Extremal particles in branching processes

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June 2016

Abstract

The purpose of this study is to investigate two related spatial branching models with the unbounded branching intensity. The objective is to describe the asymptotic behaviour of the extremal particle.

Keywords: Branching process, Galton Watson process, Poisson process, Lévy process

AMS MSC 2010: 60J80

1 Galton-Watson process on lines

1.1 Model and the result

We consider a system of particles with a spatially inhomogeneous branching rate. Each particle has an assigned position $J \in \mathbb{N}$ (dubbed as 'J-th line'). Such a particle branches with the intensity $J^\gamma$, $\gamma \in (0, 1)$ producing two offspring. Moreover, it jumps to a higher line $J' > J$ with the intensity $C J'$, $C \in (0, 1)$. The evolution of particles is independent.

The objective is to study the displacement of the top-most-one particle. Let us denote by $|X(t)|$ the number of particles in time $t$ and $X(t, i)$ the position of the $i$-th particle $1 \leq i \leq |X(t)|$. The main result of this chapter is

**Theorem 1.2.** There exist constants $A, B > 0$ such that

$$A > \limsup_{t \to \infty} \frac{\max_{1 \leq |X(t)|} X(t, i)}{t^{1 - \gamma}} \geq \liminf_{t \to \infty} \frac{\max_{1 \leq |X(t)|} X(t, i)}{t^{1 - \gamma}} > B \text{ a.s.}$$

1.3 Notation

In this section we summarize the notation used later on. For $J \in \mathbb{N}$ we define

$$Y_J := \inf \{t \geq 0 : \exists i, X(t, i) = J \}.$$

(1)

By convention $Y_J = +\infty$ if the level $J$ is never attained. The complementary event is denoted by

$$Z_J := \{Y_J < \infty \}.$$

(2)

*The research was supported by the Foundation for Polish Science under the grant HOMING PLUS/2012-6/7.
By $GW(\lambda, \mu)$ we denote the law of the Galton Watson process with the intensity of birth $\lambda$ and the intensity of death $\mu$. Particles always split into two offspring (see chapter 3, page 102 [1] for the details of definition). Moreover, we shortcut $GW(\lambda, 0)$ to $GW(\lambda)$, this will be also called a Yule process. Let $p_J$ be the first particle on the $J$-th line (arriving at time $Y_J$). By $X_J$ we denote its offspring which remain at the $J$-th line. An offspring of $p_J$ on the $J$-th line has the branching intensity equal to $J^\gamma$ and jumps higher (what we identify as death) with the intensity $C_J^{J+1}$ $1−C_J$. Thus modulo time-shift $X_J$ is distributed as the $GW(J^\gamma, C_J^{J+1})$ process.

The following times
$$t_J := C_1 J^{1−\gamma} − 1,$$
where $C_1 > 0$ is to be fixed, will play a crucial role in our analysis. They are times when typically the number of particles on the $J$-th line is large enough so that one of them performs a long jump. To formalize we denote the event
$$A_J := \{\text{number of the particles on the } J\text{-th line at the time } Y_J + t_J \geq q_J\},$$
where
$$q_J := \frac{e^{t_J(J^\gamma−C_J^{J+1})}}{J^2}.$$
We denote the related event
$$A_J^1 := \{X_J(Y_J + t_J) \geq q_J\}.$$ Note that $A_J^1 \subset A_J$ due to the fact that particles may immigrate to the $J$-th line after time $Y_J$.

1.4 General facts

In this section we gather facts used in the main proof.

**Proposition 1.5.** Let $Y(t)$ be a $GW(\lambda, \mu)$ process. Then $EY(t) = e^{t(\lambda−\mu)}$.

**Proof.** See [1] p. 108. □

**Definition 1.6.** Let $\{H_t\}_{t \geq 0}$ be $\mathbb{N}$-valued stochastic process. We define
$$f(s, t) := Es^{H_t}.$$ We say that $f$ is the generating function of the process $H$.

**Proposition 1.7.** Let $H$ be a $GW(\lambda)$ process. Then its generating function is given by
$$f(s, t) = \frac{se^{-\lambda t}}{1−(1−e^{-\lambda t})s}.$$ Moreover,
$$P(H_t = n) = e^{-\lambda t} (1−e^{-\lambda t})^{n−1}.$$ □

**Proof.** We notice that $\{P(H_t = n)\}_{n \in \mathbb{N}}$ is the geometric distribution with the parameter $e^{\lambda t}$ if $t$ is fixed. It can be checked that its generating function coincides with $f(\cdot, t)$ given by (7). The first part of the proposition can be found in [1] page 109.
The proof of the previous proposition allows us to form an observation.

**Proposition 1.8.** If an \( N \)-valued process \( H \) has the generating function \( \phi \) then \( \psi \) holds and \( \mathbb{E}H_t = e^{\lambda t} \).

**Proof.** If \( H \) has the generating function given by \( \phi \) then the distribution of \( H \) must fulfills \( \psi \), using it we compute the expected value. \( \square \)

When \( \mu > 0 \) it may happens that a process \( H \) distributed as the \( GW(\lambda, \mu) \) process becomes extinct i.e. it contains no particles. We denote such an event by \( Ext(H) \), more precisely

\[
Ext(H) := \{ \exists t \geq 0 H_t = 0 \}.
\]

We have the following proposition.

**Proposition 1.9.** Let \( H \) be a \( GW(\lambda, \mu) \) process. If \( \mu < \lambda \) then \( \mathbb{P}(Ext(H)) = \frac{\lambda}{\mu} \). If \( \mu \geq \lambda \) then \( \mathbb{P}(Ext(H)) = 1 \).

