Convergence analysis of variants of the averaged alternating modified reflections method *

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Abstract

This paper presents new variants of the averaged alternating modified reflections (AAMR) method for the best approximation problem. Under a mild constraint qualification, we first show its weak convergence and then establish a convergence rate. Furthermore, under a standard interior-point-like condition, we show that the method has a finite termination property.

Keywords: averaged alternating modified reflections method, best approximation problem, weak convergence, rate of convergence, finite termination, Hilbert space

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1 Introduction

Let $A$ and $B$ be closed convex subsets of a real Hilbert space $H$. We consider the problem of finding the closest point from a given point $x_0$ in $H$ to $A \cap B$, i.e.,

$$ \text{minimize} \ |u - x_0| \quad \text{subject to} \quad u \in A \cap B. $$

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Problem (1) is called the best approximation problem with respect to \( A \cap B \) and this problem is of considerable importance in data analysis and modeling, control system design and signal processing [3, 5, 10, 11, 15, 18, 19]. In the case when \( A \) is the set of \( N \times N \) symmetric positive semidefinite matrices \( S_N^+ \) and \( B \) is an appropriate subset of the set of \( N \times N \) symmetric matrices \( S_N \) respectively, several type matrix approximation problems can be described as (1) on space \( S_N \) (see, for instance, patterned covariance matrix problems [11, Chapter 6], controller design problems [15, Chapter 10] and well-conditioned positive definite matrix approximation problems [18, 19]).

The method discussed in this paper is the averaged alternating modified reflections (AAMR) method. The AAMR method was introduced by Aragón Artacho and Campoy [1] to solve the best approximation problem with respect to convex feasibility problems. The framework of the method for closed convex sets \( A \) and \( B \) is as follows: Given \( x_0 \in H \) and \( q \in H \),

\[
x_{n+1} = T_{A-q,B-q,\alpha,\beta}(x_n), \quad n = 0, 1, 2, \ldots,
\]

where \( \alpha, \beta \in (0, 1) \), \( T_{A-q,B-q,\alpha,\beta} : H \to H \) is the averaged alternating modified reflections operator defined by

\[
T_{A-q,B-q,\alpha,\beta} = (1 - \alpha)I + \alpha(2\beta P_B - q - I)(2\beta P_A - q - I),
\]

\( I \) denotes the identity mapping, \( C + p \) denotes a set \( C \) shifted by a point \( p \), i.e., \( C + p = \{ c + p : c \in C \} \) and \( P_C \) denotes the metric projection onto \( C \). If \( A \cap B \neq \emptyset \), under the constraint qualification

\[
q - P_{A \cap B}(q) \in (N_A + N_B)(P_{A \cap B}(q)),
\]

where \( N_A \) and \( N_B \) denote the normal cones to the sets \( A \) and \( B \), respectively, Aragón Artacho and Campoy [1, Theorem 4.1] showed that the sequence generated by (2) weakly converges to a point \( x^* \in H \), such that

\[
P_A(x^* + q) = P_{A \cap B}(q).
\]

That is, \( P_A(x^* + q) \) solves the problem (1) when \( q = x_0 \).

Assume that \( R_{A-q,B-q,\beta} = (2\beta P_B - q - I)(2\beta P_A - q - I) \) in order to simplify the notation. Since \( R_{A-q,B-q,\beta} \) is nonexpansive (see [1, Proposition 3.3]), (2) can be viewed as the Krasnosel’skiĭ-Mann fixed point iteration with respect to \( R_{A-q,B-q,\beta} \), and this method generates weakly convergent iteration sequences (see, e.g., [3, Subchapter 5.2]). Moreover, the weak cluster points of these weakly convergent iteration sequences only solve the following fixed point equation \( R_{A-q,B-q,\beta}(u) = u \). However, it is not guaranteed whether the weak cluster points solve the best approximation problem.

The goal of this paper is three-fold. First, we show an enhanced weak convergence result for a variant of the AAMR method. Second, we establish
its convergence rate. The third purpose is to analyze the finite termination property.

