Optics of a Faraday-active Mie Sphere

D. Lacoste, B. A. Van Tiggelen

Laboratoire de Physique et Modélisation des Milieux Condensés, Maison des Magistères,
B.P. 166, 38042 Grenoble Cedex 09, France

G.L.J.A Rikken, and A. Sparenberg

Grenoble High Magnetic Field Laboratory, Max-Planck Institut
für Festkörperforschung/C.N.R.S, B.P. 166, 38042 Grenoble Cedex 9, France

Abstract

We present an exact calculation for the scattering of light from a single sphere made of a Faraday-active material, into first order of the external magnetic field. When the size of the sphere is small compared to the wavelength the known T-matrix for a magneto-active Rayleigh scatterer is found. We address the issue whether or not there is a so called Photonic Hall Effect - a magneto-transverse anisotropy in light scattering - for one Mie scatterer. In the limit of geometrical optics, we compare our results to the Faraday effect in a Fabry-Perot etalon.

PACS: 42.25.Fx, 78.20.Ls

1. Introduction

Several reasons exist why one wishes to understand light scattering from a dielectric sphere made of magneto-active material. Single scattering is the building block for multiple scattering. Many experiments have been done with diffuse light in a magnetic field. Though qualitatively very useful, it turns out that a theory for point-like scatterers in a magnetic field, as first developed by MacKintosh and John and later refined by Van Tiggelen, May-
nard and Nieuwenhuizen\[4], does not always describe observations quantitatively, for the
evident reason that experiments do not contain “small” scatterers. This paper addresses
light scattering from one sphere of any size in a homogeneous magnetic field.

The model of Rayleigh scatterers has been successfully used to describe specific proper-
ties of multiple light scattering in magnetic fields such as coherent backscattering and the
Photonic Hall Effect. In the following section we present the perturbation approach on
which our work is based, which will allow us in section 3 to compute the T-matrix for this
problem. With this tool, we will be able to answer in section 4 the issue concerning the
Photonic Hall Effect.

2. Perturbation theory

We consider the light scattered by one dielectric sphere made of a Faraday-active medium
embedded in an isotropic medium with no magneto-optical properties, using a perturbative
approach to first order in magnetic field.

In this paper we set \( c_0 = 1 \). In a magnetic field, the refractive index of the sphere is a
tensor of rank two. It depends on the distance to the center of the sphere \( r \), which is given
a radius \( a \) via the Heaviside function \( \Theta(r - a) \), that equals to 1 inside the sphere and 0
outside,

\[
\varepsilon(B, r) - I = [(\varepsilon_0 - 1)I + \varepsilon_F \Phi] \Theta(|r| - a). \tag{1}
\]

In this expression, \( \varepsilon_0 = m^2 \) is the value of the normal isotropic dielectric constant of the
sphere of relative index of refraction \( m \) ( which can be complex ) and \( \varepsilon_F = 2mV_0/\omega \) is a
coupling parameter associated with the amplitude of the Faraday rotation ( \( V_0 \) being the
Verdet constant and \( \omega \) the frequency ). We introduce the antisymmetric hermitian tensor
\( \Phi_{ij} = i\varepsilon_{ijk}B_k \). Except for \( \varepsilon_F \), the Mie solution depends on the dimensionless size parameters
\( x = ka \) with \( k = \omega/c_0 \) and \( y = mx \). In this paper, we restrict ourselves to non-absorbing
media, so that \( m \) and \( \varepsilon_F \) are real-valued.
Noting that the Helmholtz equation is formally analogous to a Schrödinger equation with potential \( V(r, \omega) = [I - \varepsilon(B, r)] \omega^2 \) and energy \( \omega^2 \), the T-matrix is given by the following Born series:\[
T(B, r, \omega) = V(r, \omega) + V(r, \omega) \cdot G_0 \cdot V(r, \omega) + \ldots
\] (2)

Here \( G_0(\omega, p) = 1/(\omega^2 I - p^2 \Delta_p) \) is the free Helmholtz Green’s function and \((\Delta_p)_{ij} = \delta_{ij} - p_i p_j/p^2\). We will call \( T^0 \) the part of \( T \) that is independent of the magnetic field and \( T^1 \) the part of the T-matrix linear in \( B \). It follows from Eq. (2) that:

\[
T^1 = (1 - \varepsilon_0)\varepsilon_F \Theta T^0 \cdot \Phi \cdot (I + G_0 \cdot T^0).
\] (3)

