Coherent State Path Integrals \textit{without} Resolutions of Unity

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Abstract

¿From the very beginning, coherent state path integrals have always relied on a coherent state resolution of unity for their construction. By choosing an inadmissible fiducial vector, a set of “coherent states” spans the same space but loses its resolution of unity, and for that reason has been called a set of weak coherent states. Despite having no resolution of unity, it is nevertheless shown how the propagator in such a basis may admit a phase-space path integral representation in essentially the same form as if it had a resolution of unity. Our examples are toy models of similar situations that arise in current studies of quantum gravity.

1 Introduction

The use of coherent states in the construction of coherent state path integrals is by now a well-known story. Central to that construction is the resolution of unity afforded by a set of coherent states. By “weak coherent states” we mean a set of states that are continuously labelled and, for convenience, normalized, but, significantly, they do not admit a resolution of unity as an appropriately weighted integral of one-dimensional projection operators onto the coherent states. The example we have in mind is one for which...
the coherent states are defined by a unitary representation of a noncompact group, but the fiducial vector is not chosen from among the dense set of admissible vectors. Under circumstances, for example, that the fiducial vector is nevertheless a minimum uncertainty state, it becomes possible to define a rigorous phase-space path integral representation that is a close analog of previously defined cases involving continuous-time regularizations, even for these kinds of weak coherent states.

In the present paper, we outline the construction of such a path integral for the case of a single affine degree of freedom, i.e., for coherent states defined in terms of the affine group (often called the “ax + b group”) [3]. Such coherent states are, up to a phase factor [4], equivalent to the more commonly used SU(1,1) coherent states.

It is interesting to note that a field theory analog of the present phase-space path integral construction, which also involves a set of weak coherent states, arises in a recently developed program aimed at quantizing the gravitational field [5].

2 Kinematics and Weak Coherent States

A suitable basis for the Lie algebra of a one-dimensional affine group consists of the irreducible self-adjoint operators $Q > 0$ and $D$ that satisfy the affine commutation relation [3] (with $\hbar = 1$)

$$[Q, D] = iQ .$$  \hfill (1)

The uncertainty product for these operators reads $\Delta Q \Delta D \geq \langle Q \rangle / 2$, where $\langle Q \rangle \equiv \langle \psi | Q | \psi \rangle$, $(\Delta Q)^2 \equiv \langle (Q - \langle Q \rangle)^2 \rangle$, etc. If $|x\rangle$, $0 < x < \infty$, denote $\delta$-normalized eigenstates of $Q$, i.e., $Q|x\rangle = x|x\rangle$, then there is a two-parameter family of (real, normalized) minimum uncertainty states given by

$$\eta_{\alpha,\beta}(x) = N_{\alpha,\beta} x^\alpha e^{-\beta x}, \quad \alpha > -1/2, \quad \beta > 0 ,$$  \hfill (2)

with $N_{\alpha,\beta}$ chosen to ensure normalization. For convenience, we also set $\langle Q \rangle = 1$ which leads to a one-parameter family of minimum uncertainty states given by

$$\eta_{\beta}(x) = N_{\beta} x^{\beta-1/2} e^{-\beta x}, \quad \beta > 0 .$$  \hfill (3)
In this Schrödinger representation, \( D = -\frac{1}{2}i \left[ x (\partial / \partial x) + (\partial / \partial x) x \right] \). The affine group of interest is a two-parameter unitary group defined by

\[
U[p, q] \equiv e^{ipQ} e^{-i\ln(q)D}, \quad q > 0, \quad p \in \mathbb{R}. \tag{4}
\]

