Some Basic Aspects of Analysis on Metric and Ultrametric Spaces

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Preface

A number of topics involving metrics and measures are discussed, including some of the special structure associated with ultrametrics.
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Chapter 1

Basic notions

1.1 Metrics and ultrametrics

Let $X$ be a set. As usual, a metric on $X$ is a nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ such that $d(x, y) = 0$ if and only if $x = y$,

\[ d(x, y) = d(y, x) \tag{1.1} \]

for every $x, y \in X$, and

\[ d(x, z) \leq d(x, y) + d(y, z) \tag{1.2} \]

for every $x, y, z \in X$. If

\[ d(x, z) \leq \max(d(x, y), d(y, z)) \tag{1.3} \]

for every $x, y, z \in X$, then $d(x, y)$ is said to be an ultrametric on $X$.

Let $(X, d(x, y))$ be a metric space, and let $x \in X$ and a positive real number $r$ be given. The corresponding open ball in $X$ is defined by

\[ B(x, r) = \{ y \in X : d(x, y) < r \} \tag{1.4} \]

If $y \in B(x, r)$, then $t = r - d(x, y) > 0$, and

\[ B(y, t) \subseteq B(x, r) \tag{1.5} \]

by the triangle inequality. However, if $d(\cdot, \cdot)$ is an ultrametric on $X$, then one can check that

\[ B(y, r) \subseteq B(x, r) \tag{1.6} \]

for every $y \in B(x, r)$. In fact,

\[ B(x, r) = B(y, r) \tag{1.7} \]

for every $x, y \in X$ with $d(x, y) < r$, since we can also apply the previous argument with the roles of $x$ and $y$ reversed.
Similarly, the closed ball in a metric space $X$ centered at $x \in X$ and with radius $r \geq 0$ is defined by

$$B(x, r) = \{ y \in X : d(x, y) \leq r \}.$$ \hfill (1.8)

If $d(\cdot, \cdot)$ is an ultrametric on $X$, and if $y \in B(x, r)$, then

$$B(y, r) \subseteq B(x, r),$$ \hfill (1.9)

as before. It follows that

$$B(x, r) = B(y, r)$$ \hfill (1.10)

when $d(x, y) \leq r$, by reversing the roles of $x$ and $y$.

Let us continue to ask for the moment that $d(\cdot, \cdot)$ be an ultrametric on $X$. If $x, y, z \in X$ and $d(y, z) \leq d(x, y)$, then

$$d(x, z) \leq d(x, y),$$ \hfill (1.11)

by (1.3). Of course, we also have that

$$d(x, y) \leq \max(d(x, z), d(y, z)),$$ \hfill (1.12)

by (1.3) with the roles of $y$ and $z$ exchanged. This implies that

$$d(x, y) \leq d(x, z)$$ \hfill (1.13)

when $d(y, z) < d(x, y)$, and hence that

$$d(x, y) = d(x, z).$$ \hfill (1.14)

Put

$$V(x, r) = \{ y \in X : d(x, y) > r \}$$ \hfill (1.15)

for every $x \in X$ and $r \geq 0$, which is the same as the complement of $B(x, r)$ in $X$. If $d(\cdot, \cdot)$ is an ordinary metric on $X$ and $y \in V(x, r)$, then $t = d(x, y) - r > 0$, and one can check that

$$B(y, t) \subseteq V(x, r),$$ \hfill (1.16)

using the triangle inequality. If $d(\cdot, \cdot)$ is an ultrametric on $X$ and $y \in V(x, r)$, then we get that

$$B(y, d(x, y)) \subseteq V(x, r),$$ \hfill (1.17)

by (1.13). Similarly,

$$W(x, r) = \{ y \in X : d(x, y) \geq r \}$$ \hfill (1.18)

is the same as the complement of $B(x, r)$ in $X$ for each $x \in X$ and $r > 0$. If $d(\cdot, \cdot)$ is an ultrametric on $X$ and $y \in W(x, r)$, then we also have that

$$B(y, d(x, y)) \subseteq W(x, r),$$ \hfill (1.19)
1.2. ABSTRACT CANTOR SETS

by (1.13).

If X is any metric space, then every open ball in X is an open set in X
with respect to the topology determined by the metric. Closed balls in X are
closed sets too, which is the same as saying that V(x, r) is an open set in X for
every x ∈ X and r ≥ 0. If d(·, ·) is an ultrametric on X, then (1.9) implies that
B(x, r) is an open set in X for every x ∈ X and r > 0. In this case, W(x, r)
is an open set in X for every x ∈ X and r > 0, by (1.19), which implies that
B(x, r) is a closed set in X.

Let |x| be the absolute value of a real number x, which is equal to x when
x ≥ 0 and to −x when x ≤ 0. Thus the standard metric on the real line R is
given by |x − y|. Of course, this is far from being an ultrametric. By contrast,
the p-adic metric on the set Q of rational numbers is an ultrametric for each
prime number p. This will be discussed in Section 1.3.

1.2 Abstract Cantor sets

Let X1, X2, X3, . . . be a sequence of finite sets, each of which has at least two
elements. Also let X = ∏∞
j=1 Xj be their Cartesian product, which is the
set of sequences x = {xj}j=1∞ such that xj ∈ Xj for each j. Thus X is a
compact Hausdorff space with respect to the product topology corresponding
to the discrete topology on each factor. If x, y ∈ X and x ̸= y, then let l(x, y)
be the largest nonnegative integer such that xj = yj when 1 ≤ j ≤ l(x, y).
Equivalently, l(x, y) + 1 is the smallest positive integer j such that xj ̸= yj. If
x = y, then one can take l(x, y) = +∞. Note that

\[ l(x, y) = l(y, x) \]

for every x, y ∈ X, and that

\[ l(x, z) ≥ \min(l(x, y), l(y, z)) \]

for every x, y, z ∈ X.

Let {tl}l=0∞ be a strictly decreasing sequence of positive real numbers that
converges to 0. Put

\[ d(x, y) = t_l(x, y) \]

when x ̸= y, and d(x, y) = 0 when x = y, which corresponds to (1.22) with
t∞ = 0. It is easy to see that this defines an ultrametric on X, because of (1.20)
and (1.21), and that the topology on X determined by d(x, y) is the same as the
product topology on X corresponding to the discrete topology on each factor.
If x ∈ X and k is a nonnegative integer, then put

\[ B_k(x) = \{ y ∈ X : y_j = x_j \text{ for each } j ≤ k \}. \]

Equivalently, Bk(x) is the closed ball in X centered at x with radius tk with
respect to (1.22).

Suppose now that µj is a probability measure on Xj for each j, where all
subsets of Xj are measurable. Thus µj assigns a weight to each element of Xj,
and the sum of the weights is equal to 1. This leads to a product probability measure \( \mu \) on \( X \), where

\[
\mu(B_k(x)) = \prod_{j=1}^{k} \mu_j(\{x_j\})
\]

for each \( x \in X \) and \( k \geq 1 \). Alternatively, one can first use the \( \mu_j \)'s to define a nonnegative linear functional on the space of continuous real-valued functions on \( X \), as a limit of Riemann sums. One can then apply the Riesz representation theorem, to get a Borel probability measure on \( X \).

Let \( n_j \geq 2 \) be the number of elements of \( X_j \) for each positive integer \( j \). Also let \( \mu_j \) be the probability measure on \( X_j \) that corresponds to the uniform distribution on \( X_j \), which assigns to each element of \( X_j \) the same weight \( 1/n_j \). In this case, (1.24) reduces to

\[
\mu(B_k(x)) = 1/N_k
\]

for each \( x \in X \) and \( k \geq 1 \), where

\[
N_k = \prod_{j=1}^{k} n_j.
\]

If we put \( N_0 = 1 \), then (1.25) holds for \( k = 0 \) as well. Note that \( t_l = 1/N_l \) defines a strictly decreasing sequence of positive real numbers that converges to 0, as before.

1.3 The \( p \)-adic absolute value

Let \( p \) be a prime number, and let \( x \) be a rational number. The \( p \)-adic absolute value \( |x|_p \) of \( x \) is defined as follows. If \( x = 0 \), then \( |x|_p = 0 \), and otherwise \( x \) can be expressed as \( p^l a/b \), where \( a \), \( b \), and \( l \) are integers, and neither \( a \) nor \( b \) is an integer multiple of \( p \). In this case, we put

\[
|x|_p = p^{-l},
\]

which is not affected by any other common factors that \( a \) and \( b \) might have. It is easy to see that

\[
|x + y|_p \leq \max(|x|_p, |y|_p)
\]

and

\[
|xy|_p = |x|_p |y|_p
\]

for every \( x, y \in \mathbb{Q} \). The \( p \)-adic metric is defined on \( \mathbb{Q} \) by

\[
d_p(x, y) = |x - y|_p.
\]

This is an ultrametric on \( \mathbb{Q} \), because of (1.28).
1.4. \textit{P-Adic numbers}

If \( y \in \mathbb{Q} \) and \( n \) is a nonnegative integer, then

\[
(1 - y) \sum_{j=0}^{n} y^j = 1 - y^{n+1},
\]

by a standard computation. Here \( y^j \) is interpreted as being equal to 1 for all \( j = 0 \), as usual. If \( |y|_p < 1 \), then \( y^{n+1} \to 0 \) as \( n \to \infty \) with respect to the \( p \)-adic metric. This implies that

\[
\sum_{j=0}^{n} y^j = \frac{1 - y^{n+1}}{1 - y} \to \frac{1}{1-y}
\]

as \( n \to \infty \) with respect to the \( p \)-adic metric.

Of course, \( |x|_p \leq 1 \) for every integer \( x \). Now let \( x \in \mathbb{Q} \) with \( |x|_p \leq 1 \) be given. Thus \( x \) can be expressed as \( a/b \), where \( a \) and \( b \) are integers, \( b \neq 0 \), and \( b \) is not an integer multiple of \( p \). It is well known that there is a nonzero integer \( c \) such that \( b c \equiv 1 \mod{p} \) under these conditions. Put \( y = 1 - b c \), so that \( y \) is an integer which is divisible by \( p \), and hence \( |y|_p \leq 1/p < 1 \). It follows that

\[
x = \frac{a}{b} = \frac{a c}{b c} = \frac{a c}{1 - y}
\]

can be approximated by integers with respect to the \( p \)-adic metric, by (1.32).

1.4 \textit{p-Adic numbers}

The set \( \mathbb{Q}_p \) of \textit{p-adic numbers} can be obtained by completing \( \mathbb{Q} \) as a metric space with respect to the \( p \)-adic metric, in the same way that the real line \( \mathbb{R} \) is obtained by completing \( \mathbb{Q} \) with respect to the standard Euclidean metric. Sums and product of rational numbers can be extended to \( p \)-adic numbers in a natural way, so that \( \mathbb{Q}_p \) becomes a field. The \( p \)-adic absolute value \( |x|_p \) and \( p \)-adic metric \( d_p(x,y) \) can also be extended to \( x,y \in \mathbb{Q}_p \), in such a way that (1.28), (1.29), and (1.30) still hold. By construction, \( \mathbb{Q} \) is dense in \( \mathbb{Q}_p \) with respect to the \( p \)-adic metric, and \( |x|_p \) is an integer power of \( p \) for every \( x \in \mathbb{Q}_p \) with \( x \neq 0 \). One can show that addition and multiplication are continuous on \( \mathbb{Q}_p \) with respect to the \( p \)-adic metric, in essentially the same way as for real numbers.

The set \( \mathbb{Z}_p \) of \textit{p-adic integers} is defined by

\[
\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}.
\]

This is the same as the closed unit ball in \( \mathbb{Q}_p \), which is a closed set in \( \mathbb{Q}_p \) in particular. This is also an open set in \( \mathbb{Q}_p \) with respect to the \( p \)-adic metric, because the \( p \)-adic metric is an ultrametric, as in Section 1.1. Of course, \( \mathbb{Z}_p \) contains the set \( \mathbb{Z} \) of ordinary integers. It is easy to see that \( \mathbb{Q} \cap \mathbb{Z}_p \) is dense in \( \mathbb{Z}_p \) with respect to the \( p \)-adic metric, because \( \mathbb{Q} \) is dense in \( \mathbb{Q}_p \), and using
the ultrametric version of the triangle inequality. As in the previous section, elements of \( \mathbb{Q} \cap \mathbb{Z}_p \) can be approximated by integers with respect to the \( p \)-adic metric. Combining these statements, we get that elements of \( \mathbb{Z}_p \) can be approximated by elements of \( \mathbb{Z} \) with respect to the \( p \)-adic metric, so that \( \mathbb{Z}_p \) is the same as the closure of \( \mathbb{Z} \) in \( \mathbb{Q}_p \) with respect to the \( p \)-adic metric. Note that \( \mathbb{Z}_p \) is also closed under addition and multiplication, by (1.28) and (1.29).

Put
\[
(1.35) \quad p^l \mathbb{Z}_p = \{ p^l x : x \in \mathbb{Z}_p \} = \{ y \in \mathbb{Q}_p : |y|_p \leq p^{-l} \}
\]
for each integer \( l \). This is the same as the closed ball in \( \mathbb{Q}_p \), centered at 0 with radius \( p^{-l} \) with respect to the \( p \)-adic metric, which is also an open set in \( \mathbb{Q}_p \), as in Section 1.1. Observe that \( p^l \mathbb{Z}_p \) is a subgroup of \( \mathbb{Q}_p \) with respect to addition for each \( l \), because of (1.28). If \( l \geq 0 \), then \( p^l \mathbb{Z}_p \) is an ideal in \( \mathbb{Z}_p \) as a commutative ring, and hence the quotient \( \mathbb{Z}_p/p^l \mathbb{Z}_p \) can be defined as a commutative ring. The composition of the obvious inclusion of \( \mathbb{Z} \) in \( \mathbb{Z}_p \) with the standard quotient homomorphism from \( \mathbb{Z}_p \) onto \( \mathbb{Z}_p/p^l \mathbb{Z}_p \) leads to a ring homomorphism from \( \mathbb{Z} \) into \( \mathbb{Z}_p/p^l \mathbb{Z}_p \). The kernel of this homomorphism is
\[
(1.36) \quad \mathbb{Z} \cap (p^l \mathbb{Z}_p) = p^l \mathbb{Z},
\]
which is an ideal in \( \mathbb{Z} \). This leads to a natural injective ring homomorphism from \( \mathbb{Z}/p^l \mathbb{Z} \) into \( \mathbb{Z}_p/p^l \mathbb{Z}_p \). Every element of \( \mathbb{Z}_p \) can be expressed as the sum of elements of \( \mathbb{Z} \) and \( p^l \mathbb{Z}_p \), because \( \mathbb{Z} \) is dense in \( \mathbb{Z}_p \) with respect to the \( p \)-adic metric. Thus we get a natural ring isomorphism from \( \mathbb{Z}/p^l \mathbb{Z} \) onto \( \mathbb{Z}_p/p^l \mathbb{Z}_p \) for each nonnegative integer \( l \).

In particular, \( \mathbb{Z}_p/p^l \mathbb{Z}_p \) has exactly \( p^l \) elements for each nonnegative integer \( l \). This implies that \( \mathbb{Z}_p \) can be expressed as the union of \( p^l \) pairwise-disjoint translates of \( p^l \mathbb{Z}_p \) for each \( l \geq 0 \). It follows that \( \mathbb{Z}_p \) is totally bounded with respect to the \( p \)-adic metric, in the sense that \( \mathbb{Z}_p \) can be covered by finitely many balls of arbitrarily small radius. A well-known theorem implies that \( \mathbb{Z}_p \) is compact with respect to the topology determined on \( \mathbb{Q}_p \) by the \( p \)-adic metric, because \( \mathbb{Z}_p \) is also a closed set in \( \mathbb{Q}_p \) and \( \mathbb{Q}_p \) is complete. Of course, \( p^k \mathbb{Z}_p \) is a compact set in \( \mathbb{Q}_p \) for every integer \( k \) too, by continuity of multiplication.

### 1.5 Haar measure on \( \mathbb{Q}_p \)

If \( A \) is a locally compact commutative topological group, then it is well known that there is a nonnegative translation-invariant Borel measure on \( A \) which is finite on compact subsets of \( A \), positive on nonempty open subsets of \( A \), and which satisfies certain other regularity properties. This is known as Haar measure on \( A \), and it is unique up to multiplication by a positive real number. The real line is a commutative topological group with respect to addition and the standard topology, for instance, and Lebesgue measure on \( \mathbb{R} \) satisfies the requirements of Haar measure. Similarly, the discussion in the previous section implies that \( \mathbb{Q}_p \) is a locally compact commutative topological group with respect to addition and the topology determined by the \( p \)-adic metric. Let \( |E| \) be the corresponding Haar measure of a Borel set \( E \subseteq \mathbb{Q}_p \), normalized so that \( |\mathbb{Z}_p| = 1 \).
If $l$ is a positive integer, then it follows that
\begin{equation}
|p^l \mathbb{Z}_p| = p^{-l}.
\end{equation}
(1.37)
This uses the fact that $\mathbb{Z}_p$ can be expressed as the union of $p^l$ pairwise-disjoint translates of $p^l \mathbb{Z}_p$, as in the previous section. If $l$ is a negative integer, then $p^l \mathbb{Z}_p$ can be expressed as the union of $p^{-l}$ pairwise-disjoint translates of $\mathbb{Z}_p$, by applying the previous statement to $-l$. This implies that (1.37) also holds when $l < 0$, and hence for all $l \in \mathbb{Z}$.

If $a \in \mathbb{Q}_p$ and $E \subseteq \mathbb{Q}_p$ is a Borel set, then
\begin{equation}
a E = \{ a x : x \in E \}
\end{equation}
(1.38)
is also a Borel set in $\mathbb{Q}_p$. This is trivial when $a = 0$, and it follows from the fact that $x \mapsto a x$ is a homeomorphism on $\mathbb{Q}_p$ when $a \neq 0$. If $a \neq 0$, then $x \mapsto a x$ is an isomorphism of $\mathbb{Q}_p$ onto itself as a commutative topological group, which implies that $|a E|$ satisfies the requirements of a Haar measure on $\mathbb{Q}_p$. The uniqueness of Haar measure implies that $|a E|$ is a constant multiple of $|E|$, where the constant depends on $a$ but not $E$. To determine the constant, one can consider the case where $E = \mathbb{Z}_p$, using (1.37). If $|a|_p = p^{-l}$ for some $l \in \mathbb{Z}$, then it is easy to see that $a \mathbb{Z}_p = p^l \mathbb{Z}_p$, so that
\begin{equation}
|a \mathbb{Z}_p| = |p^l \mathbb{Z}_p| = p^{-l} = |a|_p.
\end{equation}
(1.39)
It follows that
\begin{equation}
|a E| = |a|_p |E|
\end{equation}
(1.40)
for every $a \in \mathbb{Q}_p$ and Borel set $E \subseteq \mathbb{Q}_p$, which is trivial when $a = 0$.

Let $A$ be a locally compact commutative topological group again, and let $C_{com}(A)$ be the vector space of real-valued continuous functions on $A$ with compact support. Nonnegative linear functionals on $C_{com}(A)$ correspond to nonnegative Borel measures on $A$ which are finite on compact sets and have certain other regularity properties, by the Riesz representation theorem. The existence and uniqueness of Haar measure on $A$ can also be considered in terms of Haar integrals, which are nonnegative linear functionals on $C_{com}(A)$ that are invariant under translations and positive on nonnegative elements of $C_{com}(A)$ that are positive somewhere on $A$. The ordinary Riemann integral can be used to define a Haar integral on the real line, for instance. Similarly, one can get a Haar integral on $\mathbb{Q}_p$ as a limit of suitable Riemann sums.

### 1.6 Snowflake metrics and quasi-metrics

It is well known that
\begin{equation}
(r + t)^a \leq r^a + t^a
\end{equation}
(1.41)
for all nonnegative real numbers $r, t$ when $a \in \mathbb{R}$ satisfies $0 < a \leq 1$. Indeed,
\begin{equation}
\max(r, t) \leq (r^a + t^a)^{1/a}
\end{equation}
(1.42)
for every $a > 0$, which implies that
\begin{equation}
(1.43) \quad r + t \leq (r^a + t^a) \max(r, t)^{1-a} \\
\leq (r^a + t^a)^{1+(1-a)/a} = (r^a + t^a)^{1/a}
\end{equation}
when $a \leq 1$. If $d(x, y)$ is a metric on a set $X$, then it follows that $d(x, y)^a$ is also a metric on $X$ when $0 < a \leq 1$. Similarly, if $d(x, y)$ is an ultrametric on $X$, then $d(x, y)^a$ is an ultrametric on $X$ for every $a > 0$. In both cases, $d(x, y)^a$ determines the same topology on $X$ as $d(x, y)$.

A quasi-metric on a set $X$ is a nonnegative real-valued function $d(x, y)$ on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ for every $x, y \in X$, and
\begin{equation}
(1.44) \quad d(x, z) \leq C(d(x, y) + d(y, z))
\end{equation}
for some $C \geq 1$ and every $x, y, z \in X$. Thus a quasi-metric $d(x, y)$ on $X$ is a metric on $X$ if and only if one can take $C = 1$ in (1.44). If $d(x, y)$ is a quasi-metric on $X$, then the open ball $B(x, r)$ centered at a point $x \in X$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$ can still be defined as in (1.4). One can also define a topology on $X$ corresponding to $d(\cdot, \cdot)$ in the usual way, by saying that a set $U \subseteq X$ is an open set if for each $x \in U$ there is an $r > 0$ such that $B(x, r) \subseteq X$. It is easy to see that this satisfies the requirements of a topology on $X$, but the weaker version of the triangle inequality is not sufficient to show that open balls are open sets in $X$ with respect to this topology.

If $a \in \mathbb{R}$ and $a > 1$, then it is well known that $r^a$ is a convex function on the set of nonnegative real numbers $r$. This implies that
\begin{equation}
(1.45) \quad ((r + t)/2)^a \leq (1/2) (r^a + t^a)
\end{equation}
for every $r, t \geq 0$, and hence that
\begin{equation}
(1.46) \quad (r + t)^a \leq 2^{a-1} (r + t).
\end{equation}
If $d(x, y)$ is a metric on a set $X$, then it follows that $d(x, y)^a$ is a quasi-metric on $X$ for every $a > 1$. Similarly, $d(x, y)^a$ is a quasi-metric on $X$ for every $a > 0$ when $d(x, y)$ is a quasi-metric on $X$. Of course, the topology on $X$ determined by $d(x, y)^a$ is the same as the topology on $X$ corresponding to $d(x, y)$, and in fact the open ball in $X$ centered at a point $x \in X$ and with radius $r > 0$ with respect to $d(\cdot, \cdot)$ is the same as the open ball in $X$ centered at $x$ and with radius $r^a$ with respect to $d(\cdot, \cdot)^a$.

If $d(x, y)$ is a quasi-metric on a set $X$, then one can define a uniform structure on $X$ in the usual way, by considering the subsets
\begin{equation}
(1.47) \quad U_r = \{(x, y) \in X \times X : d(x, y) < r\}
\end{equation}
of $X \times X$ for each $r > 0$. The topology on $X$ determined by this uniform structure is the same as the topology on $X$ defined in terms of open balls associated to $d(x, y)$, as before. Standard results about uniform structures imply that for each $x \in X$ and $r > 0$, $x$ is in the interior of the corresponding open ball.
1.7. SEQUENCES AND SERIES

Let \( d(x,y) \) be a quasi-metric on a set \( X \). As usual, a sequence \( \{x_j\}_{j=1}^\infty \) of elements of \( X \) is said to converge to an element \( x \) of \( X \) if for every \( \epsilon > 0 \) there is an \( L \geq 1 \) such that
\[
d(x_j, x) < \epsilon
\]
(1.48)
for every \( j \geq L \). This is equivalent to saying that \( \{x_j\}_{j=1}^\infty \) converges to \( x \) with respect to the topology on \( X \) determined by \( d(\cdot, \cdot) \). More precisely, this uses the fact that every open ball in \( X \) centered at \( x \) contains an open set that contains \( x \) as an element, as in the previous section. Similarly, a sequence \( \{x_j\}_{j=1}^\infty \) of elements of \( X \) is said to be a Cauchy sequence in \( X \) if for every \( \epsilon > 0 \) there is an \( L \geq 1 \) such that
\[
d(x_j, x_l) < \epsilon
\]
(1.49)
for every \( j, l \geq L \). In particular, it is easy to see that convergent sequences are Cauchy sequences. If \( \{x_j\}_{j=1}^\infty \) is a Cauchy sequence in \( X \), then
\[
\lim_{j \to \infty} d(x_j, x_{j+1}) = 0,
\]
(1.50)
by taking \( l = j + 1 \) in (1.49). If \( d(\cdot, \cdot) \) is an ultrametric on \( X \), and if \( \{x_j\}_{j=1}^\infty \) is a sequence of elements of \( X \) that satisfies (1.50), then one can check that \( \{x_j\}_{j=1}^\infty \) is a Cauchy sequence in \( X \).

Let \( \sum_{j=1}^\infty a_j \) be an infinite series whose terms are real numbers, complex numbers, or \( p \)-adic numbers for some prime number \( p \). If the corresponding sequence of partial sums
\[
s_n = \sum_{j=1}^n a_j
\]
(1.51)
converges in \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{Q}_p \), as appropriate, then \( \sum_{j=1}^\infty a_j \) is said to converge, and the value of the sum is defined to be the limit of \( \{s_n\}_{n=1}^\infty \). Because \( \mathbb{R} \), \( \mathbb{C} \), and \( \mathbb{Q}_p \) are complete with respect to their standard metrics, convergence of \( \sum_{j=1}^\infty a_j \) is equivalent to asking that \( \{s_n\}_{n=1}^\infty \) be a Cauchy sequence. In particular, a necessary condition for the convergence of \( \sum_{j=1}^\infty a_j \) is that \( \{a_j\}_{j=1}^\infty \) converge as a sequence to 0 in \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{Q}_p \), as appropriate. This is also a sufficient condition in the \( p \)-adic case, because the \( p \)-adic metric is an ultrametric.
An infinite series \( \sum_{j=1}^{\infty} a_j \) of real or complex numbers is said to converge absolutely if \( \sum_{j=1}^{\infty} |a_j| \) converges, where \( |a_j| \) is the absolute value of \( a_j \) in the real case, and the modulus of \( a_j \) in the complex case. It is well known that absolute convergence implies convergence, using the triangle inequality to show that the partial sums of \( \sum_{j=1}^{\infty} a_j \) form a Cauchy sequence when the partial sums of \( \sum_{j=1}^{\infty} |a_j| \) form a Cauchy sequence. One can also check that

\[
\sum_{j=1}^{\infty} a_j \leq \sum_{j=1}^{\infty} |a_j|
\]

(1.52)

when \( \sum_{j=1}^{\infty} a_j \) converges absolutely. Similarly, if \( \{a_j\}_{j=1}^{\infty} \) is a sequence of \( p \)-adic numbers that converges to 0, then

\[
\sum_{j=1}^{\infty} a_j \leq \max_{j \geq 1} |a_j|_p.
\]

(1.53)

Note that the maximum of \( |a_j|_p \) over \( j \in \mathbb{Z}_+ \) exists in this situation, because \( |a_j|_p \to 0 \) as \( j \to \infty \).

The Cauchy product of two infinite series \( \sum_{j=0}^{\infty} a_j \), \( \sum_{k=0}^{\infty} b_k \) of real, complex, or \( p \)-adic numbers is the infinite series \( \sum_{l=0}^{\infty} c_l \), where

\[
c_l = \sum_{j=0}^{l} a_j b_{l-j}.
\]

(1.54)

If \( \sum_{j=0}^{\infty} a_j \) and \( \sum_{k=0}^{\infty} b_k \) are absolutely convergent series of real or complex numbers, then it is well known that \( \sum_{l=0}^{\infty} c_l \) converges absolutely too, and that

\[
\sum_{l=0}^{\infty} c_l = \left( \sum_{j=0}^{\infty} a_j \right) \left( \sum_{k=0}^{\infty} b_k \right).
\]

(1.55)

Similarly, if \( \{a_j\}_{j=0}^{\infty} \) and \( \{b_k\}_{k=0}^{\infty} \) are sequences of \( p \)-adic numbers converging to 0, then one can check that \( \{c_l\}_{l=0}^{\infty} \) also converges to 0 in \( \mathbb{Q}_p \), using the fact that the \( p \)-adic metric is an ultrametric. It is not too difficult to verify that (1.55) holds under these conditions as well.
Chapter 2

Hausdorff measures

2.1 Diameters

Let \((M, d(x, y))\) be a metric space. As usual, the diameter of a nonempty bounded set \(A \subseteq M\) is defined by

\[
\text{diam } A = \sup\{d(x, y) : x, y \in A\}.
\]

(2.1)

It is sometimes convenient to define the diameter of the empty set to be 0, and to put \(\text{diam } A = \infty\) when \(A\) is not bounded. Note that

\[
\text{diam } A = \text{diam } \overline{A}
\]

(2.2)

for any set \(A \subseteq M\), where \(\overline{A}\) is the closure of \(A\) in \(M\).

Let \(A \subseteq M\) and \(r > 0\) be given, and put

\[
A_r = \bigcup_{x \in A} B(x, r).
\]

(2.3)

Thus \(A_r\) is an open set in \(M\), since it is a union of open sets, and \(A \subseteq A_r\). If \(w, z \in A_r\), then there are \(x, y \in A\) such that \(d(x, w), d(y, z) < r\), and hence

\[
d(w, z) < d(x, y) + 2r.
\]

(2.4)

This implies that

\[
\text{diam } A_r \leq \text{diam } A + 2r
\]

(2.5)

for each \(r > 0\).

If \(d(x, y)\) is an ultrametric on \(M\), then we can replace (2.4) with

\[
d(w, z) \leq \max(d(x, y), r),
\]

(2.6)

so that

\[
\text{diam } A_r \leq \max(\text{diam } A, r)
\]

(2.7)
for each \( r > 0 \). Of course, 
\[
\text{(2.8)} \quad \text{diam } A \leq \text{diam } A_r
\]
for every \( r > 0 \), since \( A \subseteq A_r \). Thus (2.7) implies that
\[
\text{(2.9)} \quad \text{diam } A_r = \text{diam } A
\]
when \( r \leq \text{diam } A \).

Similarly,
\[
\text{(2.10)} \quad \text{diam } B(x, r) \leq 2r
\]
for every \( x \in M \) and \( r \geq 0 \), and for any metric \( d(x, y) \) on \( M \). If \( d(x, y) \) is an ultrametric on \( M \), then
\[
\text{(2.11)} \quad \text{diam } B(x, r) \leq r
\]
for every \( x \in M \) and \( r \geq 0 \). If \( d(x, y) \) is any metric on \( M \) and \( A \) is a nonempty bounded set in \( M \), then
\[
\text{(2.12)} \quad A \subseteq B(x, \text{diam } A)
\]
for every \( x \in A \). If \( M \) is the real line with the standard metric, and if \( A \) is a nonempty bounded subset of \( \mathbb{R} \), then
\[
\text{(2.13)} \quad I_A = [\inf A, \sup A]
\]
contains \( A \) and has the same diameter as \( A \).

### 2.2 Hausdorff content

Let \((M, d(x, y))\) be a metric space again, and let \( \alpha \) be a positive real number. The \( \alpha \)-dimensional **Hausdorff content** of \( E \subseteq M \) is defined by
\[
\text{(2.14)} \quad H^\alpha_{\text{con}}(E) = \inf \left\{ \sum_j (\text{diam } A_j)^\alpha : E \subseteq \bigcup_j A_j \right\},
\]
where more precisely the infimum is taken over all collections \( \{A_j\}_j \) of finitely or countably many subsets of \( M \) such that \( E \subseteq \bigcup_j A_j \). The sum
\[
\text{(2.15)} \quad \sum_j (\text{diam } A_j)^\alpha
\]
is defined as usual as the supremum over all finite subsums when there are infinitely many \( A_j \)'s, which may be infinite. If \( A_j \) is unbounded for any \( j \), then \( \text{diam } A_j = \infty \), and (2.15) is infinite. This definition can also be used when \( \alpha = 0 \), with the conventions that \( (\text{diam } A)^0 \) is equal to 0 when \( A = \emptyset \), is equal to 1 when \( A \) is nonempty and bounded, and is equal to \( \infty \) when \( A \) is unbounded.

Note that \( H^\alpha_{\text{con}}(\emptyset) = 0 \) for every \( \alpha \geq 0 \), and that
\[
\text{(2.16)} \quad H^\alpha_{\text{con}}(E) \leq (\text{diam } E)^\alpha
\]
for every $E \subseteq M$ and $\alpha \geq 0$, by covering $E$ by itself. If $E \subseteq \bar{E} \subseteq M$, then

\begin{equation}
H_{\text{con}}^\alpha(E) \leq H_{\text{con}}^\alpha(\bar{E})
\end{equation}

for every $\alpha \geq 0$, because every covering of $\bar{E}$ in $M$ is also a covering of $E$. If $E_1, E_2, E_2, \ldots$ is any sequence of subsets of $M$, then one can show that

\begin{equation}
H_{\text{con}}^\alpha \left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} H_{\text{con}}^\alpha(E_k)
\end{equation}

for every $\alpha \geq 0$, by combining coverings of the $E_k$’s to get coverings of $\bigcup_{k=1}^{\infty} E_k$. Of course, if $H_{\text{con}}^\alpha(E_k) = \infty$ for some $k$, then the sum on the right side of (2.18) is equal to $\infty$ too, in which case the inequality is trivial. Otherwise, one can choose coverings of the $E_k$’s for which the corresponding sums (2.15) are as close as one wants to $H_{\text{con}}^\alpha(E_k)$. The main point is to do this in such a way that the sum of the errors is arbitrarily small too.

In the definition of the Hausdorff content, one might as well restrict one’s attention to coverings of $E$ by collections of closed subsets of $M$, because of (2.2). One can also restrict one’s attention to coverings by collection of open subsets of $M$, using (2.5). If $E \subseteq M$ is compact, then it follows that one can restrict one’s attention to coverings of $E$ by finitely many subsets of $M$.

