Complete Rank Theorem in Advanced Calculus 
and Frobenius Theorem in Banach Space

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Abstract The application of generalized inverses is usually neglected in pure mathematical research. However, it is very effective for this paper. We expand the famous matrix rank theorem due to R. Penrose to operators between Banach spaces. Therefore a modern perturbation analysis of generalized inverses is built. Hereby, we find and prove a complete rank theorem in advanced calculus. So a complete answer to the rank theorem problem presented by M. S. Berger is given. Let $\Lambda$ be an open set in Banach space $E$. We consider the family of subspaces $\mathcal{F} = \{M(x)\}_{x \in \Lambda}$ especially, where $\dim M(x) \leq \infty$, and investigate the necessary and sufficient condition for $\mathcal{F}$ being $c^1$ integrable at the point $x_0 \in \Lambda$. For this we introduce the concept of the co-final set $J(x_0, E_*)$ of $\mathcal{F}$ at $x_0$. Then applying the co-final set and the perturbation analysis of generalized inverses we prove the Frobenius theorem in Banach space, in the proof of which the used vector field and flow theory are avoided. The co-final set is essential to the Frobenius theorem. When $J(x_0, E_*)$ is trivial, the theorem reduces to the differential equation with initial value in Banach space. Let $B(E, F)$ be the set of all bounded linear operators from $E$ into another Banach space $F$, $\Lambda = B(E, F) \setminus \{0\}$ and $M(X) = \{T \in B(E, F) : TN(X) \subset R(X)\}$ for any $X \in \Lambda$. In 1989, V. Cafugna introduced a geometrical method for some partial differential equations and the family of subspaces $\mathcal{F} = \{M(X)\}_{x \in \Lambda}$. It is the key to this method that $\Phi_{1,1}$ is a smooth submanifold in $B(E, F)$ and tangent to $M(X)$ at any $X \in \Phi_{1,1}$, where $\Phi_{1,1}$ is the set of all Fredholm operators $T$ with $\dim N(T) = \text{codim} R(T) = 1$. However, its proof is not appeared yet until now. The co-final set of the family $\mathcal{F}$ at $A \in \Lambda$ is nontrivial in general. In section 3, applying the perturbation analysis of generalized inverses, the above property for $\Phi_{1,1}$ is expanded to more wide classes of operators $\Phi_{m,n}$, semi-Fredholm operators $\Phi_{m,\infty}$ and $\Phi_{\infty,n}$. It seems to be useful for further developing the method introduced by V. Cafagna.

Key words Rank Theorem, Locally Fine Point, Frobenius Theorem, Co-final Set, Smooth submanifold.

2000 Mathematics Subject Classification 46T99, 37C05, 53C40
0 Introduction

Let $E, F$ be two Banach spaces, $U$ an open set in $E$ and $f$ a nonlinear $c^1$ map from $U$ into $F$. Recall that $f$ is said to be locally conjugate to $f'(x_0)$ (Fréchet derivative of $f$ at $x_0$) near $x_0 \in U$ provided there exist a neighborhood $U_0(\subset U)$ at $x_0$, a neighborhood $V_0(\subset F)$ at 0, and diffeomorphisms $\varphi : U_0 \rightarrow \varphi(U_0)$ with $\varphi(x_0) = 0$, $\varphi'(x_0) = I_E$ and $\psi : V_0 \rightarrow \psi(V_0)$ with $\psi(0) = y_0(= f(x_0))$, $\psi'(0) = I_F$, such that

$$f(x) = (\psi \circ f'(x_0) \circ \varphi)(x), \quad \forall x \in U_0,$$

where $I_E$ and $I_F$ denote the identities on $E$ and $F$, respectively.

It is well known that if one of the following conditions holds: $N(f'(x_0)) = \{0\}$ and $R(f'(x_0))$ splits in $F; R(f'(x_0)) = F$, $N(f'(x_0))$ splits in $E$; and $\text{rank} f'(x) = \text{rank} f'(x_0) < \infty$ near $x_0$, then $f$ is locally conjugate to $f'(x_0)$ near $x_0$. These are important basic theorems in non-linear functional analysis (for details, see [Zie] and [Abr]). By saying the rank theorem problem one means what property of $f$ (more general than the three conditions above) ensures the equality $(\ast)$ holds. In [Beg], M. S. Berger shows that it is not yet known whether the rank theorem in advanced calculus holds even if $f$ is a Fredholm map. It is not difficult to observe that each of the three conditions above ensures that $f'(x)$ near $x_0$ has a generalized inverse $f'(x)^+$ (See Section 1) satisfying

$$\lim_{x \to x_0} f'(x)^+ = f'(x_0)^+. \quad (\ast\ast)$$

So we have in mind to seek the condition for which the equality $(\ast\ast)$ holds, try to find the answer to the rank theorem problem.

Let $X$ be a topological space, and $B(E, F)$ be the set of all bounded linear operators from $E$ into $F$. Suppose that the operator valued map $T_x : X \mapsto B(E, F)$ is continuous at $x_0 \in X$, and $T_{x_0}$ is double splitting. In Section 1, we will introduce the concept of a locally fine point of $T_x$ and prove the operator rank theorem as follows, for any generalized inverse $T_{x_0}^+$ of $T_{x_0}$ there exists a neighborhood $U_0$ at $x_0$ such that $T_x$ for any $x \in U_0$ has a generalized inverse $T_x^+$ and $T_x^+ \to T_{x_0}^+$ as $x \to x_0$. Now it is obvious that the equality $(\ast\ast)$ holds if and only if $x_0$ is a locally fine point of $f'(x)$. As expected, the following complete rank theorem in advanced calculus will be proved in this section: $f$ is locally conjugate to $f'(x_0)$, i.e., the equality $(\ast)$ holds, if and only if $x_0$ is a locally fine point of $f'(x)$. By Theorem 1.9, one observes that the complete rank theorem in advanced calculus expands widely the three results known well. Therefore the rank theorem problem presented by M. S. Berger has a complete answer. Theorems 1.10 and 1.11 are essential to the form of Frobenius theorem in Banach space and smooth differential structure for some smooth submanifolds, respectively. In Section 2, we consider the family of subspaces in $E$ as, $\mathcal{F} = \{M(x) : x \in \Lambda\}$, where $\Lambda$ is an open set in $E$, and $M(x)$ is a subspace in $E$ especially, where $\text{dim} M(x) \leq \infty$. Recall that $\mathcal{F}$ is said to be $c^1$ integrable at $x_0$ provided there exist a neighborhood $U_0$ at $x_0$ and a
\[ c^1\text{-submanifold } S \text{ in } U_0, \text{ such that } x_0 \in S \text{ and} \]
\[ M(x) = \{ \dot{c}(0) : \text{ for all } c^1\text{-curve } c(t) \subset S \text{ with } c(0) = x \}, \]

\( \forall x \in S, \text{i.e., } S \text{ is tangent to } M(x) \text{ at } x \in S. \) We introduce the co-final set \( J(x_0, E_\ast) \) of \( \mathcal{F} \) at \( x_0 \in \Lambda \) and prove if \( \mathcal{F} \) is \( c^1 \) integrable at \( x_0 \), say that \( S \) is the \( c^1 \) integral submanifold in \( E \), then there exists a neighborhood \( U_0 \) at \( x_0 \), such that \( J(x_0, E_\ast) \supset S \cap U_0 \). Moreover, we have the coordinate expression of \( M(x) \) as, \( M(x) = \{ e + \alpha e : \forall e \in M_0 \} \) for all \( x \in J(x_0, E_\ast) \), where \( M_0 = M(x_0) \) and \( \alpha \in B(M_0, E_\ast) \). Therefore we obtain the necessary and sufficient condition for \( \mathcal{F} \) being \( c^1 \) integrable at \( x_0 \), i.e., the Frobenius theorem in Banach space. Thanks to the co-final set \( J(x_0, E_\ast) \), the used vector field and flow theory are avoided in the proof of the theorem. The co-final set is essensial to the theorem. When \( J(x_0, E_\ast) \) is trivial, the theorem reduces to solve the differential equation with initial value in Banach space. Let \( \Lambda = B(E, F) \setminus \{0\} \) and \( M(X) = \{ T \in B(E, F) : TN(X) \subset R(X) \} \) for any \( X \in \Lambda \). In section 3, we investigate the family of subspaces \( \mathcal{F} = \{ M(X) \}_{X \in \Lambda} \). In fact, V. Cafagna introduces the geometrical method for some partial differential equations in [Caf], and presents the following property without a proof: \( \Phi_{1,1} \) (the set of all Fredholm operators \( T \) with \( \dim N(T) = \text{co-dim} R(T) = 1 \)) is a smooth submanifold in \( B(E, F) \) and tangent to \( M(X) \) at any \( X \in \Phi_{1,1} \), which is a key point to the method. The co-final set of \( \mathcal{F} \) at double splitting \( A \in \Lambda \), in general, is non-trivial. In order to get the integral submanifold \( S \) of \( \mathcal{F} \) at \( A \), we try its co-final set. Applying the modern perturbation analysis of generalized inverses we prove the following results: suppose that \( A \in \Lambda \) is double splitting, then \( \mathcal{F} \) is smooth integrable at \( A \); and denote by \( S \) the integral submanifold at \( A \), then \( S \) is tangent to \( M(X) \) at any \( X \in S \); let \( \Phi_{m,n}, \Phi_{m,\infty}(\Phi_{\infty,n}) \) and \( F_k \) denote the sets of all Fredholm operators \( T \) with \( \dim N(T) = m \) and \( \text{co-dim} R(T) = n \), of semi-Fredhlo operators \( T \) with \( \dim N(T) = m \) and \( \text{co-dim} R(T) = \infty(\dim N(T) = \infty \text{ and } \text{co-dim} R(T) = n) \), and of operators \( T \) with rank \( T = k < \infty \), respectively, then each of them is a smooth submanifold in \( B(E, F) \) and tangent to \( M(X) \) at any \( X \). Obviously, these expand the property for \( \Phi_{1,1} \) proposed by V. Cafagna to more wide classes of operators. It seems to be useful for further developing this method (see [An]).

1 Complete Rank Theorem and Modern Perturbation Analysis of Generalized Inverses

Recall that \( A^+ \in B(F, E) \) is said to be a generalized inverse of \( A \in B(E, F) \) provided \( A^+ A A^+ = A^+ \) and \( A = A A^+ A \), (when both \( E \) and \( F \) are Hilbert spaces, \( A^+ \) is said to be \( M \). -P. inverse of \( A \) provided \( (A A^+)^* = A A^+ \) and \( (A^+ A)^* = A^+ A \), and it is unique); \( A \in B(E, F) \) is said to be double splitting if \( R(A) \) is closed, and there exist closed subspaces \( R^+ \) in \( E \) and \( N^+ \) in \( F \) such that \( E = N(A) \oplus R^+ \) and \( F = R(A) \oplus N^+ \), respectively. It is well known that \( A \) has a generalized inverse \( A^+ \in B(F, E) \) if and only if \( A \) is double splitting.

