REGULAR HOMOTOPY CLASSES OF SINGULAR MAPS

ANDRÁS JUHÁSZ

Abstract. Two locally generic maps \( f, g : M^n \rightarrow \mathbb{R}^{2n-1} \) are regularly homotopic if they lie in the same path-component of the space of locally generic maps. Our main result is that if \( n \neq 3 \) and \( M^n \) is a closed \( n \)-manifold then the regular homotopy class of every locally generic map \( f : M^n \rightarrow \mathbb{R}^{2n-1} \) is completely determined by the number of its singular points provided that \( f \) is singular (i.e., \( f \) is not an immersion). This extends the analogous result of [4] for \( n = 2 \).

1. Introduction

This paper is a sequel to [4] in which locally generic maps of closed surfaces into \( \mathbb{R}^3 \) were classified up to regular homotopy. It turned out that for maps with at least one cross-cap point the number of singular points was the only regular homotopy invariant. The obstruction to constructing a regular homotopy was destroyed by pushing 1-cells of the surface through a singular point (see Figure 1).

In the present work we extend this result for locally generic maps of closed \( n \)-manifolds into \( \mathbb{R}^{2n-1} \) in case \( n \neq 3 \) (Theorem 1.7). However, this is not simply an adaptation of the ideas of [4]. The idea of pushing \((n-1)\)-simplices through a singular point originates from the previous work, but a lot of technical problems have to be dealt with. For example, if \( n \) is odd the Whitney-umbrella points have signs and the \((n-1)\)-dimensional obstruction is \( \mathbb{Z} \)-valued. Moreover, a new type of obstruction appears if \( n > 3 \) that was not present in the case \( n = 2 \). This is an \( n \)-dimensional obstruction and is related to the double point set of the map. In eliminating this obstruction we make essential use of a result of T. Ekholm [2].

It seems that this second type of obstruction cannot be eliminated if \( n = 3 \). In fact, in addition to the number of singular points a new \( \mathbb{Z} \)-valued invariant comes into the picture. I intend to deal with this in a separate paper.

1.1. Preliminary definitions.

Definition 1.1. Let \( M^n \) be a closed \( n \)-manifold and \( N^{2n-1} \) an arbitrary \((2n-1)\)-manifold. A map \( f : M^n \rightarrow N^{2n-1} \) is called locally generic if it is an immersion except for cross-cap (or Whitney-umbrella) singularities. Thus for any singular point \( p \) there exist coordinate systems \( (x_1, \ldots, x_n) \) about \( p \) and \( (y_1, \ldots, y_{2n-1}) \) about \( f(p) \) such that \( f \) is given near \( p \) by

\[
y_1 = x_1^2, \quad y_i = x_i, \quad y_{n+i-1} = x_1 x_i \quad (i = 2, \ldots, n).
\]
Figure 1. Pushing a 1-cell through a cross-cap point

**Notation 1.2.** Let $L(M^n, N^{2n-1})$ denote the subspace of locally generic maps, $G(M^n, N^{2n-1})$ the subspace of generic maps and $\text{Imm}(M^n, N^{2n-1})$ the subspace of immersions in $C^\infty(M^n, N^{2n-1})$ endowed with the $C^\infty$ topology. We introduce the notation $L'(M^n, N^{2n-1})$ for the space

$$L(M^n, N^{2n-1}) \setminus \text{Imm}(M^n, N^{2n-1}).$$

For $f \in L(M^n, N^{2n-1})$ the set of singular points of the map $f$ is denoted by $S(f)$.

**Remark 1.3.** Since $M^n$ is compact and the singularities of $f$ are isolated we have $|S(f)| < \infty$. It is well known that for $M^n$ closed $|S(f)|$ is an even number.

For $n > 2$ a map $f: M^n \to N^{2n-1}$ is generic iff
- it is an immersion with normal crossings except in a finite set of points
- the singular points are non-multiple cross-cap points
- $f$ has at most double crossings.

For $n = 2$ a generic $f$ might also have triple points.

**Definition 1.4.** A *regular homotopy* is a path in the space $L(M^n, N^{2n-1})$. In other words, it is a smooth map $H: M^n \times [0,1] \to N^{2n-1}$ such that $H_t \in L(M^n, N^{2n-1})$ for every $t \in [0,1]$. Here $H_t(x) = H(x,t)$ for $x \in M^n$ and $t \in [0,1]$. The regular homotopy $H$ is called *singularity fixing* if $S(H_t) = S(H_0)$ for every $t \in [0,1]$.

**Definition 1.5.** Two maps $f, g \in L(M^n, N^{2n-1})$ are called *regularly homotopic* if there exists a regular homotopy $H$ such that $H_0 = f$ and $H_1 = g$. If $H$ can be chosen to be singularity fixing then we say that $f$ and $g$ are *regularly homotopic through a singularity fixing homotopy*.

**Notation 1.6.** The fact that $f$ and $g$ are regularly homotopic is denoted by $f \sim g$. Furthermore, $f \sim_s g$ denotes that $f$ and $g$ are regularly homotopic through a singularity fixing homotopy.

1.2. **The classification theorem.**

**Theorem 1.7.** Suppose that $n > 3$ or $n = 2$. Let $M^n$ be a closed $n$-manifold and $f, g \in L'(M^n, \mathbb{R}^{2n-1})$. Then

$$f \sim g \iff |S(f)| = |S(g)|. $$

2. **Known results used in the proof of Theorem 1.7**

2.1. **Smale’s lemma and $M^n$-regular homotopies.** We will use extensively the following result of Smale (this is Theorem 1.1 in [8], the original paper of Smale is [15]). First we need a few definitions.
Theorem 2.2. Let \( D \) be the space of all \( C^\infty \) immersions of \( D^k \) in \( \mathbb{R}^m \) in the \( C^1 \) topology. Let \( \mathcal{B} = \mathcal{B}_{k,m} \) be the set of pairs \((g, g')\) where \( g : S^{k-1} \to \mathbb{R}^m \) is a \( C^\infty \) immersion and \( g' : S^{k-1} \to T\mathbb{R}^m \) is a \( C^\infty \) transversal field of \( g \). \( \mathcal{B} \) is topologized as a subspace of Cartesian product of the space of immersions \( S^{k-1} \to \mathbb{R}^m \), in the \( C^1 \) topology, with the space of continuous maps \( S^{k-1} \to T\mathbb{R}^m \), in the compact-open topology.

If \( h \in \mathcal{E} \) then let \( h' : S^{k-1} \to T\mathbb{R}^m \) be defined by \( h'(x) = \text{derivative of } h \text{ along the radius at } x \in S^{k-1} \). I.e., if \( r(x) \) is the unit tangent vector of \( D \) that is normal to \( S^{k-1} \) at \( x \) and which points away from the origin then \( h'(x) = h_*r(x) \). We define the map \( \pi : \mathcal{E} \to \mathcal{B} \) by the formula \( \pi(h) = (h|S^{k-1}, h') \). It is clear that \( \pi \) is continuous.

The following theorem is Theorem 1.1 in [3].

Theorem 2.2. If \( k < m \), then \( \pi : \mathcal{E}_{k,m} \to \mathcal{B}_{k,m} \) has the covering homotopy property.

The intuitive content of this theorem is as follows: If we are given an immersed disk \( D^k \) in \( \mathbb{R}^m \) such that \( k < m \) and we deform the boundary of the disk and the normal derivatives along the boundary, then we can deform the whole disk at the same time so as to induce the given deformation on the boundary and normal derivatives.

The following definitions were introduced by Hirsch in [3].

Definition 2.3. Let \( A \) be an arbitrary subset of the manifold \( M^n \). Let \( h : A \to Q^q \) and \( h' : TM|A \to TQ^q \) (\( Q^q \) is a manifold) be continuous maps such that \( h' \) covers \( h \). The pair \((h, h')\) is called an \( M^n \)-regular map, or \( M^n \)-immersion, of \( A \) in \( Q^q \) if the following condition is satisfied: there is a neighborhood \( V \) of \( A \) in \( M^n \) and an immersion \( l : V \to Q^q \) such that \( dl(TM|A) = h' \). It follows that \( l|A = h \). We say that \((h, h')\) is \( C^k \) if \( l \) can be chosen to be \( C^k \).

Definition 2.4. Let \( B \subset A \subset M^n \) be subsets. If \((r, r'), (s, s') : A \to Q^q \) are \( M^n \)-immersions such that \( r|B = s|B \) and \( r'|TM|B = s'|TM|B \), we say that \((r, r')\) and \((s, s')\) are tangent on \( B \), and write this as \((r, r')|B = (s, s')|B \).

Definition 2.5. Let \((r, r')\) and \((s, s')\) be \( M^n \)-immersions of \( A \) in \( Q^q \) such that for a certain (possibly empty) subset \( B \) of \( A \), \((r, r')\) and \((s, s')\) are tangent on \( B \). We say that \((r, r')\) and \((s, s')\) are \( M^n \)-regularly homotopic (rel \( B \)) if there is a path \((h_t, h'_t)\) in the space of all \( M^n \)-immersions of \( A \) in \( Q^q \) joining \((r, r')\) to \((s, s')\), such that for each \( t \), \((h_t, h'_t)|B = (r, r')|B \). Such a path is called an \( M^n \)-regular homotopy (rel \( B \)), and it is \( C^k \) if every \((h_t, h'_t)\) is \( C^k \).

Notation 2.6. The space of all \( C^\infty \) \( \mathbb{R}^n \)-immersions of \( D^k \) in \( \mathbb{R}^q \) is denoted by \( \mathcal{I}(k, q; n) \); the space of all \( C^\infty \) \( \mathbb{R}^n \)-immersions of \( S^{k-1} \) in \( \mathbb{R}^q \) is denoted by \( \mathcal{I}'(k, q; n) \).

Now let \( \pi_n : \mathcal{I}(k, q; n) \to \mathcal{I}'(k, q; n) \) be defined by

\[
\pi_n(f, f') = (f|S^{k-1}, f'|((T\mathbb{R}^n|S^{k-1})).
\]

If \( n = k \), this is the map \( \pi : \mathcal{E}_{k,q} \to \mathcal{B}_{k,q} \) defined in Definition 2.1.

For \((f, f') \in \mathcal{I}(k, q; n) \) put \( \Gamma_n(f, f') = \pi_n^{-1}(\pi_n(f, f')) \).

The following statement is Theorem 3.5 in [3]. It is a generalization of Smale’s lemma to \( M^n \)-immersions. To avoid confusion we will refer to it also as Smale’s lemma (despite the fact that it was proved by Hirsch).

Theorem 2.7. \( \pi_n \) has the covering homotopy property if \( k < q \).

Theorem 3.2 in [3] gives an alternative description of \( M^n \)-immersions:
Notation 2.8. \( \{ e_1(x), \ldots, e_n(x) \} \) denotes the standard basis of \( T_x \mathbb{R}^n \).

**Lemma 2.9.** There is a homeomorphism \( \zeta \) between the space of \( \mathbb{R}^n \)-immersions \( (h, h'): D^k \to \mathbb{R}^q \) and the space of pairs \( (l, \Psi) \), where \( l \in \text{Imm}(D^k, \mathbb{R}^q) \) and \( \Psi \) is a transversal \((n - k)\)-field along \( l \). The homeomorphism is given by \( \zeta(h, h') = (l, \Psi) \) where \( \Psi(x) = h'(e_{k+1}(x), \ldots, e_n(x)) \). Moreover, \( (f, f') \in C^k \) if and only if \( f \) and \( \Psi \) are \( C^k \).

