ALGEBRAIC HULLS OF SOLVABLE GROUPS AND EXPONENTIAL ITERATED INTEGRALS ON SOLVMANIFOLDS

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Abstract. We represent the coordinate ring of algebraic hulls (which are generalizations of the Malcev completions of nilpotent groups for solvable groups) of solvmanifolds $G/\Gamma$ by using Miller’s exponential iterated integrals (which are extensions of Chen’s iterated integrals) of invariant differential forms.

1. Introduction

Let $G$ be a simply connected Lie group and $\mathfrak{g}$ the Lie algebra of $G$. We consider the space $\bigwedge \mathfrak{g}^*_C$ of $C$-valued left $G$-invariant differential forms on $G$. We assume that $G$ has a lattice (i.e., cocompact discrete subgroup) $\Gamma$. We consider the compact homogeneous space $G/\Gamma$ and $\bigwedge \mathfrak{g}^*_C$ as a subcomplex of the de Rham complex $A^*_C(G/\Gamma)$ of $G/\Gamma$. Suppose $G$ is nilpotent. Then we have the unique unipotent algebraic group $U_\Gamma$ called the Malcev completion of $\Gamma$ such that there is a injection $\Gamma \rightarrow U_\Gamma$ with the Zariski-dense image. We can represent the coordinate ring of $U_\Gamma$ by using Chen’s iterated integrals on $G/\Gamma$ (see [2]). Since the inclusion $\bigwedge \mathfrak{g}^*_C \subset A^*_C(G/\Gamma)$ induces a cohomology isomorphism by Nomizu’s theorem [11], $\bigwedge \mathfrak{g}^*_C$ is the Sullivan minimal model of $A^*_C(G/\Gamma)$ (see [4]). This implies $H^0(B(\bigwedge \mathfrak{g}^*_C)) \cong H^0(B(A^*_C(G/\Gamma)))$ where $B(\bigwedge \mathfrak{g}^*_C)$ and $B(A^*_C(G/\Gamma))$ are the reduced bar constructions of $\bigwedge \mathfrak{g}^*_C$ and $A^*_C(G/\Gamma)$ respectively (see [3]). Hence we can represent the coordinate ring of $U_\Gamma$ by using Chen’s iterated integrals of left-invariant forms.

Suppose $G$ is solvable. Then Chen’s iterated integrals on $G/\Gamma$ does not give sufficient information on the fundamental group of $G/\Gamma$. For example, let $G = \mathbb{R} \ltimes \phi \mathbb{R}^2$ such that $\phi(t) = \left( e^t \quad 0 \quad 0 \right)$. Then $G$ has a lattice $\Gamma$ and the inclusion $\bigwedge \mathfrak{g}^*_C \subset A^*_C(G/\Gamma)$ induces a cohomology isomorphism (see [5]). Since we have $H^1(G/\Gamma, \mathbb{C}) = H^1(\bigwedge \mathfrak{g}^*_C) = \mathbb{C}$, by Chen’s results, iterated integrals represent the coordinate ring of an additive algebraic group $G_{rad} = \mathbb{C}$ (see [7]). But since $\Gamma$ is solvable and not abelian, we can’t embed $\Gamma$ in $G_{rad}$.

In [10], as an extension of the Malcev completion, Mostow constructed a Zariski-dense embedding of $\Gamma$ in an algebraic group $H_\Gamma$ called algebraic hull of $\Gamma$. In [7], Miller gave extensions of Chen’s iterated integrals called exponential iterated integrals. In this paper we represent the coordinate ring of $H_\Gamma$ by using Miller’s exponential iterated integrals of left-invariant differential forms on $G/\Gamma$.

2. Relative completions and algebraic hulls

Let $G$ be a discrete group (resp. a Lie group). We call a map $\rho : G \rightarrow GL_n(\mathbb{C})$ a representation, if $\rho$ is a homomorphism of groups (resp. Lie groups). In this paper

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we denote by $T_n(C)$ the group of $n \times n$ upper triangular matrix and denote by $U_n(C)$ the group of $n \times n$ upper triangular unipotent matrix.

2.1. **algebraic groups and pro-algebraic groups.** In this paper an algebraic group means an affine algebraic variety $G$ over $C$ with a group structure such that the multiplication and inverse are morphisms of varieties. All algebraic groups arise as Zariski-closed subgroups of $GL_n(C)$. A pro-algebraic group is an inverse limit of algebraic groups. If a pro-algebraic group is an inverse limit of unipotent algebraic groups, it is called pro-unipotent. Let $G$ be a pro-algebraic group. We denote by $U(G)$ the maximal pro-unipotent normal subgroup called the pro-unipotent radical.

