On spectral triples in quantum gravity: I

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Abstract
This paper establishes a link between noncommutative geometry and canonical quantum gravity. A semi-finite spectral triple over a space of connections is presented. The triple involves an algebra of holonomy loops and a Dirac-type operator, which resembles a global functional derivation operator. The commutation relation between the Dirac operator and the algebra has a structure related to the Poisson bracket of general relativity. Moreover, the associated Hilbert space corresponds, up to a certain symmetry group, to the Hilbert space of diffeomorphism-invariant states known from loop quantum gravity. Correspondingly, the square of the Dirac operator has, in terms of loop quantum gravity, the form of a global area-squared operator. Furthermore, the spectral action functional resembles a partition function of quantum gravity. The construction is background independent and is based on an inductive system of triangulations. This paper is the first of two papers on the subject.

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Contents

1. Introduction 2
1.1 Outline of the construction 6
2. Projective systems of simplicial complexes 8
2.1 Simplicial complexes and their loop algebras 8
2.2 Loop group homomorphisms and connections 10
2.3 The projective system 12
3. A spectral triple on a simplicial complex 13
3.1 Outline of the construction 13
3.2 Notation and basic setup 13
3.3 The Hilbert space 14
3.4 The Dirac operator 15
1. Introduction

Ever since the discovery of the standard model of particle physics, physicists have worked to understand the apparently arbitrary structure of this theory. Nature’s choice of gauge group, the Higgs sector, the 20–30 apparently unrelated parameters, etc, almost begs for a deeper explanation.

With the pioneering work of Alain Connes and co-workers on the standard model [1–8] such an explanation now appears to emerge. In Connes’ work, the standard model coupled to general relativity is expressed as a single gravitational theory. The language used for this unification is noncommutative geometry [1]. Within this framework, and under a few mathematical assumptions, the standard model coupled to gravity can be shown to be almost unique [7, 8].
Noncommutative geometry is based on the result [1, 9] that Riemannian spin geometry has an equivalent formulation in terms of commutative $\ast$-algebras and Dirac operators. In this formulation, it is the Dirac operator that carries metric information of the underlying manifold which is now written as the spectrum of the $\ast$-algebra. In total, a Riemannian spin geometry can be described in terms of a spectral triple which is the collection $(B, H, D)$ of the algebra $B$, the Dirac operator $D$ and the Hilbert space $H$ which carries a representation and action of $B$ and $D$. To obtain equivalence, the triple is required to satisfy a set of axioms of noncommutative geometry.

The language of spectral triples has a natural generalization which also includes noncommutative $\ast$-algebras and corresponding Dirac-type operators. It is this generalization which leads to the aforementioned formulation of the standard model. It turns out that the classical action of the standard model coupled to the Einstein–Hilbert action emerges from a spectral action principle applied to a specific spectral triple [3–5]. This triple involves an almost commutative $\ast$-algebra which is an algebra that factorizes into a commutative part times a matrix factor. This means that the classical action describing all fundamental physics emerges from an asymptotic expansion of the spectral action functional:

$$\text{Tr} \phi(\tilde{D}/\lambda) = \sum \lambda^n e_n.$$  

Here, $\tilde{D}$ denotes the Dirac-type operator $D$ subjected to certain inner fluctuations stemming from the noncommutativity of the algebra. Without the matrix factor, that is, for a commutative $\ast$-algebra, the same expression leads to the Einstein–Hilbert action alone. Thus, it is the inclusion of noncommutative $\ast$-algebras in the language of spectral triples that permits the formulation of all fundamental forces and particles in terms of pure gravity.

This success of noncommutative geometry as a framework to describe fundamental physics in a unified manner raises, however, a fundamental question regarding quantization. The standard model by itself is a quantum field theory. However, in its noncommutative formulation it arises as an integrated part of a purely gravitational theory. This theory is essentially classical. Quantum field theory enters the construction in a secondary step where the spectral action functional has been expanded in a gravitational sector involving only the metric field and a matter sector including all the fermionic and bosonic fields of the standard model. Quantization is applied to the latter sector only.

So this is the question: how does the quantization procedure of quantum field theory fit into the language of noncommutative geometry? Since the structure of the standard model is so readily translated into the language of noncommutative geometry one would also expect quantum field theory to have a corresponding translation. This question is further complicated by the intrinsic gravitational nature of Connes’ formulation of the standard model. If a notion of quantization does exist within this framework one would expect this to involve, at some level, quantum gravity.

This paper is motivated by these considerations. We wish to address the question regarding the inclusion of the quantization procedure in the general framework of noncommutative geometry and, in particular, in the noncommutative formulation of the standard model. We start the investigation with the assumption that the answer will involve some notion of quantum gravity. Thus, to guide our intuition we first consult canonical quantum gravity and one of its modern trends known as loop quantum gravity [11–15].

Loop quantum gravity is an approach to non-perturbative quantum gravity which is based on a formulation of general relativity in terms of the Ashtekar variables [16, 17]. These variables include a $SU(2)$ connection and its conjugate variable, an inverse densitized dreibein.

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4 Here, we give the bosonic part only.
In loop quantum gravity, one takes as the classical phase space variables the holonomy loop of the $SU(2)$ connection and its conjugate, a certain flux vector. These variables are then used in a Dirac-type quantization procedure. This involves the representation of the corresponding Poisson structure on a kinematical Hilbert space as well as the formulation of constraints encoding the symmetries of the classical theory. Here, the holonomy loops are represented as multiplication operators. This means that the kinematical Hilbert space involves functions over a space of connections.

On a technical level, loop quantum gravity exploits the holonomy formulation of gravity to recast the problem of quantizing gravity in terms of an inductive system of graphs. It turns out that the space of connections, when seen in terms of holonomies restricted to a specific graph, is a manifold. This manifold is related to the $SU(2)$ gauge group. This means that the space of connections itself is a pro-manifold, the projective limit of manifolds. This, in turn, permits a formulation of the quantization procedure on the level of finite graphs whose complexity is subsequently increased infinitely. This construction is due to Ashtekar and Lewandowski [18, 19].

In this paper, we aim to construct a model which involves elements of both noncommutative geometry and quantum gravity. We use two central elements of loop quantum gravity to obtain such a model. First, the fact that gravity can be formulated in terms of Wilson loops leads us to consider a spectral triple which involves an algebra of holonomy loops. These loop variables serve as functions on an underlying space of connections. Second, we wish to exploit the pro-manifold structure of the space of connections to describe the algebra of loops and to construct a Dirac-type operator on the space of connections. This means that we aim to construct a spectral triple over each manifold associated with graphs in the projective system. These spectral triples are required to be compatible with all embeddings between graphs. This requirement will ensure that the limit where the complexity of graphs is increased infinitely gives rise to a limit spectral triple.

This program was first initiated in [20] (also see [21]) where the authors attempted to construct such a spectral triple. There the authors found that the inductive system of graphs used in loop quantum gravity, which is the system of all piecewise analytic graphs, is too large to permit a Dirac-type operator on the space of connections. Technically, the multitude of possible embeddings of different graphs was found to be too large for a Dirac-type operator compatible with all embeddings to exist.

In the present paper, we return to this problem to now consider different systems of embedded graphs. In particular, we study the countable system of embedded graphs given by a triangulation and its barycentric subdivisions. It turns out that this restricted system of graphs does permit a Dirac-type operator on the associated projective system of manifolds. Furthermore, we find that the limit of infinitely many barycentric subdivisions gives us an accurate description of the full space of connections as well as the associated algebra of holonomy loops. The construction is general and only assumes the gauge group $G$ to be compact.

What we obtain is the following. Given a triangulation $T$ and a compact Lie group $G$, we construct a triple

\[(B_\Delta, D_\Delta, H_\Delta),\]

where $B_\Delta$ is the $*$-algebra of holonomy loops obtained via the inductive system of triangulations. The algebra is represented on the separable Hilbert space $H_\Delta$ which carries an action of the Dirac-type operator $D_\Delta$. If we denote by $\mathcal{A}$ the space of smooth connections in a trivial principal bundle $\mathcal{M} \times G$, where $\mathcal{M}$ is a manifold which corresponds to the triangulation
then we find that $A$ is densely contained in the pro-manifold associated with the algebra $B_\Delta$. This means that the triple (1) is a geometrical construction over the space of connections.

The construction of the Dirac-type operator $D_\Delta$ involves an infinite-dimensional Clifford bundle. This structure entails the canonical anticommutation relations (CARs) algebra, which appears as a tensor factor acting on the Hilbert space $\mathcal{H}_\Delta$.

Technically, the triple (1) satisfies the requirements of a semi-finite spectral triple. This is due to the fact that the infinite-dimensional Clifford bundle entails a large degeneracy of the spectrum of $D_\Delta$ which, naively, fails to have a compact resolvent. The solution to this problem is, in short, to integrate out the symmetry group related to this degeneracy. This process leads to a semi-finite spectral triple.

In addition to these rigorous mathematical results, the construction has a physical interpretation in terms of gravity.

First, the construction of the spectral triple (1) depends, as mentioned, crucially on the choice of graphs. This choice is closely related to the group of diffeomorphisms acting on the Hilbert space $\mathcal{H}_\Delta$. We find that the choice of a restricted system of graphs can be interpreted as a type of gauge fixing of the diffeomorphism group. Thus, the Hilbert space $\mathcal{H}_\Delta$ does not carry an action of any smooth diffeomorphisms. Rather, it carries an action of a discrete group of diffeomorphisms associated with the inductive system of graphs. This means that the construction reduces the diffeomorphism group to a countable group.

The spectral triple (1) has a clear interpretation in terms of a non-perturbative, background-independent quantum field theory. First of all, since the triple exists over a space of connections, the Dirac-type operator $D_\Delta$ should be interpreted as a global functional derivation operator. Also, the Hilbert space $\mathcal{H}_\Delta$ has an inner product which involves a functional integral. Next, we find that the interaction between the Dirac-type operator $D_\Delta$ and the loop algebra $B_\Delta$ reproduces the structure of the Poisson bracket of general relativity. Furthermore, the Hilbert space $\mathcal{H}_\Delta$ is found to be directly related to the Hilbert space of (spatial) diffeomorphism-invariant states known from loop quantum gravity [23]. The difference between the two is given by the group of discrete diffeomorphisms acting on $\mathcal{H}_\Delta$. Thus, we interpret the Hilbert space $\mathcal{H}_\Delta$ in terms of a partial solution to the (spatial) diffeomorphism constraint.

The square of the Dirac-type operator $D_\Delta$ can be interpreted as an integral over the underlying manifold $M$. The integrand is a quantity which, in terms of canonical quantum gravity, has an interpretation as an area-squared density operator. This operator resembles the area operators known in loop quantum gravity [24]. Furthermore, the spectral action functional of $D_\Delta$ formally resembles a Feynman integral.

The construction of the Dirac-type operator $D_\Delta$ is not unique. In fact, we find a large class of Dirac-type operators labelled by infinite sequences $\{a_n\}$ of real parameters. The operator $D_\Delta$ has a compact resolvent whenever the sequence diverges sufficiently fast. These parameters are related to the scaling behaviour of the operator and are clearly of metric origin. Thus, in order to obtain a Dirac-type operator $D_\Delta$ we are forced to choose a certain scaling behaviour.

It is clear that a correct interpretation of the sequence $\{a_n\}$ is imperative since the existence of the operator $D_\Delta$ depends crucially hereon. In this paper, we sketch one possible solution as to how these free parameters should be dealt with. We propose an extension of the spectral triple (1) to include the sequence $\{a_n\}$ as dynamical degrees of freedom. The result is a new triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ which is a fibration of spectral triples $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$. We emphasize, however, that the question as to how the sequence $\{a_n\}$ should be understood and dealt with remains open.

This paper is the first of two papers concerned with the spectral triple (1). This paper is primarily concerned with the general construction and physical interpretation of the triple.
The second paper [22] deals with the concise mathematical construction of the triple. For the exact details and proofs concerning the spectral triple (1), we therefore refer to [22].

1.1. Outline of the construction

Before we go into details we first give a brief outline of the construction. The first step is concerned with the formulation of a semi-finite spectral triple over a space of connections. The triple is, as mentioned, based on a $*$-algebra of loops which we denote by $\mathcal{B}_\Delta$. A smooth loop $l$ gives a map from the space of smooth connections, denoted by $\mathcal{A}$, into the structure group $G$:

$$l : \nabla \rightarrow \text{Hol}(\nabla, l) \in G,$$

where $\text{Hol}(\nabla, l)$ is the holonomy of the connection $\nabla \in \mathcal{A}$ along $l$ and $G$ is a compact connected Lie group. In order to describe the algebra of holonomy loops, we first restrict the loop algebra to a finite graph $\Gamma_1$ with edges $\epsilon_i$ (see figure 1). Seen from $\Gamma_1$, the connection $\nabla$ can be seen as a point in the space $G^n$:

$$\nabla = (g_1, \ldots, g_n) \in G^n = \mathcal{A}_\Gamma,$$

where $n$ is the number of edges in $\Gamma$ and $g_i = \text{Hol}(\nabla, \epsilon_i)$ is the holonomy transform along the $i$th edge. That is, a connection is given by its holonomy transforms along edges $\epsilon_i$. Clearly, $\mathcal{A}_\Gamma$ is a highly inaccurate picture of the full space $\mathcal{A}$ of connections. However, one can show [22] that for a suitable choice of embedded graphs

$$\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \ldots \subset \Gamma_n \subset \ldots,$$

the space $\mathcal{A}$ is densely contained in the limit space

$$\mathcal{A} = \lim_{n \rightarrow \infty} \mathcal{A}_{\Gamma_n}.$$  

This provides us with the strategy to construct a spectral triple over $\mathcal{A}$. With the system of graphs given by nested triangulations we construct, at the level of each graph $\Gamma_n$, a spectral triple

$$(\mathcal{B}_n, D_n, \mathcal{H}_n)_{\Gamma_n},$$

where the algebra $\mathcal{B}_n$ is generated by loops in $\Gamma_n$, the operator $D_n$ is some Dirac operator on $\mathcal{A}_{\Gamma_n} \simeq G^n$ and $\mathcal{H}_n = L^2(G^n, \text{Cl}(T^*G^n))$. This triple is almost canonical and the little choice...
one has is mostly eliminated by the requirement that the construction of the triple should be compatible with structure maps

\[ P_{nm} : A_{\tau_n} \to A_{\tau_m}, \quad n > m, \]

among the different coarse-grained spaces of field configurations.

The Dirac-type operator obtained in the limit is an operator on the space \( G^\infty \) and has the general form

\[ a_1 D_1 + a_2 D_2 + \cdots + a_k D_k + \cdots, \]

where \( D_k \) is an operator corresponding to a certain level in the projective system (2) and the infinite sequence \( \{a_k\} \) determines the weight assigned to each operator \( D_k \). We find that when the sequence \( \{a_n\} \) tends fast enough to \( \infty \), the limit triple

\[ (B_\Delta, D_\Delta, \mathcal{H}_\Delta) := \lim_{n \to \infty} (B_n, D_n, \mathcal{H}_n)_{\Gamma_n} \quad (3) \]

satisfies the requirements of a semi-finite spectral triple.

The second part of the construction is more tentative. This part is concerned with the infinite sequence \( \{a_k\} \) of free parameters which enters the construction of the semi-finite spectral triple \( (B_\Delta, D_\Delta, \mathcal{H}_\Delta) \). The sequence \( \{a_k\} \) is readily seen to carry metric data since it determines the scaling behaviour of the operator \( D_\Delta \). Also, the sequence determines the measure that arises in the square of \( D_\Delta \). Furthermore, the choice of parameters \( \{a_k\} \) is related to the invariance properties of \( D_\Delta \). Based on these observations we choose to include the sequence \( \{a_k\} \) as dynamical parameters in the construction. This means that we first construct a spectral triple

\[ (A_a, D_a, \mathcal{H}_a), \quad (4) \]

where elements in the algebra \( A_a \) are functions \( f(x_1, x_2, \ldots) \) on the moduli space of permissible sequences \( \{a_k\} \). The construction of the triple (4) is inspired by Higson and Kasparov [25]. The Hilbert space \( \mathcal{H}_a \) is an \( L^2 \) space of functions over this moduli space and \( D_a \) is a Dirac operator hereon. Next, we merge the triple (4) with the triple (3) to obtain a total triple

\[ (B_\Xi, D_\Xi, \mathcal{H}_\Xi), \quad (5) \]

where the Dirac-type operator \( D_\Xi \) combines the operators \( D_\Delta \) and \( D_a \). It is important to see that the first part of the Dirac operator, the operator \( D_\Delta \), depends on the parameters \( \{a_k\} \) which the second part, the operator \( D_a \), probes. The exact form of this interdependence, which makes the operator \( D_\Xi \) highly nontrivial, is governed by the requirement that \( \mathcal{H}_\Xi \) is a Hilbert space over an infinite-dimensional space corresponding exactly to those sequences \( \{a_k\} \) which leave the operator \( D_\Delta \) \( \Theta \)-summable.

The paper is organized as follows. Sections 2–4 give a presentation of the mathematical construction of the spectral triple \( (B_\Delta, D_\Delta, \mathcal{H}_\Delta) \). Here, section 2 introduces the basic machinery used in this paper: first, the loop algebra associated with an abstract simplicial complex and the concept of abstract connections associated with a simplicial complex and, second, projective systems of simplicial complexes. In section 3, we construct a spectral triple on the level of a simplicial complex and, in section 4, we obtain the triple \( (B_\Delta, D_\Delta, \mathcal{H}_\Delta) \) via an inductive limit of simplicial complexes. Sections 5–8 are concerned with the physical significance of the spectral triple \( (B_\Delta, D_\Delta, \mathcal{H}_\Delta) \). In section 5, we show that the spectral triple reproduces the structure of the Poisson bracket of general relativity. Then, in section 6, we give a detailed comparison between the setup presented in this paper and the setup used in loop quantum gravity and find that the two constructions, on a technical level, differ primarily in the way the diffeomorphism group is treated. Section 7 briefly reviews area operators in loop
quantum gravity and finds that the square of the Dirac operator $D_{\Delta}$ has a natural interpretation as a kind of a global-area-type operator. This naturally leads, in section 8, to an interpretation of the operator $D_{\Delta}^2$ in terms of an action. This, in turn, gives the spectral action functional of $D_{\Delta}$ a strong resemblance to a partition function related to gravity. Then, in section 9, we describe the construction of the triple $(B_t, D_t, \mathcal{H}_t)$ which includes the sequence $\{a_k\}$ as dynamical variables. Finally, in section 10.5, we mention that the Dirac-type operator $D_{\Delta}$ defines a distance on the space $\mathcal{X}^\infty$. Sections 10 and 11 contain a discussion and conclusion, respectively. Three appendices are added to first outline an extended setup which avoids the choice of a basepoint and next discuss again a notion of diffeomorphism invariance. Appendix C is concerned with certain symmetric states.

2. Projective systems of simplicial complexes

As explained in section 1, the aim is to study geometrical structures over a space of connections. To do this, we apply a dual picture of the space of connections. This means that, rather than studying the space of connections itself, we will work with an algebra of functions on the space. This algebra is an algebra of loops and has a natural interpretation in terms of holonomy loops.

The purpose of this section is to introduce the machinery needed to describe this algebra of loops. The strategy is to break up the algebra into finite parts, introduce various geometrical structures on each finite part and finally let the complexity of the construction increase infinitely to obtain the full algebra of loops.

To emphasize the purely combinatorial nature of the construction we adopt a formalism which, for the main part of the analysis, avoids any reference to an underlying manifold. This means that we will work with abstract graphs and their loop algebras. An alternative approach is to work directly with graphs on a manifold and their loop algebras. This approach, which may be intuitively clearer, is applied in [22].

2.1. Simplicial complexes and their loop algebras

We first introduce the notion of an abstract graph and its associated loop algebra. The abstract graphs we consider are given by simplicial complexes. Consider, therefore, first an abstract, finite, $d$-dimensional simplicial complex $K$ with vertices $v_i$ and directed edges $\epsilon_j : [0, 1] \rightarrow \{v_i\}$, $\epsilon_j(0) \neq \epsilon_j(1)$. 

Figure 2. An abstract simplicial complex with directed edges.
connecting the vertices, see figure 2. We shall refer to the two elements of the set \( \{0, 1\} \) as the start and endpoints of the edge. The construction which we present works for a large class of simplicial complexes. We will, however, restrict ourselves to simplicial complexes which corresponds to triangulations of \( d \)-dimensional manifolds.