Next, we choose \( |\eta\rangle \) as a normalized fiducial vector, and define our set of coherent states as composed of the continuously labelled vectors

\[
|p, q\rangle \equiv U[p, q]|\eta\rangle, \tag{5}
\]

for all \(-\infty < p < \infty, 0 < q < \infty\). The overlap function of two such coherent states is given by

\[
\langle p, q | r, s \rangle = \frac{1}{\sqrt{qs}} \int_{0}^{\infty} \eta^*(x/q) e^{-ix(p-r)} \eta(x/s) \, dx \tag{6}
\]

for a general fiducial vector. For the specific functional form (3), this expression may be evaluated and reads

\[
\langle p, q | r, s \rangle = \{ (qs)^{-1/2}/[\frac{1}{2}(q^{-1} + s^{-1}) + \frac{1}{2}i\beta^{-1}(p - r)] \}^{2\beta}. \tag{7}
\]

Observe, apart from the factor \((qs)^{-\beta}\), this expression is an analytic function of \(q^{-1} + i\beta^{-1}p\) and \(s^{-1} - i\beta^{-1}r\).

Now, we require the usual resolution of unity. Since \( dp dq \) is the (left-) invariant group measure in these coordinates, we anticipate an expression of the form

\[
c \int \langle p, q | r, s \rangle \langle r, s | t, u \rangle \, dr \, ds = \langle p, q | t, u \rangle \tag{8}
\]

for some constant \(c, 0 < c < \infty\). Indeed, for the kernels given by (7) we find that (8) is valid whenever \(\beta > 1/2\), in which case \(c = 1/\{2\pi[1 - 1/(2\beta)]\}\). However, (8) fails if \(0 < \beta \leq 1/2\). More generally, an equation such as (8) requires the fiducial vector admissibility condition

\[
\langle Q^{-1} \rangle = \int_{0}^{\infty} x^{-1} |\eta(x)|^2 \, dx < \infty. \tag{9}
\]

This condition is evidently false whenever \(0 < \beta \leq 1/2\).

The foregoing is a well-known story [3]—and, indeed, the same issue also enters into wavelet theory as well [6].
2.1 Positive-definite functions

The coherent state overlap function \((\ref{eq:6})\) clearly fulfills the condition that

\[
\sum_{j,k=1}^{N} \alpha_j^* \alpha_k \langle p_j, q_j | p_k, q_k \rangle \geq 0 \tag{10}
\]

holds for all \(N < \infty\), and for the special case of \((\ref{eq:7})\), this important positive-definite-function condition holds over the whole range of \(\beta > 0\). We recall that a continuous, positive-definite function may be adopted as a reproducing kernel, which may then be used to define a reproducing kernel Hilbert space \([\ref{eq:7}]\). In the special examples based on \((\ref{eq:7})\), such a space is a functional Hilbert space \(C_{\beta}\) composed of bounded, continuous functions. A dense set of elements in \(C_{\beta}\) is composed of functions of the form

\[
\psi(p, q) \equiv \sum_{j=1}^{J} \alpha_j \langle p, q | p_j, q_j \rangle, \quad J < \infty . \tag{11}
\]

Let another such element be denoted by

\[
\phi(p, q) \equiv \sum_{k=1}^{K} \gamma_k \langle p, q | r_k, s_k \rangle, \quad K < \infty . \tag{12}
\]

In this case, the inner product of these two elements is defined to be

\[
\langle \psi | \phi \rangle \equiv (\psi, \phi) \equiv \sum_{j=1}^{J} \sum_{k=1}^{K} \alpha_j^* \gamma_k \langle p_j, q_j | r_k, s_k \rangle . \tag{13}
\]

The completion of this space obtained by including the limit points of all Cauchy sequences in the norm \(\| \psi \| \equiv +\sqrt{(\psi, \psi)}\) determines the reproducing kernel Hilbert space \(C_{\beta}\) in each case. It is noteworthy that the resultant spaces \(C_{\beta}\) are mutually disjoint for distinct \(\beta\) values, \(\beta > 0\), save for the zero element which is common to all \(C_{\beta}\).

