S₃-permuted Frobenius Algebras

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Abstract. In the present paper by Frobenius algebra \( Y \) we mean a finite dimensional algebra \( Y \) possessing an \( Y \)-associative and invertible (nondegenerate) form (a scalar product \( \cup \)), referred to as the Frobenius \( Y \)-structure. The nondegenerate form \( \cup \) has an inverse which we will denote as \( \cap \). We drop the extra conditions of associativity and unitality of \( Y \). The Frobenius algebra \( \{Y, \cup, \cap\} \) determine a ternary \((3 \mapsto 0)\)-tensor,

\[
Y \circ \cup = \cup \circ Y \quad \text{or} \quad Y_{ij}^e \cup_{ek} = \cup_{ie} Y_{jk}^e.
\]

Frobenius algebra is formulated within the monoidal abelian category of operad of graphs \( \text{cat}(m, n) \).

Frobenius algebra allows \( S_2 \)-permuted opposite algebra to be extended to \( S_3 \)-permuted algebras. If \( \sigma \in S_3 \) denotes a permutation we can get a \( \sigma \)-permuted algebra given by

\[
\{Y, \cup, \cap\} \xrightarrow{\sigma} \cap \circ \sigma(Y \circ \cup).
\]

Operad of graphs, i.e. diagrammatic language, is used both to illustrate the construction as well as a method of proof for the main Theorem. We give two detailed examples of this construction for Clifford algebras. Our construction, however, applies to all Frobenius algebras.

2000 Mathematics Subject Classification. Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Frobenius Algebra, Nonassociative Algebra, Clifford Algebra, \( S_3 \)-permutation.

Submitted August 10, 2010. Published in Proceedings of the 4th International Conference on Mathematical Sciences for Advancement of Science and Technology, Kolkata (Calcutta), India, December 2010. Institute for Mathematics, Bio-informatics, Information-technology and Computer-science, IMBIC, www.imbic.org/index.html.

This work is supported by Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica, UNAM, Grant PAPIIT # IN104908, 2008–2010.
1. Semifields and projective planes, by Knuth in 1965

Knuth [Knuth 1965] realized that the multiplication constants $Y^k_{ij}$ of an $n$-dimensional $k$-algebra $Y \in \text{cat}(2,1)$ determine a $n \times n \times n$ cube $A$, the cube associated with the $k$-algebra $Y$, and examines the algebras that arise when the axes of the cube are permuted. Let $\sigma \in S_3$ is a permutation. Then $A^\sigma$ will represent the 3-cube $A$ with subscripts permuted by $\sigma$.

Knuth [Knuth 1965] showed that if $\pi$ is a finite projective plane coordinatized by the division ring $S$ and $A$ is the cube associated with $S$, then the six permutations of the indices of $A$ determine a series of at most six planes.

The algebra corresponding to $A^{(12)}$ is called the opposite algebra and is denoted by $Y^{\text{op}}$. Pairing each algebra with its opposite we have

\[
\begin{array}{c|c}
Y \simeq A & Y^{\text{op}} \simeq A^{(23)} \\
A^{(123)} & A^{(23)} = A^{(123)(12)} \\
A^{(132)} & A^{(13)} = A^{(132)(12)}
\end{array}
\]

We note that every algebra $Y \in \text{cat}(2,1)$ is a mixed tensor, 2-covariant and 1-contra-variant, and every $S_3$-permutation of $Y$ involve implicitly the existence of the non-degenerate scalar product. Moreover in order to assure that $S_3$-permutation leads to the unique permuted new algebra $\sigma Y \in \text{cat}(2,1)$, this implicit scalar product must be necessarily $Y$-associative. This is equivalent to say that an algebra $Y \in \text{cat}(2,1)$ must be a Frobenius algebra. In what follows we will assume that an algebra $Y \in \text{cat}(2,1)$ is a Frobenius algebra.