If $U(G) = e$, $G$ is called reductive. We denote by $C[G]$ the coordinate ring of $G$. The group structure on $G$ induces a Hopf algebra structure on $C[G]$. It is known that we have the anti-equivalence between algebras and affine varieties induces an anti-equivalence between pro-algebraic groups and reduced Hopf algebras.

**Theorem 2.1.** ([9], [6]) Let $G$ be a pro-algebraic group. Then the exact sequence

$$
1 \longrightarrow U(G) \longrightarrow G \longrightarrow G/U(G) \longrightarrow 1
$$

splits.

Let $G$ be a discrete group or Lie group. We denote by $A(G)$ the inverse limit of all representations $\phi : G \rightarrow G$ with Zariski-dense images. We call the pro-unipotent radical $U(A(G))$ of $A(G)$ the unipotent hull of $G$ and denote it $U_G$.

2.2. **Relative completion.** Let $\rho : G \rightarrow S$ be a representation of $G$ to a diagonal algebraic group $S$ with the Zariski-dense image. Let $\phi : G \rightarrow G$ be a representation of $G$ to an algebraic group $G$ with the Zariski-dense image. We call $\phi$ a $\rho$-relative representation if we have the commutative diagram

$$
1 \longrightarrow U(G) \longrightarrow G \longrightarrow S \longrightarrow 1.
$$

If $S$ is contained in an algebraic torus, for any $\rho$-relative representation $\phi : G \rightarrow G$ there exists a faithful representation $G \rightarrow T_n(C)$ such that $G \cap U_n(C) = U(G)$ (see [7]).

We denote by $\mathcal{G}_\rho(G)$ the inverse limit of $\rho$-relative representations $\phi_i : G \rightarrow G_i$. We call $\mathcal{G}_\rho(G)$ the $\rho$-relative completion of $G$. If $S$ is trivial, $\mathcal{G}_\rho(G)$ is the classical Malcev (or unipotent) completion.

2.3. **Algebraic hulls.** We define the algebraic hulls of polycyclic groups (resp. Lie groups) which constructed in [10].

A group $\Gamma$ is polycyclic if it admits a sequence

$$
\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}
$$

of subgroups such that each $\Gamma_i$ is normal in $\Gamma_{i-1}$ and $\Gamma_{i-1}/\Gamma_i$ is cyclic. For a polycyclic group $\Gamma$, we denote $\text{rank} \Gamma = \sum_{i=1}^k \text{rank} \Gamma_{i-1}/\Gamma_i$. Let $G$ be a simply connected solvable Lie group and $\Gamma$ be a lattice of $G$. Then $\Gamma$ is torsion-free polycyclic and $\dim G = \text{rank} \Gamma$.

Let $G$ be a simply connected solvable Lie group or torsion-free polycyclic group. Consider the algebraic completion $A(G)$. Then it is known that the unipotent hull
\( U_G = U(A(G)) \) is finite dimensional (see [10]). By Theorem 2.1, we have a splitting \( A(G) = (A(G)/U_G) \ltimes \phi U_G. \) Let \( K \) be the kernel of the action \( \phi : (A(G)/U_G) \to \text{Aut}(U_G). \) Then \( K \) is the maximal reductive normal subgroup of \( A(G) \) (see [10]). Denote \( H_G = A(G)/K \) and call it the algebraic hull of \( G. \)

**Theorem 2.2.** ([10], [13]) Let \( G \) be a simply connected solvable Lie group (resp. torsion-free polycyclic group). Then \( G \to H_G \) is injective and \( H_G \) is finite dimensional algebraic group such that:

1. \( \dim U(H_G) = \dim G \) (resp. = rank \( G \)).
2. The centralizer of \( U(H_G) \) in \( H_G \) is contained in \( U(H_G) \).

Conversely if an algebraic group \( H \) with an injective homomorphism \( \psi : G \to H \) with the Zariski-dense image satisfies the properties (1) and (2), then \( H \) is isomorphic to \( H_G \).