An analogous result holds for \( S^{k-1} \) (see [3], Theorem 3.3):

**Notation 2.10.** Denote the space of pairs \( (l, \Psi) \), where \( l \in \text{Imm}(S^{k-1}, \mathbb{R}^q) \) and \( \Psi \) is a transversal \((n - k + 1)\)-field along \( l \) by \( \text{Imm}_{n-k+1}(S^{k-1}, \mathbb{R}^q) \).

**Lemma 2.11.** There is a homeomorphism
\[
\chi: \mathcal{I}(k, q; n) \to \text{Imm}_{n-k+1}(S^{k-1}, \mathbb{R}^q).
\]
\( \chi \) is given as follows: Let \( \Phi \) be the normal \((n - k + 1)\)-field on \( S^{k-1} \) given by
\[
\Phi(x) = \{ r(x), e_{k+1}(x), \ldots, e_n(x) \},
\]
where \( r(x) \) is the outward unit normal to \( S^{k-1} \) in \( \mathbb{R}^k \). Then \( \chi(h, h') = (h, \Psi) \), where \( \Psi(x) = f'(\Phi(x)) \).

**2.2. The obstructions \( \tau \) and \( \Omega \).** Hirsch defined in [3] an invariant \( \tau(g') \) for each \( (g, g') \in \mathcal{I}'(k, q; n) \). The vanishing of \( \tau(g') \) implies that \( (g, g') \) comes from \( \mathcal{I}(k, q; n) \).

**Definition 2.12.** \( (f, f') \in \mathcal{I}(k, q; n) \) is said to be extendible if there is a \( (g, g') \in \mathcal{I}(k, q; n) \) such that \( \pi_k(g, g') = (f, f') \).

**Definition 2.13.** Let \( (f, f'): S^{k-1} \to \mathbb{R}^n \) be a \( C^\infty \) \( \mathbb{R}^n \)-immersion, i.e., \( (f, f') \in \mathcal{I}'(k, q; n) \). The obstruction to extending \( (f, f') \), denoted by \( \tau(f') \in \pi_{k-1}(V_{q,n}) \) is the homotopy class of the map \( S^{k-1} \to V_{q,n} \) defined by
\[
x \mapsto f'(e_1(x), \ldots, e_n(x)).
\]

The following lemma is Theorem 3.9 in [3]:

**Lemma 2.14.** If \( k < q \) and \( \tau(f') = 0 \) then \( (f, f') \) is extendible.

**Definition 2.15.** Let \( \Phi_n: \mathcal{I}(k, q; n) \to C^0(D^k, V_{q,n}) \) be as follows:
\[
\Phi_n(f, f')(x) = f'(e_1(x), \ldots, e_n(x)).
\]

Let \( (f, f'), (g, g') \in \mathcal{I}(k, q; n) \), with \( \pi_n(f, f') = \pi_n(g, g') \), so that \( (f, f') \in \Gamma_n(g, g') \). Then \( \Phi_n(f, f') \) and \( \Phi_n(g, g') \) are maps \( D^k \to V_{q,n} \) which are tangent on \( S^{k-1} \).

**Definition 2.16.** Let \( A \) be a topological space, simple in dimension \( k \). Let \( f, g: D^k \to A \) and assume that \( f(x) = g(x) \) if \( x \in S^{k-1} \). Then \( d(f, g) \in \pi_k(A) \) is represented by mapping the "top" hemisphere of \( S^k \) by \( f \) and the "bottom" one by \( g \), assuming that the orientation of \( S^k \) is given by the coordinate frame \( \{ e_1, \ldots, e_k \} \) at the "North" pole of \( S^k \).

**Definition 2.17.**
\[
\Omega(f', g') = d(\Phi_n(f, f'), \Phi_n(g, g')) \in \pi_k(V_{q,n})
\]
is called the obstruction to an \( \mathbb{R}^n \)-regular homotopy (rel \( S^{k-1} \)) between \( (f, f') \) and \( (g, g') \). (This is well defined since \( V_{q,n} \) is simple in all dimensions.)
Lemma 2.19. Let \((f,f'),(g,g'),(h,h')\) be \(C^\infty\) \(\mathbb{R}^n\)-immersions of \(\Delta^k\) in \(\mathbb{R}^q\) which are all tangent on \(\partial\Delta^k\). Then

(a) \(\Omega(f',g') + \Omega(g',h') = \Omega(f',h')\).
(b) \(\Omega(f',f') = 0\).
(c) Given \(\alpha \in \pi_k(V_{q,n})\) there exists \((g,g')\) such that \(\Omega(f',g') = \alpha\).
(d) Suppose that \(\Omega(f',g') = 0\) and \(k < q\). Let \(F: \Delta^k \times I \to V_{q,n}\) be a homotopy (rel \(\partial\Delta^k\)) between the maps \(F_0, F_1: \Delta^k \to V_{q,n}\) defined respectively by \(x \mapsto f'\{e_i(x)\}\) and \(x \mapsto g'\{e_i(x)\}, i = 1, \ldots, n\). Then there is a \(C^\infty\) \(\mathbb{R}^n\)-regular homotopy \((f_1,f_1')\) between \((f,f')\) and \((g,g')\) such that the map \(\Delta^k \times I \to V_{q,n}\) defined by \((x,t) \mapsto f_1'(e_i(x))\) is homotopic (rel \((\partial\Delta^k \times I)\cup(\Delta^k \times \partial I))\) to \(F\).

Remark 2.18. An explicit definition of \(\Omega(f',g')\) is as follows: identify the upper and lower hemispheres of \(S^k\) with \(D^k\). Let \(\omega: S^k \to V_{q,n}\) be the map \(\omega(x) = f'\{e_1(x), \ldots, e_n(x)\}\) if \(x\) is in the upper hemisphere, \(\omega(x) = g'\{e_1(x), \ldots, e_n(x)\}\) if \(x\) is in the lower hemisphere. \(\omega(x)\) is well defined on the equator because \((f,f')\) and \((g,g')\) agree on \(S^{k-1}\). Then \(\Omega(f',g')\) is the homotopy class of \(\omega\).

The following lemma is Theorem 4.3 in [3]:

Lemma 2.20. Suppose that the \(C^\infty\) \(\mathbb{R}^n\)-immersions \((f,f'),(g,g'): \partial\Delta^n \to \mathbb{R}^q\) are tangent on \((\partial\Delta^n) \setminus \text{int}(\Delta^{n-1})\), where \(\Delta^{n-1}\) is an \((n-1)\)-face of the \(n\)-simplex \(\Delta^n\). Then

\[\Omega(f'|\Delta^{n-1},g'|\Delta^{n-1}) = \tau(f') - \tau(g')\]
Corollary 2.25. If \( k \geq 2 \) then Lemma 2.24 gives a bijection
\[
b: \pi_0(\text{Imm}_1(M^n, \mathbb{R}^{n+k+1})) \to \pi_0(\text{Imm}(M^n, \mathbb{R}^{n+k})).
\]

2.4. Whitney-umbrellas. In this chapter we sum up some ideas from [10]. Fix an orientation for \( \mathbb{R}^{2n-1} \).

Definition 2.26. Suppose that \( N^n \) is an \( n \)-manifold with boundary and that \( f: N^n \to \mathbb{R}^{2n-1} \) is a generic map (thus \( f|\partial N^n \) is an embedding). Let \( \nu: \partial N^n \to T\mathbb{R}^{2n-1} \) be a transversal vector field along \( f|\partial N^n \) such that \( -\nu \) points into \( f(N^n) \). Then for \( \varepsilon \) sufficiently small define \( \mathcal{L}_f(N) \) to be the linking number
\[
\text{lk}(f|\partial N^n), f|((\partial N^n) + \varepsilon\nu).
\]

Remark 2.27. Notice that \((f, \nu)\) corresponds to the \( N^n \)-immersion \((f, df)|\partial N^n\).

As a consequence of Lemma 6 in [10] we obtain the following lemma.

Lemma 2.28. Suppose that \( f \in L(M^n, \mathbb{R}^{2n-1}) \) and \( p \in S(f) \). Choose a coordinate neighborhood \( U_p \) of \( p \) diffeomorphic to \( D^n \) such that \( f|U \) is generic and has the form \( (\mathbb{R}^{2n-1}) \). Then \( \mathcal{L}_f(U_p) = \pm 1 \), moreover, the sign does not depend on the choice of \( U_p \) if \( n \) is odd. (Recall that for \( n \) even \( \mathcal{L}_f \) is defined mod 2.)

Notation 2.29. Denote the map \((f, df)|\partial U_p \in \mathcal{I}(n, 2n - 1; n) \) by \((w, w')\).

Corollary 2.30. For \( n \) even \( \mathcal{L}_f(U_p) \equiv 1 \) (mod 2) for every neighborhood \( U_p \).

In [10] signs are defined for Whitney-umbrellas if \( n \) is odd and \( \mathbb{R}^{2n-1} \) is oriented. This can be done as follows (this is not the original definition of Whitney):

Definition 2.31. Suppose that \( f \in L(M^n, \mathbb{R}^{2n-1}) \) and \( p \in S(f) \). Choose \( U_p \) as in Lemma 2.28. The sign of the Whitney-umbrella at \( p \) is then defined as
\[
\text{sgn}(p) = \mathcal{L}_f(U_p).
\]

Notation 2.32. If \( n \) is odd and \( f \in L(M^n, \mathbb{R}^{2n-1}) \) denote the set of positive (respectively negative) cross-cap points of \( f \) by \( S_+(f) \) (respectively \( S_-(f) \)). Moreover let
\[
\#S(f) = \begin{cases} 
|S_+(f)| - |S_-(f)| & \text{if } n \text{ is odd,} \\
|S(f)| \mod 2 & \text{if } n \text{ is even,}
\end{cases}
\]
where \(|A|\) denotes the cardinality of the set \( A \).

In [10] Theorem 2 states the following:

Lemma 2.33. Let \( N^n \) be a compact \( n \)-manifold with boundary and \( f \in G(N^n, \mathbb{R}^{2n-1}) \). Then \( \mathcal{L}_f(N^n) = \#S(f) \).

Remark 2.34. If \( f \) is only locally generic, we may perturb it slightly to obtain a generic map. From this we can generalize Lemma 2.33 to locally generic maps.

As a trivial consequence we obtain

Corollary 2.35. Let \( M^n \) be a closed \( n \)-manifold and \( f \in L(M^n, \mathbb{R}^{2n-1}) \). Then for \( n \) odd the algebraic number of singular points vanishes (i.e., \(|S_+(f)| = |S_-(f)|\)), while for \( n \) even, it vanishes mod 2 (i.e., \(|S(f)| \) is even).

We will also use Theorem 7 of [10], which states the following:
Theorem 2.36. Let $M^n$ be a compact and connected manifold with boundary and $f : M^n \to \mathbb{R}^{2n-1}$ be a smooth map which is an embedding in a neighborhood of $\partial M^n$. Suppose that $\mathcal{L}_f(M^n) = 0$ if $n$ is odd and $\mathcal{L}_f(M^n) \equiv 0 \mod 2$ if $n$ is even. Then there exists an immersion $g : M^n \to \mathbb{R}^{2n-1}$ that is arbitrarily $C^0$-close to $f$ and equals $f$ in a neighborhood of $\partial M^n$.

Finally, we present an equivalent definition of the sign of a Whitney-umbrella singularity (motivated by [7]). With an additional choice we can also define the sign of a singular point if $n$ is even.

Definition 2.37. Let $f \in L(M^n, \mathbb{R}^{2n-1})$ and $p \in S(f)$. Fix an orientation of $\mathbb{R}^{2n-1}$. Then we define the sign of $p$ as follows.

