Naked Singularity in the Sultana-Dyer Space-Time

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Abstract

How to describe a black hole embedded in an expanding universe is an important topic. Some models about this issue are suggested by assuming that the metric is a conformal transformation of the Schwarzschild metric or of the isotropic black hole metric. However, there exists naked singularities in the two metrics. Recently, it is argued that the singularity in the Sultana-Dyer space-time is covered by an apparent horizon surface. But we find that such an apparent horizon does not exist if the null energy condition holds.

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1 Introduction

Isolated black holes have been investigated in great depth and detail for more than forty years. On the other hand, black holes embedded in the background of the expanding universe are also important and even more realistic situations. Some works on the issue have been carried out. In [1], McVitie found a celebrated space-time describing a black hole embedded in the Friedman-Robertson-Walker universe, which is generalized to the Reissner-Nordström case in [2]. In [3], the model of Swiss cheese black holes is shown. In [4], the author suggested the Vaidya’s space-time describing a FRW universe with a Schwarzschild-like black hole that does not expand with the rest of the universe, which is generalized to the Kerr-Newman case in [5]. In [6], the Thakurta’s

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A black hole is shown. In [7], the Sultana-Dyer black hole metric is suggested. Recently, in [8, 9], the authors suggested new solutions describing black holes embedded in the expanding universe.

Most of the models with the spherical symmetry on the topic are based on the assumption

\[ ds^2 = - \left( \frac{1 - \frac{m(t)}{2r}}{1 + \frac{m(t)}{2r}} \right)^2 dt^2 + a^2(t) \left( 1 + \frac{m(t)}{2r} \right)^4 (dr^2 + r^2 d\Omega^2), \]

or

\[ ds^2 = - \left( 1 - \frac{2m(t)}{r} \right) dt^2 + a^2(t) \left( \frac{dr^2}{1 - \frac{2m(t)}{r}} + r^2 d\Omega^2 \right), \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). Here and after, we take \( c = G = \hbar = 1 \). However, the space-time manifold described by the line element of Eq. (1) is singular at surfaces \( r = m/2 \). And there also exists a singularity in the spacetime of Eq. (2) at the surface \( r = 2m \).

Then we may ask whether the singularity in the spacetime (1) or (2) is a spacetime singularity or not? If it is a proper singularity, is the singularity naked or covered by an apparent horizon? In Ref. [11, 12], the authors claimed that, in the case of constant \( m \), the singularity is covered by an apparent horizon. However, we find such an apparent horizon does not exist if the null energy condition is satisfied.

In the paper, firstly, we show that the space-time (1) or (2) is properly singular at the surface \( r = m(t)/2 \) or \( r = 2m(t) \) respectively. Secondly, we show the apparent horizon suggested in Ref. [11, 12] does not exist. Section 4 contains conclusions and discussion.

### 2 Proper singularity

The metric of the Mcvittie space-time [1] is assumed as Eq. (1) with \( m(t) = \mu/a(t) \), where \( \mu \) is a constant parameter. In [7, 8], the metric is assumed as Eq. (1), but \( m(t) \) is taken to be constant. Here, we note that, if \( \dot{m} \equiv dm/dt = 0 \), the metrics in Eq. (2) and Eq. (1) define the same space-time. Generally, for \( \dot{m} \neq 0 \), the two metrics represent different spacetimes. In Ref. [9], the imperfect fluid stress-energy tensor is used to solve the Einstein equations. In these models, at the surface \( r = m(t)/2 \), the spacetimes are singular. Let us show this by calculating the scalar curvature of the metric in Eq. (1),

\[ R_I = \frac{1}{B^3r} \left\{ 6A^2Br \left( \frac{\ddot{a}}{a^2} + \frac{a}{a} \right) + 3A(A + 9B) \frac{\dot{a}}{a} \dot{m} + 3(4B - A) \frac{m^2}{r} + 6AB\ddot{m} \right\} \]

where

\[ A = 1 + \frac{m(t)}{2r}, \quad B = 1 - \frac{m(t)}{2r}, \]

and

\[ B = \frac{1}{2r}, \quad R_I = \frac{1}{B^3r} \left\{ 6A^2Br \left( \frac{\ddot{a}}{a^2} + \frac{a}{a} \right) + 3A(A + 9B) \frac{\dot{a}}{a} \dot{m} + 3(4B - A) \frac{m^2}{r} + 6AB\ddot{m} \right\} \]
and \( \dot{a} \equiv \frac{da}{dt} \) and \( \ddot{a} \equiv \frac{d^2a}{dt^2} \).

