A direct method of solution for the Fokas–Lenells derivative nonlinear Schrödinger equation: I. Bright soliton solutions

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Abstract

We develop a direct method of solution for finding the bright N-soliton solution of the Fokas–Lenells derivative nonlinear Schrödinger equation. The construction of the solution is performed by means of a purely algebraic procedure using an elementary theory of determinants and does not rely on the inverse scattering transform method. We present two different expressions of the solution both of which are expressed as a ratio of determinants. We then investigate the properties of the solutions and find several new features. Specifically, we derive the formula for the phase shift caused by the collisions of bright solitons.

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1. Introduction

In this study, we consider the following Fokas–Lenells (FL) derivative nonlinear Schrödinger (DNLS) equation:

\[ u_{xt} = u - 2i|u|^2u_x, \]  

(1.1)

where \( u = u(x, t) \) is a complex-valued function of \( x \) and \( t \), and subscripts \( x \) and \( t \) appended to \( u \) denote partial differentiations. The original version of the FL DNLS equation has been derived as an integrable generalization of the NLS equation using bi-Hamiltonian methods [1]. Equation (1.1) is a gauge transformed form of it. We employ the above simplified equation for convenience. Recently, the equation was shown to model the propagation of nonlinear light pulses in monomode optical fibers when certain nonlinear effects are taken into account [2]. It admits a Lax pair which enables one to solve the initial value problem by means of the inverse scattering transform (IST) method [3]. Actually, a few conservation laws and a bright one-soliton solution were obtained explicitly in [3]. Subsequently, the general bright \( N \)-soliton solution was constructed with the use of the dressing method [4], where \( N \) is an arbitrary positive integer. Note, however, that the detailed analysis of the multisoliton solutions
has not been undertaken. For the purpose of obtaining soliton solutions in a simple manner, one can employ a direct approach which is now known as Hirota’s direct method or bilinear transformation method [5, 6]. The method has been applied successfully to a large number of soliton equations including the Korteweg–de Vries, NLS and sine-Gordon equations.

The purpose of this paper is to construct the bright $N$-soliton of the FL DNLS equation within the framework of the bilinear formalism. In particular, we present two different expressions of the bright $N$-soliton solution in a simple determinantal form. The proof of the solution can be obtained by a straightforward calculation where one relies only on a few basic formulas for determinants. This paper is organized as follows. In section 2, we develop a direct method of solution. Specifically, we first transform equation (1.1) to a system of bilinear equations by introducing an appropriate dependent variable transformation. Notably, we show that one of the bilinear equations can be replaced by a trilinear equation which is easy to prove when compared with other bilinear equations. In section 3, we present the bright $N$-soliton solution of the bilinear equations. It has a simple structure expressed in terms of certain determinants. Subsequently, we perform the proof of the bright $N$-soliton solution using an elementary theory of determinants in which Jacobi’s identity will play a central role. Last, we provide an alternative expression of the bright $N$-soliton solution. The equivalence of both expressions is discussed by referring to the properties of the Cauchy matrix. We emphasize that the bright $N$-soliton solution obtained here yields the bright $N$-soliton solution of the derivative NLS equation by simply replacing the time dependence of the solution. This connection has also been found in the context of the IST [4]. In section 4, we investigate the properties of the one- and two-soliton solutions and then analyze the asymptotic behavior of the bright $N$-soliton solution in the limit of infinite time. As a result, we obtain the explicit formula for the phase shift in terms of the amplitude parameters of solitons. Section 5 is devoted to concluding remarks.

2. Exact method of solution

In this section, we develop a direct method of solution for constructing bright soliton solutions of equation (1.1) under the boundary condition $u \to 0$ as $|x| \to \infty$. Specifically, we show that equation (1.1) can be transformed to a system of bilinear equations. We also demonstrate that one of the bilinear equations can be replaced by a trilinear equation.

2.1. Bilinearization

The bilinearization of equation (1.1) is established by the following proposition.

**Proposition 2.1.** By means of the dependent variable transformation

$$u = \frac{g}{f},$$

equation (1.1) can be decoupled into the following system of bilinear equations for the tau functions $f$ and $g$:

$$D_x D_t g \cdot f = gf,$$  \hfill (2.2)

$$D_t f \cdot f^* = igg^*,$$  \hfill (2.3)

$$D_x D_t f \cdot f^* = iD_x g \cdot g^*.$$  \hfill (2.4)
Here, \( f = f(x, t) \) and \( g = g(x, t) \) are the complex-valued functions of \( x \) and \( t \), and the asterisk appended to \( f \) denotes complex conjugate and the bilinear operators \( D_x \) and \( D_t \) are defined by

\[
D^m_x D^n_t f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t)g(x', t') \bigg|_{x'=x, t'=t}, \tag{2.5}
\]

where \( m \) and \( n \) are non-negative integers.

**Proof.** Substituting (2.1) into (1.1) and rewriting the resultant equation in terms of the bilinear operators, equation (1.1) can be rewritten as

\[
\frac{1}{f^2} (D_x D_t g \cdot f - gf) - \frac{g}{f^3} (f^* D_x D_t f \cdot f + 2i g^* D_x f \cdot g) = 0. \tag{2.6}
\]

Inserting the identity

\[
f^* D_x D_t f \cdot f = f D_x D_t f \cdot f^* - 2f_i D_x f \cdot f^* + f (D_x f \cdot f^*)_x, \tag{2.7}
\]

which can be verified by direct calculation into the second term on the left-hand side of (2.6), one modifies it in the form

\[
\frac{1}{f^2} (D_x D_t g \cdot f - gf) - \frac{g}{f^3} (f (D_x D_t f \cdot f^* - i D_x g \cdot g^*))
- 2f_i (D_x f \cdot f^* - i g^*) + f (D_x f \cdot f^* - i g^*)_x = 0. \tag{2.8}
\]

By virtue of equations (2.2)–(2.4), the left-hand side of (2.8) vanishes identically. \( \square \)

It follows from (2.1) and (2.3) that

\[
|u|^2 = \frac{i}{\partial t} \ln \frac{f^*}{f}. \tag{2.9}
\]

The above formula gives the modulus of \( u \) in terms of the tau function \( f \).

### 2.2. Remark

**Proposition 2.2.** The trilinear equation for \( f \) and \( g \),

\[
f^* (g_x f - g f_x - g f) = f^*_x (g_x f - g f_x), \tag{2.10}
\]

is a consequence of the bilinear equations (2.2)–(2.4).

**Proof.** Using the definition of the bilinear operators, one can show that

\[
f^* (g_x f - g f_x - g f) - f^*_x (g_x f - g f_x)
= f^* (D_x D_t g \cdot f - gf) - \frac{g}{2} [(D_x f \cdot f^* - ig^*)_x + (D_x D_t f \cdot f^* - i D_x g \cdot g^*)]
+ g_x (D_x f \cdot f^* - ig^*). \tag{2.11}
\]

The right-hand side of (2.11) becomes zero by (2.2)–(2.4), which yields (2.10). \( \square \)

In view of proposition 2.2, we can use equation (2.10) in place of (2.2), for example in performing the proof of the bright \( N \)-soliton solution. We will see later in section 3.6 that the proof of (2.10) is simplified considerably compared with that of (2.2), even if (2.10) in not bilinear but trilinear in \( f \) and \( g \).
3. Bright N-soliton solution and its proof

In this section, we show that the tau functions \( f \) and \( g \) representing the bright \( N \)-soliton solution admit the compact determinantal expressions. This statement is proved by an elementary calculation using the basic formulas for determinants. Then, we provide an alternative form of the bright \( N \)-soliton solution. Last, we demonstrate that the tau functions \( f \) and \( g \) satisfy a trilinear equation. The implication of this result will be discussed briefly in conjunction with the bright \( N \)-soliton solution of the DNLS equation.

