Electron Bessel States in High-Energy Ionization

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Abstract. The creation of electron Bessel states of large topological charge is discussed in the context of high-energy photoionization. We show that, for short (three cycle) driving pulses of circular polarization and intensities of $5 \times 10^{16}$ Wcm$^{-2}$, it is possible to obtain electron vortex states with orbital angular momentum of few thousands units of $\hbar$. Our analysis is based on the quasi-relativistic strong-field approximation where an additional term, the relativistic mass correction, is accounted for. The effect of this modification, together with the recoil corrections, over the energy spectra of photoelectrons is also analyzed.

1. Introduction
Propagating vortex states (or beams) are characterized by screw-type wave fronts with a vanishing wave function at its centre [1, 2]. This was initially shown by Nye and Berry when analyzing the reflection of ultrasonic beams over rough surfaces [3]. The vanishing of the wave function implies that both, its real and complex parts, fade simultaneously at the screw axis. Hence, those central lines, also known as vortices or, in the terminology coined by Nye and Berry, as dislocations in wave trains, define the regions in space where the wave function’s phase is singular [1, 2]. Although the idea of quantized vortex states in quantum mechanics was already suggested by Dirac in the early 1930’s [4], the theoretical description of vortices in particle beams was offered forty years later in a series of papers by Hirschfelder et al. [5, 6]. The latter, using the hydrodynamical formulation of quantum mechanics [7], analyzed the scattering of particles by a two-dimensional barrier and determined that the incident and reflected waves interfere with the consequent formation of dislocations. Such analysis provided a better understanding of the so-called whirlpool effect in chemical reactions, i.e., it explained the formation of vortex states in molecular collisions [8, 9, 10] (for a modern analysis of atomic vortices in laser fields, see Ref. [11]). More recently, the dynamics of vortex lines in the evolution of free particles, particles in constant magnetic fields, or trapped in spherically-symmetric harmonic potentials was studied [12]. The generation of propagating electron wave packets with vortex structures (hereafter referred to as electron vortex states or EVS) was initially proposed by Bliokh et al. [13] from a non-relativistic, semiclassical treatment. Its relativistic counterpart, based on the solution of the Dirac or Klein-Gordon equations, was discussed in [14, 15, 16] (see also the reviews [1, 2] and references therein). The experimental observation of electron vortex beams was first achieved by directing coherent electron beams towards specially tailored helical phase plates [17] or by using holographic gratings [18]. More recently, it was demonstrated that
EVS are also obtained when electron plane waves interact with magnetic monopoles due to the Aharonov-Bohm effect [19].

As it was shown in [20], optical vortex beams carry a non-vanishing orbital angular momentum (OAM) equal to $m\hbar$, where $m$ is the so-called topological charge. In the context of the non-relativistic quantum mechanics, EVS are simultaneous eigenstates of the Hamiltonian and the orbital angular momentum operator. Hence, electron vortices carry a quantized OAM equal to $m\hbar$ where, in principle, $m$ can be arbitrarily large. The ‘rotating’ nature of the probability and electric currents creates magnetic fields which depend on the topological charge. Those properties suggest that vortex beams in electron microscopes can be used to probe magnetic and structural properties of matter. The first of such applications was already presented by Verbeeck and co-workers [18] where the EVS were used in obtaining the energy-loss magnetic circular dichroism spectra of an iron sample. Recently, the fast trend observed in the miniaturization of the information storage devices motivated the use of electron vortex beams at the atomic scale [21, 22] (see, also [23]). It was also shown that the unique characteristics of their helix-like wave fronts can be used to improve the resolution in electron microscopy [24, 25]. Additionally, Verbeeck et al. reported the angular momentum transfer from the vortex state to gold nanoparticles leading to their rotation [26]. Hence, EVS could play a role in the mechanical control of nanostructures. For more applications, see, e.g., Refs. [1, 2, 27].

Although the generation of EVS from traditional methods (i.e., by making use of phase plates [17], diffraction gratings [18, 24] or the Aharonov-Bohm effect [19, 27]) has been extensively documented (see, e.g., Refs. [1, 2, 28] and references therein), vortex states in photoionization were only analyzed recently [29]. In [29] we have demonstrated that it is possible to prepare EVS with orbital angular momentum of few-hundreds of units of $\hbar$ during the interaction of hydrogen-like ions with short and intense laser pulses. It is the purpose of this paper to further explore the generation and properties of vortex states in high-energy ionization.

