ON THE BRAUER-PICARD GROUP OF A FINITE SYMMETRIC TENSOR CATEGORY

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Abstract. Let $C_n$ denote the representation category of finite supergroup $\wedge k^n \rtimes \mathbb{Z}/2\mathbb{Z}$. We compute the Brauer-Picard group $\text{BrPic}(C_n)$ of $C_n$. This is done by identifying $\text{BrPic}(C_n)$ with the group of braided tensor autoequivalences of the Drinfeld center of $C_n$ and studying the action of the latter group on the categorical Lagrangian Grassmannian of $C_n$. We show that this action corresponds to the action of a projective symplectic group on a classical Lagrangian Grassmannian.

1. Introduction

Let $\mathcal{C}$ be a finite tensor category. The notion of tensor product of $\mathcal{C}$-bimodule categories [ENO] generalizes the notion of tensor product of bimodules over a ring. Bimodule categories invertible with respect to the tensor product are of particular interest. Equivalence classes of such categories form a group, called the Brauer-Picard group of $\mathcal{C}$ and denoted $\text{BrPic}(\mathcal{C})$. This group plays a crucial role in construction and classification of group-graded extensions of $\mathcal{C}$ [ENO]. When $\mathcal{C}$ is the category of finite dimensional representations of a finite dimensional Hopf algebra $H$ the group $\text{BrPic}(\mathcal{C})$ is known as the strong Brauer group of $H$, or the Brauer group of the Drinfeld double of $H$, see [COZ]. Brauer groups of Hopf algebras were extensively studied in the literature, see, e.g., [COZ, C, CC, CZ] among many other works.

Let $Z(\mathcal{C})$ denote the Drinfeld center of $\mathcal{C}$. There is a canonical isomorphism

$$\text{BrPic}(\mathcal{C}) \cong \text{Aut}^{\text{br}}(Z(\mathcal{C})), \quad (1)$$

where $\text{Aut}^{\text{br}}(Z(\mathcal{C}))$ is the group of braided tensor autoequivalences of $Z(\mathcal{C})$. This fact was first established in [ENO] for fusion categories and was later extended to finite tensor categories in [DN].

In practice it is much easier to work with the group $\text{Aut}^{\text{br}}(Z(\mathcal{C}))$ than with $\text{BrPic}(\mathcal{C})$, since the multiplication of the latter is defined by an abstract universal property while for the former it is simply the composition of functors. In addition, $\text{Aut}^{\text{br}}(Z(\mathcal{C}))$ can be viewed as a generalization of the classical orthogonal group which brings important geometric insights. In particular, one can study actions of $\text{Aut}^{\text{br}}(Z(\mathcal{C}))$ on categorical analogues of Grassmannians. This approach was used in [NR] to compute the Brauer-Picard groups of pointed fusion categories.

In this paper we use the above approach in non-semisimple setting to compute the Brauer-Picard group of the representation category $C_n$ of finite supergroup $E(n) := \wedge k^n \rtimes \mathbb{Z}/2\mathbb{Z}$ (note that $C_n$ is the most general example of a symmetric finite tensor category without non-trivial Tannakian subcategories [De]). Namely,
we show that $\text{BrPic}(\mathcal{C}_n) \cong \text{PSp}_2(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, where $\text{PSp}_2(\mathbb{Z})$ denotes the projective symplectic group (Theorem 6.11 and Corollary 6.12).

Let us explain details of our argument. We study a projective representation $\rho$ of $\text{Aut}^{br}(\mathbb{Z}(\mathcal{C}_n))$ on the $2n$-dimensional symplectic space $\text{Ext}^1_{\mathcal{Z}(\mathcal{C}_n)}(\chi, \varepsilon)$, where $\varepsilon$ and $\chi$ are the two one-dimensional representations of $\mathbb{Z}(\mathcal{C}_n)$ (this $\text{Ext}$ space is identified with a space of skew primitive elements in the Drinfeld double of $E(n)$, see Remark 2.11). The Lagrangian Grassmannian of this symplectic space turns out to be equivariantly isomorphic to the categorical Lagrangian Grassmannian $L_0(\mathcal{C}_n)$, i.e., to the set of symmetric tensor subcategories of $\mathcal{Z}(\mathcal{C}_n)$ equivalent to $\mathcal{C}_n$. The stabilizer of a point of $L_0(\mathcal{C}_n)$ is known (see Proposition 3.16 and [NR, Proposition 6.8]). Combining this analysis with the results of Carnovale and Cuadra [CC] about the Picard group of $\mathcal{C}_n$ we find the kernel of $\rho$ and show that its image consists of classes of symplectic matrices. This allows to compute $\text{Aut}^{br}(\mathbb{Z}(\mathcal{C}_n))$.

We note that the infinitesimal version of our result (which says that the Lie algebra of outer twisted derivations of the double of $E(n)$ is isomorphic to the symplectic Lie algebra $\text{sp}(2n)$) was obtained by Cuadra and Davydov in [CDa].

The paper is organized as follows.

Section 2 contains preliminary material about Hopf algebras, tensor categories, and their Drinfeld centers. Here we recall definition of the Brauer-Picard group of a tensor category We also describe the projective action of the group of tensor autoequivalences of a tensor category on certain $\text{Ext}$-spaces.

In Section 3 we recall isomorphism (1) from [ENO, DN] and discuss how braided autoequivalences of the Drinfeld center of $\text{Rep}(H)$, where $H$ is a finite dimensional Hopf algebra, can be induced from invariant twists of $H$ and from invariant 2-cocycles on $H$. We also consider the action of the group of braided autoequivalences of the center on the categorical Lagrangian Grassmannian.

In Section 4 we recall the structure of Hopf algebra $E(n) = \wedge^n k^n \times \mathbb{Z}/2\mathbb{Z}$ and explicit description of its invariant twists and 2-cocycles following Bichon and Carnovale [BC].

In Section 5 we identify the categorical Lagrangian Grassmannian of $\mathcal{C}_n$ with the classical Lagrangian Grassmannian of a symplectic form.

In Section 6 we compute $\text{Aut}^{br}(\mathbb{Z}(\mathcal{C}_n))$ using its projective representation on the $\text{Ext}$-space.

Acknowledgments. We are grateful to Juan Cuadra, Alexei Davydov, Pavel Etingof, and Bojana Femic for helpful discussions and valuable comments. The work of the second named author was partially supported by the NSA grant H98230-13-1-0236.

2. Preliminaries

2.1. General conventions. We work over an algebraically closed field $k$ of characteristic 0. Recall that a $k$-linear abelian category $\mathcal{A}$ is finite [EO] if

(i) $\mathcal{A}$ has finite dimensional spaces of morphisms;
(ii) every object of $\mathcal{A}$ has finite length;
(iii) $\mathcal{A}$ has enough projectives, i.e., every simple object of $\mathcal{A}$ has a projective cover; and
(iv) there are finitely many isomorphism classes of simple objects in $\mathcal{A}$.

All categories considered in this paper will be finite $k$-linear abelian. Any such category is equivalent to the category $\text{Rep}(\mathcal{A})$ of finite dimensional representations
of a finite dimensional $k$-algebra $A$. All functors between such categories will be additive and $k$-linear.

In this paper we freely use basic results of the theory of finite tensor categories and module categories over them [BK, EO], the theory of braided categories [JS, DGNO], and the theory of Hopf algebras [M].

For a Hopf algebra $H$ we denote by $\Delta$, $S$, $\varepsilon$, the comultiplication, antipode, and counit of $H$, respectively. We use Sweedler’s notation for comultiplication, writing $\Delta(x) = x_{(1)} \otimes x_{(2)}$, $x \in H$. We will write $\Delta^{op}$ for the opposite comultiplication, i.e., $\Delta^{op}(x) = x_{(2)} \otimes x_{(1)}$. All Hopf algebras considered in this paper are assumed to be finite dimensional.

For a positive integer $n$ we denote by $\text{Sym}_n(k)$ the additive group of symmetric bilinear forms on $k^n$. When it is convenient, we will identify $\text{Sym}_n(k)$ with the group of symmetric $n$-by-$n$ matrices. We will denote by $I_n$ the identity $n$-by-$n$ matrix.

2.2. Tensor categories and Hopf algebras. By a tensor category we mean a finite rigid tensor category whose unit object 1 is simple.

Example 2.1. For a finite dimensional Hopf algebra $H$ the category $\text{Rep}(H)$ of finite dimensional representations of $H$ is a finite tensor category. In general, a tensor category $\mathcal{A}$ is equivalent to the representation category of some Hopf algebra if and only if there exists a fiber functor (i.e., an exact faithful tensor functor) $F : \mathcal{A} \to \text{Vec}$, where $\text{Vec}$ is the tensor category of $k$-vector spaces.

Let $\mathcal{A}$ be a tensor category with the associativity constraint

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z).$$

By a tensor subcategory $\mathcal{A}$ of a tensor category $\mathcal{B}$ we mean the image of a fully faithful tensor functor (i.e., embedding) $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$. When no confusion is possible, we will simply write $\mathcal{A} \subset \mathcal{B}$ to denote a tensor subcategory.

Proposition 2.2. Let $H$ be a finite-dimensional Hopf algebra. The set of tensor subcategories of $\text{Rep}(H)$ is in bijection with the set of classes of surjective Hopf algebra homomorphisms $p : H \to K$ under the following equivalence relation: two surjective homomorphisms $p : H \to K$ and $p' : H \to K'$ are equivalent if there is a Hopf algebra isomorphism $f : K \cong K'$ such that $f \circ p = p'$.

Proof. Any surjective homomorphism $p : H \to K$ induces an embedding

$$\iota(p) : \text{Rep}(K) \to \text{Rep}(H) : V \mapsto V_H,$$

where $V_H = V$ and $hv := p(h)v$ for all $h \in H$ and $v \in V$. The image of $\iota(p)$ consists of isomorphism classes of representations of $H$ that factor through $p$ and so it does not change when $p$ is composed with an isomorphism.

Conversely, let $\iota : \mathcal{A} \to \text{Rep}(H)$ be a tensor embedding. Let $F : \text{Rep}(H) \to \text{Vec}$ be the canonical forgetful tensor functor. By Tannakian formalism, $H \cong \text{End}(F)$ and $K = \text{End}(F \circ \iota)$ is a Hopf algebra such that $\mathcal{A}$ is canonically equivalent to $\text{Rep}(K)$. The natural map $p : H = \text{End}(F) \to \text{End}(F \circ \iota) = K$ is a surjective homomorphism of Hopf algebras. It is clear that any embedding $\iota' : \mathcal{A}' \to \text{Rep}(H)$ with $\iota(\mathcal{A}) = \iota(\mathcal{A}')$ results in a homomorphism $p' : H \to K'$ equivalent to $p$. \qed

Definition 2.3. Let $\mathcal{A}$ be a tensor category and let $\mathcal{B} \subset \mathcal{A}$ be a tensor subcategory. A tensor autoequivalence $\alpha$ of $\mathcal{A}$ is called trivializable on $\mathcal{B}$ if the restriction $\alpha|_B$ is isomorphic to $\text{id}_B$ as a tensor functor.
We will denote by $\text{Aut}(\mathcal{A})$ (respectively, $\text{Aut}(\mathcal{A}, \mathcal{B})$) the group of isomorphism classes of tensor autoequivalences of $\mathcal{A}$ (respectively, tensor autoequivalences of $\mathcal{A}$ trivializable on $\mathcal{B}$).

Below we consider examples of tensor autoequivalences of $\mathcal{A} = \text{Rep}(H)$, where $H$ is a finite dimensional Hopf algebra.

**Example 2.4.** A Hopf algebra automorphism $a : H \rightarrow H$ gives rise to a tensor autoequivalence $F_a \in \text{Aut}(\text{Rep}(H))$ such that for any $H$-module $V$ one has $F_a(V) = V$ as a vector space with the action $h \otimes v \mapsto a(h) \cdot v$, $h \in H$, $v \in V$.

**Definition 2.5.** An invariant twist $\text{D}_\mathcal{A}$ on $H$ is an invertible element $J \in H \otimes H$ such that $J \Delta(h) = \Delta(h) J$ for all $h \in H$ and

$$(J \otimes 1)(1 \otimes J)(J) = (1 \otimes J)(J).$$

An invariant 2-cocycle on $H$ $\text{B}_\mathcal{A}$ is a convolution invertible linear map $\mu : H \otimes H \rightarrow k$ such that $\mu(x(1) \otimes y(1)) x(2) y(2) = \mu(x(2) \otimes y(2)) x(1) y(1)$ and

$$\mu(x(1) \otimes y(1)) \mu(x(2) y(2) \otimes z) = \mu(y(1) \otimes z(1)) \mu(x \otimes y \otimes z(2)),$$

for all $x, y, z \in H$.

