On a proper tensorial subgrid heat flux model

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Motivation

Research question:

- Can we find a nonlinear SGS heat flux model with **good physical and numerical properties**, such that we can obtain satisfactory predictions for a turbulent Rayleigh-Bénard convection?

DNS of an air-filled Rayleigh-Bénard convection at $Ra = 10^{10}$

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1F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. *On the evolution of flow topology in turbulent Rayleigh-Bénard convection*, Physics of Fluids, 28:115105, 2016.
### Motivation

Air-filled RB: $Pr = 0.7$

$$Ra = 10^8$$

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F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. *Flow topology dynamics in a 3D phase space for turbulent Rayleigh-Bénard convection*, Phys.Rev.Fluids, 5:024603, 2020.
Motivation

Air-filled RB: $Pr = 0.7$

$Ra = 10^8$

$Ra = 10^{10}$

\[\text{\textsuperscript{2}}\text{F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. Flow topology dynamics in a 3D phase space for turbulent Rayleigh-Bénard convection, Phys.Rev.Fluids, 5:024603, 2020.}\]
Motivation

Air-filled RB: $Pr = 0.7$

$Ra = 10^8$

$Ra = 10^{10}$

$Ra = 10^{11}$

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Motivation

Air-filled RB: $Pr = 0.7$

$Ra = 10^8$  
$Ra = 10^{10}$  
$Ra = 10^{11}$

208 $\times$ 208 $\times$ 400  
17.5M

768 $\times$ 768 $\times$ 1024  
607M  

1662 $\times$ 1662 $\times$ 2048  
5600M

\cite{Dabbagh:2020}

\textsuperscript{2}F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. \textit{Flow topology dynamics in a 3D phase space for turbulent Rayleigh-Bénard convection}, \textbf{Phys.Rev.Fluids}, 5:024603, 2020.
Motivation

DNS: $208 \times 208 \times 400$

Ra $= 10^8$

Ra $= 10^{10}$

Ra $= 10^{11}$
Motivation

**DNS:** $208 \times 208 \times 400$

**LES:** $80 \times 80 \times 120$

$Ra = 10^8$  $Ra = 10^{10}$  $Ra = 10^{11}$
Motivation

DNS: $208 \times 208 \times 400$

LES: $80 \times 80 \times 120$

$Ra = 10^8$, $Ra = 10^{10}$, $Ra = 10^{11}$
Motivation

DNS: $208 \times 208 \times 400$

LES: $110 \times 110 \times 168$
How to model the subgrid heat flux in LES?

\[
\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0
\]

eddy-viscosity \quad \rightarrow \quad \tau(\bar{u}) = -2\nu_T S(\bar{u})

\[
\nu_T \approx (C_m \delta)^2 D_m(\bar{u})
\]
How to model the subgrid heat flux in LES?

\[
\begin{align*}
\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} &= \nu \nabla^2 \bar{u} - \nabla \bar{p} + \bar{f} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0 \\
\text{eddy-viscosity} &\quad \longrightarrow \quad \tau(\bar{u}) = -2\nu_t S(\bar{u}) \\
\nu_t &\approx (C_m \delta)^2 D_m(\bar{u})
\end{align*}
\]

\[
\begin{align*}
\partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} &= \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u} \bar{T} - \bar{u} \bar{T} \\
\text{eddy-diffusivity} &\quad \longrightarrow \quad q \approx -\alpha_t \nabla \bar{T}
\end{align*}
\]
How to model the subgrid heat flux in LES?

\[ \partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} + \bar{f} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0 \]

eddy-viscosity \quad \longrightarrow \quad \tau(\bar{u}) = -2\nu_t S(\bar{u})

\[ \nu_t \approx (C_m \delta)^2 D_m(\bar{u}) \]

\[ \partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u}T - \bar{T} \bar{u} \]

eddy-diffusivity \quad \longrightarrow \quad q \approx -\alpha_t \nabla \bar{T}

\[ \alpha_t = \frac{\nu_t}{Pr_t} \]
How to model the subgrid heat flux in LES?

