ON THE STRUCTURE OF COVARIANT PHASE OBSERVABLES

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ABSTRACT. We study the mathematical structure of covariant phase observables. Such observables can alternatively be expressed as phase matrices, as sequences of unit vectors, as sequences of phase states, or as equivalence classes of covariant trace-preserving operations. Covariant generalized operator measures are defined by structure matrices which form a $W^*$-algebra with phase matrices as its subset. The properties of the Radon-Nikodým derivatives of phase probability measures are studied.

1. Introduction

Covariant phase observables constitute a particular solution to the problem of quantum phase (see, e.g. [1, 2]). In this paper, we study general mathematical properties of covariant phase observables and represent them as covariant trace-preserving operations (Sec. 3). We also analyze the structure matrix $W^*$-algebra of covariant generalized operator measures (Sec. 4) and the pointwise convergence of phase probability densities (Sec. 5)
Let $\mathcal{H}$ be a complex Hilbert space with a fixed basis $\{|n\rangle \in \mathcal{H} \mid n \in \mathbb{N}\}$. Define the number operator $N := \sum_{n=0}^{\infty} n \, |n\rangle \langle n|$ with its usual domain $\mathcal{D}(N) := \{\psi \in \mathcal{H} \mid n^2|\langle n|\psi\rangle|^2 < \infty\}$ and the phase shifter $R(\theta) := e^{i\theta N}$ for all $\theta \in \mathbb{R}$. Let $\mathcal{L}(\mathcal{H})$, $\mathcal{T}(\mathcal{H})$, and $\mathcal{T}(\mathcal{H})_1^+$ denote the sets of bounded operators, trace-class operators, and states (positive trace-one operators) on $\mathcal{H}$, respectively.

Let $\mathcal{B}([0, 2\pi])$ denote the $\sigma$-algebra of the Borel subsets of $[0, 2\pi)$, and consider an operator measure $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$. The measure $E$ is normalized if $E([0, 2\pi)) = I$, positive if $E(X) \geq O$ for all $X \in \mathcal{B}([0, 2\pi))$, and phase shift covariant if $R(\theta)E(X)R(\theta)^* = E(X \oplus \theta)$ for all $X \in \mathcal{B}([0, 2\pi))$ and for all $\theta \in [0, 2\pi)$, where $X \oplus \theta := \{x \in [0, 2\pi) \mid (x - \theta) \text{ (mod } 2\pi) \in X\}$. A phase shift covariant normalized positive operator measure is called a (covariant) phase observable.

In the next section we collect some known properties of covariant phase observables. The new results are contained in Sections 3-5.

### 2. The Structure of Phase Observables

Any covariant phase observable is of the (weakly convergent) form

\begin{equation}
E(X) = \sum_{n,m=0}^{\infty} c_{n,m} i_{n-m}(X) |n\rangle \langle m|, \quad X \in \mathcal{B}([0, 2\pi)),
\end{equation}

where $i_k(X) := (2\pi)^{-1} \int_X e^{it\theta} d\theta$ for all $k \in \mathbb{Z}$, and where the phase matrix $(c_{n,m})_{n,m \in \mathbb{N}}$ is a positive semidefinite (complex) matrix with...
$c_{n,n} = 1, \ n \in \mathbb{N}$ (see, e.g. Phase Theorem 2.2 of [3]). A complex matrix $(c_{n,m})$ is a phase matrix if and only if there exist a sequence $(\psi_n)_{n \in \mathbb{N}}$ of unit vectors such that $c_{n,m} = \langle \psi_n | \psi_m \rangle$, $n, m \in \mathbb{N}$ [3]. A constant sequence, e.g. $\psi_n = |0\rangle$, $n \in \mathbb{N}$, defines the canonical phase observable

$$E_{\text{can}}(X) := \sum_{n,m=0}^{\infty} i_{n-m}(X) |n\rangle \langle m|, \ X \in \mathcal{B}([0, 2\pi]),$$

whereas any orthonormal sequence, e.g. $\psi_n = |n\rangle$, $n \in \mathbb{N}$, gives the trivial phase observable

$$E_{\text{triv}}(X) := i_0(X) I, \ X \in \mathcal{B}([0, 2\pi]).$$

Next we show how any phase observable can be constructed by using a sequence of phase states (Theorem 3).

Define $\mathcal{H}_1 := \{ \psi \in \mathcal{H} \mid \sum_{n=0}^{\infty} |\langle n|\psi \rangle| < \infty \}$. A phase matrix $(c_{n,m})$ can be interpreted as a phase kernel, that is, a positive (possible unbounded in the norm of $\mathcal{H}$) sequilinear form $C : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ defined as

$$C(\varphi, \psi) := \sum_{n,m=0}^{\infty} c_{n,m} \langle \varphi | n \rangle \langle m | \psi \rangle, \ \varphi, \psi \in \mathcal{H}_1,$$

where the sum converges absolutely. Keeping this in mind, we may formally write

$$C = \sum_{n,m=0}^{\infty} c_{n,m} |n\rangle \langle m|.$$
Since $R(\theta)\mathcal{H}_1 = \mathcal{H}_1$ for all $\theta \in [0, 2\pi)$ we can define a continuous integrable function $[0, 2\pi] \to \mathbb{C}$

$$\theta \mapsto C(R(-\theta)\varphi, R(-\theta)\psi) = \sum_{n,m=0}^{\infty} c_{n,m} e^{i(n-m)\theta} \langle \varphi | n \rangle \langle m | \psi \rangle$$

for all $\varphi, \psi \in \mathcal{H}_1$, and thus a bounded positive sesquilinear form $\mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$

$$(\varphi, \psi) \mapsto E(X)_{\varphi,\psi} := \frac{1}{2\pi} \int_X C(R(-\theta)\varphi, R(-\theta)\psi) d\theta = \sum_{n,m=0}^{\infty} c_{n,m} i_{n-m}(X) \langle \varphi | n \rangle \langle m | \psi \rangle$$

for all $X \in \mathcal{B}([0, 2\pi))$. The form $(\varphi, \psi) \mapsto E(X)_{\varphi,\psi}$ has a unique bounded positive extension to $\mathcal{H} \times \mathcal{H}$ which is determined by a unique bounded operator, say, $E(X) \in \mathcal{L}(\mathcal{H})$. Operators $E(X)$, $X \in \mathcal{B}([0, 2\pi))$, constitute a covariant phase observable. The following route to define a phase observable is thus justified:

