Abstract. In this article, we study minimal isometric immersions of Kähler manifolds into product of two real space forms. We analyse the obstruction conditions to the existence of pluriharmonic isometric immersions of a Kähler manifold into those spaces and we prove that the only ones into $S^{m-1} \times \mathbb{R}$ and $H^{m-1} \times \mathbb{R}$ are the minimal isometric immersions of Riemannian surfaces. Furthermore, we show that the existence of a minimal isometric immersion of a Kähler manifold $M^{2n}$ into $S^{m-1} \times \mathbb{R}$ and $S^{m-k} \times H^k$ imposes strong restrictions on the Ricci and scalar curvatures of $M^{2n}$. In this direction, we characterise some cases as either isometric immersions with parallel second fundamental form or anti-pluriharmonic isometric immersions.

1. Introduction

This paper deals with minimal isometric immersions of a Kähler manifold into product of two real space forms. More specifically, we will be interested first with obstructions to the existence of pluriharmonic isometric immersions and secondly with restrictions on the Ricci curvature and scalar curvature of minimal Kähler isometric immersions into those spaces.

Over the years, pluriharmonic isometric immersions have been studied by several authors. In the literature, they are also called by (1, 1)-geodesic immersions and circular immersions (cf. [2, 6]). Those immersions appear as a natural extension of minimal immersions of Riemann surfaces into a target space, therefore they are minimal in the classical sense. The simplest examples of pluriharmonic isometric immersions are orientable minimal surfaces in an arbitrary Riemannian manifolds and holomorphic isometric immersions between Kähler manifolds, and it is important to notice that those immersions also have associated families of pluriharmonic isometric immersions when the ambient manifold is a Riemannian symmetric space (cf. [5]).

In real space forms, the study of pluriharmonic isometric immersions and their associated families is due to Dajczer and Gromoll (cf. [2]). They proved that for each non-holomorphic pluriharmonic isometric immersion into real space forms there exists a one-parameter family of noncongruent pluriharmonic submanifolds, likewise minimal surfaces in three-dimensional space forms. Dajczer and Rodríguez proved another interesting fact about pluriharmonic isometric immersions of Kähler manifolds into Euclidean spaces. They showed that pluriharmonic and minimal Kähler submanifolds mean the same in Euclidean spaces, although this is not obvious (cf. [3]). In addition, they proved that the only minimal isometric immersions of Kähler
manifold $M^{2n}$ into $\mathbb{H}^m$ are the minimal isometric immersions of Riemannian surfaces.

More generally for locally symmetric Riemannian manifold of non-compact type, Ferreira, Rigoli and Tribuzy showed that pluriharmonic isometric immersions and minimal isometric immersions of Kähler manifold $M^{2n}$ into those spaces are also the same objects (cf. [6]). Under some assumptions on the Ricci and scalar curvature of those target spaces, they proved additionally that the only pluriharmonic isometric immersion of Kähler manifold $M^{2n}$ into conformally flat Riemannian manifolds are the minimal isometric immersions of Riemannian surfaces.

In a seminal work, Takahashi established a necessary condition on the Ricci curvature $\text{Ric}_M$ for that a given Riemannian manifold $M^n$ admits a minimal isometric immersion into a real space form of constant sectional curvature $c$ (cf. [14]). This geometric restriction appears naturally as a consequence of Gauss equation for minimal isometric immersions and, up to normalization, the Ricci curvature must satisfy $\text{Ric}_M \leq c(n-1)$, with $n \geq 2$. In another direction of [3], Dajczer and Rodríguez proved if we want to immerse minimally in $S^m$ a Kähler manifold $M^{2n}$, this necessary condition will be more restrictive. In this case, the Ricci curvature must satisfy $\text{Ric}_M \leq nc$, with $n \geq 1$. In both works, under assumption of the existence of a minimal isometric immersion, the authors characterise the equality case as a totally geodesic isometric immersion (Takahashi theorem) and as an isometric immersion with parallel second fundamental form (Dajczer-Rodríguez theorem).

The aim of the work is to generalize these results to some products of real space forms.

First, we show that the only pluriharmonic isometric immersions of a Kähler manifold $M^{2n}$ into $S^{m-1} \times \mathbb{R}$ and $H^{m-1} \times \mathbb{R}$ are the minimal isometric immersions of Riemannian surfaces. We remark that minimal and pluriharmonic isometric immersions of a Kähler manifold into $H^{m-1} \times \mathbb{R}$ are the same objects, by Ferreira-Rigoli-Tribuzy results. Dual results are obtained for maps into $S^{m-k} \times \mathbb{H}^k$ and into warped product manifolds $I \times_R S^{m-1}$, $I \times_R H^{m-1}$, where $I \subset \mathbb{R}$ is an interval, under some additional hypotheses.

Furthermore, we discuss how the existence of a minimal isometric immersion of a Kähler manifold $M^{2n}$ into $S^{m-1} \times \mathbb{R}$ and $S^{m-k} \times \mathbb{H}^k$ can to impose strong restrictions on the Ricci curvature and the scalar curvature of $M^{2n}$. In this direction, thanks to the complex structure of $M^{2n}$, we obtain a better upper bound of Ricci curvature of minimal isometric immersions of Kähler manifolds into those manifolds, and we characterise the equality case as isometric immersions with parallel second fundamental form. We also obtain an improvement to the upper bound of scalar curvature, and we characterise the equality case as anti-pluriharmonic isometric immersions. Moreover, we observe that our technique generalize those results to isometric immersions of a Kähler manifold into conformally flat Riemannian manifolds. This case was studied by Ferreira, Rigoli and Tribuzy with certain bounds assumptions on the Ricci curvature of the ambient space.

2. Preliminaries

Let $c_1, c_2 \in \mathbb{R}$ and $n_1, n_2 \in \mathbb{N}$. We denote by $Q^n_{c_i}$ be the $n_i$-dimensional simply connected Riemannian manifold of constant sectional curvature $c_i$, for $i = 1, 2$. As usual, $Q^n_c = S^n_c$ is the $n$-Sphere for $c > 0$, $Q^n_c = \mathbb{R}^n$ is the Euclidean $n$-space for $c = 0$ and $Q^n_c = \mathbb{H}^n_c$ is the Hyperbolic $n$-space for $c < 0$. Finally, we consider
$\mathbb{Q}^m = \mathbb{Q}^{n_1}_c \times \mathbb{Q}^{n_2}_c$ be the Riemannian product manifold endowed with the product metric, denoted by $\langle \cdot , \cdot \rangle$, where $m = n_1 + n_2$, and let $\pi_i : \mathbb{Q}^m \to \mathbb{Q}^{n_i}_c$ be the projection onto the factor $\mathbb{Q}^{n_i}_c$, for $i = 1, 2$.