**Proof.** Let us define \( a_j = \mathbb{P}(a \text{ particle produces } j \text{ descendants after its death}) \). Let us consider a random variable \( Y \) with the distribution given by \( \mathbb{P}(Y = j) = a_j \) for \( j \in \mathbb{N} \). We take \( f(s) = e^{sY} \) the generating function of \( Y \). From [1] (Theorem 1 p. 108) we have that the probability of extinction of the process \( H \) is the smallest root of the equation \( f(s) = s \). By easy calculations we obtain that

\[
f(s) = \frac{\lambda s^2 + \mu}{\lambda + \mu},
\]

thus solving the equation \( f(s) = s \) leads us to the quadratic equation

\[
\lambda s^2 - s(\lambda + \mu) + \mu = 0.
\]

It has solutions

\[
s = \frac{\lambda + \mu \pm (\lambda - \mu)}{2\lambda}.
\]

By picking the smallest root we obtain the claim. \( \square \)

**Theorem 1.10.** Let \( H \) be a \( GW(\lambda, \mu) \) process for \( \lambda > \mu > 0 \). We define \( \{\bar{H}_t\}_{t \geq 0} \) to be the number of particles of \( H_t \) which have infinite number of descendants (in particular \( \bar{H}_0 = 0 \) if the process \( H \) becomes extinct). Then conditionally on \( Ext(H) \) the process \( H \) has the generating function given by

\[
\hat{f}(t, s) = \frac{se^{-(\lambda-\mu)t}}{1 - (1 - e^{-(\lambda-\mu)t})s}.
\]

**Proof.** We denote by \( \hat{f}(t, s) \) the generating function of the process in the question. From [1] p.109 we get that the generating function of \( H \) is given by

\[
f(s, t) = \mu(s - 1) - e^{(\mu-\lambda)t}(\lambda s - \mu) \quad \lambda(s - 1) - e^{(\mu-\lambda)t}(\lambda s - \mu).
\]

Further we have (see [1] Theorem 1 p. 49 and Theorem 1 p. 110)

\[
\hat{f}(s, t) = \frac{f((1 - q)s + q, t) - q}{1 - q},
\]

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where $q$ is the probability that $H$ becomes extinct. By Proposition 1.10, $q = \frac{\mu}{\lambda}$, thus

$$
\hat{f}(s, t) = \frac{f((1 - q)s + q, t) - q}{1 - q} = \frac{f((1 - \frac{\mu}{\lambda})s + \frac{\mu}{\lambda}, t) - \frac{\mu}{\lambda}}{1 - \frac{\mu}{\lambda}}
$$

$$
= \frac{\mu((1 - q)s + q - 1) - e^{(\mu - \lambda)s}}{(\lambda - \mu) - e^{(\mu - \lambda)s}} - \frac{\mu}{\lambda}
$$

$$
= \frac{(\mu - \lambda^2)s + \mu^2 - \mu - \lambda e^{(\mu - \lambda)s}((\lambda - \mu)s) - \mu}{(\lambda - \mu)(s - 1) - e^{(\mu - \lambda)s}((\lambda - \mu)s)} - \frac{\mu}{\lambda}
$$

$$
= \frac{\mu s - \mu - \lambda e^{(\mu - \lambda)s}((\lambda - \mu)s)}{(\lambda - \mu)(s - 1) - e^{(\mu - \lambda)s}((\lambda - \mu)s)} - \frac{\mu}{\lambda}
$$

$$
= \frac{e^{(\mu - \lambda)s}((\lambda - \mu)s - \mu)}{(\lambda - \mu)(s - 1 - e^{(\mu - \lambda)s})} = \frac{se^{(\mu - \lambda)s}}{1 - (1 - e^{(\mu - \lambda)s})}.
$$

\[
\square
\]

Proposition 1.11. Let $H$ be a $GW(\lambda)$ process or has the generating function given by (7). Then for any $c > 0$ we have

$$
\mathbb{P}(H_t \geq cE H_t) \geq \left( \frac{1}{10} \right)^c,
$$

if $t > \frac{K}{\lambda}$ where $K$ is an absolute constant.

Proof. Using Proposition 1.3 and 1.8 we obtain

$$
\mathbb{P}(H_t \geq cE H_t) = \sum_{n = [cE H_t]}^{\infty} e^{-\lambda t} \left( 1 - e^{-\lambda t} \right)^{n-1} = \left( 1 - e^{-\lambda t} \right)^{[cE H_t] - 1}
$$

$$
\geq \left( 1 - e^{-\lambda t} \right)^{c} \geq \left( \frac{1}{10} \right)^c,
$$

where the last inequality holds for $t > \frac{K}{\lambda}$, and large enough $K$. \[
\square
\]

1.12 Lower bound for the Galton-Watson model

In this section we will prove the bound from below in Theorem 1.2. We recall $Z_J$, $A_J$, $A^J$, $X_J$ defined in (2), (4), (8) and in Section 1.3. We start with the following proposition

Proposition 1.13. There exists $D$ such that for large enough $J$ we have

$$
\mathbb{P}(A_J \mid Z_J) \geq 1 - \frac{D}{J^2}.
$$

Proof. We denote $\bar{\mathbb{P}}_J(\cdot) = \mathbb{P}(\cdot \mid Z_J)$. Since $A^J_J \subset A_J$ thus

$$
\mathbb{P}(A_J \mid Z_J) \geq \mathbb{P}(A^J_J \mid Z_J) = \bar{\mathbb{P}}_J(A^J_J).
$$

(10)
We remind that \( \{X_j(Y + t)\}_{t \geq 0} \) conditioned on \( Z_j \) is a \( GW \left( J^*, \frac{C^{j + 1}}{1 - C} \right) \) process. By \( \tilde{X}_j(Y + t) \) we denote the reduced process as in Theorem \ref{thm1.10}. Since \( X_j \geq \tilde{X}_j \) so \( \{\tilde{X}_j(Y + t) \geq q_j\} \subset A_j^1 \) and as a result

\[
\hat{P}_j(A_j^1) \geq \hat{P}_j(\tilde{X}_j(Y + t) \geq q_j) = \hat{P}_j(\tilde{X}_j(Y + t) \geq q_j, \ Ext(X_j))' = \hat{P}_j(\tilde{X}_j(Y + t) \geq q_j | Ext(X_j))' \hat{P}_j(Ext(X_j)).
\]