To describe our goal more concretely, we introduce the following variant of the AAMR method for solving (1):

$$y_n = P_A(x_n + q), \ n = 0, 1, 2, \ldots,$$

where \( \{x_n\} \) is the sequence generated by (2). As we have mentioned, sequences generated by the AAMR method (2) are weakly convergent. But it is not clear whether the sequence \( \{y_n\} \) generated by (5) weakly converges to \( P_A(x^* + q) \) since \( P_A \) is in general not sequentially weakly continuous [20]. By using the demiclosedness principle in [2], we show that \( \{y_n\} \) weakly converges to \( P_A(x^* + q) \), without any other restrictions.

Our second purpose is to analyze the convergence rate for (5). To establish the convergence rate, we thus will use the following residual function

$$r(x) = \|P_A(x + q) - P_B(P_A(x + q))\|$$

as a measure of the convergence rate. Clearly, if \( r(x_n) = 0 \) then \( y_n = P_B(y_n) \), so \( y_n \) is in \( A \cap B \) because \( y_n \) is in \( A \) for all \( n \in \mathbb{N} \). On the other hand, if \( r(x_n) \) is large, then \( y_n \) is to be far away from the set \( B \). Therefore, the quantity \( r(x_n) \) can be viewed as a measure of the distance between the iteration \( y_n \) and the set \( B \). Recently, a comprehensive convergence rate analysis for operator splitting methods was studied in [9]. Using a useful technique established in [9, Lemma 2.1], we show that \( r(x_n) = o \left( \frac{1}{\sqrt{n}} \right) \), where the notation \( o \) means that \( s_n = o \left( \frac{1}{t_n} \right) \) if and only if \( \lim_{n \to \infty} s_n t_n = 0 \).

Our third purpose is to analyze the finite termination property of a variant of (5). Recently, under a standard interior-point-like condition \( (A \cap \text{int}B \neq \emptyset) \), finite termination of projection-type iterative methods was studied in [4, 13, 14]. Using the techniques developed in [13, 14], we show that a variant of (5) terminates finitely to a point in \( A \cap \text{int}B \).

The rest of this paper is organized as follows. In section 2, some preliminaries are presented. In section 3, we discuss the weak convergence of (5). Then, we discuss the convergence rate of (5) in section 4. Moreover, we investigate the finite termination in section 5. Finally, we make some conclusions in section 6.

## 2 Basic definitions and preliminaries

The following notations will be used in this paper: \( \mathbb{R} \) denotes the set of real numbers; \( \mathbb{N} \) denotes the set of nonnegative integers; \( H \) denotes a real
Hilbert space; for any \( x, y \in H \), \( \langle x, y \rangle \) denotes the inner product of \( x \) and \( y \); for any \( z \in H \), \( \| z \| \) denotes the norm of \( z \), i.e., \( \| z \| = \sqrt{\langle z, z \rangle} \); for any \( \{ x_n \} \subset H \), \( x_n \rightharpoonup x \) denotes weak convergence, i.e., \( \langle x_n, x^* \rangle \to \langle x, x^* \rangle \) \((n \to \infty)\) \((\forall x^* \in H)\); for any \( w \in H \) and \( A + w \) denotes \( A \) shifted by \( w \), i.e., \( A + w = \{ a + w : a \in A \} \); for any \( r > 0 \), \( B(x, r) \) denotes a closed ball with center \( x \) and radius \( r \), i.e., \( B(x, r) = \{ v \in H : \| x - v \| \leq r \} \); \( \text{int} A \) denotes the interior of set \( A \); \( A^c \) denotes the complement of \( A \); for any \( A, B \subset H \), \( \text{dist}(A, B) \) denotes the distance between two sets \( A \) and \( B \), i.e., \( \text{dist}(A, B) = \inf \{ \| x - y \| : x \in A, y \in B \} \); for any \( C \subset H \) and mapping \( U : C \to H \), \( \text{Fix}(U) \) denotes the fixed point set of \( U \), i.e., \( \text{Fix}(U) = \{ x \in C : U(x) = x \} \).