**Remark 2.6.** (i) If $\sigma : H \otimes H \rightarrow k$ is an invariant 2-cocycle on $H$ then $\sigma^*$ (viewed as an element of $H^* \otimes H^*$) is an invariant twist on $H^*$. Similarly, an invariant twist on $H$ gives rise to an invariant 2-cocycle on $H^*$.

(ii) Invariant twists on $H$ (respectively, invariant 2-cocycles on $H$) form a group under multiplication (respectively, convolution).

Let $u : H \rightarrow k$ be a convolution invertible linear map such that $u(x(1)) x(2) = x(1) u(x(2))$ for all $x \in H$ (i.e., $u$ is a central element of $H^*$). Then

$$\mu_u(x \otimes y) = u^{-1}(x(1)) u^{-1}(y(1)) u(x(2) \otimes y(2)), \quad x, y \in H,$$

is an invariant 2-cocycle on $H$. The quotient of the group of invariant 2-cocycles on $H$ by the subgroup of 2-cocycles of the form \(\text{B}_\mathcal{A}\) is called the second invariant cohomology group of $H$ and is denoted by $H^2_{\text{inv}}(H)$.

**Example 2.7.** Any invariant twist $J \in H \otimes H$ gives rise to a tensor autoequivalence $F_J \in \text{Aut}(\text{Rep}(H))$ such that $F_J = \text{id}_{\text{Rep}(H)}$ as an additive functor with the tensor structure given by

$$J_{V,U} : V \otimes U \rightarrow V \otimes U : v \otimes u \mapsto J(v \otimes u),$$

where $V, U$ are $H$-modules, $v \in V$, $u \in U$.

In a dual way, an invariant 2-cocycle $\sigma : H \otimes H \rightarrow k$ gives rise to a tensor autoequivalence $F_\sigma$ of $\text{Rep}(H^*)$ that is isomorphic to $\text{id}_{\text{Rep}(H^*)}$ as an additive functor. If we view $\text{Rep}(H^*)$ as the category of right $H$-comodules then the tensor structure of $F_\sigma$ is given by

$$\sigma_{V,U} : V \otimes U \rightarrow V \otimes U : v \otimes u \mapsto \sigma(v(1) \otimes u(1)) v(0) \otimes u(0),$$

where $V, U$ are $H$-comodules, $v \in V$, $u \in U$, and $v \mapsto v(0) \otimes v(1)$, $u \mapsto u(0) \otimes u(1)$ denote the comodule maps.

Let $\sigma_1$, $\sigma_2$ be invariant 2-cocycles of $H$. Then $F_{\sigma_1}$ and $F_{\sigma_2}$ are isomorphic autoequivalences of $\text{Rep}(H)$ if and only if $\sigma_1$ and $\sigma_2$ determine the same class in $H^2_{\text{inv}}(H)$. 
Let $\mathcal{C}$ be a braided tensor category with braiding $c_{X,Y} : X \otimes Y \overset{\sim}{\rightarrow} Y \otimes X$. For a tensor subcategory $\mathcal{D} \subset \mathcal{C}$ its centralizer $\mathcal{D}'$ is the full tensor subcategory of $\mathcal{C}$ consisting of objects $Y$ such that $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all objects $X$ in $\mathcal{C}$ [Mu].

A braided tensor category $\mathcal{C}$ is called symmetric if $\mathcal{C} = \mathcal{C}'$, i.e., if $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all objects $X, Y$ in $\mathcal{C}$.

**Remark 2.8.** By the result of Deligne [De] any finite symmetric tensor category is equivalent to the representation category of a finite supergroup. It was explained in [AEG] that any such a category can be realized as the representation category of a modified supergroup Hopf algebra $\wedge V \rtimes kG$, where $G$ is a finite group with a fixed central element $u$ such that $u^2 = 1$ and $V$ is a finite dimensional representation of $G$ on which $u$ acts by $-1$. The coalgebra structure of $\wedge V \rtimes kG$ is determined by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad g \in G,$$
$$\Delta(v) = 1 \otimes v + v \otimes u, \quad \varepsilon(v) = 0, \quad v \in V;$$

and the antipode is given by $S(g) = g^{-1}, S(v) = -v$. This category is semisimple if and only if $V = 0$.

In this paper we deal with non-semisimple symmetric tensor categories corresponding to $G = \mathbb{Z}/2\mathbb{Z}$. These are precisely the finite symmetric categories without non-trivial Tannakian subcategories. The corresponding Hopf algebras are described in Section 4.1 below.

Any finite symmetric tensor category has a unique, up to isomorphism, superfiber functor (i.e., a braided tensor functor to the category $s\text{Vec}$ of super vector spaces). This functor is identified with the forgetful tensor functor

$$\text{Rep}(\wedge V \rtimes kG) \rightarrow s\text{Vec}.$$ 

**Example 2.9.** Let $\mathcal{C} = \text{Rep}(H)$. It is well known that braidings on $\mathcal{C}$ are in bijection with quasi-triangular structures on $H$. By definition, such a structure on $H$ is an invertible element $R \in H \otimes H$ such that $\Delta^\text{op}(h) = R \Delta(h) R^{-1}$ for all $h \in H$ and

$$(\Delta \otimes \text{id})(R) = R^{13} R^{23}, \quad (\text{id} \otimes \Delta)(R) = R^{13} R^{12}.$$ 

Here we denote $R^{12} = R \otimes 1$, $R^{23} = 1 \otimes R$ and $R^{13} = \sum_i a_i \otimes 1 \otimes b_i$, where $R = \sum_i a_i \otimes b_i$.

A quasi-triangular structure is triangular if $R_{21} = R^{-1}$. Triangular structures are in bijection with symmetric braidings on $\text{Rep}(H)$.

For a braided tensor category $\mathcal{C}$ let $\text{Aut}^{br}(\mathcal{C})$ denote the group of isomorphism classes of braided autoequivalences of $\mathcal{C}$.

### 2.3. The center of a tensor category.

For any tensor category $\mathcal{A}$ let $\mathcal{Z}(\mathcal{A})$ denote its center. Recall that the objects of $\mathcal{Z}(\mathcal{A})$ are pairs $(Z, \gamma)$ where $Z$ is an object of $\mathcal{A}$ and $\gamma = \{ \gamma_X \}_{X \in \mathcal{A}}$, where

$$\gamma_X : X \otimes Z \overset{\sim}{\rightarrow} Z \otimes X,$$ 

is a natural isomorphism satisfying certain compatibility conditions. We will usually simply write $Z$ for $(Z, \gamma)$. Morphisms and tensor product in $\mathcal{Z}(\mathcal{C})$ are defined in an obvious way.
**Example 2.10.** The center of the tensor category $C = \text{Rep}(H)$, where $H$ is a Hopf algebra, is equivalent to $\text{Rep}(D(H))$, where the Hopf algebra $D(H)$ is the Drinfeld double of $H$. As a coalgebra,

$$D(H) = H^{*\text{cop}} \otimes H,$$

where $H^{*\text{cop}}$ denotes the co-opposite dual Hopf algebra of $H$. The algebra structure of $D(H)$ is given by

$$(p \otimes h)(p' \otimes h') = p\left(h_{(1)} \to p' \leftarrow S^{-1}(h_{(3)})\right) \otimes h_{(2)}h'$$

for all $p, p' \in H^*$, $h, h' \in H$, where $(h \to p \leftarrow g)(x) = p(gxh)$, for all $h, g, x \in H$, $p \in H^*$. There is a canonical quasi-triangular structure on $D(H)$:

$$\mathcal{R} = \sum_i (1 \otimes e_i) \otimes (f_i \otimes 1),$$

where $\{e_i\}$ and $\{f_i\}$ are dual bases of $H$ and $H^*$. This quasi-triangular structure corresponds to braiding $[\mathfrak{B}]$ of $Z(\text{Rep}(H))$.

It is well known that representations of $D(H)$ (i.e., objects of $Z(\text{Rep}(H))$) can be described in terms of *Yetter-Drinfeld modules over $H$*. That is, any such representation $Z$ has structures of a left $H$-module $h \otimes x \mapsto h \cdot x$ and a right $H$-comodule $\delta : Z \to Z \otimes H$, $\delta(x) = x_{(0)} \otimes x_{(1)}$, $h \in H$, $x \in Z$ satisfying the following compatibility condition:

$$\delta(h \cdot x) = h_{(2)} \cdot x_{(0)} \otimes h_{(3)}x_{(1)}S^{-1}(h_{(1)}), \quad h \in H, \ x \in Z.$$

Let $\mathcal{Y}^{D^H}$ denote the tensor category of Yetter-Drinfeld modules over $H$. The central structure $[\mathfrak{E}]$ for the Yetter-Drinfeld module $Z$ is given by

$$(8) \quad \gamma_V : V \otimes Z \to Z \otimes V, \quad \gamma_V(v \otimes x) = x_{(0)} \otimes x_{(1)} \cdot v, \quad v \in V, \ x \in Z.$$

### 2.4. The Brauer-Picard group of a tensor category.

**Let** $\mathcal{A}$ be a finite tensor category. The notion of a tensor product $\boxtimes_{\mathcal{A}}$ of $\mathcal{A}$-bimodule categories was introduced in [ENO]. With respect to this product the equivalence classes of $\mathcal{A}$-bimodule categories form a monoid. The unit of this monoid is the regular $\mathcal{A}$-category. The notion of a tensor product $\boxtimes$ corresponds to braiding $(6)$ of $\mathcal{A}$-category. The Brauer-Picard group of a tensor category $\mathcal{A}$ is the group $\text{BrPic}(\mathcal{A})$ of equivalence classes of invertible $\mathcal{A}$-bimodule categories.

The Brauer-Picard group is an important invariant of a tensor category. It is used, in particular, in the classification of extensions of tensor categories [ENO].

### 2.5. Ext-spaces and actions of autoequivalences on them.

**Let** $\mathcal{A}$ be a $k$-linear abelian category and let $V, U$ be objects of $\mathcal{A}$. An extension of $U$ by $V$ is a short exact sequence

$$(9) \quad 0 \to V \xrightarrow{i} E \xrightarrow{p} U \to 0.$$

We will consider extensions up to the following equivalence relation. Two extensions $0 \to V \xrightarrow{i} E \xrightarrow{p} U \to 0$ and $0 \to V \xrightarrow{i'} E' \xrightarrow{p'} U \to 0$ are *equivalent* if there is an
is given by the usual operation of the Baer sum. Indeed, let

$$0 \rightarrow V \rightarrow E \rightarrow U \rightarrow 0 \rightarrow 0.$$  

Equivalences classes of extensions form a $k$-vector space $\text{Ext}^1_A(U, V)$. The addition is given by the usual operation of the Baer sum. The zero element of $\text{Ext}^1_A(U, V)$ is the split extension $0 \rightarrow V \rightarrow V \oplus U \rightarrow U \rightarrow 0$. For $\lambda \in k^\times$ the $\lambda$-multiple of the class of extension (9) is the class of the extension

$$0 \rightarrow V \xrightarrow{\lambda \cdot 1} E \xrightarrow{P} U \rightarrow 0.$$

Furthermore, the extension $0 \rightarrow V \xrightarrow{\lambda \cdot 1} E \xrightarrow{P} U \rightarrow 0$, where $\lambda, \mu \in k^\times$, is equivalent to $0 \rightarrow V \xrightarrow{\lambda \mu \cdot 1} E \xrightarrow{P} U \rightarrow 0$.