By Theorem \ref{thm1.10} \( \tilde{X}_j(Y + t) \) has the generating function

\[
\hat{f}(t, s) = \frac{se^{j (1 - C^t - J^n)t}}{1 - e^{j (1 - C^t - J^n)t}s}
\]

(with respect to \( \hat{P}_j(\cdot | Ext(X_j))' \)). Thus by Proposition \ref{prop1.8} and simple calculations we obtain

\[
\mathbb{E}_{\hat{P}_j}(\tilde{X}_j(Y + t_j) | Ext(X_j))' = q_j \cdot J^2,
\]

where \( q_j \) is defined in \ref{eq:1.10}, so using Proposition \ref{prop1.11} we get

\[
\hat{P}_j(\tilde{X}_j(Y + t) \geq q_j | Ext(X_j))' = \hat{P}_j(\tilde{X}_j(Y + t) \geq q_j | Ext(X_j))' \geq \left( \frac{1}{10} \right)^{\frac{1}{J^2}}.
\]

The last inequality follows for large enough \( J \). Moreover, by Proposition \ref{prop1.9} for large enough \( J \)

\[
\hat{P}_j(Ext(X_j)) = \frac{C^{j + 1}}{J^2} \leq C^{J + 1}.
\]

Hence \( \hat{P}_j(A_j^1) \geq \left( \frac{1}{10} \right)^{\frac{1}{J^2}} (1 - C^{J + 1}) \) thus by \ref{eq:10} we have

\[
\mathbb{P}(A_j | Z_j) \geq \left( \frac{1}{10} \right)^{\frac{1}{J^2}} (1 - C^{J + 1}).
\]

For large enough \( J \) we have

\[
1 - \left( \frac{1}{10} \right)^{\frac{1}{J^2}} \geq \frac{e^{\ln 10}}{10^{\ln 10}} \cdot \frac{1}{J^2} \leq 10 \cdot \frac{\ln 10}{J^2}.
\]

So the proof is concluded.

\[\square\]

**Remark 1.14.** The considered model is equivalent to the following one. At time 0 we have one particle on the first line. A particle on the \( i \)-th line has two clocks \( Z_1 \) and \( Z_2 \), where \( Z_1 \sim \text{Exp}(i^2) \), \( Z_2 \sim \text{Exp}(\frac{C^{i + 1}}{1 - C}) \). \( Z_1 \) and \( Z_2 \) are independent. If \( Z_1 < Z_2 \) then the particle produces one additional particle on the \( i \)-th line at time \( Z_1 \). Otherwise, it jumps at \( Z_2 \) to the \( j \)-th line, where \( j > i \), with probability \( (1 - C)C^{j - i - 1} \). Particles jump and reproduce independently. In particular we see that the particle on the \( i \)-th line jumps after time \( Z_2 \), perhaps producing some additional particles before the jump.
Let 
\[ \alpha_J := \mathbb{P}(\text{a chosen particle at time } (Y_J + t_J) \text{ from the } J\text{-th line will jump to the } 2J\text{-th line in the time } [Y_J + t_J, Y_J + t_J + 1]) \]  
\[ (11) \]

**Proposition 1.15.** We have \( \alpha_J \geq C^{2J} \) and \( \lim_{J \to \infty} \alpha_J = 0 \).

*Proof.* Let \( Z_2 \sim \text{Exp}\left(\frac{C+1}{C}\right) \). We denote \( D_i := \{ \text{a chosen particle from the } i\text{-th line jumps before time } t \} \) and \( A_{i,J}(t) := \{ \text{a chosen particle from the } i\text{-th line will jump to the } J\text{-th line before time } t \} \).

\[ \mathbb{P}(A_{i,J}(t)) = \mathbb{P}(\text{jump will be to the } J\text{-th line } | \ D_i) \mathbb{P}(D_i) = (1 - C)C^{J-i-1}\mathbb{P}(D_i). \]

We have that \( \mathbb{P}(D_i) = \mathbb{P}(Z_2 < t) \) so using the cumulative distribution function of the exponential distribution we obtain

\[ \mathbb{P}(A_{i,J}(t)) = (1 - C)C^{J-i-1}\left(1 - e^{-\frac{C+1}{C}t}\right). \]

So now we see that

\[ \alpha_J = \mathbb{P}(A_{1,2J}(1)) = (1 - C)C^{2J-1}\left(1 - e^{-\frac{C+1}{C}\cdot 1}\right) \geq C^{2J}. \]

So our estimates are valid and \( \lim_{J \to \infty} \alpha_J = 0 \).

To continue we define \( I_J := [Y_J + t_J, Y_J + t_J + 1] \), a random interval of time, and

\[ B_J := \{ \text{jump from the } J\text{-th line to the } 2J\text{-th line occurs in time } I_J \}, \]

where \( Y_J, t_J \) are defined in (1), (3).

**Remark 1.16.** Set \( B_J \) is a formalized event, when a particle arrive at the \( J\text{-th line}, then we give some time (equal to } t_J \text{) for particles on that line in which they multiply. After that we investigate if any particle on that line manage to jump to the } 2J\text{-th line in time interval of length } 1.

**Proposition 1.17.** For large enough \( J \) we have

\[ \mathbb{P}(B_J | A_J) \geq 1 - \left(\frac{2}{e}\right)^J. \]

As a direct corollary of this fact and Proposition 1.15 we obtain that for large enough \( J \)

\[ \mathbb{P}(A_J \cap B_J | Z_J) \geq \left(1 - \frac{D_1}{J^2}\right) \left(1 - \frac{2}{e}\right) \geq 1 - \frac{D_1}{J^2}, \]

for some constant \( D_1 > 0 \).