When \(\beta > 1/2\) it turns out that there is an alternative (local integral) way to evaluate the inner product of any two elements in \(C_{\beta}\). In that case, the coherent states admit a resolution of unity and therefore

\[
\langle \psi | \phi \rangle = (\psi, \phi) = \int \psi(p, q)^* \phi(p, q) \, d\mu(p, q) ,
\]

\[
d\mu(p, q) \equiv dp \, dq / 2\pi [1 - 1/(2\beta)] , \quad \beta > 1/2 . \tag{14}
\]
integrated over the space $\mathbb{R} \times \mathbb{R}^+$.  

The resolution of unity can be written in a more abstract form as 

$$ \int |p, q \rangle \langle p, q| d\mu(p, q) = 1 \ , $$  

and this expression is of paramount use in deriving a coherent state path integral for the propagator along conventional lines. For comparison purposes to what follows, we recall the most common construction [2] given, e.g., by 

$$ \langle p'', q''| e^{-iT\mathcal{H}}| p', q' \rangle = \langle p'', q''| e^{-i\epsilon\mathcal{H}} \cdots e^{-i\epsilon\mathcal{H}}| p', q' \rangle 
= \lim_{\epsilon \to 0} \int \cdots \int \Pi_{l=0}^{N} \langle p_{l+1}, q_{l+1}| e^{-i\epsilon\mathcal{H}}| p_{l}, q_{l} \rangle \Pi_{l=1}^{N} d\mu(p_{l}, q_{l}) 
= \int e^i \int \Pi_{l=0}^{N} \langle p_{l+1}, q_{l+1}|(1 - i\epsilon\mathcal{H})| p_{l}, q_{l} \rangle \Pi_{l=1}^{N} d\mu(p_{l}, q_{l}) 
= \mathcal{M} \int e^{-i} \int \Pi_{l=0}^{N} \langle p_{l+1}, q_{l+1}|(\mathcal{H}+\epsilon\mathcal{H})| p_{l}, q_{l} \rangle \Pi_{l=1}^{N} d\mu(p_{l}, q_{l}) d\mu(p, q) $$

where $\epsilon \equiv T/(N + 1)$ and $-\int q dp$ (and not $\int p dq$) occurs because of our phase convention. In this expression, $p_{N+1}, q_{N+1} \equiv p''$, $q''$; $p_{0}, q_{0} \equiv p'$, $q'$, and in the last two lines we have (unjustifiably!) interchanged the integrations and the continuum limit and written for the integrand the form it would take for continuous and differentiable paths. The line before the interchange of limits generally offers a well-defined integral representation. Finally, observe that this entire construction is premised on the existence of the resolution of unity—a relation that fails to hold in the case of weak coherent states.

### 3 Continuous-Time Regularization

#### 3.1 Complex polarization

For any $\beta > 0$, all elements of $\mathcal{C}_\beta$ satisfy the relation 

$$ B \psi(p, q) \equiv [-i\beta^{-1}\partial_p + 1 + \beta^{-1}q \partial_q] \psi(p, q) = 0 \ , $$

where $\partial_p \equiv \partial/\partial p$, etc. In this sense we say that each space $\mathcal{C}_\beta$ satisfies a *complex polarization* condition [8]. Consequently, the functions in $\mathcal{C}_\beta$ also satisfy $A \psi(p, q) = 0$, where $A \equiv \frac{1}{2} \beta B^1B$. 


