Anomalies and Wess-Zumino terms in an extended, regularized Field-Antifield formalism

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Abstract
Quantization of anomalous gauge theories with closed, irreducible gauge algebra within the extended Field-Antifield formalism is further pursued. Using a Pauli-Villars (PV) regularization of the generating functional at one loop level, an alternative form for the anomaly is found which involves only the regulator. The analysis of this expression allows to conclude that recently found ghost number one cocycles with nontrivial antifield dependence can not appear in PV regularization. Afterwards, the extended Field-Antifield formalism is further completed by incorporating quantum effects of the extra variables, i.e., by explicitly taking into account the regularization of the extra sector. In this context, invariant PV regulators are constructed from non-invariant ones, leading to an alternative interpretation of the Wess-Zumino action as the local counterterm relating invariant and non-invariant regularizations. Finally, application of the above ideas to the bosonic string reproduces the well-known Liouville action and the shift \((26 - D) \rightarrow (25 - D)\) at one loop.

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1 Introduction

One of the fundamental aims of the Field-Antifield (FA) formalism is to provide a general framework for the covariant path integral quantization of gauge theories at Lagrangian level, using as principal requirement BRST invariance. In this spirit FA encompasses previous ideas and developments for quantizing gauge systems based on BRST symmetry and extends them to more complicated situations (open algebras, reducible systems, etc.).

The requirement of BRST invariance seems to define the full quantum theory by means of a single equation, the so-called quantum master equation. In a Pauli-Villars (PV) regularization scheme at one loop level was introduced to deal with this equation and anomalies were recognized to arise whenever it can not be solved in a local way. Anomalous gauge theories appear thus characterized by the breakdown of its classical gauge or BRST structure due to quantum corrections, leading to the fact that some classical pure gauge degrees of freedom become propagating at quantum level. Therefore, according to the spirit of covariant quantization of gauge theories, the convenience arises of developing an extended formalism which describes in a BRST invariant manner this phenomenon by the introduction of extra degrees of freedom already at classical level.

A proposal to consistently quantize in a BRST invariant way anomalous gauge theories with closed irreducible gauge algebras along FA ideas was considered in . There, by extending the original configuration space with the addition of extra degrees of freedom, a solution for the original regularized quantum master equation at one loop was given in terms of the antifield independent part of the anomalies. This solution turned out to be the contribution of the original and ghost fields to the Wess-Zumino term. However, questions as regularization of the extra divergent pieces coming from the new fields as well as their contribution to the Wess-Zumino action were not considered. Instead, only a formal BRST invariant measure was constructed for them.

In this paper, we further pursue the program started in along the lines sketched in . A brief account of the FA formalism, using the concepts of classical and gauge-fixed basis, is presented in sect.2. In this way, the quantum master equation naturally appears as a potential obstruction for the fulfillment of the Slavnov-Taylor identity associated to the BRST symmetry for the effective action, generalizing the original proposal of for Yang-Mills theories. The analysis is first presented in a formal fashion and, afterwards, introducing a PV type regularization at one loop level for the generating functional. The regulated BRST Ward identity yields then an alternative expression for the anomaly involving only the regulator, which is shown equivalent to that obtained in in appendix A. This alternative expression turns out to be very useful in the analysis performed in sect.3 of the form of the complete anomaly in the space of fields and antifields. An important result coming out of this analysis is that recently found ghost number one cocycles with nontrivial antifield dependence are ruled out by PV regularization. The extended formalism presented in is further extended in sect.4 by discussing how regularization for the new sector of variables should proceed for a specific type of theories. After that, in sect.5, the construction of (antifield independent) invariant regulators in the extended configuration space from non-invariant ones is described and a method to obtain Wess-Zumino actions from integration of anomalies is proposed. The procedure relies on the form of the counterterm relating the anomalies coming from different regulators analyzed in appendix B. Sect.6 deals with the application of the above ideas to the bosonic string and sect.7 with the conclusions. Finally, in appendix C we discuss the transformation properties of the regulator for the extra variables sector.
2 Regularized Field-Antifield formalism

The Field-Antifield formalism is a powerful method for the study of gauge theories. (for a review, see [14]). At the classical level, it can be seen as a general algorithm to derive a gauge-fixed action $S_\Sigma(\Phi)$ and its BRST transformation $\delta_\Sigma$ out of a given classical gauge action $S_0(\phi)$ and its associated gauge structure. At the quantum level, it provides the tools to study to what extent this classical BRST symmetry and its underlying structure are preserved (or not) by quantum corrections. This quantum BRST structure is further used to study unitarity and renormalizability issues.

2.1 Classical theory. Classical basis versus Gauge-fixed basis

Assume $S_0(\phi^i)$ to be a classical action, invariant under the (infinitesimal) gauge transformations

$$\delta\phi^i = R^i_\alpha z^\alpha, \quad i = 1, \ldots, n; \quad \alpha = 1, \ldots, m.$$  \hfill (2.1)

The BV approach starts by enlarging the original configuration space to a new manifold $\cal M$, locally coordinated by a new set of fields $\Phi^A$, $A = 1, \ldots, N$, (including, apart from the original fields, ghosts, antighosts, etc.) and their associated antifields $\Phi^*_A$, with opposite Grassmann parity. This set is often collectively denoted by $z^a = \{\Phi^A, \Phi^*_A\}$, $a = 1, \ldots, 2N$. Afterwards, $\cal M$ is endowed with an odd symplectic structure, ($\cdot, \cdot$), called antibracket and defined as

$$(X, Y) = \frac{\partial_r X}{\partial z^a} \zeta^{ab} \frac{\partial_l Y}{\partial z^b}, \quad \text{where} \quad \zeta^{ab} \equiv (z^a, z^b) = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^*_B & 0 \end{pmatrix}.$$  \hfill (2.2)

At the classical level, the fundamental object is a bosonic functional $S(z)$ with dimensions of action and verifying the so-called classical master equation,

$$(S, S) = 0,$$  \hfill (2.3)

with boundary conditions: 1) Classical limit: $S(\Phi, \Phi^*)|_{\Phi^* = 0} = S_0(\phi)$, and 2) Properness condition: $\text{rank}(S_{ab})|_{\text{on-shell}} = N$, with $S_{ab} \equiv \left( \frac{\partial_r S}{\partial z^a \partial z^b} \right)$, and where on-shell means on the surface $\{\frac{\partial_r S}{\partial z^a} = 0\}$.

In the original basis $z^a$, the expansion of $S$ in antifields

$$S(\Phi, \Phi^*) = S_0(\phi) + \Phi^*_A R^A(\Phi) + \frac{1}{2} \Phi^*_A \Phi^*_B R^{BA}(\Phi) + \ldots,$$  \hfill (2.4)

generates the structure functions of the original classical gauge algebra [14]. Besides, fulfillment of eq.(2.2) provides the relations defining its structure [16, 17]. In this sense, it is sensible to call the original basis $z^a$ classical basis [10, 11].

The gauge-fixed theory, instead, is better analyzed in terms of what can be called gauge-fixed basis [10, 11], defined in terms of a suitable gauge-fixing fermion $\Psi(\Phi)$ through the canonical transformation

$$\Phi^A \rightarrow \Phi^A, \quad \Phi^*_A \rightarrow K_A + \frac{\partial\Psi(\Phi)}{\partial \Phi^A} \equiv K_A + \Psi_A.$$  \hfill (2.5)

1This definition of classical basis differs from that given in [10, 11]. There, this concept is based in the ghost number carried by fields and antifields, while in this approach it lies on what it is obtained in the $\Phi^* = 0$ (classical) limit and in the content of $(S, S) = 0$.

2The concept of gauge-fixed basis used throughout this paper, based in [14], in which fields $\Phi^A$ do not change, is more restrictive than that considered in [11], where interchange of fields and antifields is also allowed.
Then, in the gauge-fixed basis, $z' = \{ \Phi^A, K_A \}$, the proper solution $S (2.3)$ is expressed as

$$
\hat{S}(\Phi, K) \equiv S \left( \Phi, \Phi^* = K + \frac{\partial \Psi(\Phi)}{\partial \Phi} \right) = \\
\left( S_0(\phi) + \Psi_A R^A + \frac{1}{2} \Psi_A \Psi_B R^{BA} + \ldots \right) + K_A \left( R^A + \Psi_B R^{BA} + \ldots \right) + \mathcal{O}(K^2). \tag{2.5}
$$

The antifield independent part of (2.5) is the gauge-fixed action, $S_\Sigma(\Phi)$, so that the classical limit is no longer recovered. This result provides the characterization of the gauge-fixed basis, through the boundary conditions: 1') Gauge-fixed limit: $\hat{S}(\Phi, K) \bigg|_{K=0} = S_\Sigma(\Phi)$, and 2') rank$(S_{\Sigma, AB})_{\text{on-shell}} = N$, with $S_{\Sigma, AB} \equiv \left( \frac{\partial \partial \Psi(\Phi)}{\partial \Phi^A \partial \Phi^B} \right)$. Therefore, if $\Psi$ in (2.4) is correctly chosen, propagators are well defined and the usual perturbation theory can commence.

In much the same way, the linear part in $K_A$ of (2.5) is the gauge-fixed BRST transformation of $S_\Sigma(\Phi)$

$$
\delta_\Sigma \Phi^A = (\Phi^A, \hat{S}) \bigg|_{K=0} \equiv \tilde{R}^A, \tag{2.6}
$$

the coefficients of the bilinear part are the non-nilpotency structure functions and so on. We can write

$$
\hat{S}(\Phi, K) = S_\Sigma(\Phi) + K_A \tilde{R}^A(\Phi) + \frac{1}{2} K_A K_B \tilde{R}^{BA}(\Phi) + \ldots \tag{2.7}
$$

in such a way that $\hat{S}$ (2.7) appears now as the generating functional of the structure functions which define the BRST symmetry, whereas relations derived from $(\hat{S}, \hat{S}) = 0$ characterize the structure of the classical BRST symmetry. (2.7) contains thus all the information about the underlying structure of this classical BRST symmetry.

$\hat{S}$ (2.7) is itself invariant under the off-shell nilpotent BRST symmetry

$$
\delta F(z') = (F, \hat{S}). \tag{2.8}
$$

The cohomology associated with $\delta$, usually called antibracket BRST cohomology, is related with the weak cohomology of $\delta_\Sigma$ (2.6) [17]. Both cohomologies turns out to be very important in the study of renormalization and anomaly issues.

2.2 Quantum theory. Quantum BRST transformation

The transition from the classical to the quantum theory may spoil the classical BRST structure due to quantum corrections acquired by the BRST transformations (2.4) and the higher order structure functions in (2.7). This violation indicates the presence of anomalies. The quantum aspects of the BRST formalism are most suitable studied in terms of the effective action $\Gamma$ associated through a Legendre type transformation with respect to the sources $J_A$ with the (connected part of) the generating functional

$$
Z(J, K) = \int \mathcal{D}\Phi \exp \left\{ \frac{i}{\hbar} \left[ W(\Phi, K) + J_A \Phi^A \right] \right\}, \tag{2.9}
$$

with $W(\Phi, K)$ given by

$$
W(\Phi, K) = \hat{S}(\Phi, K) + \sum_{p=1}^{\infty} \hbar^p M_p(\Phi, K), \tag{2.10}
$$
and where the local counterterms $M_p$ should guarantee the finiteness of the theory while preserving the BRST structure at quantum level as far as possible. In this way, the functional $\Gamma$ appears as the quantum analog of $\hat{S}$ \((\ref{eq:2.7})\), i.e., the coefficients in its antifield expansion

\begin{equation}
\Gamma(\Phi, K) = \Gamma(\Phi) + K_A \Gamma^A(\Phi) + \frac{1}{2} K_A K_B \Gamma^{BA}(\Phi) + \ldots,
\end{equation}

are interpreted as the quantum counterpart of the classical coefficients in $\hat{S}$: $\Gamma(\Phi)$ is the 1PI generating functional for the basic fields, including loop corrections to $S_2(\Phi)$; $\Gamma^A(\Phi)$ the quantum BRST transformations, formed by adding quantum corrections to $\hat{R}^A$ \((\ref{eq:2.6})\); $\Gamma^{AB}(\Phi)$ the quantum non-nilpotency structure functions, etc. $\Gamma(\Phi, K)$ is thus the generating functional of the structure functions characterizing the BRST symmetry at quantum level.

The quantum BRST structure and its possible violation are reflected in the (anomalous) BRST Ward identity

\begin{equation}
\frac{1}{2}(\Gamma, \Gamma) = -i\hbar (A \cdot \Gamma), \quad A \equiv \left[ \Delta W + \frac{i}{2\hbar} (W, W) \right](\Phi, K),
\end{equation}

where $(A \cdot \Gamma)$ denotes the generating functional of the 1PI diagrams with the insertion of $A$ and $\Delta$ stands for the second order differential operator

\begin{equation}
\Delta \equiv (-1)^A \frac{\partial}{\partial \Phi} \frac{\partial}{\partial \Phi^A}.
\end{equation}

Therefore, $A$ in \((\ref{eq:2.12})\) parametrizes potential departures from the classical BRST structure due to quantum corrections. In particular, in its $K_A$ expansion, the antifield independent part indicates the non-invariance of $\Gamma(\Phi)$ under the quantum BRST transformation $\Gamma^A(\Phi)$, its linear part in $K_A$ reflects an anomaly in the on-shell nilpotency of $\Gamma^A(\Phi)$, and so on.

