ON THE MAXIMUM DIAMETER OF $k$-COLORABLE GRAPHS

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Abstract. Erdős, Pach, Pollack and Tuza [J. Combin. Theory B 47 (1989), 279–285] conjectured that the diameter of a $K_{2r}$-free connected graph of order $n$ and minimum degree $\delta \geq 2$ is at most $2(\frac{r-1}{2r-1})(3r+2)\cdot \frac{n}{\delta} + O(1)$ for every $r \geq 2$, if $\delta$ is a multiple of $(r-1)(3r+2)$. For every $r > 1$ and $\delta \geq 2(r-1)$, we create $K_{2r}$-free graphs with minimum degree $\delta$ and diameter $\frac{(6r-5)n}{(2r-1)(2r+1)} + O(1)$, which are counterexamples to the conjecture for every $r > 1$ and $\delta > 2(r-1)(3r+2)(2r-3)$. The rest of the paper proves positive results under a stronger hypothesis, $k$-colorability, instead of being $K_{k+1}$-free. We show that the diameter of connected $k$-colorable graphs with minimum degree $\delta$ and order $n$ is at most $\left(3 - \frac{1}{k-1}\right)\frac{n}{\delta} + O(1)$, while for $k = 3$, it is at most $\frac{57n}{23}\delta + O(1)$.

1. Introduction

The following theorem was discovered several times [1, 5, 7, 8]:

Theorem 1. For a fixed minimum degree $\delta \geq 2$ and $n \to \infty$, for every $n$-vertex connected graph $G$, we have $\text{diam}(G) \leq \frac{3n}{\delta + 1} + O(1)$.

Note that the upper bound is sharp (even for $\delta$-regular graphs [2]), but the constructions have complete subgraphs whose order increases with $\delta$. Erdős, Pach, Pollack, and Tuza [5] conjectured that the upper bound in Theorem 1 can be strengthened for graphs not containing complete subgraphs:

Conjecture 1. [5] Let $r, \delta \geq 2$ be fixed integers and let $G$ be a connected graph of order $n$ and minimum degree $\delta$.

(i) If $G$ is $K_{2r}$-free and $\delta$ is a multiple of $(r-1)(3r+2)$ then, as $n \to \infty$,

$\text{diam}(G) \leq \frac{2(r-1)(3r+2)}{(2r^2-1)} \cdot \frac{n}{\delta} + O(1) = \left(3 - \frac{2}{2r-1} - \frac{1}{(2r-1)(2r^2-1)}\right)\frac{n}{\delta} + O(1)$.

(ii) If $G$ is $K_{2r+1}$-free and $\delta$ is a multiple of $3r-1$, then, as $n \to \infty$,

$\text{diam}(G) \leq \frac{3r-1}{r} \cdot \frac{n}{\delta} + O(1) = \left(3 - \frac{2}{2r}\right)\frac{n}{\delta} + O(1)$.

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Set \(k = 2r\) or \(k = 2r + 1\) according the cases. As connected \(\delta\)-regular graphs are \(K_{\delta+1}\)-free (apart from \(K_{\delta+1}\) itself), we need \(\delta \geq k\) (at least) to make improvement on Theorem 1. Furthermore, as the conjectured constants in the bounds are at most \(3 - \frac{2}{k}\), Theorem 1 implies that the conjectured inequalities hold trivially, unless \(\delta \geq \frac{3k}{2} - 1\).

Erdős et al. [5] constructed graph sequences for every \(r, \delta \geq 2\), where \(\delta\) satisfies the divisibility condition, which meet the upper bounds in Conjecture 1. We show these construction them in Section 2.

Part (ii) of Conjecture 1 for \(r = 1\) was proved in Erdős et al. [5]. Conjecture 1 is included in the book of Fan Chung and Ron Graham [6], which collected Erdős’s significant problems in graph theory.

No more progress has been reported on this conjecture, except that for \(r = 2\) in (ii), under a stronger hypothesis (4-colorable instead of \(K_5\)-free), Czabarka, Dankelman and Székely [3] arrived at the conclusion of Conjecture 1:

**Theorem 2.** For every connected 4-colorable graph \(G\) of order \(n\) and minimum degree \(\delta \geq 1\), \(\text{diam}(G) \leq \frac{5n}{2\delta} - 1\).

In Section 3 we give an unexpected counterexample for Conjecture 1(i) for every \(r \geq 2\) and \(\delta > 2(r - 1)(3r + 2)(2r - 3)\). The question whether Conjecture 1(ii) holds in the range \((r - 1)(3r + 2) \leq \delta \leq 2(r - 1)(3r + 2)(2r - 3)\) is still open. The counterexample leads to a modification of Conjecture 1, which no longer requires cases:

**Conjecture 2.** For every \(k \geq 3\) and \(\delta \geq \lceil \frac{3k}{2} \rceil - 1\), if \(G\) is a \(K_{k+1}\)-free (weaker version: \(k\)-colorable) connected graph of order \(n\) and minimum degree at least \(\delta\), \(\text{diam}(G) \leq (3 - \frac{2}{k}) \frac{n}{\delta} + O(1)\).

For \(k = 2r\), Conjecture 2 is identical to Conjecture 1(ii). For \(k = 2r - 1\), \(3 - \frac{2}{k} = \frac{6r - 5}{2r - 1}\), and, although the conjectured bound is likely not tight for any \(\delta\), the fraction \(\frac{6r - 5}{2r - 1}\) cannot be reduced for all \(\delta\) according to the construction in Section 3.

For the rest of the paper, we follow the restrictive approach of Czabarka, Dankelman and Székely [3], and work towards the weaker version of Conjecture 2. In other words, we use a stronger hypothesis (\(k\)-colorable instead of \(K_{k+1}\)-free) than what Erdős, Pach, Pollack, and Tuza [5] used. In our work towards upper bounds on the diameter, we only assume minimum degree at least \(\delta\), a weaker assumption than minimum degree \(\delta\). Section 4 shows that some \(k\)-colorable (in particular 3-colorable) connected graphs realizing the maximum diameter among such graphs with given order and minimum degree have some canonical properties. Hence at proving upper bounds on the diameter, we can assume those canonical properties.

Section 5 gives a linear programming duality approach to the maximum diameter problem. With this approach, proving upper bounds to the diameter boils down to solve a packing problem in a graph, such that a certain value is reached by the objective function. If a packing with that value is given, the task of checking whether the packing is feasible is trivial. Using this approach we obtain
Theorem 3. Assume $k \geq 3$. If $G$ is a connected $k$-colorable graph of minimum degree at least $\delta$, then

$$\text{diam}(G) \leq \frac{3k - 4}{k - 1} \cdot \frac{n}{\delta} + O(1) = \left(3 - \frac{1}{k - 1}\right) \cdot \frac{n}{\delta} + O(1).$$

This corroborates the conjecture of Erdős et al. in the sense that the maximum diameter among all graphs investigated in Theorem 3 is $\left(3 - \Theta\left(\frac{1}{k}\right)\right) \frac{n}{\delta}$. As a corollary, we arrive at the conclusion of Theorem 2, if the graph is 3-colorable (instead of 4-colorable).

Section 6 applies the inclusion-exclusion (sieve) formula to give upper bounds locally for the number of vertices in graphs with the canonical properties. In Section 7, we define a number of global variables that play a role in the diameter problem, and turn the upper bounds from Section 6 into linear constraints for the global variables. (This approach was motivated by the flag algebra method of Razborov [10].) A linear program of fixed size for the global variables arises, and solving this linear program proves our main positive result:

Theorem 4. For every connected 3-colorable graph $G$ of order $n$ and minimum degree at least $\delta \geq 1$,

$$\text{diam}(G) \leq \frac{57n}{23\delta} + O(1).$$

Note that as $\frac{57}{23} \approx 2.47826...$, this is an improvement on the $\frac{5}{2} \cdot \frac{n}{\delta} + O(1)$ upper bound for 4-colorable graphs (see Theorem 2 cited from [3]). In Theorem 11, in a restricted case we prove the weaker version of Conjecture 2 for $k = 3$.

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2. Clump Graphs and the Constructions for Conjecture 1

Let us be given a $k$-colorable connected graph $G$ of order $n$ and minimum degree at least $\delta$.

Take a fixed good $k$-coloring of $G$. Let layer $L_i$ denote the set of vertices at distance $i$ from $x$, and a clump in $L_i$ be the set of vertices in $L_i$ that have the same color. The number of layers is $\text{diam}(G) + 1$.

Let $c(i) \in \{1, 2, \ldots, k\}$ denote the number of colors used in layer $L_i$ by our fixed coloring. We can assume without loss of generality that in $G$, two vertices in layer $L_i$, which are differently colored, are joined by an edge in $G$, and also that two vertices in consecutive layers, which are differently colored, are also joined by an edge in $G$. We call this assumption saturation. Assuming saturation does not make loss of generality, as adding these edges does not decrease degrees, keeps the fixed good $k$-coloration, and does not reduce the diameter, while making the graph more structured for our convenience.

