(F,Dp) bound states from the boundary state

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Abstract

We use the boundary state formalism to provide the full conformal description of (F,Dp) bound states. These are BPS configurations that arise from a superposition of a fundamental string and a Dp brane, and are charged under both the NS-NS antisymmetric tensor and the (p + 1)-form R-R potential. We construct the boundary state for these bound states by switching on a constant electric field on the world-volume of a Dp brane and fix its value by imposing the Dirac quantization condition on the charges. Using the operator formalism we also derive the Dirac-Born-Infeld action and the classical supergravity solutions corresponding to these configurations.

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1 Introduction

The boundary state, originally introduced for factorizing the planar and non-planar open string one-loop diagrams in the closed string channel \[1\], has been lately very useful for describing the D branes \[2\] in the framework of string theories \[1\]. This is because the boundary state encodes all relevant properties of the D branes; in fact, as shown in Ref. \[3\], it reproduces the couplings of the D branes with the massless closed string states as dictated by the Dirac-Born-Infeld action, and also generates the large distance behavior of the classical D brane solutions of supergravity.

The boundary states which are usually considered in the literature describe simple D branes, namely extended objects that are charged under only one R-R potential. However, one expects that also more general configurations admit a stringy description by means of boundary states with a richer structure. For example, in Ref. \[5\] it has already been shown that a boundary state with an external magnetic field describes a Dp-D(p−2) bound state of two D branes. In this paper, instead, we consider boundary states with an external electric field and show that they provide the complete conformal description of the bound states between a fundamental string and a Dp brane denoted by (F,Dp) \[6\]-\[12\]. These bound states, which are a generalization of the dyonic strings introduced in Ref. \[13\], are p-dimensional extended objects which are charged under both the NS-NS two-form potential and the R-R (p + 1)-form potential of the Type II theories. Because of this property, they behave at the same time both as Dp branes and as fundamental strings. Their nature of D branes allows us to represent them by means of boundary states, while their being also fundamental strings shows up through the presence of an external electric field on their world-volume.

To describe the (F,Dp) bound states we introduce a boundary state containing two parameters: an overall constant \(x\) and the constant value of the electric field \(f\) which can always be taken to be along one longitudinal direction only. Then, by requiring the validity of the Dirac quantization condition, we show that these two parameters can be uniquely fixed in terms of a pair of integers \(m\) and \(n\) representing, respectively, the charges of the NS-NS antisymmetric tensor and of the R-R (p + 1)-form potential. By projecting this boundary state onto the massless states of the closed string spectrum \[5\], we can obtain the long distance behavior of the massless fields that characterize this configuration; then we can infer the complete classical solution describing the (F,Dp) bound states and find agreement with the results recently obtained in Ref. \[10\], which for \(p = 1\) reduce to the dyonic strings of

\[1\]For references on the use of the boundary state to study the D branes and their interactions see for example Refs. \[3, 4\].
Ref. [13]. Using the boundary state formalism, we also compute the interaction energy between two (F,Dp) bound states, and check that the no force condition holds at the full string level. This fact confirms that the boundary state with an external electric field provides the complete stringy description of the BPS bound states formed by fundamental strings and Dp branes.

This paper also contains a discussion of the limits in which one of the two charges vanishes. While the limit $m \to 0$, corresponding to a vanishing electric field, is perfectly under control because in this case the (F,Dp) bound state reduces just to a Dp brane, the other limit $n \to 0$ is more subtle because in this case the (F,Dp) bound state reduces to the fundamental string and one does not expect that the latter admits a boundary state description. Actually, when $n \to 0$ the boundary state is not well defined since it contains an overall vanishing prefactor and a divergent exponential factor involving harmonic oscillators. However, when we project it onto the massless closed string states, the vanishing and the divergent factors cancel each other and one is left with a finite and well-defined expression that exactly reproduces the large distance behavior of the fundamental string solution. Motivated by this observation, we propose a modified form of boundary state that generates the fundamental string solution in the same way as the standard boundary state does for the D branes. This operator can be regarded as an effective conformal description of the fundamental string, which, however, cannot be the complete one.

This paper is organized as follows. In Section 2 we write the boundary state with an external constant gauge field using the formalism developed in Ref. [3]. Then, by projecting it along the massless states of the closed string spectrum, we derive the Dirac-Born-Infeld action with its Wess-Zumino term. In Section 3 we consider the boundary state for a Dp brane with a constant external electric field, fix its form by imposing the Dirac quantization condition on the charges of the NS-NS and R-R potentials, and then obtain the corresponding classical solutions describing the (F,Dp) bound states. In Section 4 we use the boundary state to compute the interaction energy between two (F,Dp) bound states and show the validity of the no-force condition at the string level. In Section 5 we consider in more detail the case $p = 1$ and, after introducing non vanishing asymptotic values for the scalar fields of the Type IIB theory, show that our boundary state reproduces the dyonic string solutions. Finally, in Section 6 we discuss the limit $n \to 0$ in which the bound state (F,Dp) reduces to a fundamental string. We conclude our paper with two appendices containing more technical details. In Appendix A, we introduce the projectors along the massless closed string states and derive some of the formulas used in this paper, while in Appendix B by performing a T-duality transformation we obtain the boundary state corresponding to the bound states (W,Dp) between Kaluza-Klein waves and Dp branes recently discussed in Ref. [10].
2 The boundary state with an external field and the D-brane effective action

In this section we are going to briefly review the construction of the boundary state for a D-brane with an external field $F$ on its world-volume. We then show how to use this boundary state to derive the D-brane low-energy effective action.

2.1 The boundary state with an external field

In the closed string operator formalism the supersymmetric D$p$ branes of Type II theories are described by means of boundary states $|B\rangle$ \[14, 15\]. These are closed string states which insert a boundary on the world-sheet and enforce on it the appropriate boundary conditions. Both in the NS-NS and in the R-R sectors, there are two possible implementations for the boundary conditions of a D$p$ brane which correspond to two boundary states $|B,\eta\rangle$, with $\eta = \pm 1$. However, only the combinations

$$|B\rangle_{\text{NS}} = \frac{1}{2} \left[ |B, +\rangle_{\text{NS}} - |B, -\rangle_{\text{NS}} \right]$$

and

$$|B\rangle_{\text{R}} = \frac{1}{2} \left[ |B, +\rangle_{\text{R}} + |B, -\rangle_{\text{R}} \right]$$

are selected by the GSO projection in the NS-NS and in the R-R sectors respectively. As discussed in Ref. \[3\], the boundary state $|B, \eta\rangle$ is the product of a matter part and a ghost part

$$|B, \eta\rangle = \frac{T_p}{2} |B_{\text{mat}}, \eta\rangle |B_{g}, \eta\rangle ,$$

where

$$|B_{\text{mat}}, \eta\rangle = |B_X\rangle |B_{\psi}, \eta\rangle , \quad |B_{g}, \eta\rangle = |B_{\text{gh}}, \eta\rangle |B_{\text{sgh}}, \eta\rangle .$$

The overall normalization $T_p$ can be unambiguously fixed from the factorization of amplitudes of closed strings emitted from a disk \[16, 5\] and is the brane tension \[17\]

$$T_p = \sqrt{\pi} \left( 2\pi \sqrt{\alpha'} \right)^{3-p} .$$

The explicit expressions of the various components of $|B\rangle$ have been given in Ref. \[3\] in the case of a static D-brane without any external field on its world-volume. However, the operator structure of the boundary state does not change even when more general configurations are considered and is always of the form

$$|B_X\rangle = \exp \left[ - \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot S \cdot \tilde{\alpha}_{-n} \right] |B_X\rangle^{(0)} ,$$
and
\[ |B_\psi, \eta\rangle_{NS} = -i \exp \left[ i \eta \sum_{m=1/2}^{\infty} \psi_{-m} \cdot S \cdot \bar{\psi}_{-m} \right] |0\rangle \]
(2.7)
for the NS-NS sector, and
\[ |B_\psi, \eta\rangle_R = -\exp \left[ i \eta \sum_{m=1}^{\infty} \psi_{-m} \cdot S \cdot \bar{\psi}_{-m} \right] |B, \eta\rangle_R^{(0)} \]
(2.8)
for the R-R sector. The matrix \( S \) and the zero-mode contributions \( |B_X\rangle^{(0)} \) and \( |B, \eta\rangle_R^{(0)} \) encode all information about the overlap equations that the string coordinates have to satisfy, which in turn depend on the boundary conditions of the open strings ending on the Dp brane. Since the ghost and superghost fields are not affected by the type of boundary conditions that are imposed, the ghost part of the boundary state is always the same. Its explicit expression can be found in Ref. [3]. We do not write it again here since it will not play any significant role for our present purposes. However, we would like to recall that the boundary state must be written in the \((-1, -1)\) superghost picture in the NS-NS sector, and in the asymmetric \((-1/2, -3/2)\) picture in the R-R in order to saturate the superghost number anomaly of the disk [18, 3].

When a constant gauge field \( F \) is present on the D-brane world-volume, the overlap conditions that the boundary state must satisfy are [14]
\[ \left\{ (\mathbb{1} + \tilde{F})^{\alpha}_{\beta} \alpha^{\beta}_n + (\mathbb{1} - \tilde{F})^{\alpha}_{\beta} \bar{\alpha}^{\beta}_{-n} \right\} |B_X\rangle = 0 \]
\[ (q^i - y^i) |B_X\rangle = \left\{ \alpha^i_n - \bar{\alpha}^i_{-n} \right\} |B_X\rangle = 0 \quad n \neq 0 \]
(2.9)
for the bosonic part, and
\[ \left\{ (\mathbb{1} + \tilde{F})^{\alpha}_{\beta} \psi^\beta_m - i \eta (\mathbb{1} - \tilde{F})^{\alpha}_{\beta} \bar{\psi}^\beta_{-m} \right\} |B_\psi, \eta\rangle = 0 \]
\[ \left\{ \psi^i_m + i \eta \bar{\psi}^i_{-m} \right\} |B_\psi, \eta\rangle = 0 \]
(2.10)
for the fermionic part. In these equations, the Greek indices \( \alpha, \beta, \ldots \) label the world-volume directions 0, 1, \ldots, \( p \) along which the Dp brane extends, while the Latin indices \( i, j, \ldots \) label the transverse directions \( p+1, \ldots, 9 \); moreover \( \tilde{F} = 2\pi \alpha F \). As already noticed in Ref. [14], these equations are solved by the “coherent states” (2.6)-(2.8) with a matrix \( S \) given by
\[ S_{\mu \nu} = \left( [(\eta - \tilde{F})(\eta + \tilde{F})^{-1}]_{\alpha \beta} ; -\delta_{ij} \right) \]
(2.11)
\footnote{The unusual phases introduced in Eqs. (2.7) and (2.8) will turn out to be convenient to study the couplings of the massless closed string states with a D-brane and to find the correspondence with the classical D-brane solutions obtained from supergravity. Note that these phases are instead irrelevant when one computes the interactions between two D-branes.}
and with the zero-mode parts given by

\[ |B_X⟩^{(0)} = \sqrt{-\det(\eta + \hat{F})} \delta^{(0-p)}(q^i - y^i) \prod_{\mu=0}^9 |k^\mu = 0⟩ \]  

(2.12)

for the bosonic sector, and by

\[ |B_{\psi, \eta}⟩^{(0)}_R = \left( C T^0 \Gamma^1 \ldots \Gamma^p \frac{1 + i\eta \Gamma_{11}}{1 + i\eta} U \right)_{AB} |A⟩ |B⟩ \]  

(2.13)

for the R sector. In writing these formulas we have denoted by \( y^i \) the position of the D-brane, by \( C \) the charge conjugation matrix and by \( U \) the following matrix

\[ U = \frac{1}{\sqrt{-\det(\eta + \hat{F})}} \exp\left( -\frac{1}{2} \hat{F}_{\alpha\beta} \Gamma^\alpha \Gamma^\beta \right); \]  

(2.14)

where the symbol ; ; means that one has to expand the exponential and then antisymmetrize the indices of the \( \Gamma \)-matrices. Finally, \( |A⟩ |B⟩ \) stands for the spinor vacuum of the R-R sector.

