On the ill-posedness of the 5th-order Gardner equation

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Abstract
We present ill-posedness results for the initial value problem of the 5th-order Gardner equation. We use new breather solutions discovered for this higher order Gardner equation to measure the regularity of the Cauchy problem in Sobolev spaces $H^s(\mathbb{R})$. We find the sharp Sobolev index under which the local well-posedness of the problem is lost, meaning that the dependence of 5th-order Gardner solutions upon initial data fails to be continuous.

Keywords Gardner equation · Breather · Ill-posedness · Integrability

Mathematics Subject Classification Primary 37K15 · 35Q53; Secondary 35Q51 · 37K10

1 Introduction

In this short note we continue our work on the Gardner equation, started in [1], but this time we are going to focus on the regularity for the Cauchy problem of the 5th-order Gardner equation (5th-GE hereafter)

\[
\begin{aligned}
v_t + 10\mu^2 v_{xxx} + v_{5x} + \left[K_\mu(v)\right]_x &= 0, \quad \mu, t, x \in \mathbb{R}, \\
v(0, x) &= v_0(x),
\end{aligned}
\]

where

\[
K_\mu(v) := 10(\mu + v)v_x^2 + 20\mu v v_{xx} + 10v^2 v_{xx} + 60\mu^3 v^2 + 60\mu^2 v^3 + 30\mu v^4 + 6v^5,
\]
and \( v = v(t, x) \) is a real valued function. Note that we get (1.1) by looking for solutions of the 5th-order mKdV equation

\[
u_t + \left( u_{4x} + 10uu_x^2 + 10u^2u_{xx} + 6u^5 \right)_x = 0, \tag{1.3}
\]

in the form \( u(t, x) = \mu + v(t, x) \), \( \mu \in \mathbb{R} \), and with a suitable spatial translation. Hence for small \( \mu \ll 1 \), (1.1) can be considered as a perturbed 5th-order mKdV equation. Due to this relation with (1.3), the physical context where (1.1) appears is mainly as a perturbed model for unidirectional propagation of shallow water waves over flat surfaces, but also in wave interaction and elastic media. See [6,8,12,13] for further reading.

In this work, our aim is to extend previous results [1] on the ill-posedness for the Gardner equation

\[
v_t + (v_{xx} + 6\mu v^2 + 2v^3)_x = 0, \quad \mu, t, x \in \mathbb{R}, \tag{1.4}
\]

to the above (1.1) higher order version of the Gardner equation. Specifically, we present here one result about the ill-posedness of the 5th-GE for given data in \( H^s(\mathbb{R}) \). Our interest is to study the nonlinear evolution driven by the 5th-GE (1.1) with initial data defined by a periodic in time and spatially localized function as it was firstly introduced by Kenig et al. [7], and as we will show further.

About the well-posedness of the 5th-order mKdV Eq. (1.3), Linares by using a contraction mapping argument in [11], showed that the Cauchy problem for the Eq. (1.3) is locally well-posed at \( H^2(\mathbb{R}) \). Moreover, Kwon [10] obtained that the 5th-order mKdV Eq. (1.3) is locally well-posed at \( H^s(\mathbb{R}) \), with \( s \geq \frac{3}{2} \). About global well-posedness of the 5th-order mKdV equation, we can see that the Eq. (1.3) has this property at \( H^s(\mathbb{R}) \), if \( s \geq 1 \) (see [5,10,11] for more details). On the other hand, Alejo and Kwak [4] have obtained local and global well-posedness results for the 5th-GE (1.1) in \( H^s(\mathbb{R}) \), \( s \geq 2 \), generalizing a previous well-posedness [2] for the Gardner equation (1.4). Finally, and as far as we know, the study of the well posedness in the periodic setting for (1.1) is actually an interesting open problem, and which in fact it would generalize recent results of Kwak for the 5th-order mKdV (1.3) in the torus [9].

