Strong gravitational lensing by DHOST black holes

Javier Chagoya1·*, C Ortiz1·, Benito Rodríguez1 and Armando A Roque2·

1 Unidad Académica de Física, Universidad Autónoma de Zacatecas, 98060, México
2 Departamento de Física, Universidad de Guanajuato, 37150 León, Guanajuato, México

E-mail: javier.chagoya@fisica.uaz.edu.mx

Received 26 July 2020, revised 24 December 2020
Accepted for publication 18 January 2021
Published 15 March 2021

Abstract
The gravitational deflection of light in the strong field limit is an important test for alternative theories of gravity. However, solutions for the metric that allow for analytic computations are not always available. We implement a hybrid analytic-numerical approximation to determine the deflection angle in static, spherically symmetric spacetimes. We apply this to a set of numerical black hole solutions within the class of modified gravity theories known as degenerate higher order scalar–tensor theories (DHOST). Comparing our results to a more time consuming full numerical integration, we find that we can accurately describe the deflection angle for light rays passing at arbitrary distances from the photon sphere with a combination of two analytic-numerical approximations. Furthermore, we find a range of parameters where our DHOST black holes predict strong lensing effects whose size is comparable with the uncertainty in the properties of the supermassive black hole in M87 reported by the event horizon telescope, showing that strong lensing is a viable alternative to put constraints on these models of modified gravity.

Keywords: black holes, deflection of light, modified gravity

(Some figures may appear in colour only in the online journal)

1. Introduction
The gravitational deflection of light is one of the most studied predictions of the theory of general relativity (GR). This effect has been observed in several scenarios, from our Solar
System to massive clusters of galaxies (see [1] for a review). In many cases, a weak field approximation to gravitational deflection is enough to explain the existing phenomenology and obtain important information about the theory of gravity. For instance, post-Newtonian studies based on the weak deflection of light by the Sun show that the GR prediction is accurate up to a relative error $\sim 10^{-4}$ [2], forcing any other theory of gravity to satisfy the same constraint. For more complex systems, such as clusters of galaxies, the weak lensing theory based on the assumption that the deflection angle is small has been used to analyse the validity of different theories of gravity on cosmological scales [3, 4].

The deflection of light due to strong gravitational fields has also been studied for a long time. In 1959, Darwin obtained an exact expression for the deflection angle of light in a Schwarzschild spacetime [5], and some years later further analysis of black hole lenses were presented, in what could be considered as the beginning of black hole imaging [6, 7]. Interest in strong lensing received a boost when the possibility of obtaining an image of the area near a black hole was first discussed [8, 9]. Recently, the event horizon telescope (EHT) imaged the structure around a supermassive black hole [10], finding a ring structure with an angular diameter of $42 \pm 3 \mu$as. This offers a way not only to investigate the region around a black hole, but also to use this information to test alternative theories of gravity in the strong field regime, imposing constraints on these theories that are complementary to those obtained under weak field approximations. This has motivated several studies of the strong deflection angle in spacetimes predicted by modifications to GR (see, e.g. [11–15]). These studies rely on the existence of an analytical solution for the metric in the modified theory of gravity under consideration $^3$. The deflection angle is then studied under some approximations, in particular, Bozza [17] introduced a method that separates and carefully describes the divergent part of the deflection angle at the photon sphere from the regular part. For a Schwarzschild black hole, Bozza’s method can be compared with the exact result, giving a discrepancy in the deflection angle of about $0.06\%$. For other black holes, the results of the approximation are sometimes compared with the full numerical results, also giving good agreement near the photon sphere. By construction, Bozza’s approximation is valid only at very short distances from the photon sphere. For larger distances, a different approximation that is equally capable of handling any spherically symmetric, static, asymptotically flat spacetime was presented in [18], where it was also shown that this approximation is in good agreement with exact results for Schwarzschild and Reissner–Nördstrom black holes almost up to the photon sphere.

In this work we use the approximations mentioned above to study the deflection angle in a particular model of modified gravity. Specifically, we use a numerical solution for a static, spherically symmetric spacetime in a scalar–tensor theory that belongs to beyond Horndeski [19, 20], a generalization of Horndeski gravity [21], which is the most general scalar–tensor theory with equations of motion that are explicitly second order, thus avoiding the propagation of Ostrogradski degrees of freedom [22]. In beyond Horndeski and further generalizations, known as DHOST [23, 24] or extended scalar–tensor theories (EST) [25], higher order equations of motion are allowed as long as the Hessian matrix of the system is degenerate, thus introducing constraints that prevent the propagation of the Ostrogradski ghost.

Scalar–tensor modifications of gravity are generally motivated by their applications to cosmology (see [26] for a review). In the context of beyond Horndeski, there is a subclass of

$^3$ There are exceptions, in [16] a numerical computation is presented for the deflection angle in higher derivative gravity theories, however, the method is tested against weak deflection data.
models that give a self-tuning mechanism for solving the cosmological constant problem [27–29]. Later, in [30] it is established that this self-tuning is generic in shift-symmetric beyond Horndeski models.

On the other hand, cosmological self-tuning is not enough for guaranteeing the physical viability of a model. For ensuring this, local tests should be verified as well [31]. In other words, a screening mechanism [32–37] that suppresses the fifth force is needed at scales where deviations from GR are tightly constrained by observations. As a consequence, the solutions for the metric at these scales should resemble known GR solutions. This points out the importance of studying astrophysical solutions of the theory. Along this direction, astrophysical predictions of modified gravity models with partial breaking of the screening mechanism have been explored in several works, e.g. considering the properties of exact and numerical solutions for black holes and relativistic stars [41–50], gravitational waves [51, 52] and even the speed of sound in the atmosphere of the Earth [53]. Findings are diverse, for instance, depending on the specific model under consideration black hole solutions may or may not be the same as in GR, either exactly or asymptotically.

Another important restriction on the models that can be considered is that the relative difference between the speed of propagation of gravitational waves, \( c_{GW} \), and the speed of light, \( c \), has been measured to be below \( 10^{-16} \) [54], this is derived from the detection of gravitational waves with an electromagnetic counterpart made by LIGO, VIRGO and several other observatories [55]. Within Horndeski, only the quadratic and cubic sectors predict \( c_{GW}/c = 1 \). However, when beyond Horndeski is included, this condition can be satisfied by particular combinations of quartic and quintic Lagrangians (e.g. [56]).

The discussion above motivates the model that we study in this work. We use one of the beyond Horndeski models that is compatible with \( c_{GW}/c = 1 \). However, the static, spherically symmetric vacuum solutions of this model, studied in [46], are not exactly Schwarzschild, making it interesting to explore their observational signatures in the strong field regime. Furthermore, since these models contain an angular deficit, we discuss the constraints that can be imposed in the weak deflection limit. Not less importantly, the solutions that we use are known in analytic form only asymptotically, and numerically for the complete range of the radial coordinate, offering a non-trivial situation where we can demonstrate that the methods proposed in [17, 18] for computing (strong) deflection angles in spherically symmetric spacetimes can be implemented numerically, and the results are consistent with a full numerical computation of the deflection angle.

