Revenue Monotonicity Under Misspecified Bidders

Makis Arsenis, Odysseas Drosis, and Robert Kleinberg
Cornell University, Ithaca, NY 14853, USA.

Abstract

We investigate revenue guarantees for auction mechanisms in a model where a distribution is specified for each bidder, but only some of the distributions are correct. The subset of bidders whose distribution is correctly specified (henceforth, the "green bidders") is unknown to the auctioneer. The question we address is whether the auctioneer can run a mechanism that is guaranteed to obtain at least as much revenue, in expectation, as would be obtained by running an optimal mechanism on the green bidders only. For single-parameter feasibility environments, we find that the answer depends on the feasibility constraint. For matroid environments, running the optimal mechanism using all the specified distributions (including the incorrect ones) guarantees at least as much revenue in expectation as running the optimal mechanism on the green bidders. For any feasibility constraint that is not a matroid, there exists a way of setting the specified distributions and the true distributions such that the opposite conclusion holds.

1 Introduction

In a seminal paper nearly forty years ago [Mye81], Roger Myerson derived a beautifully precise characterization of optimal (i.e., revenue maximizing) mechanisms for Bayesian single-parameter environments. One way this result has been critiqued over the years is by noting that auctioneers may have incorrect beliefs about bidders’ values; if so, the mechanism recommended by the theory will actually be suboptimal.

In this paper we evaluate this critique by examining revenue guarantees for optimal mechanisms when a subset of bidders’ value distributions are misspecified, but the auctioneer doesn’t know which of the distributions are incorrect. Our model is inspired by the literature on semi-random adversaries in the theoretical computer science literature, particularly the work of Bradac et al. [BGSZ19] on robust algorithms for the secretary problem. In the model we investigate here, the auctioneer is given (not necessarily identical) distributions for each of n bidders. An unknown subset of the bidders, called the green bidders, draw their values independently at random from these distributions. The other bidders, called the red bidders, draw their values from distributions other than the given ones.

The question we ask in this paper is, "When can one guarantee that the expected revenue of the optimal mechanism for the given distributions is at least as great as the expected revenue that would be obtained by excluding the red bidders and running an optimal mechanism on the green subset of bidders?" In other words, can the presence of bidders with misspecified distributions in a market be worse (for the auctioneer’s expected revenue) than if those bidders were absent? Or does the increased competition from...
incorporating the red bidders always offset the revenue loss due to ascribing the wrong distribution to them?

We give a precise answer to this question, for single-parameter feasibility environments. We show that the answer depends on the structure of the feasibility constraint that defines which sets of bidders may win the auction. For matroid feasibility constraints, the revenue of the optimal mechanism is always greater than or equal to the revenue obtained by running the optimal mechanism on the set of green bidders. For any feasibility constraint that is not a matroid, the opposite holds true: there is a way of setting the specified distribution and the true distributions such that the revenue of the optimal mechanism for the specified distributions, when bids are drawn from the true distributions, is strictly less than the revenue of the optimal mechanism on the green bidders only.

The economic intuition behind this result is fairly easy to explain. The matroid property guarantees that the winning red bidders in the auction can be put in one-to-one correspondence with losing green bidders who would have won in the absence of their red competitors, in such a way that the revenue collected from each winning red bidder offsets the lost revenue from the corresponding green bidder whom he or she displaces. When the feasibility constraint is not a matroid, this one-to-one correspondence does not always exist; a single green bidder might be displaced by two or more red bidders each of whom pays almost nothing. The optimal mechanism allows this to happen at some bid profiles, because the low revenue received on such bid profiles is compensated by the high expected revenue that would be received if the red bidders had sampled values from elsewhere in their distributions. However, since the red bidders’ distributions are misspecified, the anticipated revenue from these more favorable bid profiles may never materialize.

Our result can be interpreted as a type of revenue monotonicity statement for optimal mechanisms in single-parameter matroid environments. However it does not follow from other known results on revenue monotonicity, and it is illuminating to draw some points of distinction between our result and earlier ones.

Let us begin by distinguishing pointwise and setwise revenue monotonicity results: the former concern how the revenue earned on individual bid profiles varies as the bids are increased, the latter concern how (expected) revenue varies as the set of bidders is enlarged.

- VCG mechanisms are neither pointwise nor setwise revenue monotone in general, but in single-parameter matroid feasibility environments, VCG revenue satisfies both pointwise and setwise monotonicity. In fact, Dughmi, Roughgarden, and Soundararajan [DRS09] observed that VCG revenue obeys setwise monotonicity if and only if the feasibility constraint is a matroid. The proof of this result in [DRS09] rests on a slightly erroneous characterization of matroids, and one (small) contribution of our work is to correct this minor error by substituting a valid characterization of matroids, namely Lemma 4 below.

- Myerson’s optimal mechanism is not pointwise revenue monotone, even for single-item auctions. For example, consider using Myerson’s optimal mechanism to sell a single item to Alice whose value is uniformly distributed in $[0, 4]$ and Bob whose value is uniformly distributed in $[0, 8]$. When Alice bids 0 and Bob bids 5, Bob wins and pays 4. If Alice increases her bid to 4, she wins but pays only 3.

- However, Myerson’s optimal mechanism is always setwise revenue monotone in single-parameter environments with downward-closed feasibility constraints, regardless of whether the feasibility constraint is a matroid. This is because the mechanism’s expected revenue is equal to the expectation of the maximum, over all feasible sets of winners, of the winners’ combined ironed virtual value. Enlarging the set of bidders only enlarges the collection of sets over which this maximization is performed, hence it cannot decrease the expectation of the maximum.

Our main result is analogous to the setwise revenue monotonicity of Myerson revenue, except that we are
considering monotonicity with respect to the operation of enlarging the set of bidders by adding bidders whose value distributions are potentially misspecified. We show that the behavior of Myerson revenue with respect to this stricter notion of setwise revenue monotonicity holds under matroid feasibility constraints but not under any other feasibility constraints, in contrast to the traditional setwise revenue monotonicity that is satisfied by Myerson mechanisms under arbitrarily downward-closed constraints.

