The gravitational energy-momentum pseudotensor: the cases of $f(R)$ and $f(T)$ gravity

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Abstract

We derive the gravitational energy-momentum pseudotensor $\tau^{\sigma\lambda}$ in metric $f(R)$ gravity and in teleparallel $f(T)$ gravity. In the first case, $R$ is the Ricci curvature scalar for a torsionless Levi-Civita connection; in the second case, $T$ is the curvature-free torsion scalar derived by tetrads and Weitzenböck connection. For both classes of theories the continuity equations are obtained in presence of matter. $f(R)$ and $f(T)$ are non-equivalent but differ for a quantity $\omega(T,B)$ containing the torsion scalar $T$ and a boundary term $B$. It is possible to obtain the field equations for $\omega(T,B)$ and the related gravitational energy-momentum pseudotensor $\tau^{\sigma\lambda}|\omega$. Finally we show that, thanks to this further pseudotensor, it is possible to pass from $f(R)$ to $f(T)$ and viceversa through a simple relation between gravitational pseudotensors.

Keywords: gravitational energy; conservation laws; extended theories of gravity.

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1 Introduction

The issue to extend General Relativity (GR) is related to the necessity to obtain a comprehensive picture of gravitational interaction from UV to IR scales ranging from quantum gravity, to inflation, large scale structure and today observed accelerated behavior of the Hubble flow. The standard approach is introducing dark components (dark energy and dark matter) and scalar fields (e.g. the inflaton) within the framework of Einstein’s GR. The question is today completely open because no final evidence for exotic matter and fields has been reported.

An alternative way to proceed is to modify Einstein’s theory by constructing classes of theories where GR is a particular case. This is the main issue by which Extended Theories of Gravitation
are born\(^1\). The simplest family is the so called \(f(R)\) gravity that is a generalization of the Einstein-Hilbert action which is not simply linear in the Ricci scalar \(R\). The straightforward approach is adopting a Levi Civita metric connection without torsion. In this scheme, gravity is interpreted in terms of the curvature of a spacetime manifold, the free fall of bodies happens along geodesics and the dynamical variables are the components of the metric tensor. An alternative formulation of Einstein’s gravity is the Teleparallel Equivalent of General Relativity (TEGR) \([2, 3]\) based on the parallelism of tetrads (vierbiens), an orthonormal basis in the Minkowski tangent space, at any point, of a given spacetime manifold. The geometric structure results with zero curvature and non-zero torsion. The connection is the Weitzenböck connection \([4, 5, 6, 7]\). In teleparallel gravity, the dynamical variables are the tetrads, the gravity returns to be interpreted as a force and the bodies follow the motion governed by the equations of the gravitational force produced by torsion: therefore it is the torsion to produce the gravity in TEGR; the action is equivalent to the Einstein-Hilbert one but it is built with the torsion scalar \(T\) that replaces the curvature scalar \(R\).

TEGR can be generalized in the sense of \(f(R)\), that is \(f(T)\) models can be considered where \(f(T)\) is a generic function of the torsion scalar \(T\) \([8, 9]\). However, while \(f(R)\) gravity is invariant for local Lorentz transformations and shows fourth-order field equations in the metric formalism, \(f(T)\) gravity is not invariant for local Lorentz transformations of vierbein fields and has second-order field equations. The lack of local Lorentz invariance is due to the choice of the Weitzenböck connection which implies the cancellation of the spin connection which leads to a particular tetrad called pure tetrad. We can recover this invariance by choosing a spin connection which is different from zero. Furthermore covariant \(f(T)\) gravity has been widely discussed in literature \([14, 15, 16, 17, 18, 19]\). In addition, while \(f(R)\) gravity is conformally equivalent to Einstein’s theory plus a scalar field, \(f(T)\) gravity, in general, is not conformally equivalent to TEGR plus a scalar field \([8, 9, 10, 11, 12, 13]\). It is also possible to interpret the teleparallel theories as gauge theories for the group of translations that acts on the tangent fiber bundle at each point of the manifold \([5, 6, 22]\). Finally \(f(R)\) and \(f(T)\) theories can be generalized considering derivative terms of both curvature and torsion, or assuming further geometric invariants \([23, 24, 25, 26]\).

The aim of this work is to discuss the gravitational energy-momentum pseudotensor \([27, 28]\) in \(f(R)\) and \(f(T)\) gravity, to write the related field equations, and to derive the conservation laws in presence of matter along the scheme discussed in \([29]\) (see also \([30, 31]\)). The features of gravitational energy-momentum pseudotensor point out intrinsic differences between metric and teleparallel formulations of theories of gravity. The first result is that the two theories differ for a term \(\omega(T, B)\) containing the torsion scalar \(T\) and a boundary contribution \(B\) by which the local Lorentz invariance can be restored (see also \([20, 21]\)). The gravitational energy-momentum pseudotensor, related to \(\omega(T, B)\), allows to pass from \(f(R)\) to \(f(T)\) and viceversa.

The paper is organized as follows. In Sec.2, we give a short summary on geometrical setting considering, in particular, the teleparallelism. In Sec.3 we derive the Euler-Lagrange equations and the gravitational energy-momentum pseudotensor for \(f(R)\) gravity by applying the Noether theorem to a particular continuous one-parameter group of diffeomorphisms representing rigid translations. Furthermore, we obtain the continuity equation in the presence of matter. In Sec.4 we write the field equations for \(f(T)\) gravity and obtain the gravitational energy-momentum pseudotensor. As above, we derive the continuity equation in presence of matter. Sec.5 is devoted to the gravitational pseudotensor for the \(\omega(T, B)\) term. Finally, in Sec.6 through the three obtained pseudotensors,\(^1\)We like to call these theories "Extended Gravity" instead of "Alternative Gravity" because GR is a so "well-posed", "well-formulated", and experimentally probed theory that, at present state of art, it does not need alternatives but just extensions at the various scales and comparison with new physical problems.
we discuss the relation between the two theories of gravitation. In Sec. 7, conclusions are drawn. In Appendix A, we report a list of useful formulas and, in Appendix B, a list of variations used along the paper.

2 Geometrical setting

Spacetime $\mathcal{M}$ is a pseudo-Riemannian manifold $\{\mathcal{M}, g\}$ where a metric tensor $g$ is defined. It is a non-degenerate symmetric tensor field $(0,2)$, non-positive defined with signature $(1,3) = (+, -, -, -)$. Furthermore $\mathcal{M}$ is a Hausdorff topological manifold, locally homeomorphic to a set in $\mathbb{R}^4$, equipped with a differential structure that is a maximal atlas covering it. A fiber bundle on $\mathcal{M}$ with standard fiber $\mathcal{S}$, is a quadruple $\{\mathcal{E}, \pi, \mathcal{M}, \mathcal{S}\}$, where $\mathcal{E}$, $\mathcal{M}$, $\mathcal{S}$ are differential manifolds and $\pi : \mathcal{E} \to \mathcal{M}$, is a surjective differential application, i.e. a projection, locally trivial in $\mathcal{E}$ that is $\forall p \in \mathcal{M}$ exists a set $\mathcal{U}$ in $\mathcal{M}$ and a diffeomorphism $\chi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{S}$ so that $\pi_1 \circ \chi = \pi$ where $\pi_1$ is the projection along the first coordinate. Let $G$ be a Lie group. A G-bundle structure on the fiber bundle consists of a left action $\theta : G \times \mathcal{S}$ and of a fiber bundle atlas $(\mathcal{U}_\alpha, \psi_\alpha)$ whose transition functions $\psi_{\alpha\beta}$ act on $\mathcal{S}$ via the $G$-action. A fiber bundle with a G-bundle structure is called a G-bundle. A principal (fiber) bundle $(P, \pi, \mathcal{M}, G)$ is a G-bundle with typical fiber which is a Lie group $G$, where the left action on $G$ is just the left translation that is a fiber bundle whose fiber coincides with the structure group. The frame bundle is the disjoint union, for each $p \in \mathcal{M}$, of all linear bases of the tangent space $\mathcal{T}_p$ in $p$ (fiber in $p$) which is a principal fiber bundle with a $\text{GL}(m, \mathbb{R})$ group structure acting on the fiber in $p$. A connection $\Gamma$ is a structure defined on a principal bundle $P$. It is given on a horizontal subspace $H_u$ of $\mathcal{T}_u$, the tangent space of $P$ in $u \in P$, for each $u \in P$ such that $\mathcal{T}_u = V_u \oplus H_u$, namely $\mathcal{T}(P) = V \oplus H$ and $u \to V_u$ is differentiable and invariant under the action of the structure group $G$, where $V_u$ is the vertical subspace of $\mathcal{T}_u$, namely the tangent subspace of the fiber $G_p$ in $u$. Given a connection $\Gamma$ in $P$, we define a 1-form $\omega \in g \otimes \mathcal{T}^*P$ with values in the Lie algebra $g$ of the group $G$ called connection form. In other words, for each $X \in \mathcal{T}_u$, one can define $\omega(X)$ to be the unique $A \in g$ such that $(A^*_u)$, the fundamental vector field generated by $A$, is equal to the vertical component of $X$ (see also [32]). In this perspective, the connection is a rule that allows to compare vector fields belonging to different vector spaces on the manifold [33, 34, 35, 36, 37, 38, 39, 40].

