Geometric interpretation of the large $N = 4$ index

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Abstract

We study WZW models with the large $N = 4$ superconformal symmetry. Our main result is a geometric interpretation of the large $N = 4$ index. In particular, we find that states contributing to the index belong to spectral flow orbits of special RR ground states. We use (anti-)holomorphic differentials with torsion to clarify the geometric meaning of these states in terms of differential forms on the target space.
1 Introduction

Supersymmetric 2d quantum field theories have been an area of active research for several decades. Supersymmetry is a powerful tool which gives a lot of control over the structure of these theories. In particular, it allows the construction of indices \([1, 2, 3, 4, 5]\) which provide important information about the theory at a generic point in the moduli space. For example, the elliptic genus \([2, 3, 4]\) was used to establish the relation between \(N = 2\) minimal models and Landau-Ginzburg theories. As another example, the “new index” of \([5]\) was crucial for understanding the structure of the vacua of the \(N = 2\) supersymmetric theory.

In 2d non-linear \(\sigma\)-models with superconformal symmetry an index may be understood more intuitively in terms of the geometry of the target space. For instance, the elliptic genus was interpreted in \([3]\) as the index of Dirac operator in loop spaces. The purpose of this paper is to find a geometric interpretation of the large \(N = 4\) index recently constructed in \([6]\).

The large \(N = 4\) index \(I_2\) describes theories with the large \(N = 4\) superconformal symmetry \(A_\gamma\) \([7]\). The algebra \(A_\gamma\) is different from the small \(N = 4\) superconformal algebra in that it contains two \(\hat{su}(2)\) current algebras instead of one. This higher symmetry restricts the spectrum of the theory in essential ways \([8, 9, 10, 11]\). The basic information about the spectrum, captured by the large \(N = 4\) index, was used in the search \([6, 12, 13, 14]\) for the holographic dual CFT of the string theory on \(AdS_3 \times S^3 \times S^3 \times S^1\).

The index \(I_2\) is defined by:

\[
I_2 = \left[ \frac{d}{dz} \frac{d}{d\bar{z}} Z \right]_{z_+ = -z_-, \; \bar{z}_+ = -\bar{z}_-}
\]  

(1.1)

Here \(Z\) is the RR sector \(A_\gamma\) character \([10]\)

\[Z(q, z_\pm; \bar{q}, \bar{z}_\pm) := \text{Tr}_{\text{RR}} z_\pm^{2iA_0^{-i}\lambda^3} \bar{z}_\pm^{2iA_0^{+i}\lambda^3} q^{L_0 - c/24} (1.2)\]

and \(A_0^{\pm;i}\) are zero modes of the \(\hat{su}(2)_+ \oplus \hat{su}(2)_-\) currents.

The \(N = 4\) world-sheet supersymmetry requires \([15, 16]\) the target space \(X\) of the non-linear \(\sigma\)-model to be a Hyper-Kähler-Torsion manifold \([17]\). However, realization of \(A_\gamma\) algebra is known only for the \(N = 4\) gauged WZW models which are special due to their relation with the \(N = 4\) coset CFT’s \([18, 19]\). For this reason, our strategy to reveal the geometric meaning of the large \(N = 4\) index \(I_2\) will rely on these models.

The main result of this paper is a geometric interpretation of the index \(I_2\) for the \(N = 4\) gauged WZW models. We show that the states contributing to the index \(I_2\) belong to spectral flow orbits of special RR ground states and characterize these states geometrically as certain \((p_1, p_2)\) forms on the target space \(X\). In particular, these forms satisfy

\[
\mathcal{D}C = \nu \wedge C, \quad \mathcal{D}^\dagger C = i_w C
\]

(1.3)

\[
\overline{\mathcal{D}}C = \bar{\nu} \wedge C, \quad \overline{\mathcal{D}}^\dagger C = i_{\bar{w}} C
\]

(1.4)

where \((\overline{\mathcal{D}})\) \(\mathcal{D}\) are (anti-)holomorphic differentials with torsion. Vectors \(w, \bar{w}\) and 1-forms \(\nu, \bar{\nu}\) depend on a particular RR state.
The \((p_1, p_2)\) forms, which correspond to the RR ground states relevant for the index, are further specified by a number of requirements (see Section 5). Some conditions enforce that the forms have the highest (lowest) weight under the action of the zero mode subalgebra of \(\widehat{su(2)}_+ \oplus \widehat{su(2)}_-\) on differential forms on \(X\). Other conditions state that the \((p_1, p_2)\) forms have vanishing interior (exterior) product with certain vectors (1-forms) defined in terms of torsion and a triplet of complex structures on the target space.

Our strategy to reveal the geometric meaning of the large \(N = 4\) index will be to start (Section 2) with the realization of \(A_\gamma\) algebra for the \(N = 4\) gauged WZW models, review (Section 3) the structure of ground states in a massless R-sector representation of this algebra, identify the states contributing to the index \(I_2\) (Section 4), and, finally, find a geometric interpretation of these special states (Section 5).

2 \(WZW\) models with \(A_\gamma\) symmetry.

In this section we review the realization of the \(A_\gamma\) algebra for the \(N = 4\) gauged WZW models \([15, 18, 19, 20]\). We are interested in these models for the following reason. In order to find a geometric interpretation of the large \(N = 4\) index we must start from a \(\sigma\)-model with \(A_\gamma\) symmetry and express \(A_\gamma\) generators in terms of the \(\sigma\)-model fields. At present, the \(N = 4\) gauged WZW models are the only known representatives of such \(\sigma\)-models and we will rely on them.

The key issue that makes the \(N = 4\) gauged WZW models special among \(\sigma\)-models with \(A_\gamma\) symmetry is their relation with the \(N = 4\) cosets \(G/H\). Here \(G = SU(N + 2)\) or \(G = G' \otimes U(1)\) with \(G'\) a simple group different from \(SU(M)\). In the construction of these cosets one chooses a subgroup \(H\) in such a way that \(G/(H \times U(2))\) is a Wolf space \([21]\). The simplest case \(G = SU(3), H = I\) corresponds to WZW model for the group manifold \(SU(3)\).

We recall the correspondence between general supersymmetric gauged WZW models and superconformal cosets in Section 2.1. We use the coset description of the \(N = 4\) gauged WZW models to realize the generators of \(A_\gamma\) algebra in Section 2.2.

2.1 Review of supersymmetric gauged WZW model

Here we recall basic facts about supersymmetric gauged WZW models and review their description in terms of superconformal cosets. We follow the discussion in \([22]\).

It is convenient to work in \((1,1)\) superspace with coordinates \(Z = (z, \theta)\) and \(\bar{Z} = (\bar{z}, \bar{\theta})\). Then the fields are arranged in the superfield as:

\[
G = g \left( 1 + \theta \psi + \bar{\theta} g^{-1} \bar{\psi} g - \theta \bar{\theta} \psi g^{-1} \bar{\psi} g \right) \tag{2.1}
\]

where \(g(z, \bar{z})\) is a map from Riemann surface \(\Sigma\) to the group manifold \(G\). The fermions \(\psi(z, \bar{z})\) and \(\bar{\psi}(z, \bar{z})\) take values in the Lie algebra \(\mathfrak{g}\) of \(G\). There are also gauge superfields \(A\) and \(\bar{A}\) whose components take values in the complexification of the Lie algebra \(\mathfrak{h}\) of \(H\), a subgroup of \(G\).
The $\sigma$-model action has the form:

$$S(G, A, \overline{A}) = \int d^2\theta \left\{ I(G) + \frac{k}{4\pi} \int d^2z Tr' \left( A G^{-1} D G - D G G^{-1} \overline{A} - A \overline{A} + AG^{-1} \overline{AG} \right) \right\}$$

(2.2)

where $k \in \mathbb{Z}_+$ and $D = \partial_{\theta} + \theta \partial_z$, $\overline{D} = \partial_{\overline{\theta}} + \overline{\theta} \partial_{\overline{z}}$ and we denote

$$I(G) = -\frac{k}{8\pi} \int \Sigma d^2z Tr' \left( G^{-1} D G G^{-1} \overline{D} G \right) + \int_B dt d^2z Tr' \left( \overline{G}^{-1} \partial_t \overline{G} \left[ \overline{G}^{-1} D \overline{G}, \overline{G}^{-1} \overline{D} \overline{G} \right] \right)$$

(2.3)

Here $t, z, \overline{z}$ are coordinates on a 3-manifold $B$ such that $\partial B = \Sigma$ and $\overline{G}$ is an extension of the map $G$.

We use notation $Tr' = \frac{1}{x_R} Tr_R$ where $x_R$ is the Dynkin index of a representation $R$ of the Lie algebra $g$. The hermitean generators $T_A$ satisfy

$$[T_A, T_B] = if_{A B C} T_C, \quad Tr R T_A T_B = 2x_R \delta_{A B}$$

(2.4)

where $f_{A B C}$ are anti-symmetric in all three indices. (We do not distinguish lower and upper indices.)

To realize $A_\gamma$ algebra in section 2.2 we will use the correspondence between supersymmetric gauged WZW models and superconformal cosets. This relation was established in [22] following the early proof for the bosonic CFT [23]. The idea is to fix the gauge $A = 0$ and change variables from $A$ to $H$ by parametrizing $A = -DHH^{-1}$. This procedure has to be accompanied by the introduction of ghost superfields $B, C, \overline{B}, \overline{C}$ taking values in the complexification of the Lie algebra $h$ with the action:

$$S_{\text{ghost}} = -\frac{k}{8\pi} \int d^2\theta d^2z \left\{ Tr' \left( \overline{B} \overline{D} \overline{C} + B \overline{D} C \right) \right\}$$

(2.5)

Also, using supersymmetric version of Polyakov-Wiegmann identity, one may write

$$S'(G, H) := S(G, A = -DHH^{-1}, 0) = \int d^2\theta \left( I(GH) - I(H) \right)$$

(2.6)

where $I(G)$ is defined in (2.3). The last step is to make change of variables $G \rightarrow GH^{-1}$ which has a trivial Jacobian. Then, the path integral takes the form:

$$Z = \int [dG][dH][dB][dC][d\overline{B}][d\overline{C}] e^{-\int d^2\theta \left( I(G) - I(H) \right) - S_{\text{ghost}}}$$

(2.7)

The key point in the proof of [22] is that the total energy momentum-tensor

$$T = T_g + \overline{T}_h + T_{\text{ghost}}$$

(2.8)

is equal to the one of the coset CFT up to a BRST-trivial term. Here $T_g$ corresponds to WZW model for the group $G$, the “tilde” in $\overline{T}_h$ indicates that the term with Cartan-Killing metric is taken with negative sign, and the ghost piece $T_{\text{ghost}}$ has the standard form.
Let us clarify this key issue a bit more. The components of the term \( T_g = \frac{1}{2} G_g(z) + \theta T_{\text{ghost}}(z) \) are:

\[
T_g = \frac{1}{k} \left( : E^A(z) E^A(z) : - : \Psi^A(z) \partial \Psi^A(z) : \right) \tag{2.9}
\]

\[
G_g = \frac{2}{k} \left( : \Psi^A(z) E^A(z) - \frac{i}{3k} H_{ABC} : \Psi^A(z) \Psi^B(z) \Psi^C(z) : \right) \tag{2.10}
\]

where \( H_{ABC} = \frac{1}{\sqrt{2}} f_{ABC} \) and the currents \( E^A(z), \Psi^A(z) \) are given in terms of \( \sigma \)-model fields (2.1) as:

\[
E^A(z) = - \frac{k}{\sqrt{2}} \left( \partial g g^{-1} \right)^A, \quad \Psi^A(z) = - \frac{k}{\sqrt{2}} \left( g\psi g^{-1} \right)^A \tag{2.11}
\]

They have the following non-vanishing OPE’s

\[
E^A(z) E^B(w) \sim \frac{n}{2} \frac{\delta^{AB}}{(z-w)^2} + \frac{iH_{ABC} E^C(w)}{z-w} \tag{2.12}
\]

\[
\Psi^A(z) \Psi^B(w) \sim \frac{k}{2} \frac{\delta^{AB}}{z-w} \tag{2.13}
\]

where \( n = k - c_g \) and \( c_g \) is dual coxter number of group \( G \).