We will consider based simplicial complexes which means that the complex has a preferred vertex \( v_0 \).

Given a simplicial complex \( K \), we wish to describe an algebra of based loops living on \( K \). First, a path is a finite sequence \( L = \{\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_n}\} \) of edges in \( E_K \) with the property that

\[ \epsilon_{i_k}(1) = \epsilon_{i_{k+1}}(0). \]

Next, a loop is a path satisfying

\[ \epsilon_{i_1}(0) = \epsilon_{i_n}(1), \]

and a based loop is a loop satisfying the additional requirement

\[ \epsilon_{i_1}(0) = \epsilon_{i_n}(1) = v_0. \]

By \( \epsilon_j^* \), we denote the edge \( \epsilon_j \) with the reversed direction

\[ \epsilon_j^*(\tau) = \epsilon_j(1 - \tau), \quad \tau \in \{0, 1\}. \]

A path may contain both edges and their reverse. We discard trivial backtracking by which we mean sequences that contain successions of edges \( \epsilon_i \) and their reverse \( \epsilon_i^* \). Thus, we introduce the equivalence relation

\[ \{\ldots, \epsilon_j, \epsilon_k, \epsilon_k^*, \epsilon_l, \ldots\} \sim \{\ldots, \epsilon_j, \epsilon_l, \ldots\} \]

and regard a path as an equivalence class with respect to this relation.

We define a product between two based loops \( L_i = \{\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_n}\} \), \( i \in \{1, 2\} \), simply by gluing

\[ L_1 \circ L_2 = \{\{\epsilon_{i_1}\}, \{\epsilon_{i_2}\}\}. \]

Note that this product is noncommutative.

The inversion of a based loop \( L = \{\epsilon_{j_1}, \ldots, \epsilon_{j_k}, \ldots, \epsilon_{j_n}\} \), defined by

\[ L^* = \{\epsilon_{j_1}^*, \ldots, \epsilon_{j_k}^*, \ldots, \epsilon_{j_n}^*\}, \]

is again a based loop and satisfies the requirements of an involution

\[ (L^*)^* = L, \quad (L_1 \circ L_2)^* = L_2^* \circ L_1^*. \]

Furthermore, we define the based identity loop \( L_0 \) as the equivalence class that includes the empty loop

\[ L_0 = \{\emptyset\}. \]

The based identity loop clearly satisfies

\[ L_0 \circ L' = L' \quad \forall \text{ based loops } L'. \]

This, together with the observation that

\[ L^* \circ L = L \circ L^* = L_0, \]

implies that, for based loops, the involution equals an inverse. This provides a set of based loops with a group structure. We call the group of based loops associated with \( K \) for the hoop group the holonomy loops (see [18]), denoted by \( H_G K \).
For the remaining part of this paper, we shall consider only based loops and will therefore drop the prefix ‘based’.

We finally consider formal, finite series of loops living on a complex $K$:

$$a = \sum_i a_i L_i, \quad L_i \in \mathcal{H}_G(K), \quad a_i \in \mathbb{C}. \quad (6)$$

The product between two elements $a$ and $b$ is defined by

$$a \circ b = \sum_{i,j} (a_i \cdot b_j) L_i \circ L_j,$$

and the involution of $a$ is defined by

$$a^* = \sum_{i} \bar{a}_i L_i^*.$$

The set of elements of the form (6) is a $\ast$-algebra. We denote this algebra with $L_K$.

2.2. Loop group homomorphisms and connections

We now introduce the notion of an abstract connection. Let $G$ be a compact, connected Lie group and, for later reference, fix a matrix representation of $G$.

To motivate the definition of an abstract connection, first consider a $G$-bundle over a smooth manifold $M$ and a loop $l$ in $M$. A smooth connection can be understood as a map

$$\nabla : l \mapsto \nabla(l) \in G, \quad (7)$$

which satisfies the condition

$$\nabla(l_1 \circ l_2) = \nabla(l_1) \cdot \nabla(l_2), \quad (8)$$

where $l_1$ and $l_2$ are loops in $M$. Here, the map (7) is the holonomy transform of $\nabla$ along $l$, $\nabla(l) = \text{Hol}(\nabla, l)$.

This motivates the following definition of an abstract connection as a map:

$$\nabla : \{\epsilon_j\} \to G, \quad (9)$$

which associates with each edge $\epsilon_j \in E_K$ a point $g_j \in G$. The map is required to satisfy

$$\nabla(\epsilon_j) = (\nabla(\epsilon_j^*))^{-1}.$$

Denote by $A_K$ the space of all abstract connections associated with $K$. The action of $\nabla$ is extended to a path $L = \{\epsilon_i, \epsilon_{i2}, \ldots, \epsilon_{in}\}$ simply by

$$\nabla(L) = \nabla(\epsilon_i) \cdot \nabla(\epsilon_{i2}) \cdot \ldots \cdot \nabla(\epsilon_{in}), \quad (10)$$

where the product on the rhs is matrix multiplication. This makes $\nabla$ a group homomorphism from the hoop group $\mathcal{H}_G(K)$ into $G$ which means that it satisfies

$$\nabla(L_1 \circ L_2) = \nabla(L_1) \cdot \nabla(L_2).$$

This corresponds to condition (8) and justifies the terminology abstract connection.

Via the space $A_K$, we can equip the $\ast$-algebra formed by elements of form (6) with a natural norm given by

$$\|a\| = \sup_{\forall \epsilon_i, \epsilon_{i2}, \ldots, \epsilon_{in} \in A_K} \left\| \sum_i a_i \nabla(L_i) \right\|_G, \quad (11)$$

where the norm on the rhs of (11) is the matrix norm given by the representation of $G$. This norm is in general not faithful. Let $B_K$ be the $\mathbb{C}^\ast$-algebra generated by $L_K$ in this norm. We have a $\ast$-homomorphism from $L_K$ to $B_K$. Since the norm is not faithful, this homomorphism
is not injective. We denote by $B_K$ the image of $L_K$ under this map in $B_K$. With a slight abuse of terminology, we will call $B_K$ the loop algebra.

In fact, the algebra $B_K$ is an algebra of functions over the space $A_K$ with values in the matrix representation of the group $G$. A loop $L$ gives rise to a function $f_L$ via

$$f_L(\nabla) = \nabla(L), \quad L \in \mathcal{L}_K, \quad \nabla \in A_K.$$

Note that the algebra of functions $f_L$ with the natural product

$$f_{L_1} \cdot f_{L_2} = f_{L_1 \cup L_2}$$

is noncommutative whenever $G$ is non-Abelian.

The space $A_K$ is identified as a manifold

$$A_K \simeq G^{n(K)},$$

via the bijection

$$A_K \ni \nabla \mapsto (\nabla(\epsilon_1), \ldots, \nabla(\epsilon_{n(K)})) \in G^{n(K)}, \quad (12)$$

where $n(K)$ denotes the number of edges in $V_K$. This identification gives rise to various structures on $A_K$. For example, the topological structure is given by the topological structure of $G^{n(K)}$.

A loop $L = \{\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_k}\}$ gives, according to (12), rise to a function on $G^{n(K)}$ given by

$$f_L : (g_1, g_2, \ldots, g_{n(K)}) \mapsto g_{i_1} \cdot g_{i_2} \cdot \ldots \cdot g_{i_k} \in G.$$

We think of the space $A_K$ as a ‘coarse-grained version of a space of (smooth) connections’. To clarify this interpretation, consider an embedding of the simplicial complex into a triangulation

$$\phi : \mathcal{K} \rightarrow \mathcal{T} \quad (13)$$

and a trivial principal bundle $P = M \times G$. Denote by $A$ the space of smooth connections in $P$. There is a natural map

$$\chi_K : A \rightarrow A_K, \quad \chi_K(\nabla)(\epsilon_j) = \text{Hol}(\nabla, \phi(\epsilon_j)), \quad (14)$$

where $\text{Hol}(\nabla, \phi(\epsilon_j))$ denotes the holonomy of $\nabla$ along the edge $\epsilon_j$ which now lives in $M$ via the embedding $\phi$. This means that points in $A_K$ can be understood in terms of connections. Clearly, the map (14) is not injective which is exactly what is meant by ‘coarse grained’. The idea is to gradually increase the complexity of the simplicial complex and thereby to turn the map (14) into an injection. Therefore, the next step is to introduce a refinement procedure for the simplicial complex $\mathcal{K}$. 

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**Figure 3.** The lhs shows a triangulation (dimension = 2) and the rhs its two first barycentric subdivisions.
2.3. The projective system

The key tool to refine the simplicial complex $K$ is the barycentric subdivision of simplexes. We will consider repeated barycentric subdivisions of $K$. Thus, the basic element in our analysis of geometrical structures over spaces of connections is an inductive system of simplicial complexes, see figure 3. We start with the following.

**Definition 2.3.1.** An inductive system of simplicial complexes is a countable set $\{\{K_i\}, \{I_{jk}\}\}$ of nested abstract simplicial complexes $K_i$ and embeddings

$$I_{jk} : K_j \rightarrow K_k, \quad j < k,$$

so that $I_{jk}$ is either a barycentric subdivision or an inversion of edges. The set is required to satisfy

$$\lim_i n(K_i) = \infty.$$

A barycentric subdivision of a simplicial complex $K$ is here understood as the simultaneous subdivision of all simplexes in $K$.

The simplest simplicial complex in the inductive system is called the initial complex and is denoted by $K_0$. The orientation of the initial complex is not important and one simply chooses one.

The embedding (15) gives rise to a projection between spaces of abstract connections:

$$P_{n_2,n_1} : A_{K_{n_2}} \rightarrow A_{K_{n_1}}, \quad n_2 \geq n_1,$$

which, via (12), is identified as a projection between manifolds:

$$P_{n_2,n_1} : G^{n_2} \rightarrow G^{n_1}, \quad n_2 \geq n_1.$$

The projection $P_{n_2,n_1}$ is given by composition of one or more of the following operations:

- multiplying $g_{i_1}$ with $g_{i_2}$,
- inverting $g_i$,
- leaving out some $g_i$ in $(g_1, \ldots, g_n) \in G^{n_2}$.

The embedding (15) commutes by construction with the identification $A_K \simeq G^{n(K)}$, i.e. the diagram:

$$\begin{array}{ccc}
  A_{K_{n_1+1}} & \xrightarrow{\sigma_{n_1}} & G^{n(K_{n_1+1})} \\
  \downarrow & & \downarrow \\
  A_{K_{n_1}} & \xrightarrow{\sigma} & G^{n(K_{n_1})}
\end{array}$$

where $\sigma_i$ is given by (12), commutes. This means that the limit space

$$\mathcal{X}^\Delta := \lim_{\leftarrow K} A_K$$

is a pro-manifold. That is, it is the projective limit of manifolds. This gives us immediate access to various structures on $\mathcal{X}^\Delta$. For instance, since the projections $P$ in (16) are continuous they give a topological structure on $\mathcal{X}^\Delta$. In general, the structure of a pro-manifold is a powerful tool that leads to both Hilbert space and metric structures on the limit space $\mathcal{X}^\Delta$. 
3. A spectral triple on a simplicial complex

The aim is to construct a spectral triple at the level of each simplicial complex in an inductive system of simplicial complexes. The triples are required to involve the loop algebras $B_k$ as function algebras over the associated spaces $A_k$, of abstract connections. To ensure that the limit of increased complexity is well defined, we require the spectral triples to be compatible with all projections induced by the inductive system of complexes. This means that geometrical structures over $A_k$ will converge to geometrical structures over $A$.  

3.1. Outline of the construction

Before we go into details, let us outline the construction of the spectral triple at the level of a simplicial complex $K$. The starting point is the manifold $A_K \simeq G^{\mathfrak{n}(K)}$. It is natural to first consider the Hilbert space

$$\mathcal{H}_K = L^2(G^{\mathfrak{n}(K)}),$$

where $L^2$ is with respect to the Haar measure on $G^{\mathfrak{n}(K)}$. Since we wish both to construct a Dirac operator acting on $\mathcal{H}_K$ and to have a representation of the algebra $B_k$ on $\mathcal{H}_K$, we need to equip the Hilbert space with an additional structure. Consider, therefore, instead the Hilbert space

$$\mathcal{H}_K = L^2(G^{\mathfrak{n}(K)}, Cl(T^*G^{\mathfrak{n}(K)}) \otimes M_l(\mathbb{C})), \quad (17)$$

where $l$ is the size of the representation of $G$ and $Cl(T^*G^{\mathfrak{n}(K)})$ is the Clifford bundle involving the cotangent bundle over $G^{\mathfrak{n}(K)}$. If we recall that points in $G^{\mathfrak{n}(K)}$ represent homomorphisms $\nabla$ from the hoop group into $G$, we immediately have a representation of the loop algebra $B_k$ on $\mathcal{H}_K$:

$$f_L \cdot \psi(\nabla) = (1 \otimes \nabla(L)) \cdot \psi(\nabla), \quad \psi \in \mathcal{H}_K, \quad (18)$$

where the first factor acts on the Clifford part of the Hilbert space and the second factor acts by matrix multiplication on the matrix part of the Hilbert space. Finally, we choose some Dirac operator $D_K$ on $G^{\mathfrak{n}(K)}$ and obtain the triple

$$(B_k, D_K, \mathcal{H}_K), \quad (19)$$

on the level of the simplicial complex $K$.

It is not difficult to construct the candidate (19) for a spectral triple on the space $A_K$. The crucial point, however, is to ensure that the construction is compatible with the induced projections between the simplicial complexes (16). This requirement turns out to restrict the choice of Dirac operator on $A_k$ considerably.

To ease the notation, we shall from now on write $A_i$ for $A_k$, $B_i$ for $B_k$, $H_i$ for $H_k$, $D_i$ for $D_K$, and $n_i$ for $n(K_i)$.

3.2. Notation and basic setup

Before we proceed with the construction, we need some preparations. A point $(g_1, \ldots, g_n) \in G^n$ is denoted by $\bar{g}$. Let $R_g$ denote right translation on the group $G$, i.e. $R_g(h) = hg$. Accordingly, $L_g$ denotes left translation. We also denote by $R_g$ the corresponding differential, $R_g : T_hG \to T_{hg}G$. $L_g$ likewise. Given a cotangent vector $\phi$ at the identity, we define the right translated cotangent vector field $R\phi$ by

$$R\phi(g)(v) = \phi(R_g^{-1}(v)), \quad v \in T_gG.$$
The left translated cotangent vector fields are defined equivalently. Given a projection $P : G^n \to G^m$, we denote by $P_\ast$ the corresponding differential

$$P_\ast : T_{g}G^n \to T_{P(g)}G^m$$

and by $P^\ast$ the induced map on cotangent spaces

$$P^\ast : T_{P(g)}^\ast G^m \to T_{g}^\ast G^n.$$ 

Consider the particular projection $P : G^2 \to G^1$; $(g_1, g_2) \mapsto g_1 \cdot g_2$. An element in $(v_1, v_2) \in T_{(g_1, g_2)}G^2$ transforms according to

$$P_\ast(v_1, v_2) \mapsto (R_{g_2}v_1 + L_{g_1}v_2),$$

which is best seen by writing $v_1 = \dot{\gamma_1}(\tau)|_{\tau = 0}$ where $\gamma_1(\tau) = (\gamma_{11}(\tau), g_2) \in G^2$. The same argument applies to $v_2$. Correspondingly, given an element of the cotangent bundle $\phi \in T_{g_1 \cdot g_2}^\ast G^2$ the dual maps yield

$$P^\ast\phi = (R_{g_2}\phi, L_{g_1}\phi) \in T_{(g_1, g_2)}^\ast G^2,$$

where

$$R_{g_2}\phi(v) = \phi(R_{g_2}v), \quad v \in T_{g_2}G$$

$$L_{g_1}\phi(v) = \phi(L_{g_1}v), \quad v \in T_{g_1}G.$$ 

### 3.3. The Hilbert space

To construct the Hilbert space related to a simplicial complex $K_i$ we first construct an inner product on $T_{\bar{g}}^\ast G^n$. We choose a left- and right-invariant metrics on $G$. The edges in $K$ are numbered with $1, \ldots, n_i$. In the following, we shall occasionally write $n$ instead of $n_i = \text{n}(K_i)$. Consider two elements $\phi_1, \phi_2 \in T_{\bar{g}}^\ast G^n$. We write $\phi_k = (\phi_{k1}, \ldots, \phi_{kn})$ and define the inner product by

$$\langle \phi_1 | \phi_2 \rangle_{\bar{g}} = \frac{1}{2N} \sum_{j=1}^n \langle \phi_{j1} | \phi_{j2} \rangle_{\bar{g}},$$

(21)

where $\langle \cdot | \cdot \rangle_{\bar{g}}$ is the inner product on $T_{\bar{g}}^\ast G$. In (21), $N$ is the number of barycentric subdivisions between the simplicial complex $K_i$ and the initial complex $K_o$. In [22] we prove that the inner product (21) is compatible with projections induced by embeddings (15).

With the inner product (21) on $T^\ast G^n$, we construct the Clifford bundle $\text{Cl}(T^\ast G^n)$ and define an inner product on the Hilbert space (17):

$$\langle \cdot | \cdot \rangle = \int d\mu \cdot \text{Tr} \cdot \langle \cdot | \cdot \rangle_{\text{Cl}},$$

(22)

where $d\mu$ is the Haar measure on $G^n$, $\text{Tr}$ is the trace on $M_l$ and $\langle \cdot | \cdot \rangle_{\text{Cl}}$ is the inner product of the Clifford bundle $\text{Cl}(T^\ast G^n)$. The inner product (22) is compatible with projections (15). This gives us the Hilbert space (17).

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5 There is some freedom in the choice of $N$. We could also introduce a factor $N_j$ associated with each individual edge $e_j$. Here, $N_j$ counts the number of barycentric subdivisions that lies between the edge $e_j$ and the simplest complex $K_{j'} (k < j)$ that has the edge $e_j$ as a segment of an edge in $K_{j'}$. This choice therefore associated different weights with edges in a simplicial complex according to their position in the complex.
3.4. The Dirac operator

The Dirac-like operator on each space $A_i$ is required to be compatible with the projections in the inductive system of complexes. Thus, the induced maps

$$P^*: T^*G^m \rightarrow T^*G^n, \quad m < n,$$

(23)
ge give rise to the compatibility conditions

$$P^*(D_m v)(g_1, \ldots, g_n) = D_n(P^*v)(g_1, \ldots, g_n),$$

(24)
where $v \in L^2(G^m, Cl(T^*G^m))$. These conditions largely restrict the Dirac-like operator $D_i$ on $K_i$. On the initial complex $K_0$ with the corresponding manifold $G^0$, the Dirac operator is chosen to have the canonical form\(^6\)

$$D_0 = \sum_{i,j} e^j_i \cdot \nabla^{lk}_{e^j_i},$$

(25)
where $\{e^j_i\}$ is a global basis on $T^*G$ corresponding to the $i$th copy of $G$ and where $\cdot$ denotes Clifford multiplication with the convention $e^j_i \cdot e^j_i = -1$ (no summation). $\{\hat{e}^j_i\}$ is the corresponding basis on $TG$, obtained from $\{e^j_i\}$ via the inner product (21). $\nabla^k$ is the Levi-Civita connection on $G$.

The problem is therefore to find a Dirac operator on $G^n$ compatible with projections (15). Here, it is sufficient to deal with an edge $e$ and its partitions generated by barycentric subdivisions of the complex to which $e$ belongs; see figure 4. Once the operator is constructed here, we obtain an operator on the entire system of simplicial complexes by gluing the individual operators in an obvious manner.

The generic problem is the projection

$$P : G^2 \rightarrow G; \quad (g_1, g_2) \rightarrow g_1 \cdot g_2,$$

(26)
which we now consider together with the induced map between cotangent bundles (20). Let $\{e_i\}$ be an orthonormal basis of $T^*_G G$ and denote by

$$e_i(g) = L_g e_i(id) \equiv e_i$$

the left-translated basis covectors at $g \in G$. The push-forward of the basis covector $e_i(g)$ in $T^*_{(g_1, g_2)} G$ by $P^*$ gives

$$P^* e_i (g_1 \cdot g_2) = (R_{g_2^{-1}} e_i(g), L_{g_2^{-1}} e_i(g)).$$

where $g = g_1 \cdot g_2$. This suggests a natural orthonormal basis of $T^*_{(g_1, g_2)} G^2$ given by

$$E^2_{2,i} = (E^1_i, \pm \hat{e}^i_i),$$

(27)
6 There is some freedom of choice here since the Dirac operator on $K_n$ need not involve the Levi-Civita connection. However, to obtain a self-adjoint operator, certain invariance properties must be satisfied [22].
Class. Quantum Grav. 26 (2009) 065011  J Aastrup et al

where
\[ E_1^i (g_1, g_2) = L_{g_1} R_{g_2} e_i (id), \quad E_2^i (g_2) = L_{g_2} e_i (id), \] (28)

and where \( s \) in (27) represents the appropriate sign combinations:
\[ s \in \{(+, +), (+, -)\} \]
characterizing the two orthonormal covectors. Denote by \( \hat{E}_1^i \) and \( \hat{E}_2^i,s \) the corresponding sections in \( TG \) and \( TG^2 \) respectively, defined via the inner product (21).