We would like to remark that the overlap equations (2.9) and (2.10) do not allow to determine the overall normalization of the boundary state, and not even to get the Born-Infeld prefactor of Eq. (2.12). The latter was derived in Ref. [14]. It can also more easily be obtained by boosting the boundary state and then performing a T-duality as explicitly shown in Ref. [19]. Notice that this prefactor is present only in the NS-NS component of the boundary state because in the R-R sector it cancels out if we use the explicit expression for the matrix \( U \) given in Eq. (2.14).

We end this subsection with a few comments. If \( F \) is an external magnetic field, the corresponding boundary state describes a stable BPS bound state formed by a Dp brane with other lower dimensional D-branes (like for example the Dp-D(p-2) bound state). This case was explicitly considered in Ref. [5] where the long distance behavior of the massless fields of these configurations was determined using the boundary state approach. On the contrary, if \( F \) is an external electric field, then the boundary state describes a stable bound state between a fundamental string and a Dp brane that preserves one half of the space-time supersymmetries [6, 7, 8]. This kind of bound state denoted by (F,Dp) is a generalization of the dyonic string configurations of Schwarz [13] which has recently been studied from the supergravity point of view [3, 14] and will be analyzed in detail in the following sections. However, before doing this, for completeness we show how the low-energy effective action of a D-brane is related to the boundary state we have just constructed.

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3For our conventions on \( \Gamma \)-matrices, spinors etc. see for example Refs. [3, 5].
2.2 The D-brane effective action

As we have mentioned before, the boundary state is the exact conformal description of a D-brane and therefore it contains the complete information about the interactions between a D-brane and the closed strings that propagate in the bulk. In particular it encodes the couplings with the bulk massless fields which can be simply obtained by saturating the boundary state $|B\rangle$ with the massless states of the closed string spectrum. In order to find a non-vanishing result, it is necessary to soak up the superghost number anomaly of the disk and thus, as a consequence of the superghost charge of the boundary state, we have to use closed string states in the $(-1, -1)$ picture in the NS-NS sector and states in the asymmetric $(-1/2, -3/2)$ picture in the R-R sector.

In the NS-NS sector, the states that represent the graviton $h_{\mu\nu}$, the dilaton $\phi$ and the Kalb-Ramond antisymmetric tensor $A_{\mu\nu}$ are of the form

$$\epsilon_{\mu\nu} \tilde{\psi}_{\frac{1}{2}} \psi_{\frac{1}{2}} |k/2\rangle_{-1} |\tilde{k}/2\rangle_{-1}$$

with

$$\epsilon_{\mu\nu} = h_{\mu\nu} \ , \ h_{\mu\nu} = h_{\nu\mu} \ , \ k^{\mu} h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$$

for the graviton,

$$\epsilon_{\mu\nu} = \frac{\phi}{2\sqrt{2}} (\eta_{\mu\nu} - k_{\mu} \ell_{\nu} - k_{\nu} \ell_{\mu}) \ , \ \ell^{2} = 0 \ , \ k \cdot \ell = 1$$

for the dilaton, and

$$\epsilon_{\mu\nu} = \frac{1}{\sqrt{2}} A_{\mu\nu} \ , \ A_{\mu\nu} = -A_{\nu\mu} \ , \ k^{\mu} A_{\mu\nu} = 0$$

for the Kalb-Ramond field $A_{\mu\nu}$. In order to obtain their couplings with the boundary state it is useful to first compute the quantity

$$J^{\mu\nu} \equiv -1 \langle \tilde{k}/2 | -1 \langle k/2 | \tilde{\psi}_{\frac{1}{2}} \psi_{\frac{1}{2}} |B\rangle_{\text{NS}} = -\frac{T_{p}}{2} V_{p+1} \sqrt{-\det(\eta + \hat{F})} S^{\mu\nu}$$

where $V_{p+1}$ is the (infinite) world-volume of the brane, and then to project it on the various independent fields using their explicit polarizations. We thus obtain: for the graviton

$$J_{h} \equiv J^{\mu\nu} h_{\mu\nu} = -T_{p} V_{p+1} \sqrt{-\det(\eta + \hat{F})} \left[ (\eta + \hat{F})^{-1} \right]^{\alpha\beta} h_{\beta\alpha}$$

4The factor of $1/\sqrt{2}$ in Eq. (2.18) is necessary to have a canonical normalization, see also Ref. [20].
where we have used the tracelessness of $h_{\mu \nu}$; for the dilaton

$$J_\phi \equiv \frac{1}{2\sqrt{2}} J^{\mu \nu} (\eta_{\mu \nu} - k_\mu \ell_\nu - k_\nu \ell_\mu) \phi = \frac{T_p}{2\sqrt{2}} V_{p+1} \sqrt{-\det(\eta + \hat{F})} \left[ 3 - p + \text{Tr} \left( \hat{F}(\eta + \hat{F})^{-1} \right) \right] \phi ; \quad (2.21)$$

and finally for the Kalb-Ramond field

$$J_A \equiv \frac{1}{\sqrt{2}} J^{\mu \nu} A_{\mu \nu} = -\frac{T_p}{2\sqrt{2}} V_{p+1} \sqrt{-\det(\eta + \hat{F})} \left[ (\eta - \hat{F})(\eta + \hat{F})^{-1} \right]^{\alpha \beta} A_{\beta \alpha} = -\frac{T_p}{\sqrt{2}} \sqrt{\frac{V_{p+1}}{p+1}} \xi e^{-\frac{\kappa}{2} \phi} \sqrt{-\det \left[ g + \sqrt{2} \kappa A + \hat{F} e^{-\frac{\kappa}{2} \phi} \right]} \quad (2.22)$$

where in the second line we have used the antisymmetry of $A_{\mu \nu}$.

We now show that the couplings $J_h$, $J_\phi$ and $J_A$ are precisely the ones that are produced by the Dirac-Born-Infeld action which governs the low-energy dynamics of the D-brane. In the string frame, this action reads as follows

$$S_{DBI} = -\frac{T_p}{\kappa} \int_{V_{p+1}} d^{p+1} \xi \ e^{-\frac{\kappa}{2} \phi} \sqrt{-\det \left[ G + \sqrt{2} \kappa A + \hat{F} e^{-\frac{\kappa}{2} \phi} \right]} \quad (2.23)$$

where $2\kappa^2 = (2\pi)^7 (\alpha')^4 g_s^2$ is Newton’s constant ($g_s$ being the string coupling), and $G_{\alpha \beta}$ and $A_{\alpha \beta}$ are respectively the pullbacks of the space-time metric and of the NS-NS antisymmetric tensor on the D-brane world volume.

In order to compare the couplings described by this action with the ones obtained from the boundary state, it is first necessary to rewrite $S_{DBI}$ in the Einstein frame. In fact, like any string amplitude computed with the operator formalism, also the couplings $J_h$, $J_\phi$ and $J_A$ are written in the Einstein frame (this property has been overlooked in the qualitative analysis of Ref. [7]). Furthermore, it is also convenient to introduce canonically normalized fields. These two goals can be realized by means of the following field redefinitions

$$G_{\mu \nu} = e^{\phi/2} g_{\mu \nu} \ , \ \phi = \sqrt{2} \kappa \phi \ , \ \ A_{\mu \nu} = \sqrt{2} \kappa e^{\phi/2} A_{\mu \nu} \ . \quad (2.24)$$

Using the new fields in Eq. (2.23), we easily get

$$S_{DBI} = -\frac{T_p}{\kappa} \int_{V_{p+1}} d^{p+1} \xi \ e^{-\frac{\kappa}{2} \phi} \sqrt{-\det \left[ g + \sqrt{2} \kappa A + \hat{F} e^{-\frac{\kappa}{2} \phi} \right]} \ . \quad (2.25)$$

By expanding the metric around the flat background

$$g_{\mu \nu} = \eta_{\mu \nu} + 2\kappa h_{\mu \nu} \ , \quad (2.26)$$
and keeping only the terms which are linear in $h$, $\phi$ and $A$, the action (2.23) reduces to the following expression

$$S_{DBI} \simeq - T_p \int_{V_p+1} d^{p+1}\xi \sqrt{-\det [\eta + \hat{F}]} \left\{ \left[ (\eta + \hat{F})^{-1} \right]^{\alpha\beta} h_{\beta\alpha} - \frac{1}{2\sqrt{2}} \left( 3 - p + \text{Tr} \left( \hat{F}(\eta + \hat{F})^{-1} \right) \right) \phi + \frac{1}{\sqrt{2}} \left( (\eta + \hat{F})^{-1} \right)^{\alpha\beta} A_{\beta\alpha} \right\}.$$  

(2.27)

It is now easy to see that the couplings with the graviton, the dilaton and the Kalb-Ramond field that can be obtained from this action are exactly the same as those obtained from the boundary state and given in Eqs. (2.20), (2.21) and (2.22) respectively.

Let us now turn to the R-R sector. As we mentioned above, in this sector we have to use states in the asymmetric ($-1/2, -3/2$) picture in order to soak up the superghost number anomaly of the disk. In the more familiar symmetric ($-1/2, -1/2$) picture the massless states are associated to the field strengths of the R-R potentials. On the contrary, in the ($-1/2, -3/2$) picture the massless states are associated directly to the R-R potentials which, in form notation, we denote by

$$C_{(n)} = \frac{1}{n!} C_{\mu_1...\mu_n} \, dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} \quad (2.28)$$

with $n = 1, 3, 5, 7, 9$ in the Type IIA theory and $n = 0, 2, 4, 6, 8, 10$ in the Type IIB theory. The string states $|C_{(n)}\rangle$ representing these potentials have a rather non-trivial structure. In fact, as shown in Ref. [3], the natural expression

$$|C_{(n)}\rangle \simeq \frac{1}{n!} C_{\mu_1...\mu_n} \left( C^{\mu_1...\mu_n} \frac{1 + \Gamma_{11}}{2} \right)_{AB} |A; k/2\rangle_{-1/2} |\bar{B}; \bar{k}/2\rangle_{-3/2} \quad (2.29)$$

is BRST invariant only if the potential is pure gauge. To avoid this restriction, in general it is necessary to add to Eq. (2.29) a whole series of terms with the same structure but with different contents of superghost zero-modes. However, in the present situation there exists a short-cut that considerably simplifies the analysis. In fact, one can use the incomplete states (2.29) and ignore the superghosts, whose contribution can then be recovered simply by changing at the end the overall normalizations of the amplitudes $^5$. Keeping this in mind, the couplings between the R-R potentials (2.28) and the $D_p$ brane can therefore be obtained by computing the overlap between the states (2.29) and the R-R component of the boundary state, namely

$$J_{C_{(n)}} \equiv \langle C_{(n)} | B \rangle_R \quad . \quad (2.30)$$

The evaluation of $J_{C_{(n)}}$ is straightforward, even if a bit lengthy; some details about this calculation are given in Appendix A where the complete expression for the