1. take a phase matrix $(c_{n,m})$ and define the phase kernel

$$\sum_{n,m=0}^{\infty} c_{n,m} |n\rangle \langle m|;$$

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Since for all $\varphi, \psi \in \mathcal{H}_1$ the series of continuous integrable functions

$$\sum_{n=0}^{s} \sum_{m=0}^{t} c_{n,m} e^{in\theta} \langle \varphi | n \rangle \langle m | \psi \rangle,$$

where $e_n(\theta) = e^{in\theta}$, $n \in \mathbb{N}$, $\theta \in \mathbb{R}$, converges uniformly on $[0, 2\pi]$ when $s, t \to \infty$, the function $\theta \mapsto C(R(-\theta)\varphi, R(-\theta)\psi)$ is continuous and integrable.
2. act on it by $R(\theta)$ to get
\[
R(\theta) \sum_{n,m=0}^{\infty} c_{n,m} |n\rangle \langle m| R(\theta)^* = \sum_{n,m=0}^{\infty} c_{n,m} e^{i(n-m)\theta} |n\rangle \langle m| ;
\]

3. integrate it over $X \in B([0, 2\pi))$ to get a bounded sesquilinear form $\mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C},$
\[
\frac{1}{2\pi} \int_X R(\theta) \sum_{n,m=0}^{\infty} c_{n,m} |n\rangle \langle m| R(\theta)^* d\theta = \sum_{n,m=0}^{\infty} c_{n,m} i^{n-m}(X) |n\rangle \langle m| ;
\]

4. this has a unique bounded extension $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ which defines the phase observable
\[
E(X) := \sum_{n,m=0}^{\infty} c_{n,m} i^{n-m}(X) |n\rangle \langle m| .
\]

Let $\mathcal{H}_\infty$ be a complex Banach space of vectors $\sum_{n=0}^{\infty} g_n |n\rangle$ for which the norm $\|\sum_{n=0}^{\infty} g_n |n\rangle\|_\infty := \sup \{|g_n| \mid n \in \mathbb{N}\} < \infty$. Embedding $\mathcal{H}$

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2 Equip $\mathcal{H}_1$ with the norm $\psi \mapsto \|\psi\|_1 := \sum_{n=0}^{\infty} |\langle n|\psi\rangle|$. The continuous linear mappings $\mathcal{H}_1 \to \mathbb{C}$ form a topological dual $\mathcal{H}'_1$ of $\mathcal{H}_1$. Using the Dirac notation, an element $(F) \in \mathcal{H}'_1$ can be represented in the form $(F) = \sum_{n=0}^{\infty} f_n |n\rangle$ where $(f_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ and $\sup \{|f_n| \mid n \in \mathbb{N}\} < \infty$. Defining a conjugate form $(F)$ of $(F)$ as a mapping $\mathcal{H}_1 \ni \psi \mapsto \langle F|\psi\rangle \in \mathbb{C}$ we may define the linear space $\mathcal{H}_\infty$ of conjugate forms of the elements of $\mathcal{H}'_1$. Thus, using the Dirac formalism, we may write an element $(G) \in \mathcal{H}_\infty$ of the form $(G) = \sum_{n=0}^{\infty} g_n |n\rangle$ where $(g_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ and $\sup \{|g_n| \mid n \in \mathbb{N}\} < \infty$. We can define the following norm in $\mathcal{H}_\infty$: $(G) \mapsto \|G\|_\infty := \sup \{|g_n| \mid n \in \mathbb{N}\}$. 


in \mathcal{H}_\infty we get the following triplet

\[ \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_\infty. \]

We have the following theorem [4]:

**Theorem 1.** For any phase matrix \((c_{n,m})\)

\[
\sum_{n,m=0}^{\infty} c_{n,m} \langle \varphi | n \rangle \langle m | \psi \rangle = \sum_{k=0}^{\infty} \langle \varphi | F_k \rangle (F_k | \psi \rangle),
\]

for all \(\varphi, \psi \in \mathcal{H}_1\), that is, briefly,

\[
\sum_{n,m=0}^{\infty} c_{n,m} | n \rangle \langle m | = \sum_{k=0}^{\infty} | F_k \rangle (F_k |
\]

where \(| F_k \rangle \in \mathcal{H}_\infty \) for all \(k \in \mathbb{N}\) and \(\sum_{k=0}^{\infty} | \langle n | F_k \rangle |^2 = 1 \) for all \(n \in \mathbb{N}\).

Conversely, if \((| F_k \rangle)_{k \in \mathbb{N}} \subset \mathcal{H}_\infty \) is such that \(\sum_{k=0}^{\infty} | \langle n | F_k \rangle |^2 = 1 \) then \(\sum_{k=0}^{\infty} | F_k \rangle (F_k |\) is a phase kernel.

Let \(| F \rangle \in \mathcal{H}_\infty \) and define \(| F; \theta \rangle := R(\theta) | F \rangle \) and \((F; \theta) := (F | R(\theta)^* \) for all \(\theta \in \mathbb{R}\). Since \(R(\theta') | F; \theta \rangle = | F; \theta + \theta' \rangle \) we say that \(| F; \theta \rangle \) is a phase state. It easy to see that the following sesquilinear form \(\mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}, \)

\[
(\varphi, \psi) \mapsto \frac{1}{2\pi} \int_X \langle \varphi | F; \theta \rangle (F; \theta | \psi \rangle d\theta,
\]
is positive and bounded for all $X \in \mathcal{B}([0, 2\pi])$ and it defines a covariant positive operator measure

\begin{equation}
\mathcal{B}([0, 2\pi]) \ni X \mapsto E_F(X) = \sum_{n,m=0}^{\infty} \langle n|F)(F|m)i_{n-m}(X) \langle m| \in \mathcal{L}(\mathcal{H}).
\end{equation}

