On the Equivalence Principle and a Unified Metric Description of Gravitation and Electromagnetism

Murat Özer
Department of Physics,
Faculty of Arts and Sciences,
Yıldız Technical University,
34220 Esenler, Istanbul, Turkey
E-mail: mhozer@yildiz.edu.tr

Abstract
We first investigate the form the General Relativity theory would have taken had the gravitational mass and the inertial mass of material objects been different. We then extend this analysis to electromagnetism and postulate an equivalence principle for the electromagnetic field. We argue that to each particle with a different electric charge-to-mass ratio in superimposed gravitational and electromagnetic fields there corresponds a spacetime manifold whose metric tensor $g_{\mu\nu}$ describes the dynamical actions of gravitation and electromagnetism. The electric field outside a charged sphere asserts itself independently rather than contributing to the gravitational field. The contribution of the electric field to the spacetime metric outside the charged sphere is shown to be similar to the gravitational one in the Schwartzschild metric but with a charge-to-mass ratio dependence of the test particle instead of the Reissner-Nordström metric, resulting in a unified description of gravitation and electromagnetism. We point out that there are existing experiments whose results can be explained by the equivalence principle for the electromagnetic field presented here. Additional experimental predictions of the theory are mentioned.

1 Introduction
The possibility of exhibiting gravitation and electromagnetism in a unified geometrical representation has been pursued by many mathematicians and physicists. The first one to seek for a unified explanation of gravitation and electromagnetism was Riemann (see Ref.[1]). This endeavor has really started as a full-fledged research area soon after the advent of Einstein’s general relativity theory [2]. The gauge-invariant unified theory of Weyl was based on a generalization of Riemannian geometry [3][4]. A generalization of Weyl’s theory was put forward by Eddington [5]. These unsuccessful attempts were followed by Kaluza who sought to include the electromagnetic field by increasing the number of components of the metric tensor by changing the number of dimensions to five
[6], whose work was later revived and extended by Klein [7]. Generalizations of Kaluza’s
topology were attempted by Einstein and coworkers [8, 9, 10]. Another line of approach
that produced the same field equations as in Kaluza’s theory was that of projective field
theories [11, 12]. Also worth mentioning is the work of Einstein based on Riemannian
metrics and distant parallelism [13]. Since the electromagnetic field is described by a
second rank antisymmetric tensor, the idea of employing a nonsymmetric metric tensor $g_{\mu\nu}$
whose antisymmetric part is to be associated with electromagnetism was exercised too [14].

All these attempts to obtain a unified theory of gravitation and electromagnetism in a
four dimensional Riemannian spacetime have been baffled hitherto. What lies at the root
of this bafflement is the fact that electrically charged particles do not possess a universal
charge-to-mass ratio. It is often stated that a geometric theory of electromagnetism could
have been achieved had this ratio been the same for all particles. One immediate conse-
quence of such a hypothetical unified theory through a symmetric $g_{\mu\nu}$ in four dimensions
would have been having to relax the interpretation of the $g_{\mu\nu}$ as the gravitational field since
then the components of which would have corresponded to superimposed gravitational and
electromagnetic potentials.

Having failed in these attempts, instead of yielding to a complete failure, it is our
opinion that, be it not universal, a restricted or specific geometrization of electromagnetism
should be sought. The way to achieve this seems to consider the motion of each charged
test particle one by one in a given electromagnetic field and geometrize each case seperately.
The resulting geometrization, of course, will not be of the universal type as in gravitation.
We hope to present in this work the grounds for reasons to believe that this endeavor may
be full of experimentally testable surprizes.

To this end, we shall first postulate an equivalence principle for the electromagnetic
field by way of examples and thereby conclude that the general relativity theory, after a
correction, is not only the theory of gravitation but also a unified theory of gravitation and
electromagnetism. In order to reach this conclusion we ought to emancipate ourselves from
two conceptual obstacles. First, that the equality of the gravitational mass and the inertial
mass is indispensable for the formulation of general relativity (hereafter GR). Second, that
the metrical field $g_{\mu\nu}$ represents the gravitational potentials only. Of course, it is indeed
a remarkable fact of nature that the gravitational and inertial masses associated with all
material objects are equal to a great accuracy [15, 16, 17]. As a result of this equality a
given gravitational field imparts the same acceleration to all particles at a given spacetime
point. Einstein generalized the experimental results on the equality of these two masses to
the (weak) Equivalence principle [18], that a uniform gravitational field and a uniformly
accelerating frame are locally equivalent, or stated slightly differently gravitational and
inertial forces are locally completely equivalent.

This paper is organized as follows. In Sec. 2 we consider a hypothetical world where
the gravitational and inertial masses of material objects are different. We write down the
Einstein field equations in such a world. In Sec. 3 we treat Newtonian electromagnetism in
the language of curved spacetime. The energy of a test charge in the vicinity of a charged
sphere is considered to obtain the $g_{00}$ component of the resulting spacetime metric in
Sec. 4. In Sec. 5 we consider the action integral for a charged particle in superimposed
gravitational and electromagnetic fields and obtain the same $g_{00}$ once more. In Sec. 6 we propose an elevator thought experiment for charged particles in an electromagnetic field. In Sec. 7 we propose the modified field equations for a distribution of mass and electric charge. The line element for a spherically symmetric distribution of matter and charge is presented in Sec. 8. Concluding remarks are presented in Sec. 9.