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\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} + \bar{f} - \nabla \cdot \tau(\bar{u}); \quad \nabla \cdot \bar{u} = 0
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\nu_t \approx (C_m \delta)^2 D_m(\bar{u})
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\partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u} \bar{T} - \bar{u} \bar{T}
\]

eddy-diffusivity \quad \longrightarrow \quad q \approx -\alpha_t \nabla \bar{T}

\[
\alpha_t = \frac{\nu_t}{Pr_t}
\]

\[
Pr_t?
\]
How to model the subgrid heat flux in LES?

$$Pr_t = \frac{\langle \nu_t \rangle_A}{\langle \kappa_t \rangle_A}$$

$$Ra = 10^8$$
$$Ra = 10^{10}$$
How to model the subgrid heat flux in LES?

$$\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} + \bar{f} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0$$

eddy-viscosity  \quad \longrightarrow \quad \tau(\bar{u}) = -2\nu_t S(\bar{u})

$$\nu_t \approx (C_m \delta)^2 D_m(\bar{u})$$

$$\partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u} \bar{T} - \bar{u} \bar{T}$$

eddy-diffusivity  \quad \longrightarrow \quad q \approx -\alpha_t \nabla \bar{T}$$

$$\alpha_t = \frac{\nu_t}{Pr_t}$$
How to model the subgrid heat flux in LES?

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\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla p + \bar{f} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0
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eddy-viscosity \quad \rightarrow \quad \tau(\bar{u}) = -2\nu_t S(\bar{u})

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\partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot \bar{q} \quad \text{where} \quad \bar{q} = \bar{u}\bar{T} - \bar{u}\bar{T}
\]

eddy-diffusivity \quad \rightarrow \quad \bar{q} \approx -\alpha_t \nabla \bar{T} \quad (\equiv \bar{q}^{edd})

gradient model \quad \rightarrow \quad \bar{q} \approx -\frac{\delta^2}{12} \bar{G} \nabla \bar{T} \quad (\equiv \bar{q}^{nl})
How to model the subgrid heat flux in LES?

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\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} + \bar{f} - \nabla \cdot \tau(\bar{u}) \quad ; \quad \nabla \cdot \bar{u} = 0
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\partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u} \bar{T} - \bar{u} \bar{T}
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eddy-diffusivity \quad \rightarrow \quad q \approx -\alpha_t \nabla \bar{T} \quad (\equiv q^{\text{eddy}})

gradient model \quad \rightarrow \quad q \approx -\frac{\delta^2}{12} G \nabla \bar{T} \quad (\equiv q^{\text{nl}})

\[
G \equiv \nabla \bar{u} \quad \quad q = -\frac{\delta^2}{12} G \nabla \bar{T} + O(\delta^4)
\]
A priori alignment trends

eddy-diffusivity \[ q \approx -\alpha_t \nabla \bar{T} \] (\( \equiv q^{eddy} \))

gradient model \[ q \approx -\frac{\delta^2}{12} G \nabla \bar{T} \] (\( \equiv q^{nl} \))

__References__

3F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. A priori study of subgrid-scale features in turbulent Rayleigh-Bénard convection, Physics of Fluids, 29:105103, 2017.
A priori alignment trends

eddy-diffusivity \[ q \approx -\alpha_t \nabla \overline{T} \quad (\equiv q^{\text{eddy}}) \]

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\(^3\)F. Dabbagh, F. X. Trias, A. Gorobets, A. Oliva. A priori study of subgrid-scale features in turbulent Rayleigh-Bénard convection, *Physics of Fluids*, 29:105103, 2017.
**A priori** alignment trends

eddy-diffusivity \[ q \approx -\alpha_t \nabla T \] (\( \equiv q_{\text{edd}} \))

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3 F. Dabbagh, F. X. Trias, A. Gorobets, A. Oliva. *A priori study of subgrid-scale features in turbulent Rayleigh-Bénard convection*, **Physics of Fluids**, 29:105103, 2017.
How to model the subgrid heat flux in LES?

\[ \partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} + \bar{f} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0 \]

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gradient model \quad \longrightarrow \quad q \approx -\frac{\delta^2}{12} G\nabla \bar{T} \quad (\equiv q^{nl})

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\(^4\)S. Peng and L. Davidson. *Int.J.Heat Mass Transfer*, 45:1393-1405, 2002.
How to model the subgrid heat flux in LES?