The operator measure $E_F$ is normalized, that is, a phase observable, if and only if $|\langle n|F]\rangle = 1$ for all $n \in \mathbb{N}$, that is, when

\[ |F) = \sum_{n=0}^{\infty} e^{iv_n}|n\rangle \]

where $(v_n)_{n \in \mathbb{N}} \subset [0, 2\pi)$. Let $U := \sum_{n=0}^{\infty} e^{iv_n} |n\rangle \langle n|$. Then $E_F$ is a phase observable if and only if

\[ E_F(X) = UE_{\text{can}}(X)U^*, \quad X \in \mathcal{B}([0, 2\pi]). \]

If, for two phase observables $E_1$ and $E_2$, the condition $E_1(X) = UE_2(X)U^*, \quad X \in \mathcal{B}([0, 2\pi])$, holds, we say that $E_1$ is $E_2$ up to unitary equivalence, or briefly, $E_1$ is $E_2$ (u.e.). Thus, using Theorem 1 we get a variant of Phase Theorem 2.2 of [2]:

**Theorem 2.** $E$ is a phase observable if and only if for all $X \in \mathcal{B}([0, 2\pi])$

\[ E(X) = \lim_{n \to \infty} \sum_{k=0}^{n} E_{F_k}(X) \]
where $E_{F_k}(X)$ is the bounded operator defined by a sesquilinear form

$$\frac{1}{2\pi} \int_X |F_k; \theta \rangle \langle F_k; \theta| d\theta$$

where $|F_k\rangle \in \mathcal{H}_\infty$, $k \in \mathbb{N}$, and $\sum_{k=0}^\infty |\langle n | F_k \rangle|^2 = 1$.

The phase observable $E$ is defined by a single phase state if and only if $E$ is $E_{\text{can}}$ (u.e.).

Since the sequence $n \mapsto \sum_{k=0}^n E_{F_k}(X)$ is increasing $E(X) = \text{s-lim}_{n \to \infty} \sum_{k=0}^n E_{F_k}(X)$ also.

3. Phase observables as operations

A linear mapping $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is a covariant trace-preserving operation if it is covariant $(R(\theta)\Phi(T)R(\theta)^*) = \Phi(R(\theta)TR(\theta)^*)$, $\theta \in [0, 2\pi)$, $T \in \mathcal{T}(\mathcal{H})$), trace-preserving ($\text{tr}(\Phi(T)) = \text{tr}(T)$, $T \in \mathcal{T}(\mathcal{H})$), and positive ($\Phi(\mathcal{T}(\mathcal{H})^+) \subseteq \mathcal{T}(\mathcal{H})^+$) (for the theory of operations, see e.g. [5, 6]). We prove next a theorem essentially due to Hall and Fuss [7, 8].

**Theorem 3.** A mapping $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$ is a phase observable if and only if

$$\text{tr}(TE(X)) = \text{tr}(\Phi(T)E_{\text{can}}(X))$$

for all $X \in \mathcal{B}([0, 2\pi))$ and $T \in \mathcal{T}(\mathcal{H})$ where $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is a covariant trace-preserving operation.
Proof. Let $E$ be a phase observable with the phase matrix $(c_{n,m})$. For all $T \in \mathcal{T}(\mathcal{H})$ define

$$\Theta(T) := \sum_{n,m=0}^{\infty} c_{m,n} T_{n,m} |n\rangle \langle m|.$$  

(4)

Since $T = \alpha T_\alpha - \beta T_\beta + i \gamma T_\gamma - i \delta T_\delta$ where $T_\alpha$, $T_\beta$, $T_\gamma$, and $T_\delta$ are states, and $\alpha$, $\beta$, $\gamma$, and $\delta$ are nonnegative real numbers, it suffices to consider only states. Thus, assume that $T$ is a state. Since $\sup \{ |\langle \varphi | \Theta(T) \psi \rangle| \mid \| \varphi \| \leq 1, \| \psi \| \leq 1 \} \leq \sup \{ \sum_{n,m=0}^{\infty} |T_{n,m}| |\langle \varphi |n\rangle| |\langle m| \psi \rangle| \mid \| \varphi \| \leq 1, \| \psi \| \leq 1 \}$ it follows that $\Theta(T)$ is a bounded operator. Using a decomposition $T = \sum_{j=0}^{\infty} |\phi_j\rangle \langle \phi_j|$, $\phi_j \in \mathcal{H}$, $j \in \mathbb{N}$, one sees that $\langle \psi | \Theta(T) \psi \rangle = \sum_{j=0}^{\infty} \sum_{n,m=0}^{\infty} \overline{\langle m| \phi_j \rangle\langle \psi |m\rangle} c_{m,n} \langle n| \phi_j \rangle \langle \psi |n\rangle \geq 0$ for all $\psi \in \mathcal{H}_1$ and, thus, $\Theta$ is positive. Since $\sum_{n=0}^{\infty} \langle n| \Theta(T) |n\rangle = 1$, $\Theta(T)$ is a trace-one operator. Moreover, $\text{tr}(TE(X)) = \text{tr}(\Theta(T)E_{\text{can}}(X))$, $X \in \mathcal{B}([0, 2\pi])$, and $\Theta$ is covariant. Thus, $\Theta$ is a covariant trace-preserving operation.

The converse part is trivial. \qed

There are many covariant trace-preserving operations $\Phi$ which satisfy Equation (3) for a given $E$. One such operation $\Theta$ is defined in (4). It is the identity operation in the case of the canonical phase whereas for the trivial phase it is of the form $\Theta(T) = \sum_{n=0}^{\infty} T_{n,n} |n\rangle \langle n|$. We note also that, in the case of the trivial phase, $T \mapsto T_{0,0} |1\rangle \langle 1| + T_{1,1} |0\rangle \langle 0| +$
\[ \sum_{n=2}^{\infty} T_{n,n} |n\rangle \langle n| \] is an other operation fulfilling Theorem 3. Since the diagonal elements \( T_{n,n} \) do not "contain" any phase information of the state \( T \) we see that the trivial phase "loses" all phase information. In the general case, if \( c_{n,m} = 0 \) for some \( n \neq m \), there are vector states (other than number states) \( \psi := d_n |n\rangle + d_m |m\rangle \), \( d_n, d_m \in \mathbb{C} \setminus \{0\} \), \( |d_n|^2 + |d_m|^2 = 1 \), for which the probability measure \( X \mapsto \langle \psi | E(X) \psi \rangle \) is random. Next we study the properties of \( \Theta \).