$^5$Note that this procedure is not allowed when the odd-spin structure contributes, see Ref. [3].
asymmetric R-R states is used and the contribution of the superghosts is explicitly taken into account to obtain the correct normalization. The final result is

$$J_{C(n)} = -\frac{T_p}{16\sqrt{2}n!} V_{p+1} C_{\mu_1...\mu_n} \text{Tr} \left( \Gamma^{\mu_2...\mu_1} \Gamma^0 \cdots \Gamma^p ; e^{-\frac{i}{2} F_{\alpha\beta} \Gamma^\alpha \Gamma^\beta} \right).$$  \hspace{1cm} (2.31)

It is easy to realize that the trace in this equation is non-vanishing only if $n = p + 1 - 2\ell$, where $\ell$ denotes the power of $\hat{F}$ which is produced by expanding the exponential term. Due to the antisymmetrization prescription, the integer $\ell$ takes only a finite number of values up to a maximum $\ell_{\text{max}}$ which is $p/2$ for the Type IIA string and $(p + 1)/2$ for the Type IIB string. The simplest term to compute, corresponding to $\ell = 0$, describes the coupling of the boundary state with a $(p + 1)$-form potential of the R-R sector and is given by

$$J_{C(p+1)} = \frac{\sqrt{2} T_p}{(p+1)!} V_{p+1} C_{\alpha_0...\alpha_p} \varepsilon^{\alpha_0...\alpha_p}$$  \hspace{1cm} (2.32)

where $\varepsilon^{\alpha_0...\alpha_p}$ is the completely antisymmetric tensor on the D-brane world-volume. From Eq. (2.32) we can immediately deduce that the charge $\mu_p$ of a D$p$ brane with respect to the R-R potential $C_{(p+1)}$ is

$$\mu_p = \sqrt{2} T_p$$  \hspace{1cm} (2.33)

in agreement with Polchinski’s original calculation [2].

The next term in the expansion of the exponential of Eq. (2.31) corresponds to $\ell = 1$ and yields the coupling of the D$p$ brane with a $(p - 1)$-form potential which is given by

$$J_{C(p-1)} = -\frac{\mu_p}{2(p-1)!} V_{p+1} C_{\alpha_0...\alpha_{p-2}} \hat{F}_{\alpha_{p-1}\alpha_p} \varepsilon^{\alpha_0...\alpha_p}.$$  \hspace{1cm} (2.34)

By proceeding in the same way, one can easily evaluate also the higher order terms generated by the exponential which describe the interactions of the D-brane with potential forms of lower degree. All these couplings can be encoded in the following Wess-Zumino-like term

$$S_{WZ} = \mu_p \int_{V_{p+1}} \left[ \sum_{\ell=0}^{\ell_{\text{max}}} C_{(p+1-2\ell) \wedge} e^{\hat{F}} \right]_{p+1}$$  \hspace{1cm} (2.35)

where $\hat{F} = \frac{i}{2} \hat{F}_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta$, and $C_{(n)}$ is the pullback of the $n$-form potential (2.28) on the D-brane world-volume. The square bracket in Eq. (2.35) means that in expanding the exponential form one has to pick up only the terms of total degree $(p + 1)$, which are then integrated over the $(p + 1)$-dimensional world-volume.

\footnote{Our convention is that $\varepsilon^{0...p} = -\varepsilon_{0...p} = 1.$}
In conclusion we have explicitly shown that by projecting the boundary state $|B\rangle$ with an external field onto the massless states of the closed string spectrum, one can reconstruct the linear part of the low-energy effective action of a D$p$ brane. This is the sum of the Dirac-Born-Infeld part (2.27) and the (anomalous) Wess-Zumino term (2.33) which are produced respectively by the NS-NS and the R-R components of the boundary state.

3 (F,D$p$) bound states from the boundary state

In this section we are going to show that the boundary state constructed before can be used to obtain the long distance behavior of the various fields describing the bound state (F,D$p$) formed by a fundamental string and a D$p$ brane. This type of bound state is a generalization of the dyonic string solution of Schwarz [13] and has been recently discussed from the supergravity point of view [9, 10]. As we mentioned before, the (F,D$p$) bound state can be obtained from a D$p$ brane by turning on an electric field $\hat{F}$ on its world volume [6, 7, 8]. With no loss in generality we can choose $\hat{F}$ to have non vanishing components only in the directions $X^0$ and $X^1$ so that it can be represented by the following $(p+1) \times (p+1)$ matrix

$$\hat{F}_{\alpha\beta} = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ \vdots & \vdots & \ddots \\ 0 & \cdots & 0 \end{pmatrix}.$$ (3.1)

Using this expression in Eq. (2.11) one can easily see that the longitudinal part of the matrix $S$ appearing in the boundary state is given by

$$S_{\alpha\beta} = \begin{pmatrix} -\frac{1+f^2}{1-f^2} & \frac{2f}{1-f^2} & \frac{1+f^2}{1-f^2} \\ -\frac{2f}{1-f^2} & \frac{1+f^2}{1-f^2} & 1 \\ \frac{2f}{1-f^2} & \frac{1+f^2}{1-f^2} & 1 \end{pmatrix}.$$ (3.2)

while the transverse part of $S$ is simply minus the identity in the remaining $(9-p)$ directions. Furthermore, using Eq. (3.1) one finds

$$-\det (\eta + \hat{F}) = 1 - f^2.$$ (3.3)
As we have discussed in Ref. [5], the boundary state can be used in a very efficient way to obtain the long distance behavior of the fields emitted by a D-brane and obtain the corresponding classical solution at long distances. To do so one simply adds a closed string propagator $D$ to the boundary state $B$ and then projects the resulting expression onto the various massless states of the closed string spectrum. According to this procedure, the long-distance fluctuation of a field $\Psi$ is then given by

$$\delta \Psi \equiv \langle P(\Psi) | D | B \rangle$$  \hspace{1cm} (3.4)$$

where $\langle P(\Psi) \rangle$ denotes the projector associated to $\Psi$. The explicit expressions for these projectors can be found in Appendix A (see Eqs. (A.8), (A.9), (A.10), (A.20)) for all massless fields of the NS-NS and R-R sectors.

Before giving the details of this calculation, we would like to make a few comments. Firstly, since we are not using explicitly the ghost and superghost degrees of freedom, we must take into account their contribution by shifting appropriately the zero-point energy and use for the closed string propagator the following expression

$$D = \frac{\alpha'}{4\pi} \int_{|z| \leq 1} \frac{d^2z}{|z|^2} z^{L_0 - a} \bar{z}^{\tilde{L}_0 - \bar{a}}$$ \hspace{1cm} (3.5)$$

where the operators $L_0$ and $\tilde{L}_0$ depend only on the orbital oscillators and the intercept is $a = 1/2$ in the NS-NS sector and $a = 0$ in the R-R sector. Secondly, since we want to describe configurations of branes with arbitrary R-R charge, we multiply the entire boundary state by an overall factor of $x$. Later we will see that the consistency of the entire construction will require that $x$ be an integer, and also that the electric field strength $f$ cannot be arbitrary.

Let us now begin our analysis by studying the projection (3.4) in the NS-NS sector. Since all projectors onto the NS-NS massless fields contain the following structure

$$-1 \langle \tilde{k}/2 \rangle -1 \langle k/2 \rangle \psi_{\frac{\mu}{2}}^{\nu} \bar{\psi}_{\frac{\mu}{2}} \psi_{\frac{\mu}{2}}$$ \hspace{1cm} (3.6)$$

as we can see from the explicit expressions given in Eqs. (A.8) - (A.10), it is first convenient to compute the matrix element

$$T^{\mu \nu} \equiv -1 \langle \tilde{k}/2 \rangle -1 \langle k/2 \rangle \psi_{\frac{\mu}{2}}^{\nu} \bar{\psi}_{\frac{\mu}{2}} | D | B \rangle_{\text{NS}} = -x \frac{T_{\mu}}{2} \frac{V_{n+1}}{k_{\perp}^2} \sqrt{1 - f^2} \ S^{\mu \nu}$$ \hspace{1cm} (3.7)$$

where $k_{\perp}$ is the momentum in the transverse directions which is emitted by the brane. Notice that the matrix $T^{\mu \nu}$ differs from the matrix $J^{\mu \nu}$ defined in Eq. (2.19) simply by the factor of $1/k_{\perp}^2$ coming from the insertion of the propagator, and by the overall normalization (i.e. the factor of $x$).
Using this result and the explicit form of the dilaton projector (A.8), after some straightforward algebra, we find that the long-distance behavior of the dilaton of the \((F,D_p)\) bound state is given by

\begin{equation}
\delta \phi \equiv \langle P^{(\phi)}|D|B\rangle_{\text{NS}} = \frac{1}{2\sqrt{2}} \left( \eta^{\mu\nu} - k^\mu \ell^\nu - k^\nu \ell^\mu \right) T_{\mu\nu} . \tag{3.8}
\end{equation}

Using the explicit expression for the matrix \(T_{\mu\nu}\), we get

\begin{equation}
\delta \phi = \mu_p \frac{V_{p+1}}{k_1^2} \frac{x f^2(p-5) + (3-p)}{4 \sqrt{1-f^2}} . \tag{3.9}
\end{equation}

where \(\mu_p\) is the unit of R-R charge of a D\(p\) brane defined in Eq. (2.33). Similarly, using the projector (A.10) for the antisymmetric Kalb-Ramond field, we find

\begin{equation}
\delta A_{\mu\nu} \equiv \langle P^{(A)}_{\mu\nu}|D|B\rangle_{\text{NS}} = \frac{1}{\sqrt{2}} \left( T_{\mu\nu} - T_{\nu\mu} \right) . \tag{3.10}
\end{equation}

Since with our choices the matrix \(T_{\mu\nu}\) is symmetric except in the block of the 0 and 1 directions (see Eq. (3.2)), we immediately conclude that the only non-vanishing component of the Kalb-Ramond field emitted by the \((F,D_p)\) bound state is \(A_{01}\) whose long-distance behavior is given by

\begin{equation}
\delta A_{01} = \mu_p \frac{V_{p+1}}{k_1^2} \frac{x f}{\sqrt{1-f^2}} . \tag{3.11}
\end{equation}

Finally, using Eq. (A.9) we find that the components of the metric tensor are

\begin{equation}
\delta h_{\mu\nu} \equiv \langle P^{(h)}_{\mu\nu}|D|B\rangle_{\text{NS}} = \frac{1}{2} \left( T_{\mu\nu} + T_{\nu\mu} \right) - \frac{\delta \phi}{2\sqrt{2}} \eta_{\mu\nu} \tag{3.12}
\end{equation}

which explicitly read

\begin{align*}
\delta h_{00} &= -\delta h_{11} = \mu_p \frac{V_{p+1}}{k_1^2} \frac{x f^2(p-1) + (7-p)}{8\sqrt{2}\sqrt{1-f^2}} , \\
\delta h_{22} &= \ldots \equiv \delta h_{pp} = \mu_p \frac{V_{p+1}}{k_1^2} \frac{x f^2(9-p) + (p-7)}{8\sqrt{2}\sqrt{1-f^2}} , \\
\delta h_{p+1,p+1} &= \ldots = \delta h_{99} = \mu_p \frac{V_{p+1}}{k_1^2} \frac{x f^2(1-p) + (p+1)}{8\sqrt{2}\sqrt{1-f^2}} . \tag{3.13}
\end{align*}

