Neutrino masses in a conformal multi-Higgs-doublet model

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Abstract

We construct a conformal version of a general multi-Higgs-doublet model with additional right-handed neutrino gauge-singlets. Assuming a minimal extension of the scalar sector by a real singlet field, we show that the resulting model achieves the same attractive properties as the non-conformal theory, combining the seesaw mechanism and higher-order mass production to generate naturally light neutrino masses. Starting with dimensionless couplings only, all masses and energy scales in the theory (including the heavy Majorana masses and the electroweak scale) are obtained from dimensional transmutation via the Coleman-Weinberg mechanism. A characteristic feature of the conformal model is the appearance of the “scalon” in the scalar spectrum. The mass of this particle, which can be expressed in terms of the masses of the other particles in the theory, is produced at the one-loop level. We establish a connection between the large seesaw scale and a suppression of the scalon interactions. The positivity condition for the squared scalon mass requires sufficiently large masses of the additional Higgs bosons balancing the contributions of the heavy neutrinos.

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1 Introduction

Already some time ago, W. Grimus and one of the present authors had suggested a simple extension of the standard model (SM) with an arbitrary number $n_H$ of Higgs doublets $\Phi_k$ and $n_R$ right-handed neutrino singlets $\nu_R$ [1]. In this model, the latter become massive through an explicit Majorana mass term

$$-\frac{1}{2} (\nu_R^c M_R^c + (\nu_R^c)^c M^*_R \nu_R)$$

and interact with the usual $n_L = 3$ lepton doublets $L$ of the SM via the Yukawa couplings

$$-\sum_{k=1}^{n_H} (\Phi_k^\dagger \nu_R Y_{Dk} L + \nu_L^\dagger Y^\dagger_{Dk} \nu_R \Phi_k).$$

In both (1.1) and (1.2), a vector and matrix notation is employed, such that $M_R = M_R^T$ is a complex $n_R \times n_R$ matrix and the $n_H$ complex matrices $Y_{Dk}$ are $n_R \times n_L$.

The model contains two characteristic scales related to two distinct types of parameters with non-vanishing mass dimension. First, the elements of the Majorana mass matrix $M_R$ (with their typical size denoted by $m_R$) and second, the entries $\mu^2_{ij}$ of the bilinear term in the scalar potential [2]

$$\sum_{i,j} \mu^2_{ij} \Phi_i^\dagger \Phi_j + \sum_{i,j,k,l} \lambda_{ijkl} (\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l).$$

The parameters $\mu^2_{ij}$ and $\lambda_{ijkl}$ must be tuned in such a way that the potential assumes its minimum value at $\langle \Phi_k^0 \rangle = v_k/\sqrt{2}$ generating spontaneous symmetry breaking (SSB) of the electroweak (EW) gauge group $SU(2)_L \times U(1)_Y \to U(1)_{em}$ at the EW scale

$$v = (|v_1|^2 + \ldots + |v_{n_H}|^2)^{1/2} \simeq 246 \text{ GeV}.$$

As a consequence, the Yukawa term (1.2) produces a Dirac mass term

$$-\nu_R M_D \nu_L - \nu_L M_D^\dagger \nu_R, \quad M_D = \frac{1}{\sqrt{2}} \sum_{k=1}^{n_H} v_k Y_{Dk},$$

with its typical scale denoted by $m_D$.

It was shown in [1] that for $m_D \ll m_R$ and $n_R < n_L$, the model combines the seesaw mechanism [3–7] with higher-order mass production to generate naturally small Majorana neutrino masses. Subsequently, a general analysis of the one-loop corrections to the seesaw mechanism in this model was presented in [8]. Several phenomenological aspects of the simplest version of this model with just two Higgs doublets and one right-handed neutrino were discussed in [9] and more recently in [10].

It is worth noting that only the terms with (operator) dimension two and three in the Lagrangian (i.e. the scalar mass term in (1.3) and the Majorana mass term (1.1), respectively) spoil the conformal invariance of the action of the model.
The situation is quite similar in the SM, where only the term bilinear in the Higgs doublet field (responsible for the EW scale) violates conformal symmetry explicitly. The introduction of an explicit scalar mass-term in the SM does not only break conformal invariance, but it also gives rise to the so-called hierarchy problem. In this context, Bardeen \[11\] emphasized the importance of the protective property of classical scale invariance (softly broken by the Higgs mass term) for the fine-tuning problem of the SM. Taking this perspective, the popular bugbear of frightening quadratic divergences boils down to the mere artefact of an inadequate regularization, taming the quadratic sensitivity of quantum corrections to high mass scales to a logarithmic one.

Going one step further, it is tempting to assume that (as in QCD) nature abhors the presence of dimensionful couplings in the Lagrangian of a fundamental theory and the generation of mass scales is a pure quantum effect. This idea was put forward in the seminal work of Coleman and E. Weinberg (CW) \[12\]. By employing the one-loop effective potential \[13\][14] as a convenient tool, CW demonstrated the breaking of classical scale invariance by higher order quantum corrections and the generation of a mass scale by dimensional transmutation \[12\]. Subsequently, the most efficient method to investigate the one-loop effective potential of general conformal gauge theories was developed by Gildener and S. Weinberg (GW) \[15\], which allows one to easily disentangle leading and subleading contributions.

A common property of such models is the appearance of the “scalon” \(S\), a scalar state with universal couplings and a one-loop mass \(M_S\), which can be expressed in terms of the masses of the other particles present in the theory by the relation

\[
M^2_S = \frac{1}{8\pi^2 V^2} \left( \sum_s M^4_s + 3 \sum_g M^4_g - 2 \sum_f m^4_f \right). \tag{1.6}
\]

The indices \(s\), \(g\) and \(f\) enumerate the scalars, gauge bosons and Weyl fermions (two-component spinors), respectively. The quantity \(V\) is a vacuum expectation value characterizing the scale of the model generated by dimensional transmutation \[12\].

Note that the positivity of (1.6) is the criterion for the existence of a minimum of the effective potential. This shows in hindsight why early attempts \[15\] to interpret the SM Higgs as the scalon of a conformal version of the SM (without enlarging its particle content) were doomed to fail: the relevant term \(3M_Z^4 + 6M_W^4 - 12m_t^4\) is negative because of the large mass of the top quark.

Motivated by several current puzzles in particle physics, the last few years have seen a strong revival of interest in conformal extensions of the SM (see for instance \[16\] and the references therein). It is a common feature of such models that, due to (1.6), heavy fermionic states (like the top quark or heavy Majorana neutrinos) must be balanced by the presence of bosonic states to meet the positivity requirement \(M^2_S > 0\). At the same time, all mass scales (including the EW scale) are “explained” by dimensional transmutation instead of through explicit mass terms.

In this work, we propose a conformal version of the multi-Higgs-doublet model with right-handed neutrinos. In such a scenario, the explicit mass term (1.1) is forbidden and
masses for the right-handed neutrinos can only be generated from Yukawa couplings \([17]\) after SSB. Restricting ourselves to a minimal extension of the particle content postulated in \([1]\), we introduce an additional real scalar singlet field \(\phi_0\), which interacts with \(\nu_R\) via the Yukawa couplings

\[
-\frac{\phi_0}{2} \left( \nu_R^c Y_R (\nu_R)^c + \nu_R^c Y_R^\ast \nu_R \right),
\]

where \(Y_R = Y_R^T\) is a symmetric complex \(n_R \times n_R\) matrix. The Majorana mass matrix \(M_R = \langle \phi_0 \rangle Y_R\) is now produced through the vacuum expectation value of the singlet scalar.

