Non-Markovian dynamics of the sine-Gordon solitons

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The sine-Gordon equation exhibits a gap in its linear spectrum. That gives rise to memory, or non-Markovian, effects in the soliton formation processes. The generalized variational approach is suggested to derive the model equation that governs the solitary wave evolution. The detailed analytical and numerical studies show that the soliton relaxation dynamics exhibits the main specific features of quantum emitters decay processes in photonic band gap materials. In particular, the non-Markovian effects lead to the extremely long-lived oscillations of the sine-Gordon solitons. That results in the bound state of the soliton with the radiated linear waves.

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I. INTRODUCTION

Nanostructured periodic systems that exhibit a gap in their linear spectrum give rise to a number of unique light-matter interaction processes [1]. Perhaps, the most peculiar example is the strongly modified decay dynamics of quantum emitters embedded in such media [2]. In particular, for the transition frequencies near the band edges pronounced memory, or non-Markovian, effects take place [3]. However, similar behavior is expected to be found for other excited systems as well that are coupled to a continuum of states with the band gaps.

Here, the sine-Gordon equation is considered, which represents an universal model for studies of nonlinear phenomena in many areas of physics [4]. In dimensionless form it can be written as

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

(1)

where the subscripts $t$ and $x$ stand for the partial derivatives with respect of time and space variables, respectively. The fundamental solitary wave solutions of that equation, often called kinks, are given by

$$u(x, t) = 4 \arctan \left( \frac{x - vt}{\sqrt{1 - v^2}} \right).$$

(2)

The plus sign corresponds to the kink, and the minus sign to the anti-kink solutions, respectively. These are exponentially localized solitary waves moving with $v$ group velocity. It should be stressed that any higher-order soliton solution of the sine-Gordon model consists of the certain number of fundamental solitons. For example, so-called breather solution represents a bound state of a kink and anti-kink pair [4]. Therefore, this letter concerns the relaxation dynamics of the fundamental solitons.

For the present discussion it is important that, for small amplitude waves $\sin(u) \approx u$, and therefore, the sine-Gordon model reduces to the linear Klein-Gordon equation which exhibits a band gap [3].

One of the fundamental question is to how a given initial condition evolves to the soliton solutions. The sine-Gordon equation is an integrable model and the problem can be solved by means of the inverse scattering method [4]. The general analysis shows that any initial condition asymptotically relaxes to a certain number of solitons and dispersive radiation. However, in many interesting cases, the formal evaluation of the solution meets technical difficulties, and therefore, is less useful.

In contrast, the variational approach gives simple and explicit, although approximate, expressions for the solitary wave parameters [6]. However, the original version of the method fails to account for changes caused by the dispersion of the radiation [7]. That problem was addressed in Ref. [8], where the generalized variational ansatz was suggested for studies of the nonlinear Schrödinger soliton formation processes. Later, the similar analysis was carried out for the sine-Gordon equation as well [9]. In should be noted that, as compared to the nonlinear Schrödinger equation, worse agreement between the approximate theory and the exact numerical results was achieved. Nevertheless, it was found that, an initially deformed soliton experiences damped oscillations while shedding dispersive radiation. Furthermore, far from the soliton core, the radiated waves obey the linear Klein-Gordon equation provided that the initial deformation is not too strong.

In what follows, it is demonstrated that the non-Markovian effects, caused by the gap of the linearized spectrum, play a crucial role in the sine-Gordon soliton formation processes. In particular, while relaxing to the steady state, initially deformed solitons form a bound state with the radiated linear waves. The similar behavior is known for the decay dynamics of quantum emitters embedded in nanostructures [2].

However, first let us introduce a simple model which, as will become apparent below, accounts even quantitatively for the non-Markovian relaxation effects.

