Application of microlocal analysis to an inverse problem arising from financial markets

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Abstract
One of the most interesting problems discerned when applying the Black–Scholes model to financial derivatives, is reconciling the deviation between expected and observed values. In our recent work, we derived a new model based on the Black–Scholes model and formulated a new mathematical approach to an inverse problem in financial markets. In this paper, we apply microlocal analysis to prove a uniqueness of the solution to our inverse problem. While microlocal analysis is used for various models in physics and engineering, this is the first attempt to apply it to a model in financial markets. First, we explain our model, which is a type of arbitrage model and illustrate our new mathematically applying microlocal analysis to the integral equation, we prove our uniqueness of the solution to our new mathematical model in financial markets. Finally we propose and test the numerical algorithm for our model.

Keywords: financial model, microlocal analysis, FBI transform

(Some figures may appear in colour only in the online journal)

1. Introduction

Financial derivatives are contracts wherein payment is derived from an underlying asset such as a stock, bond, commodity, interest, or exchange rate. An underlying asset $S_t$ at time $t$ is modeled by the following stochastic differential equation:

\[ dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \]
where the process \( W_t \) is Brownian motion. The parameters \( \mu(t, S) \) and \( \sigma(t, S) \) are called the real drift and the local volatility of the underlying asset, respectively.

Black and Scholes [1] first discovered how to construct a dynamic portfolio \( \Pi_t \) of a derivative security and the underlying asset. Their approach is developed in probability theory, and the hedging, and pricing theory of the derivative security is established as mathematical finance. By Ito’s lemma, the stochastic behavior of the derivative security \( u(t, S) \) is governed by the following stochastic differential equation:

\[
du = \left( \frac{\partial u}{\partial t} + \mu(t, S)S \frac{\partial u}{\partial S} + \frac{1}{2} \sigma(t, S)^2 S^2 \frac{\partial^2 u}{\partial S^2} \right) dt + \sigma(t, S)S \frac{\partial u}{\partial S} dW.
\]

In the absence of arbitrage opportunities, the instantaneous return of this portfolio must be equal to the interest rate \( r > 0 \), i.e. the return on a riskless asset such as a bank deposit. Therefore, this equality takes the form of the following partial differential equation:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sigma(t, S)^2 S^2 \frac{\partial^2 u}{\partial S^2} + (r - \delta)S \frac{\partial u}{\partial S} - ru = 0,
\]

(1.1)

where \( r \) and the dividend rate \( \delta \) are the known constants.

Their approach provides a useful, simple method of pricing inclusive of financial derivatives, risk premium, and default probability estimation under the assumption that the risky asset is log-normally distributed. However, the theoretical prices of options with different strike prices as calculated by the Black–Scholes model differ from real market prices. Specifically, when we apply the Black–Scholes model to default probability estimation, we must be careful of the deviation that arises between expected and observed values. Merton [11] has formulated a default probability estimation using a model based on [1] by considering the value of the firm instead of its stock, the firm’s debt instead of strike price, and its equity instead of option price and Boness [2] has derived the formulation of it by another method. However, as shown in deriving the Black–Scholes model (see [1]), under the no-arbitrage property of the financial market, the real drift \( \mu \) does not enter equation (1.1). In [12], taking this into account, we have derived the following new model:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sigma(t, S)^2 S^2 \frac{\partial^2 u}{\partial S^2} + \mu(t, S)S \frac{\partial u}{\partial S} - ru = 0.
\]

(1.2)

Moreover, in [12] we have established an inverse problem to reconstruct the real drift from the observable data, but only an binary option case. In Korolev, Kubo and Yagola [8], they reconstructed the unknown drift in our new model.

In this paper, we prove the uniqueness of the solution to an inverse problem with respect to the real drift by applying microlocal analysis. To give a brief description of our problem, we build upon the method in [4] and [13]. In [4], they used the standard linearization method with an option pricing inverse problem and derived the partial differential equation with the constant coefficient \( \sigma_0^2, \delta, r \) for the linear principal part \( V \). Since a change of variables means this equation is reduced to the heat equation with the right-hand side \( w(\tau, y)f(y) \), they wrote the well-known integral representation for the solution \( W \) to that heat equation with a suitable initial condition as follows:

\[
W(\tau, x) = \int_{\mathbb{R}} \int_0^\tau \frac{1}{\sqrt{2\pi(\tau - \theta)\sigma_0^2}} e^{-\frac{(\theta - x)^2}{2\sigma_0^2}} w(\theta, y)f(y) d\theta dy,
\]

(1.3)
where $f$ is a small perturbation of constant $\sigma_0$, $w(\tau, y)$ is represented by

$$w(\tau, y) = \frac{s^*}{\sqrt{2\pi \tau \sigma_0^2}} e^{-\frac{|y|^2}{2\tau \sigma_0^2}}.$$ 

Here $\tau = T - t$, $y = \log K/s^*$, $K$ is a strike price at the maturity date $T$ and $s^*$ is market price of the stock at a current time $t^*$.

For the above equation, they applied the Laplace transform to exactly evaluate an integral with respect to time. As a result, they derived the integral equation for $f$ that takes the following form

$$V(\tau, x) = \int_{\mathbb{R}} B(x, y; \tau) f(y) dy$$

with the kernel

$$B(x, y; \tau) = \frac{s^*}{\sigma_0^2 \sqrt{\pi}} \int_0^{\infty} e^{-\theta^2} d\theta$$

given by the error function, and thus proved the uniqueness for the linearized inverse problem. In our case, since our principal linear part $W$ which is derived in the same manner as [4] has the following form

$$W(\tau, x) = \int_{\mathbb{R}} \int_0^{\tau} \frac{1}{4\pi(\tau - s)\sigma_0^2} e^{-\frac{|x - y|^2}{4(\tau - s)\sigma_0^2}} w(\theta, y) f(y) d\theta dy,$$ (1.5)

where $w(\theta, y)$ takes the following form

$$w(\tau, y) = \int_0^{\infty} \frac{1}{\sqrt{4\pi \tau \sigma_0^2}} e^{-\frac{|x - y|^2}{4\tau \sigma_0^2}} dx.$$ 

Therefore we are unable to derive an integral equation by the Laplace transform as in (1.4); that is, in our case $w(\tau, y)$ is not a Gauss function but an error function. In this paper, taking this into account, we shall prove the uniqueness of the solution to the inverse problem of the real trend by applying the Fourier–Bros–Iagolnitzer (for short, FBI) transform to (1.5).

The construction of this paper is divided into seven sections. In section 2, we illustrate the linearized inverse problem of the real drift (for short, LIPD), and in section 3, we briefly describe main theorem. In section 4, we summarize basic facts on the FBI transform which plays an essential role in the proof of our main theorem. After studying properties of the integral operator $A$ in section 5, and we prove our main theorem in section 6. Finally in section 7, we propose and test the numerical algorithm for our model. By using our model, we will be able to show the existence of arbitrage opportunities in different financial markets.