Choose local coordinates $(x_1, \ldots, x_n)$ about $p$ and $(y_1, \ldots, y_{2n-1})$ about $f(p)$ such that $f$ is given near $p$ by equation (1.1). Let $D_{\varepsilon} = \{ y_1^2 + \cdots + y_{2n-1}^2 \leq \varepsilon \}$ for $\varepsilon > 0$ sufficiently small. Then Lemma 2.2 in [4] states that $B_{\varepsilon} = f^{-1}(D_{\varepsilon})$ is a closed disc neighborhood of $p$ in $M^n$. Choose an arbitrary orientation of $S_{\varepsilon} = \partial B_{\varepsilon}$. Denote by $q$ the double point of $f|S_{\varepsilon}$ and let $f^{-1}(q) = \{ q_1, q_2 \}$. Moreover, let $v = (v_1, \ldots, v_{n-1})$ be a positive basis of $T_{q_1}S_{\varepsilon}$ and $w = (w_1, \ldots, w_{n-1})$ a positive basis of $T_{q_2}S_{\varepsilon}$. Orient $\partial D_{\varepsilon}$ by its outward normal vector. Then $p$ is called positive if $(df(v), df(w))$ is a positive basis of $T_q(\partial D_{\varepsilon})$, and negative otherwise.

This definition does not depend on the choice of the orientation of $S_{\varepsilon}$. However, if $n$ is even then the sign of $p$ does depend on the ordering of the points $q_1$ and $q_2$, i.e., if we swap the two points then the sign of $p$ also changes. For $n$ odd the sign of $p$ is independent of the ordering of $q_1$ and $q_2$. Thus fixing the orientation of $\mathbb{R}^{2n-1}$ defines the sign of a Whitney-umbrella point only if $n$ is odd. If we also fix an ordering of the two branches of $f$ meeting at the open double curve ending at $f(p)$ then this defines a sign of $p$ if $n$ is even.

It is easy to verify that Definition 2.31 and Definition 2.37 are equivalent.

3. Proof of Theorem 1.7

3.1. Outline. The proof of the case $n = 2$ can be found in [4]. The generalizations of those ideas are incorporated into the present proof for the case $n > 3$.

The implication

$$f \sim g \Rightarrow |S(f)| = |S(g)|$$

is Proposition 2.4 in [4]. The proof relies on the fact that the cross-cap singularity is stable and $M^n$ is closed. Thus we will only prove the other implication.

For $n > 3$ the proof is divided into two main parts. Using Hirsch-Smale theory (for a reference see [3]) we will see that there are two obstructions to constructing a regular homotopy between two immersions from $M^n$ to $\mathbb{R}^{2n-1}$. These obstructions can be eliminated in the presence of cross-cap points.

In the first part we will construct a regular homotopy that pushes the $(n-1)$-simplices of $M^n$ through a singular point to destroy the first obstruction. In the second part we will first reduce the problem to the case $M^n = S^n$. To remove the second obstruction we will merge the closed double curves of $f$ and $g$ with the ones connecting the singular points. This can be done using a variant of the Whitney-trick. Finally, we replace $f$ and $g$ with immersions and we use an argument of [2] that shows that the Smale-invariant of an immersion of $S^n \to \mathbb{R}^{2n-1}$ is completely determined by the geometry of its self-intersection if $n \geq 4$.
3.2. Setup. From now on \( f \) and \( g \) denote the maps in the statement of Theorem 3.2. Thus \( f, g \in L'(M^n, \mathbb{R}^{2n-1}) \) and \( |S(f)| = |S(g)| \). We also suppose that \( n > 3 \) and \( \mathbb{R}^{2n-1} \) is oriented.

Using the lemma of homogeneity and Corollary 1.7 there exists a diffeotopy \( \{d_t : t \in [0,1]\} \) of \( M^n \) such that \( d_0 = id_{M^n} \), moreover

- \( d_1(S(g)) = S(f) \) if \( n \) is even,
- \( d_1(S_+(g)) = S_+(f) \) and \( d_1(S_-(g)) = S_-(f) \) if \( n \) is odd.

This implies that \( S(f \circ d_1) = S(g) \), moreover \( S_+(f \circ d_1) = S_+(g) \) and \( S_-(f \circ d_1) = S_-(g) \) if \( n \) is odd. Since \( \{f \circ d_t : t \in [0,1]\} \) provides a regular homotopy connecting \( f \) and \( f \circ d_1 \) we might suppose that \( S(f) = S(g) \), moreover \( S_+(f) = S_+(g) \) and \( S_-(f) = S_-(g) \) if \( n \) is odd.

If \( n \) is even then any two cross-caps are locally equivalent. For \( n \) odd any two cross-caps of the same sign are locally equivalent. Thus there is a neighborhood \( U \) of \( S(f) = S(g) \) such that \( f \) is regularly homotopic to \( g \) on \( U \). This regular homotopy can be extended to \( M^n \) using Smale’s lemma. Thus we may suppose that \( f|U = g|U \).

If \( U \) is chosen small enough then \( f|U = g|U \) is generic. We perturb \( f \) and \( g \) slightly outside \( U \), using a regular homotopy, to obtain generic maps. Thus we can also suppose that \( f, g \in G(M^n, \mathbb{R}^{2n-1}) \).

Choose a smooth simplicial decomposition of \( M^n \) so fine that for any \( n \)-simplex \( \Delta^n \) containing a point \( p \) of \( S(f) = S(g) \) we have \( \Delta^n \subset U \) and \( p \in \text{int}(\Delta^n) \). (The proof of the existence of a smooth simplicial decomposition can be found in [6].)

Thus \( f|\Delta^n = g|\Delta^n \) for each \( \Delta^n \) as above. For \( p \in S(f) \) we also choose \( \Delta^n \ni p \) so small that \( f|\Delta^n = g|\Delta^n \) has the canonical form \( [x^2,\ldots,x^n] \) in a coordinate neighborhood containing \( \Delta^n \) and centered at \( p \).

The fiber of the bundle \( \text{MONO}(TM, T\mathbb{R}^{2n-1}) \) is homeomorphic to the Stiefel manifold \( V_{2n-1,n} \) of \( n \)-frames in \( \mathbb{R}^{2n-1} \). It is well known that \( \pi_i(V_{2n-1,n}) = 0 \) for \( i < n - 1 \).

Thus there exists an \( M^n \)-regular homotopy connecting the \( M^n \)-immersion

\[
(f, df)|_{sk_{n-2}(M^n)} \text{ with } (g, dg)|_{sk_{n-2}(M^n)}.
\]

Using Smale’s lemma we can extend this \( M^n \)-regular homotopy to the whole manifold \( M^n \) keeping \( f|U \) fixed. So we might suppose that

\[
(f, df)|_{sk_{n-2}(M^n)} = (g, dg)|_{sk_{n-2}(M^n)}.
\]

3.3. The first obstruction. Our next task is to find an \( M^n \)-regular homotopy connecting \((f, df)\) with \((g, dg)\) on \( sk_{n-1}(M^n) \). The obstruction \( \Omega(df, dg) \) to finding a regular homotopy connecting \((f, df)\) with \((g, dg)\) on an \((n-1)\)-simplex \( \Delta^{n-1} \), fixing the boundary \( \partial \Delta^{n-1} \), lies in

\[
\pi_{n-1}(V_{2n-1,n}) \approx \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even}, \\ \mathbb{Z} & \text{if } n \text{ is odd}. \end{cases}
\]

We now prove that the sum of the obstructions on the boundary of an \( n \)-simplex on which both \( f \) and \( g \) are immersions vanishes.

**Lemma 3.1.** Suppose that the \( C^\infty \mathbb{R}^n \)-immersions \((f, f')\) and \((g, g')\) of \( \partial \Delta^n \) into \( \mathbb{R}^q \) \((q > n)\) are tangent on \( sk_{n-2}(\Delta^n) \). Denote the faces of \( \Delta^n \) by

\[
\Delta_0^{n-1}, \ldots, \Delta_n^{n-1},
\]
oriented by the outward normal vectors of $\Delta^n$ and let
\[(f_i, f'_i) = (f, f')|\Delta^{n-1}_i, \quad (g_i, g'_i) = (g, g')|\Delta^{n-1}_i \quad (i = 0, \ldots, n).\]
Then
\[
\sum_{i=0}^{n} \Omega(f'_i, g'_i) = \tau(f') - \tau(g').
\]

Proof. We define a sequence of $C^\infty \mathbb{R}^n$-immersions
\[(h_i, h'_i) : \partial \Delta^n \to \mathbb{R}^{2n-1} \quad (i = 0, \ldots, n + 1)\]
such that $(h_0, h'_0) = (f, f')$, $(h_{n+1}, h'_{n+1}) = (g, g')$ and for $0 < i < n + 1$ let
\[(h_i, h'_i)|\Delta^{n-1}_i = \begin{cases} (g_j, g'_j) & \text{if } j < i, \\ (f_j, f'_j) & \text{if } i \leq j \leq n. \end{cases}\]
Then $(h_i, h'_i)$ is $C^\infty$ since $(f, f')$ and $(g, g')$ are tangent on $sk_{n-2}(\Delta^n)$. Applying Theorem 2.20 to $(h_i, h'_i)$ and $(h_{i+1}, h'_{i+1})$ we get that
\[\Omega(f'_i, g'_i) = \tau(h'_i) - \tau(h'_{i+1}).\]
Summing up these equations we obtain the required result. \hfill \qed

Corollary 3.2. Let $q > n$ and $f, g \in \text{Imm}(\Delta^n, \mathbb{R}^q)$. Suppose that $(f, f') = (f, df)|\partial \Delta^n$ and $(g, g') = (g, dg)|\partial \Delta^n$ are tangent on $sk_{n-2}(\Delta^n)$. Using the notations of Lemma 3.1 it holds that
\[
\sum_{i=0}^{n} \Omega(f'_i, g'_i) = 0.
\]

Proof. Since both $(f, f')$ and $(g, g')$ are extendible we have that $\tau(f') = 0$ and $\tau(g') = 0$. Now Lemma 3.1 gives the statement of our corollary. \hfill \qed

Lemma 2.11 implies that there is a homeomorphism
\[\chi : \mathcal{T}(n, 2n - 1; n) \to \text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1}).\]

Notation 3.3. Let $\text{Emb}_1(S^k, \mathbb{R}^l)$ denote the space of pairs $(h, \nu)$, where $h : S^k \to \mathbb{R}^l$ is an embedding and $\nu$ is a transversal 1-field along $h$.

Definition 3.4. Two elements of $\text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ (or $\mathcal{T}(n, 2n - 1; n)$) are called regularly homotopic if they lie in the same path-component of the corresponding space. A regular homotopy is a path in $\text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ (or $\mathcal{T}(n, 2n - 1; n)$).

Proposition 3.5. For every immersion $(f, \nu) \in \text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ there exists an embedding $(h, \mu) \in \text{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1})$ such that $(f, \nu)$ is regularly homotopic to $(h, \mu)$.

Proof. The subspace of embeddings is dense open in $\text{Imm}(S^{n-1}, \mathbb{R}^{2n-1})$. Thus we can perturb $f$ by a regular homotopy to get an embedding $h$. If the perturbation is sufficiently small then $\nu$ can be deformed simultaneously as a transversal field along the regular homotopy. \hfill \qed

Definition 3.6. Suppose that $\mathbb{R}^{2n-1}$ is oriented. Let
\[
\text{lk} : \text{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1}) \to \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } n \text{ is even,} \end{cases}
\]
be defined as follows: For \((f, \nu) \in \text{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1})\) choose \(\varepsilon > 0\) sufficiently small so that \(f + \varepsilon \nu\) is an embedding. Then let \(\text{lk}(f, \nu)\) be accordingly the \(\mathbb{Z}\) or \(\mathbb{Z}_2\) linking number \(\text{lk}(f, f + \varepsilon \nu)\).

