Inspired by recent experimental studies of two-proton radioactivity in the light-medium mass region, we have employed relativistic mean-field plus state dependent BCS approach (RMF+BCS) to study the ground state properties of selected even-Z nuclei in the region $20 \leq Z \leq 40$. It is found that the effective potential barrier provided by the Coulomb interaction and that due to centrifugal force may cause a long delay in the decay of some of the nuclei even with small negative proton separation energy. This may cause the existence of proton rich nuclei beyond the proton drip-line. Nuclei $^{38}$Ti, $^{42}$Cr, $^{45}$Fe, $^{48}$Ni, $^{55}$Zn, $^{60}$Ge, $^{63}$, $^{64}$Se, $^{68}$Kr, $^{72}$Sr and $^{76}$Zr are found to be the potential candidates for exhibiting two-proton radioactivity in the region $20 \leq Z \leq 40$. The reliability of these predictions is further strengthened by the agreement of the calculated results for the ground state properties such as binding energy, one- and two-proton separation energy, proton and neutron radii, and deformation with the available experimental data for the entire chain of the isotopes of the nuclei in the region $20 \leq Z \leq 40$.

Keywords: Relativistic mean-field theory; Two-proton radioactivity; One- and Two-proton separation energy; Proton drip-lines.

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1. Introduction

The structure and decay modes of nuclei at and beyond the proton drip-line represent one of the most active areas in both experimental and theoretical studies of exotic nuclei with extreme isospin values. At the proton drip-line, further addition of protons is not possible as the nucleus becomes unbound. Beyond the drip-line, one or more valence protons may still remain confined due to Coulomb and centrifugal barriers enabling the nucleus to acquire rather long mean life time. Subsequently, it may decay by the process wherein one or more protons tunnel through the barrier leading to observation of one or more protons radioactivity. This situation is quite different from the neutron rich side of the valley of $\beta$-stability where Coulomb barrier is absent and consequently the drip-line gets extended to highly neutron rich nuclei.

Decay modes of nuclei through one- or two-proton radioactivity were theoretically proposed in the early 1960’s for the first time by Goldansky [1]. The one-proton radioactivity predicted for odd-proton nuclei was indeed observed already in the early 1980’s in experiments carried out at GSI, Darmstadt [2], and presently many nuclei (more than 30) which decay in their ground state by one-proton emission are well known [3]. However, the two-proton emission mode was experimentally verified only more than four decades after its theoretical prediction in the decay of $^{45}$Fe [4] and subsequently in other experiments in the decay of $^{54}$Zn [5] and $^{48}$Ni [6]. Many more experiments are being carried out to discover the new candidates.

Two-proton radioactivity occurs when the sequential emission of two independent protons from the nuclear ground state is energetically forbidden, but the emission of a pair of protons is allowed. In this situation, due to the gain of stability from the pairing energy, the mass of the even-Z two-proton emitter is smaller than the mass of the odd-Z one-proton daughter giving rise to the negative Q value for one-proton emission. Such a situation prohibits one-proton emission and favors two-proton radioactivity. Therefore, the simultaneous two-proton emission is energetically possible only beyond the two-proton drip-line and competes with two other decay modes, namely one-proton emission and $\beta$-decay. During the emission process, the two protons must tunnel through a wide Coulomb barrier. Hence, the emission probability strongly depends on the available Q$_{2p}$ value. For small Q$_{2p}$ values, $\beta$-decay is more probable, whereas for the large ones the life time of the given nucleus is very short leading to a fairly small window of opportunity to observe the two-proton radioactivity. Therefore, one has to look for nuclei in which one-proton decay is energetically impossible. Consequently we have chosen to study the even-Z nuclei only. Even-Z nuclei satisfying the condition S$_p > 0$ and S$_{2p} < 0$ may be the possible candidates for simultaneous two-proton emission.

Theoretical studies of one and two-proton radioactivity have been carried out within the framework of different models [8,10,11,12,13,14,15,16,17,18,19,20]. Also, the relativistic Hartree-bogoliubov and relativistic mean field (RMF) approaches [22,23,24,25,26] have been employed with reasonable success for the description and
prediction of one-proton radioactivity in the proton rich exotic nuclei in the vicinity of the proton drip-line. However, such a relativistic approach has not been used so far to describe and predict the two-proton radioactivity in the proton rich exotic nuclei as observed recently\textsuperscript{4,5,6,7,8,9}. This has motivated us to study the two-proton emitter nuclei within the framework of RMF+BCS approach. The main advantage of the RMF+BCS approach is that it provides the spin-orbit interaction in the entire mass region in a natural way\textsuperscript{27,28,29,30}. This indeed has proved to be very crucial for the study of unstable nuclei near the drip-line, since the single particle properties near the threshold are prone to large changes as compared to the case of deeply bound levels in the nuclear potential. In addition to this, the pairing properties are equally important for nuclei near the drip-line. As nuclei move away from stability and approach the drip-lines, the corresponding Fermi surface gets closer to zero energy at the continuum threshold. A significant number of the available single-particle states then form part of the continuum. Indeed the RMF+BCS scheme\textsuperscript{31,32,33,34} yields results which are in close agreement with the experimental data and with those of continuum relativistic Hartree-Bogoliubov (RCHB) and other similar mean-field calculations\textsuperscript{35,36}.

We have restricted our investigations to the proton rich isotopic chains of even Z nuclei in the medium mass region $20 \leq Z \leq 40$ as most of the available experimental data belong to this region\textsuperscript{4,5,6,7,8,9}.

2. Theoretical Formulation and Model

The relativistic mean-field (RMF) approach\textsuperscript{27,30,37} provides a description of the nuclear many-body system in terms of an effective Lagrangian containing mesonic and nucleonic degrees of freedom\textsuperscript{29}. The relativistic mean-field model of the nucleus is formulated on the basis of two approximations\textsuperscript{29,38}, (i) the mean-field assumption and (ii) the no-sea approximation.

2.1. Model Lagrangian Density

For our RMF calculations we have included apart from the photonic field, the fields corresponding to the $\sigma$, $\omega$, and $\rho$ mesons\textsuperscript{27,30,37} as shown in Table 1.

| Name of meson used in the Model | Angular momentum($J$) | Isospin($T$) | Parity($P$) |
|--------------------------------|------------------------|-------------|-------------|
| $\sigma$                       | 0                      | 0           | 1           |
| $\omega$                       | 1                      | 0           | -1          |
| $\rho$                         | 1                      | 1           | -1          |

