VARIATIONS OF KUREPA’S LEFT FACTORIAL HYPOTHESIS

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ABSTRACT. Kurepa’s hypothesis asserts that for each integer \( n \geq 2 \) the greatest common divisor of \( !n := \sum_{k=0}^{n-1} k! \) and \( n! \) is 2. Motivated by an equivalent formulation of this hypothesis involving derangement numbers, here we give a formulation of Kurepa’s hypothesis in terms of divisibility of any Kurepa’s determinant \( K_p \) of order \( p - 4 \) by a prime \( p \geq 7 \). In the previous version of this article we have proposed the strong Kurepa’s hypothesis involving a general Kurepa’s determinant \( K_n \) with any integer \( n \geq 7 \). We prove the “even part” of this hypothesis which can be considered as a generalization of Kurepa’s hypothesis. However, by using a congruence for \( K_n \) involving the derangement number \( S_{n-1} \) with an odd integer \( n \geq 9 \), we find that the integer \( 11563 = 31 \times 373 \) is a counterexample to the “odd composite part” of strong Kurepa’s hypothesis.

We also present some remarks, divisibility properties and computational results closely related to the questions on Kurepa’s hypothesis involving derangement numbers and Bell numbers.

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1. REMARKS ON KUREPA’S HYPOTHESIS

In 1971 Dj. Kurepa [10] introduced the left factorial function \( !n \) which is defined as

\[
!0 = 0, \quad !n = \sum_{k=0}^{n-1} k!, \quad n \in \mathbb{N}.
\]

\( !n \) is the Sloane’s sequence A003422 in [22].

For more details of the following conjecture and its reformulations see a overview of A. Ivić and Ž. Mijajlović [7].

Conjecture 1 (Kurepa’s left factorial hypothesis). For each positive integer \( n \geq 2 \) the greatest common divisor of \( !n \) and \( n! \) is 2.

Kurepa’s hypothesis and its equivalent formulation appear in R. Guy’s classic book [6] as problem B44 which asserts that

\[
!n \not\equiv 0 \pmod{n} \text{ for all } n > 2.
\]

Alternating sums of factorials \( \sum_{k=1}^{n-1} (-1)^{k-1}k! \) are involved in Problem B43 in [6] which was solved by M. Živković [28].

Further, Kurepa’s hypothesis was tested by computers for \( n < 1000000 \) by Mijajlović and Gogić in 1991 (see e.g., [14] and [9]). Kurepa’s left factorial hypothesis (or
in the sequel, written briefly *Kurepa’s hypothesis*) is an unsolved problem since 1971 and there seems to be no significant progress in solving it. Notice that a published proof of Kurepa’s hypothesis in 2004 by D. Barsky and B. Benzaqhou [11 Théorème 3, p. 13] contains some irreparable calculation errors in the proof of Theorem 3 of this article [2], and this proof is therefore withdrawn.

However, there are several statements equivalent to Kurepa’s hypothesis (see e.g., Kellner [8 Conjecture 1.1 and Corollary 2.3], Ivić and Mijajlović [7], Mijajlović [13 Theorem 2.1], Petojević [18] and [19 Subsection 3.3], Petojević, Žižović and S. Cvejić [20 Theorems 1 and 2], Šami [27], Stanković [23], Živković [28]). Moreover, there are numerous identities involving the left factorial function [21], Stanković [23], Živković [28]). Moreover, Kurepa’s hypothesis is equivalent to the assertion that

\[
K_d(x, y) := (2d + 1) \sum_{x \leq p \leq y} \frac{1}{p} \approx (2d + 1) \log \frac{\log y}{\log x}.
\]

(Here the second estimate is a classical asymptotic formula of Mertens [4 p. 94]). In particular, for the interval \([x, y]\) = \([23, 2^{23}]\) with \(d = 9\) the above estimate implies that \(K(23, 2^{23}) \approx 30.8977\). M. Živković [28 Table 1] verified that \(S_{p-1} := \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \not\equiv 0(\mod p)\) for all odd primes \(p < 2^{23}\). On the other hand, it follows by (2) that the expected number of such odd primes less than \(2^{23}\) is about \(\log \frac{\log 2^{23}}{\log 3} = 2.67493\); also, the expected number of such odd primes from the interval \([353, 2^{23}]\) is about \(\log \frac{\log 2^{23}}{\log 353} = 0.999729\). Accordingly to these two expected numbers of primes which would be the “counterexamples to Kurepa’s hypothesis” and in view of related Živković’s computation up to \(2^{23}\) [28], it may be of interest to determine “the probability” that \(S_{p-1} \not\equiv 0(\mod p)\) for each odd prime \(p\) such that \(x \leq p < 2^{23}\) for a given fixed \(x\). Assuming the fact that for every pair of different odd primes \(p\) and \(q\) the events \(A - “p\ satisfies the congruence \(S_{p-1} \equiv 0(\mod p)\)” and \(B - “q\ satisfies the congruence \(S_{q-1} \equiv 0(\mod q)\)” are independent, and in view of the previously mentioned heuristic probability argument, we find that the probability \(P(K[x, y])\) of
the Kurepa’s event $K[x, y]$ - “the congruence $S_{p-1} \equiv 0 \pmod{p}$ $(3 \leq x \leq y)$ is satisfied for none odd prime $p$ such that $x \leq p \leq y$” is equal to

$$P(K[x, y]) = \prod_{x \leq p \leq y} \left(1 - \frac{1}{p}\right)$$

(the above product ranges over all odd primes $p$ with $x \leq p \leq y$).

