Stationary Josephson effect in ballistic graphene junctions: effects of inhomogeneous carrier density

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Abstract. We study the stationary Josephson effect in a system of ballistic graphene in planar contact with two superconducting electrodes. Applying a quasi-classical Green’s function approach, we derive a general expression of the Josephson current which is valid whenever the chemical potential is away from the Dirac point. In its derivation we treat the mono- and bi-layer cases in a parallel manner, and take account of the fact that the carrier density is higher in the region covered by the superconductors. The behavior of the Josephson critical current is investigated at zero temperature on the basis of the obtained formula.

1. Introduction
Since the realization of a monolayer sheet of graphene, extensive studies have been devoted to uncovering its unusual electronic properties [1]. Josephson effect in graphene has been a target of intense theoretical [2-9] and experimental [10-13] studies. Although the main interest has been concentrated on how the Josephson current is affected by the unique band structure of graphene, we focus on another interesting aspect stemming from the fact that graphene is an ideal two-dimensional electron system. Graphene in a Josephson junction acquires a planar contact with a superconductor, since a natural way to create a superconductor-graphene-superconductor junction is to deposit superconductors on top of a graphene flake. Due to this contact, the carrier density in a graphene sheet becomes higher in the region covered by a superconductor. This results in a mismatch in the Fermi wave number at the interface between the covered and uncovered regions. In this paper we derive a general expression of the Josephson current in the ballistic graphene junction treating the mono- and bi-layer cases in a parallel manner, and study the influence of inhomogeneous carrier density on the Josephson critical current.

2. Formulation
We consider a planar junction consisting of a graphene sheet on which two superconductors, S₁ and S₂, of width W are deposited with separation L (see Fig. 1). The pair potential Δ(x) is assumed to be Δe^{iψ/2} in S₁ and Δe^{-iψ/2} in S₂. Let us introduce the thermal Green’s function \( \tilde{G}_j(xy, x'y'; \omega) \) for quasiparticles in the graphene sheet, where \( j = 1 \) (\( j = 2 \)) specifies the monolayer (bilayer) case and \( \omega \) is the Matsubara frequency. The Green’s function obeys [9]

\[
(i\omega\tilde{\tau}_4^{1\times4} - \tilde{H}_j - \tilde{\Sigma}_j) \tilde{G}_j(xy, x'y'; \omega) = \tilde{\tau}_4^{1\times4}\delta(x-x')\delta(y-y')
\]

(1)
We need to treat only $H$ and $H_j = \text{diag}(H_j, H_j)$. The effective Hamiltonian $H_j$ for low-energy quasiparticles is given by [1]

$$H_1 = \begin{pmatrix} -\mu + \tilde{U} & \gamma \hat{k}_- \\ \gamma \hat{k}_+ & -\mu + \tilde{U} \end{pmatrix}, \quad H_2 = \begin{pmatrix} -\mu + \tilde{U} & -\alpha \hat{k}_+^2 \\ -\alpha \hat{k}_-^2 & -\mu + \tilde{U} \end{pmatrix}$$

with $\hat{k}_\pm = -i\partial_x \pm \partial_y$, $\gamma$ and $\alpha$ are band parameters, and $\mu$ represents the chemical potential. The potential $\tilde{U}(x)$, which is assumed to be $\tilde{U}(x) = -U$ for $|x| > L/2$ and 0 for $|x| < L/2$, describes the inhomogeneity of carrier density. The planar contact with the superconductors is described by the self-energy [9]

$$\tilde{\Sigma}_j = \frac{-i\Gamma}{\sqrt{\Delta_2^2 + \omega^2}} \begin{pmatrix} \omega \chi^{(j)}_{12} & \Delta(x) \chi^{(j)}_{12} \\ \Delta(x) \ast \chi^{(j)}_{12} & -\omega \chi^{(j)}_{12} \end{pmatrix} \theta(|x| - L/2),$$

where $\Gamma$ characterizes the coupling strength, $\chi^{(1)}_{12} = \text{diag}(1,1)$, and $\chi^{(2)}_{12} = \text{diag}(1,0)$. The $(2,2)$-element of $\chi^{(2)}_{12}$ is zero because only the top layer is in contact with the superconductors in the bilayer case [14]. Assuming that our system is translationally invariant in the $y$ direction under the condition of $W \gg L$ we perform the Fourier transformation as

$$\tilde{G}_j(x, x'; q, \omega) = \int d(y - y')e^{-i\mathbf{q} \cdot (y - y')} \tilde{G}_j(xy, x'y'; \omega).$$

