On the Normalized Spectral Representation of Max-Stable Processes on a Compact Set

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Abstract: The normalized spectral representation of a max-stable process on a compact set is the unique representation where all spectral functions share the same supremum. Among the class of equivalent spectral representations of a process, the normalized spectral representation plays a distinctive role as a solution of two optimization problems in the context of an efficient simulation of max-stable processes. Our approach has the potential of considerably reducing the simulation time of max-stable processes.

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1. Introduction

Max-stable processes have become a popular tool for modeling spatial extremes, particularly in environmental sciences, see, e.g. Coles (1993), Coles and Tawn

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(1996) and Padoan, Ribatet and Sisson (2010). Let $Z = \{Z(y) : y \in K\}$ be a max-stable process with standard Fréchet margins defined on an index set $K$. Then, there exists a spectral measure $H$ defined on an appropriate set of functions $\mathbb{H}$ such that

$$Z(y) = \max_{(t,f) \in \Pi} tf(y), \quad y \in K,$$

(1.1)

where $\Pi$ is the Poisson point process on $(0, \infty) \times \mathbb{H}$ with intensity measure $t^{-2} dt H(df)$ and

$$\int_{\mathbb{H}} f(y) H(df) = 1$$

(1.2)

for all $y \in K$, see de Haan (1984); Giné, Hahn and Vatan (1990); Kabluchko (2009) and Wang and Stoev (2010), for instance. The non-negative shape functions $f$ in $\mathbb{H}$ are the spectral functions that correspond to the max-stable process $Z$.

The ensemble of spectral functions corresponding to a given max-stable process is not unique (cf. de Haan and Ferreira, 2006, Remark 9.6.2) and a choice has to be made in applications. Some specific choices may bear severe disadvantages. For instance, finite approximations based on the original definition of the Brown-Resnick process are far from the actual process, in general (Kabluchko, Schlather and de Haan, 2009). Nonetheless, the optimality of the choice of spectral functions has not been discussed in literature yet. Here, we propose a criterion for choosing spectral functions that is the solution to an optimization problem stemming from unconditional simulation of max-stable processes.

From both a theoretical and a practical point of view, it is important to be able to draw random samples from a max-stable process. While bivariate marginal distributions can be calculated frequently, higher dimensional marginal distributions do not have, in nearly all the cases, explicit formulae. Consequently, they can be addressed only by simulation. Furthermore, most applications require the estimation of characteristics of max-stable processes that cannot be explicitly calculated. That leaves simulation as the only option, see, e.g., Buishand, de Haan and Zhou (2008) and Blanchet and Davison (2011). Finally, unconditional simulation appears as part of the conditional simulation of max-stable processes (Dombry, Éyi-Minko and Ribatet, 2013; Oesting and Schlather, 2012).

According to the spectral representation (1.1), the construction of a max-stable process involves infinitely many points $(t, f) \in \Pi$. Nevertheless, since only the maximum over all functions $tf$ counts, the number of points $(t, f)$ that contribute to $Z$, i.e. $Z(y) = tf(y)$ for at least one point $y \in K$, is finite under mild conditions, see de Haan and Ferreira (2006), Cor. 9.4.4. However, their statement is a theoretical one that does not help for simulation purposes because one cannot determine ex ante which function $f$ will contribute. Assuming that $H$ is finite, Schlather (2002) suggests to start with those points $(t, f)$ that will contribute most likely to $Z$, i.e., with those that have the highest values of $t$. By
ranking the points $t$ in a descending order $t_1 > t_2, \ldots$ and assuming without loss of generality that $H$ is a probability measure, we have that $t_i = \frac{1}{\sum_{j=1}^{i} E_j}$, where $E_j$ are independent and identically distributed random variables with standard exponential distribution. Let $f_i \sim i.i.d. H$ be independent of the $E_j$ and

$$Z^{(m)}(y) = \max_{1 \leq i \leq m} \frac{1}{\sum_{j=1}^{i} E_j} f_i(y), \quad y \in K,$$  \hspace{1cm} (1.3)

a finite approximation for $Z$. Then, $Z = d Z^{(\infty)}$, i.e.

$$Z(y) = d \max_{i \geq 1} \frac{1}{\sum_{j=1}^{i} E_j} f_i(y), \quad y \in K.$$  

Therefore, if for a given $m$ we have that

$$Z^{(m)}(y) \geq \frac{1}{\sum_{j=1}^{m} E_j} \sup_{f \in H} f(y) \quad \text{for all } y \in K,$$  \hspace{1cm} (1.4)

then, obviously, $Z^{(n)}(y) = Z^{(\infty)}(y)$ for all $y \in K$ and all $n \geq m$. In other words, any spectral function $f_i$ with $i > m$ cannot contribute to $Z$. This results in a stopping rule for a “$m$-step representation” of $Z$, where $m$ is a random integer. Such a stopping rule can be applied to construct an exact simulation algorithm. In the case of Brown-Resnick processes, Oesting, Kabluchko and Schlather (2012) compare this algorithm to algorithms based on other representations. The results of Schlather (2002) imply that $m$ is finite almost surely if, for instance, $K$ is finite, the shape functions are uniformly bounded and their support is included in a fixed compact set. As a side result, we shall show that even the expectation of $m$ is finite, under rather mild conditions.

We present a toy example to clarify why the choice of spectral functions can have a major impact on the distribution of the stochastic number $m$. Consider the simplest case where $Z$ is univariate. Specializing (1.1) to $K = \{y_0\}$ and $f \equiv 1$, the random variable $Z(y_0)$ follows a univariate Fréchet distribution. It has a representation given by

$$Z(y_0) = d \max_{t \in \Pi} t$$  \hspace{1cm} (1.5)

where $\Pi$ is the Poisson point process on $(0, \infty)$ with intensity $t^{-2} \, dt$. Obviously, the right-hand side of (1.5) is fully given by the largest value of $t \in \Pi$. In other words, $m \equiv 1$. Now, let us consider the general case: $Z(y_0)$ is given by (1.1) and $f(y_0)$ is a non-degenerate random variable with expectation $\mathbb{E} f(y_0) = 1$. Then, the stochastic number $m$ is greater than 1 with positive probability. Even worse, if the right endpoint of $f(y_0)$ is infinite then $m = \infty$ almost surely. In practical applications, in particular for simulating $Z(y_0)$, the spectral representation in (1.5) would be considered as optimal. This example illustrates the optimality we intend to achieve by the choice of spectral functions for an arbitrary max-stable process.
The very general optimality problem for general index sets $K$ and arbitrary random functions $f$ seems to be rather complicated. Therefore, we shall suggest a modified optimization problem and shall demonstrate that its solution is explicit and unique for each given max-stable process and index set $K$. It can be achieved via rescaling any ensemble of spectral functions to a new ensemble of spectral functions satisfying $\sup_{y \in K} f(y) = c$, for all $f \in \mathbb{H}$. We call such a representation with all spectral functions sharing the same supremum the normalized spectral representation. This representation was initially used in constructing the spectral representation for sample-continuous max-stable processes on $K = [0,1]$, see e.g. de Haan and Lin (2001) and de Haan and Ferreira (2006), Cor. 9.4.5. Hence, in this paper, we give a theoretical justification on the optimality of the normalized spectral representation.

This paper is organized as follows. In Section 2, we revisit de Haan’s (1984) spectral representation of max-stable processes and give a formula how to transform one ensemble of spectral functions under a given spectral measure to another ensemble under a different spectral measure. We focus on a particular transformation leading to the normalized spectral representation. We state necessary and sufficient conditions on the existence and show the uniqueness of this representation. In Section 3 we define the optimization problem and give the explicit solution of the replacement problem, the normalized spectral representation. The replacement problem is evaluated and refined in Section 4. Section 5 deals with examples of the normalized spectral representation for specific cases of the max-stable process as well as the index set $K$. In Section 6, for Smith’s (1990) process, the number $m$ of considered spectral functions in the normalized spectral representation is compared to the corresponding number in the algorithm proposed by Schlather (2002) in a simulation study. The paper closes with a summary and discussion of our results.

2. The normalized spectral representation

Throughout the paper we assume that the index set $K$ is a compact Polish space. The following proposition shows how to transform one spectral representation to another one.

Proposition 2.1. Let $Z$ be a max-stable process with standard Fréchet margins defined as in (1.1) and (1.2) where the spectral functions $f$ are in some Polish space $\mathbb{H} \subset [0, \infty)^K$.

Suppose $H$ is a locally finite measure on $\mathbb{H}$. Let $g$ be some probability density on $\mathbb{H}$ w.r.t. $H$, i.e. $g \geq 0$ and $\int_{\mathbb{H}} g(f) H(df) = 1$, such that

$$H \left( \left\{ f : g(f) = 0, \sup_{y \in K} f(y) > 0 \right\} \right) = 0. \quad (2.1)$$

Then,

$$Z(y) = \max_{(t,f) \in \mathbb{H}} \frac{t f(y)}{g(f)}, \quad y \in K, \quad (2.2)$$
where \( \tilde{\Pi} \) is a Poisson point process with intensity \( t^{-2} dt g(f) H(df) \).

**Proof.** For any finite subset \( \{ y_i : i \in I \} \subseteq K \) and \( z_i > 0, i \in I \), we have
\[
\mathbb{P}(Z(y_i) \leq z_i, i \in I) = \mathbb{P}(|\Pi \cap \{(t,f) : tf(y_i) > z_i \text{ for some } i \in I\}| = 0)
\]
\[
= \exp \left( -\int_H \int_\infty^{-}\min_{i \in I} \left\{ z_i/f(y_i) \right\} t^{-2} dt H(df) \right)
\]
\[
= \exp \left( -\int_H \int_\infty^{-}\min_{i \in I} \left\{ z_i/\min_{y \in K} f(y) \right\} t^{-2} dt g(f) H(df) \right)
\]
\[
= \mathbb{P} \left( \max_{(t,f) \in \tilde{\Pi}} tf(y_i) \leq z_i, i \in I \right).
\]
\[
\square
\]

Applying Proposition 2.1, one can transform the given set of spectral functions \( \{ f \}_{(t,f) \in \Pi} \) to a new set \( \{ g(f) \}_{(t,f) \in \tilde{\Pi}} \), where \( f \) follows the transformed probability measure \( gH \) defined by
\[
gH(A) = \int_A g(f) H(df)
\]
for all measurable sets \( A \subseteq \mathbb{H} \). We will focus on a particular choice of \( g \) which leads to the normalized spectral representation as follows. Let \( f \mapsto \sup_{y \in K} f(y) \) be measurable on \( \mathbb{H} \) and assume that
\[
c := \int_\mathbb{R} \sup_{y \in K} f(y) H(df) < \infty.
\]

Then, the choice \( g = g^* \) defined by
\[
g^*(f) := c^{-1} \sup_{y \in K} f(y), \quad f \in \mathbb{H}, \quad (2.3)
\]
satisfies the assumptions of Proposition 2.1. Therefore, \( Z = d \tilde{Z} \) for
\[
\tilde{Z}(y) = \max_{t \in \Pi_0} \frac{cF_t(y)}{\sup_{y \in K} F_t(y)}, \quad y \in K, \quad (2.4)
\]
where \( \Pi_0 \) is a Poisson point process on \( (0, \infty) \) with intensity \( t^{-2} dt \) and \( F_t, t > 0 \), are independent random processes with density \( c^{-1} \sup_{y \in K} f(y) H(df) \). The modified spectral functions \( \{ cF_t/\sup_{y \in K} F_t(y) \} \) can be perceived as independent copies of a stochastic process \( F^* \) with
\[
\sup_{y \in K} F^*(y) \equiv c. \quad (2.5)
\]

**Definition 2.2.** Let \( Z \) be a max-stable process on \( K \) satisfying
\[
Z = d \max_{t \in \Pi_0} tF^*_t. \quad (2.6)
\]

Here, \( \Pi_0 \) is a Poisson point process on \( (0, \infty) \) with intensity \( t^{-2} dt \) and \( F^*_t, t > 0 \), are independent copies of a stochastic process \( F^* \) satisfying (2.5) for some \( c \in (0, \infty) \). Then, the right-hand side of (2.6) is called normalized spectral representation of \( Z \).
The choice $g = g^*$ in Proposition 2.1 leads to a valid normalized spectral representation only if $c = \int_{\mathbb{H}} \sup_{y \in K} f(y) H(df) < \infty$. Note that, in general, $c$ is not necessarily finite even though we assume $\int_{\mathbb{H}} f(y) H(df) = 1$ for all $y \in K$. However, $c$ is finite whenever $K$ consists of a finite number of points. The following proposition deals with equivalent conditions for $c < \infty$ in a more general setting, replacing $f \mapsto \sup_{y \in K} f(y)$ by an arbitrary max-linear functional.