In particular, using the fact that the greatest prime which is less than $2^{23} = 8388608$ is the 564163th prime $p_{564163} = 8388593$, applying (3) in Mathematica 8, we find that, $P(K[4, 2^{23}]) = 0.105652$. This means that the probability that $S_{p-1} := \sum_{k=0}^{p-1} (-1)^k/k! \not\equiv 0 \pmod{p}$ for all primes $p$ with $5 \leq p < 2^{23}$ is equal to 0.105652. Of course, the value 0.105652 is not sufficiently small, and hence, for the verification of the truth of Kurepa’s hypothesis, it would be useful further computations of $S_{p-1}$ modulo primes $p > 2^{23}$. For example, $P(K[2^{23}, 5000000]) = 0.899309$ shows that the probability that $S_{p-1} \equiv 0 \pmod{p}$ for at least one prime less than the 3001134th prime $p_{3001134} = 49999991$ is greater than 10%.

Furthermore, from the right part of Table 1 in [23] we also see that in the interval $[2^{23}, 2^{23}]$ there are 27 primes $p$ satisfying the condition $\min \{S_{p-1}(\text{mod } p), p - S_{p-1}(\text{mod } p)\} \leq 9$. Our computation in Mathematica 8 shows that in the interval $[1000, 100000]$ there are 118 primes $p$ satisfying the condition $\min \{S_{p-1}(\text{mod } p), p - S_{p-1}(\text{mod } p)\} \leq 99$. On the other hand, by the estimate (2) the expected number of such primes is $\approx 199 \log \frac{\log 100000}{\log 1000} = 101.654$.

Moreover, by using the mentioned heuristic argument, we might argue that the number of primes $p$ in the interval $[2^{23}, 10^{19}]$ such that $S_{p-1} \equiv 0 \pmod{p}$ is expected to be $\log \frac{\log 10^{19}}{\log 2^{23}} \approx 1.00949$. In other words, under the validity of presented heuristic arguments we have the following fact.

**Fact 1.** Under the validity of heuristic arguments presented in Remarks 1, it can be expected one prime less than $10^{19}$ which is “a counterexample” to Kurepa’s hypothesis.

2. A LINEAR ALGEBRA FORMULATION OF KUREPA’S HYPOTHESIS

Motivated by a linear algebra formulation of Kurepa’s hypothesis given in [12] we give the following definition.

**Definition 1.** For any integer $n \geq 7$ the Kurepa’s determinant $K_n$ is the determinant of order $n - 4$ defined as

$$K_n := \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 2 \\ 1 & 4 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 2 \\ 0 & 1 & 5 & 1 & 1 & \ldots & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 6 & 1 & \ldots & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 7 & \ldots & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & \ldots & 1 & 1 & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & n - 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & -4 \end{vmatrix}$$
First few values of $K_n$ are as follows: $K_7 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 2 \\ 0 & 1 & -4 \end{vmatrix} = 15$, $K_8 = -47$, $K_9 = 197$, $K_{10} = -1029$, $K_{11} = 6439$, $K_{12} = -46927$, $K_{13} = 390249$, $K_{14} = -3645737$, $K_{15} = 37792331$, $K_{16} = -430400211$ and $K_{17} = 5341017373$.

Motivated by the reformulation of Kurepa’s hypothesis given by (1) of Remarks 1, and using a linear algebra approach (working in the field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ modulo $p$), we can establish a reformulation of Kurepa’s hypothesis given by the following result.

**Theorem 1** ([12]). Let $p$ be an odd prime. Then the following statements are equivalent.

(i) Kurepa’s hypothesis holds, i.e., for each positive integer $n \geq 2$ the greatest common divisor of $!n$ and $n!$ is 2.

(ii) For each prime $p \geq 7$ the Kurepa’s determinant $K_p$ satisfies the condition $K_p \not\equiv 0 \pmod{p}$.

**Remarks 2.** In order to evaluate the Kurepa’s determinant $K_p$ modulo a prime $p \geq 11$, we apply numerous elementary transformations, and work simultaneously modulo $p$. Then we obtain the following result which in view of (1) of Remarks 1 gives an indirect proof of Theorem 1 ([12]; also see the first congruence in the proof of Proposition 4 in Section 4 with $n = p$).

**Proposition 1.** If $p$ is a prime greater than 5, then

$$K_p \equiv \frac{1}{8} \sum_{k=0}^{p-1} (-1)^k \frac{(-1)^{n-k}}{k!}.$$  

Finally, we propose the following conjecture which in view of Theorem 1 may be considered as the strong Kurepa’s hypothesis.

**Conjecture 2 (The strong Kurepa’s hypothesis).** For each integer $n \geq 7$ the Kurepa’s determinant $K_n$ is not divisible by $n$.

If in the expression (4) for $K_n$ we replace every odd element by 1 and every even element by 0, we obtain the following definition.

**Definition 2.** For any integer $n \geq 7$ the Kurepa’s binary determinant $K'_n$ is the determinant of order $n-4$ defined as

$$
K'_n := \begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & \ldots & 1 & 1 & 1 & 0 \\
: & : & : & : & : & \ldots & : & : & : & : \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & (1 - (-1)^n)/2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 
\end{vmatrix}.
$$

(Here $(1 - (-1)^n)/2 = 1$ if $n$ is odd, and $(1 - (-1)^n)/2 = 0$ otherwise.)
Remarks 3. A computation gives the following few values of Kurepa’s binary determinant $K'_n$: $K'_7 = 1$, $K'_8 = 1$, $K'_9 = -1$, $K'_{10} = -1$, $K'_{11} = 1$, $K'_{12} = 1$, $K'_{13} = -1$, $K'_{14} = -1$, $K'_{15} = 1$, $K'_{16} = 1$, $K'_{17} = -1$, $K'_{18} = -1$, which suggests the following result.

