The Structure of the Extreme Schwarzschild–de Sitter Space-time

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Abstract

The extreme Schwarzschild–de Sitter space-time is a spherically symmetric solution of Einstein’s equations with a cosmological constant $\Lambda$ and mass parameter $m > 0$ which is characterized by the condition that $9\Lambda m^2 = 1$. The global structure of this space-time is here analyzed in detail. Conformal and embedding diagrams are constructed, and synchronous coordinates which are suitable for a discussion of the cosmic no-hair conjecture are presented. The permitted geodesic motions are also analyzed. By a careful investigation of the geodesics and the equations of geodesic deviation, it is shown that specific families of observers escape from falling into the singularity and approach nonsingular asymptotic regions which are represented by special “points” in the complete conformal diagram. The redshift of signals emitted by particles which fall into the singularity, as detected by those observers which escape, is also calculated.

KEY WORDS: Black hole; cosmological constant; extreme case; global structure; geodesics

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1 INTRODUCTION

Black holes in (anti-) de Sitter space-time have attracted increased attention in the last two decades. Various aspects of the well-known Schwarzschild–de Sitter solution \[1\], \[2\] (a spherically symmetric vacuum solution with a cosmological constant \(\Lambda > 0\)) containing both the black-hole and cosmological horizons have been studied. In particular its global structure \[3\]-\[7\], geodesic motion \[10\]-\[12\], quantum effects \[13\]-\[14\], or behavior of fields in this background \[15\]-\[17\] have been clarified. Other members of a general Kerr-Newman-de Sitter family of solutions \[18\] have also been investigated including the extreme cases \[3\]-\[6\], \[8\]-\[11\], \[17\]-\[28\]. Many papers have concentrated on the interesting problem of the stability of the Cauchy horizon. There has been a growing body of evidence that (in some cases) the Cauchy horizon is classically stable to linear and even non-linear time dependent perturbations. Thus, black hole-de Sitter space-times can serve as counter-examples of the strong censorship hypothesis (see e.g. \[29\] for a detailed discussion and number of references).

Recently, exact multi-black-hole solutions in asymptotically de Sitter universe have been discovered and discussed \[30\]. These space-times describe systems of an arbitrary number of “extremally charged” black holes. Their specific properties have been investigated in subsequent papers \[31\]-\[34\]. In particular, collisions of extreme black holes and the cosmic censorship hypothesis have been studied within this framework.

There are also exact models of black-hole formation in the presence of a cosmological constant \(\Lambda\). In \[35\] it was shown that the Robinson-Trautman type II spacetimes with \(\Lambda\) converge asymptotically to a corresponding spherically symmetric Schwarzschild–(anti-) de Sitter solution for large retarded times (i.e., near the future event horizon) with the initial “perturbation” being radiated away in gravitational waves. In general, the extension of these space-times across the event horizon can only be made with a finite degree of smoothness. However, in the extreme case the horizon is smooth but nonanalytic \[36\].

Some of the results in the above mentioned papers suggest that some properties of extreme black holes are often qualitatively different from those of generic ones. This is the primary motivation of our work in which we concentrate on the solution describing extreme Schwarzschild–de Sitter black holes; we investigate and systematically summarize its properties.

In the next section we introduce the extreme Schwarzschild–de Sitter space-time. In Section 3, we present Kruskal-type null coordinates and we rigorously construct the conformal diagram using a proper conformal factor. Sections 4 and 5 are devoted to construction
of the embedding diagram and synchronous coordinates. In Section 6, a complete investigation of all geodesics (null, timelike, and spacelike) in the extreme Schwarzschild–de Sitter space-time is presented. In Section 7, relative motions described by the equation of geodesic deviation are studied. Section 8 is devoted to an analysis of interesting “asymptotic points” in the space-time, and in Section 9 the redshift viewed by observers approaching these points is calculated. The results are summarized in the concluding Section 10.

2 THE EXTREME SCHWARZSCHILD–DE SITTER SPACE-TIME

The metric of a generic Schwarzschild–de Sitter space-time in standard coordinates is

$$ds^2 = -\Phi dt^2 + \Phi^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) ,$$

where $$\Phi(r) = 1 - 2m/r - (\Lambda/3) r^2$$ with $$\Lambda > 0$$ and $$m > 0$$. For $$0 < 9\Lambda m^2 < 1$$ there exist two positive roots $$r_+$$ and $$r_{++}$$ of $$\Phi(r)$$ such that $$0 < 2m < r_+ < 3m < r_{++}$$. The root $$r_+ = (2/\sqrt{\Lambda}) \cos(\alpha/3 + 4\pi/3)$$, with $$\cos \alpha = -3m\sqrt{\Lambda}$$, describes the black-hole event horizon, and the root $$r_{++} = (2/\sqrt{\Lambda}) \cos(\alpha/3)$$ localizes the cosmological event horizon. Such space-times have been discussed in detail for example in [3]-[12] and elsewhere.

As $$\Lambda$$ approaches its extremal value, $$\Lambda \to 1/9m^2$$, the position of the black-hole horizon $$r_+$$ monotonically increases and the cosmological horizon $$r_{++}$$ decreases to the common value $$3m$$. In this paper we analyze this extreme case of the Schwarzschild–de Sitter space-time which is characterized by the condition $$9\Lambda m^2 = 1$$ (related to the Nariai solution [37], see e.g. [13]). In this case there exists only one degenerate “double” Killing horizon at $$r = 3m$$; this can be seen from the corresponding form of $$\Phi(r)$$,

$$\Phi(r) = -\frac{1}{27m^2} r^2 (r - 3m)^2(r + 6m) .$$

The surface gravity of the horizon is $$\kappa = 0$$. Also, $$\Phi \leq 0$$ everywhere, so that $$r$$ is a time coordinate, $$t$$ is a spatial coordinate, and there is no static region in the extreme Schwarzschild–de Sitter space-time.

3 GLOBAL STRUCTURE

The global structure of the extreme Schwarzschild–de Sitter space-time has already been described, for example in [3], [3] and elsewhere. However, as far as we know, the cor-
responding conformal diagram has not yet been constructed rigorously. For example, in [4], double-null coordinates were found, but the “compactifying” conformal transformation was not given, so that the diagram was only schematic. In [6], the conformal transformation was given but it cannot be applied in both regions above and below the horizon simultaneously. Here we overcome these obstacles and present an exactly constructed conformal diagram together with an explicit form of the conformal factor.

