Restriction Theorems for Principal Bundles and Some Consequences

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Abstract

The aim of this paper is to give a proof of the restriction theorems for principal bundles with a reductive algebraic group as structure group in arbitrary characteristic. Let $G$ be a reductive algebraic group over any field $k = \bar{k}$, let $X$ be a smooth projective variety over $k$, let $H$ be a very ample line bundle on $X$ and let $E$ be a semistable (resp. stable) principal $G$-bundle on $X$ w.r.t. $H$. The main result of this paper is that the restriction of $E$ to a general smooth curve which is a complete intersection of ample hypersurfaces of sufficiently high degree’s is again semistable (resp. stable).

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1 Introduction

Around 1981-82, V. Mehta and A. Ramanathan proved the following important theorem (see [3]) : Let $X$ be a smooth projective variety over $k = \bar{k}$ with a chosen polarisation and let $V$ be a torsion free coherent sheaf on it. Then the restriction of $E$ to a general, non-singular curve which is a complete-intersection of ample hypersurfaces of sufficiently high degree’s is again semistable. The corresponding theorem for principal bundles with a reductive algebraic group as structure group over a field of characteristic zero follows immediately since the semistability of a principal $G$-bundle in characteristic zero is equivalent to the semistability of its adjoint bundle and hence the semistable restriction theorem for the adjoint bundle implies the theorem for the principal $G$-bundle as well. V. Mehta and A. Ramanathan later also gave a proof of the stable restriction theorem for torsion-free coherent sheaves (see [4]). However the semistable restriction theorem in positive characteristic and the stable restriction theorem in any characteristic remain open for principal bundles. The aim of this paper is to prove these two restriction theorems. The basic idea of the proofs is similar to that of the semistable restriction theorem in [3]. The proofs given here are characteristic free. The paper is arranged as follows:
In section 2, we introduce some preliminary notions and set up some notations which will be used throughout the paper.

In section 3 we recall a degeneration argument which is central to the proofs in [3] and [4] and also draw some consequences out of it.

In section 4 we prove the semistable restriction theorem for principal bundles using the degeneration argument introduced in section 3.

In section 5 we prove the stable restriction theorem for principal bundles analogues to the stable restriction theorem for torsion-free coherent sheaves proved in [4]. The proof given here is different and substantially simpler than the proof of the stable restriction theorem in [4].

2 Preliminaries

In this section we set up some notation and recall some basic facts which will be used in the paper. Many of these have been taken from [1] with only minor changes. \(X\) will always stand for a smooth projective variety defined over a field \(k = \bar{k}\) of arbitrary characteristic. \(H\) will denote the chosen polarisation on \(X\). Let \(G \supset B \supset T\) be a reductive group, together with a chosen Borel subgroup and a maximal torus. As usual, \(X^* (T)\) and \(X_*(T)\) will respectively denote the groups of all characters and all 1-parameter subgroups of \(T\). We choose once for all, a Weyl group invariant positive definite bilinear form on \(\mathbb{Q} \otimes X^* (T)\) taking values in \(\mathbb{Q}\). This, in particular, will allow us to identify \(\mathbb{Q} \otimes X_*(T)\) with \(\mathbb{Q} \otimes X^* (T)\). Let \(\Delta \subset X^* (T)\) be the corresponding simple roots. Let \(\omega_\alpha \in \mathbb{Q} \otimes X^* (T)\) denote the fundamental dominant weight corresponding to \(\alpha \in \Delta\), so that \(\langle \omega_\alpha , \beta^\vee \rangle = \delta_{\alpha, \beta}\) where \(\beta^\vee \in \mathbb{Q} \otimes X_*(T)\) is the simple coroot corresponding to \(\beta \in \Delta\). Note that each \(\omega_\alpha\) is a non-negative rational linear combination of the simple roots \(\alpha\). Recall that the closed positive Weyl chamber \(\overline{C}\) is the subset of \(\mathbb{Q} \otimes X_*(T)\) defined by the condition that \(\mu \in \overline{C}\) if and only if \(\langle \alpha , \mu \rangle \geq 0\) for all \(\alpha \in \Delta\). The standard partial order on \(\overline{C}\) is defined by putting \(\mu \leq \pi\) if \(\omega_\alpha (\mu) \leq \omega_\alpha (\pi)\) for all \(\alpha \in \Delta\), and \(\chi (\mu) = \chi (\pi)\) for every character \(\chi\) of \(G\).

By definition, all roots and weights in \(X^* (T)\) are trivial on the connected component \(Z_0(G) \subset T\) of the center of \(G\).

Canonical reductions of principal bundles

Let \(T \subset B \subset G\) be as above. Recall that a principal \(G\)-bundle \(E\) on \(X\) is said to be semistable (resp. stable) w.r.t. \(H\) if for any reduction \(\sigma : U \to E/P\) of the structure group to a parabolic \(P \subset G\) defined on a large open set \(U \subseteq X\) (one whose complement in \(X\) has codimension at least 2) and any dominant character \(\chi : P \to k^*\), the line bundle \(\chi_* \sigma^* E\) on \(U\) has non-positive degree (resp. negative degree).

Note that although \(L_\sigma\) is only defined on \(U\), it extends uniquely (upto a unique isomorphism) to a line bundle on all of \(X\) and hence its degree is well-defined.
Note that to test the semistability (resp. stability) of a principal $G$-bundle, it suffices to consider reductions to standard maximal parabolics (i.e those containing the chosen Borel $B$).

