Long-time behavior for the Cauchy problem of the 3-component Manakov system

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Abstract
In this work, the Riemann-Hilbert problem for the 3-component Manakov system is formulated on the basis of the corresponding $4 \times 4$ matrix spectral problem. Furthermore, by applying the nonlinear steepest descent techniques to an associated $4 \times 4$ matrix valued Riemann-Hilbert problem, we can find leading-order asymptotics for the Cauchy problem of the 3-component Manakov system.

Keywords: Riemann-Hilbert problem (RHP), long-time asymptotics, the 3-component Manakov system

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that distinct approaches [1–16] had been presented to analyze the nonlinear problems, among which the inverse scattering transform (IST) [17] had been successfully applied to many important integrable systems. Particularly, Zakharov and his collaborators [18, 19] developed a Riemann-Hilbert formulation which is a modern version of the IST. The Riemann-Hilbert problem [20–29] is a nonlinear mapping between the set of smooth potentials and the corresponding set of spectral data, which is a powerful tool for the construction of soliton solutions of nonlinear evolution equations. It is also possible to investigate effective asymptotic results by performing the asymptotic analysis of the corresponding Riemann-Hilbert problems. To do asymptotics, one usually has to formulate a Riemann-Hilbert problem that has a unique solution that depends on the scattering data. At the beginning of the 1990’s, the nonlinear steepest descent technique (also called Deift-Zhou method) introduced by Deift and Zhou [30] presents a detailed rigorous proof to analyze the time asymptotic behaviors of the nonlinear integrable partial differential equations (PDEs). This technique was inspired by earlier work of Its [31] and Manakov [32]; for a detailed historical information, please see [33], further developed by Deift, Venakides, and Zhou [34]. This technique has been successfully applied in determining asymptotic formulas for the initial value problems of a large number of integrable systems associated with $2 \times 2$ and $3 \times 3$ Lax pairs such as nonlinear Schrödinger (NLS) equation [35, 36], Korteweg–de Vries (KdV) equation [37], modified KdV equation [30], sine-Gordon (SG) equation [38], Camassa-Holm (CH) equation [39], derivative NLS equation [40, 41], Sasa-Satsuma (SS) equation [42], coupled NLS equation [43], Degasperis-Procesi equation [44], modified NLS equation [45], three-component coupled modified KdV equation [46], Fokas-Lenells equation [47], Harry Dym equation [48], Kundu-Eckhaus equation [49, 50], and nonlocal NLS equation [51] etc.

In recent years, the investigation of multi-component NLS equations has been paid much attention, since they can describe a variety of complex physical phenomena and admit more abundant dynamics of localized wave solutions than ones in the scalar equations. In this work, we therefore focus on the well-known 3-component Manakov equation [52–56], whose form yields
where \( q(x, t)(i = 1, 2, 3) \) are complex-valued. Besides, \( q_1(x, 0), q_2(x, 0), q_3(x, 0) \) lie in the Sobolev space

\[
H^{1,2}(\mathbb{R}) = \{ f(x) \in L^2(\mathbb{R}) : f'(x), xf(x) \in L^2(\mathbb{R}) \}.
\]

In addition, \( q_{1,0}(x), q_{2,0}(x) \) and \( q_{3,0}(x) \) are assumed to be generic so that \( \det(\alpha(\lambda)) \) defined in the following context is nonzero in \( \mathbb{C} \). The set of such generic functions is an open dense subset of \( H^{1,2}(\mathbb{R}) \) [8, 56], which we denote by \( G \). To our knowledge, much research work has been done for multi-component models [57–59]. For example, the initial-boundary value (IBV) problem for the 3-component Manakov system \((1.1)\) on a finite interval is investigated via the Fokas method [58]. Besides, in [59], the authors derived novel dark-bright soliton solutions for the 3-component Manakov system \((1.1)\) and study the resulting soliton interactions. Recently, Tian studied the IBV problems for the two-component NLS equation on the interval and on the half-line [60, 61]. However, the long-time asymptotics for the 3-component Manakov system \((1.1)\), to the best knowledge of the authors, has never been reported up to now.

As we all know, the Deift-Zhou method is a powerful technique to analyze the long-time asymptotics for integrable nonlinear PDEs. However, since equation \((1.1)\) admits a \( 4 \times 4 \) matrix spectral problem, the RHP for equation \((1.1)\) is rather complicated to derive. The research in this work, to the best knowledge of the authors, has not been considered so far. The main purpose of the present article is to analyze the long-time asymptotics for equation \((1.1)\) by utilizing nonlinear steepest descent technique.

The structure of this article is as follows. In section 2, we convert the solution of the Cauchy problem for equation \((1.1)\) to that of a matrix RHP. In section 3, we transform the original RHP to an appropriate expression and discuss the long-time asymptotic behavior of the solution of equation \((1.1)\). Finally, some conclusions are presented in section 4.

2. Riemann-Hilbert problem

The 3-component Manakov system is of course completely integrable, it admits the result of the compatibility between the following linear differential equations [52]

\[
\begin{align*}
q_{1,t} &+ \frac{1}{2}q_{1,xx} + (|q_1|^2 + |q_2|^2 + |q_3|^2)q_1 = 0, \\
q_{2,t} &+ \frac{1}{2}q_{2,xx} + (|q_1|^2 + |q_2|^2 + |q_3|^2)q_2 = 0, \\
q_{3,t} &+ \frac{1}{2}q_{3,xx} + (|q_1|^2 + |q_2|^2 + |q_3|^2)q_3 = 0,
\end{align*}
\]

with \( \sigma = \text{diag}(-1, 1, 1, 1) \),

\[
U(x, t) = \begin{pmatrix}
0 & q_1 & q_2 \\
\bar{q}_1 & 0 & 0 \\
\bar{q}_2 & 0 & 0 \\
\bar{q}_3 & 0 & 0
\end{pmatrix}
\]

\[
q = (q_1, q_2, q_3), \quad V = \lambda U + i(U_x + U^2)\sigma/2,
\]

where the superscript \( '†' \) means Hermitian conjugate of a matrix, and \( \psi = \Psi(x, t, \lambda) \) is a column vector function of the spectral parameter \( \lambda \).

In the following, by introducing a new matrix function \( \psi(x, t; \lambda) = \mu(x, t; \lambda)e^{i(\lambda x + \lambda t)}\sigma \).

Then the spectral problem \((2.1)\) gives

\[
\begin{align*}
\mu_x(x, t; \lambda) - i\lambda[\sigma, \mu(x, t; \lambda)] &= U(x, t)\mu(x, t; \lambda), \\
\mu_t(x, t; \lambda) - i\lambda^2[\sigma, \mu(x, t; \lambda)] &= V(x, t; \lambda)\mu(x, t; \lambda).
\end{align*}
\]

We next present two eigenfunctions \( \mu_\pm(x, t; \lambda) \) of \( x \)-part of \((2.4)\) by the following Volterra type integral equations

\[
\mu_\pm = I + \int_{-\infty}^{\infty} e^{i(\lambda x - \xi \eta)}[U(\xi, t)\mu_\pm(\xi, t; \lambda)]d\xi,
\]

where \( \sigma \) represents the operators which act on a \( 4 \times 4 \) matrix \( \Omega \) by \( \sigma = [\sigma, \Omega] \). Then \( e^{\sigma} = e^{\Omega}e^{\sigma} \).