In TEGR, dynamical variables are not the metric components $g$, as in GR, but tetrads. These tetrad fields, the vierbeins $h_a (x^\mu)$, are orthonormal vector fields on $\mathcal{M}$ defining a basis for any point $p \in \mathcal{M}$ with coordinates $x^\mu$. The tangent space to $\mathcal{M}$ is Minkowski with metric $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. One can express the tetrad basis $\{h_a\}$ and its dual $\{h^a\}$ in terms of the holonomic coordinate basis $\{e_\mu\} = \{\partial_\mu\}$ and its dual basis $\{e^\mu\} = \{dx^\mu\}$ [41, 42, 43, 44]. We have

$$h_a = h_a^\mu e_\mu h^a = h^a_\mu e^\mu, \quad (2.1)$$

$$\eta_{ab} = g_{\mu\nu} h_a^\mu h_b^\nu = \eta_{ab} h^a_\mu h_b^\nu, \quad (2.2)$$

$$h^a_\mu h^\nu_a = \delta^a_\mu h^\nu_a = \delta^a_\mu = \delta^a_\mu. \quad (2.3)$$

A connection form $\omega$ (spin connection or Lorentz connection) of the given connection $\Gamma$ in the bundle of linear frames, i.e. a linear connection, is a 1-form $\omega$ with values in the Lie algebra of Lorentz group $\mathfrak{so}(1,3)$:

$$\omega = \omega_\mu dx^\mu = \frac{1}{2} \omega^a_{b\mu} e^b_a dx^\mu = \frac{1}{2} \omega^a_{b\mu} e^b_a dx^\mu, \quad (2.4)$$
with $S^a{}_b$ an appropriate representation of the Lorentz generators and $\omega^a{}_b \in A^1(M)$, where $\omega^a{}_b$ are matrices of 1-forms and $A^1(M)$, the space of all 1-forms. Defining the absolute exterior differential or the covariant exterior derivative of a given tensor $(r,s)$, valued on the $p$-forms $B^a{}_b \in A^p(M,T^r_s(M))$, as the operator $D : A^p(M,T^r_s(M)) \to A^{p+1}(M,T^r_s(M))$, we have

$$DB^a{}_b = dB^a{}_b + \omega^a{}_c \wedge B^c{}_b - \omega^d{}_b \wedge B^a{}_d,$$  

(5.5)

with the operator $d : A^p(M) \to A^{p+1}(M)$ the exterior derivative of a $p$-form. Immediately, we obtain the Cartan structure equations:

$$T^a = Dh^a = dh^a + \omega^a{}_b \wedge h^b = \frac{1}{2} T^a_b h^b \wedge h^c,$$  

(5.6)

$$R^a_b = D\omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a_b c h^c \wedge h^d,$$  

(5.7)

where $T^a$ and $R^a_b$ are the torsion and curvature 2-forms. The T and R forms are the Lie algebras of the 2-forms valued on the translations and Lorentz groups respectively.

$$T = T^a P_a = \frac{1}{2} T^a_{\mu\nu} P_a dx^\mu \wedge dx^\nu,$$  

(5.8)

$$R = \frac{1}{2} R^a_b S_a^b = \frac{1}{4} R^a_{b\mu\nu} S_a^b dx^\mu \wedge dx^\nu,$$  

(5.9)

relative to a coordinate basis $\{dx^\mu\}$ where $P_a$ are the translation generators and, as said above, $S_a^b$ are the Lorentz generators. Furthermore, torsion and curvature forms satisfy the following Bianchi identities:

$$DT^a = R^a_b \wedge h^b,$$  

First Bianchi Identity,  

(5.10)

$$DR^a_b = 0,$$  

Second Bianchi Identity.  

(5.11)

Equivalently, let $\{M,\pi,\mathcal{E}\}$ be a vector bundle on a manifold $M$. We can define a connection as a bilinear map $\nabla : T(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$, namely $(X,V) \mapsto \nabla_X V$, such that is $C^\infty(M)$-linear in $X \in T(M)$, $\mathbb{R}$-linear in $V \in \mathcal{E}(M)$ and the Leibniz formula is satisfied for $T(M)$, the set of all vector fields, and $\mathcal{E}(M)$, the space of smooth sections of the vector bundle. The covariant derivative $\nabla_X V$ of $V$ along $X$ is the section of map $\nabla$. Thus the connection coefficients $\omega^a_{bc}$ of connection $\omega$ expressed in an arbitrary anholonomic basis of tetrads $\{h_a\}$:

$$dh^a = \frac{1}{2} f^a_b c h^b \wedge h^c \iff [h_b,h_c] = f^a_b c h_a,$$  

(5.12)

$$\nabla_{h_a} h_b = \omega^a_{bc} (h_b) h_c = \omega^a_{ba} h_c.$$  

(5.13)

From the formula of 1-form exterior derivative

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]),$$  

(5.14)

we get

$$\tau(X,Y) = T^a(X,Y) h_a,$$  

(5.15)

$$\mathcal{R}(X,Y) h_b = R^a_b(X,Y) h_a,$$  

(5.16)
which define the torsion and curvature of linear connection $\omega$ on $\mathcal{M}$. A map $\tau : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$:

$$\tau (X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad (2.17)$$

with $\tau$ a tensor field of type $(1,2)$ is the torsion tensor. The Riemann (curvature) tensor $R$ is a map $R : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ defined as:

$$R \left( X,Y,Z \right) = \mathcal{R} \left( X,Y,Z \right) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\left[ X,Y \right]} Z, \quad (2.18)$$

with $R$ a tensor field of type $(1,3)$. The connection coefficients of $\omega$ on $\mathcal{M}$, in terms of the only coordinate bases, are traditionally referred to as Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, that are related to the connection coefficients $\omega^a_{\ b\ c}$ in mixed bases by:

$$\nabla_{e_{\mu}} e_{\nu} = \Gamma^a_{\nu\mu} e_{\alpha}, \quad \nabla_{e_{\mu}} h^b = \omega^a_{\ b\ c} h^c, \quad (2.19)$$

$$\Gamma^\alpha_{\nu\mu} = h^a_{\alpha} \partial_{\mu} h^b_{\nu} + h^a_{\alpha} \omega^b_{\mu c} h^c_{\nu}. \quad (2.20)$$

Eq. (2.20) is nothing else but the tetrad postulate or the absolute parallelism, that is:

$$\nabla_{\mu} h^a_{\rho} = \partial_{\mu} h^a_{\rho} - \Gamma^a_{\rho\mu} h^a_{\alpha} + \omega^a_{\mu b} h^b_{\rho} = 0. \quad (2.21)$$

