Surface state decoherence in loop quantum gravity, a first toy model

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Abstract
The quantum-to-classical transition through decoherence is a major facet of the semi-classical analysis of quantum models that are supposed to admit a classical regime, as quantum gravity should be. A particular problem of interest is the decoherence of black hole horizons and holographic screens induced by the bulk-boundary coupling with interior degrees of freedom. Here in this paper we present a first toy-model, in the context of loop quantum gravity, for the dynamics of a surface geometry as an open quantum system. We discuss the resulting decoherence and recoherence and compare the exact density matrix evolution to the commonly used master equation approximation à la Lindblad underlining its merits and limitations.

The prospect of this study is to have a clearer understanding of the boundary decoherence of black hole horizons seen by outside observers.

Keywords: loop quantum gravity, decoherence, surface dynamics, toy model, quantum surface, master equation

(Some figures may appear in colour only in the online journal)

1. Introduction

Einstein’s theory of gravitation, general relativity, describes the gravitational force as a manifestation of the curvature of space-time while the matter and energy densities tell it how to curve. No background is assumed and every field is dynamical. One of the most fascinating predictions of general relativity is the formation of black holes in a gravitational collapse. The mathematical theory of black holes is very rich and led to causality studies, global techniques to analyze space-times, to uniqueness and singularity theorems. What researchers also came to understand is the holographic nature of gravity, the fact that volume degrees of freedom can be described by some
surface degrees of freedom. This led to the statement of the holographic principle [1] which can serve as a guide for the understanding of the dynamics in quantum theories of gravity. An attractive point of view on black holes is the membrane paradigm [2–4] which describes the horizon of a black hole having a 2d surface living and evolving in 3d space. More recently, a similar point of view was developed for timelike surface [5, 6] and a dictionary with non-equilibrium thermodynamics and the hydrodynamics of viscous bubbles was established. Thus, the fact that gravitation is about geometry and that Einstein’s equation can be formulated in the language of a surface dynamics analogous to hydrodynamical equations could serve as a new avenue to understand dynamical aspects of a quantum theory of gravity such as loop quantum gravity.

Understanding gravity in the quantum regime is still an open issue and is the focus of active researches. Loop quantum gravity is a proposal of a quantum theory of gravity based on a non-perturbative canonical quantization of general relativity (for textbooks, see [7–9]). The formalism is based on a $3 + 1$ formulation of general relativity in terms of the Ashtekar–Barbero connection and a densitized triad. The spectral analysis of geometrical operators such as the area of a surface or the volume of a region led to the physical picture of a discrete space(-time) at the Planck scale. At the kinematical level, the natural basis of the state space is spanned by spin network states which are eigenstates of the area and volume operators. They live on a graph and are dressed with a spin on each edge and intertwiner on each vertex carrying, respectively, area and volume information. In this context of quantum geometry, a surface $S$ is defined by the set of edges of the spin network that it intersects, that is as the collection of the spins living on those edges, $S = (j_1, \ldots, j_N)$. Its dynamics is controlled by the Hamiltonian constraint and all the other degrees of freedom of the universe. This environment is composed for a closed surface of its exterior and interior, or more generally the rest of the spin network, and matter and field degrees of freedom. The direction we want to particularly explore is the influence of an environment on the surface dynamics.

A careful analysis of the intertwiner space led recently to a new formulation of the phase space of loop quantum gravity in terms of spinors and the $U(N)$ formalism for an $N$-valent vertex [10]. Interestingly, an $N$-valent vertex can be seen as a quantum polyhedron with $N$ faces and the $U(N)$ group appears to be the set of deformations preserving the boundary area of this geometrical quantum object [11]. The natural operators in this setting simply destroy a quantum of area in one place of the surface and recreate it at another. In a very straightforward way, this point of view on quantum geometry allows one to construct models of the dynamics of a quantum surface and opens the way to the surface dynamics viewpoint on gravitation discussed above at the quantum level.

Alongside the comprehension of the quantum dynamics of gravity through the implementation of the Hamiltonian constraint at the quantum level (for references and reviews on current researches, see [12, 13]), a major open question in loop quantum gravity is its semi-classical regime and the recovery of general relativity. A special focus is devoted to understanding the renormalization of the theory by defining properly the coarse-graining of spin network states (e.g. [11, 14–17]) and also by defining the proper notion of coherent states of the quantum geometry [18–21]. The emergence of a classical reality from a purely quantum one is an issue that dates back to the origin of quantum theory and is nowadays mostly understood thanks to decoherence, a phenomenon recently observed in (cavity) quantum electrodynamic and condensed matter experiments (for reviews and recent ideas [22–25]). The heart of the idea behind decoherence is that a system is never isolated but is an open quantum system in contact with an unmonitored environment. Information about the quantum state of the system leaks inevitably in the environment through entanglement and this information remains lost to the observer. This leads in turn to the suppression of quantum superposition and interference effects into an effectively classical mixed state. The quantum states most robust to entanglement are called pointer states and are the semi-classical states of the system.
What we propose to study in this paper is a first exploration of decoherence in loop quantum gravity by studying the dynamics of a quantum surface, modeled with a fixed number $N$ of elementary area patches. The total area, corresponding to the total spin, is supposed to be constant. This system is not isolated but in contact with an environment composed of all other degrees of freedom available, meaning primarily bulk gravitational degrees of freedom and possibly also matter degrees of freedom. Of course studying the surface dynamics in the full theory would require solving exactly the Hamiltonian constraint to obtain the true quantum dynamics which is out of reach. Still we know that the Hamiltonian couples the bulk and surface degrees of freedom. Instead, we construct effective models of the surface interacting with an environment by using the $U(N)$ deformation operators discussed above. The simplest, and quite general [26], environment we can consider is a bath of harmonic oscillators coupled bilinearly to the area preserving deformation operators. Each deformation mode is then coupled to the environment. For this first inquiry, we limit ourselves to the quantum measurement limit where the full dynamics is approximated to the interaction term only. The first toy model we look at is a surface with two patches whose dynamics can be modeled as a spin, encoding the closure defect, whose three directions are coupled to harmonic oscillators. This is a non-trivial interaction seldom explored in studies on decoherence effect. Interestingly, we obtain a decoherence phenomenon not on the value of the spin but only for certain quantum superposition of integer and half-integer spins. The decoherence time-scale appears to be independent of the spins on long time while the short time behavior maps the one studied using approximate methods. The decoherence factor decays exponentially with a decoherence time scaling as the relative distance between the spins. Those exact results contrast with master equation approaches which only capture short time behavior and predict a decoherence as long as we have a quantum superposition of different spins. The physical origin of this difference comes from the model used for the environment which is, for Markovian equation, a memory-less dynamical environment while it is considered non-dynamical for the exact toy model.

The paper is structured as follows. Section 2 reviews the basic tools used in the analysis of the models. After some reminders on decoherence in sections 3 and 4 it analyzes the exact behavior of the toy model of a two patches surface while section 5 deals with approximate master equation approaches of the complete dynamics. Section 6 concludes and opens this discussion of surface state decoherence in loop quantum gravity.

2. Surface geometry and quantization

2.1. Spin as harmonic oscillators

The geometry of a two-dimensional surface $S$ can be described from two different points of view: the intrinsic one which relies on the Riemannian curvature and the extrinsic one. The latter presupposes an embedding of the surface in a higher dimensional space like $\mathbb{R}^3$. The extrinsic curvature (also called second fundamental form) is defined as the variation of the surface normal vector $\mathbf{N} \in \mathbb{R}^3$ along the manifold $S$, see figure 1. This normal vector also gives the integration measure on the manifold. This description is privileged by the canonical quantization of geometry in loop quantum gravity.

The loop quantum gravity approach to the quantization of such a geometry is twofold. First we consider a discretization of $S$ in terms of elementary surfaces (a face or a patch) $\tilde{S}_i$. Each
patch is defined by its surface normal $R \in N_i^3$, whose norm is the area of the surface. It is further provided with a phase space defined by a Poisson Bracket
\[ \{ \}, \] that is canonically quantized to the operator commutator
\[ \{ \}, \] This phase space is then canonically quantized to the operator commutator
\[ \{ \}, \] This is the basic postulate of loop quantum gravity. The proportionality factor has the dimension of an area and is related to the Planck area
\[ \gamma \sim G \frac{c}{\hbar} \], the only dimensional quantity that appears in quantum gravity and $\gamma$ is the Immirzi parameter, a dimensionless number that fixes the scale of the theory.

The quantum state of each elementary surface patch $S_i$ is then a vector of an irreducible $SU(2)$ representation $V_{ji}$. The spin $j_i$ then gives the area of that surface in Planck units $\gamma l_P^2$. The Hilbert space of a $N$ patches surface with fixed spins $\ldots jj \ldots$ is then
\[ \mathcal{H}_{h^j} = \bigotimes_{p=1}^N V^{h_p} \] (1)

Intertwiners are defined as the $SU(2)$-invariant subspace of this Hilbert space:
\[ \mathcal{H}_{h^j}^{SU(2)} = \text{Inv}_{SU(2)}[\mathcal{H}_{h^j}] \] (2)

These singlet states are understood as the quantum counterpart of classical polyhedra [27–29]. For the purpose of the article, we will focus on a surface with fixed area $A = \sum_{p=1}^N j_p$ with Hilbert space
\[ \mathcal{H}_A = \bigoplus_{A=\sum_{p=1}^N j_p} \mathcal{H}_{h^j}^{SU(2)} \] (3)

Its $SU(2)$-invariant subspace $\mathcal{H}_A^{SU(2)}$ describes the Hilbert space of the set of all polyhedrons of area $A$. As was shown in [10], these intertwiner spaces $\mathcal{H}_A^{SU(2)}$ each carry an irreducible representation of the unitary group $U(N)$, which can be understood as the group of deformations of quantum polyhedra at fixed total boundary area $A$. We will recall the definition of the $u(N)$ generators below as the basic deformation operators for a quantum surface.

