Geometry, Isometries and Gauging of (2, 1) Heterotic Sigma-Models

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Abstract

The geometry of (2,1) supersymmetric sigma-models is reviewed and the conditions under which they have isometry symmetries are analysed. Certain potentials are constructed that play an important role in the gauging of such symmetries. The gauged action is found for a special class of models.
1 (2,1) Geometry

Heterotic sigma-models with (2,1) supersymmetry have target spaces which are hermitian manifolds with torsion \[1,2\]. They describe the target spaces of heterotic strings with (2,1) world-sheet supersymmetry \[3\], which have the remarkable property that different vacua correspond to the type IIB string and to the membrane of M-theory \[4\] and their compactifications, so that they have many potential applications to the study of M-theory, string theory and duality. The construction of the (2,1) string requires that the target space be four dimensional with signature (4,0) or (2,2) \[3\], and possess an isometry generated by a null Killing vector, which must be gauged \[3\]. For this reason, it is important to understand the geometry of gauged (2,1) sigma-models. One approach to the construction of such gauged models is given in \[3\], but for many purposes (such as the coupling to supergravity) an approach based on a conventional superspace formalism is more convenient. In this note we analyse the geometry associated with isometry symmetries of (2,1) sigma-models and construct the potentials that play a central role in the gauging. The manifestly supersymmetric gauged action is constructed for a certain class of isometry symmetries. An alternative approach to the gauging of (2,1) models was discussed in \[3\] in which only (1,1) supersymmetry was manifest, but this has a number of disadvantages; for example, the coupling to supergravity is rather inconvenient in this formalism. We instead follow here a more direct route leading to a new form of the gauged action that is manifestly (2,1) supersymmetric. Many of the results can be applied more generally to (2,p) supersymmetric sigma-models. The gauging of the general (2,1) sigma-models and their applications to string theory will be addressed in \[3\].

The geometric conditions imposed by the requirement of (2,1) world-sheet supersymmetry \[8,1\] are (i) that the target manifold \(M\) is a complex manifold with metric \(g_{ij}\) and complex structure \(J^{ij}\) satisfying

\[
J^i_j J^j_k = -\delta^i_k \\
N^{ij}_k = J^i_l J^k_{[j,l]} - J^j_l J^k_{[i,l]} = 0
\]

(1)

(ii) that \(J^i_j\) is covariantly constant

\[
\nabla_i J^j_k \equiv J^j_{k,i} + \Gamma^j_{il} J^l_k - \Gamma^j_{ik} J^l_l = 0
\]

(2)

with respect to the connection

\[
\Gamma^i_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + g^{il} H_{jkl}
\]

(3)

which differs from the usual Christoffel connection by the totally antisymmetric torsion

\[
H_{ijk} \equiv \frac{1}{2} (b_{ij,k} + b_{jk,i} + b_{ki,j})
\]

(4)
(iii) the metric $g_{ij}$ is hermitian with respect to the complex structure,

$$g_{ij}J^i_kJ^j_l = g_{kl}. \quad (5)$$

In a complex coordinate system $z^\alpha, \bar{z}^\beta = (z^\beta)^*$, in which the complex structure is constant and diagonal,

$$J^i_j = i \left( \begin{array}{cc} \delta^\beta_\alpha & 0 \\ 0 & -\delta^\beta_\alpha \end{array} \right), \quad (6)$$

these conditions imply that the torsion is given by

$$H_{\alpha\beta\gamma} = \frac{1}{2} (g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}), \quad H_{\alpha\beta\gamma} = 0, \quad (7)$$

while the metric satisfies

$$g_{\alpha[\beta,\gamma]} - g_{\delta[\beta,\delta]} = 0. \quad (8)$$

The conditions (4), (7) and (8) imply the local existence of a vector potential $k_\alpha$ such that

$$g_{\alpha\beta} = k_{\alpha,\beta} + \bar{K}_{\beta,\alpha} \quad (9)$$

$$b_{\alpha\beta} = \bar{K}_{\beta,\alpha} - k_{\alpha,\beta} \quad (10)$$

which defines the geometry locally. If the torsion $H = 0$, the manifold $M$ is Kähler with $k_\alpha = \frac{\partial}{\partial z^\alpha}K(z,\bar{z})$ where $K(z,\bar{z})$ is the Kähler potential and the (2,1) supersymmetric model in fact has (2,2) supersymmetry, while for $H \neq 0$ $M$ is a hermitian manifold with torsion.