Now, one defines the energy-momentum tensor \( T_{\text{coset}} = \frac{1}{2} G_{\text{coset}}(z) + \theta T_{\text{coset}}(z) \) for the coset CFT:

\[
T_{\text{coset}} = \frac{1}{k} : E^A E_A : + \frac{2i}{k^2} H_{ABI} E^I : \Psi^A \Psi^B : - \frac{n}{k^2} : \Psi_A \partial \Psi^A : \tag{2.14}
\]

\[
- \frac{1}{k^3} H_{ABC} H^A_{EF} : \Psi^B \Psi^C \Psi^E \Psi^F : + \frac{2}{k^2} H_{ABC} H^{ABC} : \Psi_E \partial \Psi^A : \tag{2.15}
\]

where we split the generators \( T^A = (T^A, T^I) \) (2.4) so that \( T^I, \ I = 1, \ldots, \dim(H) \) span the Lie algebra \( h \) while index “\( A \)” runs over the coset. Then,

\[
T_g - T_{\text{coset}} + \hat{T}_h + T_{\text{ghost}}
\]

was shown to be a BRST-trivial operator.

### 2.2 \( A_\gamma \) generators in terms of the gauged WZW fields.

In section 2.1 we reviewed coset description for general supersymmetric gauged WZW models. Here we recall how the coset description for the \( N = 4 \) gauged WZW models is used to realize the generators of the left-moving \( A_\gamma \) algebra.

The \( A_\gamma \) algebra consists of Virasoro current \( T(z) \), four dimension 3/2 supersymmetry generators \( G^{ai}(z) \), six generators \( A^{\pm,i} \) of the current algebra \( su(2)_+ \oplus su(2)_- \), four dimension 1/2 fermions \( Q^a(z) \) and a dimension 1 boson \( U(z) \). The central charge of this algebra is
parametrized by two natural numbers $k_+, k_-$ and has the form $c = \frac{6k_+ k_-}{k_+ + k_-}$. The OPE’s of $A_\gamma$ algebra are given in Appendix B.

The realization of $A_\gamma$ algebra for the $N = 4$ coset starts from promoting the supersymmetry current (2.15) into the four supersymmetry generators. We will omit the subscript “coset” from now on:

$$G^a = \frac{2}{k^4} \left[ \Psi^A J^a_{AB} E^B - \frac{ik}{3k} S^a_{ABC} :\Psi^A \Psi^B \Psi^C : \right]$$

(2.16)

where $J^a_{AB} = (J^i_{AB}, \delta_{AB})$, $a = (i, 4)$, $i = 1, 2, 3$. The objects $J^i_{AB}$ and $S^a_{ABC}$ will be determined shortly.

Using OPE’s (2.12)(2.13) of the currents $E^A(z), \Psi^A(z)$ we find:

$$G^a(z) G^b(w) \sim \frac{4}{k^2} \left\{ \frac{k/4}{(z-w)^3} \left[ n J^a_{AB} J^b_{AB} + \frac{1}{3} S^a_{ABC} S^b_{ABC} \right] + \frac{1}{(z-w)^2} \left[ \frac{n}{2} J^a_{AC} J^b_{BC} :\Psi^A \Psi^E : + ik J^a_{AC} J^b_{AE} H^{CEB} E^B + \frac{1}{2} S^a_{BCA} S^b_{ABC} :\Psi^A \Psi^E : \right] + \frac{i}{2} J^a_{CA} J^b_{BC} :E^A E^{\check{B}}: \Psi^A \Psi^B : + \frac{i}{4} J^a_{AC} J^b_{A} H^{\check{C}E} \partial \hat{E}^{\check{D}} - i S^{(a}_{AED} J^b_{E} ) A^B E^B :\Psi^E \Psi^D : + \right\}$$

(2.17)

We have to compare (2.17) with the corresponding relation in $A_\gamma$ algebra:

$$G^a(z) G^b(w) \sim \frac{2c}{3} \frac{\delta^{ab}}{(z-w)^3} - \frac{4k_- t^{-i}_{ab} A^{+i}(w) + 4k_+ t^{+i}_{ab} A^{-i}(w)}{k(z-w)^2}$$

$$\frac{-2k_- t^{-i}_{ab} \partial A^{+i}(w) + 2k_+ t^{+i}_{ab} \partial A^{-i}(w)}{k(z-w)} + \frac{2\delta^{ab} T(w)}{z-w}$$

(2.18)

where $t^{\pm i}_{ab} = \pm 2 \delta^i_{[a} \delta^i_{b]} + \epsilon_{iab}$ and

$$c = \frac{6k_+ k_-}{k_+ + k_-}$$

Let us first look at the terms $\sim \frac{1}{z-w}$ which are symmetric in indices $a, b$. Comparing (2.17) with (2.18) gives the following constraints:

$$J^{(a}_{AB} J^{b)}_{AC} = \delta^{ab} \delta_{AC}, \quad J^{(a}_{AC} J^{b)}_{B} = \delta^{ab} \delta_{AB}$$

(2.19)

$$J^{(a}_{AC} J^{b}_{BD} H^{CD} E^E - J^{(a}_{E} S^{b)}_{DAB} = \delta^{ab} F_{EAB}$$

(2.20)

$$J^{k}_{CD} H^{D}_{BI} = J^{kD}_{B} H^{CDI}, \quad k = 1, 2, 3$$

(2.21)

$$S^{(a}_{D[AB} S^{b)}_{EF]} = \delta^{ab} P_{ABEF}, \quad S^{(a}_{AB} S^{b)}_{EF} = \delta^{ab} v_{EF}$$

(2.22)
where tensors $F_{EAB}, P_{ABEF}, \nu_{EF}$ will be determined in a moment.

The constraints (2.19) require that $J^i_{AB}$ are almost complex structures on the coset, i.e. they are anti-symmetric\(^1\)

$$J^i_{AB} = -J^i_{BA} \tag{2.23}$$

and satisfy the algebra of imaginary quaternions

$$J^i_A B^j B^C = \varepsilon_{ijk} J^k_A C - \delta^i_j \delta^A_C \tag{2.24}$$

Let us consider (2.20). Taking first $a = i$ and $b = 4$ we find

$$S^i_{ABC} = H_{ABC}, \quad S^i_{ABC} = 3 J^i_A [A H_{BC}] D \tag{2.25}$$

Next, consider $a = 4, b = 4$ to obtain $F_{EAB} = 0$ and $a = i, b = j$ to find

$$3 J^i_C [A J^j_D B H_E]_{AB} = H_{CDE} \delta_{ij} \tag{2.26}$$

Note that (2.26) is the condition for integrability of the complex structures $J^i$. In [15] there is an explicit expression for the triplet $J^i$ satisfying (2.26). From this condition follows that $S^a_{ABC}$ can be also recast as

$$S^a_{ABC} = J^a_{A} D^1 J^a_{B} D^2 J^a_{C} D^3 H_{D_1 D_2 D_3} \tag{2.27}$$

From (2.21)(2.25) and using also:

$$H_{A[BC} H^A_{EF]} + H_{I[BC} H^H_{EF]} = 0 \tag{2.28}$$

$$H_{ABC} H^{ABD} = c_g \delta^D_C - 2 H_{IBC} H^{1BD} \tag{2.29}$$

we check that conditions (2.22) are true with

$$P_{ABEF} = H_{D[AB} H^{D}_{EF]}, \quad \nu_{EF} = H_{ECD} H_{F}^{CD} \tag{2.30}$$

Let us now look at the term $\sim \frac{1}{(z-w)^3}$ in (2.17). From (2.27) we find

$$S^a_{ABC} S^b_{ABC} = 3 d_X \delta^{ab} \tag{2.31}$$

We used that for the $N = 4$ coset $X = G/H$

$$c_h = c_g - 2, \quad d_X = 4(c_g - 1), \quad \text{dim} H = c_h^2 - 1$$

and thus

$$H_{ABC} H^{ABC} = \left( c_g d_X - 2(c_g - c_h) \text{dim} H \right) = 3d_X \tag{2.32}$$

From (2.31) we find that the term $\sim \frac{1}{(z-w)^3}$ has the correct structure as in (2.18) with central charge:

$$c = \frac{6k_+ k_-}{k}, \quad k_- = c_g - 1, \quad k_+ = n + 1 \tag{2.33}$$

\(^1\)Recall that we do not distinguish upper and lower indices.
Finally, let us consider the term \( \sim \frac{1}{(z-w)^2} \) in (2.17). We use the following helpful formulae:

\[
J_{AB}^a J_{BC}^b = -(t^{+i})_{ab} J_{BC}^i, \quad J_{EC}^a J_F^b = -(t^{-i})_{ab} J_{EF}^i
\]  
(2.34)

\[
S_{BC[A}^a S_{BC]E}^b = -(t^{-i})_{ab} J_{AE}^i + (t^{+i})_{ab} M_{AE}^i, \quad M_{AE}^i = S_{BC[A}^i H_{BC}]^E - J_{AE}^i
\]  
(2.35)

to find the correct structure as in (2.18) with

\[
A^{-i} = \frac{1}{2k_+} J_{AE}^i : \Psi^A \Psi^E :
\]
(2.36)

\[
A^{+i} = \frac{i}{2k_-} \left[ J_{AE}^i H_{AED}^i E_D + \frac{\varepsilon_{ijk} A^{+ik}}{k_-} : \Psi^A \Psi^E : \right]
\]  
(2.37)

Now let us check the OPE’s of \( su(2)_+ \oplus su(2)_- \) currents:

\[
A^{\pm i}(z) A^{\pm j}(w) \sim -\frac{k_{\pm}}{2} \frac{\delta^{ij}}{z-w} + \frac{\varepsilon_{ijk} A^{\pm ik}}{z-w}
\]
(2.38)

In order to prove (2.38), it is convenient to bring \( M_{AB}^i \) into the form

\[
M_{AB}^i = k_- J_{AB}^i - h_C^i H_{AB}^C, \quad h_F^i = J_{AB}^i H_{ABF}, \quad i = 1, 2, 3
\]  
(2.39)

Note that (2.39) follows from the general expression (2.35) by using the property of the \( N = 4 \) cosets

\[
J^k_{AB} H_{AB}^i = 0, \quad k = 1, 2, 3
\]  
(2.40)

This property is due to the specific choice of a subgroup \( H \) in the construction of the \( N = 4 \) cosets \( G/H \). We recall that for these cosets \( G = SU(N + 2) \) or \( G = G' \otimes U(1) \) with \( G' \) a simple group different from \( SU(M) \). A subgroup \( H \) is chosen in such a way that its simple roots are orthogonal to the highest root of \( SU(N + 2) \) or \( G' \).