II. THE MODEL SYSTEM

The suggested model system consists of a harmonic oscillator coupled to a semi-infinite flexible string [10]. The string is assumed to be embedded in an elastic matrix, and the reference frame is chosen such that its rest position coincides with the positive $x$-axis. Then, the transverse displacement of the string $U(x, t)$ obeys the
linear Klein-Gordon equation [11]:

\[ U_{tt} - U_{xx} + \omega_s^2 U = 0, \]  

(3)

here \( \omega_s \) is determined by the elastic matrix. The ansatz \( U(x, t) \sim \exp[-i(\omega t - kx)] \) gives the dispersion relation for the Klein-Gordon model \( \omega^2 = \omega_s^2 + k^2 \). Therefore, \( \omega_s \) fixes the band gap size.

The harmonic oscillator is located at the origin and obeys [10]:

\[ \left( \frac{d^2}{dt^2} + \omega_0^2 \right) U(0, t) = 2\gamma U_x(0, t), \]  

(4)

where \( \omega_0 \) is the oscillator eigenfrequency, and \( \gamma \) is related to the tension of the string.

Let us assume that the string is at rest initially. The solution of Eq. (3) for \( U(0, t) = f(t) \) boundary condition is well known [12]. Using the expressions derived in Ref. [12] we can write

\[ U_x(0, t) = -f'(t) - \omega_s \int_0^t \frac{J_1(\omega_s \tau)}{\tau} f(t - \tau) d\tau, \]  

(5)

where the prime denotes the time derivative of the corresponding function, and \( J_1(t) \) is the first order Bessel function of the first kind. In combination with Eq. (4) that gives

\[ f''(t) + 2\gamma f'(t) + \omega_0^2 f(t) = -2\gamma \omega_s \int_0^t \frac{J_1(\omega_s \tau)}{\tau} f(t - \tau) d\tau. \]  

(6)

This is a linear integro-differential equation which can be readily solved, for example, by means of the Laplace transform method [11]:

\[ f(t) = \left[ f(0) \frac{d}{dt} + 2\gamma f(0) + f'(0) \right] \times \left[ F(t) - \omega_s \int_0^t F(\sqrt{t^2 - \tau^2}) J_1(\omega_s \tau) d\tau \right], \]  

(7)

where

\[ F(t) = \frac{\exp(r_1 t) - \exp(r_2 t)}{r_1 - r_2}, \]

provided that \( r_1 \neq r_2 \), and

\[ r_{1,2} = -\gamma \pm \sqrt{\gamma^2 + \omega_s^2 - \omega_0^2}. \]

In addition, for \( \omega_s^2 = \omega_0^2 + \gamma^2 \), i.e. \( r_1 = r_2 \), we have

\[ F(t) = t \exp(-\gamma t). \]

The non-Markovian effects in the presented model are introduced through the right-hand side of Eq. (8). For \( \omega_s = 0 \) that term vanishes, and the governing equation describes a damped oscillator dynamics [13]. In particular, for \( \gamma \ll \omega_0 \), the system oscillates with the exponentially damped amplitude. For \( \gamma \lesssim \omega_0 \), however, the damping is so strong that the dynamics is essentially aperiodic. Finally, for \( \gamma > \omega_0 \), the system experiences exponential relaxation to the equilibrium state with no oscillations. Thus, when the gap is absent, the energy transfer from the oscillator to the string obeys the exponential law.

However, as demonstrated in Fig. 1, the situation might be completely different for \( \omega_s \neq 0 \) due to the memory effects. Note that, \( \gamma > \omega_0 \) in the considered example. In addition, it should be stressed that the oscillator eigenfrequency is outside the gap, but close to the band edge. The system dynamics exhibits the main specific features previously found for quantum emitters decay in photonic band gap materials. First of all, as compared to the free space, the spontaneous decay rate of an excited atom is enhanced for transition frequencies close to the band edge [2]. For comparison, Fig. 1 shows the relaxation of the oscillator for \( \omega_s = 0 \). In the presence of gap, for the relatively short time period \( t \sim 1 \), the system damping rate significantly exceeds that of the gapless case. Second, for the quantum emitters, the photon-atom bound state is present even when the transition frequency is outside the gap [3]. The analogous phenomenon can be identified in Fig. 1 as well. Indeed, the oscillations of the system persist even for \( t \gg 1 \), and the oscillator forms a bound state with the radiated waves. That can be shown by considering the asymptotic behavior of the considered system. In particular, taking the limit \( t \to \infty \) and assuming \( f(t) \sim \cos(\omega t) \), let us rewrite Eq. (6) as follows

\[ (\omega_0^2 - \omega^2) \cos(\omega t) - 2\gamma \omega \sin(\omega t) = -2\gamma \omega_s \int_0^\infty \frac{J_1(\omega_s \tau)}{\tau} \cos(\omega t - \omega \tau) d\tau. \]  

