A one-shot quantum joint typicality lemma

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Abstract

A fundamental tool to prove inner bounds in classical network information theory is the so-called ‘conditional joint typicality lemma’. In addition to the lemma, one often uses unions and intersections of typical sets in the inner bound arguments without so much as giving them a second thought. These arguments fail spectacularly in the quantum setting. This bottleneck shows up in the fact that so-called ‘simultaneous decoders’, as opposed to ‘successive cancellation decoders’, are known for very few channels in quantum network information theory. Another manifestation of this bottleneck is the lack of so-called ‘simultaneous smoothing’ theorems for quantum states.

In this paper, we overcome the bottleneck by proving for the first time a one-shot quantum joint typicality lemma with robust union and intersection properties. To do so, we develop two novel tools in quantum information theory which may be of independent interest. The first tool is a simple geometric idea called tilting, which increases the angles between a family of subspaces in orthogonal directions. The second tool, called smoothing, is a way of perturbing a multipartite quantum state such that the partial trace over any subset of registers does not increase the operator norm by much.

Our joint typicality lemma allows us to construct simultaneous quantum decoders for many multiterminal quantum channels. It provides a powerful tool to extend many results in classical network information theory to the one-shot quantum setting.

1 Introduction

A fundamental tool to prove inner bounds for communication channels in classical network information theory is the so-called conditional joint typicality lemma [EK12]. Very often, the joint typicality lemma is used together with implicit intersection and union arguments in the inner bound proofs. This is especially so in the construction of so-called simultaneous decoders (as opposed to successive cancellation decoders) for communication problems. In this paper, we investigate what happens when one tries to extend the classical inner bound proofs to the quantum setting. It turns out that the union and intersection arguments present a huge stumbling block in this effort. The main result of this paper is a one-shot quantum joint typicality lemma that takes care of these union and intersection bottlenecks, and allows us to extend many classical inner bound proofs to the quantum setting.

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1.1 One-shot inner bound for the classical MAC

Let us illustrate the need for an intersection argument together with a joint typicality lemma by
considering the problem of proving inner bounds for arguably the simplest multiterminal commu-
nication channel viz. the multiple access channel (MAC). We consider the one-shot classical setting.
There are two senders Alice and Bob who would like to send messages $m_1 \in [2^{R_1}]$, $m_2 \in [2^{R_2}]$ to a
receiver Charlie. There is a communication channel $\mathcal{C}$ with two inputs and one output called the
two sender one receiver MAC connecting Alice and Bob to Charlie. The two input alphabets of
$\mathcal{C}$ will be denoted by $\mathcal{X}$, $\mathcal{Y}$ and the output alphabet by $\mathcal{Z}$. Let $0 \leq \epsilon \leq 1$. On getting message
$m_1$, Alice encodes it as a letter $x(m_1) \in \mathcal{X}$ and feeds it to her channel input. Similarly on getting
message $m_2$, Bob encodes it as a letter $y(m_2) \in \mathcal{Y}$ and feeds it to his channel input. The channel $\mathcal{C}$
output a letter $z \in \mathcal{Z}$ according to the channel probability distribution $p(z|x(m_1), y(m_2))$. Charlie
now has to try and guess the message pair $(m_1,m_2)$ from the channel output. We require that
the probability of Charlie’s decoding error averaged over the channel behaviour as well as over the
uniform distribution on the set of message pairs $(m_1,m_2) \in [2^{R_1}] \times [2^{R_2}]$ is at most $\epsilon$.

Consider the following randomised construction of a codebook $\mathcal{C}$ for Alice and Bob. Fix prob-
ability distributions $p(x)$, $p(y)$ on sets $\mathcal{X}$, $\mathcal{Y}$. For $m_1 \in [2^{R_1}]$, choose $x(m_1) \in \mathcal{X}$ independently
according to $p(x)$. Similarly for $m_2 \in [2^{R_2}]$, choose $y(m_2) \in \mathcal{Y}$ independently according to $p(y)$.

We now describe the decoding strategy that Charlie follows in order to try and guess the message
pair $(m_1,m_2)$ that was actually sent. Let $0 \leq \epsilon \leq 1$. Let $p$, $q$ be two probability distributions on
the same sample space $\Omega$. A ‘classical POVM element’ or ‘test’ on $\Omega$ is defined to be a function
$f : \Omega \to [0,1]$. Intuitively, for a sample point $\omega \in \Omega$, $f(\omega)$ denotes its probability of acceptance
by the test. For two classical POVM elements $f$, $g$ on $\Omega$, we can define the ‘intersection’ classical
POVM element $f \cap g$ as follows: $(f \cap g)(\omega) := \min\{f(\omega), g(\omega)\}$. Similarly, we can define the ‘union’
classical POVM element $f \cup g$ as follows: $(f \cup g)(\omega) := \max\{f(\omega), g(\omega)\}$. Following Wang and
Renner [WR12], we define the classical \textit{hypothesis testing relative entropy} $D^r_H(p\|q)$ as follows:
\[ D^r_H(p\|q) := \max_{f : \sum f(\omega) p(\omega) \geq 1 - \epsilon} - \log \sum f(\omega) q(\omega), \]
where the maximisation is over all classical POVM elements on $\Omega$ ‘accepting’ the distribution $p$
with probability at least $1 - \epsilon$. It is easy to see that the optimising POVM element $f$ attains
equality in the constraint for $p(\cdot)$, as well as achieves the maximum in objective function for $q(\cdot)$.

Define the probability distribution $p(\cdot|x)$ on $\mathcal{Z}$ as $p(z|x) := \sum_y p(y) p(z|x,y)$ for all $z \in \mathcal{Z}$.
With a slight abuse of notation, we shall often use $p(z|x)$ to also denote the distribution $p(\cdot|x)$ on
$\mathcal{Z}$. Similarly, define probability distributions $p(\cdot|y)$, $p(\cdot)$ on $\mathcal{Z}$ in the natural fashion. Define the
probability distributions $p^{XYZ}(x,y,z) := p(x)p(y)p(z|x,y)$, $(p^{XZ} \times p^Y)(x,y,z) := p(x)p(y)p(z|x)$,
$(p^{YZ} \times p^X)(x,y,z) := p(x)p(y)p(z|y)$, $(p^X \times p^Y \times p^Z)(x,y,z) := p(x)p(y)p(z)$ on
$\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ in the natural manner. Consider classical POVM elements $f^Y$, $f^X$, $f^{X,Y}$ achieving the maximum in the
definitions of $D^r_H(p^{XYZ}||p^{XZ} \times p^Y)$, $D^r_H(p^{XYZ}||p^{YZ} \times p^X)$, $D^r_H(p^{XYZ}||p^X \times p^Y \times p^Z)$ respectively. As
a shorthand, we will use the hypothesis testing mutual information quantities $I^r_H(Y : XZ)$, $I^r_H(X : YZ)$,
$I^r_H(XY : Z)$ to denote the hypothesis testing relative entropy quantities $D^r_H(p^{XYZ}||p^{XZ} \times p^Y)$,
$D^r_H(p^{XYZ}||p^{YZ} \times p^X)$, $D^r_H(p^{XYZ}||p^X \times p^Y \times p^Z)$ respectively. Now consider the intersection classical
POVM element $f := f^X \cap f^Y \cap f^{X,Y}$. Charlie’s decoding strategy is as follows. Suppose the channel
output is $z$. Then, Charlie uses the following randomised algorithm for decoding.

For $\hat{m}_1 = 1$ to $2^{R_1}$
For $\hat{m}_2 = 1$ to $2^{R_2}$
Toss a coin with probability of HEAD being \( f(x(\hat{m}_1), y(\hat{m}_2), z) \).
If the coin comes up HEAD, declare \((\hat{m}_1, \hat{m}_2)\) as Charlie’s guess and halt.
If the coin comes up TAILS, go to next iteration.

Declare FAIL, if Charlie did not declare any guess above.

We now analyse the expectation, under the choice of a random codebook \( \mathcal{C} \), of the error probability of Charlie’s decoding algorithm. Suppose the message pair \((m_1, m_2)\) is inputted to the channel. Let \( < \) denote the lexicographic order on message pairs. Let the channel output be denoted by \( z \). Then, a decoding error occurs only if Charlie tosses a HEAD for a pair \((\hat{m}_1, \hat{m}_2) < (m_1, m_2)\) or if Charlie tosses a TAIL for \((m_1, m_2)\). The probability of this occurring, for a given codebook \( \mathcal{C} \), is upper bounded by

\[
p_e(\mathcal{C}; m_1, m_2) \\ \leq \sum_z p(z|x(m_1), y(m_2)) \sum_{(\hat{m}_1, \hat{m}_2): (\hat{m}_1, \hat{m}_2) < (m_1, m_2)} f(x(\hat{m}_1), y(\hat{m}_2), z) \\ + \sum_z p(z|x(m_1), y(m_2))(1 - f(x(m_1), y(m_2), z)) \\ \leq \sum_z p(z|x(m_1), y(m_2)) \sum_{(\hat{m}_1, \hat{m}_2): (\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)} f(x(\hat{m}_1), y(\hat{m}_2), z) \\ + \sum_z p(z|x(m_1), y(m_2))(1 - f(x(m_1), y(m_2), z)) \\ = \sum_z p(z|x(m_1), y(m_2)) \sum_{(\hat{m}_1, \hat{m}_2): \hat{m}_1 \neq m_1, \hat{m}_2 \neq m_2} f(x(\hat{m}_1), y(\hat{m}_2), z) \\ + \sum_z p(z|x(m_1), y(m_2)) \sum_{\hat{m}_2 \neq m_2} f(x(m_1), y(\hat{m}_2), z) \\ + \sum_z p(z|x(m_1), y(m_2)) \sum_{\hat{m}_1 \neq m_1} f(x(\hat{m}_1), y(m_2), z) \\ + \sum_z p(z|x(m_1), y(m_2))(1 - f(x(m_1), y(m_2), z)).
\]

The expectation, over the choice of the random codebook \( \mathcal{C} \), of the decoding error is then upper bounded by

\[
\mathbb{E}_\mathcal{C}[p_e(\mathcal{C}; m_1, m_2)] \\ \leq (2^{R_1} - 1)(2^{R_2} - 1) \sum_{x,y,z} p(x)p(y)p(z|x, y) \sum_{x',y'} p(x')p(y')f(x', y', z) \\ + (2^{R_2} - 1) \sum_{x,y,z} p(x)p(y)p(z|x, y) \sum_{y'} p(y')f(x, y', z) \\ + (2^{R_1} - 1) \sum_{x,y,z} p(x)p(y)p(z|x, y) \sum_{x'} p(x')f(x', y, z) \\ + \sum_{x,y,z} p(x)p(y)p(z|x, y)(1 - f(x, y, z)) \\ \leq (2^{R_1} - 1)(2^{R_2} - 1) \sum_{x,y,z} p(x)p(y)p(z|x, y) \sum_{x',y'} p(x')p(y')f^{X,Y}(x', y', z)
\]
Thus, for any rate pair in the region given by
\[ C \]
Then the expectation, under the choice of a random codebook \( C \), is upper bounded by
\[ \sum_{x,y,z} p(x)p(y)p(z|x,y) \sum_{y'} p(y') f_Y (x', y, z) \]
\[ + \sum_{x,y,z} p(x)p(y)p(z|x,y) \sum_{x'} p(x') f_X (x', y, z) \]
\[ + \sum_{x,y,z} p(x)p(y)p(z|x,y) ((1 - f_X (x, y, z)) + (1 - f_Y (x, y, z)) + (1 - f_X Y (x, y, z))) \]
\[ = (2^{R_1} - 1)(2^{R_2} - 1) \sum_{x',y',z} p(x')p(y')p(z) f_{X,Y} (x', y', z) \]
\[ + (2^{R_2} - 1) \sum_{x',y,z} p(y')p(x)p(z|x) f_Y (x', y, z) \]
\[ + (2^{R_1} - 1) \sum_{x',y,z} p(x')p(y)p(z|y) f_X (x', y, z) \]
\[ + \sum_{x,y,z} p(x)p(y)p(z|x,y) ((1 - f_X (x, y, z)) + (1 - f_Y (x, y, z)) + (1 - f_X Y (x, y, z))) \]
\[ \leq 2^{R_1 + R_2} 2^{-I_H (XY:Z)} + 2^{R_2} 2^{-I_H (Y:Z)} + 2^{R_1} 2^{-I_H (X:Z)} + 3\epsilon. \]

Above, we use the properties that
\[ f(x, y, z) \leq \{ f_X (x, y, z), f_Y (x, y, z), f_{X,Y} (x, y, z) \}, \]
\[ 1 - f(x, y, z) \leq (1 - f_X (x, y, z)) + (1 - f_Y (x, y, z)) + (1 - f_{X,Y} (x, y, z)) \]
for all triples \((x, y, z)\). Now define the average decoding error probability under a codebook \( C \) to be
\[ p_e(C) := 2^{- (R_1 + R_2)} \sum_{(m_1, m_2)} p_e(C; m_1, m_2). \]

Then the expectation, under the choice of a random codebook \( C \), of the average decoding error probability is upper bounded by
\[ \mathbb{E}_C [p_e(C)] \leq 2^{R_1 + R_2} 2^{-I_H (XY:Z)} + 2^{R_2} 2^{-I_H (Y:Z)} + 2^{R_1} 2^{-I_H (X:Z)} + 3\epsilon. \]

Thus, for any rate pair in the region given by
\[ R_1 \leq I_H^e (X : Y Z) - \log \frac{1}{\epsilon}, \]
\[ R_2 \leq I_H^e (Y : X Z) - \log \frac{1}{\epsilon}, \]
\[ R_1 + R_2 \leq I_H^e (XY : Z) - \log \frac{1}{\epsilon}, \]
there is a codebook \( C \) with average decoding error probability less than \( 6\epsilon \).

### 1.2 Unions and intersections in the classical setting

We now step back and discuss the intersection classical POVM element \( f := f_X \cap f_Y \cap f_{X,Y} \) used in the above proof. The intersection argument was crucial in constructing a simultaneous
decoder for Charlie. In the asymptotic iid setting for the multiple access channel, it is possible to avoid simultaneous decoding and instead use successive cancellation decoding combined with time sharing \cite{EK12}. However, in the one-shot setting time sharing does not make sense and successive cancellation gives only a finite set of achievable rate pairs. Thus, in order to get a continuous achievable rate region, we are forced to use simultaneous decoders only. There are also situations even in the asymptotic iid setting, e.g. in Marton’s inner bound with common message for the broadcast channel, where we need to use intersection arguments \cite{EK12}. Similarly, union arguments crop up in some inner bound proofs, e.g. Marton’s inner bound without common message for the broadcast channel, even in the asymptotic iid setting \cite{EK12}. In the one-shot setting union bounds occur more frequently, e.g. in the Han-Kobayashi inner bound for the interference channel \cite{Sen18}. Thus, intersection and union arguments are indispensable in network information theory.

Before we proceed to the quantum setting, we prove for completeness sake a ‘one-shot classical joint typicality lemma’. In fact, it is nothing but an application of intersection and union of classical POVM elements.

\textbf{Fact 1} (Classical joint typicality lemma). Let \(p_1, \ldots, p_t, q_1, \ldots, q_t \) be probability distributions on a set \(X\). Let \(0 \leq \{\epsilon_{ij}\}_{ij} \leq 1\), where \(i \in [t], j \in [t]\). Then there is a classical POVM element \(f\) on \(X\) such that:

1. For all \(i \in [t]\), \(\sum_x p_i(x)f(x) \geq 1 - \left\{ \sum_{j=1}^t \epsilon_{ij} \right\}\); 

2. For all \(j \in [t]\), \(\sum_x q_j(x)f(x) \leq \sum_{i=1}^t 2^{-D_H^{ij}(p_i\|q_j)}\).

\textit{Proof.} For \(i \in [t], j \in [t]\), let \(f_{ij}\) be the classical POVM element achieving the minimum in the definition of \(D_H^{ij}(p_i\|q_j)\). Define the classical POVM element \(f := \bigcup_{i=1}^t \cap_{j=1}^t f_{ij}\). Observe that for any \(x \in X\),

\[
1 - f(x) \leq \min_{i \in [t]} \left\{ \sum_{j=1}^t (1 - f_{ij}(x)) \right\}, \quad f(x) \leq \sum_{i=1}^t \min_{j \in [t]} \{ f_{ij}(x) \}.
\]

It is now easy to see that \(f\) satisfies the properties claimed above. \(\square\)

### 1.3 Extending unions and intersections to the quantum setting

We now ponder what is required to extend the above inner bound proof for the classical MAC to the setting of the one-shot classical-quantum multiple access channel (cq-MAC). In the cq-MAC, there are two senders Alice and Bob who would like to send messages \(m_1 \in [2^{R_1}], m_2 \in [2^{R_2}]\) to a receiver Charlie. There is a communication channel \(\mathcal{C}\) with two classical inputs and one quantum output connecting Alice and Bob to Charlie. The two input alphabets of \(\mathcal{C}\) will be denoted by \(X, Y\) and the output Hilbert space by \(Z\). If the pair \((x, y)\) is fed into the channel inputs, the output of the channel is a density matrix \(\rho_{x,y}\) in \(Z\). Let \(0 \leq \epsilon \leq 1\). On getting message \(m_1\), Alice encodes it as a letter \(x(m_1) \in X\) and feeds it to her channel input. Similarly on getting message \(m_2\), Bob encodes it as a letter \(y(m_2) \in Y\) and feeds it to his channel input. Charlie now has to try and guess the message pair \((m_1, m_2)\) from the channel output state \(\rho_{x(m_1),y(m_2)}\). We require that the probability of Charlie’s decoding error averaged over the uniform distribution on the set of message pairs \((m_1, m_2) \in [2^{R_1}] \times [2^{R_2}]\) is at most \(\epsilon\).

One can do a similar randomised construction of a codebook \(C\) for Alice and Bob as before. One can make use of Wang and Renner’s \cite{WR12} hypothesis testing relative entropy for a pair of
quantum states $\rho, \sigma$ in the same Hilbert space $\mathcal{H}$, which is defined as follows:

$$D'_H(\rho\|\sigma) := \max_{\Pi : \text{Tr} [\Pi \rho] \geq 1 - \epsilon} - \log \text{Tr} [\Pi \sigma],$$

where the maximisation is over all POVM elements $\Pi$ on $\mathcal{H}$ (i.e. positive semidefinite operators $\Pi, \Pi \leq \mathbb{1}_H$) ‘accepting’ the state $\rho$ with probability at least $1 - \epsilon$. Again, it is easy to see that the optimising POVM element $\Pi$ attains equality in the constraint for $\rho$, as well as achieves the maximum in objective function for $\sigma$. One can define the analogous quantum state

$$\rho^{XYZ} := \sum_{x,y} p(x)p(y) |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \rho^Z_{x,y}$$

and the tensor products of the marginals $\rho^{YZ} \otimes \rho^X, \rho^{YZ} \otimes \rho^X, \rho^X \otimes \rho^Y \otimes \rho^Z$, as well as the hypothesis testing mutual informations $I'_H(X : YZ), I'_H(Y : XZ), I'_H(XY : Z)$ in the quantum setting too. Thus, we would like to prove that any rate pair in the region given by

$$R_1 \leq I'_H(X : YZ) - \log \frac{1}{\epsilon},$$

$$R_2 \leq I'_H(Y : XZ) - \log \frac{1}{\epsilon},$$

$$R_1 + R_2 \leq I'_H(XY : Z) - \log \frac{1}{\epsilon}$$

is achievable with small average decoding error probability.

Suppose there exists a **single** POVM element $\Pi$ on the Hilbert space $\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}$ that simultaneously satisfies the following properties:

1. $\text{Tr} [\Pi \rho^{XYZ}] \geq 1 - 3\epsilon$;
2. $\text{Tr} [\Pi (\rho^{YZ} \otimes \rho^X)] \leq 2^{-I'_H(X : YZ)}$;
3. $\text{Tr} [\Pi (\rho^{XZ} \otimes \rho^Y)] \leq 2^{-I'_H(Y : XZ)}$;
4. $\text{Tr} [\Pi (\rho^X \otimes \rho^Y \otimes \rho^Z)] \leq 2^{-I'_H(XY : Z)}$.

In the classical setting, such a POVM element $\Pi$ was constructed by taking the intersections of the three POVM elements $f^X, f^Y, f^{XY}$ achieving the maximums in the definitions of the three entropic quantities $I'_H(X : YZ), I'_H(Y : XZ), I'_H(XY : Z)$. In the quantum setting, if such an ‘intersection’ POVM element $\Pi$ exists it is indeed possible to construct a decoding algorithm for Charlie with average error probability at most $O(\epsilon)$ using the ‘pretty good measurement’ $[\text{Bel75b}]$, the Hayashi-Nagaoka operator inequality $[\text{IN03}]$ and mimicking the classical analysis given above for the decoding error.

Let $\Pi^X, \Pi^Y, \Pi^{X,Y}$ be the three POVM elements achieving the maximums in the definitions of the three entropic quantities $I'_H(X : YZ), I'_H(Y : XZ), I'_H(XY : Z)$. Let us now look at various simple ideas to generalise intersection of POVM elements to the quantum setting. If the POVM elements were projectors, which can be ensured without loss of generality by embedding the quantum states into a larger Hilbert space $\mathcal{X} \otimes \mathcal{Y} \otimes (\mathcal{Z} \otimes \mathbb{C}^2)$, one can try taking the projector $\Pi$ onto the intersection of the supports of $\Pi^X, \Pi^Y, \Pi^{X,Y}$. This indeed ensures that Properties 2, 3, 4 described above hold for $\Pi$. Unfortunately, the intersection can easily be the zero subspace.
which kills all hope of satisfying Property 1 even approximately. The simplest idea to define the ‘intersection’ POVM element would be to take the product $\Pi := \Pi^X\Pi^Y\Pi^{X,Y}\Pi^Y\Pi^X$. With this definition, $\Pi$ can be shown to satisfy Property 1 with lower bound $1 - O(\epsilon)$ using the non-commutative union bound $[\text{Gao15}]$. However, it is not at all clear that the remaining three properties can be simultaneously satisfied, even approximately, because $\Pi^X$, $\Pi^Y$, $\Pi^{X,Y}$ do not commute in general. To get an idea of the difficulty, consider the expression

$$\text{Tr} [\Pi(\rho^{YZ}\otimes\rho^X)] = \text{Tr} [\Pi^X\Pi^Y\Pi^{X,Y}\Pi^Y\Pi^X(\rho^{YZ}\otimes\rho^X)]$$

that arises if one were to attempt to prove Property 2. It is true that $\text{Tr} [\Pi^Y(\rho^{YZ}\otimes\rho^X)] = 2^{-I(X;YZ)}$, but it is also possible that

$$\text{Tr} [\Pi^X\Pi^Y\Pi^{X,Y}\Pi^Y\Pi^X(\rho^{YZ}\otimes\rho^X)] - \text{Tr} [\Pi^Y(\rho^{YZ}\otimes\rho^X)] \gg 2^{-I(X;YZ)}.$$

This makes it impossible to show the achievability of the desired rate region (Equation 1) with this definition of $\Pi$. In fact, one of the main technical contributions of this paper is a robust novel notion of intersection of non-commuting POVM elements achieving the maximums in the definitions of the appropriate hypothesis testing mutual information quantities.

As a first step towards defining a robust notion of intersection of non-commuting POVM elements, we address the complementary problem of defining a robust notion of union of non-commuting POVM elements. The ‘intersection’ POVM element can then be defined to be simply the complement of the ‘union’ of the complements. Note that the complement of a POVM element $\Pi$ in a Hilbert space $\mathcal{H}$ is defined to be $1_\mathcal{H} - \Pi$. Suppose that the POVM elements were projectors, which can be ensured without loss of generality by embedding the states into a larger Hilbert space $\mathcal{H} \otimes \mathbb{C}^2$. One can then naively define the ‘union’ of a family of projectors to be the projector onto the span of the supports (the union of supports is not a vector space in general). However, this does not give us anything new as the span can indeed be the entire space and so Property 1 can fail spectacularly, that is, the lower bound in Property 1 can be as low as zero! The problem with the span idea is captured by the following example. Consider the two dimensional Hilbert space $\mathbb{C}^2$. Let $W_1$ be the one dimensional space spanned by $|0\rangle$ and $W_2$ the one dimensional space spanned by $\sqrt{1-\epsilon}|0\rangle + \sqrt{\epsilon}|1\rangle$. Consider the quantum state $\rho := |1\rangle\langle 1|$. Now the probability of $\rho$ being accepted by $W_1$, $W_2$ is 0 and $\epsilon$ respectively. However, the probability of $\rho$ being accepted by the span of $W_1$ and $W_2$ is one.

