Boolos–style proofs of limitative theorems

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Boolos’s proof of incompleteness is extended straightforwardly to yield simple “diagonalization–free” proofs of some classical limitative theorems of logic.

In his famous paper announcing the incompleteness theorem, Gödel remarked that, though his argument is analogous to the Richard and the Liar paradoxes, “Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions.” ([7] Note 14). It is interesting that, despite the fact that the soundness of arguments like Gödel’s one built on self–reference (or diagonalization) was often questioned (of course, from a philosophical not a mathematical point of view), the first attempt to support Gödel’s claim and prove the theorem using another paradox (and hence without recourse to diagonalization) came only recently. In 1989, formalizing the Berry paradox consisting in the fact that the least integer not nameable in fewer than nineteen syllables has just now been named in eighteen syllables, G. Boolos proved the semantic version of the incompleteness theorem to the effect that there are arithmetical sentences that are true but unprovable in Peano arithmetic (see [3]). The proof, as Boolos notes at the end of his paper, “unlike the usual one, does not involve diagonalization”. Not much later, in a letter, he adds “What strikes the author as of interest in the proof via Berry’s paradox is […] that it provides a different sort of reason for the incompleteness […]” (cf. [4]).

Perhaps Boolos’s proof was one of the factors that have inspired a wave of “proving old results in a new way” (see e.g. [1] and the references given there). Nevertheless, unlike the proof theoretical methods used in both Gödel’s original proof and Boolos’s one, most of these new proofs apply sophisticated model theoretical methods that can hardly be considered “finitistic”. On the other hand, Boolos’s proof can straightforwardly be extended to yield simple proofs of some fundamental theorems that are related very closely to the incompleteness theorem and to each other. The two versions of Gödel’s first incompleteness theorem (the semantical and syntactical one describing respectively the relation between truth and provability and that of provability and refutability) together with their strengthening (the Gödel–Rosser theorem), Church’s theorem on the undecidability of provability, and Tarski’s theorem on the undefinability of truth, in a sense, constitute a complete circle of mutually related statements answering some basic questions on provability and truth. The close connection between these fundamental results is also witnessed by the fact that their standard proofs have essentially the same structure: they all can be derived from a general formal version of the Liar paradox, that is, they can be considered as different formal resolutions of this paradox (cf. [11]). Now, as we shall show below, almost the same can be said if we replace the Liar paradox by Berry’s one. Actually, without any essential modification, the idea underlying Boolos’s proof of incompleteness can be used to provide “diagonalization–free” proofs of all the basic limitative theorems mentioned above.

After fixing notation and giving the definition of basic notions, we first mimic Boolos’s proof in a slightly more detailed form than that in which the original proof was given so that we can continue the proof in different directions, which is just what we shall do.

Let us first fix any one of the standard first order languages of arithmetic. By a formula (resp. sentence, term etc.) we mean a formula (resp. sentence, term etc.) of this language. Theories are arbitrary sets of sentences. Robinson arithmetic (cf. [8] I.1.1) will be denoted by $Q$. We shall denote the standard model of $Q$ (as well as its universe) by $\omega$, and say that a sentence is true (resp. a set is definable, defined etc.) if the sentence considered is true (resp. the set is
(i) For any term or formula e, let us denote by |e| the number of symbols occurring in e (we shall call this number the length of e), and let $f : \omega^2 \rightarrow \omega$ be a recursive function such that, for any formula $\mu$ and natural number $i$, $f(i, [\mu]) = [\forall v_0] (\mu \iff v_0 = i)]$. (Obviously, there exists such a function.)

(ii) For any theory $S$, let us denote by $Pr_S$ the set of G"odel numbers of sentences provable in $S$.

(iii) Let $T$ be an arbitrary theory. Let us define the relations $Fm \subseteq \omega$, and $Lh, Nm \subseteq \omega^2$ as follows:

$Fm \triangleq \{ i \in \omega : i = [\mu] \text{ for some formula } \mu \text{ with at most one free variable } v_0 \}$,

$Lh \triangleq \{ (i, j) \in \omega^2 : i = [\mu] \text{ for some formula } \mu \text{ such that } |\mu| < j \}$,

$Nm \triangleq \{ (i, j) \in \omega^2 : j \in Fm \text{ and } f(i, j) \in Pr_T \}$.

For any formula $\mu$ and number $i$, if $(i, [\mu]) \in Nm$, that is, if $\mu = \mu(v_0)$ has at most one free variable $v_0$ and $T \vdash (\forall v_0)(\mu(v_0) \iff v_0 = i)$, then we say that the formula $\mu$ names the number $i$.