Quantum BRST invariance will thus hold if the obstruction $A$ in \((\ref{eq:2.12})\) vanishes, i.e., upon fulfillment through a local object $W$ of the quantum master equation

\begin{equation}
\frac{1}{2}(W, W) - i\hbar \Delta W = 0.
\end{equation}

Eq.\((\ref{eq:2.13})\) encodes at once the classical master equation \((\ref{eq:2.2})\), satisfied by construction, plus a set of recurrent equations for the counterterms $M_p$

\begin{equation}
\begin{aligned}
(M_1, \hat{S}) &= i\Delta \hat{S}, \\
(M_p, \hat{S}) &= i\Delta M_{p-1} - \frac{1}{2} \sum_{q=1}^{p-1} (M_q, M_{p-q}), \quad p \geq 2.
\end{aligned}
\end{equation}

### 2.3 Regularized FA formalism. Pauli-Villars scheme

To make sense out of the previous calculations and expressions a regularization scheme is necessary. A prescription to regularize the FA formalism up to one loop has been considered in refs.\cite{5} \cite{6} \cite{11} \cite{18}. This proposal consists in using a Pauli-Villars (PV) regularization scheme in \((\ref{eq:2.9})\), that is, to substitute it for the one-loop regularized expression

\begin{equation}
Z_{\text{reg}}(J, K) = \int \mathcal{D}\Phi \mathcal{D}\chi \exp \left\{ \frac{i}{\hbar} \left[ \hat{S}(\Phi, K) + \hbar M_1(\Phi, K) + S_{PV}(\chi, \chi^* = 0; \Phi, K) + J_A \Phi^A \right] \right\},
\end{equation}

The effective action $\Gamma$ depends in fact on the so-called “classical fields”, $\Phi^A(J, K) = -i\hbar \frac{\partial \ln Z(J, K)}{\partial J_A}$. However, for notational simplicity, and unless confusion arise, we will denote them also by $\Phi^A$.\footnote{The effective action $\Gamma$ depends in fact on the so-called “classical fields”, $\Phi^A(J, K) = -i\hbar \frac{\partial \ln Z(J, K)}{\partial J_A}$. However, for notational simplicity, and unless confusion arise, we will denote them also by $\Phi^A$.}
where the PV fields $\chi^A$, introduced for each field $\Phi^A$, have the same statistics as their original partners, but with the path integral formally defined so that an extra minus sign occurs in front of their loops. Each PV field $\chi^A$ comes with its associated antifield $\chi^*_A$, and together can collectively be denoted as $w^a = \{\chi^A, \chi^*_A\}, \ a = 1, \ldots, 2N$. PV antifields $\chi^*_A$ have no physical significance and at the end are put to zero. Finally, the regularized theory is described by (2.15) when the cutoff mass $M$ is sent to infinity.

The regulating PV action $S_{PV}$ is determined from two requirements: i) massless propagators and couplings for PV fields should coincide with those of their partners and ii) BRST transformations for PV fields should be such that the massless part of the PV action, $S^{(0)}_{PV}$, and the measure in (2.15) be BRST invariant up to one loop. A suitable prescription for $S_{PV}$ is

$$S_{PV} = S^{(0)}_{PV} + S_M = \frac{1}{2} w^a S_{ab} w^b - \frac{1}{2} M \chi^A T_{AB} \chi^B,$$

(2.16)

with the mass matrix $T_{AB}$ arbitrary but invertible and $S_{ab}$ defined by

$$S_{ab} = \left( \frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b} \hat{S}(\Phi, K) \right).$$

(2.17)

Let us now derive an alternative form for the regularized expression of $\Delta \hat{S}$. Application of the semiclassical approximation to (2.15) yields the effective action up to one loop

$$\Gamma(\Phi, K) = \hat{S}(\Phi, K) + \hbar M_1(\Phi, K) + \frac{i\hbar}{2} \text{Tr} \ln \left[ \frac{(TR)}{(TR) - TM} \right] = \hat{S}(\Phi, K) + \hbar \Gamma_1(\Phi, K),$$

(2.18)

where Tr stands for the supertrace, $\text{Tr}(M) \equiv (-1)^A M^A_A$, and $(TR)_{AB}$ is defined from (2.17) as

$$(TR)_{AB} = \left( \frac{\partial}{\partial \Phi^A} \frac{\partial}{\partial \Phi^B} \hat{S}(\Phi, K) \right).$$

(2.19)

On the other hand, the BRST variation (2.13) of $\Gamma_1$ in (2.18) produces, by comparison with (2.12), what should be considered the regularized expression of $\Delta \hat{S}$

$$\left( \Delta \hat{S} \right)_{\text{reg}} = \delta \left\{ -\frac{1}{2} \text{Tr} \ln \left[ \frac{R}{R - M} \right] \right\} = \text{Tr} \left[ -\frac{1}{2} (R^{-1} \delta R) \frac{1}{(1 - R/M)} \right].$$

(2.20)

In appendix A, we prove the equivalence between (2.20) and the form obtained in \cite{6, 11}.

The expression we have obtained for the regularized value of $\Delta \hat{S}$ involves the BRST variation of the regulator, $\delta R$, thereby showing that (potential) anomalous symmetries are directly related with the transformation properties of $R$. In particular, if $R$ is invariant under some subset of symmetries or it transforms as $\delta R = [R, G]$ for a given $G$, (2.20) leads to a vanishing result. On the other hand, since $\left( \Delta \hat{S} \right)_{\text{reg}}$ appears also as a $\delta$-variation, its BRST variation itself vanishes, $\delta \left[ \left( \Delta \hat{S} \right)_{\text{reg}} \right] = 0$, i.e., it verifies the Wess-Zumino consistency conditions \cite{19}. Therefore, the complete expression of $\left( \Delta \hat{S} \right)_{\text{reg}}$ that is, the “anomaly” (term in $M^0$) plus the divergent terms (terms in $M^n$, $n > 0$) verify separately the Wess-Zumino consistency conditions.

Finally, and basically for computational reasons, it should be stressed that $\Delta \hat{S}$ (2.20) is equivalent \cite{3} to that obtained using the well known Fujikawa regularisation procedure \cite{20, 21}.

In other words

$$\left( \Delta \hat{S} \right)_{\text{reg}} = \text{Tr} \left[ -\frac{1}{2} (R^{-1} \delta R) \frac{1}{(1 - R/M)} \right] \sim \text{Tr} \left[ -\frac{1}{2} (R^{-1} \delta R) \exp\{R/M\} \right].$$

\footnote{For an alternative proof of the consistency of $\left( \Delta \hat{S} \right)_{\text{reg}}$, see \cite{10}.}
3 Analysis of \((\Delta \hat{S})_{\text{reg}}\)

The aim of this section is to present a detailed analysis of the regularized expression of \(\Delta \hat{S}\) (2.20) arising in the PV regularization scheme. This study is based on some results about the so-called antibracket BRST cohomology associated with the BRST operator \(\delta\) (2.8) and its relation with the weak cohomology of \(\delta \Sigma\) (2.4), which can be found in [17].

Consider the regularized expression of \(\Delta S\) (2.21). Without loss of generality, we can assume for \(R\) an expansion in antifields of the type

\[
R(\Phi, K) = R_0(\Phi) + K_A R^A + \mathcal{O}(K^2),
\]

with \(R_0(\Phi)\) invertible. \(\delta R\) becomes then in terms of \(\delta \Sigma\) (2.4)

\[
\delta R(\Phi, K) = \delta \Sigma R_0(\Phi) + (-1)^R \frac{\partial_r S_\Sigma}{\partial \Phi^A} R^A + \mathcal{O}(K).
\]

Now, upon substitution of the above expansions in (2.20), \((\Delta \hat{S})_{\text{reg}}\) acquires the form

\[
(\Delta \hat{S})_{\text{reg}} = \left( \text{Tr} \left[ -\frac{1}{2} (R_0^{-1} \delta \Sigma R_0) \frac{1}{(1 - R_0/M)} \right] - \frac{\partial_r S_\Sigma}{\partial \Phi^A} P^A \right) (\Phi) + \mathcal{O}(K). \tag{3.2}
\]

The coefficients \(P^A(\Phi)\) can be shown to be local under certain conditions, in the same way as locality of (2.20) (or even of the trace term in (2.2)) is proven [22]. In this case, the combination of the equations of motion in (2.20) can be expressed in a local way as

\[
- \frac{\partial_r S_\Sigma}{\partial \Phi^A} P^A(\Phi) + \mathcal{O}(K) = (K_A P^A, \hat{S}) + \mathcal{O}(K). \tag{3.3}
\]

On the other hand, by comparison with (2.20), the trace term in (3.2) can also be written as

\[
\text{Tr} \left[ -\frac{1}{2} (R_0^{-1} \delta \Sigma R_0) \frac{1}{(1 - R_0/M)} \right] = \delta \Sigma \left\{ -\frac{1}{2} \text{Tr} \ln \left[ \frac{R_0}{R_0 - M} \right] \right\} \equiv (\Delta \hat{S})^{(0)}_{\text{reg}}(\Phi). \tag{3.4}
\]

Collecting thus the above results (3.3) and (3.4), and absorbing the dependence on the (BRST trivial) auxiliary sector of fields through a local counterterm \(\mathcal{N}\), we can write

\[
(\Delta \hat{S})_{\text{reg}} = \mathcal{B}(\Phi_m) + (\Delta \hat{S})_{\text{reg}} + (\mathcal{M}, \hat{S}), \tag{3.5}
\]

with \((\Delta \hat{S})_{\text{reg}} \sim \mathcal{O}(K), \mathcal{M} = K_A P^A + \mathcal{N}, \{\Phi_m\}\) the minimal sector of fields and \(\mathcal{B}(\Phi_m)\) the projection to this minimal sector of \((\Delta \hat{S})_{\text{reg}}(\Phi)\) (3.4), i.e.,

\[
\mathcal{B}(\Phi_m) \equiv (\Delta \hat{S})^{(0)}_{\text{reg}}\big|_{\Phi_m} = \delta \Sigma \left\{ -\frac{1}{2} \text{Tr} \ln \left[ \frac{R_0'}{R_0 - M} \right] \right\}, \quad \text{with} \quad R_0' \equiv R_0|_{\Phi_m}. \tag{3.6}
\]

The result (3.5) has some relevant consequences when it is applied to "closed" theories. As that, we mean theories for which \(\hat{S}\) (2.7) is at most linear in the antifields, implying that \(\delta \Sigma \Phi = \delta \Phi\) and \(\delta \Sigma = 0\) "strongly", i.e., without use of the equations of motion. Indeed, in such cases, (3.6) and, as a consequence, \((\Delta \hat{S})_{\text{reg}}\) in (3.5), are seen to satisfy \(\delta[\mathcal{B}(\Phi_m)] = \delta \Sigma [\mathcal{B}(\Phi_m)] = \delta \Sigma (\cdot) = 0\) and \(\delta[(\Delta \hat{S})_{\text{reg}}] = 0\) separately. One would then be tempted to use the isomorphism between \(\delta\) and \(\delta \Sigma\) cohomologies stated in [17] and write \((\Delta \hat{S})_{\text{reg}}\) in (3.3) as \(\delta[\hat{O}(K)]\). However, as one can infer from the analysis in [10], this isomorphism holds for local functions, but not in general for integrals of local functions. A priori, thus, we can not conclude the \(\delta\) triviality of \((\Delta \hat{S})_{\text{reg}}\), which should be studied in principle case by case.

However, two important facts can be derived from the above analysis when dealing with closed theories:

\footnote{We are indebted to A. van Proeyen and W. Troost for clarifying this point to us.}
• The antifield independent part of (3.5) can be cohomologically studied separately from that containing antifields, $\hat{(\Delta \hat{S})}_{\text{reg}}$. In this way, while $\mathcal{B}(\Phi_m)$ expresses the non BRST invariance of $\Gamma(\Phi)$ in (2.11) at one loop, $\hat{(\Delta \hat{S})}_{\text{reg}}$ is likely to be related to anomalies in the nilpotency of the quantum BRST transformation and in higher order relations which define the BRST structure. Such kind of terms may be of interest for theories for which the BRST charge suffers from anomalies at quantum level (bosonic string, etc.). Their study, however, goes beyond the scope of this paper and from now on we will restrict ourselves to the analysis of the antifield independent part.

• (3.5) indicates that anomalies with non-trivial antifield dependence of the type presented in [13] are not expected to appear in this formulation. The reason behind is that they require the presence of an antifield independent part $\delta\Sigma$-invariant on-shell, while in the above regularization procedure, and under the above assumed and plausible locality of the coefficients $P^A$, the antifield independent part of (3.5), $\mathcal{B}(\Phi_m)$, results to be off-shell $\delta\Sigma$ (or $\delta$) invariant. Therefore, it is as if the regularization procedure selects from the complete set of ghost number one non-trivial cocycles a subset of “physical anomalies”, i.e., of candidates to be realized in the given theory. In any case, this mismatch between mathematical solutions and “physically” realized solutions deserves further investigation.

