On an inverse problem for a fractional semilinear elliptic equation involving a magnetic potential

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ABSTRACT. We study a class of fractional semilinear elliptic equations and formulate the corresponding Calderón problem. We determine the nonlinearity from the exterior partial measurements of the Dirichlet-to-Neumann map by using first order linearization and the Runge approximation property.

1 Introduction

The study of the fractional Calderón problem was initiated in [10] where the authors considered an inverse problem for the fractional linear operator

\[ (-\Delta)^s + q \quad (0 < s < 1). \]

See [1, 2, 3, 4, 8, 9, 23] for further studies based on [10].

Recently a fractional semilinear Calderón problem has been studied in [18]. This inverse problem can be viewed as a nonlocal analogue of the classical semilinear Calderón problem studied in [13]. In [18], the authors considered the exterior Dirichlet problem

\[ (-\Delta)^s u + a(x, u) = 0 \text{ in } \Omega, \quad u|_{\Omega_e} = g \]

where \( \Omega \) is a bounded domain with \( C^{1,1} \) boundary and \( \Omega_e := \mathbb{R}^n \setminus \bar{\Omega}, \ n \geq 2 \). Under some regularity assumptions on \( a(\cdot, \cdot) \), the authors proved that the nonlinearity \( a(\cdot, \cdot) \) can be uniquely determined from the exterior partial measurements of the Dirichlet-to-Neumann map

\[ \Lambda_a : g \to (-\Delta)^s u_g|_{\Omega_e}. \]

In this paper, we extend the earlier result in [18]. We study the generalized operator \( \mathcal{R}^s_A \), which is formally defined by

\[ \mathcal{R}^s_A u(x) := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (u(x) - R_A(x, y)u(y))K(x, y) \, dy \quad (1) \]

where \( K(x, y) = c_{n, s}/|x - y|^{n+2s} \), \( A \) is a fixed real vector-valued magnetic potential and

\[ R_A(x, y) := \cos((x - y) \cdot A(x + y/2)). \quad (2) \]
Clearly $\mathcal{R}_A^s$ coincides with $(-\Delta)^s$ when $A = 0$ and $\mathcal{R}_A^s$ is the real part of the fractional magnetic Laplacian $(-\Delta)^s_A$ formally defined by

$$(-\Delta)^s_A u(x) := 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (u(x) - e^{i(x-y) \cdot A(s+\frac{\epsilon}{x})} u(y)) K(x, y) \, dy$$

(see for instance, [3 24]) when $u$ is real-valued. We study the exterior Dirichlet problem

$$\mathcal{R}_A^s u + a(x, u) = 0 \text{ in } \Omega, \quad u|_{\partial \Omega} = g. \quad (3)$$

Our goal is to determine the nonlinearity $a(\cdot, \cdot)$ from the knowledge of the associated Dirichlet-to-Neumann map. Our inverse problem can be viewed as a semilinear analogue of the fractional Calderón problem studied in [20], which is a nonlocal analogue of the classical Calderón problem for the magnetic Laplacian studied in [7 14 21 25].

Linearization is a standard technique used in solving the nonlinear Calderón problem. See for instance, [12 20 27]. In [15], the authors applied high order linearization to prove their uniqueness theorem (see Section 3). See [15 16 19] for similar techniques used in solving the semilinear Calderón problem for local operators. In this paper, we use the first order linearization in the Sobolev space $H^s(\mathbb{R}^n)$ and the Runge approximation property obtained in [20] instead to prove our uniqueness theorem.

To ensure that the exterior Dirichlet problem (3) is well-posed for small $g$, we assume that $A$ satisfies some boundedness condition and the nonlinearity $a(x, z) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $z \to a(\cdot, z)$ is analytic with values in $C^s(\Omega)$;
(ii) $a(x, 0) = 0$ and $\partial_z a(x, 0) \geq c > 0$ for some constant $c > 0$

so we have the Taylor’s expansion

$$a(x, z) = \sum_{k=1}^\infty a_k(x) z^k, \quad a_k(x) = \partial_z^k a(x, 0) \in C^s(\Omega) \quad (4)$$

where the series converges in the Hölder space $C^s(\Omega)$ topology.

Under the assumptions above, we can define the bounded solution operator $Q_{A,a} : g \to u_g$ and the Dirichlet-to-Neumann map $\Lambda_{A,a}$ formally given by

$$\Lambda_{A,a} g := \mathcal{R}_A^s( Q_{A,a} (g) )|_{\partial \Omega} \quad (5)$$

to formulate corresponding Calderón problem.

The following theorem is the main result in this paper.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with $C^{1,1}$ boundary. Suppose $\Omega \cup \text{supp } A \subset B_r(0)$ for some constant $r > 0$ and $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}r)$, $a^{(j)}$ satisfy (i) and (ii), $W_j$ are open sets s.t. $W_j \cap B_3(0) = \emptyset$ ($j = 1, 2$). If

$$\Lambda_{A,a^{(1)}} g|_{W_2} = \Lambda_{A,a^{(2)}} g|_{W_2}, \quad g \in C^\infty_c(W_1)$$

whenever $||g||_{C^2(\mathbb{R}^n)}$ is sufficiently small, then $a^{(1)} = a^{(2)}$ in $\Omega \times \mathbb{R}$.