Throughout this work, we consider $(M^{2n}, ds^2)$ a $2n$-dimensional simply connected Kähler manifold with almost complex structure $J$, with $n \in \mathbb{N}$. This means that $M$ is a $2n$-dimensional simply connected smooth manifold, endowed with a Riemannian metric $ds^2$ (also denoted by $\langle \cdot , \cdot \rangle$), such that the almost complex structure $J$ is a parallel orthogonal tensor on the tangent bundle of $M$, i.e., $J^2 = -\text{Id}_{TM}$,

$$\langle JX, JY \rangle = \langle X, Y \rangle$$

and

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y = 0,$$

for all $X, Y \in \mathfrak{X}(M)$, where $\text{Id}_{TM}$ is the identity tensor on $TM$, $\nabla$ denotes the Riemannian connection of $M$ and $\mathfrak{X}(M) = \Gamma(TM)$ denotes the section of $TM$.

We fix the Riemann curvature tensor $\mathcal{R}$ of $M$, given by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and the Ricci tensor $\text{Ric}$ of $M$, is given by

$$\text{Ric}(X, Y) = \text{trace of the mapping } Z \mapsto \mathcal{R}(Z, X)Y,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In our convention, we consider the Ricci curvature, in the direction of a unit vector $X \in \mathfrak{X}(M)$, by the contraction of the Ricci tensor, i.e.,

$$\text{Ric}(X) = \text{Ric}(X, X),$$

and the scalar curvature function of $M$ by the trace of the Ricci curvature, i.e.,

$$\text{Scal} = \sum_{i=1}^{2n} \text{Ric}(X_i),$$

where $\{X_1, \ldots, X_{2n}\}$ is an orthonormal frame of $TM$. In particular, when $n = 1$, we have that

$$\text{Ric}(X_i) = K_M$$

and $\text{Scal} = 2K_M$, for $i = 1, 2$, where $K_M$ denotes the intrinsic curvature of $ds^2$.

We remark that the almost complex structure $J$ and the Riemann curvature tensor $\mathcal{R}$ satisfy

$$\mathcal{R}(X, Y) \circ J = J \circ \mathcal{R}(X, Y)$$

and $\mathcal{R}(JX, JY) = \mathcal{R}(X, Y)$,

and the almost complex structure $J$ and the Ricci tensor $\text{Ric}$ satisfy

$$\text{Ric}(JX, JY) = \text{Ric}(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$ (for more details, the reader may refer to [8, Chapter 9]).

Given an isometric immersion $f : M^{2n} \to \mathbb{Q}^m$, $2n < m$, we denote by $\mathcal{R}^\perp$ the curvature tensor of the normal bundle $T^\perp M$, by $\alpha$, seen as section of the bundle $T^*M \oplus T^*M \oplus T^\perp M$, the second fundamental form of $f$ and by $A_\xi$ its Weingarten operator in the normal direction $\xi \in \mathfrak{X}(M)^\perp$, given by

$$\langle A_\xi X, Y \rangle = \langle \alpha(X, Y), \xi \rangle,$$

for all $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)^\perp = \Gamma(T^\perp M)$ denotes the section of $T^\perp M$. 

In order to obtain a Bonnet-type theorem for isometric immersions into $\mathbb{Q}^m$ (and, more generally, considering the signature cases), Lira, Tojeiro and Vitório introduced in [10] the tensors $R$, $S$ and $T$ defined by

$$R = L^tL, \; S = K^tL \text{ and } T = K^tK,$$

where

$$L = d\pi_2 \circ f_* \in \Gamma(T^*M \oplus T\mathbb{Q}^{n_2}_{c_2,\mu_2}) \text{ and } K = d\pi_2|_{T^\perp M} \in \Gamma((T^\perp M)^* \oplus T\mathbb{Q}^{n_2}_{c_2,\mu_2}).$$

The tensors $R$ and $T$ are non-negative symmetric operators whose eigenvalues lie in $[0, 1]$. In particular, $\text{tr} R \in [0, n]$, as noted in [11]. In a similar way, we can define

$$\tilde{L} = d\pi_1 \circ f_* \in \Gamma(T^*M \oplus T\mathbb{Q}^{n_1}_{c_1,\mu_1}), \; \tilde{K} = d\pi_1|_{T^\perp M} \in \Gamma((T^\perp M)^* \oplus T\mathbb{Q}^{n_1}_{c_1,\mu_1}),$$

where $\tilde{R}$ and $\tilde{T}$ are nonnegative symmetric operators whose eigenvalues lie in $[0, 1]$, $\text{tr} \tilde{R} \in [0, n]$, and therefore

$$R + \tilde{R} = \text{Id}_{TM}.$$

Under these notations, in [10] the authors write the Gauss, Codazzi and Ricci equations as

\begin{equation}
\mathcal{R}(X, Y)Z = \left(c_1 (X \wedge Y - X \wedge RY - RX \wedge Y) + (c_1 + c_2)RX \wedge RY\right)Z + A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y,
\end{equation}

\begin{equation}
(\nabla^\perp_X \alpha)(Y, Z) - (\nabla^\perp_Y \alpha)(X, Z) = c_1(\langle X, Z\rangle SY - \langle Y, Z\rangle SX) + (c_1 + c_2)(\langle RY, Z\rangle SX - \langle RX, Z\rangle SY),
\end{equation}

\begin{equation}
\mathcal{R}^\perp(X, Y)\eta = \alpha(X, A_\eta Y) - \alpha(A_\eta X, Y) + (c_1 + c_2)(SX \wedge SY)\eta,
\end{equation}

where $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$, for all $X, Y, Z \in \mathfrak{X}(M)$.

In [2], Dajczer and Gromoll introduced the circular isometric immersions of a Kähler manifold. Those isometric immersions are also known in the literature as (1,1)-geodesic immersions and also as pluriharmonic immersions [6, 15]. In this work, we adopt the pluriharmonic terminology, and we recall that an isometric immersion $f : M^{2n} \to \mathbb{Q}^m$ of a Kähler manifold is said pluriharmonic if the second fundamental form of $f$ satisfies

$$\alpha(X, JY) = \alpha(JX, Y), \text{ for all } X, Y \in \mathfrak{X}(M),$$

or equivalently, if the Weingarten operator $A_\xi$ of $f$ anticommutes with the almost complex structure $J$:

$$A_\xi J + JA_\xi = 0, \text{ for any } \xi \in \mathfrak{X}(M)^\perp.$$

In particular, pluriharmonic immersions are minimal.