The non-relativistic Strong-Field Approximation (SFA) in photoionization, as originally presented by Keldysh [30], Faisal [31], and Reiss [32], is based on one fundamental assumption: the interaction of the electron with the parent ion can be neglected once the former appears in the continuum. In other words, the essence of the SFA lies in the assumption that the exact scattering state can be approximated by the Volkov solution of the electron in the laser field [33]. Note that such approximation is only well justified for the ionization of negatively-charged ions or when the asymptotic kinetic energy of the photoelectron is much larger than the ionization potential of the target. The relativistic counterpart of the SFA (RSFA), originally proposed by Reiss in the early 1990’s [34], takes advantage of the fully-relativistic analytical expression of the Volkov wave function. Such treatment accounts for the spin dynamics, radiation pressure, and other important effects (see, e.g., [35, 36, 37, 38]). The predictions arising from the SFA and RSFA differ when the intensity of the driving pulse is large enough. However, in the so-called quasi-relativistic SFA (QRSFA) the expressions for the probability amplitude of photoionization in the non-relativistic framework are modified to obtain a better agreement with the RSFA [37] (see also [39, 40]). In Ref. [29] the validity of the QRSFA is analyzed for driving field intensities up to $5 \times 10^{16}$ Wcm$^{-2}$ with the introduction of the so-called relativistic mass corrections in photoionization. Here, we will also compare the results obtained from the RSFA and QRSFA accounting for different sets of corrections. The most suitable version of the quasi-relativistic approach will be then used to determine the OAM distribution of photoelectrons in high-energy ionization.

This paper is organized in the following way. In Sec. 2 we present a geometrical description of EVS in momentum space. Also, the basis transformation from plane-wave electron states to the more suitable Bessel states is presented. In Sec. 3 we introduce the probability amplitude of photoionization in the QRSFA and the fundamental corrections used in this approach. The driving laser field is defined in Sec. 3.1. Furthermore, in Sec. 3.2 we compare the energy spectra
of photoelectrons obtained from the QRSFA and the RSFA. The conditions necessary for the
generation of EVS are discussed in Sec. 4. Additionally, the probability distribution of ionization
into vortex states and the OAM distributions are also shown there. Finally, in Sec. 5 we draw
our conclusions. Throughout this paper we keep \( \hbar = 1 \).

2. Electron vortex states in ionization
As it was mentioned above, EVS are characterized by a helix-type structure. Hence, the most
natural system of coordinates for its analysis is the cylindrical one. We start by defining a triad
of mutually orthogonal vectors

\[
\mathbf{n}_{\perp,1} = \begin{pmatrix} \cos \theta \cos \varphi_T \\ \cos \theta \sin \varphi_T \\ - \sin \theta \end{pmatrix}, \quad \mathbf{n}_{\perp,2} = \begin{pmatrix} - \sin \varphi_T \\ \cos \varphi_T \\ 0 \end{pmatrix}, \quad \mathbf{n}_\parallel = \begin{pmatrix} \sin \theta \cos \varphi_T \\ \sin \theta \sin \varphi_T \\ \cos \theta \end{pmatrix}.
\]

(1)

In fact, in the non-relativistic formulation of quantum mechanics, the free EVS are eigenstates
of the orbital angular momentum operator \( \hat{L}_\parallel = \mathbf{n}_\parallel \cdot \hat{\mathbf{L}} \), where \( \hat{\mathbf{L}} = \hat{x} \times \hat{\mathbf{p}} \), and the Hamiltonian.
Let us now write the photoelectron momentum \( \mathbf{p} \) in the coordinate system (1),

\[
\mathbf{p} = p_\parallel \mathbf{n}_\parallel + p_\perp (\mathbf{n}_{\perp,1} \cos \varphi_\mathbf{p} + \mathbf{n}_{\perp,2} \sin \varphi_\mathbf{p}).
\]

(2)