**Remark 2.11.** Let $H$ be a Hopf algebra and $\gamma$ and $\eta$ be two 1-dimensional representations of $H$. If $P_{\gamma, \eta}(H^*)$ denotes the set of $(\gamma, \eta)$-primitive elements of $H^*$, i.e., elements $\xi \in H^*$ such that $\Delta_H(\xi) = \gamma \otimes \xi + \xi \otimes \eta$, then

$$\text{Ext}^1_{\text{Rep}(H)}(\eta, \gamma) \cong P_{\gamma, \eta}(H^*)/k(\gamma - \eta).$$

Indeed, let $0 \rightarrow \gamma \xrightarrow{i} E \xrightarrow{P} \eta \rightarrow 0$ be an extension of $\eta$ by $\gamma$ and let $e_1, e_2 \in E$ be such that $e_1 = i(1)$ and $p(e_2) = 1$. Then $\{e_1, e_2\}$ is a $k$-basis of $E$ such that $h \cdot e_1 = \gamma(h)e_1$ and $h \cdot e_2 = \xi(h)e_1 + \eta(h)e_2$, for all $h \in H$ and some $\xi \in H^*$. In fact, $\xi \in P_{\gamma, \eta}(H^*)$, since

$$(hl) \cdot e_2 = h \cdot (\xi(l)e_1 + \eta(l)e_2) = (\gamma(h)\xi(l) + \xi(h)\eta(l))e_1 + \eta(hl)e_2$$

implies that $\xi(hl) = \gamma(h)\xi(l) + \xi(h)\eta(l)$, for all $h, l \in H$, whence $\Delta(\xi) = \gamma \otimes \xi + \xi \otimes \eta$. If $\xi_1'$ and $\xi_2'$ are such that $p(\xi_1') = 1$ and $h \cdot e_1 = \xi'(h)e_1 + \eta(h)e_2$, then $\xi' - \xi \in k(\gamma - \eta)$. Indeed, there exists $a \in k$ such that $e_2 - e_1' = ae_1$ and action of $h \in H$ on this relation yields $\xi - \xi' = a(\gamma - \eta)$. Similarly, the equivalence class of $\xi$ modulo $k(\gamma - \eta)$ remains the same if we pass to an extension equivalent to $E$. Thus, the map $\text{Ext}^1_{\text{Rep}(H)}(\eta, \gamma) \ni [E] \mapsto \hat{\xi} \in P_{\gamma, \eta}(H^*)/k(\gamma - \eta)$, where $\hat{\xi}$ denotes the class of $\xi$, is well defined and is easily seen to be a $k$-vector space isomorphism.

**Proposition 2.12.** Let $\mathcal{A}$ be a tensor category and let $U, V$ be simple objects of $\mathcal{A}$ such that $\alpha(V) = V$ and $\alpha(U) = U$ for all tensor autoequivalences $\alpha : \mathcal{A} \rightarrow \mathcal{A}$. Then isomorphisms

$$\rho(\alpha) : \text{Ext}^1_A(U, V) \cong \text{Ext}^1_A(\alpha(U), \alpha(V)) = \text{Ext}^1_A(U, V),$$

where the image of extension (9) under $\rho(\alpha)$ is

$$0 \rightarrow V \xrightarrow{\alpha(1)} E \xrightarrow{P} U \rightarrow 0,$$

gives rise to a projective representation of $\text{Aut}(\mathcal{A})$ on $\text{Ext}^1_A(U, V)$.

**Proof.** Let $\alpha, \alpha' : \mathcal{A} \rightarrow \mathcal{A}$ be tensor autoequivalences and let $\phi : \alpha \rightarrow \alpha'$ be a tensor isomorphism between them (so that $\alpha, \alpha'$ determine the same element of $\text{Aut}(\mathcal{A})$).
We have an isomorphism of extensions:

$$\begin{array}{c}
0 \rightarrow V \xrightarrow{\alpha(i)} \alpha(E) \xrightarrow{\alpha(p)} U \rightarrow 0 \\
\phi_V \downarrow \quad \phi_E \downarrow \quad \phi_U \\
0 \rightarrow V \xrightarrow{\alpha'(i)} \alpha'(E) \xrightarrow{\alpha'(p)} U \rightarrow 0,
\end{array}$$

where $\phi_V$, $\phi_U$ are non-zero scalars and $\phi_E$ is an isomorphism. Thus, the equivalence classes of $\rho(\alpha)$ and $\rho(\alpha')$ differ by the scalar $\phi_V \phi_U^{-1}$. Hence, the map

$$\rho : \text{Aut}(A) \rightarrow \text{PGL}(\text{Ext}_A^1(U, V))$$

is well defined. It is clear that this map is a group homomorphism. □

**Remark 2.13.** Presence of equalities rather than isomorphisms in the hypothesis of Proposition 2.12 may look unnatural. This can be resolved by replacing $A$ with an equivalent skeletal category (i.e., a category in which isomorphic objects are equal). In any event, we will only apply Proposition 2.12 in a simple situation when $U$, $V$ are one-dimensional representations (i.e., linear characters) of a Hopf algebra.

### 3. Braided autoequivalences of the center

#### 3.1. Isomorphism $\text{BrPic}(A) \simeq \text{Aut}^{br}(Z(A))$

Let $A$ be a tensor category and let $\mathcal{M}$ be an invertible $A$-bimodule category. One assigns to $\mathcal{M}$ a braided autoequivalence $\Phi_{\mathcal{M}}$ of $Z(A)$ as follows. Note that $Z(A)$ can be identified with the category of $A$-bimodule endofunctors of $\mathcal{M}$ in two ways: via the functors $Z \mapsto Z \otimes -$ and $Z \mapsto - \otimes Z$. Define $\Phi_{\mathcal{M}}$ in such a way that there is an isomorphism of $A$-bimodule functors

$$\Phi_{\mathcal{M}}(Z) \otimes - \cong - \otimes Z$$

for all $Z \in Z(A)$.

The following result was established in [ENO, DN].

**Theorem 3.1.** Let $A$ be a tensor category. The assignment

$$\Phi : \text{BrPic}(A) \xrightarrow{\sim} \text{Aut}^{br}(Z(A)) : \mathcal{M} \mapsto \Phi_{\mathcal{M}}$$

is an isomorphism.

**Example 3.2.** Let $A$ be a finite Abelian group. Then Theorem 3.1 implies that

$$\text{BrPic}(\text{Vec}_A) \cong O(A \oplus \hat{A}, q),$$

where $O(A \oplus \hat{A}, q)$ is the group of automorphisms of $A \oplus \hat{A}$ preserving the canonical quadratic form

$$q(a, \chi) = \chi(a), \quad a \in A, \chi \in \hat{A}.$$ 

Let $C$ be a braided tensor category with braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$. Then $C$ is embedded into $Z(C)$ via

$$X \mapsto (X, c_{-,X}).$$

In what follows we will identify $C$ with a tensor subcategory of $Z(C)$ (the image of this embedding). The braiding of $C$ allows to view left $C$-module categories as $C$-bimodule categories (analogously to how modules over a commutative ring
can be viewed as bimodules). Invertible left $\mathcal{C}$-module categories form a subgroup $\text{Pic}(\mathcal{C}) \subset \text{BrPic}(\mathcal{C})$ called the Picard group of $\mathcal{C}$.

**Remark 3.3.** It follows from [ENO] that $\text{BrPic}(\mathcal{C}) \cong \text{Pic}(\mathcal{Z}(\mathcal{C})).$

Let $\text{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C}) \subset \text{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ be the subgroup consisting of braided autoequivalences of $\mathcal{Z}(\mathcal{C})$ that restrict to a trivial autoequivalence of $\mathcal{C}$. The following result was established in [DN].

**Theorem 3.4.** The image of $\text{Pic}(\mathcal{C})$ under isomorphism (13) is $\text{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$.

### 3.2. Homomorphism $\text{Aut}(\mathcal{A}) \to \text{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$

For any tensor category $\mathcal{A}$ there is an induction homomorphism

$$\Gamma: \text{Aut}(\mathcal{A}) \to \text{Aut}^{br}(\mathcal{Z}(\mathcal{A})): \alpha \mapsto \Gamma_{\alpha},$$

where $\Gamma_{\alpha}(Z, \gamma) = (\alpha(Z), \gamma_{\alpha})$ and $\gamma_{\alpha}$ is defined by the following commutative diagram

$$\begin{array}{ccc}
X \otimes \alpha(Z) & \xrightarrow{\gamma_{\alpha}} & \alpha(Z) \otimes X \\
\downarrow & & \downarrow \\
\alpha(\alpha^{-1}(X)) \otimes \alpha(Z) & \xrightarrow{J_{\alpha^{-1}(X), Z}} & \alpha(Z) \otimes \alpha(\alpha^{-1}(X)) \\
\alpha(\alpha^{-1}(X) \otimes Z) & \xrightarrow{\alpha(\gamma_{\alpha^{-1}(X)})} & \alpha(Z \otimes \alpha^{-1}(X)). \\
\end{array}$$

Here $\alpha^{-1}$ is a quasi-inverse of $\alpha$ and $J_{X,Y}: \alpha(X) \otimes \alpha(Z) \xrightarrow{\sim} \alpha(X \otimes Z)$ is the tensor functor structure of $\alpha$.

For any invertible object $Z \in \mathcal{A}$ let $\text{ad}(Z)$ denote the tensor autoequivalence of $\mathcal{A}$ given by

$$\text{ad}(Z)(X) = Z \otimes X \otimes Z^*.$$

Thus, we have a homomorphism

$$\text{ad}: \text{Inv}(\mathcal{A}) \to \text{Aut}(\mathcal{A})$$

Let

$$\mathcal{F}_{\mathcal{A}}: \mathcal{Z}(\mathcal{A}) \to \mathcal{A}: (Z, \gamma) \mapsto Z$$

denote the canonical forgetful functor. Let $I_{\mathcal{A}}: \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ denote a functor right adjoint to $\mathcal{F}_{\mathcal{A}}$. It is known that $I_{\mathcal{A}}(1)$ is a commutative algebra in $\mathcal{Z}(\mathcal{A})$ and that $\mathcal{A}$ is isomorphic to the category of $I_{\mathcal{A}}(1)$-modules as a tensor category [DMNO].

**Proposition 3.5.** There is an exact sequence of group homomorphisms:

$$\text{Inv}(\mathcal{Z}(\mathcal{A})) \xrightarrow{F} \text{Inv}(\mathcal{A}) \xrightarrow{\text{ad}} \text{Aut}(\mathcal{A}) \xrightarrow{\Gamma} \text{Aut}^{br}(\mathcal{Z}(\mathcal{A})),
$$

where $F$ is induced by $\mathcal{F}_{\mathcal{A}}$.

**Proof.** For any central invertible object $(Z, \gamma)$ the functor $\text{ad}(Z)$ is isomorphic to $\text{id}_{\mathcal{A}}$ (as a tensor functor) via

$$\text{ad}(Z)(X) = Z \otimes X \otimes Z^* \xrightarrow{\gamma_{X}^{-1} \otimes \text{id}_{Z^*}} X \otimes Z \otimes Z^* \cong X \otimes 1 \cong X, \quad X \in \mathcal{A},$$
that is, \( \text{ad} \circ F \) is trivial. Suppose that \( \text{ad}(X) \) is a trivial autoequivalence of \( \mathcal{A} \) for some invertible \( X \in \mathcal{A} \). Then there is a natural tensor isomorphism

\[
X \otimes V \otimes X^* \cong V, \quad V \in \mathcal{A},
\]

which translates to a central structure on \( X \). This shows that the sequence (13) is exact at \( \text{Inv}(\mathcal{A}) \).

Next, for an invertible \( Y \in \mathcal{A} \) there is a tensor isomorphism

\[
\Gamma_{\text{ad}(Y)}(W) = Y \otimes W \otimes Y^* \xrightarrow{\eta_y \otimes \text{id}_{Y^*}} W \otimes Y \otimes Y^* \cong W,
\]

for any object \((W, \eta)\) in \( \mathcal{Z}(\mathcal{A}) \). Thus, \( \Gamma \circ \text{ad} \) is trivial. Let \( \alpha : \mathcal{A} \to \mathcal{A} \) be a tensor autoequivalence. Under isomorphism (13) \( \Gamma_\alpha \) corresponds to the \( \mathcal{A} \)-bimodule category \( \mathcal{A}_\alpha \) such that \( \mathcal{A}_\alpha = \mathcal{A} \) as a right \( \mathcal{A} \)-module category with the left action of \( \mathcal{A} \) given by

\[
(X, V) \mapsto \alpha(X) \otimes V, \quad X, V \in \mathcal{A}.
\]

It is easy to check that \( \mathcal{A}_\alpha \cong \mathcal{A} \) as an \( \mathcal{A} \)-bimodule category if and only if \( \alpha(X) = Y \otimes X \otimes Y^* \) for some invertible object \( Y \in \mathcal{A} \). Thus, the kernel of \( \Gamma \) coincides with the image of \( \text{ad} \).

**Proposition 3.6.** The image of homomorphism (13) consists of (isomorphism classes of) all braided autoequivalences \( \beta : \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A}) \) such that \( \beta(I_\mathcal{A}(1)) \cong I_\mathcal{A}(1) \) as algebras in \( \mathcal{Z}(\mathcal{A}) \).

**Proof.** It is clear from definitions that for any tensor autoequivalence \( \alpha : \mathcal{A} \to \mathcal{A} \) one has \( \alpha \circ F_\mathcal{A} \cong F_\mathcal{A} \circ \Gamma_\alpha \). Taking the adjoints of both sides and applying them to \( 1 \) we obtain algebra isomorphism \( \Gamma_\alpha(I_\mathcal{A}(1)) \cong I_\mathcal{A}(1) \).

