Configuration space and Massey products

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Abstract

The purpose of this paper is to study and compare the collapsing of two spectral sequences converging to the cohomology of a configuration space. The non collapsing of these spectral sequences is related, in some cases, to the existence of Massey products in the cohomology of the manifold $M$.

Let $\mathbb{k}$ be a field and $M$ be an oriented connected closed $m$-dimensional real manifold. We consider the configuration space of ordered $n$-tuples of points in $M$$F(M,n) = \{(x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j, \text{if } i \neq j \}$. The symmetric group $\Sigma_n$ acts freely on $F(M,n)$ and the quotient manifold $B(M,n) = F(M,n)/\Sigma_n$ is called the space of unordered $n$-tuples of distinct points in $M$.

The determination of the homology of the configuration spaces $F(M,n)$ and $B(M,n)$ has been at the origin of a lot of works: In [3], Bodigheimer, Cohen and Taylor compute the homology groups of $B(M,n)$ when $M$ is an odd-dimensional closed manifold. In [11], Löffler and Milgram compute the $\mathbb{Z}/2$-homology groups of $B(M,n)$ for any closed manifold $M$. The answers depend only on the homology groups of $M$. In [7], we have computed the rational homology of $B(M,n)$: the answer depends on the cohomology algebra of $M$, and not only on the rational Betti numbers when $M$ is even-dimensional.

There is no direct procedure to compute the cohomology algebra $H^*(F(M,n); \mathbb{k})$ in general. There are however two spectral sequences converging to $H^*(F(M,n); \mathbb{k})$. The first one, we call the Cohen-Taylor spectral sequence for $H^*(F(M,n); \mathbb{k})$, is the Leray spectral sequence for the inclusion $F(M,n) \hookrightarrow M^n$, and converges as an algebra to $H^*(F(M,n); \mathbb{k})$ (3). Its $E_1$ term is the quotient of the bigraded commutative algebra

$$H^\otimes n \otimes \wedge(x_{st})$$

where $H^i = H^i(M; \mathbb{k})$ is concentrated in bidegree $(0, i)$ and $\wedge(x_{st})$ denotes the free bigraded commutative algebra generated by the elements $x_{st}$ of bidegree $(0, m-1)$ for $1 \leq s, t \leq n, s \neq t$ by the ideal $I$ generated by the elements

$$(e_i - e_t)(a) \otimes x_{st}, \quad x_{st}^2, \quad x_{st} - (-1)^m x_{ts}, \quad x_{st} x_{su} + x_{su} x_{st} + x_{us} x_{ut},$$

where $e_i(a) = 1 \otimes \cdots \otimes a \otimes 1 \cdots 1$ ($a$ appears in the $i$th position).

The differential $d_1$ is given by

$$d_1(x_{st}) = p_{s,t}^*(\delta^M)$$

where $p_{s,t}: M^n \to M^2$ denote the projection on the $s$th and the $t$th factors and $\delta^M \in H^m(M^2; \mathbb{k}) = (H \otimes H)^m$ denotes the diagonal class.
Using a weight-filtration argument, Totaro ([14]) has proved that for a smooth projective variety $M$, the differential $d_1$ is the only non trivial differential and that $H(E_1,d_1) \cong H^*(F(M,n);k)$ as an algebra (see also [9]). He also asks for an example of a manifold for which the Cohen-Taylor spectral sequence has other nonzero differentials.

The second spectral sequence, we call the Bendersky-Gitler spectral sequence for $H^*(F(M,n);k)$, is obtained by considering the nerve of the covering of the fat diagonal

$$D^n M = \{ (x_1, \ldots, x_n) \in M^n | x_i = x_j, \text{ some pair } i \neq j \}$$

by the subspaces

$$D_{ij} = \{ (x_1, \ldots, x_n) \in M^n | x_i = x_j \} .$$

This defines a double cochain complex $C^*(X_\ast) : C^*(X_0) \xrightarrow{d_1} C^*(X_1) \xrightarrow{d_2} \cdots \xrightarrow{d_n} C^*(X_n)$ with $X_r = \Pi (D_{i_1 j_1} \cap \cdots \cap D_{i_r j_r+1} )$. As proved by Bendersky and Gitler ([2]) the first spectral sequence produced by this bicomplex is a spectral sequence of $H^*(M^n,k)$-modules converging to $H^*(M^n,D^n M)$. By Lefschetz duality we have $H^*(M^n,D^n M) \cong H_{mn-n}(F(M,n);k)$. Thus the dual of the Bendersky-Gitler spectral sequence converges to $H^{mn-n}(F(M,n);k)$.

When $k$ is a field of characteristic zero and $M$ is rationally formal, $d_1$ is the only non zero differential. Recall that a space $X$ is rationally formal if there is a sequence of quasi-isomorphisms connecting the de Rham algebra of differential forms on $M$, $\Omega^*(M)$, and its real cohomology, $(H^*(M;\mathbb{R}),0)$. A smooth projective variety is a formal space ([13]) and if $M$ is a formal space all the Massey products are trivial. Bendersky and Gitler have conjectured that higher differentials are determined by higher order Massey products.

Our main results are:

**Theorem 1.**

*The Poincaré duality of $H^*(M;k)$ induces an isomorphism of $H^*(M;k)^{\otimes n}$-modules between the $E_2$ term in the Cohen-Taylor spectral sequence and the dual of the $nm$-suspension of the $E_2$ term in the Bendersky-Gitler spectral sequence.*

Since each $H_k(F(M,n);k)$ is finite dimensional, Theorem 1 implies that the Bendersky spectral sequence for $H_{mn-n}(F(M,n);k)$ collapses at the $E_2$ term if and only if the Cohen-Taylor spectral sequence for $H^*(F(M,n);k)$ collapses at the $E_2$ term.

In particular, the collapsing theorem of Bendersky-Gitler implies the first part of the collapsing result of Totaro.

