Canonical BRST Quantisation of Worldsheet Gravities

R. Mohayee\textsuperscript{1}, C.N. Pope\textsuperscript{2\ast}, K.S. Stelle\textsuperscript{1} and K.-W. Xu\textsuperscript{2}

\textsuperscript{1} The Blackett Laboratory, Imperial College, London SW7 2BZ, England

\textsuperscript{2} Center for Theoretical Physics, Department of Physics Texas A\&M University, College Station, TX 77843–4242, U.S.A.

\begin{abstract}
We reformulate the BRST quantisation of chiral Virasoro and $W_3$ worldsheet gravities. Our approach follows directly the classic BRST formulation of Yang-Mills theory in employing a derivative gauge condition instead of the conventional conformal gauge condition, supplemented by an introduction of momenta in order to put the ghost action back into first-order form. The consequence of these simple changes is a considerable simplification of the BRST formulation, the evaluation of anomalies and the expression of Wess-Zumino consistency conditions. In particular, the transformation rules of all fields now constitute a canonical transformation generated by the BRST operator $Q$, and we obtain in this reformulation a new result that the anomaly in the BRST Ward identity is obtained by application of the anomalous operator $Q^2$, calculated using operator products, to the gauge fermion.
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1. Introduction

The BRST formalism has proven to be the most powerful approach to the quantisation of string theories. Indeed, the full spectrum of low-dimensional string and $W$-string theories can only be properly derived in the BRST formalism [1,2]. The appropriateness of the BRST formalism is owed to the control it gives in the handling of anomalies in world-sheet chiral algebras. In the case of the non-critical bosonic string, the worldsheet anomaly gives rise to a propagating Liouville mode whose presence renders the worldsheet “gravity” non-trivial. Analogous anomalies in the $W_3$ string give rise to a worldsheet $W_3$ gravity described by an $A_2$ Toda theory [3].

There exist two basic approaches to the treatment of such anomaly-induced dynamics. The standard approach of Liouville and Toda gravities is to anticipate the occurrence of the anomalies already at the classical level by introducing classically-decoupling compensating fields. These fields maintain the full worldsheet symmetries at the quantum level by construction, but the anomalies arise in this approach through an anomalous quantum-level coupling of the compensators to the other fields of the theory. The second approach is to extract the anomalous quantum dynamics directly from the anomalous Ward identities of the worldsheet symmetries. In this latter approach, non-trivial correlation functions arise at the quantum level, revealing in some cases hidden quantum symmetries such as the $SL(2,\mathbb{R})$ symmetry found by Polyakov for the bosonic string [4], or the $SL(\infty,\mathbb{R})$ symmetry found in worldsheet $W_\infty$ gravity [5,6] (which becomes a $GL(\infty,\mathbb{R})$ symmetry for $W_{1+\infty}$ gravity [6].) Because it reveals hidden symmetries, the approach of extracting dynamics from anomalous Ward identities is clearly of great importance for the non-critical theories. The symmetries so found are reminiscent of the underlying $A_N$ symmetries for the Liouville or Toda theories in the first approach, but the precise relationship between these symmetries still needs clarification.

Analysis of the anomalous Ward identities for nonlinear chiral worldsheet algebras such as $W_3$ is made more difficult by the complexity and off-diagonal nature of the anomalies. Attempts in [7] to extend the approach of [4–6] to the $W_N$ gravity case ran into the difficulty that a consistent set of conditions to impose on the background gauge fields to eliminate the anomalies could not be derived owing to their off-diagonal structure. These difficulties are presumably related to our imperfect understanding of $W_3$ geometry.

In this paper, we present a reformulation of the BRST quantisation procedure for worldsheet gravities and the derivation of anomalous Ward identities. We hope that this will prove useful for understanding the dynamics of non-critical worldsheet gravities. In our reformulation, we shall first have to choose a fully acceptable gauge condition. It is well-known that the standard conformal gauge condition employed in string theory [8] is not really an acceptable choice, because in making it one looses the Virasoro constraints as field equations. Acceptable gauges may be defined as gauge conditions that can be imposed either prior to
or after varying the action in order to obtain the classical equations of motion. The easiest way to make an acceptable gauge choice is to choose a derivative gauge.* The point is most easily expressed by comparison to Maxwell electrodynamics. In Maxwell theory, which has a local $U(1)$ gauge symmetry, the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \left[ 2F_{0i} F^{0i} + F_{ij} F^{ij} \right],$$

(1.1)

where the signature is $(-1,1,1,1)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu$ is the gauge field, and $\mu = 0, i$. The canonical momenta are defined as

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu},$$

(1.2)

namely $\pi_0 = 0$, $\pi_i = F_{0i} = -E_i$, where $E_i$ is the electric field. The equal-time Poisson Bracket (PB) is

$$\{ \pi_i(\vec{x}, t), A_j(\vec{y}, t) \}_{PB} = \delta_{ij} \delta(\vec{x} - \vec{y}).$$

(1.3)

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \pi_i \pi^i - A_0 \partial_i \pi^i,$$

(1.4)

and the canonical action for evolution from $t_0$ to $t_f$ is

$$I = \int_{t_0}^{t_f} dt \int d^3 x \left( \pi_i \partial_0 A^i - \mathcal{H} \right).$$

(1.5)

The equation of motion obtained by varying the Lagrange multiplier $A_0$ in the action (1.5) is

$$\partial_i \pi^i(\vec{x}, t) = 0.$$

(1.6)

Had we insisted on setting $A_0 = 0$ prior to varying the action we would have lost the Gauss’ law constraint $\nabla \cdot E = 0$ as an equation of motion. Consequently, we need to choose a different gauge such as the Lorentz gauge $\partial^\mu A_\mu = 0$ to fix the $U(1)$ gauge symmetry.

Another way of saying this [10] is to suppose that $A_0 = \epsilon(\vec{x}, t) \neq 0$, and to try to make a gauge transformation so as to move into $A_0 = 0$ gauge. As the gauge transformation is

$$\delta A_0 = \dot{\lambda}(\vec{x}, t),$$

(1.7)

one would have to solve the first order differential equation

$$\dot{\lambda}(\vec{x}, t) = \epsilon(\vec{x}, t).$$

(1.8)

* Earlier discussions of derivative gauges in string theory, such as the harmonic gauge, may be found in Refs [9].
Now, recall that one obtains the equations of motion by varying the fields in the action (1.5) subject to the endpoint conditions \( \delta A_\mu(t_0) = \delta A_\mu(t_f) = 0 \). This determines the Euler-Lagrange evolution of the fields between the initial and final field configurations \( A_\mu(\vec{x}, t_0) \) and \( A_\mu(\vec{x}, t_f) \). The gauge symmetries for this variational problem are defined to be those transformations that leave the action (1.5) invariant, with the same fixed initial and final times as chosen in varying the fields. Making a gauge transformation \( \delta A_\mu = \partial_\mu \lambda, \delta \pi_i = 0 \) on the fields appearing in (1.5), we find

\[
\delta I = \int d^3 x \lambda(\vec{x}, t) \partial_i \pi^i|_{t_0}^{t_f}.
\] (1.9)

Thus, requiring invariance of (1.5) for general field configurations at \( t_0 \) and \( t_f \) requires

\[
\lambda(\vec{x}, t_0) = \lambda(\vec{x}, t_f) = 0.
\] (1.10)

For the first-order differential equation (1.8), imposing the two boundary conditions on \( \lambda \) overdetermines the problem, yielding no solution. Thus, the \( A_0 = 0 \) condition is not one that can actually be achieved for the canonical action (1.5) by a gauge transformation starting from a general field configuration.

By contrast, in the covariant gauge \( \partial_\mu A^\mu = 0 \), the differential equation for the transformation parameter becomes of second order

\[
\ddot{\lambda}(\vec{x}, t) = \epsilon(\vec{x}, t),
\] (1.11)

and allows the imposition of two boundary conditions, so we can actually find a solution to move into such a gauge.

In this paper, we shall concentrate on chiral worldsheet gravities and shall study the BRST formulation and gauge fixing of a single copy of a chiral worldsheet Virasoro or \( W_3 \) algebra. One may either view these chiral worldsheet gravities as theories defined in their own right, or may take the point of view that they arise from some theory, such as a bosonic string or \( W_3 \)-string theory, that has undergone a preliminary stage of gauge fixing that includes the condition

\[
\gamma_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & h \end{pmatrix},
\] (1.12)

in complex light-cone variables \( z, \bar{z} \). This gauge condition leaves unfixed a residual gauged chiral algebra, with a remaining gauge field \( h \) (together with a spin-3 field \( B \) in the \( W_3 \) case). Whether or not the gauge condition (1.12) may legitimately be imposed in the sense of Ref. [10] in string theory, or in view of hermiticity requirements, is itself an interesting question. However those questions will lie outside the scope of the present paper, where we shall take the chiral worldsheet gravities as our starting points. We shall be more careful in our discussion of the the final stage of gauge fixing, where we shall replace the conventional conformal gauge conditions \( h = h_{\text{back}}, B = B_{\text{back}} \) by derivative gauges such as \( \partial h = \partial B = 0 \).
We shall derive the BRST charge for our construction from the BRST transformations using Noether’s theorem. It is well-known that the transformation rule for the spin-2 gauge field $h$ cannot ordinarily be obtained directly from the standard BRST charge in the conventional formulation, since the standard BRST charge contains nothing with non-vanishing commutator or anticommutator brackets with $h$. This asymmetrical treatment of the gauge fields is compounded in the $W_3$ case by the fact that the BRST transformations of $h$ and the matter fields $\varphi^i$ do not form a closed nilpotent algebra unless one uses the classical $h$ and $\varphi^i$ equations of motion [11]. Quantisation in such a situation may be handled within the context of Batalin-Vilkovisky (BV) quantisation [12], but at the price, in the $W_3$ case [13], of a significant increase in complexity with respect to the quantisation procedure that we shall present.