2 Hypothetical General Relativity with $m_g \neq m_i$

The question of the gravitational mass $m_g$ of a body not being equal to its inertial mass $m_i$ due to the possibility that the gravitational self-energy of the body contributes unequally to $m_g$ and $m_i$ was addressed in a series of papers by Nordtvedt [19]. He argues that if it is assumed

$$\frac{m_g}{m_i} = 1 + \eta \frac{G}{c^2} \int \rho(\vec{x}) \rho(\vec{x'}) \frac{d^3x d^3x'}{|\vec{x} - \vec{x}'|} / \int \rho(\vec{x}) d^3x,$$  \hspace{1cm} (1)

where $\eta$ is a dimensionless constant of order of magnitude 1, $G$ is the gravitational constant, $c$ is the speed of light, and $\rho(\vec{x})$ is the mass density of the body, then for the bodies used in the experiments of Refs. [15, 16, 17], the correction term in Eq.(1) is of order $10^{-25}$, thereby not contradicting these experiments. We would like to argue in the following, by way of a thought experiment, that the basic structure of GR would have remained intact had the gravitational mass been not equal to the inertial mass.

Consider an elevator cabin falling freely in a given gravitational field $\vec{g}$. Let there be test particles inside the cabin with different $m_g/m_i$ ratios. Let the elevator, an observer in it, and one of the particles have the same $m_g/m_i$ ratio. As the elevator falls, let the observer drop the test particles simultaneously from rest. He/she will then see the particles strike the floor or the ceiling of the elevator one by one according to their acceleration relative to the elevator (or the observer)

$$a_{rel} = \left(\frac{m_g}{m_i} - \frac{M_g}{M_i}\right) g,$$  \hspace{1cm} (2)

where $M$ is the mass of the elevator. But the test particle having the same ratio as the elevator will float motionlessly. What has happened is that the gravitational force on this particle has been cancelled by the inertial force on it due to the downward acceleration of the elevator. We, therefore, conclude that this freely falling elevator is an inertial frame only for this particular particle, but not for the others. Stated equivalently, had the $m_g/m_i$ ratio of the particles been different in a hypothetical world there would have been locally inertial but nonidentical frames unique to each particle or particles with the same $m_g/m_i$ ratio. This is in contrast to what happens when gravitational and inertial masses are equal in which case the local inertial frames in the neighborhood of each particle are identical.

---

1 Notice that the gravitational mass $m_g$ here is actually the passive gravitational mass, which is the mass acted upon by a gravitational field. Because of their equality in Newtonian mechanics, we do not distinguish in this work between the passive and active gravitational masses, the latter being the mass that gives rise to a gravitational field, and use the term gravitational mass only.
and the particles move freely in the same geometry. But, since a given point may contain
only one particle at a given time, and each particle obeys its own equation of motion, there
is no reason why particles could not have travelled in their own geometry had their $m_g$
been different from their $m_i$. Each particle, then, would have followed its own geodesic
according to
\[ \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0, \]
where the Christoffel symbols (connection coefficients) $\Gamma^\mu_{\alpha\beta}$ would have depended on the
$m_g/m_i$ ratio of the test particle and $\lambda$ is an affine parameter, such as the proper
time $\tau$ or the proper length $s$, of the geodesic. The equivalence principle then would have been
it is impossible to distinguish the fictitious inertial forces from the real gravitational forces
in a local region containing a single particle or particles with the same $m_g/m_i$ ratio. This
we shall call the single-particle equivalence principle. What would have happened to the
Einstein field equations in such a hypothetical world? By considering the Newtonian limit
it can be seen that the field equations would have taken the form
\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}, \]
where
\[ T^{\mu\nu} = \frac{m_g}{m_i} T^{\mu\nu}, \]
with $T^{\mu\nu}$ being the energy-momentum tensor of a distribution of matter or other forms of
energy. Hence the solutions of Eq. (4) would have involved $m_g/m_i$ of the test particle.
For example, the Schwarzschild exterior solution \[20\] for a static spherical distribution of
mass $M_g$ would have been
\[ ds^2 = -\left(1 - 2 \frac{m_g GM_g}{m_i c^2 r}\right) c^2 dt^2 + \left(1 - 2 \frac{m_g GM_g}{m_i c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]
Note, in such a world, that test particles with different $m_g/m_i$’s would have had different
spacetime geometries when they are in the same gravitational field.

3 Newtonian Electromagnetism in the Language of Curved Spacetime

Having presented what would have happened to GR in a hypothetical world where $m_g \neq m_i$, we can immediately draw a parallelism with electromagnetism in which the field charge is the electric charge $q$ of a particle instead of the gravitational mass $m_g$. To gain further insight into our problem and to convince ourselves that we are on the right path, let us translate Newtonian gravity in the language of curved spacetime (a la Cartan) \[21, 22\] into Newtonian electromagnetism in the language of curved spacetime. The trajectory of a charged particle in an electromagnetic field subject to the force
\[ \vec{F} = q \vec{E} + q \vec{v} \times \vec{B}, \]
Incidentally, Eqs. (4) and (5) indicate that $m_g/m_i = 1$ is imposed in Einstein’s GR.
is given in Newtonian electromagnetism by (i=1,2,3 and summation over repeated indices is implied.\footnote{From now on we denote the inertial mass \(m_i\) by \(m\), whenever there is no confusion.})

\[
\frac{d^2 x^i}{dt^2} + \frac{q}{m} \left( \frac{\partial \Phi_E}{\partial x^i} + \frac{1}{c} \frac{\partial A^i}{\partial t} \right) + \frac{q}{m} \epsilon^{ijk} k_{ln} \frac{dx^j}{dt} \frac{\partial A^n}{\partial x^l} = 0,
\]