2. Inverse problem of the real drift

In [12], we have derived a new arbitrage model and formulated an inverse option pricing problem for a reconstruction of a real drift in the binary option case. In this section, we explain how to formulate an inverse problem of our new arbitrage model and reconstruct the real drift.

Here, we consider the following problem wherein the local volatility $\sigma(t, S)$ is a positive constant $\sigma_0 > 0$ and the real drift $\mu(t, S)$ is a time-independent in our new equation (1.2) with a suitable condition:
\[ u(t, S)|_{t=T} = \max\{S - K, 0\}. \]  

By the following changes of variables and substitutions
\[ y = \log \frac{S}{K}, \quad \tau = T - t, \]
and \( a(y) = \mu(Ke^\tau), U(\tau, y) = u(T - \tau, Ke^\tau)/K, \)
the equation (1.2) and the initial data can be transformed into the following form:
\[
\begin{aligned}
\frac{\partial U}{\partial \tau} & = \frac{1}{2} \sigma_0^2 \frac{\partial^2 U}{\partial y^2} - \left( \frac{1}{2} \sigma_0^2 - a(y) \right) \frac{\partial U}{\partial y} - ru, \quad (\tau, y) \in (0, \tau^*) \times \mathbb{R}, \\
U(\tau, y)|_{\tau=0} & = \max\{e^y - 1, 0\}, \quad y \in \mathbb{R}, \\
U(\tau^*, y) & = U^*(y), \quad y \in \omega \subseteq \mathbb{R},
\end{aligned}
\]  
where \( \tau^* = T - t^* > 0, t^* \) is the current time and \( \omega \) is an interval of \( \mathbb{R} \).

Here we define that the inverse problem of the real drift (2.3) and (2.4) seeks \( a(y) \) from the given \( U^*(y) \). However, since this inverse problem is nonlinear, difficulties arise with the uniqueness and existence of the solution. Therefore, we will formulate the inverse problem of the real drift by means of the linearization method in [3] and [4].

To linearize around the constant coefficient \( \mu_0 \), we assume that
\[ a(y) = \mu_0 + f(y), \]
where \( f(y) \) denotes a small perturbation. Thus, we observe
\[ U = U_0 + V + \nu, \]
where \( U_0 \) solves the Cauchy problem (2.3) with \( a(y) \equiv \mu_0, \nu \) is quadratically small with respect to \( f \), and \( V \) is the principal part of the perturbed solution \( U \). Substituting this into the expression for \( u \) and neglecting terms of higher order with respect to \( f \), we reach the linearized inverse problem of the real drift.

### 2.1. Linearized inverse problem of the real drift (LIPD)

The parameters \( \tau^*, \mu_0, \sigma_0, \) and \( r \) are given. From the option price \( V^*(y) = \{U^*(y) - U_0(\tau^*, y)\} \), identify the perturbation \( f(y) \) satisfying
\[
\begin{aligned}
\frac{\partial V}{\partial \tau} & = \frac{1}{2} \sigma_0^2 \frac{\partial^2 V}{\partial y^2} + \left( \frac{1}{2} \sigma_0^2 - \mu_0 \right) \frac{\partial V}{\partial y} + rV = \frac{\partial U_0}{\partial y}f(y), \quad (\tau, y) \in (0, \tau^*) \times \mathbb{R}, \\
V(\tau, y)|_{\tau=0} & = 0, \quad y \in \mathbb{R}, \\
V(\tau^*, y) & = V^*(y), \quad y \in \omega.
\end{aligned}
\]  

At the end of this section, we explain in which function spaces we consider parabolic equations. Set \( \langle x \rangle = (1 + |x|^2)^{1/2} \) \( x \in \mathbb{R}^n \). For \( x, \lambda \in \mathbb{R}, \) define
\[ H^1_\lambda(\mathbb{R}^n) = \{ u \in D'(\mathbb{R}^n); \langle x \rangle^k u \in H'(\mathbb{R}^n) \}, \quad L^2_\lambda(\mathbb{R}^n) = H^0_\lambda(\mathbb{R}^n). \]
Let \( T > 0. \) Let \( a_{ij}, b_i, c \in L^\infty([0, T], L^\infty(\mathbb{R}^n)), \) and assume that \( a_0 = a_\mu \) is real-valued, and that
with some $\delta_0 > 0$. Then for every $u_0 \in L^2_x(\mathbb{R}^n)$ and $g \in L^2([0, T], H^{-1}_x(\mathbb{R}^n))$, there exists a unique $u \in L^2([0, T], H^1_x(\mathbb{R}^n)) \cap C([0, T], L^2_x(\mathbb{R}^n))$ satisfying
\[
\begin{cases}
\frac{\partial u}{\partial \tau}(t, x) - \sum_i \partial_i (a_0 u(t, x)) \partial_i u(t, x) + \sum_i b_i(t, x) \partial_i u(t, x) \\
+ c(t, x) u(t, x) = g(t, x) \text{ in } D'(\{0, T\} \times \mathbb{R}^n),
\end{cases}
\]
\[u(0, x) = u_0(x) \text{ in } L^2_x(\mathbb{R}^n).\]
Moreover, $u \in H^1(\{0, T\}, H^{-1}_x(\mathbb{R}^n))$. This claim can be proved by considering the Cauchy problem for the unknown function $v(t, x) = e^{\lambda \tau} u(t, x)$ and using the standard duality argument (i.e. the Riesz representation theorem).