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\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla p + \bar{f} - \nabla \cdot \tau(\bar{u}) \quad ; \quad \nabla \cdot \bar{u} = 0
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gradient model \quad \rightarrow \quad q \approx -\frac{\delta^2}{12} G \nabla \bar{T} \quad (\equiv q^{nl})

Peng&Davidson\(^4\) \quad \rightarrow \quad q \approx -\frac{\delta^2}{12} S \nabla \bar{T} \quad (\equiv q^{PD})

\(^4\)S.Peng and L.Davidson. **Int.J.Heat Mass Transfer**, 45:1393-1405, 2002.
A \textit{priori} alignment trends

\[ q^{nl} \equiv -\frac{\delta^2}{12} G \nabla \bar{T} \]

\[ q^{PD} \equiv -\frac{\delta^2}{12} S \nabla \bar{T} \]
How to model the subgrid heat flux in LES?

\[ \partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla p + \vec{f} - \nabla \cdot \tau(\bar{u}) ; \quad \nabla \cdot \bar{u} = 0 \]

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eddy-diffusivity \quad \rightarrow \quad q \approx -\alpha_t \nabla \bar{T} \quad (\equiv q^{eddy})

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How to model the subgrid heat flux in LES?

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\begin{align*}
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\text{eddy-viscosity} & \quad \longrightarrow \quad \tau(\bar{u}) = -2\nu_t S(\bar{u}) \\
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\end{align*}
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\partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} &= \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u} T - \bar{u} \bar{T}
\end{align*}
\]

mixed model \quad \longrightarrow \quad q \approx q^{nl} + \sigma q^{eddy} \quad (\equiv q^{mix})

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5B.J.Daly and F.H.Harlow. Physics of Fluids, 13:2634, 1970.
How to model the subgrid heat flux in LES?

\[
\partial_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = \nu \nabla^2 \bar{\mathbf{u}} - \nabla \bar{p} + \bar{\mathbf{f}} - \nabla \cdot \mathbf{\tau}(\bar{\mathbf{u}}) ; \quad \nabla \cdot \bar{\mathbf{u}} = 0
\]

eddy-viscosity \quad \rightarrow \quad \mathbf{\tau}(\bar{\mathbf{u}}) = -2\nu_t \mathbf{S}(\bar{\mathbf{u}})

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\nu_t \approx (C_m \delta)^2 D_m(\bar{\mathbf{u}})
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\partial_t \bar{T} + (\bar{\mathbf{u}} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot \mathbf{q} \quad \text{where} \quad \mathbf{q} = \bar{\mathbf{u}} \bar{T} - \bar{\mathbf{u}} \bar{T}
\]

mixed model \quad \rightarrow \quad \mathbf{q} \approx \mathbf{q}^{nl} + \sigma \mathbf{q}^{eddy} \quad (\equiv \mathbf{q}^{\text{mix}})

Daly\&Harlow^5 \quad \rightarrow \quad \mathbf{q} \approx -\mathcal{T}_{SGS} \frac{\delta^2}{12} \mathbf{G}^T \nabla \bar{T} \quad (\equiv \mathbf{q}^{DH})

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^5B.J.Daly and F.H.Harlow. *Physics of Fluids*, 13:2634, 1970.
How to model the subgrid heat flux in LES?

\[ \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = \nu \nabla^2 \vec{u} - \nabla p + \vec{f} - \nabla \cdot \tau(\vec{u}) ; \quad \nabla \cdot \vec{u} = 0 \]

eddy-viscosity \[ \rightarrow \quad \tau(\vec{u}) = -2\nu_t S(\vec{u}) \]

