On Strong Data-Processing and Majorization Inequalities with Applications to Coding Problems

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Abstract: This work provides data-processing and majorization inequalities for f-divergences, and it considers some of their applications to coding problems. This work also provides tight bounds on the Rényi entropy of a function of a discrete random variable with a finite number of possible values, where the considered function is not one-to-one, and their derivation is based on majorization and the Schur-concavity of the Rényi entropy. One application of the f-divergence inequalities refers to the performance analysis of list decoding with either fixed or variable list sizes; some earlier bounds on the list decoding error probability are reproduced in a unified way, and new bounds are obtained and exemplified numerically. Another application is related to a study of the quality of approximating a probability mass function, which is induced by the leaves of a Tunstall tree, by an equiprobable distribution. The compression rates of finite-length Tunstall codes are further analyzed for asserting their closeness to the Shannon entropy of a memoryless and stationary discrete source. In view of the tight bounds for the Rényi entropy and the work by Campbell, non-asymptotic bounds are derived for lossless data compression of discrete memoryless sources.

Keywords: Cumulant generating functions; f-divergences; list decoding; lossless source coding; Rényi entropy.

1. INTRODUCTION

Divergences are non-negative measures of dissimilarity between pairs of probability measures which are defined on the same measurable space. They play a key role in the development of information theory, probability theory, statistics, learning, signal processing, and other related fields. One important class of divergence measures is defined by means of convex functions f, and it is called the class of f-divergences. It unifies fundamental and independently-introduced concepts in several branches of mathematics such as the chi-squared test for the goodness of fit in statistics, the total variation distance in functional analysis, the relative entropy in information theory and statistics, and it is closely related to the Rényi divergence which generalizes the relative entropy. The class of f-divergences satisfies pleasing features such as the data-processing inequality, convexity, continuity and duality properties, finding interesting applications in information theory and statistics.

Majorization theory is a simple and productive concept in the theory of inequalities, which also unifies a variety of familiar bounds (see the book by Marhall et al. (2011)). The concept of majorization finds various applications in diverse fields of pure and applied mathematics, including information theory and communication.

This work, presented in the papers by Sason (2018, 2019), is focused on new data-processing and majorization inequalities for f-divergences and the Rényi entropy. The reason for discussing both types of inequalities in this work is the interplay which exists between majorization and data processing where a probability mass function P, defined over a finite set, is majorized by another probability mass function Q which is defined over the same set if and only if there exists a doubly-stochastic transformation W_{Y|X} such that an input distribution that is equal to Q yields an output distribution that is equal to P (denoted by, Q \rightarrow W_{Y|X} \rightarrow P).

We consider applications of the inequalities which are derived in this work to information theory, statistics, and coding problems. One application refers to the performance analysis of list decoding with either fixed or variable list sizes; some earlier bounds on the list decoding error probability are reproduced in a unified way, and new bounds are obtained and exemplified numerically. A second application, covered in Sason (2019), is related to a study of the quality of approximating a probability mass function, induced by the leaves of a Tunstall tree, by an equiprobable distribution. The compression rates of finite-length Tunstall codes are further analyzed for asserting their closeness to the Shannon entropy of a memoryless and stationary discrete source. A third application of our bounds relies on our tight bounds for the Rényi entropy (see Sason (2018)) and the source coding theorem by Campbell (1965) to obtain tight non-asymptotic bounds for lossless compression of discrete memoryless sources.
2. CODING PROBLEMS AND MAIN RESULTS

2.1 Bounds on the List Decoding Error Probability with f-divergences

The minimum probability of error of a random variable $X$ given $Y$, denoted by $\varepsilon_{X|Y}$, can be achieved by a deterministic function (maximum-a-posteriori decision rule) $\mathcal{L}^*: \mathcal{Y} \to \mathcal{X}$ (see Sason and Verdú (2018)):

$$\varepsilon_{X|Y} = \min_{\mathcal{L}^*: \mathcal{Y} \to \mathcal{X}} P[X \neq \mathcal{L}^*(Y)] = \max_{x \in \mathcal{X}} P[X \neq \mathcal{L}^*(Y)] = 1 - \mathbb{E} \left[ \max_{x \in \mathcal{X}} P_{X|Y}(x|Y) \right].$$

(1)

(2)

(3)

Fano’s inequality gives an upper bound on the conditional entropy $H(X|Y)$ as a function of $\varepsilon_{X|Y}$ (or, otherwise, providing a lower bound on $\varepsilon_{X|Y}$ as a function of $H(X|Y)$) when $X$ takes a finite number of possible values.