This work is organised as follows. In section 2 we give an overview of beyond Horndeski, the particular model that we use, and the black hole solutions of this model. In section 3 we outline the strong deflection limit and related lensing observables following the method of [17]. In section 4 we perform a numerical implementation of this method in order to compute the strong deflection angle in the spacetimes that we are interested in. The results of this section are applied in section 5 to compute lensing observables for two supermassive black hole candidates: Sagittarius A* and M87; in particular, we obtain the asymptotic position and separation of the lensed images. In section 6 we discuss the deflection angle far from the photon sphere, using a numerical implementation of [18], and we discuss how well the methods that we use approximate the results of full numerical computations. Section 7 is devoted to discussion and concluding remarks. We adopt geometrised units, introducing the appropriated conversion factors when treating the masses and distances of observable systems.

---

4 Scalar–tensor modifications have a dynamical scalar degree of freedom, arising from the breaking of gauge symmetries of GR. The presence in the action of terms like \( F(\phi)R \) or \( X\Box\phi \) (where \( X \) is the canonical kinetic term) implies that the scalar field mediates an extra force with the matter sector [38–40].
2. Black hole solutions in beyond Horndeski

The Horndeski Lagrangians [21] describe the most general scalar–tensor theory with equations of motion that are explicitly second order, thus guaranteeing that the system is free of Ostrogradsky instabilities [22]. Let $\phi$ be the scalar field, the Horndeski Lagrangians are given by [57, 58]

\[
\begin{align*}
\mathcal{L}_2 &= G_2, \\
\mathcal{L}_3 &= G_3 [\Phi], \\
\mathcal{L}_4 &= G_4 R + G_4 X \left\{ [\Phi]_2^2 - [\Phi_2^2] \right\}, \\
\mathcal{L}_5 &= G_5 G_{\mu\nu} \Phi^{\mu\nu} - \frac{1}{6} G_{5,X} \left\{ [\Phi]_3^3 - 3 [\Phi] [\Phi_2^2] + 2 [\Phi_3^3] \right\},
\end{align*}
\]

where $\Phi$ is a matrix with components $\nabla_\mu \nabla_\nu \phi$, $G_i$ denotes derivatives of the functions $G_i$ with respect to $X$, and

\[
X = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi, \quad [\Phi^a] = \text{tr}(\Phi^a), \quad \langle \Phi \rangle = \partial_{\mu} \phi \partial^{\mu} \phi \partial_{\nu} \phi \partial^{\nu} \phi.
\]

Assuming that the action is invariant under the shift $\phi \to \phi + \text{const}$, $G_i$ are arbitrary functions only of $X$. The condition that the equations of motion are explicitly second order can be relaxed without introducing Ostrogradsky ghosts as long as the Hessian matrix of the system, obtained by taking second derivatives of the Lagrangian with respect to velocities, is degenerate. In this case it is said that the Lagrangian is degenerate. These generalizations were first introduced as DHOST [23] or EST [25] for Lagrangians that depend quadratically on second derivatives of a scalar field, and then generalized in [24] for cubic dependence on second derivatives. The first realisations of DHOST theories were given in [19, 20], and are known as beyond Horndeski or GLPV Lagrangians.

Astrophysical systems in DHOST theories provide a way for testing these models of gravity. One feature of these theories is the presence of screening mechanisms that are usually very efficient outside matter sources, but can be broken in the interior region. For instance, in different sectors of the theory, it has been shown that outside the source the metric behaves qualitatively as the standard solutions of GR [59], that it is exactly Schwarzschild–de-Sitter [42], or that it is exactly Schwarzschild [47]. The breaking of the screening mechanism inside astrophysical bodies has been studied in these same references (see also [60]). The fact that the screening works well outside the matter source makes it more difficult to devise tests for these theories of gravity using compact objects, but tests using galactic scale systems have been proposed. On the observational side, the detection of gravitational waves from the neutron star merger GW170817 and its associated electromagnetic counterpart GRB170817A [55] put tight constraints on the speed of gravitational waves, that are satisfied by a limited class of DHOST theories. One of the models within this limited sector is given by

\[
\mathcal{L}_c = X + \mathcal{L}_4 + \mathcal{L}_4^{bH},
\]

where the quartic beyond Horndeski Lagrangian is

\[
\mathcal{L}_4^{bH} = -\frac{1}{2} \Gamma_4 \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma'} \partial_{\mu} \phi \partial_{\rho'} \phi \Phi_{\nu\rho} \Phi_{\mu'\sigma'},
\]
and \( F_4 \) is subject to \( F_4 = G_{4X}/X \). After some manipulations, \( \mathcal{L}_c \) is reduced to

\[
\mathcal{L}_c = X + G_4 R + \frac{G_4 X}{X} (\langle \Phi^2 \rangle - \langle \Phi \rangle^2). \tag{5}
\]

Static, spherically symmetric solutions to this model were studied in [46], both for black holes and relativistic stars, with

\[ G_4(X) = M_P^2 + g_4 X, \tag{6} \]

and \( g_4 \) a constant with dimensions of inverse mass squared. With this choice of \( G_4 \), the model falls within the class of shift-symmetric beyond Horndeski, thus enjoying a cosmological self-tuning. A feature of its static, spherically symmetric solutions is that the metric acquires a deficit angle as a result of a linear dependence in time of the scalar field and of the presence of the kinetic term \( X \) in the Lagrangian\(^5\). Even if the remaining components of the metric are very similar to GR solutions, the deficit angle signals a breaking of the screening mechanism outside the astrophysical source, and it opens up the possibility for testing this model with phenomenology away from the source. In order to do so, let us present in more detail the black hole solutions to the model (5). The assumptions for the scalar field and metric are

\[
\phi = M_P (\phi_0 t + \phi_1 (r)), \tag{7}
\]

\[
\text{d}s^2 = -f(r) \text{d}t^2 + h(r)^{-1} \text{d}r^2 + s_0^{-1} r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2), \tag{8}
\]

where \( \phi_0 \) is a constant that parametrizes the linear time dependence of the scalar field and \( s_0 \), also constant, introduces a deficit angle to the solution. It was shown in [46] that \( s_0 \) is required in order to obtain asymptotically locally flat solutions, while \( \phi_0 \) guarantees that this angular deficit does not induce a conical singularity. Further motivations for considering a linearly time dependent scalar field can be found in [61]. The form of \( G_4 \) and \( \phi \) is such that \( M_P \) factors out of the Lagrangian. The equations of motion fix \( s_0 = 1 - 3 g_4 \phi_0^2 \). The functions \( f(r), h(r) \) and \( \phi_1(r) \) are found asymptotically as

\[
f(r) = 1 - \frac{2M}{r} - \frac{4g_4^2 \phi_0^2 (g_4 \phi_0^2 - 2)}{r^2} + \mathcal{O} \left( \frac{1}{r^3} \right), \tag{9a}\]

\[
h(r) = 1 - \frac{2M}{r} + \frac{4g_4^2 \phi_0^2 (1 - g_4 \phi_0^2)}{r^2} + \mathcal{O} \left( \frac{1}{r^3} \right), \tag{9b}\]

\[
\phi_1(r) = \phi_0 + \frac{2M \phi_0}{r} + 2\phi_0 \left[ 2g_4^2 \phi_0^4 + (2M^2 - g_4) - 3g_4^2 \phi_0^2 \right] r^2 + \mathcal{O} \left( \frac{1}{r^3} \right), \tag{9c}\]

where \( M \) is an integration constant and \( \phi_1' = d\phi_1/dr \). We emphasise that these solutions are valid only for large \( r \), therefore, they cannot be used to study the existence of horizons nor the strong field deflection limit. It is for this reason that our analysis is based on numerical solutions, valid for arbitrary \( r \), which are obtained by imposing (9) as initial conditions and solving inwards. In figure 1 we reproduce a set of solutions presented in [46] for \( 2M = g_4 = 1 \) and different values of \( \phi_0 \). As \( \phi_0 \) increases, the horizon shrinks until it finally disappears, this defines the range of \( \phi_0 \) that admits regular black hole solutions for a given mass. In the next sections, we present the formalism for studying strong lensing and then we apply it to this set of solutions, first for arbitrary masses in order to analyse the generic properties of the model.