Now, we remember some basic concepts: by local well-posedness of the initial value problem (IVP) (1.1) we understand that there exists a unique solution \( u(t, \cdot) \) of (1.1) taking values in \( H^s \) for a time interval \([0, T)\), it defines a continuous curve in \( H^s \) and depends continuously on the initial data, that is:

for any \( \epsilon > 0 \), there exists \( \delta > 0 \), such that if \( ||u_{01} - u_{02}||_{H^s} < \delta \), then \( ||u_1 - u_2||_{H^s} < \epsilon \), with \( \delta = \delta(\epsilon, M) \), where \( ||u_{0i}||_{H^s} \leq M, \ i = 1, 2 \).

In [3], we were able to build explicit breather solutions for higher order Gardner equations. In this note, we will use the corresponding definition for the real breather solution of the 5th-GE:

**Definition 1.1 (Real breather solution of the 5th-GE)** Let \( \alpha, \beta, \mu \in \mathbb{R} \setminus \{0\} \) such that \( \Delta = \alpha^2 + \beta^2 - 4\mu^2 > 0 \). The real breather solution of the 5th-GE (1.1) is given explicitly by the formula
\[ B_{5\mu} \equiv B_{\alpha,\beta,\mu}(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{G_\mu(t, x)}{F_\mu(t, x)} \right) \right], \] (1.5)

where

\[ G_\mu(t, x) := \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - \frac{2\mu\beta[\cosh(\beta y_2) + \sinh(\beta y_2)]}{\Delta}, \] (1.6)

\[ F_\mu(t, x) := \cosh(\beta y_2) - \frac{2\mu\beta[\alpha \cos(\alpha y_1) - \beta \sin(\alpha y_1)]}{\alpha \sqrt{\alpha^2 + \beta^2 \sqrt{\Delta}}}, \] (1.7)

with

\[ y_1 := x + \delta_5 t + x_1, \quad y_2 := x + \gamma_5 t + x_2, \] (1.8)

and with velocities

\[ \delta_5 := -\alpha^4 + 10\alpha^2 \beta^2 - 5\beta^4 + 10(\alpha^2 - 3\beta^2)\mu^2 - 30\mu^4 \] (1.9)

and

\[ \gamma_5 := -\beta^4 + 10\alpha^2 \beta^2 - 5\alpha^4 + 10(3\alpha^2 - \beta^2)\mu^2 - 30\mu^4. \] (1.10)

**Remark 1.1** Note that since the functional form of this breather solution is the same as the classical breather solution of the Gardner equation [1, Def.1.1], we get similar properties on it. For instance, for each fixed time, the breather of the 5th-GE (1.5) is a function in the Schwartz class, with zero mean

\[ \int_{\mathbb{R}} B_{5\mu} = 0. \]

Furthermore, \( B_{5\mu} \) satisfies the same fourth order elliptic equation which characterizes classical Gardner breather solutions (see [3] for further reading)

\[ B_{5\mu,xx} + 2(\alpha^2 - \beta^2)(B_{5\mu,xx} + 6\mu B_{5\mu}^2 + 2B_{5\mu}^3) + (\alpha^2 + \beta^2)^2 B_{5\mu} + 10 B_{5\mu}^2 B_{5\mu,xx} + 10 B_{5\mu} B_{5\mu,x} + 6 B_{5\mu}^2 + 10\mu B_{5\mu}^2 B_{5\mu,xx} + 20\mu B_{5\mu} B_{5\mu,x} + 40\mu^2 B_{5\mu}^3 + 30\mu B_{5\mu}^4 = 0. \] (1.11)

**Remark 1.2** It is truth, from Definition 1.1, that

\[ B_{5\mu} \equiv B_{\alpha,\beta,\mu}(t, x; x_1, x_2) := \frac{2 M_\mu(t, x)}{N_\mu(t, x)} \] (1.12)
is a breather solution to the Eq. (1.1), where

\[
N_\mu(t, x) = \left[ \cosh(\beta y_2) - \frac{2\mu \beta [\alpha \cos(\alpha y_1) - \beta \sin(\alpha y_1)]}{\alpha \sqrt{\alpha^2 + \beta^2 \sqrt{\Delta}}} \right]^2 \\
+ \left[ \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - \frac{2\mu \beta [\cosh(\beta y_2) + \sinh(\beta y_2)]}{\Delta} \right]^2
\]

