Efficient Truncated Linear Regression with Unknown Noise Variance

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Abstract

Truncated linear regression is a classical challenge in statistics, wherein a label, \( y = w^T x + \epsilon \), and its corresponding feature vector, \( x \in \mathbb{R}^k \), are only observed if the label falls in some subset \( S \subseteq \mathbb{R} \); otherwise the existence of the pair \( (x, y) \) is hidden from observation. Linear regression with truncated observations has remained a challenge, in its general form, since the early works of [27, 2]. When the distribution of the error is normal with known variance, recent work of [9] provides computationally and statistically efficient estimators of the linear model, \( w \). In this paper, we provide the first computationally and statistically efficient estimators for truncated linear regression when the noise variance is unknown, estimating both the linear model and the variance of the noise. Our estimator is based on an efficient implementation of Projected Stochastic Gradient Descent on the negative log-likelihood of the truncated sample. Importantly, we show that the error of our estimates is asymptotically normal, and we use this to provide explicit confidence regions for our estimates.

1 Introduction

A common challenge facing statistical estimation is the systematic omission of relevant data from the data used to train models. Data omission is a prominent form of dataset bias, which may occur for a variety of reasons, including incorrect experimental design or data collection campaigns, which prevent certain sub-populations from being observed, instrument errors or saturation, which make certain measurements unreliable, societal biases, which may suppress the realization of certain samples, as well as legal or privacy constraints, which might prevent the use of some data. In turn, it is well-understood that failure to account for the systematic omission of data in statistical inference may lead to incorrect models, and this has motivated a long line of research in statistics, econometrics, and a range of other theoretical and applied fields, targeting statistical inference that is robust to missing data; see e.g. [12, 22, 23, 10, 29, 27, 2, 14, 15, 4, 13, 20, 3].

In this paper, we revisit the classical challenge of truncated linear regression. As in the standard linear regression setting, we assume that the world produces pairs \((x^{(i)}, y^{(i)})\), where each \( x^{(i)} \in \mathbb{R}^k \) is a feature vector drawn from some distribution \( \mathcal{P} \) and its corresponding label \( y^{(i)} \) is linearly related...
to \( x^{(i)} \) according to \( y^{(i)} = w^*^T x^{(i)} + \varepsilon^{(i)} \), for both some unknown coefficient vector \( w^* \in \mathbb{R}^k \) and random noise \( \varepsilon^{(i)} \sim \mathcal{N}(0, \sigma^2) \), where \( \sigma^2 \) is unknown and \( \varepsilon^{(i)} \) is i.i.d over \( i \). The difference to the standard setting is that we do not get to observe all pairs \((x^{(i)}, y^{(i)})\) that the world produces. Rather, a pair \((x^{(i)}, y^{(i)})\) is only observed if \( y^{(i)} \in S \), where \( S \subseteq \mathbb{R} \) is a fixed “observation set,” that is known or given to us via oracle access, i.e., we can query an oracle about whether or not some point belongs to \( S \). If \( y^{(i)} \notin S \), then \((x^{(i)}, y^{(i)})\) is removed from observation. Our goal is to estimate \( w^* \) and \( \sigma^2 \) from the pairs that survived removal, called “truncated samples.”

**Overview of our results.** While both truncated andensored linear regression are fundamental problems, which commonly arise in practice and have been studied in the literature since at least the 1950s [27, 2], there are no known statistical inference methods that are computationally efficient and enjoy statistical rates that scale with the dimension \( k \) of the problem. Recent work of [9] has obtained such methods in the simpler setting where the noise variance \( \sigma^2 \) is known. Our goal here is to extend those results in two important ways. First, we study settings where \( \sigma^2 \) is unknown, providing computationally and statistically efficient estimators for the joint estimation of \( w^* \) and \( \sigma^2 \). Moreover, we establish the asymptotic normality of our estimators, and use this to derive confidence regions for our estimators. To achieve these stronger results we make some additional assumptions compared to [9]. Namely, we assume that the covariates are sampled from some prior distributions, and we also make a stronger assumption about the survival probability of our model as we discuss below. Other works that consider learning and regression problems with censored or truncated samples are [24, 11, 21, 17].

We next provide informal statements of our results, postponing formal statements to Sections 3 and 5. It is important to note that, as shown in prior work [8, 9], for the model parameters to be identifiable from truncated samples, we need to place some assumptions that limit how aggressively the data is truncated. These assumptions are needed even without truncation (Assumption 1.2). It is important to note that, as shown in prior work [8, 9], for the model parameters to be identifiable from truncated samples, we need to place some assumptions that limit how aggressively the data is truncated. These assumptions are needed even without truncation (Assumption 1.2). Rather, a pair \((x, y)\) is only observed if \( y \in S \), where \( S \subseteq \mathbb{R} \) is a fixed “observation set,” that is known or given to us via oracle access, i.e., we can query an oracle about whether or not some point belongs to \( S \). If \( y \notin S \), then \((x, y)\) is removed from observation. Our goal is to estimate \( w^* \) and \( \sigma^2 \) from the pairs that survived removal, called “truncated samples.”

**Assumption 1.1.** We assume that \( \|w^*\|_2^2 \leq \beta \) and \( \|x^{(i)}\|_2^2 \leq 1 \) for all \( i \).

**Assumption 1.2.** For all \((x^{(i)}, y^{(i)})\) in the truncated dataset, conditioning on \( x^{(i)} \), the probability that \( w^*^T x^{(i)} + \varepsilon \in S \), with respect to the randomness in \( \varepsilon \sim \mathcal{N}(0, \sigma^2) \), is larger than some \( a \).

**Assumption 1.3.** If \((x^{(i)}, y^{(i)})_{i=1}^n\) is the truncated dataset, we assume that the feature covariance matrix, \( X = \frac{1}{n} \sum_{i=1}^n x^{(i)} x^{(i)^T} \), satisfies: \( \mathbb{E}_{x \sim \mathcal{P}}[X] = \mathbb{E}_{x \sim \mathcal{P}}[xx^T] \geq b \cdot I \) for some \( b \).

With these assumptions we prove our main results for the joint estimation of \( w^* \) and \( \sigma^2 \), listed as Informal Theorems 1.4–1.5 below, and respectively Theorems 3.2 and 5.1 in later sections.

**Informal Theorem 1.4** (Estimation). Suppose we are given \( n \) samples \((x^{(i)}, y^{(i)})_{i=1}^n\) from the truncated linear regression model with parameters \( w^* \), \( \sigma^2 \), and suppose Assumptions 1.1–1.3 hold. Then running Projected Stochastic Gradient Descent (Algorithm 1) on the conditional negative log-likelihood of the sample, using the projection set described in Definition 3.1, we obtain with probability at least \( 2/3 \) (which can be boosted to as large a constant as desirable) estimators \( \hat{w}, \hat{\sigma}^2 \) such that \( \|\hat{w} - w^*\|_2 + |\hat{\sigma}^2 - \sigma^2| \leq \varepsilon \), as long as \( n \geq C_1 \log^{\eta}(1/\varepsilon) \) or \( n \geq C_2 \varepsilon^2 \) (depending on the learning rate), where \( C_1 \) depends polynomially on \( \beta, 1/a \) and \( 1/b \), and \( C_2 \) depends linearly in \( 1/b \) and exponentially in \( \beta/a \), where \( \beta, a \) and \( b \) are respectively the constants in Assumptions 1.1, 1.2 and 1.3.

**Informal Theorem 1.5** (Asymptotic Normality). Let \( \hat{w}, \hat{\sigma}^2 \) be the estimates of Informal Theorem 1.4. There exists a matrix \( A \) that depends on \( \hat{w}, \hat{\sigma}^2 \) (and is explicitly provided in Theorem 5.1) such that the random vector \( \sqrt{n}(\hat{\Sigma}(\hat{\Sigma}^2 - (w^*)^2)) \) converges asymptotically to a normal distribution with zero mean and covariance matrix \( A \). Hence the \( 1 - \alpha \) confidence region of \((\hat{w}, \hat{\sigma}^2)\) can be computed using the matrix \( A \) and the quantities of the chi-squared distribution.

**Techniques.** Our estimates are computed by running Projected Stochastic Gradient Descent (PSGD) on the population conditional negative log-likelihood of the truncated sample; a convex function of the parameters, after an appropriate reparameterization. We provide two analyses of PSGD resulting in two different bounds that are more effective in a different range of parameters, as we discuss next.
Our first analysis hinges on establishing a lower bound on the convexity of the conditional negative log-likelihood function. We can’t however, expect that a strong bound holds over the whole parameter domain. For this reason, we add a projection step to our method, identifying a projection set wherein we show that the true parameters lie and wherein the conditional negative log-likelihood is strongly convex. This approach is similar to that of [9], except that the unknown noise variance introduces many challenges in identifying an appropriate projection set. The main result bounding the strong convexity of the conditional negative log likelihood function within the projection set is stated as Theorem 4.1. The main disadvantage of this analysis is that, when the variance is unknown, the bound on the strong convexity of the conditional negative log-likelihood becomes small and the resulting dependence of the sample size on the parameters \( \beta \) and \( 1/\alpha \) of the problem is exponential. On the other hand, the dependence of the sample size on \( 1/b \) is linear and its dependence on the required estimation accuracy is \( \Theta(1/\varepsilon^2) \), which is optimal. Our other analysis follows an approach that has not been pursued in prior work, and in particular does not rely on a global lower bound on the strong convexity of the conditional negative log-likelihood. Instead, we only bound the strong convexity at the optimum and use an upper bound on the smoothness of the objective function to prove the convergence of PSGD in terms of function value. Again, it is impossible to prove a bound on the smoothness over the whole domain, because large noise variance can lead to very bad smoothness bounds. Thus we use the same projection set as above and show that, within this set, we can get an upper bound on the smoothness that depends polynomially on all parameters, \( \beta, 1/\alpha, 1/b \), of the problem. The main drawback of this method is that the dependence on the accuracy is \( \Theta((\log^4(1/\varepsilon))/\varepsilon^4) \). It is worth noting that obtaining an algorithm whose sample complexity has an optimal \( \Theta(1/\varepsilon^2) \) dependence on the estimation accuracy, and also scales polynomially in \( 1/\alpha \) is a fundamental bottleneck, which is already present in the much simpler special case of our problem of estimating the variance of a 1-dimensional truncated Gaussian, studied in [8].

Finally, to help build intuition around the intricacies in estimating truncated linear regression models with unknown noise variance, let’s consider why natural approaches of reducing this problem to estimating truncated linear regression models with known noise variance – which can be done using the algorithm of [9] – fail. A simple such reduction might compute the empirical variance \( \sigma_n^2 \) of the truncated sample, and plug that into the algorithm of [9] to estimate the linear model, pretending that \( \sigma_n^2 \) is the true noise variance. A more sophisticated approach might grid over candidate noise variances, plug each of these candidates into the algorithm of [9] to estimate a linear model, and then select the (linear model, noise variance) pair that is most consistent, e.g. the pair whose noise variance is closest to the empirical variance of the residuals between the linear model’s predictions and the observed labels. Both of these approaches fail, as suggested by the example below. Suppose that the ground truth is \( y = w^* x + w_{0}^* + \varepsilon \), where \( x \) is one-dimensional, \( (w^*, w_{0}^*) = (0, 0) \), and the noise variance is \( \sigma^2 = 1 \). Suppose also that we create a truncated dataset by, repeatedly, sampling \( x_i \) from \( \mathcal{N}(0, 1) \), generating \( y_i \) according to the ground truth, and truncating the resulting pair \((x_i, y_i)\) unless \( y_i \geq \tau \), where \( \tau \) is some threshold. It is clear that, if we guess the true noise variance to be the empirical one, then for \( \tau \) gets bigger, the empirical variance gets smaller, thus the linear model that would be estimated by treating the empirical variance as the true noise variance would have intercept that is close to \( \tau \) rather than 0. So this naive approach would fail. Let us consider the more sophisticated one that grids over candidate noise variances. In particular, let us compare what happens when (1) we guess the noise variance to be \( \sigma^2 = \sigma_n^2 \), and run the algorithm of [9] with this guess to get an estimate \((\hat{w}, \hat{w}_0)\); and (2) when we guess the noise variance to be \( \sigma^2_n \equiv \sigma^2_n \) (the empirical variance) and run the algorithm of [9] with this guess to get an estimate \((\hat{w}^*, \hat{w}_0^*)\). It is easy to see that, as \( \tau \) gets large, \( \sigma^2_n - \frac{1}{N} \sum_{i}(y_i - \hat{w}x_i - \hat{w}_0)^2 \gg \sigma^2_n - \frac{1}{N} \sum_{i}(y_i - \hat{w}^*x_i - \hat{w}_0^*)^2 \) (here, \( N \) is the number of samples). This holds even if \( N \) goes to infinity). Thus the selection criterion proposed above will fail to prefer (1) over (2), even though the guess of the noise variance under (1) is the correct one. Our approach in this paper amounts to using a different criterion, namely selecting the pair with the largest conditional log-likelihood. But analyzing when and how well this works, as well as obtaining an efficient algorithm to find the best pair is the main contribution of this work.

**Roadmap.** We present the truncated linear regression setting and our assumptions in Section 2. We state and prove our main estimation results Section 3 and 4, and prove the asymptotic normality of our estimators in Section 5. Finally, in Section 6 we assess the performance of our methods.

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1While [8] does not explicitly analyze the dependence of the sample complexity on \( \alpha \); by tracing through their bounds, one can see that the dependence of the sample complexity on \( \alpha \) is exponential.
2 Models and assumptions

We study the truncated linear regression setting studied in [9], with the additional complexity that the variance of the noise model is unknown. In particular, let \( S \subset \mathbb{R} \) be a measurable subset of the real line, to which oracle access is provided. Assume we have \( N = 3n \) truncated samples \((x^{(i)}, y^{(i)})_{i=1}^N\) (we will split the samples into three, see 4.5), each generated according to the following procedure, for some unknown \( w^* \in \mathbb{R}^k \) and \( \sigma^* \in \mathbb{R} \) that are fixed across different \( i \)'s:

1. sample \( x^{(i)} \) from some distribution \( \mathcal{P}_0 \) with support \( \text{supp}(\mathcal{P}_0) \subseteq \mathbb{R}^k \);
2. generate \( y^{(i)} \) according to the following model, where \( \varepsilon \sim \mathcal{N}(0, \sigma^*) \):
   \[
y^{(i)} = w^T x^{(i)} + \varepsilon \tag{2.1}
\]
3. if \( y^{(i)} \in S \) then return \((x^{(i)}, y^{(i)})\) as the \( i \)th sample, otherwise repeat from step 1.

After truncation, denote the distribution of \( x \) to be \( \mathcal{P} \) with the \( \text{supp}(\mathcal{P}) = \text{supp}(\mathcal{P}_0) \). The sampled \( x \) is still \( i.i.d. \), so we could only consider the distribution \( \mathcal{P} \). As already established in prior work [8, 9, 18], some assumptions must be placed on \( S \) and its interaction with the unknown parameters \((w^*, \sigma^*)\) and the \( x^{(i)} \)'s in order for the parameters to be identifiable. We state our assumptions after a useful definition.

**Definition 2.1** (Survival Probability). Let \( S \subseteq \mathbb{R} \) be measurable, \( x, w \in \mathbb{R}^k \), and \( \sigma \in \mathbb{R} \). We define the survival probability \( \alpha(w, \sigma, x, S) \) as \( \alpha(w, \sigma, x, S) = \mathbb{P}\{Y \in S\} \) where \( Y \sim \mathcal{N}(w^T x, \sigma^2) \). When \( S \) is clear from context we may omit \( S \) from the arguments of \( \alpha \) and simply write \( \alpha(w, \sigma, x) \).

We are now ready to state our assumptions. Our first assumption gives an upper bound on the norm of the true parameters and the feature vectors.

**Assumption 2.2** (Normalization/Data Generation). Assume that \( \|w^*\|^2 \leq \beta \) and \( \|x\|^2 \leq 1 \) for all \( x \in \text{supp}(\mathcal{P}) \). We also assume oracle access for set \( S \), i.e., we have an oracle which determines, for any \( y \), whether \( y \in S \).

Our next assumption is the survival probability lower bound following [8, 9].

**Assumption 2.3** (Constant Survival Probability). For all \( x \in \text{supp}(\mathcal{P}) \), we have \( \alpha(w^*, \sigma^*, x) > a \).

Our final assumption is about the spectrum of the covariance matrix of the features and is classical in some other linear regression settings, such as in [9].

