Impact of different extended components of mean field models on transport coefficients of quark matter

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The transport coefficients like shear viscosity, electrical and thermal conductivities are critically analyzed in the framework of Nambu–Jona-Lasinio (NJL) and its different extensions like addition of vector interaction, Polyakov loop extended version (PNJL) and the entangled PNJL (EPNJL) models. We have considered the standard expressions of transport coefficients, obtained in relaxation time approximation of kinetic theory. Using a constant relaxation time, we first observe the changes of thermodynamical phase space factor for different versions of models and then their corresponding temperature dependent relaxation times are estimated. Impact of different components of model on fluidity is critically investigated.

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I. INTRODUCTION

Microscopic calculations of transport coefficients for highly dense quark matter, which may be seen in astrophysical object like compact stars, are an important input in modeling an array of astrophysical phenomena. Refs. [1–4] have gone through these microscopic estimations. Future experimental facilities like FAIR at GSI, Germany [5] and the NICA at JINR, Russia [6] are aimed to probe similar kind of high density zone in their laboratories. Transport coefficients of highly dense matter, produced there, may have influence on different phenomenological quantities like spectra, flow, which can be constructed from experimental data, measured by their detector set up.

On the other hand, a baryon free hot system can also be a matter of interest to know its transport coefficients values. It is believed that our early universe went through this state, just after few micro second from big-bang. RHIC experiments at BNL, USA and LHC experiments at CERN, Switzerland had reached this high temperature and baryon free zone and their experimental data [7–12] indicate that the matter almost behave like a nearly perfect fluid. A very small values of shear viscosity to entropy density ratio \( \frac{\eta}{s} \) corresponds to this nature and this small values of \( \frac{\eta}{s} \) has been searched as input guess values in viscous hydrodynamic model analysis during the matching experimental data of elliptic flow. [13–15] This small value of \( \frac{\eta}{s} \) from experimental side throws a challenge to the theoretical side, where microscopic calculations of \( \frac{\eta}{s} \) for quark matter can be done. Estimated values of \( \frac{\eta}{s} \) from perturbative quantum chromodynamics (pQCD) at leading order [16, 17] are found to be quite larger than its experimental value. However, Ref. [18] has recently found a significant drop of this value in next-to-leading order calculation but at the end of the article, the possibility of non-perturbative components in \( \frac{\eta}{s} \) has not been ruled out. The non-perturbative temperature domain of QCD can be well mimicked by effective QCD model calculations like Nambu–Jona-Lasinio (NJL) model and quark-meson (QM) models. Refs [19–30] have been gone through this type microscopic calculations of shear viscosity via different effective QCD models. Among them, Refs. [19–26, 29, 30] have adopted NJL model, Ref. [31] has gone through its Polyakov extension version. There are many possible additional sources by which NJL model can be modified into different versions. For example, addition of vector interaction, Polyakov loop extension, entangled Polyakov loop extensions can modify the NJL model structure. In present article, we have tried to investigate the impact of the different additional sources of NJL model on \( \frac{\eta}{s} \) calculations as well as for other transport coefficients like electrical and thermal conductivities.

The article is organized as follows. In Sec. II formalism part of different versions of NJL model has been briefly addressed and in Sec. III the expressions of different transport coefficients are derived in kinetic theory framework. Then, Sec. IV has provided the detail numerical discussion, which have explored the impact of different extensions of NJL model on \( \frac{\eta}{s} \) calculations and at last, we have summarized our studies.

II. FORMALISM OF MODEL WITH DIFFERENT EXTENSIONS

In this section we briefly discuss the mean field models that we have employed in our work. First we talk about the NJL model for two flavor case. Then we extend it by introducing the Polyakov loop field known as PNJL model, through which the deconfinement dynamics can

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be mimicked. In PNJL model the correlation between the chiral and deconfinement dynamics is weak. We impose a strong correlation between these two through Polyakov loop dependent coupling constants – this is known as entangled PNJL (EPNJL) model.