(1.13)

and

\[
M_\mu(t, x) = \left[ \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{2\mu \beta^2 [\cosh(\beta y_2) + \sinh(\beta y_2)]}{\Delta} \right] \\
\times \left[ \cosh(\beta y_2) - \frac{2\mu \beta [\alpha \cos(\alpha y_1) - \beta \sin(\alpha y_1)]}{\alpha \sqrt{\alpha^2 + \beta^2 \sqrt{\Delta}}} \right] \\
- \left[ \beta \sinh(\beta y_2) + \frac{2\mu \beta [\alpha \sin(\alpha y_1) + \beta \cos(\alpha y_1)]}{\sqrt{\alpha^2 + \beta^2 \Delta}} \right] \\
\times \left[ \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - \frac{2\mu \beta [\cosh(\beta y_2) + \sinh(\beta y_2)]}{\Delta} \right].
\]

(1.14)

We will assume here that \(\alpha, \beta > 0\), which we will call them as the *frequency* and the *amplitude* of the breather respectively. The following simplification on the breather solution will help us to analyze the regularity of the IVP for the 5th-GE. It is easy to see in (1.13)–(1.14), that selecting \(\alpha\) large, such that \(\beta/\alpha \ll 1\), the Gardner breather (1.12) reduces to

\[
v_{\alpha, \beta, \mu}(t, x) := B_{5\mu}(t, x) \approx 2\beta \cos(\alpha(x + \delta_5 t)) \text{sech}(\beta(x + \gamma_5 t))
\]

(1.15)

or simply

\[
v_{\alpha, \beta, \mu}(t, x) \approx \sqrt{2} \text{Re}[e^{i(\alpha(x + \delta_5 t))} Q_{\beta}(x + \gamma_5 t)],
\]

(1.16)

where \(Q\) denotes the solution of the nonlinear ODE

\[
Q'' - Q + Q^3 = 0,
\]

(1.17)

with

\[
Q(\xi) = \sqrt{2} \text{sech}(\xi)
\]

(1.18)
and
\[
Q_\beta(\xi) = \beta Q(\beta \xi).
\] (1.19)

## 2 Main theorem

The use of the approximation (1.16) will be a key step in the proof of the ill-posedness of the 5th-GE. The main result concerning the IVP (1.1) is the following:

**Main Theorem** (Ill-posedness of the 5th Gardner eq.) *If \( s < 3/4 \), the mapping data-solution \( v_0 \to v(t) \), with \( v(t) \) a solution of the IVP for the 5th-order Gardner equation (1.1) is not uniformly continuous.*

**Remark 2.1** The proof is simple, and it reduces to first compute the distance between two initial data with different frequencies \( \alpha_i, \ i = 1, 2 \) but the same amplitude \( \beta \). Next, we will measure the distance between two solutions at time \( t = T \) and selecting large enough frequencies \( \alpha_i, \ i = 1, 2 \), we avoid the interaction of the supports of these solutions. Finally, we obtain a lower bound to the distance of solutions at time \( t = T \). Selecting the frequencies \( \alpha_i, \ i = 1, 2 \) in a suitable way, and since \( s < 3/4 \), we will arrive to a contradiction with the continuous dependence of the mapping data-solution.

**Remark 2.2** If \( \mu = 0 \), the Eq. (1.1) becomes the 5th-order mKdV equation (1.3). Therefore, as a direct consequence, this Theorem also guarantees a result on the ill-posedness to the 5th-order mKdV Eq. (1.3) which agrees at least with the upper limit of the critical Sobolev index of the ill-posedness result for (1.3) obtained by Kwon [10]. Namely

**Corollary 2.1** (Ill-posedness of the 5th-order mKdV eq.) *If \( s < 3/4 \), the mapping data-solution \( u_0 \to u(t) \), with \( u(t) \) a solution of the IVP for the 5th-order mKdV equation (1.3) is not uniformly continuous.*