### 2.4. Direct constructions of algebraic hulls.

The idea of this subsection is based on classical works of semi-simple splitting (see [11], [12] and the references given there). Let \( g \) be a solvable Lie algebra, and \( n = \{ X \in g | \text{ad}_X \text{ is nilpotent} \} \). \( n \) is the maximal nilpotent ideal of \( g \) and called the nilradical of \( g \). Then we have \( [g, g] \subset n \). Let \( D(g) \) be the Lie algebra of derivations of \( g \). By the Jordan decomposition, we have \( \text{ad}_X = \text{ad}_{sX} + \text{ad}_{nX} \) such that \( \text{ad}_{sX} \) is a semi-simple operator and \( \text{ad}_{nX} \) is a nilpotent operator. Since we have \( d_X, n_X \in D(g) \), we have the map \( \text{ad}_s : g \to D(g) \). Since \( \text{ad} \) is trigonalizable (Lie’s theorem), this map is homomorphism with the kernel \( n \). Let \( \bar{\text{g}} = \text{Im} \text{ad}_s \ltimes g \). and \( \bar{n} = \{ X - \text{ad}_{sX} \in \bar{\text{g}} | X \in g \} \).

**Proposition 2.3.** \( \bar{n} \) is a nilpotent ideal of \( \bar{\text{g}} \) and we have a decomposition \( \bar{\text{g}} = \text{Im} \text{ad}_s \ltimes \bar{n}. \)

**Proof.** By \( \text{ad}_X - \text{ad}_{sX} = \text{ad}_X - \text{ad}_{sX} \) on \( g \), \( \text{ad}_X - \text{ad}_{sX} \) is a nilpotent operator and hence \( \bar{n} \) consists of nilpotent elements. By Lie’s theorem, we have a basis

\[
X_1, \ldots, X_l, X_{l+1}, \ldots, X_n
\]

d of \( g \otimes \mathbb{C} \) such that \( \text{ad} \) is represented by upper triangular matrices. Since the nilradical \( n \) is an ideal, \( n \otimes \mathbb{C} \) is ad-invariant subspace of \( g \otimes \mathbb{C} \). We choose \( X_1, \ldots, X_l \) a basis of \( n \otimes \mathbb{C} \). By \( [g, g] \subset n \), we have \( \text{ad}_X(g \otimes \mathbb{C}) \subset n \otimes \mathbb{C} = \langle X_1, \ldots, X_l \rangle \), and hence \( \text{ad} \) represented as

\[
\text{ad}_X = \begin{pmatrix}
 a_{11}(X) & \cdots & \cdots & a_{1l}(X) \\
 \vdots & \ddots & & \vdots \\
 a_{l1}(X) & \cdots & a_{ll}(X) & 0 \\
 0 & \cdots & 0 & \ddots \\
 \vdots & & \vdots & \ddots \\
 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

Thus we have \( \text{ad}_{sX}(X_i) = a_{11}(X)X_i \) for \( 1 \leq i \leq l \) and \( \text{ad}_{sX}(X_i) = 0 \) for \( l + 1 \leq i \leq n \). By this we have

\[
[X_i + \text{ad}_{sY}, X_j + \text{ad}_{sZ}] \in \langle X_1, \ldots, X_l \rangle = n \otimes \mathbb{C}
\]

for any \( 1 \leq i, j \leq n, Y, Z \in g \). This implies \( [\bar{\text{g}}, \bar{\text{g}}] \subset n \). By \( n \subset \bar{n} \), \( \bar{n} \) is an ideal of \( \bar{\text{g}} \) and we have \( \bar{\text{g}} = \{ \text{ad}_{sX} + Y - \text{ad}_{sY} | X, Y \in g \} = \text{Im} \text{ad}_s \ltimes \bar{n}. \) \( \square \)
By this proposition, we have the inclusion \( i : \mathfrak{g} \to D(\bar{n}) \times \bar{n} \) given by \( i(X) = \text{ad}_{\mathfrak{X}}X + X - \text{ad}_{\mathfrak{X}}X \) for \( X \in \mathfrak{g} \).

Let \( G \) be a simply connected solvable Lie group and \( \mathfrak{g} \) be the Lie algebra of \( G \). For the adjoint representation \( \text{Ad} : G \to \text{Aut}(\mathfrak{g}) \), we consider the semi-simple part \( \text{Ad}_s : G \to \text{Aut}(\mathfrak{g}) \) as similar to the Lie algebra case. Denote by \( T \) the universal covering of \( \text{Ad}_s(G) \). Let \( \bar{N} \) be the simply connected Lie group which corresponds to \( \bar{n} \). Then by Proposition 2.4 we have \( T \times G = T \times \bar{N} \). By the proof of this proposition, the action \( T \to \text{Aut}(\bar{n}) \) is the extension of the action of \( \text{Im}(\text{ad}_s) \). Hence we have \( \text{Ad}_s(G) \times G = \text{Ad}_s(G) \times \bar{N} \).