**Remark.** The nonlinear problem is reduced to the linear one when $a^{(j)}(x, z) = a^{(j)}(x) z$ ($j = 1, 2$). If this is the case, then the statement still holds after we replace $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}r)$ by the weakened assumption $A \in L^\infty(\mathbb{R}^n)$. See Theorem 1.1 in [20].
The rest of this paper is organized in the following way. In Section 2, we summarize the background knowledge. We prove that the nonlinear problem (3) is well-posed in Section 4, based on the $L^\infty$ estimate and the Hölder regularity theorem for the corresponding linear problem proved in Section 3. In Section 5, we prove the main theorem.

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2 Preliminaries

Throughout this paper

- $n \geq 2$ denotes the space dimension and $0 < s < 1$ denotes the fractional power
- $\Omega$ denotes a bounded domain with $C^{1,1}$ boundary and $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$
- $B_r(0)$ denotes the open ball centered at the origin with radius $r > 0$ and $\overline{B_r}(0)$ denotes the closure of $B_r(0)$
- $A : \mathbb{R}^n \to \mathbb{R}^n$ denotes a real vector-valued magnetic potential
- $c, C, C', C_1, \cdots$ denote positive constants (which may depend on some parameters but always independent of small constants $\epsilon, \rho$)
- $\int \cdots \int = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n}$
- $X^*$ denotes the continuous dual space of $X$ and write $\langle f, u \rangle = f(u)$ for $u \in X, f \in X^*$
- $\| \cdot \|_{C^2(\mathbb{R}^n)}$ is defined by
  $$ \| f \|_{C^2(\mathbb{R}^n)} = \sum_{|\alpha| \leq 2} \| \partial^\alpha f \|_{L^\infty(\mathbb{R}^n)}. $$

2.1 Function Spaces

Throughout this paper we refer all function spaces to real-valued function spaces. For $t \in \mathbb{R}$, we have Sobolev spaces

$$ H^t(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \int (1 + |\xi|^2)^t |\mathcal{F} u(\xi)|^2 d\xi < \infty \} $$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions. We have the natural identification $H^{-t}(\mathbb{R}^n) = H^t(\mathbb{R}^n)^*$. Let $U$ be an open set and $F$ be a closed set in $\mathbb{R}^n$,

$$ H^t(U) := \{ u|_U : u \in H^t(\mathbb{R}^n) \}, \quad H^t_F(\mathbb{R}^n) := \{ u \in H^t(\mathbb{R}^n) : \text{supp} u \subset F \}, $$

$$ \tilde{H}^t(U) := \text{the closure of } C_c^\infty(U) \text{ in } H^t(\mathbb{R}^n). $$

Since $\Omega$ is a bounded domain with $C^{1,1}$ boundary implies $\Omega$ is Lipschitz bounded, then

$$ \tilde{H}^t(\Omega) = H^t_0(\mathbb{R}^n). $$
For $0 < s < 1$, one of the equivalent forms of the norm $|| \cdot ||_{H^s(\mathbb{R}^n)}$ is

$$
||u||_{H^s(\mathbb{R}^n)} := (||u||_{L^2(\mathbb{R}^n)}^2 + \int \int \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy)^{1/2}.
$$

We have the Hölder space $C^s(U) := C^{0,s}(U)$ equipped with the standard norm given by

$$
||f||_{C^s(U)} := ||f||_{L^\infty(U)} + \sup_{x \neq y, x, y \in U} \frac{|f(x) - f(y)|}{|x - y|^s}.
$$

### 2.2 The Operator $\mathcal{R}_A^s$

In Section 1 we gave the formal pointwise definition of $\mathcal{R}_A^s$ in (1). Now we do a formal computation to motivate the bilinear form definition of $\mathcal{R}_A^s$.

It is clear from (2) that $\mathcal{R}_A(x, y) = \mathcal{R}_A(y, x)$. Hence for real-valued $u, v$, we can formally compute that

$$
2 \int \int_{|x - y| \geq \epsilon} (u(x) - \mathcal{R}_A(x, y)u(y))v(x)K(x, y) \, dy \, dx
$$

$$
= \int \int_{|x - y| \geq \epsilon} [(u(x) - \mathcal{R}_A(x, y)u(y))v(x) + (u(y) - \mathcal{R}_A(x, y)u(x))v(y)K(x, y)] \, dy \, dx
$$

$$
= \text{Re} \int \int_{|x - y| \geq \epsilon} (u(x) - e^{i(x-y) \cdot \nabla}u(y))(v(x) - e^{-i(x-y) \cdot \nabla}v(y))K(x, y) \, dx \, dy.
$$

Now let $\epsilon \to 0^+$.