1.1 Related Work

Semi-random models are a class of models studied in the theoretical computer science literature in which the input data is partly generated by random sampling, and partly by a worst-case adversary. Initially studied in the setting of graph coloring [BS95] and graph partitioning [FK01, MMV12], the study of semi-random models has since been broadened to statistical estimation [DKK+19, LRV16], multi-armed bandits [LMPL18], and secretary problems [BGSZ19]. Our work extends semi-random models into the realm of Bayesian mechanism design. In particular, our model of green and red bidders resembles in a sense that of Bradac el al. [BGSZ19] for the secretary problem which served as inspiration for this work. In both settings, green players/elements behave randomly and independently while red players/elements behave adversarially. In the secretary model of [BGSZ19], red elements can choose arbitrary arrival times while green elements’ arrival times are i.i.d. uniform in $[0, 1]$ and independent of the red arrival times. Similarly, in our setting red bidders can set their bids arbitrarily whereas green bidders sample their bids from known distributions, independently of the red bidders and one another.

Our work can be seen as part of a general framework of robust mechanism design, a research direction inspired by Wilson [Wil87], who famously wrote,

Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one agent’s probability assessment about another’s preferences or information. I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality.

This Wilson doctrine has been used to justify more robust solution concepts such as dominant strategy and ex post implementation. The question of when these stronger solution concepts are required in order to ensure robustness was explored in a research program initiated by Bergemann and Morris [BM05] and surveyed in [BM13]. Robustness and the Wilson doctrine have also been used to justify prior-free [GHK+06] and prior-independent [HR09] mechanisms as well as mechanisms that learn from samples [BDHN19, CR14, DRY15, HMR18, MR16]. A different approach to robust mechanism design assumes that, rather than being given the bid distributions, the designer is given constraints on the set of potential bid distributions and aims to optimize a minimax objective on the expected revenue. For example Azar and Micali [AM13] assume the seller knows only the mean and variance of each bidder’s distribution, Carrasco et al. [CFK+18] generalize this to sellers that know the first $N$ moments of each bidder’s distribution, Azar et al. [AMDW13] consider sellers that know the median or other quantiles of the distributions, Bergemann and Schlag [BS11] assume the seller is given distributions that are known to lie in a small neighborhood of the true distributions, and Carroll [Car17] introduced a model in which bids are correlated but the seller only knows each bidder’s marginal distribution (see [GL18, BGLT19] for further work in this correlation-robust model).

Another related subject is that of revenue monotonicity of mechanisms — regardless of the existence of adversarial bidders. Dughmi et al. [DRS09] prove a result very close in spirit to ours. They consider
the VCG mechanism in a single-parameter downward-closed environment and prove that it is revenue monotone if and only if the environment is a matroid akin to our Theorems 1 and 2.

Rastegari et al. [RCLB07] study revenue monotonicity properties of mechanisms (including VCG) for Combinatorial Auctions. Under some reasonable assumptions, they prove that no mechanism can be revenue monotone when bidders have single-minded valuations.

2 Preliminaries

2.1 Matroids

Given a finite ground set $E$ and a collection $I \subseteq 2^E$ of subsets of $E$ such that $\emptyset \in I$, we call $M = (E, I)$ a set system. $M$ is a downward-closed set system if $I$ satisfies the following property:

(I1) (downward-closed axiom) If $B \in I$ and $A \subseteq B$ then $A \in I$.

Furthermore, $M$ is called a matroid if it satisfies both (I1) and (I2):

(I2) (exchange axiom) If $A, B \in I$ and $|A| > |B|$ then there exists $x \in A \setminus B$ such that $B + x \in I$.

In the context of matroids, sets in (resp. not in) $I$ are called independent (resp. dependent). An (inclusion-wise) maximal independent set is called a basis. A fundamental consequence of axioms (I1), (I2) is that all bases of a matroid have equal cardinality and this common quantity is called the rank of the matroid. A circuit is a minimal dependent set. The set of all circuits of a matroid will be denoted by $C$. The following is a standard property of $C$.

Proposition 1 ([Oxl06, Proposition 1.4.11]). For any $C$ which is the circuit set of a matroid $M$, let $C_1, C_2 \in C, e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$. Then there exists $C_3 \in C$ such that $f \in C_3 \subseteq (C_1 \cup C_2) - e$.

For any set system $M = (E, I)$ and any given $S \subseteq E$, define $I|_S = I \cap 2^S$ and call $M|_S = (S, I|_S)$ the restriction of $M$ on $S$. Notice that restrictions maintain properties (I1), (I2) if they were satisfied already in $M$.

In what follows, we provide some examples of common matroids. The reader is invited to check that they indeed satisfy (I1), (I2). For a more in-depth study of matroid theory, we point the reader to the classic text of Oxley [Oxl06].

Uniform matroids When $I = \{S \subseteq E : |S| \leq k\}$ for some positive integer $k \leq |E|$, then $(E, I)$ is called a uniform (rank $k$) matroid.

Graphic matroids Given a graph $G = (V, E)$ (possibly containing parallel edges and self-loops) let $I$ include all subsets of edges which do not form a cycle, i.e. the subgraph $G[S] = (V, S)$ is a forest. Then $(E, I)$ forms a matroid called graphic matroid. Graphic matroids capture many of the properties of general matroids and notions like bases and circuits have their graphic counterparts of spanning trees and cycles respectively.

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1We use the shorthand $B + x$ (resp. $B - x$) to mean $B \cup \{x\}$ (resp. $B \setminus \{x\}$) throughout the paper.
**Transversal matroids** Let \( G = (A \cup B, E) \) be a simple, bipartite graph and define \( I \) to include all subsets \( S \subseteq A \) of vertices for which the induced subgraph on \( S \cup B \) contains a matching covering all vertices in \( S \). Then \( (A, I) \) is called a *transversal* matroid.

If \( M = (E, I) \) is equipped with a weight function \( w : E \to \mathbb{R}^+ \) it is called a *weighted* matroid. The problem of finding an independent set of maximum sum of weights is central to the study of matroids. A very simple greedy algorithm is guaranteed to find the optimal solution and in fact matroids are exactly the downward-closed systems for which that greedy algorithm is always guaranteed to find the optimal solution.

**Greedy** Sort the elements of \( E \) in non-increasing order of weights \( w(e_1) \geq w(e_2) \geq \ldots \geq w(e_n) \). Loop through the elements in that order adding each element to the current solution as long as the current solution remains an independent set.

**Lemma 1** ([Ox106, Lemma 1.8.3.]). Let \( M = (E, I) \) be a weighted downward-closed set system. Then Greedy is guaranteed to return an independent set of maximum total weight for every weight function \( w : E \to \mathbb{R}^+ \) if and only \( M \) is a matroid.