Conversely, assume that \( \beta : \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A}) \) is such that there is an algebra isomorphism \( \varphi : \beta(I_\mathcal{A}(1)) \cong I_\mathcal{A}(1) \). Then \( \beta \) induces a tensor autoequivalence \( \alpha \) of the category of \( I_\mathcal{A}(1) \)-modules in \( \mathcal{Z}(\mathcal{A}) \) defined by \( Z \mapsto \beta(Z) \) with the action of \( I_\mathcal{A}(1) \) given by

\[
I_\mathcal{A}(1) \otimes \beta(Z) \xrightarrow{\varphi \otimes \text{id}_{\beta(Z)}} \beta(I_\mathcal{A}(1)) \otimes \beta(Z) \cong \beta(I_\mathcal{A}(1) \otimes Z) \xrightarrow{\mu_Z} \beta(Z),
\]

where \( \mu_Z \) denotes the action of \( I_\mathcal{A}(1) \) on \( Z \). Since the category of \( I_\mathcal{A}(1) \)-modules in \( \mathcal{Z}(\mathcal{A}) \) is identified with \( \mathcal{A} \) we get \( \alpha \in \text{Aut}(\mathcal{A}) \). Since \( \mathcal{Z}(\mathcal{A}) \) is identified with the center of the category of \( I_\mathcal{A}(1) \)-modules via the free module functor \( Z \mapsto I_\mathcal{A}(1) \otimes Z \) [DMNO] we see that \( \Gamma_\alpha \cong \beta \).

Let \( \mathcal{A} = \text{Rep}(H) \), where \( H \) is a finite dimensional Hopf algebra. As in Example 2.10 we identify \( \mathcal{Z}(\text{Rep}(H)) \) with the category of Yetter-Drinfeld modules over \( H \). Below we describe braided autoequivalences of \( \mathcal{Z}(\text{Rep}(H)) \) induced from tensor autoequivalences of \( \text{Rep}(H) \), see Examples 2.7.

**Example 3.7.** Let \( \alpha : H \xrightarrow{\sim} H \) be a Hopf algebra automorphism and let \( F_\alpha \in \text{Aut}(\text{Rep}(H)) \) be the corresponding tensor autoequivalence. Then \( \Gamma_{F_\alpha}(Z) = Z \) as a vector space with the action and coaction given by

\[
h \otimes x \mapsto a(h) \cdot x, \quad \rho_\alpha(x) = x(0) \otimes a(x(1)) \quad h \in H, x \in Z.
\]

**Example 3.8.** Let \( J \in H \otimes H \) be an invariant twist and let \( F_J \in \text{Aut}(\text{Rep}(H)) \) denote the corresponding tensor autoequivalence (see Example 2.7). Then \( \Gamma_{F_J}(Z) = Z \) as an \( H \)-module with the coaction given by

\[
\rho^J(x) = (J^{-1})^2 \cdot (J^1 \cdot x)(0) \otimes (J^{-1})^3 (J^1 \cdot x)(1) J^2, \quad x \in Z.
\]
Here $J^1 \otimes J^2$ stands for $J$ and $J^{-1} \otimes J^{-2}$ for the inverse of $J$. Note that formula (20) appeared in [CZ].

**Example 3.9.** Let $\sigma \in (H \otimes H)^*$ be an invariant 2-cocycle on $H$. The dual map $\sigma^*$ can be seen as an invariant twist on $H^*$ and, as such, it gives rise to an autoequivalence $F_\sigma$ of $\text{Rep}(H^*)$. By Example 3.8 this induces an autoequivalence $\Gamma_{F_\sigma}$ of $\text{YD} \text{D}^{H*}$. Since $\text{YD} \text{D}^{H}$ and $\text{YD} \text{D}^{H*}$ are strict tensor isomorphic, via the functor that dualizes module and comodule structures, we obtain an autoequivalence of $\text{YD} \text{D}^{H}$, which we shall still denote by $\Gamma_{F_\sigma}$. If $V \in \text{YD} \text{D}^{H}$ then $\Gamma_{F_\sigma}(V) = V$ as an $H$-module, with the $H$-action given by

$$\delta V = -\sigma^{-1}(h(1) \otimes v(0)) \sigma(h(3) \otimes v(1)) (h(2) \otimes v(0)) \sigma^{-1}(h(1) \otimes v(0)), \quad h \in H, v \in V.$$  

The invertible $\text{Rep}(H)$-bimodule category $\mathcal{M}_\sigma$ corresponding to $\Gamma_{F_\sigma}$ follows from invariance of $\text{YD} \text{D}^{H}$.

**Remark 3.10.** Autoequivalences described in Examples 3.7, 3.8, 3.9 give rise to group homomorphisms

$$\begin{align*}
\iota_1 : \text{Aut}_{\text{Hopf}}(H) &\to \text{Aut}^{br}(Z(\text{Rep}(H))), \\
\iota_2 : \text{H}^2_{\text{inv}}(H) &\to \text{Aut}^{br}(Z(\text{Rep}(H))), \\
\iota_3 : \text{H}^2_{\text{inv}}(H^*) &\to \text{Aut}^{br}(Z(\text{Rep}(H))).
\end{align*}$$

Let $(H, R)$ be a quasi-triangular Hopf algebra. Then $\text{Rep}(H)$ is embedded into $\text{YD} \text{D}^{H}$ by defining the coaction on an $H$ module $V$ via

$$\delta(v) = R^2(v \otimes 1), \quad v \in V.$$  

**Proposition 3.11.** Let $(H, R)$ be a quasi-triangular Hopf algebra. Let $\sigma$ be an invariant 2-cocycle on $H$ such that

$$\sigma(R^1 \otimes h) R^2 = \varepsilon(h), \quad \sigma^{-1}(h \otimes R^1) R^2 = \varepsilon(h)$$

for all $h \in H$, where $R = R^1 \otimes R^2$. Then $\Gamma_{F_\sigma}$ belongs to the Picard group of $\text{Rep}(H)$ (corresponding to the braiding defined by $R$) via identification from Theorem 3.4.

**Proof.** It suffices to show that $\Gamma_{F_\sigma}$ belongs to $\text{Aut}^{br}(\text{YD} \text{D}^{H})$, $\text{Rep}(H))$ (recall that $Z(\text{Rep}(H)) \cong \text{YD} \text{D}^{H}$ as a braided tensor category). For any $H$-module $V$ (viewed as a Yetter-Drinfeld module via (22)) we have $\Gamma_{F_\sigma}(V) = V$ as an $H$-comodule, while its $H$-module structure is computed using (21):

$$h \cdot v = \sigma(R^1 \otimes h(1)) \sigma^{-1}(h(3) \otimes r^1) R^2 h(2) R^2 \cdot v, \quad h \in H, v \in V.$$  

Here $r$ stands for another copy of $R$. It is clear from the last equation that the invariance condition (23) implies $\Gamma_{F_\sigma} \circ \text{Rep}(H) = \text{id}_{\text{Rep}(H)}$. \qed
3.3. Induction from subcategory. Let $C$ be a braided tensor category and let $D \subset C$ be a tensor subcategory. Given an invertible $D$-module category $\mathcal{M}$ we can consider the induced $C$-module category $C \boxtimes_D \mathcal{M}$. It is clear that the latter category is invertible and that the assignment

\[(24) \quad \text{Ind}_D^C : \text{Pic}(D) \to \text{Pic}(C) : \mathcal{M} \mapsto C \boxtimes_D \mathcal{M}\]

is a group homomorphism.\(^1\)

Let $A$ be an algebra in $D$ such that $\mathcal{M}$ is equivalent to the category of $A$-modules in $D$. Then $\text{Ind}_D^C(\mathcal{M})$ is the category of $A$-modules in $C$.

Note that homomorphism (24) is not injective in general. Let

\[C = \bigoplus_{\alpha \in \Sigma} C_\alpha\]

be the decomposition of $C$ into a direct sum of $D$-module subcategories (this decomposition exists and is unique by \([EO]\)).

Let $\Sigma_0 = \{\alpha \in \Sigma \mid C_\alpha$ is an invertible $D$-module category\}.

**Proposition 3.12.** The kernel of homomorphism (24) is precisely the set of the equivalence classes of categories $C_\alpha$, $\alpha \in \Sigma_0$.

**Proof.** Suppose that $\mathcal{M}$ is an invertible $D$-module category such that

\[(25) \quad C \boxtimes_D \mathcal{M} \cong C\]

as a $C$-module category. From the $D$-module decomposition of both sides of (25) we obtain

\[\bigoplus_{\alpha \in \Sigma} (C_\alpha \boxtimes_D \mathcal{M}) \cong \bigoplus_{\beta \in \Sigma} C_\beta.\]

Taking $\alpha$ such that $C_\alpha = D$ we conclude that $\mathcal{M} \cong C_\beta$ for some $\beta \in \Sigma_0$.

Conversely, suppose that $\mathcal{M} \cong C_\beta$ for some $\beta \in \Sigma_0$. Note that

\[E = \bigoplus_{\alpha \in \Sigma_0} C_\alpha\]

is a tensor subcategory of $C$. It is group graded with the trivial component $D$. Thus, $\text{Ind}_D^C \mathcal{M} \cong E$ and, hence, $\text{Ind}_D^C \mathcal{M} = \text{Ind}_E^C(\text{Ind}_D^C \mathcal{M}) \cong C$. \(\Box\)

**Proposition 3.13.** Let $C$ be a braided tensor category and let $D \subset C$ be a tensor subcategory. The image of the composition

\[\text{Pic}(D) \xrightarrow{\text{Ind}_D^C} \text{Pic}(C) \to \text{Aut}^{br}(C)\]

is contained in $\text{Aut}^{br}(C; D')$. Here $D'$ denotes the centralizer of $D$ in $C$.

**Proof.** Let $\mathcal{M} \in \text{Pic}(D)$ be an invertible $D$-module category and let $A \in D$ be an algebra such that $\mathcal{M}$ is identified with the category of $A$-modules in $D$. When we view $A$ as an algebra in $C$ the corresponding element $\partial_M$ of $\text{Aut}^{br}(C)$ is determined by the existence of a natural tensor isomorphism

\[A \otimes X \cong \partial_M(X) \otimes A, \quad X \in C,\]

of $A$-modules \([DN]\). When $X$ centralizes $D$ (and, hence, centralizes $A$) we get a natural tensor isomorphism $\partial_M(X) \cong X$, i.e., $\partial_M|_{D'} \cong \text{id}_{D'}$. \(\Box\)

---

\(^1\)More precisely, this is a monoidal functor between monoidal groupoids.
3.4. Action on the categorical Lagrangian Grassmannian. Let $\mathcal{A}$ be a finite tensor category.

**Proposition 3.14.** There is a bijection between the set of braidings on $\mathcal{A}$ and the set of tensor embeddings $\iota : \mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$ such that $F \circ \iota(\mathcal{A}) = \mathcal{A}$. Namely, the braiding $c_{X,Z} : X \otimes Z \rightarrow Z \otimes X$ corresponds to the embedding $\iota_c : Z \hookrightarrow (Z, c_{-Z})$.

**Proof.** This is clear since every embedding $\iota : \mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$ such that $F \circ \iota(\mathcal{A}) = \mathcal{A}$ yields a braiding on $\mathcal{A}$. □

**Definition 3.15.** Let $\mathcal{A}$ be a tensor category. A tensor subcategory $\mathcal{L} \subset \mathcal{Z}(\mathcal{A})$ is called Lagrangian if $\mathcal{L}' = \mathcal{L}$.