Proposition 2. $K'_{2n} = K'_n - 1 = (-1)^n$ for all $n \geq 4$.

As an immediate consequence of Proposition 2 whose proof is given in Section 6, we get the following result.

Corollary 1. For each integer $n \geq 7$ the Kurepa’s determinant $K'_n$ is an odd integer.

Proof of Corollary 1. Using the obvious fact that $K'_n \equiv K'_n (\text{mod } 2)$ for all $n \geq 7$, by Proposition 2 we have $K'_n \equiv 1 (\text{mod } 2)$ for all $n \geq 15$. This together with the fact that $K'_n$ is odd for $7 \leq n \leq 14$ yields the assertion. \(\square\)

Obviously, Corollary 1 implies the truth of Conjecture 2 for all even integers $n \geq 8$, that is, we have the following statement.

Theorem 2. The strong Kurepa’s hypothesis holds for each even integer $n \geq 8$.

On the other hand, in Section 4 we show that “the odd composite part” of strong Kurepa’s hypothesis is not true, that is, we prove the following result.

Theorem 3. For $n = 11563 = 31 \times 373$ we have $K'_{11563} \equiv 0 (\text{mod } 11563)$. Therefore, the strong Kurepa’s hypothesis does not hold for each odd composite integer $n \geq 9$.

Hence, our results concerning the strong Kurepa’s hypothesis may be summarized as follows.

The strong Kurepa’s hypothesis can be divided into the following three parts.

• The “prime” part which asserts that $K'_p \not\equiv 0 (\text{mod } p)$ for each prime $p > 5$. This part is by Proposition 1 and Theorem 1 equivalent to Kurepa’s hypothesis (Conjecture 1).

• The “even part” which asserts that $K'_n \not\equiv 0 (\text{mod } n)$ for each even integer $n \geq 8$. This part is true by Theorem 2.

• The “odd composite part” which asserts that $K'_n \not\equiv 0 (\text{mod } n)$ for each odd composite integer $n \geq 9$. This part is disproved by Theorem 3.

3. KUREPA HYPOTHESIS AND DERANGEMENT NUMBERS

Let us consider the derangement numbers $S_n$ ($n = 0, 1, 2, \ldots$) defined as

$$S_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$  

The following result is itself interesting.

Proposition 3. Let $n \geq 4$ be a composite positive integer, and let $d \geq 2$ be any proper divisor of $n$ with $n = ad$. Then

$$S_{n-1} \equiv (-1)^{n+d} S_{d-1} \pmod{d}.$$  

Proof. Take $n = ad$ with a positive integer $a$. Notice that by (7) $S_{n-1}$ can be written as

$$S_{n-1} = \sum_{k=0}^{ad-1} (-1)^k (k+1)(k+2) \cdots (ad-1).$$

Notice that the set $D_k = \{k+1, k+2, \ldots, ad-2, ad-1\}$ contains an integer which is divisible by $d$ whenever $k+1 \leq (a-1)d$. Using this fact from (9) we find that

$$S_{n-1} \equiv \sum_{k=0}^{ad-1} (-1)^k (k+1)(k+2) \cdots (ad-1) \pmod d

\equiv \sum_{j=0}^{d-1} (-1)^{j+(a-1)d} (j+1)(j+2) \cdots (d-1) \pmod d

= (-1)^{(a-1)d} \sum_{j=0}^{d-1} (-1)^j (j+1)(j+2) \cdots (d-1)

= (-1)^{n-d} S_{d-1} = (-1)^{n+d} S_{d-1},$$

as desired. □

We are now ready to extend Theorem 2.1 of [7] and our Theorem 1 as follows.

**Theorem 4.** The following statements are equivalent:

(i) Kurepa’s hypothesis holds.

(ii) $S_{p-1} \not\equiv 0 \pmod p$ for each prime $p \geq 3$.

(iii) $S_{n-1} \not\equiv 0 \pmod n$ for each integer $n \geq 3$.

(iv) For each integer $n \geq 3$ the numerator of the fraction

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{k!}$$

written in reduced form is not divisible by $n$.

Proof. As noticed above, the equivalence $(i) \iff (ii)$ was attributed by Mijajlović [13].

The equivalence $(ii) \iff (iii)$ obviously follows from Proposition 3.

To complete the proof it is suffices to show the implications $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (ii)$. First suppose that $(iii)$ is satisfied. For any fixed $n \geq 3$ let $a$ and $b$ be relatively prime positive integers such that

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} = \frac{a}{b}.$$

Then notice that $c = (n-1)!/b$ is an integer and $S_{n-1} = (n-1)!a/b = ac$. From this and the fact that by $(iii)$, $S_{n-1} \not\equiv 0 \pmod n$, it follows that $ac$ is not divisible by $n$. Therefore, $a$ is not also divisible by $n$, which yields the assertion $(iv)$.