4 Regularised Field-Antifield formalism for anomalous gauge theories

4.1 General discussion

Let us consider an irreducible theory with closed algebra. The minimal sector of fields consists then of the classical fields $\phi^i$ and the ghosts $C^\alpha$. Under such conditions, the regulator $R$ (3.1) could be written in fact as

$$R(\Phi, K) = R(\phi) + \hat{R}(\Phi, K),$$

(4.1)

with $\mathcal{R}(\phi)$ invertible, so that (3.6) becomes, in comparison with (2.20)

$$\mathcal{B}(\Phi_m) = \text{Tr} \left[ -\frac{1}{2} (\mathcal{R}^{-1}\delta\mathcal{R}) \frac{1}{(1 - R/M)} \right] = A_\alpha(\phi)C^\alpha.$$

(4.2)

Assume now that no local counterterm $M_1(\Phi, K)$ exists satisfying (2.14), or equivalently, that no local counterterms $M_1^{(0)}(\phi)$, $M'_1(\Phi, K) \sim \mathcal{O}(K)$, exist satisfying

$$(M_1^{(0)}, \hat{\mathcal{S}}) = iA_\alpha(\phi)C^\alpha = ia_k A^k_\alpha(\phi)C^\alpha,$$

(4.3)

$$(M'_1, \hat{\mathcal{S}}) = i(\hat{\Delta \hat{S}})_{\text{reg}},$$

(4.4)

with $\{A^k_\alpha(\phi)\}$ a basis of BRST nontrivial cocycles at ghost number one and where $A_\alpha(\phi)$, from now on, will only stand for the finite pieces in (4.2) (i.e., divergent pieces are assumed to be absorbed by the BRST variation of local terms to be included in $M_1$ in (2.10)). We will restrict ourselves to the study of (3.3), postponing the analysis of antifield dependent issues, as (1.4), to the future.

The rank of the functional derivatives of the anomalies $A_\alpha(\phi)$

$$\text{rank} \left( \frac{\partial A_\alpha(\phi)}{\partial \phi^i} \right) = \text{r}(\leq m), \quad \alpha = 1, \ldots, m,$$

(4.5)
determines the number of anomalous gauge transformations and, as a byproduct, the number of pure gauge degrees of freedom that become propagating at quantum level. In the case $r < m$, it often happens that the original gauge transformations \( (2.1) \) split as
\[
\delta \phi^i = R^i_\alpha \varepsilon^\alpha = R^i_\alpha \varepsilon^\alpha + R^i_\alpha \varepsilon^\alpha = \delta(A) \phi^i + \delta(a) \phi^i, \quad a = 1, \ldots, r < m,
\]
in such a way that the regulator \( \mathcal{R}(\phi) \) in \( (4.1) \) “preserves” the \( A \) part, i.e.,
\[
\delta(A) \mathcal{R} = \begin{cases} 0 & \text{if } [\mathcal{R}, G_A \varepsilon^A] = 0, \\ [\mathcal{R}, G_A \varepsilon^A] & \text{otherwise}, \end{cases}
\]
but \( \delta(a) \mathcal{R} \neq 0 \). Then, \( (4.2) \) yields \( B(\Phi_m) = A_\alpha(\phi)C^\alpha, \) where \( A_\alpha(\phi) \) are assumed to be independent. This situation is only possible when the \( A \) part is a subgroup \( \mathfrak{g} \), while no restrictions exist for the \( a \) part. For the sake of simplicity, in the rest of the paper we consider the \( a \) part to be also a subgroup.

Let us sketch now the general ideas of the extended formalism in the case \( r = m \) \cite{21}. The proposal consists in introducing \( r = m \) new fields \( \theta^\alpha \), and demand that their gauge transformations
\[
\delta \theta^\alpha = -\hat{\mu}^\alpha_\beta(\theta, \phi) \varepsilon^\beta,
\]
are such that \( \phi^\alpha = (\phi^i, \theta^\alpha) \) constitutes a new representation of the original gauge group. In terms of the generator \( R^i_\alpha = (R^i_\alpha, -\hat{\mu}^\alpha_\beta) \), this requirement amounts to the condition that \( R^i_\alpha \) verifies the same algebra as the original ones \( R^i_\alpha \) \( (2.1) \). An explicit solution is \( \mathfrak{g} \)
\[
\hat{\mu}^\alpha_\beta(\theta, \phi) = \frac{\partial \phi^\alpha(\theta', \theta; \phi)}{\partial \theta'^\beta} \bigg|_{\theta'=0},
\]
\( \phi^\alpha(\theta'; \theta; \phi) \) being the composition functions of the gauge (quasi)group \( \mathfrak{g} \). In this way, the extension enlarges the classical physical content of the extended theory. Indeed, the finite gauge transformations of the classical fields with parameters \( \theta^\alpha, F^i(\phi, \theta) \), are gauge invariant, \( \delta F^i(\phi, \theta) = 0 \), and can thus be considered as new classical gauge invariant degrees of freedom of the extended theory. The extension procedure is then prepared to describe already at classical level the (quantum) appearence of new degrees of freedom.

In the FA framework, all these facts are summarized in the (non-proper) solution of the classical master equation in the extended space \( \mathfrak{g} \)
\[
S_{\text{ext}} = S - \theta^*_\alpha \hat{\mu}^\alpha_\beta C^\beta \equiv S + S_\theta,
\]
where \( S \) is the original proper solution and \( \theta^*_\alpha \) the antifields associated with the extra fields \( \theta^\alpha \). Its non-proper character \( \mathfrak{g} \) is just a consequence of the absence of terms related with the shift symmetry \( \delta \phi^i = 0, \delta \theta^\alpha = \sigma^\alpha \), of the classical action \( S_0(\phi) \).

To quantize the extended theory, consider an extension of \( W \) \( (2.10) \) at one loop \( \mathfrak{g} \)
\[
\tilde{W} = \tilde{S}_{\text{ext}} + h\tilde{M}_1,
\]
verifying \( (2.2) \) and \( (2.14) \) in the extended space
\[
(\tilde{S}_{\text{ext}}, \tilde{S}_{\text{ext}}) = 0, \quad (\tilde{M}_1, \tilde{S}_{\text{ext}}) = i\Delta \tilde{S}_{\text{ext}}.
\]

\footnote{For simplicity, we will consider only bosonic fields and bosonic gauge transformations, i.e., \( \epsilon(\phi^i) = 0, \epsilon(\varepsilon^\alpha) = 0 \).}

\footnote{For a complete study of the so-called quasigroup structure, as well as for further explanation of notation, we refer the reader to the original reference \( \mathfrak{g} \).}

\footnote{The gauge-fixed basis we consider in the extended theory come from gauge-fixing fermions of the type \( \Psi(\Phi) \). This implies that neither \( \theta \) nor \( \theta^* \) change under \( \mathfrak{g} \).}
In principle, one would be tempted to look for a proper solution $S_{\text{ext}}$ of (4.11) (as in ref.[24]), expressed in the gauge-fixed basis. However, since pure gauge degrees of freedom become propagating due to quantum corrections, we claim that in this proposal the classical part of $\tilde{W}$ (4.10), $\hat{S}_{\text{ext}}$, should describe only the propagation of the original fields $\Phi^A$, while propagation of the extra fields $\theta^a$ should be provided by the first quantum correction $\tilde{M}_1$. In other words, using the collective notation $\Phi^\mu = \{ \Phi^A, \theta^a \}$, we are led to use a solution $\hat{S}_{\text{ext}}$ of (4.11) for which\footnote{In (4.16) is the inverse matrix of $\mu^\alpha_{\beta} = \frac{\partial A_{\alpha}(\phi)}{\partial \phi^\beta}$.}

\[
\frac{\partial^\alpha}{\partial \phi^\beta} S_{\text{ext}}^\mu_{\nu} \bigg|_{\text{on-shell}} = N,
\]

whereas $\tilde{W}$ (4.10) should verify\footnote{In (4.15) is the inverse matrix of $\lambda^\beta_{\alpha} = \frac{\partial A_{\beta}(\phi, \theta^a)}{\partial \theta^a}$.}

\[
\frac{\partial^\alpha}{\partial \phi^\beta} W_{\mu\nu} \bigg|_{\text{on-shell}} = N + m.
\]

A convenient solution of (4.11) turns out to be (4.9) [8]. Indeed, it is non-proper, whereas (4.13) is guaranteed since $\hat{S}_{\text{ext}}$ contains the original proper solution $\hat{S}$. With respect to the quantum corrections, eq.(4.12) is (formally) specified once $\Delta \hat{S}_{\text{ext}}$ is known. A formal computation

\[
\Delta \hat{S}_{\text{ext}} = \Delta \hat{S} - \tilde{W}_{\mu\nu}^{\alpha\beta} C^\alpha
\]

indicates that the new degrees of freedom modifies $\Delta \hat{S}$. The new contribution is the unregularized logarithm of the jacobian of the BRST transformation for the $\theta^a$ fields. This fact indicates that the regularization procedure should be adapted to the extended theory in order to take into account contributions coming from the extra degrees of freedom.

However, the PV regularization program can not be applied in a direct way, basically because (4.13) implies that a “kinetic term” for $\theta^a$ is lacking in $\hat{S}_{\text{ext}}$. This drawback can be bypassed by taking the ansatz for the quantum action $\tilde{W}$ (4.10)

\[
\tilde{W} = [\hat{S}_{\text{ext}} + \hbar M_1^{(0)}] + \hbar M',
\]

with $M'$ containing the original one $M$ in (3.3) (and possibly including $\theta$ dependent terms at least of $O(K)$ solving (4.4) in a local way) and where\footnote{$\lambda^\beta_{\alpha}$ in (4.15) is the inverse matrix of $\mu^\alpha_{\beta} = \frac{\partial A_{\alpha}(\phi, \theta^a)}{\partial \phi^\beta}$.}

\[
M_1^{(0)}(\phi, \theta) = -i \int_0^1 A_{\beta}(F(\phi, \theta t)) \lambda^\beta_{\alpha}(\theta t, \phi) \theta^a dt,
\]

is the original Wess-Zumino term [19], i.e, the solution in the extended space of \[8\]

\[
(M_1^{(0)}, \hat{S}_{\text{ext}}) = i A_{\alpha}(\phi) C^\alpha.
\]

Indeed, with this choice, $\tilde{W}_{\mu\nu}$ contains now basically the original hessian $\hat{S}_{\text{AB}}$ plus a new non-diagonal block for the extra variables $\theta^a$, which essentially reads

\[
\left( \frac{\partial^2 M_1^{(0)}}{\partial \phi^\alpha \partial \theta^a} \right) = -i \left( \frac{\partial A_{\alpha}(\phi)}{\partial \phi^\alpha} \right) + O(\theta).
\]

In this way, taking into account (1.3), this ansatz gives the correct rank (4.14) for $\tilde{W}_{\mu\nu}$. Summarizing, at this first stage it seems plausible to consider as the action to regularize the extended theory

\[
S' = [\hat{S}_{\text{ext}} + \hbar M_1^{(0)}].
\]
• i) The new action $S'$ (1.18) does not verify the classical master equation

$$\langle S', S' \rangle = 2i\hbar A_\alpha(\phi)C^\alpha \neq 0,$$

(4.19)

so that the PV procedure of sect. 2 can no longer be considered.

• ii) The part providing for the propagation of the $\theta^\alpha$ fields in (4.18) contains explicitly an $\hbar^0$. This fact would ruin the usual $\hbar$ perturbative expansion and the tool to recognize one-loop anomalies.

A sensible PV regularization of the extended theory requires thus to extract from $S'$ (1.18) a classical part $W_0$ which constitutes a proper solution in the extended space, that is, satisfying

- a): $(W_0, W_0) = 0$,

- b): $\text{rank}(W_{0, \mu\nu})|_{\text{on-shell}} \equiv \text{rank}\left(\frac{\partial h_\mu W_0}{\partial \phi^{\alpha} \partial \phi^{\mu}}\right)|_{\text{on-shell}} = N + m$.

In what follows, we will see how this splitting can be implemented for certain systems through canonical transformations in the extra variables sector.

