Non-stationary poisson model of continuously functioning queuing system

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Abstract. The study of non-stationary queuing systems is related to their important applications in computing systems. Algorithms for calculating non-stationary queuing systems are quite complex. Nevertheless, the development of modelling of production processes and communications, trade processes and consumer services requires the development of methods for calculating such queuing systems. In this regard, we should mention the currently actively developing state programs "Smart city", "Digitalization of the economy" which require the development and specification of methods for calculating non-stationary queuing systems. Therefore, it is necessary to build non-stationary queuing models in such a way that their calculation would be quite simple and convenient for computing. In this paper, this can be achieved by assuming that the service time is deterministic, the input non-stationary flow of customers is Poisson, and that there are an infinite number of devices that exclude the customers from being in the queue.

1. Introduction
The study of non-stationary queuing systems is related to their important applications in computing systems (see, for example, [1]). Algorithms for calculating non-stationary queuing systems are quite complex (see, for example, [2], [3]). Nevertheless, the development of modelling of production processes and communications, trade processes and consumer services (see, for example, [4] – [6]) requires the development of methods for calculating such queuing systems. In this regard, we should mention the currently actively developing state programs "Smart city", "Digitalization of the economy" (see, for example, [7] – [9]), which require the development and specification of methods for calculating non-stationary queuing systems. Therefore, it is necessary to build non-stationary queuing models in such a way that their calculation would be quite simple and convenient for computing.

In this paper, this can be achieved by assuming that the service time is deterministic, the input non-stationary flow of customers is Poisson, and that there are an infinite number of devices that exclude the customers from being in the queue. Consider the Poisson model of a queuing system in which customers form the following Poisson flow. Such systems can be called continuous operation systems and in them the user receives the required service immediately after his arrival for a fixed time. This method of service is very convenient for the customer, because does not link it to the queuing system schedule. Therefore, such queuing systems fit perfectly, for example, in the "Smart city" program.
Along with the single-phase continuous service systems described above, various continuous conveyor queuing systems are possible. These include continuous transport lines that are included in the product labelling and packaging process; sorting lines that are designed to move goods during sorting in logistics systems; and secondary packaging lines that provide storage, protection, labelling, and transportation to storage locations; production logistics systems, combining equipment and continuous vehicles (conveyors) to participate in the production process, sorting and labelling of industrial products.

In this paper, such systems are presented as continuous service networks with deterministic service times and no queues at the nodes. To analyze such systems, we develop a special mathematical technique based on graph theory along with probabilistic calculations.

2. Mathematical model of the queuing system with continuous operation

A mathematical model of a continuous queuing system can serve as non-stationary Poisson flow of intensity $\lambda(t), t \geq 0$, of moments of users arrival, deterministic time $a$ of service and so stay of the customer in the system, as well as the number of users $n(t)$ in the system at the time of $t \geq 0$.

At the first stage, we assume that the intensity of the Poisson flow $\lambda(t), 0 \leq t \leq T$ is a continuous function of time $t$. However, for convenience of calculations, it should be assumed that for $t < 0$ and for $t > T$, the function $\lambda(t) = 0$. In this case, the number of users $n(t)$ has a Poisson distribution with the parameter $\Lambda(t) = \int_{t-a}^{t} \lambda(\tau) d\tau$. As an example, customers may be considered as visitors to a swimming pool in regime of free swimming mode.

However, there may be a situation when, along with free swimming, groups of users come to the pool at the moments $t_k, k = 1, \ldots, n$ in the swimming mode under the guidance of a coach. Maintenance of each such group is strictly regulated by $t_k$ start and $t_k + a$ end of service. If the number of users in such a group has a Poisson distribution with the parameter $\lambda_k$ and does not depend on the Poisson flow with intensity $\lambda(t)$, and the moments $t_k$ are specified in advance and therefore are deterministic, then at the time $t$ the number of users in the system has a Poisson distribution with the parameter $\Lambda(t)$, satisfying the equality

$$\Lambda(t) = \int_{t-a}^{t} \lambda(\tau) d\tau + \sum_{k=1}^{n} \lambda_k \chi(\leq T_k \leq t \leq T_k + a),$$

where $\chi(A)$ is the indicator function of the event $A$.