It can therefore be shown that the functions \( \mu_\pm \) are bounded and analytical for \( \lambda \in \mathbb{C} \), while can belong to

\[
\begin{align*}
\mu_+ : (\mathbb{C}_+, \mathbb{C}_+), \\
\mu_- : (\mathbb{C}_-, \mathbb{C}_-),
\end{align*}
\]

where \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) represent the upper and lower half complex \( \lambda \)-plane, respectively. Here taking \( i\lambda\sigma = \text{diag}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \), we then have

\[
\begin{align*}
\mathbb{C}_+ = \{ \lambda \in \mathbb{C} | \text{Re}\zeta_1 = \text{Re}\zeta_2 = \text{Re}\zeta_3 > \text{Re}\zeta_4 \}, \\
\mathbb{C}_- = \{ \lambda \in \mathbb{C} | \text{Re}\zeta_1 = \text{Re}\zeta_2 = \text{Re}\zeta_3 < \text{Re}\zeta_4 \}.
\end{align*}
\]

The solutions of differential equation \((2.4)\) can be related by a matrix independent of \( x \) and \( t \). As a result

\[
\mu_+(x, t; \lambda) = \mu_-(x, t; \lambda)e^{i(\xi x + \xi t)}\sigma(\lambda).
\]

Evaluation at \( t = 0 \) arrives at

\[
s(\lambda) = \lim_{x \to +\infty} e^{-i\lambda\sigma} \mu_+(x, 0; \lambda),
\]

i.e.,

\[
s(\lambda) = I + \int_{-\infty}^{+\infty} e^{i\lambda\sigma}[U(x, 0)\mu_+(x, 0; \lambda)]dx.
\]
The fact that $U$ is traceless together with equation (2.5) indicates
\[ \det(\mu_\pm(x, t; \lambda)) = 1. \]
As a consequence, one can obtain
\[ \det(s(\lambda)) = 1. \quad (2.8) \]
Additionally, we can know that
\[ U = -U^\dagger. \]
Then from (2.1) we find
\[ \psi^A(x, t; \lambda) = (i\lambda\sigma + U(x, t))^T_\psi^A(x, t; \lambda), \quad (2.9) \]
with $\psi^A(x, t; \lambda) = (\psi^{-1}(x, t; \lambda))^T$, where the superscript 'T' represents a matrix transpose. As a result, we have
\[ \psi^A(x, t; \bar{\lambda}) = \psi^{-1}(x, t; \lambda). \quad (2.10) \]
These relations mean that the eigenfunctions $\mu_\pm(x, t; \lambda)$ meet
\[ \mu_\pm(x, t; \bar{\lambda}) = \mu_\mp(x, t; \lambda). \quad (2.11) \]
To sum up, the matrix-valued function $s(\lambda)$ admits the following symmetry
\[ s^I(\overline{\lambda}) = s^{-1}(\lambda). \quad (2.12) \]
In the next moment, without otherwise specified, by matrix blocking we rewrite the $4 \times 4$ matrix $A$ as
\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \]
where $A_{11}$ is scalar and $A_{22}$ is a $3 \times 3$ matrix. It follows from (2.7)–(2.12) that
\[ s^I_{11}(\overline{\lambda}) = \det(s_{22}(\lambda)), \quad s^I_{21}(\overline{\lambda}) = -s_{12}(\lambda)\text{adj}(s_{22}(\lambda)), \quad (2.13) \]
where $\text{adj}(B)$ represents the adjoint matrix of matrix $B$. Because of the above expression (2.13), we can rewrite $s(\lambda)$ as
\[ s(\lambda) = \begin{pmatrix} \det(a^I(\lambda)) & b(\lambda) \\ -\text{adj}(a^I(\lambda))b^I(\lambda) & a(\lambda) \end{pmatrix}, \quad (2.14) \]
where $a(\lambda)$ is a $3 \times 3$ matrix-valued function and $b(\lambda)$ is a row vector-valued function. It follows that $a(\lambda)$ and $b(\lambda)$ meet
\[ \begin{cases} a(\lambda) = I + i \int_{-\infty}^{+\infty} q^I(\xi, 0)\mu_{12}(\xi, 0; \lambda)dx, \\ b(\lambda) = i \int_{-\infty}^{+\infty} e^{2i\xi}\eta q^I(\xi, 0)\mu_{22}(\xi, 0; \lambda)dx. \end{cases} \quad (2.15) \]
Obviously, it is not hard to see that $a(\lambda)$ is analytic in $C_+$. We have
\[ J(x, t; \lambda) = \begin{pmatrix} 1 + \gamma(\lambda)\gamma^I(\bar{\lambda}) - e^{-i\Theta(\lambda)} & -e^{-i\Theta(\lambda)} \\ -e^{i\Theta(\lambda)} & \frac{i}{\lambda} \end{pmatrix}, \quad (2.17) \]
where
\[ M_\pm(x, t; \lambda) = \lim_{\epsilon \to 0^+} M(x, t; \lambda \pm i\epsilon), \quad \Theta(x, t; \lambda) = 2\left(\frac{\lambda x}{t} + \lambda^2\right), \quad \gamma(\lambda) = b(\lambda)a^{-1}(\lambda), \quad (2.18) \]
with $\gamma(\lambda) \in \mathcal{H}^{1,1}(\mathbb{R})$ and $\sup_{\lambda \in \mathbb{R}} \gamma(\lambda) < \infty$. Then the solution of this RHP exists and is unique. Take
\[ q(x, t) = 2\lim_{\lambda \to -\infty} (\lambda M(x, t; \lambda))_{12}, \quad (2.19) \]
which can tackle the Cauchy problem of the 3-component Manakov system (1.1).

**Proof.** The existence and uniqueness for the solution of the above RHP is a consequence of a vanishing lemma for the associated RHP with the vanishing condition at infinity $M(\lambda) = O(1/\lambda)(\lambda \to \infty)$. This result holds because the jump matrix $J(x, t; \lambda)$ is positive definite (also see [62]).