**Proposition 2.3.** Assume that we are in the framework of Proposition 2.1. Furthermore, assume that the function $L : \mathbb{H} \rightarrow (0, \infty)$ is max-linear and measurable. Then the following conditions are equivalent:

1. $c_L := \int_{\mathbb{H}} L(f) H(df) < \infty$
2. $P(L(Z) \leq a) > 0$ for some $a > 0$
3. $P(L(Z) < \infty) = 1$ (or, equivalently, $P(L(Z) < \infty) > 0$).

If we additionally assume that there is some stochastic process $W$ such that

$$Z = \max_{t \in \Pi_0} tW_t,$$

where $\Pi_0$ is a Poisson point process on $(0, \infty)$ with intensity $t^{-2} dt$ and $W_t$, $t > 0$, are independent copies of $W$, we get another equivalent condition:

4. $E L(W) < \infty$.

**Proof.** The assertion follows from the following continued equality:

$$\exp \left( -\frac{c_L}{a} \right) = \exp \left( -\int_{\mathbb{H}} \int_{a/L(f)}^\infty t^{-2} dt H(df) \right) = P(L(Z) \leq a) = \exp \left( -E_W \left( \int_{a/L(W)}^\infty u^{-2} du \right) \right) = \exp \left( -a^{-1}E L(W) \right).$$

for any $a > 0$. The equivalence to the third assertion follows from the relation $P(L(Z) < \infty) = \lim_{a \to \infty} P(L(Z) \leq a)$.

**Remark 2.4.** Similar results, presenting equivalent statements for some special choices of $L$, can already be found in the literature; see, for instance, Resnick and Roy (1991), who showed the equivalence of the first and third assertion for $L(f) = \sup_{y \in K} f(y)$. For this choice of $L$, it follows that $c$ is finite if $Z$ has continuous sample paths.

While Proposition 2.3 is related to the question of the existence of a normalized spectral representation, the following proposition deals with its uniqueness.

**Proposition 2.5.** Let $Z$ be a max-stable process with a normalized spectral representation. Furthermore, let $Z^K := \sup_{y \in K} Z(y)$.

Then, we have

1. $c = -\log P(Z^K \leq 1)$
2. For any \( y_1, \ldots, y_n \in K, w_1, \ldots, w_n > 0 \), it holds
\[
\mathbb{P}(F^*(t_i) \leq w_i, 1 \leq i \leq n) = \lim_{z \to \infty} \mathbb{P} \left( \frac{Z(y_i)}{Z^K} \leq \frac{w_i}{c}, 1 \leq i \leq n \left| Z^K > z \right. \right).
\tag{2.8}
\]

Proof. The first part is a consequence of the proof of Proposition 2.3. For the proof of the second part, let \( \tilde{\Pi} = \{(t, F^*_t) : t \in \Pi_0 \} \). Then, we have
\[
\mathbb{P} \left( \tilde{\Pi} \cap \left\{ (u, w) : u > \frac{z}{c} \right\} > 0,
\right.
\]
\[
\mathbb{P} \left( \tilde{\Pi} \cap \left\{ (u, w) : u > \frac{z}{c}, 1 > \min_{1 \leq i \leq n} \frac{w_i}{w(y_i)} \right\} \right) = 0,
\]
\[
\mathbb{P} \left( \tilde{\Pi} \cap \left\{ (u, w) : u \leq \frac{z}{c}, z > \min_{1 \leq i \leq n} c w(y_i) \right\} \right) = 0
\]
\[
\leq \mathbb{P} \left( \frac{Z(y_i)}{Z^K} \leq \frac{w_i}{c}, 1 \leq i \leq n, Z^K > z \right)
\]
\[
\leq \mathbb{P} \left( \tilde{\Pi} \cap \left\{ (u, w) : u > \frac{z}{c}, \max_{1 \leq i \leq n} \frac{w(y_i)}{w_i} \leq 1 \right\} \right) > 0 \right). \tag{2.9}
\]

The lower bound in (2.9) equals
\[
\left( 1 - \exp \left( -\frac{c}{z} \mathbb{P}(F^*(y_i) \leq w_i, 1 \leq i \leq n) \right) \right)
\cdot \exp \left( -\frac{c}{z} \left( \max_{1 \leq i \leq n} \frac{F^*(y_i)}{w_i} > 1 \right) \right)
\cdot \exp \left( -\frac{c}{z} \left( \max_{1 \leq i \leq n} \frac{F^*(y_i)}{w_i} > 1 \right) \right)
\cdot \exp \left( -\frac{c}{z} \left( \max_{1 \leq i \leq n} \frac{F^*(y_i)}{w_i} - 1 \right) \right),
\]
\[
\text{while the upper bound equals}
\]
\[
1 - \exp \left( -\frac{c}{z} \mathbb{P}(F^*(y_i) \leq w_i, 1 \leq i \leq n) \right).
\]

Using \( \mathbb{P}(Z^K > z) = 1 - e^{-\frac{z}{c}} \) and taking the limit \( z \to \infty \), inequation (2.9) yields (2.8).

Remark 2.6. Proposition 2.5 shows that the law of \( F^* \) is uniquely determined by the cylinder sets
\[
\{ f \in [0, \infty)^K : f(t_i) \in B_i, 1 \leq i \leq m \},
\]
with \( t_1, \ldots, t_m \in K, B_1, \ldots, B_m \in B(\cap [0, \infty), \text{ and } m \in \mathbb{N} \). If we restrict ourselves to continuous functions, the corresponding trace \( \sigma \)-algebra is the Borel \( \sigma \)-algebra. Thus, the normalized spectral representation is unique for sample-continuous processes. Uniqueness also holds true for some more general classes of processes, e.g. càdlàg processes on \( \mathbb{R} \).
The following statement on the existence and uniqueness of the normalized spectral representation follows directly from Propositions 2.3 and 2.5.

**Corollary 2.7.** Let $Z$ be a max-stable process as defined in (1.1) such that $f \mapsto \sup_{y \in K} f(y)$ is measurable. Then, $Z$ allows for a normalized spectral representation if and only if $\sup_{y \in K} Z(y) < \infty$ a.s. In this case, the spectral process $F^*$ is unique in the sense of finite-dimensional distributions.

**Remark 2.8.** Note that the measurability of $f \mapsto \sup_{y \in K} f(y)$ is ensured if $K$ is countable or if every spectral function $f \in H$ is upper semi-continuous and its subgraph $U(f) = \{(y, z) \in K \times [0, \infty) : f(y) \leq z\}$ satisfies $U(f)^c = U(f)$, i.e. $U(f)$ equals the closure of its interior.

The normalized spectral representation is particularly useful in the context of “$m$-step representations” as in (1.4). Recall that, in the representation $Z = \max_{(t, f) \in \Pi} tf$, usually only few points in the Poisson point process $\Pi$ contribute to $Z$ as pointed out in the introduction. And the points $(t, f) \in \Pi$ satisfying $t < Z(y)/\sup_{f \in H} f(y)$ are not able to contribute to $Z(y)$, $y \in K$. This statement also holds for the transformed set of spectral functions $\{f/g(f)\}$ as constructed in Proposition 2.1. Here, points $(t, f) \in \tilde{\Pi}$ are not able to contribute if $t < Z(y)/\sup_{f \in H} (g(f)^{-1}f(y))$. (2.10)

In case of the normalized spectral representation, this implies that points $(t, f) \in \tilde{\Pi}$ cannot contribute if $t < Z(y)/c$. In particular, the number of points which are able to contribute to $Z$ is finite a.s. provided that $\inf_{y \in K} Z(y) > 0$ with probability one. If the normalized spectral representation does not exist, then, by Corollary 2.7, $\mathbb{P}(\sup_{y \in K} Z(y) = \infty) > 0$ (or, equivalently, $c = \infty$) which makes the existence of an “$m$-step representation” doubtful. Later, we will show that the expected number of points which are able to contribute to $Z$ is infinite for any spectral representation of type (2.2) in this case (see Remark 3.7). In general, however, this number depends on the choice of spectral functions as (2.10) suggests. This observation gives rise to the question of an optimal choice of spectral functions such that the number of points that are able to contribute to $Z$ is minimized. We formulate this problem in the next section and show that the normalized spectral representation solves a modified version of the optimization problem.

### 3. The optimization problem

In the following, we will always assume that we are in the framework of Proposition 2.1. Further, the process $Z$ is assumed to be almost surely strictly positive on $K$, i.e.

$$\mathbb{P}\left(\inf_{y \in K} Z(y) > 0\right) = 1.$$ (3.1)
We are interested in minimizing the number of considered spectral functions. For the transformed spectral representation \((2.2)\) with spectral functions \(\{f/g(f)\}_{(t,f) \in \Omega}\), the stopping rule \((1.4)\) can be formulated as follows. Denote

\[
Z^{(m)}(y) = \max_{1 \leq i \leq m} \frac{1}{\sum_{j=1}^{\infty} E_j} \cdot f_i(y) g(f_i), \quad y \in K, \tag{3.2}
\]

for standard exponentially distributed random variables \(E_j\) and \(f_j \sim gH\), which are all independent. Let

\[
Z^{(\infty)} = \lim_{m \to \infty} Z^{(m)}. \tag{3.3}
\]

Then, \(Z^{(\infty)} = \delta Z\) and, for fixed \(\omega \in \Omega\), we have \(Z^{(m)} = Z^{(\infty)}\) on \(K\) if

\[
\text{esssup} \sup_{f \in \mathbb{H}} \frac{f(y)}{g(f)} Z^{(m)}(y) \leq m+1 \sum_{j=1}^{\infty} E_j, \tag{3.4}
\]

where the essential supremum is taken w.r.t. the probability measure \(gH\). Thus, for fixed \(g\), we may exclude all the functions \(f \in \mathbb{H}\) with \(g(f) = 0\). By \((2.1)\), up to a set of \(H\)-measure zero, the set \(\{f \in \mathbb{H} : g(f) = 0\}\) consists of functions \(f \in \mathbb{H}\) with \(f \mid_K \equiv 0\).

For a choice of spectral functions that minimizes the number \(m\) such that \((3.4)\) holds, we need to determine at least one member of

\[Q = \arg \min_g Q_g\]

where

\[
Q_g = \mathbb{E} \min \left\{ m \in \mathbb{N} : \text{esssup} \sup_{f \in \mathbb{H}} \frac{f(y)}{g(f)} Z^{(m)}(y) \leq m+1 \sum_{j=1}^{\infty} E_j \right\}. \tag{3.5}
\]

**Remark 3.1.** By \((3.1)\), we assume that \(\inf_{y \in K} Z(y) > 0\) almost surely. If we additionally assume that \(\text{esssup}_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)} < \infty\), this guarantees that, with probability one, the stopping rule \((3.4)\) holds for some finite \(m\).

Clearly, \((3.1)\) is satisfied if \(Z\) is sample-continuous. Note that a much weaker assumption than sample-continuity already implies a much stronger statement than \((3.1)\), namely

\[
\mathbb{E} \left[ \left( \inf_{y \in K} Z(y) \right)^{-1} \right] < \infty. \tag{3.6}
\]

For \((3.6)\) to hold, it suffices that, for every \(y \in K\), there exist an open set \(U(y)\) containing \(y\), a set of spectral functions \(M(y) \subset \mathbb{H}\) with \(H(M(y)) > 0\) and a real number \(a(y) > 0\) such that \(f(x) > a(y)\) for all \(f \in M(y)\) and \(x \in U(y)\).

