The Spatial Dynamics in Kazakov–Migdal Model.

K. Zarembo

Steklov Mathematical Institute
Vavilov st.42, GSP-1, 117966 Moscow, Russia

Abstract

The spatially inhomogeneous large $N$ solutions to Kazakov–Migdal model are analyzed. The set of nonlinear differential equations is derived in the continuum limit. In one dimensional case these equations have a natural interpretation in terms of the dynamics of a Fermi gas. The multidimensional case seems to be inconsistent because of its instability related to the collapse of eigenvalues of the scalar field.
1 Introduction.

A while ago, Kazakov and Migdal [1] proposed a lattice gauge model induced by a heavy scalar field in the adjoint representation of $SU(N)$. The action of this model is the usual gauge invariant action without Yang–Mills term:

$$S = -\sum_x N \text{tr} \left[ U_0(\Phi(x)) - \frac{1}{2} \sum_{\nu=-D}^D \Phi(x)\Omega_\nu(x)\Phi(x+\nu)\Omega^\dagger_\nu(x) \right]. \quad (1.1)$$

Although there is some problems with the induction of a physical QCD [2, 3, 4, 5] it is interesting to investigate the continuum limit of this model. In this paper we study a semiclassical dynamics of the density of eigenvalues of matrix $\Phi(x)$:

$$\rho(\lambda, x) = \frac{1}{N} \text{tr} \delta(\lambda - \Phi(x)). \quad (1.2)$$

An integral equations for $\rho(\lambda, x)$ can be obtained using the technique developed by Migdal [6, 5]. When the lattice spacing goes to zero a continuum limit can be constructed about the critical potential $U_{cr}(\Phi) = D\Phi^2$. It is worth mentioning that the physical mass of scalar particles goes to zero in lattice units as the local limit is approaching in contrast to what is necessary for reproducing of QCD.

A set of nonlinear differential equations is obtained in the continuum limit. In $D = 1$ case these equations has a hydrodynamical interpretation and the translationally invariant solution is stable. When $D > 1$ the situation is qualitatively the other. The spectrum of excitations about the spatially homogeneous solution is always tachyonic. So the continuum limit in $D > 1$ case seems to be physically unacceptable because of its instability.

2 Saddle Point Equations.

Due to gauge invariance all the matrices $\Phi(x)$ can be diagonalized by gauge transformation. So, fixing the diagonal gauge and integrating over link variables one obtains the effective action depending upon $\Phi_i(x)$ – the eigenvalues of $\Phi(x)$. In the large $N$ limit WKB approximation becomes exact. The semiclassical equation of motion reads as follows

$$-U_0'(\lambda) + 2W(\lambda, x) + \sum_{\nu=-D}^D F_\nu(\lambda, x) = 0, \quad (2.1)$$

$\lambda$ varying along the support of $\rho(\lambda, x)$. The second term comes from gauge fixing determinant:

$$W(\lambda, x) = \varphi \int d\xi \frac{\rho(\xi, x)}{\lambda - \xi}. \quad (2.2)$$

$F_\nu(\lambda, x)$ is the logarithmic derivative of the Itzykson–Zuber integral:

$$F_\nu(\lambda, x) = \lim_{N \to \infty} \frac{1}{N} \frac{\partial}{\partial \Phi_i(x)} \ln \int D\Omega e^{N\text{tr} \Phi(x)\Omega \Phi(x+\nu)\Omega^\dagger} \bigg|_{\Phi_i(x) = \lambda}. \quad (2.3)$$
Of course, it depends only upon the eigenvalue densities of $\Phi(x)$ and $\Phi(x + \nu)$. Using the Schwinger–Dyson equations for Itzykson–Zuber integral Migdal obtained the following dispersion relation determining $F_\nu(\lambda, x)$ in terms of $\rho(\lambda, x)$ and $\rho(\lambda, x + \nu)$ \[6\]:

$$W(\lambda, x + \nu) = \int \frac{d\xi}{\pi} \arctan \frac{\pi \rho(\xi, x)}{\lambda - F_\nu(\xi, x) - W(\xi, x)}. \quad (2.4)$$

In spatially homogeneous case the equations (2.1), (2.2) and (2.4) were analyzed by Migdal \[6\] in some detail and were solved exactly for the Gaussian potential by Gross \[7\]. In the present paper we are interested in a spatial dynamics of the eigenvalue density.