Introducing the Kruskal-type null coordinates \( \hat{u}, \hat{v} \) by
\[
\begin{align*}
u &= \delta \cot \hat{u}, \\
v &= \delta \tan \hat{v},
\end{align*}
\] (3)
where \( \delta = -m(3 - 2 \ln 2) < 0, \ u = t - r^*, \ v = t + r^* \), and the “tortoise” coordinate
\[
r^* = \int \frac{dr}{\Phi} = \frac{9m^2}{r - 3m} + 2m \ln \left| \frac{r + 6m}{r - 3m} \right|,
\] (4)
(an additive constant was chosen such that \( r^* \to 0 \) at \( r \to \infty \)), the metric of the extreme Schwarzschild–de Sitter space-time can be written in the form
\[
ds^2 = -\frac{\delta^2}{27m^2r} \frac{(r + 6m)(r - 3m)^2}{\sin^2 \hat{u} \cos^2 \hat{v}} d\hat{u} d\hat{v} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \] (5)
The conformal diagram which can easily be obtained from Eqs. (3), (4) is drawn in Fig. 1. — the space-time may represent either extreme black holes (Fig. 1 (a)) or white holes (Fig. 1 (b)) in the de Sitter universes. These two possibilities are connected by a simple reflection \( \hat{u} \to -\hat{u}, \hat{v} \to -\hat{v} \); the parametrization (3) gives the white-hole case. The metric (5) is regular even on the horizon given by \( r = 3m \) since
\[
\lim_{r \to 3m} \left| \frac{r - 3m}{\sin \hat{u}} \right| = \lim_{r \to 3m} \left| \frac{r - 3m}{\cos \hat{v}} \right| = \frac{18m^2}{-\delta}
\] (6)
for all finite fixed \( u \) and \( v \), respectively. The causal structure is evident. Any timelike geodesic observer falling from the region \( r > 3m \) (or the infinity \( J^- \) given by \( r = \infty \)) in the black-hole space-time will either cross the horizon \( r = 3m \) and reach the singularity at \( r = 0 \), or escape to one of the “asymptotic points” \( \mathcal{P} \) given by \( u = -\infty, v = +\infty \). In the white-hole case the observers “emanate” from the singularity at \( r = 0 \) (or from the points \( \mathcal{P} \)) and, after crossing the horizon, they reach the future infinity \( J^+ (r = \infty) \) or the asymptotic points \( \mathcal{Q} \) (given by \( u = +\infty, v = -\infty \)).

Now we shall demonstrate that the infinity \( J^+ \) given by \( r = \infty \), i.e. \( \hat{u} + \hat{v} = \frac{\pi}{2} \) in the white-hole space-time (the proof for \( J^- \) in the black-hole case is analogous), is smooth.
and “de Sitter-like”. First, we introduce null coordinates $U, V$ by

$$\cot \hat{u} = \frac{\sqrt{27} m}{-\delta} \ln \left| \cot \frac{U}{2} \right|,$$

$$\tan \hat{v} = \frac{\sqrt{27} m}{-\delta} \ln \left| \cot \left( \frac{\pi}{4} - \frac{V}{2} \right) \right|,$$

in which the metric takes the form

$$ds^2 = -(1 + \frac{6m}{r}) \frac{(r - 3m)^2}{\sin U \cos V} dU dV + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $r = r(r^*)$ according to (4) and $r^* = \frac{\sqrt{27}}{2} m (\ln |\cot \frac{U}{2}| - \ln |\cot(\frac{\pi}{4} - \frac{V}{2})|)$. The relation $(\hat{u}, \hat{v}) \leftrightarrow (U, V)$ given by (7) is a one-to-one correspondence. In particular, $\hat{u} \in (0, \pi)$ is uniformly mapped to $U \in (0, \pi)$ and similarly $\hat{v} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is mapped to $V \in (-\frac{\pi}{2}, \frac{\pi}{2})$. (Moreover, by introducing $\tilde{v} = \frac{\pi}{2} - \hat{v}$ and $\tilde{V} = \frac{\pi}{2} - V$ we observe that $\tilde{v}(\tilde{V})$ is the same function as $\hat{u}(U)$.)

Now, choosing a conformal factor

$$\Omega^2 = \frac{r}{r + 6m} \frac{\sin U \cos V}{(r - 3m)^2},$$

we can write the conformal (unphysical) metric as

$$d\hat{s}^2 = \Omega^2 ds^2 = -dU dV + \frac{r}{r + 6m} \left( \frac{r}{r - 3m} \right)^2 \sin U \cos V (d\theta^2 + \sin^2 \theta d\phi^2).$$

It is straightforward to show that

1. $\Omega(J^+) = 0$,
2. $\nabla \Omega(J^+) \neq 0$ (namely, $\nabla_U \Omega = -\frac{1}{2\sqrt{27} m} = \nabla_V \Omega$ and $\nabla_{\tilde{V}} \Omega = 0 = \nabla_{\tilde{\phi}} \Omega$ on $J^+$),
3. $\tilde{g}^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega (J^+) = -\Lambda/3 < 0$,
4. $d\hat{s}^2 (J^+) = dU^2 + \sin^2 U (d\theta^2 + \sin^2 \theta d\phi^2)$.

We have thus demonstrated that $\Omega$ given by (9) is the proper conformal factor for the extreme Schwarzschild–de Sitter space-time, and that its scri is smooth with geometry and topology $S^3$, i.e. de Sitter-like.

### 4 EMBEDDING DIAGRAM

A natural time-slice in the extreme Schwarzschild–de Sitter space-time (4) is $r = r_0 = \text{const.} \neq 3m$, see Fig. 4. Introducing a coordinate $z$ by $z = \sqrt{-\Phi(r_0)} t$, the metric of the slice is given by

$$ds^2|_{r=r_0} = dz^2 + r_0^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(11)
where $z \in (-\infty, +\infty)$, $\theta \in (0, \pi)$, $\varphi \in (0, 2\pi)$. Therefore, the embedding geometry is a hyper-cylinder $R \times S^2$ with the two-spheres having a constant radius $r_0$. Assuming, without loss of generality, an equatorial section $\theta = \frac{\pi}{2}$, the embedding diagram for $r = r_0$ (in any part of the complete space-time between the points $P$ for $r_0 < 3m$ or points $Q$ for $r_0 > 3m$) is simply an infinite cylinder with axis $z \sim t$ and radius $r_0$. This is clearly similar to the embedding diagram of interior of the Schwarzschild black hole ($r_0 < 2m$ for $\Lambda = 0$).