Finally, the total Hilbert space associated with a quantum surface $S$ with $N$ patches is

\[ |j_p\rangle \in V^{h_p} \]
\[ |j_q\rangle \in V^{h_q} \]

\textbf{Figure 1.} Geometry of a 2d surface $S$ in terms of the extrinsic curvature seen as the variation of the normal. The quantum theory describes $S$ as a discretized set of patches $S_i$ whose quantum states live in $V^j$ a spin $j$, representation of $SU(2)$. 

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3The area is classically given by the norm of the normal vector $|N| = \sqrt{N^2}$. Since the normal vector $N$ is quantized into the $su(2)$ generator $J$, the squared norm $N^2$ becomes the $su(2)$ Casimir $J^2$, whose spectrum is $\sqrt{(j+1)}$ in terms of the spin $j$. So a traditional area spectrum in loop quantum gravity is $\sqrt{(j+1)}$. However, taking a square-root naturally leads to non-polynomial observables and to quantization ambiguities. For instance, in the Schwinger representation of $SU(2)$, a representation in terms of harmonic oscillators, the norm $|N|$ becomes a (quadratic) polynomial in the harmonic oscillator operators. It has a unique consistent quantization as simply the spin $j$ [10, 18]. This is also the natural area spectrum when analyzing the $SU(2)$-invariant observables and deformation algebra of intertwiners [27].
\[ \mathcal{H}_N = \bigoplus_{A \in \mathbb{N}} \mathcal{H}_N^A = \bigoplus_{\{ j_k \}} \mathcal{H}_{j_1 \ldots j_N} = \bigoplus_{\{ j_k \}} \mathcal{V}^{j_1} \otimes \ldots \otimes \mathcal{V}^{j_N}. \] (4)

In this framework, studying the dynamics is naturally done through the study of the deformations of the surface. In particular, the area of each face can change which means in the quantum theory changing the spin \( j \) attached to the face. The common \( SU(2) \) representation used in angular momentum theory is not adapted for this purpose. But the Schwinger representation of the \( su(2) \) Lie algebra in terms of harmonic oscillators is and we review its construction here [10, 30].

Let’s focus on one spin (i.e. one elementary surface patch) and introduce two harmonic oscillators \( a \) and \( b \) whose commutation relations are naturally \([a, a^\dagger] = [b, b^\dagger] = 1\). It is then straightforward to show that

\[ J_z = \frac{1}{2} (a^\dagger a - b^\dagger b), \quad J_x = J_y = a^\dagger b \] (5)

satisfy the \( su(2) \)-algebra. The total energy of the oscillators \( E = \frac{1}{2} (a^\dagger a + b^\dagger b) \) allows us to write \( J^z = E(E + 1) \), so that the total energy gives exactly the spin \( j \), i.e. the area of the elementary surface patch. Similarly, the energy difference corresponds to the magnetic quantum number \( m \). The Hilbert space we are working with is then \( \mathcal{H}_{HO} \otimes \mathcal{H}_{HO} = \oplus_{j} \mathcal{V}^j \). Using standard notations, we have the correspondence between the spin and the harmonic oscillators states

\[ |j, m\rangle = |n_a, n_b\rangle \quad j = \frac{1}{2} (n_a + n_b) \quad m = \frac{1}{2} (n_a - n_b) \] (6)

We can at once see that the action of \( a \) or \( b \) decreases the spin and thus the area by 1/2. The Schwinger representation admits natural operators allowing us to move between different spin representations of the \( su(2) \)-algebra, a feature more complicated to achieve with the standard representation.

Now consider a surface with \( N \) faces described by spins \( \{ j_i \}_{i=1,\ldots,N} \). We then indeed need \( N \) pairs of harmonic oscillators \( \{ a_i, b_i \}_{i=1,\ldots,N} \) to describe the surface state living in the Hilbert space \( \mathcal{H}_N = \mathcal{H}_{10}^{\otimes 2N} \). This representation naturally allows us to define a new set of operators that deform the surface. Following [10, 30], we define the operator \( E_{ij} \) that destroys a quantum of area at the face \( j \) and creates one at \( i \) by\(^4\):

\[ E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j \] (7)

Clearly those operators deform the surface \( \mathcal{S} \), preserve the total area and are invariant under \( SU(2) \) rotations. They then act on each space \( \mathcal{H}_N^A \) without affecting the area \( A \). The total area \( A = \sum_{p=1}^N j_p \) is related to the total energy of the oscillators as the eigenvalue of \( \hat{A} = \frac{1}{2} \sum_{p=1}^N E_{pp} \). Throughout the text we use the simplified notation \( E_{pp} = E_p \). The operators \( E_{ij} \) also satisfy the \( u(N) \) algebra [10, 27]

\[ [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{jk} \] (8)

The group \( U(N) \) can thus be seen as the group of area preserving deformations of a discrete quantum surface with \( N \) faces. The (quadratic) Casimir operator \( \tilde{C} \) of this \( u(N) \) algebra is

\(^4\) We could define operators that only destroy or create quantum of area but we do not need them yet for the present study. They are defined as \( E_{ij} = a_i b_j - a_j b_i \) and their Hermitian conjugate \( E_{ij}^\dagger \) [18] and are used to define coherent intertwiner states.
\[ C^2 = \sum_{ij} E_i^j E_i^j = 2\hat{A} + N - 2 + 2\mathbf{J} \mathbf{J} \]  

(9)

where \( \mathbf{J} = \sum_{p=1}^{N} \mathbf{J}_p \) is the total spin operator and \( \mathbf{J}_p \) are the operators for each patch defined in (5). The operators \( \mathbf{J} \) generate global SU(2) transformations on all spins simultaneously, corresponding to an overall 3d-rotation of the whole surface. When this global SU(2) Casimir vanishes, \( \mathbf{J}^2 = 0 \). We are back on the SU(2)-invariant subspace \( \mathcal{H}_0^A \) of quantum polyhedra. But in general \( \mathbf{J}^2 \) is not zero and is dubbed the ‘closure defect’ [11, 31]. This closure defect appears naturally when coarse-graining the spin network state. Nonetheless its physical significance is not yet perfectly understood but it is suspected to be related to curvature and torsion in the coarse-grained region induced by some quasi-local energy density. The important point for our concern is that the eigenstates of the operator \( \mathbf{J}^2 \) will by at the heart of our discussion of pointer states of the quantum surface.

Having now the kinematical scene for the system and natural operators to define its dynamics, we go on to discuss a very special class of states that play a central role for the semi-classical understanding of the theory.

2.2. Coherent states

Coherent states play a very special role in the understanding of the quantum/classical transition in many areas of physics and also in quantum gravity. They allow one to interpolate a classical geometry from its quantum description. SU(2) coherent states are the natural ones to use in loop quantum gravity. Following [32], a coherent state \( | j, g \rangle \) is defined by applying an SU(2) rotation \( g \) to a state analogous to the vacuum in quantum optics that minimizes the uncertainty relations such as the highest weight state \( | j, m = j \rangle \).

\[ | j, g \rangle = g | j, j \rangle, \quad g \in SU(2) \]  

(10)

A key property of coherent states is that they remain coherent under the action of a SU(2)-rotation. This follows directly from their very definition,

\[ h | j, g \rangle = | j, hg \rangle \]  

(11)

Different ways exist to index coherent states. Instead of using the SU(2) rotation \( g \), a coherent state can equivalently be labeled using spinors \( z \in \mathbb{C}^2 \). The highest weight vector is the spinor \( | \uparrow \rangle \) and can be mapped to any arbitrary unit spinor by a rotation \( g \in SU(2) \), so that a SU(2) matrix contains the same information as a unit spinor \( | \frac{\pi}{2}, \frac{\pi}{2}, 0, 0 \rangle \). Explicitly the parametrization of SU(2) coherent states by spinors goes as:

\[ | \frac{\pi}{2}, \frac{\pi}{2}, 0, 0 \rangle = | j, \uparrow \rangle, \quad | j, \downarrow \rangle = \sqrt{\langle \bar{z} \mid \bar{z} \rangle} \begin{pmatrix} z^0 & z^1 \\ z^1 & -z^0 \end{pmatrix} | j, \uparrow \rangle, \quad g | \uparrow \rangle = | e \rangle, \quad g | \downarrow \rangle = \begin{pmatrix} 1 \cos \theta & -1 \sin \theta \\ 1 \sin \theta & 1 \cos \theta \end{pmatrix} | e \rangle. \]  

(12)

\[ | j, \uparrow \rangle = | j, j \rangle, \quad | j, \downarrow \rangle = \begin{pmatrix} 1 \cos \theta & -1 \sin \theta \\ 1 \sin \theta & 1 \cos \theta \end{pmatrix} | j, \uparrow \rangle. \]  

(13)

Spinors for loop quantum gravity have been extensively studied in [10, 18, 27]. The explicit decomposition of a coherent state on the standard basis \( | j, m \rangle \) used in angular momentum theory is:
Then the norm (and scalar product) between coherent states can be calculated in terms of the simpler scalar product between spinors by the formula:

\[
|j', z'|, j, z, \rangle = \delta_{j'} \langle z'| z \rangle^{\frac{1}{2}}
\]

Such coherent states are the basic tools for the construction of more interesting states such as coherent intertwiner states or \( U(N) \) coherent states for the semi-classical analysis of loop quantum gravity.

