QChD Lite: Lark model for elementary particles

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Abstract

Larks are relativistic massless Dirac fields of spin 1/2 tensored with spinors over a Hermitian charge space of a semi-simple compact gauge Lie group. A quantum charge dynamics (QChD Lite) of larks and antilarks mediated by the gauge bosons has an infinite and relatively discrete variational energy-mass spectrum. In particular, for gauge groups SU(2) and SU(3) one gets infinite energy-mass spectra of constituent leptons and quarks. The lark model of elementary particles is non-perturbative and has no free parameters.

1 Introduction

1.1 Preamble

The ill-fated non-linear "world equation" for spinor-isospinor fields was Heisenberg’s unification project (cp. HEISENBERG[3]) intended to describe all elementary particles as bound states of these non-observable fields.

Unfortunately, the quantization in an indefinite Hilbert space on the Minkowski space met with insurmountable challenges from Pauli, Feynman et al. The conundrum inspired I. Segal’s program of non-linear quantization (SEGAL[11]) in a definite Hilbert space of Cauchy data, a cornerstone of the present paper.

Quark fields of the standard model with its gauge non-linearity can be viewed as a later vindication of the basic Heisenberg idea (who himself discussed such a connection 10 years after his controversial 1958 paper.) Yet solution of Quantum Chromodynamics (QCD) is notoriously difficult. The approximate QCD Lite of WILCZEK[12] (cp. also OKUN[10]) is the Quantum Chromodynamics of the lightest $u$ and $d$ quarks with their Lagrangian mass terms cut off.
Uncut Lagrangians of lepton and quark flavors of the standard model differ only in their phenomenological mass terms. Could it be that the flavors are quantum excitations of just one classically massless flavor? Another standing out question – how many leptonic and quark flavors are possible – also is beyond the Standard Model as well as various unifications based on different gauge groups.

This paper provides an affirmative answer to the first question via the Berezin–Toeplitz non-perturbative quantization and a demonstration of an infinite energy-mass variational spectrum of the excitations in the framework of an alternative lark model of Quantum Charge Dynamics QChD Lite (with an arbitrary semi-simple compact gauge group). Larks are relativistic massless fields of Dirac spin 1/2 fields tensored with spinors over a Hermitian charge space of a semi-simple compact gauge Lie group.

The results are based on the author’s previous description of quantum excitations of gauge bosons and a bosonization of super fields (Dynin[2] and Dynin[3]).

1.2 Synopsis

1. Generalizing QCD Lite to QChD Lite, we consider Quantum Charge Dynamics of larks, i. e., Dirac spinor 1/2 fields on Minkowski space tensored with spinors over a charge Hermitian space of a gauge semi-simple compact Lie group.

2. With restriction of the classical Yang–Mills fields to the temporal gauge, solutions of relativistic Yang–Mills–Dirac equations for larks are in one–one correspondence with their Cauchy data.

3. We apply the Schwinger’s quantum action principle as a prescription to quantize Noether’s invariant energy–mass functional (cp. Bogoliubov-Shirkov[1] Chapter II) following I. Segal’s program of non-linear quantization on Cauchy data (see, e. g., Segal[11]). Unfortunately, the conventional normal (aka Wick) quantization requires an infinite renormalization (cp. Glimm-Jaffe[5]). In the present paper such a renormalization is done via Berezin-Toeplitz (aka anti-Wick, or anti-normal, or Berezin) second quantization in a Gelfand nuclear triple (cp. Dynin[2]). For lark super fields this requires a preliminary bosonization as in Dynin[3].

4. The quantum energy-mass operator is not closable in the central Hilbert space of the nuclear triple, so that the von Neumann spectral theory of (unbounded) self-adjoint operators is not applicable.
Instead we consider a variational relative spectrum based on the mini-max principle with the caveat that the dimensions of variate subspaces are taken relative to a type I von Neumann algebra. Then, the QChD energy-mass quantum operator has an infinite relative discrete spectrum (cp. Dynin[2]).