**A. NJL**

Let us start with NJL model first. Here we are interested in two light quark flavors and we also include the isoscalar vector interaction which plays crucial role specially for system with finite density. The Lagrangian is [32, 33]:

\[
\mathcal{L}_{\text{NJL}} = \bar{\psi}(i\gamma_\mu \partial^\mu - m_0 + \gamma_0 \mu)\psi + \frac{G_S}{2} (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \tau \psi)^2 - \frac{G_V}{2} (\bar{\psi} \gamma_\mu \psi)^2, \tag{1}
\]

where, \( m_0 = m_0 \times 1 \), with 1 being the identity matrix and \( m_a = m_d = m_0 \); \( \mu \) is the chemical potential; \( \vec{\tau} \) is Pauli matrix; \( G_S \) and \( G_V \) are the four scalar and isoscalar-vector type coupling constants, respectively. The value of \( G_V \) is not fixed through parameter fitting, rather it is used as a free parameter which can take values within the range \( 0 \leq G_V / G_S \leq 1 \). With the inclusion of vector interaction we now have another condensate as quark number density \( n = \langle \bar{\psi} \gamma^0 \psi \rangle \) along with the usual chiral condensate \( \sigma = \langle \bar{\psi} \psi \rangle \). Chiral condensate will build the link between current quark mass \( m_0 \) and constituent quark mass \( M \) via the relation

\[
M = m_0 + 2G_SN_cN_f \int \frac{d^3p}{(2\pi)^3} \frac{M}{E} (1 - f_Q - f_\tilde{Q}), \tag{2}
\]

where

\[
f_Q, \tilde{Q} = \frac{1}{e^{E+\tilde{\mu}}/T + 1}, \tag{3}
\]

and quark number density make the quark chemical potential \( \tilde{\mu} \) shift to an effective chemical potential

\[
\tilde{\mu} = \mu - G_V n. \tag{4}
\]

Since, NJL is not renormalizable, we regularize the diverging vacuum integral by introducing a sharp three momentum cut-off \( \Lambda \). The energy of the quasi-quark (both up and down) of constituent mass \( M \) is given as \( E = \sqrt{p^2 + M^2} \). The chiral condensate \( \sigma \) at finite temperature depends on Fermi-Dirac distribution function, which is the function of effective chemical potential, given in Eq. (4). Hence, \( G_V \) dependence enters to the Gap equation through this thermodynamical phase space. This gap equation [2] is plotted in Fig. 1 for different values of \( G_V \) and we find a mild noticeable enhancement of \( M \) with \( G_V \) in the intermediate temperature range. Decreasing of quark chemical potential with \( G_V \) make thermal part shrink. Therefore, the contribution of [vacuum - thermal]-term in the left hand side of (self-consistent) Eq. (2) is increased, for which we are getting a increasing trend of \( M \) with \( G_V \). We can get back to the usual NJL Lagrangian by switching the vector interaction off.

With all these in hand, we can now write the thermodynamic potential using mean field approximation as

\[
\Omega_{\text{NJL}} = \frac{G_S}{2} \sigma^2 - \frac{G_V}{2} \mu^2 - 2N_fN_c \int_\Lambda \frac{d^3p}{(2\pi)^3} E \\
- 2N_fN_cT \int \frac{d^3p}{(2\pi)^3} \left[ \ln(1 + e^{-(E-\tilde{\mu})/T}) \right] + \ln(1 + e^{-(E+\tilde{\mu})/T}). \tag{5}
\]

The thermodynamic potential depends on both constituent quark mass \( M \) and the effective chemical potential \( \tilde{\mu} \).

**B. PNJL**

So far we have considered only the chiral dynamics, by which quark to hadron phase transition can be realized as restored to broken phases of chiral symmetry. Now we also incorporate the deconfinement dynamics by including Polyakov loop. It will give us another view, where we can see the quark to hadron phase transition as a confinement to deconfinement phase transition. This is formally known as PNJL model [37, 42]. Here along with the \( \sigma \) and \( n \) fields we have two more mean fields – expectation value of Polyakov loop \( \Phi \) and its conjugate \( \bar{\Phi} \). \( \Phi \) works as the order parameter for deconfinement dynamics. For two flavor the PNJL Lagrangian with vector interaction is written as

\[
\mathcal{L}_{\text{PNJL}} = \bar{\psi}(i\gamma_\mu \partial^\mu - m_0 + \gamma_0 \mu)\psi + \frac{G_S}{2} (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \tau \psi)^2 - \frac{G_V}{2} (\bar{\psi} \gamma_\mu \psi)^2 - U(\Phi, [\Phi, T]), \tag{6}
\]

where the covariant derivative \( D^\mu = \partial^\mu - igA^\mu_\alpha \lambda_\alpha / 2 \), \( A^\mu_\alpha = \delta^\mu_\alpha A^\mu_0 \) being the \( SU(3) \) background fields; \( \lambda_\alpha \)’s