The characteristic scale \(V\) of the model is the common origin of the vacuum expectation values of the neutral scalars via

\[
\langle \phi_0 \rangle = n_0 V, \quad \langle \Phi_k^0 \rangle = n_k V / \sqrt{2}
\]

with coefficients

\[
n_0 > 0, \quad n_k \in \mathbb{C}, \quad n_0^2 + \sum_{k=1}^{n_H} n_k^* n_k = 1,
\]

generating eventually all masses in the theory. This is in contrast to the non-conformal version of the model, where the origin of the EW scale \(v\) and the (heavy) Majorana scale \(m_R\) are completely unrelated. Except for unnaturally small Yukawa couplings \(Y_{D_k}\), the seesaw mass-hierarchy \(m_D \ll m_R\) requires \(|v_k| \ll v_0\) or, equivalently, \(v \ll V\), which amounts to assuming

\[
\sum_{k=1}^{n_H} n_k^* n_k \ll n_0^2 \simeq 1.
\]

In this work, we study the generic case of the proposed model with an arbitrary number \(n_H\) of Higgs doublets and a likewise unspecified number \(n_R\) of right-handed neutrino fields. Apart from gauge and conformal symmetries, no further symmetry constraints will be imposed.

The paper is organized along the following lines. In section 2 we describe the particle content of the model and construct the conformal Lagrangian. The relevant aspects of SSB via the CW mechanism are addressed in section 3. The tree-level mass matrices and the associated mass eigenfields are determined in section 4. In section 5 we list the electroweak gauge interactions of the fermions and the Yukawa couplings expressed in terms of mass eigenfields. The properties of the scalon are discussed in section 6 where we establish a connection between the large seesaw scale and the suppression of the strength of the scalon couplings and we also identify the experimentally observed 125 GeV Higgs particle in the context of our model. The calculation of the one-loop neutrino masses is carried out in section 7 elucidating the differences (and similarities) to the original non-conformal version of the multi-Higgs-doublet model. In section 8 we present the final results and an outlook.
2 Description of the model

We consider a classically scale-invariant SU(3)\(_c\) \(\times\) SU(2)\(_L\) \(\times\) U(1)\(_Y\) gauge theory with a real scalar singlet field and an arbitrary number \(n_H\) of scalar doublets,

\[
\phi_0, \quad \Phi_k = \begin{pmatrix} \Phi_k^+ \\ \Phi_k^0 \end{pmatrix}, \quad 1 \leq k \leq n_H.
\] (2.1)

The lepton sector consists of the usual SM fields

\[
L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}_r, \quad \ell_R, \quad 1 \leq r \leq n_L = 3,
\] (2.2)

furnished in addition with \(n_R\) right-handed neutrino singlets,

\[
\nu_R^{s}, \quad 1 \leq s \leq n_R.
\] (2.3)

The quark fields (transforming as triplets with respect to SU(3)\(_c\)) are denoted by

\[
Q = \begin{pmatrix} u_L \\ d''_L \end{pmatrix}_r, \quad u_R, \quad d_R, \quad 1 \leq r \leq n_L.
\] (2.4)

The transformation properties of these fields with respect to weak isospin and weak hypercharge \((T, Y/2)\) are given by \(\phi_0 \sim (0, 0), \Phi_k \sim \left(\frac{1}{2}, \frac{1}{2}\right), \ell_L \sim \left(\frac{1}{2}, -\frac{1}{2}\right), \ell_R \sim (0, -1), \nu_R^{s} \sim (0, 0), Q_r \sim \left(\frac{1}{2}, \frac{1}{3}\right), u_R \sim (0, 2/3), d_R \sim (0, -1/3)\).

The covariant derivative has the generic form

\[
D_\mu = \partial_\mu + ig_s \sum_{a=1}^{8} T^a G^a_\mu + ig \vec{T} \cdot \vec{W}_\mu + ig' \frac{Y}{2} B_\mu
\] (2.5)

with

\[
T^a = \begin{cases} \frac{\lambda_a}{2} & \text{for SU(3) triplets} \\ 0 & \text{for SU(3) singlets} \end{cases}, \quad \vec{T} = \begin{cases} \vec{\tau}/2 & \text{for SU(2) doublets} \\ 0 & \text{for SU(2) singlets} \end{cases},
\] (2.6)

where the \(\lambda_a\) and \(\vec{\tau}\) denote the Gell-Mann and Pauli matrices, respectively.

With these building blocks, the construction of the Lagrangian is now straightforward. The gauge boson part reads

\[
\mathcal{L}_g = -\frac{1}{4} \sum_{a=1}^{8} G^{a}_{\mu\nu} G^{a}_{\mu\nu} - \frac{1}{4} \bar{W}_\mu W^{\mu} - \frac{1}{4} B_\mu B^{\mu}
\] (2.7)

and the fermionic Lagrangian is given by

\[
\mathcal{L}_f = \bar{L} i \not{\partial} L + \bar{\ell}_R i \not{\partial} \ell_R + \bar{\nu}_R i \not{\partial} \nu_R + \bar{\ell}_R i \not{\partial} Q + \bar{u}_R i \not{\partial} u_R + \bar{d}_R i \not{\partial} d_R,
\] (2.8)
where the various field multiplets are now written as vectors in flavour space. The kinetic and gauge interaction terms of the scalars are contained in

$$L_s = \frac{1}{2} \partial_{\mu} \varphi_0 \partial^{\mu} \varphi_0 + \sum_{k=1}^{n_H} (D_{\mu} \Phi_k) \bar{D}^{\mu} \Phi_k.$$  

(2.9)

The Yukawa Lagrangian has the general form

$$L_Y = -\frac{\varphi_0}{2} \nu_R Y_R^c \nu_R - \sum_{k=1}^{n_H} \left[ (\Phi_k') Y_{k}^{(d)} + \bar{\Phi}_k Y_{k}^{(u)} \right] Q + \text{h.c.}$$  

(2.10)

with

$$\bar{\Phi}_k = i \tau_2 \Phi_k^* = \begin{pmatrix} \Phi_k^0 \cr -\Phi_k^- \end{pmatrix} \equiv \begin{pmatrix} \Phi_k^0 \\ -\Phi_k^- \end{pmatrix}$$  

(2.11)

and complex $n_L \times n_L$ matrices $Y_{k}^{(d)}, Y_{k}^{(u)}$. The matrices $Y_{Dk}, Y_{R}$ had already been introduced in (1.2) and (1.7), respectively.

Finally, the scalar potential is given by

$$V_0(\varphi_0, \Phi_k) = \frac{\lambda_0}{4} \varphi_0^4 + \sum_{i,j,k,l} \lambda_{ijkl} (\Phi_i^\dagger \Phi_j) (\Phi_k^\dagger \Phi_l) + \varphi_0^2 \sum_{i,j} \kappa_{ij} \Phi_i^\dagger \Phi_j,$$  

(2.12)

where hermiticity of $V_0$ and the symmetry of the second term imply the relations

$$\lambda_0 = \lambda_0^*, \lambda_{ijkl} = \lambda_{klji}, \lambda_{ijkl} = \lambda_{jilk}, \kappa_{ij} = \kappa_{ji}^*.$$  

(2.13)

The total Lagrangian

$$L = L_g + L_f + L_s + L_Y - V_0$$  

(2.14)

contains all conformally invariant and gauge-symmetric terms that can be composed of the given particle multiplets. Note that the (explicit) mass terms

$$-\frac{1}{2} \nu_R M_R (\nu_R)^c - \frac{1}{2} (\nu_R)^c M_R^\dagger \nu_R - \frac{\mu_0^2}{2} \varphi_0^2 - \sum_{i,j} \mu_{ij}^2 \Phi_i^\dagger \Phi_j$$  

(2.15)

are forbidden by scale invariance.