The list decoding setting, in which the hypothesis tester is allowed to output a subset of given cardinality, and an error occurs if the true hypothesis is not in the list, has great interest in information theory. The main idea of the successful combination of these two tools is that, given a code, it is possible to blow-up the decoding sets in a way that the probability of decoding error can be as small as desired for sufficiently large blocklengths; since the blown-up decoding sets are no longer disjoint, the resulting setup is a list decoder with sub-exponential list size (as a function of the block length).

In this section, we further study the setup of list decoding, and derive bounds on the average list decoding error probability. We first consider the special case where the list size is fixed, and then consider the more general case of a list size which depends on the channel observation. All of the following bounds on the list decoding error probability are derived in the paper by Sason (2019).

Fixed-Size List Decoding

The next result provides a generalized Fano’s inequality for fixed-size list decoding, expressed in terms of an arbitrary f-divergence. Earlier results in the literature are reproduced from the next result.

**Theorem 1.** Let $P_{X|Y}$ be a probability measure defined on $\mathcal{X} \times \mathcal{Y}$ with $|\mathcal{X}| = M$. Consider a decision rule $\mathcal{L}: \mathcal{Y} \to (\hat{X})$, where $(\hat{X})$ stands for the set of subsets of $\mathcal{X}$ with cardinality $L$, and $L < M$ is fixed. Denote the list decoding error probability by $P_e := P[X \notin \mathcal{L}(Y)]$. Let $U_M$ denote the convex probability mass function on $\mathcal{X}$. Then, for every convex function $f: (0, \infty) \to \mathbb{R}$ with $f(1) = 0$,

$$\mathbb{E} \left[ D_f(P_{X|Y}|(Y) \| U_M) \right] \geq \frac{L}{M} f \left( \frac{M(1-P_e)}{L} \right) + \left( 1 - \frac{L}{M} \right) f \left( \frac{MP_e}{M-L} \right).$$

(4)

The special case where $L = 1$ (i.e., a decoder with a single output) gives (Guntuboyina, 2011, (5)).

As consequences of Theorem 1, we first reproduce some earlier results as special cases.

**Theorem 2.** (Sason and Verdú, 2018, (139)) Under the assumptions in Theorem 1,

$$H(X|Y) \leq \log M - d \left( P_e \| \frac{L}{M} \right)$$

(5)

where $d(\cdot, \cdot) : [0, 1] \times [0, 1] \to [0, +\infty]$ denotes the binary relative entropy, defined as the continuous extension of $D[p, 1-p] = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}$ for $p, q \in (0, 1)$.

The following refinement of the generalized Fano’s inequality in Theorem 1 relies on the version of the strong data-processing inequality for $f$-divergences in (Sason, 2019, Theorem 1).

**Theorem 3.** Under the assumptions in Theorem 1, let $f: (0, \infty) \to \mathbb{R}$ be twice differentiable, and assume that there exists a constant $m_f > 0$ such that

$$f''(t) \geq m_f, \quad \forall t \in I(\xi_1, \xi_2),$$

(6)

where

$$\xi_1 := M \inf_{(x,y) \in \mathcal{X} \times \mathcal{Y}} P_{X|Y}(x|y),$$

(7)

$$\xi_2 := M \sup_{(x,y) \in \mathcal{X} \times \mathcal{Y}} P_{X|Y}(x|y),$$

(8)

and the interval $I(\cdot, \cdot)$ is the interval $I := I(\xi_1, \xi_2) = [\xi_1, \xi_2] \cap (0, \infty)$.

Let $u^+ := \max\{u, 0\}$ for $u \in \mathbb{R}$. Then,

$$\mathbb{E} \left[ D_f(P_{X|Y}|(Y) \| U_M) \right] \geq \frac{L}{M} f \left( \frac{M(1-P_e)}{L} \right) + \left( 1 - \frac{L}{M} \right) f \left( \frac{MP_e}{M-L} \right) + \frac{1}{2} m_f M \left( \mathbb{E}[P_{X|Y}(X|Y)] - \frac{1-P_e}{L} - \frac{P_e}{L-M} \right)^+. \quad (10)$$

(10)

(b) If the list decoder selects the $L$ most probable elements from $\mathcal{X}$, given the value of $Y \in \mathcal{Y}$, then (10) is strengthened to

$$\mathbb{E} \left[ D_f(P_{X|Y}|(Y) \| U_M) \right] \geq \frac{L}{M} f \left( \frac{M(1-P_e)}{L} \right) + \left( 1 - \frac{L}{M} \right) f \left( \frac{MP_e}{M-L} \right) + \frac{1}{2} m_f M \left( \mathbb{E}[P_{X|Y}(X|Y)] - \frac{1-P_e}{L} \right), \quad (11)$$

(11)

where the last term in the right side of (11) is necessarily non-negative.