**Theorem 2.** If $n \leq 3$, then both spectral sequences collapse at the $E_2$ term.

**Theorem 3.** Let $k$ be of characteristic zero and $M$ be simply connected. Suppose there exist indecomposable elements $a, b, c$ and $d \in H^*(M;k)$, such that

- $ab = ac = ad = bc = bd = cd = 0$,
- the triple Massey product $\langle b,c,d \rangle$ is a nonzero indecomposable element, and
- $a$ is not a linear combination of the elements $b, c, d$ and $\langle b,c,d \rangle$,

then the Cohen-Taylor spectral sequence for $H^*(F(M,4);k)$ does not collapse at the $E_2$ term.

**Theorem 4.** Let $k$ be of characteristic zero, $M$ be simply connected and $N = M \# (S^2 \times S^{m-2})$ be the connected sum of $M$ with a product of two spheres. If there is a triple matrix Massey product in $H^*(M;k)$ represented by an indecomposable element, then the Cohen-Taylor spectral sequence for $H^*(F(N,4);k)$ does not collapse at the $E_2$ term.

Theorems 3 and 4 answer the question of Totaro quoted above. We give explicit formulae for the differential $d_2$ of the Bendersky-Gitler spectral sequence in section 4.
This enables a lot of examples. For instance at the end of section 4 we construct a nonzero $d_2$ for $F(M, 4)$ when $M$ is the sphere tangent bundle to the manifold $S^2 \times S^2$.

The main ingredient in the text is the bicomplex $C(n, A)$ constructed in section 3 for any differential graded commutative algebra $A$ using a convenient family of oriented graphs with $n$ vertices and such that no two edges have the same range. The associated spectral sequence is isomorphic to the Bendersky-Gitler spectral sequence in characteristic zero. The simplicity of the complex allows explicit computations. In particular we prove:

**Proposition 5.** There exists a short exact sequence

$$0 \to (I_M)^* \to H^*(F(M, 3); \mathbb{k}) \to (\text{Hom}(\Omega^1_{H/\mathbb{k}}, \mathbb{k}))^{*, -1+3m} \to 0.$$ 

Here $I_M$ is the image of $H^*(M^3; \mathbb{k})$ along the map induced by the injection $F(M, 3) \hookrightarrow M^3$ and $\Omega^1_{H/\mathbb{k}}$ is the module of Kähler differentials, $H = H^*(M; \mathbb{k})$.

In the first section of the paper we describe the Bendersky-Gitler spectral sequence in terms of graphs. In section 2 we prove theorem 1. In section 3 we construct the bicomplex $C(n, A)$ and prove theorems 2. Section 4 is devoted to Massey products, the homology of $F(M, 4)$ and the proof of Theorems 3 and 4.

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1 The $E_1$ term in the Bendersky-Gitler spectral sequence

To go further in the description of the $E_1$ term of the Bendersky-Gitler spectral sequence, we introduce for any differential graded commutative algebra $(A, d)$ the differential bigraded algebra $E(n, A)$ defined as follows: As an an algebra $E(n, A)$ is the quotient of

$$A^\otimes n \otimes \wedge(e_{ij})$$

where

- the generators $e_{i,j}$, $1 \leq i < j \leq n$, have bidegree $(0, 1)$
- the elements $a_1 \otimes \cdots \otimes a_n \otimes 1$ has bidegree $(\sum |a_i|, 0)$

by the the ideal generated by the elements

$$(e_i - e_j)(a) \otimes e_{ij}$$

where $e_i(a) = 1 \otimes \cdots \otimes a \otimes 1 \otimes \cdots 1$, with $a$ appearing in the $i^{th}$ position.

The differential $d$ is defined by

$$d = d' + d''$$

where $d'$ is the multiplication by $\sum e_{ij}$ and $d''$ is the differential induced by the differential $d$ on $A$.

We are particularly interested in the case when $A$ is a commutative model of $M$ over $\mathbb{k}$. Such a commutative model exists in the following situations

- $M$ is a $\mathbb{k}$-formal space, any $\mathbb{k}$ (with $A = H^*(M; \mathbb{k})$), [3], [9].
- $M$ is $r$-connected and the characteristic $p$ of the field $\mathbb{k}$ satisfies $p > \frac{m}{r}$ ([1]),
- $\mathbb{k} = \mathbb{Q}$, (with $A = AP_L(M)$ or any Sullivan model of $M$), [13], [8],
- $\mathbb{k} = \mathbb{R}$, (with $A = \text{the de Rham complex of $M$}$).

By filtering $E(n, A)$ by the powers of the ideal generated by the $e_{ij}$, Bendersky and Gitler construct a new spectral sequence. They prove (cf. Theorems 2.7 and 3.1 in [2])

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Theorem A. For any field $k$, the cohomology $H^*(E_n, d_1)$ of the Bendersky-Gitler spectral sequence is isomorphic to $H^*(E(n, H^*(M; k)))$.

Theorem B. Let $A$ be a commutative model of $M$ over $k$. The spectral sequence associated to the double complex $E(n, A)$ is isomorphic to the Bendersky-Gitler spectral sequence for $H_{mn-*}(F(M, n); k)$ from the $E_2$ term.

Following Bendersky and Gitler [2], it is convenient to describe $E(n, A)$ in terms of oriented graphs.

Let $n$ be a fixed integer and let $G(n)$ be the set of graphs $G$ with set of vertices $V(G) = \{1, 2, \ldots, n\}$ and set of edges $E(G) \subset \{(i, j), i, j \in V(G), i < j\}$. We note $e_G = e_{i_1 j_1} \cdots e_{i_k j_k}$, if $E(G) = \{(i_1, j_1) \cdots (i_k, j_k)\}$, where the $(i_k, j_k)$ are ordered by lexicographic order.