(8)

where \(t\) is the coordinate time, \(\Phi_E\) is the electric potential, and \(\vec{A}\) is the vector potential. This can be written in curved spacetime as,

\[
\frac{d^2 t}{d\lambda^2} = 0, \quad \frac{d^2 x^i}{d\lambda^2} + \frac{q}{m} \left( \frac{\partial \Phi_E}{\partial x^i} + \frac{1}{c} \frac{\partial A^i}{\partial t} - \frac{dx^j}{dt} \frac{\partial A^n}{\partial x^l} \right) \left( \frac{dt}{d\lambda} \right)^2 = 0,
\]

(9)

where the geodesic parameter \(\lambda = at + b\), \(a\) and \(b\) being arbitrary constants. By comparing Eq. (9) with the geodesic equation (3) the nonzero connection coefficients are read off as

\[
\Gamma^i_{00} = \frac{q}{mc^2} \left( \frac{\partial \Phi_E}{\partial x^i} + \frac{1}{c} \frac{\partial A^i}{\partial t} - \frac{dx^j}{dt} \frac{\partial A^n}{\partial x^l} \right).
\]

(10)

By inserting these in the Riemann tensor

\[
R^\mu_{\nu\alpha\beta} = \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} - \frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} + \Gamma^\mu_{\gamma\alpha} \Gamma^\gamma_{\nu\beta} - \Gamma^\mu_{\gamma\beta} \Gamma^\gamma_{\nu\alpha}
\]

(11)

the nonzero components are found to be

\[
R^i_{0j0} = -R^j_{0i0} = \frac{q}{mc^2} \left( \frac{\partial^2 \Phi_E}{\partial x^i \partial x^j} + \frac{1}{c} \frac{\partial^2 A^i}{\partial t \partial x^j} - \frac{dx^n}{dt} \frac{\partial A^n}{\partial x^l} \right).
\]

(12)

The only nonzero components of the Ricci curvature tensor

\[
R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}
\]

is found to be

\[
R_{00} = \frac{q}{mc^2} \left[ \nabla^2 \Phi_E + \frac{1}{c} \frac{\partial}{\partial t} \left( \vec{\nabla} \cdot \vec{A} \right) - \vec{v} \cdot \nabla \vec{A} + \vec{v} \cdot \vec{\nabla} \left( \vec{\nabla} \cdot \vec{A} \right) \right],
\]

(14)

where \(\vec{v}\) is the velocity of the test particle. Using the equations

\[
\nabla^2 \Phi_E + \frac{1}{c} \frac{\partial}{\partial t} \left( \vec{\nabla} \cdot \vec{A} \right) = -4\pi k_e \rho_Q, \quad \vec{\nabla} \times \vec{B} = 4\pi k_m \left( \vec{J} + \vec{J}_D \right),
\]

(15)

where \(k_e\) and \(k_m\) are the electric (Coulomb) and the magnetic (Biot-Savart) constants, \(\rho_Q\) is the charge density, \(\vec{J}\) is the ordinary current density, and \(\vec{J}_D = 1/(4\pi k_e) \partial \vec{E}/\partial t\) is the displacement current density, Eq. (14) becomes

\[
R_{00} = -\frac{q}{mc^2} 4\pi \left[ k_e \rho_Q - k_m \vec{v} \cdot \left( \vec{J} + \vec{J}_D \right) \right] = -\frac{q}{mc^2} 4\pi k_e \left[ \rho_Q - \frac{1}{c^2} \vec{v} \cdot \left( \vec{J} + \vec{J}_D \right) \right],
\]

(16)

\footnote{After the use of \(\epsilon_{ijk} k_{ln} = \delta_{il} \delta_{jn} - \delta_{in} \delta_{lj} \).}
where \( k_m/k_e = 1/c^2 \) has been used. Noting that \( R_{00} = \partial \Gamma_{00}^i/\partial x^i \) and \( \Gamma_{00}^i = (g^{ii}/2)(-\partial g_{00}/\partial x^i) \) it follows that

\[
R_{00} \approx -\frac{1}{2} \nabla^2 g_{00},
\]

where we have set \( g^{11} = g^{22} = g^{33} \approx 1 \). Assuming \( \nabla^2 \vec{v} = 0 \) and \( \nabla \cdot \vec{A} = 0 \), Eqs. (16) and (17) are satisfied by

\[
g_{00} \approx -\left(1 + 2 \frac{q \Phi_E}{m c^2}\right).
\]

In a region where there are superimposed gravitational and electromagnetic fields, the above procedure would give

\[
ds^2 = -\left(1 + 2 \frac{\Phi_G}{c^2} + 2 \frac{q \Phi_E}{m c^2}\right) c^2 dt^2 + 2 \frac{q}{m} \vec{A} \cdot d\vec{x} dt,...,
\]

where \( \Phi_G \) is the gravitational potential.

Lo and behold, these equations reveal that a distribution of electric charge curves the spacetime just like a neutral mass distribution does (apart from the magnitude and a possible difference in the sense of the curvature). The motion of a test charge in an electromagnetic field is thus geometrized by connecting electromagnetic potentials to the metric of the spacetime. One distinct feature different from gravitation is that test particles with different \( q/m \)'s have their own geometries in the same electromagnetic field, whereas all test particles have the same geometry in a gravitational field irrespective of their masses.