Note also that the constant-mean-curvature foliation of the Schwarzschild–de Sitter space-time has been found in both extreme and non-extreme cases [8]. This different time-slicing is particularly suitable for numerical studies of gravitational collapse.

5 SYNCHRONOUS COORDINATES

In this section we shall present the synchronous (Lemaître-type) coordinates for the extreme Schwarzschild–de Sitter space-time. These coordinates connected with free particles moving radially outward across the white-hole horizon towards infinity ($r = \infty$) are useful for the discussion of the cosmic no-hair conjecture.

We introduce the coordinates $\tau, R$ by the relations

$$
d\tau = dt - \frac{\sqrt{1 - \Phi}}{\Phi} dr,
$$

$$
dR = -dt + \frac{1}{\Phi \sqrt{1 - \Phi}} dr,
$$

(12)

where $\Phi(r)$ is given by (2). In these coordinates the metric (1) reads

$$
ds^2 = -d\tau^2 + \left(\frac{2m}{r} + \frac{\Lambda}{3} r^2\right) dR^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
$$

(13)

with $m = 1/3\sqrt{\Lambda}$ and $r$ being given in term of $\tau$ and $R$ by

$$
r(\tau, R) = 2^{-1/3} \sqrt{\Lambda} Z (1 - Z^3)^{2/3}, \quad Z = e^{-\sqrt{\Lambda/3} (R + \tau)}.
$$

(14)

The metric (13) is clearly regular at the horizon $r = 3m$ where $2m/r + (\Lambda/3) r^2 = 1$. Privileged timelike geodesic observers with fixed $R = R_0$, $\theta = \theta_0$, $\varphi = \varphi_0$ start from the past central singularity $r = 0$ at $\tau = -R_0$ ($Z = 1$) and than move across the horizon $(Z^3 = 2 - \sqrt{3})$ to infinity $r = \infty$ as $\tau \to \infty$ ($Z = 0$).

One can easily bring the metric (13) into the form in which the de Sitter metric in “standard” coordinates arises explicitly. Introducing $\chi = (2^{1/3}/\sqrt{\Lambda}) \exp(\sqrt{\Lambda/3} R)$, the
metric reads as follows:

\[ ds^2 = -d\tau^2 + e^{2\sqrt{\Lambda/3}\tau} (1 - Z^3)^{4/3} \left[ \left( \frac{1 + Z^3}{1 - Z^3} \right)^2 d\chi^2 + \chi^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] , \]  

(15)

where \( Z \) is now given by

\[ Z = \frac{2^{1/3} e^{-\sqrt{\Lambda/3}\tau}}{\sqrt{\Lambda} \chi} . \]  

(16)

This is an exact form of the extreme Schwarzschild–de Sitter metric in the outgoing comoving coordinates describing a white hole in the de Sitter universe. Keeping the leading order terms in the expansion of (15) for \( \tau \to \infty \), we obtain

\[ ds^2 = -d\tau^2 + e^{2\sqrt{\Lambda/3}\tau} \left[ d\chi^2 + \chi^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \]  

(17)

\[ + \frac{2}{3} (\sqrt{\Lambda}/\chi)^{-3} e^{-\sqrt{\Lambda/3}\tau} \left[ 2d\chi^2 - \chi^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] + O(e^{-4\sqrt{\Lambda/3}\tau}) . \]

The “traces” of the central white hole completely disappear as \( \tau \to \infty \) in full agreement with the cosmic no-hair conjecture — asymptotically (near \( J^+ \)) we get the de Sitter metric written in standard synchronous Friedmann-Robertson-Walker form with the exponentially growing (“inflationary”) expansion factor.

6 GEODESICS

The metric of extreme Schwarzschild–de Sitter space-time (1) is spherically symmetric. So we may, without loss of generality, only consider geodesics which lie in a plane; we choose \( \theta = \frac{\pi}{2} \) here. Considering also the existence of the Killing vectors \( \partial_t \) and \( \partial_\varphi \), the geodesic equations can be written simply as

\[ \dot{t} = \frac{E}{\Phi} , \]  

(18)

\[ \dot{\varphi} = \frac{h}{r^2} , \]  

(19)

\[-\Phi \dot{t}^2 + \Phi^{-1} \dot{r}^2 + r^2 \dot{\varphi}^2 = \epsilon , \]  

(20)

where \( E \) and \( h \) are constants, \( \Phi \) is given by (2) and \( \epsilon = -1, 0, +1 \) for timelike, null and spacelike geodesics, respectively. Here \( \dot{\cdot} = d/d\tau \) where \( \tau \) is a (normalized) affine parameter along the geodesic. For geodesics \( r = \text{const.} \) we also have to consider the equation

\[ \ddot{r} + \frac{1}{2} \Phi \Phi' \dot{t}^2 - \frac{\Phi'}{2\Phi} \dot{r}^2 - r \Phi \dot{\varphi}^2 = 0 , \]  

(21)

\( (\dot{\cdot} = d/dr) \) which for \( r \neq \text{const.} \) follows from (18)-(20).
6.1 Radial geodesics

For radial motion \( \varphi = \varphi_o = \text{const.} \) so that \( h = 0 \). Therefore, (18) – (21) reduce to

\[
\dot{r}^2 = E^2 + \epsilon \Phi , \quad i = \frac{E}{\Phi} .
\]

(22) \hspace{1cm} (23)

Null geodesics \((\epsilon = 0)\) can easily be integrated:

\[
r(\tau) = \pm |E| (\tau - \tau_0) , \quad t(\tau) = t_0 \pm \text{sgn}(E) r^*(r(\tau)) ,
\]

where \( \tau_0 \) and \( t_0 \) are constants, \( r^* \) is given by (4), and \( E \neq 0 \) (for \( E = 0 \) we get orbits which will be discussed bellow). Considering (3), these null geodesics are given by \( \hat{u} = \text{const.} \) or \( \hat{v} = \text{const.} \), i.e., they are indeed represented by straight lines with inclination 45° in the conformal diagram in Fig. 1.