Formula (2) can be rewritten using the delta function construction, assuming $\hat{\lambda}(t) = \lambda(t) + \sum_{k=1}^{n} \lambda_k \delta(t - T_k)$, into $\Lambda(t) = \int_{t-a}^{t} \hat{\lambda}(\tau) d\tau$. In this case, the intensity of the input flow $\hat{\lambda}(t)$ is a generalized function (see, for example, [10, Chapter II]).

Now let’s assume that there are $r$ independent Poisson flows with intensities $\lambda_j(t), j = 1, \ldots, r$, and the deterministic service time of the customer of $j$-th flow is $\beta_j, j = 1, \ldots, r$. Then the number of customers in the system at the time of $t$ has a Poisson distribution with the parameter

$$\Lambda(t) = \sum_{j=1}^{r} \int_{t-\beta_j}^{t} \lambda_j(\tau) d\tau.$$

The mathematical model of the continuous service system proposed in this section is based on observations of a really functioning sports complex. The transition in this complex to a continuous service system significantly improved the quality of service, smoothed the load on the system in real time and allowed users to not depend on the changeable transport situation in the city. This model may be used for the "Smart city" program.
3. Mathematical model of the acyclic queuing network functioning continuously

This section discusses various continuous service networks: a multiphase system, a network with a tree structure, and a network with an acyclic structure. Such networks can occur not only when servicing users, but also in conveyor systems for processing various parts.

3.1. Multiphase queuing system of continuous action

Consider \( m \)-phase queuing system with deterministic service times \( a_i \) at \( i\)-th phase, \( i = 1, \ldots, m \). Let the Poisson input flow have a continuous intensity \( \lambda(t) \), \( 0 \leq t \leq T \), and \( \lambda(t) = 0 \) at \( t < 0 \) and at \( t > T \). Then the number of users \( n_i(t) \) at \( i\)-th phase at time \( t \) has a Poisson distribution with the parameter

\[
\Lambda_i(t) = \int_{t-a_i-a_{i-1} \ldots - a_1}^{t} \lambda(\tau) d\tau, \quad 1 < i \leq m, \quad \Lambda_1(t) = \int_{t-a_1}^{t} \lambda(\tau) d\tau.
\]

3.2. Continuous service networks with a tree structure

Suppose that the 0 root of the oriented tree \( D \) receives an input Poisson flow of intensity \( \lambda(t) \). The set of vertices \( K \) of the tree \( D \) consists of disjoint subsets \( K_l, \ l = 0, \ldots, L, \ K_0 = \{0\} \). The customer being served in the node \( k \in K_l \) at deterministic time \( a_k \), with a positive probability, \( p_{k,q} \) moves to one of the nodes of the set \( K_{l+1} \). Denote \( K_{l,k} \subseteq K_{l+1} \) a set of nodes of the set \( K_{l+1} \), where customers can come from the node \( k \in K_l \) with a positive probability, then \( \sum_{q \in K_{l,k}} p_{k,q} = 1 \), and \( K_{l,k} \cap K_{l,k'} = \emptyset, \ k, k' \in K_l, \ k \neq k' \), therefore, to the node in \( K_{l+1} \) customers can only pass from the node \( k \in K_l \).

