Lost and found: the radial quantum number of Laguerre-Gauss modes

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We introduce an operator linked with the radial index in the Laguerre-Gauss modes of a two-dimensional harmonic oscillator in cylindrical coordinates. We discuss ladder operators for this variable, and confirm that they obey the commutation relations of the su(1,1) algebra. Using this fact, we examine how basic quantum optical concepts can be recast in terms of radial modes.

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I. INTRODUCTION

An optical vortex is a light field exhibiting a pure screw phase dislocation along the propagation axis; i.e., an azimuthal phase dependence \( \exp(i\varphi) \). The integer \( \ell \) plays the role of a topological charge: the phase changes its value in \( \ell \) cycles of \( 2\pi \) in any closed circuit about the axis, while the amplitude is zero there \([1]\).

One of the most intriguing properties of vortices is that they carry orbital angular momentum (OAM). This was first realized by Allen and coworkers \([2]\) for the leading instance of Laguerre-Gauss (LG) laser modes. Furthermore, they demonstrated that these modes carry an OAM of \( \ell \hbar \) per photon along the propagation direction.

An important feature of optical OAM is that it can be easily manipulated and transferred; this has opened new horizons in various fields, ranging from mechanical micro-manipulation \([3]\) to imaging sciences \([4, 5]\), as well as potential astronomical \([6, 7]\) and communication applications \([8]\). Beyond optical wavelengths, OAM now plays a major role in electron \([9–12]\), X-ray \([13–15]\) and radio frequency engineering \([16, 18]\).

The core observation that individual photons also carry OAM brings the most exciting possibilities for employing this variable in the quantum regime, and a number of uses has already been demonstrated \([19, 24]\).

Despite this intense activity, very little attention has been paid thus far to the radial index \( p \) of the LG modes. Usually, it is stated that for \( p > 0 \), the modes are multi-ruled with \( p + 1 \) radial nodes. Beyond this short mention, no physical meaning is attached to this quantity. Two recent papers, however, have presented challenging and interesting insights into this issue \([23, 24]\). Our purpose here is to present a simple comprehensive analysis of this variable.

The two aforementioned papers considered optical modes, governed by the paraxial wave equation. These modes are ultimately suitable rescaled wave functions of the stationary states of a two-dimensional quantum oscillator, under the Schrödinger equation \([27]\). Since this latter system can properly model other interesting vortices arising in different media (as in plasmas \([28]\), superfluids \([29]\), and Bose-Einstein condensates \([29]\)), the oscillator will serve as our thread, bearing in mind that the results can be immediately translated to the optical case.

II. STATIONARY STATES OF A TWO-DIMENSIONAL OSCILLATOR

A. Cartesian coordinates

To be as self-contained as possible, we briefly review the example of an isotropic two-dimensional quantum harmonic oscillator of mass \( M \) and natural frequency \( \omega \), with coordinates in two orthogonal axes, say \( x \) and \( y \) \([30]\). The Hamiltonian can be compactly written as \( \hat{H} = \hbar \omega (\hat{n} + 1) \), where the total number operator \( \hat{n} \) is

\[
\hat{n} = \hat{n}_x + \hat{n}_y = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y,
\]

(2.1)

and the annihilation and creation operators fulfill the canonical commutation relations \( [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk} \mathbb{1} \), with \( j, k \in \{x, y\} \). Since the spectrum of \( \hat{n} \) is composed of all non-negative integers \( n \), the energies are given by \( E_{n_x, n_y} = \hbar \omega (n_x + n_y + 1) \) and these eigenvalues are \((n_x + n_y + 1)\)-fold degenerate.

Elements of the Fock basis are the common eigenvectors of \( \hat{n}_x \) and \( \hat{n}_y \):

\[
|n_x, n_y\rangle = \frac{1}{\sqrt{n_x!n_y!}} (\hat{a}_x^\dagger)^{n_x} (\hat{a}_y^\dagger)^{n_y} |0, 0\rangle,
\]

(2.2)

where \( |0, 0\rangle \) is the ground state. The stationary states of the oscillator are the product of Hermite-Gauss modes, as the oscillations in each axes are kinematically independent:

\[
\Psi_{n_x, n_y}(x, y) = C_{n_x, n_y} H_{n_x}(\alpha x) H_{n_y}(\alpha y) \exp[-\alpha^2 (x^2 + y^2)/2],
\]