Let us now turn to the R-R sector. In this case, after the insertion of the closed string propagator, we have to saturate the R-R boundary state (2.2) with the projectors on the various R-R massless fields given in Eq. (A.20). This calculation is completely analogous to the one described in the previous section and performed in detail in Appendix A to obtain the couplings of a D\(p\) brane with the R-R potentials.
The only new features are the overall factor of $x$ and the presence of the factor of $1/k_+^2$ produced by the closed string propagator. Due to the structure of the R-R component of the boundary state describing the bound state $(F,D_p)$, it is not difficult to realize that the only projectors of the form (A.20) that can give a non-vanishing result are those corresponding to a $(p+1)$-form and to a $(p-1)$-form with all indices along the world-volume directions. In particular, we find that the long distance behavior of the $(p+1)$-form is given by

$$\delta C_{01\ldots p} \equiv \langle P_{01\ldots p}^{(C)} | D | B \rangle_R = -\mu_p \frac{V_{p+1}}{k_+^2} x . \quad (3.14)$$

Similarly, given our choice of the external field, we find that the only non-vanishing component of the $(p-1)$-form emitted by the boundary state is $C_{23\ldots p}$ whose long-distance behavior turns out to be

$$\delta C_{23\ldots p} \equiv \langle P_{23\ldots p}^{(C)} | D | B \rangle_R = -\mu_p \frac{V_{p+1}}{k_+^2} x f . \quad (3.15)$$

Notice that if $p = 1$ this expression has to be interpreted as the long-distance behavior of the R-R scalar which is usually denoted by $\chi$.

In all our previous analysis, the two parameters $x$ and $f$ that appear in the boundary state seem to be arbitrary. However, this is not so at a closer inspection. In fact, they are strictly related to the electric charges of the $(F,D_p)$ configuration under the Kalb-Ramond field and the R-R $(p+1)$-form potential. It is well-known that these charges must obey the Dirac quantization condition, i.e. they must be integer multiples of the fundamental unit of (electric) charge of a $p$-dimensional extended object $\mu_p$. In our notations this quantization condition amounts to impose that the coefficients of $-\mu_p \frac{V_{p+1}}{k_+^2}$ in Eqs. (3.14) and (3.11) be integer numbers (see also Section 5 for a discussion of this issue). This implies that

$$x = n \quad \text{and} \quad -\frac{xf}{\sqrt{1-f^2}} = m \quad (3.16)$$

with $n$ and $m$ two integers. While the restriction on $x$ had to be expected from the very beginning because $x$ simply represents the number of $D_p$ branes (and hence of boundary states) that form the bound state, the restriction on the external field $f$ is less trivial. In fact, from Eq. (3.16) we see that $f$ must be of the following form

$$f = -\frac{m}{\sqrt{n^2 + m^2}} . \quad (3.17)$$

\footnote{Notice that these charges are indeed of electric type since the only non-vanishing components of the corresponding potentials have an index along the time direction.}

\footnote{This particular choice of signs is of course just a matter of convention; as we will see it leads to the results that are usually reported in the literature.}
This is precisely the same expression that appears in the analysis of Ref. [7] on the
dyonic string configurations, and is also consistent with the results of Ref. [9, 10].

Using Eq. (3.16), we can now rewrite the long distance behavior of the massless
fields produced by a (F,Dp) bound state in a more suggestive way. In doing so,
we also perform a Fourier transformation to work in configuration space. This is
readily computed by observing that, for \( p < 7 \), one has

\[
\int d^{(p+1)} x \, d^{(9-p)} y \frac{e^{ik \cdot y}}{(7-p) \, r^{7-p} \, \Omega_{8-p}} = \frac{V_{p+1}}{k^2} \tag{3.18}
\]

where \( \Omega_q = \frac{2\pi^{(q+1)/2}}{\Gamma((q+1)/2)} \) is the area of a unit \( q \)-dimensional sphere and
\( r^2 = y_i y^i \) measures the distance from the branes. For later convenience, we also
introduce the following notations

\[
\Delta_{m,n} = m^2 + n^2 \tag{3.19}
\]

and

\[
Q_p = \mu_p \frac{\sqrt{2} \kappa \Delta_{m,n}^{1/2}}{(7-p) \, \Omega_{8-p}} \tag{3.20}
\]

Then, using Eq. (3.9) and assuming for the time being that the dilaton has vanishing
vacuum expectation value, after some elementary steps, we obtain that the long-
distance behavior of the dilaton is

\[
\varphi = \sqrt{2} \kappa \mu \simeq \left( -\frac{1}{2} + \frac{5-p}{4} \frac{n^2}{\Delta_{m,n}} \right) \frac{Q_p}{r^{7-p}} \tag{3.21}
\]

Since we are going to compare our results with the standard supergravity description
of D-branes, we have reintroduced the field \( \varphi \) which differs from the canonically
normalized dilaton \( \phi \) by a factor of \( \sqrt{2} \kappa \) (see also Eq. (2.24)). Similarly, recalling
that \( g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \), from Eq. (3.13) we find

\[
g_{00} = -g_{11} \simeq -1 - \left( -\frac{3}{4} + \frac{p-1}{8} \frac{n^2}{\Delta_{m,n}} \right) \frac{Q_p}{r^{7-p}} ,
\]

\[
g_{22} = \ldots = g_{pp} \simeq 1 + \left( \frac{1}{4} + \frac{p-9}{8} \frac{n^2}{\Delta_{m,n}} \right) \frac{Q_p}{r^{7-p}} ,
\]

\[
g_{p+1,p+1} = \ldots = g_{99} \simeq 1 + \left( \frac{1}{4} + \frac{p-1}{8} \frac{n^2}{\Delta_{m,n}} \right) \frac{Q_p}{r^{7-p}} .
\]

Rescaling the Kalb-Ramond field by a factor of \( \sqrt{2} \kappa \) to obtain the standard super-
gravity normalization and using Eq. (3.11), we easily get

\[
\hat{A} = \sqrt{2} \kappa \mu \simeq -\frac{m}{\Delta_{m,n}^{1/2}} \frac{Q_p}{r^{7-p}} \, dx^0 \wedge dx^1 \tag{3.23}
\]
Finally, repeating the same steps for the R-R potentials (3.14) and (3.15) we find
\[
\hat{C}_{(p+1)} = \sqrt{2\kappa} C_{(p+1)} \simeq -\frac{n}{\Delta^{1/2}} \frac{Q_p}{r^{7-p}} \, dx^0 \wedge \ldots \wedge dx^p ,
\]
(3.24)
and
\[
\hat{C}_{(p-1)} = \sqrt{2\kappa} C_{(p-1)} \simeq \frac{m}{n} \frac{n^2}{\Delta_{m,n}} \frac{Q_p}{r^{7-p}} \, dx^2 \wedge \ldots \wedge dx^p .
\]
(3.25)

Eqs. (3.24)-(3.25) represent the leading long-distance behavior of the massless fields emitted by the (F,Dp) bound state. Of course the simple knowledge of this asymptotic behavior is not sufficient to determine the exact and complete form of the corresponding classical brane-solution. To do this, one would need to compute also the higher order terms in the large distance expansion, and eventually resum the series. These higher order terms correspond to one-point functions for the massless closed string states emitted by world-sheets with many boundaries, and thus in our formalism one would need to compute one-point amplitudes with many boundary states. Clearly these calculations become more and more involved as the number of boundary states increases, and so far only the next-to-leading term in the case of a single D3 brane has been computed [21], even if with a different formalism. However, to obtain the complete brane-solution one can follow an alternative (and easier) route, namely one can make an Ansatz on the form of the solution, use the leading long-distance behaviour to fix the parameters and finally check that the classical field equations are satisfied. In our case, it is reasonable to assume that the exact (F,Dp) brane solution can be written entirely in terms of p-dependendent powers of harmonic functions. An inspection of Eqs. (3.21)-(3.22) suggests to introduce two harmonic functions, namely
\[
H(r) = 1 + \frac{Q_p}{r^{7-p}}
\]
(3.26)
which is the usual function appearing in the D-brane solutions, and
\[
H'(r) = 1 + \frac{n^2}{\Delta_{m,n}} \frac{Q_p}{r^{7-p}}
\]
(3.27)
which has been introduced also in Ref. [10]. Then, according to our assumptions and using Eqs. (3.21)-(3.25), we can infer that the dilaton is
\[
e^{\phi} = H^{-1/2} H'^{(5-p)/4} ,
\]
(3.28)
the metric is
\[
\begin{align*}
\text{ds}^2 &= H^{-3/4} H'^{(p-1)/8} \left[ - (dx^0)^2 + (dx^1)^2 \right] \\
&+ H^{1/4} H'^{(p-9)/8} \left[ (dx^2)^2 + \cdots + (dx^p)^2 \right] \\
&+ H^{1/4} H'^{(p-1)/8} \left[ (dx^{p+1})^2 + \cdots + (dx^9)^2 \right] ,
\end{align*}
\]
(3.29)
the Kalb-Ramond 2-form is

\[ \hat{A} = \frac{m}{\Delta_{m,n}^{1/2}} \left( H^{-1} - 1 \right) dx^0 \wedge dx^1 , \]  

(3.30)

and finally the R-R potentials are

\[ \hat{C}_{(p+1)} = \frac{n}{\Delta_{m,n}^{1/2}} \left( H^{-1} - 1 \right) dx^0 \wedge \cdots \wedge dx^p , \]  

(3.31)

\[ \hat{C}_{(p-1)} = -\frac{m}{n} \left( H'^{-1} - 1 \right) dx^2 \wedge \cdots \wedge dx^p . \]  

(3.32)

Notice that the field strength associated to \( \hat{C}_{(p+1)} \) is electric, whereas the one associated to \( \hat{C}_{(p-1)} \) is magnetic. In the case \( p = 1 \), the last equation has to be replaced by

\[ \chi = -\frac{m}{n} \left( H'^{-1} - 1 \right) \]  

(3.33)

where \( \chi \) is the R-R scalar field also called axion.