To see this, we first note that a finite set \(Y = \{y_1, \ldots, y_n\} \subset K\) exists such that \(\bigcup_{i=1}^{n} U(y_i) \supset K\) as \(K\) is compact. Without loss of generality, we may assume that the corresponding sets \(M(y_1), \ldots, M(y_n)\) are pairwise disjoint. Otherwise, i.e. if \(H(M(y_i) \cap M(y_j)) > 0\) for some \(j \neq i\), the indices \(i\) and \(j\) can
be merged by considering $U(y_i) \cup U(y_j)$ instead of $U(y_i)$, $M(y_i) \cap M(y_j)$ instead of $M(y_i)$, $\min\{a(y_i), a(y_j)\}$ instead of $a(y_i)$ and $Y \setminus \{y_j\}$ instead of $Y$. Now, for any $z > 0$, we have that

$$\mathbb{P}\left(\inf_{y \in K} Z(y) \geq z\right) \geq \mathbb{P}\left(\Pi \cap (z/a(y_i), \infty) \times M(y_i)\right) > 0, \ 1 \leq i \leq n$$

where $p = \min_{1 \leq i \leq n} H(M(y_i))a(y_i) > 0$. By Bernoulli’s inequality, we obtain

$$\mathbb{P}\left(\inf_{y \in K} Z(y) \geq z\right) \geq 1 - n \exp\left(-\frac{p}{z}\right),$$

or, equivalently,

$$\mathbb{P}\left((\inf_{y \in K} Z(y))^{-1} > z\right) \leq n \exp\left(-pz\right).$$

Thus,

$$\mathbb{E}\left[(\inf_{y \in K} Z(y))^{-1}\right] \leq \int_0^\infty n \exp\left(-pz\right) dz < \infty.$$

The optimization problem (3.5) is difficult to solve because both the numerator and the denominator of

$$\frac{f(y)}{g(f)Z^{(m)}(y)}$$

depend on $y$ and the denominator is stochastic. Hence, we consider some modified versions of the optimization problem whose solutions are expected to be rather close to that of the actual problem.

### 3.1. A modified optimization problem

Recall that the stopping rule (3.4) requires that

$$\text{esssup}_{f \in \mathcal{H}} \sup_{y \in K} \frac{f(y)}{g(f)Z^{(m)}(y)} \leq \sum_{j=1}^{m+1} E_j,$$

A stronger condition than (3.4) is then

$$\text{esssup}_{f \in \mathcal{H}} \sup_{y \in K} \frac{f(y)}{g(f)\inf_{g \in K} Z^{(m)}(y)} \leq \sum_{j=1}^{m+1} E_j, \tag{3.7}$$

while a weaker condition is

$$\text{esssup}_{f \in \mathcal{H}} \sup_{y \in K} \frac{f(y)}{g(f)\sup_{g \in K} Z^{(m)}(y)} \leq \sum_{j=1}^{m+1} E_j. \tag{3.8}$$
The actual stopping rule is in between the strong and the weak condition. Suppose $T : [0, \infty)^K \to [0, \infty)$ is a functional that satisfies $T(1) = 1$ and that is max-linear, i.e.

$$T(\max\{a_1h_1, a_2h_2\}) = \max\{a_1T(h_1), a_2T(h_2)\}$$

for all $a_1, a_2 > 0$ and $h_1, h_2 : K \to [0, \infty)$. Then, we have that $T(h) \leq T(g)$ for all $h \leq g$, which leads to

$$\inf_{y \in K} h(y) \leq T(h) \leq \sup_{y \in K} h(y)$$

(3.9)

for all $h : K \to [0, \infty)$. We consider the condition

$$\text{esssup}_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{T(Z^{(m)})g(f)} \leq \sum_{j=1}^{m+1} E_j$$

(3.10)

for some suitable $T$, and regard the new condition (3.10) as a proxy for the actual stopping rule (3.4). Apparently, condition (3.10) lies in between the strong condition (3.7) and the weak condition (3.8). The corresponding modified optimization problem is then

$$Q^* = \arg \min_g Q^*_g,$$

$$Q^*_g = \mathbb{E} \min \left\{ m \in \mathbb{N} : \text{esssup}_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)T(Z^{(m)})} \leq \sum_{j=1}^{m+1} E_j \right\}.$$ 

(3.11)

In fact, Proposition 3.2 below shows that the solution of the modified problem in (3.11) is not related to the particular choice of the max-linear functional $T$. Therefore, we regard the solution of the modified optimization problem in (3.11) as a good proxy to that of the original problem in (3.5).

Examples of $T$ are $T(h) = \sup_{y \in K} h(y)$ and $T(h) = h(y_0)$ for some $y_0 \in K$. Thus, we get that minimizing

$$Q^*_g(1) = \mathbb{E} \min \left\{ m \in \mathbb{N} : \text{esssup}_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)sup_{\tilde{y} \in K} Z^{(m)}(\tilde{y})} \leq \sum_{j=1}^{m+1} E_j \right\},$$

(3.12)

or

$$Q^*_g(2)(y_0) = \mathbb{E} \min \left\{ m \in \mathbb{N} : \text{esssup}_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)Z^{(m)}(y_0)} \leq \sum_{j=1}^{m+1} E_j \right\},$$

(3.13)

are important modifications of the original optimization problem.

### 3.2. The solution of the modified optimization problem

The following proposition provides a first step to the solution of the modified optimization problem.
Proposition 3.2. Let $f \mapsto \sup_{y \in K} f(y)$ be measurable. Then,

$$Q^* = \arg \min_{g} \esssup_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f)}.$$ 

In particular, $Q^*$ does not depend on the choice of $T$.

Proof. If there exists some $g$ such that $Q^*_g$ is finite, then necessarily

$$\esssup_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f)} < \infty.$$ 

Thus, we can restrict ourselves to

$$g \in D = \left\{ g : \esssup_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f)} < \infty \right\}$$

and assume w.l.o.g. that $D \neq \emptyset$. For $c = \int_{\mathbb{H}} \sup_{y \in K} f(y) \, H(df)$ and any $g \in D$, we have

$$c \leq \int_{\mathbb{H}} \esssup_{h \in \mathbb{H}} \frac{\sup_{y \in K} h(y)}{g(h)} \, g(f) \, H(df) = \esssup_{h \in \mathbb{H}} \frac{\sup_{y \in K} h(y)}{g(h)} < \infty. \quad (3.14)$$

Thus, by (3.9), for $c_T = \int_{\mathbb{H}} T(f) \, H(df)$, we obtain $c_T \leq c < \infty$.

Next, we prove $c_T > 0$ by contradiction. Assume that $c_T = 0$. This yields $T(f) = 0$ for $H$-a.e. $f \in \mathbb{H}$ which – by the max-linearity of $Z$ – implies $T(Z) = 0$ a.s. in contradiction to $\inf_{y \in K} Z(y) > 0$ a.s. and (3.9). Thus, we conclude that $c_T \in (0, \infty)$.

Now, let $g \in D$. Using the max-stability of $T$, we have

$$Q^*_g - 1 = \sum_{m=1}^{\infty} \mathbb{P} \left( \esssup_{f \in \mathbb{H}} \frac{\sum_{j=1}^{m+1} E_j}{\sum_{j=1}^{m} E_j} > \max_{1 \leq k \leq m} \frac{1}{\sum_{j=1}^{k} E_j} \frac{T(f_k)}{g(f_k)} \right)$$

$$= \sum_{m=1}^{\infty} \mathbb{P} \left( \frac{\sum_{j=1}^{k} E_j}{\sum_{j=1}^{m+1} E_j} > \frac{T(f_k)}{g(f_k)} \right) \esssup_{f \in \mathbb{H}} \frac{f(y)}{g(f)}, \quad 1 \leq k \leq m. \quad (3.15)$$

Note that

$$\frac{T(f_k)}{g(f_k)} \esssup_{f \in \mathbb{H}} \frac{f(y)}{g(f)} \in [0, 1].$$

As the joint distribution of $(\sum_{j=1}^{k} E_j / \sum_{j=1}^{m+1} E_j)_{k=1,...,m}$ equals the joint distribution of the order statistics of $m$ independent random variables which are uniformly distributed on $[0, 1]$, we obtain

$$Q^*_g = 1 + \sum_{m=1}^{\infty} \left[ 1 - \mathbb{E} \left( \frac{T(f_1)}{g(f_1)} \esssup_{f \in \mathbb{H}} \frac{f(y)}{g(f)} \right)^m \right] \quad (3.16)$$
we get that the random variable \( M = \min\{m \in \mathbb{N} : (3.10) \text{ is satisfied}\} \) follows a geometric distribution with parameter \( \int_{\mathbb{H}} T(h)H(dh)/ \sup_{f \in \mathbb{H}} \sup_{g \in K} f(y)/g(f) \). Therefore, minimizing \( \sup_{f \in \mathbb{H}} \sup_{g \in K} f(y)/g(f) \) will not only minimize the expectation of \( M \), but also other characteristics such as \( \mathbb{P}(M > m_0) \) for \( m_0 \in \mathbb{N} \). However, this property may not hold for the stochastic number \( m \) in the actual stopping rule (3.4).

We carry on to find the density \( g \) that minimizes \( \sup_{f \in \mathbb{H}} \sup_{g \in K} f(y)/g(f) \). Instead of considering \( f \mapsto \sup_{g \in K} f(y) \) in the numerator, the following theorem deals with a broader class of functionals.

**Theorem 3.4.** Assume we are in the framework of Proposition 2.1. Let \( L : \mathbb{H} \to (0, \infty) \) be measurable and \( c_L := \int_{\mathbb{H}} L(f)H(df) < \infty \). Then,

\[
g^*(f) = c_L^{-1}L(f)
\]

is an element of

\[
Q_L^* = \arg\min_{g} \sup_{f \in \mathbb{H}} \frac{L(f)}{g(f)}.
\]

Furthermore, for every \( g \in Q_L^* \), Equation (3.17) holds for \( H \)-a.e. \( f \in \mathbb{H} \).

**Proof.** First, by contradiction, we show that the inequality

\[
\sup_{f \in \mathbb{H}} \frac{L(f)}{g(f)} \geq c_L
\]

holds for all \( g \). So, assume that (3.18) does not hold for some \( g \) considered in Proposition 2.1. Then some \( \varepsilon > 0 \) and some density \( g \) with \( \int g(f)H(df) = 1 \) exist such that, for all \( f \in \mathbb{H} \), we have \( L(f)/g(f) \leq c_L - \varepsilon \). Hence,

\[
c_L = \int_{\mathbb{H}} L(f)H(df) \leq (c_L - \varepsilon) \int_{\mathbb{H}} g(f)H(df) < c_L
\]

which is a contradiction. Hence, (3.18) is proved. Note that the choice \( g(f) = c_L^{-1}L(f) \) implies equality in (3.18). The first assertion follows.

For the proof of the second assertion, assume that there is some \( g \in Q_L^* \) such that (3.17) does not hold for \( H \)-a.e. \( f \in \mathbb{H} \). Then, as

\[
\int_{\mathbb{H}} g(f)H(df) = 1 = \int_{\mathbb{H}} c_L^{-1}L(f)H(df),
\]

we get that there is some set \( A \subset \mathbb{H} \) with \( H(A) > 0 \) such that, for all \( f \in A \), \( g(f) < c_L^{-1}L(f) \), but \( g(f) > 0 \) by (2.1). This yields \( gH(A) > 0 \) and, hence, \( \sup_{f \in \mathbb{H}} L(f)/g(f) > c_L \), which is a contradiction to \( g \in Q_L^* \). \( \square \)
The results stated above enable us to give a necessary and sufficient condition for the solvability of the optimization problem (3.11) and to describe its solution. Here, we call an optimization problem

\[
\text{arg min}_{x \in A} h(x), \quad h : A \to \mathbb{R} \cup \{\infty\},
\]
solvable if \( \inf_{x \in A} h(x) \in (-\infty, \infty) \) and if there exists some \( x_0 \in A \) such that \( h(x_0) = \inf_{x \in A} h(x) \).