Frobenius algebra is usually defined to be an associative and unital algebra $Y$ possessing a (left or right) $Y$-module isomorphism, or,
equivalently, possessing a $Y$-associative invertible scalar product, denoted here by $\cup \in \text{cat}(2,0)$, called the Frobenius structure or the Frobenius pairing, see e.g. [Eilenberg and Nakayama 1955; Caenepeel, Militaru, Zhu 2002, Definition 3 on page 32; Kock 2003, pages 95–97, Definition 2.2.5].

Let $Y \in \text{cat}(2,1)$ denote a finite dimension $k$-algebra, not necessarily associative, equipped with an $Y$-associative nondegenerate form $\cup \in \text{cat}(2,0)$. The ternary tensor (the ternary scalar product), $Y \circ \cup = \cup \circ Y \in \text{cat}(3,0)$, is defined in terms of the multiplication tensor $Y^k_{ij}$ and the form $\cup \in \text{cat}(2,0)$,

$$ (Y \circ \cup)_{ijk} = Y^e_{ij} \cup e_k. \quad (1.1) $$

Since the form $\cup$ is nondegenerate, it has an inverse, $\cap \in \text{cat}(0,2)$, and

$$ \cup \circ \cap = \cap \circ \cup \in \text{cat}(1,1), \quad (1.2) $$

$$ Y^f_{ij} \cup_{fk} \cap^{ke} = (Y \circ \cup)_{ijk} \cap^{ke} \quad (1.3) $$

$$ Y^e_{ij} = (Y \circ \cup)_{ijk} \cap^{ke}. \quad (1.4) $$

We can permute the ternary scalar tensor to get

$$ (\sigma Y)^k_{ij} = (Y \circ \cup)_{\sigma(ije)} \cap^{ek} \quad (1.5) $$

The multiplication tensor $(\sigma Y)^k_{ij}$ will determine a new $k$-algebra, that needs not to be necessarily a Frobenius algebra.

### 2. Some Nonassociative Algebra Preliminaries

We work within monoidal abelian category generated by a single object.

Since the algebras we will be working with are not necessarily associative, we will use the associator of three elements $a, b, c$ of the algebra $Y$,

$$ (a, b, c) = (ab)c - a(bc) \in \text{cat}(3,1). \quad (2.1) $$

The above associator needs that a monoidal category must be abelian.

The left nucleus of $Y$ is the set

$$ N_l = \{ l \in Y : (l, b, c) = 0 \text{ for all } b, c \in Y \}. \quad (2.2) $$

The middle nucleus of $Y$ is the set

$$ N_m = \{ m \in Y : (a, m, c) = 0 \text{ for all } a, c \in Y \}. \quad (2.3) $$

The right nucleus of $Y$ is the set

$$ N_r = \{ r \in Y : (a, b, r) = 0 \text{ for all } a, b \in Y \}. \quad (2.4) $$
The nucleus is the set \( N = N_l \cap N_m \cap N_r \).

We will denote the associator in the \( \sigma \)-permuted Frobenius algebra \( \sigma Y, \sigma \neq \text{id} \), by \((a, b, c)^{\circ}\).

An algebra \( Y \in \text{cat}(2, 1) \) is said to be flexible if for all \( a, b \in Y \),

\[
(a, b, a) = 0.
\]

The flexible property seems to be a minimum requirement in the existing studies of nonassociative real division algebras [Althoen and Kugler 1983, Benkart, Britten and Osborn 1982, Darpo 2006] and Malcev-admissible algebra [Myung 1986]. The associative, alternative and Jordan algebras all enjoy the flexible property. An even weaker association property is the association of cubes for all \( a \in Y \),

\[
(a, a, a) = 0.
\]

Clearly, the flexible property implies the association of cubes. The association of cubes appears to be a ”minimal” regularity condition that permits the analysis of an algebra. In his article Osborn [Osborn 1972] begins the study of identities on noncommutative algebras insisting that the algebras satisfies the association of cubes. We will see that the nonassociative Frobenius algebras do not satisfy this very weak associativity condition.

