Scattering of Gravitational Waves by the Weak Gravitational Fields of Lens Objects

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Abstract. We consider the scattering of gravitational waves by the weak gravitational fields of lens objects. We obtain the scattered gravitational waveform by treating the gravitational potential of the lens to first order, i.e. using the Born approximation. We find that the effect of scattering on the waveform is roughly given by the Schwarzschild radius of the lens divided by the wavelength of gravitational wave for a compact lens object. If the lenses are smoothly distributed, the effect of scattering is of the order of the convergence field $\kappa$ along the line of sight to the source. In the short wavelength limit, the amplitude is magnified by $1 + \kappa$, which is consistent with the result in weak gravitational lensing.

Key words. Gravitational lensing – Gravitational waves – Scattering

1. Introduction

Ground-based laser interferometric detectors of gravitational waves such as LIGO, VIRGO, TAMA and GEO are currently in operation to search for astrophysical sources such as neutron star binaries, black hole binaries and supernovae (e.g. Cutler & Thorne 2002). The gravitational wave signals from these binaries are extracted from the data using matched filtering with a gravitational waveform template. If the gravitational waves pass near massive compact objects or pass through intervening inhomogeneous mass distribution, the gravitational waveform is changed due to the scattering (or the gravitational lensing) by the gravitational potential of these objects. The gravitational waves do not directly interact with matter (e.g. Thorne 1987), but the gravitational lensing occurs in the same way as it does for electromagnetic waves. In this letter, we investigate the effects of scattering by lens objects on the gravitational waveform.

In the gravitational lensing of light, the scattering is discussed in terms of gravitational lensing under the geometrical optics approximation, which is valid because the wavelength is much smaller than the typical size of lens objects. But in the case of gravitational waves, since the wavelength is much larger than that of light, geometrical optics is not valid in some cases. If the wavelength is larger than the Schwarzschild radius of the lens, wave optics should be used (Peters 1974, Bontz & Haugan 1981, Thorne 1983, Deguchi & Watson 1986). This condition is rewritten as $M \ll 10^3 M_\odot (f/10^3 \text{Hz})^{-1}$, where $10^3 \text{Hz}$ is the typical frequency of gravitational waves for ground-based detectors. Hence, we use wave optics in this letter.

In the geometrical optics for the lensing of light, the strong and weak lensing are distinguished by the convergence field $\kappa$ which is the ratio of surface density of lens $\Sigma$ to a critical density $\Sigma_{\text{cr}} \sim c^2/G D_L$, where $D_L$ is the distance to the lens (Kaiser 1992, Bartelmann & Schneider 2001). In the strong lensing regime, $\kappa > \sim 1$, the multiple images of distant source are formed. But the strong lensing probability is small, $\sim 0.1\%$, for high redshift sources. Hence, the weak lensing approximation, $\kappa \ll 1$, is valid for most sources. We use the weak field approximation in the wave optics.

In the past, Peters (1974) studied the scattering by a point mass lens and a thin sheet of matter in the weak field approximation, and obtained the scattered waveform for these lens models. The gravitationally lensed waveform (which is the solution of wave equation (1)) was given in Schneider, Ehlers & Falco (1992), Sec.4.7 and 7, using the diffraction integral under the thin lens approximation. Recently, several authors have been studying wave optics in the gravitational lensing of gravitational waves using this integral (Nakamura 1998, Nakamura & Deguchi 1999, Ruffa 1999, Takahashi & Nakamura 2003, Yamamoto 2003, Macquart 2004, Takahashi 2004). In this letter, we present another method to derive the solution of Eq. (1) in the weak-field limit. We treat the gravitational field of lens to first order, i.e. using the Born approximation, and discuss its validity. We use units of $c = G = 1$. 

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Note that the above result 4 can be used if the lenses are broadly distributed between the source and the observer since the thin lens approximation is not assumed.

We use the spherical wave emitted by the source as \( \phi^0 \), then we have \( \phi^0(\mathbf{r}) = A e^{i\omega |\mathbf{r}|} / |\mathbf{r} - \mathbf{r}_s| \). In Fig. 1 \( \phi^0 \) represents the incident wave emitted by the source, while \( \phi^0 + \phi^1 \) represents the scattered wave.