Finally, if $(iv)$ is satisfied, then for any prime $p \geq 3$ set

$$\sum_{k=0}^{p-1} \frac{(-1)^k}{k!} = \frac{a}{b},$$
where \(a\) and \(b\) are relatively prime positive integers such that \(a \not\equiv 0 \pmod{p}\) and \(b \not\equiv 0 \pmod{p}\). Using this, by Wilson theorem we have

\[
S_{p-1} = (p - 1)! \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} = (p - 1)! \frac{a}{b} \equiv -\frac{a}{b} \pmod{p} \neq 0 \pmod{p}.
\]

This yields the assertion (ii) and the proof is completed. \(\square\)

**Corollary 2.** Let \(q_1, q_2, \ldots, q_l\) be odd distinct primes, let \(e_1, e_2, \ldots, e_l\) be positive integers and let \(r\) be a nonnegative integer such that \(S_{q_i^{e_i} - 1} \equiv r \pmod{q_i^{e_i}}\) for all \(i = 1, 2, \ldots, l\). Then for \(n = q_1^{e_1} q_2^{e_2} \cdots q_l^{e_l}\) there holds

\[
S_{n-1} \equiv r \pmod{n}.
\]

In particular, \(S_{n-1} \equiv 0 \pmod{n}\) if and only if \(S_{q_i^{e_i} - 1} \equiv 0 \pmod{q_i^{e_i}}\) for all \(i = 1, 2, \ldots, l\).

**Proof.** The congruence (10) immediately follows from the fact that by the congruence (8) of Proposition 3 for all \(i = 1, 2, \ldots, l\) we have

\[
S_{n-1} \equiv S_{q_i^{e_i} - 1} \pmod{q_i^{e_i}} \equiv r \pmod{q_i^{e_i}}.
\]

\(\square\)

Similarly, Proposition 3 yields the following result.

**Corollary 3.** Let \(n\) be an even positive integer with the prime factorization \(n = 2^e q_1^{e_1} q_2^{e_2} \cdots q_l^{e_l}\). If \(S_{2^e - 1} \equiv r \pmod{2^e}\) and \(S_{q_i^{e_i} - 1} \equiv -r \pmod{q_i^{e_i}}\) for all \(i = 1, 2, \ldots, l\), then

\[
S_{n-1} \equiv r \pmod{n}.
\]

Notice that Table 1 of [28] contains all primes \(p < 2^{23} = 8388608\) such that \(\min\{r_p, p - r_p\} \leq 10\), where \(r_p := p \pmod{p}\). Using Mathematica 8 we obtain only the following four prime powers \(p^e\) with \(e \geq 2\) less than 100000 such that \(\min\{r_{p^e}, p^e - r_{p^e}\} \leq 2: \{2^2, 3^2, 5^2, 7^2\}\). Using this, the set

\[
\{2, 3, 5, 7, 11, 23, 31, 67, 227, 373, 10331\}
\]

of all primes \(p\) less than 100000 of Table 1 in [28] for which \(\min\{r_p, p - r_p\} \leq 2\), Corollaries 2 and 3, we immediately obtain Table 1.
Table 1. Integers $n$ with $2 \leq n < 100000$ for which $S_{n-1} \equiv r_n \pmod{n}$ with $r_n \in \{-2, -1, 0, 1, 2\}$ and/or with related values $|r_n/n| \leq 10^{-3}$

| $n$ | factorization of $n$ | $r_n$ | $|r_n/n| \cdot 10^3$ | $n$ | factorization of $n$ | $r_n$ | $|r_n/n| \cdot 10^3$ |
|-----|----------------------|------|---------------------|-----|----------------------|------|---------------------|
| 2   | $2$                  | 0    | 0                   | 681 | $3 \times 227$       | -2   | > 1                 |
| 3   | $3$                  | 1    | > 1                 | 746 | $2 \times 373$       | -2   | > 1                 |
| 4   | $2^2$                | 2    | > 1                 | 804 | $2^2 \times 3 \times 67$ | 2    | > 1                 |
| 5   | 5                    | -1   | > 1                 | 908 | $2^2 \times 227$     | 2    | > 1                 |
| 6   | $2 \times 3$         | 2    | > 1                 | 1362| $2 \times 3 \times 227$ | 2    | > 1                 |
| 7   | 7                    | -1   | > 1                 | 1492| $2^2 \times 373$     | -2   | > 1                 |
| 8   | $2^3$                | -2   | > 1                 | 1541| $23 \times 67$       | -2   | > 1                 |
| 9   | $3^2$                | 1    | > 1                 | 2724| $2^2 \times 3 \times 227$ | 2    | 0.734214            |
| 11  | 11                   | 1    | > 1                 | 2984| $3^3 \times 373$     | -2   | 0.670241            |
| 12  | $2^2 \times 3$       | 2    | > 1                 | 3082| $2 \times 23 \cdot 67$ | 2    | 0.648929            |
| 23  | 23                   | -2   | > 1                 | 4623| $3 \times 23 \times 67$ | -2   | 0.432619            |
| 31  | 31                   | 2    | > 1                 | 5221| $23 \times 227$      | -2   | 0.383068            |
| 33  | $3 \times 11$        | 1    | > 1                 | 6164| $2^2 \times 23 \times 67$ | 2    | 0.324464            |
| 35  | $5 \times 7$         | -1   | > 1                 | 9246| $2 \times 23 \times 227$ | 2    | 0.216309            |
| 46  | $2 \times 23$        | 2    | > 1                 | 10331| $10331$           | -2   | 0.193592            |
| 49  | $7^2$                | -1   | > 1                 | 10442| $2 \times 23 \times 227$ | 2    | 0.191534            |
| 62  | $2 \times 31$        | -2   | > 1                 | 11563| $31 \times 373$     | 2    | 0.172965            |
| 67  | 67                   | -2   | > 1                 | 15209| $67 \times 227$     | -2   | 0.131501            |
| 69  | $3 \times 23$        | -2   | > 1                 | 15663| $3 \times 23 \times 227$ | -2   | 0.127689            |
| 92  | $2^2 \times 23$      | 2    | > 1                 | 18492| $2^2 \times 23 \times 227$ | 2    | 0.108154            |
| 99  | $3^2 \times 11$      | 1    | > 1                 | 20662| $2 \times 10331$    | 2    | 0.096796            |
| 124 | $2^2 \times 31$      | -2   | > 1                 | 20884| $2^2 \times 23 \times 227$ | 2    | 0.095767            |
| 134 | $2 \times 67$        | 2    | > 1                 | 23126| $2 \times 31 \times 373$ | 2    | 0.086482            |
| 138 | $2 \times 3 \times 23$ | 2    | > 1                 | 30418| $2 \times 67 \times 227$ | 2    | 0.065750            |
| 201 | $3 \times 67$        | -2   | > 1                 | 30993| $3 \times 10331$    | -2   | 0.064530            |
| 227 | 227                  | -2   | > 1                 | 31326| $2 \times 3 \times 23 \times 227$ | 2    | 0.063844            |
| 245 | $5 \times 7^2$       | -1   | > 1                 | 41324| $2^2 \times 10331$  | 2    | 0.048398            |
| 248 | $2^3 \times 31$      | -2   | > 1                 | 45627| $3 \times 67 \times 227$ | -2   | 0.043833            |
| 268 | $2^2 \times 67$      | 2    | > 1                 | 46252| $2^2 \times 31 \times 373$ | -2   | 0.043241            |
| 276 | $2^2 \times 3 \times 23$ | 2    | > 1                 | 60836| $2^2 \times 67 \times 227$ | 2    | 0.032875            |
| 373 | 373                  | 2    | > 1                 | 61986| $2 \times 3 \times 10331$ | 2    | 0.032265            |
| 402 | $2 \times 3 \times 67$ | 2    | > 1                 | 62652| $2^2 \times 3 \times 23 \times 227$ | 2    | 0.031922            |
| 454 | $2 \times 227$       | 2    | > 1                 | 91254| $2 \times 3 \times 67 \times 227$ | 2    | 0.021916            |
|     |                     |      |                     | 92504| $2^3 \times 31 \times 373$ | -2   | 0.021620            |

