Plane-parallel waves as duals of the flat background

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Abstract
We give a classification of non-Abelian T-duals (‘T’ standing for topological or toroidal) of the flat metric in \( D = 4 \) dimensions with respect to the four-dimensional continuous subgroups of the Poincaré group. After dualizing the flat background, we identify the majority of dual models as conformal sigma models in plane-parallel wave backgrounds, most of them having torsion. We give their form in Brinkmann coordinates. We find, besides the plane-parallel waves, several diagonalizable curved metrics with nontrivial scalar curvature and torsion. Using the non-Abelian T-duality, we find general solutions of the classical field equations for all the sigma models in terms of d’Alembert solutions of the wave equation.

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1. Introduction

String theory in a curved and/or time-dependent background can be formulated as a sigma model satisfying supplementary conditions. Finding solutions of equations of motion in such backgrounds is usually very complicated. That is why every solvable case attracts considerable attention. An example of such a model is a string theory in the homogeneous plane-parallel wave background, solved in [1] in terms of Bessel functions. Plane-parallel (pp-)wave backgrounds in string theory have been repeatedly investigated in the past (see e.g. references in [2]). They not only give solvable models [3], but also allow one to study the behavior of strings near spacetime singularities [4, 5]. Moreover, for extracting information about string behavior in a general curved background, one can take the Penrose limit [6], extended to
fields in string theory in [7], and study string behavior in the resulting plane-wave background.

Particular cases of the four-dimensional pp-wave background in Rosen coordinates obtained from gauged WZW (Wess–Zumino–Witten) models were given in [8, 9], and [2], as

\[ ds^2 = du^2 + \frac{g_1(u')}{g_2(u) + q^2} dx_1^2 + \frac{g_2(u)}{g_1(u') + q^2} dx_2^2, \]

\[ B_{12} = \frac{q}{g_1(u') + q^2}, \]

where \( u' = au + d \) (\( a, d = \text{const} \)) and the functions \( g_i \) can take any pair of the following values:

\[ g(u) = 1, \ u^2, \ \tanh^2 u, \ \tan^2 u, \ \cot^2 u, \ u^{-2}. \]

It is mentioned in [8] that this background is dual to the flat space for \( g_1 = 1, \ g_2 = u^2 \). We shall show that several other cases of these backgrounds are dual to the flat space as well. Moreover, we shall use this fact for finding general solutions of classical sigma model field equations in these pp-wave backgrounds. We find, beside pp-waves, several curved backgrounds with diagonalizable metrics resembling black hole [10] and cosmological [11] solutions and we check that solutions obtained by non-Abelian T-duality satisfy sigma model field equations in these backgrounds as well.

We understand the non-Abelian T-duality [12] as a special case of Poisson–Lie T-duality [13] based on the structure of the Drinfeld double. For technical reasons we shall restrict consideration to four spacetime dimensions, but the discussion can be extended to higher dimensions using the spectator fields or subgroups of the Poincaré group in higher dimensions. Investigations of the conformal invariance of pp-waves in higher dimensions can be found in e.g. [14, 15].

The plan of the paper is the following. In the next two sections we describe the method whereby the Poisson–Lie T-duality is used as a tool for the construction of dual models and their solution. In section 4, we review relevant properties concerning strings in the pp-wave background. Detailed discussion of particular examples is given in section 5. Section 6 summarizes results of dualization with respect to various subgroups of the Poincaré group. Subalgebras corresponding to these subgroups are listed in the appendix.

2. Non-Abelian T-duality

The sigma model on a manifold \( M \) is given by the classical action

\[ S_\sigma[X] = -\int d\sigma_+ d\sigma_- \left( \partial_\sigma X^\mu F_{\mu\nu}(X) \partial_\sigma X^\nu \right) \]

\[ = \frac{1}{2} \int d\sigma \left[ -\partial_\sigma X^\mu G_{\mu\nu}(X) \partial_\sigma X^\nu + \partial_\sigma X^\mu G_{\mu\nu}(X) \partial_\sigma X^\nu - 2\partial_\sigma X^\mu B_{\mu\nu}(X) \partial_\sigma X^\nu \right], \]

where \( F \) is a second-order tensor field on \( M \), with the metric and the NS–NS 2-form (torsion potential; NS stands for ‘Neveu–Schwarz’) given by the symmetric and antisymmetric parts of \( F \):

\[ G_{\mu\nu} = \frac{1}{2} \left( F_{\mu\nu} + F_{\nu\mu} \right), \quad B_{\mu\nu} = \frac{1}{2} \left( F_{\mu\nu} - F_{\nu\mu} \right). \]
The worldsheet coordinates are
\[ \sigma_+ = \frac{1}{\sqrt{2}} (\tau + \sigma), \quad \sigma_- = \frac{1}{\sqrt{2}} (\tau - \sigma). \]

The functions \( X^\mu \) are determined by the composition \( X^\mu (\tau, \sigma) = x^\mu (X(\tau, \sigma)) \), where \( X: \mathbb{R}^2 \ni (\tau, \sigma) \mapsto X(\tau, \sigma) \in M \) and \( x^\mu: U_\mu \to \mathbb{R} \) are components of a coordinate map on a neighborhood \( U_\mu \) of an element \( X(\tau, \sigma) = p \in M \).

The non-Abelian T-duality [12] of sigma models is a special case of Poisson–Lie T-duality [13, 16] that can be formulated by virtue of the Drinfeld double—a connected Lie group whose Lie algebra \( \mathfrak{g} \) can be decomposed into a pair of equally dimensional subalgebras \( \mathfrak{g}, \tilde{\mathfrak{g}} \) that are maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form \( \langle\cdot,\cdot\rangle \) on \( \mathfrak{g} \).

The Drinfeld double suited for non-Abelian T-duality is the semidirect product \( \cong \), where the group \( G \) can be taken as a subgroup of the isometry group of the background, which, in our case, will be flat. The group \( \tilde{G} \) has to be chosen Abelian in order to satisfy the conditions of dualizability of the sigma model [13]. We shall focus on the case where the isometry subgroup acts freely and transitively on the manifold, so that we can make the identification \( G \cong M \). This is usually referred to as atomic duality. Let us summarize the main points of the construction of dual models.

Given the four-dimensional subgroup \( G \) of the isometry group generated by Killing vectors of the flat metric, the tensor \( F \) is symmetric and can be written as
\[ F_{\mu\nu}(x) = G_{\mu\nu}(x) = e_0^a (g(x)) \left( E_0 \right)_{ab} e_b^b (g(x)), \]
where \( E_0 \) is a constant nonsingular symmetric matrix, and the \( e_0^a (g(x)) \) are components of the right-invariant forms \( (dg)g^{-1} \) expressed in coordinates \( \{ x^\mu \} \) on the group \( G \) and the basis of its Lie algebra \( \{ T_a \} \).

Denoting the mutually dual bases of \( \mathfrak{g} \) and Abelian \( \tilde{\mathfrak{g}} \) as \( \{ T_i \}, \{ \tilde{T}_j \} \), we construct subspaces \( e^+ = \text{Span}(T_i + E_0 \tilde{T}_j), e^- = \text{Span}(T_i - E_0 \tilde{T}_j) \) that are orthogonal w.r.t. \( \langle\cdot,\cdot\rangle \) and span the whole Lie algebra \( \mathfrak{d} \). The field equations for the sigma model on the group \( G \) can be rewritten as the equation
\[ \left( \partial_l^{-1}, e^\pm \right) = 0, \]
for mapping \( l \) from the worldsheet in \( \mathbb{R}^2 \) into the Drinfeld double \( D \).

Due to Drinfeld, there exists a unique decomposition (at least in the vicinity of the unit element of \( D \)) of an arbitrary element \( l \) of \( D \) as a product of elements from \( G \) and \( \tilde{G} \). Solutions of equation (5) and solutions of the equations of motion for the sigma model \( X^\mu (\tau, \sigma) = x^\mu (g(\tau, \sigma)) \) are related by
\[ l(\tau, \sigma) = g(\tau, \sigma) \hat{h}(\tau, \sigma) \in D, \]
where \( \hat{h}(\tau, \sigma) \in \tilde{G} \) fulfills the equations
\[ \partial_j \hat{h}_j = -v^j G_{jk} \partial_k X^\alpha, \]
\[ \partial_j \hat{h}_j = -v^j G_{jk} \partial_k X^\alpha, \]
with \( v^j \) representing components of the left-invariant fields \( v_j \) on \( G \) in the group coordinates \( x^\mu \).

The metric and the torsion potential of the non-Abelian T-dual model can be obtained from the tensor \( \tilde{F} \):
\[ F_{\mu
u}(\tilde{x}) = \left[ E_0 + \Pi(g(\tilde{x})) \right]^{-1}, \]

where the matrix \( \Pi \) is given by the adjoint representation of the Abelian subgroup \( \mathcal{G} \) on the Lie algebra of the Drinfeld double in the mutually dual bases

\[
\text{Ad}(g)^T = \begin{pmatrix} 1 & 0 \\ \Pi(g) & 1 \end{pmatrix}.
\]

The relation between the solution \( X^\mu(\tau, \sigma) \) of the equations of motion of the sigma model given by \( F \) and the solution \( \tilde{X}^\mu(\tau, \sigma) := \hat{x}^\mu(\hat{g}(\tau, \sigma)) \) of the sigma model given by \( \tilde{F} \) follows from two possible decompositions of the elements \( l \) of the Drinfeld double:

\[
g(\tau, \sigma)\hat{h}(\tau, \sigma) = \hat{g}(\tau, \sigma)h(\tau, \sigma), \quad (9)
\]

where \( g, h \in G, \hat{g}, \hat{h} \in \hat{G} \). The map \( \hat{h}: \mathbb{R}^2 \to \hat{G} \) that we need for this transformation is obtained from equations (6) and (7).

### 3. Solving the classical sigma model equations using non-Abelian T-duality

Equation (9) defines the Poisson–Lie transformation between solutions of the equations of motion of the original sigma model and its dual. Its application may be rather complicated. To use it for finding the solution of the dual model, the following three steps must be achieved:

- **Step 1:** one has to know the solution \( X^\mu(\tau, \sigma) \) for the sigma model given by \( F_{\mu\nu}(x) \).
- **Step 2:** given \( X^\mu(\tau, \sigma) \), one has to find \( \hat{h}(\tau, \sigma) \), i.e. solve the system of PDEs (6), (7).
- **Step 3:** given \( \hat{h}(\tau, \sigma) \), one has to find the dual decomposition \( \hat{l}(\tau, \sigma) = \hat{g}(\tau, \sigma)\hat{h}(\tau, \sigma) \), where \( \hat{g}(\tau, \sigma) \in \hat{G}, h(\tau, \sigma) \in G \). Functions \( \tilde{X}^\mu(\tau, \sigma) := \tilde{x}^\mu(\hat{g}(\tau, \sigma)) \) then solve the field equations of the dual sigma model.

For simplicity, we will restrict consideration to four spacetime dimensions. Our convention for the flat metric in the spacetime coordinates \((t, x, y, z)\) is

\[ \eta = \text{diag}(-1, 1, 1, 1). \]

It is easy to find solutions for the equations following from the flat metric in coordinates \( x^J \in \{t, x, y, z\} \), as they reduce to two-dimensional wave equations:

\[ \partial_\tau^2 W^J - \partial_\sigma^2 W^J = 0, \quad J = t, x, y, z. \]

However, we need to identify the group \( G \) with the manifold, i.e. find appropriate coordinate transformations between \((t, x, y, z)\) and the coordinates parameterizing the group. Choosing the parameterization of group elements as

\[ g = g(x^\mu) = e^{iT_1^\mu} e^{iT_2^\mu} e^{iT_3^\mu} e^{iT_4^\mu}, \]

where the \( T_j \) form the basis of the Lie algebra of the group, we may calculate the algebra of left-invariant fields

\[ v_j = v_j^\mu \frac{\partial}{\partial x^\mu}, \quad j = 1, \ldots, 4, \]

and compare it with the chosen four-dimensional subalgebra of Killing vectors \((K_j)\) of the flat metric in coordinates \((t, x, y, z)\). The comparison then may give the coordinate transformation.
as a solution to a set of PDEs.

