Propositional Logics of Dependence and Independence, Part I

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In this series of two papers, we study the logics of dependence and independence concepts in propositional logic. We define propositional dependence and independence logic, propositional intuitionistic dependence logic, as well as some interesting variants of them. We study the expressive power of these logics, provide deductive systems and prove completeness theorems for them.

Some of these logics satisfy downwards closure property, some do not. This first paper deals with downwards closed logics, while the second paper studies logics without the downwards closure property. This first paper is organized as follows: In Section 1, we discuss motivation and philosophical background of the propositional dependence logic, and give formal definition for it. It will turn out that propositional dependence logic, as well as other logics studied in this paper are all maximal downwards closed logics. In Section 2, we introduce propositional intuitionistic dependence logic, which is essentially equivalent to the so-called inquisitive logic [5]. A complete deductive system is given in this section as a consequence of the respective results in [5].

In Section 3, we axiomatize another natural maximal downwards closed logic, namely propositional dependence logic extended with intuitionistic disjunction. On the basis of this we axiomatize propositional dependence logic in Section 4.

Part 2 of this sequence of papers will focus on propositional independence logic.

1 Introduction

We shall investigate the logic of dependence concepts. They have been previously investigated in the context of predicate logic [14]. The fundamental concept there is the concept $=(\vec{x}, \vec{y})$ of a sequence $\vec{y}$ of variables depending on a sequence $\vec{x}$ of other variables, which is taken as a new atomic formula. The meaning of such atomic dependence formulas is given in terms of sets of assignments called teams.

Studying the logics of dependence and independence concepts in propositional logic is similar to the case of predicate logic in that we use the method of teams. A team in this case is defined to be a set of valuations. There are, however, also significant differences. Most importantly, propositional dependence and independence logics are decidable because for any given formula of the logics with $n$ propositional variables,
there are in total \(2^n\) valuations and \(2^{2n}\) teams. The method of truth tables has its analogue in these logics, but the size of such tables grows exponentially faster than in the case of traditional propositional logic, rendering it virtually inapplicable. This emphasizes the role of the axioms and the completeness theorem in providing a manageable alternative for establishing logical consequence.

Classical propositional logic is based on propositions of the form

\[ p, \neg p, p \lor q, \text{If } p, \text{ then } q \]

and more generally

\[ \text{If } p_1, \ldots, p_k, \text{ then } q. \]

We present extensions of classical propositional calculus in which one can express, in addition to the above, propositions of the form “\(q\) depends on \(p\)” and “\(q\) is independent of \(p\)”, or more generally

\[ q \text{ depends on } p_1, \ldots, p_k, \]

and

\[ p_1, \ldots, p_k \text{ are independent of } p_{j1}, \ldots, p_{jm}. \]

In our setting, both (2) and (3) are treated as atomic facts. The former is expressed formally by a new atomic formula

\[ =(p_1, \ldots, p_k, q), \]

called dependence atom, while the latter by the so-called independence atom

\[ p_1, \ldots, p_k \perp p_{j1}, \ldots, p_{jm}. \]

Intuitively, (2) means that to know whether \(q\) holds it is sufficient to consult the truth values of \(p_1, \ldots, p_k\). Note that, as in the first-order dependence logic case, (2) says nothing about the way in which \(p_1, \ldots, p_k\) are logically related to \(q\). It may be that \(p_1 \land \ldots \land p_k\) logical implies \(q\), or that \(\neg p_1 \land \ldots \land \neg p_k\) logical implies \(\neg q\), or anything in between. Technically speaking, this is to say:

\[ \text{The truth value of } q \text{ is a function of the truth values of } p_1, \ldots, p_k. \]

Given the huge amount of data available nowadays, arising from DNA, astronomical data, so-called Big Data, etc, with no clear picture what the functions in action are, it seems—and we suggest—that the propositions (2) and (3) and their logic would deserve a mathematical treatment just as the simpler propositions (1) have deserved in classical propositional logic.

Examples of natural language sentences of this kind are the following:

1. **Whether it rains depends completely on whether it is winter or summer.**
2. **Whether you end up in the town depends entirely on whether you turn here left or right.**
3. **I will be absent depending on whether he shows up or not.**
Another basic ingredient of classical propositional calculus is, as in 3, the a priori independence of the atomic propositions. Knowing the truth value of the sequence $p_{i1}, \ldots, p_{ik}$ gives no information of the truth value of $p_{j1}, \ldots, p_{jm}$. Any individual valuation $s$ fixes the true value of both $p_{i1}, \ldots, p_{ik}$ and $p_{j1}, \ldots, p_{jm}$, but if we have a set of valuations (called a team), the truth value of neither of the two sequences needs to be fixed, and we can ask whether these truth values are independent of each other in the sense that knowing one does not reveal, in the light of the given team, the other. This is, of course, the matter in the maximal team of all valuations $s$ for all relevant propositional variables. The maximal team represents the world of all logical possibilities. However, in practice we may be interested in a particular team and the manifestation of independence in that team.

For example, if we have a pool of human chromosomes arising from a group of actual people, we may ask whether certain traits are independent of each other in this pool of chromosomes. Knowing that they would be independent, if all logically possible chromosomes were present, would be of no interest what so ever. Of course, such a team of all logically possible chromosomes would densely fill the entire physical universe.

Here are examples “independence” in natural language:

1. Whether it rains is completely independent on whether it is winter or summer.
2. As to whether you end up in the town or not it makes no difference whether you turn here left or right.
3. I will decide whether I come to the party independently of whether he decides to show up or not.

The logic of “independence” will be studied in Part 2 of this series of papers. In this first paper, we give exact mathematical meaning to “dependence”. The resulting logic is called propositional dependence logic.

**Definition 1.1.** Let $p_i$ be propositional variables. Well-formed formulas of propositional dependence logic (PD) are given by the following grammar

$$
\phi ::= p_i \mid \neg p_i \mid (p_{i1}, \ldots, p_{ik}) \mid \phi \land \phi \mid \phi \otimes \phi,
$$

where $k > 0$.

The symbol $\otimes$ is called tensor (disjunction), as it corresponds to additive conjunction (instead of disjunction!) of linear logic, interested readers are referred to [1] for further discussions. In literature of logics for dependence and independence, tensor (disjunction) is usually denoted by the usual disjunction symbol $\lor$. However in this paper, we reserve $\lor$ for other use. Note also that in the syntax of PD, we only allow negation to occur in front of propositional variables, that is, all formulas are in negation normal form. Next, we define the semantics of PD.

**Definition 1.2.** (i) A valuation $s$ is a function $s : \mathbb{N} \to \{0, 1\}$. A team is a set of valuations. In particular, the empty set $\emptyset$ is a team, and the singleton $\{\emptyset\}$ of the empty valuation $\emptyset$ is a team.

(ii) For any $n \in \mathbb{N}$, an $n$-valuation $s_0$ on $N$ is the restriction of a valuation $s$ to an $n$-element subset $N \subseteq \mathbb{N}$, that is, $s_0 = s \downharpoonright N$ with $|N| = n$. An $n$-team on $N$ is a set of $n$-valuations on $N$.  

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(iii) We write $\phi(p_{i_1}, \ldots, p_{i_n})$ to mean that the propositional variables occurring in the formula $\phi$ are among $p_{i_1}, \ldots, p_{i_n}$. A formula of the form $\phi(p_{i_1}, \ldots, p_{i_n})$ is called an $n$-formula.

Let us fix an $n$-element set $N \subseteq \mathbb{N}$. There are in total $2^n$ distinct $n$-valuations, and $2^{2^n}$ distinct $n$-valuations, among which there exists a maximal team consisting of all of the $n$-valuations on $N$, denoted by $2^n$.

**Definition 1.3.** We inductively define the notion of a formula $\phi$ of PD being true on a team $X$, denoted by $X \models \phi$, as follows:

- $X \models p_i$ iff for all $s \in X$, $s(i) = 1$;
- $X \models \neg p_i$ iff for all $s \in X$, $s(i) = 0$;
- $X \models \langle p_{i_1}, \ldots, p_{i_k} \rangle$ iff for all $s, s' \in X$
  
  $$
  \langle s(i_1), \ldots, s(i_{k-1}) \rangle = \langle s'(i_1), \ldots, s'(i_{k-1}) \rangle \implies s(i_k) = s'(i_k);
  $$
- $X \models \phi \land \psi$ iff $X \models \phi$ and $X \models \psi$;
- $X \models \phi \lor \psi$ iff there exist teams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that
  
  $$
  Y \models \phi \quad \text{and} \quad Z \models \psi;
  $$

Let $L$ be the logic PD. For any formula $\phi$ of $L$, if $X \models \phi$ holds for all teams $X$, then we say that $\phi$ is valid in the logic, denoted by $\models_L \phi$ or simply $\models \phi$. If $X \models \phi \implies X \models \psi$ holds for all teams $X$, then we say that $\psi$ is a logical consequence of $\phi$, in symbols $\models \phi \vdash \psi$. If $\models \phi \vdash \psi$ and $\models \phi \vdash \phi$, then we say that $\phi$ and $\psi$ are logically equivalent, in symbols $\models \phi \equiv \psi$. When comparing the expressive powers of two logics $L_1$ and $L_2$ of dependence and independence, we write $L_1 \leq L_2$ if every formula of $L_1$ is logically equivalent to a formula of $L_2$. If $L_1 \leq L_2$ and $L_2 \leq L_1$, then we write $L_1 \equiv L_2$.

We abbreviate $p_i \land \neg p_i$ by $\bot$ and $p_i \lor \neg p_i$ by $\top$. Clearly, $\bot$ is true only on the empty team $\emptyset$ and $\top$ is true on all teams. Dependence atoms $\models (p_i)$ with a single propositional variable are of particular interests, especially they play a central role in Section 2. These atoms are called constancy dependence atoms, and clearly, for any team $X$,

- $X \models \models (p_i)$ iff for all $s, s' \in X$, $s(i) = s'(i)$,

meaning that $p_i$ has a constant value in the team $X$.

The following basic rules governing the dependence atoms are known in database theory, where they are called Armstrong’s axiom:\footnote{In full generality the Armstrong’s Axioms have arbitrary finite sequences of variables.}

1. $\models (p_{i_1}, p_i)$
2. $\models (p_{i_0}, p_{i_1}, p_{i_2}) \vdash \models (p_{i_1}, p_{i_0}, p_{i_2})$
3. $\models (p_{i_0}, p_{i_0}, p_{i_1}) \vdash \models (p_{i_0}, p_{i_1})$
4. $\models (p_{i_1}, p_{i_2}) \vdash \models (p_{i_0}, p_{i_1}, p_{i_2})$
5. $\models (p_{i_0}, p_{i_1}) \vdash \models (p_{i_1}, p_{i_2})$ 

1. $\models (p_{i_0}, p_{i_1}) \vdash \models (p_{i_0}, p_{i_2})$

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In Section 4 we show that these rules can be derived in the natural deduction system of PD.

In the syntax of PD (Definition 1.1), we did not allow negation to occur in front of an arbitrary formula, or especially in front of a dependence atom. As in the literature of dependence logic (e.g., [14]), one can easily make sense of an arbitrary negation $\neg\phi$ by viewing $\neg\phi$ as the formula obtained by pushing negation all the way to the front of atomic formulas and defining

- $X \models \neg= (p_{i_1}, \ldots, p_{i_k})$ if $X = \emptyset$.