From a graph $G$ above, we create a (weighted) clump graph $H$. Vertices of $H$ correspond to the clumps of $G$. Two vertices of $H$ are connected by an edge if there were edges between the corresponding clumps in $G$. $H$ is naturally $k$-colored and layered based on the coloration and layering of $G$. With a slight abuse of notation, we denote the layers of
$H$ by $L_i$ as well. We assign as weights to each vertex of the clump graph the number of vertices in the corresponding clump in $G$.

Given a (natural number)-weighted graph $H$, it defines a graph $G$ whose weighted clump graph is $H$ by blowing up vertices of $H$ into as many copies as their weight is. The degrees in $G$ correspond to the sum of the weights of neighbors of the vertices in $H$, $	ext{diam}(G) = \text{diam}(H)$, and the number of vertices in $G$ is the sum of the weights of all vertices in $H$.

It is convenient to describe the constructions of Erdős et al. [5] in terms of clump graphs. Any two consecutive layers of the clump graphs will form a complete graph, and, as the order of these complete graphs will be at most $2r - 1$ (resp. $2r$), the graphs will be $(2r - 1)$-colorable and $K_{2r}$-free (resp. $2r$-colorable and $K_{2r+1}$-free).

For the construction for $K_{2r}$-free graphs, when $\delta$ is a multiple of $(r - 1)(3r + 2)$: Layer $L_0$ and has one clump, for $1 \leq i \leq D$, for every odd $i$, layer $L_i$ has $r$ clumps, and for every even $i$, layer $L_i$ has $r - 1$ clumps The clump in $L_0$ gets weight $1$, and for $3 \leq i \leq D - 1$, for every odd $i$, the clumps in layer $L_i$ get and for $2 \leq i \leq D - 1$, for every even $i$, the clumps in layer $L_i$ get of weight $\frac{(r-1)\delta}{(r-1)(3r+2)}$. Use weight $\delta$ for clumps in $L_1$ and $L_D$. (In case of $r = 1$ and even $D$, use weight $\delta$ in the clumps of $L_{D-1}$ as well.)

For the construction for $K_{2r+1}$-free graphs, when $\delta$ is a multiple of $3r - 1$: Layer $L_0$ has one clump, $1 \leq i \leq D$, layer $L_i$ has $r$ clumps. The clump in $L_0$ gets weight $1$, and clumps in layers $L_i$ for $2 \leq i \leq D - 1$ get weight $\frac{1}{3r-1}$. Use weight $\delta$ for clumps in $L_1$ and $L_D$.

The diameters of these constructions obviously meet the upper bounds of Conjecture [4] within a constant term that depends on $r$.

3. COUNTEREXAMPLES

We will make use of a clump graph to create a $(2r - 1)$-colorable (and hence $K_{2r}$-free) graphs with minimum degree $\delta$ for every $r \geq 2$ that refute Conjecture [4][5].

To make our quantities slightly more palatable in the description, we make the shift $s = r - 1$, and work with $(2s + 1)$-colorable graphs for $s \geq 1$.

For positive integers $p, s$, and $\delta \geq 2s$, we will create a weighted clump graph $H_{s,\delta,p}$ with $p(6s + 1)$ layers, such that the number of vertices in two consecutive layers is at most $2s + 1$, each vertex is adjacent to all other vertices in its own layer and in the layers immediately before and after it. The layer structure of $H_{s,\delta,p}$ is basically periodic, up to a tiny modification in the weights. We are going to define a symmetric block $C_{s,\delta}$ of $6s + 1$ layers, and $H_{s,\delta,p}$ is the juxtaposition of $p$ copies of $C_{s,\delta}$, with the modification of increasing by $1$ the weight of one vertex in the second layer $L_1$ and one vertex in the next-to-last layer $L_{p(6s+1)−1}$.

Let $0 \leq d \leq 2s - 1$ be the remainder, when we divide $\delta$ by $2s$. We define $C_{s,\delta}$ by the number of points and their weights in the layers $L_m$ for $0 \leq m \leq 3s + 1$ as detailed below; for $3s + 2 \leq m \leq 6s$, $L_m$ and the weights will be the same as in $L_{6s-m}$. In layers $L_{3s+1}$, every weight will be $\lceil \frac{\delta}{2s} \rceil$ or $\lceil \frac{\delta}{2s} \rceil$ before adjustment, and in layers $L_{3s}$ the weights will be $1$. More precisely:

(A) For each $i : 0 \leq i \leq s$, let the layer $L_{3si}$ contain a single vertex with weight $1$. 
In the repetitive block $C_{1,d}$ for the weighted clump graph of the counterexample for 3-colorable/$K_4$-free graphs. The letters $X,Y,Z$ give a 3-coloration and the label above the vertex gives the weight of the vertex.

(B) For each $i : 0 \leq i \leq s - 1$, let the layer $L_{3i+1}$ contain $2s - i$ vertices, and assign them the following weights:
   
   (a) If $d = 0$, let the weight of each of these vertices be $\frac{\delta}{2s}$. The adjustment is that for a single vertex in $L_1$, whose weight is reduced to $\frac{\delta}{2s} - 1$. (By symmetry, the same adjustment happens in $L_{6s-1}$.)
   
   (b) If $d \geq 1$, then let $\min(2s - i, d - 1)$ vertices have weight $\lceil \frac{\delta}{2s} \rceil$, and the rest have weight $\lfloor \frac{\delta}{2s} \rfloor$.

(C) For each $i : 0 \leq i < s - 1$, let the layer $L_{3i+2}$ contain $i + 1$ vertices, and assign them the following weights:

   (a) If $d = 0$, let the weight of each of them be $\frac{\delta}{2s}$.
   
   (b) If $1 \leq d$, then let $d - \min(2s - i - 1, d - 1)$ vertices have weight $\lceil \frac{\delta}{2s} \rceil$, and the rest get weight $\lfloor \frac{\delta}{2s} \rfloor$.

   (D) Let layer $L_{3s-1}$ (and symmetrically layer $L_{3s+1}$) have $s$ vertices each. In these layers, let $\lfloor \frac{s}{2s} \rfloor$ vertices (resp. $\lceil \frac{s}{2s} \rceil$ vertices) have weight $\lceil \frac{\delta}{2s} \rceil$, and the remaining vertices get weight $\lfloor \frac{\delta}{2s} \rfloor$. (This weight assignment is feasible. Since $d \leq 2s - 1$, $\lfloor \frac{d}{2s} \rfloor \leq s$.)

Note that $\min(d - 1, 2s - i) = d - 1$ for $i \in \{0, 1, 2\}$. We use this minimization for $i \leq s - 2$. When $s \leq 4$, we have $s - 2 \leq 2$, consequently there is no need to use the minimization formula for $s \leq 4$. Therefore we show $C_{5,d}$ in Figure 2, which is the first instance to show all complexities of the counterexamples. The case $s = 1$, when $d \in \{0,1\}$, is even simpler: it is possible to describe the weights without reference to $d$, see Figure 1 for $C_{1,d}$.

Lemma 5. Let $p \geq 1$ and $s \geq 2$. The weighted clump graph $H_{s,d,p}$ has the following properties:

(a) $H_{s,d,p}$ is $(2s + 1)$-colorable with diameter $p(6s + 1) - 1$.

(b) The sum of the weights of all vertices is $p((2s + 1)d + 2s - 1) + 2$.

(c) For any vertex $y \in V(H_{s,d,p})$, the sum of the weights of its neighbors is at least $\delta$. 

Figure 1.
**Proof.** [a] The statement on the diameter is trivial. As the number of vertices in any two consecutive layer of $H_{s,\delta,p}$ is at most $2s+1$, we can $(2s+1)$-color $H_{s,\delta,p}$ with $(2s+1)$ colors from left to right greedily.

[b] If $W$ is the sum the weights of vertices in the block $C_{s,\delta}$, then the total sum of weights in $H_{s,\delta,p}$ is $pW + 2$ (the 2 is due to the modification), so we need to show that $W = (2s+1)\delta + 2s - 1$.