\(^5\)The exact Schwarzschild exterior metric found in [47] is for a model that does not include the standard kinetic term.
Figure 1. Numerical BH solutions for $2M = g_4 = 1$. The figure shows the inverse of the metric component $g_{rr}$ (equation (8)), i.e. $g^{rr} = h(r)$, in GR (black, dashed line) and in the beyond Horndeski model we are considering, with $\phi_0$ spanning between 0.02 and 0.32 (blue, solid lines). The size of the black hole horizon decreases as $\phi_0$ increases. The profiles for the metric component $g^{tt}$, not shown, are similar.

and then for two specific astrophysical black holes whose masses and distances are known observationally.

3. Strong deflection limit

Let us briefly review the analytical method that we use as a starting point for our numerical computations. We consider geometries described by the line element

$$\text{d}s^2 = -A(r)\text{d}t^2 + B(r)\text{d}r^2 + C(r)(\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2).$$

A dimensionless variable $x(r)$ is often introduced. If the metric components $A, B$ are asymptotically Schwarzschild, as is the case for the metrics we consider in this work, this variable is naturally defined as $x = r/2M$, where $M$ is the Schwarzschild mass. Physical units can be restored with

$$r = 2Mx \frac{G}{c^2},$$

where $G$ is the gravitational constant and $c$ is the speed of light in vacuum, with values taken from $[62]$.

We are interested in studying the deflection of light rays passing near the photon sphere, i.e. the surface where photons can orbit the black hole. The radius $x_m$ of the photon sphere is given by the outermost solution of

$$\frac{C'(x)}{C(x)} = \frac{A'(x)}{A(x)}.$$  

$\ast$The asymptotic conditions $\lim_{r \to \infty} A = \lim_{r \to \infty} B = 1$ and $\lim_{r \to \infty} C = r^2$ are sometimes imposed in order to ensure asymptotic compatibility with the Minkowski metric. In our case, only the first two conditions are true, while $\lim_{r \to \infty} C = x_0^2 r^2$, with $x_0$ defined in (8). Mathematically, the method we are using works also for this asymptotic condition.
where primes denote derivatives with respect to $x$. This equation can be obtained either by searching for the radius where the deflection angle of photons (presented below) becomes unboundedly large [63] or by analysing the behaviour of null geodesics [64]. Following the classical computation of [65], it has been shown [11] that the metric (10) predicts that the deflection angle of photons is

$$\alpha(x_0) = 2 \int_{x_0}^{\infty} dx \sqrt{\frac{B(x)}{C(x)}} \left[ \frac{C(x)A(x_0)}{A(x)C(x_0)} - 1 \right]^{-\frac{1}{2}} - \pi,$$

$$\equiv I(x_0) - \pi,$$  \hspace{1cm} (13)

where $x_0$ is the distance of closest approach of the light ray to the center of the gravitational attraction. In the strong deflection limit, the integral $I(x_0)$ diverges logarithmically [66]. Indeed, it can be separated into a divergent and a regular part,

$$I(x_0) = I_D(x_0) + I_R(x_0).$$  \hspace{1cm} (14)

After taking expansions near $x_0 = x_m$, and writing $x_0$ in terms of its associated impact parameter [17, 66]

$$b(x_0) = \sqrt{\frac{C(x_0)}{A(x_0)}},$$  \hspace{1cm} (15)

the deflection angle can be written as

$$\alpha(\beta) = -c_1 \log \left( \frac{b}{b_c} - 1 \right) + c_2 + O[(b - b_c) \log(b - b_c)],$$  \hspace{1cm} (16)

where $b_c$ is the critical impact parameter, i.e. the impact parameter for a light ray with closest approach distance $x_0 = x_m$. The parameters $c_1$ and $c_2$ depend only on $x_m$—although $x_m$ itself could depend on the parameters of the black hole solution. The divergent part of the integral $I(x_0)$ appears in the deflection angle through the logarithmic term, and its coefficient $c_1$ is equal to

$$c_1 = \sqrt{\frac{2A_m B_m}{C_m A_m - C_m A_m}},$$  \hspace{1cm} (17)

where a subscript $m$ denotes evaluation at the photon sphere, $x_m$. The regular part is left explicitly in

$$c_2 = c_1 \log \left[ x_m^2 \left( \frac{C_m}{C_m - \frac{A_m'}{A_m}} \right) + I_R(x_m) \right] - \pi.$$  \hspace{1cm} (18)

Notice that $I_R$ takes the photon sphere radius $x_m$ as parameter instead of $x_0$, this is because we are in the limit $x_0 \to x_m$, and the correction terms of order $(b - b_c) \log(b - b_c)$ or higher are all being neglected. In order to avoid integrating up to infinity, it is convenient to introduce a new coordinate $z$ defined by

$$x(z) = \frac{x_m}{1 - z}.$$  \hspace{1cm} (19)
In terms of this coordinate, $I_R$ is written as

$$I_R(x_m) = 2x_m \int_0^1 dz \left[ \sqrt{\frac{B(z)}{C(z)}} \left( \frac{C(z)A_m}{A(z)C_m} - 1 \right) - \frac{1}{1-z} \frac{c_1}{x_m^2} \right].$$

(20)

The last term comes from the definition of $I_R$, which implies subtracting from $I(x_0)$ the part that integrates to a logarithmic divergence [17].

Summing up, the method described above allows us to compute the deflection angle by performing the following steps:

(a) Use equation (12) to determine $x_m$.
(b) Obtain $c_1$ from equation (17).
(c) Compute $I_R$ from equation (20).
(d) Compute $c_2$ from equation (18).

This method has been used for studying the deflection angle of several black holes, such as Schwarzschild and Reissner–Nördstrom [17, 66], but it has also been used for a particular exact solution of Horndeski gravity [14, as well as for the Janis–Newman–Winicour (JNW) naked singularity in GR minimally coupled to a massless scalar field [11], and for a Schwarzschild black hole pierced by a cosmic string (which induces a deficit angle) [67], to mention a few examples. Before obtaining results for the deflection angle, let us define the lensing observables that we are interested in.