We consider a Dirac operator corresponding to \( G^2 \) of the form
\[ D_2 = \frac{1}{2} \sum_{i,j} E_1^j \cdot \nabla_2 \hat{E}_2^j,s, \] (29)
where \( \nabla_2 \) is a connection on \( T^*G^2 \). It turns out that the operator (29) satisfies the compatibility condition (24) if the connection \( \nabla_2 \) satisfies
\[ \nabla_2 (\hat{E}_1^i, 0) (E_1^i, 0) \equiv (\nabla^k_e \hat{E}_1^i, 0), \]
\[ \nabla_2 (0, \hat{E}_2^i) (0, E_2^i) \equiv (0, \nabla^k_e \hat{E}_2^i). \]

To obtain the general form of the Dirac-like operator on \( G^n \), we first define the following twisted covectors on \( G^n \):
\[ E_1^i (g_1) = L_{g_1} \cdots R_{g_n} R_{g_{n-1}} \cdots e_i (id) \]
\[ \vdots \]
\[ E_1^j (g_j) = L_{g_1} \cdots R_{g_{n-1}} R_{g_{n-2}} \cdots e_i (id) \]
\[ \vdots \]
\[ E_n^i (g_n) = L_{g_1} e_i (id). \] (30)

Next, we write
\[ E_{n,s}^i = (\hat{E}_1^i, \pm \hat{E}_2^i, \ldots, \pm \hat{E}_n^i), \]
where \( s = (+, \pm, \ldots, \pm) \) is the sequence of signs which characterizes the covector. Again, we denote by \( \hat{E}_{n,s}^i \) the corresponding sections in \( TG^n \). The global frames \( \hat{E}_{n,s}^i \) are found by repeated lifts of the covector \( e_i (g) \) on \( G \) to \( G^n \). Therefore, they are constructed to satisfy
\[ \nabla_2 P^* (\hat{E}_{n,s}^i) = P^* (\nabla^k_e \hat{E}_{n,s}^i), \]
where the sequence \( s \) is obtained from the sequence \( s' \) by replacing each sign in \( s' \) with the same sign twice, and where
\[ P : G^n \rightarrow G^{n/2}; \quad (g_1, \ldots, g_n) \rightarrow (g_1 \cdot g_2, \ldots, g_{n-1} \cdot g_n). \] (31)

The Dirac-like operator on \( G^n \) has the form
\[ D_n = \frac{1}{n} \sum_{i,j} E_{n,s}^i \cdot \nabla_2 \hat{E}_{n,s}^j, \] (32)
where the sum runs over \( i \) as well as all appropriate sign sequences \( s \).

We define the connections \( \nabla^n \) recursively. That is, the action of \( \nabla^n \) on basis covectors \( \hat{E}_{n,s}^i \) is given recursively and thereafter extended via linearity and the requirements of a derivation to the entire Clifford bundle. Thus, we require
\[ \nabla_{P^* (\hat{E})}^n (P^* \hat{E}) = P^* (\nabla_e \hat{E}), \]
\[ \nabla_{(P^* \hat{E})}^n (P^* \hat{E}) = 0. \] (33)
where $\mathcal{E}$ and $\hat{\mathcal{E}}$ are basis covectors and vectors, respectively, of the type $(E_1, \pm E_2, \ldots, \pm E_n/2)$ at the level $n$. In equation (33) and in the following, we denote by $(P^*\hat{\mathcal{E}})_\perp$ the general elements in the orthogonal complement to vectors (and covectors) of the form $P^*\hat{\mathcal{E}}$. We fix the remaining freedom in $\nabla^n$ with the additional condition

$$\nabla^n_{(P^*\hat{\mathcal{E}})_\perp}(P^*\hat{\mathcal{E}})_\perp = \nabla^n_{(P^*\hat{\mathcal{E}})_\perp}(P^*\bar{\mathcal{E}})_\perp = 0,$$

which is required for the construction of the trace; see section 4.1 and [22]. The properties given in (33) are again dictated by the requirement that the Dirac operator (32) is compatible with the projections.

The proof that the operator (32) satisfies the compatibility condition (24) is given in [22].

### 3.5. The general Dirac operator associated with $K_i$

The Dirac-type operator (32) is not the most general operator satisfying the requirements of compatibility with the structure maps (24). In fact, these requirements render substantial parts of the operator (32) free to modifications. This observation is closely related to the fact that the Dirac-type operator (32) will, as it stands, not descend to an operator with a compact resolvent in the inductive limit of repeated barycentric subdivisions. It turns out that the additional degrees of freedom are exactly the necessary leverage needed to obtain a well-behaved Dirac-type operator in the limit.

Let us start with the case $n = 2$. We observe that a rescaling of parts of the operator (29) according to

$$D_2 = \frac{1}{2} \sum_i \left( (E_1^i, E_2^i) \cdot \nabla^2_{(E_1^i, E_2^i)} + a_1 (E_1^i, -E_2^i) \cdot \nabla^2_{(E_1^i, -E_2^i)} \right),$$

(34)

where $a_1 \in \mathbb{R}$, does not affect the compatibility with the embedding (20). In the general case, the modification of (32) is as follows. Let $\{a_k\}$ be an infinite sequence of real numbers and put $a_0 = 1$. Consider a single edge $e$ and its subdivisions. Consider the $n$th subdivision. Let $s$ be a finite sequence of $n$ signs and denote by $\#(s)$ the number of '+'s in the beginning of the sequence. Define the number

$$m(s) = \log_2 \left( \frac{2^n}{\#(s)} \right).$$

The modified Dirac-type operator now has the form

$$D_n = \frac{1}{n} \sum_{s,i} a_{m(s)}^{n,i} \cdot \nabla^n_{E_i^s}.\]$$

(35)

Let us consider the eigenvalues of this operator. For simplicity, consider the Abelian case $G = U(1)$. If we define the product between a sequence $s = (+, \pm, \pm, \ldots)$ of signs with a sequence of real numbers $^7 (n_1, n_1, \ldots)$ by

$$(n_1, n_2, \ldots) \cdot s = n_1 \pm n_2 \pm \cdots,$$

(36)

where the signs on the rhs are read of the sequence $s$, then we can write the spectrum of the Dirac-type operator (35) as

$$\text{spec}(D) = \left\{ \pm \frac{1}{2} \sqrt{\sum_k a_{m(s)}^{2, n_1, n_2, \ldots}} \right\},$$

$^7$ Or, to be general, any sequence of objects.
where $s_k$ is the sequence of signs corresponding to the $k$th orthonormal vector $E^{n,s_k}$. In this case, the number of eigenvalues of the limit operator $D_\Delta = \lim_n D_n$ in a range $[0, \Lambda]$, $\Lambda < \infty$, is finite whenever
\[
a_n = 2^n b_n \quad \text{where} \quad \lim_{n \to \infty} b_n = \infty. \tag{37}
\]

The general case where $G$ is a non-Abelian Lie group is more complicated. The full analysis is given in [22] where we prove that for any compact Lie group $G$, there exists a sequence $\{a_i\}$ so that the number of eigenvalues within a finite range is finite up to a controllable degeneracy.

This result will be clarified in the following section.

4. The limit of the triple $(\mathcal{B}_n, D_n, \mathcal{H}_n)$

Up till now, we have considered a system of embedded, abstract simplicial complexes $\{\mathcal{K}_n\}$. To each simplicial complex $\mathcal{K}_n$, we introduced a space $\mathcal{A}_n$ which we interpreted as a coarse-grained version of a space of connections. The next step was to construct a spectral triple $(\mathcal{B}_n, D_n, \mathcal{H}_n)$ on each of the spaces $\mathcal{A}_n$. The spectral triple satisfies requirements of compatibility with the operation of barycentric subdivision. This means that we can take the limit of infinitely many barycentric subdivisions. The resulting triple, which is denoted by $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$, has the following elements.

First, the Hilbert space $\mathcal{H}_\Delta$ is constructed by adding all Hilbert spaces $\mathcal{H}_n$:
\[
\mathcal{H}_\Delta = \bigoplus_{K_n} L^2(G^{n(K)}, \text{Cl}(T^*G^{n(K)}) \otimes M_r(\mathbb{C}))/N, \tag{38}
\]
where $N$ is the subspace generated by elements of the form
\[
(\ldots, v, \ldots, -P_{ij}^*(v), \ldots),
\]
where $P_{ij}^*$ are maps between Hilbert spaces $\mathcal{H}_n$ induced by (23). The Hilbert space $\mathcal{H}_\Delta$ is the completion of $\mathcal{H}'$. The inner product on $\mathcal{H}_\Delta$ is the inductive limit inner product. This Hilbert space is manifestly separable. Next, the algebra
\[
\mathcal{B}_\Delta := \lim_{\mathcal{K}_n} \mathcal{B}_n \tag{39}
\]
contains loops defined on a simplicial complex $\mathcal{K}_n$ in $\{\mathcal{K}_n\}$ as well as their closure. Again, the algebra $\mathcal{B}_\Delta$ is separable. Finally, the Dirac-like operator $D_n$ descends to a densely defined operator on the limit space $\mathcal{H}_\Delta$:
\[
D_\Delta = \lim_{\mathcal{K}_n} D_n. \tag{40}
\]

4.1. A semi-finite spectral triple

The question is whether the triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$ satisfies the requirements of a spectral triple. Clearly, to have a true spectral triple is highly desirable, first of all to ensure that we are operating on a mathematically safe ground. Also, it gives access to the tools of noncommutative geometry.

To have a spectral triple $(B, D, H)$ where $B$ is a $C^*$-algebra represented on a Hilbert space $H$ on which an unbounded, self-adjoint operator $D$ acts means that the two requirements

(1) $(\lambda - D)^{-1}$, where $\lambda \notin \mathbb{R}$, is a compact operator;
(2) $[b, D]$, where $b \in B$, is a bounded operator and $B$ is a dense $*$-subalgebra of $B$. 

18
are satisfied. However, the triple \((B_\Delta, D_\Delta, \mathcal{H}_\Delta)\) is not spectral in this sense. The spectrum of \(D_\Delta\) involves a large degeneracy due to the infinite-dimensional Clifford bundle. This means that the resolvent of \(D\) will not be compact.

There exists, however, another sense in which a triple \((B, D, H)\) can be called spectral. If there exist a trace on the algebra containing \(B\) and the spectral projections of \(D\), then \((B, D, H)\) is called a semi-finite spectral triple if \((\lambda - D)^{-1}\) is compact with respect to this trace.

It is proven in [22] that there exists a sequence \(\{a_k\}\) with
\[
\lim_k a_k \to \infty,
\]
such that \((B_\Delta, D_\Delta, \mathcal{H}_\Delta)\) is a semi-finite spectral triple.

A semi-finite spectral triple can be thought of as a spectral triple which involves a redundant symmetry group. This symmetry group resembles a gauge group and one must integrate out the extra degrees of freedom. In the present case, the symmetry group is identified as the endomorphisms of the infinite-dimensional Clifford bundle. To deal with this redundancy, we first define a trace on the algebra containing \(B_\Delta\) as well as the spectral projections of \(D_\Delta\).

We will construct the trace at each level in the inductive system of simplicial complexes and subsequently take the limit of repeated barycentric subdivisions. For the construction we factorize the Hilbert space. Choose a global orthonormal frame in \(T^*\text{G}\). This choice gives rise to a decomposition
\[
L^2(G^n, M_I \otimes \text{Cl}(T^*G^n)) = L^2(G^n) \otimes M_I(\mathbb{C}) \otimes \text{Cl}(T^*_eG^n).
\]
We will construct the trace on the algebra
\[
C_n = K(L^2(G^n)) \otimes \text{End}(M_I) \otimes \text{End}(\text{Cl}(T^*_eG^n)),
\]
where \(K\) here denotes the compact operators.

Let \(\text{Tr}_\text{op}\) denote the operator trace on \(K(L^2(G^n))\). For each \(n\), we have the normalized trace \(\text{tr}\) on \(\text{End}(\text{Cl}(T^*_eG^n))\). We define the trace as
\[
\text{Tr} = \text{Tr}_\text{op} \otimes \text{Tr}_I \otimes \text{tr}.
\]
Note that the trace is independent of the choice of the global orthonormal frame in \(T^*\text{G}\), since a different choice of basis is given by a unitary transformation.

In [22], we prove that the limit
\[
C_\Delta := \lim C_n
\]
is a \(C^*\)-algebra and that the trace \(\text{Tr}\) gives a trace on \(C_\Delta\). Since \(B_\Delta\) is contained in the weak closure of \(C_\Delta\) the trace extends to a trace on \(B_\Delta\) as well.

The important point in this construction is the normalization of the trace \(\text{tr}\) on \(\text{End}(\text{Cl}(T^*_eG^n))\). To explain this, consider the step going from \(n\) to \(n + m\). Let \(P_\lambda\) be the spectral projection of \(D_n\) for the eigenvalue \(\lambda\). Going from \(n\) to \(n + m\) the spectral projection will, roughly speaking, be mapped to \(P_\lambda \otimes 1\). Thus, the size of the eigenspace \(\lambda\) grows in the same rate as the dimension of the Clifford bundle. To remedy this defect, we must ensure that
\[
\text{tr}(1) = 1,
\]
which is what the normalized trace does.

The proof that \((B_\Delta, D_\Delta, \mathcal{H}_\Delta)\) form a semi-finite spectral triple with respect to the trace (43) is given in [22]. It turns out that in general, the Dirac-type operator (40) may require a perturbation at each level in the inductive system. This perturbation deals with the fact that the operators \(D_n\) may, in general, have nontrivial kernels which obstructs the control of the eigenvalues of the limit operator \(D_\Delta\). These perturbations lift the entailed degeneracy, which would otherwise destroy the spectral properties of the triple. The required perturbation is, at
Figure 5. The embedding of the system \( \{ K_i \} \) into \( \mathcal{M} \) gives rise to an inductive system of triangulations \( \{ T_i \} \).

Each level in the projective system, bounded and does not affect the commutator between \( D_\triangle \) and the algebra \( \mathcal{B}_\triangle \) significantly. For the special case \( G = SU(2) \) we find that the operators \( D_n \) have trivial kernels and therefore that no perturbations are needed.

Let us end this section with a short discussion of the structure of the Hilbert space \( \mathcal{H}_\triangle \) and the role of the infinite-dimensional Clifford bundle. First, consider the decomposition in (42) and rewrite it in the suggestive form

\[
L^2(G^n) \otimes M_f(\mathbb{C}) \otimes Cl(T^*eG^n) = \mathcal{H}_{n,b} \otimes \mathcal{H}_{n,f},
\]

where \( \mathcal{H}_{n,b} = L^2(G^n) \otimes M_f(\mathbb{C}) \).

In the inductive limit, this leads to the decomposition

\[
\mathcal{H}_\Delta = \mathcal{H}_{\Delta,b} \otimes \mathcal{H}_{\Delta,f}.
\]

Next, we factorize the algebra \( \mathcal{H}_\Delta \) according to the above factorization of the Hilbert space. The result is

\[
\mathcal{C}_\Delta = \mathcal{K}(\mathcal{H}_{\Delta,b}) \otimes \mathcal{C},
\]

where

\[
\mathcal{C} = \lim_{\rightarrow n} \text{End}(Cl(T^*eG^n)),
\]

where the morphisms in this inductive system are the unital ones. In particular, \( \mathcal{C} \) is a UHF algebra and since \( Cl(T^*eG^n) \) has dimension \( 2^n \text{dim}(G) \), the \( C^* \)-algebra \( \mathcal{C} \) is isomorphic to the CAR algebra.

4.2. The space \( \overline{\mathcal{A}}^\Delta \) and generalized connections

We have already indicated that the space \( \overline{\mathcal{A}}^\Delta \) is a space of generalized connections. This means that it is a closure of the space \( \mathcal{A} \) of smooth \( G \)-connections. To see this, consider the embedding \( \phi \) of the simplicial complex \( K_i \) (see figure 5):

\[
T_i := \phi(K_i),
\]

where \( T_i \) is a triangulation of the manifold \( \mathcal{M} \). There is a natural map

\[
\chi_\Delta : \mathcal{A} \rightarrow \overline{\mathcal{A}}^\Delta, \quad \chi_\Delta(\nabla)(\phi(\epsilon_i)) = \text{Hol}(\nabla, \phi(\epsilon_i)),
\]

where \( \text{Hol}(\nabla, \phi(\epsilon_i)) \) is the holonomy of \( \nabla \) along the edge \( \phi(\epsilon_i) \) which now has a location in \( \mathcal{M} \) via the embedding \( \phi \). Further, if we are given two different connections \( \nabla_1, \nabla_2 \in \mathcal{A} \), they will differ in a point, say \( m \in \mathcal{M} \), and hence in a neighbourhood of \( m \). We can therefore choose a small, directed edge \( \phi(\epsilon_i) \) in a triangulation \( T_i \) that is sufficiently refined, so that \( \phi(\epsilon_i) \) lies in the neighbourhood of \( m \) where the connections \( \nabla_1 \) and \( \nabla_2 \) differ. Furthermore,
because the system of triangulations contains edges in all directions in $\mathcal{M}$, we can choose $\phi(\epsilon_i)$ so that

$$\text{Hol}(\nabla_1, \phi(\epsilon_i)) \neq \text{Hol}(\nabla_2, \phi(\epsilon_i)).$$

In other words, $\chi_\triangle$ is an embedding, and hence the terminology generalized connection is justified.

This means that we have successfully turned the map (14) into an injection by repeating the barycentric subdivisions of the simplicial complexes.

The identification of $\mathcal{A}_\triangle$ as a space of connections provides us with a new understanding of the spectral triple $(\mathcal{B}_\triangle, D_\triangle, \mathcal{H}_\triangle)$. First, as already mentioned, according to (44) the algebra of loops should be interpreted in terms of holonomy loops:

$$f_L(\nabla) \sim \text{Hol}(L, \nabla).$$

Since the Dirac-type operator $D_\triangle$ is a derivation on the space $\mathcal{A}_\triangle$, it should be understood in terms of a functional derivation operator:

$$D_\triangle \sim \frac{\delta}{\delta \nabla}$$

in some integrated sense which shall become clearer soon. Finally, elements in the Hilbert space $\mathcal{H}_\triangle$ are functions over field configurations of connections and the inner product in $\mathcal{H}_\triangle$ comes in the form of a functional integral:

$$\langle \Psi(\nabla) | \ldots | \Psi(\nabla) \rangle \sim \int_{\mathcal{A}_\triangle} [d\nabla] \ldots, \quad \Psi \in \mathcal{H}_\triangle,$$  \hspace{1cm} (45)

that is, an integral\(^8\) over $\mathcal{A}_\triangle$.

All together, it is clear that the construction should be interpreted in terms of quantum field theory.

Note that the diffeomorphism group $\text{diff}(\mathcal{M})$ has no natural action on $\mathcal{A}_\triangle$ or the algebra $\mathcal{B}_\triangle$. We shall comment on this fact in section 6.2.