(iv) It follows from the definition of G"odel numbering that the G"odel numbers of formulas whose variables are all among the first ones are bounded by a recursive function of their length. More precisely, in the case of any G"odel numbering, there is a recursive function $g$ (depending on the particular G"odel numbering that has been chosen) such that, for any formula $\mu$ and number $j$, whenever all the variables of $\mu$ are among the first $j$ ones (that is, for $j \geq 1$, they are all in the set $\{ v_0, v_1, \ldots, v_{j-1} \}$), $|\mu| < j$ implies that $[\mu] < g(j)$. \(^1\) Now, let us choose such a $g$, and let the relation $B \subseteq \omega^2$ be defined in the following way:

$B \triangleq \{ (i, j) \in \omega^2 : ([\mu], j) \in Lh \text{ and } (i, [\mu]) \in Nm \text{ for some formula } \mu \text{ such that } [\mu] < g(j) \}.$

\(^1\) For example, let us consider the most commonly used G"odel numbering, which (assuming that the G"odel numbers of primitive symbols of the language concerned have already been given) is defined for any sequence of symbols as follows: $([s_0, s_1, \ldots, s_j]) = p_0^{[s_0]} \cdot p_1^{[s_1]} \cdots p_j^{[s_j]}$, where $p_i$ is the $i$th prime (see e.g. [10] pp.135–6). Apart from variables, our language has only finitely many primitive symbols, so we can define $c$ to be any number that is greater than the G"odel numbers of primitive symbols except variables. Let $h(j) \triangleq \max \{ c \cup \{ v_i : 1 \leq j \} \}$ for every $j$. Then, for any formula $\mu$ such that $|\mu| < j$ and all the variables of $\mu$ are among the first $j$ ones, $[\mu] < p_{h(j)\cdot j}$. Clearly, the function $g(j) = p_{j\cdot j}$ is recursive.
(v) Obviously, \( \mathcal{F}m \) and \( \mathcal{L}k \) are \( \Delta_1 \) relations. Let us suppose that \( \mathcal{P}r_T \) is definable. (This condition is obviously satisfied if, e.g., the set of Gödel numbers of sentences belonging to \( T \) is itself definable.) Then \( \mathcal{N}m \) and \( \mathcal{B} \) are also definable.

(a) Let \( \varphi(v_0, v_1) \) be a formula (with at most the free variables \( v_0, v_1 \)) defining the relation \( \mathcal{B} \). We shall choose \( \varphi \) to be \( \Sigma_1 \) whenever \( T \) is recursively axiomatizable. This is possible since, in this case, \( \mathcal{P}r_T \) is \( \Sigma_1 \), thus both \( \mathcal{N}m \) and \( \mathcal{B} \) are also \( \Sigma_1 \). (Recall that the class of recursively enumerable relations is closed under intersection, existential quantification, and the substitution of recursive functions, cf. e.g. [12] pp. 27–8).

Note that, if a formula of recursive functions, cf. e.g. [12] pp. 27–8).

(Indeed, \( T \mu \) are provably equivalent in \( Q \) if \( \mu \) names the same number (if they name a number at all). Finally, up to provable equivalence in \( T \), there are only finitely many different formulas of length less than a given length having at most one free variable \( v_0 \). (Recall that, apart from variables, our language has only finitely many primitive symbols and see our remarks in (v) (a).) Consequently, there are only finitely many different numbers that can be named by formulas of less than a given length. Thus, there is a least number that cannot be named by a formula of length less than \( t \). Let it be denoted by \( n \).

Theorem

If \( T \) is a consistent extension of \( Q \) and \( \mathcal{P}r_T \) is definable, then \( \psi(n, t) \) is true, but \( T \not\models \psi(n, t) \).

Proof. By definition, \( n \) is the least number that cannot be named by a formula of length less than \( t \), and, again by definition, \( \psi(n, t) \) is true just in this case. Consequently,

(1) \( \psi(n, t) \) is true. On the other hand, by the definition of \( \psi \), (1) implies that

(2) \( \varphi(n, t) \) is false. Further, it is easy to see that

(3) if \( T \models \psi(n, t) \), then \( \psi(v_0, t) \) names the number \( n \).

