A critical assessment of nonlocal strain softening methods in biaxial compression

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ABSTRACT: When modelling the development of slip surfaces in strain softening soils, the finite element method can suffer from mesh dependency and a lack of convergence of the analysis. The calculation of strains using a nonlocal method avoids the independent softening of a solitary point that can cause these symptoms. Nonlocal methods calculate the strain at a point with reference to the strains at calculation points surrounding it. In this study, a biaxial compression analysis is used to compare the original and two modified nonlocal methods to local strain softening analysis. Three nonlocal parameters are introduced and
their influence on the nonlocal strain calculation is investigated. Meshes with different discretisations permit a comparison of the mesh dependency of these methods and a sensitivity analysis of nonlocal parameters. All the nonlocal methods are found to have significantly less mesh dependency than the local method. Both modified nonlocal methods exhibit less mesh dependency than the original approach, but one of the two modified methods, the Over-nonlocal method, is shown to be unstable as the mesh is refined for certain values of the nonlocal parameters.

KEYWORDS:

Strain Softening, Nonlocal, Finite Element, Mesh Dependence, Biaxial Compression
When modelling the development of slip surfaces in strain softening soils, the finite element method can suffer from mesh dependency and a lack of convergence of the analysis (Bažant & Jirásek 2002, Galavi & Schweiger 2010, Vermeer & Brinkgreve 1994).

Numerical convergence issues arise because the inclusion of a strain related soil property (i.e. strength) causes the governing partial differential equations for a static analysis to change from elliptic, where all points in the domain are subject to a change at once, to hyperbolic, where the changes are applied following a specified condition. This causes an ill posed boundary value problem (Vardoulakis & Sulem, 1995, Lu et al. 2012).

In the continuum of a typical finite element analysis, the strain associated with strength softening is calculated using the displacement information computed at the nodes of the elements. The relative location of these nodes affects both the shear band thickness and the direction of its development (Galavi & Schweiger 2010). There can be a large difference, i.e. a high gradient, between displacement and therefore strain at neighbouring nodes. This sudden change in strain is reflected in the areas of material that are modelled as undergoing strain softening. For a locally defined strain softening model, the degree of material softening is linked to the current strain at that point only. The strain calculated at any point is calculated from the displacement value assigned to that point given the current values of stress and the previous history of deformation (Bažant & Jirásek 2002). This can lead to one calculation point in the mesh independently
experiencing high strain and a corresponding reduction in strength for this level of strain. The point can appear to have softened excessively in relation to the surrounding points (Vermeer & Brinkgreve, 1994, Bažant & Jirásek, 2002). The reduction in strength reduces the load and stress that can be carried by the material at that location and this excess load and stress will be redistributed to the neighbouring material. This localises the area of strain softening and links softening geometrically to the mesh (Vermeer & Brinkgreve 1994). Through this localisation shear band formation and growth can align to the nodes of the mesh. Meshes with different size elements or arrangements of elements will create a slip surface extending in different directions and with different thicknesses (Summersgill, 2015).

The local concept of yielding and strain softening does not reflect the reality of slip surface formation and growth. In reality, plastic deformation, of which strain softening is one form, is nonlocal in character. It arises from the accumulation of dislocations and plastic flow from their movements (Eringen, 1981). The growth of shear bands is not decided by the stress and strain at the location of the highest strain, but by the release of energy from the volume surrounding that point. Furthermore, in a continuum model, such as the finite element method, it is not the ultimate strain value at the centre point of the slip surface that is calculated, but the average strain value within the representative volume, i.e. between two degrees of freedom (Bažant & Jirásek 2002). The heterogeneity caused by shear bands at a resolution below the distance between two degrees of freedom should be represented by an appropriate value and gradient between these two points (Bažant & Jirásek 2002).
Non-local models have been implemented in several numerical codes as a regularisation tool for slip surface or fracture formation. The nonlocal approach assumes that as microstructure evolves at one point, it naturally influences the surrounding points (Di Prisco et al., 2002). These models do not alter the fundamental governing equations, but introduce the calculation of strain as a nonlocal variable by spatially averaging the local variables (Lu et al., 2009). This makes this approach of nonlocal strain regularisation more straightforward to implement in an existing finite element code.