Remarks 4. If $n \geq 9$ is an odd composite integer, then by (8) for any divisor $d \geq 3$ of $n$ we have

$$S_{n-1} \equiv S_{d-1} \pmod{d}.$$  

4. THE ODD COMPOSITE PART OF STRONG KUREPA’S HYPOTHESIS IS NOT TRUE

The odd part of strong Kurepa’s hypothesis (Conjecture 2) asserts that $K_n \not\equiv 0 \pmod{n}$ for each odd composite integer $n \geq 9$. The residues $s_n := -8K_n \pmod{n}$ with $|s_n| \leq 10$ for $n < 2500$, including the corresponding residues $r_n := S_{n-1} \pmod{n}$ are presented in Table 2 (cf. Table 3).

Table 2. The odd integers $n$ with $7 \leq n < 2500$ for which $-8K_n \equiv s_n \pmod{n}$ with $s_n \in \{-10, -9, \ldots, -1, 0, 1, \ldots, 9, 10\}$ and related values $r_n := S_{n-1} \pmod{n}$
| $n$ | factorization of $n$ | $s_n$ | $r_n$ |
|-----|---------------------|-------|-------|
| 7   | $7$                 | -1    | -1    |
| 9   | $3^2$               | -1    | 1     |
| 11  | $11$                | 1     | 1     |
| 15  | $3 \times 5$       | 2     | 4     |
| 21  | $3 \times 7$       | -10   | -8    |
| 23  | $23$                | -2    | -2    |
| 27  | $3^3$               | 8     | 10    |
| 31  | $31$                | 2     | -2    |
| 33  | $3 \times 11$      | -1    | 1     |
| 35  | $3 \times 5$       | -3    | -1    |
| 39  | $3 \times 13$      | 8     | 10    |
| 49  | $7^2$               | -3    | -1    |
| 63  | $3^2 \times 7$     | -10   | -8    |
| 67  | $67$                | 1     | 1     |
| 69  | $3 \times 23$      | -2    | -2    |
| 95  | $5 \times 19$      | 7     | 9     |
| 99  | $3^2 \times 11$    | -1    | 1     |
| 117 | $3^2 \times 13$    | 8     | 10    |
| 121 | $11^2$              | 10    | 12    |
| 123 | $3 \times 41$      | 2     | 4     |
| 201 | $3 \times 67$      | -4    | -2    |
| 205 | $5 \times 41$      | 2     | 4     |
| 227 | $227$               | -2    | -2    |
| 245 | $5 \times 7^2$     | -3    | -1    |
| 351 | $3^3 \times 13$    | 8     | 10    |
| 373 | $373$               | 2     | 2     |
| 417 | $3 \times 139$     | -7    | -5    |
| 453 | $3 \times 151$     | 8     | 10    |
| 489 | $3 \times 163$     | 2     | 4     |
| 615 | $3 \times 5 \times 41$ | 2 | 4 |
| 681 | $3 \times 227$     | -4    | -2    |
| 815 | $5 \times 163$     | 2     | 4     |
| 831 | $3 \times 277$     | 5     | 7     |
| 923 | $13 \times 71$     | -5    | -3    |
| 985 | $5 \times 197$     | 7     | 9     |
| 1541| $23 \times 67$     | -4    | -2    |
| 1745| $5 \times 349$     | -8    | -6    |
Table 3. The values \( K_n, S_{n-1} \) and \((8K_n + S_{n-1})(\text{mod } n)\) for \(7 \leq n \leq 21\)