3.1. Bright \( N \)-soliton solution

The main result in this paper is given by the following theorem.

**Theorem 3.1.** The bright \( N \)-soliton solution of the system of bilinear equations (2.2)–(2.4) is expressed by the following determinants:

\[
f = |D|, \quad g = \begin{vmatrix} D & z^T \\ a & 1 \\ b & 0 \end{vmatrix}.
\] (3.1)

Here, \( D \) is an \( N \times N \) matrix and \( z, z_0, \) and \( 1 \) are \( N \)-component row vectors defined below and the symbol \( T \) denotes the transpose:

\[
D = (d_{jk})_{1 \leq j, k \leq N}, \quad d_{jk} = \frac{z_j^2 - i p_j^* x}{p_j + p_k^*}, \quad z_j = \exp \left( \frac{p_j x + \frac{1}{p_j} t + \xi_j}{1} \right), \quad (3.2a)
\]

\[
z = (z_1, z_2, \ldots, z_N), \quad \mathbf{z}_0 = (z_1/p_1, z_2/p_2, \ldots, z_N/p_N), \quad 1 = (1, 1, \ldots, 1), \quad (3.2b)
\]

where \( p_j \) and \( \xi_j \) \( (j = 1, 2, \ldots, N) \) are arbitrary complex parameters.

The bright \( N \)-soliton solution is parameterized \( 2N \) complex parameters \( p_j \) and \( \xi_j \) \( (j = 1, 2, \ldots, N) \). The parameters \( p_j \) determine the amplitude and velocity of the solitons, whereas the parameters \( \xi_j \) determine the phase of the solitons. The conditions \( p_j + p_k^* \neq 0 \) for all \( j \) and \( k \) and \( p_j \neq p_k \) for \( j \neq k \) may be imposed on the parameters to assure the regularity of the solution. We point out that the \( N \)-soliton formula given by theorem 3.1 can be shown to take the same form as that obtained in [4] by changing the soliton parameters \( p_j \) and \( \xi_j \) \( (j = 1, 2, \ldots, N) \).

3.2. Notation and basic formulas for determinants

First, we define the definition of matrices associated with the bright \( N \)-soliton solution and then provide some basic formulas for determinants. The following bordered matrices appear frequently in our analysis:

\[
D(a; b) = \begin{pmatrix} D & b^T \\ a & 0 \end{pmatrix}, \quad D(a, b; c, d) = \begin{pmatrix} D & c^T & d^T \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix},
\] (3.3)

where \( a, b, c \) and \( d \) are \( N \)-component row vectors. Let \( D_{jk} \) be the co-factor of the element \( d_{jk} \).

The following formulas are well known in the theory of determinants:

\[
\frac{\partial}{\partial x} |D| = \sum_{j,k=1}^{N} \frac{\partial d_{jk}}{\partial x} D_{jk}, \quad (3.4)
\]

\[
\begin{vmatrix} D & a^T \\ b & z \end{vmatrix} = |D| z - \sum_{j,k=1}^{N} D_{jk} a_j b_k, \quad (3.5)
\]

\[
|D(a, b; c, d)||D| = |D(a; c)||D(b; d)| - |D(a; d)||D(b; c)|. \quad (3.6)
\]
Formula (3.4) is the differentiation rule of the determinant and (3.5) is the expansion formula for a bordered determinant with respect to the last row and last column. Formula (3.6) is Jacobi’s identity.

3.3. Differentiation rules and related formulas

In terms of the notation (3.3), the tau functions $f$ and $g$ can be written as $f = |D|$ and $g = |D(1; z)|$, respectively. The differentiation rules of the tau functions with respect to $t$ and $x$ are given by the following formulas.

**Lemma 3.1.**

\[
\begin{align*}
ft & = -|D(z^*_t; z_t)|, \\
fx & = -|D(z^*; z)|, \\
fxt & = -|D(z^*_t; z)| - |D(z^*_t; z_t)| + |D(z^*_t; z_t)|,
\end{align*}
\]

(3.7)\(\text{—}\) (3.9)

\[
\begin{align*}
gt & = |D(1; z_t)|, \\
gx & = |D(1; z)| - |D(1, z^*_t; z)|, \\
gxt & = |D(1; z^*_t)| - |D(1, z^*_t; z)|.
\end{align*}
\]

(3.10)\(\text{—}\) (3.12)

**Proof.** We prove (3.7). Applying formula (3.4) to $f$ given by (3.1) with (3.2a), one obtains

\[
f_t = \sum_{j,k=1}^{N} D_{jk} \frac{z_j z^*_k}{p_j p^*_k} = \sum_{j,k=1}^{N} D_{jk} z_j z^*_k,
\]

where in passing to the second line, use has been made of the relations $z_j / p_j = z_j^*$, $z^*_k p^*_k = z^*_k$. Referring to formula (3.5) with $z = 0$ and taking into account the notation (3.3), the above expression becomes the right-hand side of (3.7). Formulas (3.8)–(3.12) can be proved in the same way if one uses (3.4), (3.5) and the relation $z_{xt} = z$ as well as some basic properties of determinants. □

The complex conjugate expressions of the tau functions $f$ and $g$ and their derivatives are expressed as follows.

**Lemma 3.2.**

\[
\begin{align*}
f^* & = |\bar{D}|, \quad \bar{D} = D^{*T}, \\
f^*_t & = -|\bar{D}(z^*_t; z_t)|, \\
f^*_x & = -|\bar{D}(z^*; z)|, \\
f^*_{xt} & = -|\bar{D}(z^*_t; z)| - |\bar{D}(z^*_t; z_t)| + |\bar{D}(z^*_t; z_t)|, \\
g^* & = |\bar{D}(z^*_t)|, \\
g^*_x & = |\bar{D}(z^*; 1)| - |\bar{D}(z^*_t; z)|, \\
g^*_{xt} & = |\bar{D}(z^*_t; 1)| - |\bar{D}(z^*_t, z^*_t; 1, z)|,
\end{align*}
\]

(3.13)\(\text{—}\) (3.18)

where $\bar{D} = (\bar{d}_{jk})_{1\leq j,k\leq N}$ is an $N \times N$ matrix with elements $\bar{d}_{jk} = d_{jk} + i$. 5
**Proof.** It follows from (3.2a) that \( d_{jk}^* = d_{kj} + i \) or equivalently, \( D^* = \bar{D}^T \), which proves (3.13) since \( f^* = |D^*| = |\bar{D}^T| = |\bar{D}| \). Formulas (3.14)–(3.18) can be proved in the same way. □

The formulas below are used to reduce the proof of the bright \( N \)-soliton solution to Jacobi’s identity.

**Lemma 3.3.**

\[
|D(1; z)| = |\bar{D}(1; z)|, \quad (3.19)
\]

\[
|D(1; z_j)| = |\bar{D}(1; z_j)| = (-1)^{N-1}c|\bar{D}(\tilde{p}^*; z)|, \quad (3.20)
\]

\[
|D(z^*; z)| = (-1)^{N-1}c|\bar{D}(z^*; z)|, \quad (3.21)
\]

\[
|D(z^*; z_n)| = (-1)^{N-1}c|\bar{D}(z^*; z)|, \quad (3.22)
\]

\[
|D(z^*; z_m; z)| = (-1)^{N-1}c|\bar{D}(z^*; z)|, \quad (3.23)
\]

\[
|D(z^*; 1; z; \tilde{p})| = (-1)^Nc|\bar{D}(z^*; \tilde{p}; z, 1)|, \quad (3.24)
\]

\[
|D(1, z^*; z, z)| = |\bar{D}(1, z^*; z, z)|, \quad (3.25)
\]

where \( \tilde{p} = (1/p_1, 1/p_2, \ldots, 1/p_N) \) and \( z_n = (p_1z_1, p_2z_2, \ldots, p_Nz_N) \) are \( N \)-component row vectors and \( c = \prod_{j=1}^N(p_j^*/p_j) \).