Let \( E \) be a phase observable with the phase matrix \( (c_{n,m}) \), and let \( \Theta(T) = \sum_{n,m=0}^{\infty} c_{n,m} T_{n,m} |n\rangle \langle m| \) for all \( T \in \mathcal{T}(\mathcal{H}) \). The dual mapping \( \Theta^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}) \) of an operation \( \Theta \) defined by the relation \( \text{tr}(T \Theta^*(A)) = \text{tr}(\Theta(T)A), A \in \mathcal{L}(\mathcal{H}), T \in \mathcal{T}(\mathcal{H}) \), is a positive linear mapping, and

\[
\Theta^*(A) = \sum_{n,m=0}^{\infty} c_{n,m} A_{n,m} |n\rangle \langle m|, \quad A \in \mathcal{L}(\mathcal{H}).
\]

From Theorem 4 one gets (weakly)

\[
\Theta(T) = \sum_{k=0}^{\infty} A_k T A_k^*, \quad T \in \mathcal{T}(\mathcal{H}),
\]

where \( A_k := \sum_{n=0}^{\infty} (F_k |n\rangle \langle n| \) for all \( k \in \mathbb{N} \) showing that \( \Theta \) is completely positive (see the First Representation Theorem of [3]). Note that \( \sum_{k=0}^{\infty} A_k A_k^* = I \) and \( \Theta^*(A) = \sum_{k=0}^{\infty} A_k^* A_k, A \in \mathcal{L}(\mathcal{H}) \).

Let \( \Theta_1^+ : \mathcal{T}(\mathcal{H})_1^+ \rightarrow \mathcal{T}(\mathcal{H})_1^+ \) be the restriction of \( \Theta \) to the set of states.
Theorem 4. 1. \( \Theta \) and \( \Theta^+ \) are injections if and only if \( c_{n,m} \neq 0 \) for all \( n, m \in \mathbb{N} \);

2. \( \Theta^+ \) is surjection if and only if \( E \) is \( E_{\text{can}} \) (u.e.);

3. \( \Theta^+ \) is bijection if and only if \( E \) is \( E_{\text{can}} \) (u.e.);

4. \( \Theta \) preserves pure states \( (\Theta(|\psi\rangle \langle \psi|)^2 = \Theta(|\psi\rangle \langle \psi|) \) for all unit vectors \( \psi \in \mathcal{H} \) if and only if \( E \) is \( E_{\text{can}} \) (u.e.).

Proof. It is easy to see that \( \Theta \) and \( \Theta^+ \) are injections if and only if 
\( c_{m,n}T_{n,m} = 0 \) for all \( n, m \in \mathbb{N} \) where \( T \in \mathcal{T}(\mathcal{H}) \) implies that \( T = O \). Thus, \( \Theta \) and \( \Theta^+ \) are injections if and only if \( c_{n,m} \neq 0 \) for all \( n, m \).

Suppose that \( \Theta^+ \) is surjection. If \( c_{m,n} = 0 = c_{n,m} \) for some \( n \neq m \) then \( \Theta^+ (T) \neq T' := (|n\rangle + |m\rangle)(\langle n| + \langle m|)/2 \) for all \( T \in \mathcal{T}(\mathcal{H})^+ \) and, thus, \( c_{n,m} \neq 0 \) for all \( n, m \) and \( \Theta^+ \) is injection and bijection.

If \( |c_{n,m}| < 1 \) for some \( n \neq m \) then there is no state \( T \) such that \( \Theta^+ (T) = T' \). Thus, \( |c_{n,m}| = 1, n, m \in \mathbb{N} \), and \( E = E_{\text{can}} \) (u.e.). This proves items (2) and (3).

Let \( \psi := \sum_{n=0}^{\infty} d_n |n\rangle \) where \( d_n > 0 \) for all \( n \) and \( \sum_{n=0}^{\infty} d_n^2 = 1 \). Now \( \Theta(|\psi\rangle \langle \psi|)^2 = \Theta(|\psi\rangle \langle \psi|) \) implies that \( \sum_{n=0}^{\infty} |c_{n,m}|^2 d_n^2 = 1 \) for all \( m \) which shows that \( |c_{n,m}| = 1, n, m \in \mathbb{N} \), and \( E = E_{\text{can}} \) (u.e.). This completes the proof. \( \square \)
4. Covariant GOMs and phase matrices

The standard way to represent an observable in quantum mechanics is to find an appropriate self-adjoint operator, or an idempotent POM, which describes that observable. However, in many cases this representation is too narrow and it is convenient to give up the idempotency (see, e.g. [9]). The strength of POMs is that they associate a probability measure to all states. If we restrict ourselves to a subset of (vector) states to be called physical states we can give up the positivity of POM and require that the operator measure gives a probability measure (via trace formula) only for physical states. Actually, we do not have to assume that the observable can even be "defined" for other states that physical ones. Hence, define a set of physical states $\mathcal{V}$. It is a linear subspace of the Hilbert space of the physical system. The linearity is assumed because of the possibility to superpose the physical states. Let $\mathcal{SL}(\mathcal{V}, \mathcal{V}; \mathbb{C})$ be the set of sesquilinear forms from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{C}$ (the first argument is antilinear). A generalized operator measure $[10]$, or a GOM, $G$ is the mapping from the $\sigma$-algebra $\mathcal{A}$ of the set of measurement outcomes $\Omega$ to $\mathcal{SL}(\mathcal{V}, \mathcal{V}; \mathbb{C})$ such that $\mathcal{A} \ni X \to [G(X)](\varphi, \psi) \in \mathbb{C}$ is a complex measure for all $\varphi, \psi \in \mathcal{V}$. It is normalized if $[G(\Omega)](\varphi, \psi) = \langle \varphi | \psi \rangle$, $\varphi, \psi \in \mathcal{V}$. 
In the case of phase, it is natural to assume that $\Omega = [0, 2\pi)$, $\mathcal{A} = \mathcal{B}([0, 2\pi))$, and $\mathcal{V}$ contains number states, coherent states, etc. Since they are elements of $\mathcal{H}_1$ we assume that $\mathcal{V} = \mathcal{H}_1$. If we study the coherent state phase measurements with the associated GOM $E : \mathcal{B}([0, 2\pi)) \to \mathcal{SL}(\mathcal{H}_1, \mathcal{H}_1; \mathbb{C})$, it is natural to assume the following phase shift covariance condition:

\[
[E(X)](|ze^{-\alpha}), |ze^{-\alpha}\rangle) = [E(X \oplus \alpha)](|z\rangle, |z\rangle)
\]

for all $X \in \mathcal{B}([0, 2\pi))$, $z \in \mathbb{C}$, and $\alpha \in [0, 2\pi)$. The following GOMs are solutions of (5):

\[
[E(X)](\varphi, \psi) = \sum_{n,m=0}^{\infty} d_{n,m}i^m_n(X)\langle \varphi | n \rangle \langle m | \psi \rangle
\]

where $(d_{n,m}) \in \mathbb{C}^{N \times N}$, $\sup \{|d_{n,m}| \mid n, m \in \mathbb{N}\} < \infty$, $X \in \mathcal{B}([0, 2\pi))$, and $\varphi, \psi \in \mathcal{H}_1$. We use the following short notation for $E$:

\[
E(X) = \sum_{n,m=0}^{\infty} d_{n,m}i^m_n(X) |n\rangle \langle m|,
\]

and we say that $E$ is a covariant GOM defined by the structure matrix $(d_{n,m})$. Note that $E([0, 2\pi)) = \sum_{n=0}^{\infty} d_{n,n} |n\rangle \langle n|$ can be extended to a unique bounded operator. If $d_{n,n} = 1$, $n \in \mathbb{N}$, then $E$ is normalized. If $(d_{n,m})$ is a phase matrix then $E$ is a phase observable. For all $\varphi, \psi \in \mathcal{H}_1$, the complex measure $X \to [E(X)](\varphi, \psi)$ has a continuous density.
which is

\[ \theta \mapsto \sum_{n,m=0}^{\infty} d_{n,m} e^{i(n-m) \theta} \langle \varphi | n \rangle \langle m | \psi \rangle. \]

Let \( \mathcal{M}_\infty \) be a set of structure matrices \((d_{n,m})_{n,m \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}},\)

\[ \sup \{|d_{n,m}| \mid n, m \in \mathbb{N}\} < \infty. \]

Since for all \((d_{n,m}) \in \mathcal{M}_\infty\) we have a unique covariant genaralized operator measure \(E\) defined in (6), we can identify \((d_{n,m})\) with \(E\). Now \( \mathcal{M}_\infty \) is a \( W^* \)-algebra (over \( \mathbb{C} \)) with the norm \[ \|(d_{n,m})\| := \sup \{|d_{n,m}| \mid n, m \in \mathbb{N}\} < \infty. \]

The summation, scalar product, and algebra product are defined pointwise. Let \(*\) be the algebra product operation, that is, \((d_{n,m}) \ast (e_{n,m}) := (d_{n,m} e_{n,m}).\)

The identity of \( \mathcal{M}_\infty \) is the canonical phase matrix \((c_{n,m})\) with \(c_{n,m} = 1,\)

\(n, m \in \mathbb{N}.\) The algebra \( \mathcal{M}_\infty \) is commutative and the involution is \((d_{n,m}) \mapsto (d_{n,m})^* := \overline{(d_{n,m})}.\)

The unique pre-dual of \( \mathcal{M}_\infty \) is the Banach space \( \mathcal{M}_1 \) of matrices \((d_{n,m})\) for which \[ \sum_{n,m=0}^{\infty} |d_{n,m}| < \infty. \]

A matrix \((d_{n,m}) \in \mathcal{M}_\infty\) has an inverse if and only if \(d_{n,m} \neq 0\) for all \(n, m \in \mathbb{N}.\)

The inverse is \((d_{n,m}^{-1}).\)

A matrix \((d_{n,m}) \in \mathcal{M}_\infty\) is positive if \(d_{n,m} \geq 0\) for all \(n, m \in \mathbb{N}.\) However, we are not interested in this standard positivity; we rather study positive semidefinitess of matrices.

The positive semidefinite matrices of \( \mathcal{M}_\infty \) form a convex cone. We denote it by \( \mathcal{M}_\infty^+ \). Any \((d_{n,m}) \in \mathcal{M}_\infty^+\) defines a covariant positive operator measure \(E\) via Equation (6).
matrices of $\mathcal{M}_\infty^+$ whose diagonal elements equal one. Let $\mathcal{C}$ be the $\sigma$-convex set of phase matrices. Phase matrices define phase observables.

The phase matrices of phase observables unitarily equivalent to $E_{\text{can}}$ are only phase matrices which have phase matrix inverses. Note that $\mathcal{C}^* = \mathcal{C}$, and for all $(c_{n,m}) \in \mathcal{C}$ the norm $\|(c_{n,m})\| = 1$, that is, all phase matrices lie on the unit ball.

We can embed the bounded operators, trace-class operators, and states in $\mathcal{M}_\infty$. We simply define

$$\mathcal{L} := \left\{ (A_{n,m}) \in \mathcal{M}_\infty \mid \sum_{n,m=0}^\infty A_{n,m} |n\rangle \langle m| \in \mathcal{L}(\mathcal{H}) \right\},$$

$$\mathcal{T} := \left\{ (T_{n,m}) \in \mathcal{M}_\infty \mid \sum_{n,m=0}^\infty T_{n,m} |n\rangle \langle m| \in \mathcal{T}(\mathcal{H}) \right\},$$

$$\mathcal{T}_1^+ := \left\{ (T_{n,m}) \in \mathcal{M}_\infty \mid \sum_{n,m=0}^\infty T_{n,m} |n\rangle \langle m| \in \mathcal{T}(\mathcal{H})_1^+ \right\}.$$

Thus, $\mathcal{T}_1^+$ contains such $(T_{n,m}) \in \mathcal{M}_\infty^+$ for which $\sum_{n=0}^\infty T_{n,n} = 1$. Note that $\mathcal{T} \cap \mathcal{C} = \emptyset$ and $\mathcal{C} \not\subseteq \mathcal{L} \neq \mathcal{M}_\infty$.