We denote by $S_1, S_2, \ldots, S_l(G)$ the components of $G$ ordered by the smallest vertex, and if $S_i$ is the ordered set $\{i_1, i_2, \ldots, i_r\}$ by $a_{S_i}$ the product $a_{i_1} a_{i_2} \cdots a_{i_r} \in A$.

Then the natural map $a_1 \otimes \cdots \otimes a_n \otimes e_G \rightarrow e_G a_{S_1} \otimes \cdots \otimes a_{S_l(G)} \otimes e_G$ ($e_G$ is the graded signature) induces an isomorphism of graded vector spaces

$$\Phi : E(n, A) \rightarrow \oplus_{G \in G(n)} A^{\otimes l(G)} e_G := A(G(n)).$$

In particular, the differential $d'$ on $E(n, A)$ induces a differential, also denoted by $d'$, on $A(G(n))$:

$$d'(a_1 \otimes a_2 \otimes \cdots \otimes a_{l(G)} \cdot e_G) =$$

$$\sum_{i < j, i,j \in S_i} a_1 \otimes \cdots \otimes a_{l(G)} \cdot e_G e_{ij}$$

$$+ \sum_{i < j, i \in S_i, j \in S_t} (-1)^{s+t} a_1 \otimes \cdots \otimes a_s a_t \otimes \cdots \hat{a_t} \otimes \cdots \otimes a_{l(G)} \cdot e_G e_{ij}$$

For instance, if $n = 3$ and $G$ the discrete graph ($e_G = 1$), then

$$d'(a \otimes b \otimes c) = (ab \otimes c)e_{12} + (a \otimes bc)e_{23} + (-1)^{|b||c|}(ac \otimes b)e_{13}.$$

It is easy to prove that the differential of $A$ induces a differential $d''$ on the graded space $A(G(n))$ such that $(d' + d'')^2 = 0$. Then Proposition 2.6 of [2] extends in:

Theorem C. For any differential graded commutative algebra $A$, the isomorphism

$$\Phi : E(n, A) \rightarrow A(G(n)).$$

is an isomorphism of bicomplexes.

For any sub-family of graphs $\Gamma \subset G(n)$ which is stable by adjonction of edges, we consider the subcomplex of $A(G(n))$

$$A\Gamma = \oplus_{G \in \Gamma} A^{\otimes l(G)} e_G.$$

We will use the above description of the bicomplex $E(n, A)$ in order to replace it by a smaller one.

Proposition 1. The ideal $J$ of $E(n, A)$ generated by the products $e_{i_r} e_{j_r}$, $i, j = 1, \ldots, n$, is a bigraded differential acyclic complex.

It follows from Proposition 1 that the canonical projection

$$E(n, A) \rightarrow \bar{E}(n, A) := E(n, A)/J$$

is a quasi-isomorphism.
In fact, as a byproduct of Proposition 1, the isomorphism \( \Phi \) induces an isomorphism of bicomplexes

\[
\Phi : E(n, A) \xrightarrow{\cong} A(\mathcal{G}(n)) := \bigoplus_{G \in \mathcal{G}} A^{\otimes l(G)} e_G,
\]

where \( \mathcal{G} \) denotes the family of graphs of \( \mathcal{G}(n) \) that do not have two edges with the same target.

**Proof.** Let \( \mathcal{G}_J \) be the set of graphs \( G \in \mathcal{G}(n) \) such that at least two edges terminate at the same vertex. Thus \( \Phi \) induces an isomorphism of differential bigraded vector spaces

\[
\Phi_J : J \to A \mathcal{G}_J.
\]

We denote by \( \mathcal{G}(n, p) \) the subset of \( \mathcal{G}(n) \) consisting of graphs with \( \leq p \) components, and by \( \mathcal{G}_J(n, p) \) the intersection \( \mathcal{G}_J(n, p) = \mathcal{G}_J \cap \mathcal{G}(n, p) \). We prove by induction on \( n \) and \( p \) that \( A \mathcal{G}_J(n, p) \) is acyclic. We suppose that the assertion has been proved for \( q < n \) and any \( p \), and we consider the disjoint union

\[
\mathcal{G}(n, 1) = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2,
\]

where

\[
\begin{align*}
\mathcal{H}_0 &= \{ G \in \mathcal{G}_J(n, 1) \mid (1, 2) \notin E(G) \}, \\
\mathcal{H}_1 &= \{ G \in \mathcal{G}_J(n, 1) \mid (1, 2) \in E(G), l(G \setminus \{(1, 2)\}) = 1 \}, \\
\mathcal{H}_2 &= \{ G \in \mathcal{G}_J(n, 1) \mid (1, 2) \in E(G), l(G \setminus \{(1, 2)\}) = 2 \}.
\end{align*}
\]

The graded vector space \( A(\mathcal{H}_1 \cup \mathcal{H}_0) \) is a sub-complex of \( A \mathcal{G}_J(n, 1) \). We filter the complex by putting \( e_{ij} \) in degree \( i + j \). The first differential is the internal one. We can therefore suppose that \( d'' = 0 \). The next nonzero differential of the associated spectral sequence, \( d_3 \), is the multiplication by \( e_{12} \), which corresponds to the addition of the edge \((1,2)\). Therefore \( d_3 : \oplus_{G \in \mathcal{H}_0} A e_G \to \oplus_{G \in \mathcal{H}_1} A e_G \) is a linear isomorphism. The complex \( A \mathcal{G}_J(n, 1) \) is thus quasi-isomorphic to the quotient complex

\[
Q := A \mathcal{G}_J(n, 1)/A(\mathcal{H}_0 \cup \mathcal{H}_1).
\]