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The classical perturbation analysis of the generalized inverse is marked with the
matrix rank theorem by R. Penrose. Let $A$ and $\Delta A$ belong to $B(\mathbb{R}^n)$, and both $A^+$
and $(A + \Delta A)^+$ be M. -P. inverses of $A$ and $(A + \Delta A)$, respectively. The theorem
says that $\lim_{\Delta A \to 0} (A + \Delta A)^+ = A^+$ if and only if $\text{rank}(A + \Delta A) = \text{rank} A$
for $\Delta A$ small enough. The condition in the theorem ensures that the computing of $A^+$ is stable.
Hereafter one will try to expand the theorem to the case of operators in $B(E,F)$.
This is very difficult because of no fixed one of generalized inverses likes M. -P. inverse
for every double splitting operator in $B(E,F)$. Thanks to a study of rank theorem
problem proposed by M. S. Berger in [Beg], we find the main concept and theorems
in modern perturbation analysis of generalized inverses, and obtain a complete rank
theorem. First will work on perturbation analysis of generalized inverse. Let $A \in$
$B(E,F)$ be double splitting, $A \neq 0$, and $A^+$ be a generalized inverse of $A$. Write
$V(A,A^+) = \{T \in B(E,F) : \|T - A\| < \|A^+\|^{-1}\}, C(A^+, T) = I_F + (T - A)A^+$, and
$D_A(A^+, T) = I_E + A^+(T - A)$. Then we have

**Theorem 1.1** The following conditions for $T \in V(A,A^+)$ are equivalent :

(i) $R(T) \cap N(A^+) = \{0\};$

(ii) $B = A^+C_A(A^+, T) = D_A(A^+, T)A^+$ is the generalized inverse of $T$ with $R(B) = R(A^+)$
and $N(B) = N(A^+);$

(iii) $R(T) \oplus N(A^+) = F;$

(iv) $N(T) \oplus R(A^+) = E;$

(v) $(I_E - A^+A)N(T) = N(A);$  

(vi) $C_A^{-1}(A^+, T)TN(A) \subset R(A);$  

(vii) $R(C_A^{-1}(A^+, T)T) \subset R(A).$

**Proof** M. Z. Nashed and X. C. Chen proved (vi) $\implies$ (ii) in [N-C]. In deed, the
inverse relation holds too. Obviously, $C_A^{-1}(A^+, T)TA^+A = A$ for all $T \in V(A,A^+)$. Hereby it follows that

$$BTB - B = 0 \quad \text{and} \quad TBT - T = -(I_F - AA^+)C^{-1}(A^+, T)T,$$

for any $T \in V(A,A^+)$. So(vi)$\iff$(ii). Evidently,

$$C_A^{-1}(A^+, T)Th = C_A^{-1}(A^+, T)TA^+Ah + C_A^{-1}(A^+, T)(I_E - A^+A)h$$

$$= Ah + C_A^{-1}(A^+, T)(I_E - A^+A)h, \quad \forall h \in E,$$

so that (vi)$\iff$(vii).

Go to the claim (v)$\iff$(vi). Assume that (vi) holds. We have for any $h \in N(A)$,
there exists $g \in R(A^+)$ such that $C_A^{-1}(A^+, T)Th = Ag$. Note $A^+Ag = g$, then
$C_A^{-1}(A^+, T)Th = Ag = C_A^{-1}(A^+, T)TA^+Ag = C_A^{-1}(A^+, T)Tg$. So $h - g \in N(T)$
and satisfies $(I_E - A^+A)(h - g) = h$. This shows that (v) holds. Conversely, assume
that (v) holds. Then we have for any $h \in N(A)$, there exists $g \in N(T)$ such that
Then the condition $A$ holds. Hereby one observes that when $A$ belongs to the corresponding class for $R$.

By (ii) in Theorem 1.1, one can conclude (iv) holds. Conversely, assume that (iv) holds. Then $B$ is the generalized inverse of $T$, this shows (i) $\iff$ (ii). So (i) $\iff$ (ii).

Go to show (i) $\iff$ (iii). Obviously, (iii) $\Rightarrow$ (i). Conversely, assume that (i) holds. Then $B$ is the generalized inverse of $T$ with $N(B) = N(A^+)$ because of the condition (i) equivalent to (ii). So (i) $\iff$ (iii).

Finally go to show (i) $\iff$ (iv). Assume that (i) holds. By the same way as showing (i) $\Rightarrow$ (ii), one can conclude (iv) holds. Conversely, assume that (iv) holds. Then $N(I_E - A^+ A)E = (I_E - A^+ A)N(T)$ i.e., (v) holds. Hence (i) holds since (v) $\iff$ (ii), $\iff$ (i). This shows (i) $\iff$ (iv).

So far we have proved the following relations (i) $\iff$ (ii), (i) $\iff$ (iii), (i) $\iff$ (iv), (ii) $\iff$ (vi), (v) $\iff$ (vi), and (vi) $\iff$ (vii). Then the proof of the theorem is completed. □

(For more information, see [H-M]).

The condition (i) looks strange, particularly, it implies many known conditions in analysis as showing in the next theorem.

Let $F_k = \{ T \in B(E,F) : \text{rank} T = k < \infty \}$, and $\Phi_{m,n} = \{ T \in B(E,F) : \dim N(T) = m < \infty \text{ and } \codim R(T) = n < \infty \}$. Let $\Phi_{n,\infty}$ be the set of all semi-Fredholm operators $T$ with $\dim N(T) = m$ and $\text{dim} R(T) = \infty$, and $\Phi_{\infty,n}$ be the set of all semi-Fredholm operators $T$ with $\text{dim} N(T) = \infty$ and $\text{dim} R(T) = n$. We have

**Theorem 1.2** Assume that $A$ belongs to any one of $F_k, \Phi_{m,n}, \Phi_{m,\infty}$ and $\Phi_{\infty,n}$. Then the condition $R(T) \cap N(A^+) = \{ 0 \}$ for $T \in V(A,A^+)$ holds if and only if $T$ belongs to the corresponding class for $A$.

**Proof** Assume that the condition $R(T) \cap N(A^+) = \{ 0 \}$ holds for $T \in V(A,A^+)$. By (ii) in Theorem 1.1, $T$ has a generalized inverse $B$ with $N(B) = N(A^+)$ and $R(B) = R(A^+)$, so that

$$N(T) \oplus R(A^+) = E = N(A) \oplus R(A^+)$$

and

$$R(T) \oplus N(A^+) = F = R(A) \oplus N(A^+).$$

Hereby one observes that when $A$ belongs to any one of $F_k, \Phi_{m,n}, \Phi_{m,\infty}$ and $\Phi_{\infty,n}, T$ belongs to the corresponding class for $A$.

Conversely, assume that $T$ belong to any one of $F_k, \Phi_{m,n}, \Phi_{m,\infty}$ and $\Phi_{\infty,n}$. By (1.1), $B = A^+ C_A^{-1}(A^+, T) = D_A^{-1}(A^+, T) A^+$ satisfies $BTB = B$. Thus $B$ and $T$ bear two projections $P_1 = BT$ and $P_2 = TB$. Indeed, $P_1^2 = BTBT = BT = P_1$, and $P_2^2 = TBTB = TB = P_2$. Note $N(B) = N(A^+)$. Clearly, $N(P_1) = N(T) + \{ e \}$
$R(T^+) : Te \in N(A^+)$}, where by the assumption, $T$ has a generalized inverse $T^+$. We next claim
\[ R(P_1) = R(A^+), R(P_2) = R(TA^+), \quad \text{and} \quad N(P_2) = N(A^+). \]

Obviously, $R(P_1) \subset R(B) = R(A^+)$. In the other hand, by $C_A^{-1}(A^+, T)TA^+ = AA^+$ we get
\[ P_1A^+e = A^+C_A^{-1}(A^+, T)TA^+e = A^+AA^+e = A^+e, \quad \forall e \in E. \]
So $R(P_1) = R(A^+)$. Obviously,
\[ R(P_2) = R(TA^+C_A^{-1}(A^+, T)) = R(TA^+). \]
\[ N(P_2) = N(TD_A^{-1}(A^+, T)A^+) \supset N(A^+), \]
and
\[ Be = BTBe = BP_2e = 0, \quad \forall c \in N(P_2). \]
So, $N(P_2) = N(A^+)$ because of $B = D_A^{-1}(A^+, T)A^{-1}$.

Thus we have
\[ F = R(P_2) \oplus N(P_2) = R(TA^+) \oplus N(A^+) \] (1.2)
and
\[ E = R(P_1) \oplus N(P_1) = R(A^+) \oplus E_* \oplus N(T), \]
where $E_* = \{e \in R(T^+) : Te \in N(A^+)\}$.

We now are in the position to end the proof.

Assume that $T \in V(A, A^+)$ satisfies $\text{rank } T < \infty$. Obviously $T$ is double splitting. Let $T^+$ be an arbitrary generalized inverse of $T$, then $E = N(T) \oplus R(T^+)$. Note $\text{dim } R(T^+) = \text{rank } T = \text{rank } A = \text{dim } R(A^+) < \infty$. According to (1.2) $\text{dim } E_* = 0$, i.e., $R(T) \cap N(A^+) = \{0\}$.

Assume that $T \in V(A, A^+)$ satisfies $\text{dim } R(T) = \text{dim } R(A) = n < \infty$, and $N(T)$ is splitting in $E$. Obviously, $T$ is double splitting. Let $T^+$ be an arbitrary generalized inverse of $T$, then $F = R(T) \oplus N(T^+)$. According to (1.2)
\[ R(TA^+) \oplus TE_* \oplus N(T^+) = F = R(TA^+) \oplus N(A^+). \]

Since $\text{dim } N(A^+) = \text{codim } R(A) = \text{codim } R(T) = \text{dim } N(T^+) = n < \infty$, one observes $\text{dim } TE_* = 0$. Note that $E_* \subset R(T^+)$ and $T|_{R(T^+)}$ is invertible in $B(R(T^+), R(T))$ then we conclude $\text{dim } E_* = 0$, i.e., $R(T) \cap N(A^+) = \{0\}$.

Assume that $T \in V(A, A^+)$ satisfies $\text{dim } N(T) = \text{dim } N(A) = m < \infty$, and $R(T)$ is splitting in $F$. Obviously, $T$ is double splitting. According to (1.2), we have
\[ R(A^+) \oplus E_* \oplus N(T) = R(A^+) \oplus N(A). \]
Hereby it is easy to observe \( \dim E_x = 0 \), i.e., \( R(T) \cap N(A^+) = \{0\} \). So far one can conclude that both \( T \) in \( V(A, A^+) \) and \( A \) belong to the same one of \( F_k, \Phi_{m,n}, \Phi_{\infty,n} \) and \( \Phi_{m,\infty} \), then \( R(T) \cap N(A^+) = \{0\} \). \( \square \)

Let \( A^\oplus \) be another generalized inverse of \( A \). \( B = A^+ AA^\oplus \), and \( \delta = \min\{\|A^+\|^{-1}, \|B\|^{-1}\} \). Moreover, we have

**Theorem 1.3** Let \( S = \{T \in V(A, A^+) : R(T) \cap N(A^+) = \{0\}\} \), and \( V_\delta = \{T \in B(E, F) : \|T - A\| < \delta\} \). Then

\[
R(T) \cap N(A^\oplus) = \{0\}, \quad \forall T \in S \cap V_\delta.
\]

**Proof** It is easy to see that \( B \) is the generalized inverse of \( A \) with \( R(B) = R(A^+) \) and \( N(B) = N(A^\oplus) \). By the conditions (i) and (iv) in Theorem 1.1, \( E = N(T) \oplus R(A^+) \), i.e., \( E = N(T) \oplus R(B) \) for all \( T \in S \). Instead of \( A^+ \) in Theorem 1.1 by \( B \), one observes that \( E = N(T) \oplus R(B) \) is equivalent to \( R(T) \cap N(B) = \{0\} \) for \( T \in V(A, B) \). So \( R(T) \cap N(A^\oplus) = \{0\} \) for \( T \in S \cap V_\delta \). \( \square \)

We now are in the position to discuss modern perturbation analysis of generalized inverses. Consider the operator valued map \( T_x \) from a topological space \( X \) into \( B(E, F) \).