In what follows we’re going to assume without loss of generality that the function \( w \) is one-to-one, meaning that no two element have the same weight. All proofs can be adapted to work in the general case using any deterministic tie breaking rule.

The following proposition provides a convenient way for updating the solution to an optimization problem under matroid constraints when new elements are added. A proof is included in the Appendix.

**Proposition 2.** Let \( M = (E, I) \) be a weighted matroid with weight function \( w : E \to \mathbb{R}^+ \). Consider the max-weight independent set \( I \) of the restricted matroid \( M_{E-x} \). Then the max-weight independent set \( I' \) of \( M \) can be obtained from \( I \) as follows: if \( (I + x) \in I \) then \( I' = I + x \), otherwise, \( I' = (I + x) - y \) where \( y \) is the minimum-weight element in the unique circuit \( C \) of \( I + x \).

### 2.2 Optimal Mechanism Design

We study auctions modeled as a *Bayesian single-parameter environment*, a standard mechanism design setting in which a *seller* (or mechanism designer) holds many identical copies of an item they want to sell. A set of \( n \) bidders (or players), numbered \( 1 \) through \( n \), participate in the auction and each bidder \( i \) has a private, non-negative value \( v_i \sim F_i \), sampled (independently across bidders) from a distribution \( F_i \) known to the seller. Abusing notation, we’ll use \( F_i \) to also denote the cumulative distribution function and \( f_i \) to denote the probability density function of the respective distribution. The value of each bidder expresses their valuation for receiving one item. Let \( V_i \) be the support of distribution \( F_i \) and define \( V = V_1 \times \ldots \times V_n \).

For a vector \( v \in V \), we use the standard notation \( v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \) to express the vector of valuations of all bidders except bidder \( i \). When the index set \( [n] \) is partitioned into two sets \( A, B \) and we have vectors \( v_A \in \mathbb{R}^A \), \( w_B \in \mathbb{R}^B \), we will abuse notation and let \( (v_A, w_B) \) denote the vector obtained by interleaving \( v_A \) and \( w_B \), i.e. \( (v_A, w_B) \) is the vector \( u \in \mathbb{R}^n \) specified by

\[
   u_i = \begin{cases} 
   v_i & \text{if } i \in A \\
   w_i & \text{if } i \in B.
   \end{cases}
\]

Similarly, when \( v \in V \), \( i \in [n] \), and \( z \in \mathbb{R} \), \( (z, v_{-i}) \) will denote the vector obtained by replacing the \( i \)th component of \( v \) with \( z \).

\(^2\)An equivalent formulation asks for a basis of maximum total weight.
A feasibility constraint $I \subseteq 2^{[n]}$ defines all subsets of bidders that can be simultaneously declared winners of the auction. We will interchangeably denote elements of $I$ both as subsets of $[n]$ and as vectors in $\{0, 1\}^n$. Of special interest are feasibility constraints which define the independent sets of a matroid. We will sometimes use the phrase matroid market to indicate this fact. Matroid markets model many real world applications. For example when selling $k$ identical copies of an item, the market is a uniform rank $k$ matroid. Another example is kidney exchange markets which can be modeled as transversal matroids ([RSU05]).

In a sealed-bid auction, each bidder $i$ submits a bid $b_i \in V_i$ simultaneously to the mechanism. Formally, a mechanism $\mathcal{A}$ is a pair $(x, p)$ of an allocation rule $x : V \rightarrow I$ accepting the bids and choosing a feasible outcome and a payment rule $p : V \rightarrow \mathbb{R}^n$ assigning each bidder a monetary payment they need to make to the mechanism. We denote by $x_i(b)$ (or just $x_i$ when clear from the context) the $i$-th component of the 0-1 vector $x(b)$ and similarly for $p$. An allocation rule is called monotone if the function $x_i(z, b_{-i})$ is monotone non-decreasing in $z$ for any vector $b_{-i} \in V_{-i}$ and any bidder $i$.

We assume bidders have quasilinear utilities meaning that bidder’s $i$ utility for winning the auction and having to pay a price $p_i$ is $u_i = v_i - p_i$ and 0 if they do not win and pay nothing. Bidders are selfish agents aiming to maximize their own utility.

A mechanism is called truthful if bidding $b_i = v_i$ is a dominant strategy for each bidder, i.e. no bidder can increase their utility by reporting $b_i \neq v_i$ regardless the values and bids of the other bidders. An allocation rule $x$ is called implementable if there exists a payment rule $p$ such that $(x, p)$ is truthful. Such mechanisms are well understood and easy to reason about since we can predict how the bidders are going to behave. In what follows we focus our attention only on truthful mechanisms and thus use the terms value and bid interchangeably.

A well known result of Myerson ([Mye81]) states that a given allocation rule $x$ is implementable if and only if $x$ is monotone. In case $x$ is monotone, Myerson gives an explicit formula for the unique payment rule such that $(x, p)$ is truthful. In the single-parameter setting we’re studying, the payment rule can be informally described as follows: $p_i$ is equal to the minimum $b_i$ that bidder $i$ has to report such that they are included in the set of winners — we’ll refer to such a $b_i$ as the critical bid of bidder $i$.

The mechanism designer, who is collecting all the payments, commonly aims to maximize her expected revenue which for a mechanism $\mathcal{A}$ is defined as $\text{Rev}(\mathcal{A}) = \mathbb{E}_{b_i \sim F_i} \left[ \sum_{i \in [n]} p_i \right]$.

**Lemma 2 ([Mye81]).** For any truthful mechanism $(x, p)$ and any bidder $i \in [n]$: $\mathbb{E}[p_i] = \mathbb{E}[\phi_i(b_i) \cdot x_i(b_i, b_{-i})]$ where the expectations are taken over $b_1, \ldots, b_n \sim F_1, \ldots, F_n$, the function $\phi_i(\cdot)$ is defined as $\phi_i(z) = z - \frac{1 - F_i(z)}{f_i(z)}$ and $\phi_i(b_i)$ is called the virtual value of bidder $i$.

The importance of this lemma is that it reduces the problem of revenue maximization to that of virtual welfare maximization. More specifically, consider a sequence of distributions $F_1, \ldots, F_n$ which have the property that all $\phi_i$ are monotone non-decreasing (such distributions are called regular). In this case, the

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3Unique up to the normalizing assumption that $p_i = 0$ whenever $b_i = 0$. 
allocation rule that chooses a set of bidders with the maximum total virtual value (subject to feasibility constraints) is monotone (a consequence of the regularity condition) and thus implementable. We’ll frequently denote this revenue-maximizing mechanism by MyerOPT.

