An Uniqueness Theorem on the Eigenvalues of Spherically Symmetric Interior Transmission Problem in Absorbing Medium

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Abstract

We study the asymptotic distribution of the eigenvalues of interior transmission problem in absorbing medium. We apply Cartwright’s theory and the technique from entire function theory to find a Weyl’s type of density theorem in absorbing medium. Given a sufficient quantity of transmission eigenvalues, we obtain limited uniqueness on the refraction index as an uniqueness problem in entire function theory.

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1 Preliminaries

In this note, we study the eigenvalues of the interior transmission problem with a twice differentiable absorbing refraction index

\[ n_1(x) := \epsilon_1(x) + i\gamma_1(x) / k; \]

\[
\begin{align*}
\Delta w + k^2(\epsilon_1(x) + i\gamma_1(x))w &= 0, & \text{in } B; \\
\Delta v + k^2(\epsilon_0 + i\gamma_0)w &= 0, & \text{in } B; \\
w &= v, & \text{on } \partial B; \\
\frac{\partial w}{\partial r} &= \frac{\partial v}{\partial r}, & \text{on } \partial B,
\end{align*}
\]

where \( r := |x| \) and \( B := \{|x| \leq 1, x \in \mathbb{R}^3\} \), \( w, v \in L^2(B), w - v \in H^2_0(B), k \in \mathbb{C} \). We consider the spherical perturbations for (1.2) by setting \( \epsilon_1(x) = \epsilon_1(r) > 0 \) and \( \gamma_1(x) = \gamma_1(r) > 0 \), \( \forall r \in [0,1]; \epsilon_0 \) and \( \gamma_0 \) are positive constants and \( n_1(r) = \epsilon_0 + i\gamma_0 / r \), when \( r \geq 1 \).

The interior transmission eigenvalues play a role in the inverse scattering theory both in numerical computation and in theoretical purpose. See Colton and Monk [6], Colton and Kress [8] and Colton, Päivärinta and Sylvester [7] for the historic and theoretical context. Moreover, the eigenvalues of the interior transmission problem is directly connected to the zeros of scattering amplitude. They are zeros of the integral average of the scattering amplitude. We refer to McLaughlin and Polyakov [14]. Moreover, it is another research interest to study the distribution of interior transmission eigenvalues in \( C \) [2, 3, 6, 8, 11, 14]. It is expected to prove a Weyl’s type of asymptotics on these eigenvalues. In this paper we use the analysis in Levin [12] to discuss the zeros of an asymptotically almost periodic function along the real axis.

Let us consider the solutions of (1.2) of the following form:

\[ v(r) = c_1 j_0(k\tilde{n}_0 r); \]

\[ w(r) = c_2 y(r) / r; \]

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where \( \hat{n}_0 := (\epsilon_0 + i \frac{\omega}{c})^{\frac{1}{2}}, j_0 \) is a spherical Bessel function of order zero and \( y(r) \) is a solution of
\[
\begin{align*}
\left\{\begin{array}{l}
y'' + k^2 (\epsilon_1(r) + i \frac{\omega r}{c}) y = 0; \\
y(0) = 0, \ y'(0) = 1.
\end{array}\right.
\tag{1.5}
\end{align*}
\]
The existence of \( c_1, c_2 \) in (1.3), (1.4) is provided by
\[
D(k) := \det \left( \begin{array}{cc} y(1) & -j_0(k\hat{n}_0) \\ \left\{y(r)\right\}'_{r=1} & -j_0'(k\hat{n}_0 r)_{r=1} \end{array} \right) = 0.
\tag{1.6}
\]
The computation on \( c \) fundamental solutions of (1.5). In this case, we use the theory provided in Erdelyi \[10, \text{p. 84}\]. In particular, we have a set of
\[
\begin{align*}
\text{By straightforward computation under (1.8) and (1.9), (1.10) is equivalent to the following system.}
\end{align*}
\]
By straightforward computation under (1.8) and (1.9), (1.10) is equivalent to the following system.
\[
\begin{align*}
\Phi'' + k^2 (\epsilon_1(r) + i \frac{\omega r}{c}) \Phi = 0, & \quad 0 \leq r \leq 1; \\
\Phi'' + k^2 (\epsilon_1 + i \frac{\omega}{c}) \Phi_0 = 0, & \quad 0 \leq r \leq 1; \\
\Phi(0) = \Phi_0(0) = 0, \Phi(1) = \Phi_0(1), \Phi'(1) = \Phi_0'(1).
\end{align*}
\tag{1.10}
\]
By straightforward computation under (1.8) and (1.9), (1.10) is equivalent to the following system.
\[
\begin{align*}
\Phi'' + k^2 (\epsilon_1(r) + i \frac{\omega r}{c}) \Phi = 0, & \quad 0 \leq r \leq 1; \\
\Phi(0) = 0, \Phi_0'(1)\Phi(1) - \Phi_0(1)\Phi'(1) = 0.
\end{align*}
\tag{1.11}
\]
The boundary condition of (1.10) or (1.11) implies each other. The determinant in (1.11) is equivalent to (1.10) as well. The zeros of the functional determinant \( D(k) \) are then the eigenvalues of (1.11). In this paper, we assume that two possibly different refraction indices have the same \( \Phi \). In general, we note
\[
\begin{align*}
e_1 &= \frac{\det \left( \begin{array}{cc} y(1) & \frac{\gamma_1 r}{\rho} \\ \left\{y(r)\right\}'_{r=1} & \frac{\gamma_1 r}{\rho} \end{array} \right)_{r=1}}{D(k)}; \\
e_2 &= \frac{\det \left( \begin{array}{cc} \frac{\gamma_1 r}{\rho} & -j_0(k\hat{n}_0) \\ \left\{\frac{\gamma_1 r}{\rho}\right\}'_{r=1} & -j_0'(k\hat{n}_0 r)_{r=1} \end{array} \right)_{r=1}}{D(k)}.
\end{align*}
\tag{1.12}
\tag{1.13}
\]
To understand the analytic behavior of the determinant \( D(k) \), we study the asymptotic solution of (1.5). In this case, we use the theory provided in Erdelyi \[10, \text{p. 84}\]. In particular, we have a set of fundamental solutions \( y_1(r), y_2(r) \) such that in a sectorial region \( S \)
\[
\begin{align*}
y_j(r; k) &= Y_j(r)\left[1 + O\left(\frac{1}{k}\right)\right]; \\
y'_j(r; k) &= Y'_j(r)\left[1 + O\left(\frac{1}{k}\right)\right].
\end{align*}
\tag{1.14}
\tag{1.15}
\]
as $|k| \to \infty$ in $S$, uniformly for $0 \leq r \leq 1$ and for arg $k$, where

$$Y_j(r) = \exp\{\beta_{0j}k + \beta_{1j}\}, \quad (1.16)$$

where $\beta_{0j}, \beta_{1j}$ satisfy

$$\begin{align*}
(\beta_{0j})^2 + \epsilon_1(r) &= 0; \\
2\beta_{0j}\beta_{1j} + i\gamma_1 + \beta_{0j}'' &= 0. 
\end{align*} \quad (1.17)$$