**Definition 1.1** Suppose that \( T_x \) is continuous at \( x_0 \in X \), and that \( T_0 \equiv T_{x_0} \) is double splitting. \( x_0 \) is said to be a locally fine point of \( T_x \) provided there exist a generalized inverse \( T_0^+ \) of \( T_0 \) and a neighborhood \( U_0 \) (dependent on \( T_0^+ \)) at \( x_0 \), such that

\[
R(T_x) \cap N(T_0^+) = \{0\}, \quad \forall x \in U_0.
\]

(1.3)

For the locally fine point, we have the following important theorem.

**Theorem 1.4** The definition of the locally fine point \( x_0 \) of \( T_x \) is independent of the choice of the generalized inverse \( T_0^+ \) of \( T_0 \).

**Proof** Assume that (1.3) holds for a generalized inverse \( T_0^+ \) of \( T_0 \) and a neighborhood \( U_0 \) at \( x_0 \). Let \( T_0^\oplus \) be another generalized inverse of \( T_0 \), \( \delta = \min\{\|T_0^+\|^{-1}, \|T_0^\oplus T_0^\oplus\|^{-1}\} \), and \( V_\delta = \{T \in B(E, F) : \|T - T_0\| < \delta\} \). Set \( V_0 = \{x \in U_0 : T_x \in V_\delta\} \). Then by Theorem 1.3, \( R(T_x) \cap N(T_0^\oplus) = \{0\} \) for all \( \forall x \in V_0 \). \( \square \)

The following theorem expands the famous matrix rank theorem by R. Penrose to operators in \( B(E, F) \).

**Theorem 1.5** (Operator rank theorem) Suppose that the operator valued map \( T_x : X \rightarrow B(E, F) \) is continuous at \( x_0 \in X \) and \( T_0 \) is double splitting. Then the following conclusion holds for arbitrary generalized inverse \( T_0^+ \) of \( T_0 \): there exists a neighborhood \( U_0 \) at \( x_0 \) such that \( T_x \) has a generalized inverse \( T_x^+ \) for \( x \in U_0 \), and \( \lim_{x \to x_0} T_x^+ = T_0^+ \), if and only if \( x_0 \) is a locally fine point of \( T_x \).

**Proof** Assume that \( x_0 \) is a locally fine point of \( T_x \). Let \( U_0 = \{x \in X : \|T_x - T_0^+\| < \|T_0^+\|^{-1}\} \) for an arbitrary generalized inverse \( T_0^+ \) of \( T_0 \), and \( T_x^+ = T_0^+ C_{T_0}^{-1}(T_0^+, T_x) \) for...
all $x \in U_0$. Since $T_x$ is continuous at $x_0 \in X$, one observes that $U_0$ is a neighborhood at $x_0$. While, it is easy to see that $T^+_x$ is the generalized inverse of $T_x$ for any $x \in U_0$ such that $\lim_{x \to x_0} T^+_x = T^+_0$. Conversely, assume that the following conclusion holds for some generalized inverse $T^+_0$ of $T_0$: there exists a neighborhood $U_0$ at $x_0$ such that $T_x$ has a generalized inverse $T^+_x$, $\forall x \in U_0$, and $\lim_{x \to x_0} T^+_x = T^+_0$. Consider the projections, $P_x = I_E - T^+_x T_x$ and $P_0 = I_E - T^+_0 T_0$. Obviously, $R(P_x) = N(T_x)$, $R(P_0) = N(T_0)$, and $P_x \to P_0$ as $x \to x_0$. Let $V_0 = \{x \in U_0 : \|P_x - P_0\| < 1\} \cap \{x \in U_0 : \|T_x - T_0\| < \|T^+_0\|^{-1}\}$. Then by Problem 4.1 in [Ka], $P_0 N(T_x) = N(T_0)$, i.e., the condition (v) in Theorem 1.1 holds for all $x \in V_0$. Thus, by the equivalence of (v) and (i) in Theorem 1.1, we conclude $R(T_x) \cap N(T^+_0) = \{0\}, \forall x \in V_0$. □

We now are going to investigate the operator rank theorem in Hilbert spaces like the matrix rank theorem by R. Penrose. Before working this, we introduce the following lemma, which is very useful in the operator theory in Hilbert space, although it is simple.

**Lemma 1.1** Suppose that $H$ is a Hilbert space, and $E_1$ is a closed subspace in $H$. Let $H = E_1 \oplus E_2$ while the relation $E_1 \perp E_2$ is not assumed, and $P_{E_1}^{E_2}$ be the projection corresponding to the decomposition $H = E_1 \oplus E_2$. Then

$$(P_{E_1}^{E_2})^* = P_{E_2}^{E_1},$$

where $E_1^\perp$ and $E_2^\perp$ denote the orthogonal complements of $E_1$ and $E_2$, respectively.

**Proof** For abbreviation, write $P = P_{E_1}^{E_2}$. It is clear that $P^*$ is also a projection since $(P^*)^2 = (P^2)^* = P^*$. So in order to show the lemma, it is enough to examine $R(P^*) = E_2^\perp$ and $N(P^*) = E_1^\perp$. Obviously

$$H = N(P) \oplus R(P^*) = E_2 \oplus R(P^*),$$

and

$$H = R(P) \oplus N(P^*) = E_1 \oplus N(P^*),$$

where $\oplus$ denotes the orthogonal direct sum. Then the lemma follows. □

With Lemma 1.1 we are going to establish the operator rank theorem in Hilbert spaces as follows.

**Theorem 1.6** Let $H_1$, $H_2$ be Hilbert spaces, and $T_x$ an operator valued map from a topological space $X$ into $B(H_1, H_2)$. Suppose that $T_x$ is continuous at $x_0 \in X$ and $R(T_0)$ is closed. Then the following conclusion holds: for M. -P inverse $T^+_0$ of $T_0$: there exists a neighborhood $U_0$ at $x_0$, such that $T_x$ has M. -P inverse $T^+_x$ (i.e., $R(T_x)$ is closed) for $x \in U_0$, and $\lim_{x \to x_0} T^+_x = T^+_0$, if and only if $x_0$ is a locally fine point of $T_x$.

**Proof** Assume that the conclusion of the theorem holds for M. -P inverse $T^+_0$ of $T_0$. By Theorem 1.5 it is immediate that $x_0$ is a locally fine point of $T_x$. Conversely, assume
that $x_0$ is a locally fine point of $Tx$. By Definition 1.1 there exists a neighborhood $U_0$ at $x_0$, such that $R(T_x) \cap N(T_0^+) = \{0\}$ for $x \in U_0$. Without loss of generality one can set $\|T_x - T_0\| < \|T_0^+\|^{-1}$ for all $x \in U_0$ since $T_x$ is continuous at $x_0$. Thus, $B_x = T_0^+ C_{T_0}^{-1}(T_0^+, T_x)$ is the generalized inverse of $T_x$ with $R(B_x) = R(T_0^+)$ and $N(B_x) = N(T_0^+)$ for all $x \in U_0$ because of the equivalence of (i) and (ii) in Theorem 1.1 and so, $R(T_x)$ is closed. Let $T_x^+$ be M.-P. inverse of $T_x$. Our goal is to verify $\lim_{x \to x_0} T_x^+ = T_0^+$. Before working on this we claim that $\lim_{x \to x_0} T_x^+ T_x = T_0^+ T_0$ and $\lim_{x \to x_0} T_x T_x^+ = T_0^+ T_0^+$. By Lemma 1.1 it is clear

$$(I_{H_x} - T_x^* B_x^*) = P_{R(T_x^+)}^{R(T_x^+)} = \forall x \in U_0. \quad (1.4)$$

In fact, $B_x T_x = P_{N(T_x^+)}^{N(T_x^+)}$ and so $(I_{H_x} - T_x^* B_x^*) = (I_{H_x} - (B_x T_x)^*) = (I_{H_x} - P_{N(T_x^+)}^{R(T_x^+)}) = P_{N(T_x^+)}^{R(T_x^+)}$.

First go to show $\lim_{x \to x_0} T_x^+ T_x = T_0^+ T_0$.

$$\|T_x^+ T_x - T_0^+ T_0\| \leq \|T_x^+ T_x (I_{H_x} - T_0^+ T_0)\| + \|(I_{H_x} - T_x^+ T_x) T_0^+ T_0\| = \|((I_{H_x} - T_0^+ T_0) T_x^+ T_x)^*\| + \|(I_{H_x} - T_x^+ T_x) T_0^+ T_0\|$$

$\leq \|(I_{H_x} - T_0^+ T_0) - (I_{H_x} - T_x^* B_x^*)\| = \|T_x^* B_x^* - T_0^+ T_0\|$

(Note that $(T_x^* T_x)^* = T_x^+ T_x$ and $(T_0^+ T_0)^* = T_0^+ T_0$ since both $T_x^+$ and $T_0^+$ are M.-P. inverse), while, by (1.4) and $R(T_x^+) = R(T_x^+)$,

$$\|(I_{H_x} - T_0^+ T_0) T_x^+ T_x\| = \|T_x^+ T_x - (I_{H_x} - T_x^* B_x^*) T_x^+ T_x - T_0^+ T_0 T_x^+ T_x\|$$

$$\leq \|(I_{H_x} - T_0^+ T_0) - (I_{H_x} - T_x^* B_x^*)\| = \|T_x^* B_x^* - T_0^+ T_0\|$$

and

$$\|(I_{H_x} - T_x^+ T_x) T_0^+ T_0\| = \|T_0^+ T_0 - (I_{H_x} - T_x^+ T_x) T_x^* B_x^* - T_x^+ T_x T_0^+ T_0\|

= \|((I_{H_x} - T_x^+ T_x) T_0^+ T_0 - (I_{H_x} - T_x^* B_x^*) T_x^* B_x^*\|$$

$$\leq \|T_0^+ T_0 - T_x^* B_x^*\|.$$
while
\[ \| (I_{H_2} - T_0T_0^+)T_xT_x^+ \| = \| (I_{H_2} - T_0T_0^+)T_xT_x^+ - (I_{H_2} - T_xB_x)T_xT_x^+ \| \]
\[ = \| (T_xB_x - T_0T_0^+)T_xT_x^+ \| \leq \| T_xB_x - T_0T_0^+ \| \]
and
\[ \| (I_{H_2} - T_xT_x^+)T_0T_0^+ \| = \| (I_{H_2} - T_xT_x^+)T_0T_0^+ - (I_{H_2} - T_xT_x^+)T_xB_x \| \]
\[ \leq \| T_0T_0^+ - T_xB_x \|. \]

Then it follows \( \lim_{x \to x_0} T_xT_x^+ = T_0T_0^+ \) from \( T_xB_x \to T_0T_0^+ \) as \( x \to x_0 \).

