Univalent Harmonic Mappings and Lift to the Minimal Surfaces

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Abstract—We construct sense-preserving univalent harmonic mappings which map the unit disk onto a domain which is convex in the horizontal direction, but with varying dilatation. Also, we obtain minimal surfaces associated with such harmonic mappings. This solves also a recent problem of Dorff and Muir. In several of the cases, we illustrate mappings together with their minimal surfaces pictorially with the help of Mathematica software.

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1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane. Shear construction of univalent harmonic mappings in $\mathbb{D}$ (see Theorem 1) motivated by Clunie and Sheil-Small [1] is instrumental in identifying harmonic analog of the classical Koebe function which and its rotation played the role of extremal for many extremal problems in the theory of univalent analytic mappings in $\mathbb{D}$. The method of shearing has been used effectively in determining several nice properties and examples of univalent harmonic mappings. Another important result for the study of surfaces using geometry and harmonic mappings is the so called Weierstrass—Enneper representation (cf. [4, p. 177–178]). The present article is essentially dealing with some applications of these two results. Similar applications are obtained in [3, 5, 8, 9, 11, 12] and thus, the present note is a continuation of these recent investigations.

Let $\mathcal{B}$ denote the class of analytic self-maps of the unit disk $\mathbb{D}$ and $\mathcal{B}_0 = \{\omega \in \mathcal{B} : \omega(0) = 0\}$. In the recent years, the class $\mathcal{H}$ of all complex-valued harmonic mappings $f = h + \overline{g}$ on $\mathbb{D}$, normalized by $h(0) = g(0) = h'(0) = 1 = 0$, where $h$ and $g$ are analytic, attracted the attention of function theorists in many different contexts. By a result of Lewy [7], $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense-preserving if and only if $J_f(z) > 0$ in $\mathbb{D}$, where $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ denotes the Jacobian of $f$. Positivity of the Jacobian is equivalent to the existence of complex dilatation $\omega \in \mathcal{B}$ such that $\omega(z) = g'(z)/h'(z)$. Let $\mathcal{S}_H$ be the class of all sense-preserving harmonic univalent mappings $f \in \mathcal{H}$ and $\mathcal{S}_H^0$, the subclass of mappings $f \in \mathcal{S}_H$ such that $f(z)(0) = 0$. Set $S = \{f = h + \overline{g} \in \mathcal{S}_H : g(z) \equiv 0\}$.

We recall that a domain $\Omega \subset \mathbb{C}$ is said to be convex in the horizontal direction (CHD) if its intersection with each horizontal line is connected (or empty). We follow the convention that $f \in \mathcal{S}_H$ is a CHD mapping if $f(\mathbb{D})$ is CHD. Now it is appropriate to recall the following theorem of Clunie and Sheil-Small [1] which is crucial in the construction of harmonic mappings.

Theorem 1. Let $f = h + \overline{g}$ be harmonic and locally univalent in $\mathbb{D}$. Then $f$ is univalent and its range is CHD if and only if $h - g$ has the same property.

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An algorithmic approach of Theorem 1 follows. For a given CHD conformal mapping \( \varphi \) of \( \mathbb{D} \) and a dilatation \( \omega \in B_0 \), the shear of \( \varphi(z) \) for the given \( \omega(z) \) is defined to be the mapping \( f = h + \overline{g} \) satisfying the pair of differential equations

\[
\begin{align*}
  h'(z) - g'(z) &= \varphi'(z), \\
  g'(z) - \omega(z)h'(z) &= 0.
\end{align*}
\]

Then a straightforward calculation gives the desired harmonic mapping \( f = h + \overline{g} \) as

\[
f(z) = \text{Re} \left\{ 2 \int_0^z \frac{\varphi'(-\zeta)}{1 - \omega(\zeta)} d\zeta - \varphi(z) \right\} + i \text{Im} \{ \varphi(z) \}. \tag{1}
\]

According to Theorem 1, \( f \) is then a CHD mapping. We recall the following version of Weierstrass–Enneper representation (cf. [4, pp. 177–178]).

**Theorem 2** (Weierstrass–Enneper representation). Let \( \Omega \subseteq \mathbb{C} \) be a simply connected domain containing the origin. If a minimal graph \( \{(u, v, F(u, v)) : u + iv \in \Omega \} \) is parameterized by sense-preserving isothermal parameters \( z = x + iy \in \mathbb{D} \), the projection onto its base plane defines a harmonic mapping \( v = u + iv = f(z) \) of \( \mathbb{D} \) onto \( \Omega \) whose dilatation is the square of an analytic function. Conversely, if \( f = h + \overline{g} \) is a harmonic univalent mapping of \( \mathbb{D} \) onto \( \Omega \) with dilatation \( \omega = g'/h' = q^2 \), the square of an analytic function \( q \), then with \( z = x + iy \in \mathbb{D} \), the parametrization

\[
X(z) = \left( \text{Re} \{ h(z) + g(z) \}, \text{Im} \{ h(z) - g(z) \}, 2\text{Im} \left\{ \int_0^z h'(-\zeta)q(\zeta)d\zeta \right\} \right)
\]

defines a minimal graph whose projection into the complex plane is \( f(\mathbb{D}) \). Except for the choice of sign and an arbitrary additive constant in the third coordinate function, this is the only such surface.

Further information about the relationship between certain harmonic mappings and the associated minimal surfaces can be found from [2, 4, 6, 11–14]. In [8], the authors considered for example the single slit CHD mapping, namely, the Koebe function \( k(z) = z/(1 - z)^2 \), and derived the following result.