3. Main results

First, we shall transform equation (2.5) into a simple form and derive an integral equation. Setting
\[a_0 = \frac{\sigma_0^2 - 2 \mu_0}{2 \sigma_0^2}, \quad b_0 = r + \frac{1}{2} \sigma_0^2 a_0^2, \quad H_a = - \left( \frac{\partial}{\partial y} - a \right)^2, \quad a = a_0 - 1,
\]
we can rewrite (2.5) as follows
\[
\begin{cases}
\left( \frac{\partial}{\partial \tau} + \frac{1}{2} \sigma_0^2 H_a \right) v(\tau, y) = f(y) w(\tau, y), (\tau, y) \in (0, \tau^*) \times \mathbb{R}, \\
v(\tau, y)|_{\tau=0} = 0, \quad y \in \mathbb{R}, \tag{3.1}
\end{cases}
\]
where $v(\tau, y) = e^{-\gamma + b_0 \tau} V(\tau, y)$ and $w(\tau, y) = e^{-\gamma + b_0 \tau} \frac{\partial u_0}{\partial y}$. The function $w$ is the solution of the following problem
\[
\begin{cases}
\left( \frac{\partial}{\partial \tau} + \frac{1}{2} \sigma_0^2 H_a \right) w(\tau, y) = 0, (\tau, y) \in (0, \tau^*) \times \mathbb{R}, \\
w(\tau, y)|_{\tau=0} = 1_{[0, \infty)}(y), \quad y \in \mathbb{R}. \tag{3.2}
\end{cases}
\]
Making the change of variables $\tau = \frac{1}{\sigma_0^2} \tilde{\tau}$ and setting
\[\tilde{\tau}(\tau, y) = \tau(\tau, y), \tilde{w}(\tau, y) = w(\tau, y), \tilde{f}(y) = \frac{2}{\sigma_0^4} f(y), \tilde{\tau}^* = \frac{\sigma_0^2}{2} \tau^*,
\]
we can transform (3.1) and (3.2) into the following simple form
\[
\begin{cases}
\left( \frac{\partial}{\partial \tilde{\tau}} + H_a \right) \tilde{v}(\tilde{\tau}, y) = \tilde{f}(y) \tilde{w}(\tilde{\tau}, y), (\tilde{\tau}, y) \in (0, \tilde{\tau}^*) \times \mathbb{R}, \\
\tilde{v}(\tilde{\tau}, y)|_{\tilde{\tau}=0} = 0, \quad y \in \mathbb{R}. \tag{3.3}
\end{cases}
\]
and
\[
\begin{cases}
\left( \frac{\partial}{\partial \tau} + H_a \right) \tilde{w}(\tilde{\tau}, y) = 0, & (\tilde{\tau}, y) \in (0, \tilde{\tau}^*) \times \mathbb{R}, \\
\tilde{w}(\tilde{\tau}, y)|_{\tilde{\tau}=0} = 1_{[0, \infty)}(y), & y \in \mathbb{R}.
\end{cases}
\]

(3.4)

Hereafter we shall consider problems (3.3) and (3.4) wherein \( \tilde{\tau}, \tilde{v}, \tilde{w}, \tilde{f} \) and \( \tilde{\tau}^* \) are rewritten as \( \tau, v, w, f \) and \( \tau^* \), respectively.

By the well-known representation of the solution to the Cauchy problem (3.3), we have the following integral equation:
\[
\tilde{v}(\tau^*, y) = \int_0^{\tau^*} U_a(\tau^* - s)[w(s, \cdot)f(\cdot)](y)dx.
\]

(3.5)

Here
\[
(U_a(\tau)\varphi)(y) = \int_{\mathbb{R}} K_a(\tau, y - x)\varphi(x)dx, \quad K_a(\tau, y) = \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{|y|^2}{4\tau} + ay}
\]

and \( w(\tau, x) \) is represented by
\[
w(\tau, x) = \int_0^\infty \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{|x - y|^2}{4\tau} + aw(x - y')}dx' = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\tau e^{\frac{\tau^2}{4}} e^{-\theta^2} d\theta.
\]

(3.6)

We now state our main result for LIPD.

**Theorem 3.1.** Let \( S \) be a nonempty closed subset of \( \mathbb{R} \) which is bounded below. Let \( f \in L^2(\mathbb{R}) \). Assume that \( \text{supp} f \subset S \) and that
\[
Af(x) := \int_0^{\tau^*} U_a(\tau^* - s)[w(s, \cdot)f(\cdot)](x)dx
\]

vanishes in a neighborhood of \( S \). Then \( f = 0 \) in \( L^2(\mathbb{R}) \).

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### 3.1. Outline of proof

Since \( Af \) is analytic in \( S' := \mathbb{R} \setminus S \), it follows that \( Af = 0 \) in \( \mathbb{R} \). Take \( L > 0 \) such that \( S \subset [-L, \infty) \), and set \( L_0 = L + 1 \). We consider the so-called Fourier–Bros–Iagolnitzer (for short, FBI) transform of \( f \), \( T_h f(x, \xi) \) \( (x, \xi \in \mathbb{R}, h > 0) \), and show that there exists \( \delta > 0 \) such that
\[
\|T_h f\|_{L^2((-L_0, \infty) \times (|\xi| > \delta))} = O(e^{-\frac{\pi}{h}}) \quad (h \to +0).
\]

This estimate implies
\[
((-L_0, \infty) \times (|\xi| > \delta)) \cap \text{WF}_a(f) = \emptyset,
\]
where \( \text{WF}_a(f) \) is the analytic wave front set of \( f \). Hence \( f \) is analytic in \((-L_0, \infty) \). Since \( f = 0 \) in \( S' \), it follows that \( f \) is analytic in \( \mathbb{R} \), which implies \( f \) is zero in \( L^2(\mathbb{R}) \).

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### 4. Properties of the FBI transform

In this section, we summarize basic facts on the FBI transform and the semiclassical pseudo-differential operators, following Martinez [10], and prove three simple propositions, not explicitly stated in [10].
**Notation.** \( \mathbf{N}_0 = \{ 0, 1, 2, 3, \ldots \} \). \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \ (\xi \in \mathbb{R}^n) \). For normed spaces \( X \) and \( Y \), \( B(X, Y) \) is the set of all bounded linear operators from \( X \) to \( Y \), and \( B(X) = B(X, X) \).

Let \( h > 0 \) be a parameter. For \( u \in S'(\mathbb{R}^n) \), the \( h \)-Fourier transform of \( u \), \( \mathcal{F}_h u \), is defined by

\[
\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\frac{\xi}{h}x} u(x) \, dx \quad \xi \in \mathbb{R}^n.
\]

\( \mathcal{F}_h \) is an isomorphism of \( S(\mathbb{R}^n) \), and is extended, by continuity, to an isomorphism of \( S'(\mathbb{R}^n) \). It is also unitary on \( L^2(\mathbb{R}^n) \). Set \( \mathcal{F} = \mathcal{F}_1 \).

**Definition 4.1.** Let \( h > 0 \) be a parameter. For \( u \in S'(\mathbb{R}^n) \), the FBI transform of \( u \), \( T_h u \), is defined by

\[
T_h u(x, \xi) = 2^{-\frac{n}{2}} (\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi / h - |x-y|^2 / 2h} u(y) \, dy, \quad (x, \xi) \in \mathbb{R}^{2n}.
\]

Here the integral is in the sense of tempered distributions. \( (T_h u(x, \xi) \) is also denoted by \( Tu(x, \xi; h) \).)

**Remark 4.1.** \( T_h u(x, \xi) \) can be written as follows

\[
T_h u(x, \xi) = \mathcal{F}_h [g_h(\cdot) u(x + \cdot)](\xi).
\]

Here \( g_h(x) = (\frac{1}{\pi h})^{\frac{n}{2}} e^{-\frac{|x|^2}{4h}} \). Note that \( \|g_h\|_{L^2(\mathbb{R}^n)} = 1 \).

**Remark 4.2.** For \( u \in L^2(\mathbb{R}^n) \), \( \|T_h u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \).

**Definition 4.2.** A distribution \( u \in S'(\mathbb{R}^n) \) is called analytic at \((x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\) if there exist \( \delta > 0 \) and a neighborhood \( V \) of \((x_0, \xi_0)\) such that

\[
\|T_h u\|_{L^2(V)} = O(e^{-\delta/h}) \text{ as } h \to +0.
\]  

(4.1)

The analytic wave front set of \( u, WF_a(u) \), is the set of all \((x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\) at which \( u \) is not analytic.