Finally, go to end the proof. Let \( G_x = T_x^+T_xB_x \) and \( Q_x = B_xT_xT_x^+ \). By computing directly
\[ G_xT_xQ_x = T_x^+T_xB_xT_xB_xT_xT_x^+ = T_x^+T_xB_xT_xT_x^+ = T_x^+, \]
meanwhile, by the two results proved above, one observes
\[ \lim_{x \to x_0} G_x = T_0^+ = \lim_{x \to x_0} Q_x. \]
Therefore
\[ \lim_{x \to x_0} T_x^+ = \lim_{x \to x_0} G_xT_xQ_x = T_0^+T_0^+ = T_0^+. \]
The proof ends. \( \square \)

This theorem is presented in the dissertation of Q. L. Huang, a post graduate student of mine; see also [H-M].

So far, a modern perturbation analysis of generalized inverses is built.

We thus have found the concept of a locally fine point during investigating of rank theorem in advanced calculus, and consequently, establish the rank theorem. Let \( f \) be a \( c^1 \) map, defined in an open set \( U \subset E \) into \( F \), and \( f'(x_0), x_0 \in U \), be double splitting. By saying the rank theorem problem we meant that the property of \( f \) can ensures the following conclusion holds: there exist neighborhoods \( U_0 \subset U \) at \( x_0, V_0 \subset F \) at 0, diffeomorphisms \( \varphi \) from \( U_0 \) onto \( \varphi(U_0) \) and \( \psi \) from \( V_0 \) onto \( \psi(V_0) \) such that \( \varphi(x_0) = 0, \varphi'(x_0) = I_E, \psi(0) = f(x_0), \psi'(0) = I \), and
\[ f(x) = (\psi \circ f'(x_0) \circ \varphi)(x), \quad \forall x \in U_0, \]
i.e., \( f \) is locally conjugate to \( f'(x_0) \) near \( x_0 \). (For details see [Beg]). Since a locally fine point \( x_0 \) of \( f'(x) \) is equivalent to \( (I_E - T_0^+N)N(f'(x)) = N(f'(x_0)) \) near \( x_0 \) (where \( T_0^+ \) is a generalized inverse of \( f'(x_0) \)), we have proved indeed the following rank theorem in [Ma.1] in 1999's. However, there are many typing mistakes in the proof and argument.
of the theorem, one can hardly read. After some modifications we rewrite the theorem as follows.

**Theorem 1.7** Let $f$ be a $c^1$ map, defined in an open set $U \subset E$ into $F$, and $f'(x_0)$ its Fréchet derivative of $f$ at $x_0 \in U$. Suppose that $f'(x_0)$ is double splitting. If $x_0$ is a locally fine point of $f'(x)$, then there exist neighborhoods $U_0$ at $x_0, V_0$ at 0, diffeomorphisms $\varphi : U_0 \to \varphi(U_0)$ and $\psi : V_0 \to \psi(V_0)$ such that $\varphi(x_0) = 0, \varphi'(x_0) = I_E, \psi(0) = f(x_0), \psi'(0) = I_F$ and

$$f(x) = (\psi \circ f'(x_0) \circ \varphi)(x), \quad \forall x \in U_0.$$ 

**Proof** It is a key point to the proof of the theorem to determine the diffeomorphisms $\varphi$ and $\psi$. Let $T_0^+$ be a generalized inverse of $T_0 = f'(x_0)$. Set

$$\varphi(x) = T_0^+(f(x) - f(x_0)) + (I_E - T_0^+T_0)(x - x_0)$$

and

$$\psi(y) = (f \circ \varphi^{-1} \circ T_0^+)(y) + (I_F - T_0^+T_0)y.$$ 

Obviously, $\varphi'(x_0) = I_E$ and $\varphi(x_0) = 0$. So there exists an open disk $D^E_{r_0}(x_0) \subset E$ with center $x_0$ and radius $r_0$ such that $\varphi : D^E_{r_0}(x_0) \to \varphi(D^E_{r_0}(x_0))$ is a diffeomorphism. Let $T_0^+D^E_{r_0}(0) \subset \varphi(D^E_{r_0}(x_0))$ where $D^E_{r_0}(0) \subset F$ is an open disk with center 0 and radius $m$. Then $\psi$ is defined on $D^E_{r_0}(0)$. Obviously, $\psi'(0) = I_F$ and $\psi(0) = f(x_0)$. So there exists an open disk in $F$, without loss of generality, still write it as $D^F_m(0)$, such that $\psi : D^F_m(0) \to \psi(D^F_m(0))$ is a diffeomorphism. Clearly, both $T_0 \circ \varphi$ and $\|f'(x) - T_0\|$ are continuous at $x_0$. Hence, there exists an open disk $D^E_{r_1}(x_0), r_1 < r_0$, such that

$$T_0 \circ \varphi(x) \in D^E_{r_1}(0) \quad \text{and} \quad \|f'(x) - T_0\| < \|T_0^+\|^{-1}, \quad x \in D^E_{r_1}(x_0).$$

Hence, $\psi \circ T_0 \circ \varphi$ is defined on $D^E_{r_1}(x_0)$. Since $x_0$ is a locally fine point of $f'(x)$, we have

$$N(f'(x)) = \varphi'(x)^{-1}N(T_0), \quad \forall x \in D^E_{r_1}(x_0),$$

which is essential to the proof. Indeed,

$$\varphi'(x) = T_0^+f'(x) + (I_E - T_0^+T_0), \quad \forall x \in D^E_{r_0}(x_0)$$

and so

$$\varphi'(x)N(f'(x)) = (I_E - T_0^+T_0)N(f'(x)).$$

Then by the equivalence of (i) and (v) in Theorem 1.1,

$$N(T_0) = (I_E - T_0^+T_0)N(f'(x)) = \varphi'(x)N(f'(x)), \quad \forall x \in D^E_{r_1}(x_0),$$

i.e., the equality (1.5) holds.

Next go to show that for any disk $D^F_l(0) \subset \varphi(D^E_{r_1}(x_0))$, there exists an open disk $D^E_{r_2}(x_0), 0 < r_2 < r_1$, such that

$$T_0^+(f(x) - f(x_0)) - (1 - t)(I_E - T_0^+T_0)(x - x_0) \in D^F_l(0),$$

(1.6)
for all $x \in D^{E}_{r_2}(x_0)$ and $t \in [0,1]$. Note that
\[
\|T_0^+(f(x) - f(x_0)) - (1-t)(I_E - T_0^+T_0)(x - x_0)\| \\
\leq \|T_0^+(f(x) - f(x_0))\| + \|(I_E - T_0^+T_0)(x - x_0)\|,
\]
and both $T_0^+(f(x) - f(x_0))$ and $(I_E - T_0^+T_0)(x - x_0)$ are continuous and vanish at $x_0$. The conclusion (1.6) is obvious. We fix some $l$ and write the corresponding $D^{E}_{r_2}(x_0)$ as $D^{E}_{r}(x_0)$. Applying (1.5) and (1.6) go to prove
\[
f \circ \varphi^{-1}(T_0^+(f(x) - f(x_0)) + (I_E - T_0^+T_0)(x - x_0)) \\
= (f \circ \varphi^{-1})(T_0^+(f(x) - f(x_0))), \forall x \in D^{E}_{r}(x_0).
\]
In view of (1.6) consider
\[
\Phi(t, x) = (f \circ \varphi^{-1})(T_0^+(f(x) - f(x_0)) - (1-t)(I_E - T_0^+T_0)(x - x_0)),
\]
for all $t \in [0,1]$ and $x \in D^{E}_{r}(x_0)$.

For abbreviation, write $x_* = T_0^+(f(x) - f(x_0)) - (1-t)(I_E - T_0^+T_0)(x - x_0)$. Directly
\[
\frac{d\Phi}{dt} = f'(\varphi^{-1}(x_*)) \cdot (\varphi^{-1})'(x_*) \cdot (T_0^+T_0 - I_E)(x - x_0).
\]
Then by (1.5), $\frac{d\Phi}{dt} = 0$ for all $t \in [0,1]$ and $x \in D^{E}_{r}(x_0)$. Thus $\Phi(1, x) = \Phi(0, x)$, i.e., the equality (1.7) holds. Finally go to prove the theorem. By computing directly
\[
(\psi \circ T_0 \circ \varphi)(x) = \psi(T_0T_0^+(f(x) - f(x_0))) \\
= (f \circ \varphi^{-1})(T_0^+(f(x) - f(x_0))) + (I_F - T_0T_0^+)(T_0T_0^+(f(x) - f(x_0))) \\
= (f \circ \varphi^{-1})(T_0^+(f(x) - f(x_0))),
\]
while by (1.7).
\[
(\psi \circ T_0 \circ \varphi)(x) = (f \circ \varphi^{-1})(T_0^+(f(x) - f(x_0)) + (I_E - T_0^+T_0)(x - x_0)) \\
= (f \circ \varphi^{-1})(\varphi(x)) = f(x), \forall x \in D^{E}_{r}(x_0).
\]
The theorem is proved. \(\square\)

By Theorem 1.5, we further have

**Theorem 1.8 (Complete Rank Theorem)**  Suppose that $f : U \subset E \rightarrow F$ is $c^3$ map, and $f'(x_0), x_0 \in U$, is double splitting. Then $f$ is locally conjugate to $f'(x_0)$ near $x_0$ if and only if $x_0$ is a locally fine print of $f'(x)$.

**Proof**  Assume that $x_0$ is a locally fine print of $f'(x)$, then by Theorem 1.7, $f$ is locally conjugate to $f'(x_0)$ near $x_0$.

Assume that $f(x) = (\psi \circ f'(x_0) \circ \varphi)(x), \forall x \in U_0$. Let $T_0^+$ be arbitrary one of the generalized inverses of $T_0 = f'(x_0)$. Without loss of generality, one can assume
for all \( x \in U_0 \). Set \( B_x = \varphi'(x)^{-1} \cdot T_0^+ \cdot \psi'(T_0 \circ \varphi(x))^{-1} \). Since \( \varphi'(x_0) = I_E \) and \( \psi'(0) = I_F \) one observes \( \lim_{x \to x_0} B_x = T_0^+ \). By computing directly,

\[
f'(x) = \psi'(T_0 \varphi(x))T_0 \varphi'(x),\]

and so \( B_x f'(x)B_x = B_x \) and \( f'(x)B_x f'(x) = f'(x) \)

for all \( x \in U_0 \). Thus by Theorem 1.5 we conclude \( x_0 \) is a locally fine point of \( f'(x) \). \( \square \)

Before the appearance of Theorem 1.7, there is only the result that if any one of the following conditions holds: \( f'(x) \in F_k \) near \( x_0 \), \( f'(x_0) \in \Phi_{0,n}(n \leq \infty) \) and \( f'(x_0) \in \Phi_{m,0}(m \leq \infty) \), then \( f \) is locally conjugate to \( f'(x_0) \) near \( x_0 \) (see [Zei]). By Theorem 1.2 we have

**Theorem 1.9** \( \) Let \( f'(x_0) \) belong to any one of the following classes: \( F_k, \Phi_{m,n}, \Phi_{\infty,n} \) \((n < \infty)\) and \( \Phi_{m,\infty}(m < \infty) \). Then \( f \) is locally conjugate to \( f'(x_0) \) near \( x_0 \), if and only if \( f'(x) \) near \( x_0 \) belongs to the class corresponding to \( f'(x_0) \).