**Theorem 3.** Let \( X \) be a minimal surface over the slit domain \( L = k(\mathbb{D}) \) with the projection \( f = h + \overline{g} \in S^0_H \), which satisfies \( h(z) - g(z) = z(1 - z)^{-2} \) and whose dilatation \( \omega = z^2 \). Then \( X = \{(u, v, F(u, v)) : u + iv \in L \} \), where

\[
\begin{align*}
  u &= \text{Re} \left\{ \frac{z(2z^2 - 3z + 3)}{3(1 - z)^3} \right\}, \\
  v &= \text{Im} \left\{ \frac{z}{(1 - z)^2} \right\}, \\
  F &= \text{Im} \left\{ \frac{z(2 - z)}{(1 - z)^2} - \frac{2z(z^2 - 3z + 3)}{3(1 - z)^3} \right\}.
\end{align*}
\]

As in the recent article of Dorff and Muir [3], we consider the generalized Koebe function \( k_c : \mathbb{D} \to \mathbb{C} \) defined by

\[
k_c(z) = \int_0^z \frac{(1 + \zeta)^{c-1}}{(1 - \zeta)^{c+1}} d\zeta = \frac{1}{2c} \left[ \left( \frac{1 + z}{1 - z} \right)^c - 1 \right] \tag{2}
\]

for \( c \in [0, 2] \), and in the case of \( c = 0 \), the function \( k_c(z) \) should be interpreted as the limiting case:

\[
k_0(z) = \lim_{c \to 0^+} k_c(z) = \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right).
\]

Obviously, \( k_1(z) = z/(1 - z) \) and \( k_2(z) = z/(1 - z)^2 \). Moreover, for \( c \in [0, 2] \), \( k_c \in S \) and \( k_c(\mathbb{D}) \) is CHD. Additionally, for \( c \in [0, 1] \), \( k_c(\mathbb{D}) \) is convex.

**Theorem 4** ([3, Theorem 3]). For \( c \in [0, 2] \), define \( f_c = h_c + \overline{g_c} \in S^0_H \) to be the harmonic mapping satisfying \( h_c(z) - g_c(z) = k_c(z) \), \( g_c(z) = z^2h_c(z) \), where \( k_c \) is given by (2). Then \( f_c(\mathbb{D}) \) is CHD, and as \( c \) varies from 0 to 2, \( f_c(\mathbb{D}) \) transforms from a strip mapping to a slit mapping.

In [3], the authors also proposed that the family of harmonic mappings given in Theorem 4 can be generalized by changing the dilatation to \( \omega(z) = z^{2m} \ (m \in \mathbb{N}) \). That is, for \( c \in [0, 2] \) and \( n = 2m \), let
The images of the unit disk

\[ F_1(\alpha; \beta_1, \beta_2; \gamma; x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_{k+l}(\beta_1)_k(\beta_2)_l}{k!l!\gamma_{k+l}} x^k y^l, \]

where \((q)_0 = 1\) for \(q \neq 0\) and for \(q \in \mathbb{C}\setminus\{0\}\), \((q)_k = q(q+1) \cdots (q+k-1) = \Gamma(q+k)/\Gamma(q)\) is the Pochhammer symbol. The function \(F_1\) can also be written as a one-dimensional Euler-type integral:

\[ F_1(\alpha; \beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_{0}^{1} \frac{t^{\alpha-1}(1-t)^{\gamma-\alpha-1}}{(1-xt)^{\beta_1}(1- yt)^{\beta_2}} dt, \]

where \(\text{Re}\gamma > \text{Re}\alpha > 0\).

2. HARMONIC MAPPINGS WITH THE DILATATION \(\omega(z) = z(z + a)/(1 + az)\)

Throughout this section, in the following examples and in Theorem 5, our aim is to construct a family of CHD mappings with the dilatation \(\omega(z) = z(z + a)/(1 + az)\), where \(-1 \leq a \leq 1\). For \(a = 1\) and \(a = 0\), \(\omega(z)\) becomes \(z\) and \(z^2\), respectively.

**Example 1.** Consider the identity mapping \(\varphi(z) = z\). Then, by (1), the shear construction produces the harmonic mappings

\[ F_a(z) = \text{Re}\{-z + (1 - a) \log(1 + z) - (1 + a) \log(1 - z)\} + i\text{Im}\{z\}. \]

The images of the unit disk \(\mathbb{D}\) under \(F_a\) for certain values of \(a \in [-1, 1]\) are shown in Fig. 1. We observe that \(F_{-a}(-z) = -F_a(z)\).

**Example 2.** Consider the strip mapping \(\varphi(z) = \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right)\) which maps \(\mathbb{D}\) onto the horizontal strip \(\{w \in \mathbb{C} : |\text{Im}\{w\}| < \pi/4\}\). Then by (1) one obtains CHD mappings

\[ F_{0,a}(z) = \text{Re}\left\{\frac{1 + a}{2} \frac{z}{1 - z} + \frac{1 - a}{2} \frac{z}{1 + z}\right\} + i\text{Im}\left\{\frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right)\right\}. \]

The images of the unit disk under \(F_{0,a}\) for certain values of \(a \in [-1, 1]\) are shown in Fig. 2. Observe that

\[ F_{0,a}(z) = -F_{0,-a}(-z) \quad \text{and} \quad F_{0,a}(e^{i\theta}) = \begin{cases} 
-\frac{\theta}{2} + i\frac{\pi}{4}, & \text{for } 0 < \theta < \pi, \\
-\frac{\theta}{2} - i\frac{\pi}{4}, & \text{for } \pi < \theta < 2\pi.
\end{cases} \]
In particular, $F_{0,a}(z)$ collapses the upper and lower semicircles into single point $(-\frac{a}{2}, \frac{\pi}{4})$ and $(-\frac{a}{2}, -\frac{\pi}{4})$, respectively.

Actually, we can show that $F_{0,a}(z)$ maps the unit disk $D$ onto the full strip $\{w \in \mathbb{C} : \|\text{Im}(w)\| < \frac{\pi}{4}\}$ for $-1 < a < 1$. We will now show that $-\infty < \text{Re}\{F_{0,a}(z)\} < +\infty$ and for this, we only need to prove that

$$-\infty < \text{Re}\left\{\frac{1 + a}{2} \frac{z}{1 - z} + \frac{1 - a}{2} \frac{z}{1 + z}\right\} < +\infty$$

for $z \in (-1, 1)$, where $z = x + iy$. Set

$$U(x) = \frac{1 + a}{2} \frac{x}{1 - x} + \frac{1 - a}{2} \frac{x}{1 + x}$$

and note that the function $U(x)$ is continuous in the interval $x \in (-1, 1)$. Fixing $a$ shows that $\lim_{x \to -1^+} U(x) = -\infty$ and $\lim_{x \to 1^-} U(x) = +\infty$. Additionally, for the cases $a = -1$ and $a = 1$, $F_{0,a}(z)$ maps the unit disk onto the half-strips

$$\left\{w : \text{Re}\{w\} < \frac{1}{2}, \|\text{Im}(w)\| < \frac{\pi}{4}\right\} \quad \text{and} \quad \left\{w : \text{Re}\{w\} > -\frac{1}{2}, \|\text{Im}(w)\| < \frac{\pi}{4}\right\},$$

respectively. This completes the proof.

