Plane contours with maximum thrust in a non-isentropic flow

A R Mustaev and N A Ostapenko
Institute of Mechanics of Lomonosov Moscow State University, 1 Michurinsky prosp.,
Moscow 119192, Russia

E-mail: sentry-cod4@mail.ru

Abstract. Plane tasks of the contour and maximum thrust nozzle modelling in the supersonic flow with shock waves are formulated and solved analytically and numerically. The first of the considered problems relates to the optimal profiling of the tail part of the body. This part consists of some corner point, followed by straight section and following him, unknown in advance curved section on which the shock interaction of the flow with the wall occurs. To improve the characteristics of the "shock" contours the exact solution of the corresponding optimization problem in the class of straight segments is considered. The second problem relates to the construction of a plane supersonic nozzle of maximum thrust in the class of contours similar to those considered in the first problem. The isentropic flow is modeled by the flow from a supersonic source. This approach allows to find the initial shape of the optimal "impact nozzle" for the subsequent improvement of its shape using a numerical method of calculation. Parametric calculations of the flow in the supersonic part of the nozzle consisting of conical and parabolic sections and its thrust are carried out which is compared with the thrust of the equivalent conical nozzle.

1. Introduction
Traditionally it is believed that the optimal supersonic nozzle with maximum thrust under other identical conditions is a nozzle with an isentropic flow in it. In this research, it is proposed to look for the nozzle (external contour of the body) of maximum thrust, by intentionally allowing the existence of shock waves in the flow. It is proposed to design the nozzle in such a way that at the initial stage, at the exit from the critical section the flow accelerates in a centered rarefaction wave near the angular point and further in the conical expanding part of the nozzle and then subjected to shock compression in the wave specially generated at the point of fracture of the nozzle contour.

Thus, in the optimization problem, the angle of inclination $\theta$ of the straight section of the contour, the position of the fracture point and the shape of the second, in the General case, curvilinear, section of the contour coming to the nozzle (contour) end, should be determined.

2. The problem of the maximum thrust contour in a supersonic flow with shock waves
Let us set the following model problem. Plane contour ABC (figure 1) consists of a straight segment AB which is an angle $\theta$ with the x axis and a curved BC which can move into each other with a fracture. The coordinates of the points A(0,$y_a$) and C($x_c,y_c$) are set. Suppose that the line $x=0$, $y<y_a$ is given a uniform flow parallel to the axis Ox with Mach number $M \geq 1$, density $\rho_\infty$ and velocity $u_\infty$. Turning around the corner point A in the rarefaction wave Prandtl-Meyer to some supersonic
speed, he then runs into a section of the BC circuit. The pressure $p_1$ on the wall BC will be determined by the formula

$$p_1 = p + \frac{1}{2} \rho u^2 C_p$$  \hspace{1cm} (2.1)\]

Where $C_p$ is the pressure coefficient, $\rho$, $p$, $u$ are, respectively, the pressure, density and velocity over the last characteristic in the Prandtl-Meyer wave. We assume that the influence of the last characteristic of the rarefaction wave does not reach the contour BC, and the resulting shock wave leaves point B. For the force acting on the contour in the direction opposite to the positive direction of the x axis, we obtain the expression

$$T = \int_{y_a}^{y_c} p(y_c - y_a) + \frac{1}{2} \rho u^2 \int_{x_a}^{x_c} C_p y dx$$  \hspace{1cm} (2.2)\]

Here $y(x)$ is a function describing the shape of the contour BC. Introducing new dimensionless variables $x = x_0 \xi$, $y = x_0 \eta + y_a$ and parameters $\lambda = (y_c - y_a) / x_c$, as well as using the known relations between the flow parameters $\rho$, $p$, $u$ and the Mach number for the isentropic motion of the particle, from (2.2) we obtain an expression for the thrust coefficient of the plane contour ABC (figure 1)

$$C_p = f(M)\frac{T}{x_0} = f(M)\frac{\gamma}{2} M^2 \int \left[ C_p \eta d \xi \right] f(M) = 1 / \left( 1 + \left( \frac{\gamma - 1}{2} \right) M^2 \right)^{\frac{\gamma}{\gamma - 1}}$$  \hspace{1cm} (2.3)\]