**Definition 2.1.** For real-valued $u, v$, we define $\mathcal{R}_A^s$ by the bilinear form

$$
\langle \mathcal{R}_A^s u, v \rangle := \text{Re} \int \int (u(x) - e^{i(x-y) \cdot \nabla}u(y))(v(x) - e^{-i(x-y) \cdot \nabla}v(y))K(x, y) \, dx \, dy
$$

$$
= 2 \int \int (u(x) - \mathcal{R}_A(x, y)u(y))v(x)K(x, y) \, dx \, dy. \quad (6)
$$

It is easy to verify that

$$
\langle \mathcal{R}_A^s u, v \rangle = \langle \mathcal{R}_A^s v, u \rangle.
$$

**Definition 2.2.** We define the magnetic Sobolev norm $|| \cdot ||_{H_A^s}$ by

$$
||u||_{H_A^s} := (||u||_{L^2}^2 + [u]_{H_A^s}^2)^{1/2}
$$

where $[u]_{H_A^s} := \langle \mathcal{R}_A^s u, u \rangle^{1/2}$.

This norm was introduced in [5, 24]. As we mentioned in Section 1, $\mathcal{R}_A^s$ is the real part of the fractional magnetic Laplacian, whose properties have been studied in [20]. In fact, Lemma 3.3 and Proposition 3.4 in [20] imply the following proposition.
Proposition 2.3. Suppose $0 < s < 1$ and $A \in L^\infty(\mathbb{R}^n)$, then we have the norm equivalence

$$|| \cdot ||_{H^s} \sim || \cdot ||_{H^s}$$

and the operator

$$\mathcal{R}_A^s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$$

is linear and bounded.

From now on we always assume $A \in L^\infty(\mathbb{R}^n)$. Corollary 5.3 in [20] implies the following proposition, which will be used later in the proof of the main theorem.

Proposition 2.4. Suppose $\Omega \cup \text{supp} A \subset B_r(0)$ for some $r > 0$, $W$ is an open set s.t. $W \setminus B_{3r}(0) \neq \emptyset$. If

$$u \in \dot{H}^s(\Omega), \quad \mathcal{R}_A^s u|_W = 0$$

then $u = 0$ in $\mathbb{R}^n$.

3 The Linear Exterior Problem

Throughout this section we assume $0 < c \leq q(x) \in L^\infty(\Omega)$.

We first recall some results in [20]. Based on Proposition 4.3 and Proposition 4.5 in [20], we have the following proposition.

Proposition 3.1. The bilinear form

$$B_{A,q}(u,v) := \langle \mathcal{R}_A^s u, v \rangle + \int_\Omega q uv$$

is coercive and bounded on $\dot{H}^s(\Omega) \times \dot{H}^s(\Omega)$. The exterior problem

$$\begin{cases} (\mathcal{R}_A^s + q)u = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e \end{cases} \quad (7)$$

has a unique (weak) solution $u_g \in H^s(\mathbb{R}^n)$ for each $g \in H^s(\mathbb{R}^n)$ and the solution operator

$$P_{A,q} : g \to u_g$$

is bounded on $H^s(\mathbb{R}^n)$.

Proposition 5.4 in [20] implies the Runge approximation property of $\mathcal{R}_A^s + q$, which will be used later in the proof of the main theorem.

Proposition 3.2. Suppose $\Omega \cup \text{supp} A \subset B_r(0)$ for some $r > 0$, $W$ is an open set s.t. $W \subset \Omega_e$ and $W \setminus \overline{B_{3r}(0)} \neq \emptyset$, then

$$S := \{ P_{A,q} f|_\Omega : f \in C^\infty_c(W) \}$$

is dense in $L^2(\Omega)$.

Next we prove an $L^\infty$ estimate and a Hölder regularity theorem, which will be useful in later sections when we deal with the nonlinear problem.
3.1 \(L^\infty\) Estimate

Lemma 3.3. If \(g \in C_c^\infty(\mathbb{R}^n)\), then \((-\Delta)^s g \in L^\infty(\mathbb{R}^n)\) and

\[
||(-\Delta)^s g||_{L^\infty(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}
]\]

Proof. For \(g \in C_c^\infty(\mathbb{R}^n)\), we have

\[
(-\Delta)^s g(x) = c_{n,s} \int \frac{2g(x) - g(x + y) - g(x - y)}{|y|^{n+2s}} dy
\]

(see for instance, Lemma 3.2 in [6]) so by using Taylor’s expansion, we have

\[
||(-\Delta)^s g(x)|| \leq c_{n,s}(\int_{|y| \leq 1} + \int_{|y| > 1}) \frac{|2g(x) - g(x + y) - g(x - y)|}{|y|^{n+2s}} dy \leq C||g||_{C^2(\mathbb{R}^n)}.
\]

\[
\Box
\]

Lemma 3.4. If \(A, g \in L^\infty(\mathbb{R}^n)\), then \((-\Delta)^s - \mathcal{R}_A^s)g \in L^\infty(\mathbb{R}^n)\) and

\[
||((-\Delta)^s - \mathcal{R}_A^s)g||_{L^\infty(\mathbb{R}^n)} \leq C||g||_{L^\infty(\mathbb{R}^n)}.
\]

Proof. Note that

\[
0 \leq 1 - R_A(x,y) = 2\sin^2(\frac{1}{2}(x-y) \cdot A(\frac{x+y}{2})) \leq C_A \min\{1, |x-y|^2\}
\]

so we have

\[
||((-\Delta)^s - \mathcal{R}_A^s)g(x)|| \leq \int (1 - R_A(x,y))K(x,y)|g(y)| dy
\]

\[
= (\int_{|y-x| \leq 1} + \int_{|y-x| > 1}) (1 - R_A(x,y))K(x,y)|g(y)| dy \leq C||g||_{L^\infty(\mathbb{R}^n)}.
\]

\[
\Box
\]

From now on we always assume \(\Omega \cup \text{supp } A \subset B_r(0)\) for some constant \(r > 0\) and \(||A||_{L^\infty(\mathbb{R}^n)} \leq \frac{\pi}{(8\sqrt{n}r)}\). This coincides with the assumption on \(A\) in the statement of Theorem 1.1.