In the reformulated BRST approach of this paper, we shall impose derivative gauge conditions and shall replace the resulting second-order ghost-sector actions with first-order actions after an introduction of momenta as auxiliary variables. The resulting gauge-fixed Virasoro and $W_3$ theories will be shown to be classically invariant under a set of BRST transformations of all fields that can now be obtained as a canonical transformation because the associated Noether charge $Q$ now properly generates the full set of BRST transformations of all fields, including the gauge fields. In deriving the BRST transformations generated by the chiral charge $Q$, we shall be treating the complex Euclidean worldsheet coordinates $z$ and $\bar{z}$ as independent and we shall treat the $\bar{\partial}$ derivative as the “evolution” derivative in the definition of momenta. As usual, however, the non-invariance of the path-integral measure for the partition function under Weyl transformations generally modifies this story at the quantum level by the presence of BRST anomalies.

A very natural but apparently new result that comes out of our reformulation of the BRST quantisation procedure is the precise relation between two notions of anomalies that one may encounter in the literature on worldsheet gravities. In conformal field theory, the notion of an anomaly is concerned with a violation of the classical BRST algebra, i.e. with a loss of nilpotence for $Q$ at the quantum level, and this is evaluated by taking a fully-contracted operator product $Q^2$, yielding a local but non-vanishing anomalous result. In ordinary field theory, the notion of an anomaly is concerned with the violation of the BRST Ward identity for the effective action $\Gamma$, i.e. the “master equation” of the Batalin-Vilkovisky (BV) formalism [12], and is evaluated by considering the 1PI diagrams, leading to an anomalous Ward identity. We shall see that the local functional expressing the anomaly in the BRST Ward identity is given by the operator product of the anomalous local operator $Q^2$ with the “gauge fermion” $\Psi$ of our reformulated gauge-fixed theory.

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† Related work on the canonical approach to the BV quantisation of string theory in derivative gauges may be found in Ref. [14], where the gauge independence of the BRST charge $Q$ is also obtained.

‡ This independence is of course naturally obtained in a Minkowski-space formulation, where $\sigma \pm \tau$ are truly independent. We shall be using the Euclidean-space formulation to facilitate later comparison with quantum operator-product calculations.
We shall calculate the chiral Virasoro anomaly at one loop and the local anomalies of chiral $W_3$ gravity at one and two loops. The Virasoro anomaly that we shall find is the same as the one derived by Polyakov by a straightforward variation of the effective action. The situation for $W_3$ is more subtle, and we shall obtain an anomaly that differs from the one given in [7] both in the values of coefficients and also in the occurrence of new terms. We shall check the correctness of our expressions for the Virasoro and $W_3$ anomalies by verifying that the Wess-Zumino consistency conditions are satisfied in each case. Whether our results for the $W_3$ anomalies are actually in conflict with those of Ref. [7] or not remains to be determined, however, since the analysis of Ref. [7] followed a different procedure of treating the gauge fields purely as external fields without gauge fixing or introduction of ghosts, and sought in this way to derive the dynamics of “induced gravity.”

The paper is organised as follows. In the next section, we shall quantise chiral Virasoro gravity with a derivative gauge condition. We shall derive the BRST charge, calculate the Virasoro anomaly and check that the Wess-Zumino consistency condition is satisfied. In section three, we shall review the conventional BV quantisation of $W_3$ gravity starting from the results of [15]. This has also recently been discussed in greater detail in [13]; the purpose of our review will be to set the stage for our reformulated treatment of the $W_3$ case, which will be given in section four. In the concluding section, we shall present the relation between anomalies in the quantum $Q^2$ algebra and in the BRST Ward identities that can be abstracted from our reformulated quantisation procedure.

2. BRST Quantisation of Virasoro gravity

In this section, we will develop our point of view on BRST quantisation by focusing on the Virasoro case. For comparison, we start by reviewing the conventional conformal-gauge BRST quantisation of worldsheet Virasoro gravity [8].

Conventional BRST quantisation

The chiral Virasoro gravity action in the preliminary gauge (1.12) is

$$I_0 = \frac{1}{\pi} \int d^2z \left( -\frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i + \frac{1}{2} h \partial \varphi^i \partial \varphi^i \right),$$

(2.1)

where the $\varphi^i$ ($i = 0, 1, \ldots, D - 1$, and $D$ is the space time dimension) are a set of matter fields and $h$ is the remaining unfixed component of the two-dimensional metric. This action is invariant under the following transformations:

$$\delta \varphi = \varepsilon \partial \varphi^i,$$

(2.2a)

$$\delta h = \bar{\partial} \varepsilon + \varepsilon \partial h - \partial \varepsilon h.$$  

(2.2b)
The standard conformal-gauge way to fix this gauge freedom is to set \( h = h_{\text{back}} \). One then obtains the gauge-fixed action

\[
I = \frac{1}{\pi} \int d^2z \left( -\frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i - b \bar{\partial} c \right. \\
\left. + \pi_h (h - h_{\text{back}}) - h (T_{\text{mat}} + T_{\text{gh}}) \right).
\] (2.3)

This action has the following BRST symmetry

\[
\delta \varphi^i = c \partial \varphi^i, \quad (2.4a)
\]
\[
\delta h = \bar{\partial} c + c \partial h - \partial c h, \quad (2.4b)
\]
\[
\delta c = c \partial c, \quad (2.4c)
\]
\[
\delta b = \pi_h, \quad (2.4d)
\]
\[
\delta \pi_h = 0. \quad (2.4e)
\]

where \( \pi_h \) is an auxiliary field and \( c, b \) denote ghost and anti-ghost fields, satisfying standard OPE relations. \( T_{\text{mat}} \) and \( T_{\text{gh}} \) are the energy-momentum tensors for the matter fields and ghost fields respectively, and are given explicitly by

\[
T_{\text{mat}} = -\frac{1}{2} \partial \varphi^i \partial \varphi^i, \quad (2.5a)
\]
\[
T_{\text{gh}} = -2b \partial c - \partial b c. \quad (2.5b)
\]

The operator products of both \( T_{\text{mat}} \) and \( T_{\text{gh}} \) close to form the OPE Virasoro algebra

\[
h^{-1} T(z) T(w) \sim \frac{\partial T}{z - w} + \frac{2T}{(z - w)^2} + h \frac{1}{2} \frac{C}{(z - w)^4},
\] (2.6)

where \( C \) is the central charge.

From the action (2.3) and the BRST transformations (2.4), one may construct the conserved charge related to this symmetry by Noether’s theorem. In this way, one obtains the standard BRST charge for chiral Virasoro gravity,

\[
Q = \int dz \ c \left( T_{\text{mat}} + \frac{1}{2} T_{\text{gh}} \right).
\] (2.7)

At this point, we encounter the difficulties with this standard procedure. In a properly-posed canonical formalism, the BRST charge should act as the generator of all of the transformations from which it was originally derived, i.e. one would like to have

\[
\delta \phi^i = \{ Q, \phi^i \},
\] (2.8)

where the \( \{\cdot, \cdot\} \) bracket is realised either classically as a Poisson bracket/antibracket or quantum-mechanically as a commutator/anticommutator. Now, this works as expected for
the fields $\varphi^i$ and $c$. If one substitutes the equations of motion, it also works for $b$ and $\pi_h$. But it can never work for the $h$ field, since nothing in (2.7) has any nontrivial commutation properties with $h$. The BRST transformation of the $h$ field can only be derived from the requirement of invariance of the action (2.3) under the BRST transformations. This is an analogue of the situation in $A_0 = 0$ gauge in Maxwell theory. Similarly to this Maxwell gauge, where Gauss’ law is lost as an equation of motion, in the Virasoro gravity case one has to remember separately to impose the Virasoro constraints by hand, since the $h$ field equation following from (2.3) implies only that $T_{\text{mat}} + T_{gh} = \pi_h$. We shall see that all of these problems are resolved naturally in our revised BRST quantisation procedure.

**Derivative-gauge BRST quantisation**

We return to the action (2.1) in the preliminary gauge choice (1.12), but now complete the gauge fixing by choosing the derivative gauge condition $\bar{\partial}h = 0$. The gauge-fixed action then becomes

$$I = \frac{1}{\pi} \int d^2z \mathcal{L} = \frac{1}{\pi} \int d^2z \left( -\frac{1}{2} \partial \varphi^i \partial \varphi^i - h T_{\text{mat}} + \pi_h \bar{\partial}h - b \bar{\partial}(\bar{\partial}c + c \partial h - \partial c h) \right).$$

(2.9)

As a result of the derivative gauge condition, the ghost action is now of second order in $\bar{\partial}$ derivatives. For comparison with conformal-field-theory operator products and also to allow us to use the canonical formalism, we next introduce auxiliary fields in order to put the ghost sector into first-order form. From the action (2.9), one sees that the fields $c$ and $b$ are no longer conjugates, so we need to define conjugate momenta

$$\pi_c = \frac{\partial \mathcal{L}}{\partial \bar{\partial}c} = -\bar{\partial}b,$$

$$\pi_b = \frac{\partial \mathcal{L}}{\partial \bar{\partial}b} = \bar{\partial}c + c \partial h - \partial c h.$$  

(2.10)

As noted in the introduction, throughout this paper we shall be treating the $\bar{\partial}$ derivative as the “evolution” derivative in the canonical formulation of the chiral theories we consider. We can rewrite the second-order action (2.9) in first-order form as

$$I = \frac{1}{\pi} \int d^2z \left( -\frac{1}{2} \bar{\partial} \varphi^i \bar{\partial} \varphi^i + \pi_h \bar{\partial}h - \pi_b \bar{\partial}b - \pi_c \bar{\partial}c - \pi_c \pi_c - h (T_{\text{mat}} + T_{gh}) \right).$$

(2.11)

From the path-integral generating functional derived from this action, we get the following OPE relations

$$\partial \varphi^i(z) \partial \varphi^i(w) \sim -\frac{\bar{h}}{(z - w)^2}, \quad \pi_h(z) h(w) \sim \frac{\bar{h}}{z - w}, \quad c(z) \pi_c(w) \sim \frac{\bar{h}}{z - w}, \quad b(z) \pi_b(w) \sim \frac{\bar{h}}{z - w}, \quad c(z) \bar{\partial}b(w) = -\bar{\partial}c(z) b(w) \sim -\frac{\bar{h}}{z - w}.$$

(2.12)
where we have introduced $\hbar$ for later convenience in counting loop orders.