5. We describe leptons and quarks as quantum spectra of larks associated with the gauge groups SU(2) and SU(3). This suggests infinitely many their generations.

2 Classical lark dynamics

2.1 Larks

We consider the Minkowski spacetime as \( \mathbb{R}^{1,3} \) as both space and time oriented with the metric \( \eta_{\mu \nu} \) of the signature \((-+, +, +, +)\). In the natural unit system the time coordinate \( x^0 = t \). Thus \( x^\mu = (t, x^k), \ k = 1, 2, 3 \).

We choose the Lichnerowicz Dirac matrices \( \gamma^\mu \) as in Lichnerowicz[9]: they are real, and in an orthonormal frame

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \eta^{\mu \nu} I, \quad \gamma^{\mu*} = -\eta_{\mu \nu} \gamma^\nu
\]

where * is the Hermitian adjoint operation.

We deal with an arbitrary gauge compact semi-simple Lie group \( G \) along with its unitary representation on a Hermitian charge space \( \mathbb{C}^n \).

\[
\ v \mapsto gv, \quad v \in \mathbb{C}^n, \quad g \in G.
\]

Let \( \Delta \) denote the spinor vector space over \( \mathbb{C}^n \) considered as the Euclidean space \( \mathbb{R}^{2n} \). By the standard exterior algebra construction (cp.Lawson-Michelsohn[8]), \( \Delta \) is identified with the complex Grassmann super space

\[
\Delta \cong \oplus_{r=0}^n \wedge^r \mathbb{C}^n.
\]

The unitary representation (2) is lifted to the even unitary representation on the spinor space \( \Delta \). We extend the notation (2) to the lifted unitary action of \( G \) on \( \Delta \). The \( G \)-module \( \Delta \) is the direct sum of the irreducible \( G \)-modules \( \wedge^r \mathbb{C}^n, \ r = 0, 1, ..., n \), of rank \( r \).

Larks \( \lambda_\mu = \lambda_\mu(x) \) are space-time Dirac spinor (spin 1/2) fields tensored with \( \Delta \). Anti-larks are conjugate spin 1/2 spacetime fields \( \lambda^*_\mu = \lambda^*_\mu(x) \) where * denotes the point-wise Grassmannian even super complex conjugation.

The corresponding super vector spaces of larks and anti-larks are denoted \( \mathcal{L} \) and \( \mathcal{L}^* \).
The homogeneous Grassmannians $\Lambda^r(\mathbb{C}^n)$ carry pointwise irreducible unitary representations of the exterior powers of $G$. The representations of $g$ are again denoted by $g$ so that the Hermitian adjoints $g^* \rightarrow \text{anti-unitary representations of } G$ on the anti-dual $\Lambda^* = \Lambda^r \mathbb{C}^{* (n+2)}$.

The twisted Dirac adjoint of a lark $\lambda$ is defined as

$$\lambda^{\ast}_{\mu} = \lambda^{\ast}_{\mu}(\gamma^0 \otimes I_\Delta). \quad (4)$$

### 2.2 Yang-Mills-Dirac (YMD) equation

The *local gauge group* $G$ is the group of infinitely differentiable $G$-valued functions $g(x)$ on $\mathbb{R}^3$ with the pointwise unitary action on the lark values. The *local gauge Lie algebra* $\text{Ad}(G)$ consists of infinitely differentiable $\text{Ad}(G)$-valued functions with the pointwise Lie bracket.

$G$ acts via the pointwise adjoint $\text{Ad}$-action on the representation of $\text{Ad}(G)$, and correspondingly via $\text{ad}$-action on the vector space $A$ of gauge fields $A = A_{\mu}(x)$ with values in $\text{Ad}(G)$.

Gauge fields $A$ define the covariant partial derivatives

$$D_{\mu}X \equiv \partial_{\mu}X - \text{ad}(A_{\mu})X, \quad X \in \text{Ad}(G), \quad D_{\mu}\text{Ad}(g) = \text{Ad}(g)D_{\mu}. \quad (5)$$

(We set a dimensionless coupling constant to 1.)