We need to introduce the unperturbed eigenfunctions \( \Psi_{\sigma, k}^\pm(r) \) of the conventional Mie problem. These eigenfunctions represent the electric field at the point \( r \) for an incident plane wave \(|\sigma, k\rangle\) along the direction \( k \) with an helicity \( \sigma \). This eigenfunction is ”outgoing” for \( \Psi_{\sigma, k}^+ \) and ”ingoing” for \( \Psi_{\sigma, k}^- \) according to the definition of the outgoing and ingoing free Helmholtz Green’s function:

\[
|\Psi_{\sigma, k}^\pm\rangle = (I + G^\pm_0 \cdot T^0) |\sigma, k\rangle.
\]

For our free Helmholtz Green’s function, this implies:\[
\Psi_{\sigma, k}^{-\ast} = (-1)^{1+\sigma} \Psi_{\sigma, k}^+(r).
\] (4)

We denote by \( k \) the incident direction and \( k' \) the scattered direction. With these notations, it is possible to obtain from Eq.(3),

\[
T^1_{k\sigma, k'\sigma'} = \varepsilon_F \omega^2 < \Psi_{\sigma, k}^- | \Theta \Phi | \Psi_{\sigma', k'}^+ >.
\] (5)

This equation can also be obtained from standard first-order Rayleigh-Schrödinger perturbation theory\[\footnote{The equation can also be obtained from standard first-order Rayleigh-Schrödinger perturbation theory.} \].

Two important symmetry relations are found to be satisfied by our T-matrix. The first one is the parity and the second one is the reciprocity:
\[ T_{-k,\sigma,-k,\sigma'}(B) = T_{k,\sigma',k,\sigma'}(B) \] (6)

\[ T_{-k',\sigma',-k,\sigma}(B) = T_{k,\sigma,k',\sigma'}(B) \] (7)

3. T matrix for Mie scattering

In order to separate the radial and the angular contributions in Eq. (5), we expand the Mie eigenfunction \( \Psi_{\sigma,k}^+ \) in the basis of the vector spherical harmonics \( \Phi_{JM} \),

\[ \Psi_{\sigma,k}^+(r) = \frac{2\pi}{\rho} i^{J+1} Y_{JM}^\lambda(\hat{r}) f^J_{\lambda,\lambda'}(r) Y_{JM}^{\lambda'}(\hat{k}) \cdot \chi_{\sigma}. \] (8)

In this definition, \( \rho = kmr \), the \( \chi_{\sigma} \) are the eigenvectors of the spin operator in the circular basis associated with the direction \( k \), and implicit summation over the repeated indices \( J, M, \lambda \) and \( \lambda' \) has been assumed (the indices \( \lambda \) and \( \lambda' \) for the components of the field can take three values, one being longitudinal and two of them being perpendicular to the direction of propagation).

Because of the presence of the function \( \Theta \) in Eq. (5), we only have to consider the field inside the sphere, whose main features are contained in the radial function \( f^J_{\lambda,\lambda'}(r) \). This matrix \( f^J_{\lambda,\lambda'} \) is known in terms of the transmission coefficients \( c_J \) and \( d_J \) of Ref. 8, and of the Ricatti-Bessel function \( u_J(\rho) \). We found the following expression:

\[ f^J(\rho) = m \begin{pmatrix}
-\imath u_J'(\rho)c_J & 0 & 0 \\
-\imath u_J'(\rho)c_J \sqrt{J(J+1)/\rho} & 0 & 0 \\
0 & 0 & u_J(\rho)d_J
\end{pmatrix}. \]

The three vectors \( \chi_{\sigma} \) - similar to the \( \chi_{\sigma}' \) but associated with the \( z \)-axis - are a convenient basis for this problem since they are the eigenvectors of the operator \( \Phi \) with eigenvalue \( -\sigma \), provided we choose the \( z \)-axis along \( B \), which we will do in what follows. Eq. (5) is simplified by this choice and the angular integration leads eventually to:

\[ \int Y_{J_1M_1}^{\lambda_1}(\hat{r}) \cdot \Phi \cdot Y_{J_2M_2}^{\lambda_2}(\hat{r}) d\Omega = \delta_{J_1J_2} \delta_{M_1M_2} Q_{\lambda_1\lambda_2}(J_1, M_1), \]
where \( Q \) is the matrix

\[
Q(J, M) = -MB \begin{pmatrix}
\frac{1}{J(J+1)} & \frac{1}{\sqrt{J(J+1)}} & 0 \\
\frac{1}{\sqrt{J(J+1)}} & 0 & 0 \\
0 & 0 & \frac{1}{J(J+1)}
\end{pmatrix},
\]