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**FIG. 1:** Temperature dependence of constituent quark masses for \( \frac{G_V}{G_S} = 0 \) (solid line), 0.5 (dotted line), 1 (dash line) at \( \mu = 0.1 \) GeV.
are the Gell-Mann matrices. One should note that here only two components of the gauge field, corresponding to \(\lambda_3\) and \(\lambda_8\), will contribute. The effective Polyakov loop gauge potential is parametrized as

\[
\mathcal{U}(\Phi, \bar{\Phi}, T) = \frac{-b_2(T)}{2} \Phi \bar{\Phi} - \frac{b_3(T)}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4(T)}{4} (\Phi \bar{\Phi})^2, \quad (7)
\]

with

\[
b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3. \quad (8)
\]

Values of different coefficients and parameters \(a_0, a_1, a_2, a_3, b_3, b_4, T_0\) and \(\kappa\) are same as those given in \[48, 49\]. We should note an important point here that in the NJL model the color trace gives us a factor of \(N_c\). In the presence of background gauge field the color trace is not straightforward. After some mathematical manipulation the color trace in PNJL model also splits out a factor of \(N_c\) along with a modified thermal distribution function for particle and antiparticle which read as \[43, 44\]

\[
f_Q = \frac{\Phi e^{-\beta(E-\mu)} + 2\Phi e^{-2\beta(E-\mu)} + e^{-3\beta(E-\mu)}}{1 + 3\Phi e^{-\beta(E-\mu)} + 3\Phi e^{-2\beta(E-\mu)} + e^{-3\beta(E-\mu)}},
\]

respectively. We get back the usual NJL results from these distribution functions by putting \(\Phi = \bar{\Phi} = 1\). Thus while calculating different transport coefficients in the ambient of these models one needs to be careful. For NJL model it will be sufficient to replace the usual mass by the effective one. But for PNJL model one also needs to incorporate the modified distribution functions (See Refs. \[28\]). With these modified distribution functions the effective mass in PNJL model reads as

\[
M = m_0 + 2G_SN_cN_f \int \frac{d^3p}{(2\pi)^3} \frac{M}{E} (1 - f_Q - f_{\bar{Q}}). \quad (10)
\]

The corresponding thermodynamic potential is written as

\[
\Omega_{\mathrm{PNJL}} = \mathcal{U}(\Phi, \bar{\Phi}, T) + \frac{G_S}{2} \sigma^2 - \frac{G_V}{2} \eta^2 - 2N_f T \int \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 3 \left( \Phi + \bar{\Phi} e^{-(E-\mu)/T} \right) e^{-(E-\mu)/T} + e^{-3(E-\mu)/T} \right] - 2N_f T \int \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 3 \left( \Phi + \bar{\Phi} e^{-(E+\mu)/T} \right) e^{-(E+\mu)/T} + e^{-3(E+\mu)/T} \right] - \kappa T^4 \ln [J(\Phi, \bar{\Phi})] - 2N_f N_c \int_{\Delta} \frac{d^3p}{(2\pi)^3} E. \quad (11)
\]

The Vandermonde determinant \(J(\Phi, \bar{\Phi})\) is given by \[43\]

\[
J(\Phi, \bar{\Phi}) = \frac{27}{24\pi^2} \left[ 1 - 6\Phi \Phi + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2 \right]. \quad (12)
\]

**C. EPNJL**

It has been confirmed through different lattice QCD simulation that chiral and deconfinement transitions take place at the same temperature \[46\] or nearly the same temperature \[47\]. Now this is not clearly understood whether it is a mere coincidence or there are some correlations between these two apparently distinct phenomena. To understand this coincidence through effective models a conjecture of strong entanglement between the chiral and deconfinement dynamics has been proposed \[43, 49\]. Because of this entanglement of two dynamics it is known as EPNJL model. This is realized by introducing Polyakov loop dependent coupling constants, where the form of the ansatz is so chosen that it is \(Z_3\) symmetric. Thus the Lagrangian in EPNJL model is same as that in \[6\] except the coupling constants \(G_S\) and \(G_V\) are now replaced by \(\tilde{G}_S(\Phi)\) and \(\tilde{G}_V(\Phi)\). They are given by

\[
\tilde{G}_S(\Phi) = G_S [1 - \alpha_1 \Phi \bar{\Phi} - \alpha_2 (\Phi^3 + \bar{\Phi}^3)], \quad (13)
\]

and

\[
\tilde{G}_V(\Phi) = G_V [1 - \alpha_1 \Phi \bar{\Phi} - \alpha_2 (\Phi^3 + \bar{\Phi}^3)]. \quad (14)
\]

If we put \(\alpha_1 = \alpha_2 = 0\) we get back usual PNJL model. The strength of the vector coupling constant is, as mentioned earlier, taken in terms of values of \(G_S\). In the
same way we can get the thermodynamic potential for EPNJL model by introducing Polyakov loop dependent coupling constants in equation 11. Now along with all the parameters in PNJL model we have two new parameters, $\alpha_1$ and $\alpha_2$ which need to be fixed. This is done and discussed in details in [51]. It is found there that the values of $(\alpha_1, \alpha_2) = (0.1, 0.1)$ allow to reproduce the coincidence of two transition temperatures to be within the range provided by lattice QCD for zero chemical potential. Except that the range provided by lattice QCD for zero chemical potential, dynamical quantities in terms of vector interaction will be $G_S$ and $G_V$ will now be replaced by $\tilde{G}_S$ and $\tilde{G}_V$ as given in equations 13 and 14, respectively.