Notice however that the above pathological phenomenon with the span occurs only because the subspaces $W_1$ and $W_2$ have ‘small angles’ between them. We overcome this problem by ‘tilting’ $W_1$, $W_2$ in orthogonal directions to form new spaces $W'_1$, $W'_2$. The Hilbert space has to be enlarged sufficiently to allow this tilting to be possible. The process of tilting increases the ‘angles’ between the subspaces in orthogonal directions allowing one to recover an upper bound for the span very close to the sum of acceptance probabilities, just like in the classical setting. This ‘tilting’ idea is formalised in Proposition 2. Thus, the ‘tilted span’ is best thought of as a robust notion of union of subspaces satisfying a well behaved union bound.

Let us define the ‘intersection’ of subspaces to be the complement of the tilted span of the complementary subspaces. Applying this recipe to $\Pi^X$, $\Pi^Y$, $\Pi^{X,Y}$ gives us a projector $\Pi$ that satisfies Property 1 with lower bound $1 - O(\sqrt{\epsilon})$ by setting $\alpha = \sqrt{\epsilon}$ in the upper bound of Proposition 2. However, it is not clear whether $\Pi$ satisfies the Properties 2, 3, 4 even approximately. This is because in order to prove, say, Property 2, one has to use the lower bound of Proposition 2 with $\alpha = \sqrt{\epsilon}$, which would give an upper bound of at least $\sqrt{\epsilon}$ for Property 2!
Nevertheless, it turns out that the tilting idea can be used as a starting point and further refined, leading finally to a proof of the desired inner bound (Equation 1) for the cq-MAC. We do so with the following sequence of steps. A full proof is given in Section 6 below.

1. Enlarge the Hilbert space $\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}$ suitably and consider a ‘perturbed’ version $\mathcal{C}'$ of the channel $\mathcal{C}$. The channel $\mathcal{C}'$ maps an input pair $(x, y)$ to a state $\rho_{x,y}'$ close to $\rho_{x,y}$. This gives a state $(\rho')^{XYZ}$ close to $\rho^{XYZ}$. A codebook achieving a certain rate point for channel $\mathcal{C}'$ with a certain error achieves the same rate point for channel $\mathcal{C}$ with only slightly more error. In fact, $(\rho')^{XYZ}$ is obtained by tilting $\rho^{XYZ}$ along a carefully chosen direction;

2. The channel $\mathcal{C}'$ is constructed in such a way that $(\rho')^{XZ} \otimes (\rho')^Y$, $(\rho')^{YZ} \otimes (\rho')^X$, $(\rho')^X \otimes (\rho')^Y$ and $(\rho')^Z \otimes (\rho')^Z$ are extremely close to the states $\rho^{XZ} \otimes \rho^Y$, $\rho^{YZ} \otimes \rho^X$, $\rho^X \otimes \rho^Y \otimes \rho^Z$ tilted in approximately orthogonal directions. This is a crucial new idea that we call smoothing of $\rho^{XYZ}$. Smoothing is possible because, though partial trace can increase the $\ell_\infty$-norm of a quantum state in general, the increase is negative or only slightly positive if the state is highly entangled between the traced out part and the untraced out part. The carefully chosen way of tilting $\rho^{XYZ}$ to get $(\rho')^{XYZ}$ ensures that $(\rho')^{XYZ}$ is highly entangled across all bipartitions of the systems $XYZ$. Nevertheless, actually achieving this smoothing is technically intricate and requires the use of a completely mixed ancilla state of sufficiently large support. We use the term augmentation to refer to this technique of using a completely mixed ancilla state for the purpose of smoothing;

3. We then observe that the tilts of $\rho^{XZ} \otimes \rho^Y$, $\rho^{YZ} \otimes \rho^X$, $\rho^X \otimes \rho^Y \otimes \rho^Z$ alluded to in the previous point are more complicated than the simple tilting idea described earlier and analysed in Proposition 2. To analyse these more complicated tilts, we define a tilting matrix $A$ and then define an $A$-tilt. We now define the ‘intersection’ POVM element $\Pi$ to be the projector onto the complement of the $A$-tilted span of the complements of the supports of $\Pi^X$, $\Pi^Y$, $\Pi^{X,Y}$. We then show in Proposition 3 that an $A$-tilt of subspaces also gives rise to a union bound which, even though not as good as the upper bound of Proposition 2, is still strong enough for the purpose of proving Property 1 with lower bound $1 - O(\epsilon)$ under the projector $\Pi$;

4. Finally, we notice from the construction of $\Pi$ that Properties 2, 3, 4 are easily satisfied by $\Pi$ for the $A$-tilted versions of $\rho^{XZ} \otimes \rho^Y$, $\rho^{YZ} \otimes \rho^X$, $\rho^X \otimes \rho^Y \otimes \rho^Z$, with upper bounds exactly the same as in the ideal case. Since the states $(\rho')^{XZ} \otimes (\rho')^Y$, $(\rho')^{YZ} \otimes (\rho')^X$, $(\rho')^X \otimes (\rho')^Y \otimes (\rho')^Z$ are extremely close to the $A$-tilted versions of $\rho^{XZ} \otimes \rho^Y$, $\rho^{YZ} \otimes \rho^X$, $\rho^X \otimes \rho^Y \otimes \rho^Z$, we finally conclude that they satisfy Properties 2, 3, 4 with upper bounds almost as good as in the ideal case under $\Pi$;

5. Thus, $\Pi$ can be used by Charlie in order to construct a decoding algorithm for the original channel $\mathcal{C}$ that achieves any rate pair in the region described in Equation 1 with average error probability at most $O(\epsilon)$.

The strategy outlined above enables us to prove the following one-shot quantum joint typicality lemma.

Let $A_1, \ldots, A_k$ be a $k$-partite quantum system with each $A_i$ isomorphic to a Hilbert space $\mathcal{H}$. Let $\rho^{A_1, \ldots, A_k}$ be a quantum state in $A_1 \ldots A_k$. For a subset $S \subseteq [k]$, let $A_S$ denote the systems $\{A_s : s \in S\}$. Let $\rho^{A_S}$ denote the marginal state on $A_S$ obtained by tracing
out the systems in $S := [k] \setminus S$ from $\rho^{A_1 \ldots A_k}$. Let $0 < \epsilon < 1$. Let $\mathcal{K}$ be a Hilbert space of dimension $\frac{2^{13(k+1)}(2|\mathcal{H}|)^6(k+1)}{(1-\epsilon)^6(k+1)}$. There exist a state $\tau^{\mathcal{K} \otimes [k]}$ independent of $\rho^{\mathcal{A}[k]}$, a state $(\rho')^{\mathcal{A}[k]}$, and a POVM element $\Pi^{\mathcal{A}[k]}$ on $A_1' \ldots A_k'$ where $A_i' \cong A_i \otimes \mathcal{K}$, with the following properties:

1. $\left\| (\rho')^{\mathcal{A}[k]} - \rho^{\mathcal{A}[k]} \otimes \tau^{\mathcal{K} \otimes [k]} \right\|_1 \leq 2^{\frac{k}{2} + 1} \epsilon \frac{k}{4}$;
2. $\text{Tr} \left[ (\Pi')^{\mathcal{A}[k]} (\rho')^{\mathcal{A}[k]} \right] \geq 1 - 2^{8(k+1)} \epsilon^{\frac{1}{4}} - 2^{\frac{k}{2} + 1} \epsilon \frac{k}{4}$;
3. For every set $S$, $\{\} \neq S \subset [k]$,
   \[
   \text{Tr} \left[ (\Pi')^{\mathcal{A}[k]} ((\rho')^{\mathcal{A}[k]} \otimes (\rho')^{\mathcal{A}[k]'}) \right] \leq 2^{-D_H(\rho^{\mathcal{A}[k]} \otimes \rho^{\mathcal{A}[k]'})}.
   \]

1.4 Related work

The bottleneck of simultaneous decoding was first pointed out by Fawzi et al. [FHS+12] in their paper on the quantum interference channel. Subsequently, Dutil [Dut11] pointed out a related bottleneck of ‘simultaneous smoothing’, which was further discussed by Drescher and Fawzi [DF13]. Simultaneous decoders have been recently used in several papers e.g. [QWW17], but the inner bounds obtained there were suboptimal compared to known inner bounds when restricted to the asymptotic iid setting.

1.5 Organisation of the paper

In the next section, we state some preliminary facts which will be useful throughout the paper. Section 3 defines and proves the union properties of the tilted span and $A$-tilted span of subspaces. In Section 4 we prove the so-called one-shot quantum joint typicality lemma, which manages to construct a robust notion of intersection of POVM elements achieving a given set of hypothesis testing mutual information quantities. The meat of the proof of the one-shot quantum joint typicality lemma is encapsulated into a proposition which we prove in Section 5. Next, we formally prove the achievability of the rate region described by Equation 1 in Section 6. Finally, we make some concluding remarks and list some open problems in Section 7.

2 Preliminaries

All Hilbert spaces in this paper are finite dimensional. The symbol $\oplus$ always denotes the orthogonal direct sum of Hilbert spaces. For a subspace $X$ of a Hilbert space $\mathcal{H}$, let $\Pi^X_\mathcal{H}$ denote the orthogonal projection in $\mathcal{H}$ onto $X$. When clear from the context, we may use $\Pi_X$ instead of $\Pi^X_\mathcal{H}$ for brevity of notation.

By a quantum state or a density matrix in a Hilbert space $\mathcal{H}$, we mean a Hermitian, positive semidefinite linear operator on $\mathcal{H}$ with trace equal to one. By a POVM element $\Pi$ in $\mathcal{H}$, we mean a Hermitian positive semidefinite linear operator on $\mathcal{H}$ with eigenvalues between 0 and 1. Stated in terms of inequalities on Hermitian operators, $0 \leq \Pi \leq 1$, where $0$, $1$ denote the zero and identity operators on $\mathcal{H}$. In what follows, we shall use several times the Gelfand-Naimark theorem which is stated below for completeness.
Fact 2 (Gelfand-Naimark). Let $\Pi$ be a POVM element in a Hilbert space $\mathcal{H}$. For any integer $d \geq 2$, there exists an orthogonal projection $\Pi'$ in $\mathcal{H} \otimes \mathbb{C}^d$ such that

$$\text{Tr} [\Pi A] = \text{Tr} [\Pi' (A \otimes |0\rangle \langle 0|^d)]$$

for all linear operators $A$ acting on $\mathcal{H}$. Above, $|0\rangle$ is a fixed vector, independent of $\Pi$ and $A$, in $\mathbb{C}^d$.

Since quantum probability is a generalisation of classical probability, one can talk of a so-called ‘classical POVM element’. Suppose we have a probability distribution $p(x)$, $x \in \mathcal{X}$. A classical POVM element on $\mathcal{X}$ is a function $f : \mathcal{X} \to [0, 1]$. The probability of accepting the POVM element $f$ is then $\sum_{x \in \mathcal{X}} p(x) f(x)$. One can continue to use the operator formalism for classical probability with the understanding that density matrices and POVM elements are now diagonal matrices.

Let $\|v\|_2$ denote the $\ell_2$-norm of a vector $v \in \mathcal{H}$. For an operator $A$ on $\mathcal{H}$, we use $\|A\|_1$ to denote the Schatten $\ell_1$-norm, aka trace norm, of $A$, which is nothing but the sum of singular values of $A$. We use $\|A\|_\infty$ to denote the Schatten $\ell_\infty$-norm, aka operator norm, of $A$, which is nothing but the largest singular value of $A$. For operators $A, B$ on $\mathcal{H}$, we have the inequality

$$|\text{Tr} [AB]| \leq \|AB\|_1 \leq \min\{\|A\|_1 \|B\|_\infty, \|A\|_\infty \|B\|_1\}.$$
Definition 1. Let $\alpha$, $\beta$ be two quantum states in the same Hilbert space. Let $0 \leq \epsilon < 1$. Then the hypothesis testing relative entropy of $\alpha$ with respect to $\beta$ is defined by

$$D_H^\epsilon(\alpha\|\beta) := \max_{\Pi : \text{Tr} [\Pi \alpha] \geq 1 - \epsilon} - \log \text{Tr} [\Pi \beta],$$

where the maximisation is over all POVM elements $\Pi$ acting on the Hilbert space.

The definition quantifies the minimum probability of ‘accepting’ $\beta$ by a POVM element $\Pi$ that ‘accepts’ $\alpha$ with probability at least $1 - \epsilon$. From the definition, it is easy to see that if $\epsilon < \epsilon'$, $D_H^\epsilon(\alpha\|\beta) < D_H^{\epsilon'}(\alpha\|\beta)$. We will require the following property of the so-called hypothesis testing mutual information.

Proposition 1. Let $0 \leq \epsilon < 1$. Let $\rho^{AB}$ be a quantum state in a bipartite system $AB$. Define the hypothesis testing mutual information $I_H^\epsilon(A : B)_\rho := D_H^\epsilon(\rho^{AB}\|\rho^A \otimes \rho^B)$. Then,

$$I_H^\epsilon(A : B)_\rho \leq 2 \log \min\{|A|, |B|\} + 3 \log \frac{1}{1 - \epsilon} + 6 \log 3 - 4.$$

Proof. Let $C$ be a third system and $\rho^{ABC}$ a purification of $\rho^{AB}$. Let $\Pi^{AB}$ be the optimising POVM element for $D_H^\epsilon(\rho^{AB}\|\rho^A \otimes \rho^B)$. Define $\Pi^{ABC} := \Pi^{AB} \otimes 1^C$. Let $|A| \leq |B|$. We will show that

$$\text{Tr} [\Pi^{ABC}(\rho^A \otimes \rho^{BC})] \geq \frac{2^4(1 - \epsilon)^3}{3^6 |A|^2}$$

which would prove the proposition.

Let $|\rho\rangle^{ABC}$ denote the pure state $\rho^{ABC}$ in ket vector form. Let

$$|\rho\rangle^{ABC} = \sum_{a=1}^{|A|} \sqrt{p_a} |x_a\rangle^A \otimes |y_a\rangle^{BC}$$

be a Schmidt decomposition for the bipartition $(A, BC)$, where $\sum_{a=1}^{|A|} p_a = 1$. Fix $0 < \delta < 1$. Define

$$\hat{A} := \left\{ a \in [|A|] : p_a \geq \delta(1 - \epsilon) \right\}.$$

Using the triangle and Cauchy-Schwarz inequalities, we get

$$\sqrt{1 - \epsilon} \leq \left\| \Pi^{ABC} |\rho\rangle^{ABC} \right\|_2$$

$$\leq \left\| \sum_{a \in \hat{A}} \sqrt{p_a} |x_a\rangle^A \otimes |y_a\rangle^{BC} \right\|_2 + \left\| \sum_{a \not\in \hat{A}} \sqrt{p_a} |x_a\rangle^A \otimes |y_a\rangle^{BC} \right\|_2$$

$$\leq \sum_{a \in \hat{A}} \sqrt{p_a} \left\| \sum_{a \not\in \hat{A}} \sqrt{p_a} |x_a\rangle^A \otimes |y_a\rangle^{BC} \right\|_2$$

$$= \sum_{a \in \hat{A}} \sqrt{p_a} \left\| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^{BC}) \right\|_2 + \sum_{a \not\in \hat{A}} p_a$$

$$= \sum_{a \in \hat{A}} \sqrt{p_a} \left\| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^{BC}) \right\|_2 + \sum_{a \not\in \hat{A}} p_a$$
\[
\sum_{a \in \hat{A}} \sqrt{p_a} \| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^BC) \|_2 + \sqrt{\delta(1-\epsilon)},
\]
which implies that
\[
\sum_{a \in \hat{A}} \sqrt{p_a} \| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^BC) \|_2 > \sqrt{1-\epsilon(1-\sqrt{\delta})}.
\]
Again using the Cauchy-Schwarz inequality, we get
\[
\sqrt{1-\epsilon(1-\sqrt{\delta})} < \sqrt{\sum_{a \in \hat{A}} p_a \| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^BC) \|_2^2}
\]
which implies that
\[
\sum_{a \in \hat{A}} \| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^BC) \|_2^2 > \sqrt{1-\epsilon(1-\sqrt{\delta})}.
\]
Now,
\[
\text{Tr} \left[ \Pi^{ABC} (\rho^A \otimes \rho^BC) \right] = \text{Tr} \left[ \Pi^{ABC} \left( \sum_{a=1}^{|A|} p_a |x_a\rangle \langle x_a| \right)^A \otimes \left( \sum_{a'=1}^{|A|} p_{a'} |y_{a'}\rangle \langle y_{a'}| \right)^{BC} \right]
\geq \sum_{a \in \hat{A}} p_a^2 \text{Tr} \left[ \Pi^{ABC} (|x_a\rangle \langle x_a|^{A} \otimes |y_a\rangle \langle y_a|^{BC}) \right]
= \sum_{a \in \hat{A}} p_a^2 \| \Pi^{ABC} (|x_a\rangle^A \otimes |y_a\rangle^BC) \|_2^2
\geq \frac{\delta^2 (1-\sqrt{\delta})^2 (1-\epsilon)^3}{|A|^2}.
\]
Taking \(\delta = 4/9\) gives the largest lower bound above, and completes the proof of the proposition. \(\square\)

We will also need the non-commutative union bound of Gao \cite{Gao15} (see also \cite{OMW18} for a recent improvement).

**Fact 3** (Noncommutative union bound). Let \(\rho\) be a positive semidefinite operator on a Hilbert space \(\mathcal{H}\) and \(\text{Tr} \rho \leq 1\). Let \(\Pi'_1, \ldots, \Pi'_k\) be orthogonal projectors in \(\mathcal{H}\). Define \(\Pi_i := \mathbb{I} - \Pi'_i\). Then,
\[
\text{Tr} \left[ \Pi'_k \cdots \Pi'_1 \rho \Pi'_1 \cdots \Pi'_k \right] \geq \text{Tr} \left[ \rho \Pi_i \right] - 4 \sum_{i=1}^k \text{Tr} \left[ \rho \Pi_i \right].
\]

### 3 Tilted span of subspaces

Let \(\mathcal{H}\) be a Hilbert space. Consider Hilbert spaces \(\mathcal{H}_j, j \in [l]\), each of dimension \(\dim \mathcal{H}\), orthogonal to each other and also to \(\mathcal{H}\). Define \(\mathcal{H} := \mathcal{H} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_l\). Let \(\mathcal{T}_j, j \in [l]\) be linear maps mapping \(\mathcal{H}\) isometrically onto \(\mathcal{H}_j\). Let \(0 \leq \alpha \leq 1\). For \(j \in [l]\), define linear maps \(\mathcal{T}_{j,\alpha} : \mathcal{H} \to \mathcal{H}\) as
\[
\mathcal{T}_{j,\alpha} := \sqrt{1-\alpha} \mathbb{I}_\mathcal{H} + \sqrt{\alpha} \mathcal{T}_j,
\]
where $\mathbb{1}_H$ is the identity embedding of $H$ into $\tilde{H}$. Observe that each $T_{j,\alpha}$ is an isometric embedding of $H$ into $\tilde{H}$.

We now define the tilted span of a collection of subspaces, formalising the intuitive description in the introduction.

**Definition 2** (Tilted span). Let $W_1, \ldots, W_l$ be subspaces of $H$. The subspace $T_{j,\alpha}(W_j) \leq H$ will be called the $\alpha$-tilt of $W_j$ along the $j$th direction. The subspace $W_\alpha$ of $\tilde{H}$ defined by

$$W_\alpha := \frac{1}{l} \sum_{j=1}^{l} T_{j,\alpha}(W_j).$$

is called the $\alpha$-tilted span of $W_1, \ldots, W_l$.

The effect of tilting the subspaces $W_1, \ldots, W_l$ along $l$ orthogonal directions is to increase the separation between them, in a sense made precise by the following proposition.

**Proposition 2.** Let $|h\rangle \in H$ be a unit vector. For $j \in [l]$, define $\varepsilon_j := \|\Pi_W |h\rangle\|_2^2$. Then,

$$(1 - \alpha) \max_{j: j \in [l]} \varepsilon_j \leq \|\Pi_{W_\alpha} |h\rangle\|_2^2 \leq \frac{1 - \alpha}{\alpha} \sum_{j=1}^{l} \varepsilon_j.$$

**Proof.** The lower bound follows trivially since the projection of $|h\rangle$ onto the $j$th summand space in the definition of $W_\alpha$ is of length $\sqrt{\varepsilon_j(1 - \alpha)}$.

We now prove the upper bound. Define $h' := \Pi_{W_\alpha} |h\rangle$. Since $h' \in W_\alpha$, let $h' = \sum_{j=1}^{l} \lambda_j |x_j\rangle$ where $\lambda_j \in \mathbb{C}$, and $|x_j\rangle$ is a unit vector in $T_{j,\alpha}(W_j)$ for $j \in [l]$. Let $|x_j\rangle = T_{j,\alpha}(|y_j\rangle)$, where $|y_j\rangle$ is a unit vector in $W_j$ and is uniquely determined by $|x_j\rangle$. Then,

$$\|h'\|_2^2 = \left\| \sum_{j=1}^{l} \sqrt{1 - \alpha} \lambda_j |y_j\rangle + \sum_{j=1}^{l} \sqrt{\alpha} \lambda_j (T_{j,\alpha}(|y_j\rangle)) \right\|_2^2 \geq \left\| \sum_{j=1}^{l} \sqrt{\alpha} \lambda_j (T_{j,\alpha}(|y_j\rangle)) \right\|_2^2 = \alpha \sum_{j=1}^{l} |\lambda_j|^2.$$

We also have

$$\|h'\|_2^2 = |\langle h | h' \rangle| = \left| \sum_{j=1}^{l} \lambda_j \langle h | x_j \rangle \right| \leq \sqrt{\sum_{j=1}^{l} |\lambda_j|^2} \cdot \sqrt{\sum_{j=1}^{l} |\langle h | x_j \rangle|^2}.$$

This implies that

$$\alpha \sqrt{\sum_{j=1}^{l} |\lambda_j|^2} \leq \sqrt{\sum_{j=1}^{l} |\langle h | x_j \rangle|^2} = \sqrt{1 - \alpha} \sqrt{\sum_{j=1}^{l} |\langle h | y_j \rangle|^2} = \sqrt{1 - \alpha} \sqrt{\sum_{j=1}^{l} |\Pi_{W_j} |h\rangle|^2} \leq \sqrt{1 - \alpha} \sqrt{\sum_{j=1}^{l} \|\Pi_{W_j} |h\rangle\|_2^2} = \sqrt{1 - \alpha} \sqrt{\sum_{j=1}^{l} \varepsilon_j}.$$
Thus,
\[
\|h'\|_2^2 \leq \sqrt{\sum_{j=1}^{l} |\lambda_j|^2} \cdot \sqrt{\sum_{j=1}^{l} |\langle h|x_j\rangle|^2} \leq \frac{1}{\alpha} \sum_{j=1}^{l} |\langle h|x_j\rangle|^2 \leq \frac{1 - \alpha}{\alpha} \sum_{j=1}^{l} \epsilon_j.
\]
This proves the desired upper bound on \(\|h'\|_2^2\) and completes the proof of the proposition. 

An easy corollary of the above proposition is a union bound for projectors that beats the union bound proved in Anshu, Jain and Warsi’s paper [AJW17a, Lemma 3] on the entanglement assisted compound quantum channel.

**Corollary 1.** Let \(\epsilon, \alpha > 0\). For \(i \in [l]\), let \(\rho_i\) be a quantum state and \(\Pi_i\) be an orthogonal projector in \(\mathcal{H}\) such that \(\text{Tr} [\Pi_i \rho_i] \geq 1 - \epsilon\). Let \(\mathcal{H}\) be defined as in Definition 2. Then there is a projector \(\hat{\Pi} \in \mathcal{H}\) such that \(\text{Tr} [\hat{\Pi} \rho_i] \geq 1 - \epsilon - \alpha\) for all \(i \in [l]\), and, for all states \(\sigma \in \mathcal{H}\),
\[
\text{Tr} [\hat{\Pi} \sigma] \leq \frac{1 - \alpha}{\alpha} \sum_{i \in [l]} \text{Tr} [\Pi_i \sigma].
\]

Suppose \(\text{Tr} [\hat{\Pi}_i \sigma] \leq \theta\) for all \(i \in [l]\). Then the upper bound promised by the above corollary is \(\frac{l \theta}{\alpha}\), whereas the upper bound given by Lemma 3 of [AJW17a] is only \(\theta(\frac{2\alpha}{l\epsilon})\log(2^\alpha)\). Thus, the above corollary leads to corresponding improvements in the achievable rates for the various settings considered in [AJW17a].