The present study aims to provide a systematic comparison of three existing non-local methods and a conventional local strain softening approach giving particular emphasis on mesh dependency and non-local parameter selection. In this study, a biaxial compression analysis of both undrained and drained numerical simulations is used to compare the original and two modified nonlocal methods to a local strain softening analysis. Three nonlocal parameters are introduced and their influence on the nonlocal strain calculation is investigated. Analyses with meshes with different discretisations permit a comparison of the mesh dependency of these methods and a sensitivity analysis of the nonlocal parameters.

THE NONLOCAL METHOD

A nonlocal method provides both a means of regulating strain calculations to avoid excessive softening and of relating strain at a point and therefore strain softening, to strains in the surrounding area. This reduces the mesh dependence
and potential numerical convergence issues associated with strain softening analyses (Summersgill 2015).

A nonlocal model is defined as a model with an averaging integral and characteristic or defined length (Bažant & Jirásek 2002). A fully nonlocal model will treat both stress and strain as nonlocal components. However, when the model is employed as a regularisation tool, it is common to adopt a partial nonlocal constitutive relation in which only the nonlocal plastic strains control the softening (Galavi & Schweiger 2010). This also makes the model easier to implement in existing finite element codes, as the nonlocal calculation can be performed as an additional calculation after the initial local calculation, with minimal modification to the existing calculations or governing equations. This format aids the calculation of nonlocal strain by providing an initial complete strain distribution for the first computation of nonlocal strain (Vermeer & Brinkgreve, 1994).

The Original Formulation

One of the first formulations for a nonlocal strain model is presented in Equation 1. This was proposed by Eringen (1981) for strain hardening applications and by Bazant et al. (1984) as a strain softening damage model.

\[
\varepsilon_p^*(x_n) = \frac{1}{V_\omega} \iiint \omega(x_n') \varepsilon_p(x_n') dx_1' dx_2' dx_3' \quad (1)
\]
where $\varepsilon^p$ is the accumulated plastic deviatoric strain, * denotes the nonlocal parameter, $x_n$ is the point at which the calculation of the nonlocal strain, $\varepsilon^{p*}$ is required, whereas $x_n'$ refers to all the surrounding locations, i.e. the location of reference strains. Therefore, $\varepsilon^p(x_n')$ equals the reference strain at the reference location. The weighting function, $\omega(x_n')$ is defined for all the reference locations, but it is centred at the location $x_n$.

The weighting function is the Gaussian or normal distribution which is shown in Equation 2 and in Figure 1. This introduces an additional parameter, the defined length, DL. A larger DL will create a wider and shorter weighting distribution, thus affecting the rate of softening, as shown in Figure 2 (a). The effect of DL on the non-local plastic strain distribution can be seen in Figure 3(a), which has been derived for a local strain equal to 1 over 0.5m of distance applied centred at the calculation point, i.e. from -0.25m to 0.25m (note that the same calculation is employed in all panels of this figure).

\[
\omega(x_n') = \frac{1}{DL\sqrt{\pi}} \exp \left[ -\frac{(x_n' - x_n)^T(x_n' - x_n)}{2DL^2} \right]
\] (2)

The weighting function is chosen such that it will not alter a uniform field of strain.

The integral of the weighting function in the three dimensions $x_1$, $x_2$ and $x_3$ is referred to as the reference volume, $V_\omega$, as shown in Equation 3. This is used to normalise the calculation of the nonlocal strain and for the Gaussian distribution is equal to approximately one. The latter attribute of the function ensures that a uniform field of strain would remain unmodified.
\[ V_\omega = \iiint \omega(x'_n) \, dx'_1 \, dx'_2 \, dx'_3 \]  

(3)

This method, which will be referred to as the “original method” herein, has been shown to have low mesh dependency when applied to strain softening analyses (Jostad & Grimstad 2011). However, the softening of the material still occurs dominantly at the centre point of the slip surface. This is where the weighting function has its maximum and therefore the largest nonlocal strain is calculated (Galavi & Schweiger 2010, Vermeer & Brinkgreve 1994, Lu et al. 2012, Jostad & Grimstad 2011). Excessive softening could still potentially occur at this point and the spread of the slip surface would therefore be constrained. In light of this, modifications have been made to the original nonlocal method to address the central high concentration of strain. Two modified methods are presented and evaluated below, the G&S method and the Over-nonlocal method.