### D. Thermodynamical quantities

Now, we see that thermal distributions, denoted by $f_{Q,\bar{Q}}$, are changing in different versions of the model. For NJL model it is FD distribution function, as given in Eq. (3). Quark chemical potential without and with vector interaction will be $\mu$ and $\tilde{\mu}$ respectively. When we come to PNJL model FD distributions transform to some modified forms, as given in Eq. (5). Aside from this construction difference, distribution of NJL and PNJL models both depend on their own constituent quark masses, which are also different. When we compare the thermal distribution functions of PNJL and EPNJL models, we notice that they have a same form but their numerical strengths become different because of different constituent quark mass $M(T)$.

Now, in general, if we denote $f_{Q,\bar{Q}}$ as thermal distribution functions, then we can present our different thermodynamical quantities in terms of $f_{Q,\bar{Q}}$, owing to the quasiparticle relation of statistical mechanics. Thermodynamical quantities like pressure $P$, the energy density $\epsilon$, and net quark/baryon density $\rho$ can be obtained from the quasiparticle relations [29]

$$P = 2N_fN_c\int \frac{d^3p}{(2\pi)^3} \mathbb{P}^2 \left[ f_Q + f_{\bar{Q}} \right] ,$$

$$\epsilon = 2N_fN_c\int \frac{d^3p}{(2\pi)^3} E \left[ f_Q + f_{\bar{Q}} \right] ,$$

$$\rho = 2N_fN_c\int \frac{d^3p}{(2\pi)^3} \left[ f_Q - f_{\bar{Q}} \right] .$$

The entropy density $s$ and the heat function $h$ are related to the above quantities through the following relations:

$$s = \frac{\epsilon + P - \mu \rho}{T} ,$$

$$h = (\epsilon + P)/\rho .$$

Heat function $h$ is an important quantity, defined by the ratio of enthalpy density $(\epsilon + P)$ to the net quark density $(\rho)$. This quantity becomes divergent (unphysical) at $\mu = 0$, where net quark density vanishes.

### III. TRANSPORT COEFFICIENTS

To calculate different transport coefficients of relativistic fluid, our necessary macroscopic quantities are energy-momentum tensor ($T^{\mu\nu}$), four dimensional current of quark/baryon charge ($N^\mu$) and electric charge ($J^\mu$). If we consider that the fluid is built up by 2-flavor quark and anti-quark, then in microscopic kinetic theory, the macroscopic quantities can be expressed as

$$T^{\mu\nu} = 2N_fN_c\int \frac{d^3p}{(2\pi)^3} p^{\mu} p^{\nu} \left( f_Q + f_{\bar{Q}} \right) ,$$

$$N^\mu = 2N_fN_c\int \frac{d^3p}{(2\pi)^3} p^\mu \left( f_Q - f_{\bar{Q}} \right) ,$$

$$J^\mu = 2N_c\sum_{u,d} \int \frac{d^3p}{(2\pi)^3} p^\mu \left( e_{u} f_Q + e_{\bar{Q}} f_{\bar{Q}} \right) ,$$

where flavor degeneracy $N_f = 2$; color degeneracy $N_c = 3$; summation stand for 2 flavor quark and anti-quark to net quark density vanishes. When we compare the thermal distribution functions of PNJL and EPNJL models, we notice that they have a same form but their numerical strengths become different because of different constituent quark mass $M(T)$.