Note that under this assumption, we have

\[
0 \leq R_A(x,y) \leq 1, \quad (x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega_c \times \Omega_e).
\]

In fact, if \((x,y) \in \Omega \times \mathbb{R}^n\), then

(i) \(y \in B_{4r}(0)\), which implies \(|x - y| \leq 4r, \ |(x - y) \cdot A(\frac{x+y}{2})| \leq \frac{\pi}{2}\),

(ii) \(y \notin B_{3r}(0)\), which implies \(|\frac{x+y}{2}| \geq r, \ R_A = 1\).

By symmetry of \(R_A\), we know the claim also holds for \((x,y) \in \mathbb{R}^n \times \Omega\).

The following two propositions generalize Proposition 3.1 and Proposition 3.3 in [17].

Proposition 3.5. Suppose \(0 < c \leq g(x) \in L^\infty(\Omega)\). If \(u \in C^{s}(\mathbb{R}^n)\) solves the exterior problem

\[
\begin{cases}
(R_A^s + q)u = f & \text{in } \Omega \\
u = g & \text{in } \Omega_c
\end{cases}
\]

for \(0 \leq f \in H^{-s}(\Omega)\) and \(0 \leq g|_{\Omega_e} \in L^\infty(\Omega_e)\), then \(u \geq 0\).
Proof. Write \( u = u^+ - u^- \) where \( u^+ = \max\{u, 0\} \) and \( u^- = \max\{-u, 0\} \). Note that

\[
|u^+(x) - u^+(y)| + |u^+(x) - u^-(y)| = |u(x) - u(y)|
\]

so \( u^+, u^- \in \mathcal{H}^s(\mathbb{R}^n) \) and \( g|_{\Omega_c} \geq 0 \) implies \( u^- \in \mathcal{H}^s(\Omega) \), so we have

\[
\langle \mathcal{R}_A^s u, u^- \rangle + \int_\Omega q u^- = f(u^-)
\]

Now write

\[
\langle \mathcal{R}_A^s u, u^- \rangle = 2 \int_\Omega (u(x) - R_A(x,y)u(y))u^-(x)K(x,y) \, dx \, dy
\]

\[
= 2 \left( \int_\Omega \int_\Omega (u(x) - R_A(x,y)u(y))u^-(x)K(x,y) \, dx \, dy \right) =: I_1 + I_2.
\]

Since \( u^- u \leq 0 \), then we have

\[
I_2 = 2 \int_{\Omega_c} \int_\Omega (u(x) - R_A(x,y)g(y))u^-(x)K(x,y) \, dx \, dy \leq 0, \quad \int_\Omega q u^- \leq 0.
\]

Note that

\[
I_1 = 2 \int_\Omega \int_\Omega (u(x) - R_A(x,y)u(y))u^-(x)K(x,y) \, dx \, dy
\]

\[
= 2 \int_\Omega \int_\Omega \left( (u^+(x) - R_A(x,y)u^+(y))u^-(x) - (u^-(x) - R_A(x,y)u^-(y))u^-(x) \right)K(x,y) \, dx \, dy
\]

\[
= -2 \int_\Omega \int_\Omega \left( R_A(x,y)u^+(y)u^-(x) + (u^-(x) - R_A(x,y)u^-)(y))u^- (x) \right)K(x,y) \, dx \, dy
\]

\[
\leq -2 \int_\Omega \int_\Omega (u^-(x) - R_A(x,y)u^-)(y))u^- (x)K(x,y) \, dx \, dy
\]

\[
= - \int_\Omega \int_\Omega \left( (u^-(x) - R_A(x,y)u^-)(y))u^- (x) + (u^- (y) - R_A(x,y)u^-)u^-(y) \right)K(x,y) \, dx \, dy
\]

\[
= - \int_\Omega \int_\Omega \left( |u^-(x)|^2 + |u^- (y)|^2 - 2R_A(x,y)u^- (x)u^- (y) \right)K(x,y) \, dx \, dy
\]

\[
\leq - \int_\Omega \int_\Omega |u^- (x) - u^- (y)|^2 K(x,y) \, dx \, dy.
\]

Since \( f(u^-) \geq 0 \), then the only possibility is

\[
\int_\Omega \int_\Omega |u^- (x) - u^- (y)|^2 K(x,y) \, dx \, dy = 0,
\]

which implies \( u^- \) is a non-negative constant \( c_0 \) in \( \Omega \). Now we show \( c_0 \) has to be 0.