3 Spontaneous symmetry breaking

Although the model appears to have a symmetric vacuum at tree level, scale invariance and gauge symmetry are actually broken as a result of higher order quantum corrections via the CW mechanism [12]. A convenient tool to study the vacuum structure of the theory is the effective potential [13,14]

$$\mathcal{V}(\varphi_0, \Phi_k) = \mathcal{V}_0(\varphi_0, \Phi_k) + \delta \mathcal{V}(\varphi_0, \Phi_k),$$  

(3.1)
where $\delta V$ contains the quantum corrections to the tree-level potential $V_0$.

Following the GW approach [15], we consider the constant field configuration

$$\varphi_0 = N_0 \in \mathbb{R}, \quad \Phi_k = \frac{1}{\sqrt{2}} N_k \in \mathbb{C}^2$$

subject to the constraint

$$N_0^2 + \sum_{k=1}^{n_H} N_k^* N_k = 1,$$

which parametrizes the $4n_H$ dimensional unit sphere $S^{4n_H}$. We are looking for a minimum of the tree potential (2.12) on $S^{4n_H}$ at a point

$$N_0 = n_0 > 0, \quad N_k = \begin{pmatrix} 0 \\ n_k \end{pmatrix}, \quad n_0^2 + \sum_{k=1}^{n_H} n_k^* n_k = 1,$$

so that the electromagnetic charge $Q_{em} = T_3 + Y/2$ remains unbroken. In this way, we obtain the stationarity condition

$$\sum_{j,k,l} \lambda_{ijkl} n_j^* n_k^* n_l + n_0^2 \sum_j \kappa_{ij} n_j - n_i \left( \sum_{k,l} \kappa_{kl} n_k^* n_l + \lambda_0 n_0^2 \right) = 0. \quad (3.5)$$

In the next step we take advantage of the fact that the parameters of the tree potential are functions of the renormalization scale $\mu$,

$$\lambda_0 = \lambda_0(\mu), \quad \lambda_{ijkl} = \lambda_{ijkl}(\mu), \quad \kappa_{ij} = \kappa_{ij}(\mu),$$

governed by the pertinent renormalization group equations. As suggested by GW [15], we adjust the renormalization scale $\mu = \Lambda_{GW}$ in such a way that the minimum value of $V_0$ on the unit sphere vanishes:

$$V_0 \left( n_0, \frac{n_k}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0. \quad (3.7)$$

In our model, the GW condition (3.7) corresponds to the equation

$$\sum_{i,j,k,l} \lambda_{ijkl} n_i^* n_j n_k^* n_l + 2n_0^2 \sum_{i,j} \kappa_{ij} n_i^* n_j + \lambda_0 n_0^4 = 0, \quad (3.8)$$

which allows us to trade one of the couplings, say $\lambda_0$, for the GW scale $\Lambda_{GW}$, being an example of dimensional transmutation [12]. In the following, we shall assume that all

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1The general form of $\delta V$ in the Landau gauge at one-loop order was obtained by CW [12] using a specific renormalization scheme. The result was subsequently confirmed in [18] using a different method. See e.g. [19] for the corresponding formula in the MS scheme.
couplings of our model are taken at $\mu = \Lambda_{GW}$ so that (3.8) holds. Note also that (3.5) and (3.8) are equivalent to the system of equations

$$\sum_j \left( \sum_{k,l} \lambda_{ijkl} n_k^* n_l + n_0^2 \kappa_{ij} \right) n_j = 0,$$

$$\sum_{i,j} \kappa_{ij} n_i^* n_j + \lambda_0 n_0^2 = 0. \quad (3.10)$$

As the tree potential is a homogeneous function of the field variable $s$, $V_0(\phi_0, \Phi_k)$ attains a minimum value zero everywhere on the ray $\phi_0 = \phi n_0, \quad \Phi_k = \phi n_0 \sqrt{2}, \quad \phi \in \mathbb{R}, \quad (3.11)$

once (3.7) holds. When the higher-order term $\delta V(\phi_0, \Phi_k)$ in (3.1) is turned on, the effective potential receives a small curvature in the radial direction, which picks out a definite value $\langle \phi \rangle = V > 0$ and also a small shift at this minimum [15]:

$$\langle \phi_0 \rangle = V n_0 + \delta \phi_0, \quad \langle \Phi_k \rangle = \frac{1}{\sqrt{2}} (V n_k + \delta \phi_k). \quad (3.12)$$

### 4 Tree-level masses

The lowest-order mass matrices of the model are obtained by expanding the scalar fields around the leading terms in (3.12),

$$\varphi_0 = v_0 + \rho_0, \quad \Phi_k = \left( \frac{\phi n_k}{\sqrt{2}} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad \phi \in \mathbb{R}, \quad (4.1)$$

with real fields $\rho_0, \rho_k$ and $\sigma_k$.

#### 4.1 Gauge bosons

From $\mathcal{L}_s$ one obtains the mass eigenvalues of the vector bosons,

$$M^2_W = v^2 g^2 / 4, \quad M^2_Z = v^2 (g^2 + g'^2) / 4, \quad v^2 = \sum_{k=1}^{n_H} v_k^* v_k, \quad (4.2)$$

with $v = (\sqrt{2} G_F)^{-1/2} \approx 246$ GeV. The associated mass eigenfields are given by

$$W^\pm_\mu = \frac{(W_1 \mp i W_2)_\mu}{\sqrt{2}}, \quad \left( \begin{array}{c} Z_\mu \\ A_\mu \end{array} \right) = \left( \begin{array}{cc} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{array} \right) \left( \begin{array}{c} W^3_\mu \\ B_\mu \end{array} \right), \quad (4.3)$$

with the Weinberg angle $\theta_W$ defined by $\sin \theta_W = g' / \sqrt{g^2 + g'^2}$. The $Z^0$ mass formula can now be rewritten as

$$M^2_Z = \frac{g^2 v^2}{4 \cos^2 \theta_W}. \quad (4.4)$$
4.2 Charged scalars

The mass matrix of the charged scalars (derived from $V_0$) reads

$$(M_+^2)_{ij} = V^2 \left( \sum_{k,l} \lambda_{ijkl} n_k^* n_l + n_0^2 \kappa_{ij} \right).$$  \hspace{1cm} (4.5)

The relation (3.9) is equivalent to the eigenvalue equation

$$M_+^2 n = 0, \quad n = (n_1 \ldots n_{n_H})^T,$$  \hspace{1cm} (4.6)

which means that the vector $n$ is an eigenvector of the hermitian matrix $M_+^2$ associated with the eigenvalue zero.

