GLOBAL ALGEBRAIC $K$-THEORY IS SWAN $K$-THEORY

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ABSTRACT. We construct an equivalence between Hausmann’s model of global stable homotopy theory in terms of symmetric spectra on the one hand and spectral Mackey functors in the sense of Barwick on a certain global effective Burnside category on the other hand.

Using this, we then give a new description of Schwede’s global algebraic $K$-theory by identifying the corresponding spectral Mackey functors with certain categorically defined Mackey functors, making precise the slogan that global algebraic $K$-theory is a coherent version of Swan $K$-theory.

INTRODUCTION

One of the defining features of equivariant homotopy theory that distinguishes it from the naive theory of $G$-objects in spaces or spectra is the existence of genuine fixed points, which are typically different from the usual homotopy fixed points. In the unstable setting, Elmendorf’s Theorem [Elm83] makes precise in which sense a $G$-space can be understood in terms of its fixed point spaces and the restriction and conjugation maps between them: for every group $G$, sending a $G$-space to the collection of its fixed point spaces together with the above structure maps provides an equivalence between unstable $G$-equivariant homotopy theory and the quasi-category of presheaves on the category of finite transitive $G$-sets.

Stably, the theory becomes more complicated: in particular, for any finite group $G$, the fixed points of a genuine $G$-spectrum come with covariant transfers along subgroup inclusions in addition to the above contravariant functoriality, and on the level of homotopy groups these two directions of functoriality are related by the so-called Mackey double coset formula, giving the homotopy groups the structure of a $G$-Mackey functor. It is then natural to ask whether we can also understand genuine $G$-spectra in terms of their fixed points together with these restriction and transfer maps. However, while the Schwede-Shipley Theorem [SS03] provides a model of genuine stable $G$-equivariant homotopy theory in terms of spectral presheaves for purely abstract reasons, the resulting indexing category is a priori very far from having an algebraic or combinatorial description, and in particular it is not clear in which sense it encodes restrictions, transfers, and a ‘coherent double coset formula’ between them.

Because of this, Guillou and May [GM11] constructed an alternative (spectrally enriched) indexing category built from a 2-category of spans of finite $G$-sets by ‘local higher group completion,’ and provided an equivalence between spectral presheaves on this indexing category and genuine stable $G$-equivariant homotopy theory. They also made precise in which sense their construction encodes fixed point spectra.
together with transfers and restrictions, allowing us to view their result as a stable analogue of Elmendorf’s Theorem.

Motivated by this, Barwick [Bar17] developed a theory of *spectral Mackey functors* formalizing the idea of functors with both covariant and contravariant functoriality in a suitable base (quasi-)category $\mathcal{F}$, together with higher homotopies encoding a Mackey double coset formula between these two directions. Specializing Barwick’s theory to the ordinary category of finite $G$-sets yields a theory of *spectral $G$-Mackey functors*, and several proofs have been given that this theory is equivalent to classical genuine stable $G$-equivariant homotopy theory, for example by Nardin [Nar17] or Clausen, Mathew, Naumann, and Noel [CMNN20].

In this note, we are concerned with the analogous story for *global homotopy theory* in the sense of [Sch18, Hau19]. Roughly and intuitively speaking, global objects are compatible families of genuine $G$-equivariant objects for all finite groups $G$, and as a concrete manifestation of this slogan they come with genuine fixed points for all finite groups $G$. In the unstable case, work of Körshgen [Kör18] and Schwede [Sch20] provides an analogue of Elmendorf’s Theorem, identifying global spaces with presheaves on the 2-category of finite groups, all homomorphisms, and conjugations (in fact, both Schwede and Körshgen work in the more general context of compact Lie groups). Similarly to the $G$-equivariant setting, the corresponding stable theory is richer: in particular, global spectra again admit covariant transfers along injective homomorphisms in addition to the contravariant functoriality inherited from the unstable world, and on the level of homotopy groups there is an analogue of the Mackey double coset formula relating the two, making the homotopy groups into so-called *global (Mackey) functors*.

It is then again natural to ask whether we can understand global stable homotopy theory in terms of fixed point spectra together with restrictions and transfers between them, and indeed there is a natural candidate for a theory of *global spectral Mackey functors*. However, while several sources [BDG+16, Ber18] mention (or implicitly assume) the existence of an equivalence between global spectral Mackey functors and global spectra, it seems that so far no proof of this has appeared in the literature. Moreover, neither of the proofs of the $G$-equivariant comparison mentioned above seem to immediately carry over to the global world.

The first purpose of this paper is to end this unpleasant state of affairs: we will explain how one can apply Barwick’s theory to the 2-category of finite groupoids to obtain a notion of *global spectral Mackey functors*, and we prove (see Theorem 5.17):

**Theorem A.** *There exists an (explicit) equivalence* $\text{mack}$ *between the quasi-category of global spectral Mackey functors and the quasi-category of global spectra.*

We moreover make precise in which sense the global spectral Mackey functor associated to a global spectrum is given by collecting the genuine fixed point spectra together with all restrictions and transfers between them, see Theorem 6.17.

While we are working higher-categorically, our proof strategy for this is closer to Guillou and May’s original argument in the equivariant setting than to the other proofs mentioned above. In particular, we will use $K$-theoretic constructions to pass from combinatorial data (namely the indexing category produced by Barwick’s machinery) to homotopy theoretic objects (namely global spectra).

**Global algebraic $K$-theory vs. Swan $K$-theory.** Having made precise in which sense a global spectrum is characterized by its fixed point spectra, restrictions and
transfers, we then apply this point of view to Schwede’s global algebraic $K$-theory \cite{Sch19b}. Namely, we provide a description of the spectral Mackey functor associated to the global algebraic $K$-theory spectrum $K_{gl}(C)$ of a symmetric monoidal category $C$ in terms of ordinary non-equivariant $K$-theory and purely categorical data as follows: associated to $C$ there is a categorical Mackey functor $\text{swan}(C)$ which collects the symmetric monoidal categories of $G$-objects in $C$ together with the evident restrictions and with transfers given in terms of a norm construction. Applying ordinary $K$-theory pointwise then gives a global spectral Mackey functor $K \circ \text{swan}(C)$, and as our main result we prove (Theorem 7.1):

**Theorem B.** There is a preferred natural equivalence $\text{mack} K_{gl}(C) \simeq K \circ \text{swan}(C)$.

This answers a question raised by Schwede \cite[Remark 6.1]{Sch19b}, and connects his concrete pointset level construction with Barwick’s higher categorical approach to equivariant algebraic $K$-theory \cite[Section 8]{BGS20}.

**Outline.** In Section 1 we recall some basic facts from global homotopy theory, and in particular we introduce Hausmann’s model of global stable homotopy theory in terms of symmetric spectra.

Afterwards, we construct the global effective Burnside category in Section 2 and compare it to a certain subcategory of the $(2,1)$-category of small symmetric monoidal categories. Section 3 is an interlude devoted to an analogue of the Schwede-Shipley Theorem applicable in our context. Using this, Section 4 gives a Mackey functor description of Schwedes ultra-commutative monoids with respect to finite groups, while Section 5 provides the corresponding description for global spectra, in particular proving Theorem B. Section 5 is then devoted to describing the Mackey functor structure in terms of the usual homotopy theoretic restrictions and transfers. Finally, we deduce Theorem B in Section 7 using essentially all of the previous results.

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1. **A reminder on global homotopy theory**

1.1. **Global spaces.** We begin by recalling several models of unstable global homotopy theory \cite[Chapter 1]{Sch18}, and more generally unstable $G$-global homotopy theory in the sense of \cite[Chapter 1]{Len20a}. Crucial to this is a certain simplicial monoid that we call the universal finite group:
Definition 1.1. Let $\mathcal{M}$ be the monoid of self-injections of the countably infinite set $\omega = \{0, 1, \ldots \}$. We let $E$ denote the right adjoint of the functor $\text{SSet} \to \text{Set}$ sending a simplicial set to its set of vertices, and we call the simplicial set $EM$ with the induced monoid structure the universal finite group.

Explicitly, $(EM)_n = M^{1+n}$ with the obvious functoriality and with pointwise multiplication. Moreover, $EM$ is canonically isomorphic to the nerve of the discrete category with object set $\mathcal{M}$, which we will again denote by $EM$.

Definition 1.2. We call a finite subgroup $H \subset \mathcal{M}$ universal if $\omega$ with $H$ acting via the restriction of the tautological $\mathcal{M}$-action is a complete $H$-set universe, i.e. every finite (or, equivalently, every countable) $H$-set embeds into $\omega$ equivariantly.

Theorem 1.3. Let $G$ be any discrete group. Then the category $EM\text{-}G\text{-SSet}$ of simplicial sets with an action of the simplicial monoid $EM \times G$ admits a unique model structure in which a map $f$ is a weak equivalence or fibration if and only if for all universal $H \subset \mathcal{M}$ and all $\varphi: H \to G$ the map $f^\varphi$ is a weak homotopy equivalence or Kan fibration, respectively; here we write $(-)^\varphi$ for the fixed points with respect to the graph subgroup $\Gamma_{H, \varphi} := \{(h, \varphi(h)) : h \in H\} \subset G \times G$.

We call this model structure the $G$-global model structure and its weak equivalences the $G$-global weak equivalences. It is simplicial, proper, combinatorial with generating cofibrations
\[ \{EM \times \varphi G \times (\partial \Delta^n \hookrightarrow \Delta^n) : H \subset \mathcal{M} \text{ universal, } \varphi: H \to G\}, \]
and filtered colimits and finite products in it are homotopical.

Proof. See [Len20a Corollary 1.2.30 and Lemma 1.1.2].

At several points we will also need another model of unstable $G$-global homotopy theory that is closer (on the pointset level) to the category of symmetric spectra:

Definition 1.4. We write $I$ for the category of finite sets and injections, and we denote by $\mathcal{I}$ the simplicial category obtained by applying $E$ to all hom sets. We will refer to enriched functors $\mathcal{I} \to \text{SSet}$ as $G$-$\mathcal{I}$-simplicial sets, and we denote the resulting simplicial functor category by $G$-$\mathcal{I}$-$\text{SSet}$.

If we have any (infinite) set $U$, then we can evaluate a $G$-$\mathcal{I}$-simplicial set $X$ at $U$ via
\[ X(U) := \colim_{A \subset U \text{ finite}} X(A), \]
and this acquires an action of the simplicial monoid $E\text{Inj}(U, U)$ by permuting the colimit terms and the enriched functoriality of $X$, see [Len20a Construction 1.4.13]. In particular, for $U = \omega$ we get a functor $ev_\omega: G$-$\mathcal{I}$-$\text{SSet} \to EM\text{-}G\text{-SSet}$.

Definition 1.5. A morphism $f: X \to Y$ of $G$-$\mathcal{I}$-simplicial sets is called a $G$-global weak equivalence if $f(\omega)$ is a $G$-global weak equivalence in $EM\text{-}G\text{-SSet}$.

Equivalently this means that for every finite group $H$, some (hence any) complete $H$-set universe $U_H$, and each homomorphism $\varphi: H \to G$ the map $f(U_H)^\varphi$ is a weak homotopy equivalence of simplicial sets.

Theorem 1.6. There is a unique combinatorial model structure on $G$-$\mathcal{I}$-$\text{SSet}$ with generating cofibrations
\[ \{\mathcal{I}(A, -) \times_\varphi G \times (\partial \Delta^n \hookrightarrow \Delta^n) : H \text{ finite group, } A \text{ finite faithful } H\text{-set, } \varphi: H \to G\} \]
and weak equivalences the $G$-global weak equivalences. We call this the $G$-global model structure. It is proper, simplicial, and filtered colimits in it are homotopical.

Moreover, the functor $ev_u: \mathcal{G-ISSet} \to E\mathcal{M-GSSet}$ is a homotopy equivalence, i.e. there is a homotopical functor $(-)[\omega^*]: E\mathcal{M-GSSet} \to \mathcal{G-ISSet}$ together with zig-zags of levelwise weak equivalences between the two composites and the respective identities.

Proof. See [Len20a, Theorem 1.4.29 and Proposition 1.4.50].

Construction 1.7. Let us make the homotopy inverse $(-)[\omega^*]$ explicit, see [Len20a, Construction 1.4.48] or [Sch19b, Constructions 3.3 and 8.2] for more details:

We pick once and for all for every non-empty finite set $A$ a bijection $\rho_A: \omega^A \cong \omega$.

If now $X$ is an $E\mathcal{M}$-object in any simplicial category with all enriched limits, then we define $X[\omega^A] = X$ for every $A \neq \emptyset$ while $X[\omega^\emptyset] = X^{E\mathcal{M}}$ (the $E\mathcal{M}$-fixed points).

If $u: \omega^A \to \omega^B$ is any injection ($A, B \neq \emptyset$), then we define $X[u]: X[\omega^A] \to X[\omega^B]$ as the map given by acting with $\rho_B u \rho_A^{-1} \in \mathcal{M}$, and analogously we get higher cells $X[u_0, \ldots, u_n]$ for all tuples of injections. On the other hand, for any injections $u_0, \ldots, u_n: \omega^\emptyset \to \omega^A$ we define $X[u_0, \ldots, u_n]$ as the degenerate $n$-simplex at the inclusion of $X^{E\mathcal{M}}$.

If now $X$ is an $E\mathcal{M}$-simplicial sets, then $X[\omega^*]$ is the $I$-simplicial set with $X[\omega^*](A) = X[\omega^A]$; a tuple of injections $i_0, \ldots, i_n: A \to B$ acts by $X[i_0, \ldots, i_n]$ where we denote for $i: A \to B$ by $i: \omega^A \to \omega^B$ the map given by extension by zero.

We omit the easy verification that this is well-defined and that $(-)[\omega^*]$ becomes a simplicially enriched functor in the evident way.

While the existence of the above model structures will be important for several arguments below, ultimately we are interested in statements about the quasi-categories they present. More precisely, whenever $\mathcal{C}$ is a category equipped with a wide subcategory $W$ of weak equivalences, we can form the quasi-localization of $\mathcal{C}$ at $W$, i.e. the universal example of a functor $\gamma: \mathcal{N}\mathcal{C} \to \mathcal{C}_W^\omega$ into some quasi-category sending $W$ into the maximal Kan complex core($\mathcal{C}_W^\omega$); one way to obtain this is to form the Hammock localization of $(\mathcal{C}, W)$ and then to apply the right derived functor of the homotopy coherent nerve, see e.g. [Hin16, 1.2]. If $W$ is clear from the context, we will just denote any quasi-localization by $\mathcal{C}_W^\omega$.

Convention 1.8. In order to avoid cumbersome notation, we agree to pick all quasi-localizations to be the respective identities on objects (which is for example possible since for the Hammock localization and the usual fibrant replacement functor in simplicial categories the resulting functor is bijective on objects), and we won’t distinguish a morphism in the 1-category from its image under the localization.

As a direct consequence of the universal property, any homotopical functor $f: \mathcal{C} \to \mathcal{D}$ induces a functor $\mathcal{C}_W^\omega \to \mathcal{D}_W^\omega$ that we will denote by $f_\omega^\omega$ or simply by $f_\omega$ again. Whenever $f$ is a homotopy equivalence, the induced functor will be an equivalence of quasi-categories; in particular, Theorem [1.6] gives us an equivalence $ev_\omega: \mathcal{G-ISSet}^{G_{\text{global}}} \to E\mathcal{M-GSSet}^{G_{\text{global}}}$. 

Remark 1.9. Specializing to $G = 1$ we get global model structures on $E\mathcal{M-GSSet}$ and $\mathcal{I-SSet}$, whose associated quasi-categories are equivalent as we have just seen. In [Sch18, Theorem 1.2.21] Schwede introduced a global model structure on the category of orthogonal spaces which contains equivariant information for all compact
Lie groups. After throwing away all information for infinite groups (i.e. after Bousfield localizing at a certain explicit class of ‘\(\mathcal{F}\)in-global weak equivalences’), this becomes equivalent to the above quasi-categories, see \[\text{[Len20a, Theorem 1.4.30 and Corollary 1.5.26]}\].

We close this subsection with a technical observation: as a simplicial model category, \(\text{EM-G-SSet}\) comes with a geometric realization functor

\[
(1.1) \quad \text{Fun}(\Delta^{\text{op}}, \text{EM-G-SSet}) \to \text{EM-G-SSet}
\]
given as the coend of the tensoring. While this functor is always left Quillen for the projective or Reedy model structure on the source and hence preserves weak equivalences between appropriately cofibrant objects, in our case we even have:

**Lemma 1.10.** The geometric realization functor \((1.1)\) preserves all \(G\)-global weak equivalences.

**Proof.** As \(\text{EM-G-SSet}\) is an enriched functor category, geometric realizations can be computed in \(\text{SSet}\), so they are simply given by taking the diagonal. The claim therefore follows from \[\text{[Len20a, Lemma 1.2.52]}\]. \(\square\)

### 1.2. Global spectra.

Our reference model of global stable homotopy theory is Hausmann’s global model structure on the category of symmetric spectra in the sense of \[\text{[HSS00]}\]. In order to define this, we need:

**Definition 1.11.** A symmetric spectrum \(X\) is called a global \(\Omega\)-spectrum if for every finite group \(H\), every finite faithful \(H\)-set \(A\), and every finite \(H\)-set \(B\) the derived adjoint structure map

\[
X(A)^H \to (R\Omega^B X(A \sqcup B))^H
\]
is a weak homotopy equivalence. Here we are deriving \(\Omega\) with respect to the usual equivariant model structure on \(\text{H-SSet}\), for example by precomposing with the singular set-geometric realization adjunction.

**Theorem 1.12.** There is a unique model structure on the category \(\text{Spectra}\) of symmetric spectra in which a map \(f\) is a weak equivalence or fibration if and only if \(f(A)^H\) is a weak homotopy equivalence or Kan fibration, respectively, for all finite groups \(H\) and all finite faithful \(H\)-sets \(A\). We call this the \(\text{global level model structure}\).

Furthermore, the global level model structure admits a Bousfield localization whose fibrant objects are precisely the level fibrant global \(\Omega\)-spectra. We call the resulting model structure the \(\text{global model structure}\) and its weak equivalences the \(\text{global weak equivalences}\).