From the symmetries of jump matrix $J(x, t; \lambda)$, one gets that $M(x, t; \lambda)$ and $(M^\dagger)^{-1}(x, t; \lambda)$ suit the same RHP (2.17). As the uniqueness for the solution of the RHP, we can get
\[ M(x, t; \lambda) = (M^\dagger)^{-1}(x, t; \lambda). \quad (2.20) \]
Note that the asymptotic expansion of $M(x, t; \lambda)$
\[ M(x, t; \lambda) = I + \frac{M_t(x, t)}{\lambda} + \frac{M_{tt}(x, t)}{\lambda^2} + \frac{M_{ttt}(x, t)}{\lambda^3} + \cdots, \quad \lambda \to \infty, \]
which shows that $(M_t)_{12} = -(M_t)_{21}$. We next derive that $q(x, t)$ determined by (2.19) solves the equation (1.1) with the help of the standard arguments of the dressing method [63, 64]. The chief idea of the dressing method is to derive two linear operators $\mathscr{M}$ and $\mathscr{N}$, that is, (I) $\mathcal{L}^N$ and $\mathcal{N}^L$ satisfy the same jump condition as $M$. (II) $\mathcal{L}^N$ and $\mathcal{N}^L$ are of $O(1/\lambda)$ as $\lambda \to \infty$. Take
\[ \begin{cases} M_t = \mathscr{M} + \mathcal{M}\sigma[M] + U\mathcal{M}, \\ M_t = \mathscr{N} + \mathcal{N}\sigma[M] + V\mathcal{N}, \end{cases} \quad (2.21) \]
where $U$ and $V$ are expressed by (2.2). A simple computation can lead to
\[ (\mathscr{M})_t = (\mathscr{M})_J, \quad (\mathcal{N})_t = (\mathcal{N})_J. \]
If $q$ can be given by (2.19), the $\mathscr{M}$ admits the homogeneous RHP
\[ \begin{cases} (\mathscr{M})_t = (\mathscr{M})_J, \\ (\mathscr{M}) = O(1/\lambda), \quad \lambda \to \infty, \end{cases} \quad (2.22) \]
which arrives at
\[ \mathscr{M} = 0. \quad (2.23) \]
Furthermore, by comparing the coefficients of $O(1/\lambda)$ in the asymptotic expansion of (2.23), one can have

$$ M^{(0)} = -i\sigma U/2, \quad [M^{(b)}]_k = i\sigma U^2/2, $$

where the superscripts $'$(O)$'$ and $'$(D)$'$ represent the off-diagonal and diagonal parts of block matrix, respectively. As a consequence, $\mathcal{N}M$ meets the following homogeneous RHP

$$
\begin{align*}
\mathcal{N}M &= \mathcal{N}M - J,
\mathcal{N}M = O(1/\lambda), \quad \lambda \to \infty,
\end{align*}
$$

which means that

$$ \mathcal{N}M = 0. \quad (2.24) $$

The compatibility condition of (2.22) and (2.24) gives the equation (1.1). This indicates that $q(x, t)$ given by (2.19) can solve the equation (1.1).

### 3. Long-time asymptotics

Inspired by earlier works of Deift and Zhou [30], the stationary point of $\Theta(\lambda)$ is given by $\lambda_0$, i.e., $d\phi/dx = 0$, where $\lambda_0 = -\kappa/2t$, as a result, $\Theta = 2(\lambda^2 - 2\lambda_0\lambda)$. Here we put our attention to physically interesting region $|\lambda_0| \leq C$, in which $C$ is a constant.

#### 3.1. Factorization of the jump matrix

We now consider that the jump matrix admits two distinct factorizations

$$ J = \begin{cases} 
1 & e^{-i\theta}\gamma(\lambda)\begin{pmatrix} 1 & 0 \\
0 & I 
\end{pmatrix} \\
1 & 0 \\
\gamma(\lambda) \begin{pmatrix} 1 & 0 \\
0 & I 
\end{pmatrix} \\
\end{cases} \begin{pmatrix} 1 & 0 \\
0 & I 
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & I 
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
0 & I 
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & I 
\end{pmatrix} $$

and the vanishing lemma [62], one can infer that $\delta$ exists and is unique. A simple and direct calculation reveals that $\det \delta$ can be solved by the Plemelj formula [62]

$$ \det(\delta(\lambda)) = e^{\chi(\lambda)}, \quad (3.3) $$

where

$$ \chi(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \left( \frac{1}{\xi - \lambda} + \frac{1}{\lambda - \lambda_0} \right) \log(1 + |\gamma(\xi)|^2) d\xi. $$

In practice, the above integral is singular as $\lambda \to \lambda_0$. We therefore write the integral in the another form

$$ \int_{-\infty}^{+\infty} \log(1 + |\gamma(\xi)|^2) d\xi = \log(1 + |\gamma(\lambda_0)|^2) \log(\lambda - \lambda_0) $$

$$ - \log(1 + |\gamma(\lambda_0)|^2) \log(\lambda - \lambda_0 + 1) $$

$$ + \int_{-\infty}^{+\infty} \frac{1}{\xi - \lambda} \log(1 + |\gamma(\xi)|^2) d\xi $$

$$ + \int_{\lambda_0}^{+\infty} \frac{1}{\xi - \lambda} \log(1 + |\gamma(\xi)|^2 - \log(1 + |\gamma(\lambda_0)|^2)) d\xi, $$

which can indicate that all the terms with the exception of the term $\log(\lambda - \lambda_0)$ are analytic for $\lambda$ in a neighborhood of $\lambda_0$. As a result, $\det \delta$ can be written as

$$ \det(\delta(\lambda)) = (\lambda - \lambda_0)^{\nu}e^{\chi(\lambda)}, $$

where

$$ \nu = -\frac{1}{2\pi} \log(1 + |\gamma(\lambda_0)|^2) < 0, $$

$$ \chi(\lambda_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{1}{\xi - \lambda} \log(1 + |\gamma(\xi)|^2) \right) d\xi $$

$$ + \int_{\lambda_0}^{+\infty} \left( \frac{1}{\xi - \lambda} \log(1 + |\gamma(\xi)|^2 - \log(1 + |\gamma(\lambda_0)|^2) \right) d\xi. \quad (3.4) $$

We next present a function $\delta(\lambda)$ as the solution of the matrix problem

$$ \begin{cases} 
\delta_0(\lambda) = (I + \gamma^\dagger\gamma)\delta(\lambda), \quad |\lambda| < \lambda_0, \\
\delta(\lambda) \to I, \quad \lambda \to \infty. 
\end{cases} \quad (3.1) $$

As the jump matrix $(I + \gamma^\dagger\gamma)$ is positive definite, the vanishing lemma arrives at the existence and uniqueness of the function $\delta(\lambda)$. Additionally, we have

$$ \begin{cases} 
\det(\delta_0(\lambda)) = (1 + |\gamma|^2) \det(\delta(\lambda)), \quad |\lambda| < \lambda_0, \\
\det(\delta(\lambda)) = 1, \quad \lambda \to \infty. 
\end{cases} \quad (3.2) $$

From the positive definiteness of the jump matrix $I + \gamma^\dagger\gamma$

$$ (\delta^\dagger(\lambda))^{-1} = \delta(\lambda). \quad (3.5) $$

In addition, for $|\lambda| < \lambda_0$, it follows from (3.1) that

$$ \lim_{\epsilon \to 0^+} (I + \gamma^\dagger(\lambda_0)\gamma(\lambda))^{-1} \lim_{\epsilon \to 0^-} (I + \gamma^\dagger(\lambda_0)\gamma(\lambda)) $$

