Derivation of statistical parameters for flexure and shear resistance of reinforced concrete beams

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Abstract. The objective of this paper is a study on factors influencing the derivation of statistical parameters of flexural resistance and shear resistance for reinforced concrete beams. A variety of RC beams under bending and shear forces are designed and analyzed, taking into consideration different cross-sectional sizes, compressive concrete strengths, reinforcement ratios, and statistical distributions. The latest version of the American Building Code ACI-318-19 and ASCE/SEI 7-16 are used. The coefficient of variation ($V$) and bias factor ($\lambda$) for the selected models of reinforced concrete beams in flexure and shear are determined using the Monte Carlo simulation technique.

1. Introduction
Reinforced concrete beams can be designed with vast range of properties. The statistical parameters of resistance available in the literature cover the basic design cases. The research conducted in this paper is supposed to extend the available database and identify factors having influence on the flexural and shear resistance parameters for reinforced concrete beams. The results could be dependent on sizes of cross-sections, concrete compressive strengths, reinforcement ratios, and assumed statistical distributions. Therefore, a variety of design cases are selected and analyzed. The nominal resistance for flexure and shear for RC beams are calculated based on the latest version of the American Building Code ACI-318-19. ASCE/SEI 7-16 Standards are used for the load and load combination factors. For the purpose of the generation of the statistical parameters of resistance, the coefficient of variation ($V$) and bias factor ($\lambda$), the Monte Carlo simulation technique is applied.

The currently available statistical parameters of resistance are developed from a normal distribution for all the parameters [1]. Since some of the researchers use a normal distribution for material strengths, while others suggest lognormal distribution [2]-[3], in this study the simulations of statistical parameters of resistance are performed for both types of distributions.

2. Statistical parameters of resistance
2.1. Resistance model
During the design process the resistance of any structural member "R" is considered as a deterministic value. In reality, the resistance "R" is a random variable due to natural and human causes. There are three sources of uncertainty in the load carrying capacity (resistance): material properties, fabrication, and analysis [4].
where:
\[ R_n \] - nominal (design) value of resistance,
\[ M \] – materials factor,
\[ F \] – fabrication factor,
\[ P \] – professional factor.

The material factor \( M \) represents the variability of material properties in strength and modulus of elasticity [4]. The material factors for concrete compressive strength and steel yield strength are based on Nowak et al. 2005 and 2011 [1, 6]. The values of bias factors are between 1.31 and 1.08, while the values of the coefficient of variation are between 0.17 and 0.11, correspondingly for concretes from 10.7 MPa (3 ksi) to 82.7 MPa (12 ksi). Both bias factor and coefficient of variation decrease for higher concrete classes. The statistical parameters for yield strength of reinforcing steel vary with the bar diameter. For reinforcing steel 414 MPa (grade 60) the bias factor varies from 1.18 to 1.12 and the coefficient of variation varies from 0.04 to 0.02.

The statistical parameters for Fabrication factor \( F \) are based on the data from Ellingwood et al. 1980. The statistical parameters for dimensions of the reinforced concrete beam are listed in table 1. For steel reinforcement the bias factor is \( \lambda = 1.0 \) and coefficient of variation \( V = 0.01 \) for rebar diameter, and \( \lambda = 1.0 \) and \( V = 0.015 \) for total steel reinforcement area \( A_s \).

### Table 1. Fabrication factor for reinforced concrete beams.

| Bar size                               | Bias factor \( \lambda \) | Coefficient of variation \( V \) |
|----------------------------------------|---------------------------|---------------------------------|
| Width of beam, cast-in-place.          | 1.01                       | 0.04                            |
| Effective depth of a reinforced beam   | 0.99                       | 0.04                            |

Professional (analysis) factor \( P \) points out the variation in the actual resistance ratio and what can be expected analytically using the strength of materials and exact dimensional values [4]. The values for professional factors were taken from Ellingwood et al. [2].