*Proof.* We recall \( q_J \) defined in (13) and check that for large enough \( J \)

\[ q_J \geq \mathcal{O}^{2J}. \]

Indeed

\[ q_J = \frac{1}{J^2}e^{(C_1J^{1-\gamma}-1)(J^\gamma - \frac{e^{J+1}}{1-e})} = \frac{1}{J^2}e^{C_1J-C_1J^{1-\gamma}\frac{e^{J+1}}{1-e}-J^\gamma+\frac{e^{J+1}}{1-e}} \geq \mathcal{O}^{2J}. \]
By the definition of $A_J$ (see (4)), at time $Y_J + t_J$ there are at least $\lceil q_J \rceil$ particles hence if we define

$$R_J := \{\text{one of the } \lceil q_J \rceil \text{ particles jumps from the } J\text{-th to } (2J)\text{-th line in time } I_J\}$$

we have that

$$\mathbb{P}(B_J \mid A_J) \geq \mathbb{P}(R_J \mid A_J)$$

$$= 1 - \prod_{i=1}^{\lceil q_J \rceil} \mathbb{P}(i\text{-th particle will not jump to } (2J)\text{-th line in time } I_J \mid A_J)$$

$$= 1 - (1 - \alpha_J)^{\lceil q_J \rceil}.$$  \hfill (14)

In the first equality we use the independence of particles. From Proposition 1.15 $\alpha_J \to 0$ as $J \to \infty$. Thus for large enough $J$ we get that $(1 - \alpha_J)^{\frac{1}{\gamma J}} < \frac{2}{\gamma}$. Moreover, combining Proposition 1.15 and the estimate for $q_J$ we obtain

$$\lceil q_J \rceil \alpha_J \geq q_J \alpha_J \geq \frac{\hat{C}_1}{\gamma J} C^{2J} = e^{J\left(\frac{\hat{C}_1}{\gamma J} + 2 \ln C\right)}.$$

Now we choose $C_1$ such that $\frac{\hat{C}_1}{2} + 2 \ln C > 0$, what implies

$$e^{J\left(\frac{\hat{C}_1}{\gamma J} + 2 \ln C\right)} > J$$

for large enough $J$.

This is constant $C_1$ introduced in first section.

So $\lceil q_J \rceil \alpha_J > J$ for large enough $J$. Thus we have that

$$(1 - \alpha_J)^{\lceil q_J \rceil} = \left((1 - \alpha_J)^{\frac{1}{\gamma J}}\right)^{\lceil q_J \rceil} \alpha_J \leq \left(\frac{2}{\gamma}\right)^J$$

what combined with (14) gives the statement of the proposition. \qed

Now we define

$$A_{k,n} := \bigcap_{l=0}^{n} A_{k2^l}, \quad B_{k,n} := \bigcap_{l=0}^{n} B_{k2^l}, \quad B_k := \bigcap_{l=0}^{\infty} B_{k2^l}.$$ 

The following proposition gives the reason to the definition above.

**Proposition 1.18.** For any $k$ we have

$$\bar{B}_k \subset \left\{ \liminf_{t \to \infty} \frac{\max_{t \leq \gamma |X(t)|} X(t,i)}{t^{\frac{1}{\gamma}}} \geq \left(\frac{2^{(2\gamma - 1) - \gamma} - 1}{2C_1}\right)^{\frac{1}{\gamma}} \right\}.$$

**Proof.** Suppose we are on $\bar{B}_k$. We wait random time $\tau$ to get to the $k$-th line (set $B_k$ ensures that this line is attained). Moreover, we know that jumping from the $k2^n$-th line to the $k2^{n+1}$-th line, where $n' \geq 0$, will not take more time than $t_{k2^n} + 1 = C_1 (k2^n)^{1-\gamma}$ (set $B_{k2^n}$ guarantees it, see Remark 1.13).

We choose arbitrary $t > \tau$ and pick $m$ such that:

$$\tau + C_1 \sum_{i=0}^{m-1} (k2^n)^{1-\gamma} = \tau + C_1 \sum_{i=0}^{m-1} (t_{k2^n} + 1)^{1-\gamma} < t.$$
\[ t \leq \tau + C_1 \sum_{i=0}^{m} (t_{k2^i} + 1)^{1-\gamma} = \tau + C_1 \sum_{i=0}^{m} (k2^i)^{1-\gamma} \]

For such \( t \) our process manages to reach the \( k \)-th line, and performs at least \( m \) jumps, \( i \)-th jump takes us from the \( k2^{i-1} \)-th line to the \( k2^i \)-th line and we wait for it at most \( t_{k2^{i-1}} + 1 \) (see Remark 1.16). So for such \( t \) we are at least on the \( k2^m \)-th line thus:

\[ \max_{i \leq |X(t)|} X(t, i) \geq \frac{k2^m}{\left( \tau + C_1 \sum_{i=0}^{m} (2^i k)^{1-\gamma} \right)^{1-\gamma}}. \]  \hfill (15)

So for RHS of (15) and large enough \( m \)

\[ \text{RHS} \geq \frac{2^m}{\left( 2C_1 \sum_{i=0}^{m} (2^{i})^{1-\gamma} \right)^{\frac{1}{1-\gamma}}} = \frac{2^m}{\left( 2C_1 \frac{2^{m(1-\gamma)} - 1}{2(1-\gamma)} \right)^{\frac{1}{1-\gamma}}} \]

\[ \geq \frac{(2^{1-\gamma} - 1)^{\frac{1}{1-\gamma}}}{(2C_1)^{\frac{1}{1-\gamma}}}. \]

From the above proposition we understand that it is important to establish bounds for the probability of \( B_k \). The following lemma will be the crucial ingredient of the proof of Theorem 1.2.