The representation of the Yang–Mills curvature tensor $F(A)$ has the usual expression

$$F(A)_{\mu\nu} \equiv \partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu} + [A_{\mu}, A_{\nu}]. \quad (6)$$

The negative Ad-invariant Killing inner product on the semi-simple Lie algebra $\text{Ad}(G)$ of $G$ implies the positive definite scalar product on $\text{Ad}(G)$:

$$X \cdot Y \equiv -\text{Trace}[\text{ad}(X)\text{ad}(Y)]. \quad (7)$$

A gauge invariant Yang-Mills-Dirac Lagrangian of QChD is

$$(1/4)F(A)^{\mu\nu} \cdot F(A)_{\mu\nu} + (1/2)\overrightarrow{\lambda} \overrightarrow{D}A\lambda, \quad (8)$$

where $\overrightarrow{\lambda} = \lambda^\ast i\gamma^0$ denotes the Dirac conjugation tensored with the Grassmann conjugation on $\Delta$, and

$$\overrightarrow{\lambda} \overrightarrow{D}A\lambda \equiv \overrightarrow{\gamma}_\mu D_{\mu}\lambda - \overrightarrow{\lambda}\gamma_\mu D_{\mu}\lambda$$

is the corresponding gauge invariant Dirac operator. The Lagrangian is even and real (cp. Lichnerowicz[9]).

**YMD equations** are the corresponding Euler–Lagrange equations

$$D_{\mu}F(A)^{\mu\nu} = i\overrightarrow{\lambda}\gamma^\nu\lambda, \quad \overrightarrow{D}A\lambda = 0. \quad (10)$$
Their solutions \((A, \lambda)\) are YMD fields.

Due to on-shell gauge invariance, a YMD field is not uniquely defined by the Cauchy data \(A(x^k) \equiv A(0, x^k), \partial_t A(x^k) \equiv A(0, x^k)\) and initial values of YMD. However, in the temporal gauge \(A_0(t, x) = 0\), then YMD equations become the essentially hyperbolic system

\[
\Box A^j - \partial^j (\partial_i A^i) + [A_i, \partial^j A^i] - [A_i, \partial^i A^j] + [A^i, [A^i, A^j]] = \overrightarrow{\nabla} A \lambda, \\
\overrightarrow{\nabla} A \lambda = 0.
\]

Essentially hyperbolic means that smooth Cauchy data \(A(x^k), \partial_t A(x^k), \lambda(x^k)\) at \(t = 0\) have finite propagation speed, and the Cauchy problem is uniquely solvable on the whole \(\mathbb{R}^{1+3}\) for any smooth Cauchy data. This follows from Gogonov-Kapitanskii [6, Theorem I] after the preliminary bosonization as in Dynin [3].

Let \(T_{YMD}^{00}(A)\) denote the Yang-Mills energy-mass functional. Then, by the Dirac-Weyl equation (10), the time-conserved Yang-Mills-Dirac energy-mass functional (cp. Lichnerowicz [9])

\[
T_{YMD}^{00}(A, \lambda) = T_{YM}^{00}(A) + (1/2)\overrightarrow{\nabla} A \lambda = T_{YM}^{00}(A).
\]

Note that the actual dependence of \(T_{YMD}^{00}(A, \lambda)\) on \(\lambda\) is in the YMD system (10).

3 Quantization of YMD energy-mass functional

3.1 YMD Bargmann–Fock Gelfand triples

Consider the Gelfand super triple (cp. Dynin [3])

\[
\mathcal{C} : \mathcal{C}^\infty \hookrightarrow \mathcal{C}^0 \hookrightarrow \mathcal{C}^{-\infty},
\]

where

- \(\mathcal{C}^\infty\) is the nuclear Schwartz space of smooth Cauchy data fields \(\theta(x^k) \equiv (A(x^k), \lambda(x^k))\) such that all their space derivatives on \(\mathbb{R}^3\) vanish faster than any power of \(x^k x_k\) as \(x^k x_k \to \infty\).