| \(n\) | \(K_n\) | \(S_{n-1}\) | \((8K_n + S_{n-1})(\text{mod } n)\) |
|------|-------|-----------|-------------------------------|
| 7    | 15    | 265       | 0                            |
| 8    | -47   | 1854      | -2                           |
| 9    | 197   | 14833     | 2                            |
| 10   | -1029 | 133496    | 4                            |
| 11   | 6439  | 133496    | 0                            |
| 12   | -46927| 14684570  | 6                            |
| 13   | 390249| 176214841 | 0                            |
| 14   | -3645737| 2290792932| 4                            |
| 15   | 37792331| 32071101049| 2                            |
| 16   | -430400211| 481066515734| -2                           |
| 17   | 5341017373| 7697064251745| 0                            |
| 18   | -71724018781| 130850092279664| 0                            |
| 19   | 1036207207363983| 2355301661033953| 0                            |
| 20   | -160241769757479| 44750731559645106| -6                           |
| 21   | 264083895859409| 895014631192902121| 2                            |

Table 2 suggests the following congruence.

**Proposition 4.** For each odd composite integer \(n \geq 9\) there holds
\[
8K_n \equiv -S_{n-1} + 2 \pmod{n}.
\]

**Proof.** It is proved in [12] that for each odd integer \(n \geq 7\)
\[
K_n \equiv -3S_n - 1 + (n - 7)! \cdot 180 \pmod{n}.
\]

If \(n \geq 9\) is an odd composite integer, then it is easy to see that \((n - 7)! \cdot 180 \equiv 0 \pmod{n}\), which substituting into above congruence yields
\[
K_n \equiv -3S_n - 1 \pmod{n},
\]
or multiplying by 8,
\[
8K_n \equiv -24S_n - 8 \pmod{n}.
\]

From the recurrence relation \(S_m = mS_{m-1} + (-1)^m\) with \(m = n - 1\) we obtain
\[
S_{n-1} = (n - 1)S_{n-2} + 1 \equiv -S_{n-2} + 1 \pmod{n}.
\]

Iterating this three times, we find
\[
8K_n \equiv -S_{n-1} + 2 \pmod{n},
\]
as desired.

**Proof of Theorem 3.** As an immediate consequence of Proposition 4, we immediately obtain that the “odd composite part” of strong Kurepa’s hypothesis is equivalent to
\[
S_{n-1} \not\equiv 2 \pmod{n} \text{ for each odd composite integer } n \geq 9.
\]

However, from Table 1 we see that \(11563 = 11 \times 373\) satisfies the congruence
\[
S_{11562} \equiv 2 \pmod{11563}
\]
which by the congruence of Proposition 4 implies that
\[
K_{11563} \equiv 0 \pmod{11563},
\]
as asserted.

**Remarks 5.** If \(n \geq 9\) is an odd composite integer with the prime factorization \(n = q_1^{e_1}q_2^{e_2} \cdots q_l^{e_l}\) then by Corollary 2, \(S_{n-1} \equiv 2 \pmod{n}\) if and only if \(S_{q_i^{e_i}-1} \equiv 2 \pmod{q_i^{e_i}}\) for each \(i = 1, 2, \ldots, l\). By [28 Table 1] we know that 31 and 373 are the only primes less than \(2^{23} = 8388608\) satisfying the congruence \(S_{p-1} \equiv 2 \pmod{p}\). Moreover, \(S_{31^{2}-1} \equiv 467 \not\equiv 2 \pmod{31^{2}}\) and \(S_{373^{2}-1} \equiv 2613 \not\equiv 2 \pmod{373^{2}}\). These
facts and Proposition 3 show that \( n = 11563 = 11 \times 373 \) is the only odd composite positive integer (i.e., a counterexample to the odd part of strong Kurepa’s hypothesis) less than \( 2^{23} = 8388608 \) for which \( S_{n-1} \equiv 2 \pmod{n} \). From Table 1 we also see that there exist 31 even positive integers less than 100000 satisfying the congruence \( S_{n-1} \equiv 2 \pmod{n} \).

5. **Kurepa’s Hypothesis and Bell Numbers**

Recall that the derangement numbers \( S_n \) considered in the previous section are closely related to the Bell numbers \( B_n \) given by the recurrence

\[
B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad n = 0, 1, 2, \ldots,
\]

with \( B_0 = 1 \) (see e.g., [5, p. 373]). \( B_n \) gives the number of partitions of a set of cardinality \( n \). This is Sloane’s sequence A000110 in [22] whose terms \( B_0, B_1, \ldots, B_8 \) are as follows: 1, 1, 2, 5, 15, 52, 203, 877, 4140.

It is known (see e.g., [25, Corollary 1.3]) that for any prime \( p \) we have

\[
(11) \quad B_{p-1} - 1 \equiv S_{p-1} \pmod{p}.
\]

The congruence (11) and the equivalence \((i) \iff (ii)\) of Theorem 3 yields that Kurepa’s hypothesis is also equivalent with the statement that

\[
B_{p-1} \not\equiv 1 \pmod{p} \quad \text{for each prime } p \geq 3.
\]