In writing these formulas we have assumed that all fields except the metric have vanishing asymptotic values. This explains why we have subtracted the 1 in the last four equations. At this point we should check that Eqs. (3.28)-(3.32) are a solution to the classical field equations. This is indeed the case since our fields agree with the ones recently derived in Ref. [10] from the supergravity point of view. Moreover, Eq. (3.33) can be shown to exactly agree with the axion field of the dyonic string solution of Schwarz [13] in the case of vanishing asymptotic background values for the scalars \( (\varphi_0 = \chi_0 = 0) \).

\[ \chi \]  

4 Interaction between two \((F, D_p)\) bound states

We now analyze some properties of the \((F, D_p)\) bound states we have described in the previous section; in particular we compute the interaction energy between two of them both from the classical and from the string point of view. To this aim, let us start by considering the contribution to the classical interaction energy due to the exchange of dilatons. The coupling \( J_\phi \) of the dilaton with the boundary state describing the \((F, D_p)\) configuration is given in Eq. (2.21) with an overall factor of

\[ \frac{9}{10} \]  

Actually, in comparing our results with those of Ref. [10], we find total agreement except for the overall sign in the Kalb-Ramond 2-form. Our sign however agrees with the dyonic string solution of Schwarz [13] when we put \( p = 1 \).
n; after using Eqs. (3.1) and (3.16), \( J_\phi \) explicitly becomes

\[
J_\phi = \frac{T_p}{2\sqrt{2}} \frac{n^2 (3 - p) - 2m^2}{\Delta_{m,n}^{1/2}} \phi .
\]  

(4.1)

Using this coupling, we can compute the potential energy density as follows

\[
U_\phi = \frac{1}{V_{p+1}} J_\phi J_\phi = \frac{T_p^2}{8} \frac{V_{p+1}}{V_{p+1}} \frac{n^2 (3 - p) - 2m^2}{\Delta_{m,n}} \phi \phi
\]

(4.2)

where

\[
\phi \phi = \frac{1}{k_\perp^2}
\]

(4.3)

is the dilaton propagator. Thus, we have

\[
U_\phi = \frac{T_p^2}{8} \frac{V_{p+1}}{V_{p+1}} \frac{n^4 (3 - p) - 2m^2}{k_\perp^2} \Delta_{m,n} \phi \phi
\]

(4.4)

Notice that \( U_\phi \) is always positive, which implies that the force between two (F,Dp) bound states due to dilaton exchanges is always attractive.

Let us now turn to the contribution to the potential energy due to graviton exchanges. The coupling of the graviton with the boundary state is given by Eq. (2.20) (again with an overall factor of \( n \)) which in our specific case becomes

\[
J_h = -T_p V_{p+1} \frac{n^2}{\Delta_{m,n}^{1/2}} V_{\alpha\beta} h_{\beta\alpha}
\]

(4.5)

where \( V_{\alpha\beta} \) is the following \((p + 1) \times (p + 1)\) matrix

\[
V_{\alpha\beta} = \left[ (\eta + \hat{F})^{-1} \right]_{\alpha\beta} = \begin{pmatrix}
-\Delta_{m,n} \frac{n^2}{n^2} & \frac{m \Delta_{m,n}^{1/2}}{n^2} \\
\frac{m \Delta_{m,n}^{1/2}}{n^2} & \Delta_{m,n} \\
1 & \ddots \\
& & 1
\end{pmatrix}
\]

(4.6)

and \( h_{\beta\alpha} \) is the graviton polarization. The gravitational potential energy is then

\[
U_h = \frac{1}{V_{p+1}} J_h J_h = T_p^2 V_{p+1} \frac{n^4}{\Delta_{m,n}} V_{\alpha\beta} V_{\gamma\delta} h_{\beta\alpha} h_{\gamma\delta}
\]

(4.7)

where

\[
h_{\beta\alpha} h_{\gamma\delta} = \left[ \frac{1}{2} (\eta_{\beta\delta} \eta_{\alpha\gamma} + \eta_{\alpha\delta} \eta_{\beta\gamma}) - \frac{1}{8} \eta_{\alpha\beta} \eta_{\gamma\delta} \right] \frac{1}{k_\perp^2}
\]

(4.8)
is the graviton propagator in the de Donder gauge. Using the explicit expression of the matrix $V$ given in Eq. (4.6) and performing some elementary algebra we obtain

$$U_h = \frac{T_p^2}{8} \frac{[n^4 (7 - p) (p + 1) + 4n^2 m^2 (7 - p) + 12m^4]}{\Delta_{m,n}} \frac{V_{p+1}}{k_\perp^2}.$$  \hspace{1cm} (4.9)

Notice that this gravitational potential energy is always positive (for $p \leq 7$), signaling the well-known fact that the exchange of gravitons always yields an attractive force.

Now let us consider the interaction between two $(F,Dp)$ bound states due to exchanges of Kalb-Ramond antisymmetric tensors. The coupling between the boundary state and the Kalb-Ramond field is given by Eq. (2.22) with an overall factor of $n$ and in our present case it becomes

$$J_A = -\frac{T_p}{\sqrt{2}} V_{p+1} \frac{n^2}{\Delta_{1/2}^{m,n}} V^{\alpha\beta} A_{\beta\alpha}.$$  \hspace{1cm} (4.10)

Thus, the corresponding potential energy density is

$$U_A = \frac{1}{V_{p+1}} \left( J_A \right) \left( J_A \right) = \frac{T_p^2}{2} V_{p+1} \frac{n^4}{\Delta_{m,n}} V^{\alpha\beta} V^{\gamma\delta} A_{\beta\alpha} A_{\delta\gamma}$$  \hspace{1cm} (4.11)

where

$$A_{\beta\alpha} A_{\delta\gamma} = (\eta_{\beta\delta}\eta_{\alpha\gamma} - \eta_{\alpha\delta}\eta_{\beta\gamma}) \frac{1}{k_\perp}$$  \hspace{1cm} (4.12)

is the propagator of an antisymmetric 2-index tensor in the Lorentz gauge. Using the explicit form (4.6) of the matrix $V$ and inserting Eq. (4.12) into Eq. (4.11), we find

$$U_A = -2 T_p^2 m^2 V_{p+1} \frac{n \, V_{p+1}}{k_\perp^2}.$$  \hspace{1cm} (4.13)

Notice that $U_A$ is always negative meaning that the corresponding force is always repulsive. This is indeed what should happen because the $(F,Dp)$ bound state carries the electric charge of the Kalb-Ramond field, and two alike charges always repel each other.

Finally, we compute the interaction energy density due to the exchange of the R-R potentials. In the case of the top (electric) form $C_{(p+1)}$, the coupling with the boundary state is given by Eq. (2.32) multiplied by $n$ and in our case it explicitly reads

$$J_{C_{(p+1)}} = \sqrt{2} T_p V_{p+1} n \, C_{01:p}.$$  \hspace{1cm} (4.14)

Then it is immediate to realize that the potential energy density is given by

$$U_{C_{(p+1)}} = \frac{1}{V_{p+1}} J_{C_{(p+1)}} \, J_{C_{(p+1)}} = 2 T_p^2 V_{p+1} n^2 C_{01:p} \, C_{01:p}$$

$$= -2 T_p^2 n \, V_{p+1} \frac{m^2}{k_\perp^2}.$$  \hspace{1cm} (4.15)
where we have used the propagator

\[ C_{01 \ldots p} C_{01 \ldots p} = - \frac{1}{k_{\perp}^2} \]  \hspace{1cm} (4.16)

which is the obvious generalization of Eq. (4.12). Since \( U_{C(p+1)} \) is negative, the corresponding force is repulsive as it should be, since the \((F,Dp)\) bound states carry the same electric charge under the \((p + 1)\)-form potential. In a similar way, we can compute the contribution to the interaction due to the exchange of the magnetic R-R potentials \( C_{(p-1)} \). In this case the coupling with the boundary state, which we can read from Eq. (2.34) with an overall factor of \( n \), is

\[ J_{C_{(p-1)}} = \sqrt{2} T_p V_{p+1} \frac{mn}{\Delta_{m,n}^{1/2}} C_{23 \ldots p} \]  \hspace{1cm} (4.17)

and hence, the corresponding potential energy density turns out to be

\[ U_{C_{(p-1)}} = \frac{1}{V_{p+1}} J_{C_{(p-1)}} J_{C_{(p-1)}} = 2 T_p^2 V_{p+1} \frac{m^2 n^2}{\Delta_{m,n}} C_{23 \ldots p} C_{23 \ldots p} \]

\[ = 2 T_p^2 \frac{m^2 n^2 V_{p+1}}{\Delta_{m,n} k_{\perp}^2} \]  \hspace{1cm} (4.18)

where we have used the propagator

\[ C_{23 \ldots p} C_{23 \ldots p} = \frac{1}{k_{\perp}^2} \]  \hspace{1cm} (4.19)

Notice that this last contribution is positive so that the associated force is always attractive as it should be for the exchange of abelian potentials of magnetic type. It is interesting to observe that not all the NS-NS (or R-R) massless fields contribute with the same sign to the interaction energy between two \((F,Dp)\) bound states. This is to be compared with what happens with simple D-branes, where the distinction between attractive and repulsive contributions coincides with the distinction between the NS-NS and R-R sectors.

In order to compute the mass density, or the tension, of the \((F,Dp)\) bound states, we follow Polchinski’s approach [17], namely we consider the total attractive potential energy and then compare it with Newton’s law in \( d = 10 \) for two \( p \)-dimensional extended objects. If we sum \( U_\phi \), \( U_h \) and \( U_{C_{(p-1)}} \), remarkable simplifications occur yielding

\[ U_{attr} = U_\phi + U_h + U_{C_{(p-1)}} = 2 T_p^2 \Delta_{m,n} \frac{V_{p+1}}{k_{\perp}^2} \]  \hspace{1cm} (4.20)

\[ \text{10In the case of simple Dp branes this amounts to consider just the contribution of the NS-NS sector; in our case however, we cannot use this identification since in the NS-NS sector there is a repulsive contribution and an attractive contribution appears in the R-R sector.} \]
Performing a Fourier transformation and using Eq. (3.18), we get

\[ U_{\text{attr}}(r) = \frac{2 T_p^2 \Delta_{m,n} \Omega_{8-p}}{(7 - p) \Omega_{8-p}} \frac{1}{r^{7-p}} \]  

(4.21)

where \( r \) is the distance between the two bound states. On the other hand, Newton’s law for two \( p \)-dimensional extended objects of mass density \( M_p \) in \( d = 10 \) reads

\[ U(r) = \frac{2 \kappa^2 M_p^2}{(7 - p) \Omega_{8-p}} \frac{1}{r^{7-p}} \]

(4.22)

where \( 2 \kappa^2 \) is Newton’s constant. By comparing Eqs. (4.21) and (4.22) we conclude that

\[ M_p = \frac{1}{\kappa} T_p \Delta_{m,n}^{1/2} \]

(4.23)

so that the tension of the (F,D\( p \)) bound state is

\[ T(m,n) = T_p \Delta_{m,n}^{1/2} = T_p \sqrt{m^2 + n^2} \]

(4.24)

This formula agrees with the one obtained in Refs. [9, 10] with ADM considerations, and explicitly shows that the (F,D\( p \)) configuration is a non-threshold bound state between a fundamental string of charge \( m \) and a D\( p \) brane of charge \( n \) (in units of \( \mu_p \)). It also makes evident the fact that elementary bound states are realized if \( m \) and \( n \) are relative prime integers.

