WARING'S PROBLEM FOR LOCALLY NILPOTENT GROUPS:
THE CASE OF DISCRETE HEISENBERG GROUPS

YA-QING HU

Abstract. Kamke [Kam21] solved an analog of Waring's problem with $n$th powers replaced by integer-valued polynomials. Larsen and Nguyen [LN19] explored the view of algebraic groups as a natural setting for Waring’s problem. This paper applies the theory of polynomial maps and polynomial sequences in locally nilpotent groups developed in a previous work [Hu21a] to solve an analog of Waring’s problem for the general discrete Heisenberg groups $H_{2n+1}(\mathbb{Z})$ for any integer $n \geq 1$.

Contents

1. Introduction 1
2. Integer-valued Polynomials and Kamke’s Generalization 4
3. Waring’s Problem for General Discrete Heisenberg Groups $H_{2n+1}(\mathbb{Z})$ 9
4. Waring’s Problem for Locally Nilpotent Groups 19
References 23

1. INTRODUCTION

Motivation. The motivation of this work is the following question of Michael Larsen:

Question. Find good notions of “polynomial sequence” and “generalized cone” so that if $G$ is a finitely generated nilpotent group and $g_0, g_1, g_2, \ldots$ is a polynomial sequence in $G$ such that no coset of any infinite index subgroup of $G$ contains the whole sequence, then there exists a positive integer $M$, a generalized cone $C \subset G$, and a subgroup $H$ of finite index in $G$ such that every element of $C \cap H$ is a product of $M$ elements of the sequence.

A previous work [Hu21a] proposed definitions for polynomial sequences $g : \mathbb{N}_0 \to G; i \mapsto g_i := g(i)$ and generalized cones and proved many desirable formal properties of polynomial sequences when the target group $G$ is locally nilpotent. The present work will answer the question in the case of the general discrete Heisenberg groups $H_{2n+1}(\mathbb{Z})$ for any integer $n \geq 1$.

Date: September 20, 2022.

2010 Mathematics Subject Classification. 11P05, 11C08, 20M14, 20F18.

Key words and phrases. Waring’s problem, polynomial maps, commutative semigroups, locally nilpotent groups, discrete Heisenberg groups.

This work was partially supported by NSF Grant [grant number DMS-1401419] and the Postdoctoral International Exchange Program of the China Postdoctoral Council [grant number Y20210319].
Background. Let $\mathbb{N}$ (resp. $\mathbb{N}_0$) be the set of positive (resp. non-negative) integers. In 1909, Hilbert [Hil09] solved the classical Waring’s problem by a difficult combinatorial argument based on algebraic identities and proved that for each positive integer $n$, there exists a bounded number $N \in \mathbb{N}$ dependent only on $n$ such that the following map given by the sum of $n$th powers of non-negative integers is surjective:

$$\mathbb{N}_0^N \to \mathbb{N}_0; \quad (x_1, x_2, \ldots, x_N) \mapsto \sum_{i=1}^N x_i^n.$$ 

This is known as the Hilbert-Waring theorem.

On the other hand, various variants of Waring’s problem have been investigated. For example, Kamke [Kam21] generalized the Hilbert-Waring theorem with $x^n$ replaced by integer-valued polynomials $f(x)$ of degree $\geq 2$. Wright [Wri34] studied the easier Waring’s problem, which seeks to determine $v(n)$, the minimum $N$ such that $\mathbb{Z}$ is the union of images of all maps of the form

$$\mathbb{N}_0^N \to \mathbb{Z}; \quad (x_1, x_2, \ldots, x_N) \mapsto \sum_{i=1}^N \varepsilon_i x_i^n,$$

for some choices of $\varepsilon_i = \pm 1$.

Moreover, the analog of Waring’s problem for (nonabelian) groups receives a great deal of attention in the last 30 years. A typical problem is to prove that every element in the group $G$ can be expressed as a short product of values of certain word map

$$w : G \times G \times \cdots \times G \to G,$$

induced by substitution of a nontrivial group word $w$ in the free group $F_d$ of rank $d$ with elements in the group $G$.

Recently, Larsen and Nguyen [LN19] explored the idea of algebraic groups as a natural setting for Waring’s problem. The work on the polynomial-valued, vector-valued and certain matrix-valued variants of Waring’s problem can naturally fit into this framework. They consider a morphism of varieties (i.e., reduced separated schemes of finite type) (resp. schemes) from $\mathbb{A}^1$ to an algebraic group $G$ defined over a field $K$ (resp. a group scheme over a number ring $O$).

At the field level, they work in the field of characteristic 0 and call a subvariety $X$ of the algebraic group $G$ generating, if there exists $n \in \mathbb{N}$ such that the product map

$$X \times \cdots \times X \to G$$

is surjective, or equivalently, every generic point of $G$ lies in the image of this product map, and call a finite collection of morphisms $f_i : X_i \to G$ generating if the finite union of Zariski closures $\overline{f_i(X_i)}$ is generating.

They are interested in the generating collections of morphisms $f_i : \mathbb{A}^1 \to G$, and for certain technical reasons, they restrict their attention to connected unipotent algebraic groups over a nonreal field $K$. (They call a field $K$ nonreal if $K$ is of characteristic 0 but not formally real, i.e., $-1$ is a sum of squares in $K$.) They prove that for any unipotent algebraic group $G$ over a nonreal field $K$ and a generating set $\{f_1, \ldots, f_n\}$ of $K$-morphisms $\mathbb{A}^1 \to G$, there exists some positive integer $M$ such that $(f_1(K) \cup \cdots \cup f_n(K))^M = G(K)$; see [LN19, Thm 2.2].
At the integral level, they work with the ring \( \mathcal{O} \) of integers of a totally imaginary number field \( K \), and a closed \( \mathcal{O} \)-subscheme \( \mathcal{G} \) of the group scheme \( \mathcal{U}_k \) of unitary \( k \times k \) matrices, and call a set \( \{ f_1, \ldots, f_n \} \) of \( \mathcal{O} \)-morphism \( \mathbb{A}^1 \rightarrow \mathcal{G} \) generating if it is generating as a \( \mathcal{K} \)-morphism. They prove that for any generating set \( \{ f_1, \ldots, f_n \} \) of \( \mathcal{O} \)-morphisms \( \mathbb{A}^1 \rightarrow \mathcal{G} \), there exists a positive integer \( M \) such that \( (f_1(\mathcal{O}) \cup \cdots \cup f_n(\mathcal{O}))^M \) is a subgroup of finite index in \( \mathcal{G}(\mathcal{O}) \); see [LN19, Thm 3.1].

This framework is convenient to work with but also has its own drawbacks. For example, in the original situation of Waring’s problem, namely for the ring \( \mathbb{Z} \), the additive group \( \mathbb{G}_a \), and the morphism \( f : \mathbb{A}^1 \rightarrow \mathbb{G}_a \) given by \( f(x) = x^n \) for \( n \geq 2 \) even, their results fall short of the Hilbert-Waring theorem. The difficulty is the ordering of \( \mathbb{Z} \), as negative integers cannot be expressed as sums of positive elements and thus the assumption of \( K \) being totally imaginary plays a vital role in their work. But this issue can be avoided in the situation of the easier Waring’s problem. Hence, they have to choose between either working over a totally imaginary number ring or doing the easier Waring problem on a general number ring. Moreover, a natural question has been raised in [LN19]: whether, for unipotent groups over general number rings \( \mathcal{O} \), one can characterize the set which ought to be expressible as a bounded product of images of the morphism \( f : \mathbb{A}^1 \rightarrow \mathcal{U}_k \) over the ring \( \mathcal{O} \).

**Strategy and main result.** To solve an analog of Waring’s problem for nilpotent groups, the original plan was to do induction over central series of the group and use Kamke’s result as the base case. [LN19, Prop 2.4] suggests a strategy for the induction step, but the ordering on \( \mathbb{Z} \) and the lack of inverses make their strategy break down for this work. In [Hu21a] we propose an alternative strategy called the *iterated symmetrization*, which enables us to find a symmetric polynomial map \( \mathbb{N}_0^L \rightarrow \mathcal{U}_k(\mathbb{Z}) \) in \( L \) variables, where the symmetric group \( S^L \) acts on such polynomial maps by permuting variables. In particular, we can make the induction method work and solve an analog of Waring’s problem for the general discrete Heisenberg groups \( H_{2n+1}(\mathbb{Z}) \). The main result of this work is the following

**Theorem 1.** Let \( G \) be the discrete Heisenberg group \( H_{2n+1}(\mathbb{Z}) \) for some positive integer \( n \) and \( g : \mathbb{N}_0 \rightarrow G \) be a polynomial sequence. If the induced polynomial sequence \( g \mod N : \mathbb{N}_0 \rightarrow G \rightarrow G/N \) is non-constant for any normal subgroup \( N \) of infinite index in \( G \), then there exists a positive integer \( M \), a finite index subgroup \( H \) of \( G \), and a proper polynomial set \( V \) of \( H \) such that every element in \( V \) can be written as a product of at most \( M \) elements in the sequence \( g_0, g_1, g_2, \ldots \).

**Organization of the paper.** In Section 2, we discuss some basic properties of integer-valued polynomials, especially on the basis for integer-valued polynomials \( \mathbb{N}_0 \rightarrow \mathbb{Z} \), state two equivalent forms of Kamke’s key theorem, briefly summarize Kamke’s generalization of Waring’s problem to integer-valued polynomials, discuss Diophantine Frobenius problem and the greatest common divisor of polynomial values, answer the question in Proposition 3 whether the commutative semigroup \( [f(\mathbb{N}_0)] \) generated by the image of an integer-valued polynomial \( f : \mathbb{N}_0 \rightarrow \mathbb{Z} \) can be covered by finitely many sumsets \( kf(\mathbb{N}_0) \), and explain in Proposition 4 why a similar result does not generalize to a vector of polynomials \( f : \mathbb{N}_0 \rightarrow \mathbb{Z} \times \cdots \times \mathbb{Z} \).

The main result of this paper is given in Section 3. As a preparation, we generalize Kamke’s key theorem and solve Waring’s problem for the abelian group \( \mathbb{Z}^m \). The fact that symmetric polynomials in several variables with rational coefficients can be written as a polynomial expression with rational coefficients in the power
sum symmetric polynomials enables us to connect the strategy of iterated symmetrization with Kamke’s result. To conquer the degenerate case, we work with a finite product of affine translations of the polynomial sequence.

In section 4, we discuss Waring’s problem for general locally nilpotent groups. In particular, we prove a similar result in Theorem (3) for the case when the degree of the polynomial sequence is exactly 1 and a primitive result in Theorem 4 for Waring’s problem for general locally nilpotent groups of degree $\geq 2$, which is still far from the question.

Acknowledgment. I thank my advisor Michael Larsen for the guidance and many helpful discussions through this work.

2. Integer-valued Polynomials and Kamke’s Generalization

Integer-valued polynomials. Let $D$ be a domain with quotient field $K$. For any subset $E$ of $K$,

$$\operatorname{Int}(E, D) := \{ f \in K[x] \mid f(E) \subseteq D \}$$

is defined to be the set of $D$-valued polynomials on $E$. Clearly, $\operatorname{Int}(E, D)$ is a subring of $K[x]$ and $\operatorname{Int}(D) := \operatorname{Int}(D, D)$ is a $D$-module and a subring of $K[x]$ that contains $D[x]$. Then we have

**Proposition 1.** The polynomials $(x^n)$ with $n \in \mathbb{N}_0$ form a basis for the $\mathbb{Z}$-module $\operatorname{Int}(\mathbb{Z})$.