### 3.1. Lensing observables

We start with the lens equation in the situation where the source and lens are almost perfectly aligned [68],

$$\beta = \theta - \frac{D_{LS}}{D_{OS}} \Delta \alpha_n,$$

(21)

where $\beta$ and $\theta$ are, respectively, the angles between the observer and the source and between the observer and the image, both measured with respect to the optical axis, $D_{LS}$ is the distance between the lens and the source plane measured along the optical axis, $D_{OS}$ is the distance between the observer and the source plane, also measured along the optical axis, and $\alpha = 2\pi n + \Delta \alpha_n$, i.e., $\Delta \alpha_n$ represents the effective deflection angle defined by Virbhadra and Ellis [69, 70]. This last relation expresses the fact that, in the strong gravity regime, the high alignment between lens and source does not imply that the deflection angle is small, instead, light rays may complete $n$ loops around the lens and then travel towards the observer with an effective deflection angle $|\Delta \alpha| \ll 1$. The number $n$ denotes the order of the image: for $n = 0$ the images are known as primary and secondary, one on each side of the optical axis, while for $n \geq 1$ the images are known as relativistic images, and can be either on the side of the primary or secondary images. The deflection angle equation (16) can be written in terms of $\theta$ by noticing that, by assumption, $\theta$ is a small angle, and can therefore be approximated as $\theta \approx b/D_{OL}$, thus

$$\alpha(\theta) = -c_1 \log \left( \frac{\theta D_{OL}}{b_c} - 1 \right) + c_2 + \ldots.$$  

(22)

As described in [68], the effective angle $\Delta \alpha_n$ can be translated to a small change in $\theta$ by expanding $\alpha(\theta)$ angle around the value $\theta_0^n$ such that $\alpha(\theta_0^n) = 2\pi n$, i.e.,

$$\Delta \theta_n = \theta - \theta_0^n,$$

(23)
with

\[ \theta_n^0 = \frac{b_c}{D_{OL}} (1 + e_n), \]
\[ e_n = \exp \left( \frac{c_2 - 2n\pi}{c_1} \right). \tag{24} \]

Notice that \( \theta_n \) decreases exponentially as \( n \) increases. Taylor expanding \( \alpha(\theta) \) near \( \theta_n^0 \) and substituting the previous result, we find

\[ \Delta \alpha_n = -\frac{c_1 D_{OL}}{b_c e_n} \Delta \theta_n. \tag{25} \]

Using this result, the lens equation (21) becomes

\[ \beta = \theta + \frac{D_{LS}}{D_{OS}} \frac{c_1 D_{OL}}{b_c e_n} \Delta \theta_n. \tag{26} \]

This can be used to compute the magnification,

\[ \mu_n = \frac{1}{\alpha'(\theta)} \bigg|_{\theta = \theta_n^0} \approx \frac{1}{\alpha'(\theta)} \bigg|_{\theta = \theta_n^0}, \tag{27} \]

where we are neglecting the correction term \( \Delta \theta_n \). After a few manipulations, this becomes

\[ \mu_n = e_n \frac{b_c^2 (1 + e_n) D_{OS}}{c_1 \beta D_{OL} D_{LS}}. \tag{28} \]

Instead of working with individual magnifications, it is more convenient to define

\[ r_\mu = \mu_1 \left( \sum_{n=2}^{\infty} \mu_n \right)^{-1}, \tag{29} \]

i.e. the ratio between the magnification of the outermost image, located at \( \theta_1 \), and the sum of the magnification of all the other images, whose position quickly approaches \( \theta_\infty \), given by the limit as \( n \to \infty \) of equation (24). The denominator in \( r_\mu \) can be summed exactly as a geometric series, and the result can be approximated under the assumption that \( c_1 \) and \( c_2 \) are of order unity, which is known to hold for Schwarzschild and can be verified for the metrics that we consider later on. The final, approximated result is

\[ r_\mu = e^{2\pi/c_1}. \tag{30} \]

Thus, obtaining \( r_\mu \) is straightforward from the results described in the previous subsection. In order to agree with the conventions used in the literature, we will report the values of

\[ r_m = 2.5 \log_{10} r_\mu. \]

The second observable that we report is the separation between the first relativistic image and the others, given by

\[ s = \theta_1 - \theta_\infty \approx \theta_1^0 - \theta_\infty. \tag{31} \]
Under the same assumptions that we made for \( r_m \), this is reduced to

\[
s \approx \theta_\infty \exp \left( \frac{c_2 - 2\pi}{c_1} \right) = \frac{b_c}{D_{OL}} \exp \left( \frac{c_2 - 2\pi}{c_1} \right).
\]  

(32)

The observables \( r_m \) and \( s \) are completely determined by \( c_1, c_2, b_c \) and \( D_{OL} \). This last quantity is fixed by observations, while the other three depend on the parameters that appear in the black hole solution under consideration, i.e. on the mass—also observed—and on the parameters that appear as a result of considering alternative models of gravity. In the next section we compute quantities that do not depend on \( D_{OL} \) for several numerical black hole solutions of beyond Horndeski, while the ones that do depend on \( D_{OL} \) are presented in section 5.

4. Numerical implementation and results

Let us study the deflection angle in the strong gravity regime for the solutions of beyond Horndeski presented in figure 1. Since these solutions are numerical, our results here are also numerical. However, the limit \( \phi_0 \to 0 \) recovers the Schwarzschild solution, thus it can be used as a reference to validate our results. Although not strictly necessary, it is convenient to introduce variables such that the black hole solution does not depend explicitly on the Schwarzschild mass \( M \). This is achieved by using \( 2M \) as unit of distance, i.e. introducing \( x = r/2M \) (physical units are restored using (11)), and redefining the constants that appear in the asymptotic solutions (which act as boundary conditions) in order to absorb any factor of \( M \). The appropriate redefinitions for equation (9) are \( g_4 = 4M^2 \tilde{g}_4 \) and \( \phi_0 = \phi_0/2M \). One should keep in mind that this redefinition is only for numerical convenience, and it is not well justified from a theoretical point of view, for instance, \( \tilde{g}_4 \) cannot be interpreted as parameters of the model since now it depends on the mass scale of the system under consideration. It is convenient to keep in mind that \( g_4 \phi_0^2 = \tilde{g}_4 \phi_0^2 \).

Using the redefined quantities \( x, \tilde{g}_4, \tilde{\phi}_0 \), the asymptotic metric takes the form

\[
f(x) = 1 - \frac{1}{x} + \frac{4\tilde{g}_4^2 \tilde{\phi}_0^2 \left( \tilde{g}_4 \tilde{\phi}_0^2 - 2 \right)}{x^2} + O \left( \frac{1}{x^4} \right),
\]

(33a)

\[
h(x) = 1 - \frac{1}{x} + \frac{4\tilde{g}_4^2 \tilde{\phi}_0^2 \left( 1 - \tilde{g}_4 \tilde{\phi}_0^2 \right)}{x^2} + O \left( \frac{1}{x^4} \right).
\]

(33b)

Strictly speaking, one should redefine \( r \) also in the line element. This would introduce a global factor of \( 4M^2 \) both in the radial and in the angular components of the redefined metric, these factors are not explicitly considered in the literature, but are correctly accounted for when reporting the impact parameter, equation (15), as \( b/2M \) instead of simply \( b \) (see, e.g., [17]). The global factors of \( 4M^2 \) cancel out in all the other quantities introduced in the previous sections.

Having clarified this, we proceed to apply the formalism of section 3, with the identifications

\[
A(x) = f(x), \quad B(x) = 1/h(x), \quad C(x) = (1 - 3\tilde{g}_4 \tilde{\phi}_0^2)^{-1}x^2.
\]

(34)

Let us describe the numerical implementation of the steps i–iv enumerated in section 3. For concreteness and numerical convenience, we make \( \tilde{g}_4 = 1 \), and we vary \( \phi_0 \) in the range \( 0 < \phi_0 \leq 0.32 \) where regular black hole solutions exist.