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where,
\[ K^\mu = (I^\mu + h\Delta^\mu_{\nu} N_\nu) \]
and \[ I^\mu = (u_\mu T_{\nu}^\sigma - hN_{\sigma}) \Delta^\mu_\nu, \]
with \[ \Delta^\mu_{\sigma} = g^\mu_{\sigma} - u^\mu u_{\sigma}. \] (28)

The transport coefficients \( \eta, \kappa \) and \( \sigma \) are basically appeared as proportional constant, linking between thermodynamical forces and currents. Their relations are given below,
\[ \pi^{ij} = \eta U^{ij}_\eta, \]
with \( U^{ij}_\eta = \left( D^i u^j + D^j u^i + \frac{2}{3} \Delta^{ij} \partial_{\nu} u^\nu \right), \] (29)
\[ I^{i} = \kappa U^{i}_\kappa, \]
with \( U^{i}_\kappa = T \Delta^{ij} \left( \frac{\nabla_i T}{T} - \frac{\nabla_j P}{h_n} \right) \), (30)
\[ J^{i}_D = \sigma^{ij} E_j. \] (31)

Here, \( D^i = \partial^i - u^i u^j \partial_j \) and \( E_j \) is electric field. Now, ow-\<ing the microscopic relations, given in Eq. (20) Eq. (21) and Eq. (22) we can get
\[ \pi^{ij} = 2N_F N_c \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j}{E} (\delta f_Q + \delta f_{\bar{Q}}), \] (32)
\[ I^{i} = T^{00} - h N_{i} = 2N_F N_c \int \frac{d^3 p}{(2\pi)^3} \frac{p_i}{E} \left\{ (p\cdot u - h)\delta f_Q + (p\cdot u + h)\delta f_{\bar{Q}} \right\}, \] (33)
\[ J^{i}_D = 2N_c \int \frac{d^3 p}{(2\pi)^3} \frac{p^i}{E} \left( \epsilon_Q \delta f_Q + \epsilon_{\bar{Q}} \delta f_{\bar{Q}} \right). \] (34)

Considering four velocity \( u = (1, 0) \) in local rest frame, we will get \( p\cdot u = E \).

The small deviation of distribution function can be assumed as
\[ \delta f_{Q,\bar{Q}} = \phi(Q,\bar{Q}) \frac{\partial f_{Q,\bar{Q}}^0}{\partial \omega}, \]
\[ = \phi(Q,\bar{Q}) \beta f_{Q,\bar{Q}}^0 (1 - f_{Q,\bar{Q}}^0), \] (35)
where \( \phi(Q,\bar{Q}) \) can be decomposed as
\[ \phi(Q,\bar{Q}) = A^{(Q,\bar{Q})}_{ij} U^{ij}_\eta + B^{(Q,\bar{Q})}_{i} U^{i}_\kappa + C^{(Q,\bar{Q})}_{i} E^{i}. \] (36)

The coefficient factors \( A_{ij}, B_i \) and \( C_i \) for different thermodynamical tensors \( U^{ij}_\eta, U^{i}_\kappa \) and \( E^{i} \), which are associated with corresponding transport coefficients \( \eta, \kappa \) and \( \sigma \) respectively, have to be found with the help of standard relaxation time approximation (RTA), form of relativistic Boltzmann’s equation
\[ p^\mu \partial_\mu f_{Q,\bar{Q}} = \frac{\delta f_{Q,\bar{Q}}}{\tau_{Q,\bar{Q}}}, \] (37)
as,
\[ A^{(Q,\bar{Q})}_{ij} = \tau_{Q,\bar{Q}} \frac{p^i p^j}{E}, \]
\[ B^{(Q,\bar{Q})}_{i} = \tau_{Q,\bar{Q}} \beta \frac{p^i}{E} (E \mp h), \]
and \[ C^{(Q,\bar{Q})}_{i} = \tau_{Q,\bar{Q}} \frac{\epsilon_{Q,\bar{Q}}}{E} p^i. \] (38)

Where \( \tau_{Q,\bar{Q}} \) are relaxation times of quark and anti quark respectively.

Now, using Eqs. (33) in Eq. (36), Eq. (35) and then in thermodynamical Eqs. (32, 33, 34), we can identify the final expressions of transport coefficients -

\[ \eta = \frac{2N_F N_c \beta}{15} \int \frac{d^3 p}{(2\pi)^3} \left( \frac{p^3}{E} \right)^2 \left\{ \tau_{Q} f_Q (1 - f_Q) + \tau_{\bar{Q}} f_{\bar{Q}} (1 - f_{\bar{Q}}) \right\}, \] (39)
\[ \kappa = \frac{2N_F N_c \beta^2}{3} \int \frac{d^3 p}{(2\pi)^3} \left( \frac{p^3}{E} \right)^2 \left\{ \tau_{Q} (E - h)^2 f_Q (1 - f_Q) + \tau_{\bar{Q}} (E + h)^2 f_{\bar{Q}} (1 - f_{\bar{Q}}) \right\}, \] (40)
\[ \sigma = \left( \frac{2N_c \beta}{3} \right) \left( \frac{5e^2}{9} \right) \int \frac{d^3 p}{(2\pi)^3} \left( \frac{p^3}{E} \right)^2 \left\{ \tau_{Q} f_Q (1 - f_Q) + \tau_{\bar{Q}} f_{\bar{Q}} (1 - f_{\bar{Q}}) \right\}. \] (41)

For simplify the notation, we have put \( f_{Q,\bar{Q}}^0 \) in last expressions instead of \( f_{Q,\bar{Q}} \). We will consider \( f_{Q,\bar{Q}} \) as equilibrium distribution function for the last expressions and also all other sections and subsections.

