Chaos and order in a finite universe

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All inhabitants of this universe, from galaxies to people, are finite. Yet the universe itself is often assumed to be infinite. If instead the universe is topologically finite, then light and matter can take chaotic paths around the compact geometry. Chaos may lead to ordered features in the distribution of matter throughout space.

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In cosmology as well as string theory, compact spaces have received renewed attention. Most discussions evade the chaos inherent in many of these spaces. Here we pursue the consequences of chaos on a compact hyperbolic space by isolating the fractal set of closed loop orbits. We also discuss the implications this may have for the distribution of large-scale structure in our own cosmos.

Compact hyperbolic spaces are known to induce chaotic mixing of trajectories as they wrap around the space. The closed loop orbits, though seemingly special, define the entire structure of the chaotic dynamics. The dense and abundant periodic orbits pack themselves into the finite space by collectively forming a fractal.

For simplicity we view the closed loop null-geodesics on a 2D finite space. Consider the double donut built by cutting a regular octagon out of a hyperbolic 2D space and identifying the opposite sides in pairs. The fundamental domain in fig.1 is drawn on the Poincaré unit sphere with the metric

$$ds^2 = -d\eta^2 + \frac{4}{(1 - r^2)^2}(dr^2 + r^2 d\phi^2).$$ (1)

Geodesics are semi-circles which are orthogonal to the boundary at $r = 1$. The shortest closed loop orbits are also drawn in fig. 1. The null geodesics are completely specified by the angular momentum $L = 4(1 - r^2)^{-2}r^2 \dot{\phi}$ and the angular coordinate $\theta$ on the boundary at which the geodesics originated. As geodesics exit and re-enter the fundamental domain, they are chaotically mixed. A re-entry map can be found given the rules for identifying the faces of the octagon. The closed loop orbits can be found systematically order by order in the number of windings around the space.

FIG. 1. The fundamental octagon, drawn above in the Poincaré unit sphere, is made finite by gluing opposite faces. The shortest closed orbits repeatedly exit one face and enter the opposite face.

In fig. 2, all of the periodic orbits are shown which execute 5 windings or less around the octagon. There are 19,624 such orbits. We find the box counting dimension of the set by covering it with boxes of size $\epsilon$ on a side and counting the growth in the number of boxes needed to cover the set as $\epsilon$ gets smaller. The dimension is found to be $D_0 = \lim_{\epsilon \to 0} \ln N(\epsilon)/\ln(1/\epsilon) = 2$. The fact that the dimension is 2 reflects the complete filling of the allowed area. The geodesics of the octagon form a self-affine fractal. We find the topological entropy, the number and location of fixed points, and the spectrum of dimensions in Ref. 2.
We suggest the underlying tangle of geodesics could be reflected in the distribution of large-scale structure \[5\]. The largest structures in the universe have their origin in quantum fluctuations. The phenomenon of scarring in quantum chaos along the periodic orbits \[6\] could lead to an enhanced filamentary structure along the shortest loops through the finite space. The web of galaxies and clusters of galaxies and the vibration modes excited by gravitational waves on the largest scales could reflect these scars. The presence of negative spatial curvature provides a natural scale with which to associate the finite topology. In flat universes, which have been extensively studied but where chaotic geodesics do not occur, there is no natural length scale on which to produce topological identifications and it is entirely ad hoc to create a fundamental topological identification scale so close to the present Hubble length. However, if a non-zero cosmological constant exists, as recent observations of distant supernova may be indicating \[7\], then the cosmological constant provides another fundamental length scale close to the current Hubble scale with which to associate topological identifications even in a zero curvature universe.

In the absence of inflation, there is no dynamical mechanism to generate large-scale fluctuations. They are simply an initial condition. A universe created finite and hyperbolic can be thought of as a realization from an ensemble of finite spaces with a spectrum of fluctuations atop a nearly constant negative curvature manifold. The spectrum of fluctuations will then be shaped according to the predictions of quantum chaos.

The tenets of quantum chaos imply that the ordered remnants of classical chaos are washed out in the transition to quantum mechanics \[8\]. This expectation is based on two conjectures. As suggested by Berry \[9\], the quantum eigenmodes are well described as concentrated on the region of phase space traced out by a typical orbit over infinite times. For a completely chaotic system the orbits cover the entire space which seems to argue for a featureless distribution of the quantum modes. The amplitude of quantum fluctuations are also conjectured to be drawn from a Gaussian random ensemble with a flat spectrum, consistent with the predictions of Random Matrix Theory. While these assumptions seem to argue for uniformity in the quantum fluctuations, they are not inconsistent with striking geometric features. Typical eigenstates in a chaotic quantum system have shown scars of enhanced probability along short period orbits \[10\]. The scars are consistent with Berry’s conjecture as typical orbits will spend the most time tracing short period loops. The scars can be related to the classical fractal of closed loops. For a completely chaotic system the fractal will fill the space with a box counting dimension equal to the dimension of the space, as we found to be the case for the compact octagon. However, if regions of the fractal are visited more frequently than others, as the shortest closed loops are in a compact space, then the scars might result \[11\].

Scars can be regions of underdensity as well as overdensity. The consequence for the build up of structure on the largest scales could be tendency to align with some short period orbits. In a 2d universe, we might see a web-like distribution of clusters aligned along the orbits of fig. \[12\] while structure on smaller scales would look featureless.

The scars would have little effect on the cosmic microwave background (CMB) although evidence of topology will be conspicuous through patterns or correlated circles \[13\]. \[14\]. The surface of last scattering is also not likely to cut right through a scar. As a result, it is reasonable that the CMB will appear smooth when the distribution of galaxies does not. The galaxies might be marking the path of the short period orbits, providing a map of the shortest route around a finite cosmos.

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**FIG. 2.** The fractal set of periodic orbits in the \((\theta, L)\) plane shown to order \(n = 5\). The \(x\)-axis corresponds to \(\theta\) in units of \(\pi/4\).
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