**The G&S Method**

This modified method was proposed by Galavi & Schweiger (2010) and will be referred to as the “G&S” method. The fundamental idea underpinning this method is that the development of the slip surface is influenced by the directly surrounding areas and not by the concentrated strain at the centre of the slip surface. Therefore this method proposes a modified weighting function, shown in Figure 1 and Equation 4, which limits the central concentration of strains, without introducing any new parameters. The greatest contributions of strains to the nonlocal calculation now stem from points that lay a little to each side of the calculation point and the strain contribution at the calculation point is
actually zero. The calculated nonlocal strain is therefore a little lower with a wider distribution, as shown in Figure 1. The G&S weighting function is influenced by the size of DL in a similar fashion to the original weighting function, as shown in Figures 2 (b) and (a) respectively. This in turn leads to a similar effect of DL on the non-local plastic strain distribution which is shown in Figure 3(b). The integral of the weighting function is still equal to one and therefore the condition of zero alteration of a uniform field is again satisfied.

\[ \omega(x'_n) = \sqrt{(x'_n - x_n)^T(x'_n - x_n)} \frac{DL^2}{DL^2} \exp \left[ -\frac{(x'_n - x_n)^T(x'_n - x_n)}{DL^2} \right] \]

(4)

The Over-nonlocal Method

Vermeer & Brinkgreve (1994) proposed the over-nonlocal method which alters the nonlocal strain formulation, as shown in Equation 5. The over-nonlocal method adopts a weighting function which prevents the formation of a concentrated peak and instead provides a more uniform value of strain across the slip surface. This introduces a new parameter, \( \alpha \) which is used to provide a contribution from the local strain at the point of calculation and to increase the nonlocal contribution to the calculation of nonlocal strain. This increases the width of the slip surface compared to the original method and reduces the concentration of strain at the centre, as shown in Figure 3(d) for \( \alpha=1.5 \) and \( DL=1.0 \).
As it can be seen with reference to Equation 5, when \( \alpha \) is greater than 1, the local strain contribution is negative. This significantly reduces the value of nonlocal strains calculated over the areas where local strain is high and increases the nonlocal strain value calculated in the areas immediately adjacent to the local strain distribution. If \( \alpha = 1.0 \) the formula reverts to the original nonlocal formulation. If \( \alpha \) is less than 1.0, the local strain part of the equation contributes by increasing the nonlocal strain to be greater than the local strain, which contradicts the aims of the formulation. The alpha parameter, \( \alpha \), must therefore always be greater than 1.0 (Vermeer & Brinkgreve, 1994).

For certain combinations of local strain input, alpha value and defined length \( DL \) there is a negative nonlocal strain. This is shown in Figure 3(c) where for an alpha of 1.5 three values of defined length are examined. For \( DL = 0.5 \), the nonlocal strain is always positive, but for \( DL = 1.0 \) and \( DL = 1.5 \) there are negative values around the local strain input. The influence of alpha on the nonlocal strain calculation is compared to the other two methods in Figure 4, with \( DL = 1.0 \). This figure shows that an increase in alpha increases the nonlocal strain outside of the area of local strain input and decreases the nonlocal strain within the area of local strain. The influence of the parameter alpha will be further investigated in biaxial compression simulations.

\[
\varepsilon^{p*}(x_n) = (1 - \alpha)\varepsilon^p(x_n) + \frac{\alpha}{V} \iiint (\omega(x'_n)\varepsilon^p(x'_n))dx'_1dx'_2dx'_3
\]  

(5)
BIAXIAL COMPRESSION ANALYSES

Biaxial compression tests have been widely used (Vermeer & Brinkgreve 1994, Lu et al 2012, Jostad & Grimstad 2011) to assess the mesh dependence of non-local methods, as they can impose severe strain localisation in the numerical model. The biaxial compression analyses compress a quadrilateral mesh from two opposing sides, whilst leaving the two other opposing sides free to deform. This creates a bifurcation and a slip surface, which therefore permits an assessment of the ability of local and nonlocal strain softening models to regulate slip surface formation for a simple analysis.