**Proof.** We prove (3.21) only. The other formulas can be proved following the same procedure as that described below. First, using the key relation which follows from (3.2a) and the definition of \( \bar{d}_{jk} \),

\[
d_{jk} - \frac{z_j^* z_k^*}{p_j} = -\frac{\rho_j}{p_j}(d_{jk} + i) = -\frac{\rho_j}{p_j}d_{jk}, \quad (3.26)
\]

the left-hand side of (3.21) can be modified into the form

\[
|D(z^*; z)| = \left| \begin{array}{c}
\left( -\frac{\rho_j}{p_j}d_{jk} \right) \\
\cdots \\
0
\end{array} \right| \tilde{z}, \quad (3.27)
\]

after multiplying the \((N+1)\)th row by \(-z_j/p_j\) and then adding the resultant expression to the \( j \)th row for \( j = 1, 2, \ldots, N \). Formula (3.21) follows from (3.27) if one extracts the factors \( 1/p_j \) and \(-p_j^*\) from the \( j \)th row and \( k \)th column, respectively, for \( j, k = 1, 2, \ldots, N \) and the factor \(-1\) from the \((N+1)\)th row. □

### 3.4. Proof of the bright \( N \)-soliton solution

#### 3.4.1. Proof of (2.2).

Let

\[
P_1 = D_1 D_2 g \cdot f - gf = g_{\alpha f} f - g_{\alpha f} f - g_{f x} + gf_{\alpha x} - gf. \quad (3.28)
\]

Substituting formulas (3.7)–(3.12) into this expression and applying Jacobi’s identity to a term \(|D(1, z^*; z, z)|D|\), \( P_1 \) reduces to

\[
P_1 = P_{11} + P_{12} + P_{13}, \quad (3.29a)
\]

with

\[
P_{11} = |D(1; z)| |D(z^*; z)| - |D(1; z)| |D(z^*; z)|. \quad (3.29b)
\]
Consider the identity

$$P_{12} = |D(1; z_1)| |D(z^*; z_1; z, z_2)| - |D(z^*_1; z_1)| |D(1; z^*_1; z, z)|,$$

(3.29c)

$$P_{13} = |D(1; z)| |D(z^*; z_0)| - |D(1; z_0)| |D(z^*; z)|.$$

(3.29d)

Referring to Jacobi’s identity, $P_{11}$ becomes

$$P_{11} = |D||D(1, z^*_2; z, z_1)|.$$

(3.30)

Consider the identity

$$
\begin{vmatrix}
|D(1; z_1)| & |D(1; z_2)| & |D(1; z)| \\
|D(z^*_1; z_1)| & |D(z^*_2; z_1)| & |D(z^*_1; z)| \\
|D(z^*_1; z_2)| & |D(z^*_2; z_1)| & |D(z^*_2; z)| \\
\end{vmatrix} = 0,
$$

(3.31)

which follows immediately since the first two columns of the determinant coincide. Expanding the above determinant with respect to the first column and using Jacobi’s identity, one finds that

$$P_{12} = |D(z^*_1; z)| |D(1, z^*_2; z, z_1)|.$$

(3.32)

Gathering up three terms (3.29d), (3.30) and (3.32), $P_1$ simplifies to

$$P_1 = \{|D| + |D(z^*_1; z_1)|\} |D(1, z^*_2; z, z_1)| + |D(1; z)| |D(z^*_2; z_1)| - |D(1; z_0)| |D(z^*_2; z)|.$$

(3.33)

It follows from formula (3.5) that

$$|D| + |D(z^*_1; z_1)| = \left| \begin{array}{cc}
D & z^*_1 \\
& 1 \end{array} \right| = (-1)^N c |\tilde{D}|,$$

(3.34)

where in passing to the last line, the procedure used for deriving lemma 3.3 has been applied. If one substitutes this result together with (3.19), (3.20), (3.23) and (3.25) into $P_1$, one can recast $P_1$ into the form

$$P_1 = (-1)^N c \{- |\tilde{D}| |D(1, z^*_2; z, z_1)| + |\tilde{D}(1; z)| |\tilde{D}(z^*_2; z_1)| - |\tilde{D}(1; z_0)| |\tilde{D}(z^*_2; z)|\},$$

(3.35)

which becomes zero by virtue of Jacobi’s identity.

3.4.2. Proof of (2.3). Let

$$P_2 = D_f \cdot f^* - i g g^* = f^* f - f f^* - i g g^*.$$

(3.36)

Substituting formulas (3.7), (3.13), (3.14) and (3.17) into this expression, $P_2$ becomes

$$P_2 = - |D(z^*_1; z_1)| |\tilde{D}| + |D(z^*_1; z_1)| - i |\tilde{D}(z^*_2; 1)| |D(1; z_0)|.$$

(3.37)

The formulas below can be derived by using the definition of the matrix $\tilde{D}$ from (3.13):

$$|\tilde{D}| = |D| - i |D(1; 1)|,$$

(3.38)

$$|\tilde{D}(z^*_2; 1)| = |D(z^*_2; 1)|,$$

(3.39)

$$|\tilde{D}(z^*_2; z_1)| = |D(z^*_2; z_1)| - i |D(z^*_2; 1; z_1)|.$$

(3.40)

After introducing (3.38)–(3.40) into (3.37), $P_2$ reduces to

$$P_2 = i |D(z^*_2; z_1)| |D(1; 1)| - |D||D(z^*_2; 1; z_1)| - |D(z^*_2; 1)| |D(1; z_1)|.$$

(3.41)

