Invariance of $\phi^4$ measure under nonlinear wave and Schrödinger equations on the plane

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Abstract

We show probabilistic existence and uniqueness for the Wick-ordered cubic nonlinear wave equation in a weighted Besov space over $\mathbb{R}^2$. To achieve this, we show that a weak limit of $\phi^4$ measures on increasing tori is invariant under the equation. We review and slightly simplify the periodic theory and the construction of the weak limit measure, and then use finite speed of propagation to reduce the infinite-volume case to the previous setup. Our argument also gives a weak (Albeverio-Cruzeiro) invariance result on the nonlinear Schrödinger equation in the same setting.

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1 Introduction

Since Jean Bourgain’s work in the 1990s, invariant measures have been an important tool in probabilistic solution theory of dispersive PDEs. Bourgain initially considered the nonlinear Schrödinger equation

\begin{equation}
\label{NLS}
i \partial_t u(x,t) + \Delta u(x,t) = \pm \lambda : u^3:
\end{equation}

on one-dimensional torus $\mathbb{T}$ [8]. He proved almost sure well-posedness in the Sobolev space $H^{1/2-\varepsilon}$ when the initial data is distributed according to a so-called $\phi^4$ measure. Later on in [9], he extended the result to $\mathbb{T}^2$. In two or more dimensions the $\phi^4$ measure is supported on distributions, and it then becomes necessary to renormalize the nonlinearity $u^3$ by Wick ordering, denoted by $: u^3 :$.

Our main subject is the defocusing massive nonlinear wave equation

\begin{equation}
\label{NLW}
\partial_{tt} u(x,t) + (m^2 - \Delta) u(x,t) = -\lambda : u^3 : \tag{NLW}
\end{equation}

on two-dimensional spatial domain $\mathbb{R}^2$. This equation was previously solved on periodic domain by Oh and Thomann [46]. The main result of this article can be stated as follows:

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Theorem 1.1 (Global existence and uniqueness). Let $\mu$ be the product of infinite-volume $\phi^4$ and white-noise measures, and $\varepsilon > 0$. For $\mu$-almost all initial data and any $T > 0$, the nonlinear wave equation (NLW) has a unique global solution in the weighted space $L^2([0,T]; H^{-2\varepsilon}(\rho))$.

Our approach is to first construct solutions in periodic domains $[-L,L]^2$, and then approximate infinite-volume solutions with them. The high-level proof strategy in a periodic domain goes back to Bourgain:

1. Define a probability distribution on the initial data.
2. Prove deterministic well-posedness for time interval $[0,\tau]$ when the initial data belongs to some set $A$ of large probability. The small time $\tau$ depends on the size of $A$.
3. Prove that the probability measure is invariant in time under the equation.
4. Intersect the sets of initial and final values, which have same probability by invariance. By iteration, the probability of blow-up by time $T = n\tau$ is bounded by $n(1 - P(A))$.
5. Use stochastic estimates to show that an increase of $P(A)$ cancels the corresponding increase of iterations $n$; thus the probability of blow-up can be made arbitrarily small.

This argument reduces the global solution theory into understanding the invariance and large deviations of a suitable probability measure. To show invariance, we use finite-dimensional approximation. Liouville’s theorem states that the Gibbs measure associated to the Hamiltonian formulation of (NLW) is invariant. These approximate measures converge in total variation to the untruncated, periodic-domain measures.

The extension to infinite volume relies on two insights. The first is the existence of uniform bounds for the measures in a polynomially weighted space [38]. This yields a convergent subsequence of measures as $L \to \infty$. The second step is to use the finite speed of propagation of (NLW) to reduce all statements about measurable events to the periodic case.

1.1 The $\phi^4$ measure

What is the natural candidate for the invariant measure? As mentioned above, Fourier-truncated versions of these equations converge the Hamiltonian $H$, with which we can define the Gibbs measure proportional to $\exp(-\beta H)$. The parameter $\beta > 0$ is called the inverse temperature. For truncated and periodic (NLW), the Gibbs measure has density\(^1\)

$$\exp \left( -\beta \int_{\mathbb{T}^d} \frac{\lambda: u^4}{4} + \frac{m^2|u|^2}{2} + |\nabla u|^2 + |\partial_t u|^2}{d\mathbb{R}^d} \right)$$ (1.1)

with respect to Lebesgue measure on the Fourier coefficients. The second term yields a Gaussian factor that can be utilized in removing the truncation.

\(^1\)In the following, we set $\beta = \lambda = 1$ as they are not too relevant for our present topic.
The continuum versions of these Gibbs measures are studied in constructive quantum field theory [23]. Stochastic quantization (see e.g. [50]) is a rigorous PDE approach for their study. In this approach the $\phi^4_2$ measure is regarded as an invariant measure for a nonlinear heat equation with white noise forcing. These equations are singular and cannot be solved classically.

The periodic $\phi^4_2$ equation was solved by Da Prato and Debussche [19]. The limit measure is absolutely continuous with respect to a Gaussian measure. Existence of infinite-volume solutions for the 2D equation was later shown by Mourrat and Weber in a polynomially weighted space [38]; see also [37]. We will rely heavily on these ideas in Section 3.

The local well-posedness theory for the more singular 3D case came in three approaches in mid-2010s: Hairer’s regularity structures [29]; Gubinelli, Imkeller and Perkowski’s [25] paracontrolled distributions; and Kupiainen’s renormalization group approach [33]. The bounds of Mourrat and Weber were then exploited by Albeverio and Kusuoka [3] and Gubinelli and Hofmanová [24] to give a self-contained construction of the $\phi^4_3$ measure.

In dimensions $d \geq 4$, the $\phi^4_d$ measures collapse to trivial Gaussian measures. The final case $d = 4$ was proved recently by Aizenman and Duminil-Copin; see their article [1] for discussion.

The $\phi^4_4$ measure is expected to be invariant under three PDEs that share essentially the same Hamiltonian: (NLS), (NLW), and the cubic stochastic nonlinear heat equation. As shown in [13, Figure 1], the periodic-domain invariance theory is almost done, with only the three-dimensional Schrödinger missing.

This theory, and hence the global well-posedness of the equations, is much less developed in the infinite volume. For wave and Schrödinger equations the previous results are limited to one dimension [10] or radial setting [56].

The largest complication is that the infinite-volume $\phi^4_d$ measures are only defined as weak limits of approximating sequences, and in particular they are no longer absolutely continuous with respect to Gaussian measure. This means that total variation convergence is no longer available and we have to prove local well-posedness for non-Gaussian initial data. Depending on the coupling constant $\lambda$, the accumulation point of the approximating sequence might not be unique.

However, we are able to show that our invariant distribution can still be coupled to a Gaussian and the perturbation term enjoys better analytic bounds. A similar fact was exploited by Bringmann and collaborators in [11, 12, 13], in situations where the singularity of the measure arises in finite volume due to short scale divergencies.

For the nonlinear Schrödinger equation the situation is even more complicated, as there is no finite speed of propagation. This means that we cannot reduce the problem to the periodic setup. However, by giving up some differentiability and thus uniqueness, we can still prove a weaker form of invariance. This sense of invariance was initially developed for Euler and Navier–Stokes equations by Albeverio and Cruzeiro [2], and was explored in the case of periodic 2D NLS in [45].

As this manuscript was being prepared, Oh, Tolomeo, Wang, and Zheng published their preprint [47] where similar ideas appear. They prove Theorem 1.1
for a more challenging equation, (NLW) with additive stochastic forcing. Their approach is based on an optimal transport argument developed in [39], whereas our globalization argument depends more heavily on finite speed of propagation. There are also slight differences in the use of stochastic quantization, and we also consider the Schrödinger equation.

1.2 Previous literature and extensions

Let us take a moment here to review the history of this question. As mentioned above, the general strategy to prove probabilistic well-posedness was developed by Bourgain [8] in context of one-dimensional periodic (NLS). This was in response to earlier work of Lebowitz, Rose, and Speer [34] in late 1980s.

Invariant measures for the one-dimensional wave equation were considered by Zhidkov [57] and McKean and Vaninsky [36]. Radially symmetric (NLW) on a three-dimensional ball was considered by Burq and Tzvetkov [17], and extended by Xu to infinite volume [56]. Recently progress has been made in three dimensions, culminating in the proof of invariance of periodic $\Phi^4_3$ under the wave equation [11, 12, 13].

NLW has also been considered with random data not sampled from the invariant measure [30, 32]. Related to the invariance of Gibbs measures is the program for showing quasi-invariance of Gaussian measures under Hamiltonian PDEs [55]; in this notion the distribution of solutions at any given time remains absolutely continuous with respect to the initial measure. For the wave equation this was carried out in [28, 49].

Another related development is the solution theory for (NLW) with additive white noise forcing. This was achieved for the 2D cubic wave equation in [26] and extended to global well-posedness in [27, 54]. The preprint [47] of Oh, Tolomeo, Wang, and Zheng considers this case. If the equation also includes dispersion, the invariant measure is moreover ergodic [53].

The nonlinearity can be replaced by a general polynomial or exponential term as in [40, 47]. It is also possible to let the solution take values in a manifold instead of $\mathbb{R}$. The invariant measures for these wave maps equations [14, 16] are known as nonlinear sigma models in the physics literature. In one dimension they can be interpreted as Brownian paths on a manifold.

For (NLS) in one dimension it is possible to consider both focusing and defocusing nonlinearities, due to the presence of an $L^2$ conservation law. Restricting to a ball in $L^2$ leads to normalizable measure if the nonlinearity is subquintic. In the quintic case the measure is normalizable only if and only if the coupling is sufficiently weak; remarkably, this threshold is known exactly [43].

In two dimensions the defocusing case can still be investigated, as was done by Bourgain [9] for the cubic case and later for general polynomial nonlinearities by Deng, Nahmod, and Yue [20]. For the focusing NLS the $L^2$ cutoff does not lead to normalizable measure anymore [15]. Quasi-invariance has also been investigated for the NLS [41, 44, 48].

In [42, 51] invariant measures of the Zakharov–Yukawa system have been studied. This is a system of coupled wave and Schrödinger equations with nonpositive Hamiltonian and an $L^2$ conservation law. Due to these properties it behaves similarly to the defocusing NLS.

The activity described above has mostly taken place on the torus. In infinite volume we mention the early result of Bourgain on one-dimensional NLS [10], as
well as the work of Cacciafesta and Suzzoni on the NLS and other Hamiltonian
equations [18]. These are in addition to the aforementioned papers [47, 56] on
two- and three-dimensional NLW.

Let us conclude this review with a comment on possible extensions of our
work and open problems. Our method extends in a straightforward way to more
general polynomial nonlinearities and to vector-valued models.

Example 1.2. The mass term \( m^2 > 0 \) in (NLW) and (NLS) is used to avoid
problems with the zero Fourier mode. Still, negative mass is interesting to
consider. One can modify a negative-mass equation
\[
\partial_t u(x, t) - (m^2 + \Delta)u(x, t) = -\lambda : u^3:
\]
by adding \( 2m^2 u(x, t) \) to both sides of the equation. Then the nonlinearity will
be of form \(-\lambda : u^3 : + 2m^2 u\), which is still dominated by the cubic term.

For the weak invariance we also expect the extension to long-range models
(with fractional Laplacian) to be straightforward, provided the resulting mea-
sures are not too singular. If one can find a suitable replacement for finite speed
of propagation in the wave case, we also expect strong invariance to follow in a
straightforward fashion.

It would be interesting to consider the strong invariance of \( P(\phi)_1 \) theories
under 1-dimensional (NLS). The \( \phi^4 \) case has been solved by Bourgain [10],
but his argument does not apply to higher-order polynomial nonlinearities. Of
course the corresponding 2D problem in the full space is also very interesting,
as well as the case of cosine and exponential nonlinearities.

Given the recent preprint [13] on invariance of three-dimensional periodic
(NLW), it is intriguing to ask about the extension to \( \mathbb{R}^3 \). While the measure-
theoretic part of our argument is dimension-independent, the analytic estimates
would require significant changes to account for the more singular behaviour.

1.3 Outline of the article

Our argument consists of mainly putting together existing pieces, hence the
majority of this article presents the techniques at a more pedestrian pace.

All the estimates happen in Besov spaces, a generalization of Sobolev spaces.
Multiplicative estimates take a particularly nice form in these spaces, and there
are many (sometimes compact) embeddings between the spaces. We collect the
main results in Section 2.

We then collect the stochastic estimates for the measures in Section 3. There
we outline the proof of existence and bounds for the \( \phi^4 \) measure over polyno-
mially weighted \( \mathbb{R}^2 \) space. In particular, we show the tightness of the sequence
on increasing tori. Some of the calculations are deferred to appendices.

We solve (NLW) on a periodic domain in Section 4. This argument is orig-
inally due to Oh and Thomann [46], but we slightly simplify the argument by
using only Besov spaces. We also present explicitly the Bourgain-style global
existence argument omitted in the cited article.

The main result in this article is presented in Section 5. We use a measur-
theoretic argument to reduce the full flow to the periodic case, and thus prove
invariance of the infinite-volume \( \phi^4 \) measure.

In Section 6, we finally consider (NLS) on \( \mathbb{R}^2 \). We prove invariance in
Albeverio–Cruzeiro sense with some weaker estimates on the solutions.
1.4 Notation

The weighted Besov spaces $B^s_{p,r}(\rho)$ are defined in Section 2 below. We abbreviate $H^s(\rho) := B^s_{2,2}(\rho)$ and $C^s(\rho) := B^s_{\infty,\infty}(\rho)$. Here $\rho$ is a polynomially decaying weight, the parameter of which may change from section to section.

We also denote by $B^s_{p,r}(A)$ the flat weight on a set $A \subset \mathbb{R}^2$, and abuse notation by writing $B^s_{p,r}(\Lambda_L)$ for the space of periodic functions. The periodic domain is denoted by $\Lambda_L := [-L,L]^2$.

We use $\mu$ for either the $\phi^4_2$ measure, or the product of $\phi^4_2$ and white noise measures; we use $\mu_L$ and $\mu_{L,N}$ for the bounded-domain and bounded-domain Fourier-truncated versions respectively. The product measure form is used in Sections 4 and 5. The $\phi^4_2$ measure is decomposed into a Gaussian free field $Z$ coupled with a more regular part $\phi$. The full product measure is supported on a space $H^s(\rho) := H^s(\rho) \times H^{s-1}(\rho)$.

Capital $\Phi_t$ is reserved for the flow of (NLW). Subscripts similar to those above are used for bounded and Fourier-truncated equations. We define linear solution operators $C_t$ and $S_t$ for (NLW) and $T_t$ for (NLS) in the corresponding sections.

The Littlewood–Paley blocks $\Delta_k$ defined below are supported on dyadic sets. We define the projection $P_N$ as a sharp Fourier cutoff to $B(0,2^N)$.

We use the general notation of $A \lesssim B$ if $A \leq cB$ for some independent constant $c$, and $A \simeq B$ for $A \lesssim B \lesssim A$. Positive constants $c,C$ may vary from line to line. The small constant $\varepsilon > 0$ appears mainly in regularity of the spaces and may change between sections (but only finitely many times). We use $\delta$ to signify other small parameters.

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2 Besov spaces

Besov spaces are a generalization of Sobolev spaces that support some useful multiplication estimates and embeddings. We collect in this section the necessary results, but largely omit the proofs. An excellent introduction to the topic is in the article of Mourrat and Weber [38]. Some results are also collected in the appendix of [24]. The textbook of Bahouri, Chemin, and Danchin [4] treats the unweighted case.

Remark 2.1. There are two conventions of weighted $L^p$ spaces in common use. [38] and [24] respectively define

$$\|f\|_{L^p_w}^p := \int f(x)^p w(x) \, dx \quad \text{and} \quad \|f\|_{L^p_w(u)}^p := \int f(x)^p w(x)^p \, dx.$$  

We use the latter convention since it lets us apply a weight also when $p = \infty$. For $p < \infty$ the conventions are interchangeable, and the statements and their
proofs require only minor changes.

**Definition 2.2** (Littlewood–Paley blocks). We fix $\Delta_k$ to be Fourier multipliers that restrict the support of $\hat{u}$ to a partition of unity. More precisely, for $k \geq 0$ they are smoothed indicators of the annuli $B(0, 2^k 8/3) \setminus B(0, 2^k 3/4)$, and for $k = -1$ of the ball $B(0, 3/4)$. The precise choice is irrelevant.

**Definition 2.3** (Weighted Besov space). We define the space $B^s_{p,r}(w)$ as the completion of $C^\infty_c(\mathbb{R}^d)$ with respect to the norm

$$
\|f\|_{B^s_{p,r}(w)} := \|2^{ks} |w(x)| |\Delta_k f(x)|\|_{L^p}
$$

where the $L^p$ norm is taken over $x \in \mathbb{R}^d$ and the $\ell^r$ norm over $k \geq -1$. We abbreviate

$$
H^s(w) := B^s_{2,2}(w) \quad \text{and} \quad C^s(w) := B^s_{\infty,\infty}(w).
$$

In particular, the space $H^s$ coincides with the usual (fractional-order) Sobolev space, where the norm is defined as the $L^2$ norm of function multiplied by $(\nabla)^s$. One can also replace the ball $B(0, 3/4)$ with annuli for all $k \in \mathbb{N}$ to define homogeneous Besov spaces: some of the following results translate to homogeneous spaces, but the zero Fourier mode is then ignored and the necessary embeddings are not available.

We will use throughout the article a nonhomogeneous polynomial weight

$$
\rho(x) := (x)^{-\alpha} := (1 + |x|^2)^{-\alpha/2}
$$

for $\alpha \geq 0$ sufficiently large. What “sufficiently large” means may vary from section to section, but the final choice is finite. In some sections we also use the unweighted space ($\alpha = 0$); this is indicated by omitting $\rho$.

The following multiplicative inequality shows that products of distributions and smooth enough functions are well-defined distributions. A recurring “trick” in the following sections is to decompose stochastic objects into distributional and more regular parts. There are also analogues of the usual $L^p$ duality and interpolation.

**Theorem 2.4** (Multiplicative inequality). Let $s_1 < s_2$ be non-zero such that $s_1 + s_2 > 0$, and let $1/p = 1/p_1 + 1/p_2$. Then

$$
\|fg\|_{B^s_{p_1}(\rho_1)} \lesssim \|f\|_{B^{s_1}_{p_1}(\rho_1)} \|g\|_{B^{s_2}_{p_2}(\rho_2)}.
$$

**Theorem 2.5** (Duality). Let $(p,p')$ and $(r,r')$ be Hölder conjugate pairs, and $\rho_1$ and $\rho_2$ polynomial weights. Then

$$
\|fg\|_{L^1(\rho_1 \rho_2)} \lesssim \|f\|_{B^s_{p_1}(\rho_1)} \|g\|_{B^{s_2}_{p_2}(\rho_2)}.
$$

**Theorem 2.6** (Interpolation). Fix $\theta \in (0,1)$, $s = \theta s_1 + (1-\theta)s_2$, and

$$
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \quad \alpha = \theta \beta + (1-\theta)\gamma.
$$

Then

$$
\|f\|_{B^{s}_{p_1}(\rho_1)} \lesssim \|f\|_{B^{s_1}_{p_1}(\rho_1)} \|f\|_{B^{s_2}_{p_2}(\rho)}.
$$
We shall use the following three embedding results. The first lets us trade smoothness for $L^p$ and $\ell^\infty$ regularity. The second plays a crucial role in the weak convergence argument by letting us pass to a convergent subsequence in a compact space.