**Proof.** See [CC97, Prop I.1.1].

**Corollary 1.** A polynomial $f$ of degree $n$ is integer-valued if and only if $f$ sends $n + 1$ consecutive integers to integers. In particular, for any $k \in \mathbb{Z}$, we have

$$\operatorname{Int}(\mathbb{Z}) = \operatorname{Int}(\mathbb{N}_0, \mathbb{Z}) = \operatorname{Int}(\mathbb{N}, \mathbb{Z}) = \operatorname{Int}(\mathbb{Z}_{\geq k}, \mathbb{Z}).$$

**Proof.** See [CC97, Cor I.1.2].

Given any set $P = \{x_0, x_2, \ldots, x_n\} \subset \mathbb{R}$, the Lagrange basis polynomials associated with $P$ are

$$\ell_i(x; P) := \prod_{0 \leq k \leq n} \frac{x - x_k}{x_i - x_k} = \frac{(x - x_0)}{(x_i - x_0)} \cdots \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \frac{(x - x_{i+1})}{(x_i - x_{i+1})} \cdots \frac{(x - x_n)}{(x_i - x_n)}.$$

Since $x_i \in P$ are distinct, $\ell_i(x; P)$ is a polynomial of degree $n$ such that $\ell_i(x_j; P) = \delta_{ij}$ is the Kronecker symbol. Then, any polynomial $f(x)$ of degree $\leq n$ can be uniquely written as

$$f(x) = \sum_{i=0}^{n} f(x_i) \ell_i(x; P).$$

In particular, for any integer $n \geq 0$, let $P_{a,n} = \{a, a + 1, \ldots, a + n\}$ be the set of $n + 1$ consecutive integers starting from any integer $a \in \mathbb{Z}$.

**Proposition 2.** The Lagrange basis polynomials $\ell_i(x; P_{a,n})$ for $0 \leq i \leq n$ form a basis for the $\mathbb{Z}$-submodule of $\operatorname{Int}(\mathbb{Z})$ consisting of integer-valued polynomials of degree at most $n$.

**Proof.** See [Hen96] or [CC97, Remark I.1.3].
Warung's Problem for Locally Nilpotent Groups

Kamke's key theorem and Kamke's generalization. According to Kamke [Kam21], the question about the simultaneous decomposition of integers into powers of integers was raised by Hilbert in a seminar. More precisely, Hilbert asked, for a given integer \( n \geq 2 \), under what as few restrictions on the positive integers \( s_1, \ldots, s_n \) as possible, there is a positive integer \( N = N(n) \), such that for each \( n \) positive integers \( s_1, \ldots, s_n \), subject to those restrictions, the system of equations

\[
s_1 = \sum_{\kappa=1}^{N} x_{\kappa}, \quad s_2 = \sum_{\kappa=1}^{N} x_{\kappa}^2, \quad \ldots, \quad s_n = \sum_{\kappa=1}^{N} x_{\kappa}^n
\]

can be simultaneously solved by integers \( x_{\kappa} \geq 0 \). To answer this question and solve Waring's problem in integer-valued polynomials, Kamke proved a theorem in two equivalent versions, which he called Kernsatz. In the sequel, we will refer to it as Kamke's key theorem.

**Theorem** (Kamke's key theorem). For each integer \( n \geq 2 \), there is an integer \( N = N(n) > 0 \), an integer \( A > 0 \) and positive numbers \( i_1 \) and \( i_\nu, J_\nu \) with

\[
0 < i_\nu < J_\nu \quad (\nu = 2, 3, \ldots, n),
\]

such that for each \( n \) integers \( s_1, s_2, \ldots, s_n \), divisible by \( A \) and subject to the following conditions

\[
i_1 < s_1; \quad i_\nu s_1^{i_\nu} < s_\nu < J_\nu s_1^{i_\nu} \quad (\nu = 2, 3, \ldots, n),
\]

the \( n \) equations

\[
s_\nu = \sum_{\kappa=1}^{N} x_{\kappa}^{i_\nu} \quad (\nu = 1, 2, \ldots, n)
\]

are simultaneously solvable by integers \( x_{\kappa} \geq 0 \).

We will refer to these conditions in Kamke's key theorem as Kamke conditions and to the following unbounded open subsets as Kamke domains:

\[
U(n, N) = \{(s_1, \ldots, s_n) \in \mathbb{R}_{\geq 0}^n \mid i_1 < s_1, i_\nu s_1^{i_\nu} < s_\nu < J_\nu s_1^{i_\nu}, \nu = 2, 3, \ldots, n\}.
\]

**Example 1.** For \( n = 2 \), Kamke's key theorem with \( N = 5, A = 2, i_1 = 7, i_2 = \frac{1}{4}, J_2 = \frac{1}{3} - \varepsilon \), \( 0 < \varepsilon < \frac{1}{12} \), is contained in [Kam21, Thm, page 2].

With the help of the key theorem, Kamke proved the following

**Theorem.** Let \( f \) be an integer-valued polynomial of degree \( \geq 2 \) with a positive leading coefficient. Then, there is an integer \( N > 0 \) and for each integer \( Z \geq 0 \) there are integers \( N' \geq 0, N'' \geq 0, x_1 \geq 0, \ldots, x_{N'} \geq 0 \), such that \( N' + N'' \leq N \) and

\[
Z = \sum_{\kappa=1}^{N'} f(x_{\kappa}) + N'';
\]

i.e., each integer \( Z \geq 0 \) is decomposable into a bounded number of elements in \( f(N_0) \) with the addition of a bounded number of units.

**Remark.** Kamke's original hypotheses include that \( f(x) \geq 0 \) for each integer \( x \geq 0 \) and coefficients of \( f \) are integers, both of which can be removed, because if the leading coefficient is greater than 0, one always has \( f(x) \geq 0 \) for sufficiently large \( x \geq B \), so instead of \( f(x) \) one may consider its translation \( f(x+B) \), and Proposition
1 guarantees that the coefficients are rational, and by multiplying with the least common denominator $A$, one could work with $Af(x)$ instead of $f(x)$.

Notice that the theorem is trivial if $f(x) = kx + b$ with $k \in \mathbb{N}$ and $b \in \mathbb{Z}$. Indeed, for each integer $Z \geq 0$ such that $Z \geq \min\{f(N_0)\}$, there is an $x \in N_0$ such that $f(x) \leq Z < f(x + 1)$.

In fact, Kamke’s theorem follows easily from the following proposition, which can be deduced from Kamke’s key theorem. This was proved in Kamke’s paper but not stated explicitly. In the sequel, we will refer to it as Kamke’s non-explicit proposition.

**Proposition.** Under Kamke’s hypotheses, there exist $B, N' \in \mathbb{N}$ such every sufficiently large number in $BN$ can be written as a sum of exactly $N'$ numbers in $f(N_0)$.

Thus, there exist positive integers $B, K, N'$, such that for all positive integers $\zeta \geq K$, we have

$$B\zeta = \sum_{\kappa=1}^{N'} f(x_{\kappa}).$$

Therefore, for all integers $Z \geq BK$, we have $Z = \sum_{\kappa=1}^{N'} f(x_{\kappa}) + b$ with $0 \leq b < B$.

Setting $N = BK + B + N'$, for each integer $Z > 0$, we obtain the following representation with $N'' \geq 0$:

$$Z = \sum_{\kappa=1}^{N'} f(x_{\kappa}) + N''$$

with $N' + N'' \leq N$.

**The semigroup generated by polynomial values.** The sumset $A + B$ (also known as the Minkowski sum) of two subsets $A$ and $B$ of an ambient abelian group $(G, +)$ is defined to be the set of all sums of one element from $A$ with the other element from $B$. Denote by $kA = A + A + \cdots + A$ the $k$-fold iterated sumset of $A$ and denote by $[A]$ the commutative semigroup generated by $A$.

It is curious to know under what assumption on an integer-valued polynomial $f : \mathbb{N}_0 \to \mathbb{Z}$, the commutative semigroup $[f(N_0)]$ can be covered by finitely many sumsets $k f(N_0)$. What if $f = (f_1, \ldots, f_n) : \mathbb{N}_0 \to \mathbb{Z}^n$ is a vector of integer-valued polynomials $f_i : \mathbb{N}_0 \to \mathbb{Z}$? Answers to these questions are the starting point of this work.

Clearly, we have the following ascending chain of unions of sumsets

\[ A \subseteq \bigcup_{k=1}^{2} kA \subseteq \cdots \subseteq \bigcup_{k=1}^{m} kA \subseteq \cdots \subseteq [A]. \]

(2.1)

If it does not stabilize for $m$ large enough, then $\bigcup_{k=1}^{m} kA$ and thus $kA$ for all $1 \leq k \leq m$ are always proper subsets of $[A]$. Further, if $A$ contains the zero element $0$ of $G$, then $[A] = \bigcup_{k=1}^{\infty} kA$ is the commutative monoid generated by $A$ and $kA \subseteq (k+1)A$ for any $k \in \mathbb{N}$ and (2.1) becomes

\[ \{0\} \subseteq A \subseteq 2A \subseteq \cdots \subseteq kA \subseteq \cdots \subseteq [A]. \]

(2.2)

In this case, $[A] = kA$ for some $k \in \mathbb{N}$ if and only if the ascending chain stabilizes for $k$ large enough.

Notice that $1$ (resp. $1$ and $0$, resp. $1$ and $-1$) can be represented by a sum of finitely many numbers in $f(N_0)$ if and only if $\mathbb{N} \subseteq [f(N_0)]$ (resp. $N_0 \subseteq [f(N_0)]$,}
resp. \( Z \subseteq [f(N_0)] \), and if this is the case, then Kamke’s result implies that each integer \( Z > 0 \) (resp. \( Z \geq 0 \), resp. \( Z \geq 0 \)) is decomposable into a bounded number of numbers in \( f(N_0) \). Therefore, we have the following

**Corollary 2** (of Kamke’s theorem). If \( f : N_0 \to Z \) is a polynomial of degree \( n \geq 1 \) with a positive leading coefficient, and \( N \subseteq [f(N_0)] \) (resp. \( N_0 \subseteq [f(N_0)] \)), then each \( Z > 0 \) (resp. \( Z \geq 0 \)) is decomposable into a bounded number of elements in \( f(N_0) \), that is, there exists a uniform \( N \in N \) such that \( N \subseteq \bigcup_{k=1}^{N} kf(N_0) \) (resp. \( N_0 \subseteq N f(N_0) \)).

The following proposition characterizes integer-valued polynomials \( f : N_0 \to Z \) such that the commutative semigroup \([f(N_0)]\) can be covered by finitely many sumsets \( kf(N_0) \).

**Proposition 3.** Let \( f : N_0 \to Z \) be an integer-valued polynomial of degree \( d \). Then, we have

\[
[f(N_0)] = \bigcup_{k=1}^{N} kf(N_0), \quad \text{for some } N = N(f) \in N,
\]

if and only if either \( d = -\infty \), i.e., \( f = 0 \), or \( d \geq 1 \) and either \( f(N_0) \subseteq N_0 \) or \( f(N_0) \subseteq -N_0 \).

To prove this proposition, we need to know when the greatest common divisor of \( f(N_0) \) is 1. The condition \( \gcd(f(N_0)) = 1 \) is equivalent to that for each prime number \( p \), there exists \( n \in N_0 \), dependent on \( p \), such that \( f(n) \not\equiv 0 \pmod{p} \).