Denote by \( \text{Sh}(p) \) the set of \( (p, n-p-2) \)-shuffles of \( \{3, 4, \ldots, n\} \), and for each \( \sigma, \tau \in \text{Sh}(p) \) the set of graphs in \( \mathcal{G}_J \) with two components, the first one containing the vertices \( 1, \sigma_1, \ldots, \sigma_p \), and the other one the vertices \( 2, \tau_1, \ldots, \tau_{n-p-2} \). Observe now that the map \( G \to G \setminus \{(1,2)\} \) induces a bijection

\[
\mathcal{H}_2 \to \prod_{p=0}^{n-2} \prod_{(\sigma,\tau) \in \text{Sh}(p)} \mathcal{G}(\sigma, \tau).
\]

For \( \sigma, \tau \in \text{Sh}(p) \) there is clearly a bijection between \( \mathcal{G}(\sigma, \tau) \) and

\[
\mathcal{G}_J \cap (\mathcal{G}(p, 1) \times \mathcal{G}(n-p-2, 1))
\]

Therefore the quotient complex \( Q \) contains the subcomplex

\[
Q_1 := \oplus_{\text{Sh}(p)} (A \mathcal{G}_J(p, 1) \otimes A \mathcal{G}(n-p-2, 1)),
\]

that is acyclic since \( p < n \). Moreover the quotient \( Q/Q_1 \) is isomorphic to

\[
\oplus_{\text{Sh}(p)} (A(\mathcal{G}(p, 1) \otimes \mathcal{G}_J(p, 1)) \otimes A \mathcal{G}(n-p-2, 1))
\]

and is acyclic since \( n-p-2 < n \). Therefore, \( A \mathcal{G}_J(n, 1) \) is also acyclic.

We suppose that \( A \mathcal{G}_J(n, p) \) is acyclic for \( p < r \), and we consider the quotient complex \( A \mathcal{G}_J(n, r)/A \mathcal{G}_J(n, r-1) \). Denote by \( \text{Sur}_J(n, 2) \) the set of surjective maps \( \{1, 2, \ldots, n\} \to \{1, 2\} \). We associate bijectively to a graph with \( r \) components in \( \mathcal{G}_J(n, r) \) two graphs \( G_1 \) and \( G_2 \), where \( G_1 \) has only one component with \( V(G_1) = \pi^{-1}(1) \), and \( G_2 \) contains exactly \( r-1 \) components with \( V(G_2) = \pi^{-1}(2) \). This produces an isomorphism of complexes

\[
A \mathcal{G}_J(n, r)/A \mathcal{G}_J(n, r-1)) =
\]

\[
\oplus_{\pi \in \text{Sur}_J(n, 2)} (A \mathcal{G}_J(\pi^{-1}(1), 1) \otimes A \mathcal{G}(\pi^{-1}(2), r-1))
\]

\[
\oplus A(\mathcal{G}(\pi^{-1}(1), 1) \otimes \mathcal{G}_J(\pi^{-1}(1), 1)) \otimes A \mathcal{G}_J(\pi^{-1}(2), r-1)).
\]

The induction hypothesis shows that this quotient complex is acyclic. \( \Box \)
2 Proof of Theorem 1

Let $H = H^*(M;\mathbb{k})$. To avoid confusion we denote by $T(n, H)$ the $E_1$ term of the Cohen-Taylor spectral sequence,

$$T(n, H) = H^⊗n \otimes (x_{ij})/I.$$  

We consider the sub-vector space, $R$, of $\wedge x_{ij}$ generated by the monomials

$$x_{i_1,j_1} \cdots x_{i_r,j_r} \quad \text{with} \quad \begin{cases} 0 \leq r \leq n \\
 i_s < j_s, \text{ for } s = 1, \ldots, r \\
 j_1 < \cdots < j_r. \end{cases}$$

The differential $H^⊗n$-module $H^⊗n \otimes R$ is a direct summand of $H^⊗n \otimes (x_{ij})$. Let $L$ be the sub $H^⊗n$-module of $H^⊗n \otimes R$ generated by the elements $(e_i - e_j) \otimes x_{ij}$, $i < j$. The quotient map $(H^⊗n \otimes R)/L \to T(n, H)$ is a bigraded isomorphism of $H^⊗n$-modules.

Since $M$ is a connected closed $m$ dimensional manifold one can choose $\omega \in H^m$ such that:

a) $H^m = \mathbb{k}\omega$

b) the relations $ab = < a; b >, a \in H^p, b \in H^q$ define non degenerated bilinear forms $H^p \otimes H^{n-p} \to \mathbb{k}$, $a \otimes b \mapsto < a; b >$.

This pairing extends to a pairing

$$(*) \quad H^⊗n \otimes H^⊗n \to \mathbb{k}, \quad < a_1 \otimes \ldots \otimes a_n; b_1 \otimes \ldots \otimes b_n = \epsilon(\sigma) < a_1; b_1 > \cdots < a_n; b_n >$$

where $\epsilon(\sigma)$ is the graded signature.

In the same fashion, the pairing

$$(\oplus_{i<j} x_{ij}) \oplus (\oplus_{s=1}^{n} e_{ij} \mathbb{k}) \to \mathbb{k}, \quad < x_{ij}; s^{1-m} e_{kl} > = \delta^k_j \delta^l_i$$

extends to a pairing

$$(**) \quad \wedge (x_{ij}) \otimes (s^{1-m} e_{kl}) \to \mathbb{k}, \quad 1 \leq i < j \leq n, 1 \leq k < l \leq n.$$  

The pairings $(*)$ and $(**)$ glue together in the pairing

$$(H^⊗n \otimes (x_{ij})) \otimes (H^⊗n \otimes (s^{1-m} e_{kl})) \to \mathbb{k}, \quad 1 \leq i < j \leq n, 1 \leq k < l \leq n.$$  

which in turn defines the $H$-linear isomorphism

$$H^⊗n \otimes (x_{ij}) \to (H^⊗n \otimes (s^{1-m} e_{kl}))^\vee, \quad z \mapsto < z; >.$$  

This isomorphism produces a $H^⊗n$-linear isomorphism

$\theta : (H^⊗n \otimes R)/L \to \mathbb{E}(n, H)$

which restricts, for each $p \geq 0$, to the isomorphisms

$$\theta^{p,q} : (H^⊗n \otimes R)/L)^{p,q} \cong (\mathbb{E}(n, H))^{p,q-pm}.$$  

Recall now that the differential $d_1$ on $T(n, H)$ is defined by $d_1(x_{ij}) = p_{i,j}^*(\delta^M)$, where $\delta^M$ denotes the diagonal class. Since $M$ is compact

$$\delta^M = \sum_{t} (-1)^{|e_t|} e_t \otimes e'_t,$$

where $\{e_t\}$ denotes a linear basis of $H^*(M;\mathbb{k})$ and $\{e'_t\}$ the Poincaré dual basis.