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\[ \partial_t \overline{T} + (\vec{u} \cdot \nabla) \overline{T} = \alpha \nabla^2 \overline{T} - \nabla \cdot \vec{q} \quad \text{where} \quad \vec{q} = \overline{uT} - \overline{u} \overline{T} \]

mixed model \[ \rightarrow \quad \vec{q} \approx q^{nl} + \sigma q^{eddyl} \quad (\equiv q^{mix}) \]

Daly\&Harlow\(^5\) \[ \rightarrow \quad \vec{q} \approx -T_{SGS} \frac{\delta^2}{12} GG^T \nabla \overline{T} \quad (\equiv q^{DH}) \]

\[ T_{SGS} = 1/|S| \]

\(^5\)B.J.Daly and F.H.Harlow. *Physics of Fluids*, 13:2634, 1970.
**A priori** alignment trends

\[ q^{nl} \equiv -\frac{\delta^2}{12} G \nabla \overline{T} \]

\[ q^{DH} \equiv -\mathcal{T}_{SGS} \frac{\delta^2}{12} GG^T \nabla \overline{T} \]

- **STABLE MODELS**
- **UNSTABLE MODELS!!!**

Angle respect to the eddy-diffusivity model, \( q^{eddy} \)
What about near-wall scaling?
What about near-wall scaling?

\[ \text{Answer: it should be } O(y^3) \]
Near-wall scaling for DH model?

\[ q^{DH} \equiv -\mathcal{T}_{SGS} \frac{\delta^2}{12} GG^T \nabla T ; \quad \mathcal{T}_{SGS} = 1/|S| \]
Near-wall scaling for DH model?

→ Answer: it is $O(y^1)$ instead of $O(y^3)$

$$q^{DH} \equiv -\mathcal{T}_{SGS} \frac{\delta^2}{12} GG^T \nabla \overline{T} ; \quad \mathcal{T}_{SGS} = 1/|S|$$

$$u = ay + O(y^2); \quad v = by^2 + O(y^3); \quad w = cy + O(y^2); \quad T = dy + O(y^2)$$

$$G = \begin{pmatrix} y & 1 & y \\ y^2 & y & y^2 \\ y & 1 & y \end{pmatrix}; \quad \nabla \overline{T} = \begin{pmatrix} y \\ 1 \\ y \end{pmatrix} \implies GG^T \nabla \overline{T} = \begin{pmatrix} y \\ y^2 \\ y \end{pmatrix} = O(y^1)$$

$$\mathcal{T}_{SGS} = 1/|S| = O(y^0)$$
Near-wall scaling for DH model?

Answer: it is $O(y^1)$ instead of $O(y^3)$

\[ q^{DH} \equiv -\mathcal{T}_{SGS} \frac{\delta^2}{12} GG^T \nabla T \quad ; \quad \mathcal{T}_{SGS} = 1/|S| \]

\[ u = ay + O(y^2) \quad ; \quad v = by^2 + O(y^3) \quad ; \quad w = cy + O(y^2) \quad ; \quad T = dy + O(y^2) \]

\[ G = \begin{pmatrix} y & 1 & y \\ y^2 & y & y^2 \\ y & 1 & y \end{pmatrix} \quad ; \quad \nabla T = \begin{pmatrix} y \\ 1 \\ y \end{pmatrix} \quad \Rightarrow \quad GG^T \nabla T = \begin{pmatrix} y \\ y^2 \\ y \end{pmatrix} = O(y^1) \]

\[ \mathcal{T}_{SGS} = 1/|S| = O(y^0) \]

Idea: build a $\mathcal{T}_{SGS}$ with the proper $O(y^2)$ scaling!!!
Building proper models for the subgrid heat flux

Let us consider models that are based on the invariants of the tensor $GG^T$

$$q \approx -C_M \left( P_{GG^T}^p Q_{GG^T}^q R_{GG^T}^r \right) \frac{\delta^2}{12} GG^T \nabla \overline{T} \quad (\equiv q^{S2})$$
Building proper models for the subgrid heat flux