It is important to point out that for $n = 1$, any orientable Riemannian surface $(\Sigma, ds^2)$ has a natural almost complex structure $J$. This structure is given by the rotation of angle $\pi/2$ on the tangent bundle $T\Sigma$ of $\Sigma$. Since for any $\xi \in \mathfrak{X}(\Sigma)^\perp$ we have $A_\xi J + JA_\xi = (\text{tr} A_\xi)J$, then $f : \Sigma \to \mathbb{Q}^m$ is pluriharmonic if and only if $\text{tr} A_\xi = 0$ for any $\xi \in \mathfrak{X}(\Sigma)^\perp$, that is, if $f$ is a minimal immersion.
3. Obstruction results

In this section, we study the conditions to a minimal isometric immersion of a Kähler manifold into $\mathbb{Q}^m$ be a pluriharmonic immersion. As a consequence, we analyze the obstruction conditions to the existence of pluriharmonic immersions of Kähler manifolds into $\mathbb{Q}^m$.

In [3], Dajczer and Rodríguez studied those kinds of problems for space forms $\mathbb{Q}^m_{c^1}$. They concluded that pluriharmonic submanifolds and minimal submanifolds are the same objects in the Euclidean space. On the other hand, in the hyperbolic case, they showed that only surfaces can be immersed under assumption of minimality; and, in the spherical case, only surfaces can be immersed under assumption of pluriharmonicity. More generally in [6], Ferreira, Rigoli and Tribuzy show that pluriharmonic submanifolds are also equivalent to minimal submanifolds in locally symmetric Riemannian manifold of non-compact type.

For a given $f : M^{2n} \to \mathbb{Q}^m$ minimal isometric immersion of a Kähler manifold, our results are based in a pluriharmonicity property: an equation that must be satisfied by the tensor $R$. Before finding this equation, we recall a general characterization of isometric immersions into slices of products of space forms proved by Mendonça and Tojeiro, [11, Proposition 8]. In terms of the trace of $R$, these submanifolds are those on which either $\text{tr } R = 0$ or $\text{tr } R = \dim M$. It is important to notice that, for their result, any assumption about almost complex structure on $M$ is required.

**Proposition 3.1.** Let $f : M^n \to \mathbb{Q}^m$ be an isometric immersion. Then $f(M^n) \subset \mathbb{Q}_{c_1}^{n_1} \times \{p\}$ for some $p \in \mathbb{Q}_{c_2}^{n_2}$ (resp. $f(M^n) \subset \{p\} \times \mathbb{Q}_{c_2}^{n_2}$ for some $p \in \mathbb{Q}_{c_1}^{n_1}$), if and only if $\text{tr } R = 0$ (resp. $\text{tr } R = n$).

**Proof.** Since $R + \tilde{R} = \text{Id}_{T_pM}$, then $\text{tr } R + \text{tr } \tilde{R} = n$. Moreover, the eigenvalues of $R$ and $\tilde{R}$ lies in $[0, 1]$, $||L||^2 = \text{tr } R$ and $||\tilde{L}||^2 = \text{tr } \tilde{R}$. Therefore, $\text{tr } R = 0$ if, and only if, $\tilde{L} = 0$; and $\text{tr } \tilde{R} = n$ if, and only if, $\text{tr } \tilde{R} = 0$, i.e., $\tilde{L} = 0$. By definition of $L$, $d\sigma_2 \circ f_{\ast} = 0$ holds if, and only if, $f(M^n) \subset \mathbb{Q}_{c_1}^{n_1} \times \{p\}$ for some $p \in \mathbb{Q}_{c_2}^{n_2}$; and by definition of $\tilde{L}$, $d\sigma_1 \circ f_{\ast} = 0$ holds if, and only if, $f(M^n) \subset \{p\} \times \mathbb{Q}_{c_2}^{n_2}$ for some $p \in \mathbb{Q}_{c_1}^{n_1}$.

In the next result, we discuss a necessary and sufficient condition to a minimal isometric immersion of a Kähler manifold into $\mathbb{Q}^m$ be a pluriharmonic immersion.

**Lemma 3.2** (Pluriharmonicity property). Let $f : M^{2n} \to \mathbb{Q}^m$ be a minimal isometric immersion of a Kähler manifold. Then $f$ is pluriharmonic if and only if the tensor $R$ satisfies the following equation

$$4c_1(n-1)(n - \text{tr } R) + (c_1 + c_2)\left(\text{tr } R^2 - ||R||^2 - \langle RJ, JR \rangle\right) = 0.$$ 

**Proof.** At a point $p \in M$, we consider an orthonormal basis $\{X_1, \ldots, X_{2n}\}$ of $T_pM$ such that $X_{2j} = JX_{2j-1}$, for $1 \leq j \leq n$. Then, at this point, by the Gauss equation
we have
\[
\langle R(X, Y), X, X_j \rangle = \langle \alpha(X, Y), \alpha(X_j, X_j) \rangle - \langle \alpha(X, X_j), \alpha(Y, X_j) \rangle \\
+ c_1 \left( \langle X, Y \rangle - \langle X, X_j \rangle \langle Y, X_j \rangle - \langle X, Y \rangle \langle RX_j, X_j \rangle \right) \\
- \langle RX, Y \rangle + \langle RX, X_j \rangle \langle Y, X_j \rangle + \langle Y, RX_j \rangle \langle X, X_j \rangle \\
+ (c_1 + c_2) \left( \langle RX_j, X_j \rangle \langle RX, Y \rangle - \langle RX, X_j \rangle \langle RX_j, Y \rangle \right).
\]

Since \( f \) is minimal, for \( X = Y = X_i \), summing in \( j \) from 1 to \( 2n \), we have
\[
(3.2) \quad \text{Ric}(X_i) = - \sum_{j=1}^{2n} \| \alpha(X_i, X_j) \|^2 \\
+ c_1 \left( 2n - 1 - \text{tr} R - 2(n - 1) \langle RX_i, X_i \rangle \right) \\
+ (c_1 + c_2) \left( \langle RX_i, X_i \rangle \text{tr} R - \| RX_i \|^2 \right),
\]
and, similarly for \( X = Y = JX_i \),
\[
(3.3) \quad \text{Ric}(JX_i) = - \sum_{j=1}^{2n} \| \alpha(JX_i, X_j) \|^2 \\
+ c_1 \left( 2n - 1 - \text{tr} R - 2(n - 1) \langle RJX_i, JX_i \rangle \right) \\
+ (c_1 + c_2) \left( \langle RJX_i, JX_i \rangle \text{tr} R - \| RJX_i \|^2 \right).
\]