**Proposition 3.7.** \(\text{lk}\) is a regular homotopy invariant, i.e., if the maps \((f, \nu), (h, \mu) \in \text{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1})\) are regularly homotopic then \(\text{lk}(f, \nu) = \text{lk}(h, \mu)\).

**Proof.** This is exactly Lemma 9 in [10]. □

Now we can extend the definition of \(\text{lk}\) from embeddings to immersions.

**Definition 3.8.** Suppose that \(\mathbb{R}^{2n-1}\) is oriented. Let

\[
\text{lk}: \text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1}) \to \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd}, \\ \mathbb{Z}_2 & \text{if } n \text{ is even}, \end{cases}
\]

be defined as follows: For \((f, \nu) \in \text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})\) choose an embedding \((h, \mu)\) in \(\text{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1})\) regularly homotopic to \((f, \nu)\) (such an embedding exists because of Proposition 3.7). Then let \(\text{lk}(f, \nu) = \text{lk}(h, \mu)\). This definition does not depend on the choice of \((h, \mu)\) because of Proposition 3.7.

**Proposition 3.9.** The function \(\text{lk}\) defined above is a regular homotopy invariant.

**Proof.** This is clear from Proposition 3.7 and Definition 3.8. □

**Notation 3.10.** Let \(I'_n = \pi_0(I'(n, 2n - 1; n))\) and \(J'_n = \pi_0(\text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1}))\).

Proposition 3.7 implies that \(\text{lk}\) defines a function on \(J'_n\), which we denote by \(\text{lk}_*\).

**Lemma 3.11.** Let \(f \in L(D^n, \mathbb{R}^{2n-1})\) and

\[
(f, f') = (f, df)|S^{n-1} \in I'(n, 2n - 1; n).
\]

Then

\[
\text{lk}(\chi(f, f')) = \#S(f).
\]

(For the definition of \(\chi\) see Lemma 2.11)

**Proof.** Perturb \(f\) using a regular homotopy to get a generic map \(h\) (thus \(h|S^{n-1}\) is an embedding). Let \((h, h') = (h, dh)|S^{n-1}\). Proposition 3.7 implies that \(\text{lk}(\chi(f, f')) = \text{lk}(\chi(h, h'))\). Moreover, the number of singular points is also a regular homotopy invariant, thus \(\#S(f) = \#S(h)\). By definition \(\text{lk}(\chi(h, h')) = L_h(D^n)\). On the other hand Lemma 2.11 implies that \(L_h(D^n) = \#S(h)\). □

**Corollary 3.12.** Let \((f, \nu) \in \text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})\). Then

\[
\text{lk}(f, -\nu) = -\text{lk}(f, \nu).
\]

**Proof.** Choose two generic maps \(f_0, f_1 \in G(D^n, \mathbb{R}^{2n-1})\) such that

\[
\chi((f_i, df_i)|S^{n-1}) = (f, (-1)^i \nu)
\]

for \(i = 0, 1\). Then \(f_0\) and \(f_1\) fit together to define a map \(h \in G(S^n, \mathbb{R}^{2n-1})\). Lemma 2.11 implies that \(\#S(f_0) + \#S(f_1) = \#S(h) = 0\). Moreover, Lemma 3.11 implies that \(\text{lk}(f, \nu) = \#S(f_0)\) and \(\text{lk}(f, -\nu) = \#S(f_1)\). Putting together these results we get that \(\text{lk}(f, \nu) + \text{lk}(f, -\nu) = 0\), as required. □

**Proposition 3.13.** \(\tau\) (see Definition 2.13) is a regular homotopy invariant. Moreover, it induces a bijection \(\tau_*: I'_n \to \pi_{n-1}(V_{2n-1, n})\).
Definition 3.14. Define the connected sum operation $\#$ on elements $x$ and $y$ of $\text{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ or $\mathcal{T}(n, 2n - 1; n)$ by joining them with a thin tube. This operation is also well defined on regular homotopy classes of maps: Suppose that $x \sim_H x_1$ and $y \sim_K y_1$. Snaue’s lemma implies that the regular homotopies $H$ and $K$ may be kept fixed on small disks. Now a regular homotopy $L$ connecting $x \# x_1$ with $y \# y_1$ is defined by taking $H$ and $K$, and joining them with a tube attached to the disks kept fixed.

Proposition 3.15. The sets $I'_n$ and $J'_n$ (introduced in Notation 3.11) endowed with the connected sum operation form abelian groups. Moreover, $\chi_*: I'_n \rightarrow J'_n$ and $\tau_*$ are group isomorphisms and $\Lambda_k$ is a group homomorphism.

Proof. It is clear that $(I'_n, \#)$ and $(J'_n, \#)$ are abelian semigroups. Let $i: D^n \rightarrow \mathbb{R}^{2n-1}$ denote the standard embedding. Then $(i, i') = (i, di)|S^{n-1}$ (respectively $\chi((i, di)|S^{n-1}) = (i, r)$, where $r$ is the outward normal field of $S^{n-1}$ in $\mathbb{R}^n$) represents the identity of $I'_n$ (respectively $J'_n$).

If $x, y \in I'_n$ then choose a representative $(f, f')$ for $x$ and $(g, g')$ for $y$. Then $\tau(f' \# g') = \tau(f') + \tau(g')$, which implies that $\tau_*(x \# y) = \tau_*(x) + \tau_*(y)$. Since $(i, i')$ is extendible, $\tau_*(i') = 0$. Proposition 3.13 states that $\tau_*$ is a bijection. Thus $\tau_*$ is a semigroup isomorphism that takes the identity of $I'_n$ to the identity of $\pi_{n-1}(V_{2n-1}, n) \cong \mathbb{Z}$ or $\mathbb{Z}_2$. This implies that $I'_n$ is a group isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$ (depending on the parity of $n$) and $\tau_*$ is a group isomorphism.

Since $\chi$ is a homeomorphism, $\chi_*: I'_n \rightarrow J'_n$ is a bijection. Moreover, from the definition of $\chi$ it is easy to see that $\chi_*$ is a semigroup homomorphism. We saw above that $\chi_*$ takes the identity $(i, i')$ of $I'_n$ to the identity $(i, r)$ of $J'_n$. Thus $J'_n$ is also a group isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$ and $\chi_*$ is a group isomorphism.

We finally show that $\Lambda_k$ is a group homomorphism from $J'_n$ to $\mathbb{Z}$ or $\mathbb{Z}_2$. Suppose that $x, y \in J'_n$ and choose representatives $(f_1, \nu_1)$ and $(f_2, \nu_2)$ for $x$, respectively $y$, such that $f_1$ and $f_2$ can be separated by a hyperplane in $\mathbb{R}^{2n-1}$. Also choose $h_1, h_2 \in G(D^n, \mathbb{R}^{2n-1})$ so that $\chi((h_i, dh_i)|S^{n-1}) = (f_i, \nu_i)$ for $i = 1, 2$. Denote the boundary sum $h_1 \sharp h_2$ with $h$. Then $\chi((h, dh)|S^{n-1}) = (f_1, \nu_1) \# (f_2, \nu_2)$ and $\# S(h) = \# S(h_1) + \# S(h_2)$. Using Lemma 3.11 we get that $\Lambda_k((f_1, \nu_1) \# (f_2, \nu_2)) = \Lambda_k(f_1, \nu_1) + \Lambda_k(f_2, \nu_2)$, i.e., $\Lambda_k(x \# y) = \Lambda_k(x) + \Lambda_k(y)$.

The following lemma is the key to connecting the works of Hirsch and Whitney.

Lemma 3.16. There exists an isomorphism

$$\alpha: \pi_{n-1}(V_{2n-1}, n) \rightarrow \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } n \text{ is even.} \end{cases}$$
such that
\[ \text{lk} \circ \chi = \alpha \circ \tau. \]

I.e., the following diagram is commutative:
\[
\begin{array}{ccc}
\mathcal{I}(n, 2n - 1; n) & \xrightarrow{\chi} & \text{Imm}_{1}(S^{n-1}, \mathbb{R}^{2n-1}) \\
\tau \downarrow & & \downarrow \text{lk} \\
\pi_{n-1}(V_{2n-1,n}) & \xrightarrow{\alpha} & \mathbb{Z} \text{ or } \mathbb{Z}_{2}.
\end{array}
\]

Remark 3.17. If \( n \) is even then \( \alpha \) might be omitted from the formula since \( \mathbb{Z}_{2} \) has only one automorphism.

Proof. We saw that \( \tau \) and \( \text{lk} \) are regular homotopy invariants, i.e., they are constant on the path-components of their domains. They define maps
\[ \tau_{*} : I'_{n} = \pi_{0}(\mathcal{I}(n, 2n - 1; n)) \to \pi_{n-1}(V_{2n-1,n}) \]
and
\[ \text{lk}_{*} : J'_{n} = \pi_{0}(\text{Imm}_{1}(S^{n-1}, \mathbb{R}^{2n-1})) \to \left\{ \begin{array}{ll} 
\mathbb{Z} & \text{if } n \text{ is odd}, \\
\mathbb{Z}_{2} & \text{if } n \text{ is even}.
\end{array} \right. \]
Proposition 3.18 implies that \( I'_{n} \) and \( J'_{n} \) are abelian groups with the connected sum operation, moreover, \( \chi_{*}, \tau_{*} \) are isomorphisms and \( \text{lk}_{*} \) is a homomorphism. We also know that \( I'_{n} \) and \( J'_{n} \) are isomorphic to \( \mathbb{Z} \) or \( \mathbb{Z}_{2} \), according to the parity of \( n \).

\[ \text{lk}_{*} \text{ is an epimorphism because of Lemma 2.28} \]
\[ \text{lk}(\chi(w, w')) = \pm 1. \]
So \( \text{lk}_{*} \) is a \( \mathbb{Z} \to \mathbb{Z} \) or \( \mathbb{Z}_{2} \to \mathbb{Z}_{2} \) epimorphism, and thus it is also an isomorphism. Now define \( \alpha \) to be \( \text{lk}_{*} \circ \chi_{*} \circ (\tau_{*})^{-1} \).

From Theorem 2.20 and Lemma 3.10 we get the following

Proposition 3.18. Suppose that \((f, \nu), (g, \mu) \in \text{Imm}_{1}(\partial \Delta^n, \mathbb{R}^{q})\) agree on \((\partial \Delta^n) \setminus \text{int}(\Delta^{n-1})\), where \(\Delta^{n-1}\) is an \((n - 1)\)-face of the \(n\)-simplex \(\Delta^n\). Then for \((f, f') = (\chi^{-1}(f, \nu))|\Delta^{n-1} \) and \((g, g') = (\chi^{-1}(g, \mu))|\Delta^{n-1}\)
\[ \alpha(\Omega(f', g')) = \text{lk}(f, \nu) - \text{lk}(g, \mu). \]

Remark 3.19. \(\Delta^{n-1}\) is co-oriented by the outward normal vector of \(\Delta^n\). Then
\[ (\chi^{-1}(f, \nu))|\Delta^{n-1} = \zeta^{-1}((f, \nu)|\Delta^{n-1}), \]
where \(\zeta\) is the homeomorphism of Lemma 2.20. Thus \(\text{lk}(f, \nu) - \text{lk}(g, \mu)\) depends only on \((f, \nu)|\Delta^{n-1}\) and \((g, \mu)|\Delta^{n-1}\).