In (2.3) $\gamma$ is the ratio of specific heats, $M$ is the Mach number behind the last characteristic of the rarefaction wave determined by isentropic formulas for given $M_\infty$ and $\theta$. We take a Newtonian model of particle collision with a wall BC. Then the pressure coefficient $C_p = 2 \sin^2 \alpha$ where $\alpha$ is the angle between the direction of the uniform flow over the last characteristic of the Prandtl-Meyer flow and the BC wall

$$\alpha = \theta - \arctan \eta$$  \hspace{1cm} (2.4)\]

Substituting (2.4) into (2.3) we find

$$C_p = f(M) f(\lambda + \gamma M^2 \int_0^1 \left( \sin \theta - \eta \cos \theta \right)^2 \eta / (1 + \eta^2) d \xi)$$  \hspace{1cm} (2.5)\]

We formulate the following variational problem: to find such a function $\eta(\xi)$ and angle $\theta$ which delivers the maximum functional $C_T$ (2.5) for a given number $M_\infty$ and parameter $\lambda = \eta(1)$. The variation of the functional considering when $\delta \xi = \xi d \theta$ written in the form

$$\delta C_T = M^2 \int_0^1 \left[ F_d - \frac{d F_d}{d \xi} \right] \delta \eta d \xi - \gamma M^2 \int_0^1 \left[ F + (tg \theta - \eta) F_\eta \right] \delta \eta d \xi \left[ f(M) \right] f(M) \eta d \xi - \gamma M^2 \int_0^1 \left[ f(M) \right] f(M) \eta d \xi - \gamma M^2 \int_0^1 \left[ f(M) \right] F_d d \xi + \gamma M^2 \int_0^1 \left[ f(M) \right] F_\eta d \xi - \gamma M^2 \int_0^1 \left[ f(M) \right] F_\eta d \xi \cos^2 \theta d \theta$$  \hspace{1cm} (2.6)\]

In (2.6) $F$ is the main function in the functional (2.5). The Euler equation admits the first integral $F - \eta F_\eta = C$ according to which the extremal segment BC is a line segment. By equating the coefficient of variation $\delta \xi b$ (2.6) to zero, two solutions can be obtained
\[ \dot{\eta} = \tan \theta \text{ and } \dot{\eta} = \tan \frac{\theta}{2} \]  

(2.7)

The first of them corresponds to a straight contour connecting points A and C which is called conical. The second corresponds to the "shock" contour, the area of existence of which is of interest. Note that the angle of inclination of the rectilinear segment BC of the extremal two times less than the angle of inclination \( \theta \) of the line segment AB.

Equating to zero the coefficient of variation \( \delta \theta \) (2.5) we obtain

\[ \lambda (M^2 - 1)^{-\frac{1}{2}} + \frac{1}{8} \left( (M^2 - 1)^{\frac{1}{2}} - (M^2 - 1)^{-\frac{1}{2}} \right) \dot{\eta}_b - 2 \left( 1 + \dot{\eta}_b^2 \right) \sin(2 \theta) (1 - \xi_b) - \frac{1}{2} \tan \theta \dot{\eta}_b \xi_b = 0 \]  

(2.8)

\[ \dot{\eta} = \tan \frac{\theta}{2}, \quad \xi_b = (\lambda - \eta_b) / (\tan \theta - \eta_b) \]

The solution of the equation (2.8) determines the desired optimal "shock" contour. Some curves of \( \theta = \text{const} \) solutions are shown in figure 2 (solid curves 1 to 5 correspond to \( \theta \) from 20° with a step of 10°). The ends of the curves (the envelopes of them are represented by dashed lines) correspond to the conical profile.