On the other hand, the Kähler structure on \( M \) implies that
\[
\langle R(X_j, X_i)X_i, X_j \rangle = \langle R(X_j, X_i)JX_i, JX_j \rangle,
\]
and therefore, by the Gauss equation, summing in \( j \) from 1 to \( 2n \), we get
\[
\text{Ric}(X_i) = \sum_{j=1}^{2n} \langle \alpha(X_i, JX_j), \alpha(X_j, JX_j) \rangle - \sum_{j=1}^{2n} \langle \alpha(X_i, JX_j), \alpha(X_j, JX_i) \rangle \\
+ c_1 \left( 1 - \langle RX_i, X_i \rangle - \langle RJX_i, JX_i \rangle \right) \\
+ (c_1 + c_2) \left( \langle RJX_i, JRX_i \rangle - \langle RX_i, JX_i \rangle \text{tr} JR \right).
\]

However, we notice that the first term of expression above is equal to zero, because of \( X_{2j} = JX_{2j-1} \), for \( 1 \leq j \leq n \). Moreover, since \( R \) is symmetric and \( J \) anti-symmetric, we have \( \text{tr} JR = 0 \). Thus,
\[
(3.4) \quad \text{Ric}(X_i) = - \sum_{j=1}^{2n} \langle \alpha(X_i, JX_j), \alpha(X_j, JX_i) \rangle \\
+ c_1 \left( 1 - \langle RX_i, X_i \rangle - \langle RJX_i, JX_i \rangle \right) + (c_1 + c_2) \langle RJX_i, JRX_i \rangle.
\]

Consider \( E = \bigoplus_{j=1}^{2n} T^*_p M \) endowed with the standard inner product. We set
\[
u_i = (\alpha(X_i, JX_1), \alpha(X_i, JX_2), \ldots, \alpha(X_i, JX_{2n})),
\]
\[
u_i = (\alpha(X_1, JX_i), \alpha(X_2, JX_i), \ldots, \alpha(X_{2n}, JX_i)).
\]
Since $\text{Ric}(JX, JY) = \text{Ric}(X, Y)$, for all $X, Y \in \mathfrak{X}(M)$ and $|u_i - v_i|^2 = |u_i|^2 + |v_i|^2 - 2\langle u_i, v_i \rangle$, by equations (3.2), (3.3) and (3.4) we have

$$|u_i - v_i|^2 = 2c_1 \left( 2(n - 1) - \text{tr} R - (n - 2)(\langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle) \right) + (c_1 + c_2) \left( \langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle \text{tr} R - \|RX_i\|^2 - \|RJX_i\|^2 - 2\langle RJX_i, JRX_i \rangle \right),$$

for $1 \leq i \leq n$. Then,

$$\frac{1}{2} \sum_{i=1}^{2n} |u_i - v_i|^2 = 4c_1(n - 1)(n - \text{tr} R) + (c_1 + c_2) \left( (\text{tr} \tilde{R})^2 - \|R\|^2 - \langle RJ, JR \rangle \right),$$

that is, $|u_i - v_i|^2 = 0$ for all $1 \leq i \leq n$ if, and only if, equation (3.1) holds. Therefore, observing that $|u_i - v_i|^2 = 0$ for all $1 \leq i \leq n$ if, and only if, $f$ is a pluriharmonic immersion, we conclude our assertion.

**Remark 3.3.** We notice that since $R + \tilde{R} = \text{Id}_T M$, we obtain an analogous Pluriharmonicy property’s Lemma for the tensor $\tilde{R}$:

$$4c_2(n - 1)(n - \text{tr} \tilde{R}) + (c_1 + c_2) \left( (\text{tr} \tilde{R})^2 - \|\tilde{R}\|^2 - \langle \tilde{R}J, J\tilde{R} \rangle \right) = 0.$$

**Remark 3.4.** By the approach used in [9], we show that the pluriharmonic property is given by

$$c_1 \left( (\text{tr} \tilde{R})^2 - \|\tilde{R}\|^2 - \langle RJ, JR \rangle \right) + c_2 \left( (\text{tr} \tilde{R})^2 - \|\tilde{R}\|^2 - \langle \tilde{R}J, J\tilde{R} \rangle \right) = 0,$$

which is equivalent to the one provided by Lemma 3.2, by the relation $R + \tilde{R} = \text{Id}_T M$. Moreover, the Pluriharmonic property’s Lemma can be generalised for minimal isometric immersions of a Kähler manifold into multiproducts of space forms $\mathbb{Q}^{n_1}_{c_1} \times \cdots \times \mathbb{Q}^{n_k}_{c_k}$. In this case, the pluriharmonic property is given by

$$\sum_{j=1}^{k} c_j \left( (\text{tr} R_j)^2 - \|R_j\|^2 - \langle R_j J, J R_j \rangle \right) = 0,$$

where $R_j$ is the $f_j$ symmetric tensor that appear in [9]. In particular, for the case $\mathbb{Q}^{n_1}_{c_1} \times \mathbb{Q}^{n_2}_{c_2}$, we have that $R_1 = \tilde{R}$ and $R_2 = R$.

As a consequence of Pluriharmonicy property’s Lemma, we analyse the obstruction conditions to the existence of pluriharmonic immersions of Kähler manifolds into $\mathbb{Q}^{m-1}_{c} \times \mathbb{R}$.

**Theorem 3.5.** Let $f : M^{2n} \to \mathbb{Q}^{m-1}_{c} \times \mathbb{R}$ be an isometric immersion of a Kähler manifold, with $c \neq 0$. Assume that

- i) either $c < 0$ and $f$ is minimal;
- ii) or $c > 0$ and $f$ is pluriharmonic.

Then $n = 1$.

