RIGID IDEALS

BRENT CODY AND MONROE ESKEW

Abstract. An ideal $I$ on a cardinal $\kappa$ is called rigid if all automorphisms of $P(\kappa)/I$ are trivial. An ideal is called $\mu$-minimal if whenever $G \subseteq P(\mu)/I$ is generic and $X \in P(\mu) \setminus V$, it follows that $V[X] = V[G]$. We prove that the existence of a rigid saturated $\mu$-minimal ideal on $\mu^+$, where $\mu$ is a regular cardinal, is consistent relative to the existence of large cardinals. The existence of such an ideal implies that GCH fails. However, we show that the existence of a rigid saturated ideal on $\mu^+$, where $\mu$ is an uncountable regular cardinal, is consistent with GCH relative to the existence of an almost-huge cardinal. Addressing the case $\mu = \omega$, we show that the existence of a rigid presaturated ideal on $\omega_1$ is consistent with CH relative to the existence of an almost-huge cardinal. The existence of a precipitous rigid ideal on $\mu^+$ where $\mu$ is an uncountable regular cardinal is equiconsistent with the existence of a measurable cardinal.

1. Introduction

An ideal $I$ on a cardinal $\kappa$ is said to be rigid if all automorphisms of the boolean algebra $P(\kappa)/I$ are trivial. Woodin proved [Woo10] that if MA $\omega_1$ holds, then every saturated ideal on $\omega_1$ is rigid. Larson [Lar02] showed that we do not need the whole of Martin’s Axiom to obtain the same conclusion; more specifically, Larson proved that if a certain cardinal characteristic of the continuum is greater than $\omega_1$, then every saturated ideal on $\omega_1$ is rigid. It is shown in [FMS88, Theorem 18], that in models of MA $\omega_1$, every saturated ideal $I$ on $\omega_1$ has an additional property: if $G \subseteq P(\omega_1)/I$ is generic and $r \in V[G] \setminus V$ is a real, then $V[r] = V[G]$. Given a cardinal $\mu$, we say that a poset $P$ is $\mu$-minimal if whenever $G \subseteq P$ is generic and $X \in P(\mu) \setminus V$, it follows that $V[X] = V[G]$. When we say that an ideal $I \subseteq P(\kappa)/I$ is $\mu$-minimal, we mean that the poset $P(\kappa)/I$ is $\mu$-minimal. Thus, under MA $\omega_1$, every saturated ideal on $\omega_1$ is rigid and $\omega$-minimal.

In this article we extend the above results on rigidity and minimality properties of ideals. We first note the following easy generalization:

Observation 1. If $\kappa > \omega_1$ is a regular cardinal carrying a saturated ideal $I$, then there is a c.c.c. forcing $P$ such that $\Vdash P \ 	ext{“The ideal generated by } I \text{ is rigid, saturated, and } \omega\text{-minimal.”}$

To prove this, we let $P$ be the Solovay-Tennenbaum forcing [ST71] to obtain MA $\kappa$. An easy application of Corollary 13 below shows that the saturation of $I$ is preserved. The arguments for rigidity and $\omega$-minimality are identical to those of [Lar02] and [FMS88]. We do not know if such ideals will be $\mu$-minimal for other $\mu < \kappa$. Nonetheless, the following theorem shows that the existence of a rigid,
saturated and $\mu$-minimal ideal on $\mu^+$, where $\mu$ is an uncountable regular cardinal, is consistent relative to large cardinals.

**Theorem 2.** Suppose GCH holds and $I$ is a saturated ideal on $\mu^+$ where $\mu$ is a regular uncountable cardinal. Then there is a $<\mu$-distributive forcing extension in which the ideal generated by $I$ is rigid, saturated and $\mu$-minimal.

The proof of Theorem 2 given below uses methods of Larson [Lar02] mentioned above which involve exploiting the fact that one may force a certain cardinal characteristic to be large, and thus in the forcing extension, $2^\mu > \mu^+$. Notice that in all the models with rigid saturated ideals mentioned thus far, GCH fails. Indeed, as we show in Section 3, GCH implies that for every regular cardinal $\mu$, there does not exist a $\mu$-minimal presaturated ideal on $\mu^+$. It is natural to wonder whether or not the situation changes if we remove the minimality requirement: is the existence of a rigid saturated ideal on some successor cardinal consistent with GCH? We will show that the existence of a rigid saturated ideal on $\mu^+$, where $\mu$ is an uncountable regular cardinal, is consistent with GCH, relative to the existence of an almost-huge cardinal.

**Theorem 3.** Suppose that $\kappa$ is an almost-huge cardinal and $\mu < \kappa$ is an uncountable regular cardinal. Then there is a $<\mu$-distributive forcing extension in which there is a rigid saturated ideal on $\mu^+$ and GCH holds.

Notice that Theorem 3 fails to address the case of ideals on $\omega_1$. We will show that it is consistent relative to the existence of an almost-huge cardinal that $\omega_1$ carries a rigid presaturated ideal while GCH holds.

**Theorem 4.** Suppose that $\kappa$ is an almost-huge cardinal and $\mu < \kappa$ is a regular cardinal. Then there is a $<\mu$-distributive forcing extension in which there is a rigid presaturated ideal on $\mu^+$ and GCH holds.

Our proofs of Theorem 3 and Theorem 4 will involve using a variation of a coding forcing introduced by Friedman and Magidor in [FM09], which they used to control the number of normal measures carried by a measurable cardinal. Assuming $\kappa$ is an almost-huge cardinal and $\mu < \kappa$ is regular, we will use a coding forcing to define a forcing $P$ such that if $G \subseteq P$ is generic over $V$, then in $V[G]$ we have $\kappa = \mu^+$, there is a saturated ideal $I$ on $\kappa$, and forcing with $P(\kappa)/I$ over $V[G]$ produces an extension $V[G * H]$ in which $H$ is the unique generic filter for $P(\kappa)/I$ over $V[G]$. Hence the ideal $I \in V[G]$ is rigid. We prove Theorem 3 and Theorem 4 below with the proof of Theorem 3 coming before that of Theorem 4 because the forcing construction for Theorem 3 is more technical. Our proof of Theorem 3 will be split into two cases: first we prove Theorem 3 for $\mu$ not the successor of a singular cardinal (see Theorem 24), then we prove the remaining case in Section 5.

We also show that a measurable cardinal will suffice to obtain a model with a precipitous rigid ideal on $\mu^+$ where $\mu$ is a regular uncountable cardinal.

**Theorem 5.** Suppose $\kappa$ is a measurable cardinal and $\mu < \kappa$ is an uncountable regular cardinal. Then there is a forcing extension in which there is a rigid precipitous ideal on $\mu^+$ and GCH holds.

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1. A variation of Friedman and Magidor’s coding forcing was used by Ben-Neria [BN15] to show that any well founded order can be realized as the Mitchell order $\prec(\kappa)$ on a measurable cardinal $\kappa$. 

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A similar result is not possible for $\omega_1$. The existence of a presaturated ideal on $\omega_1$ is known to be equiconsistent with a Woodin cardinal [Woo10]. If GCH holds and $I$ is a precipitous but not presaturated ideal on $\omega_1$, then $P(\omega_1)/I$ is forcing equivalent to $Col(\omega, \omega_2)$, which never has unique generics.

2. Preliminaries

Let us review some absorption properties of collapse forcings. Let $\mu$ be a regular cardinal. If $P$ is a $<\mu$-closed separative forcing, then for sufficiently large $\kappa$ it follows that there is a regular embedding $P \rightarrow Col(\mu, \kappa)$ and we say that $Col(\mu, \kappa)$ absorbs $P$. If $\kappa > \mu$ is an inaccessible cardinal and $P$ is any $<\mu$-closed separative forcing with $|P| < \kappa$ then there is a regular embedding $P \rightarrow Col(\mu, <\kappa)$. See [Cum10, Section 14] for more details.

In order to force the existence of a saturated ideal on $\omega_1$ starting with a model containing a huge cardinal $\kappa$, Kunen [Kun78] defined a forcing iteration $K$ of length $\kappa$ which is $\kappa$-c.c. and highly universal in the sense that many posets regularly embed into $K$, including many posets of size $\kappa$. We refer the reader to [Cox] for additional background on universal collapsing forcing. Let us review the definition of a slight variation of Kunen’s universal collapse which was used by Magidor (see [For10]), as well as some its properties that will be relevant for our proofs of Theorem 3 and Theorem 4.

Definition 6. Suppose $\kappa$ is a huge cardinal with target $\lambda$. In other words, there is an elementary embedding $j: V \rightarrow M$ with critical point $\kappa$ such that $\lambda = j(\kappa)$ and $M^\lambda \subseteq M$. Suppose $\mu < \kappa$ is a regular cardinal. Let $P = P_\kappa$ be a $<\mu$-support iteration $\langle (P_\alpha, \dot{Q}_\beta) : \alpha \leq \kappa, \beta < \kappa \rangle$ such that

1. $P_0 = Col(\mu, <\kappa)$
2. If $P_\alpha \cap V_\alpha$ is a regular suborder of $P_\alpha$ and $P_\alpha \cap V_\alpha$ is $\alpha$-c.c., we say that $\alpha$ is an active stage in the iteration, and let $\dot{Q}_\alpha$ be a $P_\alpha \cap V_\alpha$-name for $Col(\alpha, <\kappa)_{V_\alpha}^\kappa$.