The action (2.11) is invariant under the following BRST transformations:

\[
\begin{align*}
\delta \varphi^i &= c \partial \varphi^i, \\
\delta h &= \pi_b, \\
\delta c &= c \partial c, \\
\delta \pi_c &= T_{\text{mat}} + T_{\text{gh}}, \\
\delta b &= \pi_h, \\
\delta \pi_b &= 0, \\
\delta \pi_h &= 0,
\end{align*}
\]

(2.13a) - (2.13g)

where $T_{\text{gh}}$ no longer takes the form (2.5b), but is now instead

\[
T_{\text{gh}} = -2\pi_c \partial c - \partial \pi_c c.
\]

(2.14)

These transformations are all nilpotent at the classical level, i.e. at the level of Poisson antibrackets, or by taking OPEs with single contractions between fields.

Checking the invariance of the gauge-fixed action under the transformations (2.13) is made much easier by rewriting (2.11) as

\[
I = \frac{1}{\pi} \int d^2z \left( -\frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i + \pi_h \bar{\partial} h - \pi_b \bar{\partial} b - \pi_c \bar{\partial} c - \delta(h \pi_c) \right).
\]

(2.15)

From the action (2.15), we can see that the Hamiltonian density $\mathcal{H} = \delta(h \pi_c)$ is BRST trivial, and hence is invariant as a consequence of the nilpotence of the transformations (2.13). The kinetic terms constituting the remainder of (2.15) are also invariant because terms of the form $\pi_\phi \bar{\partial} \phi$ are invariant up to a total $\bar{\partial}$ derivative under arbitrary canonical transformations.*

That the transformations (2.13) are now canonical is one of the main benefits of our reformulated BRST quantisation procedure. The generator of these transformations is the BRST charge $Q$. This may be obtained from (2.13, 2.15) using Noether’s theorem, by supplying an anticommuting transformation parameter $\lambda$ which is allowed to depend on $\bar{z}$ and then collecting the factors multiplying $\bar{\partial} \lambda$ in the BRST variation of the action, giving

\[
Q = \int d\bar{z} \left( c(T_{\text{mat}} + \frac{1}{2}T_{\text{gh}}) + \pi_h \pi_b \right).
\]

(2.16)

* Note that in the worldsheet coordinates $z, \bar{z}$ that we are using, the scalar kinetic term $\bar{\partial} \varphi^i \partial \varphi^i$ is already in first-order form with respect to the “evolution” $\bar{\partial}$ derivatives, so we have not bothered to introduce conjugate momenta for the $\varphi^i$. Strictly speaking, in the canonical formalism one should introduce momenta $\pi^i$, and then should deal with the resulting constraint $\pi^i = -\frac{1}{2} \partial \varphi^i$ using the Dirac-bracket formalism. In practice, we shall find it simpler to do our explicit calculations using the quantum operator products (2.12), restricted to single contractions when discussing the classical or tree level. Further details on the canonical formalism for chiral theories may be found in Ref. [14].

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One may verify, using the OPE relations (2.12), that the desired rule (2.8) is now correctly obtained for the transformations (2.13a–g) of all fields, including the gauge field $h$.

Having a well-defined BRST formalism in hand, we now proceed to calculate the Virasoro anomaly from the BRST Ward identity.

**Virasoro Anomaly in Non-critical Dimensions**

To set the stage for our discussion of the worldsheet gravity anomalies, let us first review the BRST treatment of anomalies, following Ref. [16].

Suppose $L_{gf}$ is a BRST-invariant gauge-fixed Lagrangian density. In order to calculate correlation functions and derive the BRST form of the Ward identity, one has to introduce three kind of sources: $J_{\phi^i}$ and $K_{\phi^i}$ for the fields and their variations and $L$ for the anomaly, the extended Lagrangian density $L_{ext,\text{anom}}$ then being given by

$$L_{ext} = L_{gf} + J_{\phi^i} \phi^i + K_{\phi^i} \delta \phi^i,$$

$$L_{ext,\text{anom}} = L_{ext} + L \triangle,$$

where $\phi^i$ generically denotes all fields and $\triangle$ denotes the anomaly. The partition function with dependence upon these sources is then defined as

$$Z(J_{\phi^i}, K_{\phi^i}, L) = \int D\phi^i e^{-\int d^2 z L_{ext,\text{anom}}}.$$  

The generating function $W$ and the effective action functional $\Gamma$ are defined respectively as

$$W(J_{\phi^i}, K_{\phi^i}, L) = \ln Z(J_{\phi^i}, K_{\phi^i}, L),$$

$$\Gamma(\phi^i, K_{\phi^i}, L) = W(J_{\phi^i}, K_{\phi^i}, L) - \int d^2 z J_{\phi^i} \phi^i.$$  

The path-integral measure $D\phi$ is not generally invariant under Weyl transformations. Consequently, it is also not generally invariant under our BRST transformations, since the residual Virasoro symmetry (2.2) that is left unfixed by our preliminary gauge choice (1.12) is a composite of a Weyl transformation and compensating diffeomorphisms. In consequence, although the action $\int L_{ext}$ is BRST invariant, the partition function is generally not and so one has a BRST anomaly:

$$Z - Z' = \int D\phi^i J_{\phi^i} \delta \phi^i e^{-\int d^2 z L_{ext,\text{anom}}}$$

$$= Z J_{\phi^i} \frac{\delta W}{\delta K_{\phi^i}} = -Z \frac{\delta \Gamma}{\delta \phi^i} \frac{\delta \Gamma}{\delta K_{\phi^i}}$$

$$= -Z \triangle \cdot \Gamma = -Z \frac{\partial \Gamma}{\partial L} = -\frac{\partial Z}{\partial L},$$

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where we are using De Witt notation, so that repeated indices denote both Einstein summation and integration over arguments. From the identities (2.20), we can extract two very useful equations. The first of these is

\[
\frac{\delta \Gamma}{\delta \phi^i} \frac{\delta \Gamma}{\delta K_{\phi^i}} = \nabla \cdot \Gamma \equiv \frac{\partial \Gamma}{\partial \Delta},
\]

(2.21)

where \( \nabla \cdot \Gamma \) denotes the set of all 1PI diagrams with an insertion of the composite anomaly operator \( \Delta \). The second equation, obtained for \( L \to 0 \), is

\[
\int D\phi^i J_{\phi^i} e^{-\int d^2z L_{\text{ext}}} = -\int D\phi^i \Delta e^{-\int d^2z L_{\text{ext}}}.
\]

(2.22)

Equations (2.21) and (2.22) are different expressions of the anomalous Ward identity.

Returning to our specific case of chiral Virasoro gravity, the extended Lagrangian density \( L_{\text{ext}} \) becomes

\[
L_{\text{ext}} = -\frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i + \pi_h \bar{\partial} h - \pi_b \bar{\partial} b - \pi_c \bar{\partial} c - \pi_e \bar{\partial} e - h \left( T_{\text{mat}} + T_{gh} \right) + J_{\varphi^i} \varphi^i + J_h h + J_c c + J_b b
\]

\[
+ J_{\pi c} \pi c + J_{\pi b} \pi b + K_{\varphi^i} c \partial \varphi^i + K_h \pi h + K_b \pi b + K_c c \partial c + K_{\pi c} \left( T_{\text{mat}} + T_{gh} \right).
\]

(2.23)

Expanding loopwise in powers of \( \bar{h} \), we have

\[
\Gamma = \Gamma_0 + \bar{h} \Gamma_1 + \bar{h}^2 \Gamma_2 + \ldots, \quad \Delta = \bar{h} \Delta_1 + \bar{h}^2 \Delta_2 + \ldots
\]

(2.24)

where \( \Gamma_0 \) is the extended action \( I_{\text{ext}} \), \( \Gamma_1 \) is the one loop correction to the effective action, \( \Delta_1 \) is the one loop contribution to the anomaly, etc.