Consider an $i$ with $0 < i < s-1$. $L_{3i-1} \cup L_{3i+1}$ has $(i-1)+1+2s-i = 2s$ vertices. If $d = 0$, each of them has weight $\delta_{2s}$, otherwise $d - \min(d-1,2s-i) + \min(d-1,2s-i) = d$ of them have weight $\lceil \frac{\delta}{2s} \rceil$, and the rest have weight $\lfloor \frac{\delta}{2s} \rfloor$. So the sum of the weight of the vertices in $L_{3i-1} \cup L_{3i} \cup L_{3i+1}$ is $\delta + 1$, and so is in $L_{6s-3i+1} \cup L_{6s-3i} \cup L_{6s-3i-1}$. 
$L_{3s-1} \cup L_{3s+1}$ contains $2s$ vertices, $\left\lceil \frac s2 \right\rceil + \left\lfloor \frac s2 \right\rfloor = d$ of them has weight $\left\lceil \frac s2 \right\rceil$, the rest $\left\lfloor \frac s2 \right\rfloor$, so the sum of the weights of the vertices in $L_{3s-1} \cup L_{3s} \cup L_{3s+1}$ is also $\delta + 1$.

$L_1$ has $2s$ vertices. If $d = 0$, one of these have weight $\frac s2 - 1$ and the rest have weight $\frac s2$. If $d > 0$, $d - 1$ of the vertices have weight $\left\lceil \frac s2 \right\rceil$, the rest has weight $\left\lfloor \frac s2 \right\rceil$. The sum of the weights in $L_0 \cup L_1$ is $1 + \delta - 1 = \delta$.

So $W = 2\delta + (2s-1)(\delta + 1) = (2s+1)\delta + 2s - 1$, which finishes the proof of (b).

For (c). Let $y$ be a vertex of $H_{s,\delta,p}$. Then for some $j$ ($0 \leq j < p$), $y$ is in the $j$-th block $C_{s,\delta}$, and for some $m$ ($0 \leq m \leq 6s$), $y$ is in the layer $L_m$ of $C_{s,\delta}$. Because of symmetry, we may assume that $0 \leq m \leq 3s$. The weights in the layer $L_{3s-1}$ are less or equal than the weights in the layer $L_{3s+1}$, but may not be equal, breaking the symmetry, but still handling cases with $0 \leq m \leq 3s$ gives a $\delta$ lower bound to the degrees of all vertices of $H_{s,\delta,p}$. In addition, layer $(j,m) = (p-1, 6s-1)$, where a modification happened, is symmetric to the layer $(j,m) = (0, 1)$, where identical modification happened. Therefore checking the degrees of the vertices in the first half of the first (and modified) copy of $C_{s,\delta}$ in $H_{s,\delta,p}$ covers checking the degrees in the second half of the last (and modified) copy of $C_{s,\delta}$ in $H_{s,\delta,p}$.

If $y \in L_{3i}$ for some $0 < i \leq 2s$, then $y$ has weight $1$ and is adjacent to all vertices but itself in $L_{3i-1} \cup L_{3i} \cup L_{3i+1}$. As we have already shown in the proof of part (b) $L_{3i-1} \cup L_{3i} \cup L_{3i+1}$ has total weight $\delta + 1$, the neighbors of $y$ have total weight $\delta$.

If $y \in L_0$, then as we showed in the proof of part (b) the total weight of the vertices in $L_0$ in an unmodified block, which is not the first or the last block, is $\delta - 1$. Either $y$ is adjacent to a vertex of weight $1$ outside of its own block, or $y$ is in a modified block where the total weight of $L_1$ got increased by $1$: in both cases the sum of the weights of the neighbors of $y$ is $\delta$.

Assume now that $y$ is a vertex of $L_{3i+1} \cup L_{3i+2}$ for some $0 \leq i \leq s - 1$. Note $L_{3i+1} \cup L_{3i+2}$ contains $2s + 1$ vertices, $2s$ of which is the neighbor of $y$, plus $y$ has a neighbor of weight $1$ outside of $L_{3i+1} \cup L_{3i+2}$. We consider two cases for $d$:

If $d = 0$ and $0 < i \leq s - 1$, then each neighbor of $y$ in $L_{3i+1} \cup L_{3i+2}$ has weight $\frac s2$, so the sum of the weights of the neighbors of $y$ is $\delta + 1$. If $d = 0$ and $i = 0$, because of the adjustment, the sum of the weights of the neighbors may decrease by $1$, and is still $\delta$.

If $d > 0$, then all vertices in $L_{3i+1} \cup L_{3i+2}$ have weight at least $\left\lceil \frac s2 \right\rceil$. If $i < s - 1$, then at least $\min(d - 1, 2s - i) + d - \min(d - 1, 2s - i - 1) \geq d$ many vertices of $L_{3i+1} \cup L_{3i+2}$ have weight $\left\lceil \frac s2 \right\rceil$. If $i = s - 1$, then, as $s + 1 > \left\lceil \frac s2 \right\rceil$, we have that $\left\lceil \frac s2 \right\rceil \geq \max(1, d - (s + 1))$, so $L_{3s-2} \cup L_{3s-1}$ has at least $\min(d - 1, s + 1) + \left\lceil \frac s2 \right\rceil = d - \max(1, d - (s + 1)) + \left\lceil \frac s2 \right\rceil \geq d$ vertices with weight $\left\lceil \frac s2 \right\rceil$. Therefore, for any $0 \leq i \leq s - 1$, any $y \in L_{3i+1} \cup L_{3i+2}$ has at least $d - 1$ neighbors of weight $\left\lceil \frac s2 \right\rceil + 1$, the total weight of $y$’s neighbors is at least $2s\left\lceil \frac s2 \right\rceil + d - 1 + 1 = \delta$. This finishes the proof of (c).}

\begin{theorem}
Let $r \geq 2$, $\delta \geq 2r - 2$, and for each positive integer $p$, let $G_{r,\delta,p}$ be the graph whose weighted clump graph is $H_{r-1,\delta,p}$. Then $G_{r,\delta,p}$ is $2r - 1$ colorable (and hence $K_{2r}$-free), connected, with minimum degree $\delta$, of order $n = p((2r - 1)\delta + 2r - 3) + 2$, and of diameter \frac{(6r-5)n}{(2r-1)s+2r-3} + O(1). Consequently, Conjecture fails for every $\delta > 12r^3 - 22r^2 - 2r + 12$
2(r - 1)(3r + 2)(2r - 3). Furthermore, the difference between the coefficient of $\frac{n}{3}$ in our construction and in Conjecture [1] is $\frac{(2r^2 - 1)}{(2r - 1)\delta + 2r - 3} + o(1)$, as $\delta \to \infty$.

**Proof.** By Lemma [5] $G_{r,\delta,p}$ is $(2r-1)$-colorable with minimum degree $\delta$, diameter $p(6r-5)-1$, and it has $n = p((2r-1)\delta + 2r - 3) + 2$ vertices. Therefore its diameter is $\frac{(6r-5)n}{(2r-1)\delta + 2r - 3} + O(1)$. Consider the identity

$$\frac{(6r-5)\delta}{(2r-1)\delta + 2r - 3} - \frac{2(r-1)(3r + 2)}{(2r^2 - 1)} = \frac{1}{(2r^2 - 1)(2r - 1)} \cdot 1 - \frac{12r^3 - 22r^2 - 2r + 12}{\delta}.$$

This shows both the fact that $\frac{(6r-5)n}{(2r-1)\delta + 2r - 3} \leq \frac{2(r-1)(3r+2)}{(2r^2 - 1)} \cdot \frac{n}{\delta}$ iff $\delta \leq 12r^3 - 22r^2 - 2r + 12$, and the statement about the difference. \qed

4. **Canonical Clump Graphs**

We use the letters $X, Y, Z$ to denote three unspecified but different colors from our $k$ colors.

**Theorem 7.** Assume $k \geq 3$. Let $G'$ be a $k$-colorable connected graph of order $n$, diameter $D$ and minimum degree at least $\delta$. Then there is a $k$-colored connected graph $G$ of the same parameters, with layers $L_0, \ldots, L_D$, for which the following hold for every $i$ ($0 \leq i \leq D-1$):

(i) If $c(i) = 1$, then $c(i + 1) \leq k - 1$.

(ii) The number of colors used to color the set $L_i \cup L_{i+1}$ is $\min(k, c(i) + c(i + 1))$. In particular, when $c(i) + c(i + 1) \leq k$, then $L_i$ and $L_{i+1}$ do not share any color.

(iii) If $c(i) = k$, then $i \geq 2$ and $c(i + 1) \geq 2$.

(iv) If $|L_i| > c(i)$, i.e., $L_i$ contains two vertices of the same color, then $i > 0$ and $c(i) + \max(c(i-1), c(i+1)) \geq k$.

**Proof.** After having proved a part of the Theorem, we will assume that $G'$ itself satisfies that property when we complete the proof of the remaining parts. When we create new $G'$ graphs, they will still satisfy the already checked parts, in other words, we do not regress to issues that we already resolved. We fix a $k$-coloration of $G'$, let $x_0$ be a vertex of eccentricity $D$ in $G'$, and let $L_0, \ldots, L_D$ be the distance layering of $G'$. Without loss of generality, we assume that $G'$ is saturated.