Otherwise \( u = -c_0 < 0 \) in \( \Omega \). In this case, for \( x \in \Omega \), by pointwise definition we have

\[
\mathcal{R}_A^s u(x) = 2 \lim_{\epsilon \to 0^+} \left( \int_{\Omega \cap B_\epsilon(x)} + \int_{\Omega_c} \right) (u(x) - R_A(x,y)u(y))K(x,y) \, dy
\]

7
\[
= 2 \int_\Omega (1 - R_A(x,y))(-c_0)K(x,y)dy + 2 \int_{\Omega_c} (-c_0 - R_A(x,y)g(y))K(x,y)dy \leq 0.
\]
Both integrals converge since \(g|_{\Omega_c} \in L^\infty(\Omega_c)\) and \(0 \leq 1 - R_A(x,y) \leq C_A|x-y|^2\). Now we have got the contradiction
\[
f = R_A^*u + qu < 0 \quad \text{in } \Omega.
\]
\[\square\]

**Proposition 3.6.** Suppose \(0 < c \leq q(x) \in L^\infty(\Omega)\). If \(u \in H^s(\mathbb{R}^n)\) solves the exterior problem
\[
\begin{cases}
(R_A^* + g)u = f & \text{in } \Omega \\
u = g & \text{in } \Omega_c
\end{cases}
\]
for \(f \in L^\infty(\Omega)\) and \(g \in C_c^\infty(\Omega_c)\), then
\[
||u||_{L^\infty} \leq C||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\Omega_c)}.
\]
**Proof.** Fix a function \(\phi \in C_c^\infty(\mathbb{R}^n)\) s.t. \(0 \leq \phi \leq 1\) and \(\phi = 1\) on \(\bar{\Omega} \cup \text{supp } g\).
It is clear from the pointwise definition of \(R_A^*\) that \(R_A^*\phi \geq 0\) in \(\Omega\) so
\[
(R_A^* + q)\phi \geq c \quad \text{in } \Omega.
\]
Now let \(\tilde{\phi} := (\frac{1}{c})||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\Omega_c)}\phi\), then \(\tilde{\phi} \pm u \geq 0\) in \(\Omega_c\) and
\[
(R_A^* + q)(\tilde{\phi} \pm u) \geq 0 \quad \text{in } \Omega
\]
so \(|u| \leq \tilde{\phi}\) by the previous proposition. \[\square\]

### 3.2 Hölder Regularity

**Proposition 3.7.** (Proposition 1.1 in [22]) If \(u \in \tilde{H}^s(\Omega)\) and \((-\Delta)^s u = f \in L^\infty(\Omega)\), then \(u \in C^s(\mathbb{R}^n)\) and
\[
||u||_{C^s(\mathbb{R}^n)} \leq C||f||_{L^\infty(\Omega)}.
\]

Based on the proposition above, we now prove the Hölder regularity theorem for the linear exterior problem (7).

**Proposition 3.8.** Suppose \(\Omega \cup \text{supp } A \subset B_r(0)\) for some \(r > 0\) and \(||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{nr})\), \(0 < c \leq q(x) \in L^\infty(\Omega)\) for some \(c > 0\), \(W \cap B_{2r}(0) = \emptyset\). If \(g \in C_c^\infty(W)\), then \(u = P_{A,q}g \in C^s(\mathbb{R}^n)\)
where \(P_{A,q}\) is solution operator associated with (7) and
\[
||u||_{C^s(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}.
\]
**Proof.** Note that \(v := u - g \in \tilde{H}^s(\Omega)\) and
\[
(-\Delta)^s v = ((-\Delta)^s - R_A^*)v - R_A^*g - qu \quad \text{in } \Omega.
\]
By Proposition 3.6, \(||v||_{L^\infty} \leq C_1||g||_{L^\infty}\). By Lemma 3.4,
\[
||(R_A^* - (-\Delta)^s)v||_{L^\infty(\mathbb{R}^n)} \leq C_2||v||_{L^\infty} \leq C_3||g||_{L^\infty}.
\]
Since \( W \cap B_{3r}(0) = \emptyset \), then \( \|x+y\|^2 \geq r \), \( R_A(x,y) = 1 \) for \( x \in \Omega, y \in W \) so
\[
\mathcal{R}_Ag = (-\Delta)^s g \quad \text{in } \Omega.
\]
By Lemma 3.3, \( \|\mathcal{R}_Ag\|_{L^\infty(\Omega)} \leq C'\|g\|_{C^2(\mathbb{R}^n)} \). Hence
\[
\|(((-\Delta)^s - \mathcal{R}_A)\nu - \mathcal{R}_Ag - qu\|_{L^\infty(\Omega)} \leq C''\|g\|_{C^2(\mathbb{R}^n)}.
\]
Now by the proposition above, we have \( \|v\|_{C^2(\mathbb{R}^n)} \leq C'\|g\|_{C^2(\mathbb{R}^n)} \).