A simply connected nilpotent Lie group is considered as the real points of a unipotent \( \mathbb{R} \)-algebraic group (see [12, p. 43]) by the exponential map. We have the universal \( \mathbb{R} \)-algebraic group \( \bar{N} \) with \( \bar{N}(\mathbb{R}) = \bar{N} \). We identify \( \text{Aut}_a(\bar{N}) \) with \( \text{Aut}(\mathfrak{n}_G) \) and \( \text{Aut}_a(\bar{N}) \) has the \( \mathbb{R} \)-algebraic group structure with \( \text{Aut}_a(\bar{N})(\mathbb{R}) = \text{Aut}(N) \). So we have the \( \mathbb{R} \)-algebraic group \( \text{Aut}_a(\bar{N}) \times \bar{N} \). Then by \( \text{Ad}_s(G) \times G = \text{Ad}_s(G) \times \bar{N} \subset \text{Aut}_a(\bar{N}) \times \bar{N} \), we consider the Zariski-closure \( \mathbb{G} \) of \( G \) in \( \text{Aut}_a(\bar{N}) \times \bar{N} \). Since \( \text{Ad}_s(G) \) is a group of diagonal automorphisms, we have \( U(\mathbb{G}) = \bar{N} \). By \( \dim \mathbb{G} = \dim \bar{N} \), we can easily check that \( \mathbb{G} \) satisfies the properties (1), (2) in Theorem 2.2 and hence it is the algebraic hull \( \mathbb{H}_G \) of \( G \). Hence the inclusion \( i : \mathfrak{g} \to D(\bar{n}) \times \bar{n} \) induces the algebraic hull \( I : G \to \mathbb{H}_G \) of \( G \). Since \( i : \mathfrak{g} \to D(\bar{n}) \times \bar{n} \) is induced by the Lie algebra homomorphism \( \text{ad} : \mathfrak{g} \to D(\bar{n}) \times \bar{n} \), the composition \( G \to \mathbb{H}_G \to \mathbb{H}_G/U(\mathbb{H}_G) \) is induced by the Lie algebra homomorphism \( \text{ad}_s : \mathfrak{g} \to D(\mathfrak{g}) \) by \( U(\mathbb{G}) = \bar{N} \). Thus we have the following lemma.

**Lemma 2.4.** The algebraic hull \( G \to \mathbb{H}_G \) is an \( \text{Ad}_s \)-relative representation.

**2.5. Algebraic hulls and relative completions of solvable groups.**

**Theorem 2.5.** Let \( G \) be a simply connected Lie group. Then the algebraic hull \( \mathbb{H}_G \) is isomorphic to the \( \text{Ad}_s \)-relative completion \( S_{\text{Ad}_s}(G) \) of \( G \).

**Proof.** Consider a commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}' & \xrightarrow{\Phi} & \mathbb{H}_G \\
\downarrow & & \downarrow \\
G & \to & \mathbb{H}_G
\end{array}
\]

for some \( \text{Ad}_s \)-relative representation \( G \to \mathbb{H}' \). Since \( G \to \mathbb{H}' \) and \( G \to \mathbb{H}_G \) have Zariski-dense images, \( \Phi : \mathbb{H}' \to \mathbb{H}_G \) is surjective and the restriction \( \Phi : U(\mathbb{H}') \to U(\mathbb{H}_G) \) is also surjective. By \( U(\mathbb{H}_G) = U_G \), \( \Phi : U(\mathbb{H}') \to U(\mathbb{H}_G) \) is an isomorphism. Since \( G \to \mathbb{H}' \) and \( G \to \mathbb{H}_G \) are \( \text{Ad}_s \)-relative representations, \( \Phi \) induces the isomorphism \( \mathbb{H}'/U(\mathbb{H}') \to \mathbb{H}_G/U(\mathbb{H}_G) \). Hence \( \Phi : \mathbb{H}' \to \mathbb{H}_G \) is an isomorphism. By the definition of \( \text{Ad}_s \)-relative completion of \( G \), we have the theorem.