The Lagrangian density is written as the sum of the Lagrangian density for free nucleons $\mathcal{L}_N^{\text{free}}$, the Lagrangian density $\mathcal{L}_M^{\text{free}}$ for the free mesons $\sigma$, $\omega$ and $\rho$, and that of photon, and the Lagrangian density for the interaction between mesons and
nucleons $L_{\text{int}}$,

$$L = L_N^{\text{free}} + L_M^{\text{free}} + L_{\text{int}}$$

The interaction term includes both the linear and nonlinear couplings. For the description of nonlinear couplings we follow the treatment of Boguta and Bodmer [28] in the case of $\sigma$-mesons, and that of Sugahara and Toki [39, 40] for the $\omega$-mesons. These nonlinear interaction terms have been shown to play important role in various applications of the RMF theory to provide a quantitative description of nuclear matter and the ground state properties of nuclei [28, 37, 38, 39, 40]. As mentioned earlier this is an effective Lagrangian density to be used together with the mean-field and no-sea approximations. Thus when written in detail the model Lagrangian used in our present study has the following form,

$$L = \bar{\Psi} [i \gamma^\mu \partial_\mu - M] \Psi - \frac{1}{4} \frac{\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{1}{3} g_2 \sigma^3 - \frac{1}{4} g_3 \sigma^4}{c^2} - g_\sigma \bar{\Psi} \sigma \Psi - \frac{1}{3} \frac{g_2^2 \sigma^2 - \frac{1}{4} g_3^2 \sigma^4}{c^2} - g_\sigma \bar{\Psi} \sigma \Psi - \frac{1}{4} \frac{g_\omega^2 \omega^2}{c^2} - g_\omega \bar{\Psi} \gamma_\mu \omega^\mu \Psi - \frac{1}{4} \frac{g_\rho^2 \rho^2}{c^2} - g_\rho \bar{\Psi} \gamma_\mu \tau_a \rho^\mu \Psi R^a$$

where we have used throughout $\hbar = c = 1$. Here the field tensors, H, G and F for the vector fields due to $\omega$, $\rho$ and photon are defined through

$$H_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$$

$$G^a_{\mu\nu} = \partial_\mu R^a_\nu - \partial_\nu R^a_\mu - g_\rho R^a_\nu R^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Furthermore, the symbols $M$, $m_\sigma$, $m_\omega$, and $m_\rho$, are the masses of nucleon, and that of the $\sigma$, $\omega$, and $\rho$ mesons, respectively. The superscript 'a' labels the isospin degree of freedom and runs from 1 to 3. Similarly, $g_\sigma$, $g_\omega$, $g_\rho$ and $e^2/4\pi = 1/137$ are the coupling constants for the mesons, and the photon, respectively, whereas $\tau^a$ are the Pauli isospin matrices.

The set of parameters appearing in the effective Lagrangian include (i) the masses of the nucleons and the mesons $M$, $m_\sigma$, $m_\omega$ and $m_\rho$, (ii) the coupling constants of the meson fields to the nucleons $g_\sigma$, $g_\omega$ and $g_\rho$, and (iii) the parameters $g_2$ and $g_3$ which describe the nonlinear coupling of the $\sigma$ mesons among themselves, and the parameter $c_3$ which describes the nonlinear self coupling of the vector meson $\omega$. These have been obtained in an extensive study [39, 40] which provides a reasonably good description for the ground state of nuclei and that of nuclear matter properties. This set, termed as TMA parameters, has an A-dependence and covers the light as well as medium heavy nuclei from $^{16}O$ to $^{208}Pb$. The TMA force
parameter set determined in Ref. 39 (displayed in Table 2) has been used in the present RMF+BCS calculations.

Table 2. TMA force parameters along with the nuclear matter properties.

| Parameters | Unit          | TMA   |
|------------|---------------|-------|
| $M$        | (MeV)         | 938.9 |
| $m_\sigma$ | (MeV)         | 519.151 |
| $m_\omega$ | (MeV)         | 781.950 |
| $m_\rho$   | (MeV)         | 768.100 |
| $g_\sigma$ |               | $10.055 + 3.050/A^{0.4}$ |
| $g_\omega$ |               | $12.842 + 3.191/A^{0.4}$ |
| $g_\rho$   |               | $3.800 + 4.644/A^{0.4}$ |
| $g_2$      | $(fm)^{-1}$   | $-0.328 - 27.879/A^{0.4}$ |
| $g_3$      |               | $38.862 - 184.191/A^{0.4}$ |
| $c_3$      |               | $151.590 - 378.004/A^{0.4}$ |

| Nuclear Matter Properties |
|---------------------------|
| Saturation Density $\rho_0$ | $(fm)^{-3}$ | 0.147 |
| Bulk binding energy/nucleon $(E/A)_\infty$ | (MeV) | 16.0 |
| Incompressibility $K$ | (MeV) | 318.0 |
| Bulk symmetry energy/nucleon $a_{sym}$ | (MeV) | 30.68 |
| Effective mass ratio $m^*/m$ | | 0.635 |

In the literature 29-37 many sets of parameterizations are available for the Lagrangian density similar to that of Eqn. (1) containing the nonlinear terms for the $\sigma$ mesons but without the inclusion of nonlinear potential for the $\omega$ mesons. The earliest sets like NL1, NL2 and NL-Z etc. have been employed extensively and further improved subsequently.

2.2. Relativistic Mean-Field Equations

The equations of motion for the RMF theory are obtained from the variational principle by varying the action integral with respect to the wave functions $\psi_i (i = 1...A)$ for the nucleons, and fields $\sigma$, $\omega$ and $R$ for the mesons and the electromagnetic field $A$. In general, for a given Lagrangian density $\mathcal{L}$ the variational principle

$$\delta \int dt \int d^3x \mathcal{L}(\vec{x}, t) = 0$$

leads for the variables $q_j$, the well known E.L. equations of motion given by

$$\frac{\partial \mathcal{L}}{\partial q_j} - \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_\mu q_j)} \right\} = 0$$

Using this we can derive the equations of motion for various fields ($q_j = \psi_i$, $\sigma$, $\omega$, $R$ and $A$) appearing in our effective Lagrangian density given by Eqn. (1). Thus for the nucleons fields $\psi_i$ one obtains the Dirac equation given by

$$(\epsilon^\mu \partial_\mu - M - g_\sigma \sigma - g_\omega \gamma^\mu \omega_\mu - g_\rho \gamma^\mu \tau^a R_\mu^a - e_1 \gamma^\mu \frac{1 - \tau_3}{2} A_\mu) \psi_i = 0$$
which may be also written in the form
\[
\{\gamma^\mu (i\partial_\mu - V_\mu) - M - S\}\psi_i = 0
\]
with the relativistic fields \( S(x) \) and \( V_\mu(x) \) defined, respectively by
\[
S = g_\sigma \sigma
\]
\[
V_\mu = g_\omega \omega_\mu + g_\rho \rho^a R^a_\mu + e \frac{1 - \tau_3}{2} A_\mu
\]
Similarly one derives the equations of motions for the meson fields. For the scalar \( \sigma \)-meson one obtains the simple Klein-Gordon equation, whereas in the case of vector mesons \( \omega \) and \( \rho \), one obtains the Proca equations. However, using the Lorentz gauge for the vector mesons \( (\partial_\nu \omega^\nu = 0 \text{ etc.}) \) the Proca equations also can be transformed as Klein-Gordon equations. Thus for the mesons and photons the field equations are given by
\[
(\Box + m_\sigma^2)\sigma = -g_\sigma \rho_s - g_2 \sigma^2 - g_3 \sigma^3
\]
\[
(\Box + m_\omega^2)\omega^\mu = g_\rho \rho_\mu - c_3 \omega_\nu \omega_\nu \omega^\mu
\]
\[
(\Box + m_\rho^2)R^a_\mu = g_\rho \rho_\mu^a
\]
\[
\Box A^\mu = e \rho_{P_r}^\mu
\]
where
\[
\Box \equiv \frac{\partial^2}{\partial t^2} - \nabla^2, \quad (\hbar = c = 1)
\]
As mentioned earlier, the various densities appearing as source terms in the equations for the meson-fields are obtained in the mean-field and no-sea approximations whereby the nucleon field operator \( \hat{\psi} \) is expanded in terms of single particle wave functions \( \psi_i \). Thus the scalar density \( \rho_s \), the nucleon current density \( \rho_\mu \), the isovector current density \( \rho_{a,\mu} \) and the electromagnetic current density \( \rho_{P_r,\mu} \) are simply given by
\[
\rho_s = \sum_{i=1}^{A} w_i \bar{\psi}_i \psi_i
\]
\[
\rho_\mu = \sum_{i=1}^{A} w_i \bar{\psi}_i \gamma_\mu \psi_i
\]
\[
\rho_{a,\mu} = \sum_{i=1}^{A} w_i \bar{\psi}_i \gamma_\mu \tau_a \psi_i
\]
\[
\rho_{P_r,\mu} = \sum_{i=1}^{A} w_i \bar{\psi}_i \frac{1 - \tau_3}{2} \gamma_\mu \psi_i
\]
Here the subscript ‘Pr’ has been used to denote protons. Also we have introduced the occupation weights $w_i$ ($0 \leq w_i \leq 1$) which would facilitate the treatment of open shell nuclei. In the context of pairing correlations, these weights are identical to the BCS factors $v_i^2$ (occupation probabilities). The RMF equations given by (4), and (7) - (10) are a set of coupled equations for the nucleon fields, meson fields and the Coulomb field which are solved by iteration after applying suitable approximations.