For a braided tensor category $\mathcal{C}$ let $\mathbb{L}(\mathcal{C})$ denote the set of all Lagrangian subcategories $\mathcal{L} \hookrightarrow \mathcal{Z}(\mathcal{C})$ such that $\mathcal{L} \cong \mathcal{C}$ as a braided tensor category. This is a subset of the categorical Lagrangian Grassmannian introduced in [NR]. Note that the group $\text{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ acts on $\mathbb{L}(\mathcal{C})$ by permutation of subcategories:

$$\alpha \cdot \mathcal{L} = \alpha(\mathcal{L}), \quad \alpha \in \text{Aut}^{br}(\mathcal{Z}(\mathcal{C})), \mathcal{L} \in \mathbb{L}(\mathcal{C}).$$

**Proposition 3.16.** Let $\mathcal{C}$ be a braided tensor category with braiding $c_{X,Z} : X \otimes Z \rightarrow Z \otimes X$ and let $\iota_c : \mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$, $Z \mapsto (Z, c_{-Z})$ be the corresponding central embedding. Then the stabilizer of $\iota_c(\mathcal{C})$ in $\text{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ is

$$\text{St}(\iota_c(\mathcal{C})) \cong \text{Pic}(\mathcal{C}) \times \text{Aut}^{br}(\mathcal{C}).$$

**Proof.** The proof is the same as [NR] Proposition 6.8]. □

4. The Hopf algebra $E(n)$

4.1. Definition and basic properties. Here we recall a description of a class of non-semisimple Hopf algebras with symmetric representation categories, cf. Remark 2.8

**Definition 4.1.** Let $n$ be a positive integer and $E(n)$ denote the Hopf algebra with generators $c, x_1, \ldots, x_n$, relations

$$c^2 = 1, \quad x_i^2 = 0, \quad cx_i = -x_ic, \quad x_ix_j = -x_jx_i, \quad i, j = 1, \ldots, n,$$

and comultiplication, counit and antipode given by

$$\Delta(c) = c \otimes c, \quad \varepsilon(c) = 1, \quad S(c) = c$$

$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes c, \quad \varepsilon(x_i) = 0, \quad S(x_i) = cx_i$$

for all $i = 1, \ldots, n$.

The Hopf algebras $E(n)$ were first introduced by Nichols in [N] and studied in [BDG], [CD], [PvO1], [PvO2] and [CC]. They are pointed Hopf algebras with coradical $kC_2$. Here $C_2 = \{1, c\}$, and $k$-basis $\{c^ix_i \mid i = 0, 1, P \subseteq \{1, \ldots, n\}\}$, where, for a subset $P = \{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_s$, we denote $x_P = x_{i_1}x_{i_2}\cdots x_{i_s}$ and $x_{\emptyset} = 1$. We have $\text{dim}_k(E(n)) = 2^{n+1}$.

For $P$ as above and a subset $F = \{i_1, \ldots, i_r\}$ of $P$ define

$$S(F, P) = \begin{cases} (j_1 + \cdots + j_r) - r(r + 1)/2 & \text{if } F \neq 0 \\ 0 & \text{if } F = \emptyset. \end{cases}$$
Letting $|F|$ denote the number of elements of $F$, we have the following formula:

$$
\Delta(x_F) = \sum_{F \subseteq P} (-1)^{S(F;P)} x_F \otimes c^{[F]} x_P \setminus F.
$$

The group of Hopf automorphisms of $E(n)$ was computed in \[PvO1\]. We have

\begin{equation}
\text{Aut}_{\text{Hopf}}(E(n)) \simeq \text{GL}_n(k),
\end{equation}

with the automorphism corresponding to $T = (t_{ij}) \in \text{GL}_n(k)$, being given by $c \mapsto c$ and $x_i \mapsto \sum_j t_{ij} x_j$, for all $i = 1, \ldots, n$. The inner automorphisms of $E(n)$ correspond to $T = \pm I_n$.

Quasi-triangular structures on $E(n)$ were classified in \[PvO1\]. They are parameterized by the set $M_n(k)$ of $n$-by-$n$ matrices with coefficients in $k$ and they are given as follows. First, if $A = (a_{ij}) \in M_n(k)$ and $P = \{i_1, i_2, \ldots, i_r\}$ and $F = \{j_1, j_2, \ldots, j_s\}$ are subsets of $\{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_s$, then we denote by $[A]_{P,F}$ the $r \times r$ minor obtained from $A$ by intersecting the rows $i_1, \ldots, i_r$ with the columns $j_1, \ldots, j_s$. The $R$-matrix corresponding to $A$ was described in \[PvO1\] Remark 2 as:

$$
R_A = \frac{1}{2} (1 \otimes 1 + 1 \otimes c + c \otimes 1 - c \otimes c) + \frac{1}{2} \sum_{|P|=|F|} (-1)^{\frac{|P|(|P|-1)}{2}} [A]_{P,F} \times
$$

$$
\times (x_P \otimes c^{[P]} x_F + c x_P \otimes c^{[P]} x_F + x_P \otimes c^{[P]+1} x_F - c x_P \otimes c^{[P]+1} x_F),
$$

where the sum is over all non-empty subsets $P$ and $F$ of $\{1, \ldots, n\}$. We will use the following equivalent expression for $R_A$:

$$
R_A = \frac{1}{2} \sum_{i=0}^n (-1)^{\frac{(i+1)(i-1)}{2}} \sum_{|P|=|F|=i} [A]_{P,F} (x_P \otimes x_F + x_P \otimes c x_F +
$$

$$
+ (-1)^i c x_P \otimes x_F + (-1)^{i+1} c x_P \otimes c x_F),
$$

where the sum is over all subsets $P$ and $F$ of $\{1, \ldots, n\}$ and the convention is that $[A]_{\emptyset,\emptyset} = 1$. It was shown in \[CC\] that the quasi-triangular structure $R_A$ is triangular if and only if $A$ is symmetric.

Let $C_n := \text{Rep}(E(n))$.

**Remark 4.2.** The category $C_n$ with symmetric braiding is equivalent to the representation category of a finite supergroup $\wedge k^n \rtimes \mathbb{Z}/2\mathbb{Z}$. It is the most general example of a non-semisimple symmetric tensor category without non-trivial Tannakian subcategories.

**Proposition 4.3.** $\text{Aut}^{br}(C_n) \cong \text{GL}_n(k)/\{\pm I_n\}$.

**Proof.** By \[Dr\] the symmetric category $C_n$ has a unique, up to isomorphism, braided tensor functor to $s\text{Vec}$. Let $F$ denote the composition of this functor with the forgetful functor $s\text{Vec} \to \text{Vec}$. Then $E(n) \cong \text{End}(F)$. Since every braided tensor autoequivalence of $C_n = \text{Rep}(E(n))$ preserves $F$ it must come from a Hopf automorphism of $E(n)$. By \[B\] we have $\text{Aut}_{\text{Hopf}}(E(n)) = \text{GL}_n(k)$. Tensor autoequivalences of $C_n$ isomorphic to the identity functor come from inner automorphisms of $E(n)$. The statement follows from the observation that the group of inner Hopf automorphisms of $E(n)$ is generated by the conjugation by $c$ and is isomorphic to $\{\pm I_n\}$. 


4.2. Invariant 2-cocycles and invariant twists on $E(n)$. It was shown in [BC], that the second invariant cohomology group $H^2_{inv}(E(n))$ is isomorphic to $\text{Sym}_n(k)$, which we can identify with the additive group of lower triangular $n \times n$ matrices with entries in $k$. A representative of the cohomology class corresponding to $M = (m_{ij}) \in \text{Sym}_n(k)$ is the invariant 2-cocycle $\sigma_M : E(n) \otimes E(n) \rightarrow k$ defined by:

$$
\sigma_M(c \otimes c) = 1, \quad \sigma_M(x_i \otimes x_j) = m_{ij}, \quad i, j = 1, \ldots, n,
$$

$$
\sigma_M(x_P \otimes x_Q) = \sigma_M(cx_P \otimes cx_Q) = (-1)^{|P|}\sigma_M(x_P \otimes cx_Q) = (-1)^{|P|}\sigma_M(cx_P \otimes cx_Q)
$$

for all $P, Q \subseteq \{1, \ldots, n\}$,

$$
\sigma_M(x_P \otimes x_Q) = 0 \quad \text{if} \quad |P| \neq |Q|
$$

and some recurrence formula allowing to compute $\sigma_M(x_P, x_Q)$ when $|P| = |Q|$. In particular, we have $\sigma_M(c^i x_k \otimes c^j x_l) = (-1)^i m_{kl}$, for all $i, j = 0, 1$ and $k, l = 1, \ldots, n$.

Since $E(n)$ is self-dual, an Hopf algebra isomorphism $E(n) \rightarrow E(n)^*$ being given by $1 \mapsto 1^* + c^*$, $c \mapsto 1^* - c^*$, $x_i \mapsto x_i^* + (cx_i)^*$, where $\{(c^i x_P)^*\}_{i, P}$ is the dual basis of $\{c^i x_P\}_{i, P}$, we obtain, by duality, that the invariant dual cohomology group of $E(n)$, $H^2_{inv}(E(n)^*)$, is also isomorphic to $\text{Sym}_n(k)$. A representative for the cohomology class corresponding to $M = (m_{ij}) \in \text{Sym}_n(k)$ is the invariant twist

$$
J_M = \frac{1}{4} \sum_{i, j, P, Q} \sigma_M(c^i x_P \otimes c^j x_Q)(x_P + (-1)^i cx_P) \otimes (x_Q + (-1)^j cx_Q).
$$

4.3. The Drinfeld double of $E(n)$. Composing the two Hopf algebra isomorphisms $E(n) \rightarrow E(n)^*$, $c \mapsto 1^* - c^*$, $x_i \mapsto x_i^* + (cx_i)^*$ and $E(n)^{\text{cop}} \rightarrow E(n)$, $c \mapsto c$, $x_i \mapsto cx_i$, we obtain the isomorphism $E(n) \rightarrow E(n)^{\text{cop}}$, $c \mapsto 1^* - c^*$, $x_i \mapsto x_i^* - (cx_i)^*$. Thus, the Drinfeld double, $D(E(n))$, is generated by two copies of $E(n)$. Let $C = 1^* - c^*$ and $X_i = x_i^* - (cx_i)^*$, $i = 1, \ldots, n$. Then, viewing $E(n)$ and $E(n)^{\text{cop}}$ as Hopf subalgebras of $D(E(n))$ and taking into account [7], we see that $D(E(n))$ is generated by the grouplike elements $c$ and $C$, the $(1, c)$-primitive elements $x_1, \ldots, x_n$ and the $(1, C)$-primitive elements $X_1, \ldots, X_n$, subject to the following relations:

$$(27) \quad c^2 = 1, \quad x_i^2 = 0, \quad x_i c + cx_i = 0, \quad x_i x_j + x_j x_i = 0,$$

$$(28) \quad C^2 = 1, \quad X_i^2 = 0, \quad X_i C + CX_i = 0, \quad X_i X_j + X_j X_i = 0,$$

$$(29) \quad cC = Cc, \quad X_i c + cx_i = x_i c + CX_i = 0, \quad x_i X_j + X_j x_i = \delta_{ij}(1 - Cc),$$

for all $i, j = 1, \ldots, n$, where $\delta_{ij}$ is Kronecker’s delta. For example, we have:

$$
x_i x_j = (x_i \rightarrow X_j \leftarrow c) c + (1 \rightarrow X_j \leftarrow c) x_i + (1 \rightarrow X_j \leftarrow (-cx_i)) 1
$$

$$
= -\delta_{ij} Cc - X_j x_i + \delta_{ij} 1
$$

$$
= -X_j x_i + \delta_{ij} (1 - Cc).
$$

Lemma 4.4. If $P$ is a subset of $\{1, \ldots, n\}$ then

$$
x_P^* = (-1)^{|P|(|P|-1)} (X_P + CX_P)
$$

$$
(cx_P)^* = (-1)^{|P|(|P|+1)} (X_P - CX_P).
$$
Proof. For \( P \subseteq \{1, \ldots, n\} \) define \( Y_P = x_P^* + (cx_P)^* \). An easy argument using induction on \( |P| \) shows that, if \( P = \{i_1, \ldots, i_r\} \), with \( i_1 < i_2 < \cdots < i_r \), then \( Y_P = Y_{i_1}Y_{i_2} \cdots Y_{i_r} \). Moreover, since
\[
(1^* - c^*)(x_P^* + (cx_P)^*)(c^i x_Q) = \begin{cases} 
0 & \text{if } Q \neq P \\
(-1)^i & \text{if } Q = P
\end{cases}
\]
we have \( CY_P = x_P^* - (cx_P)^* \). In particular, \( CY_i = X_i \), for all \( i = 1, \ldots, n \), and, because \( C \) is an element of order 2 which anti-commutes with \( X_i \), we also have \( Y_i C = -CY_i \), for all \( i \). Consider now \( i \in \{0, 1\} \) and \( P = \{i_1, \ldots, i_r\} \), with \( i_1 < i_2 < \cdots < i_r \). Then
\[
C^i X_P = C^i X_{i_1}X_{i_2} \cdots X_{i_r} = C^i CY_{i_1}CY_{i_2} \cdots CY_{i_r}
\]
\[
= (-1)^\frac{e(e+1)}{2} C^{r+Y_{i_1}Y_{i_2} \cdots Y_{i_r}}
\]
\[
= (-1)^\frac{|P||P|-1}{2} C^{|P|+iY_P}
\]
\[
= (-1)^\frac{(|P||P|-1)}{2} (x_P^* + (-1)^{|P|+i}(cx_P)^*).
\]
The expressions for \( x_P^* \) and \( (cx_P)^* \) in terms of \( X_P \) and \( CX_P \) now follow easily from the above. \( \square \)

Remark 4.5. For \( i \in \{0, 1\} \) and \( P \subseteq \{1, \ldots, n\} \) we have
\[
(c^i x_P)^* = \frac{1}{2} (-1)^\frac{(|P||P|-1)}{2} (X_P + (-1)^i CX_P)
\]

Lemma 4.6. The algebra \( D(E(n)) \) has a unique non-trivial one-dimensional representation, \( \chi : D(E(n)) \to k \), defined by
\[
\chi(C) = \chi(c) = -1, \quad \chi(x_i) = \chi(X_i) = 0, \quad i = 1, \ldots, n
\]

Proof. It follows from relations \([27], [28]\) that for a one-dimensional representation \( \chi : D(E(n)) \to k \) one has \( \chi(X_i) = \chi(x_i) = 0 \) for all \( i = 1, \ldots, n \), \( \chi(c)^2 = \chi(C)^2 = 1 \), and \( \chi(c^i) = C \). This implies the claim. \( \square \)

4.4. The Picard group of \( C_n \). The group \( \text{Pic}(C_n) \) corresponding to the symmetric braiding of \( C_n \) (or, equivalently, the Brauer group of the Hopf algebra \( E(n) \) with quasi-triangular structure \( R_0 \) (see Section 4.3)) was computed by Carvalle and Cuadra in \([CC]\). This group is isomorphic to \( \text{Sym}_n(k) \times \mathbb{Z}/2\mathbb{Z} \). The torsion subgroup of \( \text{Pic}(C_n) \) consists of elements induced from \( \text{sVec} \subset C_n \) (see Section 5.3).