Remember that an outer measure on a $\sigma$-algebra $\mathcal{A}$ of subsets of $M$ is a nonnegative extended real-valued function $\mu$ on $\mathcal{A}$ such that $\mu(\emptyset) = 0$,

\begin{equation}
\mu(A) \leq \mu(B)
\end{equation}

for every $A, B \in \mathcal{A}$ with $A \subseteq B$, and $\mu$ is countably-subadditive on $\mathcal{A}$. Thus $H_{\text{con}}^\alpha$ is an outer measure on the $\sigma$-algebra of all subsets of $M$ for each $\alpha \geq 0$, for instance. Let $\mu$ be an outer measure defined on a $\sigma$-algebra $\mathcal{A}$ of subsets of $M$ that contains the Borel sets, and suppose that

\begin{equation}
\mu(A) \leq C \left( \text{diam } A \right)^\alpha
\end{equation}

for some nonnegative real numbers $C, \alpha$ and every $A \in \mathcal{A}$. If $E \in \mathcal{A}$, and if $\{A_j\}_j$ are finitely or countably many elements of $\mathcal{A}$ such that $E \subseteq \bigcup_j A_j$, then

\begin{equation}
\mu(E) \leq \sum_j \mu(A_j) \leq C \sum_j (\text{diam } A_j)^\alpha.
\end{equation}

This implies that

\begin{equation}
\mu(E) \leq C H_{\text{con}}^\alpha(E),
\end{equation}

since we can restrict our attention to coverings of $E$ by open or closed subsets of $M$ in the definition of Hausdorff content, as in the previous paragraph.
2.3 Restricting the diameters

Let \((M, d(x, y))\) be a metric space, and let \(0 \leq \alpha < \infty\) and \(0 < \delta \leq \infty\) be given. Put

\[
H_\alpha^\delta(E) = \inf \left\{ \sum_j (\text{diam } A_j)^\alpha : E \subseteq \bigcup_j A_j, \text{diam } A_j < \delta \text{ for each } j \right\}
\]

for each \(E \subseteq M\), where more precisely the infimum is taken over all collections \(\{A_j\}_j\) of finitely or countably many subsets of \(M\) such that \(E \subseteq \bigcup_j A_j\) and \(\text{diam } A_j < \delta\) for each \(j\), if there are any. If not, then put \(H_\alpha^\delta(E) = \infty\). Of course, if \(M\) is separable, then \(M\) is contained in the union of finitely or countably many balls of radius \(r\) for every \(r > 0\), and this is not a problem. This is also not a problem when \(\delta = \infty\), because every \(E \subseteq M\) is covered by a sequence of bounded subsets of \(M\).

By construction,

\[
H_{\alpha}^\delta_{\text{con}}(E) \leq H_\alpha^\delta(E) \leq H_{\alpha}^\INfty(E)
\]

for every \(\alpha \geq 0\) and \(E \subseteq M\) when \(0 < \eta < \delta \leq \infty\), since one is restricting the class of admissible coverings of \(E\) as \(\delta\) decreases. It is easy to see that

\[
H_{\alpha}^\delta_{\text{con}}(E) = H_\alpha^\INfty(E)
\]

for every \(\alpha \geq 0\) and \(E \subseteq M\), because (2.15) is infinite when \(A_j\) is unbounded for any \(j\). As before, \(H_\alpha^\delta(\emptyset) = 0\) for every \(\alpha \geq 0\) and \(\delta > 0\), and

\[
H_\delta^\alpha(E) \leq H_\delta^\alpha(\widetilde{E})
\]

when \(E \subseteq \widetilde{E} \subseteq M\). If \(E_1, E_2, E_3, \ldots\) is any sequence of subsets of \(M\), then

\[
H_\delta^\alpha \left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} H_\delta^\alpha(E_k)
\]

for every \(\alpha \geq 0\) and \(\delta > 0\), as in the previous section. Thus \(H_\delta^\alpha\) is an outer measure on the \(\sigma\)-algebra of all subsets of \(M\) for each \(\alpha \geq 0\) and \(\delta > 0\).

One might as well restrict one’s attention to coverings of \(E\) by open or closed subsets of \(M\) in (2.23), for the same reasons as before. In particular, if \(E\) is compact, then one can restrict one’s attention to coverings of \(E\) by finitely many subsets of \(M\).

Suppose that \(E_1, E_2 \subseteq M\) have the property that

\[
d(x, y) \geq \delta
\]

for some \(\delta > 0\) and every \(x \in E_1\) and \(y \in E_2\). Let \(\{A_j\}_{j \in I}\) be any collection of finitely or countably many subsets of \(M\) such that \(\text{diam } A_j < \delta\) for each \(j\) and

\[
E_1 \cup E_2 \subseteq \bigcup_{j \in I} A_j.
\]
2.4. HAUSDORFF MEASURES

Let \( I_1, I_2 \) be the set of \( j \in I \) such that \( A_j \) intersects \( E_1, E_2 \), respectively. The separation condition (2.28) implies that \( I_1 \) and \( I_2 \) disjoint subsets of \( I \), so that

\[
2.30\quad H_\delta^\alpha(E_1) + H_\delta^\alpha(E_2) \leq \sum_{j \in I_1} (\text{diam } A_j)^\alpha + \sum_{j \in I_2} (\text{diam } A_j)^\alpha
\]

\[
\leq \sum_{j \in I} (\text{diam } A_j)^\alpha.
\]

for every \( \alpha \geq 0 \). This implies that

\[
2.31\quad H_\delta^\alpha(E_1) + H_\delta^\alpha(E_2) \leq H_\delta^\alpha(E_1 \cup E_2)
\]

for every \( \alpha \geq 0 \), by taking the infimum over all such coverings \( \{A_j\}_{j \in I} \) of \( E_1 \cup E_2 \). The opposite inequality holds automatically, as in (2.27). Thus

\[
2.32\quad H_\delta^\alpha(E_1) + H_\delta^\alpha(E_2) = H_\delta^\alpha(E_1 \cup E_2)
\]

for all \( \alpha \geq 0 \) under these conditions.

2.4 Hausdorff measures

Let \( (M, d(x, y)) \) be a metric space, and let \( \alpha \geq 0 \) be given. The \( \alpha \)-dimensional Hausdorff measure of \( E \subseteq M \) is defined by

\[
2.33\quad H^\alpha(E) = \sup_{\delta > 0} H_\delta^\alpha(E),
\]

where \( H_\delta^\alpha(E) \) is as in the previous section. This can also be considered as the limit of \( H_\delta^\alpha(E) \) as \( \delta \to 0 \), since \( H_\delta^\alpha(E) \) increases monotonically as \( \delta \) decreases. As usual, \( H^\alpha(\emptyset) = 0 \) for every \( \alpha \geq 0 \), and

\[
2.34\quad H^\alpha(E) \leq H^\alpha(\overline{E})
\]

for every \( \alpha \geq 0 \) when \( E \subseteq \overline{E} \subseteq M \). If \( E_1, E_2, E_3, \ldots \) is any sequence of subsets of \( M \), then

\[
2.35\quad H^\alpha\left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} H^\alpha(E_k)
\]

for every \( \alpha \geq 0 \), by (2.27), so that \( H^\alpha \) is an outer measure on the \( \sigma \)-algebra of all subsets of \( M \) for each \( \alpha \geq 0 \).

Let \( E \subseteq M \), \( 0 \leq \alpha < \beta \), and \( \delta > 0 \) be given. If \( \{A_j\}_{j} \) is a collection of finitely or countably many subsets of \( M \) such that \( E \subseteq \bigcup_j A_j \) and \( \text{diam } A_j \leq \delta \) for each \( j \), then

\[
2.36\quad \sum_j (\text{diam } A_j)^\beta \leq \delta^{\beta-\alpha} \sum_j (\text{diam } A_j)^\alpha.
\]

This implies that

\[
2.37\quad H_\delta^\beta(E) \leq \delta^{\beta-\alpha} H_\delta^\alpha(E).
\]
If $H^\alpha(E) < \infty$, then one can pass to the limit as $\delta \to 0$, to get that $H^\beta(E) = 0$. The Hausdorff dimension of $E \subseteq M$ may be defined as the infimum of the $\alpha \geq 0$ such that $H^\alpha(E) < \infty$, if there is such an $\alpha$, and otherwise the Hausdorff dimension of $E$ is $\infty$. If $H^\alpha_{\text{con}}(E) = 0$ for some $\alpha \geq 0$ and $E \subseteq M$, then it is easy to see that $H^\beta(E) = 0$ for every $\delta > 0$, and hence that $H^\alpha(E) = 0$. The main point is that if $\{A_j\}_j$ is a collection of finitely or countable many subsets of $M$ such that $E \subseteq \bigcup_{j \in I_n} A_j$, and the corresponding sum (2.15) is small, then $\text{diam } A_j$ has to be small for each $j$.

Suppose that $H^\alpha(E) < \infty$ for some $\alpha \geq 0$ again. Thus for each positive integer $n$ there is a collection $\{A_{j,n}\}_{j \in I_n}$ of finitely or countably many open subsets of $M$ such that $E \subseteq \bigcup_{j \in I_n} A_{j,n}$, diam $A_{j,n} < 1/n$ for every $j \in I_n$, and

$$\sum_{j \in I_n} (\text{diam } A_{j,n})^\alpha < H^\alpha(E) + 1/n. \quad (2.38)$$

Put

$$\tilde{E} = \bigcap_{n=1}^{\infty} \left( \bigcup_{j \in I_n} A_{j,n} \right), \quad (2.39)$$

so that $E \subseteq \tilde{E}$, $\tilde{E}$ is the intersection of a sequence of open subsets of $M$, and $\tilde{E} \subseteq \bigcup_{j \in I_n} A_j$ for each $n$. The latter implies that $H^\alpha(\tilde{E}) \leq H^\alpha(E)$, and hence that $H^\alpha(\tilde{E}) = H^\alpha(E)$.

If $E_1, E_2 \subseteq M$ have the property that

$$d(x, y) \geq \eta \quad (2.40)$$

for some $\eta > 0$ and every $x \in E_1$ and $y \in E_2$, then (2.32) holds when $0 < \delta \leq \eta$. This implies that

$$H^\alpha(E_1) + H^\alpha(E_2) = H^\alpha(E_1 \cup E_2), \quad (2.41)$$

by taking the limit as $\delta \to 0$ in (2.32). This shows that $H^\alpha$ satisfies a well-known criterion of Carathéodory, and hence that $H^\alpha$ is countably additive on a suitable $\sigma$-algebra of measurable subsets of $M$ that includes the Borel sets. If $\alpha = 0$, then Hausdorff measure reduces to counting measure on $M$.

### 2.5 Some special cases

Suppose that $M$ is the real line, with the standard metric. As in Section 2.1, every nonempty bounded subset of $\mathbb{R}$ is contained in a closed interval with the same diameter. This implies that one may as well restrict one’s attention to coverings of $E \subseteq \mathbb{R}$ by closed intervals in the definition of $H^\alpha_\delta(E)$ for every $\alpha, \delta > 0$. If $\alpha = 0$, then one should consider the empty set as a closed interval too. One might also consider the real line itself as a closed interval, for the analogous statement for $H^\alpha_{\text{con}}(E)$, although this does not really matter.

Let us restrict our attention now to $\alpha = 1$. It is easy to see that

$$H^1_\delta(E) = H^1_{\text{con}}(E) \quad (2.42)$$
2.5. SOME SPECIAL CASES

for every $\delta > 0$ and $E \subseteq \mathbb{R}$, by subdividing intervals in $\mathbb{R}$ into finitely many arbitrarily small subintervals. It follows that

$$H^1(E) = H^1_{\text{con}}(E)$$

(2.43)

for every $E \subseteq \mathbb{R}$, which is of course the same as the Lebesgue outer measure of $E$. If $E$ is a closed interval in $\mathbb{R}$, then this is less than or equal to the diameter of $E$, which is the same as the length of $E$ as an interval, as in (2.16). As usual, one can show that $H^1(E)$ is equal to the diameter of $E$ in this case, by considering coverings of $E$ by finitely many intervals in $\mathbb{R}$.

Suppose now that $(M, d(x,y))$ is an ultrametric space. Every nonempty bounded subset of $M$ is contained in a closed ball in $M$ with the same diameter, as in Section 2.1 again. Thus one may as well restrict one’s attention to coverings of $E \subseteq M$ by closed balls in $M$ in the definition of $H^\alpha_\delta(E)$ for every $\alpha, \delta > 0$. As before, one should consider the empty set as a closed ball in $M$ when $\alpha = 0$, and one might also consider $M$ as a closed ball even when $M$ is unbounded, in the context of Hausdorff content.

In particular, these remarks can be applied to $\mathbb{Q}_p$, with the $p$-adic metric. Let us restrict our attention to $\alpha = 1$ again. Remember that $\mathbb{Z}_p$ can be expressed as the union of $p^j$ pairwise-disjoint translates of $p^l \mathbb{Z}_p$ for every positive integer $l$, as in Section 1.4. This implies that any closed ball in $\mathbb{Q}_p$ of radius $p^k$ for some $k \in \mathbb{Z}$ can be expressed as the pairwise-disjoint union of $p^l$ closed balls of radius $p^{k-l}$ for every positive integer $l$. Note that every closed ball in $\mathbb{Q}_p$ of radius $p^l$ for some $j \in \mathbb{Z}$ has diameter equal to $p^l$ too. Using this, one can check that (2.42) also holds in this case for every $E \subseteq \mathbb{Q}_p$. This implies that (2.43) holds for every $E \subseteq \mathbb{Q}_p$ as well.

If $B$ is a closed ball in $\mathbb{Q}_p$ with radius $p^j$ for some $j \in \mathbb{Z}$, then $H^1(B) \leq p^j$, by the previous discussion. Let us verify that $H^1(B) \geq p^j$ under these conditions, and hence that

$$H^1(B) = p^j.$$ (2.44)

To do this, it suffices to consider coverings of $B$ by finitely many closed balls in $\mathbb{Q}_p$, because $B$ is compact, and closed balls in $\mathbb{Q}_p$ are open sets. More precisely, it suffices to consider coverings of $B$ by finitely many closed balls of the same radius $p^{j-l}$ for some nonnegative integer $l$, by subdividing balls of different radii to get balls of the same radius. To show that $H^1(B) \geq p^j$, it is enough to check that $B$ cannot be covered by fewer than $p^j$ closed balls of radius $p^{j-l}$, for any nonnegative integer $l$. If $B = \mathbb{Z}_p$, which is the closed unit ball in $\mathbb{Q}_p$, then this follows from the discussion in Section 1.4. If $B$ is any other closed ball in $\mathbb{Q}_p$, then one can reduce to the case of $\mathbb{Z}_p$, using translations and dilations.

Now let $X_1, X_2, X_3, \ldots$ be a sequence of finite sets, where $X_j$ has exactly $n_j \geq 2$ elements for each $j$. Also let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product, as in Section 1.2. Put $t_0 = 1$, and let $t_l > 0$ be defined by

$$1/t_l = \prod_{j=1}^{l} n_j$$

(2.45)
when \( l \geq 1 \). Thus \( \{t_i\}_{i=0}^{\infty} \) is a strictly decreasing sequence of positive real numbers that converges to 0, which leads to an ultrametric \( d(x, y) \) on \( X \), as in (1.22). Remember that the closed ball in \( X \) centered at a point \( x \in X \) and with radius \( t_k \) for some nonnegative integer \( k \) is of the form \( B_k(x) \) as in (1.23). The diameter of this ball is also equal to \( t_k \). The radius of any closed ball in \( X \) with respect to \( d(x, y) \) in (1.22) can be taken to be \( t_k \) for some nonnegative integer \( k \), since these are the only positive values of \( d(x, y) \).

By construction, every closed ball \( B \) in \( X \) of radius \( t_k \) for some nonnegative integer \( k \) is the union of

\[
\prod_{j=k+1}^{l} n_j
\]

pairwise-disjoint closed balls in \( X \) of radius \( t_l \), for every integer \( l > k \). This implies that (2.42) also holds for every \( E \subseteq X \) in this situation, and hence that (2.43) holds for every \( E \subseteq X \) too. In particular,

\[
H^1(B) \leq \text{diam } B.
\]

As before, one can show that \( H^1(B) \geq \text{diam } B \), by considering coverings of \( B \) by finitely many closed balls, and subdividing the balls to get finitely many smaller balls of the same radius. This implies that

\[
H^1(B) = \text{diam } B,
\]

which corresponds exactly to (1.24), in the case where \( \mu_j \) is uniformly distributed on \( X_j \) for each \( j \).

### 2.6 Carathéodory’s construction

Let \( (M, d(x, y)) \) be a metric space, let \( \mathcal{F} \) be a collection of subsets of \( M \), and let \( \zeta \) be a nonnegative extended real-valued function on \( \mathcal{F} \). Also let \( 0 < \delta \leq \infty \) be given, and put

\[
H_\delta(E) = \inf \left\{ \sum_j \zeta(A_j) : E \subseteq \bigcup_j A_j, A_j \in \mathcal{F} \text{ for each } j, \text{ and diam } A_j < \delta \text{ for each } j \right\}
\]

for each \( E \subseteq M \). More precisely, the infimum is taken over all collections \( \{A_j\}_j \) of finitely or countably many elements of \( \mathcal{F} \) such that \( E \subseteq \bigcup_j A_j \) and \( \text{diam } A_j < \delta \) for each \( j \), if there are any. If there are no such coverings of \( E \), then we put \( H_\delta(E) = +\infty \). If \( E = \emptyset \), then we interpret \( H_\delta(E) \) as being equal to 0, using the empty covering of \( E \), and interpreting an empty sum as being 0.

As in [32], one can avoid these problems with two very mild additional hypotheses. The first is that for each \( \delta > 0 \), there be a collection \( \{A_j\}_j \) of finitely or countably many elements of \( \mathcal{F} \) such that \( \bigcup_j A_j = M \) and \( \text{diam } A_j < \delta \)
for every \( j \). If \( \mathcal{F} \) is the collection of all subsets of \( M \), then this is equivalent to asking that \( M \) be separable. This condition ensures that the coverings used in the definition of \( H_\delta(E) \) always exist. The second additional hypothesis is that for each \( \delta > 0 \), there be an \( A \in \mathcal{F} \) such that \( \text{diam } A < \delta \) and \( \zeta(A) < \delta \). This implies that \( H_\delta(\emptyset) = 0 \), without using the empty covering. In particular, this holds when \( \emptyset \in \mathcal{F} \) and \( \zeta(\emptyset) = 0 \), so that one can cover the empty set by itself.

Observe that
\[
H_\delta(E) \leq H_\delta(\tilde{E})
\]
for every \( \delta > 0 \) when \( E \subseteq \tilde{E} \subseteq M \). This simply uses the fact that every covering of \( \tilde{E} \) as in (2.49) is also a covering of \( E \), so that \( H_\delta(E) \) is the infimum of a larger collection of sums than for \( H_\delta(\tilde{E}) \). In many situations, \( \zeta \) may enjoy the monotonicity property
\[
\zeta(A) \leq \zeta(B)
\]
for every \( A, B \in \mathcal{F} \) with \( A \subseteq B \), but this is not needed to get (2.50). One can also show that \( H_\delta \) is countably subadditive for each \( \delta > 0 \), by standard arguments, so that \( H_\delta \) is an outer measure on the \( \sigma \)-algebra of all subsets of \( M \).

As before, if \( 0 < \delta < \eta \leq \infty \), then
\[
H_\eta(E) \leq H_\delta(E)
\]
for every \( E \subseteq M \), because \( H_\eta(E) \) is the infimum of a larger class of sums than for \( H_\delta(E) \). If \( E_1, E_2 \subseteq M \) satisfy \( d(x, y) \geq \delta \) for every \( x \in E_1 \) and \( y \in E_2 \), then it is easy to see that
\[
H_\delta(E_1) + H_\delta(E_2) \leq H_\delta(E_1 \cup E_2),
\]
for the same reasons as in Section 2.3. The opposite inequality holds for any \( E_1, E_2 \subseteq M \), so that equality holds in (2.53) under these conditions.

Put
\[
H(E) = \sup_{\delta > 0} H_\delta(E)
\]
for each \( E \subseteq M \), which can also be interpreted as a limit as \( \delta \to 0 \), because of (2.52). As usual, \( H(\emptyset) = 0 \), and
\[
H(E) \leq H(\tilde{E})
\]
when \( E \subseteq \tilde{E} \subseteq M \), by (2.50). Similarly, the countable subadditivity of \( H_\delta \) for each \( \delta > 0 \) implies the same property for \( H \), and hence that \( H \) is an outer measure on the \( \sigma \)-algebra of all subsets of \( M \). If \( E_1, E_2 \subseteq M \) satisfy \( d(x, y) \geq \eta \) for some \( \eta > 0 \) and every \( x \in E_1 \) and \( y \in E_2 \), then (2.53) holds when \( 0 < \delta \leq \eta \), and hence
\[
H(E_1) + H(E_2) \leq H(E_1 \cup E_2).
\]

The opposite inequality holds automatically, and it follows that \( H \) is countably additive on a suitable \( \sigma \)-algebra of measurable sets that includes the Borel sets, by Carathéodory’s criterion.
Suppose that \( E \subseteq M \) satisfies \( H(E) < \infty \), and let \( n \in \mathbb{Z}_+ \) be given. As in Section 2.4, there is a collection \( \{A_{j,n}\}_{j \in I_n} \) of finitely or countably many elements of \( \mathcal{F} \) such that \( E \subseteq \bigcup_{j \in I_n} A_{j,n} \), \( \operatorname{diam} A_{j,n} \leq 1/n \) for every \( j \in I_n \), and

\[
\sum_{j \in I_n} \zeta(A_{j,n}) < H(E) + 1/n.
\]

If we put

\[
\bar{E} = \bigcap_{n=1}^{\infty} \left( \bigcup_{j \in I_n} A_{j,n} \right),
\]

then \( E \subseteq \bar{E} \) and \( \bar{E} \subseteq \bigcup_{j \in I_n} A_{j,n} \) for each \( n \). This implies that \( H(E) = H(\bar{E}) \), since the first inclusion implies that (2.55) holds, and the opposite inequality can be derived from the second inclusion and the definition of \( H(\bar{E}) \). If every element of \( \mathcal{F} \) is a Borel set, then \( \bar{E} \) is a Borel set too.

Alternatively, one might define \( H^*_\delta(E) \) in the same way as \( H_\delta(E) \), except for replacing the requirement that \( \operatorname{diam} A_j < \delta \) for each \( j \) in (2.49) with the weaker condition that \( \operatorname{diam} A_j \leq \delta \) for each \( j \). If \( \delta = \infty \), then this condition on \( \operatorname{diam} A_j \) is vacuous, so that \( H^*_\infty \) is analogous to Hausdorff content. It is easy to see that \( H^*_\delta(E) \) satisfies the analogues of (2.50) and (2.52) for each \( \delta > 0 \), and that \( H^*_\delta(E) \) is countably subadditive, for the same reasons as before. Thus \( H^*_\delta \) is also an outer measure on the \( \sigma \)-algebra of all subsets of \( M \) for each \( \delta > 0 \). If \( E_1, E_2 \subseteq M \) satisfy \( d(x, y) > \delta \) for some \( \delta > 0 \) and every \( x \in E_1 \) and \( y \in E_2 \), then one can check that (2.53) holds, as before. Of course,

\[
H^*_\delta(E) \leq H_\delta(E)
\]

for every \( \delta > 0 \) and \( E \subseteq M \), because \( H^*_\delta(E) \) is the infimum of a larger collection of sums than for \( H_\delta(E) \). Similarly,

\[
H_\eta(E) \leq H^*_\eta(E)
\]

for every \( E \subseteq M \) when \( 0 < \delta < \eta \leq +\infty \), because \( H_\eta(E) \) is the infimum of a larger collection of sums than for \( H^*_\eta(E) \). It follows that the supremum of \( H^*_\delta(E) \) over \( \delta > 0 \) is the same as the supremum of \( H_\delta(E) \) over \( \delta > 0 \), which is equal to \( H(E) \).

If \( \mathcal{F} \) is the collection of all subsets of \( M \) and

\[
\zeta(A) = (\operatorname{diam} A)^\alpha
\]

for some \( \alpha \geq 0 \) and every \( A \subseteq M \), then \( H_\delta(E) \) is the same as \( H^*_\delta(E) \) in Section 2.3, and \( H(E) \) is the same as the \( \alpha \)-dimensional Hausdorff measure of \( E \). We have also seen that we can take \( \mathcal{F} \) to be the collection of all closed subsets of \( M \), or the collection of all open subsets of \( M \), when \( \zeta(A) \) is as in (2.61), and get the same results for \( H_\delta(E) \) and \( H(E) \). Similarly, if \( \zeta(A) \) is as in (2.61), then we can take \( \mathcal{F} \) to be the collection of all closed subsets of \( M \) and get the same result for \( H^*_\delta(E) \) as when \( \mathcal{F} \) is the collection of all subsets of \( M \), for each \( \delta > 0 \).
However, the analogous argument for open sets does not work for $H'_\delta(E)$ when $0 < \delta < \infty$, because approximations of a set $A \subseteq M$ by open sets that contain $A$ may have diameter greater than $\delta$ when $\text{diam } A = \delta$.

Let $\mathcal{F}$ and $\zeta$ be given as before, and let $\mathcal{A}$ be a $\sigma$-algebra of subsets of $M$ that contains $\mathcal{F}$. Suppose that $\mu$ is an outer measure on $\mathcal{A}$ such that

\begin{equation}
\mu(A) \leq C \zeta(A) \tag{2.62}
\end{equation}

for some nonnegative real number $C$ and every $A \in \mathcal{A}$. If $E \in \mathcal{A}$, and if $\{A_j\}_j$ are finitely or countably many elements of $\mathcal{F}$ such that $E \subseteq \bigcup_j A_j$, then

\begin{equation}
\mu(E) \leq \sum_j \mu(A_j) \leq C \sum_j \zeta(A_j) \tag{2.63}
\end{equation}

This implies that

\begin{equation}
\mu(E) \leq C H'_\infty(E) \tag{2.64}
\end{equation}

for every $E \in \mathcal{A}$, where $H'_\infty$ is the outer measure on $M$ corresponding to $\delta = \infty$ discussed earlier.

Let $\mathcal{F}$ and $\zeta$ be given again, and let $\tilde{d}(x, y)$ be another metric on $M$. Also let $\tilde{H}_\delta(E)$, $H'_\delta(E)$, and $H(E)$ be the analogues of $H_\delta(E)$, $H'_\delta(E)$, and $H(E)$, using $d(x, y)$ to define diameters of subsets of $M$ instead of $d(x, y)$. If the identity mapping on $M$ is uniformly continuous as a mapping from $M$ equipped with $d(x, y)$ to $M$ equipped with $\tilde{d}(x, y)$, then for each $\epsilon > 0$ there is a $\delta > 0$ such that

\begin{equation}
\tilde{H}_\epsilon(E) \leq H_\delta(E) \tag{2.65}
\end{equation}

for every $E \subseteq M$, and similarly for $\tilde{H}'_\epsilon(E)$ and $H'_\delta(E)$. In the limit as $\epsilon \to 0$, we get that

\begin{equation}
\tilde{H}(E) \leq H(E) \tag{2.66}
\end{equation}

for every $E \subseteq M$ under these conditions. If the identity mapping on $M$ is uniformly continuous as a mapping from $M$ equipped with $\tilde{d}(x, y)$ to $M$ equipped with $d(x, y)$, then

\begin{equation}
H(E) \leq \tilde{H}(E) \tag{2.67}
\end{equation}

for every $E \subseteq M$, for the same reasons. This implies that

\begin{equation}
\tilde{H}(E) = H(E) \tag{2.68}
\end{equation}

for every $E \subseteq M$ when $d(x, y)$ and $\tilde{d}(x, y)$ determine the same uniform structure on $M$. Of course, it is important here that we are using the same function $\zeta(A)$ for both metrics.

## 2.7 Snowflakes and quasi-metrics

Let $(M, d(x, y))$ be a metric space, and suppose that $d(x, y)^a$ is also a metric on $M$ for some $a > 0$. As in Section 1.6, this holds when $0 < a \leq 1$ and $d(x, y)$ is any
metric on $M$, and for all $a > 0$ when $d(x, y)$ is an ultrametric on $M$. It is easy to see that the diameter of $A \subseteq M$ with respect to $d(x, y)^a$ is equal to $(\text{diam } A)^a$, where $\text{diam } A$ is the diameter of $A$ with respect to $d(x, y)$. This implies that the $\alpha$-dimensional Hausdorff content of $E \subseteq M$ with respect to $d(x, y)^a$ is equal to the $(\alpha a)$-dimensional Hausdorff content of $E$ with respect to $d(x, y)$, for each $\alpha \geq 0$. Similarly, the analogue of $H_\delta^\alpha (E)$ with respect to $d(x, y)^a$ corresponds to $H_\delta^{a'} (E)$ with respect to $d(x, y)$, where $\alpha' = \alpha a$ and $\delta' = \delta^{1/a}$. It follows that the $\alpha$-dimensional Hausdorff measure of $E$ with respect to $d(x, y)^a$ is the same as the $(\alpha a)$-dimensional Hausdorff measure of $E$ with respect to $d(x, y)$. In particular, the Hausdorff dimension of $E$ with respect to $d(x, y)^a$ is equal to the Hausdorff dimension of $E$ with respect to $d(x, y)$ divided by $a$.

As in Section 1.6, $d(x, y)^a$ is a quasi-metric on $M$ for every $a > 0$ when $d(x, y)$ is a metric on $M$, or even a quasi-metric on $M$. One could define diameters, Hausdorff measures, and so on with respect to quasi-metrics, in which case the remarks in the previous paragraph would hold for all $a > 0$. However, there are some technical problems with this, related to the continuity properties of $d(x, y)$. If $d(x, y)$ is a metric on $M$, then the diameter of a set $A \subseteq M$ is the same as the diameter of the closure of $A$, and $A$ is contained open subsets of $M$ with approximately the same diameter, as in Section 2.1. Of course, this also works for quasi-metrics on $M$ of the form $d_0(x, y)^a$ for some metric $d_0(x, y)$ on $M$ and $a > 0$, by reducing to the corresponding statements for $d_0(x, y)$.

If $d(x, y)$ is a quasi-metric on $M$ of the form $d_0(x, y)^a$ for some metric $d_0(x, y)$ on $M$ and $a > 0$, then one might as well use Hausdorff measures with respect to $d_0(x, y)$ on $M$ to get Hausdorff measures with respect to $d(x, y)$, with suitable adjustments to the dimensions, as before. Alternatively, let $d(x, y)$ be a quasi-metric on $M$, and suppose that $d_1(x, y)$ is a metric on $M$ that defines the same uniform structure on $M$. This is equivalent to saying that the identity mapping on $M$ is uniformly continuous as a mapping from $M$ equipped with $d(x, y)$ to $M$ equipped with $d_1(x, y)$, and as a mapping from $M$ equipped with $d_1(x, y)$ to $M$ equipped with $d(x, y)$, where uniform continuity can be characterized in the usual way in terms of $\epsilon$’s and $\delta$’s. One can then define Hausdorff measures on $M$ with respect to $d(x, y)$ using the construction described in the previous section, where the metric $d(x, y)$ in the previous section is taken to be $d_1(x, y)$, and where $\zeta(A)$ is defined in terms of the diameter of $A$ with respect to $d(x, y)$. If $d_2(x, y)$ is another metric on $M$ that defines the same uniform structure on $M$ as $d(x, y)$, then $d_1(x, y)$ and $d_2(x, y)$ also determine the same uniform structure on $M$, and they lead to the same measures on $M$ as before.

### 2.8 Other Hausdorff measures

Let $(M, d(x, y))$ be a metric space, and let $\mathcal{F}$ be the collection of all subsets of $M$. Also let $h$ be a nonnegative real-valued function on the set $[0, +\infty)$ of nonnegative real numbers, and put

$$
(2.69) \quad \zeta(A) = h(\text{diam } A)
$$
for every bounded set $A \subseteq M$. Let us interpret this as being equal to 0 when $A = \emptyset$, which is automatic when $h(0) = 0$. One can include unbounded sets $A \subseteq M$ as well, with the convention that

$$h(+\infty) = \sup_{t \geq 0} h(t), \quad (2.70)$$

which may be infinite. This leads to outer measures $H_\delta$ and $H'_{\delta}$ on $M$ for each $\delta > 0$ as in Section 2.6, and to an outer measure $H$ on $M$, which is the Hausdorff measure associated to $h$.

Of course, this reduces to the previous situation when $h(t) = t^\alpha$ for some $\alpha \geq 0$. As usual, one can get the same results for $H_\delta$, $H'_{\delta}$, and $H$ by taking $F$ to be the collection of all closed subsets of $M$, because of (2.2). If $h(t)$ is continuous from the right at each $t \geq 0$, then one can also get the same results for $H_\delta$ and hence $H$ by taking $F$ to be the collection of all open subsets of $M$. If $M$ is the real line with the standard metric, then one can get the same results for $H_\delta$, $H'_{\delta}$, and $H$ using the collection of all closed intervals in $\mathbb{R}$, as in Section 2.5. Similarly, if $d(x,y)$ is an ultrametric on any set $M$, then one can get the same results for $H_\delta$, $H'_{\delta}$, and $H$ using the collection of all closed balls in $M$, as in Section 2.5.

Let $X_1, X_2, X_3, \ldots$ be a sequence of finite sets, where $X_j$ has exactly $n_j \geq 2$ elements for each $j$, and let $X$ be their Cartesian product, as in Section 1.2. Also let $\{t_l\}_{l=0}^\infty$ be a strictly decreasing sequence of positive real numbers that converges to 0, and let $d(x,y)$ be the corresponding ultrametric on $X$, as in (1.22). Put $\tilde{t}_0 = 1$, and let $\tilde{t}_l$ be defined for $l \geq 1$ by

$$1/\tilde{t}_l = \prod_{j=1}^l n_j, \quad (2.71)$$

so that $\{\tilde{t}_l\}_{l=0}^\infty$ is also a strictly decreasing sequence of positive real numbers that converges to 0. If $\tilde{d}(x,y)$ is the ultrametric on $X$ that corresponds to $\{\tilde{t}_l\}_{l=0}^\infty$ as in (1.22), then $\tilde{d}(x,y)$ is the same as the ultrametric considered in Section 2.5. Note that $d(x,y)$ and $\tilde{d}(x,y)$ determine the same uniform structure on $X$.