Then, OPE’s (2.38) are reproduced correctly due to the following properties of \( h_F^i \):

\[
h_A^i h_B^j H_{ABC}^k = -2k_- \varepsilon_{ijk} h_C^k
\]  
(2.41)

\[
h_C^i h_C^k = 4k_-^2 \delta_{ik}, \quad h_D^i h_F^j H_{DAB}^F H_{AB}^F = 4k_-^2 (k_- + 1) \delta_{ik}
\]  
(2.42)

Checking the rest of the OPE’s of \( A_\gamma \) algebra (see Appendix B), determines

\[
Q^b = \frac{i}{2k_-} h_F^b : \Psi^F :
\]
(2.43)

\[
U = \frac{i}{k_-} h_F^b \left( E_F^b - \frac{i}{k} H_{FED}^F : \Psi^E \Psi^D : \right)
\]
(2.44)

where we denote

\[
h_F^b = (h_F^i, h_F^i), \quad i = 1, 2, 3, \quad h_F^i = -J_{EF}^i C h_C^i \forall i
\]  
(2.45)

and \( h_F^i \) is defined in (2.39).

Note that second term in (2.44) is non-zero only for the case \( G = SU(N + 2) \). For the other choice \( G = G' \otimes U(1) \) one finds that \( U \) is a generator of the \( U(1) \) factor.

As an example of the realization of \( A_\gamma \) algebra, we present explicit expressions for \( J_{AB}^i, H_{ABC} \) and \( h_F^a, M_{AB}^i \) for the case of \( SU(3) \) WZW model in Appendix C.
3 The ground states in a massless R-sector $A_\gamma$ module.

In this section we review some of the results [8, 9, 10] about RR characters of $A_\gamma$ algebra and the structure of the ground states in a massless R-sector $A_\gamma$ module. This information will be used in Section 4 to identify the special states contributing to the index $I_2$.

Let $\mathcal{H}_{RR}$ be a representation of the RR sector algebra\(^2\) $A_\gamma^{\text{left}} \oplus A_\gamma^{\text{right}}$. The RR sector $A_\gamma$ character is defined by [10]:

$$Z(q, z_\pm; \bar{q}, \bar{z}_\mp) := Tr_{\mathcal{H}_{RR}} \frac{2i A_0^{-3}}{Z_0} \frac{2i A_0^{+3}}{Z_0} q^{L_0-c/24}$$

(3.1)

For a general unitary theory with $A_\gamma$ symmetry $Z$ has a form:

$$Z = \sum_{l,\overline{l},u} \sum_{l_+\overline{l}_-,u_\overline{u}} \sum_{l_-\overline{l}_+} \mathcal{S}Ch_{0}^{A_\gamma, R}(l_+, \overline{l}_-, u; q, z_\pm) \mathcal{S}Ch_{0}^{A_\gamma, R}(\overline{l}_+, l_-; q, z_\pm) + \ldots$$

(3.2)

where the contribution of the massless R-sector representation $r = (l_+, l_-, u)$ is given by

$$\mathcal{S}Ch_{0}^{A_\gamma, R}(l_+, l_-, u; q, z_\pm) = Tr_{r} \frac{2i A_0^{-3}}{Z_0} \frac{2i A_0^{+3}}{Z_0} q^{L_0-c/24},$$

(3.3)

and $\ldots$ stands for the massive characters which are irrelevant for us since, as shown in [6], they do not contribute to the index $I_2$.

The $A_\gamma$ character of the module $r = (l_+, l_-, u)$ can be written in a product form:

$$\mathcal{S}Ch_{0}^{A_\gamma, R}(l_+, l_-, u; q, z_\pm) = \mathcal{S}^{R}(u; q, z_\pm) \times \mathcal{S}Ch_{0}^{A_\gamma, R}(\overline{l}_+, \overline{l}_-; q, z_\pm)$$

(3.4)

Here $\mathcal{S}^{R}(u; q, z_\pm)$ denotes the R-sector character of the model $S$ which is a theory of four Majorana fermions and one free boson. The model $S$ is the simplest theory with $A_\gamma$ symmetry.

$\mathcal{S}Ch_{0}^{A_\gamma, R}(\overline{l}_+, \overline{l}_-; q, z_\pm)$ is the character of the massless R-sector representation $\overline{r} = (\overline{l}_+, \overline{l}_-)$ of the non-linear algebra $\tilde{A}_\gamma$ [24] and $\overline{l}_\pm = l_\pm - \frac{1}{2}$. We give explicit expressions for the characters $\mathcal{S}^{R}(u; q, z_\pm)$ and $\mathcal{S}Ch_{0}^{A_\gamma, R}(\overline{l}_+, \overline{l}_-; q, z_\pm)$ in Appendix D.

In what follows we will use the expressions [24] of the generators of the non-linear algebra $\tilde{A}_\gamma$ in terms of the $A_\gamma$ generators:

$$\tilde{T} = T + \frac{1}{k} U^2 + \frac{1}{k} \partial Q^a Q^a$$

(3.5)

$$\tilde{G}^a = G^a + \frac{2}{3k^2} \epsilon_{abcd} Q^b Q^c Q^d + \frac{2}{k} Q^b [t_{ba}^{+;i} \tilde{A}^{-;i} - t_{ba}^{-;i} \tilde{A}^{+;i}]$$

(3.6)

$$\tilde{A}^{\pm;ij} = A^{\pm;ij} - \frac{1}{2k} (t^{\pm;ij})^{ab} Q_a Q_b$$

(3.7)

\(^2\)We do not impose any GSO projection.
The product structure (3.4) implies the decomposition of the ground states in the module

\[ r = (l_+, l_-, u) \]

as

\[ |V\rangle \otimes |f_s\rangle \otimes |u\rangle \]  

(3.8)

Here \(|f_s\rangle \otimes |u\rangle\), \(s = 1, \ldots, 4\) are R ground states of the model \(S\), while \(|V\rangle\) are the ground states in the R-sector module \(\tilde{r} = (\tilde{l}_+, \tilde{l}_-)\) of the \(\tilde{A}_r\) algebra. (By \(|u\rangle\) we denote the ground state of the boson and by \(|f_s\rangle\) the R ground states of the four fermions in the model \(S\).)

As was shown in [10], the states \(|V\rangle\) form two irreducible representations of zero-mode algebra \(\tilde{A}_0^{\pm; i}\). The first one, let us call it \(|V_1\rangle\), is built by acting with operators \(\tilde{A}_0^{\pm; -}\) on the highest weight state \(|\tilde{\Omega}\rangle\) defined by the following equations

\[ \tilde{G}_0^+ |\tilde{\Omega}\rangle = 0, \quad K^+ |\tilde{\Omega}\rangle = 0, \]

\[ \tilde{A}_0^{\pm; +} |\tilde{\Omega}\rangle = 0, \quad i \tilde{A}_0^{\pm; 3} |\tilde{\Omega}\rangle = \tilde{l}_\pm |\tilde{\Omega}\rangle \]  

(3.9)

where \(\tilde{l}_\pm = l_\pm - \frac{1}{2}\) and we use the notations:

\[ \tilde{G}_0^\pm = \tilde{G}_0^{1 \pm} i \tilde{G}_0^{2 \pm}, \quad K^\pm = \tilde{G}_0^{3 \pm}, \quad \tilde{A}_0^{\pm; +} = \tilde{A}_0^{\pm; 1 \pm} i \tilde{A}_0^{\pm; 2 \pm} \]  

(3.10)

The second representation \(|V_2\rangle\) is built by acting with operators \(\tilde{A}_0^{\pm; -}\) on the state \(\tilde{G}_0^- |\tilde{\Omega}\rangle\) which has the properties:

\[ \tilde{A}_0^{\pm; +} \tilde{G}_0^- |\tilde{\Omega}\rangle = 0, \quad i \tilde{A}_0^{\pm; 3} \tilde{G}_0^- |\tilde{\Omega}\rangle = (\tilde{l}_\pm - \frac{1}{2}) \tilde{G}_0^- |\tilde{\Omega}\rangle \]  

(3.11)

Note, that from unitarity and (3.9) follows:

\[ \tilde{L}_0 |\tilde{\Omega}\rangle = \tilde{h} |\tilde{\Omega}\rangle, \quad \tilde{h} = \frac{(k_+ - 1)(k_- - 1)}{4k} + \frac{(\tilde{l}_+ + \tilde{l}_-)(\tilde{l}_+ + \tilde{l}_- + 1)}{k} \]  

(3.12)

Recalling also conformal dimensions of \(|f_s\rangle\) and \(|u\rangle\):

\[ h_f = \frac{1}{4}, \quad h_u = \frac{u^2}{k} \]

we compute the conformal dimension of the total R groundstate (3.8):

\[ h_{R, ground} = \tilde{h} + h_f + u = \frac{c}{24} + \frac{\mu^2}{4k} + \frac{u^2}{k} \]

where \(\mu = 2(l_+ + l_-) - 1 = 2(\tilde{l}_+ + \tilde{l}_-) + 1\).

### 4 States contributing to the index \(I_2\).

In this section we study the structure of the index \(I_2\). We identify the special RR ground states whose orbits under spectral flow generate the contribution of a massless \(r \otimes \tilde{r}\) module to the index. We will give a geometric interpretation for these states in terms of differential forms on the target space in Section 5.
First, we recall some results [6] about the structure of the index $I_2$. Using (1.1) and (3.2) the large N=4 index $I_2$ can be written as:

$$I_2 = \sum_r \sum_{\tau} N_r \tau \text{Ind}_r(q, z) \text{Ind}_{\tau}(\overline{q}, \overline{z})$$  \hspace{1cm} (4.1)$$

where $\text{Ind}_r(q, z)$ stands for the contribution of the left-moving massless R-sector $A_\gamma$ module $r = (l_+, l_-, u)$

$$\text{Ind}_r(q, z) := \left[ z_+ \frac{d}{dz_+} \text{SCh}^{A\gamma_R}_{0} l_+ l_-, u; q, z_\pm \right]_{z_- = -z, z_+ = z}$$  \hspace{1cm} (4.2)$$

and $D_0^3 = i(A_0^{+3} + A_0^{-3})$.

The factorized form (3.4) of the character $\text{SCh}^{A\gamma_R}_{0} l_+ l_-, u; q, z_\pm$ and explicit expressions for the two factors (see Appendix D) allow to compute

$$\text{Ind}_r(q, z) = (-)^{2\gamma-1} q^{u^{2}/k} \Theta_{\mu,k}(q, z), \hspace{0.5cm} \mu = 2(l_+ + l_-) - 1. \hspace{1cm} (4.3)$$

From (4.1)(4.3) and the structure of the odd theta function $\Theta_{\mu,k}(q, z)$ immediately follows that states in the module $r \otimes \tau$ which contribute to the index $I_2$ satisfy:

$$2D_0^3 = \pm \mu + 2km, \hspace{0.5cm} L_0 - c/24 = \frac{u^2}{k} + \frac{(D_0^3)^2}{k}, \hspace{0.5cm} m \in \mathbb{Z} \hspace{1cm} (4.4)$$

$$2D_0^3 = \pm \overline{\tau} + 2k\overline{m}, \hspace{0.5cm} \overline{L}_0 - c/24 = \frac{\overline{m}^2}{k} + \frac{(D_0^3)^2}{k}, \hspace{0.5cm} \overline{m} \in \mathbb{Z} \hspace{1cm} (4.5)$$

where $D_0^3 = i \left( A_0^{+3} + A_0^{-3} \right)$, $\overline{D}_0^3 = i \left( \overline{A}_0^{+3} + \overline{A}_0^{-3} \right)$ and $\overline{\mu} = 2(\overline{l}_+ + \overline{l}_-) - 1$.