$$\beta_{0j}(r) = \pm i \int_0^r \sqrt{\epsilon_1(\rho)}d\rho + E; \quad (1.19)$$

$$\beta_{1j}(r) = \mp \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho + \ln[\epsilon_1(r)]\frac{1}{\sqrt{\epsilon_1}} + F, \quad (1.20)$$

where $E, F$ are constants. The sectorial region $S \subset \mathbb{C}$ is characterized by the condition

$$\Re\{ki(\epsilon_1(r))^{1/2}\} \neq 0. \quad (1.21)$$

That is

$$S = \{ k \in \mathbb{C} | \Im k \neq 0 \}. \quad (1.22)$$

Therefore, any solution to (1.5) is of the form

$$y(r; k) = \alpha Y_1(r)[1 + O(\frac{1}{k})] + \beta Y_2(r)[1 + O(\frac{1}{k})]. \quad (1.23)$$

We use the initial condition in (1.5) to obtain

$$y(r; k) = \frac{1}{2ik[\epsilon_1(0)]^{1/2}} \exp\{ik \int_0^r \sqrt{\epsilon_1(\rho)}d\rho - \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho\} [1 + O(\frac{1}{k})]$$

$$- \frac{1}{2ik[\epsilon_1(0)]^{1/2}} \exp\{-ik \int_0^r \sqrt{\epsilon_1(\rho)}d\rho + \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho\} [1 + O(\frac{1}{k})], \quad (1.24)$$

when $|k| \to \infty$ in $S$. Similarly, we use (1.15) to obtain the asymptotics

$$y'(r; k) = \frac{1}{2i\epsilon_1(0)^{1/2}} \exp\{ik \int_0^r \sqrt{\epsilon_1(\rho)}d\rho - \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho\} [1 + O(\frac{1}{k})]$$

$$+ \frac{1}{2i\epsilon_1(0)^{1/2}} \exp\{-ik \int_0^r \sqrt{\epsilon_1(\rho)}d\rho + \frac{1}{2} \int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho\} [1 + O(\frac{1}{k})], \quad (1.25)$$

when $|k| \to \infty$ in $S$.

Let us set

$$A := \sqrt{\epsilon_0}, \quad B := \int_0^1 \sqrt{\epsilon_1(\rho)}d\rho, \quad C := \frac{1}{2} \frac{\gamma_0}{\sqrt{\epsilon_0}}, \quad D := \frac{1}{2} \int_0^1 \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho. \quad (1.26)$$

When the refraction is purely real, $\gamma_1 \equiv 0$, the advantage is to consider the Liouville transformation of $y(r) = y(r; k)$:

$$z(\xi) := [n(r)]^{1/2}y(r), \quad \text{where} \quad \xi := \int_0^r [n(\rho)]^{1/2}d\rho. \quad (1.27)$$

In particular, we define

$$B = \int_0^1 [n(\rho)]^{1/2}d\rho \quad (1.28)$$

In this case, (1.27) becomes

$$\begin{align*}
z'' + [k^2 - p(\xi)]z &= 0, \quad 0 < \xi < B; \\
z(0) &= 0; \quad z'(0) = [n(0)]^{-\frac{1}{2}}. \quad (1.29)
\end{align*}$$
where
\[ p(\xi) := \frac{n''(r)}{4(n(r))^2} = \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^4}. \] (1.30)

From Pöschel and Trubowitz [15, p.16], we review the following asymptotics.
\[ z(\xi; k) = \frac{\sin k\xi}{k} - \frac{\cos k\xi}{2k^2} Q(\xi) + \frac{\sin k\xi}{4k^3} [p(\xi) + p(0)] - \frac{1}{2} Q^2(\xi) + O(\exp[|3k\xi|/k^3]), \] (1.31)
\[ z'(\xi; k) = \cos k\xi + \frac{\sin k\xi}{2k} Q(\xi) + \frac{\cos k\xi}{4k^2} [p(\xi) - p(0)] - \frac{1}{2} Q^2(\xi) + O(\exp[|3k\xi|/k^3]). \] (1.32)

Before applying such asymptotics, we add a multiple \([n(0)]^{1/2}\) to the solutions. We have to normalize the boundary condition in (1.29) for the solution.

We state the main result of this paper.

**Theorem 1.1.** Let
\[ \Lambda_1 := \{ z \in \mathbb{C} : |\arg(z)| < \epsilon \}; \] (1.33)
\[ \Lambda_2 := \{ z \in \mathbb{C} : |\arg(z) - \pi| < \epsilon \}, \forall \epsilon > 0. \] (1.34)

Let \( n^j_{1}, j = 1, 2, \) be two unknown refraction indices and \( D^j(k) \) be the determinant corresponding to \( n^j_{1} \) and \( n^1_{1}(0) = n^2_{1}(0) \). If the zeros of \( D^1(k) \) and \( D^2(k) \) coincide in either \( \Lambda_1 \) or \( \Lambda_2 \), then \( \epsilon^j_{1}(r) \equiv \epsilon^j_{1}(r) \).

We use the vocabulary from entire function to describe the distribution of the zeros of the functional determinant \( D(k) \). We refer such a theory to Levin [12, 13].

**Definition 1.2.** Let \( f(z) \) be an entire function of order \( \rho \). We use \( N(f, \alpha, \beta, r) \) to denote the number of the zeros of \( f(x) \) inside the angle \([\alpha, \beta]\) and \(|z| \leq r\); we define the density function
\[ \Delta_f(\alpha, \beta) := \lim_{r \to \infty} \frac{N(f, \alpha, \beta, r)}{r^\rho}, \] (1.35)
and
\[ \Delta_f(\beta) := \Delta_f(\alpha_0, \beta), \] (1.36)
with some fixed \( \alpha_0 \notin E \) such that \( E \) is at most a countable set.

**Theorem 1.3.** The determinant \( D(k) \) is an entire function of order 1 and of type \( A + B \). In particular,
\[ \Delta_D(-\epsilon, \epsilon) = \Delta_D(\pi - \epsilon, \pi + \epsilon) = \frac{A + B}{\pi}. \] (1.37)

## 2 Lemmas

We need a few lemmas.