The fields $\Phi^+_k$ are related to the mass eigenfields $H^+_a$ by a unitary transformation $U$,

$$\Phi^+_k = \sum_{a=1}^{n_H} U_{ka} H^+_a, \quad U^\dagger M_+^2 U = \text{diag} (\mu_1^2, \ldots, \mu_{n_H-1}^2, 0), \quad U = U^{-1}. \hspace{1cm} (4.7)$$

The model contains $n_H - 1$ physical charged scalars $H^+_1, \ldots, H^+_n_{n_H-1}$ and one charged would-be-Goldstone-boson $H^{+_n_{n_H}} \equiv G^+$ corresponding to the last column of the unitary matrix $U$ with

$$U_{n_{n_H}} = \frac{n_k}{\sqrt{1 - n_0^2}}. \hspace{1cm} (4.8)$$

4.3 Neutral scalars

The tree-level mass term of the neutral scalars can be written in the form

$$-\frac{1}{2} \begin{pmatrix} \rho^0 & \rho^T \end{pmatrix} \begin{pmatrix} 2\lambda_{00} n_0^2 & \text{Re} k^T & \text{Im} k^T \\ \text{Re} k & A & C \\ \text{Im} k & C^T & B \end{pmatrix} \begin{pmatrix} \rho^0 \\ \rho \\ \sigma \end{pmatrix},$$  \hspace{1cm} (4.9)

with $\rho^T = (\rho_1 \ldots \rho_{n_H})$ and $\sigma^T = (\sigma_1 \ldots \sigma_{n_H})$. The vector $k \in \mathbb{C}^{n_H}$ is defined by

$$k_i = 2V^2 n_0 \sum_{j=1}^{n_H} \kappa_{ij} n_j. \hspace{1cm} (4.10)$$

The $n_H \times n_H$ matrices

$$A = \text{Re} (M_+^2 + K + K'), \quad B = \text{Re} (M_+^2 + K' - K), \quad C = \text{Im} (-M_+^2 - K' + K) \hspace{1cm} (4.11)$$

can be written in terms of $M_+^2$ and the matrices $K = K^T, K' = K'^\dagger$ introduced in [2]:

$$K_{ij} = V^2 \sum_{k,l} \lambda_{ikjl} n_k^* n_l, \quad K'_{ij} = V^2 \sum_{k,l} \lambda_{iklj} n_k n_l^*. \hspace{1cm} (4.12)$$
The two orthogonal vectors

\[
\begin{pmatrix}
  n_0 \\
  \Re n \\
  \Im n
\end{pmatrix}, \quad \begin{pmatrix}
  0 \\
  -\Im n \\
  \Re n
\end{pmatrix} \in \mathbb{R}^{2n_H+1}
\] (4.13)

are eigenvectors of $\mathcal{M}_0^2$ with eigenvalue zero. The first one is related to the scalon $S$ [15], which receives a nonvanishing mass only at one-loop order. The second one is associated with the neutral would-be-Goldstone-boson $G^0$.

The neutral scalar mass eigenfields $S_0^b$ ($b = 0, \ldots, 2n_H$) are related to $\rho_0$, $\rho$ and $\sigma$ by an orthogonal $(2n_H + 1) \times (2n_H + 1)$ matrix $\mathcal{R}$,

\[
\begin{pmatrix}
  \rho_0 \\
  \rho \\
  \sigma
\end{pmatrix} = \begin{pmatrix}
  r^T \\
  \Re R \\
  \Im R
\end{pmatrix} \begin{pmatrix}
  S_0^0 \\
  \cdots \\
  S_{2n_H}^0
\end{pmatrix}_R
\] (4.14)

with a complex $n_H \times (2n_H + 1)$ matrix $R = \Re R + i \Im R$ and $r \in \mathbb{R}^{2n_H+1}$. In index notation, we write $\mathcal{R}_{ob} = r_b$, $\mathcal{R}_{kk} = \Re R_{kb}$, $\mathcal{R}_{(k+n_H)b} = \Im R_{kb}$ with $1 \leq k \leq n_H$ and $0 \leq b \leq 2n_H$.

The orthogonality of $\mathcal{R}$ can be expressed in the form

\[
r r^T + \Re (R^\dagger R) = 1_{2n_H+1},
\] (4.15)

or, equivalently, by the relations

\[
r^T r = 1, \quad R r = 0, \quad R R^\dagger = 2 \cdot 1_{n_H}, \quad R R^T = 0.
\] (4.16)

The numbering of the mass eigenfields is chosen in such a way that $S_0^0 \equiv S$ and $S_{2n_H}^0 \equiv G^0$ so that

\[
\mathcal{R}^T \mathcal{M}_0^2 \mathcal{R} = \text{diag} \left( 0, M_1^2, \ldots, M_{2n_H-1}^2, 0 \right).
\] (4.17)

As a consequence of (4.13), the first and the last column of $\mathcal{R}$ are then obtained from $r_0 = n_0$, $R_{k0} = n_k$ (for $b = 0$) and $r_{2n_H} = 0$, $R_{k2n_H} = i n_k / \sqrt{1 - n_0^2}$ (for $b = 2n_H$), which implies

\[
\rho_0 = n_0 S + \sum_{b=1}^{2n_H-1} r_b S_b^0, \quad \rho_k + i \sigma_k = n_k S + \sum_{b=1}^{2n_H-1} R_{kb} S_b^0 + \frac{in_k}{\sqrt{1 - n_0^2}} G^0.
\] (4.18)

### 4.4 Fermions

Without loss of generality, we may assume that $\ell_{L,R}$, $u_{L,R}$ and $d_R$ are already mass eigenfields, which can always be achieved by a suitable basis transformation in flavour space. The weak eigenfield $d_L' = V_{CKM} d_L$ is related to the mass eigenfield $d_L$ by the weak mixing matrix $V_{CKM}$ [20][21]. Defining the Dirac fields $\ell = \ell_L + \ell_R$, $u = u_L + u_R$, $d = d_L + d_R$, the fermion mass terms are given by

\[
- \bar{\ell} M_{\ell} \ell - \bar{u} M_u u - \bar{d} M_d d - \bar{\nu}_L M_D \nu_L - \bar{\nu}_R M_D^\dagger \nu_R - \frac{1}{2} \bar{\nu}_R M_R (\nu_R)^c - \frac{1}{2} (\nu_R)^c M_R^\dagger \nu_R,
\] (4.19)
where

\[
M_\ell = \frac{1}{\sqrt{2}} \sum_{k=1}^{n_{\text{H}}} v_k^* Y_k^{(\ell)} = \text{diag} \left( m_e, m_\mu, m_\tau \right),
\]

\[
M_u = \frac{1}{\sqrt{2}} \sum_{k=1}^{n_{\text{H}}} v_k Y_k^{(u)} = \text{diag} \left( m_u, m_c, m_t \right),
\]

\[
M_d = \frac{1}{\sqrt{2}} \sum_{k=1}^{n_{\text{H}}} v_k^* Y_k^{(d)} V_{\text{CKM}} = \text{diag} \left( m_d, m_s, m_b \right),
\]

\[
M_D = \frac{1}{\sqrt{2}} \sum_{k=1}^{n_{\text{H}}} v_k Y_{Dk}, \quad M_R = v_0 Y_R.
\]

Introducing the left-handed field

\[
\omega_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix},
\]

the neutrino mass term can be written in the compact form

\[-\frac{1}{2} (\omega_L)^c M_\nu \omega_L + \text{h.c.},\]

with an \((n_L + n_R) \times (n_L + n_R)\) symmetric mass matrix

\[
M_\nu = \begin{pmatrix} 0 & M_T^D \\ M_D & M_R \end{pmatrix}.
\]

To obtain the (tree-level) mass eigenfields \(\hat{\omega}_L\), we perform a unitary transformation

\[
\omega_L = U \hat{\omega}_L, \quad U^\dagger = U^{-1},
\]

such that

\[
U^\dagger M_\nu U = \hat{M}_\nu
\]

is diagonal and non-negative.