### 4.2 The extended proper solution $W_0$. Background terms

In order to see how to get $W_0$, let us analyze the $\theta^\alpha$ and $\theta^*_\alpha$ dependent parts of $S'$ (1.18) by expanding them in powers of $\theta^\alpha$. Working in a canonical parametrization $(\lambda^\alpha, \phi, \theta^\alpha)$, we have for (1.18)

$$\hbar M_1^{(0)}(\phi, \theta) = -i\hbar \left[ A_\alpha(\phi)\theta^\alpha + \frac{1}{2}\theta^\alpha D_{\alpha\beta}(\phi)\theta^\beta + \frac{1}{3!}\theta^\alpha \theta^\beta \theta^\gamma (\Gamma_{\alpha \beta \gamma}) (\phi) + \ldots + \frac{1}{n!} \theta^{\alpha_1} \ldots \theta^{\alpha_n} (\Gamma_{\alpha_1 \ldots \alpha_n} D_{\alpha_{n-1} \alpha_n}) (\phi) + \ldots \right],$$

(4.20)

with $\Gamma_{\alpha}$ and $D_{\alpha\beta}$ defined by

$$\Gamma_{\alpha} = R_{\alpha}^{\dot{i}} \frac{\partial}{\partial \phi^i}, \quad D_{\alpha\beta} = \Gamma_{\beta} A_{\alpha} = \left( \frac{\partial A_{\alpha}}{\partial \phi^i} R_{\dot{i}}^{\dot{i}} \right),$$

(4.21)

while the $\theta^*_\alpha$ part in (4.9), acquires the form

$$-\theta^*_\alpha \tilde{\mu}^\beta C^\beta = -\theta^*_\alpha \left[ \delta^\beta_\beta - \frac{1}{2} T_{\beta \gamma}^\beta (\dot{\phi}) \theta^\gamma + \mathcal{O}(\theta^3) \right] C^\beta.$$

(4.22)

It seems hence reasonable to make a redefinition of $\theta^\alpha$ such that it absorbs the $\hbar$ of their kinetic term, and implement it to their antifields through a canonical transformation [24], i.e.,

$$\theta'^\alpha = \sqrt{\hbar} \theta^\alpha, \quad \theta'^*_\alpha = \frac{1}{\sqrt{\hbar}} \theta^*_\alpha.$$

(4.23)

In this way, expansions (4.20) and (1.22) become, after dropping primes

$$\hbar M_1^{(0)}(\phi, \theta) \rightarrow -i \left[ \sqrt{\hbar} A_\alpha(\phi)\theta^\alpha + \frac{1}{2} \theta^\alpha D_{\alpha\beta}(\phi)\theta^\beta + \mathcal{O}(\theta^3; 1/\sqrt{\hbar}) \right],$$

(4.24)

$$-\theta^*_\alpha \tilde{\mu}^\beta C^\beta \rightarrow -\sqrt{\hbar} \theta'^*_\alpha C^\alpha + \frac{1}{2} \theta^*_\alpha T_{\beta \gamma}^\alpha (\dot{\phi}) \theta^\gamma C^\beta + \mathcal{O}(\theta^2; 1/\sqrt{\hbar}),$$

(4.25)

\[\text{In the effective theories of the standard model, where some heavy fermions are integrated out, one should also consider Wess-Zumino terms to take into account the presence of the anomaly. In this case, however, the extra variables are present already in the classical action and the above difficulties for the propagator of the extra variables do not appear.} \]
so that, although $\hbar$ dissapears in few terms or becomes $\sqrt{\hbar}$ in higher order terms it appears in the form of negative powers of $\sqrt{\hbar}$. Therefore, it seems as if the quantum treatment of Wess-Zumino terms can only be done in a nonperturbative (in the $\hbar$ expansion sense) regime. Whether or not this is true, this perturbative treatment can at least be applied in a sensible way to models for which only the first two terms in (4.24) and (4.25) are really present. In this case, the $\hbar^0$ terms should be considered part of $W_0$, while the $\sqrt{\hbar}$ terms generalize the so-called background charges [26]. In the BV context these terms have been previously considered in [24][10]. From now on, we will call them background terms.

Let us analyze the conditions that guarantee this perturbative treatment. (4.20) stops at second order if

$$\Gamma_\gamma(D_{\alpha\beta})(\phi) = 0.$$  (4.26)

On the other hand, the gauge transformations for $\theta^\alpha$ would read, in absence of $O(\theta^2)$ terms

$$\delta \theta^\alpha = -\varepsilon^\alpha + \frac{1}{2} T_{\beta\gamma}^\alpha(\phi) \theta^\gamma \varepsilon^\beta,$$

but in this form they can only provide a representation of the original gauge algebra if $T_{\beta\gamma}^\alpha = 0$. Summarizing, when $r = m$, (4.24) and $T_{\beta\gamma}^\alpha = 0$ are sufficient conditions in order to have a sensible perturbative expansion. In this case, after having performed (4.23), $S' (4.18)$ becomes

$$S' \rightarrow \left[ \hat{S}(\Phi, K) - \frac{i}{2} \theta^\alpha D_{\alpha\beta}(\phi) \theta^\beta \right] - \sqrt{\hbar} [\theta^\alpha C^\alpha + i A_\alpha(\phi) \theta^\alpha] \equiv W_0 + \sqrt{\hbar} M_{1/2},$$

from which the form of $W_0$ can immediately be read off

$$W_0 = \hat{S}(\Phi, K) - \frac{i}{2} \theta^\alpha D_{\alpha\beta}(\phi) \theta^\beta.$$  (4.27)

Under such conditions, eq.(4.19) translates to

$$(W_0, W_0) = 0,$$  (4.28) $$\langle W_0, M_{1/2} \rangle = 0,$$  (4.29)$$\frac{1}{2} \langle M_{1/2}, M_{1/2} \rangle = i A_\alpha(\phi) C^\alpha,$$  (4.30)

thus indicating that $W_0$ (4.27) satisfies indeed the classical master equation.

Finally, properness of $W_0$ holds if rank($D_{\alpha\beta}$) = $p = m$. However, in view of (4.21), in general one can only guarantee $p \leq m$. The extremum case appears, for example, when the anomalies are gauge invariant, i.e., $D_{\alpha\beta} = 0$ and $p = 0$. For the sake of brevity, from now on we will restrict ourselves to the case $p = \text{max.}$, leaving the general case for the future.

### 4.3 Abelian anomalous subgroup

The above described situation is too restrictive and somewhat trivial. In what follows, we consider the more interesting case $r < m$ in (4.3), in which only an anomalous subgroup (the $a$ part) is abelian.

The extension procedure goes along the same lines discussed before and is based in the introduccion of $r$ new fields $\theta^a$. The subgroup character of the anomalous ($a$) part implies now that the transformation for the new fields, the extended action $S_{\text{ext}}$ and the Wess-Zumino term $M_{1/2}^{(0)}$ are obtained from (4.7), (4.9) and (4.16) by simply considering the $a$ subgroup as a group by itself; in brief, by the substitution $\alpha \rightarrow a$ and by the restriction of all the quantities to $\theta^A = 0$. 

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The only non-trivial objects of this construction relevant for our analysis are the type \( A \) gauge generators
\[
\tilde{\mu}^a_B(\theta^a, \phi) = (\tilde{\mu}^a_B + \mu^a_B \lambda^b_D \tilde{\mu}^D_B)\big|_{\theta^A = 0},
\]
with \( \tilde{\mu}^a_B, \mu^a_B, \lambda^b_D \) and \( \tilde{\mu}^D_B \) the corresponding blocks of the matrices \( \mu^a_B \) \( \text{[8]} \) and \( \mu^a_B, \lambda^b_\beta \) defined above. For a full description of the extension procedure in this case we refer the reader to \( \text{[8]} \).

It is clear thus that, as far as the \( a \) subgroup is concerned, the previous conditions \( \text{[1.26]} \) and \( T^a_{\beta\gamma} = 0 \) translates to
\[
\Gamma_c(D_{ab})(\phi) = 0, \quad \text{with} \quad D_{ab}(\phi) = \Gamma_b A_a(\phi),
\]
and \( T_{bc} = 0 \), i.e., the \( a \) subgroup should be abelian. On the other hand, the expansion for the \( A \) generators \( \text{[1.31]} \) becomes, for gauge invariant structure functions
\[
\tilde{\mu}^a_B(\theta^a, \phi) = (\tilde{\mu}^a_B + \mu^a_B \lambda^b_D \tilde{\mu}^D_B)\big|_{\theta^A = 0} = -T^a_{Bb} \theta^b + \frac{1}{4} T^b_{Dd} T^a_{Bb} \theta^b \theta^d + \frac{1}{24} T^a_{Dd} T^D_{Cc} T^C_{Bb} \theta^b \theta^c \theta^d + \ldots,
\]
thus indicating that the \( A \) transformations can be taken linear in \( \theta^a \) if
\[
T^a_{Dd} T^D_{Bb} = 0.
\]
This requirement is met, for instance, when either \( T^a_{Dd} = 0 \) and/or \( T^D_{Bb} = 0 \). In the end, a direct computation shows that
\[
\delta \theta^a = -\varepsilon^a + T^a_{Bb}(\phi) \theta^b \varepsilon^B,
\]
provide a representation of the original gauge algebra if precisely i) the structure functions are gauge invariant and ii) \( \text{[4.33]} \) holds. Summing up, conditions \( \text{[4.32]}, \text{[4.33]} \), \( T_{bc} = 0 \) and \( \Gamma_{\sigma}(T^\sigma_{\alpha\beta}) = 0 \) guarantee a sensible perturbative expansion for this extended theory. Under such conditions, the \( A \) transformation for the kinetic operator \( D_{ab} \) \( \text{[1.32]} \) reads
\[
\delta_{(A)} D_{ab} = (D_{ac} T^c_{Bb} + D_{bc} T^c_{aB}) \varepsilon^B.
\]

Now, the canonical transformation \( \text{[4.23]} \) adapted to this case brings \( S' \) \( \text{[4.18]} \) to the form
\[
S' \rightarrow \left[ \hat{S}(\Phi, K) - \frac{i}{2} \theta^a D_{ab}(\phi) \theta^b + \theta^a T^a_{Bb}(\phi) \theta^b C^B \right] - \sqrt{K} \left[ \theta^a C^a \right. + \left. i A_a(\phi) \theta^a \right] \equiv W_0 + \sqrt{K} M_{1/2},
\]
thus providing the following expression for \( W_0 \)
\[
W_0 = \hat{S}(\Phi, K) - \frac{i}{2} \theta^a D_{ab}(\phi) \theta^b + \theta^a T^a_{Bb}(\phi) \theta^b C^B.
\]
Once again, relations \( \text{[4.28]}, \text{[4.24]} \) and \( \text{[4.30]} \) hold now for \( W_0 \) and the background term \( M_{1/2} \).

A direct check of these relations gives some additional information. For example, fulfillment of \( \text{[4.28]} \) requires \( D_{ab} \) in \( \text{[4.36]} \) to be the \( a \) gauge variation of some consistent anomalies \( A_a \), while the vanishing of \( \text{[4.29]} \), instead, crucially relies on \( D_{ab} \) being precisely the \( a \) gauge variation of \( A_a \) occurring in \( M_{1/2} \). These two conditions are summarized in the relation \( D_{ab} = \Gamma_b A_a \) in \( \text{[1.32]} \). Finally, the properness condition for \( W_0 \) holds as far as \( \text{rank}(D_{ab}) = r \), which we will assume from now on.

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4.4 Regularization of the extended theory

To develop the regularization procedure, we will be mainly concerned with the last situation, since the first one–once the substitution \( a \to \alpha \) is done– can be seen as a special case of it, with no antifields \( \theta^a \) in \( W_0 \).

From the above discussion, and according to (4.15) and to the form of \( W_0 \) (4.36), it is expected that the new variables generates an extra anomaly, for which the antifield independent\( \theta \) no antifields since the first one –once the substitution

\[
B_a(\phi, \theta)C^a = B_a(\phi)C^a + \mathcal{O}(\theta) = b_k A^k_a(\phi)C^a + \mathcal{O}(\theta).
\]

(4.37)

The contribution \( \mathcal{O}(\theta) \) results to be relevant at order \( \hbar^{3/2} \) or higher. Indeed, if we undo (4.23), \( \mathcal{O}(\theta) \) become \( \mathcal{O}(\sqrt{\hbar}) \), contributing thus at higher orders in \( \hbar \). Thus, in one loop considerations, \( \mathcal{O}(\theta) \) terms can be discarded. Quantum effects of the extra variables are then expected to be realized as shifts or “renormalizations” of the original coefficients of the anomalies \( A_a(\phi) \), \( a_k \to \tilde{a}_k = (a_k + b_k) \), and, as a byproduct, of the coefficients of all the quantities directly related with them, like the Wess-Zumino term \( \tilde{M}^{1(0)} \) or the kinetic operator \( D_{ab} \).

Let us now regularize the extended theory along the lines of sect.2.3. As the analogous of the regulated generating functional (2.15), we take

\[
Z_{\text{reg}}(J, K; j, \theta^a) = \int \mathcal{D}\Phi\mathcal{D}\bar{\chi}\exp \left\{ i \hbar \left[ \hat{W} + J_A \Phi^A + j_a \theta^a \right] \right\}_{\bar{\chi}^* = 0},
\]

(4.38)

with \( \hat{W} \) given by

\[
\hat{W} = \left[ \tilde{W}_0 + W_{\text{PV}} + \sqrt{\hbar}M_{1/2} + \hbar \tilde{M} \right],
\]

(4.39)

and where \( \Phi \equiv \{ \Phi^\mu \} \) stands for the complete set of fields \( \{ \Phi^A, \theta^a \} \) and \( \bar{\chi} \equiv \{ \bar{\chi}^\mu \} \) for their associated PV fields \( \{ \chi^A, \chi^a \} \).