The concept of isotopism was introduced by Albert to provide a broader classification of the many nonassociative algebras than that of isomorphism [Tomber 1979]. Algebras \( Y \) and \( Y' \) with products \( * \) and \( \circ \) are isotopic if and only if there exists nonsingular linear transformations \( F, G, H \) from \( Y \) to \( Y' \) such that

\[
(a \circ b)H = aF * bG.
\]

The relation ”isotopic to” is an equivalence relation.

Clearly, an isotope of a division algebra is a division algebra. An algebra is said to be isotopically simple if every isotope of it is simple. We note that any isotope of a simple Clifford algebra will again be simple.

It is well known [Knuth 1965] that we can make a division algebra without a unit element into a division algebra with unit element \( e \) under a new product

\[
(a \circ e) * (e \circ b) = a \circ b.
\]

The classical introduction to the theory of nonassociative algebras is the book by Schafer [Schafer 1966]. The article [Tomber 1979] gives a history of nonassociative algebras.
3. Main Results

We work within a bi-closed monoidal category (another name a tensor category) generated by a single object. In this case the set of all objects coincides with the set of non-negative integers (with the set of natural numbers) \( \mathbb{N} \). Thus the set of all objects is, \( \text{obj} \text{cat} = \mathbb{N} \), and \( 1 \in \mathbb{N} \) is a generating object. This leads to graphical language where each morphism (an arrow of a category) is characterized by a pair of non-negative integers, morphisms are bi-graded, and we refer to this pair \{input, output\} = \{entrance, exit\}, as to the type or arity of the morphism = \{arity-in, arity-out\},

\[
\begin{align*}
\mathbb{N} &\ni m \quad \text{morphisms of (m \to n)-arity} \quad n \in \mathbb{N}.
\end{align*}
\]

**Definition 3.1** (Frobenius algebra). A not necessarily associative and not necessarily unital algebra \( Y \in \text{cat}(2, 1) \) is said to be a Frobenius algebra if exists a morphism \( \cup \in \text{cat}(2, 0) \) such that the following two conditions hold

\[
\begin{align*}
Y \circ \cup = \cup \circ Y &\in \text{cat}(3, 0) \\
\cup \circ \cap = \cap \circ \cup = | = \text{id} &\in \text{cat}(1, 1).
\end{align*}
\]

Within graphical or graphics language, the first of the above Frobenius condition (a solvable Frobenius algebra if \( \cup \neq 0 \)) is precisely the relation among two morphisms from \( \text{cat}(3, 0) \) as follows

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{align*}
\]

**Theorem 3.2.** A necessary condition that the \( k \)-algebra \( Y \) has non-degenerate \( Y \)-associative form \( \cup \) is the trace of the matrix for right multiplication by every element \( x \) of \( Y \) is the same as the trace of the matrix for left multiplication by the element \( x \). This must hold for the traces of every power of the regular, left- and right-, representations.

**Proof.** One can compose (3.2) on both sides with inverse of the Frobenius structure, \( \cap \equiv \cup^{-1} \),

\[
\begin{align*}
Y \circ \cup \circ \cap = Y \circ | = Y = \cup \circ Y \circ \cap, \\
\cap \circ \cup \circ Y = | \circ Y = Y = \cap \circ Y \circ \cup, \\
\cap \circ Y \circ \cup = Y = \cup \circ Y \circ \cap.
\end{align*}
\]
The last conditions \((3.6)\) within operad of graphs is precisely the following graphical relations

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{graph1.png}} \\
\sim \\
\text{\includegraphics[width=0.2\textwidth]{graph2.png}} \\
\sim \\
\text{\includegraphics[width=0.2\textwidth]{graph3.png}}
\end{array}
\end{equation}

(3.7)

Every algebra \(Y\) can be seen as a right \(Y\)-module and as a left \(Y\)-module, in two ways, on both dual objects. An associative algebra \(Y\) is a two-sided \(Y\)-module, known as \(Y\)-bimodule, and this \(Y\)-bimodule structure is again doubled on both dual objects.