Here, \( p_\parallel = \mathbf{p} \cdot \mathbf{n}_\parallel \), is the projection of \( \mathbf{p} \) on \( \mathbf{n}_\parallel \), while \( p_\perp = \sqrt{\mathbf{p}^2 - p_\parallel^2} \) and \( \varphi_\mathbf{p} \) define its projection onto
the \( (\mathbf{n}_{\perp,1}, \mathbf{n}_{\perp,2}) \) plane. In a basis spanned by the plane-wave states \( |\mathbf{p}\rangle \), which is most
convenient to obtain the probability amplitudes of ionization, the so-called family of twisted
momenta \( \mathbf{p}_T(\varphi) \) can be defined; namely,

\[
\mathbf{p}_T(\varphi) = p_\parallel \mathbf{n}_\parallel + p_\perp (\mathbf{n}_{\perp,1} \cos \varphi + \mathbf{n}_{\perp,2} \sin \varphi_\mathbf{p}).
\]

(3)

Its geometrical interpretation is as follows: the propagating EVS (or twisted states) are
characterized by a momentum vector of magnitude \( p_T \) circulating along a conical surface with
axis along \( \mathbf{n}_\parallel \). This, in fact, guarantees that the twisted states are eigenvectors of \( \hat{L}_\parallel \) and carry
OAM. The cone of circulation is defined by the opening angle \( 2 \beta_T \) and slant height \( p_T \). The angle \( \varphi \), also known as the twist angle, parametrizes the circulation and \( \zeta_H = \pm 1 \) determines
its direction (helicity). In our further analysis we will set \( \zeta_H = 1 \).

In the following, we shall concentrate on the generation of EVS represented as Bessel states
(BS), which are the solution to the Schrödinger equation in cylindrical coordinates (for a detailed
description, see the reviews [1, 2] and references therein). We use the notation \( |p_\parallel, p_\perp, m\rangle \) for
BS which, in position representation, are \( \langle \mathbf{x}|p_\parallel, p_\perp, m\rangle = i^m e^{ip_\parallel x_\parallel} J_m(p_\perp x_\perp) e^{i m \varphi}. \) Note that \( \mathbf{x} \) has been written in the cylindrical coordinates (1), \( J_m(z) \) is the Bessel function of the first
kind, and the integer \( m \) is the topological charge. As it was shown in Ref. [29], the collection
of states \( |p_\parallel, p_\perp, m\rangle \) constitute a basis in which the twisted momenta \( |\mathbf{p}_T(\varphi)\rangle \) can be expressed.
The latter can be Fourier decomposed such that [29]

\[
\langle \mathbf{x}|\mathbf{p}_T(\varphi)\rangle = e^{i\mathbf{x}\cdot\mathbf{p}_T(\varphi)} = \sum_{m = -\infty}^{\infty} e^{-im\varphi} \langle \mathbf{x}|p_\parallel, p_\perp, m\rangle.
\]

(4)

This, in turn, implies that \( |p_\parallel, p_\perp, m\rangle \) can also be represented in terms of \( |\mathbf{p}_T(\varphi)\rangle \), i.e.,

\[
|p_\parallel, p_\perp, m\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{im\varphi} |\mathbf{p}_T(\varphi)\rangle.
\]

(5)

Eqs. (4) and (5) allow us to perform a basis transformation from the Bessel to the plane-wave
states of twisted momenta, and vice-versa. Such property will be useful in the remaining part
of this paper.
3. Probability distribution of photoelectrons in the QRSFA

Consider a light hydrogen-like ion/atom in its ground state interacting with a finite-in-time and intense laser pulse. The later is defined by the vector potential \( \mathbf{A}(\phi) \), which vanishes outside the phase interval \( \phi = \omega t - \mathbf{k} \cdot \mathbf{x} \in [0, 2\pi] \). Here, \( \omega \) is the fundamental frequency of field oscillations, \( \mathbf{k} = \omega n / c \) is the wave vector, and \( \mathbf{n} \) represents the direction of propagation of the pulse. In the QRSFA, the probability amplitude of ionization from the bound state \( \psi = k \) phase interval \( \phi \) \( \mathbf{E}_A \) intense laser pulse. The later is defined by the vector potential

\[
\mathcal{A}(\mathbf{p}) = -i \int_{-\infty}^{\infty} dt \int d^3x \, e^{-i\mathbf{p} \cdot \mathbf{x} + iG(\phi, \mathbf{p})} \hat{H}_I(t) \psi_B(\mathbf{x}),
\]

where \( \hat{H}_I(t) \) is the interaction Hamiltonian. The latter, in the velocity gauge, reads

\[
\hat{H}_I(t) = -\frac{e}{m_e} \mathbf{A}(\phi) \cdot \dot{\mathbf{p}} + \frac{e^2}{2m_e} A^2(\phi).
\]