2.1. Geometrically thin lens

We assume that the lens objects are locally distributed at the origin. Then we have \( |\mathbf{r}'| \ll D_{L, LS} \) and \( |\mathbf{s}_i| \ll D_{L} \) in Eq. 5. Using the second-order Taylor series 2 for these small quantities \( \mathbf{r}' \) and \( \mathbf{s}_i \), \( \phi^1 \) is reduced to

\[
\frac{\partial \phi^1(\mathbf{r}_s)}{\partial \phi^0(\mathbf{r}_s)} = -\omega^2 \frac{D_S}{\pi D_L D_{LS}} \int d^3r' U(s', z') e^{i\omega d(s', s_i)},
\]

where \( D_S = D_L + D_{LS} \) and we set \( \mathbf{r}' = (s', z') \).

Here, \( t_0 \) is the geometrical time delay which is given by \( \pi \omega^2 (s', s_0) = D_S / (2D_L D_{LS}) \times |s' - (D_{LS} / D_S) s_0| \). Using the two-dimensional gravitational potential \( \psi(s') = \int d\mathbf{z}' U(s', \mathbf{z}') \), the result in Eq. 4 is reduced to

\[
\frac{\partial \phi^1(\mathbf{r}_s)}{\partial \phi^0(\mathbf{r}_s)} = -\omega^2 \frac{D_S}{2 \pi D_L D_{LS}} \int d^2s' \psi(s') e^{i\omega d(s', s_i)}.
\]

The above equation is also derived by expanding \( \psi \) to first order in the diffraction integral (see Schneider et al. [1992]). The surface density of the lens is defined as \( \Sigma(s) = \int d^2p \rho(s, z) \). We now show the lens geometry of the source, the lens and the observer in Fig. 1. The lens is distributed around the origin. The source position is \( \mathbf{r}_S = (0, -D_{LS}) \), while the observer position is \( \mathbf{r}_O = (s_O, D_L) \) where \( s_O \) is a two-dimensional vector with \( |s_O| \ll D_L \) and \( D_{LS} \) are the distances from the lens to the observer and to the source, respectively. \( \phi^0 \) is the incident wave, while \( \phi^0 + \phi^1 \) is the scattered wave.
2.2. Geometrically thick lens

We consider the lenses are broadly distributed between the source and the observer. It is easy to apply the previous result in Sect. 2.1 to this case. $\tilde{\phi}^1$ in Eq. (5) is rewritten as

$$\frac{\tilde{\phi}^1(r_o)}{\tilde{\phi}^0(r_o)} = -\frac{\omega^2}{\pi} \int d^2 s' \int_0^{D_S} \frac{D_S}{(D_S - z')} z' \times U(s', z' - D_{LS}) e^{i\omega(t(s', z', s_o))},$$

with $t_0(s', z', s_o) = D_S / [2D_S(z' - D_S)] \times |s' - (z'/D_S)s_o|^2$. If the lenses have a thickness of $\Delta z$ in the $z$ direction, the correction for this thickness in the scattered wave $\tilde{\phi}$ is of the order of $\Delta z/D_S$ from Eqs. (4) and (10). Thus, if $\Delta z \ll D_S$ the thin lens approximation is valid.

Especially, for smoothly distributed lenses the result (2) is valid but $k$ is replaced by

$$k = 4\pi \int_0^{D_S} dz' \frac{(D_S - z')}{D_S} \rho ((z'/D_S)s_o, z' - D_{LS}),$$

where $\rho$ is the mass density of the lens.

2.3. Two-point correlation function

Recently, Macquart [2004] (hereafter M04) derived the correlation function in the wave amplitude of the two detectors under the thin lens approximation. He suggested that measurement of the correlation function provides the power spectrum of the mass density fluctuation. In this section, we derive it without the thin lens approximation, but within the limit of weak gravitational fields. $^3$ We consider two observers at $r_o$ and $r_o + r_1$ with $|r_1| \ll |r_o|$. The mass fluctuation is usually characterized by the power spectrum defined as $P(k) = \int \langle \rho(r)\rho(r + r') \rangle e^{i\omega k} d^3 r'$. The correlation in the potential $U(r)$ and $U(r')$ is written as

$$\langle U(r)U(r') \rangle = \frac{2}{\pi} \int d^3 k \frac{1}{k^4} P(k) e^{-ik(r-r')}.$$ (12)

We obtain the correlation in the wave amplitudes $\tilde{\phi}^1(r_o)$ and $\tilde{\phi}^1(r_o + r_1)$ from Eqs. (5) and (7) as

$$\langle \tilde{\phi}^1(r_o)\tilde{\phi}^{1*}(r_o + r_1) \rangle = \frac{2\omega^2}{\pi} \int d^3 k \frac{1}{k} P(k) \int d^3 r' \int d^3 r''
\times e^{i\omega(k-r-\frac{r_1}{2})} \tilde{\phi}^0(r') e^{-i\omega(r-r')} \tilde{\phi}^0(r'') e^{-i\omega(r-r')},$$ (13)