The pattern of slip surfaces in a biaxial compression analyses can vary from one to three or four bifurcations depending on the dimensions of the mesh and the boundary conditions imposed. The preferential development of one or more slip surfaces will affect the load versus displacement response. In the analyses presented here, one quarter of a full problem with a square mesh is modelled as shown in Figure 5. This provides analyses that form a single slip surface each time and this enables a direct comparison of the nonlocal models under consistent conditions. It should be noted that analyses of the full problem (i.e. not making use of any symmetry in geometry) with the same boundary conditions leads to exactly the same response as the quarter model presented herein (Summersgill 2015). The initial conditions consist of a vertical stress of 50 kPa imposed on the top horizontal boundary and a 100 kPa horizontal stress applied along the right hand side lateral boundary. It was found to be unnecessary to include the stiff platens employed by Galavi & Schweiger (2010)
to introduce inhomogeneous strain. Restricting the horizontal displacement of
the top horizontal mesh boundary in Figure 5(b) and imposing an equal vertical
displacement to all nodes along this boundary was sufficient for a slip surface to
form.

The analyses presented compare the mesh dependence of the strain softening
models by performing all analyses with three meshes containing different square
element sizes. All the meshes are 1m by 1m containing 100 elements (10x10),
400 elements (20x20) or 1600 elements (40x40), Figure 6. An 80x80 mesh was
used only for the investigation of the alpha parameter of the over-nonlocal
method, which contains 0.0125m by 0.0125m square elements in a 1m square
mesh. In all cases, plane strain eight-noded isoparametric elements with reduced
Gaussian integration were used. This integration scheme was also used to
evaluate the integrals in Equations 1, 3 and 5.

The three nonlocal strain softening methods were implemented in the finite
element code ICFEP (Imperial College Finite Element Program) (Potts &
Zdravkovic, 1999), and compared with existing local strain softening models.
The latter are variants of Tresca and Mohr-Coulomb models in which the limits
for peak and residual strength can be specified by a value of percentage of plastic
deviatoric strain (E). Results for the local method are compared to the three
nonlocal methods, including two sets of over-nonlocal analyses, with \( \alpha = 1.5 \) and
\( \alpha = 2.0 \), following values used by Jostad & Grimstad (2011). An accelerated
modified Newton-Raphson scheme with a sub-stepping stress point algorithm
was employed to solve the finite element equations (Potts & Ganendra, 1994).
Both undrained and drained strain softening analyses were performed. For the undrained analyses a Tresca failure criterion is adopted. The soil has a peak undrained strength of 100kPa at 0% plastic deviatoric strain and a residual undrained strength of 50kPa at 15% plastic deviatoric strain. The drained analyses employed a Mohr Coulomb failure criterion. Similarly, the drained soil has a peak strength of $\phi' = 25^\circ$ at 0% plastic deviatoric strain which reduces linearly to residual strength of $\phi' = 10^\circ$ at 15% plastic deviatoric strain, while zero cohesional strength and zero dilation are specified. The effect of the angle of dilation is investigated separately and discussed later. The stiffness of soil is constant with a Young's Modulus, $E = 50\text{MPa}$, Poisson's ratio, $\mu = 0.49$ and $\mu = 0.2$ for undrained and drained conditions respectively.

For the nonlocal methods a defined length of $DL = 0.1\text{m}$ is used for the analyses, unless otherwise stated. This value was chosen because it is the length of the largest element employed in the analyses. This value allows the local strains surrounding the point of calculation to contribute significantly to the nonlocal strain calculations. If the value chosen for $DL$ is too small, the nonlocal strain calculation will be very similar to the local strain input and therefore the calculation process would be redundant. The influence of the defined length parameter on nonlocal strain is investigated and discussed in a following section.