- \(\mathcal{C}^0\) is the Hilbert space with the Hermitian quadratic form

\[
\theta^* \theta \equiv \int_{\mathbb{R}^3} dx^k \theta(x^k)^* \theta(x^k),
\]

(we use the bracketless notation for dualities)
• $\mathcal{C}^{-\infty}$ is the nuclear anti-dual of $\mathcal{C}^\infty$ with respect to the sesquilinear form associated with the quadratic form (15).

There is a nuclear super Bargmann–Fock Gelfand super triple over the super triple $\mathcal{C}$ (cp. dynin[3])

$$\mathcal{B} : \mathcal{B}^\infty \hookrightarrow \mathcal{B}^0 \hookrightarrow \mathcal{B}^{-\infty},$$

(16)

where

• $\mathcal{B}^\infty$ is the nuclear Frechet super space of Gâteaux entire test functionals $\Psi(\theta)$ on $\mathcal{C}^{-\infty}$ of the (topological) second order and minimal type;

• $\mathcal{B}^0$ is Bargmann–Berezin super space over $\mathcal{C}^0$, the (complete) complex Hilbert space of Gâteaux entire functionals $\Psi = \Psi(\theta^*)$ on $\mathcal{C}^{-\infty}$

$$\Psi^* = \Psi^*(\theta) \equiv \overline{\Psi(\theta^*)},$$

(17)

and integrable super Hermitian sesquilinear inner product

$$\Psi^* \Phi \equiv \int d\theta^* d\theta \, e^{-\theta^* \theta} \Psi^*(\theta) \Phi(\theta^*);$$

(18)

• $\mathcal{B}^{-\infty}$ is the nuclear anti-dual super space of $\Psi^*(\theta^*)$ on $\mathcal{C}^\infty$ is the nuclear anti-dual space of $\mathcal{B}^\infty$ with respect to the sesquilinear form associated with the quadratic form (15).

Actually, $\mathcal{B}^\infty$ and $\mathcal{B}^{-\infty}$ are locally convex topological algebras of Gâteaux entire functions with the point-wise multiplication.

3.2 Berezin–Toeplitz quantization

Bargmann–Fock Gelfand triple $\mathcal{B}$ is a subtriple of a Gauss Gelfand triple

$$\mathcal{O} : \mathcal{O}^\infty \hookrightarrow \mathcal{O}^0 \hookrightarrow \mathcal{O}^{-\infty},$$

(19)

where $\mathcal{O}^0$ is the Gauss Hilbert space of functionals that have square integrable local norms with respect to Gauss measure.

The orthogonal projector $\Pi^0$ of $\mathcal{O}^0$ is the midterm of the triple projector $\Pi \equiv (\Pi^\infty, \Pi^0, \Pi^{-\infty})$ of $\mathcal{O}$ onto $\mathcal{B}$.

Real-valued functionals $\Theta = \Theta(\vartheta) \in \mathcal{O}^{-\infty}$ are considered as classical variables. By dynin[2], the compressed products with $\Theta(\vartheta)$

$$(\hat{\Theta} \Psi)(\theta) \equiv \Pi^{-\infty}(\Theta(\vartheta) \Psi(\vartheta^*))$$

(20)
are continuous linear operators from $B^\infty$ to $B^{-\infty}$. The compression $\hat{\Theta}$ is the *Berezin–Toeplitz quantization* of the classical observable $\Theta$ (aka Berezin, Berezin–Toeplitz, anti-Wick, anti-normal, or diagonal quantization, cp. BEREZIN [?]).

The correspondence between classical and quantum observables is one-to-one.

By DYNIN [3], if $\Theta^* = \Theta$ then
\[
\inf \Theta^b(\vartheta) \leq \langle \hat{\Theta} \rangle \leq \sup \Theta^b(\vartheta),
\]
where $\Theta^b$ is the real-valued bosonization of $\Theta$.

