Light Deflection by Rotating Regular Black Holes with a Cosmological Constant

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April 22, 2022

Abstract

Using the Gauss-Bonnet theorem, we compute and examine the deflection angle of light rays by rotating regular black holes with a cosmological constant. By the help of optical geometries, we first deal with the Hayward black holes with cosmological contributions. Then, we reconsider the study of the Bardeen solutions. We inspect the cosmological constant effect on the deflection angle of light rays. Concretely, we find extra cosmological correction terms generalizing certain obtained findings. Using graphical analysis, we provide a comparative discussion with respect to the Kerr solutions. The results confirm that the non-linear electrodynamic charges affect the space-time geometry by decreasing the deflection angle of light rays by such cosmological black holes.

Keywords: Regular black holes, Deflection angle formalism, Gauss-Bonnet theorem, Cosmological constant.

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1 Introduction

Black hole physics has received a remarkable interest due to recent detections and observational findings supporting the obtained theoretical investigations. Precisely, Event Horizon Telescope (EHT) has provided an image of a black hole [1–4]. This discovery has been followed by significant efforts reserved to the investigation of various black hole aspects [5, 6]. These involve the thermodynamics and the optical behaviors encoding many interesting data of the associated physics. In connections with Anti de Sitter geometries, the thermodynamic quantities have been approached by linking the cosmological constant with the pressure [7–10]. Based on such an interplay, certain phase transitions, including the Hawking-Page one, have been dealt with extensively in many places including in [11]. It has been observed many similarities with fluid physical systems [12, 13]. Besides that, optical properties of the black holes have been also studied by focusing on two relevant notions, being the shadow and the deflection angle of light rays. The first concept has been discussed in terms of one dimensional real curves [14–17]. In this way, the corresponding size and shape have been controlled by the involved black hole parameters [18–22]. The second notion concerns the deflection angle of light rays by black holes which has been investigated using different methods [23–27]. A priori, there are two famous ones. One is relied on an elliptic formalism involving non-trivial elliptic functions including the hypergeometric ones, explored to compute the so-called total deflection angle [28, 29]. The second method, which will be exploited here, is based on the Gauss-Bonnet theorem results [30–32]. Precisely, Gibbons and Werner suggested a nice road to compute the deflection angle of light rays by black holes [33]. In this regard, many models have been discussed including regular black hole geometries [34]. These solutions have been interpreted in terms of gravitational backgrounds with non-linear electromagnetic charges. The key interest on these black holes comes from the understanding of certain singularities thanks to the pioneer works of Penrose and Hawking [35, 36]. Then, Bardeen discovered a regular black hole obeying the weak energy condition. After that, many Bardeen-like black hole solutions have been investigated where the irregularity has been linked to topological changes. This has been exploited to provide a possibility to build spaces with a maximum curvature inside the black holes [37, 38]. Moreover, other regular black hole models have been suggested by coupling the Einstein’s theory with nonlinear electrodynamics [39]. It has been shown that the regular black hole interior solutions have been found in many gravity models like Loop Quantum Gravity [40]. The black hole solutions of Bardeen and Hayward have been extensively studied by examining either thermodynamics properties or shadow optical ones [41, 42].

The aim of this work is to contribute to such activities by investigating the deflection angle of light rays by regular black holes with a cosmological constant. Using the techniques explored in [30–32], we compute and examine such an optical quantity of the rotating Hayward regular black holes and Bardeen regular black holes in terms of existing parameters. Taking the finite distance contributions, we inspect the effect of such parameters. A particular emphasis puts on the cosmological constant influence. Then, we provide a comparison
study with respect to the Kerr solutions with negative and positive cosmological constant values. This finding confirms that the non-linear electrodynamic charges affect the space-time geometry by decreasing the deflection angle of light rays near such black holes with a cosmological constant.

This paper is organized as follows. In section 2, we present the formalism needed to compute the deflection angle of light rays. In section 2, we consider the rotating regular Hayward black hole with a cosmological constant. In section 4, we deal with the rotating regular Bardeen black holes with cosmological contributions. In section 5, we provide a comparative study by elaborating graphical analysis using results of the Kerr solutions. The last section concerns concluding remarks.