Definition 3.20. For \((f, f'), (g, g') \in \mathcal{I}(n - 1, 2n - 1; n)\) that are tangent on \(\partial \Delta^n\) let
\[ o(f', g') = \alpha(\Omega(f', g')). \]

Corollary 3.21. Let \(\Delta^{n-1}\) be a face of the \(n\)-simplex \(\Delta^n\) and let \((f, f')\) and \((g, g')\) be \(C^{\infty}\) \(\mathbb{R}^{n}\)-immersions of \(\Delta^{n-1}\) into \(\mathbb{R}^{2n-1}\) tangent on \(\partial \Delta^{n-1}\). Choose extensions of \((f, f')\) and \((g, g')\) to \(\partial \Delta^n\) that are tangent on \(\partial \Delta^n \setminus \Delta^{n-1}\). Then
\[ o(f', g') = \text{lk} \circ \chi(f, f') - \text{lk} \circ \chi(g, g'). \]

Lemma 3.22. Let \((f, f')\) and \((g, g')\) be as in Corollary 3.21 and \((w, w')\) be the boundary of a Whitney-umbrella of sign \(\varepsilon\) (as in Notation 2.26). Then
\[ o(f' \# w', g') = o(f', g') + \varepsilon. \]
Proof. Using extensions as in Corollary 3.21, we get that
\[ o(f' \# w', g') = \text{lk} \circ \chi((f, f') \# (w, w')) - \text{lk} \circ \chi(g, g'). \]
Using the fact that \( \text{lk} \) and \( \chi \) are additive, we get that this equals
\[ \text{lk} \circ \chi(f, f') + \text{lk} \circ \chi(w, w') - \text{lk} \circ \chi(g, g'). \]
The definition of the sign of an umbrella implies that \( \text{lk} \circ \chi(w, w') = \varepsilon \). \( \square \)

Proposition 3.23. Let \((f, f')\) and \((g, g')\) be as in Corollary 3.21. Then \((f, f')\) and \((g, g')\) are regularly homotopic (rel \( \partial \Delta^{n-1} \)) iff \( o(f', g') = 0 \).

Proof. This is a trivial consequence of Lemma 2.19. \( \square \)

Remark 3.24. Lemma 3.22 and Proposition 3.23 show us how Whitney-umbrellas destroy obstructions to moving \( f \) to \( g \) on an \((n - 1)\)-simplex (rel boundary). We have to take the connected sum of \( f \) with several copies of the boundary of an umbrella. This will be done by a diffeotopy of \( M^n \) that pushes the \((n - 1)\)-simplex through a singular point.

Lemma 3.25. Using the notations of Lemma 3.21
\[ \sum_{i=0}^{n} o(f'_i, g'_i) = \text{lk} \circ \chi(f, f') - \text{lk} \circ \chi(g, g'). \]
In particular, if \((f, f')\) and \((g, g')\) are both extendible then
\[ \sum_{i=0}^{n} o(f'_i, g'_i) = 0. \]
Proof. This is a trivial consequence of Lemma 3.21. \( \square \)

Remark 3.26. Lemma 3.25 shows us that the obstruction \( o \) has the cocycle property, i.e., the sum of the obstructions on the boundary of an \( n \)-simplex, on which both \( f \) and \( g \) are immersions, vanishes.

3.4. Pushing \((n-1)\)-simplices through Whitney-umbrellas. Now we are going to use the apparatus developed above for the proof of Theorem 1.7. Since until now we only worked in a standard simplex in Euclidean space, we have to globalize our results to the triangulated manifold \( M^n \). We do not require \( M^n \) to be oriented.

Definition 3.27. Suppose that \( \Delta^n \) is an \( n \)-simplex of \( M^n \). Then we can define the map \( \chi = \chi_{\Delta^n} \) taking \( M^n \)-immersions of \( \partial \Delta^n \) to \( \text{Imm}_1(\partial \Delta^n, \mathbb{R}^{2n-1}) \) using the outward normal vectors \( r(x) \) of \( \partial \Delta^n \) (corners rounded off) as follows: Let \((f, f')\) be an \( M^n \)-immersion of \( \partial \Delta^n \) into \( \mathbb{R}^{2n-1} \). Then let \( \chi(f, f') = (f, \nu) \), where \( \nu(x) = f'(r(x)) \).

Definition 3.28. Suppose that \( \Delta^{n-1} \) is an \((n - 1)\)-simplex of \( M^n \) that is co-oriented. Then we can define the map \( \zeta = \zeta_{\Delta^{n-1}} \) taking \( M^n \)-immersions of \( \Delta^{n-1} \) to \( \text{Imm}_1(\Delta^{n-1}, \mathbb{R}^{2n-1}) \) as follows: Let \( r(x) \) be a positive normal field along \( \Delta^{n-1} \). If \((f, f')\) is an \( M^n \)-immersion of \( \Delta^{n-1} \) into \( \mathbb{R}^{2n-1} \) then let \( \zeta(f, f') = (f, \nu) \), where \( \nu(x) = f'(r(x)) \).

Corollary 3.21 implies that the following definition makes sense:
Definition 3.29. Suppose that \( \Delta^{n-1} \) is an \((n-1)\)-simplex of \( M^n \) that is co-oriented and let \((f, f')\) and \((g, g')\) be \(M^n\)-immersions of \( \Delta^{n-1} \) into \( \mathbb{R}^{2n-1} \) tangent on \( \partial\Delta^{n-1} \). Then the obstruction \( o(f', g') \) to finding a regular homotopy of \((f, f')\) to \((g, g')\) (rel \( \partial\Delta^{n-1} \)) is given as follows: Choose an \( n \)-simplex \( \Delta^n \) of \( M^n \) such that \( \Delta^{n-1} \subset \partial\Delta^n \). Extend \((f, f')\) and \((g, g')\) to \( \partial\Delta^n \) to be tangent on \( \partial\Delta^n \setminus \Delta^{n-1} \). Also choose a normal field \( s(x) \) along \( \partial\Delta^n \) that agrees with the co-orientation of \( \Delta^{n-1} \). Now define \( o(f', g') \) to be \( \text{lk}(f, f' \circ s) - \text{lk}(g, g' \circ s) \).

Remark 3.30. If we change the co-orientation of \( \Delta^{n-1} \) then \( o(f', g') \) changes sign. To see this we choose the same extensions of \((f, f')\) and \((g, g')\) to the boundary \( \partial\Delta^n \) of the same \( n \)-simplex \( \Delta^n \). Then we must take \(-s\) instead of \(s\) because of the changed co-orientation of \( \Delta^n \). Now using Corollary 3.12 we get that
\[
\text{lk}(f, f' \circ (-s)) - \text{lk}(g, g' \circ (-s)) = -\text{lk}(f, f' \circ s) + \text{lk}(g, g' \circ s) = -o(f', g').
\]

Let \( f \) and \( g \) be the two locally generic maps of Theorem 1.7. We have supposed that \((f, df)\) and \((g, dg)\) are tangent on \( \text{sk}_{n-2} M^n \). If \( n \) is odd choose a co-orientation \( O_{\Delta^{n-1}} \) for every \((n-1)\)-simplex \( \Delta^{n-1} \) of \( M^n \) so that \( o(df, dg) \leq 0 \). This can be done because of Remark 3.20. Let \( k = |S(f)| = |S(g)| \) and denote by \( \Delta^o_1, \ldots, \Delta^o_k \) the \( n \)-simplices of \( M^n \) that contain any point of \( S(f) = S(g) \).

We are now going to construct an oriented curve \( \gamma \) on \( M^n \) that intersects each \( n \)-simplex \( \Delta^{n-1} \) of \( M^n \) in the positive direction (according to \( O_{\Delta^{n-1}} \)) in exactly \(|o(df, dg)|\) points. Notice that if \( \Delta^{n-1} \subset \partial\Delta^o_i \) for some \( 1 \leq i \leq k \) then on \( \Delta^{n-1} \) we have \( o(df, dg) = 0 \) since \( f|\Delta^o_i = g|\Delta^o_i \). Thus \( \gamma \) avoids the simplices \( \Delta^o_i \). This may be thought of as the dual of the obstruction to deform \( f \) to \( g \) on the \((n-1)\)-skeleton of \( M^n \).

To obtain \( \gamma \) choose a set of \(|o(df, dg)|\) points \( P_{\Delta^{n-1}} \) on each \((n-1)\)-simplex of \( M^n \). Now take an \( n \)-simplex \( \Delta^n \) and denote its faces by \( \Delta^o_i, \ldots, \Delta^o_k \), moreover let \((f_i, f'_i) = (f, df)|\Delta^o_i, (g_i, g'_i) = (g, dg)|\Delta^o_i, O_i = O_{\Delta^o_i}, \) and \( P_i = P_{\Delta^o_i} \) for \( 0 \leq i \leq n \). Lemma 3.25 implies that if each \( \Delta^o_i \) is co-oriented by the outward normal vector of \( \Delta^n \) (denote this co-orientation by \( U_i \)) then
\[
\sum_{i=0}^n o(f'_i, g'_i) = 0.
\]

If we consider \( \Delta^o_i \) with the co-orientation \( U_i \) then \( o(f'_i, g'_i) < 0 \) implies that \( U_i = O_i \) and \( o(f'_i, g'_i) > 0 \) implies that \( U_i = -O_i \). Give a minus sign to each point of \( P_i \) if \( U_i = O_i \) and a plus sign otherwise. Then equation (3.1) is equivalent to the statement that the sum of the signs in \( \bigcup_{i=0}^n P_i \) equals \( 0 \). Now let \( \gamma \cap \Delta^n \) be given as follows: Make a bijection between the + and - points of \( \bigcup_{i=0}^n P_i \) and connect each pair of points with an embedded curve segment oriented from + to -. We do this so that these curve segments are pairwise disjoint. This is possible since \( n \geq 3 \).

Doing this for each \( n \)-simplex we obtain an oriented embedded curve \( \gamma \) with the required intersection property. Now we make \( \gamma \) connected by taking the connected sum of its components: Let for example \( \gamma_1 \) and \( \gamma_2 \) be two components of \( \gamma \). Then choose two points \( p_1 \in \gamma_1 \) and \( p_2 \in \gamma_2 \). Join \( p_1 \) and \( p_2 \) with an embedded curve \( \eta \) that avoids \( \Delta^n \) for \( 1 \leq i \leq k \) and also each \((n-2)\)-simplex and intersects each \((n-1)\)-simplex transversally. Then take two parallel curves \( \eta_1 \) and \( \eta_2 \) close to \( \eta \) and orient them according to the orientations of \( \gamma_1 \) and \( \gamma_2 \). If \( \eta \) intersects an \((n-1)\)-simplex \( \Delta^{n-1} \) at a point \( x \) then \( \eta_1 \) and \( \eta_2 \) will intersect \( \Delta^{n-1} \) in different directions near \( x \).
Similarly, we might modify \( \gamma \) so that it goes through exactly one point \( p \) of \( S(f) \), and let the Whitney-umbrella of \( f \) at \( p \) be positive if \( n \) is odd. To do this, choose a small embedded curve containing \( p \) and join it to \( \gamma \) as above. We will also call this modified curve \( \gamma \). Let us suppose that the \( n \)-simplex containing \( p \) is \( \Delta^n \).

Thus it will hold true that the connected embedded curve \( \gamma \) intersects algebraically each \((n-1)\)-simplex \( \Delta^{n-1} \) in \(|o(df, dg)|\) points if we consider \( \Delta^{n-1} \) with the co-orientation \( O_{\Delta^{n-1}} \). Moreover, \( \gamma \) contains the cross-cap point \( p \in \Delta^n \) and is disjoint from \( \Delta^n \) for \( 1 < i < k \).