**Example 3.** Now, consider the half-plane mapping $\varphi(z) = z/(1 - z)$. Then by (1), we obtain the harmonic mappings (see Fig. 3)

$$F_{1,a}(z) = \text{Re}\left\{\frac{1 - a}{4} \log\left(\frac{1 + z}{1 - z}\right) + \frac{1 + a}{2} \frac{z}{(1 - z)^2}\right\} + i\text{Im}\left\{\frac{z}{1 - z}\right\}.$$ 

Note that

$$\text{Re}\{F_{1,a}(re^{-i\theta})\} = \frac{1}{8} \left\{\frac{4(a + 1)r \left((r^2 + 1) \cos\theta - 2r\right)}{(r^2 - 2r \cos\theta + 1)^2}\right\}.$$
and

\[
+ (a - 1) \left( \log \left( r^2 - 2r \cos \theta + 1 \right) - \log \left( r^2 + 2r \cos \theta + 1 \right) \right) \right) = \text{Re} \{ F_{1,a}(re^{i\theta}) \},
\]

and

\[
\text{Im} \{ F_{1,a}(re^{-i\theta}) \} = - \frac{r \sin \theta}{r^2 - 2r \cos \theta + 1} = -\text{Im} \{ F_{1,a}(re^{i\theta}) \}
\]

which imply that the range \( F_{1,a}(\mathbb{D}) \) is symmetric about the real axis. Since

\[
F_{1,a}(e^{i\theta}) = \text{Re} \left\{ \frac{1 - a}{4} \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) + \frac{1 + a}{2} \frac{e^{i\theta}}{(1 - e^{i\theta})^2} \right\} + i\text{Im} \left\{ \frac{e^{i\theta}}{1 - e^{i\theta}} \right\}
\]

\[
= \frac{1 - a}{8} \log \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right) - \frac{1 + a}{4} \frac{1}{1 - \cos \theta} + i \frac{1}{2} \cot \frac{\theta}{2} =: u + iv,
\]

we easily find that

\[
u = \frac{1 - a}{8} \log (4v^2) - \frac{1 + a}{8} (4v^2 + 1).
\]

In particular, \( F_{1,1}(\partial \mathbb{D}) \) describes the parabola \( v^2 = -u - 1/4 \).

**Theorem 5.** For \( c \in [0, 2] \), and \( a \in [-1, 1] \), let \( F_{c,a} = H_{c,a} + \overline{G_{c,a}} \in \mathcal{S}_H^0 \) such that

\[
H_{c,a}(z) - G_{c,a}(z) = k_c(z) \quad \text{and} \quad \omega_a(z) = z(z + a)/(1 + az),
\]

where \( k_c(z) \) is given by (2). Then \( F_{c,a}(\mathbb{D}) \) is convex in the horizontal direction, and as \( c \) varies from 0 to 2, \( F_{c,a}(\mathbb{D}) \) transforms from a strip mapping to a slit mapping.

**Proof.** According to Theorem 1, we obtain that \( F_{c,a}(\mathbb{D}) \) is convex in the horizontal direction. Throughout the proof, it suffices to assume that \( c \in (0, 2] \setminus \{1\} \). By (3), we have

\[
H'_{c,a}(z) - G'_{c,a}(z) = \frac{1}{(1 + z)(1 - z)} \left( \frac{1 + z}{1 - z} \right)^c \quad \text{and} \quad G'_{c,a}(z) = z \frac{z + a}{1 + az} H'_{c,a}(z).
\]
Solving these two equations, we obtain

\[ H'_{c,a}(z) = \frac{1 + az}{(1 + z)^2(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^c \]

\[ = \left\{ \frac{1}{4} \left( \frac{1}{1 - z} + \frac{1}{1 + z} \right) + \frac{1 + a}{4} \frac{1}{(1 - z)^2} + \frac{1 - a}{4} \frac{1}{(1 + z)^2} \right\} \left( \frac{1 + z}{1 - z} \right)^c. \]

Straightforward integration gives

\[ H_{c,a}(z) = \frac{(1 - 2c^2 + ac) + 2c(1 - ac)z + (ac - 1)z^2}{4c(1 - c^2)(1 - z^2)} \left( \frac{1 + z}{1 - z} \right)^c - \frac{1 + ac - 2c^2}{4c(1 - c^2)} \]

and thus, we find that

\[ G_{c,a}(z) = H_{c,a}(z) - k_c(z) = \frac{(1 - 2c^2 + ac) + 2c(1 - ac)z + (ac - 1)z^2}{4c(1 - c^2)(1 - z^2)} \left( \frac{1 + z}{1 - z} \right)^c - \frac{1 + ac - 2c^2}{4c(1 - c^2)} - \frac{1}{2c} \left( \frac{1 + z}{1 - z} \right)^c - 1. \]

In order to study the mapping properties of \( F_{c,a} \), we let \( w = (1 + z)/(1 - z) \) i.e., \( z = (w - 1)/(w + 1) \), which leads to

\[ H_{c,a}(z) = \frac{1}{8} \left( \frac{a + 1}{c + 1} w^{c+1} + \frac{2}{c} w^c - \frac{a - 1}{c - 1} w^{c-1} - \frac{2(1 + ac - 2c^2)}{c(1 - c^2)} \right) \]

and

\[ G_{c,a}(z) = \frac{1}{8} \left( \frac{a + 1}{c + 1} w^{c+1} - \frac{2}{c} w^c - \frac{a - 1}{c - 1} w^{c-1} + \frac{2(1 - ac)}{c(1 - c^2)} \right). \]
so that
\[ F_{c,a}(z) = \Re \left\{ \frac{1}{4} \left( \frac{a+1}{c+1} w^{c+1} - \frac{a-1}{c-1} w^{c-1} + \frac{2(c-a)}{1-c^2} \right) \right\} + i \Im \left\{ \frac{1}{2c} (w^c - 1) \right\}. \] (4)