Next, we make an important observation. The conditions (4.4)(4.5) are invariant under symmetric spectral flow with parameters$^3 \rho = 2n, \overline{\rho} = 2\overline{n}$:

$$L_0^{(2n, 2n)} = L_0 - 2nD_0^3 + k\overline{n}, \hspace{0.5cm} D_0^{(2n, 2n)} = D_0^3 - k\overline{n}, \hspace{0.5cm} n \in \mathbb{Z} \hspace{1cm} (4.6)$$

$$L_0^{(2\overline{n}, 2\overline{n})} = \overline{L}_0 - 2\overline{n}\overline{D}_0^3 + k\overline{n}, \hspace{0.5cm} \overline{D}_0^{(2\overline{n}, 2\overline{n})} = \overline{D}_0^3 - k\overline{n}, \hspace{0.5cm} \overline{n} \in \mathbb{Z} \hspace{1cm} (4.7)$$

There are 16 ground states in the module $r \otimes \tau$ which satisfy (4.4)(4.5) for $m = 0, \overline{m} = 0$ and generate the contribution of this module to the index by means of spectral flow (4.6)(4.7).

These special RR ground states are

$$|s, \overline{s}\rangle = |g_s\rangle \otimes |\overline{g}_s\rangle, \hspace{0.5cm} s = 1, \ldots, 4, \hspace{0.5cm} \overline{s} = 1, \ldots, 4 \hspace{1cm} (4.8)$$

where $|g_s\rangle$ is one of the following special ground states in the left-moving module

$$|g_1\rangle = |\overline{\Omega}\rangle \otimes |f_1\rangle \otimes |u\rangle, \hspace{0.5cm} |g_2\rangle = |\overline{\Omega}\rangle \otimes |f_2\rangle \otimes |u\rangle \hspace{1cm} (4.9)$$

$$|g_3\rangle = \left( i\overline{\Omega}^{+;+} \right)^{2i} \left( i\overline{\Omega}^{-;-} \right)^{2i} |\overline{\Omega}\rangle \otimes |f_3\rangle \otimes |u\rangle, \hspace{0.5cm} |g_4\rangle = \left( i\overline{\Omega}^{+;+} \right)^{2i} \left( i\overline{\Omega}^{-;-} \right)^{2i} |\overline{\Omega}\rangle \otimes |f_4\rangle \otimes |u\rangle \hspace{1cm} (4.10)$$

$^3$Symmetric spectral flow with parameter $\rho$ acts as $L_0^{(\rho, \rho)} = L_0 - \rho D_0^3 + \frac{k\rho^2}{4}$, $D_0^{(\rho, \rho)} = D_0^3 - \frac{k\rho}{2}$
In (4.9)(4.10) the state $|\tilde{\Omega}\rangle$ is defined in (3.9) and R ground states $|f_s\rangle$ of the free fermions are specified as

$$
\left(Q_0^1 + iQ_0^2\right) |f_s\rangle = 0, \quad s = 1, 2,
\left(Q_0^1 - iQ_0^2\right) |f_s\rangle = 0, \quad s = 3, 4
$$

$$
\left(Q_0^3 + iQ_0^4\right) |f_s\rangle = 0, \quad s = 1, 4,
\left(Q_0^3 - iQ_0^4\right) |f_s\rangle = 0, \quad s = 2, 3
$$

(4.11)

The four right-moving states $|\tilde{g}_s\rangle$ which enter the definition of $|s, \bar{s}\rangle$ have analogous form.

The states $|s, \bar{s}\rangle$ have conformal dimensions

$$
h_{R,\text{ground}} = \frac{c}{24} + \frac{\mu^2}{4k} + \frac{u^2}{k}, \quad \overline{h}_{R,\text{ground}} = \frac{c}{24} + \frac{\overline{\mu}^2}{4k} + \frac{\overline{u}^2}{k}$$

(4.12)

and the following $iA_0^{\pm;3}$ eigenvalues

$$
m_{+}^{(1)} = l_+, \quad m_{-}^{(1)} = l_+ - \frac{1}{2}, \quad m_{+}^{(2)} = l_+ - \frac{1}{2}, \quad m_{-}^{(2)} = l_-,
$$

$$
m_{+}^{(3)} = -l_+, \quad m_{-}^{(3)} = -(l_- - \frac{1}{2}), \quad m_{+}^{(4)} = -(l_+ - \frac{1}{2}), \quad m_{-}^{(4)} = -l_-
$$

with analogous expressions for $iA_0^{\pm;3}$.

5 Geometric interpretation of the index $I_2$.

In the previous section we considered the contribution to the index $I_2$ from massless RR-sector $A_r$ module $r \otimes \mathfrak{7}$. We found that this contribution comes from spectral flow orbits of the special RR ground states $|s, \bar{s}\rangle = |g_s\rangle \otimes |\tilde{g}_s\rangle$.

Here we first clarify the geometric meaning of the left-moving states $|g_s\rangle$. Then, we combine left and right moving sectors and give a geometric interpretation of the index $I_2$.

In Section 5.1 we characterize the states $|g_s\rangle$ as Dirac spinors on the target space $X$ specified by a system of differential and algebraic equations (5.19)-(5.32).

In Section 5.2 we describe the special RR states $|s, \bar{s}\rangle$ in terms of differential forms on $X$ defined in (5.36)-(5.55).

In Section 5.3 we show that coefficients of leading terms in the index $I_2$ are sums

$$
\sum_{p_1, p_2} (-)^{p_1+p_2} n(p_1, p_2)
$$

where $n(p_1, p_2)$ is the number of $(p_1, p_2)$ forms that solve equations (5.61)-(5.70).

This is similar in spirit to the geometric description [25] of the leading contribution to the elliptic genus but the forms counted in the index $I_2$ are more special. In particular, they have either the highest or the lowest weight under the action of $su(2)_+ \oplus su(2)_-$ algebra on differential forms on the target space.
5.1 Geometric meaning of the special R ground states $|g_s\rangle$.

Here we give a geometric description of the states $|g_s\rangle$, which are left-moving parts of the special RR ground states found in Section 4.

To clarify the geometric meaning of the states $|g_s\rangle$, we proceed in the following way. First, we find the constraints on $|g_s\rangle$ in terms of the zero modes of $A_γ$ generators. Then, by representing the zero modes in terms of vector fields on the target space $X$ and Dirac matrices, we recast the constraints on $|g_s\rangle$ as a system of differential and algebraic equations on a Dirac spinor on $X$.

From the definition (4.9) of the states $|g_s\rangle$ for $s = 1, 2$ in terms of $|\hat{Ω}\rangle$, $|f_s\rangle$ we find

$$\tilde{G}^+_0|g_s\rangle = 0, \quad \tilde{G}^3_0|g_s\rangle = 0, \quad \tilde{G}^4_0|g_s\rangle = 0, \quad s = 1, 2 \quad (5.1)$$

and

$$\left( Q^1_0 + iQ^3_0 \right) |g_s\rangle = 0, \quad s = 1, 2, \quad \left( Q^3_0 + iQ^4_0 \right) |g_1\rangle = 0, \quad \left( Q^3_0 - iQ^4_0 \right) |g_2\rangle = 0. \quad (5.2)$$

As a consequence of (5.1)(5.2), $|g_s\rangle$ for $s = 1, 2$ also satisfy

$$A_0^{±;+}|g_s\rangle = 0, \quad G_0^+|g_s\rangle = 0, \quad s = 1, 2 \quad (5.3)$$

where we denote

$$A_0^{±;+} = A_0^{±;1} + iA_0^{±;2}, \quad G_0^+ := G_0^1 + iG_0^2.$$

Analogously, states $|g_s\rangle$ for $s = 3, 4$ (4.10) are specified by

$$\tilde{G}^-_0|g_s\rangle = 0, \quad \tilde{G}^3_0|g_s\rangle = 0, \quad \tilde{G}^4_0|g_s\rangle = 0 \quad (5.4)$$

and

$$\left( Q^1_0 - iQ^3_0 \right) |g_s\rangle = 0, \quad s = 3, 4, \quad \left( Q^3_0 + iQ^4_0 \right) |g_3\rangle = 0, \quad \left( Q^3_0 - iQ^4_0 \right) |g_4\rangle = 0. \quad (5.5)$$

As a consequence of (5.4)(5.5), $|g_s\rangle$ for $s = 3, 4$ also satisfy

$$A_0^{±;−}|g_s\rangle = 0, \quad G_0^-|g_s\rangle = 0, \quad s = 3, 4 \quad (5.6)$$

where

$$A_0^{±;−} := A_0^{±;1} - iA_0^{±;2}, \quad G_0^- := G_0^1 - iG_0^2.$$

In order to understand the geometric meaning of the constraints (5.1)-(5.6) on the states $|g_s\rangle$ we will represent zero modes of the operators of the $A_γ$ algebra in geometric terms. We need zero modes of the currents $E_{\hat{A}}(z)$ and $Ψ_{\hat{A}}(z)$, which were used in the realization of $A_γ$ generators in section 2.2. From the OPE (2.12) we find commutation relation for zero modes

$$[E_{\hat{0}}\hat{A}, E_{\hat{0}}\hat{B}] = iH_{\hat{A}\hat{B}\hat{C}}E_{\hat{0}}\hat{C} \quad (5.7)$$

so that $E_{\hat{0}}\hat{A}$ can be represented in terms of vector field on the group manifold $G$.

$$E_{\hat{0}}\hat{A} = -\frac{i\sqrt{k}}{2}E_{\hat{A}}\hat{M}\partial_{\hat{M}} \quad (5.8)$$
Here $X^\hat{M}$ are coordinates on $G$ and $E_A^\hat{M}$ is the inverse of the veilbein (see Appendix A). It is possible to choose the coordinates $X^M = (x^M, \phi^m)$ in such a way that $x^M, M = 1, \ldots, 4k_-$ parametrize the right coset space $X$ and $\phi^m, m = 1, \ldots, \dim H$ are coordinates on $H$. Then, $E_A^m = 0$ and $E_A^0, A = 1, \ldots, 4k_-$ are vector fields on $X$.

Note that $E_A^0$ do not form a closed Lie algebra but, using the complex structures $J_{AB}$, there are three different ways to split them into the two subsets which do so. We choose $J_{AB}^3$ and divide the generators $T_A, A = 1, \ldots, 4k_-$ into two groups $T_\alpha$ and $T_{2k_-+\alpha}$ such that $J_{2k_-+\alpha}^3 = 1, \forall \alpha = 1, \ldots, 2k_-$. Then, complex vector fields

$$V_\alpha = E_0 2k_- + \alpha - iE_0 \alpha, \quad V_\overline{\alpha} = E_0 2k_- + \alpha + iE_0 \alpha$$

(5.9)

form Lie algebras $N^\pm$:

$$[V_\alpha, V_\beta] = \mathcal{F}_{\alpha\beta}^\gamma V_\gamma, \quad [V_\alpha, V_\overline{\beta}] = \mathcal{F}_{\alpha\overline{\beta}} V_\gamma$$

(5.10)

with structure constants:

$$\mathcal{F}_{\alpha\overline{\beta}} = -H_{\alpha 2k_- + \beta} 2k_- + \gamma + H_{\beta 2k_- + \alpha} 2k_- + \gamma + iH_{\alpha 2k_- + \beta} \gamma - iH_{\beta 2k_- + \alpha} \gamma$$

(5.11)

\[\mathcal{F}_{\alpha\beta}^\gamma = -\left(\mathcal{F}_{\alpha\overline{\beta}}\right)^*\]

Next, we realize zero modes of fermions $\Psi^A$ in terms of hermitean Dirac matrices on $X$

$$\Psi_0^A = \frac{\sqrt{k}}{2} \Gamma^A, \quad A = 1, \ldots, 4k_-$$

(5.12)

and define their complex linear combinations:

$$\gamma^\alpha = \frac{1}{2} \left(\Gamma^{2k_- + \alpha} + i\Gamma^\alpha\right), \quad \overline{\gamma} = \frac{1}{2} \left(\Gamma^{2k_- + \alpha} - i\Gamma^\alpha\right)$$

\{\gamma^\alpha, \overline{\gamma}\} = \delta^\alpha \overline{\beta}$$

(5.13)

$$\rho^\alpha = iJ_{\alpha\overline{\beta}} \overline{\gamma}, \quad \overline{\rho}^\alpha = -iJ_{\overline{\alpha}\beta} \gamma$$

(5.14)

where $J_{\alpha\overline{\beta}} = J_{\alpha\overline{\beta}}^1 + iJ_{\alpha\overline{\beta}}^2$ and $J_{\overline{\alpha}\beta} = J_{\overline{\alpha}\beta}^1 - iJ_{\overline{\alpha}\beta}^2$.