If we set $f(\lambda) = (\delta^\dagger(\lambda_0))^{-1}$, we then get

$$ f_0(\lambda) = (I + \gamma^\dagger(\lambda)\gamma(\lambda))f_0(\lambda). $$

Thus, we know

$$ (\delta^\dagger(\lambda_0))^{-1} = \delta(\lambda). \quad (3.5) $$
Lemma 3.1. The vector-valued function \( \rho(\lambda) \) has a decomposition

\[
\rho(\lambda) = \mathcal{R}(\lambda) + \mathcal{H}_1(\lambda) + \mathcal{H}_2(\lambda), \quad \lambda \in \mathbb{R},
\]

where \( \mathcal{R}(\lambda) \) is piecewise rational and \( \mathcal{H}_2(\lambda) \) is analytically and continuously extended to \( \mathcal{L} \). In addition, \( \mathcal{H}_1(\lambda) \) and \( \mathcal{H}_2(\lambda) \) satisfy

\[
\begin{align*}
|e^{-i\Theta(\lambda)}\mathcal{H}_1(\lambda)| &\lesssim \frac{1}{(1+|\lambda-\lambda_0|)^{m+1}}, \quad \lambda \in \mathbb{R}, \\
|e^{-i\Theta(\lambda)}\mathcal{H}_2(\lambda)| &\lesssim \frac{1}{(1+|\lambda-\lambda_0|)^{m+1}}, \quad \lambda \in \mathcal{L},
\end{align*}
\]

where positive integer \( m \) is free. Taking the Schwartz conjugate

\[
\rho^*(\lambda) = \mathcal{H}_1^*(\lambda) + \mathcal{H}_2^*(\lambda) + \mathcal{R}^*(\lambda),
\]

leads to the same estimate for \( e^{i\Theta(\lambda)}\mathcal{H}_1^*(\lambda) \), \( e^{i\Theta(\lambda)}\mathcal{H}_2^*(\lambda) \) and \( e^{i\Theta(\lambda)}\mathcal{R}^*(\lambda) \) on the contour \( \mathbb{R} \cup \mathcal{L} \).

Proof. As \( \lambda \geq \lambda_0 \), \( \rho(\lambda) = \gamma(\lambda) \). With the help of the Taylor’s expansion, we have

\[
(\lambda - \lambda_0 - i)^{m+5} \rho(\lambda) = \sum_{j=0}^{m} \rho_j(\lambda_0)(\lambda - \lambda_0)^j
\]

for convenience. As the map \( \lambda \mapsto \Theta(\lambda) = 2(\lambda^2 - 2\lambda_0\lambda) \) is one-to-one in \( \lambda \geq \lambda_0 \), we define

\[
f(\Theta) = \begin{cases} 
\frac{(\lambda - \lambda_0 - i)^{p+2}}{(\lambda - \lambda_0 - i)^{p+2}}, & \Theta \geq -2\lambda_0^2, \\
0, & \Theta \leq -2\lambda_0^2,
\end{cases}
\]

where

\[
g(\lambda) = \frac{1}{m!} \int_{0}^{\lambda} ((\lambda - \lambda_0 - i)^{m+5} \rho(\cdot))((\lambda - \lambda_0) + \xi(\lambda - \lambda_0))(1 - q_i)\mathrm{d}\xi;
\]

which implies that

\[
\left| \frac{d^j g(\lambda)}{d\lambda^j} \right| \lesssim 1, \quad \lambda \geq \lambda_0.
\]

By making use of the Fourier transform, we have

\[
f(\Theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\Theta f(s)} \mathrm{d}s,
\]

where

\[
f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\Theta f(s)} \mathrm{d}\Theta.
\]

Thus, as \( 0 \leq j \leq p + 1 \),
\[
\int_{-2\lambda_0}^{2\lambda_0} \frac{d|f(\Theta)|^2}{d\Theta} \frac{d\Theta}{\sqrt{2\pi}} = \int_{\lambda_0}^{\infty} \left( \frac{\lambda - \lambda_0}{\lambda - \lambda_0 - i} \right)^{2p+2} \left( \frac{\lambda - \lambda_0}{\lambda - \lambda_0 - i} \right)^{2p+4} (\lambda - \lambda_0)d\lambda \leq 1.
\]

Resorting to the Plancherel theorem, we have
\[
\int_{-\infty}^{+\infty} (1 + s^2)^j|f(s)|^2ds \leq 1, \quad 0 \leq j \leq p + 1.
\]

We now present the decomposition of \( H(\lambda) \) as follows
\[
H(\lambda) = \frac{(\lambda - \lambda_0)^p}{\sqrt{2\pi} (\lambda - \lambda_0 - i)^{p+2}} \int_{-\infty}^{+\infty} e^{i\Theta f} (s)ds
+ \frac{(\lambda - \lambda_0)^p}{\sqrt{2\pi} (\lambda - \lambda_0 - i)^{p+2}} \int_{-\infty}^{\lambda} e^{i\Theta f} (s)ds
= H_0(\lambda) + H_2(\lambda).
\]

For \( \lambda \geq \lambda_0 \), we obtain
\[
|e^{-i\Theta f(\lambda)}H(\lambda)| \leq \frac{|\lambda - \lambda_0 - i|^{2-2t-p/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(s)|ds \leq |\lambda - \lambda_0 - i|^{-2t-p/2},
\]

where \( 0 \leq r \leq p + 1 \). For \( \lambda \) on the line \( \{ \lambda: \lambda_0 + e^{i\theta} \}, \theta < 0 \), we have
\[
|e^{-i\Theta f(\lambda)}H_2(\lambda)| \leq |\lambda - \lambda_0 - i|^{-2t-p/2},
\]

Here \( m = 3p + 1 \) is sufficiently large such that \( p - 1/2 > p/2 \) are greater than \( l \). The proof finishes here. Then the another case is also similar.

### 3.3. First contour deformation

In what follows, the original RHP turns into an equivalent RHP formulated on an augmented contour \( \Sigma \), where \( \Sigma = \mathcal{L} \cup \mathcal{L} \cup \mathbb{R} \) is shown in figure 2.

Note that \( J^2(x, t; \lambda) \) can be rewritten as
\[
J^2(x, t; \lambda) = (b_-)^{-1}b_+,
\]

where \( b_\pm = \mathcal{I} \pm \omega_\pm \) with
\[
b_+ = \exp(-\delta(\lambda))d(\mathcal{I} + \omega_\pm) = \begin{cases} 0 & \text{if } \delta(\lambda) > 0 \\ \exp(-\delta(\lambda)) & \text{if } \delta(\lambda) < 0 \end{cases}
\]

Let
\[
M^1(x, t; \lambda) = \begin{cases} M_1^1(x, t; \lambda), & \lambda \in \Omega_1 \cup \Omega_2, \\ M_2^1(x, t; \lambda)(b_0^\pm)^{-1}, & \lambda \in \Omega_3 \cup \Omega_4, \\ M_3^1(x, t; \lambda)(b_0^\pm)^{-1}, & \lambda \in \Omega_5 \cup \Omega_6, \end{cases}
\]

Theorem 3.3. [65] Assume \( \mu^2(x, t; \lambda) \in \mathcal{L}^2(\Sigma) + \mathcal{L}^{\infty}(\Sigma) \) satisfies
\[
\mu^2 = \mathcal{I} + C_{\mathcal{J}} \mu^2.
\]
Thus
\[ M^2(\lambda) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu^2(\xi)\omega^2(\xi)}{\xi - \lambda} d\xi, \tag{3.22} \]
is the solution of the RHP (3.18).