Obviously, the classes of operators indicated in Theorem 1.9 contain properly the three classes above. So, Theorem 1.7 and Theorem 1.8 give the complete answer to the rank theorem problem proposed by M. S. Berger in [Beg].

Besides Theorems 1.5, 1.6, 1.8 and 1.9 the concept of a locally fine point leads to expansions of the concepts of regular point, regular value, and transversality, so that it bears the generalized preimage and then the transversility theorem by R. Thom (see [Ma2] and [Ma3]). From this point of view, the concept of a locally fine point should be a nice mathematical concept.

Also, in the next sections 2 and 3, we will need the following theorems in perturbation analysis of generalized inverses.

**Theorem 1.10** \( \) Suppose that \( E_0 \) and \( E_1 \) are two closed subspaces in a Banach space \( E \) with a common complement \( E_\ast \). Then there exists a unique operator \( \alpha \in B(E_0, E_\ast) \) such that

\[
E_1 = \{ e + \alpha e : \forall \alpha \in E_0 \}. 
\]

Conversely, \( E_1 = \{ e + \alpha e : \forall \alpha \in E_0 \} \) for any \( \alpha \in B(E_0, E_\ast) \) is a closed subspace satisfying \( E_1 \oplus E_\ast = E \).

**Proof** First go to show that \( \alpha \) is unique for which \( E_1 = \{ e + \alpha e : \forall \alpha \in E_0 \} \). If \( \{ e + \alpha e : \forall \alpha \in E_0 \} = \{ e + \alpha_1 e : \forall \alpha \in E_0 \} \), then for any \( e \in E_0 \) there exists \( e_1 \in E_0 \) such that \( (e - e_1) + (\alpha e - \alpha_1 e_1) = 0 \), and so, \( e = e_1 \) and \( \alpha e = \alpha_1 e \), i.e., \( \alpha_1 = \alpha \).

This says that \( \alpha \) is unique. We now claim that \( \alpha = P_{E_0}^{E_1}P_{E_1}^{E_0} \big|_{E_0} \in B(E_0, E_\ast) \) fulfills \( E_1 = \{ e + \alpha e : \forall \alpha \in E_0 \} \).

Obviously,

\[
P_{E_0}^{E_1}P_{E_1}^{E_0} e = P_{E_0}^{E_1}(P_{E_1}^{E_0} e + P_{E_1}^{E_0} e) = P_{E_0}^{E_1} e = e, \quad \forall \alpha \in E_0, 
\]

and

\[
P_{E_1}^{E_0}P_{E_0}^{E_1} e = P_{E_1}^{E_0}(P_{E_0}^{E_1} e + P_{E_1}^{E_0} e) = P_{E_1}^{E_0} e = e, \quad \forall \alpha \in E_1.
\]
Hereby let \( \alpha = P_{E_0}^{E_*}P_{E_1}^{E_*} \), then
\[
e + \alpha e = P_{E_0}^{E_*}P_{E_1}^{E_*}e + P_{E_0}^{E_*}P_{E_1}^{E_*}e = P_{E_1}^{E_*}e \in E_1
\]
for any \( e \in E_0 \); conversely,
\[
e = P_{E_0}^{E_*}e + P_{E_*}^{E_0}e = P_{E_0}^{E_*}e + P_{E_*}^{E_0}P_{E_*}^{E_0}e
\]
for any \( e \in E_1 \). So
\[
E_1 = \{ e + \alpha e : \forall e \in E_0 \}.
\]
Conversely, assume \( E_1 = \{ e_0 + \alpha e_0 : \forall e_0 \in E_0 \} \) for any \( \alpha \in B(E_0, E_*) \). We claim that \( E_1 \) is closed. Let \( e = e_0 + \alpha e_0 \to e_* \), then
\[
P_{E_0}^{E_*}e = e_0 \to P_{E_0}^{E_*}e_* \in E_0, \text{ and } P_{E_*}^{E_0}e = \alpha e_0 \to \alpha(P_{E_0}^{E_*}e_*)
\]
since for all of \( \alpha, P_{E_0}^{E_*} \) and \( P_{E_*}^{E_0} \) are bounded. So
\[
e_* = P_{E_0}^{E_*}e_* + P_{E_*}^{E_0}e_* = P_{E_0}^{E_*}e_* + \alpha(P_{E_0}^{E_*}e_*) \in E_1.
\]
This shows that \( E_1 \) is closed. Also, one observes \( E_1 \cap E_* = \{ 0 \} \), since \( e_0 + \alpha e_0 \in E_* \) implies \( e_0 = 0 \). Obviously,
\[
e = P_{E_0}^{E_*}e + P_{E_*}^{E_0}e = (P_{E_0}^{E_*}e + \alpha(P_{E_0}^{E_*}e)) + (P_{E_*}^{E_0}e - \alpha(P_{E_0}^{E_*}e)), \quad \forall e \in E,
\]
which shows \( E_1 \oplus E_* \supset E \). So \( E_1 \oplus E_* = E \). The proof ends. \( \square \)

**Theorem 1.11** Suppose that \( A \in B(E, F) \) is double splitting and \( A \neq 0 \). Let \( A^+ \) be a generalized inverse of \( A \). Define
\[
M(A, A^+)(T) = (T - A)A^+A + C_A^{-1}(A^+, T)T, \quad \forall T \in V(A, A^+).
\]
Then \( M(A, A^+) \) is a smooth diffeomorphism from \( V(A, A^+) \) onto itself with the fixed point \( A \).

For abbreviation, write \( M(A, A^+) \) as \( M \) in the sequel.

**Proof** Note \( C_A^{-1}(A^+, T)TA^+ = AA^+ \) as indicated in the proof of Theorem 1.1. Clearly,
\[
(M(T) - A)A^+ = (T - A)A^+ + C_A^{-1}(A^+, T)TA^+ - AA^+ = (T - A)A^+
\]
so \( C_A^{-1}(A^+, M(T)) = C_A^{-1}(A^+, T) \). In addition, \( C_A^{-1}(A^+, T) \) for \( T \in V(A^+, A) \) is smooth, and \( C_A(A^+, A) = I_F \). Then one concludes \( M(T) \) is a smooth map on \( V(A, A^+) \).
into itself with the fixed point $A$. Now we merely need to show that $M$ has an inverse map on $V(A, A^+)$. Fortunately, we have

$$M^{-1}(m) = m A^+ A + C_A(A^+, m)m(I_E - A^+ A) \text{ for all } m \in V(A, A^+).$$

In what follows, we are going to examine $(M \circ M^{-1})(m) = m$ for all $m \in V(A, A^+)$ and $(M^{-1} \circ M)(T) = T$ for all $T \in V(A, A^+)$. Evidently,

$$(M^{-1}(m) - A)A^+ = (m - A)A^+ + C_A(A^+, m)m(I_E - A^+ A)A^+ = (m - A)A^+.$$ 

Hereby, one observes $C_A(A^+, M^{-1}(m)) = C_A(A^+, m)$ and so

$$C_A^{-1}(A^+, m) = C_A^{-1}(A^+, M^{-1}(m)) \text{ for any } m \in V(A, A^+).$$

Thus

$$(M \circ M^{-1})(m) = (M^{-1}(m) - A)A^+ A + C_A^{-1}(A^+, M^{-1}(m))M^{-1}(m)$$

$$= (m - A)A^+ A + C_A^{-1}(A^+, m)(m A^+ A + C_A(A^+, m)m(I_E - A^+ A))$$

$$= (m - A)A^+ A + m I_E - A^+ A + A = m, \quad \forall m \in V(A, A^+);$$

$$(M^{-1} \circ M)(T) = M(T)A^+ A + C_A(A^+, M(T))M(T)(I_E - A^+ A)$$

$$= (T - A)A^+ A + C_A^{-1}(A^+, T)T A^+ A + C_A(A^+, M(T))M(T)(I_E - A^+ A)$$

$$= T A^+ A + C_A(A^+, M(T))C_A^{-1}(A^+, T)T(I_E - A^+ A)$$

$$= T A^+ A + T(I_E - A^+ A) = T, \quad \forall T \in V(A, A^+).$$

$\square$

## 2 The co-final set and Frobenius Theorem

Let $E$ be a Banach space, and $\Lambda$ an open set in $E$. Assign a subspace $M(x)$ in $E$ for every point $x$ in $\Lambda$, especially, $\dim M(x) \leq \infty$. In this section, we consider the family $\mathcal{F}$ consisting of all $M(x)$ over $\Lambda$. We investigate the sufficient and necessary condition for $\mathcal{F}$ being $c^1$ integrable at a point $x$ in $\Lambda$. We stress that the concept of co-final set of $\mathcal{F}$ at $x_0 \in \Lambda$ is introduced.

**Definition 2.1** Suppose $E = M(x_0) \oplus E_*$ for $x_0 \in \Lambda$. The set

$$J(x_0, E_*) = \{x \in \Lambda: M(x) \oplus E_* = E\},$$

is called a co-final set of $\mathcal{F}$ at $x_0$.

The co-final set and integral submanifold of $\mathcal{F}$ at $x_0$ are connected as follows.