The function \( G(\phi, \mathbf{p}) \) in (6) comes, in part, from the analytical expression of the Volkov solution, which differs in the relativistic and non-relativistic approaches. In general, we define such function as

\[
G(\phi, \mathbf{p}) = \int_0^\phi d\phi' \left( \frac{E_{\text{kin}}(\mathbf{p}) - E_B}{\omega} - \frac{eA(\phi') \cdot \mathbf{p}}{N(\mathbf{p}, \mathbf{k})} + \frac{e^2 A^2(\phi')}{2N(\mathbf{p}, \mathbf{k})} \right).
\]

Note that two quantities, \( E_{\text{kin}}(\mathbf{p}) \) and \( N(\mathbf{p}, \mathbf{k}) \), have been introduced here. They depend on the actual corrections used in the QRSFA, which are listed below.

(i) **Retardation correction.** The relativistic phase \( \phi \) of the laser field is given by the four-vector product \( \mathbf{k} \cdot \mathbf{x} = \omega t - \mathbf{k} \cdot \mathbf{x} \). However, as it was shown in [29], the retardation correction \( \mathbf{k} \cdot \mathbf{x} \) does not play an important role when the ionization is driven by near-infrared light fields with intensities up to \( 5 \times 10^{16} \text{ Wcm}^{-2} \). Hence, in the remaining part of this paper we consider \( \phi \approx \omega t \) (no retardation).

(ii) **Recoil corrections.** The function \( N(\mathbf{p}, \mathbf{k}) \) in Eq. (8) comes from the analytical expression of the Volkov solution. When the latter is calculated from the Dirac or Klein-Gordon equations one obtains

\[
N(\mathbf{p}, \mathbf{k}) = \frac{\omega}{c} (\sqrt{p^2 + (mc)^2} - \mathbf{p} \cdot \mathbf{n}) = \omega m_e \left( 1 - \frac{\mathbf{p} \cdot \mathbf{n}}{mc} + \frac{p^2}{2(mc)^2} - \frac{p^4}{8(mc)^4} \cdots \right).
\]

In contrast, when the Schrödinger equation is considered \( N(\mathbf{p}, \mathbf{k}) = N^{(0)}(\mathbf{p}, \mathbf{k}) = \omega m_e \), which corresponds to the zeroth-order term in the expansion (9). However, a better agreement between the relativistic and non-relativistic theories can be achieved by taking higher orders in \( 1/c \) [37, 40]. This is done, for instance, in the so-called **Nordsieck correction** [39], where the function \( N(\mathbf{p}, \mathbf{k}) \) is approximated as

\[
N(\mathbf{p}, \mathbf{k}) \approx \omega m_e \left( 1 - \frac{\mathbf{p} \cdot \mathbf{n}}{mc} \right).
\]

Such modification already accounts for the recoil of the electron (radiation pressure effects) for laser field intensities up to the order of \( 10^{15} \text{ Wcm}^{-2} \). For larger intensities, the Nordsieck correction is not sufficient and even higher orders in \( 1/c \) need to be accounted for [37].
(iii) **Relativistic mass correction.** In Eq. (8) we have also introduced the function $E_{\text{kin}}(p)$, which corresponds to the kinetic energy of the photoelectron with asymptotic momentum $p$. When the fully-relativistic treatment is applied, this kinetic energy is given by

$$E_{\text{kin}}(p) = c\sqrt{p^2 + (mc)^2} - mc^2 = \frac{p^2}{2m_e} - \frac{p^4}{8m_e^2c^2} \ldots$$

In contrast, in the non-relativistic framework we obtain $E_{\text{kin}}(p) = E_{\text{kin}}^{(0)}(p) = p^2/2m_e$, which is the zeroth-order term of the expansion (11). Note that the expression $E_{\text{kin}}(\phi/\omega = E_{\text{kin}}(p)t$ appears in the integrand of the probability amplitude [see, Eqs. (6) and (8)]. Hence, higher terms in the expansion (11) can only be neglected provided that $[E_{\text{kin}}^{(0)}(p)]^2T/(2mc^2) < \pi$, where $T$ defines a characteristic time of the interaction. For our current purposes, we assume that such characteristic time is approximately equal to the duration of a single field oscillation, i.e. $T = 2\pi/\omega_L$, where $\omega_L$ is the carrier frequency of the pulse. Therefore, the non-relativistic expression for the kinetic energy is only justified provided that $[E_{\text{kin}}^{(0)}(p)]^2/(mc^2\omega_L) < 1$. By considering ionization driven by a Ti-Sapphire laser field, we find out that this condition already breaks at photoelectron kinetic energies smaller than 1 keV [29]. Hence, in the analysis of high-energy photoionization, either further terms in the expansion (11) or the fully-relativistic expression $E_{\text{kin}}(p)$ have to be used.