### 4 A Charged Test Particle in the Vicinity of a Charged Sphere

Another argument which gives the same \( g_{00} \) presented above is as follows. Consider an electrically charged metallic sphere of mass \( M \) and charge \( Q \). Let there be a test charge \( q \) of mass \( m \) in the superimposed gravitational and electrical potentials of the sphere. For simplicity, let us assume that the test charge is at rest at a position \( r \) from the center of the sphere. The energy of the test charge is given, upon factoring \( m_i c^2 \) out, by

\[
E = m_i c^2 \left(1 - \frac{m_q GM}{m_i c^2 r} - \frac{q}{m_i} \frac{k_e Q}{c^2 r}\right) \\
\approx m_i c^2 \left(1 - \frac{2 m_q GM}{m_i c^2 r} + \frac{2q}{m_i} \frac{k_e Q}{c^2 r}\right)^{1/2},
\]

where it has been assumed that the second and the third terms on the right are much smaller than one. Comparing this with the general relativistic exact equation for \( Q = 0 \) \[25\]

\[
E = mc^2 (-g_{00})^{1/2} \sqrt{1 - v^2/c^2}
\]

Note that the emergence of the term \( (\vec{A} \cdot d\vec{x}) dt \) is similar to what happens in Gravitomagnetism \[23 \ 24\] where there is a gravitomagnetic vector potential \( \vec{A}_g \) like the electromagnetic vector potential \( \vec{A} \).
we obtain the correct expression for $g_{00}$ upon putting $v = 0$. There is no reason why this purely classical argument, which is correct in the gravitational case, should fail when it is extended to include the electric potential energy of the test charge. Therefore we obtain

$$g_{00} = -\left(1 - 2\frac{GM}{c^2r} + 2\frac{q}{m} \frac{k_e}{c^2r} \right), \quad (22)$$

where the ratio $m_g/m_i$ has been set to one.

5 The Action Integral for a Charged Particle

Another supporting clue for the unified description of gravitation and electromagnetism in the manner we contemplate comes from the action integral for a charged particle moving in a region where there are superimposed gravitational and electromagnetic fields. The relativistic Lagrangian for a test particle of mass $m$ and electric charge $q$ is

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - m\Phi_G - q\Phi_E + q \vec{A}.\vec{v} \quad (23)$$

where $\Phi_G$ and $\Phi_E$ are the gravitational and electrical potentials. Even though there is no experiment to support it, the prevailing assumption in the literature dictates the action

$$I = \int_{t_1}^{t_2} \mathcal{L} dt \quad (24)$$

corresponding to $\mathcal{L}$ in eq. (23) to be written as

$$I = \int_{t_1}^{t_2} \left(-mc \frac{ds}{dt} - q\Phi_E + q \vec{A}.\vec{v}\right) dt, \quad (25)$$

where $-mc\frac{ds}{dt}$ contains only the first two terms in Eq. (23). Emancipating ourselves from this assumption and including all the terms of $\mathcal{L}$ in $-mc\frac{ds}{dt}$ entails

$$ds^2 = -\left(1 + 2\frac{\Phi_G}{c^2} + 2\frac{q}{m} \frac{\Phi_E}{c^2} - 2\frac{q}{m} \frac{\vec{v}.\vec{A}}{c^2}\right) c^2 dt^2 + ... \quad (26)$$

where terms that vanish as $c \to \infty$ have been dropped. The idea then suggests itself that the $g_{00}$ component of the metric tensor $g_{\mu\nu}$ is

$$g_{00} = -\left(1 + 2\frac{\Phi_G}{c^2} + 2\frac{q}{m} \frac{\Phi_E}{c^2}\right), \quad (27)$$

which indicates again that the electromagnetic field curves the spacetime on the same footing as the gravitational field. In the current GR theory the metric tensor $g_{\mu\nu}$ is also interpreted as the gravitational field proper. Since the components of $g_{\mu\nu}$ are determined...
sufficiently by the Einstein field equations it is believed that there is no room for the electromagnetic field in the same geometry. Our treatment of the electromagnetic field a la Cartan, the elevator experiments, and the $g_{00}$ we have obtained in Eqs. (19), (22), and (27) indicate that this interpretation may not be correct. We are motivated by Eqs. (17) and (18) to suggest that they correspond to the Newtonian limit of more fundamental tensor equations involving the Ricci tensor $R_{\mu\nu}$. Before we write down these equations, to convince ourselves more let us present the elevator cabin thought experiments by replacing the gravitational field by an electromagnetic field. The situation is very much like that in the hypothetical gravity with $m_g \neq m_i$.