**Theorem 2.7** (Besov embeddings). Let $s \in \mathbb{R}$, $1 \leq q \leq p \leq \infty$, and

$$ s' \geq s + d \left( \frac{1}{q} - \frac{1}{p} \right). $$

Then

$$ ||f||_{B^s_{p,r}(\rho)} \lesssim ||f||_{B^s_{p,r}(\rho)}. $$

The usual chain $\ell^\infty \subset \ell^1 \subset \ell^d$ applies. For any $\varepsilon > 0$ we also have

$$ ||f||_{B^s_{p,r}(\rho)} \lesssim ||f||_{B^{s+\varepsilon}_{p,r}(\rho)}. $$

Finally, $||f||_{B^s_{p,\infty}(\rho)} \lesssim ||f||_{W^{s-r}_{p,r}(\rho)} \lesssim ||f||_{B^s_{p,\infty}(\rho)}$ for all $k \in \mathbb{N}$; in particular, for $k = 0$ that corresponds to $L^p$.

**Theorem 2.8** (Compact embedding). Let $\rho_2$ and $\rho_1$ be polynomial weights with respective parameters $\alpha_2 > \alpha_1 > d$, and $s_2 < s_1$. The space $B^s_{p,r}(\rho_1)$ then embeds compactly into the less regular space $B^{s_2}_{p,r}(\rho_2)$.

For the finite-volume results, we also need periodic Besov spaces. The same multiplicative inequalities work also in this case, and furthermore the following lemma shows that we can move between periodic and polynomial-weight spaces easily. We use the Mourrat–Weber [38] definition of these spaces.

**Definition 2.9** (Periodic Besov space). Given the set $\Lambda_L := [-L, L]^d$, we define the space $B^s_{p,r}(\Lambda_L)$ as the completion of $2L$-periodic $C^\infty(\mathbb{R}^d)$ functions with respect to the Besov norm. The inner $L^p$ norm is unweighted but over $\Lambda_L$.

**Lemma 2.10** (Embedding into polynomial-weight space). Let $\rho$ be a polynomial weight with parameter $\alpha$, and let $f \in C^\infty(\mathbb{R}^d)$ be $2L$-periodic. Then

$$ ||\rho||_{L^1} ||f||_{B^s_{p,r}(\rho)} \lesssim ||f||_{B^s_{p,r}(\Lambda_L)} \lesssim L^\alpha ||f||_{B^s_{p,r}(\rho)}. $$

**Proof.** The second inequality follows from

$$ ||f||_{B^s_{p,r}(\Lambda_L)} \leq \left( \sup_{x \in \Lambda_L} \rho(x)^{-1} \right) ||f||_{B^s_{p,r}(\rho \Lambda_L)} \lesssim \rho ||f||_{B^s_{p,r}(\rho)}. $$

Similarly, if we denote by $\Lambda_L^j$ the translates $\Lambda_L + j2L$, the first inequality is

$$ ||f||_{B^s_{p,r}(\rho)} \leq \sum_{j \in \mathbb{Z}} ||f||_{B^s_{p,r}(\rho \Lambda_L^j)} \lesssim ||f||_{B^s_{p,r}(\Lambda_L)} \sum_{j \in \mathbb{Z}} \sup_{x \in \Lambda_L^j} \rho(x). $$

We must still justify that $f$ belongs to the polynomial-weight space.

Let $F$ be the restriction of $f$ to $\Lambda_L$, and let $F_n$ consist of $2n + 1$ repeats of $F$ along each axis, thus supported on $[-(2n + 1)L, (2n + 1)L]^d$. Then $F_n$ and $f$ coincide in any compact set of $\mathbb{R}^d$ for large enough $n$.

However, $F_n$ might not belong to $B^s_{p,r}(\rho)$ because we did not assume $F$ to vanish at the boundary. To fix that issue, we need to multiply $F_n$ by the
smoothed indicator of some slightly smaller set; call the result $\tilde{F}_n$. Then the difference $\tilde{F}_{n+1} - \tilde{F}_n$ consists of two disjoint $C^\infty_c$ parts, and these parts are only translated as $n$ increases.

With any $g \in C^\infty_c(\mathbb{R}^d)$, we can approximate

$$\|g\|_{B^s_{p,r}(\rho)} \lesssim \|D^{[s]}g\|_{L^p(\rho)}.$$  

These norms are clearly local, so it is easier to see that $\|\tilde{F}_{n+1} - \tilde{F}_n\|_{B^s_{p,r}(\rho)}$ is an $f$-dependent constant multiplied by $\sup_{x \in \Lambda_n} \rho(x)$. We can then use triangle inequality to estimate

$$\|\tilde{F}_{n+m} - \tilde{F}_n\|_{B^s_{p,r}(\rho)} \lesssim C_f \sum_{j=n}^{\infty} \sup_{x \in \Lambda_L} \rho(x),$$

which proves that the approximating sequence $(\tilde{F}_n)$ is Cauchy in $B^s_{p,r}(\rho)$. 

3 Stochastic quantization

In this section we construct the $\phi^4$ measure in the infinite domain $\mathbb{R}^2$ equipped with a suitable weight. This construction is well-known in the literature of stochastic quantization, and we only outline the results we will need.

We define the stochastic objects both on the periodic space $\Lambda_L := [-L,L]^2$ and the full space $\mathbb{R}^2$. The basic building block, Gaussian free field, is straightforwardly defined in both cases, whereas for the $\phi^4$ we need to take a weak limit as $L \to \infty$.

Remark 3.1. In this section we denote by $\mu$ the $\phi^4$ measure, whereas in Sections 4 and 5 we need to consider the product of $\phi^4$ and white noise measures. We go with this abuse of notation since the $\phi^4$ part is always the “interesting” measure. However, it is important to keep in mind that the NLW solution will also depend on the white noise.

3.1 Gaussian free field

Definition 3.2 (Gaussian free field). The massive GFF $\nu_L$ is the Gaussian measure on $S'((\Lambda_L)$ with covariance

$$\langle f, \phi \rangle \langle g, \phi \rangle \, d\nu_L(\phi) = \langle f, (m^2 - \Delta)^{-1}g \rangle_{L^2(\Lambda_L)}.$$  

Similarly we can introduce the infinite-volume GFF $\nu$ supported on $S'((\mathbb{R}^2)$ and with covariance

$$\langle f, \phi \rangle \langle g, \phi \rangle \, d\nu_L(\phi) = \langle f, (m^2 - \Delta)^{-1}g \rangle_{L^2(\mathbb{R}^2)}.$$  

Note that we can view $\nu_L$ as a measure on $S'((\mathbb{R}^2)$ by periodic extension. The following proposition is proved in [38, Theorem 5.1].
Theorem 3.3 (Uniform bounds for GFF). $\nu_L$ and $\nu$ have samples almost surely in $C^{-\varepsilon}(\rho)$, and for all $p < \infty$ the expectations are bounded (uniformly in $L$):

$$\sup_L \int \|\psi\|_p C^{-\varepsilon}(\rho) \, d\nu_L(\psi) < \infty, \quad \int \|\psi\|_p C^{-\varepsilon}(\rho) \, d\nu(\psi) < \infty.$$  

We will denote random variables sampled from $\nu_L, \nu$ by $Z_L, Z$. With some abuse of notation we will write the projections $Z_{L,N} = P_N Z_L$ and $Z_N = P_N Z$.

We can sample from the GFF by realizing it as

$$Z_L = \frac{1}{L} \sum_{n \in L^{-1} \mathbb{Z}^2} \frac{g_n e_n}{(m^2 + |n|^2)^{1/2}}, \quad (3.1)$$

where $g_n$ are complex Gaussians with variance 1 and $e_n(x) := \exp(2\pi n \cdot x)$. We require $g_{-n} = \overline{g_n}$ to make the field real, but otherwise $g_n$ are independent. For the full-space case we can write

$$Z = \int_{\mathbb{R}^2} \frac{\xi(y) e_n}{(m^2 + |y|^2)^{1/2}} \, dy \quad (3.2)$$

where $\xi$ is a white noise as defined below.

Definition 3.4 (White noise). The white noise $\xi$ is a Gaussian process on $\mathcal{S}'(\mathbb{R}^2)$ with covariance

$$\mathbb{E} \langle \xi, \psi \rangle_{L^2(\mathbb{R}^2)} \langle \xi, \phi \rangle_{L^2(\mathbb{R}^2)} = \langle \psi, \phi \rangle_{L^2(\mathbb{R}^2)}.$$  

The measure $\nu_L$ does not have samples of positive regularity. This means that taking powers of distributions sampled from $\nu_L$ does not make sense. Yet the Gaussian structure of the randomness allows us to still define powers of the field by so-called Wick ordering.

Definition 3.5 (Wick ordering, periodic space). Let $a_{N,L} = \mathbb{E} [Z_{L,N}(0)^2]$. We then define

$$Z_{L,N}^4 \equiv Z_{L,N}^4 - 6a_{N,L} Z_{L,N}^2 + 3a_{N,L}^2,$$

$$Z_{L,N}^3 \equiv Z_{L,N}^3 - 3a_{N,L} Z_{L,N},$$

$$Z_{L,N}^2 \equiv Z_{L,N}^2 - a_{N,L}.$$  

As $N \to \infty$, the constants $a_{N,L}$ diverge logarithmically, and the counter-terms cancel the divergence of $Z_{L,N}^k$. Further Wick powers can be defined using Hermite polynomials. Wick-ordered polynomials are defined by Wick-ordering each monomial term separately. We remark that $\mathbb{E} [Z_{L,N}(x)^2]$ does not depend on the choice of $x$ since the Gaussian free field is translation-invariant.

It will be useful to define the Wick powers with a renormalization constant that is independent of $L$. For this purpose we will use the expectation of the full-space GFF.

Definition 3.6 (Wick ordering, full space). We denote by $a_N = \mathbb{E} [Z_N(0)^2]$, and define

$$Z_{L,N}^4 \equiv Z_{L,N}^4 - 6a_N Z_{N,L}^2 + 3a_N^2,$$

$$Z_{L,N}^3 \equiv Z_{L,N}^3 - 3a_N Z_{L,N},$$

$$Z_{L,N}^2 \equiv Z_{L,N}^2 - a_N.$$
The difference between these two renormalizations is a polynomial of strictly lower degree; for the fourth Wick powers it is
\[
Z_{L,N}^4 : = -6(a_{N,L} - a_N)Z_{L,N}^2 + 3(a_{N,L}^2 - a_N^2).
\]

The next lemma asserts that the difference of renormalization constants goes to zero as \( N, L \to \infty \). This lets us always take Wick ordering with respect to the full-space GFF.

**Lemma 3.7.** For \( a_{N,L} = \mathbb{E}[Z_{L,N}(0)^2] \) and \( a_N = \mathbb{E}[Z_N(0)^2] \) we have
\[
|a_{N,L} - a_N| \lesssim \frac{1}{N} + \frac{1}{L}.
\]

For every \( L < \infty \) we can thus define \( r_L := \lim_{N \to \infty} (a_{N,L} - a_N) \lesssim \frac{1}{L} \).

**Proof.** The first renormalization constant can be written as
\[
\mathbb{E} \left| Z_{L,N}(0)^2 \right| = \frac{1}{L^2} \sum_{n \in L^{-1} \mathbb{Z}^2, |n| \leq N} \frac{1}{m^2 + |n|^2} = \sum_{n \in L^{-1} \mathbb{Z}^2, |n| \leq N} \int_{P(n)} \frac{1}{m^2 + |x|^2} \, dx,
\]
where \( P(n) \) is the rectangle \( n + [0, 1/L]^2 \). By covariance of the continuum white noise, the second renormalization constant is
\[
\mathbb{E} \left| Z_L(0)^2 \right| = \int_{|x| \leq N} \frac{1}{m^2 + |x|^2} \, dx.
\]

The difference of these integrals is estimated in two parts. Some of the rectangles \( P(n) \) extend outside the ball \( B(0, N) \); this introduces an error of order
\[
\sum_{n \in L^{-1} \mathbb{Z}^2, \text{ } P(n) \not\subset B(0, N)} \frac{1}{m^2 + |n|^2} \lesssim N \cdot \frac{1}{N^2}.
\]
Meanwhile for \( x \in P(n) \cap B(0, N) \) we can estimate
\[
\frac{1}{m^2 + |n|^2} - \frac{1}{m^2 + |x|^2} \leq \frac{|x|^2 - |n|^2}{(m^2 + |n|^2)(m^2 + |x|^2)} \leq \frac{1}{L (m^2 + |n|^2)^2}.
\]

This implies that the difference of integrals inside \( B(0, N) \) is bounded by
\[
\frac{1}{L} \sum_{n \in L^{-1} \mathbb{Z}^2, |n| \leq N} \frac{1}{m^2 + |n|^2} \lesssim \frac{1}{L}.
\]

We can now state that all Wick powers of the Gaussian free field are well-defined. This result also translates into regular enough perturbations of the GFF, in particular the \( \phi^4 \) measure defined below.
Lemma 3.8 (Moments of GFF powers). Let $Z_L$ be sampled from $\nu_L$ and let $Z_{L,N} = P_N Z_L$. Then for any $p < \infty$ and $i = 1, 2, \ldots$ we have

$$\sup_N \mathbb{E} \left[ \| Z_{L,N}^i \|^p_{C^{-\varepsilon}(\Lambda_L)} \right] < \infty$$

Furthermore: $Z_{L,N}^i$ converges almost surely in $C^{-\varepsilon}(T^2)$ and in any $L^p(\nu_L, C^{-\varepsilon}(\rho))$ to a well-defined limit. We denote this limit by $Z_L^i$ and we have

$$\sup_L \mathbb{E} \left[ \| Z_L^i \|^p_{C^{-\varepsilon}(\rho)} \right] < \infty.$$

Proof. The proof of the first statement is in [19, Lemma 3.2], and the polynomial-weight case is [38, Theorem 5.1].

Lemma 3.9 (Wick powers of perturbations). Let $\phi \in L^p(\nu_L, B^{2\varepsilon}_{p,p}(\rho))$, where $\varepsilon > 0$ and $p$ is large enough. Then

$$(Z_L + \phi)^j = \sum_{i=0}^{j} Z_L^i : \phi^{j-i}.$$  

Proof. It follows from properties of Hermite polynomials that

$$(Z_{L,N} + \phi)^j = \sum_{i=0}^{j} \binom{j}{i} Z_{L,N}^i : \phi^{j-i}.$$  

Each term is well-defined as an element of $B^{-\varepsilon}_{p/\delta, p/\delta}(\rho^{j+1})$ by Theorem 2.4; moreover the multiplication is a continuous operation. The claim then follows by passing $N \to \infty$.

The following result lets us compute covariances of Wick powers by passing to a Green’s function. For the proof, see e.g. [52, Theorem I.3].

As an application, we see that we can approximate the third Wick power by continuous maps. We use this lemma to prove that sequences of periodic solutions satisfy the PDEs also in the limit. The proof is somewhat technical, and we leave it to Appendix A.

Theorem 3.10 (Wick’s theorem). If $X$ and $Y$ are Gaussian, then

$$\mathbb{E} [ X^n : Y^n ] = n! \left( \mathbb{E} [ XY ] \right)^n.$$

Lemma 3.11 (Approximation of Wick powers). For every $\delta > 0$ and $s > 0$, there exists a continuous map $f^\delta : H^{-s}(\rho) \to L^2(\rho)$ such that the following holds. Let $Z$ (respectively $Z_L$) be sampled from the (periodic) Gaussian free field and $\phi \in L^p(\mathbb{P}, B^s_{p,p}(\rho))$. Then

$$\lim_{\delta \to 0} \mathbb{E} \left[ \left\| f^\delta(Z + \phi) - (Z + \phi)^3 \right\|_{C^{-\varepsilon}(\rho)}^p \right] = 0,$$

and respectively with $Z$ replaced by $Z_L$. 

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3.2 Coupling of the $\phi_4$ measure and the GFF

We now turn to study the $\phi_4^2$ measure that will be the invariant measure for our PDEs. We can define $\phi^4$ directly only in the periodic case; we need to take a weak limit to get to infinite volume.

Let us first recall the definition and some basic results of weak convergence of probability measures. These can be found in most probability textbooks; see for example [31, Definition 13.12].

**Theorem 3.12** (Weak convergence). A sequence of probability measures $(\mu_L)$ taking values in $\mathcal{X}$ is said to converge weakly to $\mu$ if

$$
\lim_{L \to \infty} \int_{\Omega} f(\phi) \, d\mu_L(\phi) = \int_{\Omega} f(\phi) \, d\mu(\phi) \quad \text{for all } f \in C_b(\mathcal{X}; \mathbb{R}).
$$

If $\mathcal{X}$ is a Polish space, then the weak limit is unique.

**Definition 3.13** (Tightness). A family $(\mu_L)_{L \in \mathbb{N}}$ of probability measures on $\mathcal{X}$ is called tight if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon$ such that

$$
\sup_{L \in \mathbb{N}} \mu_L(\mathcal{X} \setminus K_\varepsilon) < \varepsilon.
$$

**Lemma 3.14** (Prokhorov’s theorem; [31, Theorem 13.29]). Suppose that the sequence $(\mu_L)$ is tight and defined on a metric space $\mathcal{X}$. Then there is a subsequence $(\mu_{L_k})$ that converges weakly to a limit measure $\mu$ on $\mathcal{X}$.

We will consider a sequence of $\phi_4^2$ measures over increasingly large tori and show that it is tight over a polynomially weighted Besov space. This will give us a weak limiting measure $\phi_4^2$.

**Definition 3.15.** In finite volume $\Lambda_L$, the $\phi_4^2$ measure is given by

$$
d\mu_L(\psi) = Z_L^{-1} \exp \left( -\lambda \int_{\Lambda_L} :\psi^4(x): \, dx \right) \, d\nu_L(\psi),
$$

where $Z_L^{-1}$ is a normalization constant.

By Section 3.1 the Wick power $:\psi^4:$ makes sense as a distribution $\nu_L$-almost surely, and one can show that the exponential belongs to $L^p(\nu_L)$ for any $p < \infty$ and $L < \infty$. For our purposes, it is easier to consider $\mu_L$ as the invariant measure of the stochastic quantization equation (see [38])

$$
\partial_t u + (m^2 - \Delta) u + :u^3: = \xi, \quad u \in C(\mathbb{R}^+, H^{-\delta}(\Lambda_L)). \quad \text{(SQE)}
$$

Here $\xi$ is space-time white noise, the Gaussian process valued in $S'(\mathbb{R} \times \mathbb{R}^2)$ with covariance

$$
\mathbb{E} \left[ (\xi, f)_{L^2(\mathbb{R} \times \mathbb{R}^2)} (\xi, g)_{L^2(\mathbb{R} \times \mathbb{R}^2)} \right] = (f, g)_{L^2(\mathbb{R} \times \mathbb{R}^2)},
$$

where $f, g \in S(\mathbb{R} \times \mathbb{R}^2)$.