**Proof.** See \[\text{[Hau19, Proposition 2.5 and Theorem 2.17]}\]. \(\square\)

Global spaces and global spectra are related by a suspension spectrum-loop adjunction refining the usual adjunction between simplicial sets and symmetric spectra. We will be only interested in the left adjoint of this adjunction, which we denote by \(\Sigma^*_+\); on an \(\mathcal{I}\)-simplicial set \(X\) this is given by

\[
(\Sigma^*_+ X)(A) = S^A \wedge X(A)_+
\]
with the obvious functoriality in each variable. This functor is actually fully homotopical \[\text{[Len20a, Corollary 3.2.6]}\].
In fact, we can already define the suspension spectrum of any $I$-simplicial set $X$ (i.e. a functor $I \to \text{SSet}$) by the same formula, which we again denote by $\Sigma^* X$. The above construction then factors through the forgetful functor $\text{I-SSet} \to \text{I-SSet}$. While we will not need this below, we want to mention that there is a global model structure on $\text{I-SSet}$ (and more generally a $G$-global model structure on $G\text{-I-SSet}$) such that this forgetful functor is both the right and left half of suitable Quillen equivalences, see [Len20a, Theorems 1.4.30 and 1.4.47].

**Theorem 1.13.** The quasi-category $\text{Spectra}^\infty_{\text{global}}$ is stable and compactly generated by the global spectra $\Sigma^*_+ I(H,-)/H$ for varying finite groups $H$.

**Proof.** The first statement follows from stability of the global model structure [Hau19, Proposition 4.6-(i)], while the second one follows from [Hau19 Example 6.3]. □

2. The global effective Burnside category

We begin by explaining how one can construct the global effective Burnside category (Definition 2.4) using Barwick’s general machinery [Bar17].

**Definition 2.1.** We write $\mathcal{F}$ for the $\infty$-category of finite groupoids (i.e. the homotopy coherent or Duskin nerve of the $(2,1)$-category of finite groupoids) and $\mathcal{F}_\dagger \subset \mathcal{F}$ for the wide subcategory of faithful functors.

The canonical model structure on the category $\text{Cat}$ of small categories (see e.g. [Rez96]) restricts to a model structure on $\text{Grpd}$. Explicitly, the weak equivalences are the equivalences of groupoids, the cofibrations are the functors that are injective on objects, and the fibrations are given by the isofibrations, i.e. those functors $F: \mathcal{G} \to \mathcal{H}$ such that there exists for every $G \in \mathcal{G}$ and every $h: F(G) \to H$ in $\mathcal{H}$ a morphism $g: G \to G'$ in $\mathcal{G}$ with $F(g) = h$. This model structure is combinatorial and it is moreover simplicial with respect to the obvious enrichment. We will frequently use below that we can use this structure to compute colimits and limits in $\mathcal{F}$:

**Lemma 2.2.**

1. $\mathcal{F}$ has all finite coproducts and these can be computed in $\text{Grpd}$, i.e. the localization functor preserves finite coproducts.

2. $\mathcal{F}$ has all pullbacks and they can be computed as homotopy pullbacks in $\text{Grpd}$ in the following sense: every functor $\mathcal{G} \to \mathcal{D}$ in $\mathcal{F}$ can be represented by an equivalence followed by an isofibration $\mathcal{C} \to \mathcal{D}$ of finite groupoids, and any 1-categorical pullback

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

of finite groupoids along an isofibration $\mathcal{C} \to \mathcal{D}$ defines a pullback in $\mathcal{F}$.

**Proof.** We will prove the second statement, the first one being similar but easier.

For this we first note that the quasi-category $\mathcal{F} := N_\Delta(\text{Grpd})$ of all groupoids is complete and cocomplete and that (co)limits in it can be computed as homotopy (co)limits in $\text{Grpd}$, see [Lur09, proof of Corollary 4.2.4.8]. As $\mathcal{F}$ is a full subcategory of $\mathcal{F}$ and since ordinary pullbacks of finite groupoids are again finite groupoids, it therefore only remains to construct the above factorizations.
For this we observe that the model structure on $\text{Grpd}$ restricts to the structure of a category of fibrant objects in the sense of [Bro73, I.1], and by Brown’s Factorization Lemma (see p. 421 of op. cit.) it then suffices to show that this restricts to also make the category of finite groupoids into a category of fibrant objects. However, using again the closure under pullbacks the only non-trivial statement is the existence of path objects, for which we can simply take the standard construction:

$$G \xrightarrow{\text{const}} G^{[1]} \xrightarrow{(\text{ev}_0, \text{ev}_1)} G \times G$$

for every finite groupoid $G$.

\[\square\]

**Proposition 2.3.** The triple $(\mathcal{F}, \mathcal{F}\uparrow, \mathcal{F})$ is disjunctive in the sense of [Bar17, Definition 5.2].

**Proof.** We follow the terminology of loc. cit.

The previous lemma shows that $\mathcal{F}$ has all pullbacks and that they can be computed in terms of ordinary pullbacks. As pullbacks of faithful functors are again faithful by direct inspection, this shows that the above triple is adequate.

Another application of the previous lemma shows that $\mathcal{F}$ admits all finite coproducts and that they can be computed in $\text{Grpd}$ again. As a functor out of a coproduct is faithful if and only if it is so on each coproduct summand, and since moreover inclusions of coproduct summands are faithful, this shows that faithful functors are compatible with coproducts.

Finally, let $f: I \to K, g: J \to K$ be maps of finite sets, and assume we are given for each $(i, j) \in I \times_K J$ a pullback

\begin{equation}
\begin{array}{ccc}
A_{i,j} & \longrightarrow & B_j \\
\downarrow & & \downarrow \\
C_i & \longrightarrow & D_k
\end{array}
\end{equation}

in $\mathcal{F}$ where the horizontal maps are faithful, and where we write $k := f(i) = g(j)$. We have to show that the induced square

\begin{equation}
\begin{array}{ccc}
\coprod_{(i, j) \in I \times_K J} A_{i,j} & \longrightarrow & \coprod_{j \in J} B_j \\
\downarrow & & \downarrow \\
\coprod_{i \in I} C_i & \longrightarrow & \coprod_{k \in K} D_k
\end{array}
\end{equation}

is again a pullback in $\mathcal{F}$. For this we may assume without loss of generality that each of the diagrams (2.1) comes from a 1-categorical pullback of groupoids along an isofibration $B_j \to D_k$. By direct inspection, the above map $\coprod_{j \in J} B_j \to \coprod_{k \in K} D_k$ is again an isofibration, so it is enough to show that (2.2) is a 1-categorical pullback again. This is immediate by inspecting the standard construction of pullbacks in the 1-category of groupoids. \[\square\]

Specializing [Bar17, Definition 5.7] we can now introduce our main object of study:

**Definition 2.4.** We define $A^{\text{ef}} := A^{\text{eff}}(\mathcal{F}, \mathcal{F}\uparrow, \mathcal{F})$ and call it the global effective Burnside category.
Corollary 2.5. The quasi-category $\mathcal{A}^l$ is semiadditive, i.e. it is pointed, it has all finite coproducts and products, and the natural comparison map between the two in the homotopy category is an isomorphism.

Proof. See [Bar17] Proposition 4.3 and 5.8.

2.1. Bisets vs. correspondences of groupoids. As we will make precise later in this section, mapping spaces in $\mathcal{A}^l$ can be understood in terms of certain slices in the 2-category of groupoids. This subsection is devoted to understanding the resulting categories better, and in particular to relate them to suitable categories of bisets via the Grothendieck construction. While we can’t use their results directly, I want to mention that similar comparisons have been given by Miller [Mil17] or Dell’Ambrogio and Huglo [DH21], and that more generally the relation between the two notions has a long history in representation theory.

Definition 2.6. Let $\mathcal{G}, \mathcal{H}$ be groupoids. We call a functor $X : \mathcal{G} \times \mathcal{H} \to \mathbf{Set}$ a $\mathcal{G}$-$\mathcal{H}$-biset. We say that $X$ is $\mathcal{G}$-free if $X(G, H)$ is a free $\mathrm{Aut}_G(G)$-set for every $G \in \mathcal{G}, H \in \mathcal{H}$; equivalently: if $g, g' : G \to G'$, $h : H \to H'$ are morphisms in $\mathcal{G}$ and $\mathcal{H}$, respectively, and $x \in X(G, H)$ with $X(g, h)(x) = X(g', h)(x)$, then $g = g'$.

Construction 2.7. Let $\mathcal{G}, \mathcal{H}$ be groupoids. We write $\overline{\mathcal{F}}_{\mathcal{G}, \mathcal{H}}$ for the full 2-subcategory of the 2-categorical slice $\mathbf{Grpd} \downarrow \mathcal{G} \times \mathcal{H}$ spanned by those functors $\pi : \mathcal{X} \to \mathcal{G} \times \mathcal{H}$ for which the composition $\mathrm{pr}_\mathcal{H} \circ \pi : \mathcal{X} \to \mathcal{H}$ is faithful.

Construction 2.8. Let $\mathcal{G}$ be a groupoid. We recall that the classical Grothendieck construction (see e.g. [Lur09 2.1.1]) provides an equivalence from the 2-category $\mathrm{Fun}(\mathcal{G}, \mathbf{Grpd})$ of (say, strict) functors $\mathcal{G} \to \mathbf{Grpd}$, pseudonatural transformations, and modifications to the 2-categorical slice $\mathbf{Grpd} \downarrow \mathcal{G}$. On objects, this is given by sending $F : \mathcal{G} \to \mathbf{Grpd}$ to the map $\pi : \int F \to \mathcal{G}$, where $\int F$ is the groupoid with objects the pairs $(G \in \mathcal{G}, X \in F(G))$ and morphisms $(G, X) \to (G', X')$ the pairs $(g : G \to G', x : F(g)(X) \to X')$; composition is defined in the evident way, and the map $\pi$ is given by projection to the first factor. We will further need the definition of $\int$ on strictly natural transformations below: if $\sigma : F \Rightarrow G$ is natural, then $\int \sigma : \int F \to \int G$ is given on objects by $\int \sigma(G, X) = (G, \sigma_G(X))$ and on morphisms by $\int \sigma(g, X) = (g, \sigma_{G'}(X) : F(g)(\sigma(X)) = \sigma_{G'}(F(g)(X)) \to \sigma_{G'}(X'))$.

Lemma 2.9. Assume $\mathcal{G}$ and $\mathcal{H}$ are groupoids. Then the Grothendieck construction restricts to an equivalence $\mathrm{Fun}^\mathrm{strict}(\mathcal{G} \times \mathcal{H}, \mathbf{Set}) \to \overline{\mathcal{F}}_{\mathcal{G}, \mathcal{H}}$.

Proof. We first observe that the above is well-defined: if $F : \mathcal{G} \times \mathcal{H} \to \mathbf{Set}$ is $\mathcal{G}$-free, and $(g, h), (g', h')$ are morphisms $(G, H ; X) \to (G', H' ; X')$ in $\int F$ with $h = h'$, then $F(g, h)(X) = X' = F(g, h')(X) = F(g', h)(X)$, so $g = g'$ by freeness; thus $\mathrm{pr}_\mathcal{H} \circ \pi$ is faithful.

On the other hand, if $X$ is a functor such that $\pi : \int X \to \mathcal{G} \times \mathcal{H}$ is faithful then one easily checks that each $X(G, H)$ is equivalent to a discrete groupoid. If now $\rho : \mathcal{X} \to \mathcal{G} \times \mathcal{H}$ is any functor such that $\mathrm{pr}_\mathcal{H} \circ \rho$ is faithful, then there exists a functor $X : \mathcal{G} \times \mathcal{H} \to \mathbf{Grpd}$ such that $\rho$ is equivalent to $\pi : \int F \to \mathcal{G} \times \mathcal{H}$. As also $\mathrm{pr}_\mathcal{H} \circ \pi : \int F \to \mathcal{H}$ is faithful, we can (by the above observation) assume without loss of generality that $X$ factors through $\mathbf{Set}$. It only remains to show that $X$ is $\mathcal{G}$-free, which however just follows from running the above argument backwards.

In particular, we see that the $(2,1)$-category $\overline{\mathcal{F}}_{\mathcal{G}, \mathcal{H}}$ is actually a 1-category, i.e. the projection to its homotopy category $\mathrm{h} \overline{\mathcal{F}}_{\mathcal{G}, \mathcal{H}}$ is an equivalence.
Remark 2.10. If $X$ is any $\mathcal{G}$-$\mathcal{H}$-biset, then $\pi: \int X \to \mathcal{G} \times \mathcal{H}$ is an isofibration, and if $f: X \to Y$ is a natural transformation, then $\int f$ actually strictly commutes with the projections. Thus, the Grothendieck construction factors through the 2-subcategory $\mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso}$ whose objects are the isofibrations and whose morphisms are given by strictly commuting diagrams; in particular, $\mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso} \hookrightarrow \mathcal{F}_{\mathcal{G},\mathcal{H}}$ is an equivalence.

Remark 2.11. Write $\mathcal{T}$ for the full 2-subcategory of $\mathbf{Grpd}$ spanned by the essentially discrete groupoids. The usual straightening construction is a quasi-inverse of the Grothendieck construction that turns an object $\pi: \mathcal{X} \to \mathcal{G} \times \mathcal{H}$ of $\mathcal{F}_{\mathcal{G},\mathcal{H}}$ into a pseudofunctor $\mathcal{G} \to \mathcal{T}$ and similarly for morphisms. As the 2-category $\mathcal{T}$ is equivalent to the 1-category $\mathbf{Set}$ via the functor $\pi_0$ taking connected components, this then yields a quasi-inverse $\mathcal{L}: \pi_0^{iso} \mathcal{F}_{\mathcal{G},\mathcal{H}} \to \text{Fun}^{\mathcal{G}\text{-free}}(\mathcal{G} \times \mathcal{H}, \mathbf{Set})$ to $\mathcal{F}$ that we can describe explicitly as follows: an isofibration $\rho: \mathcal{X} \to \mathcal{G} \times \mathcal{H}$ is sent to the functor that sends $(G, H) \in \mathcal{G} \times \mathcal{H}$ to $\pi_0(\rho^{-1}(G, H))$ and a morphism $(g: G \to G', h: H \to H')$ in $\mathcal{G} \times \mathcal{H}$ to the unique map $(\mathcal{L}(\rho)(g, h): \pi_0(\rho^{-1}(G, H)) \to \pi_0(\rho^{-1}(G', H')))$ such that there exists for every $X \in \pi_0(\rho^{-1}(G, H))$ and some (hence any) choice of representatives $x$ of $X$ and $y$ of $(\mathcal{L}(\rho)(g))(X)$ a morphism $\chi: x \to y$ in $\mathcal{X}$ with $\rho(\chi) = (g, h)$.

We will now restrict our attention to finite groupoids $\mathcal{G}, \mathcal{H}$. In this case, let us write $\mathcal{F}_{\mathcal{G},\mathcal{H}} \subset \mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso}$ for the full 2-subcategory spanned by those $\mathcal{X} \to \mathcal{G} \times \mathcal{H}$ for which $\mathcal{X}$ is finite, and similarly define $\mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso} \subset \mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso}$. As any map between finite groupoids factors as a composition of an equivalence and an isofibration of finite groupoids (see Lemma 2.22), the inclusion $\mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso} \hookrightarrow \mathcal{F}_{\mathcal{G},\mathcal{H}}$ is again an equivalence.

Lemma 2.12. The Grothendieck construction $\int$ and the straightening construction $\mathcal{L}$ define mutually inverse equivalences $\text{Fun}^{\mathcal{G}\text{-free}}(\mathcal{G} \times \mathcal{H}, \mathbf{FinSet}) \rightleftarrows \mathcal{F}_{\mathcal{G},\mathcal{H}}^{iso}$.

Proof. By Lemma 2.9 and Remark 2.11 it suffices to observe that both functors restrict accordingly. 

Construction 2.13. We write $\mathcal{F}_{\mathcal{G}} := \mathcal{F}_{\mathcal{G},\ast}$; for simplicity, we agree to take the product $\mathcal{G} \times \ast$ to be actually equal to $\mathcal{G}$. With respect to this choice $\mathcal{F}_{\mathcal{G}}$ is literally equal to the 2-categorical slice $\mathcal{S} \downarrow \mathcal{G}$, where $\mathcal{S} \subset \mathcal{S}$ is the full subcategory spanned by the finite essentially discrete groupoids.

Construction 2.14. We now define a strict 2-functor $\psi: \mathcal{S} \to \mathbf{Cat}$ via $\psi(\mathcal{G}) = \text{h} \mathcal{F}_{\mathcal{G}}$, with 2-functoriality given by the usual functoriality of the 2-categorical slice. For $f: \mathcal{G} \to \mathcal{H}$ we abbreviate $f_! := \psi(f)$.

Construction 2.15. Let $f: \mathcal{G} \to \mathcal{H}$ be a faithful isofibration of finite groupoids. We construct a functor $f^*: \text{h} \mathcal{F}_{\mathcal{H}} \to \text{h} \mathcal{F}_{\mathcal{G}}$ as follows: an object $\pi: \mathcal{X} \to \mathcal{H}$ is sent to the left vertical map in the pullback square

$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{f} & \mathcal{H} \\
\downarrow{f} & & \downarrow{\pi} \\
\mathcal{X} & \xrightarrow{f \cdot} & \mathcal{X} \\
\end{array}$$

(2.3)
(note that \(\epsilon\) is faithful as a pullback of a faithful functor, so \(f^*X\) is indeed essentially discrete again). Moreover, if

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\pi & \searrow & \alpha \times G \\
& \swarrow & H \\
\end{array}
\]

represents a morphism in \(hF_{H}\), then \(f^*[\alpha, \hat{\alpha}]\) is constructed as follows: we pick for each \((X, G) \in f^*X\) an isomorphism \(\hat{\beta}_{X,G} : G \to \check{G}\) such that \(f(\hat{\beta}_{X,G}) = \alpha_X : \pi(X) \to \rho \alpha(X)\); note that this is indeed well-defined as \(\pi(X) = f(G)\) and since \(f\) was assumed to be an isofibration. We now set \(\beta(X, G) := (\alpha(X), \check{G})\), which is an element of \(f^*Y\) by definition of \(\check{G}\). There is then a unique way to extend this to a functor such that the maps \((\text{id}, \hat{\beta}_{X,G}) : (\alpha(X), G) \to (\alpha(X), \check{G})\) define a natural isomorphism filling

\[
\begin{array}{c}
f^*X \\ \downarrow f^*\pi \\
\alpha \times G \\
\end{array} \xrightarrow{\alpha \times G} \begin{array}{c}
f^*Y \\ \downarrow f^*\pi \\
\alpha \times G \\
\end{array}
\]

and we set \(f^*[\alpha, \hat{\alpha}] := [\beta, \hat{\beta}]\); we omit the easy verification that this is independent of choices and makes \(f^*\) into a functor. Moreover, one easily checks that the maps \(\epsilon\) from (2.3) assemble into a natural transformation \(f ! f^* \Rightarrow \text{id}\), and that together with the maps \(\eta : X \to f^*f!X\) induced via the universal property of the pullback from the identity of \(X\) and the original structure map \(X \to G\), this exhibits \(f^*\) as a right adjoint of \(f !\).