(a) Use equation (12) to determine \( x_m \). We use a numerical root finder in the range \( x_h < x < 2 \), where \( x_h \) is the size of the horizon obtained from the numerical solutions.
Figure 2. Critical impact parameter, $\tilde{b}_c = b_c/2M$, divided over the horizon of each solution, for $0 < \tilde{\phi}_0 < 0.26$.

Figure 3. Parameters $c_1$ and $c_2$ that determine the strong deflection limit in Bozza’s approximation, for $0 \leq \tilde{\phi}_0 \leq 0.24$. Notice that the relative change with respect to Schwarzschild ($\tilde{\phi}_0 = 0$) is much smaller in $c_1$ than in $c_2$.

for each value of $\tilde{\phi}_0$. For $\tilde{\phi}_0 \to 0$, $x_h$ recovers the horizon of a Schwarzschild black hole, $x_h = 1$, and it decreases as $\tilde{\phi}_0$ increases. Similarly, $x_m$ approaches its Schwarzschild value, $x_m = 3/2x_h = 3/2$ for small $\tilde{\phi}_0$, and it decreases as $\tilde{\phi}_0$ increases. It is interesting to note that the ratio between the photon sphere and the horizon, $x_m/x_h$, remains nearly constant as $\tilde{\phi}_0$ varies; the relative differences $1 - x_m/x_h$ are of order $10^{-3}$, i.e. for these black holes the photon sphere is relatively at the same distance from the horizon than it is for a Schwarzschild black hole. Once $x_m$ is known, it is straightforward to compute also the critical impact parameter $b_c = b(x_m)$ using equation (16). Studying the relation between $b_c$ and $x_h$ we find that their ratio increases with $\tilde{\phi}_0$: for the black holes that we are considering, the horizon is smaller than it is for a Schwarzschild black hole, but they capture photons over a larger radius in relation to the size of their horizon. This is shown in figure 2.

(b) Obtain $c_1$ from equation (17). Once we have $x_m$, it is straightforward to evaluate the argument of equation (17) as long as an interpolation of the numerical solutions and their derivatives at $x_m$ are known. If that is not the case, we have verified that a cubic polynomial interpolation on a set of data points spaced by $\Delta x = 0.01$ is enough to get values of
Assuming that $\tilde{\phi}_0$ is constrained to be very close to 0 by other types of observations, this would mean that the magnification $r_m$ would be nearly indistinguishable from the magnification due to a Schwarzschild black hole.

c_1$ with a relative difference of order $10^{-4}$ with respect to those obtained for a more precise solution with $\Delta x = 0.0005$. Our results for $c_1$ are displayed in the left panel of figure 3. We see a small relative change, around $10^{-2}$, from the Schwarzschild value $c_1 = 1$. As we notice below, this change is also small in comparison to the change in $c_2$.

(c) Compute $I_R$ from equation (20). For this step it is convenient to write the solutions for the metric in terms of the variable $z$ defined in equation (19). It is important to notice that this redefinition is different for every solution since it depends on the value of $x_m$. Nevertheless, it has the advantage of reducing the range of integration to $0 \leq z \leq 1$. Numerically we cannot use these exact limits since at $z = 0$ the argument of the integral diverges and $z \to 1$ implies $x \to \infty$. Instead, we use $z = 0.00001$ and $z = 0.997$. These limits ensure that the result of the integral does not change by more than $10^{-3}$ if the integration range is extended. One should be careful with changing the upper limit because of the inverse relation between $x$ and $1 - z$: if $z$ was very close to 1 we would need to know the numerical solution up to a very large value of $x$, for instance, $z = 0.998$ translates to $x > 500$ for all of the values of $x_m$ that we find, and $z = 0.999$ translates to $x > 1000$. The value that we use, $z = 0.997$, requires a moderate knowledge of the numerical solution, up to $x \lesssim 450$ and gives a precision of at least three decimal places in the result of $I_R$.

(d) Compute $c_2$ from equation (18). Having all the previous results at hand, this step is a simple substitution into equation (18). The results are shown in the right panel of figure 3. Notice that the relative change in $c_2$ is one order of magnitude bigger than the change in $c_1$. This is in contrast with known results for Horndeski black holes [14], and also with the case of Reissner–Nordström (e.g. [17]), where the changes in $c_1$ and $c_2$ arising from changes in the charges or parameters of each model are comparable, but is similar to results for the JNW naked singularity [17], where at leading order $c_1$ remains equal to 1 while $c_2$ experiences changes of order 1. However, it is possible to distinguish JNW from our beyond Horndeski solutions by the sign of the changes in $c_2$: for JNW, $c_2$ is larger than its Schwarzschild value, while for the beyond Horndeski solutions that we study the opposite is true.
Let us close this section by analysing the observable \( r_\mu \) defined in equation (29). This quantity depends only on \( c_1 \), therefore we can study it without making reference to particular masses or distances of an astrophysical system. As we discussed above, the deviations of \( c_1 \) with respect to its Schwarzschild value \( c_1 = 1 \) are minimal, and this is inherited to \( r_\mu \). Figure 4 shows \( r_m = 2.5 \log_{10} r_\mu \).

5. Supermassive black holes

In the previous sections we obtained all the quantities required in order to make specific predictions for observables that could be constrained, for instance, with future data from the event horizon telescope [10]. Let us focus in two astrophysical systems targeted by these observations: the supermassive black hole candidate, Sagittarius A* [71] (Sgr A*), at the center of our Galaxy, and the one at the center of the giant elliptical Galaxy M87 [72].

For Sgr A* we use the central values of the observational data reported in [73]: \( M_{\text{Sgr}} = 4.28 \times 10^6 M_\odot \), where \( M_\odot \) is the solar mass, and \( D_{\text{OL}} = 8.32 \text{ kpc} \). Both measurements have systematic and statistical uncertainties in the order of 10\(^{-1} \). Reintroducing the appropriate factors of the Newton constant \( G \) and the speed of light \( c \), we have \( 2M = 2GM_{\text{Sgr}}/c^2 = 1.26 \times 10^{10} \text{ m} \), this is used to compute \( b_\ast = b_\ast / 2M \). Then we obtain \( \theta_\infty = b_\ast / D_{\text{OL}} \), and finally we get \( s \) from equation (32). It is worth noticing the advantage of having redefined all the quantities in the metric in such a way that the mass does not appear explicitly: obtaining results for specific systems from the generic results reported in the previous section only requires introducing the appropriate values of the mass in \( \tilde{b}_\ast \). The results for \( \theta_\infty \) and \( s \) are shown in table 1.

For M87 we use the central values adopted in [74]: \( M = 6.5 \times 10^9 M_\odot \) and \( D_{\text{OL}} = 16.8 \text{ Mpc} \). In geometrized units, this leads to \( 2M = 1.92 \times 10^{13} \text{ m} \). The values of \( \theta_\infty \) and \( s \) are reported in table 2.