We shall be interested here firstly in the one-loop contributions to the anomalous Ward identity (2.21), which at order \( \bar{h} \) becomes

\[
\frac{\delta \Gamma_0}{\delta \phi^i} \frac{\delta \Gamma_1}{\delta K_{\phi^i}} + \frac{\delta \Gamma_0}{\delta \phi^i} \frac{\delta \Gamma_1}{\delta K_{\phi^i}} = \Delta_1.
\]

(2.25)

Explicit evaluation shows that the only non-vanishing contributions to the anomaly come from

\[
\frac{\delta \Gamma_0}{\delta \pi_c} \frac{\delta \Gamma_1}{\delta K_{\pi_c}} = \frac{1}{\pi^2 \bar{h}^2} \int d^2z d^2w \bar{\partial} c(z) \left( \frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i + 2 \pi_c \partial c + \partial \pi_c e \right)(z)
\]

\[
\left( (h - K_{\pi c})(\frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i + 2 \pi_c \partial c + \partial \pi_c e) \right)(w) + \ldots
\]

(2.26)
where the angle brackets $\langle \rangle$ denote the OPE of the bracketed fields and the omitted terms do not give local contributions. After some algebra, one gets the anomaly

$$
\triangle_1 = \delta \Gamma_0 \frac{\delta \Gamma_1}{\delta \pi_c \delta K_{\pi_c}} + \ldots
$$

$$
= -\frac{1}{2\pi^2} (26 - D) \int d^2 z d^2 w \frac{1}{(z - w)^4} \bar{\partial} c(z) \left( h(w) - K_{\pi_c}(w) \right)
$$

$$
= \frac{1}{12\pi} (26 - D) \int d^2 z d^2 w \frac{\partial^3}{\partial_z} \delta_{(z-w)}(z) \left( h(w) - K_{\pi_c}(w) \right)
$$

$$
= -\frac{1}{12\pi} (26 - D) \int d^2 z c \left( \partial^3 h - \partial^3 K_{\pi_c} \right).
$$

In order to check whether the form of the anomaly $\triangle$ given in (2.27) is correct, we need to verify that it satisfies the Wess-Zumino consistency condition [16]. This condition in general is simply

$$
\left( \Gamma, (\Gamma, \Gamma) \right) = 0,
$$

(2.28)

and is a consequence of the Jacobi identity for the antibracket $(\cdot, \cdot)$, which is defined as

$$
\left( A, B \right) \equiv \frac{\delta A}{\delta \phi^i} \frac{\delta B}{\delta K_{\phi^i}} + \frac{\delta A}{\delta K_{\phi^i}} \frac{\delta B}{\delta \phi^i},
$$

(2.29)

for arbitrary functionals $A$ and $B$. Note that the extended classical action is BRST invariant, as a consequence of the invariance of the gauge-fixed action (2.15) and the classical nilpotence of the transformations (2.13). In antibracket notation, this becomes $(\Gamma_0, \Gamma_0) = 0$. Then, expanding $\Gamma$ in a series in $\hbar$, the one-loop Wess-Zumino consistency condition following from (2.28) becomes

$$
\left( \Gamma_0, \triangle_1 \right) \equiv \frac{\delta \Gamma_0}{\delta \phi^i} \frac{\delta \triangle_1}{\delta K_{\phi^i}} + \frac{\delta \Gamma_0}{\delta K_{\phi^i}} \frac{\delta \triangle_1}{\delta \phi^i} = 0.
$$

(2.30)

Testing our anomaly (2.27) in this formula, we obtain

$$
\left( \Gamma_0, \triangle_1 \right) = \frac{\delta \Gamma_0}{\delta \pi_c} \frac{\delta \triangle_1}{\delta K_{\pi_c}} + \frac{\delta \Gamma_0}{\delta K_{\pi_c}} \frac{\delta \triangle_1}{\delta c} + \frac{\delta \Gamma_0}{\delta K_{\hbar}} \frac{\delta \triangle_1}{\delta \hbar}
$$

$$
= -\frac{1}{12\pi} (26 - D) \int d^2 z \left[ (\pi_b - \bar{\partial} c - c \partial h + \partial c h - c \partial K_{\pi_c} - \partial c K_{\pi_c}) \partial^3 c + c \partial c (\partial^3 h - \partial^3 K_{\pi_c}) - \pi_b \partial^3 c \right]
$$

(2.31)

$$
= 0,
$$

so we verify that the consistency condition indeed is satisfied.
3. Conventional BRST quantisation of \( W_3 \) gravity

The \( W_3 \) algebra, originally found by Zamolodchikov [17], in the conventions of Ref. [15] becomes

\[
\begin{align*}
\hbar^{-1} T_{\text{mat}}(z) T_{\text{mat}}(w) & \sim \frac{\partial T_{\text{mat}}}{z-w} + \frac{2T_{\text{mat}}}{(z-w)^2} + \hbar \frac{\frac{1}{3} C_{\text{mat}}}{(z-w)^4}, \\
\hbar^{-1} T_{\text{mat}}(z) W_{\text{mat}}(w) & \sim \frac{\partial W_{\text{mat}}}{z-w} + \frac{3W_{\text{mat}}}{(z-w)^2}, \\
\hbar^{-1} W_{\text{mat}}(z) W_{\text{mat}}(w) & \sim \frac{1}{z-w} \left( \frac{1}{15} \hbar \partial^3 T_{\text{mat}} + a \partial \Lambda \right) \\
& + \frac{1}{(z-w)^2} \left( \frac{3}{10} \hbar \partial^2 T_{\text{mat}} + 2a \Lambda \right) \\
& + \hbar \frac{\partial T_{\text{mat}}}{(z-w)^3} + \hbar \frac{2T_{\text{mat}}}{(z-w)^4} + \hbar^2 \frac{\frac{1}{3} C_{\text{mat}}}{(z-w)^6},
\end{align*}
\]

(3.1)

where \( a = \frac{16}{22+5C_{\text{mat}}} \) and \( \Lambda \) is a composite current given by the normal-ordered product\(^\dagger\)

\[
\Lambda = (T_{\text{mat}} T_{\text{mat}}) - \frac{3}{10} \partial^2 T_{\text{mat}}.
\]

(3.2)

The matter currents \( T_{\text{mat}} \) and \( W_{\text{mat}} \) represent respectively the spin 2 and spin 3 generators of the \( W_3 \) algebra; their explicit realisations in terms of \( n \) scalar fields \( \varphi^i \) are

\[
\begin{align*}
T_{\text{mat}} &= -\frac{1}{2} \partial \varphi^i \partial \varphi^i - \sqrt{\hbar} \alpha_i \partial^2 \varphi^i, \\
W_{\text{mat}} &= -\frac{1}{3} d_{ijk} \partial \varphi^i \partial \varphi^j \partial \varphi^k - \sqrt{\hbar} e_{ij} \partial \varphi^i \partial^2 \varphi^j - \hbar f_i \partial^3 \varphi^i,
\end{align*}
\]

(3.3)

where the \( \varphi^i \) satisfy the OPE

\[
\partial \varphi^i(z) \partial \varphi^j(w) \sim -\frac{\hbar \delta^{ij}}{(z-w)^2}.
\]

(3.4)

In order to obtain a realisation of the \( W_3 \) algebra\(^\ddagger\) (3.1), the constants \( \alpha_i, d_{ijk}, e_{ij} \) and \( f_i \) must satisfy the following relations found in Ref. [18]:

\[
\begin{align*}
d_{ijj} - 6e_{ij} \alpha_j + 6f_i &= 0, \\
e_{(ij)} - d_{ijk} \alpha_k &= 0,
\end{align*}
\]

(3.5)

\(^\dagger\) Normal ordering is denoted here by round brackets ( ).

\(^\ddagger\) Here, unlike in the Virasoro case, we must include background charges in order to have a multi-field realisation [18]; without background charges, one has only the original (non-critical) two-field realisation of Ref. [17].
\[ 3f_i - \alpha_j e_{ji} = 0, \quad (3.5c) \]

\[ d_{ikl} d_{jkl} + 6d_{ijk} f_k - 3e_{ik} e_{jk} = \frac{1}{2} \delta_{ij}, \quad (3.5d) \]

\[ d_{(ij}^m d_{k)l} m = \frac{1}{2} a \delta_{ij} \delta_{kl}, \quad (3.5e) \]

\[ d_{ijk} (e_{ik} - e_{kl}) + 2e_{(i}^l d_{j)kl} = a \alpha_k \delta_{ij}. \quad (3.5f) \]

Two useful consequences of this set of equations which will be useful in later calculations are

\[ e_{ii} + 12\alpha_i f_i = 0, \quad (3.5g) \]

\[ C_{\text{mat}} = -2d_{ijk} d_{ijk} - 18e_{ij} e_{ij} - 12e_{ij} e_{ji} - 360f_i^2. \quad (3.5h) \]

The central charge \( C_{\text{mat}} \) is given for the realisation (3.3) by

\[ C_{\text{mat}} = n + 12\alpha_i \alpha_i. \quad (3.6) \]

The BRST charge for this system was given by J. Thierry-Mieg [19] as follows:

\[ Q = \int dz \left( c(T_{\text{mat}} + \frac{1}{2} T_{\text{gh}}) + \gamma(W_{\text{mat}} + \frac{1}{2} W_{\text{gh}}) \right), \quad (3.7) \]

in which the ghost currents \( T_{\text{gh}} \) and \( W_{\text{gh}} \) are given by

\[ T_{\text{gh}} = -2b \partial c - \partial b c - 3\beta \partial \gamma - 2\partial \beta \gamma, \quad (3.8a) \]

\[ W_{\text{gh}} = -\partial \beta c - 3\beta \partial c - a\partial(b \gamma T_{\text{mat}}) + b \partial \gamma T_{\text{mat}} \]
\[ + \frac{(1 - 17a)}{30} h(2\gamma \partial^3 b + 9\partial \gamma \partial^2 b + 15\partial^2 \gamma \partial b + 10\partial^3 \gamma b). \quad (3.8b) \]

The ghost-antighost pairs \((c, b)\) and \((\gamma, \beta)\) correspond respectively to the \( T \) and \( W \) generators. They satisfy the following OPEs

\[ c(z)b(w) \sim \frac{h}{z - w}; \quad \gamma(z)\beta(w) \sim \frac{h}{z - w}. \quad (3.9) \]

Note that here, unlike in the Virasoro case where the ghost current \( T_{\text{gh}} \) forms a separate realisation of the Virasoro algebra, the ghost currents (3.8) do not form a separate realisation of the \( W_3 \) algebra.