It is convenient to decompose the unitary \((n_L + n_R) \times (n_L + n_R)\) matrix \(U\) as

\[
U = \begin{pmatrix} U_L & 0 \\ U_R^* & 0 \end{pmatrix}
\]

with an \(n_L \times (n_L + n_R)\) submatrix \(U_L\) and an \(n_R \times (n_L + n_R)\) submatrix \(U_R\). Defining

\[
\chi = \hat{\omega}_L + (\hat{\omega}_L)^c,
\]

the weak eigenfields \(\nu_{L,R}\) can be written as linear combinations of the neutrino mass-eigenfields

\[
\nu_L = U_L \hat{\omega}_L = U_L P_L \chi, \quad \nu_R = U_R (\hat{\omega}_L)^c = U_R P_R \chi,
\]
where $P_{L,R} = (1 \mp \gamma_5)/2$.

The unitarity of $\mathcal{U}$ is equivalent to the relations \[2\]

\[U^\dagger L U_L + U^T R U^*_R = 1_{n_L+n_R} \iff U_L U_L^\dagger = 1_{n_L}, U_R U_R^\dagger = 1_{n_R}, U_L U_R^T = 0_{n_L \times n_R} \tag{4.32}\]

for the submatrices. From (4.28), written in the form $\mathcal{U}^T \mathcal{M}_\nu = \hat{\mathcal{M}}_\nu \mathcal{U}^\dagger$, one finds

\[U^\dagger R M_D = \hat{\mathcal{M}}_\nu U^\dagger L \iff M_D^T U_R^* = U_L^* \hat{\mathcal{M}}_\nu \tag{4.33}\]

Combining the appropriate relations of (4.32) and (4.33) implies

\[U_L^* \hat{\mathcal{M}}_\nu U_L^\dagger = 0, \quad U_R \hat{\mathcal{M}}_\nu U_L^\dagger = M_D \tag{4.35}\]

and

\[\hat{\mathcal{M}}_\nu = U_R^\dagger M_D U_L + U_L^T M_D^T U^*_R + U_R^\dagger M_R U^*_R \tag{4.36}\]

Let us consider the diagonalization of the lowest-order neutrino mass matrix (4.26) in the interesting case $n_L > n_R$. Following the argument given in [1], the submatrix $M_D$ can be regarded as a linear mapping $M_D : \mathbb{C}^{n_L} \rightarrow \mathbb{C}^{n_R}$, where the relation

\[\dim \ker M_D = \dim \ker \mathbb{C}^{n_L} - \dim \ker M_D \geq n_L - n_R \tag{4.37}\]

ensures the existence of $n_L - n_R$ orthonormal vectors $u'_i \in \mathbb{C}^{n_L}$ $(1 \leq i \leq n_L - n_R)$ with $M_D u'_i = 0$. The unitary matrix $\mathcal{U}$ introduced in (4.27) can be written in the form

\[\mathcal{U} = (u_1, \ldots, u_{n_L+n_R}) \tag{4.38}\]

with $n_L + n_R$ orthonormal vectors $u_i \in \mathbb{C}^{n_L+n_R}$, where the first $n_L - n_R$ ones are given by

\[u_i = \begin{pmatrix} u'_i \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n_L - n_R. \tag{4.39}\]

Thus, the decomposition (4.29) assumes the form [1]

\[\mathcal{U} = \begin{pmatrix} U_L' & U_L'' \\ 0 & U_R'^* \end{pmatrix} \tag{4.40}\]

with the $n_L \times (n_L - n_R)$ submatrix $U_L' = (u'_1, \ldots, u'_{n_L-n_R})$, the $n_L \times 2n_R$ submatrix $U_L''$ and the $n_R \times 2n_R$ submatrix $U_R'^*$. Using (4.40) in (4.28), we finally arrive at

\[\hat{\mathcal{M}}_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M}' \end{pmatrix} \tag{4.41}\]

with a diagonal and positive $2n_R \times 2n_R$ matrix $\hat{M}'$. Thus, at tree level, there are in general $n_L - n_R$ massless Majorana neutrinos and $2n_R$ massive ones. Assuming a mass hierarchy $m_D \ll m_R$ in (4.26), the seesaw mechanism leads to $n_R$ heavy neutral fermions and $n_R$ light massive neutrinos.
5 Interaction terms

In this section we list the relevant interaction Lagrangians expressed in terms of mass eigenfields. For our present purposes, only the weak gauge couplings of the fermions and the Yukawa terms will be needed.

5.1 Gauge couplings

The fermionic gauge interactions are readily obtained from (2.8) by rewriting the covariant derivative in terms of the mass eigenfields of the vector bosons. Omitting the gluonic term, (2.5) takes the form

\[ D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}} \left( T_1 W^+ - T_2 W^- \right) + \frac{ig}{\cos \theta_W} \left( T_3 - \sin^2 \theta_W Q_{em} \right) Z_\mu + ieQ_{em} A_\mu, \]  

where \( T_\pm = T_1 \pm iT_2 \) and \( e = g \sin \theta_W \). In this manner, we obtain the charged-current Lagrangian

\[ \mathcal{L}_{cc} = -\frac{gW^+}{\sqrt{2}} \left( \chi U_L^{\dagger} \gamma^\mu P_L \ell + \pi \gamma^\mu V_{CKM} P_L d \right) + \text{h.c.} \]  

and the neutral-current Lagrangian

\[ \mathcal{L}_{nc} = -\frac{gZ}{\cos \theta_W} \left[ \frac{1}{4} \chi \gamma^\mu \left( \bar{U}_L^{\dagger} U_L P_L - \bar{U}_L^{\dagger} U_L P_R \right) \chi + \bar{\ell} \gamma^\mu \left( -\frac{1}{2} P_L + \sin^2 \theta_W \right) \ell \right. \]

\[ + \bar{\pi} \gamma^\mu \left( \frac{1}{2} P_L - \frac{2}{3} \sin^2 \theta_W \right) u + \bar{d} \gamma^\mu \left( -\frac{1}{3} P_L + \frac{1}{3} \sin^2 \theta_W \right) d \] .

5.2 Yukawa couplings

The Yukawa terms of the charged scalars (1 \( \leq a \leq n_H \)) are given by

\[ \mathcal{L}_Y(H^+_a) = H^+_a \left[ \bar{\chi} \left( \bar{\tilde{Y}}_{Da} P_L - \bar{\tilde{Y}}^{(\ell)\dagger}_a P_R \right) \ell + \bar{\pi} \left( \bar{\tilde{Y}}^{(u)}_a P_L - \bar{\tilde{Y}}^{(d)\dagger}_a P_R \right) d \right] + \text{h.c.}, \]  

where we have introduced the coupling matrices

\[ \bar{\tilde{Y}}_{Da} = U_R^{\dagger} \sum_{k=1}^{n_H} U_{ka} Y_{Dk}, \quad \bar{\tilde{Y}}^{(\ell)}_a = \sum_{k=1}^{n_H} U_{ka}^{*} Y^{(\ell)}_k U_L, \]  

\[ \bar{\tilde{Y}}^{(u)}_a = \sum_{k=1}^{n_H} U_{ka} Y^{(u)}_k V_{CKM}, \quad \bar{\tilde{Y}}^{(d)}_a = \sum_{k=1}^{n_H} U_{ka}^{*} Y^{(d)}_k. \]  