2 Deflection angle formalism

In this section, we give a concise review on the computations of the light deflection angle around a black hole. An examination shows that various methods have been proposed and exploited for many solutions. The known one has been based on the Gauss-Bonnet theorem results being extensively used to approach several black hole backgrounds [30]. In the present work, we will exploit these findings to examine such an optical quantity where the observer and the source are placed at finite distances. In particular, the calculations relay on the method developed in [31]. In this way, the deflection angle of light rays can be expressed via the relation

$$\Theta = \Psi_R - \Psi_S + \phi_{SR}$$  \hspace{1cm} (2.1)

where $\Psi_R$ and $\Psi_S$ are the angles between the light rays and the radial direction at the observer and the source position, respectively. It is denoted that $\phi_{SR}$ is the longitude separation angle, which will be specified later on [30]. To write down the associated formula, an unit tangential vector along the light will be used. This vector denoted $e^i$ is linked to the above angle as follows

$$(e^r, e^\theta, e^\phi) = \epsilon \left( \frac{d\theta}{d\phi}, 0, 1 \right)$$  \hspace{1cm} (2.2)

where $\epsilon$ is a radial quantity which can be determined from the metric black hole solutions. Putting such a metric as follows

$$ds^2 = -A(r, \theta)dt^2 + B(r, \theta)dr^2 + C(r, \theta)d\theta^2 + D(r, \theta)d\phi^2 - 2H(r, \theta)dt d\phi,$$  \hspace{1cm} (2.3)

it is possible to get its expression. Concretely, it is given by

$$\epsilon = A(r)D(r) + H^2(r) \over A(r)(H(r) + A(r)b)$$  \hspace{1cm} (2.4)

where $b$ is the impact parameter considered as a ratio of the two constants of motion derived from the orbit equation. A simplified form can be determined by considering the equatorial plane. Indeed, these two conserved quantities read as

$$E = A(r)\dot{t} + H(r)\dot{\phi} \quad L = D(r)\dot{\phi} - H(r)\dot{t}$$  \hspace{1cm} (2.5)
where the derivative with respect to the affine parameter has been used. They represent the energy and the angular momentum of the test particle, respectively. Considering a constant $t$ of the space-time metric, a 2-dimensional curved space can be built being relevant in the calculations of the light deflection angle. In the equatorial plane, its element line should take the following form

$$dl^2 \equiv \gamma_{ij} dx^i dx^j$$

(2.6)

where $\gamma_{ij}$ is a spatial metric encoding the involved physical parameters. In this space, the deflection angle of light rays can be measured from the radial direction. The above angle can be determined by taking the particular angle $\theta = \pi/2$ and using the relation

$$\cos \Psi = \gamma_{ij} e^i R^j,$$

(2.7)

where $R^j$ are the components of a radial vector given by $(\frac{1}{\sqrt{\gamma_{rr}}}, 0, 0)$. Exploiting Eq.(2.2) and Eq.(2.7), the $\sin \Psi$ term can be obtained from the following relation

$$\sin \Psi = \frac{H(r) + A(r)b}{\sqrt{A(r)D(r) + H^2(r)}}.$$  

(2.8)

With this formalism in hand, we can calculate the deflection angle of light rays by rotating regular black holes in cosmological backgrounds by considering the retrograde situation. This will be the aim of the forthcoming sections. In particular, we consider two models followed by a comparative discussion.

3 Deflection angle of light rays by cosmological Hayward black hole solutions

In this section, we investigate the deflection angle of light rays around the rotating Hayward black holes with a cosmological constant. As mentioned before, the calculations are based on the metric form. In the Boyer-Lindquist coordinates, this is given via the following line element

$$ds^2 = -\Delta_r \left( dt - a \frac{\sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\sum}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{\Sigma} \left( a dt - r^2 + a^2 \Xi d\phi \right)^2.$$  

(3.1)

The physical terms appearing in such a metric can be expressed as follows

$$\Delta_r = \left( r^2 + a^2 \right) \left( 1 - \frac{r^2 \Lambda}{3} \right) - 2M \left( \frac{r^3}{r^3 + g_h^3} \right) r, \quad \Delta_\theta = 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta,$$

$$\Xi = 1 + \frac{a^2 \Lambda}{3}, \quad \Sigma = r^2 + a^2 \cos^2 \theta.$$  

(3.2)