Let $h$ be a nonnegative real-valued function on $[0, +\infty)$ such that $h(0) = 0$ and

$$h(t_l) = \tilde{t}_l, \quad (2.72)$$

for each $l \geq 0$. If $\text{diam } A$ is the diameter of $A \subseteq M$ with respect to $d(x,y)$, then $h(\text{diam } A)$ is equal to the diameter of $A$ with respect to $\tilde{d}(x,y)$. Let $H(E)$ be the outer measure on $X$ corresponding to (2.69) and the collection $F$ of all subsets of $X$ as in Section 2.6, and let $\tilde{H}^1(E)$ be one-dimensional Hausdorff measure on $X$ with respect to $\tilde{d}(x,y)$. It is easy to see that $H(E) = \tilde{H}^1(E)$ for every $E \subseteq M$ under these conditions, using the remarks at the end of Section 2.6. Remember that $\tilde{H}^1$ can be analyzed as in Section 2.5.
2.9 Product spaces

Let \((M_1, d_1(x_1, y_1))\) and \((M_2, d_2(x_2, y_2))\) be metric spaces, and let \(M = M_1 \times M_2\) be their Cartesian product. It is easy to see that

\[
d(x, y) = \max(d_1(x_1, y_1), d_2(x_2, y_2))
\]

defines a metric on \(M\), where \(x = (x_1, x_2), y = (y_1, y_2)\). This metric has the nice property that the open ball in \(M\) centered at a point \(x = (x_1, x_2)\) and with radius \(r > 0\) is equal to the Cartesian product of the open balls in \(M_1, M_2\) centered at \(x_1, x_2\) with radii equal to \(r\). In particular, the topology on \(M\) determined by (2.73) is the same as the product topology associated to the topologies on \(M_1\) and \(M_2\) determined by the metrics \(d_1(x_1, y_1)\) and \(d_2(x_2, y_2)\), respectively. Alternatively,

\[
D_p(x, y) = (d_1(x_1, y_1)^p + d_2(x_2, y_2)^p)^{1/p}
\]

defines a metric on \(M\) when \(1 \leq p < \infty\), because of the triangle inequality for \(\ell^p\) norms. This is especially simple when \(p = 1\), and the \(p = 2\) case is very natural in the context of Euclidean geometry. Observe that

\[
d(x, y) \leq D_p(x, y) \leq 2^{1/p} d(x, y)
\]

for every \(x, y \in M\) and \(1 \leq p < \infty\), which implies that \(D_p(x, y)\) determines the same topology on \(M\) as \(d(x, y)\). This also implies analogous relations between diameters of subsets of \(M\) with respect to these metrics, and permits one to compare Hausdorff measures on \(M\) with respect to these metrics. Another nice property of (2.73) is that it is an ultrametric on \(M\) when \(d_1(x_1, y_1)\) and \(d_2(x_2, y_2)\) are ultrametrics on \(M_1\) and \(M_2\), respectively.

Let \(p_1 : M \to M_1\) and \(p_2 : M \to M_2\) be the obvious coordinate projections, so that \(p_1(x) = x_1\) and \(p_2(x) = x_2\) for every \(x = (x_1, x_2) \in M\). If \(A \subseteq M\), then

\[
diam A = \max(diam p_1(A), diam p_2(A)),
\]

where \(diam A\) is the diameter of \(A\) with respect to (2.73), and \(diam p_1(A), diam p_2(A)\) are the diameters of \(p_1(A), p_2(A)\) in \(M_1, M_2\), respectively. It follows that the diameters of \(A\) and \(p_1(A) \times p_2(A)\) with respect to (2.73) on \(M\) are the same. This implies that Hausdorff measures of a set \(E \subseteq M\) with respect to (2.73) can be defined equivalently in terms of coverings of \(E\) by products of subsets of \(M_1\) and \(M_2\). More precisely, one can restrict one’s attention to coverings of \(E\) by products of closed subsets of \(M_1\) and \(M_2\), because of (2.2).

Let \(h_1, h_2\) be monotone increasing nonnegative real-valued functions on \([0, +\infty)\), and put

\[
h_j(+\infty) = \sup_{t \geq 0} h_j(t)
\]

for \(j = 1, 2\), which may be infinite. Suppose that \(\mu_1, \mu_2\) are nonnegative Borel measures on \(M_1\) and \(M_2\) such that

\[
\mu_1(A_1) \leq C_1 h_1(diam A_1)
\]
and

\[ \mu_2(A_2) \leq C_2 h_2(\text{diam } A_2) \]  

for some nonnegative real numbers \( C_1, C_2 \) and all Borel sets \( A_1 \subseteq M_1 \) and \( A_2 \subseteq M_2 \). In particular, this ensures that \( M_1, M_2 \) are \( \sigma \)-finite with respect to \( \mu_1, \mu_2 \), so that the product measure \( \mu = \mu_1 \times \mu_2 \) can be defined on a suitable \( \sigma \)-algebra of subsets \( M \). If \( M_1 \) and \( M_2 \) are separable, then \( M \) is separable, which implies that open subsets of \( M \) can be expressed as unions of finitely or countably many products of open subsets of \( M_1 \) and \( M_2 \). In this case, open subsets of \( M \) are measurable with respect to the product measure construction, and hence Borel subsets of \( M \) are measurable too.

Put

\[ h(t) = h_1(t) h_2(t), \]

when \( 0 \leq t < \infty \), which is also a monotone increasing nonnegative real-valued function on \( [0, +\infty) \). Note that

\[ \sup_{t \geq 0} h(t) = h_1(\infty) h_2(\infty), \]  

with the convention that \( r \cdot (+\infty) = (+\infty) \cdot r \) is equal to \( +\infty \) when \( r > 0 \), and to 0 when \( r = 0 \). Thus we take \( h(+\infty) \) to be (2.81). If \( A_1 \subseteq M_1, A_2 \subseteq M_2 \) are Borel sets, \( A \subseteq M \) is measurable with respect to the product measure construction, and \( A \subseteq A_1 \times A_2 \), then

\[ \mu(A) \leq \mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2) \leq C_1 C_2 h_1(\text{diam } A_1) h_2(\text{diam } A_2) \leq C_1 C_2 h(\max(\text{diam } A_1, \text{diam } A_2)) \]

by (2.78) and (2.79). It follows that

\[ \mu(A) \leq C_1 C_2 h(\text{diam } A), \]

by taking \( A_1, A_2 \) to be the closures of \( p_1(A), p_2(A) \) in \( M_1, M_2 \), respectively, and using (2.76).
Chapter 3

Lipschitz mappings

3.1 Basic properties

Let \((M, d(x, y))\) and \((N, \rho(w, z))\) be metric spaces. A mapping \(f : M \to N\) is said to be \textit{Lipschitz} if there is a nonnegative real number \(C\) such that

\[
\rho(f(x), f(y)) \leq C d(x, y)
\]

for every \(x, y \in M\). In this case, one might also say that \(f\) is \(C\)-Lipschitz, or Lipschitz with constant \(C\), to indicate the constant \(C\). Of course, \(f\) is Lipschitz with constant \(C = 0\) if and only if \(f\) is constant. Note that the composition of two Lipschitz mappings with constants \(C_1, C_2\) is Lipschitz with constant \(C_1 C_2\).

Suppose that \(f : M \to N\) is Lipschitz with constant \(C\), and that \(A\) is a nonempty bounded subset of \(M\). Under these conditions, \(f(A)\) is a nonempty bounded set in \(N\), and

\[
diam f(A) \leq C \ diam A,
\]

where more precisely \(diam A = diam_M A\) is defined using the metric on \(M\), and \(diam f(A) = diam_N f(A)\) uses the metric on \(N\). This also works when \(A\) is unbounded, with the convention that the right side of (3.2) is infinite when \(C > 0\) and equal to \(0\) when \(C = 0\). It follows that

\[
H^\alpha_{\text{con}}(f(E)) \leq C^\alpha H^\alpha_{\text{con}}(E)
\]

for every \(E \subseteq M\) and \(\alpha \geq 0\), where \(H^\alpha_{\text{con}}(E)\) is defined using the metric on \(M\), and \(H^\alpha_{\text{con}}(f(E))\) is defined using the metric on \(N\), as before. If \(\alpha = 0\), then \(C^\alpha\) should be interpreted as being equal to \(1\) for every \(C \geq 0\).

Similarly,

\[
H^\alpha_{\delta,\text{con}}(f(E)) \leq C^\alpha H^\alpha_{\delta}(E)
\]

for every \(E \subseteq M\), \(\alpha \geq 0\), and \(\delta > 0\), at least when \(C > 0\), so that \(C \delta > 0\). This implies that

\[
H^\alpha(f(E)) \leq C^\alpha H^\alpha(E)
\]
for every $E \subseteq M$ and $\alpha \geq 0$ when $C > 0$, which also holds trivially when $C = 0$. Indeed, if $C = 0$ and $\alpha > 0$, then $H^\alpha(f(E)) = 0$ automatically. If $\alpha = 0$, then $H^\alpha$ reduces to counting measure, and the counting measure of $f(E)$ is less than or equal to the counting measure of $E$ for any mapping $f : M \to N$ and $E \subseteq M$.

A mapping $f : M \to N$ is said to be bilipschitz if there is a $C \geq 1$ such that

$$C^{-1} d(x, y) \leq \rho(f(x), f(y)) \leq C d(x, y)$$

for every $x, y \in M$. As before, one might say that $f$ is $C$-bilipschitz, or bilipschitz with constant $C$, to indicate the constant $C$. If $f$ is bilipschitz with constant $C$, then

$$C^{-1} \text{diam } A \leq \text{diam } f(A) \leq C \text{ diam } A$$

for every nonempty bounded set $A \subseteq M$. This implies that

$$C^{-\alpha} H^\alpha_{\text{con}}(E) \leq H^\alpha_{\text{con}}(f(E)) \leq C^\alpha H^\alpha_{\text{con}}(E)$$

and

$$C^{-\alpha} H^\alpha(E) \leq H^\alpha(f(E)) \leq C^\alpha H^\alpha(E)$$

for every $E \subseteq M$ and $\alpha \geq 0$. Of course, the counting measure of $E$ is equal to the counting measure of $f(E)$ for every $E \subseteq M$ when $f : M \to N$ is one-to-one.

### 3.2 Real-valued functions

Let $(M, d(x, y))$ be a metric space, and let $f$ be a real-valued function on $M$. Thus $f$ is Lipschitz with constant $C \geq 0$ with respect to the standard metric on $\mathbb{R}$ if and only if

$$|f(x) - f(y)| \leq C d(x, y)$$

for every $x, y \in X$. Of course, this implies that

$$f(x) \leq f(y) + C d(x, y)$$

for every $x, y \in M$. Conversely, if $f$ satisfies (3.11) for every $x, y \in M$, then we also have that

$$f(y) \leq f(x) + C d(x, y)$$

for every $x, y \in M$, by interchanging the roles of $x$ and $y$. It is easy to see that (3.10) is implied by (3.11) and (3.12), so that (3.10) and (3.11) are equivalent to each other.

In particular,

$$f_p(x) = d(p, x)$$

satisfies (3.11) for every $p, x, y \in M$ with $C = 1$, by the triangle inequality. This shows that (3.13) is a Lipschitz function on $M$ with constant $C = 1$ for every $p \in M$. Now let $A$ be a nonempty subset of $M$, and put

$$\text{dist}(x, A) = \inf \{d(x, z) : z \in A\}$$
for every \( x \in M \). Observe that
\[
\text{(3.15)} \quad \dist(x, A) \leq d(x, z) \leq d(x, y) + d(y, z)
\]
for every \( x, y \in M \) and \( z \in A \), which implies that
\[
\text{(3.16)} \quad \dist(x, A) \leq d(x, y) + \dist(y, A)
\]
for every \( x, y \in M \). Thus (3.14) is also a Lipschitz function on \( M \) with constant \( C = 1 \), for each nonempty set \( A \subseteq M \).

Suppose now that \( d(\cdot, \cdot) \) is an ultrametric on \( M \). In this case, we have that
\[
\text{(3.17)} \quad \dist(x, A) \leq d(x, z) \leq \max(d(x, y), d(y, z))
\]
for every \( x, y \in M \) and \( z \in A \), which is stronger than (3.15). If
\[
\text{(3.18)} \quad d(x, y) < \dist(x, A),
\]
then it follows that
\[
\text{(3.19)} \quad \dist(x, A) \leq d(y, z)
\]
for every \( z \in A \), and hence
\[
\text{(3.20)} \quad \dist(x, A) \leq \dist(y, A).
\]
Combining (3.18) and (3.20), we get that
\[
\text{(3.21)} \quad d(x, y) < \dist(y, A),
\]
so that
\[
\text{(3.22)} \quad \dist(y, A) \leq \dist(x, A),
\]
by the same argument. This shows that
\[
\text{(3.23)} \quad \dist(x, A) = \dist(y, A)
\]
when \( x, y \in M \) satisfy (3.18).

Let \( d(x, y) \) be any metric on \( M \) again, and let \( E \) be a connected subset of \( M \). If \( p, q \in E \) and \( f_p(x) \) is as in (3.13), then \( f_p(E) \) is a connected subset of \( \mathbb{R} \) that contains \( 0 \) and \( d(p, q) \), and hence contains \([0, d(p, q)]\). This implies that
\[
\text{(3.24)} \quad d(p, q) \leq H^1(f_p(E)) \leq H^1(E)
\]
for every \( p, q \in E \), so that
\[
\text{(3.25)} \quad \text{diam } E \leq H^1(E).
\]
3.3 Some examples

Let \( n_1, n_2, n_3, \ldots \) be a sequence of integers with \( n_j \geq 2 \) for each \( j \), and put

\[ X_j = \{0, 1, \ldots, n_j - 1\} \tag{3.26} \]

for each \( j \in \mathbb{Z}_+ \). Thus \( X_j \) has exactly \( n_j \) elements for each \( j \), and we let \( X = \prod_{j=1}^{\infty} X_j \) be their Cartesian product, as in Section 1.2. Also put \( N_k = \prod_{j=1}^{k} n_j \) for each \( k \in \mathbb{Z}_+ \) and \( N_0 = 1 \), and \( t_l = 1/N_l \) for every \( l \geq 0 \). This leads to an ultrametric \( d(x, y) \) on \( X \) as in (1.22), for which the corresponding one-dimensional Hausdorff measure was discussed in Section 2.5.

Observe that

\[ N_{j-1}^{-1} - N_j^{-1} = n_j N_{j-1}^{-1} - N_j^{-1} = (n_j - 1) N_j^{-1} \tag{3.27} \]

for each \( j \in \mathbb{Z}_+ \), and hence

\[ \sum_{j=k}^{l} (n_j - 1) N_j^{-1} = N_{k-1}^{-1} - N_l^{-1} \tag{3.28} \]

when \( 1 \leq k \leq l \). Put

\[ f_k(x) = \sum_{j=1}^{k} x_j N_j^{-1} \tag{3.29} \]

for each \( x \in X \) and \( k \in \mathbb{Z}_+ \), and \( f_0(x) = 0 \). Thus \( f_k(x) \) is an integer multiple of \( N_k^{-1} \) for each \( x \in X \) and \( k \geq 0 \), and

\[ 0 \leq f_k(x) \leq 1 - N_k^{-1} < 1, \tag{3.30} \]

by (3.28). One can check that every nonnegative integer multiple of \( N_k^{-1} \) strictly less than 1 can be expressed as \( f_k(x) \) for some \( x \in X \), using induction on \( k \).

If \( x, y \in X \) satisfy \( x_j \leq y_j \) for \( j = 1, \ldots, k \), then

\[ f_k(x) \leq f_k(y). \tag{3.31} \]

If \( x_j = y_j \) when \( j \leq k \) and \( k < l \), then

\[ f_l(y) \leq f_k(x) + \sum_{j=k+1}^{l} (n_j - 1) N_j^{-1} \leq f_k(x) + N_k^{-1} - N_l^{-1}, \tag{3.32} \]

by (3.28). Applying this to \( y = x \), we get that

\[ f_l(x) \leq f_k(x) + N_k^{-1} - N_l^{-1} \tag{3.33} \]

when \( k < l \). If \( x_j = y_j \) when \( j \leq k \) and \( x_{k+1} < y_{k+1} \), then

\[ f_{k+1}(x) + N_k^{-1} \leq f_{k+1}(y). \tag{3.34} \]
This implies that
\[(3.35) \quad f_l(x) + N_{l-1}^{-1} \leq f_{k+1}(y) \leq f_l(y)\]
for every \(l \geq k + 1\), because of (3.33) applied to \(k + 1\) instead of \(k\). In particular,
\[(3.36) \quad f_l(x) < f_l(y)\]
for each \(l \geq k + 1\) under these conditions. It follows that
\[(3.37) \quad f_l(x) \neq f_l(y)\]
when \(x_j \neq y_j\) for some \(j \leq l\), by considering the smallest such \(j\).

Taking the limit as \(l \to \infty\) in (3.28), we get that
\[(3.38) \quad \sum_{j=k}^{\infty} (n_j - 1) N_j^{-1} = N_{k-1}^{-1}\]
for each \(k \in \mathbb{Z}_+\), which is equal to 1 when \(k = 1\). Put
\[(3.39) \quad f(x) = \sum_{j=1}^{\infty} x_j N_j^{-1}\]
for each \(x \in X\), where the series converges by comparison with (3.38). Thus
\[(3.40) \quad 0 \leq f(x) \leq 1\]
for every \(x \in X\) and \(k \in \mathbb{Z}_+\), and
\[(3.41) \quad f(x) \leq f(y)\]
when \(x, y \in X\) satisfy \(x_j \leq y_j\) for each \(j\). If \(x_j = y_j\) when \(j \leq k\) for some \(k \geq 0\), and \(x_{k+1} < y_{k+1}\), then we also have (3.41), by taking the limit as \(l \to \infty\) in (3.35). In this case, the only way that equality can hold in (3.41) is if
\[(3.42) \quad y_{k+1} = x_{k+1} + 1, \quad x_l = n_l - 1, \quad y_l = 0\]
for each \(l \geq k + 2\).

If \(x \neq y\), then \(x_j \neq y_j\) for some \(j\), and we can choose \(k \geq 0\) as small as possible so that \(x_{k+1} < y_{k+1}\). It follows that \(f(x) = f(y)\) only when \(x = y\), or when there is a \(k \geq 0\) such that \(x_j = y_j\) for \(j \leq k\), and (3.42) holds.

Suppose again that \(x, y \in X\) satisfy \(x_j = y_j\) when \(j \leq k\) for some \(k \geq 0\). Of course, \(f_k(x) = f_k(y) \leq f(y)\), and hence
\[(3.43) \quad f_k(x) \leq f(y) \leq f_k(x) + N_k^{-1},\]
by taking the limit as \(l \to \infty\) in (3.32). In particular,
\[(3.44) \quad f_k(x) \leq f(x) \leq f_k(x) + N_k^{-1},\]
which implies that
\[(3.45) \quad |f(x) - f(y)| \leq N_k^{-1}\]
under these conditions. This shows that $f$ is Lipschitz with constant $C = 1$ as a mapping from $X$ into $\mathbb{R}$, with respect to the ultrametric $d(x, y)$ on $X$ described at the beginning of the section, and the standard metric on $\mathbb{R}$.

Let $x \in X$ and $k \geq 0$ be given, and let $B_k(x)$ be the set of $y \in X$ such that $x_j = y_j$ when $j \leq k$, as in Section 1.2. Thus

$$f(B_k(x)) \subseteq [f_k(x), f_k(x) + N_k^{-1}],$$

by (3.43). One can check that

$$f(B_k(x)) = [f_k(x), f_k(x) + N_k^{-1}],$$

for every $x \in X$ and $k \geq 0$, by standard arguments. In particular,

$$f(X) = [0, 1],$$

which is the same as (3.47) when $k = 0$.

Note that

$$H^1(B_k(x)) = N_k^{-1}$$

for every $x \in X$ and $k \geq 0$, where $H^1(B_k(x))$ is the one-dimensional Hausdorff measure of $B_k(x)$ with respect to the ultrametric $d(x, y)$ on $X$ mentioned earlier. This follows from the discussion at the end of Section 2.5. In particular,

$$H^1(X) = 1.$$

This is also consistent with the discussion of Hausdorff measure and Lipschitz mappings in Section 3.1, since the one-dimensional Hausdorff measure of an interval in the real line is the same as the length of the interval.

### 3.4 Other Lipschitz conditions

Let $(M, d(x, y))$ and $(N, \rho(w, z))$ be metric spaces, and let $a$ be a positive real number. A mapping $f : M \to N$ is said to be Lipschitz of order $a$ if there is a nonnegative real number $C$ such that

$$\rho(f(x), f(y)) \leq C \, d(x, y)^a$$

for every $x, y \in M$. As before, this condition holds with $C = 0$ if and only if $f$ is a constant mapping. If $a = 1$, then this condition is equivalent to the one discussed in Section 3.1.

If $a \leq 1$, then $d(x, y)^a$ is also a metric on $M$, as in Section 1.6. In this case, the condition described in the previous paragraph is equivalent to saying that $f$ is Lipschitz of order 1 with respect to the metric $d(x, y)^a$ on $M$, and with the same constant $C$. Similarly, if $d(x, y)$ is an ultrametric on $M$, then $d(x, y)^a$ is also an ultrametric on $M$ for every $a > 0$, and the condition in the previous paragraph is equivalent to saying that $f$ is Lipschitz of order 1 with respect to $d(x, y)^a$ on $M$. Otherwise, $d(x, y)^a$ is a quasi-metric on $M$ for every $a > 0$, as in
Section 1.6. One can define Lipschitz conditions with respect to quasi-metrics in the same way as for metrics, so that a mapping \( f : M \to N \) is Lipschitz of order \( a > 0 \) with respect to \( d(x, y) \) on \( M \) if and only if it is Lipschitz of order 1 with respect to \( d(x, y)^a \) on \( M \).

There are always a lot of real-valued Lipschitz functions of order 1 on any metric space \((M, d(x, y))\), as in Section 3.2. If \( 0 < a \leq 1 \), then \( d(x, y)^a \) is also a metric on \( M \), and the same discussion can be applied to get a lot of real-valued Lipschitz functions of order 1 on \( M \) with respect to \( d(x, y)^a \), which are the same as real-valued Lipschitz functions of order \( a \) on \( M \). Of course, the property of being Lipschitz of order \( a \) becomes stronger on bounded sets as \( a \) increases, and bounded Lipschitz functions of order 1 are also Lipschitz functions of order \( a \) when \( 0 < a \leq 1 \). If \( M \) is the real line with the standard metric, then the only Lipschitz functions of order \( a > 1 \) are constant, because the derivative of such a function must be equal to 0 at every point. Equivalently, the only Lipschitz functions of order 1 on \( \mathbb{R} \) with respect to the quasi-metric \( |x - y|^a \) are the constant functions when \( a > 1 \). If \( d(x, y) \) is any quasi-metric on a set \( M \), then there is a metric \( d(x, y) \) on \( M \) and a positive real number \( a \) such that \( d(x, y) \) is comparable to \( d(x, y)^a \), as shown in [29] and recalled in Section 1.6. This implies that there are a lot of real-valued Lipschitz functions of order 1 on \( M \) with respect to \( d(x, y) \), which are Lipschitz of order \( 1/a \) with respect to \( d(x, y) \).

Suppose that \( f : M \to N \) is Lipschitz of some order \( a > 0 \) with constant \( C \), as in (3.51). If \( A \) is a nonempty bounded subset of \( M \), then \( f(A) \) is a nonempty bounded set in \( N \), and
\[
\text{diam } f(A) \leq C \text{diam } A. 
\]

More precisely, \( \text{diam } A \) is the diameter of \( A \) with respect to the metric on \( M \), and \( \text{diam } f(A) \) is the diameter of \( f(A) \) with respect to the metric on \( N \). This implies that
\[
H^\alpha(f(E)) \leq C^\alpha H^\alpha a(E) 
\]
for every \( E \subseteq M \) and \( \alpha \geq 0 \), as in Section 3.1.

### 3.5 Subadditive functions

Let \( \sigma(t) \) be a monotone increasing real-valued function on the set \([0, +\infty)\) of nonnegative real numbers such that \( \sigma(0) = 0 \), \( \sigma(t) > 0 \) when \( t > 0 \), and
\[
\lim_{t \to 0^+} \sigma(t) = 0. 
\]

If \( d(x, y) \) is an ultrametric on a set \( M \), then \( \sigma(d(x, y)) \) is also an ultrametric on \( M \), which determines the same topology on \( M \) as \( d(x, y) \). Of course, this includes the case where \( \sigma(t) = t^a \) for some \( a > 0 \), as in Section 1.6. This is also related to the examples discussed in Section 1.2. If, in addition to the conditions just mentioned, \( \sigma(t) \) satisfies
\[
\sigma(r + t) \leq \sigma(r) + \sigma(t) 
\]
for every $r, t \geq 0$, then $\sigma(t)$ is said to be subadditive. Remember that $\sigma(t) = t^a$ is subadditive when $0 < a \leq 1$, as in Section 1.6. If $\sigma(t)$ is subadditive and $d(x, y)$ is a metric on $M$, then $\sigma(d(x, y))$ is also a metric on $M$, which determines the same topology on $M$ as $d(x, y)$.

In both cases, the identity mapping on $M$ is uniformly continuous as a mapping from $M$ equipped with $d(x, y)$ to $M$ equipped with $\sigma(d(x, y))$, because of (3.54). Similarly, the identity mapping on $M$ is uniformly continuous as a mapping from $M$ equipped with $\sigma(d(x, y))$ to $M$ equipped with $d(x, y)$. More precisely, let $\epsilon > 0$ be given, and put

$$\delta = \sigma(\epsilon) > 0.$$  

Thus $\sigma(t) \geq \delta$ when $t \geq \epsilon$, because $\sigma(t)$ is monotone increasing. Equivalently, this means that $t < \epsilon$ when $\sigma(t) < \delta$, which is exactly what we wanted.

If $\sigma(t)$ is subadditive on $[0, +\infty)$, then

$$0 \leq \sigma(r + t) - \sigma(r) \leq \sigma(t)$$

for every $r, t \geq 0$, since $\sigma(\cdot)$ is also supposed to be monotone increasing on $[0, +\infty)$. This implies that $\sigma(\cdot)$ is uniformly continuous on $[0, +\infty)$, using (3.54). Alternatively, it follows from (3.57) that $\sigma$ is Lipschitz of order 1 with constant $C = 1$ as a mapping from $[0, +\infty)$ equipped with the metric $\sigma(|x - y|)$ into the real line with the standard metric.

Put

$$\sigma(t) = \lim_{r \to t^-} \sigma(r) = \sup\{\sigma(r) : 0 \leq r < t\}$$

for each positive real number $t$, so that $\sigma(t^-) \leq \sigma(t)$ for each $t > 0$, and $\sigma(t^-)$ is monotone increasing in $t$. If the diameter of a set $A \subseteq M$ with respect to $d(x, y)$ is equal to $t$, $0 < t < \infty$, then the diameter $T$ of $A$ with respect to $\sigma(d(x, y))$ satisfies

$$\sigma(t^-) \leq T \leq \sigma(t).$$

In particular,

$$T = \sigma(t^-) = \sigma(t),$$

when $\sigma(t^-) = \sigma(t)$, which holds automatically when $\sigma$ is subadditive, as in the previous paragraph. Of course, if the diameter of $A$ with respect to $d(x, y)$ is equal to 0, then the diameter of $A$ with respect to $\sigma(d(x, y))$ is equal to 0 too.

If $A$ is unbounded with respect to $d(x, y)$, then the diameter of $A$ with respect to $\sigma(d(x, y))$ is equal to

$$\sigma(+\infty) = \sup_{r \geq 0} \sigma(r),$$

which is either a positive real number or $+\infty$.

### 3.6 Moduli of continuity

Let $(M, d(x, y))$ and $(N, \rho(w, z))$ be metric spaces, and let $\sigma(t)$ be a monotone increasing nonnegative real-valued function on $[0, +\infty)$ such that $\sigma(0) = 0$ and
\( \sigma(t) \) is continuous at 0. Suppose that \( f : M \to N \) satisfies

\[
\rho(f(x), f(y)) \leq \sigma(d(x, y)) \tag{3.62}
\]

for every \( x, y \in M \), which implies that \( f \) is uniformly continuous in particular. This includes the Lipschitz condition (3.51) as a special case, with \( \sigma(t) = C t^\alpha \).

If \( \sigma(d(x, y)) \) is a metric on \( M \), as in the previous section, then (3.62) is the same as saying that \( f \) is Lipschitz of order 1 with constant \( C = 1 \) as a mapping from \( M \) equipped with the metric \( \sigma(d(x, y)) \) into \( N \) equipped with the metric \( \rho(w, z) \).

If \( A \) is a nonempty bounded set in \( M \), and \( f : M \to N \) satisfies (3.62), then

\[
\text{diam } f(A) \leq \sigma(\text{diam } A) \tag{3.63}
\]

for every nonempty bounded set \( A \subseteq M \). Here \( \text{diam } A \) is the diameter of \( A \) with respect to \( d(x, y) \) on \( M \), and \( \text{diam } f(A) \) is the diameter of \( f(A) \) with respect to \( \rho(w, z) \) on \( N \), as usual. This also works when \( A \) is unbounded, with \( \sigma(\infty) \) defined as in (3.61). Using (3.63), one can estimate Hausdorff measures of \( f(A) \) in terms of Hausdorff measures of \( A \), where the Hausdorff measures are defined in terms of functions of diameters of sets, as in Section 2.6.

If \( f \) is any mapping from \( M \) into \( N \), then put

\[
\sigma_f(t) = \sup \{ \rho(f(x), f(y)) : x, y \in M, d(x, y) \leq t \} \tag{3.64}
\]

for each nonnegative real number \( t \), where the supremum may be equal to \( +\infty \). Thus \( \sigma_f(t) \geq 0 \) for every \( t \geq 0 \), \( \sigma_f(0) = 0 \), \( \sigma_f(t) \) is monotone increasing, and (3.62) holds with \( \sigma(t) = \sigma_f(t) \) for every \( x, y \in M \), by construction. Note that \( f \) is uniformly continuous if and only if \( \sigma_f(t) < +\infty \) when \( t \) is sufficiently small, and \( \lim_{t \to 0+} \sigma_f(t) = 0 \). The finiteness of \( \sigma_f(t) \) for every \( t > 0 \) is another matter, and is trivial when \( f(M) \) is bounded in \( N \). However, in order to estimate Hausdorff measures, it suffices to have a condition like (3.63) when the diameter of \( A \) is small.

Suppose that \( f : \mathbb{R} \to \mathbb{R} \) satisfies (3.62), where \( d(x, y) \) and \( \rho(w, z) \) are both equal to the standard metric on \( \mathbb{R} \), and \( \lim_{t \to 0+} \sigma(t)/t = 0 \). This implies that the derivative of \( f \) is equal to 0 everywhere on \( \mathbb{R} \), and hence that \( f \) is constant on \( \mathbb{R} \). This includes the case where \( f \) is Lipschitz of order \( \alpha > 1 \), as in Section 3.4. One can check that the analogous statement also holds when \( \lim_{t \to 0+} \sigma(t)/t = 0 \). If \( M \) and \( N \) are arbitrary metric spaces, \( f : M \to N \) satisfies (3.62), and \( \sigma(t) = 0 \) for some \( t > 0 \), then \( f \) is locally constant on \( M \), and in particular \( f \) is constant on \( M \) when \( M \) is connected.

### 3.7 Isometries and similarities

Let \( (M, d(x, y)) \) and \( (N, \rho(w, z)) \) be metric spaces. A mapping \( f : M \to N \) is said to be an isometry if

\[
\rho(f(x), f(y)) = d(x, y) \tag{3.65}
\]

for every \( x, y \in M \).
for every $x, y \in M$. Equivalently, $f$ is an isometry if it is a bilipschitz mapping with constant $C = 1$. Let us say that $f : M \to N$ is a similarity if there is a positive real number $\lambda$ such that

\[(3.66) \quad \rho(f(x), f(y)) = \lambda d(x, y)\]

for every $x, y \in M$. In this case, it is easy to see that

\[(3.67) \quad \text{diam } f(A) = \lambda \text{ diam } A\]

for every nonempty bounded set $A \subseteq M$, and hence that

\[(3.68) \quad H^\alpha(f(E)) = \lambda^\alpha H^\alpha(E)\]

for every $E \subseteq M$ and $\alpha \geq 0$.

Remember that a mapping $f : M \to N$ is said to be bounded if $f(M)$ is a bounded set in $N$. The space of bounded continuous mappings from $M$ into $N$ is denoted $C_b(M, N)$, and the supremum metric on $C_b(M, N)$ is defined by

\[(3.69) \quad \sup \{\rho(f(x), g(x)) : x \in M\}.\]

Note that the collection of $f \in C_b(M, N)$ such that $f(M)$ is dense in $N$ is a closed set in $C_b(M, N)$ with respect to the supremum metric. If $M$ is compact, then it follows that the collection of $f \in C_b(M, N)$ such that $f(M) = N$ is a closed set in $C_b(M, N)$ with respect to the supremum metric.

Let $\mathcal{I}(M, N)$ be the collection of isometric embeddings of $M$ into $N$. If $M$ is bounded, then $\mathcal{I}(M, N) \subseteq C_b(M, N)$, and $\mathcal{I}(M, N)$ is a closed set in $C_b(M, N)$ with respect to the supremum metric. If $M$ is complete and $f : M \to N$ is an isometry, then $f(M)$ is a closed set in $N$. In particular, $f(M) = N$ when $M$ is complete, $f : M \to N$ is an isometry, and $f(M)$ is dense in $N$. If $M$ and $N$ are compact, then $\mathcal{I}(M, N)$ is a compact set in $C_b(M, N)$ with respect to the supremum metric, by standard Arzela–Ascoli arguments.

Let $\mathcal{I}(M)$ be the collection of isometric mappings of $M$ onto itself, which is a group with respect to composition. If $M$ is bounded, then the restriction of the supremum metric to $\mathcal{I}(M)$ is invariant under left and right translations, and one can check that $\mathcal{I}(M)$ is a topological group with respect to the topology determined by the supremum metric. If $M$ is complete, then $\mathcal{I}(M)$ is the same as the collection of isometric mappings $f$ from $M$ into itself such that $f(M)$ is dense in $M$, as in the previous paragraph. If $M$ is bounded and complete, then it follows that $\mathcal{I}(M)$ is a closed subset of $\mathcal{I}(M, M)$ with respect to the supremum metric, and hence is a closed subset of $C_b(M, M)$. If $M$ is compact, then $\mathcal{I}(M)$ is also compact, with respect to the topology determined by the supremum metric, because $\mathcal{I}(M, M)$ is compact.

In fact, if $M$ is compact and $f$ is an isometry of $M$ into itself, then $f(M) = M$. To see this, suppose for the sake of a contradiction that there is an element $x_1$ of $M$ not in $f(M)$. Because $M$ is compact, $f(M)$ is compact, and hence there is an $r > 0$ such that

\[(3.70) \quad d(x_1, f(y)) \geq r\]
for every \( y \in M \). If \( \{x_j\}_{j=1}^\infty \) is the sequence of elements of \( M \) defined recursively by \( x_{j+1} = f(x_j) \) for each \( j \in \mathbb{Z}_+ \), then one can check that

\[
(3.71) \quad d(x_j, x_k) \geq r
\]

when \( j < k \), using (3.70) and the hypothesis that \( f \) be an isometry. This implies that \( \{x_j\}_{j=1}^\infty \) has no convergent subsequences, contradicting the compactness of \( M \), as desired.

As a variant of this, suppose that \( f \) and \( g \) are similarities from \( M \) into \( N \), with the same constant \( \lambda \). If \( g(M) = N \), then \( f \circ g^{-1} \) is an isometry from \( M \) into itself. If \( M \) is compact, then \( f \circ g^{-1} \) maps \( M \) onto itself, as in the previous paragraph. Thus \( f(M) = N \) under these conditions.