As we saw in the previous section, for any phase matrix $(c_{n,m})$ and a state $(T_{n,m})$ the product $(c_{n,m}) \star (T_{n,m})$ is a state. Thus, $\mathcal{C} \star \mathcal{T}_1^+ = \mathcal{T}_1^+$.

An operation $\Theta$ defined in (4) corresponds a mapping $\mathcal{T} \ni (T_{n,m}) \mapsto (c_{m,n}) \star (T_{n,m}) \in \mathcal{T}$, $(c_{n,m}) \in \mathcal{C}$, which is continuous with respect to the $\sigma$-convex means that for any sequence of phase matrices $(C_k)_{k \in \mathbb{N}}$ and for any sequence of nonnegative real numbers $(\lambda_k)_{k \in \mathbb{N}}$ for which $\sum_{k=0}^\infty \lambda_k = 1$ the series $n \mapsto \sum_{k=0}^n \lambda_k C_k$ converges to a phase matrix (with respect to the norm of $\mathcal{M}_\infty$).
trace-norm. If $\Theta_1$ and $\Theta_2$ are the operations (defined in (3)) of phase observables $E_1$ and $E_2$ with $(c^1_{n,m})$ and $(c^2_{n,m})$, respectively, then the matrix $(c^1_{m,n}) \ast (c^2_{m,n})$ corresponds the composition operation $\Theta_1 \circ \Theta_2$.

Note that $\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1$.

Let $(d_{n,m})$ and $(e_{n,m})$ be elements of $\mathcal{M}_\infty^+$. Now there exist vector sequences $(\varphi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ such that $d_{n,m} = \langle \varphi_n | \varphi_m \rangle$ and $e_{n,m} = \langle \psi_n | \psi_m \rangle$ for all $n, m \in \mathbb{N}$ [3]. Now $d_{n,m} e_{n,m} = \langle \varphi_n \otimes \psi_n | \varphi_m \otimes \psi_m \rangle$, $n, m \in \mathbb{N}$, and $(d_{n,m}) \ast (e_{n,m})$ is positive semidefinite. Hence, $\mathcal{M}_\infty^+ \ast \mathcal{M}_\infty^+ = \mathcal{M}_\infty^+$ and $\mathcal{C} \ast \mathcal{C} = \mathcal{C}$.

Let $(d_{n,m}) \in \mathcal{M}_\infty^+$. Now we can write $d_{n,m} = \sum_{k=0}^\infty d_{n,m}^{(k)}$ where $d_{n,m}^{(k)} = \langle n | F_k \rangle \langle F_k | m \rangle$ for all $n, m \in \mathbb{N}$, and $| F_k \rangle \in \mathcal{H}_\infty$, $k \in \mathbb{N}$. Hence, the finite sums of matrices $((n|F)(F|m))_{n,m \in \mathbb{N}}$, $| F \rangle \in \mathcal{H}_\infty$, form a dense subset of $\mathcal{M}_\infty^+$. Every $| F \rangle \in \mathcal{H}_\infty$ defines a covariant positive operator measure $E_F$ of Equation (2).

Following [3], we can define a certain ordering relation on $\mathcal{M}_\infty$ as follows: $(d_{n,m}) \preceq (e_{n,m})$ if $(d_{n,m}) = (e_{n,m}) \ast (f_{n,m})$ for some $(f_{n,m}) \in \mathcal{M}_\infty$. Let $(1)_{n,m \in \mathbb{N}}$ and $(\delta_{n,m})_{n,m \in \mathbb{N}}$ be the phase matrices of the canonical and the trivial phase observables, respectively. Now $(d_{n,m})_{n,m \in \mathbb{N}} \preceq (1)_{n,m \in \mathbb{N}}$ for all $(d_{n,m}) \in \mathcal{M}_\infty$ and $(\delta_{n,m}) \preceq (e_{n,m})$ for all $(e_{n,m}) \in \mathcal{C}$. Note

\[ \sum_{n,m=0}^{\infty} f_n d_{n,m} e_{n,m} f_m = \| \sum_{n=0}^{\infty} f_n | \varphi_n \otimes \psi_n \rangle \|^2 \geq 0. \]

This shows that $(d_{n,m} e_{n,m})$ is positive semidefinite.
that \( \preceq \) is not a partial ordering. It does not satisfy the antisymmetry condition.

Define the following equivalence relation in \( C \):

\[
(c_{n,m}) \simeq (d_{n,m}) \text{ if } (c_{n,m}) = (d_{n,m}) \star (e^{i(v_n - v_m)}), \quad (v_n)_{n \in \mathbb{N}} \subset [0, 2\pi).
\]

Denote the equivalence class of \( (c_{n,m}) \in C \) by \( [(c_{n,m})] \), and define a partial ordering \( \preceq \) in the set of equivalence classes as follows: \( [(c_{n,m})] \preceq [(d_{n,m})] \) if \( (c_{n,m}) = (d_{n,m}) \star (e_{n,m}) \) for some \( (e_{n,m}) \in C \). Now \( [(\delta_{n,m})] \preceq [(c_{n,m})] \preceq [(1)] \) for all \( (c_{n,m}) \in C \) and, thus, the equivalence class of the canonical phase matrix is the upper bound.