It is interesting to consider the analogue of Proposition 2 in classical probability. One can think of the subspaces \(W_j, j \in [l]\) as a set of \(l\) events, the vector \(|h\rangle\) as the classical state of a system, and \(\epsilon_j\) to be the probability of the \(j\)th event occurring. Then the probability of at least one of the events occurring is lower bounded by \(\max_{j: j \in [l]} \epsilon_j\) and upper bounded by \(\sum_{j=1}^{l} \epsilon_j\). This is nothing but the fundamental union bound of classical probability. Thus, the above proposition almost recovers the classical performance of the union bound in the context of quantum probability.

However the above proposition still falls short of defining an intersection projector satisfying Properties 1, 2, 3, 4 for the cq-MAC. For \(j \in [l]\), let \(\Pi_j\) be an orthogonal projection. We would like to define a non-trivial intersection of the projectors \(\Pi_j, j \in [l]\). Define a new projector \(\hat{\Pi}\) as the projection onto the orthogonal complement of the tilted span \(W_\alpha\) of \(W_j, j \in [l]\), where \(W_j\) is taken to be the support of \(\Pi_j\). Then it is easy to see that
\[
\text{Tr} [\hat{\Pi} \rho] \geq 1 - \frac{1 - \alpha}{\alpha} \sum_{j=1}^{l} \epsilon_j.
\]
This is tolerable for most applications of the joint typicality lemma. In fact, it allows us to prove a good enough version of Property 1 for the cq-MAC taking \(\rho = \rho^{XYZ}\), \(l = 3\), \(\Pi_1 := \Pi^X\), \(\Pi_2 := \Pi^Y\), \(\Pi_3 := \Pi^{X,Y}\).

Now consider states \(\sigma_i, i \in [l]\). Let \(2^{-k_i} := \text{Tr} [\Pi_i \sigma_i]\). For the cq-MAC, we have \(l = 3\), \(\sigma_1 := \rho^{YZ} \otimes \rho^X\), \(\sigma_2 := \rho^{XZ} \otimes \rho^Y\), \(\sigma_3 := \rho^X \otimes \rho^Y \otimes \rho^Z\). For the upper bound, the best that one can prove is
\[
\text{Tr} [\hat{\Pi} \sigma_i] \leq \alpha (1 - 2^{-k_i}) + 2^{-k_i}.
\]
The additive term of nearly \(\alpha\) makes the lower bound insufficient for applications of the joint typicality lemma. In particular, it kills any hope of proving even approximate versions of Properties 2, 3, 4 for the cq-MAC.
However, for a $k$-partite state $\rho^{A_1\ldots A_k}$, and a collection of $k$-partite states $\sigma_i := \rho^{A_{i1}\ldots A_{ik}}$, $\{\} \neq S_i \subset [k], i \in [l]$, it turns out that we can do better and indeed come up with a notion of intersection projector that is strong enough to prove a quantum joint typicality lemma. For this, as discussed in the introduction, we have to do a smoothing of $\rho$ to $\rho'$. The smoothing is achieved by a carefully chosen tilt of $\rho$. This makes $\sigma'_i := (\rho')^{A_{i1}\ldots A_{ik}}$ extremely close to a certain tilted version of $\sigma_i$. However, it turns out that the tilt of $\sigma_i$ is not only along the so-called $(S_i, \bar{S}_i)$th direction but also along the directions corresponding to subpartitions refining $(S_i, \bar{S}_i)$. Thus, by taking a linear extension of the partial order of subpartitions of $[k]$ under refinement, we are led to consider a tilting scheme where $\sigma_i$ is tilted along the directions 1, $\ldots$, $i$. This tilting scheme is described by an upper triangular tilting matrix $A$ whose $(ij)$th entry denotes the tilt along the $i$th direction for $\sigma_j$ when $i \leq j$. Hence, we have to formally define the $A$-tilted span of $W_1, \ldots, W_l$, and prove a union bound for it.

Let $0 \leq \alpha_{ij} \leq 1$, where $i, j \in [l]$. Define the $l \times l$ matrix $A := (\alpha_{ij})$. Suppose $A$ is upper triangular and diagonal dominated, that is, $\alpha_{ij} = 0$ for $i > j$ and $\alpha_{ii} \geq \alpha_{ij}$ for all $i \leq j$. Furthermore, suppose $A$ is substochastic, that is, for all $j$, $\sum_{i=1}^j \alpha_{ij} \leq 1$. If the last inequality holds with equality, we say that $A$ is stochastic. For $j \in [l]$, define linear maps $T_{j,A} : \mathcal{H} \rightarrow \mathcal{H}$ as

$$T_{j,A} := \sqrt{1 - \sum_{i=1}^j \alpha_{ij} \mathbf{1}_{\mathcal{H}}} + \sum_{i=1}^j \sqrt{\alpha_{ij}} T_i.$$ 

Observe that each $T_{j,A}$ is an isometric embedding of $\mathcal{H}$ into $\mathcal{H}$, which we shall refer to as the $A$-tilt along the $j$th direction. We shall call $A$ as the tilting matrix. A tilting matrix is always assumed to be upper triangular, diagonal dominated and substochastic.

**Definition 3** ($A$-tilted span). Let $W_1, \ldots, W_l$ be subspaces of $\mathcal{H}$. The subspace $T_{j,A}(W_j) \subseteq \mathcal{H}$ will be called the $A$-tilt of $W_j$ along the directions 1, $\ldots$, $j$. The subspace $W_A$ of $\mathcal{H}$ defined by

$$W_A := \bigcap_{j=1}^l T_{j,A}(W_j).$$

is called the $A$-tilted span of $W_1, \ldots, W_l$.

The effect of doing $A$-tilting of the subspaces $W_1, \ldots, W_l$ along $l$ orthogonal directions is to increase the separation between them, in a sense made precise by the following proposition.

**Proposition 3.** Let $A$ be an upper triangular diagonal dominated substochastic matrix. Let $|h\rangle \in \mathcal{H}$ be a unit vector. For $j \in [l]$, define $\epsilon_j := \|\Pi_{W_j}|h\rangle\|_2^2$. Then,

$$\max_{j \in [l]} \epsilon_j \left( 1 - \sum_{i=1}^j \alpha_{ij} \right) \leq \|\Pi_{W_A}|h\rangle\|_2^2 \leq \left( \sum_{j=1}^l \epsilon_j \left( \frac{1}{\sum_{k=j}^l 2^{k-j} \alpha_{kk}} \right)^{1/2} \right)^2.$$ 

In particular, if $\alpha_{jj} \geq \alpha$ for all $j \in [l]$, we can obtain an upper bound of

$$\|\Pi_{W_A}|h\rangle\|_2^2 \leq \frac{2^{l+1}}{\alpha} \sum_{j=1}^l \epsilon_j.$$
Proof. The lower bound follows trivially since the projection of $|h\rangle$ onto the $j$th summand space in the definition of $W_A$ is of length $\sqrt{\epsilon_j \left( 1 - \sum_{i=1}^{j} \alpha_{ij} \right)}$.

We now prove the upper bound. Define $h' := \Pi_{W_A} |h\rangle$. Since $h' \in W_A$, let $h' = \sum_{j=1}^{l} \lambda_j |x_j\rangle$ where $\lambda_j \in \mathbb{C}$, and $|x_j\rangle$ is a unit vector in $T_{j,A}(W_j)$ for $j \in [l]$. Let $|x_j\rangle = T_{j,A}(|y_j\rangle)$, where $|y_j\rangle$ is a unit vector in $W_j$ and is uniquely determined by $|x_j\rangle$. Since the spaces $H_j$, $j \in [l]$ are orthogonal to $H$, we have

$$
||h'||_2 = \left\| \sum_{j=1}^{l} \sqrt{1 - \sum_{i=1}^{j} \alpha_{ij}} \lambda_j |y_j\rangle + \sum_{j=1}^{l} \sum_{i=1}^{j} \sqrt{\alpha_{ij}} \lambda_j^2 T_i(|y_j\rangle) \right\|_2 \geq \left\| \sum_{j=1}^{l} \sum_{i=1}^{j} \sqrt{\alpha_{ij}} \lambda_j T_i(|y_j\rangle) \right\|_2.
$$

We now prove by backward induction on $i$ that

$$
|\lambda_i| \leq ||h'||_2 \left( \sum_{j=i}^{l} 2^{j-i} \alpha_{jj}^{-1/2} \right). \tag{2}
$$

The base case of $i = l + 1$ is vacuously true taking $\lambda_{l+1} = 0$. Suppose the claim is true for $i + 1$. We now prove it for $i$. Observe that

$$
\sqrt{\alpha_{ii}}|\lambda_i| - \sum_{j=i+1}^{l} \sqrt{\alpha_{ij}}|\lambda_j| \leq \left\| \sum_{j=i}^{l} \sqrt{\alpha_{ij}} \lambda_j T_i(|y_j\rangle) \right\|_2 \leq \left\| \sum_{i=1}^{l} \sum_{j=i}^{l} \sqrt{\alpha_{ij}} \lambda_j T_i(|y_j\rangle) \right\|_2 \leq ||h'||_2,
$$

where we used the fact that $H_i$ is orthogonal to $H_{i'}$ for all $i' \neq i$ in the second inequality. Using the induction hypothesis, we get

$$
|\lambda_i| \leq \frac{||h'||_2}{\sqrt{\alpha_{ii}}} + \sum_{j=i+1}^{l} \frac{\sqrt{\alpha_{ij}}}{\sqrt{\alpha_{ii}}} |\lambda_j| \leq \frac{||h'||_2}{\sqrt{\alpha_{ii}}} + \sum_{j=i+1}^{l} |\lambda_j| \leq ||h'||_2 \left( \alpha_{ii}^{-1/2} + \sum_{j=i+1}^{l} \sum_{k=j}^{l} 2^{k-j} \alpha_{kk}^{-1/2} \right).
$$

Rearranging, we get

$$
|\lambda_i| \leq ||h'||_2 \left( \alpha_{ii}^{-1/2} + \sum_{k=i+1}^{l} \sum_{j=i+1}^{k} 2^{k-j} \alpha_{kk}^{-1/2} \right) \leq ||h'||_2 \left( \alpha_{ii}^{-1/2} + \sum_{k=i+1}^{l} 2^{k-i} \alpha_{kk}^{-1/2} \right),
$$

which completes the proof of the inequality in (2) by induction.

We also have

$$
||h'||_2^2 = |\langle h|h'\rangle| = \left| \sum_{j=1}^{l} \lambda_j \langle h|x_j\rangle \right| \leq \left| \sum_{j=1}^{l} |\lambda_j| |\langle h|x_j\rangle| \right| = \sum_{j=1}^{l} |\lambda_j| \sqrt{1 - \sum_{i=1}^{j} \alpha_{ij}} |\langle h|y_j\rangle| \leq \sum_{j=1}^{l} |\lambda_j| \left\| \Pi_{W_j} |y_j\rangle \right\|_2 \leq \sum_{j=1}^{l} |\lambda_j| \sqrt{\epsilon_j}.
$$

Combining the above inequality with the inequality in (Equation 2), we get

$$
||h'||_2 \leq \sum_{j=1}^{l} \left( \sum_{k=j}^{l} 2^{k-j} \alpha_{kk}^{-1/2} \right) \sqrt{\epsilon_j},
$$

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which completes the proof of the proposition. The special case where \( \alpha_{jj} \geq \alpha \) for all \( j \in [l] \) follows via Cauchy-Schwarz inequality.

\[ \square \]

**Remark:** It is interesting to contrast the claims of Propositions 2 and 3. In the case where \( \alpha_{jj} = \alpha \) for all \( j \in [l] \) and \( \alpha_{jj'} = 0 \) for \( j \neq j' \), Proposition 2 gives an upper bound of \( \frac{1}{\alpha} \sum_{j=1}^{l} \epsilon_j \). On the other hand, Proposition 3 gives a worse upper bound of \( \frac{2^{l+1}}{\alpha} \sum_{j=1}^{l} \epsilon_j \). The power of Proposition 3 lies in its generality. As discussed earlier, it turns out that we need the general setting of Proposition 3 in order to construct an intersection projector strong enough to prove the one-shot quantum joint typicality lemma. In other words, Proposition 3 can be thought of as a way to do intersection of hypothesis tests in the quantum setting. Proposition 2 will be used in situations where one needs to consider union of hypothesis tests in the quantum setting. Such situations arise in several network information theoretic tasks on top of joint typicality requirements. Some concrete applications where both intersection and union of hypothesis tests are required can be found in the companion paper [Sen18].

4 The one-shot classical quantum joint typicality lemma

In this section, we state precisely our one-shot classical quantum joint typicality lemma in Theorem 1. But first, we state a technical proposition which will be used to prove our joint typicality lemma. The proof of the proposition is deferred to Section 5.

**Proposition 4.** Let \( \mathcal{H}, \mathcal{L} \) be Hilbert spaces and \( \mathcal{X} \) be a finite set. We will also use \( \mathcal{X} \) to denote the Hilbert space with computational basis elements indexed by the set \( \mathcal{X} \). Let \( c \) be a non-negative and \( k \) a positive integer. Let \( A_1 \cdots A_k \) be a \( k \)-partite system where each \( A_i \) is isomorphic to \( \mathcal{H} \). For every \( x \in \mathcal{X}^c \), let \( \rho_x \) be a quantum state in \( A_{[k]} \). Consider the augmented \( k \)-partite system \( A_1'' \cdots A_k'' \) where each \( A_i'' \) is isomorphic to \( (\mathcal{H} \otimes \mathbb{C}^2) \oplus \bigoplus_{S : \emptyset \in S \subseteq [c] \cup [k]} \mathcal{L}^{\otimes |S|} \).

View \( \rho_x^{A_{[k]}} \otimes (|0\rangle \langle 0|)^{\otimes \mathcal{H}^{[k]}} \) as a state in \( A_{[k]}'' \) under the natural embedding viz. the embedding in the \( i \)th system is into the first summand of \( A_i'' \) defined above.

Below, \( x, l \) denote computational basis vectors of \( \mathcal{X}^c \), \( \mathcal{L}^{\otimes ([c] \cup [k])} \). Let \( 0 \leq \delta \leq 1 \). For each \( x \in \mathcal{X}^c \) and each pseudosubpartition \( (S_1, \ldots, S_l) \) \( l \) \( \vdash [c] \cup [k] \), let \( 0 \leq \epsilon_x(S_1, \ldots, S_l) \leq 1 \). Let \( \epsilon_x := \sum_{(S_1, \ldots, S_l) \vdash [c] \cup [k]} \epsilon_x(S_1, \ldots, S_l) \). Then there exist:

1. States \( \rho_{x,1,\delta}'' \) in \( A_{[k]}'' \) for every \( x, l \);
2. POVM elements \( \Pi_{x,1,\delta}'' \) in \( A_{[k]}'' \) for every \( x, l \);
3. For every \( 1 \) and every \( (S_1, \ldots, S_l) \) \( l \) \( \vdash [c] \cup [k] \), \( l \) \( > 0 \), numbers \( 0 \leq \alpha(S_1, \ldots, S_l) \), \( \beta(S_1, \ldots, S_l) \), \( \delta \leq 1 \) and isometric embeddings \( T_{(S_1, \ldots, S_l),1,\delta}'' \) of \( (\mathcal{H} \otimes \mathbb{C}^2)^{\otimes k} \) into \( A_{[k]}'' \);
4. For every \( x, l \) and every \( (S_1, \ldots, S_l) \) \( l \) \( \vdash [c] \cup [k] \), \( l \) \( > 0 \), quantum states \( M_{(S_1, \ldots, S_l), x,1,\delta}'' \) and unit trace Hermitian operators \( N_{(S_1, \ldots, S_l), x,1,\delta}'' \) in \( A_{[k]}'' \);

such that:
1. \[ \left\| (\Pi^\prime_{\mathbf{x},1,\delta})_{x,1,\delta} \right\|_1 \leq (2|\mathcal{H}|)^k; \]

2. \[ \left\| M^A_{\mathbf{k}}(S_1,\ldots,S_l) \right\|_{\mathcal{L}} \leq \frac{1}{|\mathcal{H}|^k} \beta(S_1,\ldots,S_l) \delta \left\| N^A_{\mathbf{k}}(S_1,\ldots,S_l) \right\|_{\mathcal{L}} \leq \frac{3}{\sqrt{|\mathcal{L}|}}; \]

3. \[ \left\| (\rho^\prime_{\mathbf{x},1,\delta})_{\mathbf{x},1,\delta} - \rho^A_{\mathbf{x}} \otimes (|0\rangle\langle 0|)^{\otimes k} \right\|_1 \leq 2^{\frac{k+c+1}{2}} \delta; \]

4. \[ \text{Tr} \left[ (\Pi^\prime_{\mathbf{x},1,\delta})_{\mathbf{x},1,\delta}^A (\rho^A_{\mathbf{x}} \otimes (|0\rangle\langle 0|)^{\otimes k}) \right] \geq 1 - \delta^{-2k} 2^{2^{k+4}(k+1)^k} \epsilon^x; \]

5. Let \((S_1,\ldots,S_l) \vdash [c] \cup [k], l > 0\). Define \(T := [k] \setminus (S_1 \cup \cdots \cup S_l)\). Let \(\sigma^T_{\mathbf{x}}\) be a state in \(A^T\). Let \(S \subseteq [c] \cup [k], S \cap [k] \neq \{\}\); Let \(x_{[c]} \in S\), \(1_S\) be computational basis vectors in \(\mathcal{X}^{\otimes([c] \cap S)}, \mathcal{L}^{\otimes S}\). Let \(p_{[c]}S\) be a probability distribution on \(\mathcal{X}^{\otimes([c] \cap S)}\). In the following definition, let \(x_{[c]}S, Y_{S}\) range over all computational basis vectors of \(\mathcal{X}^{\otimes([c] \cap S)}, \mathcal{L}^{\otimes S}\). Define a state in \(A^\prime_{S \cap [k]}\),

\[ (\rho')_{x_{S \cap [c]}S,1_S}^{A\prime_{S \cap [k]}} := |L|^{-|S|} \sum_{x_{[c]}S} p_{[c]}S(x_{[c]}S) \ \text{Tr}_{A^\prime_{S \cap [k]}} \left[ (\rho')_{x_{S \cap [c]}S}^{A^\prime_{S \cap [k]}} x_{[c]}^{A^\prime_{S \cap [k]}}, 1_S, Y_{S}, \delta \right]; \]

Analogously define

\[ \rho^A_{x_{S \cap [c]}S} := \sum_{x_{[c]}S} p_{[c]}S(x_{[c]}S) \ \text{Tr}_{A^\prime_{S \cap [k]}} \left[ \rho^A_{x_{S \cap [c]}x_{[c]}}, 1_S, Y_{S}, \delta \right], \]

Define

\[ (\rho')_{x,1,(S_1,\ldots,S_l),\delta}^A = (\rho^A_{x_{S_1 \cap [c]}S,1_S}, \delta) A^\prime_{S_1 \cap [c]} \otimes \cdots \otimes (\rho^A_{x_{S_l \cap [c]}S,1_S}, \delta) A^\prime_{S_l \cap [c]} \otimes (\sigma^T_{\mathbf{x}} \otimes (|0\rangle\langle 0|)^{\otimes T}), \]

\[ \rho^{A}_{x,(S_1,\ldots,S_l),\delta} := A^\prime_{S_1 \cap [c]} \otimes \cdots \otimes A^\prime_{S_l \cap [c]} \otimes \sigma^T_{\mathbf{x}}. \]

Then,

\[ (\rho')_{x,1,(S_1,\ldots,S_l),\delta}^A = \alpha(S_1,\ldots,S_l,\delta) (T_{(S_1,\ldots,S_l),1,\delta}^{\prime} \rho^A_{x,(S_1,\ldots,S_l)} \otimes (|0\rangle\langle 0|)^{\otimes k}) A^\prime_{[k]} \]

\[ + \beta(S_1,\ldots,S_l,\delta) N^A_{(S_1,\ldots,S_l),1,\delta} \]

\[ + (1 - \alpha(S_1,\ldots,S_l,\delta) - \beta(S_1,\ldots,S_l,\delta)) M^A_{(S_1,\ldots,S_l),1,\delta}; \]

Moreover, the support of \(M^A_{(S_1,\ldots,S_l),1,\delta}\) is orthogonal to the support of the sum of the first two terms in the above equation, and \(\beta(S_1,\ldots,S_l,\delta) = 0\) if \(c = 0\) or [c] \(\subseteq S_i\) for some \(i \in [l] \).

6. \[ \text{Tr} \left[ (\Pi^\prime_{\mathbf{x},1,\delta})_{\mathbf{x},1,\delta}^A (T_{(S_1,\ldots,S_l),1,\delta}^{\prime} \rho^A_{x,(S_1,\ldots,S_l)} \otimes (|0\rangle\langle 0|)^{\otimes k}) A^\prime_{[k]} \right] \leq 2^{-D^H_{\mathbf{x}}(S_1,\ldots,S_l)} \epsilon^x_{\mathbf{x}} \log^A_{\mathbf{x}} \rho^A_{x,(S_1,\ldots,S_l)}. \]

Proof. Proved in Section 5 \(\square\)

We now state our one-shot classical quantum joint typicality lemma. First we state the ‘intersection case’ where we only have to take a so-called intersection of POVM elements. This can be viewed as the classical quantum version of Fact 1 when \(t = 1\) i.e. when only intersection of classical POVM elements needs to be taken, not the union.
Lemma 1 (Classical quantum joint typicality lemma, intersection case). Let $\mathcal{H}$, $\mathcal{L}$ be Hilbert spaces and $X$ be a finite set. We will also use $X$ to denote the Hilbert space with computational basis elements indexed by the set $X$. Let $c$ be a non-negative and $k$ a positive integer. Let $A_1 \cdots A_k$ be a $k$-partite system where each $A_i$ is isomorphic to $\mathcal{H}$. For every $x \in X^c$, let $\rho_x$ be a quantum state in $A_x^{[k]}$. Consider the augmented $k$-partite system $A'_1 \cdots A'_k$ where each $A'_i \equiv A''_i \otimes \mathcal{L}$, and each $A''_i$ is defined as

$$ A''_i := (H \otimes C^2) \oplus \bigoplus_{S_i \in S \subseteq [c] \cup [k]} (H \otimes C^2) \otimes \mathcal{L}^\otimes |S| $$

Also define $X' := X \otimes \mathcal{L}$.

Below, $x$, 1 denote computational basis vectors of $X^{|c|} \otimes \mathcal{L}^\otimes ([c] \cup [k])$. Let $p(\cdot)$ be a probability distribution on the vectors $x$. Define the classical quantum state

$$ \rho_{X^{|c|}A_x^{[k]}} := \sum_x p(x) |x\rangle \langle x|_{X^{|c|}} \otimes \rho_x^{A_x^{[k]}}. $$

Let $\mathcal{F}_c^{\otimes (c+k)} / |L|^{c+k}$ denote the completely mixed state on $(c+k)$ tensor copies of $L$. View $\rho_x^{A_x^{[k]}} \otimes (|0\rangle \langle 0|)^{(C^2)^\otimes k}$ as a state in $A''_i$ under the natural embedding viz. the embedding in the $i$th system is into the first summand of $A''_i$ defined above tensored with $L$. Similarly, view $\rho_{X^{|c|}A_x^{[k]}} \otimes (|0\rangle \langle 0|)^{(C^2)^\otimes k} \otimes \mathcal{F}_c^{\otimes (c+k)} / |L|^{c+k}$ as a state in $X'_i A'_i^{[k]}$ under the natural embedding.

Let $0 \leq \delta \leq 1$. For each pseudosubpartition $(S_1, \ldots, S_l) \vdash [c] \cup [k]$, let $0 \leq \epsilon(S_1, \ldots, S_l) \leq 1$. Then, there is a state $\rho'$ and a POVM element $\Pi'$ in $X'_i A'_i^{[k]}$ such that:

1. The state $\rho'$ and POVM element $\Pi'$ are classical on $X^\otimes |c| \otimes \mathcal{L}^\otimes ([c] \cup [k])$ and quantum on $A''_i^{[k]}$.