3. Decoherence and surface dynamics models

3.1. About decoherence

The destruction (or attenuation) of interference, called decoherence, of a quantum superposition through the entanglement of the system with an environment is at the heart of the modern understanding of the quantum-to-classical transition (for reviews see [22–24]). Decoherence comes from the leakage of information on the state of the system in an environment that cannot be monitored by the observer. The states most immune to this constant monitoring of the environment, that entangle least with it, are called pointer states and are in fact the natural classical states of the system. Pointer states are predictable and a quantum superposition of them evolves into a classical mixture. Pushed even further, decoherence ideas are being used to understand more deeply the emergence of a classical objective reality (quantum Darwinism approach, [25]).

Since the environment is unmonitored, the natural object to look at is the reduced density matrix of the system \( \rho(t) = \text{tr}_E(\rho(t)\rho_{SE}(0)U^{-1}(t)) \) with dynamics ruled by \( \frac{d\rho(t)}{dt} = -i[H, \rho(t)] \). In most situations this equation cannot be solved exactly.

There exist mostly two paths to analyze the open quantum dynamics ruled by the Hamiltonian \( H = H_S + H_E + H_{SE} \), with \( H_{SE} \) the free Hamiltonian and \( H_{SE} \) the interaction term. The first method is the Feynman–Vernon path integral approach [26], an exact approach but difficult to manipulate in general, and the second one is master equation approaches. Those equations have the advantage of being mathematically more accessible but rely on approximations which must be checked on the system of interest for the results to be relevant. The most used approximation is the Born–Markov approximation which, simply stated, says that initially no correlations exist between the system and the environment and that the environment has no memory (the correlation functions of the environment vanish on a timescale much smaller than any other dynamical or observational times). A particular form of Markovian master equations is the Lindblad form. This subset of equations is the most general form quantum dynamics can take (constrained by positivity and complete positivity of the reduced dynamics). Those different approaches will by used and compared in this paper for the surface dynamics we are interested in.

3.2. The general model

We are interested in the dynamics of a quantum surface \( S = (j_1 \ldots j_n) \) whose total area \( A = \sum_{p=1}^N j_p \) is supposed to be a constant of motion having in mind a black hole at equilibrium. We recall that in loop quantum gravity \( S \) is just a part of a spin network state and the remaining degrees of freedom will here be considered as its environment, see figure 2. The Hilbert space \( \mathcal{H}_S \) we work with is
Using the deformation formalism, we can use the operators $E_{ij}$ to construct a natural interaction $H_{SE}$ between $S$ and $E$ where the environment excites each deformation $E_{ij}$ through an operator $V_{ij}$ so that

$$H_{SE} = \sum_{ij} E_{ij} \otimes V_{ij}$$

(17)

All the data of the environment is encoded in the operators $V_{ij}$. For now, we do not specify the explicit form of those operators. We simply require Hermicity, $V_{ij} = V_{ji}^\dagger$.

We primarily consider as environment the bulk gravitational degrees of freedom: the bulk geometry will interact with the surface geometry, with the spins on the boundary getting excited by the fluctuations of the bulk, and we consider the interaction Hamiltonian $H_{SE}$ as encoding this interaction in an effective manner. But we could also include any matter fields (present inside or outside the region) in the environment and take them into account in the definition of the $V_{ij}$ operators.

As a first toy model, we will model the $V_{ij}$’s as a bath of harmonic oscillators. This is the typical environments considered in decoherence studies. That is what we will suppose in the remainder of the paper. The free Hamiltonian $H_E$ of the environment is thus the energy of a set of harmonic oscillators. From the Hawking radiation, thermal states for the environment seem to be the most natural. Concerning the free dynamics of the system $H_S$ we suppose that it has a contribution proportional to the area of the surface so that $H_S = \sum_{i=1}^N E_{ii}$. Since the area is fixed in our problem, such a term has no contribution to the global dynamics. Of course, it would be natural to include higher order contribution of the $E_{ii}$ operators (as for instance a Bose–Hubbard type term) but we leave this analysis for future works.

5 To remind the reader, the Bose–Hubbard Hamiltonian used in cold atom physics is of the form

$$H = -t \sum_{\langle i,j \rangle} b_i^\dagger b_j + h_i h_j + \frac{U}{2} \sum_i n_i (n_i + 1) - \mu \sum_i n_i$$

where $t$ is a hopping constant, $U$ the interaction term and $\mu$ the chemical potential. A Bose–Hubbard model for a quantum black hole would be in our setting

$$H = -t \sum_{\langle i,j \rangle} E_{ij} + \frac{U}{2} \sum_i E_i (E_i + 1).$$

Since the $\mathfrak{u}(N)$ generators are now composed of two different species $a$ and $b$ instead of a single one $b$, the physics of this model remains to be understood.

\[\text{Figure 2.} \text{ A surface } S \text{ is defined as a subset of a spin network while its environment is the remaining. The origin of the interaction between the patches comes from the structure of the graph.}\]
3.3. Decoherence from master equation

Having now set the dynamics our quantum surface, we would like to understand the influence the environment has on the evolution of states of the system, especially if decoherence occurs for certain privileged states or geometrical quantities.

The standard approach to analyze this open quantum dynamics is to formulate a master equation for the reduced density matrix involving only operators acting on the system encoding the effects of the environment. This is a non-trivial problem. Lindblad showed that under some natural assumptions (dynamical semi-group approach), in particular with a Markovian approximation, the evolution of the reduced density matrix takes the form

\[ \dot{\rho}_S = -\frac{i}{\hbar} [H_S, \rho_S] + \sum_{\mu} \left( L_\mu \rho_S L_\mu^\dagger - \frac{1}{2} (\rho_S L_\mu^\dagger L_\mu + L_\mu^\dagger L_\mu \rho_S) \right) \]

with here operators \( L_\mu \) called jump operators encoding the effect of the environment on the system and \( H_S \) the free (possibly Lamb-shifted) Hamiltonian. Following this traditional route, we would like to analyze the relevance of this equation with the deformation operators \( E_{ij} \) as jump operators for the dynamics of the quantum surface. This would be a natural choice in the light of the dynamics (17) where the deformation modes are excited by the environment.

Decoherence can be analyzed from this equation. The straightest route for this is to choose a measure of entanglement like the Von Neumann entropy or for simplicity the purity and look at the evolution of the entanglement for some quantum superposition as initial condition. The states that entangle the least with the environment are the most classical ones and called approximate pointer states.

For our problem, we can look for instance at the evolution induced by (19) on an initial quantum superposition of highest weight states \( |\psi^A_J\rangle \). They verify \( E_i |\psi^A_J\rangle = (A + J) |\psi^A_J\rangle \), \( E_J |\psi^A_J\rangle = (A - J) |\psi^A_J\rangle \) and \( E_k |\psi^A_J\rangle = 0 \) for \( k < i \) with \( A \) and \( J \), respectively, representing the area of the surface and its closure defect. They are the \( U(N) \) analogue of the \( |j_jJ\rangle \) state of \( SU(2) \). The short time evolution of the purity of the coherence of the reduced density matrix \( \text{tr}(\rho_{12} \rho_{12}) \) with initial state \( \frac{|\psi^A_J\rangle + |\psi^A_J\rangle}{\sqrt{2}} \) can be directly obtained from (19)

\[ \frac{d}{dt} \text{tr}(\rho_{12} \rho_{12}) \mid_{t=0} = - [A(N - 2) + J + J' + (J - J')^2] \text{tr} \rho_{12} \rho_{12} \]

The damping factor is always positive and composed of three terms. The last one \( (J - J')^2 \) is the one we were looking for which induces a decoherence effect on quantum superposition of geometry with different closure defect. The second shows that the states most immune to entanglement with the environment are the geometries without defect. Finally, the first shows that the greater the area is the more entangled the system will be.

This setting would be the ideal situation to study the dynamics of an open quantum surface. Nonetheless, many questions remain to be clarified from this validity to hold. What we will see in this paper from exact and approximate special cases is that

- The Markovian hypothesis must be discussed and its validity clarified. We will see that for a non-dynamical environment, this hypothesis is restricted to work only on short timescales.
Recoherence is not excluded for superposition of states with different values of the defect (because of the compactness of SU(2)). This cannot be hinted with a short time analysis. This phenomenon disappears when a large and dynamical environment is considered.

4. Analysis of a toy model

In this section, we study the open dynamics of a toy model of a quantum surface with two faces and focus on decoherence effect. The end goal is to have a clear understanding of the long time behavior of the system, to exhibit the pointer states and their physical significance. Those steps will serve as the basis for the analysis of a more realistic model of the open surface dynamics in quantum gravity. We limit our exact study to the measurement limit by neglecting the free dynamics of the environment. Thus the domain of validity of the following results is in fact limited to timescales smaller then any dynamical times of the environment.

4.1. Motivation

From the interaction $H_{SE} = \sum_{i,j=1}^{N} E_{ij} \otimes V_i$ we can motivate the introduction of the toy model by looking explicitly at the $N = 2$ patches model, see figure 3. We work in the subspace of $\mathcal{H} = \mathcal{H}_{H} \otimes \mathcal{H}_{E}$ with fixed area. By introducing the operators

$$L_{z} = \frac{E_{11} - E_{22}}{2}, \quad L_{+} = E_{12}, \quad L_{-} = L_{+}^{\dagger}$$

we can rewrite the interaction in the form

$$H_{SE} = \left( L_{z} \otimes \frac{V_{11} - V_{22}}{2} \right) + \left( L_{+} \otimes \frac{V_{12} + V_{21}}{2} \right) + \left( L_{-} \otimes \frac{V_{21} - V_{12}}{2i} \right) + \left( \frac{E_{11} + E_{22}}{2} \otimes \frac{V_{11} + V_{22}}{2} \right)$$

Using those definitions, we can check the form of the $U(N)$ Casimir operator (9) explicitly and obtain with the notation of this section that $\sum_{i,j} E_{ij}^{i} E_{ij} = \frac{E^{2}}{2} + 2L \cdot L$ with $E$ the total energy of the oscillators describing the patches. Thus we have $L^{2} = J^{2}$ and the eigenvalues of $L$ correspond exactly to the closure defect of the surface. Nonetheless, $L = J$ except in the special case where the spins are decoupled.