Consider first the case where $\beta > 1/2$. In that case, $A$ is a nonnegative, self-adjoint operator for which 0 lies in the discrete spectrum and this value is separated from the rest of the spectrum by a nonzero gap \[9\]. It follows, for any $T > 0$, that
\[
\lim_{\nu \to \infty} (e^{-\nu TA}) \delta(p - p') \delta(q - q') = \Pi(p, q; p', q'),
\]
(18)
where $\Pi(p, q; p', q')$ is the integral kernel of a projection operator onto the subspace where $A = 0$, that is, onto the space $C_{\beta}$. But that integral kernel is also given by \(2\pi[1 - 1/(2\beta)]^{-1}\langle p, q|p', q'\rangle\). Finally, we observe that $A$ is a second-order differential operator, and by means of the Feynman-Kac-Stratonovich formula, we may represent the reproducing kernel by a functional integral in the limit that the parameter $\nu \to \infty$. Specifically, we obtain the representation given by
\[
\langle p''', q'''|p', q'\rangle = \lim_{\nu \to \infty} N_{\nu} \int e^{-i\int q \dot{p} dt - (1/2\nu)\int [\beta^{-1}q^2 \dot{p}^2 + \beta q^2 \dot{q}^2] dt} Dp Dq \\
= \lim_{\nu \to \infty} 2\pi[1 - 1/(2\beta)] e^{\nu T/2} \int e^{-i\int q dp} dW_{\nu}(p, q),\]
(19)
where the first expression is formal while the second is well defined in terms of a pinned Wiener measure $W_{\nu}$ [pinned so that $p(T), q(T) = p'', q''$; $p(0), q(0) = p', q'$] on a two-dimensional space of constant negative curvature $R = -2/\beta$, and in which the stochastic integral $-\int q dp$ is interpreted in the sense of Stratonovich (midpoint rule) [10].

When $0 < \beta \leq 1/2$, the situation changes significantly. In this case, 0 is part of the continuous spectrum of $A$. The operators $\exp(-\nu TA)$ still form a semi-group and admit a Feynman-Kac-Stratonovich representation. However, as $\nu \to \infty$, we need to isolate the subspace of nonsquare-integrable functions $C_{\beta}$ that make up the desired reproducing kernel Hilbert space. As may be expected, it becomes necessary to rescale the limiting operation to extract the desired set of functions.

A very simple illustration of the desired procedure may be given with the example $\tilde{A} \equiv \frac{1}{2}\tilde{B}^2$, where $\tilde{B} = -i\partial_x$ is defined for all $x, -\infty < x < \infty$. Here, too, $\tilde{A} = 0$ lies in the continuous spectrum of $\tilde{A}$. The desired space of functions is composed of those for which $\psi(x) = \text{const.}$, namely a one-dimensional space. We can choose the reproducing kernel for this space to
be identically one. Initially, we observe that
\[
(e^{-\nu T A}) \delta(x - x') \bigg|_{x=x''} = \frac{1}{\sqrt{2\pi \nu T}} e^{-\frac{(x'' - x')^2}{2\nu T}}
\]
\[
= \int dw^\nu(x) ,
\]
where \(w^\nu(x)\) denotes a pinned Wiener measure with diffusion constant \(\nu\) [10]. Finally, the reproducing kernel of interest is obtained by the limit
\[
1 = \lim_{\nu \to \infty} e^{-\frac{(x'' - x')^2}{2\nu T}}
\]
\[
= \lim_{\nu \to \infty} \sqrt{2\pi \nu T} \int dw^\nu(x) ,
\]
a procedure which has effectively selected out the space of (nonsquare integrable) functions for which \(\partial \psi(x)/\partial x = 0\). Observe, in the present case, that the proper \(\nu\)-dependent scaling factor may also be determined self-consistently since
\[
e^{-\frac{(x'' - x')^2}{2\nu T}} = \int dw^\nu(x)/\int_{x'=0}^{x''} dw^\nu(x) .
\]
Returning to the affine group case, we assume that the proper rescaling factor may be determined self-consistently as well. (This assumption is a generalization of a conjecture of Davies [11] regarding the long-time behavior of heat kernels based on the relevant Laplace-Beltrami operator.) With this assumption, it only remains to introduce the Feynman-Kac-Stratonovich representation which leads [5] directly to
\[
\langle p'', q'' | p', q' \rangle = \lim_{\nu \to \infty} K^\nu \left( e^{-\nu T A} \right) \delta(p - p') \delta(q - q') \bigg|_{p=p'', q=q''} = \lim_{\nu \to \infty} K^\nu \int e^{-i\int q \dot{p} dt} e^{-(1/2\nu)\int [\beta q^2 + 2q \beta q - 2q^2] dt} \mathcal{D}p \mathcal{D}q
\]
\[
= \lim_{\nu \to \infty} K^\nu \int e^{-i\int q dp} dW^\nu(p, q) ,
\]
where
\[
[K^\nu]^{-1} \equiv \left. (e^{-\nu T A}) \delta(p) \delta(q - 1) \right|_{p=0, q=1} .
\]
These equations provide the sought-for path integral representation for 0 < \(\beta \leq 1/2\) when the Hamiltonian vanishes.
3.2 Introduction of dynamics