It is useful to change the integral variables to $x = r' - r''$ and $y = (r' + r'')/2$, and we assume that $x$ is much smaller than the distances $D_{LS}$. Then, expanding these small quantities $x$ and $r_1$ in the exponentials, Eq. (13) is rewritten as

$$\langle \tilde{\phi}^1(r_o)\tilde{\phi}^{1*}(r_o + r_1) \rangle = \frac{2\omega^4\Lambda^2}{\pi} \int d^3 k \frac{1}{k} P(k) \int d^3 x \int d^3 y
\times \frac{1}{[r_o - x] [r_o + x]}
\times e^{i\omega\langle (r_1 - x, r_1 - y) \rangle} e^{-i\omega x},$$ (14)

$^3$ The correlation function due to electromagnetic scattering was exactly obtained, not only for weak fluctuation but also for strong fluctuation. See references, Ishimaru [1978], Tartatski & Zavorotnyi [1980] for a detailed discussion.
where $\vec{v}$ denotes the unit vector for $v$. $A$ is the amplitude of the incident wave (see sentences after Eq. (14)). Performing the integral in Eq. (14), we obtain

$$
\langle \theta^i (r_0) \theta^j (r_0 + r_1) \rangle = 16\omega^2 \left| \theta^0 (r_0) \right|^2 \int d^2 q \frac{1}{q^2} P(q) \times \int_0^{\Delta S} d\omega e^{-i\omega \cdot r_1},
$$

where $n = (-\omega q/\omega D_S, 1)$ and $q$ is the $(x, y)$ component of the wave vector $k$. In the geometrically thin lens with $z = 0$ plane, $P(q)$ is replaced by $P_{2D}(q)\delta(z - D_{LS})$ where $P_{2D}$ is 2D power spectrum.

Let us consider a single power law model for the power spectrum as an example. This model can be used for the fluctuation of cold dark matter and gas (M04, Sec. 5.2 & 5.3). The power spectrum is $P(k) = P_0 (k/k_0)^{-n}$ for $k_{\text{min}} < k < k_{\text{max}}$, and $P(0) = 0$ otherwise. The index is $n \approx 3 - 4$, and $k_{\text{min}} \ll k_{\text{max}}$. Then, the integral in Eq. (15) is dominated by the fluctuation at the largest scale of $1/k_{\text{min}}$. The exact solution of integral (15) was given by the hypergeometric function, but we present an approximate solution for simplicity. Assuming that the separation of two detectors $|r_1|$ is much smaller than the largest scale of fluctuation $1/k_{\text{min}}$, we have

$$
\langle \theta^i (r_0) \theta^j (r_0 + r_1) \rangle = 16\omega^2 \left| \theta^0 (r_0) \right|^2 P(k_{\text{min}})k_{\text{min}}^{-2} D_S e^{-i\omega r_1} \times \left[ \frac{2\pi}{2 - n} + \frac{\pi}{6n} (k_{\text{min}}s)^2 + O(k_{\text{min}}^4) \right],
$$

where $r_1 = (s_1, z_1)$. The first term represents the dispersion of the scattered wave, while the second term is a two-point correlation function. If the lenses have a finite thickness $\Delta L$ along the $z$-axis, $D_S$ in Eq. (16) should be replaced by $\Delta L$.

### Appendix A: Derivation of Eq. (5)

Inserting $\psi(s') = 4 \int d^2 s' \Sigma(s') |n s' - s''| \psi(s'')$ into Eq. (5), we change the integral variable from $s'$ to $u \equiv s' - s''$. Then, we have

$$
\delta^i (r_0) = -4\omega^2 \frac{D_S}{D_{LS}} \int d^2 s' \Sigma(s') e^{i\omega u(s', s_0)} \times \int_0^\infty du \ln u \psi(u),
$$

where $\alpha = \omega D_S / (2D_{LS})$, $\beta = 2 |\alpha| - (D_{LS}/D_S)s_{\text{sol}}$, and $J_0$ is the 0-th order Bessel function. By using series $J_0(\beta u) = \sum_{n=0}^\infty (-\beta^2 u^2/4)^n (n!)^{-2}$, the integral in Eq. (A.1) is rewritten as,

$$
\int_0^\infty du \ln u \psi(u) = -\frac{1}{4\alpha} (\psi + (n - 2) \alpha),
$$

where $(n - 2) \alpha$ is the first term. In the previous work (M04), they use the Tatarskii, V.I. & Zavorotnyi, V.U. 1980, Progress in Optics XVIII, 207 as well.

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