For concreteness, we choose in the remaining the environment to be a bath of harmonic oscillators. We will look at the case of only one oscillator at first and then generalize to an arbitrary number. So the model Hamiltonian we consider is

$$H_{SE} = L \otimes p + \left( \frac{E_{11} + E_{22}}{2} \otimes \frac{V_{11} + V_{22}}{2} \right)$$

with $p$ the momentum operator of the harmonic oscillator. We then have a dynamics with the three directions of a spin coupled to the environment with an additional coupling to the energy of the oscillators and the spin. Since the area (i.e. the total energy of the oscillators) is fixed, the second term is non-dynamical. The first term of this interaction is the non-trivial one and involves three non-commuting observables coupled to the environment.

\[ If we had not fixed the area, this second term in $\left( \frac{E_{11} + E_{22}}{2} \otimes \left( \frac{V_{11} + V_{22}}{2} \right) \right)$ would very likely imply a decoherence of quantum superposition of the area, and thus lead to a classical notion of surface area at late time.\]
The program of this section is first to look at the potential decoherence effect induced by the interaction $L \otimes p$ with a single oscillator as the environment and then explore the consequences of a large environment.

4.2. Reduced density matrix

We focus now on the study of the spin part of the interaction $H_{GE} = L \otimes p$ with a single mode environment and want to understand the decoherence it induces. We look at a possible decoherence on the value of the total spin $j$ which is the quantum number associated with the operator $L^2$, the same $L$ that appears in the interaction. Since we have in mind the dynamics of the horizon of a black hole, we naturally consider our system to be the spin. On the other hand, it would also be legitimate to reverse the problem and focus on the induced dynamics of the oscillator describing the exterior observer. This is what is developed in appendix A for the toy model of this section.

Going back to the surface, the idea is then to study the evolution of a superposition of coherent states of the system

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|j, g\rangle + |j', g'\rangle).$$

(24)

The initial state of the harmonic environment is supposed to be the vacuum and uncorrelated to the state of the system so that the global initial state is

$$|\psi_{\text{GE}}\rangle = |\psi\rangle \otimes |0\rangle$$

(25)

The evolution of this state is obtained most simply by developing the vacuum state on the momentum basis of the environment $|0\rangle = \int_{GE} \phi(p)|p\rangle dp$ where $\phi(p) \propto e^{-\lambda p^2}$ is a Gaussian wave function (the parameter $\lambda$ is not specified explicitly for it will be easier to obtain more general results keeping it) since we have $U(t)|j, g\rangle|p\rangle = |j, e^{-i\sigma_j t} g\rangle|p\rangle$, with $\sigma_j \equiv \langle \sigma_j \rangle$ and $\sigma_i$ the Pauli matrices normalized to $\sigma_j^2 = 1$. The system remains in a coherent state when the environment is in an eigenstate of the momentum operator.

\footnote{Considering a thermal state for the environment would be more accurate from our knowledge of Hawing radiation. But since in this first investigation $H_E = 0$, this case is not relevant.}
The central object we want to calculate is the reduced density matrix of the system \( \rho(t) = \text{tr}_E( U(t) \rho_E(0) U^{-1}(t) ) \) which characterizes completely the dynamics of the system alone. Decoherence effects will be seen by analyzing the long time evolution of the \( j \neq j' \) matrix elements and by showing that they tend to zero. By introducing projection operators \( P_j \) on the subspace of spin \( j \), we can then focus on certain elements of the reduced density matrix \( \rho^{jj}(t) = P_j \rho(t) P_{j'} \). Those operators \( \rho^{jj}(t) \) contain all the information about the coherence between superposition of spins \( j \) and \( j' \).

\[
\rho^{jj}(t) = \frac{1}{\mathcal{N}^j} \text{tr}_E \left[ \int \mathbb{R}^3 \left( j, e^{-\frac{i}{\hbar} p \cdot q} \right) \left( j', e^{-\frac{i}{\hbar} p \cdot q'} \right) \rho \otimes |p\rangle\langle p| e^{-\hbar q^2 + q^2} \, dp \, dq \right] 
\]

where \( D(\theta, \hat{n}) = e^{-i\theta \hat{n} \cdot \hat{\omega}} = \cos(\theta) - i \sigma \hat{n} \sin(\theta) \) and \( \mathcal{N}^j = 2(2\lambda/\pi)^{1/2} \) is the normalization constant. The mathematical details are a bit cumbersome and before delving into them we focus on particular case that highlights the general results.

4.3. A simple calculation: decoherence for \( j' = 0 \)

We are going to calculate the reduced density matrix \( \rho^{jj}(t) \equiv \rho_j(t) \) in the simple case where \( j' = 0 \) and the coherent state is just \( |j, j\rangle \). This last restriction can be lifted to any coherent state \( |j, g\rangle \) by simply choosing the proper spherical coordinates in the following calculations.

\[
\rho_j(t) = \int \mathbb{R}^3 \left( j, \frac{1}{2\hbar} \hat{p} \right) |j, j\rangle e^{-2\lambda q^2} \, dp 
\]

The overall normalization factor \( \mathcal{N}^j \) is not written explicitly and \( \hbar = 1 \) here. Developing explicitly the action of the rotation operator, we have according to formula (14)

\[
\rho_j(t) = \int \mathbb{R}^3 \sum_{m=-j}^{j} \left( \frac{2j}{j+m} \right) (z_0^0 j+m(z_0^1 j-m) |j, j\rangle e^{-2\lambda q^2} \, dp
\]

with \( z_0^0 = \cos \left( \frac{p}{2} \right) - i \sin \left( \frac{p}{2} \right) \cos(\theta) \) and \( z_0^1 = -i \sin(\theta)e^{i\phi} \)

with \((\theta, \phi)\) the spherical coordinates. We now proceed to the explicit calculation of the integrals in this coordinate system with the measure \( dp = p^2 \sin(\theta) d\theta d\phi dp \). The integral over the \( \phi \) angle leads to a \( 2\pi \delta_{j-m,0} \) contribution. The integral over the angle \( \theta \) is then straightforward to do\(^8\), leaving us with an integral over the norm \( p \):

\(^8\)The first integral over the \( \phi \) angle gives:

\[
\rho_j(t) = 2\pi \int \cos \left( \frac{p}{2} \right) - i \sin \left( \frac{p}{2} \right) \cos(\theta) \left( \frac{2j}{j+m} \right) (z_0^0 j+m(z_0^1 j-m) |j, j\rangle e^{-2\lambda q^2} \, dp \, d\theta \, dp.
\]

The remaining integral over the angle \( \theta \) is a trigonometric integral:

\[
\int \cos \left( \frac{p}{2} \right) - i \sin \left( \frac{p}{2} \right) \cos(\theta) \, d\theta = \frac{2}{2^{|j|+1}} \frac{\sin \left( \frac{2j+1}{2} \theta \right)}{\sin \left( \frac{\theta}{2} \right)} = \frac{2}{2^{|j|+1}} \sum_{n=-j}^{j} e^{in\theta}.
\]
This last step properly highlights the decoherence effect. Indeed, we have a sum over modes of the Fourier transform of a Gaussian distribution, implying a Gaussian decay for all modes except for the zero-mode. This zero-mode gives the remaining coherence of our quantum state at late time $t \to +\infty$.

Now more explicitly, we have that

$$\rho_j^\prime(t) = 2\pi \int_0^\infty \frac{2}{2j + 1} \sin\left(\frac{(2j + 1)p}{2}\right) e^{-2\lambda p^2} dp \ |j,j\rangle \rangle$$

$$= \sum_{m=-j}^j \frac{4\pi}{2j + 1} \int_0^\infty e^{2i2mp} e^{-2\lambda p^2} p^2 dp \ |j,j\rangle$$

(29)

This last step properly highlights the decoherence effect. Indeed, we have a sum over modes of the Fourier transform of a Gaussian distribution, implying a Gaussian decay for all modes except for the zero-mode. This zero-mode gives the remaining coherence of our quantum state at late time $t \to +\infty$.

Now more explicitly, we have that

$$\sum_{m=-j}^j e^{2imp} = \begin{cases} 2 \sum_{m=1/2}^j \cos(2mtp) & \text{if } j \in \mathbb{N} + 1/2 \\ 1 + 2 \sum_{m=1}^j \cos(2mtp) & \text{if } j \in \mathbb{N} \end{cases}$$

(30)

clearly leading to a non-zero limit for integer spins:

$$\rho_j^\prime(t) = \begin{cases} \frac{8\pi}{2j + 1} \sum_{m=1/2}^j \int_0^\infty p^2 \cos(2mtp)e^{-2\lambda p^2} dp \ |j,j\rangle & \text{if } j \in \mathbb{N} + 1/2 \\ \frac{4\pi}{2j + 1} \left(\rho_j^0 + 2 \sum_{m=1}^j \int_0^\infty p^2 \cos(2mtp)e^{-2\lambda p^2} dp \ |j,j\rangle \right) & \text{if } j \in \mathbb{N} \end{cases}$$

(31)

where all the integrals can be evaluated exactly:

$$\int_0^\infty p^2 e^{-\lambda p^2} dp = \frac{\sqrt{\pi}}{4\lambda^{3/2}} \int_0^\infty p^2 \cos(mtp)e^{-\lambda p^2} dp = \frac{\sqrt{\pi}}{8} \frac{1}{\lambda^{3/2}} \left(2 - \frac{k^2 t^2}{\lambda}\right) e^{-\frac{k^2 t^2}{4\lambda}}.$$  

(32)

Beside the zero-mode, all the remaining integrals tend to zero at infinity. Let us not forget that we have omitted from the beginning the global normalization of the states to simplify the equations. Figure 5 shows the typical behavior of the reduced density matrix in the two cases of interest. The conclusion from this simplified version of the reduced density matrix shows that we cannot expect a full decoherence on the spin. Only coherence between half-integer spins is suppressed as time goes to infinity. Coherence with integer spins still exists and the limit value of the reduced density matrix is (up to a normalization factor $\frac{\sqrt{\pi}}{4\lambda^{3/2}}$)

$$\rho_j^\prime(t) \to \frac{1}{2j + 1}.$$  

(33)

Nevertheless, apart from this zero-mode contribution, all the other modes lead to Gaussian decay in $e^{\frac{k^2 t^2}{4\lambda}}$ in terms of the original Gaussian width $\lambda$ and the Fourier mode $k$. So we see a clear decoherence except for that remaining limit coherence decreasing with the spin $j$.