**Definition 1.** A canonical reduction of a principal $G$-bundle $E$ is a pair $(P, \sigma)$ where $P$ is a standard parabolic subgroup of $G$ and $\sigma : U \to E/P$ is a reduction of the structure group to $P$ such that the following two conditions hold.

1. If $\rho : P \to L = P/U$ is the Levi quotient of $P$ (where $U$ is the unipotent radical of $P$) then the principal $L$-bundle $\rho, \sigma^* E$ is semistable.

2. For any non-trivial character $\chi : P \to k^*$ whose restriction to the chosen maximal torus $T \subset B \subset P$ is a non-negative linear combination $\sum n_i \alpha_i$ of simple roots $\alpha_i \in \Delta$ with at least one $n_i \neq 0$, we have $\deg(\chi, \sigma^* E) > 0$.

It has been shown by Behrend (see [5]) (see [6] for a different, more bundle-theoretic proof when char $k = 0$) that when $X$ is a non-singular curve, each principal $G$-bundle on $X$ admits a unique canonical reduction.

Given a reduction $(P, \sigma)$ of $E$, we get an element $\mu_{(P, \sigma)} \in \mathbb{Q} \otimes X_*(T)$ defined by

$$\langle \chi, \mu_{(P, \sigma)} \rangle = \begin{cases} 
\deg(\chi, \sigma^* E) & \text{if } \chi \in X^*(P), \\
0 & \text{if } \chi \in I_P.
\end{cases}$$

If $(P, \sigma)$ is the canonical reduction of $E$, then the element $\mu_{(P, \sigma)}$ is called the HN type of $E$, and is denoted by $\text{HN}(E)$. If $\alpha \in \Delta - I_P$, then $\langle \alpha, \text{HN}(E) \rangle = \deg(\alpha, \sigma^* E) \geq 0$. As $\langle \beta, \text{HN}(E) \rangle = 0$ for all $\beta \in I_P$, we see that $\text{HN}(E)$ is in the closed positive Weyl chamber $\overline{C}$, in fact, in the facet of $\overline{C}$ defined by the vanishing of all $\beta \in I_P$.

Note that a principal bundle $E$ of type $\text{HN}(E) = \mu$ is semistable if and only if $\mu$ is central, that is, $\mu = av$ for some 1-parameter subgroup $v : k^* \to Z_0(G)$ and $a \in \mathbb{Q}$.

Given the HN-type $\mu = \text{HN}(E)$ of $E$, we can recover the corresponding standard parabolic $P$ as follows. Let $I_\mu \subset \Delta$ be the set of all simple roots $\beta$ such that $\langle \beta, \mu \rangle = 0$. Then $I_\mu$ is exactly the set of inverted simple roots which defines $P$. Alternatively, let $n \geq 1$ be any integer such that $v = n\mu \in X_*(T)$. Then the $k$-valued points of $P$ are all those $g$ for which $\lim_{t \to 0} v(t)g v(t)^{-1}$ exists in $G$.

Let $E$ be a principal $G$-bundle on $X$, let $(P, \sigma)$ be its canonical reduction, and let $(Q, \tau)$ be any reduction to a standard parabolic. Let $\text{HN}(E) = \mu_{(P, \sigma)}$ and $\mu_{(Q, \tau)}$ be the corresponding elements of $\mathbb{Q} \otimes X_*(T)$ (the element $\text{HN}(E)$ lies in the closed positive Weyl chamber $\overline{C}$, but $\mu_{(Q, \tau)}$ need not do so). Then for each $\alpha \in \Delta$ we have the inequality

$$\langle \omega_\alpha, \text{HN}(E) \rangle = \langle \omega_\alpha, \mu_{(P, \sigma)} \rangle \geq \langle \omega_\alpha, \mu_{(Q, \tau)} \rangle$$

where $\omega_\alpha \in \mathbb{Q} \otimes X_*(T)$ is the fundamental dominant weight corresponding to $\alpha$. Moreover, if each of the above inequalities is an equality then $(P, \sigma) = (Q, \tau)$.
We now recall a basic fact regarding semistability of bundles in families:

**Lemma 2.** (See [2], Proposition 5.10) Let $S$ be a finite-type irreducible scheme over $k$ and let $f : Z \to S$ be a smooth, projective morphism of relative dimension one. Let $E$ be a principal $G$-bundle on $Z$. Let $\eta$ denote the generic point of $S$ and $Z_\eta$ denote the generic fibre of $f$. If the $HN(E_\eta) = \mu$, then there exists a non-empty open subset $U \subseteq S$ such that $\forall s \in U, HN(E_s) = \mu$. In particular, if $E \mid_{Z_\eta}$ is semistable, then there exists a non-empty open set $U \subseteq X$ such that $\forall s \in U$, $E \mid_{Z_s}$ is semistable.

We now recall some facts regarding complete intersection subvarieties from [3].