\textbf{Remark 2.16.} If the above diagram (2.4) commutes strictly (i.e. \(\hat{\alpha}\) is the identity), we can pick \(\hat{\beta}\) also to be the identity, so \(f^*\alpha\) is just represented by the usual pullback of \(\alpha\) along \(f\), i.e. the restriction of \(\alpha \times G\).

\textbf{Lemma 2.17.} The functor \(\psi\) factors through the 2-subcategory \(\text{Cat}_{H}^{\text{iso}}\) of categories with finite coproducts, finite coproduct preserving functors, and all natural transformations. If \(f : G \to G'\) is faithful, then \(f!\) has a right adjoint \(f^*\), and this adjoint again preserves finite coproducts.

\textit{Proof.} It is clear that \(hF_{G} \simeq \text{FinSet} \downarrow G\) has finite coproducts for every \(G \in F\), and that these are created by the forgetful functor, so that \(f!\) preserves finite coproducts for every \(f : G \to H\). In the other hand, if \(f\) is any faithful functor, then we factor it as an equivalence \(i\) followed by a faithful isofibration \(p\) (between finite groupoids); then \(p!\) has a right adjoint by the previous construction, and any quasi-inverse to \(i!\) provides a right adjoint to it. Thus, it only remains to show that the functor \(p^*\) from the above construction preserves finite coproducts. For this, we may restrict to the functor \(p^* : hF_{H}^{\text{iso}} \to hF_{G}^{\text{iso}}\). As on these \(p^*\) is just given by the ordinary pullback while finite coproducts can be computed in the 1-category of (essentially discrete) groupoids, this is just the statement that pullbacks in groupoids preserve coproducts, also cf. the proof of Proposition 2.3 above. \(\Box\)

\textbf{Remark 2.18.} We can actually describe the restriction of \(f^*\) to \(hF_{G}^{\text{iso}} \to hF_{G}^{\text{iso}}\) for arbitrary faithful \(f : G \to G'\) as the functor \(f^\text{pb}\) given by ordinary pullback along \(f\): indeed, if \(f\) is an isofibration, this was verified in the above proof, and for an
equiv}. It is easy enough to check by hand that the composition of $f_1 f^pb$ is isomorphic to the inclusion $hF^\text{iso}_{G'} \to hF'_G$, so that precomposing $f^pb$ with a quasi-inverse to this inclusion yields a quasi-inverse of $f_1$: $hF^\text{iso} \to hF^\text{iso}_{G'}$ as claimed.

**Lemma 2.19.** Let $\mathcal{G}$ be a finite groupoid, and let $\mathcal{G}_1, \ldots, \mathcal{G}_n$ be the components of $\mathcal{G}$. Then $hF_G$ is equivalent to $\prod_{i=1}^n hF_{G_i}$, via taking fibers over $\mathcal{G}_1, \ldots, \mathcal{G}_n$ (i.e. pullback along the inclusions $\mathcal{G}_i \to \mathcal{G}$).

**Proof.** We may assume without loss of generality that each $\mathcal{G}_i$ has only one object $G_i$ and it moreover suffices to prove this for $hF^\text{iso}_G$. A basic computation then shows that the diagram

$$
\begin{array}{ccc}
hF^\text{iso}_G & \longrightarrow & \prod_{i=1}^n hF^\text{iso}_{B\text{Aut}(G_i)} \\
\downarrow & & \downarrow \\
\text{Fun}^\text{free}(\mathcal{G}, \text{FinSet}) & \cong & \prod_{i=1}^n \text{Fun}^\text{free}(B\text{Aut}(G_i), \text{FinSet})
\end{array}
$$

commutes strictly, where the top horizontal arrow is as above and the lower one is induced by the inclusions. The claim now follows from Lemma 2.12. □

**Construction 2.20.** We equip $hF_G$ with ‘the’ cocartesian symmetric monoidal structure; once we have fixed such a choice of coproducts, $\psi$ factors uniquely through the 2-category $\text{SymMonCat}$ of small symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal transformations; similarly the above adjoints can be uniquely made into symmetric monoidal functors such that they are adjoints in the 2-category $\text{SymMonCat}$. In particular, this then induces a symmetric monoidal structure on core $hF_G$ and on $f_!$ and (where applicable) $f^!$.

We moreover define a functor $\tau = \tau_G: \mathcal{G} \to \text{core } hF_{BG}$ as follows: an object $G \in \mathcal{G}$ is sent to the map $G: * \to \mathcal{G}$ classifying $G$, and a map $g: G \to G'$ is sent to

$$
\begin{array}{ccc}
* & \longrightarrow & * \\
\downarrow & \downarrow & \downarrow \\
G & \cong & G'
\end{array}
$$

**Lemma 2.21.** Let $\mathcal{C}$ be any symmetric monoidal category. Then restricting along $\tau$ defines an equivalence $\text{Fun}^\otimes(\text{core } hF_G, \mathcal{C}) \to \text{Fun}(\mathcal{G}, \mathcal{C})$, where the left hand side denotes the category of strong symmetric monoidal functors and symmetric monoidal transformations.

For the proof of the lemma, we first introduce:

**Construction 2.22.** Let $G$ be a (finite) group, and let $\mathcal{F}_G := \bigsqcup_{n \geq 0} B(\Sigma_n \wr G)$; we denote the unique object of the $n$-th summand by $n$. Then $\mathcal{F}_G$ becomes a permutative category via $n \otimes m = (n + m)$ and

$$(\sigma; g_1, \ldots, g_n) \otimes (\sigma', g_1', \ldots, g_m') = (\sigma \oplus \sigma', g_1, \ldots, g_n, g_1', \ldots, g_m')$$

where $\sigma \oplus \sigma'$ denotes the usual block sum, also see [Len20a, Construction 4.2.15].

Moreover, we then obviously have an equivalence $\mathcal{F}_G \to \text{core } hF_{BG}$ given by sending $n$ to the unique functor $n \to BG$, and a morphism $(\sigma; g_1, \ldots, g_n)$ to

$$
\begin{array}{ccc}
\leftarrow & n & \longrightarrow \\
\downarrow & \downarrow & \downarrow \\
& (g_1, \ldots, g_n) & \longrightarrow
\end{array}
$$

$BG$.
If we choose the coproduct of \( m \) and \( n \) in \( hF_{BG} \) as \( m + n \) with the inclusion \( m \hookrightarrow m + n \) and the map \( n \to m + n, i \mapsto i + m \) as structure maps, this equivalence will obviously be strict symmetric monoidal, so for a general choice of coproducts it admits a preferred strong symmetric monoidal structure.

The following lemma is well-known (and easy to check):

**Lemma 2.23.** Let \( G \) be a group and let \( C \) be any symmetric monoidal category. Then evaluating at \( 1 \in \mathcal{F}_G \) defines an equivalence \( \text{Fun}^{\otimes}(\mathcal{F}_G, C) \to G\text{-}C \), where the right hand side denotes the category of \( G \)-objects. \( \square \)

**Proof of Lemma 2.21.** As \( \tau \) is natural with respect to pushforward, we may assume without loss of generality that each component of \( \mathcal{G} \) consists of a single object, and by Lemma 2.19 we can then reduce to the case that \( \mathcal{G} = BG \) for some finite group \( G \); more precisely, if \( G_1, \ldots, G_n \) are the (pairwise non-isomorphic) objects of \( \mathcal{G} \), then the diagram

\[
\begin{array}{ccc}
\prod_{i=1}^n BG_i & \xrightarrow{\text{diag} (\tau)} & \prod_{i=1}^n hF_{BG_i} \\
\mathcal{G} & \xrightarrow{\tau} & hF_G
\end{array}
\]

with the equivalence from Lemma 2.19 on the right commutes strictly for the usual construction of fibers, and the \((2,1)\)-category \( \text{SymMonCat} \) is semiadditive.

But for \( \mathcal{G} = BG \), Construction 2.22 provided an equivalence \( \mathcal{F}_G \to \text{core} hF_{BG} \) compatible with the maps from \( BG \), so the claim follows immediately from the previous lemma. \( \square \)

### 2.2. Unfurling for 2-categories.

Our next goal is to extend \( N_\Delta(\psi) \) to a functor on \((A^{(1)})^{\text{op}}\) using the adjoints we constructed above. This is an instance of a general 2-categorical construction:

**Proposition 2.24.** Let \( \mathcal{I} \) be a strict \((2,1)\)-category, and let \( \mathcal{I}^+ \subset \mathcal{I} \) be a 2-subcategory such that \((N_\Delta(\mathcal{I}), N_\Delta(\mathcal{J}), N_\Delta(\mathcal{J}^+)) \) is an adequate triple. Let \( \mathcal{G} \) be any strict 2-category and write \( \mathcal{G}_{(2,1)} \) for its underlying \((2,1)\)-category, i.e. the 2-category obtained by throwing away all non-invertible 2-cells. Moreover, let \( \varphi : \mathcal{I} \to \mathcal{G} \) be a strict 2-functor such that for every \( i \in \mathcal{I} \) the functor \( i : \varphi(i) \) admits a right adjoint \( i^* \) and such that for every homotopy pullback diagram in \( N_\Delta(\mathcal{I}) \) as on the left in

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow j & \sigma \Downarrow & \downarrow j^* \\
C & \xrightarrow{f} & D
\end{array}
\]

\[
\begin{array}{ccc}
\varphi(A) & \xrightarrow{g} & \varphi(B) \\
\downarrow j & \sigma \Downarrow & \downarrow j^* \\
\varphi(C) & \xrightarrow{f} & \varphi(D)
\end{array}
\]

(which we drew as a diagram in \( \mathcal{I} \) by omitting the diagonal edge \( \varphi(A) \to \varphi(D) \) and already pasting the two natural isomorphisms) with vertical arrows belonging to \( \mathcal{I}^+ \), the canonical mate \( \sigma_\circ \) of \( \sigma_1 \) depicted on the right is an isomorphism.
Then there is a unique functor \( \Phi: \text{A}^{\text{eff}}(\text{N}_\Delta(\mathcal{J}), \text{N}_\Delta(\mathcal{J}), \text{N}_\Delta(\mathcal{J}^\top)) \to \text{N}_\Delta(\mathcal{C}_{(2,1)}) \) that sends a 2-simplex of the form

\[
\begin{array}{c}
\downarrow j_{02} \\
\downarrow j_{01} \\
\downarrow j_{12} \\
A \rightarrow B \rightarrow C \rightarrow D \rightarrow E
\end{array}
\]

(where we have again omitted the edge \( C \to F \) and already pasted the two transformations filling the middle square) to the 2-simplex

\[
\begin{array}{c}
\downarrow \psi(F) \\
\downarrow \psi(A) \\
\downarrow \psi(E)
\end{array}
\]

given as the pasting

\[
\begin{array}{c}
\downarrow \sigma_0 \\
\downarrow \rho^*
\end{array}
\]

where \( \rho^* \) is the total mate of \( \rho \) and \( \sigma_0 \) is defined as above.

As we will only apply this to the functor \( \psi: \mathcal{J} \to \text{Cat}^{\text{H}} \), and since we will ultimately be only interested in the postcomposition of this with core: \( \text{Cat}^{\text{H}}(2_{(2,1)} \to \text{SymMonGrpd} \), we could have also instead applied Barwick’s unfurling construction for Waldhausen bicartesian fibrations \[\text{Bar17, Section 11}\]. However, Barwick’s result first yields a (Waldhausen) cocartesian fibration, which one then has to straighten into a functor, and as we will at several places below need the above explicit description of the resulting functor on 2-simplices, proving the 2-categorical proposition directly seems to be less work than unravelling Barwick’s construction.

We moreover remark that the analogous statement for the usual span-2-category (and without any of the above strictness assumptions) appears as \[\text{BD20, Theorem 5.2.1}\]; thus, if one is willing to believe that \( A^{\text{H}} \) is equivalent to the Duskin nerve of the corresponding span 2-category in a compatible way, one could as well deduce this from \text{loc. cit.}

**Proof of Proposition 2.24**

Clearly the above defines a map \( \hat{\Phi} \) from the 2-skeleton of \( A := \text{A}^{\text{eff}}(\text{N}_\Delta(\mathcal{J}), \text{N}_\Delta(\mathcal{J}), \text{N}_\Delta(\mathcal{J}^\top)) \) to \( \text{N}_\Delta(\mathcal{C}_{(2,1)}) \). As the latter is strictly 2-coskeletal, it is then enough to show that this can be extended over 3-simplices, i.e. given any 3-simplex \( \sigma \) of \( A \), the map \( \hat{\Phi}(\partial \sigma): \partial \Delta^3 \to \text{N}_\Delta(\mathcal{C}_{(2,1)}) \) extends to \( \Delta^3 \).

To prove this, we begin by writing out some of the information encoded in such a 3-simplex \( \sigma \). Namely, this in particular consists of the following data:
(1) A diagram

\[
\begin{array}{ccc}
D & \xrightarrow{j_{01}} & E \\
\downarrow & \sigma & \downarrow \\
C & \Rightarrow & F \\
\downarrow & \downarrow & \downarrow \\
B & \Rightarrow & H \\
\downarrow & \downarrow & \downarrow \\
A & \Rightarrow & I \\
\downarrow & \downarrow & \downarrow \\
G & \Rightarrow & J \\
\end{array}
\]

in \( \mathcal{F} \), where the squares are pullback squares and we have as before already pasted the natural transformations filling them.

(2) Maps \( g_{02}, g_{03}, g_{13} \) together with 2-cells \( \gamma_{012}, \gamma_{013}, \gamma_{023}, \gamma_{123} \) such that the two pastings

\[
\begin{array}{ccc}
E & \xrightarrow{g_{12}} & F \\
\downarrow & \downarrow & \downarrow \\
D & \Rightarrow & G \\
\downarrow & \downarrow & \downarrow \\
A & \Rightarrow & B \\
\end{array}
\]

and

\[
\begin{array}{ccc}
E & \xrightarrow{g_{12}} & F \\
\downarrow & \downarrow & \downarrow \\
D & \Rightarrow & G \\
\downarrow & \downarrow & \downarrow \\
A & \Rightarrow & B \\
\end{array}
\]

agree, as well as maps \( j_{02}, j_{03}, j_{13} \) together with 2-cells \( \kappa_{012}, \kappa_{013}, \kappa_{023}, \kappa_{123} \) between the analogous composites of the \( j_{ab} \)'s, satisfying an analogous coherence condition.

(3) Maps \( f_{02} \) and \( i_{02} \) together with 2-cells \( \varphi: f_{02} \Rightarrow f_{12} \) and \( \iota: i_{02} \Rightarrow i_{12} \).

By definition of \( \hat{\Phi} \) and the construction of the Duskin nerve, the filling of \( \hat{\Phi}(\partial \sigma) \) then amounts to saying that the pastings

\[
\begin{array}{ccc}
\varphi(I) & \xrightarrow{i_{12}^*} & \varphi(H) & \xrightarrow{f_{12}^*} & \varphi(J) \\
\uparrow & \downarrow & \uparrow & \uparrow & \downarrow \\
\epsilon_1 & \xrightarrow{\iota_{02}} & \iota_{01} & \xrightarrow{\iota_{01}} & \kappa_{1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi(B) & \xrightarrow{\langle \sigma \odot \tau \rangle_0} & \varphi(E) & \xrightarrow{g_{123}} & \varphi(F) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi(A) & \xrightarrow{j_{03}^*} & \varphi(D) & \xrightarrow{\gamma_1} & \varphi(G) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\varphi(I) & \xrightarrow{i_{12}^*} & \varphi(H) & \xrightarrow{f_{12}^*} & \varphi(J) \\
\uparrow & \downarrow & \uparrow & \uparrow & \downarrow \\
\epsilon_1 & \xrightarrow{\tau_0} & f_{01}^* & \xrightarrow{\iota_{02}} & \varphi(F) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi(B) & \xrightarrow{\langle \rho \odot \sigma \rangle_0} & \varphi(C) & \xrightarrow{g_{123}} & \varphi(F) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi(A) & \xrightarrow{j_{03}^*} & \varphi(D) & \xrightarrow{\gamma_1} & \varphi(G) \\
\end{array}
\]
agree, where $\sigma \odot \tau$ denotes the pasting

\[
\begin{array}{c}
\downarrow \kappa \\
D \longrightarrow_j \ j_1 \longrightarrow C \longrightarrow_j \ j_2 \longrightarrow B \\
\downarrow g_01 \quad \downarrow f_01 \quad \downarrow \tau \\
E \longrightarrow_i \ i_01 \longrightarrow H \longrightarrow_i \ i_2 \longrightarrow I,
\end{array}
\]

$\rho \odot \sigma$ is defined analogously, and we have for readability omitted the indices of the transformations $\gamma$ and $\kappa$, as well as abbreviated $\kappa^{-\ast} = (\kappa^\ast)^{-1}$ etc.