By writing \( w = x + iy \), \( x > 0 \) and \( y \in \mathbb{R} \), from Example 2, it can be easily seen that \( F_{0,a} \) maps \( \mathbb{D} \) onto the strip \( \{ \zeta \in \mathbb{C} : |\Im \zeta| < \pi/4 \} \). If we substitute \( c = 2 \) in (4), then we have
\[ F_{2,a}(z) = \Re \left\{ \frac{1}{4} \left( \frac{1+a}{3} w^3 + (1-a)w - \frac{2(2-a)}{3} \right) \right\} + i \Im \left\{ \frac{1}{4} (w^2 - 1) \right\} = \frac{1}{4} \left( \frac{1+a}{3} (x^3 - 3xy^2) + (1-a)x - \frac{2(2-a)}{3} \right) + i \frac{x}{2} \cdot xy, \quad x > 0. \]

Observe now that each point \( z \neq 1 \) on the unit circle is carried onto a point \( w \) on the imaginary axis so that \( x = 0 \) and \( F_{2,a}(z) = -(2-a)/6 \). Similar discussion as in the case of harmonic Koebe function \( K(z) \) in [4, p. 84–86] proves that \( F_{2,a}(z) \) maps the unit disk \( \mathbb{D} \) onto the entire plane minus the real interval \( (-\infty, -(2-a)/6] \). \( \Box \)

**Remark 1.** When \( a = 0 \), \( \omega_n(z) \) becomes \( z^2 \) and thus, Theorem 5 reduces to Theorem 4, and hence, Theorem 2 is applicable. If \( a = -1 \) and \( a = 1 \), then \( \omega_n(z) \) becomes \(-z \) and \( z \), respectively, and hence, Theorem 5 is a generalization of the Theorem 4.

## 3. SHEARING CONSTRUCTION AND MINIMAL SURFACES

In this section, we use Theorem 1 to build a family of harmonic univalent mappings with a CHD range that lifts to a family of minimal surfaces as described in Theorem 2.

In 2004, Greiner [5] constructed horizontal strip harmonic mappings with dilatation \( \omega(z) = z^n \) by shearing
\[ h_{0,n}(z) - g_{0,n}(z) = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right). \]

After tedious but straightforward calculations, the shear construction produces the harmonic mappings \( f_{0,n}(z) \) defined for \( n = 2m + 1 \) \((m \in \mathbb{N}) \) by
\[ f_{0,n}(z) = \Re \left\{ h_{0,n}(z) + g_{0,n}(z) \right\} + i \Im \left\{ h_{0,n}(z) - g_{0,n}(z) \right\} = \Re \left\{ \frac{1}{n} \left( \frac{z}{1-z} - i \sum_{k=1}^{(n-1)/2} \csc \frac{2k\pi}{n} \log \left( \frac{1-ze^{-2ik\pi/n}}{1-ze^{2ik\pi/n}} \right) \right) \right\} + i \Im \left\{ \frac{2}{n} \log \left( \frac{1+z}{1-z} \right) \right\}. \]

Moreover, if \( n = 2m \) \((m \in \mathbb{N}) \), by virtue of Theorem 2, \( f_{0,n}(\mathbb{D}) \) lift to the minimal surfaces \( X_{0,n}(u, v) = (u, v, F(u, v)) \), where
\[ u = \Re \left\{ \frac{1}{n} \left( \frac{2z}{1-z^2} - i \sum_{k=1}^{n/2-1} \csc \frac{2k\pi}{n} \log \left( \frac{1-ze^{-ik\pi/n}}{1-ze^{ik\pi/n}} \right) \right) \right\}, \quad v = \Im \left\{ \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \right\}, \]
and
\[ F(u, v) = \Im \left\{ \frac{1}{n} \left( \frac{z}{1-z} + \frac{(-1)^{n/2}z}{1+z} - i \sum_{k=1}^{(n/2)-1} (-1)^k \csc \frac{2k\pi}{n} \log \left( \frac{1-ze^{-ik\pi/n}}{1-ze^{ik\pi/n}} \right) \right) \right\}. \]

**Remark 2.** For \( \omega(z) = z \), the expression for \( f_{0,1}(z) \) simplifies to
\[ f_{0,1}(z) = \Re \left\{ \frac{z}{1-z} \right\} + i \Im \left\{ \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \right\} \]
whereas for \( \omega(z) = z^2 \), we have
\[ f_{0,2}(z) = \Re \left\{ \frac{z}{1-z^2} \right\} + i \Im \left\{ \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \right\}. \]
In Fig. 4, we illustrate the minimal surfaces of the harmonic mappings $f_{0,n}(z)$ onto strip domains whenever $\omega(z) = z^n$ with $n = 4, 6, 8, 10$.

**Theorem 6.** Let $f_{1,n} = h_{1,n} + g_{1,n} \in \mathcal{S}_L^0$ such that
\[
 h_{1,n}(z) - g_{1,n}(z) = \frac{z}{1-z} \quad \text{and} \quad \omega(z) = \frac{g_{1,n}'(z)}{h_{1,n}'(z)} = z^n \quad (n \in \mathbb{N}).
\]

If $n = 2m + 1 \quad (m \in \mathbb{N})$, then we have
\[
 f_{1,n}(z) = \text{Re} \left\{ \frac{1}{n} \left( -z + \frac{z(2-z)}{1-z} - \frac{n^2-1}{6} \log(1-z) \right) \right\} + \text{Im} \left\{ \frac{z}{1-z} \right\}.
\]

If $n = 2m \quad (m \in \mathbb{N})$, then $f_{1,n}(\mathbb{D})$ lift to the minimal surfaces $X_{1,n}(u,v) = (u,v,F(u,v))$, where
\[
 u = \text{Re} \left\{ \frac{1}{n} \left( -z + \frac{z(2-z)}{1-z} - \frac{n^2-1}{6} \log(1-z) + \frac{1}{2} \log(1+z) \right) \right\} + \frac{1}{2} \sum_{k=1}^{(n-1)/2} \csc^2 \frac{k\pi}{n} \log \left( 1 - 2z \cos \frac{2k\pi}{n} + z^2 \right),
\]
\[
 v = \text{Im} \left\{ \frac{z}{1-z} \right\},
\]
and
\[
 F(u,v) = \text{Im} \left\{ \frac{1}{n} \left( -z + \frac{z(2-z)}{1-z} + \frac{(-1)^{n/2}}{2} \log(1+z) + \frac{n^2+2}{6} \log(1-z) \right) \right\}.
\]
into partial fractions. For the case of odd values of \( n \), solving these two equations, we obtain