4.4. Bath of harmonic oscillators

The previous calculations we did were with a single harmonic oscillator as an environment. We now consider a bath of $N$ harmonic oscillators and analyze the consequences on the dynamics of our system. In fact we show that only the time scales have changed by the presence of a bath but the mathematical expressions of the reduced density matrix are mostly the same as those already obtained.
The interaction is then $H_{SE} = \gamma \mathbf{L} \otimes \sum \mathbf{p}$, where $\gamma$ is a coupling constant. We again look at the evolution of the same initial state with the vacuum $|0\rangle$ being the vacuum of the whole bath

$$|\psi_{SE}\rangle = \frac{1}{\sqrt{N}}(|j\rangle + |j', g\rangle) \otimes |0\rangle$$  \hspace{1cm} (34)

We again drop the normalization in the following. Using the same method we have

$$\rho^G_S(t; N) = \text{tr}_E \left[ \int_{\mathbb{R}^N} \left| j, e^{i \pi \sigma \sum \mathbf{p}_k g} \right\langle j', e^{i \pi \sigma \sum \mathbf{p}_k g'} | \langle \mathbf{p} | e^{-\lambda (\sum p_i^2 + g^2)} | \mathbf{p} \rangle \right| \right]$$

$$\rho^G_S(t; N) = \int_{\mathbb{R}^N} \left| j, e^{i \pi \sigma \sum \mathbf{p}_k g} \right\langle j', e^{i \pi \sigma \sum \mathbf{p}_k g'} | e^{-2\lambda \sum p_i^2} \right|$$  \hspace{1cm} (35)

All that matters here is the center of mass dynamics $\sum \mathbf{p}_k$ in which the Gaussian width will be affected. To achieve this we perform a change of variable (with a Jacobian equal to one)

$$\left( \begin{array} {c} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_N \\ \end{array} \right) = \left( \begin{array} {cccc} 1 & -1 & \ldots & -1 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \\ \end{array} \right) \left( \begin{array} {c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{N-1} \\ \end{array} \right)$$  \hspace{1cm} (36)

We can then perform the Gaussian integral over the $\mathbf{u}_i$ variable without any issues. This gives again a Gaussian contribution of the form $\frac{2\pi^{(N-1)/2}}{N^{(N-1)/2}} \lambda^\frac{N-1}{2}$. Finally

$$\rho^G_S(t; N) = \frac{\pi^{N-1}}{N} \int_{\mathbb{R}^N} \left| j, e^{i \pi \sigma \sum \mathbf{p}_k g} \right\langle j', e^{i \pi \sigma \sum \mathbf{p}_k g'} | e^{-2\lambda \sum p_i^2} \right|$$  \hspace{1cm} (37)

We can then deduce the scaling law satisfied by the reduced density matrix as a function of $N$. By comparing the above formula by the one with one oscillator, reinserting the normalization, we have

$$\rho^G_S(t; N) = \frac{1}{N^\frac{1}{2}} \left( \frac{\pi^2}{2\lambda} \right)^\frac{N-1}{2} \rho^G_S(\gamma \sqrt{N} t)$$  \hspace{1cm} (38)

We conclude that in presence of $N$ harmonic oscillators instead of only one, the decoherence timescale is shortened by a factor $\sqrt{N}$. Only the convergence speed is affected, not the shape of the decoherence factor.

4.5. Studying the general spin case

The general case is more involved mathematically. We again focus on the initial coherent state with $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and suppose that $j' > j$. To understand the long time behavior of the projected reduced density matrix, we look at its norm first
\[
\text{tr}_S^\rho(t)\rho_S^{\dagger}(t) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} (D(\varrho t/2\hbar, \varrho q)g|D(\varrho t/2\hbar)g)^{\dagger}(D(\varrho t/2\hbar)g')g' e^{-2M(q^2 + p^2)} \, dp \, dq \\
= \int_{\mathbb{R}^4 \times \mathbb{R}^4} (\varphi_{\theta, \phi}^{a, b} + \varphi_{\theta, \phi}^{a, b})^{\dagger} (\varphi_{\theta, \phi}^{a, b} + \varphi_{\theta, \phi}^{a, b}) e^{-2M(q^2 + p^2)} \, dp \, dq \\
= \sum_{a} \sum_{b} \left( \begin{array}{c} 2j_a \\ a \end{array} \right) \left( \begin{array}{c} 2j_b \\ b \end{array} \right) \int_{\mathbb{R}^4 \times \mathbb{R}^4} (\varphi_{\theta, \phi}^{a, b})^{\dagger} (\varphi_{\theta, \phi}^{a, b}) e^{-2M(q^2 + p^2)} \, dp \, dq \\
\]

We perform the explicit calculation in spherical coordinates again. The integral over the azimuthal angles \( \phi \) and \( \phi' \) gives a delta function contribution \( \int_{0}^{2\pi} e^{i(a\theta - \theta')} d\phi = 2\pi \delta(a - b) \), so
\[
\text{tr}_S^\rho(t)\rho_S^{\dagger}(t) = (2\pi)^2 \sum_{a} \left( \begin{array}{c} 2j_a \\ a \end{array} \right) \left( \begin{array}{c} 2j_b \\ b \end{array} \right) \int_{\mathbb{R}^4 \times \mathbb{R}^4} (\varphi_{\theta, \phi}^{a, b})^{\dagger} (\varphi_{\theta, \phi}^{a, b}) e^{-2M(q^2 + p^2)} \, dp \, dq \\
\]

We now look at the integral over the polar angles \( \theta \). First we have
\[
(\varphi_{\theta, \phi}^{a, b})^{\dagger} (\varphi_{\theta, \phi}^{a, b}) \cos^{2j_a - a - \theta} \sin^{2j_a}(\theta' \phi) \\
\]
Gathering all the terms containing the angle \( \theta \), we need to evaluate an integral of the form
\[
\int_{0}^{\pi} \cos^{k+q}(\theta) \sin^{2a+1}(\theta) \, d\theta = 0 \quad \text{if} \quad k + q \quad \text{is odd} \\
= 2 \sum_{p=0}^{a} \left( \begin{array}{c} a \\ p \end{array} \right) \frac{(-1)^{p}}{2(p + K) + 1} \equiv C_{K}^{a} \quad \text{otherwise}, \quad k + q = 2K \\
\]

Thus we have for now
\[
\text{tr}_S^\rho(t)\rho_S^{\dagger}(t) = (2\pi)^2 \sum_{a} \left( \begin{array}{c} 2j_a \\ a \end{array} \right) \left( \begin{array}{c} 2j_b \\ b \end{array} \right) \\
\]
\[
\sum_{k+q=2K} (2j - a - \theta') \left( \begin{array}{c} 2j' - a \\ q \end{array} \right) (-1)^{K} C_{K}^{a} \int_{0}^{\pi} \cos^{2(j' + f' - 2K + a)} \left( \frac{\rho}{2} \right) \sin^{2K+a} \left( \frac{\rho}{2} \right) p^2 e^{-2Mp^2} \\
\]

There only remains now the integral over the norm variable \( p \) where all the time dependance is still hidden. In fact we are only interested on the asymptotic behavior in time of the norm in order to conclude on decoherence of the superposition or not. The important result is that we have a zero limit only on the case when \( 2(j + f') \) is an odd number, meaning that we have initially an integer/half-integer superposition. Otherwise we have a non-zero limit. From the integral
\[
\lim_{t \to \infty} \int_{0}^{\pi} \cos^{2(j' + f' - 2K + a)} \left( \frac{\rho}{2} \right) \sin^{2K+a} \left( \frac{\rho}{2} \right) p^2 e^{-2Mp^2} = \frac{\sqrt{\pi}}{4j + f' + 1} \left( \frac{2j + f'}{a + k} \right) \\
\]

Putting everything together, we finally have
This limit does not always vanish and we give a table of those limits for low spins:

\[
\lim_{t \to \infty} \text{tr} \left( \rho_2^0(t) \rho_2^{j/j'}(t) \right) = \left( \frac{2\pi}{\mathcal{N}} \right)^2 \frac{\pi}{4^2(j+j')^3+1} \left( 2(j+j') \right)^2 \sum_{a=0}^{2j} \sum_{a=0}^{2j'} \left( \frac{2j'}{j} \right) a \left( \frac{2j'}{j} \right) a \left( \frac{2j'+2}{k} \right) \sum_{k'=2a}^{2j'-a} \left( -1 \right)^k \frac{\left( \frac{2j'-a}{k} \right) \left( \frac{2j'+a}{j+k} \right) \left( \frac{j+k}{j+k} \right)}{\left( \frac{2j'+j'}{2a+j'} \right) \left( \frac{2j'+j'}{2a+j'} \right)} C_k^a
\]

(45)

This proves the statement that there exists some type of coherence as time goes to infinity. Table 1 collects numerical evaluation for different values of the spins. It is a straightforward check to recover from this generic formula the limits in the \( j' = 0 \) case given by equation (33). Figure 4 shows the general behavior of the remaining coherence as a functions of the spins.