An element of an algebra \(Y\) can be seen as a morphism \(\in \text{cat}(0, 1)\). The left- and the right- composition of \(Y \in \text{cat}(2, 1)\) with \(\text{cat}(0, 1)\) gives the two regular representations,

\begin{align}
(3.8) \quad & \text{cat}(0, 1) \circ \text{cat}(2, 1) \subset \text{cat}(1, 1) \cong \text{End}(1, 1), \\
(3.9) \quad & \text{cat}(2, 1) \circ \text{cat}(0, 1) \subset \text{cat}(1, 1) \cong \text{End}(1, 1).
\end{align}

(3.8, 3.9)

Composing \(\text{cat}(0, 1)\) with \((3.6)-(3.7)\) we arrive to the following relations among endomorphisms within \(\text{cat}(1, 1)\),

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{graph4.png}} \\
\circ \\
\text{\includegraphics[width=0.2\textwidth]{graph5.png}}
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{graph6.png}}
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{graph7.png}}
\end{array}
\end{equation}

(3.8, 3.9)
Our next aim is to calculate the traces in a ring $k \equiv \text{cat}(0, 0)$, as shown in the following equation:

\[(3.10) \quad \text{trace}(\ldots) \equiv \text{evaluation} \circ (\ldots) \circ \text{co-evaluation}.\]

Two circles in \((3.12)\) are $U$-free and they are given in terms of evaluation and co-evaluation of the dual objects, i.e. in terms of the bi-closed structure [Kelly and Laplaza 1980].

The relation \((3.12)\) is desired equality of traces of the regular representations, and this is $U$-free, i.e. it is the necessary condition for an algebra $Y$ to be the Frobenius algebra.

We left to the reader the graphical proof that not only traces of the regular representations of the Frobenius algebra must be equal, but as well as the traces of all powers of regular representations must be the same. This proves the theorem about the necessary condition for an algebra $Y \in \text{cat}(2, 1)$ to be the Frobenius algebra. \hfill \square

**Corollary 3.3.** The relations \((3.10)\) leads also to the following consequence, that is useless for our searching of the necessary condition for an algebra $Y \in \text{cat}(2, 1)$ to be a Frobenius algebra, in terms of $Y$ alone. The following relation within $\text{cat}(0, 0)$ involve the Frobenius structure besides of an algebra

\[(3.13)\]
Note that the circles in (3.12) do not have the same meaning as in (3.13). The two circles in (3.13) are \((\cap, \cup)\)-dependent, i.e. the animals in (3.13) involve the both Frobenius structures, \(\cup\) and \(\cap\).

**Corollary 3.4.** Another extra bonus corollary that follows from (3.10) is the following relation

\[
\begin{array}{c}
\bigcirc \\
\sim \\
\bigcirc
\end{array}
\] (3.14)

**Example 3.5** (Kock 2003). The algebra of real upper triangular matrices does not admit a nondegenerate associative form [Kock 2003]. Letting \(e_{11}, e_{11},\) and \(e_{11}\) be an ordered basis for this algebra, we see that the matrix for right multiplication by \(e_{11}\) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3.15)

and the matrix for left multiplication by \(e_{11}\) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (3.16)

The traces are different.

**4. Other results**

**Lemma 4.1.** Let \(Y\) denote a finite dimensional Frobenius \(k\)-algebra with the Frobenius structure \(\cup\) that is diagonal matrix in some basis. The algebra \(Y^{(12)}\) is the algebra \(Y^{op}\).