[1] Select $G = G'$ with the same $k$-coloration. The statement follows from the fact that every vertex in $L_{i+1}$ has a neighbor in $L_i$; therefore if color $X$ appears in $L_{i+1}$, then $L_i$ has at least one color different from $X$.

If [2] or [4] is not satisfied in $G'$, our general strategy is the following: create a new $k$-coloring of the vertices of $G'$ such that the set of the vertices in any layer does not change, vertices of different color will remain differently colored, and in the new coloring the already proven statements still hold. We saturate $G'$ in the new coloring (by adding new edges, if needed) to obtain $G$. Now we complete this strategy for [2] and postpone the proof of [4] till the end.

If [2] fails in $G'$, consider the smallest $i$, such that the set $L_i \cup L_{i+1}$ contains fewer than $\min(k, c(i) + c(i + 1))$ colors. By [1] $i > 0$. Observe that there are different colors $X, Y$
such that $X$ is used in both of $L_i$ and $L_{i+1}$, while $Y$ is not used in $L_i \cup L_{i+1}$. We define a new coloration by switching colors $X$ and $Y$ in all $L_j$ for all $j \geq i + 1$. This is a good coloration, in which $L_i \cup L_{i+1}$ uses one more color. Repeated application of this procedure yields a $k$-coloration where $[ii]$ holds.

The hard part of this theorem is $[iii]$. If $c(i) = k$, then by $[i]$, $i \geq 2$. If $c(i + 1) = 1$ (i.e., $c(i) < k$), then we will move clumps within $L_{i-1} \cup L_i$, and recolor of the graph, such that the resulting layered colored graph will have the same required parameters as $G'$, creating no violations of $[i]$ and $[ii]$ and reducing the number of violations of $[iii]$ in $G'$.

Let $X$ be the color used on $L_{i+1}$.

Assume first that $L_{i-2}$ contains a color different from $X$. Set $S$ be the set of vertices in $L_i$ that is colored $X$. Move the vertices of $S$ from $L_i$ to $L_{i-1}$ without recoloring them, either merging them into the $X$-colored clump of $L_{i-1}$ or creating one, if no such clump existed in $L_{i-1}$. Add new edges to achieve saturation. In the resulting graph, the layer indexed by $i$ contains $k - 1$ colors, reducing the number of violations of $[iii]$ in $G'$, and not creating any violation of $[i]$ or $[ii]$.

Hence in the following we may assume that $c(i - 2) = 1$ and $L_{i-2}$ is colored with color $X$. By $[i]$ we have $c(i - 1) \leq k - 1$. If $c(i - 2) < k - 2$, then there is a color $Y$ not used in $L_{i-2} \cup L_{i-1}$. Recolor $G'$ by switching colors $X$ and $Y$ in $L_j (0 \leq j \leq i - 2)$ and recover saturation. In the new coloring $L_{i-2}$ has a color different from $X$, and we are back to the case we already handled above.

Hence in the rest we may assume that $c(i - 2) = c(i + 1) = 1$, $c(i) = k$, $c(i - 1) = k - 1$, and both $L_{i-2}$ and $L_{i+1}$ are colored with $X$. Let $Y, Z$ be two arbitrary colors different from $X$, and $S$ be the set of vertices in $L_{i-1} \cup L_i$ colored with $X$, $Y$ or $Z$. We will repartition and recolor (only with colors $X, Y, Z$) the vertices in $S$, and possibly recolor $L_{i+1}$ from color $X$ to color $Y$. If we recolor $L_{i+1}$, then we exchange the colors $X$ and $Y$ in all layers $L_j$ for $j \geq i + 2$. After these steps, we recover saturation in $G'$. After the changes, in $G'$ both $L_{i-1}$ and $L_i$ will contain fewer than $k$ colors, and in the resulting $k$-colored graph the diameter, the order and minimum degree condition do not change, and no instances violating $[i]$ and $[ii]$ will be created, and we reduced the number of violations of $[iii]$. The difficulty is in maintaining the minimum degree condition in $G'$ along these operations. This is what we check next, and the repartitioning and recoloring of the vertices in $S$ will depend on some inequalities between certain clump sizes.

If $y$ is a vertex not in $L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1}$ or $y$ is not colored with $X$, $Y$ or $Z$ in the graph before the operations, then the neighborhood set of $y$ does not change. If $y$ is a vertex in $L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1}$ colored with one of $X, Y, Z$, then the symmetric difference between the new and old neighborhood set of $Y$ is a subset of $S$. Therefore we only need to check the minimum degree condition for vertices colored $X, Y$ or $Z$ in $L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1}$, and we have to show that after the operations they have at least as much $X, Y, Z$ colored neighbors in $L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1}$ as before the operations. For any $j (1 \leq j \leq 4)$, we will denote by $x_j, y_j$ and $z_j$ the number of vertices in $L_{i+j-3}$ colored $X, Y$ and $Z$ in $G'$ respectively, before the operations. The $k \geq 3$ assumption, together with the fact that $L_{i-1}$ has no color $X$ by $[iii]$ implies that $x_1, y_2, z_2, y_3, z_3$ and $x_4$ are positive.

We have several cases to consider:
It suffices to handle the case $x_3 \geq y_3$, as the case $x_3 \geq z_3$ can be handled similarly. Let the operations create in $L_{i-1}$ a clump of size $y_2 + y_3$ of color $Y$ and a clump of size $z_2$ of color $Z$; in $L_i$ a clump of size $x_3$ of color $X$ and a clump of size $z_3$ of color $Z$. Recolor $L_{i+1}$ with $Y$, and switch colors $X$ and $Y$ in every $L_j$ for $j \geq i + 2$, see Fig. 3. Note that, as claimed, $|L_{i-1} \cup L_i|$ did not change. We verify the minimum degree condition. Let $d(W_i)$ to denote the number of neighbors of a vertex $w$ from the clump colored $W$ in layer $L_i$ among the $X, Y, Z$ colored vertices of $L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1}$ before the operations, and $d'(W_i)$ to denote the degree of a vertex $w'$ from the clump colored $W$ in layer $L_i$ among the $X, Y, Z$ colored vertices of $L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1}$ after the operations. We have:

\[
\begin{align*}
d'(X_{i-2}) &= d(X_{i-2}) + y_3, \\
d'(Y_{i-1}) &= x_1 + z_2 + x_3 + z_3 = d(B_i), \\
d'(Z_{i-1}) &= x_1 + y_2 + z_3 + y_3 = d'(Z_{i-1}), \\
d'(X_i) &= y_2 + y_3 + z_2 + z_3 + x_4 > y_2 + y_3 + z_2 + z_3 = d(X_i), \\
d'(Z_i) &= y_2 + y_3 + x_3 + x_4 = d(Z_i), \\
d'(Y_{i+1}) &= x_3 + z_3 = d'(X_{i+1}) + (x_3 - y_3).
\end{align*}
\]

**Figure 3.** When $x_3 \geq y_3$, before and after the operations (left and right).

$x_3 < y_3$ and $x_3 < z_3$ and $(x_3 \geq y_2$ or $x_3 \geq z_2)$.

We may assume $x_3 \geq y_2$, as $x_3 \geq z_2$ can be handled similarly. Let the operations create in $L_{i-1}$ a clump of size $x_3$ of color $Y$ and a clump of size $z_2$ of color $Z$; and in $L_i$ create a clump of size $y_3 + y_2$ of color $X$ and a clump $z_3$ of color $Z$; recolor $L_{i+1}$ to color $Y$ and switch colors $X$ and $Y$ in $L_j$ for $j \geq i + 2$, see Figure 3.
ON THE MAXIMUM DIAMETER OF $k$-COLORABLE GRAPHS 11

Note that $|L_{i-1} \cup L_i|$ did not change. When we verify the minimum degree condition, we use the notation of Case 1. We have:

$$d'(X_{i-2}) = d(X_{i-2}),$$
$$d'(Y_{i-1}) = x_1 + y_2 + y_3 + z_2 + z_3 > x_1 + z_2 + x_3 + z_3 = d(Y_i),$$
$$d'(Z_{i-1}) = x_1 + x_3 + y_2 + y_3 = d(Z_{i-1}),$$
$$d'(X_i) = x_3 + x_4 + z_2 + z_3 = d(Y_i),$$
$$d'(Z_i) = x_3 + x_4 + y_2 + y_3 = d(Z_i),$$
$$d'(Y_{i+1}) = d(Y_{i+1}) + y_2.$$  

At this point, we are left with checking the statement for $x_3 < \min(y_3, z_3, y_2, z_2)$. We split this into two cases.

