Globally positive solutions of linear parabolic partial differential equations of second order with Dirichlet boundary conditions

Janusz Mierczyński
Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
PL-50-370 Wrocław
Poland
mierczyn@banach.im.pwr.wroc.pl

The purpose of this paper is to study globally positive solutions of a linear nonautonomous parabolic partial differential equation (PDE) of second order

\[ u_t = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + a_0(t,x)u, \quad t \in \mathbb{R}, x \in \Omega, \] (E)

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain, complemented with the homogeneous Dirichlet boundary conditions

\[ u(t,x) = 0, \quad t \in \mathbb{R}, x \in \partial \Omega. \] (BC)

The main result (Theorem 2.11) states that the set of solutions to (E)+(BC) that are defined and of constant sign for all \( t \in \mathbb{R} \) and \( x \in \Omega \) is a one-dimensional vector space.

S.-N. Chow, K. Lu and J. Mallet-Paret in their 1995 paper [4] were the first to address the issue for \( n = 1 \) (see also their earlier paper [3]). In fact, they proved much more: There is an invariant decomposition of the vector space of all global (in time) solutions into the direct sum of countably many one-dimensional subspaces indexed by positive integers \( l \). They obtained a characterization of the \( l \)-th subspace as the set of all global solutions for which the (Matano) lap number is constantly \( l - 1 \).

For a general space dimension \( n \), the present author in [7] has considered the problem of characterizing globally positive solutions of a linear second order parabolic PDE under Robin (regular oblique) boundary conditions. The reasoning from that paper, however, fails to carry over to general boundary conditions.

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This paper is organized as follows. The starting point is to show (in Section 1) that the solution operator for \((E)+(BC)\) and all limits of its time translates possess a sufficiently regular kernel (Green’s function). Our principal results are included in Section 2. Here, by applying ideas from the theory of linear skew-product semidynamical systems on Banach bundles, we show that globally positive solutions must be contained in some invariant subbundle of dimension one. In the Appendix generalizations to other boundary conditions are given.

1 Basic properties of solutions

We write \(\mathbb{R}^n = \{ (x_1, \ldots, x_n) : x_i \in \mathbb{R} \} \) for the \(n\)-dimensional real vector space with the Euclidean norm \(\| \cdot \| \), and \(S^{n-1}\) for the unit sphere in \(\mathbb{R}^n\).

For a function \(f : S \to Y\), where \(S \subset \mathbb{R}^n\) and \(Y\) is a Banach space, we use the notation \(D_i := \partial / \partial x_i\). If \(f\) is defined on \(S \subset \mathbb{R} \times \mathbb{R}^n = \{(t, x_1, \ldots, x_n)\}\), we write \(D_t := \partial / \partial t\). As usual, \(D_{ij}\) means \(D_i D_j\). The derivatives may be considered in the classical sense, or in the distribution sense, depending on the context. For \(v \in \mathbb{S}^{n-1}\) we understand by \(D_v\) the directional derivative in the direction of \(v\).

Given Banach spaces \(X, Y\), we write \(\mathcal{L}(X,Y)\) for the Banach space of bounded linear maps from \(X\) into \(Y\) endowed with the uniform operator topology (the norm topology), and \(\mathcal{K}(X,Y)\) for the set of compact (completely continuous) operators in \(\mathcal{L}(X,Y)\). If \(X = Y\) we write simply \(\mathcal{L}(X)\) and \(\mathcal{K}(X)\).

For a metrizable topological space \(S\) and a topological vector space \(Y\) the symbol \(C(S,Y)\) denotes the vector space of continuous functions from \(S\) into \(Y\). If \(S \subset \mathbb{R}^n\), \(C^1(S,Y)\) (resp. \(C^2(S,Y)\)) stands for the vector space of once (resp. twice) continuously differentiable functions. When \(S\) is compact and \(Y\) is a Banach space, \(C^0(S,Y)\), \(C(S,Y)\), are understood to be Banach spaces with the usual norms. If \(Y = \mathbb{R}\) we write \(C^r(S), C(S)\).

Throughout the paper \(\mathcal{A}\) defines the differential operator

\[
\mathcal{A}u := - \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u - \sum_{i=1}^{n} a_i(x) D_i u, \quad (1.1)
\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain with boundary \(\partial \Omega\) of class \(C^2\), \(\overline{\Omega} = \Omega \cup \partial \Omega\), \(a_{ij} = a_{ji} \in C^2(\overline{\Omega})\), \(\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j > 0\) for each \(x \in \overline{\Omega}\) and each nonzero \((\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\), \(a_i \in C^1(\overline{\Omega})\). If the principal part of \(\mathcal{A}\) is in the divergence form:

\[
\mathcal{A}u = - \sum_{i,j=1}^{n} D_i (a_{ij}(x) D_j u) - \sum_{i=1}^{n} \tilde{a}_i(x) D_i u,
\]

it suffices to assume \(a_{ij} \in C^1(\overline{\Omega})\).

By \(\mathcal{B}\) we denote the boundary operator

\[
\mathcal{B}u := u|_{\partial \Omega}. \quad (1.2)
\]

For \(i = 1, 2\), define \(C^i_0(\overline{\Omega}) := \{ u \in C^i(\overline{\Omega}) : \mathcal{B}u = 0 \}\). Similarly, \(C_0(\overline{\Omega}) := \{ u \in C_0 : \mathcal{B}u = 0 \}\). It is obvious that \(C^i_0(\overline{\Omega})\) (resp. \(C_0(\overline{\Omega})\)) is a closed subspace of the Banach space \(C^i(\overline{\Omega})\) (resp. \(C(\overline{\Omega})\)).
From now on we denote by $A$ the closure of $A$ in $L^2(\Omega)$ and assume that $a_0 \in L^\infty(\mathbb{R} \times \Omega)$ is fixed. Put

$$R := \text{ess sup}_{t \in \mathbb{R}, x \in \Omega} |a_0(t, x)|.$$ 

It is well known that the closed ball in $L^\infty(\mathbb{R} \times \Omega)$ with center 0 and radius $R$ endowed with the weak* topology is a metrizable compact space.

For $b \in L^\infty(\mathbb{R} \times \Omega)$ and $t \in \mathbb{R}$ by the $t$-translate, $b \cdot t$, of $b$ we understand the function $(b \cdot t)(s, x) := b(t + s, x)$ for a.e. $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Define the hull $\mathcal{B}$ of $a_0$ as the closure in the weak* topology of the set $\{a_0 \cdot t : t \in \mathbb{R}\}$.

For $b \in \mathcal{B}$ and a measurable function $u : \Omega \to \mathbb{R}$ set $N(b) u := b(0, \cdot) u$. The multiplication operator $N(b)$ is easily seen to belong to $L(L^2(\Omega))$.

By a (mild) solution to the parabolic partial differential equation

$$D_t u = \sum_{i,j=1}^n a_{ij}(x) D_{ij} u + \sum_{i=1}^n a_i(x) D_i u + b(t, x) u, \quad t > 0, x \in \Omega,$$

with the boundary condition

$$u(t, x) = 0, \quad t > 0, x \in \partial \Omega,$$

satisfying the initial condition

$$u(0, x) = u_0(x),$$

where $u_0 \in L^2(\Omega)$, and $b \in \mathcal{B}$ is considered a parameter, we understand a function $u(\cdot; b, u_0) \in C([0, \infty), L^2(\Omega))$ satisfying the integral equation

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} N(b \cdot s) u(s) \, ds \quad \text{for all } t \geq 0,$$

where $\{e^{-At}\}_{t \geq 0}$ is the holomorphic semigroup of bounded linear operators on $L^2(\Omega)$ generated by $(-A)$.