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**Figure 4.** Some numerical values of the limit of the coherence at infinity. Each value is a rational number that can be obtained by evaluating formula (45). A coherence remains only for integer or half-integer superpositions but tends rapidly to zero as the spins get higher.

**Figure 5.** Numerical evaluations of the two typical behaviors of the norm of the spin coherence (the beginning is Gaussian and has been omitted in the plots to highlights the non-trivial structures). At first the coherence tends to diminish. However, a re-coherence occurs and depending on the nature of the superposition the coherence saturates to a non-zero value (boson/boson like superposition) or tends to zero (fermion/boson like superposition).

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This limit does not always vanish and we give a table of those limits for low spins:

This proves the statement that there exists some type of coherence as time goes to infinity. Table 1 collects numerical evaluation for different values of the spins. It is a straightforward check to recover from this generic formula the limits in the \( j' = 0 \) case given by equation (33). Figure 4 shows the general behavior of the remaining coherence as a functions of the spins.
This remaining coherence at late time $t \to \infty$ decreases as the spins $j$ and $j'$ grow. So in a semi-classical regime, for large quanta of area, we could conclude for an almost-total decoherence. But in the deep quantum regime, with Planck size excitations of the geometry, this remaining coherence might play a non-trivial role.

### 4.6. On the decoherence timescale

The form of the projected reduced density matrix can be obtained exactly at any time from equation (43) and figure 4 represents its typical behavior in the two distinct cases of a boson–boson type superposition and a boson–fermion type superposition. Since the former has a non-zero limit as time goes to infinity, a coherence always remains between those states.

The short timescale behavior is dominated by a Gaussian decay with a damping time inversely proportional to the square of the distance $j - j'$ but also to their sum $j + j'$. This evolution can be obtained by a straightforward expansion at leading order in the time $t$ of equation (39),

$$\rho_S^{jj}(t) \approx \rho_S^{00}(1 - [(j + j')^2 + (j - j')^2])^\frac{1}{4}$$  \hspace{1cm} (46)

This suggests at first sight a decoherence between states with different spins. What is more, the state most immune to the interaction with the environment is the rotation invariant $j = 0$ state as suggested by the $j + j'$ damping. This is natural in the light of the interaction which couples the three rotation operators $L_i$ to the environment.

However, in the long run, a re-coherence appears in the superposition and different conclusions must be drawn. Re-coherence is a natural phenomenon when a finite size environment (with all free dynamics taken into account) is considered. The associated timescale depends on the number of modes of the environment. Only in the limit of an infinite size environment can we obtain a true decoherence but for all practical purposes the timescale can be extremely long.

As stated previously, for the problem at hand, a coherence remains between superposition of two integer or two half-integer spins. The limit value of the norm depends of course on the spins of the superposition. For instance we had the scaling law in $1/(2j + 1)^2$ for the case with $j' = 0$. For integer/half-integer spins superposition the coherence dies out as time goes to infinity but with a typical timescale completely independent of the spins. This can be straightforwardly obtained by expanding equation (43): the asymptotic behaviors are Gaussian of the form $e^{-\frac{t}{2}}$ or $e^{-\frac{t}{4j}}$, respectively, from the integer superposition and the half-integer superposition.

Finally, if we consider a non-dynamical bath of oscillators for the environment, all those coherence times and limits are rescaled by the number $N$ of oscillators given by formula (38).

### 4.7. A natural extension: how to get rid of recoherence

The analysis of the $N = 2$ patches model shows that the interaction as it is does not lead to decoherence on states with a definite closure defect value. The pointer states are not eigenstates of the operator $L^2$. A natural solution to this problem is to force the environment to couple to this operator.

Let us consider the formal interaction between $L = |L\rangle$ and $p = |p\rangle$, adding a term $L \otimes p$ to the original interaction Hamiltonian $H_{SE}$ that we postulated in (23). The analysis is quite simplified by the fact that this term in the Hamiltonian commutes with the first, so we can consider it alone. It leads to a decoherence between different spins. Consider again the superposition (25). Since $L|j, g\rangle = j|j, g\rangle$, the superposition evolves at time $t$ into the state
The states of the environment have the form $|E_j(t)\rangle = (e^{-i\hbar\eta \rho}|0\rangle)$. The decoherence factor for the non-diagonal matrix elements of the reduced density for the system is then the overlap $|\langle j,j'\rangle|$. A straightforward calculation in the momentum basis gives the explicit Gaussian behavior

$$|\langle j,j'\rangle| = e^{-\frac{1}{2\gamma_{\text{deco}}^2}(j-j')^2}$$

with the decoherence time $\gamma_{\text{deco}} = 1/|j - j'|$. There is thus a decoherence for spin superposition with a damping time inversely proportional to the distance between the spins.

The operator $\mathbf{L}^2$ can be written in terms of deformation operators. Using the relations $\mathbf{L}_i\mathbf{L}_j = \frac{1}{2}E_iE_j - \frac{1}{2}E_i - \frac{1}{2}E_j$ or in terms of the creation operators $F_{ij} = a_i a_j - a_j a_i$, $\mathbf{L}_i\mathbf{L}_j = \frac{1}{2}E_iE_j - \frac{1}{2}F_{ij}^*F_{ij}$, we can obtain the relations

$$\mathbf{L}^2 = E^2 \left(\frac{E^2}{2} + 1\right) - E_1 F_{12} = \frac{1}{4}(E_1 - E_2)^2 + \frac{1}{2}(E_{13}E_{12} + E_{12}^*E_{21})$$

Those operators could be coupled to the environment to induce a decoherence on the closure defect. In essence it amounts to coupling the Casimir operator (9) to the environment.

To conclude the discussion of the specific surface state with $N = 2$ patches with a non-dynamical environment, we have shown that the dynamics induces a decoherence effect for superposition of different spins on short timescales. A re-coherence occurs in the long run and we concluded on the damping of coherence only for superpositions of integer/half-integer spins. If we insist on having a decoherence on the closure defect, a natural solution is to introduce a new coupling to the environment.

### 5. Master equation approaches

Most models of open quantum systems and studies of decoherence are not exactly solvable and approximate methods have to be developed. Master equations based on Born–Markov approximations are the ones most commonly used for analyzing open quantum dynamics in quantum optics and condensed matter physics. They are equations for the reduced density matrix of the system taking into account the effects of the environment to first order. They are relevant for understanding the behavior of the system at a time $t$ much longer than any correlation times $\tau_c$ but still shorter than dynamical timescales $T$: $\tau_c \ll t \ll T$. This is the essence
of the Markov approximation. A large environment is needed to neglect the changes of the state of the environment due to the coupling to the system and correlations up to second order.

In the following, we apply the master equation methods to the problem of open quantum surface dynamics by first deriving the Born–Markov master equation. This step will motivate a more phenomenological approach by postulating jump operators for the Lindblad equation. The results of those different approaches are then compared to the exact results obtained previously.

5.1. Born–Markov equation

An approximate equation for the reduced density matrix of the system can be derived by an expansion of the exact equation of motion \( \frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] \) [22]. For an interaction written as \( H_{SE} = \sum_j S_j \otimes E_j \), it has the general form

\[
\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H_S, \rho_S(t)] + \frac{1}{\hbar^2} \sum_i U_i(t)\rho_S(t)S_i + S_i\rho_S(t)U_i^\dagger(t) - S_iU_i(t)\rho_S(t) - \rho_S(t)U_i^\dagger(t)S_i
\]

(50)

with the operators \( U_i(t) = \int_0^t \sum_j g_{ij}(\tau)S_j(\tau - \tau) \, d\tau \) encoding the action of the environment on the system and depending on its correlation functions \( g_{ij}(\tau) = \langle E_i(t)E_j(t - \tau) \rangle_{\rho_S} \).

To go further, the behavior in time of the correlation functions must be discussed. It depends naturally on the proper dynamics of the environment \( H_E \) and on the state \( \rho_E \). For a dynamical environment, the correlation functions decay over a timescale \( \tau_c \) called correlation time or memory time. Denoting by \( \nu \) an order of magnitude of an element of matrix of the interaction, equation (50) is an expansion in the parameter \( \nu \tau_c \). The order of magnitude of the coupling in the Born–Markov equation is \( \nu^2\tau_c/\hbar \) which is then much smaller than the memory frequency \( \tau_c^{-1} \) in the short memory time approximation. The complete Born–Markov equation is then obtained by approximating the integral in \( U_i(t) \) by its value at infinite time giving in the end a pure local in time equation of motion. However, if the environment were small or non-dynamical, the natural expansion parameter would be \( \nu t/\hbar \) and the results of the Born–Markov equation would be inaccurate on long timescales and the time dependence of the correlation functions must be kept. This is an issue we will discuss further in the section comparing the different approaches.