**Theorem 2.1** If $\mathcal{F}$ is $c^1$-integrable at $x_0 \in \Lambda$, say that $S \subset E$ is integral submanifold of $\mathcal{F}$ at $x_0$, then there exist a closed subspace $E_*$ and a neighborhood $U_0$ at $x_0$, such that

$$M(x) \oplus E_* = E, \quad \forall x \in S \cap U_0,$$

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i.e.,

\[ J(x_0, E_*) \supseteq S \cap U_0. \]

**Proof** Recall that the submanifold \( S \) in the Banach space \( E \) is said to be tangent to \( M(x) \) at \( x \in S \) provided

\[ M(x) = \{ \dot{c}(0) \mid \forall t \in [0,1] \} \subset S \text{ with } c(0) = x. \]  

By the definition of a submanifold, there exist a subspace \( E_0 \) splitting in \( E \), say \( E_0 \oplus E_1 = E \), a neighborhood \( U_0 \) at \( x_0 \), and a c\(^1\)-diffeomorphism \( \varphi : U_0 \to \varphi(U_0) \) such that \( \varphi(S \cap U_0) \) is an open set in \( E_0 \). We claim

\[ \varphi'(x)M(x) = E_0, \quad \forall x \in S \cap U_0. \]  

Let \( c(t) \) with \( c(0) = x \) be an arbitrary \( c^1 \)-curve contained in \( S \cap U_0 \), then \( \varphi'(x)c(0) \in E_0 \), and so \( \varphi'(x)M(x) \subset E_0 \). Conversely, let \( r(t) = \varphi(x) + te \subset \varphi(S \cap U_0) \) for any \( e \in E_0 \), and set \( c(t) = \varphi^{-1}(r(t)) \subset S \cap U_0 \), then \( \varphi'(x)c(0) = e \) and so \( \varphi'(x)M(x) \supset E_0 \). We now conclude that (2.2) holds. For abbreviation in the sequel, write \( M_0 = M(x_0) \). Let \( E_x = \varphi'(x)^{-1}E_1 \) and \( E_* = E_{x_0} \). We claim \( M(x) \oplus E_* = E \), \( \forall x \in S \cap U_0 \). Evidently, \( \varphi'(x)(M_0 \oplus E_*) = E_0 \oplus E_1 = E \) by (2.2), and so, \( M_0 \oplus E_* = E \). Consider the projection

\[ P_x = \varphi'(x)^{-1} \varphi'(x_0) P_{M_0}^{E_*} \varphi'(x_0)^{-1} \varphi'(x), \quad \forall x \in S \cap U_0. \]  

Obviously, \( P_x^2 = P_x \), i.e., \( P_x \) is a projection on \( E \). Next go to show \( R(P_x) = M(x) \) and \( N(P_x) = E_x \). Indeed,

\[ e \in N(P_x) \iff \varphi'(x_0)^{-1} \varphi'(x)e \in E_0 \iff \varphi'(x)e \in E_1 \iff e \in \varphi'(x)^{-1}E_1 = E_x; \]

\[ R(P_x) = \varphi'(x)^{-1} \varphi'(x_0) M_0 = \varphi'(x)^{-1}E_0 = M(x). \]

Finally applying theorem 1.5 (operator rank theorem) and Theorem 1.1 to end the proof.

Obviously, \( P_x \) is a generalized inverse of itself, and \( \lim_{x \to x_0} P_x = P_{M_0}^{E_*} \). Then by Theorem 1.5, \( x_0 \) is a locally fine point of \( P_x \). Hence there exists a neighborhood at \( x_0 \) in \( S \cap U_0 \), without loss of generality, still write it as \( S \cap U_0 \), and assume \( \| P_x - P_{M_0}^{E_*} \| < \| P_{M_0}^{E_*} \|^{-1} \) for all \( x \in S \cap U_0 \), such that \( R(P_x) \cap N(P_{M_0}^{E_*}) = \{0\} \). Moreover, by the conditions (i) and (iii) in Theorem 1.1,

\[ R(P_x) \oplus N(P_x) = E, \quad \text{i. e., } M(x) \oplus E_x = E \]

for all \( x \in S \cap U_0 \). This shows \( J(x, E_*) \supseteq S \cap U_0 \). □

Using \( M_0 \) and \( E_* \) in place of \( E_0 \) and \( E_1 \) in Theorem 2.1 respectively, we have
Lemma 2.1  With the same assumption and notations, \( E_\ast, E_0 \) and \( U_0 \) as in Theorem 2.1, the following conclusion holds: there exists a \( c^1 \) diffeomorphism \( \varphi : U_0 \to \varphi(U_0) \) with \( \varphi(x_0) = 0 \) such that \( V_0 = \varphi(S \cap U_0) \) is an open set in \( M_0 \), and
\[
\varphi'(x) M(x) = M_0, \quad \forall x \in S \cap U_0.
\]

**Proof**  Consider the \( c^1 \)-diffeomorphism \( \varphi \) in the proof of Theorem 2.1 and no loss of generality, may assume \( \varphi(x_0) = 0 \), since otherwise taking \( \varphi - \varphi(x_0) \) in place of \( \varphi \). Also, \( \varphi'(x) M(x) = E_0 \) for all \( x \in S \cap U_0 \), specially, \( \varphi'(x_0) M_0 = E_0 \). Again instead of \( \varphi \) by \( \varphi'(x_0)^{-1} \varphi \), still write it as \( \varphi \), the lemma follows. \( \square \)

By Theorem 2.1, \( J(x_0, E_\ast) \supset S \cap U_0 \) when \( F \) is integrable at \( x_0 \). Due to the co-final set \( J(x_0, E_\ast) \), we get the applicable equation of the integral submanifold under the coordinate system \( (M_0, 0, E_\ast) \) as shown in the next lemma.

**Lemma 2.2**  Suppose that \( F \) is \( c^1 \)-integrable at \( x_0 \). With the same notations, \( S, U_0, V_0 \) and \( \varphi \) as in Lemma 2.1, the following conclusions hold: there exists a neighborhood \( V_\ast \) at 0 in \( V_0 \) such that \( P^{E_\ast}_{M_0} \varphi^{-1} : V_\ast \to P^{E_\ast}_{M_0} \varphi^{-1}(V_\ast) \) is a \( c^1 \) diffeomorphism in \( M_0 \), and
\[
\varphi^{-1}(V_\ast) = (I_E + \psi)(V)
\]
(i.e., \( (I_E + \psi)(V) \) : \( \{ (x, \psi(x)) \} : \forall x \in V \) is the equation of \( S_1 = \varphi^{-1}(V_\ast) \), where \( V = P^{E_\ast}_{M_0} \varphi^{-1}(V_\ast) \) and \( \psi = (P^{E_\ast}_M \varphi^{-1}) \circ (P^{E_\ast}_{M_0} \varphi^{-1}|_{V_\ast})^{-1} \).

**Proof**  By Lemma 2.1, there are a neighborhood \( U_0 \) at \( x_0 \) and \( c^1 \) diffeomorphism \( \varphi \) from \( U_0 \) onto \( \varphi(U_0) \) with \( \varphi(x_0) = 0 \), such that \( V_0 = \varphi(S \cap U_0) \) is an open set in \( M_0 \). Consider the \( c^1 \) map \( \varphi_1 = P^{E_\ast}_{M_0} \varphi^{-1} : V_0 \to M_0 \). In order to seek \( V_\ast \) in (2.5), we are going to show that \( \varphi_1'(0) \) is invertible in \( B(M_0) \). According to (2.4) we have \( M_0 = \varphi'(x_0)^{-1}M_0 = (\varphi^{-1})'(0)M_0 \), so that \( \varphi_1'(0)M_0 = P^{E_\ast}_{M_0} \varphi^{-1}(0)M_0 = M_0 \); while
\[
\varphi_1'(0)e = P^{E_\ast}_{M_0} (\varphi^{-1})'(0)e = (\varphi^{-1})'(0)e \quad \text{for any} \quad e \in M_0,
\]
and so \( N(\varphi_1'(0)) = \{ 0 \} \) because of \( \varphi \) being a \( c^1 \) diffeomorphism. Hence \( \varphi_1'(0) \) is invertible in \( B(M_0) \). Thus by the inverse map theorem, there exists a neighborhood \( V_\ast \) at 0 in \( V_0 \) such that \( \varphi_1 : V_\ast \to \varphi_1(V_\ast) \) is a \( c^1 \) diffeomorphism. Let \( V = \varphi_1(V_\ast) \) and \( \psi = \varphi_2 \circ \varphi_1 \) where \( \varphi_2 = P^{E_\ast}_{M_0} \varphi^{-1} \). Then
\[
\varphi^{-1}(V_\ast) = \varphi_1(V_\ast) + \varphi_2(V_\ast) = (I_E + \psi)(V).
\]
Obviously, \( S_1 = \varphi^{-1}(V_\ast) \subset \varphi^{-1}(V_0) = S \cap U_0 \). The proof ends. \( \square \)

According to Theorem 1.10, for \( x \in J(x_0, E_\ast) \) \( M(x) \) has the coordinate expression
\[
M(x) = \{ e + \alpha(x)e : \forall e \in M_0 \}
\]
where \( \alpha \in B(M_0, E_\ast) \).
We now state the Frobenius theorem in a Banach space.

**Theorem 2.2** (Frobenius theorem) $\mathcal{F}$ is $c^1$ integrable at $x_0$ if and only if the following conclusions hold:

(i) $M_0$ splits in $E$, say $E = M_0 \oplus E_*$;

(ii) there exists a neighborhood $V$ at $P_{M_0}^{E_0}x_0$ in $M_0$, and a $c^1$ map $\psi : V \to E_*$, such that $x + \psi(x) \in J(x_0, E_*)$ for all $x \in V$, and $\alpha$ is continuous in $V$;

(iii) $\psi$ satisfies the following differential equation with the initial value:

$$
\begin{align*}
\psi'(x) &= \alpha(x + \psi(x)) \quad \text{for all } x \in V, \\
\psi(P_{M_0}^{E_0}x_0) &= P_{E_*}^{E_0}x_0,
\end{align*}
$$

(2.6)

where $\psi'(x)$ is Frechet derivative of $\psi$ at $x$.

**Proof** Assume that $\mathcal{F}$ is $c^1$-integrable at $x_0$. Go to prove that the conditions (i), (ii) and (iii) in the theorem hold. By Theorem 2.1, the condition (i) holds, and $J(x_0, E_*) \supset S \cap U_0$. By Lemma 2.2, there is neighborhood $V_*$ at $0$ in $V_0 = \varphi(S \cap U_0)$ such that

$$
J(x_0, E_*) \supset S \cap U_0 \supset \varphi^{-1}(V_*) = \{x + \psi(x) : \forall x \in V\},
$$

where

$$
V = P_{M_0}^{E_*} \varphi^{-1}(V_*) \quad \text{and} \quad \psi = (P_{M_0}^{E_*} \varphi^{-1}) \circ (P_{M_0}^{E_*} \varphi^{-1}|_{V_*})^{-1}.
$$

Obviously, the equation (2.6) implies that $\alpha(x + \psi(x))$ is continuous in $V$. So, it is enough to check the equation (2.6) holds for the condition (iii). First we claim

$$
M(x + \psi(x)) = (I_E + \psi'(x))M_0 \quad \forall x \in V.
$$

(2.7)

By (2.5),

$$
d(t) = (I_E + \psi)(c(t)) \quad \text{for any } c^1\text{-curve } c(t) \subset V \text{ with } c(0) = x
$$

is a $c^1$-curve $\subset \varphi^{-1}(V_*)$ with $d(0) = x + \psi(x)$. By Lemma 2.2, $\varphi^{-1}(V_*)$ is the $c^1$ integral submanifold of $\mathcal{F}$ at $x + \psi(x)$, and so, $M(x + \psi(x))$ is tangent to $\varphi^{-1}(V_*)$ at $x + \psi(x)$. Hereby, one can conclude $M(x + \psi(x)) \supset (I_E + \psi')(x)M_0$ for any $x \in V$. Conversely, let

$$
c(t) = P_{M_0}^{E_*} d(t) \quad \text{for any } c^1\text{-curve } d(t) \subset \varphi^{-1}(V_*) \text{ with } d(0) = x + \psi(x),
$$

then it follows that $c(t)$ is the $c^1$-curve $\subset V$ with $c(0) = x$ since

$$
c(t) = P_{M_0}^{E_*} \varphi^{-1}(\varphi(d(t))), \quad \varphi(d(t)) \subset V_*, \quad \text{and} \quad V = P_{M_0}^{E_*} \varphi^{-1}(V_*).
$$

Note

$$
\psi = (P_{E_*}^{M_0} \varphi^{-1}) \circ (P_{M_0}^{E_*} \varphi^{-1}|_{V_*})^{-1}.
$$
Evidently

\[(I_E + \psi)(e(t)) = P^{E_*}_{M_0}d(t) + \psi(P^{E_*}_{M_0}d(t)) = P^{E_*}_{M_0}\varphi^{-1}(\varphi(d(t))) + \psi(P^{E_*}_{M_0}\varphi^{-1}(\varphi(d(t)))) = P^{E_*}_{M_0}\varphi^{-1}(\varphi(d(t))) + P^{M_0}_{E_*}\varphi^{-1}(d(t))) = P^{E_*}_{M_0}d(t) + P^{M_0}_{E_*}d(t) = d(t).\]

So (2.7) holds.