### 2.3. Total Energy of the System

The relation for the energy of the system within the present meson field theory is derived from the Hamiltonian density $\mathcal{H}$ using the relation

$$ E_{MF} = \int d^3r \mathcal{H} $$

where the Hamiltonian density $\mathcal{H}$ is defined in terms of $\mathcal{L}$ as

$$ \mathcal{H} = \sum_{q_j} \pi_{q_j} \dot{q}_j - \mathcal{L} $$

Here the canonical momentum $\pi_{q_j}$ for the various fields ($q_j = \psi_i, \sigma, \omega^\mu, R^a_\mu$ and $A^\mu$) is given by

$$ \pi_{q_j} = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} $$

The mean-field energy $E_{MF}$ obtained above is the major portion of the energy of a nuclear system. The total energy of the system is given by

$$ E_{total} = E_{MF} + E_{Pair} - E_{CM} - Z M_p - N M_n $$

wherein $E_{Pair}$ and $E_{CM}$ are the pairing energy and the center-of-mass energy respectively.

In order to calculate the pairing energy we employ the state dependent gap equation for the single particle states wherein for the pairing interaction we have used a delta-function force. The evaluation of the pairing energy

$$ E_{Pair} = - \sum_{j>0} \Delta_j u_j v_j $$

involves finally the calculation of the pairing interaction matrix elements and the single particle pairing gaps $\Delta_j$ along with the occupation probabilities $v_j^2$ etc. For the correction to the center of mass motion we use the nonrelativistic expression

$$ E_{CM} = \frac{\langle P_{CM}^2 \rangle}{2 M A} $$

where $P_{CM}$ is the classical center of mass and $A$ is the total number of nucleons in the nucleus. A simple harmonic oscillator shell model estimate given by $E_{CM} = \frac{4}{41} \text{MeV} A^{-1/3}$ provides a reasonably good approximate description and has been used in our present study.
2.4. Specialization to Spherically Symmetric Nuclei

In the case of spherical nuclei, i.e. the systems which have rotational symmetry, the potential of the nucleon and the sources of the meson fields depend only on the radial coordinate $r$. The field equations obtained above are further simplified due to spherical symmetry. In this case, the spinors $\psi_i$ describing nucleons are characterized as usual in terms of the single particle angular momentum quantum numbers $j_i$ and $m_i$ and expressed in terms of radial functions $G_i(r)$ and $F_i(r)$ for the upper and lower components, respectively, and the spinor spherical harmonics $\mathcal{Y}_{j_i l_i m_i}$,

$$\psi_i = \frac{1}{r} \left( i G_i(r) \mathcal{Y}_{j_i l_i m_i} \right)$$  \hspace{1cm} (21)

where the spinor spherical harmonic for a given nucleon $i$ with quantum numbers $j, l$ and $m$ is defined by

$$\mathcal{Y}_{j l m} = \sum_{m_s} \langle m_s l m | l m \rangle Y_{lm}(\theta, \phi) \chi_{m_s}(s)$$  \hspace{1cm} (22)

The spinors $\psi_i$ satisfy the normalization condition

$$\int \psi_i^{\dagger} \psi_i d^3x = 1$$

which yields

$$\int dr \left\{ |G_i|^2 + |F_i|^2 \right\} = 1$$  \hspace{1cm} (23)

For the calculation of pairing energy we employ the state dependent gap equation \cite{11,12} for the single particle states

$$\Delta_{ji} = -\frac{i}{\sqrt{2j_1 + 1}} \sum_{j_2} \langle j_1^2 0^+ | V | j_2^2 0^+ \rangle \sqrt{\frac{2j_2 + 1}{\varepsilon_{j_2} - \lambda^2 + \Delta_{j_2}^2}} \Delta_{j_2}$$  \hspace{1cm} (24)

where $\varepsilon_j$ are the single particle energies, and $\lambda$ is the Fermi energy. The particle number condition is expressed in terms of the occupation probabilities for the single particle states through

$$\sum_j (2j + 1) v_j^2 = N,$$

where $N$ is the number of particles in the system, and $v_j^2$ are the occupation probabilities given by

$$v_j^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_j - \lambda}{\sqrt{\varepsilon_j - \lambda^2 + \Delta_j^2}} \right)$$  \hspace{1cm} (25)
In our calculations we use for the pairing interaction the δ-force, that is \( V = -V_0 \delta(r) \) where \( V_0 \) denotes the strength. The final result for the paring matrix element is given by

\[
\langle (j_1^2)^0^+ | V | (j_2^2)^0^+ \rangle = \frac{V_0}{8\pi} (-1)^{l_1+l_2} \sqrt{(2j_1+1)(2j_2+1)} I_R
\]

where \( I_R \) is the radial integral having the form

\[
I_R = \int dr \frac{1}{r^2} \left( G_{i_1}^* G_{j_2} + F_{i_1}^* F_{j_2} \right)^2
\]

For the purpose of our RMF+BCS calculations the value of the pairing interaction strength \( V_0 = 350 \text{ MeV fm}^3 \) was determined by obtaining a best fit to the binding energy of Ni isotopes. We use the same value of \( V_0 = 350 \text{ MeV fm}^3 \) throughout for our present studies of all the other nuclei as well. Moreover, the same strength has been used for both the protons and neutrons.