Suppose now that \( d(x,y) \) is an ultrametric on \( M \), and let \( r > 0 \) be given. Also let \( \sim_r \) be the relation on \( M \) defined by \( x \sim_r y \) when \( d(x,y) \leq r \). This is an equivalence relation on \( M \), because \( d(x,y) \) is an ultrametric on \( M \). The corresponding equivalence classes are closed balls of radius \( r \) in \( M \). If \( f \) is an isometry of \( M \) into itself, then \( f(x) \sim_r f(y) \) if and only if \( x \sim_r y \) for every \( x, y \in M \). This implies that \( f \) maps each equivalence class of \( M \) with respect to \( \sim_r \) into another equivalence class, which is the same as saying that \( f \) maps each closed ball in \( M \) with radius \( r \) into another closed ball of radius \( r \). More precisely, \( f \) maps distinct equivalence classes in \( M \) with respect to \( \sim_r \) into distinct equivalence classes in \( M \), which is the same as saying that \( f \) maps disjoint closed balls in \( M \) with radius \( r \) into disjoint closed balls with radius \( r \).

If \( M \) is totally bounded, then there are only finitely many equivalence classes in \( M \) with respect to \( \sim_r \) for each \( r > 0 \). In this case, it follows that every such equivalence class contains an element of \( f(M) \). This means that \( f(M) \) is dense in \( M \), since this holds for each \( r > 0 \). If \( M \) is complete, then we get that \( f(M) = M \). Of course, \( M \) is compact when \( f(M) = M \) and \( M \) is totally bounded.

Let \( d(x,y) \) be an arbitrary metric on \( M \) again, and suppose that \( M \) is totally bounded. Let \( r \) be a positive real number, and let \( n(r) \) be the smallest number of subsets of \( M \) with diameter less than or equal to \( r \) needed to cover \( M \). If \( E \) is any subset of \( M \) which is not dense in \( M \), then \( E \) can be covered by fewer than \( n(r) \) subsets of \( M \) with diameter less than or equal to \( r \) when \( r \) is sufficiently small, because at least one of the sets used to cover \( M \) will not intersect \( E \). If \( f \) is an isometry of \( M \) into itself, then the minimal number of sets of diameter less than or equal to \( r \) needed to cover \( f(M) \) is the same as \( n(r) \). This implies that \( f(M) \) is dense in \( M \) when \( M \) is totally bounded, and hence that \( f(M) = M \) when \( M \) is also complete and thus compact.

Suppose that \( H \) is a Hausdorff measure on \( M \), defined in terms of some function of the diameter of subsets of \( M \). Thus \( H(f(M)) = H(M) \) when \( f \) is an isometry of \( M \) into itself. If \( H(M) < +\infty \) and nonempty open subsets of \( M \) have positive measure with respect to \( H \), then it follows that \( f(M) \) is dense in \( M \), so that \( f(M) = M \) when \( M \) is compact. One can also show that \( f(M) = M \) when \( f \) is an isometry from \( M \) into itself and \( M \) is compact using compactness of \( Z(M,M) \). The covering argument in the preceding paragraph and the earlier approach using sequential compactness were suggested by students in a class, and some instances of this type of situation will be discussed in the next chapter.
Chapter 4

Functions on $\mathbb{Q}_p$

4.1 Polynomials on $\mathbb{Q}_p$

Let $p$ be a prime number, and let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with coefficients in $\mathbb{Q}_p$. Of course,

$$\begin{equation}
(x + h)^k = \sum_{j=0}^k \binom{k}{j} h^j x^{k-j}
\end{equation}$$

for every nonnegative integer $k$ and $x, h \in \mathbb{Q}_p$, where $\binom{k}{j}$ is the usual binomial coefficient. Thus

$$\begin{equation}
f(x + h) = \sum_{k=0}^n a_k (x + h)^k = \sum_{k=0}^n \sum_{j=0}^k a_k \binom{k}{j} h^j x^{k-j}
\end{equation}$$

for every $x, h \in \mathbb{Q}_p$. This implies that

$$\begin{equation}
f(x + h) - f(x) - f'(x) h = \sum_{k=1}^n \sum_{j=1}^k a_k \binom{k}{j} h^j x^{k-j},
\end{equation}$$

by subtracting the $j = 0$ terms from (4.3), and using the simple fact that $\binom{k}{0} = 1$ for each $k$.

The formal derivative of $f(x)$ is the polynomial defined by

$$\begin{equation}
f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1.
\end{equation}$$

Subtracting the $j = 1$ terms from (4.4), we get that

$$\begin{equation}
f(x + h) - f(x) - f'(x) h = \sum_{k=2}^n \sum_{j=2}^k a_k \binom{k}{j} h^j x^{k-j}
\end{equation}$$
for every $x, h \in \mathbb{Q}_p$, because $\binom{k}{1} = k$. In particular,

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x) \tag{4.7}$$

for every $x \in \mathbb{Q}_p$, since each term on the right side of (4.6) is a multiple of $h^2$.

It follows from (4.4) that $f(x)$ is Lipschitz of order 1 on bounded subsets of $\mathbb{Q}_p$. Suppose now that $a_k \in \mathbb{Z}_p$ for each $k$, so that $f$ maps $\mathbb{Z}_p$ into itself. In this case, (4.4) implies that

$$|f(x + h) - f(x)|_p \leq |h|_p \tag{4.8}$$

for every $x, h \in \mathbb{Z}_p$, since the binomial coefficients $\binom{k}{j}$ are integers. Of course, the coefficients of $f'(x)$ are elements of $\mathbb{Z}_p$ too, so that

$$|f'(x + h) - f'(x)| \leq |h|_p \tag{4.9}$$

for every $x, h \in \mathbb{Z}_p$ as well. Using (4.6), we also get that

$$|f(x + h) - f(x) - f'(x) h|_p \leq |h|^2_\mathbb{Z}_p \tag{4.10}$$

for every $x, h \in \mathbb{Z}_p$ under these conditions.

### 4.2 Hensel’s lemma (first version)

Let $f(x)$ be a polynomial with coefficients in $\mathbb{Z}_p$, so that $f(x)$ and $f'(x)$ are elements of $\mathbb{Z}_p$ for every $x \in \mathbb{Z}_p$. Suppose that $x_0 \in \mathbb{Z}_p$ satisfies $f(x_0) \in p\mathbb{Z}_p$ and $|f'(x_0)|_p = 1$. Under these conditions, Hensel’s lemma states that there is an $x \in \mathbb{Z}_p$ such that $x - x_0 \in p\mathbb{Z}_p$ and $f(x) = 0$. The proof uses Newton’s method, as follows. If $x_1 \in \mathbb{Z}_p$ is close to $x_0$, then $f(x_1)$ is approximately

$$f(x_0) + f'(x_0) (x_1 - x_0), \tag{4.11}$$

as in (4.10). In order to make this approximation equal to 0, we take

$$x_1 = x_0 - f'(x_0)^{-1} f(x_0). \tag{4.12}$$

This satisfies $x_1 - x_0 \in p\mathbb{Z}_p$, since $f(x_0) \in p\mathbb{Z}_p$ and $|f'(x_0)|_p = 1$.

Repeating the process, we shall choose a sequence of elements $x_1, x_2, x_3, \ldots$ of $\mathbb{Z}_p$ such that

$$x_j - x_{j-1} \in p\mathbb{Z}_p \tag{4.13}$$

for each $j \geq 1$. In particular, this ensures that

$$x_j - x_0 \in p\mathbb{Z}_p \tag{4.14}$$

for every $j \geq 1$, and hence that $f(x_j) - f(x_0) \in p\mathbb{Z}_p$ for every $j \geq 1$, by (4.8). Of course, this implies that

$$f(x_j) \in p\mathbb{Z}_p \tag{4.15}$$
for every \( j \geq 1 \), since \( f(x_0) \in p \mathbb{Z}_p \) by hypothesis. Similarly,
\[
(4.16) \quad f'(x_j) - f'(x_0) \in p \mathbb{Z}_p
\]
for each \( j \geq 1 \), by (4.14) and (4.9). It follows that
\[
(4.17) \quad |f'(x_j)|_p = 1
\]
for every \( j \geq 1 \), since \( |f'(x_0)|_p = 1 \) by hypothesis.
If \( x_{j-1} \) has already been chosen, then we would like to choose \( x_j \) so that
\[
(4.18) \quad f(x_{j-1}) + f'(x_{j-1})(x_j - x_{j-1}) = 0,
\]
which is the same as saying that
\[
(4.19) \quad x_j = x_{j-1} - f'(x_{j-1}) f(x_{j-1}).
\]
In particular, if \( x_{j-1} - x_0 \in p \mathbb{Z}_p \), then \( f(x_{j-1}) \in p \mathbb{Z}_p \) and \( |f'(x_{j-1})|_p = 1 \), as in the previous paragraph. This implies that (4.13) holds, so that the process can be repeated. More precisely,
\[
(4.20) \quad |x_j - x_{j-1}|_p = |f(x_{j-1})|_p.
\]
Under these conditions, we also have that
\[
(4.21) \quad |f(x_j)|_p \leq |x_j - x_{j-1}|_p^2,
\]
by applying (4.10) to \( x = x_{j-1} \) and \( h = x_j - x_{j-1} \), and using (4.18). Combining this with (4.20), we get that
\[
(4.22) \quad |f(x_j)|_p \leq |f(x_{j-1})|_p^2,
\]
for each \( j \geq 1 \). This implies that \( |f(x_j)|_p \to 0 \) as \( j \to \infty \), because \( |f(x_0)|_p < 1 \), by hypothesis. Thus \( |x_j - x_{j-1}|_p \to 0 \) as \( j \to \infty \), by (4.20) again. It follows that \( \{x_j\}_{j=1}^\infty \) is a Cauchy sequence in \( \mathbb{Z}_p \), as in Section 1.7, since the \( p \)-adic metric is an ultrametric. By completeness, \( \{x_j\}_{j=1}^\infty \) converges to an element \( x \) of \( \mathbb{Z}_p \), and in fact \( x - x_0 \in \mathbb{Z}_p \), because of (4.14). Of course, \( f(x) = 0 \), as desired, because \( f \) is continuous on \( \mathbb{Q}_p \), and \( |f(x_j)|_p \to 0 \) as \( j \to \infty \).

4.3 Hensel’s lemma (second version)

Let \( f(x) \) be a polynomial with coefficients in \( \mathbb{Z}_p \) again, and suppose that \( x_0 \in \mathbb{Z}_p \) satisfies
\[
(4.23) \quad |f(x_0)|_p < |f'(x_0)|_p^2.
\]
We would like to find an \( x \in \mathbb{Z}_p \) that is close to \( x_0 \) and satisfies \( f(x) = 0 \). Of course, \( f(x_0), f'(x_0) \in \mathbb{Z}_p \), so that \( |f(x_0)|_p, |f'(x_0)|_p \leq 1 \). If \( |f'(x_0)|_p = 1 \), then we are back in the situation discussed in the previous section. Otherwise,
Newton’s method is still applicable, but we should be a bit more careful about some of the estimates.

Let $j$ be a positive integer, and suppose that $x_{j-1} \in \mathbb{Z}_p$ has been chosen in such a way that
\begin{equation}
|x_{j-1} - x_0| < |f'(x_0)|_p
\tag{4.24}
\end{equation}
and
\begin{equation}
|f(x_{j-1})|_p \leq |f(x_0)|_p.
\tag{4.25}
\end{equation}
Thus
\begin{equation}
|f'(x_{j-1}) - f'(x_0)|_p \leq |x_{j-1} - x_0|_p < |f'(x_0)|_p,
\tag{4.26}
\end{equation}
by (4.9), which implies that
\begin{equation}
|f'(x_{j-1})|_p = |f'(x_0)|_p,
\tag{4.27}
\end{equation}
because of the ultrametric version of the triangle inequality. Let us choose $x_j \in \mathbb{Q}_p$ as in (4.19), so that
\begin{equation}
|x_j - x_{j-1}|_p = |f'(x_{j-1})|_p^{-1} |f(x_{j-1})| = |f'(x_0)|_p^{-1} |f(x_{j-1})|_p.
\tag{4.28}
\end{equation}
Combining this with (4.23) and (4.25), we get that
\begin{equation}
|x_j - x_{j-1}|_p < |f'(x_0)|_p.
\tag{4.29}
\end{equation}
It follows that
\begin{equation}
|x_j - x_0|_p < |f'(x_0)|_p,
\tag{4.30}
\end{equation}
by (4.24), and in particular that $x_j \in \mathbb{Z}_p$. This permits us to apply (4.10) with $x = x_{j-1}$ and $h = x_j - x_{j-1}$, to get that
\begin{equation}
|f(x_j)|_p \leq |x_j - x_{j-1}|_p^2,
\tag{4.31}
\end{equation}
using also (4.18). This implies that
\begin{equation}
|f(x_j)|_p \leq |f(x_{j-1})|_p,
\tag{4.32}
\end{equation}
by (4.28) and (4.29). In particular,
\begin{equation}
|f(x_j)|_p \leq |f(x_0)|_p,
\tag{4.33}
\end{equation}
by (4.25). This and (4.30) show that $x_j$ satisfies the same conditions as $x_{j-1}$, so that the process can be repeated.

More precisely, (4.28) and (4.31) imply that
\begin{equation}
|f(x_j)|_p \leq |f'(x_0)|_p^{-2} |f(x_{j-1})|_p^2
\tag{4.34}
\end{equation}
for each $j \geq 1$. Thus
\begin{equation}
|f(x_j)|_p \leq (|f'(x_0)|_p^{-2} |f(x_0)|_p) |f(x_{j-1})|_p
\tag{4.35}
\end{equation}
for each \( j \geq 1 \), by (4.25). This implies that \( |f(x_j)|_p \to 0 \) as \( j \to \infty \), by (4.23).

It follows from this and (4.28) that \( |x_j - x_{j-1}|_p \to 0 \) as \( j \to \infty \). Thus \( \{x_j\}_{j=1}^\infty \) is a Cauchy sequence in \( \mathbb{Z}_p \), as in Section 1.7, which converges to an element \( x \) of \( \mathbb{Z}_p \), by completeness. Note that

\[
|x - x_0|_p < |f'(x_0)|_p,
\]

because of the analogous condition for the \( x_j \)'s, and the fact that open balls in ultrametric spaces are closed sets. We also have that \( f(x) = 0 \), as desired, because \( f \) is continuous on \( \mathbb{Q}_p \), and \( |f(x_j)|_p \to 0 \) as \( j \to \infty \).

### 4.4 Contraction

Let \( f(x) \) be a polynomial with coefficients in \( \mathbb{Z}_p \), and suppose that \( x_0 \in \mathbb{Z}_p \) satisfies (4.23). Consider

\[
g(x) = x - (f'(x_0))^{-1} f(x + x_0),
\]

which is a polynomial with coefficients in \( \mathbb{Q}_p \) that satisfies

\[
g'(0) = 1 - (f'(x_0))^{-1} f'(x_0) = 0.
\]

More precisely, \( |f'(x_0)|_p = p^{-k} \) for some nonnegative integer \( k \), which implies that the coefficients of \( g \) are in \( p^{-k} \mathbb{Z}_p \). Using (4.23), we also get that

\[
|g(0)| = |f'(x_0)|^{-1}_p |f(x_0)| < |f'(x_0)|_p = p^{-k}.
\]

Let us now start over, and let \( k \) be a nonnegative integer and \( g(x) \) be a polynomial with coefficients in \( p^{-k} \mathbb{Z}_p \) such that

\[
g(0), g'(0) \in p^{k+1} \mathbb{Z}_p.
\]

If \( x \in p^{k+1} \mathbb{Z}_p \), then it is easy to see that

\[
g(x) \in p^{k+1} \mathbb{Z}_p
\]

and

\[
g'(x) \in p \mathbb{Z}_p.
\]

Observe that

\[
|g(y) - g(x) - g'(x) (y - x)|_p \leq p^k |x - y|_p^2
\]

for every \( x, y \in \mathbb{Z}_p \), by applying (4.10) to \( p^k y \). This implies that

\[
|g(y) - g(x)|_p \leq \max(|g'(x)|_p, p^k) |x - y|_p \leq p^{-1} |x - y|_p
\]

when \( x, y \in p^{k+1} \mathbb{Z}_p \), by (4.42).

Thus \( g \) maps \( p^{k+1} \mathbb{Z}_p \) into itself under these conditions, and the restriction of \( g \) to \( p^{k+1} \mathbb{Z}_p \) is a strict contraction, by (4.44). The contraction mapping principle implies that \( g \) has a unique fixed point in \( p^{k+1} \mathbb{Z}_p \), because \( p^{k+1} \mathbb{Z}_p \) is complete as a metric space. If \( g \) is as in (4.37), then this is the same as saying that there is a unique \( x \in p^{k+1} \mathbb{Z}_p \) such that \( f(x + x_0) = 0 \).
4.5 Local geometry

Let \( f(x) \) be a polynomial with coefficients in \( \mathbb{Z}_p \), and suppose that \( x_0 \in \mathbb{Z}_p \) satisfies \( f'(x_0) \neq 0 \). Let \( k \) be a nonnegative integer such that \( |f'(x_0)|_p = p^{-k} \), as before. If \( x \in x_0 + p^{k+1} \mathbb{Z}_p \), then

\[
|f'(x) - f'(x_0)|_p \leq |x - x_0|_p \leq p^{-k-1},
\]

by (4.9). This implies that

\[
|f'(x)|_p = |f'(x_0)|_p = p^{-k},
\]

by the ultrametric version of the triangle inequality. If \( x, y \in x_0 + p^{k+1} \mathbb{Z}_p \), then

\[
|f(y) - f(x) - f'(x)(y - x)|_p \leq |x - y|^2 \leq p^{-k-1} |x - y|_p,
\]

as in (4.10). It follows that

\[
|f(x) - f(y)|_p \leq \max(|f'(x)|_p |x - y|_p, p^{-k-1} |x - y|_p)
= p^{-k} |x - y|_p,
\]

by (4.46). Similarly, (4.46) and (4.47) also imply that

\[
p^{-k} |x - y|_p = |f'(x)|_p |x - y|_p
\leq \max(|f(x) - f(y)|_p, p^k |x - y|_p),
\]

and hence that

\[
p^{-k} |x - y|_p \leq |f(x) - f(y)|_p.
\]

This shows that

\[
|f(x) - f(y)|_p = p^{-k} |x - y|_p
\]

for every \( x, y \in x_0 + p^{k+1} \mathbb{Z}_p \) under these conditions.

Now let \( f \) be any mapping from \( x_0 + p^{k+1} \mathbb{Z}_p \) into \( \mathbb{Q}_p \) that satisfies (4.51) for every \( x, y \in p^k \mathbb{Z}_p \), where \( k \) is a nonnegative integer. In particular,

\[
f(x_0 + p^{k+1} \mathbb{Z}_p) \subseteq f(x_0) + p^{2k+1} \mathbb{Z}_p.
\]

Of course,

\[
x \mapsto p^{-k} (x - x_0) + f(x_0)
\]

is a similarity from \( x_0 + p^{k+1} \mathbb{Z}_p \) onto \( f(x_0) + p^{2k+1} \mathbb{Z}_p \) with respect to the \( p \)-adic metric, with the same similarity constant \( p^{-k} \). Because \( x_0 + p^{k+1} \mathbb{Z}_p \) is compact, one can use this to show that

\[
f(x_0 + p^{k+1} \mathbb{Z}_p) = f(x_0) + p^{2k+1} \mathbb{Z}_p,
\]

as in Section 3.7.

Remember that the one-dimensional Hausdorff measure of \( x_0 + p^{k+1} \mathbb{Z}_p \) with respect to the \( p \)-adic metric is equal to \( p^{-k-1} \), as in Section 2.5. Using this
and (4.51), it is easy to see that the one-dimensional Hausdorff measure of \( f(x_0 + p^{k+1}Z_p) \) is equal to \( p^{-2k-1} \), as in Section 3.7. Of course, this is the same as the one-dimensional Hausdorff measure of \( f(x_0 + p^{2k+1}Z_p) \). It follows that \( f(x_0 + p^kZ_p) \) is dense in \( f(x_0) + p^{2k+1}Z_p \), since every ball in \( Z_p \) with positive radius has positive one-dimensional Hausdorff measure. Note that \( f(x_0 + p^kZ_p) \) is a compact set in \( Z_p \), because \( x_0 + p^kZ_p \) is compact, and \( f \) is continuous. Thus the density of \( f(x_0 + p^kZ_p) \) in \( f(x_0) + p^{2k+1}Z_p \) implies that (4.54) holds, as before. This type of argument was also mentioned in Section 3.7, using the properties of one-dimensional Hausdorff measure in this case.

Here is an analogous but more elementary approach, which is a more explicit version of another argument in Section 3.7 in this situation. If \( n \) is a positive integer, then there is a set \( A_n \subseteq p^{k+1}Z_p \) with exactly \( p^n \) elements such that

\[
|a - b|_p \geq p^{-k-n}
\]

for every \( a, b \in A_n \) with \( a \neq b \). Equivalently, this means that the restriction of the natural quotient mapping from \( p^{k+1}Z_p \) onto \( p^{k+1}Z_p/p^{k+n+1}Z_p \) to \( A_n \) is injective. If \( f \) is as in the previous two paragraphs, then

\[
|f(x_0 + a) - f(x_0 + b)|_p = p^{-k}|a - b|_p \geq p^{-2k-n}
\]

for every \( a, b \in A_n \) with \( a \neq b \). However, \( f(x_0) + p^{2k+1}Z_p \) can be expressed as the union of \( p^n \) pairwise-disjoint closed balls of radius \( p^{-2k-n-1} \) for each positive integer \( n \). Each of these balls can contain at most one element of \( f(x_0 + A_n) \), by (4.56). It follows that each of these balls must contain an element of \( f(x_0 + A_n) \), which implies that \( f(x_0 + p^{k+1}Z_p) \) is dense in \( f(x_0) + p^{2k+1}Z_p \) under these conditions.

### 4.6 Power series

Let \( \sum_{j=0}^{\infty} a_j x_j \) be a power series with coefficients in \( Q_p \), where \( x_j \) is interpreted as being equal to 1 for every \( x \in Q_p \) when \( j = 0 \), as usual. As in Section 1.7, \( \sum_{j=0}^{\infty} a_j x^j \) converges for some \( x \in Q_p \) if and only if \( \{a_j x^j\}_{j=0}^{\infty} \) converges to 0 in \( Q_p \), which is the same as saying that

\[
|a_j x^j|_p = |a_j|_p |x|^j \to 0 \text{ as } j \to \infty.
\]

In this case, \( \sum_{j=0}^{\infty} a_j y^j \) also converges in \( Q_p \) when \( y \in Q_p \) satisfies \( |y|_p \leq |x|_p \). More precisely,

\[
\left| \sum_{j=0}^{\infty} a_j y^j - \sum_{j=0}^{n} a_j y^j \right|_p = \left| \sum_{j=n+1}^{\infty} a_j y^j \right|_p \leq \max_{j \geq n+1} |a_j y^j|_p \leq \max_{j \geq n+1} |a_j|_p |x|^j_0
\]

when \( |y|_p \leq |x|_p \), which implies that the partial sums \( \sum_{j=0}^{\infty} a_j y^j \) converge to \( \sum_{j=0}^{\infty} a_j y^j \) uniformly as \( n \to \infty \) on the set of \( y \in Q_p \) with \( |y|_p \leq |x|_p \). It follows
that $\sum_{j=0}^{\infty} a_j x^j$ defines a continuous $\mathbb{Q}_p$-valued function on the set of $x \in \mathbb{Q}_p$ for which the series converges, which is either a closed disk centered at 0 or all of $\mathbb{Q}_p$.

Now let $\sum_{j=0}^{\infty} a_j x^j$ and $\sum_{k=0}^{\infty} b_k x^k$ be infinite series with coefficients in $\mathbb{Q}_p$, and let

(4.59) \[ c_l = \sum_{j=0}^{l} a_j b_{l-j} \]

be the Cauchy product of their coefficients, as in Section 1.7. Observe that

(4.60) \[ c_l x^l = \sum_{j=0}^{n} (a_j x^j) (b_{l-j} x^{l-j}) \]

for each $x \in \mathbb{Q}_p$, so that $\sum_{l=0}^{\infty} c_l x^l$ is the Cauchy product of $\sum_{j=0}^{\infty} a_j x^j$ and $\sum_{k=0}^{\infty} b_k x^k$. In particular,

(4.61) \[ \sum_{l=0}^{\infty} c_l x^l = \left( \sum_{j=0}^{\infty} a_j x^j \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) \]

formally, collecting all of the terms which are multiples of $x^l$ for each $l \geq 0$. If $\sum_{j=0}^{\infty} a_j x^j$ and $\sum_{k=0}^{\infty} b_k x^k$ both converge for some $x \in \mathbb{Q}_p$, then it follows that $\sum_{l=0}^{\infty} c_l x^l$ converges and satisfies (4.61), as in Section 1.7.

Suppose that

(4.62) \[ f(x) = \sum_{j=0}^{\infty} a_j x^j \]

is a power series with coefficients $a_j \in \mathbb{Z}_p$ for each $j \geq 0$, and that $\{a_j\}_{j=0}^{\infty}$ converges to 0 in $\mathbb{Q}_p$. This implies that (4.62) converges for every $x \in \mathbb{Z}_p$, and that $f(x)$ defines a continuous $\mathbb{Q}_p$-valued function on $\mathbb{Z}_p$, as before. Moreover, the formal derivative

(4.63) \[ f'(x) = \sum_{j=1}^{\infty} ja_j x^{j-1} \]

also has coefficients in $\mathbb{Z}_p$ that converge to 0, and hence defines a continuous function on $\mathbb{Z}_p$ as well. It is easy to see that $f(x)$ and $f'(x)$ satisfy the same estimates (4.8), (4.9), and (4.10) as for polynomials with coefficients in $\mathbb{Z}_p$, by approximating the corresponding infinite series by their partial sums. It follows that the results for polynomials with coefficients in $\mathbb{Z}_p$ discussed in the previous sections also work for power series of this type.

In particular, the analogue of (4.10) in this context implies that the derivative of $f$ at any point $x \in \mathbb{Z}_p$ exists and is equal to $f'(x)$. There are analogous statements for any convergent power series with coefficients in $\mathbb{Q}_p$.
4.7 Linear mappings on $\mathbb{Q}_p^n$

Let $n$ be a positive integer, and let $\mathbb{Q}_p^n$ be the set of $n$-tuples $v = (v_1, \ldots, v_n)$ of elements of $\mathbb{Q}_p$. As usual, this is a vector space over $\mathbb{Q}_p$ with respect to coordinatewise addition and scalar multiplication. Put

$$||v|| = \max(|v_1|_p, \ldots, |v_n|_p)$$

for each $v \in \mathbb{Q}_p$, and observe that

$$||tv|| = |t|_p ||v||$$

for every $v \in \mathbb{Q}_p^n$ and $t \in \mathbb{Q}_p$, and that

$$||v + w|| \leq \max(||v||, ||w||)$$

for every $v, w \in \mathbb{Q}_p^n$. Thus $||v||$ is an ultranorm on $\mathbb{Q}_p^n$, which is like a norm on a real or complex vector space, except that it satisfies the ultrametric version of the triangle inequality (4.66). It follows that

$$d(v, w) = ||v - w||$$

defines an ultrametric on $\mathbb{Q}_p^n$, for which the corresponding topology on $\mathbb{Q}_p^n$ is the same as the product topology associated to the standard topology on $\mathbb{Q}_p$.

Let $e_1, \ldots, e_n$ be the standard basis vectors for $\mathbb{Q}_p^n$, so that the $j$th coordinate of $e_k$ is equal to 1 when $j = k$ and to 0 otherwise. If $T$ is a linear mapping from $\mathbb{Q}_p^n$ into itself, then put

$$||T||_{op} = \max(||T(e_1)||, \ldots, ||T(e_n)||).$$

The space of linear mappings from $\mathbb{Q}_p^n$ into itself is also a vector space over $\mathbb{Q}_p$ with respect to the usual addition and scalar multiplication of linear mappings, and it is easy to see that $||T||_{op}$ defines an ultranorm on this vector space. Each $v \in \mathbb{Q}_p^n$ can be expressed as $v = \sum_{j=1}^n v_j e_j$, and hence

$$||T(v)|| \leq \max_{1 \leq j \leq n} (|v_j|_p ||T(e_j)||) \leq ||T||_{op} ||v||.$$

Thus $||T||_{op}$ is the same as the operator norm of $T$ associated to the ultranorm $||v||$ on $\mathbb{Q}_p^n$, and

$$||T_2 \circ T_1||_{op} \leq ||T_1||_{op} ||T_2||_{op}$$

for any two linear mappings $T_1, T_2$ from $\mathbb{Q}_p^n$ into itself.

If $\{a_{j,k}\}_{j,k=1}^n$ is an $n \times n$ matrix with entries in $\mathbb{Q}_p$, then

$$(T(v))_j = \sum_{k=1}^n a_{j,k} v_k$$
defines a linear mapping from $\mathbb{Q}_p^n$ into itself, where $(T(v))_j$ is the $j$th coordinate of $T(v)$. Of course, every linear mapping from $\mathbb{Q}_p^n$ into itself can be expressed in this way, and one can check that

$$\|T\|_{op} = \max_{1 \leq j, k \leq n} |a_{j,k}|_p$$

when $T$ is as in (4.71). Note that $\|T\|_{op} \leq 1$ if and only if $a_{j,k} \in \mathbb{Z}_p$ for each $j, k = 1, \ldots n$, which happens if and only if $T$ maps $\mathbb{Z}_p^n$ into itself. If $T$ is as in (4.71), then the determinant of $T$ as a linear mapping on $\mathbb{Q}_p^n$ is the same as the determinant of the corresponding matrix $\{a_{j,k}\}_{j,k=1}^n$, and hence

$$|\det T|_p \leq \|T\|_{op}.$$  

(4.73)

Suppose that $T$ is a linear mapping from $\mathbb{Q}_p^n$ into itself that satisfies

$$\|T(v)\| = \|v\|$$

for every $v \in \mathbb{Q}_p^n$. In particular, this implies that $T$ is one-to-one, and hence that $T$ maps $\mathbb{Q}_p^n$ onto itself, by linear algebra. Thus $T$ is an invertible linear mapping on $\mathbb{Q}_p^n$, and the operator norms of $T$ and $T^{-1}$ are both equal to 1. Conversely, if $T$ is an invertible linear mapping on $\mathbb{Q}_p^n$ such that $\|T\|_{op}, \|T^{-1}\|_{op} \leq 1$, then $T$ satisfies (4.74). In this case, $T$ and $T^{-1}$ both correspond to matrices with entries in $\mathbb{Z}_p$, whose determinants are in $\mathbb{Z}_p$ as well. It follows that

$$|\det T|_p = 1,$$

(4.75)

because $\det T$ and $(\det T)^{-1} = \det T^{-1}$ are both in $\mathbb{Z}_p$. Conversely, suppose that $T$ is a linear mapping on $\mathbb{Q}_p^n$ that corresponds to an $n \times n$ matrix with entries in $\mathbb{Z}_p$, and that $T$ satisfies (4.75). This implies that $T$ is invertible, where the matrix associated to the inverse of $T$ can be expressed in terms of determinants in the usual way. More precisely, the entries of the matrix associated to $T^{-1}$ are in $\mathbb{Z}_p$ too, because of (4.75).
Chapter 5

Commutative topological groups

5.1 Haar measure

Let $G$ be a group, in which the group operations are expressed multiplicatively. If $G$ is also equipped with a topology with respect to which the group operations are continuous, then $G$ is said to be a topological group. More precisely, this means that multiplication in the group should be continuous as a mapping from $G \times G$ into $G$, where $G \times G$ is equipped with the product topology associated to the given topology on $G$. Similarly, $x \mapsto x^{-1}$ should be continuous as a mapping from $G$ onto itself. It is customary to ask also that the set containing only the identity element $e$ in $G$ be a closed set in $G$. This implies that every one-element subset of $G$ is closed, using the continuity of translations on $G$, which follows from continuity of multiplication on $G$. One can show that $G$ is Hausdorff under these conditions, and in fact regular as a topological space.

Let $G$ be a topological group which is locally compact as a topological space. It is well known that there is a nonnegative Borel measure on $G$ with suitable regularity properties that is invariant under left translations, known as Haar measure. In particular, the Haar measure of a nonempty open set in $G$ should be positive, and the Haar measure of a compact set in $G$ should be finite. This measure is unique up to multiplication by a positive real number. Similarly, there is a nonnegative Borel measure on $G$ with suitable regularity properties that is invariant under right translations, with the same type of uniqueness property. Of course, one can use the mapping $x \mapsto x^{-1}$ to switch between left and right-invariant Haar measures on $G$. If $G$ is compact, then one can show that left-invariant Haar measure on $G$ is invariant under right translations too. This is trivial when $G$ is commutative, and one can check that Haar measure on $G$ is invariant under the mapping $x \mapsto x^{-1}$ when $G$ is compact or commutative.

Using Haar measure on $G$, one gets a nonnegative linear functional on the space of continuous real or complex-valued functions with compact support on
Let $A$ be a commutative topological group, with the group operations expressed additively. A continuous homomorphism from $A$ into the multiplicative group $T$ of complex numbers with modulus equal to 1 is said to be a character on $A$. The collection of characters on $A$ forms a commutative group $\hat{A}$ with respect to pointwise multiplication, known as the dual group associated to $A$. In particular, the identity element in $\hat{A}$ is the trivial character on $A$, which is the constant function equal to 1 at every point in $A$. Note that the multiplicative inverse of $\phi \in \hat{A}$ is the same as the complex conjugate of $\phi$.

Of course, the group $\mathbb{Z}$ of integers is a commutative topological group with respect to addition the discrete topology. If $z \in T$, then

$$j \mapsto z^j$$

defines a homomorphism from $\mathbb{Z}$ into $T$, and every homomorphism from $\mathbb{Z}$ into $T$ is of this form. Similarly,

$$z \mapsto z^j$$

is a continuous homomorphism from $T$ into itself for every integer $j$, and it is
well known that every character on $\mathbb{T}$ is of this form. If $y \in \mathbb{R}$, then

$$x \mapsto \exp(i xy)$$  \hfill (5.3)

is a character on $\mathbb{R}$, where exp refers to the complex exponential function. It is also well known that every character on $\mathbb{R}$ is of this form.

If $A$ is any commutative topological group, then one can consider $\hat{A}$ equipped with the topology associated to uniform convergence on nonempty compact subsets of $A$. In particular, it is easy to see that $\hat{A}$ is also a topological group with respect to this topology. This is especially nice when $A$ is locally compact, in which case it can be shown that $\hat{A}$ is locally compact too. If $A = \mathbb{R}$ as a commutative topological group with respect to addition and the standard topology, then $\hat{A}$ is isomorphic to $\mathbb{R}$ as a commutative group, as in the previous paragraph. One can check that the dual topology on $\hat{A}$ corresponds exactly to the standard topology on $\mathbb{R}$ in this case as well.