The idea of the proof of Kurepa’s hypothesis given by D. Barsky and B. Benzaghou [1, Théorème 3, p. 13] (for a related discussion see B. Sury [26, Section 4]) is to consider what is known as the Artin-Schreier extension \( \mathbb{F}_p[\theta] \) of the field \( \mathbb{F}_p \) of \( p \) elements, where \( \theta \) is a root (in the algebraic closure of \( \mathbb{F}_p \)) of the polynomial \( x^p - x - 1 \). This is a cyclic Galois extension of degree \( p \) over \( \mathbb{F}_p \). Note that the other roots of \( x^p - x - 1 \) are \( \theta + i \) for \( i = 1, 2, \ldots, p - 1 \). The reason this field extension comes up naturally as follows. The generating series \( F(x) \) of the Bell numbers can be evaluated modulo \( p \); this means one computes a “simpler” series \( F_p(x) \) such that \( F(x) - F_p(x) \) has all coefficients multiples of \( p \), where

\[
F(x) = \sum_{n=0}^{\infty} B_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{(1-x)(1-2x) \cdots (1-nx)}
\]

is the generating function for \( B_n \)’s. Since Kurepa’s hypothesis is about the Bell numbers \( B_{p-1} \) considered modulo \( p \), it makes sense to consider \( F_p(x) \) rather than \( F(x) \). By using this idea, D. Barsky and B. Benzaghou [1, Théorème 3, p. 13] proved that \( B_{p-1} \not\equiv 1 \pmod{p} \) for any prime \( p \). However, as noticed above, this proof contains some irreparable calculation errors [2].

In view of the previous mentioned formulation of Kurepa’s hypothesis in terms of Bell numbers and the equivalence \((i) \iff (iii)\) of Theorem 3, it can be of interest to determine \( B_{n-1}(\pmod{n}) \) for composite integers \( n \). A computation in Mathematica 8 shows that there are certain positive integers \( n \) such that \( B_{n-1} \equiv 1 \pmod{n} \). All these values of \( n \) less than 20000 are 2, 4 = 2\(^2\), 16 = 2\(^4\), 28 = 2\(^2\)·7, 46 = 2·23, 134 = 2·67, 454 = 2·227, 1442 = 2·7·103, 1665 = 3\(^2\)·5·37 and 4252 = 2\(^2\)·1063. Notice also that for these values of \( n \) the residues \( S_{n-1}(\pmod{n}) \) are respectively as follows: 0, 2, 6, -6, 2, -2, 2, 568, -476 and 22.

We propose the following conjecture.
**Conjecture 3.** There are infinitely many positive integers \( n \) such that

\[
B_n \equiv 1 \pmod{n}.
\]

Recently, Z.-W. Sun and D. Zagier [25, Theorem 1.1] proved that for every positive integer \( m \) and any prime \( p \) not dividing \( m \) we have

\[
\left( \frac{B_k}{-m} \right) \equiv (-1)^{m-1} S_{m-1} \pmod{p} \tag{12}
\]

By using the congruence (12) easily follows the following result.

**Proposition 5.** An odd prime \( p \) is a counterexample to Kurepa’s hypothesis if and only if in the field \( \mathbb{F}_p \) there holds

\[
\left( \frac{B_0}{1} \right) = \left( \frac{-D_0}{1} \right) = \left( \frac{-D_1}{2} \right) = \left( \frac{-D_2}{2} \right), \tag{13}
\]

or equivalently,

\[
\left( \begin{array}{cccc}
1 & (p-1)^{p-2} & (p-1)^{p-3} & \cdots & (p-1) \\
1 & (p-2)^{p-2} & (p-2)^{p-3} & \cdots & (p-2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^{p-2} & 2^{p-3} & \cdots & 2 \\
1 & 1 & 1 & \cdots & 1 \\
\end{array} \right)
\left( \begin{array}{c}
B_0 \\
B_1 \\
B_2 \\
B_{p-2} \\
\end{array} \right)
=
\left( \begin{array}{c}
D_0 \\
-D_1 \\
D_2 \\
-D_{p-2} \\
\end{array} \right), \tag{14}
\]

**Proof.** First observe that by Fermat little theorem (cf. [25, Proof of Corollary 1.2]),

\[
\sum_{k=1}^{p-1} (p-i)^{p-k}(p-j)^{k-1} \equiv \sum_{k=1}^{p-1} \left( \frac{p-i}{p-j} \right)^{k-1} \equiv \begin{cases} -1 \pmod{p} & \text{if } i = j; \\
0 \pmod{p} & \text{if } 1 \leq i \neq j \leq p-1. \end{cases}
\]

The above congruence shows that for \((p-1) \times (p-1)\) matrices \( A = ((p-i)^{p-j})_{1 \leq i,j \leq p-1} \) and \( B = ((p-j)^{i-1})_{1 \leq i,j \leq p-1} \) from the left hand sides of (13) and (14), respectively, we have \( A \cdot B = -I_{p-1} \), where \( I_{p-1} \) is the identity matrix of order \( p-1 \). This shows that \( B = -A^{-1} \) and therefore, (13) yields (14).

In order to prove (13), by using the congruence (12) and Fermat little theorem, for any odd prime \( p \) and each \( m = 1, 2, \ldots, p-1 \) we find that

\[
\sum_{k=0}^{p-2} \frac{B_k}{(-m)^k} \equiv (-1)^{m-1} S_{m-1} + B_0 - \frac{B_{p-1}}{(-m)^{p-1}} \pmod{p} \tag{15}
\]

As noticed above, an odd prime \( p \) is a counterexample to Kurepa’s hypothesis if and only if \( B_{p-1} \equiv 1 \pmod{p} \), which substituting together with \( B_0 = 1 \) into (15) gives

\[
\sum_{k=0}^{p-2} \frac{B_k}{(-m)^k} \equiv (-1)^{m-1} S_{m-1} \pmod{p}, \quad m = 1, 2, \ldots, p-1. \tag{16}
\]
Since by Fermat little theorem, $1/(-m)^k \equiv 1/(p-m)^k \equiv (p-m)^{p-1-k} \pmod p$ for all pairs $(m, k)$ with $1 \leq m \leq p-1$ and $0 \leq k \leq p-2$, the congruences (16) can be written as

$$\sum_{k=0}^{p-2} (p-m)^{p-1-k}B_k \equiv (-1)^{m-1}S_{m-1} \pmod p, \ m = 1, 2, \ldots, p-1.$$  \hspace{1cm} (17)

Finally, observe that the set of $(p-1)$ congruences modulo $p$ given by (17) is equivalent with the matrix equality (13) in the field $\mathbb{F}_p$.