The consequences of the results displayed in tables 1 and 2 for the values of \( g_4 \) and \( \phi_0 \) have to be interpreted with some care. We are actually using the redefined quantities \( \tilde{g}_4 \) and \( \tilde{\phi}_0 \), which are given in relation to the value of \( 2M \) for each astrophysical system, so, for instance, \( \tilde{\phi}_0 = 0.02 \) implies a different value of \( \phi_0 \) for each \( M \). Furthermore, we are fixing \( \tilde{g}_4 = g_4 / (4M^2) = 1 \). This implies large values of \( g_4 \), but we also have small values of \( \phi_0 \), so that the combination \( g_4 \phi_0^2 \) remains natural. Indeed, as we noticed before, these parameters satisfy \( g_4 \phi_0^2 = \tilde{g}_4 \tilde{\phi}_0^2 \). If one wishes to fix \( g_4 \sim 1 \), a large \( \phi_0 \)—order \( 2M \)—would be required in order to recover values of \( \tilde{g}_4 \tilde{\phi}_0^2 \) similar to the ones we have used, but the terms of order 1/\( x^2 \) in the metric would still be suppressed by a factor of order \( M^2 \), leading to smaller deviations from a Schwarzschild black hole. Considering that angular measurements from EHT have an uncertainty of about 3 \( \mu \text{as} \), our results show that even for a large coupling \( g_4 \), beyond Horndeski black holes lead to lensing observables that are compatible with current data, although could be excluded in the near future. For instance, if EHT observations of M87 reach an uncertainty smaller than 1 \( \mu \text{as} \), the solutions we analyse would be constrained to \( g_4 \phi_0 \lesssim 0.14 \).

Given the current work on upgrades of EHT, this scenario is not unlikely.
6. Deflection angle for large $x_0$

As we mentioned before, Bozza’s approximation is accurate only near the photon sphere located at $x_m$. For light rays whose closest approach distance $x_0$ is away from $x_m$, we explore the deflection angle under a different approximation. We follow the methodology presented in [18]—later generalized in [75], where using the method introduced by Amore et al [76] and the principle of minimal sensitivity (PMS) [77] to minimize the error, an analytical approximation for the deflection angle is obtained. This methodology, which we briefly review below, is not based on a perturbative expansion, and it describes accurately the physics of our problem almost up to the photon sphere.

Starting from equation (13) for the deflection angle, using a new variable $\frac{\phi}{z}$, and introducing a potential defined in terms of the components of the metric,

$$V(z) = \frac{z^4}{x_0^4} \left[ \frac{C \left( \frac{z}{x_0} \right)}{B \left( \frac{z}{x_0} \right)} - \frac{C^2 \left( \frac{z}{x_0} \right) A(x_0)}{B \left( \frac{z}{x_0} \right) A \left( \frac{z}{x_0} \right) C(x_0)} \right] + \frac{z^2 A(x_0)}{C(x_0)},$$

(35)

the deflection angle can be written as,

$$\alpha(z) = 2 \int_0^1 \frac{dz}{\sqrt{V(1) - V(z)}} = \pi,$$

(36)

where $A, B, C$ are as defined in (34). A note regarding the boundary conditions is pertinent: in [18], it is assumed that $\lim_{z \to 0} C(x_0/z) = x_0^2/\epsilon^2$, while $A(z_0/z) \to 1$ and $B(z_0/z) \to 1$ in the same limit, so that $V(0) = 0$. In our case, $\lim_{z \to 0} C(x_0/z) = s_0^{-1} x_0^2/\epsilon^2$, leading to $V(0) = A(x_0)(-s_0^{-1} + s_0)$, i.e. $V(0)$ has a constant value. As we can see, this constant value cancels out in equation (36); however, it is important to take it into account in the analysis that we describe below.

Notice that this is not the same $z$ that we used in previous sections. Since the contents of these sections do not mix, we hope this is not misleading.
The integral (36) can be solved analytically for particular cases, e.g. Schwarzschild [5, 78] and Reissner–Nordström [79]. For more general cases, the methodology presented in [18] allows to find an analytical approximation. Under the assumption that the metric is locally flat at infinity, the potential (35) is approximated as a power series in $z$,

$$V(z) \approx V_k(z) = \sum_{n=0}^{k} v_n z^n, \quad (37)$$

notice that unlike [18] we are including the constant term, $v_0$, in order to account correctly for the behaviour of $V(z)$ at $z = 0$, and we are already recognising that the power series may need to be truncated at some finite value $k$.

Using a nonperturbative method based on a linear delta expansion [76, 80], Amore et al obtain (see [18, 75] for details)

$$a^{(1)}_{\text{PMS}} = \sqrt{\frac{\pi^3}{2 \rho(1)}} - \pi, \quad (38)$$

where (1) indicates the order of the approximation, and $\rho(z)$ is given by

$$\rho(z) = \sqrt{\pi} \sum_{n=0}^{k} v_n \left( \Gamma(n/2 + 1/2) \right) z^n, \quad (39)$$

where $\Gamma$ is the Gamma function.

In some cases, such as Schwarzschild and Reissner–Nördstrom, the power series for $V(z)$ is finite, and the set of coefficients $v_n$ can be exactly identified. In other cases, even if the metric is known analytically, the power series is not finite, and a truncation needs to be made,
this happens, for instance, for the JNW metric and for Einstein–Born–Infeld black holes. In our case, the metric is not known analytically, so we need to perform a numerical fit of the potential \( V(z) \) to a power series in \( z \), which we choose to truncate at \( n = 6 \). Let us explain our methodology.

- First we select a value of \( \phi_0 \), then we construct \( V(z) \) from equation (35) using the numerical solutions for the metric corresponding to this \( \phi_0 \). Since \( V(z) \) takes \( x_0 \) as a parameter, for each \( \phi_0 \) we get a set of profiles for \( V(z) \), each one associated to a value of \( x_0 \) in the range \( x_m < x_0 \lesssim 50 \), with \( x_m \) computed from equation (12). Notice that \( x_0/z \) cannot be larger than the maximum \( x \) for which the numerical solution is known, so, an upper limit on \( x_0 \) translates into a lower limit in \( z \). Now, for each \( x_0 \), we fit the numerical profile of \( V(z) \) to a power series in \( z \) truncated at \( k = 6 \), \( V_k(z) \). Figure 5 shows the results for \( \phi_0 = 0.02 \), with \( x_0 = x_m \). The relative percentage error of the numerical fit, \( |V_k(z) - V(z)|/V(z) \times 100\% \), stays below 1%.

- With the coefficients \( v_i \) at hand, we use equations (38) and (39) to compute the deflection angle for each \( x_0 \).

We repeat the steps above for different values of \( \phi_0 \) in the range \( 0 \leq \phi_0 \leq 0.32 \). Figure 6 shows the deflection angle as a function of \( x_0 \) for different choices of \( \phi_0 \). As we can observe, \( \phi_0 \)—or \( g_{45}\phi_0^2 \) as we infer from the angular part of the metric—modifies the asymptotic behaviour of the deflection angle, making it negative for sufficiently large \( x_0 \). A negative deflection angle indicates a divergent lens, this has been reported in [81–84] for wormholes and massless braneworld black holes. In [85, 86] some consequences of negative deflection angles are discussed.