The proof of the next result may be patterned after the proof of Thm. 3.3 in Chow, Lu and Mallet-Paret [4] and Thm. 1.3 in Mierczyński [7], and we do not present it here.

**Theorem 1.1.** (i) For each $b \in \mathcal{B}$ and $u_0 \in L^2(\Omega)$ there exists a unique solution $u(\cdot; b, u_0)$ to (1.3)+(1.4)+(1.5).

(ii) For each $T > 0$ the mapping

$$\mathcal{B} \times L^2(\Omega) \ni (b, u_0) \mapsto u(\cdot; b, u_0) \in C([0,T], L^2(\Omega))$$

is continuous.

(iii) For any $0 < T_1 \leq T_2$ the mapping

$$\mathcal{B} \ni b \mapsto u(\cdot; b, \cdot) \in C([T_1,T_2], K(L^2(\Omega), C^1_{\partial \Omega}))$$

is continuous.
Recall that to $(E)+(BC)$ we assign the \textit{(formally) adjoint equation}
\[
D_t v = -\sum_{i,j=1}^{n} D_{ij}(a_{ij}(x)v) + \sum_{i=1}^{n} D_i(a_i(x)v) - a_0(t,x)v, \quad t < 0, x \in \Omega, \quad (E^2)
\]
complemented with the homogeneous Dirichlet boundary conditions
\[
v(t,x) = 0, \quad t < 0, x \in \partial\Omega. \quad (BC^2)
\]
We define the \textit{adjoint differential operator},
\[
A^t v := -\sum_{i,j=1}^{n} D_{ij}(a_{ij}(x)v) + \sum_{i=1}^{n} D_i(a_i(x)v), \quad (1.7)
\]
and the \textit{adjoint boundary operator} $B^t := B$.
Let $A'$ stand for the closure of $A^t$ in $L^2(\Omega)$. For $b \in B$ and $v_0 \in L^2(\Omega)$ we say that $v(\cdot; b, v_0) \in C((-\infty, 0], L^2(\Omega))$ is a \textit{(mild) solution} to the equation
\[
D_t v = -\sum_{i,j=1}^{n} D_{ij}(a_{ij}(x)v) + \sum_{i=1}^{n} D_i(a_i(x)v) - b(t,x)v, \quad t < 0, x \in \Omega, \quad (1.8)
\]
with the boundary condition
\[
v(t,x) = 0, \quad t < 0, x \in \partial\Omega, \quad (1.9)
\]
satisfying the initial condition
\[
v(0, x) = v_0(x), \quad (1.10)
\]
if it satisfies the integral equation
\[
v(t) = e^{-A'(t)}v_0 + \int_{t}^{0} e^{-A'(s-t)}\mathcal{N}(b \cdot s)v(s) \, ds \quad \text{for all } t \leq 0, \quad (1.11)
\]
where $\{e^{-A't}\}_{t \geq 0}$ is the holomorphic semigroup of bounded linear operators on $L^2(\Omega)$ generated by $(-A')$.
Obviously we have the following counterpart to Theorem 1.1:

**Theorem 1.2.** \(\text{(i)}\) For each $b \in B$ and $v_0 \in L^2(\Omega)$ there exists a unique solution $v(\cdot; b, v_0)$ to $(1.8) + (1.9) + (1.10)$.
\(\text{(ii)}\) For each $T < 0$ the mapping
\[
B \times L^2(\Omega) \ni (b, v_0) \mapsto v(\cdot; b, v_0) \in C([T, 0], L^2(\Omega))
\]
is continuous.
\(\text{(iii)}\) For any $T_1 \leq T_2 < 0$ the mapping
\[
B \ni b \mapsto v(\cdot; b, \cdot) \in C([T_1, T_2], \mathcal{K}(L^2(\Omega), C_{0}^1(\Omega)))
\]
is continuous.
We mention here an important property.

**Proposition 1.3.** (i) For \( t_1, t_2 \geq 0, b \in \mathbb{B} \) and \( u_0 \in L^2(\Omega) \) one has
\[
  u(t_1 + t_2; b, u_0) = u(t_1; b \cdot t, u(t_2; b, u_0)) = u(t_2; b \cdot t_1, u(t_1; b, u_0)).
\]

(ii) For \( t_1, t_2 \leq 0, b \in \mathbb{B} \) and \( v_0 \in L^2(\Omega) \) one has
\[
  v(t_1 + t_2; b, v_0) = v(t_1; b \cdot t_2, v(t_2; b, v_0)) = v(t_2; b \cdot t_1, v(t_1; b, v_0)).
\]

For \( b \in \mathbb{B} \) and \( t > 0 \) define the linear operator \( \tilde{\psi}(t, b) \in \mathcal{L}(L^2(\Omega), C^1(\overline{\Omega})) \) as
\[
  \tilde{\psi}(t, b)u_0 := u(t; b, u_0).
\]
and, for \( b \in \mathbb{B} \) and \( t < 0 \) define \( \tilde{\psi}^\ast(t, b) \in \mathcal{L}(L^2(\Omega), C^1(\overline{\Omega})) \) as
\[
  \tilde{\psi}^\ast(t, b)v_0 := v(t; b, v_0).
\]

Occasionally we will consider solutions defined on open intervals. We say that \( u \in C(J, L^2(\Omega)) \), where \( J \subseteq \mathbb{R} \) is an interval, is a solution of \( (1.3) + (1.4) \) on \( J \) if for any \( T_1, T_2 \in J, T_1 \leq T_2 \), one has
\[
  u(T_2) = \tilde{\psi}(T_2 - T_1, b \cdot T_1)u(T_1).
\]
Solutions of \( (1.3) + (1.4) \) on \( J \) are defined in an analogous way.

Put \( \hat{\psi}(t, b) := i_{2} \circ \tilde{\psi}(t, b), \hat{\psi}^\ast(t, b) := i_{2} \circ \tilde{\psi}^\ast(t, b) \), where \( i_{2} \) denotes the embedding \( C^1(\overline{\Omega}) \subset L^2(\Omega) \).

An important consequence of Proposition 1.3 is the following cocycle identity.
\[
  \hat{\psi}(t_1 + t_2, b) = \hat{\psi}(t_1, b \cdot t_2) \circ \hat{\psi}(t_2, b), \quad b \in \mathbb{B}, t_1 \geq 0, t_2 \geq 0,
\] (1.12)
where we understand \( \hat{\psi}(0, b) = \text{Id}_{L^2(\Omega)} \).

As the proof of the next proposition is based in the standard way on Green’s equality, we do not present it here.

**Proposition 1.4.** For \( b \in \mathbb{B} \) and \( t > 0 \) we have
\[
  \hat{\psi}(t, b)^* = \hat{\psi}^\ast(-t, b \cdot t),
\]
where \( ^* \) represents the dual operator.

Henceforth, the symbol \( [\cdot, \cdot] \) will stand for the duality pairing between \( C^1_0(\overline{\Omega}) \) and \( C^1_0(\overline{\Omega})^* \).