In the proofs of Theorem 3 and Theorem 4 below we will need the following properties of this iteration.

Lemma 7. Suppose $P = P_\kappa$ is the iteration defined above. The following properties hold.

1. $P_\kappa$ is $<\mu$-distributive and forces $\kappa = \mu^+$;
2. $P_\kappa \subseteq V_\kappa$;
3. $P_\kappa$ is $\kappa$-c.c.;
4. for each inaccessible $\gamma < \kappa$ there is a regular embedding $e_{\gamma, \kappa}: P_\gamma \ast Col(\gamma, < \kappa) \rightarrow P_\kappa$ and
5. whenever $G \ast H$ is generic for $P_\kappa \ast Col(\kappa, < \lambda)$ over $V$, there is a regular embedding $e: Col(\mu, \kappa) \rightarrow P_\lambda/(G \ast H)$.

In the proof of Theorem 3 below we will use a different variation of Kunen’s universal collapse. The fact that the chain condition holds for this variation will follow from a result of Cox.

2In Kunen’s original definition, the Silver collapse is used at such stages $\alpha$ so that certain master conditions exist.
Theorem 8 (Cox, Theorem 39). Suppose $\kappa$ is weakly compact and $\langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) : \alpha \leq \kappa, \beta < \kappa \rangle$ is a “Kunen-style” universal iteration (see Cox, Definition 34). Suppose

1. direct limits are taken at all inaccessible $\gamma \leq \kappa$,
2. for every active $\alpha < \kappa$ we have $\models_{V_\alpha \cap \mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \subseteq V_\kappa[g_\alpha]$ and
3. each $\dot{\mathbb{Q}}_\alpha$ is forced by $V_\alpha \cap \mathbb{P}_\alpha$ to be $\kappa$-c.c.

Then $\mathbb{P}_\kappa \subseteq V_\kappa$ is “layered” on some stationary subset of

$$\Gamma := \{ W \in \mathbb{P}_\kappa(V_\kappa) : W = V_\gamma \text{ for some inaccessible } \gamma < \kappa \}.$$ 

In particular, $\mathbb{P}_\kappa$ is $\kappa$-Knaster.

Generic large cardinal properties have been extensively studied [For10], and have many applications in the form of consistency results at successor cardinals. Suppose $j : V \rightarrow M \subseteq V[G]$ is a generic elementary embedding with critical point $\kappa$, where $G$ is generic over $V$ for a forcing $\mathbb{P}$. One fundamental feature of many applications of generic embeddings is that, in certain situations, the forcing $\mathbb{P}$ which adds the embedding is forcing equivalent to $\mathbb{P}(\kappa)/I$ for a particular naturally defined ideal $I \in V$. Several notions about these kinds of ideals are:

**Definition 9.** If $\kappa$ is a regular cardinal, and $I$ is a $\kappa$-complete ideal on $\kappa$, then we say:

1. $I$ is precipitous if whenever $G \subseteq \mathbb{P}(\kappa)/I$ is generic over $V$, then $V^G / G$ is well-founded.
2. $I$ is saturated if $\mathbb{P}(\kappa)/I$ has the $\kappa^+$-c.c.
3. $I$ is presaturated if forcing with $\mathbb{P}(\kappa)/I$ preserves $\kappa^+$.

**Fact 10** (See [For10]). If $I$ is a $\kappa$-complete presaturated ideal on $\kappa$ and $2^\kappa = \kappa^+$, then

1. $I$ is precipitous.
2. If $G \subseteq \mathbb{P}(\kappa)/I$ is generic $M \cong V^G / G$ is transitive, then $M^G \cap V[G] \subseteq M$.

Foreman showed that many of these applications involving the correspondence between forcings which add generic embeddings and naturally defined ideals in the ground model, can be unified, and viewed as easy consequences of a very general theorem he called the Duality Theorem. Here we state two weak versions of Foreman’s Duality Theorem which we will use in our proofs of Theorem 3 Theorem 4 and Theorem 5

**Theorem 11** (Foreman, [For13]). Suppose $Z$ is a set and $\mathbb{P}$ is a forcing such that whenever $G \subseteq \mathbb{P}$ is generic, there is an ultrafilter $U$ on $Z$ such that $V^Z / U$ is isomorphic to a transitive class $M$. Also assume that there are functions $f_p$, $\langle f_p : p \in \mathbb{P} \rangle$ and $g$ such that $\models_{\mathbb{P}} \langle f_p \rangle_U = \mathbb{P}, \langle g_p : p \in \mathbb{P} \rangle[f_p]_U = p$ and $[g]_U = G$.” If $I = \{ Z \subseteq Z : 1 \notin \mathbb{P}[\text{id}]_U \notin j_U(X) \}$, then there is a dense embedding $d : \mathbb{P}(Z)/I \rightarrow B(\mathbb{P})$.

**Theorem 12** (Foreman, [For13]). Suppose $I$ is a precipitous $\kappa$-complete ideal on $Z$ and $\mathbb{P}$ is a $\kappa$-c.c. partial order. If $\bar{I}$ denotes the ideal generated by $I$ in $V^\mathbb{P}$, then $B(\mathbb{P} \ast \bar{\mathbb{P}}(Z)/\bar{I}) \cong B(\mathbb{P}(Z)/I \ast j(\mathbb{P}))$.

The following result of Baumgartner and Taylor [BTS2] follows immediately from Theorem 12.
Corollary 13. Suppose $\kappa$ is a successor cardinal, $I$ is a $\kappa^+$-saturated ideal on $\kappa$, and $\mathbb{P}$ is a $\kappa$-c.c. forcing. Then the ideal generated by $I$ in $V^\mathbb{P}$ is $\kappa^+$-saturated if and only if $\Vdash_\mathbb{P} j(\mathbb{P})$ is $\kappa^+$-c.c.

3. Rigidity with minimal generics

In this section, we will prove that it is consistent for $\mu^+$, the successor of a regular uncountable cardinal, to carry a rigid saturated $\mu$-minimal ideal. First we note the following obstruction.

Observation 14. If GCH holds and $\mu$ is regular, then there is no $\mu$-minimal pre-saturated ideal on $\mu^+$.

Proof. Suppose $I$ is a presaturated ideal on $\mu^+$, and $j: V \rightarrow M \subseteq V[G]$ is a generic ultrapower embedding derived from $I$. By GCH and the closure of $M$, $([\mu]^{<\kappa})^V = j([\mu]^{<\kappa}) = ([\mu]^{<\kappa})^M = ([\mu]^{<\kappa})^{V[G]}$. In $V[G]$, $|\mathcal{P}(\text{Add}(\mu))^V| = \mu$. Therefore, in $V[G]$, we can recursively choose a sequence $\langle p_\alpha : \alpha < \mu \rangle \subseteq \text{Add}(\mu)$ that generates a $V$-generic filter. If $X \subseteq \mu$ is the subset of $\mu$ coded by this sequence, then $V[X] \neq V[G]$, since $V[X]$ has the same cardinals as $V$. □

This implies that any forcing used to produce an extension with a rigid saturated $\mu$-minimal ideal, must necessarily force GCH to fail. We now prove Theorem 2 by starting with a model of GCH in which there is a saturated ideal on $\mu^+$ where $\mu$ is a regular cardinal, forcing to control a certain cardinal characteristic and then carrying out the relevant arguments of [Woo10] and [Lar02] in this context.

Proof of Theorem 2. Suppose $\mathcal{A} \cup \mathcal{B}$ is an antichain in $\mathcal{P}(\mu)/\{\text{bounded sets}\}$. Following [Kun83], we define a forcing $\mathbb{C}_{\mathcal{A}, \mathcal{B}}$: Conditions are of the form $p = (s, T)$, where $s$ is a bounded subset of $\mu$, and $T$ is a subset of $\mathcal{A}$ of size $< \mu$. We say $(s_1, T_1) \leq (s_0, T_0)$ when $s_0 \subseteq s_1$, $T_0 \subseteq T_1$, and for all $t \in T_0$, $s_1 \cap t = s_0 \cap t$. Clearly, if $G \subseteq \mathbb{C}_{\mathcal{A}, \mathcal{B}}$ is generic, and $s_G = \bigcup \{s : \exists T(s, T) \in G\}$, then $s_G \cap a$ is bounded in $\mu$ for all $a \in \mathcal{A}$.

Lemma 15. If $\mathbb{C}_{\mathcal{A}, \mathcal{B}}$, $G$, and $s_G$ are as above, then $s_G \cap b$ is unbounded in $\mu$ for all $b \in \mathcal{B}$.

Proof. Let $(s, T)$ be any condition, and let $b \in \mathcal{B}$ and $\alpha < \mu$ be arbitrary. Since $|b \cap t| < \mu$ for all $t \in T$, there is $\beta \geq \alpha$ such that $\beta \in b \setminus \bigcup T$. Then $(s \cup \{\beta\}, T) \leq (s, T)$, and this condition forces $\text{sup}(s_G \cap b) \geq \alpha$. □

Lemma 16. If $\mathbb{C}_{\mathcal{A}, \mathcal{B}}$ is as above, then it is $<\mu$-closed and $2^{<\mu}$-centered.

