Generalized Normal Forms of Two-Dimensional Autonomous Systems with a Hamiltonian Unperturbed Part

V. V. Basov\textsuperscript{1,*} and A. S. Vaganyan\textsuperscript{1,†}

\textsuperscript{1}Saint Petersburg State University

Generalized pseudo-Hamiltonian normal forms (GPHNF) and an effective method of obtaining them are introduced for two-dimensional systems of autonomous ODEs with a Hamiltonian quasi-homogeneous unperturbed part of an arbitrary degree. The terms that can be additionally eliminated in a GPHNF are constructively distinguished, and it is shown that after removing them GPHNF becomes a generalized normal form (GNF). By using the introduced method, all the GNFs are obtained in cases where the unperturbed part of the system is Hamiltonian and has monomial components, which allowed to generalize some results by Takens, Baider and Sanders, as well as Basov et al.

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1. SOME FACTS FROM THE THEORY OF NORMAL FORMS

1.1. Quasi-homogeneous polynomials and vector fields

Let $x = (x_1, x_2)$, $p = (p_1, p_2)$ and $D = (\partial_1, \partial_2)$ where $x_1, x_2$ are scalar variables, $p_1, p_2$ are nonnegative integers, and $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$.

Denote by $\langle . , . \rangle$ the standard inner product on vectors, Euclidean or Hermitian depending on the context.

Consider vector $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \in \mathbb{N}$, $\text{GCD}(\gamma_1, \gamma_2) = 1$ called weight of the variable $x$, and let $\delta = \gamma_1 + \gamma_2$.

**Definition 1** (g. d.). We call the number $\langle p, \gamma \rangle$ the generalized degree of a monomial $x_1^{p_1}x_2^{p_2}$.

**Definition 2** (QHP). Polynomial $P(x)$ is called a quasi-homogeneous polynomial of g. d. $k$ with weight $\gamma$ and is denoted by $P_\gamma^{[k]}(x)$ if it is a linear combination of monomials of g. d. $k$.  

\*Electronic address: vlvlbasov@rambler.ru
\textsuperscript{†}Electronic address: armay@yandex.ru; This research is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, St. Petersburg State University) under RF Government grant 11.G34.31.0026.; Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia
Definition 3 (VQHP). Vector polynomial $\mathcal{P}(x) = (P_1(x), P_2(x))$ is called a vector quasi-homogeneous polynomial of g. d. $k$ with weight $\gamma$ and is denoted by $\mathcal{P}^{[k]}_\gamma(x)$, if its components $P_1(x), P_2(x)$ are QHPs of g. d. $k + \gamma_1$ and $k + \gamma_2$ respectively.

Remark 1. Further, we identify each VQHP $\mathcal{P}^{[k]}_\gamma(x) = (P^{[k + \gamma_1]}_{\gamma_1}(x), P^{[k + \gamma_2]}_{\gamma_2}(x))$ with the corresponding vector field, i. e., define the action of VQHPs on formal power series:

$$\mathcal{P}^{[k]}_\gamma(f)(x) = P^{[k + \gamma_1]}_{\gamma_1}(x) \partial_1 f(x) + P^{[k + \gamma_2]}_{\gamma_2}(x) \partial_2 f(x),$$

and define the Lie bracket of two VQHPs $\mathcal{P}^{[k]}_\gamma(x) = (P^{[k + \gamma_1]}_{\gamma_1}(x), P^{[k + \gamma_2]}_{\gamma_2}(x))$ and $\mathcal{Q}^{[l]}_\gamma(x) = (Q^{[l + \gamma_1]}_{\gamma_1}(x), Q^{[l + \gamma_2]}_{\gamma_2}(x))$:

$$[\mathcal{P}^{[k]}_\gamma, \mathcal{Q}^{[l]}_\gamma](x) = (\mathcal{P}^{[k]}_\gamma(Q^{[l + \gamma_1]}_{\gamma_1}(x)) - \mathcal{Q}^{[l]}_\gamma(P^{[k + \gamma_1]}_{\gamma_1}(x)), \mathcal{P}^{[k]}_\gamma(Q^{[l + \gamma_2]}_{\gamma_2}(x)) - \mathcal{Q}^{[l]}_\gamma(P^{[k + \gamma_2]}_{\gamma_2}(x))).$$

Definition 4. Given a vector series $\mathcal{P}(x) = \sum_{k=0}^{\infty} \mathcal{P}^{[k]}_\gamma(x)$, we call the least positive integer $\chi \geq 0$ such that $\mathcal{P}^{[\chi]}_\gamma \neq 0$ the generalized order of $\mathcal{P}$. Denote the generalized order by $\text{ord}_\gamma \mathcal{P}$. If $\mathcal{P} \equiv 0$, then $\text{ord}_\gamma \mathcal{P} = +\infty$.

### 1.2. Homological equation

Consider the system

$$\dot{x} = \mathcal{P}^{[\chi]}_\gamma(x) + \mathcal{X}(x) \quad (\chi \geq 0, \mathcal{X} = (X_1, X_2), \text{ord}_\gamma \mathcal{X} \geq \chi + 1). \quad (1)$$

Let the near-identity formal change of variables

$$x = y + \mathcal{Q}(y) \quad (y = (y_1, y_2), \mathcal{Q} = (Q_1, Q_2), \text{ord}_\gamma \mathcal{Q} \geq 1) \quad (2)$$

transform it into the system

$$\dot{y} = \mathcal{P}^{[\chi]}_\gamma(y) + \mathcal{Y}(y) \quad (\mathcal{Y} = (Y_1, Y_2), \text{ord}_\gamma \mathcal{Y} \geq \chi + 1). \quad (3)$$

Then the series $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Q}$ satisfy the homological equation (see [1]):

$$[\mathcal{P}^{[\chi]}_\gamma, \mathcal{Q}^{[k]}_\gamma] = \mathcal{Y}^{[k + \chi]}_\gamma - \mathcal{Y}^{[k]}_\gamma \quad (4)$$

where $k \geq 1$, and $\mathcal{Y}^{[k + \chi]}_\gamma$ includes only the components of the VQHPs $\mathcal{Q}^{[l]}_\gamma$ and $\mathcal{Y}^{[l + \chi]}_\gamma$ with $l = 1, \ldots, k - 1$. If $\text{ord}_\gamma \mathcal{Q} = m \geq 1$, then for $k = 1, \ldots, m$, equation (4) takes the form

$$\mathcal{X}^{[l + \chi]}_\gamma = \mathcal{Y}^{[l + \chi]}_\gamma \quad (l = 1, \ldots, m - 1), \quad [\mathcal{P}^{[\chi]}_\gamma, \mathcal{Q}^{[m]}_\gamma] = \mathcal{X}^{[m + \chi]}_\gamma - \mathcal{Y}^{[m + \chi]}_\gamma. \quad (5)$$
1.3. Generalized normal form (GNF)

Denote the linear spaces of VQHPs of g. d. \( k \geq 1 \) and \( k + \chi \) in \( x \) by \( \mathcal{Q}_\gamma^{[k]} \) and \( \mathcal{Q}_\gamma^{[k+\chi]} \), and their dimensions by \( k_\gamma = \dim \mathcal{Q}_\gamma^{[k]} \) and \( (k + \chi)_\gamma = \dim \mathcal{Q}_\gamma^{[k+\chi]} \).

In the spaces \( \mathcal{Q}_\gamma^{[k]} \) and \( \mathcal{Q}_\gamma^{[k+\chi]} \), choose the standard bases \( \{(x_1^{p_1} x_2^{p_2}, 0), (0, x_1^{p_1'} x_2^{p_2'}) : \langle p, \gamma \rangle - \gamma_1 = \langle p', \gamma \rangle - \gamma_2 = q \} \), where \( q \) equals \( k \) and \( k + \chi \) respectively, order their elements lexicographically, and denote the corresponding matrix representation of the linear operator \( [\mathcal{P}_{\gamma}^{[x]}], \cdot : \mathcal{Q}_\gamma^{[k]} \to \mathcal{Q}_\gamma^{[k+\chi]} \) by \( \mathcal{P}_{\gamma,k}^{[x]} \).

Assume that the matrix \( \mathcal{P}_{\gamma,k}^{[x]} \) has rank \( r_k = k_\gamma - k_0^\gamma \) where \( 0 \leq k_0^\gamma \leq k_\gamma - 1 \).