We will use (SQE) to control the $\phi_4^2$ measure in the limit $L \to \infty$. That $\mu_L$ is indeed a stationary measure for this equation was shown by Da Prato and
Debussche [19]. We begin by decomposing the solution as \( u = Z + \phi \), where \( Z \) is the Gaussian part that solves the stationary equation

\[
\begin{cases}
\partial_t Z + (m^2 - \Delta) Z = \xi, \\
Z(0) = Z_0 \sim \text{GFF},
\end{cases}
\]  

and \( \phi \) solves

\[
\begin{cases}
\partial_t \phi + (m^2 - \Delta) \phi + : (Z + \phi)^3 : = 0, \\
\phi(0) = u(0) - Z(0).
\end{cases}
\]  

We can take \((u, Z)\) to be jointly stationary solutions to (SQE) and (3.4) so that \( Z(t) \sim \text{GFF} \); see the beginning of Section 4.3 in [24]. In particular the Wick power \( : Z^3: \) is a well-defined random distribution. Now multiplying (3.5) by \( \rho \phi \) and integrating we obtain

\[
\partial_t \| \rho^{1/2} \phi \|^2_{L^2} + \frac{1}{2}(m^2 \| \rho^{1/2} \phi \|^2_{L^2} + \| \rho^{1/2} \nabla \phi \|^2_{L^2} + \| \rho^{1/4} \phi \|^4_{L^4}) = -G(Z, \phi),
\]  

where

\[
G(Z, \phi) = 3 \int \rho : Z^3 : \phi \, dx + 3 \int \rho : Z^2 : \phi^2 \, dx \\
+ \int \rho Z \phi^3 \, dx + \int (\nabla, \rho) \phi \nabla \phi \, dx.
\]  

Here \( \nabla, \rho \) := \( \nabla (\rho \phi) - \rho \nabla \phi = (\nabla \rho) \phi \) is the commutator of the weight and the gradient. In Appendix B we show that

\[
|G(Z, \phi)| \leq Q(Z) + \frac{1}{2}(\| \rho^{1/2} \phi \|^2_{L^2} + \| \rho^{1/2} \nabla \phi \|^2_{L^2} + \| \rho^{1/4} \phi \|^4_{L^4}),
\]  

where

\[
\sup_L \mathbb{E}[|Q(Z)|^p] < \infty \quad \text{for any } p < \infty.
\]  

By moving common terms to the left-hand side of (3.6), we get the estimate

\[
\partial_t \| \rho^{1/2} \phi \|^2_{L^2} + \frac{1}{2}(m^2 \| \rho^{1/2} \phi \|^2_{L^2} + \| \rho^{1/2} \nabla \phi \|^2_{L^2} + \| \rho^{1/4} \phi \|^4_{L^4}) \leq Q(Z).
\]  

The time derivative term vanishes in expectation since \( \phi = u - Z \) was assumed to be stationary. Therefore we are left with

\[
\frac{1}{2} \mathbb{E} \left[ (m^2 \| \rho^{1/2} \phi \|^2_{L^2} + \| \rho^{1/2} \nabla \phi \|^2_{L^2} + \| \rho^{1/4} \phi \|^4_{L^4}) \right] \leq \mathbb{E} Q(Z),
\]  

which proves

\[
\sup_L \mathbb{E} \| \rho^{1/2} \phi \|^2_{H^1} < \infty,
\]  

and thus tightness in \( H^{1-\epsilon}(\rho^{1/2+\epsilon}) \). We still strengthen this in Section 3.3.
3.3 Wick powers of $\phi^4_2$

The bounds on the $\phi^4$ samples can be improved to exponential tails, which then implies $L^p$ expectations for all $p$. We defer the proof of this result to Appendix C.

**Theorem 3.16** (Exponential tails). There exists $\delta > 0$ such that

$$\int_{\Omega} \exp \left( \delta \|W_L\|^2_{C^{-\epsilon} (\rho)} \right) d\mu_L(W_L) \lesssim 1,$$

and the bound is uniform in $L$. The bound also holds in the limit $\mu$.

Since the nonlinearity in (NLW) is cubic, we will need the first three Wick powers of the $\phi^4$ field. We construct and estimate the Wick powers of $\phi^4_2$ uniformly in $L$, and thus in the $L \to \infty$ limit.

**Theorem 3.17** (Wick powers of $\phi^4$). Let $W_L = Z_L + \phi_L$ be sampled from $\phi^4_2$. Then $W^j$ is a well-defined random distribution for $j \leq 3$, and for any $\epsilon > 0$ and $p < \infty$ we have

$$\sup_L \mathbb{E} \|W^j_L\|^p_{C^{-\epsilon} (\rho^2)} < \infty.$$

Furthermore, if $W$ is sampled from the full-space $\phi^4$ measure, then

$$\mathbb{E} \|W^j\|^p_{C^{-\epsilon} (\rho^2)} < \infty.$$

**Proof.** We only do the proof in the most difficult case $j = 3$. The other cases are analogous. Recall that we have

$$W^3_L = \sum_{i=0}^3 \binom{3}{i} Z^i_L : \phi^{3-i}_L :.$$

Now for $q = 4/\epsilon$ we can estimate

$$\|Z^i_L : \phi^{3-i}_L :\|_{C^{-\epsilon} (\rho^2)} \lesssim \|Z^i_L : \phi^{3-i}_L :\|_{B^q_{\infty,2} (\rho^2)} \lesssim \|Z^i_L : \phi^{3-i}_L :\|_{B^q_{\infty,4} (\rho^2)} \lesssim \|Z^i_L : \phi^{3-i}_L :\|_{B^q_{\infty,3} (\rho^3)}.$$

The Gaussian part is bounded by Lemma 3.8. For the perturbation part, Theorem 3.16 implies that $\mathbb{E} \|W^j_L\|^p_{C^{-\epsilon} (\rho^2)} < \infty$, and we can decompose

$$\sup_L \mathbb{E} \|\phi_L\|^p_{C^{-\epsilon} (\rho^2)} \lesssim \sup_L (\mathbb{E} \|W^j_L\|^p_{C^{-\epsilon} (\rho^2)} + \mathbb{E} \|Z^i_L : \phi^{3-i}_L :\|_{B^q_{\infty,3} (\rho^3)} < \infty.$$  

This estimate provides $L^p$ regularity, whereas the estimate $\mathbb{E} \|\phi_H^2\|_{C^{-\epsilon} (\rho)} < \infty$ from Section 3.2 provides differentiability. We can interpolate between these two through

$$\|\phi_L\|_{B^{q/2}_{\infty,3} (\rho^{1/3})} \lesssim \|\phi_L\|_{C^{-\epsilon} (\rho^{1/3})} \|\phi_L\|_{H^{20}_{1/3} (\rho^{1/3})} \lesssim \|\phi_L\|_{C^{-\epsilon} (\rho^{1/3})} \|\phi_L\|_{H^{10}_{1/3} (\rho^{1/3})} + \|\phi_L\|_{C^{-\epsilon} (\rho^{1/3})}^{1-\theta},$$

where $\theta = 1 - \epsilon/12$. As we substitute this back into (3.11), we find that the final expectation is bounded. \( \square \)
From Theorem 3.17 we can bootstrap a stronger statement for the coupling. The perturbation $\phi$ is two derivatives more regular than $Z$, instead of just one derivative as showed earlier.

**Corollary 3.18 (Strong bound for regular part).** We can find random variables $Z_L, \phi_L$ such that $Z_L \sim \nu_L, Z_L + \phi_L \sim \mu_L$, and

$$\sup_L \mathbb{E} \|\phi_L\|_{H^{2-\epsilon}(\rho)}^p \lesssim 1.$$ 

**Proof.** Recall that from the stochastic quantization equation (SQE) we have

$$\phi_L(t) = \int_0^t e^{-(t-s)\Delta} (Z_L(s) + \phi_L(s))^3 : ds + e^{-t\Delta} \phi_L(0).$$

So provided $p$ is large enough that $|t - s|^{(1-\epsilon/2)p/(p-1)} \in L^1$, we can use the smoothing effect of the heat operator ([38, Proposition 5]) to estimate

$$\mathbb{E} \|\phi_L(t)\|_{H^{2-\epsilon}(\rho)}^p \lesssim \mathbb{E} \int_0^t e^{-(t-s)\Delta} (Z_L(s) + \phi_L(s))^3 : ds \|_{H^{2-\epsilon}(\rho)}^p + \mathbb{E} \|e^{-t\Delta} \phi_L(0)\|_{H^{2-\epsilon}(\rho)}^p \lesssim \mathbb{E} \int_0^t \frac{\|\phi_L(s)\|_{H^{2-\epsilon}(\rho)}^p}{s^{1-\epsilon/2}} + t^{1-\epsilon/2} \mathbb{E} \|\phi_L(0)\|_{H^{2-\epsilon}(\rho)}^p.$$ 

Since $Z_L$ and $\phi_L$ are both stationary, we may choose $t$ as we like. The integrand is then uniformly bounded by Theorem 3.17.

In total we have obtained that $\sup_L \mathbb{E} \|\phi\|_{H^{2-\epsilon}(\rho)} < \infty$. By the same compactness argument as above, $\text{Law}(Z, \phi)$ is tight on $H^{-\epsilon}(\rho) \times H^{2-\epsilon}(\rho^{1+\epsilon})$. In particular $\mu_L = \text{Law}(Z + \phi)$ is tight on $H^{-\epsilon}(\rho^{1+\epsilon})$ and has a weakly converging subsequence. We have thus proved the following:

**Theorem 3.19 ($\phi_L^2$ as a weak limit).** Let $\rho$ be a sufficiently integrable polynomial weight. The measure $\mu_L$ can be represented as

$$\mu_L = \text{Law}(Z_L + \phi_L)$$

where $Z_L$ is a GFF on $\Lambda_L$, and $\phi_L$ satisfies $\sup_L \mathbb{E} \|\phi_L\|_{H^{2-\epsilon}(\rho)}^p < \infty$. Identifying $Z_L + \phi_L$ with its periodic extension on $\mathbb{R}^2$ we have that $\mu_L = \text{Law}(Z_L + \phi_L)$ is tight on $H^{-\epsilon}(\rho^{1+\epsilon})$ and any limiting point $\mu$ satisfies

$$\mu = \text{Law}(Z + \phi)$$

where $Z$ is a Gaussian free field on $\mathbb{R}^2$ and $\mathbb{E} \|\phi\|_{H^{2-\epsilon}(\rho)}^p < \infty$.

**Proof.** We know that the limit of $\text{Law}(Z_L)$ as $L \to \infty$ is a Gaussian free field on $\mathbb{R}^2$; this follows for instance from the convergence of the covariances. It remains to show that

$$\mathbb{E} \|\phi\|_{H^{2-\epsilon}(\rho)}^p < \infty$$

but since $\|\phi^2\|_{H^{2-\epsilon}(\rho)}$ is lower semicontinuous on $H^{2-\epsilon}(\rho^{1+\epsilon})$ we have by weak convergence

$$\mathbb{E} \|\phi\|_{H^{2-\epsilon}(\rho)}^p \leq \liminf_{L \to \infty} \mathbb{E} \|\phi_L\|_{H^{2-\epsilon}(\rho)}^p < \infty.$$

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Remark 3.20. We were careful to state the preceding theorem for “any limiting point $\mu$”. When the coupling parameter $\lambda$ is large enough, there exist subsequences of $(\phi^4_{2L})$ that converge to different weak limits. This is one of the main complications in our study.

4 Invariance of periodic NLW

Let us now move on to solving the nonlinear wave equation. We fix a bounded domain $\Lambda_L = [-L, L]^2$ and consider

\[
\begin{align*}
\partial_{tt} u(x,t) + (m^2 - \Delta) u(x,t) &= -\lambda: u^3(x,t), \\
u(x,0) &= u_0(x), \\
\partial_t u(x,0) &= u'_0(x)
\end{align*}
\] (4.1)
on $\Lambda_L \times \mathbb{R}_+$. The initial value $u_0$ will be sampled from a $\phi^4$ measure and the initial time derivative $u'_0$ from a white noise measure; as remarked at the beginning of Section 3, we denote by $\mu$ now the product $(\phi^4, \text{WN})$ measure. The Wick ordering will always be taken with respect to the full space covariance, even if we start from periodic initial data.

Thanks to the finite speed of propagation (formulated more precisely in Theorem 4.1), boundary effects are visible only outside the ball $B(0, L - t)$.

By solving the equation in Fourier space, we can write the mild solution as

\[
u(x,t) = C_t u_0(x) + S_t u'_0(x) + \lambda \int_0^t [S_{t-s} u^3](x) ds,
\] (4.2)

where we use the cosine and sine operators

\[
C_t = \cos((m^2 - \Delta)^{1/2}t), \quad S_t = \frac{\sin((m^2 - \Delta)^{1/2}t)}{(m^2 - \Delta)^{1/2}}.
\] (4.3)

These are defined as Fourier multiplier operators. We see that $C_t$ preserves the $H^s(\Lambda_L)$ regularity of its argument whereas $S_t$ increases it by one.

We split the solution into nonlinear and linear parts $u = v + w$. Here $w$ solves the linear equation with the given initial data, leaving $v$ to solve the coupled equation

\[
\partial_t v(x,t) + (m^2 - \Delta)v(x,t) = \lambda: (v + w)^3:
\] (4.4)

with zero initial data. We will see that $v$ has one degree higher regularity than $w$, and its growth is controlled by $w$. The final solution will exist in $L^p([0, \tau]; B_{p,p}^s(\Lambda_L))$ up to some short, stochastic time $\tau$.

The result in this section was already proved by Oh and Thomann [46], and stated without proof by Bourgain in 1999. The argument presented below replaces the more specific Fourier restriction norm by a general Besov norm, and includes the details on convergence of solutions.

4.1 Linear part

It is a basic property of the wave equation that all wave packets travel at a fixed speed. This property applies to the Duhamel formulation (4.2) as well. The solution operators are then also bounded in weighted spaces since the weight does not change too much within a ball.
Lemma 4.1 (Finite speed of propagation). If \((u_0, u'_0)\) and \((v_0, v'_0)\) coincide on \(B(0, R)\), then the corresponding linear wave equation solutions \(u(t)\) and \(v(t)\) coincide on \(B(0, R - t)\).

Sketch of proof. By linearity, we only need to show that
\[
C_t w_0(x) + S_t w'_0(x)
\]
vanishes on \(B(0, R - t)\) if both \(w_0\) and \(w'_0\) vanish inside \(B(0, R)\). By density, we can further assume \(w_0\) and \(w'_0\) to be smooth and compactly supported. The idea is to consider the local energy
\[
E(s) := \int_{B(0,R-s)} m^2 u(x,s)^2 + |\nabla u(x,s)|^2 + |\partial_t u(x,s)|^2 \, dx.
\]
By a calculation with some vector analysis, we see that \(\partial_s E(s) \leq 0\) up to time \(t\), and \(E(0) = 0\) by assumption. See [22, Section 2.4.3] for details.

Lemma 4.2 (Boundedness of solution operators). Let \(t \leq T\). Then for \(s \in \mathbb{R}\) and \(f \in H^s(\rho)\) we have
\[
\|C_t f\|_{H^s(\rho)} \lesssim \|f\|_{H^s(\rho)} \quad \text{and} \quad \|S_t f\|_{H^{s-1}(\rho)} \lesssim \|f\|_{H^{s-1}(\rho)}.
\]

Proof. Denote by \(K\) the convolution kernel of \(C_t\). As noted in Lemma 4.1, \(K\) is supported in \(B(0,T)\). It suffices to study the case \(s = 0\) as \(C_t\) commutes with derivatives. We see that \(C_t\) is bounded on \(L^2\) with flat weight, since it is a Fourier multiplier with bounded symbol. Then
\[
\left\|C_t f\right\|_{L^2(\rho)}^2 = \int_{\mathbb{R}^2} \rho^2(x) \left[ \int_{\mathbb{R}^2} K(x-y) f(y) \, dy \right]^2 \, dx
\]
\[
\lesssim \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} K(x-y) \rho(x-y)^{-1} \rho(y) f(y) \, dy \right]^2 \, dx
\]
\[
\lesssim \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} K(x-y) \rho(y) f(y) \, dy \right]^2 \, dx
\]
\[
= \|C_t(\rho f)\|_{L^2}^2
\]
\[
\lesssim \|\rho f\|_{L^2}^2,
\]
We could estimate \(\rho(x-y)^{-1} \leq C(T)\) since \(K\) vanishes outside \(|x-y| \leq T\). For \(S_t\) the proof is identical, except that we gain a derivative of regularity.

In probabilistic terms, the linear part looks almost like the coupled \(\phi^4\) measure: there is an invariant Gaussian free field part and a more regular term with conserved norm. This yields a moment bound, which we then use to control the norm of the nonlinearity in the next section.

Lemma 4.3 (Distribution of linear part). As we substitute \(u_0 = Z_L + \phi_L\), the linear part becomes
\[
w(\cdot,t) = \left[ C_t Z + S_t u'_0 \right] + C_t \phi_L.
\]
Here the term in brackets is GFF, whereas \(C_t \phi \in H^{2-\varepsilon}(\rho)\) almost surely.
Proof. The latter part follows from preservation of Sobolev regularity by the cosine operator. To prove the first part, we need to compute the covariance. For any test functions \( \varphi, \psi \) we have
\[
\mathbb{E} \left[ \langle \varphi, C_t Z + S_t u'_0 \rangle \langle \psi, C_t Z + S_t u'_0 \rangle \right] = \mathbb{E} \left[ \langle \varphi, C_t Z \rangle \langle \psi, C_t Z \rangle \right] + \mathbb{E} \left[ \langle \varphi, S_t u'_0 \rangle \langle \psi, S_t u'_0 \rangle \right]
\]
by independence of \( Z \) and \( u'_0 \). Because \( C_t \) is a self-adjoint operator, the first term becomes
\[
\mathbb{E} \left[ \langle \varphi, C_t Z \rangle \langle \psi, C_t Z \rangle \right] = \langle C_t \varphi, \frac{C_t \psi}{m^2 - \Delta} \rangle = \left\langle \varphi, \frac{\cos(m^2 - \Delta)^{2}}{m^2 - \Delta} \psi \right\rangle.
\]
For the second term we have white noise covariance instead:
\[
\mathbb{E} \left[ \langle \varphi, S_t u'_0 \rangle \langle \psi, S_t u'_0 \rangle \right] = \langle S_t \varphi, S_t \psi \rangle = \left\langle \varphi, \frac{\sin(m^2 - \Delta)^{2}}{m^2 - \Delta} \psi \right\rangle.
\]
Now the trigonometric identity \( \sin^2 + \cos^2 = 1 \) implies
\[
\mathbb{E} \left[ \langle \varphi, C_t Z + S_t u'_0 \rangle \langle \psi, C_t Z + S_t u'_0 \rangle \right] = \left\langle \varphi, \frac{1}{m^2 - \Delta} \psi \right\rangle.
\]
\]
\[\square\]

**Lemma 4.4** (Moment bounds for linear part). Let \( L \) belong to the convergent subsequence of Theorem 3.19. Let \( w_L \) be the \( L \)-periodic linear part started from data sampled from \( L \)-periodic \( (\phi_L^1, \text{WN}) \) measure. For \( j = 1, 2, 3 \) and \( T > 0 \) we have the moment bounds
\[
\mathbb{E} \| w_L^j \|^p_{L^p([0,T]; B^s_{p,p}(\mathbb{R})]} \lesssim_{p,T,s} 1.
\]
This also implies that
\[
\mathbb{E} \| w_L^j \|^p_{L^p([0,T]; C^{-2s}(\mathbb{R})]} \lesssim_{p,T,s} 1
\]
for \( p \) sufficiently large. Both estimates are uniform in \( L \).