(iv) **Ground-state energy correction.** The relativistic ground-state energy of a hydrogen-like ion (atomic number $Z$), $E_B^{\text{rel}}$, is given by

$$E_B^{\text{rel}} - mc^2 = mc^2(\sqrt{1 - Z^2\alpha^2} - 1) = -\frac{1}{2}Z^2\alpha^2mc^2 - \frac{1}{8}Z^4\alpha^4mc^2 \ldots, \quad (12)$$

where $\alpha$ is the fine-structure constant. Its non-relativistic counterpart corresponds to the lowest-order term in (12) with respect to $\alpha$, $E_B \equiv E_B^{(0)} = -Z^2\alpha^2mc^2/2$. As it was discussed in Ref. [29], for the photoionization of light ions ($Z < 10$) by near-infrared laser fields, the approximation $E_B^{\text{rel}} - mc^2 \approx E_B$ is well justified. Hence, in the remaining part of this paper we shall use the non-relativistic form of the ground-state energy.

As we have shown, the corrections required in the QRSFA depend on the laser-field parameters and the characteristics of the ionic target. Now, let us come back to Eq. (6). Once the relevant modifications to the non-relativistic theory are determined, the probability amplitude $A(p)$ can be calculated. The latter defines the triply-differential probability distribution of photoelectrons $\mathcal{P}(p)$, which in atomic units reads

$$\mathcal{P}(p) = \frac{d^3P}{dE_{\text{kin}}d^2\Omega_p} \approx \frac{d^3P}{dE_{\text{kin}}^{(0)}d^2\Omega_p} = \frac{\alpha^2m_e^2c^2}{(2\pi)^3} |p| \cdot |A(p)|^2. \quad (13)$$

In the remaining part of this Section, and after defining the laser pulse parameters, we will compare the probability distribution in the QRSFA (13) accounting for Nordsieck, full-recoil and relativistic mass corrections, to the corresponding distribution obtained from the RSFA based on the Dirac equation. The latter is defined by Eqs. (73) to (75) and (A6) of Ref. [29]. Note, however, that for laser field intensities of the order of $5 \times 10^{16}$ Wcm$^{-2}$ the spin effects in high-energy ionization are small. Hence, we shall consider only the spin-independent probability distribution in the RSFA (Eq. (75) in the same reference).

3.1. **Laser pulse and ionic target**

We analyze the photoionization of He$^+$ ions ($Z = 2$ and $|E_B| \approx 54.4$ eV) driven by an intense, short, and circularly polarized laser pulse. The latter, which comprises of $N_{\text{osc}}$ field oscillations
Figure 1. Energy spectra of photoelectrons obtained from three forms of the QRSFA [Eq. (13)] and the relativistic Dirac equation (Eq. (75) in Ref. [29]). The reference Dirac distributions (thick solid blue lines) are compared to the quasi-relativistic ones accounting for: Nordsieck correction (10) (thin dashed magenta lines); full recoil correction (9) (thin solid red lines); and full recoil and relativistic mass corrections (11) (thick dashed cyan lines). While in the left panel the laser pulse intensity is $5 \times 10^{15}$ Wcm$^{-2}$ and the polar and azimuthal angles of detection are $\theta_p = 0.4911\pi$ and $\varphi_p = 0$, respectively, in the right panel the intensity is $10^{16}$ Wcm$^{-2}$ and the corresponding angles are $\theta_p = 0.4874\pi$ and $\varphi_p = 0$.