**Remark 4.3.**

(i) We can replace \( \|T_h u\|_{L^2(V)} \) by \( \|T_h u\|_{L^\infty(V)} \) in (4.1).

(ii) If \((x, \xi) \in WF_a(u), \) then \((x, \xi) \in WF_a(u) \) for every \( t > 0 \).

(iii) \( u \) is analytic near \( x_0 \) if and only if \( \{x_0\} \times (\mathbb{R}^n \setminus \{0\}) \cap WF_a(u) = \emptyset \).

**Definition 4.3.** Let \( m \in \mathbb{R} \). The symbol class \( S_{2m}(\langle \xi \rangle^m) \) is the set of all \( p = p(x, \xi; h) : \mathbb{R}^{2n} \times (0, 1] \to \mathbb{C} \) satisfying the following conditions:

(i) \( p(\cdot, \cdot; h) \in C^\infty(\mathbb{R}^{2n}) \) for each \( h > 0 \).

(ii) For every \( \alpha, \beta \in \mathbb{N}_0^n \), there exists \( C > 0 \) such that

\[
|\partial_x^\alpha \partial_\xi^\beta p(x, \xi; h)| \leq C \langle \xi \rangle^m \text{ for every } (x, \xi; h) \in \mathbb{R}^{2n} \times (0, 1].
\]
**Definition 4.4.** Let $t \in [0, 1]$ and $h \in (0, 1]$. For $p = p(x, \xi) \in S_{2n}((\xi)^m)$ and $u \in S^m(\mathbb{R}^n)$, $\text{Op}^t_h(p)u \in S^m(\mathbb{R}^n)$ is defined by

$$\text{Op}^t_h(p)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi/h} p((1-t)x + ty, \xi)u(y)dyd\xi.$$ 

The operator $\text{Op}^t_h(p)$ can be extended to a continuous operator on $S^m(\mathbb{R}^n)$.

**Remark 4.4.** Set $\text{Op}^t(p) = \text{Op}^t_h(p)$. In this paper, we use only the case $t = 1$.

**Theorem 4.5.** Let $t \in [0, 1]$ be fixed. Then there exists $C_t > 0$ and $M_t \in \mathbb{N}$ such that for every $p = p(x, \xi) \in S_{2n}(1)$ and $h \in (0, 1]$,

$$\|\text{Op}^t_h(p)\|_{L^2(\mathbb{R}^n)} \leq C_t \sum_{|\alpha + \beta| \leq M_t} \sup_{x, \xi \in \mathbb{R}^n} |\partial^\alpha_x \partial^\beta_\xi p(x, \xi)|.$$ 

**Definition 4.6.** Let $a > 0$, and set $\Sigma_a = \{x \in \mathbb{C}^n; |\text{Im}x| < a\}$. The symbol class $S^{\text{hol}, m}_{2n}((\xi)^m, \Sigma_a)$ is the set of all $p = p(x, \xi, h) : \Sigma_a \times \mathbb{R}^n \times (0, 1) \to \mathbb{C}$ satisfying the following conditions:

(i) $p(\cdot, \cdot; h) \in C^\infty(\Sigma_a \times \mathbb{R}^n)$ for each $h > 0$.
(ii) For each $h$ and $\xi$, the function $\Sigma_a \ni x \mapsto p(x, \xi, h) \in \mathbb{C}$ is holomorphic.
(iii) For every $\alpha, \beta \in \mathbb{N}^n_0$, there exists $C > 0$ such that

$$|\partial^\alpha_x \partial^\beta_\xi p(x, \xi; h)| \leq C(\xi)^m$$

for every $(x, \xi; h) \in \Sigma_a \times \mathbb{R}^n \times (0, 1]$.

**Remark 4.5.** For $p \in S_{2n}((\xi)^m)$ or $S^{\text{hol}, m}_{2n}((\xi)^m, \Sigma_a)$, set $p_h(x, \xi) = p(x, \xi; h)$.

The following theorem is the key to the proof of our main theorem (see [10, corollary 3.5.5] and the final remark in [10, section 3.5.5]).

**Theorem 4.7 (Microlocal exponential estimates).** Let $t \in [0, 1]$ be fixed. Let $p \in S^{\text{hol}, m}_{2n}(1, \Sigma_a)$, and set $P_h = \text{Op}^h(p_h)$. Let $\psi = \psi(\xi) \in S_{2n}(1)$ (independent of $h > 0$) be real-valued, and assume $\sup_{\xi \in \mathbb{R}^n} |\nabla \psi(\xi)| < a$. Let $f \in S_{2n}(1)$. Then there exists $C > 0$ such that for every $u \in L^2(\mathbb{R}^n)$ and $h \in (0, 1]$,

$$\|f_h e^{i\psi/h} T_h u\|_{L^2(\mathbb{R}^n)}^2 \leq C \|f_h p(x - i\nabla \psi(\xi), \xi - \nabla \psi(\xi); h) e^{i\psi/h} T_h u\|_{L^2(\mathbb{R}^n)}^2.$$ 

Moreover, $C$ can be taken uniformly in $\varepsilon \in (0, 1]$ if $\psi$ is replaced by $\varepsilon \psi$.

We shall state and prove three propositions, not explicitly stated in [10].

**Proposition 4.8.** Let $F_1, F_2$ be closed subsets of $\mathbb{R}^n$ satisfying

$$\text{dist}(F_1, F_2) = \sigma > 0.$$ 

(4.2)
Then for every \( h > 0 \) and \( u \in L^2(\mathbb{R}^n) \) with \( \text{supp} \, u \subset F_1 \),
\[
\|T_hu\|_{L^2(R^n)}^2 \leq 2^4 e^{-\frac{\sigma^2}{2}} \|u\|_{L^2(R^n)}^2.
\]  

**Proof.** By remark 4.1, we obtain
\[
\|T_hu\|_{L^2(R^n)}^2 = \int_{F_2} \int_{R^n} |g_h(x-y)|^2 |u(y)|^2 \, dy.
\]
By the assumption (4.2), we have
\[
|g_h(x-y)|^2 \leq (\pi h)^{-n/2} e^{-\frac{|x-y|^2}{h}} \quad (x \in F_2, \, y \in F_1).
\]
Therefore
\[
\|T_hu\|_{L^2(R^n)}^2 \leq (\pi h)^{-n/2} e^{-\frac{\sigma^2}{h}} \int_{F_2} \int_{R^n} e^{-\frac{|x-y|^2}{h}} |u(y)|^2 \, dy
\]
\[
\leq (\pi h)^{-n/2} e^{-\frac{\sigma^2}{h}} \int_{R^n} \int_{R^n} e^{-\frac{|x-y|^2}{h}} |u(y)|^2 \, dy
\]
\[
= 2^4 e^{-\frac{\sigma^2}{2}} \|u\|_{L^2(R^n)}^2.
\]
\[\square\]