The right-hand sides of the PDEs (6), (7), solved in step 2, are invariant w.r.t. coordinate transformation. This means that we can express the right-hand sides in terms of the coordinates \( t, x, y, z \) and use the Killing fields \( \mathbf{K}_j \) instead of the left-invariant fields on \( G \). The equations (6) and (7) then acquire the form

\[
\partial_\sigma \tilde{h}_j = - K_j^I \eta_{IJ} \partial_\tau W^J, \quad (12)
\]

\[
\partial_\tau \tilde{h}_j = - K_j^I \eta_{IJ} \partial_\sigma W^J, \quad (13)
\]

where the \( W^J \) are solutions of two-dimensional wave equation (10), and the \( K_j^I \) are the components of Killing vectors in coordinates \( t, x, y, z \).

Step 3 represents in general a rather complicated problem related to the Baker–Campbell–Hausdorff formula. Its solution simplifies substantially when the adjoint representation of the Lie algebra \( \mathfrak{g} \) is faithful. Let

\[
l = g \tilde{h} = \tilde{g} h, \quad g, h \in G, \quad \tilde{g}, \tilde{h} \in \tilde{G} = \mathbb{R}^4,
\]

and assume that the parameterizations of \( g, h, \tilde{g}, \tilde{h} \) are

\[
g = e^{t_1 T_1} e^{t_2 T_2} e^{t_3 T_3} e^{t_4 T_4}, \quad \tilde{h} = e^{h_1 \tilde{T}_1} e^{h_2 \tilde{T}_2} e^{h_3 \tilde{T}_3} e^{h_4 \tilde{T}_4},
\]

\[
\tilde{g} = e^{\tilde{h}_1 T_1} e^{\tilde{h}_2 T_2} e^{\tilde{h}_3 T_3} e^{\tilde{h}_4 T_4}, \quad h = e^{h_1 T_1} e^{h_2 T_2} e^{h_3 T_3} e^{h_4 T_4}.
\]

The variables \( x^j, \tilde{x}_k \) and \( h^k \) represent two sets of coordinates in (the vicinity of the unit of) the Drinfeld double. To express \( \tilde{x}_j, h^k \) in terms of \( x^j, \tilde{x}_k \), we can use a representation \( r \) of an element of the semi-Abelian Drinfeld double in the form of block matrices (dim \( \mathfrak{g} \) + 1) \( \times \) (dim \( \mathfrak{g} \) + 1), such that

\[
r(g) = \begin{pmatrix} \text{Ad} g & 0 \\ 0 & 1 \end{pmatrix}, \quad r(\tilde{h}) = \begin{pmatrix} 1 & 0 \\ v(\tilde{h}) & 1 \end{pmatrix},
\]

where \( v(\tilde{h}) = (\tilde{h}_1, \ldots, \tilde{h}_{\text{dim } \mathfrak{g}}) \). From equation (14), we then get

\[
r(l) = r \left( g \tilde{h} \right) = \begin{pmatrix} \text{Ad} g & 0 \\ v(\tilde{h}) & 1 \end{pmatrix} = r(\tilde{g} h) = \begin{pmatrix} \text{Ad} h & 0 \\ v(\tilde{g}) \cdot (\text{Ad} h) & 1 \end{pmatrix}.
\]

If the adjoint representation of the Lie algebra \( \mathfrak{g} \) is faithful, then the representation \( r \) of the Drinfeld double is faithful as well, and the relation (15) gives a system of equations for \( \tilde{x}_j \) and \( h^k \). If not, we can try to use the formula

\[
e^{\text{Ad} h} = e^{\exp(ad A) h} e^A
\]

to permute the elements of \( G \) and \( \tilde{G} \) in (14) and express the coordinates \( \tilde{x}_j, h^k \) in terms of \( x^j, \tilde{x}_k \).

In the following sections we shall apply the above given three steps of the Poisson–Lie transformation to solve the sigma model field equations in curved backgrounds dual to the flat metric.
4. Strings in the pp-wave background

We will be interested in the special subclass of metrics called the pp-waves. Their metric in the so-called Brinkmann coordinates \((u, v, z_3, z_4, \ldots, z_D)\) can be written as

\[
dx^2 = 2dv^2 - K(u, \vec{z}) dv^2 + d\vec{z}^2,
\]

where \(d\vec{z}^2\) is the Euclidean metric in the transversal space with coordinates \(\vec{z} = (z_3, z_4, \ldots, z_D)\). We denote the number of transversal coordinates by \(d\), such that \(D = 2 + d\). The NS–NS 2-form of particular interest to us has the form

\[
B = B_j(u, \vec{z}) du \wedge d\vec{z}_j.
\]

The metric (17) has covariantly constant null Killing vector \(\partial_t\) and particularly simple curvature properties, because the Ricci tensor has only one nonzero component

\[
R_{uu} = \frac{1}{2} \left( \partial_3^2 K + \partial_4^2 K + \cdots + \partial_D^2 K \right),
\]

and the scalar curvature vanishes. The one-loop conformal invariance conditions for the sigma model

\[
0 = R_{\mu
u} - \Gamma_\mu \Gamma_\nu \Phi - \frac{1}{4} H_{\mu \nu \lambda} H^{\nu \lambda},
\]

\[
0 = \nabla^\mu \Phi H_{\mu \nu \lambda} + \nabla^\nu \Phi H_{\mu \nu \lambda},
\]

\[
0 = R - 2 \nabla_\mu \Phi H_{\mu \nu \lambda} - \nabla_\mu \Phi \nabla^\nu \Phi - \frac{1}{12} H_{\mu \nu \lambda} H^{\mu \nu \lambda},
\]

where \(H = dB\), can be solved in some special cases. One of them is that of the model in the background resulting from the Penrose–Güven limit [6, 7], with

\[
K(u, \vec{z}) = K_j(u) z_j z_j,
\]

and the torsion

\[
H = H_j(u) du \wedge dz_i \wedge dz_j
\]

that follows from the NS–NS 2-form (18) if \(B_j(u, \vec{z})\) is linear in \(z\). The one-loop conformal invariance conditions then simplify to a solvable differential equation for the dilaton \(\Phi = \Phi(u)\):

\[
\Phi''(u) - K_{ji}(u) + \frac{1}{4} H_{ji}(u) H_{ji}(u) = 0.
\]

We are going to show that the sigma models in pp-wave backgrounds with special forms of the functions \(K_{ij}\) and \(H_{ij}\) can be obtained as non-Abelian T-duals of sigma models in a flat background. The Killing vectors of the flat metric \(\eta = \text{diag}(-1, 1, 1, 1)\) in coordinates \((t, x, y, z)\) are

\[
R_t = \partial_t, \quad P_t = \partial_t, \quad L_t = -\epsilon_{ijk} x^i \partial_k, \quad M_j = -x^i \partial_j - t \partial_j,
\]

and they form the ten-dimensional Poincaré Lie algebra. To apply the (atomic) non-Abelian T-duality on sigma models in the flat background, we shall need four-dimensional subalgebras of the Poincaré Lie algebra, classified in [17].

For \(K(u, \vec{z})\) in (17) at most quadratic in the transversal coordinates, one can find transformations that bring it to the form (22). In the following, we will be able to bring the metrics of the resulting dual models with vanishing scalar curvature to the form (17), where
\( K ( u, \bar{z}) = K_3(u)z_3^2 + K_4(u)z_4^2, \)  
\( \tag{26} \)

and the torsion is

\[ H = H(u)d\alpha \wedge dz_3 \wedge dz_4. \]

The classical field equations of the sigma model (3) in such a background then read

\[ \partial_2^2 U - \partial_4^2 U = 0, \tag{27} \]

\[ \partial_2^2 Z_3 - \partial_4^2 Z_3 = K_3(U)\left[ (\partial_\alpha U)^2 - (\partial_\beta U)^2 \right] Z_3 - H(U)\left[ \partial_\alpha Z_4 \partial_\beta U - \partial_\beta Z_4 \partial_\alpha U \right], \tag{28} \]

\[ \partial_2^2 Z_4 - \partial_4^2 Z_4 = K_4(U)\left[ (\partial_\alpha U)^2 - (\partial_\beta U)^2 \right] Z_4 + H(U)\left[ \partial_\alpha Z_3 \partial_\beta U - \partial_\beta Z_3 \partial_\alpha U \right], \tag{29} \]

\[ \partial_2^2 V - \partial_4^2 V = H(U)\left[ \partial_\alpha Z_4 \partial_\beta Z_3 - \partial_\beta Z_3 \partial_\alpha Z_4 \right] \\
+ \sum_{j=3}^{4} \left\{ 2K_j(U)Z_j\left[ \partial_\beta Z_j \partial_\alpha U - \partial_\alpha Z_j \partial_\beta U \right] \right. \\
+ \left( Z_j \right)^2 \left[ \frac{1}{2}K_j(U)\left[ (\partial_\alpha U)^2 - (\partial_\beta U)^2 \right] + K_j(U)\left( \partial_\alpha^2 U - \partial_\beta^2 U \right) \right]. \tag{30} \]

For string backgrounds, the last equation can be replaced by the so-called string conditions for \( \tilde{X}^\mu = (U, V, Z_3, Z_4) \):

\[ \partial_\sigma \tilde{X}^\mu G_{\mu \nu} (X) \partial_\sigma X^\nu + \partial_\nu \tilde{X}^\mu G_{\mu \nu} (X) \partial_\nu X^\nu = 0, \tag{31} \]

\[ \partial_\nu \tilde{X}^\mu G_{\mu \nu} (X) \partial_\sigma X^\sigma = 0. \tag{32} \]

Conditions (31), (32) for the pp-wave with the function \( K \) given by (26) yield

\[ 2\partial_\sigma U \partial_\sigma V + \sum_{j=3}^{4} \left\{ \left( \partial_\beta Z_j \right)^2 - \left( \partial_\alpha U \right)^2 K_j(U) \left( Z_j \right)^2 \right\} + (\tau \to \sigma) = 0, \]

\[ \partial_\tau U \partial_\tau V + \partial_\tau V \partial_\tau U + \sum_{j=3}^{4} \left\{ \partial_\beta Z_j \partial_\alpha Z_j - \partial_\beta U \partial_\alpha UK_j(U) \left( Z_j \right)^2 \right\} = 0. \]

Compatibility of these two first-order equations for \( V = V(\tau, \sigma) \) is guaranteed by equations (27)–(29).

Note that for nonvanishing torsion, both \( Z_3 \) and \( Z_4 \) appear in (28), (29), so even in the light-cone gauge \( U = \kappa \tau \) these equations do not separate, and it can be rather difficult to solve them in the usual way using Fourier mode expansion. Nevertheless, the T-duality gives a method for obtaining the general solution.