Let us describe the connected component of the identity \( \text{Pic}_0(C_n) \subset \text{Pic}(C_n) \), i.e.,
\[
\text{Pic}_0(C_n) := \text{Sym}_n(k).
\]
in a way suitable for our purposes. For this end, we identify \( \text{Pic}(C_n) \) with the group \( \text{Aut}^{br} (\mathbb{Z}(C_n); C_n) \) via Theorem \([34]\). Note that each \( \sigma \in H_2^{\text{inv}}(E(n)) \) satisfies condition \([28]\) and so by Proposition \([5.11]\) the assignment
\[
H_2^{\text{inv}}(E(n)) \to \text{Aut}^{br} (\mathbb{Z}(C_n)) : \sigma \mapsto \Gamma_{\sigma},
\]
where \( \Gamma_{\sigma} \) is defined in Example \([3.9]\) takes values in \( \text{Aut}^{br} (\mathbb{Z}(C_n); C_n) \). It follows from \([CC]\) Theorem 5.3] that \([31]\) restricts to an isomorphism between \( H_2^{\text{inv}}(E(n)) \) and \( \text{Pic}_0(C_n) \).
5. The categorical Lagrangian Grassmannian of $\mathcal{C}_n$

5.1. Subcategories of $\mathcal{Z}(\mathcal{C}_n)$.

Lemma 5.1. The set of surjective Hopf algebra maps $D(E(n)) \to E(n)$ is in bijection with the set of $n \times 2n$ matrices of rank $n$. The homomorphism $f$ corresponding to $(A|B) \in M_{n \times 2n}(k)$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices, is given by

\[(32) \quad f(C) = f(c) = c, \quad f(X_i) = \sum_{j=1}^{n} a_{ji} x_j, \quad f(x_i) = \sum_{j=1}^{n} b_{ji} x_j, \quad i = 1, \ldots, n\]

Proof. Let $f : D(E(n)) \to E(n)$ be a Hopf algebra map. Since $C$ and $c$ are group-like elements of $D(E(n))$, we have $f(C), f(c) \in G(E(n)) = \{1, c\}$. If $(f(C), f(c)) = (1, 1)$ then $f(X_i)$ and $f(x_i)$, $i = 1, \ldots, n$, are primitive elements of $E(n)$, so $f(X_i) = f(x_i) = 0$, for all $i = 1, \ldots, n$. Thus, $f$ is the trivial homomorphism, $f(h) = \varepsilon(h)1$, which is not surjective. If $(f(C), f(c)) = (1, c)$ then $f(X_i) = 0$, for all $i = 1, \ldots, n$. Applying $f$ to the relation $x_i X_i + X_i x_i = 1 - Cc$ we obtain $0 = 1 - c$, which is not possible. Similarly, if $(f(C), f(c)) = (c, 1)$. If $(f(C), f(c)) = (c, c)$ then $f(X_i)$ and $f(x_i)$ are $(1, c)$-primitive elements of $E(n)$, for all $i = 1, \ldots, n$. Since the space of $(1, c)$-primitive elements of $E(n)$ is $k(1-c) \oplus k x_1 \oplus \cdots \oplus k x_n$ it follows that there exist $a, b, a_{ij}, b_{ij} \in k, i, j = 1, \ldots, n$, such that $f(X_i) = a(1-c) + \sum_j a_{ji} x_j$ and $f(x_i) = b(1-c) + \sum_j b_{ji} x_j$, for all $i = 1, \ldots, n$. Using the relations $x_i c + C x_i = 0$ and $X_i C + C X_i = 0$, we readily deduce that $a = b = 0$. Since the remaining relations impose no other restrictions on the scalars $a_{ij}$ and $b_{ij}$ we are left to see under what conditions is the homomorphism associated to these scalars surjective. We claim that $f$ is surjective if and only if $f$ maps $U = \text{span}\{x_1, \ldots, x_n, x_1, \ldots, x_n\}$ onto $\text{span}\{x_1, \ldots, x_n\}$. For this, it suffices to prove that if $x_i$ is in the image of $f$ than it is in the image of the restriction of $f$ to $U$. Suppose $x_i = f(h)$, for some $h \in D(E(n))$. Since $B = \{C^j X_p c^l x_Q | j, l \in \{0, 1\}, P, Q \subseteq \{1, \ldots, n\}\}$ is a basis of $D(E(n))$ there exist $u \in U, v \in V = \text{span}\{C X_j c, C c x_j | j = 1, \ldots, n\}$ and $w \in W = \text{span}\{X_j, x_j, C X_j c, C c x_j | j = 1, \ldots, n\}$ such that $h = u + v + w$. Now $(u, v, w) \in \text{span}\{x_1, \ldots, x_n\}$ and $f(w) \in \text{span}\{c^j x_p \} \setminus \{x_1, \ldots, x_n\}$, so, from $x_i = f(u) + f(v) + f(w)$ we deduce that $f(w) = 0$. Taking into account that $f(C X_j c) = -f(X_j)$ and $f(C c x_j) = f(x_j)$, for all $j = 1, \ldots, n$, we see that $f(v) \in f(U)$, hence $x_i \in f(U)$. Thus, $f$ is surjective if and only if $f$ maps $U$ onto $\text{span}\{x_1, \ldots, x_n\}$. In terms of the scalars $a_{ij}$ and $b_{ij}$ this is equivalent to saying that the rank of the $n \times 2n$ matrix $(A|B)$, where $A = (a_{ij})$ and $B = (b_{ij})$, is $n$. This proves the lemma.

Proposition 5.2. The set $\text{L}(\mathcal{C}_n)$ of subcategories of $\mathcal{Z}(\mathcal{C}_n)$ equivalent to $\mathcal{C}_n$ as a tensor category is identified with $\text{Gr}(n, 2n)$, the Grassmannian of $n$-dimensional subspaces of a $2n$-dimensional vector space.

Proof. By virtue of Proposition 2.2, we identify $\text{L}(\mathcal{C}_n)$ with the set of equivalence classes of surjective Hopf algebra maps $D(E(n)) \to E(n)$ under the equivalence relation given by composition with automorphisms of $E(n)$. Using the description of Lemma 5.1 this equivalence relation corresponds to the equivalence relation on $M_{n \times 2n}(k)$ given by left multiplication with invertible $n$-by-$n$ matrices. The quotient set associated to the latter is $\text{Gr}(n, 2n)$. Indeed, if $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times 2n$ matrices of rank $n$ then the rows of $A$, $r_1(A), \ldots, r_n(A)$, and the rows of
\[ D \] take the canonical quasi-triangular structure of \( D \).

Set of equivalence classes of surjective Hopf algebra maps

Proposition 5.4.

L possible dimension of an isotropic subspace).

Under the correspondence of Proposition 2.2,

Proof. Let \((a_1, \ldots, a_r) = (T_1, \ldots, T_r)\)

(34) \[ \omega(a, b) = \sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i), \quad a = (a_1, \ldots, a_{2n}), \ b = (b_1, \ldots, b_{2n}), \]

and let

(35) \[ Sp_{2n}(k) = \{ T \in GL_{2n}(k) \mid \omega(T(a), T(b)) = \omega(a, b) \}, \]

(36) \[ PSp_{2n}(k) = Sp_{2n}(k)/\{ \pm I_{2n} \}. \]

be the symplectic group and the projective symplectic group, respectively. Recall that a subspace \( V \) of \( k^{2n} \) is called isotropic if \( \omega(a, b) = 0 \), for all \( a, b \in V \). An isotropic subspace \( V \) is called Lagrangian if \( \text{dim}_k(V) = n \) (which is the maximal possible dimension of an isotropic subspace).

Recall that for a braided tensor category \( \mathcal{C} \) we denote by \( L_0(\mathcal{C}) \) the set of tensor subcategories of \( Z(\mathcal{C}) \) braided equivalent to \( \mathcal{C} \).

Proposition 5.4. \( L_0(\mathcal{C}) = \text{Lag}(n, 2n) \), the Grassmanian of Lagrangian subspaces of the symplectic space \( (k^{2n}, \omega) \).

Proof. Under the correspondence of Proposition 2.2 \( L_0(\mathcal{C}) \) is identified with the set of equivalence classes of surjective Hopf algebra maps \( D(E(n)) \to E(n) \) that take the canonical quasi-triangular structure of \( D(E(n)) \) to a triangular structure.

Let \( A, B \in M_n(k) \) be such that the two block matrix \( M = (A|B) \) has rank \( n \) and let \( f : D(E(n)) \to E(n) \) be the map given by \( f \). Let \( R = \sum_{i,p} c^i x_p \otimes (c^i x_p)^* \).
be the canonical $R$-matrix of $D(E(n))$. Then, taking into account (33) and using (30), we have:

$$(f \otimes f)(R) = \frac{1}{2} \sum_{i,P} (-1)^{\frac{1 + |P|}{2} + |i|} f((c^i x_P) \otimes f(X_P + (-1)^i CX_P)$$

$$= \frac{1}{2} \sum_{i,E|E|=|F|=|P|} (-1)^{\frac{1 + |P|}{2} + |i|} [A]_{E,P}[B]_{F,P} c^i x_E \otimes (x_F + (-1)^i cx_F)$$

$$= \frac{1}{2} \sum_{|E|=|F|=|P|} (-1)^{|P|} [B]_{F,P} \left( x_E \otimes x_F + x_E \otimes cx_F + (-1)^{|P|+1} cx_E \otimes cx_F \right)$$

$$= \frac{1}{2} \sum_{j=0}^{n} (-1)^{\frac{j(j-1)}{2}} \sum_{|E|=|F|=|P|=j} [A]_{E,P}[B]_{F,P} \left( x_E \otimes x_F + x_E \otimes cx_F + (-1)^{|P|+1} cx_E \otimes cx_F \right)$$

$$= \frac{1}{2} \sum_{j=0}^{n} \left( -1 \right)^{\frac{j(j-1)}{2}} \sum_{|E|=|F|=|P|=j} \left( [A]_{E,P}[B]_{F,P} \right) \left( x_E \otimes x_F + x_E \otimes cx_F + (-1)^{|P|+1} cx_E \otimes cx_F \right)$$

$$= \frac{1}{2} \sum_{j=0}^{n} \left( -1 \right)^{\frac{j(j-1)}{2}} \sum_{|E|=|F|=|P|=j} \left( AB^i \right)_{E,F} \left( x_E \otimes x_F + x_E \otimes cx_F + (-1)^{|P|+1} cx_E \otimes cx_F \right)$$

$$= R_{AB^t}$$

where $B^t$ denotes the transpose matrix of $B$ and where we used the well known formula for the minor of a product of two matrices, $[AB]_{E,F} = \sum_{|P|=|E|} [A]_{E,P}[B]_{P,F}$. Thus, $f$ takes the canonical $R$-matrix of $D(E(n))$ to the $R$-matrix corresponding to $AB^t$. Recall that the latter is a triangular structure if and only if $AB^t$ is symmetric. This is equivalent to $AB^t = BA^t$, or, what is the same, to $\sum_{i=1}^{n} a_i b_j = \sum_{j=1}^{n} b_i a_j$, for all $i, j = 1, \ldots, n$. Subtracting the right hand term in the previous equality from the other, we obtain $\sum_{i=1}^{n} (a_i b_j - b_i a_j) = 0$, for all $i, j = 1, \ldots, n$. In terms of the matrix $M$, if we denote its rows by $r_1(M), \ldots, r_n(M)$, this condition is equivalent to saying that $\omega(r_i(M), r_j(M)) = 0$, for all $i, j = 1, \ldots, n$.