and \( B \) being the absolute value of the applied magnetic field. The linear dependence on the magnetic quantum number \( M \) can be expected for an effect like the Faraday rotation, affecting left and right circularly polarized light in an opposite way similar to Zeeman splitting. The radial integration can be done using a method developed by Bott et al.\(^9\)

\[
T_{k,k'}^1 = \frac{16\pi}{\omega} W \sum_{J,M} (-M) \left[ C_J Y_{J,M}(\hat{k}) Y_{J,M}^* \left(\hat{k}'\right) + D_J Y_{J,M}(\hat{k}) Y_{J,M}^* \left(\hat{k}'\right) \right],
\]

with the dimensionless parameter:

\[
W = V_0 B \lambda,
\]

and the coefficients:

\[
C_J = -\frac{\epsilon_j J \epsilon \epsilon^*_{\parallel}}{J(J+\infty) (\epsilon \epsilon^* - \epsilon^* \epsilon)} \left( \frac{A^*_J}{\epsilon^*_{\parallel}} - \frac{A_J}{\epsilon^*} \right),
\]

\[
D_J = -\frac{\epsilon_j J \epsilon \epsilon^*_{\parallel}}{J(J+\infty) (\epsilon \epsilon^* - \epsilon^* \epsilon)} \left( \frac{A^*_J}{\epsilon^*_{\parallel}} - \frac{A_J}{\epsilon^*} \right),
\]

with \( A_J(y) = u_J'(y)/u_J(y) \) and \( B \) the amplitude of the magnetic field directed along the unit vector \( \hat{B} \). Absorption in the sphere is still allowed. We will consider the limiting case of a perfect dielectric sphere with no absorption ( \( \Im(m) \to 0 \) ). Using l'Hospital’s rule in Eqs. (11) and (12), we obtain immediately for this case:

\[
C_J = -\frac{\epsilon_j J \epsilon \epsilon^*_{\parallel}}{J(J+\infty) (\epsilon \epsilon^* - \epsilon^* \epsilon)} \left( \frac{A^*_J}{\epsilon^*_{\parallel}} - \frac{J(J+\infty)}{\epsilon^*} + \infty + A_J^\epsilon \right),
\]

\[
D_J = -\frac{\epsilon_j J \epsilon \epsilon^*_{\parallel}}{J(J+\infty) (\epsilon \epsilon^* - \epsilon^* \epsilon)} \left( \frac{A^*_J}{\epsilon^*_{\parallel}} - \frac{J(J+\infty)}{\epsilon^*} + \infty + A_J^\epsilon \right).
\]
A. T-matrix without magnetic field

For future use, we need the on-shell T-matrix of the conventional Mie-problem. It is given by a formula analogous to Eq. (10) where $\mathcal{C}_J$ and $\mathcal{D}_J$ are replaced by the Mie coefficients $a_J$ and $b_J$, and with $M = 1$. Because of rotational invariance of the scatterer, it’s clear that the final result only depends on the scattering angle $\theta$ which is the angle between $\mathbf{k}$ and $\mathbf{k}'$ (see Fig. 1). Therefore, we get in the circular basis (associated with the indices $\sigma$ and $\sigma'$):

$$T_{\sigma\sigma'}^0 = \frac{2\pi}{i\omega} \sum_{J \geq 1} \frac{2J + 1}{J(J + 1)} (a_J^* + \sigma\sigma' b_J^*) \left[ \pi_{J,1}(\cos \theta) + \sigma\sigma' \tau_{J,1}(\cos \theta) \right]. \quad (15)$$

In this formula, the polynomials $\pi_{J,M}$ and $\tau_{J,M}$ are defined in terms of the Legendre polynomials $P_{M,J}$ by

$$\pi_{J,M}(\cos \theta) = \frac{M}{\sin \theta} P_{J,M}^M(\cos \theta), \quad \tau_{J,M}(\cos \theta) = \frac{d}{d\theta} P_{J,M}^M(\cos \theta). \quad (16)$$

B. $T_{kk'}^1$ when $\mathbf{\hat{k}} \neq \mathbf{\hat{k}}'$

It remains to express the vector spherical harmonics in Eq. (10) in terms of the natural angles of the problem in the presence of a magnetic field. The latter breaks rotational invariance. Because $T^1$ is linear in $\mathbf{\hat{B}}$, it can be constructed by considering three special cases for the direction of $\mathbf{\hat{B}}$. If $\mathbf{\hat{k}} \neq \mathbf{\hat{k}}'$, we can decompose the unit vector $\mathbf{\hat{B}}$ in the non-orthogonal but complete basis of $\mathbf{\hat{k}}$, $\mathbf{\hat{k}}'$ and $\mathbf{\hat{g}} = \mathbf{\hat{k}} \times \mathbf{\hat{k}}' / |\mathbf{\hat{k}} \times \mathbf{\hat{k}}'|$, and this results in:

$$T_{kk'}^1 = \frac{(\mathbf{\hat{B}} \cdot \mathbf{\hat{k}})(\mathbf{\hat{K}} \cdot \mathbf{\hat{k}}') - \mathbf{\hat{B}} \cdot \mathbf{\hat{k}}'}{(\mathbf{\hat{k}} \cdot \mathbf{\hat{k}}')^2 - 1} T_{\mathbf{\hat{B}} = \mathbf{\hat{k}}'}^1 + \frac{(\mathbf{\hat{B}} \cdot \mathbf{\hat{k}}')(\mathbf{\hat{K}} \cdot \mathbf{\hat{k}}) - \mathbf{\hat{B}} \cdot \mathbf{\hat{k}}}{(\mathbf{\hat{k}} \cdot \mathbf{\hat{k}}')^2 - 1} T_{\mathbf{\hat{B}} = \mathbf{\hat{k}}}^1 + (\mathbf{\hat{B}} \cdot \mathbf{\hat{g}}) T_{\mathbf{\hat{B}} = \mathbf{\hat{g}}}^1, \quad (17)$$

with

$$T_{\sigma\sigma'}^1(\mathbf{\hat{B}} = \mathbf{\hat{k}}) = \frac{2W}{\omega} \sum_{J \geq 1} \frac{2J + 1}{J(J + 1)} (-\sigma) \left( \mathcal{C}_J + \sigma\sigma' \mathcal{D}_J \right) \left[ \pi_{J,\infty}(\cos \theta) + \sigma\sigma' \tau_{J,\infty}(\cos \theta) \right], \quad (18)$$

$$T_{\sigma\sigma'}^1(\mathbf{\hat{B}} = \mathbf{\hat{k}}') = \frac{2W}{\omega} \sum_{J \geq 1} \frac{2J + 1}{J(J + 1)} (-\sigma') \left( \mathcal{C}_J + \sigma\sigma' \mathcal{D}_J \right) \left[ \pi_{J,\infty}(\cos \theta) + \sigma\sigma' \tau_{J,\infty}(\cos \theta) \right], \quad (19)$$

$$T_{\sigma\sigma'}^1(\mathbf{\hat{B}} = \mathbf{\hat{g}}) = \frac{4iW}{\omega} \sum_{J \geq 1} \sum_{M > 0} \frac{2J + 1}{J(J + 1)} M \sin(M\theta) \frac{(J - M)!}{(J + M)!} \left( \sigma\sigma' \mathcal{C}_J + \mathcal{D}_J \right) \left[ \pi_{J,M}^\infty(l) + \sigma\sigma' \tau_{J,M}^\infty(l) \right]. \quad (20)$$
C. Particular case for $T^1$ for forward scattering

The treatment in (3.2) becomes degenerate when $\hat{k}$ and $\hat{k}'$ are parallel, *i.e.* in forward scattering. In this case, $\hat{B}$ can still be expressed in a basis made of $\hat{k}$ and of two vectors perpendicular to $\hat{k}$. The contribution of these last two vectors must have the same form as in Eq. (20) for $\theta = 0$. Hence there is no such contribution and we find

$$T_{k\sigma,k'\sigma'}^1 = \delta_{\sigma\sigma'} (\hat{B} \cdot \hat{k})(-\sigma) \frac{2W}{\omega} \sum_{J \geq 1} (2J + 1) (C_J + D_J).$$

(21)

In Fig. 2 we plotted real and imaginary part of this expression in units of $W$ as a function of the size parameter $x$ for $\sigma = -1$ and $\hat{B} = \hat{k}$. The forward-scattering amplitude has an important application in inhomogeneous media, namely as the complex average dielectric constant.

D. Optical Theorem

We will check our formula on energy conservation as expressed by the Optical Theorem,

$$- \Im \frac{m(T_{k\sigma,k\sigma})}{\omega} = \sum_{\sigma'} \int d\Omega_{k'} |T_{k\sigma,k'\sigma'}^1|^2 \frac{1}{(4\pi)^2}.$$

(22)

To first order in magnetic field, the r.h.s of this equation equals

$$\frac{1}{8\pi^2} \sum_{\sigma'} \int d\Omega_{k'} \Re e (T_{0,k\sigma,k'\sigma'}^0 T_{k\sigma,k'\sigma'}^1 \ast).$$