**Theorem 3.4.** The solutions \((q_1(x, t), q_2(x, t), q_3(x, t))\) of the Cauchy problem for the 3-component Manakov system (1.1) can be expressed by
\[ q(x, t) = (q_1, q_2, q_3) = 2 \lim_{\lambda \to \infty} (\lambda M^2(\lambda))_{12}, \]

**Proof.** It follows by (3.7), (3.18), theorem 2.1 and the following expressions
\[ \left\{ \begin{array}{l}
|e^{-\rho(\lambda)} M_2(\lambda)| \lesssim |\lambda - \lambda_0 - i\tau|, \quad \lambda \in \Omega_\delta \cup \Omega_\varepsilon, \quad \lambda \to \infty, \\
|e^{-\rho(\lambda)} R(\lambda)| \lesssim |\lambda - \lambda_0 - i\tau|^{-5}, \quad \lambda \in \Omega_\delta \cup \Omega_\varepsilon, \quad \lambda \to \infty.
\end{array} \right. \]

3.4. **Second contour deformation**

We next reduce the RHP \(M^2\) on \(\Sigma\) to the RHP \(M'\) on \(\Sigma'\), where \(\Sigma' = \Sigma \setminus \mathbb{R} = \mathcal{C} \cup \mathcal{L}\) is orientated as in figure 2. In addition, we estimate the error between the RHP on \(\Sigma\) and \(\Sigma'\). Then it is proved that the solution of the equation (1.1) can be expressed in terms of \(M'\) by adding an error term.

Let \(\omega' = \omega + \omega'\), where
(I) \(\omega' = \omega\) on \(\Sigma\) and composed of terms of type \(\mathcal{H}_0(\lambda)\) and \(\mathcal{H}_1(\lambda)\);  
(II) \(\omega\) is supported on \(\mathcal{C} \cup \mathcal{L}\) and composed of the contribution to \(\omega'\) from terms of type \(\mathcal{H}_2(\lambda)\) and \(\mathcal{H}_3(\lambda)\).

Define \(\omega' = \omega' - \omega\), then \(\omega' = 0\) on \(\mathbb{R}\). Thus, \(\omega'\) is supported on \(\Sigma'\) with contribution to \(\omega'\) from \(R(\lambda)\) and \(R'(\lambda)\).

**Lemma 3.5.** For sufficiently small \(\epsilon\), as \(t \to \infty\)
\[ \begin{align*}
&\left\| \omega' \right\|_{L^2(\Sigma')} \lesssim t^{-1}, \\
&\left\| (1 - C_{\omega})(C_{\omega}^2) \omega' \right\|_{L^2(\Sigma')} \lesssim t^{-1/4}, \\
&\left\| (1 - C_{\omega})(C_{\omega}^2) \omega' \right\|_{L^2(\Sigma')} \lesssim t^{-1/4}, \\
&\left\| (1 - C_{\omega})(C_{\omega}^2) \omega' \right\|_{L^2(\Sigma')} \lesssim t^{-1/4}, \\
&\left\| (1 - C_{\omega})(C_{\omega}^2) \omega' \right\|_{L^2(\Sigma')} \lesssim t^{-1/4}.
\end{align*} \tag{3.24} \]

**Proof.** The proof of first two expressions in (3.24) follows from lemma 2.2. From the definition of \(R(\lambda)\), we have
\[ |R(\lambda)| \lesssim (1 + |\lambda - \lambda_0|^\epsilon)^{-1}, \]
on the contour \(\lambda = \lambda_0 + \lambda_0 e^{i\pi/4}; -\infty < \alpha < +\infty\).

By means of inequality (3.6), we obtain
\[ \left| e^{-\rho(\lambda)} \det(\delta(\lambda)) \delta(\lambda) R(\lambda) \right| \lesssim e^{-2\tau} \epsilon (1 + |\lambda - \lambda_0|^\epsilon)^{-1}, \tag{3.25} \]

After a direct calculation, we can obtain the last expression in (3.24).

**Lemma 3.6.** In the case \(0 < \lambda_0 \leq C\), as \(t \to \infty\), the inverse \((1 - C_{\omega})^{-1}: L^2(\Sigma) \to L^2(\Sigma)\) exists, and has uniform boundedness
\[ \left\| (1 - C_{\omega})^{-1} \right\|_{L^2(\Sigma)} \lesssim 1. \]

**Proof.** See [30] and references therein.

**Lemma 3.7.** The integral equation has estimate as \(t \to \infty\)
\[ \int_{\Sigma} ((1 - C_{\omega})^{-1} C_{\omega}^2) \omega' \xi d\xi = \int_{\Sigma} ((1 - C_{\omega})^{-1} C_{\omega}^2) \omega' \xi d\xi + O(t^{-1}). \tag{3.26} \]

**Proof.** Via a simple calculation, we derive
\[ ((1 - C_{\omega})^{-1} C_{\omega}^2) \omega' = (1 - C_{\omega})^{-1} C_{\omega}^2 \omega' + (1 - C_{\omega})^{-1} C_{\omega}^2 C_{\omega} \omega + (1 - C_{\omega})^{-1} C_{\omega} (1 - C_{\omega})^{-1} C_{\omega}^2 \omega. \]

It follows from lemma 3.5 that...
This finishes the proof of the theorem.

**Lemma 3.8.** As $t \to \infty$

$$q(x, t) = \frac{i}{\pi} \left( \int_{\Sigma} ((1 - C_{\omega'})^{-1} \mathcal{I}) \omega'(t) \bar{d}t \right)_{12} + O(t^{-1}).$$

**Proof.** A straightforward consequence of theorem 3.4 and lemma 3.8.

**Corollary 3.9.** As $t \to \infty$,

$$q(x, t) = 2 \lim_{\lambda \to \infty} \left( \lambda M'(x, t; \lambda) \right)_{12} + O(t^{-1}), \quad (3.27)$$

where $M'(x, t; \lambda)$ satisfies the RHP

$$\begin{cases}
M'(x, t; \lambda) = M'(x, t; \lambda)J'(x, t; \lambda), & \lambda \in \Sigma', \\
M'(x, t; \lambda) \to \mathcal{I}, & \lambda \to \infty,
\end{cases}$$

where

$$\omega' = \omega' + \omega', \quad b_{\omega'} = \mathcal{I} + \omega', \quad \mathcal{J'} = (b_{\omega'})^{-1}b'_{\omega'},$$

$$b'_{\omega'} = \begin{pmatrix} 1 & e^{-it\theta} \det(\delta(\lambda)) \mathcal{R}(\lambda) \delta(\lambda) \end{pmatrix}, \quad b'_{\omega'} = \mathcal{I}, \quad \text{on } \mathcal{L},$$

$$b'_{\omega'} = \mathcal{I}, \quad b'_{\omega'} = \begin{pmatrix} 1 \\
- \frac{e^{it\theta} \delta(\lambda) \mathcal{R}(\lambda)}{\det(\delta(\lambda))} \end{pmatrix}, \quad \text{on } \mathcal{L}'$$

**Proof.** Set $\mu' = (1 - C_{\omega'})^{-1} \mathcal{I}$ and

$$M'(x, t; \lambda) = \mathcal{I} + \frac{1}{2\pi i} \int_{\Sigma'} \mu'(x, t; \xi) \omega'(x, t; \xi) d\xi.$$  

Similar to theorem 3.4, we can construct this corollary 3.9 in terms of (3.4).