If we now compute the total energy density, we find a vanishing result, i.e.

\[ U_{\text{tot}} = U_{\phi} + U_h + U_A + U_{C(p+1)} + U_{C(p-1)} = 0 \]

(4.25)

meaning that the (F,D\( p \)) bound states are BPS configurations satisfying the no-force condition. Actually, this property can be proved at the full string level by computing the vacuum amplitude between two boundary states at a distance \( r \) from each other, which is defined by

\[ \Gamma = \langle B | D | B \rangle \]

(4.26)

where \( D \) is the closed string propagator (3.3). (The explicit form of the conjugate boundary state that has to be used in Eq. (4.26) can be found at the end of Appendix A.) Then following the standard methods explained in Ref. [3], it is not difficult to check that the NS-NS contribution to \( \Gamma \) is

\[ \Gamma_{\text{NS}} = \frac{V_{p+1}}{2 \pi} \frac{n^4}{\Delta_{m,n}} (8 \pi^2 \alpha')^{\frac{p+1}{2}} \int_0^\infty dt \left( \frac{\pi}{t} \right)^{9-p} e^{-r^2/(2\alpha't)} \left( f^2_3(q) - f^2_4(q) \right) \]

(4.27)
where $q = e^{-t}$ and, as usual,

$$
\begin{align*}
&f_1(q) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) \quad , \quad f_2(q) = \sqrt{2} q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 + q^{2n}) \quad , \\
&f_3(q) = q^{-\frac{1}{12}} \prod_{n=1}^{\infty} (1 + q^{2n-1}) \quad , \quad f_4(q) = q^{-\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n-1}) .
\end{align*}
$$

(4.28)

Similarly, one can show that the R-R contribution to $\Gamma$ is

$$
\Gamma_R = -V_{p+1} \frac{n^4}{2\pi} \frac{\Delta_{m,n}}{(8\pi^2 \alpha')^{\frac{p+1}{4}}} \int_0^{\infty} dt \left( \frac{\pi}{t} \right)^{\frac{2-p}{4}} e^{-t^2/(2\alpha') \pi} \frac{f_2^2(q)}{f_1^2(q)} .
$$

(4.29)

The “abstruse identity” satisfied by the $f$-functions implies that $\Gamma = \Gamma_{NS} + \Gamma_R = 0$, i.e. the BPS condition at the full string level.

We can therefore conclude that the boundary state $|B\rangle$ defined in Section 2 with an external electric field as in Eq. (3.1), really provides the complete conformal description of the BPS bound states formed by fundamental strings and $Dp$ branes.

5 Dyonic strings in the Type IIB theory

In this section we consider in more detail the (F,D1) bound states which describe the dyonic strings first introduced by J. Schwarz in Ref. [13]. Let us recall that if we set to zero the self-dual five-form of the R-R sector, the low energy effective action for the IIB string in the Einstein frame can be written as follows

$$
S_{\text{IIB}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ R + \frac{1}{4} \text{Tr} \left( \partial \mathcal{M} \partial \mathcal{M}^{-1} \right) - \frac{1}{12} F^T \mathcal{M} F \right]
$$

(5.1)

where $R$ is the scalar curvature, $\mathcal{M}$ is the $SL(2,R)$ matrix constructed out of the dilaton $\varphi$ and the axion $\chi$ according to

$$
\mathcal{M} = e^\varphi \left( \begin{array}{cc} |\lambda|^2 & \chi \\ \chi & 1 \end{array} \right)
$$

(5.2)

with

$$
\lambda = \chi + i e^{-\varphi} ,
$$

(5.3)

and finally $\mathbf{F}$ is the two-component vector $\mathbf{F} = (F_{NS}, F_R)$ formed by the field strengths $F_{NS} = d\hat{A}$ and $F_R = d\hat{C}_2$ of the two-form potentials of the NS-NS and R-R sectors. The action (5.1) has manifest invariance under global $SL(2,R)$ transformations given by

$$
\mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T \quad , \quad \mathbf{F} \rightarrow (\Lambda^T)^{-1} \mathbf{F}
$$

(5.4)
where
\[
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1 .
\] (5.5)

More explicitly, we have
\[
\lambda \to \frac{a\lambda + b}{c\lambda + d}, \quad F_{NS} \to d F_{NS} - c F_{R}, \quad F_{R} \to -b F_{NS} + a F_{R} .
\] (5.6)

The dyonic string is a classical solution of the field equations derived from the action (5.1) which is (electrically) charged under the two antisymmetric tensors of the NS-NS and R-R sectors. Let us observe that if we denote by \( J_{\mu
u} = (J_{NS\mu\nu}, J_{R\mu\nu}) \) the current which is coupled to the antisymmetric tensors, then the charge \( q = (q_{NS}, q_{R}) \) of the dyonic string is defined by
\[
q \equiv \int_{V_8} d^8 x_\perp J_{01} = \frac{1}{\sqrt{2\kappa}} \int_{V_8} d^8 x_\perp \partial_\mu \left[ \sqrt{-g} M F_{01}^\mu \right] (5.7)
\] where in the last step we have used Stoke’s theorem. In these formulas \( V_8 \) denotes the space transverse to the string world-sheet whose boundary \( \partial V_8 \) is a seven-dimensional sphere at infinity. From Eqs. (5.7) and (5.4) it is easy to realize that the electric charges transform under an \( SL(2,R) \) transformation \( \Lambda \) according to
\[
q \to \Lambda q .
\] (5.8)

As is well known, not all classical solutions are fully consistent at the quantum level; only those which carry integer charges in units of \( \mu_1 \) (see Eq. (2.33) for \( p = 1 \)) satisfy the Dirac quantization condition and are acceptable. This is precisely the case of the (F,D1) bound state discussed in Section 3. For later convenience we write explicitly the long distance behavior of the corresponding fields obtained from the boundary state and given in Eqs. (3.21)-(3.25) for \( p = 1 \), namely
\[
\delta \varphi = -\frac{m^2 - n^2}{2 \Delta_{m,n}} \frac{Q_1}{r^6}, \quad \delta \chi = \frac{mn}{\Delta_{m,n}} \frac{Q_1}{r^6},
\] (5.9)
\[
\delta g_{\mu\nu} = \text{diag} \left( \frac{3}{4} - \frac{3}{4} \frac{1}{4}, \ldots, \frac{1}{4} \right) \frac{Q_1}{r^6},
\] (5.10)
and
\[
\delta \hat{A}_{01} = -\frac{m}{\Delta_{m,n}^{1/2}} \frac{Q_1}{r^6}, \quad \delta \hat{C}_{01} = -\frac{n}{\Delta_{m,n}^{1/2}} \frac{Q_1}{r^6} .
\] (5.11)

where \( Q_1 \) is given in Eq. (3.20) for \( p = 1 \). Using Eq. (5.11) into Eq. (5.7), and remembering that in this case the asymptotic matrix \( M|_{\infty} \) is the identity, one can easily verify that
\[
q_{NS} = m \mu_1, \quad q_{R} = n \mu_1 ,
\] (5.12)
that is, as expected, the two charges are integer multiples of $\mu_1$.

We now want to generalize these considerations to the case in which the two scalar fields $\varphi$ and $\chi$ have non-vanishing asymptotic values $\varphi_0$ and $\chi_0$ respectively. This can be easily achieved by exploiting the $SL(2,R)$ invariance of the theory and “rotating” the solution given in Eqs. (5.9)-(5.11) by means of the following transformation

$$\Lambda = \begin{pmatrix} e^{-\varphi_0/2} & \chi_0 e^{\varphi_0/2} \\ 0 & e^{\varphi_0/2} \end{pmatrix}.$$  \hspace{1cm} (5.13)

Indeed, according to Eq. (5.6) we have

$$\varphi \rightarrow \tilde{\varphi} = \varphi + \varphi_0, \quad \chi \rightarrow \tilde{\chi} = e^{-\varphi_0} \chi + \chi_0,$$  \hspace{1cm} (5.14)

so that the transformed fields acquire the desired asymptotic values. Furthermore, under $\Lambda$ the antisymmetric tensors transform as follows

$$\tilde{A}_{01} \rightarrow \tilde{A}_{01} = e^{\varphi_0/2} \tilde{A}_{01}, \quad \tilde{C}_{01} \rightarrow \tilde{C}_{01} = e^{-\varphi_0/2} \tilde{C}_{01} - \chi_0 e^{\varphi_0/2} \tilde{A}_{01},$$  \hspace{1cm} (5.15)

and correspondingly the charges become

$$q_{NS} \rightarrow \tilde{q}_{NS} = e^{-\varphi_0/2} q_{NS} + \chi_0 e^{\varphi_0/2} q_R, \quad q_R \rightarrow \tilde{q}_R = e^{\varphi_0/2} q_R.$$  \hspace{1cm} (5.16)

The new configuration is acceptable only if the new charges $\tilde{q}_{NS}$ and $\tilde{q}_R$ obey the Dirac quantization condition. From Eq. (5.16) we easily see that this condition is realized if we start from a configuration like the one of Eq. (5.11) but with $m$ and $n$ replaced according to

$$m \rightarrow e^{\varphi_0/2} (m - \chi_0 n), \quad n \rightarrow e^{-\varphi_0/2} n.$$  \hspace{1cm} (5.17)

In view of the discussion of Section 3, we can say that this configuration can be obtained from a boundary state with an external field on it given by

$$\tilde{f} = -\frac{e^{\varphi_0/2} (m - \chi_0 n)}{\Delta_{m,n}^{1/2}}$$  \hspace{1cm} (5.18)

where

$$\Delta_{m,n} = e^{\varphi_0} (m - \chi_0 n)^2 + e^{-\varphi_0} n^2.$$  \hspace{1cm} (5.19)

Notice that these expressions are simply obtained from Eqs. (5.17) and (5.19) with the substitutions (5.17). Furthermore, we remark that $\tilde{f}$ is precisely the gauge field found in Ref. [7] with different considerations.

Performing the transformations on the massless fields as indicated in Eqs. (5.14) and (5.13), we finally obtain the following long distance behavior for the scalars

$$\delta \tilde{\varphi} = -\frac{e^{\varphi_0} (m - \chi_0 n)^2 - e^{-\varphi_0} n^2}{2 \Delta_{m,n}} \tilde{Q}_1 \frac{1}{r^6}, \quad \delta \tilde{\chi} = \frac{e^{-\varphi_0} (m - \chi_0 n) n}{\Delta_{m,n}} \tilde{Q}_1 \frac{1}{r^6}. $$  \hspace{1cm} (5.20)
and for the antisymmetric tensors

\[
\delta \tilde{A}_{01} = -\frac{e^{\varphi_0} (m - \chi_0 n)}{\Delta_{m,n}^{1/2}} \frac{\bar{Q}_1}{x^6}, \quad \delta \tilde{C}_{01} = -\frac{e^{\varphi_0} (|\lambda_0|^2 n - \chi_0 m)}{\Delta_{m,n}^{1/2}} \frac{\bar{Q}_1}{x^6}
\]

where \( \bar{Q}_1 \) is defined as in Eq. (3.20) with \( \bar{\Delta}_{m,n} \) in place of \( \Delta_{m,n} \). These expressions are in complete agreement with the long distance behavior of the dyonic string solution of Schwarz [13]. Thus, we can conclude that a boundary state with an external electric field \( f \) represents the exact conformal description of the dyonic configurations with arbitrary background values at the full string level.

6 Concluding remarks

In this paper we have shown that the boundary state with a constant electric field provides the complete conformal representation for the BPS bound states formed by a fundamental string and a Dp brane. At the classical level, these configurations interpolate between the pure fundamental string and the pure Dp brane. In fact the latter can be obtained by setting \( f = 0 \) (or equivalently \( m = 0 \)), while the fundamental string is realized by choosing \( f = -1 \) (or equivalently \( n = 0 \)) in Eqs.(3.21)-(3.25).