Comparing the VQHPs \( \mathcal{Q}_\gamma^{[k]}, \mathcal{Q}_\gamma^{[k+\chi]} \) and \( \mathcal{Q}_\gamma^{[k+\chi]} \) to the coefficient vectors of their expansions in the standard bases \( \mathcal{Q}_\gamma^{[k]}, \mathcal{Q}_\gamma^{[k+\chi]} \) and \( \mathcal{Q}_\gamma^{[k+\chi]} \), represent the homological equation (4) as a system of linear algebraic equations:

\[
\mathcal{P}_{\gamma,k}^{[x]} \mathcal{Q}_\gamma^{[k]} = \mathcal{Q}_\gamma^{[k+\chi]} - \mathcal{Q}_\gamma^{[k+\chi]}.
\]

From the obtained equations, select a subsystem of order \( r_k \) with nonzero determinant, and solve it with respect to \( \mathcal{Q}_\gamma^{[k]} \).

Arbitrarily fix the remaining \( k_0^\gamma \) coefficients of \( \mathcal{Q}_\gamma^{[k]} \).

Substituting \( \mathcal{Q}_\gamma^{[k]} \) in the remaining equations, we obtain \( n_k = (k + \chi)_\gamma - r_k \) linearly independent relations on coefficients of the VQHP \( \mathcal{Y}_\gamma^{[k+\chi]} \) called resonance equations:

\[
\langle a_i^k, \mathcal{Y}_\gamma^{[k+\chi]} \rangle = \langle a_i^k, \mathcal{Y}_\gamma^{[k+\chi]} \rangle, \quad a_i^k = \text{const} \quad (i = 1, n_k).
\]

Coefficients of the VQHP \( \mathcal{Y}_\gamma^{[k+\chi]} \) that really participate in at least one of the resonance equations are called resonant and the others are called nonresonant.

**Definition 5.** We call any set \( \mathcal{Y} = \bigcup_{k=1}^{\infty} \{ \mathcal{Y}_\gamma^{[k+\chi]} \}_{\gamma=1}^{n_k} \) of resonant coefficients of the VQHP \( \mathcal{Y}_\gamma^{[k+\chi]} \) such that for all \( k \geq 1 \) \( \det(\{a_{i,j}^{k,\gamma,\chi}\}_{i,j=1}^{n_k}) \neq 0 \) a resonant set.

**Definition 6 (GNF).** System (3) is called a generalized normal form if all coefficients of the vector series \( \mathcal{Y} \) are zero except for the coefficients from some resonant set \( \mathcal{Y} \) that have arbitrary values.

Thereby, each GNF is generated by a certain resonant set \( \mathcal{Y} \).

**Proposition 1 ([1, Th. 2]).** Given any system (1) and arbitrarily chosen resonant set \( \mathcal{Y} \), then there exists a near-identity formal transformation (2) that brings it to a certain GNF (3) generated by \( \mathcal{Y} \).
2. GENERALIZED PSEUDO-HAMILTONIAN NORMAL FORM (GPHNF)

2.1. Euler operator

**Definition 7.** The Euler operator with weight $\gamma$ is the VQHP $\mathcal{E}_\gamma(x) = (\gamma_1 x_1, \gamma_2 x_2)$.

Clearly, the Euler operator has zero g. d., so in the designation $\mathcal{E}_\gamma$, the g. d. is omitted.

Let $Q^{[k]}_\gamma$ be a QHP, and $\overline{Q}^{[k]}_\gamma = (Q^{[k+\gamma_1]}_{\gamma,1}, Q^{[k+\gamma_2]}_{\gamma,2})$ be a VQHP of g. d. $k \geq 0$.

**Proposition 2.** The Euler operator with weight $\gamma$ has the following properties:

1) $\mathcal{E}_\gamma(Q^{[k]}_\gamma) = k Q^{[k]}_\gamma$; 2) $[\mathcal{E}_\gamma, Q^{[k]}_\gamma] = k Q^{[k]}_\gamma$; 3) $\text{div}(Q^{[k]}_\gamma \mathcal{E}_\gamma) = (k + \delta)Q^{[k]}_\gamma$.

*Proof.* 1) The property is obvious.

2) According to 1), $[\mathcal{E}_\gamma, Q^{[k]}_\gamma] = (\mathcal{E}_\gamma(Q^{[k+\gamma_1]}_{\gamma,1}) - Q^{[k]}_\gamma(\gamma_1 x_1), \mathcal{E}_\gamma(Q^{[k+\gamma_2]}_{\gamma,2}) - Q^{[k]}_\gamma(\gamma_2 x_2)) = k Q^{[k]}_\gamma$.

3) According to 1) and the Leibniz rule, $\text{div}(Q^{[k]}_\gamma \mathcal{E}_\gamma) = \mathcal{E}_\gamma(Q^{[k]}_\gamma) + Q^{[k]}_\gamma \text{div} \mathcal{E}_\gamma = (k + \delta)Q^{[k]}_\gamma$. □

**Lemma 1.** Each VQHP $Q^{[k]}_\gamma = (Q^{[k+\gamma_1]}_{\gamma,1}, Q^{[k+\gamma_2]}_{\gamma,2})$ can be uniquely represented in the form

$$Q^{[k]}_\gamma = \mathcal{I}^{[k]}_\gamma + J^{[k]}_\gamma \mathcal{E}_\gamma \quad (\mathcal{I}^{[k]}_\gamma = (-\partial_2 I^{[k+\delta]}_{\gamma,1}, \partial_1 I^{[k+\delta]}_{\gamma,1})).$$

Herewith, $J^{[k]}_\gamma = \text{div}(Q^{[k]}_\gamma)/(k + \delta)$.

*Proof.* Take the divergence of both sides of equality (6). According to property 3) of $\mathcal{E}_\gamma$ and the fact that $\text{div}(\mathcal{I}^{[k]}_\gamma) \equiv 0$, we obtain $J^{[k]}_\gamma = \text{div}(Q^{[k]}_\gamma)/(k + \delta)$. Hence, $I^{[k+\delta]}_\gamma$ must satisfy the system of equations $\partial_1 I^{[k+\delta]}_\gamma = Q^{[k+\gamma_1]}_{\gamma,1} - \gamma_2 x_2 \text{div}(Q^{[k]}_\gamma)/(k + \delta), \quad \partial_2 I^{[k+\delta]}_\gamma = \gamma_1 x_1 \text{div}(Q^{[k]}_\gamma)/(k + \delta) - Q^{[k+\gamma_2]}_{\gamma,2}$.

By the Poincare lemma, taking into account the quasi-homogeneity, $I^{[k+\delta]}_\gamma$ exists and is unique. □

2.2. Hamiltonian resonant sets

Let $H^{[x+\delta]}_\gamma \neq 0$ be a QHP, and $\mathcal{H}^{[x]}_\gamma = (-\partial_2 H^{[x+\delta]}_\gamma, \partial_1 H^{[x+\delta]}_\gamma)$ be a VQHP of g. d. $\chi \geq 0$.

Denote the space of polynomials in $x$ by $\mathfrak{P}$, and introduce the inner-product on $\mathfrak{P}$:

$$\langle (P, Q) \rangle = P(D)\overline{Q}(x)|_{x=0} \quad (P, Q \in \mathfrak{P}, \ D = (\partial_1, \partial_2))$$

where the upper bar stands for the complex conjugation of the coefficients. Then the operator $\mathcal{H}^{[x]}_\gamma : \mathfrak{P} \to \mathfrak{P}$ conjugate to $\mathcal{H}^{[x]}_\gamma$ with respect to $\langle (\cdot, \cdot) \rangle$ has the form (see [2])

$$\mathcal{H}^{[x]}_\gamma = x_2 \cdot \partial_1 \overline{H}^{[x+\delta]}_\gamma(D) - x_1 \cdot \partial_2 \overline{H}^{[x+\delta]}_\gamma(D) = \partial_1 \overline{H}^{[x+\delta]}_\gamma(D) \cdot x_2 - \partial_2 \overline{H}^{[x+\delta]}_\gamma(D) \cdot x_1. \quad (7)$$
Definition 8. Kernel \( R_\gamma = \text{Ker} \mathcal{H}_\gamma^{[x]} \) of the operator (7) is called the space of resonant polynomials. Denote the linear space of resonant QHPs of g. d. \( k \geq 0 \) by \( R_\gamma^{[k]} \).