One sees that $P_2$ becomes zero by Jacobi’s identity.
3.4.3. Proof of (2.4). Instead of proving (2.4) directly, we differentiate (2.3) by $x$ and then prove the equation $P_3 = 0$, where
\[ P_3 = f_x f^* - f_x f^* - ig_x g^*. \]  
Substituting (3.8), (3.9), (3.11), (3.13), (3.14) and (3.17) into (3.42), $P_3$ becomes
\[
P_3 = \{ - |D(z^*_x; z)| - |D(z^*_x; z)| + |D(z^*_x, z^*_x; z, z)| \} |\tilde{D}| - |D(z^*_x; z)| |\tilde{D}(z^*_x; z)| \]
\[ - i|D(1; z)| - |D(1, z^*_x; z, z)| |\tilde{D}(z^*_x; 1)|. \]  
Using the co-factor expansion of the determinant $|D|$, 
\[
\sum_{k=1}^{N} d_{jk} D_{jk} = |D|, \quad \sum_{j=1}^{N} d_{jk} D_{jk} = |D|, 
\]  
on one obtains
\[
\sum_{j=1}^{N} \left( \frac{1}{p_j} + \frac{1}{p'_j} \right) D_{jk} = \sum_{j=1}^{N} \left( \frac{1}{p_j} + \frac{1}{p'_j} \right) |D|. \]  
The definition of $d_{jk}$ from (3.2a) gives \( \left( \frac{1}{p_j} + \frac{1}{p'_j} \right) d_{jk} = z_{jj} z_{k,j} - i \tilde{p}_j \), which, after substituting into (3.45) and using formula (3.5), yields the relation
\[
|D(z^*_x; z)| + i |D(1; \tilde{p}^*_x)| = \sum_{j=1}^{N} \left( \frac{1}{p_j} + \frac{1}{p'_j} \right) |D|. \]  
Invoking the relation $D^* = \tilde{D}^T$, the complex conjugate of this expression can be written in the form
\[
- |D(z^*_x; z)| - i |\tilde{D}(\tilde{p}^*_x; 1)| = \sum_{j=1}^{N} \left( \frac{1}{p_j} + \frac{1}{p'_j} \right) |\tilde{D}|. \]  
The following formula can be derived if one differentiates (3.46) by $x$ and notes the relation $z_{ka} = z^*_a$:
\[
- |D(z^*_x; z)| - |D(z^*_x; z_x)| + |D(z^*_x, z^*_x; z, z)| - i |D(z^*_x, 1; z, \tilde{p})| 
\]
\[ = - \sum_{j=1}^{N} \left( \frac{1}{p_j} + \frac{1}{p'_j} \right) |D(z^*_x; z)|. \]  
Multiply (3.48) by $|\tilde{D}|$ and use (3.47) to obtain
\[
\{ - |D(z^*_x; z)| - |D(z^*_x; z_x)| + |D(z^*_x, z^*_x; z, z)| \} |\tilde{D}| 
\]
\[ = i |D(1; z)| + i |\tilde{D}(\tilde{p}^*_x; 1)| |\tilde{D}(z^*_x; z)| + i |\tilde{D}(\tilde{p}^*_x; 1)| |\tilde{D}(z^*_x; 1)|. \]  
(3.49)
which, substituted into $P_3$, gives
\[
P_3 = i |\tilde{D}(z^*_x, 1; z, \tilde{p})| |\tilde{D}| + i |\tilde{D}(\tilde{p}^*_x; 1)| |\tilde{D}(z^*_x; z)| - \{ |D(1; z)| - |D(1, z^*_x; z, z)| \} |\tilde{D}(z^*_x; 1)|. \]  
(3.50)
Differentation of (3.20) with respect to $x$ yields the relation
\[
|D(1; z)| - |D(1, z^*_x; z, z)| = (-1)^{N-1} c |\tilde{D}(\tilde{p}^*_x; z, z)|. \]  
(3.51)
If one substitutes (3.21), (3.24) and (3.51) into (3.50), $P_3$ simplifies to
\[
P_3 = i (-1)^N c \{ |\tilde{D}(z^*_x, \tilde{p}^*_x; z, 1)| |\tilde{D}| - |\tilde{D}(\tilde{p}^*_x; 1)| |\tilde{D}(z^*_x; z)| + |\tilde{D}(\tilde{p}^*_x; z)| |\tilde{D}(z^*_x; 1)| \}. \]  
(3.52)
It turns out that $P_3$ becomes zero by Jacobi’s identity. \( \square \)
3.5. An alternative expression of the bright N-soliton solution

Here, we present an alternative expression of the bright N-soliton solution of equation (1.1). It is expressed by the following theorem.

**Theorem 3.2.** The tau functions \( f' \) and \( g' \) given below satisfy the system of bilinear equations (2.2)–(2.4):

\[
\begin{align*}
 f' &= \begin{vmatrix} A & 1 \\ -I & B \end{vmatrix}, \\
g' &= \begin{vmatrix} A & I \\ -I & 0 \end{vmatrix} y^T, \\
(3.53)
\end{align*}
\]

Here, \( A, B \) and \( I \) are \( N \times N \) matrices and \( y \) and \( y_i \) are \( N \)-component row vectors defined respectively by

\[
\begin{align*}
 A &= (a_{jk})_{1 \leq j,k \leq N}, \\
b_{jk} &= \frac{y_j y_k^*}{q_j + q_k^*}, \\
y_j &= \exp \left( q_j x + \frac{1}{q_j} t + \eta_j \right), \\
(3.54a)
\end{align*}
\]

\[
\begin{align*}
 B &= (b_{jk})_{1 \leq j,k \leq N}, \\
b_{jk} &= \frac{i q_k}{q_j^* + q_k}, \\
(3.54b)
\end{align*}
\]

\[
\begin{align*}
 I &= (\delta_{jk})_{1 \leq j,k \leq N} : N \times N \text{ unit matrix}, \\
y &= (y_1, y_2, \ldots, y_N), \\
y_i &= (y_1/y_1, y_2/y_2, \ldots, y_N/y_N). \\
(3.54d)
\end{align*}
\]

where \( q_j \) and \( \eta_j \) are arbitrary complex parameters.

The proof of theorem 3.2 parallels theorem 3.1 and is hence omitted. Instead, we provide an alternative proof. To this end, we first establish the following proposition.

**Proposition 3.1.** Under the parameterization \( q_j = -p^*_j \), \( \eta_j = -z^*_j + \ln c_j \) \( (j = 1, 2, \ldots, N) \), the tau functions \( f, g, f' \) and \( g' \) satisfy the relations

\[
\begin{align*}
 f' &= (-1)^N |A| f, \\
g' &= -c' |A| g, \\
(3.55)
\end{align*}
\]

where

\[
c' = \prod_{j=1}^{N} q_j, \\
c_j = \prod_{l=1}^{N} (q_j + q_l^*), \\
(3.56)
\]

and the conditions \( q_j + q_k^* \neq 0 \) for all \( j \) and \( k \) and \( q_j \neq q_k \) for \( j \neq k \) are imposed on the parameters.

**Proof.** By means of the operation of matrix multiplication, one can show that

\[
\begin{align*}
 f' &= |I + AB| = |A| |A^{-1} + B|, \\
g' &= \begin{vmatrix} I + AB & y^T \\ 1 & 0 \end{vmatrix} = |A| \begin{vmatrix} A^{-1} + B & A^{-1} y^T \\ 1 & 0 \end{vmatrix}. \\
(3.57)
\end{align*}
\]

The inverse of the Cauchy matrix \( A \) exists due to the conditions imposed on the parameters. It reads [7]

\[
A^{-1} = \begin{pmatrix} c_j c_k & 1 \\ q_j + q_k y_j y_k \end{pmatrix}_{1 \leq j,k \leq N}. \\
(3.58)
\]

The specified parametrization also leads to the relation \( y_j = c_j / z^*_j \), which, substituted into \( A^{-1} \) and \( B \), gives

\[
\begin{align*}
 A^{-1} &= \begin{pmatrix} -z_j z_k^* & 1 \\ p_j + p_k \end{pmatrix}_{1 \leq j,k \leq N}, \\
B &= \begin{pmatrix} i p_k^* \\ p_j + p_k \end{pmatrix}_{1 \leq j,k \leq N}. \\
(3.59)
\end{align*}
\]
It follows from (3.2a) and (3.59) that $A^{-1} + B = -D$. If one introduces this relation into $f'$ from (3.57), one finds the first relation of (3.55). To proceed, note that the $j$th element of the column vector $A^{-1}y_j^T$ is given by

$$\left(A^{-1}y_j^T\right)_j = \sum_{m=1}^{N}\left(A^{-1}\right)_{jm}y_{m,t} = \frac{c_j}{\gamma_j} \sum_{m=1}^{N} \frac{1}{q_m} \prod_{l=1}^{N} (q_m + q_l^*) \prod_{l=1}^{N} (q_m - q_l^*).$$

(3.60)

The sum with respect to $m$ turns out to be $(-1)^N \prod_{l=1}^{N} (q_l^*/q_l^*)/q_j^*$ by using Euler’s formula. Hence, (3.60) reduces to $(-1)^N c'/\gamma_j$ upon rewriting the resultant expression in terms of $\gamma_j$. Thus, $g'$ from (3.57) becomes

$$g' = (-1)^N c' |A| \begin{bmatrix} -D \\ 1 \end{bmatrix} \frac{z_j^T}{0} = -c'|A|g,$n$$
giving rise to the second relation of (3.55).