Also, it is possible to include strings of the form $= (\phi_1, \ldots, \phi_k)$ in the set of well-formed formulas of PD, so as to make uniform substitution a well-defined notion for PD. However, even in this case, the logic PD is not closed under Uniform Substitution rule:

- $\phi(p_{i_1}, \ldots, p_{i_n}) \vdash (\psi_1/p_{i_1}, \ldots, \psi_n/p_{i_n})$ (Sub)

since, e.g., $\models p_i \otimes \neg p_i$, whereas $\not\models = (p_i) \otimes \neg = (p_i)$.

The team semantics of propositional dependence logic is a natural adaption of the first-order team semantics, therefore, not surprisingly, many of the properties of first-order dependence logic (see [14]) are also true for PD. We invite the reader to check that PD has the empty team property, locality property, and the downwards closure property, as stated below.

**Lemma 1.4 (Empty Team Property).** PD has the empty team property, that is, $\emptyset \models \phi$ for every formula $\phi$ of the logic.

**Lemma 1.5 (Locality).** Let $\phi(p_{i_1}, \ldots, p_{i_n})$ be an $n$-formula of PD. For any teams $X, Y$ such that $X \upharpoonright \{i_1, \ldots, i_n\} = Y \upharpoonright \{i_1, \ldots, i_n\}$, we have that

- $X \models \phi \iff Y \models \phi$.

**Theorem 1.6 (Downward Closure).** For any formula $\phi$ of PD, any teams $X, Y$,

- $[X \models \phi \text{ and } Y \subseteq X] \implies Y \models \phi$.

**Corollary 1.7.** For any $n$-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of PD,

- $2^n \models \phi \iff \emptyset \models \phi$,

where $2^n$ is the maximal $n$-team on $\{i_1, \ldots, i_n\}$.

Formulas $\phi$ satisfying

- $X \models \phi \iff \forall s \in X, \{s\} \models \phi$

for all teams $X$ are said to be flat. A formula built from propositional variables and negated propositional variables by conjunction $\land$ and tensor disjunction $\otimes$ is called a classical formula. That is, a classical formula of PD is a formula that does not contain dependence atoms atoms. Classical formulas are clearly flat, and they behave classically on singleton teams, as shown in the following lemma.

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If a valuation $s$ satisfies the classical formula $\phi$, we write $s \models \phi$.
Lemma 1.8. If $\phi$ is a classical formula, then identifying tensor disjunction $\otimes$ with the disjunction $\lor$ of classical propositional logic, for any valuation $s$,

$$s \models \phi \iff \{s\} \models \phi.$$  

Proof. The lemma is proved by induction on $\phi$. We only show the non-trivial case $\phi = \psi \otimes \chi$. For any valuation $s$, we have that

$$\{s\} \models \psi \otimes \chi \iff \{s\} \models \psi \text{ or } \{s\} \models \chi \iff s \models \psi \text{ or } s \models \chi \text{ (by the induction hypothesis)} \iff s \models \psi \lor \chi.$$ 

In this series of two papers, we will also consider variants of propositional dependence logic, as well as variants of propositional independence logic, all of those are based on team semantics. In the following sections, we will prove that some of these logics, including $\mathbf{PD}$, are maximal with respect to the downwards closure property. We end this section by defining formally the notion of maximal downwards closed logic.

For each formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of a logic based on team semantics, we write $\llbracket \phi \rrbracket$ for the set of all $n$-teams on $N = \{i_1, \ldots, i_n\}$ satisfying $\phi$, i.e.,

$$\llbracket \phi \rrbracket := \{X \subseteq 2^n \mid X \models \phi\},$$

where $2^n$ is the maximal $n$-team on $N$, and $\nabla_N$ for the family of all non-empty downwards closed collections of $n$-teams on $N$, i.e.,

$$\nabla_N = \{K \subseteq 2^n \mid K \neq \emptyset, \text{ and } X \in K, Y \subseteq X \text{ imply } Y \in K\}.$$ 

Clearly, $\llbracket \phi \rrbracket \in \nabla_N$ whenever $\phi(p_{i_1}, \ldots, p_{i_n})$ is a formula of a logic $L$ with the downwards closure property and the empty team property, such as $\mathbf{PD}$. The logic $L$ is said to be a maximal downwards closed logic if for every $n$-element set $N = \{i_1, \ldots, i_n\} \subseteq \mathbb{N}$,

$$\nabla_N = \{\llbracket \phi \rrbracket : \phi(p_{i_1}, \ldots, p_{i_n}) \text{ is an } n\text{-formula of } L\}.$$

2 Propositional intuitionistic dependence logic and inquisitive logic

Before investigating propositional dependence logic, in this section, we will introduce an important example of a maximal downwards closed logic based on team semantics, namely propositional intuitionistic dependence logic ($\mathbf{PID}$), which is the propositional logic of first-order intuitionistic dependence logic, introduced in [1] and studied in [16].

Formally, the syntax of $\mathbf{PID}$ is defined as follows.

**Definition 2.1.** Well-formed formulas of propositional intuitionistic dependence logic ($\mathbf{PID}$) are given by the following grammar:

$$\phi ::= p_i \mid \neg p_i \mid \bot \mid -(p_i) \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \to \phi$$
The connectives $\lor$ and $\rightarrow$ are called *intuitionistic disjunction* and *intuitionistic implication*, respectively, and they have intuitionistic interpretations. Unexpectedly, the logic $\text{PID}$ is essentially equivalent to the so-called inquisitive logic, which was introduced in [5] with a different motivation from ours. As a corollary of this surprising connection, we will present in this section a sound and complete Hilbert style deductive system for $\text{PID}$.

To motivate the logic $\text{PID}$, consider the *classical (contradictory) negation* (denoted by $\sim$) whose team semantics is defined (naturally) as

- $X \models \sim \phi$ iff $X \not\models \phi$

for any formula $\phi$ and any team $X$. Such classical negation is clearly not definable in the downwards closed logic $\text{PD}$. This raises the question of how to define *implications*, or how to express *conditional statements* in $\text{PD}$. One natural solution is to interpret the conditional statement "if $\phi$, then $\psi$" (5) as

$$\phi \subseteq \psi := \phi \otimes \psi,$$

where $\phi^-$ stands for the *literal negation* of $\phi$, that is the formula $\neg \phi$ with negation $\neg$ pushed inside $\phi$ all the way to the front of atomic formulas. This way, for example, "if $(p \land \neg q)$, then $r$" is expressed by the formula

$$(p \land \neg q) \subseteq r := (\neg p \otimes q) \otimes r.$$

However, despite of the intuitive meaning of the resulting formula, this solution has a technical drawback: it is not able to express conditionals of dependence statements. For example, the following conditional statement

*If whether it rains depends completely on whether it is winter or summer, then whether I'll take an umbrella with me depends completely on whether it is winter or summer.*

will be interpreted as

$$=(p,q) \subseteq =(p,r) := (\neg =(p,q)) \otimes =(p,r).$$

But $\neg =(p,q)$ is (by definition) equivalent to $\bot$, thus $=(p,q) \subseteq =(p,r)$ holds, which is certainly unreasonable.

A better treatment of conditional statements is, as we suggest, to read (5) as

$$\phi \models \psi \ (\phi \text{ logically implies } \psi).$$

We will soon see that $\text{PID}$ has the downwards closure property and the Deduction Theorem holds for intuitionistic implication. Given these, the above expression will be equivalent in $\text{PID}$ to

$$\models \phi \rightarrow \psi \ (\"\phi \text{ intuitionistically implies } \psi\" \text{ is valid}).$$

In view of this, we propose to interpret (5) as $\phi \rightarrow \psi$, where $\rightarrow$ is the intuitionistic implication.

\footnote{Suppose $\phi$ is a formula of $\text{PD}$ which is equivalent to $\sim \bot$. Take any non-empty team $X$. Since $X \not\models \bot$, we have that $X \models \phi$. By the downwards closure property, $\emptyset \models \phi$ holds, implying $\emptyset \not\models \bot$: a contradiction.}
Definition 2.2. We define inductively the notion of a formula $\phi$ of PID being true on a team $X$, denoted $X \models \phi$. All the cases are the same as those of PD as defined in Definition 1.3 except the following:

- $X \models \bot$ iff $X = \emptyset$;
- $X \models \phi \lor \psi$ iff $X \models \phi$ or $X \models \psi$;
- $X \models \phi \rightarrow \psi$ iff for any team $Y \subseteq X$,
  $$ Y \models \phi \implies Y \models \psi. $$

To simplify notation, we abbreviate $\phi \rightarrow \bot$ as $\neg \phi$ for any formula $\phi$. The reader should check that with the team semantics, the negated propositional variable $\neg p_i$ from Definition 1.3 has the same meaning as $p_i \rightarrow \bot$.

It is straightforward to verify that PID has the empty team property, the locality property and the downwards closure property defined in Section 1. As with PD, the logic PID is not closed under the Uniform Substitution rule Sub either, as, e.g.,

$$ \models \neg \neg p_i \rightarrow p_i, \text{ whereas } \not\models \neg(p_i \lor \neg p_i) \rightarrow (p_i \lor \neg p_i). $$

Moreover, we invite the reader to check that PID satisfies Deduction Theorem and Disjunction Property.

Theorem 2.3 (Deduction Theorem). For any formulas $\phi$ and $\psi$ of PID,

if $\models \psi$, then $\models \phi \rightarrow \psi$.

Theorem 2.4 (Disjunction Property). For any formulas $\phi$ and $\psi$ of PID,

if $\models \phi \lor \psi$, then $\models \phi$ or $\models \psi$.

Proof. By Corollary 1.7.

In the syntax of PID, we only include constancy dependence atoms (as with first-order intuitionistic dependence logic [1] [16]). Dependence atoms in the general form can be defined by constancy dependence atoms:

$$ = (p_i_1, \ldots, p_i_k) \equiv \left( = (p_i_1) \land \cdots \land = (p_i_{k-1}) \right) \rightarrow = (p_i_k). $$

Furthermore, constancy dependence atoms are easily definable by intuitionistic disjunction:

$$ = (p_i) \equiv p_i \lor \neg p_i. $$

As a consequence, PID is equivalent to the fragment of PID which has no occurrences of dependence atoms, i.e., the fragment of PID with the same syntax as the usual propositional intuitionistic logic. Dick de Jongh and Tadeusz Litak observed that this fragment of PID is essentially equivalent to propositional inquisitive logic (InqL), defined independently with a different motivation in [5] (see also [9] and [4]).

Inquisitive semantics, introduced by Groenendijk and Roelofsen [9] and Ciardelli [4], is a new formal framework for natural language semantics based on a notion of meaning that embodies both informative and inquisitive content and as such allows for a uniform analysis of assertions (e.g., It is raining.) and questions (e.g., Is it raining?).

\footnote{In a private conversation with the first author in September 2011.}
The basic concept is the concept of a set of valuation, which we call a team. The idea of the set of valuations is an *information state*, a set which describes the state of uncertainty about a “true valuation”. This is different from the basic intuition about a team in propositional dependence logic, but this intuition can be used in dependence logic, too.

Below we present important properties and a complete axiomatization of PID obtained essentially in [5] and [4]. For simplicity, we will stick to our notations and refer to InqL only indirectly.

Fix an \(n\)-element set \(N\) of indices, the team semantics for PID given in Definition 2.2 induces an intuitionistic Kripke model, upon which the usual single-world-based Kripke semantics for PID can easily be given, therefore all axioms of intuitionistic propositional logic (IPC) are valid in PID. Interested readers are referred to [4] for detailed discussions in the context of InqL, we only mention that the induced Kripke model has \((\mathcal{P}(2^n) \setminus \{\emptyset\}, \supseteq)\) as its underlying Kripke frame, where \(2^n\) is the maximal \(n\)-team on \(N\). Such a frame is a Medvedev frame. For this reason, the schematic fragment of InqL (or PID), i.e., the fragment of the logic that is closed under Sub, is exactly Medvedev logic, see [4] for details.