### 2.5. Specialization to Axially Deformed Nuclei

The relativistic mean field description has been extended for the deformed nuclei of axially symmetric shapes by Gambhir, Ring and their collaborators \(^43\) using an expansion method. The treatment of pairing has been carried out in Ref. \(^44\) using state dependent BCS method \(^42\) as has been given by Yadav et al. \(^31\), \(^32\) for the spherical case. For axially deformed nuclei the rotational symmetry is no more valid and the total angular momentum \( j \) is no longer a good quantum number. Nevertheless, the various densities still are invariant with respect to a rotation around the symmetry axis. Here we have taken the symmetry axis to be the \( z \)-axis. Following Gambhir et al. \(^43\), it is then convenient to employ the cylindrical coordinates

\[
x = r_\perp \cos \varphi, \quad y = r_\perp \sin \varphi \quad \text{and} \quad z.
\]

The spinor \( \psi_i \) with the index \( i \) is now labeled by the quantum numbers \( \Omega_i, \pi_i \) and \( t_i \), where \( \Omega_i \) is the eigenvalue of the symmetry operator \( j_z \) (the projection of \( j_i \) on the \( z \)-axis), \( \pi_i \) indicates the parity and \( t_i \) has been used for the isospin. In terms of these quantum numbers, the spinor can now be expressed in the form:

\[
\psi_i(r, t) = \begin{pmatrix} f_i(r) \\ ig_i(r) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} f^+_i(z, r_\perp) e^{i(\Omega_i-1/2)\varphi} \\ f^-_i(z, r_\perp) e^{i(\Omega_i+1/2)\varphi} \\ ig^+_i(z, r_\perp) e^{i(\Omega_i-1/2)\varphi} \\ ig^-_i(z, r_\perp) e^{i(\Omega_i+1/2)\varphi} \end{pmatrix} \chi_i(t) \]

Here the four components \( f^\pm_i(z, r_\perp) \) and \( g^\pm_i(z, r_\perp) \) obey the Dirac equations. For the axially symmetric case the spinors \( f^\pm_i(r_\perp, z) \) and \( g^\pm_i(r_\perp, z) \) are expanded in terms of the eigenfunctions of a deformed axially symmetric oscillator potential as
has been described in the Refs. [43] and [44]. The pairing gap $\Delta_k$ appearing in the Eqn. (19) satisfies the gap equation

$$\Delta_k = \frac{1}{2} \sum_{k' > 0} \frac{\bar{V}_{kk'} |\Delta_{k'}|}{\sqrt{(\varepsilon_{k'} - \lambda)^2 + \Delta_{k'}^2}}$$  \hspace{1cm} (30)

Here the symbols $\varepsilon_{k'}$ and $\lambda$ denote the single particle and Fermi energy, whereas the pairing matrix element $\bar{V}_{kk'}$ for the symmetrically deformed case using the zero-range $\delta$-force is given by

$$\bar{V}_{ij} = <\hat{\bar{i}} | V | \bar{j}> - <\hat{\bar{i}} | V | j>$$

$$= -V_0 \int d^3r [\psi_i^\dagger \psi_i^\dagger \psi_j \psi_j - \psi_i^\dagger \psi_i^\dagger \psi_j \psi_j]$$  \hspace{1cm} (31)

$$=-V_0 \int d^3r \psi_i^\dagger \psi_i^\dagger \psi_j \psi_j$$  \hspace{1cm} (32)

A detailed description of the relativistic mean-field plus state dependent BCS approach can be found in Refs. [31, 32, 33, 43, 44, 45, 46].

3. Results and Discussion

In sec. 3.1 we have described the results of the calculated binding energies, one- and two-proton separation energies obtained by employing RMF+BCS approach including deformation degree of freedom assuming axially symmetric shapes of nuclei (referred to throughout as deformed RMF approach) to identify the nuclei which satisfy the criteria ($S_p > 0$ and $S_{2p} < 0$) in the region $20 \leq Z \leq 40$.

Nuclei satisfying the above criteria have been predicted as potential candidates for exhibiting two-proton radioactivity. It is found that barring a few cases most of the potential candidates for the two proton radioactivity are well deformed proton rich nuclei as has been discussed in sec. 3.2. A detailed examination of the results for the calculated value of the deformation obtained for the potential candidates in sec. 3.2 shows that, for example, the candidate $^{48}$Ni has spherical shape, whereas the potential nuclei $^{60}$Ge and $^{42}$Cr are slightly deformed. These spherical or near spherical potential candidates are especially useful in our theoretical studies since these can be described in detail within the spherical RMF framework in terms of spherical single particle wave functions and energies. This in turn enables us to discuss the nuclear structure aspects of these nuclei in greater details in terms of relativistic mean field (RMF) potential, wave function, proton single particle spectrum and density distributions etc. as has been elucidated in sec. 3.2.

In order to obtain the one- and two-proton separation energies using Eqns. (33) and (34) given below, systematic calculations are carried out within the framework of deformed RMF approach for the even-Z nuclei in the region $20 \leq Z \leq 40$. Thus for a nucleus with proton and neutron number $(Z, N)$, the calculations for $S_p$ and $S_{2p}$ require the deformed RMF results for the chain of isotopes with proton number Z, Z-1 and Z-2, as is evident from the form of expressions (33) and (34). In
order to proceed towards the proton drip-line and obtain the \( S_p \) and \( S_{2p} \) values for the proton rich nucleus, similar calculations are repeated in a systematic way for decreasing value of \( N \) to obtain the results for the nuclei of the isotopic chain \((Z, N-1), (Z, N-2), (Z, N-3)\) etc. The calculated deformed RMF results for the binding energies then yield the desired \( S_p \) and \( S_{2p} \) values up to the proton drip-line and beyond. These results are then plotted as a function of decreasing mass number \( A \) for different values of \( Z \) and analysed to predict the potential candidates for two-proton radioactivity as has been described in the subsection below. Calculation for neutron rich nuclei \((N > Z)\) have not been shown and discussed as they are not relevant to our study of the phenomenon of two-proton radioactivity.

3.1. Analysis of separation energy for the identification of two-proton emitters

The results of our extensive calculations obtained for the nuclei constituting the \( Z = 20 \) to \( Z = 40 \) isotopic chains by employing the deformed RMF approach have been shown in Fig. 1 as a function of increasing mass number \( A \) for all the isotopic chains of nuclei with even \( Z \) values in the region \( 20 \leq Z \leq 40 \) along with the available experimental data.\(^{47}\)

From the figure it is observed that the calculated binding energy values are very close to the available experimental data.\(^{47}\) However, due to large scales involved in this figure, small differences between the calculated and experimental values of the binding energy \((0.1 \text{ MeV} \text{ to } 3 \text{ MeV})\) are not distinctly visible in Fig. 1. The maximum difference of 3 MeV occurs in the case of few proton rich nuclei located in the vicinity of \( Z = 26 \) (Fe) isotopic chain. This difference is about 1 percent of the total binding energy of the respective nucleus in that region.

Due to close agreement between the theoretical and experimental results for the binding energy, we can accurately determine one- and two-proton separation energy values which are very crucial for making reliable prediction of the two-proton emitters. Thus, the deformed RMF approach is expected to describe the phenomena of two-proton radioactivity in the region \( 20 \leq Z \leq 40 \) in a more realistic manner as compared to other theoretical approaches wherein the accuracies are rather not really large.

As mentioned earlier, since the nuclei satisfying the condition \( S_p > 0 \) and \( S_{2p} < 0 \) might be the possible candidates for simultaneous two-proton emission, we obtain reliable and accurate calculations for the \( S_p \) and \( S_{2p} \) values. In fact these are obtained from the calculated binding energies using the expressions,

\[
S_{2p}(Z, N) = B(Z, N) - B(Z - 2, N) \quad (33)
\]

\[
S_p(Z, N) = B(Z, N) - B(Z - 1, N) \quad (34)
\]

where \( B(Z,N) \) is the binding energy of the nucleus with \( Z \) protons and \( N \) neutrons.

The calculated results of the one- and two-proton separation energy for the nuclei
Fig. 1. The results for the binding energy obtained within the framework of deformed RMF approach using the TMA force parameters for the nuclei belonging to the isotopic chains of Z = 20 to Z = 40 have been shown as open circles. These calculated results are compared with the available experimental data shown as filled circles. 

belonging to the isotopic chains of nuclei with Z = 20 to 40 have been displayed in Figs. 2 and 3, respectively, as a function of decreasing mass number A. In these figures, the filled circles denote the experimental value of one- and two-proton separation energies, whereas the open circles represent the calculated results.