Let \( \{R_{\gamma,i}^{[k]}\}_{i=1}^{s_k} \) be a basis for \( R_\gamma^{[k]} \), and \( \{\tilde{R}_{\gamma,i}^{[k]}\}_{i=1}^{\tilde{s}_k} \) be a basis for \( \tilde{R}_\gamma^{[k]} = R_\gamma^{[k]} \cap \mathcal{H}_\gamma^{[x]} (\mathfrak{P}_\gamma^{[k+\chi]}) \).

Definition 9. Sets of QHPs \( \mathfrak{S}_\gamma^{[k]} = \{S_{\gamma,j}^{[k]}\}_{j=1}^{s_k} \) and \( \tilde{\mathfrak{S}}_\gamma^{[k]} = \{\tilde{S}_{\gamma,j}^{[k]}\}_{j=1}^{\tilde{s}_k} \) are called, respectively, a Hamiltonian resonant set and a Hamiltonian reduced resonant set in g. d. \( k \), if \( \det(\{(R_{\gamma,i}^{[k]}, S_{\gamma,j}^{[k]})\}_{i,j=1}^{s_k}) \neq 0 \) and \( \det(\{(\tilde{R}_{\gamma,i}^{[k]}, \tilde{S}_{\gamma,j}^{[k]})\}_{i,j=1}^{\tilde{s}_k}) \neq 0 \). The sets \( \mathfrak{S}_\gamma = \cup_{k \geq 0} \mathfrak{S}_\gamma^{[k]} \) and \( \tilde{\mathfrak{S}}_\gamma = \cup_{k \geq 0} \tilde{\mathfrak{S}}_\gamma^{[k]} \) are called, respectively, a Hamiltonian resonant set and a Hamiltonian reduced resonant set.

Show the independence of the definition of a Hamiltonian resonant set of the choice of basis.

Since any other basis \( \{R_{\gamma,i}^{[k]}\}_{i=1}^{s_k} \) for the space \( R_\gamma^{[k]} \) is obtained from \( \{R_{\gamma,i}^{[k]}\}_{i=1}^{s_k} \) by multiplication by a nonsingular matrix: \( R_{\gamma,i}^{[k]} = \sum_{j=1}^{s_k} A_{ij} R_{\gamma,j}^{[k]} \), \( \det A \neq 0 \) (\( i = 1, s_k \)), for each Hamiltonian resonant set \( \mathfrak{S}_\gamma^{[k]} = \{S_{\gamma,j}^{[k]}\}_{j=1}^{s_k} \) in g. d. \( k \), we have \( \det(\{(R_{\gamma,i}^{[k]}, S_{\gamma,j}^{[k]})\}_{i,j=1}^{s_k}) = \det(\{\sum_{i=1}^{s_k} A_{ii}(R_{\gamma,i}^{[k]})\})_{i,j=1}^{s_k} = \det A \det(\{(R_{\gamma,i}^{[k]}, S_{\gamma,j}^{[k]})\}_{i,j=1}^{s_k}) \neq 0 \).

Similarly, the definition of a Hamiltonian reduced resonant set \( \tilde{\mathfrak{S}}_\gamma^{[k]} \) is also independent of the choice of basis.

Definition 10. We say that a Hamiltonian resonant set \( \mathfrak{S}_\gamma \) or a Hamiltonian reduced resonant set \( \tilde{\mathfrak{S}}_\gamma \) is minimal if it consists of monomials.

2.3. Definition of the GPHNF and existence theorem for the normalizing transformation

Let the unperturbed part of system (1) have the form \( \mathcal{P}_\gamma^{[x]} = \mathcal{H}_\gamma^{[x]} \) where, as in the previous subsection, \( \mathcal{H}_\gamma^{[x]} = (-\partial_2 H_\gamma^{[x+\delta]}, \partial_1 H_\gamma^{[x+\delta]}) \) is a VQHP of g. d. \( \chi \geq 0 \).

According to Lemma 1 system (1) can be uniquely represented in the form

\[
\dot{x} = \mathcal{H}_\gamma^{[x]}(x) + \sum_{k=1}^{\infty} (\mathcal{F}_\gamma^{[k+\chi]}(x) + \mathcal{G}_\gamma^{[k+\chi]}(x)E_\gamma(x)) \tag{8}
\]

where \( \mathcal{F}_\gamma^{[k+\chi]} = (-\partial_2 F_\gamma^{[k+\chi+\delta]}, \partial_1 F_\gamma^{[k+\chi+\delta]}) \). Herewith, \( \mathcal{G}_\gamma^{[k+\chi]} = \text{div}(\mathcal{X}_\gamma^{[k+\chi]})/(k + \chi + \delta) \).

Definition 11 (GPHNF). System (8) is called a generalized pseudo-Hamiltonian normal form, if for some Hamiltonian resonant set \( \mathfrak{S}_\gamma \) and Hamiltonian reduced resonant set \( \tilde{\mathfrak{S}}_\gamma \), the following conditions are satisfied:

\[
\forall k \geq 1 \quad F_\gamma^{[k+\chi+\delta]} \in \text{Lin}(\tilde{\mathfrak{S}}_\gamma^{[k+\chi+\delta]}), \quad G_\gamma^{[k+\chi]} \in \text{Lin}(\mathfrak{S}_\gamma^{[k+\chi]}).
\]
Thereby, the structure of each GPHNF is generated by certain Hamiltonian resonant and reduced resonant sets $\mathcal{S}_\gamma$ and $\bar{\mathcal{S}}_\gamma$.

**Lemma 2.**

1) Let $J^{[k]}_\gamma$ be a QHP of g. d. $k \geq 0$. Then for some QHP $K^{[k+\chi+\delta]}_\gamma$,

$$\left[\mathcal{H}^{[k]}_\gamma, J^{[k]}_\gamma \mathcal{E}^{[\gamma]}_\gamma\right] = \mathcal{K}^{[k+\chi]}_\gamma + \frac{k + \delta}{k + \chi + \delta} \mathcal{H}^{[k]}_\gamma \mathcal{J}^{[k]}_\gamma \mathcal{E}^{[\gamma]}_\gamma = (\mathcal{K}^{[k+\chi]}_\gamma = (-\partial_2 \mathcal{K}^{[k+\chi+\delta]}_\gamma, \partial_1 \mathcal{K}^{[k+\chi+\delta]}_\gamma)). \quad (9)$$

Herewith, if $\mathcal{H}^{[k]}_\gamma (J^{[k]}_\gamma) = 0$, then $\mathcal{K}^{[k+\chi]}_\gamma = -\chi J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma$ and $\mathcal{H}^{[\gamma]}_\gamma (K^{[k+\chi+\delta]}_\gamma) = 0$.

2) Conversely, given any QHP $K^{[k+\chi+\delta]}_\gamma$ such that $\mathcal{H}^{[\gamma]}_\gamma (K^{[k+\chi+\delta]}_\gamma) = 0$, then there exists unique QHP $J^{[k]}_\gamma$ such that $\mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) = 0$ and equality (9) is satisfied.

**Proof.**

1) By Lemma 1 and the identity $\text{div}(J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma) \equiv \mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma)$, it follows that

$$\left[\mathcal{H}^{[\gamma]}_\gamma, J^{[k]}_\gamma \mathcal{E}^{[\gamma]}_\gamma\right] = \mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) \mathcal{E}^{[\gamma]}_\gamma + J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma \mathcal{E}^{[\gamma]}_\gamma = \mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) \mathcal{E}^{[\gamma]}_\gamma - \chi J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma =$$

$$= \left(1 - \frac{\chi}{k + \chi + \delta}\right) \mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) \mathcal{E}^{[\gamma]}_\gamma + \mathcal{K}^{[k+\chi]}_\gamma = \frac{k + \delta}{k + \chi + \delta} \mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) \mathcal{E}^{[\gamma]}_\gamma + \mathcal{K}^{[k+\chi]}_\gamma.$$

In particular, if $\mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) = 0$, then $\mathcal{K}^{[k+\chi]}_\gamma = -\chi J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma$. Herewith, from the chain of equalities

$$\left[\mathcal{H}^{[\gamma]}_\gamma, J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma\right] = \mathcal{H}^{[\gamma]}_\gamma (J^{[k]}_\gamma) \mathcal{H}^{[\gamma]}_\gamma + J^{[k]}_\gamma \mathcal{H}^{[\gamma]}_\gamma \mathcal{H}^{[\gamma]}_\gamma = 0,$$

it follows that $\mathcal{H}^{[\gamma]}_\gamma (K^{[k+\chi+\delta]}_\gamma) = 0$.