**Remarks 6.** Notice that $(p-1) \times (p-1)$ matrix $A = ((p-i)^{p-j})_{1 \leq i, j \leq p-1}$ (in the field $\mathbb{F}_p$) on the left hand side of (13) is a Vandermonde-type matrix. Namely, interchanging $j$th column and $(p+1-j)$th column of $A$ for each $j = 2, 3, \ldots, (p-1)/2$ ($(p+1)$th column of $A$ remains fixed), the matrix $A$ becomes the Vandermonde matrix $A' = ((p-i)^j)_{1 \leq i, j \leq p-1}$.

Hence, $\det(A) = (-1)^{(p-3)/2}\det(A') = (-1)^{(p-3)/2} \prod_{1 \leq i < j \leq p-1}(p-j) - (p-i))$

whence by using Wilson theorem it can be easily show that

$$\det(A) \equiv (-1)^{(p^2-1)/8} \left( \frac{p-1}{2} \right)! \pmod p. \hspace{1cm} (18)$$

In particular, if $p \equiv 3 \pmod 4$, then by a congruence of Mordell [17], (18) implies that

$$\det(A) \equiv (-1)^{(p^2-1)/8 + (h(-p)+1)/2} \pmod p,$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$. Moreover, if $p \equiv 1 \pmod 4$, then applying Wilson theorem (18) yields

$$(\det(A))^2 \equiv -1 \pmod p.$$  \hspace{1cm} 6. Proof of Proposition 2

In order to prove Proposition 2, we will need the following lemma.

**Lemma 1.** For any integer $n \geq 3$ let $D_n$ be the determinant of order $n$ defined as

$$D_n = \left| \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 \1 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \0 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 \0 & 0 & 1 & 0 & 1 & 1 & \ldots & 1 \0 & 0 & 0 & 1 & 1 & 1 & \ldots & 1 \0 & 0 & 0 & 0 & 0 & 1 & \ldots & 1 \ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{array} \right| \equiv \left( 1 - (-1)^n \right) / 2.$$

(19)
(For example, $D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$, $D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$, $D_5 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$).

Then $D_3 = -1$ and

(20) $D_{2n} = D_{2n+1} = (-1)^n$

for all $n \geq 2$.

Proof. If $n \geq 2$, then subtracting the 2nd column from the $(2n - 1)$th column of $D_{2n}$, and thereafter expanding the determinant along the 2nd (last) column, we find that

$$D_{2n} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \ldots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -1 \end{vmatrix} = -D_{2n-1}.$$  

Similarly, if $n \geq 3$, then subtracting the $(2n - 1)$th column from the $(2n - 2)$th column of $D_{2n-1}$, then expanding the determinant along the $(2n - 1)$th (last) row, and
thereafter expanding the determinant along the \((2n - 2)\)th (last) column, we find that

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & \ldots & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
\end{vmatrix} = -D_{2n-1}
\]

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & \ldots & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\
\end{vmatrix} = -D_{2n-3}.
\]

From (22) and the fact that \(D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1\) it follows that for \(n \geq 2\)

\[
D_{2n-1} = (-1)^{n-1},
\]

which substituting in (21) gives \(D_{2n} = (-1)^n\). This shows that \(D_{2n} = D_{2n+1} = (-1)^n\) for all \(n \geq 2\), as desired. \(\square\)

**Proof of Proposition 2.** By (6) of Definition 2 with \(2n\) instead of \(n\) we have

\[
K_{2n}' = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & \ldots & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \end{vmatrix},
\]
(the determinant $K'_{2n}$ is of order $2n - 4$), whence after the expansion along the $(2n - 4)$th (last) column, and then expanding this along the $(2n - 5)$th (last) row, for all $n \geq 5$ we obtain

$$K'_{2n} = -D_{2n-6},$$

where the determinant $D_{2n-6}$ is defined by (19). Then the above equality and (20) of Lemma 1 yield

$$(23) \quad K'_{2n} = -D_{2(n-3)} = -(-1)^{n-3} = (-1)^n \quad \text{for all } n \geq 5.$$

Further, by (6) of Definition 2 with $2n - 1$ instead of $n$ we have

$$K'_{2n-1} =$$

$$(\text{the determinant } K'_{2n-1} \text{ is of order } 2n - 5), \text{ whence after the expansion along the } (2n - 5)\text{th (last) column, and then expanding this along the } (2n - 6)\text{th (last) row, for}$$
all \( n \geq 5 \) we obtain

\[
K_{2n-1}' = \begin{vmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & \ldots & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & \ldots & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
\end{vmatrix} = D_{2n-7},
\]

where the determinant \( D_{2n-7} \) is defined by (19). Then the above equality and (20) of Lemma 1 yield

\[ K_{2n-1}' = D_{2(n-4)+1} = (-1)^{n-4} = (-1)^n \quad \text{for all} \quad n \geq 5. \]

Finally, the equalities (23) and (24) and the fact that \( K_7' = K_8' = 1 \) yield the assertion of Proposition 2.

\[ \square \]

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