**Corollary 3.5.** Let \( f \mapsto \sup_{y \in K} f(y) \) be measurable. Then, the optimization problem (3.11) is solvable if and only if

\[
c = \int_{H} \sup_{y \in K} f(y) H(\text{d}f) < \infty.
\]

In this case,

\[
g^* \in Q^* = \arg \min_{g} Q_{g}^{(1)} = \arg \min_{g} Q_{g}^{(2)}(y_0)
\]

with \( g^* \) as defined in (2.3), that is, the normalized spectral representation is optimal w.r.t. (3.11). The solution is unique \( H \)-a.s.

**Proof.** If \( c = \infty \), Equation (3.14) and Proposition 3.2 yield that (3.11) is not solvable. For \( c < \infty \), the assertion follows directly from Proposition 3.2 and Theorem 3.4 with \( L(f) = \sup_{y \in K} f(y) \) and \( T(h) = \sup_{y \in K} h(y) \) and \( T(h) = h(y_0) \), respectively.

**Remark 3.6.** Note that the solution of the optimization problem (3.11) is unique in two different aspects. First, any solution \( g \in Q^* \) satisfies \( g = g^* \) \( H \)-a.s. Second, due to the uniqueness of the normalized spectral representation (Proposition 2.5), the finite-dimensional distributions of the spectral functions \( \{f/g(f)\} \) do not depend on the initial choice of the spectral functions, i.e. on the choice of the space \( \mathbb{H} \) and the measure \( H \).

**Remark 3.7.** Corollary 3.5 yields that \( Q_{g}^{(1)} = \infty \) if \( Z \) does not allow for a normalized spectral representation (or, equivalently, \( c = \infty \)). As the definitions imply that \( Q_{g}^{(1)} \leq Q_{g} \) for any \( g \), there is no spectral representation such that the expected number of points which are able to contribute to \( Z \) is finite in this case.

### 4. Evaluating the modified optimization problem

In this section, we discuss how close the relation is between the modified optimization problem and the original problem. Observing that \( Q_{g}^{(1)} \leq Q_{g}^{(2)}(y_0) \leq Q_{g} \) for all \( g \) (see the proof of Proposition 4.1), we see that the modified optimization problem is in fact minimizing a lower bound function of the mapping \( g \mapsto Q_{g} \). We will improve the lower bound and show that the normalized spectral representation also minimizes the improved lower bound function. In addition, we give a formula for calculating \( Q_{g^*} \), that is, the expected number of points which are able to
contribute to $Z$ in the normalized spectral representation regarding the actual stopping rule (3.4). In the following proposition, we especially look at $Q_g^{(1)}$, $Q_g^{(2)}(y_0)$ and $Q_g^{*}$ to get bounds for the real optimal solution.

**Proposition 4.1.**

1. The function $y_0 \mapsto Q^{(2)}_g(y_0)$ is constant on $K$.
2. $Q_g^{(1)} \leq Q_g$ and $Q_g^{(1)} \leq Q_g^{(2)}(y_0)$, $y_0 \in K$, for all $g$.
3. $Q_g^{(1)} = 1$.
4. Assuming further that there is a countable set $K_0 \subset K$ such that $\sup_{y \in K} f(y) = \sup_{y \in K_0} f(y)$ for $H$-a.e. $f \in \mathbb{H}$, we obtain that

$$Q_g^{(2)}(y_0) \leq Q_g$$

for all $g$ and all $y_0 \in K$. In particular, for any solution of the original optimization problem, $\hat{g} \in Q$, we get the bounds

$$1 = Q_g^{(1)} \leq Q_g^{(2)}(y_0) \leq Q_{\hat{g}} \leq Q_g^{*}.$$

**Proof.** First, we note that

$$\text{esssup}_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f) \sup_{y \in K} Z^{(m)}(y)} \leq \text{esssup}_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f) Z(y)}$$

and

$$\text{esssup}_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f) \sup_{y \in K} Z^{(m)}(y)} \leq \text{esssup}_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g(f) Z^{(m)}(y)}$$

for any $g$ and any $y_0 \in K$. By (1.2), we have $\int_{\mathbb{H}} f(y) H(df) = 1$ for all $y \in K$, and thus, by (3.16), $y_0 \mapsto Q_g^{(2)}(y_0)$ is constant on $K$ for any $g$. This proves the first two parts of the proposition.

To see the third assertion, we note that we have

$$\sup_{y \in K} f(y) = c \quad \text{for } g^*\text{-H-a.e. } f \in \mathbb{H}$$

(4.1)

and, thus, with (3.2), we obtain

$$\sup_{y \in K} Z^{(m)}(y) = \sup_{y \in K} \max_{i \in \mathbb{N}} \frac{1}{\sum_{j=1}^{m+1} E_j} f_i(y) \leq c E_1^{-1}$$

(4.2)

for all $m \in \mathbb{N}$. Equations (4.1) and (4.2) yield

$$Q_g^{*} = \mathbb{E} \min \left\{ m \in \mathbb{N} : \text{esssup}_{f \in \mathbb{H}} \frac{\sup_{y \in K} f(y)}{g^*(f)} \leq \sup_{y \in K} Z^{(m)}(y) \sum_{j=1}^{m+1} E_j \right\}$$

and

$$Q_g^{*} = \mathbb{E} \min \left\{ m \in \mathbb{N} : c \leq c \cdot \sum_{j=1}^{m+1} E_j / E_1 \right\} = 1.$$

For the proof of the fourth part of the proposition, we assume that there exists some countable set $K_0 \subset K$ such that $\sup_{y \in K} f(y) = \sup_{y \in K_0} f(y)$ for $H$-a.e. $f \in \mathbb{H}$. Hence, we have that

$$\text{esssup}_{f \in \mathbb{H}} \frac{f(y)}{g(f)} = \sup_{y \in K_0} \text{esssup}_{f \in \mathbb{H}} \frac{f(y)}{g(f)}.$$

(4.3)
We first consider the case that \( \operatorname{esssup}_{f \in \mathbb{H}} \sup_{g \in K} f(y)H(df) = \infty \). Then, either \( c = \int_{\mathbb{H}} \sup_{g \in K} f(y)H(df) = \infty \), which yields \( \infty = Q_g^{(1)} \leq Q_g \) (cf. Remark 3.7), or \( c < \infty \). By Proposition 2.3, the latter implies that \( \sup_{g \in K} Z(y) < \infty \) with probability one. Thus, by the definition of \( Q_g^{(1)} \) in (3.12), \( \infty = Q_g^{(1)} \leq Q_g \). The only case that remains is that \( \operatorname{esssup}_{f \in \mathbb{H}} \sup_{g \in K} f(y)H(df) < \infty \). Then, by (4.3), for every \( \varepsilon > 0 \), there exists some \( y_0(\varepsilon) \) such that

\[
\frac{1}{1 + \varepsilon} \operatorname{esssup}_{f \in \mathbb{H}} \sup_{g \in K} \frac{f(y)}{g(f)Z^{(m)}(y_0(\varepsilon))} \leq \frac{f(y_0(\varepsilon))}{\operatorname{esssup}_{f \in \mathbb{H}} \frac{f(y)}{g(f)Z^{(m)}(y_0(\varepsilon))}} \leq \operatorname{esssup}_{f \in \mathbb{H}} \frac{f(y)}{g(f)Z^{(m)}(y)}
\]

with probability one. Further, analogously to the proof of Proposition 3.2,

\[
\mathbb{E} \min \left\{ m \in \mathbb{N} : \frac{1}{1 + \varepsilon} \operatorname{esssup}_{f \in \mathbb{H}} \sup_{g \in K} \frac{f(y)}{g(f)Z^{(m)}(y_0)} \leq \sum_{j=1}^{m+1} E_j \right\}
\]

\[
= 1 + \sum_{m=1}^{\infty} \left[ 1 - \mathbb{E} \left( 1 \wedge \left( \frac{(1 + \varepsilon) \cdot f_1(y_0)}{g(f_1)} \frac{f(y_0)}{\operatorname{esssup}_{g \in K} g(f)} \right) \right) \right]^m
\]

\[
= \mathbb{E} \left( 1 \wedge \left( \frac{(1 + \varepsilon) \cdot f_1(y_0)}{g(f_1)} \frac{f(y_0)}{\operatorname{esssup}_{g \in K} g(f)} \right) \right)^{-1} \geq \frac{1}{1 + \varepsilon} Q_g^{(2)}(y_0)
\]

for all \( y_0 \in K, \varepsilon > 0 \) and all \( g \), and thus, with (4.4),

\[
(1 + \varepsilon)^{-1} \inf_{y_0 \in K} Q_g^{(2)}(y_0) \leq Q_g
\]

holds for all \( \varepsilon > 0 \). Hence, the fourth part of the proposition follows. \( \square \)

As Proposition 4.1 shows, the approximation of the optimal function value in original problem (3.5) by (3.12) and (3.13) might be quite vague. In particular, the minimum of \( Q_g^{(1)} \) always equals 1, that is, some spectral functions which in fact contribute to \( Z \) are not considered. Replacing the processes \( Z^{(m)} \) occurring in the construction by the final process \( Z^{(\infty)} \) given by (3.3) allows us to take into account all those shape functions which contribute to \( Z^{(\infty)} \). To this end, we revisit the notion of \( K \)-extremal and \( K \)-subextremal points introduced by Dombry and Éyi-Minko (2013) and Dombry and Éyi-Minko (2012). Henceforth, we suppose that the following assumption holds true which enables us to consider this problem.

**Assumption 4.2.** Let \( K \) satisfy the following conditions:

(i) There exists some countable set \( K_0 \subset K \) such that, for all \( h_1, h_2 \in H \),

\[
h_1 < h_2 \text{ on } K \iff \exists \varepsilon > 0 : h_1(y) < h_2(y) - \varepsilon \text{ for all } y \in K_0.
\]
(ii) $\mathbb{H}$ is a max-linear space, i.e.
\[ t_1 h_1 \vee t_2 h_2 \in \mathbb{H} \]
for all $h_1, h_2 \in \mathbb{H}$, $t_1, t_2 \geq 0$. Further, $1 \in \mathbb{H}$.

(iii) $\mathbb{H}$ is endowed with the $\sigma$-algebra $\mathcal{H}$ of cylinder sets, $\mathcal{H} = \sigma(\{h \in \mathbb{H} : h(y) \in B\}, y \in K, B \in \mathcal{B} \cap [0, \infty))$.

**Remark 4.3.**

1. Assumption 4.2 implies that the additional assumption in the fourth part of Proposition 4.1 holds, i.e. there exists a countable set $K_0 \subset K$ such that $\sup_{y \in K} f(y) = \sup_{y \in K_0} f(y)$ all $f \in \mathbb{H}$.

2. As $\mathcal{H}$ is the $\sigma$-algebra of cylinder sets, Assumption 4.2 ensures that the process $F^*$ in the normalized spectral representation (2.6) is uniquely determined (cf. Proposition 2.5).

**Definition 4.4.** Let $\Phi$ be some Poisson point process on $(0, \infty) \times \mathbb{H}$ with intensity measure $u^{-2} du \nu(dh)$ for some locally finite measure $\nu$ on $\mathbb{H}$. We call $(t^*, h^*) \in \Phi$ a $K$-extremal point and write $(t^*, h^*) \in \Phi^+_K$ if and only if
\[ t^* h^*(y) = \max_{(t, h) \in \Phi} th(y) \quad \text{for some } y \in K. \]

Otherwise, i.e. if $t^* h^*(y) < \max_{(t, h) \in \Phi} th(y)$ for all $y \in K$, $(t^*, h^*) \in \Phi$ is called a $K$-subextremal point and we write $(t^*, h^*) \in \Phi^-_K$.