Proposition 3.2. The tau functions $f'$ and $g'$ from (3.55) satisfy the system of bilinear equations (2.2)–(2.4) if $f$ and $g$ satisfy the same system of equations.

Proof. Let $P_1'$, $P_2'$, and $P_3'$ be

$$P'_1 = D_1D_2g' \cdot f' - g' f' = g_{a'}f' - g_{i}f_{x} - g_{f}f_{i} - g_{g}f_{a},$$

(3.61)

$$P'_2 = D_1f' \cdot f'^* - i\gamma g'^* = f_{i}f'^* - f'_{x}f_{i} - i\gamma g'^*,$$

(3.62)

$$P'_3 = f_{a}f'^* - f'_{x}f_{a} - i\gamma g'^*,$$

(3.63)

respectively. First, note that the determinant $|A|$ from (3.54a) can be modified in the form

$$|A| = \prod_{j=1}^{N} \gamma_j \prod_{k=1}^{N} \gamma_k \left| \left( \frac{1}{q_j + q_k} \right) \right| \exp \left[ \sum_{j=1}^{N} \left( q_j + q_k \right)x + \left( \frac{1}{q_j} + \frac{1}{q_k} \right) t + \eta_{j0} + \eta_{k0} \right].$$

(3.64)

It immediately follows from this expression that

$$D_1D_2|A| \cdot |A| = 0.$$  

(3.65)

Substitute (3.55) into (3.61)–(3.63) and use the relation $|c'| = 1$ to obtain

$$P'_1 = (-1)^{N-1} c'|A|^2 (D_1D_2g' \cdot f - gf) + (D_3D_1|A| \cdot |A|)gf,$$

(3.66)

$$P'_2 = |A|^2 (D_1f' \cdot f'^* - ig'^*),$$

(3.67)

$$P'_3 = |A||A|_t (D_1f' \cdot f'^* - ig'^*) + |A|^2 (f_{a}f'^* - f'_{x}f_{a} - ig'^*) + \frac{i}{2} (D_3D_1|A| \cdot |A|)ff'^*.$$  

(3.68)

Above three expressions vanish identically by virtue of (2.3), (2.4) $P_3 = 0$ with $P_3$ given by (3.42), and (3.65).

Propositions 3.1 and 3.2 lead to an alternative expression of the bright $N$-soliton solution of equation (1.1) in terms of the tau functions $f'$ and $g'$:

$$u = (-1)^{N-1} c' \frac{g'}{f'}, \quad |u|^2 = \frac{i}{\partial t} \ln \frac{f'^*}{f'}.$$  

(3.69)

Note that if $u$ satisfies equation (1.1), then $cu$, with $c$ being a complex constant satisfies the equation as well if $|c| = 1$. Hence, the factor $(-1)^{N-1} c'$ in the above expression for $u$ is irrelevant and it may be replaced simply by 1.
3.6. Remark

As already mentioned in section 2.2, one of the bilinear equations (2.2)–(2.4) can be replaced by the trilinear equation (2.10). Here, we show that the tau functions (3.1) for the bright \( N \)-soliton solution satisfy equation (2.10). The proof is quite simple. Indeed, referring to (3.7), (3.8), (3.10)–(3.14) and (3.23) and using Jacobi’s identity, we can derive the following relations:

\[
g_{xf} - g_{f}x - gf = (-1)^{N-1} c |D(1; z)| |\bar{D}(z_{i}; z_{i})| = (-1)^{N} c |D(1; z)| f^{*}, \tag{3.70}
\]

\[
g_{xf} - g_{f}x = (-1)^{N} c |D(1; z)||\bar{D}| = (-1)^{N} c |D(1; z)| f^{*}. \tag{3.71}
\]

Upon substituting (3.70) and (3.71) into (2.10), one can confirm that equation (2.10) holds identically. Note that expressions (3.70) and (3.71) have a common factor \( |D(1; z)| \).

Let \( q = (g/f)_{x} \). It then follows from (3.71) that

\[
q = (-1)^{N} c \frac{|D(1; z)| f^{*}}{f^{2}}. \tag{3.72}
\]

The form of \( q \) coincides perfectly with the \( N \)-soliton solution of the DNLS equation

\[
iq_{t} + q_{xx} + 2i(|q|^{2}q)_{x} = 0. \tag{3.73}
\]

Actually, if we replace the time dependence of \( z_{j} \) in (3.2a) as \( ip^{2}t \) instead of \( (1/p_{j})t \), then the \( N \)-soliton solution (3.72) with the tau function \( f \) from (3.1) satisfies the DNLS equation. See for example [8–10]. For completeness, we reproduce the one-soliton solution \( q_{1} \) as well as the square of its modulus \( |q_{1}|^{2} \). To this end, we put \( p_{1} = a_{1} + ib_{1} \) and \( \xi_{10} = \theta_{10} = i\chi_{10} \). Then, expression (3.72) with \( N = 1 \) yields

\[
q_{1} = \frac{2a_{1}(a_{1} - ib_{1})}{a_{1} + ib_{1}} \frac{e^{ib_{1}t + ia_{1}t} (e^{2b_{1}t} - b_{1} - ia_{1})}{(e^{2b_{1}t} - b_{1} - ia_{1})^{2}}, \tag{3.74a}
\]

\[
\theta_{1} = a_{1}(x - 2b_{1}t) + \theta_{10}, \quad \chi_{1} = b_{1}x + (a_{1}^{2} - b_{1}^{2})t + \chi_{10}. \tag{3.74b}
\]

\[
|q_{1}|^{2} = \frac{2a_{1}^{2}}{\sqrt{a_{1}^{2} + b_{1}^{2}} \cosh 2(\theta_{1} + \delta_{1}) - \frac{b_{1}}{\sqrt{a_{1}^{2} + b_{1}^{2}}}} \frac{1}{e^{2b_{1}t}} = \frac{1}{\sqrt{a_{1}^{2} + b_{1}^{2}}}. \tag{3.75}
\]

Let \( A_{1} \) and \( c_{1} \) be the amplitude and velocity of \( |q_{1}| \), respectively. We then find from (3.75) that

\[
A_{1} = \sqrt{\frac{c_{1}^{2}}{c_{1}^{2} + 4a_{1}^{2}}} \quad \text{and} \quad c_{1} = 2b_{1}. \tag{3.76}
\]

Note that if \( b_{1} > 0 \), the soliton propagates to the right, whereas if \( b_{1} < 0 \), it propagates to the left. This fact implies that both the overtaking and head-on collisions are possible in the interaction process of two solitons. It is interesting that in the case of \( b_{1} > 0(\theta_{1} < 0) \), the amplitude of the soliton is an increasing (a decreasing) function of the velocity.