6 The Elevator Thought Experiment for Charged Particles

Inasmuch as there exists a local inertial frame for every particle or particles with the same field charge-to-inertial mass ratio, we shall consider only one test particle in the following. Consider again a closed and stationary elevator cabin with an observer and a test particle in it. Let the elevator, the observer, and the test particle have the same electric charge-to-mass ratio $q/m$. For simplicity and definiteness assume that all the charges are positive. Let there be no gravitational field but an external downward uniform electric field $\vec{E}$ act on the system. When released from rest by the observer, the test particle will move downward with an acceleration $a = (q/m)E$. Now, let this system be moved into space where there are no fields of any kind to act on it, and let it be accelerated upward by an external agent with an acceleration whose magnitude is equal to that above. The floor of the elevator will accelerate towards the test particle released by the observer from rest. From the point of view of the observer the static elevator and the accelerated elevator situations are equivalent. Under these conditions he/she cannot distinguish between the existence of the electric field and the acceleration of the elevator. The single-particle equivalence principle for the electric field may thus be stated as: it is impossible to distinguish the fictitious inertial forces from the real electric forces in a local region containing a single particle or particles with the same electric charge-to-mass ratio. We are proposing this equivalence principle because it seems to correspond to reality. After all, when a collection of particles with different $q/m$’s are released from rest in a uniform electric field they will form groups as they accelerate according to their $q/m$’s. Each such group of particles will have the same spacetime manifold and thus may be taken collectively as test particles. Let us also note that in the case when the elevator is let to fall freely in the downward electric field considered above, the test particle released will float as if the elevator were motionless in free space. Again, since the acceleration of the test particle relative to the elevator (or the observer) has ceased because

$$a_{r\text{el}} = \left(\frac{q}{m} - \frac{Q}{M}\right)E = 0,$$

(28)

---

6 The rule for the direction of the acceleration of the cabin is that it be opposite the direction of motion of the test particle.

7 It is assumed, as usual, that the interactions between the particles are negligible.
where \( Q \) and \( M \) are the electric charge and the mass of the elevator, the elevator constitutes a local inertial frame.

Next, let us consider the same elevator and its contents in a region where there is only a uniform downward magnetic field \( \vec{B} \). Let the test particle be released horizontally with velocity \( v \) towards the front wall of the elevator. As the observer faces the front wall, he/she will see the particle deflect counterclockwise towards his/her left with an acceleration \( a = |(q/m)\vec{v} \times \vec{B}| = (q/m)vB \). Afterwards, let the elevator be rotated uniformly in a clockwise fashion by an external agent (e.g. a rigid rod of length \( R \) attached to the top of the elevator, the other end of the rod being fixed and serving as the rotation center.) with angular velocity \( \vec{\Omega} \) which causes a Coriolis acceleration equal to that above, i.e. \( a = |2\vec{v} \times \vec{\Omega}| = 2v\omega_L = (q/m)vB \); provided the angular frequency \( |\vec{\Omega}| = \omega_L = (q/2m)B \) is small enough so as to neglect the centrifugal acceleration \( \omega_L^2 R = (q/2m)^2 B^2 R \) compared to the Coriolis acceleration resulting in the condition \( \omega_L \ll 2v/R \). The test particle when released from rest will be seen by the observer to be moving in exactly the same way as in the first situation. Next, let the elevator, the observer, and the test particle, all having the same electric charge-to-mass ratio, be set into motion with identical velocities perpendicular to the downward magnetic field. The elevator and its contents will move in circles of the same radius but with different centers. The observer will see the test particle float and hence the elevator constitutes a local inertial frame because

\[
a_{\text{rel}} = \left( \frac{q}{m} - \frac{Q}{M} \right) vB = 0. \tag{29}
\]

We can now postulate the equivalence principle for the electromagnetic field: All effects of a uniform electromagnetic field locally on a single particle or particles with the same electric charge-to-mass ratio are identical to the effects of a uniform acceleration of the reference frame. The proposed extension of the equivalence principle to electromagnetic fields as presented here is actually already well supported by an important set of experimental evidences such as the Witteborn - Fairbank experiment to determine the acceleration of electrons in a vacuum enclosed by a copper tube in the gravitational field of the earth and the London moment in rotating superconductors, discussed in [27].

### 7 The Gravitational and Electromagnetic Field Equations

A very simple and experimentally testable unified description of gravitation with electromagnetism may be rendered possible if we give up the interpretation that \( g_{\mu\nu} \) is the gravitational field proper. We should accept the fact that \( g_{\mu\nu} \) is simply the metric tensor, to which gravitational as well as electromagnetic fields contribute separately but similarly, through which the spacetime curvature is determined. Accepting this interpretation, we can immediately write down the modified field equations. Consider a compact object with a distribution of total mass \( M_o \) and charge \( Q_o \). Let there be a distribution of charged matter with total mass \( M \) and charge \( Q \) outside this compact object. Let also a test
particle of mass $m$ and charge $q$ be moving in this region. The modified field equations outside the compact object are

$$R^\mu\nu - \frac{1}{2}g^\mu\nu R = \frac{8\pi G}{c^4} T^\mu\nu_M + \frac{8\pi k_e}{c^4} \frac{q}{m} T^\mu\nu_{CC},$$  

(30)

which should be compared with the Einstein’s equations for this case

$$R^\mu\nu - \frac{1}{2}g^\mu\nu R = \frac{8\pi G}{c^4} [T^\mu\nu_M + T^\mu\nu_{EM}(Q) + T^\mu\nu_{EM}(Q_o)].$$  

(31)