Let us consider models that are based on the invariants of the tensor $GG^T$

$$q \approx -C_M \left( P_{GG^T}^p Q_{GG^T}^q R_{GG^T}^r \right) \frac{\delta^2}{12} GG^T \nabla T \quad (\equiv q^{S2})$$

|                  | $P_{GG^T}$ | $Q_{GG^T}$ | $R_{GG^T}$ |
|------------------|------------|------------|------------|
| Formula          | $2(Q_\Omega - Q_S)$ | $V^2 + Q_G^2$ | $R_G^2$ |
| Wall-behavior    | $O(y^0)$   | $O(y^2)$   | $O(y^6)$   |
| Units            | $[T^{-2}]$ | $[T^{-4}]$ | $[T^{-6}]$ |

$$-6r - 4q - 2p = 1 \quad [T]; \quad 6r + 2q = s,$$

where $s$ is the slope for the asymptotic near-wall behavior, i.e. $O(y^s)$. 
Building proper models for the subgrid heat flux

Solutions: \( q(p, s) = -(1 + s)/2 - p \) and \( r(p, s) = (2s + 1)/6 + p/3 \)
Building proper models for the subgrid heat flux

Solutions: \( q(p, s) = -(1 + s)/2 - p \) and \( r(p, s) = (2s + 1)/6 + p/3 \)

![Graph showing solutions for q(p) and r(p)]
Building proper models for the subgrid heat flux

Solutions: \( q(p, s) = -(1 + s)/2 - p \) and \( r(p, s) = (2s + 1)/6 + p/3 \)
A priori analysis in the bulk

Estimation of the model constant, $C_{s2pqr}$

Playing with exponent $p$...

\[
q^{S2PQR} \approx -C_{s2pqr} \left( P_{GGT}^P Q_{GGT}^{-(p+3/2)} R_{GGT}^{(2p+5)/6} \right) \frac{\delta^2}{12} GG^T \nabla^T
\]

\[
\frac{\langle |q^{S2PQR}| \rangle_{bulk}}{\langle |q| \rangle_{bulk}} = 1 \rightarrow C_{s2pqr}
\]

$Ra = 10^8$  $Ra = 10^{10}$  $Ra = 10^{11}$
**A priori** analysis in the bulk

Estimation of the model constant, $C_{s2pqr}$

Playing with exponent $p...$

$$ q^{S2PQR} \approx -C_{s2pqr} \left( P_{GG}^p \ G_{GG}^{-(p+3/2)} \ R_{GG}^{(2p+5)/6} \right) \ \frac{\delta^2}{12}GG^T \nabla T $$
**A priori** analysis in the bulk

Estimation of the model constant, \( C_{s2pqr} \)

Playing with exponent \( p \)...
**A priori** analysis in the bulk

Alignment trends...

Playing with exponent $p$...

$$q^{S2PQR} \approx -C_{s2pqr} \left( P_{GG}^p Q_{GG}^{-(p+3/2)} R_{GG}^{(2p+5)/6} \right) \frac{\delta^2}{12} GG^T \nabla T$$

![Graphs showing alignment trends for different values of $p$](image)
A priori analysis in the bulk

Alignment trends...

Playing with exponent $p$...

$$q^{S2PQR} \approx -C_s 2p_{qr} \left( P_{GGT}^p Q_{GGT}^{-(p+3/2)} R_{GGT}^{(2p+5)/6} \right) \frac{\delta^2}{12} GGT \nabla T$$

$Ra = 10^8$  $Ra = 10^{10}$  $Ra = 10^{11}$
**A priori** analysis in the bulk

Alignment trends...