The intuitionistic negation (i.e., \(\phi \rightarrow \bot\), abbreviated as \(\neg \phi\)) of PID deserves comments. We leave it for the reader to check that negated formulas of PID are always flat, i.e., \(\neg \phi\) is flat for all formula \(\phi\) of PID. As all IPC axioms are valid for PID, the formula \(\phi \rightarrow \neg \neg \phi\) is always valid, whereas \(\neg \neg \phi \rightarrow \phi\) is in general not valid. It is proved in [4] that a formula \(\phi\) of PID (or InqL) is flat if and only if it satisfies the double negation law (i.e., \(\neg \neg \phi \leftrightarrow \phi\) is valid); in particular, \(\neg \neg p \rightarrow p\) is valid.

The next lemma shows that every \(n\)-team \(X\) is definable up to its subteams by an \(n\)-formula \(\Psi_X\) of PID (or InqL). Here and for all logics based on team semantics that have the empty team property, we stipulate \(\bigvee\emptyset := \bot\).

**Lemma 2.5** ([5]). Let \(X\) be an \(n\)-team on \(N = \{i_1, \ldots, i_n\}\). Define

\[\Psi_X := \neg \neg \bigvee_{s \in X} (p_{i_1}^{s(i_1)} \land \cdots \land p_{i_n}^{s(i_n)})\]

Then for any \(n\)-team \(Y\) on \(N\), we have that \(Y \models \Psi_X \iff Y \subseteq X\).

**Theorem 2.6** ([5]). PID (or InqL) is a maximal downwards closed logic.

**Proof.** It suffices to show that every \(K \in \mathcal{V}_X\) is definable by an \(n\)-formula of PID, for any \(n\)-element set \(N\) of natural numbers. Noting that \(K\) is finite (it has at most \(2^{2^n}\) elements), we obtain by Lemma 2.5 that

\[Y \models \bigvee_{X \in K} \Psi_X \iff \exists X \in K (Y \subseteq X) \iff Y \in K,\]

i.e., \([\bigvee_{X \in K} \Psi_X] = K\). \(\square\)

The above theorem also shows that every formula of PID (or InqL) is equivalent to a formula in the disjunctive-negative normal form:

\[\bigvee_{j \in J} \neg \neg \bigvee_{s \in X_j} (p_{i_1}^{s(i_1)} \land \cdots \land p_{i_n}^{s(i_n)}),\]  

(7)

where \(J\) is a finite set of indices and each \(X_j\) is an \(n\)-team on \(\{i_1, \ldots, i_n\}\). Interested readers are referred to [5] for more details on this normal form, we only remark here
that in the above formula in the normal form, each disjunct of the leftmost disjunction (a double negated formula) is flat.

\textbf{InqL} is shown in [5][4] to be equivalent to the negative variant of Maksimova’s logic [11], and to the negative variant of Kreisel-Putnam logic [10]. This is then also true for \textbf{PID}, in particular, the logic \textbf{PID} is complete with respect to the following Hilbert style deductive system:

\textbf{Definition 2.7} (A deductive system for \textbf{PID}, essentially due to [5]). We write $\vdash_{\text{PID}} \phi$ if the \textbf{PID} formula $\phi$ is derivable from the following axioms using the following rules:

\textbf{Axioms:}

1. all substitution instances of \textbf{IPC} axioms
2. $\neg\neg p_i \rightarrow p_i$ for all propositional variables $p_i$
3. all substitution instances of \textbf{ND}_k for all $k \in \mathbb{N}$:

   \begin{align*}
   (\text{ND}_k) \quad (\neg p_j \rightarrow \bigvee_{1 \leq i \leq k} \neg p_i) \rightarrow \bigvee_{1 \leq i \leq k} (\neg p_j \rightarrow \neg p_i).
   \end{align*}

or all substitution instances of the axiom \textbf{KP}:

\begin{align*}
(\text{KP}) \quad (\neg p_i \rightarrow (p_j \lor p_k)) \rightarrow ((\neg p_i \rightarrow p_j) \lor (\neg p_i \rightarrow p_k)).
\end{align*}

4. $\equiv(p_i) \leftrightarrow (p_i \lor \neg p_i)$

\textbf{Rules:}

\textbf{Modus Ponens:} \quad $\frac{\phi \rightarrow \psi}{\psi}$ \quad (MP)

\textbf{Theorem 2.8} (Strong Completeness Theorem, [5][4]). Let $\Gamma$ be a set of formulas and $\phi$ a formula of \textbf{PID}. Then $\Gamma \vdash_{\text{PID}} \phi$ if and only if $\Gamma \models_{\text{PID}} \phi$.

Propositional intuitionistic logic \textbf{PID} is our first step in the analysis of propositional dependence logic. Without having the constructive background of Brouwer’s intuitionism, it however has the connectives $\lor$ and $\rightarrow$ which obey intuitionistic rules. During the course of this paper we shall see that several different formalisms for propositional dependence logic, including \textbf{PD} and \textbf{PID}, are equivalent. What is striking about \textbf{PID} is that it arises naturally in the context of dependence logic and yet it is essentially the same as inquisitive logic, which has arisen from completely different considerations.

3 \textbf{Axiomatizing propositional dependence logic with intuitionistic disjunction}

We have seen in Section 2 that \textbf{PID} is a maximal downwards closed logic, and intuitionistic disjunction plays an important role in the normal form (Equation (7)) of a \textbf{PID}-formula. In this section, we study propositional dependence logic extended with intuitionistic disjunction (denoted by $\textbf{PD}^\lor$). We will define a natural deduction system for $\textbf{PD}^\lor$ and prove the completeness theorem. It is easy to verify that $\textbf{PD}^\lor$ has the downwards closure property and the empty team property. We show that intuitionistic disjunction facilitates us to give a very natural normal form for the logic $\textbf{PD}^\lor$, with which we will prove that $\textbf{PD}^\lor$ is also a maximal downwards closed logic.
In the context of first-order (quantified) dependence logic, intuitionistic disjunction of team semantics is uniformly definable using other connectives and quantifiers \cite{17}, therefore it clearly does not contribute to the expressive power of the logics. However, in the context of propositional logics of dependence, intuitionistic disjunction is not uniformly definable by other connectives \cite{4,11}.

First, we prove that as with \textbf{PID}, the logic \textbf{PD} also has a formula $\Theta_X$ that defines an $n$-team $X$ up to its subteams. Hereafter, we stipulate $\bigotimes \emptyset := \bot$.

**Lemma 3.1.** Let $X$ be an $n$-team on $N = \{i_1, \ldots, i_n\}$. Define 
\[
\Theta_X := \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \land \cdots \land p_{i_n}^{s(i_n)}).
\]
Then for any $n$-team $Y$ on $N$, we have that $Y \models \Theta_X \iff Y \subseteq X$.

**Proof.** “$\Rightarrow$”: Suppose $Y \models \Theta_X$. If $X = \emptyset$, then $\Theta_X := \bot$ and we must have that $Y = \emptyset = X$. Otherwise, if $X \neq \emptyset$, then for each $s \in X$, there exists $Y_s$ such that
\[
Y = \bigcup_{s \in X} Y_s \text{ and } Y_s \models p_{i_1}^{s(i_1)} \land \cdots \land p_{i_n}^{s(i_n)}.
\]
Then, either $Y_s = \emptyset$ or $Y_s = \{s\}$ implying $Y \subseteq X$.

“$\Leftarrow$”: By the downwards closure property, it suffices to show that $X \models \Theta_X$. Clearly, if $X = \emptyset$, then $\Theta_X := \bot$ and $X \models \Theta_X$. Otherwise, clearly, for each $s \in X$, we have that $\{s\} \models p_{i_1}^{s(i_1)} \land \cdots \land p_{i_n}^{s(i_n)}$, which implies that $X \models \Theta_X$. 

With the formula $\Theta_X$, we can prove that $\text{PD}^\lor$ is a maximal downwards closed logic by the same argument as that of the proof of Theorem 2.6.

**Theorem 3.2.** $\text{PD}^\lor$ is a maximal downwards closed logic.

**Proof.** For every (finite) $K \in \mathbb{N}$, let $K = \llbracket \bigvee_{X \in K} \Theta_X \rrbracket$. 

**Corollary 3.3.** $\text{PD}^\lor \equiv \text{PID} \equiv \text{InqL}$.

**Proof.** By Theorem 3.2 and Theorem 2.6.

As the case of \textbf{PID}, the proof of Theorem 3.2 shows that every $n$-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of $\text{PD}^\lor$ is logically equivalent to a formula in the normal form
\[
\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \land \cdots \land p_{i_n}^{s(i_n)}),
\]
where $F$ is a finite set of indices and each $X_f$ is an $n$-team on $\{i_1, \ldots, i_n\}$. It is worthwhile to point out that a similar normal form (of typically infinite size) for first-order dependence logic extended with intuitionistic disjunction was suggested already in [1]. The formula in the normal form (8) does not contain dependence atoms, this means that dependence atoms are expressible in $\text{PD}^\lor$. A direct translation is:
\[
=(p_{j_0}, \ldots, p_{j_k}) \equiv \bigotimes_{s \in 2^m} \left( p_{j_0}^{s(j_0)} \land \cdots \land p_{j_{k-1}}^{s(j_{k-1})} \land (p_{j_k} \lor \neg p_{j_k}) \right),
\]
where $2^m$ is the maximal $m$-team on $M = \{j_0, \ldots, j_{k-1}\}$, where $m$ is the size of $M$.

We now present a natural deduction system for $\text{PD}^\lor$ for which the normal form (8) can be obtained proof-theoretically. The main goal of the remaining part of the present section is to prove the completeness theorem for this system.
**Definition 3.4** (A natural deduction system for PD). The rules are given as follows:

1. Conjunction introduction: \( \frac{\phi \quad \psi}{\phi \land \psi} (\land I) \)
2. Conjunction elimination: \( \frac{\phi \land \psi}{\phi} (\land E) \quad \frac{\phi \land \psi}{\psi} (\land E) \)
3. Intuitionistic disjunction introduction: \( \frac{\phi}{\phi \lor \psi} (\lor I) \quad \frac{\psi}{\phi \lor \psi} (\lor I) \)
4. Intuitionistic disjunction elimination:
   \[
   \frac{\phi \quad \psi}{\chi \quad \chi} (\lor E)
   \]
5. Tensor disjunction introduction: \( \frac{\phi}{\phi \otimes \psi} (\otimes I) \)
6. Weak tensor disjunction elimination:
   \[
   \frac{\phi \quad \psi}{\chi \quad \chi} (\otimes WE)
   \]
   whenever \( \chi \) is a classical formula.
7. Tensor disjunction substitution:
   \[
   \frac{\psi}{\phi \otimes \chi \quad \chi} (\otimes Sub)
   \]
8. Commutative and associative laws for tensor disjunction: \( (\otimes \text{Com}) \)
   \( \frac{\phi \otimes (\psi \otimes \chi)}{(\phi \otimes \psi) \otimes \chi} (\otimes \text{Ass}) \)
9. Contradiction elimination: \( \frac{\phi \otimes (p_i \land \neg p_i)}{\phi} (\bot E) \)
10. Atomic excluded middle: \( \frac{p_i \otimes \neg p_i}{(EM_0)} \)
11. Dependence atom introduction: for any \( f : 2^m \rightarrow 2, \)
   \[
   \frac{\bigotimes_{s \in 2^m} (p_{j_0}^{s(j_0)} \land \cdots \land p_{j_k}^{s(j_{k-1})} \land p_{j_k}^{f(s)})}{=(p_{j_0}, \cdots, p_{j_{k-1}}, p_{j_k})} (\text{Depl})
   \]
where $2^m$ is the maximal $m$-team on the $m$-element set $\{j_0, \ldots, j_{k-1}\}$.