A radius of influence can also be specified to limit the number of elements used in the calculation of the nonlocal strain. The weighting function is an exponential function based on distance, therefore the contribution of strain diminishes
rapidly with distance, as can be seen in Figure 1. The exclusion of strains at a distance greater than 4 times DL will alter the outcome of the nonlocal strain calculation very little, but it will improve numerical efficiency. A specified radius of influence of 0.4m was therefore considered as appropriate for these analyses. The sensitivity of the analysis on the radius of influence is further investigated in the last part of this study.

**DISCUSSION OF RESULTS**

*Undrained Analyses*

Figure 7 presents the reaction load on the top of the mesh for the applied vertical displacement for all considered approaches and spatial discretisations. Clearly the load displacement curves of all the nonlocal strain softening methods show significantly less mesh dependency than the local method. As the strain limits controlling strain softening are independent of the element size in the local method, it is expected for a smaller element size to result in earlier and faster softening of the material, in agreement with the findings of previous studies (Conte et al 2010, Schädlich 2012). The residual plateau is reached later for all of the nonlocal analyses, indicating that they delay the softening of the material compared to the local analysis. The Over-nonlocal method with $\alpha = 1.5$ shows the least mesh dependency, Figure 7(c), followed by the G&S method, Figure 7(d). However, when a value of $\alpha = 2.0$ is used for the Over-nonlocal method, the 40x40 mesh produces premature softening of the material compared to the
analyses employing meshes with larger elements, but with otherwise identical arrangements, Figure 7(e).

Figure 8 compares the contour plots of accumulated plastic deviatoric strain for the local and nonlocal G&S strain softening methods for the three mesh discretisations. The width of the slip surface is less dependent on element size for the nonlocal analyses compared to the local analyses. The slip surface width reduces with element size for the local analysis, while the width of the largest contour remains similar for all three meshes when the nonlocal G&S method is employed.

The regularization of the plastic strains is shown by a diagonal cross section through the mesh, of the distribution of accumulated local plastic deviatoric strains plotted for each strain softening method in Figure 9. The nonlocal methods do not permit a high concentration of strain to develop compared to the local method. This is the case for all the non-local analyses except the Over-nonlocal with $\alpha = 2.0$ for the 40x40 mesh in Figure 9(e). The strain distribution for Figure 9(e) is more comparable to the local strain softening results in Figure 9(a). The influence of the alpha parameter is further investigated in a later section of this study.

**Drained Analyses**

The undrained biaxial compression analyses, presented in the previous section, were repeated using drained strength parameters. The local strain softening
results are compared to the results for the three nonlocal methods in Figures 10 and 11. The behaviour is similar to the undrained analyses with a reduction in mesh dependence for the nonlocal methods. The original nonlocal and over-nonlocal with $\alpha = 1.5$ showed greater mesh dependency than the equivalent undrained analyses. The G&S nonlocal method demonstrated the least mesh dependency overall. The combination of DL=0.1m, $\alpha = 2.0$ and the 40x40 mesh for the Over-nonlocal method again resulted in a sudden reduction in the reaction load, identified as premature softening in the previous section, Figure 10(d).

Furthermore, the slip surface formation is very different for the case of the local strain method compared to the nonlocal methods. As shown in the diagonal cross section of Figure 11(b), two distinct slip surfaces can be identified in the plastic deviatoric strain distribution plot of the local strain method. The differences in the predicted failure mechanisms between the local strain and some of the non-local approaches are also illustrated in the contour plots of accumulated local plastic deviatoric strain for the 40x40 configuration in Figure 12 (a, b, c) and in the incremental displacement vectors of Figure 12(d, e). The plots in panels 12 (c) and (e) demonstrate that the failure mechanism predicted by the nonlocal G&S method, in terms of contour plots and displacement vectors respectively, produces a well constrained slip surface. This was consistent for different mesh discretisations.

