POSITIVE SOLUTIONS OF A THIRD ORDER NONLOCAL BOUNDARY VALUE PROBLEM

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Abstract. We consider a nonlocal boundary value problem for a third order differential equation. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are obtained. The results are illustrated with some examples.

1. Introduction. We consider the third order nonlinear ordinary differential equation
\[ u'''(t) = g(t)f(u(t)), \quad 0 \leq t \leq 1, \] (1)
together with boundary conditions
\[ u(0) = u'(p) = \int_p^1 w(t)u''(t)dt = 0. \] (2)

Throughout the paper and with no further mention, we assume that:
(H1) \( \frac{1}{4} < p < q < 1 \) are constants;
(H2) \( g: [0, 1] \rightarrow [0, \infty) \) is a continuous function such that \( g(t) \neq 0 \) on \([0, 1]\);
(H3) \( w: [q, 1] \rightarrow [0, \infty) \) is a nondecreasing continuous function with \( w(t) > 0 \) for \( q < t \leq 1 \);
(H4) \( f: [0, \infty) \rightarrow [0, \infty) \) is continuous.

The boundary value problem (1)–(2) always has the trivial solution if \( f(0) = 0 \), but here we are only interested in positive solutions, i.e., solutions \( u(t) \) such that \( u(t) > 0 \) on \((0, 1)\).

Boundary value problems for ordinary differential equations are important from a theoretical perspective as well as for their wide variety of applications in engineering and the physical and biological sciences. The works of Dulácska [7] on the effects of soil settlement, the classic work of Love [15] and the more recent monograph by Timoshenko [18] on elasticity, and that of Mansfield [16] on the deformation of
structures are but a few examples of the rich sources of applications of these kinds of problems. Surveys of known theoretical results can be found in the monographs by Agarwal [2] and Agarwal, O’Regan, and Wong [3]. In addition, recent contributions to the literature include the papers of Anderson and Davis [1], Chu and Zhou [4], Davis et al. [5, 6], Graef and Yang [8, 9], Henderson and Wang [11], Karakostas and Tsamatos [12, 13], Nowakowski and Orpel [17], Wang [19], Webb [20], and Yang [21].

In order to prove some of our results, we will use the following fixed point theorem which is known as the Guo-Krasnosel’skii fixed point theorem [10, 14].

**Theorem 1.1.** Let $X$ be a Banach space over the reals, and let $P \subset X$ be a cone in $X$. Assume that $\Omega_1$ and $\Omega_2$ are bounded open subsets of $X$ with $0 \in \Omega_1 \subset \Omega_1 \subset \Omega_2$, and let

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that either

$(K1)$ \quad $\|Lu\| \leq \|u\|$ if $u \in P \cap \partial \Omega_1$ and $\|Lu\| \geq \|u\|$ if $u \in P \cap \partial \Omega_2$, or

$(K2)$ \quad $\|Lu\| \geq \|u\|$ if $u \in P \cap \partial \Omega_1$ and $\|Lu\| \leq \|u\|$ if $u \in P \cap \partial \Omega_2$.

Then $L$ has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

We set $X = C[0,1]$ with the norm

$$\|v\| = \max_{t \in [0,1]} |v(t)|, \quad v \in X.$$ 

Clearly, $X$ is a Banach space over the reals. We also define the constants

$$F_0 = \limsup_{x \to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \frac{f(x)}{x},$$

$$F_\infty = \limsup_{x \to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x \to +\infty} \frac{f(x)}{x}.$$ 

**2. Green’s function and estimates to positive solutions.** We need the characteristic function $\chi$ to write the expression for the Green function for the problem (1)-(2). Recall that if $I \subset R$ is an interval, then the characteristic function $\chi$ on $I$ is given by

$$\chi_t(t) = \begin{cases} 1, & \text{if } t \in I, \\ 0, & \text{if } t \notin I. \end{cases}$$

We also define

$$W(t) = \left( \int_q^1 w(v)dv \right)^{-1} \int_t^1 w(s)ds, \quad q \leq t \leq 1.$$ 