For timelike geodesics \((\epsilon = -1)\) the right hand side of (22) can not be negative and the equation can be integrated numerically. In Fig. 2 we show a typical free fall in the black-hole space-time from \( r_0 > 3m \) as a function the proper time \( \tau \). Observers with \( E \neq 0 \) will reach the curvature singularity at \( r = 0 \) in a finite proper time whereas special observers with \( E = 0 \) will never cross the horizon \( r = 3m \). Instead, they will approach the “asymptotic points” \( \mathcal{P} \) indicated in Fig. 1. Indeed, for \( E = 0 \) the equation of motion (22) can be integrated analytically (assuming \( r(\tau_0) = r_0 \)):

\[
\tau - \tau_0 = \int_r^{r_0} |\Phi(r)|^{-\frac{1}{2}} dr = 3\sqrt{3} m \int_r^{r_0} \sqrt{\frac{r}{r + 6m r - 3m}} \frac{dr}{\sqrt{r^2 + 6mr + r + 3m}}
\]

\[
= 3m \left[ \sqrt{3} \ln(\sqrt{r^2 + 6mr + r + 3m} + \ln \frac{|r - 3m|}{\sqrt{3}\sqrt{r^2 + 6mr + 2r + 3m}} \right]^{r_0}_r .
\]

(25)

Clearly, \( r = \infty \) at \( \tau = -\infty \), and for \( r \to 3m \) we get \( \tau \approx -3m \ln |r - 3m| \to \infty \). Trajectories of timelike radial geodesic observers in the conformal diagram are shown in Fig. 3. The trajectories in Fig. 3 (a) provide an interpretation of the hypersurface \( r = 3m \) as an event horizon \( \mathcal{H} \) for the observers with \( E = 0 \). Also, the expression (25) gives the proper “distance” of the horizon; it is infinite, as in the case of the extreme Reissner-Nordstrom black hole and the extreme Kerr black hole in asymptotically flat space-times (see, e.g. [39]).

Note that the trajectories for \( |E| = 1 \) can be found explicitly — they are given by (14). In fact, these are the geodesics \( R = R_0, \theta = \theta_0, \varphi = \varphi_0 \) in the synchronous coordinates
(13) (or \( \chi = \chi_0 \) in the coordinates (13)). Therefore, the trajectories for \( E = 1 \) shown in Fig. 3(b) coincide with the coordinate lines \( R = R_0 = \text{const.} \) in the conformal diagram.

**Spacelike geodesics** \((\epsilon = +1)\) are bounded for any \( E \neq 0 \), i.e. \( r(\tau) \) is “oscillating” around horizons between \( r_{MIN} < 3m < r_{MAX} \) for which the right hand side of (22) is positive. The result of a numerical integration for different values of \( E \) is presented in Fig. 4 and a typical spacelike trajectory in the conformal diagram is shown in Fig. 5. There are no analogues of such geodesics in the Schwarzschild case (for \( \Lambda = 0 \)). For \( 0 < 9\Lambda m^2 \leq 1 \) the geodesically complete Schwarzschild–de Sitter space-time consists on an infinite number of “universes” and black/white holes which are glued together in the “space-direction” and thus “tachyons” can, in principle, enter all of them.

### 6.2 Nonradial geodesics

For general geodesics \( \varphi \neq \text{const.} \) \((h \neq 0)\) it is convenient to rewrite Eq. (21) by using (18), (19) and introducing the effective potential \( V(r) \):

\[
\dot{r}^2 = E^2 - V(r),
\]

where

\[
V(r) = \Phi(r) \left( \frac{h^2}{r^2} - \epsilon \right).
\]

Note however, that (26)-(27) are not equivalent to (18)-(20) if \( \Phi = 0 \), i.e., for \( r = 3m \); the corresponding motion related to the horizon must be treated separately.

For **null geodesics** \((\epsilon = 0)\) the effective potential is shown in Fig. 3(a). For all values of \( h \) the potential grows monotonically from \( V = -\infty \) for \( r = 0 \) to its maximal value \( V = 0 \) at \( r = 3m \) and then decreases to \( V \to -h^2/27m^2 \) asymptotically as \( r \to +\infty \). Therefore, qualitatively, as null particles with \( E \neq 0 \) move from \( r = \infty \), they decelerate from the initial velocity \( |\dot{r}| = \sqrt{E^2 + h^2/27m^2} \) to \( |\dot{r}| = |E| > 0 \) at \( r = 3m \); subsequently they accelerate so that \( |\dot{r}| \to \infty \) as \( r \to 0 \). Starting from a finite \( r \), the geodesics have finite length before terminating at the physical singularity \( r = 0 \). This behaviour can be understood considering the fact that in the region \( r > 3m \) the particle is “pulled” towards \( \mathcal{J} \) by the \( \Lambda > 0 \) term whereas, for \( r < 3m \), the “attractive” influence of the black/white hole (\( m > 0 \)) becomes dominant. For \( E = 0 \) the geodesics equations can be integrated explicitly to give

\[
\tau - \tau_0 = \pm \frac{3\sqrt{3}m}{|h|} \left( \sqrt{r^2 + 6mr - 3m \ln \left| 6m\frac{\sqrt{3}r^2 + 6mr + 2r + 3m}{r - 3m} \right|} \right),
\]

(28)
so that the horizon is not reached in a finite value of the affine parameter \((\tau \to -\infty \text{ as } r \to 3m)\) in this case.

For timelike geodesics \((\epsilon = -1)\) the effective potential resembles qualitatively the potential for \(\epsilon = 0\), except that \(V \to -\infty \text{ as } r \to +\infty\); it is shown in Fig. 3 (b). Again, the particle’s velocity reaches its minimal value \(|\dot{r}| = |E|\) at \(r = 3m\). The conformal diagram is not suitable for the visualization of nonradial trajectories since it represents only the section \(\theta = \theta_0, \varphi = \varphi_0\). Therefore, in Fig. 7 we draw typical timelike nonradial geodesics of freely falling particles with different nonvanishing \(E\) and \(h\) in the “polar” diagram \(r(\varphi)\). Again, the singularity is reached in a finite proper time of any particle with \(E \neq 0\). Geodesics with \(E = 0\) will be discussed in Section 8. Note that the nature of timelike geodesics is much simpler in the extreme Schwarzschild–de Sitter space-time if compared with non-extreme cases \(0 < 9m^2\Lambda < 1\) (cf. \[12\]) — for example, there are no bounded orbits.