Theorem 1 is then a direct consequence of the commutativity of the following diagram.

$$\begin{array}{ccc}
(T(n, H))^{p,*} & \xrightarrow{\theta^p} & (\mathbb{E}(n, H)^{p,*})^\vee \\
\downarrow d_1 & & \uparrow (d_1)^\vee \\
(T(n, H))^{p+1,*} & \xrightarrow{\theta^{p+1}} & (\mathbb{E}(n, H)^{p+1,*})^\vee,
\end{array}$$

where $\mathbb{E}^\vee := Hom(\mathbb{E},\mathbb{k})$ denotes the graded dual of the graded vector space $\mathbb{E}$. □
3 The bicomplex $C(n, A)$

Let $A$ be a commutative differential graded algebra. Recall from section 1 the isomorphism of bicomplexes

$$\Phi : E(n, A) \cong A(G(n)) := \oplus_{G \in \mathcal{G}} A^{\otimes l(G)} e_G,$$

where $\mathcal{G}$ denote the family of graphs of $G(n)$ that do not have two edges with the same target. Observe that each component of $G \in \mathcal{G}$ is a tree and that the number of edges is exactly $n - l(G)$.

We denote by $\mathcal{H}$ the subset of $\mathcal{G}$ consisting of graphs such that $S_1 = \{1\}$. We suppose that $A = k \oplus A^+$, with $A^+ = \oplus_{i > 0} A^i$, and we consider the new bicomplex

$$C(n, A) = \left( \oplus_{G \in \mathcal{H}} A \otimes (A^+)^{\otimes l(G) - 1} e_G, d \right)$$

whose differential $d = d' + d''$ is defined as follows: $d'' : C(n, A)^{*, k} \rightarrow C(n, A)^{*, k+1}$ is the internal differential coming from the differential on $A$, and

$$d' : C(n, A)^{k, *} \rightarrow C(n, A)^{k+1, *}, \quad d'(a_1 \otimes \ldots \otimes a_I e_G) = \sum_{1 < i < j \leq n} \alpha_{ij} e_G e_{ij},$$

with the conditions

- the term $e_G e_{ij}$ is zero if $G \cup \{(i, j)\} \notin \mathcal{H}$;
- In the other case, denote by $S_s$ and $S_t$, $1 < s < t$, the components of $i$ and $j$, then $G' = G \cup \{(i, j)\}$ has $l - 1$ components. The element $\alpha_{ij}$ is the element of $A \otimes (A^+)^{\otimes l - 2}$ defined by

$$\alpha_{ij} = (-1)^{|a_I|(|a_s| + |a_t| - 1)} a_1 \otimes a_2 \ldots \otimes a_{s-1} \otimes a_s a_t \otimes a_{s+1} \otimes \ldots \otimes a_t \otimes \ldots \otimes a_I$$

$$- \varepsilon \cdot a_1 a_s \otimes a_2 \ldots \otimes a_{s-1} \otimes a_s a_t \otimes a_{s+1} \otimes \ldots \otimes a_t \otimes \ldots \otimes a_I$$

$$- (-1)^{|a_I|(|a_2| + |a_{i-1}|)} a_1 a_t \otimes a_2 \ldots \otimes a_{s-1} \otimes a_t \otimes a_s \otimes a_{s+1} \otimes \ldots \otimes a_t \otimes \ldots \otimes a_I,$$

with $\varepsilon = (-1)^{|a_2|(|a_2| + |a_{i-1}|) + |a_I|(|a_{i-1}| + |a_{i-1}|)}$.

For instance, for $n = 3$ and $G$ the discrete graph, we have

$$d'(a_1 a_2 a_3) = \left[ a_1 a_2 a_3 - a_1 a_2 a_3 - (-1)^{|a_1| |a_2|} a_1 a_3 a_2 \right] e_{23}.$$

**Proposition 3.** There exists a quasi-isomorphism of bicomplexes

$$\tilde{\varphi} : C(n, A) \cong E(n, A)$$

which is natural in $A$.

**Proof.** Let us consider for each $l \geq 2$ the linear map

$$\gamma_l : A^{\otimes l} \rightarrow A^{\otimes l}$$

defined by

$$\gamma_l(a_1, \ldots, a_l) = \sum (-1)^k \varepsilon(i_1, \ldots, i_k) a_1 a_{i_1} \ldots a_{i_k} a'_2 \otimes \cdots \otimes a'_l,$$

where $a'_j = a_j$ if $j \notin \{i_1, \ldots, i_k\}$ and equals 1 otherwise. The sum is taken over all the ordered (possibly empty) subsets $i_1 < i_2 < \ldots < i_k$ of $\{2, \ldots, l\}$, and $\varepsilon(i_1, \ldots, i_k)$ is the graded signature of the permutation $a_1 a_2 \ldots a_l \mapsto a_1 a_{i_1} \ldots a_{i_k} a'_2 \ldots a'_l$. We define

$$\varphi : C(n, A) \rightarrow A\mathcal{G}(n), \quad \varphi(a_1 \otimes \ldots \otimes a_I e_G) = \gamma_l(a_1, \ldots, a_I) e_G,$$

which is an injective linear map that commutes with the differentials.
Now observe that the composite $\bar{\varphi} : C(n, A) \xrightarrow{\Phi^{-1}_{\ast}} E(n, A) \to \bar{E}(n, A)$ remains injective and that for any $r \geq 2$\)

$$\text{Im} \bar{\varphi} \cdot e_r = 0.$$\)

