Essential dimension of Abelian varieties over number fields

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Abstract

We show that the essential dimension of a non-trivial Abelian variety over a number field is infinite. To cite this article: P. Brosnan, R. Sreekantan, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

Résumé

Dimension essentielle d’une variété abélienne sur un corps de nombres. On montre que la dimension essentielle d’une variété abélienne non-triviale définie sur un corps de nombres est infinie. Pour citer cet article : P. Brosnan, R. Sreekantan, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

1. Introduction

Let $k$ be a field and let $	ext{Fields}_k$ denote the category whose objects are field extensions $L/k$ and whose morphisms are inclusions $M \hookrightarrow L$ of fields. Let $F : \text{Fields}_k \to \text{Sets}$ be a covariant functor. A field of definition for an element $a \in F(L)$ is a subfield $M$ of $L$ over $k$ such that $a \in \text{im}(F(M) \to F(L))$. The essential dimension of $a \in F(L)$ is $\text{ed}a := \inf \{\text{trdeg}_k M \mid M \text{ is a field of definition for } a\}$. The essential dimension of the functor $F$ is $\text{ed} F := \sup \{\text{ed} a \mid L \in \text{Fields}_k, a \in F(L)\}$.

If $G$ is an algebraic group over $k$, we write $\text{ed} G$ for the essential dimension of the functor $L \mapsto H^1_{\text{fppf}}(L, G)$. That is $\text{ed} G$ is the essential dimension of the functor sending a field $L$ to the set of isomorphism classes of $G$-torsors over $L$. The notion of essential dimension of a finite group was introduced by J. Buhler and Z. Reichstein. The definition of the essential dimension of a functor is a generalization given later by A. Merkurjev. In [3] (which the reader could consult for further background), a notion of essential dimension for algebraic stacks was introduced. In the terminology of that paper, $\text{ed} G$ is the essential dimension of the stack $\mathcal{B}G$.

The purpose of this Note is to generalize the following result:
**Theorem 1.1.** (Corollary 10.4 [3].) Let $E$ be an elliptic curve over a number field $k$. Assume that there is at least one prime $p$ of $k$ where $E$ has semistable bad reduction. Then $\operatorname{ed} E = +\infty$.

Note that another equivalent way of stating the theorem is to say that $\operatorname{ed} E = +\infty$ for any elliptic curve $E$ over a number field such that $j(E)$ is not an algebraic integer. The result was proved by showing that Tate curves have infinite essential dimension. This method does not apply to elliptic curves with integral $j$ invariants. Nonetheless, Conjecture 10.5 of [3] guesses that $\operatorname{ed} E = +\infty$ for all elliptic curves over number fields. This conjecture is answered by the following:

**Theorem 1.2.** Let $A$ be a non-trivial Abelian variety over a number field $k$. Then $\operatorname{ed} A = +\infty$.

Note that if $A$ is an Abelian variety over $\mathbb{C}$, then $\operatorname{ed} A = 2 \dim A$. This is the main result of [2].

The theorem is an easy consequence of the following result whose formulation does not involve essential dimension. To state it, for a positive integer $m$, let $\mu_m$ denote the group scheme of $m$th roots of unity; and, for a rational prime $l$, let $\mu_l^\infty$ denote the union $\bigcup_{n \in \mathbb{Z}_+} \mu_l^n$.

**Theorem 1.3.** Let $A$ be a non-trivial Abelian variety over a number field $k$. Then there is an odd prime $\ell$ and an algebraic field extension $L/k$ such that

(i) $\mathbb{Q}_\ell / \mathbb{Z}_\ell \subset A(L)$.
(ii) $1 < |\mu_{\ell^\infty}(L)| < \infty$.

In Section 2, we derive Theorem 1.2 from Theorem 1.3. To do this, we use a result of M. Florence concerning the essential dimension of $\mathbb{Z}/\ell^n$. In Section 3, we prove Theorem 1.3. Here the main results used are those of Bogomolov and Serre on the action of the absolute Galois group $\operatorname{Gal}(k)$ on the Tate module $T_\ell A$.