(3.3)

Here, $M$ and $a$ represent the mass and the rotating parameter, respectively. $g_h$ indicates the charge of the non-linear electrodynamics and $\Lambda$ is the cosmological constant. Using the
impact parameter given by \( b = \frac{r}{u} \) and changing the variable \( r \) to \( \frac{1}{u} \), we can get

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{b^2} - \frac{4aMu}{b^4} - u^2 + 2Mu^3 + \frac{\Lambda}{3} - \frac{2a\Lambda}{3b^2u^2} - \frac{8a\Lambda M}{3b^2u} + \frac{4aMu^4g_h^3}{b^4} - 2Mu^6g_h^3
\]

\[
+ \frac{8a\Lambda Mu^2g_h^3}{3b^3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_h^4),
\]

where one has considered only the retrograde solution. In the beginning, one should calculate the separation angle integral. Putting \( \left( \frac{du}{d\phi} \right)^2 = F(u) \), this can be determined via the following computation

\[
\phi_{RS} = \int_{S}^{R} d\phi = \int_{u_S}^{u_0} \frac{1}{\sqrt{F(u)}} du + \int_{u_R}^{u_0} \frac{1}{\sqrt{F(u)}} du,
\]

where \( u_S \) and \( u_R \) are the inverse of the source and the observer distance from the black hole and where \( u_0 \) is the inverse of the closest approach \( r_0 \). Considering the weak field and the slow rotation approximations, the impact parameter can be related to \( u_0 \) as follows

\[
b = \frac{1}{u_0} + M - 2aMu_0 + \mathcal{O}(M^2, \Lambda, a^2, g_h).
\]

Performing appropriate calculations, we obtain

\[
\phi_{RS} = \phi_{RS}^{Kerr} + \left( \frac{u_R (2b^4u_R^4 + 5b^2u_R^2 - 15)}{\sqrt{1 - b^2u_R^2}} + \frac{u_S (2b^4u_S^4 + 5b^2u_S^2 - 15)}{\sqrt{1 - b^2u_S^2}} \right) \frac{Mg_h^3}{8b^3}
\]

\[
+ \left( \frac{u_R (b^2u_R^2 - 3)}{\sqrt{1 - b^2u_R^2}} + \frac{u_S (b^2u_S^2 - 3)}{\sqrt{1 - b^2u_S^2}} \right) \frac{amg_h^3}{b^4}
\]

\[
- \left( \frac{u_R (3b^4u_R^4 - 20b^2u_R^4 + 15)}{(1 - b^2u_R^2)^{3/2}} + \frac{u_S (3b^4u_S^4 - 20b^2u_S^4 + 15)}{(1 - b^2u_S^2)^{3/2}} \right) \frac{\Lambda M g_h^3}{12b^2}
\]

\[
+ (\pi - \arcsin (bu_R) - \arcsin (bu_S)) \left( -\frac{15Mg_h^3}{8b^4} + \frac{3ag_h^3}{b^5} - \frac{5\Lambda M g_h^3}{4b^5} \right)
\]

\[
+ \left( \frac{u_R}{\sqrt{1 - b^2u_R^2}} + \frac{u_S}{\sqrt{1 - b^2u_S^2}} \right) \frac{b^4\Lambda}{6} + \left( \frac{b (2 - 3b^2u_R^2)}{2 (1 - b^2u_R^2)^{3/2}} + \frac{b (2 - 3b^2u_S^2)}{2 (1 - b^2u_S^2)^{3/2}} \right) \frac{M\Lambda}{3}
\]

\[
+ \left( \frac{1 - 2b^2u_R^2}{u_R\sqrt{1 - b^2u_R^2}} + \frac{1 - 2b^2u_S^2}{u_S\sqrt{1 - b^2u_S^2}} \right) \frac{a\Lambda}{3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_h^4, Ma\Lambda, Ma\Lambda g_h^3),
\]

where the Kerr term is given by

\[
\phi_{RS}^{Kerr} = \phi_{RS}^{SW} - \left( \frac{1}{\sqrt{1 - b^2u_R^2}} + \frac{1}{\sqrt{1 - b^2u_S^2}} \right) \frac{2aM}{b^2}.
\]