Next go to verify (2.6). Due to the co-final set \(J(x_0, E_*) \supset \{x + \psi(x) : \forall x \in V\}\) as pointed at the beginning of the proof and Theorem 1.10, we have

\[M(x + \psi(x)) = \{e + \alpha(x + \psi(x))e : \forall e \in M_0\},\]

where \(\alpha(x + \psi(x)) \in B(M_0, E_*)\). Thus, it follows from the equality (2.7) that for any \(e \in M_0\), there exists \(e_0 \in M_0\) such that

\[e + \alpha(x + \psi(x))e = e_0 + \psi'(x)e_0.\]

Hence \(e = e_0\), so that

\[\alpha(x + \psi(x))e = \psi'(x)e \quad \text{for all} \quad e \in M_0,\]

i.e.,

\[\alpha(x + \psi(x)) = \psi'(x), \quad \forall x \in V.\]

Obviously,

\[\psi(P^{E_*}_{M_0}x_0) = \psi(P^{E_*}_{M_0}\varphi^{-1}(0)) = P^{M_0}_{E_*}(0)x_0.\]

Now, the necessity of the theorem is proved.

Assume that the conditions (i), (ii) and (iii) hold. Go to show that \(F\) is \(c^1\) integrable at \(x_0\). Let \(S = \{x + \psi(x) : \forall x \in V\}\),

\[V^* = \{x \in E : P^{E_*}_{M_0}x \in V\}, \quad \text{and} \quad \Phi(x) = x + \psi(P^{E_*}_{M_0}x), \forall x \in V^*.\]

Obviously, \(V^*\) is an open set in \(E\), \(\Phi(V) = S\), and \(\Phi\) is a \(c^1\) map. Moreover, we are going to prove that \(\Phi : V^* \to \Phi(V^*)\) is a diffeomorphism. Evidently, if \(\Phi(x_1) = \Phi(x_2)\), for \(x_1, x_2 \in V^*\), i.e.,

\[P^{E_*}_{M_0}(x_1 - x_2) + \psi(P^{E_*}_{M_0}x_1) - \psi(P^{E_*}_{M_0}x_2) + P^{M_0}_{E_*}(x_1 - x_2) = 0\]

then \(P^{E_*}_{M_0}x_1 = P^{E_*}_{M_0}x_2\), and so \(P^{M_0}_{E_*}x_1 = P^{M_0}_{E_*}x_2\). This shows that \(\Phi : V^* \to \Phi(V^*)\) is one-to-one. Now, in order to show that \(\Phi : V^* \to \Phi(V^*)\) is a \(c^1\) diffeomorphism, we merely show that \(\Phi(V^*)\) is an open set in \(E\). By the inverse map theorem it is enough
to examine that $\Phi'(x)$ for any $x \in V^*$ is invertible in $B(E)$. According to the condition (iii)
\[
\Phi'(x) = I_E + \psi'(P_{E_0}E^*x)P_{E_0}E^* = P_{E_0}E^* + \alpha(P_{E_0}E^*x + \psi(P_{E_0}E^*x))P_{E_0}E^* = P_{E_0}E^* + P_{E_0}M_0
\]
for any $x \in V^*$. If $\Phi'(x)e = 0$, i.e., $P_{E_0}E^*e + \psi'(P_{E_0}E^*x)P_{E_0}E^*e = P_{E_0}M_0e = 0$, then $P_{E_0}M_0e = 0$ and so, $P_{E_0}M_0e = 0$. This says $N(\Phi'(x)) = \{0\}$ for any $x \in V^*$. Next go to verify that $\Phi'(x)$ is surjective.

For abbreviation, write $M(y) = M_s$ and $y = P_{E_0}E^*x + \psi(P_{E_0}E^*x)$ for any $x \in V^*$. Obviously, $y \in S$ and by the assumption (ii), $y \in J(x_0, E_s)$, i.e., $M_s \oplus E_s = E$. Hence the following conclusion for any $e \in E$ holds: there exists $e_0 \in M_0$ such that $P_{E_0}E^*e = e_0 + \alpha(y)e_0$. Set $e_s = e_0 + P_{E_s}P_{E_0}E^*e$, then by (2.8),
\[
\Phi'(x)e_s = e_0 + \alpha(y)e_0 + P_{E_s}P_{E_0}E^*e = P_{E_0}E^*e + P_{E_s}P_{E_0}E^*e = e.
\]

This says that $\Phi'(x)$ for any $x \in V^*$ is surjective. Thus we have proved that $\Phi^{-1}$ is a $C^1$-diffeomorphism from open set $\Phi(V^*)$ onto $V^*$ and $\Phi^{-1}(S) = V$ is an open set in $M_0$, i.e., $S$ is a $C^1$ submanifold of $E$. Finally go to show that $S$ is tangent to $M(x)$ at any point $x \in S$.

Write
\[
T(x + \psi(x)) = \{\dot{e}(0) : \forall \text{ its } C^1\text{-curve}(t) \subset S \text{ with } e(0) = x + \psi(x)\}
\]
for any $x \in V$.

Repeat the process of the proof of the equality (2.2) in Theorem 2.1, one can conclude
\[
(\Phi^{-1})'(x + \psi(x))T(x + \psi(x)) = M_0, \text{ i.e., } \Phi'(x)M_0 = T(x + \psi(x))
\]
for any $x \in V$. By (2.8),
\[
\Phi'(x + \psi(x))e = e + \alpha(x + \psi(x))e, \quad \forall e \in M_0.
\]
So
\[
T(x + \psi(x)) = \Phi'(x)M_0 = M(x + \psi(x)), \quad \forall x \in V,
\]
which shows that $S$ is tangent to $M(x)$ at any $x \in S$. The proof ends. $\square$

The co-final set $J(x_0, E_s)$ is essential to the Frobenius theorem in Banach space. When $J(x_0, E_s)$ is trivial, which means that $x_0$ is an inner point of $J(x_0, E_s)$, the theorem reduces to solve the differential equation with initial value (2.6). The following example will illustrate this fact, although it is very simple.

Example Let $E = \mathbb{R}^2$, $\Lambda = \mathbb{R}^2 \setminus (0,0)$ and
\[
M(x,y) = \{(X,Y) \in \mathbb{R}^2 : Xx + Yy = 0\}, \quad \forall (x,y) \in \Lambda.
\]
Consider the family of subspaces, $\mathcal{F} = \{M(x,y) : \forall (x,y) \in \Lambda\}$. Applying Frobenius theorem in Banach space to determine the integral curve of $\mathcal{F}$ at $(0,1)$.

Set $U_0 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ and $E_* = \{(0,y) \in \mathbb{R}^2 : \forall y \in \mathbb{R}\}$. Obviously, $U_0 \subset \Lambda$, and

$$M(x,y) \oplus E_* = \mathbb{R}^2, \quad \forall (x,y) \in U_0,$$

since

$$M(x,y) \cap E_* = (0,0), \quad \forall (x,y) \in U_0.$$

Hence $J((0,1),E_*) \supset U_0$. This shows that $J((0,1),E_*)$ is trivial.

Next go to determine $\alpha$ in the equation (2.6). Note $M_0 = M(0,1) = \{(X,0) : \forall X \in \mathbb{R}\}$. Evidently,

$$M(x,y) = \{(X,-\frac{x}{y}X) : \forall X \in \mathbb{R}\} = \{(X,0) + (0 - \frac{x}{y}X) : \forall X \in \mathbb{R}\}$$

for all $(x,y) \in U_0$. By Theorem 1.10 we see $\alpha$ is unique, so

$$\alpha(x,y)(X,0) = (0,-\frac{x}{y}X), \quad \forall (x,y) \in U_0.$$

Moreover, we claim that $\alpha : U_0 \to B(M_0,E_*)$ is continuous in $U_0$. Obviously

$$\|\alpha(x + \Delta x,y + \Delta y) - \alpha(x,y))(X,0)\| = \left\| (0,\frac{x + \Delta x}{y + \Delta y})X \right\| = \frac{x + \Delta x}{y + \Delta y} - \frac{x}{y} \| (X,0)\|$$

for any $(x,y) \in U_0$, where $\|,\|$ denotes the norm in $\mathbb{R}^2$.

So

$$\|\alpha(x + \Delta x,y + \Delta y) - \alpha(x,y))\| = \left| \frac{x + \Delta x}{y + \Delta y} - \frac{x}{y} \right|,$$

where $\|,\|$ denotes the norm in $B(M_0,E_*)$. Hereby, one can conclude that $\alpha : U_0 \to B(M_0,E_*)$ is continuous in $U_0$. Let $V = \{(x,0) : |x| < 1\}$ and $\psi((x,0)) = (0,y(x))$ for any $(x,0) \in V$. In this case, the equation (2.6) reduces to $(0,-\frac{x}{y}) = (0,y'(x))$ with $y(0) = 1$, i.e.,

$$\frac{dy}{dx} = -\frac{x}{y}, \quad \forall x \in (-1,1),$$

$$y(0) = 1.$$

It is well known that the solution is $y = \sqrt{1 - x^2}$ for all $x \in (-1,1)$, which is a smooth submanifold in $U_0$. 

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When the co-final set \( J(x_0, E_\ast) \) of \( \mathcal{F} \) at \( x_0 \) is non-trivial, it is a key point to Frobenius theorem to seek \( J(x_0, E_\ast) \). In the next section, we will consider such a family of subspaces, which appears in the investigation of geometrical method for some partial differential equations (see [Caf]).

### 3 A Family of Subspaces with Non-trivial Co-Final Set and Its Smooth Integral Submanifolds

Let \( \Lambda = B(E, F) \setminus \{0\} \) and \( M(X) = \{ T \in B(E,F) : TN(X) \subset R(X) \} \) for \( X \in \Lambda \). V. Cafagna introduced the geometrical method for some partial differential equations and presented the family of subspaces \( \mathcal{F} = \{ M(X) \} X \in \Lambda \) in [Caf]. Now we take the following example to illustrate that when \( A \) is neither left nor right invertible in \( B(R^2) \), the co-final set of \( \mathcal{F} \) at \( A \) is non-trivial.