**Proposition 4.9.** Let \( F_1, F_2 \) be closed subsets of \( \mathbb{R}^n \) satisfying
\[
\text{dist}(F_1, F_2) = \sigma > 0.
\]  
Then for every \( h > 0 \) and \( u \in L^2(\mathbb{R}^n) \) with \( \text{supp} \, F_h u \subset F_1 \),
\[
\|T_hu\|_{L^2(R^n \times F_2)}^2 \leq 2^4 e^{-\frac{\sigma^2}{2}} \|u\|_{L^2(R^n)}^2.
\]

**Proof.** By [10, remark 3.4.4], we have
\[
T_hu(x, \xi) = e^{i \xi \cdot x / h} T_h F_h u(\xi, -x).
\]

Applying proposition 4.8, we can easily obtain the desired result. \[\square\]

**Proposition 4.10.** Let \( \sigma > 0 \) and \( r > 0 \). Then there exists \( \delta = \delta(\sigma, r, n) > 0 \) such that for every \( h > 0 \) and \( u \in L^2(\mathbb{R}^n) \) with \( e^{\sigma|\xi|} \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n) \),
\[
\|T_hu\|_{L^2(R^n \times \{\xi \in R^n : |\xi| > r\})}^2 \leq 2^{n/2} e^{-\frac{2\delta^2}{r}} \|e^{\sigma|\xi|} \mathcal{F}u\|_{L^2(R^n)}^2.
\]

**Proof.** Let \( u \in L^2(\mathbb{R}^n) \) with \( e^{\sigma|\xi|} \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n) \) and set \( v(x) = \mathcal{F}^{-1}[e^{\sigma|\xi|} \mathcal{F}u(\xi)](x) \). Set
\[
\delta = \inf \{ |\xi - \eta|^2 / 2 + 2\sigma |\eta| : x, \xi \in \mathbb{R}^n, \, |\xi| \geq r > 0 \}.
\]
It follows from (4.7) and remark 4.1 that
\[
\|T_hu\|^2_{L^2(R^+ \times \{|s| > r\})} = \int_{R^+} \int_{|s| > r} |g_h(x - \eta)|^2 |\mathcal{F}_h u(\eta)|^2 \, ds \, d\eta \\
= (\pi h)^{-n/2} \int_{R^+} \int_{|s| > r} e^{-\frac{|s|^2}{4h} - \frac{|\xi|^2}{4h}} |\mathcal{F}_h u(\eta)|^2 \, ds \, d\eta \\
\leq (\pi h)^{-n/2} \int_{R^+} \int_{|s| > r} e^{-\frac{|\xi|^2}{4h}} |\mathcal{F}_h u(\eta)|^2 \, ds \, d\eta \\
\leq (\pi h)^{-n/2} \int_{R^+} \int_{|s| > r} e^{-\frac{|\xi|^2}{4h}} |\mathcal{F}_h u(\eta)|^2 \, ds \, d\eta \\
= 2^{n/2} e^{-\frac{r}{4} \|v\|^2_{L^2(R^+)}}.
\]

5. Properties of the integral operator A

In this section, we prove several properties of the integral operator A
\[ Af = \int_0^\tau U_a(\tau^+ - s)w(s, \cdot) f(\cdot) \, ds \quad (f \in L^2(R)). \]

Since \( w \in L^\infty([0, \tau^*], L^\infty(R)) \), it follows easily that \( A \in B(L^2(R)) \). We shall see that \( H^*_A \in B(L^2(R)) \) by lemmas 5.2 and 5.3.

First, we fix \( \tau_0^* \in (0, \tau^*) \), and write \( Af = A_1 f + A_2 f \), where
\[
A_1 f = \int_0^{\tau_0^*} U_a(\tau^+ - s)w(s, \cdot) f(\cdot) \, ds, \\
A_2 f = \int_{\tau_0^*}^\tau U_a(\tau^+ - s)w(s, \cdot) f(\cdot) \, ds.
\]

**Lemma 5.1.**

(i) For every \( \tau > 0 \) and \( x \in R \), \( w(\tau, x) \leq e^{2\tau} \).
(ii) \( w(\tau, z) \in C^\infty([0, \infty) \times C) \). Moreover, for each \( \tau > 0 \), \( w(\tau, \cdot) \) is an entire function of \( z \).
(iii) For every \( \sigma_0 > 0 \), \( k, \alpha \in N_0 \) and \( \tau_0 \in (0, \tau^*) \), there exists \( C > 0 \) such that
\[
|\partial_z^k \partial_{\tau_0}^\alpha w(\tau, z)| \leq C \text{ for every } \tau \in [\tau_0, \tau^*] \text{ and } z \in \{ z \in C; \text{Im} z < \sigma_0 \}.
\]
(iv) For every \( \tau_0 \in (0, \tau^*) \) and \( L_0 > 0 \), there exists \( C_0 > 0 \) such that
\[
w(\tau, x) \geq C_0 \text{ for every } \tau \in [\tau_0, \tau^*] \text{ and } x \geq -L_0.
\]

**Proof.** The claims (i) \( \sim \) (iv) follow easily from the form of \( w \). \[ \square \]

**Lemma 5.2.**

(i) \( e^{2\sigma^2} \mathcal{F}H^*_A \in B(L^2(R)) \) for every \( \sigma \in (0, \tau^* - \tau_0^*) \).
(ii) There exist \( \delta > 0 \) and \( C > 0 \) such that for every \( h > 0 \) and \( f \in L^2(R) \),
\[
\|T_h H^*_A f\|^2_{L^2(R \times \{|s| > 1\})} \leq Ce^{-\frac{\tau}{4} \|f\|^2_{L^2(R)}}.
\]
Proof. By lemma 5.1 (i) and \( \|U_a(s)\|_{b(L^2(R))} \leq e^{\sigma z} \) (\( s \geq 0 \)), we obtain
\[
\int_0^\tau \|U_a(\tau_0^s - s)w(s,\cdot)f(\cdot)\|_{L^2(R)} ds \leq \tau_0^s e^{\sigma z} \|f\|_{L^2(R)},
\]
which implies \( B_1 \in B(L^2) \), where
\[
B_1f = \int_0^\tau U_a(\tau_0^s - s)[w(s,\cdot)f(\cdot)]ds.
\]
Observing \( \mathcal{F}[H_aA_1f](\xi) = (\xi + ia)^2 e^{-(\tau^s - \tau_0^s)(\xi + ia)^2} \mathcal{F}B_1f(\xi) \), we can complete the proof of (i). The claim (ii) follows from (i) and proposition 4.10.