Playing with exponent $p$...

\[
q^{S2PQR} \approx -C_{s2pqr} \left( P_{GG^T}^p Q_{GG^T}^{-\left(p+\frac{3}{2}\right)} R_{GG^T}^{\left(2p+5\right)/6} \right) \frac{\delta^2}{12} GG^T \nabla T
\]

$Ra = 10^8$  $Ra = 10^{10}$  $Ra = 10^{11}$
A priori alignment trends of S2PR in the bulk

\[ q^{nl} = -\frac{\delta^2}{12} G\nabla T \]

\[ q^{s2PR} = -C_{s2pr} P_{GGT}^{-3/2} R_{GGT}^{1/3} \frac{\delta^2}{12} GG^T \nabla T \]

STABLE MODELS

UNSTABLE MODELS!!!

Angle respect to the eddy-diffusivity model, \( q^{\text{eddy}} \)
**A priori** alignment trends of S2PR in the near-wall region

\[ q^{PD} \equiv -\frac{\delta^2}{12} S \nabla \bar{T} \]
**A priori** alignment trends of S2PR in the near-wall region

\[ q^{PD} \equiv -\frac{\delta^2}{12} S \nabla T \]

\[ \cos \beta_{model} = \frac{q^{eddy} \cdot q^{model}}{|q^{eddy}| |q^{model}|} \]
**A priori** alignment trends of S2PR in the near-wall region

\[
q^{PD} \equiv -\frac{\delta^2}{12} S \nabla T
\]

\[
\cos \beta^{PD} = \frac{\nabla T \cdot S \nabla T}{|\nabla T| \cdot |S \nabla T|} = \mathcal{O}(y^1)
\]

\[
\lim_{y \to 0^+} \frac{|q^{PD}|}{|q^{nl}|} = \lim_{y \to 0^+} \frac{|S \nabla T|}{|G \nabla T|} = \frac{1}{2}
\]

\[
\cos \beta^{model} = \frac{q^{eddy} \cdot q^{model}}{|q^{eddy}| \cdot |q^{model}|}
\]
A priori alignment trends of S2PR in the near-wall region

\[ q^{PD} \equiv -\frac{\delta^2}{12} S \nabla T \]

\[ q^{s2PR} \equiv -C_{s2pr} P^{3/2}_{GG} R^{1/3}_{GG} \frac{\delta^2}{12} GG^T \nabla T \]

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\[ \cos \beta^{S2PQR} = \frac{\nabla T \cdot GG^T \nabla T}{|\nabla T| \cdot |GG^T \nabla T|} = \mathcal{O}(y^1) \]

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**A priori** alignment trends of S2PR in the near-wall region

\[ q^{PD} \equiv -\frac{\delta^2}{12} S \nabla T \]

\[ q^{s2PR} \equiv -C_{s2pr} \frac{P}{GG^T} \frac{R_{GG^T}}{12} \frac{\delta^2}{GG^T \nabla T} \]

**Cosine angle model**: 
\[ \cos \beta_{\text{model}} = \frac{q^{\text{eddy}} \cdot q^{\text{model}}}{|q^{\text{eddy}}||q^{\text{model}}|} \]
A posteriori results?

\[ \partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = \nu \nabla^2 \bar{u} - \nabla \bar{p} + \vec{f} - \nabla \cdot \tau(\bar{u}) \; ; \quad \nabla \cdot \bar{u} = 0 \]

**eddy-viscosity** \[ \tau(\bar{u}) = -2\nu_t S(\bar{u}) \]

\[ \nu_t \approx (C_m \delta)^2 D_m(\bar{u}) \]

\[ \partial_t \bar{T} + (\bar{u} \cdot \nabla) \bar{T} = \alpha \nabla^2 \bar{T} - \nabla \cdot q \quad \text{where} \quad q = \bar{u} \bar{T} - \bar{u} \bar{T} \]
Motivation

Modeling the subgrid heat flux

Building proper models

Results

Conclusions

\section*{A posteriori results?}

\[ \partial_t \overline{u} + (\overline{u} \cdot \nabla) \overline{u} = \nu \nabla^2 \overline{u} - \nabla \overline{p} + \overline{f} - \nabla \cdot \tau(\overline{u}) ; \quad \nabla \cdot \overline{u} = 0 \]

eddy-viscosity \quad \longrightarrow \quad \tau(\overline{u}) = -2\nu_t S(\overline{u})

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\textbf{But first we need to answer the following research question:}

- Are eddy-viscosity models for momentum able to provide satisfactory results for turbulent Rayleigh-Bénard convection?
A posteriori results?