2) The converse follows from the fact that any two integrals of $\mathcal{H}^{[\gamma]}_\gamma$ are functionally dependent.

**Theorem 1.**

Given any system (8) and arbitrary Hamiltonian resonant set $\mathcal{S}_\gamma$ and Hamiltonian reduced resonant set $\bar{\mathcal{S}}_\gamma$, then there exists a near-identity formal transformation (2) that brings it to a certain GPHNF generated by $\mathcal{S}_\gamma$ and $\bar{\mathcal{S}}_\gamma$.

**Proof.**

Let $\mathcal{S}_\gamma$, $\bar{\mathcal{S}}_\gamma$ be Hamiltonian resonant and reduced resonant sets for $\mathcal{H}^{[\gamma]}_\gamma$, and let transformation (2) with ord. $\mathcal{Q} = m \geq 1$ bring system (8) into system (3) of the form

$$\dot{y} = \mathcal{H}^{[\gamma]}_\gamma \gamma(y) + \sum_{k=1}^{\infty} (\bar{\mathcal{F}}^{[k+\chi]}_\gamma \gamma(y) + \bar{\mathcal{G}}^{[k+\chi]}_\gamma \gamma(y)), \quad (10)$$

where $\bar{\mathcal{F}}^{[k+\chi]}_\gamma = (-\partial_2 \bar{\mathcal{F}}^{[k+\chi+\delta]}_\gamma, \partial_1 \bar{\mathcal{F}}^{[k+\chi+\delta]}_\gamma)$ and $\bar{\mathcal{G}}^{[k+\chi]}_\gamma = \text{div}(\gamma^{[k+\chi]}_\gamma(y))/\gamma + \chi \gamma^{[k+\chi+\delta]}_\gamma$.

By formulas (6) and (9), we get the following expression for the Lie bracket of the VQHPs $\mathcal{H}^{[\gamma]}_\gamma$ and $\mathcal{Q}^{[m]}_\gamma$:

$$\left[\mathcal{H}^{[\gamma]}_\gamma, \mathcal{Q}^{[m]}_\gamma\right] = \left[\mathcal{H}^{[\gamma]}_\gamma, \mathcal{T}^{[m]}_\gamma + J^{[m]}_\gamma \mathcal{E}^{[\gamma]}_\gamma\right] = \left(\mathcal{H}^{[\gamma]}_\gamma, \mathcal{T}^{[m]}_\gamma + K^{[m+\chi]}_\gamma\right) = \frac{m + \delta}{m + \chi + \delta} \mathcal{H}^{[\gamma]}_\gamma (J^{[m]}_\gamma) \mathcal{E}^{[\gamma]}_\gamma. \quad (11)$$

Substituting into (5) $\mathcal{P}^{[\gamma]}_\gamma = \mathcal{H}^{[\gamma]}_\gamma, \mathcal{X}^{[m+\chi]}_\gamma = \bar{\mathcal{F}}^{[m+\chi]}_\gamma + \bar{\mathcal{G}}^{[m+\chi]}_\gamma \mathcal{E}^{[\gamma]}_\gamma$, and $\mathcal{Y}^{[m+\chi]}_\gamma = \bar{\mathcal{F}}^{[m+\chi]}_\gamma + \bar{\mathcal{G}}^{[m+\chi]}_\gamma \mathcal{E}^{[\gamma]}_\gamma$ and using equality (11), we get the system of equations

$$\mathcal{H}^{[\gamma]}_\gamma (J^{[m+\delta]}_\gamma) + K^{[m+\chi+\delta]}_\gamma = \bar{\mathcal{F}}^{[m+\chi+\delta]}_\gamma - \bar{\mathcal{G}}^{[m+\chi+\delta]}_\gamma, \quad \frac{m + \delta}{m + \chi + \delta} \mathcal{H}^{[\gamma]}_\gamma (J^{[m]}_\gamma) = \bar{\mathcal{G}}^{[m+\chi]}_\gamma - \bar{\mathcal{G}}^{[m+\chi]}_\gamma. \quad (12)$$
Denote \( \mathcal{S}_\gamma^{[m+\chi]} = \{ S_{\gamma,j}^{[m+\chi]} \}_{j=1}^{s_{m+\chi}} \), \( \mathcal{S}_{\gamma}^{[m+\chi+\delta]} = \{ S_{\gamma,j}^{[m+\chi+\delta]} \}_{j=1}^{s_{m+\chi+\delta}} \).

In the linear spaces \( \mathcal{R}_\gamma^{[m+\chi]} \) and \( \mathcal{R}_{\gamma}^{[m+\chi+\delta]} = \mathcal{R}_{\gamma}^{[m+\chi+\delta]} \cap \mathcal{H}_{\gamma}^{[\chi]} (\mathcal{P}_{\gamma}^{[m+2\chi+\delta]}) \), choose bases \( \{ R_{\gamma,i}^{[m+\chi]} \}_{i=1}^{s_{m+\chi}} \) and \( \{ R_{\gamma,i}^{[m+\chi+\delta]} \}_{i=1}^{s_{m+\chi+\delta}} \) respectively.

Define matrices \( A = \{ (\langle R_{\gamma,i}^{[m+\chi]}, S_{\gamma,j}^{[m+\chi]} \rangle) \}_{i,j=1}^{s_{m+\chi}} \) and \( A = \{ (\langle R_{\gamma,i}^{[m+\chi+\delta]}, S_{\gamma,j}^{[m+\chi+\delta]} \rangle) \}_{i,j=1}^{s_{m+\chi+\delta}} \), vectors \( c = \{ (\langle R_{\gamma,i}^{[m+\chi]}, G_{\gamma,j}^{[m+\chi]} \rangle) \}_{i,j=1}^{s_{m+\chi}} \) and \( \bar{c} = \{ (\langle R_{\gamma,i}^{[m+\chi+\delta]}, F_{\gamma,j}^{[m+\chi+\delta]} \rangle) \}_{i,j=1}^{s_{m+\chi+\delta}} \), \( b = A^{-1}c \) and \( \bar{b} = A^{-1}\bar{c} \), and QHPs \( \bar{G}_{\gamma}^{[m+\chi]} = \sum_{j=1}^{s_{m+\chi}} b_j S_{\gamma,j}^{[m+\chi]} \) and \( \bar{F}_{\gamma}^{[m+\chi+\delta]} = \sum_{j=1}^{s_{m+\chi+\delta}} b_j S_{\gamma,j}^{[m+\chi+\delta]} \).

Then \( \langle R_{\gamma}^{[m+\chi]}, \bar{G}_{\gamma}^{[m+\chi]} \rangle - \langle F_{\gamma}^{[m+\chi+\delta]} \rangle = \sum_{j=1}^{s_{m+\chi}} A_{ij} b_j - c_i = 0 \) for all \( i = 1, s_{m+\chi} \).

Hence, by the Fredholm alternative, we obtain the QHP \( J_{\gamma}^{[m]} \) satisfying (12\_2), up to an integral of the VQHP \( \mathcal{H}_{\gamma}^{[\chi]} \).

To complete the proof, it is enough to consider the case where \( \bar{G}_{\gamma}^{[m+\chi]} = G_{\gamma}^{[m+\chi]} \).

We have \( \langle R_{\gamma,i}^{[m+\chi+\delta]}, \bar{F}_{\gamma}^{[m+\chi+\delta]} - F_{\gamma}^{[m+\chi+\delta]} \rangle = \sum_{j=1}^{s_{m+\chi+\delta}} A_{ij} b_j - c_i = 0 \) for all \( i = 1, s_{m+\chi+\delta} \).

Hence, by the Fredholm alternative and the definition of \( R_{\gamma,i}^{[m+\chi+\delta]} \), the QHP \( \bar{F}_{\gamma}^{[m+\chi+\delta]} - F_{\gamma}^{[m+\chi+\delta]} \) can be represented in the form (12\_1) where \( \mathcal{H}_{\gamma}^{[\chi]} (K_{\gamma}^{[m+\chi+\delta]} = 0 \). And according to Lemma 2, every such QHP \( K_{\gamma}^{[m+\chi+\delta]} \) can be obtained by choosing an appropriate \( J_{\gamma}^{[m]} \) in (11) such that \( \mathcal{H}_{\gamma}^{[\chi]} (J_{\gamma}^{[m]} = 0 \).