One soil property that has been demonstrated as influencing the location and shape of the slip surface in a stiff clay cutting is the angle of dilation, $\psi'$ (Potts et
The non-associated results (i.e. $\psi' = 0$) for G&S nonlocal analyses with a DL of 0.1 m and RI of 0.4m are compared to a set of associated analyses (i.e. $\psi' = \varphi'$) in Figure 13. The associated analyses were performed with $\psi'$ equal to $\varphi'$ at both peak and residual limits and both softening at the same rate. The associated results show the same low mesh dependence in addition to a lower imposed displacement to reach residual loading, Figure 13 (a). The strain distributions for these two sets of analyses show higher strain values than the non-associated results, Figure 13 (b). For each mesh, the shape of the associated results is a stretched version of the corresponding non-associated results. The dilation angle has not caused a change in the width of the slip surface formed, only an increase in the strains developed.

Investigation of non-local parameters

Three nonlocal parameters are introduced for the nonlocal strain softening methods. The influence of these parameters on local strain calculations was presented for a simple local strain input in Figures 3 and 4. A biaxial compression analysis provides a more complex situation in which it is important to evaluate the influence of these parameters.

The alpha parameter is only employed in the Over-nonlocal modified strain softening method. The defined length, DL, is a required input for all the nonlocal methods and influences the shape of the weighting function used to calculate nonlocal strain, as shown in Figure 2 and as a consequence the softening rate. The radius of influence, RI, is an optional parameter that improves the numerical
efficiency of analyses. The parameters DL and RI are investigated using the
nonlocal G&S method, as it was demonstrated in the previous sections that this
method had the least overall mesh dependency.

**Alpha**

As previously discussed, the parameter alpha controls the distribution of non-local strain for the Over-nonlocal method. This is further investigated parametrically in this section, examining a range of values of $\alpha > 1.0$. Since a premature softening and high concentration of strain was previously observed for the Over-nonlocal results for $\alpha = 2.0$ and the finest mesh, further undrained analyses for $\alpha = 1.75$ and $\alpha = 2.25$ were performed for the same discretisation.

The load displacement responses for $\alpha = 1.75$, $\alpha = 2.0$ and $\alpha = 2.25$ are shown in Figure 14. The result for $\alpha = 2.25$ and 40x40 mesh was unstable, but the response for $\alpha = 1.75$ and 40x40 mesh was stable. However, when an $\alpha$ equal to 1.75 was applied to a finer mesh with 80x80 elements of 0.0125m size, the load displacement response did exhibit premature softening. This is reflected in the distribution of strain over a diagonal cross section of the mesh for the three analyses with the finest mesh for each $\alpha$ value, which is shown in Figure 15.

There is a high value of strain concentrated in a smaller area similar in shape to the local strain softening method results of Figure 9 (b). The expected result for a nonlocal strain softening method is a wider spread and lower nonlocal strain value, as shown for the 10x10 and 20x20 mesh results in Figure 15. The investigation on the parameter alpha suggests that there is uncertainty in
trusting the outcome of analyses using the Over-nonlocal method when employing a high $\alpha$ and fine mesh.

Defined Length

The defined length, $DL$, defines the shape of the weighting function employed in the calculation of nonlocal strain (see Figure 2). A higher $DL$ is expected to result in a wider slip surface with a lower maximum nonlocal strain. This affects the rate of strain softening, as the same input of local strain will result in a lower nonlocal strain for a larger defined length and therefore a slower rate of softening.