Let \( \nu_\gamma \) be a thin tubular neighborhood of \( \gamma \) in \( M^n \). We are going to construct a diffeotopy \( \{ \Psi_t : 0 \leq t \leq 1 \} \) of \( \nu_\gamma \), such that \( \Psi_0 = id_{\nu_\gamma} \) and for every \( t \in [0,1] \) the diffeomorphism \( \Psi_t \) is the identity in a neighborhood of \( \partial \nu_\gamma \). Thus we can extend \( \Psi_t \) to the whole manifold \( M^n \) to be the identity outside \( \nu_\gamma \).

The diffeomorphism \( \Psi_t \) is constructed as follows: Denote by \( T \) the identity of \( \mathbb{R}^{n-1} \) if \( \gamma \) is orientation preserving, and let \( T \) be a reflection in a hyperplane of \( \mathbb{R}^{n-1} \) if \( \gamma \) is orientation reversing. Then \( \nu_\gamma \) is diffeomorphic to the factor space

\[
\Gamma = \mathbb{R} \times D_2^{n-1} / (x,y) \sim (x+1,T(y))
\]

where \( D_2^{n-1} \subset \mathbb{R}^{n-1} \) is the disc of radius 2. Denote by \( p : \mathbb{R} \times D_2^{n-1} \rightarrow \Gamma \) the projection. Let \( \lambda : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^\infty \) function such that \( \lambda(x) = 0 \) for \( x < \varepsilon \) and \( \lambda(x) = 1 \) for \( x > 1 - \varepsilon \) (where \( \varepsilon < 1 \) is a small positive constant). First we define a diffeotopy \( \{ \phi_t : 0 \leq t \leq 1 \} \) of \( \mathbb{R} \times D_2^{n-1} \) with the formula

\[
\phi_t(x,y) = \begin{cases} 
(x+t,y) & \text{if } y \leq 1, \\
(x+\lambda(s)t,y) & \text{if } y = 2 - s.
\end{cases}
\]

\( (x,y) \sim (x+1,T(y)) \) implies that \( (x+c,y) \sim (x+c+1,T(y)) \) for any number \( c \). Thus the diffeomorphism \( \phi_t \) factors through the projection \( p \) to a diffeomorphism \( \Psi_t \) of \( \Gamma \approx \nu_\gamma \). I.e., the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{R} \times D_2^{n-1} & \xrightarrow{\phi_t} & \mathbb{R} \times D_2^{n-1} \\
\downarrow p & & \downarrow p \\
\nu_\gamma & \xrightarrow{\Psi_t} & \nu_\gamma.
\end{array}
\]

Since \( \Psi_1 \) is the identity on \( p(\mathbb{R} \times D_1^{n-1}) \), the diffeomorphism \( \Psi_1 \) is the identity on a thinner tubular neighborhood of \( \gamma \). Thus \( \Psi_1(p) = p \). Moreover, let \( F \) denote a fiber of \( \nu_\gamma \) (diffeomorphic to \( D^{n-1} \)) with a normal framing \( v \) (in \( M^n \)) in the direction determined by \( \gamma \) and \( S \) a small sphere around \( p \) contained in \( \nu_\gamma \) with the outward normal framing \( r \). Then the framed submanifold \( (\Psi_1(F), d\Psi_1(v)) \) is equal to the connected sum \( (F,v)\#(S,r) \). On the other hand, the framed submanifold \( (\Psi_1(F), d\Psi_1(-v)) = (F,-v)\#(S,-r) \) (see Figure 2).

Now look at the regular homotopy \( f \circ \Psi_1 \) connecting \( f \) with \( h = f \circ \Psi_1 \). Then \( S(h) = S(f) \), moreover \( S(h) = S(\pm(f)) \) if \( n \) is odd. We also get that

\[
(h, dh)(F) = ((df, df)\#(F)) = (df, df)\#S.
\]

Notice that \( (df, df)\#S \) is the \( M^n \)-immersion \( (w, w') \) of Notation 2.24 and if we co-orient \( S \) by \( r \) then \( \text{lk} \circ \chi(w,w') = 1 \) since the sign of \( p \) is 1. Thus if we co-orient \( F \) by \( v \), as above, then Lemma 3.22 gives that \( o(dh,F, df|F) = 1 \). On the other hand, if we co-orient \( F \) by \( -v \) then \( o(dh,F, df|F) = -1 \) (see Figure 2). So if we take an
(n − 1)-simplex $\Delta^{n-1}$ of $M^n$ then $o(\partial h|\Delta^{n-1}, df|\Delta^{n-1})$ is the algebraic intersection of $\Delta^{n-1}$ and $\gamma$. From the construction of $\gamma$ we know that

$$\Delta^{n-1} \cap \gamma = \left| o(df|\Delta^{n-1}, dg|\Delta^{n-1}) \right|.$$ 

Thus on $\Delta^{n-1}$

$$o(\partial h, dg) = o(\partial h, df) + o(df, dg) = |o(df, dg)| + o(df, dg) = 0,$$

since the co-orientation $O_{\Delta^{n-1}}$ was chosen so that $o(df, dg) \leq 0$.

Consequently, there is no obstruction to finding an $M^n$-regular homotopy between $h$ and $g$ on the $(n-1)$-skeleton of $M^n$. Since $f$ is regularly homotopic to $h$, we suppose from now on that $f = h$. Therefore we can suppose that the maps $f$ and $g$ coincide on (a neighborhood of) $\text{sk}_{n-1}(M^n)$.

3.5. The second obstruction. Let us recall that $\Delta_1^n, \ldots, \Delta_k^n$ denote those $n$-simplices of $M^n$ which contain a singular point of $f$, where $k = 2l$ is even. (Each $\text{int}(\Delta_i^n)$ contains a single Whitney umbrella point. If $n$ is odd then $\Delta_{2i-1}^n, \ldots, \Delta_{2i}^n$ contain a positive Whitney umbrella, $\Delta_{2i+1}^n, \ldots, \Delta_{2i+2}^n$ a negative one for both $f$ and $g$. Moreover, $f|\Delta_i^n = g|\Delta_i^n$ for $1 \leq i \leq k$.) Let

$$U = \bigcup_{i=1}^k \text{int}(\Delta_i^n).$$

Take the manifold with boundary $N^n = M^n \setminus U$ and let $\hat{f} = f|N^n$ and $\hat{g} = g|N^n$. We know from the previous section that $\hat{f}$ and $\hat{g}$ are regularly homotopic on $\text{sk}_{n-1}(N^n)$. The secondary obstruction to finding a regular homotopy between $\hat{f}$ and $\hat{g}$, not fixing the boundary, lies in the cohomology group $H^n(N^n; \pi_n(V_{2n-1,n}))$ with twisted coefficients. However, this group is 0 since an $n$-manifold without closed components is homotopy equivalent to an $(n-1)$-dimensional simplicial complex.

So there exists a regular homotopy connecting $\hat{f}$ with $\hat{g}$, but not necessarily fixing $\partial N^n$. Using Smale’s lemma we can extend this regular homotopy to the whole manifold $M^n$, fixing a small neighborhood $V (\subset U)$ of $S(f) = S(g)$. (During this regular homotopy on the ”collar” $U \setminus V$ the map $f$ might become twisted, i.e., it may differ from $g$.)

Hence we might suppose that $f|N^n = g|N^n$ and $f|V = g|V$, but it might happen that $f|(U \setminus V) \neq g|(U \setminus V)$. If $n$ is odd then $|S_+(f)| = |S_+(g)| = l$, let us suppose...
that
\[ S_+(f) = S_+(g) \subset \bigcup_{1 \leq i \leq t} \Delta^n_i, \]
while
\[ S_-(f) = S_-(g) \subset \bigcup_{1+1 \leq i \leq k} \Delta^n_i. \]

Since up to this point we have not used the assumption \( n > 3 \), we can formulate and prove the following result that is weaker than Theorem [17] and will not be used in its proof, but will be needed in dealing with the exceptional case \( n = 3 \).

**Theorem 3.31.** Suppose that \( n > 2 \) and let \( M^n \) be a closed \( n \)-manifold. Moreover, let \( f, g \in L'(M^n, \mathbb{R}^{2n-1}) \) be two locally generic maps that satisfy \( |S(f)| = |S(g)| \).

Then there exists a map \( f' \in L'(M^n, \mathbb{R}^{2n-1}) \) and a subset \( D \subset M^n \) diffeomorphic to the disk \( D^n \) such that \( f \sim f' \), the maps \( f'|D \) and \( g|D \) are immersions, and \( f'|((M^n \setminus D) \cup f|D) = g((M^n \setminus D) \cup g|D) \).

**Proof.** Let \( U \) and \( V \) be as above (where \( S(g) \subset V \subset U \)). Let \( f \) be regularly homotopic to a map \( f' \) such that \( f'|((M^n \setminus U) = g((M^n \setminus U) \cup V \setminus f|D) \cup g|D) \). We can also suppose that \( V_i = \Delta^n_i \cap V \) is diffeomorphic to an \( n \)-disc for \( 1 \leq i \leq k \).

Let us take a small neighborhood \( U_i \) of \( \Delta^n_i \) that is diffeomorphic to \( D^n \). From now on we will denote by \( U \) the union \( \bigcup_{k=1}^k U_i \). For each \( i \) choose a 1-simplex \( e_i \) connecting a point of \( \partial U_i \) and a point of \( \partial V_i \). Let \( E_i \) be a thin tubular neighborhood of \( e_i \). Then \( D_i = U_i \setminus (V_i \cup E_i) \) is diffeomorphic to \( D^n \).

Since \( n > 2 \) there is no obstruction to constructing a regular homotopy between \( f'|E_i \) and \( g|E_i \) fixing the ends of the tube \( E_i \). Thus (using Smale’s lemma) we can suppose that \( f'|E_i = g|E_i \) for \( 1 \leq i \leq k \). Now we connect \( D_i \) and \( D_{i+1} \) for \( 1 \leq i \leq k-1 \) with a tube \( T_i \subset M^n \setminus U \) diffeomorphic to \( I \times D^{n-1} \). (Thus each \( T_i \) is an \( n \)-dimensional 1-handle attached to \( D_i \) and \( D_{i+1} \).) Let
\[ D = \left( \bigcup_{i=1}^k D_i \right) \cup \left( \bigcup_{j=1}^{k-1} T_j \right). \]

Then \( D \) is diffeomorphic to \( D^n \) and clearly \( f'|D \) and \( g|D \) are immersions, finally \( f'|((M^n \setminus D) \cup g|D) = g((M^n \setminus D) \cup f|D) \). \( \square \)

Let \( F_1, \ldots, F_l \) be pairwise disjoint subsets of \( M^n \) such that \( F_i \) is diffeomorphic to the closed \( n \)-disc \( D^n \), moreover \( F_i \) contains \( \Delta^n_i \) and \( \Delta^n_{i+1} \) in its interior for \( 1 \leq i \leq l \). We will show that for \( 1 \leq i \leq l \) the locally generic maps \( f_i = f|F_i \) and \( g_i = g|F_i \) are regularly homotopic keeping a neighborhood of \( \partial F_i \) fixed. This would finish the proof of Theorem [17]. From now on let \( 1 \leq i \leq l \) be fixed.