In this equation, the Schwarzschild term is

\[
\phi_{RS}^{SW} = \pi - \arcsin (bu_R) - \arcsin (bu_S) + \left( \frac{2 - b^2u_R^2}{\sqrt{1 - b^2u_R^2}} + \frac{2 - b^2u_S^2}{\sqrt{1 - b^2u_S^2}} \right) \frac{M}{b}.
\]
To get the remaining terms appearing in the light deflection angle expression, one should identify the $\Psi$ terms. By using the Eq.(2.8), we find

$$\sin \Psi = bu - bMu^2 + 2AMu^2 - \left( \frac{M}{6} - \frac{a}{3u} + \frac{b}{6u} \right) \Lambda + \left( bu^5 - 2au^5 + \frac{b\Lambda u^3}{6} \right) Mg_h^3$$

$$+ \mathcal{O}(M^2, \Lambda^2, a^2, g_h^4, Ma \Lambda, Ma \Lambda g_h^3).$$

(3.10)

This relation produces

$$\Psi_R - \Psi_S = \Psi^K_{err} - \Psi^K_{err} + \left( \frac{bu_R^5}{\sqrt{1 - b^2u_R^2}} + \frac{bu_S^5}{\sqrt{1 - b^2u_S^2}} \right) Mg_h^3$$

$$- \left( \frac{u_R^5}{\sqrt{1 - b^2u_R^2}} + \frac{u_S^5}{\sqrt{1 - b^2u_S^2}} \right) 2AMg_h^3 - \left( \frac{u_R^5(2b^2u_R^2 - 1)}{(1 - b^2u_R^2)^{3/2}} + \frac{u_S^5(2b^2u_S^2 - 1)}{(1 - b^2u_S^2)^{3/2}} \right) \frac{b\Lambda Mg_h^3}{6}$$

$$- \left( \frac{1}{u_R\sqrt{1 - b^2u_R^2}} + \frac{1}{u_S\sqrt{1 - b^2u_S^2}} \right) \frac{b\Lambda}{6} - \left( \frac{2b^2u_R^2 - 1}{(1 - b^2u_R^2)^{3/2}} + \frac{2b^2u_S^2 - 1}{(1 - b^2u_S^2)^{3/2}} \right) \frac{b\Lambda M}{6}$$

$$+ \left( \frac{1}{u_R\sqrt{1 - b^2u_R^2}} + \frac{1}{u_S\sqrt{1 - b^2u_S^2}} \right) \frac{a\Lambda}{3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_h^4, Ma \Lambda, Ma \Lambda g_h^3),$$

(3.11)

where one has

$$\Psi^K_{err} - \Psi^K_{err} = \Psi^{SW}_R - \Psi^{SW}_S + \left( \frac{u_R^2}{\sqrt{1 - b^2u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2u_S^2}} \right) 2AM,$$

(3.12)

and where one has found

$$\Psi^SW_R - \Psi^SW_S = (\arcsin(bu_R) + \arcsin(bu_S) - \pi) - \left( \frac{u_R^2}{\sqrt{1 - b^2u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2u_S^2}} \right) Mb. (3.13)$$

Combining the above equations, we get an expression of the light deflection angle given by

$$\alpha_h = \left( \frac{2M}{b} - \left( \frac{u_R^2}{\sqrt{1 - b^2u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2u_S^2}} \right) \frac{2AM}{b^2} \right)$$

$$- \left( \frac{1 - b^2u_R^2}{u_R\sqrt{1 - b^2u_R^2}} + \frac{1 - b^2u_S^2}{u_S\sqrt{1 - b^2u_S^2}} \right) \frac{b\Lambda}{6} + \left( \frac{1}{u_R\sqrt{1 - b^2u_R^2}} + \frac{1}{u_S\sqrt{1 - b^2u_S^2}} \right) \frac{b\Lambda M}{6}$$