**Lemma 1.19.** We have

\[ \mathbb{P}(B_k \mid Z_k) \to 1 \text{ if } k \to \infty. \]

**Proof.** For simplicity we denote \( \hat{\mathbb{P}}(\cdot) = \hat{\mathbb{P}}(\cdot \mid Z_k) \). By the continuity of probability we have that

\[ \hat{\mathbb{P}}(B_k) = \lim_{n \to \infty} \hat{\mathbb{P}}(B_{k,n}) \geq \lim_{n \to \infty} \hat{\mathbb{P}}(B_{k,n} \cap A_{k,n}). \]  \hfill (16)

Inductively, one can prove that

\[ \hat{\mathbb{P}}(B_{k,n} \cap A_{k,n}) = \prod_{l=0}^{n-1} \hat{\mathbb{P}}(B_{k2^l + 1} \cap A_{k2^l + 1} \mid A_{k,l} \cap B_{k,l}) \cdot \hat{\mathbb{P}}(A_{k} \cap B_{k}). \]  \hfill (17)

The terms above are close to 1. We notice that \( B_{k,l} \subset Z_{k2^l + 1} \) and thus by (13) and the strong Markov property we have

\[ \hat{\mathbb{P}}(B_{k2^{l+1}} \cap A_{k2^{l+1} + 1} \mid A_{k,l} \cap B_{k,l}) \geq 1 - \frac{D_1}{(k2^{l+1})^2}. \]

Directly from (13) we know that \( \hat{\mathbb{P}}(A_{k} \cap B_{k}) \geq \left( 1 - \frac{D_1}{(k2^1)^2} \right) \). By taking log in (17) we obtain

\[ \log \hat{\mathbb{P}}(B_{k,n} \cap A_{k,n}) \geq \sum_{l=0}^{n} \log \left( 1 - \frac{D_1}{(k2^l)^2} \right) \geq - \sum_{l=0}^{n} \frac{2D_1}{(k2^l)^2}. \]
where the last estimate follows by the elementary inequality $2t \leq \log(1+t)$ valid for $t < 0$ sufficiently close to 0. Recall (16) we conclude that

$$\hat{P}(B_k) \geq \lim_{n \to \infty} \exp\left(-\frac{\sum_{l=0}^{n} 2D_1}{(k2^l)^2}\right).$$

The statement of the lemma follows now by elementary considerations.

Now we are ready to prove the bound from the below in Theorem 1.2.

**Proof.** We put

$$S := \left\{ \liminf_{t \to \infty} \frac{\max_{i \leq |X(t)|} X(t, i)}{t^{1-\gamma}} \geq \left(\frac{2(1-\gamma) - 1}{2C_1}\right)^{\frac{1}{1-\gamma}} \right\}.$$

Let us fix $n \in \mathbb{N}$. Let $L_n$ denotes the number of these line bigger than $n - 1$ which was populated first. It is easy to see that $\max_{i \leq |X(t)|} X(t, i) \xrightarrow{a.s.} \infty$ thus $L_n$ is well-defined. We write

$$\mathbb{P}(S) = \sum_{k=n}^{\infty} \mathbb{P}(S \mid L_n = k) \mathbb{P}(L_n = k) \geq \sum_{k=n}^{\infty} \mathbb{P}(B_k \mid L_n = k) \cdot \mathbb{P}(L_n = k)$$

because by Proposition 1.18 we know that $B_k \subset S$.

Let us choose an arbitrary small $\varepsilon > 0$. The condition $L_n = k$ implies $Z_k$ thus, by Lemma 1.19 and the strong Markov property, choosing large enough $n$ we have $\inf_{k > n} \mathbb{P}(B_k \mid L_n = k) \geq 1 - \varepsilon$, what implies

$$\mathbb{P}(S) \geq \sum_{k=n}^{\infty} (1 - \varepsilon) \mathbb{P}(L_n = k) = 1 - \varepsilon.$$

As $\varepsilon$ is arbitrary, $\mathbb{P}(S) = 1$.

### 2 Upper bound for the Galton Watson model

The goal of this chapter is to prove the bound from above in Theorem 1.2.

We define

$$t_J := C_2 J^{1-\gamma} \text{ where } C_2 > 0 \text{ is constant to be fixed later.} \quad (18)$$

**Remark 2.1.** The considered model is equivalent to the following one. At time 0 we have one particle on the first line. A particle on the $i$-th line has clocks $Z_0, Z_1, Z_2, \ldots$ where $Z_0 \sim \text{Exp}(\gamma)$, $Z_k \sim \text{Exp}\left(\frac{\gamma + k}{\gamma - k}\right)$ for $k > 0$ and $Z_0, Z_1, Z_2, \ldots$ are independent. We denote $T := \min(Z_1, Z_2, \ldots)$. If $T > Z_0$ then the particle produces an additional particle on the $i$-th line at a time $Z_0$. If $T = Z_k < Z_0$ then it jumps to the $(i+k)$-th line at $Z_k$. In particular the particle jumps higher at $T$, perhaps producing some additional particles before the jump.