The supersymmetric sigma-model can be formulated in (2,1) superspace in terms of chiral scalar superfields $\varphi^\alpha$ satisfying

$$\mathcal{D}_+ \varphi^\alpha = 0, \quad \mathcal{D}_+ \bar{\varphi}^\beta = 0, \quad (11)$$

where $+, -$ are chiral spinor indices and the superspace conventions are as in [8]. The lowest components of the superfields, $\varphi^\alpha|_{\theta = 0} = z^\alpha$, are the bosonic complex coordinates of the space-time. The sigma-model action is then given by

$$S = i \int d^2 \sigma d\theta_+ d\bar{\theta}_+ d\theta_- \left( k_\alpha D_- \varphi^\alpha - \bar{K}_{\beta,\alpha} D_- \bar{\varphi}^\beta \right). \quad (12)$$

The action (12) is invariant under the gauge transformation

$$\delta k_\alpha = \rho_\alpha \quad (13)$$

provided $\rho_\alpha$ satisfies $\partial_\beta \rho_\alpha = i \partial_\alpha \partial_\bar{\beta} \chi$ for some arbitrary real $\chi$. This implies that $\rho$ is of the form

$$\rho_\alpha = i \partial_\alpha \chi + f_\alpha \quad , \quad \bar{\partial}_\beta f_\alpha = 0 \quad (14)$$

for some holomorphic $f_\alpha$. These transformations leave the metric and torsion invariant, but change $b_{ij}$ by an anti-symmetric tensor gauge transformation, $\delta b_{ij} = \partial_{[i} \lambda_{j]}$. 

2


2 Isometry Symmetries

We now consider the isometry symmetries of the target geometry. Let $G$ be a continuous subgroup of the diffeomorphism group of $M$. The action of $G$ on $M$ is generated by vector fields $\xi^i_a$ ($a = 1 \ldots \text{dim}G$) which satisfy the Lie bracket algebra

$$[\xi_a, \xi_b]^i = \xi_a^j \partial_j \xi_b^i - \xi_b^j \partial_j \xi_a^i \equiv \mathcal{L}_a \xi_b^i = f^i_{ab} \xi^i,$$

(15)

where $\mathcal{L}_a$ denotes the Lie derivative with respect to $\xi^i_a$ and $f^i_{bc}$ are the structure constants of the group $G$. The infinitesimal transformations of the (2,1) sigma model superfields

$$\delta \varphi^i = \lambda^a \xi^i_a$$

(16)

with constant parameters $\lambda^a$ will generate a group of proper symmetries of the sigma model field equations if the Lie derivatives with respect to the vector fields $\xi^i_a$ of the metric and torsion vanish,

$$(\mathcal{L}_a g)_{ij} = 0, \quad (\mathcal{L}_a H)_{ijk} = 0.$$ (17)

This requires that the $\xi^i_a$ are Killing vectors of the metric $g$,

$$\nabla_{(i} \xi_{j)a} = 0,$$

(18)

so that $G$ is a group of isometries of $M$, and that $\xi^i_a H_{ijk}$ is closed, so that there is a locally defined one-form $u_a$ such that $\xi^i_a H_{ijk} = \partial_{[j} u_{k]a}.$

(19)

For the transformations (16) to define a symmetry of the sigma model action, it is necessary in addition for $u_{ai}$ to be globally defined. The one-forms $u_a$ are only defined up to the addition of an exact piece:

$$u_{ai} \rightarrow u_{ai} + \partial_i \alpha_a.$$ (20)

Taking the Lie derivative of (19), we obtain that

$$D_{ba} \equiv \mathcal{L}_b u_{ai} - f^c_{ba} u_{ic}$$

(21)

is a closed one-form. If it is exact, it is often possible to use the ambiguity (20) in the definition of $u_a$ to choose it to be equivariant, i.e. to choose it so that it transforms as

$$\mathcal{L}_b u_{ai} = f^c_{ba} u_{ic}.$$ (22)