### 3.3 YMD energy-mass spectrum

The expectation of $L$ on $B^\infty$ is the functional
\[
\langle L \rangle \equiv \Psi^* L \Psi / \Psi^* \Psi, \quad \Psi \in B^\infty, \quad \Psi \neq 0.
\]

Let $\dim_W S$ denote the von Neumann dimension of a nontrivial subspace $S \subset B^\infty$ relative to $W$. Then the relative von Neumann spectral values are defined as
\[
s_0(L) \equiv \inf \langle L \rangle_\Psi, \quad s_n(L) \equiv \sup \left\{ \inf \{\langle L \rangle_\Psi \mid \Psi \in S^\perp \}, \dim_W S > n \right\}.
\]

von Neumann spectrum is called *discrete relative to* $W$ if all spectral values $s_n(L)$ have relatively finite multiplicity in it.

There is a straightforward generalization of the mini-max theorem:

*Suppose operators $L_1$ and $L_2$ are such that the expectation $\langle L_1 \rangle$ is bounded from below, $\langle L_1 \rangle \leq \langle L_2 \rangle$, and the von Neumann spectrum of $L_1$ is relatively discrete. Then the von Neumann spectrum of $L_2$ is relatively discrete, and $s_n(L_2) \geq s_n(L_1)$.*

By DYNIN [2, Section 5], the anti-Wick Yang-Mills energy-mass operator $\hat{H}_{YM}$ on Bargmann–Fock Gelfand triple $B_{YM}$ is infinite and discrete relative to the von Neumann algebra generated by the number operator $N_{YM}$.

Then the anti-Wick Yang-Mills-Dirac energy-mass operator $\hat{H}_{YMD}$ on the Bargmann–Fock Gelfand triple tensor product $B_{YMD}$
\[
B_{YMD} = B_{YM} \otimes B_L
\]
is infinite and discrete relative to the type I von Neumann algebra generated by the twisted number operator $N_{YM} \otimes I_L$ (cp. (13)). Actually, we apply the bosonization method of DYNIN [3] to apply the generalized mini-max principle to twisted number operator on the super triple $B_{YMD}$. 

7
4 Quantum spectra of leptons and quarks

The fractional electric charges of larks $\lambda(x) \in \wedge^r \mathbb{C}^n$ of rank $r$ are set to $r/n$; and of $\lambda^*(x) \in \wedge^r (\mathbb{C}^n)^*$ are set to $-r/n$.

Let $G$ be $SU(2)$ with its fundamental unitary representation on $\mathbb{C}^2$. Then larks of electric charge $Q = 1$ are singlets of rank 2. They may represent ”massless positron fields” with an infinite quantum spectrum of masses of positron flavors. Correspondingly, larks of electric charge $Q = -1$ represent ”massless electron fields” with an infinite quantum spectrum of masses of electron flavors.

Real tensor products of the values of electric charge $Q = 1$ larks and their complex conjugates represent massless Majorana neutrinos with $Q = 0$ with infinite quantum spectrum of masses of neutrino flavors.

Let $G$ be $SU(3)$ with its fundamental unitary representation on $\mathbb{C}^3$. Then larks of $Q = 1$ form singlets of rank 3 that may represent a ”massless proton” with infinite von Neumann spectrum of masses of proton flavors. Correspondingly, anti-larks of electric charge 3 form singlets with electric charge $Q = -1$ that may represent a ”massless anti-proton” with infinite quantum spectrum of masses of anti-proton flavors.

The values of larks with electric charge $Q = 2/3$, span 3-dimensional vector space. They may represent ”massless up quark” fields, and their von Neumann spectrum may represent masses of up quark generations. Similarly anti-larks of electric charge $Q = -1/3$ may represent ”massless down quark” fields, and their quantum spectrum may represent masses of infinitely many down quarks flavors.

5 Conclusion

Old programs never die (KOBZAREV-MANIN[7]).

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