It states, essentially, that the vierbiens are parallel vector fields. We can derive the components of torsion tensor $\tau$ and curvature tensor $R$ for an arbitrary connection $\omega$ with respect to anholonomic frames of the tetrad fields $\{h_a\}$ as:

$$\tau \left( h_a, h_b \right) = T^a_{\ caba} = \left( \omega^c_{\ b\ a} - \omega^c_{\ a\ b} - f^c_{\ ab} \right) h_c, \quad (2.22)$$

$$\mathcal{R} \left( h_a, h_b \right) h_c = R^d_{\ caba} h_d = \left( h_a \left( \omega^d_{\ cb} \right) - h_b \left( \omega^d_{\ ca} \right) + \omega^c_{\ d\ ab} - \omega^c_{\ a\ db} - f^c_{\ ab} \omega^d_{\ c} \right) h_d. \quad (2.23)$$

The torsion and curvature components in the mixed algebraic and spacetime indices are:

$$T^a_{\nu\mu} = \partial_{\nu} h^a_{\mu} - \partial_{\mu} h^a_{\nu} + \omega^a_{\nu c} h^c_{\mu} - \omega^a_{\mu c} h^c_{\nu}, \quad (2.24)$$

$$R^a_{\ b\nu\mu} = \partial_{\nu} \omega^a_{\ b\mu} - \partial_{\mu} \omega^a_{\ b\nu} + \omega^a_{\ b\nu} \omega^c_{\ b\mu} - \omega^a_{\ b\mu} \omega^c_{\ b\nu}. \quad (2.25)$$

On the other hand, the torsion and curvature components, in a natural basis, are given respectively by

$$\tau \left( e_{\mu}, e_{\nu} \right) = T^\rho_{\nu\mu} e_{\rho} = \left( \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} \right) e_{\rho}, \quad (2.26)$$

$$\mathcal{R} \left( e_{\lambda}, e_{\nu} \right) e_{\mu} = R^\rho_{\mu\nu\lambda} e_{\rho} = \left( \partial_{\lambda} \Gamma^\rho_{\mu\nu} - \partial_{\nu} \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\lambda} \Gamma^\sigma_{\mu\nu} \right) e_{\rho}. \quad (2.27)$$

The first Bianchi identity for torsion, in spacetime components, is:

$$\nabla_{\nu} T^\lambda_{\rho\mu} + \nabla_{\mu} T^\lambda_{\rho\nu} + \nabla_{\rho} T^\lambda_{\nu\mu} \quad (2.28)$$

$$= R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} + T^\lambda_{\rho\sigma} T^\sigma_{\mu\nu} + T^\nu_{\rho\sigma} T^\sigma_{\nu\mu} + T^\lambda_{\mu\sigma} T^\sigma_{\nu\rho}.$$

The second Bianchi identity for curvature in spacetime components is:

$$\nabla_{\nu} R^\lambda_{\sigma\rho\mu} + \nabla_{\mu} R^\lambda_{\sigma\nu\rho} + \nabla_{\rho} R^\lambda_{\sigma\mu\nu} \quad (2.29)$$

$$= R^\lambda_{\sigma\mu\rho} T^\theta_{\nu\rho} + R^\lambda_{\sigma\nu\mu} T^\theta_{\rho\mu} + R^\lambda_{\sigma\mu\rho} T^\theta_{\mu\rho}.$$
where $\nabla_\mu$ is the covariant derivative along the coordinate basis $e_\mu$ with respect to the generic connection $\omega$. In presence of torsion, the covariant derivative commutator of a scalar function does not commute as in GR. We have, for a scalar function and a vector field, the relations:

\[ [\nabla_\mu, \nabla_\nu] f = -T^\rho_{\mu\nu} \nabla_\rho f, \quad (2.30) \]

\[ [\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\alpha\mu\nu} V^\alpha - T^\alpha_{\mu\nu} \nabla_\alpha V^\rho. \quad (2.31) \]

The Weitzenböck connection is curvature-free and metric compatible. In a coordinate frame, it is defined in terms of a particular tetrad where the Lorentz connection $\omega^a_{\beta\mu}$ is vanishing. Then we have:

\[ \tilde{\Gamma}^\rho_{\mu\nu} \equiv h^a_\rho \partial_\nu h^a_\mu - h^a_\mu \partial_\nu h^a_\rho, \quad (2.32) \]

satisfying the metricity condition

\[ \tilde{\nabla}_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}^\rho_{\lambda\mu} g_{\rho\nu} - \tilde{\Gamma}^\rho_{\lambda\nu} g_{\rho\mu} = 0, \quad (2.33) \]

and the absolute parallelism postulate or tetrad postulate

\[ \tilde{\nabla}_\lambda h^a_\mu = \partial_\lambda h^a_\mu - \tilde{\Gamma}^\rho_{\mu\lambda} h^a_\rho = 0. \quad (2.34) \]

Torsion and vanishing curvature components of Weitzenböck connection in a coordinate basis are:

\[ T^\rho_{\mu\nu} \equiv \tilde{T}^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\mu\nu} = h^a_\rho \partial_\nu h^a_\mu - h^a_\mu \partial_\nu h^a_\rho - h^a_\rho \partial_\mu h^a_\nu - h^a_\mu \partial_\sigma h^a_\rho, \quad \tilde{R}^\rho_{\mu\lambda\nu} = 0, \quad (2.35) \]

\[ T^\rho_{(\mu\nu)} = 0. \quad (2.36) \]

$\tilde{\nabla}$ is the covariant derivative relative to the Weitzenböck connection. The Weitzenböck connection satisfies both the metricity condition and the tetrad postulate. In terms of parallelism, choosing the Weitzenböck connection has a straightforward interpretation: If we perform the covariant derivative of a generic vector field $V = V^\mu e_\mu = V^a h_a$ with respect to a vector field $X = X^\nu e_\nu$, we have:

\[ \tilde{\nabla}_X V = X^\nu \left[ \partial_\nu V^a + \tilde{V}^a_{\alpha\mu} \tilde{\Gamma}^\alpha_{\mu\nu} \right] e_\alpha = X^\nu \left[ \partial_\nu V^a + h^a_\alpha \tilde{\partial}_\nu h^a_\mu V^\mu \right] e_\alpha = X^\nu \left[ h^a_\alpha \partial_\nu V^a \right] e_\alpha \]

\[ \tilde{\nabla}_X V = 0 \quad \Rightarrow \quad \partial_\nu V^a = 0, \]

that is the vector $V$ is parallel transported by the Weitzenböck connection along the vector field $X$, if its components along the tetrad basis are constant. This means that the tetrad field "parallelizes" the spacetime. The Latin indices $a, b, c, \ldots$ are the "flat" or holonomic indices. They refer to tensor objects projected by tetrads on the tangent space. They can be raised or lowered by the Minkowski tensor $\eta_{ab}$, that is $F_a = \eta_{ab} F^b$. The Greek indices $\mu, \nu, \ldots$, called the "curved" or anholonomic indices, refer to tensor objects defined on the Riemannian manifold. They can be raised and lowered by the metric $g_{\mu\nu}$. In summary, the tetrad $h^a_\mu$ and its dual $h_\mu^a$ allows to project a geometric object from the Riemann manifold to the tangent space and viceversa, that is $F^a = h^a_\mu F^\mu$ and $F^\mu = h^a_\mu F_a$. The difference between the torsionless Levi-Civita connection and the Weitzenböck one is given by the contortion tensor:

\[ K^\rho_{\mu\nu} \equiv \tilde{T}^\rho_{\mu\nu} - \tilde{T}^\rho_{\nu\mu} = -\frac{1}{2} \left( T^\rho_{\mu\nu} - T^\rho_{\nu\mu} + T_{\mu\nu}^\rho \right) = h_\alpha^a \nabla_\nu h^a_\mu, \quad (2.37) \]