### 4.3. Gauge transformations

It remains to clarify whether the Dirac-type operator $D_\triangle$ is gauge invariant. To this end let $U$ be an element of the gauge group $\mathcal{G}$ of $M \times P$, i.e. $U : M \to G$ is a smooth function. Given a connection $\nabla \in \mathcal{A}$, $U$ induces a gauge-transformed connection $\tilde{\nabla}$. Given a path $L$ with startpoint $x_1$ and endpoint $x_2$, the holonomy along $L$ transforms according to

$$\text{Hol}(\nabla, L) \to U(x_1) \text{Hol}(\nabla, L) U^*(x_2).$$

To determine the properties of the Dirac-type operator $D_\triangle$ when subjected to a gauge transformation, we first consider the case $\mathcal{A}_n = G^n$ that corresponds to $n$ divisions of a single edge. Consider the general transformation

$$U_n : G^n \to G^n,$$  \hspace{1cm} (46)

$$\begin{pmatrix} g_1, g_2, \ldots, g_n \end{pmatrix} \to \begin{pmatrix} u_0 g_1 u_1^{-1}, u_1 g_2 u_2^{-1}, \ldots, u_{n-1} g_n u_n^{-1} \end{pmatrix},$$

where $u_0, u_1, \ldots, u_n$ are unitary group elements in $G$. This transformation generates a map

$$U_n : L^2(G^n, \text{Cl}(T^*G^n)) \to L^2(G^n, \text{Cl}(T^*G^n))$$

between Hilbert spaces. We need to check whether

$$D_\triangle \xi = U_n D_n U_n^* \xi, \quad \xi \in L^2(G^n, \text{Cl}(T^*G^n)).$$

\hspace{1cm} (46)

\(^8\) This integral resembles functional integrals found elsewhere in physics. First, it is similar to the inner product on the Hilbert space $\mathcal{H}_{\text{diff}}$ of diffeomorphism-invariant states in loop quantum gravity; see below. Second, it also resembles functional integrals in lattice gauge theories. Here, the main difference is the ‘lattice spacing’ $a$ in lattice gauge theories which gives the continuum limit $a \to 0$.\hspace{1cm}
In [22], we find that (46) holds whenever the connections used to construct $D_n$ satisfy a certain gauge compatibility condition. In particular, we find that the special flat connections entering the construction of $D_n$ do satisfy this condition. This, in turn, implies that the full Dirac-type operator $D_\Delta$ is gauge invariant. For the full analysis, we refer the reader to [22].

4.4. The commutator between $D_\Delta$ and the algebra $B_\Delta$

Section 5 is concerned with the relation between the Poisson algebra of general relativity and the algebra $B_\Delta$. The point is that the interaction between the Dirac-type operator $D_\Delta$ and the algebra $B_\Delta$ has the same structure as the interaction between conjugate variables of gravity. Before we show this we need to calculate the commutator between the operator (35) and a loop operator.

First, consider the simple case where $L_j$ corresponds to the function on $G^n$:

$$f_{L_j}: (g_1, \ldots, g_n) \rightarrow g_j.$$  

Thus, $L_j$ is really just the $j$th edge. We wish to calculate the commutator

$$[d\hat{E}_{ij}, f_{L_j}]\xi(g_1, \ldots, g_n),$$

where $\xi \in L^2(G^n)$. If we denote by $e_i$ the generators of the Lie algebra $g$, then we introduce the twisted generators of $g$:

$$E_{ij} = g_{j+1}g_{j+2} \cdots g_ng^{-1}e_i g^{-1}_n \cdots g_{j+1}^{-1}$$

corresponding to $\hat{E}_{ij}$. We now calculate

$$[d\hat{E}_{ij}, f_{L_j}]\xi(g_1, \ldots, g_n) = \left( d\hat{E}_{ij}f_{L_j} \right) \xi(g_1, \ldots, g_n) = \frac{d}{dt} (g_1 \cdots g_n \exp (te_i) g^{-1}_n \cdots g_{j+1}^{-1}) \xi(g_1, \ldots, g_n) = g_j E_{ij} \xi(g_1, \ldots, g_n).$$

(47)

It is important to note that product in the last line of (47) is a matrix multiplication.

Equation (47) implies that

$$[D_n, f_{L_j}] = \frac{1}{n} \sum_{s,i} \pm a_{m(s)} E_{i}^{n,s} \cdot g_j E_{ij}.$$  

(48)

The sign on the rhs of (48) corresponds to the $j$th sign in the sequence $s$. Thus, for $i = 1$ all the signs are positive or else the total number of $+$’s and $-$’s is equal.

Next, the commutator between $D_\Delta$ and a general loop $L$

$$f_{L}: (g_1, \ldots, g_n) \rightarrow g_1g_2 \cdots g_n$$

simply consists of repeated applications of (48) according to

$$[D_\Delta, f_L] = [D_\Delta, f_{L_1}] g_2 \cdots g_n + g_1 [D_\Delta, f_{L_2}] \cdots g_n + \cdots,$$

(49)

where each commutator on the rhs is calculated by inserting the appropriate operator $D_n$ corresponding to the level of refinement given by the loop $L$.

Thus, the action of $D_\Delta$ on a single loop is to insert the ‘twisted’ generators $E_{ij}$ of the Lie algebra into the loop at each vertex the loop passes through and to multiply with an appropriate element in the Clifford bundle.

9 We here consider the subdivision of an edge into $n$ segments. Therefore, $D_n$ is the operator given by (35).
Figure 6. The role of the parameters \( \{ a_i \} \) is to weight the different segments of a given loop according to the refinement of the segments.

4.5. The role of the sequence \( \{ a_n \} \)

Primarily, the role of the sequence \( \{ a_n \} \) is to shift an otherwise impossible degeneracy in the spectrum of the operator \( D_\triangle \) via condition (41). The parameters \( a_i \) introduce a hierarchy between the eigenvalues of \( D_\triangle \) on the infinitely many copies of \( G \) in \( \mathcal{A}_\triangle \). This hierarchy is closely related to the scaling behaviour of the construction.

First, consider a line segment divided into two, corresponding to the projection \( P : G^2 \rightarrow G \).

Functions on \( G^2 \) naturally fall into two classes: pull-backs of functions on \( G \)

\[
P^* : L^2(G) \rightarrow L^2(G^2)
\]

and the orthogonal complement hereof. We denote the former by \( V \) and the latter by \( V_\perp \equiv (P^*L^2(G))_\perp \). The functions in \( V \) correspond to information which is also contained in the less refined picture which involves only one copy of \( G \). This is the simplest and most coarse-grained description of parallel transports along the line segment available. The function in \( V_\perp \), on the other hand, contains additional information which cannot be traced back to the simpler picture. Each additional division of the line segment refines the picture further.

It is clear that any division of a line segment into two will involve new segments which are shorter—indepedent of any choice of metric—than the original segment. Therefore, functions in \( V_\perp \) correspond to information about \( \mathcal{A}_\triangle \) which is deeper, i.e. at a shorter distance, compared to information carried by functions in \( V \).

The role of the sequence \( \{ a_i \} \) is to take this scaling behaviour into account. The operator \( D_\triangle \) weights functions on \( \mathcal{A}_\triangle \) according to where in the projective system of spaces \( A_i \) the information originates (see figure 6). Therefore, if a function \( \psi \in L^2(\mathcal{A}_\triangle) \) is the pull-back of functions on \( L^2(A_i) \), then the corresponding eigenvalues of \( \psi \) are weighted with the appropriate parameter \( a_i \).

Thus, if we wish to probe \( \mathcal{A}_\triangle \) at a very short distance, this information will come with a very high weight factor \( a_i \) corresponding to high energy. On the other hand, if \( \mathcal{A}_\triangle \) is probed at a more coarse-grained level, then the corresponding eigenvalues are weighted with smaller weights \( a_i \).

Let us end this subsection with a curious observation. If we start with a metre, then we find that it takes about 116 subdivisions of the meter to reach the Planck length of \( 1.6 \times 10^{-35} \) m. This then corresponds to 116 of the parameters \( \{ a_i \} \). Thus, although the sequence \( \{ a_i \} \) is infinite, the number of parameters involved when probing physical scales is certainly finite.

5. Link to canonical quantum gravity I. Operator brackets

In the following two sections, we relate the construction of the spectral triple \( (B_\triangle, D_\triangle, \mathcal{H}_\triangle) \) to canonical quantization of field theories and in particular to canonical quantum gravity.
The first section is concerned with the Poisson brackets of general relativity. We first introduce the formulation of general relativity based on Ashtekar variables. Next, we show that the interaction between the Dirac-type operator $D_\triangle$ with the algebra $B_\triangle$ reproduces the structure of the Poisson brackets of general relativity when these are formulated in terms of loop variables. This means that the triple $(B_\triangle, D_\triangle, \mathcal{H}_\triangle)$ contains information which is tantamount to a representation of the Poisson algebra of general relativity.

Section 6 is primarily concerned with a more detailed comparison between the construction presented in this paper (the choice of graphs, the Hilbert space) and the study of spaces of connections within loop quantum gravity. It turns out that the key difference between the two lies in the treatment of the diffeomorphism group. We find that the Hilbert space $\mathcal{H}_\triangle$ is closely related to the Hilbert space of diffeomorphism-invariant states known from loop quantum gravity. The difference between the two seems to be a group of automorphisms of all the subdivisions of the triangulation. This means that the representation of the Poisson brackets given by the triple $(B_\triangle, D_\triangle, \mathcal{H}_\triangle)$ includes partial diffeomorphism invariance as a first principle.

For the remaining part of this section we will, unless otherwise stated, restrict ourselves to $G = SU(2)$ and dimension 3.

5.1. Canonical gravity

For an introduction to canonical gravity, see for example [28, 29]. Let us first fix some notation. We follow [29, 30]. Consider the vierbein formulation of general relativity where $E^A_\mu$ is the vierbein and $g_{\mu \nu} = E^A_\mu E^B_\nu \eta_{AB}$ is the corresponding spacetime metric where $\eta_{AB} = \{-1, 1, 1, 1\}$. Here $\mu, \nu, \ldots$ and $A, B, \ldots$ denote curved and flat spacetime indices, respectively. We assume that spacetime can be foliated according to $\mathcal{M} = \Sigma \times \mathbb{R}$ where $\Sigma$ is a spatial manifold. Let $m, n, \ldots$ and $a, b, \ldots$ denote curved and flat spatial indices, respectively. Denote by $e^m_a$ the spatial dreibein. The spin connection $\omega_{\mu AB}$ is given by

$$\omega_{\mu AB} = E^A_\nu \nabla_\mu E^B_\nu,$$

where $\nabla_\mu$ is the covariant derivative which involves the Christoffel connection.

The canonical momenta $\Pi^m_a$ corresponding to the dreibein are obtained from the Einstein–Hilbert Lagrangian $\mathcal{L}$ [29]:

$$\Pi^m_a := \frac{\delta \mathcal{L}}{\delta (\partial_t e^m_a)} = \frac{1}{2} e e^m_b (K_{ab} - \delta_{ab} K),$$

where $e = \det (e^m_a)$ and $K_{ab} = \omega_{ab0}^0$ is the extrinsic curvature. We write $K = K_{aa}$. The non-vanishing canonical Poisson brackets read as

$$\{e^m_a(x), \Pi^b_n(y)\} = \delta_{ab} \delta^m_n \delta^{(3)}(x, y).$$

The Hamiltonian formulation of gravity involves two constraints\(^{10}\) related to the symmetries of the theory: the diffeomorphism constraint corresponding to spatial diffeomorphisms within $\Sigma$ and the Hamiltonian constraint encoding the full four-dimensional diffeomorphism invariance and thus containing the dynamics of general relativity. The diffeomorphism and Hamiltonian constraints correspond to four of the ten Einstein field equations. The Hamiltonian itself is a linear combination of the two constraints.

Next, we perform a change of variables from the spatial spin connection and dreibein field to the connection

$$A_{ma} := -\frac{1}{2} \epsilon_{abc} \omega_{mbc} + \gamma K_{ma},$$

\(^{10}\)The connection formalism (see below) involves the additional Gauss constraint which corresponds to gauge invariance.
where $\epsilon_{abc}$ is the totally antisymmetric symbol, and to the inverse densitized spatial dreibein $\tilde{E}_a^m(x)$, with the Poisson brackets (all other vanish)
\[ \{ A_a^m(x), \tilde{E}_b^m(y) \} = \gamma \delta_a^b \delta^m N^3(x, y). \] (52)

Often the variables $\{ \tilde{E}, A \}$ are contracted with Pauli matrices $\tau^a$ according to
\[ \tilde{E}_a^{\alpha \beta} := \tilde{E}_a^m \tau^a \tau^{\alpha \beta}, \quad A_m^{\alpha \beta} := A_m \tau^{\alpha \beta} \]
in order to replace Lorentz indices by spinorial $SU(2)$ indices.

The variables $\{ \tilde{E}, A \}$ are the well-known Ashtekar variables [16, 17] and the parameter $\gamma \neq 0$ in (51) is known as the ‘Barbero–Immirzi parameter’ [31, 32].

The identification of a gauge connection $A_{m\alpha}$ as a primary variable of general relativity permits applications of techniques from Yang–Mills theory. In particular, one might shift focus from connections to their holonomies [11]
\[ \text{Hol}_C(A) = \mathcal{P} \exp \int_C A, \]
where $C$ is a curve in $\Sigma$, and express the Poisson brackets in terms of holonomies and a set of conjugate variables. To find a suitable choice of conjugate variables pick any two-dimensional surface $S$ in $\Sigma$ and define the flux vector
\[ F_S^a(\tilde{E}) := \int_S dF^a, \]
where the area element $\text{d}F^a$ is given by
\[ \text{d}F^a = \epsilon^{mnp} \tilde{E}_m \text{d}x^n \wedge \text{d}x^p. \] (53)

To obtain the Poisson brackets, pick a curve $C = C_1 \cdot C_2$ which intersects $S$ at the single point $C_1 \cap C_2$. Then the Poisson bracket between the new variables reads as [33]
\[ \{ F_S^a(\tilde{E}), \text{Hol}_C(A) \} = -\iota(C, S) \gamma \text{Hol}_{C_1}(A) \tau^a \text{Hol}_{C_2}(A), \] (54)
where $\iota(C, S) = \pm 1$ or 0 encodes information about the intersection of $S$ and $C$ ($\iota(C, S)$ vanishes when $C$ and $S$ do not intersect).

Note the similarity between the bracket (54) and the commutators (48) and (49) between the loop algebra $B_\Delta$ and the Dirac-type operator $D_\Delta$. Both sets of commutators prescribe an insertion of an element in the Lie algebra into the loop or curve at an ‘intersection point’: either at the intersection between the surface $S$ and the curve $C$ or at the vertex between neighbouring edges in a given simplicial complex. To investigate this correspondence we first need to consider the canonical quantization approach to quantum gravity.

### 5.2. Canonical quantum gravity

When the canonical quantization procedure is applied to gravity, the philosophy is to rewrite Poisson brackets such as (50), (52) or (54) as operator brackets and represent the canonical variables on a suitable Hilbert space as multiplication and derivation operators. Clearly, there is a freedom in choice as to which variables should be represented as multiplication operators and which should be represented as differential operators. Using the new variables, it is possible to represent the holonomies $\text{Hol}_C(A)$ as multiplication operators:
\[ \text{Hol}_C(A) \rightarrow \mathbf{C}, \]
and the corresponding triad and flux variables $\tilde{E}$ and $F$ as differential operators
\[ \tilde{E}_a^m(x) \rightarrow E_a^m(x) = \frac{\hbar}{i} \frac{\delta}{\delta A_a^m(x)}, \quad F_S^a \rightarrow F_S^a = \int_S \epsilon_{mnp} E_m^a \text{d}x^n \wedge \text{d}x^p \] (55)
on a suitable Hilbert space corresponding to a configuration space of connections. In the following, we refer to the setup used in loop quantum gravity. We postpone to section 6 the construction of the Hilbert space on which the operators $C$ and $F$ are represented within loop quantum gravity. For now, it suffices to state that the Hilbert space is based on a projective system of piecewise analytic graphs. The bracket

$$[F_a^S, C] = \pm \gamma C_1 \tau^a C_2,$$

(56)

where $C = C_1 \cdot C_2$ is piecewise analytic, is then defined as an operator bracket acting on a Hilbert space of functions over a configuration space of connections.

We wish to show that the spectral triple $(B_\Delta, D_\Delta, H_\Delta)$ reproduces the structure of the operator bracket (56). To do this, we will use the operators $F_k^i$ to construct a new operator $D$. The commutator between this operator and an operator $C$ is then shown to be identical to the commutator between the Dirac-type operator $D_\Delta$ and an element of the algebra $B_\Delta$.

To proceed, we restrict the algebra (56) to an inductive system of simplicial complexes $\{K_i\}$ and an embedding hereof $\phi(K_0) = T_i \in M$. Thus, we consider only curves $C_i$ which coincide with edges $e_j$ in some triangulation $T_i$. To simplify matters further, we first consider a single edge $e$ in the initial complex $K_0$. This edge corresponds to a sequence of edges $\{e_1, e_2, \ldots, e_{2N}\}$ in the $N$th triangulation $T_N$ arising through $N$ barycentric subdivisions; see figure 7. For simplicity, we set $\gamma = 1$. For each edge $e_j$, there exists a set of sections $\hat{E}_j^i(g_1, \ldots, g_n)$ in the tangent bundle of the $j$th copy of $G$; see section 3.4. Expand the corresponding generators $E_j^i$ in terms of Pauli matrices:

$$E_j^i = b_{i,a} \tau^a,$$

and define the new operators

$$F_j^i = \sum_a b_{i,a} F_{S_j}^a,$$

where we introduce a set of surfaces $\{S_k\}$ chosen so that $S_k$ intersects the triangulation $T_i$ at its vertices only. Choose the numbering of the surfaces so that $S_k$ intersects the edge $e_k$ at its endpoint (see figure 8). Then we obtain (no summation over repeated indices)

$$[F_j^i, C_j] = C_j E_j^i.$$

(57)

We choose $S_k$ in a way so that the sign in (57) is always positive when the orientation of the curve $C_k$ coincides with the orientation of the triangulation.

The commutator (57) has the same structure as the commutator (47) between the vector field $\hat{E}_j^i$ and the group element corresponding to the $j$th copy of $G$ in $G^n$. This means that the vector field $\hat{E}_j^i$ corresponds to the flux operator $F_j^i$:

$$F_j^i \leftrightarrow \hat{E}_j^i.$$

(58)

11 To be exact, we here assume that the triangulation $T_i$ is piecewise analytic.
Next, we define the operator
\[
D_n = \frac{1}{n} \sum s_i a_{m(s)} E_{n,s}^i \cdot \left( (F_1, F_2, F_3, \ldots) \cdot s \right),
\]
(59)
where \( s \) is the sequence of signs \((+, \pm, \pm, \ldots)\) corresponding to the vector \( E_{n,s}^i \), and \( \cdot \) in the bracket to the right is defined in (36). Then, the final commutator reads as
\[
[D_n, C_j] = \frac{1}{n} \sum s_i a_{m(s)} (\pm C_j E_{j}^i) \cdot E_{n,s}^i.
\]
(60)

In (60), the sign on the rhs is again given by the \( i \)th entries in the sequences \( s \). Commutator (60) has precisely the same form as the commutator (48). This shows that the operator \( D_n \) corresponds exactly to the Dirac operator \( D_n \) defined in equation (35):
\[
D_n \leftrightarrow D_n.
\]

Also note that the choice of surfaces \( S_k \) has no importance for the definition of (59) but only the intersection points and the vertices count. Therefore, the surfaces \( S_k \) serve merely as labels of the vertices.

The generalization to the full picture where we consider the complete set of edges \( \{ \epsilon_k \} \) in \( T_i \) is straightforward. The corresponding operator, which we again denote by \( D_n \), is simply built from the operators associated with each edge \( \epsilon_i \) in the initial triangulation \( T_0 \). Further, we denote the limiting operator with \( D_\Delta \). Again, the message is that the operator \( D_\Delta \) corresponds exactly to the Dirac-like operator \( D_n \).

This shows that the Dirac-type operator \( D_\Delta \) has, in terms of loop quantum gravity, the form of a sum of flux operators. Thus, the interaction between \( D_\Delta \) and the loop algebra \( B_\Delta \) contains information which corresponds to a representation of the Poisson algebra of general relativity.

Put differently, if we consider the loop algebra \( B_\Delta \) together with the vectors \( E_{j}^i \), then this set of variables corresponds to the conjugate set of flux and loop variables used in loop quantum gravity, with the exception that this new set of variables are restricted to a specific inductive system of graphs \( \{ T_i \} \). It is in this sense that the spectral triple \( (B_\Delta, D_\Delta, H_\Delta) \) involves information which is tantamount to a representation of the Poisson algebra of general relativity.