The above representation (5.8)-(5.14) of $E_A^0, \Psi_0^A$ allows us to express zero modes of the supersymmetry currents $G^\alpha(z)$ (2.16) in terms of vector fields $V_\alpha, V_\overline{\alpha}$ and Dirac matrices as:

$$G_0^3 + iG_0^4 = \frac{2i}{\sqrt{k}} \left[\overline{\gamma} V_\alpha + \frac{1}{2} \mathcal{F}_{\alpha\overline{\beta}}^\gamma \gamma^\beta\overline{\gamma}\right]$$

(5.15)

$$G_0^3 - iG_0^4 = -\frac{2i}{\sqrt{k}} \left[\gamma^\alpha V_\alpha + \frac{1}{2} \mathcal{F}_{\alpha\overline{\beta}}^\gamma \gamma^\beta\gamma^\delta\overline{\gamma}\right]$$

(5.16)

$$G_0^1 + iG_0^2 = \frac{2i}{\sqrt{k}} \left[\rho^\alpha V_\alpha + \frac{1}{2} \mathcal{F}_{\alpha\overline{\beta}} \rho^\beta\gamma^\delta\overline{\gamma}\right]$$

(5.17)

$$G_0^1 - iG_0^2 = -\frac{2i}{\sqrt{k}} \left[\rho^\alpha V_\alpha + \frac{1}{2} \mathcal{F}_{\alpha\overline{\beta}} \rho^\beta\overline{\gamma}\right]$$

(5.18)
Here and below the indices are lowered with $\delta_{\alpha\beta}$ or $\delta_{\alpha\bar{\beta}}$ and raised with $\delta^{\alpha\bar{\beta}}$ or $\delta^{\alpha\beta}$.

Let us now find the geometric description of the special states $|g_s\rangle$ (4.9)(4.10) relevant for the index. For this purpose, we express the constraints (5.1)-(5.6) on $|g_s\rangle$ in terms of vector fields $V_\alpha, V_{\bar{\alpha}}$ (5.9) on $X$ and Dirac matrices (5.13),(5.14). In this way the states $|g_s\rangle, s = 1,\ldots,4$ are considered as Dirac spinors on $X$ defined by the following system of equations.

I. For all $s = 1,\ldots,4$ the spinors $|g_s\rangle$ satisfy Dirac-like equations constructed out of Dirac matrices $\gamma^\alpha, \gamma^{\bar{\alpha}}$:

\[
\left[ \gamma^{\bar{\alpha}}(V_\alpha - i_\alpha) + \frac{1}{2} F_{\alpha\beta\delta} \gamma^{\delta} \gamma^{\alpha\bar{\beta}} \right] |g_s\rangle = 0 \tag{5.19}
\]
\[
\left[ \gamma^\alpha(V_{\bar{\alpha}} - i_\alpha) + \frac{1}{2} F_{\alpha\beta\delta} \gamma^{\alpha\bar{\beta}} \gamma^{\delta} \right] |g_s\rangle = 0 \tag{5.20}
\]

where

\[
i_\alpha = -\frac{i}{2k_-} \mathcal{F}_\alpha (\epsilon - iu), \quad i_\alpha = \frac{i}{2k_-} t_\alpha (\epsilon + iu) \quad \epsilon = m_+ + m_-
\]

and

\[
t_\alpha = h_\alpha^3 + i h_{2k_- + \alpha}, \quad \mathcal{F}_\alpha = (t_\alpha)^*, \quad \alpha = 1,\ldots,2k_-, \quad h_3^A = J^A_{BC} H_{ABC}
\]

II. For different $s = 1,\ldots,4$ there are different algebraic constraints on the spinors $|g_s\rangle$:

\[
x_\alpha \gamma^{\alpha} |g_s\rangle = 0, \quad s = 1,2 \quad \mathcal{F}_\alpha \gamma^{\alpha} |g_s\rangle = 0, \quad s = 3,4 \tag{5.23}
\]
\[
t_\alpha \gamma^{\alpha} |g_s\rangle = 0, \quad s = 2,3 \quad \mathcal{F}_\alpha \gamma^{\alpha} |g_s\rangle = 0, \quad s = 1,4 \tag{5.24}
\]

where

\[
x_\alpha = h_\alpha^1 + i h_\alpha^2, \quad \mathcal{F}_\alpha = (x_\alpha)^*
\]

III. For $s = 1,2$ ($s = 3,4$) the spinors $|g_s\rangle$ have the highest (lowest) weight under the action of $su(2)_+ \oplus su(2)_-$ algebra:

\[
J^+_{\alpha\beta} \gamma^{\alpha\beta} |g_s\rangle = 0, \quad y^\delta V_\delta |g_s\rangle = 0, \quad s = 1,2 \tag{5.26}
\]
\[
J^-_{\alpha\beta} \gamma^{\alpha\beta} |g_s\rangle = 0, \quad y^\delta V_\delta |g_s\rangle = 0, \quad s = 3,4 \tag{5.27}
\]

where $y^\delta = \mathcal{F}_\delta, \quad \bar{y}^\delta = x_\delta$.

The corresponding weights are $i \Lambda^\pm_0 |g_s\rangle = m_\pm |g_s\rangle$. These eigenvalue equations lead to an algebraic constraint on the spinors

\[
\frac{1}{2} (-k_- + \sum_{\alpha=1}^{2k_-} \gamma^{\alpha\bar{\alpha}}) |g_s\rangle = m_\pm |g_s\rangle \tag{5.28}
\]
and specify the action of the vector field \( f^\alpha V_\alpha - f^{\bar{\alpha}} \bar{V}_{\bar{\alpha}} \)

\[
\frac{i}{4k_-} \left[ \vec{f}_{\bar{\alpha}} \vec{V}_{\alpha} - f^\delta V_\delta + (\vec{f}_\delta \vec{F}_{\bar{\alpha} \beta} + f^\delta F_{\delta \bar{\alpha} \bar{\beta}}) \gamma^\alpha \gamma^{\bar{\beta}} \right] |g_s\rangle = \left( \frac{k_-}{2} + m_+^{(s)} + m_-^{(s)} \right) |g_s\rangle \tag{5.29}
\]

where \( f^\alpha = t^\alpha, \ \ f^\alpha = t^\alpha \).

IV. The spinors \(|g_s\rangle\) satisfy one more Dirac-like equation constructed out of Dirac matrices \( \rho^\alpha, \rho^{\bar{\alpha}} \)

\[
\left[ \rho^{\bar{\alpha}} V_\alpha + \frac{1}{2} F_{\alpha \bar{\beta}} \rho^{\alpha \bar{\beta}} \right] |g_s\rangle = 0 \quad s = 1, 2 \tag{5.30}
\]

\[
\left[ \rho^\alpha V_\alpha + \frac{1}{2} F_{\alpha \beta} \rho^{\alpha \beta} \right] |g_s\rangle = 0 \quad s = 3, 4 \tag{5.31}
\]

V. For all \( s = 1, \ldots, 4 \) the spinors \(|g_s\rangle\) have eigenvalue \( u \) under the action of \( u(1) \) generator \( iU \).

\[
-\frac{1}{4k_-} \left[ \vec{f}_{\alpha} \vec{V}_{\bar{\alpha}} - f^\delta V_\delta + (\vec{f}_\delta \vec{F}_{\alpha \bar{\beta}} - f^\delta F_{\delta \alpha \bar{\beta}}) \gamma^\alpha \gamma^{\bar{\beta}} \right] |g_s\rangle = u |g_s\rangle \tag{5.32}
\]

5.2 Geometric description of the special RR ground states \(|s, \bar{s}\rangle\).

In Section 4 we identified the RR ground states \(|s, \bar{s}\rangle\) which generate the contribution to the index \( I_2 \) of the massless \( A_\gamma \) module \( r \otimes \bar{r} \). Here we find a geometric description of these special states in terms of \((p_1, p_2)\) forms on the target space \( X \).

In order to characterize the special states \(|s, \bar{s}\rangle\) we use the left-moving \( A_\gamma \) operators written in geometric terms in Section 5.1 and supply analogous expressions for the right moving ones. Before we formulate our geometric description of the states \(|s, \bar{s}\rangle\), let us clarify the new ingredients of the construction.

The geometric realization of the four supersymmetry generators \( \overline{\mathcal{G}}_0^a \) in the right-moving sector involves vector fields \( W_\alpha, \bar{W}_{\bar{\alpha}} \) defined as follows.

\[
W_\alpha = \mathcal{E}_0^{2k_- + \alpha} - i \mathcal{E}_0^\alpha, \quad \bar{W}_{\bar{\alpha}} = \mathcal{E}_0^{2k_- + \alpha} + i \mathcal{E}_0^\alpha, \quad \alpha = 1, \ldots, 2k_- \tag{5.33}
\]

Here \( \mathcal{E}_0^A, \ A = 1, \ldots, 4k_- \) are zero modes of the currents \( \mathcal{E}_A^\bar{A}(\bar{z}) = \frac{k}{\sqrt{2}} (g^{-1} \partial g)_{\bar{A}}^\bar{A} \). They are represented as vector fields on the group manifold \( G \) as:

\[
\mathcal{E}_0^{\hat{A}} = -i \sqrt{\frac{k}{2}} \mathcal{E}_A^{\hat{A}} \partial_M, \quad \hat{A} = 1, \ldots, \dim G \tag{5.34}
\]

In (5.34) \( \mathcal{E}_A^{\hat{A}} \) is the inverse of the matrix \( \mathcal{E}_M^{\hat{A}} = -i \sqrt{\frac{k}{2}} (g^{-1} \partial_M g)^{\hat{A}}_{\hat{A}} \) and \( g \) is a parametrization of the group manifold \( G \). We also recall that in Section 5.1 we have chosen a parametrization in such a way that \( x^M, M = 1, \ldots, 4k_- \) are coordinates on the right coset \( X \) and \( \phi^m, 1, \ldots, \dim H \) are coordinates on the subgroup \( H \). In this way \( \mathcal{E}_A^\alpha = 0 \) and \( \mathcal{E}_0^A \) become vector fields on \( X \).
The other new ingredient, that appears when we combine left and right sectors, are Dirac matrices $\gamma^A$, $\gamma^\alpha$ which represent zero modes of the right-moving fermions.