### 5. Examples

#### 5.1. Example 1—subalgebra \( S_{27} \)

We shall illustrate the above-described methods of non-Abelian dualization of the flat metric on the example of Killing vectors
\[ K_1 = M_1 = -z \partial_z - t \partial_t, \]
\[ K_2 = L_2 + M_1 = -x \partial_x - (t + z) \partial_t + x \partial_z, \]
\[ K_3 = L_1 - M_2 = y \partial_y + (t + z) \partial_y - y \partial_z, \]
\[ K_4 = P_0 - P_1 = \partial_z - \partial_z \]

that span subalgebra \( S_{27} \) (see the appendix). Their nonvanishing commutation relations are

\[ [K_1, K_2] = -K_2, \quad [K_1, K_3] = -K_3, \quad [K_1, K_4] = -K_4. \] (34)

5.1.1. Duals to the flat metric. Using the parameterization (11) of the isometry subgroup \( G \), where the \( T_\mu \) are elements of its Lie algebra commuting as in (34), we get the basis of left-invariant fields on \( G \)

\[ v_1 = \partial_1 + x^2 \partial_2 + x^3 \partial_3 + x^4 \partial_4, \]
\[ v_2 = \partial_2, \quad v_3 = \partial_3, \quad v_4 = \partial_4. \]

Identifying the Killing vectors (33) with these left-invariant fields, we get the transformation of coordinates on the flat manifold

\[ t = \frac{1}{2} e^{-x^1} \left( (x^3)^2 + (x^4)^2 + 1 \right) + x^4, \quad x = -e^{-x^1} x^2, \]
\[ z = -\frac{1}{2} e^{-x^1} \left( (x^2)^2 + (x^3)^2 - 1 \right) - x^4, \quad y = e^{-x^1} x^3 \] (35)

that gives the flat metric in the group coordinates \( x^\mu \):

\[ G_{\mu \nu}(x) = F_{\mu \nu}(x) = \begin{pmatrix} 0 & 0 & 0 & e^{-x^1} \\ 0 & e^{-2x^1} & 0 & 0 \\ 0 & 0 & e^{-2x^1} & 0 \\ e^{-x^1} & 0 & 0 & 0 \end{pmatrix}. \] (36)

This can be obtained from formula (4) if one chooses

\[ E_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

The dual tensor \( \tilde{F} \) can be then found from formula (8) as

\[ \tilde{F}_{\mu \nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1 - x_4} \\ 0 & 1 & 0 & x_2 \left( \frac{x_2}{1 - x_4} \right) \\ 0 & 0 & 1 & \frac{x_3}{1 - x_4} \left( x_2^2 + x_3^2 \right) \\ \frac{1}{x_4 + 1} & -\frac{x_2}{x_4 + 1} & -\frac{x_3}{x_4 + 1} & \frac{x_2^2 + x_3^2}{x_4^2 - 1} \end{pmatrix}. \]
The scalar curvature corresponding to the metric obtained from the symmetric part of this tensor vanishes, and the Ricci tensor has only one nonvanishing component:

$$\bar{R}_{44} = -\frac{4}{(\tilde{x}_4^2 - 1)}.$$

This suggests that the dual metric could be of the pp-wave form. Indeed, the transformation of coordinates for $|\tilde{x}_4| > 1$,

$$\tilde{x}_1 = v - \frac{1}{2}\left(\tilde{z}_3^2 + \tilde{z}_4^2\right)\coth(u), \quad \tilde{x}_2 = z_3,$n$$

$$\tilde{x}_4 = \coth(u), \quad \tilde{x}_3 = z_4,$$

brings the components of the tensor $\tilde{F}$ into the form

$$\tilde{F} = \begin{pmatrix}
2\frac{z_3^2 + z_4^2}{\sinh^2(u)} & 1 - \coth(u) & \frac{z_3}{\sinh^2(u)} & \frac{z_4}{\sinh^2(u)} \\
1 + \coth(u) & 0 & 0 & 0 \\
-\frac{z_3}{\sinh^2(u)} & 0 & 1 & 0 \\
-\frac{z_4}{\sinh^2(u)} & 0 & 0 & 1
\end{pmatrix}.$$ (38)

The symmetric part yields the pp-wave metric in the Brinkmann form

$$ds^2 = 2dvdu + 2\frac{z_3^2 + z_4^2}{\sinh^2(u)}du^2 + dz_3^2 + dz_4^2.$$ (39)

The torsion obtained from the antisymmetric part vanishes, and the dilaton obtained as a solution of equation (24) acquires a rather simple form:

$$\Phi(u) = c_2 + c_1 u + 4 \log(\sinh(u)),$$

where $c_1, c_2$ are arbitrary constants.

For $|\tilde{x}_4| < 1$, the transformation

$$\tilde{x}_1 = v - \frac{1}{2}\left(\tilde{z}_3^2 + \tilde{z}_4^2\right)\tanh(u), \quad \tilde{x}_2 = z_3,$n$$

$$\tilde{x}_4 = \tanh(u), \quad \tilde{x}_3 = z_4,$$

brings the tensor $\tilde{F}$ into the form

$$\tilde{F} = \begin{pmatrix}
-2\frac{z_3^2 + z_4^2}{\cosh^2(u)} & 1 - \tanh(u) & -\frac{z_3}{\cosh^2(u)} & -\frac{z_4}{\cosh^2(u)} \\
1 + \tanh(u) & 0 & 0 & 0 \\
-\frac{z_3}{\cosh^2(u)} & 0 & 1 & 0 \\
-\frac{z_4}{\cosh^2(u)} & 0 & 0 & 1
\end{pmatrix}.$$ (40)
The metric
\[ ds^2 = 2dudv - 2\left( z_1^2 + z_2^2 \right) \frac{dv}{\cosh^2(u)} + dz_3^2 + dz_4^2. \] (41)

The torsion again vanishes and the dilaton has the form
\[ \Phi(u) = c_2 + c_1 u + 4 \log(\cosh(u)). \] (42)

We can see that the non-Abelian T-duality w.r.t. the subalgebra \( S_{2\gamma} \) produces two types of sigma models in pp-wave backgrounds, one of them singular and one regular. As we shall see, this result is obtained also from dualization w.r.t. several other subalgebras of the Poincaré algebra.

5.1.2. The solution of the classical equations of the sigma model. Our next goal is to write down the general solution of the classical field equations in the backgrounds (38) and (40). As their torsions vanish, the antisymmetric parts do not contribute to the classical field equations.

The Lagrangian for the metric (39) can be written in the form (cf. (3))
\[ L = \left[ \frac{Z_3^2 + Z_4^2}{\sinh^2(U)} \left( \partial_u U \right)^2 + \partial_u U \partial_v V + \frac{1}{2} \left( \partial_v Z_3 \right)^2 + \frac{1}{2} \left( \partial_v Z_4 \right)^2 \right] - \left[ \frac{Z_3^2 + Z_4^2}{\sinh^2(U)} \left( \partial_t U \right)^2 + \partial_t U \partial_t V + \frac{1}{2} \left( \partial_t Z_3 \right)^2 + \frac{1}{2} \left( \partial_t Z_4 \right)^2 \right]. \]

The field equations then read
\[ \partial^2_t U - \partial^2_v U = 0, \] (43)
\[ \partial^2_v Z_1 - \partial^2_v Z_1 = 2 \left( \left( \partial_v U \right)^2 - \left( \partial_t U \right)^2 \right) \frac{Z_3}{\sinh^2(U)}, \] (44)
\[ \partial^2_v Z_4 - \partial^2_v Z_4 = 2 \left( \left( \partial_v U \right)^2 - \left( \partial_t U \right)^2 \right) \frac{Z_4}{\sinh^2(U)}, \] (45)
\[ \partial^2_v V - \partial^2_v V = 4 \cosech^2(U) \left[ Z_3 \left( \partial_u U \partial_v Z_3 - \partial_t U \partial_v Z_3 \right) \right. \left. + Z_4 \left( \partial_u U \partial_v Z_4 - \partial_t U \partial_v Z_4 \right) \right] - 2 \cosech^3(U) \left[ \left( Z_3 \right)^2 + \left( Z_4 \right)^2 \right] \times \left[ \left( -\partial^2_v U + \partial^2_v U \right) \sinh(U) + \left( \left( \partial_v U \right)^2 - \left( \partial_t U \right)^2 \right) \cosh(U) \right]. \] (46)

To solve these field equations, we can follow steps 1–3 from section 2. Step 1 starts with solution of the field equations in the flat background. In the coordinates \((t, x, y, z)\) they are of the form (10) solved by
\[ W^I (\tau, \sigma) = R^I (\tau - \sigma) + L^I (\tau + \sigma), \quad I = t, x, y, z, \]
with \( R^I, L^I \) arbitrary functions. Subsequent transformation of this solution to the coordinates \( x^\mu \) by using formulas (35) produces the functions.
\[
\begin{align*}
X^1(\tau, \sigma) &= -\log \left( W^\tau + W^\sigma \right), \\
X^2(\tau, \sigma) &= -\frac{W^\tau}{W^\tau + W^\sigma}, \\
X^4(\tau, \sigma) &= \frac{2(W^\tau)^2 - (W^\tau)^2 - (W^\sigma)^2}{2(W^\tau + W^\sigma)}, \\
X^3(\tau, \sigma) &= \frac{W^\tau}{W^\tau + W^\sigma},
\end{align*}
\]

that solve the sigma model field equations in the flat background in the coordinates \(x^\mu\), i.e. in metric \((36)\).

Next, we have to perform step 2, consisting in the solution of the PDEs \((12), (13)\), with Killing fields \((33)\) on the right-hand sides. The equation \((12)\) in this case read
\[
\begin{align*}
\partial_\tau \hat{h}_1 &= W^\tau \partial_\tau W^\tau - W^\tau \partial_\sigma W^\tau, \\
\partial_\tau \hat{h}_2 &= -W^\tau \left( \partial_\tau W^\tau + \partial_\sigma W^\tau \right) + (W^\tau + W^\sigma) \partial_\sigma W^\tau, \\
\partial_\tau \hat{h}_3 &= W^\tau \left( \partial_\sigma W^\tau + \partial_\sigma W^\tau \right) - (W^\tau + W^\sigma) \partial_\sigma W^\tau, \\
\partial_\tau \hat{h}_4 &= \partial_\sigma W^\tau + \partial_\sigma W^\tau,
\end{align*}
\]
while equations \((13)\) are obtained by making the exchange \(\tau \leftrightarrow \sigma\). Compatibility of these two sets of PDEs is guaranteed by the wave equations for \(W^\theta\). Their solution is
\[
\begin{align*}
\hat{h}_1(\tau, \sigma) &= \gamma_1 + \int \left( W^\tau \partial_\tau W^\tau - W^\tau \partial_\sigma W^\tau \right) d\tau, \\
\hat{h}_2(\tau, \sigma) &= \gamma_2 - \int \left( W^\tau \left( \partial_\tau W^\tau + \partial_\sigma W^\tau \right) - (W^\tau + W^\sigma) \partial_\sigma W^\tau \right) d\tau, \\
\hat{h}_3(\tau, \sigma) &= \gamma_3 + \int \left( W^\tau \left( \partial_\sigma W^\tau + \partial_\sigma W^\tau \right) - (W^\tau + W^\sigma) \partial_\sigma W^\tau \right) d\tau, \\
\hat{h}_4(\tau, \sigma) &= \gamma_4 + \int \left( \partial_\sigma W^\tau + \partial_\sigma W^\tau \right) d\tau,
\end{align*}
\]
(47)

where \(\gamma_1, \ldots, \gamma_4\) are constants.

To get the solution of the field equations \((43)–(46)\) we have to carry out step 3. One can easily check that the adjoint representation of the algebra \((34)\) is faithful, so we can use equation \((15)\) to express the coordinates \(\tilde{x}_\mu\) in terms of \(x^\mu\) and \(\hat{h}_k\). We get
\[
\begin{align*}
\tilde{x}_1 &= \hat{h}_1 - x^x \hat{h}_2 - x^y \hat{h}_3 - x^t \hat{h}_4, \\
\tilde{x}_2 &= e^{i\gamma_1} \hat{h}_2, \\
\tilde{x}_3 &= e^{i\gamma_2} \hat{h}_3, \\
\tilde{x}_4 &= e^{i\gamma_3} \hat{h}_4.
\end{align*}
\]
(48)

Finally, we have to transform the coordinates \(\tilde{x}_\mu\) into the Brinkmann form. Composing the inverse of \((35), (48)\) and the inverse of \((37)\), we get the Brinkmann coordinates \((u, v, z_3, z_4)\) on \(\mathcal{G}\) as functions of the spacetime coordinates \((t, x, y, z)\) on the initial flat manifold and coordinates \(\hat{h}_j\) on the subgroup \(\mathcal{G}\) of the Drinfeld double:
\[
\begin{align*}
u &= \text{arccoth} \left( \frac{\hat{h}_4}{t + z} \right), \\
z_3 &= \frac{\hat{h}_2}{t + z}, \\
z_4 &= \frac{\hat{h}_3}{t + z}, \\
2(t + z) &= 2(t + z)^2 + 2(\hat{h}_3(t + z) + \hat{h}_4(-t^2 + x^2 + y^2 + z^2)) + \frac{\hat{h}_4 \hat{h}_2^2 + \hat{h}_3 \hat{h}_4}{2(t + z)^2}.
\end{align*}
\]
(49)

