COBORDISM RING OF TORIC VARIETIES

AMALENDU KRISHNA, V. UMA

ABSTRACT. We describe the equivariant algebraic cobordism rings of smooth toric varieties. This equivariant description is used to compute the ordinary cobordism ring of such varieties.

1. INTRODUCTION

Let $k$ be a field of characteristic zero. A scheme (or variety) in this text will mean a quasi-projective $k$-scheme. Based on the construction of the motivic algebraic cobordism spectrum $MGL$ by Voevodsky [25], Levine and Morel [18] invented a geometric version of the algebraic cobordism theory some years ago. They showed that the resulting cohomology theory $\Omega^\ast(-)$ is the universal oriented cohomology theory on the category of smooth varieties over $k$. Moreover, it is a universal Borel-Moore oriented homology theory on the category of all $k$-schemes. It was later shown by Levine [16] that there is isomorphism of cohomology theories $\Omega^\ast(-) \cong MGL_{2\ast\ast}(-)$. In other words, the algebraic cobordism theory of Levine and Morel computes pieces of the motivic cobordism theory of Voevodsky. Since the cobordism theory of Levine and Morel has products, one knows that for a smooth scheme $X$, the cohomology $\Omega^\ast(X)$ is a graded ring.

Since the algebraic version of the cobordism theory was discovered only recently, there are not many computations of this cohomology theory known at this stage. Levine and Morel proved a projective bundle formula for the algebraic cobordism, from which one can deduce the formula for the cobordism ring of projective spaces. In a recent work [10], Hornbostel and Kiritchenko computed the cobordism ring of the complete flag variety $GL_n/B$ using the techniques similar to the known Schubert calculus for the singular cohomology. These results were also obtained by Calmès, Petrov and Zainoulline in [5]. The rational cobordism rings of arbitrary flag varieties and flag bundles were recently described by the first author in [14] and [15]. One of the principal goals of this paper is to describe the cobordism ring of smooth toric varieties.

Our strategy of computing the ordinary cobordism ring of smooth toric varieties is to use the powerful technique of the equivariant cohomology. Such techniques have been very effective in computing the ordinary Chow ring of varieties with group action. We refer the reader to the seminal paper [3] for many results in this direction. The equivariant algebraic cobordism groups for smooth varieties were initially defined by Deshpande [6]. They were subsequently developed into a complete theory of equivariant cobordism for all $k$-schemes in [13]. This theory is based on the analogous construction of the equivariant Chow groups by Totaro [22] and Edidin-Graham [7]. All the expected basic properties of the equivariant cobordism theory were established in [13]. These properties are similar to the analogous properties of the ordinary (non-equivariant) cobordism which were established by Levine and Morel [18].

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In the first main result of this paper, we exploit the work of [13, 14] to compute the equivariant algebraic cobordism ring of smooth toric varieties. Our description of the equivariant cobordism ring is very similar to the one for the equivariant Chow ring of toric varieties in [3]. In order to apply the results of [13, 14], we first prove a decomposition theorem (cf. Theorem 6.2) for the equivariant cobordism ring of a smooth toric variety. Such a decomposition of the equivariant cobordism ring is obtained by adapting the techniques of Vezzosi and Vistoli [23] who invented these techniques in the context of the equivariant algebraic K-theory of smooth varieties with torus action. We deduce the formula for the ordinary cobordism ring of a smooth toric variety from the equivariant cobordism ring using [14, Theorem 3.4], which gives an explicit description of the ordinary cobordism ring of a variety with a torus action as a quotient of the equivariant cobordism ring.

To state our main results, let $T$ be a split torus of rank $n$ over $k$ and let $M$ denote the lattice of the one-parameter subgroups of $T$. We identify $M^\vee$ with group of characters of $T$. Let $<, >: M \times M^\vee \to \mathbb{Z}$ be the natural pairing. Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $\mathbb{R}$. We refer the reader to [9] for the basics of toric varieties. Let $\Delta_1$ denote the set of one-dimensional cones in $\Delta$. Let $\{v_\rho|\rho \in \Delta_1\}$ denote the set of primitive vectors in $M$ along the one-dimensional faces of $\Delta$. Let $\Delta_1^0$ denote the collection of subsets $S \subseteq \Delta_1$ such that $S = \{\rho_1, \ldots, \rho_s\}$ is not contained in any maximal cone of $\Delta$.

Let $S = S(T)$ denote the $T$-equivariant cobordism ring $\Omega^*_T(k)$ of the base field. Let $\Omega^*_T(X)$ and $\Omega^*(X)$ denote the $T$-equivariant and the ordinary cobordism rings of $X$ respectively. For $\rho \in \Delta_1$, the corresponding orbit closure $V_{\rho}$ is a $T$-invariant smooth closed subvariety of $X$ which is a Weil divisor on $X$. Let $[V_{\rho}]$ denote the cobordism cycle $[V_{\rho} \to X]$ in $\Omega^*(X)$.

Let $\mathbb{L}$ denote the Lazard ring. It is known that $\mathbb{L} = \bigoplus_{i \leq 0} \mathbb{L}_i$ is a graded ring which is canonically isomorphic to $\Omega^*(k)$. There is a formal (commutative) group law on $\mathbb{L}$ represented by a power series $F(u, v)$ in $\mathbb{L}[[u, v]]$ such that $(\mathbb{L}, F)$ is the universal commutative formal group law of rank one.

Let $\mathbb{L}[[t_{\rho}]]$ denote the graded power series ring over $\mathbb{L}$ in the variables $\{t_{\rho}|\rho \in \Delta_1\}$. For $x_1, \ldots, x_s \in \mathbb{L}[[t_{\rho}]]$ and $n_i \in \mathbb{Z}$, let $\sum_{i=1}^s [n_i]_F x_i$ denote the sum according to the formal group law $F$ (cf. Section 2). Let $I_\Delta$ denote the graded ideal of $\mathbb{L}[[t_{\rho}]]$ generated by the set of monomials $\{\prod_{\rho \in S} t_{\rho}|S \in \Delta_1^0\}$. Let $T_\Delta$ denote the graded ideal of the polynomial ring $\mathbb{L}[t_{\rho}]$ generated by the set of monomials $\{\prod_{\rho \in S} t_{\rho}|S \in \Delta_1^0\} \cup \{t_{\rho}^{n+1}|\rho \in \Delta_1\}$. The following results are the main features of this paper.

**Theorem 1.1.** For a smooth toric variety $X = X(\Delta)$ associated to a fan $\Delta$ in $\mathbb{R}$, there is a natural map of $S$-algebras

$$\Psi_X : \frac{\mathbb{L}[[t_{\rho}]]}{I_\Delta} \to \Omega^*_T(X)$$

which is an isomorphism.

**Theorem 1.2.** Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $\mathbb{R}$. Then the assignment $t_{\rho} \mapsto [V_{\rho}]$ defines an $\mathbb{L}$-algebra isomorphism

$$\overline{\Psi}_X : \left( T_\Delta, \sum_{\rho \in \Delta_1} [< \chi, v_{\rho}>_F t_{\rho} \right) \to \Omega^*(X),$$

where $[< \chi, v_{\rho}>_F t_{\rho}]$ denotes the formal power series in $\mathbb{L}$.

**Theorem 1.3.** Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $\mathbb{R}$. Then the assignment $t_{\rho} \mapsto [V_{\rho}]$ defines an $\mathbb{L}$-algebra isomorphism

$$\overline{\Psi}_X : \left( T_\Delta, \sum_{\rho \in \Delta_1} [< \chi, v_{\rho}>_F t_{\rho} \right) \to \Omega^*(X),$$

where $[< \chi, v_{\rho}>_F t_{\rho}]$ denotes the formal power series in $\mathbb{L}$.
where $\chi$ runs over $M^\vee$.

Since $\Omega^*(-)$ is the universal oriented cohomology theory, we see at once that all other oriented cohomology theories (e.g. $K$-theory, Chow groups, étale and singular cohomology) of smooth toric varieties can be deduced from Theorem 1.2 in a simple way.

As one would also expect, the computation of any oriented cohomology ring of toric varieties is the foundational step in computing an oriented cohomology of various spherical varieties. It is known (cf. [13, Theorem 8.7]) that the rational equivariant cobordism of a variety $X$ with the action of a reductive group is the subgroup of invariants under the Weyl group action on the equivariant cobordism of $X$ for the action of a maximal torus of $G$. This reduces most of the computations of the cobordism ring of spherical varieties to the case of the cobordism ring of toric varieties. The results of this paper will be used in [10] to compute the cobordism ring of certain spherical varieties. If $X$ is a complex toric variety, then the techniques of this paper and the known relation between the algebraic and the complex cobordism can be easily used to compute the complex cobordism of any smooth toric variety. The complex cobordism ring of toric manifolds in topology were described earlier by Buchstaber and Ray in [4].

We end the introduction with a brief outline of this paper. We recollect all the definitions and the relevant basic properties of the ordinary and the equivariant cobordism in the next section. In Section 3, we establish some preliminary results which are frequently used in this paper. We also introduce the notion of cohomological rigidity for the equivariant cobordism in this section and describe its consequences. In Sections 4 and 6, our goal is to prove our main intermediate result which describes the equivariant cobordism ring as a subring of the equivariant cobordism rings of the orbits corresponding to the maximal cones in the associated fan. This is the crucial ingredient in the proof of our main result about the equivariant cobordism ring of smooth toric varieties. In Section 7, we prove Theorem 7.1 which describes the equivariant cobordism ring in terms of a Stanley-Reisner algebra. The ordinary cobordism ring of a smooth toric variety is obtained in the final section using Theorem 7.1 and [14, Theorem 3.4].

2. Recollection of equivariant cobordism

Let $k$ be a field of characteristic zero. Since we shall be concerned with the study of schemes with group actions and the associated quotient schemes, and since such quotients often require the original scheme to be quasi-projective, we shall assume throughout this paper that all schemes over $k$ are quasi-projective.

In this section, we briefly recall the definition of equivariant algebraic cobordism and some of its main properties from [13]. Since most of the results of [13] will be repeatedly used in this text, we summarize them here for reader’s convenience. For the definition and all details about the algebraic cobordism used in this paper, we refer the reader to the work of Levine and Morel [18].

Notations. We shall denote the category of quasi-projective $k$-schemes by $\mathcal{V}_k$. By a scheme, we shall mean an object of $\mathcal{V}_k$. The category of smooth quasi-projective schemes will be denoted by $\mathcal{V}^s_k$. If $G$ is a linear algebraic group over $k$, we shall denote the category of quasi-projective $k$-schemes with a $G$-action and $G$-equivariant maps by $\mathcal{V}_G$. The associated category of smooth $G$-schemes will be denoted by $\mathcal{V}^s_G$. All $G$-actions in this paper will be assumed to be linear. Recall that this means that all $G$-schemes are assumed to admit $G$-equivariant ample line bundles. This assumption is always satisfied for normal schemes (cf. [20, Theorem 2.5], [21, 5.7]). In particular, any action of $G$ on a toric variety is linear.
Recall that the Lazard ring \( \mathbb{L} \) is a polynomial ring over \( \mathbb{Z} \) on infinite but countably many variables and is given by the quotient of the polynomial ring \( \mathbb{Z}[A_{ij} \mid (i,j) \in \mathbb{N}^2] \) by the relations, which uniquely define the universal formal group law \( F_L \) of rank one on \( \mathbb{L} \). Recall that a cobordism cycle over a \( k \)-scheme \( X \) is a family \( \alpha = [Y \xrightarrow{f} X, L_1, \ldots, L_r] \), where \( Y \) is a smooth scheme, the map \( f \) is projective, and \( L_i \)'s are line bundles on \( Y \). Here, one allows the set of line bundles to be empty. The degree of such a cobordism cycle is defined to be \( \text{deg}(\alpha) = \dim_k(Y) - r \) and its codimension is defined to be \( \dim(X) - \text{deg}(\alpha) \). If \( Z_*(X) \) is the free abelian group generated by the cobordism cycles of the above type with \( Y \) irreducible, then \( Z_*(X) \) is graded by the degree of cycles. The algebraic cobordism group of \( X \) is defined as

\[
\Omega_*(X) = \frac{Z_*(X)}{R_*(X)},
\]

where \( R_*(X) \) is the graded subgroup generated by relations which are determined by the dimension and the section axioms and the above formal group law. If \( X \) is equi-dimensional, we set \( \Omega^i(X) = \Omega_{\dim(X)-i}(X) \) and grade \( \Omega^i(X) \) by the codimension of the cobordism cycles. It was shown by Levine and Pandharipande \([19]\) that the cobordism group \( \Omega_*(X) \) can also be defined as the quotient

\[
(2.1) \quad \Omega_*(X) = \frac{Z'_*(X)}{R'_*(X)},
\]

where \( Z'_*(X) \) is the free abelian group on cobordism cycles \([Y \xrightarrow{f} X]\) with \( Y \) smooth and irreducible and \( f \) projective. The graded subgroup \( R'_*(X) \) is generated by cycles satisfying the relation of double point degeneration.