**Theorem 1.5.** For \( t > 0 \) and \( b \in \mathbb{B} \) the operators \( \hat{\psi}(t, b) \) and \( \hat{\psi}^\ast(-t, b \cdot t) \) extend respectively to the operators \( \psi(t, b) \in \mathcal{K}(C^1_0(\overline{\Omega})^*, C^1_0(\overline{\Omega})) \), \( \psi(t, b)' \in \mathcal{K}(C^1_0(\overline{\Omega})^*, C^1_0(\overline{\Omega})) \). These extensions are continuous in \( (t, b) \in (0, \infty) \times \mathbb{B} \).

Also,
\[
  [\psi(t, b)u_0|v_0] = [\psi(t, b)'v_0|u_0]
\]
for all \( t > 0, b \in \mathbb{B}, u_0 \) and \( v_0 \in C^1_0(\overline{\Omega})^* \). (Consequently, \( \psi(t, b)^* = \psi(t, b)' \))
Proof. By the cocycle identity (1.12) one has
\[ \tilde{\psi}(t, b) = \tilde{\psi}(t/2, b) \circ \tilde{\psi}(t/2, b), \]
with \( \tilde{\psi}(t/2, b) \in \mathcal{K}(L^2(\Omega)) \) and \( \tilde{\psi}(t/2, b \cdot (t/2)) \in \mathcal{K}(L^2(\Omega), C_0^1(\Omega)) \). Taking into account Proposition 1.4 one obtains
\[ \hat{\psi}(t/2, b) = \hat{\psi}(t/2, b)^* = (i_2 \circ \tilde{\psi}(t/2, b) \circ \tilde{\psi}(t/2, b))^* \]
\[ = \tilde{\psi}^*(-t/2, b \cdot (t/2))^* \circ i_2^*. \]
(1.13) yields
\[ \tilde{\psi}(t, b)^* = \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ i_2^*. \]
Again by Proposition 1.4 one obtains
\[ \hat{\psi}(t/2, b)^* = \hat{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ i_2^*. \]
We set
\[ \psi(t, b) := \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ \tilde{\psi}(t/2, b)^* \circ i_2^*. \]
The assertion on duality follows by the definition. The continuous dependence is a consequence of Theorems 1.1 and 1.2 and the continuity of the composition of linear operators in the uniform operator topology. \( \square \)

As we shall see now, an easy consequence of the above theorem is that \( \tilde{\psi}(t, b) \) is an integral operator with a fairly regular kernel \( G(t, b)(\cdot, \cdot) \) (the Green’s function).

We write \( \delta_x \) for the Dirac delta at \( x \in \overline{\Omega} \), and \( \delta'_{(x,v)} \) for the directional derivative operator at \( x \in \overline{\Omega} \) in the direction of \( v \in S^{n-1} \). Both \( \delta_x \) and \( \delta'_{(x,v)} \) are regarded as elements of \( C_0^1(\Omega)^* \).

For \( t > 0, b \in B, x \in \overline{\Omega} \) set
\[ G(t, b)(x, \cdot) := \psi(t, b)^* \delta_x, \]
and
\[ G^*(t, b)(x, \cdot) := \psi(t, b) \delta_x. \]
(1.14)
(1.15)
For \( u_0 \) and \( v_0 \) in \( L^2(\Omega) \) one has
\[ (\psi(t, b)u_0)(x) = [\psi(t, b)u_0] \delta_x \]
\[ = [\psi(t, b)^* \delta_x] u_0(x) = \int_{\Omega} G(t, b)(x, \xi) u_0(\xi) d\xi \]
(1.16)
and
\[ (\psi(t, b)^* v_0)(x) = [\psi(t, b)^* v_0] \delta_x \]
\[ = [\psi(t, b) \delta_x] v_0(x) = \int_{\Omega} G^*(t, b)(x, \xi) v_0(\xi) d\xi. \]
(1.17)
Lemma 1.6. The mappings
\[(0, \infty) \times B \times \Omega \ni (t, b, x) \mapsto G(t, b)(x, \cdot) \in C_0^1(\Omega)\]
and
\[(0, \infty) \times B \times \Omega \ni (t, b, x) \mapsto G^*(t, b)(x, \cdot) \in C_0^1(\Omega)\]
are continuous.

Proof. We prove the result only for \(G\), the proof for \(G^*\) being similar. Let \(t_k \to t > 0\), \(b_k \to b\) and \(x(k) \to x\). In particular, \(\delta_{x(k)}\) converges weakly-* to \(\delta_x\). One has
\[
\|G(t_k, b_k)(x(k), \cdot) - G(t, b)(x, \cdot)\|_{C^1(\Omega)} = \|\psi(t_k, b_k)^* \delta_{x(k)} - \psi(t, b)^* \delta_x\|_{C^1(\Omega)}
\leq \|\psi(t_k, b_k)^* \delta_{x(k)} - \psi(t, b)^* \delta_{x(k)}\|_{C^1(\Omega)} + \|\psi(t, b)^* \delta_{x(k)} - \psi(t, b)^* \delta_x\|_{C^1(\Omega)}.
\]
The first term is estimated by
\[
\|\psi(t_k, b_k)^* - \psi(t, b)^*\|_{C^1(\Omega)} \leq \|\psi(t_k, b_k)^* - \psi(t, b)\|_{L(C_0^1(\Omega), C_0^1(\Omega))},
\]
which tends, by Theorem 1.5, to 0 as \(k \to \infty\). One proves that the second term approaches 0 by noting that the dual \(\psi(t, b)^*\) of the compact operator \(\psi(t, b)\) takes weakly-* convergent sequences into (norm) convergent ones.

Lemma 1.7. For each \(t > 0\), \(b \in B\), \(x, \xi \in \Omega\) we have \(G^*(t, b)(x, \xi) = G(t, b)(\xi, x)\).

Proof. Let \(u_0, \psi_0 \in C(\Omega)\). One has
\[
[\psi(t, b)u_0]_{\psi_0} = \int_{\Omega} \left( \int_{\Omega} G(t, b)(x, \xi) u_0(x) d\xi \right) \psi_0(x) dx
= \int_{\Omega} \left( \int_{\Omega} \psi(t, b)^* u_0(x) d\xi \right) \psi_0(x) dx
\]
and
\[
[\psi(t, b)^* u_0]_{\psi_0} = \int_{\Omega} \left( \int_{\Omega} G^*(t, b)(x, \xi) \psi_0(\xi) d\xi \right) u_0(x) dx
= \int_{\Omega} u_0(x) \left( \int_{\Omega} G^*(t, b)(x, \xi) \psi_0(\xi) d\xi \right) dx.
\]
As \([\psi(t, b)u_0]_{\psi_0} = [\psi(t, b)^* u_0]_{\psi_0}\), we get
\[
\iint_{\Omega \times \Omega} G^*(t, b)(x, \xi) u_0(x) \psi_0(\xi) dx d\xi = \iint_{\Omega \times \Omega} G(t, b)(\xi, x) u_0(x) \psi_0(\xi) dx d\xi,
\]
whence we derive our assertion in the standard way.
From the definition and Lemma 1.7 it follows directly that \( G \) is a solution of (1.3) + (1.4) in \( x \) on \((0, \infty)\) and a solution of the adjoint equation (1.8) + (1.9) in \( \xi \) on \((0, \infty)\).