\[ K^{(\rho \mu \nu)} = 0, \quad (2.38) \]
where
\[
\tilde{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left( g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho} \right),
\]  
(2.39)
is the Levi Civita torsion-free connection and \( \nabla_\mu \) is the covariant derivative with respect to the Levi Civita connection. We can define the superpotential tensor as:
\[
S^\rho_{\mu\nu} \equiv \frac{1}{2} \left( K^\rho_{\mu\nu} - g^\rho_{\mu\nu} T^\sigma_{\mu\sigma} + g^\rho_{\mu\nu} T^\sigma_{\nu\sigma} \right),
\]  
(2.40)
\[
S^\rho_{\mu\nu} = 0,
\]  
(2.41)
\[
K^\mu_{\rho\mu} = -T^\mu_{\rho\mu} = S^\mu_{\rho\mu}.
\]  
(2.42)
The scalar torsion \( T \) is then:
\[
T \equiv S^\rho_{\mu\nu} T^\rho_{\mu\nu} = -2 S^\rho_{\mu\nu} K^\rho_{\mu\nu},
\]  
(2.43)
\[
= \frac{1}{4} T^\rho_{\mu\nu} T^\rho_{\mu\nu} + \frac{1}{2} T^\rho_{\mu\nu} T^\rho_{\nu\mu} - T^\rho_{\mu\nu} T^\rho_{\nu\mu} \]  
(2.44)
The Riemann tensor for the Weitzenböck connection is null:
\[
R \left[ \tilde{\Gamma} \right] = 0,
\]  
(2.45)
and then we obtain the relation between the spacetime components of curvature tensor in the Weitzenböck and Levi-Civita representations, that is:
\[
0 = \tilde{R}^\rho_{\mu\lambda\nu} = -\tilde{R}^\rho_{\mu\lambda\nu} + \nabla_\nu K^\rho_{\mu\lambda} - \nabla_\lambda K^\rho_{\mu\nu} + K^\rho_{\sigma\nu} K^\sigma_{\mu\lambda} - K^\rho_{\sigma\lambda} K^\sigma_{\mu\nu}.
\]  
(2.46)
The Ricci tensor is obtained by \( R_{\mu\nu} = R^\rho_{\mu\rho\nu} \) and then:
\[
R_{\mu\nu} = \nabla_\nu K^\rho_{\mu\rho} - \nabla_\rho K^\rho_{\mu\nu} + K^\rho_{\sigma\nu} K^\sigma_{\mu\rho} - K^\rho_{\sigma\rho} K^\sigma_{\mu\nu} = -2 \nabla_\rho S^\rho_{\mu\nu} - g_{\mu\nu} \nabla_\rho T^\rho_{\sigma\nu} - 2 S^\rho_{\nu\rho} K^\rho_{\sigma\nu}.
\]  
(2.47)
Contracting again, we get the Ricci scalar expressed with respect to the Levi Civita connection
\[
R \left[ \tilde{\Gamma} \right] (h) = -T - \frac{2}{h} \partial_\mu \left( h T^\mu_{\nu} \right) = -T - 2 \nabla_\mu T^\mu,
\]  
(2.48)
where \( h = \det \left( h^\alpha_{\mu} \right) = \sqrt{-g} \) and \( T^\mu = T^\mu_{\nu} \) is the contraction of torsion tensor with respect to the first and the third index. In TEGR, the dynamical variables are the 16 vierbien components \( h^\alpha_{\mu} \) instead of the 10 metric components \( g_{\alpha\beta} \) of GR. If we choose a tetrad basis to annihil the Lorentz connection \( \omega^a_{\mu} = 0 \), we lose the local Lorentz invariance and preserve the global one. Then we have 16 equations instead of 10 as in GR. If we restore the local Lorentz invariance by a non-trivial spin connection, it means that just 10 of the total 16 tetrad components are independents while the other 6 are fixed by the gauge. In absence of such a local symmetry, we cannot choose the tetrad unless than a global Lorentz transformation.

\footnote{In Eq. (2.47), the boundary term is defined with the minus sign according to the definition of contortion tensor (2.37). In Ref. [21], the relation between the curvature and torsion scalar is \( R = -T + B \) due to a different definition of the contortion tensor. However, the two approaches are equivalent.}
3 The gravitational energy-momentum pseudotensor of \( f(R) \) gravity

Let us consider the action:

\[
S_{f(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) ,
\]

with \( \kappa^2 = 8\pi G/c^4 \). We can calculate the variation \( \tilde{\delta} \) with respect to the metric \( g^{\mu\nu} \) and coordinates \( x^\mu \) for a generic infinitesimal transformation:

\[
x'^\mu = x^\mu + \delta x^\mu , \quad g'^{\mu\nu} (x') = g^{\mu\nu} (x) + \tilde{\delta} g^{\mu\nu} , \quad g'^{\mu\nu} (x) = g^{\mu\nu} (x) + \delta g^{\mu\nu} ,
\]

\[
\tilde{\delta} S_{f(R)} = \frac{1}{2\kappa^2} \int d^4x \left[ \delta \left( \sqrt{-g} f(R) \right) + \partial_\mu \left( \sqrt{-g} f(R) \delta x^\mu \right) \right] ,
\]

where \( \tilde{\delta} \) is the local variation and \( \delta \) is the global variation for a fixed \( x \). We obtain [1, 47, 48, 49, 50, 51]:

\[
\tilde{\delta} S_{f(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) \right] \delta g^{\mu\nu}
\]

\[
+ \int d^4x \partial_\alpha \left\{ \frac{\sqrt{-g}}{2\kappa^2} \left[ \partial_\beta f'(R) (g^{\eta\rho} g^{\alpha\beta} - g^{\alpha\eta} g^{\beta\rho}) \delta g_{\eta\rho} + f'(R) \left[ \left( \partial_\rho \eta - \partial_\eta g^{\rho\sigma} \right) \delta g_{\eta\sigma} \right] 
\right. 
\]

\[
\left. + (g^{\eta\rho} g^\tau\rho - g^{\rho\eta} g^\alpha\tau) \delta g_{\eta,\tau} \right] + f(R) \delta^\alpha_\lambda \delta x^\lambda \right\} ,
\]

where \( f'(R) = \partial f/\partial R \). Imposing the action stationarity at a fixed \( x \), that is \( \delta S_{f(R)} = 0 \) in a given domain \( \Omega \) where the total variation of both metric and its first derivatives are zero at the boundary, that is \( \delta g_{\mu\nu} \mid_{\partial\Omega} = 0 \) and \( \delta (\partial_\alpha g_{\mu\nu}) \mid_{\partial\Omega} = 0 \), the field equations in vacuum are:

\[
\Box f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) = 0 ,
\]

where \( 2\kappa^2 L_{f(R)} = \sqrt{-g} f(R) \). For an infinitesimal transformation like a rigid translation, one has:

\[
x'^\mu = x^\mu + \epsilon^\mu \Rightarrow \delta g_{\mu\nu} = -\epsilon^\lambda g_{\mu\nu,\lambda} ,
\]

because \( \partial \epsilon = 0 \). If the local variation of the action is zero and the field \( g_{\mu\nu} \) satisfies the field equations, we have:

\[
\tilde{\delta} S_{f(R)} = 0 \Rightarrow \partial_\sigma \left( \sqrt{-g} \tau^\sigma_{\lambda f(R)} \right) = 0 ,
\]

with

\[
2\kappa^2 \tau^\sigma_{\lambda f(R)} = 2 \partial_\sigma f'(R) g^{\alpha\beta} g^{\rho\gamma} / \delta g_{\rho\alpha,\beta} g_{\gamma\lambda} 
\]

\[
+ f'(R) \left[ \left( \partial_\rho \eta - \partial_\eta g^{\rho\sigma} \right) g_{\eta\sigma,\lambda} + 2 g^{\sigma\rho} g^{\gamma\tau} g_{\eta\lambda,\sigma} \right] - f(R) \delta^\alpha_\lambda
\]
This is the gravitational energy-momentum pseudotensor of \( f(R) \) gravity, with

\[
\Gamma^\rho_{\eta \sigma} = g^{\rho \nu} g^{\sigma \tau} \Gamma^\nu_{\epsilon \rho} \Gamma^\tau_{\epsilon \sigma},
\]
and

\[
\Gamma^\rho_{\eta \alpha} = g^{\alpha \tau} \Gamma^\tau_{\eta \rho}.
\]

Considering also matter fields, we have:

\[
S_m = \int_\Omega d^4x L_m,
\]
where \( L_m \) depends, at most, on first derivatives of metric \( g_{\mu \nu} \). Imposing the same variational conditions as above, we have:

\[
\delta S_m = \int_\Omega d^4x \delta L_m \delta g^{\mu \nu} = \int_\Omega d^4x \left( \frac{\sqrt{-g}}{2} T^{(m)}_{\mu \nu} \delta g^{\mu \nu} \right),
\]
where \( T^{(m)}_{\mu \nu} \) is the energy-momentum tensor of matter fields:

\[
T^{(m)}_{\mu \nu} = 2 \sqrt{-g} \delta L_m \delta g_{\mu \nu}.
\]