**Lemma 2.1.** There exists a constant \( M \) and \( k_0 > 0 \) such that
\[ \left| \frac{y(1;k)}{y'(1;k)} \right| < M; \left| \frac{j_0(k\tilde{n}_0)}{\partial_k (j_0(k\tilde{n}_0))} \right| < M, \quad |k| > k_0, \quad k \in 0 + i\mathbb{R}. \] (2.1)

**Proof.** We start with (1.24). We compute the following quantity from (1.14), (1.15) and (1.24).
\[ \frac{y(1;k)}{y'(1;k)} = \frac{e^{ikB-D}[1 + O(\frac{1}{\xi})] - e^{-ikB+D}[1 + O(\frac{1}{\xi})]}{e^{ikB-D}[ik\xi_0 - \frac{\gamma_0}{2\sqrt{\gamma_0}}] + e^{-ikB+D}[ik\xi_0 - \frac{\gamma_0}{2\sqrt{\gamma_0}}] + O(\frac{1}{\xi})}. \] (2.2)

Let \( k = i\xi \in 0 + i\mathbb{R}^+ \). Hence,
\[ \frac{y(1;i\xi)}{y'(1;i\xi)} = \frac{e^{-\xi B-D}[-\xi_0 - \frac{\gamma_0}{2\sqrt{\gamma_0}}] + e^{\xi B+D}[-\xi_0 - \frac{\gamma_0}{2\sqrt{\gamma_0}}]}{e^{-\xi B-D}[\xi_0 + \frac{\gamma_0}{2\sqrt{\gamma_0}}] + e^{\xi B+D}[\xi_0 + \frac{\gamma_0}{2\sqrt{\gamma_0}}] + O(\frac{1}{\xi})}. \]
Therefore,

\[
\frac{y(1; i\xi)}{y'(1; i\xi)} \leq \begin{cases} 
Ce^{-\xi(B+D)}[|1+O(\frac{1}{\xi})| - e^{2(\xi(B+D))}|1+O(\frac{1}{\xi})|], & \text{if } \xi > 0; \\
Ce^{\xi(B+D)}\frac{e^{-2(\xi(B+D))}|1+O(\frac{1}{\xi})| - |1+O(\frac{1}{\xi})|}{e^{-2(\xi(B+D))}|1+O(\frac{1}{\xi})|}, & \text{if } \xi < 0,
\end{cases}
\]

(2.3)

where \(C\) is a some constant. Let us consider for \(\xi > 0\),

\[
\lim_{\xi \to \infty} \frac{|1 + O(\frac{1}{\xi})| - e^{2(\xi(B+D))}|1 + O(\frac{1}{\xi})|}{|1 + O(\frac{1}{\xi})| + e^{2(\xi(B+D))}|1 + O(\frac{1}{\xi})|} \leq \lim_{\xi \to \infty} \frac{|1 + O(\frac{1}{\xi})| + e^{2(\xi(B+D))}|1 + O(\frac{1}{\xi})|}{|1 + O(\frac{1}{\xi})| - e^{2(\xi(B+D))}|1 + O(\frac{1}{\xi})|}
\]

(2.4)

\[
= \lim_{\xi \to \infty} e^{2\xi(B+D)} \lim_{\xi \to \infty} |1 + O(\frac{1}{\xi})| + \lim_{\xi \to \infty} |1 + O(\frac{1}{\xi})| - \lim_{\xi \to \infty} |1 + O(\frac{1}{\xi})|
\]

(2.5)

\[
= \frac{\lim_{\xi \to \infty} e^{2\xi(B+D)} + 1}{\lim_{\xi \to \infty} e^{2\xi(B+D)} - 1} = \frac{e^{2(B+D)} + 1}{e^{2(B+D)} - 1}
\]

(2.6)

A similar analysis holds for \(\xi < 0\). Hence in (2.3), \(|\frac{y(1; i\xi)}{y'(1; i\xi)}|\) vanishes along the imaginary axis. Hence, the first statement is proved. The second one can be proved similarly.

\[\square\]

**Definition 2.2.** Let \(f(z)\) be an integral function of finite order \(\rho\) in the angle \([\theta_1, \theta_2]\). We call the following quantity as the indicator of the function \(f(z)\).

\[
h_f(\theta) := \lim_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r^\rho}, \quad \theta_1 \leq \theta \leq \theta_2.
\]

(2.9)

**Lemma 2.3.** Let \(f, g\) be two entire functions. Then the following two inequalities hold.

\[
h_{fg}(\theta) = h_f(\theta) + h_g(\theta), \quad \text{if one limit exists};
\]

\[
h_{f+g}(\theta) \leq \max_{\theta} \{h_f(\theta), h_g(\theta)\},
\]

(2.10)

(2.11)

where if the indicator of the two summands are not equal at some \(\theta_0\), then the equality holds in (2.11).

\[\square\]

**Proof.** We can find these in [12, p.51].

**Definition 2.4.** The following quantity is called the width of the indicator diagram of entire function \(f\):

\[
d = h_f\left(\frac{\pi}{2}\right) + h_f\left(-\frac{\pi}{2}\right).
\]

(2.12)

The distribution on the zeros of an entire function is described precisely by the following Cartwright’s theorem [4, 5, 12, 13]. The following statements are from Levin [12, Ch.5, Sec.4].

**Theorem 2.5** (Cartwright). If an entire function of exponential type satisfies one of the following conditions:

the integral \(\int_0^\infty \frac{\ln |f(x) f(-x)|}{1 + x^2} \, dx\) exists, and \(h_f(0) = h_f(\pi) = 0\),

(2.13)

the integral \(\int_{-\infty}^\infty \frac{\ln |f(x)|}{1 + x^2} \, dx < \infty\).

(2.14)

the integral \(\int_{-\infty}^\infty \frac{\ln^+ |f(x)|}{1 + x^2} \, dx\) exists.

(2.15)
Proof. As we see from (1.6),

\[ |f(x)| \text{ is bounded on the real axis.} \quad (2.16) \]
\[ |f(x)| \in L^p(-\infty, \infty), \quad (2.17) \]

then

1. \( f(z) \) is of class A and is of completely regular growth and its indicator diagram is an interval on the imaginary axis.

2. all of the zeros of the function \( f(z) \), except possibly those of a set of zero density, lie inside arbitrarily small angles \( |\arg z| < \epsilon \) and \( |\arg z - \pi| < \epsilon \), where the density

\[
\Delta_f(-\epsilon, \epsilon) = \Delta_f(\pi - \epsilon, \pi + \epsilon) = \lim_{r \to \infty} \frac{N(f, -\epsilon, \epsilon, r)}{r} = \lim_{r \to \infty} \frac{N(f, \pi - \epsilon, \pi + \epsilon, r)}{r},
\]

(2.18)
is equal to \( \frac{d}{2\pi} \), where \( d \) is the width of the indicator diagram in (2.15). Furthermore, the limit \( \delta = \lim_{r \to \infty} \delta(r) \) exists, where

\[
\delta(r) := \sum_{\{|a_k| < r\}} \frac{1}{a_k};
\]

(2.19)

3. moreover,

\[
\Delta_f(\epsilon, \pi - \epsilon) = \Delta_f(\pi + \epsilon, -\epsilon) = 0,
\]

(2.20)

4. the function \( f(z) \) can be represented in the form

\[
f(z) = cz^m e^{iCz} \lim_{r \to \infty} \prod_{\{|a_k| < r\}} \left(1 - \frac{z}{a_k}\right)
\]

(2.21)

where \( c, m, B \) are constants and \( C \) is real.