**Proof.** Firstly, since $\mathbb{H}^{m-1}_{c} \times \mathbb{R}$ is a locally symmetric Riemannian manifold of non-compact type follows from [6, Proposition 1] that $f$ is also a pluriharmonic immersion. Moreover, for $\mathbb{Q}^{m} = \mathbb{Q}^{m-1}_{c} \times \mathbb{R}$ we have that $RX = (X, \partial_t^X)\partial_t^r$, where
\( \partial_t^\top \) is the projection of the unit vertical vector \( \partial_t \) (corresponding to the factor \( \mathbb{R} \)) onto \( TM \). By Pluriharmonicicy property’s Lemma, we have
\[
4(n-1)(n - \| \partial_t^\top \|^2) = 0,
\]
that is, either \( n = 1 \) or \( \| \partial_t^\top \|^2 = n \). Since \( \| \partial_t^\top \|^2 \leq 1 \), in both cases we get \( n = 1 \).
\[ \square \]

**Remark 3.6.** We point out that Theorem 3.5 was also obtained by de Almeida in her thesis [4, Theorem 3.1], using similar methods.

**Remark 3.7.** We notice that Theorem 3.5 can be extended for an isometric immersion of a Kähler manifold \( M^{2n} \) into a warped product manifold \( I \times \rho \mathbb{Q}^{m-1}_c \) endowed with the metric \( ds^2 = dt^2 + \rho(t)^2 d\theta^2 \), where \( I \subset \mathbb{R} \) is an interval, \( \rho : I \to \mathbb{R} \) is a non-constant positive smooth function and \( d\theta^2 \) denotes the metric of \( \mathbb{Q}^{m-1}_c \). Indeed, by the Gauss equation (cf. [3.1, 13]), we compute the pluriharmonicity property by
\[
4(n-1)(n\lambda(t) - \| \partial_t^\top \|^2\mu(t)) = 0,
\]
where \( \lambda(t) = \frac{c - \rho'(t)^2}{\rho(t)^2} \) and \( \mu(t) = \frac{c - \rho'(t)^2 + \rho''(t)}{\rho(t)} \). However, we observe that when either \( \rho''(t) \geq 0 \) and \( c \leq 0 \), or \( \rho''(t) \leq 0 \) and \( c > \rho'(t)^2 \), then
\[
n\lambda(t) - \| \partial_t^\top \|^2\mu(t) = (n - \| \partial_t^\top \|^2)\frac{c - \rho'(t)^2}{\rho(t)^2} - \| T \|^2\frac{\rho''(t)}{\rho(t)} = 0
\]
if, and only if, either \( \rho'(t) = 0 \) and \( \| \partial_t^\top \|^2 = n \), that is, if \( \{(t, \rho(t)) : t \in I\} \subset \mathbb{R}^2 \) is a line and \( n = 1 \), since \( \| \partial_t^\top \|^2 \leq 1 \). Therefore, if \( f : M^{2n} \to I \times \rho \mathbb{Q}^{m-1}_c \) is a pluriharmonic immersion of a Kähler manifold such that either \( \rho''(t) \geq 0 \) and \( c \leq 0 \), or \( \rho''(t) \leq 0 \) and \( c > \rho'(t)^2 \), the pluriharmonicity property (3.5) implies that \( n = 1 \).

**Corollary 3.8.** Let \( f : M^{2n} \to \mathbb{Q}^{n_1}_c \times \mathbb{Q}^{n_2}_c \) be a pluriharmonic immersion of a Kähler manifold, with \( c \neq 0 \). Then either \( tr R = n \) or \( n = 1 \).

**Proof.** Since \( c_1 + c_2 = 0 \), it follows directly of Pluriharmonicity property’s Lemma that \( 4c(n-1)(n - tr R) = 0 \).
\[ \square \]

**Corollary 3.9.** Let \( f : M^{2n} \to \mathbb{Q}^m \) be a pluriharmonic immersion of a Kähler manifold. Assume that \( R J + J R = 0 \) (resp. \( R J + J R = 2J \)). Then either \( c_1 = 0 \) or \( n = 1 \) (resp. \( c_2 = 0 \) or \( n = 1 \)).

**Proof.** Since \( R J + J R = 0 \) if, and only if, \( J^{-1} R J + R = 0 \), then \( tr R = 0 \). Moreover, \( \langle R J, J R \rangle = -\langle R J, R J \rangle = -\| R \|^2 \). By Pluriharmonicity property’s Lemma, we have \( 4c_1(n-1) = 0 \), i.e., either \( c_1 = 0 \) or \( n = 1 \). Analogously, if \( R J + J R = 2J \) then \( tr R = 2n \) and \( tr R = 0 \). Thus, by Pluriharmonicity property’s Lemma for \( \tilde{R} \), we have \( 4c_2(n-1) = 0 \), i.e., either \( c_2 = 0 \) or \( n = 1 \).
\[ \square \]

**Remark 3.10.** We observe that if \( R J + J R = 0 \) then \( tr R = 0 \), and by Proposition 3.1, we have \( f(M^{2n}) \subset \mathbb{Q}^{n_1}_c \times \{ p \} \), for some \( p \in \mathbb{Q}^{n_2}_c \). By Corollary 3.9, either \( c_1 = 0 \) and \( f(M^{2n}) \subset \mathbb{Q}^{n_1}_c \times \{ p \} \), or \( n = 1 \) and \( f(M^{2n}) \subset \mathbb{Q}^{n_1}_c \times \{ p \} \), that is, \( f \) can be seen as isometric immersion into \( \mathbb{Q}^{n_1}_c \) and then we recover a Dajczer-Rodríguez result presented in [3].
4. Curvature estimates

The goal of this section is to study upper bounds of the Ricci and scalar curvatures of Kähler manifolds, when we suppose the existence of minimal isometric immersions of these manifolds into some product of space forms.

In a classical work about minimal isometric immersions, Takahashi proved that the existence of minimal isometric immersion \( f \) of an arbitrary Riemannian manifold \( M \) into a space form \( Q^m \), \( n \geq 2 \), imposes an upper bound of the Ricci curvature of \( M \), namely,

\[
\frac{n - 1}{n} (cn - \|\alpha\|^2) \leq \text{Ric} \leq c(n - 1),
\]

where \( \|\alpha\|^2 \) denotes the norm square of the second fundamental form \([14, \text{Theorem 1}]\). The equality case holds if and only if \( f(M) \) is a totally geodesic submanifold of \( Q^m \). Dajczer and Rodríguez proved that the upper bound of the Ricci curvature of \( M \) is more restrictive for a minimal isometric immersion \( f \) of a Kähler manifold \( M \) into \( S^m \). In this case, they showed that \( \text{Ric} \leq cn \), with equality implying that \( f \) has parallel second fundamental form \([3, \text{Theorem 1.2}]\).