Let $X$ and $H$ be as before. For any non-negative integer $m$, let $S_m$ denote the projective space $\mathbb{P}(H^s(X, H^m))$. For any $r$-tuple of integers $(m_1, \cdots, m_r)$, where $0 < r < n$, let $S_m$ denote the product $S_{m_1} \times \cdots \times S_{m_r}$. Let $Z_m \subset X \times S_m$ denote the correspondence variety defined by $Z_m = \{(x, s_1, \cdots, s_r) \in X \times S_m \mid s_i(x) = 0 \forall x \in X\}$. Thus we have the following diagram:

$$
\begin{array}{ccc}
X \times S_m & \supset & Z_m \\
\downarrow p_m & & \downarrow q_m \\
X & & S_m \\
\end{array}
$$

where $p_m$ and $q_m$ are the projections. The fiber over a point $(s_1, \cdots, s_r) \in S_m$ can be thought of as a closed subscheme of $X$ embedded via $p_m$ and defined by the ideal generated by $(s_1, \cdots, s_r)$ in the homogeneous coordinate ring of $X$. Such a subscheme will be called a subscheme of type $m$. $p_m$ is a fibration with the fiber over a point $x \in X$ embedded in $S_m$ via $q_m$ as the product of hyperplanes $H_1 \times \cdots \times H_r$ where $H_i \in S_{m_i}$ consists of those sections vanishing at $x$. This shows that $Z_m$ is non-singular.

By Bertini's theorem, the generic fiber of $q_m$, say $Y_m \hookrightarrow S_m$ thought of as a closed subscheme of $X$ via $p_m$ is a geometrically irreducible, smooth, complete intersection subscheme of $X$. $Y_m$ will be called the generic subscheme of type $m$. In the case when $r = n - 1$, we call $Y_m$, the generic complete-intersection curve of type $m$.

We now recall two important propositions from [3].

**Proposition 3.** Let $\dim X = n \geq 2$. Let $m = (m_1, \cdots, m_r)$ with $1 \leq r \leq n - 1$ be a $r$-tuple of integers with each $m_i \geq 3$. Then the natural map $Pic(X) \to Pic(Y_m)$ is a bijection.
Proposition 4. (Enrique-Severi) Let $X \subseteq \mathbb{P}^n$ be a non-singular projective variety corresponding to $H$. Let $E$ be a vector bundle on $X$. Then there exists an integer $m_i$ such that if $m = (m_1, \cdots, m_r)$, $1 \leq r < n$ is an $r$-tuple of integers with each $m_i \geq m$, then for a non-empty open subset $U_m \subseteq S_m$, if $s = (s_1, \cdots, s_r) \in U_m$ is any point, then the restriction map $H^s(X, E) \to H^s(X_i, E|_{X_i})$, where $X_i$ is the closed subscheme of $X$ defined by the ideal $(s_1, \cdots, s_r)$, is a bijection.

Remark 5. (Enrique-Severi for a bounded family) It follows easily from proposition 4 that if $E_i$ is a bounded family of vector bundles on $X$, then again one can find an integer $m_i$ satisfying the property given in proposition 4.

Proposition 6. Let $A$ be a discrete valuation ring with quotient field $K$ and residue field $k$. Let $S = \text{Spec } A$. Let $Y \to S$ be a flat, projective morphism having relative dimension 1 such that $Y$ and the generic curve are both non-singular and the special curve is reduced with non-singular irreducible components $C_1, \cdots, C_r$. Let $E$ be a principal $G$-bundle on $Y$. Let $Y_K$ and $Y_i$ denote the generic and the special curve respectively. Let $E_K$ and $E_k$ denote the restrictions of $E$ to the generic and special special curve respectively. Let $E_k^i$ denote the restriction of $E_k$ to the component $C_i$. Suppose the standard parabolics underlying the canonical reduction of $E_k$ and the canonical reductions of $E_k^i$ are the same for each $i$, say $P$. Let $\chi$ be a fundamental weight corresponding to a non-inverted simple root of $P$. Let $E_P^i$ (resp. $E_{P_k}^i$) denotes the principal $P$-bundle underlying the canonical reduction of $E_k$ on $Y_K$ (resp. $E_k^i$ on $C_i$). Then we have $\deg (\chi, E_{P_k}^i) \leq \sum_{i=1}^r \deg (\chi, E_{P_k}^i)$.

Proof Since $\chi^{-1}$ is an anti-dominant character, we see that the associated line bundle $L_{\chi^{-1}}$ on $G/P$ is very ample and hence generated by global sections. Hence, the evaluation map $e : H^s(G/P, L_{\chi^{-1}}) \to L_{\chi^{-1}}|_{[eP]}$ is surjective, where $L_{\chi^{-1}}|_{[eP]}$ denoted the fiber of $L_{\chi^{-1}}$ over the identity coset $[eP] \in G/P$. Hence the dual map $L_{\chi^{-1}}|_{[eP]}^\vee : H^s(G/P, L_{\chi^{-1}})^\vee \to H^0(G/P, L_{\chi^{-1}})^\vee$ is injective and hence defines a one-dimensional subspace of $H^0(G/P, L_{\chi^{-1}})^\vee$. The scheme-theoretic stabilizer of this subspace is exactly $P$ and on it, $P$ acts by the character $\chi$. Thus we have found a representation $\rho : G \to GL(V)$ with the property that $V$ has a one-dimensional subspace whose scheme-theoretic stabilizer in $G$ is exactly $P$ and on which $P$ acts by the character $\chi$. Let $Q$ be the maximal parabolic in $GL(V)$ stabilizing this one-dimensional subspace. We thus get a closed embedding $i : G/P \hookrightarrow GL(V)/Q$.