5. the indicator function of \( f \) is of the form

\[
h_f(\theta) = \sigma |\sin \theta|. \quad (2.22)
\]

The last statement is found at Levin [13, p. 126]. We use these to compute the indicator function of \( D(k) \).

**Proposition 2.6.** \( D(k) \) is bounded over \( 0i + \mathbb{R} \) and

\[
h_D(\theta) = (A + B)|\sin \theta|, \quad \theta \in [0, 2\pi]. \quad (2.23)
\]

Proof. As we see from (1.6),

\[
D(k) = y'(1,k)\partial_r j_0(k\tilde{\eta}r)|_{r=1} \left\{ \frac{j_0(k\tilde{\eta}0)}{\partial_r j_0(k\tilde{\eta}0)|_{r=1}} - \frac{y(1; k)}{y'(1; k)}[1 + \frac{j_0(k\tilde{\eta}0)}{\partial_r j_0(k\tilde{\eta}0)|_{r=1}}]\right\}.
\]

(2.24)

Moreover, (1.7) suggests that \( D(k) \) is bounded over \( 0i + \mathbb{R} \). Hence, (2.16) is satisfied.

Now we look for (2.22). For \( k = i\xi \) in \( \mathbb{C} \) and \( |\xi| > k_0 \), we have that

\[
D(i\xi) = y'(1;i\xi)\partial_r j_0(i\xi\tilde{\eta}0)|_{r=1} \left\{ \frac{j_0(i\xi\tilde{\eta}0)}{\partial_r j_0(i\xi\tilde{\eta}0)|_{r=1}} - \frac{y(1;i\xi)}{y'(1;i\xi)}[1 + \frac{j_0(i\xi\tilde{\eta}0)}{\partial_r j_0(i\xi\tilde{\eta}0)|_{r=1}}]\right\},
\]

(2.25)

where the items inside the bracket are bounded by Lemma 2.1. Term by term, we compute

\[
h_{y'(1;k)}(\pm \frac{\pi}{2}) = \lim_{\xi \to \infty} \frac{\ln |y'(1;i\xi)|}{|\xi|} = \lim_{\xi \to \infty} \frac{\ln \left[ e^{-\xi(\xi - \frac{30}{2\sqrt{m}})}[1 + O\left(\frac{1}{\xi}\right)] + e^{\xi(\xi - \frac{30}{2\sqrt{m}})}[1 + O\left(\frac{1}{\xi}\right)]\right]}{|\xi|} = B.
\]

(2.26)
Similarly,
\[ h_{\partial \mathcal{A}(k \hat{n}_0 r)_{r=1}}(\pm \frac{\pi}{2}) = A. \]  
(2.27)

Now we use (2.10) to obtain
\[ h_D(\pm \frac{\pi}{2}) = (A + B). \]  
(2.28)

Hence, (2.22) says
\[ h_D(\theta) = (A + B) |\sin \theta|. \]  
(2.29)

Proof of Theorem 1.3. The indicator diagram of \( D(k) \) has width \( 2(A + B) \). (2.18) and (2.28) suggests that
\[ \Delta_D(-\epsilon, \epsilon) = \Delta_D(\pi - \epsilon, \pi + \epsilon) = A + B \pi. \]  
(2.30)

So we have this quantity of zeros in \( \Lambda_1 \) and \( \Lambda_2 \).

3 Stability Theorem and the Proof of Theorem 1.1

Let \( k_j \) be a common interior transmission eigenvalue of refraction index \( n_1^1(r) \) and \( n_2^1(r) \). Let \( D^i(k) \) be the corresponding functional determinant of the index \( n_1^i(r) \); \( y^i(r; k) \) be the solution. From Theorem 1.3 and the assumption of Theorem 1.1, we have
\[ B^1 = B^2, \]  
(3.1)

where
\[ B^i := \int_0^1 \sqrt{\epsilon_1^i(\rho)} d\rho. \]

Let \( \Phi^i(r) \) and \( \Phi_0(r) \) be the function defined by index \( n^i \) as in (1.10). Therefore, the boundary condition in (1.10), (1.3), (1.4), (1.12), and (1.13) imply that
\[ y^1(1; k_j) = y^2(1; k_j) \) and \( (y^1)'(1; k_j) = (y^2)'(1; k_j), \forall k_j. \]  
(3.2)

Let
\[ F(k) := y^1(1; k) - y^2(1; k), \]  
(3.3)

which is an entire function. From (1.24) and (2.9), we see \( y^i(1; k) \) is an entire function of exponential type \( B^i \) and
\[ h_{y^i}(\theta) = B^i |\sin \theta|. \]  
(3.4)

Here, we see that \( h_{y^i}(\theta) \) is a continuous function of \( \theta \). Therefore, applying (2.11) and (3.1),
\[ h_{F}(\theta) \leq B^1 |\sin \theta|. \]  
(3.5)

Hence, the indicator diagram of \( F(k) \) is equal to
\[ h_{F}(\frac{\pi}{2}) + h_{F}(\frac{\pi}{2}) \leq 2B^1. \]  
(3.6)

However, this is not possible due to the following uniqueness theorem for the entire function of the exponential type. This is Carlson’s theorem from Levin [12, p. 190].

Theorem 3.1. Let \( F(z) \) be holomorphic and at most of normal type with respect to the proximate order \( \rho(r) \) in the angle \( \alpha \leq \arg z \leq \alpha + \pi/\rho \) and vanish on a set \( N := \{a_k\} \) in this angle, with angular density \( \Delta_N(\psi) \). Let
\[ H_N(\theta) := \pi \int_0^{\alpha+\pi/\rho} \sin |\psi - \theta| d\Delta_N(\psi), \]  
(3.7)

when \( \rho \) is integral. Then, if \( F(z) \) is not identically zero,
\[ h_F(\alpha) + h_F(\alpha + \pi/\rho) \geq H_N(\alpha) + H_N(\alpha + \pi/\rho). \]  
(3.8)
In this paper, we consider $\rho \equiv 1$, $\alpha = -\frac{\pi}{2}, \frac{\pi}{2}$. Let $N$ here be the common zeros either in $\Lambda_1$ or in $\Lambda_2$. From (2.18) and (2.20), we have

$$H_N(-\frac{\pi}{2}) + H_N(\frac{\pi}{2}) = 2(A + B^1).$$

(3.9)

Therefore, we conclude from (3.2) to (3.9) and Theorem 3.1 that $F(k) \equiv 0$ and

$$y^1(1; k) \equiv y^2(1; k).$$

(3.10)

In particular,

$$B^1 = B^2; \quad D^1 = D^2.$$  

(3.11)

Similarly, we repeat the argument from (3.2) to (3.9), we can show

$$(y^1)'(1; k) \equiv (y^2)'(1; k).$$

(3.12)

They are the same entire function.