Nonradial spacelike geodesics \((\epsilon = +1)\) in extreme Schwarzschild–de Sitter space-time have a more complex structure since the corresponding effective potential depends on \(h\) in a nontrivial way. It is shown in Fig. 3 for all \(h \neq 0\), \(V \to -\infty \text{ as } r \to 0, V(3m) = 0\) and \(V \to +\infty \text{ as } r \to +\infty\).

For \(0 < |h| < 3m\) the potential has a local minimum at \(r_{\text{min}} = 3m\) and a local maximum at \(r_{\text{max}}, 0 < r_{\text{max}} < 3m\); it is given as the unique solution of the equation

\[
R^2(R^2 + 3R + 9) = 27H^2, \quad (29)
\]

where \(R = r/m\) and \(H = h/m\). The dependence of \(r_{\text{max}}\) and \(V(r_{\text{max}})\) on \(h\) is shown in Fig. 3 \((r_{\text{max}} \approx \sqrt{3}|h|\) for small \(h\)). Therefore, for all \(E^2 < E_{\text{max}}^2 = V(r_{\text{max}})\) there exist bounded orbits (“oscillations” around horizons) in the range \(r_{\text{MIN}} < 3m < r_{\text{MAX}}\); typical bounded orbits in “polar” diagram are drawn in Fig. 14. Of course, as one can see from the effective potential, there are also geodesics in the range \(0 < r < r_{\text{max}} < 3m\). For \(E^2 > E_{\text{max}}^2\) there exist a maximum value of \(r\) for all the geodesics which necessarily reach the singularity at \(r = 0\). Special geodesics \(|E| = |E_{\text{max}}|\) will be discussed in Section 8.

For \(|h| = 3m\) the potential has a point of inflexion at \(r_{\text{min}} = 3m\). Qualitatively, such spacelike “observers” with \(E \neq 0\) fall from a maximum value of \(r\) to the singularity with a growing velocity. Similar motions are found for \(|h| > 3m\) with a difference that the “observers” decelerate in the range \(3m < r < r_{\text{min}}\) since the potential now has a local maximum \(V = 0\) at \(r = 3m\) and a local minimum \(V < 0\) at \(r_{\text{min}}\) which is the solution
of (29). Geodesics with $E = 0$ approaching the “asymptotic points” $P, Q$ we analyze in Section 8.

6.3 Circular orbits

In this section we shall finally establish special nonradial geodesics in the extreme Schwarzschild-de Sitter space-time — circular orbits $r = r_0 = \text{const}$. The effective potential analysis presented above indicates that such orbits may exist only in the extreme of $V(r) \geq 0$, i.e. for $\epsilon = +1, 0 < |h| < 3m$. However, since the case $V = 0$ must be treated separately, it is better to return back to (18)-(20) and (21).

For $r = r_0 \neq 3m$ these equations imply

$$\Phi_0 \left( \frac{h^2}{r_0^2} - \epsilon \right) = E^2, \quad \epsilon r_0^2 \left( 1 - \frac{2\Phi_0}{r_0 \Phi_0'} \right)^{-1} = h^2, \quad i = \frac{E}{\Phi_0}, \quad \dot{\varphi} = \frac{h}{r_0^2}. \quad (30)$$

It can now be shown that there are no circular null or timelike geodesics of this type; there is a unique (unstable) circular spacelike geodesic $0 < r_0 < 3m$ for any $0 < |h| < 3m$ corresponding to the local maximum of the effective potential $r_{\text{max}}$ given by the solution of (29).

For $r = r_0 = 3m$ only Eqs. (19), (20) are nontrivial yielding $r_0^2 \dot{\varphi}^2 = \epsilon$ and $\dot{\varphi} = h/r_0$, respectively, so that $h^2 = \epsilon r_0^2 = 9m^2 \epsilon$. Therefore, there is only one null circular geodesic $r = 3m$ (for $h = 0$) and one spacelike circular orbit (for $|h| = 3m$). Of course, the same result we get if we start from the Kruskal-type coordinates (5).

To summarize, there are no timelike circular geodesics in the extreme Schwarzschild–de Sitter spacetime, and there is a unique null circular geodesic $r = 3m$ (with $E = 0 = h$) — the horizon (cf. [11]). For any $0 < |h| \leq 3m$ there exists a unique unstable spacelike circular orbit $0 < r_0 \leq 3m$. These results contrast with those for non-extreme cases $0 < 9\Lambda m^2 < 1$ (cf. [12]), or for the $\Lambda = 0$ case [10]. For example, in the Schwarzschild space-time the circular photon orbit at $r = 3m$ is situated outside the horizon $r = 2m$.

Note that throughout this section we have used the terms “bounded” or “circular” orbit in connection with the coordinate $r$ although in fact it is a time coordinate. However, it is reasonable to keep such a description since $r$ still measures the “distance from the singularity” situated at $r = 0$. We could alternatively associate these terms with the space coordinate $t$, but using Eq. (18) and considering $\Phi \leq 0$ we see that $t$ is always a monotone function of $\tau$. Only for $E = 0$ the geodesics are “t-circular”. 