Let \( X \) be a \( k \)-scheme of dimension \( d \). For \( j \in \mathbb{Z} \), let \( Z_j \) be the set of all closed subschemes \( Z \subset X \) such that \( \dim_k(Z) \leq j \) (we assume \( \dim(\emptyset) = -\infty \)). The set \( Z_j \) is then ordered by the inclusion. For \( i \geq 0 \), we set

\[
\Omega_i(Z_j) = \lim_{Z \in Z_j} \Omega_i(Z) \quad \text{and} \quad \Omega_*(Z_j) = \bigoplus_{i \geq 0} \Omega_i(Z_j).
\]

It is immediate that \( \Omega_*(Z_j) \) is a graded \( \mathbb{L} \)-module and there is a graded \( \mathbb{L} \)-linear map \( \Omega_*(Z_j) \to \Omega_*(X) \). We define \( F_j \Omega_*(X) \) to be the image of the natural \( \mathbb{L} \)-linear map \( \Omega_*(Z_j) \to \Omega_*(X) \). In other words, \( F_j \Omega_*(X) \) is the image of all \( \Omega_*(W) \to \Omega_*(X) \), where \( W \to X \) is a projective map such that \( \dim(\text{Image}(W)) \leq j \). One checks at once that there is a canonical niveau filtration

\[
(2.2) \quad 0 = F_{-1} \Omega_*(X) \subseteq F_0 \Omega_*(X) \subseteq \cdots \subseteq F_{d-1} \Omega_*(X) \subseteq F_d \Omega_*(X) = \Omega_*(X).
\]

2.1. Equivariant cobordism. In this text, \( G \) will denote a linear algebraic group of dimension \( g \) over \( k \). This group will be a torus for the most parts of this paper. All representations of \( G \) will be finite dimensional. Recall form \([13]\) that for any integer \( j \geq 0 \), a good pair \((V_j, U_j)\) corresponding to \( j \) for the \( G \)-action is a pair consisting of a \( G \)-representation \( V_j \) and an open subset \( U_j \subset V_j \) such that the codimension of the complement is at least \( j \) and \( G \) acts freely on \( U_j \) with quotient \( U_j/G \) a quasi-projective scheme. It is known that such good pairs always exist.
Let $X$ be a $k$-scheme of dimension $d$ with a $G$-action. For $j \geq 0$, let $(V_j, U_j)$ be an $l$-dimensional good pair corresponding to $j$. For $i \in \mathbb{Z}$, if we set
\begin{equation}
\Omega^G_i(X)_j = \Omega_{i+l-g} \left( X \times U_j \right) \left/ \Omega_{i+l-g} \left( X \times U_j \right) \right.,
\end{equation}
then it is known that $\Omega^G_i(X)_j$ is independent of the choice of the good pair $(V_j, U_j)$. It is also known that for each $j \geq 0$, $\Omega^G_* (X)_j = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i (X)_j$ is a graded $\mathbb{L}$-module.

Moreover, there is a natural surjective map $\Omega^G_* (X)_{j'} \twoheadrightarrow \Omega^G_* (X)_j$ of graded $\mathbb{L}$-modules for $j' \geq j \geq 0$.

**Definition 2.1.** Let $X$ be a $k$-scheme of dimension $d$ with a $G$-action. For any $i \in \mathbb{Z}$, we define the *equivariant algebraic cobordism* of $X$ to be
\[ \Omega^G_i(X) = \lim_{j \to -\infty} \Omega^G_i(X)_j. \]

The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega^G_i(X)$ can be non-zero for any $i \in \mathbb{Z}$. We set
\[ \Omega^G_* (X) = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i (X). \]

If $X$ is an equi-dimensional $k$-scheme with $G$-action, we let $\Omega^G_i(X) = \Omega^G_{d-i}(X)$ and $\Omega^G_* (X) = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i (X)$. It is known that if $G$ is trivial, then the $G$-equivariant cobordism reduces to the ordinary one.

**Remark 2.2.** It is easy to check from the above definition of the niveau filtration that if $X$ is a smooth and irreducible $k$-scheme of dimension $d$, then $F_j \Omega_i (X) = F^{d-j} \Omega^{d-i} (X)$, where $F^* \Omega^* (X)$ is the coniveau filtration used in [6]. Furthermore, one also checks in this case that if $G$ acts on $X$, then
\begin{equation}
\Omega^G_i(X) = \lim_{j \to -\infty} \frac{\Omega^G_i \left( X \times U_j \right)}{F^j \Omega^G_i \left( X \times U_j \right)},
\end{equation}
where $(V_j, U_j)$ is a good pair corresponding to any $j \geq 0$. Thus the above definition of the equivariant cobordism coincides with that of [6] for smooth schemes.

The following important result shows that if we suitably choose a sequence of good pairs $\{(V_j, U_j)\}_{j \geq 0}$, then the above equivariant cobordism group can be computed without taking quotients by the niveau filtration. This is often very helpful in computing the equivariant cobordism groups. It is moreover known [7] that such a sequence of good pairs always exist.

**Theorem 2.3.** (cf. [13 Theorem 6.1]) Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of $l_j$-dimensional good pairs such that
(i) $V_{j+1} = V_j \oplus W_j$ as representations of $G$ with $\dim(W_j) > 0$ and
(ii) $U_j \oplus W_j \subset U_{j+1}$ as $G$-invariant open subsets.
Then for any scheme $X$ as above and any $i \in \mathbb{Z}$,
\[ \Omega^G_i(X) \cong \lim_{\leftarrow j} \Omega^G_{i+l_j-g} \left( X^G \times U_j \right). \]

For equi-dimensional schemes, we shall write the (equivariant) cobordism groups cohomologically. The $G$-equivariant cobordism group $\Omega^*(k)$ of the ground field $k$ is denoted by $\Omega^*(BG)$ and is called the cobordism of the classifying space of $G$. We shall often write it as $S(G)$.

2.2. Change of groups. If $H \subset G$ is a closed subgroup of dimension $h$, then any $l$-dimensional good pair $(V_j, U_j)$ for $G$-action is also a good pair for the induced $H$-action. Moreover, for any $X \in \mathcal{V}_G$ of dimension $d$, $X^H \times U_j \to X^G \times U_j$ is an étale locally trivial $G/H$-fibration and hence a smooth map (cf. [2, Theorem 6.8]) of relative dimension $g-h$. This induces the pull-back map

\[ r^G_{H,X} : \Omega^G_*(X) \to \Omega^H_*(X). \]

Taking $H = \{1\}$, we get the forgetful map

\[ r^G : \Omega^G_*(X) \to \Omega_*(X) \]

from the equivariant to the non-equivariant (ordinary) cobordism groups. Since $r^G_{H,X}$ is obtained as a pull-back under the smooth map, it commutes with any projective push-forward and smooth pull-back (cf. Theorem 2.5).

The equivariant cobordism for the action of a group $G$ is related with the equivariant cobordism for the action of the various subgroups of $G$ by the following. We refer to [loc. cit., Proposition 5.5] for a proof.

**Proposition 2.4 (Morita Isomorphism).** Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then there is a canonical isomorphism

\[ \Omega^G_*(G^H \times X) \cong \Omega^H_*(X). \]

2.3. Fundamental class of cobordism cycles. Let $X \in \mathcal{V}_G$ and let $Y \xrightarrow{f} X$ be a morphism in $\mathcal{V}_G$ such that $Y$ is smooth of dimension $d$ and $f$ is projective. For any $j \geq 0$ and any $l$-dimensional good pair $(V_j, U_j)$, $[Y \xrightarrow{f_G} X_G]$ is an ordinary cobordism cycle of dimension $d+l-g$ by [loc.cit., Lemma 5.3] and hence defines an element $\alpha_j \in \Omega^G_d(X)_j$. Moreover, it is evident that the image of $\alpha_j$ is $\alpha_j$ for $j' \geq j$.

Hence we get a unique element $\alpha \in \Omega^G_d(X)$, called the $G$-equivariant fundamental class of the cobordism cycle $[Y \xrightarrow{f} X]$. We also see from this more generally that if $[Y \xrightarrow{f} X, L_1, \ldots, L_r]$ is as above with each $L_i$ a $G$-equivariant line bundle on $Y$, then this defines a unique class in $\Omega^G_{d-r}(X)$. It is interesting question to ask under what conditions on the group $G$, the equivariant cobordism group $\Omega^G_d(X)$ is generated by the fundamental classes of $G$-equivariant cobordism cycles on $X$ as $S(G)$-module. This is known to be true for torus actions by [14, Theorem 4.11].

2.4. Basic properties. The following result summarizes the basic properties of the equivariant cobordism.
Theorem 2.5. (cf. \cite{13} Theorems 5.1, 5.4) The equivariant algebraic cobordism satisfies the following properties.

(i) Functoriality: The assignment $X \mapsto \Omega^*_e(X)$ is covariant for projective maps and contravariant for smooth maps in $\mathcal{V}_G$. It is also contravariant for l.c.i. morphisms in $\mathcal{V}_G$. Moreover, for a fiber diagram

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
| & \downarrow f & | \\
Y' & \xrightarrow{g} & Y
\end{array}$$

in $\mathcal{V}_G$ with $f$ projective and $g$ smooth, one has $g^* \circ f_* = f'^* \circ g'^*$.

(ii) Localization: For a $G$-scheme $X$ and a closed $G$-invariant subscheme $Z \subset X$ with complement $U$, there is an exact sequence

$$\Omega^*_e(Z) \rightarrow \Omega^*_e(X) \rightarrow \Omega^*_e(U) \rightarrow 0.$$ 

(iii) Homotopy: If $f : E \rightarrow X$ is a $G$-equivariant vector bundle, then $f^* : \Omega^*_e(X) \xrightarrow{\cong} \Omega^*_e(E)$.

(iv) Chern classes: For any $G$-equivariant vector bundle $E \rightarrow X$ of rank $r$, there are equivariant Chern class operators $c^G_i(E) : \Omega^*_e(X) \rightarrow \Omega^*_e(X)$ for $0 \leq m \leq r$ with $c^G_0(E) = 1$. These Chern classes have same functoriality properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.

(v) Free action: If $G$ acts freely on $X$ with quotient $Y$, then $\Omega^*_e(X) \xrightarrow{\cong} \Omega^*_e(Y)$.

(vi) Exterior Product: There is a natural product map

$$\Omega^*_e(X) \otimes \Omega^*_e(X') \rightarrow \Omega^*_e(X \times X').$$

In particular, $\Omega^*_e(k)$ is a graded $\mathbb{L}$-algebra and $\Omega^*_e(X)$ is a graded $\Omega^*_e(k)$-module for every $X \in \mathcal{V}_G$. For $X$ smooth, the pull-back via the diagonal $X \hookrightarrow X \times X$ turns $\Omega^*_e(X)$ into an $S(G)$-algebra.

(vii) Projection formula: For a projective map $f : X' \rightarrow X$ in $\mathcal{V}_G^S$, one has for $x \in \Omega^*_e(X)$ and $x' \in \Omega^*_e(X')$, the formula: $f_*(x') = f_*(x') \cdot x$.

2.5. Formal group law. Let $T$ be a split torus of rank $n$ acting on a smooth variety $X$. Since $X$ is smooth, the commutative sub-$\mathbb{L}$-algebra (under composition) of $\text{End}_{\mathbb{L}}(\Omega^*_e(X))$ generated by the Chern classes of the vector bundles is canonically identified with a sub-$\mathbb{L}$-algebra of the cobordism ring $\Omega^*_e(X)$ via the identification $c^T_i(E) \mapsto c^T_i(E) = c^T_i(E) \left( X \overset{id}{\rightarrow} X \right)$. We shall denote this image also by $c^T_i(E)$ or, by $c^T_i$ if the underlying vector bundle is understood. In this paper, we shall view the (equivariant) Chern classes as elements of the (equivariant) cobordism ring of a smooth variety in this sense.

We recall from \cite{13} that the first Chern class of the tensor product of two equivariant line bundles on $X$ satisfies the formal group law of the ordinary cobordism. That is, for $L_1, L_2 \in \text{Pic}^T(X)$, one has

$$c^T_i(L_1 \otimes L_2) = F \left( c^T_i(L_1), c^T_i(L_2) \right),$$

where $F(u, v) = u + v + uv \sum_{i,j \geq 1} a_{i,j} u^{i-1} v^{j-1}$, $a_{i,j} \in \mathbb{L}_{i-j}$ is the graded power series in the graded power series ring $\mathbb{L}\left[[u, v]\right]$ which defines the universal (commutative) formal group law on $\mathbb{L}$. It is known that even though $c^T_i(L)$ is not nilpotent in
\( \Omega_T^* (X) \) for \( L \in \text{Pic}^T (X) \) (unlike in the ordinary case), \( F \left( c_1^T (L_1), c_1^T (L_2) \right) \) is a well-defined element of \( \Omega_T^1 (X) \). In particular, there is a map of pointed sets

\[
\text{Pic}^T (X) \to \Omega_T^1 (X), \quad L \mapsto c_1^T (L)
\]

such that \( c_1^T (L_1 \otimes L_1) = F \left( c_1^T (L_1), c_1^T (L_2) \right) \).

One also knows that the formal group law \( F(u,v) \) has the following properties.

(i) \( F(u,0) = F(0,u) = u \)

(ii) \( F(u,v) = F(v,u) \)

(iii) \( F(u,F(v,w)) = F(F(u,v), w) \), and

(iv) There exists (unique) \( \rho(u) \in \mathbb{L}[u] \) such that \( F(u,\rho(u)) = 0 \).

We write \( F(u,v) \) as \( u_F + v_F \). We often denote \( \rho(u) \) by \( [-1]_F u \). We often write \( F(u,v) \) as \( u_F + v_F \). Inductively, we have \( [n]_F u = [n-1]_F u + [1]_F u \) if \( n \geq 1 \) and \( [n]_F u = [-n]_F \rho(u) \) if \( n \leq 0 \). The sum \( \sum_{i=1}^{m} [n_i]_F u_i \) will mean \( [n_1]_F u_1 + [n_2]_F u_2 + \cdots + [n_m]_F u_m \) for \( n_i \in \mathbb{Z} \).