However, in general there can be obstructions to choosing an equivariant $u$ which have an interpretation in terms of equivariant cohomology $[11, 12]$. While the conditions (17) are sufficient for the isometry to be a symmetry of the (1,1) supersymmetric model, the isometry will only be compatible
with (2, 1) supersymmetry if the complex structure $J$ is invariant under the
diffeomorphisms generated by the $\xi^i_a$ [13],

$$(\mathcal{L}_a J)^i_j = 0.$$  \hfill (23)

Then the $\xi^i_a$ are Killing vectors which are holomorphic with respect to $J$, so
that

$$\partial_a \overline{\xi}^j_a = 0.$$  \hfill (24)

If the torsion vanishes, then $M$ is Kähler and for every holomorphic Killing
vector $\xi^i_a$, the one-form with components $J_{ij} \xi^j_a$ is closed so that locally there
are functions $X_a$ such that $J_{ij} \xi^j_a = \partial_i X_a$; these are the Killing potentials
which play a central role in the gauging of the supersymmetric sigma models
without torsion [10, 7]. In complex coordinates, this becomes $\xi_{aa} = -\partial_a X_a$.
When the torsion does not vanish, this generalises straightforwardly: if $\xi^i_a$ is
a holomorphic Killing vector field satisfying (19) and (23), then the one-form
with components $\omega_i \equiv J_{ij} (\xi^j_a + u^j_a)$ satisfies $\partial_a \omega_b = 0$, so that there are
generalised complex Killing potentials $Z_a \equiv Y_a + iX_a$ such that

$$\xi_{aa} + U_{aa} = \partial_a Y_a + i \partial_a X_a.$$  \hfill (25)

The $X_a$ and $Y_a$ are locally defined functions and are determined up to the
addition of constants. Note that (25) is invariant under the transformation
$U_{aa} \rightarrow U_{aa} + \partial_a \alpha_a$ provided that $Y_a$ also transforms as $Y_a \rightarrow Y_a + \alpha_a$. It will
be useful to absorb $Y$ into $u$, defining

$$u^\prime_{aa} = U_{aa} - \partial_a Y_a$$  \hfill (26)

so that $\xi_{aa} + U^\prime_{aa} = i \partial_a X_a$, as in [13].

Under the rigid symmetries (16) the variation of the Lagrangian in (12) is

$$\delta L = i \lambda^a \left( \mathcal{L}_a k_\alpha D_\varphi^\alpha - \mathcal{L}_a \overline{k}_\alpha D_{-\overline{\varphi}} \right),$$  \hfill (27)

where the Lie derivative of $k_\alpha$ is

$$\mathcal{L}_a k_\alpha = \xi^\beta_\alpha \partial_\beta k_\alpha + \xi^\beta_\alpha \partial_\beta k_\alpha + k_\beta \partial_\alpha \xi^\beta_a.$$  \hfill (28)

In general the symmetries (16) will not leave the action (12) invariant; they
will leave it invariant only up to a gauge transformation of the form (13),
which requires that

$$\mathcal{L}_a k_\alpha = i \partial_\alpha \chi_a + \vartheta_{aa}$$  \hfill (29)

for some real functions $\chi_a$ and holomorphic one-forms $\vartheta_{aa}$, $\partial_\beta \vartheta_{aa} = 0$. We
will now seek explicit forms for $\chi, \vartheta$.