If we assume that $\hat{B} \parallel \hat{k}$, we can compute this using Eqs. (18), (15), and the following orthogonality relations for the polynomials $\pi_{J1}$ and $\tau_{J1}$ (which we denote as $\pi_J$ and $\tau_J$):

$$\int d(\cos \theta) [\pi_J(\cos \theta) \pi_K(\cos \theta) + \tau_J(\cos \theta) \tau_K(\cos \theta)] = 0,$$

$$\int d(\cos \theta) [\pi_J(\cos \theta) \pi_K(\cos \theta) + \tau_J(\cos \theta) \tau_K(\cos \theta)] = \frac{2J^2(J + 1)^2}{2J + 1} \delta_{JK}.$$

(23)

The l.h.s of Eq. (22), is obtained from Eq. (21). The Optical Theorem provides us a relation between Mie coefficients, which we can actually prove from their definitions:
\[ \Re(a_j^* c_j^2 \frac{2}{k}) = \Im(c_j^2), \]  
(24)

and

\[ \Re(b_j^* d_j^2 \frac{2}{k}) = \Im(d_j^2). \]  
(25)

4. Magneto-transverse Scattering

From the knowledge of the matrix \( T^1 \), we can compute how the magnetic field affects the differential scattering cross section summed over polarizations as a function of the scattering angle. Only the diagonal part of this matrix in a basis of linear polarization will affect the scattering cross section since we consider only terms to first order in magnetic field. Therefore only the third contribution of Eq. (17) plays a role, which means that the effect will be maximum when \( \hat{B} \parallel \hat{k} \times \hat{k}' \) (the typical Hall geometry). Indeed symmetry implies that the magneto-scattering cross section should be the product of det(\( \hat{B}, \hat{k}, \hat{k}' \)) and of some function that entirely depends on the scattering angle \( \theta \).

We choose to normalize this magneto scattering cross section by the total scattering cross section in the absence of magnetic field,

\[
\frac{2 \sum_{\sigma \sigma'} \Re(T_{\sigma \sigma'}^0 T_{\sigma \sigma'}^{1*})}{\int_0^{2\pi} d\phi \int_0^{\pi} d\cos \theta \sum_{\sigma \sigma'} |T_{\sigma \sigma'}^0|^2} = -\sin \phi F(\theta)
\]  
(26)

The so called Photonic Hall effect (P.H.E) is a manifestation of a magnetically induced transverse current in the light transport which has similarities to the Hall effect known for the transport of electrons. In an experiment on the P.H.E, one measures the difference in scattered light from two opposite directions both perpendicular to the incident direction of light and to the applied magnetic field. The P.H.E is a manifestation of the anisotropy of light scattering due to a magnetic field in the regime of multiple light scattering.

Although experiments dealt with multiple scattering so far, it is interesting to see if a net magneto-transverse scattering persists for only one single scatterer. In Figs. 3,4, the
magnetic field is perpendicular to the plane of the figure, the incident light is along the \( x \)-axis. A typical measurement of the magneto-transverse scattered light is therefore associated with the projection of the curve onto the \( y \)-axis, which we define as the magneto-transverse direction.

5. Magneto-transverse scattering as a function of the size parameter

Quantitatively, the transverse light current difference is associated with a summation of the magneto scattering cross section over outgoing wavevectors and normalized to the total transverse light current. A schematic view of the geometry is displayed on Fig. 1. In our notation this is

\[
\eta \equiv \frac{I_{\text{up}}(B) - I_{\text{down}}(B)}{I_{\text{up}}(B = 0) + I_{\text{down}}(B = 0)} = \frac{2 \int_0^\pi d\phi \int_0^\pi d\cos \theta \sin \phi \sin \theta \sin \phi \sum_{\sigma\sigma'} \mathcal{R}e(T^0_{\sigma\sigma'}T^{1*}_{\sigma\sigma'})}{\int_0^\pi d\phi \int_0^\pi d\cos \theta \sin \phi \sum_{\sigma\sigma'} |T^0_{\sigma\sigma'}|^2}. \tag{27}
\]

The factor \( \sin \theta \sin \phi \) represents a projection onto the magneto transverse direction \( \hat{B} \times \hat{k} \) which is necessary since we are interested in the magneto-transverse light flux. In Fig. 3 we plotted this contribution as a function of the size parameter \( x \) for an index of refraction \( m = 1.0946 \) (the value in Ref. 2). Note the change of sign beyond \( x = 1.7 \), for which we do not have any simple explanation so far. In the range of small size parameter, \( \eta \) exhibits an \( x^5 \) dependence.