Obviously, in the limiting cases corresponding to the standard FRW universe, the standard Schwarzschild black hole or the Schwarzschild-de Sitter black hole, the scalar curvature \( R_I \) is finite. However, in this paper, we exclude these limiting cases. Then for \( \dot{m} = 0 \), \( R_I \) is infinity at the surface \( r = m/2 \). Even for \( \dot{m} \neq 0 \), we find that it is almost impossible to find a time-dependent function \( m(t) \) to avoid the divergence at \( r = m(t)/2 \). The divergence of \( R_I \) at \( r = m/2 \) implies the singularity is proper.

The Thakurta spacetime [6] is based on the line element in Eq. (2). The scalar curvature of the metric (2) is

\[
R_S = \frac{2}{C} \left\{ 3\left( \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) + 4\left( \frac{\dot{m}^2}{C^2r^2} + \frac{1}{C} \left( \frac{\dot{a}}{a} \frac{\dot{m}}{r} + \frac{\ddot{m}}{r} \right) \right) \right\},
\]

where

\[
C = 1 - \frac{2m(t)}{r}.
\]

Still, we exclude the limiting cases corresponding to the standard FRW universe, the standard Schwarzschild black hole and the Schwarzschild-de Sitter black hole. Then, similarly to the metric (1), the scalar curvature \( R_S \) at the surface \( r = 2m(t) \) is divergent. So the singularity at \( r = 2m \) is proper.

### 3 Naked Singularity

In [11, 12], the authors showed that, in the case of \( \dot{m} = 0 \), the singularity at the surface \( r = m/2 \) or \( r = 2m \) in the space-time (1) or (2) respectively is covered by an apparent horizon. In this section we will show such an apparent horizon does not exist for the case of \( \dot{m} = 0 \) if the null energy condition holds.

First, we show the outgoing light emitted from the surface \( r > m/2 \) in Eq. (1) or \( r > 2m \) in Eq. (2) can reach the future null infinity asymptotically if \( \dot{m} = 0 \). Considering the world line of a photon propagating along the direction of radius emitted from the point \((t = t_1, r = r_1)\) with constant \( \theta \) and \( \phi \), we have

\[
ds^2 = 0 = -\left( \frac{B}{A} \right)^2 dt^2 + a^2(t)A^4 dr^2,
\]

or

\[
ds^2 = 0 = -Cdt^2 + \frac{a^2(t)}{C} dr^2,
\]

For an outgoing photon which arrives \( r = r_2 > r_1 \) at the moment \( t = t_2 > t_1 \), we obtain respectively

\[
\int_{t_1}^{t_2} \frac{dt}{a} = \int_{r_1}^{r_2} \frac{(1 + \frac{m}{2r})^3}{1 - \frac{m}{2r}} dr
\]
Fig. 1: The conformal diagram of the Sultana-Dyer spacetime. The horizontal wiggly line at the bottom describes the Big Bang singularity, the wiggly line at 45 degrees denotes the $R = 2M$ null singularity, and the solid line at 45 degrees describes future null infinity. Null geodesics end at future null infinity or at the black singularity (either when it is naked if started early on, or crossing the timelike black hole apparent horizon labelled AH).

or

$$\int_{t_1}^{t_2} \frac{dt}{a} = \int_{r_1}^{r_2} \frac{dr}{1 - \frac{2m}{r}}$$ (10)

Of course, we assume $r_1 > m/2$ in Eq.(9) and $r_1 > 2m$ in Eq.(10), and $a(t)$ in the two equations.

Generally, since the Einstein equations which determine $a(t)$ and $m(t)$ are unknown, nothing can be concluded from the equation (9) and (10). But for the special case of $\dot{m} = 0$, the integral on the right-hand sides of Eq.(9) is finite for finite $r_2$. Then, in the spacetime of Eq.(11), the light emitted from the point at the surface $r = r_1 > m/2$ will reach the surface $r = r_2 > r_1$ in the finite time. This implies that outgoing photons emitted outside the surface $r = m/2$ can reach the future null infinity asymptotically in the spacetime (1) if $\dot{m} = 0$. In fact, this conclusion can be obtained easily from the conformal diagram of the Sultana-Dyer space-time, Eq.(11) with $\dot{m} = 0$, given in FIG.1 of Ref.[12] which is displayed as Fig.1 in our paper (See Ref.[12] for details). For the space-time (2), similar conclusion can be made from Eq.(10).

Here, we note that, generally, to obtain our conclusion, we should use the affine parameter $\tau$ in Eq.(11) or Eq.(10) instead of the coordinate time $t$. But, fortunately, here the coordinate time $t$ is appropriate for our conclusion, since finite $t$ always indicates finite $\tau$ along the worldline of an outgoing photon outside the surface $r = m/2$ or $r = 2m$ in the spacetime (1) or (2) respectively.