To prove the equality of (2.6) and (2.7), we first observe that applying 2-functoriality to (2.5) shows that the pastings

\[
\begin{array}{c}
\varphi(E) \xrightarrow{g_{121}} \varphi(F) \\
g_{231} \quad \gamma_1 \\
\varphi(D) \xrightarrow{g_{031}} \varphi(G)
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\varphi(E) \xrightarrow{g_{121}} \varphi(F) \\
\varphi(D) \xrightarrow{g_{031}} \varphi(G)
\end{array}
\]

agree. Arguing likewise for $\kappa$ and then appealing to the compatibility of mates with pastings moreover shows that the two pastings

\[
\begin{array}{c}
\varphi(B) \xrightarrow{j_{12}} \varphi(C) \\
\kappa^{-\ast} \quad j_{23} \\
\varphi(A) \xrightarrow{j_{03}} \varphi(D)
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\varphi(B) \xrightarrow{j_{12}} \varphi(C) \\
\kappa^{-\ast} \quad j_{02} \quad j_{23} \\
\varphi(A) \xrightarrow{j_{03}} \varphi(D)
\end{array}
\]

agree. Finally, again using 2-functoriality and the compatibility of mates with pastings shows that $(\sigma \odot \tau)_\circ$ agrees with the pasting

\[
\begin{array}{c}
\downarrow e_i \\
\varphi(I) \longrightarrow_i \ i_2 \longrightarrow \varphi(H) \longrightarrow_i \ i_1 \longrightarrow \varphi(E) \\
\downarrow f_{01} \quad \downarrow \gamma_0 \quad \downarrow g_{01} \\
\varphi(B) \longrightarrow_j \ j_2 \longrightarrow \varphi(C) \longrightarrow_j \ j_1 \longrightarrow \varphi(D) \\
\uparrow \kappa^{-\ast} \quad \uparrow \kappa^{-\ast}
\end{array}
\]

(where we have additionally rotated the diagram by $\pi$ radians for the sake of the argument below), and analogously for $(\rho \odot \sigma)_\circ$.  

Now plugging (2.10) and (2.9) into the left hand portion of (2.6) shows that the whole diagram (2.6) agrees with the pasting

\[
\begin{align*}
\varphi(I) & \xrightarrow{i_{12}^*} \varphi(H) \xrightarrow{f_{12^*}} \varphi(J) \\
e_1 & \Rightarrow \varphi(J) \\
\varphi(B) & \xrightarrow{j_{12}} \varphi(C) \xrightarrow{\varphi(E)} \varphi(F) \\
j_{23} & \xrightarrow{j_{13}} j_{01} \xrightarrow{g_{01^*}} j_{03} \xrightarrow{g_{23^*}} j_{03} \\
\varphi(A) & \xrightarrow{i_{03}^*} \varphi(D) \xrightarrow{g_{03^*}} \varphi(G).
\end{align*}
\] (2.11)

Similarly using (2.8) together with the analogue of (2.10) for \((\rho \circ \sigma)_0\) left implicit above, we see that (2.7) agrees with the pasting

\[
\begin{align*}
\varphi(I) & \xrightarrow{i_{12}^*} \varphi(H) \xrightarrow{f_{12^*}} \varphi(J) \\
e_1 & \Rightarrow \varphi(J) \\
\varphi(B) & \xrightarrow{j_{12}} \varphi(C) \xrightarrow{\varphi(E)} \varphi(F) \\
j_{23} & \xrightarrow{j_{13}} j_{01} \xrightarrow{g_{01^*}} j_{03} \xrightarrow{g_{23^*}} j_{03} \\
\varphi(A) & \xrightarrow{i_{03}^*} \varphi(D) \xrightarrow{g_{03^*}} \varphi(G).
\end{align*}
\]

However, this is in turn exactly the same as (2.11) except for the way we have embedded it into the plane, which completes the proof of the proposition. \(\square\)

### 2.3. From \(A^\text{gl}\) to symmetric monoidal categories

In order to apply the above proposition to our context we note:

**Lemma 2.25.** Let

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
j \downarrow & \sigma & \downarrow \iota \\
C & \xrightarrow{f} & D
\end{array}
\] (2.12)

be a homotopy pullback in \(\mathcal{F}\) such that the vertical functors are faithful. Then \(\sigma_0 : g_\iota f^* \Rightarrow \iota^* f_1\) is an isomorphism of functors \(\h\mathcal{F}_C \rightarrow \h\mathcal{F}_B\).

**Proof.** The claim is clear if \(i\) is an equivalence; by the compatibility of mates with pastings we may therefore assume without loss of generality that \(\sigma\) is the identity and that (2.12) is an ordinary pullback along the isofibration \(i\). In this case, we immediately see using the explicit description of the adjunctions given in Construction 2.15 that for an object \(\pi : \mathcal{X} \rightarrow \mathcal{B}\), \((\id_\mathcal{F})_\pi\) is represented by the canonical comparison map \(\mathcal{X} \times_\mathcal{B} A = \mathcal{X} \times_\mathcal{B} (\mathcal{B} \times_\mathcal{D} C) \rightarrow \mathcal{X} \times_\mathcal{D} C\), which is even an isomorphism of groupoids. \(\square\)

Thus, we may apply Proposition 2.24 to extend \(\psi\) to a functor \(\Psi : (A^\text{gl})^{\text{op}} \equiv A^\text{eff}(\mathcal{F}, \mathcal{F}, \mathcal{F}_1) \rightarrow \mathcal{N}_\Delta(\mathbf{Cat}^\text{Ht}_{(2,1)})\). Our next goal is to prove:
Theorem 2.26. The composition \( \text{core} \circ \Psi : (A^\mathsf{gl})^{\mathsf{op}} \to N_\Delta(\text{SymMonCat}_{(2,1)}) \) is fully faithful.

We denote the essential image of the above functor by \( A^\mathsf{gl} \) (and by slight abuse of notation, we will also denote the corresponding 2-subcategory of \( \text{SymMonCat} \) by the same symbol); by Lemma 2.19 it consists precisely of those symmetric monoidal categories that are equivalent to finite products of copies of \( \text{core} hF_B H \) or equivalently \( \mathcal{S}_H \) for varying finite groups \( H \).

The proof of the theorem requires some preparations.

Construction 2.27. Let \( \mathcal{G}, \mathcal{H} \) be finite groupoids. We define a functor \( \Psi' : \text{core} hF_G, H \to \text{Fun}^{\otimes}(hF_H, hF_G) \) as follows: an object \( \pi : X \to G \times H \) is sent to the functor \( \pi \circ \rho \) where \( \pi, \rho \) denote the components of \( \pi \), and the class of a morphism

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\alpha} & \mathcal{Y} \\
\pi \downarrow & \swarrow_{\alpha} & \downarrow \rho \\
\mathcal{H} & & \\
\end{array}
\]

is sent to the pasting

\[
\begin{array}{ccc}
hF_H & \xrightarrow{\pi_H} & hF_X \\
\downarrow & \downarrow (\alpha_H^{-1}) \circ & \downarrow \alpha_G \\
hF_H & \xrightarrow{\rho_H} & hF_Y \\
\end{array}
\]

where \((\alpha_H^{-1}) \circ\) again denotes the canonical mate. We omit the easy verification that this is well-defined and a functor.

Proposition 2.28. The composition \( \text{core} \circ \Psi' \) defines an equivalence

\[
\text{core} hF_G, H \simeq \text{Fun}^{\otimes}(\text{core} hF_H, \text{core} hF_G).
\]

Proof. It suffices to show this after restricting to \( \text{core} hF^{\text{iso}}_{G, H} \). In this case, the right adjoints \( \pi_H \) appearing in the construction of \( \Psi' \) can just be taken to be the ones from Construction 2.15 i.e. they are given on objects by ordinary pullback. We now agree on a specific choice of these pullbacks in one instance: namely, for any object of \( hF_H \) of the form \( \zeta : * \to H \) and any object \( \pi : \mathcal{X} \to \mathcal{G} \times H \) of \( hF^{\text{iso}}_{G, H} \), we agree to take the pullback defining \( \pi_H(\zeta) \) as the ordinary fiber of \( \pi_H \) over \( \zeta(*) \), i.e. the subgroupoid \( \pi_H^{-1}(\zeta(*)) \subset \mathcal{X} \), with structure map given by the inclusion. With these conventions, a basic computation then shows that the diagram

\[
\begin{array}{ccc}
\text{core} hF_G, H & \xrightarrow{\text{core} \circ \Psi'} & \text{Fun}^{\otimes}(\text{core} hF_H, \text{core} hF_G) \\
\downarrow & & \downarrow \tau^* \\
\text{core Fun}^{G\text{-free}}(\mathcal{G} \times H, \text{FinSet}) & \cong & \text{Fun}(H, \text{core} hF_G) \\
\end{array}
\]

actually commutes strictly, where the unlabelled isomorphism is given by currying. The claim now follows from Lemma 2.12 together with Lemma 2.21. \( \square \)
Proof of Theorem 2.26. Let $\mathcal{G}, \mathcal{H} \in \mathcal{F}$. We have to show that $\Psi$ induces a homotopy equivalence

\[
\text{Hom}^R_{\mathcal{A}^{\mathcal{F}}}(\mathcal{H}, \mathcal{G}) \to \text{Hom}^R_{\mathcal{A}^{\mathcal{F} \ni \mathcal{G}}}((\text{core } h \mathbb{F}_\mathcal{H}, \text{core } h \mathbb{F}_\mathcal{G})
\]

of the right mapping spaces. The right hand side is clearly a 1-truncated Kan-complex; on the other hand, the left hand side is a 2-truncated Kan-complex (as a subcategory of a category of functors into a 2-category), and we claim that it is in fact again 1-truncated; once we know this, it will be enough to show that (2.13) induces an equivalence of homotopy categories.

For this, let us first describe the mapping space on the left explicitly: its objects are the spans $\mathcal{H} \xrightarrow{i} \mathcal{X} \xrightarrow{f} \mathcal{G}$ such that $i$ is faithful, and a morphism from such a span to another object $\mathcal{H} \xrightarrow{j} \mathcal{Y} \xrightarrow{g} \mathcal{G}$ is given by a diagram as on the left in

\[
\begin{align*}
\pi : \mathcal{X} &\to \mathcal{G} \times \mathcal{H} \\
i &\xrightarrow{\sigma_1} \mathcal{H} \xrightarrow{i} \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\tau} \mathcal{G} \\
&\xrightarrow{\sigma_2} \mathcal{H} \xrightarrow{j} \mathcal{G} \\
\end{align*}
\]

By direct inspection using faithfulness of the contravariant legs, for any other such diagram (as depicted on the right), there is at most one homotopy between them, and such a homotopy exists if and only if there is an isomorphism $\phi : \alpha \cong \alpha'$ such that the pasting of $\phi$ with $\rho$ agrees with $\rho'$ and the pasting of $\phi$ with $(\sigma'_2)(\sigma'_1)^{-1}$ agrees with $\sigma_2\sigma_1^{-1}$. Thus, $\text{Hom}^R_{\mathcal{A}^{\mathcal{F} \ni \mathcal{G}}}((\text{core } h \mathbb{F}_\mathcal{H}, \text{core } h \mathbb{F}_\mathcal{G})$ is 1-truncated and we have an isomorphism of categories

\[
\text{core } h \mathbb{F}_\mathcal{G} \to \text{hHom}^R_{\mathcal{A}^{\mathcal{F} \ni \mathcal{G}}}((\mathcal{H}, \mathcal{G})
\]

sending $\pi : \mathcal{X} \to \mathcal{G} \times \mathcal{H}$ to the span $\mathcal{H} \xleftarrow{\pi_H} \mathcal{X} \xrightarrow{\pi_G} \mathcal{G}$, and a morphism from $\pi$ to $\rho : \mathcal{Y} \to \mathcal{G} \times \mathcal{H}$ represented by an equivalence $\alpha : \mathcal{X} \to \mathcal{Y}$ together with an isomorphism $\hat{\alpha} : \pi \cong \rho\alpha$ to the class of

\[
\begin{align*}
\pi_H &\xrightarrow{\alpha} \pi_G \\
&\xrightarrow{\hat{\alpha}} \pi_G \\
\end{align*}
\]

also cf. [Bar17, 3.7], which states (without proof) the existence of an equivalence between the mapping spaces of the effective Burnside category of a general quasi-category $\mathcal{F}$ with all pullbacks and the cores of certain slices of $\mathcal{F}$.

Similarly (but much easier), we have an isomorphism

\[
\text{hHom}^R((\text{core } h \mathbb{F}_\mathcal{H}, \text{core } h \mathbb{F}_\mathcal{G}) \cong \text{Fun}^{\otimes}((\text{core } h \mathbb{F}_\mathcal{H}, \text{core } h \mathbb{F}_\mathcal{G})
\]
that is the identity on objects and sends the class of
\[ \text{core } hF_H \Rightarrow g \]
simply to \( \sigma : f \Rightarrow g \). Using the explicit description of \( \Psi \) on 2-simplices from Proposition 2.24, we then see that the resulting composition \( \text{core } hF_H \Rightarrow \text{core } hF_G, \text{core } hF_H \Rightarrow \text{core } hF_G \) is exactly \( \text{core } \circ \Psi' \), so it is an equivalence by the previous proposition, and hence so is (2.13) as desired. \( \square \)

3. Interlude: unenriched Morita theory

As already remarked in the introduction, the Schwede-Shipley Theorem [SS03, Theorem 3.3.3] provides models of many stable model categories in terms of suitable spectrally enriched presheaves. For the proof of our main theorem, we will be interested in the following variant for stable quasi-categories, that will get around the spectral enrichment using the equivalence between the quasi-category \( \mathcal{S}_{p}^{\geq 0} \) of connective spectra and the quasi-category \( \text{CGrp}(\mathcal{S}) \) of commutative groups in \( \mathcal{S} \).

**Theorem 3.1.** Let \( \mathcal{C} \) be a locally presentable stable quasi-category with a family of compact generators \( (X_i)_{i \in I} \) such that for all \( i, j \in I \) the mapping spectrum \( F(X_i, X_j) \) is connective, i.e. \( \text{Hom}^{h\mathcal{C}}(X_i, X_j) = 0 \) for all \( r > 0 \).

We write \( \mathcal{X} \subset \mathcal{C} \) for the full subcategory spanned by the finite products of the \( X_i \)'s. Then the Yoneda embedding lifts essentially uniquely to a functor
\[ \mathcal{C} \to \text{Fun}^{ex}(\mathcal{C}^{\text{op}}, \mathcal{S}_{p}), \]
and this lift induces an equivalence \( \mathcal{C} \simeq \text{Fun}^{\oplus}(\mathcal{X}^{\text{op}}, \mathcal{S}_{p}). \)

I actually expect the above theorem to be known to experts; for example, it seems to be used implicitly by Barwick in the introduction of [Bar17] when he compares his approach to the Guillou-May spectral presheaf model. However, as I was unable to find a proof of this in the literature, I have decided to give a complete argument below.

3.1. Implicit enrichment. To make the proof of Theorem 3.1 somewhat more interesting—and at the same time to be able to obtain Mackey functor models of so-called ultra-commutative monoids in the next section—we will actually prove a much more general statement below that for example also incorporates a quasi-categorical version of Elmendorf’s Theorem. To set this up, we introduce the following terminology:

**Definition 3.2.** Let \( \mathfrak{A} \subset \text{QCAT} \) be a locally full subcategory of the large simplicial category of quasi-categories, i.e. for every \( \mathcal{C}, \mathcal{D} \in \mathfrak{A} \) the mapping object \( \text{Fun}^{\mathfrak{A}}(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D}) \) is a full subcategory closed under equivalences. We will refer to objects of \( \mathfrak{A} \) as \( \mathfrak{A} \)-categories and to morphisms of \( \mathfrak{A} \) as \( \mathfrak{A} \)-functors.

The subcategory \( \mathfrak{A} \) is called good if the following conditions are satisfied:

1. \( \mathfrak{A} \) is closed under equivalences and passing to the opposite quasi-category.
2. For all \( \mathcal{C}, \mathcal{D} \in \mathfrak{A} \), the full subcategory \( \text{Fun}^{\mathfrak{A}}(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D}) \) is closed under all small limits and colimits that exist in \( \mathcal{D} \). Moreover, it is again an \( \mathfrak{A} \)-category and for all \( X \in \mathcal{C} \) the evaluation functor \( \text{Fun}^{\mathfrak{A}}(\mathcal{C}, \mathcal{D}) \to \mathcal{D} \) is an \( \mathfrak{A} \)-functor.
Example 3.3. One easily checks that the following subcategories are good:

1. the category \( \mathfrak{A} = \text{QCAT} \) of all quasi-categories itself
2. the full subcategory \( \mathfrak{A} = \text{CAT} \) of all 1-categories
3. the subcategory \( \mathfrak{A} = \text{SADD} \) of all semiadditive quasi-categories and direct sum preserving functors,
4. the subcategory \( \mathfrak{A} = \text{ADD} \) of all additive quasi-categories and direct sum preserving functors,
5. the subcategory \( \mathfrak{A} = \text{ST} \) of all stable quasi-categories and exact functors.

Definition 3.4. Let \( \mathfrak{A} \) be as above, let \( V \in \mathfrak{A} \) be locally presentable, and let \( U : \mathcal{V} \to \mathcal{S} \) be a continuous accessible functor (or equivalently, a corepresentable functor). We say that locally small \( \mathfrak{A} \)-categories are implicitly \( V \)-enriched if the following conditions hold for every locally small \( C \in \mathfrak{A} \):

1. Postcomposition with \( U \) restricts to a fully faithful functor \( \text{Fun}^\mathfrak{A}(C^{\text{op}}, \mathcal{V}) \to \text{Fun}(C^{\text{op}}, \mathcal{S}) \) whose essential image contains all represented functors.
2. The essentially unique lift \( y^\mathfrak{A} : C \to \text{Fun}^\mathfrak{A}(C^{\text{op}}, \mathcal{V}) \) of the Yoneda embedding (see the previous condition) is an \( \mathfrak{A} \)-functor.

Example 3.5. Taking \( \mathfrak{A} = \text{QCAT} \) again and \( U : \mathcal{S} \to \mathcal{S} \) as the identity, the above conditions are satisfied for trivial reasons, so locally small quasi-categories are implicitly \( \mathcal{S} \)-enriched.

Example 3.6. Taking \( \mathfrak{A} = \text{CAT} \) and \( U : \text{Sets} \to \mathcal{S} \) the natural map, the first condition follows from the fact that \( U \) is an equivalence onto the full subcategory of discrete spaces, and the second condition is vacuous. Thus, as one would expect, locally small 1-categories are implicitly enriched in sets.

Example 3.7. We take \( \mathfrak{A} = \text{SADD} \) and \( U : \text{CMon}(\mathcal{S}) \to \mathcal{S} \) the forgetful functor; we claim that locally small semiadditive quasi-categories are implicitly enriched in \( \text{CMon}(\mathcal{S}) \). Indeed, \[ \text{GGN15, Corollary 2.5-(iii)} \] in particular verifies the first condition above, and for any semiadditive \( C \) the resulting lift of the Yoneda embedding again preserves finite products since the composition with the fully faithful functor \( \text{Fun}^\mathfrak{A}(C^{\text{op}}, \text{CMon}(\mathcal{S})) \to \text{Fun}(C^{\text{op}}, \mathcal{S}) \) does.