4 The Nonlinear Exterior Problem

Now we consider the nonlinear exterior problem
\[
\begin{aligned}
\mathcal{R}_Au + a(x,u) &= 0 \quad \text{in } \Omega \\
u &= g \quad \text{in } \Omega_e.
\end{aligned}
\tag{10}
\]
Recall that we assume the nonlinearity \( a(\cdot, \cdot) \) has the Taylor’s expansion \( (\ref{Taylor}) \) with coefficients \( a_k(x) \in C^s(\Omega) \) and we also assume \( a_1(x) \geq c > 0 \). We write
\[
R_m(x, z) := \sum_{k=m+1}^{\infty} \frac{a_k(x)}{k!} z^k.
\]
We note that \( C^s(\Omega) \) is an algebra since for \( u_1, u_2 \in C^s(\Omega) \), we have
\[
\|u_1u_2\|_{C^s(\Omega)} \leq C_0(\|u_1\|_{C^s(\Omega)}\|u_2\|_{L^\infty(\Omega)} + \|u_1\|_{L^\infty(\Omega)}\|u_2\|_{C^s(\Omega)})
\]
(see Theorem A.7 in \( (11) \)) so
\[
\|u_1u_2\|_{C^s(\Omega)} \leq 2C_0\|u_1\|_{C^s(\Omega)}\|u_2\|_{C^s(\Omega)}.
\]
Also note that by Cauchy’s estimate, we have
\[
\|a_k\|_{C^s(\Omega)} \leq \frac{k!}{R^k} \sup_{z \in C, |z|=R} ||a(\cdot, z)||_{C^s(\Omega)}, \quad R > 0.
\]
Based on the estimates above, we have the following estimates when we choose \( R = \max\{4C_0, 1\} \).

**Proposition 4.1.** If \( \|u\|_{C^s(\Omega)} \leq 1 \), then
\[
\sum_{k=m+1}^{\infty} \|a_k(x)\|_{C^s(\Omega)} \leq (\sum_{k=m+1}^{\infty} \frac{1}{2^k}) \sup_{z \in C, |z|=R} ||a(\cdot, z)||_{C^s(\Omega)}\|u\|_{C^s(\Omega)}^{m+1},
\]
\[
\sum_{k=m+1}^{\infty} \|a_k(x)\|_{C^s(\Omega)} \leq (\sum_{k=m+1}^{\infty} \frac{k}{2^k}) \sup_{z \in C, |z|=R} ||a(\cdot, z)||_{C^s(\Omega)}\|u\|_{C^s(\Omega)}^{m}.
\]

The following proposition is an analogue of Theorem 2.1 in \( (13) \).
Proposition 4.2. Suppose $\Omega \cup \text{supp } A \subset B_r(0)$ for some $r > 0$ and $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n})$, $W \cap B_{2r}(0) = \emptyset$ and $g \in C_c^\infty(W)$. There exists a small constant $\rho > 0$ s.t. if $||g||_{C^2(\mathbb{R}^n)} \leq \rho$, then the nonlinear exterior problem (10) has a unique solution $u \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ satisfying

$$(u - P_{\lambda, a_1}g) \in M := \{v \in C^s(\mathbb{R}^n) : v|_{\Omega_e} = 0, ||v||_{C^s(\mathbb{R}^n)} \leq \rho\}.$$

Moreover, we have

$$||u||_{C^s(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}.$$

Proof. Let $u_0 := P_{\lambda, a_1}g$, then by Proposition 3.8 we have

$$||u_0||_{C^s(\mathbb{R}^n)} \leq C_1||g||_{C^2(\mathbb{R}^n)},$$

and the nonlinear exterior problem (10) can be written as

$$\begin{cases}
R^s_\lambda(u - u_0) + a_1(x)(u - u_0) = -R_1(x, u) & \text{in } \Omega \\
u - u_0 = 0 & \text{in } \Omega_e.
\end{cases}$$

(11)

Now for $f \in L^\infty(\Omega)$, we consider the solution operator $J : f \to u_f \in \dot{H}^s(\Omega)$ of

$$R^s_\lambda u + a_1(x)u = f \text{ in } \Omega.$$

We write

$$(-\Delta)^s u = ((-\Delta)^s - R^s_\lambda)u - a_1(x)u + f \text{ in } \Omega,$$

so by Lemma 3.4, Proposition 3.6 and 3.7, we have $J(f) \in C^s(\mathbb{R}^n)$ and

$$||J(f)||_{C^s(\mathbb{R}^n)} \leq C_2||f||_{L^\infty(\Omega)}.$$

We define maps $G, F$ by

$$G(v) := R_1(x, u_0 + v), \quad F := J \circ G.$$

We will show that $F$ is a contraction map on the complete metric space $M$ for small $\rho$, which will be chosen later. In fact, for small $\rho$ and $v \in M$, we have

$$||F(v)||_{C^s(\mathbb{R}^n)} \leq C_2||G(v)||_{L^\infty(\Omega)} = C_2||R_1(x, u_0 + v)||_{L^\infty(\Omega)} \leq C_2^2||u_0 + v||^2_{C^s(\Omega)} \leq C_2^2\rho^2.$$

Here we use the first estimate in Proposition 4.1 and the constant $C_2^2$ is independent of $\rho$.