We generalize a result given in Dombry and Éyi-Minko (2013) and show that the random sets $\Phi^+_K$ and $\Phi^-_K$ are point processes on $(0, \infty) \times \mathbb{H}$, i.e. $\Phi^+_K(C)$ and $\Phi^-_K(C)$ are random variables for any bounded set $C \in \mathcal{B} \times \mathcal{H}$. In contrast to Dombry and Éyi-Minko (2013), we are interested in tuples $(t, h)$ instead of the product $th$ and we do not restrict to continuous functions. Nevertheless, the proof runs analogously.

**Proposition 4.5.** $\Phi^+_K$ and $\Phi^-_K$ are point processes on $(0, \infty) \times \mathbb{H}$.

**Proof.** First, we note that, for $\mathbb{H}_0 = (0, \infty) \times \mathbb{H}$, the mapping $\phi : \mathbb{H}_0 \to \mathbb{H}$, $(t, h) \mapsto th(\cdot)$ is measurable. Therefore, events of the type $\{\omega \in \Omega : \Phi((t, h) \in \mathbb{H}_0 : th \in C) = k\}$ are measurable for any $C \in \mathcal{H}$ and $k \in \mathbb{N}_0$.

Now, let $u_0 > 0$, $C \in \mathcal{H}$ with $\nu(C) < \infty$ and $k \in \mathbb{N}_0$. Furthermore, let $K = \{x_1, x_2, \ldots\}$ be as in Assumption 4.2. Then, the event
\[ \{\omega \in \Omega : \Phi^-_K((u_0, \infty) \times C) \geq k\} \]
\[ = \bigcup_{\varepsilon \in \mathbb{Q}_+} \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}^+_n} \left\{\omega \in \Omega : \Phi\left(\{(t, f) \in \mathbb{H}_0 : tf(x_i) > q_i\}\right) \geq 1, 1 \leq i \leq n, \right. \]
\[ \left. \Phi\left(\{(u_0, \infty) \times C) \cap \{(t, f) \in \mathbb{H}_0 : tf(x_j) \leq q_j - \varepsilon, 1 \leq j \leq n\}\right) \geq k\right\} \]
is measurable. Thus, $\Phi^+_K$ and $\Phi^-_K = \Phi \setminus \Phi^-_K$ are point processes. \qed

For applying the theory of extremal and subextremal points to the construction of the process $Z^{(\infty)}$, we define the Poisson point process
\[ \Phi = \{(t, f/g(f)) : (t, f) \in \Pi\}. \]
Then, similarly to the proof of Lemma 3.2 in Dombry and Éyi-Minko (2012), the following result on $\tilde{\Pi}_k$ can be shown.

**Lemma 4.6.** Conditional on $Z^{(\infty)}$, the point process $\Phi_k^-$ is a Poisson point process on $(0, \infty) \times \mathbb{H}$ with intensity measure

$$\tilde{\Lambda}^-(dt, dh) = t^{-2} \times \int_{\mathbb{H}} \int_T 1_{f(\cdot) \in dh} \cdot 1_{t f(\cdot) / g(f) < Z^{(\infty)}(\cdot)} g(f) H(df) \, dt.$$  

In the following, we will mainly focus on the first component of the point process $\Phi$, i.e. we consider the point processes

$$\Pi_{0,k}^+ = \{ t : (t, f/g(f)) \in \Phi_k^+ \}$$

and

$$\Pi_{0,k}^- = \{ t : (t, f/g(f)) \in \Phi_k^- \},$$

respectively. Obviously, any $t^* \in \Pi_{0,k}^+$ is taken into account by the definition of $Q_g$ in (3.5). Thus, we can rewrite (3.5) as

$$Q_g = \mathbb{E}|\Pi_{0,k}^+| + \mathbb{E} \left( \left\{ t \in \Pi_{0,k}^- : \text{ess sup}_{f \in \mathbb{H}} \frac{f(y)}{g(f) Z^{(\infty)}(y)} > \frac{1}{t} \right\} \right). \quad (4.5)$$

Including all the points of $\Pi_{0,k}^+$, i.e. all the spectral functions that finally contribute to $Z^{(\infty)}$, and replacing $Z^{(m)}$ by $Z^{(\infty)}$, we analogously obtain refined versions of (3.11), (3.12) and (3.13) as

$$\tilde{Q}_g^* = \mathbb{E}|\Pi_{0,k}^+| + \mathbb{E} \left( \left\{ t \in \Pi_{0,k}^- : \text{ess sup}_{f \in \mathbb{H}} \frac{f(y)}{g(f) T(Z^{(\infty)})} > \frac{1}{t} \right\} \right), \quad (4.6)$$

$$\tilde{Q}_g^{(1)} = \mathbb{E}|\Pi_{0,k}^+| + \mathbb{E} \left( \left\{ t \in \Pi_{0,k}^- : \text{ess sup}_{f \in \mathbb{H}} \frac{\sup_{y \in \mathbb{H}} f(y)}{g(f) T(Z^{(\infty)}(y))} > \frac{1}{t} \right\} \right), \quad (4.7)$$

$$\tilde{Q}_g^{(2)}(y_0) = \mathbb{E}|\Pi_{0,k}^+| + \mathbb{E} \left( \left\{ t \in \Pi_{0,k}^- : \text{ess sup}_{f \in \mathbb{H}} \frac{\sup_{y \in \mathbb{H}} f(y)}{g(f) Z^{(\infty)}(y_0)} > \frac{1}{t} \right\} \right). \quad (4.8)$$

By definition, obviously $\tilde{Q}_g^{(1)} \geq Q_g^{(1)}$ and $\tilde{Q}_g^{(2)} \geq Q_g^{(2)}$ for all $g$. Hence, $\tilde{Q}_g^{(1)}$ and $\tilde{Q}_g^{(2)}$ are improved lower bounds of $Q_g$ (see Proposition 4.11 below). The optimization of these bounds is facilitated by the following result on $\Pi_{0,k}^+$.

**Lemma 4.7.** We have

$$\mathbb{E}|\Pi_{0,k}^+| = \mathbb{E} \left( \int_{\mathbb{H}} \sup_{y \in \mathbb{H}} \frac{f(y)}{Z(y)} H(df) \right)$$

which does not depend on the choice of $g$.

**Proof.** Let $B = [t_0, \infty)$ with $t_0 > 0$. Then, we have

$$\mathbb{E} \left| \Pi_{0,k}^+ \cap B \right| = \mathbb{E} \left| \Pi \cap (B \times \mathbb{H}) \right| - \mathbb{E} \left| \Pi_{0,k}^- \cap B \right|.$$
Conditioning on \( Z^{(\infty)} \), Lemma 4.6 yields

\[
\mathbb{E}\left| \Pi_{0,K}^+ \cap B \right| = \int_\mathbb{H} \int_0^\infty t^{-2} 1_{t \geq t_0} \, dt \, g(f) H(df) \\
- \mathbb{E}_{Z^{(\infty)}} \left( \int_\mathbb{H} \int_0^\infty t^{-2} 1_{t \geq t_0} \left[ \sup_{y \in K} \frac{f(y)}{g(f)Z^{(\infty)}(y)} \right] \, dt \, g(f) H(df) \right) \\
= \mathbb{E}_Z \left( \int_\mathbb{H} \int_0^\infty t^{-2} 1_{t \geq t_0} \left[ \sup_{y \in K} \frac{f(y)}{g(f)Z^{(\infty)}(y)} \right] \, dt \, g(f) H(df) \right). 
\]

Considering a monotone sequence \( t_{0,n} \downarrow 0 \) as \( n \to \infty \), the monotone convergence theorem yields

\[
\mathbb{E}\left| \Pi_{0,K}^+ \right| = \mathbb{E}_Z \left( \int_\mathbb{H} \int_0^\infty t^{-2} t_{1/n} \sup_{y \in K} \frac{f(y)}{g(f)Z^{(\infty)}(y)} \, dt \, g(f) H(df) \right) \\
= \mathbb{E}_Z \left( \int_\mathbb{H} \sup_{y \in K} \frac{f(y)}{g(f)Z(y)} H(df) \right),
\]

which completes the proof. \( \square \)

The results stated in Lemma 4.6 and 4.7 facilitate the calculation of \( Q_g \) and allow us to relate the minimizer of (4.6) to the solution of our previously modified optimization problem, \( g^* \in Q^* \).

**Proposition 4.8.**

1. For any \( g \), we have

\[
Q_g = \mathbb{E}_Z \left( \esssup_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)Z(y)} \right).
\]  

2. Assume that \( f \mapsto \sup_{y \in K} f(y) \) is measurable. Then, with \( \tilde{Q}_g \) as in (4.6), for any max-linear function \( T \), it holds

\[
\arg\min_g \tilde{Q}_g^* \supset \arg\min_g \esssup_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)} = Q^*,
\]

where \( Q^* = \arg\min_g Q_g^* \).

**Proof.** Let \( Z^{(\infty)} \) be given by (3.3). By Lemma 4.7, we obtain

\[
Q_g = \mathbb{E}_Z \int_\mathbb{H} \sup_{y \in K} \frac{f(y)}{Z(y)} H(df) \\
+ \mathbb{E} \left( \left\{ \left. t \in \Pi_{0,K}^- : \sup_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)Z^{(\infty)}(y)} > t^{-1} \right\} \right) \right). 
\]

Conditioning on \( Z^{(\infty)} \), Lemma 4.6 yields

\[
\mathbb{E} \left( \left\{ \left. t \in \Pi_{0,K}^- : \sup_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g(f)Z^{(\infty)}(y)} > t^{-1} \right\} \right) \right)
\]
leads to two implications in application. Firstly, it allows for a sufficient condition, this yields that $Q_{g^*} < \infty$ if $c < \infty$. In other words, the expectation

$$Q_{g^*} \leq c \cdot E\left[\left(\inf_{y \in K} Z(y)\right)^{-1}\right].$$

(4.11)
of the stochastic number \( m \) from (3.4) is then finite for the normalized representation.

We further evaluate when \( Q_{g^*} \) reaches its upper bound as in (4.11). Note that equality in (4.11) holds if and only if

\[
\mathrm{esssup}_{f \in \mathbb{H}} \sup_{y \in K} \frac{f(y)}{g^*(f)Z(y)} = \frac{c}{\inf_{\tilde{y} \in K} Z(\tilde{y})} \quad \text{a.s.} \tag{4.12}
\]

Furthermore, by Assumption 4.2, \( \sup_{y \in K} \) in (4.12) may be replaced by \( \sup_{y \in K_0} \) for some countable set \( K_0 \). The fact that \( K_0 \) is countable allows for interchanging \( \mathrm{esssup}_{f \in \mathbb{H}} \) and \( \sup_{y \in K_0} \), i.e. Equation (4.12) is equivalent to

\[
\sup_{y \in K_0} \mathrm{esssup}_{f \in \mathbb{H}} \frac{f(y)}{g^*(f)Z(y)} = \frac{c}{\inf_{\tilde{y} \in K} Z(\tilde{y})} \quad \text{a.s.} \tag{4.13}
\]

Thus, condition (4.12) holds if and only if, with probability one, there exists a sequence \( (y_n)_{n \in \mathbb{N}} \) in \( K_0 \) such that

\[
\lim_{n \to \infty} \mathrm{esssup}_{f \in \mathbb{H}} \frac{f(y_n)}{\sup_{\tilde{y} \in K} f(\tilde{y})} = \lim_{n \to \infty} \frac{Z(y_n)}{\inf_{\tilde{y} \in K} Z(\tilde{y})}. \tag{4.14}
\]

Note that the left-hand side of (4.14) is bounded from above by 1, while the right-side is bounded from below by 1. Condition (4.14) can be reformulated in the following way: For every \( \varepsilon > 0 \) and almost every sample path of \( Z \), there exists some \( y \in K \) with

\[
Z(y) < (1 + \varepsilon) \inf_{\tilde{y} \in K} Z(\tilde{y}) \tag{4.15}
\]

and

\[
H \left( \left\{ f \in \mathbb{H} : f(y) > (1 - \varepsilon) \sup_{\tilde{y} \in K} f(\tilde{y}) \right\} \right) > 0. \tag{4.16}
\]

Analogously to Equation (4.11), where \( \mathbb{E}m \) is bounded from above, the number \( m \) can be bounded from above a.s. by

\[
\min \left\{ \tilde{m} \in \mathbb{N} : c/ \inf_{\tilde{y} \in K} Z(\tilde{y}) \leq \sum_{j=1}^{\tilde{m}+1} E_j \right\} \tag{4.17}
\]

and, again, \( m \) equals (4.17) a.s. if and only if (4.15) and (4.16) hold.