Let $F$ denote the principal $GL(V)$-bundle obtained by extension structure group of $E$ via $\rho$ and let $F(V)$ denote the rank $n$ vector bundle corresponding to $F$. Let $F_K$ (resp. $F_k$) denote the restriction of $F$ to $Y_K$ (resp. $Y_k$). Similarly let $F(V)_K$ (resp. $F(V)_k$) denote the restrictions of $F(V)$ to $Y_K$ (resp. $Y_k$). Corresponding to the embedding $i$, we get a closed embedding of $E/P \hookrightarrow F/Q$. Since $Q$ is a stabilizer of a 1-dimensional subspace of $V$, it
follows that $F/Q$ is naturally the same as $\mathbb{P}(F(V))$. Since any reduction of structure group of $E$ to $P$ corresponds functorially to a section of $E/P$, any $P$-reduction of $E$ naturally induces a $Q$-reduction of $F$, or equivalently, gives a line sub-bundle of $F(V)$.

Let $\alpha_K: Y_K \to E_K/P$ denote the canonical reduction of $E_K$ over $Y_K$. By the valuative criterion of properness applied to the projection $E/P$ over $S$, we see that this reduction spreads to a reduction $\alpha: Y \setminus F \to E/P$ where $F$ is a finite subset of $Y_k$. Let $\alpha_k$ denote the restriction of $\alpha$ to $Y_k \setminus F$. Again by applying the valuative criterion to the projection $E_k/P \to Y_k$, we see that the reduction $\alpha_k$ extends to a reduction, call it $\gamma_k$, over all of $Y_k$. Let $E^{\alpha_k}_P$ denote the restrictions of the principal $P$-bundle on $Y_k$ corresponding to $\gamma_k$ to the curves $C_i$. Let $\overline{\alpha}_K$ (resp. $\overline{\gamma}_k$) denote the corresponding $Q$-reduction on $F_K$ (resp. $F_k$) obtained by composing the reduction $\alpha_K$ (resp. $\gamma_k$) via the closed embedding $E_K/P \hookrightarrow F_K/Q$ (resp. $E_k/P \hookrightarrow F_k/Q$). Let $L_{\overline{\gamma}_k}$ (resp. $L_{\overline{\gamma}_i}$) denote the line sub-bundles of $F(V)_K$ (resp. $F(V)_k$) corresponding to $\overline{\alpha}_K$ (resp. $\overline{\gamma}_k$). It is easy to see that $L_{\overline{\gamma}_k}$ is isomorphic to $\chi_s E_{P_k}$. By the properness of the Quot scheme, the short exact sequence $0 \to L_{\overline{\gamma}_k} \to F(V)_k \to F(V)_K/L_{\overline{\gamma}_k} \to 0$ over $Y_K$ can be completed to an exact sequence $0 \to L_{\overline{\gamma}_k} \to F(V) \to F(V)/L_{\overline{\gamma}_k} \to 0$ over $Y$ such that $F(V)/L_{\overline{\gamma}_k}$ is flat over $S$. Let $L_{\overline{\gamma}_k}$ denote the restriction of $L_{\overline{\gamma}_k}$ to $X_k$ which can be thought of as a rank 1 subsheaf of $F(V)_k$. Let $\tilde{L}_{\overline{\gamma}_k}$ denote the saturation of $L_{\overline{\gamma}_k}$, i.e. the line sub-bundle of $F(V)_k$ obtained by pulling up the torsion on $X_k$ in the cokernel $F(V)_k/L_{\overline{\gamma}_k}$. Then it is easy to see that $\tilde{L}_{\overline{\gamma}_k}$ is the line bundle obtained by extension of structure group of $\gamma_k$ via $\chi$. Clearly $\deg L_{\overline{\gamma}_k} = \deg L_{\overline{\gamma}_k} \leq \deg \tilde{L}_{\overline{\gamma}_k} = \sum_{i=1}^r \deg (\chi_s E_{P_i}^{\overline{\gamma}_k}) \leq \sum_{i=1}^r \deg (\chi_s E_{P_i}^i)$, where the last inequality is by the definition of the canonical reduction. Hence $\deg (\chi_s E_{P_k}) \leq \sum_{i=1}^r \deg (\chi_s E_{P_i}^i)$, thereby completing the proof of the lemma. \hfill $\square$

Remark 7. : The above proof also shows that if $E_k$ is semistable restricted to every irreducible component of $Y_k$, then $E_K$ is also semistable. It also follows from the proof of the above proposition that if the inequality in the above proposition is actually an equality, then the canonical reduction $E_{P_k}$ on the generic curve spreads to give a $P$-reduction on all of $Y$ and whose restriction to every irreducible component of the special curve coincides with the canonical reduction on that component.

3 A degeneration argument

In this section we recall a basic result from [3] regarding degenerating family of curves and draw some easy consequences from it.
Proposition 8. (See [3], Proposition 5.2)
Let \( l = m + r \). Let \( U_m \subseteq S_m \) and \( U_1 \subseteq S_1 \) be non-empty open subsets. Then there exists a point \( s \in S_1 \) and a smooth curve \( C \) contained in \( S_1 \) passing through \( s \) such that:

i) \( C - s \subseteq U_l \)

ii) \( q_1^{-1}(C) \) is non-singular and \( q_1^{-1}(C) \to C \) is flat.

iii) The fibre \( q_1^{-1}(s) \) is a reduced curve with \( \alpha' \) non-singular components \( C_1, \cdots, C_{\alpha'} \) intersecting transversally and such that at most two of them pass through any point of \( X \) and such that each \( C_i \) is a fibre of \( q_m \) over a point of \( U_m \).

Lemma 9. If \( E \mid_{Y_m} \) is semistable, then \( E \mid_{Y_l} \) is semistable for any \( l \geq m \).