(8)

This expression determines the allowed values for \( \omega \) in the stationary regime. In particular, we have

\[ \omega^2 = \omega_0^2 - 2\gamma^2 \pm 2\gamma \sqrt{\gamma^2 + \omega_s^2 - \omega_0^2}, \]  

(9)

provided that \( \omega \) is real and \( \omega \leq \omega_s \). Therefore, only those
with the Lagrangian density for the sine-Gordon model.

The Lagrange function of the deformed soliton is \( \mathcal{L}_{\text{nl}} = \frac{\pi^2}{3} \left( \frac{w'}{w} \right)^2 - \frac{4}{w} - 4w. \) (14)

The linear waves are radiated due to the nonlinear processes in the core region of the soliton \( x \in [-\varepsilon, +\varepsilon]. \) Therefore, the linear waves are generated at \( x = \pm \varepsilon \) and the corresponding Lagrange function reads \( \delta J \).

Now, according to the variational approach, we insert Eq. (11) in the Lagrange function for the deformed soliton, to obtain \( \delta J = \int_{-\infty}^{+\infty} \mathcal{L}_{\text{nl}} \, dt + \int_{-\infty}^{+\infty} \mathcal{L}_{\text{lin}} \, dt. \) (11)

The first term on the right-hand side of this expression corresponds to the soliton part, while the second and third terms result from the variation of the action functional we obtain \( \delta J = \int_{-\infty}^{+\infty} \left[ \frac{\pi^2}{3} \left( \frac{w'}{w} \right)^2 - \frac{2\pi^2}{3w^3} + \frac{4}{w^2} - 4 \right] \delta w \, dt \)

\( + \int_{-\infty}^{+\infty} U_x(\varepsilon, t) \delta U(\varepsilon, t) \, dt \)

\( - \int_{-\infty}^{+\infty} U_x(-\varepsilon, t) \delta U(-\varepsilon, t) \, dt. \) (18)

The linear waves obey Eq. (17), for the variation of the action functional we obtain \( \delta J = \int_{-\infty}^{+\infty} \left[ \frac{\pi^2}{3} \left( \frac{w'}{w} \right)^2 - \frac{2\pi^2}{3w^3} + \frac{4}{w^2} - 4 \right] \delta w \, dt \)

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It is clear that $\varepsilon \sim w_s$. Nevertheless, the variational method treats the deformed soliton as a point particle $\gamma$. For that reason, the actual value for $\varepsilon$ is not needed for our purposes. Therefore, for the sake of convenience, formally we can choose $\varepsilon = 0$. Furthermore, note that $U(x, t)$ obeys Eq. (17), and so, $U_\varepsilon(x = 0, t)$ is given by Eq. (5) with $\omega_s = 1$ and $f(t) = \alpha \delta w(t)$. Finally, since $w = 1 + \delta w$, the left-hand side of Eq. (21) can be linearized with respect of $\delta w$ to give

$$\delta w''(t) + 2\gamma \delta w'(t) + \omega_0^2 \delta w(t) =$$

$$-2\gamma \int_0^t \frac{J_1(\tau)}{\tau} \delta w(t - \tau) d\tau,$$

(22)

with $\omega_0 = 2\sqrt{3}/\pi \approx 1.1$, and $\gamma = 1.5\alpha^2/\pi^2$.

