The importance of being discrete - life always wins on the surface

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Many systems in chemistry, biology, finance and social sciences present emerging features which are not easy to guess from the elementary interactions of their microscopic individual components.

In the past, the macroscopic behavior of such systems was modeled by assuming that the collective dynamics of microscopic components can be effectively described collectively by equations acting on spatially continuous density distributions. It turns out that quite contrary, taking into account the actual individual/discrete character of the microscopic components of these systems is crucial for explaining their macroscopic behavior.

In fact, we find that in conditions in which the continuum approach would predict the extinction of all the population (respectively the vanishing of the invested capital or of the concentration of a chemical substance, etc), the microscopic granularity insures the emergence of macroscopic localized sub-populations with collective adaptive properties which allow their survival and development. In particular it is found that in 2 dimensions "life" (the localized proliferating phase) always prevails.

In addition to physics, an increasing range of sciences: chemistry, biology, ecology, finance, urban and social planning have passed in the last century to quantitative mathematical methods.

Along with the obvious benefits, it turns out that the traditional differential equations approach has brought some fallacy into their study.

We present here a very simple generic model which contains proliferating (and dying) individuals and we show that in reality it behaves very differently than its representation in terms of continuum density distributions: In conditions in which the continuum equations predict the population extinction, the individuals self-organize in spatio-temporally localized adaptive patches which insure their survival and development.

This phenomenon admits multiple interpretations in various fields:

- if the individuals are interpreted as interacting molecules, the resulting chemical system emerges spatial patches of high density which evolve adaptively in a way similar with the first self-sustaining systems which might have anticipated living cells.

- if the individuals are the carriers of specific genotypes represented in the genetic space, the patches can be identified with species, which rather than becoming extinct, evolve between various genomes (locations in the genetic space) by abandoning regions of low viability in favor of more viable regions. This adaptive speciation behavior emerges in spite of the total randomness we assume for the individuals motions in the genetic space (mutations).

- interpreted as financial traders, the individuals develop a "herding" behavior in spite of the fact that we do not introduce communication or interaction between them. This leads to the flourishing of markets which the continuum analysis would doom to extinction.

All these phenomena have in common the emergence of large, macroscopic structures from apparently uniform background due to the amplification of small, microscopic fluctuations which originate in the individualized character of the elementary components of the system. This mechanism insures in particular that on large enough 2 dimensional surfaces, even if the average growth rate is negative (due to very large death rate), adaptive structures always emerge and flourish.

Imagine an area inhabited by a population of eternal agents A which are spread out uniformly with average density $n_A$ and move around randomly, with diffusion coefficient $D_A$. Imagine now a race of mortals, B, which are also spread over this area, with initial uniform density $n_B(0)$. The B agents die at a constant rate, $\mu$, $(B \xrightarrow{\mu} \emptyset)$ and proliferate (divide) when they meet the "catalysator", A, with rate $\lambda (B + A \xrightarrow{\lambda} B + B + A)$. The B’s are diffusive, hopping at the rate $D_B$. What will happen?

The naive lore based on macroscopic continuity assumptions will predict that A reaches a spatially homogeneous distribution, $n_A(x) = n_A$, while the B time variation $\frac{\partial n_B}{\partial t}$ is represented by the linear differential partial differential equation:

$$\frac{\partial n_B}{\partial t} = D_B \nabla^2 n_B + (\lambda n_A - \mu)n_B$$

The first term represents the uniformization effect of B diffusion while the $\mu B$ term represents the fact that a certain fraction of B’s die per unit time. The crucial term $\lambda n_A n_B$ represents the proliferation of B’s in the presence of the "life giving" A’s. Note that the equation is linear in $n_B$ and that for initial spatially uniform $n_A$ and $n_B$ distributions it has the time exponential solution

$$n_B(t) = n_B(0)e^{(\lambda n_A - \mu)t}.$$  

In particular Eq. (2), predicts that if the macroscopic proliferation rate $\lambda n_A$ is lower than the death rate $\mu$, the B population will uniformly decrease to extinction.