### Table 2. Professional factor for reinforced concrete beams.

| Bar size | Bias factor \( \lambda \) | Coefficient of variation \( V \) |
|----------|---------------------------|---------------------------------|
| Beam, flexure | 1.020                     | 0.06                            |
| Beam, shear  | 1.075                     | 0.10                            |

### 2.2. Selection of Design Cases

The design cases considered in the study are selected in such way to cover all the statistical data that was found for RC statically determinate beam components and be close to realistic values. First, three different dimensions for beam cross section are chosen. Secondly, for each cross section a different rebar diameter and stirrup spacing are selected. Thirdly, according to the design code limitations different flexural and shear reinforcement ratios are taken into consideration. For flexure it is maximum and minimum reinforcement area. For shear it is maximum, minimum and no shear reinforcement. A cross-section without shear reinforcement means that the concrete cross-section is carrying half of the shear. Then, for each model, thirteen different compressive strengths of concrete are used. In total 78 design cases for flexure and 117 design cases for shear are identified and analyzed to obtain resistance parameters. Moreover, for each case two types of distribution (normal and lognormal) for material strength are considered. For simulation of the dimensions a normal distribution is assumed [5]. All design cases are listed in table 3.
Table 3. Selected design cases.

| Resistance | Beam | Dimensions mm (in) | Main bar diameter (Ø) | Stirrup diameter (Ø) | Reinforcement area | $f'_c$ MPa (ksi) | $f_y$ MPa (ksi) |
|------------|------|--------------------|-----------------------|----------------------|--------------------|-----------------|----------------|
| flexure    | I    | 508×406 (20×16)   | 19.05mm (#6)          | 9.52mm (#3)          | maximum            | 20.7 to 82.7    | 414 (3 to 12)  |
|            | II   | 813×508 (32×20)   | 25.4mm (#8)           | 9.52mm (#3)          | minimum            | 82.7 (3 to 12)  | 414 (60)       |
|            | III  | 406×305 (16×12)   | 32.25mm (#10)         | 9.52mm (#3)          | maximum            | 82.7 (3 to 12)  | 414 (60)       |
| shear      | I    | 508×406 (20×16)   | 19.05mm (#6)          | 9.52mm (#3)          | no shear reinforcement | 20.7 to 82.7    | 414 (3 to 12)  |
|            | II   | 813×508 (32×20)   | 25.4mm (#8)           | 9.52mm (#3)          | no shear reinforcement | 82.7 (60)       |                |
|            | III  | 406×305 (16×12)   | 32.25mm (#10)         | 9.52mm (#3)          | no shear reinforcement | 82.7 (60)       |                |

2.3. Flexural Resistance of RC Beams

To calculate the nominal flexural resistance for the reinforced concrete beam $R_n$, the design formula (2) is used [10]. Based on the Monte Carlo simulation technique statistical parameters of resistance for all the selected design cases are determined. First, a normal distribution is used for all parameters such as $A_s$, $f_y$, $d$, $b$, and $f'_c$. Then, a lognormal distribution is used for material strength $f_y$ and $f'_c$, while other parameters remained normally distributed.

$$R_n = A_s \cdot f_y \cdot \left( d - \frac{a}{2} \right)$$  \hspace{1cm} (2)

Where equivalent compression zone is:

$$a = \frac{A_s \cdot f_y}{0.85 \cdot f'_c \cdot b}$$  \hspace{1cm} (3)

Statistical parameters for compressive strength of concrete are taken from Nowak et al. 2005 and 2011 [1, 6] based on cylindrical test data. The value of $f'_c$ obtained from cylinder tests are reduced by 10% for calculating nominal moment resistance because the average in-place strength is lower than a standard cylinder test [1, 6]. The reinforcement ratios for all models are taken from minimum and maximum allowed by the ACI requirements. For computation of minimum reinforcement equations (4) and (5) are used. Values of maximum reinforcement are taken from allowable design phase. A reinforcement limitation is changed for different models by using a different material strength:

$$\rho_{min} = \frac{0.25 \sqrt{f'_c}}{f_y}$$  \hspace{1cm} (4)

$$\rho_{min} = \frac{1.4}{f_y}$$  \hspace{1cm} (5)
2.4. Shear Resistance of RC Beams