**Assumption 2.4** (Thickness of Feature Covariance). Let \( X = \frac{1}{n} \sum_{i=1}^{n} x^{(i)} x^{(i)T} \) be the \( k \times k \) feature vector covariance matrix. We assume that, for some positive real number \( b \), we have \( \mathbb{E}_{x \sim \mathcal{P}}[X] = \mathbb{E}_{x \sim \mathcal{P}}[xx^T] \geq b \cdot I \). Here we abuse the notation that the \((x^{(i)}, y^{(i)})_{i=1}^N\) are samples used for PSGD.

3 Main Result

When the variance parameter \( \sigma^2 \) is unknown, the conditional log-likelihood function of the plain linear regression becomes non-concave, and is therefore not straightforward to optimize. For this reason, we follow the common practice of reparameterizing the problem with respect to the parameters \( v = w/\sigma^2 \) and \( \lambda = 1/\sigma^2 \). The algorithm that we use is Projected SGD on the population negative log-likelihood function \( \ell(v, \lambda) \) as presented in Algorithm 1. We formally define the loss function \( \ell \) in Section 4.1.

Our goal is to apply the above algorithm to the conditional negative log-likelihood function of the truncated linear regression model. To execute the above algorithm in polynomial time, we need to solve the following algorithmic problems:

(a) **initial feasible point:** compute an initial feasible point in some projection set \( D \),
(b) **unbiased gradient estimation:** sample an unbiased estimate of \( \nabla \ell(v^{(t-1)}, \lambda^{(t-1)}) \),
(c) **efficient projection:** design an algorithm to project to the set \( D \).
Algorithm 1 Projected SGD on \( \ell \)

**Input:** \((x^{(i)}, y^{(i)})\) for \(i \in [n]\), learning rates \(\eta_t\)

\begin{align*}
1: & \quad (w_0, \sigma_0^2) \leftarrow \text{OLS estimates using other } n \text{ samples; } (q, \lambda^{(0)}) \leftarrow (w_0/\sigma_0^2, 1/\sigma_0^2) \\
2: & \quad \text{In terms of } \sigma_0^2 \text{ define convex set } D \text{ over pairs } (v, \lambda) \text{ as described in Definition 3.1} \\
3: & \quad v^{(0)} \leftarrow \arg\min_{v, (v, \lambda^{(0)}) \in D} \|v - q\|_2 \\
4: & \quad \text{for } t = 1, \ldots, n \text{ do} \\
5: & \quad \text{Sample gradient } u^{(t)} \text{ such that } \mathbb{E}[u^{(t)}(v, \lambda)_{t-1}] = \nabla \ell(v_{t-1}, \lambda_{t-1}) \\
6: & \quad (v^{(t)}, \lambda^{(t)}) \leftarrow (v^{(t-1)}, \lambda^{(t-1)}) - \eta_t \cdot u^{(t)} \\
7: & \quad (v^{(t)}, \lambda^{(t)}) \leftarrow \arg\min_{(v, \lambda) \in D} \|v - (v^{(t-1)}, \lambda^{(t-1)})\|_2 \\
8: & \quad \text{end for} \\
9: & \quad (\hat{v}, \hat{\lambda}) \leftarrow (v^{(n)}, \lambda^{(n)}); \text{ return } (\hat{w}, \hat{\sigma}^2) \leftarrow (\hat{v}/\hat{\lambda}, 1/\hat{\lambda})
\end{align*}

Solving the computational problems (a)-(c) is the first step in the proof of our main result. First we define the appropriate projection set to use in our algorithm.

**Definition 3.1 (Projection Set).** We define the projection set to be the \((v, \lambda)\) satisfying

\[
D = \left\{ (v, \lambda) = \left( w, 1 \right) \in \mathbb{R}^k \times \mathbb{R} \left| \frac{1}{8(5 + 2 \log(1/\alpha))} \leq \frac{\sigma^2 v^2}{\sigma_0^2} \leq \frac{96}{\alpha^2} \cdot \|w\|_2^2 \leq \beta \right. \right\}
\]

where \(\sigma_0^2\) is the estimated variance from solving the classical linear regression problem (using \(n\) samples) on the data ignoring truncation.

This projection set contains two parts: the first part that restricts the weights and the second part that restricts the variance. For the second part, we use the variance predicted using ordinary least squares.

Now we are ready to formally state our main theorem about the estimation of the parameters \(w\) and \(\sigma\) from truncated samples under the Assumptions 2.2-2.4.

**Theorem 3.2.** Let \((x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})\) be \(n\) samples from the linear regression model (2.1) with ground truth \(w^*, \sigma^*\). If Assumptions 2.2-2.4 hold, then we can instantiate the learning rates used in Algorithm 1 such that, with success probability \(\geq 2/3\), the output estimates \(\hat{w}, \hat{\sigma}^2\) satisfy either one of the following bounds:

\[
\|\hat{w} - w^*\|_2 + |\hat{\sigma}^2 - \sigma^2| \leq \frac{\text{poly}(\sigma, \frac{1}{\sigma^*} \cdot \frac{\log(n)}{n^{1/4}}) \cdot \log(n)}{\sqrt{n}} \quad (3.1)
\]

\[
\|\hat{w} - w^*\|_2 + |\hat{\sigma}^2 - \sigma^2| \leq \frac{\text{poly}(\sigma^*, \sqrt{\varepsilon}) \cdot \exp\left(\frac{\text{poly}(\frac{\beta}{\sigma^*})}{\sqrt{n}}\right)}{\sqrt{n}} \quad (3.2)
\]

for some polynomials that we make explicit in the detailed proof of the theorem (Section B.7).

**Remark 3.3.** It is easy to see that the probability of success in Theorem 3.2 can be boosted to \(1 - \delta\) with an additional cost of order \(\log(1/\delta)\) in the rates. We can achieve this using a folklore boosting technique for parameter estimation problems. In particular, we can run the algorithm of Theorem 3.2 \(\log(1/\delta)\) times independently and then we can pick the estimate that is \(\varepsilon\) close to at least the half of the rest of the estimates, where \(\varepsilon\) is the target estimation error. It is easy to see that with this boosting technique we can get error at most \(2\varepsilon\) with probability of success at least \(1 - \delta\).

To prove Theorem 3.2 we provide two different analyses of the Algorithm 1. The first analysis uses only the fact that the stochastic gradients of the population conditional negative log-likelihood function have bounded second moment and is based on the following theorem.

**Theorem 3.4 (Theorem 2 of [26]).** Let \(\ell\) be a convex function with a minimizer \((v^*, \lambda^*)\) and suppose that there exists a constant \(\rho\) such that

(i) **bounded step variance:** \(\mathbb{E}\left[\|u^{(t)}\|_2^2\right] \leq \rho^2\) where \(u^{(t)}\) is the sampled step variance in algorithm 1,
We define the distributions

We provide an overview of the proof of Theorem 3.2, postponing most technical details to the supplementary material. The main steps of the proof are the following.

1. In Section 4.1 we derive the negative population conditional log-likelihood, its gradient, and its Hessian matrix.
2. In Section 4.2 we discuss the computational problems (a) - (c).
3. In Section 4.3 we formally state our results that prove the properties (i), (ii), and (iv).
4. In Section 4.4 we state our results for property (iii), the feasibility of the optimal solution.
5. Finally, in Section 4.5 we use all the above results to complete the proof of 3.2.

4 Overview of the proof of Theorem 3.2.

We provide an overview of the proof of Theorem 3.2, postponing most technical details to the supplementary material. The main steps of the proof are the following.

In this section we define the objective function that we use to apply our PSGD algorithm. This objective function is derived by taking the negative expected value of the log-likelihood of \( y \) conditional on the value of \( x \). Given a sample \((x, y)\), the negative conditional log-likelihood that \( y \) is a sampled from the truncated linear regression with parameters \( w, \sigma^2 \) given the value of \( x \) is equal to

\[
\ell(w, \sigma^2; x, y) = \frac{1}{2\sigma^2} (y^2 - 2yu^T x) + \log \left( \int_S \exp\left(-\frac{1}{2\sigma^2}(z^2 - 2zw^T x)\right)dz \right)
\]

If we reparameterize \( \ell \) with \( \lambda = 1/\sigma^2 \) and \( v = w/\sigma^2 \), we have

\[
\ell(v, \lambda; x, y) = \frac{1}{2} (\lambda y^2 - 2yu^T x) + \log \left( \int_S \exp\left(-\frac{1}{2}(\lambda z^2 - 2zv^T x)\right)dz \right) \tag{4.1}
\]

We define the distributions \( F_x = \mathcal{N}(w^T x, \sigma^2, S) \) and \( Q_x = \mathcal{N}(w^T x, \sigma^2, S) \), and the negative population conditional log-likelihood function \( \bar{\ell} \) as follows

\[
\bar{\ell}(v, \lambda) = \mathbb{E}_{z \sim P} \left[ \mathbb{E}_{y \sim F_x}[\ell(v, \lambda; x, y)] \right]. \tag{4.2}
\]
We can then easily compute
\[
\frac{\partial \bar{\ell}}{\partial v} = \mathbb{E}_{x \sim \mathcal{P}} \left[ \mathbb{E}_{y \sim F_x} [z \cdot x] - \mathbb{E}_{y \sim F_x} [y \cdot x] \right] \quad \frac{\partial \bar{\ell}}{\partial \lambda} = \frac{1}{2} \mathbb{E}_{y \sim F_x} \left[ \mathbb{E}_{z \sim Q_x} [z^2] - \mathbb{E}_{z \sim Q_x} [z^2] \right].
\]

From these we also compute the Hessian matrix which is
\[
\mathbf{H}(v, \lambda) = \mathbb{E}_{x \sim \mathcal{P}} \left[ \begin{pmatrix} \text{Var}_{z \sim Q_x} [z] xx^T & \text{Cov}_{z \sim Q_x} \left[ \frac{1}{2} z^2, z \right] x^T \\ \text{Cov}_{z \sim Q_x} \left[ \frac{1}{2} z^2, z \right] x^T & \text{Var}_{z \sim Q_x} \left[ \frac{1}{2} z^2 \right] \end{pmatrix} \right]
\] (4.3)

Here, \( \mathbf{H} \) is the expectation of covariance matrices, which is positive definite. Thus, \( \bar{\ell} \) is convex.

### 4.2 Computational Problems

The computational problems (a) - (c) can be tackled in the following way

(a) We start with computing \((w_0, \sigma_0)\) by applying OLS to the truncated data.

(b) As we see from the expression of the gradient of \(\bar{\ell}\), we can compute an unbiased estimate of the gradient using one of our samples \((x^{(i)}, y^{(i)})\), our sample access to the distribution \(\mathcal{P}\) of \(x\)’s, and by doing rejection sampling until we find a point with \((x, z)\) with \(z \in S\). To bound the time needed for this step, we need to understand the mass of the survival set \(S\) for some \(x \sim \mathcal{P}\) and some vector of parameters \((v, \lambda) \in D\). See Supplementary Material B.2.

(c) We show in the supplementary material that \(D\) is a convex set and define an efficient algorithm to project our estimates to \(D\). It is elaborated in Supplementary Material A.2.

For more details on the way we solve the computational problems (a) - (c) we refer to the supplementary material.

### 4.3 Bounded Step Variance, Bounded Domain, and Strong Convexity

In this section we present the statements that establish the properties (i), (ii), (iv) that we need in order to apply Theorem 3.4 and Theorem 3.5.

**Theorem 4.1.** For every \((v, \lambda) \in D\), we have
\[
\mathbb{E} \left[ \| (y - z) x \|^2 \right] \leq \text{poly}(1/a)(1 + \beta^2 + \sigma_0^2), \quad \mathbf{H}(v, \lambda) \geq b \cdot \exp \left( - \text{poly}(1/a)(1 + \beta^2/\sigma_0^2) \right) \begin{pmatrix} \sigma_0^2 I & 0 \\ 0 & \sigma_0^2 \end{pmatrix}
\]
where, \(x \sim \mathcal{P}\), \(y \sim F_x\) and \(z \sim Q_x\) and \(\mathbf{H}\) from (4.3).

The proof of Theorem 4.1 can be found in Supplementary Material B.3. Also, it is easy to see that the diameter of \(D\) is bounded: \(\|v\|_2 < \|w\|_2/\sigma^2 \leq 8(5 - 2 \log a)\sqrt{\beta/\sigma_0^2}\) and \(|\lambda| < 1/\sigma^2 \leq 8(5 - 2 \log a)/\sigma_0^2\).

### 4.4 Feasibility of Optimal Solution

We need to calculate the probability for the optimal point to be in the projection set, i.e., \(w^*, \sigma^* \in D\).

**Lemma 4.2.** Under Assumptions 2.3 and 2.2 we have that, \(\mathbb{P}(w^*, \sigma^*) \in D) \geq 13/16\).

The proof of this Lemma can be found in Supplementary Material B.4.

### 4.5 Proof of Theorem 3.2

The algorithm that proves Theorem 3.2 starts with splitting the samples \(n\) into three parts of equal size in order to avoid dependence. The first part of \(n/3\) samples is used to compute the OLS estimates \(w_0\), where \(w_0\) is used in the initialization step. The second part of \(n/3\) samples is used to compute the OLS estimate of \(\sigma_0\), where \(\sigma_0\) is used both to initialize and define the projection set \(D\), and is used at every step where a projection to \(D\) is needed. The number of sample to define \(D\) is \(\text{poly}(\frac{\beta}{a^2}, \frac{1}{\sigma^2})\), and our \(n\) itself will reach this bound (see Section B.4.). Finally, the third part of the \(n/3\) samples is used in our main PSGD algorithm. For the rest of the proof we abuse the notation and we use
\( n \leftarrow n/3 \) for simplicity. The next thing that we need to decide is the learning rates that we are going to use for the PSGD. We will analyze PSGD under two possible choices of its learning rate schedule, namely \( \eta_t = c/\sqrt{t} \) and \( \eta_t = 1/(\zeta \cdot t) \), and will show that it produces estimates for the ground-truth parameters that scale as \( A_1 = \frac{\text{poly}(\frac{\sigma^2}{\sigma^*}, \frac{1}{\sqrt{n}}) \cdot \log(n)}{n^{1/2}} \) and \( A_2 = \frac{\text{poly}(\sigma^3 \cdot \frac{\sigma^*}{\sqrt{n}})}{b_\gamma n} \) respectively, for some polynomials that we will make explicit in Section B.7.

**Analysis for learning schedule** \( \eta_t = c/\sqrt{t} \). First, by combining Theorem 3.4 and Theorem 4.1, and noticing the relationship between \( \sigma_0 \) and \( \sigma^* \) in the projection set, we get the following corollary whose proof can be found in Section B.7.

**Corollary 4.3.** Suppose that \((v^*, \lambda^*) = (w^*/\sigma^* + 1/\sigma^2) \in D \) and \( \bar{\ell} \) is defined as in (4.2). Then, the output \((\hat{v}, \hat{\lambda})\) of Algorithm 1 satisfies \( \mathbb{E}\left[ \bar{\ell}(\hat{v}, \hat{\lambda}) - \bar{\ell}(v^*, \lambda^*) \right] \leq \frac{\text{poly}(\frac{\sigma^2}{\sigma^*}, \frac{1}{\sqrt{n}})}{n^{1/2}} \).

Our next step is to utilize the above bound to so that our estimates achieve small errors in the parameter space. For this we use the strong convexity of \( \ell \) at the optimum.

**Lemma 4.4.** The Hessian at the true \((v^*, \lambda^*)\) satisfies \( H(v^*, \lambda^*) \succeq \frac{\text{poly}(\sigma^*/\alpha, \beta/\sigma^* \cdot \sqrt{\delta})}{\text{poly}(1/\sqrt{n}, \beta/\sigma^*)} \cdot \log(n) \).