A comparison of the results obtained in the deformed RMF approach with the available experimental data for the one- and two-proton separation energies shows that in general theoretical results are in fairly good agreement with the measurements, though for some cases the calculated values are rather appreciably different from the experimental data. In a few cases this marked difference between the calculated results and the data can be attributed to the large uncertainty in the measurements shown by the error bars in Figs. 2 and 3.

It is seen from Fig. 2 for the two-proton separation energy that nuclei $^{45}$Fe, $^{48}$Ni and $^{54}$Zn which have been shown to be two-proton emitters in recent experiments are located beyond the two-proton drip-line with negative two-proton separation energy values -0.90 MeV, -1.49 MeV and -1.97 MeV, respectively. Out of these, nuclei $^{45}$Fe and $^{48}$Ni have positive one-proton separation energy ($S_p > 0$) and therefore fulfill the criteria of being a two-proton emitter ($S_p > 0$ and $S_{2p} < 0$). In contrast to the above mentioned two nuclei, $^{45}$Fe and $^{48}$Ni, for the nucleus $^{54}$Zn both one- and two-proton separation energies are found to be negative. Interestingly, the nucleus $^{55}$Zn which belongs to the Z = 30 isotopic chain is found to be a two-proton emitter according to the results of our deformed RMF calculations.

Moreover, the result of our extensive calculations further show that the nuclei
Fig. 2. Calculated results for the two-proton separation energy $S_{2p}$ obtained in the deformed RMF approach using the TMA force parameters are shown by open circles. The lines connecting different isotopes with proton number lying between $Z = 20$ to $Z = 40$ have been drawn to guide the eyes. The filled circles denote the corresponding available experimental data with error bars for the two-proton separation energy $S_{2p}$.

$^{38}$Ti, $^{42}$Cr, $^{60}$Ge, $^{63,64}$Se, $^{68}$Kr, $^{72}$Sr and $^{76}$Zr satisfy the criteria $S_p > 0$ and $S_{2p} < 0$. These nuclei are, therefore, expected to be the potential candidates for exhibiting the two-proton radioactivity. It may be emphasized that the experimental data for the separation energies for the nuclei identified above (with the exception of $^{55}$Zn and $^{60}$Ge) are consistent with the criteria $S_p > 0$ and $S_{2p} < 0$. The difference between the calculated and measured results for nuclei $^{55}$Zn and $^{60}$Ge can be attributed to the appreciable errors in the experimental binding energies. Results of the one- and two-proton separation energies for the nuclei identified above along with the available experimental data have been listed in Table 3 to facilitate an easy comparison.

It may be mentioned that the separation energy values near the drip-line are close to zero and therefore the sign of the separation energy value is very sensitive and at times model dependent.

In order to check the force parameter dependence, we have carried out the RMF+BCS calculations for some nuclei using the NL-SH force parameters which are equally popular for the relativistic mean-field calculations. It is found that generally our RMF+BCS results for the two force parameters are similar though there are always some differences in finer details. However, it is worthwhile to note that our predictions of potential two-proton emitters are not affected if we use NL-SH force parameters instead of TMA force parameters, as evident from
Fig. 3. Calculated results for the one-proton separation energy $S_p$ obtained in our deformed RMF approach using the TMA force parameters have been depicted by open circles. The lines connecting the isotopes for the nuclei with proton number ranging from $Z = 20$ to $Z = 40$ have been drawn to guide the eyes. Available experimental data have been depicted by the filled circles for the purpose of comparison.

Table 4.
Table 4. Comparison between one-proton separation energy ($S_p$) and two-proton separation energy ($S_{2p}$) obtained by using TMA and NL-SH force parameters in respect of some selected nuclei.

| Nucleus | RMF+BCS(TMA) | RMF+BCS(NL-SH) |
|---------|--------------|----------------|
|         | $S_p$ (MeV) | $S_{2p}$ (MeV) | Whether 2p emission condition fulfilled | $S_p$ (MeV) | $S_{2p}$ (MeV) | Whether 2p emission condition fulfilled |
| $^{45}$Fe | 0.43 | -0.90 | Yes | 0.88 | -0.51 | Yes |
| $^{48}$Ni | 0.08 | -1.49 | Yes | 0.54 | -1.19 | Yes |
| $^{54}$Zn | -0.34 | -1.97 | No | -1.79 | -4.31 | No |

3.2. Representative Examples of Two-Proton Emitters Located Beyond the Two-Proton Drip-Line

The results of the calculated quadrupole deformation values of the predicted two-proton emitters and their respective daughter nuclei are summarized below Table 5, especially to highlight and identify the nuclei which are spherical in shape as these may be studied in greater detail within the framework of spherical RMF.

Table 5. Results of the quadrupole deformation parameter for the matter density distribution $\beta_{2m}$ for the predicted two-proton emitters and their corresponding daughter nuclei.

| Parent Nucleus | $\beta_{2m}$ | Daughter Nucleus | $\beta_{2m}$ |
|----------------|-------------|-----------------|-------------|
| $^{38}$Ti      | 0.22        | $^{36}$Ca       | 0.00        |
| $^{42}$Cr      | -0.17       | $^{40}$Ti       | -0.15       |
| $^{45}$Fe      | 0.00        | $^{43}$Cr       | -0.07       |
| $^{48}$Ni      | 0.00        | $^{46}$Fe       | 0.00        |
| $^{55}$Zn      | 0.25        | $^{53}$Ni       | 0.14        |
| $^{60}$Ge      | 0.13        | $^{58}$Zn       | 0.10        |
| $^{63}$Se      | -0.23       | $^{61}$Ge       | 0.19        |
| $^{64}$Se      | 0.24        | $^{62}$Ge       | 0.23        |
| $^{68}$Kr      | -0.29       | $^{66}$Se       | -0.25       |
| $^{72}$Sr      | -0.27       | $^{70}$Kr       | -0.31       |
| $^{76}$Zr      | -0.33       | $^{74}$Sr       | -0.35       |

From the table it is seen that nuclei $^{45}$Fe, $^{48}$Ni and $^{60}$Ge as well as their corresponding daughter nuclei have spherical or near spherical shapes. In contrast, the nuclei $^{55}$Zn and $^{64}$Se, and also their respective daughter nuclei are seen to have prolate deformation, whereas the nuclei $^{42}$Cr, $^{68}$Kr, $^{72}$Sr and $^{76}$Zr, and their respective daughter nuclei have oblate shapes. From these results, it is evidently seen that besides the nuclei $^{38}$Ti and $^{63}$Se, all other identified two-proton emitters and their corresponding daughter nuclei have similar shapes. Thus, it appears that the shape of the two-proton emitter nucleus is almost preserved in the process of two-proton emissions.

From amongst the predicted two-proton radioactive nuclei mentioned above, we have chosen the nuclei $^{42}$Cr, $^{48}$Ni and $^{60}$Ge which have spherical or near spherical shapes as can be seen from Table 4, to describe their detailed properties within the
framework of the spherical RMF approach in order to obtain greater insight into
the structural properties of these two-proton emitters.