Using the form of the Christoffel symbols

$$\left\{ \frac{\gamma}{\alpha \beta} \right\} = -g^{\alpha \overline{\pi}} H_{\alpha a \overline{\pi}}, \quad \left\{ \frac{\gamma}{\alpha \beta} \right\} = g^{\alpha \overline{\pi}} H_{\alpha a \overline{\pi}},$$  \hfill (30)
we find that the Killing equation (18) becomes
\[ 0 = \nabla_{(\alpha} \xi_{\beta)} = \partial_{(\alpha} \xi_{\beta)a} - H_{\alpha\beta} \xi_a + H_{\alpha\beta} \xi, \tag{31} \]
Comparing with (19), which yields
\[ \partial_{[\alpha} \xi_{\beta]} = H_{\alpha\beta} \xi_a + H_{\alpha\beta} \xi, \tag{32} \]
we find the relation
\[ 2H_{\alpha\beta} \xi_a = \partial_{[\alpha} \xi_{\beta]} - \partial_{(\alpha} \xi_{\beta)a}. \tag{33} \]
Furthermore, eq. (25) gives
\[ \xi_{\alpha a} = \partial_{\alpha} (Y_a + iX_a) - u_{\alpha a}, \tag{34} \]
which implies that (33) can be rewritten in the following two equivalent ways
\[ 2H_{\alpha\beta} \xi_a = \partial_{\alpha} (\xi_{\beta} - \partial_{\beta} Y_a) \]
\[ = -\partial_{\alpha} (\xi_{\beta} + i\partial_{\beta} X_a). \tag{35} \]
Moreover, it follows from (24) and eqs. (7) and (9) that
\[ H_{\alpha\beta} \xi_a = \partial_{\alpha} (\xi_{\beta} + i\partial_{\beta} X_a), \tag{36} \]
Hence, substituting this in (35) and integrating, we find
\[ - (\xi_{\beta} + i\partial_{\beta} X_a) = 2\xi_{\alpha} k_{[\beta]} - \xi_{\beta}. \tag{37} \]
for some antiholomorphic function. The holomorphy of
\[ \vartheta_{\alpha a} = 2\xi_{\alpha} \partial_{\gamma} k_{[\alpha]} + \xi_{\alpha} - i\partial_{\alpha} X_a, \tag{38} \]
which follows from the above construction, can also be checked by direct calculation using eqs. (24) and (34).

Similarly, (33) also yields an expression for \( u_{\alpha a} \): subtituting (34) in (33), we find
\[ u_{\alpha a} = \partial_{\alpha} Y_a + 2\xi_{\alpha} \partial_{\gamma} k_{[\alpha]} - \vartheta_{\alpha a}. \tag{39} \]
It is easily checked that the expression (39) of \( u_{\alpha a} \) is compatible with the geometric condition (19).

Note that the right hand side of (34) is invariant under \( u_{\alpha a} \to u_{\alpha a} + \partial_{\alpha} \alpha_a \) and \( Y_a \to Y_a + \alpha_a \), as it should be. Absorbing \( Y \) into \( u \), as in (25) the one-form
\[ D'_{\alpha ba} \equiv \mathcal{L}_{\nu} u'_{\nu a} - f_{ba} u'_{ic}, \tag{40} \]
is closed, which implies the local existence of a real potential \( E_{ba} \) such that
\[ D'_{baa} = i\partial_{\alpha} E_{ba}. \tag{41} \]
(note that $E_{ba}$ is only defined up to the addition of real constants). In turn, the potential $E_{ba}$ is determined by the imaginary part of the generalised Killing potential. This is seen by taking the Lie derivative of eq. (34) and integrating, which yields

$$E_{ba} = \mathcal{L}_b X_a - f_{ba}^c X_c + e_{ba}$$

(42)

where the $e_{ba}$ are real constants which we henceforth absorb into the definition of $E_{ba}$.

Note that the ambiguity $X_a \rightarrow X_a + C_a$ in the definition of $X_a$ (for some constant $C_a$) does not affect $\vartheta$. Under the transformations (13), (14), both $\vartheta$ and $\chi$ undergo certain shifts, as can be checked using the forms (38) and (44).

Now, using the relations (10), (24) and (38), we find that the Lie derivative of $k_\alpha$ can be written in the following way:

$$\mathcal{L}_a k_\alpha = \xi^\beta_\alpha \left( \partial_\beta k_\alpha + \partial_\alpha k_\beta \right) - \partial_\alpha \left( \xi^\beta_\alpha k_\beta - \xi^\beta_\alpha k_\beta \right) + 2 \xi^\beta_\alpha \partial_\beta k_\alpha$$

(43)

and we have found that $\vartheta$ is given by (38), while

$$\chi_a \equiv X_a + i \left( \xi^\beta_\alpha k_\beta - \xi^\beta_\alpha k_\beta \right).$$

(44)