The result (3.72) also follows from the viewpoint of the IST. Indeed, if we identify \( q \) with \( u_{x} \), then the \( x \)-part of the Lax pair for the FL DNLS equation coincides with that of the DNLS equation [2]. It turns out that the bright \( N \)-soliton solution of the DNLS equation can be derived from that of the FL DNLS equation via the relation \( q = u_{x} \) and replacing the time dependence of the exponential functions \( z_{j} \), as demonstrated here.
4. Properties of the bright N-soliton solution

In this section, we investigate the properties of the bright N-soliton solution particularly focusing on the one- and two-soliton solutions and then address the asymptotic behavior of the N-soliton solution for large time. To this end, we first parametrize the complex constants \( p_j \) and \( \xi_0 \) by the real constants \( a_j, b_j, \theta_0 \) and \( \chi_0 \) as

\[
p_j = a_j + ib_j, \quad \xi_0 = \theta_0 + i\chi_0, \quad a_j > 0, \quad j = 1, 2, \ldots, N, \tag{4.1}
\]

and introduce the new independent variables \( \theta_j \) and \( \chi_j \) in accordance with the relations

\[
\theta_j = a_j(x + c_j t) + \theta_0, \quad c_j = \frac{1}{a_j^2 + b_j^2}, \quad j = 1, 2, \ldots, N, \tag{4.2a}
\]

\[
\chi_j = b_j(x - c_j t) + \chi_0, \quad j = 1, 2, \ldots, N. \tag{4.2b}
\]

In terms of these variables, \( z_j \) is expressed as

\[
z_j = e^{\theta_j + i\chi_j}, \quad j = 1, 2, \ldots, N. \tag{4.2c}
\]

4.1. Bright one-soliton solution

The tau functions \( f = f_1 \) and \( g = g_1 \) corresponding to the bright one-soliton solution follow from (3.1) and (3.2) with \( N = 1 \). They read

\[
f_1 = \frac{z_1 z_1^* - ip_1^*}{p_1 + p_1^*}, \quad g_1 = -\frac{z_1}{p_1}. \tag{4.3}
\]

In terms of the new parameters defined by (4.1) and (4.2), the one-soliton solution \( u_1 = u_1(\theta_1, \chi_1) \) takes the form of an envelope soliton

\[
u_1 = -\frac{p_1 + p_1^*}{p_1} \frac{z_1}{z_1 z_1^* - ip_1^*} = -\frac{2a_1}{a_1 + ib_1} e^{\theta_1 + i\chi_1}. \tag{4.4}
\]

The square of the modulus of \( u_1 \) can be written as

\[
|u_1|^2 = \frac{2a_1^2}{(a_1^2 + b_1^2)^2} \frac{1}{\cosh^2 2(\theta_1 + \delta_1) - \frac{4b_1}{\sqrt{a_1^2 + b_1^2}}}, \quad e^{2\delta_1} = \frac{1}{\sqrt{a_1^2 + b_1^2}}, \tag{4.5a}
\]

showing that \(|u_1|\) has the amplitude \( A_1 \) given by

\[
A_1 = \sqrt{\frac{2(\sqrt{a_1^2 + b_1^2} + b_1)}{\sqrt{a_1^2 + b_1^2}}}. \tag{4.5b}
\]

It represents a localized pulse (or a bright soliton) moving to the left at a constant velocity \( c_1 = 1/(a_1^2 + b_1^2) \). The amplitude–velocity relation follows immediately from (4.5b) to give:

\[
A_1 = \sqrt{2(\sqrt{c_1 + b_1})}. \tag{4.6}
\]

See figure 1. The characteristic of the solution depends crucially on the signature of the parameter \( b_1 \), as we shall see now.

For a fixed value of \( b_1, A_1 \) is an increasing function of \( c_1 \) if \( b_1 > 0 \). If, on the other hand, \( b_1 < 0 \), then \( A_1 \) becomes an increasing function of \( c_1 \) in the interval \( 0 < c_1 \leq \frac{1}{4b_1} \) and a decreasing function of \( c_1 \) in the interval \( \frac{1}{4b_1} < c_1 < \frac{1}{2b_1} \). In the case of \( b_1 > 0 \), the amplitude \( A_1 \) remains finite in the limit of \( a_1 \to 0 \) (i.e. infinite width) as opposed to the behavior of the
usual soliton for which the amplitude tends to zero in the limit of infinite width. It then turns out that the bright soliton \( u_1 \) from (4.4) reduces to an algebraic soliton of the form

\[
u_1 = \frac{ic_1^2 \exp \left[ i \left( \frac{1}{\sqrt{\c_1^2}} x - \sqrt{\c_1^2 + \chi_{10}} \right) \right]}{x + \c_1 t + x_0 - i \sqrt{\c_1^2 + \chi_{10}}}, \quad c_1 = \frac{1}{\c_1^2}, \tag{4.7a}\]

and \( |u_1|^2 \) from (4.5) becomes

\[
|u_1|^2 = \frac{c_1^2}{(x + \c_1 t + x_0)^2 + 4} \tag{4.7b}
\]

where we have put \( \theta_{10} = \alpha_1 x_0 - \delta_1 \). The similar peculiar structure of the solution has been found for the bright soliton solution of the DNLS equation [11].

In the case of \( b_1 < 0 \), we can observe that when the velocity lies in the interval \( \frac{1}{4\c_1^2} < \c_1 < \frac{1}{\c_1} \), the solution exhibits an interesting feature. Specifically, a large soliton propagates slower than a small soliton, as will be illustrated graphically in the interaction process of two bright solitons (see figure 3). The amplitude–velocity relation described above is in striking contrast to that of the one-soliton solution of the DNLS equation. Actually, we see from (3.76) that \( A_1 \) becomes a decreasing function of \( |\c_1| \) when \( b_1 < 0 \).

Last, it is instructive to compare solutions (4.4) and (4.5) with that derived by the IST. We define the new parameters \( \Delta_1 \) and \( \gamma_1 \) by the relations \( a_1 = \Delta_1 \sin \gamma_1, \quad b_1 = -\Delta_1 \cos \gamma_1, \quad 0 < \gamma_1 < \pi \), and then replace \( \theta_{10} \) by \( \theta_{10} + \ln \Delta_1 \) (or equivalently, \( \theta_1 \) by \( \theta_1 + \ln \Delta_1 \)). We find that expressions (4.4) and (4.5) recast respectively to

\[
u_1 = -\frac{1}{\Delta_1} \frac{2ie^{i\theta_1 + i\gamma_1} \sin \gamma_1}{e^{2i\theta_1 + i\gamma_1} + 1}, \tag{4.8a}\]

\[
|u_1|^2 = \frac{1}{\Delta_1^2} \frac{2 \sin^2 \gamma_1}{\cosh 2\theta_1 + \cos \gamma_1}, \tag{4.8b}
\]

which is just the one-soliton solution presented in [3]. Note in these expressions that \( A_1 = 2 \sin \frac{\gamma_1}{2}/\Delta_1 \) and \( c_1 = 1/\Delta_1^2 \). The algebraic soliton is generated from (4.8) if one puts \( \gamma_1 = \pi - \epsilon \) and takes the limit \( \epsilon \to +0 \) [3].