**Proof.**

\[
(Y \circ \cup)_{ijk} = Y^{k}_{ij} \cup_{kk}.
\] (4.1)

\[
(\sigma Y)^{k}_{ij} = Y^{k}_{ji} \cap_{kk} = Y^{k}_{ji} \cup_{kk} \cap_{kk}.
\]

\[
(\sigma Y)^{k}_{ij} = Y^{k}_{ji}.
\]

\[
\square
\]

**Lemma 4.2.** The algebra \(Y^{(23)}\) will always have a left identity and will have an identity if and only if \(\cup\) is the identity matrix.
PROOF. The computation for the left identity goes as follows:

\[(Y \circ \cup)_{ii} = Y^i_i \cup_{ii} = \cup_{ii}.\]  \hspace{1cm} (4.2)

\[(\sigma Y)^i_{ii} = (Y \circ \cup)_{ii} \cap_{ii} = \cup_{ii} \cap_{ii} = 1\]  \hspace{1cm} (4.3)

since \(\cup\) and \(\cap\) are inverses and \(\cup\) is a diagonal matrix.

The algebra \(Y^{(23)}\) will have a right identity if and only if

\[(\sigma Y)^i_{ii} = (Y \circ \cup)_{ii} \cap_{ii} = 1\]  \hspace{1cm} (4.4)

But \((Y \circ \cup)_{ii} = Y^{11}_{ii} \cup_{ii} = \cup_{ii}\) and then

\[(\sigma Y)^i_{ii} = \cup_{ii} \cap_{ii} = 1.\]  \hspace{1cm} (4.5)

This last forces \(\cup_{ii} = 1\) for all \(i\). \qed

**Theorem 4.3.** Cubes do not associate in the algebra \(Y^{(23)}\) if there is some \(\cup_{xx} = -1\).

PROOF. We compute the associator \((b_x, b_x, b_x)^\circ\).

\[b_x \circ b_x = (\sigma Y)^k_{xx} = (Y \circ \cup)_{xk} \cap_{xx}.\]  \hspace{1cm} (4.6)

Since \((Y \circ \cup)_{xk} = Y^k_{xx} \cup_{xx}\) and \(Y^k_{xx}\) is nonzero if and only if \(k = 1\) in which case it has the value 1, \(b_x \circ b_x = -E\).

\[\circ_{xk} = (b_x \circ b_x) \circ b_x - b_x \circ (b_x \circ b_x)\]
\[= -E \circ b_x + b_x \circ E\]  \hspace{1cm} (4.7)

By the above Lemma we know that \(-E \circ b_x = -b_x\)
\[(\sigma Y)^k_{x1} = (Y \circ \cup)_{xk} \cap_{x1}\] where \((Y \circ \cup)_{xk} = Y^1_{xk} \cup_{x1}\) as before \(Y^1_{xk}\) is nonzero if and only if \(k = 1\) in which case it has the value \(-1\), and \(b_x \circ E = -bx\). \qed

The algebra \(Y^{(13)}\) will always have a right identity element; we omit the corresponding computations for this algebra.
5. The Complex Numbers

The complex numbers are the Clifford algebra $Cl_{0,1}$ with multiplication table given by

\[
\begin{array}{c|cc}
   & E & I \\
\hline
E & E & I \\
I & I & -E
\end{array}
\]

(5.1)

\[
Y^1_{11} = Y^2_{12} = Y^2_{21} = 1, \quad Y^1_{22} = -1
\]

(5.2)

This algebra possesses a family of Frobenius structures generated by two forms

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\quad \text{and} \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

(5.3)

The one nondegenerate $\mathbb{C}$-associative form $\cup$ is

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(5.4)

We want to compute the algebra $Cl_{0,1}^{(23)}$

The ternary tensor is $(Y \circ \cup)_{ijk} = Y^e_{ij} \cup^e_{ek} = Y^k_{ij} \cup_{kk}$.