More precisely, the MyerOPT mechanism works as follows:

- Collect bids $b_i$ from every bidder $i \in [n]$.
- Compute $\phi_i(b_i)$ and discard all bidders whose virtual valuation is negative.
- Solve the optimization problem $S^* = \arg\max_{S \subseteq I} \sum_{i \in S} \phi_i(b_i)$.
- Allocate the items to $S^*$ and charge each bidder $i \in S^*$ their critical bid.

Handling non-regular distributions is possible using the standard technique of ironing. Very briefly, it works as follows. So far, we’ve been expressing virtual value functions: if we sample values, we sample quantiles $z$ for a given sample $F$. Now, suppose that $F$ is not affected when we let $q_i = \Pr_{b_i \sim F_i}[b_i \geq z]$. Another way to think of this is, instead of sampling values, we sample quantiles $q_i$ distributed uniformly at random in the interval $[0, 1]$ which are then transformed into values $v_i(q_i) = F_i^{-1}(1 - q_i)$. Let $R_i(q_i) = q_i \cdot v_i(q_i)$ and notice that $\phi_i(v_i(q_i)) = \frac{dR_i}{dq_i}|_{q=q_i}$. Now, since $v_i(\cdot)$ is a non-increasing function we have that $\phi_i(\cdot)$ is monotone if and only if $R$ is concave.

Now, suppose that $F_i$ is such that $R_i$ is not concave. One can consider the concave hull of $\overline{R}_i$ of $R_i$ which replaces $R_i$ with a straight line in every interval where $R_i$ was not following that concave hull. The corresponding function $\overline{\phi}_i(\cdot) = \frac{\overline{R}_i}{dq}$ is called ironed virtual value function.

**Lemma 3** (Har [Theorem 3.18]). For any monotone allocation rule $x$ and any virtual value function $\phi_i$ of bidder $i$, the expected virtual welfare of $i$ is upper bounded by their expected ironed virtual value welfare.

$$\mathbb{E}[\phi_i(v_i(q_i)) \cdot x_i(v_i(q_i), v_{-i}(q))] \leq \mathbb{E}[(\overline{\phi}_i(v_i(q_i))) \cdot x_i(v_i(q_i), v_{-i}(q))]$$

Furthermore, the inequality holds with equality if the allocation rule $x$ is such that for all bidders $i$, $x_i(\cdot) = 0$ whenever $\overline{R}_i(q) > R_i(q)$.

As a consequence, consider the monotone allocation rule which allocates to a feasible set of maximum total ironed virtual value. On the intervals where $\overline{R}_i(q) > R_i(q)$, $\overline{R}_i$ is linear as part of the concave hull so the ironed virtual value function, being a derivative of a linear function, is a constant. Therefore, the allocation rule is not affected when $q$ ranges in such an interval.

A crucial property of any (ironed) virtual value function $\phi$ corresponding to a distribution $F$ is that $z \geq \phi(z)$ for all $z$ in the support of $F$. This is obvious for $\phi$ as defined in Lemma 2. We claim it also holds for ironed virtual value functions: if $z$ lies in an interval where $\overline{\phi} = \phi$ it holds trivially. Otherwise, if $z \in [a, b]$ for some interval where $\phi$ needed ironing (i.e. $\overline{R}(q) > R(q)$ in the quantile space), we have: $z \geq a \geq \phi(a) = \overline{\phi}(a) = \overline{\phi}(z)$. We’ve thus proven:

**Proposition 3.** Any (possibly non-regular) distribution $F$ having an ironed virtual value function $\overline{\phi}$ satisfies:

$$z \geq \overline{\phi}(z)$$

for any $z$ in the support of $F$.

**Remark 1.** For simplicity, in the remainder of the paper we’ll use $\phi$ and $\overline{\phi}$ interchangeably and we will refer to $\phi$ as virtual value function. The reader should keep in mind that if the associated distribution is non-regular, then ironed virtual value functions should be used instead.

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4In general, $v_i(q_i) = \min\{v \mid F_i(v) \geq q_i\}$. 
3 Revenue Monotonicity on Matroid Markets

We extend the standard single-parameter environment to allow for bidders with misspecified distributions. Formally, the \( n \) bidders are partitioned into sets \( G \) and \( R \); the former are called green and the latter red. The color of each bidder (green or red) is not revealed to the mechanism designer at any point. Green bidders sample their values from their respective distribution \( F_i \) but red bidders are sampling \( v_i \sim F'_i \) for some \( \{F'_i\}_{i \in R} \) which are completely unknown to the mechanism designer and can be adversarially chosen.

In this section we are interested in studying the behavior of Myerson’s optimal mechanism when designed under the (wrong) assumption that \( v_i \sim F_i \) for all \( i \in [n] \). Specifically, we ask the question of whether the existence of the red bidders could harm the expected revenue of the seller compared to the case where the seller was able to identify and exclude the red bidders, thus designing the optimal mechanism for the green bidders alone. The following definition makes this notion of revenue monotonicity more precise.

**Definition 1 (RMMB).** Consider a single-parameter, downward-closed market \( M = (E, I) \) of \( |E| = n \) bidders. A mechanism \( \mathcal{A} \) is Revenue Monotone under Misspecified Bidders (RMMB) if for any distributions \( F_1, \ldots, F_n \), any number \( 1 \leq k \leq n \) of green bidders and any fixed misspecified bids \( b'_R \in \mathbb{R}^R \) of the red bidders:

\[
\mathbb{E} [\text{Rev}(\mathcal{A}(b_G, b_R))] \geq \mathbb{E} [\text{Rev}(\mathcal{A}(b_G))] \tag{1}
\]

where both expectation are taken over \( b_G \sim \prod_{i \in G} F_i \).

An alternative definition of the revenue monotonicity property allows red bidders to have stochastic valuations drawn from distributions \( F'_i \neq F_i \) instead of fixed bids. We note that the two definitions are equivalent: if \( \mathcal{A} \) is RMMB according to Definition 1 then inequality (1) holds pointwise for any fixed misspecified bids and thus would also hold in expectation. For the other direction, if inequality (1) holds in expectation over the red bids, regardless of the choice of distributions \( \{F'_i \mid i \in R\} \) then we may specialize to the case when each \( F'_i \) is a point-mass distribution with a single support point \( b_i \) for each \( i \in R \), and then Definition 1 follows.