It is apparent from the choice of \( F_i \) that \( S(f_i) = S(g_i) \) and \( |S(f_i)| = |S(g_i)| = 2 \). Moreover, if \( n \) is odd we have that \( S_+(f_i) = S_+(g_i) \) and \( S_-(f_i) = S_-(g_i) \) are both 1-element sets. Thus \( \#S(f_i) = 0 \) and \( \#S(g_i) = 0 \). Lemma [2.14] implies that \( \mathcal{L}_{f_i}(F_i) = 0 \) and \( \mathcal{L}_{g_i}(F_i) = 0 \). Using Lemma [2.16] we get that
\[ 0 = \mathcal{L}_{f_i}(F_i) = \text{lk} \circ \chi((f_i, df_i)|\partial F_i) = \alpha \circ \tau((f_i, df_i)|\partial F_i). \]

Since \( \alpha \) is an isomorphism, \( \tau((f_i, df_i)|\partial F_i) = 0 \) and from Lemma [2.14] we can see that the map \( (f_i, df_i)|\partial F_i \) is extendible. Similarly, \( (g_i, dg_i)|\partial F_i \) is also extendible. (This also follows from Theorem [2.8].)
Let $S^n_+$ (respectively $S^n_-$) denote the northern (respectively southern) hemisphere of $S^n$ and we identify $S^n_+ + F_i$. Then we get from the previous paragraph that there exist two locally generic maps $f'_i, g'_i ∈ L(S^n, ℝ^{2n-1})$ such that $f'_i|S^n_+ = f_i$, $g'_i|S^n_+ = g_i$, moreover $f'_i|S^n_- = g'_i|S^n_-$ is an immersion. Thus it is sufficient to prove the following theorem:

**Theorem 3.32.** Let $f, g ∈ L(S^n, ℝ^{2n-1})$ be locally generic maps and suppose that $f|S^n_+ = g|S^n_+$ are immersions, $|S(f)| = |S(g)| = 2$ and $S(f) = S(g)$, moreover $S_+(f) = S_+(g)$ if $n$ is odd. Then there exists a singularity fixing regular homotopy connecting $f$ and $g$ that is fixed on $S^n_+$, i.e., $f ∼_s g$ (rel $S^n_+$).

We will use the following lemma in the proof of Theorem 3.32.

**Lemma 3.33.** Let $f, g: S^n → ℝ^{2n-1}$ be locally generic maps such that $S(f) = S(g) ⊂ int(S^n_+)$ and $f|S^n_- = g|S^n_-$. Moreover, a singularity fixing regular homotopy connecting $f$ and $g$ can be extended to $S^n$.

Proof. Choose a point $x ∈ ∂S^n_+$ and denote the differential $(df)_x = (dg)_x$ by $j$. Let $S = S(f) = S(g)$ and

$$B = \{ h ∈ Imm(S^n_+, ℝ^{2n-1}): (dh)_x = j \},$$

moreover

$$E = \{ h ∈ L(S^n, ℝ^{2n-1}): S(h) = S \text{ and } (dh)_x = j \}. $$

We define a map $p: E → B$ by the formula $p(h) = h|S^n_-$ for $h ∈ E$.

We will now show that $p$ has the covering homotopy property. For this end choose a subset $B ⊂ S^n_+$ such that $B$ is diffeomorphic to an open $n$-disc centered at the North pole $N$ of $S^n$, moreover $S ⊂ B$. Denote by $R$ the intersection of $S^n_+ \setminus B$ with the geodesic arc connecting $N$ and $x$. Let $K = S^n_+ \setminus (B ∪ R)$, then $int(K)$ is diffeomorphic to an open $n$-disc. Let $P$ be a polyhedron, $ψ: P × I → B$ and $ϕ_0: P × \{0\} → E$, such that $ψ(P × \{0\}) = p ∘ ϕ_0$. We must extend $ϕ_0$ to $ϕ: P × I → E$ so that $ψ = p ∘ ϕ$, i.e., the following diagram is commutative:

$$
\begin{array}{cc}
P × I & \xrightarrow{ϕ} & E \\
\downarrow{id} & & \downarrow{p} \\
P × I & \xrightarrow{ψ} & B.
\end{array}
$$

For $r ∈ P$ and $t ∈ I$ we define $ϕ(r, t)|[(B ∪ R)]$ to be equal to $ϕ_0(r)|[(B ∪ R)]$. Moreover, let $(dϕ(r, t))|R = (dϕ_0(r))|R$. This is possible since $(B ∪ R) ∩ S^n_+ = \{x\}$ and $(dψ(r, t))_x = j$ for every $r ∈ P$ and $t ∈ I$. The map $ϕ_0(r)|K$ is an immersion for every $r ∈ P$ and $ϕ(r, t)$ is already defined (together with normal derivatives) on $∂K$ for every $r ∈ P$ and $t ∈ I$. Thus using Smale’s lemma (Theorem 2.12) $ϕ(r, t)$ can be extended to $K$ as an immersion for every $r ∈ P$ and $t ∈ I$.

It is well known that the space $B$ is contractible, thus its homotopy groups are all trivial. Take the fiber of $p$ defined by $F = p^{-1}(f|S^n_+)$, then by assumption $f, g ∈ F$. Denote by $i$ the inclusion of $F$ into $E$. Let us look at the following part of the homotopy exact sequence of the fibration $p: E → F → B$:

$$π_1(B) → π_1(F) i_∗ → π_0(E) p_∗ → π_0(B).$$

Since $π_1(B) = 0$ and $π_0(B) = 0$, the morphism $i_*$ is an isomorphism. By assumption $f ∼_s g$, moreover, a singularity fixing regular homotopy $H$ connecting $f$ and $g$ can be chosen so that $(dH_t)_x = j$ for every $t ∈ [0, 1]$. Thus $i(f)$ and $i(g)$ lie in the same
path-component of $E$, i.e., they represent the same element of $\pi_0(E)$. Using the fact that $i_*$ is a monomorphism we get that $f$ and $g$ lie in the same path-component of $F$. This means that there exists a singularity fixing regular homotopy connecting $f$ and $g$ that is fixed on $S^n$. □

3.6. Merging double curves. The idea of the procedure of eliminating the second obstruction (and achieving the coincidence of $f$ and $g$ – after a regular homotopy – also on the last part of $M^n$, i.e., on $U$) is as follows: In [2] Ekholm showed that the regular homotopy class of an immersion $S^n \to \mathbb{R}^{2n-1}$ is completely determined by the behavior of the map in a small neighborhood of the double curves if $n \geq 4$. We are going to reduce the problem to this result of Ekholm. First we make $f$ and $g$ coincide in a neighborhood of their singular sets. Then using Theorem 2.36 we will get to the case of the immersions of Ekholm’s theorem. To manipulate the double curves of $f$ and $g$ we will use the following construction.

In the presence of Whitney umbrella singular points we can merge the closed double curves with the double curves connecting the singular points. The merging is done by using the following construction motivated by the Whitney-trick (for a reference see [5] or [9]). (Let us recall that in the Whitney trick one defines a standard model and then one shows that it can be embedded into the manifold under consideration.)

The construction. In our case the standard model $S$ is chosen to be the standard model of the Whitney-trick (two arcs in $D^2$ that intersect in two points) times the interval $[-1,1]$ (see the left side of Figure 3). Thus we have two surfaces $P_1$ and $P_2$ in $D^2 \times [-1,1]$, both diffeomorphic to the square, that intersect in two line segments $l_1$ and $l_2$. The regular homotopy in this model is the identity near $\partial(D^2 \times [-1,1])$ and is the isotopy of the standard model of the Whitney trick ($l_2$ is pushed above $l_1$) near $D^2 \times \{0\}$. As a result of this deformation the double points are removed near $D^2 \times \{0\}$, and thus the double curves of the final arrangement coincide with the connected sum $l_1 \# l_2$ (see the right side of Figure 3).

Let $p$ and $q$ be two double points of the generic map $f: N^n \to \mathbb{R}^{2n-1}$, where $N^n$ is any compact connected manifold with boundary. All we have to do is to embed $D^2 \times [-1,1] \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$ into $\mathbb{R}^{2n-1}$ so that $P_1 \times \mathbb{R}^{n-2} \times \{0\}$ and $P_2 \times \{0\} \times \mathbb{R}^{n-2}$ map to $\text{Im}(f)$, moreover $l_1 \times \{0\}$ maps on the double curve through $p$ and $l_2 \times \{0\}$ maps on the double curve through $q$.

Let $f^{-1}(p) = \{p_1, p_2\}$ and $f^{-1}(q) = \{q_1, q_2\}$. Choose an embedded curve $c_i$ in $N^n$ connecting $p_i$ and $q_i$ and let $C_i = f(c_i)$ for $i = 1, 2$. We might suppose that

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The standard model}
\end{figure}
$C_1 \cap C_2 = \{p, q\}$ and no other double or singular point of $f$ lies on $c_1$ and $c_2$. Thus $f|c_1$ and $f|c_2$ are embeddings. Let $D_1$ be an embedded (tubular) neighborhood of $C_1$ in $f(N^n)$ diffeomorphic to $D^n$ for $i = 1, 2$. The intersection $D_1 \cap D_2 = L_1 \sqcup L_2$, where $p \in L_1$ and $q \in L_2$.

Next we choose a Whitney disk $W$ containing $C_1$ and $C_2$ as in [5]. Since $\dim(\text{Im}(f)) + \dim(W) = n + 2 < 2n - 1$ (because $n \geq 4$), if $W$ is in general position then $W \cap \text{Im}(f) = C_1 \cup C_2$. We might also suppose that $W$ is transversal to $L_1$ and $L_2$. Since from now on we will work only in a neighborhood of $W$, we might restrict our attention to $D_1 \cup D_2$ and forget about the rest of $\text{Im}(f)$.

Construct a Riemannian metric on $\mathbb{R}^{2n-1}$ for which $D_1 \perp D_2$. Choose an orientation of $D_1$ and a co-orientation of $D_2$ in $\mathbb{R}^{2n-1}$. Let $\alpha_1, \ldots, \alpha_{n-1} \in \nu_p(D_2 \subset \mathbb{R}^{2n-1}) \subset T_p D_1$ be a positive normal frame of $D_2$. Let $\tilde{\nu}(p) \in T_p L_1$ be taken so that $\tilde{\nu}(p), \alpha_1, \ldots, \alpha_{n-1}$ is a positive basis of $T_p D_1$. Similarly, let $\beta_1, \ldots, \beta_{n-1} \in \nu_q(D_2 \subset \mathbb{R}^{2n-1}) \subset T_q D_1$ be a positive normal frame of $D_2$ and $\tilde{\nu}(q) \in T_q L_2$ be chosen so that $\tilde{\nu}(q), \beta_1, \ldots, \beta_{n-1}$ is a negative basis of $T_q D_2$.

We can extend $\tilde{\nu}$ to $C_1 \cup C_2$ so that $\tilde{\nu}|C_i$ is tangent to $D_i$ since $n \geq 3$. Next we extend $\nu$ from $C_1 \cup C_2$ to a normal field of $W$; Since $\nu(W \subset \mathbb{R}^{2n-1})$ is the trivial bundle and $C_1 \cup C_2$ is homotopy equivalent to $S^1$, the obstruction to extend $\tilde{\nu}$ is an element of $\pi_1(V_{2n-3,1}) = \pi_1(S^{2n-4})$. But this group is 0 since $2n - 4 > 1$ if $n \geq 3$. Taking the exponential map along $\tilde{\nu}$ on $W$ we can embed the standard model $S = D^2 \times [-1, 1]$ into $\mathbb{R}^{2n-1}$. Denote the image of $S$ by $U$.

Orient $C_1$ from $p$ to $q$ and denote by $\tau(x)$ the positive unit tangent vector of $C_1$ at $x$. Moreover, let $\nu(p) = \tau(p)$ and $\nu(q) = -\tau(q)$. Since $n \geq 4$ we can extend $\nu$ along $C_2$ as a normal field of $D_2$.