Also for small $\rho$ and $v_1, v_2 \in M$, we have

$$||F(v_1) - F(v_2)||_{C^s(\mathbb{R}^n)} \leq C_2||G(v_1) - G(v_2)||_{L^\infty(\Omega)}$$

$$= C_2||R_1(x, u_0 + v_1) - R_1(x, u_0 + v_2)||_{L^\infty(\Omega)}$$

$$\leq ||v_1 - v_2||_{L^\infty(\Omega)} \sum_{k=2}^\infty \frac{||a_k(x)||}{(k-1)!} ||u_0 + v_1|^{k-1} + |u_0 + v_1|^{k-1}||_{L^\infty(\Omega)}$$

$$\leq C_3||v_1 - v_2||_{L^\infty(\Omega)}(||u_0 + v_1||_{C^s(\Omega)} + ||u_0 + v_2||_{C^s(\Omega)})$$

$$\leq C_4\rho||v_1 - v_2||_{L^\infty(\Omega)}.$$
and then \( u \) coincides the Dirichlet-to-Neumann map defined in Subsection 2.3 in [18]. By Proposition 4.1 and 4.2, we have

\[
\|v_0\|_{C^r(\mathbb{R}^n)} = \|F(v_0)\|_{C^r(\mathbb{R}^n)} \leq C_2\|u_0 + v_0\|_{C^r(\Omega)}^2 \leq C_3\rho \|u_0\|_{C^r(\Omega)} + \|v_0\|_{C^r(\Omega)}^2
\]

where the constant \( C_3 \) is independent of \( \rho \). Hence, for small \( \rho < 1/(2C'_3) \), we have

\[
\|v_0\|_{C^r(\mathbb{R}^n)} \leq 2C'_3\rho \|u_0\|_{C^r(\mathbb{R}^n)}
\]

and then \( u := u_0 + v_0 \) satisfies

\[
\|u\|_{C^r(\mathbb{R}^n)} \leq C\|g\|_{C^2(\mathbb{R}^n)}.
\]

### 5 The Inverse Problem

From now on, we denote the bounded solution operator associated with (10) by \( Q_{A,a} \).

Proposition 4.2 ensures that the Dirichlet-to-Neumann map \( \Lambda_{A,a} \) given by (5) is well-defined for \( g \) satisfying the condition assumed in the statement of the proposition. We remark that \( \Lambda_{A,a} \) coincides the Dirichlet-to-Neumann map defined in Subsection 2.3 in [13] when \( A = 0 \).

The first order linearization in \( H^s(\mathbb{R}^n) \) will be useful when we prove Theorem 1.1 later.

**Proposition 5.1.** Suppose \( \Omega \cup \text{supp} A \subset B_r(0) \) for some \( r > 0 \) and \( \|A\|_{L_\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}) \), \( W \cap B_{3r}(0) = \emptyset \) and \( g \in C_c^\infty(W) \), then

\[
Q_{A,a}(\varepsilon g)/\varepsilon \to P_{A,a_1}g
\]

in \( H^s(\mathbb{R}^n) \) as \( \varepsilon \to 0 \).

**Proof.** Write \( u_{\varepsilon,g} := Q_{A,a}(\varepsilon g) \) and \( u_g := P_{A,a_1}g \) for sufficiently small \( \varepsilon \).

Note that \( v_{\varepsilon,g} := u_g - \frac{u_{\varepsilon,g}}{\varepsilon} \in H^s(\Omega) \) and we have

\[
R_A^s v_{\varepsilon,g} + a_1(x)v_{\varepsilon,g} = \frac{1}{\varepsilon} R_1(x,u_{\varepsilon,g}) \quad \text{in} \quad \Omega.
\]

Now choose \( v_{\varepsilon,g} \) as a test function, then by Proposition 2.3 we have

\[
\langle R_A^s v_{\varepsilon,g} + a_1 v_{\varepsilon,g}, v_{\varepsilon,g} \rangle \geq |v_{\varepsilon,g}|^2_{H^s_A} + C\|v_{\varepsilon,g}\|^2_{L^2(\Omega)} \geq C''\|v_{\varepsilon,g}\|^2_{H^s}
\]

and by Proposition 4.1 and 4.2, we have

\[
\frac{1}{\varepsilon} |\langle R_1(x,u_{\varepsilon,g}) , v_{\varepsilon,g} \rangle| \leq \frac{C}{\varepsilon} \| R_1(x,u_{\varepsilon,g}) \|_{L^\infty(\Omega)} \| v_{\varepsilon,g} \|_{L^2(\Omega)}
\]

\[
\leq \frac{C''}{\varepsilon} \| u_{\varepsilon,g} \|_{C^r(\mathbb{R}^n)} \| v_{\varepsilon,g} \|_{L^2(\Omega)} \leq C'''\varepsilon\|g\|^2_{C^2(\mathbb{R}^n)} \| v_{\varepsilon,g} \|_{L^2(\Omega)}.
\]

Hence we have

\[
\|v_{\varepsilon,g}\|_{H^s} \leq C'''\varepsilon\|g\|^2_{C^2(\mathbb{R}^n)}.
\]

Now it is clear that \( v_{\varepsilon,g} \to 0 \) in \( H^s(\mathbb{R}^n) \) as \( \varepsilon \to 0 \).
Now we are ready to prove Theorem 1.1.

Proof. Write \( u^{(j)}_{e,g} := Q_{A,a^{(j)}}(eg) \) and \( u^{(j)}_{g} := P_{A,a^{(j)}}(g) \) for \( g \in C^\infty_c(W_1) \) and sufficiently small \( \epsilon \).