A. Rayleigh scatterers

For Rayleigh scatterers, the formulas (17) to (21) simplify dramatically because one only needs to consider the first partial wave of \( J = 1 \) and the first terms in a development in powers of \( y \) (since \( y \ll 1 \)). From Eqs. (13) and (14), we find:

\[
C_\infty = -\varepsilon^{\uparrow \downarrow}/\varepsilon^\varepsilon(\varepsilon + \varepsilon^\varepsilon)^\varepsilon
\]

and

\[
D_\infty = -\nabla^\varepsilon/\nabla^\varepsilon \Delta^{\varepsilon} \Delta
\]

so that we can keep only \( C_\infty \) and drop \( D_\infty \) as a first approximation. Adding all contributions of Eqs. (17) and (13), and changing from circular basis to linear basis of polarization we find
\[ T_{k,k'} = \begin{pmatrix} t_0 \hat{k} \cdot \hat{k}' + it_1 \hat{B} \cdot (\hat{k} \times \hat{k}') & it_1 \hat{B} \cdot \hat{k} \\ -it_1 \hat{B} \cdot \hat{k}' & t_0 \end{pmatrix}, \] (28)

with \( t_0 = -6i\pi a_1^* / \omega \) the conventional Rayleigh T-matrix and \( t_1 = 6C_\infty W / \omega \). This form agrees with the Rayleigh point-like scatterer model discussed in Ref. We note that Eqs. (18) and (19) give off-diagonal contributions in Eq. (28) whereas Eq. (20) gives a diagonal contribution. This is a general feature that persists also beyond the regime of Rayleigh scatterers.

For a Rayleigh scatterer the magneto cross section of Fig. 3 exhibits symmetry, since the positive and negative lobes of the curve are of the same size but of opposite sign. Hence no net magneto-transverse scattering exists for one Rayleigh scatterer. In fact Eq. (28) provides the following expression for \( F(\theta) \):

\[ F_{\text{Rayleigh}}(\theta) = \frac{3m x^3}{4\pi^2 (m^2 + 2)^2} \cos \theta \sin \theta. \] (29)

As the size of the sphere gets bigger, the magneto corrections become asymmetric, as seen in Figure 4. When the size is further increased, new lobes start to appear in the magnetic cross section corresponding to higher spherical harmonics. These lobes do seem to have a net magneto-transverse scattering.

One single Rayleigh scatterer does not induce a net magneto-transverse flux. It is instructive to consider the next simplest case, namely two Rayleigh scatterers positioned at \( r_1 \) and \( r_2 \). If their mutual separation well exceeds the wavelength, it is easy to show that the collective cross-section simply equals the one-particle cross-section multiplied by an interference factor \( S(k, k') = |\exp(i(k - k') \cdot r_1) + \exp(i(k - k') \cdot r_2)|^2 \). This interference factor changes the angular profile of the scattering cross-section and makes sure that a net magneto-transverse flux remains. The estimate for two Rayleigh particles with an incident wave vector along the inter-particle axis, is found to be:

\[ \eta \sim \frac{V_0 B}{k} x^3 \left( \frac{\sin(k r_{12})}{k r_{12}} \right)^2 \quad \text{if} \quad k r_{12} \gg 1 \] (30)
This simple model suggests that the “Photonic Hall Effect” is in fact a phenomenon generated by interference of different light paths. In Fig. 6 we show how differential cross-sections of two particles change in a magnetic field. More scattering is now directed into the forward direction and as a result the cancellation of the net magneto-transverse flux in Fig. 3 is removed. One Mie sphere mimics qualitatively this simple model and should on the basis of the principle outlined above exhibit a magneto-transverse current. The model also suggests that the regime of Rayleigh-Gans scattering - the Born approximation for one sphere but contrary to Rayleigh scattering still allowing interferences of different scattering events - should exhibit a net magneto-transverse flux. Indeed, explicit calculations in this regime confirm this statement, with $\eta \sim x^5$, independent of the index of refraction $m$ of the sphere.

**B. Geometrical Optics**

In the regime of large size parameters, the Mie solution can be obtained from Ray Optics. Apart from Fraunhofer diffraction (persisting for any finite geometry), the Mie solution for a ray with central impact should be equivalent to the one for a slab geometry. In our magnetic-optic approach, this means to study Faraday rotation in a Fabry-Perot cavity. This model is of special interest because the Fabry-Perot cavity is known to enhance the Faraday rotation, i.e., the rotation is additive in the total traversed path length as opposed to the case of rotary power.