In which sense does this alternative representation of the Poisson algebra (54) given by the triple \( (B_\Delta, D_\Delta, H_\Delta) \) differs by the representation used in loop quantum gravity? To answer this question, we need a precise comparison of the general setup used in this paper with the setup used in loop quantum gravity. This will be the topic of the following section.
For later reference, we need an expression for the square of the Dirac-like operator (59). We write
\[ D^2_\Delta = \sum_{ij} c_{ij} F^i_j F^i_j + \text{lower order}, \]
where \( c_{ij} \) are constants depending on the parameters \( \{a_i\} \) and where we, by lower order, refer to terms which are linear in \( F^i_j \).

6. Link to canonical quantum gravity: II. Diffeomorphisms

In this section, we first introduce the inductive system of piecewise analytic graphs applied in loop quantum gravity. This system entails a space of generalized connections much alike the space \( \mathcal{A}_\Delta \) constructed in this paper. We find that the essential difference between the two spaces is separability. This issue is closely related to which subgroups of the diffeomorphism group are acting on the two spaces respectively.

It is therefore important to understand the role of the diffeomorphism constraint in our setting and to relate it to the diffeomorphism constraint in loop quantum gravity. We find that the Hilbert space \( \mathcal{H}_\Delta \) resembles, up to an automorphism group of the infinite triangulation, the Hilbert space of diffeomorphism-invariant states in loop quantum gravity. However, at present we do not know how mathematically rigorously to define this group of automorphisms.

6.1. Functional spaces of connections in loop quantum gravity

The following is a review of material known in loop quantum gravity. In particular, we refer to the publications [18, 19, 34].

Let \( M \) be a real analytic manifold and let \( \mathcal{P} \) be the space of piecewise analytic-directed paths on \( M \). We will consider two paths in \( \mathcal{P} \) to be the same if they differ by trivial backtracking. \( \mathcal{P} \) has a product simply by composing paths. We define the space of generalized \( G \)-connections by
\[ \mathcal{A}^\mathcal{P} = \text{Hom}(\mathcal{P}, G), \]
where \( \text{Hom} \) means maps \( \nabla \) from \( \mathcal{P} \) to \( G \) satisfying
\[ \nabla(P_1 P_2) = \nabla(P_1) \nabla(P_2). \]
The space \( \mathcal{A}^\mathcal{P} \) was studied in [19].

Again, we consider the space \( \mathcal{A} \) of smooth \( G \)-connections. Like in section 4.2: there is a natural embedding \( \chi_a : \mathcal{A} \to \mathcal{A}^\mathcal{P} \) given by
\[ \chi_a(\nabla)(P) = \text{Hol}(\nabla, P), \]
where \( \text{Hol}(\nabla, P) \) denoted the holonomy of \( \nabla \) along \( P \).

6.1.1. Projective structure on \( \mathcal{A}^\mathcal{P} \). On \( M \), we will consider the system of connected piecewise analytic graphs. This system is directed under inclusions of graphs. Given a piecewise connected analytic graph \( \Gamma \) in \( M \), we denote by \( \{\epsilon_1, \ldots, \epsilon_{n(\Gamma')}\} \) the edges and by \( \{v_1, \ldots, v_{m(\Gamma')}\} \) the vertices. Let \( \mathcal{P}_\Gamma \) be the set of paths in \( \Gamma \) and define
\[ \mathcal{A}_\Gamma = \text{Hom}(\mathcal{P}_\Gamma, G). \]

An inclusion of graphs \( \Gamma' \subset \Gamma \) induces a projection
\[ P_{\Gamma'\Gamma} : \mathcal{A}_\Gamma \to \mathcal{A}_{\Gamma'}. \]
Since every analytic path is a path in a piecewise analytic graph and vice versa, we get

$$\mathcal{A}^\circ = \lim_{\Gamma} A^\Gamma,$$

where the projective limit is taken over all piecewise analytic graphs.

If we choose an orientation of the edges \(\{\ell_1, \ldots, \ell_n(\Gamma)\}\), we again obtain an identification

$$A^\Gamma \simeq G^n(\Gamma)$$

via the map

$$A^\Gamma \ni \nabla \mapsto (\nabla(\ell_1), \ldots, \nabla(\ell_n(\Gamma))) \in G^n(\Gamma).$$

This identification gives rise to various structures on \(A^\Gamma\) and therefore also on \(\mathcal{A}^\circ\), for example the topological structure given by the topological structure of \(G^n(\Gamma)\). Also, the projections \(P_{\Gamma^\Gamma}\) are continuous and hence give a topological structure on the projective limit, i.e. \(\mathcal{A}^\circ\). We note that \(\mathcal{A}\) is dense in \(\mathcal{A}^\circ\); see [22] and references therein.

Another structure arising from this identification is the measure or the more relevant Hilbert space structure. Since \(G\) is a compact group, we can equip \(G^n(\Gamma)\) uniquely with the product Haar measure and define

$$L^2(A^\Gamma) = L^2(G^n(\Gamma)).$$

The projections \(P_{\Gamma^\Gamma}\) induce embeddings of Hilbert spaces:

$$P_{\Gamma^\Gamma}^* : L^2(A^\Gamma) \to L^2(A^\Gamma).$$

With this, we then define

$$L^2(\mathcal{A}^\circ) = \lim_{\rightarrow} L^2(A^\Gamma),$$

which is non-separable.

The Hilbert space \(L^2(\mathcal{A}^\circ)\) is the Hilbert space that carries the representation of the Poisson algebra (56) used in loop quantum gravity. It is known as the kinematical Hilbert space.

6.1.2. Actions of the diffeomorphism group. Since an analytic diffeomorphism of \(M\) maps \(\mathcal{P}\) to \(\mathcal{P}\) we get an action of the group of analytic diffeomorphisms \(\text{Diff}_a(M)\) on \(\mathcal{A}^\circ\). On the other hand, the group of all diffeomorphisms \(\text{Diff}(M)\) acts on \(\mathcal{A}\) but does not extend to an action on \(\mathcal{A}^\circ\). There are several different ways of completing \(\mathcal{A}\) depending on the choice of ‘lattice’, for example piecewise analytic paths like in the case described above, various kinds of smooth paths, where one gets an action of the full diffeomorphism group. See [35, 36] for results and a thorough discussion. All the different completions contain \(\mathcal{A}\), but the crucial difference seems to be the size of the symmetry group.

6.2. Comparing the spaces \(\overline{\mathcal{A}}^\circ\) and \(\overline{\mathcal{A}}\)

So far we have introduced the three spaces \(\mathcal{A}\), \(\overline{\mathcal{A}}^\circ\) and \(\overline{\mathcal{A}}\). \(\mathcal{A}\) is the space of smooth \(G\)-connections, and we demonstrated that

$$\mathcal{A} \hookrightarrow \overline{\mathcal{A}}^\circ, \quad \mathcal{A} \hookrightarrow \overline{\mathcal{A}},$$

which means that both \(\overline{\mathcal{A}}^\circ\) and \(\overline{\mathcal{A}}\) are spaces of generalized connections. Furthermore, since it is known that \(\mathcal{A}\) is dense in \(\overline{\mathcal{A}}\), and since there exists a natural surjection

$$\sigma : \overline{\mathcal{A}} \to \overline{\mathcal{A}}^\circ,$$
we know that $A$, too, is dense in $\mathcal{A}^\triangle$. To see how the surjection $\sigma$ works, recall that both $\mathcal{A}^\triangle$ and $\mathcal{A}^a$ are spaces of homomorphisms:

$$\mathcal{A}^\triangle = \text{Hom}(\mathcal{P}_{\triangle}, G), \quad \mathcal{A}^a = \text{Hom}(\mathcal{P}_a, G),$$

and since$^{12}$\{\mathcal{P}_{\triangle}\} $\hookrightarrow$ \{\mathcal{P}_a\}, $\sigma$ is just the corresponding surjection between the spaces (64) of homomorphisms.

To be precise, we should state that we now think of the space $\mathcal{A}^\triangle$ in terms of an embedding \(\phi : K_i \to T_i\) of the projective system \{\mathcal{K}_i\}, and that we therefore identify

$$\mathcal{A}^\triangle = \lim_{\rightarrow} A_\mathcal{T}.$$  

In the following, we will demonstrate the relationship between these three spaces of connections.

First, it is important to see that the shift from $A$ to a larger space of generalized connections is necessary to equip $A$ with topological and Hilbert space structures. The identification of $\mathcal{A}^\triangle$ and $\mathcal{A}^a$ as pro-manifolds is the key step to obtain this. The space $A$ itself is not a pro-manifold.

The key difference between the two spaces $\mathcal{A}^\triangle$ and $\mathcal{A}^a$ is that $\mathcal{A}^a$ has an action of the (analytic) diffeomorphism group; $\mathcal{A}^\triangle$ does not. Further, the Hilbert space structure associated with the two spaces are respectively non-separable for $\mathcal{A}^a$ and separable for $\mathcal{A}^\triangle$. That is, the way $A$ is completed is decisive for how large a symmetry group remains. This shows that there is a direct link between separability of the Hilbert space structure and the action of the diffeomorphism group.

The following diagram illustrates the relationship among the three spaces of connections:

\[
\begin{array}{ccc}
\mathcal{A}^\triangle & \xleftarrow{\text{diff}(\mathcal{M})} & \mathcal{A}^a \\
& \sigma \downarrow & \\
\mathcal{A} & \xleftarrow{\text{diff}(\triangle)} & \mathcal{A}^\triangle
\end{array}
\]

where $\text{diff}(\triangle)$ is an automorphism group of the infinite triangulation underlying $\mathcal{A}^\triangle$. Below, we will discuss what properties this group should have.

Before we return to $\text{diff}(\triangle)$ we find it worth noting that the construction of the Dirac-like operator (40) requires a shift from $\mathcal{A}^a$ to $\mathcal{A}^\triangle$. In [20], the authors attempted to construct such an operator on $\mathcal{A}^a$ and a corresponding Hilbert space. The attempt was unsuccessful exactly because the number of possible projections (63) is too large to permit a Hilbert space structure carrying a Dirac-like operator similar to (40). The shift from $\mathcal{A}^a$ to $\mathcal{A}^\triangle$ reduces sufficiently the type of embeddings between graphs. We appear to find the following hierarchy:

\[
\begin{array}{cc}
A: & \text{No Hilbert space structure} \\
& \text{No Dirac-like operator} \\
& \text{Action of $\text{diff}(M)$} \\
\mathcal{A}^\triangle: & \text{Hilbert space structure, non-separable} \\
& \text{No Dirac-like operator} \\
& \text{Action of analytic diffeomorphisms}
\end{array}
\]

$^{12}$ We assume here that the edges in the triangulations $T_i$ are analytic. This depends on the embedding $\phi : K_i \to T_i$ but is no real restriction.
\( \mathcal{A}^\triangle \): Hilbert space structure, separable

Dirac-like operator

Action of \( \text{diff}(\triangle) \).

One way to think of the space \( \mathcal{A}^\triangle \) is to see it as the space of (smooth) connections subjected to a sort of gauge fixing of the diffeomorphism group\(^{13}\). Here, gauge fixing is meant in the sense that a symmetry group is (partly) removed while the integration at hand still involves the entire space, in this case the space of smooth connections \( \mathcal{A} \). In this sense, the inner product (45) resembles a functional integral over \( \mathcal{A} \) ‘up to diffeomorphisms’.

Another way to think of the space \( \mathcal{A}^\triangle \) is to relate it to the construction of the Hilbert space \( \mathcal{H}_{\text{diff}} \) of diffeomorphism-invariant states in loop quantum gravity, see [23]. To do this, we first note that each triangulation in particular is a piecewise analytic graph, provided \( T \) is a piecewise analytic triangulation, and we hence get an embedding

\[
\text{lim} \ L^2(\mathcal{A}_T) \xrightarrow{\iota} \text{lim} \ L^2(\mathcal{A}_T).
\]

On the other hand, consider the Hilbert space formally defined by

\[
\mathcal{H}_{\text{diff}} = \text{diffeomorphism-invariant states in } \text{lim} \ L^2(\mathcal{A}_T).
\]

The elaborate definition is written in [23]. There is a surjection

\[
\text{lim} \ L^2(\mathcal{A}_T) \xrightarrow{q} \mathcal{H}_{\text{diff}}
\]

given by meaning vectors over the action of the diffeomorphism group. We thus get a map

\[
\text{lim} \ L^2(\mathcal{A}_T) \xrightarrow{\Xi} \mathcal{H}_{\text{diff}}
\]

by composing. This map is also going to be a surjection since each graph can, via a diffeomorphism, be mapped into a triangulated graph\(^{14}\). The map is not injective because of the symmetries of the triangulated graph.

We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{lim} \ L^2(\mathcal{A}_T) & \xrightarrow{i} & L^2(\mathcal{A}_T) \\
\downarrow & & \downarrow q \\
\text{lim} \ L^2(\mathcal{A}_T) & \xrightarrow{\Xi} & \mathcal{H}_{\text{diff}}
\end{array}
\]

The amount by which the map \( \Xi \) fails to be injective should exactly be the group \( \text{diff}(\triangle) \). Thus, one property of \( \text{diff}(\triangle) \) should be that whenever two subgraphs of the infinite triangulation can be mapped to each other with an analytic diffeomorphism, then there shall be an automorphism in \( \text{diff}(\triangle) \) doing the same.

In this respect, we can think of the Hilbert space \( L^2(\mathcal{A}_T) \), and thereby also \( \mathcal{H}_{\triangle} \), as Hilbert spaces of diffeomorphism-invariant states up to the group \( \text{diff}(\triangle) \). In this picture one should therefore think of a loop associated with a simplicial complex as a kind of equivalence class of smooth loops where the equivalence is with respect to diffeomorphisms.

It is interesting that the construction of \( \mathcal{H}_{\triangle} \) seems to rely on choosing a subgroup of the diffeomorphism group which compromises between two opposite considerations, as follows.

---

\(^{13}\) In fact, the space \( \mathcal{A}^\triangle \) is, in this line of thinking, subjected to another partial gauge fixing by reducing the symmetry group from the group of all diffeomorphisms to the group of analytical diffeomorphisms.

\(^{14}\) This is strictly speaking not correct since it depends on the choice of ‘analytic’ diffeomorphisms. The rough picture is that the combinatorics of every piecewise analytic graph can be represented by a triangulated graph. This does however not mean that there exists an analytic diffeomorphism mapping it to a triangulated graph. Therefore, one needs to enlarge the class of analytic Diffeomorphisms; see, for example, [23]
• For the Hilbert space $\mathcal{H}_\Delta$ to carry a representation of the loop algebra, it must have an action of at least parts of the diffeomorphism group since the loop algebra itself is not diffeomorphism invariant. This requirement works to maximize the size of the remaining symmetry group $\text{diff}(\Delta)$. 

• The construction of a Dirac-type operator acting on $\mathcal{H}_\Delta$ requires a strict control over permissible embeddings of type (15). This requirement works to minimize the size of $\text{diff}(\Delta)$. 

Therefore, the identification of the symmetry group $\text{diff}(\Delta)$ is important for the physical interpretation of the spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$. 

Let us now return to the question regarding the different representations of the Poisson algebras (52) and (54), the representation used in loop quantum gravity and the representation contained in the triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$. We find that these different representations correspond to different choices of holonomy loops in (54). The representation used in loop quantum gravity involves the non-separable Hilbert space $L^2(\mathcal{A})$ whereas the representation contained in the spectral triple involves the separable Hilbert space $L^2(\mathcal{A})$. The central difference between these two approaches therefore lies in the corresponding symmetry groups. The larger the class of loops is, the larger is the symmetry group.

7. Area operators

The area operators constitute an important set of operators in loop quantum gravity [24]. It turns out that the area operator also exists within the framework described by the spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$. These operators are, however, not very natural objects to consider within this framework. However, we find that the square of the Dirac-like operator $D_\Delta$ has a natural interpretation in terms of a global-area-type-squared operator.

7.1. Area operators in loop quantum gravity

Classically, the area of a two-dimensional surface $S$ in $\Sigma$ is given by

$$A(S) = \int_S \sqrt{\text{det} F} \cdot d F^a.$$

where the area element $d F^a$ was defined in (53). To convert this expression into a form suitable for quantization, consider a partition of $S$ into $N$ smaller surfaces $S_n$ such that for any $N$, we have $\bigcup_n S_n = S$. Then the area of $S$ can be written as

$$A(S) = \lim_{N \to \infty} \sum_{n=1}^{N} \sqrt{F^i_{S_n} F^j_{S_n} \delta_{ij}}.$$  \hspace{1cm} (66)

In loop quantum gravity, the area operator is constructed by substituting the classical flux variable $F^i_{S_n}$ with the corresponding operators $F^i_{S_n}$:

$$A(S) = \lim_{N \to \infty} \sum_{n=1}^{N} \sqrt{F^i_{S_n} F^j_{S_n} \delta_{ij}}.$$  \hspace{1cm} (67)

The point here is that this operator, at the level of a given graph, is well defined. This is due to the fact that the number of intersections of any graph with the surface $S$ is finite. Continued subdivisions of $S$ are obsolete once the resolution is reached where each $S_n$ contains one intersection point. If the area element $S_n$ does not involve an intersection with the graph $\Gamma$ then the corresponding operator $F^i_{S_n}$ vanishes. This means that the surface $S$ has a minimal
subdivision into elementary cells \( S_n \), each containing one intersection point with the graph. The area operator obtains the form

\[
A(S) = \sum_i A(p_i),
\]

where the sum runs over intersection points \( p_n \) and where

\[
A(p_n) = \sqrt{F_j S_n F_k S_n \delta_{jk}}.
\]  

(68)

The operator \( A(p_n) \) basically assigns an area to the intersection point \( p_n \). The spectrum of the area operator \( A(S) \), at the level of a given graph \( \Gamma \), can be computed and reads as \([15]\)

\[
\text{Spec}(A(S)) = \left\{ 4\pi r_p^2 \sqrt{j_p(j_p + 1)} \right\},
\]

where \( p \) are intersection points between \( S \) and the graph \( \Gamma \) and \( j_p \) are positive half-integers (for details and subtleties, we refer to \([15]\)). \( r_p \) is the Planck length.

7.2. A global area operator

The spectral triple \((B_{\Delta}, D_{\Delta}, \mathcal{H}_{\Delta})\) contains information of another representation of the operator algebra \((56)\), now based on curves in the inductive system of triangulations. Therefore, it is possible to repeat the line of reasoning from loop quantum gravity and define a second area operator, now based on the Dirac-like operator \(D_{\Delta}\) and the vectors \(\hat{E}_j^i\).

Recall the interpretation \((58)\) which relates the vectors \(\hat{E}_j^i\) to the flux operators \(F_j^i\). This leads to an area operator

\[
A_{\Delta}(S) = \lim_{N \to \infty} \sum_n \sqrt{F_j^i S_n^i \hat{E}_j^i \delta_{ij}},
\]  

(69)

where \( F_j S_n^i \) are as explained in section 5.2.

The difference between the operators \((67)\) and \((69)\) lies not in the expression of the operators themselves but rather in the Hilbert spaces on which they act. The former acts on a non-separable Hilbert space whereas the latter acts on the separable Hilbert space \(\mathcal{H}_{\Delta}\).

The area operator \((69)\), however, does not appear to be a natural object in the construction presented here. It involves a surface \(S\) which has no natural place in this setting.

We can nevertheless apply the line of reasoning from loop quantum gravity to obtain a better understanding of the operator \(D_{\Delta}\). Consider again the interpretation of \(D_{\Delta}\) in terms of conjugate variables of canonical gravity. If we combine equation \((61)\) with equation \((68)\) we obtain

\[
D^2 = \sum_i c_i A(v_i)^2 + \cdots,
\]  

(70)

where, in the limit, the sum runs over all vertices in the inductive system of triangulations \([77]\). It is clear that the set of all vertices is a dense set in \(\mathcal{M}\). Therefore, the sum over all vertices, weighted with the sequence \(\{a_i\}\) and a multiplicity factor given by the valency of the vertex, defines an integral over \(\mathcal{M}\):

\[
\sum_v \cdots \to \int_{\mathcal{M}} \text{d Vol}.
\]

This provides us with a surprising interpretation of \(D^2\) and thereby of \(D^2_{\Delta}\). According to \((70)\) it is an operator which is related to the area of the entire manifold \(\mathcal{M}\):

\[
D^2_{\Delta} \sim \int_{\mathcal{M}} (A(x))^2 + \cdots.
\]  

(71)

It follows from the construction of \(D_{\Delta}\) that the spectrum of \(D^2_{\Delta}\) is discrete.
8. The spectral action functional

The identification of the operator $D^2\triangle$ as a global operator related to the area of the underlying manifold leads us to a new interpretation of the spectral triple $(B, D, H)$. In this Section, we suggest that the operator $D^2\triangle$ should be thought of as a kind of action. Subsequently, we point out that the spectral action of $D\triangle$ resembles a partition function. The argumentation presented in the following is tentative.