Further information into the relation of the isometry subgroup $G$ of $M$ to its geometry can be obtained by deriving the action of the Lie bracket algebra on $k_\alpha$. First, note that (43) and (44) imply

$$\mathcal{L}_b (X_a - X_a) = f_{ba}^c (X_c - X_c) + i \left[ \xi^\beta_\alpha \left( \partial_\beta X_a + \vartheta^\beta_\alpha \right) - \xi^\beta_\alpha \left( \partial_\beta X_a + \vartheta^\beta_\alpha \right) \right]$$

$$= f_{ba}^c (X_c - X_c) + \mathcal{L}_a \chi_b + i \left( \xi^\beta_\alpha \vartheta^\beta_\alpha - \xi^\beta_\alpha \vartheta^\beta_\alpha \right)$$

(45)

so that

$$\mathcal{L}_b \chi_a - \mathcal{L}_a \chi_b - f_{ba}^c \chi_c = \mathcal{L}_b X_a - f_{ba}^c X_c + i \left( \xi^\beta_\alpha \vartheta^\beta_\alpha - \xi^\beta_\alpha \vartheta^\beta_\alpha \right).$$

(46)

Then, taking the Lie derivative of (43) with respect to the isometry generated by $\xi^\alpha_\alpha$ and substracting the resulting equation with group indices interchanged, we find

$$[\mathcal{L}_b, \mathcal{L}_a] k_\alpha = f_{ba}^c \mathcal{L}_c k_\alpha + i \partial_\alpha (\mathcal{L}_b \chi_a - \mathcal{L}_a \chi_b - f_{ba}^c \chi_c)$$

$$+ (\mathcal{L}_b \vartheta^\alpha_\alpha - \mathcal{L}_a \vartheta^\alpha_\alpha - f_{ba}^c \vartheta_\alpha) \right),$$

(47)

which can be rewritten as

$$[\mathcal{L}_b, \mathcal{L}_a] k_\alpha = f_{ba}^c \mathcal{L}_c k_\alpha + i \partial_\alpha (\mathcal{L}_b X_a - f_{ba}^c X_c)$$

$$+ \partial_\alpha \left( \xi^\beta_\alpha \vartheta^\beta_\alpha \right) + (\mathcal{L}_b \vartheta^\alpha_\alpha - \mathcal{L}_a \vartheta^\alpha_\alpha - f_{ba}^c \vartheta_\alpha) \right) \right),$$

(48)
upon substituting eq. (46). On the other hand, the Lie derivatives satisfy
the Lie algebra of \( G \), so that

\[
[L_a, L_b]k_\alpha = f^c_{\alpha b}L_c k_\alpha
\]  

(49)

Thus the sum of the last three terms on the right hand side of (48) must
vanish. We will now show that this is indeed the case.

First, taking the Lie derivative of \( \vartheta_{\alpha a} \) in (38) with respect to
\( \xi^b_{\beta} \), we find

\[
L_b \vartheta_{\alpha a} = f^c_{\alpha b} \vartheta_{\alpha c} + 2 \xi^\gamma_{a \beta} \partial_\gamma \vartheta_{\alpha b} - D'_{ba},
\]  

(50)

where we have used eq. (43), the relation (42) and the definition (41). The
second term on the right hand side of (50) can be rewritten as

\[
2 \xi^\gamma_{a \beta} \partial_\gamma \vartheta_{\alpha b} = \xi^\gamma_{a \beta} \partial_\gamma \vartheta_{\alpha b} + \partial_\gamma \vartheta_{\alpha b} \xi^\gamma_{a \beta} - \partial_\alpha (\xi^\gamma_{a \beta} \vartheta_{\alpha b}) = L_a \vartheta_{\alpha b} - \partial_\alpha (\xi^\gamma_{a \beta} \vartheta_{\alpha b})
\]  

(51)

using the holomorphy of \( \vartheta_{\alpha b} \). Substituting (51) in (50), we find the relation

\[
L_b \vartheta_{\alpha a} - L_a \vartheta_{\alpha b} = f^c_{\alpha b} \vartheta_{\alpha c} - \partial_\alpha (\xi^\gamma_{a \beta} \vartheta_{\alpha b} + i E_{ba}).
\]  

(52)

Inserting (42) and (52) in (48), we find that the sum of the last three terms
on the right hand side of (48) explicitly cancels when (41) and (42) are used,
so that (48) indeed reduces to (49).