Then the Green function $G : [0,1] \times [0,1] \to [0,\infty)$ for the equation

$$u''''(t) = 0$$

with the boundary condition (2) is given by

$$G(t, s) = -t(p-s)\chi_{[0,p]}(s) + \frac{(t-s)^2}{2} \chi_{[0,1]}(s)$$

$$+ \frac{t(2p-t)}{2} W(s) \chi_{[p,1]}(s) + \frac{t(2p-t)}{2} \chi_{[0,1]}(s).$$
The problem (1)–(2) is equivalent to the integral equation
\[ u(t) = \int_0^1 G(t, s)g(s)f(u(s))ds, \quad 0 \leq t \leq 1 \] (3)
in the sense that if \( u \) is a solution of the boundary value problem (1)–(2), then it is a solution of the integral equation (3), and conversely. We will take different cases to prove the positivity of the Green function. If \( s \geq p \), then
\[ G(t, s) = \frac{(t-s)^2}{2} \chi_{[0,q)}(s) + \frac{t(2p-t)}{2} W(s) \chi_{(q,1)}(s) + \frac{t(2p-t)}{2} \chi_{[0,q]}(s). \]
If \( s \leq p \) and \( s \geq t \), then
\[ G(t, s) = \frac{t(2s-t)}{2}. \]
If \( s \leq p \) and \( s \leq t \), then
\[ G(t, s) = \frac{s^2}{2}. \]
It is easy to see from the above expressions that \( G(t, s) > 0 \) for \( t, s \in (0,1) \).

The following two lemmas provide information about functions that satisfy the boundary conditions.

**Lemma 2.1.** If \( u \in C^3[0,1] \) satisfies (2), and
\[ u'''(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq 1, \] (4)
then \( u(t) \geq 0 \) for \( 0 \leq t \leq 1 \).

*Proof.* The lemma follows easily from the fact that
\[ u(t) = \int_0^1 G(t, s)u'''(s)ds \geq 0 \]
for \( 0 \leq t \leq 1 \). \( \square \)

**Lemma 2.2.** If \( u \in C^3[0,1] \) satisfies (2) and (4), then \( u(p) = \|u\| \).

*Proof.* Suppose that \( u \in C^3[0,1] \) satisfies (2) and (4). Since
\[ \int_q^1 w(t)u''(t)dt = 0, \]
there exists \( r \in (q,1) \) such that \( w(r)u''(r) = 0 \), which implies that \( u''(r) = 0 \). Note that (4) implies that \( u'' \) is nondecreasing on \([0,1]\). Therefore, we have
\[ u''(t) \leq 0 \quad \text{for} \quad t \in [0,r] \quad \text{and} \quad u''(t) \geq 0 \quad \text{for} \quad t \in [r,1]. \]
Therefore, \( u'(t) \) is nonincreasing on \([p,q] \subset [0,r]\). Since \( u'(p) = 0 \), we have \( u'(q) \leq 0 \) and \( u'(0) \geq 0 \). Since \( w(t) \) is nondecreasing, we have
\[ (w(t) - w(r))u''(t) \geq 0 \quad \text{for} \quad q \leq t \leq 1. \]
Thus,
\[
0 = \int_q^1 w(t)u''(t)dt \\
= \int_q^1 w(r)u''(t)dt + \int_q^1 (w(t) - w(r))u''(t)dt \\
\geq \int_q^1 w(r)u''(t)dt \\
= w(r)(u'(1) - u'(q)),
\]
which implies that \(u'(1) \leq u'(q)\). Therefore, \(u'(1) \leq 0\). Since \(u'(t)\) is concave upward on \([0, 1]\), we have
\[
u'(t) \geq 0 \text{ on } [0, p] \quad \text{and} \quad u'(t) \leq 0 \text{ on } [p, 1].
\]
Therefore, \(u(t)\) attains its maximum at \(p\), which completes the proof of the lemma.

In the remainder of the paper, we will make use of the function \(a : [0, 1] \to [0, 1]\) defined by
\[
a(t) = \frac{t(2p - t)}{p^2}.
\]
It is easy to see that
\[
a(t) \geq \min\{t, 1 - t\} \quad \text{for} \quad t \in [0, 1].
\]

Our next lemma provides a lower estimate on \(u(t)\).