\[
(Y \circ \cup)_{111} = Y^1_{11} \cup_{11} = 1
\]

\[
(Y \circ \cup)_{122} = Y^2_{12} \cup_{22} = -1
\]

\[
(Y \circ \cup)_{212} = Y^2_{21} \cup_{22} = -1
\]

\[
(Y \circ \cup)_{221} = Y^1_{22} \cup_{11} = -1.
\]

The new multiplication constants will be $(\sigma Y)_{ij}^k = (Y \circ \cup)_{ikj} \cap^{ij}$.

\[
(\sigma Y)_{11}^1 = (Y \circ \cup)_{111} \cap^{11} = 1 \times 1 = 1
\]

\[
(\sigma Y)_{12}^2 = (Y \circ \cup)_{122} \cap^{22} = -1 \times -1 = 1
\]

\[
(\sigma Y)_{21}^2 = (Y \circ \cup)_{221} \cap^{11} = -1 \times 1 = -1
\]

\[
(\sigma Y)_{22}^1 = (Y \circ \cup)_{212} \cap^{22} = -1 \times -1 = 1
\]

The multiplication in the new algebra will be denoted by $\circ$. The multiplication table will be given by

\[
\begin{array}{c|cc}
   & E & I \\
\hline
E & E & I \\
I & -I & E
\end{array}
\]

(5.5)

This algebra has a left identity, $E$; furthermore,

\[
(\alpha E + \cup I) \circ (\alpha E + \cup I) = (\alpha^2 + \cup^2) E.
\]

(5.6)

The left nucleus is the field generated by the element $E$;

\[
N_m = N_r = (0)
\]

(5.7)
This algebra does not support a nondegenerate associative form $B$ because the matrix for right multiplication by $E$ is \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\] and the matrix for left multiplication by $E$ is \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix};
\] the traces are not the same.

We make this algebra into the complex numbers under the product (5.8) \[(a \circ e) \star (e \circ b) = a \circ b.\]

Any division algebra 2-dimensional over the field $\mathbb{R}$ will be the complex numbers [Dickson 1906].

### 6. The Quaternion Division Algebra

The multiplication tensor of the real quaternion division algebra $\text{Cl}_{0,2}$ is given in

\[
\begin{array}{c|c|c|c|c}
E & I & J & K \\
\hline
E & E & I & J & K \\
I & I & -E & K & -J \\
J & J & -K & -E & I \\
K & K & J & -I & -E \\
\end{array}
\]

(6.1)

The nondegenerate associative form $B$ is given by

(6.2) \[
\cup_{ij} = \begin{cases} 
  b_i^2 & \text{if } i = j \\
  0 & \text{otherwise}
\end{cases}
\]

The five new algebras that arise from our construction are given below. Each can be shown to be an isotope of the quaternion algebra or its opposite under the product (6.3) \[(a \circ e) \star (e \circ b) = a \circ b.\]

The relations among the various algebras follow those given in the table in the Introduction. Only the quaternion algebra and its opposite are associate. The four nonassociative algebras satisfy

(6.4) \[(a_0E + \alpha_1I + \alpha_2J + \alpha_3K)^2 = (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) E\]

for all $\alpha_0E + \alpha_1I + \alpha_2J + \alpha_3K$ in the algebra. The nonassociative algebras possess only a one sided identity. There is no in-depth study of the nonflexible division algebras in the literature.
Lemma 4.1 applied to the element $E$ in each of the nonassociative algebras implies that none has admits an $\mathbb{Q}$-associative, nondegenerate form.

7. Conclusion

Theorem 3.2 gives a necessary condition for the existence of a non-degenerate $Y$-associative form $\in \text{cat}(2,0)$ in terms of the $Y \in \text{cat}(2,1)$ alone. Can we find a sufficient condition using only the algebra $Y$?

The examples beg the question: "Are all 4-dimensional real division algebras in which each element squares to a scalar multiple of the one sided identity an isotope of the quaternion division algebra?"

The concepts developed in the paper can be applied to any finite dimensional Frobenius algebra $\{Y, \cup, \cap\}$.

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