To get the general solution of the classical field equations \((43)–(46)\) in the curved background with the metric \((39)\), we have to replace the coordinates \((t, x, y, z)\) in \((49)\) by the solutions \(W^\tau = W^\tau(\tau, \sigma)\) of the wave equation \((10)\), and the \(\hat{h}_\mu\) by the solutions \((47)\) of the PDEs \((12), (13)\). We obtain
The expression for the function $V(\tau, \sigma)$ is rather extensive, but can be easily read from (49). String-type solutions in the light-cone gauge (see e.g. [1, 18]), i.e.,

$$U(\tau, \sigma) = \kappa \tau, Z_3(\tau, \sigma) = \sum_{n=-\infty}^{\infty} Z_3^n(\tau) e^{2i\sigma n}, Z_4(\tau, \sigma) = \sum_{n=-\infty}^{\infty} Z_4^n(\tau) e^{2i\sigma n},$$  

are obtained if

$$W'(\tau, \sigma) + W'(\tau, \sigma) = e^{\kappa \tau} \sinh(\kappa \tau),$$

$$W'(\tau, \sigma) = \sinh(\kappa \tau) \sum_{n=-\infty}^{\infty} e^{2i\sigma n} (2in + \kappa) \int Z_3^n(\tau) \cosech(\kappa \tau) \, d\tau,$$

$$W'(\tau, \sigma) = \sinh(\kappa \tau) \sum_{n=-\infty}^{\infty} e^{2i\sigma n} (2in + \kappa) \int Z_4^n(\tau) \cosech(\kappa \tau) \, d\tau,$$

where $Z_3^n(\tau)$ and $Z_4^n(\tau)$ solve the differential equation

$$Z'^n(\tau) + \left( 4 n^2 - 2 \kappa^2 \cosech^2(\kappa \tau) \right) Z(\tau) = 0.$$

The solution of the classical field equations in the curved background with the metric (41) is obtained from the solution (50) when $\text{arccoth}$ is replaced by $\text{arctanh}$.

5.2. Example 2—subalgebra $S_{17}$

The second example will deal with the subalgebra

$$S_{17} = \text{Span} \left[ \mathcal{K}_1 = L_3 + \epsilon (P_0 + P_3), \mathcal{K}_2 = P_1, \mathcal{K}_3 = P_2, \mathcal{K}_4 = P_0 - P_3 \right], \epsilon = \pm 1,$$

which produces the dual model with torsion and whose representation is not faithful. The commutation relations of this subalgebra are

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_2.$$

The transformation of coordinates in the flat background

$$t = x^1 + x^4, \quad x = x^2, \quad y = x^3, \quad z = x^1 - x^4$$  

yields components of the flat metric in the group coordinates as

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & -2\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2\epsilon & 0 & 0 & 0 \end{pmatrix}.$$
The dual background in this case is

\[
\tilde{F}_{\mu\nu}(x) = \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{2e} \\
0 & 1 & 0 & \frac{x_3}{2e} \\
0 & 0 & 1 & -\frac{x_2}{2e} \\
-\frac{1}{2e} - \frac{x_3}{2e} & \frac{x_2}{2e} & -\frac{x_2^2 + x_3^2}{4e^2}
\end{pmatrix},
\]

and transformation to the Brinkmann coordinates,

\[
\tilde{x}_1 = -v, \quad \tilde{x}_2 = z_3, \quad \tilde{x}_3 = z_4, \quad \tilde{x}_4 = 2e \, u,
\]

brings the dual metric into the homogeneous and isotropic form

\[
dx^2 = 2du \, dv - \left( z_3^2 + z_4^2 \right) du^2 + dz_3^2 + dz_4^2.
\]

The torsion in Brinkmann coordinates is constant:

\[
H = -2du \land dz_3 \land dz_4,
\]

and the dilaton is

\[
\Phi(u) = c_1 + c_2 \, u.
\]

To find general solutions of the field equations of the dual sigma model with torsion, we have to express the coordinates \(\tilde{x}_\mu\) in terms of \(x^\nu\) and \(\tilde{h}_k\). As the adjoint representation of \(S_{17}\) is not faithful, we have to use formula (16) to solve equation (9) for coordinates of \(\tilde{g}\). We get

\[
\tilde{x}_1 = \tilde{h}_1 + x^2 \tilde{h}_3 - x^3 \tilde{h}_2, \quad \tilde{x}_2 = \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1, \\
\tilde{x}_3 = \tilde{h}_4, \quad \tilde{x}_4 = \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1.
\]

Like in the previous section, combining this with (53) and (52), we find the general solution of the field equations of the sigma model with metric (54) and torsion (55) as

\[
U(\tau, \sigma) = \frac{\tilde{h}_4(\tau, \sigma)}{2e}, \quad V(\tau, \sigma) = -\tilde{h}_1(\tau, \sigma) - \tilde{h}_3(\tau, \sigma) W^x(\tau, \sigma) + \tilde{h}_2(\tau, \sigma) W^y(\tau, \sigma), \\
Z_3(\tau, \sigma) = \cos (\Omega(\tau, \sigma)) \tilde{h}_2(\tau, \sigma) - \sin (\Omega(\tau, \sigma)) \tilde{h}_3(\tau, \sigma), \\
Z_4(\tau, \sigma) = \cos (\Omega(\tau, \sigma)) \tilde{h}_3(\tau, \sigma) + \sin (\Omega(\tau, \sigma)) \tilde{h}_2(\tau, \sigma),
\]

where the \(W^I(\tau, \sigma)\) are solutions of the wave equations (10),

\[
\Omega(\tau, \sigma) = \frac{W^I + W^\perp}{2e},
\]

and the \(\tilde{h}_\mu\) are solutions of the PDEs (12), (13):

\[
\tilde{h}_1 = \gamma_1 + \int \left[ e \left( \partial_\tau W^x - \partial_\sigma W^\perp \right) + W^x \partial_\tau W^y - W^y \partial_\tau W^x \right] d\sigma, \\
\tilde{h}_2 = \gamma_2 - \int \partial_\sigma W^x \, d\sigma, \quad \tilde{h}_3 = \gamma_3 - \int \partial_\sigma W^y \, d\sigma,
\]

\[
\tilde{h}_4 = \gamma_4 + \int \left[ e \left( \partial_\tau W^x - \partial_\sigma W^\perp \right) + W^y \partial_\tau W^x - W^x \partial_\tau W^y \right] d\sigma.
\]
String-type solutions in the light-cone gauge (51) are obtained if we choose

\[ W^I(r, \sigma) + W^\tau(r, \sigma) = 2 \epsilon \kappa \sigma, \]

\[
W^\tau(r, \sigma) = \sum_{n=-\infty}^{\infty} e^{2i \alpha n} \int Z_3^n(\tau) (\kappa \sin(\kappa \sigma) - 2in \cos(\kappa \sigma))
- Z_4^n(\tau) (\kappa \cos(\kappa \sigma) + 2in \sin(\kappa \sigma)) d\tau,
\]

\[
W^\tau(r, \sigma) = \sum_{n=-\infty}^{\infty} e^{2i \alpha n} \int Z_3^n(\tau) (\kappa \cos(\kappa \sigma) + 2in \sin(\kappa \sigma))
+ Z_4^n(\tau) (\kappa \sin(\kappa \sigma) - 2in \cos(\kappa \sigma)) d\tau.
\]

where \( Z_3^n(\tau) \) and \( Z_4^n(\tau) \) solve the system of differential equations

\[
Z_3^n(\tau) + \left( 4n^2 + \kappa^2 \right) Z_3^n(\tau) - 4in \kappa Z_4^n(\tau) = 0,
\]

\[
Z_4^n(\tau) + \left( 4n^2 + \kappa^2 \right) Z_4^n(\tau) + 4in \kappa Z_3^n(\tau) = 0.
\]

5.3. Example 3—subalgebra \( S_{19} \)

The third example will deal with the subalgebra

\[ S_{19} = \text{Span} \left[ \mathcal{K}_1 = L_3 + \alpha P_3, \mathcal{K}_2 = P_3, \mathcal{K}_3 = P_4, \mathcal{K}_4 = P_0 \right], \quad \alpha \neq 0, \]

which produces a diagonalizable dual metric with nonvanishing scalar curvature and torsion.

The commutation relations of this subalgebra

\[
[ \mathcal{K}_1, \mathcal{K}_2 ] = \mathcal{K}_3, \quad [ \mathcal{K}_1, \mathcal{K}_3 ] = -\mathcal{K}_2
\]

are equal to those in the previous example, but the subalgebras of Killing vectors cannot be transformed into one another by an element of the group of proper orthochronous Poincaré transformations (see [17]), and the representations of the commutation relations in Killing vector fields on \( M \) are different. This leads to a different transformation of coordinates in the flat background, namely,

\[
x^1 = \frac{z}{\alpha}, \quad x^2 = x, \quad x^3 = y, \quad x^4 = t.
\]

The components of the flat metric in the group coordinates then read

\[
F_{\mu \nu} = \begin{pmatrix}
\alpha^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]
The dual background in this case is

\[
\mathcal{F}^\nu_{\mu} = \begin{pmatrix}
1 & \frac{\delta_3}{\alpha^2 + \delta_2^2 + \delta_1^2} & \frac{\delta_1}{\alpha^2 + \delta_2^2 + \delta_3^2} & 0 \\
-\frac{\delta_3}{\alpha^2 + \delta_2^2 + \delta_1^2} & \frac{\delta_2}{\alpha^2 + \delta_2^2 + \delta_3^2} & 0 & 0 \\
\frac{\delta_1}{\alpha^2 + \delta_2^2 + \delta_3^2} & 0 & \frac{\delta_2}{\alpha^2 + \delta_1^2 + \delta_3^2} & 0 \\
0 & 0 & 0 & \frac{1}{\alpha^2 + \delta_1^2 + \delta_3^2}
\end{pmatrix}.
\]

and its symmetric part gives a metric with nonvanishing scalar curvature

\[
\tilde{R} = -\frac{4(\delta_2^2 + \delta_3^2)}{(\alpha^2 + \delta_2^2 + \delta_3^2)^2}.
\]

This means that it cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the form

\[
dx^2 = -dy_1^2 + dy_2^2 + \frac{y_2^2 \alpha^2}{y_2^2 + \alpha^2} dy_3^2 + \frac{1}{y_2^2 + \alpha^2} dy_4^2,
\]

via

\[
\delta_1 = y_4, \quad \delta_2 = y_2 \cos y_3, \quad \delta_3 = y_2 \sin y_3, \quad \delta_4 = y_1,
\]

The torsion then acquires the form

\[
H = \frac{2y_2 \alpha^2}{(y_2^2 + \alpha^2)^2} dy_2 \wedge dy_3 \wedge dy_4,
\]

and the dilaton satisfying (19)–(21) is

\[
\Phi = \log \left(\frac{y_2^2 + \alpha^2}{\delta_2^2 + \delta_3^2}\right) + \text{const}.
\]

To find general solutions of the field equations of this dual sigma model, we have to express the coordinates \(\delta_\mu\) in terms of \(x^\nu\) and \(h_2\). As the adjoint representation of \(S_{19}\) is not faithful, we have to use formula (16) to solve equation (9) for the coordinates of \(g\). We get

\[
\delta_1 = \hat{h}_1 + \hat{x}^2 \hat{h}_3 - \hat{x}^2 \hat{h}_2, \quad \delta_2 = \hat{h}_2 \cos x^1 - \hat{h}_3 \sin x^1,
\]

\[
\hat{x}_2 = \hat{h}_2 \sin x^1 + \hat{h}_3 \cos x^1.
\]

Like in the previous section, combining this with (58) and (56), we find the general solution of the field equations of the sigma model with metric (57) and torsion (59) as

\[
Y_1(\tau, \sigma) = \hat{h}_1(\tau, \sigma),
\]

\[
Y_2(\tau, \sigma) = \sqrt{\hat{h}_2(\tau, \sigma)^2 + \hat{h}_3(\tau, \sigma)^2},
\]

\[
Y_3(\tau, \sigma) = \arctan \left(\frac{\cos (\Omega(\tau, \sigma))\hat{h}_3(\tau, \sigma) + \sin (\Omega(\tau, \sigma))\hat{h}_2(\tau, \sigma)}{\cos (\Omega(\tau, \sigma))\hat{h}_2(\tau, \sigma) - \sin (\Omega(\tau, \sigma))\hat{h}_3(\tau, \sigma)}\right),
\]

\[
Y_4(\tau, \sigma) = \hat{h}_1(\tau, \sigma) + \hat{h}_3(\tau, \sigma) W^1(\tau, \sigma) - \hat{h}_2(\tau, \sigma) W^3(\tau, \sigma),
\]

\[
\hat{h}_2(\tau, \sigma) = \frac{\delta_2}{\alpha^2 + \delta_2^2 + \delta_3^2}, \quad \hat{h}_3(\tau, \sigma) = \frac{\delta_3}{\alpha^2 + \delta_2^2 + \delta_3^2}.
\]
where the $W^I(\tau, \sigma)$ are solutions of the wave equations (10), $\Omega(\tau, \sigma) = \frac{W^I(\tau, \sigma)}{\sigma}$, and the $\tilde{h}_\mu$ are solutions of the PDEs (12), (13):

$$\tilde{h}_1 = \gamma_1 - \int [a \, \partial_\tau W^z + W^\sigma \partial_\sigma W^z - W^z \partial_\tau W^\sigma] \, d\sigma,$$

$$\tilde{h}_2 = \gamma_2 - \int \partial_\tau W^z \, d\sigma, \quad \tilde{h}_3 = \gamma_3 - \int \partial_\sigma W^z \, d\sigma,$$

$$\tilde{h}_4 = \gamma_4 + \int \partial_\tau W^z \, d\sigma.$$

As this background is not of the pp-wave form, the light-cone gauge cannot be implemented [4]. Nevertheless, the field equations are solvable.