Next we write \( H_aA_2f(x) \) in the following form
\[
H_aA_2f(x) = \int_0^\tau H_aU_a(\tau^s - s)[w(s,\cdot)f(\cdot)](x) ds = Op^1(p) f(x),
\]
where
\[
p(x, \xi) = (\xi + ia)^2 \int_0^\tau e^{-(\tau^s - s)(\xi + ia)^2} w(s,x) ds
\]
\[
= \int_0^\tau \frac{\partial}{\partial s} e^{-(\tau^s - s)(\xi + ia)^2} w(s,x) ds
\]
\[
= w(\tau^s, x) - e^{-(\tau^s - \tau_0^s)(\xi + ia)^2} w(\tau_0^s, x) - \int_0^\tau e^{-(\tau^s - s)(\xi + ia)^2} \frac{\partial w}{\partial s}(s,x) ds.
\]
Take \( \chi_1(\xi) \in C_0^\infty (R) \) such that \( \chi_1 = 0 \) if \( |\xi| < \frac{1}{2}, \chi_1 = 1 \) if \( |\xi| > \frac{1}{2} \), and set
\[
p_j(x, \xi; h) = p_j(x, \xi) = p(x, \xi/h) \chi_j(\xi) (j = 1, 2),
\]
where \( \chi_2(\xi) = 1 - \chi_1(\xi) \). Then we can write
\[
H_aA_2f(x) = Op^1(p) f(x) = Op^1(p_1,h) f(x) + Op^1(p_2,h) f(x).
\]

Lemma 5.3.
(i) For every \( \sigma_0 > 0 \), \( \alpha, \beta \in \mathbf{N}_0 \), there exists \( C > 0 \) such that for every \( z \in C \) with \( |\text{Im}z| < \sigma_0 \) and \( \xi \in \mathbf{R} \),
\[
|\partial_{\xi}^\alpha \partial_z^\beta p(z, \xi)| \leq C(\xi)^{-|\beta|}.
\]
(ii) There exists \( C > 0 \) such that for every \( x, \xi \in \mathbf{R} \),
\[
|p(x, \xi) - w(\tau^s, x)| \leq C(\xi)^{-2}.
\]
(iii) \( H_aA_2 \in B(L^2(\mathbf{R})) \).

Proof. The claims (i), (ii) follow from lemma 5.1 and the definition of \( p \). Since \( p \in S_2(1) \) by (1), it follows that \( H_aA_2 \in B(L^2(\mathbf{R})) \).
Lemma 5.4.

(i) \( p_1(x, \xi; h) \in S_2^{\text{bol}}(1, \Sigma_{0}) \) for every \( \sigma_0 > 0 \).

(ii) There exists \( C > 0 \) such that for every \( x, \xi \in \mathbb{R} \) and \( h \in (0, 1] \)

\[
|p_1(x, \xi; h) - w(\tau^*, x)\chi_1(\xi)| \leq Ch^2.
\]

Proof. The claims (i) and (ii) follow from lemma 5.3 (1) and (2), respectively. \( \square \)

Lemma 5.5. There exists \( C > 0 \) such that for every \( f \in L^2(\mathbb{R}) \) and \( h \in (0, 1] \)

\[
\|T_h\|f(p_{2,h})f\|L^2(\mathbb{R}_x, \{\xi > 1\}) \leq Ce^{-\frac{\tau}{h}}\|f\|L^2(\mathbb{R}). \tag{5.5}
\]

Proof. Since

\[
\text{supp } F^{-1}(p_{2,h})f(x) = (2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-y)\xi/h}p_{2,h}(y, \xi)f(y)dyd\xi,
\]

it follows that

\[
\mathcal{F}_h[\text{supp } F^{-1}(p_{2,h})f(x)](\xi) = (2\pi h)^{-1} \int_{\mathbb{R}} e^{-i\xi/h}p_{2,h}(y, \xi)\chi_1(\xi)f(y)dy.
\]

By the definition of \( p_{2,h} \), we have

\[
\text{supp } F^{-1}(p_{2,h})f(x) \subset \{\xi \leq 1/2\}.
\]

Set \( q_h(x, \xi) = p(x, \xi)\chi_2(h\xi) \). Using theorem 4.5, we can find \( C > 0 \), independent of \( h \in (0, 1] \), such that

\[
\|\text{supp } F^{-1}(p_{2,h})f\|L^2(\mathbb{R}) = \|\text{supp } F^{-1}(q_h)\|L^2(\mathbb{R}) \leq C\|f\|L^2(\mathbb{R}).
\]

Applying proposition 4.9, we can complete the proof. \( \square \)

6. Proof of theorem 3.1

Let \( S \) be a nonempty closed subset of \( \mathbb{R} \) which is bounded below. Let \( f \in L^2(\mathbb{R}) \). Assume that \( \text{supp } f \subset S \) and that \( Af = 0 \) in a neighborhood of \( S \).

First, we shall prove \( Af = 0 \) in \( \mathbb{R} \). Recall that

\[
Af(x) = -\int_{0}^{\tau^*} \int_{\mathbb{R}} K_\alpha(\tau^* - s, x - y)w(s, y)f(y)dyds, \quad x \in \mathbb{R},
\]

with

\[
K_\alpha(\tau, z) = \frac{1}{\sqrt{4\pi} e^{\frac{-|z|^2+\tau^2}{4}}} e^{-z \tau^* + \tau}, \quad \tau > 0, z \in \mathbb{C}.
\]

Then \( K_\alpha(\tau, z) \) is holomorphic in \( z \in \mathbb{C} \), and for every compact subset \( K \) of \( U = \{z \in \mathbb{C}; |\text{Im } z| < \text{dist(Re } z, S)\} \), there exist \( C, C', \alpha > 0 \) such that the following estimates hold for every \( 0 < \tau \leq \tau^*, z \in K, y \in \mathbb{R} \):
Using Morera’s theorem and Fubini’s theorem, we see that $Af$ can be extended to a holomorphic function in $U$, which implies that $Af$ is analytic in $S$. Since $Af = 0$ in a neighborhood of $S$, $Af$ is analytic in $R$. Therefore, $Af = 0$ in $R$.