If $z \in \mathbb{T}$ has nonnegative real part and $z \neq 1$, then the real part of $z^j$ is negative for some integer $j$. This implies that the only subgroup of $\mathbb{T}$ consisting of $z \in \mathbb{T}$ with nonnegative real part is the trivial subgroup $\{1\}$. If $A$ is a commutative topological group and $\phi \in \hat{A}$ has the property that the real part of $\phi(x)$ is nonnegative for every $x \in A$, then it follows that $\phi$ is the trivial character on $A$. In particular, if $\phi \in \hat{A}$ satisfies

$$|\phi(x) - 1| \leq 1$$  \hfill (5.4)

for every $x \in A$, then $\phi$ is the trivial character on $A$. If $\phi, \psi \in \hat{A}$ satisfy

$$|\phi(x) - \psi(x)| \leq 1$$  \hfill (5.5)

for every $x \in A$, then $\phi(x) = \psi(x)$ for every $x \in A$, since we can apply the previous argument to $\phi/\psi$. Equivalently, the distance between any two distinct elements of $\hat{A}$ with respect to the supremum metric is greater than 1. If $A$ is compact, then the topology on $\hat{A}$ mentioned in the previous paragraph is the same as the topology determined by the supremum metric on $\hat{A}$, and hence $\hat{A}$ is discrete with respect to this topology.

### 5.3 Compact commutative groups

Let $A$ be a compact commutative topological group, with the group operations expressed additively. As in Section 5.1, there is a unique translation-invariant nonnegative regular Borel measure $H$ on $A$ that satisfies $\hat{H}(A) = 1$, which is the normalized Haar measure on $A$. Let $L^2(A)$ be the usual space of complex-valued square-integrable functions on $A$ with respect to $H$, with the inner product

$$\langle f, g \rangle = \int_A f(x) \overline{g(x)} \, dH(x).$$  \hfill (5.6)
If $\phi$ is a character on $A$, then
\begin{equation}
\int_A \phi(x) \, dH(x) = \int_A \phi(x + a) \, dH(x) = \phi(a) \int_A \phi(x) \, dH(x)
\end{equation}
for every $a \in A$, using the translation-invariance of $H$ in the first step. If $\phi(a) \neq 1$ for some $a \in A$, then it follows that
\begin{equation}
\int_A \phi(x) \, dH(x) = 0.
\end{equation}
This implies that
\begin{equation}
\langle \phi, \psi \rangle = \int_A \phi(x) \overline{\psi(x)} \, dH(x) = 0
\end{equation}
when $\phi$ and $\psi$ are distinct elements of $\hat{A}$, by applying the previous argument to $\phi(x) \overline{\psi(x)}$, which is a nontrivial character on $A$. The normalization $H(A) = 1$ implies that each element of $\hat{A}$ has $L^2$ norm equal to 1, so that the elements of $\hat{A}$ are orthonormal in $L^2(A)$.

Let $C(A)$ be the algebra of complex-valued continuous functions on $A$, equipped with the supremum norm. If $E$ is the linear span of $\hat{A}$ in $C(A)$, then it is easy to see that $E$ is a sub-algebra of $C(A)$ which is invariant under complex conjugation and contains the constant functions on $A$. It is well known that $\hat{A}$ separates points in $A$, which implies that $E$ separates points in $A$. It follows that $E$ is dense in $C(A)$ with respect to the supremum norm, by the Stone–Weierstrass theorem. In particular, $E$ is dense in $L^2(A)$, so that $\hat{A}$ is an orthonormal basis for $L^2(A)$.

Similarly, if $E_1$ is a subgroup of $\hat{A}$, then the linear span $E_1$ of $E_1$ in $C(A)$ is a sub-algebra of $C(A)$ that is invariant under complex-conjugation and contains the constant functions. If $E_1$ separates points in $A$, then $E_1$ separates points in $A$ too, and hence $E_1$ is dense in $C(A)$, by the Stone–Weierstrass theorem again. If $\phi$ is any character on $A$ not in $E_1$, then $\phi$ is orthogonal to every element of $E_1$ with respect to the $L^2$ inner product, which implies that $\phi$ is orthogonal to every element of $E_1$. This implies that $\phi = 0$ when $E_1$ is dense in $C(A)$, which is a contradiction. It follows that $E_1 = \hat{A}$ when $E_1$ is a subgroup of $\hat{A}$ that separates points in $A$.

If $\phi \in \hat{A}$ and the real part of $\phi(x)$ is nonnegative for each $x \in A$, then $\phi$ is the trivial character on $A$, as in the previous section. Alternatively, if $\phi$ is a nontrivial character on $A$, then the integral of $\phi$ with respect to $H$ is equal to 0, as in (5.8). If the real part of $\phi(x)$ is nonnegative for every $x \in A$, then it follows that the real part of $\phi(x)$ is equal to 0 for every $x \in A$, contradicting the fact that $\phi(0) = 1$. One can also use the orthonormality of characters on $\hat{A}$ to get that
\begin{equation}
\int_A |\phi(x) - \psi(x)|^2 \, dH(x) = 2
\end{equation}
when $\phi, \psi$ are distinct elements of $\hat{A}$.
5.4 Cartesian products

Let $A_1, \ldots, A_n$ be finitely many commutative topological groups, and consider their Cartesian product $A = \prod_{j=1}^n A_j$. It is easy to see that $A$ is a commutative topological group as well, where the group operations are defined coordinate-wise, and using the corresponding product topology. If $\phi_1, \ldots, \phi_n$ are characters on $A_1, \ldots, A_n$, respectively, then

\begin{equation}
\phi(x) = \prod_{j=1}^n \phi_j(x_j)
\end{equation}

defines a character on $A$. Conversely, one can check that every character on $A$ is of this form.

Now let $I$ be an infinite set, and suppose that $A_j$ is a commutative topological group for each $j \in I$. As before, $A = \prod_{j \in I} A_j$ is a commutative topological group with respect to coordinatewise addition and the product topology. Let $j_1, \ldots, j_n$ be finitely many distinct elements of $I$, and let $\phi_{j_l}$ be a character on $A_{j_l}$ for each $l = 1, \ldots, n$. Clearly

\begin{equation}
\phi(x) = \prod_{l=1}^n \phi_{j_l}(x_{j_l})
\end{equation}

defines a character on $A$, where $x_j \in A_j$ denotes the $j$th coordinate of $x \in A$ for each $j \in I$. Conversely, suppose that $\phi$ is a character on $A$. Thus the set $V$ of $x \in A$ such that the real part of $\phi(x)$ is positive is an open set in $A$ that contains $0$. By definition of the product topology on $A$, there are open sets $U_j \subseteq A_j$ for each $j \in I$ such that $0 \in U_j$ for each $j$, $U_j = A_j$ for all but finitely many $j \in I$, and $\prod_{j \in I} U_j \subseteq V$. Put $B_j = A_j$ when $U_j = A_j$, and $B_j = \{0\}$ otherwise. If $B = \prod_{j \in I} B_j$, then $B$ is a subgroup of $A$ contained in $V$, so that the real part of $\phi(x)$ is positive when $x \in B$. It follows that $\phi(x) = 1$ for every $x \in B$, as in Section 5.2. This implies that $\phi(x)$ depends only on the finitely many coordinates $x_j$ of $x$ such that $B_j = \{0\}$ for each $x \in A$, and hence that $\phi$ can be expressed as in (5.12), as in the case of finite products.

Let $A = \prod_{j=1}^n A_j$ be the product of finitely many commutative topological groups again. If $A_j$ is locally compact for each $j = 1, \ldots, n$, then $A$ is locally compact too, and Haar measure on $A$ basically corresponds to the product of the Haar measures on $A_1, \ldots, A_n$. More precisely, if there is a countable base for the topology of $A_j$ for each $j$, then one can use the standard construction of product measures. Otherwise, one should use a version of product measures for Borel measures with suitable regularity properties. Equivalently, one can get a Haar integral on $A$ using Haar integrals on the $A_j$’s.

If $A = \prod_{j \in I} A_j$ is the product of infinitely many compact commutative topological groups, then $A$ is also a compact commutative topological group with respect to the product topology, by Tychonoff’s theorem. Haar measure on $A$ again basically corresponds to the product of the Haar measures on the $A_j$’s, normalized so that the measure of $A_j$ is equal to 1 for each $j \in I$. As
before, this is simpler when \( I \) is countably infinite, and there is a base for the topology of \( A_j \) with only finitely or countably many elements for each \( j \in I \), which implies that there is a base for the topology of \( A \) with only finitely or countably many elements. At any rate, one can look at the Haar integral on \( A \), in terms of the Haar integrals on the \( A_j \)'s. Using compactness, one can show that continuous functions on \( A \) can be approximated uniformly by continuous functions on \( A \) that depend on only finitely many coordinates, for which the Haar integral is much easier to define.

### 5.5 Discrete commutative groups

Let \( A \) be a commutative group equipped with the discrete topology, so that every homomorphism from \( A \) into \( T \) is continuous and hence a character. Note that the collection \( T^A \) of all mappings from \( A \) into \( T \) is a commutative group with respect to pointwise multiplication, and that \( ˆA \) is a subgroup of \( T^A \). More precisely, \( T^A \) can be considered as a Cartesian product of copies of \( T \) indexed by \( A \), equipped with the product topology corresponding to the standard topology on \( T \), and \( T^A \) is a compact topological group with respect to this topology, as in the previous section. One can check that \( ˆA \) is a closed subgroup of \( T^A \) with respect to the product topology, so that \( ˆA \) becomes a compact topological group with respect to the induced topology. This topology on \( ˆA \) is the same as the one mentioned in Section 5.2 in this case, because compact subsets of \( A \) are finite when \( A \) is equipped with the discrete topology.

Similarly, if \( E \) is any nonempty set, then the collection \( T^E \) of mappings from \( E \) into \( T \) is a compact commutative topological group with respect to pointwise multiplication and the product topology that corresponds to the standard topology on \( T \), as in the previous paragraph. If \( E \subseteq A \), then there is an obvious homomorphism from \( T^A \) onto \( T^E \), that sends each mapping from \( A \) into \( T \) to its restriction to \( E \). This homomorphism is continuous with respect to the corresponding product topologies, and the restriction of this homomorphism to \( ˆA \) is a continuous homomorphism from \( ˆA \) into \( T^E \). Suppose that \( E \) is a set of generators of \( A \), in the sense that every element of \( A \) can be expressed as a sum of finitely many elements of \( E \) and their inverses, where elements of \( E \) may be repeated. Under these conditions, the homomorphism from \( ˆA \) into \( T^E \) just mentioned is a homeomorphism of \( ˆA \) onto its image in \( T^E \), with respect to the topology on the image of \( ˆA \) in \( T^E \) induced by the product topology on \( T^E \).

Let \( B \) be a subgroup of \( A \), let \( x \in A \setminus B \) be given, and let \( B(x) \) be the subgroup of \( A \) generated by \( B \) and \( x \). If \( \phi \) is a homomorphism from \( B \) into \( T \), then it is well known that \( \phi \) can be extended to a homomorphism from \( B(x) \) into \( T \). If \( A \) is generated by \( B \) and finitely or countably many other elements of \( A \), then one can repeat the process to get an extension of \( \phi \) to a homomorphism from \( A \) into \( T \), and otherwise one can use Zorn's lemma or the Hausdorff maximality principle. Using this, one can show that for each \( a \in A \) with \( a \neq 0 \) there is a homomorphism \( \phi \) from \( A \) into \( T \) such that \( \phi(a) \neq 1 \). This implies that characters on \( A \) separate points in \( A \).
5.6. CHARACTERS ON $\mathbb{Z}_p$

Put
\begin{equation}
\Psi_a(\phi) = \phi(a)
\end{equation}
for each $a \in A$ and $\phi \in \hat{A}$, so that $\Psi_a$ maps $\hat{A}$ into $T$. More precisely, $\Psi_a$ is a continuous homomorphism from $\hat{A}$ into $T$, with respect to the topology on $\hat{A}$ discussed earlier. Thus $\Psi_a$ is an element of the dual $\hat{\hat{A}}$ of the dual $\hat{A}$ of $A$, and it is easy to see that
\begin{equation}
a \mapsto \Psi_a
\end{equation}
defines a homomorphism from $A$ into $\hat{\hat{A}}$. Note that $\hat{\hat{A}}$ should be equipped with the discrete topology, because $\hat{A}$ is compact, so that (5.14) is automatically continuous. This mapping (5.14) is also one-to-one, because $\hat{A}$ separates points in $A$, as in the previous paragraph. Of course, the collection of elements of $\hat{\hat{A}}$ of the form $\Psi_a$ for some $a \in A$ is a subgroup of $\hat{\hat{A}}$. This subgroup of $\hat{\hat{A}}$ automatically separates points in $\hat{A}$, because $\phi \in \hat{A}$ is not the trivial character exactly when (5.13) is not equal to 1 for some $a \in A$. It follows that every element of $\hat{\hat{A}}$ is of the form $\Psi_a$ for some $a \in A$, as in Section 5.3, because $\hat{A}$ is compact.

5.6 Characters on $\mathbb{Z}_p$

Let us begin with some remarks about cyclic groups. Let $n$ be a positive integer, let $n\mathbb{Z}$ be the subgroup of $\mathbb{Z}$ consisting of integer multiples of $n$, and let $\mathbb{Z}/n\mathbb{Z}$ be the corresponding quotient group, which is a cyclic group of order $n$. Also let $w \in \mathbb{C}$ be an $n$th root of unity, so that $w^n = 1$, which implies that the modulus of $w$ is equal to 1. The mapping from $j \in \mathbb{Z}$ to $w^j \in T$ is equal to 1 on $n\mathbb{Z}$, and hence determines a group homomorphism from $\mathbb{Z}/n\mathbb{Z}$ into $T$. Every homomorphism from $\mathbb{Z}/n\mathbb{Z}$ into $T$ is of this form, which implies that the dual group associated to $\mathbb{Z}/n\mathbb{Z}$ is also a cyclic group of order $n$.

Now let $p$ be a prime number, and let $\phi$ be a continuous homomorphism from $\mathbb{Z}_p$ as a commutative topological group with respect to addition into $T$. The continuity of $\phi$ implies that there is a nonnegative integer $k$ such that the real part of $\phi(x)$ is positive for every $x \in p^k\mathbb{Z}_p$. This implies that $\phi(x) = 1$ for every $x \in p^k\mathbb{Z}_p$, as in Section 5.2, because $p^k\mathbb{Z}_p$ is a subgroup of $\mathbb{Z}_p$. It follows that $\phi$ determines a homomorphism from $\mathbb{Z}_p/p^k\mathbb{Z}_p$ into $T$. We have also seen in Section 1.4 that $\mathbb{Z}/p^k\mathbb{Z}_p$ is isomorphic as a group to $\mathbb{Z}/p^k\mathbb{Z}$, so that the induced homomorphism from $\mathbb{Z}_p/p^k\mathbb{Z}_p$ into $T$ is of the form described in the preceding paragraph. Conversely, every homomorphism from $\mathbb{Z}_p/p^k\mathbb{Z}_p$ into $T$ leads to a homomorphism from $\mathbb{Z}_p$ into $T$, by composition with the canonical quotient mapping from $\mathbb{Z}_p$ onto $\mathbb{Z}_p/p^k\mathbb{Z}_p$. Any homomorphism from $\mathbb{Z}_p$ into $T$ of this type is automatically continuous, because $p^k\mathbb{Z}_p$ is an open subgroup of $\mathbb{Z}_p$ for each $k \geq 0$.

Let $n$ be a positive integer again, and consider the space of complex-valued functions on $\mathbb{Z}/n\mathbb{Z}$. This is an $n$-dimensional vector space, which may be equipped with a translation-invariant inner product as in Section 5.3. Of course,
normalized Haar measure on \( \mathbb{Z}/n \mathbb{Z} \) is simply the measure that assigns the value \( 1/n \) to each element of \( \mathbb{Z}/n \mathbb{Z} \). As before, characters on \( \mathbb{Z}/n \mathbb{Z} \) are orthonormal with respect to this inner product. It follows that the characters on \( \mathbb{Z}/n \mathbb{Z} \) form an orthonormal basis for the space of complex-valued functions on \( \mathbb{Z}/n \mathbb{Z} \), since there are exactly \( n \) characters on \( \mathbb{Z}/n \mathbb{Z} \).

Similarly, characters on \( \mathbb{Z}_p \) are orthonormal with respect to the \( L^2 \) inner product associated to normalized Haar measure on \( \mathbb{Z}_p \). Note that there are \( p^k \) characters on \( \mathbb{Z}_p \) obtained from characters on \( \mathbb{Z}_p/p^k \mathbb{Z}_p \) for each nonnegative integer \( k \). The linear span of these \( p^k \) characters consists of the complex-valued functions on \( \mathbb{Z}_p \) that are constant on the cosets of \( p^k \mathbb{Z}_p \) in \( \mathbb{Z}_p \), which is a vector space of dimension \( p^k \). The linear span of the set of all characters on \( \mathbb{Z}_p \) is the space of complex-valued functions on \( \mathbb{Z}_p \) that are constant on the cosets of \( p^k \mathbb{Z}_p \) in \( \mathbb{Z}_p \) for some nonnegative integer \( k \). In particular, this implies that the linear span of the characters on \( \mathbb{Z}_p \) is dense in the space of all continuous complex-valued functions on \( \mathbb{Z}_p \) with respect to the supremum norm, and hence in \( L^2(\mathbb{Z}_p) \) as well.

### 5.7 The quotient group \( \mathbb{Q}_p/\mathbb{Z}_p \)

Let \( p \) be a prime number again, and consider the quotient \( \mathbb{Q}_p/\mathbb{Z}_p \) of \( \mathbb{Q}_p \) as a commutative group with respect to addition by its subgroup \( \mathbb{Z}_p \). Also let \( \mathbb{Z}[1/p] \) be the collection of rational numbers of the form \( p^{-j}x \), where \( x \in \mathbb{Z} \), and \( j \) is a nonnegative integer. This is a dense subgroup of \( \mathbb{Q}_p \) with respect to addition and the \( p \)-adic metric, because \( \mathbb{Z} \) is dense in \( \mathbb{Z}_p \), as in Section 1.4. It follows that the image of \( \mathbb{Z}[1/p] \) under the canonical quotient mapping from \( \mathbb{Q}_p \) onto \( \mathbb{Q}_p/\mathbb{Z}_p \) is all of \( \mathbb{Q}_p/\mathbb{Z}_p \), since \( \mathbb{Z}_p \) is an open subgroup of \( \mathbb{Q}_p \). Thus we get a homomorphism from \( \mathbb{Z}[1/p] \) onto \( \mathbb{Q}_p/\mathbb{Z}_p \) whose kernel is equal to

\[
\mathbb{Z}[1/p] \cap \mathbb{Z}_p = \mathbb{Z}.
\]

This leads to a group isomorphism from \( \mathbb{Z}[1/p]/\mathbb{Z} \) onto \( \mathbb{Q}_p/\mathbb{Z}_p \). Because \( \mathbb{Z}_p \) is an open subgroup of \( \mathbb{Q}_p \), we take \( \mathbb{Q}_p/\mathbb{Z}_p \) to be equipped with the discrete topology.

Alternatively, observe that

\[
\mathbb{Z}[1/p] = \bigcup_{j=0}^{\infty} p^{-j} \mathbb{Z}
\]

and

\[
\mathbb{Q}_p = \bigcup_{j=0}^{\infty} p^{-j} \mathbb{Z}_p,
\]

which imply that

\[
\mathbb{Z}[1/p]/\mathbb{Z} = \bigcup_{j=0}^{\infty} (p^{-j} \mathbb{Z})/\mathbb{Z}.
\]
5.7. THE QUOTIENT GROUP $\mathbb{Q}_p/\mathbb{Z}_p$

and

$$(5.19) \quad \mathbb{Q}_p/\mathbb{Z}_p = \bigcup_{j=0}^{\infty}((p^{-j}\mathbb{Z}_p)/\mathbb{Z}_p).$$

Of course, $(p^{-j}\mathbb{Z}_p)/\mathbb{Z}_p$ is isomorphic as a group to $\mathbb{Z}/p^j\mathbb{Z}$ for each nonnegative integer $j$. We also have that $p^{-j}\mathbb{Z}_p/\mathbb{Z}_p$ is isomorphic as a group to $\mathbb{Z}_p/p^j\mathbb{Z}_p$, which is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$ when $j \geq 0$, as in Section 1.4. The obvious inclusion of $p^{-j}\mathbb{Z}$ in $p^{-j}\mathbb{Z}_p$ leads more directly to a group homomorphism from $(p^{-j}\mathbb{Z})/\mathbb{Z}$ into $(p^{-j}\mathbb{Z}_p)/\mathbb{Z}_p$, which is actually an isomorphism, for the usual reasons. It is easy to see that the isomorphism from $\mathbb{Z}[1/p]/\mathbb{Z}$ onto $\mathbb{Q}_p/\mathbb{Z}_p$ described in the previous paragraph sends $(p^{-j}\mathbb{Z})/\mathbb{Z}$ onto $(p^{-j}\mathbb{Z}_p)/\mathbb{Z}_p$ in this way for each nonnegative integer $j$.

We can also consider $\mathbb{Z}[1/p] \subseteq \mathbb{Q}$ as a subgroup of $\mathbb{R}$ with respect to addition, so that $\mathbb{Z}[1/p]/\mathbb{Z}$ can be identified with a subgroup of $\mathbb{R}/\mathbb{Z}$. If $\exp z$ is the complex exponential function, then

$$(5.20) \quad r \mapsto \exp(2\pi i r)$$

defines a continuous homomorphism from $\mathbb{R}$ as a commutative topological group with respect to addition onto $\mathbb{T}$ as a compact commutative group with respect to multiplication. The kernel of this homomorphism is equal to $\mathbb{Z}$, which leads to an isomorphism from $\mathbb{R}/\mathbb{Z}$ onto $\mathbb{T}$. This isomorphism sends $\mathbb{Z}[1/p]/\mathbb{Z}$ onto the subgroup of $\mathbb{T}$ consisting of all roots of unity with order equal to $p^l$ for some nonnegative integer $l$. Using the isomorphism between $\mathbb{Z}[1/p]/\mathbb{Z}$ and $\mathbb{Q}_p/\mathbb{Z}_p$ described earlier, we get a homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$ with kernel $\mathbb{Z}_p$.

Equivalently,

$$(5.21) \quad E_p(x') = \exp(2\pi i x')$$

defines a homomorphism from $\mathbb{Z}[1/p]$ as a commutative group with respect to addition into $\mathbb{T}$ as a commutative group with respect to multiplication, with kernel equal to $\mathbb{Z}$. If $x \in \mathbb{Q}_p$, then there is an $x' \in \mathbb{Z}[1/p]$ such that $x - x' \in \mathbb{Z}_p$, because $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$, as before. If $x'' \in \mathbb{Z}[1/p]$ also satisfies $x - x'' \in \mathbb{Z}_p$, then $x' - x'' \in \mathbb{Z}_p$, and hence $x' - x'' \in \mathbb{Z}$, as in (5.15). This implies that $E_p(x') = E_p(x'')$, so that we can extend $E_p$ to a mapping from $\mathbb{Q}_p$ into $\mathbb{T}$ by putting

$$(5.22) \quad E_p(x) = E_p(x')$$

when $x \in \mathbb{Q}_p$, $x' \in \mathbb{Z}[1/p]$, and $x - x' \in \mathbb{Z}_p$. If $x, y \in \mathbb{Q}_p$ and $x', y' \in \mathbb{Z}[1/p]$ satisfy $x - x', y - y' \in \mathbb{Z}_p$, then $x' + y' \in \mathbb{Z}[1/p]$ and

$$(5.23) \quad (x + y) - (x' + y') = (x - x') + (y - y') \in \mathbb{Z}_p,$$

so that

$$(5.24) \quad E_p(x + y) = E_p(x' + y') = E_p(x')E_p(y') = E_p(x)E_p(y).$$

Thus the extension of $E_p$ to $\mathbb{Q}_p$ is a homomorphism from $\mathbb{Q}_p$ as a commutative group with respect to addition into $\mathbb{T}$ as a commutative group with respect to multiplication. It is easy to see that the kernel of this homomorphism is equal to $\mathbb{Z}_p$, since the kernel of $E_p$ on $\mathbb{Z}[1/p]$ is equal to $\mathbb{Z}$. 

5.8 Characters on $\mathbb{Q}_p$

Let $p$ be a prime number, and let $\phi$ be a continuous homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$. Thus the restriction of $\phi$ to $\mathbb{Z}_p$ is a continuous homomorphism from $\mathbb{Z}_p$ into $\mathbb{T}$, and hence there is a nonnegative integer $k$ such that $\phi(x) = 1$ for every $x \in p^k \mathbb{Z}_p$, as in Section 5.6. This implies that $\phi$ determines a homomorphism from $\mathbb{Q}_p/p^k \mathbb{Z}_p$ into $\mathbb{T}$. Conversely, every homomorphism from $\mathbb{Q}_p/p^k \mathbb{Z}_p$ into $\mathbb{T}$ leads to a homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$, by composition with the canonical quotient mapping from $\mathbb{Q}_p$ onto $\mathbb{Q}_p/p^k \mathbb{Z}_p$. Any homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$ obtained in this way is continuous, because $p^k \mathbb{Z}_p$ is an open subgroup of $\mathbb{Q}_p$.

Let $E_p$ be the group homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$ discussed in the previous section, and put

\[ \phi_y(x) = E_p(xy) \]

for each $x, y \in \mathbb{Q}_p$. Thus $\phi_y$ is a group homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$ for each $y \in \mathbb{Q}_p$, which is trivial when $y = 0$. Otherwise, if $y \neq 0$, then the kernel of $\phi_y$ is equal to $p^k \mathbb{Z}_p$. In particular, $\phi_y$ is continuous as a mapping from $\mathbb{Q}_p$ into $\mathbb{T}$ for every $y \in \mathbb{Q}_p$. This implies that $\phi_y$ is an element of the dual $\hat{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as a commutative topological group with respect to addition, and it is easy to see that the mapping from $y \in \mathbb{Q}_p$ to $\phi_y \in \hat{\mathbb{Q}}_p$ is a group homomorphism.

Similarly, the restriction of $\phi_y$ to $\mathbb{Z}_p$ is a continuous homomorphism from $\mathbb{Z}_p$, as a commutative topological group with respect to addition into $\mathbb{T}$, and the mapping from $y \in \mathbb{Q}_p$ to the restriction of $\phi_y$ to $\mathbb{Z}_p$ is a group homomorphism from $\mathbb{Q}_p$ into the dual $\hat{\mathbb{Z}}_p$ of $\mathbb{Z}_p$. As before, the restriction of $\phi_y$ to $\mathbb{Z}_p$ is the trivial character on $\mathbb{Z}_p$ if and only if $y \in \mathbb{Z}_p$. This implies that the mapping from $y \in \mathbb{Q}_p$ to the restriction of $\phi_y$ to $\mathbb{Z}_p$ leads to an injective homomorphism from $\mathbb{Q}_p/\mathbb{Z}_p$ into $\hat{\mathbb{Z}}_p$. One can check that every continuous group homomorphism from $\mathbb{Z}_p$ into $\mathbb{T}$ is equal to the restriction of $\phi_y$ to $\mathbb{Z}_p$ for some $y \in \mathbb{Q}_p$, so that $\hat{\mathbb{Z}}_p$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ as a group. More precisely, if $\phi$ is a homomorphism from $\mathbb{Z}_p$ into $\mathbb{T}$ whose kernel contains $p^k \mathbb{Z}_p$ for some nonnegative integer $k$, then $\phi$ is equal to the restriction of $\phi_y$ to $\mathbb{Z}_p$ for some $y \in p^{-k} \mathbb{Z}_p$.

If $y \in \mathbb{Z}_p$, then the kernel of $\phi_y : \mathbb{Q}_p \rightarrow \mathbb{T}$ contains $\mathbb{Z}_p$, and hence $\phi_y$ determines a homomorphism $\psi_y$ from $\mathbb{Q}_p/\mathbb{Z}_p$ into $\mathbb{T}$. As in the previous section, we consider $\mathbb{Q}_p/\mathbb{Z}_p$ to be equipped with the discrete topology, so that every homomorphism from $\mathbb{Q}_p/\mathbb{Z}_p$ into $\mathbb{T}$ is automatically continuous. Thus $\psi_y$ is an element of the dual $(\mathbb{Q}_p/\mathbb{Z}_p)$ of $\mathbb{Q}_p/\mathbb{Z}_p$ as a commutative topological group with respect to the discrete topology for each $y \in \mathbb{Z}_p$. Note that $\psi_y$ is the trivial character on $\mathbb{Q}_p/\mathbb{Z}_p$ if and only if $\phi_y$ is the trivial character on $\mathbb{Q}_p$, which happens if and only if $y = 0$. It is easy to see that $y \mapsto \psi_y$ defines a group homomorphism from $\mathbb{Z}_p$ into $(\mathbb{Q}_p/\mathbb{Z}_p)$, as usual.

If $\psi$ is any homomorphism from $\mathbb{Q}_p/\mathbb{Z}_p$ into $\mathbb{T}$, then one can check that there is a $y \in \mathbb{Z}_p$ such that $\psi = \psi_y$. This is the same as saying that if $\phi$ is a homomorphism from $\mathbb{Q}_p$ into $\mathbb{T}$ whose kernel contains $\mathbb{Z}_p$, then there is a $y \in \mathbb{Z}_p$ such that $\phi = \phi_y$. To see this, one can start by showing that for each
nonnegative integer \( j \), there is a \( y_j \in \mathbb{Z}_p \) such that \( \phi = \phi_{y_j} \) on \( p^{-j} \mathbb{Z}_p \). The image of \( y_j \) in \( \mathbb{Z}_p/p^i \mathbb{Z}_p \) is uniquely determined by this property, which implies that \( \{y_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{Z}_p \). Thus \( \{y_j\}_{j=1}^{\infty} \) converges to an element \( y \) of \( \mathbb{Z}_p \), by completeness of the \( p \)-adic metric, and one can verify that \( \phi = \phi_{y} \).

It follows that \( y \mapsto \psi_y \) is a group isomorphism of \( \mathbb{Z}_p \) onto \( (\mathbb{Q}_p/\mathbb{Z}_p) \). Because \( \mathbb{Q}_p/\mathbb{Z}_p \) is equipped with discrete topology, \( (\mathbb{Q}_p/\mathbb{Z}_p) \) is compact with respect to the usual topology on the dual group. One can also check that \( y \mapsto \psi_y \) is a homeomorphism with respect to the topology on \( \mathbb{Z}_p \) determined by the \( p \)-adic metric and the usual topology on the dual group \( (\mathbb{Q}_p/\mathbb{Z}_p) \).

If \( \phi \) is any continuous homomorphism from \( \mathbb{Q}_p \) into \( T \), then the kernel of \( \phi \) contains \( p^k \mathbb{Z}_p \) for some integer \( k \). Under these conditions, there is a \( y \in p^{-k} \mathbb{Z}_p \) such that \( \phi = \phi_y \) on \( \mathbb{Q}_p \). This follows from the previous discussion when \( k = 0 \), and otherwise it is easy to reduce to that case. This implies that \( y \mapsto \phi_y \) defines a group isomorphism from \( \mathbb{Q}_p \) onto its dual. It is not too difficult to verify that this mapping is also a homeomorphism with respect to the topology on \( \mathbb{Q}_p \) defined by the \( p \)-adic metric and the corresponding topology on \( \mathbb{Q}_p \).
Chapter 6

$r$-Adic integers

6.1 $r$-Adic absolute values

Let $r = \{r_j\}_{j=0}^{\infty}$ be a sequence of positive integers, with $r_j \geq 2$ for each $j$. Put

$$R_l = \prod_{j=1}^{l} r_j$$

for each positive integer $l$, and $R_0 = 1$, so that $\{R_l\}_{l=0}^{\infty}$ is a strictly increasing sequence of positive integers. Note that $R_{l+1} \mathbb{Z}$ is a proper subset of $R_l \mathbb{Z}$ for each $l \geq 0$, and that $\bigcap_{l=0}^{\infty} R_l \mathbb{Z} = \{0\}$. If $a$ is a nonzero integer, then let $l_r(a)$ be the largest nonnegative integer such that $a \in R_{l_r(a)} \mathbb{Z}$, and put $l_r(0) = +\infty$. Equivalently, $l_r(a) + 1$ is the smallest positive integer such that $a \not\in R_{l_r(a)+1} \mathbb{Z}$ when $a \neq 0$. It is easy to see that

$$l_r(a + b) \geq \min(l_r(a), l_r(b))$$

and

$$l_r(ab) \geq \max(l_r(a), l_r(b))$$

for every $a, b \in \mathbb{Z}$. In particular, $l_r(-a) = l_r(a)$ for each $a \in \mathbb{Z}$. If $r$ is a constant sequence, then

$$l_r(a - b) \geq l_r(a) + l_r(b)$$

for every $a, b \in \mathbb{Z}$, and equality holds when $r_1$ is a prime number.

Let $t = \{t_l\}_{l=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers that converges to 0, with $t_0 = 1$. Put

$$|a|_r = t_{l_r(a)}$$

for each nonzero integer $a$, and $|0|_r = 0$, which corresponds to (6.5) with $t_\infty = 0$. Let us call $|a|_r$ the $r$-adic absolute value of $a \in \mathbb{Z}$, although it also depends on $t$. If $p$ is a prime number, $r_j = p$ for each $j \geq 1$, and $t_l = p^{-l}$ for each $l \geq 0$, then this reduces to the usual $p$-adic absolute value on $\mathbb{Z}$, as in Section 1.3.
Observe that
\[
|a + b|_r \leq \max(|a|_r, |b|_r)
\]
and
\[
|a b|_r \leq \min(|a|_r, |b|_r)
\]
for every \(a, b \in \mathbb{Z}\), by (6.2) and (6.3). If \(r\) is a constant sequence, and if \(t\) is submultiplicative in the sense that
\[
t_{k+l} \leq t_k t_l
\]
for every \(k, l \geq 0\), then
\[
|a b|_r \leq |a|_r |b|_r
\]
for every \(a, b \in \mathbb{Z}\), by (6.4). If \(p\) is a prime number, \(r_j = p\) for each \(j \geq 1\), and \(t_l = (t_1)^l\) for each \(l \geq 0\), then equality holds in (6.9) for each \(a, b \in \mathbb{Z}\). Of course, this reduces to the case of the \(p\)-adic absolute value when \(t_1 = 1/p\), and otherwise \(|a|_r\) would be the same as a positive power of the \(p\)-adic absolute value of \(a\).