Discussions and numerical experimentation of these proposed bounds are provided in the paper by Sason (2019), showing the obtained improvement over Fano’s inequality.

Variable-Size List Decoding

In the more general setting of list decoding where the size of the list may depend on the channel observation, Fano’s inequality has been generalized as follows.

**Theorem 4.** (Ahlswede (1975) and (Raginsky and Sason, 2019, Appendix 3.E)) Let $P_{X|Y}$ be a probability measure defined on $\mathcal{X} \times \mathcal{Y}$ with $|\mathcal{X}| = M$. Consider a decision rule $\mathcal{L}: \mathcal{Y} \to 2^L$, and let the (average) list decoding
error probability be given by \( P_L := \mathbb{P}[X \notin \mathcal{L}(Y)] \) with \(|\mathcal{L}(y)| \geq 1\) for all \( y \in \mathcal{Y} \). Then,
\[
H(X|Y) \leq h(P_L) + \mathbb{E}[\log |\mathcal{L}(Y)|] + P_L \log M, \tag{12}
\]
where \( h: [0, 1] \to [0, \log 2] \) denotes the binary entropy function. If \(|\mathcal{L}(Y)| \leq N\) almost surely, then also
\[
H(X|Y) \leq h(P_L) + (1 - P_L) \log N + P_L \log M. \tag{13}
\]

By relying on the data-processing inequality for \( f \)-divergences, we derive in the following an alternative to \( D_{KL} \) divergences, we derive in the following an alternative

**Theorem 5.** Under the assumptions in (12), for all \( \gamma \geq 1 \),
\[
P_L \geq \frac{1 + \gamma}{2} - \frac{\gamma \mathbb{E}[|\mathcal{L}(Y)|]}{M} - \frac{1}{2} \mathbb{E} \left[ \sum_{x \in X} P_{X|Y}(x|Y) - \frac{\gamma}{M} \right]. \tag{14}
\]

Let \( \gamma \geq 1 \), and let \(|\mathcal{L}(y)| \leq \frac{M}{\gamma}\) for all \( y \in \mathcal{Y} \). Then, (14) holds with equality if, for every \( y \in \mathcal{Y} \), the list decoder selects the \( |\mathcal{L}(y)|\) most probable elements in \( X \) given \( Y = y \); if \( x(y) \) denotes the \( \ell \)-th most probable element in \( X \) given \( Y = y \), where \( \ell \) in probabilities are resolved arbitrarily, then (14) holds with equality if
\[
P_{X|Y}(x|y) = \begin{cases} a(y), & \forall \ell \in \{1, \ldots, |\mathcal{L}(y)|\}, \\ 1 - a(y) |\mathcal{L}(y)|/M & \forall \ell \in \{|\mathcal{L}(y)| + 1, \ldots, M\}, 
\end{cases}
\]
with \( a: \mathcal{Y} \to [0, 1] \) being an arbitrary function which satisfies \( \gamma \leq a(y) \leq \frac{1}{|\mathcal{L}(y)|} \), \( \forall y \in \mathcal{Y} \). \tag{15}

As an example, let \( X \) and \( Y \) be random variables taking their values in \( X = \{0, 1, 2, 3, 4\} \) and \( Y = \{0, 1\} \), respectively, and let \( P_{XY} \) be their joint probability mass function, which is given by
\[
P_{XY}(0,0) = P_{XY}(1,0) = P_{XY}(2,0) = \frac{1}{4}, \quad P_{XY}(3,0) = P_{XY}(4,0) = \frac{1}{16}, \quad P_{XY}(0,1) = P_{XY}(1,1) = P_{XY}(2,1) = \frac{1}{32}, \quad P_{XY}(3,1) = P_{XY}(4,1) = \frac{3}{32}. \tag{17}
\]