$$+ \left( \frac{1}{u_R\sqrt{1 - b^2u_R^2}} + \frac{1}{u_S\sqrt{1 - b^2u_S^2}} \right) 2a\Lambda + \frac{u_R^2(2b^4u_R^4 + b^2u_R^2 - 3)}{\sqrt{1 - b^2u_R^2}} + \frac{u_S^2(2b^4u_S^4 + b^2u_S^2 - 3)}{\sqrt{1 - b^2u_S^2}}$$

$$\times \left( \frac{5Mg_h^3}{8b^3} + \frac{a\Lambda g_h^3}{b^4} + \frac{(\arcsin(bu_R) - \arcsin(bu_S))}{\sqrt{1 - b^2u_R^2}} \right) \left( \frac{15Mg_h^3}{8b^4} + \frac{3aMg_h^3}{b^5} - \frac{5\Lambda Mg_h^3}{4b^2} \right)$$

$$+ \mathcal{O}(M^2, \Lambda^2, a^2, g_h^4, Ma \Lambda, Ma \Lambda g_h^3).$$

(3.14)
This form can be reduced to a simplified one using certain convenient approximations. Taking \( u_S \ll 1 \) and \( u_R \ll 1 \), we can get an expression involving divergent terms coupled to the cosmological contributions. These terms should be existed to show the cosmological background dependence. The desired deflection angle of light rays is found to be

\[
\alpha_h = \frac{4M}{b} - \frac{4aM}{b^2} - \frac{15\pi Mg_h^3}{8b^4} + \frac{3\pi aMg_h^3}{b^5} - \frac{5\Lambda\pi Mg_h^3}{4b^2} + \frac{b\Lambda M}{3} - \left( \frac{1}{u_R} + \frac{1}{u_S} \right) \frac{b\Lambda}{6}
+ \left( \frac{1}{u_R} + \frac{1}{u_S} \right) \frac{2a\Lambda}{3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_h^4, Ma\Lambda, Ma\Lambda g_h^3).
\]

(3.15)

An examination shows that this expression recovers many previous findings. In the absence of the cosmological contributions, we get the results of the ordinary Hayward solutions investigated in [34]. Moreover, the Schwarzschild black hole and the Kerr black hole results can be obtained by sending the extra parameters to zero [43].

4 Deflection angle of light rays by cosmological Bardeen black holes

In this section, we deal with other regular black holes. Precisely, we study the deflection angle of light rays around the rotating Bardeen black holes with a cosmological constant. To start, we give the associated metric in the Boyer-Lindquist coordinates. In this way, the line element of such a metric reads as

\[
ds^2 = -\frac{\Delta_r}{\Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left( adt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2.
\]

(4.1)

This involves the physical terms

\[
\Delta_r = (r^2 + a^2) \left( 1 - \frac{r^2 \Lambda}{3} \right) - 2M \left( \frac{r^2}{r^2 + g_b^2} \right)^{\frac{3}{2}} r, \quad \Delta_\theta = 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta,
\]

(4.2)

\[
\Xi = 1 + \frac{a^2 \Lambda}{3}, \quad \Sigma = r^2 + a^2 \cos^2 \theta
\]

where \( M \) and \( a \) denote the mass and the rotating parameter of the black hole, respectively. \( g_b \) represents the charge of the nonlinear electrodynamics. To compute the associated deflection angle, the orbit equation is needed being relevant as indicated in the previous section. A close examination provides the following relation

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{b^2} - \frac{4aM}{b^3} - \frac{a^2 + 2Mu^3 + \frac{\Lambda}{3} - \frac{2a\Lambda}{3b^2u^2} - \frac{8a\Lambda M}{3b^3u} - 3Mu^5}{b^5} g_b^2
+ \frac{6aMv^2}{b^3} + \frac{4a\Lambda M u^2}{b^5} + \mathcal{O}(M^2, \Lambda^2, a^2, g_b^3).
\]