Let \( C \) be a closed and convex subset of \( H \). A mapping \( U : C \to H \) is said to be

(i) **firmly nonexpansive** if
\[
\| U(x) - U(y) \|^2 \leq \langle x - y, U(x) - U(y) \rangle \quad (x, y \in C);
\]

(ii) **nonexpansive** if
\[
\| U(x) - U(y) \| \leq \| x - y \| \quad (x, y \in C);
\]

(iii) **\( \alpha \)-averaged** for \( \alpha \in (0, 1) \), if there exists a nonexpansive mapping \( R : C \to H \) such that
\[
U = (1 - \alpha)I + \alpha R.
\]

The **metric projection** of a point \( x \in H \) onto \( C \), denoted by \( P_C(x) \), is defined as a unique solution to problem

\[
\text{minimize } \| x - y \| \quad \text{subject to } y \in C.
\]

We know that \( P_C \) is (firmly) nonexpansive and satisfies \( P_{x+C}(y) = P_C(y-x) + x \) for all \( x, y \in H \). See [3], [10], [16] and [11] for further information on metric projections. The **normal cone** to \( C \) at \( x \) is defined by
\[
N_C(x) = \{ v \in H : \langle v, y - x \rangle \leq 0 \text{ for all } y \in C \}.
\]

Let \( A \) and \( B \) be nonempty, closed and convex subsets of \( H \). Given \( \alpha, \beta \in (0, 1) \), we define the **averaged alternating modified reflections (AAMR)** operator \( T_{A,B,\alpha,\beta} : H \to H \) as
\[
T_{A,B,\alpha,\beta} = (1 - \alpha)I + \alpha (2\beta P_B - I)(2\beta P_A - I).
\]

Assume that \( R_{A,B,\beta} = (2\beta P_B - I)(2\beta P_A - I) \). We list the following useful properties of \( T_{A,B,\alpha,\beta} \) and \( R_{A,B,\beta} \):
(1) \((2\beta P_A - I)\) (resp. \((2\beta P_B - I)\)) is nonexpansive and \(T_{A,B,a,\beta}\) is \(\alpha\)-averaged;

(2) For any \(q \in H\),

(a) \(\text{Fix}(T_{A-q,B-q,a,\beta}) = \text{Fix}(R_{A-q,B-q,\beta})\);

(b) \(\text{Fix}(T_{A-q,B-q,a,\beta}) \neq \emptyset\) if and only if \(A \cap B \neq \emptyset\) and \(q\) satisfies (3).

See [1, Sections 3 and 4] for more details.

Let \(C\) and \(D\) be two closed and convex subsets of \(H\). The condition (3) is important to guarantee the existence of fixed points of \(T_{A-q,B-q,a,\beta}\). The following notion is closely related to (3). The pair of sets \(\{C, D\}\) is said to have the strong conical hull intersection property (strong CHIP) at \(x \in C \cap D\) if \(N_{C \cap D}(x) = N_C(x) + N_D(x)\). We say \(\{C, D\}\) has the strong CHIP if it has the strong CHIP at each \(x \in C \cap D\). In particular, it was shown in [1, Proposition 4.1] that, for all \(q \in H\), \(q\) satisfies (3) if and only if \(\{A, B\}\) has the strong CHIP. A well-known sufficient condition for the strong CHIP is the following standard interior-point-like condition, \(A \cap \text{int}B \neq \emptyset\). For more general sufficient conditions for the strong CHIP, see [7, 10]. The condition \(A \cap \text{int}B \neq \emptyset\) and the following result will be useful in Section 5.

Lemma 2.1. Let \(A\) and \(B\) be nonempty sets in \(H\). If \(A \cap \text{int}B \neq \emptyset\), then for any \(e \in H\), there exists \(\gamma > 0\) such that \(A \cap \text{int}(B + \gamma e) \neq \emptyset\).