(2.3)

with \( \alpha = \sqrt{M \omega / \hbar} \) and \( C_{n_x, n_y} = \alpha / \sqrt{\pi^{n_x+n_y} n_x! n_y!} \) a normalization constant. To recover the equivalent beam solutions, one needs to take \( \alpha = \sqrt{2} \), since the paraxial wave equation (in adimensional coordinates) coincides with the Schrödinger equation for the oscillator when \( M = 2\hbar \) and \( \omega = 1 \).
For our purposes, the solution at $t = 0$ is enough. The wavefunction at any other time can be obtained in a simple way by using the explicit form of the propagator. For beams, where the role of time is played by the coordinate $z$ along the symmetry axis, this propagation brings about additional interesting points, such as the Gouy phase.

B. Cylindrical coordinates

The axes $x$ and $y$ do not enjoy a privileged role in the problem. Since the energy is invariant under rotations in the $xy$ plane, we could as well have chosen any other rotated reference frame. To take a better advantage of this symmetry, we consider the $z$-component of the angular momentum, $\hat{L}_z = \hbar \hat{\ell}$, with $\hat{\ell} = i(\hat{a}_x \hat{a}_y - \hat{a}_y \hat{a}_x)$, and use the rotated bosonic operators \([31]\)

$$
\hat{a}_\pm = \frac{1}{\sqrt{2}}(\hat{a}_x \mp i \hat{a}_y), \quad \hat{a}_\mp^* = \frac{1}{\sqrt{2}}(\hat{a}_x \pm i \hat{a}_y),
$$

where $[\hat{a}_j, \hat{a}_k^*] = 0$ with $j,k \in \{+, -\}$. We can then check that

$$
\hat{n} = \hat{n}_+ + \hat{n}_-, \quad \hat{\ell} = \hat{n}_+ - \hat{n}_-,
$$

whose interpretation is direct: the system can be envisioned now as consisting of “quanta” with positive (counterclockwise rotation around $z$) and negative (clockwise rotation around $z$) orbital angular momentum.

The Fock basis $\{|n_+, n_-\}$ of the common eigenvectors of $\hat{n}_+$ and $\hat{n}_-$ can be constructed much in the same way as in Eq. (2.2). However, it will prove useful to consider instead the continuous set

$$
|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta \hat{a}_+^\dagger - \eta^* \hat{a}_-^\dagger + \hat{a}_+ \hat{a}_-\right)|0,0\rangle,
$$

parametrized by the complex number $\eta = r \exp(-i \phi)$. The states $|\eta\rangle$ constitute an orthonormal basis, whose properties have been reviewed in depth in Ref. \([32]\). In the representation they generate $[\Psi(\eta) = \langle \eta | \psi \rangle]$, the action of the basic operators is

$$
\hat{a}_+ \psi(\eta) = \left(\frac{\eta}{2} + \frac{\partial}{\partial \eta^*}\right) \psi(\eta),
$$

$$
\hat{a}_- \psi(\eta) = -\left(\frac{\eta^*}{2} + \frac{\partial}{\partial \eta}\right) \psi(\eta),
$$

while for the adjoints we have

$$
\hat{a}_+^\dagger = \hat{a}_- - \eta^*, \quad \hat{a}_-^\dagger = \hat{a}_+ - \eta.
$$

Note that since the exponential acting on the vacuum in Eq. (2.6) is not unitary, the creation and destruction operators are not conjugate one of the other under the usual boson conjugation.

Given the above, $\hat{n}$ and $\hat{\ell}$ act in this space as

$$
\hat{n} \mapsto r^2 - 1 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}\right) - 1,
$$

$$
\hat{\ell} \mapsto -i \frac{\partial}{\partial \phi}.
$$

As $[\hat{\ell}, \hat{n}] = 0$, the basis $\{|n_+, n_-\}$ can be also reinterpreted as common eigenvectors of $\hat{n}$ and $\hat{\ell}$, with eigenvalues $n_+ + n_-$ and $\ell = n_+ - n_-$, respectively. The stationary states in this basis can be readily obtained using Eqs. (2.9); the final result reads

$$
\Psi_{nl}(r, \phi) = C_{nl} e^{-\alpha^2 r^2/2} (\alpha r)^l L_n^{\ell}(\alpha^2 r^2) e^{i \ell \phi},
$$

where $L_n^\ell(x)$ are the generalized Laguerre polynomials and $C_{nl} = \sqrt{\pi} (p + |\ell|)! / \sqrt{\pi} p^{|\ell|} \alpha^{|\ell|}$ a normalization constant, with $p = (n - |\ell|)/2$. The probability distribution $|\Psi_{nl}(r, \phi)|^2$ shows $p$ dark concentric rings.