12. Dependence atom elimination:

$$= (p_{j_0} \cdots p_{j_{k-1}}, p_{j_k})$$

$$(\text{DepE})$$

where $2^m$ is the maximal $m$-team on the $m$-element set $\{j_0, \ldots, j_{k-1}\}$.

13. Distributive laws:

$$\phi \otimes (\psi \lor \chi)$$

$$(\text{Dstr} \otimes \lor)$$

$$\phi \land (\psi \otimes \chi)$$

$$(\text{Dstr} \land \otimes)$$

If a formula $\phi$ of $\mathbf{PD}^{\lor}$ is derivable in the system, then we write $\vdash_{\mathbf{PD}^{\lor}} \phi$ or simply $\vdash \phi$. If $\phi \vdash \psi$ and $\psi \vdash \phi$, then we say that $\phi$ and $\psi$ are provably equivalent, in symbols $\phi \equiv \psi$.

In the above system, all of the substitution, commutative, associative and distributive rules involving tensor disjunction $\otimes$ are necessary, as we only have the weak elimination rule for tensor disjunction ($\otimes$WE), whereas the strong elimination rule:

$$\phi \otimes (\psi \lor \chi)$$

$$(\text{Dstr} \otimes \lor)$$

is not valid (since e.g. $=(p_i) \otimes =(p_i) \not \equiv =(p_i)$). Moreover, not all usual distributive laws are valid in $\mathbf{PD}^{\lor}$, for example, the following distributive law:

$$\phi \lor (\psi \otimes \chi)$$

$$(\text{Dstr} \lor \otimes)$$

is not valid even for classical formulas, since, e.g., $(p_i \lor \neg p_i) \otimes (p_i \lor \neg p_i) \not \equiv p_i \lor (\neg p_i \otimes \neg p_i)$.

Interesting derivable rules of the deductive system are listed in the next corollary.

**Corollary 3.5.** The following are derivable rules:

1. *Ex falso:* $\frac{p_i \land \neg p_i}{\phi}$ (ex falso)

2. *Distributive laws:*

$$\frac{\phi \otimes (\psi \lor \chi)}{(\phi \otimes \psi) \land (\phi \otimes \chi)}$$

$$(\text{Dstr} \otimes \land)$$

$$\frac{\phi \lor (\psi \otimes \chi)}{(\phi \lor \psi) \otimes (\phi \lor \chi)}$$

$$(\text{Dstr} \lor \otimes)$$

$$\frac{(\phi \otimes \psi) \land (\phi \otimes \chi)}{\phi \otimes (\psi \land \chi)}$$

$$(\ast)$$

$$(\text{Dstr}^{\ast} \otimes \land)$$

$$\frac{\phi \land (\psi \lor \chi)}{\phi \land (\psi \lor \chi)}$$

$$(\ast)$$

$$(\text{Dstr}^{\ast} \land \land)$$
3. Commutative and associative rules for conjunction and intuitionistic disjunction:

\[
\begin{align*}
\frac{\phi \land \psi}{\psi \land \phi} & \quad \text{(Com)} \\
\frac{\phi \lor \psi}{\psi \lor \phi} & \quad \text{(Com)} \\
\frac{(\phi \land \psi) \land \chi}{\phi \land (\psi \land \chi)} & \quad \text{(Ass)} \\
\frac{(\phi \lor \psi) \lor \chi}{\phi \lor (\psi \lor \chi)} & \quad \text{(Ass)}
\end{align*}
\]

4. Substitution rules for intuitionistic disjunction and conjunction:

\[
\begin{align*}
\frac{[\psi]}{\phi \lor \chi} & \quad \text{(\lor Sub)} \\
\frac{\phi \land \psi}{\chi} & \quad \text{(\land Sub)}
\end{align*}
\]

5. Distributive laws for intuitionistic disjunction and conjunction:

\[
\begin{align*}
\frac{\phi \land (\psi \lor \chi)}{(\phi \land \psi) \lor (\phi \land \chi)} & \quad \text{(Dstr)} \\
\frac{\phi \lor (\psi \land \chi)}{(\phi \lor \psi) \land (\phi \lor \chi)} & \quad \text{(Dstr)} \\
\frac{\phi \lor (\psi \lor \chi)}{(\phi \lor \psi) \lor (\phi \lor \chi)} & \quad \text{(Dstr)}
\end{align*}
\]

Proof. Items 3-5 are derived as usual. It remains to derive all the other rules.

For (ex falso):

\[
\begin{align*}
\frac{p_i \land \neg p_i}{\phi} & \quad \text{(\land I)} \\
\frac{\phi \land \psi}{\psi} & \quad \text{(\land E)}
\end{align*}
\]

For (Dstr \land):

\[
\begin{align*}
\frac{\phi \land (\psi \land \chi)}{\phi \land \psi} & \quad \text{(\land E)} \\
\frac{\phi \land \psi}{\phi \land \chi} & \quad \text{(\land E)}
\end{align*}
\]

For (Dstr \lor)

\[
\begin{align*}
\frac{\phi \lor \psi}{\phi \lor \chi} & \quad \text{(\lor I)} \\
\frac{\phi \lor \psi \land (\phi \lor \chi)}{(\phi \lor \psi) \lor (\phi \lor \chi)} & \quad \text{(\lor E)}
\end{align*}
\]

For (Dstr \land \lor):

\[
\begin{align*}
\frac{(\phi \land \psi) \land (\phi \land \chi)}{(\phi \land \psi) \land \phi} & \quad \text{(\land E, \land Sub)} \\
\frac{(\phi \land \psi) \land \chi}{(\phi \land \psi) \land \phi} & \quad \text{(\land E, \land Sub)}
\end{align*}
\]

For (Dstr \land \lnot):

\[
\begin{align*}
\frac{(\phi \land \psi) \land \lnot p_i}{\phi \land \psi} & \quad \text{(\land I, \lnot p_i)}
\end{align*}
\]

For (Dstr \lor \land):

\[
\begin{align*}
\frac{(\phi \lor \psi) \land (\phi \lor \chi)}{(\phi \lor \psi) \land \phi} & \quad \text{(\lor I, \land Sub)} \\
\frac{(\phi \lor \psi) \land \chi}{(\phi \lor \psi) \land \phi} & \quad \text{(\lor I, \land Sub)}
\end{align*}
\]

(*) whenever \(\phi\) is a classical formula.
For (Dstr$^\land \land$): If $\phi$ is a classical formula, then we have the following derivation:

\[
(\phi \land \psi) \circ (\phi \land \chi) \quad (\landE, \circSub)
\]
\[
\phi \circ \phi \quad (\circWE)
\]
\[
\phi \quad (\circSub)
\]
\[
\phi \land (\psi \circ \chi) \quad (\landI)
\]

Next, we prove the soundness theorem for the above deductive system.

**Theorem 3.6 (Soundness Theorem).** For any formulas $\phi$ and $\psi$ of $\mathbf{PD}^V$,

\[
\phi \vdash \psi \implies \phi \models \psi.
\]

**Proof.** It suffices to show that all of the deductive rules are valid. The rules 1-5, 7-10 are easy to verify. The validity of (Depl) and (DepE) follows from Equation (9). It remains to verify the validity of the rules 6 and 13.

For (\circWE), it suffices to show that $[\phi \models \chi$ and $\psi \models \chi] \implies \phi \circ \psi \models \chi$, whenever $\chi$ is a classical formula (thus flat). For any team $X$ such that $X \models \phi \circ \psi$, there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$. Since $\phi \models \chi$ and $\psi \models \chi$, we have that $Y \models \chi$ and $Z \models \chi$, which imply that $X \models \chi$, as $\chi$ is flat.

For (Dstr $\circ \lor$), it suffices to show that $\phi \circ (\psi \lor \chi) \models (\phi \circ \psi) \lor (\phi \circ \chi)$. For any team $X$ such that $X \models \phi \circ (\psi \lor \chi)$, there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi \lor \chi$. It follows that $Y \cup Z \models \phi \circ \psi$ or $Y \cup Z \models \phi \circ \chi$, hence $X \models (\phi \circ \psi) \lor (\phi \circ \chi)$.

For (Dstr $\land \circ$), it suffices to show that $\phi \land (\psi \circ \chi) \models (\phi \land \psi) \circ (\phi \land \chi)$. For any team $X$ such that $X \models \phi \land (\psi \circ \chi)$, we have that $X \models \phi$ and $X \models \psi \circ \chi$. The latter implies that there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \psi$ and $Z \models \chi$. By the downwards closure property, $Y \models \phi$ and $Z \models \phi$. It follows that $Y \models \phi \land \psi$ and $Z \models \phi \land \chi$, thus $X \models (\phi \land \psi) \circ (\phi \land \chi)$.

For (Dstr $\circ \lor$), it suffices to show that $(\phi \circ \psi) \lor (\phi \circ \chi) \models (\phi \circ \psi \lor \chi)$. For any team $X$ such that $X \models \phi \circ \psi \lor \chi$, we have that $X \models \phi \circ \psi$ or $X \models \phi \circ \chi$. In the former case, there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$, which implies that $Z \models \psi \lor \chi$, thereby $X \models \phi \circ (\psi \lor \chi)$. By a similar argument, one derives $X \models \phi \circ (\psi \lor \chi)$ in the latter case as well.

Next, we show that every $\mathbf{PD}^V$ formula is provably equivalent to a formula in the intended normal form (8).

**Theorem 3.7.** Any $n$-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of $\mathbf{PD}^V$ is provably equivalent to a formula of the normal form

\[
\bigvee_{f \in F} \bigotimes_{x \in X_f} (p^{s_{f(i_1)}}_{i_1} \land \cdots \land p^{s_{f(i_n)}}_{i_n}),
\]

where $F$ is a finite set of indices, and each $X_f$ is an $n$-team on $N = \{i_1, \ldots, i_n\}$.

**Proof.** We first prove the following claim:

**Claim.** If $\{i_1, \ldots, i_n\} \subseteq \{j_1, \ldots, j_n\} \subseteq \mathbb{N}$, then any $m$-formula $\psi(p_{j_1}, \ldots, p_{j_m})$ in the normal form is provably equivalent to an $n$-formula $\theta(p_{j_1}, \ldots, p_{j_n})$ in the normal form.
Proof of the claim. Without loss of generality, we may assume that $N = \{j_1, \ldots, j_n\} = \{i_1, \ldots, i_m, i_{m+1}, \ldots, i_n\}$. If $n = m$, then the claim holds trivially. Now assume $n > m$ and

$$
\psi(p_{i_1}, \ldots, p_{i_m}) = \bigvee_{f \in F} \bigotimes_{s \in X_f} (p^{s(i_1)} \land \ldots \land p^{s(i_m)}),
$$

where $F$ is a finite set of indices, and each $X_f$ is an $m$-team on $M = \{i_1, \ldots, i_m\}$. Let

$$
\theta(p_{i_1}, \ldots, p_{i_m}) = \bigvee_{f \in F} \bigotimes_{s \in 2^n} (p^{s(i_1)} \land \ldots \land p^{s(i_m)}),
$$

where $2^n$ is the maximal $n$-team on $N$.