2.6. Cobordism ring of classifying spaces. Let \( A = \bigoplus_{j \in \mathbb{Z}} A_j \) be a commutative graded \( R \)-algebra with \( R \subset A_0 \) and \( d \geq 0 \). Let \( S^{(n)} = \bigoplus_{i \in \mathbb{Z}} S_i \) be the graded ring such that \( S_i \) is the set of formal power series in variables \( t = (t_1, \ldots, t_n) \) of the form \( f(t) = \sum_{m(t) \in \mathcal{C}} a_{m(t)} m(t) \), where \( a_{m(t)} \) is a homogeneous element in \( A \) of degree \( |a_{m(t)}| \) such that \( |a_{m(t)}| + |m(t)| = i \). Here, \( \mathcal{C} \) is the set of all monomials in \( t = (t_1, \ldots, t_n) \) and \( |m(t)| = i_1 + \cdots + i_n \) if \( m(t) = t_1^{i_1} \cdots t_n^{i_n} \). One often writes this graded power series ring as \( A[[t]]_{\text{gr}} \) to distinguish it from the usual formal power series ring \( A[[t]] \).

Notice that if \( A \) is only non-negatively, then \( S^{(n)} \) is nothing but the standard polynomial ring \( A[t_1, \ldots, t_n] \) over \( A \). It is also easy to see that \( S^{(n)} \) is indeed a graded ring which is a subring of the formal power series ring \( A[[t_1, \ldots, t_n]] \). The following result summarizes some basic properties of these rings. The proof is straightforward and is left as an exercise.

**Lemma 2.6.** (i) There are inclusions of rings \( A[t_1, \ldots, t_n] \subset S^{(n)} \subset A[[t_1, \ldots, t_n]] \), where the first is an inclusion of graded rings.

(ii) These inclusions are analytic isomorphisms with respect to the \( t \)-adic topology. In particular, the induced maps of the associated graded rings

\[
A[t_1, \ldots, t_n] \to \text{Gr}_t S^n \to \text{Gr}_t A[[t_1, \ldots, t_n]]
\]

are isomorphisms.

(iii) \( S^{(n-1)}[[t_i]]_{\text{gr}} \cong S^{(n)} \).

(iv) \( \frac{S^{(n)}}{(t_{i_1} \cdots t_{i_r})} \cong S^{(n-r)} \) for any \( n \geq r \geq 1 \), where \( S^{(0)} = A \).

(v) The sequence \( \{t_1, \ldots, t_n\} \) is a regular sequence in \( S^{(n)} \).

(vi) If \( A = R[x_1, x_2, \ldots] \) is a polynomial ring with \( |x_i| < 0 \) and \( \lim_{i \to \infty} |x_i| = -\infty \), then \( S^{(n)} \cong \lim_{i} A[x_1, \ldots, x_i][[t]]_{\text{gr}} \).

Since we shall mostly be dealing with the graded power series rings in this text, we make the convention of writing \( A[[t]]_{\text{gr}} \) as \( A[[t]] \), while the standard formal power series ring will be written as \( A[[t]] \). We shall also write \( S^{(n)} \) simply as \( S \) when the number of \( t \)-parameters is fixed.
It is known [13, Proposition 6.5] that if $T$ is a split torus of rank $n$ and if \{\chi_1, \ldots, \chi_n\} is a chosen basis of the character group $\hat{T}$, then there is a canonical isomorphism of graded rings

$$L[[t_1, \cdots, t_n]] \xrightarrow{\cong} \Omega^*(BT), \quad t_i \mapsto e_i^*(L_{\chi_i}).$$

Here, $L_\chi$ is the $T$-equivariant line bundle on $\text{Spec}(k)$ corresponding to the character $\chi$ of $T$. We shall write $S(T) = \Omega^*(BT)$ simply as $S$ in this text if the underlying torus is fixed. One also has isomorphisms

$$\Omega^*(BGL_n) \cong L[[\gamma_1, \cdots, \gamma_n]] \quad \text{and} \quad \Omega^*(BSL_n) \cong L[[\gamma_2, \cdots, \gamma_n]]$$

of graded $L$-algebras, where $\gamma_i$'s are the elementary symmetric polynomials in $t_1, \cdots, t_n$ that occur in $\Omega^*(BT)$.

2.7. Comparison with equivariant Chow groups. In this paper, we fix the following notation for the tensor product while dealing with inverse systems of modules over a commutative ring. Let $A$ be a commutative ring with unit and let \{\{L_n\}\} and \{\{M_n\}\} be two inverse systems of $A$-modules with inverse limits $L$ and $M$ respectively. Following [22], one defines the topological tensor product of $L$ and $M$ by

$$(2.12) \quad L \hat{\otimes} A M := \lim_{n} (L_n \otimes_A M_n).$$

In particular, if $D$ is an integral domain with quotient field $F$ and if \{\{A_n\}\} is an inverse system of $D$-modules with inverse limit $A$, one has $A \hat{\otimes} D F = \lim_{n} (A_n \otimes_D F)$.

The examples $\mathbb{Z}_{(p)} = \lim_{n} \mathbb{Z}/p^n$ and $\mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q} \to \lim_{n} \mathbb{Z}_{(x^n)} [[x]] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[x]]$ show that the map $A \otimes_D F \to A \hat{\otimes} D F$ is in general neither injective nor surjective.

If $R$ is a $\mathbb{Z}$-graded ring and if $M$ and $N$ are two $R$-graded modules, then recall that $M \otimes_R N$ is also a graded $R$-module given by the quotient of $M \otimes R_0 N$ by the graded submodule generated by the homogeneous elements of the type $ax \otimes y - x \otimes ay$ where $a, x$ and $y$ are the homogeneous elements of $R, M$ and $N$ respectively. If all the graded pieces $M_i$ and $N_i$ are the limits of inverse systems \{\{M_i\}\} and \{\{N_i\}\} of $R_0$-modules, we define the graded topological tensor product as $M \hat{\otimes}_R N = \bigoplus_{i \in \mathbb{Z}} (M \hat{\otimes}_R N)_i$, where

$$(2.13) \quad (M \hat{\otimes}_R N)_i = \lim_{\lambda} \left( \bigoplus_{j, j' = i \lambda} \frac{M_j \otimes R_0 \ N_j'}{(ax \otimes y - x \otimes ay)} \right).$$

Notice that this reduces to the ordinary tensor product of graded $R$-modules if the underlying inverse systems are trivial.

It is known that there is a natural map $\Phi_X : \Omega^G_*(X) \to CH^G_*(X)$ of graded $\mathbb{L}$-modules, where $CH^G_*(X)$ are the $G$-equivariant Chow groups of $X$ defined by Totaro [22] and Edidin-Graham [7]. The above map is in fact a morphism of $S(G)$-modules, which is a ring homomorphism if $X$ is smooth. The following is the analogue of the corresponding result of Levine and Morel [18] for the equivariant cobordism.

**Theorem 2.7.** (cf. [13, Proposition 7.1]) The map $\Phi_X$ induces an isomorphism of graded $\mathbb{L}$-modules

$$\Phi_X : \Omega^G_*(X) \hat{\otimes}_L \mathbb{Z} \xrightarrow{\cong} CH^G_*(X).$$
This is a ring isomorphism if \( X \) is smooth.

3. Preliminary results

In this section, we collect some preliminary results about the intersection theory of the cobordism cycles and their equivariant analogues. We also recall the notion of cohomological rigidity for a regular embedding of \( G \)-schemes and its consequence for the structure of the equivariant cobordism ring of smooth \( G \)-schemes.

Lemma 3.1. Let

\[
\begin{array}{ccc}
W & \overset{g'}{\longrightarrow} & Z \\
\downarrow{f'} & & \downarrow{f} \\
Y & \overset{g}{\longrightarrow} & X
\end{array}
\]

be a Cartesian square of closed immersions in \( \mathcal{V}^S_k \) such that \( f \) and \( g \) are transverse to each other. Then \( g^* \circ f_* = f'_* \circ g'^*: \Omega^*(Z) \to \Omega^*(Y) \).

Proof. Let \( \alpha = [Z' \overset{p}{\rightarrow} Z] \) be a cobordism cycle on \( Z \) such that \( Z' \) is smooth and irreducible and \( p \) is projective. It suffices to show that the result holds for \( \alpha \) (cf. (2.1)). By [17, Lemma 7.1], we can assume that \( p \) is transverse to \( g' \). In particular, \( f \circ p \) is transverse to \( g \). Setting \( W' = W \times_Z Z' \), we see that \( W' \) is smooth and there is a commutative diagram

\[
\begin{array}{ccc}
W' & \overset{g''}{\longrightarrow} & Z' \\
\downarrow{p'} & & \downarrow{p} \\
W & \overset{g'}{\longrightarrow} & Z \\
\downarrow{f'} & & \downarrow{f} \\
Y & \overset{g}{\longrightarrow} & X
\end{array}
\]

where all the squares are Cartesian. We now have

\[
g^* \circ f_* (\alpha) = g^* \left( \left[ Z' \overset{f'\circ p'}{\longrightarrow} X \right] \right) \\
= [W' \overset{f'\circ p'}{\longrightarrow} Y] \\
= f'_* \left( [W' \overset{p'}{\rightarrow} W] \right) \\
= f'_* \circ g'^* \left( \left[ Z' \overset{p}{\rightarrow} Z \right] \right) \\
= f'_* \circ g'^* (\alpha),
\]

where the second and the fourth equality hold by the transversality of the vertical and the horizontal arrows in the bottom and top squares (cf. [18, 5.1.3]). This completes the proof. \( \square \)

Corollary 3.2. Let \( G \) be a linear algebraic group over \( k \) and let

\[
\begin{array}{ccc}
W & \overset{g'}{\longrightarrow} & Z \\
\downarrow{f'} & & \downarrow{f} \\
Y & \overset{g}{\longrightarrow} & X
\end{array}
\]
be a Cartesian square of closed immersions in $\mathcal{V}_G^S$ such that $f$ and $g$ are transverse to each other. Then $g^* \circ f_* = f'_* \circ g'^* : \Omega^*_G(Z) \to \Omega^*_G(Y)$.

**Proof.** We choose a sequence $\{(V_j, U_j)\}_j$ of $l_j$-dimensional good pairs as in Theorem 2.3 and for any $F \in \mathcal{V}_G^S$, let $F_j$ denote the mixed quotient $F \times U_j$. We then see that

$$W_j \xrightarrow{g'_j} Z_j \xrightarrow{f'_j} Y_j \xrightarrow{g_j} X_j$$

is a Cartesian square of closed immersions in $\mathcal{V}_G^S$ such that $f_j$ and $g_j$ are transverse to each other. Applying Lemma 3.1, we get

$$g'^* \circ f_j^* = f'_j \circ g^* : \Omega^*_G(Z_j) \to \Omega^*_G(Y_j) \text{ for each } j \geq 0.$$  

Since the push-forward and pull-back maps on equivariant cobordism groups are defined by taking the limit over these maps on the sequence of mixed quotients, taking the inverse limit over $j \geq 0$ in (3.4) and applying Theorem 2.3, we get our desired assertion.

**Lemma 3.3.** Let $G$ be as above and let $Y \xrightarrow{f} X$ be a closed immersion in $\mathcal{V}_G^S$. Then $f_* \circ f^*(\alpha) = \alpha f^*(1)$ in $\Omega^*_G(X)$.

**Proof.** Although this follows from the more general projection formula of Theorem 2.5, we give a simple proof of special case of closed immersion. We first prove this in the non-equivariant case. As in the proof of Lemma 3.1, we can assume that $\alpha = [Z \xrightarrow{g} X]$, where $g$ is transverse to $f$. In that case, we have

$$f_* \circ f^*(\alpha) = [Z \times_X Y \to X] = \left([Z \xrightarrow{f} X] \cdot [Z \xrightarrow{g} X]\right),$$

where the second equality holds by the transversality condition and the definition of intersection of cobordism cycles (cf. [18, Sections 5,6]). The equivariant case is reduced to the non-equivariant one exactly as in the proof of Corollary 3.2.

The following is the equivariant version of the self-intersection formula for cobordism. We refer the reader to [14, Proposition 3.1] for a proof.

**Proposition 3.4** (Self-intersection formula). Let $G$ be as above and let $Y \xrightarrow{f} X$ be a regular $G$-equivariant embedding in $\mathcal{V}_G^S$ of pure codimension $d$ and let $N_{Y/X}$ denote the equivariant normal bundle of $Y$ inside $X$. Then one has for every $y \in \Omega^*_G(Y)$, $f^* \circ f_*(y) = c_d^G(N_{Y/X}) \cdot y$.