**Example** Let \( \Lambda = B(R^2) \setminus \{0\}, \) and the matrix \( A = \{a_{i,j}\}_{i,j=1}^2 \) and the subspace \( E_\ast \) in \( B(R^2) \) be as follows

\[
a_{i,j} = 0 \text{ except } a_{1,1} = 1, \quad \text{and} \quad \{ T = \{t_{i,j}\} \in B(R^2) : t_{i,j} = 0 \text{ except } t_{2,2} \},
\]

respectively. To verify that \( J(A, E_\ast) \) is non-trivial. Evidently, \( N(A) = \{(0, y) : \forall y \in R\} \) and \( R(A) = \{(x, 0) : \forall x \in R\} \). So

\[
M(A) = \{ T = \{t_{i,j}\} \in B(R^2) : t_{2,2} = 0 \}
\]

and

\[
M(A) \oplus E_\ast = B(R^2).
\]

However, there is an invertible operator \( A_\varepsilon = \{a_{i,j}\} \) with \( a_{1,1} = 1, a_{1,2} = a_{2,1} = 0, \) and \( a_{2,2} = \varepsilon, \) such that \( M(A_\varepsilon) = B(R^2) \) for any \( \varepsilon \neq 0. \) This says that \( J(A, E_\ast) \) is non-trivial since \( \dim M(A) = 3, \dim M(A_\varepsilon) = 4, \) and \( \lim_{\varepsilon \to 0} A_\varepsilon = A. \)

In this case, for seeking integral submanifold of \( \mathcal{F} \) at \( A \), we try the co-final set \( J(A, E_\ast) \) according to Theorem 2.1.

**Lemma 3.1** Suppose that \( X \in \Lambda \) is double splitting, say that \( X^+ \) is a generalized inverse of \( X. \) Then

\[
M(X) = \left\{ P_{N(X)}^N T + P_{N(X)}^{R(X)} T P_{N(X)}^N : \forall T \in B(E,F) \right\},
\]

and one of its complements is

\[
E_X = \left\{ P_{N(X)}^{R(X)} T P_{N(X)}^{R(X)} : \forall T \in B(E,F) \right\}.
\]
**Proof** Obviously

\[ T = P_{N(X^+)}^N(X^+) + P_{R(X^+)}^N(X^+) + P_{R(X^+)}^{R(X^+)} P_{N(X^+)}^R(X^+) \]  \tag{3.3}

So the equality (3.1) follows from the definition of \( M(X) \) and (3.3). \( \square \)

As a corollary of the lemma we have

**Corollary 3.1** With the assumption and notations in Lemma 3.1 we have

\[ \mathbb{E}_X = \{ T \in B(E, F) : R(T) \subset N(X^+) \text{ and } N(T) \subset R(X^+) \} \]  \tag{3.4}

This is immediate from (3.1) and (3.2).

Let \( \mathbb{E}_* = \{ T \in B(E, F) : R(T) = N(A^+) \text{ and } N(T) = R(A^+) \} \). By Lemma 3.1 and Corollary 3.1, \( M(A) \oplus \mathbb{E}_* = B(E, F) \). In the same way, one can assert that \( M(X) \oplus \mathbb{E}_* = B(E, F) \) for double splitting \( X \in \Lambda \) with \( R(X^+) = R(A^+) \) and \( N(T^+) = N(A^+) \), i.e., \( X \in J(A, \mathbb{E}_*) \). Let \( S = \{ T \in V(A, A^+) : R(T) \cap N(A^+) = \{0\} \} \). By Theorem 1.1,

\[ S = \{ T \in V(A, A^+) : R(T^+) = R(A^+) \text{ and } N(T^+) = N(A^+) \} \subset J(A, \mathbb{E}_*) \cap V(A, A^+) \]

Let \( X \in \Lambda \) be double splitting. Consider the smooth diffeomorphism from \( V(A, A^+) \) onto itself with the fixed point \( X \),

\[ M(X, X^+)(T) = (T - X)X^+X + C^{-1}_X(X^+, T)T \]

as defined in Theorem 1.11.

By computing directly,

\[ M(X, X^+)'(T)\Delta T = \Delta TX^+X + C^{-1}_X(X^+, T)\Delta T - C^{-1}_X(X^+, T)\Delta TX^+C^{-1}_X(X^+, T)T \]  \tag{3.5}

for any \( T \in V(A, A^+) \), where \( M(X, X^+)'(T) \) denotes the Fréchet derivative of \( M(X, X^+)(T) \) at \( T \).

**Theorem 3.1** Suppose that \( A \in \Lambda \) is double splitting. Then \( S \) is a smooth submanifold in \( B(E, F) \) and tangent to \( M(X) \) at any \( X \in S \).

**Proof** First we show that \( S \) is a smooth submanifold in \( B(E, F) \). By Theorem 1.11, \( M(A, A^+)(T) \) is a smooth diffeomorphism from \( V(A, A^+) \) onto itself with the fixed point \( A \), and hence we merely claim

\[ M(A, A^+)(S) = M(A) \cap V(A, A^+) \]  \tag{3.6}

Evidently,

\[ M(A, A^+)(T)N(A) = C^{-1}_A(A^+, T)TN(A) \subset R(A) \]
because of the equivalence of the conditions (v) and (i) in Theorem 1.1. This shows $M(A, A^+)(S) \subset M(A) \cap V(A, A^+)$. Conversely, go to verify

$$m = M(A, A^+)^{-1}(T) \in S \quad \forall T \in M(A) \cap V(A, A^+).$$

Note the equality shown in the proof of Theorem 1.1, $C_A^{-1}(A^+, T) = C_A^{-1}(A^+, m)$. Let $m = M(A, A^+)^{-1}(T)$, for any $T \in M(A) \cap V(A, A^+)$. Then

$$C_A^{-1}(A^+, T) = C_A^{-1}(A^+, M(A, A^+)(m)) = C_A^{-1}(A^+, m).$$

Hereby,

$$C_A^{-1}(A^+, m)mN(A) = C_A^{-1}(A^+, T)M(A, A^+)^{-1}(T)N(A) = TN(A) \subset R(A),$$

so that $m \in S$ because of the equivalence of (v) and (i) in Theorem 1.1. This says that $S$ is a smooth submanifold in $B(E, F)$. Finally go to show that $S$ is tangent to $M(X)$ at any $X \in S$. Clearly, $C(X, X^+, X) = I_F$ and so, $C^{-1}(X, X^+, X) = I_F$. Hereby

$$M(X, X^+)/\Delta T = \Delta T, \quad \forall \Delta T \in B(E, F)$$

(3.7)

according to (3.5). In what follows, we claim that $S$ is tangent to $M(A)$ at $A$. From (3.6) and (3.7) it follows that $M(A, A^+)(A)\dot{c}(0) = \dot{c}(0) \in M(A)$ for any $c^1$-curve $c(t) \subset S$ with $c(0) = A$. This shows that the set of all tangent vectors of $S$ contains in $M(A)$. Conversely, let $c(t) = M(A, A^+)^{-1}(d(t))$ for any $c^1$-curve $d(t) \subset M(A) \cap V(A, A^+)$ with $d(0) = A$. By (3.6), $c(t) \subset S$ and $c(0) = A$. While $d(t) = M(A, A^+)(c(t))$ and so, $\dot{d}(0) = M(A, A^+)(A)\dot{c}(0) = \dot{c}(0).$ This shows that $M(A)$ contains in the set of all tangent vectors of $S$. Now it is proved that $S$ is tangent to $M(A)$ at $A$. Next turn to the proof for any $X \in S$. Let $X^+$ be its generalized inverse with $N(X^+) = N(A^+)$ and $R(X^+) = R(A^+)$, and $S_1 = \{T \in V(X, X^+) : R(T) \cap N(X^+) = \{0\}\}$. In the same way as the proof of (3.6) one can prove $M(X, X^+)(S_1) = M(X) \cap V(X, X^+)$. Since $N(X^+) = N(A^+), S \cap V(X, X^+) = S_1 \cap V(A, A^+) = S_1 \cap S$ and $X \in S \cap S_1$. Hence

$$M(X, X^+)(S \cap S_1) = M(X) \cap V(A, A^+) \cap V(X, X^+).$$

(3.8)

Repeat the process of the proof of $S$ being tangent to $M(A)$ at $A$. Let $c(t)$ be any $c^1$ curve\(\subset S \cap S_1\ with c(0) = X\). Then it follows from (3.8) and (3.7) that

$$M(X, X^+)/\dot{c}(0) = \dot{c}(0) \quad \text{so that} \quad \dot{c}(0) \in M(X).$$

Conversely, let $d(t)$ be any $c^1$-curve\(\subset M(X) \cap V(A, A^+) \cap V(X, X^+)\) with $d(0) = X$ and $c(t) = M(X, X^+)^{-1}(d(t))$. Then $d(t) = M(X, X^+)(c(t))$. By (3.7),

$$\dot{d}(0) = M(X, X^+)/\dot{c}(0) = \dot{c}(0).$$

Combining the two results above one concludes that $S$ is tangent to $M(X)$ at $X$.\[\square\]
**Theorem 3.2** Each of $F_k, \Phi_{m,n}, \Phi_{m,\infty}$ and $\Phi_{\infty,n}$ is a smooth submanifold in $B(E,F)$ and tangent to $M(X)$ at any $X$ in it.

**Proof** It is well known that $X \in F_k \cup \Phi_{m,n} \cup \Phi_{m,\infty} \cup \Phi_{\infty,n}$ is double splitting. By Theorem 3.1, $S = \{T \in V(X,X^+) : R(T) \cap N(X^+) = \{0\}\}$ is a smooth submanifold in $B(E,F)$ tangent to $M(T)$ at any $T \in S$, and $M(X,X^+)(S) = M(X) \cap V(X,X^+)$. Write $\Phi$ as any one of $F_k, \Phi_{m,n}, \Phi_{m,\infty}$ and $\Phi_{\infty,n}$. By Theorem 1.2 and (3.6),

$$S = \{T \in V(X,X^+) : R(T) \cap N(X^+) = \{0\}\} = \Phi \cap V(X,X^+)$$

and

$$M(X,X^+)(S) = M(X,X^+)(\Phi \cap V(X,X^+)) = M(X) \cap V(X,X^+). \quad (3.9)$$

Obviously, $(M(X,X^+),V(X,X^+),B(E,F))$ is a smooth admissible chart of $B(E,F)$ at $X \in \Phi$. While $M(X,X^+)(\Phi \cap V(X,X^+)) = M(X) \cap V(X,X^+)$ is an open set in $M(X)$ because of (3.9) and $M(X) \oplus \mathbb{E}_X = B(E,F)$ by Lemma 3.1. So we conclude that $\Phi$ is a smooth submanifold in $B(E,F)$. Finally, by Theorem 3.1, $\Phi$ is tangent to $M(X)$ at any $X \in \Phi$. The theorem is proved. □

Theorem 3.2 expands the result for $\Phi_{1,1}$ to more wide classes of operators. It seems to be nice to further developing of the method by V. Cafagna in [Caf]. (see [An]).

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