Since \( f(n) \equiv f(n \mod p) \pmod{p} \), we can choose a sufficiently large \( n \) among the equivalence class \( n \mod p \). Hence, we have

**Lemma 1.** For any integer-valued polynomial \( f \in \text{Int}(Z) \), the following conditions are all equivalent:

\[
\gcd(f(Z)) = 1 \iff \gcd(f(N_0)) = 1 \iff \gcd(f(N)) = 1 \iff \gcd(f(Z_{\geq k})) \text{ for any } k \in Z.
\]

Furthermore, the following three lemmas give shortcuts to verify these conditions:

**Lemma 2.** By Proposition 1, any integer-valued polynomial \( f(x) \) of degree \( d \geq 0 \) can be written as

\[
f(x) = a_d \binom{x}{d} + \cdots + a_1 \binom{x}{1} + a_0 \binom{x}{0}
\]

with \( a_i \in Z \) for all \( i = 0, \cdots, d \). Then, for any \( k \in Z \), we have

\[
\gcd(f(Z)) = \gcd(f(N_0)) = \gcd(f(N)) = \gcd(f(Z_{\geq k})) = \gcd\{a_0, a_1, \cdots, a_d\}.
\]

**Proof.** Trivial. \( \square \)

**Lemma 3.** Let \( P_{a,d} = \{a, a+1, \cdots, a+d\} \) be the set of \( d+1 \) consecutive integers starting from any integer \( a \in Z \). For any integer-valued polynomial \( f(x) \) of degree \( d \geq 0 \) and any \( k \in Z \), we have

\[
\gcd(f(Z)) = \gcd(f(N_0)) = \gcd(f(N)) = \gcd(f(Z_{\geq k})) = \gcd(f(P_{a,d})).
\]

**Proof.** By Proposition 2. \( \square \)

**Lemma 4.** For any integer-valued polynomial \( f(x) \) of degree \( d \geq 0 \) and any \( k \in Z \), we have

\[
\gcd(f(Z)) = \gcd(f(N_0)) = \gcd(f(N)) = \gcd(f(Z_{\geq k})) = 1,
\]

if and only if there exist \( m_1, m_2 \in Z_{\geq k} \) such that \( \gcd\{f(m_1), f(m_2)\} = 1 \).
**Proof.** By Chinese remainder theorem. □

For any finite number of vectors \(v_1, v_2, \ldots, v_n\) in \(\mathbb{R}^N\), a vector of the form \(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n\) with \(\alpha_i \in \mathbb{R}_{\geq 0}\) is called a conical sum of \(v_1, v_2, \ldots, v_n\), and is called an integer conical sum if in addition \(\alpha_i \in \mathbb{N}_0\). Given a set \(S = \{a_1, a_2, \ldots, a_n\}\) of positive integers such that \(\gcd(S) = 1\), the Diophantine Frobenius problem asks to determine the largest integer that cannot be expressed as an integer conical sum of these numbers. This integer, usually denoted by \(g(a_1, a_2, \ldots, a_n)\) or \(g(S)\), is called the Frobenius number of the set \(S\). The following folk result (see [Alf05, Thm 1.0.1]) guarantees the finiteness of this number:

**Theorem.** Given a set \(S = \{a_1, a_2, \ldots, a_n\}\) of positive integers, if \(\gcd(S) = 1\), then there exists an integer \(N\) such that any integer \(s \geq N\) is representable as an integer conical sum of \(a_1, \ldots, a_n\).

Now we are ready to prove Proposition 3.

**Proof.** If \(d = -\infty\), i.e., \(f = 0\), then \(N = 1\) and \([f(N_0)] = f(N_0)\); if \(d = 0\), i.e., \(f(N_0) = \{a\}\) for some \(0 \neq a \in \mathbb{Z}\), then \([f(N_0)] = a\mathbb{N}\) cannot be covered by finitely many sumsets.

Without loss of generality, we may assume that \(f : \mathbb{N}_0 \to \mathbb{Z}\) is a polynomial of degree \(d \geq 1\) with a positive leading coefficient. There exists \(b \in \mathbb{N}\) such that \(f(x) > 0\) for all \(x > b\), so \(f(x)\) achieves its minimum value \(B \in \mathbb{Z}\) in \([0, b) \cap \mathbb{N}_0\).

If \(B < 0\), then \(BN \subset [f(N_0)]\) cannot be covered by finitely many sumsets. So we only need to consider the case when \(B \geq 0\), i.e., when \(f(N_0) \subseteq \mathbb{N}_0\). If necessary, we can multiply \(f\) by a nonzero rational number and assume that \(\gcd(f(N_0)) = 1\). By the discussion above, the condition on \(f\) is necessary, and it suffices to prove the assertion when \(d \geq 1\) such that \(f(N_0) \subseteq \mathbb{N}_0\) and \(\gcd(f(N_0)) = 1\).

If \(d = 1\), then by Lemma 2 or 3, we must have \(f(x) = ax + b\), where \(a \in \mathbb{N}\) and \(b \in \mathbb{N}_0\) and \(\gcd\{a, b + a\} = \gcd(f(N_0)) = 1\). If \(a = 1\), then \([f(N_0)] = \mathbb{Z}_{\geq b} = f(N_0)\) and \(N = 1\). If \(b = 0\), then \(a = 1\), thus \([f(N_0)] = N_0 = f(N_0)\) and \(N = 1\). If \(b = 1\), then \([f(N_0)] = \mathbb{N} = \bigcup_{k=1}^a k\cdot f(N_0)\) and \(N = a\). So we may assume \(a \neq 1\) and \(b \neq 0, 1\).

Each integer greater than the Frobenius number \(g(a + b, b)\) belongs to the subgroup \([f(N_0)]\), since \(a + b, b \in f(N_0)\). Then, from each equivalence class \(i + a\) of \(\mathbb{Z}/a\), where \(i \in \{0, 1, \ldots, a - 1\}\), we can pick the smallest integer \(Z_i \in [f(N_0)]\). Then, let \(N_i\) be the smallest number such that each \(Z_i\) is decomposed into the sum of exactly \(N_i\) elements of \(f(N_0)\). Let \(M = \max\{Z_i \mid 0 \leq i \leq a - 1\}\). Then, for any \(Z \geq M + b\), there exists exactly one way to represent

\[Z = aK + b + Z_i, \text{ for some } i.\]

From \(aK + b + Z_i \geq M + b\), we obtain \(aK \geq M - Z_i \geq 0\) and thus \(K \geq 0\). Hence, \(Z\) can be decomposed into the sum of exactly \(1 + N_i\) elements of \(f(N_0)\). Let \(N'\) be the least integer such that each \(Z < M + b\) in \([f(N_0)]\) can be decomposed into the sum of at most \(N'\) elements of \(f(N_0)\). Then, we can choose \(N = \max\{N', 1 + N_i \mid i = 0, 1, \ldots, a - 1\}\).

For \(d \geq 2\), a similar proof follows from Kamke’s non-explicit proposition. □

But in general Proposition 3 does not generalize to a vector of polynomials \(f : \mathbb{N}_0 \to \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}\). The failure has to do with the boundary of the convex cone spanned by vectors of the polynomial values. Here is an example when it happens.
Example. Consider the following example
\[ f : \mathbb{N}_0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}; \quad x \mapsto (x, x^2, \ldots, x^n), \]
which is a vector of polynomials of degree \( \leq n \). Since \( f(0) = (0, 0, \ldots, 0) \), we have the following ascending chain of sumsets
\[ f(\mathbb{N}_0) \subseteq 2f(\mathbb{N}_0) \subseteq \cdots \subseteq kf(\mathbb{N}_0) \subseteq \cdots \subseteq [f(\mathbb{N}_0)]. \]
But this chain does not stabilize for \( k \) large enough. Indeed, we have
\[ (N + 1, N + 1, \ldots, N + 1) \in (N + 1)f(\mathbb{N}_0), \]
which can only be the sum of \((1, 1, \ldots, 1)\)'s and \((0, 0, \ldots, 0)\)'s, and we need at least \((N + 1) (1, 1, \ldots, 1)\)'s. Hence, \( kf(\mathbb{N}_0) \) is always a proper subset of \([f(\mathbb{N}_0)]\) for any \( k \in \mathbb{N} \).

More generally, motivated by the above example, we have the following result.

**Proposition 4.** Consider the following map of a vector of polynomials
\[ f : \mathbb{N}_0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}; \quad x \mapsto (f_1(x), f_2(x), \ldots, f_n(x)), \]
where at least two \( f_i : \mathbb{N}_0 \rightarrow \mathbb{Z} \) have degree \( \geq 0 \) and at least two \( f_j : \mathbb{N}_0 \rightarrow \mathbb{Z} \) are not proportional to each other. Then, for any \( m \in \mathbb{N} \), \( \bigcup_{k=1}^{m} kf(\mathbb{N}_0) \) and thus \( kf(\mathbb{N}_0) \) for all \( 1 \leq k \leq m \) are always proper subsets of \([f(\mathbb{N}_0)]\).

**Proof.** The proof is essentially done by contradiction and the strategy has been illustrated in the previous example and Proposition 3. \( \square \)

**Remark.** Consider the ring \( \mathcal{O} \) of integers of a totally imaginary number field \( K \), and a closed commutative \( \mathcal{O} \)-subscheme \( \mathcal{G} \) of the group scheme \( \mathcal{U}_k \) of unitary \( k \times k \) matrices, and a generating \( \mathcal{O} \)-morphism \( f : \mathbb{A}^1 \rightarrow \mathcal{G} \), i.e., \( f \) is generating as a \( K \)-morphism. Then, \([LN19, \text{ Thm 3.1 or Prop 3.6}]\) implies that there exists some positive integer \( M \) such that \( Mf(\mathcal{O}) \) is a subgroup of finite index in \( \mathcal{G}(\mathcal{O}) \). But if we are working in a real number field \( K \), then Proposition 4 implies that an analogous statement will no longer hold in general, for example, when \( K = \mathbb{Q} \), \( \mathcal{O} = \mathbb{Z} \), \( \mathcal{G} = \mathbb{A}^2 \) as a closed commutative \( \mathbb{Z} \)-subscheme of \( \mathcal{U}_3 \) and \( f : \mathbb{A}^1 \rightarrow \mathbb{A}^2 \) defined by \( x \mapsto (x^2, x^4) \).

### 3. Waring’s Problem for General Discrete Heisenberg Groups \( H_{2n+1}(\mathbb{Z}) \)

The main result of this paper will be given in this section. But let us first recall definitions of polynomial maps, polynomial sequences and polynomial sets. For any map \( f : S \rightarrow G \) from a nonempty semigroup \( S \) to a group \( G \) and for any \( s \in S \), we define the following left and right forward finite differences
\[ L_s(f) : S \rightarrow G \quad \text{and} \quad R_s(f) : S \rightarrow G \]
\[ t \mapsto f(s + t)f(t)^{-1}, \quad t \mapsto f(t)^{-1}f(s + t). \]
Then, \( f : S \rightarrow G \) is a called a polynomial map of degree \( \leq d \), if for any \( s_1, s_2, \ldots, s_{d+1} \in S \),
\[ D_{s_1}D_{s_2} \cdots D_{s_{d+1}} f = 1_G, \]
where each \( D \) is arbitrarily taken to be \( L \) or \( R \). The minimal \( d \) with this property is called the degree of \( f \). In particular, a map \( f : S \rightarrow G \) is a polynomial map of degree \( -\infty \) if \( f \) maps \( S \) to the identity element \( 1_G \) of \( G \), and \( f \) is a polynomial map
of degree 0 if it is a constant \( \neq 1_G \). A polynomial sequence is a polynomial map of the form \( f : \mathbb{N}_0 \to G \).

A polynomial set \( U = \text{Im} f \) in a path-connected nilpotent Lie group \( N \) is given by the image of a continuous polynomial map \( f : \mathbb{R}_{\geq 0}^n \to N \) for some \( n \), and \( U \) is called open (resp. closed, resp. proper) if \( U \) is open (resp. is closed, resp. has nonempty interior). A nonempty subset \( V \) of a nilpotent group \( G \) is called a polynomial set, if it is the inverse image \( \phi^{-1}(U) \) of a polynomial set \( U \) of a nilpotent Lie group \( N \) along some group homomorphism \( \phi : G \to N \), and \( V = \phi^{-1}(U) \) is called open (resp. closed, resp. proper) in \( G \) provided that \( U \) has the same property in \( N \). In this work, generalized cones in nilpotent groups are given by proper polynomial sets.

**Case in which \( \langle g \rangle \subset \mathbb{Z}^m \).** The following theorem is a generalization of Kamke’s key theorem.

**Theorem 2.** Let \( g : \mathbb{N}_0 \to \mathbb{Z}^m \) be a polynomial sequence. If \( g \) mod \( N \) is non-constant for any subgroup \( N \) of infinite index in \( \mathbb{Z}^m \), then there exists \( M \in \mathbb{N} \), a subgroup \( H \) of finite index in \( \mathbb{Z}^m \) and a proper polynomial set \( V \) of \( H \) such that each element in \( V \) can be written as a sum of exactly \( M \) elements in \( g(\mathbb{N}_0) \).

**Remark.** In this case, \( g = (g_1, \ldots, g_m) \) is just a vector of polynomials \( g_i : \mathbb{N}_0 \to \mathbb{Z} \) and \( V = H \cap \text{Im} f \) is the intersection of \( H \) with the image of some proper continuous polynomial map \( f = (f_1, \ldots, f_m) : \mathbb{R}_{\geq 0}^n \to \mathbb{R}^m \) along the embedding \( \varphi : H \hookrightarrow \mathbb{R}^m \), where each \( f_i : \mathbb{R}_{\geq 0}^n \to \mathbb{R} \) is a polynomial in \( n \) variables; cf. [Hu21a, Cor 13 and Cor 17].