Using microscopic representation techniques, one finds that populations of discrete proliferating agents are much more resilient than one would first guess based on macroscopic or continuum (PDE) treatment (Fig. 1).
FIG. 1. The lower half of the figure shows the exponential growth of the average B population as a function of time, in the actual simulation (solid blue line) compared to the exponential decrease predicted by the continuum approximation (dashed red line). \(< A >\) is the average number of A reactants per site. Other symbols are defined in the text. The snapshots in the upper half show the spatial configuration of A and B reactants. B reactants are seen to be localized in islands (notice that what is plotted is the logarithm of B concentration at the A site, this is justified when the B concentration in the A site is extremely low since there is only one in the whole simulation space, thus \(\lambda n_A - \mu \approx -\mu\)). The dashed red line shows the exponential growth with coefficient \(\epsilon_0 = 2d\lambda A \kappa\) where \(\kappa = \log_e(\lambda/D_B)\), is the slope of the island (this slope is exhibited in the main graph, and can be derived from a simple approximate calculation). \(\epsilon_0\) is \(\lambda - \mu - 2dD_B\) derived similarly.

Let us consider first the simplest situation of a single A agent jumping randomly (with a diffusion coefficient \(D_A\)) between the locations in an infinite d-dimensional space. In between A jumps the B density at the A location grows exponentially as \(n_B(t) \sim n_B(0)e^{(\lambda - \mu - 2dD_B)t}\). Where \(\lambda\), \(\mu\), and \(2dD_B\) stand for the proliferation, death, and the loss due to diffusion, respectively. The estimation is made by neglecting the flow of B’s from neighbouring site to the A site, this is justified when the B concentration in the neighbouring sites is much lower than on the A site. In the same limit, the ratio between the height of the B density at the A location and the height of the B density on a neighboring site is easily estimated: \(\lambda/D_B\). Consequently, each A jump corresponds to a sudden downwards jump by a factor of \(\lambda/D_B\) in the height of the B hill. As there are in average \(2dD_A\) such jumps per unit time, the net effect of proliferation, diffusion and death, gives the B concentration at the A site as a function of time:

\[
n_B(t) = n_B(0)e^{(\lambda - \mu - 2dD_B - 2dD_A \log_e(\lambda/D_B))t}
\]

The approximation is in good agreement with the simulation shown in Fig. 2. The slope of the island, on a log scale, is indeed seen to be \(\log_e(\lambda/D_B)\), the the time dependence of the height of the B island in between A jumps is indeed given approximately by an exponent, \((\lambda - \mu - 2dD_B)t\). Consequently the dashed red line (in the

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1The above analysis turns void if \(D_B = 0\), where the spatial dimensions of the island do not grow at all. We do not consider this singular case in this paper.
inset) which represents Eq. 3 follows closely the actual growth seen in the simulation (blue line). The difference between the theory and simulation is mainly due to cases

where two or more A jumps follow each other rapidly, in this case the island’s shape does not stabilize before another A jump is made, these rather rare events, modify somewhat the actualy result 1.

One may ask what is the situation in the case, when single colonies are unstable (i.e where the exponent in Eq. 3 is negative). One possibility is that in such a situation the continuum approximation is valid and the B concentration decays to zero. Another possibility is that, although single isolated colonies are unstable, global effects such as islands growing, joining and splitting give us back the survival feature. In particular, since large colonies are more stable than small colonies, one may expect the typical size of an “active” colony to grow with time. This behavior is demonstrated in Fig. 3 which show the active clusters in a two dimensional system developing in time. Evidently, the small clusters either decay or merge into larger and larger clusters.

FIG. 3. The spatial distribution of B’s for different times (earlier times to the left), where concentrations greater than 10 are colored red and concentrations below 10 are color coded.