11
7 GEODESIC DEVIATION

Let us consider an arbitrary radial timelike observer in the extreme Schwarzschild–de Sitter space-time \((1), (2)\). We can set up an orthonormal parallelly propagated frame

\[
\vec{e}(0) = \dot{t} \partial_t + \dot{r} \partial_r, \quad \vec{e}(1) = \frac{1}{r} \partial_\theta, \quad \vec{e}(2) = \frac{1}{r \sin \theta} \partial_\phi, \quad \vec{e}(3) = \frac{\dot{r}}{\Phi} \partial_t + \Phi \dot{t} \partial_r,
\]

(31)

where \(\vec{e}(0)\) is the four-velocity of the observer; all the coefficients must be evaluated at a given proper time \(\tau\) of the geodesic observer, \(r = r(\tau)\) etc. Projecting the curvature tensor of the space-time onto the frame \((31)\) we get a coordinate-independent form of the equation of geodesic deviation

\[
\ddot{Z}^{(1)} = \frac{\Lambda}{3} Z^{(1)} - \frac{m}{r^3} Z^{(1)},
\]

\[
\ddot{Z}^{(2)} = \frac{\Lambda}{3} Z^{(2)} - \frac{m}{r^3} Z^{(2)},
\]

\[
\ddot{Z}^{(3)} = \frac{\Lambda}{3} Z^{(3)} + \frac{2m}{r^3} Z^{(3)},
\]

(32)

where \(Z^{(i)} = Z^{\mu} e^{(i)}_{\mu}, i = 1, 2, 3\), are frame components of the vector connecting two nearby free test particles. For \(r \to \infty\) the cosmological constant \(\Lambda\) dominates in \((32)\) so that asymptotically \(Z^{(i)} \approx \exp(\sqrt{\frac{\Lambda}{3}} \tau)\) as the observers approach exponentially expanding de Sitter-like infinity (cf. \((17)\)). On the other hand, falling to the singularity at \(r = 0\), the \(\Lambda\)-terms become negligible; the observers are stretched by tidal forces in the radial direction and are squeezed in the perpendicular directions as in the Schwarzschild black-hole spacetime.

Exact solutions can be obtained numerically by a simultaneous integration of \((22)\) and \((32)\). In this section, however, we concentrate on analytic investigation of the behaviour of particles approaching the asymptotic regions given by the “points” \(P\) and \(Q\) along timelike radial geodesics with \(E = 0\). These are given explicitly by \((25)\) so that \(r \approx 3m + A \exp(-\tau/3m)\) as \(\tau \to \infty\), where \(A\) is a constant (positive if approaching \(P\) and negative if approaching \(Q\)). The asymptotic form of \((32)\) is then

\[
\ddot{Z}^{(j)} = a e^{-\tau/3m} Z^{(j)},
\]

\[
\ddot{Z}^{(3)} = \left(\Lambda - 2a e^{-\tau/3m}\right) Z^{(3)},
\]

(33)

where \(j = 1, 2\) and \(a = A\Lambda/3m\). Performing substitutions

\[
T = 6m \sqrt{|a|} e^{-\tau/6m},
\]

\[
\tilde{T} = 6m \sqrt{|2a|} e^{-\tau/6m},
\]

(34)
these equations go over to the Bessel equation so that general solutions of (33) are

\[ Z^{(j)}(T) = A_j J_0(T) + B_j K_0(T), \]
\[ Z^{(3)}(\tilde{T}) = C J_2(\tilde{T}) + D N_2(\tilde{T}), \]  

(35)

for particles approaching \( P \) and

\[ Z^{(j)}(T) = A_j J_0(T) + B_j N_0(T), \]
\[ Z^{(3)}(\tilde{T}) = C J_2(\tilde{T}) + D K_2(\tilde{T}), \]  

(36)

for those approaching \( Q \), where \( A_j, B_j, C, D \) are constants. Expansions of (35) and (36) for \( \tau \to \infty \) give

\[ Z^{(j)} \approx \alpha_j + \beta_j \tau + \gamma_j e^{-\tau/3m} + \cdots, \]  
\[ Z^{(3)} \approx \gamma e^{-\tau/3m} \left(1 - \frac{2A}{9m} e^{-\tau/3m} + \cdots\right) + \delta e^{\tau/3m} \left(1 + \frac{2A}{9m} e^{-\tau/3m} + \cdots\right), \]  

(37)

where \( \alpha_j, \beta_j, \gamma, \delta \) are constants. Therefore, relative motion in the perpendicular directions \( \vec{e}_1, \vec{e}_2 \) is uniform as \( \tau \to \infty \); for a special choice of initial conditions, \( \beta_j = 0 \), we get \( Z^{(j)} \to \text{const} \). Similarly, for \( \delta = 0 \) the motion in the radial direction \( \vec{e}_3 \) is given by \( Z^{(3)} \to 0 \). Thus, relative motion close to \( P \) and \( Q \) is nonsingular. This supports our physical interpretation of \( P \) and \( Q \) as “asymptotic points” representing regions which do not belong to the singularity at \( r = 0 \) or to the de Sitter-like infinity \( r = \infty \), although they seem to “lie” on the same lines in the conformal diagram shown in Fig. 1.

8 NATURE OF THE “ASYMPTOTIC POINTS” \( P, Q \)

In this section we shall look at these regions represented by “points” \( P, Q \) more closely. The “points” \( P \) in the conformal diagram are given by \( u = -\infty, v = +\infty \) whereas \( Q \) are given by \( u = +\infty, v = -\infty \). It was demonstrated in the previous section that relative motions of observers approaching \( P, Q \) differ significantly from those corresponding to \( r \to 0 \) or \( r \to \infty \). In fact, \( P \) and \( Q \) represent asymptotic regions which are reached by a family of special observers such that \( r \to 3m \) as \( \tau \to \infty \); observers from \( r \geq 3m \) reach \( P \), and observers from \( r \leq 3m \) reach \( Q \). Since \( V(r \to 3m) \to 0 \), we see from (26) that these geodesic observers must have \( E = 0 \) (otherwise they would reach \( r = 3m \) in a finite \( \tau \)). Eq. (18) then gives \( t = t_0 = \text{const} \). (note again that \( t \) is a space coordinate) and the trajectories of all such observers in the conformal diagram coincide with those presented in Fig. 3 (a). The effective potentials shown in Figs. 3 (a), (b) and Fig. 8 for
\( \epsilon = 0, -1, +1 \), respectively, indicate that all geodesic observers with \( E = 0 \) can reach \( r = 3m \) asymptotically except for spacelike radial observers \( (\epsilon = +1, h = 0) \). Therefore, the family of geodesics approaching \( \mathcal{P} \) consists of the following:

1. **null radial** \( (\epsilon = 0, h = 0) \); this is the circular orbit on the horizon (see Section 6.3).

2. **timelike radial** \( (\epsilon = -1, h = 0) \); these are given by (23) so that \( r^* \approx 1/(r - 3m) \approx \exp(\tau/3m) \), i.e., \( u = t_0 - r^* \to -\infty, v = t_0 + r^* \to +\infty \) as \( \tau \to \infty \).