The first indication that $(D\triangle)^2$ may be understood in terms of an action is, as already mentioned, that it has the form of an integral over an underlying manifold.

The second indication is directly related to the classical Einstein–Hilbert action. Within the language of noncommutative Geometry, the Einstein–Hilbert action has a natural interpretation as an area of the underlying manifold. With the previous section in mind, this suggests that the operator $(D\triangle)^2$ is somehow related to the Einstein–Hilbert action.

Let us go into some details. Consider a four-dimensional, Riemannian Spin geometry described by the real, even spectral triple $(B, D, H)$, where $B$ is the algebra of smooth functions on $M$, $D$ is the Dirac operator and $H$ is the Hilbert space of square integrable spinors. The Euclidean Einstein–Hilbert action on $M$ can be computed directly from $D$. It is proportional to the Wodzicki residue of the inverse square of the Dirac operator $D^2$ [37]. If we denote by $\text{Tr}^+$ the Dixmier trace and define the noncommutative integral $\int f$ by

$$\int f := \text{Tr}^+ f |D|^{-4},$$

one has

$$\int D^2 = -\frac{1}{48\pi^2} \int_M R \sqrt{g} \, d^4x,$$

where $dv = \sqrt{g} \, d^4x$ is the volume form and $R$ is the scalar curvature. It is in this sense that one may interpret the Einstein–Hilbert action as the ‘two-dimensional measure of a four manifold’, the ‘area’ of $M$ [38].

This resembles the findings of the previous section where we saw that the operator $D^2\triangle$ has a natural interpretation in terms of a global area-squared operator over an underlying manifold. If we ignore the obvious issue of ‘area’ versus ‘area-squared’ one may speculate whether there is some relation between the operator $(D\triangle)^2$ and the Einstein–Hilbert action.

These considerations entail an interesting interpretation of the spectral action functional of $D\triangle$. Consider the quantity

$$\text{Tr} \exp (-s D^2\triangle),$$

where $s$ is a real parameter, and let us perform a formal calculation to obtain a better understanding of this quantity. We write

$$\text{Tr} \exp (-s D^2\triangle) = \sum_{\psi_n} \langle \psi_n | \exp (-s D^2\triangle) | \psi_n \rangle$$

$$= \int_{\mathcal{X}} [d\nabla] \langle \delta \nabla | \exp (-s D^2\triangle) | \delta \nabla \rangle$$

$$= \int_{\mathcal{X}} [d\nabla] \left( \int_{\mathcal{X}} [d\nabla_1] \delta \nabla(\nabla_1) \exp (-s D^2\triangle) (\nabla_1) \right)$$

$$= \int_{\mathcal{X}} [d\nabla] \exp (-s D^2\triangle) (\nabla),$$

where $\{\delta \nabla\}$ is the orthogonal set of delta functions on the space $\mathcal{X}$ and $[d\nabla]$ denotes the measure on $\mathcal{X}$ introduced in section 4. The final expression in (73) may, strictly speaking,
not make mathematical sense. However, we know that the initial quantity (72) is well defined. This formal calculation shows that the spectral action (72) has the form of a formal Feynman integral where the expression

\[ D^2 \delta \nabla \]

plays the role of an action.

### 9. Including the sequences \( \{a_n\} \) as dynamical variables

Up till this point, we have constructed the spectral triple \((B_\Delta, D_\Delta, H_\Delta)\) and related it to quantum gravity. We have shown that it contains information of the Poisson brackets of canonical gravity and that it partly incorporates diffeomorphism invariance. Two questions remain to be addressed as follows.

1. The spectral triple relies on the divergent sequence \( \{a_i\} \). What structure does this sequence represent and how do we deal with it?
2. How do we incorporate the remaining diffeomorphisms contained in the group \( \text{diff}(\Delta) \)?

One solution to the first question would be to fix the sequence \( \{a_i\} \) in a manner which counts a partition of an edge with a factor \( \frac{1}{2} \). However, we will argue that there may be another solution which is more natural. This section is concerned with this issue.

The fact that the sequence \( \{a_i\} \) carries metric information was pointed out in the discussion in subsection 4.5 where we relate the sequence \( \{a_i\} \) to the scaling behaviour of the construction. This suggests, as a possible solution to the first question, that we should include the sequence \( \{a_i\} \) in the construction as dynamical variables. If the sequence \( \{a_i\} \) has a metric origin, then it seems natural to try to integrate over these degrees of freedom.

Therefore, we extend the spectral triple \((B_\Delta, D_\Delta, H_\Delta)\) to include the sequence \( \{a_k\} \) as dynamical variables. By doing this we obtain a new triple, denoted by \((B_t, D_t, H_t)\), which is a fibration of triples \((B_\Delta, D_\Delta, H_\Delta)\) over the space of permissible sequences \( \{a_k\} \). It turns out that there is a way to obtain this which leaves the spectral triple \((B_t, D_t, H_t)\) with very few free parameters.

This section gives a presentation of ideas and methods used to construct the triple \((B_t, D_t, H_t)\). A detailed account of the construction will be presented elsewhere.

To emphasize the dependence on the sequence \( \{a_i\} \), we shall in the following write \( D_\Delta(\{a_i\}) \) instead of \( D_\Delta \).

### 9.1. The space of permissible sequences \( \{a_i\} \)

We would like to think of the parameters \( a_i \) as a coordinate in a space \( H \) of permissible sequences \( \{a_i\} \) (that is, sequences for which \((B_\Delta, D_\Delta, H_\Delta)\) is a semi-finite spectral triple) and to define a Dirac operator on this space. Clearly \( H \subset \mathbb{R}^\infty \). The sequence \( \{a_i\} \) is characterized by the condition that it diverges sufficiently fast. This roughly means that the inverse sequence \( \{a_i^{-1}\} \) converges sufficiently fast towards zero:

\[ \lim_{i} a_i^{-1} = 0 \text{ sufficiently fast.} \quad (74) \]

This convergency condition is easier to work with than the condition on \( (a_i) \)'s (41) since convergent sequences can be understood in terms of a Hilbert space structure. Therefore, in a first step we consider \( L^2 \)-sequences. That is, sequences

\[ \{x_k\} = (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty, \]
which satisfy
\[ \| \{ x_k \} \|^2 := \sum_k x_k^2 < \infty. \] (75)

The space of sequences satisfying (75) is an infinite dimensional, real, separable Hilbert space which we also denote by \( H \).

The exact relationship between vectors \( \{ x_k \} \) in \( H \) and the admissibly sequences \( \{ a_k \} \) will be determined in the following.

In order to construct a Dirac operator on \( H \) we first need a Hilbert space structure over \( H \).

The techniques used to construct \( L^2(H) \) and the corresponding spectral triple are essentially the same techniques we used to construct the space \( \mathcal{A} \) and the triple \((\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)\). The following is based on ideas by Higson and Kasparov [25].

The strategy is to first consider finite-dimensional subspaces \( H_n \) of \( H \), corresponding Hilbert spaces \( L^2(H_n) \) and structure maps \( P^*_{m,n} : L^2(H_n) \to L^2(H_m), \quad n \leq m \), between Hilbert spaces. The next step is to construct a spectral triple over each space \( H_n \), ensure compatibility with the structure maps \( P^*_{m,n} \) and finally obtain a triple over \( H \) by taking the limit \( n \to \infty \).

There are two potential difficulties with this strategy, which are as follows.

- The spaces \( H_n \) are non-compact. The construction of the spectral triple \((\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)\) relies strongly on the fact that \( G \) is compact. This is a necessary condition to construct the structure maps \( P^* \) between Hilbert spaces. Specifically, we used the fact that the constant function is square integrable. This is no-longer the case when the underlying space is non-compact.
- The Dirac-type operator, which we aim to construct, will have a non-compact resolvent. This is the same problem we encountered when we constructed the operator \( D_\Delta \) and a problem which will always arise when one constructs a Dirac-like operator on an infinite-dimensional space (also see [25]). This problem was solved in section 3 by introducing the parameters \( \{ a_i \} \), whose presence is the very reason why we now wish to construct the extended triple \((\mathcal{B}_\epsilon, D_\epsilon, \mathcal{H}_\epsilon)\). Therefore, it appears that we are in danger of a circular argument. Certainly, we do not wish to introduce yet another infinite sequence to ensure a well-behaved Dirac operator acting on \( L^2(H) \) entailing yet another extension of \((\mathcal{B}_\epsilon, D_\epsilon, \mathcal{H}_\epsilon)\) and so forth.

It turns out that there exists a way to construct the triple \((\mathcal{B}_\epsilon, D_\epsilon, \mathcal{H}_\epsilon)\) which avoids these two technical pitfalls.

The first problem is related to the fact the canonical map from \( \mathcal{H}_n \) to \( \mathcal{H}_m \) is given by \( \xi \mapsto \xi \otimes 1_{n,m} \), where \( 1_{n,m} \) is the identity function on the \( m-n \) dimensional orthogonal complement to \( \mathcal{H}_n \) in \( \mathcal{H}_m \). However, the function \( 1_{n,m} \) is not a \( L^2 \)-function and therefore the canonical map is not a map between Hilbert spaces. We can remedy this defect by finding a suitable \( L^2 \)-function \( \phi_{n,m} \) to replace the constant function \( 1_{n,m} \). The choice of function \( \phi_{n,m} \) is restricted by the requirement that it must lie in the kernel of the Dirac-type operator, which we wish to construct, when this is restricted to the orthogonal complement to \( \mathcal{H}_n \) in \( \mathcal{H}_m \).

The second problem is related to the fact that we deal with an infinite-dimensional space. To solve this problem, we modify the canonical algebra of functions on \( \mathcal{H}_n \) to include operators which project onto finite-dimensional subspaces.
9.2. A spectral triple over \( \mathbb{R}^\infty \)

We first consider finite-dimensional subspaces \( H_n = \mathbb{R}^n \) embedded in \( H \) by

\[
H_n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots) \in H.
\]

Before we proceed with the construction, we introduce some notation and definitions. The Gaussian functions on \( \mathbb{R} \)

\[
\phi_\alpha(x) = \left(\frac{\pi \alpha}{4}\right)^{1/4} \exp\left(-\frac{x^2}{2\alpha}\right),
\]

where \( \alpha \) is a real positive number, is a square integrable normalized function on \( \mathbb{R} \). The product

\[
\phi_\alpha(x_1) \cdot \ldots \cdot \phi_\alpha(x_n) = \left(\frac{\pi \alpha}{4}\right)^{n/4} \exp\left(-\sum_{i=1}^{n} \frac{x_i^2}{2\alpha}\right)
\]

is square integrable and normalized on \( \mathbb{R}^n \). Also define

\[
\phi_{n,m}(x_{n+1}, \ldots, x_m) = \phi_\alpha(x_{n+1}) \cdot \ldots \cdot \phi_\alpha(x_m).
\]

This is a square integrable and normalized function on the orthogonal complement \( H_{n,m} \) of \( H_n \) in \( H_m \).

Given a function \( \xi \in L^2(H_n) \), define a function \( P^*_n,m(f), n < m \) in \( L^2(H_m) \) by

\[
P^*_n,m(f)(x_1, \ldots, x_m) = f(x_1, \ldots, x_n) \cdot \phi_{n,m}.
\]

Note that \( P^*_n,m \) defines an embedding of Hilbert spaces. We can therefore define the limit

\[
L^2(H) := \lim_{n \to \infty} L^2(H_n),
\]

which is a separable Hilbert space over \( H \).

Note that, as mentioned in the previous section, the Gauss distributions play the same role in the construction as the constant function did in the construction of \( \mathcal{H}_\triangle \). This solves the first pitfall mentioned above.

Since we wish to construct a Dirac-type operator, we need the Clifford bundle. We define

\[
L^2(H, \text{Cl}(H)) := \lim_{n \to \infty} L^2(H_n, \text{Cl}(H_n)),
\]

where we have used the unital embedding \( \text{Cl}(H_n) \to \text{Cl}(H_m), n < m \).

We identify \( \text{Cl}(H_n) \) with the exterior product \( \Lambda^* H_n \) and define the operators

\[
\text{ext}(e) = \text{exterior multiplication by } e \text{ on } \Lambda^* H_n,
\]

\[
\text{int}(e) = \text{interior multiplication by } e \text{ on } \Lambda^* H_n,
\]

\[
c_\pm(e) = \text{ext}(e) \pm \text{int}(e),
\]

which means that \( c_- (e) \) is Clifford multiplication with \( e \) and \( c_+ (e) \) is Clifford multiplication with \( e \) except that \( e^2 = 1 \) and not \( e^2 = -1 \). Furthermore, one easily checks that \( \{c_+ (e), c_- (e)\} = 0 \).

Define \( e_i \) to be the vector which is 1 and the \( i \)th place and zero elsewhere. Define the operator \( D_n \) on \( L^2(H_n, \text{Cl}(H_n)) \) by

\[
D_n = \sum_{i=1}^{n} \left( c_- (e_i) \frac{\partial}{\partial x_i} + c_+ (e_i) \frac{x_i}{\alpha} \right).
\]

(77)

This operator is known as the Bott–Dirac operator and its construction is due to Higson and Kasparov [25]. It is proven in [25] that it is self-adjoint with a compact resolvent. It satisfies

\[
D_n \phi_{1,n}(x_1, \ldots, x_n) = 0.
\]
In fact, the kernel of $D_n$ is one dimensional and is given by the function $\phi_1(x_1, \ldots, x_n)$. From this, we get that

$$P^*_{n,n}(D_n(\epsilon)) = D_n(\epsilon),$$

and we obtain a self-adjoint unbounded operator $D$ on $L^2(H, \text{Cl}(H))$.

The square of $D_n$ is

$$D_n^2 = \sum_i \left(-\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} + \frac{x_i x_i}{\alpha^2}\right) - \frac{n}{\alpha} + \frac{2}{\alpha} N,$$

(78)

where $N$ is the operator which assigns a differential form its degree. The first part of (78) is the harmonic oscillator Hamiltonian which has the spectrum $0, 2/\alpha, 4/\alpha, \ldots$.

As anticipated, the operator $D$ does not have a compact resolvent since the group of all finite perturbations of $N$ acts on its spectrum. A possible solution to this problem is to modify the canonical algebra of functions acting on $L^2(H, \text{Cl}(H))$. Instead of functions on $L^2(H, \text{Cl}(H))$, we consider functions on finite subspaces of $L^2(H, \text{Cl}(H))$ and extend these to the full Hilbert space by tensoring them with projection onto Gauss functions. Thus, effectively we include cut-off’s in the algebra. This will ensure that the resolvent of $D$, understood in terms of an interaction with the algebra, is compact.

Define

$$\Phi_{n,m} := 1_{H_n} \otimes P_{\phi_{n,m}} \in B(L^2(H_m), \text{Cl}(m)),$$

where $P_{\phi_{n,m}}$ is the orthogonal projection onto $\phi_{n,m}$ in $H_{n,m}$. Let $A_n$ be the algebra of operators in $L^2(H_m, \text{Cl}(H_m))$ generated by tensor products of elements in $C_0(\mathbb{R}) \otimes \text{Cl}(1)$ and $\Phi_{k,l}$. Note that $A_n$ embeds in $A_{n,m}$ by $A_n \ni a \mapsto a \otimes 1_{H_{n,m}} \cdot \Phi_{n,m}$.

Define

$$A = \lim_{\to} A_n.$$

In particular, $A$ consist of operators on $L^2(H, \text{Cl}(H))$. Since the operators in $A$ contain a cut-down to the Gauss distribution from a certain point on, we see that $(A, D, L^2(H, \text{Cl}(H)))$ is a spectral triple since

$$a(\lambda - D)^{-1}, \quad a \in A, \quad \lambda \notin \mathbb{R}$$

is a compact operator.

Note that the semi-finite trick with the Clifford bundle, which worked for the spaces of connections, does not seem to work in this case, since the kernel of $D$ is one dimensional. Therefore, there is no symmetry which can be discarded.

9.3. $\theta$-summability

We would like to interpret condition (75) in terms of $\theta$-summability of the operator $D_{\Delta(\alpha)}$. That is, we wish to establish a relation between sequences $\{x_i\}$ satisfying (75) and sequences $\{a_i\}$ which leave

$$\text{Tr} \exp \left(-D^2_{\Delta(\alpha)}\right)$$

finite. Thus, we seek a function

$$f : \mathbb{R}^\infty \to \mathbb{R}^\infty$$

which satisfies

$$\text{Tr} \exp \left(-D^2_{\Delta(\alpha)}\right) < \infty \quad \Leftrightarrow \quad f(\{a_i\}) \in l^2(\mathbb{N}).$$

(80)
This emphasis on $\theta$-summability has a clear physical motivation. In section 8, we noted that the quantity (79) has the form of a path integral. In this light, $\theta$-summability simply means that this path integral is finite.

9.4. The $U(1)$-case

As a toy example, let us consider the $U(1)$-case and the simplified operator $D'_{\Delta(a_i)}$ on $U(1)^\infty$ which is the limit of operators of the form

$$D'_n := D_1 + a_2 D_2 + \cdots + a_n D_n,$$

where $D_i$ is the Dirac operator on the $i$th copy of $G$. With this operator we consider expression (79) and calculate

$$\text{Tr} \exp \left( -(D'_\Delta)^2 \right) = \prod_n \text{Tr} \exp \left( -a_n^2 D_1^2 \right),$$

where $D_1$ is the Dirac operator on $U(1)$. Taking the logarithm, we obtain

$$\ln \text{Tr} \exp \left( -(D'_\Delta)^2 \right) = \sum_n \ln \text{Tr} \exp \left( -a_n^2 D_1 \right) = \sum_n \ln \sum_k \exp \left( -a_n^2 k^2 \right).$$

Thus,

$$\text{Tr} \exp \left( -(D'_\Delta)^2 \right) < \infty$$

if and only if

$$\sum_n \sum_k \exp \left( -a_n^2 k^2 \right) < \infty.$$

If we define the function

$$f(a) = \text{sgn}(a) \sqrt{\sum_{k=1}^\infty \exp(-a^2 k^2)},$$

we see that, for $G = U(1)$ and for the modified operator, $\theta$-summability of $D'_{\Delta(a_i)}$ is directly related to the Hilbert space condition

$$\sum_n f(a_n)^2 < \infty.$$

This means that the sequence $\{a_i\}$ is related to the Hilbert state $\{x_i\} \in H$ through the relation

$$x_i = f(a_i).$$

9.5. The triple $(B_t, D_t, \mathcal{H}_t)$

We are now ready to combine the two spectral triples $(B_{\Delta}, D_{\Delta}, \mathcal{H}_{\Delta})$ and $(A, D, L^2(H, \text{Cl}(H)))$. A point $\{x_n\}$ in $H$ gives a $\theta$-summable spectral triple on $B_{\Delta}$ via the Dirac operator on $\mathcal{H}_{\Delta}$ defined by the sequence $f^{-1}(\{x_n\})$. Let us denote this Dirac operator by $D_{f^{-1}(\{x_n\})}$. We define the operator $D_t$, acting on

$$\mathcal{H}_t := L^2(H, \text{Cl}(H)) \otimes \mathcal{H}_{\Delta}$$

by

$$D_t(\xi \otimes \eta)(\{x_n\}) = D(\xi) \otimes \eta(\{x_n\}) + (-1)^{\text{deg}(\xi)} \xi \otimes D_{f^{-1}(\{x_n\})}(\eta)(\{x_n\}).$$

where $\text{deg}(\xi)$ means the degree of $\xi$ with respect to the degree in $\text{Cl}(H)$.
One of the requirements that this defines a semi-finite spectral triple over $A \otimes B_\triangle := B_t$ is that
\[
\frac{a}{1 + D_t^2}, \quad \text{for all } a \in B_t,
\]
is $\tau$-compact, where $\tau$ is a trace defined in (43) tensored with the trace on $B(L^2(H, \text{Cl}(H)))$, where $B$ denotes the bounded operators.

We see that this requirement collides with the following symmetry property: all $a \in B_t$ are from a certain step of the form
\[
a \otimes \mathbb{1} \otimes P_{\phi_n},
\]
and therefore the operator
\[
\frac{a}{1 + D_t^2}
\]
is invariant under symmetries permuting all coordinates bigger than $n$.