6. Results for other subalgebras

The classification of subalgebras of the Poincaré algebra in [17] was carried out up to a group of inner automorphisms of the connected component of the Poincaré group (proper orthochronous Poincaré transformations). There are 35 inequivalent four-dimensional subalgebras of the Poincaré algebra generated by Killing vectors (25).

Only the subgroups corresponding to the subalgebras $S_1, S_2, S_4, S_5, S_6, S_7, S_{17}, S_{18}, S_{19}, S_{23}, S_{25}$–$S_{29}, S_{31}, S_{33}$, listed in the Appendix, act transitively and freely on the flat spacetime and can be used for the atomic non-Abelian T-duality. Non-Abelian duals generated by the subalgebras $S_1, S_2, S_6$ give backgrounds with flat metric and vanishing torsion. We will not discuss them further. Dual backgrounds obtained from duality w.r.t. the subalgebras $S_{17}, S_{18}, S_{19}$ have nontrivial scalar curvature. The others are pp-waves, most of them with nonzero torsion, as we shall see from the following list of results. We do not repeat the results for subalgebras $S_{27}, S_{17}, S_{19}$ described in section 5.

6.1. The pp-waves

6.1.1. Subalgebras $S_7, S_8$. The non-isomorphic subalgebras

$$S_7 = \text{Span}[K_1 = 2M_1 + \alpha P_1, \ K_2 = L_2 + M_1, \ K_3 = P_0 - P_3, \ K_4 = P_2],$$

$$S_8 = \text{Span}[K_1 = M_1, \ K_2 = L_2 + M_1, \ K_3 = P_0 - P_3, \ K_4 = P_2]$$

differ only in the value of the parameter $\alpha$: it is positive for $S_7$, while $\alpha = 0$ for $S_8$ [17]. The commutation relations for $S_7$ are

$$[K_1, K_2] = -2K_2 - \alpha K_3, \quad [K_1, K_3] = -2K_3.$$

The transformation of coordinates in the flat background

$$t = x^1 x^2 (\alpha) + \frac{1}{2} e^{-2i} \left((x^2)^2 - 1\right) + x^3, \quad x = x^1 e^{-2i} (x^2),$$

$$z = x^1 x^2 \alpha - \frac{1}{2} e^{-2i} \left((x^2)^2 - 1\right) - x^3, \quad y = x^4,$$
gives components of the flat metric in the group coordinates:

\[
F_{\mu \nu}(x) = \begin{pmatrix}
\alpha^2 & -e^{-2\alpha} \left( 2\chi^1 + 1 \right) & 2e^{-2\alpha} & 0 \\
-e^{-2\alpha} \left( 2\chi^1 + 1 \right) & e^{-4\alpha} & 0 & 0 \\
2e^{-2\alpha} & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

The dual background in this case is

\[
\tilde{F}_{\mu \nu}(x) = \begin{pmatrix}
0 & 0 & \frac{1}{2 - 2\tilde{x}_1} & 0 \\
0 & 1 & \frac{\tilde{x}_3\alpha + \alpha + 2\tilde{x}_2}{2 - 2\tilde{x}_1} & 0 \\
\frac{1}{2\tilde{x}_3 + 2} & \frac{-\tilde{x}_3\alpha - \alpha - 2\tilde{x}_2}{2\tilde{x}_3 + 2} & \frac{(2\tilde{x}_2 + \alpha\tilde{x})^2}{4(\tilde{x}_3^2 - 1)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

and the torsion vanishes.

The transformation to Brinkmann coordinates valid for \( |\tilde{x}_3| < 1 \),

\[
\tilde{x}_1 = -2v - \frac{1}{4} \left( a^2u + 4z_3\alpha + 2z_3\alpha \log \left( 1 - \tanh^2(u) \right) \right),
\]

\[
+ \frac{1}{16} \left[ \tanh(u) \left[ \alpha^2 \log^2 \left( 1 - \tanh^2(u) \right) + 4\alpha^2 \log \left( 1 - \tanh^2(u) \right) + 16z_3^2 + 4\alpha^2 \right] \right].
\]

\[
\tilde{x}_2 = z_3 - \frac{1}{4} \alpha \tanh(u) \log \left( 1 - \tanh^2(u) \right),
\]

\[
\tilde{x}_3 = -\tanh(u), \quad \tilde{x}_4 = z_4,
\]

brings the dual metric and dilaton to the forms

\[
ds^2 = 2dudv - 2 \frac{z_3^2}{\cosh^2(u)} du^2 + dz_3^2 + dz_4^2,
\]

\[\Phi(u) = c_1 + c_2 u + 2 \log(\cosh(u)).\]

The transformation for \(|\tilde{x}_3| > 1\) obtained by making the replacement \(\tanh \rightarrow \coth\) gives the dual metric and dilaton in Brinkmann coordinates:

\[
ds^2 = 2dudv + 2 \frac{z_3^2}{\sinh^2(u)} du^2 + dz_3^2 + dz_4^2,
\]

\[\Phi(u) = c_1 + c_2 u + 2 \log(\sinh(u)).\]

These results are independent of \(\alpha\) and valid for both \(S_7\) and \(S_8\); hence we can restrict consideration to the simpler case of \(S_8\). Even though the adjoint representation of \(S_8\) is not faithful, we can solve equation (9) for coordinates of \(\tilde{g}\):

\[
\tilde{x}_1 = \tilde{h}_1 - x^1\tilde{h}_2 - x^1\tilde{h}_3,
\]

\[
\tilde{x}_2 = e^{i\tilde{h}_2}, \quad \tilde{x}_3 = e^{i\tilde{h}_3}, \quad \tilde{x}_4 = \tilde{h}_4.
\]
Like in the previous section, transformations (60) and (63) enable us to find general solutions of field equations of the sigma models with metrics (61) and (62).

6.1.2. Subalgebra $S_{23}$

\[ S_{23} = \text{Span}\{\mathcal{K}_1 = L_2 + M_1 - \frac{1}{2}(P_0 + P_3), \mathcal{K}_2 = L_1 - M_2 + \alpha P_4, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = P_2\}, \quad \alpha > 0. \]

The commutation relations are

\[ [\mathcal{K}_1, \mathcal{K}_2] = \alpha \mathcal{K}_3 - \mathcal{K}_4, \quad [\mathcal{K}_2, \mathcal{K}_4] = -\mathcal{K}_3, \quad \alpha > 0. \]

The transformation of coordinates in the flat background is

\[ t = \frac{1}{6} \left( -x^3 + 3\left( x^2 + 1 \right)x + 6x^3 \right), \quad x = x^2 + \frac{x^4}{2}, \]
\[ z = \frac{1}{6} \left( -x^3 + 3\left( x^2 - 1 \right)x - 6x^3 \right), \quad y = x^4 - x^3x^2. \]

The flat metric in the group coordinates reads

\[ F_{\mu\nu}(x) = \begin{pmatrix} 0 & \alpha x^4 & 1 - x^2 \\ \alpha x^4 & \alpha^2 + (x^4)^2 & 0 - x^4 \\ -x^2 & 0 & 0 \end{pmatrix}, \]

and the dual background is given by

\[ \tilde{F}^\kappa_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & \tilde{x}_4 - \alpha \tilde{x}_3 & -\tilde{x}_3 \\ \alpha^2 + \tilde{x}_3^2 & \tilde{x}_4 - \alpha \tilde{x}_3 & 0 & \tilde{x}_4(\tilde{x}_4 - \alpha \tilde{x}_3) \\ 1 & \alpha \tilde{x}_3 - \tilde{x}_4 & \tilde{x}_3(\tilde{x}_4 - \alpha \tilde{x}_3) & \tilde{x}_4^2 + \tilde{x}_3^2 \end{pmatrix}. \]

The dual metric in Brinkmann coordinates

\[ \tilde{x}_1 = \frac{1}{24(1 + u^2)^{3/2}} \left( -12(4 + u^2)\left( 1 + u^2 \right)\alpha z_3 - 12u\sqrt{1 + u^2}(2 + u^2)\tilde{z}_3^2 \right. \]
\[ + \sqrt{1 + u^2}\left( 1 + u^2\right)(24v + u(-48 + 28u^2 - 3u^4)\alpha^4) - 12u\tilde{z}_3^2 \right), \]
\[ \tilde{x}_2 = \sqrt{1 + u^2}\alpha z_4, \quad \tilde{x}_3 = ua, \quad \tilde{x}_4 = \frac{1}{2}u(-4 + u^2)\alpha^2 + \sqrt{1 + u^2} z_3 \]
then has the form
\[ ds^2 = 2dudv + \frac{(2u^2 - 1)\xi^2 + 3\zeta^2}{(u^2 + 1)^2} du^2 + dz_1^2 + dz_2^2 . \]

while the torsion and the dilaton are
\[ H = \frac{2}{1 + u^2} d\alpha \wedge dz_3 \wedge dz_4, \quad \Phi(u) = c_1 + c_2 u + \log \left(1 + u^2\right) . \]

To find general solutions of the field equations of the dual sigma model, we have to express the coordinates \( \tilde{x}_\mu \) in terms of \( x^\nu \) and \( \tilde{h}_k \). We get
\[ \alpha = \alpha = - \frac{1}{2} x^2 \tilde{h}_3 - \tilde{h}_4, \quad \tilde{x}_3 = \tilde{h}_3, \]
\[ \tilde{x}_2 = \tilde{h}_2 - x^4 \left(\alpha \tilde{h}_3 - x^2 \tilde{h}_3 - \tilde{h}_4\right), \quad \tilde{x}_4 = x^2 \tilde{h}_3 + \tilde{h}_4 . \]

6.1.3. Subalgebra \( S_{25} \).

\( S_{25} = \text{Span}\left[\mathcal{K}_1 = L_2 + M_1 - \varepsilon P_2, \mathcal{K}_2 = P_0 + P_3, \mathcal{K}_3 = P_1, \mathcal{K}_4 = P_0 - P_1\right], \quad \varepsilon = \pm 1 . \)

The commutation relations are
\[ [\mathcal{K}_1, \mathcal{K}_2] = 2 \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_4 . \]

The transformation of coordinates in the flat background
\[ t = x^2 + x^4, \quad x = x^3, \quad y = -e x^1, \quad z = x^2 - x^4 \]
yields the flat metric in the group coordinates:
\[ F_{\mu\nu}(x) = \begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} . \]