Proof By the openness of semistability (see lemma 2), we see that since \( E \mid_{Y_m} \) is semistable, there exists a non-empty open set \( U_m \) of \( S_m \) such that \( E \mid_{q_m^{-1}(s)} \) is semistable for any \( s \in U_m \). By proposition 8 we can degenerate \( Y_l \) into a reduced curve such that every irreducible component of the special curve belongs to \( U_m \). As in the proof of proposition 6 we see that any \( P \)-reduction on \( E \mid_{Y_l} \) contradicting the semistability of \( E \mid_{Y_l} \) induces a \( P \)-reduction on the special curve whose restriction to each of the irreducible components contradicts the semistability of \( E \) restricted to those components and hence by the choice of \( U_m \), contradicts the semistability of \( E \mid_{Y_m} \).

Lemma 10. Let \( X, H \) be as before. Let \( E \) be a principal \( G \)-bundle on \( X \). Let \( C \) be a smooth curve on \( X \). Then \( E \) is semistable (resp. stable) if its restriction to \( C \) is semistable (resp. stable).

Proof Suppose \( E \) is not semistable (resp. stable). Let \( \sigma : U \to E/P \) be a parabolic reduction of \( E \) contradicting the semistability (resp. stability) of \( E \), where \( U \) is a large open subset of \( X \). Let \( C' \) be any smooth curve contained in \( U \). Let \( C' \) be any smooth curve contained in \( U \). Then \( \sigma \) restricts to give a \( P \)-reduction on \( C' \) contradicting the semistability (resp. stability) of \( E \mid_C \). By constructing a degenerating family of curves as in proposition 6 with the generic curve contained in \( U \) and with the special curve \( C \), we see as in proposition 6 that this reduction induces a \( P \)-reduction on \( C \) as well which contradicts the semistability (resp. stability) of \( E \mid_C \).

Lemma 11. Let \( X, H \) be as before. Let \( (\alpha_1, \cdots, \alpha_t) \) be any \( t \)-tuple of integers with \( 1 \leq t \leq n - 1 \) and each \( \alpha_i \geq 3 \). Let \( \{X_i\} \) be a bounded family of closed subschemes of \( X \). Then there exists an integer \( m_0 \) such that \( \forall m \geq m_0 \), the generic complete-intersection subvariety \( Y_m \) of type \( (\alpha_1^m, \alpha_2^m, \cdots, \alpha_t^m) \) (having codimension \( t \) in \( X \)) is not contained in \( X_i \) for any \( i \).

Proof The proof is by induction on \( t \). Without loss of generality one may assume that \( X_i \)'s are all hypersurfaces. Since they form a bounded family, their degree’s are bounded above by some integer \( d \). Hence the lemma clearly holds for \( t = 1 \). Let us assume that the
result holds for \( t - 1 \). Let \( Y_m = H_1 \cap H_2 \cap \cdots \cap H_t \) be the complete-intersection of generic hypersurfaces of degree’s \( \alpha_1^m, \ldots, \alpha_t^m \) respectively. Suppose \( Y_t \subset X_t \), for some \( X_t \in \{ X_t \} \). By the induction hypothesis, \( H_1 \cap H_2 \cap \cdots \cap H_{t-1} \) is not contained in \( X_t \) and hence by irreducibility of \( H_1 \cap H_2 \cap \cdots \cap H_{t-1}, H_1 \cap H_2 \cap \cdots \cap H_{t-1} \cap X_t \) is a closed subscheme of \( H_1 \cap H_2 \cap \cdots \cap H_{t-1} \) of codimension atleast \( t \) in \( X \). Since it contains \( Y_m \) as a closed subscheme, its codimension in \( X \) is exactly \( t \). Thus it follows that \( Y_m \) is an irreducible component of \( H_1 \cap H_2 \cap \cdots \cap H_{t-1} \cap X_t \). But simply by comparing degree’s we see that when \( m \) exceeds \( d \), this is not possible. This completes the proof of the lemma \( \forall t \).

**Notation:** For the remainder of the paper, we fix an \((n-1)\)-tuple of integers \((\alpha_1, \ldots, \alpha_{n-1})\) with each \( \alpha_i \geq 3 \). Let \( \alpha = \alpha_1 \cdots \alpha_{n-1} \). For a positive integers \( m \), we denote by \( \mathbf{m} \) the sequence \((\alpha_1^m, \ldots, \alpha_{n-1}^m)\). Henceforth we will denote by \( Y_m \), the complete-intersection curve of type \((\alpha_1^m, \ldots, \alpha_{n-1}^m)\). As remarked earlier, this is a geometrically irreducible, non-singular curve of degree \( \alpha^m \).

### 4 Semistable Restriction Theorem

We now state and prove the semistable restriction theorem for principal bundles.

**Theorem 12.** Let \( X \) and \( H \) be as before. Let \( E \) be a principal \( G \)-bundle on \( X \). Then there exists an integer \( m_o \) such that \( \forall m \geq m_o, E \mid_{Y_m} \) is again semistable.