**Proof.** By Lemma 4.3 we can decompose \( w_L = w_{\text{stat}} + \psi \) where \( \psi(t) = C_t \phi_L \) and \( w_{\text{stat}} \) has distribution \( w_L \) for every time. We thus have by stationarity
\[
\int_0^T \mathbb{E} \| w_{\text{stat}}^j \|_{L^p_{B^{s}_{p,p}(\mathbb{R})]}^p \, dt = T \mathbb{E} \| (Z(0))^j \|_{L^p_{B^{s}_{p,p}(\mathbb{R})]}^p.
\]
When \( j = 3 \), we can expand the binomial and estimate
\[
\mathbb{E} \| w_{\text{stat}}^3 \|_{L^p_{B^{s}_{p,p}(\mathbb{R})]}^p + \mathbb{E} \| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p + \mathbb{E} \| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p + \mathbb{E} \| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p
\]
at a fixed time. The middle terms can be expanded with Theorem 2.4 like in
\[
\| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p \lesssim \| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p + \mathbb{E} \| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p + \mathbb{E} \| w_{\text{stat}}^3 \|_{B^{s}_{p,p}(\mathbb{R})]}^p.
\]
The other cases \( j = 1 \) and \( j = 2 \) are analogous. Now we can deduce the claim since the norm of \( w_{\text{stat}} \) is in \( L^p([0,T]) \) by stationarity, and \( \psi \) is in \( C([0,T], B^s_{p,p}(\mathbb{R})]) \) by Corollary 3.18 uniformly for a sequence of \( L \to \infty \).

The second claim follows from the embedding stated in Theorem 2.7 by choosing \( p \geq 2/\varepsilon \). \( \square \)
We can now show that powers of the linear parts converge as $L \to \infty$. We will use this result as we pass to the full space in Section 5.

**Lemma 4.5 (Convergence of linear parts).** Let $1 \leq p < \infty$. As $L \to \infty$, $w^L_t :$ converges in probability to $w^t :$ in $L^p([0,T], C^{-\varepsilon}(\rho^3))$, where $w$ is a linear solution started from initial data $Z + \phi$.

**Proof.** The convergence of $C_t \phi_L$ to $C_t \phi$ in $H^{2-\varepsilon}(\rho)$ follows from continuity of $C_t$ in $H^{2-\varepsilon}(\rho)$. We need to show that $w^L_{st,L} : \to : w^t_{st} :$ in $L^p([0,T], C^{-\varepsilon}(\rho^3))$. Then continuity of Besov product from $C^{-\varepsilon} \times H^{2-\varepsilon}$ to $C^{-\varepsilon}$ implies convergence of $(w^L_{st,L} + \phi_L)^3$.

We have that $w^L_{st,L} \to w_{st}$ in $C([0,T], H^{-\varepsilon}(\rho^3))$ by continuity of the linear operators. Now with $f^\delta$ as in Lemma 3.11 we have

$$w^L_{st,L} : - : w^t_{st} : = [w^L_{st,L} : - f^\delta(w^L_{st,L})]$$

$$\quad + [f^\delta(w^L_{st,L}) - f^\delta(w_{st})] + [f^\delta(w_{st}) - : w^t_{st} :].$$

The middle term goes to 0 since $f^\delta$ is continuous from $H^{-\varepsilon}(\rho)$ to $C^{-\varepsilon}(\rho^3)$, and for the first and last term we have by stationarity

$$E \left[ \int_0^T \| f^\delta(w^L_{st}) \!-\! : w^t_{st} : \|_{C^{-\varepsilon}(\rho^3)} \, ds \right] = T \| f^\delta(w^L_{st}(0)) \!-\! : w^t_{st}(0) : \|_{C^{-\varepsilon}(\rho^3)}.$$

This goes to 0 uniformly in $L$ by Lemma 3.11.

### 4.2 Fixed-point iteration

We now use the standard fixed-point argument for

$$v(x,t) = -\lambda \int_0^t [S_{t-s} : (v + w)^3 :](x) \, ds. \quad (4.5)$$

As usual, we need to check boundedness and contractivity of the solution operator. We do the iteration in the periodic Besov space $L^\infty([0,\tau]; H^{1-\varepsilon}(\Lambda_L))$. The weight must be “flat” because we need it to be the same on both sides of the multiplicative estimates.

This argument is completely deterministic. We control the growth of $v$ by assuming bounds on the linear part $w$; these bounds will be verified by stochastic estimates in Section 4.3.

**Lemma 4.6 (Boundedness).** Let $M = \max_{j=1,2,3} \| w^j : \|_{L^4([0,1]; C^{-\varepsilon}(\rho))}$. The operator

$$(Fv)(x,t) := -\lambda \int_0^t [S_{t-s} : (v + w)^3 :](x) \, ds$$

maps a ball of radius $R$ into a ball of radius $C_{\Lambda} \lambda \tau^{1/2} M (1 + R^3)$ in the periodic space $L^\infty([0,\tau]; H^{1-\varepsilon}(\Lambda_L))$. 

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Proof. We can commute the Fourier multiplier and apply Jensen’s inequality in

\[
\|Fv\|_{L^p_t H^{1-\varepsilon}(\Lambda_L)} = \lambda \sup_{0 \leq t \leq \tau} \left[ \int_{\mathbb{R}^2} \left| \left( \nabla \right)^{1-\varepsilon} \int_0^t S_{t-s} : (v + w)^3 \, ds \right|^2 \, dx \right]^{1/2}
\]

\[
\leq \lambda^{1/2} \sup_{0 \leq t \leq \tau} \left[ \int_{\mathbb{R}^2} \left| \left( \nabla \right)^{1-\varepsilon} S_{t-s} : (v + w)^3 \right|^2 \, ds \, dx \right]^{1/2}
\]

\[
= \lambda^{1/2} \left[ \int_0^\tau \int_{\mathbb{R}^2} \left| \left( \nabla \right)^{-\varepsilon} : (v + w)^3 \right|^2 \, ds \, dx \right]^{1/2}
\]

\[
= \lambda^{1/2} \| (v + w)^3 \|_{L^\infty_t H^{1-\varepsilon}(\Lambda_L)}.
\]

In the second-to-last step we used the increase in Besov regularity from $S_t$. We can now expand the binomial power by triangle inequality and estimate each term separately. First, $\|w^3\|_{L^p_t H^{1-\varepsilon}(\Lambda_L)} \lesssim L^p \text{M by assumption and Jensen.}$

The second term is estimated as

\[
\|w^2 \|_{L^\infty_t H^{1-\varepsilon}(\Lambda_L)} \lesssim \|w\|_{L^p_t C^{\varepsilon}(\Lambda_L)} \|v\|_{L^p_t H^{2\varepsilon}(\Lambda_L)},
\]

and for the third one we use Theorem 2.4 twice:

\[
\|w^3\|_{L^\infty_t H^{1-\varepsilon}(\Lambda_L)} \lesssim \|w\|_{L^p_t C^{\varepsilon}(\Lambda_L)} \|w^2\|_{L^p_t H^{2\varepsilon}(\Lambda_L)} \lesssim L^p \|w\|_{L^p_t B^{3\varepsilon}_{p,q}(\Lambda_L)}^2.
\]

We also perform the a similar multiplicative estimate for the $v^3$ term. Thus we have estimated

\[
\| (v + w)^3 \|_{L^\infty_t H^{1-\varepsilon}(\Lambda_L)} \lesssim L^p \text{M} \left[ 1 + \|v\|_{L^p_t H^{2\varepsilon}(\Lambda_L)} + \|v\|^2_{L^p_t B^{3\varepsilon}_{p,q}(\Lambda_L)} + \|v\|^3_{L^p_t B^{4\varepsilon}_{p,q}(\Lambda_L)} \right],
\]

which yields the required bound after embedding $H^{1-\varepsilon}$ into $B^{3\varepsilon}_{p,0}$ by Theorem 2.7. With the estimates above, this is possible for $\varepsilon < 1/12$. \qed

Lemma 4.7 (Contraction). In the setting of Lemma 4.6, we also have

\[
\|Fv - F\tilde{v}\|_{L^\infty(0,\tau], H^{1-\varepsilon})} \lessgtr C_\lambda \lambda^{1/2} M (1 + R^2)\|v - \tilde{v}\|_{L^\infty(0,\tau], H^{1-\varepsilon})}.
\]

Proof. We can begin as in Lemma 4.6 to get the upper bound

\[
\lambda^{1/2} \| (v + w)^3 : - (\tilde{w} + \tilde{v})^3 \|_{L^\infty_t H^{1-\varepsilon}(\Lambda_L)}.
\]

When we again expand the binomials, we get three terms to estimate (since the $w^3$ terms cancel each other). First,

\[
\|w^2 : (v - \tilde{v})^2 \|_{L^\infty_t H^{1-\varepsilon}(\Lambda_L)} \lesssim \|w^2\|_{L^p_t C^{\varepsilon}(\Lambda_L)} \|v - \tilde{v}\|_{L^p_t H^{2\varepsilon}(\Lambda_L)} \lesssim M \|v - \tilde{v}\|_{L^p_t H^{1-\varepsilon}(\Lambda_L)}.
\]

In the second term we additionally need to expand

\[
\|v^2 - \tilde{v}^2\|_{L^p_t H^{2\varepsilon}(\Lambda_L)} \lesssim \|v - \tilde{v}\|_{L^p_t B^{3\varepsilon}_{p,q}(\Lambda_L)} \|v + \tilde{v}\|_{L^p_t B^{3\varepsilon}_{p,q}(\Lambda_L)} \lesssim 2R \|v - \tilde{v}\|_{L^p_t H^{1-\varepsilon}(\Lambda_L)}.
\]
In the final term, the corresponding expansion is
\[
\|v^3 - \tilde{v}^3\|_{L^4 H^{2\varepsilon}} \\
= \|(v - \tilde{v})(v^2 + v\tilde{v} + \tilde{v}^2)\|_{L^4 H^{2\varepsilon}} \\
\lesssim \|v - \tilde{v}\|_{L^2 B_{1,4}^{\varepsilon}} \left( \|v\|_{L^1 B_{1,4}^{\varepsilon}} \|\tilde{v}\|_{L^1 B_{1,4}^{\varepsilon}} + \|\tilde{v}\|_{L^1 B_{1,4}^{\varepsilon}} \right) \\
\lesssim \|v - \tilde{v}\|_{L^2 H^{1-\varepsilon}(\Lambda L)} R^2.
\]

All together, we get the claimed inequality for \(\varepsilon\) small.

**Theorem 4.8.** The nonlinear equation (4.5) has a unique solution
\[
v \in L^\infty([0, \tau]; H^{1-\varepsilon}(\Lambda L)).
\]
The norm of \(v\) depends only on \(M\) as given in Lemma 4.6, and the solution time \(\tau\) depends on both \(M\) and the period length of \(\Lambda L\).

**Proof.** It only remains to choose \(R\) and \(\tau\) such that
\[
\begin{align*}
C_\lambda \lambda \tau^{1/2} M(1 + R^4) &\leq R, \\
C_\lambda \lambda \tau^{1/2} M(1 + R^4) &\leq \frac{1}{2}.
\end{align*}
\]
We can select \(R = \max\{1, M\}\) and \(\tau = C_\lambda^{-2}(4\lambda R^3)^{-2}\).

### 4.3 Extension to global time

The analysis of previous sections also applies to the truncated equation
\[
\begin{cases}
\partial_t u(x, t) + (m^2 - \Delta) u(x, t) = -\lambda P_N : P_N u^3 : , \\
u(x, 0) = P_N u_0(x), \\
\partial_t u(x, 0) = P_N u'_0(x)
\end{cases}
\]
where \(P_N\) truncates the Fourier series to terms with frequency at most \(2^N\) in absolute value.\(^2\) The estimates are only changed by a constant factor since the projection operators \(P_N\) are bounded uniformly in Besov norm, and the linear operators \(C_t\) and \(S_t\) do not change the Fourier support.

The reason to pass to (4.6) is that the state space now consists of finitely many Fourier modes. Because the equation is still Hamiltonian, a theorem of Liouville automatically implies invariance of the corresponding Gibbs measure.

**Definition 4.9** (Truncated Gibbs measure). The measure \(\mu_{L,N}\) is supported on the subset of \(\mathcal{H}^{-\varepsilon}(\rho)\) that contains \(2L\)-periodic functions Fourier-truncated to \([-2^N, 2^N]^2\), and is given by the density
\[
f(u_0, u'_0) = \exp \left( -\lambda \int :P_N u_0^3 : dx \right)
\]
with respect to the periodic, truncated (Gaussian free field, white noise) measure.
\(^2\)Recall that we define the Besov space with a full-space Fourier transform; the Fourier transform is a linear combination of Dirac deltas in this case.
Theorem 4.10 (Local-in-time invariance). The flow of (4.6) is well-defined up to time $\tau$, and the Fourier-truncated measure $\mu_{L,N}$ is invariant under the flow.

Proof. Existence of solution follows from the previous sections, and invariance of measure from Liouville’s theorem.

The invariance of measure allows us to probabilistically extend the solution to arbitrary time. If the local time $\tau$ would be a conserved quantity, we could simply restart the flow from $u(\tau)$ and get a solution up to time $2\tau$. This is the case for $L^2$ solutions of (NLS). Such a conservation law does not exist here, but the growth of Besov norm can be controlled in a high-probability set. This is because the solutions at time $\tau$ are distributed identically to the initial data.

Definition 4.11 (Bounded-moment set). Let us recall that we denote by $H^{-\varepsilon}(\rho)$ the space of all initial data $H^{-\varepsilon}(\rho) \times H^{-1-\varepsilon}(\rho)$. We define

$$B_M := \left\{ (u_0, u'_0) \in H^{-\varepsilon}(\rho) : \|w_j\|_{L^2([0,1]; C^{-\varepsilon/2}(\rho))} \leq M \text{ for } j = 1, 2, 3 \right\},$$

where $w$ is the $L$-periodic linear solution to (4.1) with data $(u_0, u'_0)$.

Let us denote the truncated nonlinear flow by $\Phi_{N,t}^{-1}$, and let $\tau$ be the solution time from Theorem 4.8 with $M$ as above. Then the set $B_M$ contains initial data that lead to a well-defined solution in $[0, \tau]$. Correspondingly, $\Phi_{N,\tau}^{-1} B_M$ contains such data that the solution exists in $[-\tau, 0]$; by intersection with $B_M$, we thus get a solution in $[\tau, 2\tau]$. By iterating this $m$ times, we get a set of initial data that supports the flow up to time $T = m\tau$.

By overlapping these solution intervals, we can guarantee uniqueness of the solution. The dependency on period length $L$ is contained in the solution time $\tau$ as chosen in Theorem 4.8, and thus the next growth bound is uniform in $L$.

However, the number of intersections $m$ depends on $L$ and moment bound $M$ via $\tau$.

Lemma 4.12 (Growth bound). Let us define

$$B_{M,L,N} := B_M \cap \Phi_{N,\tau/2}^{-1} B_M \cap \cdots \cap \Phi_{N,\tau/2}^{-2m} B_M.$$  

For $\mu_{L,N}$-almost all $(u_0, u'_0) \in B_{M,L,N}$, there exists a unique solution $u$ to Eq. (4.6) up to time $T = 2m\tau$, and

$$\|w_j\|_{L^2([0,T]; H^{-\varepsilon}(\Lambda_L))} \lesssim T^{1/2} M^j$$

for $j = 1, 2, 3$. The constant is independent of $N$ and $L$.

Proof. Although the definition of $B_M$ uses the non-truncated linear equation, we may pass to the truncated equation by $C_t(P_N u_0) = P_N(C_t u_0)$ and boundedness of $P_N$ on the periodic space.

That the solution is unique follows from the fact that the local solution intervals overlap and each local solution is unique. It thus remains to verify the moment bounds.

For $j = 1$ the claim follows immediately from writing $u = v + w$; the nonlinear part $v$ has $L^\infty([k\tau, (k+1)\tau], H^{1-\varepsilon}(\Lambda_L))$ norm bounded by $M$ in Theorem 4.8, whereas the linear part satisfies

$$\|w\|_{L^2([0,T]; H^{-\varepsilon}(\Lambda_L))} \lesssim T^{1/4} M$$
by Hölder and the definition of $B_M$. For $j = 2$ we are to estimate
\[ \|w^2\|_{L^2([0,T];H^{-\epsilon}(\Lambda_L))} + 2\|vw\|_{L^2([0,T];H^{-\epsilon}(\Lambda_L))} + \|v^2\|_{L^2([0,T];H^{-\epsilon}(\Lambda_L))}. \]
Here the only relevant difference is estimating
\[ \|vw\|_{L^2([0,T];H^{-\epsilon}(\Lambda_L))} \lesssim \|v\|_{L^4([0,T];H^{2\epsilon}(\Lambda_L))} \|w\|_{L^4([0,T];H^{-\epsilon}(\Lambda_L))}. \]
with Besov multiplication and Hölder. Thanks to regularity of $v$, we have
\[ \|v\|_{L^2([0,T];H^{-\epsilon}(\Lambda_L))} \lesssim T^{1/2} \|v\|_{L^\infty([0,T];H^{1-\epsilon}(\Lambda_L))}. \]
The case $j = 3$ follows similarly.

Moreover, this set of initial data has high probability. Here we use the finite-dimensional invariance to bound the probabilities.

**Lemma 4.13** (Data has high probability). Given $k \in \mathbb{N}$, there exists $M_k$ such that $\mu_{L,N}(B_{M_k,L,N}) \geq 1 - 2^{-k}$. The value of $M_k$ may depend on $L$ but not $N$.