**Remark 4.9.** If \( Z \) is represented by a stochastic process, i.e. \( Z \) is defined as in (2.7) (cf. Penrose, 1992, for example), \( H \) is a probability measure, namely the law of \( W \). In this case, condition (4.16) is equivalent to

\[
\mathbb{P} \left( W(y) > (1 - \varepsilon) \sup_{\tilde{y} \in K} W(\tilde{y}) \right) > 0. \tag{4.18}
\]

Secondly, Proposition 4.8 implies that minimizing \( \hat{Q}_{g^*} \) can be achieved by any \( g^* \in Q^* \). We thus obtain the following corollary.
Corollary 4.10. Let the mapping \( f \mapsto \sup_{y \in K} f(y) \) be measurable. Then, the optimization problem given in (4.6) is solvable if and only if the optimization problem (3.11) is solvable (cf. Corollary 3.5). In this case, we have

\[
Q^* \subset \arg \min_g \tilde{Q}^{(1)}_g = \arg \min_g \tilde{Q}^{(2)}_g (y_0).
\]

In particular, the normalized spectral representation is optimal w.r.t. (4.6).

Analogously to Proposition 4.1, the following result can be shown.

Proposition 4.11. For any \( \tilde{g} \in Q \), we have

\[
1 \leq \tilde{Q}^{(1)}_{g^*} \leq \inf_{y_0 \in K} \tilde{Q}^{(2)}_{g^*} (y_0) \leq Q_{g^*},
\]

where \( g^* \in Q^* \) is given by (2.3).

Remark 4.12. In view of Proposition 4.8, it appears promising to replace \( g^* \) by an element of

\[
Q^*_0 = \arg \min_g \left\{ \text{esssup}_f \mathbb{E}_Z \left( \sup_{y \in K} \frac{f(y)}{g(f)Z(y)} \right) \right\}
\]

to improve the partial minimization of \( Q \). If the functional

\[
L_0 : \mathbb{H} \to (0, \infty), \quad f \mapsto \mathbb{E}_Z \left( \sup_{y \in K} Z(y)^{-1} f(y) \right)
\]

is measurable and if

\[
c_0 = \int_{\mathbb{H}} \mathbb{E}_Z \left( \sup_{y \in K} Z(y)^{-1} f(y) \right) H(df) < \infty,
\]

then, by Theorem 3.4, an element of \( Q^*_0 \) is given by

\[
g_0(f) = (c_0(f))^{-1} \mathbb{E}_Z \left( \sup_{y \in K} Z(y)^{-1} f(y) \right).
\]

Conversely, for every \( g \in Q^*_0 \), (4.19) holds for \( H \)-a.e. \( f \in \mathbb{H} \).

Note that the calculation of \( g_0 \) is much more laborious than the calculation of \( g^* \), as the former one requires the computation of the expectation of \( \sup_{y \in K} (Z(y)^{-1} f(y)) \). However, computational experiments in case of the original Smith process (Smith, 1990) on finite intervals \([-b, b] \subset \mathbb{R}\) indicate that \( Q_{g_0} \) is not significantly smaller than \( Q_{g^*} \). Thus, the usage of \( g^* \) seems to be preferable over the usage of \( g_0 \) due to its accessibility.

5. Examples for the normalized spectral representation

In this section, we will investigate some specific cases for the process \( Z \) from Proposition 2.1 and for the index set \( K \). Under the general assumption that the mapping \( f \mapsto \tilde{f} = \sup_{y \in K} f(y) \) is measurable, we consider the normalized spectral representation \( \tilde{Z} = d \) \( Z \) in (2.4). For some examples, we explicitly calculate
and, thus, the normalized spectral representation (K case, where \( f(y_0) = f(y_0) \) and, thus, the normalized spectral representation (2.4) simplifies to \( \tilde{Z}(y_0) = \max_{t \in \Pi_0} t \) as \( c = \int_{\mathbb{E}} f(y_0) \, H(df) = 1 \). Thus, only the largest point of \( \Pi \) needs to be considered for a realization of \( \tilde{Z}(y_0) \) as discussed in the introduction. Next, we deal with more sophisticated examples.

### 5.1. Mixed moving maxima

Let \( Z \) be a mixed moving maxima process on some compact set \( K \subset \mathbb{R}^d \), that is, \( f(y) = h(y-x) \) for some random shift \( x \in S \subset \mathbb{R}^d \) and a random function \( h: \mathbb{R}^d \to [0, \infty) \) and

\[
H(C) = (\Lambda \times \pi)\{(x, h) : h(\cdot - x) \in C\}, \tag{5.1}
\]

\( C \subset \mathbb{H} \subset [0, \infty)^K \), where \( \Lambda \) is locally finite measure on \( S \) and \( \pi \) is a probability measure on some Polish space \( P \subset [0, \infty)^{2d} \). Then, the law \( g^*H \) of \( F_t = h_t(\cdot -X_t) \) in (2.4) can be decomposed in the following way: First, we consider \( h_t \) with distribution \( \mathbb{P}(h_t \in dh) = \xi(h) \pi(dh), h \in P \), where \( \xi(h) = c^{-1} \int_{\mathcal{S}} \sup_{y \in K} h(y-z) \Lambda(dz) \) and \( c = \int_{\mathbb{P}} \int_{\mathcal{S}} \sup_{y \in K} g(y-z) \Lambda(dz) \pi(dg) \). Then, \( X_t \mid h_t \) follows the law \( \mathbb{P}(X_t \in dx \mid h_t \in dh) = \mu(x, h) \) where

\[
\mu(x, h) = \left( \int_S \sup_{y \in K} h(y-z) \Lambda(dz) \right)^{-1} \sup_{y \in K} h(y-x), \quad x \in S, \ h \in P.
\]

Here, \( t \in \Pi_0 \) cannot contribute to \( \tilde{Z} \) if

\[
t < \inf_{y \in K} \left( \tilde{Z}(y) / \text{esssup}_{(x, h) \in S \times P} \frac{h(y-x)}{\sup_{y \in K} h(y-x)} \right).
\]

The right-hand side equals \( \inf_{y \in K} \tilde{Z}(y) / c \ a.s. \) if and only if conditions (4.15) and (4.16) are met. In case of a mixed moving maxima process, (4.16) is equivalent to

\[
(\Lambda \times \pi)\{(x, h) \in S \times P : \frac{h(y-x)}{\sup_{y \in K} h(y-x)} > 1 - \varepsilon \} > 0. \tag{5.2}
\]

**Remark 5.1.** Note that the decomposition of \( g^*H \) relies on the fact that \( H \) is the push forward measure of the product measure \( \Lambda \times \pi \). This procedure can be generalized for the case that \( H \) is the push-forward measure of a product measure of the form \( \nu_1 \times \ldots \times \nu_n \).

Note that the results for mixed moving maxima processes can also be applied if \( Z \) is a stationary process with a representation by a stochastic process as in (2.7). In this case, we may introduce some “random shifting”. The following proposition can be shown in exactly the same way as Theorem 2 in Oesting, Kabluchko and Schlather (2012).
Proposition 5.2. Let \( \{W(y), y \in \mathbb{R}^d\} \) be a stochastic process such that the max-stable process \( \{Z(y), y \in \mathbb{R}^d\} \) given by (2.7) is stationary. Furthermore, let \( S \subset \mathbb{R}^d \). Then, for any probability measure \( \Lambda \) on \( S \), we have that

\[
Z(\cdot) \overset{d}= \max_{t \in \Pi_0} tW_t(\cdot - X_t),
\]

where \( X_t \sim \text{i.i.d.} \Lambda, t \in (0, \infty) \). Equivalently,

\[
Z(\cdot) = \max_{(t,x,f) \in \Pi} tf(\cdot - x),
\]  

(5.3)

where \( \Pi \) is a Poisson point process on \( (0, \infty) \times S \times P \) with intensity measure \( t^{-2} dt \times \Lambda(dx) \times \pi(df) \) with \( \pi \) being the law of \( W \) and \( P \subset [0, \infty)^{\mathbb{R}^d} \) being a Polish space.

Thus, using representation (5.3) for \( Z|_{K} \), where \( K \subset \mathbb{R}^d \) is compact, with an arbitrary probability measure \( \Lambda \) on some set \( S \subset \mathbb{R}^d \), we are in the framework of a mixed moving maxima process. By Proposition 2.3, the number \( m \) of considered spectral functions in the normalized spectral representation \( \tilde{Z} \) is finite a.s. if and only if \( \mathbb{E} \sup_{y \in K} W(y) < \infty \). Recall that the normalized spectral representation is unique (cf. Proposition 2.5). Thus, the representation as well as the number \( m \) of considered spectral functions do not depend on the choice of \( \Lambda \). However, different choices of \( \Lambda \) may lead to different ways of decomposing the measure \( g^*H \).

5.2. Monotone, radial symmetric shape function and \( K \) a convex, compact set

Let \( Z \) be a stationary moving maxima process on \( \mathbb{R}^d \) restricted to a convex compact set with a radial symmetric shape function, that is, let \( Z \) be defined as in Proposition 2.1 where \( H \) is given by \( H(A) = \lambda(\{|x \in \mathbb{R}^d : f_0(\|x\|) \in A\}) \) for any measurable set \( A \subset [0, \infty)^{\mathbb{R}^d} \), \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}^d \) and \( f_0 : [0, \infty) \to [0, \infty) \). Further, we assume that \( f_0 \) is monotonically decreasing. Then,

\[
\tilde{f}_0(x) := \sup_{y \in K} f_0(\|y - x\|) = f_0(d(x, K))
\]

with \( d(x, K) = \min_{y \in K} \|y - x\| \) and thus \( g^* \) as defined in (2.3) satisfies

\[
g^*(f_0(\|\cdot - x\|)) = c^{-1} \tilde{f}_0(x) = c^{-1} f_0(d(x, K)),
\]

where \( c = \int_{\mathbb{R}^d} \tilde{f}_0(x) \, dx = \int_{\mathbb{R}^d} f_0(d(x, K)) \, dx \). If \( f_0 \) is continuous at the origin, then condition (5.2) is met, which implies \( Q_y^* = \mathbb{E}\left( c / \inf_{y \in K} \tilde{Z}(y) \right) \).