When $\beta > 1/2$, the procedure to introduce a nonzero Hamiltonian has been worked out previously \cite{3}. In particular, when $\beta > 1/2$, it follows that

$$
\langle p'', q'' | e^{-i\hat{H}T} | p', q' \rangle = \lim_{\nu \to \infty} N_\nu \int e^{-i\int [q\dot{p} + h(p,q)] \, dt} e^{-(1/2\nu)\int [\beta^{-1} q^2 \dot{p}^2 + \beta q^{-2} \dot{q}^2] \, dt} \, \mathcal{D}p \, \mathcal{D}q
$$

$$
= \lim_{\nu \to \infty} 2\pi [1 - 1/(2\beta)] e^{\nu T/2} \int e^{-i\int [q\dot{p} + h(p,q)] \, dt} \, dW_\nu(p, q), \quad (25)
$$

where $\mathcal{H}$ and $h(p,q)$ are related by

$$
\mathcal{H} \equiv \int h(p,q) \, |p,q\rangle \langle p,q| \, d\mu(p,q) \quad (26)
$$

or equivalently by

$$
\langle p'', q''| \mathcal{H}|p', q'\rangle = \int \langle p'', q''| p,q \rangle \, h(p,q) \, \langle p,q| p', q'\rangle \, d\mu(p,q). \quad (27)
$$

Observe that this phase-space path integral involves $h(p,q)$, which is a different symbol associated with the operator $\mathcal{H}$ than the previously used symbol $H(p,q)$. These two symbols are connected by

$$
H(p', q') = \langle p', q'| \mathcal{H}|p', q'\rangle = \int |\langle p', q'| p,q \rangle|^2 \, h(p,q) \, d\mu(p,q). \quad (28)
$$

Equations (25) and (26) [or (27)] provide a satisfactory solution for a continuous time regularized, coherent state path-integral representation for the propagator when the coherent states possess a resolution of unity, i.e., in the present case whenever $\beta > 1/2$.

We now turn our attention to the case where $0 < \beta \leq 1/2$ and no resolution of unity exists. First, we offer several simple examples to show, nevertheless, that a solution to this problem may possibly exist. Initially, observe that if $h(p,q) = k$, a constant, then the expected path-integral representation [cf. (23) and (25)] yields the correct answer even though the standard relations (26) or (27) are meaningless!

For the next set of examples, with $R$ and $S$ arbitrary $c$-number parameters, we assert that

$$
\langle p'', q''| e^{-i(RQ+SD)T} | p', q' \rangle = \langle p'' e^{ST} + (R/S)(e^{ST} - 1), q'' e^{-ST}| p', q' \rangle
$$
\[ \lim_{\nu \to \infty} K_{\nu} \int_{p', q'} e^{ST+(R/S)(e^{ST}-1)} e^{-i \int q dp} dW^\nu(p, q) \]
\[ = \lim_{\nu \to \infty} K_{\nu} \int_{p', q'} e^{-i \int [q dp+(rq+spq) dt]} dW^\nu(p, q). \quad (29) \]