By the assumption, we have

\[
R_A^* u^{(1)}_{e,g} = R_A^* u^{(2)}_{e,g} \quad \text{in } W_2.
\]

Since \( u^{(1)}_{e,g} = u^{(2)}_{e,g} = eg \) in \( \Omega_\epsilon \), then by Proposition 2.4 we have \( u^{(1)}_{e,g} = u^{(2)}_{e,g} =: u_{e,g} \) in \( \mathbb{R}^n \) so

\[
R_A^* u_{e,g} + a^{(i)}(x,u_{e,g}) = 0 \quad \text{in } \Omega \quad (j = 1,2),
\]

which implies

\[
(a^{(1)}_1(x) - a^{(2)}_1(x))u_{e,g} = R^{(2)}_1(x,u_{e,g}) - R^{(1)}_1(x,u_{e,g}) \quad \text{in } \Omega.
\]

Now note that

\[
||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^2(\Omega)} \leq ||(a^{(1)}_1(x) - a^{(2)}_1(x))(1 - \frac{u_{e,g}}{\epsilon})||_{L^2(\Omega)} + \frac{1}{\epsilon}||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^2(\Omega)} ||u_{e,g}||_{L^2(\Omega)}
\]

\[
\leq ||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^\infty(\Omega)} ||1 - \frac{u_{e,g}}{\epsilon}||_{L^2(\Omega)} + \frac{1}{\epsilon}||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^2(\Omega)} ||u_{e,g}||_{L^2(\Omega)} \quad (12)
\]

For given \( \delta > 0 \), by Proposition 3.2 we can choose \( g \in C^\infty_c(W_1) \) s.t.

\[
||1 - u_{g}||_{L^2(\Omega)} \leq \delta.
\]

For this chosen \( g \), we have

\[
||1 - \frac{u_{e,g}}{\epsilon}||_{L^2(\Omega)} \leq 2\delta
\]

for small \( \epsilon \) by Proposition 5.1 and we also have

\[
\frac{1}{\epsilon}||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^2(\Omega)} \leq \frac{C}{\epsilon}||R^{(2)}_1(x,u_{e,g}) - R^{(1)}_1(x,u_{e,g})||_{L^\infty(\Omega)} \leq \frac{C'}{\epsilon}||u_{e,g}||_{L^2(\Omega)}^2 \leq C''\epsilon ||g||_{L^2(\mathbb{R}^n)}^2
\]

for small \( \epsilon \) by Proposition 4.1 and 4.2.

Now let \( \epsilon \to 0 \) in (12), then we have

\[
||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^2(\Omega)} \leq 2\delta ||a^{(1)}_1(x) - a^{(2)}_1(x)||_{L^\infty(\Omega)}.
\]

Since \( \delta \) is arbitrary, then \( a^{(1)}_1 = a^{(2)}_1 =: a_1 \).

Iteratively, once we have shown \( a^{(1)}_j = a^{(2)}_j \) (\( 1 \leq j \leq l - 1 \)), then we have

\[
\frac{1}{\prod}(a^{(1)}_1(x) - a^{(2)}_1(x))u_{e,g}^l = R^{(2)}_l(x,u_{e,g}) - R^{(1)}_l(x,u_{e,g}) \quad \text{in } \Omega.
\]

Now note that

\[
|||a^{(1)}_l(x) - a^{(2)}_l(x)||||_{L^2(\Omega)} \leq |||a^{(1)}_l(x) - a^{(2)}_l(x)||||_{L^\infty(\Omega)} ||1 - \frac{u_{e,g}}{\epsilon}||_{L^2(\Omega)} + \frac{1}{\epsilon}||a^{(1)}_l(x) - a^{(2)}_l(x)||_{L^2(\Omega)}.\]

12
For given $\delta > 0$, we can choose $g \in C^\infty_c(W_1)$ s.t.
\[ ||1 - u_g||_{L^2(\Omega)} \leq \delta, \]
and for this chosen $g$
\[
\frac{1}{\epsilon} |||a^{(1)}_1(x) - a^{(2)}_1(x)|||_{L^2(\Omega)} \leq \frac{C}{\epsilon} ||R^{(2)}_1(x, u_{e,g}) - R^{(1)}_1(x, u_{e,g})||_{L^\infty(\Omega)} ^{\frac{1}{\epsilon}} \\
\leq \frac{C'}{\epsilon} ||u_{e,g}||_{C^\epsilon(\Omega)} \leq C'' \epsilon^{\frac{1}{\epsilon}} \eta^{\frac{1}{\epsilon}} ||g||_{C^2(\mathbb{R}^n)} ^{\frac{1}{\epsilon}}
\]
for small $\epsilon$ by Proposition 4.1 and 4.2.
Now let $\epsilon \to 0$, then we have
\[ |||a^{(1)}_i(x) - a^{(2)}_i(x)|||_{L^2(\Omega)} \leq 2\delta |||a^{(1)}_i(x) - a^{(2)}_i(x)|||_{L^\infty(\Omega)}. \]
Since $\delta$ is arbitrary, then $a^{(1)}_i = a^{(2)}_i$.

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