**Theorem 2.6.** Let \( G \) be a simply connected solvable Lie group and \( \Gamma \) a lattice of \( G \). Then the algebraic hull \( \mathbb{H}_\Gamma \) of \( \Gamma \) is isomorphic to \( \text{Ad}_s(\Gamma) \)-relative completion \( S_{\text{Ad}_s(\Gamma)}(\Gamma) \) of \( \Gamma \).

**Proof.** For the algebraic hull \( \psi : G \to \mathbb{H}_G \) of \( G \), we consider the Zariski-closure of \( \psi(\Gamma) \) in \( \mathbb{H}_G \). Then by \( \dim G = \text{rank} \Gamma \) we can easily check that this algebraic group satisfies (1) and (2) in Theorem 2.2 and hence it is the algebraic hull \( \mathbb{H}_\Gamma \) of \( \Gamma \). By
the above theorem, $\Gamma \to \mathbf{H}_\Gamma$ is a $\text{Ad}_{\mathbf{ad}}$-relative representation. As similar to the above proof, we have the theorem.

\section{Exponential iterated integral on solvmanifolds}

In this section we consider Miller’s exponential iterated integrals. Let $M$ be a $C^\infty$-manifold and $\Omega_x M$ be a space of piecewise smooth loops $\lambda : [0, 1] \to M$ with $\lambda(0) = x$. For 1-forms $\omega_1, \ldots, \omega_n \in A^1_\mathbb{C}(M)$, the iterated integral $\int \omega_1 \omega_2 \cdots \omega_n : \Omega_x M \to \mathbb{C}$ is defined by

$$
\int_\lambda \omega_1 \omega_2 \cdots \omega_n = \int_{0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1} F(t_1) F(t_2) \cdots F(t_n) dt_1 dt_2 \cdots dt_n \lambda \in \Omega_x M
$$

where $F(t) dt = \lambda^* \omega_1 \in A^1([0, 1])$. In [7], for $\delta_1, \delta_2, \ldots, \delta_n, \omega_1, \omega_2, \omega_3, \ldots, \omega_{n-1} \in A^1_\mathbb{C}(M)$ Miller defined the exponential iterated integral $e^{\delta_1 \omega_1 e^{\delta_2 \omega_2} e^{\delta_3 \omega_3} \cdots e^{\delta_{n-1} \omega_{n-1} e^{\delta_n} \omega_n}} : \Omega_x M \to \mathbb{C}$ as

$$
\int_\lambda e^{\delta_1 \omega_1 e^{\delta_2 \omega_2} e^{\delta_3 \omega_3} \cdots e^{\delta_{n-1} \omega_{n-1} e^{\delta_n} \omega_n}} = \sum_{m_1, m_2, \ldots, m_n \geq 0} \int_\lambda \delta_1 \omega_1 e^{\delta_2 \omega_2} \cdots e^{\delta_{n-1} \omega_{n-1} e^{\delta_n} \omega_n} \delta_{n} \ldots \delta_n.
$$

Then this infinite sum converges (see [7]). Let $L \subset A^1_\mathbb{C}(M)$ be a finitely generated $\mathbb{Z}$-module of 1-forms such that $dL = 0$. We denote $E^L(M, x)$ the $\mathbb{C}$-vector space of functions on $\Omega_x M$ generated by

$$
\{ \int e^{\delta_1 \omega_1 e^{\delta_2 \omega_2} e^{\delta_3 \omega_3} \cdots e^{\delta_{n-1} \omega_{n-1} e^{\delta_n} \omega_n}} | \delta_1, \ldots, \delta_n \in L, \omega_1, \omega_2, \omega_3, \ldots, \omega_{n-1} \in A^1_\mathbb{C}(M) \}.
$$

If $I \in E^L(M, x)$ is constant on homotopy classes of loops $\lambda : [0, 1] \to M$ relative to $\{0, 1\}$, we call $I$ a closed exponential iterated integral. Let $H^0(E^L(M, x))$ denote the subspace of closed exponential iterated integrals. Take a $\mathbb{Z}$-basis $\{ \delta_1, \delta_2, \ldots, \delta_n \}$ of $L$. Then we have the diagonal representation $\rho : \pi_1(M, x) \to D_n(\mathbb{C})$ such that $\rho(\lambda) = \text{diag}(\int_\lambda e^{\delta_1}, \ldots, \int_\lambda e^{\delta_n})$ for $\lambda \in \pi_1(M, x)$. Consider the $\rho$-relative completion $\mathcal{S}_\rho(\pi_1(M, x))$ of the fundamental group of $M$. Miller showed the following theorem.