**3.5. Reduction of the Riemann-Hilbert problems**

In this subsection, we localize the jump matrix of the RHP to the neighborhood of the stationary phase point $\lambda_0$. Under suitable scaling of the spectral parameter, the RHP is reduced to a RHP with constant jump matrix which can be solved explicitly.

Let $\Sigma_0$ denote the contour $[\lambda = \alpha e^{\pm \frac{\pi}{4}}; \alpha \in \mathbb{R}]$. Denote the restriction of $\Sigma$ to the contour labeled by $j$.

Figure 4. The contour $\mathcal{L}$ is defined by (3.45).

Figure 2. The orient jump contour $\Sigma = \mathcal{L} \cup \mathcal{J} \cup \mathbb{R}$ and domains $\{\Omega_1\}^4$ and $\{\Omega_j\}^6$ denote the restriction of $\Sigma$ to the contour labeled by $j$.

Figure 3. The oriented contour $\Sigma_0$ (\Sigma_0 denote the contour $[\lambda = \alpha e^{\pm \frac{\pi}{4}}; \alpha \in \mathbb{R}]$).

where

$$\omega' = \omega' + \omega', \quad b_{\omega'} = \mathcal{I} + \omega', \quad J' = (b_{\omega'})^{-1}b'_{\omega'},$$

$$b'_{\omega'} = \begin{pmatrix} 1 & e^{-it\theta} \det(\delta(\lambda)) \mathcal{R}(\lambda) \delta(\lambda) \end{pmatrix}, \quad b'_{\omega'} = \mathcal{I}, \quad \text{on } \mathcal{L},$$

$$b'_{\omega'} = \mathcal{I}, \quad b'_{\omega'} = \begin{pmatrix} 1 \\
- \frac{e^{it\theta} \delta(\lambda) \mathcal{R}(\lambda)}{\det(\delta(\lambda))} \end{pmatrix}, \quad \text{on } \mathcal{L}',$$

and set $\tilde{\omega} = N\omega'$. A direct change-of-variable argument means that

$$C_{\omega'} = N^{-1}C_{\omega} N,$$

where the operator $C_{\omega'}$ is a bounded map from $L^2(S_0)$ into $L^2(S_0)$.

One can infer that

$$\tilde{\omega} = \tilde{\omega} = \begin{pmatrix} 0 & (N_{S_1}(\lambda)) \\
0 & 0 \end{pmatrix},$$

on $\mathcal{L}_c = \{\lambda = \alpha e^{\pm \frac{\pi}{4}}; -\infty < \alpha < +\infty\}$, and

$$\tilde{\omega} = \mathcal{I}, \quad \text{on } \mathcal{L}_c$$

Phys. Scr. 95 (2020) 065226 X-B Wang and B Han
on \( \hat{\mathcal{L}} \), where
\[
s_1(\lambda) = e^{-\bar{\theta}(\lambda)}(\det \delta(\mathcal{R})(\lambda)\delta(\lambda)),
\]
\[
s_2(\lambda) = \frac{e^{\bar{\theta}(\lambda)}\delta^{-1}(\lambda)\mathcal{R}'(\lambda)}{\det(\delta(\lambda))}.
\]

**Lemma 3.10.** As \( t \to \infty \), and \( \lambda \in \hat{\mathcal{L}} \), for an arbitrary positive integer \( l \), then
\[
|\langle N(\hat{\delta})\rangle(\lambda)| \lesssim t^{-l},
\]  
where
\[
\hat{\delta}(\lambda) = e^{-\bar{\theta}(\lambda)}\mathcal{R}(\lambda)[\delta(\lambda) - (\det \delta(\lambda))\mathcal{I}].
\]

**Proof.** It follows from (3.1) and (3.2) that \( \hat{\delta} \) satisfies the following RHP
\[
\begin{align*}
\hat{\delta}_+(\lambda) &= \delta(\lambda)((1 + |\gamma(\lambda)|^2) + e^{\bar{\theta}(\lambda)}f(\lambda), \quad \lambda \in (-\infty, \lambda_0), \\
\hat{\delta}_-(\lambda) &= 0, \quad \lambda \to \infty,
\end{align*}
\]
where \( f(\lambda) = [\mathcal{R}(\gamma^\gamma - |\gamma|^2)\delta.](\lambda) \). The solution for the above RHP can be expressed by
\[
\hat{\delta}(\lambda) = X(\lambda) \int_{-\infty}^{\lambda} e^{\bar{\theta}(\lambda)}f(\xi)\mathcal{R}(\xi - \lambda) - \lambda d\xi,
\]
\[
X(\lambda) = \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\lambda} \frac{\log(1 + |\gamma(\xi)|^2)}{\xi - \lambda} d\xi \right).
\]

Observe that
\[
\mathcal{R}^{-1} - \gamma \mathcal{R} = (\mathcal{R} - \rho)\gamma - |\gamma|^2(\mathcal{R} - \rho).
\]

Similar to lemma 3.2, \( f(\lambda) \) can be decomposed into two parts: \( f_1(\lambda) \) and \( f_2(\lambda) \), where \( f_2(\lambda) \) admits an analytic continuation to \( \mathcal{L}_t \) satisfying
\[
\begin{align*}
\left| e^{\bar{\theta}(\lambda)}f_1(\lambda) \right| &\lesssim \frac{1}{(1+1-\lambda_0+1/t)^m}, \quad \lambda \in \mathbb{R}, \\
\left| e^{\bar{\theta}(\lambda)}f_2(\lambda) \right| &\lesssim \frac{1}{(1+1-\lambda_0+1/t)^m}, \quad \lambda \in \mathcal{L}_t,
\end{align*}
\]
where (see figure 4)
\[
\mathcal{L}_t : \{ \lambda = \lambda_0 - 1/t + \alpha e^{\lambda_0}, \quad 0 \leq \alpha < +\infty \}.
\]

As \( \lambda \in \hat{\mathcal{L}} \), we have
\[
\langle N(\hat{\delta})\rangle = X(\lambda_0 + \frac{\lambda}{2\sqrt{t}}) \int_{-\infty}^{\lambda} e^{\bar{\theta}(\xi)}f(\xi)\mathcal{R}(\xi - \lambda) - \lambda d\xi
\]
\[
+ X(\lambda_0 + \frac{\lambda}{2\sqrt{t}}) \int_{-\infty}^{-\infty} e^{\bar{\theta}(\xi)}f(\xi)\mathcal{R}(\xi - \lambda) - \lambda d\xi
\]
\[
+ X(\lambda_0 + \frac{\lambda}{2\sqrt{t}}) \int_{-\infty}^{\lambda} e^{\bar{\theta}(\xi)}f(\xi)\mathcal{R}(\xi - \lambda) - \lambda d\xi
\]
\[
= I_1 + I_2 + I_3.
\]

Following a similar way, as a result of Cauchy’s theorem, we can find \( I_3 \) along the contour \( \mathcal{L}_t \) take the place of the interval \((-\infty, \lambda_0 - 1/t)\) and get \( |I_3| \lesssim t^{-l+1} \). As a result, it is not hard to see that (3.29) holds.