**Proof.** We may first use the triangle inequality and union bound to estimate the probability of the complement (non-existence of local solution) as
\[ \mathbb{P}\left( \max_{j=1,2,3} \|w_N^j\|_{L^4([k\tau,k\tau+1];C^{-\epsilon}(\rho))} > M \right) \leq \sum_{j=1}^3 \sum_{k=0}^m \mathbb{P}\left( \|w_N^j\|_{L^4([k\tau,k\tau+1];C^{-\epsilon}(\rho))} > M \right). \]
This in turn is bounded with Markov’s inequality and invariance:
\[ \sum_{k=0}^m \mathbb{E}\|w_N^j\|_{L^4([k\tau,k\tau+1];C^{-\epsilon}(\rho))}^p \lesssim m \frac{\mathbb{E}\|w_N^j\|_{L^p([0,1];C^{-\epsilon}(\rho))}^p}{M^p}. \]
The expectation is bounded by Lemma 4.4 for any large $p$; this estimate is uniform in $N$. Now we substitute $m = T/\tau$ and $\tau = C_L M^{-C}$ from Theorem 4.8. By choosing $p$ large enough, our final estimate is
\[ \mathbb{P}(w_N \notin \mathcal{D}_{M,N}) \lesssim L, p, TM^{C-p}, \]
which vanishes as $M$ is chosen to be large.

### 4.4 Invariance of non-truncated measure

Let us use Lemma 4.13 to rename the sets of initial data defined above:

**Definition 4.14** (High-probability set of data). We define the set $\mathcal{D}_{k,L,N}$ to equal $\mathcal{B}_{M_k,L,N}$ where $M_k$ is chosen with Lemma 4.13 such that $\mu_{L,N}(\mathcal{D}_{k,L,N}) \geq 1 - 2^{-k}$.
We can now take a limit of these sets and get a high-probability set of initial data without any Fourier truncation. We follow here the argument of Burq and Tzvetkov [17, Section 6]. We need to drop the regularity a bit in order to use the compact embedding (Theorem 2.8).

To define the limit measure $\mu_L$, we remove the truncation in Definition 4.9. By construction, the measures $\mu_{L,N}$ and $\mu_L$ are all absolutely continuous with respect to the non-truncated (GFF, WN) measure, and $\mu_{L,N} \to \mu_L$ in total variation.

**Theorem 4.15** (Limiting set of initial data). We define a subset of $H^{-\varepsilon}(\rho)$ by

$$D_{k,L} := \left\{ (u_0, u'_0) = \lim_{m \to \infty} (u_{0,N_m}, u'_{0,N_m}) \in D_{k,L,N_m} \text{ for some } N_m \to \infty \right\},$$

where the limit is taken in $H^{-2\varepsilon}(\rho)$. Then $\mu_L(D_{k,L}) \geq 1 - 2^{-k}$. The linear parts of solutions started from $(u_0, u'_0) \in D_{k,L}$ satisfy the same moment bound as in Lemma 4.13.

**Proof.** It follows from the definition that

$$\limsup_{N \to \infty} D_{k,L,N} \subset D_{k,L},$$

and then Fatou’s lemma implies

$$\mu_L(D_{k,L}) \geq \mu_L \left( \limsup_{N \to \infty} D_{k,L,N} \right) \geq \limsup_{N \to \infty} \mu_L(D_{k,L,N}) = \limsup_{N \to \infty} \mu_{L,N}(D_{k,L,N}) \geq 1 - 2^{-k}.$$

Here the equality holds by convergence of $\mu_{L,N} \to \mu_L$ in total variation. The bound for moments follows from continuity of $S_t$ and $C_t$ and the matching bound for $D_{k,L,N}$, uniform in $N$.

To show invariance of the limiting measure as $N \to \infty$, we need to approximate full solutions by Fourier-truncated solutions. The next lemma gives convergence in a qualitative sense.

**Lemma 4.16** (Limit solves NLW). If $u_m \in L^p([0,T]; H^{-\varepsilon}(\rho))$ are solutions to the truncated equation (4.6) with data $(u_{0,N_m}, u'_{0,N_m})$ as in Theorem 4.15, then $u(x,t) := \lim_{m \to \infty} u_m(x,t)$ solves the non-truncated equation (4.1) with the limiting initial data $(u_0, u'_0) \in D_{k,L}$.

**Proof.** As the solution operators $(C_t, S_t) : H^{-\varepsilon}(\Lambda_L) \to H^{-\varepsilon}(\Lambda_L)$ are continuous, the linear part converges:

$$w(t) = C_t u_0 + S_t u'_0 = \lim_{m \to \infty} (C_t u_{0,N_m} + S_t u'_{0,N_m}).$$

Let us then consider the integral part in (4.2). We need to show that

$$\lim_{m \to \infty} \int_0^t S_{t-s} (P_{N_m} : P_{N_m} u_m^3 : - : u^3:) \, ds = 0.$$
We write the inner term as $P_{N_m} (w_{N_m};-; :u^3:) - (1 - P_{N_m}) : u^3:$ and note that

$$\int_0^T \|(1 - P_{N_m}) : u^3:\|_{H^{-2\varepsilon}} \, ds \leq \int_0^T 2^{-N_m \varepsilon} \|P_{\geq N} : u^3\|_{H^{-\varepsilon}} \, ds$$

goes to 0 as $N_m \to \infty$ by the definition of Besov norm. Since $P_{N_m}$ is bounded (uniformly in $N_m$) it is sufficient to show that

$$:P_{N_m} : u^3_{N_m} : \to : u^3:\text{ in } L^1([0,T]; H^{-2\varepsilon}(\Lambda_L)).$$

We write $u_{N_m} = w_{N_m} + v_{N_m}$ where $\|v_{N_m}\|_{H^{1-\varepsilon}} \lesssim M^3$ by construction. This means that we can extract a subsequence of $v_{N_m}$ with a limit $v = u - w$. Assume that $:P_{N_m}^j w^j_{N_m} : \to : w^j:$ for $j = 1, 2, 3$ in $L^p([0,T], C^{-\varepsilon}(\Lambda_L))$. Then $:(P_{N_m} u_{N_m})^3 : \to : u^3:$ follows from

$$:(P_{N_m} u)^3 : = \sum_{j=0}^3 : (P_{N_m} w)^j : : v^{3-j}:$$

and continuity of the products.

We observe that $(u_0, u_0^0) \to : w^j:$ is a measurable map from $B_{p,\varepsilon}^{-1} \times B_{p,\varepsilon}^{1-\varepsilon}$ to $L^p([0,T], B_{p,\varepsilon}^{-\varepsilon})$. Lusin’s theorem implies for any $\delta > 0$ there exists a set $A_\delta$ such that $\nu_L(A_\delta) < \delta$ and $: w^j :$ depends continuously on $(u_0, u_0^0)$. Furthermore $:(P_{N_m} w)^j : \to : w^j:$ almost surely, at least up to subsequence. By Egorov’s theorem we can find a set $A_\delta$ such that $:(P_{N_m} w)^j : \to : w^j:$ uniformly and $\nu_L(A_\delta) < \delta$. Then on the complement of $\bar{A} := A_\delta \cup \bar{A}_\delta$ we have that (note that $w$ depends on the initial data $u_0$)

$$\lim_{m \to \infty} \| : (P_{N_m} w_{N_m})^j : - : w^j : \|_{L^p([0,T], B_{p,\varepsilon}^{-\varepsilon}(\Lambda_L))}$$

$$\leq \lim_{m \to \infty} \sup_{u_0 \in \bar{A}_\delta} \| : (P_{N_m} w)^j : - : w^j : \|_{L^p([0,T], B_{p,\varepsilon}^{-\varepsilon}(\Lambda_L))}$$

$$+ \| : w^j_{N_m} : - : w^j : \|_{L^p([0,T], B_{p,\varepsilon}^{-\varepsilon}(\Lambda_L))}$$

Now the first term goes to 0 by uniform convergence and the second by continuity. This implies that on the complement of $\bar{A}_\delta$ the equation holds. Thus it holds also on $\cup_{\delta > 0} \bar{A}_\delta$, which is a set of probability 1.

To quantify the convergence, we derive pointwise bounds in the bounded set $D_{k,L}$. These pointwise results follow from Fourier projections in Besov spaces, although we need to drop the regularity of our target space by $\varepsilon$. Again, this change is irrelevant since $\varepsilon$ is arbitrarily small.

**Theorem 4.17** (Invariance of finite-volume measure). We have

$$\lim_{N \to \infty} \sup_A |\mu_L(A) - \mu_{L,N}(A)| = 0,$$

where the supremum is taken over all measurable subsets of $H^{-2\varepsilon}(\Lambda_L)$. This implies that $\mu_L(\Phi_t A) = \mu_L(A)$ for all $t \in [0,T]$.

**Proof.** The first statement extends total variation convergence to the product measure, which holds since the first marginal converges in total variation. We may assume $A$ to be a subset of $D_{k,L}$ for $k$ large, since the complement of $D_{k,L}$
has probability $2^{-k}$. By the same reasoning, we may also intersect with the set of probability $1 - 2^{-k}$ to be defined below. Moreover, we can bound the difference $|\mu_L(\Phi_t A) - \mu_L(A)|$ by

$$
|\mu_{L,N}(\Phi_t^N P_N A) - \mu_{L,N}(P_N A)| + \sum_{s \in \{0,t\}} |\mu_L(\Phi_s A) - \mu_{L,N}(\Phi_s^N P_N A)|.
$$

Here the first term vanishes by finite-dimensional invariance from Theorem 4.10. The second term can be split as

$$
|\mu_{L}(\Phi_s A) - \mu_{L,N}(\Phi_s A)| + |\mu_{L,N}(\Phi_s A) - \mu_{L,N}(\Phi_s^N P_N A)|,
$$

where the first part vanishes uniformly in $A$ due to convergence in total variation. The latter part is the measure of a symmetric difference, which can be bounded by the measure of $B(0,R) \times H^{-1-2\varepsilon}(\Lambda_L)$. Here the radius of $B(0,R) \subset H^{-2\varepsilon}(\Lambda_L)$ must satisfy

$$
R = \sup_{u_0 \in A} \|\Phi_s u_0 - \Phi_s^N P_N u_0\|_{H^{-2\varepsilon}(\Lambda_L)}.
$$

This pointwise bound is crude but suffices since we assume $(u_0, u_0') \in D_{k,L}$. In particular, we do not need to care about estimating the time derivatives, as the product measure of the difference is already small by the first factor.

By the uniform bounds on $D_{k,L}$, we know that the full solution $u$ and the Fourier-truncated solution $u_N$ are well-defined up to time $T$. We split the difference $(u - u_N)(t)$ further into three parts in Lemmas 4.18, 4.19, and 4.20 below. Thanks to the uniform moment bounds given in Theorem 4.15, we get

$$
R = \sup_{u_0 \in A} \|\Phi_s u_0 - \Phi_s^N P_N u_0\|_{H^{-2\varepsilon}(\Lambda_L)}\leq 2^{-\varepsilon N}(M^3 + \exp(CM^3)) + H_N \exp(CM^3).
$$

This bound is uniform in $A$ and the constant does not depend on $N$. The first term vanishes as we take $N \to \infty$. The inside set of probability $1 - 2^{-k}$ from Lemma 4.20 we have

$$
0 \leq H_N \leq \kappa_N \xrightarrow{N \to \infty} 0,
$$

so the second term goes to 0 as well. This means that the ball vanishes in the limit, and correspondingly $\mu_L(B(0,R) \times H^{-1-2\varepsilon}(\Lambda_L)) \to 0$.

The first two estimates are deterministic and use Besov space properties; however, the low frequencies of nonlinear parts may have more complicated interactions. The probabilistic terms are due to presence of the GFF. Again, the moment bounds for GFF allow us to control the nonlinear part $v$ in sets of arbitrarily high probability.

**Lemma 4.18** (Fourier approximation, linear part). The linear parts satisfy

$$
\|w(t) - w_N(t)\|_{H^{-2\varepsilon}(\Lambda_L)} \leq C_k 2^{-\varepsilon N}
$$

with probability $1 - 2^{-k}$.
Proof. Since the operators defining \( w \) and \( w_N \) are Fourier multipliers, the difference can be written as a projection:
\[
\|w(t) - w_N(t)\|_{H^{-2\varepsilon}(\mathcal{A}_L)} = \|P_{>N}w(t)\|_{H^{-2\varepsilon}(\mathcal{A}_L)}.
\]
It now follows from the definition of Besov space that
\[
\|P_{>N}w(t)\|_{H^{-2\varepsilon}(\mathcal{A}_L)} \lesssim 2^{-\varepsilon N}\|w(t)\|_{H^{-\varepsilon}(\mathcal{A}_L)}.
\]
The bounds in Lemma 4.4 are \( L^p \) in time and thus inapplicable here, but by stationarity it is enough to consider \( w(0) \). Since the moments of \( \phi^4 \) are bounded by Lemma 3.17, we can choose \( C_k \) such that \( \mathbb{P}(\|w(t)\|_{H^{-\varepsilon}(\mathcal{A}_L)} > C_k) < 2^{-k} \).

**Lemma 4.19** (Fourier approximation, high-frequency nonlinearity). We have for all \( t \leq T \) the estimate
\[
\left\| \int_0^t P_{>N}S_t-u(x,t)^3 : ds \right\|_{H^{-2\varepsilon}(\mathcal{A}_L)} \lesssim 2^{-\varepsilon N}M^3.
\]

Proof. By Jensen
\[
\left\| \int_0^t P_{>N}S_t-u(x,t)^3 : ds \right\|_{H^{-2\varepsilon}}^2 \leq t \int_0^t \left\| P_{>N}S_t-u(x,t)^3 : ds \right\|_{H^{-2\varepsilon}}^2.
\]
Now we can again use the boundedness and pay a little regularity to get
\[
\left\| P_{>N}S_t-u(x,t)^3 : H^{-2\varepsilon} \right\| \lesssim 2^{-\varepsilon N}\|u(x,t)^3 : H^{-1-\varepsilon}\|.
\]
Now the estimate follows from Lemma 4.12. By continuity, the lemma holds also in the limit.

**Lemma 4.20** (Fourier approximation, low-frequency nonlinearity). There exist random variables \( H_N : H^{-\varepsilon}(\mathcal{A}_L) \to \mathbb{R}_+ \) such that on \( \mathcal{D}_{k,L} \) we have
\[
\left\| \int S_t-u^3(s) - P_N(s)u^3(s) : ds \right\|_{H^{-2\varepsilon}(\mathcal{A}_L)} \lesssim \exp(CM^3)(2^{-\varepsilon N} + H_N),
\]
and for any \( k \in \mathbb{N} \) there exists a sequence \( \kappa_N \to 0 \) such that
\[
\nu(H_N \geq \kappa_N) \leq 2^{-k}.
\]

Proof. Let us rewrite the left-hand side as \( \|v - v_N\|_{H^{1-\varepsilon}} \), where
\[
v_N = \int_0^t S_t-u^3(s) : ds, \quad \text{and} \quad v = \int_0^t S_t-u^3 : ds.
\]
By Lemma 4.12 we have
\[
\|v_N(t)\|_{H^{1-\varepsilon}} = \left\| \int_0^t S_t-u^3(s) : ds \right\|_{H^{1-\varepsilon}} \lesssim M^3,
\]
and similarly for the limit \( v \). We can split
\[
\|v - v_N\|_{H^{1-2\varepsilon}}
\]
\[
= \left\| \int_0^t S_{t-s} \left[ :w^3: (s) - P_N : (P_N u_N)^3 : (s) \right] \, ds \right\|_{H^{1-2\varepsilon}}
\]
\[
\leq \int_0^t \| :w^3: (s) - (P_N u_N)^3 : (s)\|_{H^{1-2\varepsilon}} \, ds + \left\| \int_0^t P_{> N} : (P_N u_N)^3 : (s) \, ds \right\|_{H^{1-2\varepsilon}}
\]
\[
\lesssim \int_0^t \| :w^3: (s) - (P_N u_N)^3 : (s)\|_{H^{1-2\varepsilon}} \, ds + 2^{-\varepsilon N} M^3,
\]
where the second term is bounded as in Lemma 4.19. We expand the remaining integrand to get
\[
:w^3: - : (P_N u_N)^3 : = \sum_{j=0}^3 \left( \begin{array}{c} 3 \\ j \end{array} \right) (w^j : w^{3-j} - : w_N^j : (P_N v_N)^{3-j}),
\]
and further split it into
\[
:w^j : w^{3-j} - : w_N^j : (P_N v_N)^{3-j} = (w^j : - : w_N^j : (P_N v_N)^{3-j}
\]
\[
+ : w^j : (w^{3-j} - (P_N v_N)^{3-j}).
\]
Now the first term is bounded by
\[
\| : w^j : - : w_N^j : (P_N v_N)^{3-j} \|_{H^{1-2\varepsilon}} \lesssim \| : w^j : - : w_N^j : \|_{H^{1-2\varepsilon}} M^3,
\]
and in the nontrivial cases \( j = 1, 2 \) we can bound the second term by
\[
\| : w^j : (w^{3-j} - (P_N v_N)^{3-j}) \|_{H^{1-2\varepsilon}}
\]
\[
\lesssim \| : w^j : \|_{C^{\kappa}} \| w^{3-j} - (P_N v_N)^{3-j} \|_{H^{1-2\varepsilon}}
\]
\[
\lesssim \| : w^j : \|_{C^{\kappa}} \| v - P_N v_N \|_{H^{1-2\varepsilon}} M^3.
\]
By again separating the high-frequency part, the final factor equals
\[
\|v - P_N v_N\|_{H^{1-2\varepsilon}} \lesssim \|v - v_N\|_{H^{1-2\varepsilon}} + 2^{-\varepsilon N} M^3.
\]
Now setting
\[
H_N = \sup_{\substack{j \leq 3 \, \, \, j \neq 1, 2}} \int_0^T \| : w^j : - : w_N^j : (s) \|_{C^{\kappa-\varepsilon}} \, ds,
\]
we finally obtain
\[
\|(v - v_N)(t)\|_{H^{1-2\varepsilon}} \lesssim M^3 \left[ 2^{-\varepsilon N} + H_N + \int_0^t \sum_{j=1}^2 \| : w^j : (s) \|_{C^{\kappa-\varepsilon}} \|(v - v_N)(s)\|_{H^{1-2\varepsilon}} \, ds \right].
\]
Then Grönwall’s lemma gives that
\[
\|(v - v_N)(t)\|_{H^{1-2\varepsilon}} \lesssim M^3 (2^{-\varepsilon N} + H_N) \exp \left( \int_0^T \sum_{j=1}^2 \| : w^j : (s) \|_{C^{\kappa-\varepsilon}} \, ds \right)
\]
\[
\lesssim \exp(C M^3)(2^{-\varepsilon N} + H_N).
\]
5 Global invariance of NLW

In any bounded region the behaviour of (NLW) only depends on the light cone, and we are free to use the periodic solution theory. Within this bounded region, it is impossible to distinguish between flows of different period length (as long as the period is sufficiently long). We use this property to pass the period length to limit.

In Section 5.1 we show that all statements about measurable events can be reduced back to the periodic case. We then show in Section 5.2 that the unperiodic flow can be approximated by periodic solutions started from periodic data. The main results are finally proved in Section 5.3.

Let us first reference some more properties of weak convergence, in addition to those defined in Section 3.2. We use these implicitly in the following.