Proof. If $\langle (s_\alpha, T_\alpha) : \alpha < \beta \rangle$ is a descending sequence with $\beta < \mu$, then $(\bigcup_{\alpha} s_\alpha, \bigcup_{\alpha} T_\alpha)$ is the infimum of the sequence. If $(s, T_0)$ and $(s, T_1)$ are two conditions, then $(s, T_0 \cup T_1)$ is their infimum. □

Let $\mathbb{P}_{\kappa^+}$ be a $<\mu$-support forcing iteration $\langle (\mathbb{P}_\alpha, \hat{Q}_\alpha) : \alpha < \kappa^+ \rangle$ satisfying the following properties.

1. For each $\alpha$, $\Vdash_\alpha \hat{Q}_\alpha = \mathbb{C}_{\mathcal{A}, \mathcal{B}}$ for some pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} \cup \mathcal{B}$ is an antichain in $\mathcal{P}(\mu)/\{\text{bounded sets}\}$.
2. For every $\alpha < \kappa^+$ and every $\mathbb{P}_\alpha$-name $\sigma$ for such a pair, there is $\beta \geq \alpha$ such that $\Vdash_\beta \hat{Q}_\beta = C_\sigma$.
3. Every $\mathbb{P}_{\kappa^+}$-name $\tau$ for such a pair, there is $\alpha < \kappa^+$ and a $\mathbb{P}_\alpha$-name $\sigma$ such that $\Vdash_{\kappa^+} \tau = \sigma$. 

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An iteration satisfying (1) and (2) can be defined using a suitable bookkeeping function because inductively we have $\models_\alpha 2^\kappa = \kappa^+$.

Furthermore, (3) is a consequence of the fact that the entire iteration is $\kappa$-c.c.

Any $<\mu$-support iteration of $<\mu$-closed posets is $<\mu$-closed. Therefore, there is a dense set of conditions $p$ such that at all $\alpha < \kappa^+$, there is $s \in V$ with $p \upharpoonright \alpha \models _\alpha p(\alpha) = (s, T)$ for some name $T$. We may assume that we force with this dense suborder.

We show by induction that for all $\alpha < \kappa$, $|P\alpha| = \kappa$, $P\alpha$ is $\kappa$-c.c., and $P\alpha$ preserves GCH. The base case and successor steps are easy. The cardinality claim at limit stages follows from the fact that $\kappa^{<\mu} = \kappa$. To show the chain condition, let $\{p_\beta : \beta < \kappa\} \subseteq P\alpha$ and let $A \in [\kappa]^\kappa$ be such that $\{\supp p_\alpha : \alpha \in A\}$ is a $\Delta$-system and such that the bounded sets of $\mu$ mentioned on the root are all the same. The chain condition and cardinality together imply that GCH is preserved going forward. In the end, $\models_{\kappa^+} 2^\mu = \kappa^+$, but the $\kappa$-c.c. holds of the whole iteration for the same reason as above.

To get the desired consistency result, we use Corollary 13. In our situation, if $j : V \rightarrow M \subseteq V[G]$ is a generic ultrapower embedding derived from a saturated normal ideal $I$ on $\kappa = \mu^+$ then it follows by elementarily that in $M$, the forcing $j(P_{\kappa^+})$ is a $<\mu$-support iteration of $\mu$-centered forcings of length $j(\kappa^+)$. Since $M^\mu \cap V[G] \subseteq M$, this holds in $V[G]$ as well. Thus in $V[G]$, we can carry out the same $\Delta$-system argument to show that $j(P_{\kappa^+})$ has the $\kappa^+$-c.c. It follows from Theorem 13 that $\bar{I}$, the ideal generated by $I$, is saturated $V^{P_{\kappa^+}}$.

Let $H \subseteq P_{\kappa^+}$ be generic. To show $\bar{I}$ is $\mu$-minimal in $V[H]$, suppose $G \subseteq P(\kappa)/\bar{I}$ is generic over $V[H]$, and $x \subseteq \mu$ is in $V[H][G] \setminus V[H]$. Let $\tau$ be a name for $x$. By Proposition 2.12, there is a function $f : \kappa \rightarrow P(\mu)$ in $V[H]$ such that $\models_{\bar{I}} [\check{f}]_{\check{G}} = \tau$. Since $j(f)(\kappa) = x$ and $x \neq f(\alpha)$ for any $\alpha < \kappa$, $f$ is one-to-one on a $\bar{I}$-measure-one set, which we may assume is all of $\kappa$ by adjusting $f$ off this large set. Since $2^{<\mu} = \mu$ in $V[H]$, each $f(\alpha)$ is coded by a branch through the complete binary tree of height $\mu$, and so the range of $f$ corresponds to a collection of almost-disjoint subsets of this tree. Since $V[H]$ is a forcing extension by $P_{\kappa^+}$, it follows that for any $Y \subseteq \kappa$ in $V[H]$, there is $y \subseteq \mu$ such that $y \cap f(\alpha)$ is unbounded in $\mu$ if and only if $\alpha \in Y$. Thus we have $Y \in G$ iff $\kappa \in j(Y)$ iff $y \cap j(f)(\kappa)$ is unbounded in $\mu$. This implies that from $x$ we may recover $G$.

Now we show that $\bar{I}$ is rigid, following [Lar02].

**Lemma 17.** If $f, g$ are one-to-one functions with respective disjoint domains $A, B$ contained in a regular cardinal $\kappa$, then there are nonstationary $A', B'$ such that $f[A \setminus A'] \cap g[B \setminus B'] = \emptyset$.

**Proof.** Let $C = f[A] \cap g[B]$, and let $\pi : f^{-1}[C] \rightarrow g^{-1}[C]$ be a bijection such that $f(\alpha) = g(\pi(\alpha))$ for all $\alpha \in f^{-1}[C]$. Neither $\pi$ nor $\pi^{-1}$ can be regressive on a stationary set. Let $A' = \{\alpha : \pi(\alpha) < \alpha\}$ and $B' = \{\beta : \pi^{-1}(\beta) < \beta\}$. If $x = f(\alpha) = g(\beta)$, then $\beta = \pi(\alpha)$. Either $\beta < \alpha \in A'$, or $\alpha < \beta \in B'$. \qed

If $\bar{I}$ is nonrigid then whenever $G_0 \subseteq P(\kappa)/\bar{I}$ is generic, there is a different generic $G_1$ such that $V[G_0] = V[G_1]$. The corresponding generic ultrapower embeddings $j_0 : V[H] \rightarrow M_0 = V[H][G_0]$ and $j_1 : V[H] \rightarrow M_1 = V[H][G_1]$ are distinct, yet $P(\mu)M_0 = P(\mu)M_1 = P(\mu)V[H][G_0]$. Let $x \in P(\mu)V[H][G_0] \setminus P(\mu)V[H]$. Let $f,g$ be such that $j_0(f)(\kappa) = j_1(g)(\kappa) = x$. Since $G_0 \neq G_1$, we can pick disjoint $A, B \subseteq \kappa$
such that \( \kappa \in j_0(A) \) and \( \kappa \in j_1(B) \). As before, since \( x \notin V[H] \), we may assume \( f \) and \( g \) are one-to-one on \( A \) and \( B \).

Using Lemma 17, we may also assume \( f[A] \cap g[B] = \emptyset \). Since \( H \) is \( \mathbb{P} \)-generic over \( V \), there is \( y \subseteq \mu \) such that \( |y \cap z| = \mu \) for each \( z \in f[A] \) and \( |y \cap z| < \mu \) for each \( z \in g[B] \). By elementarity, \( M_0 \models |y \cap x| = \mu \) and \( M_1 \models |y \cap x| < \mu \). But this is an absolute property between the models, so we have a contradiction. \( \square \)

4. RIGIDITY WITH GCH

Suppose \( \kappa \) is an inaccessible cardinal and \( \mu < \kappa \) is regular. All of the standard posets used to force \( \kappa = \mu^+ \), such as the Levy collapse, Silver collapse, etc., have many nontrivial automorphisms. Hence, if \( \mathbb{P} \) is one of these standard collapse forcings and \( G \) is generic for \( \mathbb{P} \) over \( V \), then in \( V[G] \) there are many distinct \( V \)-generic filters for \( \mathbb{P} \). We will show that there is a forcing \( \mathbb{Q} \) such that if \( G \) is generic for \( \mathbb{Q} \) over \( V \), then \( V[G] \models \kappa = \mu^+ \) and, in \( V[G] \), there is a unique \( V \)-generic filter for \( \mathbb{Q} \). We will use a variation of the coding forcing introduced by Friedman and Magidor [FM09] to add a club which will code the generic for a collapsing forcing, as well as the generic for the coding forcing itself into the stationarity of subsets of \( \kappa \).

Suppose \( \mathbb{P} \) is some \(<\mu\)-closed forcing such that \( \mathbb{P} \) is \( \kappa \)-c.c., \( |\mathbb{P}| = \kappa \) and \( \Vdash \kappa = \mu^+ \).