**Lemma 2.3.** If \(u \in C^3[0, 1]\) satisfies \((2)\) and \((4)\), then
\[
u(t) \geq \|u\|a(t) \quad \text{for} \quad t \in [0, 1].
\]

**Proof.** Suppose that \(u(t) \in C^3[0, 1]\) satisfies \((2)\) and \((4)\). Without loss of generality, we may assume that \(\|u\| = u(p) = 1\). If we define
\[
y(t) = u(t) - a(t) = u(t) - \frac{2t}{p} + \frac{t^2}{p^2},
\]
then
\[
y'(t) = u'(t) - \frac{2}{p} + \frac{2t}{p^2}, \quad y''(t) = u''(t) + \frac{2}{p^2}, \quad \text{and} \quad y'''(t) = u'''(t) \geq 0.
\]
Clearly, we have
\[
y(0) = 0, \quad y(p) = 0, \quad \text{and} \quad y'(p) = 0.
\]
By the Mean Value Theorem, there exists \(r_1 \in (0, p)\) such that \(y'(r_1) = 0\). Since \(y'(t)\) is concave upward, we have
\[
y'(t) \geq 0 \text{ on } [0, r_1], \quad y'(t) \leq 0 \text{ on } [r_1, p], \quad \text{and} \quad y'(t) \geq 0 \text{ on } [p, 1].
\]
This, together with the fact that \(y(0) = y(p) = 0\), implies that
\[
y(t) \geq 0 \text{ on } [0, 1],
\]
and completes the proof of the lemma.

The next theorem is a direct consequence of Lemmas 2.1, 2.2, and 2.3.

**Theorem 2.4.** If \(u \in C^3[0, 1]\) is a nonnegative solution to the problem \((1)-(2)\), then
\[
a(t)u(p) \leq u(t) \leq u(p) \quad \text{on} \quad [0, 1].
\]
3. Existence of Positive Solutions. In this section, we present our main existence results. We begin with some notations. Define the constants
\[ A = \int_0^1 G(p, s)g(s)a(s) \, ds \quad \text{and} \quad B = \int_0^1 G(p, s)g(s) \, ds, \]
and let
\[ P = \{ x \in X : x(p) \geq 0, \ a(t)x(p) \leq x(t) \leq x(p) \quad \text{on} \quad [0, 1] \}. \]
We see that \( P \) is a positive cone in the Banach space \( X \). Define the operator \( T : P \rightarrow X \) by
\[ Tu(t) = \int_0^1 G(t, s)g(s)f(u(s))ds, \quad 0 \leq t \leq 1. \]
It is well known that \( T : P \rightarrow X \) is a completely continuous operator, and the same type of arguments as those used in Section 2 shows that \( T(P) \subset P \). Now, finding a positive solution of the integral equation (3) is equivalent to solving
\[ Tu = u, \quad u \in P. \]
That is, in order to solve the problem (1)–(2), we only need to find a fixed point of \( T \) in \( P \).

Our first existence result is the following.

**Theorem 3.1.** If \( BF_0 < 1 < Af_\infty \), then the boundary value problem (1)–(2) has at least one positive solution.

**Proof.** Choose \( \varepsilon > 0 \) such that \( (F_0 + \varepsilon)B \leq 1 \). There exists \( H_1 > 0 \) such that
\[ f(x) \leq (F_0 + \varepsilon)x \quad \text{for} \quad 0 < x \leq H_1. \]
For each \( u \in P \) with \( \| u \| = H_1 \), we have
\[
(Tu)(p) = \int_0^1 G(p, s)g(s)f(u(s)) \, ds \\
\leq (F_0 + \varepsilon) \int_0^1 G(p, s)g(s)u(s) \, ds \\
\leq (F_0 + \varepsilon)\| u \| \int_0^1 G(p, s)g(s)ds \\
= (F_0 + \varepsilon)\| u \| B \\
\leq \| u \|,
\]
which means \( \| Tu \| \leq \| u \| \). Hence, if we let
\[ \Omega_1 = \{ u \in X : \| u \| < H_1 \}, \]
then
\[ \| Tu \| \leq \| u \| \quad \text{for} \quad u \in P \cap \partial \Omega_1. \]

Now choose \( c \in (0, 1/4) \) and \( \delta > 0 \) such that
\[ \left( f_\infty - \delta \right) \int_c^{1-c} G(p, s)g(s)a(s) \, ds > 1. \]
There exists \( H_3 > 0 \) such that
\[ f(x) \geq (f_\infty - \delta)x \quad \text{for} \quad x \geq H_3. \]
Let 
\[ H_2 = \max \left\{ \frac{H_3}{c}, 2H_1 \right\}. \]