Here $T^\mu\nu_M$ is the matter energy-momentum tensor of the matter outside the compact object. $T^\mu\nu_{EM}(Q)$ and $T^\mu\nu_{EM}(Q_o)$ are the energy-momentum tensors of the electromagnetic fields due to $Q$ and $Q_o$, respectively. $T^\mu\nu_{CC}$ is a tensor such that, in the absence of the object and neglecting gravity, $R^00_{CC} = (8\pi k_e/c^4)q/m (T^00_{CC} + T_{CC}/2)$ reduces to Eq. (16) in the Newtonian limit, with $R^00_{CC}$ being the contribution of the second term on the right in Eq. (30) to $R^00$, and $T_{CC}$ being the trace of $T^\mu\nu_{CC}$. The tensor $T^\mu\nu_{CC}$ may be called the charge-current tensor. It replaces $T^\mu\nu_{EM}(Q)$ in Eq. (31) with a different coupling. It is the source of the electromagnetic field outside the object and does not contribute to the gravitational field. One such tensor is

$$T^\mu\nu_{CC} = \frac{1}{3} v^\alpha J^\alpha \left( \frac{1}{c^2} U^\mu U^\nu + g^\mu\nu \right),$$

(32)

where $v^\alpha = (\gamma, \vec{v})$ is the four-velocity of the test particle, $J^\alpha = (c\rho, \vec{J} + \vec{J}_D)$, and $U^\mu = (\gamma u, \vec{u})$ is the four-velocity of the charge distribution, and $\gamma = (1 - v^2/c^2)^{-1/2}$. It is clear that $T^\mu\nu_{CC}$ should not be confused with an energy-momentum tensor. In the Newtonian limit, when $v/c << 1$, $T^00_{CC} = 0$. Notice that the second term on the right-hand side of Eq. (31) is replaced by the $T^\mu\nu_{CC}$ term in Eq. (30) while the third term on the right-hand side of Eq. (31) which is due to the electromagnetic field of the object does not appear in our scheme. This is due to the same reason that the energy-momentum tensor of the gravitational field of the compact object does not appear on the right-hand sides of the Eqs. (30) and (31). In our scheme the effects of the free gravitational and electromagnetic fields (which are due to the compact object) on the spacetime geometry are already implicitly included by the left-hand side of the equations. To avoid any confusion, let us emphasize once again that the first term on the right-hand side of Eq. (30) is the matter tensor due to mass-energy and is the only source of gravity. The second term, with a different coupling from the first, is the charge-current tensor and contributes to the electromagnetic field only. This is because, in our scheme, the unified GR described by Eq. (30) is the metric theory of gravitation and electromagnetism. Of course, there is the possibility that it might also be the metric theory of weak and strong interactions. In the presence of weak and strong charges the right-hand side of Eq. (30) may actually be containing terms whose forms we do not know at present. However, for this to happen

---

9We emphasize that mass has the same meaning here as in Einstein’s GR. Gravitational, electromagnetic, nuclear, and other forms of energy may contribute to mass. Massless particles like photons have an effective mass in a gravitational field given by $E/c^2$, with $E$ being the energy of individual photons.
the form of these interactions must be similar to those of gravity and electromagnetism. Currently, this seems not to be the case and is an open question that deserves further study.

We emphasize that the trajectory of a charged particle moving in superimposed gravitational and electromagnetic fields due to a given source only is not described, in our scheme, by the equation

\[
d\frac{2}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{q}{mc^2} F^\mu_{\alpha} \frac{dx^\alpha}{ds},
\]

where \( F^\mu_{\alpha} \) is the electromagnetic field strength tensor. The correct equation for the trajectory of such a particle is the geodesic equation \( (3) \) in which, contrary to Einstein’s GR, the coefficients \( \Gamma^\mu_{\alpha\beta} \) get direct contribution from the electromagnetic field. In Einstein’s GR, in the absence of gravity or in the presence of gravity but locally, the equation of motion of a charged test particle in an electromagnetic field, due to the vanishing of the \( \Gamma^\mu_{\alpha\beta} \), is the special relativistic equation

\[
d\frac{2}{ds^2} = \frac{q}{mc^2} F^\mu_{\alpha} \frac{dx^\alpha}{ds}.
\]

In our scheme, the equation of motion of a charged particle in such a case is

\[
d\frac{2}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0,
\]

which is the geodesic equation \( (3) \). But now the coefficients \( \Gamma^\mu_{\alpha\beta} \) get contribution from the electromagnetic field only. Hence, in our scheme Eq. \( (34) \) is not exact but approximate.

8 The Line Element for a Spherically Symmetric Distribution of Matter and Charge

To gain further insight into our scheme, let us consider the field equations describing the empty space \( \text{\textsuperscript{11}} \) external to a distribution of total mass \( M \) and charge \( Q \) (the subscript \( o \) has been dropped now). They are

\[
R^\mu\nu = 0,
\]

in our scheme, as opposed to

\[
R^\mu\nu = \frac{8\pi G}{c^4} T_{EM}^\mu\nu(Q),
\]

which are the Einstein field equations in this case. In Newtonian gravity and electromagnetism the equations satisfied by the potentials \( \phi_G \) and \( \phi_E \) in a region where there are gravitational and electromagnetic fields but devoid of matter and electric charge are

\( \text{\textsuperscript{10}} \)

It seems that all the interactions related to forces representable by potentials curve the spacetime independently.