By
\[
\frac{\partial}{\partial x} G(t, b)(x, \xi) \quad \text{(resp. } \frac{\partial}{\partial \xi} G(t, b)(x, \xi)\text{)}
\]
we denote the directional derivative of \( G \) in \( x \) (resp. in \( \xi \)) in the direction of \( v \in S^{n-1} \). The next result shows that the above derivatives are solutions of the corresponding equations on \((0, \infty)\).

**Proposition 1.8.**

(i) For \( t > 0, b \in \mathbb{B}, x \in \overline{\Omega} \) and \( v \in S^{n-1} \) we have
\[
\frac{\partial}{\partial x} G(t, b)(x, \cdot) = \psi(t, b)^* \delta'(x, v).
\]

(ii) For \( t > 0, b \in \mathbb{B}, \xi \in \overline{\Omega} \) and \( v \in S^{n-1} \) we have
\[
\frac{\partial}{\partial \xi} G(t, b)(\cdot, \xi) = \psi(t, b) \delta'(\xi, v).
\]

**Proof.** We consider only (i) as the case (ii) is treated similarly. It suffices to prove that
\[
[\frac{\partial}{\partial v_x} G(t, b)(x, \cdot)|u_0] = [\psi(t, b)^* \delta'(x, v)|u_0]
\]
for each \( u_0 \in C(\overline{\Omega}) \). The right-hand side of (1.18) can be written as
\[
[\psi(t, b)^* \delta'(x, v)|u_0] = [\psi(t, b)u_0|\delta'(x, v)] = \frac{\partial}{\partial v_x} \left( \int_\Omega G(t, b)(x, \xi) u_0(\xi) \, d\xi \right).
\]

Because by Lemma 1.6 \( (\partial/\partial v_x)G(t, b) \) is a continuous function of \((x, \xi) \in \overline{\Omega} \times \overline{\Omega}\), we can interchange differentiation and integration in the right-hand expression. But obviously
\[
\int_\Omega \frac{\partial}{\partial v_x} G(t, b)(x, \xi) u_0(\xi) \, d\xi = [u_0|\frac{\partial}{\partial v_x} G(t, b)(x, \xi)].
\]

\( \square \)

The arguments of the proof of Lemma 1.6 easily extend to show the following.

**Lemma 1.9.** The mappings
\[
(0, \infty) \times \mathbb{B} \times \overline{\Omega} \times S^{n-1} \ni (t, b, x, v) \mapsto (\partial/\partial v_x)G(t, b)(x, \cdot) \in C^1_0(\overline{\Omega})
\]
and
\[
(0, \infty) \times \mathbb{B} \times \overline{\Omega} \times S^{n-1} \ni (t, b, \xi, v) \mapsto (\partial/\partial v_\xi)G(t, b)(\cdot, \xi) \in C^1_0(\overline{\Omega})
\]
are continuous.
2 Positive solutions

For a Banach space $X$ consisting of real functions (resp. of equivalence classes, modulo measure zero sets, of real functions) defined on $\Omega$, by $X_+$ we denote the (standard) cone of nonnegative functions (resp. of equivalence classes of functions nonnegative a.e.). It is straightforward that $X_+$ is a closed convex set such that
\begin{itemize}
  \item[a)] For each $u \in X_+$ and $\alpha \geq 0$ one has $\alpha u \in X_+$, and
  \item[b)] $X_+$ does not contain any one-dimensional subspace.
\end{itemize}

If $X = L^2(\Omega)$, or $X = C(\Omega)$, or $X = C^1(\Omega)$, the cone $X_+$ is reproducing, that is, $X_+ + X_+ = X$.

For $X = L^2(\Omega)$ or $X = C(\Omega)$, let $\leq$ denote the partial ordering induced on $X$ by the cone $X_+$:
\[
  v \leq u \quad \text{if } u - v \in X_+.
\]
We write $v < u$ if $v \leq u$ and $v \neq u$. The reverse symbols are used in the standard way.

The cone $C_0(\Omega)_+$ (resp. $C^1_0(\Omega)_+$) consists of those $u \in C_0(\Omega)$ (resp. $u \in C^1_0(\Omega)$) for which $u(x) \geq 0$ at each $x \in \Omega$. The cone $C_0(\Omega)_+$ has empty interior, whereas it is easy to verify that the interior of $C^1_0(\Omega)_+$ equals
\[
  C^1_0(\Omega)_{++} := \{ u \in C^1_0(\Omega) : u(x) > 0 \text{ for } x \in \Omega \}.
\]
where $D_\nu$ is the derivative in the direction of the unit normal vector field $\nu : \partial \Omega \to \mathbb{R}^n$ pointing out of $\Omega$. The partial ordering induced on $C^1_0(\Omega)$ by $C^1_0(\Omega)_+$ is denoted by $\leq_1$. For $u, v \in C^1_0(\Omega)$ we write $u <_1 v$ if $u \leq_1 v$ and $u \neq v$, and $u \ll_1 v$ if $v - u \in C^1_0(\Omega)_{++}$.

Let $\mathbf{e}$ be the (nonnegative) principal eigenfunction of the Laplacian on $\Omega$ with Dirichlet boundary conditions normalized so that $\|\mathbf{e}\|_{C(\Omega)} = 1$. The standard regularity theory and the maximum principle for elliptic PDE’s yield the following.

**Lemma 2.1.** (i) $\mathbf{e} \in C^1(\Omega)$.
(ii) $\mathbf{e}(x) > 0$ for $x \in \Omega$.
(iii) $\mathbf{e}(x) = 0$ and $D_\nu \mathbf{e}(x) < 0$ for $x \in \partial \Omega$.
(In other words, $\mathbf{e} \in C^1_0(\Omega)_{++}$.)

For $u \in C(\Omega)$ define the order-unit norm of $u$ as
\[
  \|u\|_\mathbf{e} := \inf\{ \alpha > 0 : -\alpha \mathbf{e}(x) \leq u(x) \leq \alpha \mathbf{e}(x) \text{ for each } x \in \Omega\}.
\]
By Amann [1], the set
\[
  C_\mathbf{e}(\Omega) := \{ u \in C(\Omega) : \|u\|_\mathbf{e} < \infty \}
\]
is a Banach space with the norm $\|\cdot\|_\mathbf{e}$, partially ordered by the relation $\leq_\mathbf{e}$ induced by the cone $C_\mathbf{e}(\Omega)_+ := C_\mathbf{e}(\Omega) \cap C(\Omega)_+$ with nonempty interior $C_\mathbf{e}(\Omega)_{++}$. Evidently $C_\mathbf{e}(\Omega)$ embeds (set-theoretically) in $C(\Omega)$. For $u, v \in C_\mathbf{e}(\Omega)$ we write $u <_\mathbf{e} v$ if $u \leq_\mathbf{e} v$ and $u \neq v$, and $u \ll_\mathbf{e} v$ if $v - u \in C_\mathbf{e}(\Omega)_{++}$.