For more elaborate derivation of these RTA expressions, reader can see Refs. [53, 54]. One can get exactly same expressions in Kubo approach (see Refs. [56, 55]).

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IV. RESULTS

In Sec. [I] we have discussed about the formalism of different extension components of NJL models like (a) vector interaction [II], (b) PNJL [III] and (c) EPNJL [IV]. Present article is intended to investigate the comparative role of these different extensions of NJL models on transport coefficients of quark matter. If we notice
the expressions of transport coefficients in Eqs \([9], [11], [40]\), then we can identify two parts, carrying temperature \((T)\) and chemical potential \((\mu)\) dependent information. One is relaxation time of medium constituent and another is the thermodynamical part, influenced by its Fermi-Dirac distribution function as well as \(T, \mu\) dependent mass. To know the impact of latter, let us first keep relaxation time as a free parameter and taking a constant value of it, we will obtain the values of transport coefficients for different extensions of NJL models. These are discussed one by one.

A. Impact of vector interaction

![Shear viscosity \(\eta\) (a), entropy density \(s\) (b) and \(\eta/s\) (c) as a function of \(T\) at different values of \(G_V\) in NJL model.](image1)

Here, we will see the effect of vector interaction on thermodynamical phase-space factor of transport coefficients. For this purpose, we have generated transport coefficients \(\eta, \sigma\) and \(\kappa\) for different strength of vector interaction. Another point is that transition temperature for two cases will be different. The transition temperature \((T_c)\) of NJL model, where we have only chiral dynamics, is 177 MeV for \(\mu = 0\). As we increase \(\mu\) the transition temperature keeps on decreasing. As we introduce the vector interaction the transition temperature gets modified for a given chemical potential – it starts increasing with the strength of \(G_V\), which basically couples to the chemical potential through the relation \(\tilde{\mu} = \mu - G_V n\), \(\tilde{\mu}\) being the effective chemical potential. Now it is evident that as we increase the value of \(G_V\) the value of effective chemical potential decreases, thus the transition temperature decreases.

In Figs. \([3]\) (a)-(c), shear viscosity \(\eta\), entropy density \(s\) and \(\eta/s\) are plotted as a function of \(T\). From graph it is clear that values of all of \(\eta, s\) and \(\eta/s\) decrease with increasing strength of vector interaction but their quantitative changes are different. Their reduction due to vector interaction is more noticeable near the transition temperature. The reason for reduction of transport coefficient with vector interaction can be realized as follows. We have seen already in Sec. \([1]\) and in Fig. \([1]\), the
The constituent mass $M$ is slightly enhanced with $G_V$ near the transition temperature. On the other hand effective chemical potential $\mu$ decreases with $G_V$. These increasing $M$ and decreasing $\mu$ make thermodynamical phase space part of $\eta$ reduce. This is also true for $s$, which also carries its own thermodynamical phase space part, constructed by FD distribution function. Their ratio $\eta/s$ follows the similar decreasing nature, which hints that vector interaction can push the fluid towards a (nearly) perfect fluid nature. However its impact on this action may not be exceed more than 10%.

Similar phenomena is also observed in Fig. 4(a) for electrical conductivity ($\sigma$), but totally different variation can be found for thermal conductivity, as shown in Fig. 4(b). For thermal conductivity, heat function $h$, or more precisely enthalpy density per net baryon/quark density plays an important role. Its temperature dependence is shown in Fig. 4(c), where we see that $h$ increases with $G_V$ at high temperature, which is dominantly appeared in $\kappa$. Now, the reason for increasing $h$ with $G_V$ can be understood as follows. Increasing of $G_V$ make $\mu$ decrease and so, $\rho$ decreases. Hence $h \propto 1/\rho$ increases.