5. On the pointwise convergence of phase kernels

As we have seen, a phase observable \( E \) is determined uniquely by a phase matrix \( (c_{n,m}) \) via Equation (1). For any trace-class operator \( T \) we can define a complex measure \( X \mapsto p^E_T(X) := \text{tr}(TE(X)) \) which is absolutely continuous with respect to the normalised Lebesgue measure and, thus, has a Radon-Nikodým derivative \( g^E_T \) such that

\[
p^E_T(X) = (2\pi)^{-1} \int_X g^E_T(\theta) d\theta, \quad X \in \mathcal{B}([0, 2\pi]).
\]

Following Equation (1) it is tempting to write \( g^E_T(\theta) = \sum_{n,m=0}^{\infty} T_{m,n} c_{n,m} e^{i(n-m)\theta} \) where the summation converges pointwise for \( d\theta \)-almost all \( \theta \in \mathbb{R} \). But is it possible? In this section we study this problem.
Let us start with the simplest case. Let $E$ be the canonical phase, and let $T = |\varphi\rangle \langle \psi|$ where $\varphi, \psi \in \mathcal{H}$. From Carleson Theorem we know that any $L^2$-Fourier series converges pointwise for almost all $\theta \in \mathbb{R}$. Thus, we get

$$g^E_{|\varphi\rangle \langle \psi|} (\theta) = \sum_{n=0}^{\infty} \langle \psi | n \rangle e^{i n \theta} \sum_{m=0}^{\infty} \langle m | \varphi \rangle e^{-i m \theta} = \sum_{n,m=0}^{\infty} \langle m | \varphi \rangle \langle \psi | n \rangle e^{i (n-m) \theta}$$

for almost all $\theta \in \mathbb{R}$. Let then $T$ be an arbitrary trace-class operator, and let $E$ be any phase observable with the covariant trace-preserving operation $\Phi$ of Theorem 3. Now we can write $\Phi(T) = T_\alpha - T_\beta + iT_\gamma - iT_\delta$ where the operators $T_u$ are positive trace-class operators with decompositions $T_u = \sum_{k=0}^{\infty} |\varphi_k^{(u)} \rangle \langle \varphi_k^{(u)} |$, $\varphi_k^{(u)} \in \mathcal{H}$, $k \in \mathbb{N}$, where $u = \alpha, \beta, \gamma, \delta$. Thus,

$$g^E_T (\theta) = g^E_{T_\alpha} (\theta) - g^E_{T_\beta} (\theta) + ig^E_{T_\gamma} (\theta) - ig^E_{T_\delta} (\theta)$$

and, by monotonic convergence,

$$g^E_{T_u} (\theta) = \sum_{k=0}^{\infty} \sum_{n,m=0}^{\infty} \langle m | \varphi_k^{(u)} \rangle \langle \varphi_k^{(u)} | n \rangle e^{i (n-m) \theta}$$

for all $u = \alpha, \beta, \gamma, \delta$ and for almost all $\theta \in \mathbb{R}$. We will get a similar equation without using the operation $\Phi$. Namely, by using Theorem one gets for any $|\varphi\rangle \langle \varphi|$

$$g^E_{|\varphi\rangle \langle \varphi|} (\theta) = \sum_{k=0}^{\infty} \sum_{n,m=0}^{\infty} \langle \varphi | n \rangle \langle F_k | m \rangle \langle F_k | \varphi \rangle \langle m | \varphi \rangle e^{i (n-m) \theta}$$
for almost all $\theta \in \mathbb{R}$. A problem of Equations (7) and (8) is that it is not clear that we can change the order of $k$- and $(n,m)$-sums. So we have to consider other methods.

First we prove a simple proposition. Let $B$ be a complex Banach space, and let $S : \mathcal{H} \times \mathcal{H} \to B$ be a bounded sesquilinear form (the first argument is antilinear), that is, $\|S\| := \sup \{\|S(\varphi, \psi)\| \mid \|\varphi\| \leq 1, \|\psi\| \leq 1\} < \infty$. Note that $\mathcal{T}(\mathcal{H})$ is equipped with the trace norm.

**Proposition 1.** Denote $S_{n,m} := S(|n\rangle, |m\rangle)$ for all $n, m \in \mathbb{N}$. Then for all $\varphi, \psi \in \mathcal{H}$

$$S(\varphi, \psi) = \lim_{s,t \to \infty} \sum_{n=0}^{s} \sum_{m=0}^{t} S_{n,m} \langle \varphi | n \rangle \langle m | \psi \rangle,$$

that is,

$$S = \sum_{n,m=0}^{\infty} S_{n,m} |n\rangle \langle m|$$

weakly, and $S$ can be uniquely extended to a continuous linear mapping $\tilde{S} : \mathcal{T}(\mathcal{H}) \to B$,

$$T \mapsto \tilde{S}(T) := \sum_{n,m=0}^{\infty} S_{n,m} T_{m,n} := \lim_{s,t \to \infty} \sum_{n=0}^{s} \sum_{m=0}^{t} S_{n,m} T_{m,n}$$

where $T_{m,n} := \langle m | T | n \rangle$, $n, m \in \mathbb{N}$. Clearly, $\tilde{S}(|\psi\rangle \langle \varphi|) = S(\varphi, \psi)$ for all $\varphi, \psi \in \mathcal{H}$. 

Proof. For \( \varphi, \psi \in \mathcal{H} \) one gets

\[
\|S(\varphi, \psi) - S(P_s \varphi, P_t \psi)\| \leq \|S\| \|\varphi\| \|\psi - P_t \psi\| + \|S\| \|\varphi - P_s \varphi\| \|P_t \psi\| \to 0
\]

when \( s, t \to 0 \) where \( P_s := \sum_{n=0}^{s} |n\rangle \langle n| \). Fix \( T \in \mathcal{T}(\mathcal{H})_1^+ \). One can write \( T = \sum_{k=0}^{\infty} \lambda_k \varphi_k \langle \varphi_k | \) where \( \lambda_k \in [0, 1], \sum_{k=0}^{\infty} \lambda_k = 1, \varphi_k \in \mathcal{H}, \) and \( \|\varphi_k\| = 1 \) for all \( k \in \mathbb{N} \). Define \( \alpha^T := \sum_{k=0}^{\infty} \lambda_k S(\varphi_k, \varphi_k) \) and \( \alpha_{s,t}^T := \sum_{k=0}^{s} \lambda_k S(P_s \varphi_k, P_t \varphi_k) = \sum_{n=0}^{s} \sum_{m=0}^{t} S_{n,m} T_{m,n} \) which exist since \( \sum_{k=0}^{\infty} \lambda_k = 1 \) and \( \|S(\varphi, \psi)\| \leq \|S\| \) for all vectors \( \varphi, \psi \) with \( \|\varphi\| \leq 1, \|\psi\| \leq 1 \). Also we see that \( \|\alpha^T\| \leq \|S\| \). By the dominated convergence theorem