**Lemma 3.5.** Let $G$ be as above and let $Y \in V_G^S$. Let $G$ act diagonally on $X = Y \times \mathbb{P}_k^n$ by acting trivially on $\mathbb{P}_k^n$. There is a canonical isomorphism

$$\bigoplus_{p=0}^n \Omega^{i-p}_G(Y) \to \Omega^i_G(X)$$

$$(a_0, \ldots, a_n) \mapsto \sum_{j=0}^n \pi^*(a_p)\zeta^p$$
where \( \pi : X \to Y \) is the projection and \( \zeta \) is the class of the tautological line bundle on \( \mathbb{P}^n_k \). In particular, there is an \( S(G) \)-algebra isomorphism

\[
\Omega^*_G(X) \overset{\cong}{\to} \Omega^*_G(Y) \boxtimes_B \Omega^*(\mathbb{P}^n).
\]

**Proof.** We choose a sequence \( \{(V_j, U_j)\}_j \) of good pairs for the \( G \)-action as in Theorem 2.3. We then have \( X_j = X \times U_j = Y_j \times \mathbb{P}^n \) and hence the projective bundle formula for the non-equivariant cobordism gives an isomorphism

\[
\bigoplus_{p=0}^n \Omega^{i-p}(Y_j) \to \Omega^i(X_j).
\]

Taking the inverse limit over \( j \geq 0 \) and using [13, Theorem 6.1], we conclude the proof. \( \square \)

### 3.1. Cohomological Rigidity in cobordism

Recall that for homomorphisms \( A_i \overset{f_i}{\to} B, i = 1, 2 \) of abelian groups, \( A_1 \times A_2 \) denotes the fiber product \( \{(a_1, a_2) | f_1(a_1) = f_2(a_2)\} \). Let \( Y \hookrightarrow X \) be a closed embedding in \( \mathcal{V}_G^S \) of codimension \( d \geq 0 \) and let \( N_{Y/X} \) denote the normal bundle of \( Y \) in \( X \). Following [23] and [12], we shall say that \( Y \) is cohomologically rigid inside \( X \) if \( c^G_d(N_{Y/X}) \) is a non-zero divisor in the cobordism ring \( \Omega^*_G(Y) \). The cohomological rigidity has the following important consequence.

**Proposition 3.6.** Let \( Y \hookrightarrow X \) be a closed embedding in \( \mathcal{V}_G^S \) such that \( Y \) is cohomologically rigid inside \( X \). Let \( Y \overset{i}{\hookrightarrow} X \) and \( U \overset{j}{\hookrightarrow} X \) be the inclusion maps, where \( U \) is the complement of \( Y \) in \( X \). Then:

(i) The localization sequence

\[
0 \to \Omega^*_G(Y) \overset{i_*}{\longrightarrow} \Omega^*_G(X) \overset{j^*}{\longrightarrow} \Omega^*_G(U) \to 0
\]

is exact.

(ii) The restriction ring homomorphisms

\[
\Omega^*_G(X) \overset{(i^* j^*)}{\longrightarrow} \Omega^*_G(Y) \times \Omega^*_G(U)
\]

give an isomorphism of rings

\[
\Omega^*_G(X) \overset{\cong}{\longrightarrow} \Omega^*_G(Y) \times \Omega^*_G(U),
\]

where \( \Omega^*_G(Y) = \Omega^*_G(Y) / (c^G_d(N_{Y/X})) \), and the maps

\[
\Omega^*_G(Y) \to \Omega^*_G(Y), \ \Omega^*_G(U) \to \Omega^*_G(Y)
\]

are respectively, the natural surjection and the map

\[
\Omega^*_G(U) = \frac{\Omega^*_G(X)}{i_* (\Omega^*_G(Y))} \overset{i_*}{\longrightarrow} \Omega^*_G(Y) / (c^G_d(N_{Y/X})) = \Omega^*_G(Y),
\]

which is well-defined by Proposition 3.4.

**Proof.** Cf. [14, Proposition 4.1]. \( \square \)
4. Stratification of toric varieties

Recall that a split diagonalizable group $G$ over $k$ is a commutative linear algebraic group whose identity component is a split torus. Let $G$ be a split diagonalizable group acting on a smooth variety $X$. Following the notations of [23], for any $s \geq 0$, we let $X_{\leq s} \subseteq X$ be the open subset of points whose stabilizers have dimension at most $s$. We shall often write $X_{\leq s-1}$ also as $X_{<s}$. Let $X_s = (X_{\leq s} \setminus X_{<s})$ denote the locally closed subset of $X$, where the stabilizers have dimension exactly $s$. We think of $X_s$ as a subspace of $X$ with the reduced induced structure. It is clear that $X_{\leq s}$ and $X_s$ are $G$-invariant subspaces of $X$. Let $N_s$ denote the normal bundle of $X_s$ in $X_{\leq s}$, and let $N^0_s$ denote the complement of the 0-section in $N_s$. Then $G$ clearly acts on $N_s$. The following result of loc. cit. describes some properties of these subspaces.

**Proposition 4.1.** Let $s$ be non-zero integer.

(i) There exists a finite number of $s$-dimensional subtori $T_1, \ldots, T_r$ in $G$ such that $X_s$ is the disjoint union of the fixed point spaces $X_{s}^{T_j}$.

(ii) $X_s$ is smooth locally closed subvariety of $X$.

(iii) $N^0_s = (N_s)_{<s}$.

**Proof.** This is a special case of the more general result [23, Proposition 2.2] which holds for regular $G$-schemes over any connected and separated Noetherian base scheme. \qed

Since $X$ is smooth, it follows from Proposition 4.1 that $X_s$ is a smooth and closed subvariety of $X_{\leq s}$, which is itself smooth.

In this paper, we are interested in applying Proposition 4.1 to smooth toric varieties. For such varieties, the various strata $X_s$ have simple description in terms of the torus orbits. Since this description will be crucial for our results, we sketch it in some detail here and refer the reader to [9, Section 3] for proofs.

So let $T$ be a split torus of rank $n$ over $k$. Let $M = \text{Hom}(G_m, T)$ be its lattice of one-parameter subgroups and let $M^\vee$ be the character lattice of $T$. Let $\Delta$ be a fan in $M_\mathbb{R}$ and let $\Delta_1$ and $\Delta_{\text{max}}$ denote the subsets of the one-dimensional cones and the maximal cones in $\Delta$ respectively.

Let $X = X(\Delta)$ be the unique toric variety associated to the fan $\Delta$. We assume that every positive dimensional cone of $\Delta$ is generated by it edges such that the primitive vectors along these edges form a subset of a basis of $M$. This is equivalent to saying that $X$ is smooth. In this case, there is an one-to-one correspondence between the $T$-orbits in $X$ and the cones in $\Delta$. For every cone $\sigma \in \Delta$, the corresponding orbit $O_\sigma$ is isomorphic to the torus $T/T_\sigma$, where $T_\sigma$ is associated to the sublattice $M_\sigma$ of $M$ generated by $\sigma \cap M$. Under this isomorphism, the origin (identity point) of $T/T_\sigma$ corresponds to the distinguished $k$-rational point $x_\sigma$ of $O_\sigma$. In particular, for every $0 \leq s \leq n$, $X_s$ is of the form

\begin{equation}
X_s = \prod_{\dim(\sigma) = s} O_\sigma \cong \prod_{\dim(\sigma) = s} T/T_\sigma.
\end{equation}

We shall write $\tau \leq \sigma$ if $\tau$ is a face of $\sigma$ as cones in $\Delta$. The orbit closure $V_\sigma$ of $O_\sigma$ is the toric variety associated to the fan $\ast(\sigma) = \{ \tau \in \Delta | \sigma \leq \tau \}$, called the star of $\sigma$. Moreover, it is clear from the characterization of the smoothness of toric varieties that $V_\sigma$ is also smooth and is the union of all orbits $O_\tau$ such that $\tau$ is a face of $\sigma$. In particular, $O_\sigma$ is closed in $X$ if and only if $\sigma \in \Delta_{\text{max}}$. The following general result will play a crucial role in our study of the equivariant cobordism ring of toric varieties.

**Lemma 4.2.** Let $G$ be a split diagonalizable group and let $T \subseteq G$ be a subtorus of rank $r \geq 1$. Let $G$ act on $X = (G/T) \times \mathbb{P}^n_k$ diagonally by acting trivially on $\mathbb{P}^n_k$. Then $X$ is smooth and $X_s$ is a smooth locally closed subvariety of $X$ for any $s \geq 0$. The following result of loc. cit. describes some properties of these subspaces.

**Proof.** This is a special case of the more general result [23, Proposition 2.2] which holds for regular $G$-schemes over any connected and separated Noetherian base scheme. \qed

Since $X$ is smooth, it follows from Proposition 4.1 that $X_s$ is a smooth and closed subvariety of $X_{\leq s}$, which is itself smooth.

In this paper, we are interested in applying Proposition 4.1 to smooth toric varieties. For such varieties, the various strata $X_s$ have simple description in terms of the torus orbits. Since this description will be crucial for our results, we sketch it in some detail here and refer the reader to [9, Section 3] for proofs.

So let $T$ be a split torus of rank $n$ over $k$. Let $M = \text{Hom}(G_m, T)$ be its lattice of one-parameter subgroups and let $M^\vee$ be the character lattice of $T$. Let $\Delta$ be a fan in $M_\mathbb{R}$ and let $\Delta_1$ and $\Delta_{\text{max}}$ denote the subsets of the one-dimensional cones and the maximal cones in $\Delta$ respectively.

Let $X = X(\Delta)$ be the unique toric variety associated to the fan $\Delta$. We assume that every positive dimensional cone of $\Delta$ is generated by it edges such that the primitive vectors along these edges form a subset of a basis of $M$. This is equivalent to saying that $X$ is smooth. In this case, there is an one-to-one correspondence between the $T$-orbits in $X$ and the cones in $\Delta$. For every cone $\sigma \in \Delta$, the corresponding orbit $O_\sigma$ is isomorphic to the torus $T/T_\sigma$, where $T_\sigma$ is associated to the sublattice $M_\sigma$ of $M$ generated by $\sigma \cap M$. Under this isomorphism, the origin (identity point) of $T/T_\sigma$ corresponds to the distinguished $k$-rational point $x_\sigma$ of $O_\sigma$. In particular, for every $0 \leq s \leq n$, $X_s$ is of the form

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X_s = \prod_{\dim(\sigma) = s} O_\sigma \cong \prod_{\dim(\sigma) = s} T/T_\sigma.
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**Lemma 4.2.** Let $G$ be a split diagonalizable group and let $T \subseteq G$ be a subtorus of rank $r \geq 1$. Let $G$ act on $X = (G/T) \times \mathbb{P}^n_k$ diagonally by acting trivially on $\mathbb{P}^n_k$. Then $X$ is smooth and $X_s$ is a smooth locally closed subvariety of $X$ for any $s \geq 0$. The following result of loc. cit. describes some properties of these subspaces.

**Proof.** This is a special case of the more general result [23, Proposition 2.2] which holds for regular $G$-schemes over any connected and separated Noetherian base scheme. \qed
and via its natural action on the quotient $G/T$. Let $E$ be a $G$-equivariant vector bundle of rank $d$ on $X$ such that in the eigenspace decomposition of $E$ with respect to $T$, the submodule corresponding to the trivial character is zero. Then $c^*_d(E)$ is a non-zero divisor in $\Omega^*_G(X)$.

Proof. Set $D = G/T$ and let $q : X = D \times \mathbb{P}^n \to \text{Spec}(k)$ be the structure map. We choose a splitting (not necessarily canonical) $G = T \times D$ and apply [21] Lemma 3.5 to see that $E$ has a unique direct sum decomposition

$$E = \bigoplus_{i=1}^m E_{\chi_i} \otimes q^*(L_{\chi_i}).$$

Here, each $E_{\chi_i}$ is a $D$-bundle on $X$ and $L_{\chi_i}$ is the line bundle in $\text{Pic}_T(k)$ corresponding to an $1$-dimensional representation $\chi_i$ of $T$. Since $\text{rank}(E) = d$, we have by the Whitney sum formula, $c^*_d(E) = \prod_{i=1}^m c^*_d(E_{\chi_i} \otimes L_{\chi_i})$, where $d_i = \text{rank}(E_{\chi_i})$.

We can thus assume that $E = E_{\chi} \otimes L_{\chi}$, where $\chi$ is not a trivial character of $T$ by our assumption.

Since the $D$-equivariant vector bundles on $X$ are equivalent to ordinary vector bundles on $\mathbb{P}^n$, we can identify $E_{\chi}$ with an ordinary vector bundle of rank $d$ on $\mathbb{P}^n$. Moreover, we can apply Proposition 2.4 and Lemma 3.5 (cf. [14] Lemma 6.1) to see that there is a ring isomorphism

$$\Omega^*_G(X) \cong \Omega^*_T(\mathbb{P}^n) \cong \Omega^*(\mathbb{P}^n)[[t_1, \cdots, t_r]].$$

Thus, we are reduced to showing that $c^*_d(E_{\chi} \otimes L_{\chi})$ is a not a zero divisor in $\Omega^*(\mathbb{P}^n)[[t_1, \cdots, t_r]]$ if $E_{\chi}$ is a vector bundle of rank $d$ on $\mathbb{P}^n$ and $\chi$ is non-trivial character of $T$.

By [14], Lemma 6.2, there is a morphism $Y \xrightarrow{p} \mathbb{P}^n$ which is a composition of the affine and projective bundles such that $p^*(E_{\chi})$ is a direct sum of line bundles and the $S$-algebra map $\Omega^*_T(\mathbb{P}^n) \xrightarrow{p^*} \Omega^*_T(Y)$ is injective. We can thus replace $\mathbb{P}^n$ by $Y$ and assume that $E_{\chi}$ is a direct sum of line bundles. Set $E_{\chi} = \bigoplus_{i=1}^s L_i$ and $v_i = c_1(L_i)$. Since $\Omega^{> \dim(Y)}(Y) = 0$, we see that each $v_i$ is nilpotent in $\Omega^*(Y)$.