Two sets of biaxial compression analyses were performed to assess the influence of the defined length, in both undrained and drained conditions. The examined values of $DL$ are restricted by the size of the elements in the meshes employed. The minimum value employed for $DL$ should be equal to the maximum element size found in the meshes. This permits the nonlocal calculations to have local reference strains at an appropriate distance from the calculation point for them to influence the nonlocal strain. It should be noted that the local strains are plastic deviatoric strains which are calculated at the elements' Gauss points. Without some local strain values providing a meaningful contribution to the nonlocal strain calculated, the nonlocal strain value would be very similar to the local strain, making the process of calculating it redundant. The values for $DL$ were chosen for a ratio of the element length to defined length of 1:1, 1:2, 1:4 and 1:8.
The results for the undrained and drained analyses are presented in Figure 15 in terms of load-displacement curves. They confirm that as DL increases the imposed displacement required to reach a residual reaction load increases, indicating a reduction in the softening rate. The change in DL seems to control only the softening rate, as the peak and residual reaction loads are not affected. The undrained results for the reaction load versus imposed displacement show consistent and almost mesh independent behaviour. For the undrained case only a minor mesh dependency can be depicted for DL equal to 0.1, which is related to the use of a 1:1 to ratio. In the drained analyses, the results for a DL of 0.05 are not the same and require further investigation. The distributions of strains for a diagonal cross section of the drained analyses are presented in Figure 17 by the mesh employed. Clearly, there is a reduction in the peak nonlocal plastic deviatoric strain as DL increases. Furthermore, the mesh has an influence on the width of the slip surface; as element size decreases so does the slip surface width for the same value of DL. In order to identify a suitable ratio of element length to defined length, the non-conforming results are re-examined. The results for the 20x20 mesh and DL equal to the element size (ratio 1:1) show the formation of two slip surfaces. Therefore the 1:1 ratio leads to a mesh dependency in both the undrained and drained analyses. Given the difference exhibited in these analyses, a ratio of at least 1:2 would be preferable for the maximum element size to defined length.

Radius of Influence
The radius of influence, RI, is an optional parameter that limits the distance from the calculation point of the neighbouring strains that will be included in the nonlocal strain calculation. A variety of analyses were performed to validate the use of a RI value of 4 times the defined length, employed in the analyses presented above. These analyses evaluate the RI value for a 1m by 1m mesh for both drained and undrained soil properties, employing DL values of 0.1m or 0.2m, and mesh discretisations of 10x10 and 20x20 elements. The results for a set of drained analyses employing a 20x20 mesh and DL = 0.1m are presented in Figure 18 and this set is representative of all the RI investigations presented in Summersgill (2015). The value of RI has some influence on the results for a ratio of 1:1 and 1:2 of defined length to RI. For a value of RI three times DL or greater, the reaction load response and strain distribution is almost indistinguishable from an analysis performed with no RI specified, i.e. where all neighbouring strains contribute to the calculation of a nonlocal strain no matter how far away they are from the position for which the nonlocal strain is being evaluated.

The time saving that RI provides make the use of this parameter a sensible decision. For the analysis presented in Figure 18, the RI = 0.3m analysis is performed in 28% of the time of an analysis with no RI specified (i.e. RI equal to infinity). An RI = 0.4m still provides a time saving, requiring 41% of the time for an analysis with no RI specified. These results are for a simple mesh and relatively simple problem. The potential time saving for a more complex mesh or analysis could be significant.
CONCLUSIONS

This paper assesses the performance of three nonlocal strain softening methods (namely the original formulation of Eringen (1981), the Over-nonlocal of Vermeer & Brinkgreve (1994) and the G&S of Galavi & Schweiger (2010)) with comparison to each other and to a local strain softening method. The mesh dependency of each of the methods is evaluated with biaxial compression analyses that employ three meshes with different element discretisations. All the nonlocal methods are found to have significantly lower mesh dependency than the examined local strain softening method, for both drained and undrained strength parameters. Given the lower mesh dependency, the nonlocal strain softening method is a more viable option for boundary value problems with a range of element sizes and shapes. Both modified methods (i.e. the Over-nonlocal and the G&S) exhibit less mesh dependency than the original nonlocal approach. However, a parametric study on the performance of the Over-nonlocal method for a range of alpha values showed that the method becomes unstable for high values of alpha and fine meshes. Therefore, among the considered approaches, the G&S method provides the best compromise between low mesh dependency and consistency of results.

A sensitivity investigation of the results for the nonlocal parameters of defined length DL and radius of influence RI, was also undertaken. The results show that the defined length parameter, DL influences the rate of strain softening, while its value is also constrained by the size of the elements in the mesh employed. DL must be sufficiently large that the local strains referenced for calculating the
nonlocal strain will contribute significantly to the nonlocal calculation. The
selection of nonlocal parameters, defined length and radius of influence required
for use of the nonlocal G&S method should consider both the size of the elements
in the mesh employed and the impact of DL on the rate of strain softening.