Proof. Let \(u \in A \cap \text{int}B\). Then, there exists \(r > 0\) such that \(B(u, r) \subset B\). We can choose sufficiently small \(\gamma > 0\) to make the following holds:

\[
\|u - (u - \gamma e)\| = \gamma\|e\| \leq r.
\]

This implies that \(u - \gamma e \in B(u, r) \subset B\) and hence \(u - \gamma e \in \text{int}B\). Since \((\text{int}B + \gamma e) \subset \text{int}(B + \gamma e)\) (see, e.g., [17]), we can therefore conclude that \(u \in A \cap (\text{int}B + \gamma e) \subset A \cap \text{int}(B + \gamma e)\). \(\square\)

3 Weak convergence result

This section shows the weak convergence of the modification of the AAMR method.

We consider the following iterative method. Choose \(x_0, q \in H\) and \(\alpha, \beta \in (0, 1)\) and consider the iterative scheme

\[
\begin{cases}
  y_n = P_A(x_n + q) \\
  x_{n+1} = T_{A-q,B-q,a,\beta}(x_n), \quad n = 0, 1, 2, \ldots.
\end{cases}
\] (7)

Before we proceed with the convergence analysis of (7), we introduce the following result.
Proposition 3.1. [2, Theorem 2.10] Set \( I = \{1, 2, \ldots, m\} \), where \( m \) is an integer greater than or equal to 2. Let \( \{F_i\}_{i \in I} \) be a family of firmly nonexpansive mappings on \( H \), and let, for each \( i \in I \), \( \{z_{i,n}\} \) be a sequence in \( H \) such that for all \( i, j \in I \),

\[
\begin{align*}
  z_{i,n} &\rightharpoonup z_i \text{ and } F_i(z_{i,n}) \rightharpoonup x, \\
  \sum_{i \in I} (z_{i,n} - F_i(z_{i,n})) &\to -mx + \sum_{i \in I} z_i, \\
  F_i(z_{i,n}) - F_j(z_{j,n}) &\to 0.
\end{align*}
\]

Then \( F_i(z_i) = x \), for every \( i \in I \).

The first main result is stated as follows.

Theorem 3.1. Let \( A \) and \( B \) be closed and convex sets in \( H \) and let \( \{y_n\} \) be the sequence generated by (7). If \( A \cap B \neq \emptyset \) and \( q \in (N_A + N_B)(P_{A \cap B}(q)) \), then \( \{y_n\} \) weakly converges to \( P_{A \cap B}(q) \).

Proof. Using [1, Remark 3.2 and Corollary 4.1], we have \( \text{Fix}(R_{A-q,B-q,\beta}) \neq \emptyset \). Let \( u \in \text{Fix}(R_{A-q,B-q,\beta}) \). Since \( \{x_n\} \) can be viewed as the Krasnosel’ski˘ı-Mann fixed point iteration with respect to nonexpansive mapping \( R_{A-q,B-q,\beta} \), by virtue of [3, Theorem 5.14], we have that, for any \( n \in \mathbb{N} \),

\[
\begin{align*}
\alpha (1-\alpha) \| (I - R_{A-q,B-q,\beta})(x_n) \|^2 &\leq \| x_n - u \|^2 - \| x_{n+1} - u \|^2 \quad (8) \\
\| (I - R_{A-q,B-q,\beta})(x_n) \|^2 &\leq \| (I - R_{A-q,B-q,\beta})(x_n) \|^2. \quad (9)
\end{align*}
\]

Moreover,

\[
\| (I - R_{A-q,B-q,\beta})(x_n) \| \to 0 \quad (n \to \infty). \quad (10)
\]