We can write $f = c^*_d(L_{\chi}) = \sum_{j=1}^r [m_j] F t_j$ with $m_j \neq 0$ for some $j$. The Whitney sum formula then yields $c^*_d(E_{\chi}) = \prod_{i=1}^s F(f, v_i)$. Moreover, as $Y$ is obtained from $\mathbb{P}^n$ by a composition of the affine and projective bundles, it follows from the projective bundle formula and the homotopy invariance of the ordinary cobordism that $\Omega^*(Y)$ is torsion-free. We now apply [14], Lemma 5.3 with $R = \mathbb{Z}$ and $A = \Omega^*(Y)$ to conclude that $c^*_d(E_{\chi})$ is a non-zero divisor in $\Omega^*_T(Y)$.

\[ \square \]

5. Specialization maps in cobordism

Our aim in the next two sections is to prove a decomposition theorem for the equivariant cobordism rings of smooth toric varieties. This decomposition theorem is motivated by the similar results for the equivariant $K$-theory in [23] and for the equivariant Chow groups in [12]. As for the Chow groups and $K$-theory, we prove our decomposition theorem using some specialization maps between certain equivariant cobordism rings. We construct these maps in this section. The main technical tool to do this is the deformation to the normal cone method. Since this technique will be used at many steps in the proofs, we briefly recall the construction from [8], Chapter 5 for reader’s convenience.
Let $X$ be a smooth $k$-scheme and let $Y \xrightarrow{f} X$ be a smooth closed subscheme of codimension $d \geq 1$. Let $M$ be the blow-up of $X \times \mathbb{P}^1$ along $Y \times \infty$. Then $Bl_Y(X)$ is a closed subscheme of $\widetilde{M}$ and one denotes its complement by $M$. There is a natural map $\pi : M \to \mathbb{P}^1$ such that $\pi^{-1}(\mathbb{A}^1) \cong X \times \mathbb{A}^1$ with $\pi$ the projection map and $\pi^{-1}(\infty) \cong X'$, where $X'$ is the total space of the normal bundle $N_{Y/X}$ of $Y$ in $X$. One also gets the following diagram, where all the squares and the triangles commute.

\begin{equation}
(5.1)
\begin{array}{c}
\xymatrix{ Y \ar[r]^{p_Y} \ar[d]_{f} & Y \times \mathbb{P}^1 \ar[r]^{i_{\infty}} & Y \\
\ar[r]^{u'} & \ar[r]^{j'} & F \\
X \ar[r]^{h} \ar[u]_{i} & M \ar[u]_{i} \ar[u]_{F'} & X' \ar[u]_{f'} \ar[u]_{j} \\
X \times \mathbb{A}^1. \ar[u]_{u} & \ar[u]_{i} & }
\end{array}
\end{equation}

In this diagram, all the vertical arrows are the closed embeddings, $i_0$ and $i_{\infty}$ are the obvious inclusions of $Y$ in $Y \times \mathbb{P}^1$ along the specified points, $i$ and $j$ are inclusions of the inverse images of $\infty$ and $\mathbb{A}^1$ respectively under the map $\pi$, $u$ and $f'$ are are zero section embeddings and $p_Y$ is the projection map. In particular, one has $p_Y \circ i_0 = p_Y \circ i_{\infty} = i_Y$.

We also make the observation here that in case $X$ is a $G$-variety and $Y$ is $G$-invariant, then by letting $G$ act trivially on $\mathbb{P}^1$ and diagonally on $X \times \mathbb{P}^1$, one gets a natural action of $\widetilde{G}$ on $M$, and all the spaces in the above diagram become $\widetilde{G}$-spaces and all the morphisms become $\widetilde{G}$-equivariant. This observation will be used in what follows.

5.1. Specialization maps. Let $T$ be a split torus of rank $n$ and let $X = X(\Delta)$ be a smooth toric variety corresponding to a fan $\Delta$ in $M = \text{Hom}(\mathbb{G}_m, T)$. For the rest of this paper, we fix a split torus $T$ and all toric varieties will be associated to fans in the lattice $M$ of one-parameter subgroups in $T$. There is a filtration of $X$ by $T$-invariant open subsets

$$\emptyset = X_{\leq -1} \subset X_{\leq 0} \subset \cdots \subset X_{\leq n} = X,$$

where $X_{\leq s}$ and $X_s$ are defined before. In particular, $T$ acts freely on $X_{\leq 0}$ and trivially on $X_{n}$. In fact $X_{\leq 0}$ is the dense orbit of $X$ isomorphic to $T$. We fix $1 \leq s \leq n$ and let $X_s \xrightarrow{f_s} X_{\leq s}$ and $X_{\leq s} \xrightarrow{g_s} X_{\leq s}$ denote the closed and the open embeddings respectively. Let $\pi : M_s \to \mathbb{P}^1$ be the deformation to the normal cone for the embedding $f_s$ as above. We have already observed that for the trivial action of $T$ on $\mathbb{P}^1$, $M_s$ has a natural $T$-action. Moreover, the deformation diagram (5.1) is a diagram of smooth $T$-spaces. For $0 \leq t \leq s$, we shall often denote the open subspace $(M_s)_{\leq t}$ of $M_s$ by $M_{s,t}$. The terms like $M_{s,t}$ and $M_{s,t}$ (and also for $N_s$) will have similar meaning in what follows. Since $T$ acts trivially on $\mathbb{P}^1$, it acts on $M_s$ fiberwise, and one has $N_s = \pi^{-1}(\infty)$ and

\begin{equation}
(5.2)
M_{s_{\leq t}} \cap \pi^{-1}(\mathbb{A}^1) = X_{\leq t} \times \mathbb{A}^1; \ M_{s,t} \cap \pi^{-1}(\mathbb{A}^1) = X_t \times \mathbb{A}^1.
\end{equation}
Let $N_{s,t} \hookrightarrow M_{s,t}$ and $X_{\leq t} \times \mathbb{A}^1 \xrightarrow{j_{s,t}} M_{s,t}$ denote the obvious closed and open embeddings. We define $i_{s,t}$ and $j_{s,t}$ similarly. Let $N_{s,t} \xrightarrow{g_{s,t}} N_{s,t}$ and $M_{s,t} \xrightarrow{g_{s,t}} M_{s,t}$ denote the other closed embeddings. One has a commutative diagram

$$
\begin{array}{ccc}
X_{\leq t} & \xrightarrow{g_{s,t}} & X_{\leq t} \times \mathbb{A}^1 \\
\downarrow f_{s,t} & & \downarrow f_{t}
\end{array}
\begin{array}{ccc}
X_{\leq s} & \xrightarrow{g_{s,t}} & X_{\leq s} \times \mathbb{A}^1 \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
M_{s} & \xrightarrow{j_{s,t}} & M_{s} \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
X_{\leq s} \times \mathbb{P}^1 & \xrightarrow{\delta_{s,t}} & X_{\leq s}
\end{array}
$$

where $g_{s,t}$ is the 0-section embedding, and the composite of all the maps in the bottom row is identity. This gives us the diagram of equivariant cobordism groups

$$
\begin{array}{ccc}
\Omega^*_T (N_{s,t}) & \xrightarrow{i_{s,t}^*} & \Omega^*_T (M_{s,t}) \\
\downarrow \eta_{s,t} & & \downarrow \delta_{s,t} \\
\Omega^*_T (N_{s,t}) & \xrightarrow{i_{s,t}^*} & \Omega^*_T (M_{s,t}) \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
\Omega^*_T (X_{\leq s} \times \mathbb{A}^1) & \xrightarrow{g_{s,t}^*} & \Omega^*_T (X_{\leq s}) \\
\downarrow f_{s,t} & & \downarrow f_{t}
\end{array}
$$

where the left and the middle squares commute by Theorem 2.5 (i) and the right square commutes by Corollary 3.2. Since the last horizontal maps in both rows are natural isomorphisms by the homotopy invariance, we shall often identify the last two terms in both rows and use $j_{s,t}^*$ and $(j_{s,t} \circ g_{s,t})^*$ interchangeably.

**Lemma 5.1.** For every $1 \leq s \leq n$ and $0 \leq t \leq s$, one has

$$
N_{s,t} \simeq \prod_{\sigma \in \Delta} \prod_{\tau \leq \sigma} O_{\tau} \quad \text{and} \quad M_{s,t} \simeq \prod_{\sigma \in \Delta} \prod_{\tau \leq \sigma} (O_{\tau} \times \mathbb{P}^1).
$$

**Proof.** This follows from [23, Lemma 6.1] by very little modifications. We only give the sketch. We first assume that the fan $\Delta$ consists of all the faces of a cone $\sigma$ of dimension $s \geq 1$ such that $X_{\leq s} = U_{\sigma}$. In this case, $X_{\leq s}$ is a $T$-equivariant vector bundle over $O_{\sigma}$ in such a way that the zero-section is precisely the inclusion $O_{\sigma} \subset X_{\leq s}$. In particular, one has $X_{\leq s} \simeq N_s$ and $M_{\leq s} \simeq X_{\leq s} \times \mathbb{P}^1$. We now see from (4.11) that

$$
N_{s,t} \simeq \prod_{\tau \leq \sigma} O_{\tau} \quad \text{and} \quad M_{s,t} \simeq \prod_{\tau \leq \sigma} (O_{\tau} \times \mathbb{P}^1).
$$

For the general case, we notice that $X_s = \prod_{\dim(\sigma) = s} O_{\sigma}$ by (4.11) and for an $s$-dimensional cone $\sigma \in \Delta$, the intersection of $X_s$ with $U_\sigma$ is precisely $O_{\sigma}$. Hence, we reduce the special case above.

**Corollary 5.2.** Let $X = X(\Delta)$ be a smooth toric variety. Then for $s \geq 1$, $X_s$ is cohomologically rigid inside $X_{\leq s}$. The same holds for the inclusions $N_{s,t} \subset N_{s,t}$ and $M_{s,t} \subset M_{s,t}$.

**Proof.** Let $d_s$ be the codimension of $X_s$ inside $X_{\leq s}$. By (4.11), it suffices to show that $O_\sigma = T/T_\sigma$ is cohomologically rigid inside $X_{\leq s}$, where $\sigma$ is a cone of $\Delta$ of dimension $s$. Since $s \geq 1$, it follows from the parts (i) and (iii) of Proposition 4.1 that the conditions of Lemma 4.2 are satisfied for $D = O_\sigma$ and $n = 0$. We conclude that
Such that \( i(5.5) \Omega \)

The diagram

There are ring homomorphisms

Theorem 5.3. There are ring homomorphisms

\[
S_{p_X, s}^\leq t : \Omega^*_T(X_{\leq t}) \to \Omega^*_T(N_{s, \leq t});
\]

\[
S_{p_X, s}^t : \Omega^*_T(X_t) \to \Omega^*_T(N_{s, t})
\]

such that \( i_{s, \leq t}^* = S_{p_X, s}^\leq t \circ j_{s, \leq t}^* \) and \( i_{s, t}^* = S_{p_X, s}^t \circ j_{s, t}^* \). Moreover, both the squares in the diagram

\[
(5.5)
\]

\[
\begin{array}{ccc}
\Omega^*_T(X_{\leq t}) & \xrightarrow{f^*} & \Omega^*_T(X_t) \\
S_{p_X, s}^\leq t & \downarrow & S_{p_X, s}^t \\
\Omega^*_T(N_{s, \leq t}) & \xrightarrow{\eta_{s, t}^*} & \Omega^*_T(N_{s, t})
\end{array}
\]

commute.

Proof. We consider the commutative diagram

\[
(5.6)
\]

\[
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
\Omega^*_T(N_{s, t}) & \xrightarrow{i_{s, t}^*} & \Omega^*_T(M_{s, t}) \\
& \eta_{s, t}^* & \downarrow \\
\Omega^*_T(N_{s, \leq t}) & \xrightarrow{i_{s, \leq t}^*} & \Omega^*_T(M_{s, \leq t}) \\
& \delta_{s, t}^* & \downarrow \\
\Omega^*_T(N_{s, \leq t-1}) & \xrightarrow{i_{s, \leq t-1}^*} & \Omega^*_T(M_{s, \leq t-1}) \\
& \downarrow \\
& & 0
\end{array}
\]

All rows in this diagram are exact by Theorem 2.5(ii), and all columns are exact by Corollary 5.2 and Proposition 3.6.

We apply Proposition 3.4 to the inclusions \( i_{s, \leq t} \) and \( i_{s, t} \) to see that the composites \( i_{s, \leq t}^* \circ i_{s, \leq t}^* \) and \( i_{s, t}^* \circ i_{s, t}^* \) are multiplication by the first Chern class \( c_1^T \) of the corresponding normal bundles. But these normal bundles are the inverse images of a line bundle on \( \mathbb{P}^1 \). It follows that these normal bundles are trivial, because the restriction of any line bundle on \( \mathbb{P}^1 \) to \( \infty \in \mathbb{P}^1 \) and hence on the fiber over \( \infty \) is clearly trivial. We conclude that the composites \( i_{s, \leq t}^* \circ i_{s, \leq t}^* \) and \( i_{s, t}^* \circ i_{s, t}^* \) are zero.