A ray with central impact is characterized by $J = 1$ in Mie theory. We assume that $x \gg 1$ and $y \gg 1$ which allows some simplification in the expression of the Mie coefficients. We find for the Mie coefficients $c_1$ and $d_1$ the following behavior:

$$c_1 = \frac{2 e^{i(x-y)}}{(m+1) (1 + re^{-2iy})},$$

$$d_1 = \frac{2 e^{i(x-y)}}{(m+1) (1 - re^{-2iy})}. \quad (31)$$

In this formula, $r = (m-1)/(m+1)$ is the complex Fresnel reflection coefficient. Putting this expression into Eqs. (13), (14) and (21), we can compute the exact behavior of the T-
matrix in the forward direction. We note $T_{\text{scatt}}^0$ the part of $T^0$ due to scattering only, which is obtained from Eq. (15) by replacing the Mie coefficients $a_J$ and $b_J$ by $a_J - \frac{1}{2}$ and $b_J - \frac{1}{2}$, since the terms $\frac{1}{2}$ are associated with the Fraunhofer diffraction, which does not exist for the slab geometry.

Since for a small perturbation, we have

$$T = T_{\text{scatt}}^0 + T^1 \simeq T_{\text{scatt}}^0 e^{T^1/T_{\text{scatt}}^0}, \quad (32)$$

we see that the change in phase $\delta \phi$ due to the magnetic field is in fact related to the imaginary part of $T^1/T_{\text{scatt}}^0$. This change in phase can be interpreted as the Faraday effect.

From Eq. (31), we find in the basis of circular polarization:

$$\Im \left( \frac{T^1}{T_{\text{scatt}}^0} \right)_{\sigma \sigma'} = \delta \phi (-\sigma) \hat{B} \cdot \hat{k} \delta_{\sigma \sigma'}, \quad (33)$$

with

$$\delta \phi = 2aV_0 B \frac{1 + R}{1 - R} \frac{1}{1 + \mathcal{M} \sin(\varepsilon \hat{\tau}) \varepsilon}. \quad (34)$$

with $\mathcal{M} = \Delta \mathcal{R}/(\infty - \mathcal{R}) \varepsilon$ and the reflectivity $R = r^2$. We note that the quantity $(-\sigma) \hat{B} \cdot \hat{k}$ is conserved for a given ray, which generates the accumulation of the Faraday rotation. The function $\delta \phi$ tends to $2aV_0 B$ as $R \to 0$, since it represents the normal Faraday rotation in an isotropic medium of length $2a$, as it should be for our geometry. When $R$ is large, two new factors come into play: $(1 + R)/(1 - R)$ which is the maximum gain factor of the Faraday rotation due the multiple interference in the Fabry-Perot cavity and

$$\mathcal{A}(\hat{\tau}) = \frac{\infty}{\infty + \mathcal{M} \sin(\varepsilon \hat{\tau}) \varepsilon}$$

which is an Airy function of width $4/\sqrt{\mathcal{M}}$. The finesse of the cavity is then $\mathcal{F} = \pi \sqrt{\mathcal{M}}/\varepsilon$.

At resonance, the Faraday rotation is maximally amplified - assuming no losses - relative to single-path Faraday rotation. We stress that one needs $\delta \phi \ll 1$, in order for Eq. (32) to apply. The Faraday rotation has the effect of splitting each transmission peak in the Fabry-Perot cavity into two peaks of smaller amplitude both associated with a different state of helicity.
The amplification of the Faraday rotation is a consequence of the amplified path length of the light. In other words, the Faraday rotation measures the time of interaction of the light with the magnetic field. This time is found to be the dwell time $\tau$ of the light in the cavity for this 1D problem. The change in phase follows the simple relation:

$$\delta \phi = V_0 B \frac{\tau c_0}{m}$$

(35)

where $c_0/m$ is recognized as the speed of light in the sphere.

The dwell time of the light in the cavity varies between a maximum value of

$$\tau_{\text{dwell}}^{\text{max}} = (1 + m^2) a/c_0,$$

and a minimum value of

$$\tau_{\text{dwell}}^{\text{min}} = 4 m^2/(1 + m^2) a/c_0.$$

These typical oscillations are visible in the plot of the change of phase $\delta \phi$ of Fig. 7.

6. Summary and Outlook

In this paper we addressed the Faraday effect inside a dielectric sphere. We have shown that this theory is consistent with former results concerning the predictions of the light scattered by Rayleigh scatterers in a magnetic field. It is possible to get from this perturbative theory quantitative predictions concerning the Photonic Hall Effect for one single Mie sphere, such as the scattering cross section, the dependence on the size parameter or on the index of refraction.

We will start experiments addressing single Mie scattering in a magnetic field. A second challenge is to implement our Mie solution into a multiple scattering theory.