\[
\|\alpha^T - \alpha_{s,t}^T\| \leq \sum_{k=0}^{\infty} \lambda_k \|S(\varphi_k, \varphi_k) - S(P_s \varphi_k, P_t \varphi_k)\|
\]

\[
\leq \|S\| \sum_{k=0}^{\infty} \lambda_k (\|\varphi_k - P_t \varphi_k\| + \|\varphi_k - P_s \varphi_k\|) \to 0
\]

when \( s, t \to \infty \). As we can easily see from the beginning of the proof, the matrix elements \( T_{n,m} \) define the operator \( T \) uniquely and, thus, \( \tilde{S}(T) := \alpha^T \) is well-defined. Since any \( T \in \mathcal{T}(\mathcal{H}) \) can be uniquely written in the form \( T = \alpha T_\alpha - \beta T_\beta + i \gamma T_\gamma - i \delta T_\delta \) where \( T_\alpha, T_\beta, T_\gamma, \) and \( T_\delta \) are states, and \( \alpha, \beta, \gamma, \) and \( \delta \) are nonnegative real numbers we can define \( \tilde{S}(T) := \alpha \tilde{S}(T_\alpha) - \beta \tilde{S}(T_\beta) + i \gamma \tilde{S}(T_\gamma) - i \delta \tilde{S}(T_\delta) \). The rest of the proof follows immediately. \( \square \)
Note that it follows from Proposition 1 that any bounded operator $A$ can be written in the form $A = \sum_{n,m=0}^{\infty} A_{n,m} |n\rangle \langle m|$ (weakly) and $\text{tr}(AT) = \sum_{n,m=0}^{\infty} A_{n,m} T_{m,n}$ for any $T \in \mathcal{T}(\mathcal{H})$ where $A_{n,m} := \langle n|A|m\rangle$, $n, m \in \mathbb{N}$.

Let $E$ be a phase observable with $(c_{n,m})$, and let $g^E_T$ be a Radon-Nikodým derivative of the complex measure $p^E_T$ associated to $T \in \mathcal{T}(\mathcal{H})$. The sesquilinear mapping $\mathcal{H} \times \mathcal{H} \ni (\varphi, \psi) \mapsto g^E_T|\varphi\rangle\langle\psi| \in L^1[0,2\pi]$ is bounded since by using the polarization identity and the parallelogram law $(2\pi)^{-1} \int_{0}^{2\pi} |g^E_T|\varphi|\langle\psi|\theta\rangle|\,d\theta \leq \|\psi\|^2 + \|\varphi\|^2$ for all $\psi, \varphi \in \mathcal{H}$.

From Proposition 1 one gets

$$g^E_T = \sum_{n,m=0}^{\infty} T_{m,n} c_{n,m} e^{-i(n-m)\theta}$$

where $e_{n}(\theta) = e^{in\theta}$ and the double series converges with respect to the $L^1$-norm. This implies [12, Theorem 3.12, p. 68] the following theorem:

**Theorem 5.** There exists a subsequence $\mathbb{N} \ni k \mapsto n_k \in \mathbb{N}$, $n_1 < n_2 < n_3 < \ldots$, such that

$$g^E_T(\theta) = \lim_{k \to \infty} \sum_{n,m=0}^{n_k} T_{m,n} c_{n,m} e^{i(n-m)\theta}$$

for almost all $\theta \in \mathbb{R}$.

It can be shown [12, Theorem 7.8, p. 140] that

$$g^E_T(\theta) = 2\pi \frac{dp^E_T[0,x]}{dx} \bigg|_{x=\theta}$$
for almost all \( \theta \in [0, 2\pi) \). Thus, by direct calculation one gets
\[
g_E^T(\theta) = \lim_{\epsilon \to 0^+} \sum_{n,m=0}^{\infty} T_{n,m} c_{n,m} e^{i(n-m)\theta} f_{n-m}^{(1)}(\epsilon)
\]
for almost all \( \theta \in \mathbb{R} \) where \( f_k^{(1)}(\epsilon) := (e^{ik\epsilon} - 1)/(ik\epsilon) \), \( k \neq 0 \), and \( f_0^{(1)}(\epsilon) = 1 \). Thus, \( \lim_{\epsilon \to 0^+} f_k^{(1)}(\epsilon) = 1 \) for all \( k \in \mathbb{Z} \). Also, by a theorem of Fatou [13, p. 34], one can show that
\[
g_E^T(\theta) = \lim_{\epsilon \to 0^+} \sum_{n,m=0}^{\infty} T_{n,m} c_{n,m} e^{i(n-m)\theta} f_{n-m}^{(2)}(\epsilon)
\]
for almost all \( \theta \in \mathbb{R} \) where \( f_k^{(2)}(\epsilon) := (1-\epsilon)^{|k|} \) for all \( k \in \mathbb{Z} \) and the double series converges absolutely when \( \epsilon \in (0, 1] \). Also \( \lim_{\epsilon \to 0^+} f_k^{(2)}(\epsilon) = 1 \) for all \( k \in \mathbb{Z} \).

Since the operators
\[
C^{(j)}_\epsilon := \sum_{n,m=0}^{\infty} c_{n,m} f_{n-m}^{(j)}(\epsilon) |n\rangle\langle m|, \quad j = 1, 2,
\]
are bounded with \( \|C^{(1)}_\epsilon\| \leq 2\pi/\epsilon \) and \( \|C^{(2)}_\epsilon\| \leq 2/\epsilon - 1 \) for each \( \epsilon \in (0, 1] \) it follows that
\[
g_E^T(\theta) = \lim_{\epsilon \to 0^+} \text{tr} [TR(\theta)C^{(j)}_\epsilon R(\theta)^*], \quad j = 1, 2,
\]
for almost all \( \theta \in \mathbb{R} \).

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