The Yukawa interactions of the neutral scalars (0 \( \leq b \leq 2n_H \)) take the form

\[ \mathcal{L}_Y(S^0_b) = -\frac{S^0_b}{\sqrt{2}} \left[ \ell \left( \bar{\tilde{Y}}^{(\ell)}_b P_L + \bar{\tilde{Y}}^{(\ell)\dagger}_b P_R \right) \ell + \frac{1}{2} \chi \left( F_b P_L + F_b^{\dagger} P_R \right) \chi \right. \]

\[ + \left. \bar{\pi} \left( \bar{\tilde{Y}}^{(u)}_b V_{CKM} P_L + \bar{\tilde{Y}}^{(u)\dagger}_b P_R \right) d + \bar{d} \left( \bar{\tilde{Y}}^{(d)}_b P_L + \bar{\tilde{Y}}^{(d)\dagger}_b P_R \right) u \right], \]
where
\[ F_b = U_R^\dagger \hat{Y}_{Db} U_L + U_L^T \hat{Y}_{Db}^T U_R^\dagger + \sqrt{2} r_b U_R^\dagger Y U_R^* \] (5.8)
and
\[ \hat{Y}_{b}^{(f)} = \sum_{k=1}^{n_H} Y_k^{(f)} R_{kb}^*, \quad \hat{Y}_{Db} = \sum_{k=1}^{n_H} Y_{Db} R_{kb}, \] (5.9)
\[ \hat{Y}_{b}^{(d)} = \sum_{k=1}^{n_H} Y_k^{(d)} R_{kb}^*, \quad \hat{Y}_{b}^{(u)} = \sum_{k=1}^{n_H} Y_k^{(u)} R_{kb}. \] (5.10)

Note the presence of the extra piece with \( r_b \) and the additional scalon coupling (for \( b = 0 \)) in (5.8) compared to the corresponding Yukawa couplings of the model described in [1].

6 Scalon properties

The general (model-independent) features of the scalon interactions had already been noticed in [15]. They follow from the fact that the fundamental order parameter \( V \) and the scalon \( S \) enter in the Lagrangian only via the combination \( \phi = V + S \) parametrizing the flat direction of the tree potential.

As a consequence, the Yukawa couplings of the scalon are obtained by simply multiplying the fermion mass terms by the factor \( S/V \),

\[ \mathcal{L}_Y(S) = -\frac{S}{V} \left( 7M_\ell \ell + \frac{1}{2} M_{\mu} \mu + M_{d} d + \overline{\pi M_u u} \right), \] (6.1)

which can also be checked by using (5.7-5.9) for \( b = 0 \) together with (4.20-4.23) and (4.36).

Similarly, the scalon couplings generated by (2.9) are related to the mass terms of the vector bosons through the interaction Lagrangian

\[ \left( M_W^2 W_{\mu}^+ W_{-\mu} + \frac{1}{2} M_Z^2 Z_{\mu} Z^\mu \right) \left( \frac{2S}{V} + \frac{S^2}{V^2} - \frac{2n_0}{v^2} \sum_{b=1}^{2n_H-1} r_b S^0_b S \right), \] (6.2)

where Goldstone boson terms have not been included.

The extraction of the scalon couplings from the scalar tree-potential is simplified by organizing (2.12) in powers of \( V + S \). Terms proportional to \((V + S)^4\) and \((V + S)^3\) are both absent because of the GW condition (3.7) and the minimum condition (3.5). The piece with \((V + S)^2\) contains the scalar mass terms together with the associated scalon couplings. Therefore, the latter take the form

\[ - \left( \sum_{a=1}^{n_H-1} \mu_a^2 H_a^+ H_a^- + \frac{1}{2} \sum_{b=1}^{2n_H-1} M_b^2 (S^0_b)^2 \right) \left( \frac{2S}{V} + \frac{S^2}{V^2} \right). \] (6.3)

\(^2\)See (3.11), (3.12), (4.1) and (4.18) in the case of our model.
Finally, the term proportional to $V + S$ relates cubic scalar couplings (without the scalon) to quartic scalar couplings with exactly one scalon. As they will not be needed in the following, we refrain from displaying them here.

A glance at (6.1-6.3) reveals that the scalon decays predominantly into a pair of the heaviest possible states [15]. There remains the question whether the state $S$ can be identified with the $H^0$ particle discovered at a mass of 125 GeV at the LHC [22,23]. Studies of the $H^0$ decay properties performed so far are consistent with the SM predictions for the Higgs particle. The $H^0$ signal strength (for combined final states) of $1.10 \pm 0.11$ [24] can serve as a quantitative measure of possible (small) deviations from the SM. Comparison of the SM Higgs couplings with (6.1) and (6.2) shows that $S \equiv H^0$ would actually correspond to $V \simeq v$. However, as we envisage to implement the seesaw mechanism in our model, the necessary mass hierarchy in (4.23) requires $|v_k| \ll v_0$ or, equivalently, $v \ll V$. Thus, in our preferred scenario, the identification of the scalon with the scalar boson observed at 125 GeV is excluded and one of the states $S^0_b$ ($1 \leq b \leq 2n_H - 1$) has to play the role of the $H^0$ particle.

In this way, we arrive at the prediction of a new neutral scalar $S$ with couplings (6.1-6.3) suppressed by a factor $v/V$ compared to the corresponding interactions of the SM Higgs $H^0$. Inserting the contributions of the relevant states in (1.6), the scalon mass formula of our model reads

$$M^2_S = \frac{1}{8\pi^2V^2} \left( M_{H^0}^4 + 3M_Z^4 + 6M_W^4 - 12m_t^4 + \sum_{S^0_b \neq H^0} M_b^4 + 2\sum_a \mu_a^4 - 2 \text{Tr} \hat{M}_\nu^4 \right),$$

(6.4)

where the tiny contributions from quarks lighter than the top have not been included. The neutrino term is practically exclusively determined by the masses of the heavy neutral fermions with $\text{Tr} \hat{M}_\nu^4 \simeq \text{Tr} (M_R^* M_R)^2$. Compared to the masses of the heavy Higgs and fermion states, the scalon mass is suppressed by the loop factor present in (6.4). A further suppression occurs through the (partial) cancellation of bosonic and fermionic contributions in (6.4), still allowing a wide range of possible scalon mass values.

The positivity condition $M^2_S > 0$ leads to the mass inequality

$$\sum_{S^0_b \neq H^0} M_b^4 + 2\sum_a \mu_a^4 > -M_{H^0}^4 - 3M_Z^4 - 6M_W^4 + 12m_t^4 + 2 \text{Tr} \hat{M}_\nu^4, \quad (317 \text{ GeV})^4,$$

(6.5)

which can only be fulfilled by sufficiently large masses of the additional Higgs fields compensating the contribution of the heavy neutrinos.

7 Neutrino masses at one-loop order

The one-loop corrections generate an $n_L \times n_L$ submatrix $\delta M_L$ [11] in the upper left corner of the tree-level mass matrix (4.26), where at tree level there is a zero submatrix. As a consequence, the sum of all one-loop graphs contributing to $\delta M_L$ must be finite and the
The final result can be expressed in terms of the (tree-level) parameters of the theory. The submatrices $M_D$ and $M_R$ also receive corrections, in this case from one-loop graphs as well as from counterterms. Altogether, $M_\nu$ gets a shift

$$M_\nu \to M_\nu^{(1)} = M_\nu + \delta M_\nu, \quad \delta M_\nu = \begin{pmatrix} \delta M_L & \delta M_D^T \\ \delta M_D & \delta M_R \end{pmatrix}. \tag{7.1}$$

As we are only interested in the masses of those neutrino states that are massless at tree level, it is sufficient to compute the submatrix $\delta M_L$. To this end, we consider the neutrino self-energy matrix in the basis of the tree-level mass eigenfields. We use the decomposition

$$\Sigma(p) = A_L(p^2)\phi P_L + A_R(p^2)\phi P_R + B_L(p^2)P_L + B_R(p^2)P_R, \tag{7.2}$$

where $p$ is the neutrino four-momentum. The dispersive parts of the coefficients satisfy

$$A_L^\dagger = A_L, \quad A_R^\dagger = A_R, \quad B_L^\dagger = B_R. \tag{7.3}$$