ACKNOWLEDGMENTS

We wish to thank Y. Castin for very stimulating and valuable discussions.
REFERENCES

1. F. A. Erbacher, R. Lenke, and G. Maret, "Multiple Light Scattering in Magneto-optically active media," *Europhys. Lett.*, 21(5):551–556, 1993.

2. G. L. J. A. Rikken and B. A. van Tiggelen, "Observation of magnetically induced transverse diffusion of light," *Nature*, 381:54–55, 1996.

3. A. Sparenberg, G. L. J. A. Rikken, and B. A. van Tiggelen, "Observation of Photonic Magneto-resistance," *Phys. Rev. Lett.*, 79(4):757-760, 1997.

4. F. C. MacKintosh and S. John, "Coherent backscattering of light in the presence of time-reversal-noninvariant and parity-non conserving media," *Phys. Rev. B*, 37(4):1884–1897, 1988.

5. B. A. van Tiggelen, R. Maynard, and T. M. Nieuwenhuizen, "Theory for multiple light scattering from Rayleigh scatterers in magnetic fields," *Phys. Rev. E*, 53(3):2881–2908, 1996.

6. R. G. Newton, *Scattering Theory of Waves and Particles*. Springer Verlag, New York, 1982.

7. L. Landau and E. Lifchitz, *Quantum Mechanics*. Mir, Moscow, 1967.

8. H. C. van de Hulst, *Light Scattering by Small Particles*. Dover, New York, 1980.

9. A. Bott and W. Zdunkowski, "Electromagnetic energy within dielectric spheres," *J. Opt. Soc. Am. A*, 4(8):1361–1365, 1987.

10. H. Y. Ling, "Theoretical investigation of transmission through a Faraday-active Fabry-Perot étalon," *J. Opt. Soc. Am. A*, 11:754–758, 1994.

11. R. Rosenberg, C.B. Rubinstein, and D.R. Herriott, "Resonant Optical Faraday Rotator," *Appl. Opt.*, 3:1079–1083, 1964.
12. V. Gasparian, M. Ortuno, J. Ruiz, and E. Cuevas, "Faraday Rotation and Complex-Valued Traversal Time for Classical Light Waves," *Phys. Rev. Lett.*, 75(12), 1995.
Fig. 1. Schematic view of the magneto-scattering geometry. Generally, $\theta$ denotes the angle between incident and outgoing wave vectors; $\phi$ is the azimuthal angle in the plane of the magnetic field and the $y$-axis. The latter is by construction the magneto-transverse direction defined as the direction perpendicular to both magnetic field and incident wave vector. The angle $\alpha$ coincides with the angle $\theta$ in the special but relevant case that the incident vector is normal to the magnetic field.

Fig. 2. The solid line is the real part and the dashed line the imaginary part of the magneto forward scattering matrix $T_{kk}^{1}$ in the circular basis of polarization, plotted versus the size parameter $x$ for an index of refraction of $m = 1.33$ in units of $W = V_{0}B\lambda$.

Fig. 3. Magneto-transverse scattering cross section $F(\theta)$ for a Rayleigh scatterer with index of refraction $m = 1.1$ and size parameter $x = 0.1$. The solid line is a positive correction and the points denote a negative correction. The curve has been normalized by the parameter $W$. No net magneto-transverse scattering is expected in this case because the projection onto the $y$-axis of these corrections cancel.

Fig. 4. Magneto-transverse scattering cross section $F(\theta)$ for a Mie scatterer of size parameter $x = 5$ and of index of refraction $m = 1.1$. The curve has been normalized by the parameter $W$. The solid line is for positive correction and the points denote a negative correction. In this case a net magneto-transverse scattering is expected because the projection onto the $y$-axis of these corrections do not cancel.

Fig. 5. Normalized magneto-transverse light current $\eta$ as a function of the size parameter $x$ for an index of refraction of $m = 1.0946$. The curve is displayed in units of $W$.

Fig. 6. Magneto-cross section for two Rayleigh scatterers each of size parameter $ka = 0.1$ and separated by a distance corresponding to a size parameter of $kr_{12} = 5$. In this case the enhanced forward scattering leads also to a net magneto-transverse current along the vertical axis.
Fig. 7. Magnetically induced change of phase $\delta \phi$ - similar to Fabry-Perot modes of a cavity - as a function of size parameter $x$ for the partial wave of $J = 1$ - “central impact” -. The curve is for $m = 10$, and has been normalized by the value $2aV_0B$. 
\[ \hat{z} = B \]

\[ \hat{k} \cdot \hat{k}' = \cos \theta \]

\[ d\hat{k}' = d\phi \sin \alpha \, d\alpha \]

\[ \hat{y} = \text{magneto-transverse} \]