In order to improve the upper bounds of the Ricci and scalar curvatures of minimal Kähler submanifolds into some products space forms, we study firstly a general upper bound of the scalar curvature of those submanifolds in \( Q^m \times Q^m \), given in terms of the tensor \( R \). For this purpose, we said that an isometric immersion \( f \) of a Kähler manifold \( M \) is an anti-pluriharmonic immersion if the second fundamental form of \( f \) satisfies

\[
\alpha(X, JY) = -\alpha(JX, Y), \quad \text{for all } X, Y \in \mathfrak{X}(M),
\]

or equivalently, if the Weingarten operator \( A_\xi \) of \( f \) commutes with the almost complex structure \( J \):

\[
A_\xi J = JA_\xi, \quad \text{for any } \xi \in \mathfrak{X}(M)^\perp.
\]

Anti-pluriharmonic immersions into Euclidean space were firstly studied by Retberg in \([12]\) and by Ferus in \([7]\), where was proved that anti-pluriharmonic immersions into \( Q^m \) have parallel second fundamental forms. We notice that, in an analogous way of holomorphic immersions, if the target space is also a Kähler manifold, then anti-holomorphic isometric immersions are anti-pluriharmonic immersions.

**Lemma 4.1.** Let \( f : M^{2n} \rightarrow Q^m \) be a minimal isometric immersion of a Kähler manifold. Then the scalar curvature of \( M \) satisfies

\[
\text{Scal} \leq 2nc_1(n - \text{tr} R) + \frac{(c_1 + c_2)}{2} \left( (\text{tr} R)^2 - \|R\|^2 + \langle R J, JR \rangle \right).
\]

Moreover, the equality holds if, and only if, \( f \) is an anti-pluriharmonic immersion.

**Proof.** At a point \( p \in M \), we consider an orthonormal basis \( \{X_1, \cdots, X_{2n}\} \) of \( T_pM \) such that \( X_{2j} = JX_{2j-1} \), for \( 1 \leq j \leq n \). Consider \( E = \bigoplus_{j=1}^{2n} T_p^{\perp}M \) endowed with the standard inner product. We set

\[
u_i = (\alpha(X_i, JX_1), \alpha(X_i, JX_2), \cdots, \alpha(X_i, JX_{2n})),
\]

\[
u_i = (\alpha(X_1, JX_i), \alpha(X_2, JX_i), \cdots, \alpha(X_{2n}, JX_i)).
\]
Since \( \text{Ric}(JX, JY) = \text{Ric}(X, Y) \), for all \( X, Y \in \mathfrak{X}(M) \), by equations (3.2), (3.3) and (3.4), and the Parallelogram identity, we get that

\begin{equation}
|u_i + v_i|^2 = -4\text{Ric}(X_i) + 2A_i + B_i + C_i,
\end{equation}

for \( 1 \leq i \leq 2n \), where the coefficients \( A_i \), \( B_i \) and \( C_i \) are given by

\begin{align*}
A_i &= c_1 \left( 1 - \langle RX_i, X_i \rangle - \langle RJX_i, JX_i \rangle \right) + (c_1 + c_2)\langle RJX_i, JRX_i \rangle, \\
B_i &= c_1 \left( (2n - 1) - \text{tr } R - 2(n - 1) \langle RX_i, X_i \rangle \right) \\
&\quad + (c_1 + c_2) \left( \langle RX_i, X_i \rangle \text{tr } R - \|RX_i\|^2 \right), \\
C_i &= c_1 \left( (2n - 1) - \text{tr } R - 2(n - 1) \langle RJX_i, JX_i \rangle \right) \\
&\quad + (c_1 + c_2) \left( \langle RJX_i, JX_i \rangle \text{tr } R - \|RJX_i\|^2 \right).
\end{align*}

Thus, by equation (4.1), we obtain

\begin{equation}
\text{Ric}(X_i) \leq \frac{1}{4}(2A_i + B_i + C_i).
\end{equation}

On the other hand, we compute \( 2A_i + B_i + C_i \) by

\begin{align*}
2A_i + B_i + C_i &= 2c_1 \left( 2n - \text{tr } R - n \left( \langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle \right) \right) \\
&\quad + (c_1 + c_2) \left( \left( \langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle \right) \text{tr } R \\
&\quad - \|RX_i\|^2 - \|RJX_i\|^2 + \langle RJX_i, JRX_i \rangle \right).
\end{align*}

Therefore, summing in \( i \) from 1 to \( 2n \), we obtain

\begin{equation*}
4\text{Scal} \leq \sum_{i=1}^{2n} (2A_i + B_i + C_i) = 8nc_1(n - \text{tr } R) + (c_1 + c_2) \left( 2(\text{tr } R)^2 - 2\|R\|^2 - 2\langle RJ, JR \rangle \right),
\end{equation*}

that conclude our assertion.

Note that the equality holds if, and only if

\[ \sum_{i=1}^{2n} |u_i + v_i|^2 = 0, \]

that is, \( u_i = -v_i \), for \( 1 \leq j \leq n \), i.e.,

\[ \alpha(X, JY) + \alpha(JX, Y) = 0 \]

for \( X, Y \in \mathfrak{X}(M) \), therefore, if, and only if, \( f \) is anti-pluriharmonic.

\[ \square \]

In our next results, we show that the existence of a minimal isometric immersion of a Kähler manifold \( M^{2n} \) into either \( S^{m-1}_c \times \mathbb{R} \) or \( S^{m-k}_c \times \mathbb{H}^{k} \) imposes strong restrictions on the Ricci curvature and the scalar curvature of \( M^{2n} \).