### III. QUANTUM OPTICS WITH RADIAL MODES

A. The radial number operator

Since the number of dark rings is $p = (n - |\ell|)/2$, the operator

$$
\hat{\rho} = \frac{1}{2}(\hat{n} - |\ell|) = \begin{cases} 
\hat{n}_- & \text{for } \ell > 0, \\
\hat{n}_+ & \text{for } \ell < 0,
\end{cases}
$$

seems to be a sensible definition for the radial-number operator of the Laguerre-Gauss modes. According to Eq. (2.9), in differential form it reads

$$
\hat{\rho} \mapsto -\frac{1}{4} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}\right) + \frac{i}{2} \frac{\partial}{\partial \phi} + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} - \frac{1}{2}.
$$

Incidentally, it coincides with the operator found in Ref. \([25]\) by setting $r \mapsto \alpha r$.

To simplify the following discussion, let us, for the time being, relabel the stationary states $|n, \ell\rangle$ as $|p, \ell\rangle$, where $p$ indicates the radial mode eigenvalue; i.e.,

$$
\hat{\rho} |p, \ell\rangle = p |p, \ell\rangle.
$$

As heralded in the Introduction, we are interested in exploring the Hilbert space associated with the radial number $p$, while keeping the OAM $\ell$ fixed. At first sight, one might look for the canonical conjugate variable to $\hat{\rho}$. Since, according to Eq. (3.1), $\hat{\rho} \mapsto \hat{n}_- \text{ or } \hat{n}_+$ (depending on the sign of $\ell$), such a variable would be a phase $\phi_- \text{ or } \phi_+$. This means that if we denote by $\hat{\phi} = \exp(i\hat{\rho}_\pm)$ the exponential of such a putative phase, the corresponding commutation relation would read \([33]\)

$$
[\hat{\phi}, \hat{\rho}] = \hat{\rho}.
$$
This immediately implies
\[ \hat{\epsilon} = \sum_{p=0}^{\infty} |p\rangle\langle p+1|, \] (3.5)
so that
\[ \hat{\epsilon} |p, \ell\rangle = |p-1, \ell\rangle, \quad \hat{\epsilon}^\dagger |p, \ell\rangle = |p+1, \ell\rangle. \] (3.6)
Whereas the spectrum of \( \hat{\epsilon} \) is unbounded, including all the in-
teger numbers, the spectrum of \( \hat{P} \) is semiunitary, as it com-
prises only non-negative integers. This indicates that the action
of \( \hat{\epsilon} \) as a ladder operator fails at \( p = 0 \), and consequently
it cannot be unitary:
\[ \hat{\epsilon}^\dagger \hat{\epsilon} = \hat{I}, \quad \hat{\epsilon} \hat{\epsilon}^\dagger = \hat{P} \] (3.7)
where \( \hat{P} = |0, \ell\rangle\langle 0, \ell| \) is the projector on the "vacuum".