According to the above discussion, in the new proper classical action \( \hat{W}_0 \) in (4.39), the \( \theta^a \) kinetic operator should be taken as

\[
\hat{D}_{ab} = c_k D^k_{ab}, \quad \text{with} \quad D^k_{ab} = \left( \frac{\partial A^k_a}{\partial \theta^b} \hat{R}^k_b \right),
\]

(4.40)

i.e., the original coefficients \( a_k \) in (1.3) has been relaxed to \( c_k \) and should be determined in the regularization procedure. \( M_{1/2} \), on the other hand, is expected to be related with the renormalization of the original background term \( M_{1/2} \) in (4.35), while \( \tilde{M} \) is a suitable counterterm taking care of possible dependences on the auxiliar sector and on the antifields.

Finally, the PV action \( W_{PV}|_{\chi^* = 0} \) in (4.39) is determined from \( \hat{W}_0 \) in (4.39) by means of (2.16)

\[
W_{PV}|_{\chi^* = 0} = W_{PV}^{(0)}|_{\chi^* = 0} + W_M = \frac{1}{2} \chi^\mu (\hat{T} \hat{R})_{\mu\nu} \chi^\nu - \frac{1}{2} M \hat{\chi}^\mu \hat{T}_{\mu\nu} \hat{\chi}^\nu,
\]

with \( (\hat{T} \hat{R})_{\mu\nu} \) defined according to (2.17) as

\[
(\hat{T} \hat{R})_{\mu\nu} = \left( \frac{\partial_l}{\partial \Phi^\mu} \frac{\partial_r}{\partial \Phi^\nu} \hat{W}_0 \right),
\]

(4.41)

and where we choose the mass term \( W_M \) with no mixing between the original PV fields \( \chi^A \) and the extra ones \( \chi^a \) and containing the mass matrix \( T_{AB} \) used in the regularization of the original theory, i.e.,

\[
W_M = -\frac{1}{2} M \hat{\chi}^\mu \hat{T}_{\mu\nu} \hat{\chi}^\nu = -\frac{1}{2} M \left( \chi^A T_{AB} \chi^B + \chi^a T_{ab} \chi^b \right).
\]

(4.42)
Now, an straightforward application of the semiclassical expansion to (1.38) yields the effective action of the extended theory up to one loop

\[ \hat{\Gamma} = \hat{W}_0 + \sqrt{\hbar} \hat{M}_{1/2} + \hbar \hat{M} + \frac{i\hbar}{2} \text{Tr} \ln \left( \frac{(\hat{T} \hat{R})}{(\hat{T} R) - TM} \right), \]

(4.43)

while from its corresponding BRST Ward identity (2.12) the following anomaly arises

\[ -i\hbar \hat{A} = \sqrt{R}(\hat{M}_{1/2}, \hat{W}_0) - i\hbar \left[ (\Delta \tilde{W}_0)_{\text{reg}} + i(\hat{M}, \hat{W}_0) + \frac{i}{2}(\hat{M}_{1/2}, \tilde{M}_{1/2}) \right]. \]

(4.44)

Finally, the regularized expression of \((\Delta \tilde{W}_0)\) (or, equivalently, of \((\Delta \tilde{S}_{\text{ext}})\) since both share the same unregularized form) turns out to be, once again, the BRST variation of the trace part in (4.43)

\[ (\Delta \tilde{W}_0)_{\text{reg}} = \delta \left\{ -\frac{1}{2} \text{Tr} \ln \left( \frac{R}{R - M} \right) \right\} = \text{Tr} \left[ -\frac{1}{2}(\tilde{R}^{-1} \bar{\delta} \tilde{R}) \frac{1}{1 - R/M} \right], \]

(4.45)

where \(\bar{\delta}\) is now the BRST transformation in the extended space generated by \(\hat{W}_0, \bar{\delta}^R = (F, \tilde{W}_0)\).

Let us now investigate the form of \(\hat{R}\) along the lines of sect.3 to gain insight into the structure of (4.45). By direct inspection of \(\hat{W}_0\) in (4.39), \((\hat{T} \hat{R})_{\mu\nu}\) (4.41) is seen to be

\[ (\hat{T} \hat{R})_{\mu\nu} = \left( TR_{AB}(\Phi, K) \begin{array}{c} \begin{array}{c} -i \hat{D}_{ab}(\phi) \\
\end{array} \end{array} + \mathcal{O}(\theta) + \mathcal{O}(\theta^*), \right. \]

with \((TR)_{AB}\) the original massless kinetic operator (2.13). Then, the inverse of the new mass matrix in (4.42) leads to the extended regulator \(\hat{R}\)

\[ (\hat{T}^{-1})^{\mu\nu}(\hat{T} \hat{R})_{\mu\nu} \equiv \hat{R}^{\mu\nu} = \left( \begin{array}{cc} R^A_B \begin{array}{c} \begin{array}{c} -i(T^{-1})^{ac} \hat{D}_{cb}(\phi) \\
\end{array} \end{array} + \mathcal{O}(\theta) + \mathcal{O}(\theta^*). \right. \]

Using now the expansion (4.1) for the original regulator \(R\) and assuming an expansion for \(T_{ab}\) of the form \(T_{ab} = T_{0,ab}(\phi) + \ldots\), with \(T_{0,ab}(\phi)\) invertible, we obtain a similar expansion for \(\hat{R}\)

\[ \hat{R} = [\hat{R}(\phi) + \mathcal{O}(\theta)] + \hat{R}, \quad \text{with} \quad \hat{R}^{\mu\nu}(\phi) = \left( \begin{array}{cc} R^A_B(\phi) \begin{array}{c} \begin{array}{c} -i(T^{-1})^{ac} \hat{D}_{cb}(\phi) \end{array} \end{array} + \mathcal{O}(\theta) + \mathcal{O}(\theta^*). \right. \]

(4.46)

and where \([\hat{R}(\phi) + \mathcal{O}(\theta)]\) plays now an analogous role as that of \(\mathcal{R}(\phi)\) (4.1) in the original theory. Similar expansions are shared by \(\hat{R}^{-1}\) and \(\bar{\delta} \hat{R}\) due to the linearity in \(\theta^a\) of \(\bar{\delta} \theta^a\). In the end, plugging all these results in \((\Delta \tilde{W}_0)_{\text{reg}}\) (4.45), the following expression is obtained

\[ (\Delta \tilde{W}_0)_{\text{reg}} = \left\{ \text{Tr} \left[ -\frac{1}{2}(\hat{R}^{-1} \bar{\delta} \hat{R}) \frac{1}{1 - \hat{R}/M} \right] + \mathcal{O}(\theta) \right\} + (\Delta \tilde{W}_0)_{\text{reg}} + (\mathcal{M}', \hat{W}_0), \]

(4.47)

with \((\Delta \tilde{W}_0)_{\text{reg}} = \mathcal{O}(K, \theta^*)\) and where each one of the terms in (4.47) is \(\bar{\delta}\) invariant by itself.

Let us restrict now to the study of the antifield independent part in (4.47). \(\mathcal{O}(\theta)\), as argued below, can simply be discarded. On the other hand, the diagonal structure of \(\hat{R}(\phi)\) (4.46) yields the following decomposition for the trace in eq. (4.47)

\[ \text{Tr} \left[ -\frac{1}{2}(\hat{R}^{-1} \bar{\delta} \hat{R}) \frac{1}{1 - \hat{R}/M} \right] = \text{Tr} \left[ -\frac{1}{2}(R^{-1} \delta R) \frac{1}{1 - R/M} \right] + \text{Tr} \left[ -\frac{1}{2}(R_{\theta}^{-1} \delta R_{\theta}) \frac{1}{1 - R_{\theta}/M} \right], \]

(4.48)
The one-loop renormalized action up to one loop in the antifield independent sector. Implementation of these conditions leads to the independent part of \( \tilde{\mathcal{A}} \) with extra degrees of freedom realized as a shift or renormalization of the coefficients \( \delta \phi^i = \hat{\delta} \phi^i = R^i_\alpha C^\alpha \).

The first trace in the right-hand side of (4.43) is the original anomaly, \( A_a(\phi)C^a \), whereas the second trace should be considered as the antifield independent contribution (4.37) to the anomaly coming from the extra fields. This second term could produce type \( A \) anomalies unless an \( A \) invariant regulator \( \mathcal{R}_\theta \) (4.49) is found. In appendix C we argue that the transformation property of \( T_{0,ab}(\phi) \) under the \( A \) subgroup

\[
\delta_{(A)} T_{0,ab} = (T_{0,ac} T_{bA}^c + T_{0,bc} T_{aA}^c) \varepsilon^A,
\]

is a sufficient condition to get this result. Assuming then that such a mass matrix has been found, the final form for \( (\Delta \tilde{W}_0)_{\text{reg}} \) (4.47) reads

\[
(\Delta \tilde{W}_0)_{\text{reg}} = (A_a + B_a)(\phi)C^a + (\tilde{\Delta} \tilde{W}_0)_{\text{reg}} + (\mathcal{M}', \tilde{W}_0).
\]

It is straightforward now to determine \( \tilde{M}_{1/2} \) and \( \tilde{W}_0 \) which yield the vanishing of the antifield independent part of \( \tilde{\mathcal{A}} \) (4.44). Indeed, the \( h \) part of (4.44) vanishes for

\[
\tilde{M}_{1/2} = - \left[ \theta^a_\alpha C^a + i(A_a + B_a)(\phi)\theta^a \right] = - \left[ \theta^a_\alpha C^a + i\tilde{A}_a(\phi)\theta^a \right],
\]

with \( \tilde{A}_a = (a_k + b_k)A_k^a \), whereas vanishing of the \( \sqrt{h} \) term is acquired for \( \tilde{D}_{ab} \) (4.40) of the form

\[
\tilde{D}_{ab} = \left( \frac{\partial \tilde{A}_a}{\partial \phi^i} \right) = \tilde{D}_{ab}.
\]

Equations (4.50) and (4.51) express hence the conditions for the vanishing of the antifield independent part of \( \tilde{\mathcal{A}} \) and, as a consequence, for the (partial) fulfillment of (2.12) for \( \tilde{\Gamma} \) (4.43) up to one loop in the antifield independent sector. Implementation of these conditions leads to the one-loop renormalized action

\[
\tilde{W}_0 + \sqrt{h} \tilde{M}_{1/2} + h \tilde{M} = \left[ \tilde{S}(\Phi, K) - \frac{i}{2} \theta^a_\alpha \tilde{D}_{ab} \theta^b + \theta^a_\alpha T_{Bb}^a \theta^b C^B \right] - \sqrt{h} \left[ \theta^a_\alpha C^a + i\tilde{A}_a \theta^a \right] + h \tilde{M},
\]

which, in terms of the original extra variables (4.23) becomes the solution (4.10) of the regularized quantum master equation at one-loop level in the extended formalism, i.e.,

\[
\left[ \tilde{S}(\Phi, K) - \theta^a_\alpha \left( C^a - T_{Bb}^a \theta^b C^B \right) \right] - i h \left[ \tilde{A}_a \theta^a + \frac{1}{2} \theta^a \tilde{D}_{ab} \theta^b \right] + h \tilde{M} = \tilde{S}_{\text{ext}} + h \tilde{M}_{1} \equiv \tilde{W}.
\]

In summary, from (4.52) it is concluded that, at source independent level, the effect of the extra degrees of freedom is realized as a shift or renormalization of the coefficients \( a_k \) of the original Wess-Zumino term to \( (a_k + b_k) \), as argued before. This ends the description of the regularization procedure in the extended theory.

\[\text{Invariance of regulators should be understood up to terms of the form \([\mathcal{R}, \mathcal{G}]\), i.e., they are invariant as far as they yield vanishing anomalies.}\]
5 Invariant Pauli-Villars regularization in the extended configuration space

Anomalous gauge theories are known to suffer from the absence of BRST (or gauge) invariant regulators in the original configuration space. Within the above extended formalism, instead, they give rise to BRST invariant theories up to one loop in the antifield independent sector. This fact suggests the existence of PV invariant regulators \( \tilde{R}'(\phi, \theta) \) in the extended formalism. Such possibility has been considered in [24], although this formulation differs in spirit from ours. In this section we shall show, first, how to construct such invariant PV regulators and second, how a natural interpretation arises of the Wess-Zumino action as the local counterterm interpolating between invariant and noninvariant regularizations.

5.1 Completely anomalous gauge theory

To illustrate the construction, let us consider first of all the case \( r = m \) in \((4.5)\). In the extended theory the combinations \( F_i(\phi, \theta) \) result to be gauge (or BRST) invariant. Therefore, an invariant regulator \( R'(\phi, \theta) \) can be built up from a non-invariant one \( R(\phi) \) by the simple rule of substituting the fields \( \phi \) in \( R \) by their gauge transformed \( F_i(\phi, \theta) \), that is,

\[
R'(\phi, \theta) \equiv R(F(\phi, \theta)) \Rightarrow \delta R' = 0. \tag{5.1}
\]

The construction of invariant regulators in this way turns out to be a useful tool to “integrate” anomalies and obtain the Wess-Zumino action. This observation is based on the following facts.