**Theorem 4.2.** Let \( f : M^{2n} \to S^{m-1}_c \times \mathbb{R} \) be a minimal isometric immersion of a Kähler manifold. Then the Ricci curvature of \( M \) satisfies \( \text{Ric} \leq c(2n - \|\partial_t\|^2)/2 \), with equality implying that \( f(M^{2n}) \subset S^{m-1}_c \times \{t\} \) for some \( t \in \mathbb{R} \), and that \( f \) has parallel second fundamental form.
Proof. In the case of $S^{n-1} \times \mathbb{R}$, we have that $RX = (X, \partial_i^T)\partial_i^T$, where $\partial_i^T$ is the projection of the unit vertical vector $\partial_i$ (corresponding to the factor $\mathbb{R}$) onto $TM$. Then, the coefficients $A_i$, $B_i$ and $C_i$ are given by

\[
A_i = c \left( 1 - \langle \partial_i^T, X_i \rangle^2 - \langle \partial_i^T, JX_i \rangle^2 \right),
\]
\[
B_i = c \left( 2n - 1 - \|\partial_i^T\|^2 - 2(n-1)\langle \partial_i^T, X_i \rangle^2 \right),
\]
\[
C_i = c \left( 2n - 1 - \|\partial_i^T\|^2 - 2(n-1)\langle \partial_i^T, JX_i \rangle^2 \right),
\]

for $1 \leq i \leq 2n$. We compute $2A_i + B_i + C_i$ by

\[
2A_i + B_i + C_i = 2c \left( 2n - \|\partial_i^T\|^2 - n\langle \partial_i^T, X_i \rangle^2 + \langle \partial_i^T, JX_i \rangle^2 \right).
\]

Thus, by equation (4.2), we obtain

\[
(4.3) \quad \text{Ric}(X_i) \leq \frac{c}{2} \left( 2n - \|\partial_i^T\|^2 - n\langle \partial_i^T, X_i \rangle^2 + \langle \partial_i^T, JX_i \rangle^2 \right),
\]

and therefore,

\[
(4.4) \quad \text{Ric}(X_i) \leq \frac{c}{2} (2n - \|\partial_i^T\|^2),
\]

for $1 \leq i \leq 2n$ and $n \geq 1$.

If the equality holds on (4.4), then by inequality (4.3), we have that $\langle \partial_i^T, X_i \rangle^2 + \langle \partial_i^T, JX_i \rangle^2 = 0$, for $1 \leq i \leq 2n$, i.e., $\partial_i^T = 0$. Thus, $f(M^{2n})$ lies into a slice $S^{m-1} \times \{t\}$, for some $t \in \mathbb{R}$, and satisfies $\text{Ric} = nc$. Therefore, by [3, Theorem 1.2], $f$ has parallel second fundamental form.

Remark 4.3. Given a minimal isometric immersion $f: M^n \to Q^{m-1}_c \times \mathbb{R}$ of an arbitrary manifold $M^n$, $n \geq 2$, the Gauss equation provides a natural bound for the Ricci curvature, controlled by $\|\partial_i^T\|^2$, on which $\|\partial_i^T\|^2 \leq 1$, in the following sense:

\[
\text{Ric} \leq c(n-1 - \|\partial_i^T\|^2), \quad \text{for } c > 0,
\]
\[
\text{Ric} \leq c(n-1)(1 - \|\partial_i^T\|^2), \quad \text{for } c < 0.
\]

In both cases, the equality case holds if and only if either $f(M^2)$ is a totally geodesic surface in $Q^{m-1}_c \times \mathbb{R}$, or $f(M^n)$ is a totally geodesic submanifold that lies into a slice of $Q^{m-1}_{c-1} \times \mathbb{R}$.

When we assume that $M^{2n}$ is a Kähler manifold, with $n > 1$, and $c > 0$, this upper bound is less restrictive than the one provides by Theorem 4.2. However for $n = 1$, the upper bound provided by the Gauss equation is more restrictive than the one provides our Theorem 4.2.

Remark 4.4. We point out that the upper bound provides by Theorem 4.2 also holds in $Q^{m-1}_c \times \mathbb{R}$, with $c \in \mathbb{R}$. Moreover, when $c = 0$, this upper bound is the same obtained by Gauss equation. For $c < 0$, by the previous remark, we can check that this upper bound is less restrictive than the one provides by Gauss equation; and we recall that surfaces are the only minimal Kähler submanifolds in $\mathbb{H}^{m-1} \times \mathbb{R}$ (Theorem 3.5).

Corollary 4.5. Let $f: M^{2n} \to S^{m-1}_c \times \mathbb{R}$ be a minimal isometric immersion of a Kähler manifold. Then the scalar curvature of $M$ satisfies $\text{Scal} \leq 2nc(n - \|\partial_i^T\|^2)$. The equality holds if, and only if, $f$ is an anti-pluriharmonic immersion.
Proof. Since $RX = \langle X, \partial_i^R \partial^R_i \rangle$, by a direct computation we get that $\langle RJ, JR \rangle = 0$ and $\|R\|^2 = ||\partial_i^R||^2 = (\text{tr} R)^2$. Therefore, by Lemma 4.1, we obtain our assertion. \hfill \blacksquare

**Remark 4.6.** Theorem 4.2 give us an upper bound of the scalar curvature of $M^{2n}$, precisely $\text{Scal} \leq nc(2n - ||\partial_i^R||^2)$. However, this upper bound is less restrictive than the one provides by Corollary 4.5.

**Corollary 4.7.** Let $f : M^{2n} \rightarrow S^{m-k}_c \times \mathbb{H}^k_{-c}$ be a minimal isometric immersion of a Kähler manifold. Then $\text{Ric} \leq c(2n - \text{tr} R)/2$, with equality implying $f(M) \subset S^{m-k}_c \times \{p\}$ for some $p \in \mathbb{H}^k_{-c}$, $\text{Ric} = cn$ and that $f$ has parallel second fundamental form.

**Proof.** Since $c_1 = -c_2 = c$, then the coefficients $A_i$, $B_i$ and $C_i$ are given by

$$A_i = c\left(1 - \langle RX_i, X_i \rangle - \langle RJX_i, JX_i \rangle\right),$$

$$B_i = c\left(2n - 1 - \text{tr} R - 2(n - 1)\langle RX_i, X_i \rangle\right),$$

$$C_i = c\left(2n - 1 - \text{tr} R - 2(n - 1)\langle RJX_i, JX_i \rangle\right),$$

for $1 \leq i \leq 2n$. We compute $2A_i + B_i + C_i$ by

$$2A_i + B_i + C_i = 2c\left(2n - \text{tr} R - n(\langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle)\right).$$

Thus, by equation (4.2), we obtain

$$\text{Ric}(X_i) \leq 2c\left(2n - \text{tr} R - n(\langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle)\right),$$

and therefore,

$$\text{Ric}(X_i) \leq \frac{c}{2}(2n - \text{tr} R),$$

for $1 \leq i \leq 2n$, since $R$ is a non-negative operator.