Phenomenologically, the weak field deflection angle is constrained by Solar System observations (e.g. VLBI [2]) to be within \( \sim 10^{-4} \) of the GR prediction. This would imply a tight
Figure 7. Comparison between the PMS-analytic (L. approx., equation (41), dashed lines) and the PMS-numerical approaches (solid lines) for $\tilde{\phi}_0 = 0.1$ and $\tilde{\phi}_0 = 0.2$. For large values of $x_0$ both approximations are consistent.

Given that the asymptotic deflection angle is controlled by the angular part of the metric (34), it is possible to use a further analytical approximation upon the PMS first order result by replacing equation (34) with a Schwarzschild metric, but with the angular component rescaled by $s_0^{1/2}$. Following [18], this approach leads to

$$\alpha_{\text{PMS}}^{(1)} = \pi \left( \frac{\sqrt{s_0}}{1 - \frac{2}{x_0}} - 1 \right).$$

(40)

We remind the reader that $s_0 = 1 - 3\tilde{g}_{44}\tilde{\phi}_0^2$. If $x_0 \gg 1$, we can write

$$\alpha_{\text{PMS}}^{(1)} \approx \pi \left( \sqrt{s_0} - 1 \right) + \frac{2\sqrt{s_0}}{x_0}.$$

(41)

Notice that for $s_0 = 0$ the weak field GR deflection angle is recovered. Figure 7 shows that this analytical approach is consistent with the results of a numerical PMS approximation.

To conclude this section, let us put together the two numerical methods to compute the deflection angle presented so far. In Bozza’s approximation, we can obtain the deflection angle for a given $x_0$ with the help of equations (15) and (16). The results are shown in figure 8 for Schwarzschild and for $\tilde{\phi}_0 = 0.2$. On the other hand, using the PMS method described in this section, we can also obtain the deflection angle for a given $x_0$, the results are shown in the same figure. As expected, these approximations disagree close to the photon sphere and
also for large distances. However, it is interesting to note that, together, they describe accurately the deflection angle over all the range of \( x_0 \): when Bozza’s approximation begins to fail, the PMS method starts to give good results. To see this, we include in figure 8 the exact solution for Schwarzschild, and a full numerical result for \( \tilde{\phi}_0 = 0.2 \) obtained by direct integration of equation (13). It is worth mentioning that direct integration is more computationally expensive than the numerical implementation of Bozza’s method: for each \( \tilde{\phi}_0 \), Bozza’s method reduces to computing only one integral, while a full numerical result requires one integral for each \( x_0 \).

**7. Discussion**

Gravitational deflection of light is an important test for modified theories of gravity. Here we have focussed on models of modified gravity that fit within the DHOST category. Thanks to screening mechanisms, some of these theories admit an exact Schwarzschild solution, thus automatically recovering the basic predictions of GR for weak and strong deflection of light. However, on general grounds, the DHOST Lagrangian may contain terms that lead to modifications of the Schwarzschild metric, and in many cases to solutions that cannot be obtained in exact form but only under some approximations or numerical treatment. In this work we studied one of these models, which besides modifying the radial and time components of the Schwarzschild metric, also modifies the angular part.

The gravitational deflection angle in the strong field limit is given by an integral that diverges at the photon sphere, and that can be solved analytically only in a few cases. This motivated the development of strong field approximations. In particular, Bozza’s approximation has two advantages that are relevant for our work: first, it is given in terms of coefficients that are
directly related to observables, second, the same set of coefficients—that depend on one integral—is accurate over a certain interval for $x_0$ away from the photon sphere. Numerically, we could integrate the exact expression for the deflection angle, but this can be quite inefficient, since one integral is required for each closest approach distance $x_0$ that we want to investigate.

In view of the above, we investigated a hybrid analytic-numerical method based on Bozza’s approach. In the Schwarzschild limit, we verified that our results near the photon sphere agree with exact and fully analytic approximations. For the non-Schwarzschild solutions of the DHOST model we consider, we compared our results to full numerical integration, finding good agreement as well. These results are relevant because they show that a hybrid method is a good replacement for full integration. Moreover, this method can be used for any static, spherically symmetric, asymptotically locally flat solutions that are known only numerically, as is the case for several solutions in modified gravity. In this spirit, we also investigated a hybrid method to compute the deflection angle away from the photon sphere, this time with the analytic part based on the PMS approach. We verified that for $x_0 \gtrsim 1.5 x_m$, this method is consistent with exact or full numerical results. In summary, we have shown that it is possible to compute the deflection angle (13) over the entire range of $x_0$ as the union of two hybrid approximations,

$$\alpha(x_0) = \begin{cases} 
\text{Bozza-numerical approach} & \text{for } x_m \lesssim x_0 \lesssim 1.5 x_m \\
\text{PMS-numerical approach} & \text{for } x_0 \gtrsim 1.5 x_m.
\end{cases} \quad (42)$$

Furthermore, we have shown that for $x_0 \gtrsim 20 x_m$, an analytic weak field approximation based on the Schwarzschild metric with a rescaled angular component correctly accounts for the effects of a constant angular deficit in the metric.

Regarding the phenomenology of the DHOST model we consider in this work, using the strong deflection results we calculated the angular position $\theta_\infty$ where lensed images accumulate around a supermassive black hole, as well as the separation $s$ between these images and the outermost one. Specifically, we considered Sagittarius A* and M87, finding that, for the range of parameters that we use, the deviations of $\theta_\infty$ from its Schwarzschild value are in the order of micro arc seconds. Current observations by EHT constrain the angular diameter of M87’s shadow with an uncertainty of $\pm 3 \mu$as. In the near future, this type of observations could impose constraints on non-perfectly screened modified gravity black holes at order $10^{-1}$ in the relative size of the corrections. Although this is weaker than the constraints in the Solar System—order $10^{-4}$, strong deflection tests a completely different regime of the theory and provides complementary information that can be used to further reduce the space of viable modified gravity models.

Acknowledgments

AR and BR acknowledge support by CONACyT graduate scholarships Nos. 570326 and 733442.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.
ORCID iDs

Javier Chagoya 🐦 https://orcid.org/0000-0001-6780-5127
C Ortiz 🐦 https://orcid.org/0000-0003-3220-4203
Armando A Roque 🐦 https://orcid.org/0000-0002-3290-6072