3. **null nonradial** \( (\epsilon = 0, h \neq 0) \); they are given by Eq. (28) so that \( r^* \approx \exp(C_1 \tau) \), where \( C_1 = |h|/9\sqrt{3}m^2 \), implying \( u \to -\infty, v \to +\infty \).

4. **timelike nonradial** \( (\epsilon = -1, h \neq 0) \); introducing \( \xi \equiv r - 3m \) the Eqs. (29), (27) can be written for \( r \to 3m \) as \( \xi \approx -C_2 \xi \), where \( C_2 = \sqrt{9m^2 + h^2}/9m^2 \). Therefore, \( r^* \approx 1/\xi \approx \exp(C_2 \tau) \to \infty \) so that \( u \to -\infty, v \to +\infty \).

5. **spacelike nonradial** \( (\epsilon = +1, h \neq 0) \); from Fig. 8 it follows that geodesics of this type with \( E = 0 \) and \( r \geq 3m \) can exist only if \( |h| \geq 3m \). As in the case 4., for \( r \to 3m \) we get \( r^* \approx 1/\xi \approx \exp(C_3 \tau) \to \infty \), where \( C_3 = \sqrt{h^2 - 9m^2}/9m^2 \), i.e., \( u \to -\infty, v \to +\infty \) as \( \tau \to \infty \). Note that trajectories of such observers in the conformal diagram (in contrast to all the previous cases) do not extend to \( r = \infty \). Instead, they “make loops” around \( \mathcal{P} \): tachyons moving “outward” would reach the maximum value of \( r \) (which is \( r = |h| \)) and then they approach the same \( \mathcal{P} \) again asymptotically as \( \tau \to \infty \).

Similarly, the geodesics approaching \( \mathcal{Q} \) are:

1. **null radial** ; this is the circular orbit on the horizon.

2. **timelike radial** ; from (25) it follows that \( r^* \approx 1/(r - 3m) \approx -\exp(\tau/3m) \), i.e., \( u \to +\infty, v \to -\infty \) as \( \tau \to \infty \).

3. **null nonradial** ; Eq. (28) gives \( r^* \approx -\exp(C_1 \tau) \) so that \( u \to -\infty, v \to +\infty \).

4. **timelike nonradial** ; Eqs. (26) and (27) for \( \eta \equiv 3m - r \) give \( \dot{\eta} \approx -C_2 \eta \) so that \( r^* \approx -1/\eta \approx -\exp(C_2 \tau) \to \infty \). Again, \( u \to -\infty, v \to +\infty \) as \( \tau \to \infty \).

5. **spacelike nonradial** ; these geodesics exist for \( |h| \geq 3m \) only. If \( |h| > 3m \) we get \( r^* \approx -\exp(C_3 \tau) \to -\infty \). If \( |h| = 3m \) the geodesics with \( E = 0 \) are approaching the point of inflexion of \( V(r) \) at \( r = 3m \) and in such a case Eqs. (26) and (27) give \( \dot{\eta} \approx \sqrt{2/27m^3}\eta^{3/2} \) so that \( r^* \approx -\tau^2 \to -\infty \). In both cases, \( u \to -\infty, v \to +\infty \).
It can be shown that asymptotic motion such that \( r \to r_0 = \text{const.} \) as \( \tau \to \infty \) is possible only if \( E = V(r_0) \) where \( r_0 \) is either a local maximum or a point of inflexion of the effective potential. Therefore, the above geodesics represent the only geodesic motion approaching \( r = 3m \) asymptotically; i.e. regions \( \mathcal{P} \) and \( \mathcal{Q} \).

Note also that for \( h \) such that \( 0 < |h| < 3m \) there exists a special class of nonradial spacelike geodesics with \( |E| = |E_{\text{max}}| \) (cf. Sec. 6.2) which asymptotically approach \( r_{\text{max}} \) as \( \tau \to \infty \). Here, \( r_{\text{max}} \) depends on \( h \), as indicated in Fig. 9, and represents the local maximum of \( V(r) \), \( 0 < r_{\text{max}} < 3m \). Eq. (23) gives \( t \approx \pm \left( |E_{\text{max}}|/\Phi(r_{\text{max}}) \right) \tau \to \pm \infty \), i.e., these geodesics also seem to approach \( \mathcal{P}, \mathcal{Q} \) in the conformal diagram. However, they converge to \( r_{\text{max}} \) which is different from \( r = 3m \) corresponding to \( \mathcal{P}, \mathcal{Q} \). In fact, \( u = t - r^*(r_{\text{max}}) \to \pm \infty \) and \( v = t + r^*(r_{\text{max}}) \to \pm \infty \).

9 REDSHIFT

Finally, we shall investigate the redshift of signals emitted by a source falling into the singularity \( r = 0 \) in the extreme Schwarzschild–de Sitter space-time representing a black hole, see Fig. 1(a). We shall assume that the source follows a timelike radial geodesic \((\epsilon = -1, h = 0)\) given by Eqs. (22) and (23) with \( E > 0 \). Geodesics of this type are shown in Fig. 3(a); they start in the region \( r > 3m \) and reach the horizon \( r = 3m \), \( t = -\infty \) in a finite proper time. At an event \((t_e, r_e > 3m)\) the source emits a signal with frequency \( \omega_e \) which propagates along the null radial geodesic \( \hat{u} = \text{const.} \) given by Eq. (24). Our goal here is to investigate the redshift \( z = \omega_o/\omega_e - 1 \), where \( \omega_o \) is the frequency of the signal as measured by an observer remaining outside the black-hole horizon. In asymptotically flat black-hole space-times it is standard to choose static distant outer observer, \( r = \text{const.} \to \infty \). Unfortunately, for principal reasons we can not make this natural choice here. The extreme Schwarzschild–de Sitter space-time is not asymptotically flat and contains no static region (\( r \) is a time coordinate, and \( r = \infty \) represents \( J^- \), i.e. time and null past infinity). Moreover, as we have observed in Section 6, most observers starting at \( r > 3m \) cross the horizon and fall into the singularity. The only “reasonable” outer observers are those approaching the nonsingular asymptotic region given by \( \mathcal{P} \). Therefore, we shall assume that \( \omega_o \) is detected by an observer moving along a timelike radial geodesic with \( E = 0 \) given \( r_o = -\sqrt{-\Phi(r_o)}, t_o = \text{const.}, \) with \( \tau \) being observer’s proper time (see Eqs. (22), (23)); typical trajectories of this type are shown in Fig. 3(a).