Nominal shear resistance for the reinforced concrete beam ($R_n$) with shear reinforcement can be calculated by formula (6). Monte Carlo simulations are used to determine resistance of components for all selected design cases. First, a normal distribution for all components ($A_V, f_y, f'_c, S, b$ and $d$) is assumed. Then, a lognormal distribution is used to calculate strength of materials such as ($f_y$ and $f'_c$), while other parameters remained normally distributed.

$$ R_n = V_n = V_c + V_s $$

(6)

where, the shear resistance provided by concrete is calculated by following formula (7)

$$ V_c = \frac{1}{6} \sqrt{f'_c} \cdot b \cdot d $$

(7)

and shear resistance provided by shear reinforcement (8)

$$ V_s = \frac{A_V \cdot f_s \cdot d}{S} $$

(8)

Parameters for compressive strength of concrete are assumed from Nowak et al. 2005 [6]. Concrete compressive strengths $f'_c$ are reduced 5% for calculating nominal shear resistance for the reinforced concrete beam [7]. Three design situations are taken into consideration for amount of shear reinforcement: maximum shear reinforcement, minimum shear reinforcement, and no shear reinforcement (the cross-section carries half of shear). Based on the ACI requirement for stirrup spacing, three equations (9), (10), and (11) are used to calculate maximum and minimum shear reinforcement area.

Maximum shear reinforcement

$$ A_{v_{max}} = \frac{4}{6} \sqrt{f'_c} \cdot \frac{b}{f_y} \cdot d \cdot S $$

(9)

$$ A_{v_{min}} = 0.062 \cdot \sqrt{f'_c} \cdot \frac{b \cdot S}{f_y} $$

(10)

But not less than

$$ A_{v_{min}} = 0.35 \cdot \frac{b \cdot S}{f_y} $$

(11)

The stirrup spacing is assumed to vary between 200 mm to 305 mm (6 in to 12 in), according to beam dimension and materials. For the cross-sectional dimension, compressive strength and yield strength components are taken to be the same as flexural resistance.

3. Results of statistical parameters

3.1. Monte Carlo Simulations

Based on Monte Carlo simulations, the resistance of reinforced concrete beams for flexure and shear are computed. In order to derive statistical parameters of flexural resistance, one thousand simulations are performed for each of the selected design cases.

3.2. Results for flexure resistance parameters (beam I, II and III)

The variability of the results for flexural resistance can be observed in figure 1 to figure 8 plotted below. The statistical parameters for flexural resistance are gathered in table 4 and are presented in accordance to reinforcement ratios.
Figure 1. Bias factor vs. material strength for flexure, normal distribution.

Figure 2. Bias factor vs. material strength for flexure, lognormal distribution.

Figure 3. Bias factor vs. material strength for flexure, normal distribution.

Figure 4. Bias factor vs. material strength for flexure, lognormal distribution.
Figure 5. C.O.V. vs. material strength for flexure, normal distribution.

Figure 6. C.O.V. vs. material strength for flexure, lognormal distribution.

Figure 7. C.O.V. vs. material strength for flexure, normal distribution.

Figure 8. C.O.V. vs material strength for flexure, lognormal distribution.
Table 4. Statistical parameters for flexural resistance.

| Flexural Resistance | Normal Distribution | Strength Lognormal Distribution |
|---------------------|---------------------|---------------------------------|
|                     | $\lambda$ | $V$ | $\lambda$ | $V$ |
| beam I              |           |     |           |     |
| $\rho_{\text{max}}$ | 1.123    | 0.080 | 1.126    | 0.079 |
| $\rho_{\text{min}}$ | (0.34-0.55) % | | (0.34-0.55) % | |
| beam II             |           |     |           |     |
| $\rho_{\text{max}}$ | 1.136    | 0.079 | 1.137    | 0.080 |
| $\rho_{\text{min}}$ | (0.34-0.55) % | | (0.34-0.55) % | |
| beam III            |           |     |           |     |
| $\rho_{\text{max}}$ | 1.130    | 0.081 | 1.132    | 0.081 |
| $\rho_{\text{min}}$ | (0.34-0.55) % | | (0.34-0.55) % | |

3.3. Results for shear (beam I, II and III)

The variability of the results for shear resistance can be observed in figure 9 to figure 12 plotted below. The statistical parameters for shear resistance are presented in table 5 according to concrete compressive strength and shear reinforcement due to their straight impact on shear resistance [9].