Actually, we have to show that \( T \models \psi(n, t) \) implies \( T \models (\forall v_0)(\psi(v_0, t) \iff v_0 = n) \). Clearly, in one direction, the formal implication is trivial: \( T \models \psi(n, t) \land v_0 = n \iff \psi(v_0, t) \). The other direction, in turn, follows from the fact that \( T \) is an extension of \( Q \) since \( Q \models v_0 \not\models v \lor i \leq v_0 \) (cf. [8] I.1.6 (5)), which, in turn, implies a weak kind of provable uniqueness of least elements; more precisely, for any formula \( \mu(v_0) \) and number \( i \),

\[ Q \models \neg\mu(i) \land (\forall v_2 < i)\mu(v_2) \Rightarrow (\forall v_0)[\neg\mu(v_0) \land (\forall v_2 < v_0)\mu(v_2) \Rightarrow v_0 = i]. \]

Now, it follows from \( k > 3 \) that \( 18k < 8k^2 \). Thus the definition of \( t \) implies that \( \mid \psi(v_0, t) \mid \leq \mid \psi(v_0, v_1) \mid + k_2 |t| = k_1 + k_2(15 + k + 1 + k + 1) = k_1 + k_2(17 + 2k) \leq k + k(17 + 2k) = 18k + 2k^2 < 10k^2 = t \). So we have \( |\psi(v_0, t)| < t \), which, together with (3), shows that, if \( T \models \psi(n, t) \), then \( \psi(v_0, t) \) is actually a formula witnessing the truth of \( \varphi(n, t) \). But \( \varphi(n, t) \) is false by (2). Consequently, \( T \not\models \psi(n, t) \).
Now we can give the semantical incompleteness theorem in the usual formulation. The theory $T$ is called to be \textit{sound} if all the sentences belonging to $T$ are true.

**Corollary 1** (Semantic version of Gödel’s first incompleteness theorem)

\textit{Let $Pr_T$ be definable (in particular, let $T$ be recursively axiomatizable). If $T$ is sound, then $T$ is incomplete.}

\textbf{Proof.} First of all, for any theory $S$, let us denote by $\text{Ded } S$ the set of all sentences provable in $S$, and let $T' \equiv Q \cup T$. Then obviously, $\text{Ded } T' = \text{Ded } (Q \cup \text{Ded } T)$, that is, $Pr_{T'} = Pr_{Q \cup \text{Ded } T}$.

Since $Q$ is finite and $Pr_T$ is definable by our assumption, the set of Gödel numbers of the sentences in $Q \cup \text{Ded } T$ is again definable, which, in turn, implies the definability of the set $Pr_{Q \cup \text{Ded } T} = Pr_{T'}$. Further, $Q$ is sound by definition, thus $T'$ is also sound. Soundness, in turn, implies consistency. Consequently, $T'$ satisfies the conditions of the Theorem. Therefore $T' \not\vdash \psi(n, t)$ and $\neg \psi(n, t)$ is false. Thus, on the one hand, $T \not\vdash \psi(n, t)$ follows from the fact that $T \subseteq T'$, on the other hand, sound theories cannot prove false sentences.

So far we have only reiterated Boolos’s proof with some minor modifications that open up the possibility to make a few steps farther along the lines set by the original proof, and have formulated its most immediate consequence.\footnote{The detailed exposition, however, has its reward. The Theorem is a slightly more general version of the \textit{semantical} incompleteness theorem than the usual one. Indeed, it seems that the standard proofs (cf. e.g. [6] p.229 or [2] p.100), being essentially based on the diagonal lemma in one way or other, yield the theorem in such a form in which the condition of soundness of the theory concerned (which is, of course, a much stronger requirement than that of its consistency) inevitably appears; see the proof of the abstract version of this theorem in [11]. For that matter, if we had followed Boolos’s proof word by word, then we could have weakened even the condition that $T$ is an extension of $Q$. Actually, in order to define $n$, it is enough to suppose that the sentences $ς_i \neq ς_j$, $i, j \in \omega$ are all theorems of $T$. Then we can proceed as follows. $ψ(ς_0, t)$ defines $n$ as the least element of a non-empty set of natural numbers. Since the least element of such a set is unique, the sentence $ψ(∀ς_0)(ψ(ς_0, t) \iff ς_0 = n)$, expressing the uniqueness of this element and the fact that this element is just $n$, is true. On the other hand, the provability of $n$ in $T$ would entail that $ψ(n, t)$ is true (recall that $|ψ(ς_0, t)| < t$), contradicting the truth of $ψ(n, t)$. As far as the third condition of the Theorem is concerned, the usual strong assumption of recursive axiomatizability of $T$ can obviously be weakened to the definability of $Pr_T$ in the classical proofs as well.}\n
For $T$ is sound, if all the sentences provable in $T$ are true, then $T'$ satisfies the conditions of the Theorem. Therefore $T' \not\vdash \psi(n, t)$ and $\neg \psi(n, t)$ is false. Thus, on the one hand, $T \not\vdash \psi(n, t)$ follows from the fact that $T \subseteq T'$, on the other hand, sound theories cannot prove false sentences.