Now we give a geometric interpretation of the states $|s, \bar{s}\rangle$ relevant for the index $I_2$ in terms of differential forms on the target space $X$:

$$|s, \bar{s}\rangle = \frac{1}{p_1 p_2} C_{\alpha_1 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p} (s) \gamma^{\alpha_1 \ldots \alpha_p} \bar{\beta}_1 \ldots \bar{\beta}_p |hw\rangle, \quad \gamma^\alpha |hw\rangle = 0, \quad \bar{\gamma}^\alpha |hw\rangle = 0 \quad (5.35)$$

Here $C_{\alpha_1 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p}$ are components of a $(p_1, p_2)$ form $C_{\alpha \bar{\beta}}$ on $X$ (with respect to a basis of veilbein 1-forms).

For all $s, \bar{s} = 1, \ldots, 4$ the forms $C_{\alpha \bar{\beta}}$ satisfy

$$\mathcal{D} C_{\alpha \bar{\beta}} = \nu^s \wedge C_{\alpha \bar{\beta}}, \quad \mathcal{D}^\dagger C_{\alpha \bar{\beta}} = i w^s C_{\alpha \bar{\beta}} \quad (5.36)$$

where symbol $i w^s$ denotes the interior product with vector $w$ and (anti-)holomorphic differential $(\bar{\mathcal{D}}) \mathcal{D}$ acts on a $(p_1, p_2)$ form as follows:

$$(\mathcal{D} C_{\alpha \bar{\beta}})_{\alpha_1 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p} := V_{\alpha_1} \left(C_{\alpha_2 \ldots \alpha_p+1 \bar{\beta}_1 \ldots \bar{\beta}_p}\right) + \frac{p_1}{2} F_{\alpha_1 \alpha_2} C_{\delta \alpha_3 \ldots \alpha_p+1 \bar{\beta}_1 \ldots \bar{\beta}_p}$$

$$(\mathcal{D}^\dagger C_{\alpha \bar{\beta}})_{\alpha_1 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p} := V^\beta \left(C_{\alpha \alpha_2 \ldots \alpha_p-1 \bar{\beta}_1 \ldots \bar{\beta}_p}\right) - \frac{p_1 - 1}{2} F^\delta_{\alpha \alpha_2 \ldots \alpha_p-1} C_{\delta \alpha_3 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p}$$

$$(\bar{\mathcal{D}} C_{\alpha \bar{\beta}})_{\alpha_1 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p} := W_{\alpha_1} \left(C_{\alpha_2 \ldots \alpha_p+1 \bar{\beta}_1 \ldots \bar{\beta}_p}\right) + \frac{p_2}{2} F_{\alpha_1 \bar{\beta}_2} C_{\alpha \alpha_2 \ldots \alpha_p+1 \bar{\beta}_1 \ldots \bar{\beta}_p}$$

$$(\bar{\mathcal{D}}^\dagger C_{\alpha \bar{\beta}})_{\alpha_1 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p} := W^\beta \left(C_{\alpha \alpha_2 \ldots \alpha_p-1 \bar{\beta}_1 \ldots \bar{\beta}_p}\right) - \frac{p_2 - 1}{2} F^\delta_{\alpha \alpha_2 \ldots \alpha_p-1} C_{\alpha \alpha_3 \ldots \alpha_p \bar{\beta}_1 \ldots \bar{\beta}_p}$$

Here $F_{\alpha \beta} \delta$ (5.11) is torsion and vector fields $V_{\alpha}, W_{\alpha}, V_{\bar{\alpha}}, W_{\bar{\alpha}}$ are defined in (5.9),(5.33).

The 1-forms $\nu^s, \nu^\alpha, \bar{\nu}^\alpha$ and vectors $w^s, \bar{w}^\alpha$ have the following flat components:

$$\nu^s = \frac{i}{2 k_-} t_\alpha (\epsilon^s + i u), \quad \nu^\alpha = -\frac{i}{2 k_-} f^\alpha (\epsilon^\alpha - i u) \quad (5.39)$$

$$w^s = -\frac{i}{2 k_-} f^\alpha (\epsilon^s - i u), \quad \bar{w}^\alpha = \frac{i}{2 k_-} t_\alpha (\epsilon^\alpha + i u) \quad (5.40)$$

where $\epsilon^s = m^s_+ + m^s_-$, $\bar{\epsilon}^\alpha = \bar{m}^\alpha_+ + \bar{m}^\alpha_-$ and $t_\alpha = h^3_\alpha + i h^3_{2k_-+\alpha}$, $f^\alpha = h^3_\alpha - i h^3_{2k_-+\alpha}$.

We recall also the definition of $h^3_A = (h^3_\alpha, h^3_{2k_-+\alpha})$ in terms of torsion and complex structures $h^3_F = J^s AB H_{ABF}$.

The degrees of the forms $C_{\alpha \bar{\beta}}$ describing RR ground states $|s, \bar{s}\rangle$ are given by:

$$p_1^s = k_- + 2m_-^s, \quad p_2^s = k_- - 2\bar{m}_-^s \quad (5.41)$$
The structure of the form \( C_{(p_1, p_2)}^{(s, \bar{s})} \) is further constrained as

\[
C_{(p_1, p_2)}^{(s, \bar{s})} = *(J^-, J^+) B_{(2k_--p_1, 2k_--p_2)}^{(s)}
\]

(5.42)

where \( B_{(2k_--p_1, 2k_--p_2)}^{(s)} \) is a \((2k_--p_1, 2k_--p_2)\) form which is in the kernel of \( \mathcal{D}^\dagger \) for \( s = 1, 2 \) \((\mathcal{D} \text{ for } s = 3, 4)\) as well as in the kernel of \( \mathcal{D}^\dagger \) for \( \bar{s} = 1, 2 \) \((\bar{\mathcal{D}}^\dagger \text{ for } \bar{s} = 3, 4)\).

The symbol \( *(J^-, J^+) \) stands for Hodge-like duality operation which maps \((n_1, n_2)\) forms into \((2k_--n_1, 2k_--n_2)\) forms and is defined in components as

\[
\left( *(J^-, J^+) B_{(n_1, n_2)} \right)_{\alpha_1 \ldots \alpha_{2k_--n_1} \beta_1 \ldots \beta_{2k_--n_2}} = \kappa_{n_1} \kappa_{n_2} B_{\delta_1 \ldots \delta_{n_1} \bar{\alpha}_1 \ldots \bar{\alpha}_{n_2}} \times
\]

(5.43)

\[
J^- \delta_1 \beta_1 \ldots J^- \delta_{n_1} \beta_{n_1} J^+ \delta_1 \bar{\beta}_1 \ldots J^+ \delta_{n_2} \bar{\beta}_{n_2} \epsilon_{\beta_1 \ldots \beta_{n_1} \alpha_1 \ldots \alpha_{2k_--n_1}} \epsilon_{\bar{\beta}_1 \ldots \bar{\beta}_{n_2} \bar{\alpha}_1 \ldots \bar{\alpha}_{n_2}}
\]

where \( \kappa_{n_1} = (-)^{n_1+1(i)n_1} n_1! \), \( \kappa_{n_2} = (-)^{n_2+1(-i)n_2} n_2! \) and \( J^- \alpha \beta = J_{\alpha \beta}^1 - iJ_{\alpha \beta}^2 \), \( J^+ \bar{\alpha} \bar{\beta} = J_{\bar{\alpha} \bar{\beta}}^1 + iJ_{\bar{\alpha} \bar{\beta}}^2 \).

The forms \( C_{(p_1, p_2)}^{(s, \bar{s})} \) are further specified by a number of equations. Below we list these constraints for different values of \( s, \bar{s} \), i.e. for the forms describing different states \( |s, \bar{s} > \) (5.35):

\[
x \wedge C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad (5.44)
\]

s = 1, 2

\[
J^+_{[\alpha \beta} C_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2]}^{(s, \bar{s})} = 0, \quad \bar{\gamma} \delta V_{\delta}^{\bar{s}}(C^{(s, \bar{s})}_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2}}) = 0 \quad (5.45)
\]

s = 3, 4

\[
i y C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad (5.46)
\]

\[
J^- \alpha \beta C_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2}}^{(s, \bar{s})} = 0, \quad \gamma \delta V_{\delta}^{\bar{s}}(C^{(s, \bar{s})}_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2}}) = 0 \quad (5.47)
\]

\[
\bar{s} = 1, 2
\]

\[
i \bar{y} C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad (5.48)
\]

\[
J^+ \bar{\alpha} \bar{\beta} C_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2}}^{(s, \bar{s})} = 0, \quad \bar{\gamma} \bar{\delta} W_{\bar{\delta}}^{\bar{s}}(C^{(s, \bar{s})}_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2}}) = 0 \quad (5.49)
\]

\[
\bar{s} = 3, 4
\]

\[
x \wedge C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad (5.50)
\]

\[
J^-_{[\alpha \beta} C_{\beta_1 \ldots \beta_{p_2]}^{(s, \bar{s})} = 0, \quad \gamma \delta W_{\delta}^{\bar{s}}(C^{(s, \bar{s})}_{\alpha_1 \ldots \alpha_{p_1} \beta_1 \ldots \beta_{p_2}}) = 0 \quad (5.51)
\]

\[
i f C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad \text{for} \quad s = 1, 4, \quad t \wedge C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad \text{for} \quad s = 2, 3, \quad (5.52)
\]

\[
\bar{t} \wedge C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad \text{for} \quad \bar{s} = 1, 4, \quad i \bar{f} C_{(p_1, p_2)}^{(s, \bar{s})} = 0 \quad \text{for} \quad \bar{s} = 2, 3 \quad (5.53)
\]

where the flat components of the 1-forms \( x, \bar{x}, t, \bar{t}\) and vectors \( y, \bar{y}, f, \bar{f}\) are given by

\[
x_{\alpha} = h_{\alpha}^1 + ih_{\alpha}^2, \quad t_{\alpha} = h_{\alpha}^3 + ih_{2k--)_{\alpha}^3, \quad y_{\alpha} = h_{\alpha}^1 - ih_{\alpha}^2, \quad f_{\alpha} = h_{\alpha}^3 - ih_{2k--)_{\alpha}^3 \quad (5.54)
\]
\[ \mathbf{x}^\alpha = (x^\alpha)^*, \quad \mathbf{t}^\alpha = (t^\alpha)^*, \quad \mathbf{y}^\alpha = (y^\alpha)^*, \quad \mathbf{f}^\alpha = (f^\alpha)^* \]

We also recall the definition of \( h_i^A = (h_i^1, h_i^{2k_+ + \alpha}) \), \( \alpha = 1, \ldots, 2k_- \) in terms of torsion and complex structures \( h_i^A = J^i AB H_{ABF} \).