The Majorana nature $\chi^c = \chi$ of the neutrino field implies the consistency condition

$$\Sigma(p) = C\Sigma(-p)^TC^{-1} \quad \Rightarrow \quad A_L = A_R^T, \quad B_L = B_R^T, \quad B_R = B_L^T. \tag{7.4}$$

For our purposes, it suffices to consider only $B_L(p^2)$ at $p^2 = 0$. Transforming back to the original basis one arrives at

$$\delta M_L = U_L^* B_L(0)U_L^\dagger = U_L^* \left[ B_L^{(Z^0)}(0) + \sum_{b=0}^{2n_H} B_L^{(s_b^0)}(0) \right] U_L^\dagger, \tag{7.5}$$

receiving contributions from one-loop graphs with the $Z^0$ boson and neutral scalars. Following the relation $U_L^* \mathcal{M}_\nu U_L^\dagger = 0$ from (4.35) is used. Employing also $U_L U_L^\dagger = 1_{n_H}$ displayed in (4.32), one obtains the (gauge dependent) expression

$$U_L^* B_L^{(Z^0)}(0) U_L^\dagger = \frac{M_Z^2}{(4\pi)^2 v^2} U_L^* \left\{ 4\hat{\mathcal{M}}_\nu \int_0^1 d\alpha \ln \left( \alpha M_Z^2 + (1 - \alpha)\hat{\mathcal{M}}_\nu^2 \right) \right. \tag{7.6}$$

$$+ \frac{\hat{\mathcal{M}}_\nu^3}{M_Z^2} \int_0^1 d\alpha \left[ \ln \left( \alpha \xi Z M_Z^2 + (1 - \alpha)\hat{\mathcal{M}}_\nu^2 \right) - \ln \left( \alpha M_Z^2 + (1 - \alpha)\hat{\mathcal{M}}_\nu^2 \right) \right] \left. \right\} U_L^\dagger.$$

In the scalar contribution to (7.5) we encounter the products $U_L^* F_b$ and $F_b U_L^\dagger$, respectively. Using the unitarity relations (4.32), these terms can be rewritten as

$$U_L^* F_b = Y_D^T U_R^*, \quad F_b U_L^\dagger = U_R^* \hat{Y}_D b, \tag{7.7}$$
which shows that the additional pieces proportional to $r_b$, present in (5.8), do not appear in (7.5). The sum of all scalar loops is finite because of the last relation in (4.16) and we obtain

$$U_L^\dagger \sum_{b=0}^{2n_H} B_L^{(S)} (0) U_L = \frac{1}{2(4\pi)^2} \sum_{b=0}^{2n_H} Y_{Db}^T U_R^{*} \hat{\mathcal{M}}_\nu \int_0^1 d\alpha \ln \left( \alpha M_b^2 + (1 - \alpha) \hat{M}_\nu^2 \right) U_R^\dagger Y_{Db}. \quad (7.8)$$

Using $\hat{Y}_{D2n_H} = i\sqrt{2}M_D/v$ and (4.33), the contribution of the neutral Goldstone boson $G^0 \equiv S_{2n_H}$ can be recast into the form

$$-\frac{1}{(4\pi)^2 v^2} U_L^* \hat{\mathcal{M}}_\nu^3 \int_0^1 d\alpha \ln \left( \alpha \xi Z M_Z^2 + (1 - \alpha) \hat{M}_\nu^2 \right) U_L^\dagger, \quad (7.9)$$

and we see that this term cancels exactly the gauge-dependent part of (7.6).

The total result for $\delta M_L$ is now given by

$$\delta M_L = \frac{3}{(4\pi)^2 v^2} U_L^* \hat{\mathcal{M}}_\nu^3 \frac{\ln(\hat{\mathcal{M}}_\nu^2/M_Z^2)}{\hat{M}_\nu^2/M_Z^2 - 1} U_L^\dagger + \frac{1}{(4\pi)^2 V^2} U_L^* \hat{\mathcal{M}}_\nu^3 \frac{\ln(\hat{\mathcal{M}}_\nu^2/M_b^2)}{M_\nu^2/M_b^2 - 1} U_L^\dagger + \frac{1}{2(4\pi)^2} \sum_{b=1}^{2n_H-1} Y_{Db}^T U_R^{*} \hat{\mathcal{M}}_\nu \ln(\hat{\mathcal{M}}_\nu^2/M_b^2) \hat{M}_\nu^2/\hat{M}_b^2 - 1 U_R^\dagger Y_{Db}. \quad (7.10)$$

The first term is the residual $Z^0$ contribution once all cancellations have been taken into account. The second one represents the scalon contribution $\delta M_L^{(S)}$ obtained from the summand with $b=0$ in (7.8) by using $\hat{Y}_{D0} = \sqrt{2}M_D/V$ and (4.33). The third term contains the contributions of all physical scalars except the scalon.

Recalling (5.9), the Yukawa matrices in (7.10) are related to the ones introduced in (1.2) by $Y_{Db} = \sum_k Y_{Dk} R_{kb}$. The orthogonality relation $RR^T = 0$ of the transformation matrix $R$ given in (4.16) can be rewritten as

$$\sum_{b=1}^{2n_H-1} R_{kb} R_{\ell b} = \frac{v_k v_\ell}{v^2} \left( 1 - \frac{v^2}{V^2} \right), \quad (7.11)$$

modifying the corresponding formula in the original version of the model [11] only by the extra factor $1 - v^2/V^2 \simeq 1$.

Strictly speaking, the scalon contribution to $\delta M_L$ vanishes at the order we are working. To be fully consistent at one-loop, we have to insert tree-level masses in (7.10) corresponding to $M_S = 0$ in the case of the scalon, which leads to $\delta M_L^{(S)} = 0$ in this limit. In this sense and apart from the small modification in (7.11), we recover the same one-loop result.
for $\delta M_L$ that had already been obtained in the non-conformal model without additional scalar singlet \[8\].

The diagonalization of the one-loop mass matrix $\hat{M}^{(1)}_\nu$ was described in \[1\]. As a net result, the diagonal tree-level mass matrix (4.41) with zero entries in the left upper block is replaced by the one-loop matrix

$$\hat{M}^{(1)}_\nu = \left( \begin{array}{cc} M_0 & 0 \\ 0 & M' \end{array} \right), \quad \hat{M}_0 = U_L^T \delta M_L U_L', \quad (7.12)$$

where the $n_L - n_R$ orthonormal vectors in $U'_L = (u'_1, \ldots, u'_{n_L-n_R})$, so far only subject to the constraint $M_D u'_i = 0$, can now be chosen in such a way that $\hat{M}_0$ becomes diagonal and non-negative.

The $Z^0$ as well as the scalon contribution in (7.10) have the general structure

$$U^*_L \hat{M}_\nu f(\hat{M}_\nu) \hat{M}_\nu U_L = M_D^T U_R^* f(\hat{M}_\nu) U_R^T M_D, \quad (7.13)$$

where the form on the right-hand side follows from (4.33). Because of $M_D U'_L = 0$, both terms do not contribute to the one-loop induced neutrino masses in $\hat{M}_0$. The explicit expression is given by

$$\hat{M}_0 = \frac{1}{2(4\pi)^2} U_L'^T \left( \sum_{b=1}^{2n_H-1} \hat{Y}_{Db}^T U_R^* \hat{M}_\nu \frac{\ln(\hat{M}_\nu^2/M_b^2)}{\hat{M}_\nu^2/M_b^2 - 1} U_R^T \hat{Y}_{Db} \right) U_L'. \quad (7.14)$$

Neglecting terms suppressed by a factor $m_D/m_R$, (7.14) can be rewritten as

$$\hat{M}_0 \simeq \frac{1}{2(4\pi)^2} U_L'^T \left( \sum_{b=1}^{2n_H-1} \hat{Y}_{Db}^T U_R^* \frac{\ln(M_R M_b^*/M_b^2)}{M_R M_b^*/M_b^2 - 1} \hat{Y}_{Db} \right) U_L', \quad (7.15)$$

which is just the form of the result given in formula (3.10) of \[1\].