It should be emphasized that though the nuclei $^{42}$Cr and $^{60}$Ge are slightly
deformed, these have been treated here within the spherical RMF approach only as
an approximation. On the other hand not only $^{48}$Ni is spherical but most of the Ni
isotopes are found to be spherical in shape indicating $Z = 28$ remains a good magic
number for all the isotopes with neutron number ranging from $N = 20$ to $N = 70$. It
should also be stated that the nuclei $^{42}$Cr, $^{48}$Ni and $^{60}$Ge which have been identified
as two-proton emitters are extremely proton rich nuclei located beyond the two-
proton drip-line. Since the Ni isotopes are found to be spherical, these are ideally
suited to be studied within the spherical RMF approach without any approximation.
Thus these isotopes have been studied in rather great detail in order to show that the
physical characteristics exhibited by bound proton rich isotopes remain intact even
beyond the two-proton drip-line. It is found that the nucleus $^{48}$Ni which lies beyond
the two-proton drip-line and thus is unbound, due to the centrifugal and Coulomb
barrier develops finite life time and eventually decays via two-proton radioactivity.
In the following we have also presented some detailed results for the proton rich
$^{48-56}$Ni isotopes to demonstrate a systematic variation in the mean-field potential,
wave function, energy of the single particle states and the proton density with
increasing neutron number as we move towards the line of stability. The most
interesting result of these study of Ni isotopes is that the highest proton single
particle state $1f_{7/2}$, for example, which remains completely filled and bound for all
the Ni isotopes up to the proton drip-line, preserves its characteristics even beyond
the drip-line when the nucleus $^{48}$Ni becomes unbound as it exhibits negative two-
proton separation energy. This explains the long life time of this isotope even when it lies beyond the two-proton drip-line and eventually decays. This is manifested
in the form of two proton radioactivity. This conclusion is reinforced by observing
a similar characteristics of the radial density distribution for the $^{48}$Ni isotope as
compared to those of the bound isotopes $^{50-56}$Ni. The shell closure or magicity of
the $^{48}$Ni isotope also remains preserved as for the bound isotopes $^{50-56}$Ni. These are
significant characteristics explaining why the nucleus $^{48}$Ni even while lying beyond
the two-proton drip-line has positive one-proton separation energy and decays via
two-proton radioactive mode.

3.2.1. The nucleus $^{42}_{24}$Cr$_{18}$
The predicted two-proton emitter nucleus $^{42}_{24}$Cr$_{18}$ is found to be somewhat deformed
with $\beta_{2\pi} = -0.17$. However, as mentioned above, in order to learn its detailed
structure in terms of spherical single particle energies and wave functions, we have
studied it within the framework of the spherical RMF approach. This approximate
treatment, though not fully justified, is expected to shed light as regard to its finite
life time even though it is located beyond the two-proton drip-line.

As mentioned earlier, this nucleus acquires long life time against two-proton
Fig. 4. The RMF potential energy (sum of the scalar and vector potentials) for the nucleus $^{42}\text{Cr}_{18}$ as a function of radius is shown by the solid line. The long dashed line represents the sum of RMF potential energy and the centrifugal barrier energy for the proton resonant state $^{1}f_{7/2}$. Decay due to the combined barrier provided by the Coulomb and centrifugal effects, even though it has negative two proton separation energy. In order to demonstrate the physical situation of proton rich $^{42}\text{Cr}_{18}$ nucleus, we have plotted in Fig. 4 the RMF potential and the single particle energy spectrum for the bound proton single particle states. The figure also depicts a few positive energy proton states in the continuum including the resonant states $^{1}f_{7/2}$ at energy 0.77 MeV. We have also shown in Fig. 4 the total mean-field potential for the resonant $^{1}f_{7/2}$ state, obtained by adding the centrifugal potential energy.

It may be emphasized that besides the resonant state $^{1}f_{7/2}$, other positive energy proton single particle states do not play any significant role in contributing to the total pairing energy as only this state has substantial overlap with the bound states near the Fermi level. This can be inferred from Fig. 5 wherein we have displayed the radial wave functions of some of the proton single particle states close to the Fermi surface, the proton Fermi energy being $\lambda_{p} = 0.80$ MeV. These include the bound states $^{2}s_{1/2}$ and $^{1}d_{3/2}$, and the continuum states $^{2}p_{1/2}$ and $^{3}s_{1/2}$, in addition to the resonant state $^{1}f_{7/2}$.

The wave function for the $^{1}f_{7/2}$ state plotted in Fig. 5 is clearly seen to be confined within a radial range of about 7 fm and has a decaying component outside this region, characterizing a resonant state. In contrast, the main part of the wave function for the non-resonant states, e.g. $^{2}p_{1/2}$ and $^{3}s_{1/2}$, is seen to be spread over outside the potential region, though a small part is also contained inside the potential range. This type of states thus has a poorer overlap with the bound states near the Fermi surface leading to small value for the pairing gap $\Delta_{J}$. Further, the positive energy states lying much higher from the Fermi level, for example, $^{1}g_{7/2}$,
Fig. 5. Radial wave functions of a few representative proton single particle states with energy close to the Fermi surface for the nucleus $^{42}_{24}$Cr$_{18}$. The resonant state (1f$^7/2$) similar to the bound states (2s$_{1/2}$ and 1d$^3/2$) is mostly confined within the potential region.

1g$^9/2$ etc. have a negligible contribution to the total pairing energy of the system.

Fig. 6. Pairing gap energy $\Delta_j$ of proton single particle states with energy close to the Fermi surface for the nucleus $^{42}_{24}$Cr$_{18}$. The proton resonant single particle state 1f$^7/2$ at energy 0.77 MeV has the gap energy of about 1 MeV which is close to that of bound states 1d$^5/2$, 1d$^3/2$ and 2s$_{1/2}$. In this respect the resonant state 1f$^7/2$ behaves similar to that of a bound state such as 1d$^5/2$, 1d$^3/2$, 2s$_{1/2}$ etc.
This is also clearly evident from Fig. 6 which shows the calculated single particle pairing gap energy \( \Delta_1 \) for some of the proton single particle states in the nucleus \(^{42}_{24}\text{Cr}_{18}\). However, we have not shown in the figure the single particle states having negligibly small \( \Delta_1 \) values as these do not contribute significantly to the total pairing energy. One observes indeed in Fig. 6 that the proton resonant single particle state \( 1f_{7/2} \) at energy 0.77 MeV has the pairing gap energy of about 1 MeV which is close to that of the bound states \( 1d_{5/2} \), \( 1d_{3/2} \) and \( 2s_{1/2} \). Also, Fig. 6 shows that the pairing gap value for the non-resonant states \( 2p_{1/2} \) and \( 3s_{1/2} \) lying in the continuum is negligibly small.