**Remark.** Proposition 4 implies that \( \langle g \rangle \) in general cannot be covered by finitely many sumsets \( kg(\mathbb{N}_0) \). So the best possible result is that points of certain proper polynomial set of a finite index subgroup \( H \) of \( \langle g \rangle \) can be covered by finitely many sumsets \( kg(\mathbb{N}_0) \).

**Proof.** Notice that \( h = g - g(0) \) is another polynomial sequence satisfying our hypothesis: for any subgroup \( N \) of infinite index in \( \mathbb{Z}^m \), if \( h \) mod \( N \) is a constant, then so is \( g \) mod \( N \). So we may assume that \( g(0) = (0, \ldots, 0) \). Indeed, suppose we have proven the theorem for \( h \) by finding the desirable positive integer \( M \), subgroup \( H \) of finite index in \( \mathbb{Z}^m \) and proper polynomial set \( V \) of \( H \). Then, we have

\[
\sum_{i=1}^M g(x_i) - \sum_{i=1}^M h(x_i) = Mg(0).
\]

Since \( h(0) = (0, \ldots, 0) \), we can find a larger \( M \) so that \( Mg(0) \) lies in the lattice \( H \) and translate the proper polynomial set \( V \) of \( H \) by a certain scalar multiple of \( g(0) \) to obtain a proper polynomial set that works for \( g \). Moreover, since \( g \) mod \( N \) is non-constant for any subgroup \( N \) of infinite index in \( \mathbb{Z}^m \), \( \langle g \rangle \) must have finite index in \( \mathbb{Z}^m \). Then, there exists a nonsingular matrix \( D \in M(m, \mathbb{Z}) \) such that \( \langle g \rangle = D\mathbb{Z}^m \). Replacing \( g \) by \( D^{-1}g \) if necessary, we may also assume that \( \langle g \rangle \cong \mathbb{Z}^m \).

Write \( g(x) = (g_1(x), \ldots, g_m(x)) \in \mathbb{Z}^m \), where \( g_i(x) = \sum_{j=1}^d c_{ji}x^j \), \( d = \max\{\deg g_i : 1 \leq i \leq m\} \), and \( c_{ji} \in \mathbb{Q} \). Since \( \mathbb{Z}^m \) is generated by \( g \), there exist \( M_i \in \mathbb{N} \), \( \varepsilon_{ki} \in \{\pm 1\} \) and \( x_{ki} \in \mathbb{N}_0 \) such that

\[
e_{i} = \left( \sum_{k=1}^{M_i} \varepsilon_{ki}g_1(x_{ki}), \ldots, \sum_{k=1}^{M_i} \varepsilon_{ki}g_m(x_{ki}) \right),
\]
where \( e_1 = (1,0,\ldots,0), \ldots, e_m = (0,\ldots,0,1) \) are the standard generators of \( \mathbb{Z}^m \).

Rewrite the above equation as follows:

\[
e_i = \left( \sum_{k=1}^{M_i} \varepsilon_{ki} x_{ki}, \ldots, \sum_{k=1}^{M_i} \varepsilon_{ki} x_{ki}^d \right) \left( \begin{array}{c} c_{11} \cdots c_{1m} \\ \vdots \vdots \\ c_{d1} \cdots c_{dm} \end{array} \right).
\]

For this system of linear equations to be solvable, the coefficient matrix

\[
C = \left( \begin{array}{ccc} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{d1} & \cdots & c_{dm} \end{array} \right) \in M_{d,m}(\mathbb{Q})
\]

must have full column rank \( m \) and thus \( d \geq m \).

If \( m = 1 \), then Kamke’s non-explicit proposition covers the case in which \( d \geq 2 \); if \( m = 1 = d \), then \( g(x) = \pm x \) gives rise to a trivial case. So we may assume that \( m \geq 2 \) and thus \( d \geq 2 \). Let \( M \) be \( N(d) \) as in Kamke’s key theorem and \( \tilde{g} : \mathbb{N}^M_{\geq 0} \to \mathbb{Z}^m \) be defined by

\[
(x_1, \ldots, x_M) \mapsto \sum_{k=1}^{M} g(x_k) = \left( \sum_{k=1}^{M} g_1(x_k), \ldots, \sum_{k=1}^{M} g_m(x_k) \right).
\]

Then, we can write

\[
\tilde{g}(x_1, \ldots, x_M) = \left( \sum_{j=1}^{d} c_{j1}s_j, \ldots, \sum_{j=1}^{d} c_{jm}s_j \right) = (s_1, \ldots, s_d)C,
\]

where \( s_j = \sum_{k=1}^{M} x_k^j \) for \( j = 1, \ldots, d \). Let \( A \) be as in Kamke’s key theorem. Replace \( A \) by \( kA \) for some \( k \in \mathbb{N} \) if necessary so that \( AC \in M_{d,m}(\mathbb{Z}) \). Now we define a linear polynomial map

\[
p : \mathbb{R}^d_{\geq 0} \to \mathbb{R}^m ; (s_1, \ldots, s_d) \mapsto (s_1, \ldots, s_d)C
\]

By [Hu21a, Lem 8], a Kamke domain always contains a proper polynomial set given by some continuous polynomial map \( q : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^d \). Let \( f = p \circ q \). Then, \( H = p(\mathbb{A}^d) = D\mathbb{Z}^m \) is a lattice and thus a finite index subgroup in \( \mathbb{Z}^m \) for some nonsingular \( D \in M(m, \mathbb{Z}) \). Let \( U(d, M) \) be the Kamke domain. Then, we have the following relation of sets

\[
\tilde{g}(\mathbb{N}^M_{\geq 0}) \supset p(U(d, M) \cap A\mathbb{N}^d_0) = p(U(d, M) \cap A\mathbb{Z}^d) \supset \text{Im} f \cap H.
\]

Let \( \phi : H \hookrightarrow \mathbb{R}^m \) be the inclusion map. Then, \( U = \text{Im} f \) (resp. \( V = U \cap H = \phi^{-1}(U) \)) is a proper polynomial subset in \( \mathbb{R}^m \) (resp. \( H \)) and each element of \( V \) can be written as a sum of exactly \( M \) elements in the sequence \( g_0, g_1, g_2, \ldots \). \( \Box \)

Remark. Notice that each entry \( c_{ji} \) the coefficient matrix \( C \) can be thought as \( \tilde{g}_i^{(j)}(0)/j! \). So it is natural to consider the following matrix associated with \( g = (g_1, \ldots, g_m) \):

\[
J(g)(x) = \begin{pmatrix}
g_1^{(1)}(x) & \cdots & g_m^{(1)}(x) \\
\vdots & \ddots & \vdots \\
g_1^{(d)}(x) & \cdots & g_m^{(d)}(x)
\end{pmatrix} = \text{diag}(1!, 2!, \ldots, d!)C.
\]
Then, $C$ and $J(g)(0)$ have the same rank $m$. Moreover, for any pair $(a, b) \in \mathbb{N} \times \mathbb{N}_0$, by the chain rule, the associated matrix of $g(ax + b)$ satisfies the following equation

\[ J(g(ax + b))(0) = \text{diag}(a, a^2, \ldots, a^d)J(g)(b). \]

If $g$ satisfies the hypothesis of Theorem 2, then so does $g(ax + b)$. Then, the same argument shows $J(g(ax + b))(0)$ and thus $J(g)(b)$ have the same rank $m$.

**A simple observation.** Consider a polynomial sequence of the following form

\[
g = \begin{pmatrix}
1 & g_{1,2} & \cdots & g_{1,n} \\
1 & g_{2,2} & \cdots & g_{2,n} \\
& \ddots & \ddots & \ddots \\
& & \ddots & g_{n-1,n} \\
& & & 1
\end{pmatrix} : \mathbb{N}_0 \rightarrow \mathcal{U}_n(\mathbb{Z}).
\]

By [Hu21a, Thm 11], each $g_{i,j}$ is a polynomial of degree $d_{i,j}$ with rational coefficients. Let $L$ be an arbitrary natural number and $x_1, x_2, \ldots, x_L$ be variables in $\mathbb{N}_0$. Consider the ordered product

\[ \hat{g} := \bigcirc_{i=1}^{L} g : \mathbb{N}_0^L \rightarrow \mathcal{U}_n(\mathbb{Z}); \quad (x_1, \ldots, x_L) \mapsto g(x_1) \cdots g(x_L). \]

Let $\hat{d}_{i,j}(L)$ be the total degree of the polynomial map $\hat{g}_{i,j}(x_1, \ldots, x_L)$ in the $L$ variables. Then, the set $\{\hat{d}_{i,j}(L) \mid 1 \leq i < j \leq n, L \in \mathbb{N}\}$ has an upper bound $B$, which satisfies:

\[
B \geq \max_{1 \leq i < j \leq n} \{d_{i,j}, d_{i,1} + d_{i,j}, \ldots, d_{i,1} + d_{i,2} + \cdots + d_{i,m,j}\},
\]

\[
i + 1 \leq l \leq j - 1, \ldots, i + 1 \leq l_1 < l_2 < \cdots < l_m \leq j - 1\}.
\]

Indeed, each entry $\hat{g}_{i,j}(x_1, \ldots, x_L)$ is of the form

\[
\sum_{1 \leq k \leq L} g_{i,j}(x_k) + \sum_{1 \leq k_1 < k_2 \leq L} \left( \sum_{i+1 \leq l \leq j-1} g_{i l}(x_{k_1})g_{i j}(x_{k_2}) \right) + \cdots 
\]

\[
+ \sum_{1 \leq k_1, \ldots, k_m \leq L} \left( \sum_{i+1 \leq l_1 < l_2 < \cdots < l_m \leq j-1} g_{i l_1}(x_{k_1})g_{i l_2}(x_{k_2}) \cdots g_{i m j}(x_{k_m}) \right) + \cdots.
\]

Since the corresponding terms in the above equation vanish for $m \geq j - i$, the total degree of each entry $\hat{g}_{i,j}$ has a universal upper bound $B$, which satisfies (3.1).

**Case in which** $\langle g \rangle \subset H_{2n+1}(\mathbb{Z})$. Recall that for any $n \times n$ upper unitriangular matrix $X$ in the Lie group $\mathcal{U}_n(\mathbb{R})$, its logarithm is defined by

\[
\log X := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{(X - I)^k}{k} = \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} (X - I)^k,
\]

which lies in the nilpotent Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices. In this case, the logarithm and the exponential

\[
\exp(Y) := \sum_{k=0}^{\infty} \frac{Y^k}{k!} = \sum_{k=0}^{n-1} \frac{Y^k}{k!}, \quad \forall Y \in \mathfrak{g}
\]
are mutually inverse to each other.

As illustrated in the following proposition, the strategy of iterated symmetrization will be applied to solve Waring’s problem in general discrete Heisenberg groups.

**Proposition 5.** Let $X_1, \ldots, X_N$ be any matrices in the continuous Heisenberg group $H_{2n+1}(\mathbb{R})$ for some integer $n \geq 1$. Then, there exists $M \in \mathbb{N}$ and a sequence $a_1, a_2, \ldots, a_M \in \{1, 2, \ldots, N\}$, only dependent on $N$, such that

\[
X_{a_1} X_{a_2} \cdots X_{a_M} = \exp \left( \frac{M}{N} \sum_{i=1}^{N} \log X_i \right).
\]

In particular, each $a_1, a_2, \ldots, a_M \in \{1, 2, \ldots, N\}$ appears exactly $M/N$ times.