Put
\[
d_r(a, b) = |a - b|_r
\]
for every \(a, b \in \mathbb{Z}\), which we shall call the \(r\)-adic metric on \(\mathbb{Z}\), although it also depends on \(t\), as before. This is symmetric in \(a\) and \(b\), because \(l_r(-c) = l_r(c)\) for every \(c \in \mathbb{Z}\), and hence \(|-c|_r = |c|_r\). Using (6.6), we get that
\[
d_r(a, c) \leq \max(d_r(a, b), d_r(b, c))
\]
for every \(a, b, c \in \mathbb{Z}\), so that \(d_r(\cdot, \cdot)\) is an ultrametric on \(\mathbb{Z}\). As usual, this reduces to the \(p\)-adic metric on \(\mathbb{Z}\) when \(r_j = p\) for some prime number \(p\) and every \(j \geq 1\), and \(t_l = p^{-l}\) for each \(l \geq 0\).

### 6.2 An embedding

Let \(r = \{r_j\}_{j=1}^{\infty}\) and \(R_l\) be as in the previous section, and consider the Cartesian product
\[
X = \prod_{l=1}^{\infty}(\mathbb{Z}/R_l \mathbb{Z}),
\]
consisting of the sequences \(x = \{x_l\}_{l=1}^{\infty}\) with \(x_l \in \mathbb{Z}/R_l \mathbb{Z}\) for each \(l\). As in Section 1.2, this is a compact Hausdorff topological space with respect to the product topology that corresponds to the discrete topology on \(X_l = \mathbb{Z}/R_l \mathbb{Z}\) for each \(l\). This is also a commutative ring with respect to coordinatewise addition and multiplication, using the standard ring structure on \(\mathbb{Z}/R_l \mathbb{Z}\) for each \(l\). It is easy to see that the ring operations are continuous on \(X\), so that \(X\) is a topological ring.

Let \(q_l\) be the canonical quotient mapping from \(\mathbb{Z}\) onto \(\mathbb{Z}/R_l \mathbb{Z}\) for each \(l \geq 1\), which is a ring homomorphism with kernel \(R_l \mathbb{Z}\). Thus
\[
q(a) = \{q_l(a)\}_{l=1}^{\infty},
\]
is an element of $X$ for each $a \in \mathbb{Z}$, so that $q$ defines a mapping from $\mathbb{Z}$ into $X$. This mapping is an injective ring homomorphism, because $\bigcap_{l=1}^{\infty} R_l \mathbb{Z} = \{0\}$. If $l(x, y)$ is defined for $x, y \in X$ as in Section 1.2, and $l_r(a)$ is as in the previous section, then

$$l(q(a), q(b)) = l_r(a - b)$$

(6.14)

for every $a, b \in \mathbb{Z}$.

Let $t = \{t_l\}_{l=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers that converges to 0 and with $t_0 = 1$, as before. This leads to an ultrametric $d(x, y)$ on $X$ as in (1.22), and to an $r$-adic metric $d_r(a, b)$ on $\mathbb{Z}$, as in (6.10). Under these conditions,

$$d(q(a), q(b)) = d_r(a, b)$$

(6.15)

for every $a, b \in \mathbb{Z}$, using also (6.5). By construction, $d_r(a, b)$ is invariant under translations on $\mathbb{Z}$, and in fact $d(x, y)$ is invariant under translations on $X$ as a commutative group with respect to addition as well.

Note that $X$ is complete with respect to $d(x, y)$, because $X$ is compact. One can also check this directly from the definitions. It follows that the completion of $\mathbb{Z}$ with respect to $d_r(a, b)$ can be identified with the closure of $q(\mathbb{Z})$ in $X$. We shall discuss this further in the next section.

### 6.3 Coherent sequences

Let us continue with the notation and hypotheses in the previous sections. Observe that there is a natural ring homomorphism from $\mathbb{Z}/R_{l+1} \mathbb{Z}$ onto $\mathbb{Z}/R_{l} \mathbb{Z}$ for each $l \geq 1$, because $R_{l+1} \mathbb{Z} \subseteq R_{l} \mathbb{Z}$. An element $x = \{x_l\}_{l=1}^{\infty}$ of $X$ is said to be a coherent sequence if the image of $x_{l+1}$ under the natural homomorphism from $\mathbb{Z}/R_{l+1} \mathbb{Z}$ onto $\mathbb{Z}/R_{l} \mathbb{Z}$ is equal to $x_l$ for each $l$. Let $Y$ be the subset of $X$ consisting of all coherent sequences, which is a sub-ring of $X$ with respect to termwise addition and multiplication. It is easy to see that $Y$ is also a closed set in $X$ with respect to the product topology, which implies that $Y$ is compact, since $X$ is compact.

If $a \in \mathbb{Z}$, then the image of $q_{l+1}(a)$ under the natural mapping from $\mathbb{Z}/R_{l+1} \mathbb{Z}$ onto $\mathbb{Z}/R_{l} \mathbb{Z}$ is equal to $q_l(a)$ for each $l$, so that $q(a) = \{q_l(a)\}_{l=1}^{\infty}$ is a coherent sequence. Thus $q(\mathbb{Z}) \subseteq Y$, and one can check that

$$\overline{q(\mathbb{Z})} = Y,$$

(6.16)

where $\overline{q(\mathbb{Z})}$ is the closure of $q(\mathbb{Z})$ with respect to the product topology on $X$. More precisely, let $x \in Y$ and $n \in \mathbb{Z}_+$ be given, and let $a$ be an integer such that $q_n(a) = x_n$. This implies that $q_l(a) = x_l$ when $l \leq n$, because $x = \{x_l\}_{l=1}^{\infty}$ is a coherent sequence. It follows that $x \in \overline{q(\mathbb{Z})}$, as desired, since $n$ is arbitrary.

The space $\mathbb{Z}_r$ of $r$-adic integers can be obtained initially as a metric space by completing $\mathbb{Z}$ with respect to the $r$-adic metric. Using the isometric embedding $q$ of $\mathbb{Z}$ in $X$, $\mathbb{Z}_r$ can be identified with the set $Y$ of coherent sequences in $X$, equipped with the restriction of the metric $d(x, y)$ on $X$ to $Y$. This identification is very convenient for showing that addition and multiplication on $\mathbb{Z}$ can be
extended continuously to $\mathbb{Z}_r$, so that $\mathbb{Z}_r$ is a compact commutative topological group. Similarly, the $r$-adic absolute value can be extended to $\mathbb{Z}_r$, by taking the distance to 0 in $\mathbb{Z}_r$, and it satisfies properties like those on $\mathbb{Z}$.

If $t' = \{t'_l\}_{l=0}^\infty$ is another sequence of positive real numbers that converges to 0, then we get another $r$-adic absolute value function $|a|_{t'}$ on $\mathbb{Z}$ as in (6.5), a corresponding $r$-adic metric $d'_r(a,b)$ on $\mathbb{Z}$ as in (6.10), and another metric $d'(x,y)$ on $X$ as in (1.22). However, the embedding $q$ of $\mathbb{Z}$ into $X$ and the set $Y$ of coherent sequences in $X$ do not depend on $t$, and the metric $d'(x,y)$ also determines the product topology on $X$. The identity mapping on $\mathbb{Z}$ is uniformly continuous as a mapping from $\mathbb{Z}$ equipped with $d_r(a,b)$ onto $\mathbb{Z}$ equipped with $d'_r(a,b)$, as well as in the other direction, and there are analogous statements for the identity mapping on $X$ and the metrics $d(x,y)$ and $d'(x,y)$. In particular, the completion $\mathbb{Z}_r$ of $\mathbb{Z}$ does not depend on the choice of $t$ as a topological ring.

### 6.4 Haar measure on $\mathbb{Z}_r$

Let us continue with the same notation and hypotheses as before. In particular, let us identify the ring $\mathbb{Z}_r$ of $r$-adic integers with the set $Y$ of coherent sequences in $X$. Let $n$ be a positive integer, and put

$$Y_n = \{ x = \{ x_l \}_{l=1}^\infty \in Y : x_n = 0 \} = \{ x = \{ x_l \}_{l=1}^\infty \in Y : x_l = 0 \text{ for each } l \leq n \},$$

where the second step uses the fact that $x \in Y$ is a coherent sequence. This is a closed set in $X$ with respect to the product topology, and an ideal in $Y$ as a commutative ring. This is also a relatively open set in $Y$, because $\mathbb{Z}/R_n \mathbb{Z}$ is equipped with the discrete topology. It is easy to see that

$$\overline{q(R_n \mathbb{Z})} = Y_n,$$

for essentially the same reasons as in (6.16). Let $\pi_n$ be the mapping from $Y$ into $\mathbb{Z}/R_n \mathbb{Z}$ defined by

$$\pi_n(x) = x_n,$$

which is a ring homomorphism from $Y$ into $\mathbb{Z}/R_n \mathbb{Z}$ whose kernel is equal to $Y_n$. Of course,

$$\pi_n(q(a)) = q_n(a)$$

for every $a \in \mathbb{Z}$, so that $\pi_n$ maps $Y$ onto $\mathbb{Z}/R_n \mathbb{Z}$.

Let $H$ be Haar measure on $Y$, normalized so that $H(Y) = 1$. Observe that

$$H(Y_n) = 1/R_n$$

for each positive integer $n$, because $Y$ can be expressed as the disjoint union of $R_n$ translates of $Y_n$, by the discussion in the preceding paragraph. In this situation, it is easy to define the Haar integral of a continuous real or complex-valued function on $Y$ directly as a limit of Riemann sums, with the measure
of $Y_n$ and its translates equal to $1/R_n$. This leads to a translation-invariant
regular Borel measure on $Y$, by the Riesz representation theorem, which is
Haar measure on $Y$.

Let $t = \{t_l\}_{l=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers
that converges to 0 and with $t_0 = 1$, as in Section 6.1. This leads to an $r$-adic absolute value function $|a|_r$ on $\mathbb{Z}$ as in (6.5), an $r$-adic metric $d_r(a, b)$ on $\mathbb{Z}$ as in (6.10), and a metric $d(x, y)$ on $X$ as in (1.22). The $r$-adic absolute value function and metric can be extended to $\mathbb{Z}_r$ in a natural way, as in the previous section, and the extension of the $r$-adic metric on $\mathbb{Z}_r$ corresponds exactly to the restriction of $d(x, y)$ to $Y$. By construction, these metrics are invariant under translations, and the diameter of $Y$ is equal to $t_0 = 1$. Similarly, the diameter of $Y_n$ is equal to $t_n$ for each positive integer $n$.

Let $H_1^{\delta}(E), H_1^\alpha(E)$, and $H_1^{\alpha \cap E}$ be defined for $\alpha \geq 0$, $0 < \delta \leq \infty$, and $E \subseteq Y$ as in Chapter 2, using the restriction of $d(x, y)$ to $Y$. Because this is an ultrametric on $Y$, one may as well use coverings of $E \subseteq Y$ by closed balls in $Y$ in the definitions of $H_1^{\alpha \cap E}$ and $H_1^\alpha(E)$, as in Section 2.5. More precisely, one should consider the empty set as a closed ball in $Y$ when $\alpha = 0$, but we are mostly interested in $\alpha > 0$ here. Otherwise, the closed balls in $Y$ are $Y$ itself and the translates of $Y_n$ for each positive integer $n$.

Let us now restrict our attention for the rest of this section to the case where
$\alpha = 1$ and $t_l = 1/R_l$
for each $l \geq 0$, which satisfies the usual conditions on $t$. Remember that
$H_1^{\delta}(E) \leq H_1^1(E)$ for every $E \subseteq Y$ and $\delta > 0$, by construction. In the present situation, one can check that
\begin{equation}
H_1^1(E) = H_1^1(\con(E))
\end{equation}
for every $E \subseteq Y$ and $\delta > 0$, as in Section 2.5. This uses the fact that $Y_n$ can be expressed as the union of $R_k/R_n$ translates of $Y_k$ when $k \geq n$. It follows that
\begin{equation}
H_1^1(E) = H_1^1(\con(E))
\end{equation}
for every $E \subseteq Y$ under these conditions, as before.

In particular,
\begin{equation}
H_1^1(Y) = H_1^1(\con(Y)) \leq \text{diam } Y = 1.
\end{equation}

In order to show that
\begin{equation}
H_1^1(Y) = 1,
\end{equation}
it suffices to verify that $H_1^1(Y) \geq 1$ for every $\delta > 0$. Because $Y$ is compact, one might as well consider only coverings of $Y$ by finitely many sets in the definition of $H_1^1(Y)$, as in Section 2.3. In fact, it is enough to consider only coverings of $Y$ by closed balls, as mentioned earlier. One can then reduce to coverings of $Y$ by finitely many balls of the same diameter, by subdividing the balls in a covering of $Y$ as necessary. Thus one gets either a covering of $Y$ by itself, or by finitely many translates of $Y_k$ for some $k \geq 1$. The first case is trivial, and in the second
6.5. SOME RELATED GROUPS

Case, we have that $Y$ cannot be covered by fewer than $R_k$ translates of $Y_k$. This implies that $H^1_k(Y) \geq 1$ for every $\delta > 0$, and hence that (6.26) holds.

Similarly,

$$H^1(Y_n) = 1/R_n$$

for each $n \geq 1$, which implies that $H^1(U) > 0$ when $U$ is a nonempty open subset of $Y$. Of course, any Hausdorff measure on $Y$ with respect to a translation-invariant metric on $Y$ is invariant under translations as well.

6.5 Some related groups

If $k$ is a positive integer, then let $k^{-1} \mathbb{Z}$ be the set of integer multiples of $1/k$, which is a subgroup of the group $\mathbb{Q}$ of rational numbers with respect to addition. Note that $\mathbb{Z} \subseteq k^{-1} \mathbb{Z}$, so that the quotient group

$$\frac{(k^{-1} \mathbb{Z})}{\mathbb{Z}}$$

can be defined in the usual way. Of course, (6.28) is isomorphic to $\mathbb{Z}/k \mathbb{Z}$.

Let $r = \{r_j\}_{j=1}^{\infty}$ and $R_l$ be as in Section 6.1, and observe that

$$R_l^{-1} \mathbb{Z} \subseteq R_{l+1}^{-1} \mathbb{Z}$$

for each $l \geq 0$. Thus

$$\bigcup_{l=0}^{\infty} R_l^{-1} \mathbb{Z}$$

is also a subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$, so that the quotient group

$$\left(\bigcup_{l=0}^{\infty} R_l^{-1} \mathbb{Z}\right)/\mathbb{Z}$$

is defined. If we consider $(R_l^{-1} \mathbb{Z})/\mathbb{Z}$ as a subgroup of (6.31), then

$$\frac{(R_l^{-1} \mathbb{Z})/\mathbb{Z}}{(R_{l+1}^{-1} \mathbb{Z})/\mathbb{Z}}$$

for every $l$, because of (6.29), and (6.31) is the same as

$$\bigcup_{l=0}^{\infty} (R_l^{-1} \mathbb{Z})/\mathbb{Z}.\)$$

If $p$ is a prime number, and $r_j = p$ for each $j$, then (6.30) is the same as $\mathbb{Z}[1/p]$, as in Section 5.7.

Remember that $\exp(2\pi iw)$ defines a continuous homomorphism from $\mathbb{R}$ as a commutative topological group with respect to addition onto $\mathbb{T}$ with kernel $\mathbb{Z}$, which leads to an isomorphism from $\mathbb{R}/\mathbb{Z}$ onto $\mathbb{T}$. The image of (6.31) under this isomorphism consists of the $z \in \mathbb{T}$ such that

$$z^{R_l} = 1$$

(6.34)
for some \( l \geq 0 \). In fact, every homomorphism from (6.31) into \( T \) takes values in this subgroup of \( T \). It follows that homomorphisms from (6.31) into \( T \) correspond exactly to homomorphisms from (6.31) into itself composed with the embedding of (6.31) into \( T \) obtained from the complex exponential function.

Let \( \theta \) be a homomorphism from (6.31) into itself. Note that \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) consists of exactly the elements \( a \) of (6.31) such that \( R_l \cdot a \) is equal to 0 in (6.31). This implies that
\[
\theta((R^{-1}I)\mathbb{Z})/\mathbb{Z}) \subseteq (R^{-1}I)\mathbb{Z}/\mathbb{Z}
\]
for each \( l \). Let \( \theta_l \) be the restriction of \( \theta \) to \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) for each \( l \geq 1 \).

Because \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) is a cyclic group, \( \theta_l \) is determined by its value at the generator of \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) for each \( l \). This permits \( \theta_l \) to be expressed in terms of multiplication by an integer. This integer is determined by \( \theta_l \) modulo \( R_l \), so that \( \theta_l \) corresponds to an element \( x_l \) of \( \mathbb{Z}/R_l \mathbb{Z} \). Conversely, every element of \( \mathbb{Z}/R_l \mathbb{Z} \) determines a homomorphism from \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) into itself in this way.

By construction, \( \theta_l \) is equal to the restriction of \( \theta_{l+1} \) to \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) for each \( l \). This means exactly that \( x_l \) is the image of \( x_{l+1} \) under the natural homomorphism from \( \mathbb{Z}/R_{l+1} \mathbb{Z} \) onto \( \mathbb{Z}/R_l \mathbb{Z} \). Thus \( x = \{x_l\}_{l=1}^{\infty} \) is a coherent sequence, which is to say that \( x \) is an element of the group \( Y \) discussed in Section 6.3. Conversely, every element of \( Y \) leads to a sequence of homomorphisms \( \theta_l \) from \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) into itself such that \( \theta_l \) is the restriction of \( \theta_{l+1} \) to \((R^{-1}I)\mathbb{Z})/\mathbb{Z}\) for each \( l \). This leads in turn to a homomorphism \( \theta \) from (6.31) into itself, since (6.31) is the same as (6.33).

The collection of homomorphisms from (6.31) into itself is a commutative group with respect to addition. The discussion in the previous paragraphs determines a one-to-one correspondence between this group and \( Y \), which is a group isomorphism. Here we consider (6.31) to be equipped with the discrete topology, so that the corresponding dual group consists of all homomorphisms from (6.31) into \( T \). Because of the correspondence between homomorphisms from (6.31) into \( T \) and homomorphisms from (6.31) into itself mentioned earlier, we get an isomorphism between \( Y \) and the dual group associated to (6.31). One can check that this isomorphism is also a homeomorphism with respect to the usual topology on the dual of (6.31) as a discrete commutative group.

### 6.6 Characters on \( \mathbb{Z}_r \)

Let us continue with the same notation and hypotheses as before, and let \( \phi \) be a continuous homomorphism from \( Y \) as a commutative topological group with respect to addition into \( T \). Thus the set of \( x \in Y \) such that the real part of \( \phi(x) \) is positive is an open set in \( Y \) that contains 0, and hence contains \( Y_n \) for some positive integer \( n \). This implies that \( \phi(x) = 1 \) for every \( x \in Y_n \), as in Section 5.2, because \( Y_n \) is a subgroup of \( Y \). If \( \pi_n \) is the homomorphism from \( Y \) onto \( \mathbb{Z}/R_n \mathbb{Z} \) in (6.19), then there is a homomorphism \( \psi \) from \( \mathbb{Z}/R_n \mathbb{Z} \) as a commutative group with respect to addition into \( T \) such that
\[
\phi = \psi \circ \pi_n,
\]
because the kernel of \( \pi_n \) is equal to \( Y_n \). Conversely, if \( \psi \) is a homomorphism from \( \mathbb{Z}/R_n \mathbb{Z} \) as a commutative group with respect to addition into \( T \), then (6.36) defines a continuous group homomorphism from \( Y \) into \( T \).

Alternatively, we have seen in the previous section that \( Y \) is isomorphic as a commutative topological group to the dual group associated to (6.31), where (6.31) is equipped with the discrete topology. As in Section 5.5, it follows that each element of (6.31) determines a character on \( Y \), and in fact that this defines an isomorphism between (6.31) and the dual of \( Y \). The natural topology on the dual of \( Y \) is discrete, because \( Y \) is compact, so that this isomorphism is automatically a homeomorphism. In this case, one can also check that the dual of \( Y \) is isomorphic to (6.31) using the remarks in the previous paragraph, and the descriptions of the group homomorphisms from \( \mathbb{Z}/R_n \mathbb{Z} \) into \( T \) at the beginning of Section 5.6.

As in Section 5.3, characters on \( Y \) are orthonormal with respect to the usual \( L^2 \) inner product associated to Haar measure on \( Y \). There are \( R_n \) characters on \( Y \) of the form (6.36) for each positive integer \( n \), which are constant on the cosets of \( Y_n \) in \( Y \). The linear span of these characters consists of all functions on \( Y \) that are constant on the cosets of \( Y_n \) in \( Y \), since every function on \( \mathbb{Z}/R_n \mathbb{Z} \) can be expressed as a linear combination of characters on \( \mathbb{Z}/R_n \mathbb{Z} \). The linear span of all characters on \( Y \) consists of functions on \( Y \) that are constant on the cosets of \( Y_n \) in \( Y \) for some \( n \).

### 6.7 Topological equivalence

Let \( r = \{r_j\}_{j=1}^\infty \) be as in Section 6.1, and let \( r' = \{r'_j\}_{j=1}^\infty \) be another sequence of integers with \( r'_j \geq 2 \) for each \( j \). Also let \( R_l \) be associated to \( r \) as before, and put

\[
R'_l = \prod_{j=1}^{l} r'_j
\]

when \( l \geq 1 \), and \( R'_0 = 1 \). If for each \( l \in \mathbb{Z}_+ \) there is an \( n \in \mathbb{Z}_+ \) such that \( R'_n \) is an integer multiple of \( R_l \), then we put

\[
r \prec r'.
\]

It is easy to see that this relation is reflexive and transitive, and that (6.38) holds if and only if every open subset of \( \mathbb{Z} \) with respect to the \( r \)-adic topology is an open set with respect to the \( r' \)-adic topology as well. Similarly, if we put

\[
r \sim r'.
\]

when \( r \prec r' \) and \( r' \prec r \), then we get an equivalence relation on the set of these sequences, which holds exactly when the \( r \)-adic and \( r' \)-adic topologies on \( \mathbb{Z} \) are the same.

As in Section 6.2, consider the Cartesian product

\[
X' = \prod_{l=1}^{\infty} (\mathbb{Z}/R'_l \mathbb{Z})
\]
associated to \( r' \), which is a compact commutative topological ring with respect to coordinatewise addition and multiplication, and using the product topology corresponding to the discrete topology on \( \mathbb{Z}/R'_l\mathbb{Z} \) for each \( l \). Let \( q'_l \) be the canonical quotient mapping from \( \mathbb{Z}/R'_l\mathbb{Z} \) for each \( l \), and put

\[
q'(a) = \{ q'_l(a) \}_{l=1}^\infty
\]

for each \( a \in \mathbb{Z} \), which defines an injective ring homomorphism from \( \mathbb{Z} \) into \( X' \). As in Section 6.3, there is a natural ring homomorphism from \( \mathbb{Z}/R'_i\mathbb{Z} \) onto \( \mathbb{Z}/R_l\mathbb{Z} \) for each \( l \), and \( x' = \{ x'_l \}_{l=1}^\infty \in X' \) is said to be a coherent sequence if \( x'_l \) is the image under this homomorphism of \( x'_{l+1} \) for each \( l \). Let \( Y' \) be the set of coherent sequences in \( X' \), which is a closed sub-ring of \( X' \). This is the same as the closure of \( q'(\mathbb{Z}) \) in \( X' \), and the topological ring \( \mathbb{Z}_{r'} \) of \( r'\)-adic integers can be identified with \( Y' \).

If \( r \prec r' \), then there is a natural continuous ring homomorphism from \( Y' \) onto \( Y \), defined as follows. Let \( x' \in Y' \) and \( l \in \mathbb{Z}_+ \) be given, and remember that there is an \( n = n(l) \in \mathbb{Z}_+ \) such that \( R'_n \) is an integer multiple of \( R_l \). Of course, this implies that \( R'_k \) is an integer multiple of \( R_l \) for every \( k \in \mathbb{Z}_+ \) with \( k \geq n \), and hence that there is a natural ring homomorphism from \( \mathbb{Z}/R'_k\mathbb{Z} \) onto \( \mathbb{Z}/R_l\mathbb{Z} \) when \( k \geq n \). Let \( x_l \) be the image of \( x'_k \) under this homomorphism, which one can check is the same for all \( k \geq n \), because \( x'_k \) is a coherent sequence. One can also check that \( x = \{ x_l \}_{l=1}^\infty \) is a coherent sequence in \( X \), so that

\[
x' \mapsto x
\]

leads to a natural mapping from \( Y' \) into \( Y \). This mapping is a continuous ring homomorphism, with respect to the topologies induced on \( Y \) and \( Y' \) by the product topologies on \( X \) and \( X' \), respectively. If \( a \in \mathbb{Z} \), then (6.42) sends \( q'(a) \) to \( q(a) \), so that (6.42) may be considered as an extension of the identity mapping on \( \mathbb{Z} \) to a continuous ring homomorphism from \( \mathbb{Z}_{r'} \) into \( \mathbb{Z}_r \). Because \( Y' \) is compact, (6.42) maps \( Y' \) onto a compact set in \( Y \), and onto a closed set in \( Y \) in particular. This implies that (6.42) maps \( Y' \) onto \( Y \), since \( q'(\mathbb{Z}) \) is mapped onto \( q(\mathbb{Z}) \), which is dense in \( Y \). If \( r \prec r' \), then (6.42) is an isomorphism from \( Y' \) onto \( Y \) as topological rings.

As in Section 6.5,

\[
\bigcup_{l=0}^\infty (R'_l)^{-1} \mathbb{Z}
\]

is a subgroup of \( \mathbb{Q} \) with respect to addition that contains \( \mathbb{Z} \). Observe that \( r \prec r' \) if and only if the analogous subgroup (6.30) of \( \mathbb{Q} \) associated to \( r \) is contained in (6.43), and that \( r \sim r' \) if and only if (6.30) is the same as (6.43). Similarly, the quotient group

\[
\left( \bigcup_{l=0}^\infty (R'_l)^{-1} \mathbb{Z} \right)/\mathbb{Z}
\]

and its analogue (6.31) for \( r \) may be considered as subgroups of \( \mathbb{Q}/\mathbb{Z} \). Clearly (6.30) is contained in (6.43) if and only if (6.31) is contained in (6.44), and
(6.30) is equal to (6.43) if and only if (6.31) is equal to (6.44). Thus \( r \preceq r' \) if and only if (6.31) is contained in (6.44), and \( r \sim r' \) if and only if (6.31) is the same as (6.44).
Chapter 7

Some geometric conditions

7.1 A class of isometries

Let $X_1, X_2, X_3, \ldots$ be a sequence of sets, each of which has at least two elements, and let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product, as in Section 1.2. Also let \( \{t_i\}_{i=0}^{\infty} \) be a strictly decreasing sequence of positive real numbers, and let $d(x, y)$ be the ultrametric on $X$ defined as in (1.22). Thus $x, y \in X$ satisfy

\[ d(x, y) \leq t_k \]

for some nonnegative integer $k \geq 0$ if and only if $x_j = y_j$ when $j \leq k$. Suppose that $\phi : X \to X$ is a Lipschitz mapping of order 1 with constant $C = 1$ with respect to $d(\cdot, \cdot)$, so that

\[ d(\phi(x), \phi(y)) \leq d(x, y) \]

for every $x, y \in X$. Let us express $\phi(x)$ as

\[ \phi(x) = \{\phi_j(x)\}_{j=1}^{\infty}, \]

where $\phi_j : X \to X_j$ for each $j$. If $x, y \in X$ satisfy $x_j = y_j$ for $j \leq k$ and some $k$, then it follows that

\[ d(\phi(x), \phi(y)) \leq t_k, \]

and hence that $\phi_j(x) = \phi_j(y)$ for $j \leq k$. Equivalently, this means that for each $k \geq 1$, $\phi_k(x)$ only depends on $x_j$ with $j \leq k$. Conversely, if $\phi_k : X \to X_k$ has this property for each $k \geq 1$, then $\phi : X \to X$ defined as in (7.3) satisfies (7.2) for every $x, y \in X$.

If $x, y \in X$ satisfy

\[ x_j = y_j \text{ for } j < k \text{ and } x_k \neq y_k \]

for some $k \in \mathbb{Z}_+$, then $d(x, y) = t_{k-1}$, by construction. Suppose that $\phi_k$ has the property mentioned in the preceding paragraph for each $k \geq 1$, and that

\[ \phi_k(x) \neq \phi_k(y) \]
when \( x, y \in X \) satisfy (7.5). This implies that

\[
d(\phi(x), \phi(y)) = t_{k-1}
\]

when \( x, y \in X \) satisfy (7.5), because \( \phi_j(x) = \phi_j(y) \) when \( j < k \). It follows that \( \phi : X \to X \) is an isometry with respect to \( d(\cdot, \cdot) \) under these conditions. Conversely, if \( \phi : X \to X \) is an isometry with respect to \( d(\cdot, \cdot) \), then one can check that \( \phi \) has these properties.

In particular, if \( \phi_k(x) \) depends only on \( x_k \) for each \( k \), then \( \phi \) satisfies (7.2). In this case, \( \phi : X \to X \) is an isometry with respect to \( d(\cdot, \cdot) \) if and only if \( \phi_k(x) \) corresponds to a one-to-one mapping from \( X_k \) into itself for each \( k \). Similarly, if \( \phi_k(x) \) corresponds to a mapping from \( X_k \) onto itself for each \( k \), then \( \phi \) maps \( X \) onto itself.

### 7.2 Some isometric equivalences

Let \( r = \{r_j\}_{j=1}^\infty \) be a sequence of positive integers with \( r_j \geq 2 \) for each \( j \), and let \( R_k \) be as in Section 6.1. Also let \( X \) and \( Y \) be as in Sections 6.2 and 6.3, respectively. Put

\[
\tilde{X}_j = \{0, 1, \ldots, r_j - 1\}
\]

for each \( j \geq 1 \), and

\[
\tilde{X} = \prod_{j=1}^{\infty} \tilde{X}_j.
\]

If \( t = \{t_j\}_{j=0}^\infty \) is a strictly decreasing sequence of positive real numbers, then we get corresponding ultrametrics \( d(x, y) \) on \( X \) and \( d'(x', y') \) on \( \tilde{X} \), as in Section 1.2. As before, a mapping \( \psi : Y \to \tilde{X} \) corresponds exactly to a sequence of mappings \( \psi_j : Y \to \tilde{X}_j \), \( j \in \mathbb{Z}_+ \), with

\[
\psi(x) = \{\psi_j(x)\}_{j=1}^\infty
\]

for each \( x \in Y \).

Suppose that \( \psi : Y \to \tilde{X} \) is Lipschitz of order 1 with constant \( C = 1 \) with respect to the restriction of \( d(x, y) \) to \( x, y \in Y \) and \( d'(x', y') \) on \( \tilde{X} \), so that

\[
d'(\psi(x), \psi(y)) \leq d(x, y)
\]

for every \( x, y \in Y \). If \( x, y \in Y \) satisfy \( x_j = y_j \) for \( j \leq k \) and some \( k \), then we get that \( \psi_j(x) = \psi_j(y) \) when \( j \leq k \), as in the previous section. This is the same as saying that \( \psi_k(x) \) depends only on \( x_k \) for each \( k \geq 1 \), because the elements of \( Y \) are coherent sequences. Conversely, if \( \psi_k(x) \) depends only on \( x_k \) for each \( k \geq 1 \), then \( \psi : Y \to \tilde{X} \) satisfies (7.11).

Suppose now that \( \psi_k(x) \) depends only on \( x_k \) for each \( k \geq 1 \), and that

\[
\psi_k(x) \neq \psi_k(y)
\]
when $x, y \in Y$ satisfy

$$x_{k-1} = y_{k-1} \text{ and } x_k \neq y_k.$$  

(7.13)

If $k = 1$, then we interpret (7.13) as meaning simply that $x_1 \neq y_1$. Under these conditions, $\psi : Y \to \tilde{X}$ is an isometry, for the same reasons as before. Conversely, any isometry from $Y$ into $\tilde{X}$ has these properties.

If $\theta_k$ is a mapping from $\mathbb{Z}/R_k\mathbb{Z}$ into $\tilde{X}_k$, then

$$\psi_k(x) = \theta_k(x_k)$$

(7.14)

defines a mapping from $Y$ into $\tilde{X}_k$. We would like to choose such a mapping $\theta_k$ for each $k \in \mathbb{Z}_+$ so that the corresponding mapping $\psi_k$ satisfies (7.12) for every $x, y \in Y$ for which (7.13) holds. If $k = 1$, then we can use any one-to-one mapping from $\mathbb{Z}/R_1\mathbb{Z}$ onto $\tilde{X}_1$, because $R_1 = r_1$ and $\tilde{X}_1$ has exactly $r_1$ elements. Suppose now that $k \geq 2$, and remember that there is a natural ring homomorphism from $\mathbb{Z}/R_k\mathbb{Z}$ onto $\mathbb{Z}/R_{k-1}\mathbb{Z}$, because $R_k\mathbb{Z} \subseteq R_{k-1}\mathbb{Z}$. The kernel of this homomorphism is equal to

$$R_{k-1}\mathbb{Z}/R_k\mathbb{Z},$$

(7.15)

which has exactly $r_k$ elements. Of course, $\mathbb{Z}/R_k\mathbb{Z}$ can be partitioned into translates of (7.15). The property of $\psi_k$ that we want is equivalent to saying that the restriction of $\theta_k$ to any translate of (7.15) in $\mathbb{Z}/R_k\mathbb{Z}$ is injective. It is easy to choose $\theta_k$ in this way, because (7.15) has exactly $r_k$ elements, which is the same as the number of elements of $\tilde{X}_k$. This leads to a sequence of mappings $\psi_k : Y \to \tilde{X}_k$ as in (7.14), and hence a mapping $\psi : Y \to \tilde{X}$ as in (7.10), which is an isometry. Note that $\theta_k$ also maps every translate of (7.15) in $\mathbb{Z}/R_k\mathbb{Z}$ onto $\tilde{X}_k$, by construction. Using this, one can check that $\psi$ maps $Y$ onto $\tilde{X}$ as well.

### 7.3 Doubling metrics

A metric $d(x, y)$ on a set $M$ is said to be doubling if there is a positive real number $C$ such that every open ball in $M$ with radius $r > 0$ can be covered by $\leq C$ open balls of radius $r/2$. In this case, one might also say that the metric space $(M, d(x, y))$ is doubling, or simply that $M$ is doubling, if the choice of the metric is clear. If $M$ is doubling, then we can apply the condition repeatedly to get that every open ball in $M$ with radius $r$ can be covered by $\leq C^k$ open balls of radius $2^{-k}r$ for every $k \in \mathbb{Z}_+$. In particular, this implies that bounded subsets of $M$ are totally bounded. If $M$ is doubling and complete, then it follows that subsets of $M$ that are both closed and bounded are compact as well.

If $M$ is doubling, then the iterated condition mentioned in the previous paragraph implies that every closed ball in $M$ with radius $r > 0$ can be covered by a bounded number of closed balls of radius $r/2$. Similarly, every subset of $M$ with diameter $\leq r$ can be covered by a bounded number of sets with diameter $\leq r/2$. This implies that the restriction of $d(x, y)$ to any subset of $M$ is a doubling metric too. One can also use the iterated version of the doubling
condition to show that if $M$ is is bilipschitz equivalent to a metric space that is doubling, then $M$ is doubling.