The following lemma is the key technical result of this section

**Lemma 2.2.** We have

$$\sum_{J=1}^{\infty} \mathbb{P}\left(\max_{i \leq |X(t_J)|} X(t_J, i) \geq J\right) < \infty.$$
Proof. We assume the same notation as in Remark 2.1. If all particles are below the $J$-th line then they have smaller birth intensity than $J^\gamma$, thus they reproduce slower than a $GW(J^\gamma)$ process. Now we estimate the time the particle needs to jump to the $J$-th line or higher. Let us define $T_1 := \min(Z_j, Z_{j+1}, \ldots)$. Particle on the $i$-th line $i < J$, jumps to the $J$-th line or higher if $T = T_1$. If $T < T_1$ it jumps lower than the $J$-th line, and still its birth intensity is bounded from the above by $J^\gamma$. Elementary calculations gives us that $T_1 \sim Exp \left( \frac{C'}{1-C} \right)$, thus jumping to the $J$-th line or higher has intensity $\frac{C'}{1-C}$ (in particular it is independent of the line in which our particle is). Let us consider $X'$ a $GW \left( J^\gamma, \frac{C'}{1-C} \right)$ process. Before our model reaches the $J$-th line, it reproduces slower than $X$. Furthermore the intensity of jumping to the $J$-th line or higher in our model equals to the intensity of the death in the process $X'$. We conclude these observations in

\[
\mathbb{P} \Big( \text{we have a death in the process } X' \text{ until time } t \Big) \geq \mathbb{P} \left( \max_{i \leq \chi(t)} X(t, i) \geq J \right).
\]

To analyse the probability of death we introduce $X''(t)$ a Galton-Watson process obtained from $X'$ in such a way that we ignore deaths. It is easy to see that $X''$ has the same intensity of the birth as $X'$, thus it is a $GW(J^\gamma)$ process. Let us denote $N(t) := \text{number of deaths in the process } X'$ until time $t$. We see that

\[
\chi_{N(t) \geq 1} \leq X''(t) - X'(t).
\]

So from (19) we get

\[
\mathbb{P} \left( \max_{i \leq \chi(t)} X(t, i) \geq J \right) \leq \mathbb{P} \left( \text{we have a death in the process } X' \text{ until time } t_j \right)
= \mathbb{E} \chi_{N(t) \geq 1} \leq \mathbb{E} (X''(t) - X'(t)).
\]

We recall $t_j$ defined in (18). By Proposition 1.3 the right-hand side of (20) is equal to

\[
e^{\gamma \sum_{j=1}^{t_j} (J^\gamma - \frac{C'}{1-C})} = e^{\gamma \sum_{j=1}^{t_j} (1 - e^{-\gamma (\frac{C'}{1-C})})} \leq e^{\gamma \sum_{j=1}^{t_j} C'} \frac{e^{\gamma t_j C'}}{1-C} = \frac{C_2 J^{1-\gamma} \gamma}{(1-C)} (CeC_2)^J.
\]

So we proved that

\[
\mathbb{P} \left( \max_{i \leq \chi(t)} X(t, i) \geq J \right) \leq \frac{C_2 J^{1-\gamma}}{(1-C)} (CeC_2)^J.
\]

Now it is enough to choose $C_2$ such that $CeC_2 < 1$.  

We are ready to prove the bound from above in Theorem 1.2. The Borel-Cantelli Lemma and Lemma 2.2 imply that there exists a random variable $J_0$ such that $\max_{i \leq \chi(t)} X(t, i) < J$ for $J > J_0$. Further, for $t_{j-1} < t < t_j$, $J > J_0$ we have

\[
\frac{\max_{i \leq \chi(t)} X(t, i)}{t_j^{1-\gamma}} \leq \frac{J}{t_j^{1-\gamma}} = \frac{J}{(C_2 (J-1)^{1-\gamma})^{1-\gamma}}
= \frac{J}{J-1} \left( \frac{1}{C_2} \right)^{\frac{1}{1-\gamma}} \xrightarrow{J \to \infty} \left( \frac{1}{C_2} \right)^{\frac{1}{1-\gamma}}.
\]
3 Branching model with the heavy tailed Poisson process and unbounded intensity of branching

3.1 Model and notation

In this chapter we consider a system of branching particles in $\mathbb{R}$. Particles move accordingly to the compound Poisson process with the jump measure $\nu$ and branch with the intensity $f(x) = (\log(|x| + 1))^{\gamma}$, $\gamma \in (0, 1)$. During branching a particle dies and produces two offspring at its location, which execute the same dynamics. Particles are independent. At time $t = 0$ the system is initialized with a single particle located at 0. We assume two conditions on $\nu$

\[ \nu((-\infty, 0)) = 0, \tag{21} \]

and

\[ \lim_{x \to \infty} \frac{\nu([x, +\infty))}{x^{-\alpha}L(x)} = 1 \tag{22} \]

for some $\alpha > 0$ and a slowly varying function $L : (0, +\infty) \mapsto (0, +\infty)$.

**Corollary 3.2.** The following description is equivalent to the considered model.

We start with a single particle at the point $(0, 0)$. At time $t$ the particle has a position $(t, X_t)$, where $X_t$ is the compound Poisson process with jump measure $\nu$, which fulfills (21) and (22). We take the branching intensity $f(x) = (\log(|x| + 1))^{\gamma}$, $\gamma \in (0, 1)$. The particle dies producing two descendants at the same location at time $t := \inf\{t' \mid \int_0^{t'} f(X_s)ds > Y\}$, where $Y \sim \text{Exp}(1)$ is independent of $X_t$ and $t_0$ is time at which the particle was born. Particles are independent.

We denote by $|X(t)|$ the number of particles at time $t$, $X(t, i)$ the position of $i$-th particle at time $t$, $i \leq |X(t)|$,

\[ M(t) := \max_{i \leq |X(t)|} X(t, i). \tag{23} \]

The main goal of this chapter is the following theorem.

**Theorem 3.3.** There exist constants $A, B > 0$ such that

\[ A > \lim_{t \to \infty} \frac{\log M(t)}{t^{1/\gamma}} \geq \lim_{t \to \infty} \frac{\log M(t)}{t^{1/\gamma}} > B. \]

We start with two technical lemmas. The first one follows directly by the Karamata representation theorem (see Corollary 2.1 [2]).

**Lemma 3.4.** Let $L$ be a slowly varying function. Then for every $\eta > 0$ and large enough $x$ we have $L(x) \leq x^\eta$ and $L(x) \geq x^{-\eta}$.