In summary, by minimizing the total action \( S_T = S_{f(R)} + S_m \), one obtains the following field equations in presence of matter:

\[
f(R) P_{\mu \nu} = \kappa T^{(m)}_{\mu \nu}.
\]

From the contracted Bianchi identities, we have:

\[
\nabla^\nu G_{\mu \nu} = 0 \leftrightarrow \nabla^\nu \left( f(R) P_{\mu \nu} \right) = 0 \leftrightarrow \nabla^\nu T^{(m)}_{\mu \nu} = 0,
\]
where we adopted the formula (A.1) in Appendix A. Finally, from the variation (3.6) for a rigid translation and from the matter field equations (3.12), we have:

\[
\delta L_{f(R)} + \partial_\sigma \left( L_{f(R)} \delta x^\sigma \right) = \frac{\sqrt{-g}}{2 \kappa} P^{\mu \nu}_{f(R)} \delta g_{\mu \nu} - \partial_\sigma \left( \sqrt{-g} T^{\sigma \lambda} \right) \epsilon^\lambda = \left[ \frac{1}{2} \sqrt{-g} T^{\mu \nu}_{(m)} g_{\mu \nu, \lambda} - \partial_\sigma \left( \sqrt{-g} T^{\sigma \lambda} \right) \right] \epsilon^\lambda.
\]

From the following identity which holds because \( T^\eta_{\alpha} \) is symmetric, one has:

\[
\sqrt{-g} \nabla^\eta T^\sigma_{\alpha} = \partial_\eta \left( \sqrt{-g} T^\sigma_{\alpha} \right) - \frac{1}{2} g_{\rho \sigma, \alpha} T^{\rho \sigma} \sqrt{-g}.
\]

It is:

\[
\delta L_{f(R)} + \partial_\sigma \left( L_{f(R)} \delta x^\sigma \right) = \left[ -\partial_\sigma \left( \sqrt{-g} T^\sigma_{\alpha} \right) + \sqrt{-g} T^\sigma_{\lambda, \sigma} - \partial_\sigma \left( \sqrt{-g} T^{\sigma \lambda} \right) \right] \epsilon^\lambda.
\]

If, for rigid translations, the local variation is zero, one has:

\[
\delta L_{f(R)} + \partial_\sigma \left( L_{f(R)} \delta x^\sigma \right) = 0 \rightarrow \partial_\sigma \left[ \sqrt{-g} (T^\sigma_{\lambda} + T^{\sigma \lambda}) \right] = \sqrt{-g} \nabla^\sigma T^{\sigma \lambda},
\]
and then, from the contracted Bianchi identity (3.13), we obtain the total conservation law:

\[
\partial_\sigma \left[ \sqrt{-g} \left( T^\sigma_{\lambda f(R)} + T^{\sigma \lambda} \right) \right] = 0.
\]

Similar considerations can be developed also in the case of \( f(T) \) gravity as we are going to do below.
4 The gravitational energy-momentum pseudotensor of \( f(T) \) gravity

In TTEGR and its generalizations, vierbiens are the fields that have to be considered for variations. Let us take into account a generic analytic \( f(T) \) Lagrangian depending on the torsion scalar \( T \).

The action is:

\[
S_{f(T)} = \frac{1}{2\kappa^2} \int d^4 x f(T), \tag{4.1}
\]

where \( h = \det \left(h^\alpha_\alpha\right) \) is the tetrad determinant and, as above, \( \kappa^2 = 8\pi G/c^4 \). Varying the action with respect to \( h^\alpha_\rho \) at a fixed \( x \) and imposing stationarity, we have:

\[
\delta S_{f(T)} = \frac{1}{2\kappa^2} \int d^4 x \left[ h f_T \delta T + f(T) \delta h \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4 x \left\{ 4\partial_\sigma (h f_T S^\sigma_\alpha) - 4 h f_T \delta T \nu a S^\nu_\mu \rho + f(T) h h^\alpha_\rho \right\} = 0.
\]

Adopting Eqs. (A.2), (A.3), (B.2), (B.3) and imposing the total variation \( \delta h^\alpha_\rho |_{\partial \Omega} = 0 \) at the boundary, we obtain the field equations in vacuum:

\[
4 h^{-1} \partial_\alpha \left[ h f_T S^\alpha_\rho \right] - 4 f_T \delta T \nu a S^\nu_\mu \rho + f(T) h h^\alpha_\rho = 0. \tag{4.2}
\]

Let us vary now the action \( S_{f(T)} \) without fixing the domain:

\[
\tilde{\delta} S_{f(T)} = \frac{1}{2\kappa^2} \int d^4 x \left[ \delta (h f(T)) + \partial_\mu (h f(T) \delta x^\mu) \right], \tag{4.3}
\]

we obtain, for a rigid translation \((3.6)\) and imposing that the tetrads satisfy \((4.2)\):

\[
\delta h^\alpha_\rho = -\epsilon^\nu_\mu \partial_\mu h^\alpha_\rho, \tag{4.4}
\]

\[
\tilde{\delta} S_{f(T)} = 0 \iff \partial_\sigma \left( h T^\sigma_\lambda f(T) \right) = 0, \tag{4.5}
\]

where

\[
2\kappa^2 T^\sigma_\lambda f(T) = -4 f_T S^\sigma_\alpha h^\alpha_\rho \lambda - f(T) \delta^\sigma_\lambda \tag{4.6}
\]

this is the the gravitational energy-momentum pseudotensor of \( f(T) \) gravity. Furthermore, it is \( S^\alpha_\rho = h^\alpha_\mu S^\mu_\rho \). For TTEGR, one has:

\[
S_{TTEGR} = \frac{1}{2\kappa^2} \int d^4 x h T \tag{4.7}
\]

and then the gravitational energy-momentum pseudotensor is

\[
T^\sigma_\lambda |_{TTEGR} = \frac{2}{\kappa^2} S^\sigma_\eta \Gamma^\eta_{\rho \lambda} - \frac{T}{2\kappa^4} \delta^\sigma_\lambda \tag{4.8}
\]

where \( \Gamma \) is now the Weitzenbök connection. In presence of matter, by varying the action, we have:

\[
2 h^{-1} \partial_\alpha \left[ h f_T S^\alpha_\rho \right] - 2 f_T \delta T \nu a \Theta^\nu_\mu \rho + \frac{1}{2} f(T) h h^\alpha_\rho = \kappa^2 T^{(m)}_a \rho, \tag{4.9}
\]
where the energy-momentum tensor of matter fields is
\[
\mathcal{T}^{(m)}_{\alpha \rho} = -\frac{1}{h} \frac{\delta L_m}{\delta h^\alpha_{\rho}}.
\] (4.10)

Using Eqs. (2.16) and (2.17), Eqs. (4.9) assume the form:
\[
H_{\mu \nu} = f T G_{\mu \nu} + \frac{1}{2} g_{\mu \nu} \left[ f (T) - f_T T \right] + 2 f_T S_{\rho \mu \sigma} \nabla^\sigma T = \kappa^2 \mathcal{T}_{\mu \nu},
\] (4.11)

where \(G_{\mu \nu}\) is the Einstein tensor and \(\mathcal{T}_{\mu \nu} = h^\alpha_{\mu} \mathcal{T}^{(m)}_{\alpha \nu}\). From the divergence of the first term of (4.11), i.e. \(H_{\mu \nu}\), using Eqs. (2.16) and the anti-symmetry of contortion \(K_{(\rho \mu) \nu} = 0\), we have:
\[
\nabla^\mu H_{\mu \nu} = -2 f_T S_{\rho \mu \sigma} K_{\rho \nu} \nabla^\sigma T = -H_{\lambda \alpha} K^\lambda_{\alpha \nu} = -H_{(\lambda \alpha)} K_{\lambda \alpha \nu} - H_{[\lambda \alpha]} K^{\lambda \alpha}_{\nu}.
\] (4.12)