**Proof** The proof of the theorem is by contradiction. For any non-negative integer \( m \), let \( E_m \) denote the restriction of \( E \) to \( Y_m \). As remarked earlier in lemma\[9\] if \( E_m \) is semistable for some \( m \), then it is semistable \( \forall l \geq m \). So assume that the restriction \( E_m \) is non-semistable for all \( m \). Since there are only finitely many standard parabolics, we can find a sequence of integers \( \{m_k\} \) such that the standard parabolic underlying the canonical reduction of \( E \mid_{Y_{m_k}} \) is the same \( \forall k \), say \( P \). By proposition\[2\] for each \( m \in \{m_k\} \), we can find a non-empty open subset \( U_m \subseteq S_m \) such that the \( HN \mid_{q_m^{-1}(s)} \) is constant for all \( s \in U_m \). For any \( m \in \{m_k\} \), let \( E_{mp} \) denote the principal \( P \)-bundle underlying the canonical reduction of \( E_m \). Fix a fundamental weight \( \chi \) of \( P \). Let \( \chi, E_{mp} \) denote the line bundle obtained by extension of structure group of \( E_{mp} \) by \( \chi \). Let \( L_m \) denote the extension of \( \chi, E_{mp} \) to all of \( X \) (see proposition\[3\]). Let \( d_m = \deg L_m \). Note that \( \deg L_m = \alpha^m \cdot \deg \chi, E_{mp} \).

Claim: The sequence of integers \( d_{m_k} \) is a decreasing function of \( k \). Proof of claim: Let \( l = m + r \) with \( m, l \in \{m_k\} \) and \( r > 0 \). As before, degenerate \( Y_l \) into a reduced curve with \( \alpha^r \) non-singular, irreducible components \( C_1, \ldots, C_{\alpha^r} \), with \( C_i = q_{m+l}^{-1}(s_i) \) for some \( s_i \in U_m \), intersecting transversally such that atmost two components intersect at any given point. Let \( E_p \) denote the canonical reduction of \( E \mid_{C_i} \). By proposition\[6\] we see that \( \deg (\chi, E_{lp}) \leq \sum_{i=1}^{\alpha^r} \deg (\chi, l_{p_i}) = \alpha^r \deg (\chi, E_{mp}) \), where the last equality is
because of the choice of $U_m$. This immediately implies that $d_l \leq d_m$. Thus we see that $d_{m_k}$ is decreasing function of $k$ thereby completing the proof of the claim.

Since by the definition of canonical reduction $d_{m_k} \geq 0$, we conclude that the set of line bundles $L_{m_k}$ form a bounded family.

Consider the closed embedding $i : E/P \hookrightarrow F/Q$ described before in lemma\textsuperscript{[6]}. where $F$ is a principal $GL(V)$-bundle and $Q$ is a maximal parabolic in $GL(V)$ stabilizing a one-dimensional subspace on which $P$ acts by the character $\chi$. Thus we have for each $m$, $F/Q = \mathbb{P}(F(V)) = \mathbb{P}(L'_m \otimes F(V))$. Since the line bundles $L_{m_k}$ form a bounded family, by remark\textsuperscript{[5]} there exists an integer $m_1$ such that for a curve of type $m$ in $\{m_k\}$ with $m \geq m_1$, the restriction map $\text{Hom} (L_m, F(V)) \rightarrow \text{Hom} (L_m \mid_{Y_m} \rightarrow F(V) \mid_{Y_m})$ is a bijection. For each $m \in \{m_k\}$, consider the schematic inverse images of the cone over $E/P$ inside $L'_m \otimes F(V)$ defined by the embedding $i$ via all the global sections of $L'_m \otimes F(V)$. Since $L_{m_k}$'s form a bounded family, we see that this collection of closed subschemes of $X$ for all sections and for all possible $k$ form a bounded family and hence by lemma\textsuperscript{[11]} it follows that there exists an integer $m_o \geq m_1$ such that this collection cannot contain any curve of type $\geq m_o$ unless that subscheme containing this curve equals all of $X$.

Let $m \geq m_o$ be any integer in $\{m_k\}$ and consider the the section $\varphi_m : Y_m \rightarrow E/P$ corresponding to the canonical reduction of $E_m$. Via the embedding $i$, this may be viewed as a section $Y_m \rightarrow F/Q$ and hence corresponds to a line sub-bundle of $F(V) \mid_{Y_m}$ which can be easily seen to be isomorphic to $L_m \mid_{Y_m}$ and hence we get a section, say $\varphi_m$ of $\text{Hom} (L_m \mid_{Y_m}, F(V) \mid_{Y_m})$. Lift this to a section $\tilde{\varphi}_m$ of $\text{Hom} (L_m, F(V))$. Let $V_m$ be the open set where this lifted section is non-zero. On $V_m$, we get a line sub-bundle of $F(V)$ and hence a section $\tilde{\varphi}_m : V_m \rightarrow F/Q$ which extends the section $\varphi_m$ on $Y_m$. Note that since $V_m$ contains a curve which is a complete-intersection of ample hypersurfaces, namely $Y_m$, it follows that $V_m$ is a large open set. Let $X_m$ denote the scheme-theoretic inverse image of the cone over $E/P$ inside $L'_m \otimes F(V)$ via $\tilde{\varphi}_m$. It is easy to see that $X_m$ is a closed subscheme of $X$ containing $Y_m$ and hence by the choice of $m$, equals all of $X$. This shows that on $V_m$, the section $\tilde{\varphi}_m$ actually factors through a section $V_m \rightarrow E/P$ via the embedding $i$ lifting the canonical reduction on $Y_m$. This section has the property that the line bundle obtained by extension of structure group via $\chi$ has positive degree thereby contradicting the semistability of $E$. This contradiction shows that $E \mid_{Y_m}$ is semistable thereby completing the proof of the theorem. \hfill $\Box$

In fact in the above proof it can be shown that the line bundles $L_m$ with $m \gg 0$ are all isomorphic. Although, we do not need this fact for the proof of the above theorem or for the rest of the paper, it is an interesting fact in itself.