The zeros of $y^i(1; k)$ are the eigenvalues of the equation

$$\begin{cases}
(y^i)'' + k^2n_1(r)y^i = 0, & 0 \leq r \leq 1; \\
y^i(0) = 0, & y^i(1; k) = 0,
\end{cases}$$

(3.13)

while the zeros of $(y^i)'(1; k)$ are the eigenvalues of the equation

$$\begin{cases}
(y^i)'' + k^2n_1'(r)y^i = 0, & 0 \leq r \leq 1; \\
y^i(0) = 0, & (y^i)'(1; k) = 0.
\end{cases}$$

(3.14)

Let us understand more on the structure of the zero set.

**Definition 3.2.** Let us define

$$\mathcal{Y}(r; k) := y^1(r; k) - y^2(r; k).$$

(3.15)

The following lemma holds.

**Lemma 3.3.** Let $k$ be an interior transmission eigenvalue for index $n_1^1$ and $n_2^1$. Then, $\mathcal{Y}(1; k)$ satisfies the following boundary value problem.

$$\begin{cases}
\mathcal{Y}''(r; k) + (k^2\epsilon_1^1 + ik\gamma_1^1)\mathcal{Y}(r; k) + (k^2\epsilon_d + ik\gamma_d)\mathcal{Y}'(r; k) = 0, & 0 \leq r \leq 1; \\
\mathcal{Y}(0) = 0; & \mathcal{Y}'(1; k) = 0;
\end{cases}$$

(3.16)

(3.17)

where $\epsilon_d := \epsilon_1^1 - \epsilon_1^2; \gamma_d := \gamma_1^1 - \gamma_1^2$.  

(3.18)

For any $k \in \mathbb{C}$ that satisfies

$$\begin{cases}
y^1(1; k) = y^2(1; k); \\
(y^1)'(1; k) = (y^2)'(1; k).
\end{cases}$$

(3.19)

is an interior transmission eigenvalues of problem (3.16) - (3.17).

**Proof.** We note that (3.18) and (3.19) are satisfied as in (3.2); (3.16) and (3.17) are derived from (1.5). \qed

The system (3.16) - (3.19) has a perturbation theory to merely finitely many eigenvalues. Such a theory is established in [2] Sec.2. Let us review as follows.

Let us define

$$u := w - v.$$  

(3.21)

We rewrite (1.2) as

$$\begin{align*}
\Delta u + (k^2\epsilon_1 + ik\gamma_1)u + (k^2\epsilon_e + ik\gamma_e)v &= 0; \\
\Delta v + (k^2\epsilon_0 + ik\gamma_0)v &= 0; \\
u &= 0, \quad \text{on } \partial B; \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial B, \quad \text{where } \epsilon_e := \epsilon_1 - \epsilon_0; \gamma_e := \gamma_1 - \epsilon_0.
\end{align*}$$

(3.22)

(3.23)

(3.24)

(3.25)
The equation makes sense for $u \in H^2_0(B)$ and $v \in L^2(B)$ such that $\Delta v \in L^2(B)$.

Setting

$$X(B) := H^2_0(B) \times \{v \in L^2(B) | \Delta v \in L^2(B)\},$$

we can define the linear operators $A, B_{\gamma}$ and $D_\epsilon$:

$$D_\epsilon : L^2(B) \times L^2(B) \mapsto L^2(B) \times L^2(B)$$

by

$$A := \begin{pmatrix} \Delta_{00} & 0 \\ 0 & \Delta \end{pmatrix};$$

$$B_{\gamma} := \begin{pmatrix} i\gamma_1 & i\gamma_c \\ 0 & i\gamma_0 \end{pmatrix};$$

$$D_\epsilon := \begin{pmatrix} \epsilon_1 & \epsilon_c \\ 0 & \epsilon_0 \end{pmatrix},$$

where $\Delta_{00}$ is the Laplacian acting on function in $H^2_0(B)$, i.e. with the zero Cauchy data on $\partial B$. Let

$$p := \begin{pmatrix} u \\ v \end{pmatrix}$$

and note that the domain of $A$ is $X(B)$ and $A$ is an unbounded densely defined operator in $L^2(B) \times L^2(B)$.

Using the results in [2], we can easily show that

$$D_\epsilon^{-1} = \frac{1}{\epsilon_0\epsilon_1} \begin{pmatrix} \epsilon_0 & -\epsilon_c \\ 0 & \epsilon_1 \end{pmatrix}$$

and that the transmission eigenvalues of (1.2) is the quadratic eigenvalues of the equation

$$Ap + kB_{\gamma}p + k^2D_\epsilon p = 0, \quad p = \begin{pmatrix} u \\ v \end{pmatrix} \in L^2(B) \times L^2(B).$$

According to [2], the eigenvalue problem (1.2) is equivalent to the eigenvalue problem for the closed unbounded operator

$$T_{\epsilon,\gamma} := I_{\epsilon,\gamma}^{-1}K,$$

where

$$K := \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

and

$$I_{\epsilon,\gamma} := \begin{pmatrix} -B_{\gamma} & -I \\ D_\epsilon & 0 \end{pmatrix},$$

where $1 : L^2(B) \times L^2(B) \mapsto L^2(B) \times L^2(B)$ is the identity operator. Moreover,

$$I_{\epsilon,\gamma}^{-1} = D_\epsilon^{-1} \begin{pmatrix} 0 & -I \\ -D_\epsilon & -B_{\gamma} \end{pmatrix},$$

which is bounded in $L^2(B) \times L^2(B)$. Therefore, $T_{\epsilon,\gamma}$ is closed in $[L^2(B) \times L^2(B)]^2$.

Now we define the operator

$$P := T_{\epsilon,\gamma} - T_{\epsilon,\gamma=0}.$$ 

Consequently,

$$P = \begin{pmatrix} 0 & 0 \\ 0 & -D_\epsilon^{-1}B_{\gamma} \end{pmatrix}.$$ 

There is a stability for finitely many interior transmission eigenvalues whenever the generalized norm

$$\delta(T_{\epsilon,\gamma}, T_{\epsilon,\gamma=0})$$
is controlled. We apply the (24) in [2].

\[
\delta(T_{\epsilon, \gamma}, T_{\epsilon, \gamma} = 0) \leq \|P\| \leq \|D_{\epsilon}^{-1}B_{\gamma}\|.
\] (3.38)

In [2], they perturb the non-absorbing media, \((\gamma_1 = 0, \gamma_0 = 0)\), to conclude the existence of the finitely many interior transmission eigenvalues in absorbing media, in which \((\gamma_1, \gamma_0)\) is small. In particular, they explain as follows.