The above diagram now automatically defines the specializations \( S_{p_X, s}^\leq t \) and \( S_{p_X, s}^t \) and gives the desired factorization of \( i_{s, \leq t}^* \) and \( i_{s, t}^* \). Since \( i_{s, t}^* \) and \( j_{s, t}^* \) are ring homomorphisms, and since the latter is surjective, we deduce that \( S_{p_X, s}^t \) is also a ring homomorphism. The map \( S_{p_X, s}^\leq t \) is a ring homomorphism for the same reason.
We are now left with the proof of the commutativity of (5.5). To prove that the
right square commutes, we consider the following diagram.

\[
\begin{array}{c}
\Omega^r_T(M_{s,t}) \xrightarrow{j^*_{s,t}} \Omega^r_T(M_{s,\leq t}) \\
\downarrow \quad \downarrow \\
\Omega^r_T(X_t) \xrightarrow{\eta^*_{s,t}} \Omega^r_T(X_{s,\leq t}) \\
\downarrow \quad \downarrow \\
\Omega^r_T(N_{s,t}) \xrightarrow{\eta^*_{s,t}} \Omega^r_T(N_{s,\leq t}) \\
\end{array}
\]

It is easy to check that \(N_{s,\leq t}\) and \(M_{s,t}\) are transverse over \(M_{s,\leq t}\) (cf. [24, Lemma 1]) and hence the back face of the above diagram commutes by Corollary 3.2. The upper face commutes by diagram (5.6). Since \(j^*_{s,t}\) is surjective, a diagram chase shows that the lower face also commutes, which is what we needed to prove.

Finally, since we have shown in diagram (5.6) that \(\eta^*_{s,t}\) is injective and since the right square commutes, it now suffices to show that the composite square in (5.5) commutes in order to show that the left square there commutes.

By Lemma 3.3, the composite maps \(f^*_{ts} \circ f^*_t\) and \(\eta^*_{s,t} \circ j^*_{s,t}\) are multiplication by \(f^*_{ts}(1)\) and \(\eta^*_{s,t}(1)\) respectively. Since \(Sp^l_{X,s}\) and \(Sp^l_{X,s}\) are ring homomorphisms, it suffices to show that

\[
Sp^l_{X,s} (f^*_{ts} \circ j^*_{s,t}(1)) = Sp^l_{X,s} (f^*_{ts}(1)) = \eta^*_{s,t}(1).
\]

But this follows directly from the commutativity of the right square in (5.5). □

6. Decomposition for equivariant cobordism ring

We need the following intermediate step for the decomposition theorem for the equivariant cobordism rings of smooth toric varieties. Let \(X = X(\Delta)\) be a smooth toric variety as above.

**Proposition 6.1.** The restriction maps

\[
\Omega^r_T(X_{\leq s}) \xrightarrow{(f^*_{ts},\eta^*_{s,t})} \Omega^r_T(X_s) \times \Omega^r_T(X_{<s})
\]

define an isomorphism of rings

\[
\Omega^r_T(X_{\leq s}) \xrightarrow{\cong} \Omega^r_T(X_s) \times \Omega^r_T(N^0_s),
\]

where \(\Omega^r_T(X_s) \xrightarrow{\eta^*_{s,\leq s-1}} \Omega^r_T(N^0_s)\) is the pull-back

\[
\Omega^r_T(X_s) \xrightarrow{\cong} \Omega^r_T(N_s) \rightarrow \Omega^r_T(N^0_s)
\]

and

\[
\Omega^r_T(X_{\leq s}) \xrightarrow{Sp^l_{X,s}} \Omega^r_T(N_{s,\leq s-1}) = \Omega^r_T(N^0_s)
\]
is the specialization map of Theorem 5.3.
Proof. We only need to identify the pull-back and the specialization maps with the appropriate maps of Proposition 3.6. In the diagram

\[
\begin{array}{c}
0 \longrightarrow \Omega_T^*(X_s) \xrightarrow{f_{s,\infty}} \Omega_T^*(N_s) \xrightarrow{\eta_{s,\leq s-1}} \Omega_T^*(N_s^0) \longrightarrow 0
\end{array}
\]

where \(f_{s,\infty} : X_s \to N_s\) is the 0-section embedding, the top sequence is exact by Proposition 4.1 and Corollary 5.2, and the lower triangle commutes by Proposition 3.4. Since \(f_{s,\infty}\) is an isomorphism, this immediately identifies the pull-back map of the proposition with the quotient map \(\Omega_T^*(X_s) \to \Omega_T^*(X_s)/(\eta_{s,\leq s-1})\).

Next, we consider the diagram

\[
\begin{array}{c}
\Omega_T^*(X_{\leq s}) \xrightarrow{f_{s,\leq s-1}^*} \Omega_T^*(X_{<s}) \\
\downarrow \quad \quad \downarrow \\
\Omega_T^*(N_s) \xrightarrow{f_{s,\leq s-1}} \Omega_T^*(N_s^0) \\
\downarrow \\
\Omega_T^*(X_s)
\end{array}
\]

Since the top horizontal arrow in the above diagram is surjective, we only need to show that \(Sp_{X,s}^{\leq s} \circ j_{s,\leq s-1}^* = \eta_{s,\leq s-1} \circ f_{s,\leq s-1}^*\) in order to identify \(Sp_{X,s}^{\leq s-1}\) with the map \(j^*\) of Proposition 3.6. It is clear from the diagram (5.0) and the definition of the specialization maps that the top square above commutes. We have just shown above that the lower triangle also commutes. This reduces us to showing that

\[f_{s,\infty}^* \circ Sp_{X,s}^{\leq s} = f_{s,\leq s-1}^*\]

If \(X_s \times \mathbb{P}^1 \xrightarrow{F_s} M_s\) denotes the embedding (cf. (5.1)), then for \(x \in \Omega_T^*(X_{\leq s})\), we can write \(x = j_{s,\leq s}^*(y)\) by the surjectivity of \(j_{s,\leq s}^*\) (see the proof of Theorem 5.3). Then

\[
\begin{align*}
(f_{s,\infty}^* \circ Sp_{X,s}^{\leq s} \circ j_{s,\leq s}^*)(y) &= f_{s,\infty}^* \circ i_{s,\leq s}^*(y) = g_{\infty,\leq s}^* \circ F_s^*(y) = g_{0,\leq s}^* \circ F_s^*(y) \\
&= f_{s}^* \circ j_{s}^*(y) = f_{s}^*(x),
\end{align*}
\]

where the second equality follows from Corollary 3.2. This proves (6.1) and the proposition. \(\square\)

**Theorem 6.2.** For a smooth toric variety \(X = X(\Delta)\), the ring homomorphism

\[
\Omega_T^*(X) \longrightarrow \prod_{s=0}^{n} \Omega_T^*(X_s)
\]

is injective. Moreover, its image consists of the \(n\)-tuples \((\alpha_s)\) in the product with the property that for each \(s = 1, \ldots, n\), the pull-back of \(\alpha_s \in \Omega_T^*(X_s)\) in \(\Omega_T^*(N_{s,s-1})\) is same as \(Sp_{X,s}^{\leq s-1}(\alpha_{s-1}) \in \Omega_T^*(N_{s,s-1})\). In other words, there is a ring isomorphism

\[
\Omega_T^*(X) \overset{\cong}{\longrightarrow} \Omega_T^*(X_n) \times \Omega_T^*(X_{n-1}) \times \cdots \times \Omega_T^*(X_0).
\]
\textbf{Proof.} We prove by the induction on the largest integer \( s \) such that \( X_s \neq \emptyset \).

If \( s = 0 \), there is nothing to prove. If \( s > 0 \), we have by induction
\begin{equation}
\Omega_T^r(X_{<s}) \xrightarrow{\sim} \Omega_T^r(X_{s-1}) \times \cdots \times \Omega_T^r(N_{s-1,s-2}) \times \Omega_T^r(N_{1,0}).
\end{equation}

Using this and Proposition \( \ref{prop:restriction} \) it suffices to show that if \( \alpha_s \in \Omega_T^r(X_s) \) and if \( \alpha_{<s} \in \Omega_T^r(X_{<s}) \) with the restriction \( \alpha_{s-1} \in \Omega_T^r(X_{s-1}) \) are such that \( \alpha_s \mapsto \alpha^0_s \in \Omega_T^r(N^0_s) \) and \( \alpha_s \mapsto \alpha_{s,s-1} \in \Omega_T^r(N_{s,s-1}) \), then
\[ Sp_X^{s-1}(\alpha_{<s}) = \alpha^0_s \quad \text{iff} \quad Sp_X^{s-1}(\alpha_{s-1}) = \alpha_{s,s-1}. \]

Using the commutativity of the left square in Theorem \( \ref{thm:main} \) this is reduced to showing that the restriction map
\begin{equation}
\Omega_T^r(N^0_s) \to \Omega_T^r(N_{s,s-1})
\end{equation}
is injective.

It follows from \( \eqref{eq:projective} \) and Lemma \( \ref{lem:restriction} \) that
\[ N_s = \coprod_{\sigma \in \Delta, \dim(\sigma) = s} N_{O_\sigma/X_{\leq s}} \quad \text{and} \quad N_{s,s-1} = \coprod_{\sigma \in \Delta, \dim(\sigma) = s} \coprod_{\tau \in \partial \sigma} O_\tau. \]

Moreover, for every cone \( \sigma \in \Delta \), we have \( N_{O_\sigma/X_{\leq s}} = N_{O_\sigma/U_\sigma} \), where \( U_\sigma \) is the open toric subvariety of \( X \) defined by the fan consisting of all faces of \( \sigma \). Thus it suffices to prove \( \eqref{eq:injectivity} \) when \( X \) is of the form \( U_\sigma \), where \( \sigma \) is an \( s \)-dimensional cone in \( \Delta \).

In this case, \( U_\sigma \cong N_s \) is the \( T \)-equivariant vector bundle over \( O_\sigma = T/T_\sigma \) of the form \( (T \times V_\sigma)/T_\sigma \to T/T_\sigma \), where \( V_\sigma \) is the \( k \)-vector space spanned by the part \( B \) of a basis of \( M \) which generates \( \sigma \). In particular, \( N_s = \bigoplus_{j=1}^s L_{\chi_j} \), where \( \{\chi_1, \ldots, \chi_s\} \) is a part of a basis of \( M \) and \( L_{\chi} \) is the \( T \)-equivariant line bundle on \( O_\sigma \) associated to the character \( \chi \) of \( T \).

Next, it follows from Proposition \( \ref{prop:restriction} \) that
\[ \operatorname{Ker}(\Omega_T^r(X_s) \to \Omega_T^r(N_{s,s-1})) = \bigcap_{i=1}^s (c_T^r(L_{\chi_i})) \quad \text{and} \quad \operatorname{Ker}(\Omega_T^r(X_s) \to \Omega_T^r(N^0_s)) = (c_T^r(N_s)). \]

Setting \( \gamma_i = c_T^r(L_{\chi_i}) \) and \( \gamma = c_T^r(N_s) \), we see from the surjectivity \( \Omega_T^r(X_s) \to \Omega_T^r(N^0_s) \) that showing the injectivity of the map in \( \eqref{eq:injectivity} \) is equivalent to showing that
\begin{equation}
(\gamma) = \left( \bigcap_{i=1}^s \gamma_i \right) = \bigcap_{i=1}^s (\gamma_i)
\end{equation}
in \( \Omega_T^r(X_s) \cong \Omega_T^r(k) = \mathbb{L}[t_1, \ldots, t_s] \). Since \( B = \{\chi_1, \ldots, \chi_s\} \) can be identified with a basis of \( T_\sigma \), we have an isomorphism \( \Omega_T^r(N_{O_\sigma}) = \mathbb{L}[t_1, \ldots, t_s] \), where \( t_i = c_T^r(L_{\chi_i}) \) for \( 1 \leq i \leq s \) \((\text{cf. } \eqref{eq:toric})\). The equality of \( \eqref{eq:injectivity} \) is now easy to check and follows also from \( \eqref{eq:projective} \) Lemma \( \ref{lem:restriction} \). \( \square \)

\textbf{Lemma 6.3.} For \( 1 \leq s \leq n \), there is a canonical isomorphism
\[ N_{s,s-1} = \prod_{\sigma \in \Delta, \dim(\sigma) = s} \prod_{\tau \in \partial \sigma} O_\tau. \]
Furthermore, for each \( s \)-dimensional cone \( \sigma \) and \( \tau \in \partial \sigma \), the composition of the map

\[
Sp_{X,s}^{-1} : \Omega_T^* (X_{s-1}) = \prod_{\tau \in \Delta \atop \dim(\tau) = s-1} \Omega_T^* (O_\tau) \rightarrow \Omega_T^* (N_{s,s-1}) = \prod_{\sigma \in \Delta \atop \dim(\sigma) = s} \prod_{\tau \in \partial \sigma} \Omega_T^* (O_\tau)
\]

with the projection

\[
Pr_{\sigma,\tau} : \prod_{\sigma \in \Delta \atop \dim(\sigma) = s} \prod_{\tau \in \partial \sigma} \Omega_T^* (O_\tau) \rightarrow \Omega_T^* (O_\tau)
\]

is the projection

\[
\prod_{\tau \in \Delta \atop \dim(\tau) = s-1} \Omega_T^* (O_\tau) \rightarrow \Omega_T^* (O_\tau).
\]

Proof. The first assertion is already shown in Lemma 5.1. The second assertion for the \( K \)-theory is shown in [23, Lemma 6.1]. The same proof goes through here as well in verbatim in view of our description of the specialization maps in Theorem 5.3. □

The following is our first result in the description of the equivariant cobordism ring of a smooth toric variety. This is analogous to the similar description of the equivariant \( K \)-theory in [23, Theorem 6.2] and equivariant Chow groups in [3, Theorem 5.4]. Recall that \( S \) denotes the ring \( S(T) = \Omega_T^* (k) \).