**Proof.** If we set $Y_i = \log X_i$ for $1 \leq i \leq N$, then it is equivalent to proving

\[
\exp(Y_{a_1}) \exp(Y_{a_2}) \cdots \exp(Y_{a_M}) = \exp \left( \frac{M}{N} \sum_{i=1}^{N} Y_i \right)
\]

for some sequence $a_1, a_2, \ldots, a_M \in \{1, 2, \ldots, N\}$. From the Baker-Campbell-Hausdorff formula $\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y])$, we obtain that

\[
\exp(Y_{a_1}) \exp(Y_{a_2}) \cdots \exp(Y_{a_M}) = \exp \left( \sum_{i=1}^{M} Y_{a_i} + \frac{1}{2} \sum_{1 \leq i < j \leq M} [Y_{a_i}, Y_{a_j}] \right)
\]

Hence, if we choose the sequence to be $1, 2, \ldots, N-1, N, N, N-1, \ldots, 2, 1$, then the main terms $\sum_{i=1}^{M} Y_{a_i}$ becomes $\exp \left( 2 \sum_{i=1}^{N} Y_i \right)$ and all the error terms, i.e., the commutators, cancel with each other, since $[Y_i, Y_j] + [Y_j, Y_i] = 0$ for all $1 \leq i, j \leq n$.

From [Hu21a, Theorem 14] or the proposition above, we obtain the following

**Corollary 3.** Let $\hat{g} = \bigcirc_{i=1}^{N} g : \mathbb{R}_{\geq 0}^{N} \to H_{2n+1}(\mathbb{R})$ be the ordered product obtained from any continuous polynomial map $g : \mathbb{R}_{\geq 0} \to H_{2n+1}(\mathbb{R})$. Then, there is a finite natural number $M$, only dependent on $N$, and a sequence $\sigma_1, \sigma_2, \ldots, \sigma_M$ in the permutation group $S_N$ such that the product

\[
\hat{g} = \prod_{i=1}^{M} \sigma_i(\hat{g}) = \sigma_1(\hat{g}) \sigma_2(\hat{g}) \cdots \sigma_M(\hat{g})
\]

is a symmetric continuous polynomial map of the form

\[
\hat{g}(x_1, \ldots, x_N) = \exp \left( \sum_{i=1}^{N} \log g(x_i) \right).
\]

**Proof of the main result.** For any $n \in \mathbb{N}$, we denote by $G := H_{2n+1}(\mathbb{Z})$ the general discrete Heisenberg group. Let $D$ be a nonnegative integer, $\Gamma$ be a subgroup of $G$ and

\[
\pi_D : \Gamma \to G \to H_{2n+1}(\mathbb{Z}/D)
\]

be the reduction homomorphism. The principal congruence subgroup of level $D$ in $\Gamma$ is defined to be the kernel of $\pi_D$, and is usually denoted by $\Gamma(D)$ and a congruence subgroup of $\Gamma$ is defined to be any group containing a principal congruence subgroup. It is immediate that $\Gamma(D)$ is a normal subgroup of finite index in $\Gamma$ and
all congruence subgroups containing $\Gamma(D)$ have finite index in $\Gamma$ and correspond to subgroups of the finite groups $\pi_1(\Gamma)$. Therefore,

$$H_{2n+1}(D\mathbb{Z}) := \ker(G \rightarrow H_{2n+1}(\mathbb{Z}/D))$$

is a normal subgroup of finite index in $G$.

For ease of notation, we can identify $H_{2n+1}(\mathbb{R})$ with $\mathbb{R}^{2n+1}$ via the map

$$\begin{pmatrix} 1 & a & c \\ I_n & b \\ 1 \end{pmatrix} \mapsto (a, b, c),$$

where $I_n$ is the identity matrix of size $n$, with group law given by matrix multiplication:

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + a \cdot b').$$

Similarly, we can identify the Lie algebra $\mathfrak{h}_{2n+1}(\mathbb{R})$ of $H_{2n+1}(\mathbb{R})$ with $\mathbb{R}^{2n+1}$ via the map

$$\begin{pmatrix} 0 & a & c \\ 0_n & b \\ 0 \end{pmatrix} \mapsto (a, b, c).$$

Note that depending on the context the same notation $(a, b, c)$ may denote different matrices. Let $\pi_{[1,2n]} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n}$ or $\pi_{[1,2n]} : H_{2n+1}(\mathbb{R}) \cong \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n}$ be the projection of the first $2n$ coordinates and $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be the symplectic form given by the nonsingular, skew-symmetric matrix $\Omega = \left( \begin{array}{cc} I_n & -I_n \end{array} \right)$. Then, it is easy to verify the following formulas

$$(a, b, c)^{-1} = (-a, -b, -c + a \cdot b),$$

$$(a, b, c)(a', b', c')(a, b, c)^{-1} = (a', b', c' + a \cdot b' - a' \cdot b),$$

$$(a, b, c)(a', b', c')(a, b, c)^{-1}(a', b', c')^{-1} = (0, 0, a \cdot b' - a' \cdot b) = (0, 0, \omega((a, b), (a', b'))).$$

Notice that the center of $H_{2n+1}(\mathbb{R})$ is $Z = \{(0, 0, c) \in H_{2n+1}(\mathbb{R}) | c \in \mathbb{R} \} \cong \mathbb{R}$, and that the matrix exponential $\exp : \mathfrak{h}_{2n+1}(\mathbb{R}) \rightarrow H_{2n+1}(\mathbb{R})$ and the matrix logarithm $\log : H_{2n+1}(\mathbb{R}) \rightarrow \mathfrak{h}_{2n+1}(\mathbb{R})$ are diffeomorphisms inverse to each other.

Let $g : N_0 \rightarrow G$ be a polynomial sequence. We can write

$$g(x) = (a(x), b(x), c(x)) = (0, 0, d(x))(a(x), b(x), \frac{1}{2}a(x) \cdot b(x)),$$

where $a = (g_{1,2}, \ldots, g_{1,n+1})$ is a vector of polynomial sequences $g_{1,i} : N_0 \rightarrow \mathbb{Z},$ $b = (g_{2,n+2}, \ldots, g_{n+1,n+2})$ is a vector of polynomial sequences $g_{i,n+2} : N_0 \rightarrow \mathbb{Z},$ $c = g_{1,n+2} : N_0 \rightarrow \mathbb{Z}$ is a polynomial sequence, and $d = c - \frac{1}{2}a \cdot b : N_0 \rightarrow \mathbb{Z}$ is another polynomial sequence. We may replace $N_0$ by $\mathbb{Z}_{\geq N}$ for some sufficiently large $N$ so that $g_{i,j}(x)$ does not change its sign for all $x \geq N$. Since $(0, 0, d(x))$ lies in the center of $G$, we have

$$\log g(x) = \log(0, 0, d(x)) + \log(a(x), b(x), \frac{1}{2}a(x) \cdot b(x)),$$

$$= (0, 0, d(x)) + (a(x), b(x), 0) = (a(x), b(x), d(x)).$$
Consider the ordered product
\[
\hat{g} := \bigodot_{i=1}^{L} g : \mathbb{N}_0^L \to G; \quad (x_1, \ldots, x_L) \mapsto g(x_1) \cdots g(x_L).
\]

By the Baker-Campbell-Hausdorff formula, we have
\[
\hat{g}(x_1, \ldots, x_L) = \exp \left( \sum_{i=1}^{L} \log g(x_i) \right) \exp \left( \frac{1}{2} \sum_{1 \leq i < j \leq L} [\log g(x_i), \log g(x_j)] \right)
\]
\[
= \left( \sum_{i=1}^{L} a(x_i), \sum_{i=1}^{L} b(x_i), \sum_{i=1}^{L} d(x_i) + \frac{1}{2} \left( \sum_{i=1}^{L} a(x_i) \right) \cdot \left( \sum_{i=1}^{L} b(x_i) \right) \right)
\]
\[
= \left( 0, 0, \frac{1}{2} \sum_{1 \leq i < j \leq L} \omega((a(x_i), b(x_i)), (a(x_j), b(x_j))) \right).
\]

Then, by the proof of Proposition 5, the product
\[
\hat{g}(x_1, \ldots, x_L) = g(x_1) \cdots g(x_L) g(x_L) \cdots g(x_1)
\]
is a symmetric polynomial map in \( L \) variables \( x_1, x_2, \ldots, x_L \) of the form
\[
\exp \left( \sum_{i=1}^{L} \log g(x_i) \right) = \left( \sum_{i=1}^{L} 2a(x_i), \sum_{i=1}^{L} 2b(x_i), \sum_{i=1}^{L} 2d(x_i) + 2 \left( \sum_{i=1}^{L} a(x_i) \right) \cdot \left( \sum_{i=1}^{L} b(x_i) \right) \right).
\]

Let \( B \) be the least upper bound as in the simple observation and \( a_{i,j} \) be the coefficient of the term \( x^k \) in the polynomial \( g_{i,j}(x) \). Then, we can write
\[
g_{i,j}(x) = \sum_{k=1}^{B} a_{i,j}^k x^k = (a_{i,j}^{0}, a_{i,j}^{1}, \ldots, a_{i,j}^{B}) \cdot (1, x, \ldots, x^B).
\]

To ease the notation, we write \( a_{i,j}^B = (a_{i,j}^{0}, a_{i,j}^{1}, \ldots, a_{i,j}^{B}) \) and \( s_B = (s_1, s_2, \ldots, s_B) \), where \( s_j = \sum_{i=1}^{L} x_i^j \) for \( j = 0, 1, \ldots, B \). Each \( \hat{g}_{i,j} \) is a symmetric polynomial of total degree \( \leq B \) and thus can be written as a polynomial expression with rational coefficients in the power sum symmetric polynomials \( s_1, s_2, \ldots, s_B \), i.e.,
\[
\hat{g}_{i,j}(x_1, \ldots, x_L) = p_{i,j}(s_1, \ldots, s_B),
\]

where \( p_{i,j} \) is an integer-valued polynomial in \( B \) variables \( s_1, \ldots, s_B \in \mathbb{N}_0 \) with rational coefficients of the following forms:
\[
p_{1,1}(s_1, \ldots, s_B) = 2a_{i,j}^B \cdot s_B, \quad p_{i,n+2}(s_1, \ldots, s_B) = 2a_{i,n+2}^B \cdot s_B,
\]
\[
p_{1,n+2}(s_1, \ldots, s_B) = 2a_{1,n+2}^B \cdot s_B - \sum_{k=2}^{n+1} \sum_{j=0}^{B} a_{1,k}^j a_{n+2,k}^{j-1} s_j + 2 \sum_{k=2}^{n+1} (a_{1,k}^B \cdot s_B) (a_{n+2,k}^B \cdot s_B).
\]

Therefore, we can define the following continuous polynomial map in \( B \) variables \( s_1, \ldots, s_B \in \mathbb{R}_{\geq 0} \):
\[
p := \left( p_{1,2}, \ldots, p_{1,n+1}, p_{2,n+2}, \ldots, p_{n+1,n+2}, p_{1,n+2} \right) : \mathbb{R}_{\geq 0}^B \to H_{2n+1}(\mathbb{R}).
\]

Then, \( \log p(s_1, \ldots, s_B) \) is given by
\[
\left( 2a_{1,2}^B \cdot s_B, \ldots, 2a_{1,n+1}^B \cdot s_B, 2a_{2,n+2}^B \cdot s_B, \ldots, 2a_{2,n+1,n+2}^B \cdot s_B, 2a_{1,n+2}^B \cdot s_B - \sum_{k=2}^{n+1} \sum_{j=0}^{B} a_{1,k}^j a_{n+2,k}^{j-1} s_j \right),
\]
whose Jacobian matrix with respect to the \( s_j \)’s \( (j = 1, \ldots, B) \) is given by
\[
\mathbf{J} = \begin{pmatrix}
2a_{1,2}^1 & \cdots & 2a_{1,n+1}^1 & 2a_{2,n+2}^1 & \cdots & 2a_{n+1,n+2}^1 & 2a_{1,2}^1 - \sum_{k=2}^{n+1} \sum_{j=0}^{1} a_{1,k}^j a_{1,n+2}^{1-l}
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
2a_{1,2}^B & \cdots & 2a_{1,n+1}^B & 2a_{2,n+2}^B & \cdots & 2a_{n+1,n+2}^B & 2a_{1,2}^B - \sum_{k=2}^{n+1} \sum_{j=0}^{B} a_{1,k}^j a_{1,n+2}^{B-l}
\end{pmatrix}.
\]

Let \( \mathbf{J}_0 \) be the matrix formed by the first \( 2n \) columns of \( \mathbf{J} \) and let \( \mathbf{J}_1 \) be the matrix which has the constant vector
\[
(3.9) \quad (2a_{1,2}^0, \ldots, 2a_{1,n+1}^0, 2a_{2,n+2}^0, \cdots, 2a_{n+1,n+2}^0, 2a_{1,2}^0 - \sum_{k=2}^{n+1} a_{1,k}^0 a_{1,n+2}^{0-l})
\]
as its first row and \( \mathbf{J} \) as the remaining rows.