**Note.** There exists a similar estimate
\[
|\langle N(\hat{\delta})\rangle| \lesssim t^{-l}, \quad t \to \infty, \quad \lambda \in \hat{\mathcal{L}},
\]
where
\[
\hat{\delta}(\lambda) = e^{i\theta(\lambda)}[\delta^{-1}(\lambda) - (\det(\delta))^{-1}(\lambda\mathcal{I})] \mathcal{R}'(\lambda).
\]

**Theorem 3.11.** As \( t \to \infty \)
\[
q(x, t) = \frac{1}{\sqrt{t}} \lim_{\lambda \to +\infty} (\lambda M^0(x, t; \lambda))_{12} + O\left( \frac{\log t}{\sqrt{t}} \right),
\]
where
\[
M^0(x, t; \lambda) \text{ meets the RHP}
\]
\[
\begin{align*}
M_0^0(\lambda) &= M_0^0(\lambda)J^0(\lambda), \quad \lambda \in \Sigma^0, \\
M_0^0(\lambda) &\to \mathcal{I}, \quad \lambda \to \infty.
\end{align*}
\]

Here \( J^0 = (\mathcal{I} - \omega^0)(\mathcal{I} + \omega^0) \) and
\[
\begin{align*}
\omega^0 &= \omega^0 \ \begin{pmatrix}
0 & \eta^2\lambda^2 e^{-\frac{4\lambda^2}{\bar{T}} - \gamma(\lambda_0)} \\
0 & 0
\end{pmatrix} \quad \lambda \in \Sigma^0, \\
\omega^0 &= \omega^0 \ \begin{pmatrix}
0 & 0 \\
0 & -\eta^2\lambda^2 e^{-\frac{4\lambda^2}{\bar{T}} - \gamma(\lambda_0)}
\end{pmatrix} \quad \lambda \in \Sigma^0,
\end{align*}
\]
and

\[
\omega^0 = \omega^0 = \begin{cases} 
\eta^2 - 2\nu e^{-\frac{1}{2}j\gamma^2} \in \Sigma^0_2, \\
0 - \eta^2 - 2\nu e^{\frac{1}{2}j\gamma^2} \in \Sigma^0_4,
\end{cases}
\]

with

\[
\eta = (4t)^{-\frac{1}{2}} e^{i\beta^0 t + \chi}(t).
\]

**Proof.** It follows from (3.29) and lemma 3.35 in [30] that

\[
\|\hat{\omega} - \omega^0\|_{x^* \in (\Sigma^0_2 \cap x^* \in (\Sigma^0_4) \cap x^* \in (\Sigma^0_2)} \lesssim \frac{\log t}{\sqrt{t}}.
\]

As a result,

\[
\int_{\Sigma^0_2} ((1 - C_\omega^{-1})T)(\xi)\omega^0(\xi) d\xi \\
= \int_{\Sigma^0_2} ((1 - C_\omega^{-1})N(T)(\xi)\omega^0(\xi) d\xi \\
= \int_{\Sigma^0_2} ((1 - C_\omega^{-1})T)(\xi)\omega^0(\xi) d\xi \\
= \frac{1}{\sqrt{t}} \int_{\Sigma_0} ((1 - C_\omega^{-1})T)(\xi)\omega^0(\xi) d\xi + O\left(\frac{\log t}{t}\right).
\]

For \( \lambda \in \mathbb{C} \setminus \Sigma^0_2 \), let

\[
M^0(\lambda) = T + \frac{1}{2\pi i} \int_{\Sigma_0} ((1 - C_\omega^{-1})T)(\xi)\omega^0(\xi) d\xi - \frac{1}{\xi - \lambda}.
\]

Then \( M^0 \) can deal with the above RHP. From above expressions and lemma 3.8, it is straightforward to derive this theorem.

**Note.** Particularly, if

\[
M^0(\lambda) = T + \frac{M^0}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \infty,
\]

then

\[
q(x, t) = \frac{1}{\sqrt{t}} \left(\frac{\hat{\omega}}{1} + \frac{\psi(t)}{1}\right).
\]

### 3.6. Solving the model problem

In order to give \( M^0 \) explicitly, it is worth considering the following transformation

\[
\psi(\lambda) = H(\lambda)\lambda^{-\theta} e^{i\beta t + \chi}, \quad H(\lambda) = \eta^2 M^0(\lambda)\eta^{-\sigma},
\]

which indicates that

\[
\psi(\lambda) = \psi(\lambda)\nu(\lambda), \quad \nu = \lambda^{i\beta} e^{i\beta t + \chi} M^0.
\]

As the jump matrix is constant along each ray, we have

\[
\frac{d\psi(\lambda)}{d\lambda} = \frac{d\psi(\lambda)}{d\lambda} v(\lambda),
\]

from which it follows that \( \frac{d\psi(\lambda)}{d\lambda} \psi^{-1}(\lambda) \) has no jump discontinuity along any of the rays. Additionally, from the relation between \( H(\lambda) \) and \( \psi(\lambda) \), we have

\[
\frac{d\psi(\lambda)}{d\lambda} = \frac{dH(\lambda)}{d\lambda} H^{-1}(\lambda) + \frac{1}{2} i\lambda H(\lambda) \sigma H^{-1}(\lambda) \\
- \frac{i\nu}{\lambda} H(\lambda) \sigma H^{-1}(\lambda) \\
= O\left(\frac{1}{\lambda}\right) + \frac{1}{2} i\lambda \sigma - \frac{1}{2} i\nu[\sigma, M^0] \eta^{-\sigma}.
\]

It follows from the Liouville's theorem that

\[
\frac{d\psi(\lambda)}{d\lambda} = \frac{1}{2} i\lambda \sigma \psi(\lambda) + \beta \psi(\lambda),
\]

where

\[
\beta = -\frac{1}{2} i\nu[\sigma, M^0] \eta^{-\sigma} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.
\]

Particularly,

\[
(M^0) = -i\nu[\sigma, M^0].
\]

It is further possible to find that the solution of the RHP for \( M^0(\lambda) \) is unique, and therefore we have an identity

\[
(M^0(\lambda)) = (M^0(\lambda))^{-1},
\]

which implies that \( \beta_{12} = -\beta_{21} \). Let

\[
\psi(\lambda) = \begin{pmatrix} \psi_1(\lambda) & \psi_2(\lambda) \\ \psi_3(\lambda) & \psi_4(\lambda) \end{pmatrix}
\]

From (3.53) we obtain

\[
\frac{d^2\psi_1(\lambda)}{d\lambda^2} = \frac{\beta_{12} \beta_{21} - i - \frac{\lambda}{2}}{4} \psi_{11},
\]

\[
\frac{\beta_{21} \psi_{21}(\lambda)}{d\lambda} = \frac{d\psi_1(\lambda)}{d\lambda} + \frac{i}{2} \lambda \psi_1(\lambda),
\]

\[
\frac{d^2\beta_{12} \psi_{22}(\lambda)}{d\lambda^2} = \frac{(\beta_{12} \beta_{21} + i - \frac{\lambda}{2})}{4} \beta_{12} \psi_{22}(\lambda),
\]

\[
\psi_{12}(\lambda) = \frac{1}{\beta_{12} \beta_{21}} \left(\frac{d\beta_{12} \psi_{22}(\lambda)}{d\lambda} - \frac{i}{2} \lambda \beta_{12} \psi_{22}(\lambda)\right).
\]