Now let’s assume that, outside the surface $r = m/2$ in the spacetime (1) or the surface $r = 2m$ in the spacetime (2) respectively, there exists an apparent horizon defined by the equation $f(t, r) = 0$, if $\dot{m} = 0$. Then at the moment $t = t_0$, a space-like two surface is defined by $f(t_0, r) = 0$, and the expansion of the null generators of the future causal boundary of the two surface, $\theta$, is
zero at the two surface. Due to the analysis in the last paragraph, we know outgoing photons emitted from the surface \( f(t_0, r) = 0 \) will reach the future null infinity asymptotically if \( \dot{m} = 0 \). Then, naturally, the outgoing future causal boundary of the two surface intersects the future null infinity.

However, on the other hand, it is known that the Raychaudhuri equation [15, 16, 17] implies that the future causal boundary of the two surface \( f(t_0, r) = 0 \) cannot intersect the future null infinity because \( \theta \) is zero at the two surface \( f(t_0, r) = 0 \) and is nonincreasing if the null energy condition holds

\[
T_{ab}l^al^b \geq 0. \tag{11}
\]

\( T_{ab} \) is the energy-momentum tensor and \( l^a \) is a null vector.

Then we obtain two contrary conclusions from the assumption of the existence of the apparent horizon. This indicates that in the spacetime (1) or (2), outside the surface \( r = m/2 \) or \( r = 2m \) respectively, there does not exist any apparent horizon and the singularity at \( r = m/2 \) or \( r = 2m \) is naked, if the null energy condition holds and \( \dot{m} = 0 \).

In particular, in the model of the Sultana-Dyer black hole [7], the energy-momentum tensor is

\[
T_{ab} = T_{ab}^{(I)} + T_{ab}^{(II)} \tag{12}
\]

where \( T_{ab}^{(I)} = \rho_{I}u_au_b \) describes an ordinary massive dust and \( T_{ab}^{(II)} = \rho_{n}k_ak_b \) describes a null dust with density \( \rho_{n} \) and \( k_ak_b = 0 \). Obviously, the null energy condition is satisfied for any null vector \( l^a \). Then we conclude that in the space time of the Sultana-Dyer black hole, there is no apparent horizon outside the surface \( r = m/2 \) in Eq.(1) and the singular surface \( r = m/2 \) is naked. The conclusion is different to that in [12].

4 Conclusion and Discussion

The issue on the solutions of the Einstein equation describing black holes embedded in the FRW universe is very important. Some spacetime models on this issue [8, 9, 19] have been suggested. Most of the models are based on the assumption of Eq.(2) or Eq.(1). However, there exists singularity in the spacetime of Eq.(2) or Eq.(1). The singularity cannot be eliminated even for the imperfect fluid stress-energy tensor to be used to solve the Einstein equations [9]. But it is argued that the singularity in the spacetime (1) at \( r = m/2 \) is gravitationally weak [13] in the sense that it does not crush extended bodies which falls into it to zero volume [14]. In this view, the line elements (1) and (2) may still make some sense in physics.

In [12], the authors claimed that, for \( \dot{m} = 0 \), there exists a regular apparent horizon \( R_{bh} \) inside which the surface \( r = m/2 \) in Eq.(1) is contained. However, our analysis above shows that photons emitted outside the surface \( r = m/2 \) can reach the future null infinity if \( \dot{m} = 0 \). So the outgoing future causal boundary of the apparent horizon will intersect the future null infinity. However, if
the null energy condition holds, it is impossible for the future causal boundary of the apparent horizon to intersect the future null infinity because of the non-decreasing expansion of the null generators of the causal boundary indicates by the Raychaudhuri equation [15, 16, 17]. Then such an apparent horizon does not exist, and the singular surface is naked still if $\dot{m} = 0$. The conclusion in [11, 12] may be due to the special coordinate $(T, R, \theta, \phi)$. Actually, the coordinate transformation between $(T, R, \theta, \phi)$ and the ones in Eq. (1) is not diffeomorphic globally, which is singular at the apparent horizon $R_{bh}$.

Our conclusion for the case of $\dot{m} = 0$ cannot be applied to that of $\dot{m} \neq 0$ directly. However, since our analysis is based on two conditions: the null energy condition holds and outgoing photons emitted from the apparent horizon (if exists) can reach the future null infinity. Then it is still possible for the existence of the apparent horizon outside the surface $r = m/2$ or $r = 2m$ in the space-time (1) or (2) respectively, if the null energy condition is violated or outgoing photons emitted from the apparent horizon (if exists) cannot reach the future null infinity.

Acknowledgments

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