In order to remedy this, we modify the construction of the spectral triple $(A, D, L^2(H, \text{Cl}(H)))$ slightly. This modification is based on the observation that the parameter $\alpha$ in the Gauss distribution $\phi_\alpha(x)$ is in fact a free parameter. Further, there is an infinite number of these free parameters in the construction of the triple $(A, D, L^2(H, \text{Cl}(H)))$, one for each dimension in $H$. Let us denote these with $\{\alpha_i\}$. By choosing these parameters carefully, we can solve the problem mentioned above.

For $D_t$ to have a Tr-compact resolvent, its eigenvalues must approach infinity. However, for each copy of $G$ we know that its eigenvalues scale with $(\alpha_i)^{-1}$. Thus, if we require the sequence $\{\alpha_i\}$ to satisfy
\[
\lim_{i \to \infty} \alpha_i = 0,
\]
then the triple $(B_t, D_t, \tau_t)$ may be spectral.

Let us pause to comment on this idea. It appears that we have simply exchanged one infinite sequence of free parameters $\{\alpha_i\}$ with another infinite sequence $\{\alpha_i\}$. However, we suggest that the sequence $\{\alpha_i\}$ is not arbitrary but should be determined completely from symmetry considerations. The idea is that $\alpha_i$’s represent an ‘average background’ over which the parameters $\alpha_i$ are allowed to fluctuate. Therefore, we suggest to choose a specific sequence $\{\alpha_i\}$ satisfying the following requirements:

- copies of $G$ which correspond to the same level in the projective system of graphs should be assigned the same $\alpha_i$’s;
- given two copies of $G$ where one copy $G_2$ corresponds to a subdivision of the edge associated with the other copy $G_1$, then the corresponding parameters $\alpha_1, \alpha_2$ should satisfy
\[
\alpha_2 = \alpha_1^2.
\]

The first point reflects a rotational symmetry within each graph. The second point is related to the scaling properties of the construction: a division of a line segment corresponds to a factor $\frac{1}{2}$ (see section 4.5 and the discussion of the role of the parameters $\{\alpha_i\}$).

To implement this, we enumerate the barycentric subdivisions with $k$ and introduce the modified Gauss distributions:
\[
\phi_k(x) = (\pi \alpha_k)^{1/4} \exp \left( -\frac{x^2}{2\alpha_k} \right),
\]
where

\[ \alpha_k = 2^{-k} \alpha. \quad (84) \]

This means that we associate with each edge in the simplicial complexes a parameter \( \alpha_k \). In the initial complex, each edge is associated with the parameter \( \alpha_0 = \alpha \). Edges arising through the first barycentric subdivision are then associated with the parameter \( \alpha_1 = \alpha/2 \) and so forth.

Consider now the triple \((A, D, L^2(H, Cl(H)))\) constructed with these new Gauss distributions. This means that the Hilbert space embeddings (76) are modified together with the Bott–Dirac operator (77) which is now of the form

\[ D_n = \sum_{k=0}^{N} \sum_{i=1}^{n_k} \left( c_- (e_i) \frac{\partial}{\partial x_i} + c_+ (e_i) \frac{x_i}{\alpha_k} \right), \]

where \( n = \sum_k n_k \) and the first sum runs over the number of barycentric subdivisions involved in the graph with which \( D_n \) is associated, \( N \) being the total number of barycentric subdivisions. The second sum runs over the number of edges within a barycentric subdivision.

One now repeats the above construction leading to the operator (83), now with the modified Bott–Dirac operator.

The details concerning the spectral properties of the triple \((B_t, D_t, \mathcal{H}_t)\) will appear elsewhere.

10. Discussion

Before we end this paper, we give few remarks concerning the physical interpretation of the constructions presented.

10.1. The quantization scheme

As already mentioned, the shift to Ashtekar variables permits a formulation of general relativity which is close to ordinary Yang–Mills theory. In the Hamilton formulation, the theory involves a configuration space of connections corresponding to ordinary \( SU(2) \) Yang–Mills theory. The essential difference between general relativity and Yang–Mills theory lies in the algebra of constraints which encode the symmetries of the theory. For general relativity, the constraints encode diffeomorphism invariance. These constraints are closely related to the foliation of the spacetime \( \mathcal{M} \) according to

\[ \Sigma \times \mathbb{R} \rightarrow \mathcal{M}, \quad (85) \]

where \( \Sigma \) is a three-dimensional hypersurface. Loosely stated, the constraints encode information about diffeomorphisms within and perpendicular to \( \Sigma \). Therefore, the key question in any quantization procedure of gravity is the implementation of diffeomorphism invariance in the construction.

To quantize a constrained theory, there are in general two standard approaches, as follows.

(a) To first eliminate, at a classical level, the constraints and thereby find the reduced phase space of the theory. The quantization procedure is applied to the reduced phase space.

(b) To first construct quantum kinematics for the full phase space by ignoring the constraints, then construct operators corresponding to the classical constraints and finally solve the quantum constraints to obtain the physical Hilbert space.

Most formulations of loop quantum gravity follow the second approach which is due to Dirac (see [42] for a detailed exposition). In section 5.2, we demonstrated that the spectral triple
The spectral triple \((\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)\) contains information which is tantamount to a representation of the Poisson algebra \((\mathcal{P})\) of general relativity in the sense that the interaction between the Dirac-type operator and the loop algebra reproduces the Poisson bracket between the loop and flux variables. In our interpretation, the construction should be understood as a quantization scheme which lies somewhere between the approaches (a) and (b).

First of all, there is nothing in the geometrical construction that corresponds directly to the constraint algebra of general relativity. However, in section 6 we related the Hilbert space \(\mathcal{H}_\Delta\) to the diffeomorphism-invariant Hilbert space \(\mathcal{H}_{\text{diff}}\) of loop quantum gravity. The difference between \(\mathcal{H}_\Delta\) and \(\mathcal{H}_{\text{diff}}\) is the group \(\text{diff}(\Delta)\) of certain automorphisms of the system of simplicial complexes. Therefore, we think of the spectral triple \((\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)\) as a geometrical construction describing a quantization procedure of gravity on a Hilbert space which corresponds to a partial solution to the (spatial) diffeomorphism constraint. What remains then, in this picture, is the implementation of the remaining diffeomorphisms encoded in the group \(\text{diff}(\Delta)\).

The second point in which our construction differs from canonical quantum gravity is the issue of time. Clearly, a foliation of the manifold is essential for canonical gravity since the Hamilton formulation is based on a choice of an explicit time direction. In contrast to this, the construction presented here does not a priori involve a foliation of spacetime. Further, the dimension of the underlying space is completely free. The only restriction is the choice of the symmetry group, which, for now, has to be compact\(^{15}\). In particular, we can consider triangulations of a four-dimensional manifold.

These observations suggest two possible interpretations of the construction: First, the construction may be interpreted as a completely covariant construction in four dimensions, involving the full, four-dimensional group of diffeomorphisms. This interpretation leaves no room for a foliation and thus no Hamilton constraint. Therefore, a dynamical principle replacing the Hamilton constraint must be sought elsewhere. This interpretation implies that the group \(G\) corresponds to four-dimensional local Lorentz rotations. Next, if the loop algebra lives on a four-dimensional manifold, then the interpretation of the Dirac-type operator \(D_\Delta\) in terms of canonical quantum gravity must involve a flux operator \(F_S\), where \(S\) is now a three-dimensional hypersurface. Further, the interaction between \(D_\Delta\) and the algebra of loops should be understood as a four-dimensional operator algebra containing a representation of the Poisson algebra of general relativity as a kind of subalgebra.

Alternatively, we may interpret the construction as fundamentally three-dimensional, with \(G\) equal to \(SU(2)\). This interpretation does not necessarily mean that we consider a foliation of spacetime according to \((85)\). Rather, we would like time to emerge naturally from the construction.

Indeed, let us end this discussion with the remark that the philosophy of noncommutative geometry is to seek a time within the algebraic construction rather than imposing it a priori. Here we think of the Tomita–Takesaki theory \([43]\) which identifies a one-parameter automorphism group uniquely up to inner automorphisms. It is an interesting question whether this group of modular automorphisms is nontrivial in this or perhaps some similar construction. If the answer is in the affirmative, then this might provide us with an alternative to the foliation and thereby with a new dynamical principle.

10.2. The group \(\text{diff}(\Delta)\) and the question of constraints

Regardless of how one interprets the spectral triple \((\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)\), it is clear that one should seek to obtain invariance under the group \(\text{diff}(\Delta)\) of graph-preserving diffeomorphisms.

\(^{15}\) The case \(G = SO(1, 3)\), for example, is not permitted.
Traditionally, given some space $M$, invariance under a symmetry group $G$ is obtained by taking the quotient $M/G$. However, noncommutative geometry gives an alternative approach to quotient spaces. A simple example is the space of two points identified. Within noncommutative geometry, this setup is described via two-by-two matrices where the off-diagonal entries represent the identification of the points. This entails a noncommutative algebra and an additional structure.

In the case of gravity, the relevant symmetry group is the diffeomorphism group. This group is probably too large for an application of the noncommutative approach. However, in the present setup we have instead the much smaller group $\text{diff}(\triangle)$ and one could speculate if the machinery of noncommutative geometry of quotient spaces could be successfully applied. This idea differs fundamentally from a Dirac-type quantization procedure.

Thus, what we suggest is to obtain a formal diffeomorphism invariance by considering the semi-direct product of the loop algebra with the group $\text{diff}(\triangle)$:

$$B_\triangle \rtimes \text{diff}(\triangle),$$

and thereafter building a spectral triple with an associated Hilbert space $L^2(A) \otimes L^2(\text{diff}(\triangle))$. Presumably, such a construction will give rise to additional degrees of freedom through fluctuations around the Dirac-type operator.

### 10.3. Background independence

In section 6.2 we argued that the space $\mathcal{X}_0$ should be viewed as a space of connections subjected to a gauge fixing of the diffeomorphism group. Let us comment on what background structures this gauge fixing and the entire construction depends on.

The construction in section 2 depends primarily on the initial simplicial complex $K_0$. The idea is that the basic data entering the construction are the topology of the corresponding manifold. The initial complex gives rise to an initial triangulation $T_0$ via a homeomorphism $h : K \rightarrow M$. This initial triangulation introduces a metric structure on $M$. The embedding $A \hookrightarrow \mathcal{X}_0$ will, of course, depend crucially on the triangulation and in particular on the homeomorphism $h$. However, the construction and spectrum of the Dirac-type operator is independent of $h$ and so is its interaction with the algebra and the construction of the Hilbert space. In fact, the notion of a manifold may be left out altogether. It is only the identification of $\mathcal{X}_0$ as a space of generalized connections that requires a manifold.

Another question is the dependence on the sequence $\{a_i\}$. The Dirac-type operator $D_\triangle$ depends crucially on this sequence as does its spectrum. Thus, one may argue that some degree of background dependence enters the construction with the sequence $\{a_i\}$. In section 9, we attempted to free the construction of this dependence by making $a_i$’s dynamical. The price paid, however, is the introduction of a new sequence $\{\alpha_i\}$ determining the Gauss distributions. Therefore, it seems that the construction, whether we consider the triple $(B_\triangle, D_\triangle, \mathcal{H}_\triangle)$ or the triple $(B_t, D_t, \mathcal{H}_t)$, does possess some degree of dependence on a parameter space which one may interpret in terms of a background. The exact nature and implication of this dependence are to be clarified.

### 10.4. Additional degrees of freedom

The construction of the triples $(B_\triangle, D_\triangle, \mathcal{H}_\triangle)$ and $(B_t, D_t, \mathcal{H}_t)$ takes canonical quantum gravity as its point of departure. Thus, both spectral triples are a priori of purely gravitational origin.
However, there are several indications that the framework presented in this paper contains additional degrees of freedom, both bosonic and fermionic.

First, recall that the framework of noncommutative geometry generally involves fermionic degrees of freedom since it involves a Dirac-type operator acting on a Hilbert space. The ‘fermions’ involved in the spectral triple \((B_\Delta, D_\Delta, H_\Delta)\) are clearly very different from the fermions of the standard model since they live on a space of connections. However, this space of connections is of course linked to an underlying manifold. A classical limit will presumably involve some kind of delta function on the space of connections (see the following section for a discussion hereof) and therefore leave only spacetime degrees of freedom for the fermions. Therefore, the interesting question is what structures may emerge from these ‘fermions’ in a classical limit.

Furthermore, during the construction of the triple \((B_\Delta, D_\Delta, H_\Delta)\) we found that the CAR algebra emerged in an almost canonical fashion. It seems plausible that almost any construction of a Dirac-type operator on an infinite-dimensional space will naturally entail an infinite-dimensional Clifford bundle which, in turn, leads to the CAR algebra. Thus, it seems that this framework provides us with a mechanism which equips a purely gravitational setting, the construction of a Dirac-type operator on an infinite dimensional space of field configurations, with basic elements of fermionic quantum field theory.

Finally, let us consider the possibility that the noncommutativity of the loop algebra will generate an additional bosonic sector through the inner automorphisms of the algebra. To explain this, first consider a finite-dimensional, real spectral triple \((A, D, H)\). A noncommutative algebra \(A\) contains inner automorphisms of the form

\[
\alpha_u(x) = u xu^* \quad \forall x \in A,
\]

where \(u\) is an arbitrary element of the unitary group, \(uu^* = u^*u = 1\). A change of representation of the algebra \(A\) from \(\pi\) to \(\pi \circ \alpha_u^{-1}\) is equivalent to the replacement of the Dirac operator \(D\) by

\[
\tilde{D} = D + A' + JA'J^{-1},
\]

where \(A' = u[D, u^*]\) is a noncommutative 1-form and \(J\) is a real structure. In the noncommutative formulation of the standard model coupled to gravity the entire bosonic sector of the standard model, including the Higgs boson, is generated by this type of fluctuations of the Dirac operator by a general 1-form \(A' = \sum a_i[D, b_i]\) with \(a_i, b_i \in A\).

This mechanism is general. The interesting question is what kind of bosonic sector the inner automorphisms of the spectral triple \((B_\Delta, D_\Delta, H_\Delta)\) will generate.

The general idea behind these remarks is that pure quantum gravity may, as a ‘free’ spin-off, contain matter degrees of freedom. Thus, we believe that one should not attempt to couple matter degrees of freedom to the construction presented in this paper but rather hope to see matter emerge dynamically.

10.5. Distances on \(\tilde{A}\)

On a Riemannian spin geometry, the Dirac operator \(D\) contains the geometrical information of the manifold \(\mathcal{M}\). In particular, distances can be formulated purely algebraically due to Connes. Given two points \(x, y \in \mathcal{M}\), their distance is given by

\[
d(x, y) = \sup_{f \in C^\infty(\mathcal{M})} \{||f(x) - f(y)||[D, f]|| \leq 1\}. \tag{87}
\]

For a commutative algebra, the underlying space is a Riemannian geometry and the Dirac operator acts on a Hilbert space of spinors. In the noncommutative formulation of the standard model, the Hilbert space is labelled by the fermions in the standard model.
On a noncommutative geometry, the state space replaces the notion of points. It is possible to extend the notion of distance to the state space by generalizing (87).

In the present case, it is natural to ask whether the Dirac-like operator $D_\Delta$ introduces a metric structure over the space $\mathcal{A}^{\Delta}$ and if so, how distances should be interpreted in this setting.

The possibility for a distance between smooth connections in $\mathcal{A}$ and some closure hereof was first considered in [20]. There, however, the distance between two smooth connections was found to be infinite.

In the present setting the situation is different due to the sequence $\{a_i\}$. The role of $a_i$'s is to assign a weight to each copy of $G$ and therefore to scale the corresponding distance on $G$.

Therefore, given two points in $\mathcal{A}^{\Delta}$ their distance

$$d(\nabla_1, \nabla_2) = \sup_{a \in B_\Delta} \{ \| \nabla_1(a) - \nabla_2(a) \| \| D_\Delta, a \| \leq 1 \}, \quad \nabla_1, \nabla_2 \in \mathcal{A}^{\Delta},$$

(88)

where we choose $\| \cdot \|$ as the operator norm on matrices in $M_l(\mathbb{C})$, is well defined and finite, even for $\nabla_1, \nabla_2 \in \mathcal{A}$.

To be precise, one should note that the algebra $B_\Delta$ is not large enough to contain all connections in its state space. Thus, for (88) to make sense one might have to consider a larger algebra instead. The point is that the Dirac operator $D_\Delta$ does give a distance on $\mathcal{A}^{\Delta}$.

The distance (88), when it is applied to smooth connections, is not independent of the embedding $h$ of simplicial complexes into triangulations.

With the possibility of a metric structure on the space $\mathcal{A}^{\Delta}$, it is natural to ask which connections are ‘close’ to each other. Clearly, this depends on the choice of the sequence $\{a_i\}$. However, it is possible to give some general remarks independent of this choice.

Recall that the space $\mathcal{A}^{\Delta}$ is the limit space $\lim_n G^n$ in an appropriate sense. The distance between $\nabla_1, \nabla_2 \in \mathcal{A}_k$ is simply the sum of geodesic distances on each copy of $G$, weighted with the appropriate parameters $a_i$. In the previous section, we found that the role of the parameters $\{a_i\}$ is to scale the different copies of $G$ according to their location in the projective system of simplicial complexes. The larger $a_i$'s correspond to short distances, now with respect to some manifold. This means that the distance between $\nabla_1$ and $\nabla_2$ is larger if the connections differ mostly on those copies of $G$ which are assigned small $a_i$'s. Correspondingly, the distance between $\nabla_1$ and $\nabla_2$ is smaller if $\nabla_1$ and $\nabla_2$ differ mostly on those copies of $G$ which are assigned large $a_i$'s. Therefore, in general, the two generalized connections will be ‘close’ to each other if they differ mostly at short scales.

Via Levi-Civit`a connections, we can interpret equation (88) as a distance between geometries. Again, we can say that the distance between two geometries depends on the scale on which they differ.

Let us consider the Abelian case $G = U(1)$ and two points $\nabla_1, \nabla_2$ in $\mathcal{A}^{\Delta}$. In this case, $\nabla_1$ and $\nabla_2$ are given by sequences of angles $\{\theta_1^i\}$ and $\{\theta_2^i\}$ where each angle $\theta_i^j$ corresponds to points $\exp(2\pi i \theta_i^j)$ on the $i$th copies of $U(1)$. Let us for simplicity choose a coordinate system on $G^n$ so that $\theta_i^j$ corresponds to the parameter $a_i$. Thus, for example, if we consider $G \times G$ parametrized by angles $\phi_1$ and $\phi_2$, we choose

$$\theta_1 = \phi_1 + \phi_2, \quad \theta_2 = \phi_1 - \phi_2$$

17 In this case, however, the distance formula will be degenerate since different geometries may have identical Levi-Civit`a connections.
and so forth. Then the distance between $\nabla_1$ and $\nabla_2$ reads as
\[
d(\nabla_1, \nabla_2) = \sum_k \frac{2^k}{a_k 2\pi} |\theta^1_k - \theta^2_k|,
\]
which should be read together with condition (74).

The notion of a distance on a space of connections is not a new one but was already discussed by Feynman [39] and Singer [40]. Also see [41] and references herein. However, these papers all deal with distances between gauge equivalence classes of connections. The construction presented in this paper is quite different from these previous works since two connections which differ by a gauge transformation will, in general, have a non-vanishing distance.

11. Conclusion and outlook

In this paper, we establish a link between the mathematics of noncommutative geometry and the field of canonical quantum gravity. We construct a spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ over a space of connections and show that the Poisson structure of general relativity, formulated in terms of loop variables, is encoded in the interaction between the Dirac-type operator $D_\Delta$ and the loop algebra $B_\Delta$. The Hilbert space $\mathcal{H}_\Delta$ can be interpreted as a partial solution of the diffeomorphism constraint of canonical gravity. The inner product of $\mathcal{H}_\Delta$ involves a functional integral over a space of connections and the Dirac-type operator $D_\Delta$ has the form of a global functional derivation. Consequently, we interpret the triple in terms of a non-perturbative, background-independent, quantum field theory.

The construction is based on a projective system of simplicial complexes. The simplicial complexes are related through repeated barycentric subdivisions. The triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ is the limit of spectral triples formulated at the level of finite simplicial complexes. Since the operation of barycentric subdivision is countable, the limit triple is separable and spectral.