If \( u \in P \) with \( \|u\| = H_2 \), then for \( c \leq t \leq 1 - c \), we have
\[ u(t) \geq \min \{t, 1-t\} \|u\| \geq cH_2 \geq H_3. \]

Thus, for \( u \in P \) with \( \|u\| = H_2 \), we have
\[ (Tu)(p) = \int_c^{1-c} G(p, s)g(s)f(u(s))ds \geq (f_\infty - \delta) \int_c^{1-c} G(p, s)g(s)u(s)ds \geq (f_\infty - \delta)\|u\| \int_c^{1-c} G(p, s)g(s)a(s)ds \geq \|u\|, \]
which means that \( ||Tu|| \geq \|u\| \). If we let
\[ \Omega_2 = \{ u \in X : \|u\| < H_2 \}, \]
then \( \overline{\Omega_1} \subset \Omega_2 \) and 
\[ ||Tu|| \geq \|u\| \text{ for } u \in P \cap \partial \Omega_2. \]

Therefore, condition (K1) of Theorem 1.1 is satisfied, and so there exists a fixed point of \( T \) in \( P \). This in turn implies that there is a positive solution to the boundary value problem (1)-(2) and completes the proof of the theorem.

The following theorem is a companion result to Theorem 3.1.

**Theorem 3.2.** If \( BF_\infty < 1 < A_0 \), then the boundary value problem (1)-(2) has at least one positive solution.

**Proof.** Choose \( \varepsilon > 0 \) such that \( (f_0 - \varepsilon)A \geq 1 \). There exists \( H_1 > 0 \) such that 
\[ f(x) \geq (f_0 - \varepsilon)x \text{ for } 0 < x \leq H_1. \]

For \( u \in P \) with \( \|u\| = H_1 \), we have
\[ (Tu)(p) = \int_0^1 G(p, s)g(s)f(u(s))ds \geq (f_0 - \varepsilon) \int_0^1 G(p, s)g(s)u(s)ds \geq (f_0 - \varepsilon)\|u\| \int_0^1 G(p, s)g(s)a(s)ds = A(f_0 - \varepsilon)\|u\| \geq \|u\|, \]
so \( ||Tu|| \geq \|u\| \). If we let
\[ \Omega_1 = \{ u \in X : \|u\| < H_1 \}, \]
then
\[ ||Tu|| \geq \|u\| \text{ for } u \in P \cap \partial \Omega_1. \]

To construct \( \Omega_2 \), we choose \( \delta \in (0, 1) \) such that 
\[ ((F_\infty + \delta)B + \delta) < 1. \]
There exists an \( H_3 > 0 \) such that
\[
    f(x) \leq (F_\infty + \delta)x \quad \text{for} \quad x \geq H_3.
\]
Let \( M = \max_{0 \leq x \leq H_3} f(x) \). Then,
\[
    f(x) \leq M + (F_\infty + \delta)x \quad \text{for} \quad x \geq 0.
\]
Now set
\[
    K = M \int_0^1 G(p, s)g(s) \, ds
\]
and let
\[
    H_2 = \max\{2H_1, K(1 - (F_\infty + \delta)B)^{-1}\}. \tag{5}
\]
Note that (5) implies that \( K + (F_\infty + \delta)B H_2 \leq H_2 \). For each \( u \in P \) with \( \|u\| = H_2 \), we have
\[
    (Tu)(p) = \int_0^1 G(p, s)g(s)f(u(s)) \, ds
    \leq \int_0^1 G(p, s)g(s)(M + (F_\infty + \delta)u(s)) \, ds
    \leq K + (F_\infty + \delta) \int_0^1 G(p, s)g(s)u(s) \, ds
    \leq K + (F_\infty + \delta)H_2 \int_0^1 G(p, s)g(s) \, ds
    = K + (F_\infty + \delta)H_2 B
    \leq H_2,
\]
which means \( \|Tu\| \leq \|u\| \). Thus, if we let
\[
    \Omega_2 = \{ u \in X : \|u\| < H_2 \},
\]
then \( \overline{\Omega_1} \subset \Omega_2 \) and
\[
    \|Tu\| \leq \|u\| \quad \text{for} \quad u \in P \cap \partial \Omega_2.
\]
We can then conclude from part (K2) of Theorem 1.1 that the problem (1)-(2) has at least one positive solution.