**Lemma 3.5.** There exists constant $D > 0$ such that for any $n, k \in \mathbb{N}$ (so that $n, k \geq 1$) the following inequality holds

\[ \nu\left((e^{n+k} - e^n, \infty)\right) \geq e^{-(k+n)D\alpha}, \]

where $\alpha$ is the same as in (22).
Proof. We take arbitrary $\eta > 0$. By (22) and Lemma 3.5 we have

$$\nu \left( (e^{n+k} - e^n, \infty) \right) \geq (1 - \varepsilon)(e^{k+n} - e^n)^{-\alpha}(e^{k+n} - e^n)^{-\eta},$$

for large enough $n + k$. For the rest of $n, k$ (there are only finite number of them) the measure above is bounded away from 0. So there exists a constant $F$ such that

$$\nu \left( (e^{n+k} - e^n, \infty) \right) \geq F \cdot (e^{k+n} - e^n)^{-\alpha - \eta}$$

for any $n, k \in \mathbb{N}$. Now the proof follows by simple calculations. \hfill \Box

3.6 Bound from below

The proof strategy is to map the model to the one studied in the previous section. To this end we divide the space into strips $U_n = [e^n, e^{n+1}]$. A particle in $U_n$ behaves similarly to a particle on the $n$-th line in the model investigated in the previous chapter. Let us consider a particle being in the strip $U_n$. By assumption (21) it can only go higher thus

$$\int_t^{t+h} f(X(s,i))ds \geq \int_t^{t+h} f(e^n)ds = \int_t^{t+h} \ln^\gamma(1 + e^n)ds \geq h \ln^\gamma(e^n) = hn^\gamma.$$

Let us take $Y \overset{d}{=} \text{Exp}(1)$. We have that

$$\mathbb{P} \left( \text{a particle that is in } U_n \text{ splits before time } h \right) \geq \mathbb{P} \left( \int_t^{t+h} f(X(s,i))ds > Y \right) \geq \mathbb{P} \left( hn^\gamma > Y \right) = 1 - e^{-hn^\gamma}. \quad (24)$$

So if a particle is in $U_n$, then it does not wait longer than $\text{Exp}(n^\gamma)$ to reproduce.

Let us consider a particle (denoted by $i$) at time $t_0$ who is in $U_n$, in particular $X(t_0, i) > e^n$. We decompose the process $t \mapsto X(t + t_0, i) - X(t_0, i)$ into $X_t^n$ containing jumps smaller than $e^{n+k} - e^n$ and $X_t^2$ with the rest of the jumps. Now we will estimate time needed to for the particle reach $\bigcup_{m=k}^\infty U_{n+m}$ at time $t_0 + t$. By the assumption (21) the process $X^1$ is non-negative, thus a sufficient condition is that $X_t^2 \geq e^{n+k} - e^n$. This is equivalent to $X_t^2 > 0$. Put formally

$$\mathbb{P} \left( \text{a particle from } U_n \text{ will reach } \bigcup_{m=k}^\infty U_{n+m} \text{ before time } h \right) \geq \mathbb{P} \left( X_t^2 > 0 \right). \quad (25)$$

We know that $X_t^2 = \sum_{i=1}^{N_t} \xi_i$, where $N_t$ is the Poisson process with the intensity $\nu \left( (e^{n+k} - e^n, \infty) \right)$, therefore time to the first jump is distributed like $\text{Exp} \left( \nu(e^{n+k} - e^n, \infty) \right)$, which by Lemma 3.5 is stochastically bounded from above by $\text{Exp} \left( e^{-(n+k)D_\alpha} \right)$.

Now we introduce a model considered in the first section, denoted by $Y$. It starts with one particle at line 1. A particle on the $n$-th line branches with intensity $n^\gamma$. Moreover it jumps to the $n'$-th line ($n' > n$) or higher with the intensity $e^{-n'D_\alpha}$. We conclude observing that particles in the Poisson model which are in $U_n$, reproduce faster than the ones of $Y$ on the $n$-th line and they reach $\bigcup_{m=k}^\infty U_{n+m}$ faster than particles from the $n$-th line reach the $(n + k)$-th. Thus, the process $\ln (M(t))$ (see (23)) is stochastically bounded from below by $\max_{i \leq |Y(t)|} Y(t, i)$. This implies

$$\liminf_{t \to \infty} \frac{\ln M(t)}{t^{1-\gamma}} \geq \liminf_{t \to \infty} \frac{\max_{i \leq |Y(t)|} Y(t, i)}{t^{1-\gamma}} \geq C, \text{ a.s.},$$

for some $C > 0$. The last inequality comes from Theorem 1.2
3.7 Bound from above

The aim of this section is to prove the bound from above in Theorem 3.3.

We put \( t_J := J^{1-\gamma} \). We will prove that \( \sum \mathbb{P}(M(t_j) > e^J) < \infty \). Once this is done, the same argument as in the proof of Theorem 1.2 in Section 2 can be used to extend to any time.

Let us consider \( Y \) a branching process in which particles move accordingly to the compound Poisson process with the same jump measure as the process \( X \) has and branch with the intensity \( f(x) = 2J^\gamma, \gamma \in (0,1) \). During branching a particle dies and produces two offspring at its location, which execute the same dynamics. Particles are independent. At time \( t = 0 \) the system is initialized with a single particle located at \( 0 \). \( Y \) has constant branching intensity \( 2J^\gamma \), which implies that \( |Y(t)| \) is a GW(2J\(^\gamma\)) process. We recall that \( |Y(t)| \) is equal to the number of particles in time \( t \). Before the system \( X \) reaches level \( e^J \) particles have smaller branching intensity than \( \ln^\gamma (1+e^J) \leq 2J^\gamma \).