In what follows we assume bidders always submit bids that fall within the support of their respective distribution. Green bidders obviously follow that rule and red bidders should do as well, otherwise the mechanism could recognize they are red and just ignore them.

Consider first the simpler case of selling a single item. This corresponds to a uniform rank 1 matroid market. Intuitively when the item is allocated to a green bidder, the existence of the red bidders is not problematic and in fact could help increase the critical bid and thus the payment of the winner. On the other hand, when a red bidder wins one has to prove that they are not charged too little and thus risk bringing the expected revenue down.

Let \( m = \max(\max_{i \in G} \phi_i(b)), 0 ) \) be the random variable denoting the highest non-negative virtual value in the set of green bidders. Let also \( X \) be an indicator a random variable which is 1 if the winner belongs to \( G \) and \( Y \) denote an indicator random variable which is 1 if the winner belongs to \( R \). For the mechanism \( \text{MyerOPT} \) have:

\[
\mathbb{E} [\text{revenue from green bidders}] = \mathbb{E} [m \cdot X] \tag{2}
\]
\[
\mathbb{E} [\text{revenue from red bidders}] \geq \mathbb{E} [m \cdot Y] \tag{3}
\]

where (2) follows from Myerson’s lemma and (3) follows from the observation that the winner of the optimal auction never pays less than the second-highest virtual value. To see why the latter holds, let \( f_s \)
be the second highest virtual value, \( r \) be the red winner and \( g \) is the green player with the highest virtual value. The critical bid of the red winner is at least \( \phi_r^{-1}(\phi_r(x)) \geq \phi_r^{-1}(\phi_g(b_g)) \geq \phi_g(b_g) \) where we applied the fact that \( x \geq \phi(x) \) to the virtual value function \( \phi = \phi_r \) and the value \( x = \phi_r^{-1}(\phi_g(b_g)) \).

Summing (2) and (3) and using the fact that \( X + Y = 1 \) whenever \( m > 0 \), we find:

\[
E \left[ \text{Revenue from all bidders } 1, \ldots, n \right] \geq E \left[ m \cdot X \right] + E \left[ m \cdot Y \right] = E \left[ m \right] = E \left[ \text{revenue of MyerOPT on } G \right]
\]

We therefore concluded that Myerson’s optimal mechanism is RMMB in the single-item case. We are now ready to generalize the above idea to any matroid market.

**Theorem 1.** Let \( M = (E, I) \) be any matroid market. Then MyerOPT in \( M \) is RMMB.

**Proof.** Call \( G \) the set of green bidders and \( R \) the set of red bidders. Let \((x, p)\) denote the allocation and payment rules for the mechanism MyerOPT that runs Myerson’s optimal mechanism on all \( n \) bidders, using the given distribution of each. Let \((x', p')\) denote the allocation and payment rules for the mechanism MyerOPT\(_G\) that runs Myerson’s optimal mechanism in the bidder set \( G \) only. For a set \( S \subseteq [n] \), let \( T_S \) be the random variable denoting the independent subset of \( S \) that maximizes the sum of ironed virtual values. In other words, \( T_S \) is the set of winners chosen by Myerson’s optimal mechanism on bidder set \( S \).

By Myerson’s Lemma, the revenue of MyerOPT\(_G\) satisfies:

\[
E \left[ \sum_{i \in G} p_i'(b) \right] = E \left[ \sum_{i \in G} x'_i(b) \cdot \phi_i(b) \right] \tag{4}
\]

By linearity of expectation, we can break up the expected revenue of MyerOPT into two terms as follows:

\[
E \left[ \sum_{i \in [n]} p_i(b) \right] = E \left[ \sum_{i \in G} p_i(b) \right] + E \left[ \sum_{i \in R} p_i(b) \right] \tag{5}
\]

The first term on the right side of (5) expresses the revenue original from the green bidders. Using Myerson’s Lemma, we can equate this revenue with the expectation of the green winners’ combined virtual value:

\[
E \left[ \sum_{i \in G} p_i(b) \right] = E \left[ \sum_{i \in G} x_i(b) \cdot \phi_i(b) \right] \tag{6}
\]

To express the revenue coming from the red bidders in terms of virtual valuations, we provide the argument that follows. One way to derive \( T_{G+R} \) from \( T_G \) is to start with \( T_G \) and sequentially add elements of \( T_{G+R} \cap R \) in arbitrary order while removing at each step the least weight element in the circuit that potentially forms (repeated application of Proposition \( 2 \)). Let \( e \) be the \( i \)-th red element we’re adding. If no circuit forms after the addition, then \( e \) pays the smallest value in its support which is a non-negative quantity. Otherwise, let \( C \) be the unique circuit that forms after that addition. Let \( f \) be the minimum weight element in \( C \) and let \( b_f \) be the associated bid made by player \( f \). Notice that if \( f \) must be green; by assumption, every red element we’re adding is part of the eventual optimal solution so it cannot be removed at any stage of this process.
The price charged to \( e \) is their critical bid which we claim is at least \( \phi_{e}^{-1}(\phi_{f}(b_f)) \). The reason is that \( e \) is part of circuit \( C \) and \( f \) is the min-weight element of that circuit. The min-weight element of a circuit is never in the max-weight independent set so if bidder \( e \) bids any value \( v \) such that \( \phi_{e}(v) < \phi_{f}(b_f) \) they will certainly not be included in the set of winners, \( T_{G+R} \). By Proposition it follows that \( \phi_{e}^{-1}(\phi_{f}(b_f)) \geq \phi_{f}(b_f) \) thus \( p_e(b) \geq \phi_f(b_f) \).

The above reasoning allows us “charge” each red bidder’s payment to a green player’s virtual value in \( T_G \setminus T_{G+R} \):

\[
\mathbb{E} \left[ \sum_{i \in R} p_i(b) \right] \geq \mathbb{E} \left[ \sum_{i \in T_G \setminus T_{G+R}} \phi_i(b_i) \right] = \mathbb{E} \left[ \sum_{i \in G} (x'_i(b) - x_i(b)) \cdot \phi_i(b_i) \right] \tag{7}
\]

The second line is justified by observing that for \( i \in G, x'_i(b) = x_i(b) \) unless \( i \in T_G \setminus T_{G+R} \), in which case \( x'_i(b) - x_i(b) = 1 \).