**Theorem 3.1.** ([7 Theorem 6.1]) The space $H^0(E^L(M, x))$ is a Hopf algebra and we have a Hopf algebra isomorphism

$$
H^0(E^L(M, x)) \cong \mathbb{C}[\mathcal{S}_\rho(\pi_1(M, x))].
$$

**Remark 3.1.** For any $\rho$-relative representation $\phi : \pi_1(M, x) \to G$, Miller showed that $\phi$ is the monodromy of a flat connection $\omega$ on $M \times \mathbb{C}^n$ whose connection form is an upper triangular matrix. Then the monodromy of $\omega$ is given by $I + \sum_{i=1}^\infty \int \omega_i \cdots \omega$ and its matrix entries are exponential iterated integrals. In the proof of Theorem 6.1 of [7], Miller showed that these matrix entries generate the coordinate ring $\mathbb{C}[\mathbf{G}]$.

Consider a simply connected solvable Lie group $G$ with a lattice $\Gamma$. Take a diagonalization of the semi-simple part $\text{ad}_s$ of the adjoint representation $\text{ad}$ on $\mathfrak{g}$. Write $\text{ad}_s = \text{diag}(\delta_1, \ldots, \delta_n)$ where $\delta_1, \ldots, \delta_n$ are characters of $\mathfrak{g}$. By $\delta_1, \ldots, \delta_n \in \text{Hom}(\mathfrak{g}, \mathbb{C})$, we regard $\delta_1, \ldots, \delta_n$ as left-invariant closed 1-forms. Let $L$ be the $\mathbb{Z}$-module generated by $\delta_1, \ldots, \delta_n$. Consider the algebraic hull $\mathbf{H}_\Gamma$ of $\Gamma$. Since we have $\pi_1(G/\Gamma, x) \cong \Gamma$, by Theorem 2.7, we have:
Corollary 3.2. We have a Hopf algebra isomorphism
\[ H^0(E^L(G/\Gamma, x)) \cong \mathbb{C}[H_\Gamma]. \]

Let \( E^L(g^c) \) denote the subvector space of \( E^L(G/\Gamma, x) \) generated by
\[ \{ \int e^{\delta_i} \omega_{12} \cdots \omega_{n-1n} e^{\delta_n} | \delta_1, \ldots, \delta_n \in L \ \omega_{12}, \omega_{23}, \ldots, \omega_{n-1n} \in g^c \}. \]

Studying the proof of [7, Lemma 5.1], we can see that \( E^L(g^c) \) is closed under the multiplication. We define the subring
\[ H^0(E^L(g^c)) = E^L(g^c) \cap H^0(E^L(G/\Gamma, x)) \]
of \( H^0(E^L(G/\Gamma, x)) \).

Theorem 3.3. We have \( H^0(E^L(g^c)) = H^0(E^L(G/\Gamma, x)) \).