The proof of Lemma 4.4 can be found in Supplementary Material B.3.1. We also utilize the following lemma to calculate an upper bound on \( \ell(v^*, \lambda^*) \):

**Lemma 4.5.** (Lemma 3 in [9]) Let \( x, w \in \mathbb{R}^k \), \( \sigma > 0 \), then \( \mathbb{E}_{y \sim \mathcal{N}(w^T x, \sigma^2, S)} (y - w^T x)^2 \leq (4 - 2 \log \alpha(w, \sigma, x)) \sigma^2. \)

By the above lemma, we have:

\[
\bar{\ell}(v^*, \lambda^*) = \mathbb{E}_{x \in P} \mathbb{E}_{y \sim \mathcal{N}(v, \sigma, S)} \left[ \frac{(y - w^T x)^2}{2\sigma^2} + \log \alpha(w^*, \sigma^*, x) \right] \leq 4 - 2 \log \min_{x \in P} \alpha(w^*, \sigma^*, x) \leq 4 - 2 \log a
\]

Applying Markov’s inequality we have

\[
\mathbb{P}\left[ \bar{\ell}(\hat{v}, \hat{\lambda}) - \bar{\ell}(v^*, \lambda^*) > \frac{1}{7} \right] \leq \frac{1}{7}.
\]

The inequality implies that, with at least 6/7 probability, the actual difference is at most 7 times its expectation. Also, the probability that \((v^*, \lambda^*) \in D\) is at least 13/16 by Lemma 4.2. Combining these two, we have at least 13/16 - 1/7 > 2/3 probability to hold:

\[
\bar{\ell}(\hat{v}, \hat{\lambda}) - \bar{\ell}(v^*, \lambda^*) < \frac{\text{poly}(\frac{\sigma^*}{\alpha}, \beta, \frac{1}{\sigma^*}) \cdot \log(n)}{\delta \cdot n^{1/2}}
\]

(4.5)

Now applying Lemma 4.4, since \( \hat{v}, \hat{\lambda} \) converge to \( v^*, \lambda^* \), when they have small distance, we have

\[
\|\hat{v} - v^*\|^2 + (\hat{\lambda} - \lambda^*)^2 \leq \frac{2}{\delta} \left( \bar{\ell}(\hat{v}, \hat{\lambda}) - \bar{\ell}(v^*, \lambda^*) \right) \leq \frac{\text{poly}(\frac{\sigma^*}{\alpha}, \beta, \frac{1}{\sigma^*}) \cdot \log(n)}{\delta \cdot n^{1/2}}
\]

(4.6)

where \( H(v^*, \lambda^*) \succeq \delta I \) and \( \delta^{-1} = \text{poly} \left( \frac{\sigma^*}{\alpha \cdot \beta} \cdot \frac{1}{\sigma^*} \right) \).

Finally, doing some simple calculations for the reparameterization—see Section B.7 in supplementary material—we have that

\[
\|\hat{w} - w^*\|^2 + |\hat{\sigma}^2 - \sigma^*|^2 \leq \sqrt{2 \left( \|\hat{w} - w^*\|^2 + |\hat{\sigma}^2 - \sigma^*|^2 \right)^2} \leq \text{poly}(1/\alpha, \sigma^*) \left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|_2
\]

(4.7)

Combining inequalities (4.4), (4.5), (4.6), (4.7), inequality (3.1) of Theorem 3.2 follows.

**Analysis for learning schedule** \( \eta_t = 1/(\zeta \cdot t) \). We combine Theorem 3.5 and Theorem 4.1 to get the following corollary whose proof can be found in Section B.6.

**Corollary 4.6.** Suppose the true parameters \((v^*, \lambda^*) = (w^*/\sigma^* + 1/\sigma^2) \in D \) and \( \bar{\ell} \) is defined as equation (4.2). Then, for \( \hat{v}, \hat{\lambda} \) being the output of Algorithm 1, it holds that

\[
\mathbb{E}\left[ \left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|_2^2 \right] \leq \text{poly} \left( \frac{\beta}{\sigma^* + \frac{1}{\sigma^*}}, \frac{1}{\alpha} \right) \times \exp \left( \text{poly}(1/\alpha) \left( 1 + \frac{\beta}{\sigma^* + \frac{1}{\sigma^*}} \right) \right) \frac{1}{\delta \cdot n^{1/2}}.
\]
We now prove the asymptotic normality of our estimates (Theorem 5.1), which we use to obtain their confidence regions (Corollary 5.2). For the proofs we refer to the Section B.8.

**Theorem 5.1.** Let \((\hat{w}, \hat{\sigma}^2)\) be the output of Algorithm 1 with \(\eta_b = 1/(\zeta \cdot t)\). Then, the vector \(\sqrt{n}S^{-1/2}((\hat{w}, \hat{\sigma}^2)) - (w^*, \sigma^*)\), asymptotically converges to \(\mathcal{N}(0, I_{k+1})\), where

\[
S = \frac{1}{\zeta} R^T \Sigma R, \quad R = \begin{pmatrix} 1/\hat{\sigma}^2 I & -\hat{w}/\hat{\sigma}^4 \\ 0 & -1/\hat{\sigma}^4 \end{pmatrix},
\]

- \(\Sigma\) is the matrix that satisfies \((\hat{H} - \frac{2}{\zeta} I) \Sigma + \Sigma (\hat{H} - \frac{2}{\zeta} I) = \Gamma(\hat{v}, \hat{\lambda}) = \Gamma \left( \frac{w}{\sigma^2}, \frac{1}{\sigma^2} \right)\).
- \(\hat{H}\) is the empirical estimation of the Hessian matrix defined by (4.3) at the point \((\hat{v}, \hat{\lambda})\).
- \(\Gamma(v, \lambda) = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \ell(v, \lambda; x^{(i)}, y^{(i)})}{\partial (v, \lambda)} \frac{\partial \ell(v, \lambda; x^{(i)}, y^{(i)})}{\partial (v, \lambda)^T}\).

**Corollary 5.2.** Let \(q_{\alpha}\) be the \(1 - \alpha\) quantile of distribution \(\chi^2_{k+1}\). Then the \(1 - \alpha\) asymptotic confidence region of \((w, \sigma^2)\) is \(\left( \begin{pmatrix} \hat{w} \\ \hat{\sigma}^2 \end{pmatrix} - \begin{pmatrix} w^* \\ \sigma^* \end{pmatrix} \right)^T R^T \Sigma R \left( \begin{pmatrix} w^* \\ \sigma^* \end{pmatrix} - \begin{pmatrix} \hat{w} \\ \hat{\sigma}^2 \end{pmatrix} \right) \leq \frac{q_{\alpha}}{\zeta n}.

### 6 Synthetic Experiments

In Figure 1, we show how our algorithm performs against models with large noise variances. For this experiment, we sample a 10 dimensional ground-truth weight vector \(w^* \sim U(-1, 1)\) and generate 10000 \((\hat{x}^{(i)}, \hat{y}^{(i)})\) pairs according to \(x^{(i)} \sim U(-5, 5)\) and \(\hat{y}^{(i)} = w^* x^{(i)} + \varepsilon^{(i)}\) where \(\varepsilon^{(i)} \sim \mathcal{N}(0, \sigma^*).\) We fix \(w^*\) and \(x^{(0)}, \ldots, x^{(n)}\) across all trials, but vary \(\sigma^*\). After adding Gaussian noise to our ground-truth predictions, we left truncate at zero, removing all of the pairs who’s \(\hat{y}^{(i)}\) is negative; retaining approximately 50% of the original samples. We vary \(\sigma^2\) over the interval \([1, 10]\) and evaluate how well our procedure recovers \(w^*\) and \(\sigma^*\) in comparison to OLS, [9], given \(\sigma_{\text{true}}\), and truncreg, an R package for truncated regression. All three methods remove a significant amount of bias in the presence of truncation. We emphasize though that truncreg cannot be applied in settings where the truncation is more complicated that a simple interval whereas our method can still be applied.

In Figure 2, we show that our theoretical error bounds hold in practice. For this experiment, we set \(w^*\) to a ten dimensional weight vector of ones, sample \(n\) \((\hat{x}^{(i)}, \hat{y}^{(i)})\) pairs, where \(x^{(i)} \sim \mathcal{N}(0, I)\) and \(\varepsilon^{(i)} \sim \mathcal{N}(0, 1)\), and right truncate at zero. We generate our data once. However, for each experiment, we randomly select a \(k\) samples that exceed zero, and run our experiment. We vary \(k\) over the interval \([10, 5000]\). Our results show that the error bound asymptotically approaches zero at a rate that reaffirms our theoretical analysis.
7 Semi-Synthetic Experiment

Here we show results for an additional experiment that we conducted on semi-synthetic data. For this experiment, we used the [1] dataset. The [1] dataset was originally collected by the Norwegian Public Roads Administration for a study of air pollution at a road in Oslo, Norway. The dataset consists of 500 observations. Interestingly enough, it is common for environmental data to be truncated because of problems in reliably measuring low concentrations.

For this experiment, we apply left truncation to the dataset, varying a truncation parameter $C$ over the interval $[1.0, 4.0]$. We run our procedure a total of 10 times and retain the trial that has the smallest gradient as our algorithm’s prediction. We conduct the experiment with the same hyperparameters as for the synthetic data experiments. Below, we report our results.

In Figure 3 (a), we show the $L^2$ distance between the model parameters of each method and the ground-truth, which we take to be the OLS estimation without truncation. In this figure, we show that our method performs very comparably to [7]. We also note that, at high values of the truncation, both methods exhibit non-monotonicity in their estimation error, which is presumably due to model misspecification and the truncation being too aggressive.

In Figure 3 (b), we report the $L^1$ distance between the predicted noise variance and the ground-truth noise variance. Again, our method performs very comparably to [7], and we observe non-monotonicity at high values of the truncation.

Figure 3: Our method’s performance on the semi-synthetic PM10 dataset.

Acknowledgements

CD and PS were supported by NSF Awards CCF-1901292, DMS-2022448 (FODSI) and DMS-2134108, by a Simons Investigator Award, by the Simons Collaboration on the Theory of Algorithmic
Fairness, by a DSTA grant, and by the DOE PhILMs project (No. DE-AC05-76RL01830). MZ was supported by NSF DMS-2023505 (FODSI).

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A Projection Set

If it is not specified, all the norms for vectors are $L^2$ norms, by default.

A.1 Proof for the convexity of the projection set

Recall that the projection set is

$$D_r = \{ (v, \lambda) = (w, \frac{1}{\sigma^2}) \in \mathbb{R}^k \times \mathbb{R} \mid \frac{1}{8(5-2 \log a)} \leq \frac{\sigma^2}{\sigma_0^2} \leq \frac{96}{a^2}, \|w\|^2 \leq \beta \}$$

So we have

$$\|w\|^2 = w^Tw = (v/\lambda)^T(v/\lambda) = v^Tv/\lambda^2 = \|v\|^2 / \lambda^2$$

Therefore, if we transform it into a direct representation with $v, \lambda$, we get

$$D_r = \{ (v, \lambda) \in \mathbb{R}^k \times \mathbb{R}_{>0} \mid \frac{a^2}{96\sigma_0^2} \leq \lambda \leq \frac{8(5-2 \log a)}{\sigma_0^2}, \|v\|^2 \leq \beta \lambda^2 \}$$

Where $\mathbb{R}_{>0}$ denotes the set of positive real numbers. We prove that this is a convex set by showing that it is an (infinite) intersection of convex sets.

The first constraint $\frac{a^2}{96\sigma_0^2} \leq \lambda \leq \frac{8(5-2 \log a)}{\sigma_0^2}$ is a region between two hyper planes, which is also a convex constraint.

We prove that the second constraint $\|v\|^2 \leq \beta \lambda^2$ is convex. Suppose we have two vectors such that $\|v_1\|^2 \leq \beta \lambda_1^2$ and $\|v_2\|^2 \leq \beta \lambda_2^2$. We need to prove that $\|\mu v_1 + (1-\mu)v_2\|^2 \leq \beta (\mu \lambda_1 + (1-\mu)\lambda_2)^2$ for all $\mu \in (0, 1)$. With this condition and knowing that $\lambda_1, \lambda_2 > 0$, we know that $\|v_1\| \leq \sqrt{\beta} \lambda_1$ and $\|v_2\| \leq \sqrt{\beta} \lambda_2$. By the triangle inequality, we have $\|\mu v_1 + (1-\mu)v_2\|^2 \leq \mu \|v_1\|^2 + (1-\mu)\|v_2\|^2 \leq (\mu \sqrt{\beta} \lambda_1 + (1-\mu) \sqrt{\beta} \lambda_2)^2 = \beta (\mu \lambda_1 + (1-\mu)\lambda_2)^2$. So, we proved the convexity of the third constraint.

Since the three constraints are all convex and the projection set is the intersection of the three constraints, the projection set is convex.

A.2 Algorithm for projecting to the projection set

We have an explicit formula for projecting to the projection set. For convenience, we denote $\lambda_{min} = \frac{a^2}{96\sigma_0^2}$ and $\lambda_{max} = \frac{8(5-2 \log a)}{\sigma_0^2}$. Thus, the projection formula projects $(v, \lambda) \in D_r$ by minimizing $\| (v_0, \lambda_0) - (v, \lambda) \|$ as below:

$$\begin{cases} 
(v_0, \lambda_0) & \lambda_{min} \leq \lambda_0 \leq \lambda_{max}, \|v_0\|^2 \leq \beta \lambda_0^2 \\
(v_0, \lambda_{max}) & \lambda_0 \geq \lambda_{max}, \|v_0\|^2 \leq \beta \lambda_{max}^2 \\
(\sqrt{\frac{3}{2}} \lambda_{min}, \frac{\|v_0\|}{\sqrt{\|v_0\|^2 + \lambda_{min}}}) & \lambda_0 \geq \lambda_{max}, \|v_0\|^2 \geq \beta \lambda_{max}^2 \\
(v_0, \lambda_{min}) & \lambda_0 \leq \lambda_{min}, \|v_0\|^2 \leq \beta \lambda_{min}^2 \\
(\sqrt{\frac{3}{2}} \lambda_{max}, \frac{\|v_0\|}{\sqrt{\|v_0\|^2 + \lambda_{max}}}) & 0 \leq \lambda_{min}, \sqrt{3} \lambda_{min} \leq \|v_0\| \leq \sqrt{3} \lambda_{max} + \frac{\lambda_{min} - \lambda_0}{\sqrt{3}} \\
(\frac{\|v_0\|}{\beta(\|v_0\|^2 + \lambda_{max})} v_0, \sqrt{\|v_0\|^2 + \lambda_{max}}) & \lambda_0 \leq \lambda_{max}, \|v_0\| \geq \sqrt{3} \lambda_{max} + \frac{\lambda_{max} - \lambda_0}{\sqrt{3}} \\
(\beta(\|v_0\|^2 + \lambda_{max}) \frac{\|v_0\|}{\|v_0\|^2 + \lambda_0}, \sqrt{\|v_0\|^2 + \lambda_0}) & \text{Otherwise}
\end{cases}$$

B Missing Proofs

If it is not specified, all the norms for vectors are $L^2$ norms, by default.

B.1 Sampling the Gradient of the Objective Function

Let $D^*$ be the joint distribution of the observed pairs $(x, y)$, where $(w^*, \sigma^*)$ are the ground truth parameters. Notice that we have a sampler that generates samples $(x, y)$ from $D^*$. Also let’s define $D_{(w, \sigma)}$ as the joint distribution of the pairs $(x, y)$, when the vector of parameters is $(w, \sigma)$. We recall that $P$ is the marginal distribution of $x$ of the joint distribution $D^*$. Also, $P_0$ is the original distribution of $x$ before truncation.
So, after truncation, we have

\[ D'(\bar{x}, \bar{y}) = \frac{1\{\bar{y} \in S\} \cdot \mathcal{N}(w^T \bar{x}, \sigma^2; \bar{y}) \cdot \mathcal{P}_0(\bar{x})}{\mathbb{E}_{x \sim \mathcal{P}_0}[\alpha(w^*, \sigma^*, \bar{x}; S)]} \]

We note that \( \mathcal{N}(w^T \bar{x}, \sigma^2; \bar{y}) \) is the probability density of \( \bar{y} \); that is, \( \exp \left( -\frac{(\bar{y} - w^T \bar{x})^2}{2\sigma^2} \right) \). To make it explicit, let’s say that the relationship between \( \mathcal{P} \) and \( \mathcal{P}_0 \) is,

\[ \mathcal{P}(\bar{x}) = \int_{\mathbb{R}} \frac{1\{\bar{y} \in S\} \cdot \mathcal{N}(w^T \bar{x}, \sigma^2; \bar{y}) \cdot \mathcal{P}_0(\bar{x})}{\mathbb{E}_{x \sim \mathcal{P}_0}[\alpha(w^*, \sigma^*, \bar{x}; S)]}d\bar{y} = \frac{\alpha(w^*, \sigma^*, \bar{x}; S)}{\mathbb{E}_{x \sim \mathcal{P}_0}[\alpha(w^*, \sigma^*, \bar{x}; S)]} \cdot \mathcal{P}_0(\bar{x}) \]

Now we can sample the gradient. The gradient is

\[ \frac{\partial \bar{\ell}}{\partial \nu} = \mathbb{E}_{x \sim \mathcal{P}} \left[ \mathbb{E}_{y \sim \mathcal{N}(w^T x, \sigma^2, S)} [y \cdot x] - \mathbb{E}_{y \sim \mathcal{F}_x} [y \cdot x] \right] \]

\[ \frac{\partial \bar{l}}{\partial \lambda} = \frac{1}{2} \mathbb{E}_{x \sim \mathcal{P}} \left[ \mathbb{E}_{y \sim \mathcal{F}_x} [y^2] - \mathbb{E}_{z \sim \mathcal{Q}_x} [z^2] \right], \]

where \( \mathcal{F}_x = \mathcal{N}(w^T x, \sigma^2, S) \) and \( \mathcal{Q}_x = \mathcal{N}(w^T x, \sigma^2, S) \). Further, note that

\[ \mathbb{E}_{x \sim \mathcal{P}} \mathbb{E}_{y \sim \mathcal{N}(w^T x, \sigma^2, S)} [y \cdot x] = \int x \int_S (y \cdot x) \frac{\mathcal{N}(w^T x, \sigma^2; y)}{\mathcal{N}(w^T x, \sigma^2; S)} dy \cdot \mathcal{P}(x) \cdot dx \]

At each gradient step, we sample a pair of \( x, y \), and multiply them together. Similarly we sample \( \frac{1}{2} y^2 \), where \( x \sim \mathcal{P} \) and \( y \sim \mathcal{F}_x \), and ignore \( x \) after sampling. For the second term, we use rejection sampling to generate the data \( z \cdot x \) for \( z \sim \mathcal{Q}_x \) and \( x \sim \mathcal{P} \). Here, since we know the \( w \) and \( \sigma \), the generation process is simple. First, we generate \( x \) using \( D' \) (we discard the \( y \) value) and then sample \( z = w^T x + \varepsilon \) where \( \varepsilon = \mathcal{N}(0, \sigma^2) \). If \( z \notin S \), we continue to sample \( \varepsilon \) until we get \( z \in S \). Similarly, we can sample \( \frac{1}{2} z^2 \) where \( x \sim \mathcal{P}, z \sim \mathcal{Q}_x \) in this way, and we can ignore \( x \) after sampling.