By [1, Theorem 4.1],

\[
x_n \rightharpoonup x^* \quad (n \to \infty), \quad (11)
\]

such that \( P_A(x^* + q) = P_{A \cap B}(q) \). Since \( P_{A-q} \) is firmly nonexpansive and \( \{x_n\} \) is bounded, \( \{P_{A-q}(x_n)\} \) is bounded. Then, there exists a subsequence \( \{P_{A-q}(x_{n_k})\} \) of \( \{P_{A-q}(x_n)\} \) such that \( \{P_{A-q}(x_{n_k})\} \) weakly converges to some \( x \in H \) and hence

\[
P_{A-q}(x_{n_k}) \rightharpoonup x \quad (k \to \infty). \quad (12)
\]

To simplify the notation, define

\[
w_n = 2\beta P_{A-q}(x_n) - x_n, \quad n = 0, 1, 2, \ldots.
\]
From the definition of $R_{A-q,B-q,\beta}$, we have
\[
I - R_{A-q,B-q,\beta} = I - (2\beta P_{B-q} - I)(2\beta P_{A-q} - I)
= I - 2\beta P_{B-q}(2\beta P_{A-q} - I) + 2\beta P_{A-q} - I
= 2\beta(P_{A-q} - P_{B-q}(2\beta P_{A-q} - I)).
\]
This together with (10) yields
\[
2\beta\|P_{A-q}(x_n) - P_{B-q}(w_n)\| \to 0 \quad (n \to \infty),
\]
and hence
\[
\|P_{A-q}(x_n) - P_{B-q}(w_n)\| \to 0 \quad (n \to \infty). \tag{13}
\]
This implies that \{\(P_{B-q}(w_{n_k})\)\} weakly converges to \(x\) and hence
\[
P_{B-q}(w_{n_k}) \rightharpoonup x \quad (k \to \infty). \tag{14}
\]
Using (12) and (14), we have
\[
w_{n_k} \to 2\beta x - x^* \quad (k \to \infty), \tag{15}
\]
and set \(w^* = 2\beta x - x^*\). Using (11), (12), (14) and (15), we have
\[
x_{n_k} - P_{A-q}(x_{n_k}) + w_{n_k} - P_{B-q}(w_{n_k}) \to -2x + x^* + w^* \quad (k \to \infty). \tag{16}
\]
Therefore, the assumptions of Proposition 3.1 are satisfied at this theorem by taking
\[
z_{1,k} = x_{n_k}, F_1(z_{1,k}) = P_{A-q}(x_{n_k}), z_{2,k} = w_{n_k}, F_2(z_{2,k}) = P_{B-q}(w_{n_k}),
\]
and we have that
\[
P_{A-q}(x^*) = x.
\]
Since \(x\) is an arbitrary weak cluster point of \{\(P_{A-q}(x_n)\)\}, we conclude that
\[
P_{A-q}(x_n) \rightharpoonup P_{A-q}(x^*) \quad (n \to \infty).
\]
This together with the property of \(P_A\) yields
\[
P_A(x_n + q) \to P_A(x^* + q) \quad (n \to \infty).
\]
Remark 3.1. Since \( P_A(x^* + q) = P_{A \cap B}(q) \) (see [1, Proposition 3.4]), (7) generates a sequence weakly converging to the unique solution to the best approximation problem (1). That is, (7) can directly be applied to solve problem (1). Moreover, we can also show that

\[
P_B(2\beta P_A(x_n + q) - x_n) \rightharpoonup P_A(x^* + q) \quad (n \to \infty).
\]

The proof is much the same as that of Theorem 3.1.

Remark 3.2. When \( H \) is finite-dimensional, \( \{x_n\} \) strongly converges, and hence \( \{y_n\} \) strongly converges to \( P_A(x^* + q) \). Numerical results of (7) were presented in [1, Section 7] to demonstrate the efficiency in comparison with existing algorithms. However, in infinite-dimensional Hilbert space, the weak convergence of \( \{y_n\} \) was not guaranteed because \( P_A \) may fail to be sequentially weakly continuous [2, 20]. We showed weak convergence of \( \{y_n\} \), without any other restrictions.