Proof. Consider the algebraic hull \( \psi : G \to H_G \) of \( G \). Since \( \psi : G \to H_G \) is \( \text{Ad}_s \)-relative, we can assume \( H_G \subset T_r(\mathbb{C}) \) and \( U_r(\mathbb{C}) \cap H_G = U(H_G) \) as in Section 2.2. Let \( \psi_s : g \to t_r(\mathbb{C}) \) be the derivative of \( \psi \) where \( t_r(\mathbb{C}) \) is the Lie algebra of \( T_r(\mathbb{C}) \). We write
\[ \psi_s = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1r} \\ \vdots & \ddots & \ddots & \vdots \\ \omega_{r-1r} & \cdots & \omega_{rr} \\ \omega_{rr} \end{pmatrix} \]
as we consider \( \psi_s \in \text{Hom}(g, \mathbb{C}) \otimes T_r(\mathbb{C}) \). Then we have
\[ (d\psi_s - \psi_s \wedge \psi_s)(X, Y) = \psi_s([X, Y]) - [\psi_s(X), \psi_s(Y)] = 0 \]
for \( X, Y \in g \). Hence we have the flat connection \( d - \psi_s \) on the vector bundle \( G \times \mathbb{C}^r \). Consider the parallel transport \( T = I + \sum_{i=1}^{\infty} \int \psi_s \cdots \psi_s \) of this connection. Let \( P_rG \) be the space of the paths \( \gamma : [0, 1] \to G \) with \( \gamma(0) = e \) where \( e \) is the identity element of \( G \). We consider the spaces \( P_rG/\sim \) of homotopy classes of \( \gamma \in P_rG \) relative to \( \{0, 1\} \). Since \( G \) is simply connected, we have \( P_rG/\sim = G \). It is easily seen that the parallel transport \( T \) on \( P_rG/\sim = G \) is a homomorphism whose derivative is equal to \( \psi_s \). Hence we can identify the parallel transport \( T \) on \( P_rG/\sim \) with the representation \( \psi_s \). Since \( \psi_s \) is \( \text{Ad}_s \)-relative and the diagonal entries of \( T \) are \( \int e^{\omega_{11}}, \ldots, \int e^{\omega_{rr}} \), we have \( \omega_{11}, \ldots, \omega_{rr} \in L \). By the proof of Theorem 2.6, the injection \( \phi : \Gamma \to H_\Gamma \) is the restriction of \( \psi \) on \( \Gamma \). Thus the representation \( \phi \) is the monodromy \( I + \sum_{i=1}^{\infty} \int \psi_s \cdots \psi_s \) of the left-invariant flat connection \( d - \psi_s \) on the vector bundle \( G/\Gamma \times \mathbb{C}^r \). By Remark 3.1, the ring \( \mathbb{C}[H_\Gamma] \) is generated by matrix entries of \( I + \sum_{i=1}^{\infty} \int \psi_s \cdots \psi_s \). Hence the theorem follows from Corollary 3.2. \( \square \)

4. An Example and a remark

Let \( N \) be a simply connected nilpotent Lie group and \( n \) the Lie algebra of \( N \). We suppose that \( G \) has a lattice \( \Gamma \). Then we can represent the coordinate ring of the Malcev completion of \( \Gamma \) by using Chen’s iterated integral of left-invariant forms on \( N \). In this paper we give another representation of the Malcev completion of the fundamental group of some nilmanifold.
Consider the solvable Lie group $G = \mathbb{R} \ltimes \mu \mathbb{C}^2$ such that $\mu(t) = \begin{pmatrix} e^{i\pi t} & te^{i\pi t} \\ 0 & e^{i\pi t} \end{pmatrix}$.

We have the lattice $\Gamma = 2\mathbb{Z} \ltimes (\mathbb{Z} + i\mathbb{Z})$. We consider the inclusion $\bigwedge g^* \subset A^*(G/\Gamma)$. The map $H^*(\bigwedge g^*) \to H^*(G/\Gamma, \mathbb{C})$ induced by this inclusion is injective (see [13]).

By $(\bigwedge g^*)^0 = \mathbb{C}$ and $(\bigwedge g^*)^1 \cap dA^0(G/\Gamma) = 0$, we have an isomorphism $H^0(B(\bigwedge g^*, x)) \cong H^0(\overline{B}(\bigwedge g^*))$ where $H^0(B(\bigwedge g^*, x))$ is the space of closed Chen's iterated integrals of the left-invariant forms on the based loop space $\Omega_x G/\Gamma$ and $H^0(\overline{B}(\bigwedge g^*))$ is the reduced bar construction (see [2]). Since we have $H^1(\bigwedge g^*) \cong \mathbb{C}$, we have an isomorphism $H^0(B(\bigwedge g^*, x)) \cong \mathbb{C}[G_{ad}]$.

On the other hand, let $L$ be the sub $\mathbb{Z}$-module of $g^*$ generated by $\{i\pi dt\}$. Then by Corollary 3.2 and Theorem 3.3 we have an isomorphism $H^0(E_L(g^*)) \cong \mathbb{C}[H_{\Gamma}]$.

Since we have $\mu(2t) = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$ for $t \in \mathbb{Z}$, $\Gamma$ is nilpotent. Hence $H_{\Gamma}$ is the Malcev completion of $\Gamma$. Since two compact solvmanifolds having the same fundamental group are diffeomorphic (see [8] or [13]), $G/\Gamma$ is diffeomorphic to a nilmanifold. By these arguments we notice:

**Remark 4.1.** By closed Chen's iterated integrals of the 1-forms $g^*_c$ on $G/\Gamma$, we can not represent the coordinate ring of Malcev completion of the fundamental group of the nilmanifold $G/\Gamma$. But the closed $L$-exponential iterated integrals of $g^*_c$ enable us to represent it.

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