**Case 3**: $x_3 < \min(y_3, z_3, y_2, z_2)$ and $z_2 \geq y_3$.

The operations are shown in Figure 5. When we check the degrees, we use the notation introduced in Case 1.

$$d'(X_{i-2}) = d(X_{i-2}) + x_3,$$
$$d'(Y_{i-1}) = x_1 + z_2 + y_3 + z_3 > x_1 + z_2 + x_3 + z_3 = d(Y_{i-1}),$$
$$d'(Z_{i-1}) = x_1 + y_2 + x_3 + y_3 + z_3 > x_1 + y_2 + y_3 + x_3 = d(Z_{i-1}),$$
$$d'(X_i) = z_2 + y_2 + x_3 + x_4 \geq y_2 + y_3 + x_3 + x_4 = d(Z_i),$$
$$d'(Y_{i+1}) = d(Y_{i+1}).$$

**Case 4**: $x_3 < \min(y_3, z_3, y_2, z_2)$ and $z_2 < y_3$.

First note that the clump colored $X$ in $L_i$ can simply be moved into $L_{i-1}$ keeping the minimum degree condition. After this move on the left side of Figure 6, we see the mirror image of the left side of Figure 5 just the numbers are different. The operations for this case, which are the “mirror image”, of the operations in the previous case, is shown in Figure 6. Because of the symmetry, we do not delve into the details.
Figure 5. The case $x_3 \leq \max(y_2, y_3, z_2, z_3)$ and $z_2 \geq y_3$, before and after the operations.

Figure 6. The case $x_3 \leq \max(y_2, y_3, z_2, z_3)$ and $z_2 < y_3$, before and after the operations.

This concludes the proof of (iii).

Finally, to prove part (iv) take the least $i$, such that $|L_i| > c(i)$, but $c(i) + \max(c(i - 1), c(i + 1)) < k$. As $|L_0| = c(0) = 1$, this means $i > 0$. First we show that we may assume that $L_{i-1} \cup L_i \cup L_{i+1}$ misses some color $X$. Indeed, if all colors appear in $L_{i-1} \cup L_i \cup L_{i+1}$, let $X$ be a color not used in $L_{i-1} \cup L_i$ and $Y$ be a color not used in $L_i \cup L_{i+1}$. Create a new coloring of $G$ by switching the colors $X,Y$ in $L_j$ for all $j \geq i + 1$. After the switch, $X$ is missing from $L_{i-1} \cup L_i \cup L_{i+1}$.

Since $|L_i| > c(i)$, there are two vertices $x,y$ in $L_i$ that are colored the same. Recolor $x$ with color $X$. This is a valid coloring, in which $L_i$ contains one more color than before. Repeating this procedure produces a coloring, in which $|L_i| = c_i$ or $c(i) + \max(c(i - 1), c(i + 1)) = k$. Repeating this procedure recursively for the next least $i$, we can eliminate one after the other the $i$’s that fail (iv), not creating any instances where the first three statements would fail.

□

Definition 1. We call a $k$-colored weighted clump graph $H$ canonical, if there is a graph $G$, whose clump graph is $H$, and $H$ satisfies the four statements in Theorem 7, i.e., $H$ has $D + 1$ layers $L_0, L_1, \ldots, L_D$, where $D = \text{diam}(H)$, and for each $1 \leq i < D$ we have

(i) If $|L_i| = 1$, then $L_{i+1} \leq k - 1$. 

}\end{document}
(ii) The number of colors used to color the set $L_i \cup L_{i+1}$ is $\min(k, c(i) + c(i + 1))$. In particular, when $c(i) + c(i + 1) \leq k$, then $L_i$ and $L_{i+1}$ do not share any color.

(iii) If $|L_i| = k$, then $i \geq 2$ and $|L_{i+1}| \geq 2$.

(iv) If $L_i$ has a weight that is bigger than 1, then $i > 0$ and $|L_i| + \max(|L_{i-1}|, |L_{i+1}|) \geq k$.

Note that (ii) implies that the edges missing between $L_i$ and $L_{i+1}$ form a matching of size $\max(0, |L_i| + |L_{i+1}| - k)$. In particular, when $|L_i \cup L_{i+1}| \leq k$ then all edges between $L_i$ and $L_{i+1}$ are present.

Corollary 8. In the canonical clump graph of a 3-colored connected graph, the following color sets are possible in two consecutive layers:

(1) $X, Y, X|YZ, YZ|X, XY|XZ, XY|XYZ, XYZ|XY, XY|XYZ$.

5. Duality

In this Section, $k$ is fixed. Look differently at our diameter problem: assume that the diameter $D$, and the lower bound $\delta$ for the degrees of the graph are fixed (in addition to $k$), how small $n$ can be, such that connected $k$-colorable graphs of order $n$, minimum degree at least $\delta$, and diameter $D$ exist? Let $\mathcal{H}$ denote the family of canonical clump graphs of diameter $D$ that arises from connected $k$-colorable graphs with diameter $D$ and minimum degree at least $\delta$, of unspecified order. Fix an $H \in \mathcal{H}$, and consider the following packing problem for $H$: assign non-negative real dual weights $u(y) \geq 0$ to $y \in V(H)$, and

Maximize $\delta \cdot \sum_{y \in V(H)} u(y)$,

subject to condition

(2) $\forall x \in V(H) \sum_{y \in V(H): xy \in E(H)} u(y) \leq 1$.

Theorem 9. Assume that there exist constants $\tilde{u} > 0, C \geq 0$, such that for all $D$ and $\delta$, and all $H \in \mathcal{H}$, in the linear program (2) the optimum is at least

(3) $\tilde{u}\delta(D + 1) - C\delta$.

Then, for any $H \in \mathcal{H}$, we have

(4) $D \leq \frac{1}{\tilde{u}} \cdot \frac{n}{\delta} + C$.

Proof. Fix $H \in \mathcal{H}$. Then $H$ is the clump graph of saturated graph $G$. $G$ can be reconstructed by assigning $w(x) \geq 1$ integer weights for all vertices of $H$, such that we assign 1 to the vertex in $L_0$. Now $n = |V(G)| = \sum_{x \in V(H)} w(x)$. Consider the optimization problem

Minimize $\sum_{x \in V(H)} w(x)$,
subject to condition

\[ \forall y \in V(H) \quad \sum_{x \in V(H) : xy \in E(H)} w(x) \geq \delta. \quad (5) \]

We face the trivial inequality of the duality of linear programming [4]: namely, for any \( u \) and \( w \) feasible solutions, by (5) and (2), we have:

\[ \delta \sum_{y \in V(H)} u(y) \leq \sum_{y \in V(H)} u(y) \sum_{x \in V(H) : xy \in E(H)} w(x) \]
\[ = \sum_{x \in V(H)} w(x) \sum_{y \in V(H) : xy \in E(H)} u(y) \leq \sum_{x \in V(H)} w(x). \]

As the objective function reaches \( \delta \sum_{y \in V(H)} u(y) \geq \tilde{u}\delta(D+1) - C\delta \), the theorem follows. \( \square \)

**Proof of Theorem**. Assume \( k \geq 3 \). Consider a \( k \)-colorable canonical clump graph \( H \) with layers \( L_0, \ldots, L_D \). We are going to find a good packing \( u \) on the vertices of \( H \) as required to use Theorem 9. The dual weighting \( u \) will take at most \( 2k - 2 \) different values, and every layer will get the same total dual weight.

Let \( i \) be an integer, \( 0 \leq i \leq D \). Set \( L_{i-1} = L_{D+1} = \emptyset \).

If \( |L_i| \leq k - 1 \), then assign the dual weight \( \frac{k-1}{(3k-4)|L_i|} \) to every \( v \in L_i \). This makes the total dual weight of \( L_i \) exactly \( \frac{k-1}{3k-4} \), and the dual weight of every \( v \in L_i \) at least \( \frac{1}{3k-4} \).

If \( |L_i| = k \), let \( X_i \) be the (possibly empty) set of vertices in \( L_i \) connected to every vertex in \( L_{i-1} \cup L_{i+1} \), and set \( Y_i = L_i \setminus X_i \). As \( |X_i| \leq k - |L_{i-1}| \) and \( H \) is canonical, by Definition 1 we have \( 0 \leq |X_i| \leq k - 2 \), and \( Y_i \neq \emptyset \).

Set the dual weight of every \( v \in X_i \) to \( \frac{1}{3k-4} \), and the dual weight of every \( v \in Y_i \) to \( \frac{|X_i|}{3k-4} + (k - |X_i|) \left( \frac{1}{3k-4} - \frac{1}{(3k-4)(k - |X_i|)} \right) = \frac{k-1}{3k-4} \).

Moreover, as \( k - |X_i| \geq 2 \), the dual weight of \( v \in Y_i \) is at least \( \frac{1}{2(3k-4)} \).

Now take vertex \( x \) of \( H \). Then \( x \in L_j \) for some \( 0 \leq j \leq D \). We are going to check that the neighbors of \( x \) have a total dual weight of at most 1.