In the following, we calculate \( \tilde{f}_0(x), x \in \mathbb{R}^d \), and \( c \) for different cases of \( K \). First, consider the case that \( K \) is a \( d \)-dimensional ball \( b(0, R) \) centered at the origin with radius \( R \), i.e. \( K = b(0, R) = \{ x \in \mathbb{R}^d : \|x\| \leq R \} \), we get that
\[ \tilde{f}_0(x) = f_0(0) \mathbf{1}_{\|x\| \leq R} + 1_{\|x\| > R} f_0(\|x\| - R). \] Assume that the random variable \( X \) follows the probability density that is proportional to \( \tilde{f}_0 \). Then we get

\[ P(\|X\| \leq r) = c^{-1} \left( f_0(0)(r \wedge R)^d + d f_0^{(r-R)\vee 0}(\tilde{r} + R)^{d-1} f_0(\tilde{r}) d\tilde{r} \right) \]

with \( c = f_0(0) R^d + d f_0^{\infty}(\tilde{r} + R)^{d-1} f_0(\tilde{r}) d\tilde{r} < \infty. \)

Second, consider the case that \( K \) is a \( d \)-dimensional cube is of particular interest, i.e. the case that \( K = [-R, R]^d \) for some \( R > 0 \). Then, we get

\[ \tilde{f}_0((x_1, \ldots, x_d)) = f_0 (\|(x_1| - R) \vee 0, \ldots, (|x_d| - R) \vee 0\|). \]  

We consider the subcases \( d = 1 \) and \( d = 2 \) to derive explicit formulae. If \( d = 1 \), then \( K = [-R, R] = b(0, R) \), and, according to the formulae above, we get that

\[ \tilde{f}_0(x) = 1_{|x| \leq R} f_0(0) + 1_{|x| > R} f_0(|x| - R) \]

and thus,

\[ c = \int_R \tilde{f}_0(x) dx = 2 R f_0(0) + \int_{|x| > 0} f_0(|x|) dx = 2 R f_0(0) + 1. \]

If \( d = 2 \), we obtain

\[ \tilde{f}_0(x) = 1_{|x_1| \vee |x_2| \leq R} f_0(0) + 2 \cdot 1_{|x_1| \wedge |x_2| \leq R, |x_1| \vee |x_2| > R} f_0 (\|(x_1| \wedge |x_2|) - R) + 1_{|x_1| \wedge |x_2| > R} f_0 (\|(x_1| - R, |x_2| - R\|). \]

Thus,

\[ c = (2R)^2 \cdot f_0(0) + 2 \cdot 2R \cdot \int_R f_0(|x|) dx + \int_{\mathbb{R}^2} f_0(|x|) dx 
\]

\[ = 4 R^2 f(0) + 4R \int_R f_0(|x|) dx + 1. \]

Next, we further specify explicit examples on the function \( f_0 \), under which the constant \( c \) can be further calculated.

**Example 5.3.**

1. **Indicator function**
   We consider the case that the shape function is the indicator function of a ball \( b(0, r) \) with radius \( r > 0 \) centered at the origin, i.e. \( f_0(\|x\|) = 1_{\|x\| \leq r} \). In this case we have \( \tilde{f}_0(x) = 1_{K \oplus b(0, r)}(x) \) and \( c = \text{vol}(K \oplus b(0, r)) \) where \( \oplus \) denotes morphological dilation and \( \text{vol} \) the \( d \)-dimensional volume. Here, all the finite approximations derived from the normalized spectral representation coincide with the corresponding approximations resulting from the algorithm proposed by Schlather (2002).

2. **Smith model**
   As the second example, we consider the Gaussian extreme value process (Smith, 1990) where \( f_0 \) is a Gaussian density function. Here, for simplicity, we assume the shape function to be the density of a multivariate normal random vector \( Y \sim \mathcal{N}(0, \sigma^2 I_d) \) with \( \sigma > 0 \). Thus, it is a radial symmetric monotone function. Let \( K = [-R, R]^d \) for some \( R > 0 \). Then,
by the considerations above, we get that $f_0$ is of type (5.4) and for $d=1,2$, we obtain

$$c = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{R}{\sigma} + 1, & d = 1 \\ \frac{2}{\pi} \left(\frac{R}{\sigma}\right)^2 + 2\sqrt{\frac{2}{\pi}} \frac{R}{\sigma} + 1, & d = 2. \end{cases}$$

Remark 5.4. By the considerations in Subsection 5.1, all these results can be generalized for the case that the shape function is not deterministic, but random with law $\pi$, i.e.

$$H(A) = (\lambda \times \pi)(\{(x, f_0) \in \mathbb{R}^d \times [0, \infty)^{(0, \infty)} : f_0(\|x\|) \in A\}).$$

Now, for random shape functions with law $\pi$, we consider the case that $K$ grows unboundedly. For simplicity, we assume that $K = b(0, R) \subset \mathbb{R}^d$ with $R \to \infty$. Here, by the considerations above, we have $\hat{f}_0(x) = f_0(0) 1_{r \leq R} + 1_{r > R} f_0(r - R)$ for every $f_0 \in \text{supp}(\pi)$. Then, as a special case of Subsection 5.1, we get that $\xi(h) = (\int f_0(0) \pi(df_0) - 1(h(0) + o(1)))$ and $\mu(x, h) = \frac{R^d}{|b(0, 1)|} + o(R^{-d})$. Thus, we obtain the representation

$$\hat{Z}(y) = \max_{t \in \Pi_d} \frac{|b(0, R)| \int h(0) \pi(dh) h_t(y, X_t)}{h_t(0)} + o(1), \quad \|y\| \leq R,$$

where $h_t \sim \xi(h) \pi(dh)$ and $X_t \mid h_t \sim \mu(x, h_t) dx$, $t > 0$, are all independent.

This representation is very similar to the standard mixed moving maxima representation used for the simulation algorithm proposed in Schlather (2002). The main difference, however, is that the shape functions $h_t$ are transformed to have the same value at the origin and are drawn with modified law $\mathbb{P}(h_t \in \mathbb{R}^d)$ instead of $\pi$. This difference also causes a different asymptotic behavior of the number of considered shape functions. While $Q_{g^*} = \mathbb{E}(\int f_0(0) \pi(df_0) \cdot |b(0, 1)| R^d + o(R^d)) / \inf_{y \in b(0, R)} Z(y)$, the expected number of spectral functions taken into account in Schlather’s (2002) algorithm equals $\mathbb{E}(\text{esssup}_{f_0 \in \text{supp}(\pi)} f_0(0) \cdot |b(0, 1)| R^d + o(R^d)) / \inf_{y \in b(0, R)} Z(y)$. Thus, by using the normalized spectral representation, the number is asymptotically decreased by a factor $\int f_0(0) \pi(df_0) / \text{esssup}_{f \in \text{supp}(\pi)} f(0)$.

For details on Schlather’s (2002) algorithm and the number of considered spectral functions, see Section 6.

5.3. Brown-Resnick Processes

We consider a Brown-Resnick process

$$Z(y) = \max_{t \in \Pi_d} t \exp(B_t(y) - \sigma^2(y)/2), \quad y \in K,$$

on a compact set $K \subset \mathbb{R}^d$, where $\Pi_d$ is a Poisson point process on $(0, \infty)$ with intensity measure $t^{-2} dt$ and $B_t$, $t > 0$, are independent copies of a stochastic process $\{B(y), y \in K\}$. Here, $B$ is a zero-mean Gaussian process with
stationary increments, variogram $\gamma$, and variance $\sigma^2(\cdot)$. Note that $Z$ is stationary and its law only depends on $\gamma$ (cf. Kabluchko, Schlather and de Haan, 2009). As the representation (5.5) is of type (2.7), we can use the fourth condition of Proposition 2.3 for the existence of the normalized spectral representation. Thus, the number $m$ from the stopping rule (3.4) is finite if and only if $\mathbb{E}(\sup_{y \in K} \exp (B(y) - \sigma^2(y)/2)) < \infty$.

However, if $\gamma$ tends to infinity fast enough, the original definition turns out to provide inappropriate finite approximations and the mixed moving maxima representation is a promising option (cf. Oesting, Kabluchko and Schlather, 2012). Thus, we aim to derive the normalized spectral representation starting with a stationary moving maxima representation, i.e. $H$ is defined by (5.1), where $A$ is the Lebesgue measure on $S = \mathbb{R}^d$. By Kabluchko, Schlather and de Haan (2009), such a representation exists if $B(y) - \sigma^2(y)/2 \to -\infty$ a.s. for $\|y\| \to \infty$ and $B$ has continuous sample paths. In this case, the random variables $\tau = \arg \max (B(\cdot) - \sigma^2(\cdot)/2)$ and $v = \max \exp (B(\cdot) - \sigma^2(\cdot)/2)$ are well-defined and, by Engelke et al. (in press), the shape function $h \sim \pi$ is given by

$$h(\cdot) \stackrel{d}{=} \left( \int_{\mathbb{R}^d} \int_{C(\mathbb{R}^d)} f(t) \mathbb{P}_{h_0}(df) \, dt \right)^{-1} h_0(\cdot),$$

where $h_0$ has the law

$$\mathbb{P}_{h_0}(A) = \frac{\int_0^{\infty} y^\tau W(\cdot + \tau) \in A, \tau \in [0,1]^d | v = y) \mathbb{P}_v(dy)}{\int_0^{\infty} y^\tau \mathbb{P}(\tau \in [0,1]^d | v = y) \mathbb{P}_v(dy)},$$

with $W(\cdot) = \exp(B(\cdot) - \sigma^2(\cdot)/2)$. Thus, we have $\arg \max h = 0$ and $\max h = (\int_{\mathbb{R}^d} \int_{C(\mathbb{R}^d)} f(t) \mathbb{P}_{h_0}(df) \, dt)^{-1}$ a.s.

Furthermore, as $B$ has continuous sample paths, we have that, for any compact set $K \subset \mathbb{R}^d$, $\mathbb{P}(\sup_{y \in K} (B(y) - \sigma^2(y)/2 < \infty) = 1$ and thus, by Theorem 2.1.2 in Adler and Taylor (2007),

$$\mathbb{E}(\sup_{y \in K} \exp (B(y) - \sigma^2(y)/2)) < \infty.$$ 

Hence, by Proposition 2.3, we obtain $c < \infty$, i.e. the existence of the normalized spectral representation. As $S = \mathbb{R}^d$ and $h$ is continuous at the origin, we get that (5.2) holds, and thus, by the stopping rule (3.4), a point $t \in \Pi_0$ cannot contribute to $\{\tilde{Z}(y), y \in K\}$ if $t < c^{-1} \inf_{y \in K} \tilde{Z}(y)$. Hence, we have a valid stopping rule for Brown-Resnick processes, as $\inf_{y \in K} \tilde{Z}(y) > 0$ a.s.

Remind that the normalized spectral functions $F_t = c F_t / \sup_{y \in K} F_t(y)$ in (2.4) are uniquely determined by (2.5) with

$$c = \mathbb{E}(\sup_{y \in K} \exp (B(y) - \sigma^2(y)/2) = \int_{\mathbb{R}^d} \sup_{y \in K} f(y - x) \, dx \, \pi(df),$$

(cf. Proposition 2.5). In particular, the normalized spectral representation for the representation (5.5) is the same as for the equivalent mixed moving maxima representation. However, the representations provide different ways to decompose the distribution $g^*H$ of $F_t$. 


6. Comparison to the algorithm proposed in Schlather (2002)

In this section, we compare the number of spectral functions considered in the normalized spectral representation to that considered in Schlather’s (2002) algorithm for mixed moving maxima processes. First, we present the algorithm proposed by Schlather (2002) and calculate the number of considered shape functions in the general case. In Subsections 6.1 and 6.2, we compare this number to the corresponding number for the normalized spectral representation in case of the Smith process (Smith, 1990) theoretically and in a simulation study.

Let \( \{ Z(y) : y \in \mathbb{R}^d \} \) be a stationary mixed moving maxima process, i.e. \( H \) is given by (5.1) and \( \Lambda \) is the Lebesgue measure on \( \mathbb{R}^d \). In Schlather (2002), a simulation algorithm is proposed which is shown to be exact if the shape functions \( h \in \text{supp}(\pi) \) are jointly bounded and have joint support, i.e. \( \pi(\{ h : b(x) < C \text{ for all } x \in \mathbb{R}^d \}) = 1 \) for some \( C > 0 \) and \( \pi(\{ b : \text{supp}(h) \subset b(0, r) \}) = 1 \) for some \( r > 0 \) (Schlather, 2002, Thm. 4). In this case,

\[
Z(y) =_{d} |K \oplus b(0, r)| \cdot \max_{1 \leq n \leq M} \frac{F_n(y - U_n)}{\sum_{k=1}^n \xi_k}, \quad y \in K,
\]

where \( \xi_k \) are independent and identically distributed random variables with standard exponential distribution, \( F_k \) follow the law \( \pi \), \( U_k \) are uniformly distributed on \( K \oplus b(0, r) \) and all these random variables are independent. Further, \( M \) is a random number defined by

\[
M = \min \left\{ m \in \mathbb{N} : \frac{C}{\sum_{k=1}^{m+1} \xi_k} \leq \inf_{y \in K} \max_{1 \leq n \leq m} \frac{F_n(x - U_n)}{\sum_{k=1}^n \xi_k} \right\}.
\]

Here, analogously to Proposition 4.8, the following result can be shown.