within a $\sin^2$ envelope, is defined by the vector potential $\mathbf{A}(\phi) = A_0[f_1(\phi)\mathbf{\varepsilon}_1 + f_2(\phi)\mathbf{\varepsilon}_2]$, where $\mathbf{\varepsilon}_j$, $j = 1, 2$, are two normalized polarization vectors such that $\varepsilon_1 \times \varepsilon_2 = \mathbf{n}$, and

$$f_j(\phi) = - \int_0^\phi d\phi' F_j(\phi').$$

(14)

Here, $F_j(\phi)$ are the electric-field shape functions, given by

$$F_j(\phi) = N\omega \sin^2\left(\frac{\phi}{2}\right) \sin(N_{\text{osc}}\phi + \delta_j) \cos(\delta + \delta_j),$$

(15)

for $\phi \in [0, 2\pi]$ and zero outside this interval. The phases $\delta$ and $\delta_j$ define the polarization of the pulse [for circular polarization we choose $\delta = \pi/4$ and $\delta_j = (j - 1)\pi/2$] and $N = \sqrt{8/3}N_{\text{osc}}$ is a normalization constant [38]. For our numerical illustrations, we consider a three-cycle ($N_{\text{osc}} = 3$) Ti-Sapphire laser pulse (carrier frequency $\omega_L = N_{\text{osc}}\omega = 1.5498$ eV) with intensities equal to $5 \times 10^{15}$ Wcm$^{-2}$, $10^{16}$ Wcm$^{-2}$ or $5 \times 10^{16}$ Wcm$^{-2}$.

3.2. Energy spectra of photoelectrons in the QRSFA and RSFA

We present now the probability distribution of photoelectrons in the QRSFA [Eq. (13)] and compare it to the spin-independent distribution obtained from the RSFA (Eq. (75) in Ref. [29]). While the retardation and ground-state energy corrections are neglected, we analyze the effects of the recoil and relativistic mass corrections. In Fig. 1 we show the probability distribution in the RSFA (thick solid blue lines) and the distributions in the QRSFA accounting for three different modifications, as described in the caption. The energy spectra shown in the left and right panels are obtained for the laser pulse intensity of $5 \times 10^{15}$ Wcm$^{-2}$ and $10^{16}$ Wcm$^{-2}$, respectively. As it can be seen from the left panel, the three quasi-relativistic approaches lead to qualitatively similar distributions as compared to the relativistic one. However, important quantitative discrepancies are observed when the Nordsieck (10) or the full-recoil correction (9) are used together with the non-relativistic form of the kinetic energy (thin dashed magenta and thin solid red lines, respectively). Such differences increase rapidly with the laser field intensity.
and photoelectron kinetic energies, as it can be seen from the right panel. In contrast, when the full-recoil and relativistic-mass corrections are accounted for simultaneously (thick dashed cyan line), the QRSFA coincides almost exactly with the relativistic distribution for both intensities presented in this figure. This corroborates that the retardation and ground-state corrections are negligible for the parameters considered in this paper.

4. Spiral of ionization and generation of vortex states

In the previous Section we have defined the probability amplitude of ionization in the QRSFA (6) for the transition between the bound and the plane-wave states. Now, we shall discuss the generation of EVS by performing a transformation to a basis of BS. In doing so, and by making use of the property (4), we obtain that the probability amplitude of ionization into the twisted momentum $p_T(\varphi)$ is given by

$$A(p_T(\varphi)) = \sum_{\ell=0}^{\infty} e^{i\ell\varphi}A_\ell(p_{\parallel}, p_{\perp}).$$

However, as it was shown in Refs. [41, 42, 43], the high-energy photoionization of light ions, when driven by short and circularly- or elliptically-polarized laser pulses, is only observed with sufficiently large probability around a well-defined path in momentum space. Such path, the so-called three-dimensional momentum spiral $p_S(\phi)$, follows from the saddle-point analysis of the time-integral in Eq. (6). Specifically, the curve $p_S(\phi)$ is a real vector function which approximately satisfies the saddle-point equation $G'(\phi, p) = 0$ (prime indicates the derivative over $\phi$) and it is given by

$$p_S(\phi) = p^\parallel(\phi) + p^\perp(\phi)n = eA(\phi) + \frac{e^2A^2(\phi)}{2mc\sqrt{1-Z^2\alpha^2}}n,$$

where the relativistic form of the bound state energy (12) has been used. While the spiral is defined for the whole duration of the pulse ($\phi \in [0, 2\pi]$), it only has a physical meaning when the kinetic energy of the photoelectron is large enough [29, 41, 43]. For the parameters considered in this paper, it can be estimated that Eq. (17) is valid when $E_{\text{kin}}(0) > 10|E_B| = 0.54$ keV [29, 43], which is within the range of applicability of the SFA. Note that in (17), $p^\parallel(\phi)$ and $p^\perp(\phi)$ represent the parallel and perpendicular components of the asymptotic momentum with respect to the propagation direction of the laser field $n$, respectively. The superscripts are used to differentiate them from the components $p_{\parallel}$ and $p_{\perp}(\varphi)$ of the twisted momentum, as defined in (3).