Example 3.8. We take \( \mathfrak{A} = \text{ADD} \) and \( U : \text{CGrp}(\mathcal{S}) \to \mathcal{S} \) the forgetful functor. Then \[ \text{GGN15, Corollary 2.10-(iii)} \] together with the previous argument implies that locally small additive quasi-categories are implicitly \( \text{CGrp}(\mathcal{S}) \)-enriched. Equivalently, we can consider the functor \( U = \Omega^\infty : \mathcal{S} p^{\geq 0} \to \mathcal{S} \), i.e. locally small additive quasi-categories are implicitly enriched in connective spectra.

Example 3.9. Taking \( \mathfrak{A} = \text{ST} \) and \( U = \Omega^\infty : \mathcal{S} p \to \mathcal{S} \), \[ \text{Lur18, Corollary 1.4.23} \] together with the argument from Example 3.7 shows that locally small stable quasi-categories are implicitly \( \mathcal{S} p \)-enriched.

Remark 3.10. While we will not prove this here, I want to remark that the \( \mathfrak{A} \)-category \( \mathcal{V} \) together with the representable functor \( U \) is actually essentially uniquely determined by \( \mathfrak{A} \): namely, if \( v \in \mathcal{V} \) corepresents \( U \), then \( (\mathcal{V}, v) \) is the initial example of a locally presentable \( \mathfrak{A} \)-category together with a chosen object.

Building on this one can show that as soon as all left adjoint functors between \( \mathfrak{A} \)-categories are \( \mathfrak{A} \)-functors (which is true in all of our examples), \( \mathcal{V} \) together with the functor \( \mathcal{S} \to \mathcal{V} \) left adjoint to \( U \) will be a mode in the sense of \[ \text{CSY20, MS21} \], and our setup gets a somewhat similar flavour to the theory developed in...
op. cit. (although their work is far more extensive). I am not aware of an analogue of Theorem 3.1 or Theorem 3.19 below in the context of general modes.

In the following let us fix \( \mathfrak{A} \) and \( U : \mathcal{Y} \to \mathcal{S} \) as above. If now \( \mathcal{C} \) is any locally small \( \mathfrak{A} \)-category, then we pick once and for all a functor \( y_{\mathfrak{A}} : \mathcal{C} \to \text{Fun}_{\mathfrak{A}}(\mathcal{C}^{\text{op}}, \mathcal{Y}) \) together with an equivalence \( \psi : y_{\mathfrak{A}} \simeq U \circ y_{\mathfrak{A}}^{\mathfrak{A}} \). Applying the same to \( \mathcal{C}^{\text{op}} \) gives us a functor \( y_{\mathfrak{A}}^{\mathfrak{A}} : \mathcal{C}^{\mathfrak{A}} \to \text{Fun}_{\mathfrak{A}}(\mathcal{C}, \mathcal{Y}) \) together with an equivalence \( \psi_{\mathfrak{A}} : y_{\mathfrak{A}}^{\mathfrak{A}} \simeq U \circ y_{\mathfrak{A}}^{\mathfrak{A}} \). By assumption, both \( y_{\mathfrak{A}} \) and \( y_{\mathfrak{A}}^{\mathfrak{A}} \) are \( \mathfrak{A} \)-functors, and so are \( y_{\mathfrak{A}}^{\mathfrak{A}}(X) \) and \( y_{\mathfrak{A}}^{\mathfrak{A}}(X) \) for every \( X \in \mathcal{C} \).

**Remark 3.11.** The above two functors are compatible in the following sense: if \( \mathcal{C} \) is any locally small \( \mathfrak{A} \)-category and \( X \in \mathcal{C} \) is arbitrary, then \( y_{\mathfrak{A}}^{\mathfrak{A}}(X) \) is an \( \mathfrak{A} \)-functor and so is the composition

\[
\mathcal{C}^{\mathfrak{A}} \xrightarrow{y_{\mathfrak{A}}^{\mathfrak{A}}} \text{Fun}_{\mathfrak{A}}(\mathcal{C}, \mathcal{Y}) \xrightarrow{\text{ev}} \mathcal{Y}
\]

as evaluation is an \( \mathfrak{A} \)-functor.

On the other hand, we have equivalences \( \psi(X) : U \circ y_{\mathfrak{A}}^{\mathfrak{A}}(X) \simeq y(X) : \mathcal{C}^{\mathfrak{A}} \to \mathcal{S} \) as well as \( \text{ev}_{X} \circ \psi_{\mathfrak{A}} : U \circ \text{ev}_{X} \circ y_{\mathfrak{A}}^{\mathfrak{A}} \simeq \text{ev}_{X} \circ y_{\mathfrak{A}}^{\mathfrak{A}} = y(X) : \mathcal{C}^{\mathfrak{A}} \to \mathcal{S} \). It follows by full faithfulness of postcomposition with \( U \) that there exists an essentially unique pair of an equivalence \( \theta : y^{\mathfrak{A}}(X) \simeq \text{ev}_{X} \circ y_{\mathfrak{A}}^{\mathfrak{A}} \) together with a homotopy filling

\[
\begin{array}{c}
\begin{array}{ccc}
U \circ y_{\mathfrak{A}}(X) & \xrightarrow{\psi(X)} & y(X) \\
\downarrow & & \downarrow \\
U \circ y_{\mathfrak{A}}^{\mathfrak{A}}(X) & \xrightarrow{\theta} & U \circ \text{ev}_{X} \circ y_{\mathfrak{A}}^{\mathfrak{A}}.
\end{array}
\end{array}
\]

**Definition 3.12.** Let \( \mathcal{C} \in \mathfrak{A} \); a subcategory \( \mathcal{B} \subset \mathcal{C} \) is called an \( \mathfrak{A} \)-subcategory if \( \mathcal{B} \) is an \( \mathfrak{A} \)-category and the inclusion \( \mathcal{B} \to \mathcal{C} \) is an \( \mathfrak{A} \)-functor.

**Remark 3.13.** The above lifts of the Yoneda embedding are compatible with passing to full \( \mathfrak{A} \)-subcategories (though not necessarily with passing to general subcategories) in the following sense:

Let \( \mathcal{C} \) be a locally small \( \mathfrak{A} \)-category, let \( \mathcal{X} \subset \mathcal{C} \) be any full \( \mathfrak{A} \)-subcategory, and let \( X \in \mathcal{X} \). Then there exists an equivalence \( \rho : y_{\mathcal{X}}(X) \simeq y_{\mathcal{X}}(X)|_{\mathcal{X}^{\mathfrak{A}}} \) (unique up to non-canonical homotopy) sending the component of \( \text{id}_{X} \) in \( y_{\mathcal{X}}(X)(X) \) to the component of \( \text{id}_{X} \) in \( y_{\mathcal{X}}(X)(X) \). As \( y_{\mathcal{X}}(X)|_{\mathcal{X}^{\mathfrak{A}}} \) is again an \( \mathfrak{A} \)-functor, the same argument as in Remark 3.11 then provides us with an essentially unique pair of an equivalence \( \bar{\rho} : y_{\mathcal{X}}^{\mathfrak{A}}(X) \simeq y_{\mathcal{X}}^{\mathfrak{A}}(X)|_{\mathcal{X}^{\mathfrak{A}}} \) together with a homotopy filling

\[
\begin{array}{c}
\begin{array}{ccc}
y_{\mathcal{X}}(X) & \xrightarrow{\rho} & y_{\mathcal{X}}(X)|_{\mathcal{X}^{\mathfrak{A}}} \\
\downarrow & & \downarrow \\
y_{\mathcal{X}}^{\mathfrak{A}}(X) & \xrightarrow{\bar{\rho}} & y_{\mathcal{X}}^{\mathfrak{A}}(X)|_{\mathcal{X}^{\mathfrak{A}}}.
\end{array}
\end{array}
\]

In particular \( U \bar{\rho} \) sends the component of \( \psi_{\mathcal{X}}(\text{id}_{X}) \) to the component of \( \psi_{\mathcal{X}}(\text{id}_{X}) \).

**Definition 3.14.** Let \( \mathcal{C} \) be an \( \mathfrak{A} \)-category and let \( F : \mathcal{C} \to \mathcal{Y} \) be an \( \mathfrak{A} \)-functor. We say that \( F \) is corepresented by an object \( X \in \mathcal{C} \) and a class \( i \in \pi_{0}(UF(X)) \) if \( U \circ F \) is corepresented by \( (X, i) \) in the usual sense, i.e. for every \( Y \in \mathcal{C} \) the composition

\[
\text{maps}(X, Y) \xrightarrow{UF} \text{maps}(UF(X), UF(Y)) \xrightarrow{i} \text{maps}(*, UF(Y)) \cong UF(Y)
\]

is an isomorphism in \( \mathcal{S} \).
Example 3.15. For any locally small $\mathcal{A}$-category $\mathcal{C}$, the functor $y^\mathcal{A}_\mathrm{op}(X) : \mathcal{C} \to \mathcal{V}$ is corepresented by $(X, [\psi_\mathrm{op}(\mathrm{id}_X)])$. In fact, it is not hard to show using the Yoneda Lemma and full faithfulness of $\text{Fun}^\mathcal{A}(\mathcal{C}, U)$ that if $F : \mathcal{C} \to \mathcal{S}$ is any $\mathcal{A}$-functor corepresented by $(X, \iota)$, then there is an equivalence $y^\mathcal{A}(X) \simeq F$ characterized uniquely up to homotopy by the property that the induced transformation of functors $h^\mathcal{C} \to h^\mathcal{S}$ sends $[\psi_\mathrm{op}(\mathrm{id}_X)]$ to $\iota$.

Example 3.16. By Remark 3.11 we conclude from the previous example that similarly the $\mathcal{A}$-functor $\mathcal{E} X \circ y^\mathcal{A} : \mathcal{C} \to \mathcal{V}$ is corepresented by $(X, [\psi_\mathrm{op}(\mathrm{id}_X)])$.

The following result provides a generalization of the Yoneda Lemma to our context:

Lemma 3.17. Let $X$ be a small $\mathcal{A}$-category and let $\mathcal{X} \in X$. Then the $\mathcal{A}$-functor $\mathcal{E} X : \text{Fun}^\mathcal{A}(X^\mathrm{op}, \mathcal{V}) \to \mathcal{V}$ is corepresented by $(y^\mathcal{A}(X), [\psi(\mathrm{id}_X)])$.

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
\text{maps}(y^\mathcal{A}(X), F) & \xrightarrow{U \circ \mathcal{E} X} & \text{maps}(U(y^\mathcal{A}(X)(X)), UF(X)) \\
U \circ & & \downarrow \psi^* \\
\text{maps}(U \circ y^\mathcal{A}(X), U \circ F) & \xrightarrow{\psi^*} & \text{maps}(y(X), U \circ F) \\
\psi^* & & \downarrow [\psi(\mathrm{id}_X)]^* \\
\text{maps}(y(X), U \circ F) & \xrightarrow{\mathcal{E} X} & \text{maps}(y(X), UF(X))
\end{array}
$$

in $h^\mathcal{S}$, and we have to show that the (top right) composite from the upper left to the lower right corner is an isomorphism. But the top left vertical map is an isomorphism as $U \circ -$ is fully faithful, the lower left vertical map is an isomorphism as $\psi$ is an equivalence, and so is the bottom composition by the usual Yoneda Lemma, see [Lur09, Lemma 5.1.5.2] or [Cis19, Theorem 5.8.9]. The claim follows by 2-out-of-3. □

Proposition 3.18. Let $\mathcal{X}$ be a small $\mathcal{A}$-category. Then $\text{Fun}^\mathcal{A}(\mathcal{X}^\mathrm{op}, \mathcal{V})$ is locally presentable and generated under colimits by the functors $y^\mathcal{A}(X) : \mathcal{X}^\mathrm{op} \to \mathcal{V}$ for varying $X \in \mathcal{X}$.

Proof. It is clear that $\text{Fun}^\mathcal{A}(\mathcal{X}^\mathrm{op}, \mathcal{V})$ is cocomplete and locally small. By [Lur09, Theorem 5.5.1.1] it therefore suffices to show that the objects $y^\mathcal{A}(X)$ for $X \in \mathcal{X}$ are small and that they generate $\text{Fun}^\mathcal{A}(\mathcal{X}^\mathrm{op}, \mathcal{V})$ under colimits.

For the first statement, it suffices to observe that by the previous lemma $y^\mathcal{A}(X)$ corepresents the composition

$$
\text{Fun}^\mathcal{A}(\mathcal{X}^\mathrm{op}, \mathcal{V}) \xrightarrow{\mathcal{E} X} \mathcal{V} \xrightarrow{U} \mathcal{S}
$$

(in the usual sense), which is accessible by assumption on $U$ and since colimits on the left are computed pointwise.

For the second statement, we let $F \in \text{Fun}^\mathcal{A}(\mathcal{X}^\mathrm{op}, \mathcal{V})$ arbitrary, and we recall that $\text{Fun}(\mathcal{X}^\mathrm{op}, \mathcal{S})$ is generated by the corepresented functors $y(X), X \in \mathcal{X}$. By
full faithfulness of the Yoneda embedding, we therefore find a small quasi-category $I$ and a functor $X_\bullet: I \to \mathcal{X}$ such that $U \circ F \simeq \text{colim } y \circ X_\bullet$.

We now claim that $F \simeq \text{colim } y^{\mathcal{A}} \circ X_\bullet$, which will then complete the proof of the proposition. Indeed, by construction $U \circ F \simeq \text{colim } U \circ y^{\mathcal{A}} \circ X_\bullet$, so that the colimit on the right hand side in particular lies in the essential image of the forgetful functor $\text{Fun}(\mathcal{A}(\mathcal{X}^{\text{op}}, \mathcal{V})) \to \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{S})$. But as the forgetful functor is fully faithful by assumption, this already implies that $U \circ \text{colim } y^{\mathcal{A}} \circ X_\bullet \simeq \text{colim } U \circ y^{\mathcal{A}} \circ X_\bullet$ and hence altogether $U \circ F \simeq U \circ \text{colim } y^{\mathcal{A}} \circ X_\bullet$. The claim follows from another application of full faithfulness of $U \circ -$.

Now we can finally prove:

**Theorem 3.19.** Let $\mathcal{A} \subset \text{QCAT}$ be good, let $\mathcal{V} \in \mathcal{A}$ be locally presentable, and let $U: \mathcal{V} \to \mathcal{S}$ be continuous and accessible such that locally small $\mathcal{A}$-categories are implicitly $\mathcal{V}$-enriched.

Moreover, let $\mathcal{C} \in \mathcal{A}$ be locally presentable and let $\mathcal{X} \subset \mathcal{C}$ be a small $\mathcal{A}$-subcategory satisfying the following conditions:

1. For every $X \in \mathcal{X}$ the functor $y^{\mathcal{A}}_{\text{op}}(X): \mathcal{C} \to \mathcal{V}$ is cocontinuous.
2. For varying $X \in \mathcal{X}$, the functors $y^{\mathcal{A}}_{\text{op}}(X)$ are jointly conservative.

Then the composition $\Upsilon: \mathcal{C} \xrightarrow{y^{\mathcal{A}}_{\text{op}}} \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) \xrightarrow{\text{restriction}} \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{V})$ is an equivalence.

In the situation of Example 3.5, this can be viewed as a quasi-categorical analogue of Elmendorf’s Theorem, also cf. [Lur09, Corollary 5.1.6.11]. Note that while we can also apply this to the stable world (Example 3.9), this is not quite the statement we are after; instead, we will show in the next subsection how to deduce Theorem 3.19 from the specialization of the above theorem to the case of additive quasi-categories (Example 3.8).

**Proof.** We begin with the following observation:

**Claim.** $\Upsilon$ is continuous and cocontinuous.

**Proof.** Let us first show that $\Upsilon$ is continuous. As fully faithful functors reflect limits, it is enough for this to show that the composition $\mathcal{C} \xrightarrow{\Upsilon} \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) \xrightarrow{\text{restriction}} \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{V})$ is continuous, for which it is in turn enough to show this after postcomposing with $\text{ev}_X$ for all $X \in \mathcal{X}$. However, by definition of $y^{\mathcal{A}}_{\text{op}}$ the resulting functor $\mathcal{C} \to \mathcal{V}$ is equivalent to $y_{\text{op}}(X): \mathcal{C} \to \mathcal{V}$, i.e. the functor corepresented by $X$ in the usual sense, hence continuous.

To show that $\Upsilon$ is cocontinuous, we first observe that colimits in $\text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{V})$ can be computed in $\text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{S})$ again, so that it suffices by Remark 3.11 that all the functors $y_{\text{op}}^{\mathcal{A}}(X) \simeq \text{ev}_X \circ \Upsilon: \mathcal{C} \to \mathcal{V}$ are cocontinuous, which however holds by assumption.

As source and target of $\Upsilon$ are locally presentable (Proposition 3.18), the Adjoint Functor Theorem provides us with a left adjoint $\Lambda$ to $\Upsilon$. As $\Upsilon$ is conservative by assumption, it only remains to show that the unit $F \to \Upsilon \Lambda F$ is an equivalence for every $F \in \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{V})$, and as both $\Upsilon$ and $\Lambda$ are cocontinuous, it is by another
application of Proposition 3.18 enough to show this for \( F = \Upsilon(X) \simeq y^\text{ADD}_X(X), \) \( X \in \mathcal{X}. \) By abstract nonsense about adjunctions we are then further reduced to showing that the counit \( \Lambda \Upsilon X \to X \) is an equivalence for all \( X \in \mathcal{X}, \) for which it is in turn enough that \( \Upsilon : \text{maps}(X, T) \to \text{maps}(\Upsilon X, \Upsilon T) \) is an isomorphism in \( \mathcal{K} \) for all \( T \in \mathcal{C}. \)

For this we consider the commutative diagram

\[
\begin{array}{ccc}
\text{maps}(X, T) & \xrightarrow{\Upsilon} & \text{maps}(\Upsilon(X), \Upsilon(T)) \\
U_{\text{oev}_X} \circ \Upsilon & \downarrow & U_{\text{oev}_X} \\
\text{maps}(U\Upsilon(X)(X), \Upsilon(T)(X)) & \xrightarrow{[\psi^*_{\text{id}_\mathcal{X}}]} & \text{maps}(U\Upsilon(X)(X), \Upsilon(T)(X)) \\
\text{maps}(*, U\Upsilon(T)(X)) & \xrightarrow{[\psi^*_{\text{id}_\mathcal{X}}]} & \text{maps}(*, U\Upsilon(T)(X))
\end{array}
\]

in \( \mathcal{K}. \) The left hand vertical composite is an isomorphism by Example 3.16 and so is the right hand vertical composition by Lemma 3.17 together with Remark 3.13.