References

[1] Ellis R S 2010 Gravitational lensing: a unique probe of dark matter and dark energy Phil. Trans. R. Soc. A 368 967–87
[2] Shapiro S S, Davis J L, Lebach D E and Gregory J S 2004 Measurement of the solar gravitational deflection of radio waves using geodetic very-long-baseline interferometry data, 1979–1999 Phys. Rev. Lett. 92 121101
[3] Schmidt F 2008 Weak lensing probes of modified gravity Phys. Rev. D 78 043002
[4] Pratten G, Munshi D, Valageas P and Brax P 2016 3D weak lensing: modified theories of gravity Phys. Rev. D 93 103524
[5] Galton Darwin C 1959 The gravity field of a particle Proc. R. Soc. A 249 180–94
[6] Cunningham C T and Bardeen J M 1972 The optical appearance of a star orbiting an extreme Kerr black hole Astrophys. J. 173 L137
[7] Luminet J-P 1979 Image of a spherical black hole with thin accretion disk Astron. Astrophys. 75 228–35
[8] Falcke H, Melia F and Agol E 2000 Viewing the shadow of the black hole at the galactic center Astrophys. J. 528 L13–6
[9] Doeleman S S et al 2001 Structure of Sagittarius A* at 86 GH[CLC]z[CLC] using VLBI closure quantities Astron. J. 121 2610
[10] Akiyama K et al 2019 First M87 event horizon telescope results. I. The shadow of the supermassive black hole Astrophys. J. 875 L1
[11] Virbhadra K S, Narasimha D and Chitre S M 1998 Role of the scalar field in gravitational lensing Astron. Astrophys. 337 1–8
[12] Wei S-W, Yang K and Liu Y-X 2015 Black hole solution and strong gravitational lensing in Eddington-inspired Born–Infeld gravity Eur. Phys. J. C 75 253
[13] Zhao S-S and Xie Y 2016 Strong field gravitational lensing by a charged Galileon black hole J. Cosmol. Astropart. Phys. 2016007
[14] Bozza V 2002 Gravitational lensing in the strong field limit Phys. Rev. D 66 103001
[15] Amore P and Arceo Diaz S 2006 Analytical formulas for gravitational lensing Phys. Rev. D 73 083004
[16] Zumalacárregui M and García-Bellido J 2014 Transforming gravity: from derivative couplings to matter to second-order scalar–tensor theories beyond the Horndeski Lagrangian Phys. Rev. D 89 064046
[17] Horndeski G W 1974 Second-order scalar–tensor field equations in a four-dimensional space Int. J. Theor. Phys. 10 363–84
[18] Woodard R 2015 Ostrogradsky’s theorem on Hamiltonian instability Scholarpedia 10 32243
[19] Langlois D and Noui K 2016 Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability J. Cosmol. Astropart. Phys. 2016 034
[20] Ben Achour J, Crisostomi M, Koyama K, Langlois D, Noui K and Tasinato G 2016 Degenerate higher order scalar–tensor theories beyond Horndeski up to cubic order J. High. Energy Phys. JHEP12(2016)100
[25] Crisostomi M, Koyama K and Tasinato G 2016 Extended scalar–tensor theories of gravity *J. Cosmol. Astropart. Phys.* **JCAP04**(2016)044

[26] Clifton T, Ferreira P G, Padilla A and Skordis C 2012 Modified gravity and cosmology *Phys. Rep.* **513** 1–189

[27] Charmousis C, Copeland E J, Padilla A and Saffin P M 2012 Self-tuning and the derivation of a class of scalar–tensor theories *Phys. Rev. D* **85** 104040

[28] Charmousis C, Copeland E J, Padilla A and Saffin P M 2012 General second order scalar–tensor theory, self tuning, and the Fab Four *Phys. Rev. Lett.* **108** 051101

[29] Babichev E, Charmousis C, Langlois D and Saito R 2015 Beyond fab four *Class. Quantum Grav.* **32** 242001

[30] Babichev E and Esposito-Farese G 2017 Cosmological self-tuning and local solutions in generalized Horndeski theories *Phys. Rev. D* **95** 024020

[31] Will C M 2006 The confrontation between general relativity and experiment *Living Rev. Rel.* **9** 3

[32] Khoury J and Weltman A 2004 Chameleon fields: awaiting surprises for tests of gravity in space *Phys. Rev. Lett.* **93** 171104

[33] Khoury J and Weltman A 2004 Chameleon cosmology *Phys. Rev. D* **69** 044026

[34] Hinterbichler K and Khoury J 2010 Symmetron fields: screening long-range forces through local symmetry restoration *Phys. Rev. Lett.* **104** 231301

[35] Hinterbichler K, Khoury J, Levy A and Matas A 2011 Symmetron cosmology *Phys. Rev. D* **84** 103521

[36] Babichev E, Deffayet C and Ziour R 2009 k-Mouflage gravity *Int. J. Mod. Phys.* **D18** 2147–54

[37] Lam H, Nicolis A and Stubbs C 2009 Equivalence principle implications of modified gravity models *Phys. Rev. D* **80** 104002

[38] Damour T and Nordtvedt K 1993 Tensor–scalar cosmological models and their relaxation toward general relativity *Phys. Rev. D* **48** 3436–50

[39] Fujii Y and Maeda K 2007 *The Scalar–Tensor Theory of Gravitation* (Cambridge Monographs on Mathematical Physics vol 7) (Cambridge: Cambridge University Press)

[40] Kase R, Tsujikawa S and De Felice A 2016 Cosmology with a successful Vainshtein screening in theories beyond Horndeski *Phys. Rev. D* **93** 024007

[41] Cisterna A, Delsate T and Rinaldi M 2015 Neutron stars in general second order scalar–tensor theory: the case of nonminimal derivative coupling *Phys. Rev. D* **92** 044050

[42] Babichev E, Koyama K, Langlois D, Saito R and Sakstein J 2016 Relativistic stars in beyond Horndeski theories *Class. Quantum Grav.* **33** 235014

[43] Babichev E, Charmousis C and Lehébel A 2016 Black holes and stars in Horndeski theory *Class. Quantum Grav.* **33** 154002

[44] Maselli A, Silva H O, Minamitsuji M and Berti E 2016 Neutron stars in Horndeski gravity *Phys. Rev. D* **93** 124056

[45] Babichev E, Charmousis C and Lehébel A 2017 Asymptotically flat black holes in Horndeski theory and beyond *J. Cosmol. Astropart. Phys.* JCAP04(2017)027

[46] Chagoya J and Tasinato G 2018 Compact objects in scalar–tensor theories after GW170817 *J. Cosmol. Astropart. Phys.* JCAP08(2018)006

[47] Kobayashi T and Hiramatsu T 2018 Relativistic stars in degenerate higher-order scalar–tensor theories after GW170817 *Phys. Rev. D* **97** 104012

[48] Ben Achour J and Liu H 2019 Hairy Schwarzschild-(A)dS black hole solutions in degenerate higher order scalar–tensor theories beyond shift symmetry *Phys. Rev. D* **99** 064042

[49] Achour J B, Liu H and Mukohyama S 2020 Hairy black holes in DHOST theories: exploring disformal transformation as a solution generating method *J. Cosmol. Astropart. Phys.* JCAP02(2020)023

[50] Ben Achour J, Liu H, Motohashi H, Mukohyama S and Noui K 2020 On rotating black holes in DHOST theories *J. Cosmol. Astropart. Phys* JCAP11(2020)001

[51] Crisostomi M and Koyama K 2018 Vainshtein mechanism after GW170817 *Phys. Rev. D* **97** 021301

[52] Hirano S, Kobayashi T and Yamauchi D 2019 Screening mechanism in degenerate higher-order scalar–tensor theories evading gravitational wave constraints *Phys. Rev. D* **99** 104073

[53] Babichev E and Lehébel A 2018 The sound of DHOST *J. Cosmol. Astropart. Phys.* JCAP12(2018)027
[84] Eiroa E F and Sendra C M 2012 Gravitational lensing by massless braneworld black holes *Phys. Rev. D* 86 083009

[85] Izumi K, Hagiwara C, Nakajima K, Kitamura T and Asada H 2013 Gravitational lensing shear by an exotic lens object with negative convergence or negative mass *Phys. Rev. D* 88 024049

[86] Kitamura T, Nakajima K and Asada H 2013 Demagnifying gravitational lenses toward hunting a clue of exotic matter and energy *Phys. Rev. D* 87 027501