If the equality holds, we have that $\langle RX_i, X_i \rangle + \langle RJX_i, JX_i \rangle = 0$, for $1 \leq i \leq 2n$, i.e., $\text{tr} R = 0$, and thus $M$ satisfies $\text{Ric} = cn$. By Proposition 3.1, $f(M^{2n})$ lies into a slice $S^{m-k}_c \times \{p\}$ for some $p \in \mathbb{H}^k_{-c}$ and, therefore, by [3, Theorem 1.2], $f$ has parallel second fundamental form. \hfill \blacksquare

**Remark 4.8.** Given a minimal isometric immersion $f : M^n \rightarrow Q^{m-k}_c \times Q^k_{-c}$ of an arbitrary manifold $M^n$, $n \geq 2$ and $c \neq 0$, the Gauss equation provides a natural bound for the Ricci curvature, controlled by $\text{tr} R$, on which $0 \leq \text{tr} R \leq n$, in the following sense:

$$\text{Ric} \leq c(n - 1 - \text{tr} R), \quad \text{for } c > 0,$$

$$\text{Ric} \leq c(n - 1)(1 - \text{tr} R), \quad \text{for } c < 0 \text{ and } \text{tr} R \leq 1,$$

$$\text{Ric} \leq c(1 - \text{tr} R), \quad \text{for } c < 0 \text{ and } \text{tr} R > 1.$$

For either $c > 0$ or $c < 0$ and $\text{tr} R < 1$, the equality case holds if and only if either $f(M^2)$ is a totally geodesic surface in $Q^{m-k}_c \times Q^k_{-c}$, or $f(M^n)$ is a totally geodesic submanifold that lies into a slice $Q^{m-k}_c \times \{p\}$, for some $p \in Q^k_{-c}$. For $c < 0$ and $\text{tr} R > 1$, the equality case holds if and only if either $f(M^2)$ is a totally geodesic surface in $\mathbb{H}^{m-1}_c \times S^k_c$, or $f(M^n)$ is a totally geodesic submanifold that lies into a slice $\{p\} \times S^k_c$, for some $p \in \mathbb{H}^{m-k}_c$. Finally, for $c < 0$ and $\text{tr} R = 1$, the equality case holds if and only if $f(M^2)$ is a totally geodesic surface in $\mathbb{H}^{m-k}_c \times S^k_{-c}$.
When we assume that $M^{2n}$ is a Kähler manifold, with $n > 1$ and $c > 0$, this upper bound is less restrictive than the one provides by Theorem 4.2. However for $n = 1$, the upper bound provided by the Gauss equation is more restrictive than the one provides our Theorem 4.2.

**Remark 4.9.** We point out that the upper bound provides by Corollary 4.7 also holds in $\mathbb{H}^{m-k}_c \times S^k_{-c}$. However, by the previous remark, we can check that this upper bound is less restrictive than the one provides by Gauss equation.

**Corollary 4.10.** Let $f : M^{2n} \to Q^{m-k}_c \times Q^k_{-c}$ be a minimal isometric immersion of a Kähler manifold, with $c \neq 0$. Then the scalar curvature of $M$ satisfies $\text{Scal} \leq 2nc(n - \text{tr} R)$.

**Remark 4.11.** In the previous section, we see that minimal isometric immersions into $Q^{m-k}_c \times Q^k_{-c}$ with $\text{tr} R = n$ are not necessarily surfaces (Pluriharmonici property’s Lemma and Corollary 3.8). However, the latest corollaries give us some information about what occurs in this case; we obtain that $\text{Ric} \leq nc/2$ and $\text{Scal} \leq 0$.

**Remark 4.12.** Let $f : M^{2n} \to \tilde{M}^m$ be a minimal isometric immersion of a Kähler manifold into an arbitrary Riemannian manifold $\tilde{M}^m$ and denote by $\tilde{R}$ the Riemann curvature tensor of $\tilde{M}$. In the general case, with the conventions used in the proofs of Lemma 3.2 and Lemma 4.1, our results are obtained by the study of the quantities $\omega_{i, -} = |u_i - v_i|^2$ and $\omega_{i, +} = |u_i + v_i|^2$, given by

$$
\omega_{i, \pm} = -2(\text{Ric}(X_i) \pm \text{Ric}(X_i)) + \sum_{j=1}^{2n} \left( \langle \tilde{R}(X_j, X_i)X_i, X_j \rangle 
\pm 2\langle \tilde{R}(X_j, X_i)jX_i, jX_j \rangle + \langle \tilde{R}(X_j, jX_i)jX_i, X_j \rangle \right),
$$

where $\{X_1, \cdots, X_{2n}\}$ is an orthonormal basis of $T_pM$, such that $X_{2j} = jX_{2j-1}$, for $1 \leq j \leq n$. Then $\sum_{i=1}^{2n} \omega_{i, -} = 0$ is the pluriharmonici property and $\omega_{i, +} \geq 0$ provides the Ricci and scalar estimates for $M^{2n}$. When $\tilde{M}^m$ is a conformally flat Riemannian manifold, its Riemann curvature tensor is given by

$$
\langle \tilde{R}(X, Y)Z, W \rangle = S(X, W)(Y, Z) + S(Y, Z)(X, W)$$

$$- S(X, Z)(Y, W) - S(Y, W)(X, Z),$$

where $S$ is the Schouten tensor of $\tilde{M}^m$, defined by

$$
S(X, Y) = \frac{1}{m-2} \left( \text{Ric}(X, Y) - \frac{\text{Scal}}{2(m-1)}(X, Y) \right),
$$

for $X, Y, Z, W \in \mathcal{X}(\tilde{M})$. If $S|_{TM}$ denotes the restriction of $S$ to $TM \times TM$, then

$$
\sum_{i=1}^{2n} \omega_{i, -} = 8(n - 1)\text{tr} S|_{TM},
$$

$$
\sum_{i=1}^{2n} \omega_{i, +} = 4\left(2n\text{tr} S|_{TM} - \text{Scal} \right).$$
Therefore, if \( f : M^{2n} \rightarrow \tilde{M}^m \) is a minimal isometric immersion of a Kähler manifold into a conformally flat Riemannian manifold \( \tilde{M}^m \) then \( \text{Scal} \leq 2n \text{tr} S|_{TM} \), where the equality holds if, and only if, \( f \) is an anti-pluriharmonic immersion. Moreover, if \( f \) satisfies \( \text{tr} S|_{TM} \neq 0 \) then it is pluriharmonic if, and only if, \( n = 1 \). We observe that special cases satisfying this trace assumption were studied by Ferreira, Rigoli and Tribuzy in [6].

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