Hereafter, we denote by $\delta_i$ and $C_j$ positive constants independent of $\varepsilon \in (0, 1]$ and $h \in (0, 1]$, and by $\| \cdot \|$ the norm $\| \cdot \|_{L^2(R)}$. Take $L > 0$ such that $S \subset [-L, \infty)$, and set $L_0 = L + 1$. Since $\text{supp} \ f \subset S \subset [-L, \infty)$, it follows from proposition 4.8 that

$$
\| T_h f \|^2_{L^2([-\infty, -L] \times R)} \leq C_0 e^{-\varepsilon^2}. \tag{6.1}
$$

Write

$$
0 = H_{\varepsilon} Af = H_{\varepsilon} A_1 f + \text{Op}_h^1(p_{1,h}) f + \text{Op}_h^1(p_{2,h}) f.
$$

By virtue of lemmas 5.2 and 5.5,

$$
\| T_h H_{\varepsilon} A_1 f \|^2_{L^2(R \times \{|\xi| \leq 1\})} \leq C_1 e^{-\varepsilon^2},
$$

$$
\| T_h \text{Op}_h^1(p_{2,h}) f \|^2_{L^2(R \times \{|\xi| \leq 1\})} \leq C_2 e^{-\varepsilon^2}.
$$

Combining these estimates, we have

$$
\| T_h \text{Op}_h^1(p_{1,h}) f \|^2_{L^2(R \times \{|\xi| \leq 1\})} \leq C_3 e^{-\varepsilon^2}. \tag{6.2}
$$

Take a real-valued function $\psi \in C^\infty(R)$ such that $\psi = 0$ if $|\xi| < 1$, $\psi = 1$ if $|\xi| > 2$, and $0 \leq \psi \leq 1$. Let $\varepsilon \in (0, 1]$ be a small parameter to be fixed later. Define

$$
T^\varepsilon_h u = e^{\psi(\xi)/h} T_h u.
$$

Using theorem 4.5 and lemma 5.4, we obtain

$$
\| T^\varepsilon_h \text{Op}_h^1(p_{1,h}) f \|^2_{L^2(R \times \{|\xi| \leq 1\})} = \| T_h \text{Op}_h^1(p_{1,h}) f \|^2_{L^2(R \times \{|\xi| \leq 1\})} \leq \| \text{Op}_h^1(p_{1,h}) f \|^2_{L^2(R)} \leq C_4. \tag{6.3}
$$

Applying theorem 4.7, we obtain our main microlocal exponential estimate:

$$
\| T^\varepsilon_h \text{Op}_h^1(p_{1,h}) f \|^2 \geq \| p_1(x - i \varepsilon \partial_x \psi(\xi), \xi - \varepsilon \partial_\xi \psi(\xi); h) T_h f \|^2 - C_5 h \| T_h f \|^2.
$$

Using Taylor’s formula and lemma 5.4, we obtain

$$
\| T^\varepsilon_h \text{Op}_h^1(p_{1,h}) f \|^2 \geq \| p_1(y, \xi; h) T_h f \|^2 - C_6(\varepsilon + h) \| T_h f \|^2 \geq \| w(\tau^*, x) \chi_1 T_h f \|^2 - C_7(\varepsilon + h) \| T_h f \|^2 \geq 2^{-1} \| w(\tau^*, x) T_h f \|^2 - 2 \| w(\tau^*, x) \chi_2 T_h f \|^2 - C_7(\varepsilon + h) \| T_h f \|^2.
$$

Setting $c_0 = \inf_{x \geq -L_0} w(\tau^*, x)^2/2 > 0$, we have

$$
\| T^\varepsilon_h \text{Op}_h^1(p_{1,h}) f \|^2 \geq (c_0 - C_7(\varepsilon + h)) \| T_h f \|^2_{L^2([-L_0, \infty) \times R)} \geq C_7(\varepsilon + h) \| T_h f \|^2_{L^2([-\infty, -L_0] \times R)} - C_8. \tag{6.4}
$$

where we used $\psi(\xi) = 0$ if $|\xi| < 1$. By virtue of (6.1)–(6.4), fixing $\varepsilon$ such that $0 < \varepsilon < \min\{1/16, \delta_3/2, c_0/(2C_7)\}$, we obtain

$$
|K_n(\tau, z - y)| \leq \frac{C}{\sqrt{T}} e^{-\|\xi\|_C^2} \leq C' e^{-\|\xi\|_C^2} \leq C' e^{-\|\xi\|_C^2}. 
$$
Since \( \psi(\xi) = 1 \text{ if } |\xi| > 2 \), we conclude
\[
\|T_h f\|_{L^2((-L_0, \infty) \times \{ |\xi| > 2 \})} = O(e^{-\frac{2}{h^2}}) \quad (h \to +0).
\] (6.5)

This estimate implies
\[
((-L_0, \infty) \times \{ |\xi| > 2 \}) \cap \text{WF}_a(f) = \emptyset.
\]

Hence \( f \) is analytic in \((-L_0, \infty)\). Since \( f = 0 \) in \((-\infty, -L)\), it follows that \( f \) is analytic in \( \mathbb{R} \), which implies \( f \) is zero in \( L^2(\mathbb{R}) \). This completes the proof.

7. Numerical algorithm and its testing

In this section, we propose and numerically test our algorithm for reconstructing a real drift from several discrete option prices.

Here, our algorithm provides a method to reconstruct a trend coefficient from observed data using integral equation (3.5), which we derived in section 3 above. A discrete representation of integral equation (3.5) is obtained by using the method presented in [7] as
\[
v(\tau^*, x_i) \approx \sum_{j=1}^{n} K(x_i, y_j; \tau^*) f(y_j) \Delta y,
\] (7.1)

where \( x_i (i = 1, \cdots, m) \) are measurements points, \( y_j (j = 1, \cdots, n) \) are points on interval \( \omega \) that divide \( \omega \) into an \((n-1)\) grid, and \( \Delta y = \frac{|y_n - y_1|}{n-1} \). Further, we have
\[
K(x_i, y_j; \tau^*) = \int_0^{\tau^*} \frac{1}{\sqrt{2\pi(\tau^* - s)\sigma^2}} \exp \left( -\frac{|x_i - y_j|^2}{2(\tau^* - s)\sigma^2} \right) w_0(x, y_j) ds
\] (7.2)

where \( w_0(s, y) \) is
\[
w_0(s, y) = \frac{2}{\pi} e^{(1-a_0)^2 s^2 + (1-a_0) s} \int_{\frac{\pi}{2} + \tau_0}^{\infty} e^{-\theta^2} d\theta.
\]

Given the above, we next consider the problem of finding trend coefficient \( f \) that satisfies equation
\[
V = Kf
\] (7.3)

where
\[
V = (v(x_1, \tau^*), v(x_2, \tau^*), \cdots, v(x_m, \tau^*))',
\] (7.4)

\[
f = (f(y_1), f(y_2), \cdots, f(y_n))'
\] (7.5)

and \( K \) are \( m \times n \) matrices in which each \((i, j)\)-entry is expressed by (7.2).

Using the above, we consider the following minimization problem by applying the Tikhonov regularization method (see [5, 6] and [9]) i.e. we find \( f \) that minimizes the functional
\[
\|Kf - V\|^2_0 + \lambda \|f\|^2_0,
\] (7.6)
for given regularization parameter $\lambda$ and measured (noisy) data $V^\delta$, where $V^0 = V$, i.e. we set the noiseless data $V$ parameter to be $V^0$, and the above norms $\| \cdot \|_a, \| \cdot \|_b$ will be selected depending on the available data. To verify the effectiveness of our algorithm, we solve minimization problem (7.6) with measured data $\nu^\delta(\tau^*, x_i)$ equal to $e^{-a_0 x - b_0 \tau^*} (U^0(\tau^*, x_i) - U_0(\tau^*, x_i))$, where $U^0$ solves parabolic equation (2.3) with $a(y)$, i.e. $U^0$ represents noiseless data $U$, and we set 1% and 5% relative random noise data $U_0.01$ and $U_0.05$, respectively, as data contaminated with noise by generating a normal random number in MATLAB. Further, $U_0$ solves parabolic equation (2.3) with $\mu_0$ instead of $a(y)$.