The order-unit norm $\|\cdot\|_\mathbf{e}$ is monotonic, that is, $0 \leq_\mathbf{e} u \leq_\mathbf{e} v$ implies $\|u\|_\mathbf{e} \leq \|v\|_\mathbf{e}$.
Since, as can be easily verified from the definition, \( \|u\|_{C(\overline{\Omega})} \leq \|u\|_{e} \) for each \( u \in C_{e}(\overline{\Omega}) \), the embedding of \( C_{e}(\overline{\Omega}) \) into \( C_{0}(\overline{\Omega}) \) is continuous. Moreover, it embeds the cone \( C_{e}(\overline{\Omega})_{+} \) into the cone \( C_{0}(\overline{\Omega})_{+} \).

Before proceeding further, we introduce a concept that will be useful in the sequel. Define the mapping \( g : \partial \Omega \times [0,1) \to \mathbb{R}^{n} \) by the formula \( g(x,r) := x - r\nu(x) \). This mapping is easily seen to be of class \( C^{1} \). Further, for each \( x \in \partial \Omega \) its derivative at \((x,0)\) is an isomorphism. By the Inverse Function Theorem there is a neighborhood \( U \) of \( \partial \Omega \times \{0\} \) in \( \partial \Omega \times [0,1) \) such that \( g|_{U} \) is a \( C^{1} \) diffeomorphism onto its image \( V \). Taking \( V \) smaller if necessary we can assume \( V \subset \overline{\Omega} \). Now we define

\[
r := \pi_{1} \circ g^{-1},
\]

where \( \pi_{1} \) is the projection onto the first coordinate. It is straightforward that \( r \) is a \( C^{1} \) retraction of \( V \) onto \( \partial \Omega \).

**Proposition 2.2.**  
(i) The natural embedding \( i : C_{0}^{1}(\overline{\Omega}) \subset C_{e}(\overline{\Omega}) \) is continuous.

(ii) \( i(C_{0}^{1}(\overline{\Omega})_{++}) \subset C_{e}(\overline{\Omega})_{++} \).

**Proof.** To prove (i), suppose by way of contradiction that there is a sequence \( (u_{k})_{k=1}^{\infty} \subset C_{0}^{1}(\overline{\Omega}) \) such that \( \|u_{k}\|_{C^{1}([\overline{\Omega}])} = 1 \) and \( \|u_{k}\|_{C_{0}(\overline{\Omega})} \to k \) for each \( k \in \mathbb{N} \). Passing to a subsequence if necessary we can assume that \( u_{k} \) converge in the \( C(\overline{\Omega}) \)-norm to some \( \tilde{u} \in C_{0}(\overline{\Omega}) \). By the definition of \( C_{e}(\overline{\Omega}) \), for each \( k \in \mathbb{N} \) there is \( x_{(k)} \in \overline{\Omega} \) such that \( |u_{k}(x_{(k)})| \to ke(x_{(k)}) \). Obviously, no such \( x_{(k)} \) can belong to \( \partial \Omega \). Further, there is no subsequence of \( (x_{(k)}) \) converging to \( \tilde{x} \in \Omega \), since otherwise \( |\tilde{u}(\tilde{x})| \to ke(\tilde{x}) \) for each \( k \in \mathbb{N} \), which contradicts \( e(\tilde{x}) > 0 \). Now we take a subsequence (denoted again by \( (x_{(k)}) \)) converging to \( \tilde{x} \in \partial \Omega \). The continuity of the retraction \( r \) implies \( \lim_{k \to \infty} r(x_{(k)}) = r(\tilde{x}) = \tilde{x} \). For each sufficiently large \( k \) we have

\[
\frac{e(x_{(k)}) - e(r(x_{(k)}))}{\|x_{(k)} - r(x_{(k)})\|} < \frac{1}{k} \frac{|u_{k}(x_{(k)}) - u_{k}(r(x_{(k)}))|}{\|x_{(k)} - r(x_{(k)})\|}.
\]

The expression on the left-hand side tends, as \( k \to \infty \), to \( -D_{e}e(\tilde{x}) \), which is positive by Lemma 2.1. The Mean Value Theorem tells us that for each \( k \in \mathbb{N} \) there is \( \tilde{x}(k) \) \( \in \Omega \) belonging to the line segment with endpoints \( x_{(k)} \) and \( r(x_{(k)}) \), such that

\[
u_{k}(x_{(k)}) - u_{k}(r(x_{(k)})) = -\frac{\partial u_{k}}{\partial r(x_{(k)})}(\tilde{x}(k)) \cdot \|x_{(k)} - r(x_{(k)})\|.
\]

As the derivatives of all \( u_{k} \)'s are bounded, this yields \( D_{e}e(\tilde{x}) = 0 \), a contradiction.

Assume now that \( u \in C_{0}^{1}(\overline{\Omega})_{++} \). It is straightforward that \( u \in C_{e}(\overline{\Omega})_{+} \). In order to prove that \( u \in C_{e}(\overline{\Omega})_{++} \) it is enough to show that there is \( K > 0 \) such that \( u(x) \geq Ke(x) \) for each \( x \in \overline{\Omega} \). Suppose not. Then there is a sequence \( (x_{(k)})_{k=1}^{\infty} \) such that \( u(x_{(k)}) < ke(x_{(k)}) \) for each \( k \in \mathbb{N} \). Evidently each \( x_{(k)} \) belongs to \( \Omega \). As in the proof of (i) we prove that each convergent subsequence of \( (x_{(k)})_{k=1}^{\infty} \) tends to some \( \tilde{x} \in \partial \Omega \), from which it follows that \( D_{e}e(\tilde{x}) = 0 \).  

\( \square \)
We proceed now to investigating order-preserving properties of $\psi(t, b)$. The following theorem is a consequence of the parabolic strong maximum principle (compare problems 3.3.6–3.3.8 in Henry’s book [5]).

**Theorem 2.3.** (i) For $t > 0$, $b \in \mathbb{B}$ and a nonzero $u_0 \in L^2(\Omega)_+$ we have $\psi(t, b)u_0 \in C^1_0(\overline{\Omega})_{++}$.

(ii) For $t > 0$, $b \in \mathbb{B}$ and a nonzero $v_0 \in L^2(\Omega)_+$ we have $\psi(t, b)^*v_0 \in C^1_0(\overline{\Omega})_{++}$.

For $t > 0$, $b \in \mathbb{B}$ put

$$
\psi(t, b) := i \circ \psi(t, b) \circ I,
$$

where $i : C^1_0(\overline{\Omega}) \subset C_e(\overline{\Omega})$, $I : C_e(\overline{\Omega}) \subset C^1_0(\overline{\Omega})^*$ are the natural embeddings. The cocycle identity (see (1.12)) takes the form

$$
\psi(t_1 + t_2, b) = \psi(t_1, b \cdot t_2) \circ \psi(t_2, b), \quad b \in \mathbb{B}, t_1 > 0, t_2 > 0. \tag{2.1}
$$

Set $\mathcal{W}$ to be the product Banach bundle $\mathbb{B} \times C_e(\overline{\Omega})$ with base space $\mathbb{B}$ and model fiber $C_e(\overline{\Omega})$. We define an endomorphism $\Psi_e$ of the bundle $\mathcal{W}$ by the formula

$$
\Psi_e(b, u) = (b \cdot 1, \psi_e(b)u), \quad b \in \mathbb{B}, u \in C(\overline{\Omega}),
$$

where $\psi_e(b) := \psi_e(1, b)$.