The claim follows by 2-out-of-3. \( \square \)

3.2. The stable case. In this subsection we will finally prove Theorem 3.1. For this let us begin with the following counterpart of Proposition 3.18.

**Proposition 3.20.** Let \( \mathcal{C} \) be stable and let \( \mathcal{X} \subset \mathcal{C} \) be a full subcategory closed under direct sum such that for all \( X, Y \in \mathcal{X} \) the mapping spectrum \( F(X, Y) \) is connective. Then the exact functor \( \text{ev}_X : \text{Fun}_0^\mathcal{X}(\mathcal{X}^{\text{op}}, \mathcal{P}) \to \mathcal{P} \) is corepresented by \( (y^\text{ST}_\mathcal{X}(X)|_{\mathcal{X}^{\text{op}}}, [\psi^\text{ST}_{\mathcal{X}}(\text{id}_\mathcal{X})]) \) for every \( X \in \mathcal{X}. \)

**Proof.** Let \( i : \mathcal{P}^{\geq 0} \to \mathcal{P} \) denote the inclusion. Then \( y^\text{ST}_\mathcal{X}(X)|_{\mathcal{X}^{\text{op}}} \) factors through \( i \) for every \( X \in \mathcal{X} \) by assumption, and arguing as in Remark 3.13 we get an equivalence \( y^\text{ST}_\mathcal{X}(X)|_{\mathcal{X}^{\text{op}}} \simeq i \circ y^\text{ADD}_\mathcal{X}(X) : \mathcal{X}^{\text{op}} \to \mathcal{P} \) for all \( X \in \mathcal{X} \) sending the component \( [\psi^\text{ST}_{\mathcal{X}}(\text{id}_\mathcal{X})] \) to \( [\psi^\text{ADD}_{\mathcal{X}}(\text{id}_\mathcal{X})] \). It therefore suffices to show that \( \text{ev}_X \) is corepresented by \( (i \circ y^\text{ADD}_\mathcal{X}, [\psi^\text{ADD}_{\mathcal{X}}(\text{id}_\mathcal{X})]) \).

Now let \( F \in \text{Fun}_0^\mathcal{X}(\mathcal{X}^{\text{op}}, \mathcal{P}) \) arbitrary. Then we have a commutative diagram

\[
\begin{array}{ccc}
\text{maps}(y^\text{ADD}_\mathcal{X}(X), \tau \circ F) & \xrightarrow{\Omega^\infty \circ i} & \text{maps}(\Omega^\infty i y^\text{ADD}_\mathcal{X}(X)(X), \Omega^\infty i \tau F(X)) \\
\text{maps}(i \circ y^\text{ADD}_\mathcal{X}(X), i \tau \circ F) & \xrightarrow{\epsilon} & \text{maps}(\Omega^\infty (i \circ y^\text{ADD}_\mathcal{X}(X)), \Omega^\infty (i \tau F(X))) \\
\text{maps}(i \circ y^\text{ADD}_\mathcal{X}(X), F) & \xrightarrow{\Omega^\infty \circ i} & \text{maps}(\Omega^\infty i y^\text{ADD}_\mathcal{X}(X)(X), \Omega^\infty F(X)) \\
\end{array}
\]

in \( \mathcal{K}, \) where \( \tau : \mathcal{P} \to \mathcal{P}^{\geq 0} \) is right adjoint to the inclusion and \( \epsilon \) denotes the counit of the adjunction. In this, the left hand vertical composition is an isomorphism by adjointness, and so is the right hand vertical map as \( \Omega^\infty \epsilon \) is an equivalence. On the other hand, Lemma 3.17 (for \( \mathcal{A} = \text{ADD} \) for \( X \in \mathcal{X} \)) shows that the top horizontal composite is an isomorphism, and hence so is the lower horizontal one by 2-out-of-3, which is precisely what we wanted to prove. \( \square \)

**Corollary 3.21.** In the above situation, \( \text{Fun}_0^\mathcal{X}(\mathcal{X}^{\text{op}}, \mathcal{P}) \) is locally presentable with generators (in the stable sense) given by the functors \( y^\text{ST}_\mathcal{X}(X)|_{\mathcal{X}^{\text{op}}} \) for \( X \in \mathcal{X}. \)
Proof. As Fun^{\oplus}(\mathcal{X}^{\text{op}}, \mathcal{S}) is stable, locally small, and has small coproducts, it is enough by [Lur18, Corollary 1.4.4.2] to show that the \( y^{\mathcal{ST}}(X)|_{\mathcal{X}^{\text{op}}} \) together with their shifts detect zero objects (on ordinary hom-sets in the homotopy category). This is however immediate from the previous proposition. □

Proof of Theorem 3.1. Again we begin by observing:

Claim. \( \Upsilon \) is continuous and cocontinuous.

Proof. The same argument as in the proof of Theorem 3.19 shows that \( \Upsilon \) is continuous. To prove that it is also cocontinuous, we note that since \( \Upsilon \) is a continuous functor of stable quasi-categories, it is exact, and hence it preserves finite colimits. Thus, it only remains to show that it also preserves filtered colimits.

For this, let us write \( \mathcal{Y} \supseteq \mathcal{X} \) for the smallest stable subcategory of \( \mathcal{C} \) containing \( X \); note that \( \mathcal{Y} \) again consists of compact objects. We will show that the functor \( \bar{\Upsilon}: \mathcal{C} \to \text{Fun}^{\text{ex}}(\mathcal{Y}^{\text{op}}, \mathcal{S}) \) defined analogously preserves filtered colimits, which will then in particular imply the claim.

Indeed, \( \Omega_{\infty} \circ \phi: \text{Fun}^{\text{ex}}(\mathcal{Y}^{\text{op}}, \mathcal{S}) \to \text{Fun}(\mathcal{Y}^{\text{op}}, \mathcal{S}) \) is fully faithful by Example 3.9, so it suffices that \( \Omega_{\infty} \circ \bar{\Upsilon} \) preserves filtered colimits, for which it is again enough to prove this after evaluating at each \( Y \in \mathcal{Y} \). But as before the resulting composition \( \mathcal{C} \to \mathcal{I} \) is just equivalent to \( y(Y) \), so the claim follows from compactness of \( Y \). △

Next we note that more generally all mapping spectra \( F(X,Y) \) for \( X,Y \in \mathcal{X} \) are connective as taking mapping spectra preserves direct sums in each variable. In particular, we can apply the previous corollary, which provides us with a left adjoint \( \Lambda \) of \( \Upsilon \), and by the same arguments as before it will then be enough to show that the counit \( \Lambda \Upsilon X \to X \) is an equivalence for every \( X \in \mathcal{X} \). This is however again just a formal consequence of the corepresentability statement given in Proposition 3.20. □

4. Mackey functors and ultra-commutativity

4.1. A reminder on global \( \Gamma \)-spaces. Similarly to the equivariant situation, the correct notion of 'commutative monoids up to globally coherent homotopies' turns out to be more subtle than just commutative monoids in the quasi-category of global \( \Gamma \)-spaces, a phenomenon for which Schwede coined the term \textit{ultra-commutativity}.

The model we will use here is based on a generalization of Segal’s \( \Gamma \)-spaces [Seg74, BF78], which for us will mean the quasi-localization of the 1-category \( \Gamma \text{-SSet}_\ast \) of reduced functors \( \Gamma \to \text{SSet}_\ast \), at the levelwise weak equivalences. Note that this is canonically equivalent to the analogous subcategory \( \text{Fun}_\ast(\text{NG}, \mathcal{S}) \subset \text{Fun}(\text{NG}, \mathcal{S}) \) (which most people would call the quasi-category of \( \Gamma \)-spaces) as a consequence of [Lur09, Proposition 4.2.4.4].

Inside the \( \Gamma \)-spaces, we have the full subcategory of special \( \Gamma \)-spaces, i.e. those \( \Gamma \)-spaces \( X \) for which the Segal maps \( X(S_+) \to X(1^+) \) are weak homotopy equivalences. By general nonsense [Len20a, Proposition A.1.15] the chosen quasi-localization \( \Gamma \text{-SSet}_\ast \to \Gamma \text{-SSet}_{\text{special}}^\infty \) restricts to exhibit the full subcategory of the quasi-category \( \Gamma \text{-SSet}_{\text{special}}^\infty \) spanned by the special \( \Gamma \)-spaces as a quasi-localization of the 1-category \( \Gamma \text{-SSet}_{\text{special}}^\infty \) of global \( \Gamma \)-spaces. In particular, we see that the above equivalence \( \Gamma \text{-SSet}_{\text{special}}^\infty \simeq \text{Fun}_\ast(\text{NG}, \mathcal{S}) \) identifies \( (\Gamma \text{-SSet}_{\text{special}})^\infty \) with \( \text{CMon}(\mathcal{S}) \).

Now we are ready to introduce the corresponding global theory:
Theorem 4.1. There is a unique model structure on the category $\Gamma\text{-}\mathcal{E}\mathcal{M}\text{-}\mathbf{SSet}_*$ of $\Gamma$-spaces with $\mathcal{E}\mathcal{M}$-action in which a map $f: X \to Y$ is a weak equivalence or fibration if and only if $f(S_+)$ is a $\Sigma_S$-global weak equivalence for all finite sets $S$; here we equip $X(S_+)$ and $Y(S_+)$ with the $\Sigma_S$-action induced by the functoriality of $X$ in $\Gamma$.

We call this model structure the global level model structure. It is simplicial, combinatorial with generating cofibrations

$$\{ (\Gamma(S_+, -) \wedge \mathcal{E}\mathcal{M}_+)/H \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+: H \text{ finite group, } S \text{ finite } H\text{-set} \},$$

and filtered colimits in it are homotopical.

Proof. See [Len20a, Theorem 2.2.22]. □

We will usually refer to the above objects simply as ‘global $\Gamma$-spaces.’

Just like non-equivariantly, we will mostly be interested in those global $\Gamma$-spaces that satisfy an additional specialness condition:

Definition 4.2. A global $\Gamma$-space is called special if the usual Segal map $X(S_+) \to X(1^+) \times S$ is a $\Sigma_S$-global weak equivalence for every finite set $S$. Here $\Sigma_S$ acts on the left as before and on the right via permuting the factors.

We write $\Gamma\text{-}\mathcal{E}\mathcal{M}\text{-}\mathbf{SSet}_{*\text{special}}$ for the full subcategory spanned by the global $\Gamma$-spaces.

Remark 4.3. Let $H \subset \mathcal{M}$ be universal. Then restricting to trivial homomorphisms $H \to \Sigma_S$ shows that taking $H$-fixed points sends global level weak equivalences of global $\Gamma$-spaces to level weak equivalences of ordinary $\Gamma$-spaces. Similarly, using that fixed points commute with products, we see that $X^H$ is a special $\Gamma$-space in the usual sense for every special global $\Gamma$-space $X$.

Arguing precisely as in [Len20a, Theorem 2.2.52] one gets by general abstract nonsense:

Theorem 4.4. The model structure from Theorem 4.1 admits a Bousfield localization whose fibrant objects are the globally level fibrant special global $\Gamma$-spaces. We call this the global special model structure and its weak equivalences the (global) special weak equivalences. □

In particular, we see that the inclusion $\Gamma\text{-}\mathcal{E}\mathcal{M}\text{-}\mathbf{SSet}_{*\text{special}} \hookrightarrow \Gamma\text{-}\mathcal{E}\mathcal{M}\text{-}\mathbf{SSet}_*$ induces an equivalence

$$(\Gamma\text{-}\mathcal{E}\mathcal{M}\text{-}\mathbf{SSet}_{*\text{special}})_{\text{level}}^{\infty} \simeq (\Gamma\text{-}\mathcal{E}\mathcal{M}\text{-}\mathbf{SSet}_*)^{\infty}_{\text{special}}$$

between the quasi-localization of the special global $\Gamma$-spaces at the level weak equivalences and the quasi-localization of all global $\Gamma$-spaces at the special weak equivalences; this additional flexibility will come in handy at several points below.

Remark 4.5. Schwede originally introduced so-called ultra-commutative monoids [Sch18, Chapter 2] as his model for global coherent commutativity. While these have the advantage of containing meaningful information for all compact Lie groups, as long as one is only interested in global homotopy theory for finite groups (as we are here), there is no difference to the homotopy theory of global special $\Gamma$-spaces, see [Len20a, Corollary 2.3.17].

Again there is an analogous story based on $\mathcal{I}$-simplicial sets:
Theorem 4.6. There is a unique model structure on \( \Gamma \text{-I-SSet}_+ \) in which a map \( f \) is a weak equivalence or fibration if and only if \( f(S+) \) is a \( \Sigma_S \)-global weak equivalence or fibration, respectively, for every finite set \( S \). We call this the global level model structure and its weak equivalences the global level weak equivalences. It is combinatorial with generating cofibrations

\[
\{(\Gamma(S+, -) \wedge I(A, -)) + H \wedge (\partial \Delta^n \hookrightarrow \Delta^n)_+ : H \text{ finite group, } S \text{ finite } H\text{-set}, \\
A \text{ finite faithful } H\text{-set}\}
\]

and moreover simplicial.

Proof. See [Len20a, Theorem 2.2.28]. \( \square \)

Theorem 4.7. The homotopical functors \( \text{ev}_\omega \) and \( (-)[\omega^+] \) from Theorem 1.6 assemble into mutually inverse homotopy equivalences \( \Gamma \text{-I-SSet}_+ \rightleftarrows \Gamma \text{-EM-SSet}_+ \), and analogously for the subcategories of special global \( \Gamma \)-spaces, where we call \( X \in \Gamma \text{-I-SSet}_+ \) special iff \( X(\omega) \) is special.

Proof. See [Len20a, Theorem 2.2.30 and Corollary 2.2.50]. \( \square \)

Arguing as in Remark 4.3 we get:

Lemma 4.8. Let \( H \) be a finite group and let \( \mathcal{U}_H \) be a complete \( H\)-set universe. Then the functor \( \Gamma \text{-I-SSet}_+ \rightarrow \Gamma \text{-SSet}_+ \) given by \( X \mapsto X(\mathcal{U}_H)^H \) (with the evident functoriality in each variable) sends global level weak equivalences to level weak equivalences. Moreover, if \( X \in \Gamma \text{-I-SSet}_+ \) is special, then \( X(\mathcal{U}_H)^H \) is a special \( \Gamma\)-space.

Proof. See [Len20a Corollary 2.2.50 and Theorem 2.2.61]. \( \square \)

4.2. A very brief reminder on parsummable categories. There is yet another approach to ‘global coherent commutativity’ based on so-called parsummable categories [Sch19b] that will play a supporting role in several computations below. We will only recall the necessary definitions and statements here and refer the reader to op. cit. for more details.

Definition 4.10. Let \( C \) be an \( EM\)-category, and let \( x \in C \) be any object. We say that \( x \) is supported on a finite set \( A \subset \omega \) if \( u.x = x \) for all \( u \in M \) that restrict to the identity on \( A \). We call \( x \) finitely supported if it is supported on some finite set; in this case the support of \( x \) is defined as the intersection of all finite sets on which \( x \) is supported. We call \( C \) tame if all its objects are finitely supported, and we denote by \( EM\text{-Cat}^\tau \subset EM\text{-Cat} \) the full subcategory of (small) tame \( EM\)-categories.

Definition 4.11. Let \( C, D \) be tame \( EM\)-categories. Their box product \( C \boxdot D \) is the full subcategory of \( C \times D \) spanned by those pairs \( (x, y) \) such that the support of \( x \) is disjoint from the support of \( y \).

It is not hard to show that \( C \times D \) is again a tame \( EM\)-category, and that the functoriality and structure maps of the usual cartesian symmetric monoidal structure restrict to make the box product into the tensor product of a preferred symmetric monoidal structure on \( EM\text{-Cat}^\tau \) [Sch19b Proposition 2.35].
**Definition 4.12.** A parsummable category is a commutative monoid in $EM \text{-} Cat^r$ with respect to the box product.

Thus, a parsummable category is a small tame $EM$-category $C$ equipped with a specified object $0$ (which has empty support) and a strictly commutative, associative, and unital operation $+: C \boxtimes C \to C$.

**Example 4.13.** Let $H$ be a finite group and let $A$ be a finite $H$-set. Then we have a tame $EM$-category

$$\prod_{n \geq 0} \text{EInj}(n \times A, \omega)/\Sigma_n \wr H$$

where the wreath product acts on $n \times A$ in the obvious way. This becomes a parsummable category for the unique injection $0 \times A \to \omega$ as zero object, and the sum induced by juxtaposition; see [Sch19b, Example 4.6] for more details. This is actually the free parsummable category on $\text{EInj}(A, \omega)$.

We will only be interested in the case where $A$ is non-empty and faithful, in which case we denote this parsummable category by $F_H$.

**Remark 4.14.** Associated to any parsummable category $C$ we have a special global $\Gamma$-space $N \varpi(C)$ with $N \varpi(C)(1^+) = NC$, see [Sch19b, Construction 4.3] for details, where this is denoted $\gamma$. The corresponding functor from parsummable categories to special global $\Gamma$-spaces is actually an equivalence with respect to a certain notion of global weak equivalences of parsummable categories [Len20a, Theorem 4.3.3].

### 4.3. Global $\Gamma$-spaces vs. global Mackey functors.

In order to compare global $\Gamma$-spaces to global Mackey functors, we want to apply the results of the previous section (for $A = \text{SADD}$) to the various fixed point functors. This requires some preparations:

**Proposition 4.15.** Let $H \subset M$ be universal. Then the functor induced by $(-)^H: \Gamma \text{-} EM \text{-} SSet^s_{\text{special}} \to \Gamma \text{-} SSet^s_{\text{special}}$ on associated quasi-categories is cocontinuous.

**Proof.** We will show that the induced functor preserves filtered colimits, $\Delta^{op}$-shaped colimits, and finite products. As both sides are semiadditive (see Theorem 4.9 for the left hand side), this will then suffice to prove the claim.