Combining Equations (4) - (7) we get:

\[
\mathbb{E} \left[ \sum_{i \in [n]} p_i(b) \right] \geq \mathbb{E} \left[ \sum_{i \in G} x_i(b) \cdot \phi_i(b_i) \right] + \mathbb{E} \left[ \sum_{i \in G} (x'_i(b) - x_i(b)) \cdot \phi_i(b_i) \right] = \mathbb{E} \left[ \sum_{i \in G} x'_i(b) \cdot \phi_i(b_i) \right] = \mathbb{E} \left[ \sum_{i \in G} p'_i(b) \right]
\]

In other words, the expected revenue of MyerOPT is greater than or equal to that of MyerOPT\(_G\).

\[\square\]

4 General Downward-Closed Markets

When the market is not a matroid, the existence of red bidders can do a lot of damage to the revenue of the mechanism as shown in the following simple example.

Example 1. Consider a 3-element downward-closed set system on \( E = \{a, b, c\} \) with maximal feasible sets: \( \{a, b\} \) and \( \{c\} \). Let \( c \) be a green bidder with a deterministic value of 1 and \( a, b \) be red bidders each with a specified value distribution given by the following cumulative distribution function \( F(x) = 1 - (1 + x)^{1-N} \) for some parameter \( N \). Note that the associated virtual value function is:

\[
\phi(x) = x - \frac{1 - F(x)}{f(x)} = x - \frac{(1 + x)^{1-N}}{(N - 1)(1 + x)^N} = x - \frac{1 + x}{N-1} = (1 - \frac{1}{N-1}) x - \frac{1}{N-1}.
\]

For this virtual value function we have \( \phi^{-1}(0) = \frac{1}{N-2}, \phi^{-1}(1) = \frac{N}{N-2} \).

Consider the revenue of Myerson’s mechanism when the red bidders, instead of following their specified distribution, they each bid \( \phi^{-1}(1) \) — and the green bidder bids 1, the only support point of their distribution.

---

\(5\)This is a consequence of the optimality of the Greedy algorithm since the min-weight element of a circuit is the last to be considered among the elements of the circuit and its inclusion will violate independence.
The set \( \{a, b\} \) wins over \( \{c\} \) since the former sums to a total virtual value of \( 2 \) over the latter’s virtual value \( 1 \) so bidders \( a, b \) pay their critical bid.

To compute that, notice that each of the bidders \( a, b \) could unilaterally decrease their bid to any \( \varepsilon > \frac{1}{N-2} \) and they would still win the auction since the set \( \{a, b\} \) would still have a total virtual value greater than \( 1 \). Therefore, each of \( a, b \) pays \( \frac{1}{N-2} \) for a total revenue of \( \frac{2}{N-2} \).

On the other hand, the same mechanism when run on the set \( \{c\} \) of only the green bidder, always allocates an item to \( c \) and collects a total revenue of \( 1 \).

Letting \( N \to \infty \) we see that the former revenue tends to zero while the latter remains \( 1 \), violating the revenue monotonicity property of Definition\(^1\) by an unbounded multiplicative factor.

To generalize the above idea to any non-matroid set system we need the following lemma.

**Lemma 4.** A downward-closed set system \( S = (E, \mathcal{I}) \) is not a matroid if and only if there exist \( I, J \in \mathcal{I} \) with the following properties:

1. For every \( K \in \mathcal{I}|_{I\cup J} \), if \( |K| \geq |I| \) then \( K \supseteq I \setminus J \).
2. \( |J\setminus I| \geq 1 \).
3. \( I \) is a maximum cardinality element of \( \mathcal{I}|_{I\cup J} \).

**Proof.** For the forward direction, suppose \( S \) is not a matroid and let \( V \) be a minimum-cardinality subset of \( E \) that is not a matroid. Since \( \mathcal{I}|_V \) is downward-closed and non-empty, it must violate the exchange axiom. Hence, there exist sets \( I, J \in \mathcal{I}|_V \) such that \( |I| > |J| \) but \( J + x \notin I \) for all \( x \in I \setminus J \). Note that \( V = I \cup J \), since otherwise \( I \cup J \) is a strictly smaller subset of \( E \) satisfying the property that \( (I \cup J, \mathcal{I}|_{I\cup J}) \) is not a matroid.

Observe that \( J \) is a maximal element of \( \mathcal{I}|_V \). The reason is that \( V = I \cup J \), so every element of \( V \setminus J \) belongs to \( I \). By our assumption on the pair \( I, J \), there is no element \( y \in I \) such that \( J + y \in I \setminus J \). Since \( \mathcal{I}|_V \) is downward-closed, it follows that no strict superset of \( J \) belongs to \( \mathcal{I}|_V \).

We now proceed to prove that \( I, J \) satisfy the required properties of the lemma:

1. Let \( K \in \mathcal{I}|_V \) with \( |K| \geq |I| \). It follows that \( |K| > |J| \) but \( J \) is maximal in \( \mathcal{I}|_V \), so \( K \) and \( J \) must violate the exchange axiom. Thus, \( \mathcal{I}|_{K\cup J} \) is not a matroid. By the minimality of \( V \), this implies \( K \cup J = V \) hence \( K \supseteq I \setminus J \).
2. If \( J \setminus I = \emptyset \) then \( J \subseteq I \) which contradicts the fact that \( I, J \) violate the exchange axiom.
3. Suppose there exists \( I' \in \mathcal{I}|_V \) with \( |I'| > |I| \), then by property 1 we have \( I' \supseteq I \setminus J \). Remove elements of \( I \setminus J \) from \( I' \) one by one, in arbitrary order, until we reach a set \( K \in \mathcal{I}|_V \) such that \( |K| = |I| \). This is possible because after the entire set \( I \setminus J \) is removed from \( I' \), what remains is a subset of \( J \), hence has strictly fewer elements than \( I \). The set \( K \) thus constructed has \( |K| = |I| \) but \( K \notin I \setminus J \), violating property 1.