The BRST charge (3.7) corresponds to the conventional BRST action for \( W_3 \) gravity [15]

\[ I = \frac{1}{\pi} \int d^2 z \mathcal{L} \]
\[ = \frac{1}{\pi} \int d^2 z \left( -\frac{1}{2} \partial \phi^i \partial \phi^i - h T_{\text{mat}} - B W_{\text{mat}} + \delta [b (h - h_{\text{back}}) + \beta (B - B_{\text{back}})] \right), \quad (3.10a) \]
\[ = \frac{1}{\pi} \int d^2 z \left( -\frac{1}{2} \partial \phi^i \partial \phi^i - b \partial c - \beta \partial \gamma + \pi_h (h - h_{\text{back}}) + \pi_B (B - B_{\text{back}}) \right) \]
\[ - h (T_{\text{mat}} + T_{\text{gh}}) - B (W_{\text{mat}} + W_{\text{gh}}), \quad (3.10b) \]

\[ \]
where the conventional gauge conditions for the spin-2 and spin-3 gauge fields \( h \) and \( B \) are \( h = h_{\text{back}} \) and \( B = B_{\text{back}} \), imposed by the Lagrange multipliers \( \pi_h \) and \( \pi_B \). The transformation rules of the fields in (3.10) are given by

\[
\delta \varphi^i = c \partial \varphi^i + d_{ijk} \gamma \partial \varphi^j \partial \varphi^k + a b \gamma \partial \varphi^i + a \alpha_i \partial (b \gamma \partial \varphi^i) + h f_i \partial^2 \gamma, \tag{3.11a}
\]

\[
\delta h = \partial c + c \partial h - \frac{a}{2} (\gamma \partial B - \partial \gamma B) \partial \varphi^i \partial \varphi^i - a \sqrt{h} (\gamma \partial B - \partial \gamma B) \alpha_i \partial^2 \varphi^i
\]

\[
+ \frac{1-17a}{30} h (2 \gamma \partial^3 B - 3 \partial \gamma \partial^2 B + 3 \partial^2 \gamma \partial B - 2 \partial^3 \gamma B), \tag{3.11b}
\]

\[
\delta B = \partial \gamma + c \partial B - 2 \partial c B + 2 \gamma \partial h - \partial \gamma h, \tag{3.11c}
\]

\[
\delta c = c \partial c - \frac{a}{2} \gamma \partial \gamma \partial \varphi^i \partial \varphi^i - a \sqrt{h} \alpha_i \gamma \partial \gamma \partial^2 \varphi^i + \frac{1-17a}{30} h (2 \gamma \partial^3 \gamma - 3 \partial \gamma \partial^2 \gamma), \tag{3.11d}
\]

\[
\delta \gamma = c \partial \gamma - 2 \partial c \gamma, \tag{3.11e}
\]

\[
\delta b = \pi_h, \quad \delta \beta = \pi_B, \quad \delta \pi_h = 0, \quad \delta \pi_B = 0. \tag{3.11f, g, h, i}
\]

In saying that (3.10) corresponds to (3.7), one needs to be careful about the logical status of this correspondence. The currents \( T_{\text{mat}}, W_{\text{mat}}, T_{\text{gh}} \) and \( W_{\text{gh}} \) certainly appear to be the natural elements to be extracted from the structure of the full quantum BRST charge (3.7) for constructing the renormalized action (3.10), once one has realised that a consistent Lagrangian quantisation must be based upon the original quantum \( W_3 \) algebra (3.1) instead of some deformation of this algebra. Indeed, the work of Ref. [15] started from this natural guess. But the relation between (3.10) and (3.7) is not algorithmic. The results of Ref. [15] may be summarised as the demonstration, by explicit Feynman-diagram evaluation of the low-order anomalies, that for values of the background charges corresponding to a realisation of (3.1) with \( C_{\text{mat}} = 100 \) the guess (3.10) actually works, i.e. that all the matter-dependent and universal anomalies do cancel. The renormalized action (3.10) is itself invariant under the transformations (3.11) only up to order \( \hbar^{1/2} \). It fails to be invariant at order \( \hbar \) precisely as needed so as to cancel the local anomalous terms arising in the BRST Ward identity from variations of non-local contributions to the effective action \( \Gamma \). The invariance of (3.10) under (3.11) at orders \( \hbar^0 \) and \( \hbar^{1/2} \) allows one to investigate a partially-anomalous generalisation of the results of [15] with \( C_{\text{mat}} \neq 100 \). In this partially-anomalous case, the status of the central charge \( C_{\text{mat}} \) as an independent free parameter related to the order \( \hbar^{1/2} \) background charges \( \alpha_i \) via (3.5, 3.6) makes it possible to abandon the requirement of cancellation of the universal anomalies (depending only on \( h \) and \( B \) and the ghosts and antighosts) while still insisting nonetheless upon cancellation of all anomalies depending on the matter fields \( \varphi^i \). Implementing this noncritical scheme is possible but very cumbersome in the theory defined by the conventional BRST action (3.10) [13], but the implementation will become much more transparent in our reformulated BRST quantisation procedure.
It is clear upon inspection of the BRST charge (3.7) and the transformation rules (3.11) that the transformation rules for the gauge fields $h$ and $B$ cannot be obtained directly from the BRST charge (3.7). This is as in the Virasoro case, and one can only obtain the transformations of $h$ and $B$ by requiring the invariance of the theory. This is what was done in the critical $C_{\text{mat}} = 100$ theory [15], yielding the result that the gauge fields must transform in the coadjoint representation of the algebra. Clearly, however, the choice of transformations for $h$ and $B$ is trickier in the noncritical $C_{\text{mat}} \neq 100$ theory, since one obviously cannot require invariance of the theory in this case, and the renormalisation corrections to the $h$ and $B$ transformations must be determined by requiring the absence of matter-dependent anomalies in the anomalous Ward identity. This requirement presumably leads once again to the requirement that the gauge fields transform in the coadjoint representation as in (3.11), but we are not aware of a direct verification of this fact in the conventional formulation (3.10) of the noncritical theory. Once again, this issue will become greatly simplified in our reformulated quantisation procedure, where $h$ and $B$ will be treated in the same way as all other fields.

Before leaving the conventional BRST formulation of $W_3$ gravity, we recall [11] that the BRST transformations (3.11) have a structure that is not directly obtained by the standard simple prescription of replacing the parameters of the classical $W_3$ gauge transformations by ghosts. The classical chiral $W_3$ gravity action is [20]

$$I_{\text{class}} = \frac{1}{\pi} \int d^2 z \left( -\frac{1}{2} \partial \phi^i \partial \phi^i + \frac{1}{2} h \partial \phi^i \partial \phi^i + \frac{1}{3} B d_{ijk} \partial \phi^i \partial \phi^j \partial \phi^k \right),$$  \hspace{1cm} (3.12)

and it is invariant under the following infinitesimal transformations

$$\delta \phi^i = \varepsilon \partial \phi^i + d_{ijk} \eta \partial \phi^j \partial \phi^k; \hspace{1cm} (3.13a)$$
$$\delta h = \bar{\partial} \varepsilon + \varepsilon \partial h - \partial \varepsilon h - \frac{a}{2} (\eta \partial B - \partial \eta B ) \partial \phi^i \partial \phi^i; \hspace{1cm} (3.14b)$$
$$\delta B = \bar{\partial} \eta + \varepsilon \partial B - 2 \partial \varepsilon B + 2 \eta \partial h - \partial \eta h, \hspace{1cm} (3.14c)$$

where $\varepsilon$ and $\eta$ are infinitesimal parameters for the Virasoro and spin-3 transformations, respectively. One would normally expect the corresponding BRST transformations to be obtained simply by replacing $\varepsilon$ by the spin-2 ghost $c$ and $\eta$ by the spin-3 ghost $\gamma$. However, comparison with the transformation (3.10a) of $\phi^i$ shows that the actual transformation contains an unexpected term $b \gamma \partial \gamma \partial \phi^i$. As a consequence, the classical action (3.12) is not invariant under the BRST transformations (3.11).*

* It is, of course, still true that the $h$-independent terms in the complete action (3.10b) are invariant under the $h$-independent terms in the BRST transformation rules (3.11). The slightly unusual new feature in a case such as $W_3$ gravity, where the algebra is non-linear, is that the $h$-independent terms in the ghost kinetic terms and gauge-fixing terms in the total action play a rôle in ensuring invariance under BRST transformations at the classical level. One could consider a rather trivial linear classical algebra, in which the Poisson bracket of $W_{\text{mat}}$ with $W_{\text{mat}}$ were zero, rather than being proportional to $(T_{\text{mat}})^2$. This would correspond to setting the constant $a$ to zero in the classical action and BRST transformation rules, under which circumstances the unusual terms that do not come from a simple replacement of parameters by ghosts would disappear.
The non-standard term in the $\delta \varphi^i$ transformation is related to another peculiarity of the BRST transformations (3.11). Taking the $h \to 0$ limit of (3.11) and calculating $\delta^2$ on the various fields, one finds that the transformations fail to be nilpotent on $h$ and $\varphi^i$. For example, one finds after some algebra that

$$
\begin{align*}
\delta^2 h &= a \gamma \partial \gamma \partial \varphi^i \left[ - \partial \partial \varphi^i + h \partial^2 \varphi^i + \partial h \partial \varphi^i \\
&\quad + d_{ijk} \partial B \partial \varphi^j \partial \varphi^k + d_{ijk} B \partial (\partial \varphi^j \partial \varphi^k) + a \partial^2 \gamma b B \partial \varphi^i \right].
\end{align*}
\tag{3.15}
$$

Similarly, in $\delta^2 \varphi^i$ one has, amongst other terms, a term $a \pi_h \gamma \partial \gamma \partial \varphi^i$ that is nowhere canceled. Both of these non-closure expressions, however, vanish upon use of the classical field equations. For example, the equation of motion for $\varphi^i$ is

$$
\frac{\partial L}{\partial \varphi^i} - \partial \frac{\partial L}{\partial \partial \varphi^i} = \partial \partial \varphi^i - h \partial^2 \varphi^i - \partial h \partial \varphi^i - d_{ijk} \partial B \partial \varphi^j \partial \varphi^k - d_{ijk} B \partial (\partial \varphi^j \partial \varphi^k) + a \partial (B b \partial \gamma \partial \varphi^i) = 0,
\tag{3.16}
$$

which causes the terms in the square brackets in (3.15) to vanish. Similarly, the equation of motion for the gauge field $h$ is

$$
\pi_h - (T_{\text{mat}} + T_{\text{gh}}) = 0,
\tag{3.17}
$$

which causes the non-closure terms in $\delta^2 \varphi^i$ to vanish. Nonetheless, the transformations (3.11) do express an invariance of the gauge-fixed action (3.10) because the terms arising from the variation of $h$ in $\delta (b h)$ in (3.10a) due to the $\delta^2 h$ off-shell non-closure are canceled by the extra term $a b \gamma \partial \gamma \partial \varphi^i$ in the transformation of $\varphi^i$. The off-shell non-closure of the BRST algebra and the corresponding complication of the BRST transformations is strongly reminiscent of the BRST formulation of supergravity theories prior to the introduction of auxiliary fields. As in that case [21], curing these problems will require us to find a new formalism that achieves full off-shell closure of the BRST algebra.