\[
h'_1, n(z) = (1 - z)^{-2}(1 - z^{n})^{-1},
\]

which has a pole of order 3 at \( z = 1 \) and simple poles at the other \( n \)-th roots of unity when \( n = 2m + 1 \) (\( m \in \mathbb{N} \)). Similarly, it has a pole of order 3 at \( z = 1 \) and simple poles at \( z = -1 \) and at the other \( n \)-th roots of unity when \( n = 2m \) (\( m \in \mathbb{N} \)). In view of these observations, we can decompose \( h'_1, n(z) \) into partial fractions. For the case of odd values of \( n \), we rewrite (7) as

\[
h'_1, n(z) = \frac{\kappa_1}{1 - z} + \frac{\kappa_2}{(1 - z)^2} + \frac{\kappa_3}{(1 - z)^3} + \sum_{k=1}^{(n-1)/2} \frac{\alpha_k}{1 - ze^{-i\frac{2k\pi}{n}}} + \sum_{k=1}^{(n-1)/2} \frac{\beta_k}{1 - ze^{i\frac{2k\pi}{n}}}
\]

and the constants may be computed by using the residue theorem:

\[
\begin{align*}
\kappa_1 &= \frac{n^2 - 1}{12n}, \\
\kappa_2 &= \frac{n - 1}{2n}, \\
\kappa_3 &= \frac{1}{n}, \\
\alpha_k &= \frac{1}{n(1 - e^{-i\frac{2k\pi}{n}})^2}, \\
\beta_k &= \frac{1}{n(1 - e^{i\frac{2k\pi}{n}})^2}.
\end{align*}
\]

By integrating the previous expression we arrive at the expression for the case of odd values of \( n \):

\[
h_{1, n}(z) = \frac{n - 1}{2n} \frac{z}{1 - z} + \frac{z(2 - z)}{2n(1 - z)^2} - \frac{n^2 - 1}{12n} \log(1 - z)
+ \frac{1}{4n} \sum_{k=1}^{(n-1)/2} \csc^{2} \frac{k\pi}{n} \log \left( 1 - 2z \cos \frac{2k\pi}{n} + z^2 \right).
\]

In the case of even values of \( n \) one can write (7) into partial fractions as

\[
h'_1, n(z) = \frac{\lambda_1}{1 - z} + \frac{\lambda_2}{(1 - z)^2} + \frac{\lambda_3}{(1 - z)^3} + \frac{\lambda_4}{1 + z} + \sum_{k=1}^{(n/2)-1} \frac{\gamma_k}{1 - ze^{-i\frac{2k\pi}{n}}} + \sum_{k=1}^{(n/2)-1} \frac{\delta_k}{1 - ze^{i\frac{2k\pi}{n}}}.
\]

Again, using the residue theorem or otherwise, we find that

\[
\begin{align*}
\lambda_1 &= \frac{n^2 - 1}{12n}, \\
\lambda_2 &= \frac{n - 1}{2n}, \\
\lambda_3 &= \frac{1}{n}, \\
\lambda_4 &= \frac{1}{4n}, \\
\gamma_k &= \frac{1}{n(1 - e^{-i\frac{2k\pi}{n}})^2}, \\
\delta_k &= \frac{1}{n(1 - e^{i\frac{2k\pi}{n}})^2}.
\end{align*}
\]

and we arrive at the expression

\[
h_{1, n}(z) = \frac{n - 1}{2n} \frac{z}{1 - z} + \frac{z(2 - z)}{2n(1 - z)^2} - \frac{n^2 - 1}{12n} \log(1 - z) + \frac{1}{4n} \log(1 + z)
+ \frac{1}{4n} \sum_{k=1}^{(n/2)-1} \csc^{2} \frac{k\pi}{n} \log \left( 1 - 2z \cos \frac{2k\pi}{n} + z^2 \right).
\]

In both cases, the corresponding function \( g_{1, n}(z) \) may be computed using the first relation (5) and the above two cases. Finally, the desired harmonic mapping \( f_{1, n}(z) \) follows from writing \( f_{1, n}(z) \) as

\[
f_{1, n}(z) = u + iv = \text{Re}\{h_{1, n}(z) + g_{1, n}(z)\} + i\text{Im}\{h_{1, n}(z) - g_{1, n}(z)\}.
\]

Consequently, \( f_{1, n}(z) \) for the case of odd values of \( n \) is given by

\[
f_{1, n}(z) = \text{Re}\left\{ \frac{1}{n} \left( \frac{-z}{1 - z} + \frac{z(2 - z)}{(1 - z)^2} - \frac{n^2 - 1}{6} \log(1 - z) \right) \right\}.
\]
and \( f_{1,n}(z) \) for the case of even values of \( n \) takes the form \( f_{1,n} = u + iv \), where \( u \) is given by (6) and \( v(x, y) = \text{Im} \{ z/(1 - z) \} \). In view of Theorem 2, \( f_{1,n}(D) \) for the case of even \( n \) lifts to the minimal surfaces \( X_{1,n}(u, v) = (u, v, F(u, v)) \), where \( u \) is given by (6), \( v = v(x, y) = \text{Im} \{ z/(1 - z) \} \) and

\[
F(u, v) = 2\text{Im} \left\{ \int_0^z \sqrt{\omega_n(\zeta)} h_{1,n}(\zeta) d\zeta \right\} = 2\text{Im} \left\{ \int_0^\zeta \frac{\zeta^{n/2}}{(1 - \zeta)^2(1 - \zeta^n)} d\zeta \right\} - \frac{1}{2n} \sum_{k=1}^{(n/2) - 1} (-1)^k \csc^2 \frac{k\pi}{n} \log \left( 1 - 2z \cos \frac{2k\pi}{n} + z^2 \right) \cdot \}

The proof is complete. \( \square \)

**Remark 3.** If \( \omega(z) = z \) in Theorem 6, then the expression for \( f_{1,1}(z) \) simplifies to \( f_{1,1}(z) = \text{Re} \{ k(z) \} + i\text{Im} \{ l(z) \} \), where \( k(z) = z/(1 - z)^2 \) and \( l(z) = z/(1 - z) \). Here we may compare \( f_{1,1}(z) \) with the well known harmonic half-plane mapping \( L(z) \) defined by \( L(z) = \text{Re} \{ l(z) \} + i\text{Im} \{ k(z) \} \). For \( \omega(z) = z^2 \), the expression of \( f_{1,2}(z) \) is given by

\[
f_{1,2}(z) = \text{Re} \left\{ \frac{1}{2} \frac{z}{(1 - z)^2} + \frac{1}{4} \log \left( \frac{1 + z}{1 - z} \right) \right\} + i\text{Im} \left\{ \frac{z}{1 - z} \right\}.
\]

For this special case, see also [3, Theorem 3]. In Fig. 5, we have drawn the images of \( D \) under the harmonic mappings \( f_{1,n}(z) \) with the dilatation \( \omega(z) = z^n, \quad n = 3, 4, 5, 6 \). In Fig. 6, we have drawn the minimal surfaces associated with the harmonic mappings \( f_{1,n}(z) \) if \( \omega(z) = z^n, \quad n = 4, 6, 8, 10 \) (see Theorem 2).