### 3 The Quadratic Potential.

Before studying the general equations it is instructive to consider a more simple model with purely quadratic potential $U_0(\Phi) = \frac{1}{2}m_0^2 \Phi^2$. The saddle point equations can be simplified in this case due to the observation \[7\] that translationally invariant semi-circular distribution of eigenvalues solves (2.1), (2.2), (2.4). The semi-circular ansatz is useful in the case with a spatial fluctuations too:

$$\rho(\lambda, x) = \sqrt{\mu(x) - \frac{1}{4} \mu^2(x)} \lambda^2, \quad F_\nu = \frac{1}{2} f_\nu(x) \lambda. \quad (3.1)$$

Substituting (3.1) into (2.2) and (2.4) and doing the integrals we express $W(\lambda, x)$ and $f_\nu(x)$ in terms of $\mu(x)$. All the functions $U_0'(\lambda)$, $W(\lambda, x)$ and $F_\nu(\lambda, x)$ are proportional to $\lambda$ with coefficient of proportionality depending on the $\mu(x)$ only, so the $\lambda$-dependence in the equation (2.1) can be eliminated, that gives

$$\mu(x) = m_0^2 - \frac{1}{2} \sum_{\nu = -D}^{D} \left[ \sqrt{\mu^2(x) + 4 \frac{\mu(x)}{\mu(x + \nu)} - \mu(x)} \right]. \quad (3.2)$$

This equation has translationally invariant solution \[7\]

$$\mu_\pm = \frac{m_0^2(D - 1) \pm D \sqrt{m_0^4 - 4(2D - 1)}}{2D - 1}. \quad (3.3)$$

It is interesting that (3.2) has no strongly fluctuating antiferromagnetic solutions.

Now we are going to take the continuum limit of the equation (3.2), so we rescale $m_0^2 = m^2 a^2 + 2D$. At that moment a difference between $D = 1$ and $D > 1$ cases appears. Really, in the former case $\mu_+$ vanishes as the lattice spacing goes to zero. From (3.3) we see that $\mu_+$ scales as $2ma$, so rescaling $\mu(x) \to \mu(x)a$ and expanding (3.2) in $a$ up to the second order we get for

$$\sqrt{\frac{1}{N} \text{tr} \Phi^2(x)} = \mu^{-1/2}(x) \equiv \phi(t), \quad t = ix$$

the equation

$$\ddot{\phi} - m^2 \phi + \frac{1}{4\phi^3} = 0. \quad (3.4)$$
This describes oscillations with doubled frequency $2m$ about the static solution $\phi = (2m)^{-1/2}$.

In multidimensional case $\mu_-$ vanishes in the local limit and scales as $-m^2a^2/(D-1)$, so after rescaling $\mu(x) \to \mu(x)a^2$ we obtain from (3.2) the following equation (in the Minkowski space):

$$\Box \phi - m^2 \phi - \frac{D-1}{\phi} = 0.$$  \hspace{1cm} (3.5)

The effective potential for $\phi$ is unbounded from below, so the collapse of eigenvalues taking place in the naive (unregularized) continuum limit is inevitable. Translationally invariant solution $\phi = \sqrt{-(D-1)/m^2}$ corresponds to a maximum of the effective potential, so it is unstable.

4 The Continuum Limit for an Arbitrary Potential.

Let us consider the one dimensional case first. The canonical scaling dimensions in (2.1), (2.2) and (2.4) are recovered by the substitution $\lambda \to \lambda a^{-1/2}$, $\xi \to \xi a^{-1/2}$, $U_0'(\lambda) \to 2\lambda a^{1/2} + aU'(\lambda)$, $\rho(\xi, x) \to \rho(\xi, x)a^{1/2}$. For $F_\pm(\lambda, x)$ we can write $F_\pm(\lambda, x) = \lambda a^{-1/2} + [v_\pm(\lambda, x) - W(\lambda, x)] a^{1/2} + G_\pm(\lambda, x)a^{3/2}$. The first term in this expression is chosen so that (2.4) becomes the identity at the vanishing lattice spacing. The first and the second order terms in (2.4) reads

$$\int d\xi \rho(\xi, x) v_\pm(\xi, x) \frac{(\lambda - \xi)^2}{\lambda - \xi} = \pm \frac{\partial}{\partial x} W(\lambda, x)$$  \hspace{1cm} (4.1)

$$\int d\xi \rho(\xi, x) \left[ G_\pm(\xi, x) + \frac{v_\pm^2(\xi, x) - \frac{1}{3} \rho^2(\xi, x)}{\lambda - \xi} \right] = \frac{1}{2} \frac{\partial^2}{\partial x^2} W(\lambda, x).$$  \hspace{1cm} (4.2)