4. Canonical BRST quantisation of $W_3$ gravity

In this section, we shall quantise chiral $W_3$ gravity using our reformulated BRST construction based upon derivative gauge conditions for the spin-2 and spin-3 gauge fields. We shall calculate all the anomalies at order $h$ together with the local anomaly at order $h^2$, which contains the spin-3 anomaly. We shall show that these anomalies satisfy the Wess-Zumino consistency condition as required. In this discussion, we shall present the BRST formulation for a general model based upon a realisation of the $W_3$ algebra with an arbitrary value of the matter central charge $C_{\text{mat}}$, adjustable by an appropriate choice of the background-charge terms. Of course, from the point of view of noncritical $W_3$ gravity, the most natural choice
for the values of these central charges might be considered to be zero, which is also covered by the general discussion that we shall give.

The gauge conditions that we shall choose for chiral $W_3$ gravity are $\bar{\partial}h = 0$ and $\bar{\partial}B = 0$, naturally generalising our discussion in the Virasoro-gravity case. This BRST quantisation procedure in the $W_3$ case proceeds now along lines strictly parallel to our Virasoro discussion. Accordingly, we now shorten our $W_3$ discussion by presenting directly the fully-renormalized gauge-fixed chiral $W_3$-gravity action, including background-charge terms with parameters $\alpha^i$ chosen so as to be consistent with (3.5) but in general corresponding to $C_{\text{mat}} \neq 100$:

$$I = \frac{1}{\pi} \int d^2 z L$$

$$= \frac{1}{\pi} \int d^2 z \left( -\frac{1}{2} \bar{\partial} \phi^i \partial \phi^i - h T_{\text{mat}} - B W_{\text{mat}} + \pi_h \bar{\partial}h - b \bar{\partial} [\bar{\partial} c + c \bar{\partial} h - \partial c h - \frac{a}{2} (\gamma \bar{\partial} B - \partial \gamma B) \partial \phi^i \partial \phi^i - a \sqrt{h} (\gamma \bar{\partial} B - \partial \gamma B) \alpha_i \partial^2 \phi^i + \frac{1 - 17a}{30} h (2 \gamma \partial^3 B - 3 \partial \gamma \partial^2 B + 3 \partial^2 \gamma \partial B - 2 \partial^3 \gamma B)] - \beta \bar{\partial} [\bar{\partial} \gamma + c \bar{\partial} B - 2 \partial c B + 2 \gamma \bar{\partial} h - \partial \gamma h] \right),$$

where $T_{\text{mat}}$ and $W_{\text{mat}}$ are given in (3.3). As in the Virasoro case, we now reduce this action to first-order form by introducing momenta conjugate to $c, b, \gamma, \beta$:

$$\pi_c = \frac{\partial L}{\partial \bar{\partial} c} = -\bar{\partial} b$$

$$\pi_b = \frac{\partial L}{\partial \bar{\partial} b} = \bar{\partial} c + c \bar{\partial} h - \partial c h - \frac{a}{2} (\gamma \bar{\partial} B - \partial \gamma B) \partial \phi^i \partial \phi^i - a \sqrt{h} (\gamma \bar{\partial} B - \partial \gamma B) \alpha_i \partial^2 \phi^i + \frac{1 - 17a}{30} h (2 \gamma \partial^3 B - 3 \partial \gamma \partial^2 B + 3 \partial^2 \gamma \partial B - 2 \partial^3 \gamma B)$$

$$\pi_\gamma = \frac{\partial L}{\partial \bar{\partial} \gamma} = -\bar{\partial} \beta$$

$$\pi_\beta = \frac{\partial L}{\partial \bar{\partial} \beta} = \bar{\partial} \gamma + c \bar{\partial} B - 2 \partial c B + 2 \gamma \bar{\partial} h - \partial \gamma h.$$ (4.2)

Using these definitions, the gauge-fixed action (4.1) may be put into first-order form (once again considering the off-diagonal kinetic term $\bar{\partial} \phi^i \partial \phi^i$ to be of first order in $\bar{\partial}$ derivatives):

$$I = \frac{1}{\pi} \int d^2 z \left( -\frac{1}{2} \bar{\partial} \phi^i \partial \phi^i + \pi_h \bar{\partial} h + \pi_B \bar{\partial} B - \pi_b \bar{\partial} b - \pi_c \bar{\partial} c - \pi_\beta \bar{\partial} \beta - \pi_\gamma \bar{\partial} \gamma - \frac{1}{h} (T_{\text{mat}} + T_{\text{gh}}) - B (W_{\text{mat}} + W_{\text{gh}}) \right),$$

$\dagger$ In the case of the minimal two-scalar realisation of the $W_3$ algebra with $\phi^1$ playing the rôle of the Virasoro Liouville field and $\phi^2$ playing the analogous rôle for the spin-3 symmetry, equations (3.5) require $\alpha_1 = \sqrt{3} \alpha_2$ in order to obtain a realisation of the $W_3$ algebra. This relation leaves undetermined one background-charge parameter, corresponding to the unfixed value of the matter-sector central charge $C_{\text{mat}}$. 

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where $T_{gh}$ and $W_{gh}$ are no longer given by (3.8a,b) but now take forms involving the conjugate momenta,

\begin{align}
T_{gh} &= -2\pi_c \partial c - \partial \pi_c c - 3\pi_\gamma \partial \gamma - 2\partial \pi_\gamma \gamma, \\
W_{gh} &= -\partial \pi_c c - 3\pi_\gamma \partial c - a[\partial (\pi_c \gamma T_{\text{mat}}) + \pi_c \partial \gamma T_{\text{mat}}] \\
&\quad + \frac{(1 - 17a)}{30} \hbar (2\gamma \partial^3 \pi_c + 9\partial \gamma \partial^2 \pi_c + 15\partial^2 \gamma \partial \pi_c + 10\partial^3 \gamma \pi_c). 
\end{align}

(4.4a, 4.4b)

From the path-integral generating functional derived from (4.3) we obtain the following OPE relations:

\begin{align}
\partial \varphi^i(z)\partial \varphi^j(w) &\sim -\frac{\hbar}{(z-w)^2}; & \pi_h(z)h(w) &\sim \frac{\hbar}{z-w} \\
c(z)\pi_c(w) &\sim \frac{\hbar}{z-w}; & b(z)\pi_b(w) &\sim \frac{\hbar}{z-w} \\
\pi_B(z)B(w) &\sim \frac{\hbar}{z-w}; & \gamma(z)\pi_\gamma(w) &\sim \frac{\hbar}{z-w} \\
\beta(z)\pi_\beta(w) &\sim \frac{\hbar}{z-w}; & c(z)\bar{\partial}b(w) &\sim -\frac{\hbar}{z-w}; & \gamma(z)\bar{\partial}\beta(w) &\sim -\frac{\hbar}{z-w}. 
\end{align}

(4.5)

The BRST transformations corresponding to the action (4.3) are

\begin{align}
\delta \varphi^i &= c \partial \varphi^i + d_{ijk} \gamma \partial \varphi^j \partial \varphi^k + a \pi_c \gamma \partial \gamma \partial \varphi^i \\
&\quad + \sqrt{\hbar} \left( -\alpha_i \partial c + (e_{ij} - e_{ji}) \gamma \partial^2 \varphi^j - e_{ji} \partial \gamma \partial \varphi^j - a \alpha_i \partial (\pi_c \gamma \partial \gamma) \right) + \hbar f_i \partial^2 \gamma_i, \\
\delta h &= \pi_b, & \delta B &= \pi_\beta, \\
\delta c &= c \partial c - a \gamma \partial c \partial \varphi^i \partial \varphi^j - a \sqrt{\hbar} \alpha_i \gamma \partial \gamma \partial^2 \varphi^i + \frac{1 - 17a}{30} \hbar (2\gamma \partial^3 \gamma - 3\partial \gamma \partial^2 \gamma), \\
\delta \gamma &= c \partial \gamma - 2\partial c \gamma, \\
\delta b &= \pi_h, & \delta \beta &= \pi_B, & \delta \pi_c &= T_{\text{mat}} + T_{gh}, & \delta \pi_\gamma &= W_{\text{mat}} + W_{gh}, \\
\delta \pi_h &= 0, & \delta \pi_B &= 0, & \delta \pi_c &= 0, & \delta \pi_\beta &= 0, 
\end{align}

(4.6a, 4.6b, 4.6c, 4.6d, 4.6e, 4.6f, 4.6g, 4.6h, 4.6i, 4.6j, 4.6k, 4.6l, 4.6m)

where $d_{ijk}$, $\alpha_i$, $e_{ij}$, $f_i$ are chosen so as to satisfy (3.5).

Taking the limit $\hbar \to 0$ in (4.2, 4.3, 4.4, 4.6) and replacing (4.5) by the corresponding classical Poisson bracket/antibracket relations, one obtains a classical BRST system that repairs all of the deficiencies of classical $W_3$ gravity as outlined in section three. In particular, the classical BRST transformations (4.6) obtained in the limit $\hbar \to 0$ are now fully nilpotent without use of classical equations of motion. This may be verified directly by evaluating $\delta^2$ on the various fields using (4.6); the checks for $\pi_c$ and $\pi_\gamma$ are the most algebraically involved.