Fix a bijection \( b : \kappa \to \mathbb{P} \) and let \( G \) be generic for \( \mathbb{P} \) over \( V \). Working in \( V \), let \( W, X, Y \) and \( Z \) be increasing functions from \( \kappa \) to \( \kappa \) such that the ranges of \( W, X, Y \) and \( Z \) are each cofinal in \( \kappa \) and together form a disjoint partition of \( \kappa \). Let \( \langle \eta_\alpha : \alpha < \kappa \rangle \) be an increasing enumeration of the regular cardinals in the interval \([\mu^+, \kappa)\). For each \( \alpha < \kappa \) let \( E_\alpha = \text{cof}(\eta_\alpha)^V \cap (\eta_\alpha, \kappa) \) and let \( \bar{E}_\kappa = \langle E_\alpha : \alpha < \kappa \rangle \).

Notice that in \( V[G] \) each set in the sequence \( \bar{E}_\kappa \) remains stationary since \( \mathbb{P} \) is \( \kappa \)-c.c. Working in \( V[G] \), let \( Q = \text{Code}(G, \bar{E}_\kappa) \) be the set of all closed bounded \( c \subseteq \kappa = (\mu^+)^{V[G]} \) such that for \( i < \kappa \),

- if \( b(i) \in G \) then \( c \cap E_{W(i)} = \emptyset \) and
- if \( b(i) \notin G \) then \( c \cap E_{X(i)} = \emptyset \).

Conditions in \( Q \) are ordered by setting \( d \leq c \) iff:

1. \( d \) is an end extension of \( c \) and
2. for \( i \leq \max(c) \), if \( i \in c \) then \((d \setminus c) \cap E_{Y(i)} = \emptyset \) and if \( i \notin c \) then \((d \setminus c) \cap E_{Z(i)} = \emptyset \).

This defines the coding poset \( \text{Code}(G, \bar{E}_\kappa) \in V[G] \).

**Lemma 18.** The poset \( \text{Code}(G, \bar{E}_\kappa) \) defined above is \(<\mu\)-closed and \(<\kappa\)-distributive in \( V[G] \).

**Proof.** First we show that \( \text{Code}(G, \bar{E}_\kappa) \) is \(<\mu\)-closed in \( V[G] \). Suppose \( \gamma < \mu \) and \( \langle c_i : i < \gamma \rangle \in V[G] \) is a decreasing sequence of conditions in \( \text{Code}(G, \bar{E}_\kappa) \). Let \( \delta = \sup\{\max(c_i) : i < \gamma \} \). Then \( \text{cof}(\delta)^V \leq \gamma \), which implies \( \text{cof}(\delta)^V \leq \gamma < \mu \) and since every element of \( \bigcup_{\alpha < \kappa} E_\alpha \) has cofinality greater than \( \mu \) in \( V \), it follows that \( \bigcup_{i < \gamma} c_i \cup \{\delta\} \in \text{Code}(G, \bar{E}_\kappa) \) is a lower bound of the sequence.

Next we show that \( \text{Code}(G, \bar{E}_\kappa) \) is \(<\kappa\)-distributive in \( V[G] \). Since \( \kappa = (\mu^+)^{V[G]} \), it will suffice to show that \( \text{Code}(G, \bar{E}_\kappa) \) is \( \leq \mu \)-distributive in \( V[G] \). Fix a sequence \( \bar{D} = \langle D_i : i < \mu \rangle \) of open dense subsets of \( \text{Code}(G, \bar{E}_\kappa) \) in \( V[G] \) and a condition \( c \in \text{Code}(G, \bar{E}_\kappa) \). Let \( S_\mu^c = (\text{cof}(\mu) \cap \kappa)^V \) and notice that \( S_\mu^c \) does not appear on
the sequence $\bar{E}_0 = \langle E_\alpha : \alpha < \kappa \rangle$. Since $P$ is $\kappa$-c.c., it follows that $S^\kappa_\mu$ is a stationary subset of $\kappa$ in $V[G]$. Thus, working in $V[G]$ we may fix some large regular cardinal $\theta$ and a well-ordering $<_\theta$ of $H_\theta$ and an elementary submodel $N \prec (H_\theta, \in, <_\theta)$ such that

- $c, \text{Code}(G, \bar{E}_\kappa), S^\kappa_\mu, \bar{D} \in N$
- $|N|^{V[G]} = \mu$
- $N \cap \kappa \in S^\kappa_\mu$
- $N^{<\mu} \cap V[G] \subseteq N$

Working in $V[G]$, let $\langle \beta_i : i < \mu \rangle$ be an increasing, continuous and cofinal sequence of ordinals in $\delta = N \cap \kappa$. Using the well-ordering $<_\theta$ and elementarity, we may build a decreasing sequence of conditions $\langle c_i : i < \mu \rangle$ such that $c_0 = c$ and for each $i < \mu$ we have (1) $c_{i+1} \leq c_i$, (2) $c_{i+1} \in D_i$, (3) $\beta_i \leq \max(c_i)$ and (4) $c_i \in N$. At limit stages we make use of the fact that $\text{Code}(G, \bar{E}_\kappa)$ is $<\mu$-closed in $V[G]$ and $N^{<\mu} \cap V[G] \subseteq N$. Since the ordinal $\delta = N \cap (\mu^+)^{V[G]} = \sup\{\max(c_i) : i < \mu\}$ has cofinality $\mu$ in $V$, it follows that $\delta \notin \bigcup_{\alpha < \kappa} E_\alpha$, and thus $c_\infty = \bigcup_{i < \mu} c_i \cup \{\delta\} \in \text{Code}(G, \bar{E}_\kappa)$ is a lower bound of the sequence $\langle c_i : i < \mu \rangle$. □

**Lemma 19.** Suppose $\kappa$ is an inaccessible cardinal and $\mu < \kappa$ is regular with $\mu^{<\mu} = \mu$. Let $P$ be a forcing notion such that $b : \kappa \to P$ is a bijection and $\Vdash_P \kappa = \mu^+$. Suppose $G \ast H \subseteq P \ast \text{Code}(G, \bar{E}_\kappa)$ is generic over $V$ and let $C = \bigcup H$. Then in $V[G \ast H]$, we have

1. For $i < \kappa$, $b(i) \in G$ iff $E_{W(i)}$ is nonstationary and $b(i) \notin G$ iff $E_{X(i)}$ is nonstationary.
2. For $i < \kappa$, $i \in C$ iff $E_{Y(i)}$ is nonstationary and $i \notin C$ iff $E_{Z(i)}$ is nonstationary.
3. There is a unique $V$-generic filter for $P \ast \text{Code}(G, \bar{E}_\kappa)$ (in $V[G \ast H]$).

**Proof.** The proof is similar to [FM09] Lemma 8.

1. It follows from the definition of extension in $\text{Code}(G, \bar{E}_\kappa)$ that if $b(i) \in G$ then $E_{W(i)}$ is nonstationary. Conversely, we will prove that if $b(i) \notin G$ then $E_{W(i)}$ remains stationary in $V[G \ast H]$. Suppose $b(i) \notin G$ and that $c \Vdash \bar{D} \subseteq \kappa$ is club. It will suffice to find an extension $c_\infty \leq c$ with $c_\infty \Vdash \bar{D} \cap E_{W(i)} \neq \emptyset$. Working in $V[G]$, since $E_{W(i)}$ is a stationary subset of $\kappa = (\mu^+)^{V[G]}$, it follows that for some large enough regular cardinal $\theta$, we may let $<_\theta$ be a well-order of $H_\theta$ and find $N \prec (H_\theta, \in, <_\theta)$ such that

- $c, \text{Code}(G, \bar{E}_\kappa), E_{W(i)}, \bar{D} \in N$
- $|N|^{V[G]} = \mu$
- $N \cap \kappa \in E_{W(i)}$
- $N^{<\mu} \cap V[G] \subseteq N$

We have $E_{W(i)} \subseteq \kappa \cap \text{cof}(\mu)^{V[G]}$ and thus, working in $V[G]$, we may fix a sequence of ordinals $\langle \beta_i : i < \mu \rangle$ which is increasing, continuous and cofinal in $\delta = N \cap \kappa$. Using the well-ordering $<_\theta$ and elementarity, we recursively construct a decreasing sequence of conditions $\langle c_i : i < \mu \rangle$ and a sequence of ordinals $\langle \eta_i : i < \mu \rangle$ such that $c_0 = c$ and for each $i < \mu$ we have (1) $c_{i+1} \Vdash \eta_{i+1} \in \bar{D}$, (2) $\beta_i \leq \max(c_i, \eta_{i+1})$, (3) $\eta_i \leq \eta_{i+1}$ and (4) $c_i \in N$. At limit stages we make use of the facts that $\text{Code}(G, \bar{E}_\kappa)$ is $<\mu$-closed in $V[G]$ and $N^{<\mu} \cap V[G] \subseteq N$. Thus $\delta = N \cap \kappa = \sup\{\max(c_i) : i < \mu\} = \sup\{\eta_i : i < \mu\}$. Let $c_\infty = \bigcup\{c_i : i < \omega\} \cup \{\delta\}$. Since $\delta \in E_{W(i)}$ and $f(i) \notin G$, it
follows that $c_\infty$ is a condition in $\text{Code}(G, \vec{E}_\kappa)$ and that $c_\infty$ extends each $c_i$. Hence $c_\infty \Vdash \delta \in D \cap E_{W(i)}$. This completes the proof of (1).