The doubling condition can be defined in the same way for quasi-metrics, with the same type of properties as before. If $d(x, y)$ is a quasi-metric on a set $M$ and $a$ is a positive real number, then we have seen that $d(x, y)^a$ is quasi-metric on $M$ too, as in Section 1.6. It is easy to see that $d(x, y)$ is doubling if and only if $d(x, y)^a$ is doubling.

The real line $\mathbb{R}$ is doubling with respect to the standard metric, and similarly $\mathbb{R}^n$ is doubling with respect to the standard metric for every positive integer $n$. Moreover, one can use the invariance of the standard metric under translations and dilations to reduce the doubling condition to the case of the unit ball, which then follows from the fact that the unit ball is totally bounded. Similarly, the set $\mathbb{Q}_p$ of $p$-adic numbers is doubling with respect to the $p$-adic metric, for every prime number $p$.

Let $X_1, X_2, X_3, \ldots$ be a sequence of sets, each of which has at least two elements, and let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product. Also let $\{t_l\}_{l=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers, and let $d(x, y)$ be the corresponding ultrametric on $X$, as in (1.22). If $X_j$ has only finitely many elements for each $j$, and if $\{t_l\}_{l=0}^{\infty}$ converges to 0, then $X$ is totally bounded with respect to $d(x, y)$. Conversely, one can check that these conditions are necessary for $X$ to be doubling with respect to $d(x, y)$.

Of course, $X$ is bounded with respect to $d(x, y)$ by construction. If $X$ is doubling with respect to $d(x, y)$, then $X$ is totally bounded in particular, and hence the number of elements of $X_j$ has to be finite for each $j \geq 1$, as in the previous paragraph. In fact, the number of elements of $X_j$ has to be uniformly bounded in this case.

Similarly, if $X$ is totally bounded with respect to $d(x, y)$, then we have seen that $\{t_l\}_{l=0}^{\infty}$ converges to 0, which implies that for each $l \geq 0$, the number of $j \geq l$ such that $t_j \geq t_l/2$ is finite. If $X$ is doubling with respect to $d(x, y)$, then the number of $j \geq l$ such that $t_j \geq t_l/2$ is uniformly bounded over $l$. Conversely, if the number of elements of $X_j$ is uniformly bounded in $j$, and if the number of $j \geq l$ such that $t_j \geq t_l/2$ is uniformly bounded in $l$, then $X$ is doubling with respect to $d(x, y)$.

Now let $r = \{r_j\}_{j=1}^{\infty}$ be a sequence of positive integers with $r_j \geq 2$ for each $j$, and let $t = \{t_l\}_{l=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers. This leads to an $r$-adic ultrametric on $\mathbb{Z}$, as in Section 6.1. As before, $\mathbb{Z}$ is totally bounded with respect to this $r$-adic metric if and only if $\{t_l\}_{l=0}^{\infty}$ converges to 0. One can check that $\mathbb{Z}$ is doubling with respect to this $r$-adic metric if and only if the $r_j$’s are bounded and the number of $j \geq l$ such that $t_j \geq t_l/2$ is uniformly bounded in $l$. Remember that the set $\mathbb{Z}_r$ of $r$-adic integers is obtained by completing $\mathbb{Z}$ as a metric space with respect to the $r$-adic metric. Under these same conditions on $r$ and $t$, $\mathbb{Z}_r$ is doubling with respect to the corresponding extension of the $r$-adic metric. This also follows from the earlier discussion of Cartesian products, using the isometric equivalence described at the end of the preceding section.
CHAPTER 7. SOME GEOMETRIC CONDITIONS

7.4 Doubling measures

A nonnegative Borel measure \( \mu \) on a metric space \((M, d(x, y))\) is said to be doubling if the measure of every open ball in \( M \) is positive and finite, and if there is a positive real number \( C \) such that

\[
\mu(B(x, 2r)) \leq C \mu(B(x, r)) 
\]  

(7.16)

for every \( x \in M \) and \( r > 0 \). It is easy to see that Lebesgue measure on \( \mathbb{R}^n \) is doubling with respect to the standard metric on \( \mathbb{R}^n \) for each positive integer \( n \), and that Haar measure on \( \mathbb{Q}_p \) is doubling with respect to the \( p \)-adic metric for every prime number \( p \). Some other examples will be discussed later in the section. If \( \mu \) satisfies (7.16) on \( M \), then

\[
\mu(B(x, 2^k r)) \leq C^k \mu(B(x, r)) 
\]  

(7.17)

for every \( x \in M \), \( r > 0 \), and \( k \in \mathbb{Z}_+ \). Using this, one can check that if the measure of some open ball in \( M \) is positive or finite with respect to \( \mu \), then every open ball in \( M \) has the same property, because of (7.16).

Suppose that \( \mu \) is a doubling measure on a metric space \((M, d(x, y))\), and let \( x \in M \) and \( r > 0 \) be given. Also let \( y_1, \ldots, y_n \) be finitely many elements of \( B(x, r) \) such that

\[
d(y_j, y_l) \geq r/2 \]  

(7.18)

when \( j \neq l \). We would like to show that

\[
n \leq C_1 
\]  

(7.19)

for some positive real number \( C_1 \) that depends only on the doubling constant for \( \mu \). It follows from (7.18) and the triangle inequality that

\[
B(y_j, r/4) \cap B(y_l, r/4) = \emptyset 
\]  

(7.20)

when \( j \neq l \), and hence

\[
\sum_{j=1}^n \mu(B(y_j, r/4)) = \mu \left( \bigcup_{j=1}^n B(y_j, r/4) \right). 
\]  

(7.21)

Using the triangle inequality again, we have that

\[
B(y_j, r/4) \subseteq B(x, 5r/4) 
\]  

(7.22)

for each \( j \), so that

\[
\mu \left( \bigcup_{j=1}^n B(y_j, r/4) \right) \leq \mu(B(x, 5r/4)). 
\]  

(7.23)

In the other direction,

\[
B(x, 5r/4) \subseteq B(y_j, 9r/4) 
\]  

(7.24)
7.4. DOUBLING MEASURES

for each \( j \), because \( d(x, y_j) \leq r \) by hypothesis. Thus \( \mu(B(x, 5r/4)) \) is bounded by a constant times \( \mu(B(y_j, r/4)) \) for each \( j \), by the doubling condition. Combining this with (7.21) and (7.23), we get (7.19), as desired.

Suppose now that \( n \) is the largest positive integer for which there are \( n \) elements \( y_1, \ldots, y_n \) of \( B(x, r) \) satisfying (7.18). If \( y \) is any element of \( B(x, r) \), then it follows that

\[
d(y, y_j) < r/2 \quad \text{(7.25)}
\]

for some \( j = 1, \ldots, n \), since otherwise \( y_1, \ldots, y_n \) together with \( y \) would be \( n + 1 \) elements of \( B(x, r) \) with the same property. This shows that

\[
B(x, r) \subseteq \bigcup_{j=1}^{n} B(y_j, r/2) \quad \text{(7.26)}
\]

and hence that \( M \) is doubling as a metric space, because \( x \in M \) and \( r > 0 \) are arbitrary, and \( n \) is uniformly bounded.

If \( d(x, y) \) is an ultrametric on \( M \), then the proof of (7.19) can be improved somewhat. In this case, we can replace (7.20) with

\[
B(y_j, r/2) \cap B(y_l, r/2) = \emptyset \quad \text{(7.27)}
\]

when \( j \neq l \). Of course, we should then use the analogue of (7.21) with \( r/4 \) replaced by \( r/2 \). We also have that

\[
B(y_j, r/2) \subseteq B(x, r) \quad \text{(7.28)}
\]

for each \( j \), by the ultrametric version of the triangle inequality, so that

\[
\mu\left(\bigcup_{j=1}^{n} B(y_j, r/2)\right) \leq \mu(B(x, r)), \quad \text{(7.29)}
\]

which is the substitute for (7.23). Instead of (7.24), we can use the fact that

\[
B(x, r) \subseteq B(y_j, r) \quad \text{(7.30)}
\]

for each \( j \), by the ultrametric version of the triangle inequality, and then continue as before.

Let \( X_1, X_2, X_3, \ldots \) be a sequence of finite sets, each of which has at least two elements, and let \( X = \prod_{j=1}^{\infty} X_j \) be their Cartesian product. Also let \( \{t_i\}_{i=0}^{\infty} \) be a strictly decreasing sequence of positive real numbers that converge to 0, and let \( d(x, y) \) be the corresponding ultrametric on \( X \), as in (1.22). As in the previous section, \( X \) is doubling with respect to \( d(x, y) \) if and only if the number of elements of \( X_j \) is uniformly bounded in \( j \), and the number of \( j \geq l \) such that \( t_j \geq t_l/2 \) is uniformly bounded in \( l \). Let \( \mu_j \) be a probability measure on \( X_j \) for each \( j \), where all subsets of \( X_j \) are measurable, and let \( \mu \) be the corresponding product measure on \( X \), as in Section 1.2. If \( \mu \) is a doubling measure on \( X \) with respect to \( d(x, y) \), then there is a \( c > 0 \) such that

\[
\mu_j(\{x_j\}) \geq c \quad \text{(7.31)}
\]
for every $j \geq 1$ and $x_j \in X_j$. This implies that $X_j$ has $\leq 1/c$ elements for each $j$, because $\mu_j(X_j) = 1$. Conversely, if there is a $c > 0$ such that (7.31) holds for every $j \geq 1$ and $x_j \in X_j$, and if the number of $j \geq l$ such that $t_j \geq t_l/2$ is uniformly bounded in $l$, then one can check that $\mu$ is a doubling measure on $X$.

Similarly, let $r = \{r_j\}_{j=1}^\infty$ be a sequence of positive integers with $r_j \geq 2$ for each $j$, and let $t = \{t_j\}_{j=0}^\infty$ be a strictly decreasing sequence of positive real numbers that converges to $0$. If Haar measure on $\mathbb{Z}_r$ is doubling with respect to the $r$-adic metric on $\mathbb{Z}_r$ associated to $r$ and $t$, then it is easy to see that the $r_j$’s have to be uniformly bounded in $j$. Conversely, if the $r_j$’s are uniformly bounded in $j$, and if the number of $j \geq l$ such that $t_j \geq t_l/2$ is uniformly bounded in $l$, then one can check that Haar measure on $\mathbb{Z}_r$ is doubling with respect to the $r$-adic metric associated to $r$ and $t$. One can also look at this in terms of an isometric equivalence of $\mathbb{Z}_r$ with a Cartesian product $X$, as in Section 7.2. More precisely, Haar measure on $\mathbb{Z}_r$ corresponds to a product measure $\mu$ on $X$ with respect to this isometric equivalence, using the probability measures $\tilde{\mu}_j$ that are uniformly distributed on each factor $\tilde{X}_j$ in (7.9), in the sense that $\mu_j(\{x_j\})$ is the same for each $x_j \in X_j$.

If $d(\cdot, \cdot)$ is a quasi-metric on $M$, then one can define the notion of a doubling measure on $M$ in the same way as before, at least if open balls in $M$ are Borel sets. In particular, open balls in $M$ with respect to $d(\cdot, \cdot)$ are open sets when $d(\cdot, \cdot)$ is continuous with respect to the topology on $M$ that it determines. At any rate, this is normally not a problem, and there are various ways to deal with it. One can check that the arguments in this and the next sections have suitable versions for quasi-metrics, with different constants, as appropriate.

If $d(\cdot, \cdot)$ and $d'(\cdot, \cdot)$ are quasi-metrics on $M$ such that each is bounded by a constant multiple of the other, then it is easy to see that doubling measures on $M$ with respect to $d(\cdot, \cdot)$ are the same as doubling measures on $M$ with respect to $d'(\cdot, \cdot)$, aside from measurability issues as in the previous paragraph. Similarly, if $d(\cdot, \cdot)$ is a quasi-metric on $M$ and $a$ is a positive real number, then $d(\cdot, \cdot)^a$ is a quasi-metric on $M$, as in Section 1.6, and doubling measures on $M$ with respect to $d(\cdot, \cdot)$ are the same as doubling measures on $M$ with respect to $d(\cdot, \cdot)^a$, aside from measurability issues again. As in Section 1.6, if $d(\cdot, \cdot)$ is a quasi-metric on $M$, it is shown in [29] that there is a metric $\tilde{d}(\cdot, \cdot)$ on $M$ and a positive real number $a$ such that $d(\cdot, \cdot)$ and $\tilde{d}(\cdot, \cdot)^a$ are each bounded by constant multiples of the other. It follows that doubling measures on $M$ with respect to $d(\cdot, \cdot)$ are the same as doubling measures with respect to $\tilde{d}(\cdot, \cdot)$, aside from the usual measurability issues.

### 7.5 Another doubling condition

Let $h(r)$ be a monotone increasing nonnegative real-valued function on the set $[0, +\infty)$ of nonnegative real numbers. If there is a nonnegative real number $C$ such that

\begin{equation}
(7.32) \quad h(2r) \leq C \cdot h(r)
\end{equation}
7.5. ANOTHER DOUBLING CONDITION

for every \( r \geq 0 \), then we say that \( h \) satisfies a doubling condition. Using the monotonicity of \( h \), we can reformulate (7.32) as saying that

\[
(7.33) \quad h(r + t) \leq h(2 \max(r,t)) \leq C \max(h(r), h(t))
\]

for every \( r, t \geq 0 \). As usual, we can also iterate (7.32), to get that

\[
(7.34) \quad h(2^k r) \leq C^k h(r)
\]

for every \( r \geq 0 \) and positive integer \( k \). Note that \( h(r) = r^a \) satisfies these conditions with \( C = 2^a \) for each \( a \geq 0 \).

Let \((M, d(x,y))\) be a metric space, and let \( \mu \) be a nonnegative Borel measure on \( M \). Suppose that the measure of every open ball in \( M \) with respect to \( \mu \) is finite, and put

\[
(7.35) \quad h_x(r) = \mu(B(x,r))
\]

for every \( x \in M \) and \( r > 0 \). Thus \( h_x(r) \) is a monotone increasing nonnegative real-valued function on \([0, +\infty)\) for each \( x \in M \), and in fact \( h_x(r) \) is also left-continuous at each \( r > 0 \) from the left for every \( x \in M \), because of the countable additivity of \( \mu \). Clearly \( \mu \) satisfies the doubling condition (7.16) for some \( C \geq 0 \) and every \( x \in M \) and \( r > 0 \) if and only if \( h_x(r) \) satisfies the doubling condition (7.32) with the same constant \( C \) for every \( x \in M \) and \( r \geq 0 \).

As in [32], \( \mu \) is said to be uniformly distributed on \( M \) if (7.35) does not depend on \( x \), so that there is a function \( h(r) \) on \([0, +\infty)\) such that \( h(0) = 0 \) and

\[
(7.36) \quad \mu(B(x,r)) = h(r)
\]

for every \( x \in M \) and \( r > 0 \). If \( X = \prod_{j=1}^{\infty} X_j \) and \( \mu \) are as in the previous section, for instance, then \( \mu \) has this property when \( \mu_j \) is uniformly distributed on \( X_j \) for each \( j \), in the sense that \( \mu_j(\{x_j\}) \) is the same for each \( x_j \in X_j \). If there is a transitive group of isometries on \( M \) that preserve \( \mu \), then \( \mu \) is uniformly distributed on \( M \) in the sense of (7.36). In particular, this includes the case of Haar measure on a topological group equipped with a translation-invariant metric. If \( \mu \) is uniformly distributed on \( M \), and if \( M \) is doubling as a metric space, then one can check that \( \mu \) is a doubling measure on \( M \).

Now let \( d(x,y) \) be a quasi-metric on a set \( M \), and let \( h(r) \) be a monotone increasing nonnegative real-valued function on \([0, +\infty)\) such that \( h(0) = 0 \) and \( h(r) > 0 \) when \( r > 0 \). If \( h(r) \) also satisfies a doubling condition as in (7.32), then it is easy to see that \( h(d(x,y)) \) is a quasi-metric on \( M \) as well. If, in addition,

\[
(7.37) \quad \lim_{r \to 0^+} h(r) = 0,
\]

then \( h(d(x,y)) \) determines the same topology on \( M \) as \( d(x,y) \), and indeed they determine the same uniform structure on \( M \). This is a variant of the situation in Section 3.5.
CHAPTER 7. SOME GEOMETRIC CONDITIONS

7.6 Some variants

Let $\mu$ be a doubling measure on a metric space $(M, d(x, y))$, and let $x, y \in M$ be given, with $x \neq y$. Put $t = d(x, y) > 0$, and observe that

$$B(x, t/2) \cap B(y, t/2) = \emptyset$$  \hspace{1cm} (7.38)

and

$$B(x, t/2) \cup B(y, t/2) \subseteq B(x, 3t/2),$$  \hspace{1cm} (7.39)

by the triangle inequality. Thus

$$\mu(B(x, t/2)) + \mu(B(y, t/2)) \leq \mu(B(x, 3t/2)).$$  \hspace{1cm} (7.40)

We also have that

$$B(x, t/2) \subseteq B(y, 3t/2),$$  \hspace{1cm} (7.41)

because $\mu$ is a doubling measure on $M$. Then $\mu(B(y, 3t/2))$ is bounded by a constant multiple of $\mu(B(y, t/2))$, and hence

$$\mu(B(x, t/2)) \leq c_1 \mu(B(y, 3t/2)),$$  \hspace{1cm} (7.42)

under these conditions. In particular, if $x$ is a limit point of $M$, then one can use this to show that $\mu(\{x\}) = 0$. Similarly, if $M$ is unbounded, then one can check that $\mu(M) = +\infty$.

Suppose now that $d(\cdot, \cdot)$ is an ultrametric on $M$, and let $\mu$ be a nonnegative Borel measure on $M$ such that the measure of every open ball is open and finite. Instead of the doubling condition (7.16), let us ask that

$$\mu(B(w, r)) \leq C_2 \mu(B(w, r))$$  \hspace{1cm} (7.43)

for some $C_2 \geq 1$ and every $w \in M$ and $r > 0$. Let $w \in M$ and $r > 0$ be given, and let $z_1, \ldots, z_n$ be finitely many elements of $B(w, r)$ such that

$$d(z_j, z_l) = r$$  \hspace{1cm} (7.44)

when $j \neq l$. Thus the open balls $B(z_j, r)$ are pairwise-disjoint subsets of $B(w, r)$, so that

$$\sum_{j=1}^{n} \mu(B(z_j, r)) \leq \mu(B(w, r)).$$  \hspace{1cm} (7.45)

We also have that

$$B(z_j, r) = B(w, r)$$  \hspace{1cm} (7.46)

for each $j$, because $d(w, z_j) \leq r$ for each $j$, and using the ultrametric version of the triangle inequality. This implies that

$$\mu(B(w, r)) = \mu(B(z_j, r)) \leq C_2 \mu(B(z_j, r))$$  \hspace{1cm} (7.47)
for each \( j \), by hypothesis. Averaging over \( j \), we get that

\[
\mu(B(w, r)) \leq \frac{C_2}{n} \sum_{j=1}^{n} \mu(B(z_j, r)) \leq \frac{C_2}{n} \mu(B(w, r)),
\]

using (7.45) in the second step. It follows that

\[
n \leq C_2.
\]

If we take \( n \) to be the largest positive integer for which there are \( n \) elements \( z_1, \ldots, z_n \) of \( B(w, r) \) satisfying (7.44) when \( j \neq l \), and if \( z \) is any element of \( B(w, r) \), then \( d(z_j, z) < r \) for some \( j \), since otherwise there would be \( n + 1 \) elements of \( B(w, r) \) with this property. This shows that

\[
B(w, r) \subseteq \bigcup_{j=1}^{n} B(z_j, r),
\]

and hence

\[
B(w, r) = \bigcup_{j=1}^{n} B(z_j, r),
\]

because \( B(z_j, r) \subseteq B(w, r) \) for each \( j \), as in (7.46).

Let \( x, y \in M \) be given again, with \( x \neq y \), and put \( t = d(x, y) > 0 \). Because \( d(\cdot, \cdot) \) is an ultrametric, we have that

\[
B(x, t) \cap B(y, t) = \emptyset
\]

and

\[
B(x, t) \cup B(y, t) \subseteq B(x, t).
\]

instead of (7.38) and (7.39). This implies that

\[
\mu(B(x, t)) + \mu(B(y, t)) \leq \mu(B(x, t)),
\]

which replaces (7.40). Under these conditions, \( B(x, t) \) is the same as \( B(y, t) \), so that

\[
\mu(B(x, t)) = \mu(B(y, t)) \leq C_2 \mu(B(y, t)),
\]

by (7.43). Combining this with (7.54), we get that

\[
\mu(B(x, t)) \leq \mu(B(x, t)) - \mu(B(y, t)) \\
\leq \mu(B(x, t)) - (1/C_2) \mu(B(x, t)) \\
= (1 - (1/C_2)) \mu(B(x, t)),
\]

in place of (7.42).

Of course, the condition (7.50) that a closed ball in \( M \) with radius \( r \) can be covered by a bounded number of open balls of radius \( r \) is weaker than the usual doubling condition for metrics, as in Section 7.3. If \( X \) is a Cartesian product
as in Section 1.2, then this condition corresponds to asking that the number
of elements of the $X_j$’s be bounded, without any additional condition on the
sequence $\{t_l\}_{l=0}^{\infty}$ used to define the ultrametric as in (1.22). Similarly, (7.43)
is weaker than the doubling condition (7.16) in Section 7.4. Let $X$ be as in
Section 1.2 again, and let $\mu$ be the probability measure on $X$ corresponding to
the product of probability measures $\mu_j$ on $X_j$ for each positive integer $j$, as
before. In this case, it is easy to see that $\mu$ satisfies (7.43) if and only if

$$\mu_j(\{x_j\}) \geq 1/C_2 \quad (7.57)$$

for every $j \geq 1$ and $x_j \in X_j$, without additional conditions on the $t_l$’s. If
$r = \{r_j\}_{j=1}^{\infty}$ is a sequence of positive integers, with $r_j \geq 2$ for each $j$, then Haar
measure on the group $\mathbb{Z}_r$ of $r$-adic integers satisfies (7.43) with respect to an $r$
if and only if the $r_j$’s are uniformly bounded. As usual, this can also be seen in
terms of a suitable isometric equivalence with a Cartesian product, as in Section
7.2. If $\mu$ is a uniformly distributed Borel measure on an ultrametric space $M$,
and if $M$ satisfies the covering condition (7.50) with (7.49), then $\mu$ also satisfies
(7.43), as in the previous section.

### 7.7 Separability

Let $(M, d(x, y))$ be a metric space. If the metric $d(x, y)$ is doubling, then bounded subsets of $M$ are totally bounded, as in Section 7.3. This implies
that $M$ is separable, by expressing $M$ as a countable union of balls, each of
which is totally bounded and thus has a countable dense subset. In particular,
if there is a doubling measure $\mu$ on $M$, then $d(x, y)$ is a doubling metric on $M$, as
in Section 7.4, and hence $M$ is separable. Suppose now that $\mu$ is a nonnegative Borel measure on $M$ such that every open ball in $M$ has positive finite measure
with respect to $\mu$, and let us check that $M$ is separable.

Let $x \in M$ and $r, t > 0$ be given, with $t \leq r$, and let $A$ be a subset of $B(x, r)$
such that

$$d(y, z) \geq t \quad (7.58)$$

for every $y, z \in A$ with $y \neq z$. Thus the balls $B(y, t/2)$ with $y \in A$ are pairwise
disjoint, and

$$B(y, t/2) \subseteq B(x, 3r/2) \quad (7.59)$$

for each $y \in A$. If $y_1, \ldots, y_n$ are finitely many elements of $A$ such that

$$\mu(B(y_j, t/2)) \geq a \quad (7.60)$$

for some $a > 0$ and $j = 1, \ldots, n$, then

$$na \leq \sum_{j=1}^{n} \mu(B(y_j, t/2)) = \mu\left(\bigcup_{j=1}^{n} B(y_j, t/2)\right) \leq \mu(B(x, 3r/2)) \quad (7.61)$$
7.7. SEPARABILITY

since $\bigcup_{j=1}^n B(y_j, t/2) \subseteq B(x, 3r/2)$, by (7.59). This shows that $n$ is uniformly bounded under these conditions, and hence that there are only finitely many $y \in A$ such that

$$\mu(B(y, t/2)) \geq a.$$  

(7.62)

Applying this to a sequence of $a$’s converging to 0, we get that $A$ has only finitely or countably many elements.

Suppose now that $A$ is a maximal subset of $B(x, r)$ such that (7.58) holds for every $y, z \in A$ with $y \neq z$, which exists by Zorn’s lemma or the Hausdorff maximality principle. If $w$ is any element of $B(x, r)$, then

$$d(w, y) < t$$

(7.63)

for some $y \in A$, since otherwise $A \cup \{w\}$ would be a larger set with the same property. This implies that

$$B(x, r) \subseteq \bigcup_{y \in A} B(y, t).$$

(7.64)

where $A$ has finitely or countably many elements, as before. It follows that $B(x, r)$ has a dense subset with only finitely or countably many elements, by considering a sequence of $t$’s converging to 0. Thus $M$ is separable under these conditions, since it can be expressed as the union of a sequence of open balls.

Alternatively, let $k$ be a positive integer, and let $A_k$ be a subset of $B(x, r)$ that satisfies (7.58) for every $y, z \in A_k$ with $y \neq z$, and

$$\mu(B(y, t/2)) \geq 1/k$$

(7.65)

for every $y \in A_k$. The earlier argument shows that $A_k$ is a finite set with a bounded number of elements, and so we suppose now that $A_k$ is a maximal set with these properties, for each $k \in \mathbb{Z}_+$. Thus

$$A = \bigcup_{k=1}^{\infty} A_k$$

(7.66)

has only finitely or countably many elements, although this set $A$ does not normally satisfy (7.58) for every $y, z \in A$ with $y \neq z$.

If $w$ is any element of $B(x, r)$, then

$$\mu(B(w, t/2)) \geq 1/k$$

(7.67)

for some $k \in \mathbb{Z}_+$, because $\mu(B(w, t/2)) > 0$ by hypothesis. It follows that (7.63) holds for some $y \in A_k$, since otherwise $A_k \cup \{w\}$ would be a larger set with the same properties as $A_k$. In particular, (7.63) holds for some $y \in A$, so that (7.64) holds again in this situation. This implies that $B(x, r)$ has a dense subset with only finitely or countably many elements, and hence that $M$ is separable, for the same reasons as before.

Similarly, if there is an $a > 0$ such that (7.62) holds for every $y \in B(x, r)$, then the previous argument shows that the number of elements of a set $A$ as
before is bounded. This implies that \( B(x, r) \) can be covered by finitely many balls of radius \( t \), as in (7.64). If for each \( t \in (0, r] \) there is an \( a > 0 \) with this property, then it follows that \( B(x, r) \) is totally bounded.

As usual, these arguments can be simplified when \( d(\cdot, \cdot) \) is an ultrametric on \( M \). In this case, if \( y, z \in B(x, r) \) and \( 0 < t \leq r \), then either \( d(y, z) < t \), and hence \( B(y, t) = B(z, t) \), or \( d(y, z) \geq t \), which implies that

\[
B(y, t) \cap B(z, t) = \emptyset.
\]

Of course, \( B(y, t) \subseteq B(y, r) = B(x, r) \) for every \( y \in B(x, r) \) when \( t \leq r \). It follows that for each \( a > 0 \) and \( t \in (0, r] \), there cannot be more than

\[
\mu(B(x, r))/a
\]

distinct open balls \( B(y, t) \) contained in \( B(x, r) \) such that

\[
\mu(B(y, t)) \geq a.
\]

In particular, for each \( t \in (0, r] \), there are only finitely or countably many distinct open balls \( B(y, t) \) contained in \( B(x, r) \).
Chapter 8

Maximal functions

8.1 Definitions

Let \((X, d(x, y))\) be a metric space, and let \(\mu\) be a nonnegative Borel measure on \(X\) such that the measure of any open ball in \(X\) is positive and finite. If \(f\) is a locally integrable function on \(X\) with respect to \(\mu\), then put

\[
M(f)(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu
\]

for each \(x \in X\), which may be infinite. More precisely, the supremum is taken over all open balls \(B = B(y, r)\) in \(X\) that contain \(x\) as an element. This is the uncentered version of the Hardy–Littlewood maximal function associated to \(f\) with respect to \(\mu\) on \(X\). Similarly, if \(\nu\) is a nonnegative Borel measure on \(X\), then the corresponding maximal function is defined by

\[
M(\nu)(x) = \sup_{B \ni x} \frac{\nu(B)}{\mu(B)}
\]

for each \(x \in X\). Of course, this reduces to (8.1) when \(\nu\) is given by

\[
\nu(A) = \int_A |f| \, d\mu
\]

for every Borel set \(A \subseteq X\). If \(\nu\) is a real or complex Borel measure on \(X\), then \(M(\nu)\) is defined to be the same as \(M(|\nu|)\), where \(|\nu|\) is the total variation measure associated to \(\nu\).

If \(f\) and \(g\) are locally integrable functions on \(X\) with respect to \(\mu\), then

\[
M(f + g)(x) \leq M(f)(x) + M(g)(x)
\]

for every \(x \in X\). Similarly,

\[
M(tf)(x) = |t| M(f)(x)
\]
for every \( x \in X \) and real or complex number \( t \), as appropriate, so that the mapping from \( f \) to \( M(f) \) is sublinear. There are analogous statements for maximal functions of Borel measures, as in the previous paragraph.

Let \( \nu \) be a nonnegative Borel measure on \( X \), and let \( t \) be a nonnegative real number. If
\[
M(\nu)(x) > t
\]
for some \( x \in X \), then there is an open ball \( B \) in \( X \) such that \( x \in B \) and
\[
\frac{\nu(B)}{\mu(B)} > t,
\]
by the definition of \( M(\nu) \). Conversely, if \( B \) is an open ball in \( X \) that satisfies (8.7), then \( M(\nu)(y) > t \) for every \( y \in B \). Thus
\[
V_t = \{ x \in X : M(\nu)(x) > t \}
\]
is the same as the union of the open balls \( B \) in \( X \) that satisfy (8.7). In particular, (8.8) is an open set in \( X \) for each \( t \geq 0 \).

If \( f \) is a bounded Borel measurable function on \( X \), then
\[
\sup_{x \in X} M(f)(x) \leq \|f\|_{\infty},
\]
where \( \|f\|_{\infty} \) denotes the \( L^\infty \) norm of \( f \) with respect to \( \mu \). We shall consider other estimates for maximal functions in the next sections.

### 8.2 Three covering arguments

Let \( I, I', \) and \( I'' \) be three intervals in the real line, which may be open, closed, or half-open and half-closed. If
\[
I \cap I' \cap I'' \neq \emptyset,
\]
then it is easy to see that one of these intervals is contained in the union of the other two. Now let \( I_1, I_2, \ldots, I_n \) be finitely many intervals in \( \mathbb{R} \), which may again be open, closed, or half-open and half-closed. Using the previous argument repeatedly, one can find indices \( 1 \leq j_1 < j_2 < \cdots < j_r \leq n \) such that
\[
\bigcup_{l=1}^{r} I_{j_l} = \bigcup_{k=1}^{n} I_k
\]
and no element of \( \mathbb{R} \) is contained in more than two of the \( I_{j_l} \)'s.

Suppose instead that \( d(x, y) \) is an ultrametric on a set \( X \), and let \( B_1, \ldots, B_n \) be finitely many distinct balls in \( X \) with respect to \( d(x, y) \), which may be open or closed. In this case, there are indices \( 1 \leq j_1 < j_2 < \cdots < j_r \leq n \) such that
\[
\bigcup_{l=1}^{r} B_{j_l} = \bigcup_{k=1}^{n} B_k,
\]
and the balls $B_j$ are pairwise disjoint. To see this, one can take the $B_j$’s to be maximal among $B_1, \ldots, B_n$ with respect to inclusion. This uses the fact that if $B$ and $B'$ are two open or closed balls in $X$, then either $B \subseteq B'$, $B' \subseteq B$, or $B \cap B' = \emptyset$.

Suppose now that $d(x, y)$ is any metric on a set $X$, and let $B_j = B(x_j, r_j)$ be the open ball in $X$ centered at a point $x_j \in X$ with radius $r_j > 0$ for $j = 1, \ldots, n$. By rearranging the indices if necessary, we may also ask that $r_j$ be monotone decreasing in $j$. Put $j_1 = 1$, and let $j_2$ be the smallest integer such that $2 \leq j_2 \leq n$ and
\begin{equation}
B_{j_1} \cap B_{j_2} = \emptyset,
\end{equation}
if there is one. Similarly, if $1 = j_1 < j_2 < \cdots < j_l < n$ have been chosen, then let $j_{l+1}$ be the smallest integer such that $j_l < j_{l+1} \leq n$ and
\begin{equation}
B_{j_{k}} \cap B_{j_{l+1}} = \emptyset
\end{equation}
for each $k = 1, \ldots, l$, if there is one. This process has to stop in a finite number $r$ of steps, and the corresponding balls $B_{j_l}$ are pairwise disjoint, by construction.

If an integer $i$, $1 \leq i \leq n$, is not equal to $j_l$ for some $l$, then there is an $l$ such that $j_l < i$ and $B_i \cap B_{j_l} \neq \emptyset$. This implies that
\begin{equation}
B_i \subseteq B(x_{j_l}, 3r_{j_l}),
\end{equation}
since the radius $r_i$ of $B_i$ is less than or equal to $r_{j_l}$. It follows that
\begin{equation}
\bigcup_{i=1}^{n} B_i \subseteq \bigcup_{l=1}^{r} B(x_{j_l}, 3r_{j_l}).
\end{equation}
Essentially the same argument works when the $B_j$’s are closed balls, or a mixture of open and closed balls. If $d(x, y)$ is a quasi-metric on $X$, then the radius $3r_{j_l}$ in (8.15) and (8.16) should be replaced with another constant multiple of $r_{j_l}$, depending on the constant in the quasi-metric condition for $d(x, y)$.