Let \( \mathcal{L}(0) := \{0, 1, 2\} \) and \( \mathcal{L}(1) := \{3, 4\} \) be the lists in \( X \), given the value of \( Y \in \mathcal{Y} \). We get \( P_Y(0) = P_Y(1) = \frac{1}{2} \), so the conditional probability mass function of \( X \) given \( Y \) satisfies \( P_{X|Y}(x|y) = 2P_{XY}(x,y) \) for all \((x,y) \in X \times Y \). It can be verified that, if \( \gamma = \frac{1}{2} \), then \( \max(|\mathcal{L}(0)|, |\mathcal{L}(1)|) = 3 \leq \frac{M}{\gamma} \), and also (15) and (16) are satisfied (here, \( M := |X| = 5 \), \( a(0) = \frac{1}{4} = \frac{1}{32} \), and \( \gamma = \frac{1}{2} \leq \frac{1}{32} \)). By Theorem 5, it follows that (14) holds in this case with equality, and the list decoding error probability is equal to \( P_L = 1 - \mathbb{E}[a(Y) |\mathcal{L}(Y)|] = \frac{1}{2} \) (i.e., it coincides with the lower bound in the right side of (14) with \( \gamma = \frac{1}{2} \)). On the other hand, the generalized Fano’s inequality in (12) gives that \( P_L \geq 0.1206 \) (the left side of (12) is \( H(X|Y) = \frac{1}{2} \log 2 - \frac{1}{4} \log 3 = 2.1038 \text{ bits} \)); moreover, by letting \( N := \max_y |\mathcal{L}(y)| = 3 \), (13) gives the looser bound \( P_L \geq 0.0939 \). This exemplifies a case where the lower bound in Theorem 5 is tight, whereas the generalized Fano’s inequalities in (12) and (13) are looser.

### 2.2 Lossless Source Coding

For uniquely-decodable (UD) source codes, Campbell (1965) proposed the cumulant generating function of the codeword lengths as a generalization to the frequently used design criterion of average code length. The motivation in the paper by Campbell (1965) was to control the contribution of the longer codewords via a free parameter in the cumulant generating function; if the value of this parameter tends to zero, then the resulting design criterion becomes the average code length per source symbol; on the other hand, by increasing the value of the free parameter, the penalty for longer codewords is more severe, and the resulting code optimization yields a reduction in the fluctuations of the codeword lengths.

We introduce the coding theorem by Campbell (1965) for lossless compression of a discrete memoryless source (DMS) with UD codes, which serves for our analysis (see Sason (2018)).

**Theorem 6.** Consider a DMS which emits symbols with a probability mass function \( P_X \) defined on a (finite or countably infinite) set \( X \). Consider a UD fixed-to-variable source code operating on source sequences of \( k \) symbols with an alphabet of the codewords of size \( D \). Let \( \ell(x^k) \) be the length of the codeword which corresponds to the source sequence \( x^k := (x_1, \ldots, x_k) \in X^k \). Consider the scaled cumulant generating function of the codeword lengths:
\[
\Lambda_k(\rho) := \frac{1}{k} \log_D \left( \sum_{x^k \in X^k} P_X(x^k) D^{\rho \ell(x^k)} \right), \quad \rho > 0 \tag{18}
\]
where
\[
P_{X^k}(x^k) = \prod_{i=1}^{k} P_X(x_i), \quad \forall x^k \in X^k. \tag{19}
\]

Then, for every \( \rho > 0 \), the following hold:

a) **Converse result:**
\[
\frac{\Lambda_k(\rho)}{\rho} \geq \frac{1}{\log D} H_{\ell_{\max}}(X). \tag{20}
\]

b) **Achievability result:** there exists a UD source code, for which
\[
\frac{\Lambda_k(\rho)}{\rho} \leq \frac{1}{\log D} H_{\ell_{\max}}(X) + \frac{1}{k}. \tag{21}
\]

The bounds in Theorem 6, expressed in terms of the Rényi entropy, imply that for sufficiently long source sequences, it is possible to make the scaled cumulant generating function of the codeword lengths approach the Rényi entropy as closely as desired by a proper fixed-to-variable UD source code; moreover, the converse result shows that there is no UD source code for which the scaled cumulant generating function of its codeword lengths lies below the Rényi entropy. By invoking L’Hôpital’s rule, one gets from (18)
\[
\lim_{\rho \downarrow 0} \frac{\Lambda_k(\rho)}{\rho} = \frac{1}{k} \sum_{x^k \in X^k} P_X(x^k) \ell(x^k) = \frac{1}{k} \mathbb{E}[\ell(X^k)]. \tag{22}
\]
Hence, by letting $\rho$ tend to zero in (20) and (21), it follows
that Campbell’s result in Theorem 6 generalizes the well-known bounds on the optimal average length of UD fixed-to-variable source codes:

$$\frac{1}{\log D} H(X) \leq \frac{1}{k} E[\ell(X^k)] \leq \frac{1}{\log D} H(X) + \frac{1}{k} \\ \text{and (23) is satisfied by Huffman coding. Campbell’s result therefore generalizes Shannon’s fundamental result for the average codeword lengths of lossless compression codes, expressed in terms of the Shannon entropy.}

Following the work by Campbell (1965), non-asymptotic bounds were derived by Courtade and Verdú (2014) for the scaled cumulant generating function of the codeword lengths for $P_X$-optimal variable-length lossless codes. These bounds were used by Courtade and Verdú (2014) to obtain simple proofs of the asymptotic normality of the distribution of codeword lengths, and the reliability function of memoryless sources allowing countably infinite alphabets.