We will first show the corresponding statements for $(-)^H: \Gamma \text{-} EM \text{-} SSet^\infty_{\text{special}} \to \Gamma \text{-} SSet^\infty_{\text{special}}$. Indeed, filtered colimits in the 1-category of global $\Gamma$-spaces are homotopical by Theorem 4.1, and so are geometric realizations and finite products by Lemma 1.10 and Theorem 1.3, respectively, applied levelwise. It follows directly that filtered colimits and finite products in the quasi-category of global $\Gamma$-spaces can be computed in the underlying 1-category, while [Lur09, Corollary A.2.9.30] shows that $\Delta^{op}$-shaped homotopy colimits can be computed as geometric realizations. However, taking $H$-fixed points clearly preserves products and filtered colimits on the pointset level, and it also commutes with geometric realization as the latter is just given by taking diagonals. This completes the proof that the induced functor $\Gamma \text{-} EM \text{-} SSet^\infty_{\text{special}} \to \Gamma \text{-} SSet^\infty_{\text{special}}$ preserves filtered colimits, $\Delta^{op}$-shaped colimits, and finite products.

It only remains to show that $(\Gamma \text{-} EM \text{-} SSet^s_{\text{special}})^\infty$ is closed under all of these operations, for which it is then in turn enough to show this on the pointset level.
Construction 4.16. We refer the reader to [SS79 Definition 2.1] for the construction of the special \( \Gamma \)-category \( \Gamma(\mathcal{C}) \) associated to a small symmetric monoidal category \( \mathcal{C} \); all that we will need below is that this construction is functorial in the 1-category \( \text{SymMonCat}^0 \) of small symmetric monoidal categories and \textit{strictly unital} strong symmetric monoidal functors, and that its underlying category is naturally isomorphic to \( \mathcal{C} \); for simplicity we will suppress this natural isomorphism below and pretend that \( \text{ev}_{1^+} \circ \Gamma \) is equal to the forgetful functor \( \text{SymMonCat}^0 \rightarrow \text{Cat} \).

Using this, we now simply define

\[
\Gamma_{\text{gl}} := N \circ \text{Fun}(EM, \cdot) \circ \Gamma : \text{SymMonCat}^0 \rightarrow \Gamma\text{-}\text{EM-SSet}_*,
\]

where \( EM \) acts on itself from the right in the obvious way, inducing a left \( EM \)-action on \( \text{Fun}(EM, \cdot) \).

Just like \( N \circ \Gamma \) lands in special \( \Gamma \)-spaces, \( \Gamma_{\text{gl}} \) factors through the subcategory of special global \( \Gamma \)-spaces, see [Len20a Example 2.2.49].

Remark 4.17. The fact that \( \Gamma \) (and hence \( \Gamma_{\text{gl}} \)) is only functorial in \textit{strictly unital} symmetric monoidal functors is just a minor annoyance; in particular, the inclusion \( \text{SymMonCat}^0 \hookrightarrow \text{SymMonCat} \) into the 1-category of symmetric monoidal categories and all symmetric monoidal functors is a homotopy equivalence, see e.g. [Len20b proof of Proposition 6.7]. As moreover the inclusion of the 1-category of small symmetric monoidal categories into the corresponding \( (2,1) \)-category is a simplicial localization by a standard argument, cf. [Len20a Proposition A.1.10], the composite

\[
N(\text{SymMonCat}^0) \xrightarrow{N(\Gamma_{\text{gl}})} N(\Gamma\text{-}\text{EM-SSet}_*^{\text{special}}) \xrightarrow{\text{loc}} (\Gamma\text{-}\text{EM-SSet}_*^{\text{special}})^\infty
\]

factors through a functor \( N_\Delta(\text{SymMonCat}^{(2,1)}) \rightarrow (\Gamma\text{-}\text{EM-SSet}_*^{\text{special}})^\infty \) in an essentially unique way; we pick one such factorization and denote it by \( \Gamma_{\text{gl}} \) again.

Theorem 4.18. Let \( H \subset \mathcal{M} \) be universal. The functor

\[
(\Gamma\text{-}\text{EM-SSet}_*^{\text{special}})^\infty \rightarrow \mathcal{S}
\]

induced by \( X \mapsto X(1^+)^H \) is corepresented by \( (\Gamma_{\text{gl}}(\mathfrak{F}_H), \tau) \) where

\[
\tau \in \pi_0 \Gamma_{\text{gl}}(\mathfrak{F}_H)(1^+) = \pi_0(\text{Fun}(EM, \mathfrak{F}_H)^H)
\]

denotes the isomorphism class of some (hence any) \( H \)-fixed functor \( T : EM \rightarrow \mathfrak{F}_H \) whose restriction to \( EH \) is given by \( T(h_2, h_1) = (\id; h_2 h_1^{-1}) : 1 \rightarrow 1 \).

Proof. We first note that \( \tau \) is well-defined: namely, the functor \( EH \rightarrow \mathfrak{F}_H \) specified above is an \( H \)-fixed object of \( \text{Fun}(EH, \mathfrak{F}_H) \) by direct inspection, and as \( EH \hookrightarrow EM \) is a right \( H \)-equivariant equivalence of categories (i.e. an equivalence in the 2-category of right \( H \)-categories, right \( H \)-equivariant functors, and right \( H \)-equivariant natural transformations), it follows that this admits an essentially unique extension to an object \( T \in \text{Fun}(EM, \mathfrak{F}_H)^H \).

As \( \Gamma(1^+, \cdot) \wedge EM/H_+ \) corepresents the functor \( h \Gamma(\Gamma\text{-}\text{EM-SSet}_*^{\infty}) \rightarrow h\mathcal{S} \) sending an arbitrary global \( \Gamma \)-space \( X \) to \( X(1^+)^H \) via evaluation at the component of \( \id_{1^+} \wedge [1] \) (since it is cofibrant and does so in the simplicially enriched sense on the pointset level), it suffices to construct a special weak equivalence \( \Gamma(1^+, \cdot) \wedge EM/H_+ \rightarrow \Gamma_{\text{gl}}(\mathfrak{F}_H) \) sending the isomorphism class of \( \id_{1^+} \wedge [1] \) to \( \tau \).
For this, we pick a free $H$-orbit $A \subset \omega$, and we recall the parsunmable category $\mathcal{F}_H$ from Example [4.13] we confuse $E \text{Inj}(A, \omega)/H$ with $E \text{Inj}(1 \times A, \omega)/\Sigma_1 H \subset \mathcal{F}_H$. 

If now $\mu : 2 \times \omega \to \omega$ is any injection, then [Sch19b, Construction 5.6] associates to $\mathcal{F}_H$ a symmetric monoidal category $\mu^* \mathcal{F}_H$ with the same underlying category, and [Len20a, Proposition 4.2.19] gives a symmetric monoidal equivalence $j : \mathcal{F}_H \to \mu^* \mathcal{F}_H$ sending $h : 1 \to [h|_{A_i}, \text{incl}]$ in $E \text{Inj}(A, \omega)/H \subset \mathcal{F}_H$. 

On the other hand, Remark [3.14] associates to $\mathcal{F}_H$ a special global $\Gamma$-space $N_F(\mathcal{F}_H)$ that is given in degree $1^+$ by $N_F(\mathcal{F}_H)$, and [Len20a, proof of Theorem 4.1.33] yields an equivalence $N_F(\mathcal{F}_H) \to \Gamma_{gl}(\mu^* \mathcal{F}_H)$ that is induced in degree $1^+$ by the unit of the adjunction forget: $E \mathbf{-Cat} \rightleftarrows \text{Fun}(E \mathbf{M}, -)$. Finally, the unique map $\Gamma(1^+, -) \wedge E \mathcal{M}/H_+ \to N_F(\mathcal{F}_H)$ that sends $1^+ \wedge [1]$ to $[\text{incl}] \in E \text{Inj}(A, \omega)/H$ is a global special weak equivalence as a consequence of [Len20a, Proposition 4.2.1].

It is then trivial to check that the composition

$$\Gamma(1^+, -) \wedge E \mathcal{M}/H_+ \simeq N_F(\mathcal{F}_H) \simeq \Gamma_{gl}(\mu^* \mathcal{F}_H) \simeq \Gamma_{gl}(\mathcal{F}_H)$$

of the above equivalences sends the component of $\text{id}_{1^+} \wedge [1] \in (\Gamma(1^+, 1^+) \wedge E \mathcal{M}/H_+)^H$ to $\tau$, which completes the proof of the theorem.

**Proposition 4.19.** Write $A_{gl} \subset (\Gamma-E \mathbf{M}-\text{SSet}_*^{\text{special}})^\infty$ for the full subcategory spanned by the finite products of the global special $\Gamma$-spaces of the form $\Gamma_{gl}(\mathcal{F}_H)$ for finite groups $H$. Then the essentially unique lift of the Yoneda embedding induces an equivalence of quasi-categories

$$(\Gamma-E \mathbf{M}-\text{SSet}_*^{\text{special}})^\infty \simeq \text{Fun}^\oplus(A_{gl}^{op}, \text{CMon}(\mathcal{S}))$$

**Proof.** By Theorem [4.19] (for $\mathcal{A} = \text{SADD}$, see Example [5.7]) it suffices to show that $y_{op}^{\text{SADD}}(X)$ is cocontinuous for each $X \in A_{gl}$, and that they are jointly conservative.

However, for $X = \Gamma_{gl}(\mathcal{F}_H)$, $y_{op}^{\text{SADD}}(X)$ is equivalent to the composition

$$((\Gamma-E \mathbf{M}-\text{SSet}_*^{\text{special}})^\infty \simeq (\Gamma-E \mathbf{M}-\text{SSet}_*^{\text{special}})^\infty \simeq \text{CMon}(\mathcal{S}))$$

(where the unlabelled equivalence is as above) since the postcompositions with the forgetful functor to $\mathcal{S}$ are equivalent by Theorem [4.18] and as [4.11] preserves finite products. Thus, $y_{op}^{\text{SADD}}(X)$ is cocontinuous in this case by Proposition [4.14].

Moreover, we immediately see that already the $y_{op}^{\text{SADD}}(\Gamma_{gl}(\mathcal{F}_H))$'s are jointly conservative (even as functors to $\mathcal{S}$).

For general objects of $A_{gl}$ we note that $y_{op}^{\text{SADD}}(A \times B) \simeq y_{op}^{\text{SADD}}(A) \times y_{op}^{\text{SADD}}(B)$ for all $A, B$, as they become equivalent after postcomposing with the forgetful functor by semiadditivity. However, again using semiadditivity, $y_{op}^{\text{SADD}}(A) \times y_{op}^{\text{SADD}}(B)$ is equivalently the coproduct, and cocontinuous functors are stable under all colimits. Thus, we inductively see that $y_{op}^{\text{SADD}}(X)$ is cocontinuous for all $X \in A_{gl}$. This completes the proof of the proposition.

**Remark 4.20.** As $A_{gl}$ is semiadditive, [CGN15, Corollary 2.5-(iii)] shows that the forgetful functor $\text{CMon}(\mathcal{S}) \to \mathcal{S}$ yields an equivalence $\text{Fun}^\oplus(A_{gl}^{op}, \text{CMon}(\mathcal{S})) \simeq \text{Fun}^\times(A_{gl}^{op}, \mathcal{S})$. In particular, we conclude that the usual Yoneda embedding induces an equivalence $(\Gamma-E \mathbf{M}-\text{SSet}_*^{\text{special}})^\infty \simeq \text{Fun}^\times(A_{gl}^{op}, \mathcal{S})$, i.e. we may identify global special $\Gamma$-spaces with the non-abelian derived category $\mathcal{P}_\Sigma(A_{gl})$ in the sense of [Lur09, Definition 5.5.8.8].
It remains to relate $A_{gl}^{op}$ to the global effective Burnside category $A^{gl}$. For this we will prove more generally:

**Proposition 4.21.** For each finite group $H$ and each symmetric monoidal groupoid $\mathcal{C}$, the functor $\Gamma_{gl}$ induces an isomorphism

\[(4.2) \quad \text{maps}(\tilde{\mathcal{H}}_H, \mathcal{C}) \to \text{maps}(\Gamma_{gl}\tilde{\mathcal{H}}_H, \Gamma_{gl}\mathcal{C})\]

in $h\mathcal{S}$.  

**Proof.** We may assume without loss of generality that $H$ is a universal subgroup of $\mathcal{M}$. Then Theorem 4.15 shows that $(\Gamma_{gl}(\tilde{\mathcal{H}}_H), \tau)$ corepresents the functor sending a special global $\Gamma$-space $X$ to $X(1^+)^H$.  

On the other hand, Lemma 2.23 implies that $\mathcal{C} \to N(H\mathcal{C})$ is corepresented on $h\mathcal{N}_\Delta(\text{SymMonGrpd})$ by $(\tilde{\mathcal{H}}_H, \rho)$, where $\rho \in \pi_0N(H\tilde{\mathcal{H}}_H)$ denotes the component of the functor $P$ sending $h \in BH$ to $h: 1 \to 1$. Finally, we have an equivalence $\text{Fun}(EM, \mathcal{C})^H \to H\mathcal{C}$ sending an $H$-fixed functor $F$ to $F(1)$ with $h \in H$ acting via $F(h, 1)$, while sending a natural transformation $\tau: F \Rightarrow F'$ to $\tau_1$.  

Now a straightforward computation shows that the composition of the above natural isomorphisms

\[(4.3) \quad \text{maps}(\tilde{\mathcal{H}}_H, -) \cong \text{NH}(-) \cong \text{NFun}(EM, -)^H \cong \text{maps}(\Gamma_{gl}\tilde{\mathcal{H}}_H, \Gamma_{gl}(-))\]

in $h\mathcal{S}$ sends the component of the identity of $\tilde{\mathcal{H}}_H$ to the component of the identity of $\Gamma_{gl}(\tilde{\mathcal{H}}_H)$. Thus, the enriched Yoneda Lemma for $h\mathcal{N}_\Delta(\text{SymMonGrpd})$ shows that (4.2) agrees with the composite (4.3), so it in particular is an isomorphism. \(\square\)

**Theorem 4.22.** There is an equivalence of quasi-categories

\[(\GammaEMSSet_{\text{special}}^{\text{op}})^\infty \simeq \text{Fun}^\oplus(A^{gl}, \text{CMon}(\mathcal{S})).\]

Arguing as in Remark 4.20 we then also get an equivalence to $\text{Fun}^\times(A^{gl}, \mathcal{S})$. \(\square\)

**Proof.** The $(2,1)$-category of symmetric monoidal categories is semiadditive and $\Gamma_{gl}$ clearly preserves products. Thus, the previous proposition implies that $\Gamma_{gl}$ restricts to an equivalence $A_{gl} \to A_{gl}$. On the other hand, Theorem 2.24 gives an equivalence $A^{gl} \simeq A_{gl}^{op}$; thus, the claim follows from Proposition 4.19. \(\square\)

4.4. **Global categorical Mackey functors.** Next, we will discuss yet another model of ultra-commutative monoids, this time in terms of categorical Mackey functors, i.e. additive functors $A^{gl} \to \text{SymMonCat}^\infty$ where we consider the $1$- or $(2,1)$-category $\text{SymMonCat}$ as equipped with the (non-equivariant) weak homotopy equivalences. The crucial ingredient for this (apart from the above results) will be the following comparison:

**Theorem 4.23.** The functor $\Gamma_{gl}: N_\Delta(\text{SymMonCat}_{(2,1)}) \to (\GammaEMSSet_{\text{special}}^{\text{op}})^\infty$ descends to an equivalence of quasi-categories when we equip the source with the global weak equivalences [Sch19a, Definition 3.2], i.e. those symmetric monoidal functors $f: \mathcal{C} \to \mathcal{D}$ such that the induced functor $G\mathcal{C} \to G\mathcal{D}$ is a weak homotopy equivalence for every finite group $G$. \(\square\)

**Proof.** See [Len20a, Theorem 4.3.7].

**Proposition 4.24.** (1) The forgetful functor $\text{SymMonCat}_{(2,1)} \to \text{Cat}_{(2,1)}$ factors essentially uniquely through an equivalence $N_\Delta(\text{SymMonCat}_{(2,1)}) \simeq \text{CMon}(N_\Delta(\text{Cat}_{(2,1)}))$. 


The forgetful functor $\text{SymMonCat}^\infty_{\text{global}} \to \text{Cat}^\infty_{\text{global}}$ factors essentially uniquely through an equivalence $\text{SymMonCat}^\infty_{\text{global}} \simeq \text{CMon}(\text{Cat}^\infty_{\text{global}})$.

**Proof.** For the first statement it suffices to show that $\Gamma : \text{SymMonCat}^\infty_{\text{categorical}} \to (\text{Γ-Cat}_{\text{special}}^\infty)^\ast$ is an equivalence. This seems to be a folklore result, which appears at several places in the literature without proof or with a sketch proof only, see e.g. [Lur18, discussion after Remark 2.0.0.6] for the latter. For a full proof we argue as follows: by MacLane’s Coherence Theorem, the inclusion $\text{PermCat} \hookrightarrow \text{SymMonCat}^\infty$ of the subcategory of small permutative categories and strict symmetric monoidal functors becomes an equivalence after localizing at the symmetric monoidal equivalences, see e.g. [Len21, Theorem 1.19] where this argument is spelled out. On the other hand, [Sha20, Corollary 6.19] shows that $\Gamma$ is part of a Quillen equivalence between a model structure on $\text{PermCat}$ whose weak equivalences are the symmetric monoidal equivalences and a model structure on $\text{Fun}(\Gamma, \text{Cat})$ whose fibrant objects are the special ones (i.e. those functors such that all the Segal maps are equivalences of categories) and whose weak equivalences between fibrant objects are precisely the levelwise equivalences. The latter is then equivalent to $(\text{Γ-Cat}_{\text{special}}^\infty)^\ast$ via the functor that divides out the zeroth category.

For the second statement, it suffices to prove this after postcomposing with the equivalence $N : \text{Cat}^\infty_{\text{w.e.}} \to \text{SSet}^\infty = \mathcal{S}$, i.e. we want to show that $N \circ \Gamma$ induces an equivalence $\text{SymMonCat}^\infty_{\text{w.e.}} \simeq (\text{Γ-SSet}_{\text{special}}^\infty)^\ast$. This was originally proven by Mandell [Man10, Theorem 1.4] (on the level of homotopy categories), the quasi-categorical version can be found e.g. as [Len20a, Theorem 4.3.8].