Let denote by $(\Gamma, d)$ the cokernel of $\bar{\varphi}$:

$$(\Gamma, d) = (\bigoplus_{G \in \mathcal{G}} C(G)e_G, d),$$

with

$$C(G) = \left\{ \begin{array}{ll}
A^i, & \text{if } G \text{ contains an edge } (1, r) \text{ for some } r \\
\bigoplus_{s=1}^{i-1} A^{i-s} \otimes \mathbb{k} \otimes A^{s-1}, & \text{otherwise}
\end{array} \right.$$\)

The differential $d$ induced by $d$ is the sum $d = d'' + \bar{d}_0 + \bar{d}_1$, where $d''$ is the internal differential, $\bar{d}_0$ consists into the multiplication by $\sum_s e_{1s}$ and $\bar{d}_1$ comes from the multiplication by $\sum_{2 \leq \ell < s} e_{r,s}$.

We prove that $H^\ast(\Gamma, d_0) = 0$, which implies the result by an elementary spectral sequence argument. Let $\mathcal{H}$ defined as before, and consider the partition

$$\mathcal{G} = \mathcal{H} \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \ldots \cup \mathcal{G}_n,$$

where $\mathcal{G}_r, r \geq 2$, denotes the set of graphs $G$ such that $r$ is the smallest vertex $\neq 1$ in the connected component of $1$. We denote by $\mathcal{G}'_r$ the set of graphs $G$ in $\mathcal{G}$ such that the components of $1$ and $r$ are different and do not contain any of the vertices $2, \ldots, r-1$. This implies a direct sum decomposition of $\Gamma$:

$$\Gamma = \bigoplus_{r=2}^n (\Gamma_r \oplus \Gamma'_r),$$

where $\Gamma_r = \bigoplus_{G \in \mathcal{G}_r} D(G)e_G$ with $D(G) = D_1(G) \otimes \ldots \otimes D_l(G)$ and

$$D_1(G) = \left\{ \begin{array}{ll}
A^+ & \text{if } S_1 \text{ contains one of the vertex } 2, \ldots, r-1 \\
A & \text{otherwise,}
\end{array} \right.$$\)

and where, $\Gamma'_r = \bigoplus_{G \in \mathcal{G}'_r} D'(G)e_G$, with $D'(G) = D'_1(G) \otimes \ldots \otimes D'_l(G)$, and

$$D'_1(G) = \left\{ \begin{array}{ll}
A^+ & \text{if } S_1 \text{ contains one of the vertex } 2, \ldots, r-1 \\
\mathbb{k} & \text{if } S_1 \text{ contains the vertex } r \\
A & \text{otherwise.}
\end{array} \right.$$\)

To establish the decomposition $\Gamma = \bigoplus_{r=2}^n (\Gamma_r \oplus \Gamma'_r)$, let $a = a_1 \otimes \ldots \otimes a_l e_G$ be an element in $\Gamma$. When $G \in \mathcal{H}$, we denote by $r$ the smallest vertex contained in a component $S_i$ such that $a_i \in k$. In this case $a \in \Gamma_r$. Suppose now that $G \notin \mathcal{G}_r$. If for each $i$, $2 \leq i < r$, the vertex $i$ belongs to a component $C_s$ with $a_s \in A^+$, then $a \in \Gamma_r$. In the other case, denote by $t$ the smallest vertex in a component $C_s$ with $a_s \notin k$, then $a \in \Gamma'_t$.

The vector space $\Gamma_2 \oplus \Gamma'_2$ is a subcomplex of $\Gamma$, and the multiplication by $e_{12}$ induces an isomorphism $\Gamma_2 \xrightarrow{\cong} \Gamma_2$; in the quotient $\Gamma/(\Gamma_2 \oplus \Gamma'_2)$, the vector space $\Gamma_3 \oplus \Gamma'_3$ is a subcomplex and the multiplication by $e_{13}$ is an isomorphism $\Gamma_3 \xrightarrow{\cong} \Gamma_3$. More generally, $\Gamma_r \oplus \Gamma'_r$ is a subcomplex in the quotient $\Gamma/((\oplus_{s \leq r} \Gamma_s) \oplus (\oplus_{s < r} \Gamma'_s))$, the vector space $\Gamma_r \oplus \Gamma'_r$ is a subcomplex and the multiplication by $e_{1r}$ is an isomorphism $\Gamma_r \xrightarrow{\cong} \Gamma_r$. This shows that $(\Gamma, d_0)$ is acyclic. \hfill $\square$

Theorem 2 of the Introduction follows directly from the next Proposition

**Proposition 4.** Let $M$ be a simply connected compact oriented manifold. Then $E_{n-1} = E_{\infty}$ in the Bendersky-Gitler spectral sequence for $H^\ast(F(M; n); \mathbb{k})$.