**Corollary 2** (Tarski’s theorem on the undefinability of arithmetical truth)

\textit{The set of Gödel numbers of true sentences is not definable.}

\textbf{Proof.} Let $Tr$ be the set of Gödel numbers of true sentences and let us suppose that $Tr$ is definable. Choose $T$ in the Theorem to be the set of all true sentences, that is, let $T = \{σ : σ$ is a true sentence\}. Then, clearly, $T$ is deductively closed, i.e. the sentences provable in $T$ are all in $T$. Consequently, $Pr_T = Tr$, so that $T$ is consistent and $Pr_T$ is definable. Moreover, by definition, $T \supseteq Q$. So we can apply the Theorem: $ψ(n, t)$ is true but unprovable in $T$. But $Pr_T = Tr$ implies that this is impossible because it means that, for any sentence $σ$, $σ$ is true iff $σ$ is provable in $T$.\n
\textbf{Proof.} First of all, for any theory $S$, let us denote by $\text{Ded } S$ the set of all sentences provable in $S$, and let $T' = Q \cup T$. Then obviously, $\text{Ded } T' = \text{Ded } (Q \cup \text{Ded } T)$, that is, $Pr_{T'} = Pr_{Q \cup \text{Ded } T}$.

Since $Q$ is finite and $Pr_T$ is definable by our assumption, the set of Gödel numbers of the sentences in $Q \cup \text{Ded } T$ is again definable, which, in turn, implies the definability of the set $Pr_{Q \cup \text{Ded } T} = Pr_{T'}$. Further, $Q$ is sound by definition, thus $T'$ is also sound. Soundness, in turn, implies consistency. Consequently, $T'$ satisfies the conditions of the Theorem. Therefore $T' \not\vdash \psi(n, t)$ and $\neg \psi(n, t)$ is false. Thus, on the one hand, $T \not\vdash \psi(n, t)$ follows from the fact that $T \subseteq T'$, on the other hand, sound theories cannot prove false sentences.
Kikuchi has modified Boolos’s notion of naming to obtain the syntactic version of the first incompleteness theorem for suitable extensions of Peano arithmetic and the second incompleteness theorem (see [9]).\(^3\) As far as the first incompleteness theorem is concerned, in fact, this modification is not needed. What is more important, with the help of making some plausible additional observations, we can derive the syntactic incompleteness theorem from the previous results for a considerably weaker theory than Peano arithmetic:

**Corollary 3** (Syntactic version of Gödel’s first incompleteness theorem)

*Let \( T \) be a recursively axiomatizable extension of \( Q \).*

(i) *If \( T \) is consistent, then \( T \not\vdash \varphi(v_0, v_1) \).*

(ii) *If \( T \) is \( \omega \)-consistent, then \( T \not\vdash \varphi(n, t) \).*

**Proof.** Since \( T \) is recursively axiomatizable, \( \varphi(v_0, v_1) \) is, by definition, a \( \Sigma_1 \) formula. Further, since \( \omega \)-consistency implies consistency, the conditions of the theorem hold in both cases.

(i) Since \( \varphi(v_2, t) \) is \( \Sigma \), the sentence \( (\forall v_2 < n)\varphi(v_2, t) \) is also \( \Sigma \). Moreover, it is true. (This follows from the definition of \( \psi \) and (1) in the proof of the theorem.) Therefore, by \( \Sigma \) completeness, \( T \vdash (\forall v_2 < n)\varphi(v_2, t) \). Hence \( T \vdash \neg \varphi(n, t) \) would imply \( T \vdash \psi(n, t) \), contradicting the theorem. Thus \( T \not\vdash \neg \varphi(n, t) \).\(^4\)

(ii) Suppose that \( T \) is \( \omega \)-consistent. Since \( \varphi(v_0, v_1) \) is now \( \Sigma_1 \), there is a \( \Delta_0 \) formula \( \mu(v_0, v_1, v_i) \) such that \( \varphi(v_0, v_1) = (\exists v_i)\mu(v_0, v_1, v_i) \). As we have already seen, \( \varphi(n, t) \) is false (cf. (2) in the proof of the theorem). It follows from this that, for any number \( j \), \( \neg \mu(n, t, j) \) is a true \( \Delta_0 \) sentence. Using \( \Sigma \) completeness, we have \( T \vdash \neg \mu(n, t, j) \) for every \( j \), which, by the definition of \( \omega \)-consistency,\(^5\) implies that \( T \not\vdash (\exists v_i)\mu(n, t, v_i) \), i.e. \( T \not\vdash \varphi(n, t) \).