**Remark 1.4.** The recent preprint [7] of Karpenko and Merkurjev provides another way to show that the essential dimension of an Abelian variety over a number field is infinite. To be precise, by generalizing the results of that paper slightly, one can use them to compute the essential dimension of the group scheme $A[n]$ of $n$-torsion points of an Abelian variety. In fact, using this idea one can show that the essential dimension of an Abelian variety over a $p$-adic field is also infinite. However, the present proof of Theorem 1.2 is shorter than a proof using [7] would be and we hope that Theorem 1.3 is independently interesting.

2. **Theorem 1.3 implies Theorem 1.2**

As mentioned above, we will use the following result [6, Theorem 4.1] of M. Florence:

**Theorem 2.1.** Let $\ell$ be an odd prime and $r$ a positive integer. Let $L / \mathbb{Q}$ be a field such that $|\mu_{\ell^\infty}(L)| = \ell^r$. Then, for any positive integer $k$,

$$\operatorname{ed}_L \mathbb{Z}/\ell^k = \max \{ 1, \ell^{k-r} \}.$$  

**Corollary 2.2.** Let $A$ be an Abelian variety over a field $L$ of characteristic 0. Let $\ell$ be an odd prime and suppose that the statements in the conclusion of Theorem 1.3 are satisfied; i.e.:

(i) $\mathbb{Q}_\ell / \mathbb{Z}_\ell \subset A(L)$.
(ii) $1 < |\mu_{\ell^\infty}(L)| < \infty$.

Then $\operatorname{ed} A = +\infty$.

**Proof.** Since $L$ satisfies (ii), $\operatorname{ed}_L \mathbb{Z}/\ell^n \to \infty$ as $n \to \infty$. By (i), there is an injection $(\mathbb{Z}/\ell^n)_L \to A$. Therefore, by [1, Theorem 6.19], $\operatorname{ed} A \geq \operatorname{ed}_L \mathbb{Z}/\ell^n - \dim A$ for all $n$. Letting $n$ tend to $\infty$, we see that $\operatorname{ed} A = +\infty$. \qed
Proof of Theorem 1.2 assuming Theorem 1.3. Let $A$ be a non-trivial Abelian variety over a number field $k$. Using Theorem 1.3 and Corollary 2.2, we can find a field extension $L/k$ such that $ed A_L = +\infty$. This implies that $ed A = +\infty$ (by [1, Proposition 1.5]).

3. Galois representations and the proof of Theorem 1.3

Let $A$ be a non-trivial Abelian variety over $k$ as in Theorem 1.3. Before proving Theorem 1.3, we fix some (standard) notation. We write $\text{Gal}(k) := \text{Gal}(\bar{k}/k)$ for the absolute Galois group of the number field $k$. For a rational prime $\ell$, we write $T_\ell A$ for the Tate-module $\lim A[\ell^n]$ of the Abelian variety $A$. We write $V_\ell A$ for $T_\ell A \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell$. For an integer $n$, we write $\mathbb{Z}/n(1)$ for $\mu_n$, and for $j \in \mathbb{Z}$, we write $\mathbb{Z}/n(j)$ for $\mu_n^j$. We write $\mathbb{Z}_\ell(j) := \varprojlim \mathbb{Z}/l^m(j)$.

For any prime $p$ of $k$ where $A$ has good reduction, write $T_p$ for the corresponding Frobenius torus. (For this notion see [4, Definition 3.1 and p. 326] or [9].) Since $A$ is non-trivial, $T_p$ contains a rank 1 torus $D \cong \mathbb{G}_m$ such that, for every rational prime $\ell \notin p$, $D(\mathbb{Q}_\ell) \subset \text{GL}(V_\ell A)$ is the set of homotheties (i.e. scalar matrices) [4, Proposition 3.2].

**Lemma 3.1.** Let $p$ be a prime of $k$ such that the reduction $A/p$ of $A$ at $p$ is good but not supersingular. Then the rank of $T_p$ is strictly greater than 1.