Another important consequence of (52) follows from symmetrization with
respect to group indices: this yields

\[
\partial_\alpha (\xi^\gamma_{(a} \vartheta_{\gamma b)} + i E_{(ba)}) = 0,
\]  

(53)

which upon integration implies that

\[
\hat{d}_{(ab)} \equiv -\xi^\gamma_{(a} \vartheta_{\gamma b)} + i E_{(ba)}
\]  

(54)

is an antiholomorphic function, \( \hat{d}_{ab} = \hat{d}_{ab}(\bar{z}) \). Then, defining \( c_{(ab)} \) as the real
part of \( \hat{d}_{(ab)} \), we find

\[
c_{(ab)} \equiv \hat{d}_{(ab)} + \bar{d}_{(ab)} = -\xi^\gamma_{(a} \vartheta_{\gamma b)}. \]  

(55)

We now show that the \( c_{(ab)} \) are real constants. Substituting the explicit
expression (38) for \( \vartheta_{\alpha a} \) and using the relation (39), we find

\[
\xi^\gamma_{a \beta} \vartheta_{\alpha b} = 2 \xi^\gamma_{a \beta} \xi^\beta_{(a} \partial_{\gamma b)} - \xi^\gamma_{a \beta} u'_{\gamma b}.
\]  

(56)

Then, symmetrization with respect to group indices yields

\[
\xi^\gamma_{(a} \vartheta_{\gamma b)} = -\xi^\gamma_{(a} u'_{\gamma b)},
\]  

(57)

Hence, we find that (38) can be rewritten as

\[
c_{(ab)} = \xi^\gamma_{(a} u'_{\gamma b)}. \]  

(58)
This is precisely the definition of \( c_{(ab)} \) given in ref. [11], where it was shown that they are real constants whose vanishing is a necessary condition for the gauging of the sigma model to be possible [11].

The equivariance condition on the imaginary part of the generalised Killing potential,

\[
\mathcal{L}_b X_a = f_{ba}^c X_c,
\]

was found in [11] to be another necessary condition for the gauging of the isometries generated by the \( \xi_i^a \) to be possible. If (59) holds, then it follows from (42) that the potential \( E_{ba} \) defined in (11) is a constant and can be chosen to vanish,

\[
E_{ba} = 0,
\]

and that eqs. (52), (53) and (54) simplify. The equations (40), (41) then imply that \( u' \) is equivariant.

Summarizing, the action of a group \( G \) generated by the vector fields \( \xi^i_a \) as in (16) is a symmetry provided the \( \xi^i_a \) are holomorphic Killing vectors, i.e. eqs. (24) and (18) hold, so that the metric and complex structure are invariant, and in addition the torsion is invariant, i.e. eqs. (17) and (23) hold. In general, the isometry symmetries will not leave the action (12) invariant, but will leave it invariant up to a gauge transformation of the form (13). The geometry and Killing potentials then determine the quantity \( \mathcal{L}_a k_\alpha \) appearing in the gauge transformation to take the form (13), with \( \chi, \vartheta \) as in (14) and (38). It is then found that the Lie bracket algebra of the function \( k_\alpha \) which determines the geometry closes. Also, the quantities defined in (55) are the real constants \( c_{(ab)} \) of ref. [5]. When the imaginary part of the generalised Killing potential is chosen to be equivariant, i.e. when (59) holds, it is found that the potential \( E_{ba} \) defined in (11) vanishes. Then the one-forms \( u'_a \) defined in (26) are equivariant and the geometry simplifies.

We note the result of ref. [5], where it was shown that the equivariance condition (59) on the imaginary part of the generalised Killing potential must hold in order for the gauging of the supersymmetric sigma model to be possible.

The discussion given here also applies to the geometry and isometries of the target space of (2,0) heterotic strings. The corresponding formulae can be obtained from those given in the foregoing by appropriate truncation of the (2,1) superfields.