The following derivation proves $\psi \vdash \theta$:

1. $\bigvee_{f \in F} \bigotimes_{s \in X_f} (p^{s(i_1)} \land \ldots \land p^{s(i_m)})$

2. $(p_{i_{m+1}} \land \neg p_{i_{m+1}}) \land \ldots \land (p_n \land \neg p_n)$ (EM0, $\land$)

3. $\bigvee_{f \in F} \left( (\bigotimes_{s \in X_f} (p^{s(i_1)} \land \ldots \land p^{s(i_m)})) \land (p_{i_{m+1}} \land \neg p_{i_{m+1}}) \land \ldots \land (p_n \land \neg p_n) \right)$

   (1), (2), $\land$, Dstr

4. $\bigvee_{f \in F} \bigotimes_{s \in 2^n} (p^{s(i_1)} \land \ldots \land p^{s(i_m)} \land p^{s(i_{m+1})} \land \ldots \land p^{s(i_n)})$ (Dstr $\land$, $\otimes$, $\otimes$ Sub)

Conversely, $\theta \vdash \psi$ is proved by the following derivation:

1. $\bigvee_{f \in F} \bigotimes_{s \in 2^n} (p^{s(i_1)} \land \ldots \land p^{s(i_m)} \land p^{s(i_{m+1})} \land \ldots \land p^{s(i_n)})$

2. $\bigvee_{f \in F} \bigotimes_{s \in 2^n} (p^{s(i_1)} \land \ldots \land p^{s(i_m)})$ ($\land$, $\otimes$ Sub)

3. $\bigvee_{f \in F} \bigotimes_{s \in X_f} (p^{s(i_1)} \land \ldots \land p^{s(i_m)})$ ($\otimes$ WE)

We now prove the theorem by induction on $\phi(p_{i_1}, \ldots, p_{i_n})$.

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = p_{i_k}$. Obviously $p_{i_k} \vdash \bigotimes_{s \in \{1\}} p^{s(i_k)}$, where the function $1: \{i_k\} \to \{0, 1\}$ is defined as $1(i_k) = 1$. Then by the Claim, the 1-formula $p_{i_k}$ is provably equivalent to an $n$-formula $\theta(p_{i_1}, \ldots, p_{i_n})$ in the normal form.

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = \neg p_{i_k}$ is proved analogously to the above case.

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = \rightarrow(p_{j_0}, \ldots, p_{j_k})$. Let $\{j_0, \ldots, j_{k-1}\} = \{\xi_1, \ldots, \xi_m\} (m \leq k)$, where $\xi_1, \ldots, \xi_m$ are $m$ distinct indices. By the Claim, it suffices to prove that the $m$ or $(m + 1)$-formula $\rightarrow(p_{j_0}, \ldots, p_{j_k})$ is provably equivalent to a formula $\eta$ in the normal form.
We first show that \( (p_{j_0}, \cdots, p_{j_k}) \vdash \eta_0 \), where

\[
\eta_0 := \bigvee_{f \in 2^m} \bigotimes_{s \in 2^m} \left( p_{j_0}^{s_{j_0}} \land \cdots \land p_{j_{k-1}}^{s_{j_{k-1}}} \land p_{j_k}^{f(s)} \right),
\]

where \( 2^m \) is the maximal \( m \)-element set on the \( m \)-element set \( \{ \xi_1, \ldots, \xi_m \} \). The direction \( \eta_0 \vdash (p_{j_0}, \cdots, p_{j_k}) \) is derived easily by applying (Depl) to each \( \lor \)-disjunct of \( \eta_0 \) and (\( \lor \)E). For the other direction \( (p_{j_0}, \cdots, p_{j_k}) \vdash \eta_0 \), we have the following derivation:

1. \( \bigotimes_{s \in 2^m} (p_{j_0}^{s_{j_0}} \land \cdots \land p_{j_{k-1}}^{s_{j_{k-1}}} \land (p_{j_k} \lor \neg p_{j_k})) \) (DepE)
2. \( \bigotimes_{s \in 2^m} (p_{j_0}^{s_{j_0}} \land \cdots \land p_{j_{k-1}}^{s_{j_{k-1}}} \land (p_{j_k} \land \neg p_{j_k})) \) (Dstr)
3. \( \bigotimes_{s \in 2^m} (p_{j_0}^{s_{j_0}} \land \cdots \land p_{j_{k-1}}^{s_{j_{k-1}}} \land (p_{j_k}^{s_{j_k}} \land \cdots \land p_{j_{k-1}}^{s_{j_{k-1}}} \land \neg p_{j_k})) \) (Dstr)
4. \( \bigotimes_{s \in 2^m} (p_{j_0}^{s_{j_0}} \land \cdots \land p_{j_{k-1}}^{s_{j_{k-1}}} \land p_{j_k}^{f(s)}) \) (Dstr \( \otimes \) \( \lor \))

Note that \( \eta_0 \) is not yet in the required normal form, so in the next step, we turn \( \eta_0 \) into a formula \( \eta \) in the normal form. If \( j_k \notin \{ \xi_1, \ldots, \xi_m \} \), then put \( \xi_{m+1} = j_k \) and let

\[
\eta := \bigvee_{f \in 2^m} \bigotimes_{s \in 2^m} (p_{j_1}^{s_{j_1}} \land \cdots \land p_{j_{m}}^{s_{j_{m}}}) \land p_{j_{m+1}}^{f(s)},
\]

where \( 2^m \) can easily be viewed as an \( (m + 1) \)-team \( X_f = \{ s \cup \{ (\xi_{m+1}, f(s) \} \mid s \in 2^m \} \) on the \( (m + 1) \)-element set \( \{ \xi_1, \ldots, \xi_{m+1} \} \). Now, one derives \( \eta_0 \vdash \eta \) by (\( \land \)E), (\( \land \)I), (\( \otimes \)Sub).

If \( j_k \in \{ \xi_1, \ldots, \xi_m \} \), w.l.o.g. we assume \( j_k = \xi_m \) and let

\[
\eta := \bigvee_{s(\xi_m) = f(s)} \bigotimes_{s(\xi_m) = f(s)} (p_{j_1}^{s_{j_1}} \land \cdots \land p_{j_{m-1}}^{s_{j_{m-1}}} \land p_{j_m}^{f(s)}).
\]

Now, one derives \( \eta_0 \vdash \eta \) by (\( \lor \)E), (\( \lor \)Sub), and \( \eta \vdash \eta_0 \) is derived by (\( \otimes \)I), (\( \land \)I), (\( \otimes \)Sub).

Case \( \phi(p_{i_1}, \ldots, p_{i_n}) = \psi(p_{i_1}, \ldots, p_{i_n}) \lor \chi(p_{i_1}, \ldots, p_{i_n}) \). By induction hypothesis, we have that

\[
\psi(p_{i_1}, \ldots, p_{i_n}) \vdash \bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s_{i_1}(f)} \land \cdots \land p_{i_n}^{s_{i_n}(f)}),
\]

\[
\chi(p_{i_1}, \ldots, p_{i_n}) \vdash \bigvee_{g \in G} \bigotimes_{s \in X_g} (p_{i_1}^{s_{i_1}(g)} \land \cdots \land p_{i_n}^{s_{i_n}(g)}),
\]

where each \( X_f, X_g \subseteq 2^m \). If \( F, G \neq \emptyset \), then it follows from (\( \lor \)E) and (\( \lor \)I) that

\[
\psi \lor \chi \vdash \bigvee_{h \in F \lor G} \bigotimes_{s \in X_h} (p_{i_1}^{s_{i_1}(h)} \land \cdots \land p_{i_n}^{s_{i_n}(h)}).
\]

If \( F = \emptyset \) or \( G = \emptyset \), then \( \psi \vdash \neg p_{i_1} \lor \neg p_{i_1} \) or \( \chi \vdash \neg p_{i_1} \land \neg p_{i_1} \). In the former case, we derive by (ex falso), (\( \lor \)E) and (\( \lor \)I) that

\[
\psi \lor \chi \vdash \bigvee_{g \in G} \bigotimes_{s \in X_g} (p_{i_1}^{s_{i_1}(g)} \land \cdots \land p_{i_n}^{s_{i_n}(g)}).
\]

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Similarly, in the latter case, we derive $\psi \lor \chi \vdash \bigvee_{f \in F, s \in X_f} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)})$.

Similarly, in the latter case, we derive $\psi \lor \chi \vdash \bigvee_{f \in F, s \in X_f} (p_{i_1}^{s_0(1)} \land \cdots \land p_{i_n}^{s_0(n)})$.

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = \psi(p_{i_1}, \ldots, p_{i_n}) \otimes \chi(p_{i_1}, \ldots, p_{i_n})$. By induction hypothesis, we have (11). If $F = \emptyset$ or $G = \emptyset$, then $\psi \vdash p_{i_1} \land \neg p_{i_1}$ or $\chi \vdash p_{i_1} \land \neg p_{i_1}$. In the former case, we derive by ($\land E$) and ($\otimes I$) that

$$
\psi \otimes \chi \vdash (p_{i_1} \land \neg p_{i_1}) \otimes \left( \bigvee_{g \in G, s \in X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)}) \right)
$$

Similarly, in the latter case, we derive $\psi \otimes \chi \vdash \bigvee_{f \in F, s \in X_f} (p_{i_1}^{s_0(1)} \land \cdots \land p_{i_n}^{s_0(n)})$.

Now, assume $F, G \neq \emptyset$. We show that $\psi \otimes \chi \vdash \theta$, where

$$
\theta := \bigvee_{(f, g) \in F \times G, s \in X_f \cup X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)}).
$$

For $\psi \otimes \chi \vdash \theta$, we have the following derivation:

1. $\psi \otimes \chi$
2. $\left( \bigvee_{f \in F, s \in X_f} (p_{i_1}^{s_0(1)} \land \cdots \land p_{i_n}^{s_0(n)}) \right) \otimes \left( \bigvee_{g \in G, s \in X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)}) \right)$
3. $\bigvee_{f \in F} \left( \bigotimes_{s \in X_f} (p_{i_1}^{s_0(1)} \land \cdots \land p_{i_n}^{s_0(n)}) \otimes \Big( \bigvee_{g \in G, s \in X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)}) \Big) \right)$

(Dstr $\otimes \lor$)

4. $\bigvee_{f \in F, g \in G} \left( \bigotimes_{s \in X_f \cup X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)}) \otimes \bigotimes_{s \in X_f \cup X_g} (p_{i_1}^{s_0(1)} \land \cdots \land p_{i_n}^{s_0(n)}) \right)$

(Dstr $\otimes \lor$)

5. $\bigvee_{(f, g) \in F \times G, s \in X_f \cup X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)})$ ($\otimes WE$)

The other direction $\theta \vdash \psi \otimes \chi$ is proved conversely using ($\otimes I$) and (Dstr $\otimes \lor \otimes$).

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = \psi(p_{i_1}, \ldots, p_{i_n}) \land \chi(p_{i_1}, \ldots, p_{i_n})$. By induction hypothesis, we have (11). If $F = \emptyset$ or $G = \emptyset$, then $\psi \vdash p_{i_1} \land \neg p_{i_1}$ or $\chi \vdash p_{i_1} \land \neg p_{i_1}$. In this case, we derive $\psi \land \chi \vdash \emptyset$, i.e., $\psi \land \chi \vdash \emptyset$, by ($\land E$) and (ex falso).

Now, assume $F, G \neq \emptyset$. We show that $\psi \land \chi \vdash \theta$, where

$$
\theta := \bigvee_{(f, g) \in F \times G, s \in X_f \cup X_g} (p_{i_1}^{s_1(1)} \land \cdots \land p_{i_n}^{s_1(n)}).
$$

For $\psi \land \chi \vdash \theta$, we have the following derivation:

1. $\psi \land \chi$
Now, we are in a position to prove the completeness theorem.

The next lemma is crucial in proof of the completeness theorem.