First of all, eq.\((4.17)\) for \( M_1(0)(\phi, \theta) \) can be interpreted as the expression relating the anomalies

\[
iB_1(\phi, C^\alpha) = 0, \quad iB_0(\phi, C^\alpha) = iA_\alpha(\phi)C^\alpha,
\]

arising in the invariant \((1)\) and non-invariant \((0)\) regularizations, through the BRST variation of a local counterterm in the extended configuration space, i.e.,

\[
iB_1 - iB_0 = -iA_\alpha(\phi)C^\alpha = \delta(-M_1(0)(\phi, \theta)) \iff (M_1(0), \tilde{S}_{\text{ext}}) = iA_\alpha(\phi)C^\alpha.
\]

On the other hand, these two regularizations are connected by the interpolation

\[
R(t) = R(F(\phi, \theta t)), \quad t \in [0, 1]. \tag{5.2}
\]

Under such conditions, we can apply the results in appendix B and take for \( M_1(0)(\phi, \theta) \) expression \((B.4)\) adapted to this case, namely

\[
M_1(0)(\phi, \theta) = -i \int_0^1 dt \text{Tr} \left\{ -\frac{1}{2} \left[ \tilde{R}^{-1}(F(\phi, \theta t)) \partial_t \tilde{R}(F(\phi, \theta t)) \right] \frac{1}{(1 - \tilde{R}(F(\phi, \theta t))/M)} \right\}. \tag{5.3}
\]

Now, explicit computation of the \( \partial_t \) derivative of \( R(t) \) \((5.2)\) yields, after use of the Lie equation for \( F_i(\phi, \theta) \) \[8\]

\[
\partial_t R(F(\phi, \theta t)) = \left( \frac{\partial R}{\partial \phi^i} \right)(F) \partial_t F^i(\phi, \theta t) = \left( \frac{\partial \tilde{R}}{\partial \phi^i} \tilde{R}_i^\alpha(F) \right) \lambda^\beta_\alpha(\theta t, \phi) \theta^\alpha = (\delta_{\beta \alpha} R)(F) \lambda^\beta_\alpha(\theta t, \phi) \theta^\alpha,
\]

\[\text{Invariant PV regularizations of this type were earlier considered, for chiral gauge theories, in [23]. More recently, a similar invariant PV regularization has also been used in [28] in the quantization of the two dimensional chiral Higgs model.}\]
where \((\delta \beta R)\) stands for the BRST (or gauge) variation of \(R\) having dropped out the ghosts \(C^\beta\) (or the gauge parameters \(\varepsilon^\beta\)). Upon substitution of this result in (5.3)

\[
M_1^{(0)}(\phi, \theta) = -i \int_0^1 dt \text{Tr} \left[ -\frac{1}{2} \frac{1}{(R^{-1} \delta \beta R)(1 - R/M)} \right] (F) \lambda_\alpha ^\beta (\theta t, \phi) \theta^\alpha, \tag{5.4}
\]

we recognize in the trace factor the form (1.2) for the anomaly with argument \(F^i(\phi, \theta t)\), so that (5.4) acquires the form (4.16) of the Wess-Zumino term for the original theory. This expression was previously derived in [8] using a different approach.

Therefore, from this construction a new interpretation [24] of the Wess-Zumino term arise: it is the local counterterm giving the interplay between the original, non-invariant regularization and the new invariant one (5.1).

### 5.2 Anomalous free gauge subgroup

The situation just considered is very restrictive since certain theories possess regulators preserving a subgroup (the \(A\) part) of the gauge transformations. Therefore, a modification of the above proposal should be considered. We restrict our analysis to the case in which the anomalous (\(a\)) sector is a subgroup.

Assume then that the original regulator \(R(\phi)\) satisfies (4.6). The analogous of the invariant objects \(F^i(\phi, \theta)\) considered in the previous case are now the combinations \(\tilde{F}^i(\phi, \theta^a) = F^i(\phi, \theta^a, \theta^A = 0)\), with transformation laws

\[
\delta_{(a)} \tilde{F}^i = 0, \quad \delta_{(A)} \tilde{F}^i = R^i_B (\tilde{F}) M^B_A \varepsilon^A,
\]

and with \(M^B_A\) an invertible matrix whose form is irrelevant for our purposes [8]. Then, the desired invariant regulator \(R'(\phi, \theta)\) turns out to be in terms of the original one

\[
R'(\phi, \theta^a) = R(\tilde{F}(\phi, \theta)). \tag{5.5}
\]

Indeed, invariance of \(R'(\phi, \theta)\) under \(a\) transformations comes from the \(a\) invariance of \(\tilde{F}^i\), \(\delta_{(a)} \tilde{F}^i = 0\), as in the previous case, while for the \(A\) part it is

\[
\delta_{(A)} R' = \left( \frac{\partial R}{\partial \theta^A} R^i_A \right) (\tilde{F}) \varepsilon^A = (\delta_{(A)} R)(\tilde{F}) \varepsilon^A = \left\{ \begin{array}{ll}
0 & [R, G_B](\tilde{F}) M^B_A \varepsilon^A
\end{array} \right\
\]

the result being now a direct consequence of the invariance (4.6) of \(R\) under the \(A\) subgroup.

Finally, the Wess-Zumino term \(M_1^{(0)}(\phi, \theta^a)\), interpreted again as the counterterm relating the anomalies

\[
iB_1(\phi, C^a) = 0, \quad iB_0(\phi, C^a) = iA_a(\phi) C^a,
\]

obtained using invariant (1) and non-invariant (0) regulators, can be constructed along the previous lines simply by substituting the above quantities by those corresponding to the \(a\) subgroup as a group by itself. In particular, the interpolating regularization between \(R(\phi)\) and \(R'(\phi, \theta^a)\) reads now

\[
R(t) = R(\tilde{F}(\phi, \theta^a t)), \quad t \in [0, 1], \tag{5.6}
\]

yielding at the end the same form for \(M_1^{(0)}(\phi, \theta^a)\) worked out in [8] for this particular case.
6 Example: the Bosonic String

In this section we illustrate the use of the extended formalism by applying it to the bosonic string. In this way we will see that a natural interpretation arises of the well-known shift of the numerical coefficient \(26 - D\) in front of the Liouville action to \(25 - D\), in agreement with \[31\]. This model will also serve to exemplify the method proposed in sect.5 for constructing invariant regulators and Wess-Zumino actions in the extended configuration space.

6.1 Regularization of the original theory

The bosonic string is an example of a gauge theory in which part of the gauge group can be kept anomaly free while the anomalous part can be chosen to be an abelian subgroup. The classical action for this system

\[ S_0 = \int d^2 \xi \left[ -\frac{1}{2} \sqrt{-\mathbf{g}} g^{\alpha \beta} \partial_\alpha \xi \partial_\beta \xi \right], \quad \text{with} \quad g \equiv -\det g_{\alpha \beta}, \]

describes \(D\) bosons \(X^\mu(\xi)\) coupled to the gravitational field \(g_{\alpha \beta}\) in two dimensions and possesses the following (infinitesimal) gauge transformations

\[
\delta X^\mu = v^\alpha \partial_\alpha X^\mu, \\
\delta g_{\alpha \beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha + \lambda g_{\alpha \beta},
\]

which split into two subgroups: Weyl transformations \((\lambda)\) and diffeomorphisms \((v^\alpha)\).

Direct application of the FA formalism yields as proper solution of \((6.2)\)

\[
S = S_0 + \int d^2 \xi \left[ X^\star C^\alpha \partial_\alpha X + g^{\alpha \beta} \left( \nabla_\alpha C_\beta + \nabla_\beta C_\alpha + C g_{\alpha \beta} \right) - C_{\beta}^\alpha C^\beta \partial_\alpha C - C^\star C^\alpha \partial_\alpha C + b^{\alpha \beta} d^{\alpha \beta} \right],
\]

(6.1)

where \(\{C^\alpha, C\}\) are the diffeomorphisms and Weyl ghosts, respectively, \(\{X^\star, g^{\alpha \beta}, C_\alpha, C^\star\}\) the antifields of the minimal sector fields and \(\{b^{\alpha \beta}, d^{\alpha \beta}, b_{\alpha \beta}^*, d_{\alpha \beta}^*\}\) the fields and antifields of the auxiliary sector.

A usual gauge-fixing fermion is \(\Psi = -1/2 b^{\alpha \beta} (g_{\alpha \beta} - h_{\alpha \beta})\), with \(h_{\alpha \beta}\) a given background metric. Using now \((6.2)\), the antifields acquiring a shift are

\[
g^{\alpha \beta} \to g^{\alpha \beta} - \frac{1}{2} b^{\alpha \beta}, \quad b_{\alpha \beta}^* \to b_{\alpha \beta}^* - \frac{1}{2} (g_{\alpha \beta} - h_{\alpha \beta}),
\]

so that in the new gauge-fixed basis, \(S\) \((6.1)\) adopts the form

\[
\hat{S}(\Phi, K) = \int d^2 \xi \left\{ \frac{1}{2} X \Box X - \frac{1}{2} b^{\alpha \beta} \left( \nabla_\alpha C_\beta + \nabla_\beta C_\alpha + C g_{\alpha \beta} \right) - \frac{1}{2} d^{\alpha \beta} (g_{\alpha \beta} - h_{\alpha \beta}) \right\} + \left[ X^\star g^{\alpha \beta} \partial_\alpha X + g^{\alpha \beta} \left( \nabla_\alpha C_\beta + \nabla_\beta C_\alpha + C g_{\alpha \beta} \right) - C_{\beta}^\alpha C^\beta \partial_\alpha C - C^\star C^\alpha \partial_\alpha C + b_{\alpha \beta}^* d^{\alpha \beta} \right]\}
\]

\[= S_\Sigma(\Phi) + K_A R^A, \quad (6.2)\]

with \(R^A\) the BRST transformation of the field \(\Phi^A\) and where, for simplicity, antifields \(\Phi_\alpha^\star\) and BRST sources \(K_A\) are identified. Also, the kinetic operator for the matter fields \(X^\mu\) in \((3.2)\) is defined by \(\Box = \partial_\alpha (\sqrt{gg^{\alpha \beta}} \partial_\beta) = \sqrt{gg^{\alpha \beta}} \nabla_\alpha \nabla_\beta\).

\(^{13}\)Quantization of the bosonic string as an anomalous gauge theory along the lines of the hamiltonian BRST formalism \(\[20\]\) has recently been considered in \(\[30\]\). 

\(^{14}\)For earlier comments about this shift, see ref.\(\[32\]\).
The form of the gauge-fixed action $S_\Sigma$ in (6.2) suggests some field redefinitions in order to distinguish propagating and non-propagating fields. Indeed, by introducing a new symmetric, traceless field $b^{\alpha\beta}$ and a new pair of ghosts $\tilde{b}, \tilde{C}$, related to the old ones $\tilde{b}^{\alpha\beta}, C$ by

$$b^{\alpha\beta} = \tilde{b}^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} \tilde{b}, \quad C = \tilde{C} - \nabla_\alpha C^\alpha,$$

the gauge-fixed action adopts the form

$$S_\Sigma(\Phi) = \int d^2\xi \left[ \frac{1}{2} \Box X - \frac{1}{2} \tilde{b}^{\alpha\beta} (\nabla_\alpha C_\beta + \nabla_\beta C_\alpha) - \frac{1}{2} \tilde{b} \tilde{c} - \frac{1}{2} d^{\alpha\beta} (g_{\alpha\beta} - h_{\alpha\beta}) \right],$$

and allows to identify the ghosts $\tilde{b}, \tilde{C}$ and the fields $d^{\alpha\beta}, g_{\alpha\beta}$ as non-propagating. These fields will not occur in loops and their contribution to the anomaly is expected to vanish.