**Theorem 7.** For \( n \in \mathbb{N} \), let \( f_{2,n} = h_{2,n} + g_{2,n} \in \mathcal{S}_H^0 \) such that

\[
h_{2,n}(z) - g_{2,n}(z) = z(1 - z)^{-2} \quad \text{and} \quad \omega(n) = g_{0,n}(z)/h_{0,n}(z) = z^n. \tag{8}
\]

If \( n = 2m + 1 (m \in \mathbb{N}) \), then we have

\[
f_{2,n}(z) = \text{Re} \left\{ \frac{-z}{(1 - z)^2} + \frac{(n - 1)(n - 2)}{3n} - \frac{z}{1 - z} + \frac{n - 2z(2 - z)}{n} - \frac{4z(z^2 - 3z + 3)}{(1 - z)^3} \right\} + \frac{i}{2n} \sum_{k=1}^{(n/2) - 1} \cot \frac{k\pi}{n} \csc^2 \frac{k\pi}{n} \log \left( 1 - ze^{-\frac{2ik\pi}{n}} \right) + i\text{Im} \left\{ \frac{z}{1 - z} \right\}.
\]

If \( n = 2m (m \in \mathbb{N}) \), then \( f_{2,n}(D) \) lift to the minimal surfaces \( X_{2,n}(u, v) = (u, v, F(u, v)) \), where

\[
u = \text{Im} \left\{ \frac{z}{(1 - z)^2} \right\},
\]

\[
F(u, v) = \text{Im} \left\{ \frac{4 - n^2}{6n} - \frac{2z(2 - z)}{n} - \frac{4z(z^2 - 3z + 3)}{(1 - z)^3} \right\}.
\]
By using the residue theorem or otherwise, one can easily see that
\[
\frac{1}{2\pi i} \oint_C \frac{1}{z^n - z^m} \, dz = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}
\]

Now, consider the case \( n = 2m + 1 \) \((m \in \mathbb{N})\). Since \( h'_{2,n}(z) \) has a pole of order 4 at \( z = 1 \) and simple poles at other \( n \)-th roots of unity, \( h'_{2,n}(z) \) can be represented using partial fractions as follows:
\[
h'_{2,n}(z) = \frac{\lambda_1}{1-z} + \frac{\lambda_2}{(1-z)^2} + \frac{\lambda_3}{(1-z)^3} + \frac{\lambda_4}{(1-z)^4} + \sum_{k=1}^{(n-1)/2} \frac{A_k}{1-ze^{-\frac{2k\pi}{n}}} + \sum_{k=1}^{(n-1)/2} \frac{B_k}{1-ze^{\frac{2k\pi}{n}}}.
\]

By using the residue theorem or otherwise, one can easily see that
\[
\lambda_1 = 0, \quad \lambda_2 = \frac{(n-1)(n-2)}{6n}, \quad \lambda_3 = \frac{n-2}{n}, \quad \lambda_4 = \frac{2}{n},
\]
and
\[
A_k = \frac{1}{n} \frac{1+e^{\frac{2k\pi}{n}}}{(1-e^{\frac{2k\pi}{n}})^3}, \quad B_k = \frac{1}{n} \frac{1+e^{-\frac{2k\pi}{n}}}{(1-e^{-\frac{2k\pi}{n}})^3}.
\]

Using these values, we arrive at the expression
\[
h'_{2,n}(z) = \frac{(n-1)(n-2)}{6n} \frac{1}{(1-z)^2} + \frac{n-2}{n} \frac{1}{(1-z)^3} + \frac{2}{n} \frac{1}{(1-z)^4}
\]
\[
+ \frac{1}{n} \left( \sum_{k=1}^{(n-1)/2} \frac{1+e^{\frac{2k\pi}{n}}}{1-e^{\frac{2k\pi}{n}}} \frac{1}{(1-ze^{-\frac{2k\pi}{n}})^3} + \sum_{k=1}^{(n-1)/2} \frac{1+e^{-\frac{2k\pi}{n}}}{1-e^{-\frac{2k\pi}{n}}} \frac{1}{(1-ze^{\frac{2k\pi}{n}})^3} \right).
\]

**Fig. 5.** \( f_{1,n}(D) \) for various values of \( n = 3, 4, 5, 6 \).
Integration from 0 to \( z \) gives
\[
h_{2,n}(z) = \frac{(n-1)(n-2)}{6n} \frac{z}{1-z} + \frac{n-2}{2n} \frac{z(2-z)}{(1-z)^2} + \frac{2}{3n} \frac{z(z^2 - 3z + 3)}{(1-z)^3} \\
+ \frac{i}{4n} \sum_{k=1}^{(n-1)/2} \cot \frac{k\pi}{n} \csc^2 \frac{k\pi}{n} \log \left( \frac{1 - ze^{-i\frac{2k\pi}{n}}}{1 - ze^{i\frac{2k\pi}{n}}} \right),
\]
and, as a consequence of it, \( g_{2,n}(z) \) can be written explicitly by using the first relation in (8). Then the desired harmonic mappings \( f_{2,n} \in S^0_H \) for odd values of \( n \) are given by
\[
f_{2,n}(z) = \operatorname{Re} \left\{ -\frac{z}{1-z} + \frac{(n-1)(n-2)}{3n} \frac{z}{1-z} + \frac{n-2}{n} \frac{z(2-z)}{(1-z)^2} + \frac{4}{3n} \frac{z(z^2 - 3z + 3)}{(1-z)^3} \\
+ \frac{i}{2n} \sum_{k=1}^{(n-1)/2} \cot \frac{k\pi}{n} \csc^2 \frac{k\pi}{n} \log \left( \frac{1 - ze^{-i\frac{2k\pi}{n}}}{1 - ze^{i\frac{2k\pi}{n}}} \right) \right\} + i \operatorname{Im} \left\{ \frac{z}{1-z} \right\}.
\]
For even values of \( n \), we have \( f_{2,n}(z) = u + iv \), where \( u \) is given by (9) and \( v = \operatorname{Im} \left\{ \frac{z}{1-z} \right\} \).