Thus before \( X \) reaches a strip \( x \geq e^J \), the system \( Y \) reproduces faster and jumps in the same way as \( X \), therefore (we recall \( M \) defined in (23))

\[
\mathbb{P}(M(t_J) > e^J) \leq \mathbb{P}(\max_{i \leq |Y(t_J)|} Y(t_J,i) > e^J). \quad (26)
\]

We define \( A_J = \{\max_{i \leq |Y(t_J)|} Y(t_J,i) > e^J\} \) and write

\[
\mathbb{P}(A_J) \leq \mathbb{P}(A_J, |Y(t_J)| < \lceil e^J \rceil) + \mathbb{P}(|Y(t_J)| \geq \lceil e^J \rceil) = I_1 + I_2.
\]

Since \( |Y(t)| \) is the GW(2J\(^\gamma\)) process, so by the Markov inequality and Proposition 1.5

\[
I_2 \leq \frac{\mathbb{E}[|Y(t_J)|]}{\lceil e^J \rceil} \leq \frac{e^{J2J^\gamma}}{\lceil e^J \rceil} = e^{-\gamma J}.
\]

For \( I_1 \) we first use the conditional expectation

\[
I_1 \leq \mathbb{E} \left[ \chi_{\{\lceil Y(t_J) \rceil < \lceil e^J \rceil\}} \sum_{i=1}^{\lceil Y(t_J) \rceil} \chi_{\{Y(t_J,i) > e^J\}} \right]
\]

\[
= \mathbb{E} \left[ \chi_{\{\lceil Y(t_J) \rceil < \lceil e^J \rceil\}} \sum_{i=1}^{\lceil Y(t_J) \rceil} \mathbb{E} \left[ \chi_{\{Y(t_J,i) > e^J\}} \mid Y(t_J) \right] \right]
\]

Using the fact that the conditional expectation above does not depend on \( i \) we estimate

\[
I_1 \leq \mathbb{E} \left[ \chi_{\{\lceil Y(t_J) \rceil < \lceil e^J \rceil\}} \sum_{i=1}^{\lceil e^J \rceil} \mathbb{E} \left[ \chi_{\{Y(t_J,i) > e^J\}} \mid Y(t_J) \right] \right] \leq \lceil e^J \rceil \mathbb{P}(Y(t_J,1) > e^J).
\]

(27)

The process \( t \mapsto Y(t,1) \) is a compound Poisson process with the jump measure \( \nu \). Now we do some simple calculations

\[
\mathbb{P}(Y(t_J,1) > e^J) = \sum_{i=1}^{\infty} \mathbb{P}(Y(t_J,1) > e^J \mid \text{we have } i \text{ jumps}) \mathbb{P}(\text{we have } i \text{ jumps})
\]

\[
= \sum_{i=1}^{\infty} \mathbb{P} \left( \sum_{j=1}^{i} \xi_j > e^J \right) e^{-t_J \eta} \frac{(t_J \eta)^i}{i!}
\]

\[13\]
We use (22) in the last equality, for large enough \( J \) function \( e^J \). Obviously \( \sum_{j=1}^{\infty} e^{-t_j} = 1 \) and \( \sum_{j=1}^{\infty} j e^J = R \). First, we estimate \( R \) for \( e^J \) and \( e^J \). Summarizing our estimates we get

\[
R_1 = \sum_{i=1}^{J} \nu \left( \frac{e^J}{\eta} \right) e^{-\eta_j} \frac{(t_j \eta_j)^i}{i!} \leq \sum_{i=1}^{J} \frac{2i}{\eta} \left( e^J \right)^{-\alpha} L \left( e^J \right) e^{-\eta_j} \frac{(t_j \eta_j)^i}{i!}.
\]  
(28)

We use (22) in the last equality, for large enough \( J \). Now by Lemma 2.4 applied to a slowly varying function \( \frac{2}{\eta} L(x) \) we have for an arbitrary \( \epsilon > 0 \) and large enough \( J \)

\[
R_1 \leq \sum_{i=1}^{J} \left( \frac{e^J}{i} \right)^{-\alpha + \epsilon} e^{-\eta_j} \frac{(t_j \eta_j)^i}{i!} \leq e^{-J(\alpha - \epsilon)} \cdot J^{\alpha - \epsilon + 1} \sum_{i=1}^{J} e^{-\eta_j} \frac{(t_j \eta_j)^i}{i!}
\]

\[
\leq e^{-J(\alpha - \epsilon)} \cdot J^{\alpha + \epsilon}
\]

For \( J \) large enough we can estimate \( R_2 \) using the Stirling formula

\[
R_2 \leq 2 \sum_{i=1}^{\infty} \frac{e^J}{i} \leq 2 \sum_{i=1}^{\infty} q^i = Const \cdot q^J.
\]  
(29)

where \( q \) is a constant which can be made arbitrary close to 0 since \( t_j \cdot J^{-1} \to 0 \) (we remind that \( t_j := J^{1 - \gamma} \)). Due to (22) and previous calculations we showed that \( \mathbb{P}(Y(t_J, 1) > e^J) \leq e^{-J(\alpha - \epsilon)} \cdot J^{\alpha + \epsilon} + Const \cdot q^J \). Summarizing our estimates we get

\[
\mathbb{P}(M(t_J) > e^J) \leq e^{\frac{J}{2} \epsilon} \left( e^{-J(\alpha - \epsilon)} \cdot J^{\alpha + \epsilon} + Const \cdot q^J \right) + e^{-\frac{J}{2} \epsilon}.
\]

Obviously \( \sum_{j=1}^{\infty} e^{-\frac{J}{2} \epsilon} < \infty \). If we take \( \epsilon = \frac{\alpha}{100} \) and \( q \) such that \( e^\alpha \cdot q < 1 \) then also

\[
\sum_{i=1}^{\infty} e^{\frac{J}{2} \epsilon} \left( e^{-J(\alpha - \epsilon)} \cdot J^{\alpha + \epsilon} + Const \cdot q^J \right) < \infty,
\]

and the proof of the upper estimate in Theorem 3.3 is concluded.

Acknowledgment I would like to thank dr P. Miloš for all comments, which improved the presentation of the proof.

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