B. Transport coefficients in NJL, PNJL, EPNJL models

In this section we are interested to investigate the effect of the deconfinement dynamics in different transport quantities. The transition temperature ($T_\tau$) of NJL model, where we have only chiral dynamics, is 177 MeV for $\mu = 0$. As we increase $\mu$ the transition temperature keeps on decreasing. On the other hand in PNJL model we have both chiral and deconfinement dynamics. So essentially we have two phase transitions – one for the chiral phase transition and the other for the deconfinement phase transition ($T_\Phi$). In PNJL model, for zero chemical potential $T_\sigma = 233$ MeV, where as $T_\Phi = 228$ MeV (for $\mu = 0$, $\Phi = \Phi$, so we have $T_\Phi = T_\Phi$). As we increase $\mu$ both transition temperatures decrease and also there is now differences between $T_\sigma$ and $T_\Phi$ for nonzero $\mu$, though very small. We take average of the two temperatures ($T_\sigma + T_\Phi$) to denote as deconfinement temperatures for nonzero values of $\mu$. Since the chiral transition temperature is always very close to the deconfinement transition temperature, we use the average of the two ($T_\sigma + T_\Phi$) to denote as the critical temperature in PNJL model.

In Fig. 5(a-c), the temperature dependence of $\eta/T^3$, $s/T^3$ and $\eta/s$ are plotted for NJL (solid red line), PNJL (green dotted line) and EPNJL (blue dashed line) models. The critical temperature for different models are different, which can be seen from different melting pattern of $M(T)$ in Fig. 2. The rate of increment of the quantities $\eta/T^3$, $s/T^3$ and $\eta/s$ face a dominant changes at those transition temperatures. Hence, we may get an indirect mapping of quark-hadron phase transition via these quantities. In the intermediate temperature zone, near to quark hadron transition, the values of $\eta/s$ assign the ranking - NJL > EPNJL > PNJL. So as one goes from NJL to EPNJL to PNJL, fluid behavior of quark matter approaches towards a (nearly) perfect fluid nature.

$\sigma/T$ and $\kappa/T^2$ in Figs. 6(a) and (b) also show similar type of pattern qualitatively. During transition from NJL to EPNJL to PNJL, we notice that $\eta$, $s$ and $\sigma$ face dominant suppression but $\eta/s$ and $\kappa$ face little mild suppression. Reason for $\eta/s$ is because of ratio, which determine a competitive changes between $\eta$ and $s$. On the other hand, $\kappa$ carry its own $T$ dependent statistical weight factor as well as additional thermodynamical quantity - heat function $h$, shown in Fig. 6(c). Their collective impact ultimately provide a reduction of $\kappa$ as one goes from NJL to EPNJL to PNJL.
C. Temperature dependent relaxation time

So far, we have presented our results of transport coefficients for fixed values of \( \tau \) but it can also be a temperature dependent quantity, if one attempts to calculate it microscopically. From experimental side, \( \eta/s \) of quark matter created at RHIC is found to be very close to its lower bound \( \frac{1}{4\pi} \), based on viscous hydrodynamic model analysis of elliptic flow [12]. We may get an rough idea about the values of \( \tau \), for which our estimated \( \eta/s \) will be close to the lower bound. This restriction also give us a temperature dependent \( \tau \) instead of its constant value. For massless spin 1/2 particle, \( \tau = \frac{5}{4\pi T} \) gives us \( \eta/s = \frac{1}{4\pi} \). This is shown as the black line in the Fig. 7. Imposing same restriction of \( \eta/s = \frac{1}{4\pi} \) in NJL, PNJL and EPNJL model calculations, we get required relaxation time \( \tau(T) \), displayed by dotted, dashed and dash-dotted lines in Fig. 7. Let us analyze these curves. We know that (approximately) massless quark can only be expected at very high temperature but as we decrease the temperature, the non-zero quark condensate will form, for which constituent quark mass also grows up. Mapping this fact via gap equation in NJL model, thermodynamical part of \( \eta \) become suppressed in low temperature domain with respect to massless case. This lower value of thermodynamical part can be compensated by little higher values of \( \tau \) for getting same values of \( \eta/s = \frac{1}{4\pi} \) as obtained in massless case. Therefore red dotted line (\( \tau \) of NJL model) is quite larger than black solid line (\( \tau \) of massless case) in low temperature domain. Above the transition temperature, both curves are merged as condensate melts down completely. When we transit to PNJL model, the confinement picture has been taken into consideration (statistically) via modified thermal distribution function, which has lower statistical weight than FD distribution. So, with respect to NJL case, PNJL has lower strength for thermodynamical part of \( \eta \), so it needs little larger values \( \tau \) for getting KSS limits of \( \eta/s \). This is also seen by green dash line in Fig. 7. The EPNJL curve sits in between NJL and PNJL curves as expected from their \( M(T) \) pattern in Fig. 2.