So, we have proved the existence of the QHPs \( J_{\gamma}^{[m+\delta]} \) and \( J_{\gamma}^{[m]} \) that satisfy (12), and hence, there exists a VQHP \( \bar{Q}_{\gamma}^{[m]} \) such that the transformation (2) with \( \text{ord}_{\gamma} Q = m \geq 1 \) takes system (8) to the view (10) with \( \bar{F}_{\gamma}^{[m+\chi+\delta]} \in \text{Lin}(\mathcal{S}_{\gamma}^{[m+\chi+\delta]} \) and \( \bar{G}_{\gamma}^{[m+\chi]} \in \text{Lin}(\mathcal{S}_{\gamma}^{[m+\chi]} \).

Hence, step by step increasing \( m \), we find the required transformation as a composition of the transformations obtained on each step. \( \square \)

3. REDUCTION OF THE GPHNF TO GNF

Consider GPHNF (10) \( \dot{y} = \mathcal{H}_{\gamma}^{[\chi]} (y) + \sum_{k=1}^{\infty} \left( \bar{F}_{\gamma}^{[k+\chi]} (y) + \bar{G}_{\gamma}^{[k+\chi]} (y) \mathcal{E}_{\gamma} (y) \right) \) generated by arbitrary minimal sets \( \mathcal{S}_{\gamma}, \bar{\mathcal{S}}_{\gamma} \).

Coefficients of its perturbation \( \mathcal{Y} \) can be expressed in terms of \( \bar{F} \) and \( \bar{G} \) as follows:

\[
Y_{1}^{(p_1+1,p_2)} = -(p_2+1)\bar{F}^{(p_1+1,p_2+1)} + \gamma_1 \bar{G}^{(p_1,p_2)}, \quad Y_{2}^{(p_1,p_2+1)} = (p_1+1)\bar{F}^{(p_1+1,p_2+1)} + \gamma_2 \bar{G}^{(p_1,p_2)}. \tag{13}
\]

Note that if \( y_1^{p_1} y_2^{p_2} \notin \mathcal{S}_{\gamma} \), then there exists a vector \( q = (q_1, q_2) \) with integer nonnegative components such that \( \langle p - q, \gamma \rangle = \chi \) and at least one of the coefficients \( [\mathcal{H}_{\gamma}^{[\chi]} , y_1^{q_1} y_2^{q_2} \mathcal{E}_{\gamma}]^{(p_1+1,p_2)} \) and \( [\mathcal{H}_{\gamma}^{[\chi]} , y_1^{q_1} y_2^{q_2} \mathcal{E}_{\gamma}]^{(p_1,p_2+1)} \) is nonzero.

Define a subset \( \mathcal{Y} \) of the set of coefficients of \( \mathcal{Y} \), element by element, as follows:

i) \( Y_{1}^{(p_1+1,p_2)}, Y_{2}^{(p_1,p_2+1)} \in \mathcal{Y} \), if \( y_1^{p_1+1} y_2^{p_2+1} \in \bar{\mathcal{S}}_{\gamma}, y_1^{p_1} y_2^{p_2} \in \mathcal{S}_{\gamma} \).
ii) either $Y_1^{(p_1, p_2)} \in \mathcal{E}_2$ or $Y_2^{(p_1, p_2)} \in \mathcal{E}_2$, if $y_1^{p_1+1}y_2^{p_2+1} \notin \mathcal{E}_2$, $y_1y_2 \in \mathcal{E}_2$;

iii) either $Y_1^{(p_1, p_2)} \in \mathcal{E}_2$ or $Y_2^{(p_1, p_2)} \in \mathcal{E}_2$, if $y_1^{p_1+1}y_2^{p_2+1} \notin \mathcal{E}_2$, $y_1y_2 \notin \mathcal{E}_2$ and there exists a vector $q = (q_1, q_2)$ with nonnegative integer components such that $\langle p - q, \gamma \rangle = \chi$ and, respectively, $[\mathcal{H}_\gamma, y_1^{q_1}y_2^{q_2}\mathcal{E}_\gamma]^{(p_1, p_2+1)} \neq 0$ or $[\mathcal{H}_\gamma, y_1^{q_1}y_2^{q_2}\mathcal{E}_\gamma]^{(p_1+1, p_2)} \neq 0$.

**Theorem 2.** Given any system (1) with a Hamiltonian unperturbed part $\mathcal{P}_\gamma^{[k]} = \mathcal{H}_\gamma^{[k]}$ where $\mathcal{H}_\gamma^{[k]} = (-\partial y_1, \partial y_2^{[k+\delta]})$, and given arbitrary Hamiltonian resonant set $\mathcal{S}_\gamma$, Hamiltonian reduced resonant set $\mathcal{S}_\gamma$, and the set $\mathcal{E}_2$ constructed by the above rules, then there exists a near-identity formal transformation (2) that brings it to the form (3), where all coefficients of the perturbation are zero, except for the coefficients from $\mathcal{E}_2$ that have arbitrary values. Moreover, the obtained system is a GNF.

**Proof.** According to Proposition 1, it is enough to show that the set $\mathcal{E}_2$ is a resonant set for the unperturbed part $\mathcal{H}_\gamma^{[k]}$.

Denote $\mathcal{E}_\gamma^{[k+\delta]} = \{Y_1^{(p_1, p_2)}, Y_2^{(p_1, p_2+1)} \in \mathcal{E}_2 : \langle p, \gamma \rangle = k+\chi\}$, $\gamma_k = |\mathcal{E}_\gamma^{[k+\delta]}|$, $r_k = |\mathcal{S}_\gamma^{[k+\delta]}|$, and $\bar{r}_k = |\mathcal{S}_\gamma^{[k+\delta]}|$.

First, show that for all $k \geq 1$, the number $\gamma_k$ is equal to the number of independent resonance equations in g. d. $k + \chi$, i. e., $\gamma_k = \dim \mathcal{E}_\gamma^{[k+\delta]} - \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]})$, where $\mathcal{E}_\gamma^{[k+\delta]}$ denotes the linear space of VQHPs in g. d. $k + \chi$, and $[\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]}]$ denotes the image of the linear operator $[\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]}] : \mathcal{E}_\gamma^{[k]} \rightarrow \mathcal{E}_\gamma^{[k+\delta]}$.

Indeed, by construction, $\gamma_k = r_k + \bar{r}_k$. In turn, it follows from the definition of the sets $\mathcal{S}_\gamma$ and $\mathcal{S}_\gamma$ that $\gamma_k = \dim \mathcal{E}_\gamma^{[k+\delta]} - \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]})$ and $\gamma_k = \dim \mathcal{E}_\gamma^{[k+\delta]} - \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]}) + \dim \mathcal{S}_\gamma^{[k+\delta]}$. According to (6) and (9), $\dim \mathcal{E}_\gamma^{[k+\delta]} = \dim \mathcal{E}_\gamma^{[k+\delta]} + \dim \mathcal{E}_\gamma^{[k+\delta]}$, and $\dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]}) = \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]}) + \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]}) + \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]})$. Hence, $\gamma_k = r_k + \bar{r}_k = \dim \mathcal{E}_\gamma^{[k+\delta]} - \dim(\mathcal{H}_\gamma^{[k]}, \mathcal{E}_\gamma^{[k]})$.