In view of Theorem 2, for \( n = 2m \) (\( m \in \mathbb{N} \)), \( f_{2,n}(\mathbb{D}) \) lift to the minimal surfaces \( X_{2,n}(u, v) = (u, v, F(u, v)) \), where \( u \) is given by (9), \( v = \operatorname{Im} \left\{ \frac{z}{1-z} \right\} \), and \( F(u, v) \) is obtained from
\[
F(u, v) = 2 \operatorname{Im} \left\{ \int_0^z \sqrt{\omega_n(\zeta)} h'_{2,n}(\zeta) d\zeta \right\} = 2 \operatorname{Im} \left\{ \int_0^z \frac{(1+\zeta)^{n/2}}{(1-\zeta)^3(1-\zeta^n)} d\zeta \right\} \\
= \operatorname{Im} \left\{ \frac{4n^2}{6n} \frac{z}{1-z} - \frac{2}{n} \frac{2z(2-z)}{(1-z)^2} + \frac{4}{3n} \frac{z(z^2 - 3z + 3)}{(1-z)^3} \\
+ \frac{i}{2n} \sum_{k=1}^{(n/2)-1} (-1)^k \cot \frac{k\pi}{n} \csc^2 \frac{k\pi}{n} \log \left( \frac{1 - ze^{-i\frac{2k\pi}{n}}}{1 - ze^{i\frac{2k\pi}{n}}} \right) \right\}.
\]
The proof is complete. \( \square \)

**Remark 4.** For \( \omega(z) = z \) in Theorem 7, the resulting function \( f_{2,1}(z) \) is the well-known harmonic Koebe function.

In Fig. 7, we have illustrated the harmonic mappings \( f_{2,n}(z) \) of the unit disk \( \mathbb{D} \) onto slit domains whenever \( \omega(z) = z^n \) for \( n = 3, 4, 5, 6 \). In Fig. 8, we have drawn the minimal surfaces of the harmonic mappings \( f_{2,n}(z) \) onto slit domains whenever \( \omega(z) = z^n \) for \( n = 4, 6, 8, 10 \).

**Theorem 8.** For \( c \in [0, 2] \) and \( n \in \mathbb{N} \), consider the harmonic mappings \( f_{c,n} = h_{c,n} + \overline{g_{c,n}} \in S^0_H \) which satisfy the conditions
\[
h_{c,n}(z) - g_{c,n}(z) = k_c(z) \quad \text{and} \quad g_{c,n}(z) = z^n h'_{c,n}(z), \tag{10}
\]
where \( k_c(z) \) is given by (2). Then \( f_{c,n}(\mathbb{D}) \) is convex in the horizontal direction, and as \( c \) varies from 0 to 2, \( f_{c,n}(\mathbb{D}) \) transforms from a strip mapping to a wave plane and then to a slit mapping. In particular, \( f_{c,n}(\mathbb{D}) \) lift to minimal surfaces when \( n \) is an even positive integer.

**Proof.** For each \( c \in [0, 2] \), \( k_c \in S \) and \( k_c(\mathbb{D}) \) is a domain convex in the horizontal direction. Thus, by Theorem 1, \( f_{c,n}(\mathbb{D}) \) is a CHD domain. It remains to consider the mapping properties of the function \( f_{c,n} \). Solving the two equations in (10), one can easily find that
\[
h'_{c,n}(z) = \left( \frac{1+z}{1-z} \right)^c \frac{1}{(1-z^2)(1-z^n)}. \tag{11}
\]
As in the proof of earlier theorems, for the case \( n = 2m + 1 (m \in \mathbb{N}) \), we may rewrite \( h_{c,n}^\prime (z) \) as

\[
\begin{align*}
    h_{c,n}^\prime (z) &= \left( \frac{1+z}{1-z} \right)^c \left[ \frac{1}{4} \left( \frac{1}{1-z} + \frac{1}{1+z} \right) + \frac{1}{2n} \frac{1}{(1-z)^2} \right. \\
    &+ \frac{1}{n} \left( \sum_{k=1}^{(n-1)/2} \frac{1}{(1-e^{i\frac{2k\pi}{n}})(1-ze^{-i\frac{2k\pi}{n}})} + \sum_{k=1}^{(n-1)/2} \frac{1}{(1-e^{-i\frac{2k\pi}{n}})(1-ze^{i\frac{2k\pi}{n}})} \right) \\
\end{align*}
\]

Integrating the last equation from 0 to \( z \) and then analyzing the resulting expression carefully, one obtains