Now we proceed for estimating relaxation time \( \tau \) or
thermal width $\Gamma$ from the simplest relation,

$$\Gamma(T, \mu, \vec{k}_a) = \frac{1}{\tau} = \sigma_{ab} \int \frac{d^3k_b}{(2\pi)^3} v_{ab} f_b , \quad (42)$$

where

$$v_{ab} = \frac{((E_a + E_b)^2 - 4M^2)^{1/2}(E_a + E_b)}{2E_a E_b} \quad (43)$$

is relative velocity with $E_{a,b} = \{\vec{k}_{a,b}^2 + M^2\}^{1/2}$ and $\sigma_{ab} = \pi r^2$ is hard sphere scattering cross section with radius $r$. For numerical purpose, we will take $r = 1$ fm to mimic roughly strong interaction scale. We see that thermal width $\Gamma$ is the function of $T$, $\mu$ and momentum $\vec{k}_a$ of probe particle $a$. Taking thermal average, we get

$$\Gamma_{av}(T, \mu) = \frac{1}{\tau_{av}} = \int \frac{d^3k_a}{(2\pi)^3} \Gamma(T, \mu, \vec{k}_a) f_a / \int \frac{d^3k_a}{(2\pi)^3} f_a , \quad (44)$$

which becomes a momentum independent function.

Fig. 9 shows average relaxation times, obtained from Eq. (44) in NJL (red dotted line), PNJL (green dashed line), EPNJL (blue dash-dotted line) models. The massless relaxation curve (black solid line) of Fig. 7 is also shown in Fig. 8 to see how far the model-estimated curves are from the lower possible $\tau(T)$. We see that they are quite far from massless curve at low temperature domain but quite close in high temperature zone. It hints about the possibility of getting a (nearly) perfect fluid nature at high temperature zone. We also find a mild transition in rate of decreasing $\tau_{av}(T)$, which may be visualized as indirect mapping of phase transition.

Next, using the temperature dependent relaxation time, we have repeated our transport coefficients estimations, where $\eta$ and $\sigma$ become almost similar for NJL, PNJL and EPNJL models with temperature dependent relaxation time. When one goes from NJL to EPNJL to PNJL models, the suppression of thermodynamical part and enhancement of relaxation time are compensating to each other and ultimately, curves of $\eta(T)$ and $\sigma(T)$ are merged. For quantities like $\eta/s$ and $\kappa$, which carry thermodynamical quantities like $s$, $h$, the picture will be different. Being independent of $\tau$, the quantities $s$ and $h$ remain different in NJL, PNJL and EPNJL models, for which corresponding $\eta/s$ and $\kappa$ curves also become different as shown in Figs. 9(a) and (b) respectively. The straight horizontal line in Fig. 8(a) indicates the KSS values of $\eta/s$ and we notice that all curves are quite close to KSS line at high temperature. It is again hinting the possibility of getting (nearly) perfect fluid quark matter at high temperature. When one compare between Fig. 8(e) and Fig. 8(a), some interesting message can be found. Increasing $\eta/s(T)$ for constant $\tau$ is converted to decreasing $\eta/s(T)$ for estimated $\tau(T)$ as latter one is dominantly tuned by decreasing $\tau(T)$. Again for constant $\tau$, NJL to PNJL model push the medium towards lower $\eta/s$ but it appears to be opposite for estimated $\tau(T)$.

V. SUMMARY

In this paper we have mainly focused in calculating the transport coefficients in relaxation time in different scenarios using different mean field models and have a comparison among them. First we briefly sketch the models and discuss their main characteristics. To start with we describe the NJL model where we have only chiral dynamics. We also incorporate the vector interaction which becomes important in the presence of chemical potential. We calculate transport properties like $\eta$, $\sigma$, $\kappa$ and $\eta/s$ and investigate the impact of vector interaction on these properties.

Then to mimic QCD further closely we incorporate the deconfinement dynamics, along with the chiral one, by taking into account the background gauge field through PNJL model. The transport coefficients get affected in presence of such background field. We also incorporate a strong entanglement between the chiral and deconfinement dynamics to enforce the chiral and deconfinement transitions coincide within the range provided by the QCD. This is achieved through EPNJL model. Then we investigate the effects of the entanglement on the aforementioned transport coefficients. Finally we compare the behavior of different transport coefficients found in these three different models.

At the end, we have estimated relaxation time via simple dilute gas relation with hard sphere scattering. Interestingly the temperature variations of relaxation time
in different models make the transport coefficients like $\eta$ and $\sigma$ almost coincide and they behave almost the same in those three models. But for coefficients like $\eta/s$ and $\kappa$ we found differences in their behaviors in different models.

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