The action (4.3) and transformation rules (4.6) that we have given are fully renormalized and include arbitrary background charges consistent with (3.5). In considering this general
case, one needs to know, in addition to the invariance of (4.3) at order $\hbar^0$, that the variation of (4.3) under (4.6) also vanishes at order $\hbar^{1/2}$, and correspondingly that the nilpotence of the transformations (4.6) is obtained also at order $\hbar^{1/2}$. These low-order observations will provide the basis of the Wess-Zumino consistency condition that we shall discuss shortly.

As we found in (3.10a) for the Virasoro case, the action (4.3) may be rewritten using the BRST transformations (4.6) in “canonical BRST” form

$$I = \frac{1}{\pi} \int d^2z \left(-\frac{1}{2} \overline{\partial} \phi^i \partial \phi^j + \pi_h \overline{\partial} h + \pi_B \overline{\partial} B - \pi_b \overline{\partial} b - \pi_c \overline{\partial} c - \pi_\beta \overline{\partial} \beta - \pi_\gamma \overline{\partial} \gamma - \delta(h \pi_c + B \pi_\beta) \right),$$

identifying the “gauge fermion” for our formulation as

$$\Psi = h \pi_c + B \pi_\gamma.$$  \hspace{1cm} (4.7)

The form (4.7) of the $W_3$ gravity action makes the invariance of the gauge-fixed action at orders $\hbar^0$ and $\hbar^{1/2}$ manifest, since the BRST transformations (4.6) now constitute a canonical transformation, and all the “kinetic” $\pi^a \overline{\partial} q^a$ terms in (4.7) (including the $\overline{\partial} \phi^i \partial \phi^j$ kinetic term) are invariant under arbitrary canonical transformations, while the Hamiltonian density $H = \delta \Psi$ is BRST trivial and thus is invariant under (4.6) up to the order to which (4.6) is nilpotent, i.e. at orders $\hbar^0$ and $\hbar^{1/2}$.

The canonical generator of the transformations (4.6) is the BRST charge $Q$ of our reformulated quantisation procedure:

$$Q = \int dz \left( c (T_{\text{mat}} + \frac{1}{2} T_{\text{gh}}) + \gamma (W_{\text{mat}} + \frac{1}{2} W_{\text{gh}}) + \pi_h \pi_b + \pi_B \pi_\beta \right)$$

$$= \int dz \left( c T_{\text{mat}} + \gamma W_{\text{mat}} + \pi_h \pi_b + \pi_B \pi_\beta + \pi_\gamma (c \partial \gamma - 2 \partial c \gamma) \right)$$

$$+ \pi_c (c \partial c + a \gamma \partial \gamma) T_{\text{mat}} + \frac{1}{30} (1 - 17a) \hbar (2 \gamma \partial^3 \gamma - 3 \partial \gamma \partial^2 \gamma).$$ \hspace{1cm} (4.9a)

This BRST charge differs from that of [12] not only in that it contains the momenta of our first-order formulation, but also in containing the final two terms of (4.9a), which are new. These new terms generate the transformations of the spin-2 and spin-3 gauge fields, so (4.9) is now fully canonical.

**The $W_3$ anomaly**

We formulate the BRST Ward identity of $W_3$ gravity as in section two by introducing sources $K_{\phi^i}$ for the nonvanishing variations (4.6). Here we denote all fields $\phi^i$ and the sources of their variations $K_{\phi^i}$ as

$$\phi^i = (\phi^i, h, \pi_h, B, \pi_B, b, \pi_b, c, \pi_c, \beta, \pi_\beta, \gamma, \pi_\gamma),$$

$$K_{\phi^i} = (K_{\phi^i}, K_h, K_B, K_b, K_\beta, K_\gamma, K_c, K_{\pi_c}, K_{\pi_\gamma}).$$ \hspace{1cm} (4.10a)

(4.10b)
The extended classical Lagrangian is the sum of (4.3) plus the source terms,
\[
\mathcal{L}_{\text{ext}} = \mathcal{L}_{\text{gf}} + K_{\phi}^i \delta \phi^i \\
= - \frac{1}{2} \partial \phi^i \partial \phi^i + \pi_h \partial \bar{h} + \pi_B \partial \bar{B} - \pi_b \partial \bar{b} - \pi_c \partial \bar{c} - \pi_\beta \partial \bar{\beta} - \pi_\gamma \partial \bar{\gamma} - \pi_b \pi_c - \pi_\beta \pi_\gamma \\
- h (T_{\text{mat}} + T_{\text{gh}}) - B (W_{\text{mat}} + W_{\text{gh}}) + K_{\phi}^i [c \partial \phi^i + d_{ijk} \gamma \partial \phi^j \partial \phi^k + a \pi_c \partial (\pi_\gamma \partial \phi^i)] + h f_i \partial^2 \gamma \\
+ K_h \pi_b + K_B \pi_\beta + K_h \partial \phi^i - a \sqrt{h} \alpha_i \gamma \partial \gamma \partial^2 \phi^i + \frac{1}{30} (1 - 1773) h (2 \gamma \partial^3 \gamma - 3 \partial \gamma \partial^2 \gamma) \\
+ K_{\pi_c} (T_{\text{mat}} + T_{\text{gh}}) + K_{\pi_\gamma} (W_{\text{mat}} + W_{\text{gh}}).
\]

Since we have already included the background charges \( \alpha^i \) and the attendant other factors of \( \sqrt{h} \) and \( h \) in the currents appearing in (4.11), we shall have to expand the effective action \( \Gamma \) into a half-step series in \( h \):
\[
\Gamma = \Gamma_0 + h \frac{1}{2} \Gamma_1 + h \Gamma_1 + h \frac{3}{2} \Gamma_2 + h^2 \Gamma_2 + \cdots,
\]
where \( \Gamma_0 \) contains all the terms in the effective action of order \( h^0 \). These effective-action terms in \( \Gamma_0 \) are just the \( h^0 \) terms appearing in (4.11), so we may write \( \Gamma_0 = S_0 \), where by \( S_n \) we mean the terms of order \( h^n \) in the extended classical action, \( I_{\text{ext}} = \sum_m h^{m/2} S_m/2 \). At order \( h^{1/2} \), we have a similar situation, \( \Gamma_{1/2} = S_{1/2} \). At order \( h \) we encounter for the first time loop corrections, so \( \Gamma_1 \) can be divided into two parts, \( \Gamma_1 = S_1 + \Gamma_{1, nl} \), where \( \Gamma_{1, nl} \) contains the (non-local) loop contributions. At order \( h^{3/2} \), \( \Gamma_{3/2} \) is obtained from one-loop corrections to \( S_{1/2} \). At order \( h^2 \), we have three different kinds of loop contributions: one-loop contributions involving two terms from \( S_{1/2} \) together with terms from \( S_0 \), one-loop contributions involving a term from \( S_1 \) together with terms from \( S_0 \), and finally the true two-loop contributions, involving only terms from \( S_0 \).

As we have already stated, the gauge-fixed action (4.3) is invariant under the transformations (4.6) at orders \( h^0 \) and \( h^{1/2} \), and the transformations (4.6) are nilpotent to these same orders. Expressing these statements in antibracket notation (2.29), we have
\[
\left( S_0, S_0 \right) = 0; \quad \left( S_0, S_{1/2} \right) = 0.
\]

We must also expand the anomaly \( \triangle \) in a half-step series in \( h \),
\[
\triangle = h \triangle_1 + h \frac{1}{2} \triangle_2 + h^2 \triangle_2 + \cdots.
\]
Now we can write the anomalous Ward identity (2.21) at orders $\bar{h}, h^{3/2}$ and $h^2$ as the separate equations

\[
\mathcal{A}_1 = \Delta_1 = \frac{\delta S_0}{\delta \phi^i} - \frac{\delta \Gamma_1}{\delta K_{\phi^i}} + \frac{\delta S_0}{\delta K_{\phi^i}} + \frac{\delta S_1}{\delta \phi^i} + \frac{\delta S_1}{\delta K_{\phi^i}},
\]

\[
\mathcal{A}_2 = \Delta_2 + \mathcal{A}_{2,\text{nl}} = \frac{\delta S_0}{\delta \phi^i} - \frac{\delta \Gamma_2}{\delta K_{\phi^i}} + \frac{\delta S_0}{\delta K_{\phi^i}} + \frac{\delta S_1}{\delta \phi^i} + \frac{\delta S_1}{\delta K_{\phi^i}} + \frac{\delta S_2}{\delta \phi^i} + \frac{\delta S_2}{\delta K_{\phi^i}},
\]

where $\mathcal{A}_n$ represents the total (local plus non-local) anomaly at order $h^n$, while $\Delta_n$ represents the local order-$h^n$ anomaly that remains after non-local “dressings” of lower-order anomalies are separated off, in accordance with (2.21). The term $\mathcal{A}_{2,\text{nl}}$ is just such a dressing of the order-$h$ anomaly.

Using the extended Lagrangian (4.11) and the relations (3.5, 3.6), we find the local anomalies after a straightforward but somewhat tedious calculation:

\[
\Delta_1 = \frac{16}{30\pi} (1 - 17a) - \frac{a}{12\pi} C_{\text{mat}} \int d^2z \left( \gamma \pi_c \partial \gamma (\partial^3 h - \partial^3 K_{\pi_c}) + \partial^3 c [\gamma K_c \partial \gamma - \pi_c (\partial \gamma B - \gamma \partial B - \partial \gamma K_{\pi_c} + \gamma \partial K_{\pi_c})] \right)
\]

\[
+ \frac{1}{12\pi} (100 - C_{\text{mat}}) \int d^2z c (\partial^3 h - \partial^3 K_{\pi_c}),
\]

\[
\Delta_2 = -\frac{29}{50\pi} (1 - 17a) + \frac{C_{\text{mat}}}{360\pi} \int d^2z \gamma (\partial^5 B - \partial^5 K_{\pi_c}).
\]

Note that the above results contain only universal, i.e. purely gauge-field-dependent and ghost-dependent anomalies. We recall that this is because we are employing in (4.11) the renormalisations already found in Ref. [15] for the cancellation of matter-dependent anomalies. We note also that the second term in $\Delta_1$ is same as in the Virasoro case (2.27) except for its coefficient, which is not surprising since the Virasoro algebra is a subalgebra of the $W_3$ algebra. Since we now have an additional pair of spin-3 ghosts, the coefficient changes from 26 to 100.