### 3 Gauging the Isometries

We now turn to the gauging of the (2,1) sigma-model. The aim is to promote the rigid isometry symmetries (16) to local ones in which the scalar fields transform as

\[
\delta \varphi^\alpha = \Lambda^a \xi^\alpha_a, \quad \delta \varphi^\alpha = \bar{\Lambda}^\alpha \bar{\xi}^\alpha_a,
\]

(61)
where the parameters $\Lambda^a, \bar{\Lambda}^a$ satisfy the chirality conditions
\[ \overline{D}_+ \Lambda^a = 0, \quad D_+ \bar{\Lambda}^a = 0. \] (62)

Under a finite transformation,
\[ \varphi \rightarrow \varphi' = e^{L_{\Lambda} \xi} \varphi, \quad \bar{\varphi} \rightarrow \bar{\varphi}' = e^{L_{\bar{\Lambda}} \bar{\xi}} \bar{\varphi}, \] (63)

where
\[ \Lambda \cdot \xi \equiv \Lambda^a \xi^a \frac{\partial}{\partial \varphi^a} \] (64)
and $L_{\Lambda} \xi \varphi^\alpha$ denotes the action of the infinitesimal diffeomorphism with parameter $\Lambda \cdot \xi$,
\[ L_{\Lambda} \xi \varphi^\alpha \equiv \Lambda \cdot \xi^\alpha, \] (65)
and acts on tensors as the Lie derivative with respect to $\Lambda \cdot \xi$. To construct the gauged action, we couple the sigma-model to the (2,1) supersymmetric gauge multiplet [5]. The constraints for this multiplet can be solved to express the superconnections in terms of a prepotential $V$, which transforms as
\[ e^V \rightarrow e^{V'} = e^{L_{\Lambda} V} e^{-\bar{\xi}}, \] (66)
and a spinorial connection $A^a_-$, with the infinitesimal transformation
\[ \delta A^a_- = D_- \Lambda^a + [A_-, \Lambda]^a. \] (67)

We choose a chiral representation in which the right-handed covariant derivatives are
\[ \nabla_+ = \overline{D}_+, \quad \nabla_+ = e^V D_+ e^{-V}, \] (68)
while the left-handed covariant derivative is defined by
\[ \nabla_- \varphi^\alpha = D_- \varphi^\alpha - A^a_- \xi_a^\alpha. \] (69)

Now let us define (following [1])
\[ \tilde{\varphi} = e^{L_{\nu} \bar{\xi} \bar{\varphi}}, \] (70)
where
\[ L_{\nu} \bar{\xi} = V^a \xi^a \frac{\partial}{\partial \varphi^a}. \] (71)

Then the fields $\varphi, \tilde{\varphi}$ satisfy the covariant chiral constraints
\[ \nabla_+ \varphi^\alpha = 0, \quad \nabla_+ \bar{\varphi}^\alpha = 0, \] (72)
and transform as
\[ \delta \varphi^\alpha = \Lambda^a \xi^a, \quad \delta \tilde{\varphi}^\alpha = \Lambda^a \xi^a(\tilde{\varphi}). \] (73)

Note that the $\tilde{\varphi}$ transformation involved the parameter $\Lambda$ while that for $\bar{\varphi}$ involved $\bar{\Lambda}$. The left-handed covariant derivative of $\tilde{\varphi}$ is
\[ \nabla_- \tilde{\varphi}^\alpha = D_- \tilde{\varphi}^\alpha - A^a_- \xi_a^\alpha(\tilde{\varphi}). \] (74)
In general, the potential $k_\alpha$ is only gauge invariant up to a transformation of the form \( (13) \), so that its Lie derivative is given by \( (29) \) for some $\chi, \vartheta$. Consider first the special case in which $L_a k_\alpha = 0$, so that the action \( (12) \) is invariant under the rigid transformations \( (16) \). Then the gauged sigma-model is obtained by minimal coupling. This coupling is achieved by replacing $\bar{\phi}$ with $\tilde{\phi}$ and replacing the supercovariant derivative $D_-$ with the gauge covariant derivative $\nabla_-$ defined in \( (69) \) and \( (74) \). This gives the Lagrangian

\[
L_0 = i \left( k_\alpha (\varphi, \bar{\varphi}) \nabla_- \varphi^\alpha - \tilde{k}_\alpha (\varphi, \bar{\varphi}) \nabla_- \varphi^\alpha \right). \tag{75}
\]

This is indeed invariant under the transformations \( (66), (67) \) and \( (73) \) provided $L_a k_\alpha = 0$.