More precisely, $\rho'$, $\Pi'$ can be expressed as

$$ (\rho')_{X^{|c|}A_x^{[k]}} = |L|^{-(c+k)} \sum_{x,1} p(x) |x\rangle \langle x|_{X^{|c|}} \otimes |1\rangle |L|^{\otimes ([c] \cup [k])} \otimes (\rho')_{x,1}^{A''_x^{[k]}}, $$

$$ (\Pi')_{X^{|c|}A_x^{[k]}} = \sum_{x,1} |x\rangle \langle x|_{X^{|c|}} \otimes |1\rangle |L|^{\otimes ([c] \cup [k])} \otimes (\Pi')_{x,1}^{A''_x^{[k]}}, $$

where $(\rho')_{x,1}^{A''_x^{[k]}}, (\Pi')_{x,1}^{A''_x^{[k]}}$ are quantum states and POVM elements respectively for all computational basis vectors $x \in X^{|c|}$, $1 \in \mathcal{L}^\otimes ([c] \cup [k])$;

2. $$ \left\| (\rho')_{X^{|c|}A_x^{[k]}} - \rho_{X^{|c|}A_x^{[k]}} \otimes (|0\rangle \langle 0|)^{(C^2)^\otimes k} \otimes \mathcal{F}_c^{\otimes (c+k)} / |L|^{c+k} \right\|_1 \leq 2^{c+k-1} \delta; $$

3. $$ \text{Tr} [(\Pi')_{X^{|c|}A_x^{[k]}} (\rho')_{X^{|c|}A_x^{[k]}}] \geq 1 - \delta - 2^{2k+4} \sum_{(S_1, \ldots, S_l) \vdash [c] \cup [k]} \epsilon(S_1, \ldots, S_l) \geq 2^{c+k-1} \delta. $$
4. Let \((S_1, \ldots, S_l) \mapsto [c] \cup [k], l > 0\). Define \(T := [k] \setminus (S_1 \cup \cdots \cup S_l)\). Let \(\sigma_A^T\) be a state in \(A_T\). Let \(S \subseteq [c] \cup [k]\), \(S \cap [k] \neq \emptyset\), Let \(x[S]\) be computational basis vectors in \(X^\otimes [c] \otimes S\). Let \(p_{[c]}(\mathcal{S}, \cdot)\) be a probability distribution on \(X^\otimes [c] \setminus S\). In the following definition, let \(x'[x]\) range over all computational basis vectors of \(X^\otimes [c] \setminus S\), \(|S|\). Define a state in \(A''_{S[x]}\),

\[
(\rho')^{A''_{x[S]}[k]}_{x[S]} := |\mathcal{L}|^{-|S|} \sum_{x'[x]} p_{[c]}(\mathcal{S}, x'[x]) \text{Tr} \left[ (\rho')^{A''_{x[\cdot][c]}[k]}_{x[\cdot][c]} \right] \cdot \rho^{A_{x[\cdot][c]}[k]}_{x(S_1, \ldots, S_l), \delta}.
\]

Analogously define

\[
\rho^{A_{x[\cdot][c]}[k]}_{x(S_1, \ldots, S_l)} := \sum_{x'[x]} p_{[c]}(\mathcal{S}, x'[x]) \text{Tr} \left[ \rho^{A_{x'[x][c]}[k]}_{x'[x][c]} \right].
\]

Define

\[
(\rho')^{A''_{x[c]}[k]}_{x(c, \ldots)} := (\rho_{x'(S_1, \ldots, S_l), \delta})^{A''_{x(c)[c]}[k]} \otimes \cdots \otimes (\rho_{x(S_1, \ldots, S_l), \delta})^{A''_{x[c]}[k]} \otimes (\sigma^T_A) \otimes (|0\rangle \langle 0|)^{\otimes |T|},
\]

\[
\rho_{x(S_1, \ldots, S_l)} := \rho_{x(S_1, \ldots, S_l), \delta} \otimes \cdots \otimes \rho_{x(S_1, \ldots, S_l)} \otimes \sigma^T_A.
\]

Let \(q(S_1, \ldots, S_l)(\cdot)\) be a probability distribution over vectors \(x\). Define

\[
(\rho')^{X[c]}_{S} := |\mathcal{L}|^{-(c+k)} \sum_{x, l} q(S_1, \ldots, S_l)(x) |x\rangle \langle x|^{X[c]} \otimes |l\rangle \langle l|^{X[c]} \otimes (\rho')^{A''_{x[c]}[k]}_{x(c, \ldots)} \cdot \rho_{x(S_1, \ldots, S_l)}^{A/c[k]}.
\]

Then,

\[
\text{Tr} \left[ (\Pi')^{A''_{x[c]}[k]}_{x(c, \ldots)} (\rho')^{X[c]}_{S} \right] \leq \max \left\{ 2^{-d} \bar{d}_{H}(S_1, \ldots, S_l) \rho X[c][k]_{S_1, \ldots, S_l}, 3(2|H|)^k \right\}.
\]

**Proof.** The lemma is an easy consequence of Proposition \([3]\). Consider first the augmented \(k\)-partite system \(A''_1 \cdots A''_k\). For computational basis vectors \(x, 1\) let \((\rho_{x(1), \delta})^{A''_{1[k]}}\) be the quantum state and \((\Pi')^{A''_{1[k]}}_{x(1), \delta}\) be the POVM element in \(A''_{1[k]}\) guaranteed by Existence Statements 1 and 2 of Proposition \([3]\). These quantities are used to define the state \(\rho'\) and POVM element \(\Pi'\) in \(X''_{1[k]}\) as in Claim 1 of the lemma.

From Claim 3 of Proposition \([3]\),

\[
\left\| (\rho')^{X[c]}_{[c]} - \rho X[c][k]_{S_1, \ldots, S_l} \otimes (|0\rangle \langle 0|)^{\otimes k} \otimes \frac{1}{|\mathcal{L}|^{k}} \right\|_1 \leq 2^{-d} \bar{d}_{H}(S_1, \ldots, S_l) \rho X[c][k]_{S_1, \ldots, S_l}, 3(2|H|)^k.
\]

\[
\left\| (\rho')^{A''_{1[k]}_{x(1), \delta}} \right\|_1 \leq 2^{c+k} + 1 \delta,
\]

where \(H = [H] \otimes [k]\).
which proves Claim 2 of the lemma.

For each pseudosubpartition \((S_1, \ldots, S_l) \vdash [c] \sqcup [k]\), let \(\Pi^{A_k}_{x, (S_1, \ldots, S_l)}\) be the optimising POVM element in the definition of \(D_H^{e_x(S_1, \ldots, S_l)} (\rho^{x[A_k][A_k]}_{(S_1, \ldots, S_l)})\). Without loss of generality, it is of the form

\[
\Pi^{x[A_k]}_{x, (S_1, \ldots, S_l)} = \sum_{x} |x\rangle \langle x^{[c]}| \otimes \Pi^{A_k}_{x, (S_1, \ldots, S_l)},
\]

where for each \(x\), \(\Pi_{x,(S_1,\ldots,S_l)}\) is a POVM element in \(A_k\). Define \(\epsilon_{x,(S_1,\ldots,S_l)} := 1 - \text{Tr} [\Pi^{A_k}_{x,(S_1,\ldots,S_l)} \rho^{x[A_k]}_{x,(S_1,\ldots,S_l)}]\). Then, \(\Pi^{A_k}_{x,(S_1,\ldots,S_l)}\) is the optimising POVM element in the definition of \(D_H^{e_x(S_1,\ldots,S_l)} (\rho^{x[A_k]}_{x,(S_1,\ldots,S_l)})\).

This implies that

\[
\sum_{x} q_{(S_1,\ldots,S_l)}(x)^2 D_H^{e_x(S_1,\ldots,S_l)} (\rho^{x[A_k]}_{x,(S_1,\ldots,S_l)}) = 2 - D_H^{e_x(S_1,\ldots,S_l)} (\rho^{x[A_k]}_{x,(S_1,\ldots,S_l)}),
\]

(3)

From Claims 4, 3 of Proposition [4]

\[
\text{Tr} [(\Pi')^{x[A_k]}_{x'[A_k]} (\rho^{x[A_k]})] \\
= |L|^{-(c+k)} \sum_{x_1} p(x) \text{Tr} [(\Pi')^{A_k}_{x_1,\delta} (\rho^{A_k}_{x_1,\delta})] \\
\geq |L|^{-(c+k)} \sum_{x_1} p(x) \left( \text{Tr} [(\Pi')^{A_k}_{x_1,\delta} (\rho^{A_k}_{x_1,\delta} \otimes (|0\rangle \langle 0|)^{c+k} \otimes k)] - \| (\rho^{A_k}_{x_1,\delta} - \rho^{A_k}_{x_1,\delta} \otimes (|0\rangle \langle 0|)^{c+k} \|_1 \right) \\
\geq 1 - \sum_{x} p(x) \delta - 2^c 2^{2^{c+k+4}(c+k+1)} \epsilon_{x,(S_1,\ldots,S_l)} - 2^{\frac{k+c}{2} + 1} \delta \]

which proves Claim 3 of the lemma.

Using claims 5, 6, 1, 2 of Proposition [4] and Equation [3] we get

\[
\text{Tr} [(\Pi')^{x[A_k]}_{x'[A_k]} (\rho^{x[A_k]})] \\
= |L|^{-(c+k)} \sum_{x_1} q_{(S_1,\ldots,S_l)}(x) \text{Tr} [(\Pi')^{A_k}_{x_1,\delta} (\rho^{A_k}_{x_1,\delta})] \\
= \alpha_{(S_1,\ldots,S_l),\delta} |L|^{-(c+k)} \sum_{x_1} q_{(S_1,\ldots,S_l)}(x) \text{Tr} [(\Pi')^{A_k}_{x_1,\delta} (\rho^{A_k}_{x_1,\delta})] \\
+ \beta_{(S_1,\ldots,S_l),\delta} |L|^{-(c+k)} \sum_{x_1} q_{(S_1,\ldots,S_l)}(x) \text{Tr} [(\Pi')^{A_k}_{x_1,\delta} (\rho^{A_k}_{x_1,\delta})] \\
+ (1 - \alpha_{(S_1,\ldots,S_l),\delta} - \beta_{(S_1,\ldots,S_l),\delta}) |L|^{-(c+k)} \sum_{x_1} q_{(S_1,\ldots,S_l)}(x) \text{Tr} [(\Pi')^{A_k}_{x_1,\delta} (\rho^{A_k}_{x_1,\delta})] \\
\leq \alpha_{(S_1,\ldots,S_l),\delta} |L|^{-(c+k)} \sum_{x_1} q_{(S_1,\ldots,S_l)}(x) 2^{-D_H^{e_x(S_1,\ldots,S_l)} (\rho^{x[A_k]}_{x,(S_1,\ldots,S_l)})}
\]

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classical quantum joint typicality lemma, intersection case. That is why we call it the one-shot conditional joint typicality lemma with intersection, where the conditioning system continues to then take the intersection of all the tests. The preceding lemma is a generalisation of the classical probability distribution

\[ p(x_{[c]}, y) p(x_{[c]} | x_{[c]} = 0) p(x_{[c]} | x_{[c]} = 1) \]

which should be accepted with as low probability as possible. The distribution \( p(x_{[c]}) \) is a product of the marginals on \( X_S \) and \( X_{[k]} \setminus S \) conditioned on \( X_{[c]} \), which gives rise to the name of conditional joint typicality lemma. For each \( S \subseteq [k] \), one can take the optimal test for the above task, and then take the intersection of all the tests. The preceding lemma is a generalisation of the classical conditional joint typicality lemma with intersection, where the conditioning system continues to be classical but the remaining systems are allowed to be quantum. That is why we call it the one-shot classical quantum joint typicality lemma, intersection case.

In the important special case where there is no classical system i.e. \( c = 0 \), we get the following corollary of Lemma 1.

**Corollary 2** (Quantum joint typicality lemma, intersection case). Let \( \mathcal{H}, \mathcal{L} \) be Hilbert spaces. Let \( k \) be a positive integer. Let \( A_1 \cdots A_k \) be a \( k \)-partite system where each \( A_i \) is isomorphic to \( \mathcal{H} \). Let \( \rho \) be a quantum state in \( A_{[k]} \). Consider the augmented \( k \)-partite system \( A_1' \cdots A_k' \) where each \( A_i' \cong A_i'' \otimes \mathcal{L} \), and each \( A_i'' \) is defined as

\[ A_i'' := (\mathcal{H} \otimes \mathbb{C}^2) \oplus \bigoplus_{S \ni i \in [k]} (\mathcal{H} \otimes \mathbb{C}^2) \otimes \mathcal{L}^{\otimes |S|}. \]

Below, \( 1 \) denotes a computational basis vector of \( \mathcal{L}^{\otimes [k]} \). Let \( \frac{\mathcal{L}^{\otimes k}}{|\mathcal{L}|} \) denote the completely mixed state on \( k \) tensor copies of \( \mathcal{L} \). View \( \rho^{A_{[k]} \otimes (|0\rangle\langle 0|)^{\otimes k}} \otimes \frac{\mathcal{L}^{\otimes k}}{|\mathcal{L}|} \) as a state in \( A_{[k]}' \) under the natural embedding viz. the embedding in the \( i \)th system is into the first summand of \( A_i' \) defined above tensored with \( \mathcal{L} \).

Let \( 0 \leq \epsilon, \delta \leq 1 \). Then, there is a state \( \rho' \) and a POVM element \( \Pi' \) in \( A_{[k]}' \) such that:
1. The state $\rho'$ and POVM element $\Pi'$ are classical on $L^{[k]}$ and quantum on $A''_k$. More precisely, $\rho'$, $\Pi'$ can be expressed as

$$(\rho')_{A'[k]} = |L|^{-k} \sum_1 |I\rangle \langle I| L^{[k]} \otimes (\rho')_{A''_1,\delta}. \tag{1}$$

$$(\Pi')_{A'[k]} = \sum_1 |I\rangle \langle I| L^{[k]} \otimes (\Pi')_{A''_1,\delta}. \tag{2}$$

where $(\rho')_{A''_1,\delta}$, $(\Pi')_{A''_1,\delta}$ are quantum states and POVM elements respectively for all computational basis vectors $I \in L^{\otimes[k]}$;

2. \[
\left\| (\rho')_{A[k]} - \rho^{A[k]} \otimes (0 \langle 0 \rangle) (C^2)^{\otimes k} \otimes \frac{I_{L^{k}}}{|L|^{k}} \right\|_1 \leq 2^\delta + 1 \delta; \tag{3}
\]

3. \[
\text{Tr} \left[ (\Pi')_{A[k]} (\rho')_{A[k]} \right] \geq 1 - \delta - 2^{17(k+1)^2} \epsilon - 2^\delta + 1 \delta; \tag{4}
\]

4. Let $(S_1, \ldots, S_l) \vdash [k]$, $l > 0$. Define $T := [k] \setminus (S_1 \cup \cdots \cup S_l)$. Let $\sigma^A_T$ be a state in $A_T$. Let $\{\}$ $\neq S \subseteq [k]$. Let $1_S$ be computational basis vectors in $L^{\otimes S}$. In the following definition, let $1_S$ range over all computational basis vectors of $L^{\otimes[S]}$. Define a state in $A''_S$,

$$(\rho')_{1_S,\delta} = |L|^{-|S|} \sum_{1_S} \text{Tr}_{A''_S} [((\rho')_{A[k]}])_{1_S,\delta}. \tag{5}$$

Analogously define $\rho^A_S := \text{Tr}_{A_S} [\rho^{A[k]}].$ Define

$$(\rho')_{A''_1,1(S_1,\ldots,S_l),\delta} := (\rho'_{1S_1,\delta})_{A''_1} \otimes \cdots \otimes (\rho'_{1S_l,\delta})_{A''_1} \otimes (\sigma^A_T \otimes (0 \langle 0 \rangle) (C^2)^{\otimes |T|})), \tag{6}$$

$$(\rho')_{A''_1} := \rho^{A_S} \otimes \cdots \otimes \rho^{A_S} \otimes \sigma^A_T. \tag{7}$$

Define

$$(\rho')_{(S_1,\ldots,S_l)} := |L|^{-k} \sum_{x \subseteq 1} |I\rangle \langle I| L^{[k]} \otimes (\rho')_{1(S_1,\ldots,S_l),\delta}. \tag{8}$$

Then,

$$\text{Tr} \left[ (\Pi')_{A[k]} (\rho')_{(S_1,\ldots,S_l)} \right] \leq \max \left\{ 2^{-D_H^e(\rho^{A[k]} \| \rho^{A_S}) \frac{3(2H)^k}{\sqrt{|L|}} } \right\}. \tag{9}$$

Remark: The statement of the joint typicality lemma given at the end of Section 1.3 easily follows from the more general statement in Corollary 2 using Proposition 1. We first observe by Proposition 1 that for any state $\rho^{A[k]}$ and any subset $\{\} \neq S \subset [k]$, \[
D_H^e(\rho^{A[k]} \| \rho^{A_S} \otimes \rho^{A_S}) \leq k \log |H| + 3 \log \frac{1}{1 - \epsilon} + 6 \log 3 - 4. \tag{10}
\]
Choose $\mathcal{L}$ of dimension $\binom{3^{13}|\mathcal{H}|}{2^{n(1-\varepsilon)^2}}$ in Corollary \[ \text{Choose the Hilbert space } \mathcal{K} \text{ to be of dimension} \]

$$2|\mathcal{L}|^{k+1} < \frac{2^{13(k+1)}(2|\mathcal{H}|)^{4k(k+1)}}{(1-\varepsilon)^{6(k+1)}}.$$ 

This makes the dimension of $\mathcal{H} \otimes \mathcal{K}$ large enough to contain $A'_t$. Choose $\delta := \varepsilon^\frac{1}{4}$. Define $\tau^{\mathcal{K}\otimes[k]} := (|0\rangle\langle0|)^{(\mathbb{C}^2)^\otimes[k]} \otimes \frac{\mathbb{I}^{\otimes[k]}}{|\mathcal{K}|^k}$. The statement in the abstract now follows easily from Corollary \[ \text{We now prove the one-shot classical quantum joint typicality lemma, general case} \]

i.e. we have to take union of intersection of POVM elements. This can be viewed as the classical quantum version of Fact \[ \text{Theorem 1 (Classical quantum joint typicality lemma, general case). Let } \mathcal{H}, \mathcal{L} \text{ be Hilbert spaces and } \mathcal{X} \text{ be a finite set. We will also use } \mathcal{X} \text{ to denote the Hilbert space with computational basis elements indexed by the set } \mathcal{X}. \text{ Let } \mathcal{C} \text{ be a non-negative and } k \text{ a positive integer. Let } A_1 \cdots A_k \text{ be a } k\text{-partite system where each } A_i \text{ is isomorphic to } \mathcal{H}. \text{ Let } t \text{ be a positive integer. Let } \mathbf{x}^t \text{ denote a } t\text{-tuple of elements of } \mathcal{X}^s; \text{ we shall denote its } i\text{th element by } \mathbf{x}^t(i). \text{ Consider the extended } k\text{-partite system } A_1 \cdots A_k \text{ where each } A_i \cong A'_t \otimes \mathbb{C}^2 \otimes \mathbb{C}^{t+1}, A'_t \cong A''_t \otimes \mathcal{L}, \text{ and each } A''_t \text{ is defined as} \]

$$A''_t := (\mathcal{H} \otimes \mathbb{C}^2) \oplus \bigoplus_{s_i \in S} (\mathcal{H} \otimes \mathbb{C}^2) \otimes \mathcal{L}^{\otimes|S|}.$$ 

Also define $\hat{X} := \mathcal{X} \otimes \mathcal{L}$. Below, $\mathbf{x}$ denotes computational basis vectors of $\mathcal{X}^{[c]}$, and $\mathbf{1}$ denotes computational basis vectors of $\mathcal{L}^{\otimes s}$ where $s$ will be clear from the context. Let $p(\cdot)$ denote a probability distribution on the vectors $\mathbf{x}$. Let $p(1;\cdot), \ldots, p(t;\cdot)$ denote probability distributions on $\mathbf{x}^t$ such that the marginal of $p(i;\mathbf{x}^t)$ on the $i$th element is $p(\mathbf{x}^t(i))$. For $i \in [t]$, define the classical quantum states

$$\rho^{(\mathcal{X}^{[c]})^t,A'_t}(\mathbf{x}^t(i)) := \sum_{\mathbf{x}^t} p(i;\mathbf{x}^t) |\mathbf{x}^t\rangle \langle \mathbf{x}^t|^{(\mathcal{X}^{[c]})^t} \otimes \rho^{A'_t}.$$ 

Let $\frac{\mathcal{L}^{\otimes k}}{|\mathcal{L}|^k}$ denote the completely mixed state on $(c+k)$ tensor copies of $\mathcal{L}$. View $\rho^{A'_t}(\mathbf{x}^t(i)) \otimes (|0\rangle\langle0|)^{(\mathbb{C}^2)^\otimes k} \otimes \frac{\mathcal{L}^{\otimes k}}{|\mathcal{L}|^k} \otimes (|0\rangle\langle0|)^{(\mathbb{C}^{t+1})^\otimes k} \otimes (|0\rangle\langle0|)^{(\mathbb{C}^{t+1})^\otimes k}$ as a state in $\mathcal{A}_k$ under the natural embedding viz. the embedding in the $t$th system is into the first summand of $A''_t$ defined above tensored with $\mathcal{L} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{t+1}$. Similarly, view $(\rho(i))^{(\mathcal{X}^{[c]})^t,A'_t}(i) \otimes (|0\rangle\langle0|)^{(\mathbb{C}^2)^\otimes k} \otimes \frac{\mathcal{L}^{\otimes (ct+k)}}{|\mathcal{L}|^{ct+k}} \otimes (|0\rangle\langle0|)^{(\mathbb{C}^{t+1})^\otimes k} \otimes (|0\rangle\langle0|)^{(\mathbb{C}^{t+1})^\otimes k}$ as a state in $(\mathcal{X}^{[c]})^t\hat{A}_t$ under the natural embedding.