![Figure 2. Some extremal curves \( \theta = \text{const} \).](image1)

![Figure 3. Break point behavior.](image2)

The position of the fracture point \( \xi_b \) on extremals depending on \( M_{\infty} \) and \( \theta \) can be traced in figure 3. Solid curves 1-5 correspond to the same values of the angle \( \theta \) as in figure 2. On curves 4 and 5, the symbol 1 shows the boundary with the values of the number \( M_{\infty} \) to the right of which the arc extremals BC (figure 1) the influence of the centered Prandtl-Meyer wave begins to propagate through its last characteristic and the corresponding characteristic in the shock layer near the second rectilinear segment of the extremal BC.

The ratio of the thrust of the optimal shock contour \( T \) to the thrust of the equivalent in size conical contour \( T_C \) is shown in figure 4 depending on \( M_{\infty} \) at different angles of inclination \( \theta \) of the first extremal segment (solid curves 1-5 correspond to the same values of the angle \( \theta \) as in figure 2).

Symbols 1 on solid curves 4 and 5, as in figure 3 indicate the beginning of the influence of the Prandtl-Meyer wave at the second part of the extremal, is built on the theory of Newton. The left ends of solid curves 1-5, coming to the abscissa axis, indicate that for smaller numbers \( M_{\infty} \) and corresponding \( \theta \), the extremal will be a conical contour. When the number \( M_{\infty} \) tends to the limit value when at \( \theta = \text{const} \) for a reversal around the point A, the flow with the number \( M = \infty \) is realized (the right ends of the solid curves in figure 2), \( T / T_C \rightarrow \infty \). Note that \( T \) was calculated on "Newtonian" contours by exact solution.
3. Exact solution of the optimization problem in the class of contours consisting of two straight segments

Natural development of the problem considered in section 2, is the search for shock contours in the class of straight segments in the exact formulation. In other words, it is proposed not to use Newton's theory to describe the interaction of a supersonic homogeneous flow flowing after a turn in the Prandtl-Meyer dilution wave along the first straight segment of the extremal AB with the second straight segment BC, but to consider the exact flow around the segment BC as a flow around a wedge with an angle α by a supersonic flow with the Mach number M.

Without giving the necessary conditions of the extremum and their analysis, we present some data on the solution of this problem. The curves $\theta = \text{const}$ corresponding to optimal shock solutions are shown in figure 2, the dashed lines 1-5. The corresponding dependences for the position of the fracture point $\xi_b$ extremals, as well as $T/T_C$ are shown in figures 3 and 4, respectively (dashed lines 1-5). An important conclusion, which follows from the analysis of the exact solution of the optimization problem, is the absence of the influence of the rarefaction wave on the extremal area BC, in contrast to the solution within the Newton's theory. The transition points of the solution from the optimal shock contours to the conical contours at $T/T_C = 1$ are indicated by the symbol 2 in figure 4 for $\theta = 20^\circ$ and $30^\circ$ (dashed curves 1 and 2, respectively).

Figure 5 shows data on the value of the angle of fracture of the extremal $\alpha$ under the same notation of curves in figure 2. As can be seen in the exact solution angle $\alpha$ is not a constant value, as is the case in Newton's theory (2.8). However, for $M_\infty$ tending to the limit value at each $\theta = \text{const}$, when the Mach number behind the Prandtl-Meyer wave $M \rightarrow \infty$, the angle $\alpha$ asymptotically tends, in each case, to some constant value close to $\theta/2$, which is a confirmation of the result following from the solution of the variational problem using Newton's theory. In general, in the exact formulation, the extremal-
circuit implementing the flow with the shock wave exists in a wide range of changes in the defining parameters, which indicates the feasibility of finding appropriate solutions in supersonic nozzles.

4. The problem of a plane nozzle of maximum thrust with shock waves in a supersonic flow

The optimal plane impact nozzle is similar to the problem considered in sections 2 and 3, is sought in the class of circuits consisting of a straight section AB, constituting an angle \( \theta \) with the nozzle axis, during which the isentropic acceleration of the flow, and following it, unknown in advance curvilinear section BC, described by the function \( \eta(\xi) \), on which the shock interaction of the flow with the wall (figure 6). The isentropic flow in the nozzle will be modeled by the flow from a supersonic source, centered at point E (figure 6), whose position on the nozzle axis is determined by the angle \( \theta \) and the size of the critical section \( l_0 \) so that the wall of the nozzle AB coincides with the streamline of the source. In this case, the sound line of the source AD (figure 6, dashed line) must pass through point A of the critical section. To solve the optimization problem of the shape of the curvilinear section of the nozzle, assume that the pressure coefficient on the wall BC is determined by the Newton’s formula.