Choose, as we may by the proof of lemma above, an integer $s$ so that $\forall m \geq s, d_m$ is constant. Choose any $l, m \geq s$. As in the above proof, consider a degenerating family $D \rightarrow S$, where $S$ is a dvr and the generic fiber is $Y_l$ and the special fiber has $\alpha^{l-m}$ irreducible,
non-singular components in $U_m$ (in the notation of the proof). By the remark following lemma 6, it follows that the canonical reduction $E_P$, spreads to a $P$-reduction $\tilde{E}_P$, on all of $D$ and whose restriction to every irreducible component of the special fiber coincides with the canonical reduction there. Thus $L_I|_D$ and $\chi^*, \tilde{E}_P$ are two line bundles on $D$ which are isomorphic restricted to the generic fiber and have the same degree restricted to every irreducible component of the special fiber. Hence they are isomorphic on all of $D$. This implies that $L_I$ is isomorphic to $L_m$ on all the components of $D_k$ and hence by proposition 3 they are isomorphic on $X$.

**Proposition 13. (Openness of Semistability in higher relative dimensions)**

Let $\pi : Z \rightarrow S$ be a smooth, projective morphism, where $S$ is a finite-type $k$-scheme. Let $\mathcal{O}_{Z/S}(1)$ be a relatively very ample line bundle on $Z$. Let $E$ be a principal $G$-bundle on $Z$. Let $Z_\eta$ denote the generic fiber of $\pi$. Suppose $E|_{Z_\eta}$ is semistable w.r.t. $\mathcal{O}_{Z_\eta}(1)$. Then there exists a non-empty open subset $U \subseteq S$ such that $\forall s \in U$, $E|_{Z_s}$ is semistable w.r.t $\mathcal{O}_{Z_s}(1)$.

**Proof** Choose a closed subscheme $C \hookrightarrow Z$ such that the restricted morphism $\pi' : C \rightarrow S$ is a smooth, projective morphism of relative dimension 1 and such that the generic curve $C_\eta \hookrightarrow Z_\eta$ is a complete-intersection of general ample hypersurfaces in $Z_\eta$ of sufficiently high enough degree’s so that by theorem 12, the restriction of $E$ to $C_\eta$ is again semistable. Hence by lemma 2 there exists a non-empty open subset $U \subset S$ such that $\forall s \in U$, $E|_{\pi'^{-1}(s)}$ is again semistable. Then by lemma 10 it follows that $E|_{Z_s}$ is also semistable $\forall s \in U$. □

**5 Stable Restriction Theorem**

The aim of this section is to prove the stable restriction theorem for principal bundles. The stable restriction theorem for torsion-free sheaves was proved in [4] and is as follows:

**Theorem 14. ([4])** With notation as in theorem 12 let $E$ be a stable torsion-free coherent sheaf on $X$ w.r.t. $H$. Then there exists an integer $m_o$ such that for any $m \geq m_o$, the restriction of $E$ to $Y_m$ is again stable.

Once again, as in the case of the semistable restriction theorem, by openness of stability, it follows that there exists a non-empty open subset $U_m \subseteq S_m$ such that for any $s \in U_m$, $E|_{q_m^{-1}(s)}$ is again stable.

We begin by proving the openness of stability for a principal $G$-bundle defined over a smooth family of curves.

**Lemma 15. (Openness of Stability for a family of curves)** Let $S$ be a scheme of finite type over $k$ and let $f : Z \rightarrow S$ be a smooth, projective morphism of relative dimension
one. Let $E$ be a principal $G$-bundle on $Z$. Let $\eta$ denote the generic point of $S$ and $Z_\eta$ denote the generic fibre of $f$. Suppose $E|_{Z_\eta}$ is stable. Then there exists a non-empty open subset $U \subseteq S$ such that $\forall s \in U$, $E|_{Z_s}$ is stable.

**Proof** Since the family of curves is flat, the genus is constant in the family, say $g$. By openness of semistability (see lemma 2), we know that there exists a neighbourhood $V$ of $\eta$ such that for any $s \in V$, the restriction $E_s$ is again semistable. Hence the open subset of $U$ parametrizing stable bundles is the set of points $s \in V$ for which $E_s$ admits a reduction to some maximal parabolic $P$ such that the line bundle obtained by extension of structure group via the unique fundamental weight of $P$ has degree zero. By Riemann-Roch, the Hilbert polynomial of $Z_s$ for any $s \in S$ w.r.t. such a line bundle is $n + 1 - g$.

Fix a maximal parabolic $P \subseteq G$. Let $L$ denote the line bundle on $E/P$ corresponding to the fundamental weight of $P$. Consider the Hilbert scheme $\text{Hilb}_{E/P/Z/S}^{n+1-g,L}$ which represents the functor from $S$-schemes to sets, associating to any $S$-scheme $T$, the set of all closed subschemes of $E_T/P$ which are flat over $T$ and whose restriction to every schematic-fibre has Hilbert polynomial $n + 1 - g$. There exists an open subset $\text{Hilb}_{E/P/Z/S}^{\alpha n+1-g,L}$ which parametrizes those subschemes of $E/P$ which are sections with this property. By properness of $\text{Hilb}_{E/P/Z/S}^{n+1-g,L}$ over $S$, its image is a closed subspace of $S$. Since $E|_{Z_\eta}$ is stable, it follows that the image of $\text{Hilb}_{E/P/Z/S}^{\alpha n+1-g,L}$ is a locally closed subset of $S$ which misses the generic point. The union of all these locally closed subschemes for all standard maximal parabolics is a locally closed subset in $S$ whose complement in $V$ contains an open set $U$ parametrizing stable bundles.