All these problems thus place this interpretation on shaky
ground. For this reason, we prefer to follow an alternative
route. To this end, we observe that to increase (resp. decrease)
the radial number by one unit, with \( \ell \) unchanged, we need to
create (resp. annihilate) one positive quantum and one negative
quantum, namely
\[ \hat{k}_+ |p, \ell\rangle = \hat{a}_-^\dagger \hat{a}_+^\dagger |p, \ell\rangle \propto |p+1, \ell\rangle, \]
\[ \hat{k}_- |p, \ell\rangle = \hat{a}_- \hat{a}_+ |p, \ell\rangle \propto |p-1, \ell\rangle. \] (3.8)
One can check that
\[ [\hat{k}_+, \hat{k}_-] = -2\hat{k}_z, \quad [\hat{k}_-, \hat{k}_z] = \hat{k}_+, \quad [\hat{k}_+, \hat{k}_z] = -\hat{k}_-, \] (3.9)
with \( \hat{k}_z = (\hat{n} + \hat{\mathbb{I}})/2 \). This means that if we define \( \hat{k}_\pm = \hat{k}_z \pm i\hat{k}_y \), we have
\[ [\hat{k}_+, \hat{k}_-] = -2\hat{k}_z, \quad [\hat{k}_-, \hat{k}_x] = \hat{k}_+, \quad [\hat{k}_+, \hat{k}_x] = \hat{k}_-, \] (3.10)
that is, they are the generators of the \( su(1, 1) \) algebra, as first
noticed in Ref. [25].

\section{Radial coherent states}

To explore the issue in more detail, it is convenient to give
some basic background on some well-known irreducible
representations (irreps) of \( SU(1, 1) \), which are excellently re-
viewed in Ref. [34] and whose role in quantum optics is dif-
cult to underestimate [35, 44].

The Casimir operator for this group is \( \hat{K}^2 = \hat{K}_z^2 - \hat{K}_x^2 - \hat{K}_y^2 \), which can be expressed as \( \hat{K}^2 = k(k-1) \hat{\mathbb{I}} \), where the
Bargmann index \( k \) labels the different irreps (this index plays
the role of spin for rotations). In our case, a simple calculation
shows that \( k = (|\ell| + 1)/2 \), so that \( k = 1/2, 1, 3/2, \ldots \), which
corresponds to the so-called positive discrete series, for which
\( \hat{k}_z \) is diagonal and has a discrete spectrum. In the Fock basis
\( \{|n_+, n_-\} \), the basis states of the irrep \( k \) are \( \{|k, k+p\} \), with
\( p = 0, 1, \ldots \), and hence
\[ \hat{k}_z |k, k+p\rangle = (k+p) |k, k+p\rangle, \] (3.11)
while the ladder operators act as
\[ \hat{k}_+ |k, k+p\rangle = \sqrt{(2k+p)(p+1)} |k, k+p+1\rangle, \]
\[ \hat{k}_- |k, k+p\rangle = \sqrt{r(2k+p-1)} |k, k+p-1\rangle. \] (3.12)
Note that we can make the identification \( |p, \ell\rangle \leftrightarrow |k, k+p\rangle \),
provided \( k = (|\ell| + 1)/2 \).

Since \( \hat{k}_- |k, k\rangle = 0 \), this state can be taken as the "vacuum".
Indeed, \( \hat{D}(\xi) = \exp(\xi \hat{k}_+ - \xi^* \hat{k}_-) \) are truly displacement
operators, so according to Perelomov prescription [45], the set
\[ |\zeta\rangle = \hat{D}(\xi) |k, k\rangle \] (3.13)
constitutes a family of \textit{bona fide} coherent states parame-
trized by the pseudo-Euclidean unit vector \( \mathbf{n} = (\sin \omega \cos \phi, \sin \omega \sin \phi, \cosh \omega) \), with \( \xi = (\omega/2) \exp(i\phi) \)
and \( \zeta = \tanh(\omega/2) \exp(-i\phi) \).

By expanding the exponential and employing the disan-
ting theorem, we get the decomposition
\[ |\zeta\rangle = (1 - |\zeta|^2)^k \sum_{p=0}^{\infty} \frac{\Gamma(2k+p)}{p! \Gamma(2k)} \zeta^p |k, k+p\rangle, \] (3.14)
and projecting over the complete basis \( |\eta\rangle \) we get the corre-
spending wave function \( \Psi_\xi(r, \phi) \) in the transverse parameters.
The expression can be simplified into an exponential form using
the identity
\[ \exp \left( \frac{\gamma r}{\gamma - 1} \right) = (1 - \gamma)^{1+|\ell|} \sum_{p=0}^{\infty} \gamma^p L_p^{2|\ell|}(x), \] (3.15)
so the final result is
\[ \Psi_\xi(r, \phi) = \sqrt{\frac{\alpha^2}{\pi |\ell|!}} \sum_{p=0}^{\infty} \frac{\gamma^p L_p^{2|\ell|}(x)}{(1 - |\zeta|^2)^{1+|\ell|}} e^{\xi r/2} e^{i\phi}. \] (3.16)

We see that the Perelomov coherent states are polynomial-
Gauss modes at \( t = 0 \); a subfamily of Hypergeometric-Gauss
modes upon evolution, as discussed in Ref. [46]. They are also eigenstates of the OAM, and shape invariant in the time
evolution.