Notice that if \( w, \sigma = w^*, \sigma^* \), we have \( \mathcal{Q}_x = \mathcal{F}_x \), so

\[ \frac{\partial \bar{\ell}}{\partial \nu}(w^*, \sigma^2) = \mathbb{E}_{x \sim \mathcal{P}} [z \cdot x] - \mathbb{E}_{y \sim \mathcal{F}_x} [y \cdot x] = 0 \]

and

\[ \frac{\partial \bar{l}}{\partial \lambda} = \frac{1}{2} \mathbb{E}_{x \sim \mathcal{P}} \left[ \mathbb{E}_{y \sim \mathcal{F}_x} [y^2] - \mathbb{E}_{z \sim \mathcal{Q}_x} [z^2] \right] = 0. \]

### B.2 Auxiliary Lemmas for Survival Probability of Feasible Points

For many of the following proofs, we require an estimation of \( \alpha(w, \sigma, x) \) for a feasible \( w \).

**Lemma B.1.** Let \( x, w, w' \in \mathbb{R}^k \) and \( \sigma, \sigma' > 0 \). Then,

\[ \log \left( \frac{1}{\alpha(w, \sigma, x)} \right) \leq \max \left( 1, \frac{2\sigma'^2}{\sigma^2} \right) \log \left( \frac{1}{\alpha(w', \sigma', x)} \right) + \max \left( 0, \frac{4\sigma'^2}{\sigma^2} - 2 \right) + \frac{2\sigma'^2}{\sigma^2} \left( \frac{(w^T x - w'^T x)^2}{2\sigma^2} \right) \]

**Proof.** Now, let’s define \( D, D', \) and \( D_S' \) as \( D = \mathcal{N}(w^T x, \sigma), D' = \mathcal{N}(w'^T x, \sigma'), \) and \( D_S' = \mathcal{N}(w'^T x, \sigma', S) \). Then, we have:

\[ \alpha(w, \sigma, x) = \mathbb{E}_{y \sim D} [1_{y \in S}] = \mathbb{E}_{y \sim D'} [1_{y \in S} \exp \left( \frac{(y - w^T x)^2}{2\sigma^2} - \frac{(y - w'^T x)^2}{2\sigma^2} \right)] \]

\[ = \alpha(w', \sigma', x) \cdot \mathbb{E}_{y \sim D_S'} [\exp \left( \frac{(y - w^T x)^2}{2\sigma^2} - \frac{(y - w'^T x)^2}{2\sigma^2} \right)] \]

\[ \geq \alpha(w', \sigma', x) \cdot \exp \left( \frac{(y - w^T x)^2}{2\sigma^2} - \frac{(y - w'^T x)^2}{2\sigma^2} \right) . \]
Now, from Cauchy-Schwarz we have that
\[
\frac{(y - w^T x)^2}{2\sigma^2} \leq \frac{2\sigma^2}{\sigma^2} \left( \frac{(y - w'^T x)^2}{2\sigma^2} + \frac{(w^T x - w'^T x)^2}{2\sigma^2} \right)
\]
and if we apply this to the above expression and take the logarithm of both sides of the inequality we get
\[
\log \alpha(w, \sigma, x) \geq \log \alpha(w', \sigma', x) + \mathbb{E}_{y \sim D_\delta} \left[ \left( 1 - \frac{2\sigma^2}{\sigma^2} \right) \frac{(y - w'^T x)^2}{2\sigma^2} - \mathbb{E}_{y \sim D_\delta} \left[ \frac{2\sigma^2}{\sigma^2} \left( \frac{(w^T x - w'^T x)^2}{2\sigma^2} \right) \right] \right]
\]
\[
\geq \log \alpha(w', \sigma', x) + \mathbb{E}_{y \sim D_\delta} \left[ \min \left( 0, 1 - \frac{2\sigma^2}{\sigma^2} \right) \left( 2 - \log \alpha(w', \sigma', x) \right) - \frac{2\sigma^2}{\sigma^2} \left( \frac{(w^T x - w'^T x)^2}{2\sigma^2} \right) \right]
\]
\[
= \min \left( -1, -\frac{2\sigma^2}{\sigma^2} \right) (- \log \alpha(w', \sigma', x)) - \min \left( 0, 2 - \frac{4\sigma^2}{\sigma^2} \right) - \frac{2\sigma^2}{\sigma^2} \left( \frac{(w^T x - w'^T x)^2}{2\sigma^2} \right)
\]
The second to last line is followed from Lemma 4.5 and the Lemma follows.

\[
\square
\]

### B.3 Proof of Theorem 4.1 and Lemma 4.4

Denote \( s = \max_{\sigma \in \Sigma} \min_{\sigma' \in \Sigma} \frac{\sigma - \sigma'}{\sigma} = \sqrt{8(5 - 2 \log \alpha) \frac{\alpha}{\sigma}} \) be the largest possible difference between variances, where \( D \) is the projection set defined in 3.1.

Next, to prove our strong convexity result, we use the following anti-concentration bound.

**Theorem B.2** (Theorem 8 in [6]). There is an absolute constant \( C \) such that if \( p: \mathbb{R}^n \to \mathbb{R} \) is a polynomial of degree at most \( d \), \( 0 < q < \infty \), and \( \mu \) is a log-concave probability measure on \( \mathbb{R}^n \), then
\[
\left( \int |p(x)|^q d\mu(x) \right)^{1/q} \leq C q \alpha^{-1/d} \cdot \mu \{ x \in \mathbb{R}^n : |p(x)| \leq \alpha \} \leq C q
\]

We also need the following lemma for bounding the survival probability for any parameters in \( D \).

**Lemma B.3.** For \( w, \sigma \in D \), we can find a lower bound for the survival probability: \( \frac{1}{\alpha(w, \sigma, x)} \leq \exp(\text{poly}(1/\alpha)(1 + \frac{\beta}{\sigma^2})) \)

**Proof.** From Lemma B.1, when we plug \( \sigma' = \sigma^*, w' = w^* \), we can derive that
\[
\log \left( \frac{1}{\alpha(w, \sigma, x)} \right) \leq \max \left( 1, \frac{2\sigma^2}{\sigma^2} \right) \log \left( \frac{1}{\alpha(w^*, \sigma^*, x)} \right) + \max \left( 0, \frac{4\sigma^2}{\sigma^2} \sigma^* \right) + \frac{2\sigma^2}{\sigma^2} \left( \frac{(w^T x - w^* T x)^2}{2\sigma^2} \right)
\]

Since \( \sigma, \sigma^* \in D \), we have \( \frac{\sigma^2}{\sigma^2} \leq s^2 \). Thus, we have
\[
\log \left( \frac{1}{\alpha(w, \sigma, x)} \right) \leq \max \left( 1, \frac{2\sigma^2}{\sigma^2} \right) \log \left( \frac{1}{\alpha(w^*, \sigma^*, x)} \right) + \max \left( 0, \frac{4\sigma^2}{\sigma^2} \sigma^* \right) + \frac{2\sigma^2}{\sigma^2} \left( \frac{(w^T x - w^* T x)^2}{2\sigma^2} \right)
\]
\[
\leq 2s^2 \left( \frac{1}{\alpha(w^*, \sigma^*, x)} + 2 + \frac{(w^T x - w^* T x)^2}{2\sigma^2} \right)
\]
\[
\leq 2s^2 \left( - \log \alpha + 2 \right)
\]
\[
\leq 2s^2 \left( - \log \alpha + 2 + \frac{2\beta}{\sigma^2} \right)
\]

3
Since $\sigma_0$ and $\sigma^*$ only have a polynomial difference, we have $\log \left( \frac{1}{\alpha(u, \sigma, x)} \right) \leq \text{poly}(1/a)(1 + \frac{\beta}{\sigma^*})$ which finishes the proof.

**Lemma B.4.** Let $z \sim N(\mu, \sigma^2, S)$ be a truncated normal variable. Assume $\alpha(\mu, \sigma^2, S) = a$. Then, we have that

$\text{Var}(z^2) = 2\mu^2\sigma^2(O(\log(a))) + \sigma^4O(\log(a)^2)$

**Proof.** Denote an affine transformation $S' = \{ \frac{x - \mu}{\sigma} | x \in S \}$. So $D = N(\mu, \sigma, S)$ is transformed into $D' = N(0, 1, S')$, while the survival probability is maintained. We then have

$\text{Var}_{z \sim D}(z^2) = \text{Var}_{y \sim D'}((\mu + \sigma y)^2) = \text{Var}_{y \sim D'}(2\sigma\mu y + \sigma^2 y^2) \leq 8(\sigma\mu)^2\text{Var}_{y \sim D'}(y) + 2(\sigma)^4\text{Var}_{y \sim D'}(y^2)$

By Lemma 4.5 we have $\text{Var}_{y \sim D'}[y] \leq \mathbb{E}[y^2] \leq 4 - 2 \log a$, and

$\text{Var}_{D'}(y^2) \leq \mathbb{E}_{D'}(y^4) \leq \frac{\int_q^\infty x^4 e^{-\frac{x^2}{2}} dx}{\int_q^\infty e^{-\frac{x^2}{2}} dx} = \frac{\int_q^\infty x^4 e^{-\frac{x^2}{2}} dx}{a/2}$

where $q$ satisfies $\int_q^\infty e^{-\frac{z^2}{2}} dz = a/2$.

Now, all we need to prove is that $q = O(\sqrt{-\log a})$. Notice that if $q > 2\sqrt{-\log a}$ (in the case $a < \sqrt{2}/2$ and $q > 1$), then we have

$a/2 = \int_q^\infty e^{-z^2/2} dz = \int_{q^2/2}^\infty \frac{1}{\sqrt{z}} e^{-z/2} dz < \frac{\sqrt{2}}{q} e^{-q^2/2} < \sqrt{2}a^2/q < a$

This is a contradiction.

By integration by parts, we obtain

$\int_q^\infty z^4 e^{-z^2/2} dz = q^3 e^{-q^2/2} + 3 \int_q^\infty z^2 e^{-z^2/2} dz = q^3 e^{-q^2/2} + 3 q e^{-q^2/2} + 3 \int_q^\infty e^{-z^2/2} dz$

and

$\frac{\int_q^\infty z^4 e^{-z^2/2} dz}{\int_q^\infty e^{-z^2/2} dz} = 3 + \frac{q^3 e^{-q^2/2} + 3 q e^{-q^2/2}}{\int_q^\infty e^{-z^2/2} dz} \leq 3 + \frac{q^3 e^{-q^2/2} + 3 q e^{-q^2/2}}{\int_q^{q+1/q} e^{-(q^2+3)/2} dz} \leq 3 + 15q^2 + 5q^4$

Since when $a < \sqrt{2}/2$, we have both $q < 2\sqrt{-\log a}$ and $\text{Var}_{D'}(y^2) = O(\log a^2)$. The lemma follows. \qed

**B.3.1 Strong Convexity**

Now, let’s deal with the Hessian Matrix.

First, we consider the case without truncation, that is,

$\mathbb{E}_P \left[ \begin{pmatrix} \text{Var}_x[z | xx^T] & -\text{Cov}_x \left[ \frac{1}{2} z^2, z \right] x \\ -\text{Cov}_x \left[ \frac{1}{2} z^2, z \right] x^T & \text{Var}_x \left[ \frac{1}{2} z^2 \right] \end{pmatrix} \right]$

The variance (Var$_x$) and covariance (Cov$_x$) are calculated from the untruncated normal $z \sim N(w^T x, \sigma^2)$. For all $x$, we have

$\begin{pmatrix} v \\ \lambda \end{pmatrix}^T \begin{pmatrix} \text{Var}_x[z | xx^T] & -\text{Cov}_x \left[ \frac{1}{2} z^2, z \right] x \\ -\text{Cov}_x \left[ \frac{1}{2} z^2, z \right] x^T & \text{Var}_x \left[ \frac{1}{2} z^2 \right] \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \sigma^2(x^T v)^2 - 2\sigma^2 w^T x x^T v \lambda + \frac{1}{2} \sigma^2 (2(w^T x)^2 + \sigma^2) \lambda^2 = I(x)$
Denote \( \bar{X} = E_{x \sim \mathcal{P}} \left[ \sum x x^T \right] \). By assumption 2.4, we have \( \bar{X} \succeq b I \), and thus we have

\[
\begin{align*}
\mathbb{E}_P[I(x)] &= \mathbb{E}_P[\sigma^2 (x^T v)^2 - 2\sigma^2 w^T x x^T v \lambda \frac{1}{2} \sigma^2 (2(w^T x)^2 + \sigma^2) \lambda^2] \\
= &\sigma^2 v^T X v - 2\sigma^2 w^T X v \lambda + \frac{1}{2} \sigma^2 (2w^T X w + \sigma^2) \lambda^2 \\
= &\sigma^2 (x - w \lambda)^T X (x - w \lambda) + \frac{1}{2} \sigma^4 \lambda^2 \\
\geq &b\sigma^2 \|x - w \lambda\|^2 + \frac{1}{2} \sigma^4 \lambda^2
\end{align*}
\]

So, it is larger than

\[
\frac{b}{2b(1 + \|w\|^2/\sigma^2) + 1} (\sigma^2 \|v\|^2 + \sigma^4 \lambda^2)
\]

Now we consider each summand again for the untruncated sum: we can write it as an integral in 2 variables with degree at most 4, as:

\[
I(x) = \left(\begin{array}{c} v \\ \lambda \end{array}\right)^T \left(\begin{array}{cc} \text{Var}_x[z x x^T] & -\text{Cov}_x \left[\frac{1}{2} z^2, z\right] x \\ -\text{Cov}_x \left[\frac{1}{2} z^2, z\right] x^T & \text{Var}_x \left[\frac{1}{2} z^2\right] \end{array}\right) \left(\begin{array}{c} v \\ \lambda \end{array}\right)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} p(y, z) dy dz
\]

Where \( \mathbb{P}_x \) is the untruncated joint normal distribution \( \mathcal{N}(w^T x, \sigma^2)^{\otimes 2} \). This can easily be verified as a log-concave measure. Next, we define polynomial \( p(y, z) \) as

\[
p(y, z) = \frac{1}{2}(v^T x)^2(y - z)^2 - \frac{1}{2} \lambda v^T x(y - z)^2(y + z) + \frac{1}{8} \lambda^2(y - z)^2(y + z)^2
\]

Let \( \alpha = I(x) \alpha(w, \sigma, x)^8/2^{12} \mathcal{C}^4 \) where \( \mathcal{C} \) is the constant defined in Theorem B.2. Since joint normal is a log-concave distribution, using Theorem B.2, plug in \( d = q = 4 \) and \( \alpha \) we get

\[
I(x)^{1/4} \alpha^{-1/4} \mathbb{P}\{p(y, z) \leq \alpha\} \leq 4\mathcal{C}
\]

Hence we have \( \mathbb{P}\{p(y, z) \leq \alpha\} \leq \frac{\alpha(w, \sigma, x)^2}{\alpha(w, \sigma, x)} \). Notice that the polynomial \( p \) is non-negative, therefore, we have a probability of \( \mathbb{P}\{p(y, z) > \alpha\} > 1 - \frac{\alpha(w, \sigma, x)^2}{2} \). Since \( \mathbb{P}\{p(y, z) \in S \times S\} = \alpha(w, \sigma, x)^2 \), we have \( \mathbb{P}\{p(y, z) > \alpha|y, z \in S \times S\} > \frac{1}{2} \). So we can estimate

\[
\frac{1}{\alpha(w, \sigma, x)^2} \mathbb{E}[p(y, z)] \geq \frac{1}{\alpha(w, \sigma, x)^2} \alpha(w, \sigma, x)^2 \frac{\alpha(w, \sigma, x)^8}{2^{12} \mathcal{C}^4} I^{(i)}
\]

Here, the variance (\( \text{Var}'_x \)) and covariance (\( \text{Cov}'_x \)) are calculated from the truncated normal distribution \( z \sim \mathcal{N}(w^T x, \sigma^2, S) \).