We conclude therefore that the surjective Hopf algebra maps $D(E(n)) \to E(n)$ which take the canonical quasi-triangular structure of $D(E(n))$ to a triangular structure of $E(n)$ correspond to $n \times 2n$ matrices, of rank $n$, with entries from $k$, such that the symplectic form $\omega$ on $k^{2n}$ vanishes on the subspace generated by their rows. Equivalence classes of such maps have, as their correspondent in $\text{Gr}(n, 2n)$, those subspaces on which the symplectic form vanishes, whence the assertion in the statement. \qed
6. Computation of $\text{Aut}^\text{br}((\mathcal{C}_n))$

6.1. $\text{Ext}^1_{(\mathcal{C}_n)}(\chi, \varepsilon)$ as a symplectic space. Recall that $\mathcal{Z}(\mathcal{C}_n) = \text{Rep}(D(E(n)))$ has precisely two invertible objects: the trivial representation $\varepsilon$ and the one-dimensional representation $\chi$ from Lemma 4.6.

**Proposition 6.1.** The space $\text{Ext}^1_{(\mathcal{C}_n)}(\chi, \varepsilon)$ of equivalence classes of extensions of the one-dimensional representation $\chi$ by the trivial representation $\varepsilon$ is isomorphic to $k^{2n}$. The equivalence class corresponding to $a = (a_1, \ldots, a_{2n}) \in k^{2n}$ is the one associated to the extension

$$0 \to \varepsilon \overset{i}{\to} V_a \overset{p}{\to} \chi \to 0$$

where $V_a = k^2$ is the 2-dimensional $D(E(n))$-module with basis $\{v_1 = (1, 0), v_2 = (0, 1)\}$, $D(E(n))$-action given in matrix form by

\[
\begin{align*}
C, c &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
X_i &\mapsto \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, \\
x_i &\mapsto \begin{pmatrix} 0 & a_{n+i} \\ 0 & 0 \end{pmatrix}, i = 1, \ldots, n
\end{align*}
\]

and the maps $i$ and $p$ are such that $i(1) = v_1$ and $p(v_2) = 1$.

**Proof.** Let $V$ be an extension of $\chi$ by $\varepsilon$. Then $V$ comes equipped with two maps $i$ and $p$ such that

$$0 \to \varepsilon \overset{i}{\to} V \overset{p}{\to} \chi \to 0$$

is an exact sequence. Let $v_1 = i(1)$ and choose $v_2 \in V$ such that $p(v_2) = 1$. Then $\{v_1, v_2\}$ is a $k$-basis of $V$ on which the elements of $D(E(n))$ act by $h \cdot v_1 = \varepsilon(h)v_1$ and $h \cdot v_2 = f(h)v_1 + \chi(h)v_2$, for all $h \in D(E(n))$, and for some linear map $f \in \text{Hom}(D(E(n)), k)$. Consider now another extension

$$0 \to \varepsilon \overset{i'}{\to} V' \overset{p'}{\to} \chi \to 0$$

of $\chi$ by $\varepsilon$ and associate to $V'$, as above, a basis $\{v'_1, v'_2\}$ and a linear map $f' \in \text{Hom}(D(E(n)), k)$. We claim that if there exists a homomorphism of extensions $\varphi : V \to V'$ then $f$ and $f'$ differ by a multiple of $\chi - \varepsilon$. Indeed, if $\varphi$ is such a map then, from $\varphi \circ i = i'$ and $p' \circ \varphi = p$, we readily deduce that $\varphi(v_1) = v'_1$ and $\varphi(v_2) = \lambda v'_1 + v'_2$, for some $\lambda \in k$. Letting $h \in D(E(n))$ act on the latter relation and taking into account that $\varphi$ commutes with the action of $D(E(n))$, we arrive at the equality $(f(h) + \lambda \chi(h))v'_1 + \chi(h)v'_2 = (\lambda \varepsilon(h) + f'(h))v'_1 + \chi(h)v'_2$, which shows that $f' - f = \lambda(\chi - \varepsilon)$. In particular, if we take $V' = V$ and $\varphi = \text{id}_V$, we see that the 2$n$-tuple $(f(X_1), \ldots, f(X_n), f(x_1), \ldots, f(x_n))$ does not depend on the choice of $v_2$. Also, the above discussion shows that the same 2$n$-tuple depends only on the equivalence class of $V$. We can, thus, define a map $\text{Ext}^1_{(\mathcal{C}_n)}(\chi, \varepsilon) \to k^{2n}$ sending the equivalence class of $V$ to $(f(X_1), \ldots, f(X_n), f(x_1), \ldots, f(x_n))$. This map is easily seen to be one-to-one and onto, sending the equivalence class of the extension $V_a$, in the statement, to $a \in k^{2n}$. \qed

**Lemma 6.2.** Let

$$0 \to \varepsilon \overset{i}{\to} V_a \overset{p}{\to} \chi \to 0$$

be the extension of $\chi$ by $\varepsilon$ associated to the 2$n$-tuple $a = (a_1, \ldots, a_{2n}) \in k^{2n}$ and let $\{v_1, v_2\}$ be a basis of $V_a$ such that the action of $D(E(n))$ on $V_a$ is given by $[\mathbf{37}]$. Then, as a Yetter-Drinfeld module over $E(n)$, $V_a$ has the action given by the
restriction of action of $D(E(n))$ to the copy of $E(n)$ generated by $c$ and $\{x_i\}$, and coaction given by

$$\rho(v_1) = v_1 \otimes 1 \quad \text{and} \quad \rho(v_2) = \sum_{j=1}^{n} a_j v_1 \otimes x_j + v_2 \otimes c.$$ 

Proof. We only need to verify the formulas for the coaction. This is given by

$$\rho(v) = \sum_{i,p} (c^i x_P)^* \cdot v \otimes c^i x_P$$

for all $v \in V_n$, so, taking into account (30) and the following easy to check relations

$$(C^i X_P) \cdot v_1 = \begin{cases} v_1 & \text{if } P = \emptyset \\ 0 & \text{if } P \neq \emptyset \end{cases} \quad \text{and} \quad (C^i X_P) \cdot v_2 = \begin{cases} (-1)^i v_2 & \text{if } P = \emptyset \\ a_j v_1 & \text{if } P = \{j\} \\ 0 & \text{if } |P| \geq 2 \end{cases}$$

the verification is straightforward. \qed

Lemma 6.3. Let $V_n$ be the underlying object of an extension of $\chi$ by $\varepsilon$. Then $V_n$ centralizes $\chi$, i.e., the squared braiding $c_{V_n,\chi} \circ c_{\chi,V_n}$ is the identity morphism.

Proof. As a Yetter-Drinfeld module over $E(n)$, $\chi$ has the module structure given by $c \cdot 1 = -1$, $x_i \cdot 1 = 0$, for all $i = 1, \ldots, n$, and the comodule structure $1 \mapsto 1 \otimes c$. Let $\{v_1, v_2\}$ be a basis of $V_n$ such that the action of $D(E(n))$ on $V_n$ is given by (37). Then, from Lemma \[6.2\] and \[6.3\], we deduce that

$$c_{V_n,\chi} \circ c_{\chi,V_n}(1 \otimes v_1) = c_{V_n,\chi}(v_1 \otimes 1) = 1 \otimes c \cdot v_1 = 1 \otimes v_1$$

and

$$c_{V_n,\chi} \circ c_{\chi,V_n}(1 \otimes v_2) = c_{V_n,\chi} \left( \sum_{j=1}^{n} a_j v_1 \otimes x_j \cdot 1 + v_2 \otimes c \cdot 1 \right)$$

$$= -c_{V_n,\chi}(v_2 \otimes 1)$$

$$= -1 \otimes c \cdot v_2$$

$$= 1 \otimes v_2$$

whence the claim. \qed

Corollary 6.4. Let $\tau \in \text{Aut}_{br}(\mathcal{Z}(\mathcal{C}_n))$ be the image of the non-trivial element of $\text{Pic}(\text{sVec}) \cong \mathbb{Z}/2\mathbb{Z}$ under the composition

$$\text{Pic}(\text{sVec}) \xrightarrow{\text{Ind}_{\text{vec}(\mathcal{C}_n)}} \text{Pic}(\mathcal{Z}(\mathcal{C}_n)) \cong \text{Aut}_{br}(\mathcal{Z}(\mathcal{C}_n)).$$

Then $\tau(V_n) \cong V_n$.

Proof. This follows from Proposition \[6.2\] and Lemma \[6.3\]. \qed

Proposition 6.5. Let $V_a$ and $V_b$ be two extensions of $\chi$ by $\varepsilon$ associated to $a = (a_1, \ldots, a_{2n})$ and $b = (b_1, \ldots, b_{2n})$ and suppose that $\{v_1, v_2\}$ and $\{v_1', v_2'\}$ are bases of $V_a$ and $V_b$, respectively, on which $D(E(n))$ acts as in (37). Then the matrix of $c_{V_b,V_a} \circ c_{V_a,V_b} : V_a \otimes V_b \to V_b \otimes V_a$ in the basis $\{v_1 \otimes v_1', v_1 \otimes v_2', v_2 \otimes v_1', v_2 \otimes v_2'\}$ is

$$\begin{pmatrix}
1 & 0 & 0 & \omega(b,a) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$
Proposition 6.6. The assignment $\mathcal{L} \to \mathcal{L} \cap \text{Ext}_{Z(C_n)}^1(\chi, \varepsilon)$ is a bijection between $L_0(C_n)$ and $\text{Lag}(n, 2n)$.

Proof. We saw in Proposition 3.3 that $L_0(C_n) = \text{Lag}(n, 2n)$. Let $U \in \text{Lag}(n, 2n)$ and $A = (a_{ij}), B = (b_{ij})$ be $n$-by-$n$ matrices such that the rows of $M = (A|B) \in M_{n \times 2n}(k)$, $r_1(M), \ldots, r_n(M)$, form a basis of $U$. Then the subcategory of $Z(C_n)$, braided equivalent to $C_n$, corresponding to $U$ is $\mathcal{L}_U$, where $\mathcal{L}_U$ is the image of the restriction functor associated to $f: D(E(n)) \to E(n)$, $f(C) = f(c) = c$, $f(X_i) = \sum_{j=1}^n a_{ij}x_j$, $f(x_i) = \sum_{j=1}^n b_{ij}x_j$, $i = 1, \ldots, n$. We will show that, under the isomorphism of Proposition 6.1, $\mathcal{L}_U \cap \text{Ext}_{Z(C_n)}^1(\chi, \varepsilon) = U$, which will prove the claim.

Let

$$0 \to \varepsilon \xrightarrow{i} V \xrightarrow{p} \chi \to 0$$

be an element of $\mathcal{L}_U \cap \text{Ext}_{Z(C_n)}^1(\chi, \varepsilon)$ and let $\{v_1, v_2\}$ be a basis of $V$ such that $v_1 = i(1)$ and $p(v_2) = 1$. If $\{v_1^i, v_2^i\}$ is the dual basis of $\{v_1, v_2\}$ then the element of $k^{2n}$ corresponding to $V$, under the isomorphism of Proposition 6.1 is

$$a_V = (v_1^i(X_1 \cdot v_2), \ldots, v_1^i(x_n \cdot v_2), v_1^i(x_1 \cdot v_2), \ldots, v_1^i(X_1 \cdot v_2))$$

Since $v_1^i(X_1 \cdot v_2) = v_1^i(f(X_1)v_2) = v_1^i(\sum_j a_{ij}x_j v_2) = \sum_j a_{ij}v_1^i(x_j v_2)$ and $v_1^i(x_1 \cdot v_2) = v_1^i(f(x_1)v_2) = v_1^i(\sum_j b_{ij}x_j v_2) = \sum_j b_{ij}v_1^i(x_j v_2)$, for all $i = 1, \ldots, n$, we deduce that

$$a_V = \sum_{j=1}^n v_1^i(x_j v_2)(a_{j1}, \ldots, a_{jn}, b_{j1}, \ldots, b_{jn}) = \sum_{j=1}^n v_1^i(x_j v_2)r_j(M)$$

Thus, $a_V \in U$, for all $V \in \mathcal{L}_U \cap \text{Ext}_{Z(C_n)}^1(\chi, \varepsilon)$. To complete the proof we need only show that $V_{r_i(M)} \in \mathcal{L}_U$, for all $i = 1, \ldots, n$. A quick check shows that the
representation $V_{r,(M)}$ is obtained from the following matrix representation of $E(n)$:

$$E(n) \rightarrow M_d(k), \quad c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_j \mapsto \begin{pmatrix} 0 & \delta_{i,j} \\ 0 & 0 \end{pmatrix}, \quad j = 1, \ldots, n$$

by restriction of scalars via $f$. □

6.2. Representation of $\text{Aut}^{br}(Z(C_n))$ on $\text{Ext}^1_{Z(C_n)}(\varepsilon, \chi)$. Recall from Proposition 2.12 that there is a group homomorphism

$$\rho : \text{Aut}^{br}(Z(C_n)) \rightarrow PGL(\text{Ext}^1_{Z(C_n)}(\varepsilon, \chi)) = PGL_{2n}(k).$$

Note that the projective symplectic group $P Sp_{2n}(k) = Sp_{2n}(k)/\pm \mathbb{I}_{2n}$ can be viewed as a subgroup of $PGL_{2n}(k)$.