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⚠️ But first we need to answer the following research question:

- Are **eddy-viscosity models** for momentum able to provide satisfactory results for turbulent Rayleigh-Bénard convection?

**Idea:** let’s do an LES for momentum and a DNS for temperature!
DNS at very low $Pr$ number

**Why?** scale separation grows as $\eta_K/\eta_T = Pr^{3/4}$.

$\eta_T$: Obukhov-Corrsin scale; $\eta_K$: Kolmogorov scale
DNS at very low $Pr$ number

**Why?** scale separation grows as $\eta_K/\eta_T = Pr^{3/4}$. Here: $\eta_T \approx 53.2\eta_K$

$\eta_T$: Obukhov-Corrsin scale; $\eta_K$: Kolmogorov scale

DNS of a RB at $Ra = 7.14 \times 10^6$ and $Pr = 0.005$ (liquid sodium)

$488 \times 488 \times 1280 \approx 305M$
DNS at very low $Pr$ number

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LES\textsuperscript{6} results at very low $Pr$ number

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\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig.png}
\end{figure}

\textsuperscript{6}F.X.Trias, D.Folch, A.Gorobets, A.Oliva. Building proper invariants for eddy-viscosity subgrid-scale models, Physics of Fluids, 27: 065103, 2015.
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![Graph showing computational cost vs. number of grid points for different models: LES-S3QR, LES-WALE, LES-Vreman, LES-QR, No model, DNS. The graph indicates the computational cost in CPU core-hours for various Nusselt numbers and grid point numbers.]
**LES results at very low Pr number**

RB at \( Ra = 7.14 \times 10^6 \) and \( Pr = 0.005 \) (DNS \( \rightarrow 488 \times 488 \times 1280 \approx 305M \))

\[
\begin{align*}
64 \times 32 \times 32 \\
96 \times 52 \times 52
\end{align*}
\]
LES results at very low $Pr$ number

RB at $Ra = 7.14 \times 10^6$ and $Pr = 0.005$ (DNS $\rightarrow 488 \times 488 \times 1280 \approx 305M$)

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Concluding remarks

- A new tensor-diffusivity model has been proposed\(^7\)
  \[
  q^{s2PR} = -C_{s2pr} P^{-3/2}_{GG^T} R_{GG^T}^{1/3} \frac{\delta^2}{12} GG^T \nabla T
  \]

- Locally defined, unconditionally stable and vanishes for 2D flows √
- Good *a priori* alignment trends and proper near-wall scaling √
- Eddy-viscosity models work for RB √

---

\(^7\)F.X.Trias, F.Dabbagh, A.Gorobets, C.Oliet. *On a proper tensor-diffusivity model for LES of buoyancy-driven turbulence*, *Flow Turbul Combust*, 105:393-414, 2020.

\(^8\)F.X.Trias, A.Gorobets, M.Silvis, R.Verstappen, A.Oliva. *A new subgrid characteristic length for turbulence simulations on anisotropic grids*, *Phys.Fluids*, 26:115109, 2017.
Concluding remarks

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Future research:
- *A posteriori* tests using \(q^{s2PR}\) for RB

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Future research:
- \textit{A posteriori} tests using \(q^{s2PR}\) for RB
- How \(\delta\) should be defined for highly anisotropic grids\(^8\)?

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\(^7\) F.X. Trias, F. Dabbagh, A. Gorobets, C. Oliet. \textit{On a proper tensor-diffusivity model for LES of buoyancy-driven turbulence}, \textit{Flow Turbul Combust}, 105:393-414, 2020.

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Thank you for your virtual attendance