Now, by taking both the minimum survival probability, \( \alpha(w, \sigma, x) \) and the sum of the inequality above, we show that the lower bound of the Hessian Matrix is:

\[
r \begin{pmatrix} \sigma^2 I & 0 \\ 0 & \sigma^4 \end{pmatrix} := \min_{\alpha(w, \sigma, x)^8} \frac{b}{2b(1 + \|w\|^2/\sigma^2) + 1} \begin{pmatrix} \sigma^2 I & 0 \\ 0 & \sigma^4 \end{pmatrix}
\]

From the lower bound in Lemma B.3, we have

\[
r \geq \exp \left( -\text{poly}(1/a)(1 + \frac{\beta}{\sigma_0^2}) \right) \frac{b}{2b(1 + \|w\|^2/\sigma^2) + 1}
\]
Since $\sigma > \text{poly}(1/a)\sigma_0$ and $\|w\|^2 < \beta$, we have
\[
H \geq b \exp \left( -\text{poly}(1/a) \left( 1 + \frac{\beta}{\sigma_0^2} \right) \right) \left( \frac{\sigma_0^2 I}{\sigma_0^4} \right)
\]
Using the same argument, we can derive Lemma 4.4, calculating the lower bound for the strong convexity of the minimum point. We can put $\sigma = \sigma^*$, $w = w^*$ and $\min_i \alpha(w, x) \geq a$ due to the assumptions.
\[
H(w^*, \lambda^*) \geq b \frac{\sigma^8}{2^{13}C^2} \frac{1}{3 + 2b\|w^*\|^2/\sigma^*^2} \left( \frac{\sigma^*^2 I}{\sigma^*^4} \right)
\]
**Remark:** By Lemma B.3, we have $\alpha(w, \sigma, x) \geq \exp \left( -2s^2 \left( -\log a + 2 + \frac{2\beta}{\sigma^2} \right) \right)$. Notice that $\sigma^2 \geq \frac{1}{18(5-2\log\sigma_0^2)}$, we have
\[
\frac{b}{2b(1 + \|w\|^2/\sigma^2) + 1} \geq \frac{b}{2b(1 + \beta/\sigma^2) + 1} = \Omega\left( \frac{b}{b(1 + \beta/\sigma^2) + 1} \right)
\]
Therefore, we can yield a bound of
\[
H \geq \Omega \left( \exp \left( -16s^2 \left( -\log a + 2 + \frac{2\beta}{\sigma^2} \right) \right) \right)
\]
Here, we can ignore the constant factor, since it is unknown what the $C$ is in [5]. Also, we did not transform a general $\sigma$ in the projection set to $\sigma_0$ on purpose.

**B.3.2 Bounded Step Variance**

Now, let’s focus on the upper bound of the squared variance. To eliminate confusion let $y$ be the value of the dependent variable for which we are computing the gradient.

Let the $y \sim \mathcal{N}(w^T x, \sigma^2)$ and $z \sim \mathcal{N}(w^T x, \sigma^2)$. So, we have
\[
\mathbb{E}_P \left[ \mathbb{E} \left[ \| (y - z) x \|^2 \right] \right] \leq 4 \cdot \mathbb{E}_P \left[ \mathbb{E} \left[ \| (y - w^T x) x \|^2 + \| (z - w^T x) x \|^2 \right] \right] + 4 \cdot \| x^T (w - w^*) x \|^2
\]

For the first term, we have that by Lemma B.1, it holds that
\[
4 \mathbb{E}_P \left[ \mathbb{E} \left[ \| (y - w^T x) x \|^2 \right] \right] \leq 4 \mathbb{E}_P \left[ \mathbb{E} \left[ (y - w^T x)^2 \right] \right] \| x \|^2
\]
\[
\leq 4(\sigma^*)^2 (2 - 4 \log(\alpha(w^*, x, \sigma^*))) \| x \|^2 \leq 4(2 - 4 \log a)(\sigma^*)^2 \leq 4(2 - 4 \log a)(s\sigma_0)^2
\]

Similarly, we have
\[
4 \mathbb{E}_P \left[ \| (z - w^T x) x \|^2 \right] \leq 4 \mathbb{E}_P \left[ (z - w^T x)^2 \right] \| x \|^2
\]
\[
\leq 4\sigma^2 \mathbb{E}_P \left[ 2 - 4 \log(\alpha(w, x, \sigma)) \right] \| x \|^2
\]
\[
= 8\sigma^2 + 16 \mathbb{E}_P \left[ -\sigma^2 \log(\alpha(w, x, \sigma)) \right] \leq 8\sigma^2 + 16 \mathbb{E}_P \left[ -\sigma^2 \log(\alpha(w, x, \sigma)) \right]
\]

By Lemma 4.5, we have
\[
-\sigma^2 \log(\alpha(w, x, \sigma)) \leq \max \left( \sigma^2, 2\sigma^2 \right) \log \frac{1}{\alpha(w^*, \sigma^*, x)} + \max \left( 0, 4\sigma^2 - 2\sigma^2 \right) + 2\sigma^2 \left( w^T x - w^T x \right)
\]
\[
\leq \sigma^2 (-s^2 \log a + 4 + 2\beta)
\]

Hence, we have
\[
4 \mathbb{E}_P \mathbb{E} \left[ \| (z - w^T x) x \|^2 \right] \leq 8\sigma^2 + 16 \mathbb{E}_P \left[ -\sigma^2 \log(\alpha(w, x, \sigma)) \right] \| x \|^2
\]
\[
\leq 8\sigma^2 \sigma_0^2 + 16\sigma_0^2 (4 + 2\beta - s^2 \log a)
\]
And for the last term, we have
\[ 4 \cdot \|x^T(w - w^*)x\|^2 \leq 16\beta^2 \]
because \( \|w\|^2, \|w^*\|^2 \leq \beta^2 \) and \( \|x\| \leq 1 \).

Now, let’s deal with the squared gradient of \( \lambda \). Notice that for \( y \sim \mathcal{N}(w^T x, \sigma^2, S) \), we have:
\[
E[y^4] = (E[y^2])^2 + \text{Var}(y^2)
\]
\[
= 2(E[(y - w^T x)^2] + (w^T x)^2)^2 + \text{Var}(y^2)
\]
\[
\leq 2((2 - 4 \log \alpha(w^*, \sigma^*, x))\sigma^2 + (w^T x)^2)^2 + 2(\log \alpha(w^*, \sigma^*, x))^2 O(\log \alpha(w^*, \sigma^*, x)) + \sigma^4 O(\log \alpha(w^*, \sigma^*, x))^2
\]
\[
\leq 2(O(\log a)poly(1/a)\sigma_0^2 + \beta)^2 + O(\log a)poly(1/a)\beta\sigma_0^2 + \sigma_0^4 O(\log a)poly(1/a)
\]
\[
= \text{poly}(1/a)(\sigma_0^4 + \beta^2) \leq \text{poly}(1/a)(\sigma_0^4 + \beta^2)
\]
The first inequality is due to 4.5 and B.4. The second inequality comes from the following three facts: because of the projection set, we have \( \sigma^* = \text{poly}(1/a)\sigma_0, (w^T x)^2 \leq \|w\|^2 \|x\|^2 \leq \beta \) by 2.2, and \( \alpha(w^*, \sigma^*, x) \geq a \) by our assumption 2.3. The third inequality combines the term and by AM-GM inequality: \( \beta\sigma_0 \leq (\beta^2 + \sigma_0^4)/2 \).

Now we are dealing with \( z \sim \mathcal{N}(w^T x, \sigma^2, S) \), which is more complicated, but still has the same order bound.
\[
E[z^4] = (E[z^2])^2 + \text{Var}(z^2)
\]
\[
= 2(E[(z - w^T x)^2] + (w^T x)^2)^2 + \text{Var}(z^2)
\]
\[
\leq 2((2 - 4 \log \alpha(w, \sigma, x))\sigma^2 + (w^T x)^2)^2 + 2(\log \alpha(w, \sigma, x))^2 O(\log \alpha(w, \sigma, x)) + \sigma^4 O(\log \alpha(w, \sigma, x))^2
\]
The inequalities hold for the same reason as bounding \( E[y^4] \). By B.3 We know that
\[- \log \alpha(w, \sigma, x^{(i)}) \leq \text{poly}(1/a)(1 + \frac{\beta}{\sigma_0^2}) \]. Since we are estimating the parameters inside the projection set, we can derive \( \sigma = \text{poly}(1/a)\sigma_0 \) and by 2.3, we have \( (w^T x)^2 \leq \|w\|^2 \|x\|^2 \leq \beta \), giving us
\[
2((2 - 4 \log \alpha(w, \sigma, x))\sigma^2 + (w^T x)^2)^2
\]
\[
\leq 2((2 + \text{poly}(1/a)(1 + \frac{\beta}{\sigma_0^2}))^2 \sigma^2) + (w^T x)^2)
\]
\[
= 2((2 + \text{poly}(1/a)(1 + \frac{\beta}{\sigma_0^2}))^2 \sigma_0^2)\text{poly}(1/a) + \beta)
\]
\[
= \text{poly}(1/a)(\sigma_0^4 + \beta^2)
\]
The last inequality is due to the AM-GM inequality, since \( \beta\sigma_0^2 \leq 1/2(\beta^2 + \sigma_0^4) \) and
\[
2(\log \alpha(w, \sigma, x))^2 O(\log \alpha(w, \sigma, x))^2 + \sigma^4 O(\log \alpha(w, \sigma, x))^2
\]
\[
\leq 2\beta\sigma_0^2 O(\text{poly}(1/a)(1 + \frac{\beta}{\sigma_0^2})) + \text{poly}(1/a)\sigma_0^4 (\text{poly}(1/a)(1 + \frac{\beta}{\sigma_0^2}))^2
\]
\[
= 2\beta O(\text{poly}(1/a)(\beta + \sigma_0^2)) + \text{poly}(1/a)((\sigma_0^2 + \beta)^2)
\]
\[
= \text{poly}(1/a)(\beta^2 + \beta\sigma_0^2 + \sigma^4) = \text{poly}(1/a)(\sigma_0^4 + \beta^2)
\]
Combining these, we get the result:
\[
\mathbb{E}_P \left[ y^2 / 2 - z^2 / 2 \right]^2 \leq \frac{1}{4} \mathbb{E}_P \left[ (y^4) + E(z^4) \right]
\]
\[
\leq \mathbb{E}_P \left[ \text{poly}(1/a)(\sigma_0^4 + \beta^2) \right] = \text{poly}(1/a)(\sigma_0^4 + \beta^2) = \text{poly}(1/a)(1 + \sigma_0^4 + \beta^2)
\]
The gradient of \( v \) is also within this bound. Thus, the bounded step variance is proved.

**Remark:** From the proof we can say
\[
\mathbb{E}[y^4] = O(1 - \log a)(\sigma^4 + \beta^2)
\]
And similarly, we have the bound for $\mathbb{E}(z^2)$ using Lemma B.3
\[
\mathbb{E}[z^4] \leq 2((2 - 4 \log \alpha(w, \sigma, x))\sigma^2 + (w^T x)^2)^2 + 2(w^T x)^2\sigma^2 O(\log \alpha(w, \sigma, x))
\]
\[
+ \sigma^4 O(\log \alpha(w, \sigma, x)^2)
\]
\[
= 2((2 + 8\alpha^2(-\log a + 2 + \frac{2\beta}{\sigma^2}))\sigma^2 + (w^T x)^2)^2
\]
\[
+ 2(w^T x)^2\sigma^2(2\sigma^2(\log a + 2 + \frac{2\beta}{\sigma^2}))^2 + \sigma^4 O(4\alpha^4(\log a + 2 + \frac{2\beta}{\sigma^2})^2)
\]

Notice that because $\sigma^2/\sigma^2 \leq s^2 = O(\frac{1-\log a}{a^4})$, and also $(w^T x)^2 \leq \|w\|^2\|x\|^2 \leq \beta$, the highest “degree” containing $a$ is $a^{-6}(1 - \log a)^{10}$ \((a^{-4}(1 - \log a)^4 \text{ comes from } s^4, \text{ additional } (1 - \log a)^2 \text{ from the coefficients, and additionally } a^{-4}(1 - \log a)^4 \text{ comes from a hidden } s^4 \text{ produced by } \sigma^2/\sigma^4).$

Also, we can write $(w^T x)^2\sigma^2 = O(\sigma^4 + (w^T x)^4)$. Finally, we can write
\[
\mathbb{E}[z^4] \leq O\left(\frac{(1 - \log a)^{10}}{a^8}(\beta^2 + \sigma^4)\right)
\]

The terms in $\mathbb{E}[\|(y - z)x\|^2]$ have lower degrees of both in $1/\alpha, \beta$, and $\sigma^*$. So, we can only add one to $\beta^2 + \sigma^4$ as in the proof above. Therefore, we can write the total gradient as a bound of
\[
O\left(\frac{(1 - \log a)^{10}}{a^8}(1 + \beta^2 + \sigma^4)\right)
\]

This bound is greater than the bounded domain.

### B.4 Proof of Lemma 4.2

We estimate the projection set with the following procedure:

1. We use OLS on $m$ samples $(x^{(i)}, y^{(i)})$, and get a estimated weight $w_0$
2. Then we take another $m$ samples $(\bar{x}^{(i)}, \bar{y}^{(i)})$ and calculate the mean $\sigma_0^2 = \frac{1}{m}\sum_{i=1}^{m}(\bar{y}^{(i)} - w_0^T \bar{x}^{(i)})^2$.

First, we try to figure out the lower bound of $\sigma_0^2$. We prove a claim: $m$ is large (at least $m \geq 3$), for any $w$, there are at least $n/6$ (with probability $> 15/16$) samples such that $|\bar{y}^{(i)} - w^T \bar{x}^{(i)}| > \frac{\alpha}{4}\sigma^*$. Notice that $\sigma_0^2 = \frac{1}{m}\sum_{i=1}^{m}|\bar{y}^{(i)} - w^T \bar{x}^{(i)}|^2$ for some empirical $w = w_0$. To finish the proof, we scale the whole distribution by multiplying $\frac{1}{\sigma^*}$. Since we proved this, we have $\sigma_0^2 = \frac{1}{m}\sum_{i=1}^{m}|\bar{y}^{(i)} - w^T \bar{x}^{(i)}|^2 \geq \frac{\sigma^2}{16}\sigma^*$. Now we can show that $\sigma^* = \frac{1}{2}$ WLOG. In the proof of Lemma 9 in the paper [9], we notice that in the worst case, $yi - w^T x^{(i)}$ are part of the distribution $D(a)$ (where $D(a)$ dominates $\text{Unif}(a/2, a/2) = U(a)$). For all $t > 0$ and $D \sim D(a)$ and $U \sim U(a)$, $\mathbb{P}(\|D\| < t) \leq \mathbb{P}(\|U\| < t)$ holds. Here, this means that even for the densest $[-\frac{2}{3}, \frac{2}{3}]$ it only take 1/2. So, any $a/2$ window of $y$ takes at most 1/2 of the probability. So we have
\[
\mathbb{P}\left(|y^{(i)} - w^T x^{(i)}| > \frac{\alpha}{4}\right) > \frac{1}{2}
\]

Now, all we have left to prove that if $X_1, X_2 \sim \text{Ber}(1/2)$, then $\mathbb{P}(X_1 + \cdots + X_n < n/6) < 1/16$ if $n > 4$. Thus, we need to prove that $\left(\frac{n}{6}\right) + \cdots + \left(\frac{n}{6}\right) < 2^n/16$ where $r = \lfloor(n - 1)/6\rfloor$.