**Theorem 6.4.** Let \( X = X(\Delta) \) be a smooth toric variety associated to a fan \( \Delta \) in \( M_\mathbb{R} \). There is an injective homomorphism of \( S \)-algebras

\[
\Phi_X : \Omega_T^* (X) \rightarrow \prod_{\sigma \in \Delta_{\text{max}}} S (T_\sigma),
\]

where \( \Delta_{\text{max}} \) is the set of maximal cones in \( \Delta \). An element \((a_\sigma) \in \prod_{\sigma \in \Delta_{\text{max}}} S (T_\sigma)\) is in the image of this homomorphism if and only if for any two maximal cones \( \sigma_1 \) and \( \sigma_2 \), the restrictions of \( a_{\sigma_1} \) and \( a_{\sigma_2} \) to \( S (T_{\sigma_1 \cap \sigma_2}) \) coincide.

**Proof.** Since every proper face of a cone \( \sigma \) is contained in \( \partial \sigma \), it follows immediately from (4.1), Theorem 6.2, Lemma 6.3 and the isomorphism \( \Omega_T^* (O_\sigma) \cong S (T_\sigma) \) that \( \Omega_T^* (X) \) is a subring of \( \prod_{\sigma \in \Delta} S (T_\sigma) \) consisting of elements \((a_\sigma)\) with the property that the restriction of \( a_\sigma \in S (T_\sigma) \) to \( S (T_\tau) \) coincides with \( a_\tau \) whenever \( \tau \leq \sigma \). The theorem now follows from the fact that every cone in \( \Delta \) is contained in a maximal cone in \( \Delta \). □

If \( X = X(\Delta) \) is a smooth projective toric variety, then all the maximal cones in \( \Delta \) are \( n \)-dimensional and the closed orbits \( O_\sigma \) are all \( k \)-rational points. We conclude from Theorem 6.4 that

**Corollary 6.5.** For a smooth projective toric variety \( X = X(\Delta) \), there is an injective homomorphism of \( S \)-algebras

\[
\Omega_T^* (X) \hookrightarrow \prod_{\sigma \in \Delta_{\text{max}}} S \cong S^{\lvert \Delta_{\text{max}} \rvert}
\]

whose image is the set of elements \((a_\sigma)\) such that for any two adjacent maximal cones \( \sigma_1 \) and \( \sigma_2 \), the restrictions of \( a_{\sigma_1} \) and \( a_{\sigma_2} \) to \( S (T_{\sigma_1 \cap \sigma_2}) \) coincide.
7. Stanley-Reisner presentations

In this section, we use Theorem \[6.4\] to give a description of the equivariant cobordism ring of a smooth toric variety which is analogous to the stanley-Reisner presentation of the equivariant $K$-theory (cf. \[23, Theorem 6.1\]) and equivariant cohomology (cf. \[1, Theorem 8\]). This will prove Theorem 1.2. We follow the notations of \[23\] in this description.

Let $T$ be a split torus of rank $n$ and let $M$ denote the lattice of the one-parameter subgroups of $T$. Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $M_\mathbb{R}$. For $r \geq 1$, let $\Delta_r$ denote the set of $r$-dimensional cones in $\Delta$. For $\sigma \in \Delta_{\text{max}}$, let $M_\sigma$ denote the group of one-parameter subgroups of $T_\sigma$ so that $\hat{T}_\sigma = M_\sigma^\vee$ as an abelian group. For any $\rho \in \Delta_1$, let $v_\rho$ denote the generator of the monoid $\rho \cap M$. Note that if $\{\rho_1, \ldots, \rho_s\}$ is the set of one-dimensional faces of $\sigma \in \Delta_{\text{max}}$, then the smoothness of $X$ implies that $\{v_{\rho_1}, \ldots, v_{\rho_s}\}$ is a basis of $M_\sigma$. Let $\{v_{\rho_1}^\vee, \ldots, v_{\rho_s}^\vee\}$ denote the dual basis of $M_\sigma^\vee$.

We also recall (cf. \[2.9\]) that for $\sigma \in \Delta$, there is a canonical embedding of abelian groups $\hat{T}_\sigma \hookrightarrow (\Omega^1_{T_\sigma}(k), F)$ given by $\chi \mapsto c_1^{T_\sigma}(L_\chi)$, if we consider the addition in $\Omega^1_{T_\sigma}(k)$ according to the formal group law of $L$. In what follows, $\hat{T}_\sigma$ will be thought of as a subgroup of degree one elements in $S(T_\sigma) = \Omega^*_{T_\sigma}(k)$ in this sense.

For each $\rho \in \Delta_1$, we define an element $u_\rho = (u_\rho^\sigma) \in \prod_{\sigma \in \Delta_{\text{max}}} S(T_\sigma)$ such that

$$u_\rho^\sigma = \begin{cases} v_\rho^\vee & \text{if } \rho \leq \sigma \\ 0 & \text{otherwise} \end{cases}$$

(7.1)

Then $u_\rho$ has the property that for all $\sigma_1, \sigma_2 \in \Delta_{\text{max}}$, the restrictions of $u_{\rho_1}^\sigma \in \hat{T}_{\sigma_1}$ and $u_{\rho_2}^\sigma \in \hat{T}_{\sigma_2}$ in $\hat{T}_{\sigma_1 \cap \sigma_2}$ coincide. It follows from Theorem \[6.4\] that each $u_\rho$ is an element of $\Omega^*_{T}(X)$.

If $S$ is a subset of $\Delta_1$ which is not contained in any maximal cone of $\Delta$, then for any given $\sigma \in \Delta_{\text{max}}$, there is one $\rho \in S$ such that $\rho \leq \sigma$. This implies in particular that $u_\rho^\sigma = 0$. We conclude from this that the elements $u_\rho$ satisfy the relation

$$\prod_{\rho \in S} u_\rho = 0 \text{ in } \Omega^*_{T}(X)$$

(7.2)

whenever $S \subseteq \Delta_1$ is such that it is not contained in any maximal cone of $\Delta$. We shall denote the collection of all such subsets of $\Delta_1$ by $\Delta^0_1$.

Let $L[[t_\rho]]$ denote the graded power series ring over $L$ in the variables $\{t_\rho | \rho \in \Delta_1\}$ and let $I^\Delta$ denote the graded ideal generated by the set of monomials $\{\prod_{\rho \in \Delta_1} t_\rho | S \in \Delta^0_1\}$. Since $\Omega^*_T(X)$ is a subring of $\prod_{\sigma \in \Delta_{\text{max}}} S(T_\sigma)$ and since this latter term is a product of graded power series rings over $L$, it follows from (7.1) and (7.2) that there is a $L$-algebra homomorphism

$$\Psi_X : \frac{L[[t_\rho]]}{I^\Delta} \to \Omega^*_T(X)$$

(7.3)

$$t_\rho \mapsto u_\rho.$$

**Theorem 7.1.** For a smooth toric variety $X = X(\Delta)$ associated to a fan $\Delta$ in $M_\mathbb{R}$, the homomorphism $\Psi_X$ is an isomorphism.
Proof. We prove by the induction on the number of cones in $\Delta$. Suppose $\Delta = \{ \sigma \}$ is a singleton set. In that case, $\sigma$ is the only maximal cone and we have seen in the proof of Theorem 6.2 that $X = U_\sigma$ is a $T$-equivariant vector bundle over $O_\sigma$ such that the inclusion $O_\sigma \hookrightarrow X$ is the zero-section embedding. Hence, there is an isomorphism $\Omega^*_T(X) \xrightarrow{\cong} \Omega^*_T(O_\sigma) \cong S(T_\sigma) = \mathbb{L}[t_1, \cdots, t_s]$, where $s$ is the dimension of $\sigma$. It is also clear in this case that the ideal $I_\Delta$ in (7.3) is zero. Hence, we have isomorphism

$$\mathbb{L}[t_1, \cdots, t_s] \xrightarrow{\cong} \Omega^*_T(X) \xrightarrow{\cong} S(T_\sigma).$$

We consider now the general case. We assume that $|\Delta| \geq 2$ and choose a maximal cone $\sigma$ of dimension $s \geq 1$ in $\Delta$. Let $X' = X'(\Delta')$ be the toric variety associated to the fan $\Delta' = \Delta \setminus \{ \sigma \}$. Note that $O_\sigma$ is a closed $T$-orbit in $X$ and $X'$ is the complement of $O_\sigma$ in $X$. Let $U_\sigma \subset X$ be the principal open set associated to the fan consisting of all faces of $\sigma$ and let $U'$ be the complement of $O_\sigma$ in $U_\sigma$. Then $U'$ is nothing but the complement of the zero-section in the $T$-equivariant vector bundle $U_\sigma \to O_\sigma$. Let $i_\sigma : O_\sigma \hookrightarrow X$ and $j_\sigma : X' \hookrightarrow X$ denote the closed and open embeddings respectively. Let $S_\sigma = \{ \rho_1, \cdots, \rho_s \}$ be the set of one-dimensional faces of $\sigma$ and set

$$x_\sigma = \prod_{j=1}^s t_{\rho_j} \in \mathbb{L}[t_{\rho}]_{I_\Delta} \quad \text{and} \quad y_\sigma = \prod_{j=1}^s u_{\rho_j} \in \Omega^*_T(X).$$

Since $N_{O_\sigma/X} = N_{O_\sigma/U_\sigma}$ and since the latter is of the form $\bigoplus_{j=1}^s L_{\chi_j}$, where $\{ \chi_1, \cdots, \chi_s \}$ is a basis of $\tilde{T}_\sigma$, it follows from the proof of Lemma 4.2 and the definition of the elements $u_{\rho}$ that

$$\epsilon^T_\sigma (N_{O_\sigma/X}) = y_\sigma \in \Omega^*_T(X).$$

We consider the diagram

$$\mathbb{L}[t_{\rho_1}, \cdots, t_{\rho_s}] \xrightarrow{x_\sigma} \Omega^*_T(O_\sigma) \xrightarrow{\cong} S(T_\sigma)$$

$$\xrightarrow{\Lambda}[t_{\rho}]_{I_\Delta} \xrightarrow{\Psi_X} \Omega^*_T(X) \xrightarrow{\Phi_X} \prod_{\tau \in \Delta_{\max}} S(T_{\tau})$$

where the horizontal maps on the top are the obvious isomorphisms taking $t_{\rho_j}$ to $u_{\rho_j}$. The left and the right vertical maps are the multiplication by the indicated elements in the target rings. We claim that all the vertical arrows are injective and the left square in this diagram commutes.

To prove the claim, notice that the composite outer square clearly commutes by the definition of $x_\sigma$ and $y_\sigma$ and the map $\Psi_X$. Since $\Phi_X$ is injective by Theorem 6.4, we only need to show that the right square commutes and the right vertical arrow is injective to prove the claim.

We first observe that the right vertical arrow is the multiplication by $y_\sigma$ on the factor $S(T_\sigma)$ and is zero on the other factors of $\prod_{\tau \in \Delta_{\max}} S(T_{\tau})$. Thus the required injectivity is equivalent to showing that the multiplication by $y_\sigma$ is injective in $S(T_\sigma)$. But this is obvious since $S(T_\sigma) \cong \mathbb{L}[t_{\rho_1}, \cdots, t_{\rho_s}]$ is an integral domain and $y_\sigma$ is clearly non-zero (cf. [14, Lemma 5.3]).
To show the commutativity of the right square, we observe from the proof of Theorem 6.4 that $\Phi_X$ is simply the product of the pull-back maps $i^*_{\tau}: \Omega^*_T(O_{\tau}) \to \Omega^*_T(X)$, $\tau \in \Delta_{\text{max}}$. Hence the composite $\Phi_X \circ i_{\sigma}* = i^*_{\tau} \circ i_{\sigma}* $ on the factor $S(T_{\tau})$ and zero on the other factors of $\prod_{\tau \in \Delta_{\text{max}}} S(T_{\tau})$. Since we have just seen that the composite $S(T_{\tau}) \xrightarrow{y_{\sigma}} \prod_{\tau \in \Delta_{\text{max}}} S(T_{\tau})$ is of similar type, we are reduced to showing that the triangle

\[
\begin{array}{ccc}
\Omega^*_T(O_{\sigma}) & \xrightarrow{i_{\sigma}} & \Omega^*_T(X) \\
\downarrow \psi \downarrow & & \downarrow \psi' \\
\Omega^*_T(O_{\sigma}) & \xrightarrow{j_{\sigma}} & \Omega^*_T(X')
\end{array}
\]

commutes. But this follows immediately from Proposition 3.4 and (7.3).