One may wonder whether the same interpolation can be done at the full conformal level, i.e. directly on the boundary state. While there is of course no problem in switching off the electric field to obtain the boundary state for a pure Dp brane, the other limit, \( f \to -1 \) (or equivalently \( n \to 0 \)), is singular and not well defined on the boundary state. The reason is that the Dirac-Born-Infeld prefactor \( n\sqrt{1 - f^2} \) vanishes in this limit, while the longitudinal part of the S matrix (see Eq. (3.2)) diverges. However, these two effects compensate each other whenever the boundary state is saturated with closed string states which contain at most one left and one right oscillator with longitudinal index. This is precisely the structure of the massless states of the NS-NS sector. Using this observation and the fact that the R-R sector does not play any role in the fundamental string solution, we can introduce an effective operator which generates the classical fundamental string in the same way as the boundary state does for the D brane. Such operator is equal in structure to a boundary state without R-R sector and with a NS-NS sector defined by a longitudinal \((2 \times 2)\) S matrix given by

\[
S_{\alpha\beta} = \begin{pmatrix}
-2 & -2 \\
2 & 2
\end{pmatrix}
\]
and no transverse $S$ matrix. This effective operator is perfectly well defined and, when it is projected onto the dilaton, the graviton and the antisymmetric tensor, it yields the fundamental string solution. Furthermore, it appears to describe a BPS object, because the vacuum amplitude between two such states is identically vanishing. Therefore, it can be considered as an effective conformal representation of the fundamental string which is valid at very large distances, i.e. only for the massless fields. However, it cannot be the complete conformal description of the fundamental string solution. In fact, a “coherent state” structure, like the one given for instance in Eq. (2.6), always enforces an identification between left and right oscillators, which is appropriate only when the string world sheet has a boundary; this is certainly not the case for the pure fundamental string which only couples to closed string states and therefore must have independent left and right sectors. Finding the complete conformal description of the fundamental string remains therefore an open problem.

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Appendix A

In this appendix we give some details about the massless closed string states and define the corresponding projectors. In the $(-1,-1)$ superghost picture of the NS-NS sector, the massless states are

\begin{align}
|\phi\rangle &= \frac{\phi}{\sqrt{8}} (\eta_{\mu\nu} - k_\mu \ell_\nu - k_\nu \ell_\mu) \tilde{\psi}_-^{\mu} \tilde{\psi}_-^{\nu} |k/2\rangle_{-1} |\overline{k/2}\rangle_{-1}, \\
|h\rangle &= h_{\mu\nu} \tilde{\psi}_-^{\mu} \tilde{\psi}_-^{\nu} |k/2\rangle_{-1} |\overline{k/2}\rangle_{-1}, \\
|A\rangle &= \frac{A_{\mu\nu}}{\sqrt{2}} \tilde{\psi}_-^{\mu} \tilde{\psi}_-^{\nu} |k/2\rangle_{-1} |\overline{k/2}\rangle_{-1}
\end{align}

where $\phi$, $h_{\mu\nu}$ and $A_{\mu\nu}$ are the dilaton, graviton and Kalb-Ramond polarizations respectively. The corresponding conjugate states are

\begin{align}
\langle \phi | &= -1 \langle \overline{k/2} |_{-1} \langle k/2 |_{-1} \psi_+^{\nu} \tilde{\psi}_+^{\mu} \frac{\phi}{\sqrt{8}} (\eta_{\mu\nu} - k_\mu \ell_\nu - k_\nu \ell_\mu) ,
\end{align}

25
\[ \langle h \rangle = -1 \langle k/2 \rangle -1 \langle k/2 \rangle \psi^{\mu}_{1/2} \tilde{\psi}^{\mu}_{1/2} h_{\mu\nu} \quad \text{(A.5)} \]
\[ \langle A \rangle = -1 \langle k/2 \rangle -1 \langle k/2 \rangle \psi^{\mu}_{1/2} \tilde{\psi}^{\mu}_{1/2} \frac{A_{\mu\nu}}{\sqrt{2}} \quad \text{(A.6)} \]

The normalization of these states has been chosen in such a way that their norms

\[ \langle \phi | \phi \rangle = \phi^2 \quad , \quad \langle h|h \rangle = h^{\mu\nu} h_{\mu\nu} \quad , \quad \langle A|A \rangle = \frac{1}{2} A^{\mu\nu} A_{\mu\nu} \quad \text{(A.7)} \]

correspond to canonically normalized fields.

It is useful to define also projection operators that, when applied to an arbitrary massless state of the closed string, select the graviton, the dilaton and the Kalb-Ramond field components contained in that state. Given the form of the massless states (A.1)-(A.6), it is not difficult to verify that these projectors are

\[ \langle P^{(\phi)} | \phi \rangle = \phi \quad , \quad \langle P^{(h)}_{\mu\nu} | h_{\mu\nu} \rangle = h_{\mu\nu} \quad , \quad \langle P^{(A)}_{\mu\nu} | A \rangle = A_{\mu\nu} \quad \text{(A.11)} \]

Indeed, they satisfy the following properties with all other overlaps being zero.

Let us now consider the massless states of the R-R sector. As we mentioned in Section 2, in order to have a non vanishing overlap with the boundary state, we must work in the asymmetric \((-1/2, -3/2)\) picture so that the superghost number anomaly of the disk is soaked up. In Eq. (2.29) we wrote an expression for the state representing a \(n\)-form R-R potential \(C_{(n)}\). However, that expression has to be interpreted as an “effective” and simplified description which can be used only when parity violating terms do not contribute to scattering amplitudes. The complete expression for the massless R-R states instead involves infinite terms with different superghost number which combine to give

\[ |C_{(n)}\rangle = \frac{1}{2\sqrt{2} n!} C_{\mu_1...\mu_n} \left[ (C T^{\gamma_1...\gamma_n} \Pi_+)_{AB} \cos(\gamma_0 \tilde{\beta}_0) + (C T^{\gamma_1...\gamma_n} \Pi_-)_{AB} \sin(\gamma_0 \tilde{\beta}_0) \right] |A; k/2\rangle_{-1/2} |B; \tilde{k}/2\rangle_{-3/2} \quad \text{(A.12)} \]
where $\Pi_{\pm} = (1 \pm \Gamma_{11})/2$, and $\beta_0$ and $\gamma_0$ are the superghost zero-modes. The corresponding conjugate state is
\[
\langle C(n) | = -1/2 \langle B, \tilde{k}/2| -3/2 \langle A, k/2| (CT^{\mu_1 \ldots \mu_n} \Pi_-)_A B_2 \cos(\beta_0 \tilde{\gamma}_0) \\
+ (CT^{\mu_1 \ldots \mu_n} \Pi_+)_{AB} \sin(\beta_0 \tilde{\gamma}_0) \right] \frac{(-1)^n}{2\sqrt{2}n!} C_{\mu_1 \ldots \mu_n} .
\]

We now show that these states are correctly normalized. Due to the presence of the superghost zero modes, the norm $\langle C(n)|C(n)\rangle$ is naively divergent and a suitable regularization is necessary \[22, 3\]. This can be performed by inserting in the scalar product the operator $x^{2F_0 - 2\gamma_0 \beta_0}$, where $F_0$ is the zero-mode part of the world-sheet fermion number, and letting $x \to 1$ at the end. Keeping this in mind, let us first compute the superghost contribution. Using the equation
\[
\langle -1/2| \langle -3/2| e^{i\eta_0 \beta_0} x^{-2\gamma_0 \beta_0} e^{i\eta_2 \gamma_0 \bar{\beta}_0} | -1/2 \rangle | -3/2 \rangle = \frac{1}{1 - \eta_1 \eta_2 x^2} , \quad (A.14)
\]
we can easily obtain
\[
J \equiv \langle -1/2| \langle -3/2| \cos(\beta_0 \tilde{\gamma}_0) x^{-2\gamma_0 \beta_0} \cos(\gamma_0 \bar{\beta}_0) | -1/2 \rangle | -3/2 \rangle \\
= \frac{1}{2} \left( \frac{1}{1 - x^2} + \frac{1}{1 + x^2} \right) , \quad (A.15)
\]
\[
K \equiv \langle -1/2| \langle -3/2| \sin(\beta_0 \tilde{\gamma}_0) x^{-2\gamma_0 \beta_0} \sin(\gamma_0 \bar{\beta}_0) | -1/2 \rangle | -3/2 \rangle = \\
= \frac{1}{2} \left( \frac{1}{1 - x^2} - \frac{1}{1 + x^2} \right) , \quad (A.16)
\]
and also check that analogous expressions with one sine and one cosine are vanishing.

Then, recalling that the fermionic vacua are such that $\langle A|B\rangle = \langle A|\bar{B}\rangle = (C^{-1})^{AB}$, we get
\[
\langle C(n)|C(n)\rangle = \frac{(-1)^n}{8(n!)^2} C_{\mu_1 \ldots \mu_n} C_{\nu_1 \ldots \nu_n} \\
\times \lim_{x \to 1} \left\{ J \text{Tr} \left[ x^{2F_0} \left( CT^{\mu_1 \ldots \mu_n} \Pi_- C^{-1} \right)^T \Gamma^{\nu_1 \ldots \nu_n} \Pi_+ \right] \\
+ K \text{Tr} \left[ x^{2F_0} \left( CT^{\mu_1 \ldots \mu_n} \Pi_+ C^{-1} \right)^T \Gamma^{\nu_1 \ldots \nu_n} \Pi_- \right] \right\} . \quad (A.17)
\]

Using the transposition properties of the $\Gamma$ matrices and exploiting Eqs. (A.15) and (A.16), after some simple algebra we obtain
\[
\langle C(n)|C(n)\rangle = \frac{1}{16(n!)^2} C_{\mu_1 \ldots \mu_n} C_{\nu_1 \ldots \nu_n} \lim_{x \to 1} \left[ \text{Tr} \left( x^{2F_0} \Gamma^{\mu_n \ldots \mu_1} \Gamma_{\nu_n \ldots \nu_1} \right) \frac{1}{1 + x^2} \\
+ \text{Tr} \left( x^{2F_0} \Gamma^{\mu_n \ldots \mu_1} \Gamma_{\nu_n \ldots \nu_1} \Gamma_{11} \right) \frac{1}{1 - x^2} \right] . \quad (A.18)
\]
The parity violating term containing $\Gamma_{11}$ is identically zero, so that we can safely take the limit $x \to 1$ and get

$$
\langle C(n) | C(n) \rangle = \frac{1}{32 (n)!^2} C_{\mu_1 \ldots \mu_n} C_{\nu_1 \ldots \nu_n} \text{Tr} (\Gamma_{\mu_1 \ldots \mu_1} \Gamma_{\nu_1 \ldots \nu_1})
$$

$$
= \frac{1}{n!} C^{\mu_1 \ldots \mu_n} C_{\mu_1 \ldots \mu_n} . \quad (A.19)
$$

This result shows that the state (A.12) is canonically normalized. Now we can define the projection operator associated to $|C(n)\rangle$ which turns out to be

$$
\langle P_{\mu_1 \ldots \mu_n}^{(C)} | C(n) \rangle = C_{\mu_1 \ldots \mu_n} . \quad (A.20)
$$