Considering the symmetry of the energy-momentum tensor \(T_{\mu \nu}\), the anti-symmetry of the first two indices of the contortion, \(K_{(\rho \mu) \nu} = 0\), and \(H_{[\mu \nu]} = 0\), we obtain:
\[
\nabla^\mu H_{\mu \nu} = 0.
\] (4.13)

This means that, also in teleparallelism, the following Bianchi relations hold:
\[
\nabla^\nu G_{\mu \nu} = 0 \leftrightarrow \nabla^\mu H_{\mu \nu} = 0 \leftrightarrow \nabla^\nu T_{\mu \nu} = 0.
\] (4.14)

The local variation of the \(f (T)\) action for a rigid translation, considering also the matter fields (4.9), gives:
\[
0 = \frac{\hbar}{\kappa^2} \left( P_{f (T)} \right)_{\alpha \rho} \delta h^\alpha_{\rho} - \partial_\sigma (h \tau^\sigma_{\lambda}) \epsilon^\lambda = [-h \mathcal{T} \quad h^\alpha_{\rho,\lambda} - \partial_\sigma (h \tau^\sigma_{\lambda})] \epsilon^\lambda,
\] (4.15)

where
\[
\left( P_{f (T)} \right)_{\alpha \rho} = \frac{\kappa^2 \delta L_{f (T)}}{\delta h^\alpha_{\rho}}.
\] (4.16)

From the symmetry of \(T_{\alpha \rho}\):
\[
h \nabla^\eta T^\eta_{\alpha} = \partial_\eta (h T^\eta_{\alpha}) - hh^\alpha_{\rho,\eta} T_{\alpha \rho},
\] (4.17)

and from (4.15), we get:
\[
h \nabla^\eta T^\eta_{\alpha} = \partial_\eta (h T^\eta_{\alpha}) - \partial_\rho (h \tau^\rho_{\lambda}) = 0 \Rightarrow \partial_\rho [h (T^\rho_{\lambda} + \tau^\rho_{\lambda})] = h \nabla^\eta T^\eta_{\lambda}.
\] (4.18)

Finally, from the Bianchi identities (4.14), we obtain the conservation of the matter energy-momentum tensor and the gravitational energy-momentum pseudotensor, that is:
\[
\partial_\sigma \left[ h \left( \tau^\sigma_{\lambda f (T)} + T^\sigma_{\lambda} \right) \right] = 0
\] (4.19)

Eq. (4.19) can be derived also by using the anti-symmetry of the superpotential \(S_{\mu \nu \rho}\) after writing the field Eqs. (4.9) in presence of matter and adopting only the spacetime indices, that is:
\[
2h^{-1} \partial_\sigma \left[ h f_T S^\sigma_{\lambda} \right] + 2 f_T \tilde{\Gamma}^\nu_{\eta \lambda} S^\eta_{\mu} \tau^\rho_{\mu} + \frac{1}{2} f (T) \delta^\rho_{\lambda} = \kappa^2 T^\rho_{\lambda}.
\] (4.20)
By this approach, we have the anti-symmetry of superpotential and the commutativity of partial derivatives:

\[ \partial_\rho \partial_\sigma \left[ h f_T S^{\rho\sigma}_\lambda \right] = - \partial_\rho \partial_\sigma \left[ h f_T S^{\rho\sigma}_\lambda \right] = 0, \quad (4.21) \]

that is

\[ 2 \partial_\rho \partial_\sigma \left[ h f_T S^{\rho\sigma}_\lambda \right] = \partial_\rho \left[ h \kappa^2 T^\mu_\lambda + h \kappa^2 \left( - \frac{2}{\kappa^2} f_{\mu} S^\eta_\mu \hat{\Gamma}^\mu_\eta_\lambda - \frac{f(T)}{2 \kappa^2} \delta^\rho_\lambda \right) \right] = 0, \quad (4.22) \]

from which we obtain the conservation law \((4.19)\).

The gravitational energy-momentum pseudotensors in metric and tetradic formalisms can be connected considering the role of boundary terms. By these further terms, it is possible to construct a further gravitational pseudotensor by which it is possible to connect the two representations.

5 The gravitational energy-momentum pseudotensor for the \(\omega(T, B)\) term

Theories like \(f(R)\) and \(f(T)\) are not equivalent as GR and TEGR, that is the linear cases in the curvature \(R\) and torsion \(T\) scalars (see [13] for a discussion). In fact, while it is possible to formulate \(f(R)\) gravity by a conformal transformation of the metric as GR plus a scalar field, this is not possible, in general, for \(f(T)\) (see [14]). Furthermore, in a pure metric formalism, \(f(R)\) dynamics is given by fourth-order field equations while \(f(T)\) field equations are second order. Finally, \(f(R)\) gravity is invariant under local Lorentz transformations while teleparallel \(f(T)\) gravity depends on the chosen tetrad frame. However, we can always define a teleparallel equivalent version of \(f(R)\) gravity considering

\[ f(R(h)) = f(T) + \omega(T, B), \quad (5.1) \]

where \(T\) is the torsion scalar and \(B = (2/h) \partial_\mu (h T^\mu) = 2 \nabla_\mu T^\mu\) is the boundary term by which we have \(R(h) = -T - B\) [20, 21].

Let us calculate now the gravitational energy-momentum pseudotensor related to the boundary term \(B\) taking into account the action \(S_B\) which will allow us to pass from the gravitational energy-momentum pseudotensor \(\tau^\alpha_{\lambda f(R)}\), coming from \(f(R)\) gravity, to the gravitational energy-momentum pseudotensor \(\tau^\alpha_{\lambda f(T)}\), coming from \(f(T)\) gravity. We start from

\[ S_B = \frac{1}{2 \kappa^2} \int_{\Omega} d^4 x h \omega(T, B). \quad (5.2) \]

Let us vary the action \(S_B\) with respect to the tetrads \(h^a_\rho\) at a fixed \(x\) by using the variations in Appendix [15] and the relations \((A.7), (A.8), (A.9)\). We have

\[ \delta S_B = \frac{1}{2 \kappa^2} \int_{\Omega} d^4 x \left[ \omega(T, B) \delta h + h \omega_T \delta T + h \omega_B \delta B \right], \quad (5.3) \]
and then
\[
2\kappa^2 \delta L_B = \left[ 4\partial_\sigma (\hbar \omega_T S^\rho_\sigma) - 4\hbar \omega_T T^\rho_\nu a S_\mu^\nu \rho + \omega \hbar h_a^\rho - B \omega_B h h_a^\rho + 2 h h_a^\rho \Box \omega_B \\
- 2 h h_a^\rho \nabla_\sigma \nabla_\rho \omega_B + 4 h \partial_\lambda \omega_B S^\lambda_\rho \right] \delta h^\rho_\rho - \partial_\sigma \left[ 4\hbar \omega_T S^\rho_\sigma + 2 h \partial_\lambda \omega_B \left( h_a^\rho g^\lambda_\sigma - h_a^\sigma g^\lambda_\rho \right) \\
- 2 \omega_B T^\sigma h h_a^\rho + 2 \omega_B h \left( T^\rho_\sigma a + h_a^\sigma T^\rho + g^\rho_\sigma T_a \right) \right] \delta h^\rho_\rho - 2 \partial_\rho \omega_B \left( h_a^\rho g^\sigma_\eta - h_a^\sigma g^\rho_\eta \right) \delta \left( \partial_\eta h_a^\rho \right) \right],
\]
(5.4)

where \( \omega_T = \partial \omega / \partial T \), \( \omega_B = \partial \omega / \partial B \), and \( 2\kappa^2 L_B = \hbar \omega (T, B) \). Imposing that, at the boundary of the domain, tetrads and first derivative variations, \( \delta h^\rho_\rho \) and \( \delta \left( \partial_\eta h_a^\rho \right) \), are zero, we obtain the field equations relative to the boundary terms, that is

\[
4\partial_\sigma [\hbar \omega_T S^\rho_\sigma] - 4\hbar \omega_T T^\rho_\nu a S_\mu^\nu \rho + \omega (T, B) \hbar h_a^\rho - \hbar \omega_B \hbar h_a^\rho \\
+ 2 \hbar h_a^\rho \Box \omega_B - 2 \hbar h_a^\rho \nabla_\sigma \nabla_\rho \omega_B + 4 \partial_\rho \omega_B S^\rho_\sigma = 0.
\]
(5.5)