### 4.2. Bright 2-soliton solution

The bright two-soliton solution exhibits a variety of interesting features. Here, we investigate the interaction process of bright solitons focusing on the asymptotic behavior of the solution.
As a result, we obtain the formulas for the phase shifts for each soliton. The corresponding tau functions $f_2$ and $g_2$ are given by (3.1) and (3.2) with $N = 2$. Explicitly,

$$f_2 = -\frac{p^*_1 p^*_2 (p_1 - p_2) (p_2 - p_1)^2}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)(p_2 + p_2^*)} - \frac{ip^*_2}{(p_1 + p_2^*) (p_2 + p_2^*)} z_2 z_2^*  
- \frac{-ip^*_1}{(p_1 + p_1^*) (p_2 + p_1^*)} z_2 z_2^* + \frac{-ip^*_2}{(p_1 + p_2^*) (p_2 + p_2^*)} z_2 z_2^* 
+ \frac{ip^*_1}{(p_1 + p_2^*) (p_2 + p_1^*)} z_2 z_2^* + \frac{ip^*_2}{(p_1 + p_2^*) (p_2 + p_2^*)} z_2 z_2^*. \tag{4.9a}$$

$$g_2 = -\frac{ip_2 (p_1^* - p_2^*)}{p_1 (p_2 + p_1^*)(p_2 + p_2^*)} z_1 + \frac{ip_1 (p_1^* - p_2^*)}{p_2 (p_1 + p_1^*)(p_1 + p_2^*)} z_2 
- \frac{p_1^2 (p_1 - p_2)}{p_1 p_2 (p_1 + p_1^*)(p_2 + p_2^*)} z_1 z_2 z_2^* + \frac{p_1^2 (p_1 - p_2)}{p_1 p_2 (p_1 + p_1^*)(p_2 + p_2^*)} z_1 z_2 z_2^*. \tag{4.9b}$$

Now, we order the magnitude of the velocity of each soliton in the $(x, t)$ coordinate system as $c_1 > c_2$. We first take the limit $t \to -\infty$ with $\theta_1$ being fixed. Since in this limit $|z_1| = \text{finite}$ and $|z_2| \to \infty$, the leading-order asymptotics of $f_2$ and $g_2$ are found to be

$$f_2 \sim \frac{z_2 z_2^*}{(p_1 + p_1^*) (p_2 + p_2^*)} \left\{ \frac{(p_1 - p_2) (p_1^* - p_2^*)}{(p_1 + p_1^*) (p_2 + p_2^*)} z_1^* - ip_1^* \right\}, \tag{4.10a}$$

$$g_2 \sim \frac{p_2^* (p_1 - p_2)}{p_1 p_2 (p_1 + p_1^*)(p_2 + p_2^*)} z_1 z_2 z_2^*. \tag{4.10b}$$

The asymptotic form of the two-soliton solution $u_2$ evaluated by (4.10) is expressed as

$$u_2 \sim \frac{p_1 + p_1^*}{p_1} \left\{ \frac{z_1}{z_1 z_2^* - ip_1} \right\}, \quad \chi_1 = z_1 \exp \left\{ -\ln \left\{ \frac{p_2 (p_1 + p_1^*)}{p_2^* (p_1 - p_2)} \right\} \right\}. \tag{4.11}$$

Thus, the asymptotic of $u_2$ takes the same profile as that of the one-soliton solution (4.4) except the phase shifts. Specifically,

$$u_2 \sim u_1 (\theta_1 + \Delta \theta_1^{(-)}, \chi_1 + \Delta \chi_1^{(-)}), \tag{4.12a}$$

$$\Delta \theta_1^{(-)} = -\ln \left\{ \frac{p_1 + p_1^*}{p_1 - p_2} \right\}, \quad \Delta \chi_1^{(-)} = -\arg \frac{p_1 + p_1^*}{p_1 - p_2} - \arg \frac{p_2^*}{p_2} \mp \pi. \tag{4.12b}$$

Next, we take the limit $t \to +\infty$, with $\theta_1$ being fixed. In this limit, $|z_1| = \text{finite}$ and $|z_2| \to 0$. The expressions corresponding to (4.10) and (4.11) are given by

$$f_2 \sim -\frac{ip_2^* (p_1 - p_2) (p_1^* - p_2^*)}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)(p_2 + p_2^*)} \left\{ \frac{(p_1 + p_2^*) (p_2 + p_1^*)}{(p_1 - p_2) (p_1^* - p_2^*)} z_1^* - ip_1^* \right\}, \tag{4.13a}$$

$$g_2 \sim -\frac{ip_2 (p_1^* - p_2^*)}{p_1 (p_2 + p_1^*)(p_2 + p_2^*)} z_1, \tag{4.13b}$$

$$u_2 \sim \frac{p_1 + p_1^*}{p_1} \left\{ \frac{z_1^*}{z_1^* z_2^* - ip_1} \right\}, \quad \chi_1 = z_1 \exp \left\{ \ln \left\{ \frac{p_2 (p_1 + p_1^*)}{p_2^* (p_1 - p_2)} \right\} \right\}. \tag{4.14}$$

It follows from (4.14) that

$$u_2 \sim u_1 (\theta_1 + \Delta \theta_1^{(+)}, \chi_1 + \Delta \chi_1^{(+)}), \tag{4.15a}$$
We can perform the similar asymptotic analysis while keeping $\theta_2$ fixed. Hence, we quote the final result. As $t \to -\infty$, the expression corresponding to (4.12) reads
\[
\Delta \theta_2^{(+)} = \ln \left| \frac{p_2 + p_1^*}{p_2 - p_1} \right|,
\]
\[
\Delta \chi_2^{(+)} = \ar
g \frac{p_2 + p_1^*}{p_2 - p_1} + \ar \frac{p_1}{p_1^*} + \pi.
\] (4.17b)

whereas as $t \to +\infty$, the expression corresponding to (4.15) reads
\[
\Delta \theta_2^{(-)} = -\ln \left| \frac{p_2 + p_1^*}{p_2 - p_1} \right|,
\]
\[
\Delta \chi_2^{(-)} = -\ar \frac{p_2 + p_1^*}{p_2 - p_1} - \ar \frac{p_1}{p_1^*} + \pi.
\] (4.18b)

Thus, the total phase shifts are given by the formulas
\[
\Delta \chi_2 = -\frac{2}{a_2} \ln \left| \frac{p_2 + p_1^*}{p_2 - p_1} \right|,
\]
\[
\Delta \chi_2 = -2\ar \frac{p_2 + p_1^*}{p_2 - p_1} - 2\ar \frac{p_1}{p_1^*}.
\] (4.19)

Figure 2 shows the time evolution of a bright two-soliton solution $U = |u|$ with positive $b_j$ ($j = 1, 2$). See the solid line in figure 1 which plots the amplitude–velocity relation for $b_1 = 1$. In the present example, $A_1 = 1.99, A_2 = 1.44, c_1 = 0.99, c_2 = 0.20$ and the phase
Figure 3. Time evolution of a bright two-soliton solution with the parameters $a_1 = 0.5$, $a_2 = 1.2$, $b_1 = -1.0$, $b_2 = -1.0$.

shifts are given by $\Delta x_1 = 2.00$, $\Delta x_2 = -0.10$. The feature of the interaction process of solitons is similar to that of the usual solitons, namely the large soliton propagates faster than the small soliton and the former one suffers a positive phase shift whereas the latter one suffers a negative phase shift.

Figure 3 depicts the similar plot for negative $b_j$ ($j = 1, 2$) where $A_1 = 0.43, A_2 = 0.69, c_1 = 0.80, c_2 = 0.41$ and $\Delta x_1 = 3.55, \Delta x_2 = -1.48$.

In this case, the velocities $c_j$ ($j = 1, 2$) are chosen to satisfy the inequality $\frac{1}{2 \beta_j} < c_j < \frac{1}{\beta_j}$. Consequently, the velocity of each soliton becomes a decreasing function of the amplitude, as indicated by the broken line in figure 1. Unlike the previous example, there appears a peculiar feature in the interaction process of solitons. Indeed, as evidenced from figure 3, the large soliton is seen to propagate slower than the small soliton and suffers a negative phase shift. On the other hand, the small soliton suffers a positive phase shift.