We briefly describe the procedure to obtain the Green’s function for the monolayer case. The bilayer case can be treated in a similar manner. Firstly we consider the uncovered region of $|x| < L/2$, where the self-energy $\tilde{\Sigma}_j$ and the potential $\tilde{U}$ vanish. For a given $q$, the quasiparticle state at $\mu$ is characterized by $k_\pm = (\pm k, q)$, where $k = \sqrt{k_F^2 - q^2}$ with $k_F \equiv \mu/\gamma$. We define $\phi \equiv \text{arg}\{k + iq\}$ in terms of which we can express $k = k_F \cos \phi$ and $q = k_F \sin \phi$. It is convenient to express the elements of the Green’s function as $[\tilde{G}]_{11} = -[\tilde{G}]_{22} = g$, $[\tilde{G}]_{12} = f$, and $[\tilde{G}]_{21} = f^\dagger$. We need to treat only $g$ and $f^\dagger$, which are expressed within a quasiclassical approximation as

$$g(x, x'; q, \omega) = e^{ik(x-x')}\Lambda_{++}[-i\theta(x-x')/v_x + c_{++}] + e^{ik(x+x')}\Lambda_{-+}c_{+-} + e^{-ik(x-x')}\Lambda_{+-}c_{-+} + e^{-ik(x+x')}\Lambda_{--}[-i\theta(x'-x)/v_x + c_{-+}],$$

$$f^\dagger(x, x'; q, \omega) = e^{i(kx-kx')}\Lambda_{++}d_{++} + e^{i(kx+kx')}\Lambda_{--}d_{-+} + e^{-i(kx-kx')}\Lambda_{-+}d_{+-} + e^{-i(kx+kx')}\Lambda_{+-}d_{--},$$

where $v_x = \gamma \cos \phi$, $k^\pm = k \pm i\omega/v_x$, and

$$\Lambda_{\pm\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm e^{\mp i\phi} \\ \pm e^{\mp i\phi} & 1 \end{pmatrix}, \quad \Lambda_{\pm\mp} = \frac{1}{2} \begin{pmatrix} e^{\mp i\phi} & \mp 1 \\ \pm 1 & -e^{\mp i\phi} \end{pmatrix}. \tag{7}$$

Secondly we introduce the boundary condition for $g$ and $f^\dagger$ at $x = \pm L/2$ to determine the coefficients $c_{\sigma\sigma'}$ and $d_{\sigma\sigma'}$. Let us consider the covered region of $|x| > L/2$. In this region,
the quasiparticle state at $\mu$ is characterized by $p_{\pm} = (\pm p, q)$, where $p = \sqrt{p_F^2 - q^2}$ with $p_F = (\mu + U)/\gamma$. We define $\theta = \arg\{p + iq\}$ in terms of which we can express $p = p_F \cos \theta$ and $q = p_F \sin \theta$. Solving the Bogoliubov-de Gennes equation for an imaginary energy $i\omega$,

$$\left(\omega \tau^z_{4\times 4} - \hat{H}_j - \Sigma_j\right) \hat{\Psi}_j(xy) = 0,$$  \hspace{1cm} (8)

by setting $\hat{\Psi}_j(xy) = e^{iqy}\hat{\psi}_j(x)$, we obtain two linearly independent solutions. We decompose $\hat{\psi}_j$ into the electron and hole components as $\hat{\psi}_j = \hat{\psi}_{j}^{e}(\hat{\psi}_{j}^{h})$. From the resulting solutions we can show that the electron and hole components satisfy

$$\psi_{j}^{e}(\pm L/2) = M_{\pm} \psi_{j}^{h}(\pm L/2).$$  \hspace{1cm} (9)

The matrix $M_{\pm}$ is given by

$$M_{\pm} = \frac{e^{\pm i\omega/2}}{\Delta \cos \theta} \begin{pmatrix} \hat{\omega} \cos \theta \mp i\tilde{\Omega} \sin \theta & \pm\tilde{\Omega} \\ \pm\tilde{\Omega} & \hat{\omega} \cos \theta \mp i\tilde{\Omega} \sin \theta \end{pmatrix},$$  \hspace{1cm} (10)

where $\tilde{\Delta} = \eta_\omega \Delta$ and $\tilde{\omega} = (1 + \eta_\omega)\omega$ with $\eta_\omega = \Gamma/\sqrt{\Delta^2 + \omega^2}$, and $\tilde{\Omega} = \sqrt{\Delta^2 + \omega^2}$. Combining (5), (6) and (9), we can determine $c_{\sigma\sigma'}$ and $d_{\sigma\sigma'}$. Note that the influence of a mismatch in the Fermi wave number is taken into account via the boundary condition (9).