The mapping $\Psi_e$ is an endomorphism of the Banach bundle $\mathcal{W}$, covering the homeomorphism $\phi$, $\phi(b) = b \cdot 1$, of the base space $\mathbb{B}$. Its iterates $\Psi^k_e$, $k \in \mathbb{N}$, form the linear skew-product semidynamical system $\{\Psi^k_e\}_{k=1}^{\infty}$ on $\mathcal{W}$. For $k \in \mathbb{N}$ we write

$$
\Psi^k_e(b, u) = (b \cdot k, \psi^{(k)}_e(b)u),
$$

where we denote

$$
\psi^{(k)}_e(b) := \psi_e(b \cdot (k - 1)) \circ \psi_e(b \cdot (k - 2)) \circ \cdots \circ \psi_e(b \cdot 1) \circ \psi_e(b).
$$

The bundle endomorphism $\Psi^*_e$ dual to $\Psi_e$,

$$
\Psi^*_e(b, v) := (b \cdot (-1), \psi_e(b)^*v), \quad b \in \mathbb{B}, v \in C_e(\overline{\Omega})^*;
$$

acts on the dual bundle $\mathcal{W}^* = \mathbb{B} \times C_e(\overline{\Omega})^*$ and covers the inverse homeomorphism $\phi^{-1}$, $\phi^{-1}(b) = b \cdot (-1)$. Its iterates form the linear skew-product dynamical system $\{\Psi^*_e\}_{k=1}^{\infty}$ dual to $\{\Psi^k_e\}_{k=1}^{\infty}$.

As a consequence of Theorem 2.3 we have that $\psi_e(b)u \gg_e 0$ for each $b \in \mathbb{B}$ and each $u \in C_e(\overline{\Omega})$, $u >_e 0$. We refer to this property as the strong monotonicity of the linear skew-product semidynamical system $\{\Psi^*_e\}$.

The next theorem is based on results of P. Poláčik and I. Tereščák [14] (for an earlier result compare the present author’s unpublished manuscript [6]). We begin with introducing some notation: for a subbundle $\mathcal{W}_1$ of $\mathcal{W}$, we denote $\mathcal{W}_1(b) = \{u \in C_e(\overline{\Omega}) : (b, u) \in \mathcal{W}_1\}$ (in other words, $\{b\} \times \mathcal{W}_1(b)$ is the fiber of $\mathcal{W}_1$ over $b \in \mathbb{B}$). A subbundle $\mathcal{W}_1$ is called invariant if for each $b \in \mathbb{B}$ from $u \in \mathcal{W}_1(b)$ it follows that $\psi_e(b)u \in \mathcal{W}_1(b \cdot 1)$.

**Theorem 2.4.** There exists a decomposition of $\mathcal{W}$ into a direct sum $\mathcal{W} = \mathcal{S} \oplus \mathcal{T}$ of invariant subbundles having the following properties:
(i) There is a continuous mapping \( \mathcal{B} \ni b \mapsto w_b \in C_0(\overline{\Omega})^+ \) such that \( \|w_b\|_e = 1 \) and \( S(b) = \{ \omega_b : \alpha \in \mathbb{R} \} \). In particular, \( S \) has dimension one and \( S(b) \setminus \{0\} \subseteq C_0(\overline{\Omega})^+ \cup -C_0(\overline{\Omega})^+ \) for any \( b \in \mathcal{B} \).

(ii) There is a continuous mapping \( \mathcal{B} \ni b \mapsto \tilde{w}_b \in C_0(\overline{\Omega})^+ \) such that \( \|\tilde{w}_b\|_e = 1 \) and \( T(b) = \{ u \in C_0(\overline{\Omega}) : \int_{\Omega} u(x) \; w_b(x) \, dx = 0 \} \). In particular, \( T \) has codimension one and \( T(b) \cap C_0(\overline{\Omega})^+ = \{0\} \) for any \( b \in \mathcal{B} \).

(iii) The mapping \( \Psi_e|_S \) is a bundle automorphism.

(iv) There are constants \( D \geq 1 \) and \( 0 < \lambda < 1 \) such that

\[
\frac{\|\psi_e^{(k)}(b) u_1\|_e}{\|\psi_e^{(k)}(b) u_2\|_e} \leq D \lambda^k \frac{\|u_1\|_e}{\|u_2\|_e}
\]

for each \( b \in \mathcal{B} \), \( u_1 \in T(b) \), \( u_2 \in S(b) \setminus \{0\} \), \( k \in \mathbb{N} \).

Proof. According to [10], \( W \) decomposes into a direct sum of invariant sub-bundles \( S \) and \( T \) satisfying (i), (iii) and (iv). Moreover, there is a continuous mapping \( \mathcal{B} \ni b \mapsto \tilde{w}_b \in C_0(\overline{\Omega})^+ \) such that

(a) \( \|\tilde{w}_b\|_{C_0(\overline{\Omega})^+} = 1 \),

(b) \( [u|\tilde{w}^*_b]_e > 0 \) for each \( b \in \mathcal{B} \) and each nonzero \( u \in C_0(\overline{\Omega})^+ \),

(c) \( T(b) = \{ u \in C_0(\overline{\Omega}) : [u|\tilde{w}^*_b]_e = 0 \} \),

(d) for each \( b \in \mathcal{B} \) there is \( \kappa(b) > 0 \) such that \( \psi_e^*(b) \tilde{w}_b^* = \kappa(b) \tilde{w}_b^*(-1) \),

where \( [\cdot]_e \) denotes the duality pairing between \( C_0(\overline{\Omega}) \) and \( C_0(\overline{\Omega})^+ \).

From (d) we derive by Theorem 2.3 that \( \tilde{w}_b^* \in C_0(\overline{\Omega})^+ \) for each \( b \in \mathcal{B} \). Property (b) implies \( \tilde{w}_b^* \in C_0(\overline{\Omega})^+ \setminus \{0\} \). By Theorem 2.3 and (d), \( \tilde{w}_b^* \in C_0(\overline{\Omega})^+ \) for each \( b \in \mathcal{B} \). Putting \( w_b^* := \tilde{w}_b^*/\|\tilde{w}_b^*\|_e \), completes the proof.

Define the linear operator \( P(b) \in \mathcal{L}(C_0(\overline{\Omega})) \) as

\[
P(b)u := [u|w_b^*]_ew_b = \left( \int_{\Omega} u(x) \; w_b^*(x) \, dx \right) w_b.
\]

It is easy to verify that \( P(b) \) has kernel \( T(b) \), range \( S(b) \), and \( P(b) \circ P(b) = P(b) \). Hence \( P(b) \) is a projector of \( C_0(\overline{\Omega}) \) onto \( S(b) \). By the continuity of \( w_b \) and \( w_b^* \) in \( b \) it follows that the assignment \( \mathcal{B} \ni b \mapsto P(b) \in \mathcal{L}(C_0(\overline{\Omega})) \) is continuous. The family \( P := \{ (b, P(b)) : b \in \mathcal{B} \} \) is a bundle projection with kernel \( T \) and range \( S \).

The property described in Theorem 2.3(iv) is called exponential separation. With the help of the projection \( P \) we can write it as

\[
\frac{\|\text{Id} - P(b \cdot k)\|_e \psi_e^{(k)}(b) u\|_e}{\|P(b \cdot k) \psi_e^{(k)}(b) u\|_e} \leq D \lambda^k \frac{\|\text{Id} - P(b)\|_e \|u\|_e}{\|P(b) u\|_e} \quad (2.2)
\]

for \( b \in \mathcal{B} \), \( u \in C_0(\overline{\Omega}) \setminus T(b) \), \( k \in \mathbb{N} \).