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REFERENCES

Bazant, Z.P. Belytschko, T.B. & Chang, T.P. (1984) Continuum model for strain softening. J. Engng. Mech. 110(12): 1666-1692, doi: 10.1061/(ASCE)0733-9399(1984)110:12(1666)

Bazant, Z.P. & Jirasek, M. (2002). Nonlocal integral formulations of plasticity and damage: Survey of progress. J. Engng. Mech. 128(11): 1119-1149, doi: 10.1061/(ASCE)0733-9399(2002)128:11(1119)

Conte, E., Silvestri, F. & Troncone, A. (2010) Stability analysis of slopes in soils with strain-softening behaviour. Computers and Geotechnics. 37 (5), 710-722, doi: 10.1016/j.compgeo.2010.04.010

Di Prisco, C. Imposimato, S. & Aifantis, E.C. (2002). A visco-plastic constitutive model for granular soils modified according to non-local and gradient approaches. Int. J. Numer. Anal. Meth. Geomech. 26(2):121-138, doi: 10.1002/nag.195

Eringen, A.C. (1981) On nonlocal plasticity. Int. J. Eng. Sci., 19: 1461-1474, doi: 10.1016/0020-7225(81)90072-0

Galavi, V. & Schweiger, H.F. (2010) Nonlocal multilaminate model for strain softening analysis. Int. J. Geomech. 10: 30-44, doi: 10.1061/(ASCE)1532-3641(2010)10:1(30)
Jostad, H.P. & Grimstad, G. (2011) Comparison of distribution function for the non-local strain approach. In (ed.). Computational geomechanics: COMGEO II: proceedings of the 2nd International symposium on computational geomechanics, Cavtat: Dubrovnik, 27-29 April 2011

Lu, X. Bardet, J & Huang, M. (2009) Numerical solutions of strain localization with nonlocal softening plasticity. Computer Methods in Applied Mechanics and Engineering. 198 (47-48), 3702-3711.

Lu, X. Bardet, J. & Huang. M. (2012) Spectral analysis of non-local regularization in two-dimensional finite element models. Int. J. Numer. Anal. Meth. Geomech. 36: 219-235, doi: 10.1002/nag.1006

Potts, D.M. & Ganendra, D. (1994). An evaluation of substepping and implicit stress point algorithms. Comput. Meth. Appl. Mech. Engng 119: 341-354. doi: 10.1016/0045-7825(94)90094-9

Potts, D. M., Kovacevic, N. & Vaughan, P. R. (1997) Delayed collapse of cut slopes in stiff clay. Géotechnique. 47 (5), 953-982.

Potts, D.M. & Zdravković, L. (1999) Finite Element analysis in geotechnical engineering: Theory Thomas Telford, London
Schädlich, B. (2012) A multilaminate constitutive model for stiff soils. PhD. Technische Universität Graz.

Summersgill, F.C. (2015) Numerical modelling of stiff clay cut slopes with nonlocal strain regularisation. PhD Thesis Imperial College London.

Vardoulakis, I.G. & Sulem, J. (1995) Bifurcation Analysis in Geomaterials, Chapman & Hall.

Vermeer, P.A. & Brinkgreve, R.B.J. (1994). A new effective non-local strain measure for softening plasticity. In R. Chambon, J. Desrues & I. Vardoulakis (eds), Localization and bifurcation theory for soils and rocks: 89-100, Rotterdam: Balkema, The Netherlands.

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(a) Original

(b) G&S

(c) Over-nonlocal

(d) DL=1.0m
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Figure 5

(a) Full problem

Δu = 0

lines of symmetry

(b) 1/4 of problem

Δu = 0

Δd

Δd

v

u
Figure 6

100 elements
10 x 10
0.1m x 0.1m
(a)

400 elements
20 x 20
0.05m x 0.05m
(b)

1600 elements
40 x 40
0.025m x 0.025m
(c)
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(a) Local

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(a) 10x10, DL=0.1m, Ratio 1:1
    20x20, DL=0.05m, Ratio 1:1
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(b) 10x10, DL=0.2m, Ratio 1:2
    20x20, DL=0.1m, Ratio 1:2
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