Suppose that \( g \mod N \) is non-constant for any normal subgroup \( N \) of infinite index in \( G \). Since any infinite index normal subgroup of \( G/[G, G] \) corresponds to an infinite index normal subgroup of \( G \) containing \([G, G]\), we see that the induced map
\[
g \mod [G, G] : \mathbb{N}_0 \to \mathbb{Z}^{2n}; \quad x \mapsto (\mathbf{a}(x), \mathbf{b}(x)),
\]
satisfies the hypotheses of Theorem 2. By the proof of Theorem 2, \( g \mod [G, G] \) has degree at least \( 2n \) and \( \mathbf{J}_0 \) has rank \( 2n \) and \( p_{1,2}, \ldots, p_{1,n+1}, p_{2,n+2}, \ldots, p_{n+1,n+2} \) are linearly independent in variables \( s_1, \ldots, s_B \) with \( B \geq 2n \). In this case, the rank of \( \mathbf{J} \) is \( 2n \), if and only if the last column of \( \mathbf{J} \) is a \( \mathbb{Q} \)-linear combination of the first \( 2n \) columns of \( \mathbf{J} \), if and only if \( d(x) \) has the form
\[
(3.10) \quad d(x) = \mathbf{u} \cdot \mathbf{a}(x) + \mathbf{v} \cdot \mathbf{b}(x) + w,
\]
where \( \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Q}^n \), \( \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Q}^n \), and \( w \in \mathbb{Q} \).

Just like the abelian case, we consider the following matrix associated to \( \log g \):
\[
\mathbf{J}(\log g)(x) = \begin{pmatrix}
g_{1,2}^{(1)} & \cdots & g_{1,n+1}^{(1)} & g_{2,n+2}^{(1)} & \cdots & g_{n+1,n+2}^{(1)} & d^{(1)}
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
g_{1,2}^{(B)} & \cdots & g_{1,n+1}^{(B)} & g_{2,n+2}^{(B)} & \cdots & g_{n+1,n+2}^{(B)} & d^{(B)}
\end{pmatrix}(x).
\]

Notice that entries of \( \mathbf{J}(\log g)(0) \) are nothing but positive scalar multiples of coefficients of \( \log g(x) \) or the corresponding entries of the above matrix \( \mathbf{J} \). Similarly, let \( \mathbf{J}_0(\log g)(x) \) be the matrix formed by the first \( 2n \) columns of \( \mathbf{J}(\log g)(x) \) and let \( \mathbf{J}_1(\log g)(x) \) be the matrix which contains
\[
(3.11) \quad (g_{1,2}(x), \cdots, g_{1,n+1}(x), g_{2,n+2}(x), \cdots, g_{n+1,n+2}(x), d(x))
\]
as its first row and \( \mathbf{J}(g) \) as the remaining rows.

For any pair \((a, b) \in \mathbb{N} \times \mathbb{N}_0\), we can also consider the polynomial sequence \( g(ax + b) \), i.e., the composition of \( g(x) \) with the affine translation \( ax + b \). By the chain rule of derivatives, the matrix \( \mathbf{J}(\log g(ax + b))(x) \) is given by
\[
\begin{pmatrix}
ag_{1,2}^{(1)} & \cdots & ag_{1,n+1}^{(1)} & ag_{2,n+2}^{(1)} & \cdots & ag_{n+1,n+2}^{(1)} & ad^{(1)}
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
ag_{1,2}^{(B)} & \cdots & ag_{1,n+1}^{(B)} & ag_{2,n+2}^{(B)} & \cdots & ag_{n+1,n+2}^{(B)} & ad^{(B)}
\end{pmatrix}(ax + b).
\]
If $g_i(x) = (a_i(x), b_i(x), c_i(x)) : \mathbb{N}_0 \to G, 1 \leq i \leq m$, are polynomial sequences, then the logarithm of their ordered product is given by

$$\log(g_1g_2 \cdots g_m) = \sum_{i=1}^{m} \log g_i + \frac{1}{2} \sum_{1 \leq i < j \leq m} [\log g_i, \log g_j]$$

and

$$J(g_1g_2 \cdots g_m)(x) = \sum_{i=1}^{m} J(\log g_i)(x) + \frac{1}{2} \sum_{1 \leq i < j \leq m} J(\log g_i, \log g_j)(x),$$

where

$$J(\log g_i, \log g_j)(x) = \begin{pmatrix} 0 & (a_i \cdot b_j^{(1)} - a_j \cdot b_i^{(1)}) & \cdots & (a_i \cdot b_j^{(1)}) \cdot b_i^{(1)} - a_j \cdot b_i^{(1)} \cdot b_i^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \sum_{k=0}^{B} (B)_k (a_i^{(k)} \cdot b_j^{(B-k)} - a_j^{(k)} \cdot b_i^{(B-k)}) \cdot b_i^{(B-k)}(x) \end{pmatrix}.$$

We need the following lemma to handle the degenerate case.

**Lemma 5.** If $g \mod N : \mathbb{N}_0 \to G \to G/N$ is non-constant for any normal subgroup $N$ of infinite index in $G$, then there exists a finite number of pairs $(a_i, b_i) \in \mathbb{N}_0 \times \mathbb{N}_0, 1 \leq i \leq m$, such that the associated matrix $J(\log h)(y)$ of the polynomial sequence

$$h(x) := g(a_1x + b_1) \cdots g(a_mx + b_m) : \mathbb{N}_0 \to G$$

has rank $2n + 1$.

**Proof.** Since $g \mod [G,G] = (a, b) : \mathbb{N}_0 \to G/[G,G] \cong \mathbb{Z}^{2n}$ satisfies the hypotheses of Theorem 2, by the remark after the proof of that theorem, we can conclude $J_0(\log g)(b) = J_0((a, b))(b)$ has rank $2n$ for any $b \in \mathbb{N}_0$. If rank $J(\log g)(b) = 2n + 1$ for some $b \in \mathbb{N}_0$, then we are done. Suppose rank $J(\log g)(b) = \text{rank} J_0(\log g)(b) = 2n$ for all $b \in \mathbb{N}_0$, i.e., $d(x)$ is of the form (3.10).

For any tuple $(a_1, \ldots, a_m, b) \in \mathbb{N}_m \times \mathbb{N}_0$, consider the ordered product

$$h(x) = g(a_1x + b)g(a_2x + b) \cdots g(a_mx + b).$$

Then, we have

$$J(\log h)(0) = \sum_{i=1}^{m} J(\log g(a_i, x + b))(0) + \frac{1}{2} \sum_{1 \leq i < j \leq m} J(\log g(a_i, x + b), \log g(a_j, x + b))(0)$$

$$= \text{diag} \left( \sum_{i=1}^{m} a_i, \sum_{i=1}^{m} a_i^2, \ldots, \sum_{i=1}^{m} a_i^B \right) J(\log g)(b)$$

$$+ \frac{1}{2} \sum_{1 \leq i < j \leq m} \sum_{l=2}^{n+1} \begin{pmatrix} 0 & \sum_{k=0}^{1} (B)_k (a_i^{k}a_j^{1-k} - a_j^{k}a_i^{1-k}) \cdot g_{1,l}^{(1-k)}g_{1,n+2}^{(1-k)}(b) \\ \vdots & \vdots \\ 0 & \sum_{k=0}^{n} (B)_k (a_i^{k}a_j^{B-k} - a_j^{B-k}a_i^{B-k}) \cdot g_{1,l}^{(B-k)}g_{1,n+2}^{(B-k)}(b) \end{pmatrix}.$$
is a linear combination of the first $2n$ columns of $\sum_{i=1}^{m} J(\log (a_i x + b))(0)$. Since $g_{i,j}(x)$ are polynomials in $\mathbb{Q}[x]$, we obtain a system of $B$ polynomial equations in $\mathbb{Q}[a_1, \ldots, a_m]$ of degree at most $B$. So $S_m(b)$ consists of tuples $(a_1, \ldots, a_m) \in \mathbb{N}^m$ that are solutions to these polynomial equations. For $m > 1$, we need to analyze when $S_m(b) = \mathbb{N}^m$, i.e., when these $B$ polynomials vanish on all of $\mathbb{N}^m$. However, comparing similar terms, we see that this can happen if and only if the last column of $\frac{1}{2} \sum_{1 \leq i < j \leq m} J([\log g(a_i x + b), \log g(a_j x + b)])(0)$ is a vector of zero polynomials in variables $a_1, \ldots, a_m$. Therefore, as the coefficient of $a_i^k a_j^{L-k}$,

$$\frac{1}{2} \sum_{l=2}^{n+1} \binom{L}{k} \left( g_{1,l} g_{l,n+2} (b) - g_{1,l} (L-k) g_{l,n+2} (b) \right)$$

must vanish for all $L = 1, \ldots, B$ and $k = 0, \ldots, L$. Since this works for all $b \in \mathbb{N}_0$, we see that

$$(a \cdot b)^{(L)}(x) = \sum_{l=2}^{n+1} \binom{L}{k} g_{1,l} g_{l,n+2} (x)$$

vanishes for any odd $L = 1, \ldots, B$, and thus $a \cdot b$ must be a constant $C$. However, if $a \cdot b = C$, then $(a, b)$ cannot satisfy the hypotheses of [Hu21b, Thm 2]. So for sufficiently large $m$, we must have $S_m(b) \neq \mathbb{N}^m$ for some $b \in \mathbb{N}_0$ and thus the proof is complete. \hfill \Box

In the proof of the main result, we will need the following maps of sets

$$\delta : H_{2n+1} (\mathbb{R}) \to H_{2n+1} (\mathbb{R}), \quad \iota : H_{2n+1} (\mathbb{R}) \to H_{2n+1} (\mathbb{R}),$$

$$(a, b, c) \mapsto (a, b, c - \frac{1}{2} a \cdot b), \quad (a, b, c) \mapsto (a, b, c + \frac{1}{2} a \cdot b),$$

which are bijective and mutually inverses to each other.

Proof of Theorem 1. Since $g$ mod $N$ is non-constant for any normal subgroup $N$ of infinite index in $G$, $\langle g \rangle$ must have finite index in $G$. Set $B$ as before and let $L' \in \mathbb{N}$ be the least number such that

$$\max \{ d_i, j(L) | 1 \leq i < j \leq n, 1 \leq L \leq L' \} = B.$$ 

By Kamke’s key theorem, there exist positive integers $A, L''$, and positive numbers $i_1$ and $i_\nu$, $J_\nu$ with $0 < i_\nu < J_\nu$, $\nu = 2, 3, \ldots, B$, such that for each $B$ integers $s_1, \ldots, s_B$, divisible by $A$ and subject to the following conditions

$$s_1 \in (i_1, \infty); \quad s_\nu / s_\nu^{'} \in (i_\nu, J_\nu) \quad \text{for} \quad \nu = 2, 3, \ldots, B,$$

the $B$ equations $s_\nu = \sum_{k=1}^{L''} x_k$, $\nu = 1, 2, \ldots, B$, are simultaneously solvable by integers $x_\kappa \geq 0$. Let $L$ be the least integer $\geq \max \{ L', L'' \}$ divisible by $A$.

By [Hu21a, Cor 8], the subgroup $\langle \hat{g} \rangle = \langle \bigcup_{i=1}^{L} \hat{g} \rangle$ has finite index in $\langle g \rangle$. Since $\langle g \rangle$ is finitely generated and nilpotent, by a result due to Mal’tsev (cf. [CMZ17, Thm 2.23]), $\langle \hat{g} \rangle |_{N_0} = (\langle L \rangle)^{-1}$ has finite index in $\langle \hat{g} \rangle$ and thus in $\langle g \rangle$. By [Hu21a, Thm 15], $\langle \hat{g} \rangle$ has finite index in $\langle \hat{g} \rangle$ and thus also in $\langle g \rangle$. Then, we have $\langle \hat{g}(N_0^B) \rangle \subseteq \langle p(N_B^0) \rangle \subseteq \langle p(R_{Z \geq 0}^B) \rangle \subseteq H_{2n+1} (\mathbb{R})$. Since $\langle \hat{g}(N_0^B) \rangle$ has finite index in $G$, $\langle p(R_{Z \geq 0}^B) \rangle$ contains a neighborhood of the identity of $H_{2n+1} (\mathbb{R})$ and thus $\langle p(R_{Z \geq 0}^B) \rangle = H_{2n+1} (\mathbb{R})$. In particular, $\langle p(U) \rangle = H_{2n+1} (\mathbb{R})$ for any subset $U$ of $R^B_{Z \geq 0}$ with nonempty interior.