(2) is similar to (1).

For (3), suppose that in $V[G*H]$ there is a $V$-generic filter for $\mathbb{P} * \text{Code}(G, \vec{E}_\kappa)$, call it $G^*H$. Then $V \subseteq V[G^*H] \subseteq V[G*H]$. Suppose $G^* \neq G$. Without loss of generality, suppose that for some $i < \kappa$ we have $f(i) \in G^* \setminus G$. Then by (1), it follows that $E_{W(i)}$ is nonstationary in $V[G^*H]$, but becomes stationary in $V[G*H]$, which is impossible. The rest of the cases for (3) are similar. \hfill \square

With the above technique of coding a generic for a collapse forcing, we are ready to prove Theorem 4 that is, we will show that if $\kappa$ is an almost-huge cardinal and $\mu < \kappa$ is regular, then there is a forcing extension in which there is a rigid presaturated ideal on $\mu^+$ and GCH holds.

**Proof of Theorem 4.** Suppose $j : V \to M$ is an elementary embedding with critical point $\kappa$ such that $\lambda = j(\kappa)$, $M^{<\lambda} \subseteq M$ and $j$ is the ultrapower by an almost-huge tower (see [CZ14]). Without loss of generality, assume GCH holds. Let $\mu < \kappa$ be a regular cardinal and let $\mathbb{P} = \mathbb{P}_\kappa$ be Magidor’s variation of Kunen’s universal collapse for forcing $\kappa = \mu^+$, as given in Definition 3 above. Let $\vec{Q} = \text{Col}(\kappa, < \lambda)$ be a $\mathbb{P}$-name for the Levy-collapse below $\lambda$. Assume $G*H$ is generic for $\mathbb{P} * \vec{Q}$ over $V$.

Working in $V$, let $\vec{E}_\kappa$ be the sequence of stationary subsets of $\kappa$ in the definition of the coding forcing above. It follows that each set in the sequence $\vec{E}_\kappa$ remains stationary in $V[G*H]$ and the poset $\text{Code}(G, \vec{E}_\kappa)$ is the same whether defined in $V[G]$ or $V[G*H]$. We will prove that if $K$ is generic for $\text{Code}(G, \vec{E}_\kappa)$ over $V[G*H]$, then in $V[G*H]$ there is a rigid presaturated ideal on $\mu^+$.

First we argue that the embedding $j$ can be generically extended to have domain $V[G*(H \times K)]$. Since the poset $\text{Code}(G, \vec{E}_\kappa)$ is $<\mu$-closed and has size $\kappa$ there is a regular embedding $\text{Code}(G, \vec{E}_\kappa) \to \text{Col}(\mu, \kappa)$ and by property 5 of the universal collapse $\mathbb{P}$ listed above, there is a regular embedding $\text{Col}(\mu, \kappa) \to j(\mathbb{P})/G*H$. Thus we may let $\vec{G}$ be generic for the quotient $j(\mathbb{P})/(G*H \times K)$ and extend the elementary embedding to $j : V[G] \to M[\vec{G}]$. By elementarity, $j(\mathbb{P})$ is $\lambda$-c.c. and thus $M[\vec{G}]^{<\lambda} \cap V[G] \subseteq M[G]$.

Next we must lift $j$ to have domain $V[G*H]$. In $M[\vec{G}]$, for each $\alpha < \lambda$, $j[H \upharpoonright \alpha]$ is a directed subset of $\text{Col}(\lambda, < j(\lambda))$ of size $< \lambda$, thus $m_\alpha = \inf j[H \upharpoonright \alpha]$ is a condition in $\text{Col}(\lambda, < j(\lambda))$. In $V[\vec{G}]$, let $\{D_\alpha : \alpha < \lambda\}$ enumerate the dense open subsets of $\text{Col}(\lambda, < j(\lambda))$ that live in $M[\vec{G}]$. We inductively build a chain $\{q_\alpha : \alpha < \lambda\}$ such that

$$\hat{H} = \{ q \in \text{Col}(\lambda, j(\lambda)) : (\exists \alpha < \lambda) q_\alpha \leq q \}$$

is an $M[\vec{G}]$-generic filter and $m_\alpha \in \hat{H}$ for all $\alpha < \lambda$. Assume that we have constructed a sequence of conditions $\{q_i : i < \alpha\}$ such that for each $i < \alpha$, $q_i \leq m_i$, $q_i$ is compatible with all $m_\beta$ and $q_i \in D_i$. Then $q_\alpha'$ is compatible with all $m_\beta$. Let $\gamma < j(\lambda)$ be such that $D_\alpha \upharpoonright \gamma = \text{def} D_\alpha \cap \text{Col}(\lambda, < \gamma)$ is dense in $\text{Col}(\lambda, < \gamma)$ and $q_\alpha' \in D_\alpha \upharpoonright \gamma$. Choose $q_\alpha \in D_\alpha \upharpoontright \gamma$ below $q_\alpha' \wedge m_\gamma$. Then $q_\alpha$ is compatible with all $m_\beta$ since if $\beta \geq \gamma$, $m_\beta \upharpoontright \gamma = m_\gamma$. This completes the construction of $H$. Since $j[H] \subseteq \hat{H}$ the embedding extends to $j : V[G*H] \to M[\vec{G} * \hat{H}]$.

Now let $c^* = \bigcup j[K \cup \{\kappa\}]$. We will check that $c^*$ is a condition in $j(\text{Code}(G, \vec{E}_\kappa)) = \text{Code}(j(G), j(\vec{E}_\kappa))^{M[\vec{G}]} = \text{Code}(j(G), j(\vec{E}_\kappa))^V[\vec{G}]$ extending every element of $j[K] = K$. It suffices to show that $\kappa$ is not in any of the stationary sets on the sequence
j(\bar{E}_\alpha) = (\bar{E}_\alpha : \alpha < j(\kappa)) where \bar{E}_\alpha = (\text{cof}(\eta_\alpha) \cap [\eta_\alpha^+, j(\kappa)])^V. If \alpha \neq \kappa then clearly \kappa \notin \bar{E}_\alpha. If \alpha = \kappa then \kappa \notin \bar{E}_\alpha = \bar{E}_\kappa = (\text{cof}(\kappa) \cap [\kappa^+, j(\kappa)])^V. Thus c^* is a master condition. Let \bar{K} be generic for j(\text{Code}(G, \bar{E}_\kappa)) over M[\bar{G} \ast \bar{H}] with c^* \in \bar{K}. Then j lifts to j : V[G \ast (H \times K)] \rightarrow M[\bar{G} \ast (\bar{H} \times \bar{K})].

Let U be the ultrafilter on \mathcal{P}(\kappa)^V[G^*(H \times K)] induced by this extended embedding:

\[\{X \in \mathcal{P}(\kappa)^V[G^*(H \times K)] : \kappa \in j(X)\}.\]

Consider the natural commutative diagram

\[
\begin{array}{ccc}
V[G \ast (H \times K)] & \xrightarrow{j} & M[\bar{G} \ast (\bar{H} \times \bar{K})] \\
\downarrow{j_U} & & \downarrow{k} \\
V[G \ast (H \times K)]^\kappa / U
\end{array}
\]

where j_U is the ultrapower embedding and k([f]_U) = j(f)(\kappa). Since k is an elementary embedding, it follows that the ultrapower V[G \ast (H \times K)]^\kappa / U is well-founded and can thus be identified with its transitive collapse N. Note that crit k > \kappa.

Let us now show that N = M[\bar{G} \ast (\bar{H} \times \bar{K})]. Recall that the original embedding j : V \rightarrow M is the ultrapower by an almost-largie tower, and thus M is the direct limit of a directed system of \alpha-supercompactness embeddings j_\alpha : V \rightarrow M_\alpha for \alpha < \lambda. Every member of M_\alpha is of the form j_\alpha(f)(j_\alpha[\alpha]) for some function f \in V with dom(f) = [\alpha]^{<\kappa}. If k_\alpha : M_\alpha \rightarrow M is the factor map such that j = k_\alpha \circ j_\alpha, then the critical point of k_\alpha is above \alpha, so k_\alpha(x) = k_\alpha(x) whenever M_\alpha \models |x| \leq \alpha. Since M is the direct limit of this system of supercompactness embeddings it follows that for all x \in M there is some \alpha < \lambda and some f \in V such that

\[x = k_\alpha([f]) = k_\alpha(j_\alpha(f)(j_\alpha[\alpha])) = j(f)(k_\alpha(j_\alpha[\alpha])) = j(f)(j[\alpha]).\]

Let \beta be any ordinal. There is some \alpha with \kappa \leq \alpha < \lambda and some f \in V such that \beta = j(f)(j[\alpha]). Let \lambda' : \kappa \rightarrow \alpha be a bijection in V[G \ast (H \times K)]. Then \beta = j(f)(j[\lambda'])[\kappa] = k(j_U(f)(j_U[\lambda'])[\kappa])). Thus \beta \in \text{range}(k). This implies that k does not have a critical point and thus N = M[\bar{G} \ast (\bar{H} \times \bar{K})].