**Proof.** The $E_2$ term of the Bendersky-Gitler spectral sequence is isomorphic as a bigraded vector space to the $E_2$ term of the spectral sequence associated to the bicomplex $C(n, A)$. Since $C(n, A)^{\geq n-1, \ast} = 0$, we get $E^{p,q}_2 = 0$ for $p > n - 2$. \hfill $\square$
If $H$ is a differential graded commutative algebra then $C(3, H)$ is the complex $(d^n = 0)$

$$(H \otimes H^+ \otimes H^+) \xrightarrow{d_1} (H \otimes H^+) \cdot e$$

where $e$ is a variable of degree 1, and

$$d_1(a \otimes b \otimes c) = a \otimes bc - ab \otimes c - (-1)^{|c||b|} ac \otimes b.$$  

When $H = H^*(M; \mathbb{k})$, the kernel of $d_1$, $E_{20}^*$, is the image $I_M$ of the map

$$H^*(M^3; \mathbb{k}) \rightarrow H^*(F(M, 3); \mathbb{k})$$

induced by the inclusion $F(M, 3) \hookrightarrow M^3$. The cokernel of $d_1$ is the first Hochschild homology group of the unital algebra $H$ with coefficients in the bimodule $H$, $HH_1(H; H)$, and by (III, 1.1.10) is isomorphic to the $H$-module of Kähler differentials $\Omega^1_{H/\mathbb{k}}$ on $H$. It follows from the Lefschetz duality $H^p(M^n, D^n M) \cong H_{nm-p}(F(M, n); \mathbb{k})$ that

**Proposition 5.** For a $m$-dimensional oriented closed manifold $M$ there is a short exact sequence

$$0 \rightarrow (I_M)^* \rightarrow H^*(F(M, 3); \mathbb{k}) \rightarrow (\text{Hom}(\Omega^1_{H/\mathbb{k}}, \mathbb{k}))^{*+1-3m} \rightarrow 0.$$

### 4 $F(M, 4)$ and Massey products

For a commutative differential graded algebra $(A, d)$, the double complex $C(4, A)$ has the form

$$0 \rightarrow A \otimes (A^+)\otimes^3 \xrightarrow{d'} C(4, A)^{1,*} \xrightarrow{d'} C(4, A)^{2,*} \rightarrow 0,$$

with

$$C(4, A)^{1,*} = (A \otimes A^+ \otimes A^+) e_{23} + (A \otimes A^+ \otimes A^+) e_{24} + (A \otimes A^+ \otimes A^+) e_{34},$$

$$C(4, A)^{2,*} = (A \otimes A^+) e_{23} e_{24} + (A \otimes A^+) e_{23} e_{34}$$

$$d'(a \otimes b \otimes c \otimes d) = (a \otimes bc \otimes d - ab \otimes c \otimes d - (-1)^{|b||c|} ac \otimes b \otimes d) e_{23}$$

$$+ (a \otimes b \otimes cd - (-1)^{|b||c|} ac \otimes b \otimes d - (-1)^{|d|(|b||c|)} ad \otimes b \otimes c) e_{34}$$

$$+ ((-1)^{|d||c|} a \otimes bd \otimes c - (-1)^{|d||c|} ab \otimes d \otimes c - (-1)^{|d|(|b||c|)} ad \otimes b \otimes c) e_{24},$$

$$d'((a \otimes b \otimes c) e_{23}) = \Delta(a, b, c) (e_{23} e_{34} + e_{23} e_{24})$$

$$d'((a \otimes b \otimes c) e_{24}) = -\Delta(a, b, c) e_{23} e_{24}$$

$$d'((a \otimes b \otimes c) e_{34}) = -\Delta(a, b, c) e_{23} e_{34},$$

where $\Delta(a, b, c) = (a \otimes bc - ab \otimes c - (-1)^{|b||c|} ac \otimes b)$

From Theorem B we deduce

**Proposition 6.** Let $\mathbb{k}$ be a field of characteristic zero, the $E_2$ term of the Bendersky-Gitler spectral sequence for $H^*(F(M, 4); \mathbb{k})$ satisfies

$$E_2^{2,*} = \Omega^1_{H/\mathbb{k}} e_{23} e_{24} \otimes \Omega^1_{H/\mathbb{k}} e_{23} e_{34},$$

with $H = H^*(M; \mathbb{k})$.  

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In order to detect nonzero element in $E_2^{2,*}$, we now describe a special quotient of $E_2^{2,*}$. Denote $H = H^*(M; \mathbb{k})$ and $Q(H) = H^+/(H^+ \cdot H^+)$, the space of indecomposable elements, and let

$$\psi : Q(H) \otimes Q(H) \to Q(H) \otimes Q(H)$$

be the linear map defined by $\psi(a \otimes b) = a \otimes b - (-1)^{|a||b|} b \otimes a$. The image of $\psi$ is the sub-vector space of $\Sigma_2$-invariants elements in $Q(H) \otimes Q(H)$, where the action of the generator $\tau$ of $\Sigma_2$ is defined by

$$\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

We now consider the composition

$$H \otimes H^+ \xrightarrow{\cong} (\mathbb{k} \otimes H^+) \oplus (H^+ \otimes H^+) \to (\mathbb{k} \otimes Q(H)) \oplus (Q(H) \otimes Q(H)) \xrightarrow{\psi} (\mathbb{k} \otimes Q(H)) \oplus (Q(H) \otimes Q(H))\Sigma_2.$$

By definition of $d'$, this morphism induces a surjective map from $H^{2,*}(C(4, H), d_1)$ onto

$$[([\mathbb{k} \otimes Q(H)) \oplus (Q(H) \otimes Q(H))\Sigma_2]) e_{23} e_{34} + [([\mathbb{k} \otimes Q(H)) \oplus (Q(H) \otimes Q(H))\Sigma_2]) e_{23} e_{24}.$$

Recall now the definition of a triple matrix Massey product ([12]). Consider three matrices $L, B, C$

$$L = (a_1, \ldots, a_r), \quad B = \begin{pmatrix} b_{11} & \ldots & b_{1s} \\ \ldots & \ldots & \ldots \\ b_{r1} & \ldots & b_{rs} \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix},$$

whose entries are cocycles in $A$ such that

$$L \cdot B = (d(x_1), \ldots, d(x_s)) \quad \text{and} \quad B \cdot C = \begin{pmatrix} d(y_1) \\ \vdots \\ d(y_r) \end{pmatrix}.$$

The triple matrix Massey product $\langle L, B, C \rangle$ is defined, up to some indeterminacy, as the class of the cocycle $\sum_i x_i c_i - \sum_j (-1)^{|a_j|} a_j y_j$. Nonetheless the residual class of $\langle L, B, C \rangle$ in the quotient $Q(H^*(A))$ is uniquely defined.