From the characteristics of the proton resonant \( 1f_{7/2} \) single particle state, it is evidently clear that this state behaves similar to a bound state. This property also enables the nucleus \(^{42}_{24}\text{Cr}_{18}\) to have finite life time even with negative two-proton separation energy while lying beyond the two-proton drip-line as is elucidated below. In the nucleus \(^{42}_{24}\text{Cr}_{18}\), the last 4 protons outside the closed s-d shells occupy the \( 1f_{7/2} \) single particle state. If we consider the neighboring isotones of \(^{42}_{24}\text{Cr}_{18}\), that is \(^{38}_{20}\text{Ca}_{18}\) and \(^{40}_{22}\text{Ti}_{18}\) with increasing number of protons in the \( 1f_{7/2} \) proton single particle state, calculations show that while the nucleus \(^{38}_{20}\text{Ca}_{18}\) for which the \( 1f_{7/2} \) state is completely empty, is bound and has spherical shape, the nucleus \(^{40}_{22}\text{Ti}_{18}\) is also bound though slightly deformed. On further addition of two more protons to \(^{40}_{22}\text{Ti}_{18}\), the next isotope \(^{42}_{24}\text{Cr}_{18}\) is formed in which now the \( 1f_{7/2} \) resonant state is occupied by four protons. Due to the resonant nature of the proton single particle \( 1f_{7/2} \) state the pairing energy is increased and the nucleus tends to remain bound, whereas the Coulomb interactions amongst the increasing number of protons acts in a disruptive way to make the nucleus unbound with negative two-proton separation energy. However, the centrifugal barrier provided by high angular momentum resonant \( 1f_{7/2} \) state together with the Coulomb barrier enables the nucleus \(^{42}_{24}\text{Cr}_{18}\) to gain finite life time, and eventually it decays by two proton emission. It should be emphasized that the contribution of pairing energy plays an important role for the stability of nuclei and consequently in deciding the position of the neutron and proton drip-lines. Also it is remarkable that in contrast to the proton rich nucleus such as \(^{42}_{24}\text{Cr}_{18}\), if we consider the neutron rich nucleus like \(^{60}_{20}\text{Ca}_{40}\), it is found that further addition of neutrons while approaching the extremely neutron rich nucleus \(^{76}_{20}\text{Ca}_{50}\), the single particle neutron states \( 3s_{1/2}, 1g_{9/2}, 2d_{5/2}, 2d_{3/2} \) which lie near the Fermi level gradually come down close to zero energy, and subsequently the \( 1g_{9/2} \) and \( 3s_{1/2} \) neutron single particle states even become bound states. This helps in accommodating more and more neutrons which are just bound. In the case of neutrons we do not have the disruptive Coulomb force anymore. Similar explanation holds good for other two-proton emitters like \(^{38}_{22}\text{Ti}_{16}\) and \(^{60}_{32}\text{Ge}_{28}\).

### 3.2.2. The nucleus \(^{48}_{28}\text{Ni}_{20}\)

In order to illustrate the case of proton rich \(^{48}\text{Ni}\) nucleus, we have plotted in Fig. 7, the RMF potential and the wave functions of its few representative proton single
Fig. 7. Upper panel: Radial wave functions of a few representative proton single particle states for the nucleus $^{48}$Ni. It is seen that the bound single particle states such as $1f_{7/2}$, $2s_{1/2}$ and $1d_{3/2}$ are mostly confined inside the potential region. In contrast the $2p_{3/2}$ state which is not a bound state has a large spread outside the potential region as well. Lower panel: The potential energy (sum of the scalar and vector potentials), for the nucleus $^{48}$Ni as a function of radius is shown by the solid line. The dashed line represents the sum of potential energy and the centrifugal barrier energy for the proton single particle state $1f_{7/2}$. The effective potential indeed prohibits a rapid decay of $^{48}$Ni even though this nucleus is unbound and lies beyond the two-proton drip-line.

Particle states in the lower and upper panels respectively. The lower panel also shows the spectrum of bound proton single particle states along with the positive energy state $2p_{3/2}$ in the continuum. We have also depicted in lower panel of Fig. 7 the total mean-field potential for the highest bound and fully occupied proton single particle state $1f_{7/2}$, obtained by adding the centrifugal potential energy. The combined effect of Coulomb barrier and centrifugal barrier prevents the protons from quickly leaving the proton rich nucleus $^{48}$Ni located beyond the two-proton drip-line. The delay associated with the tunneling process allows for the observation of two-proton radioactivity. In the upper panel of Fig. 7 we have displayed the radial wave functions of some proton single particle states, the proton Fermi energy being $\lambda_p = 2.86$ MeV. These include the bound $1d_{3/2}$, $2s_{1/2}$, $1f_{7/2}$ and the continuum $2p_{3/2}$ proton single particle state.

It is seen that the radial wave function of the proton single particle state $1f_{7/2}$ remains mainly confined to the region of the potential well. In contrast, the wave
function for a typical continuum state $2p_{3/2}$ is spread over to large distances outside of the potential region. Therefore, the important outcome from the above discussions is that the highest proton single particle state $1f_{7/2}$ remains completely occupied and bound even for the nucleus $^{48}$Ni lying beyond the two-proton drip-line with negative two-proton separation energy.

Since $^{48-56}$Ni isotopes have the spherical shape, it would be interesting to employ spherical RMF approach to investigate the behaviour of single particle spectrum, potential, wave function and density distribution of Ni isotopes with increasing $Z/N$ ratio. With this in view, first we plot in Fig. 8 the RMF potential (lower panel), and radial wave function of the proton single particle state $1f_{7/2}$ for the $^{48-56}$Ni isotopes as a function of radius in two different scales (middle and upper panel). We note from the lower panel of Fig. 8 that although the two-proton separation energy of nucleus $^{48}$Ni is negative, its potential behaves similar to the $^{50-56}$Ni isotopes which have positive two-proton separation energy.

![Fig. 8. Lower Panel: The RMF potential energy (sum of the scalar and vector potentials), for the isotopes $^{48-56}$Ni as a function of radial distance is shown by the solid line. Middle Panel: Radial wave functions (in a linear scale) of proton single particle state $1f_{7/2}$ for the isotopes $^{48-56}$Ni. Upper Panel: Radial wave functions of proton single particle state $1f_{7/2}$ for the isotopes $^{48-56}$Ni shown in a logarithmic scale in order to demonstrate the difference in the wave function for various isotopes at large distances. It is remarkable that within the potential region this difference is indeed not large.](image-url)
Moreover, it is clearly seen that similar to the case of $^{50-56}$Ni isotopes, the proton single particle state $1f_{7/2}$ in the nucleus $^{48}$Ni remains confined to its potential region. In order to demonstrate the difference in the wave function for various isotopes at large distances, we have plotted in the upper panel of Fig. 8 the radial wave function of proton single particle state $1f_{7/2}$ for the $^{48-56}$Ni isotopes in logarithmic scale. It is seen that the wave function of the proton single particle state $1f_{7/2}$ starts spreading slightly outside the potential region with increasing $Z/N$ ratio. Nevertheless, a large part of the wave functions is contained inside the potential range. From these results it is clear that the nucleus $^{48}$Ni lying beyond the two-proton drip-line behaves similar to the $^{50-56}$Ni isotopes for which the two-proton separation energy is positive.

![Graph](image)

Fig. 9. Variation of the proton single particle energies obtained in the spherical RMF+BCS calculations with the TMA force for the $^{48-56}$Ni isotopes with decreasing neutron number $N$ (or in other words with increasing $Z/N$ ratio). Position of the proton Fermi level has been shown by solid circles. The energy levels have been connected by dashed line only to guide the eyes. It is interesting to observe that the nucleus $^{48}$Ni being unbound preserves the large gap between the $1f_{5/2}$ and $1f_{7/2}$ proton single particle states and the magicity appears to remain intact. This characteristics is unusual for an unbound nucleus.

In Fig. 9 we have shown the variation of the proton single particle levels and position of the Fermi level of $^{48-56}$Ni isotopes with decreasing neutron number $N$ (or in other words with increasing $Z/N$ ratio). It is readily seen from the figure that the existence of large energy gap between proton single particle levels $1f_{7/2}$ and $1f_{5/2}$ explains the tradition proton shell closure at $Z = 28$ in $^{48-56}$Ni isotopes. The
proton Fermi energy which lies at $\epsilon_f = -4.91$ MeV in the $^{56}$Ni nucleus moves to $\epsilon_f = 2.86$ MeV in the nucleus $^{48}$Ni having maximum $Z/N$ value.