\footnote{Here, $| \cdot |$ is the standard restriction symbol in \LaTeX.}
Case 1: If \( d(x) \) is not of the form (3.10), then the rank of \( \mathbf{J} \) is \( 2n+1 \). Let \( U(B, L) \) be the Kamke domain, which by [Hu21a, Lem 8] contains a proper polynomial set given by some continuous polynomial map \( q \), and set \( f = p \circ q \). Therefore, the dimension of \( p(\mathbb{R}_2^B) \) or the proper polynomial set \( U = \text{Im} \, f \) is \( 2n+1 \) as desired. Moreover, we have
\[
g(\mathbb{N}_0^\mathbb{L}) \supset p(U(B, L) \cap A\mathbb{N}_0^B) = p(U(B, L) \cap AZ^B) \supset U \cap p(AZ^B).
\]
Then, \( \delta \circ p \) is affine linear in \( s_1, \ldots, s_B \). It is easy to see that the Jacobian matrix of \( \delta \circ p \) with respect to \( s_1, \ldots, s_B \) is the same as the one of \( \log p \). If necessary, we can replace \( A \) by \( kA \) for some \( k \in \mathbb{N} \) so that \( p(AZ^B) \in G \). Since the first row vector (3.9) of \( \mathbf{J}_1 \) is a \( \mathbb{Q} \)-linear combination of the row vectors of \( \mathbf{J} \), we can replace \( L \) by \( k'L \) for some \( k' \in \mathbb{N} \) so that \( \delta \circ p(AZ^B) \) is a lattice with integral entries in the underlying Euclidean space of \( H_{2n+1}(\mathbb{R}) \) and the subgroup \( H = H_{2n+1}(D\mathbb{Z}) \) lies in \( \delta \circ p(AZ^B) \) for some \( D \in 2\mathbb{N} \). Thus, \( \iota(H) \) lies in \( p(AZ^B) \). But \( \iota(H) = H \) as sets, since \( D \) is even and divides \( D : D/2 \). Then, \( H \) is a finite index subgroup of \( G \) such that \( H \subset p(AZ^B) \). Let \( \phi : H \hookrightarrow H_{2n+1}(\mathbb{R}) \) be the inclusion map. Then, \( V = \phi^{-1}(U) = U \cap H \subset g(\mathbb{N}_0^{2L}) \) is a proper polynomial subset in \( H \) and each element of \( V \) can be written as a product of exactly \( 2L \) elements in the sequence \( g_0, g_1, g_2, \ldots \).

Case 2: If \( d(x) \) is of the form (3.10), then the rank of \( \mathbf{J} \) is \( 2n \). Then, by Lemma 5, we can find a finite number of pairs \((a_i, b_i) \in \mathbb{N} \times \mathbb{N}_0, 1 \leq i \leq m \), such that the associated matrix \( \mathbf{J}(\log h)(0) \) of the polynomial sequence \( h(x) := g(a_1 x + b_1) \cdots g(a_m x + b_m) : \mathbb{N}_0 \to G \) has rank \( 2n+1 \). Working with \( h(x) \) instead of \( g(x) \), we are reduced to the previous case.

### 4. Waring’s Problem for Locally Nilpotent Groups

The basic setting of Waring’s problem for locally nilpotent groups is as follows: Let \( G \) be a locally nilpotent group, \( g : \mathbb{N}_0 \to G \) be an arbitrary polynomial sequence of degree \( d \), and \( [g] \) (resp. \( \langle g \rangle \)) be the semigroup (resp. locally nilpotent subgroup) generated by the polynomial sequence \( g_0, g_1, g_2, \ldots \). By [Hu21a, Prop 2], we may assume that \( G = \langle g \rangle \) is finitely generated and nilpotent.

The whole discussion of Waring’s problem for finitely generated nilpotent groups is divided into different cases, according to the degree of \( g \), and the cardinality and nilpotency class of the group \( \langle g \rangle \). But we only state and prove some nontrivial cases here.

We may assume that \( \langle g \rangle \) is infinite. Otherwise, one can easily show that \( [g] = \langle g \rangle \) is an open polynomial set such that each element in \( [g] \) can be written as a product of at most \( M \) elements in the sequence \( g_0, g_1, g_2, \ldots \) for some \( M \in \mathbb{N} \).

We may as well assume that \( d \geq 1 \). If \( d \leq 0 \), i.e., \( g \) is constant, then it is easy to prove that there exists \( M \in \mathbb{N} \) and a proper polynomial set \( V \) in \( \langle g \rangle \) such that each element in \( V \cap [g] \) can be written as a product of at most \( M \) elements in the sequence \( g_0, g_1, g_2, \ldots \).

**Case in which** \( d = 1 \). The following example sheds some light on the proof of next theorem.

**Example.** Consider the quotient group \( G \) of \( H_3(\mathbb{Z}) = \langle x, y \mid [x, [x, y]] = [y, [x, y]] = 1 \rangle \) given by
\[
G = \langle x, y \mid [x, [x, y]] = [y, [x, y]] = 1 = x^ny^{-m} \rangle, \text{ for some } n \neq 0.
\]
Then, $G^{ab} = \langle \bar{x}, \bar{y} \mid n\bar{x} - m\bar{y} = 0 \rangle$ has rank 1. Consider the polynomial sequence $g : \mathbb{N}_0 \to G; i \mapsto xy^i$ of degree 1. Then, $g_0 = x, g_1 = xy,$ and $g_0^{-1}g_1 = y$. Hence, $\langle g \rangle = G$ but $\langle g \rangle^{ab}$ has rank 1.

**Theorem 4.** Let $g : \mathbb{N}_0 \to G$ be a polynomial sequence of degree $d = 1$ such that $\langle g \rangle$ is infinite. If $g_0 \neq 1_G$ and $g \mod N$ is non-constant for any normal subgroup $N$ of infinite index in $\langle g \rangle$, then there exists $M \in \mathbb{N}$, a subgroup $H$ of finite index in $\langle g \rangle$, and a proper polynomial set $V$ in $H$ such that each element in $V \cap \langle g \rangle$ can be written as a product of at most $M$ elements in the sequence $g_0, g_1, g_2, \ldots$.

**Proof.** Since $\mathbb{N}_0$ is a commutative monoid, by [Hu21a, Prop 1], $g$ is affine multiplicative, and $l := g_0^{-1}g$ and $r := gg_0^{-1}$ are multiplicative, i.e., $l_i = l_1^i$ and $r_i = r_1^i$ for all $i \in \mathbb{N}_0$. Then,

$$\langle g \rangle = \langle g_0, g_1, g_2, \ldots \rangle = \langle g_0, l_1 \rangle = \langle g_0, r_1 \rangle = \langle g_0, g_1 \rangle$$

and thus $\langle g \rangle^{ab} = \langle g \rangle/[\langle g \rangle, \langle g \rangle] = \langle \bar{g}_0, \bar{l}_1 \rangle$ has rank 0, 1 or 2.

If $\langle g \rangle^{ab}$ has rank 0, then it is finite. Then, $\langle g \rangle$ is also finite, which contradicts our assumption. Suppose that $\langle g \rangle^{ab} = \langle g_0, l_1 \rangle$ has rank 2. Then, we have $\langle g \rangle^{ab} = \langle g_0, l_1 \rangle$ and $\langle l_1 \rangle$ is a subgroup of infinite index in $\langle g \rangle^{ab}$. Let $N$ be the normal subgroup of $\langle g \rangle$ such that $(\langle g \rangle/N) \cong (\langle g \rangle^{ab}/\langle l_1 \rangle)$. Since $(\langle g \rangle^{ab}/\langle l_1 \rangle) \cong \mathbb{Z}$, $N$ has infinite index in $\langle g \rangle$.

Since $g_1 = g_0l_1 = g_0l_1^i \in g_0N$, it follows that $g \mod N$ is a constant, which also contradicts our assumption.

So $\langle g \rangle^{ab} = \langle \bar{g}_0, \bar{l}_1 \rangle$ must have rank 1. If $\langle l_1 \rangle$ is a subgroup of infinite index in $\langle g \rangle$, then by [Hu21a, Lem 6] there exists a normal subgroup $N$ of infinite index in $\langle g \rangle$ containing $\langle l_1 \rangle$. Since $g_i = g_0l_i = g_0l_1^i \in g_0N$, it follows that $g \mod N$ is a constant. Hence, $\langle l_1 \rangle$ must be a subgroup of finite index in $\langle g \rangle$ and thus $l_1$ has infinite order. Let $H = \langle l_1 \rangle \cong \mathbb{Z}$ be a subgroup of $\langle g \rangle$.

If $g_0$ has finite order, say $n$, then we consider $h := g_0^{-n}g$. Then, $h_i = g_0^{-n}g_0l_1^i = l_1^i$ and thus $h : \mathbb{N}_0 \to G$ is a polynomial map of degree 1 with $h_0 = 1_G$. Thus, $H = \langle h \rangle = \langle l_1 \rangle \cong \mathbb{Z}$ is an infinite cyclic group generated by $l_1$. Then, we have $[h] = \{l_1^i \mid i \in \mathbb{N}_0\} \cong \mathbb{N}_0$.

If $g_0$ has infinite order, then we claim that there exist $n, m \in \mathbb{Z} \setminus \{0\}$ such that $g_0^n = l_1^m$. Indeed, if cosets $g_0^i(l_1)$ and $g_0^j(l_1)$ had no intersection whenever $i \neq j$, then this would imply that $\langle l_1 \rangle$ had infinite index in $\langle g \rangle$. So for some $i > j$, the $g_0^i(l_1)$ and $g_0^j(l_1)$ have nonempty intersection and thus must coincide. Then, $g_0^i = g_0^j l_1^m$ implies that $g_0^{i-j} = l_1^m$. Then, we can take $n = i - j \geq 1$. Again, we consider $h := g_0^{-n}g$. Then, $h_k = g_0^{-n}g_0^{k}l_1^k = l_1^{m+k}$ and $\langle h \rangle = \langle l_1 \rangle \cong \mathbb{Z}$ is a subgroup of finite index in $\langle g \rangle$. Then, we have $[h] = \{l_1^{m+i} \mid i \in \mathbb{N}_0\} \cong \mathbb{Z}_{\geq m}$.

In either case, we consider the group homomorphism $\phi : H \cong \mathbb{Z} \to \mathbb{R}$ and the proper polynomial set $\mathbb{R}_{\geq k}$ for some large $k$ in $\mathbb{R}$. Then, $\phi^{-1}(\mathbb{R}_{\geq k})$ is a proper polynomial set in $H$ such that each element in $[h]$ can be written as a product of at most 1 element in the sequence $h_0, h_1, h_2, \ldots$ and thus at most $n$ element in the sequence $g_0, g_1, g_2, \ldots$.

**The most general case.** Below is a primitive result of the most general case of Waring’s problem for locally nilpotent groups.

**Theorem 4.** If $g$ has degree $d \geq 2$ and $g \mod N$ is non-constant for any normal subgroup $N$ of infinite index in $\langle g \rangle$, then there exist $A, M \in \mathbb{N}$, a subgroup $H$ of finite index in $\langle g \rangle$, a polynomial subset $V$ of $H$ and a polynomial map $p : \mathbb{N}_0^B \to \mathcal{U}_n(\mathbb{Z})$
such that every element in \( V \cap p(A^B_0) \) can be written as a product of at most \( M \) elements in the sequence \( g_0, g_1, g_2, \ldots \).

We first prove the fundamental case when \( \langle g \rangle \) is finitely generated, torsion-free and nilpotent.