For the “only if” direction, supposing that \( S \) is a matroid, we must show that no \( I, J \in \mathcal{I} \) satisfy all three properties. To this end, suppose \( I \) and \( J \) satisfy 2 and 3. Since \( S|_{I\cup J} \) is a matroid, there exists \( K \supseteq J \) such that \( K \in \mathcal{I}|_{I\cup J} \) and \( |K| = |I| \). By property 2, we know that no \( |I| \)-element superset of \( J \) contains \( I \setminus J \) as a subset. Therefore, the set \( K \) violates property 1. \( \square \)

We are now ready to generalize Example\(^1\) to every non-matroid set system.
Theorem 2. For any $\mathcal{M} = (E, I)$ which is not a matroid, MyerOPT is not RMMB.

Proof. Consider a downward-closed $\mathcal{M} = (E, I)$ which is not a matroid. We are going to show there exists a partition of players into green and red sets and a choice of valuation distributions and misspecified red bids such that the RMMB property is violated.

Let $I, J \subseteq E$ be the subsets whose existence is guaranteed by Lemma 4. Define $G = J$ to be the set of green bidders, $R = I \setminus J$ to be the set of red bidders. All other bidders are irrelevant and can be assumed to be bidding zero. Set the value of each green bidder to be deterministically equal to 1. For each red bidder $r$, the specified value distribution has the same cumulative distribution function $F(x) = 1 - (1 + x)^{1-N}$ defined in Example 1.

Now let’s consider the expected revenue of Myerson’s mechanism when every bidder in $R$ bids $\phi^{-1}(1)^{6}$

Every bidder’s virtual value is 1, so the mechanism will choose any set of winners with maximum cardinality which, according to Lemma 4, property (3), is $|I|$. For example, the set of winners could be $J$.

A consequence of Lemma 4, property (1) is that for every red bidder $r$ there is no set of bidders disjoint from $\{r\}$ with combined virtual value greater than $|I| - 1$. Thus each red bidder pays $\phi^{-1}(0)$. Elements of $I \cap J$ correspond to green bidders who win the auction and pay 1, because a green bidder pays 1 whenever they win. There are $|I \cap J|$ such bidders. Thus, the Myerson revenue is $|I \cap J| + \frac{1}{N-2}|J \setminus I|$. The optimal auction on the green bidders alone charges each of these bidders a price of 1, receiving revenue $|J| = |I \cap J| + |J \setminus I|$. This exceeds $|I \cap J| + \frac{1}{N-2}|J \setminus I|$ as long as

\[(N - 2) \cdot |J \setminus I| > |I \setminus J|.\]  

This inequality is satisfied, for example, when $N = |I \setminus J| + 3$, because $J \setminus I$ has at least one element (Lemma 4, property (2)).

5 Open Questions

The previous section concluded with a proof that for any non-matroid system, the ratio $r = \frac{E[\text{Rev}(G)]}{E[\text{Rev}(G \cup R)]}$ for Myerson’s optimal mechanism can be greater than 1. An interesting question is whether that ratio can be made arbitrarily large as in Example 1. If the sets $I, J$ in the above proof are such that $I \cap J = \emptyset$, then the ratio can be made unbounded with the same construction. We do not know if possibly another choice of red/green bidders and their distributions can give an unbounded ratio for all non-matroid system.

A broader question our work leaves unanswered is whether it’s possible to design other mechanisms (potentially non-truthful) that, in the presence of red and green bidders in non-matroid downward-closed market, can always guarantee a constant approximation to Myerson’s revenue on the green bidders alone. For instance, in Example 1 one could possibly consider randomized mechanisms that ignore a random bidder in the set \{a, b\} before running Myerson’s auction.

6Members of $G$ bid as well, but it hardly matters, because their bid is always 1 — the only support point of their value distribution — so the auctioneer knows their value without their having to submit a bid.

7In general, the mechanism might choose any set $W$ of winners such that $|W| = |I|$. The way to handle this case is similar to the one used in the proof of Theorem 3 in the Appendix.
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A Missing proofs

Here we provide a proof of Proposition 2 for updating the optimal solution of a weighted matroid:

**Proof.** Consider running the Greedy algorithm in parallel on both $M_{|E|-1}$ and $M$ and call these executions $E^-, E$ respectively.

In case $(I+x) \in I$, the downward-closed property of $M$ guarantees that both executions will make identical decisions on elements other than $x$ and element $x$ will be included in the optimal solution of $M$, hence $I^* = I + x$.

For the other case, suppose first that $x = y$, i.e. $x$ is the min-weight element on $C$. At the time $x$ is inspected, all other elements of $C$ have already been inspected and added to the current solution, hence $x$ is not included since it would violate independence. Therefore, both executions proceed making identical decisions in every step and arrive to the same solution $I^* = (I + x) - x = I$.

Now suppose that $x \neq y$. At the time element $x$ is considered in $E$, it can be safely included in the solution. The reason is that if adding $x$ resulted in a circuit $C'$, then $C' \subseteq C$ violating the minimality of $C$. The next step at which the two executions will diverge again is when considering $y$—if they diverged at a previous step it would again mean that $x$ is part of a circuit $C' \subseteq C$—at this point $E$ ignores $y$.

Finally, suppose that the two executions diverge at a later step on an element $e$ with $w(e) < w(y)$. Denote by $J$ the current solution $E^-$ is maintaining and thus $(J + x) - y$ is the current solution of $E$. There are two reasons the executions might diverge:

- $(J + e) \in I$ but $((J + x) - y) + e \notin I$.
  
  In this case, there must exist circuit $C' \subseteq ((J + x) - y) + e$ such that $x, e \in C'$ and $y \notin C'$. Therefore, by Proposition 1 there exists circuit $C''$ such that $e \in C'' \subseteq (C' \cup C) - x$. This is a contradiction because $C''$ is a circuit of $J + e$ which was assumed to be independent.

- $(J + e) \notin I$ but $((J + x) - y) + e \in I$.

  This case is similar and the proof is omitted.

\[\blacksquare\]

B A note on revenue monotonicity of VCG

A revenue monotonicity result similar to ours is proven for VCG in matroid markets in [DRS09]. We noticed that one of the propositions used in the proof of that theorem is incorrect. Here we provide a counter-example and offer an alternative proof using our Lemma 4.

Proposition 2.9 of [DRS09] claims that “A downward-closed set system $(U, I)$ with $I \neq \emptyset$ is a matroid if and only if for every pair $A, B$ of maximal sets in $I$ and $y \in B$, there is some $x \in A$ such that $A \setminus \{x\} \cup \{y\} \in I$.”

Here we notice that the “backward” direction of this proposition does not hold.