For a generic variation of fields and coordinates, it is:

\[
\delta L_B + \partial_\rho (L_B \delta x^\rho) = (P_B)_a^\rho \delta h_a^\rho + \partial_\sigma \left( h J^\sigma \right)
\]
(5.6)

where \( J^\sigma \) is the Noether current and \( (P_B)_a^\rho \) is:

\[
(P_B)_a^\rho = \frac{1}{\hbar} \frac{\delta L_B}{\delta h_a^\rho}.
\]
(5.7)

Then we have

\[
- 2 \kappa^2 J^\sigma = \left[ 4\omega_T S^\rho_\sigma + 2 \partial_\lambda \omega_B \left( h_a^\rho g^\lambda_\sigma - h_a^\sigma g^\lambda_\rho \right) - 2 \omega_B T^\sigma h_a^\rho \\
+ 2 \omega_B \left( T^\rho_\sigma a + h_a^\sigma T^\rho + g^\rho_\sigma T_a \right) \right] \delta h_a^\rho - 2 \omega_B \left( h_a^\rho g^\sigma_\eta - h_a^\sigma g^\rho_\eta \right) \delta \left( \partial_\eta h_a^\rho \right) - \omega \delta x^\sigma.
\]
(5.8)

If the action is invariant under rigid translations and tetrads verify the field equations \( 5.5 \), we have, assuming \( \delta \left( \partial_\eta h_a^\rho \right) = -h_a^\rho \rho, \lambda \epsilon^\lambda, \) for translations

\[
\delta S_B = 0 \Rightarrow \partial_\sigma \left( h T^\sigma_{\mu \nu} \right) = 0,
\]
(5.9)

where \( J^\sigma = -\tau^\sigma_{\lambda \nu} \epsilon^\lambda \):

\[
2\kappa^2 \tau^\sigma_{\lambda \nu} (T, B) = - \left[ 4\omega_T S^\rho_\sigma + 2 \omega_B \left( T^\rho_\sigma a + h_a^\sigma T^\rho + g^\rho_\sigma T_a \right) - T^\sigma h_a^\rho \\
+ 2 \partial_\lambda \omega_B \left( h_a^\rho g^\sigma_\eta - h_a^\sigma g^\rho_\eta \right) \right] \epsilon^\lambda h_a^\rho + 2 \omega_B \left( h_a^\rho g^\sigma_\eta - h_a^\sigma g^\rho_\eta \right) \epsilon^\lambda h_a^\rho - \omega \delta x^\sigma
\]
(5.10)

with \( \tau^\sigma_{\lambda \nu} \) the gravitational energy-momentum pseudotensor of \( \omega (T, B) \). This object is due to the boundary terms. The calculation of pseudotensor \( \tau^\sigma_{\lambda \nu} \) could be performed directly from derivatives
instead of using the variations. Let us express the boundary term $B$ in terms of tetrads $h_a^\rho$ and its derivatives:

$$B = 4 \left[ h_a^{\rho} g^{\sigma \mu} \partial_\mu h_a^\sigma + (h_a^\rho h_b^{[\tau \gamma \lambda]} - h_a^{[\lambda} h_b^{\tau \gamma]) g^{\mu \rho} - h_a^{\mu} h_b^{[\tau} g^{\lambda]} \partial_\mu h_a^\rho \right].$$  \hspace{1cm} (5.11)

Using the gravitational energy-momentum pseudotensor for a generic Lagrangian depending up to second order tetrad derivatives (see [29]) and adopting the formulas (A.4), (A.5), (A.6), i.e.

$$h^{\tau \sigma}_\mu = \left( \frac{\partial L_B}{\partial \partial_\sigma h_a^\rho} - \partial_\lambda \left( \frac{\partial L_B}{\partial \partial_\lambda h_a^\rho} \right) \right) h^a_{\rho, \mu} + \frac{\partial L_B}{\partial \partial_\lambda h_a^\rho} h^a_{\rho, \mu \lambda} - \delta^a_{\mu} L_B,$$  \hspace{1cm} (5.12)

we have

$$2\kappa^2 \tau^\sigma_\mu|_\omega = \left[ \omega_T \frac{\partial T}{\partial \partial_\sigma h_a^\rho} + \omega_B \frac{\partial B}{\partial \partial_\sigma h_a^\rho} - h^{-1} \partial_\lambda \left( \omega_B \frac{\partial B}{\partial \partial_\lambda h_a^\rho} \right) \right] h^a_{\rho, \mu}$$

$$+ \omega_B \frac{\partial B}{\partial \partial_\lambda h_a^\rho} h^a_{\rho, \mu \lambda} - \omega (T, B) \delta^a_\mu L_B.$$  \hspace{1cm} (5.13)

Considering the following derivatives of $B$ and $T$, one has

$$\frac{\partial B}{\partial \partial_\sigma h_a^\rho} = 2 \left( h_a^{\rho} T^\sigma - T_a g^{\sigma \rho} - h_a^\sigma T^\rho - T^\rho a \right) + 4 \left( h_a^{\rho} h_b^{[\sigma} g^{\lambda]} + h_a^{[\sigma} h_b^{\tau ] g^{\lambda]} \right)$$

$$- h_a^{[\rho} h_b^{\sigma]} g^{\tau \lambda} - h_a^{[\rho} h_b^{\lambda} g^{\tau]} \partial_\sigma h^b_{\tau},$$  \hspace{1cm} (5.14)

and

$$\frac{\partial T}{\partial \partial_\sigma h_a^\rho} = 4 \delta^a_\mu.$$  \hspace{1cm} (5.15)

Combining these results, it is straightforward to obtain the expression (5.10).

6 From $f(R)$ gravity to $f(T)$ gravity and viceversa by $\tau^\sigma_\lambda|_\omega(T,B)$

Also if the two theories of gravity are not equivalent, it is possible to pass from $f(R)$ to $f(T)$ (and viceversa) by means of the pseudotensor $\tau^\sigma_\lambda|_\omega(T,B)$ being:

$$\tau^\sigma_\lambda|_\omega(f(R)) = \tau^\sigma_\lambda|_\omega(f(T)) + \tau^\sigma_\lambda|_\omega(T,B).$$  \hspace{1cm} (6.1)

The demonstration of this statement can be easily achieved as follows. Let us locally vary the action for $f(R)$ with respect to the metric $g_{\mu \nu}$ and the two actions for $f(T)$ and $\omega(T, R)$ with respect to
the tetrads $h^\alpha{}\rho$, we have:

$$
\tilde{\delta}_{g,x} S_{f(R)} = \frac{1}{2\kappa^2} \int_{\Omega} d^4x \left[ \sqrt{-g} P_{f(R)}^{\mu\nu} \delta g_{\mu\nu} + \partial_\lambda \left( 2\kappa^2 \sqrt{-g} J^\lambda_{f(R)} \right) \right],
$$

$$
\tilde{\delta}_{h,x} S_{f(T)} = \frac{1}{2\kappa^2} \int_{\Omega} d^4x \left[ 2h \left( P_{f(T)} \right)_{a}^\rho \delta h^a_\rho + \partial_\lambda \left( 2\kappa^2 h J^\lambda_{f(T)} \right) \right],
$$

$$
\tilde{\delta}_{h,x} S_{\omega(T,B)} = \frac{1}{2\kappa^2} \int_{\Omega} d^4x \left[ 2h \left( P_\omega \right)_{a}^\rho \delta h^a_\rho + \partial_\lambda \left( 2\kappa^2 h J^\lambda_{\omega(T,B)} \right) \right],
$$

where $J^\lambda$ are the Noether currents. From

$$
S_{f(R)}(h) = S_{f(T)} + S_{\omega(T,B)},
$$

we have

$$
\tilde{\delta}_{g,x} S_{f(R)} = \tilde{\delta}_{h,x} S_{f(T)} + \tilde{\delta}_{h,x} S_{\omega(T,B)};
$$

that is

$$
\sqrt{-g} P_{f(R)}^{\mu\nu} \delta g_{\mu\nu} + \partial_\lambda \left( 2\kappa^2 \sqrt{-g} J^\lambda_{f(R)} \right) = 2h \left[ \left( P_{f(T)} \right)_{a}^\rho + \left( P_\omega \right)_{a}^\rho \right] \delta h^a_\rho
$$

$$
+ \partial_\lambda \left[ 2\kappa^2 h \left( J^\lambda_{f(T)} + J^\lambda_{\omega(T,B)} \right) \right].
$$