If \( |L_j| \leq k - 1 \), or \(|L_j| = k \) and \( x \notin X_j \), then the weight of \( x \) is at least \( \frac{1}{3k-4} \). Since the open neighborhood of \( x \) is a subset of \( (L_{j-1} \cup L_j \cup L_{j+1}) \setminus \{x\} \), the sum of the weight of its neighbors is at most \( \frac{3(k-1)}{3k-4} - \frac{1}{3k-4} = 1 \).

If \( |L_j| = k \) and \( x \in X_j \), then there is a \( y \in L_{j-1} \cup L_{j+1} \) such that the open neighborhood of \( x \) is contained by \( (L_{j-1} \cup L_j \cup L_{j+1}) \setminus \{x, y\} \). As the sum of the weights of \( x \) and \( y \) is at least \( \frac{1}{3k-4} \), the total weight of the neighbors of \( x \) at most \( \frac{3(k-1)}{3k-4} - \frac{1}{3k-4} = 1 \).

The total dual weight of the vertices in \( H \) is \( \frac{k-1}{3k-4}(D+1) \). Now Theorem 8 follows from Theorem 9. \( \square \)

Using \( k = 3 \), we get a weaker version of Theorem 2.
Corollary 10. If $G$ is connected 3-colorable graph of order $n$ and minimum degree $\delta \geq 1$, then
\[ \text{diam}(G) \leq \frac{5n}{2\delta} + O(1). \]

6. Inclusion-Exclusion (Sieve)

Let us be given a 3-colorable saturated connected graph $G$ of order $n$ and minimum degree at least $\delta$, which maximizes the diameter $D$ among such graphs. By Theorem 10, we may assume without loss of generality that the clump graph of $G$ is canonical. Furthermore, Corollary 1 tells what kind of color sets can be in consecutive layers. We often use these facts without explicit reference in the future. Let $\ell_i = |L_i|$ denote the cardinality of the $i$th layer of $G$. As we are about to prove Theorem 4, we can assume without loss of generality that $\ell_i \leq 2\delta$. Indeed, if $G$ does not satisfy this inequality, eliminate vertices from clumps with excess above $\delta$, to obtain the graph $G'$ on $n'$ vertices. $G'$ still satisfies the conditions of Theorem 4 and therefore its conclusion with $n'$ replacing $n$. Hence $G$ also satisfies the conclusion of Theorem 4. We are going to build lower bounds for the sum of a couple of consecutive $\ell_i$'s, from which we derive lower bounds for $n$. The key tool is the inclusion-exclusion formula for the size of the union of the open neighborhoods of some vertices. Note that a vertex in $L_i$ can have neighbors only in $L_{i-1}, L_i, L_{i+1}$. We denote the open neighborhood of vertex $z$ by $N(z)$. In Subsection 6.1 we do this approach when the vertices are taken from different clumps from the same $L_i$, in Subsection 6.2 we do this for vertices taken from two consecutive layers. Recall that $c(i)$ denotes the number of clumps in $L_i$. Let $S = \{i : c(i) = 1\}$ be the set of singles. We use the notation $x_i, y_i, z_i$ to represent vertices in the clumps with color $X_i, Y_i, Z_i$, respectively. Here $X_i, Y_i, Z_i$ can be any of the colors $A, B, C$, but they must be different colors. For the ease of computation we introduce $L_{-1} = L_{D+1} = \emptyset$, so $\ell_{-1} = \ell_{D+1} = 0$.

6.1. Sieve for neighborhoods of vertices from one layer.

Here we assume $0 \leq i \leq D$.

Case 1. $c(i) = 1$. We obviously have $\ell_{i-1} + \ell_{i+1} \geq \delta$, which we prefer to write as
\[ 2\ell_{i-1} + 2\ell_i + 2\ell_{i+1} \geq 2\delta + 2\ell_i. \]

Case 2. We have $2\ell_{i-1} + \ell_i + 2\ell_{i+1} \geq 2\delta$ from the fact that vertices from either color in the $i$th layer have at least $\delta$ neighbors. We prefer to write this as
\[ 2\ell_{i-1} + 2\ell_i + 2\ell_{i+1} \geq 2\delta + \ell_i. \]

6.2. Sieve by two consecutive layers.

Now we assume $0 \leq i < D$, so $i+1 \leq D$.

Case 1. $i \in S, i+1 \in S$. We have
\[ \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} \geq 2\delta. \]

Case 2. $i \in S, i+1 \notin S$. Assume $X_i = L_i$, which implies $L_{i+1} = Y_{i+1} \cup Z_{i+1}$. Apply (6) to $L_i$ to obtain $2\ell_{i-1} + 2\ell_{i+1} \geq 2\delta$, apply (7) to $L_{i+1}$ to obtain $2\ell_i + \ell_{i+1} + 2\ell_{i+2} \geq 2\delta$, and
average into

\[ \ell_{i-1} + \ell_i + \frac{3}{2} \ell_{i+1} + \ell_{i+2} \geq 2\delta. \]

**Case 3.** \(i \not\in \mathcal{S}, i + 1 \in \mathcal{S}\). Like in Case 2, we obtain

\[ \ell_{i-1} + \frac{3}{2} \ell_i + \ell_{i+1} + \ell_{i+2} \geq 2\delta. \]

**Case 4.** \(i \not\in \mathcal{S}, i + 1 \not\in \mathcal{S}\). In this case \(L_i\) and \(L_{i+1}\) must share a color, and their union must use all 3 colors. We can assume without loss of generality that none of \(X_i, Y_i, X_{i+1}, Z_{i+1}\) is empty. Take \(x_i \in X_i, y_i \in Y_i, x_{i+1} \in X_{i+1}, z_{i+1} \in Z_{i+1}\). Considering the neighborhood of \(x_i\), we have

\[ \delta \leq \ell_{i-1} + |Y_i| + |Z_i| + |Y_{i+1}| + |Z_{i+1}|, \]

considering the neighborhood of \(x_{i+1}\), we have

\[ \delta \leq \ell_{i+2} + |Y_i| + |Z_i| + |Y_{i+1}| + |Z_{i+1}|, \]

considering the neighborhood of \(y_i\), we have

\[ \delta \leq \ell_{i-1} + |X_i| + |Z_i| + |X_{i+1}| + |Z_{i+1}|, \]

considering the neighborhood of \(z_{i+1}\), we have

\[ \delta \leq \ell_{i+2} + |X_i| + |Y_i| + |X_{i+1}| + |Y_{i+1}|. \]

Weighting (11) and (12) with 1/3, (13) and (14) with 2/3, and summing them up, we obtain

\[ \ell_{i-1} + \frac{4}{3} (\ell_i + \ell_{i+1}) + \ell_{i+2} \geq 2\delta. \]

Adding up (8), (9), (10), (15) for \(i = 1, \ldots, D - 1\), we obtain

\[ 4n + \sum_{(i,j):i \not\in \mathcal{S}, j \not\in \mathcal{S}, i-j=1} \frac{1}{3} \ell_i + \sum_{i:i+1 \in \mathcal{S}, i \not\in \mathcal{S}} \frac{1}{2} \ell_i + \sum_{i:i-1 \in \mathcal{S}, i \not\in \mathcal{S}} \frac{1}{2} \ell_i \geq 2D\delta + O(\delta). \]

The \(O(\delta)\) error term arises from the fact that certain \(\ell_i\) terms, at the front and at the end, do not arise four times, as many times they are counted in \(4n\).

### 6.3. Sieve for neighborhoods of vertices from three consecutive layers.

We are going to give lower bounds to

\[ 2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) = 2|L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1} \cup L_{i+2}| \]

using inclusion-exclusion, based on a case analysis of the color content of \(L_{i-1}, L_i, L_{i+1}\).

**Case 1.** \(i - 1 \not\in \mathcal{S}, i \not\in \mathcal{S}, i + 1 \not\in \mathcal{S}\). This boils down to two subcases:

- **Subcase 1.1.** \(L_{i-1}\) and \(L_{i+1}\) share at least two colors. We may assume in this case that none of \(X_{i-1}, Y_{i-1}, X_i, Z_i, X_{i+1}, Y_{i+1}\) is empty. Take \(y_{i-1} \in Y_{i-1}, z_i \in Z_i, x_{i+1} \in X_{i+1}\). Using inclusion-exclusion we have
  
  \[ |N(y_{i-1}) \cup N(z_i) \cup N(x_{i+1})| \geq 3\delta - (|X_{i-1}| + \ell_i + |Y_{i+1}|). \]
Similarly, take \( x_{i-1} \in X_{i-1}, z_i \in Z_i, y_{i+1} \in Y_{i+1} \) and use inclusion-exclusion to get
\[
|N(x_{i-1}) \cup N(z_i) \cup N(y_{i+1})| \geq 3\delta - (|Y_{i-1}| + \ell_i + |X_{i+1}|).
\]
Combining the two inequalities above we obtain
\[
(18) \quad 2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) \geq 6\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}.
\]

**Subcase 1.2.** \( L_{i-1} \) and \( L_{i+1} \) share only one color. We may assume that \( L_{i-1} = X_{i-1} \cup Y_{i-1} \); \( L_{i+1} = Y_{i+1} \cup Z_{i+1} \) where \( X_{i-1}, Y_{i-1}, Y_{i+1}, Z_{i+1} \neq \emptyset \), and \( X_i, Z_i \neq \emptyset \).