\( \text{\textsuperscript{11}} \) Empty space in our scheme means that there is neither neutral nor charged matter present and no physical fields except the gravitational and electromagnetic fields.
\[ \nabla^2 \phi_G = 0 \text{ and } \nabla^2 \phi_E = 0. \] Of the two general relativistic equations, Eqs. \((36)\) and \((37)\), it is the former one that upholds both these Newtonian equations. To find the solution of Eq. \((36)\) for a static and spherical distribution of matter of mass \(M\) and electric charge \(Q\) with a test particle of mass \(m\) and electric charge \(q\) in the vicinity, we write the spacetime metric in the standard form (see, for example \([28]\))

\[ ds^2 = -A(r)c^2 dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{38} \]

where \(A(r)\) and \(B(r)\) are functions of \(r\) to be obtained by solving the field equations, Eq. \((36)\). Following the same steps given in textbooks, the metric is obtained to be

\[ ds^2 = -\left(1 + \frac{k}{r}\right)c^2 dt^2 + \left(1 + \frac{k}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{39} \]

where \(k\) is an integration constant to be determined. Far from the source, \(r\) is approximately the radial distance and \(-g_{00} = 1 + h_{00}\). In this region \(h_{00}\) is small and equals \(k/r\). Since there are superimposed Newtonian potentials given by \(\Phi = (m_g/m_i)\Phi_G + (q/m)\Phi_E\) outside the source, equations \((19)\) and \((26)\) suggest that \(h_{00} = 2\Phi/c^2\) and \(k = -2GM/c^2 + 2(q/m)k_Q/c^2\). The solution is thus found to be \(12\)

\[ ds^2 = -\left(1 - 2\frac{GM}{c^2 r} + 2\frac{q}{m} \frac{k_Q}{c^2 r}\right)c^2 dt^2 + \left(1 - 2\frac{GM}{c^2 r} + 2\frac{q}{m} \frac{k_Q}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{40} \]

When \(Q = 0\), or \(q = 0\) (but \(m \neq 0\)) Eq. \((40)\) reduces correctly to the Schwarzschild solution \(13\) \([20]\). When \(Q \neq 0\), and \(q \neq 0\), Eq. \((40)\) replaces the Reissner-Nordström solution \(29\) \([30]\) of Eq. \((37)\) which we believe does not describe the actual physics correctly. It is given by \(14\)

\[ ds^2 = -\left(1 - 2\frac{GM}{c^2 r} + \frac{Gk_Q^2}{c^4 r^2}\right)c^2 dt^2 + \left(1 - 2\frac{GM}{c^2 r} + \frac{Gk_Q^2}{c^4 r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{41} \]

Comparison of the third terms in \(g_{00}\) of Eqs. \((40)\) and \((41)\) reveals the philosophy of our unification. In Eq. \((41)\), the electric field of the charge distribution contributes to the gravitational field of the matter. Whereas in our scheme, which we call unified general relativity (hereafter UGR), there is an equivalence principle for the electromagnetic field as well, and the right-hand side of Eq. \((36)\) is zero, as opposed to Eq. \((37)\) of standard GR;

\(12\)Note that, if desired, the factor \(q/m\) may be set to +1 or -1 by choosing the units appropriately. For example, we may choose for electrons \(e = m_e\), where \(e = 1.6022 \times 10^{-19} C\) and \(m_e = 9.1095 \times 10^{-31} kg\), which gives \(1C = 5.6856 \times 10^{-12}kg\). This makes \(q_e/m_e = -1\) at the expense of changing \(k_e = 8.9875 \times 10^9 Nm^2C^{-2}\) to \(k'_e = 2.7803 \times 10^{12} Nm^2kg^{-2}\), and \(k_m = 10^{-7}Ns^2 C^{-2}\) to \(k'_m = 3.0935 \times 10^{15} Ns^2kg^{-2}\). If gravitation and electromagnetism are described together as we contemplate here, such a system of units seems to be more natural. Note also that one could have measured the mass in terms of the electric charge. This would have given for electrons, \(1kg = 1.7588 \times 10^{11} C\), and \(k_e, k_m, \text{ and } G\) would have changed to \(k'_e = 1.5807 \times 10^{33} m^3 s^{-2} C^{-1}\), \(k'_m = 1.7588 \times 10^4 mC^{-1}\), and \(G' = 3.7935 \times 10^{-22} m^3 s^{-2} C^{-1}\).

\(13\)For neutral massless particles like photons it might so happen that \(q/m = 1\).

\(14\)It should be noted, however, that the Reissner-Nordström solution is for a neutral test particle. The trajectory of a charged test particle in this case is given by Eq. \((39)\).
the electric field does not contribute to the gravitational field, it asserts itself separately. One of the most salient features of the UGR theory is that it is a multi-metric theory. There is a distinct metric for every test particle due to the charge-to-mass ratio, as there must be so as to agree with classical electrodynamics in the Newtonian limit.

9 Concluding Remarks

We have tried in this work a new route to geometrize the motion of a test charge in an electromagnetic field. By considering the elevator cabin experiments we were led to postulate an equivalence principle for the electromagnetic field. Our geometrization was further supported by the treatment of Newtonian electromagnetism in the language of curved spacetime and by the action integral of a test charge. Within the precision of the present experiments, the equivalence principle and the geometrization of the motions of test particles in a given gravitational field are universal. All test particles are effected universally. The equivalence principle and the geometrization of the motion of test charges in a given electromagnetic field, however, are specific. Different test charges are effected specifically. This is, of course, what is observed in nature.