The forcing used to extend the embedding to have domain V[G \ast (H \times K)] was \mathbb{R} =\text{det} j(\mathcal{P})/\mathcal{P} \ast (\text{Code}(j(G), j(\bar{E}_\kappa))/c^*). We will use Foreman’s Duality Theorem (Theorem 14 above) to show that \mathbb{R} is forcing-equivalent to forcing with \mathcal{P}(\kappa)/I where I \in V[G \ast (H \times K)] is defined as

\[I = \{X \in \mathcal{P}(\kappa)^V[G^*(H \times K)] : 1 \Vdash_{\mathbb{R}} \text{id}_U \notin j(X)\}\]

where \text{id}_U is an \mathbb{R}-name for the ultrafilter on \kappa derived from the generic embedding. We need to verify that the ultrapower N satisfies the hypotheses of Foreman’s theorem.

Let us show that there is a regular embedding \epsilon : \mathbb{P} \ast (\text{Col}(\kappa, <\lambda) \times \text{Code}(\bar{G}, \bar{E})) \rightarrow j(\mathcal{P}) in the ultrapower N of the form j((\epsilon_\alpha : \alpha < \kappa)(\kappa)), where \{\epsilon_\alpha : \alpha < \kappa\} \in V[G \ast (H \times K)] is a sequence of regular embeddings. Fix an increasing sequence \langle \kappa_\alpha : \alpha < \kappa \rangle of inaccessible cardinals which is cofinal in \kappa. By the absorption properties of the universal collapse \mathbb{P} = \mathbb{P}_\kappa, there exist regular embeddings \epsilon_\alpha : \mathbb{P}_{\kappa_\alpha} \ast (\text{Col}(\kappa_\alpha, <\kappa) \times \text{Code}(\bar{G}_{\kappa_\alpha}, \bar{E}_{\kappa_\alpha})) \rightarrow \mathbb{P}_{\kappa_\alpha} where the forcing \text{Col}(\kappa_\alpha, <\kappa) \times \text{Code}(\bar{G}_{\kappa_\alpha}, \bar{E}_{\kappa_\alpha}) is defined in \mathbb{V}_{\mathbb{P}_{\kappa_\alpha}} just as \text{Col}(\kappa, <\lambda) \times \text{Code}(\bar{G}, \bar{E}) was defined in \mathbb{V}_{\mathbb{P}_{\kappa}}. It follows that \epsilon is represented in the ultrapower N as j((\epsilon_\alpha : \alpha < \kappa)(\kappa)). Thus N computes the quotient algebra from \bar{G} = j(G) and \epsilon, and this is represented by a function with domain \kappa. For each p \in j(\mathcal{P}), there is an ordinal
\( \alpha < \lambda \) and a function \( f_p \in V \) such that \( p = j(f_p)(j[\alpha]) = j_\alpha(f_p)(j_\alpha[\alpha]). \) Using bijections \( b_0 : \kappa \to (\alpha < \kappa)^V \) and \( b_1 : \kappa \to \alpha \) in \( V[G \ast (H \times K)] \), we can build a function \( f'_p : \kappa \to P \) that represents \( p \) in the ultrapower \( N \). Thus, the hypotheses of Foreman’s Duality Theorem are met.

By Lemma 19 there is in \( V[\hat{G}*(\check{H} \times \check{K})] \) a unique generic for \( P_\lambda * \text{Code}(j(G), j(\check{E}_\kappa)). \) Since there is a dense embedding \( P(\kappa)/I \to j(P)/(G*(H \times K)) * \text{Code}(j(G), j(\check{E}_\kappa))/c^*, \) \( P(\kappa)/I \) is rigid. Since \( P_\lambda * \text{Code}(j(G), j(\check{E}_\kappa)) \) preserves \( \kappa^+ \), it follows that \( I \) is presaturated. Thus, in \( V[G*(H \times K)] \) there is a rigid presaturated ideal on \( \kappa = \mu^+ \) and GCH holds. This completes the proof of Theorem 4.

In order to prove Theorem 3 we will need the following notion found in [AS83].

**Definition 20.** A \( S \subseteq \kappa \) is called fat stationary if \( S \) is stationary and for every club \( C \subseteq \kappa \) and every \( \alpha < \kappa \), there is a closed \( c \subseteq C \cap S \) of order-type \( \alpha \).

Harvey Friedman [Fri74] showed that every stationary subset of \( \omega_1 \) is fat. Obviously this cannot be true of larger \( \kappa \) because of the existence of more than one cofinality below \( \kappa \). We now introduce a related notion.

**Definition 21.** A sequence \( \langle a_\alpha : \alpha < \kappa \rangle \) is called a fat diamond sequence (\( \spadesuit_{\kappa} \)-sequence) if for every \( X \subseteq \kappa \), \( \{ \alpha : X \cap \alpha = a_\alpha \} \) is fat stationary.

**Lemma 22.** If \( \kappa \) is a regular uncountable cardinal, then the forcing \( \text{Add}(\kappa) \) to add a Cohen subset to \( \kappa \) introduces a \( \spadesuit_{\kappa} \)-sequence.

**Proof.** Let \( g : \kappa \to 2 \) be a generic Cohen function on \( \kappa \). For each \( \alpha < \kappa \), let \( a_\alpha = \beta < \alpha : g(\alpha + \beta) = 1 \). Let \( X \) be a name for a subset of \( \kappa \), \( \dot{C} \) a name for a club, \( \xi < \kappa \) an ordinal, and \( q_0 \) a condition. Construct a descending sequence of conditions \( \langle p_\alpha : \alpha < \kappa \rangle \) below \( q_0 \) and an increasing sequence of ordinals \( \langle \gamma_\alpha : \alpha < \kappa \rangle \) with the following properties:

1. The domain of each condition is an ordinal.
2. \( p_{\alpha + 1} \) decides \( X \cap \text{dom} p_\alpha \).
3. \( p_{\alpha + 1} \models \text{dom} p_\alpha < \gamma_{\alpha + 1} \in \dot{C} \).
4. If \( \alpha \) is a limit, then \( \gamma_\alpha = \sup_{\beta < \alpha} \gamma_\beta \).
5. If \( \alpha \) is a limit, then letting \( q_\alpha = \bigcup_{\beta < \alpha} p_\beta \), we have \( \text{dom} p_\alpha = \alpha \cdot 2 \), \( p_\alpha \models \alpha = q_\alpha \), and \( p_\alpha(\alpha + \beta) = 1 \) iff \( q_\alpha \models \beta \in X \).

Let \( \{ \delta_\beta : \beta < \xi \} \) be the first \( \xi \) limit ordinals. Then \( p_{\delta_\xi} \models \{ \gamma_{\delta_\beta} : \beta < \xi \} \subseteq \dot{C} \wedge (\forall \beta < \xi) \hat{X} \cap \gamma_{\delta_\beta} = \delta_{\delta_\beta} \).

**Lemma 23.** \( \spadesuit_{\kappa} \)-sequences are preserved by \( \kappa \)-closed forcing.

**Proof.** Let \( \{ a_\alpha : \alpha < \kappa \} \) be a \( \spadesuit \)-sequence. Let \( P \) be a \( \kappa \)-closed forcing, let \( p_0 \in P \) be any condition, let \( \dot{C} \) be a name for a club, and \( X \) a name for a subset of \( \kappa \). Construct a descending sequence of conditions \( \langle p_\alpha : \alpha < \kappa \rangle \) below \( p_0 \) and an increasing sequence of ordinals \( \langle \gamma_\alpha : \alpha < \kappa \rangle \) with the following properties:

1. \( p_{\alpha + 1} \) decides \( X \cap \gamma_\alpha \).
2. \( p_{\alpha + 1} \models \gamma_\alpha \in \dot{C} \).
3. If \( \alpha \) is a limit, then \( \gamma_\alpha = \sup_{\beta < \alpha} \gamma_\beta \).

Let \( Y = \{ \alpha : (\exists \beta p_\beta \models \alpha \in X \} \), and let \( D = \{ \gamma_\alpha : \alpha < \kappa \} \). For any \( \xi < \kappa \), there is a closed set \( c \) of order-type \( \xi \) contained in \( \{ \alpha \in D : Y \cap \alpha = a_\alpha \} \). Therefore, \( p_\xi \models c \subseteq \{ \alpha \in \dot{C} : X \cap \alpha = a_\alpha \} \).
We are now ready to prove the following special case of Theorem 3 in which \( \mu \) is not the successor of a singular cardinal.

**Theorem 24.** Suppose that \( \kappa \) is an almost-huge cardinal and \( \mu < \kappa \) is an uncountable regular cardinal which is not the successor of a singular cardinal. Then there is a \(<\mu\)-distributive forcing extension in which there is a rigid saturated ideal on \( \mu^+ \) and GCH holds.

**Proof.** Suppose \( \kappa \) is an almost-huge with target \( \lambda \), and let \( \mu < \kappa \) be regular uncountable. We must show that there is a forcing extension in which GCH holds and there is a rigid saturated ideal on \( \mu^+ \). Without loss of generality we can assume that GCH holds and that there is a \( \diamond \)-sequence \( \langle a_\alpha : \alpha < \mu \rangle \) in \( V \).