Figure 9. Bias factor vs. material strength for shear resistance, normal distribution.

Figure 10. Bias factor vs. material strength for shear resistance, lognormal distribution.
4. Analysis of results

It is recognized that flexural resistance is carried out by reinforcement [8]. Therefore, the reinforcement ratio can control statistical parameters for resistance more than other components. The flexural resistance parameters were calculated for the maximum and minimum reinforcement ratio. From table 4, it was observed that the bias factor ($\lambda$) decreases for higher reinforcement ratios. The change in bias factor is not significant, about 1% for beam I and beam III, while for beam II the change was lower than 1%. It is because the maximum reinforcement ratio obtained from the design phase is close to the minimum reinforcement ratio provided by ACI, or sometimes less than the minimum. Figure 5 and figure 6 show a slightly higher coefficient of variation $(V)$ for higher reinforcement ratios for beams I, II and III. However, the differences in the coefficients of variation are very small and they can be assumed to be equal (table 4).

The shear reinforcement ratio can have a large impact on the statistical parameters for shear resistance. The statistical parameters that are calculated for maximum shear reinforcement are significantly different from those without shear reinforcement (up to 10% in high strength concrete). Figure 9 and figure 10 show a higher value of bias factor for maximum shear reinforcement. The coefficient of variation decreases for maximum shear reinforcement (figure 11 and figure 12). Another observation is that higher compressive strength of concrete results in decreased statistical parameters of shear resistance, bias and coefficient of variation (table 5). It was proven that statistical
parameters for shear resistance can be affected by the shear reinforcement ratio more than compressive strength of concrete, when a higher shear reinforcement ratio is used. But for minimum and no shear reinforcement concrete compressive strength can control statistical parameters rather than shear reinforcement ratio. This fact was also proven by Nowak et al. (2011) [1].