The dual background is given by
\[ \bar{F}^\mu_{\nu}(\tilde{x}) = \begin{pmatrix} 1 & 0 & \tilde{x}_4 & -\tilde{x}_3 \\ 0 & \frac{\tilde{x}_4}{\tilde{x}_4 + e^2} & \frac{-\tilde{x}_3}{\tilde{x}_4 + e^2} & \frac{-1}{2} \\ 0 & 0 & \frac{e^2}{\tilde{x}_4 + e^2} & \frac{\tilde{x}_3 \tilde{x}_4}{\tilde{x}_4 + e^2} \\ \tilde{x}_3 & \frac{1}{\tilde{x}_4 + e^2} & \frac{-\tilde{x}_3 \tilde{x}_4}{\tilde{x}_4 + e^2} & \frac{-\tilde{x}_3^2}{\tilde{x}_4 + e^2} \end{pmatrix} . \]

The transformation to Brinkmann coordinates
\[ \tilde{x}_1 = e \sqrt{u^2 + 1} z_4, \quad \tilde{x}_3 = \sqrt{u^2 + 1} z_3 , \]
\[ \tilde{x}_2 = \frac{1}{e(u^2 + 1)} \left[u(u^2 + 2)z_3^2 + uz_2^2\right] - 2e v, \quad \tilde{x}_4 = e u . \]
gives the dual metric
\[ dz^2 = 2du \, dv + \frac{(2u^2 - 1)z_1^2 - 3z_3^2}{(u^2 + 1)^2} \, du^2 + dz_3^2 + dz_4^2. \]

The torsion and dilaton then read
\[ H = -\frac{2}{1 + u^2} \, du \wedge dz_3 \wedge dz_4, \quad \Phi(u) = c_1 + c_2 \, u + \log \left( 1 + u^2 \right). \]

To find general solutions of field equations of the dual sigma model we have to express the coordinates \( \tilde{x}_\mu \) in terms of \( x^\nu \) and \( \tilde{h}_k \). We get
\begin{align*}
\tilde{x}_1 &= \tilde{h}_1 + 2x^2 h_3 + x^3 h_4, \\
\tilde{x}_2 &= \tilde{h}_2 - x^1(2\tilde{h}_3 - x^1 h_4), \\
\tilde{x}_3 &= \tilde{h}_3 - x^1 h_4, \\
\tilde{x}_4 &= \tilde{h}_4.
\end{align*}

6.1.4. Subalgebras \( S_{26}, S_{27} \). The subalgebras \( S_{26}, S_{27} \) differ once again only in the value of parameter \( \alpha \): it is positive for \( S_{26} \), while \( \alpha = 0 \) for \( S_{27} \) [17].
\begin{align*}
S_{26} &= \text{Span} \left[ \mathcal{K}_1 = M_1 + \alpha P_3, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = L_1 - M_2, \mathcal{K}_4 = P_0 - P_3 \right], \\
S_{27} &= \text{Span} \left[ \mathcal{K}_1 = M_3, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = L_1 - M_2, \mathcal{K}_4 = P_0 - P_3 \right].
\end{align*}

Their commutation relations are
\[ [\mathcal{K}_1, \mathcal{K}_2] = -\mathcal{K}_2 - \alpha \mathcal{K}_4, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4. \]

The transformation of coordinates in the flat background
\begin{align*}
t &= -\alpha x^1 x^2 + \frac{1}{2} e^{-2i} \left( (x^1)^2 + (x^2)^2 \right) + x^4, \quad x = x^1 - e^{-i} x^2, \\
z &= \alpha x^1 x^2 - \frac{1}{2} e^{-2i} \left( (x^1)^2 + (x^2)^2 - 1 \right) - x^4, \quad y = e^{-i} x^3
\end{align*}
gives the flat metric in the group coordinates:
\[ F_{\mu \nu}(x) = \begin{pmatrix}
\alpha^2 & -e^{-2i} \alpha (x^1 + 1) & 0 & e^{-2i} \\
-e^{-2i} \alpha (x^1 + 1) & e^{-2i} & 0 & 0 \\
0 & 0 & e^{-2i} & 0 \\
e^{-2i} & 0 & 0 & 0
\end{pmatrix}. \]
In the dual background,

\[
\bar{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix}
0 & 0 & 0 & \frac{1}{1 - \tilde{x}_4} \\
0 & 1 & 0 & \frac{\tilde{x}_4 \alpha + \alpha + \tilde{x}_2}{1 - \tilde{x}_4} \\
0 & 0 & 1 & \frac{\tilde{x}_3}{1 - \tilde{x}_4} \\
\frac{1}{\tilde{x}_4 + 1} & -\frac{\tilde{x}_4 \alpha + \alpha - \tilde{x}_2}{\tilde{x}_4 + 1} & \frac{\tilde{x}_3}{\tilde{x}_4 + 1} & \frac{\tilde{x}_3^2 + 2\alpha \tilde{x}_4 \tilde{x}_2 + \tilde{x}_3^2 + \alpha^2 \tilde{x}_4^2}{\tilde{x}_4^2 - 1}
\end{pmatrix},
\]

the torsion vanishes.

The transformation to Brinkmann coordinates

\[
\bar{x}_1 = -\nu + \frac{1}{8} \left( -4 \alpha^2 + \tanh(u) \left( 4 \left( z_3^2 + z_4^2 + \alpha^2 \right) + \alpha^2 \log \left( 1 - \tanh^2(u) \right) \left( \log \left( 1 - \tanh^2(u) \right) + 4 \right) \right) - 4z_3 \alpha \left( \log \left( 1 - \tanh^2(u) \right) + 2 \right) \right),
\]

\[
\bar{x}_2 = z_3 - \frac{1}{2} \alpha \tanh(u) \log \left( 1 - \tanh^2(u) \right),
\]

\[
\bar{x}_3 = z_4,
\]

\[
\bar{x}_4 = -\tanh(u),
\]

for \(|\bar{x}_1| < 1\), brings the dual metric and dilaton to forms independent of \(\alpha\):

\[
d\bar{s}^2 = 2d\bar{u}d\bar{v} - 2 \frac{z_3^2 + z_4^2}{\cosh^2(u)} d\bar{u}^2 + dz_3^2 + dz_4^2,
\]

\[
\Phi(u) = c_1 + c_2 \ u + 4 \log (\cosh (u)).
\]

A similar transformation (see section 5.1) gives the dual metric and dilaton for \(|\bar{x}_1| > 1\) in Brinkmann coordinates:

\[
d\bar{s}^2 = 2d\bar{u}d\bar{v} + 2 \frac{z_3^2 + z_4^2}{\sinh^2(u)} d\bar{u}^2 + dz_3^2 + dz_4^2,
\]

\[
\Phi(u) = c_1 + c_2 \ u + 4 \log (\sinh (u)).
\]

The solution of the field equations of the dual sigma models was found in section 5.1.

6.1.5. Subalgebra \(S_{28}\)

\(S_{28} = \text{Span}\left[ K_1 = L_3 - \beta M_3, K_2 = L_2 + M_1, K_3 = L_1 - M_2, K_4 = P_0 - P_1 \right], \ \beta \neq 0.\)

The commutation relations are

\[
[K_1, K_2] = \beta K_2 - K_3, \quad [K_1, K_3] = K_2 + \beta K_3, \quad [K_1, K_4] = \beta K_4, \quad \beta \neq 0.
\]
The transformation of coordinates in the flat background
\[
\begin{align*}
t &= \frac{1}{2} \left( (x^2)^2 + (x^3)^2 + 1 \right) e^{\xi_\beta} + x^4, \quad x = x^2 (e^{\xi_\beta}), \\
z &= -\frac{1}{2} \left( (x^2)^2 + (x^3)^2 - 1 \right) e^{\xi_\beta} - x^4, \quad y = x^3 e^{\xi_\beta}
\end{align*}
\]
gives the flat metric in the group coordinates:
\[
F_{\mu\nu} (x) = \begin{pmatrix}
0 & 0 & 0 & -e^{\xi_\beta} \\
0 & e^{2\xi_\beta} x^i & 0 & 0 \\
0 & 0 & e^{2\xi_\beta} x^i & 0 \\
-e^{\xi_\beta} x^i & 0 & 0 & 0
\end{pmatrix}
\]
After the transformation
\[
\begin{align*}
\tilde{x}_1 &= \frac{1}{2} \left( 2\nu - \tanh (u) (z_3^2 + z_4^2) \right), \\
\tilde{x}_2 &= z_3 \cos \left( \frac{\log (\cosh (u))}{\beta} \right) + z_4 \sin \left( \frac{\log (\cosh (u))}{\beta} \right), \\
\tilde{x}_3 &= z_4 \cos \left( \frac{\log (\cosh (u))}{\beta} \right) - z_3 \sin \left( \frac{\log (\cosh (u))}{\beta} \right), \\
\tilde{x}_4 &= -\tanh (u)
\end{align*}
\]
of the dual background,
\[
\tilde{F}_{\mu\nu} (\tilde{x}) = \begin{pmatrix}
0 & 0 & 0 & \frac{1}{\beta (\tilde{x}_4 - 1)} \\
0 & 1 & 0 & \frac{\beta \tilde{x}_2 - \tilde{x}_3}{\beta - \beta \tilde{x}_4} \\
0 & 0 & 1 & \frac{\tilde{x}_2 + \beta \tilde{x}_3}{\beta - \beta \tilde{x}_4} \\
\frac{1}{\tilde{x}_4 \beta + \beta} & \frac{\tilde{x}_3 - \beta \tilde{x}_2}{\beta (\tilde{x}_4 + 1)} & -\frac{\tilde{x}_2 + \beta \tilde{x}_3}{\tilde{x}_4 \beta + \beta} & \frac{\beta^2 + 1}{\beta^2 (\tilde{x}_4 - 1)}
\end{pmatrix}
\]
the dual metric and the dilaton for \(|\tilde{x}_4| < 1\) are expressed in the Brinkmann coordinates as
\[
d\tilde{s}^2 = 2dudv - \frac{\left( z_3^2 + z_4^2 \right) \left( 1 + 2\beta^2 \cosech^2 (u) \right)}{\beta^2} du^2 + dz_3^2 + dz_4^2,
\]
\[
\Phi (u) = c_1 + c_2 u + 4 \log (\cosh (u)).
\]
The dual metric and the dilaton for \(|\tilde{x}_4| > 1\) in Brinkmann coordinates are
\[
d\tilde{s}^2 = 2dudv - \frac{\left( z_3^2 + z_4^2 \right) \left( 1 - 2\beta^2 \coth^2 (u) \right)}{\beta^2} du^2 + dz_3^2 + dz_4^2,
\]
\[
\Phi (u) = c_1 + c_2 u + 4 \log (\sinh (u)).
\]
In both cases the torsion is of the form

\[ H = -\frac{2}{\beta} du \wedge dz_3 \wedge dz_4. \]

To find the solution of the equations of the dual sigma model, we also need \( \tilde{x}_j, h^k \) expressed in terms of \( x^i \) as

\[ \begin{align*}
\tilde{x}_1 &= x^2 \tilde{h}_2 + x^3 \tilde{h}_3 + x^4 \tilde{h}_4 + \tilde{h}_1 + x^3 \tilde{h}_2 - x^2 \tilde{h}_3, \\
\tilde{x}_2 &= e^{-\beta \tilde{t}} \left( \tilde{h}_3 \sin \left( x^1 \right) + \tilde{h}_2 \cos \left( x^1 \right) \right), \\
\tilde{x}_3 &= e^{-\beta \tilde{t}} \left( -\tilde{h}_2 \sin \left( x^1 \right) + \tilde{h}_3 \cos \left( x^1 \right) \right), \\
\tilde{x}_4 &= \tilde{h}_4 e^{-\beta \tilde{t}}.
\end{align*} \]

### 6.1.6. Subalgebra \( S_{29} \)

\[ S_{29} = \text{Span} \left[ \mathcal{K}_1 = L_3 - \beta M_3, \mathcal{K}_2 = R_0 - P_3, \mathcal{K}_3 = P_3, \mathcal{K}_4 = P_2 \right], \quad \beta \neq 0. \]