We use the following variation of Kunen’s collapse. Let \( P_\kappa \) be an iteration \( \langle (P_\alpha, Q_\alpha) : \alpha \leq \kappa, \beta < \kappa \rangle \) with \(<\mu\)-support defined as follows.

1. \( P_0 = \Col(\mu, <\kappa) \)
2. If \( (P_\alpha \cap V_\alpha) \triangleleft P_\alpha \) and is \( \alpha \)-c.c., then \( Q_\alpha \) is a \( (P_\alpha \cap V_\alpha) \)-name for \( \Col(\alpha, <\kappa) \times \Col(\mu, \alpha) \).

By Theorem 8 \( P = P_\kappa \) is \( \kappa \)-c.c. We have \( j(P) = P(j(\kappa)) \), and \( j(P) \cap V_\kappa = P \).

By construction, \( P \ast \Col(\kappa, < j(\kappa)) \triangleleft j(P) \). By same arguments as for Theorem 4 if \( G \ast H \subseteq P \ast \Col(\kappa, < j(\kappa)) \), then in \( V[G \ast H] \), \( \kappa = \mu^+ \), and there is a \( \kappa^+ \)-saturated ideal on \( \kappa \) with quotient isomorphic to \( B(j(P)/(G \ast H)) \). As in the proof of Theorem 4 a generic embedding for this ideal will always extend the ground model almost-huge embedding.

Now we define the forcing which turns the ideal into a rigid one. Let \( f : \kappa \to V_\kappa \) be a bijection in \( V \) such that for all inaccessible \( \alpha < \kappa \), \( f \restriction \alpha \) is a bijection with \( V_\alpha \). Let \( A \subseteq \kappa \) be the set of active stages of \( P \). At each \( \alpha \in A \), \( G \) gives a generic surjection \( g_\alpha : \mu \to \alpha \). Using a canonical coding and \( f \restriction \alpha \), let \( X_\alpha \in \P(\mu)^{V[G]} \) code the pair \( (g_\alpha, G \cap V_\alpha) \).

The key feature of these sets is that if we have another \( P \)-generic \( G' \in V[G] \), and \( \alpha \) is some ordinal such that \( (g'_\alpha, G' \cap V_\alpha) \neq (g_\alpha, G \cap V_\alpha) \), then \( X'_\alpha \neq X_\beta \) for any \( \beta < \kappa \). Let \( b : \kappa \times \mu \to A \) be a bijection, and for each \( (\alpha, \beta) \in \kappa \times \mu \), let \( S^{\beta}_\alpha = \{ \gamma < \mu : X^{b(\alpha, \beta)}_\gamma \cap \gamma = \alpha \} \). Each \( S^{\beta}_\alpha \) is fat, and if \( (\alpha, \beta) \neq (\alpha', \beta') \), then \( |S^{\beta}_\alpha \setminus S^{\beta'}_{\alpha'}| < \mu \). If necessary, slightly adjust the sets so that \( S^{\beta}_\alpha \setminus S^{\gamma}_{\alpha} = \emptyset \) for \( \beta \neq \gamma \).

Let \( C \) be the collection of partial functions \( p : \kappa \to \P(\mu) \) with the following properties:

1. \( |p| < \mu \).
2. For all \( \alpha < \kappa \), \( p(\alpha) \) is a closed bounded subset of \( \mu \setminus 1 \).
3. For all \( \alpha < \kappa \), if \( f(\alpha) \in G \), then \( p(\alpha) \cap S^{\beta}_{\alpha} = \emptyset \).
4. For all \( \alpha < \kappa \), if \( f(\alpha) \notin G \), then \( p(\alpha) \cap S^{1}_{\alpha} = \emptyset \).

We let \( q \leq p \) when:

1. \( \dom p \subseteq \dom q \).
2. For all \( \alpha \in \dom p \), \( q(\alpha) \cap (\max(p(\alpha)) + 1) = p(\alpha) \).
3. If \( \beta \in p(\alpha) \), then \( (q(\alpha) \setminus p(\alpha)) \cap S^{\beta}_{\alpha} = \emptyset \).
4. If \( \beta \in [1, \max(p(\alpha))] \setminus p(\alpha) \), then \( (q(\alpha) \setminus p(\alpha)) \cap S^{\beta+1}_{\alpha} = \emptyset \).

A standard \( \Delta \)-system argument establishes the following.

**Lemma 25.** \( C \) is \( \kappa \)-c.c.

By Theorem 12 if \( I \) denotes the ideal generated by \( I \) in \( V[G \ast H]^C \), then \( B(C \ast \P(\kappa)/I) \cong B(P(\kappa)/I \ast j(C)) \). Since we can carry out the \( \Delta \)-system argument in
Let \( \nu \) name for a function from \( p \) to the ordinals, \( \bar{\nu} \) the unique generics over \( V \). This implies \( \bar{I} \) is forced to be rigid, because

\[
\begin{align*}
\hskip-1cm j(P \ast \dot{C}) & \sim j(P) \ast j(\dot{C}) \\
& \sim (P \ast \text{Col}(\kappa, <\lambda) \ast \frac{j(P)}{G \ast H}) \ast j(\dot{C}) \\
& \sim P \ast \text{Col}(\kappa, <\lambda) \ast \mathcal{P}(\kappa)/I \ast j(\dot{C}) \\
& \sim P \ast \text{Col}(\kappa, <\lambda) \ast \mathcal{C} \ast \mathcal{P}(\kappa)/\bar{I}.
\end{align*}
\]

Note that we have GCH in \( V[G \ast H] \), and standard name-counting arguments show that this is preserved by \( \mathcal{C} \). For the moment we will assume \( \mu \) is not the successor of a singular cardinal; that case requires a bit more care and will be dealt with afterwards.

**Lemma 26.** Let \( \mu \) be as above and assume for all cardinals \( \alpha < \beta < \mu, \beta^\alpha < \mu \).

Let \( K \subseteq \mathcal{C} \) be generic over \( V[G \ast H] \).

1. For all \( \alpha, C_\alpha = \bigcup_{p \in K} p(\alpha) \) is club in \( \mu \).
2. \( \mathcal{C} \) is \( <\mu \)-distributive.
3. If \( S \subseteq \mu \) is fat stationary and almost-disjoint with every \( S^\beta_\alpha \), then \( S \) remains stationary in \( V[G \ast H \ast K] \).
4. If \( S \subseteq \mu \) is fat stationary and almost-disjoint with every \( S^\beta_\alpha \), then \( S \) remains stationary in \( V[G \ast H \ast K] \).
5. In \( \mathcal{V}[G[H \ast K]] \), \( \beta \in C_\alpha \setminus 1 \Rightarrow S^{2,\beta+1}_\alpha \) is stationary.

**Proof.**

For (1), let \( p \in \mathcal{C}, \alpha < \kappa, \) and \( \beta < \mu \) be arbitrary. Let \( \gamma < \mu \) be greater than \( 2 \cdot \delta + 1 \) for all \( \delta \in p(\alpha) \). \( \mu \setminus \bigcup_{\xi < \gamma} S^\beta_\alpha \) is unbounded in \( \mu \), so let \( \delta > \beta \) be in this set. Then \( (p \setminus (\alpha, p(\alpha))) \cup \{(\alpha, p(\alpha) \cup (\theta))\} \) is a condition stronger than \( p \). This shows that \( C_\alpha \) is forced to be unbounded. It is closed because all initial segments are closed.

We will show (2)–(4) with one construction. Let \( \nu < \mu \) be a regular cardinal, \( \dot{\gamma} \) a name for a function from \( \nu \) to the ordinals, \( \dot{\mathcal{D}} \) a name for a club in \( \mu, p_0 \in \mathcal{C}, S \) any fat set which is almost-disjoint from all \( S^\beta_\alpha \), and \( \xi < \kappa \). Let \( \theta \) be a sufficiently large regular cardinal, and let \( (M_\alpha : \alpha < \mu) \) be an increasing sequence of elementary submodels of \( H_\theta \), each of size \( < \mu \) and having transitive intersection with \( \mu \), with \( \{\nu, \dot{\gamma}, \dot{\mathcal{D}}, p_0, \mathcal{C}, \xi\} \in M_0 \), and such that if \( \alpha \) is a successor, then \( M^{<\nu}_\alpha \subseteq M_\alpha \).