**Theorem 4.** Let $\mathbb{k}$ be of characteristic zero, $M$ be simply connected and $N = M \#(S^2 \times S^{m-2})$ be the connected sum of $M$ with a product of two spheres. If there is a triple matrix Massey product in $H^*(M; \mathbb{k})$ whose residual class in $Q(H^*(M; \mathbb{k}))$ is nonzero, then the Cohen-Taylor spectral sequence for $H^*(F(N, A); \mathbb{k})$ does not collapse at the $E_2$-term.

**Proof.** Let us take a Sullivan model $(A, d)$ for $M$ over the field $\mathbb{k}$ that satisfies the following properties: $A$ is finite dimensional, $A^0 = \mathbb{k}$, $A^1 = 0$, $A^m = \mathbb{k} \omega$, and $A^{>m} = 0$. We choose for the product $S^2 \times S^{m-2}$ the model $((x, y)/(x^2, y^2), d = 0)$ with $x$ in degree 2 and $y$ in degree $m - 2$. Therefore, ([3]), a model for the connected sum $N = (S^2 \times S^{m-2})\# M$ is given by the differential graded algebra

$$(A \times \mathbb{k} \wedge (x, y)/(x^2, y^2))/\omega - xy, d).$$

Suppose that $A$ admits a triple matrix Massey product $(L, B, C)$ that is non zero in $H^+(M)/(H^+(M) \cdot H^+(M))$, then with the above notation for $L$, $B$ and $C$, we consider the element

$$u = \sum_{ij} x \otimes a_i \otimes b_{ij} \otimes c_j - \sum_{ij} (-1)^{|c_j| + |b_{ij}| + |c_j| + |a_i| + |b_{ij}| + |a_i|} x \otimes c_j \otimes b_{ij} \otimes a_i.$$
Then $d''u = 0$ and $d_1([u]) = 0$ in the first spectral sequence of the bicomplex $C(4, A)$. More precisely,

$$d'(u) = \left( \sum_j x \otimes d(x_j) \otimes c_j - (-1)^{|c_j|+|a_i|+|b_j||a_i|} \sum_i x \otimes d(y_i) \otimes a_i \right) e_{23}$$

$$+ \left( \sum_i (-1)^{|a_i|} x \otimes a_i \otimes d(y_i) - (-1)^{|c_j|+|c_j|+|b_j|+|c_j|} \sum_j x \otimes c_j \otimes d(x_j) \right) e_{34}$$

Therefore

$$d_2([u]) = [x \otimes (L, B, C)] e_{23} e_{24} + 2[x \otimes (L, B, C)] e_{23} e_{34}.$$ 

The element $d_2([u])$ is nonzero in $E_2$ because its image is nonzero in

$$\left( (Q(H) \otimes Q(H))^\Sigma_2 \right) e_{23} e_{24}.$$ 

\[\square\]

Consider now the case where $a, b, c$ and $d$ are elements in $H^*(M; \mathbb{k})$, such that $ab = ac = ad = bc = bd = cd = 0$. In the first spectral sequence of the bicomplex $C(4, H)$ we obtain the formula

$$d_2([a \otimes b \otimes c \otimes d]) = \left( (-1)^{|a|} a \otimes (b, c, d) + (a, b, c) \otimes d \right)$$

$$+ \left( (-1)^{|a|+|b|+|c|} a \otimes (b, c, d) \right) e_{23} e_{34}$$

$$+ \left( (-1)^{|a|+|b|+|c|} a \otimes (b, c, d) \right) e_{23} e_{34}$$

We deduce:

**Theorem 3.** Let $\mathbb{k}$ be of characteristic zero and $M$ be simply connected. Suppose there exist indecomposable elements $a, b, c$ and $d \in H^*(M; \mathbb{k})$, such that

- $ab = ac = ad = bc = bd = cd = 0$,
- the triple Massey product $\langle b, c, d \rangle$ is a nonzero indecomposable element in cohomology, and
- $a$ is not a linear combination of the elements $b, c, d$ and $\langle b, c, d \rangle$,

then the Cohen-Taylor spectral sequence for $H^*(F(M, 4); \mathbb{k})$ does not collapse at the $E_2$-term.

**Proof.** By formula (*), the image of $d_2([a \otimes b \otimes c \otimes d])$ in $(Q(H) \otimes Q(H))^\Sigma_2$ is nonzero because the element $a$ is not a linear combination of the elements $b, c, d$ and $\langle b, c, d \rangle$. \[\square\]

**Example.** Let $M$ be the sphere tangent bundle to the manifold $S^2 \times S^2$. A Sullivan minimal model for $M$ is given by $(\wedge (x, y, u, v, t), d)$ with $|x| = |y| = 2$, $|u| = |v| = |t| = 3$, $d(u) = x^2$, $d(v) = y^2$ and $d(t) = xy$ (\textcircled{3}). A basis of the cohomology is given by $1, [x], [y], [tx - uy], [ty - vx], [txy - uy^2]$. The cohomology classes in degree 5 are usual triple Massey products represented by indecomposable elements in cohomology. Formula (*) gives

$$d_2([x] \otimes [x] \otimes [y] \otimes [y]) = ([x] \otimes [ty - vx] - [ty - vx] \otimes [x] - 2[tx - uy] \otimes [y]) e_{23} e_{24} + \mu e_{23} e_{34}.$$ 

The component of $e_{23} e_{34}$ is not symmetric, hence not zero in $E_2^{2,2}$. The Bendersky-Gitler spectral sequence for $H_{m,n-*}(F(M, 4); \mathbb{Q})$ does not collapse at the $E_2$ term.
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