By construction, all the $\gamma_k$ elements of the set $\mathcal{E}_\gamma^{[k+\delta]}$ can be uniquely expressed from the system of $\gamma_k$ linear equations (13) in terms of coefficients of the QHPs $\mathcal{F}_\gamma^{[k+\delta]}$, $\mathcal{G}_\gamma^{[k+\delta]}$ from (10) (all but the $\gamma_k$ coefficients of the perturbation of the GPHNF are zero in g. d. $k + \chi$). Therefore, it remains to show transformations that eliminate the coefficients that do not belong to $\mathcal{E}_2$, in cases ii) and iii).

ii) Let $Y_1^{(p_1, p_2)} \in \mathcal{E}_2$, $y_1^{p_1+1}y_2^{p_2+1} \notin \mathcal{E}_2$, $y_1y_2 \in \mathcal{E}_2$. Thereby, $Y_2^{(p_1, p_2+1)} \notin \mathcal{E}_2$ and $(Y_1^{(p_1, p_2)}, y_1^{p_1+1}y_2^{p_2+1}) = \mathcal{G}^{(p_1, p_2)}y_1^{p_1}y_2^{p_2}\mathcal{E}_\gamma$. It follows then from the expansion

$$y_1^{p_1}y_2^{p_2}\mathcal{E}_\gamma = \frac{\gamma_2}{p_1 + 1}(-(p_2 + 1)y_1^{p_1+1}y_2^{p_2}, (p_1 + 1)y_1^{p_1}y_2^{p_2+1}) + \frac{\gamma_1(p_1 + 1) + \gamma_2(p_2 + 1)}{p_1 + 1}(y_1^{p_1+1}y_2^{p_2}, 0)$$
that the coefficient $Y_2^{(p_1, p_2+1)}$ can be set to zero by adding the term $-\gamma_2 \check{G}(p_1, p_2)(p_1+1)^{-1} y_1^{p_1+1} y_2^{p_2+1}$ to $\check{F}^{[k+\chi+\delta]}$ in the proof of Theorem 1, which is possible, since $y_1^{p_1+1} y_2^{p_2+1} \not\in \check{S}_\gamma$.

Similarly, in case where $Y_2^{(p_1, p_2+1)} \in \mathcal{Y}$, $y_1^{p_1+1} y_2^{p_2+1} \not\in \check{S}_\gamma$, $y_1^{p_1} y_2^{p_2} \in \check{S}_\gamma$, we eliminate the coefficient $Y_1^{(p_1+1, p_2)} \not\in \mathcal{Y}$.

iii) Let $Y_1^{(p_1+1, p_2)} \in \mathcal{Y}$, $y_1^{p_1+1} y_2^{p_2+1} \in \check{S}_\gamma$, $y_1^{p_1} y_2^{p_2} \not\in \check{S}_\gamma$, and let there exist a vector $q = (q_1, q_2)$ with nonnegative integer components $q_1, q_2$ such that $(p - q, \gamma) = \chi$, $[\check{H}_\gamma^{[x]}, y_1^{q_1} y_2^{q_2} E_\gamma]^{(p_1, p_2+1)} \neq 0$. Then, $Y_2^{(p_1, p_2+1)} \not\in \mathcal{Y}$ and

$$
\left( Y_1^{(p_1+1, p_2)} y_1^{p_1+1} y_2^{p_2}, Y_2^{(p_1+1, p_2)} y_1^{p_1} y_2^{p_2+1} \right) = \check{F}(p_1+1, p_2+1) \left( -(p_2+1) y_1^{p_1+1} y_2^{p_2}, (p_1+1) y_1^{p_1} y_2^{p_2+1} \right).
$$

Then the coefficient $Y_2^{(p_1, p_2+1)}$ can be set to zero by using a transformation of the form (2) where $Q = C y_1^{q_1} y_2^{q_2} E$ with an appropriate coefficient $C$. Herewith, all the extra terms can be taken into account by adding in a suitable way the terms that do not belong to $\check{S}_\gamma$ and $\check{S}_\gamma$ to the QHPs $\check{F}^{[k+\chi+\delta]}$ and $\check{G}^{[k+\chi]}$ respectively (see the proof of Theorem 1).

Similarly, if $Y_2^{(p_1, p_2+1)} \in \mathcal{Y}$, $y_1^{p_1+1} y_2^{p_2+1} \in \check{S}_\gamma$, $y_1^{p_1} y_2^{p_2} \not\in \check{S}_\gamma$, and $[\check{H}_\gamma^{[x]}, y_1^{q_1} y_2^{q_2} E_\gamma]^{(p_1, p_2+1)} \neq 0$, we eliminate the coefficient $Y_1^{(p_1+1, p_2)} \not\in \mathcal{Y}$.

**Remark 2.** If there are several pairs of coefficients $Y_1^{(p_1+1, p_2)}, Y_2^{(p_1, p_2+1)}$ in one and the same g. d. that fit case iii), then there exists the same number of monomials $y_1^{q_1} y_2^{q_2}$ that satisfy the conditions described for this case, and the coefficients for the transformations are obtained from an algebraic system with nonzero determinant that expresses the equality to zero of the corresponding coefficients of $\mathcal{Y}$.

### 4. GNFS of Systems the Hamiltonian unperturbed part of which has monomial components

In this section, using Theorems 1 and 2, we compute GNFS for systems with a Hamiltonian unperturbed part represented by a vector monomial, i. e. a vector with the monomial components. The results are compared with the known GNFS, obtained earlier by Takens [7], Baider and Sanders [6], Basov et al. [1, 3–5].

#### 4.1. The unperturbed part $(x_2^{m-1}, 0)$ with $m \geq 2$

Consider system (8) where $\mathcal{H}_\gamma^{[x]} = (x_2^{m-1}, 0), \ H_\gamma^{[x+\delta]} = -\frac{x_2^m}{m}, \ \gamma = (1, 1), \ \chi = m - 2, \ m \geq 2$. 
In this case, the space of resonant polynomials has the view (see [2])

\[ \mathfrak{R}_\gamma = \text{Lin}(\{x_1^{p_1}x_2^{p_2} : \ p_2 = 0, m - 2 \}). \]

\( \mathfrak{R}_\gamma \) does not contain integrals of \( \mathcal{H}_\gamma^{|x|} \) of degree higher than \( m \), that is, every Hamiltonian reduced resonant set is a Hamiltonian resonant set, in degrees higher than \( m \).

**Corollary 1.** Given any system (1) with the unperturbed part \((x_2^{m-1}, 0)\) where \( m \geq 2 \), then there exists a near-identity formal transformation (2) that brings it into the GNF

\[
\begin{align*}
\dot{y}_1 &= y_2^{m-1} + \sum_{i=m}^{\infty} \sum_{j=0}^{m-3} Y_1^{(i-j,j)} y_1^{i-j} y_2^j + y_1^2 y_2^{m-2} \sum_{i=0}^{\infty} Y_1^{(i+2,m-2)} y_1^{i}, \\
\dot{y}_2 &= \sum_{i=m}^{\infty} \sum_{j=0}^{m-2} Y_2^{(i-j,j)} y_1^{i-j} y_2^j + y_1 y_2^{m-1} \sum_{i=0}^{\infty} Y_2^{(i+1,m-1)} y_1^{i}
\end{align*}
\]  
(14)

where for each \( i \geq 0 \), we take either \( Y_1^{(i+2,m-2)} = 0 \) or \( Y_2^{(i+1,m-1)} = 0 \).

**Proof.** For all \( k > m \), choose the Hamiltonian resonant sets

\[ \mathfrak{G}_\gamma^{[k]}, \mathfrak{G}_\gamma^{[k]} = \{ x_1^{p_1}x_2^{p_2} : \ p_2 = 0, m - 2; \ |p| = k \}. \]

Then GPHNF (10) takes the form (14) with \( Y_1^{(i+2,m-2)} = Y_2^{(i+1,m-1)} = \tilde{G}^{(i+1,m-2)} \). Hence, the corollary follows from Theorem 2. \( \square \)

For \( m = 2 \), formula (14) gives the Takens normal form [7], and for \( m = 3 \), it gives [4, Th. 11].

4.2. The unperturbed part \((-mx_1^l x_2^{m-1}, lx_1^{l-1} x_2^m)\) with \( l > m \geq 1 \)

Consider system (8) where \( \mathcal{H}_\gamma^{|x|} = (-mx_1^l x_2^{m-1}, lx_1^{l-1} x_2^m) \), \( \mathcal{H}_\gamma^{|x+\delta|} = x_1^l x_2^m \), \( \gamma = (1, 1) \), \( \chi = l + m - 2, \ l > m \geq 1 \), and \( \text{GCD}(l,m) = d \).

In this case, the space of resonant polynomials has the view (see [2])

\[ \mathfrak{R}_\gamma = \text{Lin}(\{x_1^{p_1}x_2^{p_2} : \ p_1 = \overline{0,l-2}, \ \text{or} \ p_2 = \overline{0,m-2}, \ \text{or} \ p_1 = rl/d - 1, \ p_2 = rm/d - 1, \ r \geq d \}). \]

\( \mathfrak{R}_\gamma \) does not contain integrals of \( \mathcal{H}_\gamma^{|x|} \) of degree higher than \( l + m \), that is, every Hamiltonian reduced resonant set is a Hamiltonian resonant set, in degrees higher than \( l + m \).