**Lemma 16.** With notation as above, if there exists some $m$ such that $E_m$ is stable, then $\forall l \geq m$, $E_l$ is also stable.

**Proof** By lemma 15, let $U_m$ denote the non-empty open subset of $S_m$ consisting of points $s \in S_m$ such that the $E|_{q_\alpha'(s)}$ is again stable. Let $l = m + r$. Degenerate $Y_l$ into a reduced curve $C$ with $\alpha'$ many components, $C_1, \cdots, C_{\alpha'}$, intersecting transversally, each of which is in $U_m$ and such that atmost two of them intersect at any point. Suppose $E_l$ is not stable. Since $E_l$ is semistable (see remark following lemma 6), there exists a $P$-reduction $E_{lp}$ of $E_l$ and a dominant character $\chi$ of $P$ such that line bundle obtained by extension of structure group has degree zero. As in the proof of lemma 6, we see this $P$-reduction on $Y_l$ induces a $P$-reduction on $C$. Let $E_p^{l}$ denote its restriction to $C_l$. Then as in the proof of lemma 6 we see that $\deg (\chi, E_{lp}) \leq \sum_{i=1}^{\alpha'} \deg (\chi, E_p^{l})$. Since by the choice of $U_m$, $E|_{C_l}$ is stable we immediately see that the right hand right of this inequality is strictly less than zero and hence so is the left hand side. This contradiction shows that $E_l$ is stable as well.
Theorem 17. ( Stable Restriction Theorem) Let $X, H$ and $G$ be as before and let $E$ be a principal $G$-bundle on $X$ which is stable w.r.t. $H$. Then there exists an integer $m_0$ such that $\forall m \geq m_0$, the restriction $E|_{Y_m}$ is again stable.

Consequently by lemma [15] there exists a non-empty open subset $U_m \subseteq S_m$ such that for any $s \in U_m$, the restriction of $E$ to $q_m^{-1}(s)$ is again stable.

Proof By theorem [12] there exists an integer $m_1$ such that $\forall m \geq m_1$, $E|_{Y_m}$ is again semistable. The proof now is by contradiction. By lemma [15] we see that if the restriction $E_m$ is stable for some $m$ then it is stable for all $l \geq m$. So assume that $E_m$ is not stable for any $m$. Since there are only finitely many standard parabolics, choose a sequence of increasing integers $\{m_k\}$ with $m_k \geq m_1$ such that there exists a standard maximal parabolic, say $P$, with the property that $\forall k$, there exists a $P$-reduction $\psi_k : Y_{m_k} \rightarrow E/P$ contradicting the stability of $E_{m_k}$. Let $\chi$ denote the fundamental weight corresponding to $P$. Since $E_{m_k}$ is semistable $\forall k$, we see that the line bundle $\chi, \psi_k(E)$ has degree 0. As in the proof of lemma [6] by using the Chevalley semi-invariant lemma, we get a $G$-equivariant embedding of fiber-bundles $i : E/P \hookrightarrow F/Q$, where $F$ is a principal $GL(V)$-bundle and $Q$ is a maximal parabolic in $GL(V)$ stabilizing a 1-dimensional subspace of $V$ on which $P$ acts by the character $\chi$. Via the embedding $i$, we can think of $\psi_k$ as sections of $F/Q$ and hence we get line sub-bundles of $F(V)|_{Y_m}$.

By lemma [3] extend these to line bundles on all of $X$. Since these line bundles all have degree zero, they form a bounded family and hence by choosing these $m_k$’s to be large enough, we can assume that the restriction map $\text{Hom}(L_{m_k}, F(V)) \rightarrow \text{Hom}(L_{m_k}|_{Y_m}, F(V)|_{Y_m})$ is a bijection. The rest of the proof is similar to that of theorem [12]. We just sketch it briefly for the sake of completeness: By taking schematic inverse images of the cone over $E/P$ in $L_{m_k} \otimes F(V)$ defined by the embedding $i$, via all the global sections of $L_{m_k} \otimes F(V)$ for all possible $k$, we get a bounded family of closed subschemes of $X$ and hence by lemma [11] it cannot contain curves of arbitrarily large types. Thus eventually for some $m_0 \gg 0$, the reduction $\psi_{m_0}$, thought of as a section of $\text{Hom}(L_{m_0}|_{Y_{m_0}}, F(V)|_{Y_{m_0}})$ via $i$ has a lift to a section of $\text{Hom}(L_{m_0}, F(V))$ which on the large open subset $V \subseteq X$ where it is non-zero (containing $Y_{m_0}$) actually comes from a section $V \rightarrow E/P$ lifting the reduction $\psi_{m_0}$ on $Y_{m_0}$. This section naturally has the property that the line bundle obtained by extension of structure group via the unique fundamental weight corresponding to $P$ has degree zero thereby contradicting the stability of $E$. This contradiction shows that $E|_{Y_{m_0}}$ is stable thereby completing the proof of the theorem. $\Box$

Remark 18. (Openness of stability in higher relative dimensions) The stable restriction theorem immediately implies the openness of stability for a principal $G$-bundle over a smooth family of projective schemes over a finite-type $k$-scheme. The proof is the same as the proof of lemma [13] with theorem [12] and lemma [2] in the proof replaced by theorem [17] and lemma [15] respectively.
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