**Theorem 3.4** (Cakoni, Colton, Haddar). Let \(\epsilon_0 \in L^\infty([0, 1])\) and satisfy \(\epsilon_0(r) \geq \theta_0 > 0\), \(\epsilon_1(r) \geq \theta_1 > 0\) and \(\epsilon_{\epsilon} := \epsilon_1 - \epsilon_0 > 0\) almost every in \([0, 1]\). Let \(k_j, j = 0, \cdots, l\) be the first \(l + 1\) real positive transmission eigenvalues, counted according to its multiplicity, corresponding to (1.2) in absorbing media. Then, for any \(\sigma > 0\), there exists a

\[
\eta := \frac{\sup_{[0, 1]} \epsilon_0 + \sup_{[0, 1]} \epsilon_1}{4 \inf_{[0, 1]} \epsilon_0 \inf_{[0, 1]} \epsilon_1} \{\sup \gamma_0 + \sup \gamma_1\} > 0,
\] (3.39)

depending on \(\sigma\), such that there exists \(l + 1\) transmission eigenvalues of (1.2) in absorbing media, \(\epsilon_1 > 0\), \(\gamma_1 > 0\), in the \(\sigma\)-neighborhood of \(k_j, j = 0, \cdots, l\), whenever \(\eta\) is small enough.

For the application in this paper, we consider the perturbation operator

\[
P' := T_{\epsilon, \gamma} - T_{\epsilon, \gamma'}.
\] (3.40)

We note that the existence of the transmission interior eigenvalue to (1.2) and its distributional rule is already described by Cartwright’s theory as in Theorem 1.3. Hence, we consider the perturbation from one absorbing index \(n_1 = \epsilon_1 + i\frac{\gamma_1}{\epsilon_1}\) to the other one, with \(\epsilon_1\) fixed, \(n_1' = \epsilon_1 + i\frac{\gamma_1'}{\epsilon_1}\). Now we compute

\[
\|P'\| = \|T_{\epsilon, \gamma} - T_{\epsilon, \gamma'}\| \leq \|D_{\epsilon}^{-1}(B_{\gamma'} - B_{\gamma})\|.
\] (3.41)

Moreover,

\[
\|D_{\epsilon}^{-1}(B_{\gamma'} - B_{\gamma})\| = \|\frac{1}{\epsilon_0 \epsilon_1} \left( \begin{array}{cc} \epsilon_0 & -\epsilon_0 \\ 0 & \epsilon_1 \end{array} \right) \left( \begin{array}{cc} i\gamma_1 & i\gamma_1' \\ 0 & i\gamma_0 \end{array} \right) - \left( \begin{array}{cc} i\gamma_1 & i\gamma_0 \\ 0 & i\gamma_0' \end{array} \right) \right)\|
\] \[
\leq \|\frac{1}{\epsilon_0 \epsilon_1} \left( \begin{array}{cc} \epsilon_0 & -\epsilon_0 \\ 0 & \epsilon_1 \end{array} \right) \left( \begin{array}{cc} i(\gamma_1' - \gamma_1) & i(\gamma_1' - \gamma_1) \\ 0 & 0 \end{array} \right) \right)\|
\] \[
= \|\frac{1}{\epsilon_0 \epsilon_1} \left( i\epsilon_0(\gamma_1' - \gamma_1) & i\epsilon_0(\gamma_1' - \gamma_1) \\ 0 & 0 \end{array} \right) \right)\|
\] \[
\leq \frac{2}{\inf_{[0, 1]} \epsilon_0 \epsilon_1} \sup |\gamma_1' - \gamma_1|.
\] (3.42)

We proved the following stability among the interior transmission eigenvalues.

**Proposition 3.5.** Let \(n_1\) be a refraction index satisfying the assumption in (12). Let \(k_j, j = 0, \cdots, l\) be any \(l + 1\) interior transmission eigenvalues, counted according to its multiplicity, corresponding to (1.2) in absorbing media, \(\epsilon_1 > 0\), \(\gamma_1 > 0\). Then, for any \(\sigma > 0\), there exists a

\[
\eta := \frac{1}{\inf_{[0, 1]} \epsilon_0 \epsilon_1} \sup_{[0, 1]} |\gamma_1' - \gamma_1|,
\] (3.46)

depending on \(\sigma\), such that there exists \(l + 1\) interior transmission eigenvalues of (1.2) in absorbing media, \(\epsilon_1 > 0\), \(\gamma_1 > 0\), in the \(\sigma\)-neighborhood of \(k_j, j = 0, \cdots, l\), whenever \(\eta = \frac{1}{\inf_{[0, 1]} \epsilon_0 \epsilon_1} \sup_{[0, 1]} |\gamma_1' - \gamma_1|\) is small enough.

A finite collection of interior transmission eigenvalues moves continuously with respect to \(\gamma_1\). This explains the perturbation theory for finitely many eigenvalues. To study the asymptotic behavior, we analyze the asymptotic behavior of zeros of \(y(1; k)\). We need the following counting lemma.

**Lemma 3.6.** If \(|z - j\pi| \geq \delta > 0\), \(j \in \mathbb{Z}\), then

\[
\exp\{|3z|\} < \frac{O(1)}{\delta} |\sin z|.
\] (3.47)
Proof. We modify the one in [15]. Let \(|z| \geq \frac{\pi}{2n}\), where \(n\) is to be chosen. We discuss two cases: \(0 \leq x \leq \frac{\pi}{2n}\) and \(\frac{\pi}{2n} \leq x \leq \frac{\pi}{2}\).

Now \(x^2 + y^2 = |z|^2\) and \(0 \leq x \leq \frac{\pi}{2n}\). It implies that

\[
y^2 = |z|^2 - x^2 \geq \left(\frac{\pi}{n}\right)^2 - \left(\frac{\pi}{2n}\right)^2 = \frac{3\pi^2}{4n^2}.
\]

Hence,

\[
cosh^2 y \geq 1 + y^2 \geq 1 + \frac{3\pi^2}{4n^2} \geq \left[1 + \frac{1}{4}(\frac{\pi}{n})^2\right] \cos^2 x.
\]

For \(\frac{\pi}{2n} \leq x \leq \frac{\pi}{2}\), we see \(\cos \frac{\pi}{2n} \geq \cos x \geq \cos \frac{\pi}{2} = 0\) and

\[
\frac{1}{\cos x} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \geq 1 + \frac{x^2}{2}, \quad \text{when } x \text{ is small}.
\]

Hence, squaring on both sides,

\[
\frac{1}{\cos^2 x} \geq 1 + x^2 + \frac{x^4}{4}, \quad \text{when } x \text{ is small}.
\]

Using this with \(\cosh^2 y \geq 1\),

\[
cosh^2 y \geq \frac{1}{\cos^2 \frac{\pi}{2n}} \cos^2 x \geq [1 + (\frac{\pi}{2n})^2 + \frac{1}{4}(\frac{\pi}{2n})^4] \cos^2 x \geq [1 + \frac{1}{4}(\frac{\pi}{n})^2] \cos^2 x.
\]

To (3.49) and (3.52), we use \(|\sin z|^2 = \cosh^2 y - \cos^2 x\). Hence,

\[
|\sin z|^2 \geq \cosh^2 y - (1 + \frac{\pi^2}{4n^2})^{-1} \cos^2 y \geq (1 - (1 + \frac{\pi^2}{4n^2})^{-1}) \frac{\cosh^2 y}{4}.
\]