\[
\begin{align*}
    h_{c,n} (z) &= \left( \frac{1+z}{1-z} \right)^c \left[ \frac{1}{4c} + \frac{1+z}{4n(1+c)(1-z)} \right. \\
    &+ \left. \frac{(n-1)/2}{n(1-c)(1+z)^c} \sum_{k=1}^{(n-1)/2} \left( 1 - c; -c, 1; 2 - c; \frac{1-z}{2}, \frac{1-ze^{i\frac{2k\pi}{n}}}{1-e^{i\frac{2k\pi}{n}}} \right) \right] \\
\end{align*}
\]
we can write
\[
\frac{\text{Re}\{h + g\} + i\text{Im}\{h - g\}}{\text{Re}\{2h(z) - k_c\} + i\text{Im}\{k_c\}}.
\]
and as a consequence of it and (10), the resulting harmonic mappings \( f_{c,n}(z) \) for the case of odd values of \( n \) have the form
\[
f_{c,n}(z) = \text{Re}\{2h_{c,n}(z) - k_c(z)\} + i\text{Im}\{k_c(z)\} = \text{Re}\left\{ \frac{1}{2c} - 2N_1 + \left( \frac{1 + z}{1 - z} \right)^c \left[ \frac{1 + z}{2n(1 + c)(1 - z)} \right] \right\} + \sum_{k=1}^{(n-1)/2} 2^{2c+1}(1-z) e^{i\frac{2k\pi}{n}} F_1 \left( 1 - c; -c, 1; 2 - c; \frac{1 - z}{2}, \frac{1 - z}{1 - e^{i\frac{2k\pi}{n}}} \right)
\]
\[
\left( \frac{1 - z}{1 - e^{i\frac{2k\pi}{n}}} \right)
\]
where
\[
N_1 = \frac{1}{4c} + \frac{1}{4n(1 + c)} + \sum_{k=1}^{(n-1)/2} 2^{2c} e^{i\frac{2k\pi}{n}} F_1 \left( 1 - c; -c, 1; 2 - c; \frac{1}{2}, \frac{1}{1 - e^{i\frac{2k\pi}{n}}} \right)
\]
\[
\frac{n(1 - c)(1 + z)^c \left( 1 - e^{i\frac{2k\pi}{n}} \right)}{\left( 1 - e^{-i\frac{2k\pi}{n}} \right)}
\]
As before, we need to deal with the two cases. Observe that if \( f = u + iv = h + \overline{g} \) and \( h - g = k_c \), then we can write
\[
f = \text{Re}\{h + g\} + i\text{Im}\{h - g\} = \text{Re}\{2h(z) - k_c\} + i\text{Im}\{k_c\},
\]
Fig. 7. Slit images of \( f_{2,n}(D) \) for various values of \( n = 3, 4, 5, 6 \).
Fig. 8. \( f_{2,n}(\mathbb{D}) \) lift to the minimal surfaces for various values of \( n = 4, 6, 8, 10 \).

\[
- \sum_{k=1}^{(n-1)/2} 2^{c-1}(1-z) F_1 \left( 1-c; -c, 1; 2-c; \frac{1-z}{2}, \frac{1-z}{1-e^{-i\frac{2k\pi}{n}}} \right) \right] + \text{Im} \left\{ \frac{1}{2c} \left[ \left( \frac{1+z}{1-z} \right)^c - 1 \right] \right\}.
\]

Similarly, if \( n = 2m \ (m \in \mathbb{N}) \), then from (11) one can easily see that

\[
h'_{c,n}(z) = \left( \frac{1+z}{1-z} \right)^c \left[ \frac{1}{4} \left( \frac{1}{1-z} + \frac{1}{1+z} \right) + \frac{1}{2n} \left( \frac{1}{(1-z)^2} + \frac{1}{(1+z)^2} \right) \right.
\]

\[
+ \frac{1}{n} \left( \sum_{k=1}^{(n/2)-1} \frac{1}{(1-e^{i\frac{2k\pi}{n}})(1-ze^{i\frac{2k\pi}{n}})} + \sum_{k=1}^{(n/2)-1} \frac{1}{(1-e^{-i\frac{2k\pi}{n}})(1-ze^{-i\frac{2k\pi}{n}})} \right)
\]
and thus, integrating it from 0 to $z$ gives

$$h_{c,n}(z) = \left( \frac{1+z}{1-z} \right)^c \left[ \frac{1}{4c} + \frac{1+z}{4n(1+c)(1-z)} - \frac{1-z}{4n(1-c)(1+z)} \right]$$

$$+ \sum_{k=1}^{(n/2)-1} 2^c (1-z) e^{i \frac{2k\pi}{n}} F_1 \left( 1-c; -c, 1; 2-c; \frac{1+z}{2}, \frac{1-z}{1-e^{i \frac{2k\pi}{n}}} \right)$$

$$- \sum_{k=1}^{(n/2)-1} 2^c (1+z) e^{i \frac{2k\pi}{n}} F_1 \left( 1-c; -c, 1; 2-c; \frac{1+z}{2}, \frac{1-z}{1-e^{-i \frac{2k\pi}{n}}} \right) - N_2,$$

where

$$N_2 = \frac{1}{4c} + \frac{c}{2n(1-c^2)} + \sum_{k=1}^{(n/2)-1} 2^c e^{i \frac{2k\pi}{n}} F_1 \left( 1-c; -c, 1; 2-c; \frac{1}{2}, \frac{1-e^{i \frac{2k\pi}{n}}}{1-e^{-i \frac{2k\pi}{n}}} \right)$$

$$- \sum_{k=1}^{(n/2)-1} 2^c F_1 \left( 1-c; -c, 1; 2-c; \frac{1}{2}, \frac{-e^{i \frac{2k\pi}{n}}}{1-e^{-i \frac{2k\pi}{n}}} \right).$$

Fig. 9. Images of $f_{c,x}(D)$ for various values of $c \in [0, 2]$. 
Again, using the observation made in the case of odd values of \( n \), the resulting harmonic mappings \( f_{c,n}(z) \) for even values of \( n \) are given by

\[
f_{c,n}(z) = \text{Re}\{2h_{c,n}(z) - k_c(z)\} + i\text{Im}\{k_c(z)\} = u + iv,
\]

where \( u \) and \( v \) in this case take the form

\[
u = \text{Im}\left\{\frac{1}{2c} \left[\left(\frac{1+z}{1-z}\right)^c - 1\right]\right\},
\]

respectively. Note that, by Theorem 2, for even values of \( n \), the harmonic mappings \( f_{c,n}(\mathbb{D}) \) lift to the
minimal surfaces $X_{c,n}(u,v) = (u,v,F(u,v))$, where $u,v$ are as above and

$$F(u,v) = \text{Im} \left\{ \left( \frac{1+z}{1-z} \right)^c \left[ \frac{(1+i^n)(1+z)}{2n(1+c)(1-z)} \right] 
+ \sum_{k=1}^{(n/2)-1} (-1)^k 2^{c-1}(1-z)e^{i\frac{2k\pi}{n}} F_1 \left( \frac{1-c}{1+c}, 1; 2-c; \frac{1-z}{2}, \frac{1-z}{1-e^{i\frac{2k\pi}{n}}} \right) \right\}.$$ 

Remark 5. If we take $n = 2$, then Theorem 7 reduces to Theorem 3 in [3].

Figures 9 and 10 are the graphs of $f_{c,n}(z)$ for various values of $c \in [0,2]$. We see that $f_c(\mathbb{D})$ transforms from strip region to wave plane for various values of $c \in [0,1]$.

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