Hence, concerning the neutrino mass spectrum, we encounter essentially the same situation as in the case of the conventional multi-Higgs-doublet model with right-handed neutrinos. In general (i.e. without any relations among the Yukawa matrices $\hat{Y}_{Db}$),

$$\nu_0 = \max (0, n_L - n_H n_R) \quad (7.16)$$

of the $n_L - n_R$ neutrinos with vanishing tree-level masses remain massless at the one-loop level \[1\], whereas the other $n_L - n_R - \nu_0$ states receive masses from (7.14).

**8 Conclusions**

The purpose of this work was the construction of a conformal SU(2)$_L \times$ U(1)$_Y$ multi-Higgs-doublet model being able to reproduce the attractive features of the SM extension proposed in \[1\], where the neutrino mass spectrum results from a combination of the seesaw
mechanism at the tree-level with higher-order mass production. Starting from the particle content described in [1] (with \( n_L \) generations of SM fermions, \( n_R \) right-handed neutrino gauge-singlets and \( n_H \) Higgs doublets), already the simplest possible enlargement of the scalar sector by only one real singlet field turned out to be sufficient to attain the goal.

The conformal symmetry of the model allows only terms with (operator) dimension four in the Lagrangian. Therefore, the associated coupling parameters must be dimensionless and, in contrast to the model suggested in [1], explicit mass terms for the right-handed fermion singlets or the scalars as well as trilinear scalar-couplings are forbidden. However, already at the one-loop level, the scale invariance of the classical theory is broken by quantum corrections: the CW mechanism generates a mass scale by dimensional transmutation [12]. The order parameter \( V \), which determines the characteristic scale of the model, is the common origin of all dimensionful quantities in the EW sector, including all vacuum expectation values (in particular the EW scale \( v \approx 246 \text{ GeV} \)) and a fortiori of the masses of quarks and leptons. In this framework, the EW interaction is put on an equal footing with QCD, where the characteristic scale \( \Lambda_{QCD} \) of the strong interaction is also a consequence of dimensional transmutation.

The seesaw mechanism [3–7] is an essential ingredient of our model. In the absence of explicit mass terms in the Lagrangian, the Yukawa interaction of the scalar singlet with the right-handed neutrino singlets produces a Majorana mass term at a scale \( m_R \sim V \), once the scalar field receives its vacuum expectation value through the CW mechanism. At the same time, the Yukawa terms of the Higgs doublets produce Dirac mass terms at a scale \( m_D \), where the necessary mass hierarchy \( m_D \ll m_R \) (being in general equivalent to \( v \ll V \)) can be accommodated (though not “explained”) in the framework of this model.

One encounters an analogous situation in the description of the mass spectra of quarks and charged leptons: the observed mass values serve as input parameters in some Yukawa coupling matrices, but cannot be deduced from the theory.

The scalar spectrum of the model comprises \( n_H - 1 \) charged physical scalars and \( 2n_H \) neutral ones. Two of the neutral scalars stand out by their specific properties: the scalon \( S \) [15] and the Higgs particle \( H^0 \) observed at 125 GeV [22,23]. The scalon becomes massive at one-loop order, where its mass can be expressed in terms of the masses of all other fields. The scalon couplings to the other particles can be derived in a model-independent way [15]. They are flavour-diagonal but differ by a factor \( v/V \) from the corresponding interactions of the SM Higgs field. This observation together with the experimentally measured decay rates of the \( H^0 \) [24], being in good agreement with the predictions of the SM, inhibit the identification of the scalon with the \( H^0 \) particle if \( v/V \ll 1 \) is assumed. Thus, the existence of a particle with the properties of the scalon is a genuine prediction of our model.

The positivity condition for the squared scalon mass (ensuring also the stability of the theory) leads to a mass inequality, which can only be satisfied by sufficiently heavy scalar fields with masses of comparable size to those of the heavy neutrinos, suggesting a natural reason for the suppression of flavour-changing neutral interactions by neutral scalar exchange.

Compared to the heavy masses (typically of the order \( V \)), the scalon mass is suppressed by a one-loop factor and, in addition, by opposite signs of bosonic and fermionic contri-
butions in its mass formula. Still, a wide range of possible scalon mass values (with quite different phenomenological scenarios) remains open.

An important part of this work was dedicated to the exploration of the neutrino mass spectrum in the framework of the conformal theory. Extending and adapting the calculation performed in [5] for our purposes, we have computed the dominant one-loop corrections to the tree-level mass matrix of the light neutrinos. We have explicitly checked the finiteness and the gauge independence (in a general $R_\xi$ gauge) of these contributions. Applying our results to the interesting case $n_L > n_R$, we recover the same features of the neutrino mass spectrum as in [1], demonstrating that also the conformal version of the model is capable of combining the seesaw mechanism with higher-order mass production. Assuming a sufficiently high scale $V \gg v$, there are $n_R$ heavy neutral leptons, $n_R$ neutrinos being light because of the seesaw mechanism and $n_L - n_R$ massless neutrinos at the tree level. At the one-loop level, $\nu_0 = \max(0, n_L - n_H n_R)$ states stay massless, whereas the remaining $n_L - n_R - \nu_0$ ones become massive with naturally small and calculable masses generated by the exchange of neutral physical scalars, where the scalon contribution was shown to vanish to the order considered. As already emphasized in [9], the dominant one-loop corrections to the seesaw mechanism are quadratic in the Yukawa couplings (just like the masses of the light neutrinos at tree-level) and the one-loop corrections are smaller than the tree-level results only because of the appearance of the loop factor $1/(4\pi)^2$.

Finally, the simplest non-trivial version of the model with just two Higgs doublets and a single right-handed neutrino ($n_H = 2$, $n_R = 1$) may serve as an illustration of our findings. In this case, the spectrum of the theory contains four Majorana masses with a natural hierarchy. One heavy and one light mass are obtained by the seesaw mechanism, the next one appears at the one-loop level and the last one at two loops. The scalar sector consists of a single charged Higgs field $H^\pm$ and four neutral ones ($H^0, S, H^0_1, H^0_2$). The scalon mass formula takes the form

$$M_S^2 = \frac{1}{8\pi^2 V^2} \left( M^4_{H^0} + 3 M^4_Z + 6 M^4_W - 12 m_t^4 + M^4_{H^0_1} + 2 M^4_{H^\pm} - 2 M^4_R \right),$$

(8.1)

where the positivity condition $M_S^2 > 0$ is equivalent to the mass inequality

$$M^4_{H^0_1} + M^4_{H^0_2} + 2 M^4_{H^\pm} > -M^4_{H^0} - 3 M^4_Z - 6 M^4_W + 12 m_t^4 + 2 M^4_R.$$  

(8.2)

As a consequence, the large Majorana mass contribution on the right-hand side of (8.2) requires sufficiently heavy states in the scalar sector.

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3A detailed phenomenological analysis of this minimal version of the model is in preparation.
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