In this relation, \( r(t) \) and \( s(t), 0 \leq t \leq T, T > 0, \) are any smooth functions for which
\[ T e^{-i \int_0^T [r(t)Q+s(t)D] dt} \equiv e^{-i(RQ+SD)T}, \quad (30) \]
where \( T \) denotes the time-ordering operator. In deriving the last line of (29) we have changed variables in a well-defined integral and observed that the associated changes that appear in the measure will all effectively disappear when \( \nu \to \infty. \) In particular, the required change of variables is that given by
\[ p(t) \to p(t)e^{t \overline{\sigma}(t)} + [\overline{\sigma}(t)/\overline{\sigma}(t)][e^{t \overline{\sigma}(t)} - 1], \]
\[ q(t) \to q(t)e^{-t \overline{\sigma}(t)}, \]
\[ t \overline{\sigma}(t) \equiv \int_0^t s(u) du, \quad t \overline{\sigma}(t) \equiv \int_0^t r(u) du, \]
\[ \overline{\sigma}(T) \equiv S, \quad \overline{\sigma}(T) \equiv R. \quad (31) \]

Equation (29) serves to evaluate the propagator in the case that \( H = RQ + SD \) and by linearity and completeness of the basic operators (with \( T = 1 \) \( \exp[-i(RQ+SD)] \)), Eq. (29) implicitly evaluates the propagator for a much wider class of Hamiltonians.

More generally, we assert in the case \( 0 < \beta \leq 1/2 \) that the following path integral representation holds:
\[ \langle p^{''}, q^{''} | e^{-iH\nu} | p', q' \rangle \]
\[ = \lim_{\nu \to \infty} \nu \int e^{-i \int [q dp + h(p, q)] dt} e^{-(1/2\nu) \int [\beta^{-1} q^2 \hat{p}^2 + \beta q^{-2} \hat{q}^2] dt} Dp Dq \]
\[ = \lim_{\nu \to \infty} K_{\nu} \int e^{-i \int [q dp + h(p, q)] dt} dW^\nu(p, q), \quad (32) \]
where, in the present case, the connection between \( H(p, q) \) is implicitly given by
\[ \langle p^{''}, q^{''} | H | p', q' \rangle = \lim_{\nu \to \infty} K_{\nu} \int e^{-i \int q dp} h(p(u), q(u)) dW^\nu(p, q) \quad (33) \]
for any \( u, 0 < u < T \). Indeed—thanks to analyticity—the diagonal matrix elements determine the operator uniquely in the present case, and we may therefore assert that

\[
H(p'', q'') \equiv \langle p''', q'''| \mathcal{H}| p''', q'''angle
= \lim_{\nu \to \infty} K_\nu \int e^{-i \oint q dp} h(p(u), q(u)) dW_\nu(p, q).
\]

Finally, we conjecture that these relations hold, at least, for the class of self-adjoint Hamiltonian operators \( \mathcal{H} \) composed of semi-bounded polynomials of the basic operators \( Q \) and \( D \).

4 Conclusion

In Eqs. (32) and (33) [or (34)] we have arrived at our desired goal of representing the propagator in terms of a phase-space path integral that determines the propagator as coherent-state matrix elements of the evolution operator for a special class of weak coherent states that do not admit a conventional coherent state resolution of unity. It is also noteworthy that Eqs. (32) and (33) [or (34)] also hold when \( \beta > 1/2 \), and thus these equations characterize the entire range of \( \beta, \beta > 0 \).

Finally, we again remark that in recent studies of quantum gravity [5], weak coherent states are used in an essential way. And, analogous to the elementary examples presented in the present paper, a phase-space functional integral representation is constructed and used despite the fact that no conventional coherent state resolution of unity exists.

Dedication

It is a pleasure to dedicate this article to Martin C. Gutzwiller. Martin has been and continues to be an inspiration to us all in his dedication to quality research carried out in a careful way. One may even suppose that working at a Major Industrial Laboratory for a good part of his scientific career has been of particular value in shaping his life’s work in such a desirable manner!
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