Consider now the more general case in which \( (29) \) holds, but with $\vartheta = 0$, so that

\[
L_a k_\alpha = i \partial_\alpha \chi_a. \tag{76}
\]

The action based on \( (75) \) is no longer gauge invariant; using \( (66), (67), (69) \) and the infinitesimal variation of the fields \( (73) \), we find

\[
\delta L_0 = i \Lambda^a D_- X_a (\varphi, \bar{\varphi}) + \Lambda^a A_b^b \left( \partial_\alpha \chi_a \xi_b^\alpha + \partial_\alpha \chi_a A^b_b \xi^\alpha_b \right) (\varphi, \bar{\varphi}). \tag{77}
\]

This can be cancelled by adding the following term to $L_0$:

\[
\hat{L}_0 = - A^a_- X_a (\varphi, \bar{\varphi}). \tag{78}
\]

The definition \( (44) \) implies that the terms multiplying the gauge connection $A_-$ combine to yield the generalised Killing potential $X$:

\[
L_0^{(g)} = L_0 + \hat{L}_0 = i \left( k_\alpha D_- \varphi^\alpha - \tilde{k}_\alpha D_- \varphi^\alpha \right) (\varphi, \bar{\varphi}) - A^a_- X_a (\varphi, \bar{\varphi}). \tag{79}
\]

The Lagrangian $L_0^{(g)}$ in \( (79) \) is the full gauge-invariant action for the gauged (2,1) model in the special case where $\vartheta = 0$ provided that the generalised Killing potential $X$ transforms covariantly under the isometries \( (61) \), i.e.

\[
\delta X_a = f^a_{bc} \Lambda^b X_c. \tag{80}
\]

To see this, note that the variation of the first term in \( (79) \) is given by

\[
\delta \left[ i \left( k_\alpha D_- \varphi^\alpha - \tilde{k}_\alpha D_- \varphi^\alpha \right) (\varphi, \bar{\varphi}) \right] = - \Lambda^a D_- X_a (\varphi, \bar{\varphi}) + i D_- \Lambda^a \left( \xi^\alpha_b \bar{k}_a - \xi^\alpha_b k_a \right) (\varphi, \bar{\varphi})
\]

\[
= - D_- \Lambda^a X_a (\varphi, \bar{\varphi}), \tag{81}
\]

the manipulations being similar to those which lead to the expression \( (77) \); the last identity follows upon integrating by parts, discarding a surface term and using the definition \( (44) \) (notice that the second term on the right-hand side of \( (44) \) has cancelled). On the other hand, the variation of the second term in \( (79) \) is

\[
\delta \left[ - A^a_- X_a (\varphi, \bar{\varphi}) \right] = - D_- \Lambda^a X_a (\varphi, \bar{\varphi}) - A^a_- \left( \delta X_a + f^a_{bc} \Lambda^b X_c \right) (\varphi, \bar{\varphi}), \tag{82}
\]
where we have used the variations (67) of the gauge connection. Adding (81) and (82), a cancellation occurs, and one is left with

$$\delta L_g^{(0)} = -A^a_\mu \left( \delta X_a + f^c_{ba} \Lambda^b X_c \right)(\phi, \tilde{\phi}),$$  \hspace{1cm} (83)

which vanishes if the equivariance condition (80) is satisfied. Furthermore, it is easily seen that (80) must be imposed if the variation (83) is to vanish, as no simple Lorentz-invariant object can be constructed with variation given by the gauge connection. As seen in the last section, the condition (80) implies the equivariance of $u'$ while $c_{(ab)} = 0$ as a result of (55) and the assumption that $\vartheta = 0$. Thus the obstructions to gauging found in [11, 5] are all absent.

Summarizing, we find that, in the special case where $\vartheta = 0$, the action (12) for the (2,1) model can be gauged provided the same geometric condition as that found in ref. [5] is satisfied, namely the equivariance of the generalized Killing potential $X$. Moreover, if (80) holds, then the gauged (2,1) sigma-model action in this case is the superspace integral of the gauge invariant Lagrangian (73). The general case in which $\vartheta \neq 0$ and the Lie derivative of $k_\alpha$ is given by (29) is more complicated and will be treated in [5], using the methods of [5].

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