Of Bombs and Boats and Mice and Men:
a random tour through some scaling laws

Niall MacKay
Department of Mathematics
The University of York
G I Taylor and the Bomb
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Shock wave radius depends on energy $ML^2 T^{-2}$, air density $ML^{-3}$ and time $T$. 
G I Taylor and the Bomb

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Dimensional analysis implies radius

$$R \sim \left( \frac{Et^2}{\rho} \right)^{1/5}$$
This is a scale-invariant process or scaling law:

\[ y \sim x^a \quad \text{or} \quad y \approx Cx^a \quad \text{or} \quad \log y \sim a \log x + \log C \]

Rescaling, multiplying \( x \) by \( \lambda \) and \( y \) by \( \mu \), changes only the constant,

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where \( C' = C\lambda^a/\mu; \)
Scaling Laws

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*the form of the law remains the same and there is no preferred scale.*
Trees

1'

100'
Yachts

14’

109’
Yachts

14’

Speed $v$ depends on length $l$ and acceleration due to gravity, $g$;

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$$v \sim \sqrt{gl}.$$
How does yacht price scale with length?
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\[ P \sim L^{3.8 \pm 0.2}, \quad \sum R^2 = 0.71 \]
How does yacht price scale with length?

\[ P \sim L^{3.5 \pm 0.1} e^{-0.03 \text{(age/yr)}} \quad \sum R^2 = 0.87 \]
Roasting times

1 kg

5 kg
Roasting times

Roasting time $t$ depends on mass $[m] = M$, density $[\rho] = ML^{-3}$, diffusion coefficient $[\kappa] = TL^{-2}$.

So $t \sim \kappa (m/\rho)^{2/3}$. If the partridge takes 1 hr, the turkey takes 3 hrs.
If mass $M \sim H^3$ and height $H \mapsto 2H$ then $M \mapsto 8M$, so if BMI $= \frac{M}{H^2}$ then BMI $\mapsto 2$ BMI.
Body Mass Index
Body Mass Index

NJM, *Journal of Biomechanics* 2010, arXiv:0910.5834
UK: $M \sim H^{2.70 \pm 0.05}$, $\sum R^2 = 0.996$
Body Mass Index

Hong Kong: \( M \sim H^{2.66 \pm 0.05} \), \( \sum R^2 = 0.997 \)
Do mammals obey scaling laws?

20 g  200 kg
Do mammals obey scaling laws?

*On Being the Right Size*, J. B. S. Haldane, 1928:

‘You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft....’
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\[ ma = mg - kAv^2 \]

where mass \( m \sim L^3 \), cross-sectional area \( A \sim L^2 \), so terminal velocity (at which \( a = 0 \)) \( v^2 \sim L^3/L^2 = L \) and \( v \sim L^{1/2} \sim m^{1/6} \).

If the mouse’s \( v = 20 \) mph, the bear’s \( v = 100 \) mph.
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‘...A rat is killed, a man is broken, a horse splashes.’
Metabolic Scaling

height $H \mapsto 2H$, mass $M \mapsto 8M$, power $P \mapsto 4P$, $P \sim M^{2/3}$
Marsupials: $P \sim M^{0.75 \pm 0.01}$, $\sum R^2 = 0.99$
Kleiber’s Law

\[ P \sim M^{3/4} \]
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is a property of branching networks (West, Brown, Enquist 1997)
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...or \( M \sim H^{8/3} \), \( P \sim H^2 \) \Rightarrow \( P \sim M^{3/4} \)
Eutheria: $P \sim M^{0.72\pm0.01}$, $\sum R^2 = 0.96$
Metabolic Scaling

Eutheria residuals
Metabolic Scaling

Kolokotrones et al., *Curvature in metabolic scaling*, Nature 2010

![Graph showing metabolic scaling with equation and R^2 value](image)

**Eutheria:** $P \sim M^{0.72\pm0.01}$, $\sum R^2 = 0.96$
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A scaling law

\[ P = CM^{\beta_1} \]

may be written

\[ \log P = \beta_0 + \beta_1 \log M \quad \text{where} \quad C = e^{\beta_0}. \]
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Multiplying \( M \) by \( \lambda \), or (equivalently) adding \( \log \lambda \) to \( \log M \), just changes the constant \( C \) to \( C' = Ce^\lambda \):

the power \( \beta_1 \) is unchanged, while \( \beta_0 \mapsto \beta_0 + \log \lambda \).
Metabolic Scaling

\[ \log(P) = \beta_0 + \beta_1 \log(M) + \beta_2 \log^2(M) \]
Metabolic Scaling

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\log (P) = \beta_0 + \beta_1 \log (M) + \beta_2 \log^2 (M)
\]

This is no longer scale-invariant:

Multiplying \( M \) by \( \lambda \), or (equivalently) adding \( \log \lambda \) to \( \log M \), changes the linear term and thus the form of the law.
Metabolic Scaling

\[
\log \left( \frac{P}{P_0} \right) = \beta_0 + \beta_1 \log \left( \frac{M}{M_0} \right) + \beta_2 \log^2 \left( \frac{M}{M_0} \right).
\]
Metabolic Scaling

\[ \log \left( \frac{P}{P_0} \right) = \beta_0 + \beta_1 \log \left( \frac{M}{M_0} \right) + \beta_2 \log^2 \left( \frac{M}{M_0} \right). \]

Suppose now that we choose a different scale \( M'_0 \).
Let \( \mu = \log(M'_0/M_0) \).
Then
\[ \log \left( \frac{P}{P_0} \right) = \beta_0 + \beta_1 \mu + \beta_2 \mu^2 + (\beta_1 + 2\beta_2 \mu) \log \left( \frac{M}{M'_0} \right) + \beta_2 \log^2 \left( \frac{M}{M'_0} \right). \]
Metabolic Scaling

NJM, Comment on Kolokotrones et al., *Journal of Theoretical Biology* 2011

![Graph showing log(BMR/kJ/hr) vs. log(M/g)]
The quadratic curve is fitted with no preferred choice of origin, so the values of the linear and constant terms are meaningless.
The quadratic curve is fitted with no preferred choice of origin, so the values of the linear and constant terms are meaningless. A quadratic of any fixed curvature can approximate a line arbitrarily well over a finite interval if that interval is sufficiently far from the turning point.
...or $M \sim H^{8/3}$, $P \sim aH^{5/3} + bH^2 \Rightarrow P \sim aM^{5/8} + bM^{3/4}$
...or $M \sim H^{8/3}$, $P \sim aH^{5/3} + bH^2 \Rightarrow P \sim aM^{5/8} + bM^{3/4}$

The End