The following theorem is crucial in proving our main results in the next subsection.
Theorem 2.5. There is a constant $L > 0$ such that for each $b \in B$ and each $u \in C_e(\overline{\Omega})_+$ we have

$$\frac{\| (\text{Id} - P(b \cdot 1)) \psi_u(b) u \|_e}{\| P(b \cdot 1) \psi_u(b) u \|_e} \leq L.$$ 

Before proving the above theorem we need a series of auxiliary results.

Proposition 2.6. (i) $G(t, b)(x, \xi) > 0$ for each $(x, \xi) \in \Omega \times \Omega$ and each $t > 0$, $b \in B$.

(ii) $G(t, b)(x, \xi) = 0$ for each $(x, \xi) \in (\partial \Omega \times \Omega) \cup (\partial \Omega \times \Omega)$ and each $t > 0$, $b \in B$.

(iii) $(\partial / \partial \nu_x) G(t, b)(x, \xi) < 0$ for each $(x, \xi) \in \partial \Omega \times \Omega$, $t > 0$ and $b \in B$.

(iv) $(\partial / \partial \nu_\xi) G(t, b)(x, \xi) < 0$ for each $(x, \xi) \in \Omega \times \partial \Omega$, $t > 0$ and $b \in B$.

(v) $(\partial^2 / \partial \nu_x \partial \nu_\xi) G(t, b)(x, \xi) < 0$ for each $(x, \xi) \in \partial \Omega \times \partial \Omega$, $t > 0$ and $b \in B$.

Proof. This is a consequence of (1.14), (1.15), Lemma 1.7, Proposition 1.8, Lemma 1.9 and Theorem 2.3. 

For $b \in B$ define the function $\overline{G}_b : \overline{\Omega} \rightarrow C^1_0(\overline{\Omega})$ as

$$\overline{G}_b(\xi) := G(1, b)(\cdot, \xi),$$

and the function $\overline{G}_b : \overline{\Omega} \rightarrow C_e(\overline{\Omega})$ as $\overline{G}_b := \text{i} \circ \overline{G}_b$ (recall that $\text{i}$ denotes the embedding $C^1_0(\overline{\Omega}) \subset C_e(\overline{\Omega})$).

Lemma 2.7. (i) The assignment $B \ni b \mapsto G_b \in C^1(\overline{\Omega}, C_e(\overline{\Omega}))$

is continuous.

(ii) $G_b(\xi) \gg_e 0$ for each $b \in B$ and $\xi \in \Omega$.

(iii) $(\partial / \partial \nu) G_b(\xi) \ll_e 0$ for each $b \in B$ and $\xi \in \partial \Omega$.

Proof. The corresponding properties hold for the function $\overline{G}_b$ by Lemmas 1.6, 1.7 and 1.9 (part (ii)), and Proposition 2.6 (parts (ii) and (iii)). Now it suffices to make use of Proposition 2.7. 

For $b \in B$ and $\xi \in \Omega$ we define

$$m(b, \xi) := \sup \{ \alpha \geq 0 : G_b(\xi) \gg_e \alpha e \}.$$ 

It is easily checked that $m$ is a lower semicontinuous function of $(b, \xi) \in B \times \Omega$.

Lemma 2.8. The number

$$\gamma := \inf \left\{ \frac{m(b, \xi)}{\| G_b(\xi) \|_e} : b \in B, \xi \in \Omega \right\}$$

is positive.
Proof. Recall that \( r \) is a retraction of a neighborhood \( V \) of \( \partial \Omega \) in \( \overline{\Omega} \) onto \( \partial \Omega \). By taking the neighborhood \( V \) smaller (if necessary) we can assume that
\[
\frac{\partial}{\partial \nu_{r(\xi)}} G_b(\xi) \ll e \quad \text{for all } \xi \in V.
\]
As \( m \) is lower semicontinuous, there is \( \epsilon_1 > 0 \) such that
\[
\frac{\partial}{\partial \nu_{r(\xi)}} G_b(\xi) \ll -\epsilon_1 e \quad \text{for all } b \in B \text{ and all } \xi \in V.
\]
By integration we get \( G_b(\xi) \gg e^{\epsilon_1} \| \xi - r(\xi) \| \), therefore
\[
m(b, \xi) \geq \epsilon_1 \| \xi - r(\xi) \|.
\]
On the other hand, by continuity there is \( \epsilon_2 > 0 \) such that
\[
\frac{\partial}{\partial \nu_{r(\xi)}} G_b(\xi) \gg e^{-\epsilon_2} e \quad \text{for all } b \in B \text{ and all } \xi \in V,
\]
from which it follows that \( \| G_b(\xi) \|_e \leq e^{\epsilon_2} \| \xi - r(\xi) \| \). Consequently,
\[
m(b, \xi) \| G_b(\xi) \|_{C_e(\Omega)} \geq e^{\epsilon_1} \epsilon_2 \quad \text{for all } b \in B \text{ and } \xi \in \Omega \setminus \text{int} \Omega.
\]
Now, as \( G_b(\xi) \gg e \) for \( \xi \) in the compact set \( \Omega \setminus \text{int} \Omega \) and the function \( m \) is lower semicontinuous and positive on \( \Omega \setminus \text{int} \Omega \), there is \( \gamma_1 > 0 \) such that
\[
m(b, \xi) \| G_b(\xi) \|_{C_e(\Omega)} \geq \gamma_1 \quad \text{for all } b \in B \text{ and } \xi \in \Omega \setminus \text{int} \Omega.
\]
We have thus proved that \( \gamma \geq \min\{\epsilon_1/\epsilon_2, \gamma_1\} > 0 \).

Proposition 2.9. For each \( b \in B \) and \( u \in C_e(\Omega)_+ \) we have
\[
\gamma \| \psi_e(b) u \|_{C_e(\Omega)} \leq e \| \psi_e(b) u \|_e \leq \| \psi_e(b) u \|_{C_e(\Omega)}.
\]
Proof. The inequality \( \psi_e(b) u \leq e \| \psi(b) u \|_{C_e(\Omega)} \) follows by the definition of the \( C_e(\Omega) \)-norm. Further, one has
\[
\psi_e(b) u = \int_{\Omega} u(\xi) G_b(\xi) \, d\xi \geq \left( \int_{\Omega} m(b, \xi) u(\xi) \, d\xi \right) e,
\]
and, by Lemma 2.8
\[
\left( \int_{\Omega} m(b, \xi) u(\xi) \, d\xi \right) e \geq e \left( \int_{\Omega} \| G_b(\xi) \|_{C_e(\Omega)} u(\xi) \, d\xi \right) e,
\]
whereas
\[
\int_{\Omega} \| G_b(\xi) \|_{C_e(\Omega)} u(\xi) \, d\xi = \int_{\Omega} u(\xi) G_b(\xi) \, d\xi \geq \| \psi_e(b) u \|_{C_e(\Omega)},
\]
from which our assertion follows.
Theorem 2.5 gives the set of global solutions forms a vector subspace of $C$. For the remainder of the proof we drop the symbol of the base point in the Banach spaces will notice that Proposition 2.9 implies that (Hilbert’s) projective $P$ by $\psi_e$ for all $k$ solutions of $(E)+(BC)$. In particular, the null solution belongs to Proposition 2.10. If $u \in P$ then $u(t, \cdot) \in S(a_0 \cdot t)$ for all $t \in \mathbb{R}$.