Let \( D \subseteq J \) be the club set of \( \alpha \) where \( M_\alpha \cap \mu = \alpha \). To show (3) and (4) we use the same argument. Either let \( T = S \) or \( S^\alpha_\xi \), where we let \( n = 1 \) if \( f(\xi) \in G \) and \( n = 0 \) if \( f(\xi) \notin G \). Let \( (\alpha_i : i \leq \nu + \omega + 1) \) be a continuous increasing sequence contained in \( D \cap T \). Choose a descending chain of conditions below \( p_0 \) as follows: Given \( p_i \in M_{\alpha_i+1}, \) let \( p_{i+1} \) be such that \( \alpha_i \leq \max p_{i+1}(\gamma) < \alpha_{i+1} \) for all \( \gamma \in \text{dom}(p_{i+1}) \). This is possible by the argument for (1) and elementarity. Also, let \( p_{i+1} \) decide \( \dot{\gamma}(i) \) and some \( \beta_{i+1} \in \dot{\mathcal{D}} \) such that \( \alpha_i < \beta_i \). If \( i \) is a limit, let \( p_i(\gamma) = \bigcup_{j<i} p_j(\gamma) \cup \{\alpha_i\} \) for each \( \gamma \in \bigcup_{j<i} \text{dom} p_j \). This is a condition because \( M_{\alpha_i} \), knows \( T \) is almost-disjoint with all (other) \( S^\beta_\alpha \), so their intersections are bounded below \( \alpha_i \). It forces that \( \beta_i = \sup_{j<i} \beta_j \in \dot{\mathcal{D}} \). The construction continues because for each limit \( i < \nu, (p_j : j < i) \in M_{\alpha_i+1} \). In the end, \( p_\mu \) is a condition deciding \( \dot{\gamma} \) and forcing \( T \cap \dot{\mathcal{D}} \neq \emptyset \).

To show (5), let \( 0 < \beta < \mu, \xi < \kappa, q_0 \in \mathcal{C}, \) and let \( \dot{\mathcal{C}} \) be a name for a club. Take \( q_1 \subseteq q_0 \) such that \( \max(q_1(\xi)) > \beta \). Let \( \gamma = 2 \cdot \beta + 1 \) if \( \beta \in q_1(\xi) \), and \( \delta = 2 \cdot \beta \).
if \( \beta \notin q(\xi) \). Construct a sequence of models as before, and take the analogous club \( D \). Let \((\alpha_i : i \leq \omega)\) be a continuous increasing sequence contained in \( D \cap S^\delta_\xi \).

Choose a descending chain of conditions below \( q_1 \) such that \( \alpha_\omega \in q_\omega(\gamma) \) for all \( \gamma \in \text{dom} \, q_\omega \), and \( q_\omega \models \alpha_\omega \in \dot{C} \cap S^\delta_\xi \).

**Lemma 27.** If \( G * K \) is \( \mathbb{P} \ast \dot{C} \)-generic, then there is no other \( \mathbb{P} \ast \dot{C} \)-generic filter in \( V[G][K] \).

The argument for the above lemma is the same as for Lemma 13 part (3). This concludes the proof of Theorem 24. \( \square \)

Finally, we sketch the proof of Theorem 5 which closely follows the above arguments. Let \( \kappa \) be measurable with normal measure \( U \), and let \( \mu < \kappa \) be regular. Assume \( \dot{\kappa} \) holds. Force with \( \text{Col}(\mu, <\kappa) \), and let \( I \) be the ideal generated by the dual of \( U \). By the Duality Theorem, \( \mathcal{P}(\kappa)/I \) is forcing-equivalent to \( \text{Col}(\mu, <\iota(\kappa)) \).

Therefore, forcing with it preserves \( \dot{\kappa} \)-sequences. We then force over this model with \( \dot{C} \), where we code the generic for \( \text{Col}(\mu, <\kappa) \). The generated ideal will be rigid, and the key reason is the following: If \( G \subseteq \mathcal{P}(\kappa)/I \) is generic and \( j : V \rightarrow M \subseteq V[G] \) is the generic ultrapower embedding, then \( j(\dot{C}) \) is the same whether defined in \( M \) or \( V[G] \), since it uses the same \( \dot{\kappa} \)-sequence. Thus the stationarity of the relevant sets is absolute between \( M \) and \( V[G] \), even though \( M \) is not \( \mu \)-closed.

5. Near singular cardinals

In this section, we show how to modify the previous arguments to obtain \( \text{GCH} \) along with rigid saturated ideals on double successors of singulars, which is the remaining case of Theorem 5. The key issue to address is that we cannot assume the elementary submodels of Lemma 26 are sufficiently closed. We do not need to singularize any large cardinals, but only collapse our almost-huge \( \kappa \) to be \( \mu^+ \), where \( \mu \) is the successor of a singular in the ground model. We then apply the same forcing \( \dot{C} \) but we must work harder to prove the version of Lemma 26 without the assumption that \( \nu^{<\nu} < \mu \) for \( \nu < \mu \). Our argument uses ideas from the proof of Theorem 2 in [ASS83].

Let \( \mu = \nu^+ \), and fix a function \( f : \mu \times \mu \rightarrow \nu \) such that \( f(\alpha, \cdot) \upharpoonright \alpha \) is an injection of \( \alpha \) into \( \nu \) for each \( \alpha < \mu \). Let \( \dot{C} \) be a name for a club, \( T \) a fat subset of \( \mu \) like in Lemma 26 and \( p_0 \in \mathbb{C} \). Let \( \dot{g} \) be a name for a function from some \( \delta < \nu \) to the ordinals, and assume \( \delta > \text{cf}(\nu) \). Let \( \theta \) be a sufficiently large regular cardinal, and let \( \langle M_\alpha : \alpha < \mu \rangle \) be a continuous increasing sequence of \( \nu \)-sized elementary submodels of \( (\mathbb{H}_\theta, \in, _{\theta}, p_0, \mathbb{C}, T, \dot{g}) \), where \( _{\theta} \) is a well-order. Build the models such that each \( \alpha, M_\alpha \cap \mu \in \mu \) and \( \langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1} \). We assume \( \text{GCH} \) holds, which implies that \( \mathcal{P}(\alpha) \subseteq M_0 \) for all \( \alpha < \nu \).

Let \( D = \{ \alpha : M_\alpha \cap \mu = \alpha \} \), and let \( A \subseteq D \cap T \) be a closed subset of order-type \( \delta^+ \). Fix a cofinal sequence \( \langle \gamma_i : i < \text{cf}(\nu) \rangle \) in \( \nu \), where \( \gamma_0 \geq \delta \). Let \( h : [A]^2 \rightarrow \text{cf}(\nu) \) be defined as \( h(\langle \alpha, \beta \rangle) = \text{the least } i \text{ such that } f(\alpha, \beta) < \gamma_i \), when \( \alpha > \beta \). Using the Erdős-Rado Theorem, let \( B \subseteq A \) have order-type \( \delta \) and be homogeneous in the coloring \( h \), say of color \( \xi \).

Note for every \( X \in [\kappa]^{\omega} \), the set of conditions \( q \) such that \( X \subseteq \text{dom} \, q \) is dense. Now we construct a descending chain \( \langle p_i : i \leq \delta \rangle \) below \( p_0 \) as before. Let \( \langle \alpha_i : i \leq \delta \rangle \) enumerate the closure of \( B \), except that we skip the first successors of limits. Given \( p_i \in M_{\alpha_{i+1}} \), let \( p_{i+1} \) be the \( _{\theta} \)-least condition such that:
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(1) \( \alpha_i \leq \max p_{i+1}(\gamma) \leq \alpha_{i+1} \) for all \( \gamma \in \text{dom}(p_{i+1}) \).

(2) \( p_{i+1} \) decides \( g(i) \) and some \( \beta_{i+1} \in C \) such that \( \alpha_i \leq \beta_{i+1} < \alpha_{i+1} \).

(3) \( M_i \cap \kappa \subseteq \text{dom } p_{i+1} \).

At limit \( i \), we define \( \beta_i \) and \( p_i \) as before. The key thing to check is that at such a stage, \( p_i \in M_{i+1} \). Since we skipped successors of limit points of \( B \), there is some \( \alpha^* \in M_{i+1} \cap B \setminus \alpha_i \). For each \( \gamma \in \text{dom } p_i \), \( f(\alpha^*, \cdot)(B \cap \alpha_i) \) is a subset of \( \gamma \). The sequence of all of them can be coded as a subset of \( \gamma \). Also, by condition (3) above \( \text{supp } p_i = (M_i \cap \kappa) \in M_{i+1} \). Therefore the condition \( p_i \) is definable from parameters in \( M_{i+1} \).

6. Questions

Question 1. Is it consistent that \( \text{CH} \) holds and there is a rigid saturated ideal on \( \omega_1 \)?

Question 2. Is it consistent that there is a rigid presaturated ideal on the successor of a singular strong limit cardinal?

Question 3. Let us say that an ideal \( I \subseteq \mathcal{P}(Z) \) is \( \alpha \)-rigid if whenever \( G \) is generic for \( \mathcal{P}(Z)/I \) over \( V \) there are exactly \( \alpha \) generic filters for \( \mathcal{P}(Z)/I \) in \( V[G] \). Is it consistent that a \( 2 \)-rigid precipitous ideal exists on \( \omega_1 \)?

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(Brent Cody) Virginia Commonwealth University, Department of Mathematics and Applied Mathematics, 1015 Floyd Avenue, PO Box 842014, Richmond, Virginia 23284
E-mail address, B. Cody: bmcody@vcu.edu
URL: http://www.people.vcu.edu/~bmcody/

(Monroe Eskew) Virginia Commonwealth University, Department of Mathematics and Applied Mathematics, 1015 Floyd Avenue, PO Box 842014, Richmond, Virginia 23284
E-mail address, M. Eskew: mbeskew@vcu.edu