**Proof. (of the Main Theorem.)** We consider the IVP for the 5th-order Gardner equation with initial data given by the breather solution (1.5),
\[
\begin{align*}
  v_t + 10\mu^2 v_{xxx} + v_{5x} + [K_\mu(v)]_x &= 0, \\
  v(0, x) &= v_{\alpha, \beta, \mu}(0, x),
\end{align*}
\] (2.1)

where \( K_\mu(v) \) is defined in (1.2). With \( \mu \) fixed, we take the parameter \( \alpha \) large enough, such that \( \frac{\beta}{\alpha} \ll 1 \). Then, from (1.16), the initial data reads
\[
v_{\alpha, \beta, \mu}(0, x) \approx \sqrt{2} \text{Re}[e^{i\alpha x} Q_\beta(x)],
\] (2.2)

with \( Q_\beta \) defined in (1.19). We take
\[
\beta = \alpha^{-2s} \quad \text{and} \quad \alpha_1, \alpha_2 \sim \alpha.
\] (2.3)
Observe that \( \hat{Q}_\beta(\cdot) \) concentrates in the ball \( V_\beta(0) = \{ \xi \in \mathbb{R}; \ |\xi| < \beta \} \). First, we calculate the \( H^s \)-norm of two different initial data for the 5th-GE in the regime with \( \alpha \) large enough, such that \( \frac{\beta}{\alpha} \ll 1 \):

\[
\| v_{\alpha_1, \beta, \mu}(0) \|_{H^s}^2 \approx \| (1 + |\xi|^2)^{s/2} \hat{Q}_\beta(\xi - \alpha_1) \|_{L^2}^2 \approx C \alpha^{2s} \beta = C, \quad j = 1, 2,
\]

(2.4)

where \( C \) denotes a constant. Second, we now compute the distance between these initial data

\[
\| v_{\alpha_1, \beta, \mu}(0) - v_{\alpha_2, \beta, \mu}(0) \|_{H^s}^2 \approx \| (1 + |\xi|^2)^{s/2} (\hat{Q}_\beta(\xi - \alpha_1) - \hat{Q}_\beta(\xi - \alpha_2)) \|_{L^2}^2 \leq C \alpha^{2s} \int_{-\infty}^{+\infty} \left[ \int_{\xi - \alpha_1}^{\xi - \alpha_2} \frac{d}{d\rho} \hat{Q}_\beta(\rho) d\rho \right]^2 d\xi
\]

(2.5)

\[
\leq C \alpha^{2s} \frac{|\alpha_1 - \alpha_2|}{\beta^2} \int_{-\infty}^{+\infty} \int_{\xi - \alpha_1}^{\xi - \alpha_2} |\hat{Q}_\beta'(|\rho|)|^2 d\rho d\xi
\]

\[
\leq C \alpha^{2s} \frac{|\alpha_1 - \alpha_2|}{\beta^2} \left( \int_{\rho + \alpha_1}^{\rho + \alpha_2} d\xi \right) \int_{-\infty}^{+\infty} |\hat{Q}_\beta(\rho)|^2 d\rho
\]

\[
\leq C \alpha^{2s} \frac{(\alpha_1 - \alpha_2)^2}{\beta^2} \beta = C \alpha^{2s} (\alpha_1 - \alpha_2)^2 \alpha^{2s} = C (\alpha^{2s} (\alpha_1 - \alpha_2))^2.
\]

Next, we consider the corresponding solutions \( v_{\alpha_1, \beta, \mu}(t) \) and \( v_{\alpha_2, \beta, \mu}(t) \) at the time \( t = T \). We can see that

\[
\| v_{\alpha_1, \beta, \mu}(T) - v_{\alpha_2, \beta, \mu}(T) \|_{H^s}^2 \approx \alpha^{2s} \| v_{\alpha_1, \beta, \mu}(T) - v_{\alpha_2, \beta, \mu}(T) \|_{L^2}^2.
\]

(2.6)

From (1.16), if \( \alpha \) is large enough,

\[
v_{\alpha_j, \beta, \mu}(T, x) \approx \sqrt{2} Re[e^{i(\alpha_j(x + \gamma_5 T))} \beta Q(\beta(x + \gamma_5 T))], \quad j = 1, 2.
\]

(2.7)

Moreover, note that

\[
\gamma_5 = -\beta^4 + 10\alpha^2 \beta^2 - 5\alpha^4 + 10(3\alpha^2 - \beta^2)\mu^2 - 30\mu^4 \sim -5\alpha^4
\]

(2.8)

and

\[
\alpha_1^4 - \alpha_2^4 \sim (\alpha_1^2 + \alpha_2^2)(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2) \sim (\alpha_1 - \alpha_2)\alpha^3.
\]