Note that equations (5.45)(5.49) state that the form \( C^{(s, \overline{s})}_{(p_1, p_2)} \) for \( s = 1, 2 \) or \( \overline{s} = 1, 2 \) is the highest weight state under the action of \( su(2)_+ \otimes su(2)_- \) on \( (*, p_2) \) or \( (p_1, *) \) forms on \( X \). Analogously, conditions (5.47)(5.51) imply that the form \( C^{(s, \overline{s})}_{(p_1, p_2)} \) for \( s = 3, 4 \) or \( \overline{s} = 3, 4 \) is the lowest weight state. The corresponding \( su(2)_- \) weights \( iA^{-3} = m_-(s) \), \( i\overline{A}^{-3} = m_-(\overline{s}) \) specify the degrees (5.41) of the form. The following equations ensure that the \( su(2)_+ \) weights are \( iA^{+3} = m_+(s) \), \( i\overline{A}^{+3} = m_+(\overline{s}) \) and the \( u(1) \) eigenvalues are \( iU = u \), \( i\overline{U} = \overline{u} \).

\[
\mathbf{f}^\delta \left( V_{\delta}(C^{(s, \overline{s})}_{\alpha_1 \ldots \alpha_{p_1} \overline{\beta}_1 \ldots \overline{\beta}_{p_2}}) + p_1 \mathbf{F} \right)_{\delta \alpha_1} C^{(s, \overline{s})}_{\alpha_2 \ldots \alpha_{p_1} \overline{\beta}_1 \ldots \overline{\beta}_{p_2}} = -ik_- (\epsilon^{(s)} + 2(\epsilon^{(s)} - iu)) C^{(s, \overline{s})}_{\alpha_1 \ldots \alpha_{p_1} \overline{\beta}_1 \ldots \overline{\beta}_{p_2}}
\]

\[
\mathbf{f}^\delta \left( W_{\delta}(C^{(s, \overline{s})}_{\overline{\beta}_1 \ldots \overline{\beta}_{p_2} \alpha_1 \ldots \alpha_{p_1}}) + p_2 \mathbf{F} \right)_{\delta \beta_1} C^{(s, \overline{s})}_{\beta_2 \ldots \beta_{p_2} \alpha_1 \ldots \alpha_{p_1}} = ik_- (\epsilon^{(s)} - 2(\overline{\epsilon^{(\overline{s})}} - i\overline{u})) C^{(s, \overline{s})}_{\beta_1 \ldots \beta_{p_2} \alpha_1 \ldots \alpha_{p_1}}
\]

where \( \epsilon^{(s)} = m_+^{(s)} + m_-^{(s)} \), \( \overline{\epsilon^{(s)}} = m_+^{(s)} + m_-^{(s)} \).

### 5.3 The geometric meaning of the leading terms in the index

In Section 5.2 we found the geometric interpretation of the RR states \(|s, \overline{s}\rangle \) whose orbits under spectral flow (4.6)(4.7) generate the contribution of \( r \otimes \overline{r} \) module to the index \( I_2 \). These states were described as \((p_1, p_2)\) forms on the target space specified by the system of equations (5.36)-(5.55).

Now we clarify the geometric meaning of the leading terms in the index \( I_2 \). Ignoring the contribution of the excited states in each of the \( r \otimes \overline{r} \) modules, we recast the expression (4.1) for the index as

\[
I_2 = \sum_u \sum_{\overline{\mu}} q^{k_1 \overline{k}} q^{k_1 \overline{k} - 1} \sum_{\mu = 1}^{k_1} q^{k_1 \overline{k} - 1} \left( z^{\mu} \overline{z}^{\overline{\mu}} d_{++}(\mu, \overline{\mu}) + z^{\mu} \overline{z}^{\overline{\mu}} d_{--}(\mu, \overline{\mu}) \right) + \ldots
\]

In (5.56) we denote:

\[
d_{++}(\mu, \overline{\mu}) = \sum_{p_1 = \max(\mu + k_- + k_+, k_-)}^\min(k_- + k_-, 2k_- - 1) \sum_{p_2 = \max(\overline{\mu} - 1, 1)}^\min(k_- + k_-, \overline{k}_- + 1) (-)^{p_1 + p_2} n(p_1, +; p_2, +) \]
\[ d_{-+}(\mu, \overline{\nu}) = - \sum_{p_1 = \text{max}(\mu + k_+ - k_-)}^{\text{min}(k_+, \mu - 1, 2k_+, 2k_- - 1)} \sum_{p_2 = \text{max}(\mu + k_- - k_+, k_-)}^{\text{min}(k_-, k_+-1, 2k_-, 2k_- - 1)} (-)^{p_1 + p_2} n(p_1, +; p_2, -) \quad (5.58) \]

\[ d_{+-}(\mu, \overline{\nu}) = - \sum_{p_1 = \text{max}(k_+, k_- - 1)}^{\text{min}(k_-, k_-)} \sum_{p_2 = \text{max}(\mu + k_- - k_+, k_-)}^{\text{min}(k_-, k_+-1, 2k_-, 2k_- - 1)} (-)^{p_1 + p_2} n(p_1, -; p_2, +) \quad (5.59) \]

\[ d_{-=}(\mu, \overline{\nu}) = \sum_{p_1 = \text{max}(\mu + k_+ - k_-)}^{\text{min}(k_+, \mu - 1, 2k_+, 2k_- - 1)} \sum_{p_2 = \text{max}(\mu + k_- - k_+, k_-)}^{\text{min}(k_-, k_+-1, 2k_-, 2k_- - 1)} (-)^{p_1 + p_2} n(p_1, -; p_2, -) \quad (5.60) \]

Here \( n(p_1, +; p_2, +) \) is the number of \((p_1, p_2)\) forms which have the following special properties:

**I.**

\[ \mathcal{D}C = \nu \wedge C, \quad \mathcal{D}^i C = 0, \quad \overline{\mathcal{D}} C = 0, \quad \overline{\mathcal{D}}^i C = i_{\overline{w}} C \quad (5.61) \]

where (anti-)holomorphic differential \( \overline{\mathcal{D}} \) with torsion is defined in (5.38). The symbol \( i_{\overline{w}} \) denotes the interior product with vector \( \overline{w} \). The 1-form \( \nu \) and vector \( \overline{w} \) have flat components

\[ \nu_\alpha = \frac{i}{2k_-} t_\alpha (\mu + iu), \quad \overline{w}_\alpha = i \overline{f^\alpha} (\overline{\mu} + i\overline{\nu}) \quad (5.62) \]

where \( \overline{f^\alpha} = t_\alpha = h_\alpha^{3} + ih_\alpha^{3k_+ + \alpha}, \quad \alpha = 1, \ldots, 2k_- \) and we recall that \( h_\alpha^i = (h_\alpha^{i}, h_\alpha^{j_2k_+ + \alpha}) \)

is defined as \( h_\alpha^i = J^i BC H_{ABC} \).

**II.**

\[ C = \star_{(J^{-1}, J^+)} B, \quad \mathcal{D}^i B = 0, \quad \overline{\mathcal{D}} B = 0 \quad (5.63) \]

where Hodge-like duality operation \( \star_{(J^{-1}, J^+)} \) is defined in (5.43).

**III.**

\[ J^{[\alpha \beta \gamma C_{\alpha_1 \ldots \alpha_{p_1}} \beta_{1 \ldots \beta_{p_1}} \gamma_{p_2}} = 0, \quad \overline{f^\gamma} V_\gamma (C_{\alpha_1 \ldots \alpha_{p_1}} \beta_{1 \ldots \beta_{p_1}} \gamma_{p_2}) = 0, \]

\[ J^{+ \alpha \beta \gamma \lambda C_{\alpha_1 \ldots \alpha_{p_1}} \beta_{1 \ldots \beta_{p_1}} \gamma_{p_2} \alpha_1 \ldots \alpha_{p_1} \lambda} = 0, \quad \overline{f^\nu} W_\nu (C_{\beta_{1 \ldots \beta_{p_1}} \gamma_{p_2} \alpha_1 \ldots \alpha_{p_1}}) = 0 \quad (5.64) \]

where vector fields \( V_\alpha, V_\overline{\alpha}, W_\alpha, W_\overline{\alpha} \) are given in (5.9, 5.33) and \( \overline{f^\alpha} = h_\alpha^{1} + ih_\alpha^{2} \).

**IV.**

\[ x \wedge C = 0, \quad i_{\overline{g}} C = 0, \quad \overline{t} \wedge C = 0, \quad i_{f} C = 0 \quad (5.65) \]

where 1-forms \( x, \overline{t} \) and vectors \( \overline{g}, f \) have flat components

\[ x_\alpha = \overline{f^\alpha}, \quad \overline{t}_\alpha = f_\alpha = h_\alpha^{3} - ih_\alpha^{3k_+ + \alpha} \]
The equations in group III state that the form counted in \( n(p_1, +; p_2, +) \) has the highest weight under the action of \( su(2)_+ \oplus su(2)_- \) algebra on \((*, p_2)\) and \((p_1, *)\) forms on \(X\). The corresponding weights are:

\[
iA^{-;3}_0 = \frac{p_1 - k_-}{2}, \quad \overline{iA^{-;3}}_0 = \frac{k_- - p_2}{2}, \quad iA^{+;3}_0 = \frac{\mu - p_1 + k_-}{2}, \quad \overline{iA^{+;3}}_0 = \frac{\overline{\mu} - k_- + p_2}{2}
\]

The eigenvalues of the form under the \( u(1) \) action are

\[
iU = u, \quad \overline{iU} = \overline{u}
\]

These eigenvalue equations have the structure (5.55) with substitution \( \epsilon^{(s)} \rightarrow \mu, \quad \overline{\epsilon^{(s)}} \rightarrow \overline{\mu} \)

Analogously, \( n(p_1, +; p_2, -) \) is the number of \((p_1, p_2)\) forms which have the following special properties:

I.

\[
\mathcal{D} C = \nu' \wedge C, \quad \mathcal{D}^\dagger C = 0, \quad \overline{\mathcal{D}} C = \overline{\nu'} \wedge C, \quad \overline{\mathcal{D}}^\dagger C = 0
\]

where

\[
\nu'_{\alpha} = \frac{i}{2k_-} t_{\alpha}(\mu + iu), \quad \overline{\nu'}_{\alpha} = -\frac{i}{2k_-} f^\alpha(-\overline{\mu} - i\overline{u}),
\]

II.

\[
C = *(J^-, J^+) B', \quad \mathcal{D}^\dagger B' = 0, \quad \overline{\mathcal{D}}^\dagger B' = 0
\]

III.

\[
J^+_{[\alpha_\beta]} C_{\alpha_1...\alpha_{p_1}} \overline{\beta_1...\beta_{p_2}} = 0, \quad \overline{g}^\delta V^\gamma(C_{\alpha_1...\alpha_{p_1}} \overline{\beta_1...\beta_{p_2}}) = 0,
\]

\[
j^-_{[\alpha_\beta]} C_{\overline{\alpha_1...\alpha_{p_1}} \beta_1...\beta_{p_2}} \overline{\alpha_1...\alpha_{p_1}} = 0, \quad g^\delta W^\gamma(C_{\overline{\beta_1...\beta_{p_2}} \alpha_1...\alpha_{p_1}}) = 0
\]

IV.

\[
x \wedge C = 0, \quad \overline{x} \wedge C = 0, \quad i\overline{f} C = 0, \quad i f C = 0
\]

The equations in group III state that the form counted in \( n(p_1, +; p_2, -) \) has the highest weight under the action of \( su(2)_+ \oplus su(2)_- \) algebra on \((*, p_2)\) forms and the lowest weight under the action on \((p_1, *)\) forms. The corresponding weights are:

\[
iA^{-;3}_0 = \frac{p_1 - k_-}{2}, \quad \overline{iA^{-;3}}_0 = \frac{k_- - p_2}{2}, \quad iA^{+;3}_0 = \frac{\mu - p_1 + k_-}{2}, \quad \overline{iA^{+;3}}_0 = \frac{\overline{\mu} - k_- + p_2}{2}
\]

The eigenvalues of the form under the \( u(1) \) action are

\[
iU = u, \quad \overline{iU} = \overline{u}
\]

These eigenvalue equations have the structure (5.55) with substitution \( \epsilon^{(s)} \rightarrow \mu, \quad \overline{\epsilon^{(s)}} \rightarrow -\overline{\mu} \)

The other two quantities \( n(p_1, -; p_2, +) \) and \( n(p_1, -; p_2, -) \) can be defined in a similar way.
6 Conclusion

The main result of this paper is a geometric interpretation of the index $I_2$ for the $N = 4$ gauged WZW models. In particular, we showed that the states contributing to the index $I_2$ belong to spectral flow orbits (4.6)(4.7) of special RR ground states. We characterized these states geometrically as $(p_1, p_2)$ forms on the target space specified by equations (5.36)-(5.55).