Let us now compute the overlap of $\langle P^{(C)} |$ with the R-R boundary state. Since the projector (A.21) contains only zero-modes, it is enough to consider the zero-mode part of the R-R boundary state, which is explicitly given by [3]

$$
|B, \eta\rangle_R^{(0)} = -\frac{T_p}{2} \delta^{(9-p)} (q^j - y^i) e^{i \eta \gamma_0 \tilde{\gamma}_0} \mathcal{M}_{AB}^{(\eta)} |A, 0\rangle_{-1/2} |\tilde{B}, \tilde{0}\rangle_{-3/2} \quad (A.22)
$$

where

$$
\mathcal{M}_{AB}^{(\eta)} = \left[ \Gamma^0 \ldots \Gamma^p \frac{1 + i \eta \Gamma_{11}}{1 + i \eta} ; e^{-1/2 F_{\alpha \beta} \Gamma^\alpha \Gamma^\beta} ; \right]_{AB} \quad (A.23)
$$

Then, using Eq. (A.20), we obtain

$$
\langle P_{\mu_1 \ldots \mu_n}^{(C)} | B, \eta\rangle_R = -V_{p+1} T_p \frac{(1)^n}{4 \sqrt{2}} \lim_{x \to 1} \left\{ -1/2 |\tilde{D}|_{-3/2} \langle C | \times \right.
$$

$$
\left. \left[ (\Gamma_{\mu_1 \ldots \mu_n} \Pi_-)_{CD} \cos(\beta_0 \tilde{\gamma}_0) + (\Gamma_{\mu_1 \ldots \mu_n} \Pi_+)_{CD} \sin(\beta_0 \tilde{\gamma}_0) \right] x^{2 F_{0-2 \beta_0} \mathcal{M}_{AB}^{(\eta)} e^{i \eta \gamma_0 \tilde{\gamma}_0} |A\rangle_{-1/2} |\tilde{B}\rangle_{-3/2} \right\} . \quad (A.24)
$$

Let us compute first the superghost contribution. Using Eq. (A.14), we get

$$
\langle -\frac{1}{2} | -3/2 \rangle \cos(\beta_0 \tilde{\gamma}_0) x^{-2 \gamma_0 \beta_0} e^{i \eta \gamma_0 \tilde{\gamma}_0} | -1/2 \rangle | -3/2 \rangle = \frac{1}{2} \left[ \frac{1}{1 - \eta x^2} + \frac{1}{1 + \eta x^2} \right] ,
$$

$$
\langle -\frac{1}{2} | -3/2 \rangle \sin(\beta_0 \tilde{\gamma}_0) x^{-2 \gamma_0 \beta_0} e^{i \eta \gamma_0 \tilde{\gamma}_0} | -1/2 \rangle | -3/2 \rangle = \frac{1}{2i} \left[ \frac{1}{1 - \eta x^2} - \frac{1}{1 + \eta x^2} \right] ;
$$
then, after some simple manipulations, Eq. \((A.24)\) becomes
\[
- V_{p+1} T_p \frac{1}{8\sqrt{2}} \left\{ \text{Tr} \left( x^{2F_0} \Pi_+ \Gamma_{\mu_0...\mu_1} C^{-1} \mathcal{M}^{(\eta)} \right) \left( \frac{1}{1 - \eta x^2} + \frac{1}{1 + \eta x^2} \right) + 
- i \text{Tr} \left( x^{2F_0} \Pi_- \Gamma_{\mu_0...\mu_1} C^{-1} \mathcal{M}^{(\eta)} \right) \left( \frac{1}{1 - \eta x^2} - \frac{1}{1 + \eta x^2} \right) \right\} . \tag{A.25}
\]
Finally, summing over the two R-R spin-structures to perform the GSO projection, we get
\[
\langle P^{(C)}_{\mu_1...\mu_n} | B \rangle_R = - V_{p+1} T_p \frac{1 - (-1)^{n+p}}{16\sqrt{2}} \times \lim_{x \to 1} \left\{ \text{Tr} \left( x^{2F_0} \Gamma_{\mu_0...\mu_1} \Gamma_0 \ldots \Gamma^p ; e^{-1/2\hat{F}_{\alpha\beta} \Gamma_\alpha \Gamma_\beta} \right) \frac{1}{1 + x^2} + \text{Tr} \left( x^{2F_0} \Gamma_{\mu_0...\mu_1} \Gamma_0 \ldots \Gamma^p \Gamma_{11} ; e^{-1/2\hat{F}_{\alpha\beta} \Gamma_\alpha \Gamma_\beta} \right) \frac{1}{1 - x^2} \right\} . \tag{A.26}
\]
If the indices $\mu_1 \ldots \mu_n$ are all along the world-volume of the D$p$-brane (which is precisely the case of interest in this paper), then the term containing $\Gamma_{11}$ in Eq. \((A.27)\) vanishes; therefore, taking the limit $x \to 1$ and observing that $n$ and $p$ must have opposite parity in order to have a non-zero result, we finally obtain
\[
\langle P^{(C)}_{\mu_1...\mu_n} | B \rangle_R = - V_{p+1} T_p \frac{1}{16\sqrt{2}} \text{Tr} \left( \Gamma_{\mu_0...\mu_1} \Gamma_0 \ldots \Gamma^p ; e^{-1/2\hat{F}_{\alpha\beta} \Gamma_\alpha \Gamma_\beta} \right) . \tag{A.27}
\]
Taking into account the relation
\[
\langle P^{(C)}_{\mu_1...\mu_n} | \rangle = \frac{C^{\mu_1...\mu_n}}{n!} = \langle C^{(n)} | , \tag{A.28}
\]
we see that Eq. \((A.27)\) reproduces Eq. \((2.31)\).

In the final part of this appendix we explicitly write the form of the conjugate boundary state that has to be used in Eq. \((4.26)\). For the bosonic coordinate we have
\[
\langle B_X \rangle = \langle 0 \rangle \langle B_X | \exp \left[ - \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \cdot S \cdot \tilde{\alpha}_n \right] , \tag{A.29}
\]
while for the fermionic part we have
\[
\text{NS} \langle B_\psi, \eta \rangle = i \langle 0 | \exp \left[ -i \eta \sum_{m=1/2}^{\infty} \psi_m \cdot S \cdot \tilde{\psi}_m \right] \tag{A.30}
\]
in the NS-NS sector, and
\[
\text{R} \langle B_\psi, \eta \rangle = - \langle 0 | \langle B, \eta | \exp \left[ -i \eta \sum_{m=1}^{\infty} \psi_m \cdot S \cdot \tilde{\psi}_m \right] \tag{A.31}
\]
in the R-R sector, where
\[ \langle 0 \rangle_R^{(a)} | B, \eta \rangle = (-1)^p | A \rangle \langle B | \left( C \Gamma^0 \Gamma^1 \ldots \Gamma^p \frac{1 - i \eta \Gamma_{11}}{1 + i \eta} U \right)_{AB} \] (A.32)
with $U$ given in Eq. (2.14).

\section*{Appendix B}

In this appendix we show how to derive from the boundary state the classical solution corresponding to the bound state $(W,D_p)$ between a Kaluza-Klein wave $W$ and a $D_p$ brane. The $(W,D_p)$ bound state can be easily obtained from a $(F,D_{(p+1)})$ configuration by performing a T-duality along the longitudinal spatial direction of the fundamental string \textit{i.e.} along $X^1$ in our conventions. As shown in Ref. [5], a T-duality transformation in a given direction is simply realized by changing the sign in the corresponding row of the $S$ matrix which defines the boundary state. Therefore, in our case, we can start from the $(p+2) \times (p+2)$ longitudinal $S$ matrix of the $(F,D(p+1))$ configuration given in Eq. (3.2), and then change the sign of the second row corresponding to $X^1$ to get
\[
S_{\alpha\beta} = \begin{pmatrix}
-\frac{1+f^2}{1-f^2} & \frac{2f}{1-f^2} \\
\frac{2f}{1-f^2} & -\frac{1+f^2}{1-f^2} \\
\vdots & \ddots & \ddots \\
1 & \ddots & \ddots & 1
\end{pmatrix}.
\] (B.1)
Notice that the matrix (B.1) is symmetric, as opposed to the one of Eq. (3.2) which is antisymmetric.

Following the same procedure described in Section 3, we can obtain the long distance behavior of the massless fields emitted by the $(W,D_p)$ bound state. For simplicity, we will concentrate on the massless fields of the NS-NS sector. When written in terms of $T_{\mu\nu}$ (see Eq. (3.7)), the long distance behavior of these fields is formally equal to the one of the $(F,D_{p+1})$ bound state and is given by Eqs. (3.8), (3.10) and (3.12). However, since the new matrix $T_{\mu\nu}$ is symmetric, we can immediately conclude the no Kalb-Ramond field is emitted by the $(W,D_p)$ bound state. On the contrary, there is an off-diagonal component in $T_{\mu\nu}$ which gives rise to an off-diagonal component in the metric. This is the distinctive feature of this configuration.

Writing the boundary state with the matrix (B.1) where the external field $f$ is given by Eq. (3.17), and performing a Fourier transformation, we then get the
following long distance behavior for the dilaton

$$\varphi \simeq \frac{3 - p}{4} \frac{n^2}{\Delta_{m,n}} \frac{Q_{p+1}}{r^{6-p}}, \quad (B.2)$$

and for the metric components

$$g_{00} \simeq -1 - \left(1 + \frac{p + 1}{8} \frac{n^2}{\Delta_{m,n}}\right) \frac{Q_{p+1}}{r^{6-p}},$$

$$g_{01} = g_{10} \simeq - \frac{m}{\Delta_{m,n}^{1/2}} \frac{Q_{p+1}}{r^{6-p}},$$

$$g_{11} \simeq 1 + \left(1 + \frac{p - 7}{8} \frac{n^2}{\Delta_{m,n}}\right) \frac{Q_{p+1}}{r^{6-p}},$$

$$g_{22} = \ldots \simeq g_{pp} \simeq 1 + \frac{p - 7}{8} \frac{n^2}{\Delta_{m,n}} \frac{Q_{p+1}}{r^{6-p}}, \quad (B.3)$$

$$g_{p+1,p+1} = \ldots \simeq g_{99} \simeq 1 + \frac{p + 1}{8} \frac{n^2}{\Delta_{m,n}} \frac{Q_{p+1}}{r^{6-p}}.$$

Using these expressions and assuming that the complete solution of the (W,Dp) bound state can be written in terms of the harmonic functions $H$ and $H'$ defined in Eqs. (3.26) and (3.27) (with $p$ replaced by $(p + 1)$), we can infer that the dilaton is

$$e^{\varphi} = \frac{H'^{(3-p)}/4}{}, \quad (B.4)$$

and the metric is

$$ds^2 = - H^{-1} H'^{(p+1)/8} (dx^0)^2 + H H'^{(p-7)/8} (dx^1 + A_0 dx_0)^2$$

$$+ H'^{(p-7)/8} \left[ (dx^2)^2 + \cdots + (dx^p)^2 \right]$$

$$+ H'^{(p+1)/8} \left[ (dx^{p+1})^2 + \cdots + (dx^9)^2 \right], \quad (B.5)$$

where the Kaluza-Klein vector potential $A_0$ is

$$A_0 = \frac{m}{\Delta_{m,n}^{1/2}} \left( H^{-1} - 1 \right). \quad (B.6)$$

This solution agrees with the one presented in Ref. [10], except that our Kaluza-Klein vector has a different sign and a vanishing asymptotic value. For the R-R potentials, after performing the T-duality, one proceeds as we discussed in Section 3 and obtains two R-R (p+1)-forms, one of electric type and one of magnetic type, whose expressions are similar to those of Eqs. (B.31) and (B.32) with obvious changes.
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