For $n \leq 18$ it is easy to see this result by checking one by one. Regardless, for $13 \leq n \leq 18$, we have $(r + 1)^{1/n} < 2^n/16$. For $n \geq 19$, we use induction to prove the stronger bound of $(r + 1)^{1/n} < 2^n/16$. Now, suppose that this stronger statement holds for $n$, and we want to prove the same bound for $n + 6$. We already know that $(r + 1)^{1/n} < 2^n/16$. So, if we want to prove $(r + 2)^{1/n} < 2^{n+6}/16$, we need to show that
\[
\frac{r + 2}{r + 1}^{1/n} \frac{(n + 6)\cdots(n + 1)}{(r + 1)^2(n - r + 5)\cdots(n + r + 1)} < 64
\]
Further, we know that
\[
\frac{r + 2}{r + 1} \leq \frac{4}{3} < 2
\]
\[
\frac{n + 1}{r + 1} \leq \frac{n + 1}{n/6} \leq 6 \times \frac{14}{13} < 7
\]
\[
\frac{(n + 6) \cdots (n + 2)}{(n - r + 5) \cdots (n - r + 1)} \leq \frac{(n + 6)(n + 5)(n + 4)(n + 3)(n + 2)}{(5n/6 + 5)(5n/6 + 4)(5n/6 + 3)(5n/6 + 2)(5n/6 + 1)}
\]
\[
19 \times 18 \times 17 \times 16 \times 15
\]
\[
15 \times 14 \times 13 \times 12 \times 11 < 4
\]
So, we get
\[
\frac{r + 2}{(r + 1)^2} \frac{(n + 6) \cdots (n + 1)}{(n - r + 5) \cdots (n - r + 1)} \leq 2 \times 7 \times 4 < 64
\]
Therefore, we prove the claim. Now we turn to the upper bound on the \( \sigma_0^2 \). Denote the least square estimator of \( m \) samples to be \( w_1 \). So we have
\[
E[\sigma_0^2] = E \left[ \frac{1}{m} \sum_i (y^{(i)} - u_0^T x^{(i)})^2 \right] \leq \frac{1}{2} E \left[ \sum_i (y^{(i)} - w_0^T x^{(i)})^2 + (w_0^T x^{(i)} - w_1^T x^{(i)})^2 \right]
\]
Since \( w_1 \) is the OLS estimator, we have
\[
E \left[ \frac{1}{m} \sum_i (y^{(i)} - u_1^T x^{(i)})^2 \right] \leq E \left[ \frac{1}{m} \sum_i (y^{(i)} - w_0^T x^{(i)})^2 \right] \leq \frac{1}{m} \sum_i (4 - 2 \log a)\sigma^2 = (4 - 2 \log a)\sigma^2
\]
The first inequality is because of OLS, and the second inequality is due to Lemma 4.5. Now we need to bound \( E[\frac{1}{m} \sum_i (w_0^T x^{(i)} - w_1^T x^{(i)})^2] \). Notice that
\[
(w_0^T x^{(i)} - w_1^T x^{(i)})^2 = ((w_0^T - w_1^T) x^{(i)})^2 \leq \|w_0^T - w_1^T\|_2^2 \|x^{(i)}\|_2^2 \leq \|w_0^T - w_1^T\|^2
\]
So, we need this value to have a small norm. Also since we are doing OLS on the same distribution of data \((x, y)\), we can prove that there is a concentration of samples around the empirical estimator.

Moreover, notice that the weight formula for linear regression is \( \left( \sum_i x^{(i)} x^{(i)^T} \right)^{-1} \sum_i x^{(i)} y^{(i)} \) (here we abuse the notation, we are talking about general \( m \) samples.) Or, we do an average, yielding \( \frac{1}{m} \sum_i x^{(i)} x^{(i)^T} \). Therefore, if we have infinite samples, we get a final estimator \( w_\infty = (E(x x^T))^{-1}E(y x) \) for \((x, y) \sim D^*\), where \( D^* \) is defined in Section B.1. We also have \( w_0, w_1 \to w_\infty \) if \( m \to \infty \). Now, we are going to show how will they converge.

First, we investigate \( \frac{1}{m} \sum_i w^{(i)}(i)_{x^{(i)}}^T \). We need a theorem in \([28]\).

**Theorem B.5.** (Combination of Theorem 1.6.2 and Section 1.6.3 in \([28]\)) Let \( x_1, \ldots, x_n \) be i.i.d. random vectors with dimension \( p \). Let \( Y = \frac{1}{n} \sum x_i x_i^T \) and \( A = E[x_i x_i^T] \). Assume that each one is uniformly bounded \( \|x_k\| \leq B \) for each \( k = 1, \ldots, n \). Introduce the sum \( Z = Y - A \) Then \( P(\|Z\|_2 \geq t) \leq 2p \cdot \exp \left( -\frac{n t^2/2}{\|A\|_2 + 3B t/3} \right) \) for all \( t \geq 0 \). Here \( \|A\|_2 \) is the spectral norm.

Let \( X_0 = E(x x^T) \). We can choose a big enough \( m \) such that with probability \( 1 - \frac{1}{m^2} \), the spectral norm of \( \frac{1}{m} \sum_i x^{(i)} x^{(i)^T} - X_0 \) does not exceed \( \delta; \delta \) is another constant determined later. This \( \delta \) is smaller than \( b \) and \( X_0 \geq bI \) by the assumptions. So, \( \frac{1}{m} \sum_i x^{(i)} x^{(i)^T} \) is positive definite and we can take the inverse.

Now, let’s calculate the inverse. Let \( \hat{X} = \frac{1}{m} \sum_i x^{(i)} x^{(i)^T} \). Notice that
\[
\hat{X}^{-1} - X_0^{-1} = X_0^{-1}(X_0 - \hat{X})X_0^{-1}
\]
has spectral norm at most \( \frac{\delta}{b(\delta - b)} \). Since \( X_0 \) has a spectral norm \( \geq b \), \( X_0 - \tilde{X} \) has spectral norm \( \leq \delta \) therefore \( \tilde{X} \) has spectral norm \( \geq b - \delta \).

Then we investigate \( \tilde{x} = \frac{1}{m} \sum_i y(i)x(i) \). Let \( x_0 = \mathbb{E}[yx] \). We use Chebyshev’s inequality in the vector form. Notice that

\[
\mathbb{P}(\|\tilde{x} - x_0\| \geq \delta') \leq \frac{\mathbb{E}[\|\tilde{x} - x_0\|^2]}{\delta'^2}
\]

In our case, we know that all \( y, x \) are independently chosen. So, we have

\[
\mathbb{E}[\|\tilde{x} - x_0\|^2] = \frac{1}{m} \mathbb{E}[\|y \cdot x\|^2]
= \frac{1}{m} \left( \mathbb{E}\left[ \left( \| (y - w^T x)^2 \cdot x \|^2 + \| (w^T x) x \|^2 \right) \right] \right) \leq \frac{1}{2m} ((4 - 2 \log a) \sigma^* + \beta)
\]

Here, we use both Lemma 4.5 and assumption 1.1 that \( \|x\| \leq 1, \|w\|^2 \leq \beta \). From this we can also derive that

\[
\mathbb{E}[\|x_0\|] \leq \sqrt{\mathbb{E}[\|x_0\|^2]} = \sqrt{(4 - 2 \log a) \sigma^* + \beta}
\]

We also make \( m \) large enough to have the probability of at most 1/64, so that \( \|\tilde{x} - x_0\| \geq \delta' \).

Notice that we have the estimator \( w = \tilde{X}^{-1}\tilde{x}_0 \). This gives us

\[
\|\tilde{X}^{-1}\tilde{x} - X_0^{-1}x_0\| = \|\tilde{X}^{-1}(\tilde{x} - x_0) + (\tilde{X}^{-1} - X_0^{-1})x_0\|
\leq \|\tilde{X}^{-1}(\tilde{x} - x_0)\| + \|(\tilde{X}^{-1} - X_0^{-1})x_0\| \leq \frac{1}{b - \delta} \delta' + \frac{\delta}{b(b - \delta)} \sqrt{(4 - 2 \log a) \sigma^* + \beta}
\]

(Notice for any vector \( v \) and square matrix \( A \) if \( \|v\| \leq a, \|A\| \leq b \), then \( \|Av\| \leq ab \). This is because the \( L_2 \) norm for a vector equals to its spectral norm, and we have \( \|AB\| \leq \|A\|_2 \|B\|_2 \).

Since there is a 1/64 probability that \( \|\tilde{x} - x_0\| \leq \delta' \) does not hold, and there is also a 1/64 probability that \( \|\tilde{X} - X_0\| \leq \delta' \) does not hold, this \( \|\tilde{X}^{-1}\tilde{x} - X_0^{-1}x_0\| \leq \frac{1}{b - \delta} \delta' + \frac{\delta}{b(b - \delta)} \sqrt{(4 - 2 \log a) \sigma^* + \beta} \) holds with a probability of at least 1 - 1/32. That means, \( w_0 \) and \( w_1 \) each have a 1 - 1/32 probability of being close to the convergence limit, and a probability of at most 1 - 1/16 that both of them are close.

Take \( \delta = \frac{\sigma^*^2}{4(4 - 2 \log a) \sigma^* + \beta} \) and \( \delta' = b \sigma^*/4 \). So, we have at most 1 - 1/16 probability that both \( w_0 \) and \( w_1 \) are close to \( w_\infty \). Therefore, the norm between \( w_0 \) and \( w_1 \) is at most \( \sigma^* \).

To achieve \( \delta' \) difference of \( \tilde{X} - X_0 \), we may need \( 2p \cdot \exp \left( \frac{-\min(\delta/2, 3)}{\log(\log(128k))} \right) \leq 1/64 \). In our case, we have \( p = k, \delta, B = 1 \) and \( \|A\|_2 \) also has an upper bound 1. So, we need at least \( \log(128k) \frac{1+\delta/3}{\delta^2} \leq \log(128k) \frac{2(4 - 2 \log a) \sigma^* + \beta}{\delta^2} \) samples.

Also, to achieve \( \delta' \) difference of \( \tilde{x} - x_0 \), we may need \( \frac{1}{2m} ((4 - 2 \log a) \sigma^* + \beta) \frac{1}{(b \sigma^*/4)^2} \leq 1/64 \). So, we need at least \( \frac{512 ((4 - 2 \log a) \sigma^* + \beta)}{b^2 \sigma^*^2} \) samples.

These two bounds of samples are covered with "poly" constraints of the informal theorem 1.4 and the first inequality of theorem 3.2.

Overall, we have

\[
\mathbb{E}[\sigma_0^2] = \mathbb{E}\left[ \frac{1}{m} \sum_i (y(i) - w_0^T x(i))^2 \right] \leq \frac{1}{2} \mathbb{E}\left[ \sum_i (y(i) - w_0^T x(i))^2 + (w_0^T - w_1^T)^2 \right] \leq \frac{1}{2} ((4 - 2 \log a) \sigma^* + \beta^2) = \frac{1}{2} ((5 - 2 \log a) \sigma^* + \beta^2)
\]
By Markov’s Inequality, we have a probability of 1/16 that \( \sigma_0^2 \leq 16(\mathbb{E}(\sigma_0^2)) \). Now, we have 1/8 probability (1/16 for Markov, 1/16 for the concentration) to hold \( \sigma_0^2 \leq 16(\mathbb{E}(\sigma_0)) \leq 8(5 - 2\log a)\sigma^2 \). Therefore, for the whole projection set, it takes \( \geq 1 - 1/8 - 1/16 = 13/16 \) probability to hold.

### B.5 Proof of the Smoothness

The smoothness can be given by this bound:

**Theorem B.6.** The Hessian of \( \ell(v, \lambda) \) satisfies \( H^2(v, \lambda) \leq \gamma^2 I \) where \( \gamma = \text{poly}(1/a, \frac{1}{\sigma_0}, \sigma_0, \beta) \).

Hence \( \ell \) is \( \gamma \) smooth.

Denote that \( s = \frac{\max_{x \in D, \sigma} \sigma}{\min_{x \in D, \sigma}} = \sqrt{8(5 - 2\log a)\frac{56}{27}} \) is the largest possible difference between the variances. Since we already have \( H \) is symmetric and positive definite, \( H^2 \) is symmetric and positive definite. So, by the Cauchy-Schwarz inequality, we have

\[
H^2 = \left( E_{x \sim \mathcal{D}} [H_x] \right)^2 \preceq E_{x \sim \mathcal{D}} [H_x^2]
\]

Since for each sample \( (x^{(i)}) \) we can calculate

\[
H_x^2 = \begin{pmatrix}
Ax^T & -Bx \\
-Bx^T & C
\end{pmatrix}^2
\]

\[
= \begin{pmatrix}
(A^2 + B^2\|x\|^2)xx^T & -(A + C)B\|x\|^2 x \\
-(A + C)B\|x\|^2 x^T & (B\|x\|^2)I + C^2
\end{pmatrix}
\]

\[
\preceq 2\begin{pmatrix}
(A^2 + B^2\|x\|^2)xx^T & 0 \\
0 & (B\|x\|^2)I + C^2
\end{pmatrix}
\]

\[
\preceq 2\begin{pmatrix}
(A^2 + B^2I) & 0 \\
0 & B^2 + C^2
\end{pmatrix}
\]

\[
\preceq 2\begin{pmatrix}
(A^2 + AC)I & 0 \\
0 & AC + C^2
\end{pmatrix}
\]

where

\[
A = \text{Var}_D[z], B = \text{Cov}_D\left[\frac{1}{2}z^2, z\right], C = \text{Var}_D\left[\frac{1}{2}z^2\right]
\]

and the distribution \( D = \mathcal{N}(w^Tx, \sigma^2, S) \), we can deduce the first inequality this way. Since the matrix \( H^2_x \) is positive definite, it is also positive semi-definite. Further, if the matrix \( \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \) is positive semi-definite, the matrix \( \begin{pmatrix} A & -B \\ -B^T & C \end{pmatrix} \) is also positive semi-definite. Thus, we conclude

\[
\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \preceq 2\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}
\]

The second inequality comes from \( B^2 > 0 \) and \( d = \|x\|_2^2 < 1 \). Finally, the third inequality comes from \( B^2 < AC \).

Thus, we can write

\[
\gamma^2 = 2E_p [A + C]^2 = 2E_p \left[ \text{Var}_{D(x)}(\|z\| + \text{Var}_{D(x)}[\frac{1}{2}z^2]) \right]^2
\]

where \( D(x) = \mathcal{N}(w^Tx, \sigma^2, S) \). Also, by Lemma B.4, we have

\[
(A + C)^2 = \text{poly}\left( \log \frac{1}{\alpha(w, \sigma)}\sigma^2 + \log^2 \frac{1}{\alpha(w, \sigma)}\sigma^4 + \log \frac{1}{\alpha(w, \sigma)}(w^T x)^2\sigma^2 \right)
\]
Since $\log \frac{1}{\sigma(w, x)}$ can be written as $\text{poly}(1/a)(1 + \beta/\sigma_0)$ by Lemma B.3, $\sigma$ can be written as $\text{poly}(\sigma_0, 1/a)$, and $|w^T x| \leq \beta$. Then finally we can write

$$\gamma^2 = \text{poly} \left( \frac{1}{a}, \beta, \sigma_0 + \frac{1}{\sigma_0} \right)$$

Now, we will prove the smoothness. For any two vectors $\theta_1 = (v_1, \lambda_1)$ and $\theta_2 = (v_2, \lambda_2)$, we define the unit vector with same direction of $(v_2, \lambda_2) - (v_1, \lambda_1)$ as $u$. Since the projection set written in $\nu, \lambda$ is convex, we claim that all of the points along the line segment $(v_2, \lambda_2)(v_1, \lambda_1)$ are within the projection set. Hence,

$$\left\| \nabla \ell(v_2, \lambda_2) - \nabla \ell(v_1, \lambda_1) \right\| = \left\| \int_0^{\theta_2 - \theta_1} \frac{\partial \nabla \ell}{\partial u} \right|_{(v_1, \lambda_1) + ut} \left\| u dt \right\|$$

$$\leq \int_0^{\theta_2 - \theta_1} \left\| \frac{\partial \nabla \ell}{\partial u} \right|_{(v_1, \lambda_1) + ut} \left\| u dt \right\| = \int_0^{\theta_2 - \theta_1} \sqrt{u^T H((v_1, \lambda_1) + ut) H((v_1, \lambda_1) + ut) u dt}$$

$$\leq \int_0^{\theta_2 - \theta_1} \sqrt{u^T \gamma^2 I u dt} = \gamma \|\theta_2 - \theta_1\|$$

Thus, we derive that $L_D$ is $\gamma$-smooth.