As described in the subsections that follow, we verify that we can reconstruct function $f(y)$ from $\nu^\delta(\tau^*, x_i)$ by using our algorithm, particularly in the case of $f(y) = y$.

7.1. Direct problem

First, we assume $\sigma_0 = 1$, $\mu_0 = 1$ and $r = 0$. The direct problem for (2.3) is then solved numerically, using the Crank–Nicholson scheme, as

$$a_j U_{i+1,j+1} + (1 + b_j) U_{i+1,j} + c_j U_{i+1,j-1} = -a_j U_{i,j+1} + (1 - b_j) U_{i,j} - c_j U_{i,j-1},$$

(7.7)

where $U_{i,j} = U(t_i, y_j)$, and

$$a_j = -\frac{\Delta \tau}{4 (\Delta y)^2} \left( 1 + \Delta y \left( \frac{1}{2} + y_j \right) \right),$$

$$b = \frac{\Delta \tau}{2 (\Delta y)^2},$$

$$c_j = -\frac{\Delta \tau}{4 (\Delta y)^2} \left( 1 - \Delta y \left( \frac{1}{2} + y_j \right) \right).$$

Here, we take a uniform grid

$$\mathcal{\tilde{\omega}} = \{(\tau_i, y_j) : \tau_i \in (0, \tau^*), y_j \in I_15 = (-15, 15), i = 1, 2, \ldots, M, j = 1, 2, \ldots, 800\}$$

with artificial zero Dirichlet boundary conditions at $y = -15$ and 15, and $\Delta \tau = \tau_{i+1} - \tau_i = 0.001$, $\Delta y = y_{i+1} - y_i = \frac{30}{799}$.

From this, the discrete representation of direct problem (2.3) can be written as

$$u_{i+1} = A^{-1} B u_i$$

(7.8)

where $u_i = (U_{i,2}, U_{i,3}, \ldots, U_{i,799})^T$, $e_{798} = (0, 0, \ldots, 0, 1)^T$ and

$$A = \begin{pmatrix}
1 + b & a_2 & 0 & 0 & \cdots & 0 \\
a_3 & 1 + b & a_3 & 0 & \cdots & 0 \\
0 & c_4 & 1 + b & a_4 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & c_{798} & 1 + b & a_{798} & 0 \\
0 & \cdots & 0 & c_{799} & 1 + b & 0
\end{pmatrix}.$$
where \( \tilde{b} = b + \Delta r \). Next, we aim to solve for \( U_0 \) in parabolic equation (2.3) with \( \mu_0 \) instead of \( a(y) \). Substituting \( W_0(\tau, y) = e^{\alpha \tau - \beta y}U_0(\tau, y) \) transforms (2.3) with \( \mu_0 \) instead of \( a(y) \) into

\[
\begin{align*}
\frac{\partial W_0}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 W_0}{\partial y^2} &= 0, \\
W(\tau, y)|_{\tau=0} &= e^{-a_0 y} \max\{e^y - 1, 0\}. 
\end{align*}
\]

(7.9)

Using the formula for the fundamental solution to heat equation (7.9), we obtain

\[
U_0(\tau^*, y) = 2 \pi \frac{e^{\frac{1}{4} \tau \sigma^2 + \tau \mu_0}}{\sqrt{2 \tau \sigma_0^2}} \text{erfc}(q_1) + 2 \pi \frac{e^{\frac{1}{4} \tau \sigma^2 - \tau \mu_0}}{\sqrt{2 \tau \sigma_0^2}} \text{erfc}(q_2),
\]

where

\[
q_1 = -\frac{x + \frac{1}{4} \tau \sigma^2 + \tau \mu_0}{\sqrt{2 \tau \sigma_0^2}}, \quad q_2 = -\frac{x - \frac{1}{4} \tau \sigma^2 + \tau \mu_0}{\sqrt{2 \tau \sigma_0^2}}.
\]

Figure 1. Noiseless.
Figure 2. 1% relative random noise.

Figure 3. 5% relative random noise.
and

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} \, dt. \]

7.2. Inverse problems

By using the solution of the direct problem defined in equation (7.8) with artificial boundary conditions and equation (7.10), combined with the Tikhonov regularization method, we can inversely compute solution \( f(y) = y \) to equation (3.5). To verify whether our numerical algorithm for reconstructing the trend coefficient is valid and stable, if the data is contaminated with noise, we use measured data, noiseless data, 1% and 5% relative random noise data. Further, we use \( \omega_x = (-0.8, 0.8) \) as the interval of measured data \( v(\tau, x) \), where \( x_i \) \( (i = 1, 2, \ldots, 42) \) are points of \( \omega_x \) and their collection coincides with \( y_j \) \( (j = 1, 2, \ldots, 42) \), which are points of \( \omega \) used in solving the direct problem. Also, since the prices near the strike are the most interest for practitioners, we consider results the reconstruction of the trend coefficient on interval \((-0.15, 0.15)\), to investigate the recovered function around \( y = 0 \) \((S = K)\).

Figure 1 shows reconstruction results using noiseless data, where we use 6 different observation times \( \tau^* = 0.4, 0.475, 0.55, 0.625, 0.7, \) and 0.775. From the figure, we observe that the reconstruction is numerically near-perfect around \( y = 0 \) when \( \tau^* = 0.55 \). For larger time \( \tau^* = 0.775 \), the reconstructed \( f \) is better near \( y = -0.1 \), but conversely, for smaller time \( \tau^* = 0.4 \), the reconstructed \( f \) is near-perfect on interval \((0.1, 0.15)\), as shown in figure 1.

Thus, owing to the selection of \( \tau^* \) and \( \lambda \), the change of the position of \( y \), at which \( f(y) \) is reconstructed, depending on values of \( \tau^* \) is seen in the reconstruction, but we can see that the reconstruction is near-perfect on interval \((-0.15, 0.15)\).

Next, we compare the reconstructed trend coefficients using the noiseless data with the 1% and 5% relative random noise data. Figures 2 and 3 show results of the reconstruction using the 1% and 5% relative random noise data, respectively.

From figure 2, we observe that the reconstruction of the 1% relative random noise data is almost identical to the reconstruction using noiseless data shown in figure 1. From figure 3, we observe that the reconstruction of the 5% relative random noise data has a noticeable gap when compared to the reconstruction of the noiseless data. Nonetheless, even in this case, the form of the reconstruction of the noiseless data is maintained, and we think, changes depending on the size of the error (i.e. noise).

From the above results, we conclude that our numerical algorithm for reconstructing trend coefficients is indeed stable for data containing the 1% and 5% relative random noise data.

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