(4.3)
After computations, we obtain

\[
\phi_{RS} = \phi_{RS}^{Kerr} + \left( \frac{b^4 u_R^4 + 4b^2 u_R^2 - 8}{2b^3 \sqrt{1 - b^2 u_R^2}} + \frac{b^4 u_S^4 + 4b^2 u_S^2 - 8}{2b^3 \sqrt{1 - b^2 u_S^2}} \right) Mg_6^2 \\
- \left( \frac{3b^4 u_R^4 - 12b^2 u_R^2 + 8}{(1 - b^2 u_R^2)^{3/2}} + \frac{3b^4 u_S^4 - 12b^2 u_S^2 + 8}{(1 - b^2 u_S^2)^{3/2}} \right) \frac{\Lambda Mg_6^2}{4} + \left( \frac{6 - 3b^2 u_R^2}{\sqrt{1 - b^2 u_R^2}} + \frac{6 - 3b^2 u_S^2}{\sqrt{1 - b^2 u_S^2}} \right) \frac{aM g_6^2}{b^4}
\]
\[
+ \left( \frac{u_R}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S}{\sqrt{1 - b^2 u_S^2}} \right) \frac{b^3 \Lambda}{6} + \left( \frac{b (2 - 3b^2 u_R^2)}{2 (1 - b^2 u_R^2)^{3/2}} + \frac{b (2 - 3b^2 u_S^2)}{2 (1 - b^2 u_S^2)^{3/2}} \right) \frac{M \Lambda}{3}
\]
\[
+ \left( \frac{1 - 2b^2 u_R^2}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1 - 2b^2 u_S^2}{u_S \sqrt{1 - b^2 u_S^2}} \right) \frac{a \Lambda}{3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_6^3, Ma \Lambda, Ma \Lambda g_6^2).
\]

To get the remaining terms appearing in the deflection angle expression, one should find the \( \Psi \) terms. Concretely, we find

\[
\sin \Psi = bu - bMu^2 + 2aMu^2 + \left( \frac{3bu^4}{2} - 3au^4 + \frac{b \Lambda u^2}{4} \right) Mg_6^2 - \left( \frac{bM}{6} - \frac{a}{3u} + \frac{b}{6u} \right) \Lambda
\]
\[
+ \mathcal{O}(M^2, \Lambda^2, a^2, g_6^3, Ma \Lambda, Ma \Lambda g_6^2).
\]

This relation leads to

\[
\Psi_R - \Psi_S = \Psi_R^{Kerr} - \Psi_S^{Kerr} + \left( \frac{u_R^4}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^4}{\sqrt{1 - b^2 u_S^2}} \right) \frac{3bMg_6^2}{2}
\]
\[
- \left( \frac{u_R^4}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^4}{\sqrt{1 - b^2 u_S^2}} \right) 3aMg_6^2 - \left( \frac{u_R^2 (2b^2 u_R^2 - 1)}{(1 - b^2 u_R^2)^{3/2}} + \frac{u_S^2 (2b^2 u_S^2 - 1)}{(1 - b^2 u_S^2)^{3/2}} \right) \frac{b \Lambda M g_6^2}{4}
\]
\[
- \left( \frac{1}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1}{u_S \sqrt{1 - b^2 u_S^2}} \right) b \Lambda - \left( \frac{2b^2 u_R^2 - 1}{(1 - b^2 u_R^2)^{3/2}} + \frac{2b^2 u_S^2 - 1}{(1 - b^2 u_S^2)^{3/2}} \right) \frac{b \Lambda M}{6}
\]
\[
+ \left( \frac{1}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1}{u_S \sqrt{1 - b^2 u_S^2}} \right) \frac{a \Lambda}{3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_6^3, Ma \Lambda, Ma \Lambda g_6^2).
\]