Hence, letting \(C_n = 2[1 - (1 + \frac{\pi^2}{4n^2})^{-1}] - \frac{\pi}{2} = \frac{2\pi \sqrt{n+1}}{n+1} = O(n)\), we conclude

\[
\exp(|\Im y|) < C_n |\sin z|.
\]

Considering \(\delta = \frac{1}{n}\), this proves the lemma.

\[\square\]

**Proposition 3.7.** Let \(z_j\) be the zeros of \(y(1;k)\) and \(z_j'\) be the zeros of \(y'(1;k)\). The following asymptotics hold.

\[
\frac{j\pi}{B} = \frac{iD}{B} + O\left(\frac{1}{j}\right), \quad j \in \mathbb{Z};
\]

\[
\frac{j\pi}{B} = \frac{(j - \frac{1}{2})\pi}{B} = \frac{iD}{B} + O\left(\frac{1}{j}\right), \quad j \in \mathbb{Z}.
\]

**Proof.** We consider the zeros of \(k[\epsilon_1(0)\epsilon_1(1)]^{\frac{j}{2}}y(1;k)\) instead. We observe from [12] that

\[
|k[\epsilon_1(0)\epsilon_1(1)]^{\frac{j}{2}}y(1;k) - \sin\{kB + iD\}| = O\left(\frac{1}{|k|}\right)O(\exp(|\Im k|B))
\]

\[
= O\left(\frac{1}{|k|}\right)\exp(|\Im k|B), \quad \Im k \neq 0.
\]

Firstly we apply the Rouché’s theorem inside the strip with boundary: \(Re k = \frac{(-\frac{1}{2})\pi}{B}, \quad Re k = \frac{(j + \frac{1}{2})\pi}{B}\). For this purpose, we choose \(M\) large in the

\[
C_{j/M} = O(j/M); \quad |k| = j
\]

such that (3.54) and (3.55) imply

\[
|k[\epsilon_1(0)\epsilon_1(1)]^{\frac{j}{2}}y'(1;k) - \sin\{kB + iD\}| < |\sin\{kB + iD\}|, \quad \Im k \neq 0.
\]
The contour applies even without the behavior at \( \Re k = 1 \). Hence, we know \( y^1(1; k) \) has a zero inside the strip.

Secondly, we apply Rouché’s theorem again over a contour as the boundary of the vertical strip with a punctured hole: \( \Re k = (j\pi - \frac{1}{2}B), \Re k = (j\pi + \frac{1}{2}B) \), and \( |k - (\frac{j\pi}{B} - i\frac{D}{2})| = \rho \). We may choose \( j \) large and some \( M \) such that

\[
\rho > \frac{M}{jB} \quad \text{and} \quad |C_j/M O(\frac{1}{j})| < 1. \tag{6.31}
\]

In such a punctured strip,

\[
|k[\epsilon_1(0)\epsilon_4(1)]^\frac{1}{4} y^1(1; k) - \sin\{kB + iD\}| < |\sin\{kB + iD\}|, \Re k \neq 0, \tag{6.32}
\]

by Lemma 3.6. Hence, the zeros of \( k[\epsilon_1(0)\epsilon_4(1)]^\frac{1}{4} y^1(1; k) \) are the same ones of \( \sin\{kB + iD\} \) inside the strip, but outside the \( \rho \)-ball centered at \( \frac{j\pi}{B} - i\frac{D}{2} \), for \( j \) large. There is no zero there.

Therefore, we rewrite this as

\[
z_j = \frac{j\pi}{B} - \frac{iD}{B} + O\left(\frac{1}{j}\right), \quad j \in \mathbb{Z}. \tag{6.33}
\]

Similarly, from (1.25) we have

\[
|k\frac{\epsilon_1(0)}{\epsilon_4(1)} y^1(1; k) - \cos\{kB + iD\}| = O\left(\frac{1}{|k|}\right) \exp\{|3k|B\}, \Re k \neq 0. \tag{6.34}
\]

We apply Lemma 3.6 as follows: \( |z + \frac{\pi}{2} - j\pi| \geq \delta > 0, \quad j \in \mathbb{Z} \), then

\[
\exp\{|3z|\} < O\left(\frac{1}{\delta}\right) |\cos z|. \tag{6.35}
\]

Apply Rouché’s theorem again with (3.63) and (3.64), we prove the asymptotics (6.56). \( \square \)

The following lemma is classic [15, Ch. 2].

**Lemma 3.8.** When \( n_1(r) = \epsilon_4(r) \), then all of zeros of \( y(1; z), y'(1; z) \) are real.

**Proof.** We apply the theorem in [15] Theorem 1]. Since

\[
y(1; z) = \frac{1}{2iz[\epsilon_1(0)\epsilon_0]^\frac{1}{4}} \exp\{izB\}[1 + O\left(\frac{1}{z}\right)] - \frac{1}{2iz[\epsilon_1(0)\epsilon_0]^\frac{1}{4}} \exp\{-izB\}[1 + O\left(\frac{1}{z}\right)]. \tag{6.66}
\]

Therefore, we rewrite this as

\[
z[\epsilon_1(0)\epsilon_0]^\frac{1}{4} y(1; z) := \sin zB - f(z), \tag{6.67}
\]

where

\[
f(z) = -\frac{1}{2i} e^{izB} O\left(\frac{1}{z}\right) + \frac{1}{2i} e^{-izB} O\left(\frac{1}{z}\right). \tag{6.68}
\]

It is easy to deduce that \( f(z) \) is an exponential function of at most type \( B \). By the uniqueness of (1.8), \( y(1; z) \) is real on the real axis. Hence, (6.67) implies that \( f(z) \) is real on the real axis. Without loss of generality, we assume \( |f(z)| \leq 1 \) on the real. If not, we scale (6.67) with respect to \( z \) as follows. In particular, on the real axis,

\[
|f(zM)| \leq \frac{1}{2} O\left(\frac{1}{zM}\right) + \frac{1}{2} O\left(\frac{1}{zM}\right) = O\left(\frac{1}{Mz}\right) = \frac{1}{M} O\left(\frac{1}{z}\right). \tag{6.69}
\]

We choose \( M > 1 \) large such that

\[
|f(zM)| \leq 1, \quad \text{on the real axis.} \tag{6.70}
\]

Therefore, [15] Theorem 1] implies that

\[
zM[\epsilon_1(0)\epsilon_0]^\frac{1}{4} y(1; zM) = \sin zMB - f(zM) \tag{6.71}
\]

has only real zeros or vanishes identically. Thus, \( y(1; z) \) has only real zero or vanishes identically. The proof for \( y'(1; z) \) is similar. \( \square \)
Now we return to the result (3.10) and (3.72). We set the absorbing part of index \( n_i' := \epsilon_i + i \frac{\pi}{2} \), \( i = 1, 2 \), to be zero. Let \( y^{j,0}(1; z) \) be the solution to such an index. They are represented by the construction of (1.23) and (1.25) or (1.31) and (1.32). However, we can merely deduce that

\[
y^{j,0}(1; z) = \frac{1}{2iz[e_1(0)e_2]} \exp\{izB^j[1 + O(\frac{1}{z})]\} - \frac{1}{2iz[e_1(0)e_2]} \exp\{-izB^j[1 + O(\frac{1}{z})]\},
\]

where \( B^1 = B^2 \). We need to prove \( y^{1,0}(1; z) \equiv y^{2,0}(1; z) \).