We next shall show that these anomalies satisfy the Wess-Zumino consistency condition. In order to do this, we shall need parts of $\mathcal{A}_{2,\text{nl}}$ at order $h^2$ that, although non-local in structure, will make local contributions to the consistency condition when antibracketed with $S_0$:

\[
\mathcal{A}_{2,\text{nl}} = \frac{1}{2} \left( -\frac{29}{50\pi} (1 - 17a) + \frac{C_{\text{mat}}}{360\pi} \right) \int d^2z K_{\pi_c} \frac{\partial^5}{\partial} (B - K_{\pi_c})
\]

\[
- \frac{1}{2} \left( \frac{100 - C_{\text{mat}}}{12\pi} \right) \int d^2z K_{\pi_c} \frac{\partial^3}{\partial} (h - K_{\pi_c}) + \mathcal{A}'_{2,\text{nl}},
\]

(4.17)
where \( A_{2,\text{nl}}' \) denotes the remaining non-local parts of \( A_2 \).

The Wess-Zumino consistency condition for the order-\( h \) anomaly \( \Delta_1 \) is obtained by taking an antibracket with \( S_0 \),

\[
\left( S_0, \Delta_1 \right) = \left( S_0, \left( S_0, \Gamma_1 \right) \right) + \frac{1}{2} \left( S_0, \left( S_1, S_1 \right) \right),
\]

and then using the Jacobi identity for the antibracket together with the low-order relations (4.13). Thus, we obtain the condition

\[
\left( S_0, \Delta_1 \right) = 0,
\]

similarly to (2.30), despite the presence of the order \( h^{1/2} \) terms in the extended Lagrangian (4.11). Insertion of the calculated anomaly (4.16a) then shows after some algebra that (4.19) is indeed satisfied, confirming the correctness of the form of (4.16a).

In continuing on to check the Wess-Zumino consistency condition at order \( h^2 \), one needs to take into account the dressing \( A_{2,\text{nl}} \) of the order-\( h \) anomaly that is present in (4.16c). In order to show how this works, we shall for simplicity consider just the local terms appearing in the order-\( h^2 \) consistency condition; the non-local terms will then have to cancel separately. Taking an antibracket with \( S_0 \), we have at order \( h^2 \)

\[
\left( S_0, A_2 \right) = \left( S_0, \left( S_0, \Gamma_2 \right) \right) + \left( S_0, \left( S_1, \Gamma_3 \right) \right) + \left( S_0, \left( \Gamma_1, \Gamma_1 \right) \right).
\]

Use of the Jacobi identity, (4.13) and (4.16b), and restricting attention to the local contributions in (4.20) then gives

\[
\left( S_0, \Delta_2 \right) + \left( S_0, A_{2,\text{nl}} \right)_{\text{loc}} + \left( S_1, \Delta_1 \right) = 0,
\]

where the second term is restricted to local contributions after taking the antibracket. Insertion of the calculated results (4.16, 4.17) shows that the consistency condition (4.21) indeed is satisfied, confirming our form for the anomaly at order \( h^2 \).

The results (4.16) confirm once again that for a matter system (3.3) with \( C_{\text{mat}} = 100 \), all the anomalies cancel [19]. Using (4.16) for a non-critical matter system opens the way to an investigation of anomaly-induced dynamics in the correlation functions of \( W_3 \) gravity along lines generalising the work of Ref. [4].

5. Conclusion: operator-product versus Ward-identity anomalies

In this paper, we have reformulated the BRST quantisation procedure for chiral world-sheet gravities by the adoption of a derivative gauge condition and the introduction of momenta in order to put the ghost sector of the theory back into first-order form. These simple
changes to the BRST formalism for worldsheet gravities render the formalism canonical in the sense that the BRST transformations of all fields now arise as canonical transformations generated by the BRST charge $Q$.

A very simple, but apparently so far unnoticed, consequence of this canonical structure is the following relation between the notion of an anomaly in the BRST operator algebra, i.e. the failure of $Q^2$ to vanish at the quantum level, and the anomalies (2.27, 4.16) in the BRST Ward identities.

In the case of Virasoro gravity, interpreting the BRST charge $Q$ (2.16) as a normal-ordered quantum operator, one may calculate $Q^2$ by standard operator-product techniques. Writing (2.16) as the integral of a normal-ordered operator current,

$$Q = \oint \frac{dz}{2\pi i} J_B(z), \quad (5.1)$$

where the integral in complex worldsheet coordinates is now interpreted as a closed loop around the origin and in order to recover equivalence to the standard mode-expansion result, one should collect the simple poles in (5.1) using Cauchy’s theorem, hence the factor of $(2\pi i)^{-1}$ in the measure. Calculating $Q^2$ using standard operator-product rules, one obtains

$$Q^2 \sim \hbar^2 \oint \frac{dz}{2\pi i} \langle J_B J_B \rangle_1(z); \quad (5.2a)$$

$$\langle J_B J_B \rangle_1 = \frac{1}{6} (26 - D) c \partial^3 c. \quad (5.2b)$$

In evaluating (5.2), we have taken as usual the operator product $J_B(z) J_B(w)$ and have extracted the residue of the first-order pole $(z - w)^{-1}$ in the resulting Laurent series; the result of this procedure is here denoted $\langle J_B J_B \rangle_1$.

Equation (5.2) expresses the BRST anomaly as it is understood in conformal field theory. Its relation to the Ward-identity anomaly (2.27) may now be simply stated in our reformulated BRST procedure as

$$\hbar \Delta_1 = -\frac{1}{2\pi} \int d^2 z \left\langle \langle J_B J_B \rangle_1, \left( \pi_c (h - K \pi_c) \right) \right\rangle_1. \quad (5.3)$$

This can also be expressed using the gauge fermion for the extended Virasoro-gravity action (2.23),

$$\Psi_{\text{Vir, ext}} = \pi_c h + \sum_i (-1)^{[i]} \phi^i K_{\phi^i}, \quad (5.4)$$

where $[i]$ takes the values $(0, 1)$ for (bose, fermi) variables; variation of (5.4) produces all of the non-kinetic terms in (2.23), as one can see from (2.15) and remembering that $\delta K_{\phi^i} = 0$. Using (5.4), the relation (5.3) can be written

$$\hbar \Delta_1 = -\frac{1}{2\pi} \int d^2 z \left\langle \langle J_B J_B \rangle_1, \Psi_{\text{Vir, ext}} \right\rangle_1. \quad (5.5)$$

\[\uparrow\] Note, however, that the only part of $\Psi_{\text{Vir, ext}}$ that contributes in (5.5) is $\pi_c (h - K \pi_c)$.
In the case of $W_3$ gravity, for the BRST operator (4.9) we find the operator-product anomaly
\[
\langle J_B J_B \rangle_1 = \frac{1}{6} (100 - C_{\text{mat}}) c \partial^3 c + \left( - \frac{16}{15} (1 - 17a) + \frac{a}{6} C_{\text{mat}} \right) \gamma \pi_c \partial \gamma \partial^3 c \\
+ \bar{h} \left( \frac{29}{25} (1 - 17a) - \frac{C_{\text{mat}}}{180} \right) \gamma \bar{\partial} \gamma.
\] (5.6)

Using this, we can express the non-vanishing Ward-identity anomalies (4.15) as
\[
\bar{h} \triangle_1 + h^2 \triangle_2 = -\frac{1}{2\pi} \int d^2 z \left\{ \langle J_B J_B \rangle_1, \left( \pi_c (h - K_{\pi_c}) + \pi_\gamma (B - K_{\pi_\gamma}) - cK_c \right) \right\}_1,
\] (5.7)
or, in terms of the gauge fermion for the extended $W_3$ action (4.11),
\[
\Psi_{W_3, \text{ext}} = \pi_c h + \pi_\gamma B + \sum_i (-1)^{|i|} \phi^i K_{\phi^i},
\] (5.8)
we can write
\[
\bar{h} \triangle_1 + h^2 \triangle_2 = -\frac{1}{2\pi} \int d^2 z \left\{ \langle J_B J_B \rangle_1, \Psi_{W_3, \text{ext}} \right\}_1.
\] (5.9)

Consequently, by applying $\langle J_B J_B \rangle_1$ to the gauge fermion, we have reproduced all of the local $W_3$ anomalies. Note that the order-$\bar{h}$ local anomaly $\triangle_2$ arises in this relation from the terms in $\langle J_B J_B \rangle_1$ that carry an explicit factor of $\bar{h}$; these terms arise from at most double contractions, but involve the order-$\bar{h}$ renormalisation terms in (4.8). Thus, the local operator-product anomalies for $W_3$ all arise from processes involving at most double contractions, in contrast to the Feynman-diagram calculation, where genuine two-loop diagrams are involved.

The simplicity of the relations (5.5, 5.9) suggests a general result for the relation between operator-product $Q^2$ anomalies as calculated in conformal field theory and the anomalies occurring in the BRST Ward identities. In order to obtain this relation, it appears to be necessary to take care, as we have in this article, to use a legitimate gauge choice, so that the BRST transformations of all fields arise as canonical transformations generated by the BRST charge $Q$. It remains an interesting problem to show whether this result is obtained in general, as well as finding its analogue in field theories in other dimensions, such as Yang-Mills theory.
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