Proof. Suppose by way of contradiction that there is $\tau \in \mathbb{R}$ such that $u(\tau, \cdot) \notin S(a_0 \cdot \tau)$. By invariance of $S$ one has $u(\tau - k, \cdot) \notin S(a_0 \cdot (\tau - k))$ for any $k \in \mathbb{N}$. Theorem 2.9 gives

$$\frac{\|(\Id - P(a_0 \cdot (\tau - k)))u(\tau - k, \cdot)\|_e}{\|P(a_0 \cdot (\tau - k))u(\tau - k, \cdot)\|_e} \leq L \quad \text{for all } k \in \mathbb{N} \cup \{0\}.$$  

For the remainder of the proof we drop the symbol of the base point in the projection $P$. Put $M := \|(\Id - P)u(\tau, \cdot)\|_e/\|Pu(\tau, \cdot)\|_e$. Take a positive integer $k_0$ not less than $\log(M/2DL)/\log \lambda$. Exponential separation yields

$$\frac{\|(\Id - P)u(\tau, \cdot)\|_e}{\|Pu(\tau, \cdot)\|_e} = \frac{\|\psi_e(k_0, a_0 \cdot (\tau - k_0))(\Id - P)u(\tau - k_0, \cdot)\|_e}{\|\psi_e(k_0, a_0 \cdot (\tau - k_0))Pu(\tau - k_0, \cdot)\|_e} \leq DL^{k_0} \frac{\|(\Id - P)u(\tau - k_0, \cdot)\|_e}{\|Pu(\tau - k_0, \cdot)\|_e} \leq DL^{k_0} L \leq \frac{M}{2}.$$
Let \( \mathcal{P} : \mathcal{P} \rightarrow \mathcal{S}(a_0) \) be the linear operator

\[ \mathcal{P} u := u(0, \cdot). \]

**Theorem 2.11.** \( \mathcal{P} \) is a linear isomorphism. Therefore, \( \dim \mathcal{P} = 1 \).

**Proof.** Let \( u \in \mathcal{P} \) be such that \( u(0, \cdot) = 0 \). Obviously \( u(t, \cdot) = 0 \) for all \( t \geq 0 \). Suppose to the contrary that \( u(\tau, \cdot) \neq 0 \), say \( u(\tau, \cdot) \in C_0(\Omega) \setminus \{0\} \), for some \( \tau < 0 \). By Theorem 2.3 \( u(0, \cdot) \not\geq 0 \), a contradiction. This proves that \( \mathcal{P} \) is a monomorphism. To prove that \( \mathcal{P} \) is an epimorphism it suffices to find a globally positive solution \( \bar{u} \) such that \( \bar{u}(0, \cdot) = w_{a_0} \). For \( k \in \mathbb{N} \) we define

\[
\bar{u}(k, \cdot) := \psi_{\bar{a}}^{(k)}(a_0)w_{a_0},
\bar{u}(-k, \cdot) := (\psi_{\bar{a}}^{(k)}(a_0 \cdot (-k)))^{-1}w_{a_0}(-k).
\]

As a result of the cocycle identity (2.1) \( \bar{u}(k_2, \cdot) = \psi_e(k_2 - k_1, a_0 \cdot k_1)\bar{u}(k_1, \cdot) \) for any two integers \( k_1 < k_2 \). Now, set

\[
\bar{u}(t, \cdot) := \psi_e(t - [t], a_0 \cdot [t])\bar{u}([t], \cdot)
\]

for each \( t \in \mathbb{R} \setminus \mathbb{Z} \). Observe that for such a \( t \) we have (again by the cocycle identity)

\[
\bar{u}([t] + 1, \cdot) = \psi_e([t] + 1, a_0 \cdot t)\bar{u}(t, \cdot)
= \psi_e([t] + 1, a_0 \cdot t)\psi_e(t - [t], a_0 \cdot [t])\bar{u}([t], \cdot)
= \psi_e(1, a_0 \cdot [t])\bar{u}([t], \cdot),
\]

hence

\[
\bar{u}(t_2, \cdot) = \psi_e(t_2 - t_1, a_0 \cdot t_1)\bar{u}(t_1, \cdot)
\]

for any \( t_1 < t_2 \).

**Appendix: Other boundary conditions**

Here we briefly outline how the results presented in the main body of the paper carry over to some other boundary conditions (at least for the case that the principal part of \( \mathcal{A} \) is in the divergence form).

Assume that the boundary \( \partial \Omega \) of \( \Omega \) is a disjoint union of two closed sets, \( \partial \Omega = \partial_D \Omega \cup \partial_R \Omega \), where \( \partial_R \Omega \) is of class \( C^3 \). The boundary conditions will be the following

\[
u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial_D \Omega,
\frac{\partial u}{\partial \beta}(t, x) + c(x)u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial_R \Omega, \tag{BC'}
\]

where \( \beta \in C^1(\partial_R \Omega, \mathbb{R}^n) \) is a nontangential vector field pointing out of \( \Omega \), and \( c \in C^1(\partial_R \Omega) \) is a nonnegative function. We assume for the sake of definiteness that \( \partial_D \Omega \neq \emptyset \), since otherwise the results can be obtained in a much simpler way by working in the Banach space \( L^1(\Omega) \), as shown by the author in [7].
The boundary operator $B$ is defined by
\[ B u(x) := \begin{cases} u(x) & \text{for } x \in \partial_D \Omega, \\ \frac{\partial u}{\partial \beta}(x) + c(x) u(x) & \text{for } x \in \partial_R \Omega. \end{cases} \]

In place of $C^0_0(\Omega)$ we take $C_B(\Omega)$ defined as $C_B(\Omega) := \{ u \in C(\Omega) : u(x) = 0 \text{ for } x \in \partial_D \Omega \}$. The symbols $C_B(\Omega)$, $i = 1, 2$, are defined in an analogous way.

The first of the major modifications concerns the definition of the adjoint boundary operator $B^*$
\[ B^* v(x) := \begin{cases} v(x) & \text{for } x \in \partial_D \Omega, \\ \frac{\partial v}{\partial \beta^*}(x) + c^*(x) v(x) & \text{for } x \in \partial_R \Omega, \end{cases} \]
where the nontangential vector field $\beta^*$ and the nonnegative function $c^*$ are chosen so as for the Green’s identity to be fulfilled. For explicit formulas the reader is referred to Amann [2].

In Section 2 the function $e$ must be defined as the (normalized) nonnegative principal eigenvalue of the Laplacian on $\Omega$ with the boundary condition $B u = 0$.

Lemma 2.1 takes the form

**Lemma A.** (i) $e \in C^1(\Omega)$.
(ii) $e(x) > 0$ for $x \in \Omega \cup \partial_R \Omega$.
(iii) $e(x) = 0$ and $D_\nu e(x) < 0$ for $x \in \partial_D \Omega$.

Accordingly, in further results $\Omega$ (resp. $\partial \Omega$) should be replaced by $\Omega \cup \partial_R \Omega$ (resp. $\partial_D \Omega$).

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