4. Nonexistence results. In this section, we give some results that ensure that the problem (1)-(2) has no positive solutions.

**Theorem 4.1.** If \( B f(x) < x \) for all \( x \in (0, +\infty) \), then the boundary value problem (1)-(2) has no positive solutions.

**Proof.** Assume to the contrary that \( x(t) \) is a positive solution of the problem (1)-(2). Then, \( x \in P, \, x(p) > 0 \), and
\[
    x(p) = \int_0^1 G(p, s)g(s)f(x(s)) \, ds
    < B^{-1} \int_0^1 G(p, s)g(s)x(s) \, ds
    \leq B^{-1} x(p) \int_0^1 G(p, s)g(s) \, ds
    \leq x(p),
\]
which is a contradiction. \( \square \)
The proof of the following theorem is quite similar to that of Theorem 4.1, and in the interest of space, the details will be omitted.

**Theorem 4.2.** If \( Af(x) > x \) for all \( x \in (0, +\infty) \), then the problem (1)–(2) has no positive solutions.

5. **Examples.** We conclude this paper with some examples that illustrate the above theorems.

**Example 1.** Consider the third order boundary value problem
\[
\begin{align*}
  u'''(t) &= g(t)f(u(t)), \quad 0 < t < 1, \\
  u(0) &= u'(2/3) = u'(3/4) - u'(1) = 0,
\end{align*}
\]
where
\[
  g(t) = (1 + t)/10, \quad 0 \leq t \leq 1,
\]
\[
  f(u) = \lambda u(1 + 3u)/(1 + u), \quad u \geq 0,
\]
and \( \lambda > 0 \) is a parameter. It is easy to see that \( F_0 = f_0 = \lambda \) and \( F_\infty = f_\infty = 3\lambda \). Calculations using Maple or other appropriate software show that
\[
  A = \frac{363853}{24883200} \quad \text{and} \quad B = \frac{203}{12960}.
\]
From Theorem 3.1, we see that if
\[
  22.7961 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 63.8423,
\]
then problem (6)–(7) has at least one positive solution. From Theorems 4.1 and 4.2, we see that if
\[
  \lambda < \frac{1}{3B} \approx 21.2807 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 68.3881,
\]
then the problem (6)–(7) has no positive solutions.

**Example 2.** Consider the third order boundary value problem
\[
\begin{align*}
  u'''(t) &= g(t)f(u(t)), \quad 0 < t < 1, \\
  u(0) &= u'(2/3) = \int_{3/4}^{1} tu''(t) \, dt = 0,
\end{align*}
\]
where
\[
  g(t) = (1 + t)/10, \quad 0 \leq t \leq 1,
\]
\[
  f(u) = \lambda u(1 + 3u)/(1 + u), \quad u \geq 0,
\]
and \( \lambda > 0 \) is a parameter. Once again, \( F_0 = f_0 = \lambda \) and \( F_\infty = f_\infty = 3\lambda \). Calculations show that
\[
  A = \frac{2585581}{174182400} \quad \text{and} \quad B = \frac{2887}{181440}.
\]
From Theorem 3.1, we have that if
\[
  22.4557 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 62.8472,
\]
then the problem (8)–(9) has at least one positive solution. From Theorems 4.1 and 4.2, we see that if
\[
  \lambda < \frac{1}{3B} \approx 20.9490 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 67.3669,
\]
then the problem (8)–(9) has no positive solutions.
These examples show that our results are quite sharp.

In conclusion, we wish to point out that we have not required that \( f_0 = F_0 = 0 \) and \( f_\infty = F_\infty = +\infty \) (the superlinear case), or that \( f_0 = F_0 = +\infty \) and \( f_\infty = F_\infty = 0 \) (the sublinear case), or that \( f(u)/u \) even has a limit at 0 or \( \infty \). Notice that if \( f \) is superlinear, then Theorem 3.1 applies, while if \( f \) is sublinear, then Theorem 3.2 should be used. Also, we do not require that \( g(t) \) not vanish identically on any subinterval of \([0, 1]\) as is often done.

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Received September 2006; 1st revision March 2007; 2nd revision August 2007.

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