Exploiting the above equations, we obtain the deflection angle of light rays

\[
\alpha_b = \left( \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) \frac{2M}{b} - \left( \sqrt{1 - b^2 u_R^2} - \sqrt{1 - b^2 u_S^2} \right) \frac{2aM}{b^2}
\]
\[
- \left( \frac{1 - b^2 u_R^2}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{1 - b^2 u_S^2}{u_S \sqrt{1 - b^2 u_S^2}} \right) b \Lambda + \left( \frac{1}{\sqrt{1 - b^2 u_R^2}} + \frac{1}{\sqrt{1 - b^2 u_S^2}} \right) b \Lambda M
\]
\[
+ \left( \frac{\sqrt{1 - b^2 u_R^2}}{u_R} + \frac{\sqrt{1 - b^2 u_S^2}}{u_S} \right) \frac{2a \Lambda}{3} + \left( \frac{b^4 u_R^4 + b^2 u_R^2 - 2}{\sqrt{1 - b^2 u_R^2}} + \frac{b^4 u_S^4 + b^2 u_S^2 - 2}{\sqrt{1 - b^2 u_S^2}} \right) \frac{2M g_6^2}{b^3}
\]
\[
- \left( \frac{b^4 u_R^4 + b^2 u_R^2 - 2}{\sqrt{1 - b^2 u_R^2}} + \frac{b^4 u_S^4 + b^2 u_S^2 - 2}{\sqrt{1 - b^2 u_S^2}} \right) \frac{3aM g_6^2}{b^4} + \left( \frac{5b^2 u_R^2 - 8}{\sqrt{1 - b^2 u_R^2}} + \frac{5b^2 u_S^2 - 8}{\sqrt{1 - b^2 u_S^2}} \right) \frac{\Lambda M g_6^2}{4b}
\]
\[
+ \mathcal{O}(M^2, \Lambda^2, a^2, g_6^3, Ma \Lambda, Ma \Lambda g_6^2).
\]
This relation can be simplified by considering acceptable approximations. Vanishing \( u_S b \) and \( u_R b \), we can get an expression involving the divergent terms coupled to the geometric contributions showing the AdS background dependence. The final light deflection angle expression is found to be

\[
\alpha_b = \frac{4M}{b} - \frac{4aM}{b^2} - \frac{8Mg_b^2}{b^3} - \frac{4\Lambda M g_b^2}{b^4} + \frac{12aM g_b^2}{b^4} + \frac{b\Lambda M}{3} - \left( \frac{1}{u_R} + \frac{1}{u_S} \right) \frac{b\Lambda}{6} + \left( \frac{1}{u_R} + \frac{1}{u_S} \right) \frac{2a\Lambda}{3} + \mathcal{O}(M^2, \Lambda^2, a^2, g_b^2, Ma\Lambda, Ma\Lambda g_b^2).
\]

(4.9)

As the above black hole solution, we can recover the previous findings by removing the extra physical parameters. The present expressions can generalize the previous results \([34, 43]\).

Having obtained the deflection angle of light rays by regular rotating black holes in the presence of the cosmological constant, we move to graphically analysis the corresponding optical behaviors. In particular, we vary the involved parameters including the cosmological constant. Different regions of the space parameter have been inspected and compared with the Kerr solutions.

5 Results and discussions

To inspect the variation aspects of the investigated optical quantity, the moduli space coordinated by the involved parameters is needed. These behaviors will be illustrated by considering certain regions of such a parameter space involving cosmological contributions. Fixing first the black hole mass and taking a negative cosmological constant, we will vary the remaining ones. For the Hayward black hole solutions, these aspects are presented in Fig.(1).

For small values of the impact parameter, it has been remarked that the deflection angle of light rays is a decreasing function with a remarkable \( g_b \) effect. Indeed, it decreases with such a parameter. For large values of the impact parameter, however, the deflection angle of light rays becomes increasing functions with coinciding one dimensional real curves for generic values of \( g_b \). Moreover, we observe similar effect contributions appearing in the ordinary black hole solutions where the deflection angle decreases by increasing the rotation parameter. The obtained results confirm that the space-time of the ordinary solutions could be affected by \( g_b \). To analyze the cosmological constant effect, the deflection angle in terms of \( b \) for various values of \( \Lambda \) will be inspected. The geometry pushes one to deal with different backgrounds by considering negative and positive values of \( \Lambda \). These behaviors are presented in Fig.(2). The analysis depends on two regions associated with the impact parameter values. To do so, we consider first small values of the impact parameter. For small negative values of the cosmological constant, we have remarked a critical behavior where the deflection angle takes a minimal value which augments by decreasing the cosmological constant \( \Lambda \). For positive values, however, the deflection angle remains a decreasing function of \( b \). For large values of the impact parameter, we have observed that the cosmological constant effect is
relevant. Among others, we remark a linear effect of the deflection angle of light rays.