Proof. By [Hal57, Thm 7.5], every finitely generated torsion-free nilpotent group \( \langle g \rangle \) is isomorphic to a subgroup of \( U_n(\mathbb{Z}) \) for some \( n = n(\langle g \rangle) \). Let \( B \) be the least upper bound given in the simple observation and \( L' \in \mathbb{N} \) be the least number such that

\[
B \in \{ d_{i,j}(L) \mid 1 \leq i < j \leq n, 1 \leq L \leq L' \}.
\]

Then, \( B \) cannot be \( -\infty \) (resp. 0), otherwise \( g \) has degree \( -\infty \) (resp. 0), and \( B \) cannot be 1, otherwise Inequalities (3.1) and [Hu21a, (4.5)] imply that

\[
d \leq \max \{ d_{k_1,k_2} + \cdots + d_{k_{n-1},k_n} \mid 1 = k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n = n \} \leq B = 1.
\]

Since \( B \geq 2 \), by Kamke’s key theorem, there exist positive integers \( A, L'' \), and positive numbers \( i_1 \) and \( i_\nu, J_\nu \) with \( 0 < i_\nu < J_\nu, \nu = 2, 3, \ldots, B \), such that for each \( B \) integers \( s_1, \ldots, s_B \), divisible by \( A \) and subject to the conditions

\[
s_1 \in (i_1, \infty); \quad s_\nu/s'_1 \in (i_\nu, J_\nu) \quad \text{for} \quad \nu = 2, 3, \ldots, B,
\]

the \( B \) equations \( s_\nu = \sum_{n=1}^{L''} x_n^\nu, \nu = 1, 2, \ldots, B \) are simultaneously solvable by integers \( x_n \geq 0 \).

Let \( L = \max \{ L', L'' \} \) and consider the ordered product

\[
\hat{g} := \bigcap_{i=1}^{L} g : \mathbb{N}_0^L \to U_n(\mathbb{Z}); \quad (x_1, \ldots, x_L) \mapsto g(x_1) \cdots g(x_L).
\]

By [Hu21a, Cor 8], \( \langle \hat{g} \rangle \) has finite index in \( \langle g \rangle \). Then, by [Hu21a, Thm 14], there exists a finite natural number \( M' \) and a sequence \( \sigma_1, \sigma_2, \ldots, \sigma_{M'} \in S_L \), such that the product

\[
\tilde{g} = \prod_{i=1}^{M'} \sigma_i(\hat{g}) = \sigma_1(\hat{g})\sigma_2(\hat{g}) \cdots \sigma_{M'}(\hat{g})
\]

is a symmetric polynomial map in \( L \) variables \( x_1, x_2, \ldots, x_L \).

Since \( \langle g \rangle \) is finitely generated and nilpotent, by a result due to Mal’tsev (cf. [CMZ17, Thm 2.23]), \( \langle \tilde{g} \mid_{\mathbb{N}_0} \rangle = \langle \hat{g}^L \rangle \) has finite index in \( \langle g \rangle \) and thus in \( \langle \hat{g} \rangle \). By [Hu21a, Thm 15] \( \langle \hat{g} \rangle \) has finite index in \( \langle \tilde{g} \rangle \) and thus in \( \langle g \rangle \).

Each entry \( \tilde{g}_{i,j} \) is a symmetric polynomial of total degree \( \leq B \) and can be written as a polynomial expression with rational coefficients in the power sum symmetric polynomials \( s_1, s_2, \ldots, s_B \), i.e.,

\[
\tilde{g}_{i,j}(x_1, \cdots, x_N) = p_{i,j}(s_1, \ldots, s_B),
\]

where \( p_{i,j} \) is an integer-valued polynomial in \( B \) variables \( s_1, \ldots, s_B \in \mathbb{N}_0 \) with rational coefficients. Therefore, we can define the following continuous polynomial map in \( B \) variables \( s_1, \ldots, s_B \in \mathbb{R}_\geq 0 \):

\[
p = \begin{pmatrix}
1 & p_{1,2} & p_{1,3} & \cdots & p_{1,n} \\
1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n} \\
& & \ddots & \ddots & \vdots \\
& & & p_{n-1,n} & 1
\end{pmatrix} : \mathbb{R}^B_\geq 0 \to U_n(\mathbb{R}).
\]
Thus, the goal of studying the image of $\tilde{g} : \mathbb{N}_0^N \to \mathcal{U}_n(\mathbb{Z})$ is reduced to studying the image of the restriction of $p : \mathbb{R}_+^B \to \mathcal{U}_n(\mathbb{R})$ on $U(B, N) \cap AN_0^B$. Consider the homomorphism

$$\phi : H = \langle \tilde{g} \rangle \to \langle g \rangle \to \mathcal{U}_n(\mathbb{Z}) \to \mathcal{U}_n(\mathbb{R}).$$

Clearly, each element in $\phi^{-1}(p(U(B,N))) \cap \phi^{-1}(p(AN_0^B))$ is a product of exactly $M = LM'$ elements in the sequence $g_0, g_1, g_2, \ldots$.

Let $q : \mathbb{R}_+^B \to \mathbb{R}^B$ be the continuous polynomial map as in [Hu21a, Lem 8] such that its image has nonempty interior in $U(B,N)$. Then, $f := p \circ q : \mathbb{R}_+^B \to \mathcal{U}_n(\mathbb{R})$ a continuous polynomial map. Hence, $U = f(\mathbb{R}_+^B)$ is a polynomial set in $\mathcal{U}_n(\mathbb{R})$ and $V = \phi^{-1}(U)$ is a polynomial set in $\langle g \rangle$ such that each element in $V \cap \phi^{-1}(p(AN_0^B))$ is a product of exactly $M = LM'$ elements in the sequence $g_0, g_1, g_2, \ldots$. □

Remark. It is not guaranteed that $U = f(\mathbb{R}_+^B)$ is open or proper in $\mathcal{U}_n(\mathbb{R})$; see the degenerate cases in the proof of Theorem 1.

Next, we will show how to reduce the most general case to the fundamental case above.

**Proof.** The goal is to reduce to the case when $G$ is finitely generated torsion free nilpotent, in particular, when $G = \mathcal{U}_n(\mathbb{Z})$. There are two possible ways to achieve this. If $g_i$ has finite order for all $i$, then $G$ is a finite nilpotent group. So we may assume that not all $g_i$ have finite order.

An old result of Hirsch implies that every finitely generated nilpotent group $G$ is isomorphic to a finite index subgroup of a direct product of a finite nilpotent group $T$ and a finitely generated torsion-free nilpotent group $F$, cf. [Hir38, Thm 2.22] and [Hir46, Thm 3.21 and Thm 3.23]. Let $p_1 : G \to T \times F \to T$ and $p_2 : G \to T \times F \to F$ be canonical projections onto direct summands of $T \times F$. We have two induced polynomial sequences $t = p_1 \circ g : \mathbb{N}_0 \to T$ and $f = p_2 \circ g : \mathbb{N} \to F$ such that $g = (t, f)$.

The first idea is to take a periodic subsequence. By [Hu21a, Prop 7], $t$ is periodic, say, of period $P \in \mathbb{N}$. Then, we can take any periodic subsequence $s_i := g_{iP+b}$ for some $0 \leq b < P$, such that $p_1(s_i)$ is a constant. Then, $s : \mathbb{N}_0 \to G$ is a polynomial subsequence of degree not larger than the degree of $g$, such that $p_1(s_i)$ is a constant of finite order in $T$. By [Hu21a, Thm 13], $\langle s \rangle$ is a finite index subgroup of $G$. Instead of the polynomial sequence $g$, we work with the periodic polynomial subsequence $s$. So we may assume that $s : \mathbb{N}_0 \to \mathbb{Z}/m \times \mathcal{U}_n(\mathbb{Z})$ for some $m$ and $p_1(s) = 1 \in \mathbb{Z}/m$.

Form the following ordered product

$$\hat{s} := \bigotimes_{i=1}^L s : \mathbb{N}_0^L \to \mathbb{Z}/m \times \mathcal{U}_n(\mathbb{Z}); \quad (x_1, \ldots, x_L) \mapsto s(x_1) \cdots s(x_L),$$

and require that $L$ is divisible by $m$. Then, we have $p_1(\hat{s}) = \hat{0} \in \mathbb{Z}/m$.

The second idea is to replace $g$ by $g^{[T]} = (t^{[T]}, f^{[T]}) = (1_T, f^{[T]})$, where $|T|$ is the cardinal of $T$. Since $G$ is finitely generated and nilpotent, by a result due to Mal’tsev (cf. [CMZ17, Thm 2.23]), $\langle g^{[T]} \rangle$ is finitely generated torsion free nilpotent and has finite index in $G$. But notice that one may not be able to compare the degree of $g^{[T]}$ with the degree of $g$.

Suppose that there were a normal subgroup $N'$ of infinite index in $\langle s \rangle$ (resp. $\langle g^{[T]} \rangle$) such that $s \mod N'$ (resp. $g^{[T]} \mod N'$) is constant. (Notice that $N'$ is not necessarily normal in $G$, as normality is not a transitive relation.) Since $N'$ is
a subgroup of infinite index in $G$, by [Hu21a, Lem 6], $N'$ is contained in a normal subgroup $N$ of infinite index in $G$. This implies that $s \mod N$ (resp. $g^{[T]} \mod N$) is also constant, which is a contradiction.

Hence, it suffices to work with $s$ or $g^{[T]}$, which generates a finitely generated torsion free nilpotent subgroup of finite index subgroup in $G$. □

Remark. If $(g)$ is infinite and has nilpotency class $\leq 2$, then the idea given in the previous proof also allows us to reduce the problem to Theorem 2 or Theorem 1 and we can show that if $g \mod N$ is non-constant for any normal subgroup $N$ of infinite index in $(g)$, then there exists $M \in \mathbb{N}$, a subgroup $H$ of finite index in $(g)$ and a proper polynomial set $V$ of $H$ such that every element in $V$ can be written as a product of at most $M$ elements in the sequence $g_0, g_1, g_2, \ldots$.

References

[Alf05] J.L.R. Alfonsín. The Diophantine Frobenius problem. Oxford Lecture Series in Mathematics and Its Applications. OUP Oxford, 2005.

[CC97] Paul-Jean Cahen and Jean-Luc Chabert. Integer-valued Polynomials. American Mathematical Society Translations. American Mathematical Society, 1997.

[CMZ17] A. E. Clement, S. Majewicz, and M. Zyman. The Theory of Nilpotent Groups. Birkhäuser Basel, 2017.

[Hal57] P. G. Hall. The Edmonton notes on nilpotent groups. Queen Mary College Mathematics Notes. Queen Mary College Department of Mathematics, 1957.

[Hen96] Kurt Hensel. Ueber den grössten gemeinsamen Theile r aller Zahlen, welche durch eine ganze Function von $n$ Veränderlichen darstellbar sind. Journal für die reine und angewandte Mathematik, 116:350–356, 1896.

[Hil09] David Hilbert. Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl $r$ter Potenzen (Waring’sches Problem), dem andenken an hermann minkowski gewidmet. Mathematische Annalen, 67:281–300, Sep 1909.

[Hir38] A. K. Hirsch. On Infinite Soluble Groups (II). Proceedings of the London Mathematical Society, s2-44(1):336–344, Jan 1938.

[Hir46] A. K. Hirsch. On Infinite Soluble Groups (III)*. Proceedings of the London Mathematical Society, s2-49(1):184–194, Jan 1946.

[Hu21a] Ya-Qing Hu. Polynomial maps and polynomial sequences in groups, 2021. arXiv:2105.08000.

[Hu21b] Ya-Qing Hu. Waring’s problem for locally nilpotent groups: The case of discrete heisenberg groups, 2021. arXiv:2011.06683.

[Kam21] E. Kamke. Verallgemeinerungen des Waring-Hilbertschen Satzes. Mathematische Annalen, 83(1):85–112, Mar 1921.

[LN19] Michael Larsen and Dong Quan Ngoc Nguyen. Waring’s problem for unipotent algebraic groups. Annales de l’Institut Fourier, 69(4):1857–1877, 2019.

[Wri34] E. Maitland Wright. An easier Waring’s problem. Journal of the London Mathematical Society, s1-9(4):267–272, 1934.