**Corollary 2.** Given any system (1) with the unperturbed part \((-mx_1^l x_2^{m-1}, lx_1^{l-1} x_2^m)\) where \( l > m \geq 1 \) and \( \text{GCD}(l,m) = d \), then there exists a near-identity formal transformation (2) that
brings it into the GNF

\[
\dot{y}_1 = -my_1^l y_2^m - 1 + \sum_{k=l+m}^{l-2} \left( \sum_{i=0}^{l-2} Y_1^{(i,k-i)} y_i y_2^{k-i} + \sum_{j=0}^{m-3} Y_1^{(k-j,j)} y_1^{k-j} y_2^j \right) + \\
+ \sum_{i=0}^{l-2} Y_1^{(i+l+2,m-2)} y_i^l + Y_1^{(l-1,m+1)} \sum_{j=0}^{\infty} Y_1^{l-1,m+j+1} y_j^2 + \\
+ \sum_{r=d+1}^{\infty} Y_1^{(r+l,rm/d-1)} y_1^{r/l} y_2^{m/r} + \sum_{s=d+1+\left[3/r+1/m\right]}^{\infty} Y_2^{(sl/d-1,sm/d-2)} y_1^{s/l} y_2^{m/s} 
\]

(15)

where for each \(i, j \geq 0, r \geq d + 1\) and \(s \geq d + 1 + \left[3/r+1/m\right]\), we take either \(Y_1^{(i+l+2,m-2)} = 0\) or \(Y_2^{(i+l+1,m-1)} = 0\), either \(Y_1^{(l-1,m+j+1)} = 0\) or \(Y_2^{(l-2,m+j+2)} = 0\), either \(Y_1^{(r+l,rm/d-1)} = 0\) or \(Y_2^{(r+l,rm/d-1)} = 0\), and in case \(m = 1\), we take \(Y_2^{(sl/d-2,sm/d-1)} = 0\), and in case \(m \geq 2\) we take either \(Y_1^{(sl/d-1,sm/d-2)} = 0\) or \(Y_2^{(sl/d-2,sm/d-1)} = 0\).

Proof. For all \(k > l + m\), choose the Hamiltonian resonant sets

\[
\mathcal{G}_\gamma[k], \tilde{\mathcal{G}}_\gamma[k] = \{x_1^{p_1}, x_2^{p_2} : p_1 = 0, l - 2, \text{ or } p_2 = \overline{m - 2}, \text{ or } p_1 = r/l - 1, p_2 = rm/d - 1, r \geq d; \ |p| = k \}
\]

Then GPHNF (10) takes the form (15) where \(Y_1^{(l-1,m+j+1)} = Y_2^{(l-2,m+j+2)} = \mathcal{G}^{(l-2,m+j+1)}\), \(Y_1^{(l-1,m+j+1)} = Y_2^{(l-2,m+j+2)} = \mathcal{G}^{(l-2,m+j+1)}\), \(Y_1^{(l-1,m+j+1)} = Y_2^{(l-2,m+j+2)} = \mathcal{G}^{(l-2,m+j+1)}\), \(Y_1^{(l-1,m+j+1)} = Y_2^{(l-2,m+j+2)} = \mathcal{G}^{(l-2,m+j+1)}\), and \(Y_1^{(l-1,m+j+1)} = Y_2^{(l-2,m+j+2)} = \mathcal{G}^{(l-2,m+j+1)}\).

Transformation (2), with \(Q = C_1 y_1^{sl/d-1} y_2^{sm/d-1}\) \(\mathcal{G} \) where \(C = \text{const.}\) changes the coefficients \(Y_1^{(l-1,m+j+1)}\) and \(Y_2^{(l-2,m+j+2)}\) by \(C(1-m)(l+m)\) and \(C(l-1)(l+m)\) respectively.

Hence, the corollary follows from Theorem 2.

For \(l = 2, m = 1\), formula (15) gives [3, Th. 7] for \(\alpha = -1/2\).

4.3. The unperturbed part \((-x_1^m x_2^{m-1}, x_1^{m-1} x_2^m)\) with \(m \geq 1\)

Consider system (8) where \(\mathcal{H}_\gamma\) \(=-x_1^m x_2^{m-1}, x_1^{m-1} x_2^m\), \(\mathcal{H}_\gamma^{\chi+\delta} = x_1^m x_2^m / m, \ \gamma = (1,1), \ \chi = 2m - 2, \text{ and } m \geq 1\).
In this case, the space of resonant polynomials has the view (see [2])

$$\mathcal{R}_\gamma = Lin\{x_1^{p_1}x_2^{p_2} : p_1 = 0, m - 2, \text{ or } p_2 = 0, m - 2, \text{ or } p_1 = p_2 \}.$$ 

Since monomials $x_1^k x_2^l$ ($k \geq 0$) are integrals of the unperturbed part, such monomials are absent in the Hamiltonian reduced resonant set, in degrees higher than $2m$.

**Corollary 3.** Given any system (1) with the unperturbed part $(-x_1^m x_2^{m-1}, x_1^{m-1} x_2^m)$ where $m \geq 1$, then there exists a near-identity formal transformation (2) that brings it into the GNF

$$\dot{y}_1 = -y_1^m y_2^{m-1} + \sum_{k=2m}^{\infty} \left( \sum_{i=0}^{m-2} Y_1^{(i,k-i)} y_1^i y_2^{k-i} + \sum_{j=0}^{m-3} Y_1^{(k-j,j)} y_1^{k-j} y_2^j \right) +$$

$$+ y_1^{m+2} y_2^{m-2} \sum_{i=0}^{\infty} Y_1^{(i+m+2,m-2)} y_1^i + y_1^{m+1} y_2 y_1^{m+1} \sum_{j=0}^{\infty} Y_1^{(m-1,m+j+1)} y_2^j +$$

$$+ y_1^m y_2^{m} \sum_{r=0}^{\infty} Y_1^{(r+m+1,r+m)} y_1^r y_2^r,$$

$$\dot{y}_2 = y_1^{m-1} y_2^m + \sum_{k=2m}^{\infty} \left( \sum_{i=0}^{m-3} Y_2^{(i,k-i)} y_1^i y_2^{k-i} + \sum_{j=0}^{m-2} Y_2^{(k-j,j)} y_1^{k-j} y_2^j \right) +$$

$$+ y_1^{m+1} y_2^{m-1} \sum_{i=0}^{\infty} Y_2^{(i+m+1,m-1)} y_1^i + y_1^{m-2} y_2^{m+2} \sum_{j=0}^{\infty} Y_2^{(m-2,m+j+2)} y_2^j +$$

$$+ y_1^m y_2^{m} \sum_{r=0}^{\infty} Y_2^{(r+m+1,r+m)} y_1^r y_2^r,$$

where for each $i, j, r \geq 0$, we take either $Y_1^{(i+m+2,m-2)} = 0$ or $Y_2^{(i+m+1,m-1)} = 0$, either $Y_1^{(m-1,m+j+1)} = 0$ or $Y_2^{(m-2,m+j+2)} = 0$, and either $Y_1^{(r+m+1,r+m)} = 0$ or $Y_2^{(r+m,r+m+1)} = 0$.

**Proof.** For all $k > 2m$, choose

$$\mathcal{G}_\gamma^{[k]} = \{x_1^{p_1} x_2^{p_2} : p_1 = 0, m - 2, \text{ or } p_2 = 0, m - 2, \text{ or } p_1 = p_2; |p| = k \},$$

$$\mathcal{G}_\gamma^{[k]} = \{x_1^{p_1} x_2^{p_2} : p_1 = 0, m - 2 \text{ or } p_2 = 0, m - 2; |p| = k \}.$$ 

Then GPHNF (10) takes the form (16) where $Y_1^{(i+m+2,m-2)} = Y_2^{(i+m+1,m-1)} = G^{(i+m+1,m-2)},$ $Y_1^{(m-1,m+j+1)} = Y_2^{(m-2,m+j+2)} = G^{(m-2,m+j+1)},$ $Y_1^{(r+m+1,r+m)} = Y_2^{(r+m,r+m+1)} = G^{(r+m,r+m)}.$

Hence, the corollary follows from Theorem 2. 