Stipulating that the components of the metric tensor $g_{\mu\nu}$ correspond to the gravitational and electromagnetic potentials in Riemannian spacetime, as opposed to, for example, Weyl’s generalization of Riemannian spacetime, we were led to give up the interpretation that the $g_{\mu\nu}$ is the gravitational field only. This enabled us to postulate the field equations describing the motion of a charged test particle in superimposed gravitational and electromagnetic fields. The new feature of the modified field equations is the presence of what we call the charge-current tensor $T_{\mu\nu}^{CC}$. This term on the right-hand side of the field equations is the source of the electromagnetic field. It is not an energy-momentum tensor and does not contribute to the gravitational field. The expression we have presented for $T_{\mu\nu}^{CC}$ in this work has the correct Newtonian limit. For a compact massive and charged object the space outside it is not empty according to Einstein’s GR; there is the electromagnetic field contributing to the gravitational field through its energy-momentum tensor. Whereas in our scheme, the space outside such an object is empty because now there is an equivalence principle not only for the gravitational field but also for the electric field. Two implications of the Principle of Equivalence for electromagnetic fields are (1) the Principle of General Covariance is extended to include electromagnetic fields and (2) the laws of Special Relativity hold locally in a coordinate system with vanishing gravitational and electromagnetic fields.

Physics is an experimental science. It is incumbent on a new theory that possesses unorthodox predictions that it be confronted with experiment. Unlike some other unsuccessful unification schemes that had no new experimental predictions, our scheme has several distinct predictions different from that of standard GR. The deflection of an electron beam in the vicinity of a charged spherical mass in a vacuum chamber can be decisive to choose the correct line element among from the one we have proposed in this work and the Reissner-Nordström line element which reduces to the Minkowski line element for a laboratory-size charged sphere. An immediate prediction of the unified theory intro-
duced is that there should be electrical geometry waves similar to gravitational geometry waves also known as gravitational waves. There are other predictions of the present theory. For example, if one considers an electron moving radially away from a positively charged sphere and applies the conservation of energy and then replaces the escape velocity from a radius $r$ with the speed of light, one obtains the radius of the object from which electrons cannot escape. Such an object can be rightly called an electrical black hole and can be built in the laboratory. The general relativistic theory to predict and give the radius of such an object turns out to be the present theory. This and other predictions of the present theory will be presented in our forthcoming publications.

Acknowledgements
We are grateful to Prof. Mahjoob O. Taha for invaluable discussions. Suggestions by Clovis Jacinto de Matos of ESA, Paris to improve the presentation are greatly appreciated.

References
[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman and Company, 1973.
[2] A. Einstein, Ann. d. Phys. 49, 769 (1916).
[3] For a quick account of unified theories, see the book by P. G. Bergmann, Dover Publications Inc. New York.
[4] H. Weyl, Sitzungsber. d. Preuss. Akad. d. Wiss. 465,(1918); Ann. d. Phys. 59, 101 (1919); Space, Time, Matter, Dover Publications Inc. New York.
[5] A. S. Eddington, Proc. Roy. Soc. A 99, 104 (1921).
[6] Th. Kaluza, Sitzungsber. d. Preuss. Akad. d. Wiss. 966 (1921).
[7] O. Klein, Z. Phys. 37, 903 (1926).
[8] A. Einstein and Mayer, Berl. Ber. 541 (1931) and 130 (1932).
[9] A. Einstein and P. Bergmann, Ann. of Math. 39, 683 (1938).
[10] A. Einstein, V. Bargmann, and P. G. Bergmann, Theodore von Kármán Anniversary Volume, Pasadena 212 (1941).
[11] O. Veblen, Projektive Relativitätstheorie, Berlin, Springer, (1933).
[12] W Pauli, Ann. d. Phys. 18, 305 (1933); 18, 337 (1933).
[13] A. Einstein, Ann. d. Math. 102, 685 (1930).

15 cease in the year 2000.
[14] E. Schrödinger, Space-Time Structure, Cambridge University Press, 1960.

[15] E. v. Eötvös, D. Pekar, and E. Fekete, Ann. d. Phys. 68, 11 (1922).

[16] P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (U.S.A) 26, 442 (1964).

[17] V. G. Braginsky and V. I. Panov, Sov. Phys. JETP 34, 464 (1971).

[18] A. Einstein, Ann. d. Phys. 35, 898 (1911).

[19] K. Nordtvedt, Jr., Phys. Rev. 169, 1017 (1968); Phys. Rev. 180, 1293 (1969); Intern. J. Theoret. Phys. 3, 133 (1970); Phys. Rev. D 3, 1683 (1971).

[20] K. Schwarzschild, Berl. Ber. 189 (1916).

[21] A. Trautman, F. A. E. Pirani, and H. Bondi, Lectures on General Relativity, Vol. 1, Brandeis Summer Institute in Theoretical Physics, Prentice-Hall Inc., 1964. Trautman’s lectures, pp.104-121.

[22] See Ch. 12 of Ref.[1] and references therein.

[23] B. Mashhoon, F. Gronwald, H. I. M. Lichtenegger, Lect. Notes Phys. 562, 83 (2001).

[24] S. J. Clark and R. W. Tucker, Class. Quan. Grav. 17, 4125 (2000).

[25] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Pergamon Press, 4th Ed.

[26] B. Mashhoon, Phys. Let. A. 173, 347 (1993).

[27] C. Jacinto de Matos, M. Özer, G. L. Izworski, physics.gen-ph/1712.04347 and references therein.

[28] J. Foster and J. D. Nightingale, A Short Course in General Relativity, Springer, 2nd Ed.

[29] H. Reissner, Ann. d. Phys. 50, 106 (1916).

[30] G. Nordström, Proc. Kon. Ned. Akad. Wet.

[31] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley & Sons.