4 Convergence rate result

We next establish the convergence rate of (7). To estimate the convergence rate, we consider the following residual function

\[
r(x) = \|P_A(x + q) - P_B(P_A(x + q))\|.
\]  

(17)

Let \( \{x_n\} \) be a sequence generated by (2). Then, from the definition of (17), \( r \) has the following properties:

- \( r(x) \geq 0 \ (x \in H) \);
- \( r(x_n) = \|y_n - P_B(y_n)\| \);
- \( r(x) = 0 \) if and only if \( P_A(x + q) = P_B(P_A(x + q)) \in A \cap B \).

The next lemma is useful to our proof of the convergence rate theorem.

Lemma 4.1. [9, Lemma 1.2] Let \( \{\alpha_n\} \) be the sequence in \( \mathbb{R} \) such that

1. \( \alpha_n \geq 0 \);
2. \( \sum_{i=0}^{\infty} \alpha_i < \infty \);
3. \( \{\alpha_n\} \) is monotonically non-increasing,

then \( \alpha_n = o\left(\frac{1}{n}\right) \), where the notation \( o \) means that \( \alpha_n = o\left(\frac{1}{n}\right) \) if and only if \( \lim_{n \to \infty} \alpha_n \cdot n = 0 \).
The second main result is stated as follows.

**Theorem 4.1.** Let \( A \) and \( B \) be closed and convex sets in \( H \) and let \( \{y_n\} \) be the sequence generated by (7). If \( A \cap B \neq \emptyset \) and \( q - P_{A \cap B}(q) \in (N_A + N_B)(P_{A \cap B}(q)) \), then \( r(x_n) = o\left(\frac{1}{\sqrt{n}}\right) \).

**Proof.** Let \( u \in \text{Fix}(R_{A-q,B-q,\beta}) \). By (8) in the proof of Theorem 3.1, we have, for any \( n \in \mathbb{N} \),

\[
\alpha(1 - \alpha)\| (I - R_{A-q,B-q,\beta})(x_n) \|^2 \leq \| x_n - u \|^2 - \| x_{n+1} - u \|^2.
\]

Summing up from \( j = 0 \) to \( k \),

\[
\alpha(1 - \alpha) \sum_{j=0}^{k} \| (I - R_{A-q,B-q,\beta})(x_j) \|^2 \leq \| x_0 - u \|^2 - \| x_{k+1} - u \|^2 \leq \| x_0 - u \|^2,
\]

and hence

\[
\sum_{j=0}^{\infty} \| (I - R_{A-q,B-q,\beta})(x_j) \|^2 < \infty.
\]

Obviously, \( \| (I - R_{A-q,B-q,\beta})(x_n) \|^2 \geq 0 \), using the above result and (9), the assumptions of Lemma 4.1 are satisfied at this theorem by taking

\[
\alpha_n = \| (I - R_{A-q,B-q,\beta})(x_n) \|^2,
\]

and hence

\[
\| (I - R_{A-q,B-q,\beta})(x_n) \|^2 = o\left(\frac{1}{n}\right).
\]

This implies that

\[
n \| (I - R_{A-q,B-q,\beta})(x_n) \|^2 \to 0 \ (n \to \infty),
\]

and hence

\[
\sqrt{n} \| (I - R_{A-q,B-q,\beta})(x_n) \| \to 0 \ (n \to \infty).
\]  

Using \( I - R_{A-q,B-q,\beta} = 2\beta(P_{A-q} - P_{B-q}(2\beta P_{A-q} - I)) \) and the property of the metric projection, we have

\[
\| (I - R_{A-q,B-q,\beta})(x_n) \| = 2\beta \| (P_{A-q} - P_{B-q}(2\beta P_{A-q} - I))(x_n) \|
\]

\[
= 2\beta \| P_{A-q}(x_n) - P_{B-q}(2\beta P_{A-q}(x_n) - x_n) \|
\]

\[
= 2\beta \| P_{A}(x_n + q) - q - P_{B}(2\beta P_{A}(x_n + q) - q - x_n + q) + q \|
\]

\[
= 2\beta \| y_n - P_{B}(2\beta y_n - x_n) \|.
\]
This together with (18) implies that
\[ 2\beta\sqrt{n}\|y_n - P_B(2\beta y_n - x_n)\| \to 0 \quad (n \to \infty). \]