Lemma 3.8. For any non-empty finite collections \( \{ X_f \mid f \in F \} \), \( \{ Y_g \mid g \in G \} \) of \( n \)-teams on an \( n \)-element set \( N \subseteq \mathbb{N} \), the following are equivalent:

(a) \( \bigvee_{f \in F} \Theta X_f \models \bigvee_{g \in G} \Theta Y_g \);

(b) for each \( f \in F \), we have that \( X_f \subseteq Y_{g_f} \) for some \( g_f \in G \).

Proof. (a)\( \Rightarrow \) (b): For each \( f_0 \in F \), by Lemma [3.1], \( X_{f_0} \models \Theta X_{f_0} \), thus \( X_{f_0} \models \bigvee_{f \in F} \Theta X_f \), which by (a) implies that \( X_{f_0} \models \bigvee_{g \in G} \Theta Y_g \). It follows that there exists \( g_{f_0} \in G \) such that \( X_{f_0} \models \Theta Y_{g_{f_0}} \). Hence by Lemma [3.1], \( X_{f_0} \subseteq Y_{g_{f_0}} \).

(b)\( \Rightarrow \) (a): Suppose \( X \) is any \( n \)-team on \( N \) satisfying \( X \models \bigvee_{f \in F} \Theta X_f \). Then \( X \models \Theta X_f \) for some \( f \in F \), which by Lemma [3.1] and (b) means that \( X \subseteq X_f \subseteq Y_{g_f} \) for some \( g_f \in G \). Since by Lemma [3.1], \( Y_{g_f} \models \Theta Y_{g_f} \), it follows from the downwards closure property that \( X \models \Theta Y_{g_f} \), thereby \( X \models \bigvee_{g \in G} \Theta Y_g \), as required.

The next lemma is crucial in proof of the completeness theorem.

Theorem 3.9 (Completeness Theorem). For any \( PD^V \) formulas \( \phi \) and \( \psi \),

\[ \phi \models \psi \iff \phi \vdash \psi. \]
Proof. Suppose $\phi \models \psi$, where $\phi = \phi(p_{i_1}, \ldots, p_{i_n})$ and $\psi = \psi(p_{i_1}, \ldots, p_{i_n})$. By Theorem 3.7, we have that

$$\phi \vdash \bigvee_{f \in F} \Theta_{X_f}, \quad \psi \vdash \bigvee_{g \in G} \Theta_{Y_g}.$$ 

for some finite sets $\{X_f \mid f \in F\}$ and $\{Y_g \mid g \in G\}$ of $n$-teams on $\{i_1, \ldots, i_n\}$. Then, by the soundness theorem, we have that

$$\bigvee_{f \in F} \Theta_{X_f} \models \bigvee_{g \in G} \Theta_{Y_g}.$$ 

If $F = \emptyset$, then $\phi \vdash \neg p_{i_1}$, thus, by (ex falso), we obtain that $\phi \vdash \psi$. If $G = \emptyset$, then $\psi \vdash \neg p_{i_1}$. By Lemma 3.1, either $F = \emptyset$ or $X_f = \emptyset$ for all $f \in F$. In both cases, we have that $\phi \vdash \neg p_{i_1}$, hence $\phi \vdash \psi$.

If $F, G \neq \emptyset$, then by Lemma 3.8 for each $f \in F$, we have that $X_f \subseteq Y_{f|f}$ for some $g_{f|f} \in G$. Thus, we have the following derivation:

1. $\bigwedge_{s \in X_f} (p^{s(i_1)} \land \cdots \land p^{s(i_n)})$
2. $\bigwedge_{s \in Y_{f|f}} (p^{s(i_1)} \land \cdots \land p^{s(i_n)})$ (⊗1)
3. $\bigwedge_{s \in Y_{f|f}} (p^{s(i_1)} \land \cdots \land p^{s(i_n)})$
4. $\bigwedge_{s \in Y_{f|f \setminus X_f}} (p^{s(i_1)} \land \cdots \land p^{s(i_n)})$
5. $\bigwedge_{g \in G} \Theta_{Y_g}$ (⊗1)
6. $\bigvee_{g \in G} \Theta_{Y_g}$

Thus $\Theta_{X_f} \vdash \bigvee_{g \in G} \Theta_{Y_g}$ for each $f \in F$, which by (∨E) implies $\bigvee_{f \in F} \Theta_{X_f} \vdash \bigvee_{g \in G} \Theta_{Y_g}$, namely $\phi \vdash \psi$.

Theorem 3.10 (Strong Completeness Theorem). Let $\Gamma$ be a set of formulas and $\phi$ a formula of $\text{PD}^\lor$. Then

$$\Gamma \vdash \phi \iff \Gamma \models \phi.$$ 

Proof. Since $\text{PID}$ is compact [4] and $\text{PD}^\lor \equiv \text{PID}$ (by Corollary 3.3), we know that $\text{PD}^\lor$ is also compact. Now, assuming $\Gamma \models \phi$, by compactness we have that $\Delta \models \phi$ for some finite $\Delta \subseteq \Gamma$, which by Theorem 3.9 implies $\Delta \vdash \phi$, thereby $\Gamma \vdash \phi$.

Propositional dependence logic with disjunction, $\text{PD}^\lor$, has now been proved equivalent to $\text{PID}$, and hence equivalent to inquisitive logic. However, it is built on different logical operations, namely the operations of the “basic” propositional dependence logic $\text{PD}$ plus intuitionistic disjunction. Accordingly it has its own proof system centred around its own connectives. We now turn our attention to $\text{PD}$ and show that it is likewise equivalent to $\text{PID}$ and it has, likewise, its own characteristic complete deductive system.
4 Axiomatizing propositional dependence logic

In this section, we study the expressive power of propositional dependence PD logic, show that it is equivalent to PID, and give it a complete axiomatization based on the method used in the previous section for PD∨.

Apparently PD is less expressive than PD∨, as it has less connectives. However, PD is in fact also a maximal downwards closed logic. This result is due to Taneli Huuskonen.

Theorem 4.1 (T. Huuskonen). PD is a maximal downwards closed logic.

Proof. We present the proof of Huuskonen with his kind permission. It suffices to show that for every set \(N = \{i_1, \ldots, i_n\} \subseteq \mathbb{N}\), every collection \(K \in \nabla_N\), there is an \(n\)-formula \(\phi(p_{i_1}, \ldots, p_{i_n})\) of PD such that \(K = \llbracket \phi \rrbracket\).

Define formulas \(\alpha_m\) for each \(m \in \mathbb{N}\) as follows:

- \(\alpha_0 := p_{i_1} \land \neg p_{i_1}\);
- \(\alpha_1 := (p_{i_1}) \land \cdots \land (p_{i_n})\);
- \(\alpha_m := \bigotimes_{i=1}^m \alpha_1\), for \(m \geq 2\).

Claim 1. \(X \models \alpha_m \iff |X| \leq m\), for any \(n\)-team \(X\) on \(N\).

Proof of Claim 1. Clearly, \(X \models \alpha_0 \iff X = \emptyset \iff |X| \leq 0\), and \(X \models \alpha_1 \iff |X| \leq 1\). If \(m > 1\), then we have that

\[
X \models \alpha_m \iff X = \bigcup_{i=1}^m X_i\text{ with } X_i \models \alpha_1\text{ for each } i
\]

\[
\iff X = \bigcup_{i=1}^m X_i\text{ with } |X_i| \leq 1\text{ for each } i
\]

\[
\iff |X| \leq m.
\]

Let \(Y\) be a non-empty \(n\)-team with \(|Y| = k + 1\) and \(2^n\) the maximal \(n\)-team on \(N\).

By Lemma 3.1

\[
X \models \Theta_2^n \iff X \subseteq 2^n \setminus Y \iff X \cap Y = \emptyset.
\]

\[(12)\]

Define \(\Theta_Y := \alpha_k \otimes \Theta_2^n \setminus Y\).

Claim 2. \(X \models \Theta_Y \iff Y \subseteq X\), for any \(n\)-team \(X\) on \(N\).

Proof of Claim 2. We have that

\[
X \models \Theta_Y \iff X = U \cup V\text{ such that } U \models \alpha_k\text{ and } V \models \Theta_2^n \setminus Y
\]

\[
\iff X = U \cup V\text{ such that } |U| \leq k\text{ and } V \cap Y = \emptyset
\]

(by Claim 1 and (12)).

(*)
If $Y \not\subseteq X$, then $|Y \setminus X| \geq 1$. We have that $X = (Y \cap X) \cup (X \setminus Y)$. Clearly $(X \setminus Y) \cap Y = \emptyset$. On the other hand,

$$|Y \cap X| = |Y \setminus (Y \setminus X)| = |Y| - |Y \setminus X| \leq (k + 1) - 1 = k.$$  

Thus, by $(*)$, we conclude that $X \models \Theta^*_Y$.

Conversely, suppose $X \models \Theta^*_Y$. By $(*)$, $X = U \cup V$ such that $|U| \leq k < k + 1 = |Y|$ and $V \cap Y = \emptyset$. It follows that there exists $s \in Y$ such that $s \notin U \cup V = X$, thus $Y \not\subseteq X$, as required.

Now, let $K \in \mathbb{N}_N$. Consider the finite collection

$$2^n \setminus K = \{ Y_j \mid j \in J \}$$

of $n$-teams on $N$ which are not in $K$.

**Claim 3.** For any $n$-team $X$ on $N$,

$$X \in K \iff Y_j \not\subseteq X \text{ for all } j \in J.$$

**Proof of Claim 3.** If $X \notin K$, then by definition, $X = Y_{j_0}$ for some $j_0 \in J$, so $Y_{j_0} \subseteq X$. Conversely, if $X \in K$, then as $K$ is downwards closed, for all $Y \subseteq X$, $Y \in K$. Thus for all $Y_j \notin K$, we must have that $Y_j \not\subseteq X$.

Finally, since $\emptyset \in K$, we have that $Y_j \neq \emptyset$ for any $j \in J$. Hence by Claim 2 and Claim 3, we obtain that for any $n$-team $X$ on $N$,

$$X \models \bigwedge_{j \in J} \Theta^*_Y \iff Y_j \not\subseteq X \text{ for all } j \in J \iff X \in K,$$

i.e., $K = \{ \bigwedge_{j \in J} \Theta^*_Y \}$, as required. \qed

**Corollary 4.2.** $\text{PD} \equiv \text{PD}^\forall \equiv \text{PID} \equiv \text{InqL}$.

**Proof.** By Theorem 4.1 and Corollary 3.3. \qed

Propositional logic can be viewed as first-order logic over a first-order model $2$ with a two-element domain $\{0, 1\}$. An $n$-valuation on $N = \{i_1, \ldots, i_n\}$ can be viewed as a first-order assignment from an $n$-element set $\{x_{i_1}, \ldots, x_{i_n}\}$ of first-order variables into the first-order model $2$. In this sense, Corollary 4.2 shows that for a fixed number $n$, propositional dependence logic (viewed as a restricted first-order dependence logic) can characterize all downwards closed collections $K$ of first-order teams of $2$ with an $n$-element domain. These collections $K$ are called $(2, n)$-suits in [3], and the function value $f(2, n)$ represents the number of distinct collections $K$ for the fixed $n$. This way the counting result in [3] shows that there is no compositional semantics for PD in which the semantic truth set of an $n$-formula $\phi$ is a subset of $2^n$, justifying that the team semantics given in this paper is indeed an appropriate compositional semantics for PD.

The proof of Theorem 4.1 shows that every $n$-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of PD is logically equivalent to a formula in the normal form $\bigwedge_{j \in J} \Theta^*_Y$, where $J$ is a finite set of indices and each $Y_j$ is an $n$-team on $\{i_1, \ldots, i_n\}$. Using this normal form, one can define a natural deduction system for PD and prove the completeness theorem. However, as this normal form is complex, we will not take this approach to the axiomatization of
PD. Instead, we define a similar natural deduction system with that of PD* and prove the completeness theorem by a similar technique.