4.3. Bright N-soliton solution

The asymptotic analysis for the bright N-soliton solution can be performed straightforwardly following the two-soliton case. Hence, without entering into the detailed calculation, we describe the main results only. To proceed, let us order the magnitude of the velocity of each soliton as $c_1 > c_2 > \cdots > c_N$. We take the limit $t \to -\infty$ with $\theta_n$ being finite. Since in this limit $|z_j| \to 0$ for $j < n$ and $|z_j| \to \infty$ for $n < j$, we find that the leading-order asymptotic of the tau function $f = f_N$ from (3.1) and (3.2a) can be written in the form

\begin{equation}
    f_N \sim d(1, 2, \ldots, n)d(n + 1, n + 2, \ldots, N) \prod_{k=1}^{n-1} (-ip_k) \prod_{j=n+1}^{N} (z_j z_j^*) (d_n z_n z_n^* - ip_n^*), \quad (4.20a)
\end{equation}

where

\begin{equation}
    d(m, m + 1, \ldots, n) = \left( \frac{1}{p_j + p_k^*} \right)_{m \leq j, k \leq n} = \frac{\prod_{m \leq j, k \leq n} (p_j - p_k^*) (p_j^* - p_k^*)}{\prod_{m \leq j, k \leq n} (p_j + p_k^*)}, \quad (4.20b)
\end{equation}
is the Cauchy determinant and
\[
d_n = \frac{d(1, 2, \ldots, n-1)d(n, n+1, \ldots, N)}{d(1, 2, \ldots, n)d(n+1, n+2, \ldots, N)} = \prod_{j=n+1}^{N} \frac{(p_n - p_j)(p_n^* - p_j^*)}{(p_n + p_j)(p_n^* + p_j^*)}.
\]

(4.20c)

Similarly, \( g_N \) has the asymptotic form
\[
g_N \sim \frac{(-1)^N}{p_n} \frac{d(1, 2, \ldots, n-1)d(n, n+1, \ldots, N)}{d(1, 2, \ldots, n)} \times \prod_{k=1}^{n-1} \left( -ip_k^* \right) \prod_{j=1}^{n-1} \frac{p_j(p_n^* - p_j^*)}{p_j(p_n^* + p_j^*)} \prod_{k=n+1}^{N} p_k(p_n + p_k^*) \prod_{j=n+1}^{N} (z_j z_j^*) z_n.
\]

(4.21)

Taking into account the relation
\[
d(1, 2, \ldots, n) = \frac{1}{p_n + p_n^*} \prod_{j=1}^{n-1} (p_n - p_j)(p_n^* - p_j^*) d(1, 2, \ldots, n-1),
\]

we obtain from (4.20) and (4.21) the asymptotic form of the \( N \)-soliton solution
\[
u_N \sim (-1)^N p_n + p_n^* \prod_{j=1}^{n-1} \frac{p_j(p_n + p_j^*)}{p_j(p_n^* + p_j^*)} \prod_{k=n+1}^{N} \frac{p_k(p_n - p_k)}{p_k(p_n + p_k^*)} z_n z_n^* - ip_n^*,
\]

(4.23)

This expression can be rewritten in terms of the one-soliton solution as
\[
u_N \sim u_1(\theta_n + \Delta \theta_n^{(-)}, \chi_n + \Delta \chi_n^{(-)}),
\]

(4.24a)

with
\[
\Delta \theta_n^{(-)} = \sum_{j=1}^{n-1} \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right| - \sum_{j=n+1}^{N} \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right|, \quad \Delta \chi_n^{(-)} = \sum_{j=1}^{n-1} \left\{ \arg \left( \frac{p_n + p_j^*}{p_n - p_j} \right) + \arg \left( \frac{p_j}{p_j^*} \right) \right\} - \sum_{j=n+1}^{N} \left\{ \arg \left( \frac{p_n + p_j^*}{p_n - p_j} \right) + \arg \left( \frac{p_j}{p_j^*} \right) \right\} + N \pi.
\]

(4.24b)

As \( t \to +\infty \), on the other hand, the asymptotic form of \( u_N \) is found to be
\[
u_N \sim (-1)^N p_n + p_n^* \prod_{j=1}^{n-1} \frac{p_j(p_n + p_j^*)}{p_j(p_n^* + p_j^*)} \prod_{k=n+1}^{N} \frac{p_k(p_n - p_k)}{p_k(p_n^* + p_k^*)} d_n z_n z_n^* - ip_n^*,
\]

(4.25)

which leads to the expression
\[
u_N \sim u_1(\theta_n + \Delta \theta_n^{(+)}, \chi_n + \Delta \chi_n^{(+)})
\]

(4.26a)

with
\[
\Delta \theta_n^{(+)} = -\Delta \theta_n^{(-)}, \quad \Delta \chi_n^{(+)} = -\Delta \chi_n^{(-)} + 2N \pi.
\]

(4.26b)

We see from (4.24) and (4.26) that in the rest frame of reference, the asymptotic form of the bright \( N \)-soliton solution can be represented by a superposition of \( N \) independent bright one-soliton solutions, with the only difference being the phase shifts. It follows from (4.24) and (4.26) that the formulas for the total phase shifts of the \( n \)th soliton are given by
\[
\Delta \chi_n = \frac{2}{a_n} \left\{ \sum_{j=n+1}^{N} \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right| - \sum_{j=1}^{n-1} \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right| \right\}.
\]

(4.27a)
\[ \Delta \chi_n = 2 \sum_{j=n+1}^{N} \left\{ \arg \left( \frac{p_n + p_j^*}{p_n - p_j} \right) + \arg \left( \frac{p_j}{p_j^*} \right) \right\} - 2 \sum_{j=1}^{n-1} \left\{ \arg \left( \frac{p_n + p_j^*}{p_n - p_j} \right) + \arg \left( \frac{p_j}{p_j^*} \right) \right\}. \]

(4.27b)

The above formulas reduce to (4.16) and (4.19) for the special case of \( N = 2 \). They clearly show that each soliton has pairwise interactions with other solitons, namely there are no many-particle collisions among solitons.

5. Concluding remarks

In this paper, we have presented two different expressions of the bright \( N \)-soliton solution of the FL DNLS equation. The exact method of solution developed here is purely algebraic which does not recur to the IST. The \( N \)-soliton solution given by theorem 3.1 has been obtained by the dressing method [4], while the form presented by theorem 3.2 is new. The equivalence of both expressions has been demonstrated by a straightforward computation using the properties of the Cauchy matrix. We have also shown that the bright \( N \)-soliton solution of the DNLS equation can be derived from that of the FL DNLS equation through a simple relation. This fact has stemmed from the analysis of a trilinear equation among the tau functions. Lastly, we have investigated in detail the properties of the solutions and found some new features.

In conclusion, we briefly comment on the dark soliton solutions. The construction of soliton solutions under nonvanishing boundary conditions is difficult to perform when compared with that under vanishing boundary conditions. Specifically, the analysis by means of the IST requires the delicate discussion on the spectral problem. On the other hand, the bilinear transformation method is easily applicable to obtain soliton solutions. In part II of this study, we will address the construction of the dark \( N \)-soliton solution of the FL DNLS equation on a background of a plane wave. The system of bilinear equations for the equation is almost the same as that of the bright soliton case. However, a difficulty arises due to the constraint imposed on the complex parameters \( p_j \) \((j = 1, 2, \ldots, N)\), as in the case of the similar problem for the NLS equation in which \( p_j \) lie on a circle in the complex plane [12]. In particular, this fact must be used explicitly to prove the bilinear equation corresponding to equation (2.2). We will show that the trilinear equation analogous to (2.10) plays an important role in the proof and the inspection of the equation also provides the dark \( N \)-soliton solution of the DNLS equation.

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