Let $0 \leq \alpha, \varepsilon, \delta \leq 1$. For each pseudosubpartition $(S_1, \ldots, S_t) \mapsto [c] \cup [k]$ and each $i \in [t]$, let $0 \leq \epsilon_{i,(S_1, \ldots, S_t)} \leq 1$. Then, there are states $\rho'(1), \ldots, \rho'(t)$ and a POVM element $\hat{\Pi}$ in $(\mathcal{X}^{[c]})^t\hat{A}_t$ such that:

1. The states $\rho'(1), \ldots, \rho'(t)$ and POVM element $\hat{\Pi}$ are classical on $\mathcal{X}^{[c]t} \otimes \mathcal{L}^{[c]t\cup[k]}$ and quantum on $A''_t \otimes (\mathbb{C}^2)^\otimes[k] \otimes (\mathbb{C}^{t+1})^\otimes[k]$. More precisely, $\rho'(i), i \in [t]$, $\hat{\Pi}$ can be expressed as

$$\rho'(i) = |\mathcal{L}|^{-(ct+k)} \sum_{\mathbf{x}^t,i} p(i;\mathbf{x}^t) |\mathbf{x}^t\rangle \langle |(\mathcal{X}^{[c]})^t \otimes |\mathbf{1}\rangle \langle |\mathbf{1}|^{[c]t\cup[k]}$$

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4. Let $(\rho')_{x_i, i \in \Pi} A'_{[k]} \otimes (|0\rangle \langle 0|)^{(C^2)^{\otimes k}} \otimes (|0\rangle \langle 0|)^{(C^t+1)^{\otimes k}},$

$$(\Pi')^\dagger(\hat{X}_c')y \hat{A}_{[k]} = \sum_{x, l} [x^t](x_i)^t \otimes |l\rangle (I|^L_{[c] \cup [k]} \otimes (\Pi')_{x^t, l, \delta} \hat{A}_{[k]}),$$

where $(\rho')_{x_i, i \in \Pi} A'_{[k]}$ are quantum states for all computational basis vectors $x \in X^{\otimes [c]},$ $1 \in L^{\otimes ([c] \cup [k])}$ and $(\Pi')_{x^t, l, \delta} A'_{[k]}$ are POVM elements for all computational basis vectors $x^t \in X^{\otimes [c]},$ $1 \in L^{\otimes ([c] \cup [k])};$

2. For all $i \in [t],$

$$\left\| (\rho'(i))(x_i)^t \hat{A}_{[k]} - (\rho(i))(x_i)^t \hat{A}_{[k]} \otimes (|0\rangle \langle 0|)^{(C^2)^{\otimes k}} \otimes \frac{I}{|L|^k} \otimes (|0\rangle \langle 0|)^{(C^t+1)^{\otimes k}} \otimes (|0\rangle \langle 0|)^{(C^t+1)^{\otimes k}} \right\| \leq \frac{2^{c+k+1} + 1}{\delta};$$

3. For all $i \in [t],$

$$\text{Tr} \left[ (\hat{\Pi})^\dagger(x_i)^t \hat{A}_{[k]} (\rho'(i))(x_i)^t \hat{A}_{[k]} \right] \geq 1 - \delta^{-2k} 2^{c+k+4(k+1)^k} \sum_{(S_1, \ldots, S_t) \in \mathcal{S} \otimes [c] \cup [k]} \epsilon_{i, (S_1, \ldots, S_t)} - \frac{2^{c+k+1} + 1}{\delta} - \alpha;$$

4. Let $(S_1, \ldots, S_t) \mapsto [c] \cup [k], l > 0$. Define $T := [k] \setminus (S_1 \cup \cdots \cup S_t)$. Let $\sigma^T_x$ be a state in $A_T$. Let $S \subseteq [c] \cup [k], S \cap [k] \neq \emptyset$. Let $x_i^{[c]} \in S, 1_S$ be computational basis vectors in $X^{\otimes ([c] \setminus S), L^\otimes S}$. Let $p_{[c]}|S(\cdot)$ be a probability distribution on $X^{\otimes ([c]\setminus S)}$. In the following definition, let $x_i^{[c]}$, $1_S'$ range over all computational basis vectors of $X^{\otimes ([c]\setminus S)}$, $L^\otimes S$. Define a state in $A^T_{S^c\cup[k]},$

$$(\rho')_{x_{S^c\cup[k]}, 1_S, \delta} := |L|^{-|S|} \sum_{x_{[c]} \setminus S} p_{[c]}|S(x_{[c]}|S) \text{Tr}_{A^T_{S^c\cup[k]}} \left[ (\rho')_{x_{S^c\cup[k]}x_{[c]}|S} 1_S' \delta \right].$$

Analogously define

$$\rho_{x_{S^c\cup[k]}} := \sum_{x_{[c]} \setminus S} p_{[c]}|S(x_{[c]}|S) \text{Tr}_{A^T_{S^c\cup[k]}} \left[ \rho_{x_{S^c\cup[k]}x_{[c]}|S} 1_S' \delta \right].$$

Define

$$(\rho')_{x_{1}(S_1, \ldots, S_t), \delta} := (\rho'x_{S_1 \cup \{0\}, S_1, \delta}) A^T_{S_1 \cap [k]} \otimes \cdots \otimes (\rho'x_{S_t \cup \{0\}, S_t, \delta}) A^T_{S_t \cap [k]} \otimes (\sigma^T_x \otimes (|0\rangle \langle 0|)^{C^t} \otimes |T\rangle),$$

$$\rho_{x_{1}(S_1, \ldots, S_t), \delta} := \rho_{x_{S_1 \cup \{0\}}} \otimes \cdots \otimes \rho_{x_{S_t \cup \{0\}}} \otimes \sigma^T_x.$$
\[ \rho_{s_{i}(S_{1}, \ldots , S_{l})}^{(X_{0})^{\dagger}A_{[k]}} := \sum_{x} q_{i}(S_{1}, \ldots , S_{l}) (x^{t})^{(X_{0})^{\dagger}A_{[k]} \otimes \rho_{x_{i}(S_{1}, \ldots , S_{l})}^{(X_{0})^{\dagger}A_{[k]}}} \]

Then,

\[ \text{Tr} \left[ (\hat{\Pi})^{(X_{0})^{\dagger}A_{[k]}} (\rho_{s_{i}(S_{1}, \ldots , S_{l})}^{(X_{0})^{\dagger}A_{[k]}}) \right] \leq \frac{1 - \alpha}{\alpha} \sum_{j=1}^{t} \max \left\{ 2^{-D_{B}^{\tau_{i}}(S_{1}, \ldots , S_{l})} (\rho_{j}^{(X_{0})^{\dagger}A_{[k]}} \| \rho_{s_{i}(S_{1}, \ldots , S_{l})}^{(X_{0})^{\dagger}A_{[k]}}) , \frac{3(2|\mathcal{H}|)^{k}}{\sqrt{|\mathcal{L}|}} \right\} . \]

**Proof.** For each \( x \in X_{c}, 1 \in \mathcal{L}_{c} \cup [k] \), construct the state \((\rho_{\tau_{i}}^{A_{[k]}^{[t]}})_{x,1,\delta}^{(X_{0})^{\dagger}A_{[k]}}\) and POVM element \((\Pi'(i))_{x,1,\delta}^{(X_{0})^{\dagger}A_{[k]}}\) as in Claim 1 of Lemma \[1\]. For each \( x^{t} \in (X_{c})^{t}, 1 \in \mathcal{L}_{c} \cup [k], i \in [t] \) define the POVM element

\[ (\Pi'(i))_{x^{t},1,\delta}^{(X_{0})^{\dagger}A_{[k]}} := (\Pi')_{x^{t}(i)[c],[1],\delta}^{(X_{0})^{\dagger}A_{[k]}} . \]

By Fact \[2\] there exists an orthogonal projector \((\hat{\Pi}(i))_{x^{t},1,\delta}^{(X_{0})^{\dagger}A_{[k]} \otimes (C^{2})^{[k]}c} \) such that

\[ \text{Tr} [ (\hat{\Pi}(i))_{x^{t},1,\delta}^{(X_{0})^{\dagger}A_{[k]} \otimes (C^{2})^{[k]}c} (\tau_{A_{[k]}^{[t]}} \otimes (0,1))^{(C^{2})^{[k]}c} ] = \text{Tr} [ (\Pi'(i))_{x^{t},1,\delta}^{(X_{0})^{\dagger}A_{[k]}} \tau_{A_{[k]}^{[t]}} ] \]

for all states \( \tau_{A_{[k]}^{[t]}} \). The projector \((\hat{\Pi}(i))_{x^{t},1,\delta}^{(X_{0})^{\dagger}A_{[k]} \otimes (C^{2})^{[k]}c} \), \( i \in [t] \) using Proposition \[2\]. This settles Claim 1 of the theorem.

Claims 2 and 3 of the theorem follow from the corresponding Claims 2 and 3 of Lemma \[1\].

Claim 4 of the theorem follows from Claim 4 of Lemma \[1\] and Proposition \[2\], combined with the observation that averaging over \( x \in X_{c}, 1 \in \mathcal{L}_{c} \cup [k] \) does not affect the \((S_{1}, \ldots , S_{l}) \) ‘structure’ of the state \((\rho_{\tau_{i}}^{A_{[k]}^{[t]}})_{x,1,\delta}^{(X_{0})^{\dagger}A_{[k]}}\) \[Q.E.D.\]

The above theorem is a generalisation of Fact \[1\] to the classical quantum setting involving ‘union of intersection of POVM elements’. However, it has the shortcoming that it can only handle classical quantum ‘\( q(\cdot) \)’ states corresponding to pseudosubpartitions of \( c \cup [k] \), unlike Fact \[1\] which can handle any classical ‘\( q(\cdot) \)’ state. Overcoming this shortcoming remains an important open problem.

In both Lemma \[1\] and Theorem \[1\] in the important special case of \( k = 1 \), it is not necessary to augment the register \( A_{1} \) with \( \mathcal{L} \). This is because a pseudosubpartition of \( c \cup [1] \) consists of only one subset which must contain the register \( A_{1} \). This simplifies the proofs of Lemma \[1\] and Theorem \[1\]. In fact, we can prove the following two joint typicality lemmas which are the ones most used in applications to network information theory. These lemmas are stated in a form that allows the ‘negative hypotheses’ to be an arbitrary fixed quantum state in some cases instead of just the marginals.

**Corollary 3** (Useful joint typicality lemma, intersection case). Let \( \mathcal{H}, \mathcal{L} \) be Hilbert spaces and \( X \) be a finite set. We will also use \( X \) to denote the Hilbert space with computational basis elements indexed by the set \( X \). Let \( c \) be a non-negative integer. Let \( A \) denote a quantum register with Hilbert space \( \mathcal{H} \). For every \( x \in X^{c} \), let \( \rho_{x} \) be a quantum state in \( A \). Consider the extended quantum system

\[ A' := (\mathcal{H} \otimes C^{2}) \oplus \bigoplus_{S: \{S\} \neq \emptyset \subseteq [c]} (\mathcal{H} \otimes C^{2}) \otimes \mathcal{L}^{\otimes |S|} . \]
Also define the augmented classical system \( \mathcal{X}' := \mathcal{X} \otimes \mathcal{L} \).

Below, \( \mathbf{x}, \mathbf{l} \) denote computational basis vectors of \( \mathcal{X}^{[c]} \), \( \mathcal{L}^{[c]} \). Let \( p(\cdot) \) be a probability distribution on the vectors \( \mathbf{x} \). Define the classical quantum state

\[
\rho_{\mathcal{X}^{[c]} A}^{c} := \sum_{\mathbf{x}} p(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}|_{\mathcal{X}^{[c]}} \otimes \rho_{\mathbf{x}}^{A}.
\]

Let \( \frac{\mathcal{L}^{[c]}}{|\mathcal{L}|} \) denote the completely mixed state on \( c \) tensor copies of \( \mathcal{L} \). View \( \rho_{\mathbf{x}}^{A} \otimes (|0\rangle \langle 0|)^{C^{2}} \) as a state in \( A' \) under the natural embedding viz. the embedding is into the first summand of \( A' \) defined above. Similarly, view \( \rho_{\mathcal{X}^{[c]} A}^{c} \otimes (|0\rangle \langle 0|)^{C^{2}} \otimes \frac{\mathcal{L}^{[c]}}{|\mathcal{L}|} \) as a state in \( \mathcal{X}_{[c]} A' \) under the natural embedding.

Let \( 0 \leq \varepsilon, \delta \leq 1 \). Choose \( \mathcal{L} \) to have dimension \( |\mathcal{L}| = \frac{3\beta^{4}}{2^{\beta} 1^{\varepsilon}} \). Then, there is a state \( \rho' \) and a POVM element \( \Pi' \) in \( \mathcal{X}_{[c]} A' \) such that:

1. The state \( \rho' \) and POVM element \( \Pi' \) are classical on \( \mathcal{X}^{[c]} \otimes \mathcal{L}^{[c]} \) and quantum on \( A' \). More precisely, \( \rho' \), \( \Pi' \) can be expressed as

\[
(\rho')_{\mathcal{X}^{[c]} A}^{[c]} = |\mathcal{L}|^{-c} \sum_{\mathbf{x}, \mathbf{l}} p(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}|_{\mathcal{X}^{[c]}} \otimes |\mathbf{l}\rangle \langle \mathbf{l}|_{\mathcal{L}^{[c]}} \otimes (\rho')_{\mathbf{x}, \mathbf{l}}^{A'},
\]

\[
(\Pi')_{\mathcal{X}^{[c]} A}^{[c]} = \sum_{\mathbf{x}, \mathbf{l}} |\mathbf{x}\rangle \langle \mathbf{x}|_{\mathcal{X}^{[c]}} \otimes |\mathbf{l}\rangle \langle \mathbf{l}|_{\mathcal{L}^{[c]}} \otimes (\Pi')_{\mathbf{x}, \mathbf{l}}^{A'},
\]

where \( (\rho')_{\mathbf{x}, \mathbf{l}}^{A'} \), \( (\Pi')_{\mathbf{x}, \mathbf{l}}^{A'} \) are quantum states and POVM elements respectively for all computational basis vectors \( \mathbf{x} \in \mathcal{X}^{[c]}, \mathbf{l} \in \mathcal{L}^{[c]} \);

2. \[
\left\| (\rho')_{\mathcal{X}^{[c]} A}^{[c]} - \rho_{\mathcal{X}^{[c]} A}^{c} \otimes (|0\rangle \langle 0|)^{C^{2}} \right\|_{1} \leq 2^{\varepsilon+1} \delta;
\]

3. \[
\text{Tr} \left[ ((\Pi')_{\mathcal{X}^{[c]} A}^{[c]} (\rho')_{\mathcal{X}^{[c]} A}^{[c]} \right) \geq 1 - \delta^{-2} 2^{2\varepsilon+5} 3\varepsilon - 2^{\varepsilon+1} \delta;
\]

4. Let \( S \subseteq [c] \). Let \( \mathbf{x}_{S}, \mathbf{l}_{S} \) be computational basis vectors in \( \mathcal{X}^{\otimes S}, \mathcal{L}^{\otimes S} \). In the following definition, let \( \mathbf{x}'_{S}, \mathbf{y}'_{S} \) range over all computational basis vectors of \( \mathcal{X}^{\otimes ([c] \setminus S)}, \mathcal{L}^{\otimes ([c] \setminus S)} \). Define a state in \( A' \);

\[
(\rho')_{\mathbf{x}_{S}, \mathbf{l}_{S}}^{A', S} := |\mathcal{L}|^{-|S|} \sum_{\mathbf{x}'_{S}, \mathbf{y}'_{S}} p(\mathbf{x}'_{S} | \mathbf{x}_{S}) (\rho')_{\mathbf{x}_{S}, \mathbf{x}'_{S}}^{A'} \mathbf{l}_{S}, \mathbf{y}'_{S}, \delta.
\]

Analogously define

\[
\rho_{\mathbf{x}_{S}}^{A} := \sum_{\mathbf{x}'_{S}} p(\mathbf{x}'_{S} | \mathbf{x}_{S}) \rho_{\mathbf{x}_{S}, \mathbf{x}'_{S}}^{A'}.
\]

Let \( (S_{1}, S_{2}, S_{3}) \vdash [c] \). Let \( \sigma^{A} \) be a fixed quantum state in \( A \). If \( S_{1}, S_{3} = \emptyset \) and \( S_{2} = [c] \), define

\[
(\rho')_{(\emptyset, [c], \emptyset)}^{\mathcal{X}^{[c]} A} := |\mathcal{L}|^{-c} \sum_{\mathbf{x}} p(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}|_{\mathcal{X}^{[c]}} \otimes |\mathbf{l}\rangle \langle \mathbf{l}|_{\mathcal{L}^{[c]}} \otimes (\sigma^{A} \otimes |0\rangle \langle 0|)^{C^{2}},
\]

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Let $\mathcal{H}, \mathcal{L}$ be Hilbert spaces and $\mathcal{X}$ be a finite set. We will also use $\mathcal{X}$ to denote the Hilbert space with computational basis elements indexed by the set $\mathcal{X}$. Let $c$ be a non-negative integer. Let $A$ denote a quantum register with Hilbert space $\mathcal{H}$. For every $x \in \mathcal{X}^c$, let $\rho_x$ be a quantum state in $A$. Let $t$ be a positive integer. Let $x^t$ denote a $t$-tuple of elements of $\mathcal{X}^c$; we shall denote its $i$th element by $x^t(i)$. Consider the extended quantum system $\hat{A}$ where $\hat{A} \cong A^t \otimes \mathbb{C}^2 \otimes \mathbb{C}^{t+1}$, and $A^t$ is defined as

$$A^t := (\mathcal{H} \otimes \mathbb{C}^2) \oplus \bigoplus_{S:t \subseteq S \subseteq [c] \cup \{t\}} (\mathcal{H} \otimes \mathbb{C}^2) \otimes \mathcal{L}^{\otimes |S|}.$$ 

Also define the augmented classical system $\hat{\mathcal{X}} := \mathcal{X} \otimes \mathcal{L}$.

Below, $x$, $1$ denote computational basis vectors of $\mathcal{X}^c$, $\mathcal{L}^{\otimes c}$. Let $p(\cdot)$ denote a probability distribution on the vectors $x$. Let $p(1; \cdot), \ldots, p(t; \cdot)$ denote probability distributions on $x^t$ such that the marginal of $p(i; x^t)$ on the $i$th element is $p(x^t(i))$. For $i \in [t]$, define the classical quantum states

$$\rho^{(\mathcal{X}^c)^t A}(i) := \sum_{x^t} p(i; x^t) |x^t\rangle \langle x^t|^{(\mathcal{X}^c)^t} \otimes \rho_{x^t(i)}^{A^t}.$$ 

Let $\frac{\mathcal{L}^{\otimes c}}{2^c}$ denote the completely mixed state on $c$ tensor copies of $\mathcal{L}$. View $\rho_X^A \otimes (|0\rangle\langle 0|)^{\mathcal{L}^2} \otimes (|0\rangle\langle 0|)^{\mathcal{L}^{t+1}}$ as a state in $\hat{A}$ under the natural embedding viz. the embedding is into the
first summand of $A'$ defined above tensored with $\mathbb{C}^2 \otimes \mathbb{C}^{t+1}$. Similarly, view $\rho^{(X_c)_i} A'(i) \otimes (|0\rangle \langle 0|)^{C^2} \otimes \frac{L \otimes (|0\rangle \langle 0|)^{C^2} \otimes (|0\rangle \langle 0|)^{C^{t+1}}}{2^{2(1-\epsilon)\epsilon}}$ as a state in $(\hat{X}_c)^{ct} \hat{A}$ under the natural embedding.

Let $0 \leq \alpha, \epsilon, \delta \leq 1$. Choose $L$ to have dimension $|L| = \frac{3^{1+\|H\|}}{2^{2(1-\epsilon)\epsilon}}$. Then, there are states $\rho'(1), \ldots, \rho'(t)$ and a POVM element $\hat{\Pi}$ in $(\hat{X}_c)^{ct} \hat{A}$ such that:

1. The states $\rho'(1), \ldots, \rho'(t)$ and POVM element $\hat{\Pi}$ are classical on $X^{\otimes [c]} \otimes L^{[c]}$ and quantum on $\hat{A}$. More precisely, $\rho'(i)$, $i \in [t]$, $\hat{\Pi}$ can be expressed as

$$
\rho'(i) = |L|^{-ct} \sum_{x^t, l'^t} p(i; x^t)|x^t\rangle\langle x^t| (X_c)^{\otimes t} \otimes |l^t\rangle\langle l^t| (L_c)^{\otimes t} \\
\propto (\rho'(l'^t), i, \delta) \otimes (|0\rangle \langle 0|)^{C^2} \otimes (|0\rangle \langle 0|)^{C^{t+1}},
$$

$$
\hat{\Pi} = \sum_{x^t, l'^t} |x^t\rangle\langle x^t| (X_c)^{\otimes t} \otimes |l^t\rangle\langle l^t| (L_c)^{\otimes t} \otimes \hat{\Pi}_{x^t, l'^t, \delta},
$$

where $(\rho'(l'^t))_{x^t, l'^t, \delta}$ are quantum states for all computational basis vectors $x \in X^{\otimes [c]}$, $l \in L^{[c]}$ and $(\hat{\Pi})_{x^t, l'^t, \delta}$ are POVM elements for all computational basis vectors $x^t \in X^{\otimes [c]}$, $l' \in L^{[c]}$.

2. For all $i \in [t]$, 

$$
\left\| (\rho'(i)) (\hat{X}_c)^{\hat{A}} - (\rho(i)) (X_c)^{\hat{A}} \otimes (|0\rangle \langle 0|)^{C^2} \otimes \frac{L^{\otimes t}}{|L|^{ct}} \otimes (|0\rangle \langle 0|)^{C^2} \otimes (|0\rangle \langle 0|)^{C^{t+1}} \right\|_1 \leq 2^{\frac{c+1}{t}+1}\epsilon;
$$

3. For all $i \in [t]$, 

$$
\text{Tr} \left[ \left( \hat{\Pi} (\hat{X}_c)^{\hat{A}} (\rho'(i)) (\hat{X}_c)^{\hat{A}} \right) \right] \geq 1 - \delta - 2^{c+5} 3^{c+5} \epsilon - 2^{\frac{c+1}{t}+1}\epsilon - \alpha;
$$

4. Let $S \subseteq [c]$. Let $x_S$, $1_S$ be computational basis vectors in $X^{\otimes S}$, $L^{\otimes S}$. In the following definition, let $x'_S$, $y'_S$ range over all computational basis vectors of $X^{\otimes (c\setminus S)}$, $L^{\otimes (c\setminus S)}$. For $S \neq \emptyset$, define states in $A'$,

$$
\rho_{x_S}^{A'} := \sum_{x'_S, y'_S} p(x'_S|x_S) (\rho (x'_S, y'_S))_{x_S x'_S, y_S y'_S}.
$$

Analogously define

$$
\rho_{x_S}^{A} := \sum_{x'_S} p(x'_S|x_S) \rho_{x_S x'_S}.
$$

Let $\sigma^A$ be a fixed quantum state in $A$. For $S = \emptyset$, define

$$
(\rho')_{\emptyset} := \sigma^A \otimes |0\rangle \langle 0|^{C^2}, \quad \rho^{\emptyset} := \sigma^A.
$$

For $i \in [t], S \subseteq [c]$, let $q_{i,S}(\cdot)$ be a probability distribution on $x^t$. Define

$$
(\rho')_{i,S} := |L|^{-ct} \sum_{x^t} q_{i,S}(x^t)|x^t\rangle\langle x^t| (X_c^{\otimes [c]} \otimes |l^t\rangle\langle l^t| (L_c^{\otimes [c]} \otimes (\rho')_{x^t(i), l^t(i), \delta}^{A'}.
$$
Observe that the terms in the above summation have orthogonal range spaces. It can be estimated as follows:

\[
\rho_{l;S}^{(x_i)^TA} := \sum_{x^t} q_{i;S}(x^t) x^t(x^t)^{X^\otimes[c]} \otimes \rho_{x^t(i)S}^A.
\]

Then,

\[
\text{Tr} \left[ (\hat{\Pi})(\hat{x}_i)^A (\rho_{l;S}^{(x_i)^TA}) (\hat{x}_i)^A \right] \leq \frac{1 - \alpha}{\alpha} \sum_{j=1}^{t} 2^{-\frac{\alpha}{\alpha}} \| \rho_{l;S}^{(x_i)^TA} \|_{\rho_{x^t(i)S}^A}.
\]

5 Proof of Proposition 4

We prove Proposition 4 in this section. We first show how to construct the objects whose existence is promised by the proposition. We then proceed to prove its claims. All the while, we use the notation of the proposition: \( x, l \) will denote computational basis vectors of \( X^\otimes[c], L^\otimes([c] \cup [k]), 0 \leq \delta \leq 1 \), and \( 0 \leq \epsilon_{x, (S_1, \ldots, S_t)} \leq 1 \).

5.1 Constructing \((\rho_{l;S}^{A''})_{x, l, S, \delta}\)

Let \( S \subseteq [c] \cup [k], S \cap [k] \neq \{\} \). Define \( \hat{S} := ([c] \cup [k]) \setminus S \). Let \( I_S \) be a computational basis vector of \( L^\otimes S \). We define an isometric embedding \( T_{S, I_S} \) of \((H \otimes C^2)^\otimes(\hat{S} \cap [k])\) into \((H \otimes C^2) \otimes L^\otimes[S] \otimes (S \cap [k])\) as follows:

\[
(T_{S, I_S}(h, S \cap [k]))_x := (h_x, I_S)
\]

where \( s \in S \cap [k] \), \( h_{S \cap [k]} \) is a computational basis vector of \((H \otimes C^2)^\otimes(S \cap [k])\), \( h_x \), \( (T_{S, I_S}(h_{S \cap [k]}))_x \) are the entries in the \( s \)th coordinate of \( h_{S \cap [k]} \), \( T_{S, I_S}(h_{S \cap [k]}) \). Observe that \( T_{S, I_S} \) maps computational basis vectors of \((H \otimes C^2)^\otimes(S \cap [k])\) to computational basis vectors of \((H \otimes C^2) \otimes L^\otimes[S] \otimes (S \cap [k])\). Thus, \( T_{S, I_S} \) is an isometric embedding of \((H \otimes C^2)^\otimes(S \cap [k])\) into \((H \otimes C^2) \otimes L^\otimes[S] \otimes (S \cap [k])\) which further embeds into \( A''_{S \cap [k]} \) in the natural fashion. Let \( 1^{(H \otimes C^2)^\otimes(S \cap [k])} \) denote the identity embedding of \((H \otimes C^2)^\otimes(S \cap [k])\) into \( A''_{S \cap [k]} \). Observe that the range spaces of \( T_{S, I_S} \otimes 1^{(H \otimes C^2)^\otimes(S \cap [k])} \), as \( S \) ranges over subsets of \([c] \cup [k]\) intersecting \([k]\) non-trivially, embed orthogonally into \( A''_{[k]} \) under the natural embedding. Also, for a fixed \( S \subseteq [c] \cup [k], S \cap [k] \neq \{\} \) the range spaces of \( T_{S, I_S} \), as \( I_S \) ranges over computational basis vectors of \( L^\otimes S \), embed orthogonally into \( A''_{S \cap [k]} \) under the natural embedding.

We now define an isometric embedding \( T_{S, I_S, \delta} \) of \((H \otimes C^2)^\otimes(S \cap [k])\) into \( A''_{S \cap [k]} \) as follows:

\[
T_{S, I_S, \delta} := \frac{1}{N(S, \delta)} \left( 1^{(H \otimes C^2)^\otimes(S \cap [k])} + \sum_{(S_1, \ldots, S_t) \neq S, I_S > 0} \delta^t T_{S_1, I_{S_1}} \otimes \cdots \otimes T_{S_t, I_{S_t}} \otimes 1^{(H \otimes C^2)^\otimes((S \cap [k]) \setminus (S_1 \cup \cdots \cup S_t))} \right),
\]

where the normalisation factor \( N(S, \delta) \) is put in to make the embedding preserve length of vectors. Observe that \( N(S, \delta) \) is independent of the vector in \((H \otimes C^2)^\otimes(S \cap [k])\) on which \( T_{S, I_S, \delta} \) acts since the terms in the above summation have orthogonal range spaces. It can be estimated as follows:

\[
N(S, \delta) := 1 + \sum_{(S_1, \ldots, S_t) \neq S, I_S > 0} \delta^t \leq 1 + \sum_{l=1}^{\frac{|S \cap [k]|}{1}} \delta^t \frac{2^{|S \cap [k]|}(l + 1)|S \cap [k]|}{l} \leq 1 + \sum_{l=1}^{\frac{|S \cap [k]|}{1}} \frac{2^{|S \cap [k]|} |S|}{l} < e^{2^{|S|}}. \tag{5}
\]
It is clear that \( N(S, \delta) \geq 1 \) and \( N(T_1, \delta) \cdots N(T_m, \delta) \leq N(S, \delta) \) if \((T_1, \ldots, T_m) \vdash S\).