From (2.1) we have

$$v_+(\lambda, x) + v_-(\lambda, x) = 0$$  \hspace{1cm} (4.3)

$$U'(\lambda) = G_+(\lambda, x) + G_-(\lambda, x).$$  \hspace{1cm} (4.4)

The functions $v_+(\xi, x)$ and $v_-(\xi, x)$ differs only by sign, so we denote $v(\xi, t) = -iv_+(\xi, x) = iv_-(\xi, x)$, where $t = ix$. After integration by parts in the l.h.s. of (4.1) we obtain the following equation for $v$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \xi} (\rho v) = 0.$$  \hspace{1cm} (4.5)

It is well known, that $\rho(\xi, t)$ may be interpreted as the density of a Fermi gas in an external potential [8]. So the equation (4.5) has a natural interpretation as the hydrodynamical continuity condition, $v$ being the velocity of a Fermi gas flow. The equation of motion for it is obtained as follows. First, differentiate (4.3) in $t$, substitute the result in (4.2) and integrate by parts in the r.h.s. Consequently integrating by parts the second
term in the square brackets. The last step is the elimination of \( G_{\pm} \) by adding of the two equations (4.2) and using (4.4). The result reads

\[
\frac{\partial}{\partial t} (\rho v) + \frac{\partial}{\partial \xi} \left[ \rho \left( U + v^2 + \frac{1}{3} \pi^2 \rho^2 \right) \right] - U \frac{\partial \rho}{\partial \xi} = 0. \tag{4.6}
\]

The present consideration is a generalization of a method used by Gross \[7\] to reproduce the well known fermionic solution \[8\] to one dimensional matrix model:

\[
\rho_0(\xi) = \frac{1}{\pi} \sqrt{2E - 2U(\xi)}. \tag{4.7}
\]

This is just the static solution to (4.5), (4.6). Constant \( E \), the Fermi level, is determined by the normalization condition for \( \rho_0 \).

In the acoustical approximation – \( \rho(\xi, t) = \rho_0(\xi) + u(\xi, t), |u(\xi, t)| \ll \rho_0(\xi) \),

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial \xi} \left[ \pi \rho_0 \frac{\partial}{\partial \xi} (\pi \rho_0 u) \right] = 0. \tag{4.8}
\]

The local velocity of sound, \( \pi \rho_0(\xi) \), is equal to the Fermi momentum, as one might expect.

Now let us turn to the multidimensional case. From the analysis of the Gaussian potential we learned that the density of eigenvalues scales as \( a^1 \). So we write \( \xi \to \xi a^{-1}, \lambda \to \lambda a^{-1}, \rho(\xi, x) \to \rho(\xi, x)a, U'_0(\lambda) \to 2D \lambda a^{-1} + U'(\lambda), F_\nu(\lambda, x) = \lambda a^{-1} + v_\nu(\lambda, x) + \left[ G_\nu(\lambda, x) - W(\lambda, x) \right] a \). It can be verified directly that this is the only way to obtain the sensible continuum limit of (2.1), (2.2) and (2.4). All the steps in a derivation of the equations for \( \rho \) and \( v_\nu \) are the same as in the one dimensional case. After the Wick rotation \( x^D \to -i x^0, v_\nu \to iv_\nu \) we get

\[
\frac{\partial \rho}{\partial x^\nu} + \frac{\partial}{\partial \xi} (\rho v_\nu) = 0 \tag{4.9}
\]

\[
\frac{\partial}{\partial x^\nu} (\rho v_\nu) + \rho \left[ U' + 2(D - 1)W \right] + \frac{\partial}{\partial \xi} (\rho v_\nu^2) = 0. \tag{4.10}
\]

For the Gaussian potential the substitution of (3.1) with \( v_\nu(\xi, x) \) linear in \( \xi \) reduces these equations to (3.3).