Let us define

\[
Q(z) := \frac{y^{1,0}(1; z)}{y^{2,0}(1; z)}.
\]

Firstly, we apply Lemma 3.8. \( Q(z) \) is holomorphic outside \( 0i + \mathbb{R} \). We use (3.65) and (3.60) to find a sequence of real numbers \( \rho_j \) such that \( \rho_j \sim O(\frac{1}{j}) \) and that each zero of \( y^{1,0}(1; z) \) and \( y^{2,0}(1; z) \) is located inside the ball \( \Omega_j := \{ z \mid |z - \frac{\pi}{B^1}| \leq \rho_j \} \) for all \( j \). We substitute the asymptotics (1.31) into \( Q(z) \). We obtain

\[
Q(z) = \frac{\sin B^1z + O\left(\frac{\epsilon}{y^1 \|B^1\|}ight)}{\sin B^2z + O\left(\frac{\epsilon}{y^2 \|B^2\|}\right)} = \frac{\sin B^1z[1 + O(\frac{1}{j})]}{\sin B^2z[1 + O(\frac{1}{j})]} |z - \frac{j\pi}{B^1}| = \rho_j,
\]

where we have applied the (3.51). Since \( B^1 = B^2 \),

\[
Q(z) = \frac{\sin B^1z[1 + O(\frac{1}{j})]}{\sin B^2z[1 + O(\frac{1}{j})]} = 1 + O(\frac{1}{j}), \quad |z - \frac{j\pi}{B^1}| = \rho_j.
\]

Thus,

\[
\sup_{|z - \frac{j\pi}{B^1}| = \rho_j} |Q(z) - 1| = O(\frac{1}{j}).
\]

We consider the maximum principle in complex analysis inside \( \Omega_j \)'s for large \( j \). Hence, the limit on the left vanishes as \( j \to \infty \). Therefore,

\[
Q(z) \to 1, \quad \text{as} \quad z \to 0i \pm \infty \quad \text{on} \quad 0i + \mathbb{R}.
\]

Thus, there are only finitely many zeros or poles of \( Q(z) \) on \( 0i + \mathbb{R} \). Let \( \{ \sigma_j \}_{k=1}^{N_1}, \{ \xi_k \}_{k=1}^{N_2} \) be the zeros and poles of \( Q(z) \) on \( 0i + \mathbb{R} \). They must be zeros of \( y^{1,0}(1; k) \), \( y^{2,0}(1; k) \). Rouche’s theorem and (3.60) imply that \( N_1 = N_2 \). Henceforth, \( y^{1,0}(1; k) \) and \( y^{2,0}(1; k) \) have all common zeros after their \( N_1 \)-th zero. Let us denote them as \( \{ \tau_1, \tau_2, \ldots \} \).

Let us define

\[
q(z) := Q(z) \prod_{k=1}^{N_1} \frac{z - \xi_k}{z - \sigma_k}.
\]

Hence, \( q(z) \) is entire and

\[
|q(z)| \to 1, \quad \text{as} \quad z \to 0i \pm \infty.
\]

Moreover, we apply (2.10) to find

\[
h_Q(\theta) = h_{y^{1,0}}(\theta) - h_{y^{2,0}}(\theta) = B^1 \sin \theta - B^2 \sin \theta = 0;
\]

\[
h_q(\theta) = h_q(\theta).
\]

We consider the (3.77), (3.78) and (3.80) under Phragmén-Lindelöf’s theorem [12, p. 38] and (16) to deduce that \( q(z) \) is uniformly bounded in \( \mathbb{C}^+ \). A similar argument works in \( \mathbb{C}^- \). Therefore, Liouville’s theorem implies that \( q(z) \) is a constant that is \( q(z) \equiv 1 \). From (3.77),

\[
\prod_{k=1}^{N_1} (z - \xi_k) y^{1,0}(1; z) \equiv \prod_{k=1}^{N_1} (z - \sigma_k) y^{2,0}(1; z).
\]
We repeat every equation since (3.73) similarly to conclude that
\[
N_1' \prod_{l=1}^{N_1'} (z - \xi'_l) (y^{1,0}'(1; z)) \equiv \prod_{l=1}^{N_1} (z - \sigma'_l)(y^{2,0}'(1; z)), \tag{3.82}
\]
where \(\{\xi'_l\}\) is the zeros of \((y^{2,0})'(1; z)\); \(\{\sigma'_l\}\) of \((y^{1,0})'(1; z)\). Besides these finite exceptional points, \(\{\tau'_m\}\) is the common zero set.

To prove a sharper identity, we apply Proposition 3.5 under the equation (3.10) and (3.12) by letting \(\gamma_1 \downarrow 0\): There exists \(\{\eta_1, \eta_2, \ldots, \eta_{M_1}\}\) such that they satisfy
\[
y^{1,0}(1; \eta_j) = y^{2,0}(1; \eta_j) \neq 0; \tag{3.83}
\]
\[
(y^{1,0})'(1; \eta_j) = (y^{2,0})'(1; \eta_j) \neq 0, \tag{3.84}
\]
where \(\eta = 1, \ldots, \eta_{M_1}\) as the interior transmission eigenvalues for index \(n_1^1 = \epsilon_1^1\) and \(n_1^2 = \epsilon_1^2\). We note that the density of Dirichlet eigenvalues is \(\frac{4\pi}{\lambda^3}\), but the density of transmission eigenvalues is \(\frac{4\pi}{\lambda^{3-}}\). We have more interior eigenvalues. Let us choose \(M_1 > N_1\) and \(M_1 > N_1'\). Therefore,
\[
\prod_{k=1}^{N_1} (z - \xi_k) \equiv \prod_{k=1}^{N_1} (z - \sigma_k); \tag{3.85}
\]
\[
\prod_{l=1}^{N_1'} (z - \xi'_l) \equiv \prod_{l=1}^{N_1'} (z - \sigma'_l), \tag{3.86}
\]
by the fundamental theorem of algebra. Thus,
\[
y^{1,0}(1; z) \equiv y^{2,0}(1; z); (y^{1,0})'(1; z) \equiv (y^{2,0})'(1; z). \tag{3.87}
\]
As (3.87) has shown, we have reduced an absorbing problem to a non-absorbing one. The rest of proof is the application of the inverse rod density problem which we refer to the Corollary 2.9 in [1]. This proves the Theorem 1.1.

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