The map \( T_{S, l, 1, \delta} \) extends to subspaces and density matrices in \((H \otimes \mathbb{C}^2)^{\otimes (S \cap [k])}\) in the natural way. We now define the quantum state

\[
(\rho')_{x,1,\delta}^{A'[k]} := T_{[c] \cap [k], 1, \delta} \left( \rho_x^{A[k]} \otimes (|0\rangle \langle 0|)^{\otimes k} \right)
\]

in \(A_1' \cdots A_k'\), where \(\rho_x^{A[k]} \otimes (|0\rangle \langle 0|)^{\otimes k}\) can be thought of as a density matrix in \((H \otimes \mathbb{C}^2)^{\otimes [k]}\) in the natural fashion.

### 5.2 Constructing \((\Pi')_{x,1,\delta}^{A'[k]}\)

For each \(x, (S_1, \ldots, S_l) \vdash [c] \cup [k], l > 0\), let \(\Pi'_{x,(S_1,\ldots,S_l)}\) be the POVM element in \(A_1 \cdots A_k\) with the property that

\[
\begin{align*}
\text{Tr} \left[ (\Pi'_{x,(S_1,\ldots,S_l)})^{A[k]} \right] &\geq 1 - \epsilon_x(S_1,\ldots,S_l), \\
\text{Tr} \left[ (\Pi'_{x,(S_1,\ldots,S_l)})^{A[k]} \right] &\leq 2^{-D^{x}([S_1,\ldots,S_l])} \epsilon_x(S_1,\ldots,S_l).
\end{align*}
\]

By Fact 2 there exists an orthogonal projection \(\Pi_{x,(S_1,\ldots,S_l)}\) in \((A_1 \cdots A_k) \otimes (\mathbb{C}^2)^{\otimes k} \cong (H \otimes \mathbb{C}^2)^{\otimes [k]}\) such that

\[
\begin{align*}
\text{Tr} \left[ (\Pi_{x,(S_1,\ldots,S_l)})^{A[k]} \otimes (|0\rangle \langle 0|)^{\otimes k} \right] &\geq 1 - \epsilon_x(S_1,\ldots,S_l), \\
\text{Tr} \left[ (\Pi_{x,(S_1,\ldots,S_l)})^{A[k]} \right] &\leq 2^{-D^{x}([S_1,\ldots,S_l])} \epsilon_x(S_1,\ldots,S_l).
\end{align*}
\]

Let \(Y_{x,(S_1,\ldots,S_l)}\) denote the orthogonal complement of the support of \(\Pi_{x,(S_1,\ldots,S_l)}\) in \((H \otimes \mathbb{C}^2)^{\otimes [k]}\).

Identify \((H \otimes \mathbb{C}^2)^{\otimes [k]}\) with the Hilbert space \(H\) of Proposition 3. Arrange all the non-empty pseudosubpartitions \((T_1, \ldots, T_m) \vdash [c] \cup [k], m > 0\) into a linear order extending the refinement partial order \(\preceq\). Define the tilting matrix \(A\), whose rows and columns are indexed by non-empty pseudosubpartitions of \([c] \cup [k]\) that intersect \([k]\) non-trivially, as follows:

\[
A_{(S_1,\ldots,S_l),(T_1,\ldots,T_m)} := \frac{\delta^{2l}}{N(T_1, \delta) \cdots N(T_m, \delta)} \quad \text{if} \quad (S_1, \ldots, S_l) \preceq (T_1, \ldots, T_m), \quad \text{otherwise.}
\]

Observe that \(A\) is upper triangular, diagonal dominated and substochastic. The diagonal dominated property of \(A\) follows from the fact that \(N(T_1, \delta) \cdots N(T_m, \delta) \leq N(W_1, \delta) \cdots N(W_n, \delta)\) if \((T_1, \ldots, T_m) \preceq (W_1, \ldots, W_n)\). The reason \(A\) is substochastic in general, and not stochastic, is because the empty pseudosubpartition is not included amongst the rows and columns of \(A\). More precisely,

\[
\sum_{(S_1,\ldots,S_l),(S_1,\ldots,S_l)\preceq(T_1,\ldots,T_m),l>0} \delta^{2l} = N(T_1, \delta) \cdots N(T_m, \delta) - 1.
\]

For \((S_1, \ldots, S_l) \vdash [c] \cup [k], l > 0\) define an isometric embedding \(T_{(S_1,\ldots,S_l),1,\delta}\) of \((H \otimes \mathbb{C}^2)^{\otimes [k]}\) into \(A'[k]\) as follows:

\[
T_{(S_1,\ldots,S_l),1,\delta} := T_{S_1,1,\delta} \otimes \cdots \otimes T_{S_l,1,\delta} \otimes I^{(H \otimes \mathbb{C}^2)^{\otimes [k]\setminus [S_1 \cup \cdots \cup S_l]}}.
\]
Observe that the $A$-tilt of $Y_{x,(S_1,...,S_l)}$ along the $(S_1,...,S_l)$th direction is nothing but the action of the isometric embedding $\mathcal{T}_{(S_1,...,S_l)},1,\delta$:
\[
Y'_{x,(S_1,...,S_l),1,\delta} := (\mathcal{T})_{(S_1,...,S_l),A}(Y_{x,(S_1,...,S_l)}) = \mathcal{T}_{(S_1,...,S_l),1,\delta}(Y_{x,(S_1,...,S_l)}).
\]

Define the $A$-tilted span
\[
Y'_{x,1,\delta} := (Y'_{x,1})_A = \sum_{(S_1,...,S_l)=+[c] \cup [k], l>0} Y'_{(S_1,...,S_l),1,\delta}.
\]

View $Y'_{x,1,\delta}$ as a subspace of $A''_1 \ldots A''_k$. Let $(\Pi')_{Y''_{x,1,\delta}}$ denote the orthogonal projection in $A''_1 \ldots A''_k$ onto $Y'_{x,1,\delta}$.

Let $(\Pi')_{(\mathcal{H} \otimes \mathbb{C}^2)^{\otimes [k]}}$ denote the orthogonal projection in $A''_1 \ldots A''_k$ onto $(\mathcal{H} \otimes \mathbb{C}^2)^{\otimes [k]}$. We finally define the POVM element $(\Pi')_{x,1,\delta}$ in $A''_1 \ldots A''_k$ to be
\[
(\Pi')_{x,1,\delta} := (\mathbb{1}^{A''_1} - (\Pi')_{Y''_{x,1,\delta}})\cdot ((\Pi')_{(\mathcal{H} \otimes \mathbb{C}^2)^{\otimes [k]}}(\mathbb{1}^{A''_k} - (\Pi')_{Y''_{x,1,\delta}}))
\]

5.3 Constructing $\alpha_{(S_1,...,S_l),\delta}, \beta_{(S_1,...,S_l),\delta}, \mathcal{T}_{(S_1,...,S_l),1,\delta}$

Recall that the isometric embedding $\mathcal{T}_{(S_1,...,S_l),1,\delta}$ of $(\mathcal{H} \otimes \mathbb{C}^2)^{\otimes [k]}$ into $A''_{[k]}$ has already been defined in Equation 9 above. Also define
\[
\alpha_{(S_1,...,S_l),\delta} := \frac{N(S_1,\delta)N(\bar{S}_1 \cup [c],\delta)}{N([c] \cup [k],\delta)} \ldots \frac{N(S_l,\delta)N(\bar{S}_l \cup [c],\delta)}{N([c] \cup [k],\delta)},
\]
\[
\alpha_{(S_1,...,S_l),\delta} + \beta_{(S_1,...,S_l),\delta} := \frac{N(S_1 \cup [c],\delta)N(\bar{S}_1 \cup [c],\delta)}{N([c] \cup [k],\delta)} \ldots \frac{N(S_l \cup [c],\delta)N(\bar{S}_l \cup [c],\delta)}{N([c] \cup [k],\delta)}.
\]

Since $(S,\bar{S} \cup [c]) \leq (S \cup [c],\bar{S} \cup [c]) \vdash [c] \cup [k]$ for any $S \subseteq [c] \cup [k], \{\} \neq S \cap [k] \subset [k], N(S,\delta)N(\bar{S} \cup [c],\delta) \\ N([c] \cup [k],\delta)$, it follows that $0 \leq \alpha_{(S_1,...,S_l),\delta}, \beta_{(S_1,...,S_l),\delta} \leq 1$.

5.4 Constructing $M''_{(S_1,...,S_l),1,\delta}, N''_{(S_1,...,S_l),1,\delta}$

Let $S \subseteq [c] \cup [k], \{\} \neq S \cap [k] \subset [k]$.

Define $\bar{S} := ([c] \cup [k]) \setminus S$. For a subset $T \subseteq [c] \cup [k], T \cap [k] \neq \{\}$ we say that $T$ crosses $S$ if $T \cap S \cap [k] \neq \{\}$. We define a pseudosubpartition $(T_1,...,T_l) \vdash [c] \cup [k]$ to cross $S$ if there exists an $i \in [l]$ such that $T_i$ crosses $S$.

We use the notation $(T_1,...,T_l) \models_S S$ to denote that pseudosubpartition $(T_1,...,T_l)$ crosses $S$. We define the $S$-signature of a pseudosubpartition $(T_1,...,T_l)$ to be the pseudosubpartition $(T_1,...,T_{l'})$ where $l' \leq l$ and $T_{l},...,T_{l'}$ are the subsets that actually cross $S$. We shall denote pseudosubpartitions $(T_1,...,T_l) \vdash [c] \cup [k]$ where for all $i \in [l], T_i$ crosses $S$, by the notation $(T_1,...,T_l) \models_S S$. If pseudosubpartition $(T_1,...,T_l)$ has $S$-signature $(T_1,...,T_{l'})$, we shall denote it by $(T_1,...,T_l) \models_S (T_1,...,T_{l'})$. Observe that $(T_1,...,T_{l'}) \models_S (T_1,...,T_{l'})$. For $S_1,...,S_l$
From Equation (4) we observe that

\[ T_{[c] \cup [k], 1, \delta} = \sqrt{\frac{N(S \cup [c], \delta)N(S \cup [c], \delta)}{N([c] \cup [k], \delta)}} T_{S \cup [c], 1, S \cup [c], \delta} \otimes T_{S \cup [c], 1, S \cup [c], \delta} + \frac{1}{\sqrt{N([c] \cup [k], \delta)}} \sum_{(T_1, \ldots, T_l) \in S(T_1, \ldots, T_l) \cap S(T_1, \ldots, T_l)} \delta^l (T_{T_1, T_1} \otimes \cdots \otimes T_{T_l, T_l} \otimes 1(H \otimes \mathbb{C}^2 \otimes ([k] \setminus (T_1 \cup \cdots \cup T_l))) \right) \]  

(13)

We will denote the second term of the summation above by \( T_{\times S, 1, \delta} \). Observe that \( T_{\times S, 1, \delta} \) is a scaled isometric embedding of \( (H \otimes \mathbb{C}^2) \otimes [k] \) into \( A^\delta_{[k]} \).

Let \( |h\rangle \) be a unit length vector in \( (H \otimes \mathbb{C}^2) \otimes [k] \). Let the Schmidt decomposition of \( |h\rangle \) with respect to \( (S \cap [k], S \cap [k]) \) be

\[ |h\rangle (H \otimes \mathbb{C}^2) \otimes [k] = \sum_i \sqrt{p_i} |\alpha_i\rangle (H \otimes \mathbb{C}^2) \otimes |\beta_i\rangle (H \otimes \mathbb{C}^2) \otimes (S \cap [k]), \]

where \( \sum_i p_i = 1 \). Let the Schmidt decomposition of \( T_{\times S, 1, \delta} (|h\rangle) \) with respect to \( (S \cap [k], S \cap [k]) \) be

\[ (T_{\times S, 1, \delta} (|h\rangle))^A_{[k]} = \sum (T_1, \ldots, T_l) \in \times S (T_1, \ldots, T_l) \sum_j \sqrt{q_j(T_1, \ldots, T_l)} |\gamma_j(T_1, \ldots, T_l)\rangle A^\delta_{S \cap [k]} \otimes |\delta_j(T_1, \ldots, T_l)\rangle A^\delta_{S \cap [k]} \]  

(14)

where \( \sum (T_1, \ldots, T_l) \in \times S (T_1, \ldots, T_l) \sum_j q_j(T_1, \ldots, T_l) \gamma_j(T_1, \ldots, T_l) \) may be less than one reflecting the fact that the length of \( T_{\times S, 1, \delta} (|h\rangle) \) may be less than one. For a pseudosubpartition \( (T_1, \ldots, T_l) \equiv \times S \), the expression

\[ \sum_j \gamma_j(T_1, \ldots, T_l)^A_{S \cap [k]} \otimes |\delta_j(T_1, \ldots, T_l)\rangle A^\delta_{S \cap [k]} \]  

is the Schmidt decomposition of

\[ \frac{1}{\sqrt{N([c] \cup [k], \delta)}} \sum_{(T_1, \ldots, T_l) \in S(T_1, \ldots, T_l) \cap S(T_1, \ldots, T_l)} \delta^l ((T_{T_1, T_1} \otimes \cdots \otimes T_{T_l, T_l} \otimes 1(H \otimes \mathbb{C}^2 \otimes ([k] \setminus (T_1 \cup \cdots \cup T_l))) (|h\rangle)) A^\delta_{[k]} \]

with respect to \( (S \cap [k], S \cap [k]) \). We observe that the span of the vectors \( \{ |\gamma_j(T_1, \ldots, T_l)\rangle A^\delta_{S \cap [k]} \}_j \) is orthogonal to the span of the vectors \( \{ |\delta_j(S_1, \ldots, S_m)\rangle A^\delta_{S \cap [k]} \}_j \) for different pseudosubpartitions \( (T_1, \ldots, T_l), (S_1, \ldots, S_m) \equiv \times S \). Similarly, the span of \( \{ |\delta_j(S_1, \ldots, S_m)\rangle A^\delta_{S \cap [k]} \}_j \) is orthogonal to the span of \( \{ |\gamma_j(S_1, \ldots, S_m)\rangle A^\delta_{S \cap [k]} \}_j \). Thus, Equation (14) is indeed a valid Schmidt decomposition for \( T_{\times S, 1, \delta} (|h\rangle) \).

Now,

\[ (T_{[c] \cup [k], 1, \delta} (|h\rangle))^A_{[k]} = \sqrt{\frac{N(S \cup [c], \delta)N(S \cup [c], \delta)}{N([c] \cup [k], \delta)}} \sum_i \sqrt{p_i} (T_{S \cup [c], 1, S \cup [c], \delta} (|\alpha_i\rangle))^A_{S \cap [k]} \otimes (T_{S \cup [c], 1, S \cup [c], \delta} (|\beta_i\rangle))^A_{S \cap [k]} + \sum (T_1, \ldots, T_l) \in \times S (T_1, \ldots, T_l) \sum_j \sqrt{q_j(T_1, \ldots, T_l)} |\gamma_j(T_1, \ldots, T_l)\rangle A^\delta_{S \cap [k]} \otimes |\delta_j(T_1, \ldots, T_l)\rangle A^\delta_{S \cap [k]} \]

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Observe that this is a Schmidt decomposition of $\left(\sum_{i} T_{c[i]}1,\delta(\langle h|)\right)_{A''_{[k]}}$, that is, for all $i$, $(T_{1}, \ldots, T_{l}) \models x \times S$, $j(T_{1}, \ldots, T_{l})$, $T_{S\cup\{c\},1,\delta}(|\alpha_{i}\rangle) \perp |\gamma_{j(T_{1}, \ldots, T_{l})}\rangle$ and $T_{S\cup\{c\},1,\delta}(|\beta_{i}\rangle) \perp |\delta_{j(T_{1}, \ldots, T_{l})}\rangle$.

Thus,

$$
\text{Tr}_{A''_{S\cap[k]}} \left[ (\sum_{i} T_{c[i]}1,\delta(\langle h|)\right)_{A''_{[k]}}
$$

$$
= \frac{N(S \cup \{c\},\delta) N(S \cup \{c\},\delta)}{N([c] \cup [k],\delta)} \sum_{i} p_{i} \left( T_{S\cup\{c\},1,\delta}(|\alpha_{i}\rangle)_{A''_{S\cap[k]}} \right)
$$

$$
+ \sum_{(T_{1}, \ldots, T_{l}) \models x \times S} \sum_{j(T_{1}, \ldots, T_{l})} q_{j(T_{1}, \ldots, T_{l})} \langle \gamma_{j(T_{1}, \ldots, T_{l})} | \gamma_{j(T_{1}, \ldots, T_{l})} \rangle_{A''_{S\cap[k]}}
$$

$$
= \frac{N(S \cup \{c\},\delta) N(S \cup \{c\},\delta)}{N([c] \cup [k],\delta)} \left( T_{S\cup\{c\},1,\delta} \left( \text{Tr}_{(\mathcal{H} \otimes \mathbb{C}^{2}) \otimes S_{\cap[k]}}(\langle h|\rangle) \right) \right)_{A''_{S\cap[k]}}
$$

$$
+ \sum_{(T_{1}, \ldots, T_{l}) \models x \times S} \sum_{j(T_{1}, \ldots, T_{l})} q_{j(T_{1}, \ldots, T_{l})} \langle \gamma_{j(T_{1}, \ldots, T_{l})} | \gamma_{j(T_{1}, \ldots, T_{l})} \rangle_{A''_{S\cap[k]}}
$$

Express $\rho_{x}^{A''_{[k]}} \otimes (\langle 0|\langle 0|\otimes k$ in terms of its eigenbasis

$$
\rho_{x}^{A''_{[k]}} \otimes (\langle 0|\langle 0|\otimes k = \sum_{i} s_{i} (h(i)) \langle h(i)| (\mathcal{H} \otimes \mathbb{C}^{2}) \otimes k, (15)
$$

where $\sum_{i} s_{i} = 1$. Let $|\gamma_{j(T_{1}, \ldots, T_{l})}(i)\rangle$, $q_{j(T_{1}, \ldots, T_{l})}(i)$ be the appropriate vectors and coefficients for $|\langle h(i)|$ as defined in Equation 14. Then using Equation 6 we get

$$
\text{Tr}_{A''_{S\cap[k]}} \left[ (\rho_{x}^{A''_{[k]}} \right]
$$

$$
= \frac{N(S \cup \{c\},\delta) N(S \cup \{c\},\delta)}{N([c] \cup [k],\delta)} \left( T_{S\cup\{c\},1,\delta} \left( \text{Tr}_{A''_{S\cap[k]}} \left[ \rho_{x}^{A''_{[k]}} \otimes (\langle 0|\langle 0|\otimes k) \right] \right) \right)_{A''_{S\cap[k]}}
$$

$$
+ \sum_{i} s_{i} \sum_{(T_{1}, \ldots, T_{l}) \models x \times S} \sum_{j(T_{1}, \ldots, T_{l})} q_{j(T_{1}, \ldots, T_{l})}(i) \langle \gamma_{j(T_{1}, \ldots, T_{l})}(i) | \gamma_{j(T_{1}, \ldots, T_{l})}(i) \rangle_{A''_{S\cap[k]}}
$$

$$
= \frac{N(S \cup \{c\},\delta) N(S \cup \{c\},\delta)}{N([c] \cup [k],\delta)} \left( T_{S\cup\{c\},1,\delta} \left( \rho_{x}^{A''_{S\cap[k]}} \otimes (\langle 0|\langle 0|\otimes |S_{\cap[k]}\rangle) \right) \right)_{A''_{S\cap[k]}} + (M'')_{S,x,1,\delta}
$$

where $(M'')_{S,x,1,\delta}$ is defined to be the second term in the summation in the first equality above. Note that $(M'')_{S,x,1,\delta}$ is a positive semidefinite matrix with support orthogonal to the support of the first matrix in the summation above.

We say that a pseudosubpartition $(T_{1}, \ldots, T_{l}) \models S \cup \{c\}$ leaks out of $S$ if there exists an $i \in [l]$ such that $T_{i} \cap S \cap \{c\} \neq \emptyset$. We use the notation $(T_{1}, \ldots, T_{l}) \models_{\perp} S$ to denote that pseudosubpartition $(T_{1}, \ldots, T_{l})$ leaks out of $S$. Observe that

$$
T_{S\cup\{c\},1,\delta} = \sqrt{\frac{N(S,\delta)}{N(S \cup \{c\},\delta)}} T_{S,1,\delta}
$$

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We use $\mathcal{T}_{\|s\|S\cup[c],\delta}$ to denote the second term in the sum above. The map $\mathcal{T}_{\|s\|S\cup[c],\delta}$ is a scaled isometric embedding of $(\mathcal{H} \otimes \mathbb{C}^2)^{\otimes |S\cap[k]|}$ into $A^\nu_{S\cap[k]}$. Let $|h\rangle \in (\mathcal{H} \otimes \mathbb{C}^2)^{\otimes |S\cap[k]|}$ be a unit length vector. Define the Hermitian matrix
\[
(N''_{S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]} := (\mathcal{T}_{S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]} - \frac{N(S,\delta)}{N(S \cup [c],\delta)} (\mathcal{T}_{S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]}.
\]
Then,
\[
(N''_{S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]} = \sqrt{\frac{N(S,\delta)}{N(S \cup [c],\delta)}} (\mathcal{T}_{S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]} + (\mathcal{T}_{\|s\|S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]}.
\]
Looking at Equation 15 we define the Hermitian matrix
\[
(N''_{S\cup[c],\delta}|h\rangle \langle h|) A^\nu_{S\cap[k]} := \sum_i s_i (N''_{S\cup[c],\delta}|h(i)\rangle \langle h(i)|) A^\nu_{S\cap[k]}.
\]
Note that $(N''_{S\cup[c],\delta}) A^\nu_{S\cap[k]} = 0$ if $c = 0$. Thus,
\[
(\mathcal{T}_{S\cup[c],\delta}\rho_x A^\nu_{S\cap[k]} \otimes (|0\rangle \langle 0|^{\otimes |S\cap[k]|})) A^\nu_{S\cap[k]} = \frac{N(S,\delta)}{N(S \cup [c],\delta)} (\mathcal{T}_{S\cup[c],\delta}\rho_x A^\nu_{S\cap[k]} \otimes (|0\rangle \langle 0|^{\otimes |S\cap[k]|})) A^\nu_{S\cap[k]} + (N''_{S\cup[c],\delta}) A^\nu_{S\cap[k]},
\]
and
\[
\text{Tr} [A^\nu_{S\cap[k]}] = \frac{N(S,\delta)N(S \cup [c],\delta)}{N([c] \cup [k],\delta)} (\mathcal{T}_{S\cup[c],\delta}\rho_x A^\nu_{S\cap[k]} \otimes (|0\rangle \langle 0|^{\otimes |S\cap[k]|})) A^\nu_{S\cap[k]} + (N''_{S\cup[c],\delta}) A^\nu_{S\cap[k]}.
\]
Observe that $(M''_{S\cup[c],\delta}) A^\nu_{S\cap[k]}$ has support orthogonal to the sum of the first two terms in the above equation and $\text{Tr} [(M''_{S\cup[c],\delta}) A^\nu_{S\cap[k]}] = 1 - \frac{N(S\cup[c],\delta)|S\cup[c],\delta|}{N([c] \cup [k],\delta)}$. Also note that the sum of the first two terms is a positive semidefinite matrix.