The square of the Dirac-type operator has, in terms of canonical quantum gravity, a natural interpretation as a global operator related to the area operators known in loop quantum gravity. We interpret the operator $(D_\Delta)^2$ in terms of an action and show that the spectral action of $D_\Delta$ has the form of a Feynman path integral. Thus, at the core of the construction we find an object which resembles a partition function related to quantum gravity.

The construction of the spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ differs from a traditional canonical quantization procedure of general relativity in the way the group of diffeomorphisms is treated. Rather than encoding the symmetries of the classical theory in a set of constraints, the construction works directly on a Hilbert space $\mathcal{H}_\Delta$ which can be interpreted as a partial solution of the (spatial) diffeomorphism constraint. This means that the Poisson algebra of general relativity is represented on the separable Hilbert space $\mathcal{H}_\Delta$.

The Dirac-type operator $D_\Delta$ depends on an infinite sequence of parameters $\{a_i\}$. These parameters determine the scaling behaviour of the construction. We believe that a correct understanding and treatment of these parameters are essential. In this paper, we propose a possible way to treat the sequence $\{a_i\}$. Since the sequence is seen to have metric origin, we propose to include it as a dynamical variable in the construction. This leads to a new triple, denoted by $(B_t, D_t, \mathcal{H}_t)$, which includes the spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ as well as a sector permitting the sequence $\{a_i\}$ to vary. The triple $(B_t, D_t, \mathcal{H}_t)$ is constructed to ensure that the operator $D_\Delta$ is $\theta$-summable. Thus, we permit only sequences $\{a_i\}$ which leave the spectral action of $D_\Delta$ finite.

Furthermore, the Dirac-type operator $D_\Delta$ defines a distance on the underlying space of connections. Clearly, this distance depends strongly on the sequence $\{a_i\}$. However, we find
that a general feature of this distance function is that two connections are ‘close’ if they differ mostly at short scales.

It is possible to read this paper in a more conservative way, discarding the role of noncommutative geometry and reading it as a reformulation of loop quantum gravity. If we ignore the construction of the Dirac-type operator $D_\Delta$ and focus instead on the algebra $\mathcal{B}_\Delta$ and the Hilbert space $\mathcal{H}_\Delta$, without the Clifford bundle, and consider the algebra of the vectors $\hat{E}_i$, then, as already mentioned, we obtain a representation of the Poisson algebra of general relativity on a separable Hilbert space. Therefore, this Hilbert space, which we denote by $\mathcal{H}_\Delta'$, replaces the otherwise non-separable Hilbert space $L^2(\mathcal{A})$ known as the kinematical Hilbert space in loop quantum gravity. In loop quantum gravity, the kinematical Hilbert space is the Hilbert space on which the constraints of general relativity are defined. Thus, one could formulate the complete set of constraints of general relativity in terms of operators acting on the Hilbert space $\mathcal{H}_\Delta'$. First, as explained in section 6, the Hilbert space $\mathcal{H}_\Delta'$ has an action of the reduced set of diffeomorphisms $\text{diff}(\Delta)$. Therefore the spatial diffeomorphism constraint should be formulated in terms of the group $\text{diff}(\Delta)$. Next, one may likewise formulate the Gauss and Hamiltonian constraints of loop quantum gravity on $\mathcal{H}_\Delta'$. The central message here is that it seems possible to formulate loop quantum gravity in terms of a separable kinematical Hilbert space. It is an interesting question whether this observation will have an impact on any of the challenges facing loop quantum gravity.

Let us finally remind the reader that the focus of this paper is the physical significance of the spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$. The detailed mathematical analysis of the triple is given in [22].

**Outlook**

More analysis is needed to understand the physical and mathematical significance of the spectral triple $(\mathcal{B}_\Delta, D_\Delta, \mathcal{H}_\Delta)$. First of all it is imperative to understand the role and proper treatment of the sequence $\{a_i\}$ since this sequence is central to the existence of the triple. The sequence $\{a_i\}$ should be understood in connection with the group of diffeomorphisms $\text{diff}(\Delta)$ since elements hereof are given by rearrangement of the parameters $a_i$. Other issues which we think deserves attention are as follows.

- First, we have seen that the spectral action resembles a Feynman path integral. We believe that the computation and analysis of the spectral action functional is the most interesting task to address at the present stage of the project. Here, one should consider the Dirac operator which involves the inner fluctuations described in section 10.
- A prime issue for any theory or framework for non-perturbative quantum gravity is the formulation of a semi-classical limit. In the present case, the aim is to obtain a classical limit which involves not only a smooth geometry—characterized by a commutative $*$-algebra—but to obtain a limit which includes an additional matrix factor of the type that characterizes the almost commutative algebra in Connes’ formulation of the standard model. Indeed, since Connes’ geometrical realization of the standard model is so attractive and powerful as it stands, it remains to understand why the algebra which lies at the heart of this formulation should have this particular noncommutativity. We suggest that the source of this noncommutativity lies in pure quantum gravity. Recall that the loop algebra $\mathcal{B}_\Delta$ is essentially an almost commutative algebra over the space $\mathcal{A}$ of connections. That is, it is a product of smooth functions over $\mathcal{A}$ and a matrix factor $M_l(\mathbb{C})$. If we think of a classical limit as the emergence of a single geometry, it seems reasonable to expect something close to a delta function peaked around a connection $\nabla$. However, when we
apply the loop algebra on a delta function it reduces to the matrix factor $M_l(C)$ or some subalgebra hereof. One could speculate that this corresponds to the matrix factor of an almost commutative algebra. Furthermore, if we keep in mind that the construction of the spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ involves a choice of a basepoint\(^{18}\), then it seems possible that a similar construction which does not involve this basepoint will entail a smearing of the matrix factor over the manifold. Thus, in total this may add up to an almost commutative algebra. Thus, we speculate that the matrix factor behind the noncommutative formulation of the standard model is related to a noncommutative algebra of holonomy loops.

We suspect this issue to be related to the calculation of the spectral action. However, one should also investigate whether ideas from loop quantum gravity concerning coherent states \([44]\) can be applied.

- The question about time is fundamental to any general covariant theory since such theories have no preferred time flow. This situation is even more complicated when one attempts to include quantum theory since this will presumably lead to a theory which involves some notion of superpositions of geometries and thus does not permit any notion of a predetermined time. A possible solution to this problem has been proposed by Connes (see for example \([38]\); also see \([45]\) for similar ideas developed by Connes and Rovelli). The idea is that the concept of time is intimately linked to the noncommutativity of the algebra of observables of quantum gravity. Specifically, it is a fundamental property of von Neumann algebras that they possess a one-parameter family of automorphisms which is unique up to inner automorphisms. Thus, the idea is that this group of automorphisms, the modular group, should be understood in terms of a time. It is therefore natural to ask whether the noncommutative $\ast$-algebras introduced in this paper give rise to a nontrivial modular group and whether this can be interpreted as a time flow.

Alternatively, one might try to exploit the fact that the construction presented in this paper is basically quantum mechanics on the group $G$ taken infinitely many times. Here, each copy of $G$ corresponds to a degree of freedom originating somewhere on the underlying manifold. Thus, one could consider the time evolution, with respect to one copy of $G$, given by the operator $\exp(i\Delta t)$ where $\Delta$ is the Laplace operator. This suggests that the unitary operator

$$U(t) := \exp(i(D_\Delta)^2 t)$$

may be thought of as a time evolution operator.

- The construction of the spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ starts with a simplicial complex. It is an important question to determine the exact dependence of the triple on the choice of the initial complex. For instance, consider two simplicial complexes $K_1$ and $K_2$ chosen so that their union is the barycentric subdivision of yet another simplicial complex $K_3$. Does the construction depend on whether one chooses $K_3$ or the union of $K_1$ and $K_2$ as the initial complex? The answer is, a priori, yes, since the two choices will come with different sequences of parameters $\{a_i\}$. The exact nature of this dependence needs to be clarified. Clearly, the idea is that the spectral triple should depend only on topological data coming from the underlying manifold. A related issue is the merging of different spectral triples based on different simplicial complexes. By gluing complexes, it should be possible to move from one topological setting to another. These issues are all connected with the sequence $\{a_i\}$ and the group of diffeomorphisms in $\text{diff}(\Delta)$.

- A related issue is the possibility of obtaining a similar construction based on a different projective system of graphs. The choice of simplicial complexes (or triangulations) and

\(^{18}\) See appendix A for an extension of the triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ which avoids this choice of basepoint.
barycentric subdivisions seems natural but is not compulsory. In [22] we provide certain
necessary requirements for a system of graphs to be suitable for the construction of a
spectral triple. These requirements leave room for projective systems of graphs which are
not simplicial complexes. Again, more analysis is needed to determine the dependence
of the final construction on different choices of graphs. It is clear that a projective
system of graphs must be countable in order to permit the construction of a spectral
triple.

• It is desirable to be also able to deal with a non-compact gauge group such as $SO(3, 1)$.

At present, the compactness of the gauge group is essential for the construction of the
Hilbert space to work. Basically, we need the identity to be an $L^2$-function with respect to
the Haar measure. One could speculate whether the techniques of Higson and Kasparov
[25] might be applied to resolve this issue. Here, the trick is to use a Dirac operator with a
nontrivial square-integrable kernel and to define embeddings between Hilbert spaces via
this kernel.

• One should further clarify the relation between the construction presented here and loop
quantum gravity. In particular, it would be interesting to understand if there is a relation
between the operator $D_{\triangle}$ and the Hamilton constraint. Here one should most likely
consider the fluctuated version of $D_{\triangle}$, with respect to inner automorphisms, since this
operator also involves the loop algebra.

• It is an interesting question whether the construction presented in this paper has anything to
say about nonperturbative gauge theory. Consider a single line segment and the sequence
$a_n = a_02^n$, where $n$ corresponds to the number of subdivisions of the segment. This is
a natural sequence to consider but we know that the asymptotic behaviour $a_n \sim 2^n$ is
too weak for the Dirac-type operator to have a compact resolvent. If we consider instead
the sequence $a_n = a_0(2 + |\epsilon|)^n$, then $D_{\triangle}$ will have a compact resolvent as long as $\epsilon$
is non-zero (at least in the $U(1)$ case). This setup is then extended to all line segments in the
projective system of graphs. As long as $\epsilon$ is non-zero, the corresponding spectral action
functional of $D_{\triangle}$ is well defined and resembles a Feynman path integral over a space
of connections. The interesting question is what theory this object represents. It seems
clear that it should be understood as a non-perturbative quantum field theory involving a
gauge field. The setup resembles lattice gauge theory with the crucial difference that a
lattice spacing is absent and that one does not have the freedom to choose an action. It
would be interesting to calculate the spectral action and, subsequently, to take the limit
$\epsilon \to 0$. Presumably, this limit will lead to divergences since the operator $D_{\triangle}$ ceases to
have a compact resolvent when $\epsilon = 0$. One can speculate whether these divergences
might correspond to the divergences found in perturbative quantum field theory.

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Appendix A. A spectral triple without the basepoint

The diffeomorphism invariance obtained so far only includes diffeomorphisms which preserve the basepoint introduced in section 2. The role of the basepoint is to equip the algebra of loops with a product. Because the choice of basepoint partly breaks diffeomorphism invariance, we would like to obtain a structure which avoids the basepoint. It turns out that such a construction does exist. Instead of the group structure of loops, this more general construction is based on a groupoid structure of path.

Once again, we start with an abstract simplicial complex $\mathcal{K}$ with vertices $\{v_i\}$ and edges $\{\epsilon_j\}$. Consider the Hilbert space $\mathcal{H}_\mathcal{K} = L^2(\{v_i\} \times \mathcal{A}_\mathcal{K}, \mathcal{M}_l(\mathbb{C}))$, where we recall that $\mathcal{A}_\mathcal{K} = G^{\mathcal{A}(\mathcal{K})}$. For now, we omit the Clifford bundle which is not necessary for the construction and representation of the algebra.

In fact, there are two natural algebras to consider. Denote by $\Omega_1$ the set of loops in $\mathcal{K}$ with an arbitrary basepoint and consider first the algebra generated by these loops equipped with the product

$$f_{L_1} \cdot f_{L_2} = \begin{cases} 0 & \text{if basepoints differ} \\ f_{L_1 \cdot L_2} & \text{if basepoints coincide.} \end{cases}$$

Again, we can construct a norm via the matrix norm on $G$ and we obtain a $C^*$-algebra which we denote by $\mathcal{B}_{\Omega_1}$.

The second option, which is perhaps more natural, is to consider paths in $\mathcal{K}$. Denote by $\mathcal{P}$ the set of paths in $\mathcal{K}$ and denote by $\mathcal{B}_\mathcal{P}$ the algebra generated by such paths with a natural product. Clearly, $\mathcal{B}_{\Omega_1} \hookrightarrow \mathcal{B}_\mathcal{P}$. Concretely, we let the algebra $\mathcal{B}_\mathcal{P}$ be given by its representation which is as follows. Given a path $p$ that starts in $v_1$ and ends in $v_2$ we write

$$(f_p \cdot \xi)(v_j, \nabla) = \begin{cases} 0 & \text{if } v_j \neq v_2 \\ \nabla(p) \cdot \xi(v_1, \nabla) & \text{if } v_j = v_2, \end{cases}$$

where $\nabla(p)$ is defined as in (10), just with a path instead of a loop.

In this setup, a path can now be seen as an operator which combines the holonomy along the path with a matrix structure

$$|v_m\rangle \langle v_n|$$

and we note that loops with an arbitrary basepoint are found on the diagonal of this matrix structure. Thus, a natural trace will pick out all the loops and thereby, in terms of holonomy loops, the gauge-covariant elements.

The inner product on the Hilbert space $\mathcal{H}_\mathcal{K}$ also involves a sum over vertices. The rest of the construction can be carried out in a manner similar to the construction of the triple $(\mathcal{B}_\Delta, \mathcal{D}_\Omega, \mathcal{H}_\Delta)$.

This formulation seems better suitable for a semi-classical limit since it involves the points of the manifold, which, presumably, should emerge in such a limit.

Appendix B. On diffeomorphism invariance

Although the Hilbert space $L^2(\mathcal{A}^\mathcal{K})$ does not have an action of the full diffeomorphism group, it is possible to introduce a notion of analytic diffeomorphisms on $L^2(\mathcal{A}^\mathcal{K})$ via the space $L^2(\mathcal{A})$. This, in turn, allows us to extend the action of certain operators on $L^2(\mathcal{A}^\mathcal{K})$ to the larger Hilbert space $L^2(\mathcal{A})$ and thereby introduce a notion of (analytic) diffeomorphism invariance for these operators. This setup involves an embedding of general piecewise analytic...
graphs $\Gamma$ into triangulations $T_i$, coming from the projective system $\{K_i\}$. Therefore, the action of these operators on $L^2(\mathcal{A}^\infty)$ will depend on a choice of embedding. It is important to realize that this construction does not work for the operator $D_\triangle$ since we do not have a Clifford bundle over the space $L^2(\mathcal{A}^\infty)$.

The following should be read as a rough idea or strategy as to how one introduces a notion of diffeomorphism invariance on $L^2(\mathcal{A}^\infty)$, rather than a complete analysis.

Given an element $\xi_\Gamma \in L^2(\mathcal{A}^\infty)$ associated with a piecewise analytic graph $\Gamma$, we choose an embedding of a suitable simplicial complex $K_i$:

$$\phi : K_i \rightarrow T_i,$$

where $T_i$ is a triangulation in $\mathcal{M}$, so that $\Gamma$ lies in $T_i$. Consider an operator $O$ on $L^2(\mathcal{A}^\infty)$.

This could, for example, be the Laplace operator. The action of $O$ on $\xi_\Gamma$ is defined as

$$O(\xi_\Gamma) = \phi(O(\phi^{-1}\xi_\Gamma)).$$

If we map $d : \xi_\Gamma \rightarrow \xi_{d(\Gamma)}$ with a diffeomorphism $d$, then the action of $O$ changes accordingly:

$$O(\xi_{d(\Gamma)}) = \phi'(O((\phi')^{-1}\xi_\Gamma)),$$

where $\phi'$ is a new embedding. We obtain the diagram

$$\begin{array}{cccc}
\xi_\Gamma & \xrightarrow{\phi^{-1}} & \xi_\Delta & \xrightarrow{\phi} & O\xi_\Delta & \xrightarrow{d} & (O\xi_\Delta)_\Gamma \\
d & \downarrow & d_\Delta & \downarrow & d_\Delta & d & \\
\xi_{d(\Gamma)} & \xrightarrow{(\phi')^{-1}} & \xi'_\Delta & \xrightarrow{\phi'} & O\xi'_\Delta & \xrightarrow{d} & (O\xi'_\Delta)_{d(\Gamma)}
\end{array}
$$

where $d_\Delta$ belongs to $\text{diff}(\Delta)$. To have a diffeomorphism-invariant state means that the state is invariant under $\text{diff}(\Delta)$. This is exactly the space $\mathcal{H}_{\text{diff}}$ of diffeomorphism-invariant states mentioned in the previous subsection. A diffeomorphism-invariant operator $O$ is an operator for which the centre part of the diagram (B.1) commutes:

$$\begin{array}{c}
\xi_\Delta \\
d_\Delta \\
\xi'_{d(\Gamma)}
\end{array} \xrightarrow{O} \begin{array}{c}
O(\xi_\Delta) \\
O(\xi'_\Delta)
\end{array}$$

This reflects the simple observation that a loop is, by itself, not an observable since it is not self-adjoint. Only the combination $L + L^{-1}$ is self-adjoint, exactly because it is invariant with respect to the symmetry group of the loop.

Appendix C. Symmetric states

In quantum field theory, the vacuum state is defined as the unique translational-invariant state. In the present setting, there are certain states which display a high degree of symmetry and may be thought in terms of a ground state.

First, consider the spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ and the two states

$$\psi_0(\nabla) = \psi_0(g_1, g_2, \ldots, g_n, \ldots) = \delta_{G_1}(id) \cdot \delta_{G_2}(id) \cdot \ldots \cdot \delta_{G_n}(id) \cdot \ldots,$$

and

$$\phi_0(\nabla) = 1.$$

Here, $\delta_{G_n}$ is the delta function on the $n$th copy of $G$. The action of a loop $L$ on the first state reads as

$$f_L \cdot \psi_0(\nabla) = \psi_0(\nabla)$$
for any $L$ in $B_\Delta$. This means that the entire algebra $B_\Delta$ collapses into the identity on the state $\psi_0$. This state, however, does not lie in the Hilbert space nor in the domain of the operator $D_\Delta$. In contrast to this, the state $\phi_0$ lies in the kernel of $D_\Delta$. Both states $\psi_0$ are invariant under $\text{diff}(\Delta)$ since they are invariant under any permutation of the argument $(g_1, \ldots, g_n, \ldots)$.

If we consider instead the spectral triple $(B_\Delta, D_\Delta, \mathcal{H}_\Delta)$ there are again two states which are highly symmetric. First, consider
\[
\mathcal{H}_\Delta \ni \psi_0(\nabla, \bar{x}) = \phi_0(\nabla) \otimes \xi_0(\bar{x}),
\]
where
\[
\xi_0(\bar{x}) = \xi_0(x_1, x_2, \ldots, x_n, \ldots) = \delta_{x_1}(\alpha_1) \cdot \delta_{x_2}(\alpha_2) \cdot \ldots \cdot \delta_{x_n}(\alpha_n) \cdot \ldots.
\]
This state fixes the sequence $\{\alpha_i\}$ on the ‘background’ sequence $\{\alpha_i\}$ which was, in section 84, fixed according to relation (84). Again, this states does not lie in the domain of the Dirac operator $D_\Delta$. Note, however, that by choosing the parameters $\alpha_i$’s according to relation (84) one obtains a Dirac-type operator $D_\Delta$ which is, up to an overall factor, invariant under a shift in the parameters $\alpha_i$. Such a shift corresponds to a change of scale.

Next, we may also consider the state
\[
\mathcal{H}_\Delta \ni \Phi_0(\nabla, \bar{x}) = \phi_0(\nabla) \otimes \eta_0(\bar{x}),
\]
where
\[
\eta_0(\bar{x}) = \phi_1(x_1) \cdot \phi_2(x_2) \cdot \ldots \cdot \phi_n(x_n) \cdot \ldots.
\]
Here $\phi_n(x_n)$ is the Gauss distribution which we used to construct the triple $(B_\Delta, D_\Delta$, $\mathcal{H}_\Delta$). Since the state $\eta_0$ lies in the kernel of $D_\Delta$, the state $\Phi_0$ lies in the kernel of $D_\Delta$.

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