Choose vectors $\xi_1(p), \ldots, \xi_{n-2}(p)$ in $T_p D_1$ so that $\tilde{\nu}(p), \nu(p), \xi_1(p), \ldots, \xi_{n-2}(p)$ is a positive basis of $T_p D_1$. Using parallel translation extend $\xi_1, \ldots, \xi_{n-2}$ to $C_1$. Then $\tilde{\nu}(q), \nu(q), \xi_1(q), \ldots, \xi_{n-2}(q) \in T_q D_1$ is a negative basis since $\nu(q) = -\tau(q)$. Thus from the definition of $\tilde{\nu}(q)$ we get that $\nu(q), \xi_1(q), \ldots, \xi_{n-2}(q)$ is a positive basis of $\nu_q(D_2 \subset \mathbb{R}^{2n-1})$. Moreover, $\nu(p), \xi_1(p), \ldots, \xi_{n-2}(p)$ is a positive basis of $\nu_p(D_2 \subset \mathbb{R}^{2n-1})$ and $\nu|C_2$ is a continuous vector field. All this together implies that the $(n-2)$-frames $\xi(p)$ and $\xi(q)$ can be extended onto $C_2$ to be transversal to $D_2$ and $\nu$.

We can extend the frame $\xi$ to $U$: The pair $(U, C_1 \cup C_2)$ is homotopy equivalent (deformation retracts onto) $(\overline{D}^2, S^1)$. Thus the obstruction lies in $\pi_1(V_{2n-4,n-2}) = 0$ since $n \geq 4$ implies that $1 < 2(n - 4) - (n - 2)$.

Finally we construct an $(n-2)$-frame $\eta$ on $U$ that is tangent to $D_2$ on $C_2$. Look at the bundle of orthogonal $(n-2)$-frames on $U$ that are perpendicular to $U$ and $\xi$. Since $U$ is contractible, this bundle is trivial; let $\eta$ be any section. Notice that $\nu(U \cap D_2 \subset U)$ is spanned by the vector field $\nu|C_2$ (The subset $U \cap D_2$ corresponds to $P_2$ in the standard model $S$). Because $\eta|C_2 \perp \xi, U$, we get that $\eta|C_2 \perp (\xi, \nu|C_2) = \nu(D_2 \subset \mathbb{R}^{2n-1})|C_2$.

Thus $\eta|C_2$ is tangent to $D_2$, as required.
Using the exponential map on the fields $\xi$ and $\eta$ along $U$ we can embed $S \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$ into $\mathbb{R}^{2n-1}$ as described above. $\square$

The regular homotopy constructed above is clearly singularity fixing.

3.7. The proof of Theorem 3.32. Using Lemma 3.33 we only have to prove that $f \sim g$. With a singularity fixing regular homotopy we can perturb $f$ and $g$ to obtain generic maps. Let us recall (see the beginning of the previous section) that the main idea of the proof is to reduce the problem to the case of immersions and then apply Ekholm’s theorem.

Let $l_f$ (respectively $l_g$) denote the closure of the double curve of $f$ (respectively $g$) that connects the two singular points of $f$ (respectively $g$). With the aid of the above construction we merge using a singularity fixing regular homotopy all closed double curves of $f$ (respectively $g$) with $l_f$ (respectively $l_g$). From now on we will suppose that $l_f$ is the only double curve of $f$ and $l_g$ is the only double curve of $g$. Let $m_f = f^{-1}(l_f)$ and $m_g = g^{-1}(l_g)$. Since $n \geq 4$ there exists a diffeotopy $\{d_t : t \in [0, 1]\}$ of $S^n$ that fixes $S(f) = S(g)$ and takes $m_g$ to $m_f$. Composing $f$ with $\{d_t\}$ we can suppose that $m_f = m_g$. Let $m = m_f = m_g$ and $S(f) = S(g) = \{p, q\}$. If $n$ is even we have to be careful when choosing the diffeotopy $\{d_t\}$: If we order the two components of $m \setminus \{p, q\}$ then this defines signs of the singular points of $f$ and $g$ (see Definition 3.27), and it might happen that $S_+(f) \neq S_+(g)$ (the equality $\#S(f) = \#S(g) = 0$ still holds). In this case we compose $f$ with a diffeotopy of $S^n$ that fixes $p$ and $q$ and swaps the two components of $m \setminus \{p, q\}$ in order to swap $S_+(f)$ and $S_+(f)$. This is possible if $n \geq 4$. Thus if $n$ is even we also fix an ordering of the components of $m \setminus \{p, q\}$ and suppose that for the induced signs $S_\pm(f) = S_\pm(g)$.

Next we compose $f$ with a diffeotopy of $\mathbb{R}^{2n-1}$ that moves $l_f$ to $l_g$. Thus we can suppose that $f|m = g|m$ and let $l = l_f = l_g$. Using Lemma 2.14 of [1] we can straighten $f$ and $g$ close to their self intersection int($l$), i.e., we can achieve that they agree with their normal derivatives in a neighborhood of $m$.

We now sketch the idea of the construction that follows: The singular points of $f$ and $g$ coincide and have the same signs if $n$ is odd. The sign is the only local isotopy invariant a Whitney-umbrella singularity can have. Moreover, any two bundles over an interval are equivalent. Since $m$ is an orientation preserving curve in $S^n$ the two cross-caps at the end of $l_f$ and $l_g$ are in the same “relative position”. So $f$ and $g$ are isotopic if restricted to a small neighborhood of $m$, fixing $S(f) = S(g)$. This isotopy can be extended to an ambient isotopy of $\mathbb{R}^{2n-1}$. So we might suppose that $f$ and $g$ agree in a small open tubular neighborhood $V$ of their common double locus $m$. The exact details are as follows:

Since $S^n$ is orientable, the normal bundle $\nu(m \subset S^n)$ is trivial. Thus we can choose a normal framing $\nu$ of $m$ in $S^n$. Notice that $\ker(df_p) = \ker(dg_p) = T_p(m)$ and $\ker(df_q) = \ker(dg_q) = T_q(m)$, so $df(\nu_p)$ and $dg(\nu_q)$ are $(n-1)$-frames in $T_{f(p)}\mathbb{R}^{2n-1} = T_{g(p)}\mathbb{R}^{2n-1}$. Similarly, $df(\nu_q)$ and $dg(\nu_p)$ are $(n-1)$-frames in $T_{f(q)}\mathbb{R}^{2n-1} = T_{g(q)}\mathbb{R}^{2n-1}$. Moreover, $\nu_p = df(\nu(m \setminus \{p, q\})$ and $\nu_q = dg(\nu(m \setminus \{p, q\}))$ provide $(2n-2)$-frames along int($l$).

Suppose that $S_+(f) = S_+(g) = \{p\}$ and $S_-(f) = S_-(g) = \{q\}$. Let $a = f(p) = g(p)$ and $b = f(q) = g(q)$, then $a$ and $b$ are the endpoints of the curve $l$. Orient $l$ from $a$ to $b$, this induces an orientation of $\nu(l \subset \mathbb{R}^{2n-1})$. (Recall that we fixed an orientation of $\mathbb{R}^{2n-1}$ to define the signs of the Whitney-umbrellas.) Then
from Definition 2.37 of the signs of the singular points we can see that \( \{ \nu_f, \nu_g \} \) provides a negative basis of \( \mu = \nu \left( \text{int}(l) \subset \mathbb{R}^{2n-1} \right) \). Thus there exists a homotopy \( T: \text{int}(l) \times I \to GL_+(2n-2, \mathbb{R}) \) such that for every \( x \in \text{int}(l) \) the transformation \( T(x, 0) = \text{id}_{\mathbb{R}^{2n-2}} \) and \( (\nu_f)_x T(x, 1) = (\nu_g)_x \). The homotopy \( T \) can be extended to an ambient isotopy of \( f \) that takes \( f \) to \( g \) on a small open tubular neighborhood \( V \) of \( m \), and this isotopy is identical outside a neighborhood of \( l \). Thus we can suppose that \( f[V] = g[V] \).

Let \( D \subset V \) be diffeomorphic to the closed \( n \)-disc \( D^n \), so that \( p, q \in \text{int}(D) \) and \( f|\partial D = g|\partial D \) is an embedding. Since the only double locus of \( f \) and \( g \) is \( m \subset V \), the distances \( d_1 = d(f(D), f(S^n \setminus V)) \) and \( d_2 = d(g(D), g(S^n \setminus V)) \) are positive. Let \( 0 < \varepsilon < \min(d_1, d_2) \). Using Theorem 2.36 choose a generic immersion \( h: D \to \mathbb{R}^{2n-1} \) that agrees with \( f|\partial D \) in a neighborhood of \( \partial D \), moreover the \( C^0 \)-distance of \( h \) and \( f|D \) is \( < \varepsilon \). (This is possible since \( \# S(f[D]) = 0 \) and \( f|\partial D \) is an embedding.) Denote by \( f_1 \) (respectively \( g_1 \)) the immersion of \( S^n \) in \( \mathbb{R}^{2n-1} \) that agrees with \( f \) (respectively \( g \)) on \( S^n \setminus D \) and with \( h \) on \( D \). From the choice of \( h \) we can see that \( f(S^n \setminus V) \cap h(D) = \emptyset \) and \( g(S^n \setminus V) \cap h(D) = \emptyset \). Moreover, since \( f|(S^n \setminus V) \) and \( g|(S^n \setminus V) \) are embeddings that do not intersect \( f(V \setminus D) = g(V \setminus D) \), the double loci of \( f_1 \) and \( g_1 \) are contained in \( V \). But \( f_1[V] = g_1[V] \), thus \( f_1 \) and \( g_1 \) agree in a neighborhood of their common double locus. Theorem 1 in [2] implies that the regular homotopy class of a generic immersion \( S^n \to \mathbb{R}^{2n-1} \) can be expressed in terms of the geometry of the self intersection if \( n \geq 4 \). Thus \( f_1 \sim g_1 \).

Since \( f_1[D] = g_1[D] \), the regular homotopy connecting \( f_1 \) and \( g_1 \) can be chosen to be constant on \( D \), i.e., \( f_1 \sim g_1 \) (rel \( D \)). This gives us a regular homotopy \( f \sim g \) (rel \( D \)), thus \( f \sim_s g \).

3.8. An application. In part 4 of [4] the \( n = 2 \) case of Theorem 1.7 was applied to projections of immersions of surfaces into \( \mathbb{R}^4 \). Knowing Theorem 1.7 the same results can be generalized without modification of the proof.

Let us first recall some definitions.

**Definition 3.34.** Let

\[
\text{Imm}_\pi(M^n, \mathbb{R}^{2n}) = \{ F \in \text{Imm}(M^n, \mathbb{R}^{2n}) : \pi \circ F \in L(M^n, \mathbb{R}^{2n-1}) \}
\]

be the subspace of \( \text{Imm}(M^n, \mathbb{R}^{2n}) \) formed by those immersions whose projection in \( \mathbb{R}^{2n-1} \) is locally generic. Two immersions \( F, G \in \text{Imm}_\pi(M^n, \mathbb{R}^{2n}) \) are called \( \pi \)-homotopic (denoted by \( F \sim_\pi G \)) if they are in the same path-component of \( \text{Imm}_\pi(M^n, \mathbb{R}^{2n}) \).

The generalization of Theorem 4.7 of [4] is then the following.

**Theorem 3.35.** If \( n = 2 \) or \( n > 3 \) and \( F, G \in \text{Imm}_\pi(M^n, \mathbb{R}^{2n}) \) then

\[
F \sim_\pi G \Leftrightarrow [F \sim G \text{ and } \pi \circ F \sim \pi \circ G].
\]

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Department of Analysis, Eötvös Loránd University, Pázmány Péter sétány 1/C, Budapest, Hungary 1117

E-mail address: juhasz.68@dpf.hu