To complete the proof of the theorem, we now consider the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{I_{[t_\rho]} \cdot \cdots \cdot I_{[t_{\rho_n}]}} & \mathbb{L}_{[I_{t_{\rho.n}}]} \\
\downarrow \cong & & \downarrow \psi_X \\
0 & \xrightarrow{\Omega^*_T(O_{\sigma})} & \Omega^*_T(X) \\
\downarrow i_{\sigma*} & & \downarrow j_{\sigma} \\
0 & \xrightarrow{\Omega^*_T(O_{\sigma})} & \Omega^*_T(X') \\
\end{array}
\]

(7.6)

where $j_{\sigma}$ is the natural quotient map by the ideal $(x_{\sigma})$ in $\frac{\mathbb{L}_{[I_{t_{\rho.n}}]}}{I_{t_{\rho.n}}}$. Note that the image of the first map in the top row is the ideal $(x_{\sigma})$ because the product of $x_{\sigma}$ with any $t_{\rho}, \rho \notin \{\rho_1, \cdots, \rho_n\}$ is zero. The left square in this diagram commutes and the first maps in both the rows are injective by the above claim. The bottom row is exact by Theorem 2.5. Since $\sigma$ is not a cone of $\Delta'$, the element $x_{\sigma}$ is zero in $\frac{\mathbb{L}_{[I_{t_{\rho.n}} \cdot \rho \in \Delta'_{\rho}]}{I_{t_{\rho.n}}}$ and hence the map $j_{\sigma*} \circ \psi_X$ has a factorization:

\[
\mathbb{L}_{[I_{t_{\rho.n}}]} \xrightarrow{I_{t_{\rho.n}}} \mathbb{L}_{[I_{t_{\rho.n}} \cdot \rho \in \Delta'_{\rho}]} \xrightarrow{\psi_{X'}} \Omega^*_T(X'),
\]

where the middle arrow is the natural map of the Stanley-Reisner power series rings induced by the inclusion of the fans $\Delta' \subset \Delta$. Letting $\psi_{X'}$ denote the composite $\frac{\mathbb{L}_{[I_{t_{\rho.n}} \cdot \rho \in \Delta'_{\rho}]}{I_{t_{\rho.n}}}$, we see that the right square in the above diagram also commutes.

If all the cones of $\Delta$ are at most one-dimensional, then $x_{\sigma} = t_{\rho}$, where $\rho = \sigma$ and it is obvious that $\frac{\mathbb{L}_{[I_{t_{\rho.n}}]}{I_{t_{\rho.n}}}$ is the Stanley-Reisner ring associated to the fan $\Delta'$. If $\Delta$ has a cone of dimension at least two, we can assume that $\sigma$ is of dimension at least two. In that case, we have $\Delta' = \Delta_1$ and the natural inclusion $\Delta^0_{\sigma} \subset \Delta^0_1$ gives the equality $\Delta^0_{\sigma} = \Delta^0_1 \coprod \{S_{\sigma}\}$. In particular, we have

\[
\mathbb{L}_{[I_{t_{\rho.n}} \cdot \rho \in \Delta'_{\rho}]} \xrightarrow{I_{t_{\rho.n}}} \mathbb{L}_{[I_{t_{\rho.n}} \cdot \rho \in \Delta'_{\rho}]} \xrightarrow{\psi_{X'}} \Omega^*_T(X'),
\]

On the other hand, $\Delta'$ is a fan with smaller number of cones than in $\Delta$ and $X' = X'(\Delta')$. Hence the map $\frac{\mathbb{L}_{[I_{t_{\rho.n}} \cdot \rho \in \Delta'_{\rho}]}{I_{t_{\rho.n}}}$ is an isomorphism by induction. We conclude that the map $\psi_{X'}$ in the diagram (7.6) is an isomorphism. A diagram chase shows that $\psi_X$ is also an isomorphism. This completes the proof of the theorem. \qed
8. Cobordism ring of toric varieties

Let $T$ be a split torus of rank $n$ with the group of one-parameter subgroups $M$. We now describe the ordinary cobordism ring of a smooth toric variety using Theorem 7.1 and the following result which explicitly describes the ordinary cobordism ring as a quotient of the equivariant cobordism ring. Recall from Section 2 that for any linear algebraic group $G$ acting on a scheme $X$, there is a natural forgetful map $r^G_X: \Omega^*_G(X) \to \Omega^*(X)$ which is a ring homomorphism if $X$ is smooth.

**Theorem 8.1.** (cf. [14, Theorem 3.4]) Let $T$ be a split torus acting on a $k$-variety $X$. Then the forgetful map $r^T_X$ induces an isomorphism

$$
\Omega^*_T(X) \otimes_S \mathbb{L} \xrightarrow{\cong} \Omega^*_S(X).
$$

If $X$ is smooth, this is an $\mathbb{L}$-algebra isomorphism.

Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $M_\mathbb{R}$. Let $<,> : M \times M^\vee \to \mathbb{Z}$ denote the natural pairing. Recall that for an one-dimensional cone $\rho \in \Delta$, the symbol $v_\rho$ denotes the generator of the monoid $M \cap \rho$. It is well known that the associated orbit closure $V_\rho = \overline{O_\rho}$ is also a smooth toric variety which is a $T$-equivariant Weil divisor on $X$. In particular, $[V_\rho \to X]$ is a $T$-equivariant cobordism cycle on $X$. Let $[V_\rho] \in \Omega^*_T(X)$ denote its fundamental class (cf. Subsection 2.3). We denote the cobordism cycle $[V_\rho \to X] \in \Omega^*(X)$ also by $[V_\rho]$.

Let $\mathbb{L}[t_\rho] = \mathbb{L}[t_\rho, \rho \in \Delta_1]$ be the graded polynomial ring over $\mathbb{L}$ with each $t_\rho$ homogeneous of degree one. Let $T_\Delta$ denote the graded ideal in $\mathbb{L}[t_\rho]$ generated by the set of monomials $\{ \prod_{\rho \in S} t_\rho | S \in \Delta^0_1 \} \cup \{ t_\rho^{n+1} | \rho \in \Delta_1 \}$. It is clear that the sum

$$
\sum_{\rho \in \Delta_1} [ < \chi, v_\rho > ]_F t_\rho \quad (\text{cf. Subsection 2.5})
$$

is a well-defined homogeneous element of degree one in the graded ring $\mathbb{L}[t_\rho] / T_\Delta$ for every $\chi \in M^\vee$.

**Theorem 8.2.** Let $X = X(\Delta)$ be a smooth toric variety associated to a fan $\Delta$ in $M_\mathbb{R}$. Then the assignment $t_\rho \mapsto [V_\rho]$ defines an $\mathbb{L}$-algebra isomorphism

$$
\Psi_X : \mathbb{L}[t_\rho] \to \Omega^*(X), \quad \left( T_\Delta, \sum_{\rho \in \Delta_1} [ < \chi, v_\rho > ]_F t_\rho \right)
$$

where $\chi$ runs over $M^\vee$.

**Proof.** We first show that there is an $\mathbb{L}$-algebra isomorphism

$$
\widetilde{\Psi}_X : \mathbb{L}[[t_\rho]] \to \Omega^*(X), \quad \left( I_\Delta, \sum_{\rho \in \Delta_1} [ < \chi, v_\rho > ]_F t_\rho \right)
$$

where $\chi$ runs over $M^\vee$. Since the orbit closure $V_\rho$ associated to an one-dimensional cone $\rho$ is the union of the orbits $O_\sigma$ such that $\rho \leq \sigma$, it is clear from our definition of $u_\rho$ in (7.1) that it is precisely the class of the cycle $V_\rho$ in $\prod_{T \in \Delta_\max} S(T)$ and hence in $\Omega^*_T(X)$. Thus, the isomorphism in (8.2) follows immediately from Theorems 7.1 and 8.1 once we know that the class of $c^T_1(L_\chi)$ in $\prod_{T \in \Delta_\max} S(T)$ is $\sum_{\rho \in \Delta_1} [ < \chi, v_\rho > ]_F u_\rho$

for a character $\chi$ of $T$. 

Let $V_\Delta \subseteq \prod_{\sigma \in \Delta_{\text{max}}} \hat{T}_{\sigma}$ be the subgroup consisting of elements $(x^\sigma)$ with the property that for all $\sigma_1, \sigma_2 \in \Delta_{\text{max}}$, the restrictions of $x^{\sigma_1} \in \hat{T}_{\sigma_1}$ and $x^{\sigma_2} \in \hat{T}_{\sigma_2}$ in $\hat{T}_{\sigma_1 \cap \sigma_2}$ coincide. It is easy to check that the elements $u_\rho$ form a basis of $V_\Delta$ (cf. [23 Proposition 6.3]). Here, $\hat{T}_{\sigma}$ is thought of as a subgroup of degree one elements in $S(T_{\sigma})$ as in Section 7. As an element of $\prod_{\sigma \in \Delta_{\text{max}}} \hat{T}_{\sigma}$, we have

$$\chi = \sum_{\sigma \in \Delta_{\text{max}}} \sum_{\rho \leq \sigma} \langle \chi, v_\rho \rangle = \sum_{\rho \in \Delta_1} \sum_{\rho \leq \sigma} \langle \chi, v_\rho \rangle = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle > 0.$$ 

In particular, we get $c_1^T(I_\chi) = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle > 0$. This proves the isomorphism of (8.2).

To complete the proof of the theorem, we only have to show that the inclusion $\mathbb{L}[t_\rho] \subset \mathbb{L}[\{t_\rho\}]$ descends to an isomorphism of graded rings

$$\tag{8.3} \frac{\mathbb{L}[t_\rho]}{I_\chi, \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle > F \ t_\rho} \cong \frac{\mathbb{L}[\{t_\rho\}]}{I_\chi, \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle > F \ t_\rho}.$$

Let $A$ and $B$ denote the rings on the left and the right hand sides of (8.3) respectively. We first observe that $\tilde{\Psi}_X(t_\rho) = \{v_\rho\}$, which is a homogeneous element of degree one in $\Omega^\ast(X)$. Since $\Omega^{>\ast}(X) = 0$, we conclude from (8.2) that $t_\rho^{n+1} = 0$ in $B$ for each $\rho \in \Delta_1$. Since $\mathbb{L}[t_\rho]/(t_\rho^{n+1}) \cong \mathbb{L}[\{t_\rho\}]/(t_\rho^{n+1})$, the isomorphism of (8.3) is now immediate.

**Examples 8.3.** We now illustrate Theorem 8.2 by using it to verify the known formulae for cobordism rings of some standard toric varieties. If $X = X(\Delta)$, where $\Delta$ consists of all the faces of a single cone $\sigma$ in $M_\mathbb{R}$, then the smoothness of $X$ implies that the set of primitive vectors $\{v_1, \ldots, v_s\}$ corresponding to the one-dimensional faces $\{\rho_1, \ldots, \rho_s\}$ of $\sigma$ can be extended to a basis $\{v_1, \ldots, v_s, v_{s+1}, \ldots, v_n\}$ of $M$. Setting $\chi_i = v_i^\vee$ for $1 \leq i \leq s$, we get $I_\chi = 0 = c_1^T(L_\chi)$ for $s + 1 \leq i \leq n$ and $c_1^T(L_\chi) = t_\rho_i$ for $1 \leq i \leq s$. In particular, we obtain

$$\Omega^\ast(X) \cong \frac{\mathbb{L}[t_{\rho_1}, \ldots, t_{\rho_s}]}{I_\chi, \sum_{\rho \in \Delta_1} \langle \chi_i, v_\rho \rangle > F t_\rho} = \frac{\mathbb{L}[t_{\rho_1}, \ldots, t_{\rho_s}]}{(t_{\rho_1}, \ldots, t_{\rho_s})} = \mathbb{L}.$$

Next we consider the case of $\mathbb{P}^n = X(\Delta)$. We take $M = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n$ and let $\{\rho_1, \ldots, \rho_{n+1}\}$ be the set of one-dimensional cones in $\Delta$, where $\rho_i$ is the edge along the vector $v_i$ for $1 \leq i \leq n$ and $\rho_{n+1}$ is the edge along the primitive vector $v_{n+1} = -(v_1 + \cdots + v_n)$. Set $\chi_i = v_i^\vee$ for $1 \leq i \leq n$. It is then easy to see that if $c_i^T$ denotes the element $\sum_{\rho \in \Delta_1} \langle \chi_i, v_\rho \rangle > F t_\rho$ in $\mathbb{L}[t_{\rho_1}, \ldots, t_{\rho_{n+1}}]$, then $c_i^T = t_{\rho_i} - F t_{\rho_{n+1}}$. 

for $1 \leq i \leq n$. In particular, we get
\[ c^T_i = t_{p_i} - t_{p_{n+1}} + t_{p_i} t_{p_{n+1}} f \left( t_{p_i}, t_{p_{n+1}} \right) = t_{p_i} - t_{p_{n+1}} \left( 1 + t_{p_i} g \left( t_{p_i}, t_{p_{n+1}} \right) \right) = t_{p_i} - u_i t_{p_{n+1}}, \]
where $u_i$ is an invertible homogeneous element of degree zero in $\mathbb{L}[t_{p_1}, \ldots, t_{p_{n+1}}]$.

Since $I_\Delta = \left( \prod_{i=1}^{n+1} t_{p_i} \right)$, we get
\[ \Omega^* \left( \mathbb{P}^n \right) \cong \mathbb{L} \left[ [t_{p_1}, \ldots, t_{p_{n+1}}] \right] \frac{t_{p_i} - t_{p_{n+1}}}{t_{p_i} - t_{p_{n+1}} + \left( 1 + t_{p_i} g \left( t_{p_i}, t_{p_{n+1}} \right) \right)} \]
\[ \cong \mathbb{L} \left[ [t_{p_1}, \ldots, t_{p_{n+1}}] \right] \frac{\prod_{i=1}^{n+1} t_{p_i} - u_1 t_{p_{n+1}} - \ldots - u_n t_{p_{n+1}}}{\prod_{i=1}^{n+1} t_{p_i} - t_{p_{n+1}}^{n+1}}. \]

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai, India.

E-mail address: amal@math.tifr.res.in

Department of Mathematics, IIT Madras, Chennai, India

E-mail address: vuma@iitm.ac.in