Apply inclusion-exclusion for the neighborhoods of \( x_{i-1} \in X_{i-1}, z_i \in Z_i, y_{i+1} \in Y_{i+1} \) we get
\[
|N(x_{i-1}) \cup N(z_i) \cup N(y_{i+1})| \geq 3\delta - (|Y_{i-1}| + \ell_i + |Z_{i+1}|),
\]
and doing it again for \( y_{i-1} \in Y_{i-1}, x_i \in X_i, z_{i+1} \in Z_{i+1} \) we get
\[
|N(y_{i-1}) \cup N(x_i) \cup N(z_{i+1})| \geq 3\delta - (|Z_{i-1}| + \ell_i + |Y_{i+1}|).
\]
We obtain \([18]\), like in the previous subcase.

**Case 2.** \( i - 1 \notin S, i \in S, i + 1 \notin S \). We may assume \( L_i = Z_i \), and for \( j \in \{i - 1, i + 1\} \) \( L_j = X_j \cup Y_j \), where none of \( X_{i-1}, Y_{i+1}, X_{i}, Y_{i-1}, Z_i \) is empty. This can be handled like Subcase 1.1 to obtain \([18]\).

**Case 3.** \( i - 1 \in S, i \in S, i + 1 \in S \). We can assume \( L_{i-1} = X_{i-1}, L_i = Y_i, L_{i+1} = Z_{i+1} \) (in case \( L_{i+1} = X_{i+1} \), switch colors \( X \) and \( Z \) in layers \( L_j \) for \( j \geq i + 1 \)). Select \( x_{i-1} \in X_{i-1}, y_i \in Y_i, z_{i+1} \in Z_{i+1} \), and apply inclusion-exclusion for \( |N(x_{i-1}) \cup N(y_i) \cup N(z_{i+1})| \) to obtain
\[
2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) \geq 6\delta - 2\ell_i.
\]

**Case 4.** \( i - 1 \in S, i \in S, i + 1 \notin S \). As the clump graph is canonical, \( c(i + 1) = 2 \). Hence we can assume \( L_{i-1} = X_{i-1}, L_i = Y_i, L_{i+1} = X_{i+1} \cup Z_{i+1} \). Applying inclusion-exclusion for the neighborhoods of representative elements, we obtain
\[
|N(x_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3\delta - \ell_i - |Z_{i+1}|
\]
and
\[
|N(x_{i-1}) \cup N(y_i) \cup N(z_{i+1})| \geq 3\delta - \ell_i - |X_{i+1}|.
\]
Combining the last two displayed formulae, we obtain
\[
2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) \geq 6\delta - 2\ell_i - \ell_{i+1},
\]
which is even stronger than \([18]\).

**Case 5.** \( i - 1 \notin S, i \in S, i + 1 \in S \). This is a mirror image of Case 4, so we have
\[
2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) \geq 6\delta - \ell_{i-1} - 2\ell_i.
\]

**Case 6.** \( i - 1 \in S, i \notin S, i + 1 \in S \). We may assume \( X_{i-1} = L_{i-1}, Y_i \cup Z_i = L_i, X_{i+1} = L_{i+1}, \) where \( X_{i-1}, Y_i, Z_i, X_{i+1} \) are nonempty. Select \( x_{i-1} \in X_{i-1}, y_i \in Y_i, z_i \in Z_i, x_{i+1} \in X_{i+1} \).

Clearly
\[
|L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1} \cup L_{i+2}| \geq |N(x_{i-1}) \cup N(x_{i+1})| + |N(y_i) \cup N(z_i)| - |(N(x_{i-1}) \cup N(x_{i+1})) \cap (N(y_i) \cup N(z_i))| \geq (2\delta - \ell_i) + (2\delta - \ell_{i-1} - \ell_{i+1}) - \ell_i = 4\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}.
\]
We conclude

\begin{equation}
2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) \geq 8\delta - 4\ell_i - 2\ell_{i-1} - 2\ell_{i+1}.
\end{equation}

Case 7. \(i - 1 \in S, i \notin S, i + 1 \notin S\). We may assume \(X_{i-1} = L_{i-1}, Y_i \cup Z_i = L_i, \) where \(X_{i-1}, Y_i, Z_i\) and \(X_{i+1}\) are nonempty. Select \(x_i \in X_{i-1}, y_i \in Y_i, z_i \in Z_i, x_{i+1} \in X_{i+1}\). Clearly

\[
|L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1} \cup L_{i+2}| \geq |N(x_{i-1}) \cup N(x_{i+1}) \cup N(y_i) \cup N(z_i)|
\]

\[
\geq (2\delta - \ell_i) + (2\delta - \ell_{i-1} - \ell_{i+1}) - \ell_i
\]

\[
= 4\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}.
\]

We conclude \([19]\) again.

Case 8. \(i - 1 \notin S, i \notin S, i + 1 \in S\). As this is the mirror image of Case 7', we arrive at the same conclusion \([19]\), as the conclusion is symmetric.

For \(1 \leq i \leq D - 1\), we call a triplet of consecutive layers \((i - 1, i, i + 1)\) singular, if \(i \notin S\) and \((i + 1 \in S\) or \(i - 1 \in S\)). Let \(s\) denote the number of singular triplets. Summing up the lower bounds to \((17)\) obtained in the 8 cases, we have

\[
10n \geq 6\delta D - 2n + O(\delta) - \sum_{i \in S, i \neq i+1} (\ell_{i-1} + \ell_{i+1}) - \sum_{i \in S, i \neq i+1} (\ell_{i-1} + \ell_{i+1})
\]

\[
- \sum_{i \in S, i = i+1} \ell_{i+1} - \sum_{i \in S, i = i+1} \ell_{i-1} - \sum_{i \in S, i = i+1} (-2\delta + 2\ell_i + \ell_{i-1} + \ell_{i+1})
\]

\[
= 6\delta D - 2n + 2s\delta + O(\delta) - \sum_{i \in S, i \neq i+1} \ell_i - \sum_{i \in S, i \neq i+1} \ell_i - \sum_{i \in S, i \neq i+1} \ell_i - \sum_{i \in S, i \neq i+1} \ell_i
\]

\[
- \sum_{i \in S, i = i+1} \ell_i - \sum_{i \in S, i = i+1} \ell_i - 2 \sum_{i \in S, i = i+1} \ell_i - \sum_{i \in S, i = i+1} \ell_i - \sum_{i \in S, i = i+1} \ell_i
\]

Now we use that

\[
\sum_{i \in S, i \neq i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i \leq \sum_{i \in S, i \neq i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i + 2 \sum_{i \in S, i = i+1} \ell_i
\]

and

\[
2 \sum_{i \in S} \ell_i = \left( \sum_{i \in S, i = i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i \right)
\]

\[
+ \left( \sum_{i \in S, i = i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i + \sum_{i \in S, i = i+1} \ell_i \right)
\]
to obtain
\[
12n \geq 6\delta D + 2s\delta + O(\delta) - 2 \sum_{i:j \notin S} \ell_i - 2 \sum_{i-1 \in S \land i+1 \in S} \ell_i - 2 \sum_{i-1 \in S \land i+1 \in S} \ell_i
\]
\[
= 6\delta D + 2s\delta - 2n + O(\delta) - 2 \sum_{i-1 \in S \land i+1 \in S} \ell_i.
\]

This gives
\[
(20) \quad 7n \geq 3\delta D + s\delta + O(\delta) - \sum_{i-1 \in S \land i+1 \in S} \ell_i.
\]

7. Optimization

Figure 7. Visual representation for some variables denoted with Greek letters. Layers with black filled circles represent the layers whose vertices we count, the empty circles show how many colors are present in the nearby layers. Gray filled circles represent a third color that may or may not be present in the layer.