The average number of sharp rings in the state \( |\zeta\rangle \) is
\[ \bar{\rho} = \frac{|\zeta|^2}{|\zeta|^2 - 1} (|\ell| + 1), \] (3.17)
and the statistical distribution of rings \( W_\rho = |\langle \rho, \ell | \zeta \rangle|^2 \) is
\[ W_\rho = \frac{(|\ell| + 1)^{|\ell|+1}(p + |\ell|)!}{p! |\ell|!} \frac{\bar{\rho}^p}{(\bar{\rho} + |\ell| + 1)^{p+|\ell|+1}}. \] (3.18)
In Fig. 1 we have plotted this distribution for a coherent state
with \( \ell = 1 \) and different values of \( \bar{\rho} \). In Fig. 2 we have plotted
\(|\Psi_\xi(r, \phi)|^2 \) for a coherent state with \( \ell = 1 \) and \( \bar{\rho} = 1 \), and
the corresponding distribution for the Laguerre-Gauss eigenstate
with \( p = 1 \). The striking differences can be appreciated at a
glance.
This equation can be solved in the state uncertain relation which appears as a reasonable generalization of the standard of coherent states, due to Barut and Girardello \[47\]:

\[
\hat{k} - |\zeta\rangle_{BG} = |\zeta\rangle_{BG} \tag{3.19}
\]

which appears as a reasonable generalization of the standard coherent states as eigenstates of the annihilation operator. This equation can be solved in the \(|p, \ell\rangle\) basis, yielding

\[
|\zeta\rangle_{BG} = \frac{|\zeta|^{\ell/2}}{L(2|\zeta|)} \sum_{p=0}^{\infty} \frac{\zeta^p}{\sqrt{p!(p+|\ell|)!}} |p, \ell\rangle, \tag{3.20}
\]

where \(L(x)\) is the modified Bessel function. Projecting again in the transverse coordinates we get, after some calculations,

\[
\Psi_{\zeta, BG}(r, \varphi) = \sqrt{\frac{\alpha^2}{\pi L(2|\zeta|)}} e^{(\zeta^2 - \alpha^2 r^2)/2} J_\ell(2\zeta r) e^{i\varphi}. \tag{3.21}
\]

This set of coherent states are thus realized as Bessel-Gauss modes. However, these solutions are not shape invariant, which runs against the notion of coherence.

### C. Radial intelligent and squeezed states

The commutation relations \(3.29\) imply that these operators cannot be measured simultaneously, which is reflected by the uncertainty relation

\[
\Delta \hat{\lambda}_x \Delta \hat{\lambda}_y \geq \frac{1}{2} |\langle \hat{\lambda} \rangle|, \tag{3.22}
\]

where \(\Delta \hat{\lambda} = [\langle \hat{\lambda}^2 \rangle - \langle \hat{\lambda} \rangle^2]^{1/2}\) stands for the variance. According to the standard definition, squeezing occurs whenever \(48\)

\[
(\Delta \hat{\lambda}_x)^2 \leq \frac{1}{2} |\langle \hat{\lambda} \rangle| \quad \text{or} \quad (\Delta \hat{\lambda}_y)^2 \leq \frac{1}{2} |\langle \hat{\lambda} \rangle|. \tag{3.23}
\]

Intelligent states are those for which \(3.22\) holds as an equality. The coherent states \(3.10\) and \(3.21\) are intelligent but not squeezed.