As we all know, the Weber's equation

\[
\frac{d^2g(\zeta)}{d\zeta^2} + \left(\alpha + \frac{1}{2} \frac{\lambda^2}{4}\right) g(\zeta) = 0,
\]

admits the solution

\[
g(\zeta) = c_1 D_{\alpha}(\zeta) + c_2 D_{\alpha}(-\zeta),
\]
where $D_a(\cdot)$ represents the standard parabolic-cylinder function and satisfies
\[
\frac{dD_a(\zeta)}{d\zeta} + \frac{\zeta}{2} D_a(\zeta) - aD_{a-1}(\zeta) = 0,
\]
\[
D_a(\pm i\zeta) = \frac{\Gamma(1 + a) e^{\pi i/4}}{\sqrt{2\pi}} D_{a-1}(\pm i\zeta)
+ \frac{\Gamma(1 + a) e^{-\pi i/4}}{\sqrt{2\pi}} D_{a-1}(\mp i\zeta).
\] (3.58)

As $\zeta \to \infty$, from [62] we have
\[
D_a(\zeta) = \begin{cases} 
\zeta^a e^{-\zeta^2/4} \left(1 + O \left(\frac{1}{\zeta}\right)\right), & |\arg\zeta| < \frac{3\pi}{4}, \\
\zeta^a e^{-\zeta^2/4} \left(1 + O \left(\frac{1}{\zeta}\right)\right) - \frac{\sqrt{\pi}}{\Gamma(a-\zeta/2)} e^{\pi i/4 + \zeta^2/4} \left(1 + O \left(\frac{1}{\zeta}\right)\right), & -\frac{\pi}{4} < |\arg\zeta| < \frac{5\pi}{4}, \\
\zeta^a e^{-\zeta^2/4} \left(1 + O \left(\frac{1}{\zeta}\right)\right) + \frac{\sqrt{\pi}}{\Gamma(a-\zeta/2)} e^{-\pi i/4 + \zeta^2/4} \left(1 + O \left(\frac{1}{\zeta}\right)\right), & -\frac{5\pi}{4} < |\arg\zeta| < \frac{3\pi}{4}.
\end{cases}
\] (3.59)

Following the ray $\arg\lambda = -\frac{\pi}{4}$,
\[
\Psi_1(\lambda) = \Psi_1(\lambda) \left(1 + O \left(\frac{1}{\lambda}\right)\right),
\]
\[
\beta_{12} e^{i(\lambda-3\lambda/4)D_{a-1}(i\lambda)} = e^{4\pi i/4} D_a(e^{\pi i/4}) \gamma(\lambda_0).
\]

It follows from (3.58) that
\[
D_a(e^{\pi i/4}) = \Gamma(1 + a) e^{\pi i/2} D_{a-1}(e^{3\pi i/4})
+ \frac{\Gamma(1 + a) e^{-\pi i/2}}{\sqrt{2\pi}} D_{a-1}(e^{-\pi i/4}).
\]

We then analyze the coefficients of the two independent functions and get
\[
\beta_{12} = \frac{e^{\pi i/4} \Gamma(1 + a) e^{\pi i/2} \gamma(\lambda_0)}{\sqrt{2\pi}}
\]
\[
\beta_{12} = \frac{e^{\pi i/4} \Gamma(1 + a) e^{\pi i/2} \gamma(\lambda_0)}{\sqrt{2\pi}}.
\] (3.60)

Summarizing the above results, the following theorem 3.12 can be easily established.

Theorem 3.12. Suppose that $(q_1, q_2, q_3)$ possesses the solution for the Cauchy problem of the 3-component Manakov system (1.1) with $q_1(0), q_2(0), q_3(0) \in \mathcal{G}$. Then $\frac{\log |\lambda_0|}{C} \leq C$, the leading asymptotics of $(q_1, q_2, q_3)$ admits the following explicit form
\[
q(x, t) = (q_1, q_2, q_3) = \frac{i}{2} \sqrt{\pi} \Gamma(i\nu)(4t)^{\nu/2} \gamma(\lambda_0) e^{2\lambda_0^2 + 3(\lambda_0 + \frac{2}{\lambda_0} - \frac{3}{2})},
\] (3.61)

where $\lambda_0 = -\frac{i}{\lambda_0}$. $C$ is a constant, $\Gamma(\cdot)$ represents a Gamma function, the vector-valued function $\gamma(\lambda)$ is expressed by (2.18). Additionally, $\nu$ and $\chi(\lambda_0)$ are determined by (3.4).

4. Conclusions

In this paper, by generalizing the nonlinear steepest descent method, we have studied the long-time behavior for the Cauchy problem of the 3-component Manakov system (1.1) with a $4 \times 4$ Lax pair. The nonlinear steepest descent technique is good for exploring the long-time asymptotics of integrable...
equations with Lax pairs and is also good for integrable systems with higher-order spectral problems. Here there are two reasons for choosing the 3-component Manakov system (1.1) as a model problem: (I) due to its physical interest. To well model important types of nonlinear physical phenomena in a proper way, there is a necessity to go beyond the standard NLS description. A crucial development consists of the investigation of coupled nonlinear models, as many physical systems comprise interacting wave components of distinct modes, polarizations or frequencies. In recent years, the multi-components coupled NLS equations have become a topic of intense research in the field of mathematical physics, since the components are usually more than one practically for many physical phenomena. (II) There have been the asymptotics for several nonlinear integrable equations with \(2 \times 2\) and \(3 \times 3\) Lax pairs, for instance, KdV equation, NLS equation, mKdV equation, Camassa-Holm equation, derivative NLS equation, Sasa-Satsuma equation, coupled NLS equations etc. However, there is just a little of literature about nonlinear integrable equations with \(4 \times 4\) Lax pairs. As a result, how to discuss the asymptotic behavior of nonlinear integrable equations with \(4 \times 4\) Lax pairs is interesting and meaningful. In [30], the function \(\delta\) can be solved explicitly by the Plemelj formula because \(\delta\) meets a scalar RHP. However, the function \(\delta\) in our work admits \(3 \times 3\) matrix RHP. The unsolvability of the \(3 \times 3\) matrix function \(\delta\) is a challenge for us. Noticing that our purpose is to investigate the asymptotic behavior for solution of the 3-component Manakov system (1.1), a natural idea is using the available function \(\det \delta\) to approximate \(\delta\) by error control.

Finally, we remark that the nonlinear steepest descent technique can be used to analyze the Long-time asymptotics of many nonlinear integrable equations with nonzero boundary conditions. Thus, it is essential to discuss whether the Long-time asymptotics of the 3-component Manakov system with nonzero boundary conditions can be obtained by using the nonlinear steepest descent technique? These will be left for future discussions.

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