Now define
\[
(M''_{S\cup[c],\delta}) A^\nu_{S\cap[k]} := |\mathcal{L}|^{-\frac{1}{2}} \sum_{x'\cup[c],S} p_{x}[S(x'[c],S)(M''_{S\cup[c],\delta}) A^\nu_{S\cap[k]}_{S\cup[c],\delta}] S_{S\cap[k]} x',\delta;
\]
\[
(N''_{S\cup[c],\delta}) A^\nu_{S\cap[k]} := |\mathcal{L}|^{-\frac{1}{2}} \frac{N(S\cup[c],\delta)N(S\cup[c],\delta)}{N([c] \cup [k],\delta)} \sum_{x'\cup[c],S} p_{x}[S(x'[c],S)(N''_{S\cup[c],\delta}) A^\nu_{S\cap[k]}_{S\cup[c],\delta}] S_{S\cap[k]} x',\delta.
\]

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Obviously, $(M')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta}$ is a positive semidefinite matrix and $(N')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta}$ is a Hermitian matrix. Moreover, $(N')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta} = 0$ if $c = 0$. Recalling the definition of $(\rho')^{A''_{S\cap[k]}}_{x_{S\cap[r]};1S,\delta}$ in Claim 5 of Proposition [4], we see that

$$(\rho')^{A''_{S\cap[k]}}_{x_{S\cap[r]};1S,\delta} = \frac{N(S,\delta)N(S \cup [c],\delta)}{N([c] \cup [k],\delta)} (\mathcal{T}_{S,1S,\delta}^{A''_{S\cap[r]}(\rho_{x_{S\cap[r]}} \otimes (|0\rangle\langle0|^{\mathbb{C}^2}) \otimes |S\cap[k]\rangle\langle S\cap[k]|)))^{A''_{S\cap[k]}} + (N')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta} + (M')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta}.$$  

Observe that $(M')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta}$ has support orthogonal to that of the sum of the first two terms in the above equation and $\text{Tr} [(M')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta} + (N')^{A''_{S\cap[k]}}_{S,x_{S\cap[r]};1S,\delta}] = 1 - \frac{N(S|\cup[c],\delta)N(S,\delta)}{N([c] \cup [k],\delta)}$. Also note that the sum of the first two terms is a positive semidefinite matrix.

Now recalling the definition of $(\rho')^{A''_{S\cap[k]}}_{x_{1,1(1\ldots,1)},\delta}$ from Claim 5 of Proposition [4], we see that

$$(\rho')^{A''_{S\cap[k]}}_{x_{1,1(1\ldots,1)},\delta} = \left(\frac{N(S,\delta)N(S,\cup[c],\delta)}{N([c] \cup [k],\delta)} \cdots \frac{N(S,\delta)N(S,\cup[c],\delta)}{N([c] \cup [k],\delta)} \mathcal{T}_{S,1S,\delta}^{A''_{S\cap[r]}(\rho_{x_{S\cap[r]}} \otimes (|0\rangle\langle0|^{\mathbb{C}^2}) \otimes |S\cap[k]\rangle\langle S\cap[k]|))}^{A''_{S\cap[k]}} \cdots \right) + \text{Other Terms I + Other Terms II}$$

$$= \alpha_{S_1,\ldots,S_l,\delta}(\mathcal{T}_{S_1,\ldots,S_l,1\delta}^{\alpha_{[1]}(\rho_{x_{S_1,\ldots,S_l}} \otimes (|0\rangle\langle0|^{\mathbb{C}^2}) \otimes |S_1\cap[k]\rangle\langle S_1\cap[k]|))}^{A''_{[1]}} + \text{Other Terms I + Other Terms II}.$$  

Above, the notation “Other Terms II” denotes a $(2' - 1)$-fold sum of tensor products of $(l + 1)$ matrices where one multiplicand is $\sigma_{x_{S\cap[k]}(S_1,\ldots,S_l \cup [-c] \cup [k])}$, at least one multiplicand is of the form $(M')^{A''_{S\cap[r]}}_{S,x_{S\cap[r]};1S,\delta}$, and the remaining multiplicands are of the form

$$\frac{N(S_j,\delta)N(S_j \cup [c],\delta)}{N([c] \cup [k],\delta)} (\mathcal{T}_{S_j,1S_j,\delta}^{A''_{S\cap[r]}(\rho_{x_{S\cap[r]}} \otimes (|0\rangle\langle0|^{\mathbb{C}^2}) \otimes |S_j\cap[k]\rangle\langle S_j\cap[k]|))}^{A''_{S\cap[r]} + (N')^{A''_{S\cap[r]}}_{S,x_{S\cap[r]};1S_j,\delta}$$

having $\ell_1$-norm at most one. We use $(M')^{A''_{[1]}}_{S_1,\ldots,S_l,x_{1,\delta}}$ to denote the “Other Terms II”. It is clear that $(M')^{A''_{[1]}}_{S_1,\ldots,S_l,x_{1,\delta}}$ is a positive semidefinite matrix with trace $1 - \alpha_{S_1,\ldots,S_l,\delta} - \beta_{S_1,\ldots,S_l,\delta}$. Define

$$M^{A''_{[1]}}_{S_1,\ldots,S_l,x_{1,\delta}} := \frac{(M')^{A''_{[1]}}_{S_1,\ldots,S_l,x_{1,\delta}}}{1 - \alpha_{S_1,\ldots,S_l,\delta} - \beta_{S_1,\ldots,S_l,\delta}}$$

It is now clear that $M^{A''_{[1]}}_{S_1,\ldots,S_l,x_{1,\delta}}$ is a positive semidefinite matrix with unit trace with support orthogonal to that of the sum of the first two terms in Equation [13]. The notation “Other Terms
\( \Gamma \) denotes a \((2^l - 1)\)-fold sum of tensor products of \((l + 1)\) matrices where one multiplicand is 
\[
\sigma_x^{A[k]}(S_{T,\ldots,S_T}) \otimes (|0\rangle\langle 0|)^{C^2} \otimes |\text{[S]}\rangle\langle \text{[S]}|),
\]
and one multiplicand is of the form 
\[
(N^\prime)_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]},
\]
and the remaining multiplicands are of the form 
\[
\frac{N(S_{T,\ldots,S_T},\delta)N(S_{T,\ldots,S_T},\delta)}{N([c] \cup [k], \delta)} (T_{S_{T,\ldots,S_T},i,1,\delta}^{\sigma_x^{A[k]}(S_{T,\ldots,S_T}) \otimes (|0\rangle\langle 0|)^{C^2} \otimes |\text{[S]}\rangle\langle \text{[S]}|)} A''_{S_{T,\ldots,S_T},i,1,\delta}
\]
with \(\ell_1\)-norm at most one. We use \((N^\prime)_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}\) to denote the “Other Terms \(\Gamma\)”. It is clear that 
\[
(N^\prime)_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}
\]
is a Hermitian matrix with trace \(\beta_{S_{T,\ldots,S_T},x,i,1,\delta}\). Define 
\[
N_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]} := \begin{cases} 
\frac{(N^\prime)_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}}{|A''[k]|} & \text{if } \beta_{S_{T,\ldots,S_T},x,i,1,\delta} = 0 \\
0 & \text{otherwise} 
\end{cases}
\]
It is now clear that \(N_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}\) is a Hermitian matrix with unit trace. Also, \(\beta_{S_{T,\ldots,S_T},x,i,1,\delta} = 0\) if \(c = 0\).

Thus, 
\[
(\rho')_{x,1}(S_{T,\ldots,S_T},x,i,1,\delta) = \alpha_{S_{T,\ldots,S_T},x,i,1,\delta}(T_{S_{T,\ldots,S_T},x,i,1,\delta}^{A[k]} \otimes (|0\rangle\langle 0|)^{C^2} \otimes |\text{[S]}\rangle\langle \text{[S]}|)) A''_{S_{T,\ldots,S_T},x,i,1,\delta} + \beta_{S_{T,\ldots,S_T},x,i,1,\delta} N_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]} + (1 - \alpha_{S_{T,\ldots,S_T},x,i,1,\delta} - \beta_{S_{T,\ldots,S_T},x,i,1,\delta}) M_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}.
\]

### 5.5 Proving \(\|\Pi_{x,i,1,\delta}^{A''[k]}\| \leq (2|\mathcal{H}|)^k\)

Using Equation [12] we have 
\[
\|\Pi_{x,i,1,\delta}^{A''[k]}\|_1 \leq \left\| (\Pi_{x,i,1,\delta}^{A'[k]} - \Pi_{x,i,1,\delta}^{A''[k]}) \right\|_\infty \left\| (\Pi_{x,i,1,\delta}^{A'[k]} - \Pi_{x,i,1,\delta}^{A''[k]}) \right\|_1 \leq (|\mathcal{H} \otimes \mathbb{C}^2|)^{\otimes k} = (2|\mathcal{H}|)^k,
\]
where we use the fact that 
\[
\left\| (\Pi_{x,i,1,\delta}^{A'[k]} - \Pi_{x,i,1,\delta}^{A''[k]}) \right\|_\infty \leq 1
\]
as \(\Pi_{x,i,1,\delta}^{A''[k]}\) is the orthogonal projection onto the complement of the subspace \(\mathbb{Y}_{x,i,1,\delta}\).

### 5.6 Proving \(\|M_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}\|_\infty \leq \frac{1}{|T|} \beta_{S_{T,\ldots,S_T},x,i,1,\delta} \|N_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[k]}\|_\infty \leq \frac{3}{|\mathcal{L}|^2}\)

Let \(S \subseteq [c] \cup [k]\), \(\emptyset \neq S \cap [k] \neq [k]\). Let \(|h_x(i)\rangle \in (\mathcal{H} \otimes \mathbb{C}^2)^{\otimes k}\) be the \(i\)th eigenvector of 
\[
\rho_x^{A[k]} \otimes (|0\rangle\langle 0|)^{C^2} \otimes |\text{[S]}\rangle\langle \text{[S]}|).
\]
Fix a partition \((T_1, \ldots, T_r) =_{\times 1,\delta} S\) and an index \(j_{(T_1, \ldots, T_r)}\) in the Schmidt decomposition of \(T_{x,S_{T,\ldots,S_T},x,i,1,\delta}(h_x(i))\) with respect to \((S \cap [k], S \cap [k])\) given in Equation [14]. Define 
\[
(M^\prime)_{S_{T,\ldots,S_T},x,i,1,\delta}^{A''[S_{T,\ldots,S_T}]}(i) := |\mathcal{L}|^{-|S|} \sum_{I_{S_{T,\ldots,S_T}}} |\gamma_{j_{(T_1, \ldots, T_r)}}(I_{S_{T,\ldots,S_T}})\rangle \langle \gamma_{j_{(T_1, \ldots, T_r)}}(I_{S_{T,\ldots,S_T}})|^{A''[S_{T,\ldots,S_T}]}(i),
\]

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where we write $|\gamma_j(t_1, \ldots, t_{\nu}) \cdot x \cdot \mathbf{i}_S \mathbf{i}_S'(i)|$ in order to emphasise the dependence on $x$ and $1 := \mathbf{i}_S \mathbf{i}_S'$. From the definition of $(M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'}$ in Equation 18 it is easy to see that proving

$$\left\| (M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'}(t_1, \ldots, t_{\nu}) \right\|_\infty \leq \frac{1}{|\mathcal{L}|}$$

for all subpartitions $(t_1, \ldots, t_{\nu}) \vDash_{\times} S$, indices $j(t_1, \ldots, t_{\nu})$ and $i$ suffices to show that

$$\left\| (M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'} \right\|_\infty \leq \frac{1}{|\mathcal{L}|} \text{Tr} \left( (M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'} \right).$$

Now it is easy to see using Equations 19, 20 that this implies that

$$\left\| (M')_{(S_1, \ldots, S_l), x, \mathbf{i}_S, \mathbf{i}_S'} \right\|_\infty \leq \frac{1}{|\mathcal{L}|} \text{Tr} \left( (M')_{(S_1, \ldots, S_l), x, \mathbf{i}_S, \mathbf{i}_S'} \right) \implies \left\| M_{(S_1, \ldots, S_l), x, \mathbf{i}_S, \mathbf{i}_S'} \right\|_\infty \leq \frac{1}{|\mathcal{L}|}.$$

It only remains to show

$$\left\| (M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'}(t_1, \ldots, t_{\nu}) \right\|_\infty \leq \frac{1}{|\mathcal{L}|}$$

for any subpartition $(t_1, \ldots, t_{\nu}) \vDash_{\times} S$, index $j(t_1, \ldots, t_{\nu})$ and $i$. In fact, we will prove the stronger statement that

$$\left\| (M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'}(t_1, \ldots, t_{\nu}) \right\|_\infty \leq \frac{1}{|\mathcal{L}|(|T_{\nu} \cup \cdots \cup \mathcal{T}_{\nu}) \cap S|}.$$

Since $(T_1 \cup \cdots \cup T_{\nu}) \cap \bar{S} \neq \{\}$, this would complete the proof of the first part of Claim 2 of Proposition 4.

By triangle inequality, we have

$$\left\| (M')_{S, x, \mathbf{i}_S, \mathbf{i}_S'}(t_1, \ldots, t_{\nu}) \right\|_\infty \leq |\mathcal{L}|^{-|\bar{S} \cap (T_{\nu} \cup \cdots \cup \mathcal{T}_{\nu})|} \frac{|\mathcal{L}|^{-|\bar{S} \cap (T_{\nu} \cup \cdots \cup \mathcal{T}_{\nu})|}}{2} \sum_{l : (T_1 \cup \cdots \cup T_{\nu})} \sum_{v : (T_1 \cup \cdots \cup T_{\nu})} |\gamma_j(t_1, \ldots, t_{\nu}) \cdot x \cdot \mathbf{i}_S \mathbf{i}_S'(i)| \gamma_j(t_1, \ldots, t_{\nu}) \cdot v \mathbf{i}_S \mathbf{i}_S'(i) \mathbf{i}_S \mathbf{i}_S'(i),$$

where we use the fact that

$$\left| \gamma_j(t_1, \ldots, t_{\nu}) \cdot x \cdot \mathbf{i}_S \mathbf{i}_S'(i) \right| \leq \left| \gamma_j(t_1, \ldots, t_{\nu}) \cdot x \cdot \mathbf{i}_S \mathbf{i}_S'(i) \right| \gamma_j(t_1, \ldots, t_{\nu}) \cdot x \cdot \mathbf{i}_S \mathbf{i}_S'(i) \mathbf{i}_S \mathbf{i}_S'(i),$$

for $\mathbf{i}_S \mathbf{i}_S'(i) \neq \mathbf{i}_S \mathbf{i}_S'(i)$ in the equality above. This follows from the observation that for distinct computational basis vectors $\mathbf{i}_S \mathbf{i}_S'(i)$, $\mathbf{i}_S \mathbf{i}_S'(i)$, there exists an $i \in [l']$, a coordinate $a \in S \cap T_i$ such that $\mathbf{i}_a \neq \mathbf{i}_a$ which implies that

$$\left| \gamma_j(t_1, \ldots, t_{\nu}) \cdot x \cdot \mathbf{i}_S \mathbf{i}_S'(i) \right| \mathbf{i}_S \mathbf{i}_S'(i) \mathbf{i}_S \mathbf{i}_S'(i).$$
lie in the orthogonal subspaces 

\((\mathcal{H} \otimes \mathbb{C}^2) \otimes |S \cap \mathcal{S}|, b \otimes A''_b (S \cap [k]) \{ b \}, ((\mathcal{H} \otimes \mathbb{C}^2) \otimes |T_i \cap \mathcal{S}|, b \otimes A''_b (S \cap [k]) \{ b \})\) 

\(b \in S \cap [k] \cap T_i\), where the first multiplicands in the two tensor products are embedded into \(A''_b\).

This completes the proof of the first part of Claim 2 of Proposition [a].

Again, let \(S \subseteq [c] \cup [k], \{k\} \neq S \cap [k] \neq [k]\). If \([c] \subset S\), then \(N''_{S \cup [c], \delta}(|h \rangle \langle h|) = 0\). Suppose \(S \cap [c] \neq [c]\). Suppose one were to show that

\[
\left\| \sum_{v_{[c]} \setminus S} (N''_{S \cap [c], \delta} (|h \rangle \langle h|))^{A''_b (S \cap [k])} \right\| \leq 3|L|\left| |c| \setminus S \right|^{-1/2}
\]

for all unit length vectors \(|h \rangle \in (\mathcal{H} \otimes \mathbb{C}^2) \otimes (S \cap [k])\). From Equations 17, 18 and the triangle inequality, in either case, it will follow that

\[
\left\| (N''_{S \times S \cap [c], \delta}) \right\| \leq \frac{N(S \cup [c], \delta)N(S \cap [c], \delta)}{N([c] \cup [k], \delta)} \cdot \frac{3}{\sqrt{|L|}}
\]

Now it is easy to see using Equations 19, 21 that this implies that

\[
\beta(S_1, \ldots, S_l, \delta) \left\| (N''_{(S_1, \ldots, S_l), \delta}) \right\| \leq \frac{3}{\sqrt{|L|}}
\]

It only remains to show for \(S \cap [c] \neq [c]\) that \(\left\| \sum_{v_{[c]} \setminus S} (N''_{S \cap [c], \delta} (|h \rangle \langle h|))^{A''_b (S \cap [k])} \right\| \leq 3|L|\left| |c| \setminus S \right|^{-1/2}\)

for any unit length vector \(|h \rangle \in (\mathcal{H} \otimes \mathbb{C}^2) \otimes (S \cap [k])\). By Equation 16 it suffices to show that

\[
\left\| \sum_{v_{[c]} \setminus S} T_{\leftarrow_{S \cup [c], \delta}} (|h \rangle \langle h|) \right\| \leq |L|\left| |c| \setminus S \right|^{-1/2}, \quad \left\| \sum_{v_{[c]} \setminus S} T_{\leftarrow_{S \cup [c], \delta}} (|h \rangle \langle h|) \right\| \leq |L|\left| |c| \setminus S \right|^{-2/2}
\]

Since the range spaces of the summands in the definition of \(T_{\leftarrow_{S \cup [c], \delta}}\) are orthogonal, it suffices to show, for any \((T_1, \ldots, T_l) \mapsto S \cup [c], (T_1, \ldots, T_l) \mapsto S, 1, 1_{v_{[c]} \setminus (S \cup [c])} \) that

\[
\left\| \sum_{v_{[c]} \setminus S \cap [(T_1, \ldots, T_l)]} (T_{\leftarrow_{T_1, T_1 \cap (S \cup [c])}} \otimes \cdots \otimes T_{\leftarrow_{T_l, T_l \cap (S \cup [c])}} \otimes 1_{(\mathcal{H} \otimes \mathbb{C}^2) \otimes (S \cup [c]) \setminus (T_1 \cup \cdots \cup T_l)}) (|h \rangle \langle h|) \right\| = \sqrt{|L|\left| |c| \setminus S \right| \cap (T_1 \cup \cdots \cup T_l)}
\]

and

\[
\left\| \sum_{v_{[c]} \setminus S \cap (T_1 \cup \cdots \cup T_l)} (T_{\leftarrow_{T_1, T_1 \cap (S \cup [c])}} \otimes \cdots \otimes T_{\leftarrow_{T_l, T_l \cap (S \cup [c])}} \otimes 1_{(\mathcal{H} \otimes \mathbb{C}^2) \otimes (S \cup [c]) \setminus (T_1 \cup \cdots \cup T_l)}) (|h \rangle \langle h|) \right\| = 1.
\]

The last two equalities arise from the fact that for any two distinct computational basis vectors \(v_{[c]} \setminus S \cap (T_1 \cup \cdots \cup T_l) \neq v_{[c]} \setminus S \cap (T_1 \cup \cdots \cup T_l)\) the range spaces of

\[
T_{\leftarrow_{T_1, T_1 \cap (S \cup [c])}} \otimes \cdots \otimes T_{\leftarrow_{T_l, T_l \cap (S \cup [c])}} \otimes 1_{(\mathcal{H} \otimes \mathbb{C}^2) \otimes (S \cup [c]) \setminus (T_1 \cup \cdots \cup T_l)}
\]
are orthogonal. This follows from the observation that there exists an \( i \in [l] \), a coordinate \( a \in [c] \cap S \cap T_i \) such that \( Y'_a \neq Y''_a \) which implies that the two range spaces embed into the orthogonal spaces

\[
((\mathcal{H} \otimes \mathbb{C}^2) \otimes |I_{T_i \cap S \cap [c]}^Y \rangle \langle b^Y|) b \otimes A''_{S \cap [k]} \{ b \}, ((\mathcal{H} \otimes \mathbb{C}^2) \otimes |I_{T_i \cap S \cap [c]}^Y \rangle \langle b^Y|) b \otimes A''_{S \cap [k]} \{ b \},
\]

\( b \in S \cap [k] \cap T_i \), where the first multiplicands in the two tensor products are embedded into \( A''_b \).

This completes the proof of the second part of Claim 2 of Proposition 4.

**5.7 Proving** \( \left\| (\rho^A)^{[k]}_{x,1,\delta} - \rho^{[k]} \otimes (|0\rangle \langle 0|^{\mathbb{C}^2}) \otimes k \right\|_1 \leq 2^{k+1} \delta \)

Let \( |h\rangle \) be a unit length vector in \( (\mathcal{H} \otimes \mathbb{C}^2) \otimes k \). From Equation 4 and Inequality 5 we get

\[
\langle T_{[c] \cap [k],1,\delta} | h \rangle = \frac{1}{\sqrt{N([c] \cup [k], \delta)}} \geq e^{-\varepsilon_{2c+k-1}}.
\]

Thus,

\[
\left\| T_{[c] \cap [k],1,\delta} | h \rangle \right\| - \left| \langle h | h \rangle \right\| \leq 2 \left\| T_{[c] \cap [k],1,\delta} | h \rangle \right\| \leq 2 \sqrt{2 - 2e^{-\varepsilon_{2c+k-1}}} < 2^{k+1} \delta,
\]

where we used the fact that \( e^{-x} \geq 1 - x \) in the last inequality above. Recalling Equation 6 and applying the above inequality to the eigenvectors of \( \rho^{[k]} \otimes (|0\rangle \langle 0|^{\mathbb{C}^2}) \otimes k \) allows us to prove the desired Claim 3 of Proposition 4.

**5.8 Proving** \( \text{Tr} \left[ (\Pi'_{Y,1,\delta})^{A''_{[k]}} (\rho^{[k]} \otimes (|0\rangle \langle 0|^{\mathbb{C}^2}) \otimes k) \right] \geq 1 - \delta^{-2k} \sum_{c} \epsilon_{X,(T_1,\ldots,T_m)} \left( |h\rangle \right) \}

Let \( |h\rangle \) be an eigenvector of \( \rho^{[k]} \otimes (|0\rangle \langle 0|^{\mathbb{C}^2}) \otimes k \). For a pseudosubpartition \((T_1,\ldots,T_m) \uplus [c] \uplus [k] \), define

\[
\epsilon_{x,(T_1,\ldots,T_m)}(|h\rangle) := \left\| \Pi'_{Y,x,(T_1,\ldots,T_m)} \left( |h\rangle \right) \right\|_2
\]

where the subspace \( Y_{x,(T_1,\ldots,T_m)} \leq (\mathcal{H} \otimes \mathbb{C}^2) \otimes [k] \) is defined just after Equation 7 above. Recall that the subspace \( Y'_{x,1,\delta} \) defined in Equation 11 is the \( A \)-tilted span of \( \{ Y_{x,(T_1,\ldots,T_m)} : (T_1,\ldots,T_m) \uplus [c] \uplus [k], m > 0 \} \) where the tilting matrix \( A \) is defined in Equation 8 above. Recall that \( A \) is upper triangular, substochastic and diagonal dominated, and the diagonal entries of \( A \) satisfy

\[
A_{(T_1,\ldots,T_m),(T_1,\ldots,T_m)} = \frac{\delta^{2m}}{N(T_1,\delta) \cdots N(T_m,\delta)} \geq \frac{\delta^{2k}}{N([c] \cup [k], \delta)} \geq e^{-\varepsilon_{2c+k-1}} \delta^{2k},
\]

for all \((T_1,\ldots,T_m) \uplus [c] \uplus [k] \). By Proposition 3

\[
\left\| (\Pi')^{A''_{[k]}}_{Y,x,1,\delta} \left( |h\rangle \right) \right\|_2 \leq \delta^{2c+k-1} \delta^{-2k} \sum_{(T_1,\ldots,T_m) \uplus [c] \uplus [k], m > 0} \epsilon_{x,(T_1,\ldots,T_m)}(|h\rangle).
\]

Using Equation 7 and applying this inequality to the eigenvectors of \( \rho^{[k]} \otimes (|0\rangle \langle 0|^{\mathbb{C}^2}) \otimes k \), we get

\[
\text{Tr} \left[ (\Pi')^{A''_{[k]}}_{Y,x,1,\delta} \left( \rho^{[k]} \otimes (|0\rangle \langle 0|^{\mathbb{C}^2}) \otimes k \right) \right] \leq \delta^{2c+k-1} \delta^{-2k} \sum_{(T_1,\ldots,T_m) \uplus [c] \uplus [k], m > 0} \epsilon_{x,(T_1,\ldots,T_m)}.
\]
Finally, using Fact 3 and Equation 12, we get

\[
\text{Tr} \left( \left( \frac{A''_{[k]}}{A''_{[k]}} \right) \otimes (0 \langle 0 | C^2 \rangle \otimes k) \right)
\]

\[
= \text{Tr} \left( \left( \frac{A''_{[k]}}{(H \otimes C^2) \otimes k} \right) \otimes (0 \langle 0 | C^2 \rangle \otimes k) \right) (1 A''_{[k]} - (\Pi') Y''_{[k]}(1 A''_{[k]} - (\Pi') Y''_{[k]}(1 \otimes (0 \langle 0 | C^2 \rangle \otimes k))) + 1 - \frac{\text{Tr} \left( (\Pi') A''_{[k]}(H \otimes C^2) \otimes k \right) (\rho_{A''_{[k]}} \otimes (0 \langle 0 | C^2 \rangle \otimes k))}{1 - 4(\delta - 2k + 6) \epsilon_k^{(k+1)k}}
\]

5.9 Proving \((\rho''_{[k]})_{x,1,1,1,1,\delta} = (\alpha_{(s_1, \ldots, s_1), \delta} T_{(s_1, \ldots, s_1), 1, \delta} (\rho'_{(s_1, \ldots, s_1)} \otimes (0 \langle 0 | C^2 \rangle \otimes k))) A''_{[k]} + \beta_{(s_1, \ldots, s_1), \delta} N'_{(s_1, \ldots, s_1), x, 1, \delta} + (1 - \alpha_{(s_1, \ldots, s_1), \delta} - \beta_{(s_1, \ldots, s_1), \delta}) M'_{(s_1, \ldots, s_1), x, 1, \delta}
\]

This claim and the accompanying claims of orthogonality and vanishing of \(\beta_{(s_1, \ldots, s_1), \delta}\) are proved at the end of Section 5.3 above.