**4.4. The unperturbed part $(\pm x_2^{m-1}, x_1^{l-1})$ with $l \geq m \geq 2$**

Consider system (8) where $\mathcal{H}_\gamma^{[\chi]} = (\pm x_2^{m-1}, x_1^{l-1}),$ $\mathcal{H}_\gamma^{[\chi+\delta]} = x_1^l / l \mp x_2^m / m,$ $\gamma = (m/d, l/d),$ $\chi = (lm - l - m) / d,$ $l \geq m \geq 2,$ and $d = \text{GCD}(l, m).$
Lemma 3. Minimal Hamiltonian resonant and reduced resonant sets in g. d. \( k > lm/d \) have the view

\[
\mathcal{S}_\gamma^{[k]} = \{ x_1^{p_1-r_p} x_2^{p_2+r_p m} : p_1 \neq -1 \mod l, \ p_2 = 0, m - 2 \},
\]

\[
\tilde{\mathcal{S}}_\gamma^{[k]} = \{ x_1^{p_1-r_p} x_2^{p_2+r_p m} : p_1 \neq -1, 0 \mod l, \ p_2 = 0 \text{ or } p_1 \neq -1 \mod l, \ p_2 = \overline{m - 2} \}
\]

where \( \langle p, \gamma \rangle = k \), and \( r_p, \bar{r}_p \) are arbitrary integers such that \( 0 \leq r_p, \bar{r}_p \leq \lfloor p_1/l \rfloor \).

**Proof.** Let \( R = \sum_{p_1, p_2=0}^\infty R(p_1, p_2) x_1^{p_1} x_2^{p_2} \in \mathcal{R}_\gamma \).

According to formula (7) \( H^{[\chi]} = x_2 (\partial^{l-1}/\partial x_1^{l-1}) + x_1 (\partial^{m-1}/\partial x_2^{m-1}) \), thus

\[
\sum_{i=1-l}^\infty \sum_{j=0}^\infty (l-1)! C_i^{l-1} R(i,j) x_1^{i+l+1} x_2^{j-1} + \sum_{i=0}^\infty \sum_{j=m-1}^\infty (m-1)! C_j^{m-1} R(i,j) x_1^{i+1} x_2^{-m+1} = 0,
\]

hence, setting the coefficients of \( x_1^{i+l+1}, x_2^{j+1} \), and \( x_1^{i+1} x_2^{j+1} \) to zero, we obtain the equations

\[
R^{(i,m-1)} = 0, \quad R^{(l-1,j)} = 0, \quad (l-1)! C_i^{l+1} R(i+l,j) + (m-1)! C_j^{m-1} R(i,j+m) = 0 \quad (i, j \geq 0).
\]

(17)

Hence, by induction, we find that \( R^{(i,km-1)}, R^{(kl-1,j)} = 0 \) for all \( k \geq 1 \).

It also follows from equations (17) that any resonant polynomial \( R \) is uniquely defined by its coefficients \( R^{(p_1-r_p,l,p_2+r_p m)} \) of the monomials \( x_1^{p_1-r_p} x_2^{p_2+r_p m} \) where \( p_2 \leq m - 2 \) \( (0 \leq r_p \leq \lfloor p_1/l \rfloor) \). Thus, set \( \mathcal{S}_\gamma^{[k]} \) of such monomials is a minimal Hamiltonian resonant set.

For the Hamiltonian reduced resonant set, the lemma follows from the fact that any quasi-homogeneous polynomial integral for \( H^{[\chi]} \) is a power of \( H^{[\chi+\delta]}_\gamma \).

For each given \( k \in \mathbb{Z} \), denote \( \theta[k] = 0 \) if \( k < 0 \), and \( \theta[k] = 1 \) if \( k \geq 0 \).

**Corollary 4.** Given any system (1) with the unperturbed part \( (\pm x_2^{m-1}, x_1^{l-1}) \) where \( l \geq m \geq 2 \), then there exists a near-identity formal transformation (2) that brings it into the GNF

\[
\dot{y}_1 = \pm y_2^{m-1} + \sum_{j=1}^{m-2} y_2^{j-1} \left( \sum_{i \neq 0, -1 \mod l} Y_1^{(i,j-1)} y_1^i + \sum_{r=1+\theta[m-j]}^{\infty} Y_1^{(r l-1,j-1)} y_1^{r l-1} + \sum_{s=1}^{\infty} Y_1^{(s l,j-1)} y_1^s \right) + \sum_{i \neq 0 \mod l} Y_1^{(i,m-2)} y_1^i y_2^{m-2},
\]

(18)

\[
\dot{y}_2 = y_1^{l-1} + \sum_{j=1}^{m-2} y_1^{j-1} \left( \sum_{i \neq 0, -1 \mod l} Y_2^{(i-1,j)} y_1^{i-1} + \sum_{r=1+\theta[m-j]}^{\infty} Y_2^{(r l-2,j)} y_1^{r l-2} + \sum_{s=1}^{\infty} Y_2^{(s l-1,j)} y_1^{s l-1} \right) + \sum_{i \neq 0 \mod l} Y_2^{(i-1,0)} y_1^{i-1} y_2 + \sum_{i \neq 0 \mod l} Y_2^{(i-1,m-1)} y_1^{i-1} y_2^{m-1}
\]
where for each \( i \neq 0 \mod l, \ i > l/m, \ j = \frac{1}{l}m - 2, \ r \geq 1 + \theta[m - jl], \) and \( s \geq 1, \) we take either \( Y_1^{(i,m-2)} = 0 \) or \( Y_2^{(i-1,m-1)} = 0, \) either \( Y_1^{(r-1,j-1)} = 0 \) or \( Y_2^{(r-2,j)} = 0, \) either \( Y_1^{(sl,j-1)} = 0 \) or \( Y_2^{(sl-1,j)} = 0, \) except for the case where \( l = m, \) in which for \( j = m - 2, \) we take \( Y_1^{(sm,m-3)} = 0 \) in the pairs \( \{Y_1^{(sm,m-3)}, Y_2^{(sm-1,m-2)}\}.\)

Proof. For all \( k > lm/d, \) choose the Hamiltonian resonant and reduced resonant sets given by Lemma 3 with all \( r_p, \tilde{r}_p = 0: \)

\[
\mathcal{S}_\gamma[k] = \{x_1^{p_1}x_2^{p_2} : p_1 \not\equiv -1 \mod l, \ p_2 = 0, \ m - 2; \ \langle p, \gamma \rangle = k \}, \\
\mathcal{\bar{S}}_\gamma[k] = \{x_1^{p_1}x_2^{p_2} : p_1 \not\equiv -1, \ 0 \mod l, \ p_2 = 0 \text{ or } p_1 \not\equiv -1 \mod l, \ p_2 = 1, \ m - 2; \ \langle p, \gamma \rangle = k \}.
\]

Then GPHNF (10) takes the form (18) where \( Y_1^{(i,m-2)} = Y_2^{(i-1,m-1)} = \widetilde{G}^{(i-1,m-2)}, \ Y_1^{(r-1,j-1)} = Y_2^{(r-2,j-1)} = \widetilde{G}^{(r-2,j-1)}, \ Y_1^{(sl,j-1)} = -\widetilde{F}^{(sl,j)}, \) and \( Y_2^{(sl-1,j)} = s\widetilde{F}^{(sl,j)}. \)

Transformation (2) with \( Q = Cy_1^{sl-l}y_2^jE \) where \( C = \text{const} \) changes the coefficients \( Y_1^{(sl,j-1)} \) and \( Y_2^{(sl-1,j)} \) by \(-Cj\) and \( C(l - j - 2)\) respectively. Hence, for \( l > m, \) as well as for \( l = m \) and \( j = 1, m - 3, \) we can set either coefficient in the pair \( \{Y_1^{(sl,j-1)}, Y_2^{(sl-1,j)}\} \) to zero by choosing \( C. \)

And only in case where \( l = m \) and \( j = m - 2, \) the coefficient \( Y_2^{(sm-1,m-2)} \) does not change under this transformation. In this case, we zero the coefficient \( Y_1^{(sm,m-3)} \) in system (18) by choosing \( C = Y_1^{(sm,m-3)}/(m - 3). \) Hence, the corollary follows from Theorem 2.

For \( m = 2, \) GNF (18) is the so called second order Takens-Bogdanov normal form that was obtained by Baider and Sanders in [6]. In particular, for \( l = 3, m = 2, \) formula (18) agrees with [5, Th. 4], and in case \( l = 4, m = 2, \) it is consistent with [1, Th. 3].
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