Hereafter we derive an expression of the Josephson current $I_j(\varphi)$ for the monolayer case ($j = 1$) and the bilayer case ($j = 2$). Using $c_{++}$ and $c_{--}$ we can express

$$I_j(\varphi) = 2\pi e v_j N_j(0) W \int_{-\pi/2}^{\pi/2} d\phi \cos \phi T \sum_{\omega} \Re\{c_{++} - c_{--}\},$$  \hspace{1cm} (11)

where the density of states and the Fermi velocity are given by $N_1(0) = \mu/(\pi \gamma^2)$ and $v_1 = \gamma$ in the monolayer case, and $N_2(0) = 1/(2\pi \alpha)$ and $v_2 = 2\sqrt{\mu \alpha}$ in the bilayer case. We present the expression of $I_j(\varphi)$ in the short-junction limit (i.e., $L$ is much shorter than the thermal length and the superconducting coherence length) of experimental importance,

$$I_j(\varphi) = 2\pi e v_j N_j(0) W \int_{-\pi/2}^{\pi/2} d\phi \cos \phi T \sum_{\omega} \Pi_j(\phi, \omega, \varphi),$$  \hspace{1cm} (12)

where

$$\Pi_1 = \tilde{\Delta}^2 \cos^2 \phi \cos^2 \theta \sin \varphi \left[\tilde{\omega}^2 \cos^2 \phi \cos^2 \theta + \tilde{\Omega}^2 (1 - \sin \phi \sin \theta)^2 \right. $$

$$- \tilde{\Omega}^2 (\sin \phi - \sin \theta)^2 \cos 2kL + \tilde{\Delta}^2 \cos^2 \phi \cos^2 \theta \sin \varphi \left],$$  \hspace{1cm} (13)

$$\Pi_2 = \tilde{\Delta}^2 \sin^2 2\phi \sin^2 2\theta \sin \varphi \left[\tilde{\omega}^2 \sin^2 2\phi \sin^2 2\theta + \tilde{\Omega}^2 (1 - \cos 2\phi \cos 2\theta)^2 \right. $$

$$- \tilde{\Omega}^2 (\cos 2\phi - \cos 2\theta)^2 \cos 2kL + \tilde{\Delta}^2 \sin^2 2\phi \sin^2 2\theta \sin \varphi \right].$$  \hspace{1cm} (14)

We note that (12) is reduced to the result reported in [9] in the limit of $U \rightarrow 0$.

### 3. Critical current at zero temperature

We now consider the influence of inhomogeneous carrier density on the critical current $I_c^{(j)} \equiv \max_{\varphi}\{I_j(\varphi)\}$. For simplicity we focus on the strong coupling limit of $\Gamma/\Delta \gg 1$ at zero temperature. It is convenient to rewrite the critical current as $I_c^{(j)} = e\Delta N_c C_{\max}/2$ with
\( N_j = 2 v_j N_j(0) W \) being the number of conducting channels. The numerical result for \( C_{\text{max}}^{(j)} \) is shown in Fig. 2, where the following parameters are employed: \( L = 200 \text{ nm} \), \( \Delta = 120 \mu \text{eV} \), and \( \mu = 32 \text{ meV} \). We observe that the suppression of \( I_c^{(j)} \) with increasing \( U \) is more pronounced in the bilayer case than in the monolayer case. Indeed, \( C_{\text{max}}^{(1)} \) converges to a constant value nearly equal to 1.22 [3] with increasing \( U \), while \( C_{\text{max}}^{(2)} \) vanishes in the limit of \( U \to \infty \). This reflects the chiral structure of quasiparticle wavefunctions which completely differs between the mono- and bi-layer cases.

![Figure 2. \( C_{\text{max}} \) as a function of \( U/\mu \) for the monolayer (solid line) and bilayer (broken line) cases.](image)

4. Summary
To study the stationary Josephson effect in a planar junction of graphene, we have derived a general expression of the Josephson current by applying a quasi-classical Green’s function approach. The obtained expression can be applied to both the mono- and bi-layer cases with arbitrary \( \Gamma \) and \( U \). In the strong-coupling limit, we have investigated the influence of inhomogeneous carrier density on the Josephson critical current \( I_c \) at zero temperature. We have found that \( I_c \) is suppressed by the carrier inhomogeneity, particularly in the bilayer case.

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