### B.6 Proof of Corollary 4.6

**Proof.** By Theorem 4.1, we can choose $\rho$ to be $\text{poly}(\beta, \sigma_0 + \frac{1}{\sigma_0}, \frac{1}{\sigma_0})$, and $\zeta = \exp \left( -\text{poly}(1/a)(1 + \beta/\sigma_0) \right) \min\{\sigma_0^2, \sigma_0^3\}$. Then using Section 4.4 the assumptions of Theorem 3.5 are satisfied and applying Theorem 3.5 the Lemma follows. \hfill \square

### B.7 Detailed proof of Theorem 3.2

The notation is the same as in Section 4.5, and we present the details of the two cases considered in that section. Here, the estimation below does not contain the number of samples for $n$ to to define projection $D$. In Section 4.2, we need the number of samples as $O\left( \frac{(1 - \log n)}{\sigma_0^2 + \beta} \right)$. This is bounded above by the right hand side of 3.2: $n$ must be at the order of $\text{poly}\left( \frac{\sigma^*}{\sigma_0}, \frac{\beta}{1 - \sigma_0} \right)$ and the order of $\text{poly}\left( \frac{\beta}{1 - \sigma_0} \right)$.

**Case $\eta_t = c/\sqrt{t}$.** First, combining Theorem 3.4 and Theorem 4.1 and the projection set, we get the corollary 4.3. We can derive the corollary as follows: by Theorem 4.1 and the rest of Section 4.3, we know that the step variance and the domain is bounded by some polynomial of $\sigma_0, \beta, \frac{1}{\sigma_0}, 1/\sigma_0$.

By Theorem 3.4, if the final $(w^*, \sigma^*) \in D$, we have, after $n$ steps, $\mathbb{E} \left[ \tilde{\ell}(\tilde{v}, \tilde{\lambda}) - \ell(w^*, \lambda^*) \right] \leq \text{poly}\left( \frac{\sigma^*}{\sigma_0}, \frac{1}{\sigma_0}, \log(n) \right)$. Notice that from the definition of the projection set, $\sigma_0/\sigma^*$ and $\sigma^*/\sigma_0$ both are bounded by $\text{poly}(1/a)$. Therefore, we can rewrite this bound to $\mathbb{E} \left[ \tilde{\ell}(\tilde{v}, \tilde{\lambda}) - \ell(w^*, \lambda^*) \right] \leq \text{poly}\left( \frac{\sigma^*}{\sigma_0}, \frac{1}{\sigma_0}, \log(n) \right)$.

Our next step is to transform the optimality in function values to closeness to the true parameters. To do so we use the strong convexity of $\tilde{\ell}$ at the optimum.

By Lemma 4.5, we have:

$$\tilde{\ell}(v^*, \lambda^*) = \mathbb{E}_{x \in \mathcal{P}} \mathbb{E}_{y} \left[ \frac{(y - w^* x)^2}{2\sigma^2} + \log \alpha(w^*, \sigma^*, x) \right] \leq 4 - 2 \log \min_{x \in \mathcal{P}} \alpha(w^*, \sigma^*, x) \leq 4 - 2 \log a$$

We receive the first inequality by ignoring the $\log \alpha(w^*, \sigma^*, x)$ term and the fact that the expectation is smaller then the maximum. By applying Markov’s inequality we get

$$\mathbb{P} \left( \tilde{\ell}(\tilde{v}, \tilde{\lambda}) - \tilde{\ell}(v^*, \lambda^*) > \delta \mathbb{E} \left[ \tilde{\ell}(\tilde{v}, \tilde{\lambda}) - \tilde{\ell}(v^*, \lambda^*) \right] \right) < \frac{1}{7}. \tag{B.1}$$
The inequality implies that, with a probability of at least 6/7, the actual difference is at most 7 times its expectation. The constant 7 can be absorbed by the poly term. Also, the probability that \((v^*, \lambda^*) \in D\) is at least 13/16 by Lemma 4.2. Combining these two, we have at least 13/16 − 1/7 > 2/3 probability to hold:

\[
\mathbb{E} \left[ \ell(\hat{v}, \hat{\lambda}) - \ell(v^*, \lambda^*) \right] \leq \frac{\text{poly}(\frac{\sigma^* \cdot \beta}{ab} \cdot \frac{1}{a \cdot \sigma^*}) \cdot \log(n)}{n^{1/2}}
\]

(B.2)

Now by applying Lemma 4.4, we transform the difference in functions to the L2 norm distance. Since \(\hat{v}, \hat{\lambda}\) converge to \(v^*, \lambda^*\), when they have small distance, we can approximate the rate using the convexity of the minimum. Therefore, we have

\[
\|\hat{v} - v^*\|^2 + (\hat{\lambda} - \lambda^*)^2 \leq \frac{2}{\delta} \left( \mathbb{E} \left[ \ell(\hat{v}, \hat{\lambda}) - \ell(v^*, \lambda^*) \right] \right) \leq \frac{\text{poly}(\frac{\sigma^* \cdot \beta}{ab} \cdot \frac{1}{a \cdot \sigma^*}) \cdot \log(n)}{\delta \cdot n^{1/2}}
\]

(B.3)

where

\[
\mathbf{H}(v^*, \lambda^*) \geq \frac{b}{2^{13}C^2} \frac{a^8}{3 + 2\|w^*\|^2 / \sigma^*} \begin{pmatrix} \sigma^* I & 0 \\ 0 & \sigma^* I \end{pmatrix} \geq \delta I
\]

Notice that \(\|w\|^2 \leq \beta\). Therefore, \(\delta^{-1} = \text{poly}(\frac{\sigma^* \cdot \beta}{ab} \cdot \frac{1}{a \cdot \sigma^*})\), and this term can be absorbed into the term \(\text{poly}(\frac{\sigma^* \cdot \beta}{ab} \cdot \frac{1}{a \cdot \sigma^*})\).

Finally, by Cauchy-Schwartz Inequality, we have

\[
\|\hat{\omega} - w^*\| + |\hat{\sigma}^2 - \sigma^*|^2 \leq \sqrt{2} \left( \|\hat{\omega} - w^*\|^2 + |\hat{\sigma}^2 - \sigma^*|^2 \right)
\]

(B.4)

And for the \(w\) part, we parameterize back and get

\[
\|\hat{\omega} - w^*\|^2 \leq \left\| \hat{\omega} - \frac{w^*}{\lambda^*} \right\|^2 \leq 2 \left( \left\| \hat{\omega} - \frac{w^*}{\lambda^*} \right\|^2 + \left\| \hat{\omega} - \frac{w^*}{\lambda^*} \right\|^2 \right)
\]

(B.5)

The last inequality is because of the projection set, we have \(\hat{\omega}, \lambda^* = \text{poly}(1/a)\sigma^*\). Also, by Assumption 2.2 we have \(\|\hat{\omega}\| = \|\hat{\omega}\|^2 \leq \sqrt{\beta/(\text{poly}(1/a)\sigma^*)}\). For the \(\sigma\), we have

\[
|\hat{\sigma}^2 - \sigma^*|^2 = \left\| \frac{1}{\lambda^*} - \frac{1}{\lambda^*} \right\|^2 \leq \frac{\text{poly}(1/a, \sigma^*)}{\|\hat{\omega} - (v^*, \lambda^*)\|^2}
\]

(B.6)

The last inequality is also derived from the projection set. Combining inequality (B.1) with inequality (B.6), we have, with a probability of at least 2/3,

\[
\|\hat{\omega} - w^*\| + |\hat{\sigma}^2 - \sigma^*|^2 \leq \sqrt{2} \left( \|\hat{\omega} - w^*\|^2 + |\hat{\sigma}^2 - \sigma^*|^2 \right)
\]

\[
\leq \sqrt{2 \text{poly}(1/a, \sigma^*, \beta) \|\hat{\omega} - (v^*, \lambda^*)\|^2} \leq \sqrt{2 \text{poly}(1/a, \sigma^*, \beta) \frac{\text{poly}(\frac{\sigma^* \cdot \beta}{ab} \cdot \frac{1}{a \cdot \sigma^*}) \cdot \log(n)}{\delta \cdot n^{1/2}}}
\]

\[
= \frac{\text{poly}(\frac{\sigma^* \cdot \beta}{ab} \cdot \frac{1}{a \cdot \sigma^*}) \cdot \log(n)}{n^{1/4}}
\]

And therefore, inequality (3.1) of Theorem 3.2 follows.

Remark: We can derive in the theorem 2 in [26] that the bound is \(O(\rho^2 \frac{1 + \log n}{\sqrt{n}})\). We have already written the bound for \(H\) and step variance in the form of \(\sigma^*\). Therefore, the right hand side of inequality B.2 is

\[
O \left( \frac{(1 - \log a)^{10}}{a^8} \left( 1 + \beta^2 + \sigma^*^4 \right) \frac{\log n}{\sqrt{n}} \right)
\]
And \( \delta^{-1} = \frac{3+2\beta/\sigma^2}{ba^3 \min(\sigma^2, \sigma^4)} = O\left(\frac{(\sigma^2 + \beta)(1 + \sigma^2)}{ba^3 \sigma^4}\right) \). Here, we used a trick that \( \min(x^2, x^4) = \Omega\left(\frac{x^4}{1 + \sigma^2}\right) \).

By equations B.5 and B.6, the factor to be multiplied from \( v, \lambda \) to \( w \), is \( \max(\frac{1}{\lambda^{x^2}}, \frac{1}{\lambda^{x^2}x^2}) = \max(\sigma^4, \sigma^4 \|\hat{w}\|^2) = O((\sigma^4)^4(1 + \beta)) \) the factor to be multiplied from \( \lambda \) to \( \sigma \) is \( \frac{1}{\lambda^{x^2}} = \sigma^2 \delta^2 = O(\sigma^4 s^2) = O(\sigma^4 1 - \frac{\log \sigma}{a^3}(1 + \beta)) \).

Therefore, by multiplying these bounds and square root, we have a bound of

\[
O\left(\sqrt{\frac{(1 - \log a)^{10}}{a^8}} (1 + \beta^2 + \sigma^4) \log n (\sigma^2 + \beta)(1 + \sigma^2) \frac{1 - \log a}{\sigma^4} (1 + \beta)\right)
\]

Which can be simplify to

\[
O\left(\frac{(1 - \log a)^{5.5}}{a^8} \frac{\sqrt{n}}{\sqrt{b}} (1 + \beta^2)(\sigma^3 + \sigma^{-1})\right)
\]

**Case** \( \eta = 1/(\zeta t) \). We just plug in the result in Theorem 4.1 into Theorem 3.5 and in Section B.6 we have proved Corollary 4.6. That is, if \( w^*, \sigma^* \in D \), we have

\[
E \left[ \left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|^2 \right] \leq \text{poly} \left(\beta, \sigma_0 + \frac{1}{a^3}, \frac{1}{a} a \times \exp \left(\text{poly}(1/a) \left(1 + \frac{\beta}{\sigma_0}\right)\right)\right) \cdot \frac{1}{b^2 n}.
\]

Applying Markov’s inequality again as inequality (B.1), we have

\[
\mathbb{P} \left[ \left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|^2 > 7 \mathbb{E} \left[ \left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|^2 \right] \right] < 1/7 \quad (B.7)
\]

Similarly to the proof of inequality (3.1) above, of Theorem 3.2, we have that with probability at least \( 2/3 \) it holds that

\[
\left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|^2 \leq \frac{\text{poly}(\sigma_0) \cdot \exp \left(\text{poly}(\frac{\beta}{\sigma_0})\right)}{b^2 \cdot n}
\]

Again, using the same inequality (B.4) to (B.6), we can transform the bound from square bound to a linear one:

\[
\|\hat{w} - w^*\| + \|\sigma^2 - \sigma^*^2\| \leq \sqrt{2\text{poly}(1/a, \sigma^*, \beta) \left\| (\hat{v}, \hat{\lambda}) - (v^*, \lambda^*) \right\|^2}
\]

\[
\leq \sqrt{2\text{poly}(1/a, \sigma^*, \beta) \text{poly}(\sigma_0) \cdot \exp \left(\text{poly}(\frac{\beta}{\sigma_0})\right)} \cdot \frac{1}{b^2 \cdot n}
\]

We get the last inequality because the \( \text{poly}(1/a, \sigma^*, \beta) \) can be absorbed into \( \text{poly}(\sigma_0) \) and exponential term of \( \text{poly}(\beta/\sigma_0) \). Therefore, the inequality (3.2) of Theorem 3.2 follows.

**Remark:** Again, we can calculate the bound by plugging Theorem 4.1 into 3.5. This gives us a difference of \( \frac{3\rho^2}{\sigma^2} \), where \( \rho \) is the bound on the step variance in Section B.3.2 and \( \zeta \) is the lower bound in the B.3.1. Specifically, we have

\[
\rho^2 = O\left(\frac{(1 - \log a)^{10}}{a^8} (1 + \beta^2 + \sigma^4)\right)
\]

and from the remark in the B.3.1, we have

\[
\zeta = \Omega \left(\exp \left(-16s^2 \left(-\log a + 2 + \frac{2\beta}{\sigma^2}\right)\right) \frac{b}{b(1 + \beta/\sigma^2) + 1}\right) \min(\sigma^2, \sigma^4)
\]
for all the possible $\sigma$ in the projection set. Since we know that $\sigma^2/\sigma^2$ we may say $\sigma^2/\sigma^2 \geq O\left(\frac{a^2}{1-\log a}\right)$. Also $s = \frac{96b(5-2\log a)}{a^2}$. For $b(1+\beta/\sigma^2)^{1-\log a}$, the numerator is $O(1 + \beta/\sigma^2) = O\left(1 + \frac{\beta(1-\log a)/\sigma^2}{\sigma^2}\right)$. Therefore, we can write $\frac{1}{\xi}$ as

$$O\left(\exp\left(\frac{9 \times 2^{14}(1 - \log a)}{a^2} - \log a + 2 + \frac{2\beta}{\sigma^2}\right)\frac{1 + \beta/\sigma^2}{\beta\sigma^2} \left(1 + \frac{\beta}{2\sigma^2}\right)^{\left(1 - \log a\right)^3}\right)$$

Since we know that $\rho^2 = O\left(\frac{1-\log a}{\sigma^2}\right)$ and the coefficient $O\left(\frac{1 - \log a}{\sigma^2}\right)$ is equal to the total gradient. $(\exp\left(\frac{3 \times 2^{12}(5-2\log a)}{a^2} (1 - \log a + 2 + \frac{2\beta}{\sigma^2})\right) \frac{(1 - \log a)^8.5}{a^{15}} (1 + \frac{\beta}{2\sigma^2}) (\sigma^{-1}) \frac{1}{\sigma^{-1}}$)

**B.8 Proof of Theorem 5.1 and Corollary 5.2**

We cite a lemma in the multivariate version of Proposition 2 in [19].

**Definition B.7.** A stochastic algorithm is a sequence $(x_k)_{k \geq 0}$ of random variables defined in a probability space $(\Omega, \mathcal{F}, P)$ and valued in $\mathbb{R}^d$. Define $(\mathcal{F}_k)_{k \geq 0}$ as the natural $\sigma$-field associated to the stochastic algorithm $(x_k)_{k \geq 0}$, i.e., $\mathcal{F}_k = \sigma(x_0, x_1, \ldots, x_k), k \geq 0$. A policy is a sequence of random probability measures $(P_k)_{k \geq 0}$, each defined on a measurable space $(S, \mathcal{S})$ that are adapted to $\mathcal{F}_k$.

**Definition B.8.** Given a policy $(P_k)_{k \geq 0}$ and a learning rates sequence $(\alpha_k)_{k \geq 1}$ of positive numbers, the SGD algorithm is defined by the update rule $x_k = x_{k-1} - \alpha_k C_k g(x_{k-1}, \xi_k)$ where $\xi_k \sim P_{k-1}$ with $g : \mathbb{R}^d \times S \rightarrow \mathbb{R}^d$.

**Theorem B.9.** Proposition 2 in [19]. If we suppose that the assumptions below are fulfilled:

1. The gradient estimation of $F$ is unbiased, that is, for $k \geq 1$, the expected sampled gradient is equal to the total gradient. $(\mathbb{E}(g(x_{k-1}, \xi_k)|\mathcal{F}_{k-1}) = \nabla F(x_{k-1}))$.
2. The sequence $(\alpha_k)_{k \geq 1}$ is positive, decreasing to 0, and satisfies the Robbins-Monro condition: $\sum_{k \geq 1} \alpha_k = +\infty$ and $\sum_{k \geq 1} \alpha_k^2 < \infty$.
3. The objective function is $L$-smooth.
4. The objective function has only one minimum point $x^*$ and $\lim_{\|x\| \rightarrow \infty} F(x) = \infty$.
5. With probability 1, there exist $0 \leq \mathcal{L}, \sigma^2 < \infty$ such that $\forall x \in \mathbb{R}^d, \forall k \in \mathbb{N}, \mathbb{E}(\|g(x_{k-1}, \xi_k)\|^2|\mathcal{F}_{k-1}) \leq 2\mathcal{L}(F(x) - F(x^*)) + \sigma^2$.
6. The sequence of step-size is equal to $\alpha_k = \alpha k^{-\beta}$ with $\beta \in (1/2, 1]$.
7. $H = \nabla^2 F(x^*)$ is positive definite and $x \mapsto \nabla^2 F(x)$ is continuous at $x^*$.
8. Denote $w_k = \nabla F(x_{k-1}) - g(x_{k-1}, \xi_k)$ and $\gamma_k = \mathbb{E}(w_{k+1}w^T_{k+1}|\mathcal{F}_k)$. There is a positive definite matrix $\Gamma_k$ such that $\lim_{k \rightarrow \infty} k \rightarrow \infty$.
9. And there exist $\delta, \varepsilon > 0$ such that almost surely $\sup_{k \geq 0} \mathbb{E}(\|w_k\|^2 + \|g(x_{k-1}, \xi_k)\|^2) \mathbb{I}_{\|x - x^*\| \geq \varepsilon} < \infty$.