Table 5. Statistical parameters for flexural resistance

| f\text{c}' | Normal distribution | Strength lognormal distr. |
|---|---|---|
| | \(\lambda\) | \(V\) | \(\lambda\) | \(V\) |
| **Maximum shear reinforcement** | | | | |
| 20.7 MPa (3 ksi) | 1.205 | 0.117 | 1.208 | 0.120 |
| 24.1 MPa (3.5 ksi) | 1.204 | 0.116 | 1.205 | 0.118 |
| 27.6 MPa (4 ksi) | 1.199 | 0.117 | 1.205 | 0.117 |
| 31.0 MPa (4.5 ksi) | 1.195 | 0.117 | 1.204 | 0.116 |
| 34.5 MPa (5 ksi) | 1.197 | 0.118 | 1.197 | 0.118 |
| 37.9 MPa (5.5 ksi) | 1.193 | 0.119 | 1.201 | 0.117 |
| 41.4 MPa (6 ksi) | 1.196 | 0.117 | 1.197 | 0.119 |
| 44.8 MPa (6.5 ksi) | 1.189 | 0.114 | 1.189 | 0.114 |
| 48.3 MPa (7 ksi) | 1.193 | 0.115 | 1.191 | 0.115 |
| 55.2 MPa (8 ksi) | 1.183 | 0.118 | 1.187 | 0.119 |
| 62.0 MPa (9 ksi) | 1.182 | 0.116 | 1.186 | 0.113 |
| 69.0 MPa (10 ksi) | 1.188 | 0.117 | 1.188 | 0.116 |
| 82.7 MPa (12 ksi) | 1.186 | 0.115 | 1.191 | 0.115 |
| **Minimum shear reinforcement** | | | | |
| 20.7 MPa (3 ksi) | 1.198 | 0.126 | 1.201 | 0.130 |
| 24.1 MPa (3.5 ksi) | 1.182 | 0.127 | 1.190 | 0.126 |
| 27.6 MPa (4 ksi) | 1.178 | 0.128 | 1.184 | 0.124 |
| 31.0 MPa (4.5 ksi) | 1.165 | 0.123 | 1.169 | 0.124 |
| 34.5 MPa (5 ksi) | 1.162 | 0.121 | 1.160 | 0.124 |
| 37.9 MPa (5.5 ksi) | 1.149 | 0.121 | 1.159 | 0.121 |
| 41.4 MPa (6 ksi) | 1.144 | 0.118 | 1.146 | 0.123 |
| 44.8 MPa (6.5 ksi) | 1.141 | 0.119 | 1.146 | 0.122 |
| 48.3 MPa (7 ksi) | 1.135 | 0.123 | 1.141 | 0.118 |
| 55.2 MPa (8 ksi) | 1.129 | 0.118 | 1.138 | 0.121 |
| 62.0 MPa (9 ksi) | 1.126 | 0.117 | 1.126 | 0.117 |
| 69.0 MPa (10 ksi) | 1.126 | 0.118 | 1.130 | 0.119 |
| 82.7 MPa (12 ksi) | 1.117 | 0.119 | 1.120 | 0.117 |
| **No shear reinforcement** | | | | |
| 20.7 MPa (3 ksi) | 1.198 | 0.146 | 1.206 | 0.142 |
| 24.1 MPa (3.5 ksi) | 1.175 | 0.141 | 1.186 | 0.140 |
| 27.6 MPa (4 ksi) | 1.160 | 0.138 | 1.171 | 0.135 |
| 31.0 MPa (4.5 ksi) | 1.149 | 0.136 | 1.156 | 0.135 |
| 34.5 MPa (5 ksi) | 1.141 | 0.134 | 1.142 | 0.132 |
| 37.9 MPa (5.5 ksi) | 1.133 | 0.132 | 1.136 | 0.129 |
| 41.4 MPa (6 ksi) | 1.122 | 0.132 | 1.129 | 0.132 |
| 44.8 MPa (6.5 ksi) | 1.115 | 0.129 | 1.120 | 0.133 |
| 48.3 MPa (7 ksi) | 1.113 | 0.128 | 1.116 | 0.128 |
| 55.2 MPa (8 ksi) | 1.101 | 0.131 | 1.105 | 0.126 |
| 62.0 MPa (9 ksi) | 1.099 | 0.129 | 1.099 | 0.130 |
| 69.0 MPa (10 ksi) | 1.088 | 0.129 | 1.096 | 0.128 |
| 82.7 MPa (12 ksi) | 1.086 | 0.130 | 1.095 | 0.127 |
By comparison of results for different beam sizes (figure 1 to 12), it can be concluded that changes in the beam cross-section did not have significant influence on statistical parameters for flexure and shear resistance. This confirms the assumption by Ellingwood et al. 1980 [2], and Nowak et al. 2011 [1]. It was also observed that the type of the assumed distribution of the material strength does not have significant influence on the results.

More detailed explanation of the results can be found in Zheer 2020 [11].

5. Conclusions
The objective of this paper was to extend the available database and identify factors influencing the statistical parameters of resistance (bias factor and coefficient of variation) for rectangular reinforced concrete beams under flexure and shear. The study was performed for a variety of design cases with different cross sections, concrete strengths, reinforcement ratios, and types of statistical distribution for material strength. ACI 318-19 Code and ASCE/SEI 7-16 Standards were used. The simulation of statistical parameters was based on the Monte Carlo technique. It is concluded that reinforcement ratio and bar diameter causes a small effect on the statistical parameters for flexural resistance, and they affect mostly the bias factor. The statistical parameters for shear resistance are governed by the shear reinforcement ratio and concrete compressive strength. Generally, the coefficient of variation of shear resistance is larger than the coefficient of variation of flexural resistance, because shear failure is much harder to predict. It was found that the element size and probability distribution type of the material strength does not significantly influence the results.

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