**Proposition 6.7.** The image of the group homomorphism $\rho$ belongs to $P Sp_{2n}(k)$.

**Proof.** Let

$$0 \rightarrow \varepsilon \xrightarrow{\iota} V_a \xrightarrow{p} \chi \rightarrow 0$$

$$0 \rightarrow \varepsilon \xrightarrow{\iota'} V_{a'} \xrightarrow{p'} \chi \rightarrow 0$$

be a pair of elements in $\text{Ext}^1_{Z(C_n)}(\chi, \varepsilon)$. By Proposition 6.5, we have

$$c_{V_a', V_a} \circ c_{V_a, V_{a'}} = \text{id}_{V_a \otimes V_{a'}} + \omega(a, a') (p \otimes p') \circ (i \otimes i').$$

Here $p \otimes p'$ is a morphism from $V_a \otimes V_{a'}$ to $\chi \otimes \chi = \varepsilon$. For $\alpha \in \text{Aut}^{br}(Z(C_n))$ let $\alpha(a) \in \text{Ext}^1_{Z(C_n)}(\chi, \varepsilon)$ be such that $\alpha(V_a) = V_{\alpha(a)}$. Applying $\alpha$ to both sides of (39) we obtain

$$\alpha(c_{V_a', V_a} \circ c_{V_a, V_{a'}}) = \text{id}_{V_{\alpha(a)} \otimes V_{\alpha(a')}} + \omega(a, a') \alpha((p \otimes p') \circ (i \otimes i')).$$

On the other hand,

$$\alpha(c_{V_a', V_a} \circ c_{V_a, V_{a'}})$$

$$= J_{V_a', V_a} \circ (c_{\alpha(a'), V_{\alpha(a)}} \circ c_{V_{\alpha(a)}, V_{\alpha(a')}}) \circ J_{V_a, V_{a'}}^{-1}$$

$$= \text{id}_{V_{\alpha(a)} \otimes V_{\alpha(a')}} + \omega(\alpha(a), \alpha(a')) J_{V_a, V_{a'}} \circ (\alpha(p) \otimes \alpha(p')) \circ (\alpha(i) \otimes \alpha(i')) \circ J_{V_a, V_{a'}}^{-1},$$

where $J_{X,Y} : \alpha(X) \otimes \alpha(Y) \xrightarrow{\sim} \alpha(X \otimes Y)$ denotes the tensor functor structure of $\alpha$.

Thus, $\omega(\alpha(a), \alpha(a')) = \omega(a, a')$, as required. □

Recall the homomorphisms

$$\iota_1 : \text{Aut}_{\text{Hopf}}(H) \rightarrow \text{Aut}^{br}(Z(\text{Rep}(H))),$$

$$\iota_2 : H^2_{\text{inv}}(H) \rightarrow \text{Aut}^{br}(Z(\text{Rep}(H))),$$

$$\iota_3 : H^2_{\text{inv}}(H^*) \rightarrow \text{Aut}^{br}(Z(\text{Rep}(H))).$$

from Remark 3.10. In the next Proposition we compute the images of the compositions of these homomorphisms with $\rho$.

**Proposition 6.8.** We have

(i) $\rho \circ \iota_1 (\text{Aut}_{\text{Hopf}}(E(n))) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in GL_{2n}(k) \right\}$,

(ii) $\rho \circ \iota_2 (H^2_{\text{inv}}(E(n))) = \left\{ \begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix} \mid B = B^t \right\}$,

(iii) $\rho \circ \iota_3 (H^2_{\text{inv}}(E(n)^*)) = \left\{ \begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix} \mid B = B^t \right\}$. 

Here each matrix denotes the class in $PSp_{2n}(k)$.

Proof. (i) This is clear in view of Proposition 4.3.

(ii) Let $\sigma_M \in (E(n) \otimes E(n))^\ast$ be the invariant 2-cocycle associated to $M = (m_{ij}) \in \text{Sym}_n(k)$, $\Gamma_{\sigma_M}$ the autoequivalence of $\text{Rep}(D(E(n)))$ induced by $\sigma_M$ and $V_a \in \text{Ext}^1_{\mathcal{Z}(\mathcal{C}_n)}(\chi, \varepsilon)$, where $a = (a_1, \ldots, a_{2n}) \in k^{2n}$. Viewing $V_a$ as a Yetter-Drinfeld module over $E(n)$, by Lemma 6.2 and regarding $\Gamma_{\sigma_M}$ as an autoequivalence of $E(n)\mathcal{YD}^{E(n)}$, we have, according to Example 4.9 that $\Gamma_{\sigma_M}(V_a) = V_a$ as an $E(n)$-comodule, with the $E(n)$-module structure given by

$$h \cdot v = \sigma_M^{-1}(h)(h(2) \cdot z(0))(1) \otimes h(1) \sigma_M(h(3) \otimes z(1))(h(2) \cdot z(0))(0), \quad h \in E(n), v \in V_a.$$  

Let $\{v_1, v_2\}$ be a basis of $V_a$ such that the $D(E(n))$-module structure of $V_a$ is given by (37). Then, a straightforward computation shows that, $h \cdot v_1 = \varepsilon(h)v_1$, for all $h \in E(n)$, $c \cdot v_2 = -v_2$ and

$$x_i \cdot v_2 = a_{n+j} + \sum_{i=1}^n (m_{ij} + m_{ji})a_i$$

for all $i = 1, \ldots, n$. Thus, $\Gamma_{\sigma_M}(V_a) = V_a'$, where

$$a^n = \begin{pmatrix} I_n & 0 \\ M + Mt & I_n \end{pmatrix} a^t$$

and the result follows.

(iii) Let $M = (m_{ij}) \in \text{Sym}_n(k)$ and let

$$J_M = \frac{1}{4} \sum_{i,j \neq P,Q} \sigma_M(c^i x_P \otimes c^j x_Q)(x_P + (-1)^i cx_P) \otimes (x_Q + (-1)^j cx_Q)$$

be the invariant twist associated to $M$. Observe that

$$J_M = 1 \otimes 1 + \sum_{j,l=1}^n m_{jl} x_j \otimes cx_l + L$$

where $L$ is a linear combination of $c^i x_P \otimes c^j x_Q$, with $i, j \in \{0, 1\}$ and $P, Q \subseteq \{1, \ldots, n\}$ such that $|P| \geq 2$ or $|Q| \geq 2$. Let $V_a \in \text{Ext}^1_{\mathcal{Z}(\mathcal{C}_n)}(\chi, \varepsilon)$ and $\Gamma_{J_M}$ be the autoequivalence of $\text{Rep}(D(E(n)))$ induced by $J_M$. As a Yetter-Drinfeld module over $E(n)$, $\Gamma_{J_M}(V_a)$ is $V_a$ as an $E(n)$-module, with the comodule structure given by

$$\rho_{J_M}(v) = (J_M^{-1})^2 \cdot (J_M^{-1} \cdot v)(0) \otimes (J_M^{-1})^2 \cdot (J_M^{-1} \cdot v)(1) J_M^2,$$

for all $v \in V_a$. If $\{v_1, v_2\}$ is a basis for $V_a$ such that the action of $D(E(n))$ on $V_a$ is given by (37), then one can easily check, using Lemma 6.2 that $\rho_{J_M}(v_1) = v_1 \otimes 1$ and

$$\rho_{J_M}(v_2) = \sum_{i=1}^n a_i v_1 \otimes x_i + \sum_{i=1}^n \left( \sum_{j=1}^n a_{n+j}(m_{ij} + m_{ji}) \right) v_1 \otimes cx_i + v_2 \otimes c.$$  

Taking into account that the induced $E(n)^\ast$-module structure of $\Gamma_{J_M}(V_a)$ is $f \cdot v = \sum f(v(1))v(0)$, for all $f \in E(n)^\ast$ and $v \in \Gamma_{J_M}(V_a)$ we readily deduce the $D(E(n))$-module structure of $\Gamma_{J_M}(V_a)$. We have $C \cdot v_1 = v_1, C \cdot v_2 = -v_2, X_1 \cdot v_1 = 0$.
and

$$X_i \cdot v_2 = (x_i^* - (cx_i)^*) \cdot v_2 = \left(a_i - \sum_{j=1}^n a_{n+j}(m_{ij} + m_{ji})\right) v_1.$$ 

Thus, $\Gamma_J M(V_a) = V_{a'}$, where

$$a' = \left( \begin{array}{cc} I_n & -(M + M^t) \\ 0 & I_n \end{array} \right) a^t$$

which concludes the proof. □

**Corollary 6.9.** The image of homomorphism (38) is $PSp_{2n}(k)$.

**Proof.** The three subgroups from Proposition 6.8 generate $PSp_{2n}(k)$, so the statement follows from Proposition 6.7. □

Recall from Corollary 6.4 that $\tau \in Aut^{br}(Z(C_n))$ denotes the autoequivalence induced from the non-trivial element of $Pic(sVec)$, where $sVec$ is viewed as a tensor subcategory of $Z(C_n)$.

**Proposition 6.10.** (i) The kernel of homomorphism (38) is $\langle \tau \rangle$ (and so is isomorphic to $Z/2Z$).

(ii) The restriction of homomorphism (38) on the subgroup of $Aut^{br}(Z(C_n))$ generated by the images of $Aut_{Hopf}(E(n))$, $H^2_{inv}(E(n))$, $H^2_{inv}(E(n)^*)$ is injective.

**Proof.** (i) By Theorem 3.4 and Proposition 6.6 the kernel of $\rho$ is a subgroup of $Pic(C_n) = Pic_0(C_n) \times \langle \tau \rangle$. By Corollary 6.4 the subgroup $\langle \tau \rangle \subset Pic(C_n)$ acts trivially on the (projective) Ext space and so belongs to the kernel of $\rho$.

In Section 4.4 we discussed an isomorphism between $H^2_{inv}(E(n)^*)$ and $Pic_0(C_n)$ (both groups are isomorphic to $Sym_n(k)$). Combining this with Proposition 6.8(iii) we deduce that the kernel of $\rho$ coincides with $\langle \tau \rangle$.

(ii) Every matrix $M \in Sp_{2n}(k)$ can be uniquely written as $M = XYZ$, where $X, Y, Z$ are matrices from parts (i), (ii), and (iii) of Proposition 6.8 respectively. This implies the injectivity statement. □

**Theorem 6.11.** We have

(40) $Aut^{br}(Z(C_n)) \simeq PSp_{2n}(k) \times Z/2Z$.

The action of $Aut^{br}(Z(C_n))$ on $L_0(C_n)$ corresponds to the action of $PSp_{2n}(k)$ on the Lagrangian Grassmannian $Lag(n, 2n)$.

**Proof.** That $Z/2Z = \langle \tau \rangle$ is a central subgroup of $Aut^{br}(Z(C_n))$ and that the quotient by it is isomorphic to $PSp_{2n}(k)$ follows from Corollary 6.9 and Proposition 6.10(i). Thus we have a central extension

(41) $1 \to \langle \tau \rangle \to Aut^{br}(Z(C_n)) \to PSp_{2n}(k) \to 1$.

This extension splits by Proposition 6.10(ii).

Finally, the Lagrangian equivariance follows from Proposition 6.6. □

**Corollary 6.12.** $BrPic(C_n) \cong PSp_{2n}(k) \times Z/2Z$. 


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