It is now straightforward to calculate the redshift using the well known formula \( z = (k_\mu u^\mu)_e/(k_\mu u^\mu)_o - 1 \), where \( u^\mu_e = (E/\Phi(r_e), -\sqrt{E^2 - \Phi(r_e)}, 0, 0) \) is the four-velocity of
the source, \( u_\mu^o = (0, -\sqrt{-\Phi(r_o)}, 0, 0) \) is the four-velocity of the observer, and \( k^\mu = (-1/\Phi(r), -1, 0, 0) \) is the null vector tangent to the photon trajectory; we get

\[
z = \sqrt{-\Phi(r_o)} \frac{E + \sqrt{E^2 - \Phi(r_e)}}{-\Phi(r_e)} - 1.
\] (38)

For a source approaching the horizon, \( r_e \to 3m \), we have also \( r_o \to 3m \) (since \( 3m < r_o < r_e \)). Near the horizon, \( \Phi(r) \) given by (2) can be written as \( \Phi(r) \approx -\left(r - 3m\right)^2/9m^2 \), so that we can express (38) in the form \( z \approx 6mE(r_o - 3m)/(r_e - 3m)^2 - 1 \). It only remains to find a relation between \( r_e \) and \( r_o \). Since the emission and observation events are connected by a photon trajectory \( u = \text{const.} \), we have \( t_e - r^*(r_e) = t_o - r^*(r_o) \), where \( r^* \) is given by (4). The relation between \( t_e \) and \( r_e \) follows from Eqs. (22) and (23), \( t_e + \text{const.} = -E \int [\Phi(r_e)\sqrt{E^2 - \Phi(r_e)}]^{-1} dr_e \approx -\int \Phi^{-1}(r_e) dr = -r^*(r_e) \). Thus, \( 2r^*(r_e) = r^*(r_o) + \text{const.} \), implying \( (r_e - 3m) \approx 2(r_o - 3m) \). Considering finally \( r_o - 3m \approx \exp(-\tau/3m) \) (see (23)), we arrive at the formula

\[
z \approx \exp\left(\frac{\tau}{3m}\right)\). \] (39)

The redshift grows exponentially with the characteristic e-folding time \( \tau_e = 3m = 1/\sqrt{\Lambda} \). This result may seem somewhat surprising since, for extreme Reissner-Nordstrom and extreme Kerr black holes, the redshifts are given by power laws \( \text{[39], [41]} \). However, we should emphasize again that these redshifts were calculated with respect to distant observers “\( r = \infty \)” in asymptotically flat solutions, contrary to our case where \( r \to 3m \) for observers approaching the point \( \mathcal{P} \) in non-asymptotically flat extreme Schwarzschild-de Sitter black hole space-time.

10 CONCLUSION

We have analyzed the extreme Schwarzschild-de Sitter space-time describing a spherically symmetric black (or white) holes in the de Sitter universe characterized by the condition \( 9\Lambda m^2 = 1 \). Coordinates suitable for rigorous discussion of the global structure and the cosmic no-hair conjecture have been introduced. All possible geodesic motions have also been investigated and, with the help of the equation of geodesic deviation, the nature of specific nonsingular “asymptotic points” \( \mathcal{P}, \mathcal{Q} \) in the conformal diagram has been studied. It has been demonstrated that they represent whole asymptotic regions for large classes of geodesics (\( \mathcal{P} \) separate singularities of different black/white holes and \( \mathcal{Q} \) separate different de Sitter-like past/future infinites). Observers approaching these regions radially detect exponentially growing redshift of signals emitted by particles falling to the singularity.
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Figure 1: Conformal diagram of the extreme Schwarzschild–de Sitter space-time with $9\Lambda m^2 = 1$. (a) The singularity $r = 0$ in future, corresponding to black holes. The maximal analytic extension of the geometry is obtained by glueing an infinite number of regions shown in the figure, or joining a finite number of regions via identification of events along two horizons $r = 3m$. (b) The time-reversed diagram $(\dot{u} \rightarrow -\dot{u}, \dot{v} \rightarrow -\dot{v})$, corresponding to white holes.

Figure 2: A geodesic timelike observer falling radially will reach the horizon $r = 3m$ and the black-hole singularity $r = 0$ in a finite proper time $\tau$. Only if the constant of motion $E$ vanishes, $\tau \rightarrow \infty$ as $r \rightarrow 3m$. Here we assume $r = r_0 = 10m$ at $\tau = 0$.

Figure 3: Trajectories of typical timelike radial geodesics in the conformal diagram for observers: (a) with $E = 0$ which are approaching asymptotic points $P, Q$, (b) with $E \neq 0$ (here we assume $E = 1$).

Figure 4: Spacelike radial geodesics oscillate around $r = 3m$ corresponding to different horizons. The amplitude of oscillations grows with a growing value of $E$.

Figure 5: Typical trajectory of a spacelike radial geodesic in the conformal diagram of the extreme Schwarzschild-de Sitter space-time (here we assume $E = 1$).

Figure 6: Effective potentials $V(r)$ for nonradial motions in the extreme Schwarzschild–de Sitter space-time for (a) null geodesics, (b) timelike geodesics.

Figure 7: Typical timelike nonradial geodesics drawn in the “polar” diagram, $r(\varphi)$, for different values of $E$, (a) for $h = 0.2m$, (b) for $h = m$. Starting from $r = 10m$ they cross the horizon $r = 3m$ and reach the singularity $r = 0$ in a finite value of the proper time.

Figure 8: Effective potentials $V(r)$ for nonradial spacelike geodesics depend significantly on $h$. See the text for more details.

Figure 9: Plots of the local maximum $r_{\text{max}}$ of the effective potential and $V(r_{\text{max}})$ for nonradial spacelike geodesics with $0 < |h| < 3m$ as a function of $h$.

Figure 10: Typical spacelike nonradial geodesics drawn in the “polar” diagram, $r(\varphi)$, for different values of $E$, (a) for $h = 0.2m$, (b) for $h = 0.5m$. 
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