(2.9)

The information above shows that \( v_{\alpha_j, \beta, \mu}(T), \ j = 1, 2, \) concentrates in \( V_{\beta^{-1}}(5\alpha_j^4 T), \ j = 1, 2 \). So, we basically have disjoint supports if

\[
\alpha^3 (\alpha_1 - \alpha_2)T \gg \beta^{-1} = \alpha^{2s}.
\]

(2.10)
Under this condition, we have that
\[
\| v_{\alpha_1, \beta, \mu}(T) - v_{\alpha_2, \beta, \mu}(T) \|_{L^2}^2 \approx \| v_{\alpha_1, \beta, \mu}(T) \|_{L^2}^2 + \| v_{\alpha_2, \beta, \mu}(T) \|_{L^2}^2 \approx \beta
\]  
(2.11)
and
\[
\| v_{\alpha_1, \beta, \mu}(T) - v_{\alpha_2, \beta, \mu}(T) \|_{H^s}^2 \geq C \alpha^{2s} \beta = C.
\]  
(2.12)

If we select
\[
\alpha_1 = \alpha + \frac{\delta}{2\alpha^{2s}}, \; \alpha_2 = \alpha - \frac{\delta}{2\alpha^{2s}}, \; \alpha_1 - \alpha_2 = \frac{\delta}{\alpha^{2s}},
\]  
(2.13)
we have that
\[
(\alpha^{2s}(\alpha_1 - \alpha_2))^2 = \delta^2
\]  
(2.14)
and, from (2.10),
\[
\alpha^3 \frac{\delta}{\alpha^{2s}} T \gg \alpha^{2s}.
\]  
(2.15)
Finally, from (2.15),
\[
T \gg \frac{\alpha^{4s-3}}{\delta}.
\]  
(2.16)

Since \( s < \frac{3}{4} \), given \( \delta, T > 0 \), we can choose \( \alpha \) so large that (2.16) is still valid, and then (2.12) does not satisfy uniform continuity and we conclude. □

References

1. Alejo, M.A.: On the ill-posedness of the Gardner equation. J. Math. Anal. Appl. 396(1), 256–260 (2012)
2. Alejo, M.A.: Well-posedness and stability results for the Gardner equation. Nonlinear Differ. Equ. Appl. NoDEA 19(4), 503–520 (2012)
3. Alejo, M.A., Cardoso, E.: Dynamics of breathers in the Gardner hierarchy: universality of the variational characterization. preprint, arXiv:1901.10409
4. Alejo, M.A., Kwak, C.: The initial value problem for the 5th order Gardner equation (submitted). arXiv:1901.03350
5. Grünrock, A.: On the hierarchies of higher order mKdV and KdV equations. Cent. Eur. J. Math. 8(3), 500–536 (2010)
6. Kakutani, T.: Weakly nonlinear hydromagnetic waves in a cold collision free plasma. J. Phys. Soc. Jpn. 26, 5 (1969)
7. Kenig, C., Ponce, G., Vega, L.: On the Ill-posedness of some canonical dispersive equations. Duke Math. J. 106(3), 617–633 (2001)

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8. Kichenassamy, S., Olver, P.J.: Existence and non-existence of solitary wave solutions to higher order model evolution equations. SIAM J. Math. Anal. 3, 1141–1166 (1992)
9. Kwak, C.: Low regularity Cauchy problem for the fifth-order modified KdV equations on $\mathbb{T}$. J. Hyperb. Differ. Equ. 15(3), 463–557 (2018)
10. Kwon, S.: Well posedness and Ill-posedness of the fifth-order modified KdV equation. Electron. J. Differ. Equ. 1, 1–15 (2008)
11. Linares, F.: A higher order modified Korteweg–de Vries equation. Comput. Appl. Math. 14(3), 253–267 (1995)
12. Marchant, T.R., Smyth, N.F.: The extended Korteweg–de Vries equation and the resonant flow of a fluid over topography. J. Fluid Mech. 221, 263–288 (1990)
13. Olver, P.J.: Hamiltonian perturbation theory and water waves. Contemp. Math. 28, 231–249 (1984)

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