Now, let us pass to analyze the regularized expression of $\Delta S$. The regulator $R(\Phi, K)$ in (2.20) is determined from the PV massless kinetic operator $(TR)_{AB}$ (2.19) and the PV mass matrix $T_{AB}$. In the basis $\{X, \tilde{b}^{\alpha\beta}, C^\alpha, d^{\alpha\beta}, g_{\alpha\beta}, \tilde{b}, \tilde{C}\} \equiv \{p; np\}$, $(TR)_{AB}$ adopts the form

$$(TR)_{AB}(\Phi, K) = \begin{pmatrix} (TR)_p(g_{\alpha\beta}) & (TR)_{np} \\ (TR)_{np}^T & O(X) \end{pmatrix} + O(X) + (TR)_{AB}(\Phi, K),$$

where the corresponding *invertible* blocks for propagating and non-propagating fields read

$$(TR)_p(g_{\alpha\beta}) = \begin{pmatrix} \Box & (TR)_{\alpha\beta\gamma} \\ -(TR)_{\alpha\beta\gamma} & (TR)_{\alpha\beta\gamma} \end{pmatrix}, \quad (TR)_{np} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}.$$
We can thus restrict the analysis of (6.4) to the contribution coming from $R(g_{\alpha\beta})$. First of all, the constant block $R_{np}$ in (6.4) produces, upon the $\delta$ variation of $R$, a vanishing entry for this part. The relevant part in (6.4) then becomes

$$\text{Tr} \left[ -\frac{1}{2} (R_{p}^{-1} \delta R_{p}) \frac{1}{(1 - R_{p}/M)} \right],$$

(6.5)

yielding a vanishing contribution of the non-propagating fields to the anomaly. On the other hand, the diagonal block structure of $R_{p}$ (6.3) splits (6.5) into two separate contributions

$$\text{Tr} \left[ -\frac{1}{2} (R_{m}^{-1} \delta R_{m}) \frac{1}{(1 - R_{m}/M)} \right] + \text{Tr} \left[ -\frac{1}{2} (R_{gh}^{-1} \delta R_{gh}) \frac{1}{(1 - R_{gh}/M)} \right],$$

(6.6)

coming each one from the matter field sector and from the ghost fields ($\tilde{b}^{\alpha\beta}, C^{\alpha}$) sector, respectively, and with the matter regulator $R_{m}$ given by

$$R_{m} = \frac{1}{\sqrt{g}} \partial_{\alpha} (\sqrt{g} g^{\alpha\beta} \partial_{\beta}) = g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta}.$$  

(6.7)

Now, for definiteness, let us analyze the matter contribution in (6.6) as an example of the use of the form (2.20) for $(\Delta S)_{\text{reg}}$. First, the BRST variation of $R_{m}$ (6.7) reads

$$\delta R_{m} = -R_{m} C - \left[ R_{m}, G \right], \quad G = C^{\alpha} \partial_{\alpha},$$

so that, whereas it transforms in the “appropriate” way under diffeomorphisms, its Weyl variation neither is zero nor it can be written in commutator form. The whole gauge group splits thus into two subgroups, one of them, diffeomorphisms, anomaly free and the other, the abelian Weyl subgroup, anomalous. Then, upon substitution of this result in the first term of (6.6), we obtain

$$\text{Tr} \left[ -\frac{1}{2} (R_{m}^{-1} \delta R_{m}) \frac{1}{(1 - R_{m}/M)} \right] = \text{Tr} \left[ \frac{1}{2} C \left( 1 - \frac{\Box}{\sqrt{g} M} \right)^{-1} \right] \sim \text{Tr} \left[ \frac{1}{2} C \exp \left\{ \frac{\Box}{\sqrt{g} M} \right\} \right],$$

(6.8)

which turns out to be, using well-known results on the calculation of the Weyl anomaly

$$-i \int d^{2}\xi \left[ \left( \frac{D}{8\pi} \right) \sqrt{g} R - \left( \frac{D}{48\pi} \right) \sqrt{g} R \right] C \equiv A_{m}(g_{\alpha\beta}) \cdot C,$$  

(6.9)

where $R$ is the scalar curvature and $A_{m}(g_{\alpha\beta})$ stands for the contribution of the matter sector to the Weyl anomaly. Contributions to (6.5) coming from the ghost sector can be treated in the same way using the regulator $R_{gh}$ obtained from $(TR)_{\alpha\beta\gamma}$ and $T_{gh}$ proposed in [33]. The net effect of these contributions is a shift in the above numerical coefficients by

$$-\frac{D}{48\pi} \rightarrow \frac{26 - D}{48\pi}, \quad \frac{D}{8\pi} \rightarrow \frac{D - 2}{8\pi}.$$  

Finally, the divergent pieces arising after regularization in (6.6) (partly displayed in (6.9)) are seen to be absorbed by the BRST variation of a suitable local counterterm

$$\int d^{2}\xi \left[ \left( \frac{D - 2}{8\pi} \right) \sqrt{g} C \right] \delta \left\{ \int d^{2}\xi \left[ \left( \frac{D - 2}{8\pi} \right) \sqrt{g} \right] \right\},$$

(6.10)

so that in the end, the antifield independent part of the original anomaly to deal with becomes

$$\int d^{2}\xi \left[ \left( \frac{26 - D}{48\pi} \right) \sqrt{g} R \right] C \equiv i A(g_{\alpha\beta}) \cdot C.$$

(6.11)

15Traces over continuous indices involved in expressions like (6.8) should be computed in the euclidian space. Coming back to the Minkowsky space causes the appearence of $-i$ factors, as the one in front of (6.9).
6.2 Extended theory, invariant regulator and Wess-Zumino term

Once the anomalous character of the theory has been verified, the construction of the extended field-antifield formalism goes as follows [8]. Since only the one parametric Weyl group is anomalous, no rank troubles arise and we are led to introduce a new scalar field $\theta$. Its transformation under the action of the whole gauge group reads [8]

$$\delta \theta = v^\alpha \partial_\alpha \theta - \lambda,$$

yielding thus the following form of the extended action $S_{\text{ext}}$

$$S_{\text{ext}} = S + \int d^2 \xi \left[ \theta^* (C^\alpha \partial_\alpha \theta - C) \right].$$

The Wess-Zumino term which corresponds to the original Weyl anomaly (6.11) can now be evaluated either by integrating the BRST variation of the Wess-Zumino term giving (6.11), or by interpreting it as the counterterm (modulo divergent pieces) relating invariant and non-invariant regularizations. The first possibility was contemplated in [8]. As for the second, we exemplify it by considering only the matter sector. Obviously, since the ghost contribution is exactly the same up to numerical coefficients, the Wess-Zumino term coming from this sector should also share the same functional form.

Let us first construct an invariant regulator $R'_m$ for the matter sector. The rule (5.5) yields

$$R_m = \frac{1}{\sqrt{g}} \Box \rightarrow R'_m = \frac{1}{(\sqrt{g})'} \frac{e^{-\theta}}{\sqrt{g}} \Box = e^{-\theta} R_m.$$

This invariant and the non-invariant regulator (6.7) are related by the interpolation (5.6)

$$R_m(t) = e^{-\theta t} R_m = e^{-\theta t} \frac{e^{-\theta}}{\sqrt{g}} \Box, \quad t \in [0, 1].$$

Now, expression (5.3) of the searched-for counterterm gives in this case

$$M_1^{(0)}(g_{\alpha\beta}, \theta) = -i \int_0^1 dt \, \text{Tr} \left[ \frac{1}{2} \theta \left( 1 - e^{-\theta t} R_m/M \right)^{-1} \right].$$

The integrand in (6.15) is expression (6.8) evaluated with the Weyl transformed of the original regulator (6.7). Hence, we have

$$\text{Tr} \left[ \frac{1}{2} \theta \left( 1 - e^{-\theta t} R_m/M \right)^{-1} \right] \sim \text{Tr} \left[ \frac{1}{2} \theta \exp \{ R'_m/M \} \right] = A_m(g_{\alpha\beta}(t)) \cdot \theta,$$

with $A_m(g_{\alpha\beta})$ given by (5.3) and where $R'_m, g_{\alpha\beta}'(t)$ stands for the (finite) Weyl transformed of the regulator $R_m$ and of the metric field with parameter $\theta t$, i.e., eq.(6.14) and $g_{\alpha\beta}'(t) = e^{\theta t} g_{\alpha\beta}$.

Substituting all these expressions in (6.15) and performing the integration over $t$, we get [8]

$$M_1^{(0)}(g_{\alpha\beta}, \theta) = - \int d^2 \xi \left\{ \left[ \frac{D}{48\pi} \left( \frac{1}{2} \theta \Box + \sqrt{g} \theta \right) \right] - \left[ \frac{D}{8\pi} M \sqrt{g} \right] + \left[ \frac{D}{8\pi} M \sqrt{g} e^\theta \right] \right\}.$$

Each one of the three pieces above deserves a different interpretation. The first one is really the contribution of the matter sector to the Wess-Zumino term. Substitution of the coefficient $(-D)$ by $(26 - D)$ in it yields thus the complete Wess-Zumino action. The second is the part of the local counterterm (6.10) whose BRST variation gives the divergent term in (5.3). Finally,
the third term is a BRST (or gauge) invariant counterterm playing no role, so that it can be simply dropped out. In summary, the original Wess-Zumino term reads

$$M_1^{(0)}(g_{\alpha \beta}, \theta) = \int d^2 \xi \left\{ \left( \frac{D - 26}{48 \pi} \right) \left[ \frac{1}{2} \theta \Box \theta + \sqrt{g} \theta \right] \right\},$$

(6.16)

and it can be interpreted as the tree level Liouville action for the bosonic string, $\theta$ being thus the Liouville field.

To conclude, a direct inspection to the form of the $\theta$ transformations (6.12) and of the Wess-Zumino term (6.16) indicates that this system fits the requirements described in sect. 4. Indeed, since the commutator of a Weyl transformation and a diffeomorphism

$$[\delta_R (v^\alpha), \delta_W (\lambda)] = \delta_W (-v^\alpha \partial^\alpha \lambda),$$

does not contain diffeomorphisms, the structure constants $T_{AB}^C$ vanish and condition (4.33) is trivially satisfied, while condition (4.32) is also seen to hold due to the Weyl invariance of the kinetic operator $\Box$ in (6.16).

## 6.3 Extended proper solution, background term and extended regularization

Having verified that the application of the regularization process described in sect. 4 to this model is sensible, let us pass now to implement it. First of all, from $S_{\text{ext}}$ (6.13) and the Wess-Zumino action (6.16), we should recognize the relevant extended proper solution $W_0$ and the background term $M_{1/2}$. The canonical transformation (4.23) adapted to this case performs this task and determines them to be

$$W_0 = \hat{S}(\Phi, K) + \int d^2 \xi \left[ \left( \frac{D - 26}{48 \pi} \right) \left( \frac{1}{2} \theta \Box \theta \right) + \theta^* C^\alpha \partial_\alpha \theta \right],$$

(6.17)

$$M_{1/2} = - \int d^2 \xi \left[ \theta^* C - \left( \frac{D - 26}{48 \pi} \right) \sqrt{g} \theta \right],$$

which are seen to fulfill relations (4.28), (4.29) and (4.30).

The modified extended proper solution $\hat{W}_0$, obtained from $W_0$ by leaving undetermined the numerical coefficients of the kinetic operator for the extra variables, is obtained through the substitution $a = \left( \frac{D - 26}{48 \pi} \right) \rightarrow \hat{a}$ in (6.17), i.e.,

$$\hat{W}_0 = \hat{S}(\Phi, K) + \int d^2 \xi \left[ \frac{\hat{a}}{2} \theta \Box \theta + \theta^* C^\alpha \partial_\alpha \theta \right].$$

(6.18)

The regulator $R_\theta$ (4.49) for the $\theta$ sector is determined from the new $\theta$ kinetic operator in (6.18) once an explicit mass matrix is chosen. In this case, the similarity of the kinetic operators for the matter and $\theta$ sector suggests to use a similar mass matrix for the PV field of the extra variable

$$T_\theta = T_{\theta,0} = \hat{a} \sqrt{g},$$

from which, using expression (4.49), the following regulator $R_\theta$ is obtained

$$R_\theta = \frac{1}{\sqrt{g}} \Box,$$

i.e., exactly the same regulator as for the matter part (6.7), with the only difference that now only one scalar field is involved. Therefore, the contribution of the extra sector to the antifield independent part of the anomaly is just expression (6.11) with numerical coefficient $-1$. A first
effect of the extra degree of freedom at one loop is thus a "renormalization" of the original coefficient $a = \left( \frac{D-26}{48 \pi} \right)$ of the anomaly (6.11) to $\left( \frac{D-25}{48 \pi} \right)$ [4].

Now, cancelation of the antifield independent sector of the $\bar{h}$ part in (4.44) is acquired for

$$\tilde{M}_{1/2} = - \int d^2 \xi \left[ \theta^* C - \left( \frac{D-25}{48 \pi} \right) \sqrt{g} R \theta \right],$$

while the $\bar{h}$ equation $(\tilde{M}_{1/2}, \tilde{W}_0) = 0$ is fulfilled by taking $\tilde{a} = \left( \frac{D-25}{48 \pi} \right)$ in (6.18). These conditions together lead to the vanishing of the antifield independent part of the obstruction $\tilde{A}$ (4.44) in the extended theory.

Finally, coming back to the original variables $\theta, \theta^*$, these effects appear realized together as a shift of the original coefficient of the Wess-Zumino term (6.16) in the same amount. The final, one-loop renormalized Wess-Zumino action reads then

$$\tilde{M}_{1}^{(0)}(g_{\alpha\beta}, \theta) = \int d^2 \xi \left\{ \left( \frac{D-25}{48 \pi} \right) \left[ \frac{1}{2} \square \theta + \sqrt{g} R \theta \right] \right\}.$$ 

This result was earlier obtained in [31] and further reproduced in [35], by using a heat kernel regularization procedure for the non-trivial, gauge invariant measure of the Liouville field, and in [34] through the application of the background field method. An alternative approach, based in a canonical quantization of the regularized system taking into account changes in the type of constraints, can also be found in [36].