Besides, we showed that the coefficients of the various leading terms in the index can be obtained by counting with $(-1)^{p_1+p_2}$ sign the numbers $n(p_1, p_2)$ of $(p_1, p_2)$ forms with certain properties (5.61)-(5.70). This is similar in spirit to the geometric description of the leading contribution to the elliptic genus but the forms counted in the index $I_2$ are more special. In particular, they have either the highest or the lowest weight under the action of $su(2)_+ \oplus su(2)_-$ algebra on differential forms on the target space.

Although we have used the structure of the $N = 4$ cosets in realization of $A_\gamma$ algebra, our results are formulated in a rather general form in terms of the torsion and the triplet of complex structures on the target space. Therefore, it is a natural question if our geometric interpretation is valid for arbitrary $\sigma$-models with $A_\gamma$ symmetry. It is hard to answer this question. The major problem stems from the fact that realization of $A_\gamma$ algebra is known only for $N = 4$ gauged WZW models. One can construct new theories with $A_\gamma$ symmetry by considering symmetric product orbifolds of these WZW models, but then one has to deal with another problem of smoothing the target space into a manifold by resolving the orbifold singularities. Finding a general $\sigma$-model with $A_\gamma$ symmetry remains an open problem.

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7 Appendix A

Here we give the component form of the action for the supersymmetric gauged WZW model. As we review in section 2 the path integral for this theory takes the form:

$$Z = \int [dG][dH][dB][dC][d\bar{B}][d\bar{C}]e^{-\int d^2\theta \left(I(G) - I(H)\right) - S_{\text{ghost}}}$$

where the $\sigma$-model fields are arranged in superfields as:

$$G = g (1 + \theta \psi + \overline{\theta} g^{-1} \overline{\psi} g - \overline{\theta} \overline{\psi} g^{-1} \overline{\psi} g)$$

$$H = h (1 + \theta \eta + \overline{\theta} h^{-1} \overline{\eta} h - \overline{\theta} \overline{\eta} h^{-1} \overline{\eta} h)$$

where $g(h)$ is a map from Riemann surface $\Sigma$ to the group $G(H)$. The fermions $\psi$, $\overline{\psi}(\eta, \overline{\eta})$ take values in the Lie algebra of $G(H)$.

The component form of the action is given by:

$$S = I_{\text{bos}}(g) - I_{\text{bos}}(h) + I_{\text{ferm}}(\psi, \overline{\psi}, g) - I_{\text{ferm}}(\eta, \overline{\eta}, h) + S_{\text{ghost}}$$
where
\[ I_{bos}(g) = -\frac{k}{8\pi} \left\{ \int_{\Sigma} d^2 z T r' \left[ g^{-1} \partial g^{-1} \partial g \right] + \int_{\Sigma} dt d^2 z T r' \left( \tilde{g}^{-1} \partial \tilde{g} \left[ \tilde{g}^{-1} \partial \tilde{g}, \tilde{g}^{-1} \partial \tilde{g} \right] \right) \right\} \tag{7.75} \]

and
\[ I_{ferm}(\psi, \bar{\psi}, g) = \frac{k}{8\pi} \int_{\Sigma} d^2 z \left\{ T r' \left( \psi \partial g \bar{\psi} \right) + T r' \left( \bar{\psi} \partial g \psi \right) \right\} \tag{7.76} \]

Here \( \nabla_g = \partial - [\partial g^{-1}, -] \), \( \nabla_g = \bar{\partial} - [\partial g, -] \) are the covariant derivatives in the adjoint representation of the Lie algebra of \( G \).

Introducing coordinates \( X^M \) on \( G \) we define a veilbein:
\[ E^A_M = -i \sqrt{\frac{k}{2}} (\partial_M g g^{-1})^A, \quad E^A_M E^B_N = \delta^A_B \tag{7.77} \]

so that \( I_{bos}(g) \) has a standard \( \sigma \)-model form (with \( \alpha' = 1 \))
\[ I_{bos}(g) = \frac{1}{2\pi} \int_{\Sigma} d^2 z \left( g_{MN} + B_{MN} \right) \partial X^M \partial X^N \tag{7.78} \]

Here \( g_{MN} = E^A_M E^B_N \delta_{AB} \) and \( B_{MN} \) is defined as
\[ \frac{1}{\sqrt{k}} E^A_M E^B_N E^C_P H_{ABC} = \frac{3}{2} \partial_M B_{NP} \tag{7.79} \]

## 8 Appendix B

Here we summarize the OPE’s of the \( A_\gamma \) algebra:
\[ G^a(z) G^b(w) \sim \frac{2c}{3} \frac{\delta^{ab}}{(z - w)^3} - \frac{4k_- t_{ab}^+ i A^{+i}(w) + 4k_+ t_{ab}^- i A^{-i}(w)}{k(z - w)^2} \tag{8.80} \]
\[ - \frac{2k_- t_{ab}^+ \partial A^{+i}(w) + 2k_+ t_{ab}^- \partial A^{-i}(w)}{k(z - w)} + \frac{2\delta^{ab} T(w)}{z - w} \]
\[ A^\pm(z) A^\pm(w) \sim -\frac{k_\pm}{2} \frac{\delta^{ij}}{(z - w)^2} + \frac{\varepsilon_{ijk} A^k}{z - w} \tag{8.81} \]
\[ Q^a(z) G^b(w) \sim \frac{\delta^{ab} U(w)}{(z - w)} + \frac{t_{ab}^i A^{+i}(w) - t_{ab}^- i A^{-i}(w)}{(z - w)} \tag{8.82} \]
\[ A^\pm(z) G^a(w) \sim \frac{k_\pm}{k} \frac{t_{ab}^i Q^b(w)}{(z - w)^2} + \frac{1}{2} \frac{t_{ab}^i G^b(w)}{z - w} \tag{8.83} \]
\[ A^\pm(z) Q^a(w) \sim \frac{1}{2} \frac{t_{ab}^i Q^b(w)}{(z - w)} \tag{8.84} \]
\[ Q^a(z)Q^b(w) \sim -\frac{k}{2} \frac{\delta^{ab}}{(z-w)} \] (8.85)

\[ U(z)G^a(w) \sim \frac{Q^a(w)}{(z-w)^2} \] (8.86)

\[ U(z)U(w) \sim -\frac{k}{2} \frac{1}{(z-w)^2} \] (8.87)

where \( t^i_{ab} = \pm 2 \delta^{i}_{[a} \delta^i_{b]} + \epsilon_{iab} \) and

\[ c = \frac{6k_+ k_-}{k}, \quad k = k_+ + k_- \]

Comment: The name \( A_{\gamma} \) originated from the fact that the OPE’s of the algebra can be parametrized by two parameters \( k \) and \( \gamma = \frac{k_-}{k} \).

9 Appendix C

Here we give explicit expressions for \( H_{ABC}, J^i_{AB}, h^i_F, M^i_{AB} \) in the example of \( SU(3) \) WZW model. The hermitean generators of \( su(3) \) are:

\[
T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (9.88)

\[
T^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \] (9.89)

\[
T^5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \] (9.90)

\[
T^7 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T^8 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (9.91)

They satisfy \( [T^A, T^B] = if_{ABC}T^C \), \( TrT^A T^B = \delta^{AB} \).

The non-zero components of torsion \( H_{ABC} = \frac{1}{\sqrt{2}} f_{ABC} \):

\[
H_{127} = -\frac{1}{2}, \quad H_{128} = -\frac{\sqrt{3}}{2}, \quad H_{347} = -\frac{1}{2}, \quad H_{348} = \frac{\sqrt{3}}{2}, \] (9.92)

\[
H_{135} = -\frac{1}{2}, \quad H_{146} = -\frac{1}{2}, \quad H_{245} = \frac{1}{2}, \quad H_{236} = -\frac{1}{2}, \quad H_{567} = -1 \] (9.93)

The non-zero components of complex structures \( J^i_{AB} \):

\[
J_{12}^3 = -1, \quad J_{34}^3 = -1, \quad J_{56}^3 = -1, \quad J_{78}^3 = -1 \] (9.94)
The non-zero components of $h^0_F$ defined in (2.45):

$$h^1_F = -4\delta_{F,6}, \quad h^2_F = 4\delta_{F,5}, \quad h^3_F = 4\delta_{F,7}, \quad h^4_F = -4\delta_{F,8}$$

(9.97)

The non-zero components of $M^i_{AB}$ defined in (2.39):

$$M^3_{78} = -2, \quad M^3_{56} = 2, \quad M^1_{57} = 2, \quad M^1_{68} = 2$$

$$M^2_{58} = -2, \quad M^2_{67} = 2$$

(9.98)

(9.99)

10 Appendix D

Here we give explicit expressions for the characters which appear in Section 3.

The $A_\gamma$ character of the model $S$ is given by:

$$\mathcal{S}^R(u; q, z_\pm) = q^{u^2/k+1/8} F^R(q, z_\pm) \times \prod_{n=1}^\infty (1 - q^n)^{-1}(z_+ + z_+ z_+^{-1} + z_-^{-1})$$

(10.100)

where

$$F^R(q, z_\pm) = \prod_{n=1}^\infty (1 + z_+ z_- q^n)(1 + z_+^{-1} z_- q^n)(1 + z_+ z_-^{-1} q^n)(1 + z_+ z_-^{-1} q^n)$$

(10.101)

The character of the massless R-sector representation $\tilde{r} = (\tilde{l}_+, \tilde{l}_-)$ of the $\tilde{A}_\gamma$ algebra has the following form:

$$Ch^0_{A_\gamma, R} (\tilde{l}_+, \tilde{l}_-; q, z_\pm) = q^{\tilde{h}-\tilde{c}/24} F^R(q, z_\pm) \times$$

$$\prod_{n=1}^\infty (1 - q^n)^{-2}(1 - z_+^2 q^n)^{-1}(1 - z_+^{-2} q^n)^{-1}(1 - z_-^2 q^n)^{-1}(1 - z_-^{-2} q^n)^{-1}(1 - z_+ z_-^{-1} q^n)^{-1}(1 - z_+ z_- q^n)^{-1}(1 - z_+^{-1} z_- q^n)^{-1}(1 - z_+^{-1} z_-^{-1} q^n)^{-1} \times$$

$$\sum_{m, n = -\infty}^\infty q^{nk_+ + 2nk_+ + m^2 k_- + 2m l_-} \sum_{\epsilon_+, \epsilon_- = \pm 1} \epsilon_+ \epsilon_-^{2\epsilon_+ (\tilde{l}_+ + nk_+) \tilde{l}_- (\tilde{l}_- + mk_-)} (z_+^{\epsilon_+} q^{-n} + z_-^{\epsilon_-} q^{-m})^{-1}$$

(10.102)

where

$$\tilde{h} = \frac{(k_+ - 1)(k_- - 1)}{4k} + \frac{(\tilde{l}_+ + \tilde{l}_-)(\tilde{l}_+ + \tilde{l}_- + k_+)}{k},$$

$$\tilde{c} = \frac{6k_+ k_-}{k} - 3, \quad k = k_+ + k_-.$$
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