The commutation relations are

\[ \begin{align*}
\left[ \mathcal{K}_1, \mathcal{K}_2 \right] &= \beta \mathcal{K}_2, \\
\left[ \mathcal{K}_1, \mathcal{K}_3 \right] &= \mathcal{K}_4, \\
\left[ \mathcal{K}_1, \mathcal{K}_4 \right] &= -\mathcal{K}_3, \\
\beta \neq 0.
\end{align*} \]

The transformation of coordinates in the flat background

\[ t = -\frac{1}{2} e^{\sqrt{\beta}} + x^2, \quad x = x^3, \quad y = x^4, \quad z = -\frac{1}{2} \left( e^{\sqrt{\beta}} - x^2 \right) \]

gives the flat metric in the group coordinates:

\[ F_{\mu \nu}(x) = \begin{pmatrix}
0 & e^{\beta x^1} & 0 & 0 \\
e^{\beta x^1} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \]

The dual background is

\[ \tilde{F}_{\mu \nu}(\tilde{x}) = \begin{pmatrix}
0 & \frac{1}{\tilde{x}_2 \beta + \tilde{\beta}} & 0 & 0 \\
\frac{1}{\beta - \beta \tilde{x}_2} & \frac{\beta^2 (\tilde{x}_2^2 - 1)}{\beta - \beta \tilde{x}_2} & \tilde{x}_4 & \tilde{x}_3 \\
0 & -\frac{\tilde{x}_4}{\tilde{x}_2 \beta + \beta} & 1 & 0 \\
0 & \frac{\tilde{x}_3}{\tilde{x}_2 \beta + \beta} & 0 & 1
\end{pmatrix}. \]

The dual metric, dilaton and torsion in Brinkmann coordinates are the same as in section 5.2:

\[ \begin{align*}
ds^2 &= 2 du dv - (z_3^2 + z_4^2) du^2 + dz_3^2 + dz_4^2, \\
\Phi(u) &= c_1 + c_2 u, \\
H &= -2 du \wedge dz_3 \wedge dz_4.
\end{align*} \]

To find the solution of the equations of motion of the dual sigma model, we also need \( \tilde{x}_j, h^k \) expressed in terms of \( x^i \) as
\[ x_1 = x^2 \tilde{h}_3 \beta + \tilde{h}_1 - x^2 \tilde{h}_3 + x^3 \tilde{h}_4, \quad x_3 = \tilde{h}_3 \cos (x^1) - \tilde{h}_4 \sin (x^1), \]
\[ x_2 = \tilde{h}_2 e^{x^1(-\beta)}, \quad x_4 = \tilde{h}_3 \sin (x^1) + \tilde{h}_4 \cos (x^1). \]

6.1.7. Subalgebras \( S_{31}, S_{33} \). The subalgebras

\[ S_{31} = \text{Span} \left[ \mathcal{K}_1 = M_3, \mathcal{K}_2 = R + \beta P_2, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = L_2 + M_1 \right], \]
\[ S_{33} = \text{Span} \left[ \mathcal{K}_1 = M_3 + \alpha P_2, \mathcal{K}_2 = P_1 + \beta P_2, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = L_2 + M_1 \right] \]

differ only in the value of the parameter \( \alpha \): it is positive for \( S_{33} \), while \( \alpha = 0 \) for \( S_{31} \). In both cases, \( \beta \neq 0 \). The subalgebras are isomorphic even though they are not equivalent under conjugacy through proper orthochronous Poincaré transformations. Their commutation relations are

\[ [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4, \quad [\mathcal{K}_2, \mathcal{K}_4] = -\mathcal{K}_3. \]

The transformation of coordinates in the flat background

\[ t = x^3 - x^2 x^4 - \frac{1}{2} e^{-x^1} \left( x^4 \right)^{2} + 1, \quad x = x^2 + e^{-x^1} x^4, \]
\[ z = -x^3 + x^2 x^4 + \frac{1}{2} e^{-x^1} \left( x^4 \right)^{2} - 1, \quad y = x^4 + x^2 \beta \]
gives the flat metric in the group coordinates as

\[
F_{\mu \nu}(x) = \begin{pmatrix}
\alpha^2 & \alpha \beta & -e^{-x^1} \xi^2 & e^{-x^1} \xi^2 \\
\alpha \beta & \beta^2 + 1 & 0 & e^{-x^1} \\
-e^{-x^1} & 0 & -\xi^2 & 0 \\
e^{-x^1} \xi^2 & e^{-x^1} & 0 & -e^{-2x^1}
\end{pmatrix}.
\]

For the dual background

\[
\vec{F}_{\mu \nu}(\xi) = \begin{pmatrix}
0 & 0 & -\frac{1}{\tilde{x}_3 + 1} & 0 \\
0 & \frac{1}{\beta^2 + \tilde{x}_3^2} & \frac{a \beta (\tilde{x}_3 + 1) \tilde{x}_4}{(\tilde{x}_3 + 1)(\beta^2 + \tilde{x}_3^2)} & -\frac{\tilde{x}_3 + 1}{\beta^2 + \tilde{x}_3^2} \\
0 & \frac{1}{\beta^2 + \tilde{x}_3^2} & \frac{-a \beta - \tilde{x}_3 \tilde{x}_4 + \tilde{x}_4}{(\tilde{x}_3 - 1)(\beta^2 + \tilde{x}_3^2)} & \frac{\alpha \beta (\tilde{x}_3 + 1) - (\beta^2 + 1) \tilde{x}_4}{(\tilde{x}_3 - 1)(\beta^2 + \tilde{x}_3^2)} \\
0 & \frac{1}{\beta^2 + \tilde{x}_3^2} & \frac{-a \beta - \tilde{x}_3 \tilde{x}_4 + \tilde{x}_4}{(\tilde{x}_3 - 1)(\beta^2 + \tilde{x}_3^2)} & \frac{\alpha \beta (\tilde{x}_3 + 1) - (\beta^2 + 1) \tilde{x}_4}{(\tilde{x}_3 - 1)(\beta^2 + \tilde{x}_3^2)}
\end{pmatrix}
\]

we can find a rather complicated coordinate transformation that enables us to eliminate the dependence on \( \alpha \) of the background. The dual metric, torsion and dilaton for \( |\tilde{x}_3| < 1 \) in Brinkmann coordinates are
\[ ds^2 = 2 du dv + \left[ \frac{\text{sech}^4(u) \left( 2 (\beta^2 + 1) \sinh^2(u) - \beta^2 \right)}{\left( \tanh^2(u) + \beta^2 \right)^2} - \frac{\beta^2 \text{sech}^4(u) \left( 2 (\beta^2 + 1) \cosh^2(u) + 1 \right)}{\left( \tanh^2(u) + \beta^2 \right)^2} \right] du^2 + dz^2_1 + dz^2_1, \]

\[ H = \frac{2 \beta}{\beta^2 \cosh^2(u) + \sinh^2(u)} \, du \wedge dz_1 \wedge dz_4, \]

\[ \Phi(u) = c_1 + c_2 u + \log \left( (\beta^2 + 1) \cosh(2u) + \beta^2 - 1 \right). \]

The dual metric, torsion and dilaton for \(|x_3| > 1\) in Brinkmann coordinates are

\[ ds^2 = 2 du dv + \left[ \frac{\beta^2 \text{sech}^4(u) \left( 2 (\beta^2 + 1) \sinh^2(u) - 1 \right)}{\left( \coth^2(u) + \beta^2 \right)^2} - \frac{\text{cosech}^4(u) \left( 2 (\beta^2 + 1) \cosh^2(u) + \beta^2 \right)}{\left( \coth^2(u) + \beta^2 \right)^2} \right] du^2 + dz^2_1 + dz^2_1, \]

\[ H = -\frac{2 \beta}{\cosh^2(u) + \beta^2 \sinh^2(u)} \, du \wedge dz_3 \wedge dz_4, \]

\[ \Phi(u) = c_1 + c_2 u + \log \left( (\beta^2 + 1) \cosh(2u) - \beta^2 + 1 \right). \]

To find general solutions of field equations of the dual sigma model it is sufficient to express the coordinates \(x_\mu\) in terms of \(x^i\) and \(h_i\) for \(\alpha = 0\). We get

\[ \tilde{x}_1 = h_1 - x^3 h_3 - x^4 h_4, \quad \tilde{x}_2 = h_2 - x^4 h_3, \]

\[ \tilde{x}_3 = e^{i \tau} h_3, \quad \tilde{x}_4 = e^{i \tau} \left( x^2 h_3 + h_4 \right). \]

6.2. Diagonalizable metrics with nontrivial scalar curvature

6.2.1. Subalgebra \(S_{11}\)

\[ S_{11} = \text{Span} \left[ \mathcal{K}_1 = M_3 + \alpha P_2, \mathcal{K}_2 = P_3, \mathcal{K}_3 = P_3, \mathcal{K}_4 = P_1 \right], \quad \alpha > 0. \]

The commutation relations are

\[ [\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_2. \]

The flat metric in the group coordinates

\[ x^1 = \frac{y}{\alpha}, \quad x^2 = t, \quad x^3 = z, \quad x^4 = x. \]
reads

\[ F_{\mu\nu} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

The dual background

\[
\tilde{F}_{\mu\nu} = \begin{pmatrix}
\frac{1}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & -\frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & -\frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & 0 \\
\frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & \frac{1}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & -\frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & 0 \\
-\frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & -\frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & \frac{1}{\alpha^2 - \tilde{x}_3^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

gives a metric with nonvanishing scalar curvature:

\[
\tilde{R} = 2 \frac{2\tilde{x}_2^2 - 2\tilde{x}_3^2 - 5\alpha^2}{(\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2)^2},
\]

so it cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the time-dependent form

\[
ds^2 = -dy_1^2 + dy_2^2 + \frac{y_1^2 \alpha^2}{y_1^2 + \alpha^2} \, dy_3^2 + \frac{1}{y_1^2 + \alpha^2} \, dy_4^2, \tag{64}
\]

via

\[
\tilde{x}_1 = y_3, \quad \tilde{x}_2 = y_1 \cosh y_3, \quad \tilde{x}_3 = y_1 \sinh y_3, \quad \tilde{x}_4 = y_2.
\]

The torsion then acquires the form

\[
H = -\frac{2y_1 \alpha^2}{(y_1^2 + \alpha^2)^2} \, dy_4 \wedge dy_3 \wedge dy_4,
\]

and the dilaton satisfying (19)–(21) is

\[
\Phi = \log \left( \frac{y_1^2 + \alpha^2}{\alpha^2} \right) + \text{const}.
\]

To find the solution of the equations of this dual sigma model we need the above transformation between \(y_j\) and \(\tilde{x}_j\), and also \(\tilde{x}_j\) expressed in terms of \(x^i\), \(\tilde{h}_i\):

\[
\tilde{x}_1 = \tilde{h}_1 + x^2 \tilde{h}_3 + x^3 \tilde{h}_2, \quad \tilde{x}_2 = \tilde{h}_2 \cosh x^1 - \tilde{h}_3 \sinh x^1, \quad \tilde{x}_3 = \tilde{h}_3 \cosh x^1 - \tilde{h}_2 \sinh x^1.
\]

6.2.2. Subalgebra \(S_{18}\)

\[ S_{18} = \text{Span}\left[ \mathcal{K}_1 = L_3 + \alpha P_0, \mathcal{K}_2 = P_1, \mathcal{K}_3 = P_2, \mathcal{K}_4 = P_3 \right], \quad \alpha > 0. \]
The commutation relations are the same as for $S_{17}$ and $S_{19}$:

\[
[K_1, K_2] = K_3, \quad [K_1, K_3] = -K_2.
\]

The flat metric in the group coordinates

\[
x^1 = \frac{t}{a}, \quad x^2 = x, \quad x^3 = y, \quad x^4 = z
\]

reads

\[
\alpha = \begin{pmatrix}
-\alpha^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The dual background

\[
\tilde{F}_{\mu\nu} = \begin{pmatrix}
1 & \tilde{x}_3 & -\tilde{x}_2 \\
-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2 & -\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2 & 0 \\
-\tilde{x}_1 & -\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2 & 0 \\
0 & 0 & 0 \\
-\tilde{x}_2^2 + \tilde{x}_3^2 & -\tilde{x}_2^2 + \tilde{x}_3^2 & 0
\end{pmatrix}
\]

gives a metric with nonvanishing scalar curvature:

\[
\tilde{R} = -\frac{10\alpha^2 + 4\left(\tilde{x}_2^2 + \tilde{x}_3^2\right)}{\left(\tilde{x}_2^2 + \tilde{x}_3^2 - \alpha^2\right)^2},
\]

so it cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the form

\[
d\tilde{s}^2 = \frac{1}{\tilde{x}_3^2 - \alpha^2} dy_1^2 + \frac{y_3^2\alpha^2}{\alpha^2 - \tilde{x}_3^2} dy_2^2 + dy_3^2 + dy_4^2,
\]

via

\[
\tilde{x}_1 = y_1, \quad \tilde{x}_2 = y_3 \cos y_2, \quad \tilde{x}_3 = y_3 \sin y_2, \quad \tilde{x}_4 = y_4.
\]

Note the singularity on the surfaces $y_3 = \pm \alpha$. For $|y_3| < \alpha$ the time-like direction is given by the vector $\partial_{y_3}$, whereas for $|y_3| > \alpha$ the time-like vector is $\partial_{\tilde{x}_3}$.