The analysis which leads to the following result for lossless source compression with uniquely-decodable (UD) codes is provided in the paper by Sason (2018).

Let $X_1, \ldots, X_n$ be i.i.d. symbols which are emitted from a DMS according to a probability mass function $P_X$ whose support is a finite set $X$ with $|X| = n$. In order to cluster the data, suppose that each symbol $X_i$ is mapped to $Y_i = f(X_i)$ where $f \in \mathcal{F}_{n,m}$ is an arbitrary deterministic function (independent of the index $i$) with $m < n$. Consequently, the i.i.d. symbols $Y_1, \ldots, Y_n$ take values on a set $\mathcal{Y}$ with $|\mathcal{Y}| = m < |X|$. Consider two UD fixed-to-variable source codes: one operating on the sequences $x^k \in X^k$, and the other one operates on the sequences $y^k \in \mathcal{Y}^k$; let $D$ be the size of the alphabets of both source codes. Let $\ell(x^k)$ and $\ell(y^k)$ denote the length of the codewords for the source sequences $x^k$ and $y^k$, respectively, and let $\Lambda_k(\cdot)$ and $\overline{\Lambda}_k(\cdot)$ denote their corresponding scaled cumulant generating functions (see (18)).

Relying on our tight bounds on the Rényi entropy (of any positive order) in (Sason, 2018, Theorems 1, 2, and Theorem 6, we obtain upper and lower bounds on $\frac{\Lambda_k(\rho) - \overline{\Lambda}_k(\rho)}{\rho}$ for all $\rho > 0$ (see Sason, 2018, Theorem 5). To that end, for $m \in \{2, \ldots, n-1\}$, if $P_X(1) < \frac{1}{m}$, let $\tilde{X}_m$ be the equiprobable random variable on $\{1, \ldots, m\}$; otherwise, if $P_X(1) \geq \frac{1}{m}$, let $\tilde{X}_m \in \{1, \ldots, m\}$ be a random variable with the probability mass function

$$P_{\tilde{X}_m}(i) = \begin{cases} \frac{P_X(i)}{1 - P_X(1)}, & i \in \{1, \ldots, n^*\}, \\ \frac{1}{m - n^*}, & j = n^* + 1, \ldots, m, \end{cases}$$

where $n^*$ is the maximal integer $i \in \{1, \ldots, m-1\}$ such that

$$P_X(i) \geq \frac{1}{m - i} \sum_{j=i+1}^{m} P_X(j). \quad (24)$$

The result in (Sason, 2018, Theorem 5) is of interest since it provides upper and lower bounds on the reduction in the cumulant generating function of close-to-optimal UD source codes as a result of clustering data, and (Sason, 2018, Remark 11) suggests an algorithm to construct such UD codes which are also prefix codes. For long enough sequences (as $k \to \infty$), the upper and lower bounds on the difference between the scaled cumulant generating functions of the suggested source codes for the original and clustered data almost match, being roughly equal to $\rho \left( H(X) - H(X_m^{\tilde{X}_m}) \right)$ (with logarithms on base $D$, which is the alphabet size of the source codes), and as $k \to \infty$, the gap between these upper and lower bounds is less than $0.08607 \log_D 2$. Furthermore, in view of (22),

$$\lim_{\rho \downarrow 0} \frac{\Lambda_k(\rho) - \overline{\Lambda}_k(\rho)}{\rho} = \frac{1}{k} \left( E[\ell(X^k)] - E[\ell(Y^k)] \right), \quad (25)$$

so, it follows from (Sason, 2018, Theorem 5) that the difference between the average code lengths (normalized by $k$) of the original and clustered data satisfies

$$\frac{1}{k} \leq \frac{E[\ell(X^k)] - E[\ell(Y^k)]}{k} - \frac{H(X) - H(X_m^{\tilde{X}_m})}{\log D} \leq 0.08607 \log_D 2, \quad (26)$$

and the gap between the upper and lower bounds is small.

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