It is evident from the Fig. 9 that despite the fact that the nucleus $^{48}$Ni is located beyond the two-proton drip-line, the energy gap between proton single particle levels $1f_{7/2}$ and $1f_{5/2}$ remains significant enough to maintain the $Z = 28$ shell closure. This is due to the fact that the highest proton single particle state $1f_{7/2}$ preserves its characteristics beyond the two-proton drip-line. A similar conclusion can be drawn from the variation of the proton density distribution as a function of radial distances. Result of such a variation have been depicted in Fig. 10 for Ni isotopes with $N = 20$ to 28. As the proton shell in $^{48-56}$Ni isotopes remains closed we observe sharply falling asymptotic density distribution for these isotopes. It is remarkable that the proton density distribution of the unbound $^{48}$Ni isotope has the radial dependence similar to the other isotopes $^{50-56}$Ni which are bound. From the various results presented for the Ni isotopes, it is observed that the unbound proton rich nucleus preserves properties very similar to the other bound states and this enables it to have finite life time and decay by emission of two protons.

Fig. 10. The radial density dependence of the proton density distributions for the $^{48-56}$Ni isotopes obtained in spherical RMF approach. The interesting result is that the proton density distribution of the unbound $^{48}$Ni isotope has the radial dependence similar to the other isotopes $^{50-56}$Ni which are bound, this causes the unbound nucleus $^{48}$Ni to have finite decay time.
3.2.3. The nucleus $^{60}_{32}\text{Ge}_{28}$

![Diagram of the RMF potential energy for the unbound nucleus $^{60}_{32}\text{Ge}_{28}$ as a function of radius. The long dashed line represents the sum of RMF potential energy and the centrifugal barrier energy for the proton resonant state $1f_{5/2}$. It also shows the energy spectrum of some proton single particle states including the important resonant states $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ at 0.73, 1.13 and 2.11 MeV, respectively.](image)

Fig. 11. The RMF potential energy for the unbound nucleus $^{60}_{32}\text{Ge}_{28}$ as a function of radius is shown by the solid line. The long dashed line represents the sum of RMF potential energy and the centrifugal barrier energy for the proton resonant state $1f_{5/2}$. It also shows the energy spectrum of some proton single particle states including the important resonant states $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ at 0.73, 1.13 and 2.11 MeV, respectively.

The predicted two-proton emitter nucleus $^{60}_{32}\text{Ge}_{28}$ is found to have slightly deformed shape with $\beta_{2m} = 0.13$ and has been described here in an approximate manner within the framework of spherical RMF approach as in the case of nucleus $^{42}_{24}\text{Cr}_{18}$. The calculated two-proton separation energy for the nucleus $^{60}_{32}\text{Ge}_{28}$ is found to be negative, though very close to the zero energy. Thus, theoretically this nucleus appears to be unbound against the two-proton decay. On the other hand, through the measured value of the two-proton separation energy for this nucleus is also found to be close to zero, it has positive sign but with large error bars.

In the case of proton rich $^{60}_{32}\text{Ge}_{28}$ nucleus the proton resonant states are found to be $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ as has been shown in Fig. 11. The figure displays together the RMF potential and the spectrum for the bound proton single particle states. We have also depicted in Fig. 11 the total mean-field potential for the resonant $1f_{5/2}$ state, obtained by adding the centrifugal potential energy. It may be emphasized that amongst the many proton single particle states in the continuum only the resonant state has sizable overlap with the bound single particle states. Thus the resonant states $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ make significant contribution to pairing energy.

This can be inferred from Fig. 12 wherein we have displayed the radial wave
functions of some of the proton single particle states close to the Fermi surface, the proton Fermi energy being $\lambda_p = 0.64$ MeV. These include the bound states $2s_{1/2}$ and $1f_{7/2}$, and the continuum states $3p_{3/2}$ and $3s_{1/2}$, in addition to the resonant $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ single particle states. The wave functions for the $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ states plotted in Fig. 12 are clearly seen to be confined within a radial range of about 7 fm and have a small decaying component outside this region, characterizing the resonant states. In contrast, the main part of the wave function for the non-resonant states, e.g. $3s_{1/2}$ and $3p_{3/2}$, is seen to be spread outside the potential region, though a small part is also contained inside the potential range. This type of states thus has a poorer overlap with the bound states near the Fermi surface leading to small value for the pairing gap $\Delta_j$.

This is clearly evident from Fig. 13 which shows the calculated single particle pairing gap energy $\Delta_j$ for some of the proton states in the nucleus $^{60}_{32}$Ge$_{28}$. One observes indeed in Fig. 13 that the gap energies for the resonant $1f_{5/2}$, $2p_{3/2}$ and $2p_{1/2}$ states have values close to 1 MeV which is quantitatively similar to that of bound states $1f_{7/2}$ and $2s_{1/2}$. Also, Fig. 13 shows that the pairing gap value for the non-resonant states $2d_{5/2}$ and $3s_{1/2}$ lying in the continuum is negligibly small.

The explanation of this nucleus acquiring long life time despite lying close to the two-proton drip-line with negative two-proton separation energy can be given by considering the neighboring isotones like $^{58}_{28}$Ni$_{28}$, $^{56}_{30}$Zn$_{28}$ in a manner similar to the case of nucleus $^{42}_{24}$Cr$_{18}$ described earlier.
In order to show the effect of addition of protons to a fixed number of neutrons which is equivalent to highlighting the changes as one moves towards the proton drip-line and beyond, we have shown in Fig. 14, the variation in radial density distributions for the isotones corresponding to \( N = 20 \) covering the \( Z \) range from 20 to 28.

Fig. 14. Results for the radial density distributions obtained in spherical RMF approach using TMA force parameters for \( N = 20 \) isotones with proton number \( Z \) ranging from 20 to 28. The hatched area shows the neutron density distributions for the nucleus \( ^{48}\text{Ni}_{20} \) corresponding to \( N = 20 \). The proton density distributions for different isotones have been shown by solid lines.
It is evidently seen that the proton density has a long extended tail indicating loosely bound protons. Further increase in number of protons make the system unbound and in some cases such as $^{42}_{24}$Cr, $^{48}_{28}$Ni, $^{54}_{20}$Ge discussed here, we have the phenomena of two proton radioactivity.

4. Summary
To summarize, we have employed deformed relativistic mean-field plus state dependent BCS (RMF+BCS) approach to study the ground state properties of proton rich nuclei in the region $20 \leq Z \leq 40$. We have found that the potential barrier provided by the Coulomb interaction and that due to centrifugal force may cause a long delay in the decay of some of the nuclei with small negative proton separation energy. This may cause the existence of proton rich nuclei beyond the proton drip line. Nuclei $^{38}_{16}$Ti, $^{44}_{24}$Cr, $^{48}_{28}$Ni, $^{55}_{20}$Zn, $^{60}_{28}$Ge, $^{63,64}_{28}$Se, $^{68}_{28}$Kr, $^{72}_{28}$Sr and $^{76}_{28}$Zr are expected to be the possible candidates for exhibiting two-proton radioactivity in the region $20 \leq Z \leq 40$. The calculated two-proton separation energy and other relevant ground state properties are found to be in good agreement with the available experimental data. This demonstrates the validity and usefulness of the deformed RMF+BCS approach for the description of proton rich nuclei near the drip lines.

The above mentioned results of our investigations for the two-proton radioactivity are expected to provide additional impetus for more experimental studies to verify the potential candidates predicted above for the two-proton radioactivity in the near future.

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