Now for the specific problem we are interested in, we use the interaction (17) and express the Born–Markov equation. The equation is here simplified by the fact that we neglect the proper dynamics of the surface. In particular, the operators \( U_{ij} \) have the simple form

\[
\int_0^t g_{ij,k}(\tau) S_k(\tau - \tau) \, d\tau
\]

(51)

To go further we have to specify the form of the correlation functions. First it is natural to expect the correlation functions to be symmetric in time. To be more specific, let us imagine we have a harmonic environment and that the operator \( V_{ij} \) creates a photon at \( j \) with creation
operator $\gamma^j_\dagger$ (a quanta of area is destroyed) and absorbs one at $i$ with destruction operator $\gamma_i$ (a quanta of area is created) so $V_{ij} = \gamma^j_\dagger \gamma_i$. For the environment in the vacuum state or thermal state (any Gaussian states), the Wick theorem applies and allows us to develop the correlation functions. Replacing those correlation functions into the master equation is then straightforward. For an isotropic, homogeneous non-dynamical environment, we obtain the simplest form of the equation

$$\frac{d\rho_S}{dt} = \sum_{i,j=1}^N \kappa(t) \left[ E_{ij} \rho_S E_{ij}^\dagger - \frac{1}{2} (E_{ij}^\dagger E_{ij}^\dagger \rho_S + \rho_S E_{ij}^\dagger E_{ij}^\dagger) \right]$$

(52)

where $\kappa(t) = t\kappa$ with $\kappa$ a constant function of the correlation function. This master equation has the Lindblad form. In the full Born–Markov approximation, $\kappa(t)$ would be independent of time and a decoherence would be expected a priori with an exponential decay $e^{-t/\tau_d}$ with $\tau_d$ a decoherence timescale. Here however, the linear time dependence caused by the non-dynamical character of the environment (non-Markovianity) leads to a decoherence with a Gaussian behavior $e^{-t^2/\tau_d^2}$. This form is in full agreement with the short time exact calculations (46).

5.2. Lindblad approach

Once again, we focus on the simplest $N = 2$ patches model with the spin interaction part and take a phenomenological approach to it with the Lindblad master equation. The jump operators (Lindblad operators) are the spin $L_i$ operators and no free dynamics is supposed to occur for the system. We should not forget that we really consider the Schwinger representation here and that we work not in the Hilbert space at a given spin $j$. Superposition of states with different values of the spin $j$ are permitted. Let us emphasize there are some subtleties regarding the correlation functions and the definition of the jump operators in order to compare those master equation approaches to the exact dynamics proposed in the last section due to the hypothesis of a non-dynamical environment $H_E = 0$. We keep in mind this important point but discuss now in a phenomenological way a Lindblad equation with $J_i$ jump operators as done traditionally in quantum optics models.

The master equation we propose to study is thus

$$\frac{d\rho_S}{dt} = \sum_{i=x,y,z} L_i \rho_S L_i - \frac{1}{2} (L_i L_i \rho_S + \rho_S L_i L_i) = \sum_{i=x,y,z} L_i \rho_S L_i - \frac{1}{2} (L^2 \rho_S + \rho_S L^2)$$

(53)

For the surface dynamics we are ultimately interested in, we want to understand if there is a decoherence phenomenon on a superposition with different values of the spin $j$. Since $L^2$ commutes with the jump operators, the environment does not induce transitions between states with different spins and no dissipation occurs. To focus on coherence between different spin states, we can look again at the projection of the reduced density matrix $\rho_{kl} = P_k \rho P_l$ with $P_{kl}$ the projection operator on the subspace of spin $k$ and $l$, respectively.

$$\frac{d\rho_{kl}}{dt} = \sum_{i=x,y,z} L_i \rho_{kl} L_i - \frac{1}{2} ((k(k+1) + l(l+1))\rho_{kl}$$
Searching for pointer states (approximate pointer states generally) requires us to evaluate a measure of entanglement such as the Von Neumann entropy or the purity of the states $\rho(t)$. For our purpose we will mostly focus on the purity of the projected reduced density matrix.

$$\frac{d}{dt} \rho_{kl}^\dagger = \sum_{i=k, l} 2\epsilon L_i \rho_{kl} - (k(k+1) + l(l+1)) \rho_{kl}$$  \hspace{1cm} (54)

Let us for instance look at the short time evolution of the superposition $|\psi\rangle = \frac{\frac{1}{\sqrt{2}} |k\rangle + |l\rangle}{\|}|\psi\rangle$, \hspace{1cm} (55)

We thus qualitatively see that a superposition of different spin states leads to a more rapid entanglement with the environment than a state with definite spin. Moreover, we see that only a rotation invariant state is immune to entanglement (at first order) with the environment, whereas even a state with a definite spin gets entangled with its environment (the higher the spin the more entangled). This behavior can be generalized to an arbitrary initial pure state (for the proof see appendix B)

$$\frac{d}{dt} \rho_{kl}^\dagger \bigg|_{t=0} = -[k-l]^2 + (k+l) \rho_{kl}^\dagger$$  \hspace{1cm} (56)

Let us discuss now the relations between the different approaches and in particular why the conclusions appear not to be the same. We have explored in two different ways a possible decoherence effect for quantum superposition of states with different spins associated with the closure defect. The first method was based on an exact calculation for the $N=2$ patches model and the second used the traditional methods of Markovian master equations.

- The ingredient for master equations to work is to have a large enough dynamical environment for its correlation functions to vanish on a timescale smaller than any relaxation or observational times. Qualitatively said, the environment is without memory. In this context, we have shown that the surface (approximate) pointer states are those with a given value of the closure defect and that the decoherence factor has an exponential decay $e^{-\tau_2 t}$. The difference between Gaussian and exponential decay is thus traced back to the memory of the environment controlled by its dynamics.

9 The choice of one or the other should in a proper limit give the same approximate results. In fact, with the purity is associated a linear entropy $S_{\text{lin}} = 1 - \rho^2$. This is the first term in the expansion of the Von Neumann entropy. Computing the trace of the successive powers of the density matrix, Renyi entropies can be defined and finally using the replica trick the Von Neumann entropy can be obtained.
The predictions on the decoherence effect differ for the two methods and only match on a short timescale. In particular the exact analysis shows that a recoherence occurs with a non-zero limit (a limit still approaching zero as the spins get higher). If as expected the closure defect is associated with a quasi-local energy density and the curvature or torsion it generates, the spin is also expected to be high enough for a black hole. Thus for all practical purposes, we can conclude on an effective decoherence on the closure defect.

6. Conclusion

Decoherence is now a cornerstone of quantum physics to clarify the quantum-to-classical transition. In a theory of quantum gravity, the geometry is a dynamical and fluctuating field and quantum superposition of geometry are perfectly allowed states. Their non-observability in the classical regime remains to be clarified in the semi-classical analysis of loop quantum gravity. Our first investigation focus on the open dynamics of a quantum surface coupled to an environment comprising all the other gravitational and matter degrees of freedom of the Universe. This bulk-boundary coupling induces a decoherence and our aim was to understand the emergence of some geometrical super-selection sector.

Through the deformation formalism of quantum geometry, we proposed toy models for the open dynamics of a quantum surface in the context of loop quantum gravity and a natural coupling between a bath of harmonic oscillators and the deformations of the surface. We looked for a decoherence on the closure defect of a surface with fixed area using two different methods: one exact method analyzing the physical effect on a superposition of the interaction part of the Hamiltonian (quantum measurement limit) and the other using master equations approaches under Born–Markov approximations. The two approaches agree on the short timescale and indeed conclude on a decoherence of quantum superposition of states with different spins associated with the closure defect. The decoherence factor is here a Gaussian decaying with a timescale inversely proportional to the spin difference. However, due to the
different treatment of the structure of the environment, the conclusions differ as time goes to infinity. The exact treatment neglects the proper dynamics of the environment which thus has an infinite correlation time (constant correlation functions) and leads to a re-coherence of integer/integer or half-integer/half-integer superpositions. Nonetheless this non-zero limit is for all practical purposes irrelevant when large spins are considered, which is potentially the case for black holes.

To sum up, the lesson for (loop) quantum gravity to draw from the analysis of decoherence in our very simple toy models is that bulk (interior and exterior) degrees of freedom of the gravitational field (and matter field) will likely induce a decoherence of geometrical observables on the boundary (see figure 6 for a sum up). For instance, in our simplified setting, we observe a decoherence on the closure defect, which can be seen as a measure of curvature of the interior region bounded by the surface. This decoherence is well-understood using a Markovian dynamic but, depending on the precise bulk/boundary interaction and the structure of the environment, non-Markovian effects could be important and lead for instance to re-coherence effects.

From the present construction, we can see a few lines along which we could further develop the surface dynamics model and refine our study of decoherence:

- The free dynamics can be properly taken into account. This would allow for instance to rigorously verify that an environment without memory would lead to a full decoherence on the spins since the compactness of the SU(2) group could not be seen by the environment.
- A drawback of the current approach is that we are considering a dynamics and a decoherence of a geometry evolving in a given classical time. To be more true to the relativistic point of view, it would be most interesting to have a quantum model where a classical notion of time would emerge from a decoherence process along with the decoherence on geometric properties. We would then look at the flow of correlations between two observables of a system and a quantum clock. Some relationships between an intrinsic decoherence induced by a (discrete) quantum time have been explored in [33, 34].
- Before considering even the coupling of the boundary surface and the bulk, and instead of the natural harmonic oscillators dynamics for the system, we could consider a more involved model for the free boundary such as a Bose–Hubbard model. The horizon of the black hole would then be seen as an interacting gas of punctures [35, 36]. The phase diagram as a function of the mass and temperature could then be studied, checking that at high mass there exists a superfluid phase and Bose gas phase at small mass, respectively characterized by a diffusive and a ballistic response to local perturbations.

The semi-classical analysis of loop quantum gravity has mostly up to now been focused on understanding coherent states interpolating a quantum and classical geometry and on the coarse-graining of spin network states. Still, it is an important and non-trivial issue to understand in a quantum theory of gravity the quantum-to-classical transition through a decoherence mechanism and poses some conceptual questions. From the perspective of describing quantum gravity from quantum information, for instance by computing entanglement entropy, a proper definition on the separation between bulk, boundary and exterior degrees of freedom in quantum gravity has to be found [37]. The subtleties come from the gauge invariance or diffeomorphism invariance. The state of an exterior observer is then obtained by tracing out the bulk degrees of freedom composed of matter and gravitational degrees of freedom. This step raises questions again in light of the holographic principle from which we learn that the bulk degrees of freedom are fully encoded on the boundary. The very meaning of tracing out bulk degrees of freedom is quite ambiguous. After clarifying those conceptual issues, we could then investigate the existence of some decoherence phenomena seen by an outside observer
on the horizon induced by the bulk-boundary coupling and identify the semi-classical states (pointer states) selected by the bulk or better understand the relationship between coarse-graining methods and tracing out degrees of freedom.