7 Conclusions and Outlook

The aim of this paper has been to further study the formulation of the extended Field-Antifield formalism for anomalous gauge theories, previously proposed and developed in [8] [9], by means of the incorporation of quantum one-loop effects coming from the extra variables sector. To do that, an extension of the PV scheme proposed in [5] [6] [11] has been constructed to explicitly take into account at once the regularization of both the original and the extra fields, maintaining as far as possible the features characterizing the (regularization of the) original theory. In this fashion, background terms, known to appear in other formulations and giving rise to the anomalies in a different way, as well as a new proper solution in the extended space, directly arise from the combination $(\tilde{S} + \hbar \tilde{M}_{1}^{(0)})$ of the extended, non proper solution and of the Wess-Zumino term as a result of a canonical transformation in the extra field sector. Unfortunately, only a certain type of theories (bosonic string, abelian chiral Schwinger model, etc.) seems to admit the perturbative description we present, indicating that maybe a quantum treatment of Wess-Zumino terms goes beyond the scope of the usual $\hbar$ perturbative expansion. In any case, our proposal works for the restricted theories we study, yielding in the end cancellation of the antifield independent part of the complete anomaly and thus BRST invariance of the extended theory up to one-loop in this restricted sector. Furthermore, in the particular example we present –the bosonic string– the application of this general framework leads to a natural interpretation of the well-known shift $(26 - D)$ to $(25 - D)$ as a one-loop renormalization of the Wess-Zumino term due to the extra (Liouville) field quantum effects. In any case, however, it should be stressed that the physical character of an anomalous gauge theory quantized in this fashion is not answered from the above construction, although it would be of interest to analyze the unitarity of the extended theory along the lines of [37]. There, it has been shown that, under certain conditions on the generators of the gauge algebra, unitarity relies in the norm of the classical gauge invariant degrees of freedom. In the present case this would amount to study the norm of the "classical" degrees of freedom associated with the new proper solution $W_0$. 

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The extended field-antifield formalism, on the other hand, presents some other interesting features. Its covariance, for instance—in the sense that the role of the anomalous propagating degrees of freedom, played in previous approaches by the pure gauge modes of the gauge fields (e.g., the conformal mode of the metric field in the bosonic string), is now taken over by the extra degrees of freedom—allows to get rid of these pure gauge fields by gauge-fixing it as in the usual non-anomalous theories. Related to this covariance, it would be of interest to elucidate the relationship between our proposal and the description given in [11], where the anomalous propagating degrees of freedom are associated with some original antifields. Moreover, the extended formalism also permits a complete determination of the transformation properties of the extra variables and of the Wess-Zumino action from the underlying (quasi)group structure, although the locality of such objects, which relies on the locality of the quantities defining the (quasi)group, remains as an open problem worth to be investigated.

Finally, we would like to stress the importance that throughout our developments the alternative expression for the anomaly (2.20) involving only the regulator has found. Apart for its simplicity and the fact that it establishes a clear relationship between anomalies and transformation properties of the regulator, the combination of its form together with the expansion (5.1) for a general regulator yields the result that for “closed” theories (i.e., $\delta^2 \Sigma = 0$) anomalies obtained in the PV scheme split into two $\delta$ off-shell invariant parts, one of them antifield independent while the other one carries all the antifield dependence. Algebraic counterexamples to this result has recently been given in [13] in the form of solutions of the Wess-Zumino consistency conditions with a non-trivial dependence on the antifields. Our result excludes the appearance of such type of anomalies and indicates that the regularization procedure acts as a sort of “selection rule” identifying from the complete set of algebraic solutions of the consistency conditions a subset of “physically” realized anomalies. Further understanding of this point clearly deserves future investigation.

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A Alternative form of $(\Delta \hat{S})_{\text{reg}}$

Our main purpose here is to show that expression (2.20) for $(\Delta \hat{S})_{\text{reg}}$ is completely equivalent to the form obtained in [3 11]

$$(\Delta \hat{S})_{\text{reg}} = \left[ K_B^A + \frac{1}{2} (T^{-1})^{AC} \delta T_{CB} (-1)^B \left( \frac{1}{1 - R/M} \right)^B_A \right], \quad (A.1)$$

with $K_B^A$ defined from (2.17) to be

$$K_B^A = \left( \frac{\partial}{\partial K_A} \frac{\partial}{\partial \Phi^B} \hat{S}(\Phi, K) \right).$$
The idea consists in using some relations between \( K_B^A \), \( T_{AB} \) and \( R_B^A \) and express [A.1] in terms of the regulator \( R_B^A \) alone. First, the symmetry properties of \( T_{AB} \) and the definition of the supertranspose of \( K_B^A \)

\[
T_{BA} = (-1)^{A+B+AB} T_{AB}, \quad (K^T)^B_A = (-1)^{A(B+1)} K_B^A,
\]

allows to rewrite [A.3] in the equivalent form

\[
(\Delta \hat{S})_{\text{reg}} = \left\{ \frac{1}{2} \left[ (K + R^{-1}T^{-1}K^TTR)^A_B + (T^{-1})^{AC} \delta T_{CB}(-1)^B \right] \left( \frac{1}{1 - R/M} \right)^B_A \right\}, \quad (A.2)
\]

Now, differentiation of the classical master equation for \( \hat{S} \)

\[
\frac{\partial}{\partial \Phi^A} \frac{\partial}{\partial \Phi^B} (\hat{S}, \hat{S}) = 0,
\]

provides the following identity between \( K_B^A \) and \( TR_{AB} \)

\[
(TRK)_{AB} + (TRK)_{BA}(-1)^{AB} + (-1)^B \delta (TR)_{AB} = 0.
\]

This relation can now be used to express the combination \( (K + R^{-1}T^{-1}K^TTR)^A_B \) in [A.2] as

\[
(K + R^{-1}T^{-1}K^TTR)^A_B = -(R^{-1})^{AC} C \delta R^B_B(-1)^B - (R^{-1})^{AC} \delta T_{CD}(-1)^D R^D_B,
\]

which substituted in [A.2] eliminates its explicit dependence on \( K_B^A \) and \( T_{AB} \). In the end, use of the definition of the supertrace as well as its cyclic property in the resulting expression yields [2.20], as we would like to show.

### B Counterterms

The particular expression of the anomaly is not unique. Its form depends both on the intermediate regularization scheme (i.e., on the mass part of \( S_{PV} \) [2.16], for example) and on the form of the counterterm \( M_1 \). Different expressions of the consistent anomaly, or equivalently, of \( (\Delta \hat{S})_{\text{reg}} \), are thus expected to be related by the BRST variation of a local counterterm. The form of this counterterm when the regularization schemes are connected by a continuous interpolating regulator was conjectured in [3]. Here, we derive the expression of this counterterm\(^{16}\).

Consider two different regularizations defined in terms of the mass matrices and regulators \( (T_0, R_0) \) and \( (T_1, R_1) \), satisfying \( T_0 R_0 = T_1 R_1 = (TR) \). In each one, [2.21] reads

\[
(\Delta \hat{S})_{\text{reg},0(1)} = \delta \left\{ -\frac{1}{2} \text{Tr} \ln \left[ \frac{(TR)}{T_0(1)} M - (TR) \right] \right\}. \quad (B.1)
\]

Assume now that there exists a continuous path \( T(t), \, t \in [0,1] \), interpolating from the first mass matrix, \( T_0 = T(0) \), to the second, \( T_1 = T(1) \). This interpolation induces in turn another interpolation between \( R_0 \) and \( R_1 \)

\[
T(t)R(t) = (TR) \Rightarrow R(t) = T^{-1}(t)(TR). \quad (B.2)
\]

The difference between the two regularized expressions of \( \Delta \hat{S} \) [B.1] is then

\[
(\Delta \hat{S})_{\text{reg},1} - (\Delta \hat{S})_{\text{reg},0} = \delta \left\{ -\frac{1}{2} \text{Tr} \ln \left[ \frac{(TR)}{T_1 M - (TR)} \right] + \frac{1}{2} \text{Tr} \ln \left[ \frac{(TR)}{T_0 M - (TR)} \right] \right\} = \delta M_0(\Phi, K),
\]

\(^{16}\)An alternative proof of this conjecture can be found in [3].
from which we can read off the form of the counterterm relating them. Further use of $T(t)$
allows to rewrite $M_0$ as an integral over $t$

$$M_0(\Phi, K) = \int_0^1 dt \partial_t \left\{ -\frac{1}{2} \text{Tr} \ln \left[ \frac{(TR)}{(T(t)M - (TR))} \right] \right\} = \int_0^1 dt \text{Tr} \left[ \frac{1}{2} (T^{-1}(1) \delta T(t)) \frac{1}{(1 - R(t)/M)} \right].$$

which exactly coincides with the expression conjectured in [5].

In view of the results of sect.5, it is convenient to rewrite (B.4) as an expression involving
only $R(t)$. This can be achieved by considering the independence of the kinetic term $(TR)$ for
the PV fields on $t$ (B.2). This property induces the relation

$$\partial_t [T(t)R(t)] = \partial_t (TR) = 0 \Leftrightarrow (T^{-1}(1) \delta T(t)) = -(\partial_t R(t))^{-1}(1),$$

which, when substituted in (B.3), yields the desired expression

$$M_0(\Phi, K) = \int_0^1 dt \text{Tr} \left[ -\frac{1}{2} (R^{-1}(1) \delta R(t)) \frac{1}{(1 - R(t)/M)} \right].$$

(C) Transformation properties of the regulator $R_\theta$

In sect.4.4 we argued that the introduction of extra variables could lead to type A anomalies
unless an invariant regulator $R_\theta(\phi)$ is used. The purpose of this appendix is the obtention of
the conditions ensuring the vanishing of these extra anomalies.

The transformation properties of the regulator $(R_\theta)_{ab}^\phi(\phi) = -i(T^{-1})^{ac} \hat{D}_{cb}(\phi)$, (4.5) can be obtained from those of the kinetic term $\hat{D}_{cb}(\phi)$ (4.10) and the mass matrix $(T_0^{-1})^{ab} \equiv (T^{-1})^{ab}$. In particular, since $\hat{D}_{cb}(\phi)$ (4.11) only differs from the original one in its numerical coefficients, the transformation properties of the former are completely determined from that of the latter
(4.32), (4.34). We have thus

$$\delta_{(a)} \hat{D}_{ab} = 0, \quad \delta_{(A)} \hat{D}_{ab} = \left( \hat{D}_{ac} T_{ab}^{cB} + \hat{D}_{bc} T_{ab}^{cA} \right) \epsilon^B. \tag{C.1}$$

It is obvious then that the vanishing of type A anomalies lies entirely on the A transformation
to $T_{ab}$. Indeed, taking into account (C.1), the a transformation of the regulator reads

$$\delta_{(a)}(R_\theta)^a_b = -i\delta_{(a)}(T^{-1})^{ac} \hat{D}_{cb}. \tag{C.2}$$

In this way, new contributions to the original $A_\mu(\phi)$ anomalies only come from the noninvariance
of $T_{ab}$ under the a part. On the other hand, the A transformations for $R_\theta$ are

$$\delta_{(A)}(R_\theta)^a_b = -i\delta_{(A)}(T^{-1})^{ac} \hat{D}_{cb} - i(T^{-1})^{ac} \delta_{(A)} \hat{D}_{cb}$$

$$= i(T^{-1})^{ac} \delta_{(A)} T_{ed}(T^{-1})^{dc} \hat{D}_{cb} - i(T^{-1})^{ac} \left( \hat{D}_{cd} T_{cB}^{d} + \hat{D}_{bd} T_{cB}^{d} \right) \epsilon^B. \tag{C.3}$$

Hence, to avoid new anomalies, the above transformation should be zero or, at least, of the
form $[R_\theta, G]$. This is precisely the case if

$$\delta_{(A)} T_{ab} = (T_{ac} T_{aB}^{c} + T_{bc} T_{aB}^{c}) \epsilon^A, \tag{C.4}$$

yielding in this way the transformation rule

$$\delta_{(A)}(R_\theta)^a_b = [R_\theta, G]^a_b \equiv T_{ab}^a \epsilon^B.$$  

In summary, (C.3) is the suitable transformation property of $T_{ab}$ considered in sect.4.4 guaran-
teeding that no new symmetries become anomalous upon introduction of the extra variables.
With respect to the completely anomalous case (i.e., the abelian theory) the transformation of the regulator is as in (C.2) with the substitution $a \rightarrow \alpha$, i.e.,

$$
\delta(R_\theta)^\alpha_\beta = -i[\delta(T^{-1})^\alpha_\gamma] \hat{D}_\gamma \beta,
$$

so that an extra contribution to the original anomaly is likely to appear if $T_{\alpha\beta}$ is not invariant.

In the usual cases, a constant mass matrix without any dependence on the fields can be chosen. Then, $\delta(T^{-1})^\alpha_\beta = 0$, and no new contributions arises. In any case, the existence of such invariant and/or constant mass matrix should be analyzed model by model.

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