### 8.3 Weak-type estimates

Let $(X, d(x, y))$ be a metric space, and let $\mu$ be a nonnegative Borel measure on $X$ such that the measure of any open ball in $X$ is positive and finite. Also let $\nu$ be a nonnegative Borel measure on $X$ such that $\nu(X) < +\infty$, and let $V_t$ be as in (8.8) for each $t \geq 0$. Under suitable conditions, we would like to show that
\begin{equation}
\mu(V_t) \leq C_1 t^{-1} \nu(X)
\end{equation}
for some positive real number $C_1$ and every $t > 0$, where $C_1$ does not depend on $\nu$ or $t$. As in Section 8.1, $V_t$ is the same as the union of the open balls $B$ in $X$ that satisfy (8.7), for each $t > 0$. Note that $X$ is separable, as in Section 7.7, which implies that there is a base for the topology of $X$ with only finitely or countably many elements. It follows that $V_t$ can be expressed as the union of
finitely or countably many open balls $B$ in $X$ that satisfy (8.7) for each $t > 0$, by Lindelöf’s theorem in topology. Let $B_1, \ldots, B_n$ be finitely many distinct open balls in $X$ that satisfy (8.7) for some $t > 0$, so that

$$\mu(B_j) < t^{-1} \nu(B_j)$$

for $j = 1, \ldots, n$. In order to obtain an estimate of the form (8.17), it suffices to show that

$$\mu\left(\bigcup_{j=1}^{n} B_j\right) \leq C_1 t^{-1} \nu(X),$$

where $C_1 > 0$ does not depend on $t$, $\nu$, or $B_1, \ldots, B_n$, and in particular where $C_1$ does not depend on $n$.

Suppose first that $d(x, y)$ is an ultrametric on $X$. In this case, there are indices $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ such that (8.12) holds, and the balls $B_{j_l}$ are pairwise disjoint, as in the preceding section. This implies that

$$\mu\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{l=1}^{r} \mu(B_{j_l}) < t^{-1} \sum_{l=1}^{r} \nu(B_{j_l}) \leq t^{-1} \nu(X),$$

using (8.18) in the first inequality, and pairwise-disjointness of the $B_{j_l}$’s in the second inequality. Thus (8.19) holds with $C_1 = 1$, as desired.

Now let $X$ be the real line with the standard metric, so that the open balls $B_j$ are open intervals. As before, there are indices $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ such that (8.12) holds, and no element of $X = \mathbb{R}$ is contained in more than two of the $B_{j_l}$’s. Let $1_A(x)$ be the characteristic or indicator function associated to a set $A \subseteq X$, which is equal to 1 when $x \in A$ and to 0 when $x \in X \setminus A$. The condition that no point belong to more than two of the $B_{j_l}$’s implies that

$$\sum_{l=1}^{r} 1_{B_{j_l}}(x) \leq 2$$

for every $x \in X = \mathbb{R}$. It follows that

$$\sum_{l=1}^{r} \nu(B_{j_l}) = \int_{\mathbb{R}} \left(\sum_{l=1}^{r} 1_{B_{j_l}}(x)\right) d\nu(x) \leq 2 \nu(\mathbb{R}).$$

Using this, we get that

$$\mu\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{l=1}^{r} \mu(B_{j_l}) < t^{-1} \sum_{l=1}^{r} \nu(B_{j_l}) \leq 2 t^{-1} \nu(\mathbb{R}),$$

as desired.
as in the previous situation. This gives (8.19), with $C_1 = 2$.

Suppose that $d(x, y)$ is any metric on a set $X$, and that $B_j = B(x_j, r_j)$ for some $x_j \in X$ and $r_j > 0$, $j = 1, \ldots, n$. The third argument in the preceding section implies that there are indices $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ such that (8.16) holds and the balls $B_{j_i}$ are pairwise disjoint. If $\mu$ is a doubling measure on $X$, then it follows that

$$\mu\left( \bigcup_{k=1}^n B_k \right) \leq \mu\left( \bigcup_{l=1}^r B(x_{j_l}, 3r_{j_l}) \right) \leq C_1 \sum_{l=1}^r \mu(B_{j_l}),$$

for a suitable constant $C_1$. Combining this with (8.18), we get that

$$\mu\left( \bigcup_{k=1}^n B_k \right) \leq C_1 \sum_{l=1}^r \mu(B_{j_l}) < C_1 t^{-1} \sum_{l=1}^r \nu(B_{j_l}) \leq C_1 t^{-1} \nu(X),$$

using also the fact that the $B_{j_l}$'s are pairwise disjoint in the last step. Thus the same type of estimate holds when $\mu$ is a doubling measure on any metric space.

### 8.4 Distribution functions

Let $(X, A, \mu)$ be a measure space, and let $g$ be a measurable function on $X$ with values in the set $[0, +\infty]$ of nonnegative extended real numbers. The corresponding distribution function is defined on $[0, +\infty)$ by

$$\lambda(t) = \mu(\{x \in X : g(x) > t\}),$$

which is a monotone decreasing function on $[0, +\infty)$ with values in $[0, +\infty]$. Of course, $\lambda(t) \leq \mu(X)$ for every $t \geq 0$, and

$$t^p \lambda(t) \leq \int_{\{x \in X : g(x) > t\}} g(x)^p \, d\mu(x) \leq \int_X g(x)^p \, d\mu(x)$$

for every $p, t > 0$. In particular, $\lambda(t) < +\infty$ for every $t > 0$ when $g \in L^p(X)$ for some $p \in (0, +\infty)$.

Let us suppose from now on in this section that $X$ is at least $\sigma$-finite with respect to $\mu$. Note that the set

$$\{x \in X : g(x) > 0\}$$

is measurable and $\sigma$-finite when $\lambda(t) < +\infty$ for each $t > 0$, so that we could simply replace $X$ with (8.28) in this situation, if necessary. Let us also consider $[0, +\infty)$ as a $\sigma$-finite measure space with respect to Lebesgue measure, so that $X \times [0, +\infty)$ is a $\sigma$-finite measure space as well, with respect to the usual product measure construction. If $X$ is a topological space too, then we may consider...
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$X \times [0, +\infty)$ as a topological space with respect to the product topology, using
the topology induced on $[0, +\infty)$ by the standard topology on $\mathbb{R}$.

Put

$$U_r = \{(x, t) \in X \times [0, +\infty) : t < r < g(x)\}$$

for each $r \in (0, +\infty)$, and

$$U = \{(x, t) \in X \times [0, +\infty) : t < g(x)\}.$$

Observe that

$$U = \bigcup_{r \in \mathbb{Q}_+} U_r,$$

where $\mathbb{Q}_+ = \mathbb{Q} \cap (0, +\infty)$ is the set of all positive rational numbers. This
implies that $U$ is a measurable set in $X \times [0, +\infty)$, since it can be expressed as
a countable union of measurable rectangles. If $X$ is a topological space, and if

$$\{x \in X : g(x) > t\}$$

is an open set in $X$ for each $t \geq 0$, then it is easy to see that $U$ is an open set
in $X \times [0, +\infty)$. In fact, (8.31) shows that $U$ can be expressed as a countable union of open subsets of $X$ and $[0, +\infty)$ in this case.

Let $p > 0$ be given, and let $1_U(x, t)$ be the indicator function associated to
$U$ on $X \times [0, +\infty)$. Observe that

$$pt^{p-1} 1_U(x, t)$$

is a measurable function on $X \times [0, +\infty)$, because $U$ is a measurable set. Clearly

$$\int_X \left( \int_{[0, +\infty)} pt^{p-1} 1_U(x, t) \, dt \right) \, d\mu(x)$$

by elementary calculus, and

$$\int_{[0, +\infty)} \left( \int_X pt^{p-1} 1_U(x, t) \, d\mu(x) \right) \, dt$$

$$= \int_0^\infty pt^{p-1} \left( \int_{x \in X : g(x) > t} d\mu(x) \right) \, dt$$

$$= \int_0^\infty pt^{p-1} \lambda(t) \, dt.$$
Remember that a monotone function on an interval in the real line continuous at all but at most finitely or countably many points, which simplifies questions of measurability and integrability. At any rate, it follows from (8.34) and (8.35) that
\[
\int_X g(x)^p \, d\mu(x) = \int_0^\infty p \, t^{p-1} \lambda(t) \, dt
\]
for every \( p > 0 \), by Fubini’s theorem.

### 8.5 \( L^p \) Estimates

Let \((X, d(x, y))\) be a metric space again, and let \( \mu \) be a nonnegative Borel measure on \( X \) for which the measure of every open ball in \( X \) is positive and finite. Suppose that there is a positive real number \( C_1 \) such that
\[
\mu(\{x \in X : M(f)(x) > t\}) \leq C_1 \, t^{-1} \int_X |f(x)| \, d\mu(x)
\]
for every integrable function \( f \) on \( X \) with respect to \( \mu \). This is the same as (8.17) in Section 8.3, when \( \nu \) corresponds to \( f \) as in (8.3) in Section 8.1. Let a real number \( p \geq 1 \) be given, and suppose now that \( f \in L^p(X) \) with respect to \( \mu \). We would like to show that \( M(f) \in L^p(X) \) when \( p > 1 \).

Put
\[
f_t(x) = \begin{cases} f(x) & \text{when } |f(x)| \leq t \\ 0 & \text{when } |f(x)| > t \end{cases}
\]
for each \( t > 0 \). Thus \( f_t \) is a bounded measurable function on \( X \), with \( \|f\|_\infty \leq t \), so that
\[
M(f)(x) \leq t
\]
for every \( x \in X \), as in (8.9) in Section 8.1. Observe that
\[
M(f)(x) \leq M(f_{at})(x) + M(f - f_{at})(x) \leq at + M(f - f_{at})(x)
\]
for every \( x \in X \) and \( a, t > 0 \). If \( 0 < a < 1 \) and \( M(f)(x) > t \), then it follows that \( M(f - f_{at})(x) > (1 - a) \, t \), which is to say that
\[
\{x \in X : M(f)(x) > t\} \subseteq \{x \in X : M(f - f_{at})(x) > (1 - a) \, t\}.
\]
This implies that
\[
\mu(\{x \in X : M(f)(x) > t\}) \\
\leq \mu(\{x \in X : M(f - f_{at})(x) > (1 - a) \, t\}) \\
\leq C_1 (1 - a)^{-1} \, t^{-1} \int_X |f(x) - f_{at}(x)| \, d\mu(x) \\
\leq C_1 (1 - a)^{-1} \, t^{-1} \int_{\{x \in X : |f(x)| > at\}} |f(x)| \, d\mu(x).
\]
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More precisely, this uses (8.37) in the second step, applied to \( f - f_{at} \) instead of \( f \), and \((1 - a)t\) instead of \( t \). In the third step, we have used the fact that \( f(x) - f_{at}(x) \) is equal to \( f(x) \) when \(|f(x)| > at\), and is 0 otherwise. Note that \( f - f_{at} \) is integrable on \( X \) for every \( a, t > 0 \), even when \( p > 1 \).

Let us restrict our attention now to the case where \( p > 1 \). The integral of \( M(f)(x)^p \) with respect to \( \mu \) can be expressed as in (8.36) with \( g = M(f) \), and we can use (8.42) to estimate \( \lambda(t) \) as in (8.26). This implies that

\[
(8.43) \int_X M(f)(x)^p \, d\mu(x) \\
\leq \int_0^\infty pt^{p-1} \left( C_1 (1 - a)^{-1} t^{-1} \int_{\{x \in X : |f(x)| > at\}} |f(x)| \, d\mu(x) \right) \, dt \\
= pC_1 (1 - a)^{-1} \int_0^\infty t^{p-2} \left( \int_{\{x \in X : |f(x)| > at\}} |f(x)| \, d\mu(x) \right) \, dt.
\]

Interchanging the order of integration, we get that

\[
(8.44) \int_X M(f)(x)^p \, d\mu(x) \\
\leq pC_1 (1 - a)^{-1} \int_X \left( \int_0^{(|f(x)|/a)^{p-1}} t^{p-2} \, dt \right) |f(x)| \, d\mu(x) \\
= pC_1 (1 - a)^{-1} \int_X (p - 1)^{-1} (|f(x)|/a)^{p-1} |f(x)| \, d\mu(x) \\
= pC_1 (1 - a)^{-1} (p - 1)^{-1} a^{1-p} \int_X |f(x)|^p \, d\mu(x).
\]

It follows that \( M(f) \in L^p(X) \), with \( L^p \) norm bounded by the \( L^p \) norm of \( f \) times a constant that depends on \( p \) when \( p > 1 \), by taking the \( p \)th root of both sides of (8.44). This works using any \( a \in (0, 1) \), so that one can choose an optimal \( a \) for each \( p \). In particular, it is better to take \( a \) close to 1 as \( p \) increases.

8.6 Conditional expectation

Let \((X, \mathcal{A}, \mu)\) be a probability space, so that \( X \) is a set, \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets of \( X \), and \( \mu \) is a probability measure on \( X \), which is to say a nonnegative countably-additive measure on \( \mathcal{A} \) such that \( \mu(X) = 1 \). Suppose that \( f \) is a real or complex-valued function on \( X \) that is measurable with respect to \( \mathcal{A} \), and integrable with respect to \( \mu \). Thus

\[
(8.45) \nu_f(A) = \int_A f \, d\mu
\]

is defined for each \( A \in \mathcal{A} \), and determines a countably-additive real or complex-valued measure on \( \mathcal{A} \), as appropriate.

Let \( \mathcal{B} \) be another \( \sigma \)-algebra of subsets of \( X \) contained in \( \mathcal{A} \), so that \( \mathcal{B} \) is a \( \sigma \)-subalgebra of \( \mathcal{A} \). The restriction of \( \nu_f \) to \( \mathcal{B} \) is a countably-additive real or
8.6. CONDITIONAL EXPECTATION

A complex-valued measure on \((X, \mathcal{B})\), which is absolutely continuous with respect to the restriction of \(\mu\) to \(\mathcal{B}\). The Radon–Nikodym theorem implies that there is a measurable function \(f_B\) on \(X\) with respect to \(\mathcal{B}\) which is also integrable with respect to \(\mu\) such that

\[
\int_A f_B \, d\mu = \nu_f(A) = \int_A f \, d\mu
\]

for every \(A \in \mathcal{B}\). This function \(f_B\) is known as the conditional expectation of \(f\) with respect to \(\mathcal{B}\), which may be denoted \(E(f | \mathcal{B})\) as well. If \(f'_B\) is any other function on \(X\) that satisfies the same properties as \(f_B\), then it is easy to see that \(f_B = f'_B\) almost everywhere on \(X\) with respect to \(\mu\). Of course, if \(\mathcal{B} = \mathcal{A}\), then we can simply take \(f_B = f\). The conditional expectation of \(f\) with respect to \(\mathcal{B}\) may also be denoted \(E_{\mathcal{A}}(f | \mathcal{B})\), to indicate the initial \(\sigma\)-algebra \(\mathcal{A}\) explicitly.

As a basic class of examples, let \(\mathcal{P}\) be a partition of \(X\) into finitely or countably many measurable sets with positive measure with respect to \(\mu\). Thus \(\mathcal{P}\) is a collection of finitely or countably many pairwise-disjoint elements of \(\mathcal{A}\) such that \(\mu(A) > 0\) for each \(A \in \mathcal{P}\), and the union of the elements of \(\mathcal{P}\) is equal to \(X\). Also let \(\mathcal{B}(\mathcal{P})\) be the collection of subsets of \(X\) that can be expressed as a union of elements of \(\mathcal{P}\), which is interpreted as including the empty set. It is easy to see that \(\mathcal{B}(\mathcal{P})\) is a \(\sigma\)-subalgebra of \(\mathcal{A}\), and that a function \(f\) on \(X\) is measurable with respect to \(\mathcal{B}(\mathcal{P})\) if and only if \(f\) is constant on each \(A \in \mathcal{A}\).

If \(f\) is a measurable function on \(X\) with respect to \(\mathcal{A}\) which is integrable with respect to \(\mu\), then the conditional expectation \(f_B\) of \(f\) with respect to \(\mathcal{B}\) is given by

\[
f_B(x) = \frac{1}{\mu(A)} \int_A f \, d\mu
\]

for every \(A \in \mathcal{B}\) and \(x \in A\).

As another class of examples, let \((X_1, \mathcal{A}_1, \mu_1)\) and \((X_2, \mathcal{A}_2, \mu_2)\) be probability spaces, and consider their Cartesian product \(X = X_1 \times X_2\). The standard product measure construction leads to a \(\sigma\)-algebra \(\mathcal{A}\) on \(X\), and a probability measure \(\mu\) defined on \(\mathcal{A}\). Let \(\mathcal{B}_1\) be the collection of subsets of \(X\) of the form \(A \times X_2\), where \(A \in \mathcal{A}_1\). This is a \(\sigma\)-subalgebra of \(\mathcal{A}\), and a function \(f(x) = f(x_1, x_2)\) on \(X\) is measurable with respect to \(\mathcal{B}_1\) if and only if \(f(x_1, x_2)\) only depends on \(x_1\), and this function of \(x_1\) is measurable with respect to \(\mathcal{A}_1\) as a function on \(X_1\). If \(f\) is a function on \(X\) which is measurable with respect to \(\mathcal{A}\) and integrable with respect to \(\mu\), then the conditional expectation \(f_{B_1}\) of \(f\) with respect to \(\mathcal{B}_1\) is given by

\[
f_{B_1}(x_1, x_2) = \int_{X_2} f(x_1, y_2) \, d\mu_2(y_2),
\]

essentially by Fubini’s theorem.

Let \(\mathcal{A}\) be any \(\sigma\)-algebra of subsets of a set \(X\) again, and let \(\nu\) be a real or complex measure defined on \(\mathcal{A}\). Remember that the corresponding total
variation measure $|\nu|$ is defined on $\mathcal{A}$ by

\begin{equation}
|\nu|(A) = \sup_{j=1}^{\infty} \sum_{j=1}^{\infty} |\nu(A_j)|,
\end{equation}

where the supremum is taken over all sequences $A_1, A_2, A_3, \ldots$ of pairwise-disjoint measurable subsets of $X$ whose union is equal to $A$. It is well known that $|\nu|$ is a countably-additive nonnegative measure defined on $\mathcal{A}$, and that $|\nu|(X) < +\infty$.

Suppose that $\mathcal{B}$ is a $\sigma$-subalgebra of $\mathcal{A}$, and let $\nu_B$ be the restriction of $\nu$ to $\mathcal{B}$, which is a countably-additive real or complex measure defined on $\mathcal{B}$. Observe that

\begin{equation}
|\nu_B|(A) \leq |\nu|(A)
\end{equation}

for every $A \in \mathcal{B}$, where $|\nu|$ is the total variation of $\nu$ as a measure on $\mathcal{A}$, and $|\nu_B|$ is the total variation of $\nu_B$ as a measure on $\mathcal{B}$. More precisely, if $A \in \mathcal{B}$, then $|\nu_B|(A)$ is the supremum of the same type of sums as in (8.49), but where the $A_j$’s are required to be in $\mathcal{B}$. Thus $|\nu|(A)$ is given by a supremum of sums that includes the sums whose supremum is equal to $|\nu_B|(A)$ when $A \in \mathcal{B}$, which implies (8.50).

Let $\mu$ be a probability measure defined on $\mathcal{A}$, and let $f$ be a real or complex-valued function on $X$ which is measurable with respect to $\mathcal{A}$ and integrable with respect to $\mu$. If $\nu = \nu_f$ is as in (8.45), then it is well known that

\begin{equation}
|\nu|(A) = \int_A |f| \, d\mu
\end{equation}

for every $A \in \mathcal{B}$. Let $\mathcal{B}$ be a $\sigma$-subalgebra of $\mathcal{A}$, and let $\nu_B$ be the restriction of $\nu$ to $\mathcal{B}$, as in the previous paragraph. If $f_B$ is the conditional expectation of $f$ with respect to $\mathcal{B}$, then $\nu_B(A)$ is equal to the integral of $f_B$ over $A$ with respect to $\mu$ for every $A \in \mathcal{B}$, as in (8.46), and hence

\begin{equation}
|\nu_B|(A) = \int_A |f_B| \, d\mu
\end{equation}

for every $A \in \mathcal{B}$, as in (8.51). It follows that

\begin{equation}
\int_A |f_B| \, d\mu \leq \int_A |f| \, d\mu
\end{equation}

for every $A \in \mathcal{B}$, because of (8.50).

Even if a real or complex measure $\nu$ defined on $\mathcal{A}$ is not absolutely continuous with respect to $\mu$, it may be that the restriction $\nu_B$ of $\nu$ to a $\sigma$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is absolutely continuous with respect to the restriction of $\mu$ to $\mathcal{A}$. Under these conditions, the Radon–Nikodym theorem again implies that $\nu_B$ can be represented on $\mathcal{B}$ by integration of a function $f_B$ on $X$ that is measurable with respect to $\mathcal{B}$ and integrable with respect to $\mu$. As before, one can combine (8.50) and (8.52) to get that

\begin{equation}
\int_A |f_B| \, d\mu \leq |\nu|(A)
\end{equation}
8.7. ADDITIONAL PROPERTIES

for every $A \in \mathcal{B}$. If $\mathcal{B} = \mathcal{B}(\mathcal{P})$ is the $\sigma$-algebra generated by a partition $\mathcal{P}$ of $X$ into finitely or countably many elements of $\mathcal{A}$, each of which has positive measure with respect to $\mu$, then every measure defined on $\mathcal{B}$ is absolutely continuous with respect to the restriction of $\mu$ to $\mathcal{B}$. In this case, we have that

\begin{equation}
(8.55) \quad f_{\mathcal{B}}(x) = \frac{\nu(A)}{\mu(A)}
\end{equation}

for every $A \in \mathcal{B}$ and $x \in A$, instead of (8.47).

Suppose now that $\mathcal{B}$ and $\mathcal{C}$ are $\sigma$-subalgebras of $\mathcal{A}$, with $\mathcal{B} \subseteq \mathcal{C}$, and let $\nu$ be a real or complex-valued measure defined on $\mathcal{A}$. If $\nu_\mathcal{B}$ and $\nu_\mathcal{C}$ are the restrictions of $\nu$ to $\mathcal{B}$, $\mathcal{C}$, respectively, then $\nu_\mathcal{B}$ is also the same as the restriction of $\nu_\mathcal{C}$ as a measure defined on $\mathcal{C}$ to a measure on $\mathcal{B}$. This is basically trivial, but it has the following nice interpretation for conditional expectation. Let $f$ be a real or complex-valued function on $X$ that is measurable with respect to $\mathcal{A}$ and integrable with respect to $\mu$, and let $f_\mathcal{B} = E_\mathcal{A}(f \mid \mathcal{B})$, $f_\mathcal{C} = E_\mathcal{A}(f \mid \mathcal{C})$ be the conditional expectations of $f$ with respect to $\mathcal{B}$, $\mathcal{C}$, respectively. Also let

\begin{equation}
(8.56) \quad (f_\mathcal{C})_\mathcal{B} = E_\mathcal{C}(f_\mathcal{C} \mid \mathcal{B})
\end{equation}

be the conditional expectation of $f_\mathcal{C}$ with respect to $\mathcal{B}$, where $f_\mathcal{C}$ is considered as a measurable function with respect to $\mathcal{C}$ instead of $\mathcal{A}$. Under these conditions, it is easy to see that

\begin{equation}
(8.57) \quad (f_\mathcal{C})_\mathcal{B} = f_\mathcal{B}.
\end{equation}

This follows from the previous statement about measures, applied to $\nu = \nu_f$ as in (8.45). In particular, if $f$ is already measurable with respect to $\mathcal{C}$, then

\begin{equation}
(8.58) \quad E_\mathcal{A}(f \mid \mathcal{B}) = E_\mathcal{C}(f \mid \mathcal{B}),
\end{equation}

as in the case of $f_\mathcal{C}$ in (8.56).

8.7 Additional properties

Let $(X, \mathcal{A}, \mu)$ be a probability space again, and let $\mathcal{B}$ be a $\sigma$-subalgebra of $\mathcal{A}$. Also let $f$ be a real or complex-valued function on $X$ that is measurable with respect to $\mathcal{A}$ and integrable with respect to $\mu$, and let $f_\mathcal{B}$ be the conditional expectation of $f$ with respect to $\mathcal{B}$. Of course, $|f|$ is a nonnegative real-valued integrable function on $X$, and so the conditional expectation $(|f|)_\mathcal{B}$ of $|f|$ with respect to $\mathcal{B}$ is real-valued and nonnegative as well. Observe that

\begin{equation}
(8.59) \quad \int_A |f_\mathcal{B}| \, d\mu \leq \int_A |f| \, d\mu = \int_A (|f|)_\mathcal{B} \, d\mu
\end{equation}

for every $A \in \mathcal{B}$, by (8.53) and the definition of $(|f|)_\mathcal{B}$. It follows that

\begin{equation}
(8.60) \quad |f_\mathcal{B}| \leq (|f|)_\mathcal{B}
\end{equation}
almost everywhere on $X$ with respect to $\mu$, since both sides of the inequality are measurable with respect to $B$.

Now let $p \in (1, +\infty)$ be given, and suppose that $|f|^p$ is integrable on $X$ with respect to $\mu$. Thus the conditional expectation $(|f|^p)_B$ of $|f|^p$ with respect to $B$ can be defined as before, and is real-valued and nonnegative. If $A \in B$ and $\mu(A) > 0$, then

$$
(8.61) \quad \left( \frac{1}{\mu(A)} \int_A |f_B|^p \, d\mu \right)^p \leq \left( \frac{1}{\mu(A)} \int_A |f|^p \, d\mu \right)^p \leq \frac{1}{\mu(A)} \int_A |f|^p \, d\mu = \frac{1}{\mu(A)} \int_A (|f|^p)_B \, d\mu.
$$

This uses (8.53) in the first step, Jensen’s or Hölder’s inequality in the second step, and the definition of $(|f|^p)_B$ in the third step. One can check that this implies that

$$
(8.62) \quad (|f_B|^p)^p \leq (|f|^p)_B
$$

almost everywhere on $X$, because $|f_B|$ and $(|f|^p)_B$ are both measurable with respect to $B$. It follows that

$$
(8.63) \quad \int_X (|f_B|^p) \, d\mu \leq \int_X (|f|^p)_B \, d\mu = \int_X |f|^p \, d\mu,
$$

using the definition of $(|f|^p)_B$ in the second step. In particular, $|f_B|^p$ is integrable with respect to $\mu$ as well.

Let $L^p(X, A, \mu)$ be the usual space of real or complex-valued functions $f$ on $X$ such that $f$ is measurable with respect to $A$ and $|f|^p$ is integrable with respect to $\mu$, for $1 \leq p < \infty$. More precisely, $L^p(X, A, \mu)$ consists of equivalence classes of such functions, which are equal to each other almost everywhere with respect to $\mu$ on $X$. It is well known that $L^p(X, A, \mu)$ is complete with respect to the metric associated to the the $L^p$ norm

$$
(8.64) \quad \|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}.
$$

Similarly, $L^\infty(X, A, \mu)$ consists of equivalence classes of functions on $X$ that are measurable with respect to $A$ and essentially bounded on $X$. The $L^\infty$ norm $\|f\|_\infty$ is defined to be the essential supremum of $|f|$ on $X$, and $L^\infty(X, A, \mu)$ is complete with respect to the metric associated to this norm.

Of course, if $f$ is measurable with respect to $B$, then $f$ is measurable with respect to $A$ too. This leads to a natural linear mapping from $L^p(X, B, \mu)$ into $L^p(X, A, \mu)$ for each $p$, $1 \leq p \leq \infty$, which is an isometry with respect to the $L^p$ norm. Note that the image of $L^p(X, B, \mu)$ under this mapping is a closed linear subspace of $L^p(X, A, \mu)$, because $L^p(X, B, \mu)$ is complete.
It is easy to see that \( f \mapsto f_B \) is a linear mapping from \( L^1(X, \mathcal{A}, \mu) \) into \( L^1(X, \mathcal{B}, \mu) \), using the uniqueness of \( f_B \). Moreover,

\[(8.65) \quad \int_X |f_B| \, d\mu \leq \int_X |f| \, d\mu \]

for every \( f \in L^1(X, \mathcal{A}, \mu) \), by (8.53) with \( A = X \). If \( f \in L^p(X, \mathcal{A}, \mu) \) and \( 1 < p < \infty \), then \( f_B \in L^p(X, \mathcal{B}, \mu) \) and

\[(8.66) \quad \|f_B\|_p \leq \|f\|_p, \]

by (8.63). Similarly, if \( f \in L^\infty(X, \mathcal{A}, \mu) \), then

\[(8.67) \quad \int_A |f_B| \, d\mu \leq \int_A |f| \, d\mu \leq \|f\|\infty \mu(A) \]

for every \( A \in \mathcal{B} \), using (8.53) in the first step. This implies that \( f_B \) is essentially bounded on \( X \) as well, and that (8.66) holds when \( p = \infty \), because \( f_B \) is measurable with respect to \( \mathcal{B} \).

Let \( f \in L^1(X, \mathcal{A}, \mu) \) and \( B \in \mathcal{B} \) be given, and let \( 1_B(x) \) be the characteristic or indicator function on \( X \) associated to \( B \), which is equal to 1 when \( x \in B \) and to 0 otherwise. If \( A \in \mathcal{B} \), then \( A \cap B \in \mathcal{B} \) too, and hence

\[(8.68) \quad \int_A f_B 1_B \, d\mu = \int_{A \cap B} f_B \, d\mu = \int_{A \cap B} f \, d\mu = \int_A f \, d\mu, \]

using the definition of \( f_B \) in the second step. Of course, \( 1_B \) is measurable with respect to \( \mathcal{B} \) on \( X \), because \( B \in \mathcal{B} \), and \( f_B \) is measurable with respect to \( \mathcal{B} \) by construction, so that \( f_B 1_B \) is measurable with respect to \( \mathcal{B} \) as well. This shows that \( f_B 1_B \) is equal to the conditional expectation of \( f 1_B \) with respect to \( \mathcal{B} \), since it satisfies the requirements of the conditional expectation.

Similarly, if \( g \in L^\infty(X, \mathcal{A}, \mu) \), then

\[(8.69) \quad (fg)_B = f_B g. \]

This reduces to the discussion in the previous paragraph when \( g = 1_B \) for some \( B \in \mathcal{B} \), which implies that (8.69) holds when \( g \) is a simple function on \( X \) that is measurable with respect to \( \mathcal{B} \), by linearity. One can use this to get that (8.69) holds for every \( g \in L^\infty(X, \mathcal{B}, \mu) \), because simple functions on \( X \) that are measurable with respect to \( \mathcal{B} \) are dense in \( L^\infty(X, \mathcal{B}, \mu) \) with respect to the \( L^\infty \) norm. If \( f \in L^p(X, \mathcal{A}, \mu) \) for some \( p \), \( 1 \leq p \leq \infty \), and if \( q \) is the exponent conjugate to \( p \), in the sense that \( 1 \leq q \leq \infty \) and \( 1/p + 1/q = 1 \), then one can check that (8.69) holds for every \( g \in L^q(X, \mathcal{B}, \mu) \). This also uses Hölder's inequality, and the fact that \( f_B \in L^p(X, \mathcal{B}, \mu) \) when \( f \in L^p(X, \mathcal{A}, \mu) \) and \( 1 \leq p \leq \infty \), as before.

### 8.8 Another maximal function

Let \((X, \mathcal{A}, \mu)\) be a probability space, and let \( \mathcal{B}_1, \ldots, \mathcal{B}_n \) be finitely many \( \sigma \)-subalgebras of \( \mathcal{A} \), with \( \mathcal{B}_j \subseteq \mathcal{B}_{j+1} \) for \( j = 1, \ldots, n - 1 \). Also let \( f \in L^1(X, \mathcal{A}, \mu) \)
be given, and let
\[ f_j = f_{B_j} = E(f \mid B_j) \]
be the conditional expectation of \( f \) with respect to \( B_j \) for each \( j \). Put
\[ f^*_l(x) = \max_{1 \leq j \leq l} |f_j(x)| \]
for each \( l = 1, \ldots, n \), and observe that \( f^*_l \) is measurable with respect to \( B_l \) for each \( l \). If \( g \in L^1(X, \mathcal{A}, \mu) \) too, then it is easy to see that
\[ (f + g)^*_l(x) \leq f^*_l(x) + g^*_l(x) \]
for each \( l \). Similarly,
\[ (t f)^*_l(x) = |t| f^*_l(x) \]
for each \( l \) and real or complex number \( t \), as appropriate, so that the mapping from \( f \) to \( f^*_l \) is sublinear.

Let \( t > 0 \) be given, and put
\[ A_l(t) = \{ x \in X : f^*_l(x) > t \} \]
for each \( l = 1, \ldots, n \), which is an element of \( B_l \), because \( f^*_l \) is measurable with respect to \( B_l \). Note that \( f^*_1 = |f_1| \), so that
\[ A_1(t) = \{ x \in X : |f_1(x)| > t \}, \]
and hence
\[ \mu(A_1(t)) \leq \mu(A_1(t)) \leq t^{-1} \int_{A_1(t)} |f_1| \, d\mu \leq t^{-1} \int_{A_1(t)} |f| \, d\mu, \]
using (8.54) in the second step. If \( l > 1 \), then
\[ A_l(t) \setminus A_{l-1}(t) = \{ x \in X : f^*_l(x) \leq t, f^*_l(x) > t \} = \{ x \in X : f^*_l(x) \leq t, |f_l(x)| > t \}, \]
by the definition of \( f^*_l \), and in particular \( |f_l(x)| > t \) on (8.77). Thus
\[ \mu(A_l(t) \setminus A_{l-1}(t)) \leq t^{-1} \int_{A_l(t) \setminus A_{l-1}(t)} |f_l| \, d\mu \leq t^{-1} \int_{A_l(t) \setminus A_{l-1}(t)} |f| \, d\mu, \]
again using (8.53) in the second step, and the fact that \( A_l(t) \setminus A_{l-1}(t) \in B_l \), since \( A_l(t) \in B_l \) and \( A_{l-1}(t) \in B_{l-1} \subseteq B_l \).

By construction, \( f^*_l \) is monotone increasing in \( l \), which implies that
\[ A_l(t) \subseteq A_{l+1}(t) \]
for $l = 1, \ldots, n - 1$. It follows that the sets $A_j(t) \setminus A_{j-1}(t)$ are pairwise disjoint for $j \geq 2$, and disjoint from $A_1(t)$. Using (8.76) and (8.78), we get that

\begin{align}
\mu(A_l(t)) &= \mu(A_1(t)) + \sum_{j=2}^{l} \mu(A_j(t) \setminus A_{j-1}(t)) \\
&\leq t^{-1} \int_{A_1(t)} |f| \, d\mu + \sum_{j=2}^{l} t^{-1} \int_{A_j(t) \setminus A_{j-1}(t)} |f| \, d\mu \\
&= t^{-1} \int_{A_1(t)} |f| \, d\mu \leq t^{-1} \int_{X} |f| \, d\mu
\end{align}

for each $l$, with the obvious simplifications when $l = 1$.

If $f \in L^\infty(X, \mathcal{A}, \mu)$, then $f^*_l \in L^\infty(X, \mathcal{B}_l, \mu)$ and

\begin{align}
\|f^*_l\|_\infty \leq \|f\|_\infty
\end{align}

for each $l$, by the $p = \infty$ version of (8.66). Using this and the weak-type estimate on $L^1$ in (8.80), one can get $L^p$ estimates for $f^*_l$ when $1 < p < \infty$, as in Section 8.5, with $C_1 = 1$. 

\section{Another Maximal Function}
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