The torsion acquires the form

\[
H = \frac{2y_3\alpha^2}{\left(y_3^2 - \alpha^2\right)^2} dy_1 \wedge dy_2 \wedge dy_3
\]

and the dilaton satisfying (19)–(21) is

\[
\Phi = \log\left(y_3^2 - \alpha^2\right) + \text{const}.
\]

To find the solution of the equations of this dual sigma model, we need the above transformation between $y_j$ and $\tilde{x}_j$, and also $\tilde{x}_j$ expressed in terms of $x^l, \tilde{h}_k$. As the
commutation relations are the same as for \( S_{17} \) and \( S_{19} \), we get
\[
\tilde{x}_1 = \tilde{h}_1 + x^2 \tilde{h}_3 - x^3 \tilde{h}_2, \quad \tilde{x}_2 = \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1, \\
\tilde{x}_4 = \tilde{h}_4, \quad \tilde{x}_3 = \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1.
\]

7. Conclusion

We have classified all atomic non-Abelian duals of the four-dimensional flat spacetime with respect to four-dimensional subgroups of the Poincaré group. As a result, we have obtained 14 different kinds of exactly solvable sigma models in the four-dimensional curved backgrounds. Due to the non-Abelian T-duality, one can find general solutions of the classical field equations for all of these dual models in terms of d’Alembert solutions of the wave equation. The method of obtaining the solutions is described in section 3 and examples are given in sections 5 and 6. One-loop beta equations for all of the dual backgrounds yield simple ordinary differential equations for dilatons. Their solutions are given in sections 5 and 6.

Eleven of the dual backgrounds are plane-parallel waves whose metrics can be brought to the Brinkmann form
\[
ds^2 = 2 du \, dv - \left[ K_3(u) z_3^2 + K_4(u) z_4^2 \right] du^2 + dz_3^2 + dz_4^2.
\]

The torsion is then
\[H = dB = H(u) du \wedge dz_3 \wedge dz_4\]
Depending on the chosen subgroup, functions \( K_3(u) \), \( K_4(u) \), \( H(u) \) acquire various forms, as follows:
\[
K_3(u) = K_4(u) = 1, \quad H(u) = -2, \tag{66}
\]
\[
K_3(u) = \frac{3}{(u^2 + 1)^2}, \quad K_4(u) = -\frac{2u^2 - 1}{(u^2 + 1)^2}, \quad H(u) = \pm \frac{2}{u^2 + 1}, \tag{67}
\]
\[
K_3(u) = 2 \sech^2(u), \quad K_4(u) = 2 \delta \sech^2(u), \quad \delta = 0, 1, \quad H(u) = 0, \tag{68}
\]
\[
K_3(u) = -2 \cosech^2(u), \quad K_4(u) = -2 \delta \cosech^2(u), \quad \delta = 0, 1, \quad H(u) = 0, \tag{69}
\]
\[
K_3(u) = K_4(u) = \frac{1 + 2\beta^2 \sech^2(u)}{\beta^2}, \quad H(u) = -\frac{2}{\beta}, \tag{70}
\]
\[
K_3(u) = K_4(u) = \frac{1 - 2\beta^2 \cosech^2(u)}{\beta^2}, \quad H(u) = -\frac{2}{\beta}. \tag{71}
\]
\[ K_3(u) = -\frac{\text{sech}^4(u) \left( 2 \left( \beta^2 + 1 \right) \sinh^2(u) - \beta^2 \right)}{(\tanh^2(u) + \beta^2)^2}, \]
\[ K_4(u) = \frac{\beta^2 \text{sech}^4(u) \left( 2 \left( \beta^2 + 1 \right) \cosh^2(u) + 1 \right)}{(\tanh^2(u) + \beta^2)^2}, \]
\[ H(u) = \frac{2\beta}{\beta^2 \cosh^2(u) + \sinh^2(u)}, \] (72)
\[ K_3(u) = \frac{\text{cosech}^4(u) \left( 2 \left( \beta^2 + 1 \right) \cosh^2(u) + \beta^2 \right)}{(\coth^2(u) + \beta^2)^2}, \]
\[ K_4(u) = -\frac{\beta^2 \text{cosech}^4(u) \left( 2 \left( \beta^2 + 1 \right) \sinh^2(u) - 1 \right)}{(\coth^2(u) + \beta^2)^2}, \]
\[ H(u) = -\frac{2\beta}{\cosh^2(u) + \beta^2 \sinh^2(u)}, \] (73)

where \( \beta \in \mathbb{R} \setminus \{0\} \).

Even though the \( B \)-fields obtained by T-duality are usually not of the form
\[ B = B_1(u) du \wedge dz_1, \]
they are gauge equivalent to
\[ B' = H(u) du \wedge (z_3 dz_4 - z_4 dz_3), \]
and the corresponding sigma models are exactly conformal [3]. Except for (70), (71), these pp-wave backgrounds can be transformed to the gauged WZW background forms (1) by the standard transformation from Brinkmann to Rosen coordinates [19]. In most of the transformed backgrounds the function \( g_1 \) acquires the form \( g_1(u) = 1 \) and the function \( g_2 \) acquires the form of one of the functions (2), but some other combinations of functions \( (g_1, g_2) \) also arise, namely \( (u^{-2}, \tanh^2 u), (u^{-2}, \coth^2 u), (\tanh^2 u, \tanh^2 u) \) and \( (\coth^2 u, \coth^2 u) \).

Consequently, the pp-waves of the form (1) are duals of the flat metric not only for \( g_1(u) = 1 \) and \( g_2(u) = u^2 \), as mentioned in section 1, but also for many other combinations of functions \( g_1, g_2 \) from the set (2).

It is a remarkable fact that duals with respect to subgroups corresponding to non-isomorphic algebras may lead to the same backgrounds (up to a coordinate transformation). These are the cases of the metric (66) produced by subalgebras \( S_{17} \) and \( S_{29} \), and also metrics (68), (69) obtained from \( S_7 \) and \( S_8 \) for \( \delta = 0 \), and from \( S_{26} \) and \( S_{27} \) for \( \delta = 1 \). The metric (66) is apparently a homogeneous exactly solvable model with nontrivial constant torsion. On the other hand, isomorphic (but not equivalent under proper orthochronous Poincaré transformation) algebras \( S_{23} \) and \( S_{25} \) give the same metric, namely (67), but opposite torsions. Isomorphic algebras \( S_{31} \) and \( S_{33} \) give the same metrics and torsions, namely (72), (73).

We also get, besides the pp-waves, dual metrics with nonvanishing scalar curvature and torsion:
\[ ds^2 = -dy_1^2 + dy_2^2 + \frac{y_1^2}{y_1^2 + \alpha^2} dy_3^2 + \frac{1}{y_1^2 + \alpha^2} dy_4^2, \]
\[ H = -\frac{2y_1\alpha}{(y_1^2 + \alpha^2)^2} dy_1 \wedge dy_3 \wedge dy_4, \]
(74)

\[ ds^2 = \frac{1}{y_3^2 - \alpha^2} dy_1^2 + \frac{y_3^2}{\alpha^2 - y_3^2} dy_2^2 + dy_3^2 + dy_4^2, \]
\[ H = \frac{2y_3\alpha}{(y_3^2 - \alpha^2)^2} dy_1 \wedge dy_2 \wedge dy_3, \]
(75)

\[ ds^2 = -dy_1^2 + dy_2^2 + \frac{y_2^2}{y_2^2 + \alpha^2} dy_3^2 + \frac{1}{y_2^2 + \alpha^2} dy_4^2, \]
\[ H = \frac{2y_2\alpha}{(y_2^2 + \alpha^2)^2} dy_2 \wedge dy_3 \wedge dy_4. \]
(76)

They are obtained as non-Abelian duals with respect to \( S_{11}, S_{18}, S_{19} \). Note that isomorphic (but not equivalent under proper orthochronous Poincaré transformation) subalgebras \( S_{17} \) (respectively \( S_{18}, S_{19} \)) lead to backgrounds with vanishing (respectively nonvanishing) curvature.

The metrics (74)–(76) remind us of the black hole [10] and cosmological backgrounds [11] rewritten in [2] into diagonal forms depending again on particular functions \( s_1, s_2 \). The difference from (74)–(76) lies in these functions.

Appendix. Poincaré subalgebras

We summarize the four-dimensional Poincaré subalgebras that act freely and transitively on the flat manifold. The numbering of the subalgebras follows from the order introduced in [17], table IV.

\[ S_1 = \text{Span}[R_0, P_1, P_2, P_3], \]
\[ S_2 = \text{Span}[M_3, R_0 - P_0, P_1, P_2], \]
\[ S_6 = \text{Span}[L_2 + M_1 - \frac{1}{2} (R_0 + P_0), P_1, R_0 - P_3, P_2], \]
\[ S_7 = \text{Span}[2M_3 + \alpha P_1, L_2 + M_1, R_0 - P_3, P_2], \quad \alpha > 0, \]
\[ S_8 = \text{Span}[M_3, L_2 + M_1, R_0 - P_3, P_2], \]
\[ S_{11} = \text{Span}[M_3 + \alpha P_2, P_0, P_3, P_1], \quad \alpha > 0, \]
\[ S_{17} = \text{Span}[L_3 + \epsilon (R_0 + P_0), P_1, P_2, (P_0 - P_3)], \quad \epsilon = \pm 1, \]
\[ S_{18} = \text{Span}[L_3 + \alpha R_0, P_0, P_2, P_3], \quad \alpha > 0, \]
$S_{19} = \text{Span} \left[ L_3 + \alpha P_1, P_1, P_2, P_0 \right], \quad \alpha \neq 0,$
$S_{23} = \text{Span} \left[ L_2 + M_1 - \frac{1}{2} (P_0 + P_3), L_1 - M_2 + \alpha P_1, P_0 - P_3, P_2 \right], \quad \alpha \neq 0,$
$S_{25} = \text{Span} \left[ L_2 + M_1 - e P_3, P_0 + P_1, P_0 - P_1 \right], \quad e = \pm 1,$
$S_{26} = \text{Span} \left[ M_3 + \alpha P_1, L_2 + M_1, L_1 - M_2, P_0 - P_3 \right], \quad \alpha > 0,$
$S_{27} = \text{Span} \left[ M_3, L_2 + M_1, L_1 - M_2, P_0 - P_3 \right], \quad \beta \neq 0,$
$S_{28} = \text{Span} \left[ L_3 - \beta M_3, L_2 + M_1, L_1 - M_2, P_0 - P_3 \right], \quad \beta \neq 0,$
$S_{29} = \text{Span} \left[ L_3 - \beta M_3, P_0 - P_3, P_1, P_2 \right], \quad \beta \neq 0,$
$S_{31} = \text{Span} \left[ M_3, P_1 + \beta P_3, P_0 - P_3, L_2 + M_1 \right], \quad \beta \neq 0,$
$S_{33} = \text{Span} \left[ M_3 + \alpha P_2, P_1 + \beta P_3, P_0 - P_3, L_2 + M_1 \right], \quad \alpha > 0, \beta \neq 0.$

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