**Lemma 5.1** (Weak limits in product spaces; [7, Theorem 2.8]). Assume that \( X \times X' \) is separable. Then \( (\mu_L \times \mu'_L) \) converges weakly to \( \mu \times \mu' \) if and only if \( (\mu_L) \) and \( (\mu'_L) \) converge weakly to \( \mu \) and \( \mu' \) respectively.

**Lemma 5.2** (Skorokhod’s theorem; [7, Theorem 6.7]). Suppose that \( (\mu_L) \) converge weakly to \( \mu \) supported on a separable space. Then there exist a common probability space \( \tilde{P} \) and random variables \( X_L, X \) such that \( \text{Law}(X_L) = \mu_L, \text{Law}(X) = \mu \), and \( X_L \to X \) almost surely.

5.1 Reduction to bounded domain

The Borel \( \sigma \)-algebra of \( \mathbb{R}^2 \) can be generated by compact sets, or even just closed balls. We will show below an analogous result for the Borel \( \sigma \)-algebra of \( \mathcal{H}^{-2\varepsilon}(\rho) \): the \( \sigma \)-algebra is generated by restrictions of distributions to bounded domains.

**Theorem 5.3** (\( \sigma \)-algebra generated by bounded-domain functions). Let \( s, s' \in \mathbb{R} \), and let \( \mathcal{A}^s \) be the family of sets that where inclusion only depends on restrictions to compact domains:

\[
\mathcal{A}^s := \{ A \subset H^s(\rho) : \exists \text{ compact } D \text{ s.t. } f \in A \iff f|_D \in A \ \forall f \in H^s(\rho) \}.
\]

That is, \( 1_A(f) = g_A(f)|_D \) for some \( g_A : \mathcal{H}^{-2\varepsilon}(D) \to \{0, 1\} \). Then

1. the closed ball \( \bar{B} = \bar{B}(f, R) \subset H^s(\rho) \) can be constructed with \( \sigma \)-closed operations from sets in \( \mathcal{A}^s \);

2. the Borel \( \sigma \)-algebra of \( H^s(\rho) \) is a sub-\( \sigma \)-algebra of \( \sigma(\mathcal{A}^s) \);

3. the Borel \( \sigma \)-algebra of \( H^s(\rho) \times H^{s'}(\rho) \) is a sub-\( \sigma \)-algebra of \( \sigma(\mathcal{A}^s) \times \sigma(\mathcal{A}^{s'}) \).

**Proof.** By the definitions of Borel \( \sigma \)-algebra and product \( \sigma \)-algebra, it is enough
to verify the first statement for arbitrary $B = \overline{B}(f, R)$ and $s \in \mathbb{R}$. We can write

$$B = \left\{ g \in H^s(\rho) : \int_{\mathbb{R}^2} \rho(x) x^2 \left( (1 - \Delta)^{s/2} (f - g) \right)^2 \, dx \leq R^2 \right\}$$

$$= \left\{ g \in H^s(\rho) : \int_{\mathbb{R}^2} \rho(x) x^2 \left[ \int_{\mathbb{R}^2} K_s(x - y) (f - g)(y) \, dy \right]^2 \, dx \leq R^2 \right\}$$

$$= \lim_{N \to \infty} \left\{ g \in H^s(\rho) : \sum_{l,m,n=1}^{N} \int_{A_{lmn}} \rho(x) x^2 \left[ \int_{A_m} K_s(x - y) (f - g)(y) \, dy \right]^2 \, dx \leq R^2 \right\}.$$  

Here we denote by $K_s$ the convolution kernel of $(1 - \Delta)^{s/2}$ and by $(A_{lmn})_{l,m,n \in \mathbb{N}}$ some partitioning of $\mathbb{R}^2$, e.g. by unit squares. For finite $N$, the set thus depends on $K_s$, $f$, and $g$ only inside the compact set $\bigcup_{l,m,n=1}^{N} (A_{lmn} + A_m)$. Since $\limsup$ is a $\sigma$-closed operation, this proves that $B$ can be constructed from sets in $A^s$.  

We need to combine this result with the definition of weak limit. If we a priori assume the weak limit to exist, we can show uniqueness by testing against a much smaller class of test functions. This strategy of adapting the test functions to the specific model is very common; see the book of Ethier and Kurtz [21, Section 3.4] for details.

**Corollary 5.4** (Reduction to bounded domains). Let $\mathcal{F}$ be the set of bounded Lipschitz functions $\varphi : \mathcal{H}^{-2\varepsilon}(\rho) \to \mathbb{R}$ that depend only on the restriction of argument to some compact domain: for any $\varphi \in \mathcal{F}$, there exists a compact $D \subset \mathbb{R}^2$ such that $\varphi(f) = \varphi(f|D)$ for all $f \in \mathcal{H}^{-2\varepsilon}(\rho)$.

Assume that the sequence $(\mu_t)$ has a weak limit $\mu^*$, and let $\mu$ be a Borel probability measure on $\mathcal{H}^{-2\varepsilon}(\rho)$. If

$$\lim_{t \to \infty} \int_{\Omega} \varphi(f) \, d\mu_t(f) = \int_{\Omega} \varphi(f) \, d\mu(f)$$

for all $\varphi \in \mathcal{F}$, then the weak limit $\mu^*$ equals $\mu$.

**Proof.** Fix two distinct points $(f, f')$ and $(g, g')$ in $\mathcal{H}^{-2\varepsilon}(\rho)$. By the general theory of distributions, there exist $\alpha, \beta \in C_0^\infty(\mathbb{R}^2)$ such that $\langle \alpha, f - g \rangle \neq 0$ or $\langle \beta, f' - g' \rangle \neq 0$. Then

$$\eta(f, f') := |\langle \alpha, f \rangle| + |1 + \langle \alpha, f \rangle| + |\langle \beta, f' \rangle| + |1 + \langle \beta, f' \rangle|$$

depends only on the compact domain $\text{Supp} \alpha \cup \text{Supp} \beta$, is continuous, takes only non-negative values, and $h(f, f') \neq h(g, g')$. Continuity follows from linearity and Theorem 2.5. Furthermore

$$\tilde{\eta}(f, f') := \frac{\eta(f, f')}{1 + \eta(f, f')}$$

is bounded and composed of Lipschitz functions; thus it belongs to $\mathcal{F}$.

As $(f, f')$ and $(g, g')$ were arbitrary, we have shown $\mathcal{F}$ to separate points. Then [21, Theorem 3.4.5] implies that $\mathcal{F}$ is a separating family, meaning that

$$\int_{\Omega} \varphi(f) \, d\mu^*(f) = \int_{\Omega} \varphi(f) \, d\mu(f)$$

for all $\varphi \in \mathcal{F}$ implies $\mu^* = \mu$. The claim follows from the uniqueness of weak limit.  

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5.2 Invariance of measure

Assuming that the period is large enough, the solution in a bounded domain $D$ is independent of the choice of periodization. However, the initial data sampled from $\mu_L$ still depends on the period length. In this section we quantify the convergence of solutions.

Let us first construct a probabilistic solution set associated with the region $D \subset \mathbb{R}^2$. This argument is analogous to Theorem 4.13, but with a twist: by Theorem 4.8 the growth bound in $D$ is independent of the periodization, but the local solution time $\tau$ is not. However, at discrete times $\{k\tau\}$ we can use the invariance of measure; this property is qualitative and holds for all period lengths.

**Theorem 5.5 (Limiting set of data, polynomial weight).** Fix a bounded domain $D \subset \mathbb{R}^2$. There exists a set $\mathcal{E}_k \subset \mathcal{H}^{-2\varepsilon}(\mu)$ and $M_k > 0$ such that $\mu(\mathcal{E}_k) > 1 - 2^{-k}$ and for almost all $(u_0, u'_0) \in \mathcal{E}_k$ the flow $\Phi_t$ of (NLW) is bounded by $C T M_k^2$ in $L^2([0, T]; H^{-2\varepsilon}(D))$.

**Proof.** Let us consider periodic initial data $(u_{0,L}, u'_{0,L})$ sampled from $\mu_L$, and define the probabilistic data sets

$$\mathcal{E}_{L,M}(D) := \left\{ \|\Phi_{L,t} u_{0,L}\|_{L^2([0,T]; H^{-2\varepsilon}(D))} \leq M \right\}.$$

We extend this definition as $\mathcal{E}_{\infty,M}$ for non-periodic data sampled from $\mu$. By Skorokhod’s lemma we may assume that $u_{0,L} \rightarrow u_0$ almost surely. To estimate the probability of this limit, we first use Fatou’s lemma:

$$\mathbb{P}(\mathcal{E}_{\infty,M}) = \tilde{\mathbb{P}}(\mathcal{E}_{\infty,M}) \geq \tilde{\mathbb{P}}(\limsup_{L \to \infty} \mathcal{E}_{L,M}) \geq \limsup_{L \to \infty} \tilde{\mathbb{P}}(\mathcal{E}_{L,M}).$$

Let us now bound the norm of the flow. We have $D + B(0, T) \subset \Lambda_R, \Lambda_{L}$ for $R$ sufficiently large and $L \geq R$; thus the restriction norm is not able to distinguish between $\Phi_{R,t}$ and $\Phi_{L,t}$. By moving between these two flows, we can extract the solution time $\tau$ to depend on $R$ (fixed) and not $L$ (divergent).

First, we have the following bound for any $\tau > 0$:

$$\|\Phi_{L,t} u_{0,L}\|_{L^2([0,T]; H^{-2\varepsilon}(D))}^2 \leq \|\Phi_{R,t} u_{0,L}\|_{L^2([0,T]; H^{-2\varepsilon}(D))}^2$$

$$= \sum_{k=0}^{T/\tau} \|\Phi_{R,t} u_{0,L}\|_{L^2([k\tau,(k+1)\tau]; H^{-2\varepsilon}(D))}^2$$

$$= \sum_{k=0}^{T/\tau} \|\Phi_{R,t} \Phi_{L,k\tau} u_{0,L}\|_{L^2([0,\tau]; H^{-2\varepsilon}(D))}^2$$

Let us show that the $R$-periodic flow is valid. Assume that

$$\|\Phi_{L,k\tau} u_{0,L}\|_{L^2([0,\tau]; H^{-2\varepsilon}(D))} \leq M$$

for $j = 1, 2, 3$ and all $k \leq T/\tau$, where $\tau$ is chosen as per Theorem 4.8 for the domain $\Lambda_R$ (fixed) and moment bound $M$. Then the local growth bound in Lemma 4.12 gives that $\Phi_R$ is valid up to time $\tau$ and

$$\|\Phi_{R,t} \Phi_{L,k\tau} u_{0,L}\|_{L^2([0,\tau]; H^{-2\varepsilon}(D))} \leq 2M.$$
Correspondingly the full sum satisfies
\[ \| \Phi_{L,t} u_{0,L} \|^2_{L^2([0,T]; H^{-2\epsilon}(\mathbb{R}^2))} \lesssim \frac{TM^2}{\tau} \lesssim TM^{2+c}. \]
The assumption made above is probabilistic, and as before our estimate for the sum holds with probability at least
\[ 1 - P \left( \max_{j=1,2,3} \|: (\Phi_{\text{lin},k\tau} u_{0,L})^j :\|^2_{L^2([0,1]; H^{-2\epsilon}(\mathbb{D}))} > M \right). \]
It is here that we use the invariance of \( \mu_L \) under \( \Phi_L \). As in Lemma 4.13, we bound the probability from below by
\[ 1 - CTM^c \frac{E \|: (\Phi_{\text{lin}} u_{0,L})^j :\|^p_{L^p([0,1]; H^{-2\epsilon}(\mathbb{D}))}}{M^p}. \]
The expectation is bounded by Lemma 4.4 uniformly in \( L \). Again we choose \( p \) large enough to make the power on \( M \) negative and then pass to \( M \) large enough.

As we approximate full-space solutions by periodic ones, we also need to estimate the error made. The following result is analogous to Lemma 4.20. In contrast to Theorem 4.17, now we also need to estimate the time derivatives. However, the estimate for time derivatives is bootstrapped from the one for the \( \phi^4 \) component.

**Lemma 5.6** (Stability, \( \phi^4 \) component). Assume that
\[ \max_{j=1,2,3} \|: u_L^j :\|_{L^4([0,1]; C^{-2\epsilon}(\rho))} \leq M \]
holds for all \( L \) and in the limit. Fix a bounded domain \( D \subset \mathbb{R}^2 \), and define the spatial cutoff \( \chi_t \) as a mollified indicator of \( D \) and \( \chi_{t-s} \) as that of \( D + B(0,s+1) \). Then the random solutions satisfy
\[ \| \chi_t (u_L - u)(t) \|_{H^{-2\epsilon}(\rho)} \lesssim \| \chi_0 (u_{0,L}, u_0') - \chi_0 (u_0, u_0') \|_{H^{-2\epsilon}(\rho)} + \exp(CM^3) H_L, \]
where \( H_L \) are random variables satisfying: for any \( k \in \mathbb{N} \) there exist \( \kappa_L \to 0 \) such that \( \mathbb{P}(H_L \geq \kappa_L) \leq 2^{-k} \).

**Proof.** The first term on the right-hand side comes from continuity of the linear solution operators (Lemma 4.2). For the nonlinear term, our goal is to show
\[ \chi_t \int_0^t S_{t-s} : u(s)^3 : ds = \lim_{L \to \infty} \chi_t \int_0^t S_{t-s} : u_L(s)^3 : ds. \]
We can first apply finite speed of propagation to \( S_{t-s} \) to reduce
\[ \int_0^t \| \chi_t S_{t-s} : u(s)^3 : \|_{H^{-2\epsilon}(\rho)} \lesssim \int_0^t \| \chi_{t-s} : u(s)^3 : \|_{H^{-2\epsilon}(\rho)}. \]
We then perform the same manipulations as in Lemma 4.20, only replacing the Fourier cutoff $N$ by the period length $L$ and keeping track of the weights. Thanks to the bounded domain, we can always re-introduce $\rho$ such as in

$$
\| : w^2(s) : (v - v_L)(s) \chi_{t-s} \|_{H^{-2\varepsilon}(\rho)} \\
\lesssim \| \chi' : w^2(s) : \|_{C^{1}(-1)} \| \chi_{t-s} (v - v_L)(s) \|_{H^{-2\varepsilon}(\rho)} \\
\lesssim \| \chi' \|_{C^{1}(-1)} \| : w^2(s) : \|_{C^{1}(-1)} \| \chi_{t-s} (v - v_L)(s) \|_{H^{-2\varepsilon}(\rho)},
$$

where $\chi'$ is a smooth cutoff of some larger domain. We also define the stochastic term

$$
H_L := \sup_{j \geq 3} \int_0^t \left\| : w^j : - : w^j_L : (s) \right\|_{C^{2\varepsilon}(\rho)} \, ds
$$

as a variation of that in Lemma 4.20. Thus we have bounded

$$
\| \chi_{t} (v - v_L)(t) \|_{H^{-2\varepsilon}(\rho)} \\
\lesssim M^3 \left[ H_L + \int_0^t \sum_{j=1}^2 \| : w^j : \|_{C^{2\varepsilon}(\rho)} \| \chi_{t-s} (v - v_L)(t-s) \|_{H^{-2\varepsilon}(\rho)} \right],
$$

and again Grönwall gives

$$
\| \chi_{t} (v - v_L(t)) \|_{H^{-2\varepsilon}(\rho)} \lesssim \exp(CM^3) H_L.
$$

**Lemma 5.7 (Stability, time derivative).** Fix $k, \ell \in \mathbb{N}$, and assume that

$$
\lim_{L \to \infty} \| \chi (u_L - u)(t) \|_{H^{-2\varepsilon}(\rho)} = 0,
$$

where $\chi$ is any cutoff. Then we have

$$
\| \chi \partial_t (u_L - u)(t) \|_{H^{-1-2\varepsilon}(\rho)} \leq 2^{-\ell}
$$

in a set of probability at least $1 - 2^{-k}$.

**Proof.** Let us first note that by passing to the mild formulation we have

$$
\lim_{s \to 0} \frac{u(t+s) - u(t)}{s} = \lim_{s \to 0} \frac{1}{s} \int_0^s S_{t-r} : u(r)^3 : \, dr + \frac{C_{t+s} - C_t}{s} u_0 + \frac{S_{t+s} - S_t}{s} u_0.
$$

The first term converges almost surely to $: u(t)^3 :$. The limit of the last two terms is a bounded linear operator $L$ from $\mathcal{H}^{-2\varepsilon}(\rho)$ to $\mathcal{H}^{-1-2\varepsilon}(\rho)$, as can be seen by considering the Fourier multiplier symbols.

We estimate the difference of Wick powers with Lemma 3.11:

$$
\| : u_L(t)^3 : - : u(t)^3 : \|_{H^{-2\varepsilon}(\rho)} \leq J_L + \| f^d (u_L(t)) - f^d (u(t)) \|_{H^{-2\varepsilon}(\rho)} + J_\infty,
$$

where

$$
J_L := \| u_L(t)^3 - f^d(u_L(t)) \|_{H^{-2\varepsilon}(\rho)}
$$

and $J_\infty$ is defined analogously. By Lemma 3.11 the expectations of $J_L$ and $J_\infty$ vanish as $\delta \to 0$; we get two $2^{-\ell/3}$ terms with probability $1 - 2^{-k}$ by fixing $\delta$ small enough. We then bound the middle term by $2^{-\ell/3}$ by continuity of $f^d$; choosing $L$ to be large enough.

\[\square\]
5.3 Invariance proof finished

**Theorem 5.8** (Global invariance). We have $\mu \circ \Phi_t = \mu$ for all $0 \leq t \leq T$.

*Proof.* We know a priori that the pushforward measure $\mu \circ \Phi_t$ exists as $\Phi_t$ is a measurable map. (By Theorem 5.3, we only need to check restrictions to bounded domains. There $\Phi_t$ is almost surely defined as a composition of small-time periodic flows.)

By the weak limit and finite-volume invariance, we also have that for all bounded and continuous $f: \mathcal{H}^{-2\ell}(\rho) \to \mathbb{R}$,

$$\int_{\mathcal{H}} f(u_0, u_0') \, d\mu = \lim_{L \to \infty} \int_{\mathcal{H}} f(\Phi_{Lt}(u_{L,0}, u_{L,0}')) \, d\mu_L.$$ 

Since the weak limit is unique, we only need to show that

$$\lim_{L \to \infty} \int_{\mathcal{H}} f(\Phi_{Lt}(u_{L,0}, u_{L,0}')) \, d\mu_L = \int_{\mathcal{H}} f(\Phi_t(u_0, u_0')) \, d\mu.$$

Corollary 5.4 lets us assume that $f$ is Lipschitz and depends on the restriction of its arguments to some $\Lambda_R$. We can further pass to a common probability space by Lemma 5.2.