By the definition of \( P_B \), we have
\[ \|y_n - P_B(y_n)\| \leq \|y_n - P_B(2\beta y_n - x_n)\| \]
and hence
\[ 2\beta\sqrt{n}\|y_n - P_B(y_n)\| \to 0 \quad (n \to \infty). \]

We can therefore conclude that
\[ r(x_n) = o\left(\frac{1}{\sqrt{n}}\right). \]

\[ \square \]

**Remark 4.1.** The worst-case convergence rates of the Krasnosel’skii-Mann iterations have been analyzed in [8, 12]. We estimated that \( r(x_n) \) converges to zero at a rate of \( o\left(\frac{1}{\sqrt{n}}\right) \). On the other hand, it is not guaranteed whether the weak cluster points of the Krasnosel’skii-Mann iterations solve the best approximation problem. We showed that (7) generates a sequence weakly converging to the solution to problem (1).

## 5 Finite termination result

In this section, we investigate finite termination of a modification of (7). We make the following assumptions.

**Assumption 5.1.**

(A1) \( B \) is closed and convex cone;

(A2) \( A \cap \text{int}B \neq \emptyset \).

**Remark 5.1.** Assumption (A2) implies that \( \text{int}B \neq \emptyset \). Using Lemma 2.1, for any \( e \in \text{int}B \), \( A \cap (B + \gamma e) \neq \emptyset \) for sufficiently small \( \gamma > 0 \).

We know the following lemma, due to Rami, Helmke and Moore [14].

**Lemma 5.1.** [14, Lemma 2.3] Let \( C \) be a closed and convex cone in \( H \) such that \( \text{int}C \neq \emptyset \). If \( e \in \text{int}C \), then it holds
\[ \text{dist}(C + e, (\text{int}C)^c) > 0. \] (19)
Remark 5.2. An example of \( C \) satisfying (19) is \( S_+^N \).

- Since \( S_+^N \) is a closed and convex cone and \( \delta I_N \in \text{int}\, S_+^N (= S_+^N) \) for \( \delta > 0 \), \( \text{dist}(S_+^N + \delta I_N, (S_+^N)^c) > 0 \), where \( I_N \) is the \( N \times N \) identity matrix and \( S_+^N \) is the set of \( N \times N \) symmetric positive definite matrices.

- For any \( \delta > 0 \), the lower bound of \( \text{dist}(S_+^N + \delta I_N, (S_+^N)^c) \) can be estimated by \( \delta \), i.e.,

\[
\text{dist}(S_+^N + \delta I_N, (S_+^N)^c) \geq \delta
\]

(see [13, Section 4]).

Suppose that Assumption 5.1. Let \( e \in \text{int}\, B \) and \( \gamma > 0 \) such that \( A \cap (B + \gamma e) \neq \emptyset \). The existence of \( e \) and \( \gamma \) are guaranteed by (A2) and Lemma 2.1. We consider the following modification of (7). Choose \( z_0 \) and \( \alpha, \beta \in (0, 1) \) and consider the iterative scheme

\[
\begin{align*}
  w_n &= P_A(z_n) \\
  z_{n+1} &= T_{A,B+\gamma e,\alpha,\beta}(z_n), \quad n = 0, 1, 2, \ldots.
\end{align*}
\]

Remark 5.3. In theorems 3.1 and 4.1, we used the metric projections onto the sets shifted by \(-p\) satisfying (3). The condition (3) is automatically satisfied when \( A \cap \text{int}\, B \neq \emptyset \) holds (see [1]).

The third main result is stated as follows.

