde{A}^\mu = \frac{\mu_0}{c} \int J^\mu(ct', x', y', z') \frac{\delta(R - c(t - t'))}{R} d(ct') dx' dy' dz'.$$
Using the symmetry between the primed and unprimed coordinates $\partial'_\mu \left[ \frac{\delta(R-c(t-t'))}{R} \right] = -\partial_\mu \left[ \frac{\delta(R-c(t-t'))}{R} \right]$, we switch the derivative to the prime coordinates and then integrate by parts. Using the product rule for differentiation we find that $\partial'_\mu \left[ J^\mu \frac{\delta(R-c(t-t'))}{R} \right] = J^\mu \partial'_\mu \left[ \frac{\delta(R-c(t-t'))}{R} \right] + \partial'_\mu \left[ J^\mu \frac{\delta(R-c(t-t'))}{R} \right]$, where the second term is zero in the middle equation because of postulate 5, Eq. (12). Therefore,

$$\partial_\mu \tilde{A}^\mu = \frac{\mu_0}{c} \int \partial'_\mu \left[ J^\mu(c'\lambda; \mathbf{x}) \frac{\delta(R-c(t-t'))}{R} \right] d^4x'.$$

(15)

This volume integral evaluates as a surface integral in 4-space of the argument of the differential by an extension of the divergence theorem, where the surface is at plus or minus infinity in space and time. As long as $J^\mu$ is localized such that it goes to zero faster than $1/R$ in the limit that $R$ goes to infinity then the value of Eq. (15) = 0. Therefore the gauge of $\tilde{A}^\mu$ in Eq. (11) is the Lorentz gauge $\partial_\mu \tilde{A}^\mu = 0$.

Generalizing Eq. (11) to an arbitrary gauge is straight-forward since $A^\mu = \tilde{A}^\mu + \partial^\mu f$ where $A^\mu$ is the potential in an arbitrary gauge determined by $f$. Plugging $\tilde{A}^\mu = A^\mu - \partial^\mu f$ into Eq. (11) (using $\nu$ instead of $\lambda$) and using $\partial_\nu A^\nu = \partial_\nu \partial^\nu f$ gives

$$\partial_\nu \left[ \partial^\nu A^\mu - \partial^\mu A^\nu \right] = \mu_0 J^\mu = -\partial_\nu F^{\mu\nu}.$$  

(16)

Using the definition of the field tensor, it is easily verified that this leads to the four inhomogeneous Maxwell equations.$^{21}$

V. CONCLUSIONS

Eighty years after quantum mechanics has shown that the potentials are important fields of electrodynamics corresponding to the 4-momentum per unit charge transferred by the fields, the potentials still don’t get the respect they deserve in the undergraduate electricity and magnetism course. This is largely due to gauge invariance. Here, we have shown that the Lorentz force law and the homogeneous Maxwell’s equations of electrodynamics are a natural consequence of the 4-potential representing the potential energy and momentum per unit charge. Furthermore, we have derived the entire set of Maxwell’s equation from the 4-potential in a fully transparent way, explicitly showing all the necessary assumptions that are built into the equations. As part of this discussion we have demonstrated that the phenomenon of gauge invariance is a necessary and important property of energy and
momentum that affects all systems and therefore the scalar and vector potentials should be seen as being just as ‘physical’ as energy and momentum.

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1. A.C.T. Wu, “Evolution of the concept of the Vector Potential in the description of fundamental interactions”, Int. J. of Mod. Phys. A., 41, No. 16 3235-3277 (2006).
2. H. Weyl, Zeit. fur Physik 56, 330 (1929) [See A.C.T. Wu p. 3261 for relevant quote and O’Raifeartaigh for English translation]
3. L. O’Raifeartaigh, *The Dawning of Gauge Theory*, Princeton University Press (1997).
4. W. Ehrenberg and R. E. Siday, “The Refractive Index in Electron Optics and the Principles of Dynamics”, Proc. Phys. Soc. B 62, 8–21 (1949).
5. Y. Aharonov and D. Bohm, “Significance of electromagnetic potentials in quantum theory”, Phys. Rev. 115, 485-491 (1959).
6. E. J. Konopinski, “What the electromagnetic vector potential describes”, Am. J. Phys. 46 (5), 499–502 (1978).
7. Mark D. Semon and John R. Taylor, “Thoughts on the magnetic vector potential”, Am. J. Phys. 64 (11), 1361–1369 (1996).
8. Dragan V Redzic, “Faraday’s law via the magnetic vector potential”, Eur. J. Phys. 28, N7–N10 (2007).
9. Toshimi Adachi, Shigeru Sasabe, Toshio Inagaki, and Masao Ozaki, “The vector potential revisited”, Electrical Engineering in Japan, 113 (6), 11–16 (1992).
10. David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. Prentice Hall (1999). [Compare pp. 235–238 to Semon & Taylor.]
11. Richard P. Feynman, Robert B. Leighton, and Matthew Sands *The Feynman Lectures on Physics* Volume II, Addison-Wessley Publishing Company (1964), p. 15–8 and 15–14. [In particular ‘... : E and B are slowly disappearing from the modern expression of physical laws; they are being replaced by A and φ’ (p. 15–14).]
12. John D. Jackson, *Classical Electrodynamics*, 2nd ed. John Wiley & Sons (1975), section 12.1. [Most textbooks like Jackson will derive the Hamiltonian from the Lorentz Force law. I contend that the Hamiltonian is more fundamental since it is valid for quantum mechanics. The results
are the same, though. The Lorentz force implies the existence of \( A \) and \( \Phi \) and the existence of the \( A \) and \( \Phi \) will lead to the Lorentz force.]

13 Emil J. Konopinski, *Classical Description of Motion*, W. H. Freeman and Company (1969), pp. 173–175.

14 Herbert Goldstein, *Classical Mechanics*, 2nd ed., Addison-Wesley Publishing Company (1980), pp. 349–350. [Here Goldstein discusses a simpler example of a cart with a mass attached to a spring. He derives two different sets of \((H, p)\) that are related by the gauge transformation using \( f = mv_o x \).]

15 O. D. Jefimenko, “Causal equations for electric and magnetic fields and Maxwell’s equations: comment on paper by Heras”, Am. J. Phys. 76 (2), pp. 101–102 (2008).

16 O. D. Jefimenko, “Presenting electromagnetic theory in accordance with the principle of causality” Eur. J. Phys. 25, pp. 287–296 (2004)

17 J. A. Heras, “Can Maxwell’s equations be obtained from the continuity equation?”, Am. J. Phys. 75, pp. 652–657 (2007).

18 David H. Frisch and Lawrence Wilets, “Development of the Maxwell-Lorentz Equations from Special Relativity and Gauss’s Law”, Am. J. Phys. 24, pp. 574–579 (1956)

19 John D. Jackson, *Classical Electrodynamics*, 2nd ed. John Wiley & Sons (1975), section 12.2.

20 John D. Jackson, *Classical Electrodynamics*, 2nd ed. John Wiley & Sons (1975), section 12.11.

21 David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. Prentice Hall (1999), p. 539.