Since the following identities hold by expressing the r.h.s. in tetrad terms

$$
\sqrt{-g} P_{f(R)}^{\mu\nu} \delta g_{\mu\nu}[h] = 2h \left[ \left( P_{f(T)} \right)_{a}^\rho + \left( P_\omega \right)_{a}^\rho \right] \delta h^a_\rho,
$$

we have

$$
\partial_\lambda \left( \sqrt{-g} J^\lambda_{f(R)} \right) = \partial_\lambda \left[ h \left( J^\lambda_{f(T)} + J^\lambda_{\omega(T,B)} \right) \right],
$$

from which Eq. (6.1) holds for rigid translations thanks to the Noether currents. In other words, by this formalism based on pseudotensors, it is possible to relate metric and tetradic pictures of $f(R)$ and $f(T)$ theories of gravity.

### 7 Conclusions

The gravitational energy-momentum pseudotensor $\tau^\alpha{}\lambda$ is an important feature of gravitational interaction capable of discriminating among theories of gravity. In this paper, we discussed this pseudotensor comparing its derivation in GR, in TEGR, and in their straightforward generalizations, namely $f(R)$ and $f(T)$ gravity. The considerations have been developed in both metric and vierbien formalisms. The main result is that the two pseudotensors are related defining a further pseudotensor, $\tau^\alpha{}\lambda_{\omega(T,B)}$, derived from the boundary term $B$ that connects the curvature and the torsion scalars as $\tilde{R} = -T - B$. This characteristic could assume a relevant role in tests of gravity aimed to put in evidence further terms and degrees of freedom with respect to GR. In particular, as discussed in detail in [52], the gravitational pseudotensor is important to point out differences in quadrupolar gravitational radiations coming from GR and $f(R)$ gravity. Using a similar approach in metric and teleparallel gravity, discrimination of theories can be achieved considering gravitational radiation: further polarizations and energy-momentum contributions can be selected and classified in view of possible observations. For example, as discussed in [53, 54], gravitational waves
in $f(T)$ gravity and its generalizations are substantially different with respect to gravitational waves in $f(R)$ gravity so detecting further polarization modes of gravitational radiation could be a way to discriminate among theories. The issue of degrees of freedom in $f(T)$ gravity is discussed in detail in [55].

In a forthcoming paper, we will discuss the weak field limit of gravitational pseudotensor in the various formulations in view of selecting physical observables and discussing possible experimental limits on gravitational radiation.

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**A Appendix: Useful formulae**

\[
\nabla^\nu \nabla_\mu \nabla_\nu f(R) = R^\alpha_\mu \nabla_\alpha f(R) + \nabla_\mu \square f(R) \tag{A.1}
\]

\[
\frac{\partial h_{\nu}}{\partial h^\alpha_\rho} = -h_{\alpha}^\nu h_\rho^\alpha
\tag{A.2}
\]

\[
\frac{\partial g^{\mu\nu}}{\partial h^\alpha_\rho} = -g^{\mu\nu} h_\alpha^\rho - g^{\rho\mu} h_\nu^\nu
\tag{A.3}
\]

\[
\partial_\alpha h_\rho^\alpha = -h_\alpha^\mu h_\rho^\beta \partial_\alpha h_\beta^\beta = -\check{\Gamma}_\rho^\alpha_\mu h_\alpha^\lambda
\tag{A.4}
\]

\[
\frac{\partial h}{\partial h^\alpha_\rho} = h_\alpha^\rho h
\tag{A.5}
\]

\[
\partial_\mu h = h\check{\Gamma}_{\mu\nu} - hK_{\nu\rho} = h\check{\Gamma}_{\nu\mu} = h h_\alpha^\rho h_\rho^\alpha
\tag{A.6}
\]

\[
\partial_\sigma(hF^{\rho\sigma}) = h\nabla_\sigma F^{\rho\sigma} \quad \text{so} \quad F^{(\rho\sigma)} = 0
\tag{A.7}
\]

\[
\nabla_\alpha h_\rho^\beta = -h_\alpha^\eta K^{\rho}_{\eta\sigma} \nabla_\sigma h_\alpha^\sigma = T_\alpha
\tag{A.8}
\]

\[
K^{\rho}_{\alpha\lambda} = K^{\rho\lambda}_{\alpha} + T^{\rho\lambda}_{\alpha}
\tag{A.9}
\]

**B Appendix: Variations**

\[
\delta h_\beta^\rho = -h_\alpha^\mu h_\beta^\rho \delta h_\mu^\alpha
\tag{B.1}
\]

\[
\delta h = h h_\alpha^\rho \delta h_\rho^\alpha = -h h_\rho^\alpha \delta h_\alpha^\rho
\tag{B.2}
\]

\[
\delta T = -4T^\mu_\nu S_\mu^\nu \rho \delta h_\rho^\alpha - 4S_\alpha^\rho \delta \left(\partial_\alpha h^\alpha_\rho\right)
\tag{B.3}
\]

\[
\delta T^{\lambda}_{\alpha} = -\left(T^{\lambda}_{\alpha} + h_\alpha^\lambda T^\rho + g^{\lambda\rho} T_\alpha\right) \delta h^\rho_\rho + \left(h_\alpha^\rho \delta g^{\lambda\rho} - h_\alpha^\sigma g^{\lambda\rho}\right) \delta \left(\partial_\sigma h^\sigma_\rho\right)
\tag{B.4}
\]
\[ \delta B = \delta \left[ \frac{2}{h} \partial_\sigma (hT^{\sigma \rho}) \right] = -\frac{B}{h} \delta h + \frac{2}{h} \partial_\mu [T^\mu \delta h + h \delta T^\mu] \] (B.5)

\[ \begin{align*}
\hbar & \omega T \delta T = \left[ -4 \hbar \omega T T^\mu_{\rho \sigma} S_{\mu \rho}^{\ \sigma} + 4 \partial_\sigma (\hbar \omega T S_{\rho}^{\ \sigma}) \right] \delta \hbar^\rho \rho - \partial_\sigma \left( 4 \hbar \omega T S_{\rho}^{\ \sigma} \delta \hbar^\rho \rho \right) \\
\hbar & \omega_B \delta B = \left[ -B \hbar h a^\rho + 2 hh a^\rho \Box \omega_B - 2 hh a^\sigma \nabla_\sigma \nabla^\rho \omega_B + 4 h \partial_\lambda \omega_B S_{\lambda \rho} \right] \delta \hbar^\rho \rho \\
& - \partial_\sigma \left[ 2 h \partial_\lambda \omega_B \left( h a^\rho g^{\lambda \sigma} - h a^\sigma g^{\lambda \rho} \right) \delta \hbar^\rho \rho - 2 \omega_B (T^\sigma \delta h + h \delta T^\sigma) \right]
\end{align*} \] (B.6)

\[ \hbar \omega_B \delta B = \left[ -B \hbar h a^\rho + 2 hh a^\rho \Box \omega_B - 2 hh a^\sigma \nabla_\sigma \nabla^\rho \omega_B + 4 h \partial_\lambda \omega_B S_{\lambda \rho} \right] \delta \hbar^\rho \rho \\
& - \partial_\sigma \left[ 2 h \partial_\lambda \omega_B \left( h a^\rho g^{\lambda \sigma} - h a^\sigma g^{\lambda \rho} \right) \delta \hbar^\rho \rho - 2 \omega_B (T^\sigma \delta h + h \delta T^\sigma) \right] \] (B.7)

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