5.10 Proving \(\text{Tr} \left( (\Pi') A''_{[k]}(T_{(s_1, \ldots, s_1), 1, \delta} (\rho''_{(s_1, \ldots, s_1)} \otimes (0 \langle 0 | C^2 \rangle \otimes k))) A''_{[k]} \right) \leq 2^{-D H} (\rho_{\rho_{[k]}}(\rho''_{[k]})_{(s_1, \ldots, s_1)} A''_{[k]})
\]

Using Equations 12 11 10 7 we get

\[
\text{Tr} \left( (\Pi') A''_{[k]}(T_{(s_1, \ldots, s_1), 1, \delta} (\rho''_{(s_1, \ldots, s_1)} \otimes (0 \langle 0 | C^2 \rangle \otimes k))) A''_{[k]} \right)
\]

\[
= \text{Tr} \left( (\Pi') A''_{[k]}(H \otimes C^2) \otimes k \right) (1 A''_{[k]} - (\Pi') Y''_{[k]}(1 A''_{[k]} - (\Pi') Y''_{[k]}(1 \otimes (0 \langle 0 | C^2 \rangle \otimes k))) + 1 - \frac{\text{Tr} \left( (\Pi') A''_{[k]}(H \otimes C^2) \otimes k \right) (\rho''_{(s_1, \ldots, s_1)} \otimes (0 \langle 0 | C^2 \rangle \otimes k))}{1 - 4(\delta - 2k + 6) \epsilon_k^{(k+1)k}}
\]

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lies in the vector space $T$ henceforth. In the asymptotic iid setting, Winter \cite{Win01} used a standard inner bound for the multiple access channel with classical inputs and quantum output, called 6 One-shot inner bound for the classical quantum MAC definition of $Y$ is an isometric embedding of $(H \otimes C^2)^{\otimes k}$ of $X$.

In the second inequality above, we used the fact that $T_2$ with two classical inputs and one quantum output called a cq-MAC connecting Alice and Bob to Charlie. The two input alphabets for the cq-MAC with an arbitrary number of senders, but their achievable rates restricted to the asymptotic iid setting are inferior to the optimal rates obtained by Winter. Thus, for more than two senders a simultaneous decoder for the cq-MAC achieving optimal rates was hitherto unknown even in the asymptotic iid setting. In this section, we construct a simultaneous decoder for the cq-MAC for any number of senders using Lemma \ref{lem:claim}. We illustrate the method by considering a cq-MAC with two senders, but the method works for any number of senders. We obtain an inner bound which turns out to be the natural one-shot analogue of Winter’s inner bound, as well as the natural quantum analogue of the one-shot classical inner bound described in Section \ref{sec:one-shot-classical}. In other words, our one-shot inner bound reduces to Winter’s optimal inner bound in the asymptotic iid setting. A one-shot outer bound nearly matching our inner bound was shown recently by Anshu, Jain and Warsi \cite{AJW18}. That paper also gives another inner bound for the cq-MAC which, though nearly optimal in the one-shot setting, is not known to reduce to the standard asymptotic iid inner bound of Winter unlike our one-shot inner bound.

There are two senders Alice and Bob who would like to send classical messages $m_1 \in [2^{R_1}]$, $m_2 \in 2^{R_2}$ to a receiver Charlie. There is a communication channel $\mathcal{C}$ with two classical inputs and one quantum output called a cq-MAC connecting Alice and Bob to Charlie. The two input alphabets of $\mathcal{C}$ will be denoted by $\mathcal{X}$, $\mathcal{Y}$ and the output Hilbert space by $Z$. Let $0 \leq \epsilon \leq 1$. On getting
message \( m_1 \), Alice encodes it as a letter \( x(m_1) \in \mathcal{X} \) and feeds it to her channel input. Similarly on getting message \( m_2 \), Bob encodes it as a letter \( y(m_2) \in \mathcal{Y} \) and feeds it to his channel input. The channel \( \mathcal{C} \) outputs a quantum state \( \rho^{Z(m_1),y(m_2)}_{x} \) in \( \mathcal{Z} \). Charlie now has to try and guess the message pair \((m_1, m_2)\) from the channel output. We require that the probability of Charlie’s decoding error averaged over the uniform distribution on the set of message pairs \((m_1, m_2) \in [2^{R_1}] \times [2^{R_2}]\) is at most \( \epsilon \).

Define a new sample space \( \mathcal{U} \) and put on it a probability distribution \( p(u) \). The resulting random variable \( U \) plays the role of the so-called ‘time sharing random variable’ in our one-shot setting. Fix conditional probability distributions \( p(x|u), p(y|u) \) on sets \( \mathcal{X}, \mathcal{Y} \). Consider the classical-quantum state

\[
\rho^{UXYZ}_{u} := \sum_{u,x,y} p(u)p(x|u)p(y|u)\langle u,x,y,z \rangle_{u,x,y,z}^{UXY} \otimes p^{Z}_{z}. 
\]

This state ‘controls’ the encoding and decoding performance for the channel \( \mathcal{C} \).

Consider a new alphabet, as well as Hilbert space, \( \mathcal{L} \) and define the augmented systems \( \mathcal{U}' := \mathcal{U} \otimes \mathcal{L}, \mathcal{X}' := \mathcal{X} \otimes \mathcal{L}, \mathcal{Y}' := \mathcal{Y} \otimes \mathcal{L} \), and the extended system \( \mathcal{Z}' \) defined in Corollary 3 with \( c = 3 \). Consider a ‘variant’ cq-MAC \( \mathcal{C}' \) with input alphabets \( \mathcal{U}', \mathcal{X}', \mathcal{Y}' \) and output Hilbert space \( \mathcal{Z}' \). On input \((u,l_u), (x,l_x), (y,l_y)\) to \( \mathcal{C}' \), the output of \( \mathcal{C}' \) is the state \( \rho^{Z'}_{x,y} \otimes |0\rangle\langle 0|^{C^2} \). The output of \( \mathcal{C}' \) is taken to embed into the first summand in the definition of \( \mathcal{Z}' \) in Corollary 3. The classical quantum state ‘controlling’ the encoding and decoding for \( \mathcal{C}' \) is nothing but \( \rho^{UYZ} \otimes |0\rangle\langle 0|^{C^2} \otimes \frac{1_{C^3}}{|C^3|} \). The channel \( \mathcal{C}' \) can be trivially obtained from channel \( \mathcal{C} \). The expected average decoding error for \( \mathcal{C}' \) is the same as the expected average decoding error for \( \mathcal{C} \) for the same rate pair \((R_1, R_2)\). In fact, an encoding-decoding scheme for \( \mathcal{C}' \) immediately gives an encoding-decoding scheme for \( \mathcal{C} \) with the same rate pair \((R_1, R_2)\) and the same decoding error.

Let \( 0 \leq \delta \leq 1 \). Next, consider a ‘perturbed’ cq-MAC \( \mathcal{C}'' \) with the same input alphabets and output Hilbert space as in \( \mathcal{C}' \). However, on input \((u,l_u), (x,l_x), (y,l_y)\) the output is the state \( \rho^{Z'}_{x,y} \otimes |0\rangle\langle 0|^{C^2} \otimes \frac{1_{C^3}}{|C^3|} \) provided by setting \( c = 3 \) in Corollary 3. Consider the classical quantum state

\[
(p')^{U'X'Y'Z'} := |C|^{-3} \sum_{u,x,y,l_u,l_x,l_y} p(u)p(x|u)p(y|u)\langle u,l_u \rangle_{U'} \otimes |x,l_x \rangle_{X'} \otimes |y,l_y \rangle_{Y'} \otimes (p')^{Z'}_{u,x,y,l_u,l_x,l_y}. 
\]

This state ‘controls’ the encoding and decoding performance for the channel \( \mathcal{C}'' \). By Claim 2 of Lemma 1

\[
\left\| (p')^{U'X'Y'Z'} - \rho^{XYZ} \otimes |0\rangle\langle 0|^{C^2} \otimes \frac{1_{C^3}}{|C^3|} \right\|_1 \leq 2^3 \delta. 
\]

Thus, the expected average decoding error for \( \mathcal{C}'' \) is at most the expected average decoding error for \( \mathcal{C}' \), which is also the same as the expected average decoding error for \( \mathcal{C} \), plus \( 2^3 \delta \), for the same rate pair \((R_1, R_2)\), and the same decoding strategy.

Consider the following randomised construction of a codebook \( \mathcal{C} \) for Alice and Bob for communication over the channel \( \mathcal{C}'' \). Fix probability distributions \( p(u), p(x|u), p(y|u) \) on sets \( \mathcal{U}, \mathcal{X}, \mathcal{Y} \). Choose a sample \((u,l_u)\) according to the product of the distribution \( p(u) \) on \( \mathcal{U} \) and the uniform distribution on computational basis vectors of \( \mathcal{L} \). For all \( m_1 \in [2^{R_1}] \), choose \((x,l_x)(m_1) \in \mathcal{X} \times \mathcal{L}\) independently according to the product of the distribution \( p(x|u) \) on \( \mathcal{X} \) and the uniform distribution on \( \mathcal{L} \). Similarly for all \( m_2 \in [2^{R_2}] \), choose \((y,l_y)(m_2) \in \mathcal{Y} \times \mathcal{L}\) independently according to the product of the distribution \( p(y|u) \) on \( \mathcal{Y} \) and the uniform distribution on \( \mathcal{L} \).
We now describe the decoding strategy that Charlie follows in order to try and guess the message pair \((m_1, m_2)\) that was actually sent, given the output of \(\mathcal{C}'\). Let \((\Pi')^Z_{u,x,l_u,l_x,y,l_y,d}\) be the POVM elements provided by Lemma 11. Charlie uses the *pretty good measurement* [Bel75b, Bel75a] constructed from the POVM elements
\[
(\Pi')^Z_{(u,l_u),(x,l_x)}(m_1),(y,l_y)(m_2),d,
\]
where \((m_1, m_2) \times [2^R_1] \times [2^R_2]\).

We now analyse the expectation, under the choice of a random codebook \(\mathcal{C}\), of the error probability of Charlie's decoding algorithm. Suppose the message pair \((m_1, m_2)\) is inputted to \(\mathcal{C}'\). The output of \(\mathcal{C}'\) is the state \((\rho)'_{(u,l_u),(x,l_x),(m_1),(y,l_y),(m_2)}. Let \(\Lambda_{\tilde{m}_1,\tilde{m}_2}^Z\) be the POVM element corresponding to decoded output \((\tilde{m}_1, \tilde{m}_2)\) arising from the pretty good measurement. By the Hayashi-Nagaoka inequality [HN03], the decoding error for \((m_1, m_2)\) is upper bounded by
\[
\begin{align*}
\text{The expectation, over the choice of the random codebook } \mathcal{C}, \text{ of the error probability of Charlie's decoding algorithm. Suppose the message pair } (m_1, m_2) \text{ is inputted to } \mathcal{C}'. \text{ The output of } \mathcal{C}' \text{ is the state } (\rho)'_{(u,l_u),(x,l_x),(m_1),(y,l_y),(m_2)}. \text{ Let } \Lambda_{\tilde{m}_1,\tilde{m}_2}^Z \text{ be the POVM element corresponding to decoded output } (\tilde{m}_1, \tilde{m}_2) \text{ arising from the pretty good measurement. By the Hayashi-Nagaoka inequality [HN03], the decoding error for } (m_1, m_2) \text{ is upper bounded by}
\end{align*}
\]
Define the POVM element
\[
(\Pi')^U'X'Y'Z' := \sum_{u,x,y,l_u,l_x,l_y} |u,l_u\rangle\langle u,l_u| ^U' \otimes |x,l_x\rangle\langle x,l_x| ^X' \otimes |y,l_y\rangle\langle y,l_y| ^Y' \otimes (\Pi')^Z_{(u,l_u),(x,l_x),(y,l_y)}\delta
\]
as in Corollary 3. From Corollary 3 recall the hypothesis testing conditional mutual information quantities \(I_H^U(X : Z|UY), I_H^U(Y : Z|UX), I_H^U(XY : Z|U)\) calculated with respect to the state \(\rho_{UYZ}^U\) corresponding to the original channel \(\mathcal{C}\). The expectation, over the choice of the random codebook \(\mathcal{C}\), of the decoding error for \((m_1, m_2)\) is upper bounded by
\[
\begin{align*}
The expectation, over the choice of the random codebook \(\mathcal{C}\), of the decoding error for \((m_1, m_2)\) is upper bounded by
\end{align*}
\]
\[ = 2|\mathcal{L}|^{-3} \sum_{u,x,y,l,a,l_x,l_y} p(u)p(x|u)p(y|u) \text{Tr} \left[ (1^{Z''} - (\Pi')^{(u,l_a),(x,l_x),(y,l_y)}) (\rho')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} \right] \\
+ 4(2R_1 - 1)|\mathcal{L}|^{-4} \sum_{u,l_a,x,l_x,y,l_y} p(u)p(x|u)p(x'|u)p(y|u) \\
\text{Tr} \left[ (\Pi')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} (\rho')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} \right] \\
+ 4(2R_2 - 1)|\mathcal{L}|^{-4} \sum_{u,l_a,x,l_x,y,l_y} p(u)p(x|u)p(y|u)p(y'|u) \\
\text{Tr} \left[ (\Pi')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} (\rho')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} \right] \\
+ 4(2R_1 - 1)(2R_2 - 1)|\mathcal{L}|^{-5} \sum_{u,l_a,x,l_x,y,l_y} p(u)p(x|u)p(x'|u)p(y|u)p(y'|u) \\
\text{Tr} \left[ (\Pi')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} (\rho')^{Z''}_{(u,l_a),(x,l_x),(y,l_y)} \right] \\
= 2 \text{Tr} \left[ (1^{U''X'Y''Z''} - (\Pi')^{U''X'Y''Z''} (\rho')^{U''X'Y''Z''} \right] \\
+ 4(2R_1 - 1) \text{Tr} \left[ (\Pi')^{U''X'Y''Z''} (\rho')^{U''X'Y''Z''} \right] \\
+ 4(2R_2 - 1) \text{Tr} \left[ (\Pi')^{U''X'Y''Z''} (\rho')^{U''X'Y''Z''} \right] \\
+ 4(2R_1 - 1)(2R_2 - 1) \text{Tr} \left[ (\Pi')^{U''X'Y''Z''} (\rho')^{U''X'Y''Z''} \right] \\
\leq 2(\delta - 2\delta^{135} + 2\delta) + 2^{R_1 + 2 - I_H(X:YZ|U)} + 2^{R_2 + 2 - I_H(Y:ZX|U)} \\
+ 2^{R_1 + R_2 + 2 - I_H(XZ:Y|U)} \\
\text{where we used Claims 3, 4 of Corollary 3 in the last inequality above.}

Taking \( \delta = \epsilon^{1/3} \) and choosing a rate pair \((R_1, R_2)\) satisfying
\[ R_1 \leq I_H(X:YZ|U) - 2 - \log \frac{1}{\epsilon}, \]
\[ R_2 \leq I_H(Y:ZX|U) - 2 - \log \frac{1}{\epsilon}, \]
\[ R_1 + R_2 \leq I_H(XY:Z|U) - 2 - \log \frac{1}{\epsilon}, \]
ensures that the expected average decoding error for channel \( C'' \) is at most \( 2^{135} \epsilon^{1/3} \). This implies that the expected average decoding error for the original channel \( C \) is at most \( 2^{135} \epsilon^{1/3} \). Thus there exists a codebook \( \mathcal{C} \) with average decoding error for \( C \) at most \( 2^{135} \epsilon^{1/3} \). By a standard technique of taking maps from classical symbols to arbitrary quantum states, we can then prove the following theorem.

**Theorem 2.** Let \( C : X'Y' \rightarrow Z \) be a quantum multiple access channel. Let \( U, X, Y \) be three new sample spaces. For every element \( x \in X \), let \( \sigma^x \) be a quantum state in the input Hilbert space \( X' \) of \( C \). Similarly, for every element \( y \in Y \), let \( \sigma^y \) be a quantum state in the input Hilbert space \( Y' \) of \( C \). Let \( p(u)p(x|u)p(y|u) \) be a probability distribution on \( U \times X \times Y \). Consider the classical quantum state
\[ \rho^{UXYZ} := \sum_{u,x,y} p(u)p(x|u)p(y|u)|u,x,y\rangle\langle u,x,y|^{UXY} \otimes (C(\sigma^x \otimes \sigma^y))^{Z}. \]

Let \( R_1, R_2, \epsilon, \) be such that
\[ R_1 \leq I_H(X:Y|Z|U) - 2 - \log \frac{1}{\epsilon}, \]
$$R_2 \leq I'_H(Y : XZ|U)_{\rho} - 2 - \log \frac{1}{\epsilon},$$

$$R_1 + R_2 \leq I'_H(XY : Z|U)_{\rho} - 2 - \log \frac{1}{\epsilon}. $$

Then there exists an $$(R_1, R_2, 2^{135}\epsilon^{1/3})$$-quantum MAC code for sending classical information through $C$.

A similar bound can be proved for sending classical information through an entanglement assisted q-MAC by using the position based coding technique of Anshu, Jain and Warsi [AJW17]. Earlier, Qi, Wang and Wilde [QWW17] had constructed a one-shot simultaneous decoder for this problem which also used position based coding, but their inner bound is suboptimal when reduced to the asymptotic iid setting. In contrast, our one-shot inner bound reduces to the best known inner bound in the asymptotic iid setting which was proved by Hsieh, Devetak and Winter [HDW08] using successive cancellation arguments.

**Theorem 3.** Let $C : X'Y' \rightarrow Z$ be a quantum multiple access channel. Let $U, X, Y$ be three new Hilbert spaces. Let $\psi^{UXX'Y'Y'}$ be a classical quantum state with the following structure:

$$\psi^{UXX'Y'Y'} = \sum_u p(u)|u\rangle\langle u| \otimes \psi^{XX'}_u \otimes \psi^{YY'}_u.$$

Consider the classical quantum state

$$\rho^{UXYZ} := \sum_u p(u)|u\rangle\langle u| \otimes (\varphi^{X'Y' \rightarrow Z} \circ \psi^{XX'}_u \otimes \psi^{YY'}_u)^{XY Z}.$$

Let $R_1, R_2, \epsilon$, be such that

$$R_1 \leq I'_H(X : YZ|U)_{\rho} - 2 - \log \frac{1}{\epsilon},$$

$$R_2 \leq I'_H(Y : XZ|U)_{\rho} - 2 - \log \frac{1}{\epsilon},$$

$$R_1 + R_2 \leq I'_H(XY : Z|U)_{\rho} - 2 - \log \frac{1}{\epsilon}. $$

Then there exists an entanglement assisted $$(R_1, R_2, 2^{135}\epsilon^{1/3})$$-quantum MAC code for sending classical information through $C$.

### 7 Conclusion and open problems

In this work, we have proved a one-shot classical quantum joint typicality lemma that not only extends the iid classical conditional joint typicality lemma to the one-shot and quantum settings, but also extends intersection and union arguments that are ubiquitous in classical network information theory to the quantum setting. We introduced two novel tools in the process of proving our joint typicality lemma viz. tilting and smoothing, which should be useful elsewhere. Our lemma allows us to transport many packing arguments arising in proofs of inner bounds from the classical to the quantum setting. We illustrated this by constructing a simultaneous decoder for the one-shot classical quantum MAC. More of such applications can be found in the companion paper [Sen18].
However, the statement of our quantum joint typicality lemma is not as strong as the classical statement. It can only handle negative hypothesis states that are a tensor product of marginals and at most one arbitrary quantum state. Proving a quantum lemma that can handle arbitrary negative hypothesis states, as in the classical setting, is an important open problem.

Another drawback of our quantum lemma is that it cannot handle covering arguments and their unions and intersections that often arise in source coding applications. This is unlike the classical case. Addressing this deficiency is another important open problem.

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References

[AJW17a] Anshu, A., Jain, R., and Warsi, N. A hypothesis testing approach for communication over entanglement assisted compound quantum channel. Arxiv 1706.08286, 2017.

[AJW17b] Anshu, A., Jain, R., and Warsi, N. One-shot entanglement assisted classical and quantum communication over noisy quantum channels: A hypothesis testing and convex-split approach. Arxiv 1702.01940, 2017.

[AJW18] Anshu, A., Jain, R., and Warsi, N. On the near-optimality of one-shot classical communication over quantum channels. Arxiv 1804.09644, 2018.

[BD10] Brandao, F. and Datta, N. The quantum capacity of channels with arbitrarily correlated noise. *IEEE Transactions on Information Theory*, 56(3):1447–1460, 2010.

[BD11] Buscemi, F. and Datta, N. One-shot rates for entanglement manipulation under non-entangling maps. *IEEE Transactions on Information Theory*, 57(3):1754–1760, 2011.

[Bel75a] Belavkin, V. *Radiotekhnika i Electronika*, 20(6):1177–1185, 1975. English translation: “Optimal distinction of non-orthogonal quantum signals”, *Radio Engineering and Electronic Physics*, 20, pp. 39–47, 1975.

[Bel75b] Belavkin, V. Optimal multiple quantum statistical hypothesis testing. *Stochastics*, 1:315–345, 1975.

[DF13] Drescher, L. and Fawzi, O. On simultaneous min-entropy smoothing. In Proc. *IEEE Int. Symp. Inf. Theory (ISIT)*, July 2013.
[Dut11] Dutil, N. *Multiparty quantum protocols for assisted entanglement distillation*. PhD thesis, McGill University, 2011. Also arXiv:1105.4657.

[EK12] El Gamal, A. and Kim, Y. *Network Information Theory*. Cambridge University Press, 2012.

[FHS+12] Fawzi, O., Hayden, P., Savov, I., Sen, P., and Wilde, M. Classical communication over a quantum interference channel. *IEEE Transactions on Information Theory*, 58(6):3670–3691, 2012.

[Gao15] Gao, J. Quantum union bounds for sequential projective measurements. *Physical Review A*, 92(5):(052331–1)–(052331–6), 2015.

[HDW08] Hsieh, M-H., Devetak, I., and Winter, A. Entanglement-assisted capacity of quantum multiple-access channels. *IEEE Transactions on Information Theory*, 54(7):3078–3090, 2008.

[HN03] Hayashi, M. and Nagaoka, H. General formulas for capacity of classical-quantum channels. *IEEE Transactions on Information Theory*, 49:1753–1768, 2003.

[OMW18] Oskouei, S., Mancini, S., and Wilde, M. Union bound for quantum information processing. Arxiv 1804.08144, 2018.

[QWW17] Qi, H., Wang, Q., and Wilde, M. Applications of position-based coding to classical communication over quantum channels. Arxiv 1704.01361, 2017.

[Sen12] Sen, P. Achieving the Han-Kobayashi inner bound for the quantum interference channel. In *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, July 2012. Also arXiv:1109.0802.

[Sen18] Sen, P. Inner bounds via simultaneous decoding in quantum network information theory. In preparation, 2018.

[Win01] A. Winter. The capacity of the quantum multiple-access channel. *IEEE Transactions on Information Theory*, 47(7):3059–3065, 2001.

[WR12] Wang, L. and Renner, R. One-shot classical-quantum capacity and hypothesis testing. *Physical Review Letters*, 108:200501–200505, May 2012.