Let $\mathcal{F}_{L,M}$ be the intersection of those data sets such that both the full and $L$-periodic flow have moment bound $M$ in $\Lambda_R$ by Theorem 5.5, that $H_L \leq 2^{-\ell}$ in Lemma 5.6, and that the $2^{-\ell}$ bound in Lemma 5.7 holds. This set has probability at most $C2^{-k}$ when $M$ and $L$ are large enough. Let us recall that $M$ only depends on $R$ and not $L$. We can then estimate

$$\lim_{L \to \infty} \mathbb{E} \left[ |f(\Phi_{Lt}(u_{L,0}, u_{L,0}')) - f(\Phi_t(u_0, u_0'))| \right] \leq \lim_{L \to \infty} \mathbb{E} \left[ |\mathbf{1}_{\mathcal{F}_{L,M}} [f(\Phi_{Lt}(u_{L,0}, u_{L,0}')) - f(\Phi_t(u_0, u_0'))] + 2^{-k} \|f\|_\infty \right]$$

$$= \lim_{L \to \infty} \mathbb{E} \left[ |\mathbf{1}_{\mathcal{F}_{L,M}} [f(\Phi_{Lt}(u_{L,0}, u_{L,0}')) - f(\Phi_t(u_0, u_0'))] + 2^{-k} \|f\|_\infty \right]$$

$$\leq \lim_{L \to \infty} \text{Lip}_f \mathbb{E} \left[ |\mathbf{1}_{\mathcal{F}_{L,M}} |(\Phi_{Lt}(u_{L,0}, u_{L,0}')) - (\Phi_t(u_0, u_0'))| \|_{\mathcal{H}^{-2\ell}(\rho)} + 2^{-k} \|f\|_\infty \right].$$

Here we used respectively boundedness, that $f$ cannot distinguish between $\Phi_t$ and $\Phi_{Lt}$ once $L \geq R + T$, and Lipschitz continuity. The spatial cutoff $\chi$ depends on $f$ through $R$. 

The two components of $\mathcal{H}^{-2\ell}(\rho)$ are now estimated with Lemmas 5.6 and 5.7: The first component is bounded by

$$\lim_{L \to \infty} \mathbb{E} \left[ |\mathbf{1}[\chi(\tilde{u}_{L,0}, \tilde{u}'_{L,0}) - \Phi_t(u_0, u_0')]| \right]_{\mathcal{H}^{-2\ell}(\rho)} + \exp(CM^3)2^{-\ell}.$$ 

The initial data converges almost surely and uniform integrability allows us to commute the limit and expectation. By increasing $\ell$ and $L$ as necessary, we can make this term arbitrarily small. This then allows us to bound the second component.

We can rephrase the invariance as a global existence and uniqueness result as in the introduction. The same result holds for the second and third powers of the solution since the corresponding $\phi^3$ moments are finite.

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Proof of Theorem 1.1. We have now showed that a limit exists in any compact region of $\mathbb{R}^2$, and that the product measure $\mu$ is invariant under the flow. The latter implies that

$$E \| \Phi_t u_0 \|^2_{L^2([0,T]; H^{-2\epsilon}(\rho))} = \int_0^T E \| u_0 \|^2_{H^{-2\epsilon}(\rho)} dt \lesssim T,$$

meaning that the norm is almost surely finite. We still need to check that the limit really solves (NLW).

The convergence of linear parts follows from Lemma 4.5. We only need to show that

$$v(t) = \int_0^t S_{t-s} : u(s)^3 : ds.$$

Equivalently, we can show that

$$\lim_{L \to \infty} \| v - v_L \|_{H^{1-2\epsilon}(\rho)} \lesssim \lim_{L \to \infty} \int_0^t \| S_{t-s} : u(s)^3 : - : u_L(s)^3 : \|_{H^{1-2\epsilon}(\rho)} ds$$

vanishes as $L \to \infty$. Since $S_{t-s}$ is bounded by Lemma 4.2 and $: u^3 :$ is assumed to belong to $H^{-2\epsilon}(\rho)$, we can use dominated convergence. Then we are left with estimating

$$\lim_{L \to \infty} \left\| \sum_{j=0}^3 \binom{3}{j} [ : w(s)^j : v(s)^{3-j} : - : u_L(s)^3 : - : w_L(s)^j : v_L(s)^{3-j} : :] \right\|_{H^{-2\epsilon}(\rho)}.$$

As $w_L$ converges in $L^p([0,T]; H^{-2\epsilon}(\rho))$ and $v_L$ converges in $C([0,T]; H^{1-2\epsilon}(\rho))$, and the product is a continuous operator, we can pass $w_L$ and $v_L = u_L - w_L$ to the limit. This shows that $v$ satisfies the mild formulation.

### 6 Weak invariance of NLS

The situation is more complicated for the nonlinear Schrödinger equation

$$\begin{cases}
i \partial_t u_L + (m^2 - \Delta) u_L = -\lambda : u_L^3 : , \\
u_L(0) \sim \phi_{2,L}^4, \end{cases}$$

on $\Lambda_L \times \mathbb{R}$. Invariance of the periodic $\phi_2^4$ measure under this equation was shown already by Bourgain [9] – see also the more pedagogic review [45] – but our preceding extension argument is broken for two reasons.

The linear solution operator

$$T_t u := \exp(it\Delta) u := \mathcal{F}^{-1} \left[ \exp(-it|\xi|^2) \hat{u}(\xi) \right]$$

(6.2)

does not increase the regularity of its argument. Therefore the mild solution

$$u(t) = T_t u(0) + \lambda \int_0^t [T_{t-s} : u(s)^3 : ](x) ds$$

(6.3)

is not amenable to the fixpoint argument of Section 4. Moreover, NLS does not possess finite speed of propagation: wave packets propagate at a speed
proportional to their frequency squared. This means that the argument in Section 5 is not applicable either.

However, if we can accept some loss of regularity, we can still use the previous tightness argument. That allows us to approximate full-space solutions by (a subsequence of) periodic solutions. This sense of invariance was introduced by Albeverio and Cruzeiro [2] in the context of Navier–Stokes equations.

**Remark 6.1.** Unlike with the wave equation, here the initial data is complex-valued. Our construction of the $\phi^4$ measure holds also in this setting and all results from Section 3 carry over.

Compactness is given by a version of the usual embedding theorem for Hölder-continuous functions:

**Lemma 6.2** (Arzelà–Ascoli theorem). The Hölder space $C^\alpha([0,T];H^s(\rho))$ is defined by the norm

$$\|f\|_{C^\alpha([0,T];H^s(\rho))} := \sup_{0 \leq s \neq t \leq T} \frac{\|f(t) - f(s)\|_{H^s(\rho)}}{|t - s|^\alpha}.$$ 

Then the embedding

$$C^{2\varepsilon}([0,T];H^s(\rho)) \hookrightarrow C^\varepsilon([0,T];H^{s-\varepsilon}(\rho^{1+\varepsilon}))$$

is compact.

We first collect a lemma needed for the tightness proof. By giving up some differentiability, we can get a dispersive estimate that is independent of time.

**Lemma 6.3** (Dispersive estimate). Fix $1 \leq p, q \leq \infty$, and let us endow $\mathbb{R}^d$ with weight $\rho(x) = (1 + |x|^2)^{\alpha/2}$, where $|\alpha| > d$. The Schrödinger propagator $T_t$ then satisfies for all $s \in \mathbb{R}$ the estimate

$$\|T_t f\|_{B^s_{p,q}(\rho)} \lesssim \|f\|_{B^s_{p,q}(\rho)}.$$ 

**Proof.** Let us first assume $\alpha > d$. We will estimate the $L^p$ norm inside

$$\|T_t f\|_{H^s_{p,q}(\rho)} = \left\|2^{ks}\|\Delta_k T_t f\|_{L^p(\rho)}\right\|_{L^q}. $$

We can write $\Delta_k T_t = \Delta_k \Delta'_k T_t$ where $\Delta'_k$ is a smooth indicator of a larger annulus, given by multiplier symbol $\varphi(2^{-k} \cdot)$. Let $K_k$ be the convolution kernel of $\Delta_k T_t$; by weighted Young’s inequality [24, Lemma A.8] we then have

$$\|K_k * (\Delta_k f)\|_{L^p(w)} \leq \|K_k\|_{L^1(w^{-1})} \|\Delta_k f\|_{L^p(w)} \leq \|1\|_{L^1(w^{-1})} \|K_k\|_{L^\infty} \|\Delta_k f\|_{L^p(w)}.$$ 

Since $w^{-1}$ is integrable, the first factor is finite. The second factor satisfies the scaling relation

$$\int_{\mathbb{R}^d} e^{i x \cdot \xi} \varphi(2^{-k} \xi) e^{-i |\xi|} \, d\xi \leq 2^{kd} \int_{\mathbb{R}^d} |\varphi(\eta)| \, d\eta,$$

where $\varphi$ is compactly supported. This yields the required bound. The case $k = -1$ (ball in Fourier space) also gives a constant factor.
For \( \alpha < -d \), we proceed by duality and skew-adjointness of the operator. By Theorem 2.5 and the above we have

\[
\| T_t f \|_{B^s_{p',q}(\rho)} = \sup_g \langle T_t f, g \rangle \\
\leq \sup_g \| f \|_{B^s_{p,q}(\rho)} \| T_{t-s} g \|_{B^{-s}_{p',q'}(\rho^{-1})} \\
\leq \sup_g \| f \|_{B^s_{p,q}(\rho)} \| g \|_{B^{-s}_{p',q'}(\rho^{-1})} \\
= \sup_g \| f \|_{B^s_{p,q}(\rho)},
\]

where the supremum is taken over all \( g \) with \( B^{-s}_{p',q'}(\rho^{-1}) \) norm equal to 1.

**Remark 6.4.** We can also swap \( p \) and 1 in the weighted Young’s inequality to get

\[
\| T_t f \|_{B^s_{p,q}(\rho)} \lesssim \| f \|_{B^s_{1,q}(\rho)}.
\]

In the unweighted case this Besov estimate follows from Theorem 2.7 and conservation of \( L^2 \) norm. These results are reminiscent of the standard dispersive estimate

\[
\| T_t f \|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2} \| f \|_{L^1(\mathbb{R}^d)},
\]

and by interpolation one can find “exchange rates” between spatial regularity and integrability in time. Interpolation of weighted Besov spaces is explored in [35]; see especially Theorem 4 there.

**Theorem 6.5** (Tightness). The sequence of periodic solutions \( u_L \) is tight in \( C^{\alpha-\varepsilon}([0,T]; H^{-4-2\varepsilon}(\rho^{1+\varepsilon})) \) for any \( \alpha < 1/2, \varepsilon > 0 \), and integrable weight \( \rho \).

**Proof.** We will show that

\[
\sup_L \mathbb{E} \| u_L \|_{C^{\alpha}([0,T]; H^{-4-\varepsilon}(\rho))}^2 < \infty.
\]

This implies tightness in a slightly less regular space by Lemma 6.2.

From the mild formulation of the equation we obtain that

\[
u_L(t) - u_L(s) = (T_t - T_s)u_L(0) + \int_s^t T_{t-r}\cdot:u_L(l)^3 \, dr,
\]

so we will need to estimate

\[
\left\| \int_s^t T_{t-r}\cdot:u_L(r)^3 \, dr \right\|_{H^{-4-\varepsilon}(\rho)} + \|(T_t - T_s)u(0)\|_{H^{-4-\varepsilon}(\rho)}.
\]

For the first term, we can use Cauchy–Schwarz to exchange the integrals:

\[
\left\| \int_s^t T_{t-r}\cdot:u_L(r)^3 \, dr \right\|_{H^{-4-\varepsilon}(\rho)} \\
\leq |t-s|^{1/2} \left[ \int_s^t \| T_{t-r}\cdot:u_L(r)^3 \|_{H^{-4-\varepsilon}(\rho)}^2 \, dr \right]^{1/2} \\
\lesssim |t-s|^{1/2} \left[ \int_0^T \| :u_L(r)^3 \|_{H^{-2-\varepsilon}(\rho)}^2 \, dr \right]^{1/2}.
\]

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Here we used the uniform bound from Lemma 6.3. The Wick power is bounded in expectation by Theorem 3.17, and the bound is uniform in $L$.

For the second term we use the functional derivative
\[
(e^{-it\Delta} - e^{-is\Delta}) f = \int_s^t (-i\Delta) e^{-ir\Delta} f \, dr
\]
and fundamental theorem of calculus to compute
\[
\| (T_t - T_s) u(0) \|_{H^{-4-\varepsilon}(\rho)} = \left\| \int_s^t \Delta T_r u(0) \, dr \right\|_{H^{-4-\varepsilon}(\rho)}
\leq |t - s|^{1/2} \left( \int_0^T \| \Delta T_r u(0) \|_{H^{-4-\varepsilon}(\rho)}^2 \, dr \right)^{1/2}
\leq |t - s|^{1/2} \left( \int_0^T \| u(0) \|_{H^{-\varepsilon}(\rho)}^2 \, dr \right)^{1/2}.
\]

Again the expectation is bounded. All of these estimates are uniform in $L$ and hold for all $t, s \in [0, T]$.

By changing the probability space with Skorokhod’s theorem (Lemma 5.2) we can assume that $u_L \to u$ almost surely. We still need to establish that $u$ satisfies the limiting equation.

**Theorem 6.6 (Limit solves NLS).** There exists a probability space $\tilde{\mathbb{P}}$ and random variable $\tilde{u} \in L^2(\tilde{\mathbb{P}}, C^\varepsilon([0, T]; H^{-4-\varepsilon}(\rho)))$ such that
\[
\tilde{u}(t) = T_t \tilde{u}(0) + \int_0^t T_{t-s} : \tilde{u}(s)^3 : \, ds
\]
and $\text{Law}(\tilde{u}(t)) = \mu$ for all $t \in [0, T]$.

**Proof.** For clarity we omit the tildes in the proof. In addition to almost sure convergence of $u_L$ and $:u_L^3:$; we have by tightness
\[
u_L \to u \quad \text{in} \quad L^2(\tilde{\mathbb{P}}, C^\varepsilon([0, T]; H^{-4-\varepsilon}(\rho))).
\]
Hölder continuity implies that we also have convergence of
\[
u_L(t) \to u(t) \quad \text{in} \quad L^2(\tilde{\mathbb{P}}, H^{-4-\varepsilon}(\rho))
\]
for all $t \in [0, T]$.

We repeat the approximation argument of Lemma 4.5. Let $f^\varepsilon(u)$ approximate $:u^3:$ as in Lemma 3.11. Then
\[
\int_0^t T_{t-s} ( : u(s)^3 : - : u_L(s)^3 : ) \, ds = \int_0^t T_{t-s} ( : u(s)^3 : - f^\varepsilon(u(s))) \, ds
+ \int_0^t T_{t-s} (f^\varepsilon(u(s))) \, ds
+ \int_0^t T_{t-s} ( : u_L(s)^3 : - f^\varepsilon(u_L(s))) \, ds.
\]
Here the first and last term vanish in $H^{-4-\varepsilon}(\rho)$ norm as $\delta \to 0$. This follows directly from the approximation result and is uniform in $L$.

To bound the second term, we use boundedness of $f^\delta$:

$$E \| T_{t-s} f^\delta(u(s)) \|_{H^{-2}(\rho)} \lesssim E \| f^\delta(u(s)) \|_{L^2(\rho)} \lesssim E \| u(s) \|_{H^{-4-\varepsilon}(\rho)},$$

and the same for $u_L$. This lets us use dominated convergence in

$$\lim_{L \to \infty} E \int_0^t T_{t-s} (f^\delta(u(s)) - f^\delta(u_L(s))) \, ds = 0.$$

Thus for any fixed $\delta > 0$ and $\kappa > 0$, we have for all large $L$ that

$$E \int_0^t T_{t-s} (f^\delta(u(s)) - f^\delta(u_L(s))) \, ds < \kappa.$$

In summary, we first pass to $\delta$ small and then to $L$ large. This gives that

$$\lim_{L \to \infty} E \int_0^t T_{t-s} (u(s)^3 : - u_L(s)^3 : ) \, ds = 0.$$

By passing to a further subsequence, we then have almost sure convergence of these nonlinear terms. This finishes the proof. 

\section*{A Proof of Lemma 3.11}

Let us state the lemma in a slightly different way. We remark that this lemma is not used in the construction of $:(Z + \phi)^3:$; so the expressions are valid.

\textbf{Lemma A.1.} Let us fix a Fourier cutoff $\chi \in C_c^\infty(\mathbb{R}^2)$ and define $\chi_\delta(x) = \chi(\delta x)$. Let $Z^\delta = \chi_\delta(\langle \nabla \rangle) Z$ and $Z^\delta_L = \chi_\delta(\langle \nabla \rangle) Z_L$. Define

$$f^\delta(Z^\delta) := (Z^\delta)^3_{\delta} := (Z^\delta)^3 - 3a_4 Z^\delta,$$

where $a_4 = E \| (Z^\delta)^2 \|$. Then if $\phi \in L^p(\mathbb{P}, B_{p,p,\rho}(\rho))$, we have

$$\lim_{\delta \to 0} E \left\| \langle (Z + \phi)^3 : - (Z^\delta + \phi)^3_{\delta} \rangle \right\|_{L^\infty(\rho)} = 0, \text{ and}$$

$$\lim_{\delta \to 0} \sup_L E \left\| \langle (ZL + \phi)^3 : - (Z^\delta_L + \phi)^3_{\delta} \rangle \right\|_{L^\infty(\rho)} = 0.$$  \hfill (A.1)

Since $f^\delta$ restricts the Fourier support of its argument to a bounded interval, it follows that its image is in $L^2$ and the map is continuous.