Translationally invariant solution to (4.9), (4.10) is determined by the following condition

\[
 \Phi \int d\lambda \frac{\rho_0(\lambda)}{\xi - \lambda} = -\frac{U'(\xi)}{2(D - 1)}. \tag{4.11}
\]

It coincides with the saddle point equation for one matrix model with a potential \(-U(\xi)/(D-1)\). The generic solution to it is \[8, 9\]

\[
\rho_0(\xi) = \frac{1}{2\pi} \sqrt{Q(\xi) - \frac{1}{(D - 1)^2} [U'(\xi)]^2}, \tag{4.12}
\]

where \( Q(\xi) \) is a polynomial of degree less than that of \( U'(\xi) \). However, this solution is unstable. The reason is that the r.h.s. of (4.11) has a negative sign, so the eigenvalues of
Φ(x) are accumulated not in a minimum, but in a maximum of the potential. The instability of the translationally invariant solution, of course, can be demonstrated explicitly by linearization of (4.9), (4.11) near ρ₀(ξ). The linearized equation reads

$$\Box u + 2(D - 1) \frac{\partial}{\partial \xi} \left[ \rho_0 \varphi \int d\lambda \frac{u(\lambda, x)}{\xi - \lambda} \right] = 0. \quad (4.13)$$

Substitution \( u(\xi, x) = e^{-ik\cdot x} u_0(\xi) \) leads to the following eigenvalue problem

$$Af(\xi) \equiv -2(D - 1) \varphi \int d\lambda \frac{[\rho_0(\lambda)f(\lambda)]'}{\xi - \lambda} = k^2 f(\xi). \quad (4.14)$$

There \( f(\lambda) = 1/\rho_0(\lambda) \int_0^\lambda d\xi u_0(\xi) \). Operator \( A \) is self-adjoint with respect to the scalar product \( (f_1, f_2) = \int d\lambda \rho_0(\lambda)f_1^*(\lambda)f_2(\lambda) \), all its eigenvalues being negative, because

$$(f_1, Af_2) = -(D - 1) \int d\xi d\lambda \frac{[\rho_0(\xi)f_1^*(\xi) - \rho_0(\lambda)f_1^*(\lambda)] [\rho_0(\xi)f_2(\xi) - \rho_0(\lambda)f_2(\lambda)]}{(\xi - \lambda)^2}. \quad (4.15)$$

So all the spectrum of excitations is tachyonic.

5 Conclusions.

In multidimensional case we does not find such fine physical picture as in one dimension. Translationally invariant solution to the semiclassical equations of motion corresponds to a maximum of the potential, the fluctuations about it are unstable. Although we can not rule out the existence of a more complicated stable vacuum, we do not see the physical reasons for this possibility. So it seems that without the valuable changes, like an inclusion of fermions \([3, 10]\), Kazakov-Migdal model is unstable in the continuum limit.

6 Acknowledgements.

I would like to thank L.Chekhov for introducing me to this domain and for useful discussions.

References

[1] V.A.Kazakov and A.A.Migdal, *Induced QCD at Large N*, Princeton preprint PUPT-1322 (May, 1992).

[2] I.Kogan, G.Semenoff, and N.Weiss, *Induced QCD and Hidden Local Z_N Symmetry*, UBC preprint UBCTP-92-022 (June, 1992).
[3] S.Khokhlachev and Yu.Makeenko, The Problem of Large-N Phase Transition in Kazakov-Migdal Model of Induced QCD, Moscow preprint ITEP-YM-5-92 (July, 1992)

[4] Yu.Makeenko, Reduction, Master Field and Loop Equations in Kazakov-Migdal Model, Moscow preprint ITEP-YM-6-92 (August, 1992).

[5] A.A.Migdal, Mixed Model of Induced QCD, Paris preprint LPTENS-92-23 (August, 1992).

[6] A.A.Migdal, Exact Solution of Induced Lattice Gauge Theory at Large N, Princeton preprint PUPT-1323 (June, 1992).

[7] D.J.Gross, Some Remarks about Induced QCD, Princeton preprint PUPT-1335 (August, 1992).

[8] E.Brézin, C.Itzykson, G.Parisi, and J.B.Zuber, Commun. Math. Phys., 59 (1978) 35.

[9] F.David, Nucl. Phys., B348 (1991) 507.

[10] S.Khokhlachev and Yu.Makeenko, Adjoint Fermions Induce QCD, Moscow preprint ITEP-YM-7-92 (August, 1992).