For the Bardeen solutions, the behaviors of the light deflection angle are plotted in Fig. (3). It has been observed similar global behaviors as the previous black hole solutions, where we have identified the same parameter effects. In particular, the finding results match perfectly with the fact that the space-time of the ordinary solutions could be affected by $g_h$. However, to unveil certain local distinctions between the regular black hole backgrounds, we present a comparative analysis. In terms of the Kerr solutions with cosmological contributions, indeed,
the deflection angle of light rays can be reduced to

$$\alpha_{Kerr}^{\Lambda} \sim \frac{4M}{b} - \frac{4aM}{b^2} + \frac{b\Lambda M}{3} - \left( \frac{1}{u_R} + \frac{1}{u_S} \right) \frac{b\Lambda}{6} + \left( \frac{1}{u_R} + \frac{1}{u_S} \right) \frac{2a\Lambda}{3}. \quad (5.1)$$

In Fig.(4), we illustrate the behavior of the deflection angle of the light rays for the regular black hole solutions as a function of the impact parameter, compared with the Kerr solutions. It has been revealed that small ranges of $b$ are relevant in such an analysis for positive and negative cosmological constant values. However, large values do not bring notable distinctions due to coinciding real curves. It has been remarked that the negative values of the cosmological constant increase the light deflection angle compared with positive ones. Indeed, all solutions involve the same deflection angle of light rays. In this way, the
Figure 4: Deflection angle variations of Bardeen, Hayward and Kerr black holes with a cosmological constant.

Discussion concerns the first range. Indeed, it has been remarked that the deflection angle of light rays by the Bardeen black holes is smaller than the one obtained for Hayward solutions. Moreover, the light deflection angle of both solutions is smaller than the Kerr solution one. This result agrees with the previous one showing that the non-linear electrodynamic charges affect the space-time geometry. Such parameters decrease the deflection angle of light rays. We could anticipate that such an angle for both solutions goes to the one of the ordinary rotating black holes with the cosmological constant. In particular, it has been checked that we can recover other findings by sending the cosmological constant to zero [33,34,43].

6 Conclusion

In this work, we have studied the weak gravitational lensing for the rotating regular black holes with cosmological contributions. Considering the photon motion in the equatorial plane framework, we have obtained the optical metric needed to provide the associated orbit equations. Using the Gauss-Bonnet theorem results, we have first elaborated the light deflection angle relation of the Hayward solutions by using convenient approximations. The obtained expression has recovered the previous findings. Similar computations have been applied to rotating regular Bardeen black hole solutions. The cosmological relevant relations have been established to derive the deflection angle of light rays. As the first regular model, we have generalized certain previous works. In particular, it has been remarked that the non-trivial electrodynamics charges affect the space-time geometry. The obtained results have confirmed the previous works.

Moreover, some other works have been recovered by removing such geometrical contributions. To examine the deflection angle of light rays by such regular black holes with a
cosmological constant, we have provided a graphical analysis by varying the involved relevant parameters. Fixing the mass parameter, certain regions of the associated moduli space have been used by considering negative and positive values of the cosmological constant. We have obtained extra corrections terms being considered as extended contributions. Vanishing these terms, we have recovered certain optical behaviors. Finally, we have given a comparative observation concerning regular black hole solutions with respect to the Kerr solutions. It has been shown that large values of $b$ do not involve significant distinctions. In the small ranges of $b$, however, the deflection angles of the Bardeen and the Hayward black holes are smaller than the one of the Kerr solutions with a cosmological constant. This finding matches perfectly with the previous results indicating that the non-linear electrodynamic charges affect the space-time geometry by decreasing the deflection angle of light of rays.

This work comes up with certain open questions. One of them may concern other methods being used to compute the deflection angle of light rays. It would be of interest to investigate such an optical data by applying the elliptic formalism exploited in many places. We hope comeback to such an issue in future works.

Acknowledgments

The authors would like to thank N. Askour, Y. Hassouni, K. Masmar and M. B. Sedra for collaborations on related topics. They would like also to thank the editor and the anonymous referee for comments and suggestions. This work is partially supported by the ICTP through AF.

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