One crucial idea in the axiomatization of PD* is that every formula in the logic is semantically equivalent to a formula in the disjunctive normal form \( \lor_{f \in \Phi} \Theta_X \), and the deductive system of PD* allows us to derive the normal form proof-theoretically as well. To axiomatize PD, we will also make use of this disjunctive normal form, but apparently this formula is not a formula in the language of PD, as intuitionistic disjunction is not a legitimate connective for PD. Nevertheless, we will develop a natural deduction system for PD in which one derives in effect the disjunctive normal form for every formula without actually using the intuitionistic disjunction.

A valid PD-formula \( \phi \) is also a valid formula of the logic PD*, thus it can be derived in the natural deduction system of PD*. If \( \phi \) does not contain any dependence atom, then one derives in the natural deduction system of PD* that \( \phi \vdash \Theta_X \) for some \( n \)-team \( X \), and intuitionistic disjunction does not occur in any of the deduction steps. Only when \( \phi \) contains dependence atoms, do deductive rules involving intuitionistic disjunction apply in the derivation of \( \phi \). In other words, dependence atoms are the only source of the intuitionistic disjunctions in the disjunctive normal of PD-formulas.

Let us take a closer look at a dependence atom. We have that \( =\{p_{j_0}, \ldots, p_{j_k}\} \) is both semantically and proof theoretically (in PD* by the deductive rules (Depl) and (DepE)) equivalent to the formula

\[
\bigvee_{f \in 2^m} (\{p_{j_0}, \ldots, p_{j_k}\})^f,
\]

where \( 2^m \) is the maximal \( m \)-team on the \( m \)-element set \( \{j_0, \ldots, j_{k-1}\} \) and

\[
(\{p_{j_0}, \ldots, p_{j_k}\})^f = \bigotimes_{s \in 2^m} \left( p_{s(j_0)} \land \cdots \land p_{s(j_{k-1})} \land p_{f(s)} \right).
\]

On the other hand, the team semantics dictates that if a team \( X \) satisfies the dependence atom \( =\{p_{j_0}, \ldots, p_{j_k}\} \), then there exists a function \( f : 2^m \rightarrow 2 \) such that for all \( s \in X \),

\[
s(j_k) = f(s) \upharpoonright \{j_0, \ldots, j_{k-1}\}.
\]

The function \( f \) realizes the dependence atom, and clearly Equation (13) is expressed by the formula \( (\{p_{j_0}, \ldots, p_{j_k}\})^f \). Let \( \phi^*_f \) denote the formula obtained from a PD-formula \( \phi \) by replacing a particular occurrence of the dependence atom \( =\{p_{j_0}, \ldots, p_{j_k}\} \) by the formula \( (\{p_{j_0}, \ldots, p_{j_k}\})^f \). We shall see that \( \phi \) is equivalent to the formula

\[
\bigvee_{f \in 2^m} \phi^*_f,
\]

and therefore each \( \phi^*_f \) can be viewed as a realization of \( \phi \). Roughly speaking, we will use this observation to push the intuitionistic disjunction created by dependence atoms in a formula in one step to the front of the formula. This way we will be able to avoid handling intuitionistic disjunction in the deductive system of PD.

More precisely and more generally, suppose the following are all the occurrences of all dependence atoms in a formula \( \phi \) of PD:

\[
=\{p_{j_0}^{e_1}, \ldots, p_{j_{k_1}}^{e_1}\}, \ldots, =\{p_{j_0}^{e_c}, \ldots, p_{j_{k_c}}^{e_c}\}.
\]

Define a realization sequence \( \Omega = \langle f_1, \ldots, f_c \rangle \) of \( \phi \) to be a sequence such that for each \( 1 \leq \xi \leq c, f_\xi : 2^{m_{\xi}} \rightarrow 2 \) is a function from the maximal \( m_{\xi} \)-team on the \( m_{\xi} \)-element
all formulas $\psi$ where $\Omega \in \Lambda$, we obtain from (17) that

$$A, B, C$$

where $\Lambda$ consist of all the $f_k$'s such that the dependence atoms $(=p_{i_0} \ldots, p_{i_k})$ occur in $\psi$ and $\chi$, respectively;

- $(\psi \land \chi)_{\Omega} := \psi_{\Omega}^* \land \chi_{\Omega}^*$, where $\Omega^0$ and $\Omega^1$ are subsequences of $\Omega$ consisting of all the $f_k$'s such that the dependence atoms $(=p_{i_0} \ldots, p_{i_k})$ occur in $\psi$ and $\chi$, respectively;

- $(\psi \lor \chi)_{\Omega} := \psi_{\Omega}^* \lor \chi_{\Omega}^*$, where $\Omega^0$ and $\Omega^1$ are as above.

Next, we show that every PD formula is logically equivalent to the intuitionistic disjunction of all its realizations.

**Lemma 4.3.** Let $\phi$ be a formula of PD and $\Lambda$ the set of all its realizing sequences. Then

$$\phi \equiv \bigvee_{\Omega \in \Lambda} \phi_{\Omega}^*.$$  

*Proof.* We prove by induction on $\phi$ that $\phi_{\Omega}^* \models \phi$ for each $\Omega \in \Lambda$, and $\phi \models \bigvee_{\Omega \in \Lambda} \phi_{\Omega}^*$.

Case $\phi = p_i$, or $\phi = \neg p_i$ is trivial, as $\psi_{\Omega}^* \equiv (p_i) \equiv p_i$ and $\bigvee_{\Omega \in \Lambda} (\neg p_i) \equiv (\neg p_i) \equiv \neg p_i$. Case $\phi = \psi \lor \chi$ follows from the proof of Theorem 3.7.

Case $\phi = \psi \land \chi$. By induction hypothesis, we have that

$$\psi_{\Omega}^* \models \psi, \chi_{\Omega}^* \models \chi, \quad \text{for each } \Omega^0 \in \Lambda^0 \text{ and each } \Omega^1 \in \Lambda^1,$$  

(16)

$$\psi \models \bigvee_{\Omega^0 \in \Lambda^0} \psi_{\Omega}^* \text{ and } \chi \models \bigvee_{\Omega^1 \in \Lambda^1} \chi_{\Omega}^*,$$  

(17)

where $\Lambda^0$ is the set of all $\Omega^0$'s which are subsequences of some $\Omega \in \Lambda$ consisting of all realizing functions of the dependence atoms occurring in $\psi$, and $\Lambda^1$ is obtained in the same way for $\chi$.

It follows from (16) that $\psi_{\Omega^0}^* \land \chi_{\Omega}^* \models \psi \land \chi$, namely $(\psi \land \chi)_{\Omega}^* \models \psi \land \chi$ for each $\Omega \in \Lambda$. On the other hand, since $A \land (B \lor C) \models (A \land B) \lor (A \land C)$ for all formulas $A, B, C$, we obtain from (17) that

$$\psi \land \chi \models \left( \bigvee_{\Omega^0 \in \Lambda^0} \psi_{\Omega^0}^* \right) \land \left( \bigvee_{\Omega^1 \in \Lambda^1} \chi_{\Omega}^* \right) \models \bigvee_{\Omega^0 \in \Lambda^0, \Omega^1 \in \Lambda^1} (\psi_{\Omega^0}^* \land \chi_{\Omega}^*),$$

where

$$\bigvee_{\Omega^0 \in \Lambda^0, \Omega^1 \in \Lambda^1} (\psi_{\Omega^0}^* \land \chi_{\Omega}^*) \equiv \bigvee_{\Omega \in \Lambda} (\psi_{\Omega^0}^* \land \chi_{\Omega}^*) \models (\psi \land \chi)_{\Omega},$$

as required.

Case $\phi = \psi \lor \chi$. By induction hypothesis, we have (16) and (17). That $(\psi \lor \chi)_{\Omega}^* \models \psi \lor \chi$ for each $\Omega \in \Lambda$ follows from (16). That $\psi \land \chi \models \bigvee_{\Omega \in \Lambda} (\psi \land \chi)_{\Omega}$ is proved analogously to the above case using the fact that $A \land (B \lor C) \models (A \land B) \lor (A \land C)$ for all formulas $A, B, C$. \hfill \Box
We shall view the formula \( \forall \Omega \in \Lambda \phi^* \Omega \) (which is not in the language of PD) or the sequence \( \langle \phi^* \Omega \rangle \Omega \in \Lambda \) as a weak normal form for formulas of PD. We now define a natural deduction system for PD which will enable us to derive in effect the weak normal form \( \forall \Omega \in \Lambda \phi^* \Omega \) for every formula \( \phi \).

**Definition 4.4 (A natural deduction system for PD).** The deductive rules are given as follows:

1. The rules (\( \land I \)), (\( \land E \)), (\( \otimes I \)), (\( \otimes WE \)), (\( \otimes \text{Sub} \)), (\( \text{Com} \otimes \)), (\( \text{Ass} \otimes \)), (\( \bot E \)), (\( \text{EM}_0 \)), (\( \text{Dstr} \land \otimes \)), (\( \text{DepI} \)) as in Definition 3.4.

2. Realization transition:

\[
\begin{align*}
\left[ \phi^*_{\Omega_1} \right] & \quad \left[ \phi^*_{\Omega_m} \right] \\
\vdots & \quad \vdots \\
\theta & \quad \theta & \quad \phi \quad (\text{RTr})
\end{align*}
\]

where \( \{ \Omega_1, \ldots, \Omega_m \} \) is the set of all realizing sequences of \( \phi \).

The realization transition rule (RTr) has the same effect as the combination of intuitionistic disjunction elimination rule (\( \lor E \)) and following rule

\[
\frac{\phi}{\lor_{i=1}^n \phi^*_{\Omega_i}}.
\]

In particular, the rules (RTr) and (RE) imply in effect that \( \lor_{\Omega \in \Lambda} \phi^* \Omega \vdash \phi \), for any PD formula \( \phi \), where \( \Lambda \) is the set of all realizing sequences of \( \phi \). However, in the above deductive system for PD, we avoid the use of intuitionistic disjunction by taking the rule (RTr) instead.

**Corollary 4.5.** The following are derivable rules:

1. All rules in Corollary 3.5 which do not involve intuitionistic disjunction \( \forall \).

2. Realization Elimination: for any realizing sequence \( \Omega \) of \( \phi \),

\[
\frac{\phi^*_{\Omega}}{\phi} \quad (\text{RE})
\]

**Proof.** The rules in Item 1 are derived in the same way as in Corollary 3.5. We now proceed to derive the rule (RE) by induction on \( \phi \). The case \( \phi = p_i \) or \( \neg p_i \) is trivial. The case \( \phi = (p_{i_0}, \ldots, p_{i_k}) \) is derived by (DepI).

Case \( \phi = \psi \lor \chi \). By induction hypothesis, we have that \( \psi^*_{\Omega^0} \vdash \psi \) and \( \chi^*_{\Omega^i} \vdash \chi \), where \( \Omega^0 \) and \( \Omega^i \) are as before. By (\( \otimes \text{Sub} \)), we derive that \( \psi^*_{\Omega^0} \otimes \chi^*_{\Omega^i} \vdash \psi \otimes \chi \), namely \( (\psi \otimes \chi)^*_{\Omega} \vdash \psi \otimes \chi \).

The case \( \theta = \psi \land \chi \) is proved similarly using (\( \land \text{Sub} \)).

Next, we prove the soundness theorem for the above deductive system.