Therefore, the deformed soliton dynamics in the sine-Gordon model is governed by Eq. (6), with the solution given by Eq. (7). In this case, $\gamma$ represents the coupling constant of the linear radiation to the soliton core. However, since the soliton and dispersive radiation are the constituents of the same solution of the sine-Gordon equation, it is expected that the order of magnitude of $\gamma$ is 1. Moreover, it is clear that, in the absence of the gap the soliton relaxation process would be exponential. That is, according to the discussion given above, $\gamma > \omega_0 \approx 1.1$. In fact, the choice $\gamma = 2$ guarantees very good agreement with the numerical simulations of the sine-Gordon equation.

IV. DISCUSSION AND CONCLUSIONS

There is a subtle point which appears important for comparison of the model calculations with the exact solutions of Eq. (1). In particular, Eq. (5) is obtained by solving a boundary value problem with no linear waves at $t = 0$. However, the evolution of the deformed solitons for Eq. (1) represents an initial value problem. Therefore, in general, we need to solve a mixed boundary and initial value problem for Eq. (3). Nevertheless, in many interesting cases, the presented boundary value problem for Eq. (3) still can be utilized to obtain the relevant solution for the comparison. For example, consider the initial value problem for Eq. (1) defined by Eq. (10) with sufficiently small $\delta w(0) \neq 0$ and $\delta w''(0) = 0$. Note that, the initial deformation extends over the whole space. In the sine-Gordon equation such a perturbation represents linear radiation everywhere but the soliton core region. In contrast, Eqs. (5) and (7) show that the linear waves are generated for $t > 0$. The considered initial condition for Eq. (1) corresponds to a solution of Eq. (6) at certain $t = \phi_0$. To show that, the initial phase shift $\phi_0$ between the model calculations and the numerical solution of the sine-Gordon equation must be introduced. Simultaneously, the choice of $f(0)$ must guarantee that $f(\phi_0) = \delta w(0)$. Note that, $\delta w'(0)$ is an arbitrary constant in general. Thus, it is necessary to adjust $f'(\phi_0)$ to $\delta w'(0)$. For this purpose we should solve Eq. (6) numerically. Then, the solution of the given boundary value problem with the appropriately chosen $\phi_0$, results in the relevant mixed boundary and initial value problem at $t = \phi_0$. Figure 3 shows an example of such calculation. Note that, since $f'(\phi_0) = \delta w'(0)$ is enforced, the solution of Eq. (6) exhibits a kink at $t = \phi_0$. An excellent quantitative agreement between the numerical solutions of Eqs. (1) and (6) should be emphasized. The approach developed in Ref. 9 gives only qualitative agreement with the exact numerical results.

In general, according to the discussion given above, the phase shift $\phi_0$ depends on the initial perturbation. However, as demonstrated in Fig. 4 for the considered initial value problem $\phi_0 = 1.25$ for other values of $\delta w(0)$.
The studied soliton oscillations at $t \gg 1$ have different origin as compared to the wobbling kink solutions of the sine-Gordon model, the kink oscillations are due to the linear discrete eigenmode of the system on the static kink background. The sine-Gordon kinks do not possess such eigenstates. The wobbling kink of the sine-Gordon model represents a three-soliton solution, and so, is an intrinsically nonlinear excitation. In contrast, the extremely long-lived oscillations of the sine-Gordon solitons studied here are governed by the linear equation. The deformed solitons asymptotically tend to the stationary oscillatory state which is caused solely by the forbidden band gap. That represents the bound state of the soliton with the radiated linear waves. Therefore, in that limiting case of the sine-Gordon equation, it should not be surprising to find the effects reminiscent of non-Markovian dynamics.

In conclusion, a generalized variational approach is developed to model the soliton formation processes for the sine-Gordon model. It is shown that, if the initial soliton deformation is not too strong, the system dynamics is governed by the linear integro-differential equation. The detailed analytical and numerical examination of the suggested model indicates that the soliton relaxation dynamics exhibits the main specific features of quantum emitters decay processes in photonic band gap materials. The presented results unveil the significant role of the non-Markovian effects in the sine-Gordon soliton formation dynamics.

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