From the analytical expression of the three-dimensional spiral (17), several properties of the probability distribution can be determined. For instance, at a fixed phase $\phi$, the ratio $p^\parallel|/|p_{\parallel}|$ defines the polar angle of photoelectron detection $\theta_p$, i.e., $\cos \theta_p = p^\parallel|/|p_{\parallel}|$. The corresponding azimuthal angle $\varphi_p$ is given by $\varphi_p = \arg[p_{\perp} \cdot (\epsilon_1 + i\epsilon_2)]$. Furthermore, the relativistic kinetic energy at which the probability distribution is peaked, can be approximated as $E_{\text{kin}}(p) \approx cp^\parallel$, provided that the target ions are sufficiently light ($Z\alpha \ll 1$) [41].

In order to obtain EVS with sufficiently large probability, at least two conditions need to be met. Namely, the asymptotic momentum must be described by the function $p_T(\varphi)$ [Eq. (3)] and the cone of twisted momenta has to approach the ionization spiral $p_S(\phi)$ [Eq. (17)]. This is fulfilled by setting a vector $p_0$ which starts at the origin of coordinates and ends at one point belonging to the spiral, i.e., $p_0 = p_S(\phi_0)$, where $\phi_0$ is a fixed phase. Such vector determines the polar and azimuthal angles $\theta_0$ and $\varphi_0$, respectively. Now, the symmetry axis of the conical surface, $n_{\parallel}$, is set by adding two small increments to the aforementioned angles, i.e., we define $\theta_T = \theta_0 + \delta\theta_T$ and $\varphi_T = \varphi_0 + \delta\varphi_T$. This, in turn, defines the system of coordinates (1) and the
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Figure 2. Distribution of photoelectrons into vortex states, $|A(p_T(\varphi))|^2$ [Eq. (16)] (left panel), and the phase derivative, $\Phi'(p_T(\varphi))$ [Eq. (19)] (right panel), as functions of $\varphi$. The results from the relativistic treatment and the ones from the QRSFA (full-recoil and mass corrections included) are shown as solid and dashed red lines, respectively. The driving pulse, which is described in Sec. 3.1, consists of three field oscillations ($N_{osc} = 3$) with the intensity $I = 5 \times 10^{16}$Wcm$^{-2}$. For the parameters defining the cone of twisted momenta, see the text.

projections $p_{\parallel 0} = p_0 \cdot n_\parallel$ and $p_{\perp 0} = \sqrt{p_0^2 - p_{\parallel 0}^2}$. With this in mind, we write

$$p_T(\varphi) = (p_{\parallel 0} + \delta p_{\parallel}) n_\parallel + (p_{\perp 0} + \delta p_{\perp}) (n_{\perp,1} \cos \varphi + \zeta_H n_{\perp,2} \sin \varphi),$$

where we have also introduced the increments $\delta p_{\perp}$ and $\delta p_{\parallel}$. Note that, according to this construction, the half-opening angle of the cone is $\delta_T = \arccos(p_0 \cdot n_\parallel/|p_0|)$. In the following, we shall concentrate on the case when the cone and spiral are tangent to each other at $\varphi = 0$ and $\phi = \pi$. This happens by setting $\delta p_{\perp} = \delta p_{\parallel} = \delta \varphi_T = 0$ and, therefore, $\beta_T = |\delta \varphi_T|$ for $|\delta \varphi_T| < \pi/2$. This particular choice of parameters guarantees that the EVS are generated efficiently.