We first establish (A.1) in the case $\phi = 0$. In this case it is enough to prove that

$$\sup_\delta E \| f^\delta(Z) \|_{L^{\infty}(\rho)} < \infty,$$

and that for any fixed $\varepsilon > 0$ we have

$$\lim_{\delta \to 0} E \left\| \chi_{\varepsilon}(\langle \nabla \rangle) \langle (Z^\delta)^3_{\delta} - Z^3 \rangle \right\|_{L^2(\rho)}^2 = 0,$$

which will imply convergence in tempered distributions $S'$.  \hfill (A.4)
Let us first verify (A.3). By Besov embedding it is enough to establish
\[
\sup_{\delta} \mathbb{E} \left\| (Z^\delta)^3 : \delta \right\|_{B_{p,\rho}^{\ell/2}(\rho)}^p < \infty
\]
for \( p \) large. We use the stationarity of \( Z \), integrability of \( \rho \), and hypercontractivity to estimate
\[
\mathbb{E} \parallel (Z^\delta)^3 : \delta \parallel_{B_{p,\rho}^{\ell/2}(\rho)}^p = \sum_{k \geq 1} 2^{-k(\ell/2)} \int_{\mathbb{R}^3} \mathbb{E} |\Delta_k : (Z^\delta)^3 : \delta(x)|^p \rho(x) \, dx \\
= \sum_{k \geq 1} 2^{-k(\ell/2)} \int_{\mathbb{R}^3} \mathbb{E} |\Delta_k : (Z^\delta)^3 : \delta(0)|^p \rho(x) \, dx \\
\lesssim \sum_{k \geq 1} 2^{-k(\ell/2)} \left( \mathbb{E} |\Delta_k : (Z^\delta)^3 : \delta(0)| \right)^{p/2}.
\]
Now we can use Wick’s theorem to get
\[
\mathbb{E} |\Delta_k : (Z^\delta)^3 : \delta(0)|^2 \\
= \int_{\mathbb{R}^3+2} \mathbb{E} [K_k(x_1) : (Z^\delta)^3 : (x_1) K_k(x_2) : (Z^\delta)^3 : (x_2)] \, dx_1 \, dx_2 \\
= \int_{\mathbb{R}^3+2} K_k(x_1) K_k(x_2) G^\delta(x_1, x_2)^3 \, dx_1 \, dx_2,
\]
where \( K_k \) is the convolution kernel of \( \Delta_k \) and \( G^\delta(x_1, x_2) := \mathbb{E} [Z^\delta(x_1) Z^\delta(x_2)] \). It is known from [23, Chapter 7.2] that
\[
|G^\delta(x_1, x_2)| \lesssim (C + |\log(|x_1 - x_2|)|) \exp(-m^2|x_1 - x_2|).
\]
This implies that \( \sup_{\delta > 0} \parallel G^\delta(x_1, x_2) \parallel_{L^p(dx_1 \, dx_2)} < \infty \) for any \( p < \infty \). In particular
\[
\left| \int_{\mathbb{R}^3+2} K_k(x_1) K_k(x_2) G^\delta(x_1, x_2)^3 \, dx_1 \, dx_2 \right| \lesssim \parallel K_k(x_1) K_k(x_2) \parallel_{L^q(dx_1 \, dx_2)} \\
\lesssim \parallel K_k \parallel_{L^q}^2
\]
for any \( q > 1 \). Since \( K_k \) has constant \( L^1 \) norm for all \( k \) and its \( L^\infty \) norm scales as \( 2^{k} \), we can take \( \parallel K_k(x_1) \parallel_{L^q} \lesssim 2^{k/3} \) by choosing \( q \) close enough to 1. Substituting this back to the Besov norm, we get a convergent geometric sum. This proves (A.3).

Let us then turn to (A.4). Let \( \tilde{K}_\epsilon = F^{-1} \chi^2_\epsilon \). We have \( \tilde{K}_\epsilon \in L^\infty \) since \( \chi^2_\epsilon \) is a
Schwartz function. Then we compute
\[
\mathbb{E} \| \chi_{\{\nabla\}} (Z^{\delta}) \|_{L^2(\rho)}^2 = \mathbb{E} \int \rho(x)^2 \tilde{K}(x - y) \cdot (Z^{\delta})^3 \cdot \phi \, dx \, dy
\]
\[
+ 2 \mathbb{E} \int \rho(x)^2 \tilde{K}(x - y) \cdot ((Z^{\delta})^3 \cdot \phi)(y) \, dx \, dy
\]
\[
- 2 \mathbb{E} \int \rho(x)^2 \tilde{K}(x - y) \cdot (Z^3 : \phi)(x) \, dx \, dy
\]
\[
= \int \rho(x)^2 \tilde{K}(x - y) G(x)^3 \, dx \, dy + \int \rho(x)^2 \tilde{K}(x - y) G^2(x, y)^3 \, dx \, dy
\]
\[
- 2 \int \rho(x)^2 \tilde{K}(x - y) \bar{G}^3(x, y)^3 \, dx \, dy,
\]
where \( \bar{G}^3(x, y) = \mathbb{E} [Z^3(x)Z(y)] \). We will have that the right-hand side goes to 0 as soon as we can show that
\[
\lim_{\delta \to 0} G^3(x, y) = \lim_{\delta \to 0} \bar{G}^3(x, y) = G(x, y) \quad \text{in } L^1(\mathbb{R}^2).
\]
Since
\[
\bar{G}^3(x, y) = \int K^3(x - z) \mathbb{E} [Z(z)Z(y)] \, dz = \int K^3(x - z) G(z, y) \, dz,
\]
and \( K^3 \) goes to a Dirac measure, the convergence holds. The argument for \( G^3(x, y) \) is the same.

We still need to check the \( \phi \neq 0 \) case. Here
\[
:(Z^\delta + \phi)^3 : \delta = \sum_{i=0}^3 : (Z^\delta)^3 : \delta \phi^{3-i} \to \sum_{i=0}^3 : (Z)^i : \phi^{3-i} = :(Z + \phi)^3 : \delta
\]
where the convergence holds by convergence of \( : (Z^\delta)^3 : \delta \) in \( C^{-\infty} (\rho) \) and continuity of the product.

For the periodic case we again start with \( \phi = 0 \). We may pass from \( : (\cdot)^3 : \delta \) to \( : (\cdot)^3 : a_{\delta, L} \) where \( : (\cdot)^3 : a_{\delta, L} = : (\cdot)^3 - a_{\delta, L} (\cdot) \) and \( a_{\delta, L} = \mathbb{E} Z_{\delta, L}^3 \), since we can show in the same way as in Lemma 3.7 that \( \sup_{a_{\delta, L}} |a_{\delta, L} - a_{\delta}| < \infty \). Then repeating the computations from the first case we arrive at having to estimate
\[
\left\| \int K^\delta (\cdot - z) G_L(z, y) \, dz - G_L(\cdot, y) \right\|_{L^1} \leq \delta \| G_L(\cdot, y) \|_{W^{1,1}}
\]
and the right-hand side is known to be bounded uniformly in \( L \) [23, Chapter 7.3]. The \( \phi \neq 0 \) case follows analogously as above.

B Computations for Section 3.2

In order to prove Theorem 3.19, we needed to bound the absolute value of
\[
3 \int_{\mathbb{R}^2} \rho \phi Z^3 : dx + 3 \int_{\mathbb{R}^2} \rho \phi^2 Z^2 : dx + \int_{\mathbb{R}^2} \rho \phi^3 Z \, dx + \int_{\mathbb{R}^2} \phi (\nabla \rho \cdot \nabla \phi) \, dx
\]
\[
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\]
with a term like

\[ Q(Z) + \frac{1}{2} \left( m^2 \| \phi \|^2_{L^2(\rho^{1/2})} + \| \phi \|^2_{H^1(\rho^{1/2})} + \| \phi \|^2_{L^4(\rho^{1/4})} \right), \]

where \( Q(Z) \) is bounded in expectation.

We bound each of the integrals in the following lemmas, selecting \( Q(Z) \) to consist of norms of Wick powers of \( Z \). The norms have bounded expectation by Lemma 3.8. In each proof we apply the multiplicative inequality (Theorem 2.4) and Besov duality (Theorem 2.5). These calculations are originally due to Mourrat and Weber [38].

**Lemma B.1** ([24, Lemma A.7]). Let \( \rho_1 \) and \( \rho_2 \) be polynomial weights and \( s, \varepsilon > 0 \). We have the following two estimates:

\[
\begin{align*}
\| f^2 \|_{B^s_{1,1}(\rho_1, \rho_2)} & \lesssim \| f \|_{L^2(\rho_1)} \| f \|_{H^{s+\varepsilon}(\rho_2)} \\
\| f^3 \|_{B^s_{1,1}(\rho_1, \rho_2)} & \lesssim \| f \|^2_{L^2(\rho_1)} \| f \|_{H^{s+\varepsilon}(\rho_2)}.
\end{align*}
\]

**Lemma B.2.** Assume that \( \rho^{1/2} \in L^1(\mathbb{R}^2) \) and \( \varepsilon < 1/4 \). Then for any \( \delta > 0 \) there exists a constant \( C > 0 \) such that

\[
\left| \int_{\mathbb{R}^2} \rho^{1/2} Z \, dx \right| \leq C \| Z \|_{C^{-s}(\rho^{1/16})}^8 + \delta \left( \| \phi \|_{L^4(\rho^{1/4})}^4 + \| \phi \|^2_{H^1(\rho^{1/2})} \right).
\]

**Proof.** We first use duality and Lemma B.1 to estimate

\[
\int_{\mathbb{R}^2} |\rho^{1/2} Z| \, dx \lesssim \| \rho^{1/2} Z \|_{B^s_{1,1}(\rho^{1/16})} \| \rho^{1/16} \|_{C^{-s}(\rho^{1/16})}
\]

\[
\lesssim \| \phi \|^2_{L^4(\rho^{1/4})} \| \phi \|_{H^{s+\varepsilon}(\rho^{1/16})} \| Z \|_{C^{-s}(\rho^{1/16})}.
\]

Inside the middle Besov norm, we can trade off some weight via

\[
\| \rho^{s/16} \Delta_j \phi \|_{L^2} \lesssim \| \rho^{1/16} \|_{L^4} \| \rho^{s/16} \Delta_j \phi \|_{L^{9/3}}.
\]

We can also increase the regularity from \( 2\varepsilon \) to \( 1/2 \). This simplifies the interpolation

\[
\| \phi \|_{B^s_{1,1}(\rho^{1/4})} \lesssim \| \phi \|_{B^s_{2,\infty}(\rho^{1/4})} \| \phi \|_{B^s_{2,\infty}(\rho^{1/2})} \lesssim \| \phi \|_{L^4(\rho^{1/4})} \| \phi \|_{H^1(\rho^{1/2})}.
\]

We substitute this back into the original product. Finally, we use Young’s product inequality twice to first extract \( \| Z \|_{C^{-s}(\rho^{1/16})} \) and then to separate the \( L^4 \) and \( H^1 \) norms.

**Lemma B.3.** Assume that \( \rho^{1/4} \in L^1(\mathbb{R}^2) \) and \( \varepsilon < 1/2 \). Then for every \( \delta > 0 \) there exists a constant \( C > 0 \) such that

\[
\left| \int_{\mathbb{R}^2} \rho^{1/2} Z^2 \, dx \right| \leq C \| Z \|_{C^{-s}(\rho^{1/16})} \| \rho^{1/2} \|_{B^s_{1,1}(\rho^{1/16})} \| Z^2 \|_{C^{-s}(\rho^{1/16})}.
\]

**Proof.** Again, duality and Lemma B.1 give

\[
\int_{\mathbb{R}^2} |\rho^{1/2} Z^2| \, dx \lesssim \| \phi \|^2_{L^2(\rho^{7/16})} \| \phi \|_{H^{s+\varepsilon}(\rho^{1/2})} \| Z^2 \|_{C^{-s}(\rho^{1/16})}.
\]
We can again trade off some weight in
\[ \| \rho^{7/16} \phi \|_{L^2} \leq \| \rho^{3/16} \|_{L^{4/3}} \| \rho^{1/4} \phi \|_{L^4}. \]
We can increase the regularity in the middle term make the weight larger in
the last term to make them match the statement. Young’s product inequality
finishes the proof. \[ \square \]

**Lemma B.4.** Assume that \( \rho^{1/4} \in L^4(\mathbb{R}^2) \) and \( \varepsilon < 1 \). Then for every \( \delta > 0 \) there exists a \( C > 0 \) such that
\[
\left| \int_{\mathbb{R}^2} \rho \phi: Z^3 : dx \right| \leq C \| Z^3 : \|_{C_{-\varepsilon}^{1/(\rho^{1/4})}} + \delta \| \phi \|_{H^{1/2}(\rho^{1/4})}^2.
\]

**Proof.** By duality
\[
\int_{\mathbb{R}^2} | \rho \phi : Z^3 : | dx \lesssim \| \phi \|_{B^1_{1,1}(\rho^{3/4})} \| Z^3 : \|_{C(\rho^{1/4})}.
\]
Then we do a series of tradeoffs in
\[
\| \phi \|_{B^1_{1,1}(\rho^{3/4})} \lesssim \| \phi \|_{B^1_{1,2}(\rho^{3/4})} \lesssim \| \phi \|_{B^1_{2,2}(\rho^{1/2})}
\]
and finish with Young’s product inequality. \[ \square \]

**Lemma B.5.** Assume that \( \rho^{1/4} \in L^4(\mathbb{R}^2) \). Then there exists \( C > 0 \) such that
\[
\left| \int_{\mathbb{R}^2} \phi(\nabla \rho \cdot \nabla \phi) \ dx \right| \leq C + \delta \left( \| \phi \|_{L^4(\rho^{1/4})}^4 + \| \phi \|_{H^1(\rho^{1/2})}^2 \right).
\]

**Proof.** Let us observe that we can write the dot product components as
\[
(\partial_j \rho)(\partial_j \phi) = (\partial_j [1 + x_1^2 + x_2^2]^{-\alpha/2})(\partial_j \phi) = \frac{\alpha x_j \rho(x)}{1 + x_1^2 + x_2^2}(\partial_j \phi).
\]
The factor in front is uniformly bounded by \( \rho(x) \). Thus
\[
\int_{\mathbb{R}^2} | \phi(x)(\nabla \rho \cdot \nabla \phi)(x) | dx \leq \alpha \int_{\mathbb{R}^2} \rho(x) | \phi(x) | \| \nabla \phi(x) | dx
\]
\[
\leq \alpha \| \phi \|_{L^2(\rho^{1/2})} \| \nabla \phi \|_{L^2(\rho^{1/2})}
\]
\[
\leq C + \delta \left( \| \phi \|_{L^4(\rho^{1/4})}^4 + \| \phi \|_{H^1(\rho^{1/2})}^2 \right),
\]
where we did again a weight–\( L^p \) tradeoff and applied Young twice. \[ \square \]

**C Exponential tails**

In order to prove the existence of Wick powers, we needed to establish exponen-
tial tails of some weighted Besov norms for \( \phi^k_{L,L} \) uniformly in \( L \). More concretely
we will define the measure
\[
\hat{\mu}_L(A) = \frac{1}{Z_L} \int_{A} \exp(h(\| \langle \nabla \rangle^{-\gamma} \psi \|_{L^p(\rho)})) \ d\mu_L(\psi), \quad (C.1)
\]
where \( h : \mathbb{R} \to \mathbb{R} \) is a smooth function, constant near 0 and growing linearly at infinity, and \( Z_L \) is the associated normalization constant. We will prove that \( \sup_L Z_L < \infty \).

We begin with the following lemma. In finite volume the Gaussian tails of \( \mu_L \) are not difficult to establish; see [6, Section 3]. This \textit{a priori} bound means that the assumptions of the lemma are satisfied. The lemma then makes the uniform bound easier to derive.

**Lemma C.1** ([5, Lemma A.7]). Let \((\Omega, F)\) be a measurable space and \( \nu \) be a probability measure on \( \Omega \). Let \( \Sigma : \Omega \mapsto \mathbb{R} \) be a measurable function such that \( \exp(\Sigma) \in L^1(d\nu) \).

Define \( d\nu \Sigma = \frac{1}{\int \exp(\Sigma) d\nu} \exp(\Sigma) d\nu \). Then

\[
\int \exp(\Sigma) d\nu \leq \exp\left( \int \Sigma(x) d\nu \right).
\]

**Proof.** Multiplying both sides of

\[
d\nu \Sigma = \frac{1}{\int \exp(\Sigma) d\nu} \exp(\Sigma) d\nu
\]

by \( \exp(-\Sigma) \) and integrating we obtain

\[
\left( \int \exp(-\Sigma) d\nu \right) \left( \int \exp(\Sigma) d\nu \right) = 1.
\]

Then it remains to apply Jensen’s inequality to the first factor. \( \square \)

We choose \( \Sigma = h(\|\rho(\nabla)^{-\varepsilon}\|_{L^p}) \) and \( \nu = \mu_L \) in the lemma. Then the claim follows if we can find a uniform estimate for

\[
\int_{H^{-\varepsilon}(\rho)} h(\|\rho(\nabla)^{-\varepsilon}\|_{L^p}) d\mu_L.
\]

(C.2)

We then have the following lemma:

**Lemma C.2.** The measure \( \bar{\mu}_L \) is an invariant measure for the equation

\[
\partial_t \bar{u} + (m^2 - \Delta)\bar{u} + :\bar{u}^3: = V(\bar{u}) + \xi,
\]

where \( \xi \) is space-time white noise.
Proof. Note that $V$ is continuous on $C^{-\delta}(\Lambda_L)$. With this in mind the proof becomes a minor modification of the proof of Da Prato and Debbussche [19, Section 4] and we omit it.

Again performing the Da Prato–Debussche trick, i.e. decomposing $\bar{u} = Z + \bar{\phi}$, we obtain that $\bar{\phi}$ satisfies
\[
\partial_t \bar{\phi} + (m^2 - \Delta) \bar{\phi} + : (Z + \bar{\phi})^3 : = V(Z + \bar{\phi}).
\] (C.5)

We again test the equation with $\rho \bar{\phi}$ to obtain
\[
\partial_t \int \rho \bar{\phi}^2 \, dx + m^2 \int \rho \bar{\phi}^2 \, dx + \int |\nabla \bar{\phi}|^2 \, dx + \int \bar{\phi}^4 \, dx + G(Z, \bar{\phi}) = \int \rho V(Z + \bar{\phi}) \bar{\phi} \, dx.
\] (C.6)

From the definitions and Hölder’s inequality
\[
\int \rho V(Z + \bar{\phi}) \bar{\phi} \, dx 
\leq \frac{h'}{\| \langle \nabla \rangle^{\frac{\epsilon}{2}} (Z + \bar{\phi}) \|_{L^p(\rho)}^{p-1}} \int (\rho \langle \nabla \rangle^{-\epsilon} (Z + \bar{\phi}))^{p-1} \rho \langle \nabla \rangle^{-\epsilon} (\rho \bar{\phi}) \, dx 
\leq \frac{1}{\| \langle \nabla \rangle^{-\epsilon} (Z + \bar{\phi}) \|_{L^p(\rho)}^{p-1}} \| \langle \nabla \rangle^{-\epsilon} (Z + \bar{\phi}) \|_{L^p(\rho)}^{p-1} \| \langle \nabla \rangle^{-\epsilon} (\rho \bar{\phi}) \|_{L^p(\rho)} 
\leq \| \rho \|_{H^{-\epsilon}(\mathbb{R}^2)} \| \bar{\phi} \|_{H^1(\rho)} 
\leq C + \frac{1}{2} \| \bar{\phi} \|^2_{H^1(\rho)}.
\] We thus have that
\[
\int \rho V(Z + \bar{\phi}) \bar{\phi} \, dx \leq m^2 \int \rho \bar{\phi}^2 + \int |\nabla \bar{\phi}|^2 \, dx + C. \quad (C.7)
\]

We also apply the reasoning from Section 3.2 to the remainder term $G(Z, \bar{\phi})$. Upon taking an expectation, the time derivative and white noise integrals vanish. This implies again the boundedness of $H^1$ norm. Sobolev embedding then gives $\sup_L \mathbb{E}[\| \bar{\phi} \|^2_{L^p(\rho)}] < \infty$, which gives the statement for exponential tails in $L^p$ norm.

Corollary C.3. As $p$ was arbitrary, we also have
\[
\sup_L \int \exp(\| \psi \|_{C^{-2\epsilon}(\rho)}) \, d\mu_L < \infty.
\]
This implies that $\int \| \psi \|^p_{C^{-2\epsilon}(\rho)} \, d\mu_L$ is finite uniformly in $L$ for any $p < \infty$.

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