Let us formulate the following variational problem:

Find such a function \( \eta(\xi) \) and angle \( \theta \), which deliver a maximum of the corresponding functional \( C_T \) given the parameter \( l_0 \) and \( \eta(1) = l_1 \). Writing out the first variation of the functional and equating to zero the coefficient of variation \( \delta \xi_b \), get \( \eta(\xi) = \tan\left(\frac{\varphi}{2}\right) \). Here \( \varphi \) is the inclination angle of the streamline of the source (figure 6). The result indicates that to achieve the maximum thrust, it is necessary that the angle between the direction \( \varphi \) of the streamline of the source and the wall of the aircraft was half. If we further assume that the optimal contour is a continuous differentiable function, then it must satisfy the equation \( \eta = \tan\left(\frac{\varphi}{2}\right) \), the solution of which is a parabola with focus in the center of the source. Thus, the solution of the variational problem for a plane nozzle in the model formulation allows us to proceed to the numerical determination of the shape of the optimal shock nozzle.

With the help of a stationary analogue of the Godunov scheme [1], parametric calculations of the flow in the supersonic part of the nozzle consisting of a conical with an angle \( \theta \) and parabolic sections with the number of nodes along the ordinate axis 300 are carried out. In the critical section of the OA (figure 6) was set uniform flow with Mach number=1.01. The figure 7, as an example, shows the results of calculations of the thrust ratio of the optimal impact nozzle \( T \) to the thrust of the conical nozzle equivalent in size \( T_C \) at \( l_0 = 0.1 \) depending on \( \theta \) for different sizes of the nozzle output section \( l_1 = 0.35, 0.4, 0.5 \) (symbols 1 with approximating curves 1, 2, 4, respectively). To improve the accuracy, calculations were also carried out for \( l_1 = 0.35, 0.4, 0.45 \) and 0.5 with the number of nodes 1000 with a generalization of the Kolgan modification [2], described in [3] (symbols 2 and approximating curves 1-4, respectively).

The maximum advantage of the optimal impact nozzle with respect to the equivalent conical nozzle is achieved at \( \theta = 22.5^\circ \), \( l_1 = 0.4 \) and is more than 6% in calculations with increased accuracy.
The figures 8 and 9 respectively shows the flow pattern in the optimal impact nozzle and the equivalent conical nozzle (lines on which the Mach number is a constant and streamlines) when $\theta = 22.5^\circ$, $l_0 = 0.1$, $l_1 = 0.4$ and with the number of nodes 1000.

**Figure 8.** Lines on which the Mach number is a constant and streamlines in the optimum impact nozzle.

**Figure 9.** Lines on which the Mach number is a constant and streamlines in the equivalent conical nozzle.

5. Conclusion

Plane problems on the contour and nozzle of maximum thrust in a supersonic flow with shock waves are set and solved analytically and numerically. In the problem of the contour generating a shock wave at the fracture point, it is found that each angle of the flux deviation $\theta$ in the Prandtl-Meyer dilution wave in the parameter plane ($M, \lambda$) corresponds to a bounded curve, the ends of which correspond to a conical profile – a line segment connecting the end points of the contour. It is found that the ratio of the thrust forces of the equivalent shock and conical contours can vary from 1 to $\infty$. In the optimal shock nozzle problem in the framework of Newton's theory, it is shown that the second extremal segment generating the shock wave is a parabola. Using the constructed computational code, which uses the modified Godunov method, parametric calculations of the flow in the supersonic part of the optimal shock nozzle obtained in the model problem with a supersonic source interacting with the second extremal arc according to Newton's theory are carried out. It is found that depending on the determining parameters of the problem, the thrust of the optimal nozzle consisting of a straight segment and a parabola segment mating with it can exceed by several percent the thrust of the equivalent conical nozzle.

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