Let’s denote \( p_k \) the probability of receiving an input flow customer to the vertex \( k \in K_l \) and let’s put \( T_k \) the total time of the request’s stay in the network until it leaves the vertex \( k \). From the definition of a network with a tree structure, it follows that there is a single path from the vertex 0 to the vertex \( k \in K_l \)

\[\gamma_k = \{0, s(1), s(2), \ldots, s(l-1), s(l) = k\}, \ s(1) \in K_1, \ldots, s(l-1) \in K_{l-1}, \ k \in K_l \]

and hence the following equalities are performed

\[p_k = p_{0,s(1)} \cdot p_{s(1),s(2)} \cdot \ldots \cdot p_{s(l-1),k}, \ T_k = a_0 + a_s(1) + a_s(2) + \ldots + a_s(l-1) + a_k.\] (3)

Using Formula (3), we get that at the moment of time \( t \) a random number of customers in the vertex \( k \) has a Poisson distribution with the parameter

\[
\Lambda_k(t) = \int_{t-T_k}^{t} p_k \lambda(\tau) d\tau.
\]

Note that Poisson random flows arriving at the vertices \( k \in K_l \), are independent this means that the Poisson distributed random variables that characterize the number of customers at the vertices \( k \in K_l \) at the moment \( t \geq 0 \) are also independent. Therefore, if \( K_{l}^* \subseteq K_l \), then the number of customers that are in the vertex set \( K_l^* \) at the time \( t \) has a Poisson distribution with the parameter

\[
\Lambda_{K_l^*}(t) = \sum_{k \in K_l^*} \Lambda_k(t).
\] (5)
3.3. Acyclic queuing networks of continuous action

Now suppose that the structure of a continuous queuing network is defined by an acyclic digraph $F$ with an input vertex $0$. We assume that on the set $K$ of vertices of an acyclic digraph $F$ a partial order between vertices $i \succeq j$ is defined if and only if there is a path from vertex $i$ to vertex $j$. Let the vertex $0$ is the only maximal in terms of order $\succeq$ in the set $K$ and so it is possible to hold the path from $0$ to any other vertex. In [11], an algorithm for calculating the maximum (by the number of edges) length $L(i)$ of the path from the vertex $0$ to any other vertex $i$ is constructed. Then the set $K$ can be divided into disjoint subsets $K_0, K_1, \ldots, K_L$, by the rule $K_l = \{i : L(i) = l\}$, $l = 0, 1, \ldots, L$. And any edge of the digraph $F$ goes from the vertex of the set $i$ to the vertex $j$ only if $L(i) < L(j)$. If $L(j) - L(i) = r > 1$, then in the edge $(i, j)$ we can enter dummy. As a result, all edges are from the set $K_{l-1}$ to the set $K_l$, $l = 1, \ldots, L$. And the transition from vertex $i$ in the following fictitious vertex $j$ is implemented with unit probability, and the time of stay of the customer in a fictitious vertex $j$ satisfies the equality $a_j = 0$, and a digraph $F_1$, and like a digraph $F$ is also acyclic.

![Figure 1](image_url1)  
Figure 1. Converting a network $F$ (left) to a network $F_1$ (right).

If the vertex $j$ of the digraph $F_1$ includes several incoming edges, then the vertex can be divided into several vertices $j_1, \ldots, j_s$, so that each of them contains only one edge. Such a transformation of the acyclic digraph $F_1$ into digraph $F_2$ does not change the process of customer servicing in any way, since there are no delays for customers in queues before servicing in the network. Therefore, the service network defined by the digraph $F_2$, is tree-like and Formula (5) can be applied to it. Moreover, at the moment $t > 0$, random numbers of orders in the dummy vertices $j_1, \ldots, j_s$, have Poisson distributions and are independent. Therefore, to define a Poisson distribution parameter that specifies the number of points in the vertex $j \in F_1$ at $t > 0$, we can use Formula (4).

![Figure 2](image_url2)  
Figure 2. Converting acyclic network $F_1$ into a tree-type network $F_2$.

4. Conclusion

The acyclic queuing networks of continuous action may be extended to acyclic queuing networks with the assembly and disassembly of customers and deterministic service times. Such model may be used in a description of conveyor systems.
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