The inequalities (16) and (20) are key constraints for our linear program. The linear program is in global variables, which are mostly the fraction of vertices of \( G \) in certain type of layers, which live in a neighborhood of certain type of layers. The global variables, denoted by Greek letters, will be:

\[
\mu = \frac{1}{n} \sum_{i \in (i) = 1} \ell_i
\]

\[
\alpha_1 = \frac{1}{n} \sum_{i:0 < i < D, c(i) = 2, i-1 \in S, i+1 \notin S} \ell_i
\]

\[
\alpha_2 = \frac{1}{n} \sum_{i:0 < i < D, c(i) = 2, i-1 \notin S, i+1 \in S} \ell_i + \frac{1}{n} \sum_{i:0 < i < D, c(i) = 2, i+1 \in S, i-1 \notin S} \ell_i
\]

\[
\phi = \frac{D\delta}{n}
\]

\[
\psi = \frac{s\delta}{n}
\]
Figure 7 illustrate the variables whose definition involves sums. Clearly, all variables are non-negative. We use Corollary 8 on what kind of layers can be consecutive. From the definitions, it easily follows that

\[(21) \quad \mu + \alpha_1 + \alpha_2 \leq 1 \quad \text{and} \quad \psi \leq \frac{2}{3}.\]

We have

\[
\sum_{(i,j) : i \notin S, j \notin S, |i-j|=1} \ell_i = 2n(1 - \alpha_1 - \alpha_2 - \mu) + n\alpha_2 + O(\delta) = n \left(2 - 2\mu - 2\alpha_1 - \alpha_2 + 2 + \frac{\delta}{n}\right),
\]

since (except possibly for \(i = D\)) \(\ell_i\)'s accounted for in the definition of \(\mu\) and \(\alpha_i\) do not contribute to the sum on the left side, \(\ell_i\)'s accounted for in \(\alpha_2\) appear once, and all other \(\ell_i\)'s appear twice. In addition,

\[
\sum_{i: i+1 \in S, i \notin S} \ell_i + \sum_{i: i-1 \in S, i \notin S} \ell_i = n \left(2\alpha_1 + \alpha_2 + \frac{\delta}{n}\right).
\]

Using these observations, simple algebra derives from (20)

\[(22) \quad 12\phi + 4\mu - 2\alpha_1 - \alpha_2 \leq 28 + O\left(\frac{\delta}{n}\right).
\]

From (20) using

\[
\sum_{i: i \notin S \lor i \notin S} \ell_i = n(\alpha_1 + \alpha_2)
\]

we get

\[(23) \quad 7 \geq 3\phi + \psi - \alpha_1 - \alpha_2 + O\left(\frac{n}{\delta}\right).
\]

Let \(D\) denote the set of layers with 2 colors, with singles on both side. (Their cardinalities added up to \(\alpha_1\).) Let \(E\) denote the set of layers that are adjacent to at least one layer from \(D\). Hence all layers in \(E\) are singles. Let \(F\) denote the set of remaining layers, i.e. not in \(D \cup E\). First note that

\[(24) \quad |D| + |E| + |F| = D + 1.\]

By the minimum degree condition, for all \(i : 0 < i < D\) we have \(\delta \leq \ell_{i-1} + \ell_i + \ell_{i+1}\). Hence, \(\delta |F| \leq \sum_{i \in F}(\ell_{i-1} + \ell_i + \ell_{i+1}) \leq 3n(1 - \alpha_1) + O(\delta)\) and

\[(25) \quad |F| \leq 3(1 - \alpha_1)\frac{n}{\delta} + O(1).
\]

It is not difficult to see that \(|D| + |E| \leq 3s\). Using this observation with (24) and (25), we obtain

\[
3\phi = 3\frac{\delta}{n}(|D| + |E| + |F| - 1) \leq \frac{3\delta}{n}(3s) + \frac{3\delta}{n}|F| + O\left(\frac{\delta}{n}\right).
\]
and hence

\[(26) \quad \phi \leq 3\psi + 3(1 - \alpha_1) + O\left(\frac{\delta}{n}\right).\]

We tried to use more inequalities and more variables, splitting \(\alpha_1\) further, based on the number of colors in the layers before and after. Removing redundant variables and conditions, we finalized our linear program based on constraints \((21), (22), (23)\) and \((26)\) as follows:

**Maximize** \(\phi = \frac{D\delta}{n}\)

subject to

\[
\begin{align*}
\mu + \alpha_1 + \alpha_2 & \leq 1 \\
3\psi & \leq 2 \\
12\phi + 4\mu - 2\alpha_1 - \alpha_2 & \leq 28 + O\left(\frac{\delta}{n}\right) \\
3\phi + \psi - \alpha_1 - \alpha_2 & \leq 7 + O\left(\frac{\delta}{n}\right) \\
\phi - 3\psi + 3\alpha_1 & \leq 3 + O\left(\frac{\delta}{n}\right)
\end{align*}
\]

\(\phi, \mu, \psi, \alpha_1, \alpha_2 \geq 0\)

Let \(x = (x_1, x_2, x_3, x_4, x_5)^T = (\phi, \mu, \psi, \alpha_1, \alpha_2)^T\), let \(A\) be the \(5 \times 5\) coefficient matrix above, \(b = (1, 2, 28, 7, 3)^T\), and \(h\) be any concrete error term in the constraint column within the \(O\left(\frac{\delta}{n}\right)\) bounds. Let \(y = (y_1, y_2, y_3, y_4, y_5)\). Consider now four closely related linear programs:

\[(27) \quad Ax \leq b + h; \quad x \geq 0; \quad \text{maximize } x_1;\]

\[(28) \quad Ax \leq b; \quad x \geq 0; \quad \text{maximize } x_1;\]

\[(29) \quad yA \geq (1, 0, 0, 0, 0, 0, 0); \quad y \geq 0; \quad \text{minimize } y(b + h)^T;\]

\[(30) \quad yA \geq (1, 0, 0, 0, 0, 0, 0); \quad y \geq 0; \quad \text{minimize } yb^T.\]

Our standard reference to linear programming is [1]. Note that \((27)\) is identical to the displayed linear program, and that \((27)\) and \((29)\), and \((28)\) and \((30)\) are dual linear programs, respectively, and the Duality Theorem of Linear Programming applies to them. Utilizing the open source online tool [9], we solved \((28)\) with optimum \(\phi = \frac{57}{23}\) attained at \((\frac{22}{23}, 0, \frac{22}{23}, \frac{22}{23}, \frac{22}{23})^T\). By duality, \(\frac{57}{23}\) is the optimum of \((30)\) as well. The polytope defined by the constraints of \((28)\) has a feasible solution \(x^*\), for which inequalities in the 3rd, 4th and 5th constraints hold strictly—just modify the optimal solution by reducing \(\phi\) a bit. We want to show that \((27)\) has a finite optimum, if \(n\) is sufficiently large. By the first constraint in \((27)\), \(\phi \leq 3\) for \(n\) sufficiently large. Our only concern is whether \((27)\) has a feasible solution at all, as negative error terms might eliminate it. Clearly \(x^*\) is a feasible solution, if \(n\) is sufficiently large. By the Duality Theorem, \((29)\) has a finite minimum value, which is equal to the maximum value for \((27)\). As the polytopes of \((29)\) and \((30)\)
are the same, this finite minimum is achieved in one of the finitely many vertices of this polytope, say $y^{(1)}, \ldots, y^{(m)}$, as these linear programs only differ in their objective functions. Now we have
\[
\max \ x_1 \ \text{in (27)} = \min_{y \geq 0} y^T (b + h) = \min_{i=1}^m y^{(i)}(b + h)^T \\
\geq \ \min_{i=1}^m y^{(i)}b^T + \min_{i=1}^m y^{(i)}h^T = \frac{57}{23} + O \left( \frac{\delta}{n} \right).
\]

On the other hand,
\[
\max \ x_1 \ \text{in (27)} = \min_{y \geq 0} y^T (b + h) = \min_{i=1}^m \left( y^{(i)}b^T + y^{(i)}h^T \right) \\
\leq \ \min_{i=1}^m \left( y^{(i)}b^T + \max_{i=1}^m y^{(i)}h^T \right) = \min_{i=1}^m y^{(i)}b^T + \max_{i=1}^m y^{(i)}h^T \\
= \frac{57}{23} + O \left( \frac{\delta}{n} \right).
\]

We concluded the proof of Theorem 4. The linear programming arguments above should be well-known, but we were unable to find a reference.

The following theorem proves the weaker version of Conjecture 2 for $k = 3$, in a restricted case of no single layers:

**Theorem 11.** For every connected 3-colorable graph $G$ of order $n$ and minimum degree at least $\delta \geq 1$, such that in the canonical clump graph of $G$ no layer $L_i$ is a single for $0 < i < D$, we have
\[
\text{diam}(G) \leq \frac{7n}{3\delta} + O(1).
\]

**Proof.** If there are no single color layers besides $L_0$ and $L_D$, in (16) the second and third sums are zero, and the first is upper bounded by $\frac{2}{3}n$. This yields $14n/3 \geq 2D\delta + O(\delta)$. An alternative proof of the theorem is from [20] in which $s = 0$ and the sum is $O(\delta)$ in this case.

The theorem also holds if the number of single layers is bounded as $n \to \infty$. We are not aware of constructions getting close to this upper bound without single layers.

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