**Proposition 6.1.** The expectation of \( M \), defined as above, equals

\[
\mathbb{E} M = \mathbb{E} \left( \frac{|K \oplus b(0, r)| \cdot C}{\inf_{y \in K} Z(y)} \right).
\]

If the shape functions are not jointly compactly supported, the max-stable process \( Z \) can be approximated using shape functions which are cut off outside a compact set \( J \), i.e. \( \tilde{F}_n(x) = F_n(x) \cdot 1_{x \in J} \). Then, with \( \tilde{U}_k \sim_{\text{i.i.d.}} \text{Unif}(K \oplus \tilde{J}) \) where \( \tilde{J} = \{-x : x \in J\} \), for the process \( Z_J(\cdot) \) defined by

\[
Z_J(y) = |K \oplus \tilde{J}| \cdot \max_{n \in \mathbb{N}} \frac{\tilde{F}_n(y - \tilde{U}_n)}{\sum_{k=1}^n \xi_k}, \quad y \in K,
\]

the number \( M \) of shape functions that need to be considered for the exact process \( \{ Z_J(y), y \in K \} \) is finite a.s., and, by Proposition 6.1, its expectation equals \( \mathbb{E} \left( \frac{|K \oplus \tilde{J}| \cdot C}{\inf_{y \in K} Z_J(y)} \right) \).
6.1. Theoretical Comparison in the Case of the Smith Process

In order to compare the aforementioned two numbers of spectral functions, we consider the Smith process described in Example 5.3 on a rectangle $[-R, R]^d$ for $d = 1, 2$. By Example 5.3 and Equations (4.11)-(4.16), the expected number of spectral functions considered in the normalized spectral representation $\tilde{Z}$ equals

$$Q_g = \begin{cases} \left( \frac{\sqrt{2}R}{\pi} + 1 \right) \mathbb{E} \left( \sup_{y \in [-R, R]} Z(y)^{-1} \right), & d = 1 \\ \left( \frac{2}{\pi} \left( \frac{R}{\sigma} \right)^2 + 2 \sqrt{\frac{2}{\pi} R + 1} \right) \mathbb{E} \left( \sup_{y \in [-R, R]^d} Z(y)^{-1} \right), & d = 2. \end{cases}$$

For the simulation algorithm of Schlather (2002), we need an approximation as described above. Here, a natural choice for cutting off the shape function is $L = [-k\sigma, k\sigma]^d$ for some $k \in \mathbb{N}$. Then, the expected number of considered shape functions equals

$$\mathbb{E}M_k = \sqrt{\frac{2^d}{\pi}} \left( \frac{R}{\sigma} + k \right)^d \mathbb{E} \left( \sup_{y \in [-R, R]^d} Z_{[-k\sigma, k\sigma]^d}(y)^{-1} \right).$$

Thus, the ratio between the two expected numbers of considered spectral functions, $Q_g / \mathbb{E}M_k$, can be written as a product

$$\frac{Q_g}{\mathbb{E}M_k} = A_{R,k} P_{R,k} \quad (6.1)$$

where

$$A_{R,k} = \begin{cases} \frac{R + \sqrt{\pi/2\sigma}}{R^2 + \sqrt{2\pi R + \pi/2} + k^2}, & d = 1, \\ \frac{2}{\pi} \left( \frac{R}{\sigma} \right)^2 + 2 \sqrt{\frac{2}{\pi} R + 1}, & d = 2. \end{cases}$$

and

$$P_{R,k} = \frac{\mathbb{E} \left( \sup_{y \in [-R, R]^d} Z(y)^{-1} \right)}{\mathbb{E} \left( \sup_{y \in [-R, R]^d} \tilde{Z}(y)^{-1} \right)}.$$

As we have $Z_{[-k\sigma, k\sigma]^d} \rightarrow_d \tilde{Z}$ as $k \to \infty$, the relative number of considered spectral functions asymptotically equals $Q_g / \mathbb{E}M_k = A_{R,k}(1 + o(1))$ as $k \to \infty$. Note that $A_{R,k} < 1$ if and only if $k > \sqrt{\pi}$. In addition, as $Z_{[-k\sigma, k\sigma]^d}$ is constructed via the cut off shape functions $\tilde{F}_n(\cdot) \leq F_n(\cdot)$, we have that $P_{R,k} \leq 1$. Thus, in the product (6.1), the first factor $A_{R,k}$ basically refers to the area to which the points of the Poisson point process belong, and the second factor $P_{R,k}$ refers to the exactness of the approximation by Schlather’s (2002) algorithm.

6.2. Simulation Study for the Smith Process

We will now verify the theoretical considerations above in a simulation study. To this end, for $\sigma = 1$, we simulate $Z$ and $Z_{[-k, k]^d}$ for $k = 2, 3$ on a grid.
Table 1

Results for simulations of $\tilde{Z}$ and $Z_{[-k,k]}$, $k = 2, 3$, on \{-R, -R + 0.1, \ldots, R - 0.1, R\} for different $R$. For each case, $A_{R,k}$ and the estimates for $Q_{g^*}$, $\mathbb{E}M_k$ and $P_{R,k}$ as defined in Subsection 6.1 are displayed, based on $N = 5000$ simulations of each process.

| $R$ | $Q_{g^*}$ | $\mathbb{E}M_2$ | $Q_{g^*}$ | $\mathbb{E}M_2$ | $A_{R,2}$ | $\hat{P}_{R,2}$ | $\mathbb{E}M_3$ | $Q_{g^*}$ | $\mathbb{E}M_3$ | $A_{R,3}$ | $\hat{P}_{R,3}$ |
|-----|----------|----------------|----------|----------------|----------|----------------|----------------|----------|----------------|----------|----------------|
| 1   | 3.12     | 4.38           | 0.71     | 0.75           | 0.94     | 5.46           | 0.57           | 0.56     | 1.00           |         |                |
| 2   | 5.73     | 7.57           | 0.76     | 0.81           | 0.94     | 8.93           | 0.64           | 0.65     | 0.98           |         |                |
| 5   | 15.82    | 18.82          | 0.84     | 0.89           | 0.95     | 19.98          | 0.79           | 0.78     | 1.02           |         |                |
| 10  | 35.63    | 40.57          | 0.88     | 0.94           | 0.94     | 41.16          | 0.87           | 0.87     | 1.00           |         |                |
| 50  | 239.75   | 257.61         | 0.93     | 0.99           | 0.94     | 247.35         | 0.97           | 0.97     | 1.00           |         |                |
| 100 | 540.44   | 579.11         | 0.93     | 0.99           | 0.94     | 550.70         | 0.98           | 0.98     | 1.00           |         |                |

Table 2

Results for simulations of $\tilde{Z}$ and $Z_{[-k,k]}$, $k = 2, 3$, on \{-R, -R + 0.25, \ldots, R - 0.25, R\} for different $R$. For each case, $A_{R,k}$ and the estimates for $Q_{g^*}$, $\mathbb{E}M_k$ and $P_{R,k}$ as defined in Subsection 6.1 are displayed, based on $N = 2500$ simulations of each process.

| $R$ | $Q_{g^*}$ | $N_2$ | $Q_{g^*}$ | $N_2$ | $A_{R,2}$ | $\hat{P}_{R,2}$ | $N_3$ | $Q_{g^*}$ | $N_3$ | $A_{R,3}$ | $\hat{P}_{R,3}$ |
|-----|----------|-------|----------|-------|----------|----------------|-------|----------|-------|----------|----------------|
| 1   | 8.14     | 14.86 | 0.55     | 0.56  | 0.96     | 26.37         | 0.31  | 0.32     | 0.96  |         |                |
| 2   | 26.32    | 40.17 | 0.66     | 0.66  | 1.00     | 61.07         | 0.43  | 0.42     | 1.03  |         |                |
| 5   | 150.89   | 189.83| 0.79     | 0.80  | 0.99     | 247.10        | 0.61  | 0.61     | 1.00  |         |                |
| 10  | 636.63   | 727.33| 0.87     | 0.88  | 0.99     | 839.55        | 0.76  | 0.75     | 1.01  |         |                |

$K = \{-R, -R + h, \ldots, R - h, R\}^d$, $d = 1, 2$. The density function $g^*$, however, is chosen as if $K$ was the rectangle $[-R, R]^d$.

In the case $d = 1$, for $h = 0.1$ and $R \in \{1, 2, 5, 10, 50, 100\}$ we simulate each process $N = 5000$ times. The values of $Q_{g^*}$ and $\mathbb{E}M_k$ are estimated via the corresponding empirical means denoted by $\hat{Q}_{g^*}$ and $\hat{\mathbb{E}}M_k$. For estimation of $P_{R,k}$ we use the plug-in estimator $\hat{P}_{R,k}$ based on the empirical means of $\sup_{y \in K} \tilde{Z}(y)^{-1}$ and $\sup_{y \in K} Z_{[-k,k]}(y)^{-1}$. The results of the simulation study are shown in Table 1.

First, we note that -- in accordance to Equation (6.1) -- $Q_{g^*}$ is generally smaller than $\mathbb{E}M_k$. For instance, for $R = 1$, the number of considered shape functions is decreased by 29\% ($k = 2$) and 43\% ($k = 3$), respectively. Furthermore, we observe that $\hat{P}_{R,k}$ seems to be almost constant in $R$, namely $\hat{P}_{R,2} \approx 0.95$ and $\hat{P}_{R,3} \approx 1$ which shows that the approximation of $\tilde{Z}$ by $Z_{[-3,3]}$ is largely good for $h = 0.1$. Thus, the behavior of $Q_{g^*}/\mathbb{E}M_k$ is basically driven by $A_{R,k}$ which tends to 1 as $R \to \infty$. For large $R$, $Q_{g^*}/\mathbb{E}M_k \approx P_{R,k}$. Thus, we get the surprising fact that $\mathbb{E}M_2 > \mathbb{E}M_3$ even though the approximation of $\tilde{Z}$ by $Z_{[-2,2]}$ is less accurate than by $Z_{[-3,3]}$.

For $d = 2$, $R \in \{1, 2, 5, 10\}$ and $h = 0.25$, each process is simulated $N = 2500$ times. The results are shown in Table 2. In general, the results are similar to our observations for $d = 1$. However, for $d = 2$ the improvements compared to Schlather’s (2002) algorithm are even more distinct. In the case $R = 1$, the
number of considered spectral functions is decreased by 45% ($k = 2$) and 69% ($k = 3$), respectively. However, the results of the algorithm by Schlather (2002) seem to be quite accurate even for $k = 2$ as $P_{R,k}$ suggests.

7. Summary and Discussion

Whilst in the definition of a max-stable process an infinite number of spectral functions is involved, the minimal number of spectral functions that are actually to be considered in a simulation is an open problem. We consider two substitution problems, problems (3.11) and (4.6), and show that the unique normalized spectral representation is a solution in both cases. Although we feel that problem (4.6) is rather close to the original problem (3.5), it remains unclear whether the normalized spectral representation is also the solution to the original one. It is even not known whether different initial choices of the spectral representation in (1.1) may lead to the same solution via renormalizations $g$ in (2.2) and whether the solution is unique. This is left for future research.

Section 6 reveals two remarkable facts: (i) the potential of the approach based on the normalized spectral representation to improve the algorithm of Schlather (2002) and (ii) the occasional occurrence of a smaller number of considered shape functions in a better approximation. Neither a careful coding that exploits our fundamental results seems to be straightforward nor are the implications on the real running times foreseeable. This is also left for future research.

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