Theorem 5.1. Suppose that Assumption 5.1 holds. Let \{\( w_n \)\} be the sequence generated by (20), where \( e \in \text{int}\, B \) and \( \gamma > 0 \) such that \( A \cap (B + \gamma e) \neq \emptyset \). Then \{\( w_n \)\} terminates finitely to some point \( w \in A \cap \text{int}\, B \).

Proof. Using the assumption (A2) and Lemma 2.1, \( \{A, B+\gamma e\} \) has the strong CHIP (see, e.g., [1,7]). Using [1, Remark 3.2 and Theorem 3.1], we have \( \text{Fix}(R_{A,B+\gamma e,\beta}) \neq \emptyset \). Let \( u \in \text{Fix}(R_{A,B+\gamma e,\beta}) \). Since \{\( z_n \)\} can be viewed as the Krasnosel’skiĭ-Mann fixed point iteration with respect to nonexpansive mapping \( R_{A,B+\gamma e,\beta} \), by virtue of [3, Theorem 5.14], we have that, for any \( n \in \mathbb{N} \),

\[
\alpha(1 - \alpha)\| (I - R_{A,B+\gamma e,\beta})(z_n) \|^2 \leq \| z_n - u \|^2 - \| z_{n+1} - u \|^2.
\]

By summing up (21) from \( j = 0 \) to \( k \),

\[
\alpha(1 - \alpha) \sum_{j=0}^{k} \| (I - R_{A,B+\gamma e,\beta})(z_j) \|^2 \leq \| z_0 - u \|^2 - \| z_{k+1} - u \|^2 \leq \| z_0 - u \|^2.
\]
Using \( I - R_{A,B+\gamma e,\beta} = 2\beta(P_A - P_{B+\gamma e}(2\beta P_A - I)) \) and the similar arguments as in the proof of Theorem 4.1, we can show that
\[
\|(P_A - P_{B+\gamma e}(2\beta P_A - I))(z_n)\| = o\left(\frac{1}{\sqrt{n}}\right). \tag{22}
\]

On the other hand, using Lemma 5.1, we have
\[
\text{dist}(B + \gamma e, A \cap (\text{int}B)^c) \geq \text{dist}(B + \gamma e, (\text{int}B)^c) > 0.
\]

Using (22), there exists \( l_0 \in \mathbb{N} \) such that
\[
\|(P_A - P_{B+\gamma e}(2\beta P_A - I))(z_l)\| < \text{dist}(B + \gamma e, A \cap (\text{int}B)^c) \tag{23}
\]
for all \( l \geq l_0 \). Let \( l \in \mathbb{N} \) with \( l \geq l_0 \). If \( w_l = P_A(z_l) \notin \text{int}B \), then \( w_l \in A \cap (\text{int}B)^c \). From the definition of \( \text{dist}(B + \gamma e, A \cap (\text{int}B)^c) \), we can see that
\[
\|P_A(z_l) - P_{B+\gamma e}(2\beta P_A - I)(z_l)\| \geq \text{dist}(B + \gamma e, A \cap (\text{int}B)^c),
\]
and this is a contradiction to (23). Therefore, \( w_l \in A \cap \text{int}B \) for all \( l \geq l_0 \). \( \square \)

**Remark 5.4.** Finite termination of projection-type iterative methods was established in [4, 13, 14]. The techniques used in Theorem 5.1 can be found in [13, 14].

### 6 Conclusion

In this paper, we have studied variants of the AAMR method for solving the best approximation problem in an infinite-dimensional Hilbert space. In particular, its theoretical properties such as global weak convergence, an \( o\left(\frac{1}{\sqrt{n}}\right) \) rate and finite termination are established. Our variant has a few advantages. First, the method can directly be applied to solve the best approximation problem. Second, it guarantees a convergence rate of \( o\left(\frac{1}{\sqrt{n}}\right) \).

Although no numerical results are given here, the behavior of (7) can be estimated from the computational experience reported in [1, Section 7], since the method in [1, Section 7] is essentially the same as that considered in this paper in the finite-dimensional setting.

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