One of the standard ways to prove the Gödel–Rosser theorem is to show that it is a direct consequence of the Church theorem. We shall also follow this route, that is, using the previous results, we first show that no consistent extension of \( Q \) is decidable:

**Corollary 4** (Church’s theorem on the undecidability of arithmetic)

*If \( T \) is a consistent extension of \( Q \), then \( T \) is undecidable.*

**Proof.** Suppose, for sake of contradiction, that \( Pr_T \) is a recursive relation. It follows from this that \( Nm \) is also a recursive one, hence \( B \) is again recursive since, on the one hand, \( Fm \) and \( Lh \) are recursive, on the other, the class of recursive relations is closed under intersection, bounded quantification, and the substitution of recursive functions (cf. e.g. [12] pp.27–8). Consequently, the formula \( \varphi \) defining \( B \) can now be chosen to be \( \Delta \). Since \( T \) is supposed to be a consistent extension of \( Q \) and \( Pr_T \) to be recursive (which, of course, implies the recursive axiomatizability of \( T \)), we can apply Corollary 3 (i). Consequently, \( T \not\vdash \neg \varphi(n, t) \). This, however, leads to a contradiction since, by (2) in the proof of the theorem, \( \neg \varphi(n, t) \) is true, i.e. it is a true \( \Sigma \) sentence. Its truth, in turn, by \( \Sigma \) completeness, implies its provability in \( T \), that is, \( T \vdash \neg \varphi(n, t) \).\(^3\)

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\(^3\)There is, however, a minor mistake in his proof of the first incompleteness theorem (see the proof of Theorem 2.2 (ii) in [9]). Indeed (using the notation of [9]), \( Q(m, \rho) \) is obviously not \( \Sigma_1 \). What is needed, therefore, in order for that proof given in [9] to go through, is the simple fact (to be shown, needless to say, without using the soundness of \( PA \) that there is a \( \Sigma_1 \) sentence \( Q^*(m, \rho) \) satisfying not only the requirement that \( Q(m, \rho) \) implies \( Q^*(m, \rho) \) in \( PA \), but also the additional one that the truth of \( Q^*(m, \rho) \) implies the same for \( Q(m, \rho) \).

\(^4\)Note that \( \neg \varphi(n, t) \) is yet another sentence that is true but unprovable.

\(^5\)\( T \) is \( \omega \)-consistent if for any formula \( \eta(v_i) \), it follows from \( T \vdash (\exists v_i)\eta(v_i) \) that \( T \not\vdash \eta(j) \) for some number \( j \) (cf. e.g. [10] p. 142).
Corollary 5 (Rosser–Gödel incompleteness theorem)

If $T$ is a consistent and recursively axiomatizable extension of $Q$, then $T$ is incomplete.

Proof. Let us suppose that, on the contrary, $T$ is complete. Let $\text{Snt}(v_0)$ and $\text{Neg}(v_0, v_1)$ denote $\Delta$ formulas defining, respectively, the set of Gödel numbers of sentences and the relation that holds between the Gödel number of a sentence and that of its negation. Further, let $\text{Pr}_T(v_0)$ a $\Sigma$ formula defining the set $\text{Pr}_T$. Now we set

$$\text{Pr}_{cT}(v_0) \equiv -\text{Snt}(v_0) \lor (\exists v_1)(\text{Pr}_T(v_1) \land \text{Neg}(v_0, v_1)).$$

Then $\text{Pr}_{cT}(v_0)$ is a $\Sigma$ formula. On the other hand, it follows from the completeness and consistency of $T$ that $\text{Pr}_{cT}(v_0)$ defines just the complement of $\text{Pr}_T$. Therefore, both $\text{Pr}_T$ and its complement are $\Sigma$, so that $\text{Pr}_T$ is recursive, contradicting the previous corollary.

The proofs we have given demonstrate that the Boolos–style formalization of Berry’s paradox is, in fact, a proof schema. Indeed, in order to obtain the proofs of Gödel’s semantical incompleteness theorem, Tarski’s theorem, Gödel’s syntactical incompleteness theorem, and Church’s theorem, we have simply applied the common conceptual framework given implicitly by Boolos’s incompleteness proof to four kinds of formal theories of arithmetic, namely, to theories for which the set $\text{Pr}_T$ is, respectively, definable, the set of (Gödel numbers of) all true sentences, recursively enumerable, and recursive.

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