Consider the following counter-example. Let $U = \{a, b, c, d, e\}$ and define $I_3$ to be the independent sets of the uniform rank 3 matroid on the 4-element subset $\{a, b, c, d\}$. Now let $I$ be the downwards-closed closure of $I_3 \cup \{\{a, c, e\}, \{b, d, e\}\}$. We claim $S = (U, I)$ violates the above proposition.

- $(U, I)$ is not a matroid: This is easy to see as $I = \{a, c, e\}, J = \{d, e\}$ violate the exchange property.
• Nevertheless, for every pair \( A, B \) maximal sets in \( I \) and \( y \in B \), there is some \( x \in A \) such that \( A \setminus \{ x \} \cup \{ y \} \in I \).

First, notice that is suffices to show this for all \( y \in B \setminus A \). Otherwise, if \( y \in A \cap B \) then set \( x = y \) in which case \( A \setminus \{ x \} \cup \{ y \} = A \in I \).

A second observation is that if both \( A, B \) are contained in \( \{ a, b, c, d \} \) then the statements holds; after all the forward direction of the proposition holds and the restriction of \( S \) on that 4-element subset is a matroid.

Finally, notice that \( S \) is symmetric under a permutation that swaps the roles of \( a \leftrightarrow b \) and \( c \leftrightarrow d \).

The following table summarizes a case analysis on the choice of \( A, B, y \) and provides a choice of \( x \) for each that satisfy the aforementioned property. The cases that are missing are equivalent to one of the cases in the table under symmetry.

| \( A \)     | \( B \)     | \( y \) | \( x \) |
|------------|------------|--------|--------|
| \( \{a, c, e\} \) | \( \{b, d, e\} \) | \( b \) | \( e \) |
| \( \{a, c, e\} \) | \( \{a, c, d\} \) | \( d \) | \( e \) |
| \( \{a, c, e\} \) | \( \{b, c, d\} \) | \( b \) | \( e \) |
| \( \{a, c, d\} \) | \( \{a, c, e\} \) | \( e \) | \( d \) |
| \( \{b, c, d\} \) | \( \{a, c, e\} \) | \( a \) | \( b \) |
| \( \{b, c, d\} \) | \( \{a, c, e\} \) | \( e \) | \( c \) |

We now turn into providing a proof of the “only if” direction of Theorem 4.1 in [DRS09].

We use the notation \( 1[S] \) for any subset \( S \subseteq U \) of bidder to mean the bidding profile where every bidder in \( S \) bids 1 and every bidder in \( U \setminus V \) bids 0.

**Theorem 3** ("only if" direction of [DRS09 Theorem 4.1]). Let \( (U, I) \) be a downward-closed set system that is not a matroid, then there exists a set \( V \subseteq U \) and an element \( x \in V \) such that the revenue of VCG on bid profile \( 1[V \setminus x] \) exceed the revenue of VCG on bid profile \( 1[V] \).

**Proof.** By Lemma 3 there exist \( I, J \in I \) with properties 1-3. Let \( V = I \cup J \) and let \( x \) be an arbitrary element of \( I \setminus J \). We will prove that the revenue of VCG on \( 1[V] \) is less than the revenue of VCG on \( 1[V \setminus x] \).

Consider the VCG payment of every bidder in the bid profile \( 1[V] \). Since \( I \) is a maximum cardinality element of \( I|V \) (Lemma 3 property 3), we may choose \( I \) as the set of winners. Let \( W \) denote the intersection of all elements of \( I|V \) that have cardinality \( |I| \). By property 1 of Lemma 3, \( I \setminus J \subseteq W \). Every element of \( W \) pays zero, because for \( y \in W \) the maximum cardinality elements of \( I|V \) have size \( |I| - 1 \), hence \( y \) could bid zero and still belong to a winning set. On the other hand, every element \( y \in I - W \) pays 1, because by the definition of \( W \) there is a set \( K \in I|V \) such that \( |K| = |I| \) but \( y \not\in K \). If \( y \) lowers its bid below 1, then \( K \) rather than \( I \) would be selected as the set of winners, hence \( y \) must pay 1 in the VCG mechanism. Finally, bidders not in \( I \) pay zero because they are not winners. The VCG revenue on bid profile \( 1[V] \) is therefore \( |I \setminus W| \).

Now recall that \( x \) denotes an arbitrary element of \( I \setminus J \), and consider the VCG payment of every bidder in the bid profile \( 1[V \setminus x] \). Since \( V \setminus x \) is a proper subset of \( V \), \( (V \setminus x, I|V \setminus x) \) is a matroid. The rank of this matroid is \( |I| - 1 \), since Lemma 3 property 1 implies that \( I|V \setminus x \) contains no sets of size \( |I| \). We may assume that \( I \setminus x \) is chosen as the set of winners of VCG. Let \( J' \) denote a superset of \( J \) that is a basis of \( (V \setminus x, I|V \setminus x) \).

If \( y \) is an element of \( (I \setminus J') \setminus x \), the set \( I \setminus x \setminus y \) has strictly fewer elements than \( J' \) so the exchange axiom implies there is some \( z \in J' \) such that \( I \setminus x \setminus y + z \in I \). This set \( I \setminus x \setminus y + z \) is a basis of \( (V \setminus x, I|V \setminus x) \) that...
does not contain $y$, hence the VCG payment of any $y \in (I \setminus J') - x$ is 1. Now consider any $y \in I \setminus W$. By the definition of $W$, there is a set $K \in I|V$ such that $|K| = |I|$ but $y \notin K$. Then $K - x$ is a basis of $(V - x, I|_{V-x})$ but $y \notin K$, implying that $y$’s VCG payment is 1.

We have shown that in the bid profile $1[V]$, the bidders in $I \setminus W$ pay 1 and all other bidders pay zero, whereas in the bid profile $1[V - x]$, the bidders in $I \setminus W$ still pay 1 and, in addition, the bidders in $(I \setminus J') - x$ pay 1. Furthermore the set $(I \setminus J') - x$ is non-empty. To see this, observe that $|J'| = |I| - 1 = |I - x|$, but $J' \neq I - x$ because then $J$ would be a subset of $I$, contrary to our assumption that $I, J$ satisfy Lemma 4 property (2). Hence, $I - x$ contains at least one element that does not belong to $J'$, meaning $(I \setminus J') - x$ is nonempty. We have thus proven that the VCG revenue of $1[V - x]$ exceeds the VCG revenue of $1[V]$ by at least $|I - J' - x|$, which is at least 1.

□