The arguments and computer experiments mentioned above show convincingly that the individuals based life is much more resilient than a hypothetical life density spread continuously across spatial regions.

These conclusions should suffice to induce professionals in biology, finance and social sciences to consider seriously the regime in which their systems are naively non-viable (decay to extinction) when viewed macroscopically but perfectly viable in reality (and when simulated correctly at the microscopic individual level).

In particular, most of the species in nature could be in this regime: negative naive average macroscopic growth rate but actual survival and proliferation. Similarly, markets which might look unappealing when averaging over the various investing possibilities might prove lucrative enough (at least for the lucky investors which hit profit opportunities A) as to maintain them in the competitive range. In fact this line of thought might provide an explanation to the emergence of life from the random chemistry of its component molecules in spite of the formal extreme improbability of the event. Equally it might explain the paradoxes in finance between the efficient market hypothesis (absence of systematic profit opportunities in equilibrium markets) and the actual profits which investors extract daily from the market.

In order to obtain a more rigorous bound on the parameter range in which life overcomes the gloom prognosis of the macroscopic analysis we used the renormalization group (RG) analysis which indicates that on large enough surfaces, life always wins. For higher dimensions, the domination of life still extends to arbitrary low n_A densities, but a minimal finite \lambda value is required.

In RG, the collective behavior of the system is identified by integrating out the small length scale, short time fluctuations, leaving us with an effective theory for the large scale objects. Here, these are the large, stable islands shown in Fig. 3. The new, effective theory is characterized by renormalized coupling constants, i.e., modified numerical values of the effective rates (growth rate, death rate, hopping etc.) on large length scale. The process of decimating small fluctuations is then iterated again and again, giving us flow line which reflect the evolution of the effective values of the coupling constants as one integrates larger and larger scales l.

Fig. 4 shows the flow lines of m (m = \mu - \lambda n_A) and \lambda due to the iteration of the decimation process 2. The flow is given by the equations:

\[
\frac{dm}{dl} = 2m - \frac{\lambda^2 n_A}{2\pi D}
\]

\[
\frac{d\lambda}{dl} = \lambda[2 - d + \frac{\lambda}{2\pi D}]
\]

For d \leq 2 we see that for large length and time scales (that is, after many iterations of the decimation process), \lambda grows without limit while m eventually becomes negative. This implies that on the large scale, the system actually behaves as if \lambda n_A > \mu, and life always wins.

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2 The details of this RG analysis, which involves the presentation of the exact Master equation of the process as a field integral and the \epsilon-expansion around the critical dimension d_c = 2 are out of the scope of this report and will be presented elsewhere.

3 D \equiv D_A + D_B is the effective diffusion constant.
FIG. 4. Lower panel shows flow lines for $d > 2$. Shaded region flows to negative mass ("life"). Upper panel shows flow lines for $d \leq 2$, the whole parameter space flows to negative mass.

In higher dimensions ($d > 2$) Fig. 4a indicates a dynamical phase transition where for part of the parameter space the system flows to negative $m$ (life) and for another part the system flows to positive $m$ (death).

It should be noted that the flow portrayed in Fig. 4 is associated with larger and larger length scales: For a finite system, the flows should be truncated and the size of the system may be crucial: simulations with parameters identical to that of Fig. 3, lead to extinction when carried out on a system size 4 times smaller.

In conclusion, our results suggest that the dimensionality of the system and its size are crucial features for its capability to emerge and sustain life. This may explain the fact that most of the ecological systems are two dimensional. Reinterpreting in the genome space, the present results provide the conceptual link between the atomized structure of the life building blocks and the explosive Darwinian tandem, noise + proliferation.

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3 See, e.g., Mattis D. C. and Glasser, M. L., Rev. Mod. Phys. 70, 979 (1998) and references therein.
4 Winder, N. "Jonah and the Flatworm", Human Ecology Review, to appear.
5 For formally similar effects in the quantum static regime see Anderson, P. W., Phys. Rev. 109 (1958) 1492.