As the probability amplitude of ionization is a complex function, it can be described by its modulus $|A(p_T(\varphi))|$ and phase $\Phi(p_T(\varphi)) = \arg A(p_T(\varphi))$. For our further purposes, we also introduce its phase derivative with respect to the twist angle

$$\Phi'(p_T(\varphi)) = \frac{d}{d\varphi} \Phi(p_T(\varphi)).$$

This derivative is of special interest in the analysis of the orbital angular momentum probability distribution $P_m(p_{\parallel}, p_{\perp}) = p_{\perp} |A_m(p_{\parallel}, p_{\perp})|^2/(2\pi)^2$ [29]. In fact, it can be shown that large values of $\Phi'(p_T(\varphi))$ lead to OAM distributions concentrated around points $|m| \gg 1$. Hence, vortex states with large topological charges are expected to appear. Additionally, non-vanishing second (or larger) derivatives relate to chirped OAM structures, similarly to what is observed in Compton scattering [29, 44].

In Fig. 2 we present the distribution of photoelectrons into vortex states, $|A(p_T(\varphi))|^2$ [Eq. (16)] (left panel) and $\Phi'(p_T(\varphi))$ [Eq. (19)] (right panel), as functions of $\varphi$. Here, we have chosen the intensity of the laser pulse to be $I = 5 \times 10^{16}$ Wcm$^{-2}$. The increments in momentum are, as stated before, $\delta p_{\perp} = \delta p_{\parallel} = \delta \varphi_T = 0$ and we set $\delta \varphi_T = -0.1\pi$. As the photoelectrons are emitted with large probability when the laser pulse acquires its maximum strength, we have chosen the phase $\phi = \pi$. Also, for those parameters and taking into account the spiral of ionization we set $\varphi_T = 0$ and $\theta_T = 0.37\pi$. The perpendicular and parallel components of the asymptotic momentum are $p_{\perp} = 0.055m_ec$ and $p_{\parallel} = 0.17m_ec$, respectively. While in both panels of Fig. 2 the solid blue curves come from the RSFA treatment, the dashed red ones are the results from the QRSFA accounting for the full-recoil and relativistic mass corrections. Again, both
treatments agree very well with each other for the parameters considered here. From the left panel, it can be seen that the largest values of the distribution are obtained at twisted angles close to zero. This is actually expected, as $p_T(\varphi)$ is closest to the spiral in such region. In the right panel we observe that $\Phi'(p_T(\varphi))$ strongly depends on the twist angle. Hence, it is expected that higher-order derivatives do not vanish and the resulting OAM structure should exhibit a chirp [29]. Also, the maximum value achieved by $\Phi'(p_T(\varphi))$ is approximately 3300, which suggests that EVS with topological charges $m \approx 3300$ can be observed with large probability.

In Fig. 3 we present the OAM distributions for the same parameters as in Fig. 2 in both the linear (left panel) and logarithmic (right panel) scales. For visual purposes, the discrete values have been joined by the thin-blue lines. This time, we restrict our calculations to the QRSFA with recoil and mass corrections accounted for. In the left panel we observe a chirped structure characterized by a long plateau followed by a dense series of peaks. The maximum of the distribution is found at $m \approx 3270$, which coincides very well with our previous estimates. From the right panel one can see the formation of a supercontinuum in OAM, where the distribution does not change drastically in a range of tenths to hundreds of units of $\hbar$. Note that a similar situation, i.e., the formation of long supercontinua, is observed in the energy spectra of photoelectrons when the driving pulse is sufficiently short and intense [38, 41, 42, 43]. We expect that the formation of such structure in the OAM distribution can be used in the synthesis of very short-in-time electron wave packets with large angular momenta. This could open a new door to the development of experimental techniques for the characterization of magnetic properties of matter in very short time scales.

5. Conclusions

We have analyzed the role played by the retardation, recoil, ground state energy, and relativistic mass corrections in the QRSFA. According to our calculations, a proper description of photoionization driven by near-infrared laser fields with intensities close to $5 \times 10^{16}$ Wcm$^{-2}$ requires the use of the full recoil and relativistic mass corrections. Only in such case, a good agreement between the QRSFA and the RSFA based on the Dirac equation can be achieved.

On the other hand, we have proven that EVS with topological charges of few thousands can be observed during the ionization of light ions/atoms by intense, short, and circularly-polarized laser fields. This happens when the family of twisted states approaches the three-dimensional momentum spiral. In such case, the OAM distribution exhibits a long angular-momentum supercontinuum followed by series of peaks and a sharp cutoff. We expect that those properties would be of interest for the generation of EVS with large topological charges and short durations.
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