Indeed, these intelligent states are solutions of the eigenvalue problem \(49\)

\[
(\hat{k}_s - i\lambda \hat{k}_y)|\Psi_\lambda\rangle = \Lambda |\Psi_\lambda\rangle, \quad \lambda \in \mathbb{R}. \tag{3.24}
\]

Although they have been investigated from various perspectives \(50–52\), we follow here the comprehensive approach of Ref. \(53\), which starts by noting that the coherent state \(\exp(i\tau \hat{k}_y)|k, k\rangle\) is intelligent provided \(\lambda = \cosh \tau\) [with eigenvalue \(\Lambda = -(k + M) \sinh \tau\), and \(M = 0, 1, \ldots\) an integer number]. Then, the most general intelligent state can be written as

\[
|\Psi_{\ell M}(\tau)\rangle = \exp(i \tau \hat{k}_y) |\kappa_M^\ell(\tau)\rangle, \tag{3.25}
\]

where \(\tau\) is the squeezing parameter and the seed state \(|\kappa_M^\ell\rangle\) can be expressed as

\[
|\kappa_M^\ell(\tau)\rangle = \sum_{p=0}^{M} c_p^\ell(\tau) |k, k+p\rangle. \tag{3.26}
\]

The coefficients \(c_p^\ell\) can be obtained as a recursion relation; the final result reads

\[
c_p^\ell = \left( \frac{M}{p} \right) \left( \frac{(-1)^p}{(2k+p-1)^{1/2}} \right) \tanh^p(\tau) c_0^\ell, \tag{3.27}\]

and \(c_0^\ell\) is fixed by the normalization of the state. The infinite family \(3.25\) of states parametrized by \(k\) and \(M\) is actually squeezed.

Next we need to project \(\exp(i \tau \hat{k}_y)|\kappa_M^\ell(\tau)\rangle\) on the basis \(|k, k+p\rangle\). To this end, we recall that the action of \(\exp(i \tau \hat{k}_y)\) on a basis state \(|k, k+p\rangle\) is given in terms of SU(1, 1) Wigner \(d\)-functions \(54\)

\[
\exp(i \tau \hat{k}_y)|k, k+p\rangle = \sum_{p=0}^{\infty} d_{k+k+p}^{k+p}(-1)^p \tanh^p(\tau) c_0^\ell, \tag{3.28}
\]

In this way, we get, expressed in transverse coordinates

\[
\Psi_{\ell M}(r, \varphi, \tau) = \sum_{p' = 0}^{\infty} \sum_{p = 0}^{M} d_{k+k+p}^{k+p'}(\tau) c_p^\ell(\tau) f_{2k+1, p}(r) e^{i\varphi}, \tag{3.29}
\]

\[\text{FIG. 1. (Color online). Probability distribution } W_p \text{ for a coherent state } |\zeta\rangle \text{ with } \ell = 1 \text{ as as function of } p \text{ and the ring average number } \bar{p}.\]

\[\text{FIG. 2. (Color online). Intensity profiles } |\Psi(r, \varphi)|^2 \text{ for a radial coherent state } (3.16), \text{ with } |p\rangle = 1 \text{ (blue) and for an eigenstate } |p, \ell\rangle \text{ with } p = 1 \text{ and } \ell = 1 \text{ (red). In the inset, we show the corresponding density plots.}\]
The effect of increasing \( \ell \), for fixed \( p \) and \( \tau \), is illustrated by plotting the intensity profile \( |\Psi_{\ell M}(r, \varphi, \tau)|^2 \) as a function of \( r \) in Fig. 3: this intensity tends to a Gaussian-like shape. The effect of increasing \( \tau \) for fixed \( \ell \) and \( M \) is illustrated in Fig. 4 and leads to the appearance of rings as we increase \( \tau \).

For large values of \( \ell \) (more concretely, for \( k \) and \( p \) large, but \( p/k \ll 1 \)), one has at hand a compact asymptotic approximation to the \( d \) functions, namely

\[
d_{k+p,k}(\beta) \simeq (-1)^p \frac{1}{[(k+p)^2-k^2]^{1/4}} e^{-k(\beta - p\psi)/2}, \tag{3.31}
\]

whose Gaussian nature is evident.

IV. CONCLUDING REMARKS

In summary, we have provided a handy toolbox to deal with the radial index of Laguerre-Gauss modes and how it can be used to construct a consistent quantum theory of this variable. We stress that this is more than an academic curiosity, since recent experiments in our laboratory\,[56] have confirmed that the radial degree of freedom of single photons can be manipulated individually in a quantum regime. It is our hope that the results presented here will inspire novel quantum protocols and algorithms using such a “forgotten quantum number”.

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