**Proof.** This follows directly from [4, Proposition 3.3].

The following lemma was suggested to us by N. Fakhruddin:

**Lemma 3.2.** Let $V$ be an $n$-dimensional vector space over a field $F$, and let $T$ be an $F$-split torus in $\text{GL}_V$ of rank at least 2 containing the homotheties. Then there is a non-zero vector $v \in V$ and a rank 1 subtorus $S$ of $T$ such that

(i) $S$ fixes $v$;
(ii) the determinant map $\det : S \to \mathbb{G}_m$ is surjective.

**Proof.** The proof is elementary linear algebra with the character lattice, $X^*(T)$.

We can find a basis $e_1, \ldots, e_n$ of $V$ and characters $\lambda_1, \ldots, \lambda_n \in X^*(T)$ such that $t e_i = \lambda_i(t) e_i$ for $t \in T$, $i \in \{1, \ldots, n\}$. Since $T$ contains the homotheties, $\det$ is a non-trivial character of $T$. Moreover, since $T \subset \text{GL}_V$, the $\lambda_i$ generate $X^*(T)$. Since $\text{dim } X^*(T) \otimes \mathbb{Q} \geq 2$, it follows that there exists $i$ such that $\lambda_i^\perp \not\subset \text{det}^\perp$. Thus we can find a cocharacter $\nu$ such that $\langle \nu, \lambda_i \rangle = 0$ but $\langle \nu, \text{det} \rangle \neq 0$. Set $S$ equal to the image of $\nu$ in $T$ and $v = e_i$.

**Proof of Theorem 1.3.** Let $A$ be a non-trivial Abelian variety over a number field $k$. We can find a prime $p$ in $k$ such that $A$ has good reduction at $p$ but $A/p$ is not supersingular. (This is well known if $\text{dim } A = 1$: the case where $A$ has CM is standard and otherwise it follows from the exercise on page IV-13 of [8].) When $\text{dim } A > 1$ it can be proved by adapting the exercise as Ogus does in Corollary 2.8 of his notes in [5].) Thus the Frobenius torus $T_p$ has rank at least 2. Using Tchebotarev density, it is easy to see that $T_p \otimes \mathbb{Q}_\ell$ is a split torus for all rational primes $\ell$ in a set of positive density. Thus, we can find an odd rational prime $\ell$ such that $\ell \notin p$ and $T_p \otimes \mathbb{Q}_\ell$ is split. Now, set $F = k(\zeta_\ell)$ where $\zeta_\ell$ is a primitive $\ell$th root of unity. Note that $T_p$ is the Frobenius torus for $A_F$ as Frobenius tori are invariant under finite extension of the ground field.

Now, using Lemma 3.2, we can find a rank 1 subtorus $S \subset T_p \otimes \mathbb{Q}_\ell$ and a vector $v \in T_{\ell} A_F$ such that $S$ fixes $v$ and $\det : S \to \mathbb{G}_m$ is surjective. Let $\rho : \text{Gal}(F) \to \text{Aut}(V_{\ell} A_F)$ denote the Galois representation on the Tate module and let $H = \{ g \in \text{Gal}(F) \mid \rho(g)v = v \}$. By a theorem of Bogomolov [4, Theorem B] (and the fact that $S$ fixes $v$), it follows that

$$\text{Lie}(S) \subset \text{Lie}(\rho(H))$$

where $\text{Lie}(S)$ denotes the Lie algebra of $S$ as an algebraic group and $\text{Lie}(\rho(H))$ denotes the Lie algebra as an $\ell$-adic group. Therefore the intersection of $S(\mathbb{Q}_\ell)$ with $\rho(H)$ contains an open neighborhood of the identity in $S(\mathbb{Q}_\ell)$. In particular, $\text{det}(H)$ contains a neighborhood of the identity in $\mathbb{Q}_\ell^\times$. Set $L := \hat{F}^H$. Then, from the fact that $v$ is fixed by $H$, it follows that $\mathbb{Q}_\ell / \mathbb{Z}_\ell \subset A(L)$. On the other hand, since $\wedge^{\text{dim } A} T_{\ell} A \cong \mathbb{Z}_\ell(\text{dim } A)$, the fact that $\text{det}(H)$ contains a neighborhood of the identity in $\mathbb{Q}_\ell^\times$ implies that $\mu_{\ell \infty}(L)$ is finite. This completes the proof of Theorem 1.3. \qed
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