Assume that $(H - \kappa I)$ is positive definite where $\kappa = 1/\beta=1/2\alpha$. Let $(x_k)_{k \geq 0}$ be obtained by the SGD rule, then $\frac{1}{\sqrt{n}}(x_k - x^*)$ weakly converge to a multivariate normal $\mathcal{N}(0, \Sigma)$ where $\Sigma$ satisfy the following equation:

$$(H - \kappa I)\Sigma + \Sigma(H^T - \kappa I) = \Gamma$$
In our settings, we have $x_k = (v \lambda)^{(k)}$ and the $\xi_k$ is new data $(x_k, y_k)$. The gradient $g(x_k, \xi_k)$ is $\nabla - \ell(v^{(k)}, \lambda^{(k)}; x^{(k)}, y^{(k)})$. Also, we have set $C_k = I, \alpha = \frac{1}{\zeta}, \beta = 1$ where $\zeta I$ is the lower bound of the Hessian matrix $H$ at $(v^*, \lambda^*)$. Here, we can see that $H - \frac{1}{2} I$ is positive definite, since $H \succeq \frac{1}{2} I > \frac{1}{2 \zeta} I$. In the section below, we prove that the nine assumptions hold in our settings. Also, in our settings, $H$ is symmetric, so $H^T = H$. We will omit the transpose of $H$ in the future content.

B.8.1 Assumptions 1-7

- Assumption 1 holds for our algorithm: each time the sampled $u_i$ is an unbiased estimator of $\nabla \ell((v^{(t-1)}, \lambda^{(t-1)})$.
- Assumption 2 holds because of our chosen parameter. The step size is $\eta_i = 1/\zeta \cdot t$ which also satisfies assumption 6.
- Assumption 3 is proved in the subsection B.5.
- Assumptions 5 and 7 are implied by Theorem 4.1 for the bounded step variance and strong convexity.
- From the strong convexity of the Hessian Matrix, we derive that the solution of $\nabla \ell$ is unique since $\ell$ has an unique minimum. Also, since the Hessian matrix is strongly convex everywhere, it can be deduced that when $\| (v, \lambda) \| \to \infty$, the $\ell$ goes to infinity. This proves assumption 4.

B.8.2 Assumptions 8 & 9

First, we propose another proposition, to state that $(v, \lambda)$ converge almost surely to $(v^*, \lambda^*)$:  

**Theorem B.10.** Proposition 1 in [19]: Assumptions 1 to 5 are fulfilled. Then the sequence of iterates $(x_k)_{k \geq 0}$ obtained by the SGD rule in Definition B.8 converges almost surely towards the minimizer $x^* \to x^*$

Since $(v, \lambda)$ converge almost surely to $(v^*, \lambda^*)$, the $\ell$ is twice differentiable, and $x^{(i)}, y^{(i)}$ are i.i.d. then we have that the distribution of $w_k$ converges to the case when $v, \lambda \to v^*, \lambda^*$. So, $\mathbb{E}[w_k w_k^T]$ converges to some matrix $\Gamma$. The matrix $\Gamma$ is positive definite, since it is a positive combination of some matrices of form $vv^T$. Since $n \to \infty$ the parameter $(v, \lambda)$ converge to $(v^*, \lambda^*)$ almost surely, and the gradient of $\ell$ at the true parameter is zero, we have

$$
\Gamma = \mathbb{E}_{x \sim P} \mathbb{E}_{y}[\left(\frac{\partial l(x, y; v, \lambda)}{\partial (v, \lambda)} - \nabla \ell \right) \left(\frac{\partial l(x, y; v, \lambda)}{\partial (v, \lambda)} - \nabla \ell \right)^T]
$$

when $(v, \lambda) = (v^*, \lambda^*)$. Since the assumption 2.4 gives $\mathbb{E}_{x \sim P} x x^T \succeq b I$, the rank of $x^{(i)}$ is full. Therefore, the gradients will not all be on the same linear subspace (since the gradient is some scalar times $x^{(i)}$ and $y^{(i)}$, which itself is generated with noise.) Hence, $\Gamma$ is positive definite.

For assumption 9, notice that if we bound $v, \lambda$ by the projection set, the $w, \sigma$ are both bounded. Thus, $\forall (v, \lambda) \in D_r$, the gradient is bounded if we set $\varepsilon$ such that $\| (v, \lambda) - (v^*, \lambda^*) \|$ to be inside the projection set. Also $P$ is bounded. So, if $y \sim F_x = \mathcal{N}(w^* T x, \sigma^2, S)$, we have

$$
\frac{\partial \ell}{\partial v} = (\mathbb{E}_{z \sim Q_x} [z] - y) x, \quad \frac{\partial \ell}{\partial \lambda} = \frac{1}{2} (y^2 - \mathbb{E}_{z \sim Q_x} [z^2])
$$

Where $Q_x = \mathcal{N}(w^* T x, \sigma^2, S)$. Next, we show that $\mathbb{E}_y \left[ \left\| \frac{\partial \ell}{\partial v} \right\|^4 + \left( \frac{\partial \ell}{\partial \lambda} \right)^4 \right]$ is bounded. We know that the expression can be written as

$$
\mathbb{E}_y \left[ \left\| \frac{\partial \ell}{\partial v} \right\|^4 + \left( \frac{\partial \ell}{\partial \lambda} \right)^4 \right] = \mathbb{E}_{y \sim \mathcal{N}(w^* T x, \sigma^2)} \left[ \frac{1}{\alpha^4} \left( \left\| \frac{\partial \ell}{\partial v} \right\|^4 + \left( \frac{\partial \ell}{\partial \lambda} \right)^4 \right) \right]
$$

$$
\leq \frac{1}{\alpha^4} \mathbb{E}_{y \sim \mathcal{N}(w^* T x, \sigma^2)} \left[ \left\| \frac{\partial \ell}{\partial v} \right\|^4 + \left( \frac{\partial \ell}{\partial \lambda} \right)^4 \right]
$$

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Because $\mathbb{E}[z]$ and $\mathbb{E}[z^2]$ are bounded, and $x$ is also bounded. Also, $w^* T x$ and $\sigma^*$ are bounded. the expression inside the expectation can be written as a polynomial of $y$ of degree $\leq 8$. Notice that for the polynomial $P(t)$, if $t$ is a given normal distribution $\mathcal{N}(\mu, \sigma)$ where $\mu$ and $\sigma$ are bounded, $\mathbb{E}_y(P(t))$ is bounded. Thus, we have $\mathbb{E}_y \left[ \frac{\partial L}{\partial \theta} \right]^4 + \left( \frac{\partial L}{\partial \sigma} \right)^4$ is bounded if we choose $x$ in the projection set.

Therefore, $\|w_k\|^4 = \| \frac{\partial L(v, x, y)}{\partial (v, \lambda)} - \nabla \tilde{E} \|^4 \leq 8 \| \frac{\partial L(v, x, y)}{\partial (v, \lambda)} \|^4 + \| \nabla \tilde{E} \|^4$ is also bounded, since the bound of the latter term can be derived by the bounded step variance in 4.1.

Therefore, when we set $\varepsilon$ such that $\| (v, \lambda) - (v^*, \lambda^*) \|$ to be inside the projection set, and $\delta = 2$, we can maintain a bounded expectation.

### B.8.3 Final Estimation

Since all assumptions hold, we know that the difference of the estimator and true parameter 
$$
\left( \sqrt{k^{-1/\zeta}} \right)^{-1} (v, \lambda) - (v^*, \lambda^*) \text{ is asymptotically normal, and that the variance is given by equation} 
$$
$$
(H - \kappa I) \Sigma + \Sigma (H^T - \kappa I) = \Gamma
$$

Where $H$ is the Hessian matrix at $(v^*, \lambda^*)$.

Notice that the $(v, \lambda)$ converge to $(v^*, \lambda^*)$ almost surely, so we can apply the plug-in confidence interval. We can substitute the true estimates by the empirical estimates as justified by the Slutsky’s theorem. Notice that the parameter $\hat{H} = H(\hat{v}, \hat{\lambda})$ converges to the true parameter $H$ (because $v, \lambda$ is converging by Theorem B.10) And $\hat{\Gamma}$ (defined below) converges to $\Gamma$ (by the last subsection.) Therefore, we can apply Slutsky’s Theorem and we can substitute the estimated parameters for true parameters to calculate the confidence region. We can substitute $H$ to $\hat{H}$ where $\hat{H} = H(\hat{v}, \hat{\lambda})$ is the empirical Hessian matrix evaluated at $(\hat{v}, \hat{\lambda})$. Also, the empirical $\hat{\Gamma}$ can be evaluated by $\hat{\Gamma}(\hat{v}, \hat{\lambda})$, where

$$
\hat{\Gamma}(v, \lambda) = \frac{1}{n} \sum_{i=1}^n \frac{\partial L(v, x, y)}{\partial (v, \lambda)} \frac{\partial L(v, x, y)}{\partial (v, \lambda)}^T
$$

That is, we can write

$$
\sqrt{n}(\hat{\theta} - \theta') \rightarrow \mathcal{N}(0, \zeta^{-1} \Sigma)
$$

Where $(H - \kappa I) \Sigma + \Sigma (H - \kappa I) = \Gamma(\hat{v}, \hat{\lambda})$, and $\theta$ is the joined vector of $v, \lambda$. Notice that here we are converging to a random matrix, so we interpret this estimation as

$$
\zeta^{1/2} \Sigma^{-1/2} \sqrt{n}(\hat{\theta} - \theta') \rightarrow \mathcal{N}(0, I)
$$

By the delta method, since we have $(w, \sigma) = (v/\lambda, 1/\lambda)$, we know that

$$
\nabla(\theta) = J^T = \frac{\partial (w, \sigma^2)}{\partial (v, \lambda)} \bigg|_{v=\hat{v}, \lambda=\hat{\lambda}} \begin{pmatrix} 1/\lambda \hat{I} \\ -\hat{w}^T/\hat{\lambda}^2 \\ -1/\hat{\lambda}^2 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}^2 I \\ -\hat{w}^T \hat{\sigma}^2 \\ -\hat{\sigma}^4 \end{pmatrix}
$$

So we have (now $w, \sigma^2$ are written in terms of $\theta$)

$$
\sqrt{n}(\hat{\theta} - \theta) \sim \mathcal{N}(0, J(\zeta^{-1} \Sigma) J^T)
$$

Or, in other terms, let $S = J(\zeta^{-1} \Sigma) J^T$, we have

$$
\sqrt{n}S^{-1/2}(\hat{\theta} - \theta) \sim \mathcal{N}(0, I)
$$

Thus, Theorem 5.1 is proved. In addition, this yields

$$
(\hat{\theta} - \theta)^T \frac{1}{n} J(\zeta^{-1} \Sigma) J^T (\hat{\theta} - \theta) \sim \chi_{k+1}
$$

Let $q_\alpha$ is the $1 - \alpha$ quantile of the distribution $\chi_{k+1}$, so we have the confidence region

$$
(\hat{\theta} - \theta)^T (J(\zeta^{-1} \Sigma) J^T)^{-1}(\hat{\theta} - \theta) \leq q_\alpha/n
$$
Where we can calculate

\[ J^{-1} = \begin{pmatrix} 1/\hat{\sigma}^2 I & -\hat{\omega}/\hat{\sigma}^4 \\ 0 & -1/\hat{\sigma}^4 \end{pmatrix} \]

To make it clearer, we define \( R = J^{-1} \). So, or, write back to \( \hat{\omega}, \hat{\sigma} \), as

\[
\left( \begin{pmatrix} \hat{\omega} \\ \hat{\sigma}^2 \end{pmatrix} - \begin{pmatrix} \hat{\omega} \\ \hat{\sigma}^2 \end{pmatrix} \right)^T \Sigma^{-1} R \left( \begin{pmatrix} \omega \\ \sigma^2 \end{pmatrix} - \begin{pmatrix} \hat{\omega} \\ \hat{\sigma}^2 \end{pmatrix} \right) < \frac{q_\alpha}{\zeta n}
\]

\[
C \quad \text{Experiment Setup}
\]

In our main paper, we provide theoretical guarantees for truncated linear regression with unknown noise variance and test our procedure on synthetic data. Here, we give an overview of how we set up these experiments. In our experimental section, we mention that we perform each experiment for an specified number of trials. For each algorithm a trial means something different. For the [7] experiments, we write the dataset of interest to a csv file, which is then read in an R script and the procedure is run. For [9] and our algorithm, a trial is considered complete in a of couple ways. First off, every 100 steps, we check the \( L^2 \) norm of the validation set’s gradient. If its magnitude is less than \( 1e^{-1} \), we terminate the procedure, and return the current estimates. However, if after taking 2500 gradient steps, the validation set’s gradient \( L^2 \) norm is greater than \( 1e^{-1} \), we re-run the stochastic process. We do this a maximum of three times, and return the trial with the smallest gradient.

For these experiments, we use a PyTorch SGD optimizer, starting our procedure with a learning rate of \( 1e^{-1} \), and decaying it at a rate of \( .9 \) every 100 gradient steps.

All of the experiments performed in this paper were performed on a 15-inch MacBookPro with a 2.2 GHz 6-Core Intel Core i7 with 16 GB of memory. It is of note to mention that 2500 gradients steps with batch size 10 ran in a maximum of 3 seconds for the experiments.
### D Code

We provide code for our experiment at the following GitHub repository: https://github.com/pstefanou12/Truncated-Regression-With-Unknown-Noise-Variance-NeurIPS-2021.

Below, we provide the gradient that we used for conducting all of our experiments.

```python
class TruncatedUnknownVarianceMSE(ch.autograd.Function):
    
    Computes the gradient of negative population log likelihood for truncated linear regression with unknown noise variance.
    
    @staticmethod
    def forward(ctx, pred, targ, lambda_, phi):
        ctx.save_for_backward(pred, targ, lambda_)
        ctx.phi = phi
        return 0.5 * (pred.float() - targ.float()).pow(2).mean(0)

    @staticmethod
    def backward(ctx, grad_output):
        pred, targ, lambda_ = ctx.saved_tensors
        # calculate std deviation of noise distribution estimate
        sigma = ch.sqrt(lambda_.inverse())
        stacked = pred[None, ...].repeat(args.num_samples, 1, 1)
        # add noise to regression predictions
        noised = stacked + sigma * ch.randn(stacked.size())
        # filter out copies that fall outside of truncation set
        filtered = ctx.phi(noised)
        z = noised * filtered
        lambda_grad = .5 * (targ.pow(2) - (z.pow(2).sum(dim=0) /
                             (filtered.sum(dim=0) + args.eps))
                             # multiply the v gradient by lambda, because autograd computes v_grad*x*variance, thus need v_grad*(1/variance) to cancel variance factor
                             )
                             out = z.sum(dim=0) / (filtered.sum(dim=0) + args.eps)
                             return lambda_ * (out - targ) / pred.size(0),
                             targ / pred.size(0), lambda_grad / pred.size(0), None
```

Listing 1: Truncated version of mean squared-error loss

### E Future Work

There are many ways to build upon our work. One thing to point out is that, we use an SGD framework, while [7] uses an analytic gradient and Hessian provided by the [16] package and the Newton-Raphson method. Since we do not explicitly calculate integrals in our method, our framework could be used to train non-linear models, including neural networks. Further, now that there are two methods to learn from truncated linear models, it would be interesting to explore an actual example where a dataset has been **truncated** or **censored** due to uncontrollable circumstances. With multiple methods for learning from truncated samples, it would interesting to see what results each of the method’s give. An interesting field to explore would be environmental sciences, as there are a lot of examples where there is truncation due to measurement instrumentation failure or natural causes. One last to ponder would be what theoretical guarantees can be derived for an algorithm in the **censored** setting. We emphasize that our algorithm for truncated data will work in the censored setting, but with additional knowledge of all the covariate features, can we design an algorithm with better error and/or run-time bounds?