Appendix A. Reduced density matrix of the environment

In the core of this paper, we analyzed the reduced density matrix of the surface in contact with an unmonitored environment. We could also look at the behavior of the reduced density matrix of the environment. We recall that the environment is composed of all the degrees of freedom (matter or gravitational) in the Universe except those associated with the system.

Consider the state
\[
|\psi_{SE}\rangle = \frac{1}{\sqrt{N}} \left| j, \sigma \right\rangle \otimes \left( |p\rangle + |q\rangle \right)
\]  
(A.1)

Since we have \( U(t) |j, \sigma\rangle |p\rangle = |j, e^{-i\pi j/8} \sigma\rangle |p\rangle \), the state at time \( t \) is simply
\[
|\psi_{SE}(t)\rangle = \frac{1}{\sqrt{N}} \left( |j, D(tp/2, \hat{p})\sigma\rangle |p\rangle + |j, D(tq/2, \hat{q})\sigma\rangle |q\rangle \right)
\]  
(A.2)

Tracing over the surface, we can obtain the reduced density matrix of the environment. The coherence terms are modulated by a decoherence factor which is the overlap
\[
\int |\langle j, D(tp/2, \hat{p})\sigma| j, D(tp/2, \hat{q})\sigma\rangle|^2
\]  
(A.3)

We specify the calculation to the spin up case \( g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) to have an explicit form of the overlap
\[
\langle D(tp/2, \hat{q})\sigma| D(tp/2, \hat{p})\sigma\rangle^2 = \langle \cos(tp/2) \cos(tq/2) + \sin(tp/2) \sin(tq/2) \hat{p} \cdot \hat{q} \\
- i(\cos(tp/2) \sin(tq/2) \hat{q} \cdot \sigma + \cos(tq/2) \sin(tp/2) \hat{p} \cdot \sigma \\
+ \sin(tp/2) \sin(tq/2)(\hat{p} \wedge \hat{q}) \cdot \sigma) \rangle^2
\]  
(A.4)

A more general state for the system could be considered as a superposition on the spins \( \sum_j \alpha_j |j, \sigma\rangle \), thus generalizing the overlap (A.4) to
\[
O_{pq}(t) = \sum_j |\alpha_j|^2 \langle D(tp/2, \hat{q})\sigma| D(tp/2, \hat{p})\sigma\rangle^2
\]  
(A.5)

Let us consider a particular superposition with amplitude \( \alpha_j = 1/\sqrt{2} \). This simplifies the overlap to an exponential
\[
O_{pq}(t) = e^{i\langle D(tp/2, \hat{q})\sigma| D(tp/2, \hat{p})\sigma\rangle}
\]  
(A.6)

The phase of this overlap corresponds to some relaxation, whereas the modulus is the decoherence factor \( D_{pq}(t) \) of the superposition that has the simple form
\[
D_{pq}(t) = e^{i\cos(tp/2) \cos(tq/2) \sin(tp/2) \sin(tq/2) \hat{p} \cdot \hat{q}}
\]  
(A.7)

This decoherence factor is periodic in time and thus does not lead to a proper decoherence in the momentum as one could have expected from the interaction form. The origin of this periodicity can be traced back to the compact structure of \( SU(2) \sim S^3 \).
Let us show that for small time, we recover a decoherence in the momentum comparable to
the one obtained in ‘the flat case interaction’ \( H_{SE} = \mathbf{x} \otimes \mathbf{p} \). For this interaction, it is straightforward to show that the decoherence factor has the form \( D_{pq}(t) = e^{-\frac{(p^2 + q^2)}{2}t} \). Doing the expansion in time of equation (A.7), we have

\[
D_{pq}(t) \propto e^{-\frac{(p^2 + q^2)}{2}t} + r p q \]

As long as the structure of the rotation group is not explored completely, we obtain the same decoherence effect as in the flat case. This is a consequence of the local flatness of SU(2).

**Appendix B. Proof the the bound on the purity evolution**

**Proposition.** For an initially pure state of the system, the short time behavior of the purity evolve according to

\[
\frac{d \text{tr} \rho_{kl} \rho_{kl}^\dagger}{dt} \bigg| _{t=0} \leq - [(k - l)^2 + (k + l)] \text{tr} \rho_{kl} \rho_{kl}^\dagger \tag{B.1}
\]

**Proof.** We want to obtain a differential inequality on the norm of the coherence for the reduced density matrix of the system. Consider then a pure state \( |\psi\rangle \) of the system and develop it on the coherent states basis.

\[
\rho_S = |\psi\rangle \langle \psi| = \sum_j \int_{S^2} \psi_j(\hat{n}) |j, \hat{n}\rangle \langle j, \hat{n}| \ d^2n
\]

With this decomposition, the projected reduced density matrix and its norm are

\[
\text{tr} L_i \rho_S L_i \rho_S = \sum_j \int_{S^2} \psi_j(\hat{n}) \psi_j(\hat{n}') \langle j| L_i |j\rangle \langle \hat{n}'| L_i |\hat{n}'\rangle \ d^2n d^2n' \tag{B.2}
\]

The overlap between two coherent states of the spin operator \( L_i \) for arbitrary spin can be obtained using the special case of the spin 1/2

\[
\langle k, \hat{n}'|L_i|k, \hat{n}\rangle = k \langle \hat{n}'| \sigma_i |\hat{n}\rangle \langle \hat{n}'| \hat{\sigma}_i |\hat{n}\rangle^{2k-1} \tag{B.3}
\]

We then write the evolution equation and isolate the contribution we are interested in. The aim is then to obtain an inequality on the remaining term:

\[
\frac{d \text{tr} \rho_{kl} \rho_{kl}^\dagger}{dt} = - [(k - l)^2 + (k + l)] \text{tr} \rho_{kl} \rho_{kl}^\dagger + 2|\alpha_k|^2 |\alpha_l|^2 kl \\
\times \int_{S^2} \psi_k(\hat{n}) \psi_l(\hat{n}) \psi_k(\hat{n}') \psi_l(\hat{n}') \langle \hat{n}'| \hat{n}\rangle^{2k-1} \langle \hat{n}'| \hat{n}\rangle^{2l-1} \\
\times \left( \sum_{\sigma, \tilde{\sigma}} \langle \sigma| \tilde{\sigma} \rangle \langle \tilde{\sigma}| \sigma\rangle - \langle \sigma| \tilde{\sigma}\rangle \langle \tilde{\sigma}| \sigma\rangle \right) \ d^2n d^2n' d^2n'' d^2n''' \tag{B.4}
\]
Clearly we need to show the integral is not negative to obtain the required result. This integral is first of all real. Then by using $\sum_{p=x,y,z} \text{tr} \sigma_i A \sigma_i B = 2tAuB - tAB$ with $A$ and $B$ two $2 \times 2$ matrices, we can evaluate the sum on the coordinates,

$$\sum_{i=x,y,z} \langle \hat{n}| \sigma_i |\hat{n}\rangle \langle \hat{n}| \sigma_i |\hat{n}\rangle = 2\langle n'|m\rangle \langle m|n \rangle - \langle n'|\hat{n}\rangle \langle \hat{n}|n \rangle$$ (B.5)

The integral has for now the following form

$$2 \int_{S^2} \psi(\hat{n}) \psi^*(\hat{n'}) \langle \hat{n}| \sigma_i |\hat{n}\rangle \langle \hat{n}| \sigma_i |\hat{n}\rangle (\hat{n}|\hat{n'}) 2k - 1 |(n'|m\rangle \langle m|n \rangle - \langle n'|\hat{n}\rangle \langle \hat{n}|n \rangle) \, d^2nd^2m'd^2m'$$

(B.6)

The Cauchy–Schwarz inequality will allow us to conclude. To see this, we write the first term of the integral as a trace

$$\int_{S^2} \psi(\hat{n}) \psi^*(\hat{n'}) \langle \hat{n}| \sigma_i |\hat{n}\rangle \langle \hat{n}| \sigma_i |\hat{n}\rangle (\hat{n}|\hat{n'}) 2k - 1 |(n'|m\rangle \langle m|n \rangle - \langle n'|\hat{n}\rangle \langle \hat{n}|n \rangle) \, d^2nd^2m'd^2m'$$

$$= \text{tr} \int_{S^2} \psi(\hat{n}) \psi^*(\hat{n'}) \langle \hat{n}| \sigma_i |\hat{n}\rangle \langle \hat{n}| \sigma_i |\hat{n}\rangle (\hat{n}|\hat{n'}) 2k - 1 |(n'|m\rangle \langle m|n \rangle - \langle n'|\hat{n}\rangle \langle \hat{n}|n \rangle) \, d^2nd^2m'd^2m'$$

(B.7)

The two operators $O_k$ and $O_l$ are Hermitians and positives. With $0 \leq \text{tr} O_k O_l \leq \text{tr} O_l \text{tr} O_k$ we have

$$\text{tr} O_k O_l \leq \int_{S^2} \psi(\hat{n}) \psi^*(\hat{n'}) \langle \hat{n}| \sigma_i |\hat{n}\rangle \langle \hat{n}| \sigma_i |\hat{n}\rangle (\hat{n}|\hat{n'}) 2k - 1 |(n'|m\rangle \langle m|n \rangle - \langle n'|\hat{n}\rangle \langle \hat{n}|n \rangle) \, d^2nd^2m'd^2m'$$

(B.8)

This concludes the proof that the integral in (B.4) is always negative and also the differential inequality we conjectured.

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