**Theorem 4.6 (Soundness Theorem).** For any formulas \( \phi \) and \( \psi \) of PD,

\[
\phi \vdash \psi \implies \phi \models \psi.
\]
Proof. It suffices to show that all of the deductive rules are valid. The validity of (RTr) follows from Lemma 4.3 and the validity of all the other rules follows from the proof of Theorem 3.6. □

We now proceed to prove the completeness theorem for PD using the above weak normal form. First, we show a completeness theorem for the dependence atom-free fragment of PD (which consists of all classical formulas of PD).

Proposition 4.7. For any classical formulas \( \phi, \psi \) of PD, \( \phi \vdash_{\text{CPL}} \psi \iff \phi \vdash_{\text{PD}} \psi \). In particular, \( \phi \models \psi \implies \phi \vdash_{\text{PD}} \psi \).

Proof. For classical formulas (which can be identified with formulas of classical propositional logic CPL), PD has the same deductive rules as CPL. □

Theorem 4.8 (Completeness Theorem). For any formulas \( \phi \) and \( \psi \) of PD,

\[
\phi \models \psi \implies \phi \vdash \psi.
\]

Proof. Suppose \( \phi \models \psi \). By Lemma 4.3 we have that

\[
\phi \equiv \bigvee_{\Omega \in \Lambda} \phi^\ast_\Omega \text{ and } \psi \equiv \bigvee_{\Delta \in \Lambda'} \psi^\ast_\Delta,
\]

where \( \Lambda \) and \( \Lambda' \) are the sets of all realizing sequences of \( \phi \) and for \( \psi \), respectively.

If \( \Lambda, \Lambda' \neq \emptyset \), then for each \( \Omega \in \Lambda \), we have that \( \phi^\ast_\Omega \models \bigvee_{\Delta \in \Lambda'} \psi^\ast_\Delta \). Since \( \phi^\ast_\Omega(p_{i_1}, \ldots, p_{i_n}) \) and each \( \psi^\ast_\Delta(p_{i_1}, \ldots, p_{i_n}) \) are classical, by Propositional 4.7 one can turn these formulas into the disjunctive normal form in the proof system of CPL or PD, i.e.,

\[
\phi^\ast_\Omega \vdash_{\text{PD}} \Theta_X \text{ and } \psi^\ast_\Delta \vdash_{\text{PD}} \Theta_Y,
\]

where \( X_\Omega, Y_\Delta \) are some \( n \)-teams on \( \{i_1, \ldots, i_n\} \), and the \( n \)-formula \( \Theta_X(p_{i_1}, \ldots, p_{i_n}) \) is defined as in Lemma 3.1. By Soundness Theorem, we obtain that \( \Theta_X \models \bigvee_{\Delta \in \Lambda'} \Theta_Y \). Now, by the same argument as that in the proof of Theorem 3.9 one derives that \( \Theta_X \models \Theta_{\Delta_0} \) for some \( \Delta_0 \in \Lambda' \), which by (18) implies that \( \phi^\ast_\Omega \vdash_{\text{PD}} \psi^\ast_{\Delta_0} \). Hence we obtain by (RE) that \( \phi^\ast_\Omega \vdash_{\text{PD}} \psi \) for all \( \Omega \in \Lambda \), which by (RTr) yields \( \phi \vdash \psi \) as desired.

If \( \Lambda, \Lambda' = \emptyset \), then neither of \( \phi \) and \( \psi \) contains any dependence atoms, i.e., both \( \phi \) and \( \psi \) are classical. In this case, \( \phi \vdash \psi \) follows from Proposition 4.7.

If \( \Lambda = \emptyset \) and \( \Lambda' \neq \emptyset \), then \( \phi(p_{i_1}, \ldots, p_{i_n}) \) is classical, whereas \( \psi(p_{i_1}, \ldots, p_{i_n}) \) is not. In this case, \( \phi \vdash_{\text{PD}} \Theta_X \) and \( \psi^{\Delta_0} \vdash_{\text{PD}} \Theta_Y \) for each \( \psi^\ast_\Delta \), where \( X, Y_\Delta \) are some \( n \)-teams on \( \{i_1, \ldots, i_n\} \). Thus \( \Theta_X \models \bigvee_{\Delta \in \Lambda'} \Theta_Y \) and \( \phi \vdash \psi \) is proved by a similar argument with the above.

If \( \Lambda \neq \emptyset \) and \( \Lambda' = \emptyset \), then \( \psi \) is classical, whereas \( \phi \) is not. In this case, \( \phi \vdash \psi \) is proved by a similar argument. □

Theorem 4.9 (Strong Completeness Theorem). Let \( \Gamma \) be a set of formulas and \( \phi \) a formula of PD. Then

\[
\Gamma \vdash \phi \iff \Gamma \models \phi.
\]

Proof. Since PD \( \equiv \text{PID} \), we know that PD is also compact. Therefore the theorem follows. □

We end this section with an application of the above given natural deduction system for PD.
Example 4.10. The following formulas, known as Armstrong’s axioms \([2]\), are derivable in \(PD\):

(i) \(=(p_i, p_i)\)

(ii) \(=(p_{i_0}, p_{i_1}, p_{i_2}) \vdash (p_{i_1}, p_{i_0}, p_{i_2})\)

(iii) \(=(p_{i_0}, p_{i_0}, p_{i_1}) \vdash (p_{i_0}, p_{i_1})\)

(iv) \(=(p_{i_1}, p_{i_2}) \vdash (p_{i_0}, p_{i_1}, p_{i_2})\)

(v) \(=(p_{i_0}, p_{i_1}), (p_{i_1}, p_{i_2}) \vdash (p_{i_0}, p_{i_1})\)

Proof. The derivations are as follows:

(i) \[
\frac{p_1 \otimes \neg p_1}{(p_1 \land p_1) \otimes (\neg p_1 \land \neg p_1)} \quad \text{(AI, \otimes Sub)}
\]
\[
\overset{s \in 2^1}{\forall (p_1 \otimes p_1) \otimes (p_4^{(s)} \land p_1^{(s)})} \quad \text{(Depl)}
\]

where \(2^1\) is the maximal 1-team on \(\{i\}\) and \(f : 2^1 \to 2\) is defined as \(f(s) = s(i)\).

(ii) \[
\forall f : 2^2 \to 2 \quad [(=(p_{i_0}, p_{i_1}, p_{i_2})^*)_f] \quad \text{ Com, \otimes Sub}
\]
\[
\overset{s \in 2^2}{\forall (p_1^{(s)} \land p_1^{(s)} \land p_2^{(s)})} \quad \text{(Depl)}
\]
\[
\overset{p_{i_0} \otimes (p_1^{(s)} \land p_1^{(s)} \land p_2^{(s)})}{=} (p_{i_1}, p_{i_0}, p_{i_2}) \quad \text{(RTr)}
\]

where \(2^2\) is the maximal 2-team on \(\{i_0, i_1\}\).

(iii) \[
\forall f : 2^1 \to 2 \quad [(=(p_{i_0}, p_{i_0}, p_{i_1})^*)_f] \quad \text{ (\&E, \otimes Sub)}
\]
\[
\overset{s \in 2^1}{\forall (p_1^{(s)} \land p_1^{(s)} \land p_2^{(s)})} \quad \text{(Depl)}
\]
\[
\overset{p_{i_0} \otimes (p_1^{(s)} \land p_1^{(s)})}{=} (p_{i_0}, p_{i_1}) \quad \text{(RTr)}
\]

where \(2^1\) is the maximal 1-team on \(\{i_0\}\).

(iv)
\( \forall f : 2^I \rightarrow 2 \left[ \left( \left( p_{i_0}, p_{i_1} \right) \right)_f \right] \)

\[ \bigotimes_{s \in 2^I} \left( p_{i_1}^{s(i_1)} \land p_{i_2}^{f(s)} \right) \quad p_{i_0} \otimes p_{i_0} \quad (\text{EM}_0) \]

\[ \left( \bigotimes_{s \in 2^I} \left( p_{i_1}^{s(i_1)} \land p_{i_2}^{f(s)} \right) \right) \land \left( p_{i_0} \otimes p_{i_0} \right) \quad (\land I) \]

\[ \bigotimes_{s \in 2^I} \left( p_{i_0}^{s(i_0)} \land p_{i_1}^{s(i_1)} \land p_{i_2}^{f(s)} \right) \quad (\text{Distr} \land \otimes) \]

\[ \left( p_{i_0}, p_{i_1}, p_{i_2} \right) = \left( p_{i_0}, p_{i_1}, p_{i_2} \right) \quad (\text{Depl}) \]

\[ \left( p_{i_0}, p_{i_1}, p_{i_2} \right) = \left( p_{i_0}, p_{i_1}, p_{i_2} \right) \quad (\text{RT}) \]

where \( 2^I, 2^I \) are the maximal 1-team on \( \{ i_1 \} \) and the maximal 2-team on \( \{ i_0, i_1 \} \), respectively, and \( g : 2^I \rightarrow 2 \) is defined as \( g(s) = f(s \mid \{ i_1 \}) \).

(v) We first derive that for each \( f_0 : 2^I \rightarrow 2 \) and \( f_1 : 2^I \rightarrow 2 \),

\[ \left( \left( p_{i_0}, p_{i_1} \right) \land \left( p_{i_1}, p_{i_2} \right) \right)_{(f_0, f_1)} = \left( p_{i_0}, p_{i_2} \right), \]

where \( 2^I \) and \( 2^I \) are maximal 1-teams on \( \{ i_0 \} \) and on \( \{ i_1 \} \), respectively.

Now, we derive \( = \left( p_{i_0}, p_{i_1} \right), = \left( p_{i_1}, p_{i_2} \right) \models = \left( p_{i_0}, p_{i_2} \right) \) as follows:
∀f₀ : 2¹ → 2, ∀f₁ : 2¹ → 2

\[
\frac{(p_{i0}, p_{i1}) ∧ (p_{i1}, p_{i2})}{(RTr)} \quad \frac{(p_{i0}, p_{i1})}{(\text{by } (\ast))} \quad \frac{(p_{i1}, p_{i2})}{(\lor I)}
\]

\[
\frac{(p_{i0}, p_{i1}) ∧ (p_{i1}, p_{i2})}{(\text{by } (\ast))} \quad \frac{(p_{i0}, p_{i1})}{(\lor I)} \quad \frac{(p_{i1}, p_{i2})}{(\text{by } (\ast))}
\]

5 Summary

We have shown that there is a robust propositional dependence logic for reasoning about dependencies on the propositional level. This logic can be fairly called robust because it has four different equivalent formulations: PD, PD⁺, PID and InqL. The different formulations have their respective complete axiomatisations.

There are known results on the computational complexity of propositional dependence logic. For the downwards closed logics PD, PD⁺, PID and InqL, a formula \( \phi \) being satisfiable by some team is equivalent to \( \phi \) being satisfiable by some singleton team. Since over singleton team dependence atoms are always true and the logics have the same semantics as classical propositional logic, the satisfiability problem (SAT) for all these logics has the same complexity as SAT for classical propositional logic, which is well-known to be NP-complete. As for model checking problem (MC), [6] shows that MC for PD and PD⁺ are NP-complete, and it follows from [7] that MC for PID and InqL are coNP-complete. The computational complexity of the validity problem of these logics is unknown. On the other hand, quantified propositional dependence logic (with an obvious definition) has a connection with Dependency Quantified Boolean Formulas, which is proved in [12] to be NEXPTIME-complete.

There are also many recent results on modal dependence logic (introduced in [15]), as well as on first order (quantified) dependence logic. Another recent development is the rise of independence logic (introduced in [8]), which is, a priori, stronger than dependence logic. The second part of this paper will focus on propositional independence logic.

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