**Construction 4.25.** We first define a strict 2-functor $\tilde{\text{swan}} : \text{SymMonCat}^\infty_{\text{(2,2)}} \to \text{Fun}(\text{A}^\text{op}_{\text{gl}}, \text{SymMonCat}^\infty_{\text{(2,1)}})$ as the composition of the enriched Yoneda embedding with restriction along $\text{A}^\text{op}_{\text{gl}} \hookrightarrow \text{SymMonCat}^\infty_{\text{(2,2)}}$, i.e.

$$\tilde{\text{swan}}(C)(T) = \text{Fun}^\otimes(T, C)$$

for any $C \in \text{SymMonCat}, T \in \text{A}^\text{op}_{\text{gl}}$, with the evident functoriality in both variables.

Using this, we now define the functor $\text{swan}$ as the composition

$$N_{\Delta}(\text{SymMonCat}^\infty_{\text{(2,1)}}) \xrightarrow{N_{\Delta}(\tilde{\text{swan}})} N_{\Delta}(\text{Fun}(\text{A}^\text{op}_{\text{gl}}, \text{SymMonCat}^\infty_{\text{(2,1)}})) \xrightarrow{\text{loc}} \text{Fun}(\text{A}^\text{op}_{\text{gl}}, \text{SymMonCat}^\infty_{\text{w.e.}}) \xrightarrow{\Psi^*} \text{Fun}(\text{A}^\text{op}_{\text{gl}}, \text{SymMonCat}^\infty_{\text{w.e.}})$$

where the unlabelled arrow is the canonical comparison map.

**Theorem 4.26.** The functor $\text{swan}$ descends to an equivalence of quasi-categories $\text{SymMonCat}^\infty_{\text{global}} \simeq \text{Fun}^\otimes(\text{A}^\text{op}_{\text{gl}}, \text{SymMonCat}^\infty_{\text{w.e.}})$.

In order to prove this we will need:

**Proposition 4.27.** Let $\mathcal{I}$ be any quasi-category which admits all sifted colimits, and let us write $\text{Fun}^{\text{gl}_{\mathcal{I}}}_\Sigma(N_{\Delta}(\text{SymMonCat}^\infty_{\text{(2,1)}}), \mathcal{I})$ for the full subcategory of $\text{Fun}(N_{\Delta}(\text{SymMonCat}^\infty_{\text{(2,1)}}), \mathcal{I})$ spanned by those functors that invert global weak equivalences and such that the induced functor $\text{SymMonCat}^\infty_{\text{global}} \to \mathcal{I}$ preserves sifted colimits. Then the restriction functor

$$\text{Fun}^{\text{gl}_{\mathcal{I}}}_\Sigma(N_{\Delta}(\text{SymMonCat}^\infty_{\text{(2,1)}}), \mathcal{I}) \to \text{Fun}(\text{A}^\text{op}_{\text{gl}}, \mathcal{I})$$

is an equivalence.
Proof. Fix a localization $\gamma : N_\Delta(\text{SymMonCat}_{(2,1)}) \to \text{SymMonCat}^\infty_{\text{global}}$. By the universal property of localization together with Theorem 4.23 it is then enough to show that restriction along $\Gamma_{\text{gl}} \circ \gamma|_{A_{\text{gl}}} : A_{\text{gl}} \to (\Gamma-\text{EM-SSet}^\text{special})^\infty$ induces an equivalence $\text{Fun}^\Sigma(\text{SymMonCat}^\infty_{\text{global}}, \mathcal{F}) \simeq \text{Fun}(A_{\text{gl}}, \mathcal{F})$. However, by Proposition 4.21 together with Remark 4.20 the map $\Gamma_{\text{gl}} \circ \gamma|_{A_{\text{gl}}}$ agrees up to conjugation by equivalences with the Yoneda embedding $A_{\text{gl}} \to \mathcal{P}(\mathcal{C}(A_{\text{gl}}) = \text{Fun}^\Sigma(A_{\text{gl}}^\text{op}, \mathcal{F})$. Thus, the claim follows from [Lur09, Proposition 5.5.8.15]. □

Proof of Theorem 4.26. We begin with the following observation, which is a coherent way of saying that the mapping spaces $\text{maps}(F, C)$ in $\text{SymMonCat}^\infty_{\text{global}}$ can be computed as the ordinary symmetric monoidal functor categories:

**Claim.** The diagram

$$
\begin{array}{ccc}
N_\Delta(\text{SymMonCat}_{(2,1)}) & \xrightarrow{\text{forget oswan}} & \text{Fun}^\times(A_{\text{gl}}^{\text{op}}, \mathcal{C}^{\text{w.e.}}) \\
\downarrow\text{loc} & & \downarrow N \\
\text{SymMonCat}^\infty_{\text{gl}} & \xrightarrow{\Psi \circ y} & \text{Fun}^\times(A_{\text{gl}}^{\text{op}}, \mathcal{F})
\end{array}
$$

commutes up to natural equivalence.

Proof. Let $G$ be a finite groupoid, and let $G_1, \ldots, G_r$ be a system of representatives of isomorphism classes of objects of $G$. Then $\text{swan}(C)(G) \simeq \prod_{i=1}^r \text{Aut}_G(G_i)\cdot C$ naturally for any symmetric monoidal category $C$, so the upper path through (4.4) descends to a functor from $\text{SymMonCat}^\infty_{\text{gl}}$. However, this factors up to equivalence as the composite

$$
\begin{array}{ccc}
\text{SymMonCat}^\infty_{\text{gl}} & \xrightarrow{\Gamma_{\text{gl}}} & (\Gamma-\text{EM-SSet}^\text{special})^\infty \\
\downarrow\text{loc} & & \downarrow y \circ \text{ev} \\
\text{SymMonCat}^\infty_{\text{gl}} & \xrightarrow{\Gamma_{\text{gl}}} & (\Gamma-\text{SSet}^\text{special})^\infty \xrightarrow{\text{ev}} \text{SSet}
\end{array}
$$

(where we have secretly identified $G$ with a universal subgroup of $\mathcal{M}$ isomorphic to it) so that the claim follows from Theorem 4.23 together with Proposition 4.15.

We now claim that this induced functor preserves sifted colimits, which amounts to saying that $C \mapsto N(G\cdot C) : \text{SymMonCat}^\infty_{\text{gl}} \to \mathcal{F}$ preserves sifted colimits for every finite group $G$. However, this factors up to equivalence as the composite

$$
\begin{array}{ccc}
\text{SymMonCat}^\infty_{\text{gl}} & \xrightarrow{\Gamma_{\text{gl}}} & (\Gamma-\text{EM-SSet}^\text{special})^\infty \\
\downarrow\text{loc} & & \downarrow y \circ \text{ev} \\
\text{SymMonCat}^\infty_{\text{gl}} & \xrightarrow{\Gamma_{\text{gl}}} & (\Gamma-\text{SSet}^\text{special})^\infty \xrightarrow{\text{ev}} \text{SSet}
\end{array}
$$

We now consider the diagram

$$
\begin{array}{ccc}
A_{\text{gl}} & \xrightarrow{\Gamma_{\text{gl}}} & A_{\text{gl}} \\
\downarrow\text{loc} & & \downarrow \\
\text{SymMonCat}^\infty_{\text{gl}} & \xrightarrow{\Gamma_{\text{gl}}} & (\Gamma-\text{EM-SSet}^\text{special})^\infty
\end{array}
$$

commuting up to (canonical) equivalence. Proposition 4.21 and Theorem 4.23 show that the top and bottom horizontal arrows are equivalences. Together with Remark 4.20 and Theorem 2.26 we therefore conclude that the lower horizontal arrow in (4.4) is an equivalence.

As the nerve $\mathcal{C}^{\text{w.e.}} \to \mathcal{F}$ is an equivalence, we conclude from the claim that swan induces an equivalence $\text{SymMonCat}^\infty_{\text{gl}} \to \text{Fun}^\times(A_{\text{gl}}^{\text{op}}, \mathcal{C}^{\text{w.e.}})$. To finish
the proof, we now simply observe that the forgetful functor $\text{SymMonCat}_{w.e.}^\infty \to \text{Cat}_{w.e.}^\infty$ factors through an equivalence $\text{SymMonCat}_{w.e.}^\infty \simeq \text{CMon}(\text{Cat}_{w.e.}^\infty)$ by Proposition 4.24, so that the forgetful functor $\text{Fun}^\otimes(\text{A}^{\otimes}, \text{SymMonCat}_{w.e.}^\infty) \to \text{Fun}^\otimes(\text{A}^{\otimes}, \text{Cat}_{w.e.}^\infty)$ is an equivalence. □

Finally, we will give a more explicit description of some of the data of the functor $\tilde{\text{swan}}(\mathcal{C})$ for a given symmetric monoidal category $\mathcal{C}$ in line with the usual construction of Swan $K$-theory. For this we first recall that we have for every finite groupoid $G$ and every symmetric monoidal $\mathcal{C}$ an equivalence $\tau^*_G: \tilde{\text{swan}}(\mathcal{C})(\text{core } hF_G) \simeq G\text{-}\mathcal{C}$ by Lemma 2.21.

**Proposition 4.28.** Let $\varphi: \mathcal{H} \to \mathcal{G}$ be a map of finite groupoids. Then the diagram

$$
\begin{array}{ccc}
\text{Fun}^\otimes(\text{core } hF_G, \mathcal{C}) & \xrightarrow{\text{Fun}^\otimes(\varphi^! \otimes \cdot, \cdot)} & \text{Fun}^\otimes(\text{core } hF_H, \mathcal{C}) \\
\tau^*_G & \downarrow & \tau^*_H \\
\mathcal{G}\text{-}\mathcal{C} & \xrightarrow{\varphi^*} & \mathcal{H}\text{-}\mathcal{C}
\end{array}
$$

in $\text{SymMonCat}$ commutes strictly for every symmetric monoidal category $\mathcal{C}$.

Put differently, the restrictions in $\text{swan}(\mathcal{C})$ (i.e. the functoriality in the contravariant legs of $\text{A}^{\otimes}$) are given up to conjugation by the chosen equivalences by the usual categorical restriction functors.

**Proof.** Both paths through the diagram (4.5) are actually strict symmetric monoidal functors as the symmetric monoidal structure on $\text{Fun}^\otimes$ is pointwise. Thus, it suffices to show that this commutes as a diagram in $\text{Cat}$. While this is easy enough to do directly, we will instead appeal to the $\text{Cat}$-enriched Yoneda Lemma, which reduces this to proving that for $\mathcal{C} = \text{core } hF_G$ both paths through the diagram send the identity to the same object. Plugging in the definitions, this amounts to the relation

$$
\tau_G \circ \varphi = \varphi^! \circ \tau_H
$$

which one immediately verifies by direct inspection. □

The statement for transfers will require significantly more work. Fortunately, however, we will still be able to bypass most of the computations by a similar Yoneda argument.

For this we first have to recall the usual symmetric monoidal norm construction. In order to avoid making formulas too complicated we will restrict to the case of subgroup inclusions (as opposed to faithful functors between general groupoids).

**Construction 4.29.** Let $G$ be a finite group and let $H \subset G$ be any subgroup. We pick a system of representatives $g_1, \ldots, g_r$ of the set of right $H$-cosets $G/H$. We now define for every $g \in G$ a permutation $\sigma(g)$ as well as group elements $h_1(g), \ldots, h_r(g)$ via the formula $gg_i = g\sigma(i)h_i(g)$. One easily checks that we then have an injective group homomorphism $\iota: G \to \Sigma_r \wr H, g \mapsto (\sigma(g); h_1(g), \ldots, h_r(g))$, see e.g. [Sch18, Construction 2.2.29].

We now define for any permutative category $\mathcal{C}$ a functor $\text{tr}_G^H: \mathcal{H}\text{-}\mathcal{C} \to G\text{-}\mathcal{C}$ as follows: we send an $H$-object $X$ to the tensor product $\bigotimes_{i=1}^r X$ and likewise on morphisms; the $G$-action on $\text{tr}_G^H X$ is given by restricting the $\Sigma_r \wr H$-action coming from the permutation action and the original $H$-action along $\iota$. 
Clearly, $\text{tr}_G^H$ is natural in strict symmetric monoidal functors (and in fact even 2-natural with respect to symmetric monoidal isomorphisms between these). It therefore extends essentially uniquely to a natural transformation of endofunctors of $N_\Delta(\text{SymMonCat}_{(2,1)})$, that we again denote by $\text{tr}_G^H$.

**Example 4.30.** Let $\mathcal{F}_H$ be the permutative category from Construction 2.22, and let $t_H : BH \to \mathcal{F}_H$ be the ‘tautological $H$-object,’ i.e. $t_H(h) = h : 1 \to 1$.

Then plugging in the definitions shows that $\text{tr}_G^H(t_H) = r$ with $G$-action given by $\iota : G \to \Sigma_r : H = \text{Aut}(r)$.

**Example 4.31.** Using the previous example together with the equivalence from Construction 2.22 we can now compute $\text{tr}_G^H(\tau_{BH}) \in G-(\text{core } F_BH)$ up to isomorphism: namely, naturality of $\text{tr}_G^H$ shows that $\text{tr}_G^H(\tau_{BH})$ is isomorphic the unique functor $G/H \to BH$ with $g \in G$ acting as

\[
\begin{array}{ccc}
G/H & \xrightarrow{g \cdot} & G/H \\
& \searrow & \\
& BH & \leftarrow
\end{array}
\]

where $h$ denotes the natural transformation with $h_{[g/1]}(g) = h_i(g)$.

**Proposition 4.32.** Let $G$ be a finite group, let $H \subset G$ be a subgroup, and let $\text{incl} : H \hookrightarrow G$ denote the inclusion. Then the diagram

$\begin{array}{ccc}
\text{Fun}^\otimes(\text{core } F_{BH}, C) & \xrightarrow{\text{Fun}^\otimes(\text{incl}^*, C)} & \text{Fun}^\otimes(\text{core } F_{BG}, C) \\
\tau_H & \downarrow & \tau_G \\
H-C & x\xrightarrow{\text{tr}_G^H} & G-C
\end{array}$

commutes up to symmetric monoidal isomorphism for every symmetric monoidal category $C$. Moreover, for varying $C$ these isomorphisms can be chosen to assemble into a homotopy between the corresponding natural transformations of endofunctors of $N_\Delta(\text{SymMonCat}_{(2,1)})$.

Put differently, up to higher coherences the transfers of $\text{swan}(C)$ correspond under our chosen equivalences $\text{swan}(C)(G) \simeq G-C$ to the usual categorically defined norms.

**Proof.** We will prove the second statement directly. Since all endofunctors of $N_\Delta(\text{SymMonCat}_{(2,1)})$ in question are additive, it suffices by Proposition 4.24 to prove this after postcomposing with the forgetful functor $N_\Delta(\text{SymMonCat}_{(2,1)}) \to N_\Delta(\text{Cat}_{(2,1)})$. Moreover, we may restrict to the nerve of the 1-category of permutative categories and strict symmetric monoidal functors in the source.

For this it will then be enough to construct an invertible modification filling the diagram

$\begin{array}{ccc}
\text{Fun}^\otimes(\text{core } F_{BH}, C) & \xrightarrow{\text{Fun}^\otimes(\text{incl}^*, C)} & \text{Fun}^\otimes(\text{core } F_{BG}, C) \\
\tau_H & \downarrow & \tau_G \\
H-C & x\xrightarrow{\text{tr}_G^H} & G-C
\end{array}$
of strict 2-functors \( \textbf{PermCat}_{(2,2)} \to \textbf{Cat}_{(2,2)} \) and 2-natural transformations. To this end, we let \( \text{incl}^* : \mathfrak{S}_G \to \mathfrak{S}_H \) denote the essentially unique symmetric monoidal functor such that \( \text{incl}^* \circ \iota_G \cong \text{tr}_G^H(\iota_H) \).

**Claim.** The diagram

\[
\begin{array}{ccc}
\mathfrak{S}_G & \xrightarrow{\text{incl}^*} & \mathfrak{S}_H \\
\downarrow & & \downarrow \\
\text{core}_H \mathbb{F}_G & \xrightarrow{\text{incl}^*} & \text{core}_H \mathbb{F}_H
\end{array}
\]

(where the horizontal arrows are as in Construction 2.22) commutes up to symmetric monoidal isomorphism.

**Proof.** As \( \mathfrak{S}_G \) corepresents the functor taking \( G \)-objects (via restriction along \( \iota_G \)), it suffices to show that the images of \( \iota_G : BG \to \mathfrak{S}_G \) under the two paths in this diagram are isomorphic. Unravelling definitions, this amounts to showing that \( \text{tr}_G^H(\iota_H) \cong \text{incl}^* \circ \iota_G \) in \( \text{G-core} \mathbb{F}_{BH} \). The left hand side was computed up to isomorphism in Example 4.31. We will now compute the right hand side:

To this end, we first replace \( \iota_G \) by an isomorphic \( G \)-object with values in \( \text{core}_H \mathbb{F}_{BH} \) as follows: consider \( p : EG \to BG, p(g_2, g_1) = g_2^{-1}g_2^{-1}g_1 \) with \( G \)-action via left multiplication; as \( p(gg_2, gg_1) = g_2^{-1}g_2^{-1}gg_1 = g_2^{-1}g_1 \) this is indeed well-defined.

We now have an isomorphism from \( \ast \to BG \) to \( p \) via the map \( \ast \to EG \) classifying \( 1 \), and we claim that this is \( G \)-equivariant, i.e. that the two composites

\[
\begin{array}{ccc}
\ast & \xrightarrow{1} & EG \\
\downarrow & & \downarrow g \\
BG & \xrightarrow{p} & EG
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\ast & \xrightarrow{1} & EG \\
\downarrow & & \downarrow g \\
BG & \xrightarrow{p} & EG
\end{array}
\]

are related by an invertible 2-cell in \( \mathbb{F}_{BG} \) for every \( g \in G \). Indeed, the edge \( (1, g) \) of \( EG \) defines the desired natural isomorphism.

As on \( \text{core}_H \mathbb{F}_{BG} \) the functor \( \text{incl}^* \) is just given by ordinary pullback, we conclude that \( \text{incl}^* \circ \iota_G \) is isomorphic to the action groupoid \( G//H \) (i.e. the subcategory of \( EG \) consisting of all morphisms \( (g_2, g_1) \) such that \( g_2 \in g_1H \)) with structure map given by \( p(g_2, g_1) = g_2^{-1}g_1 \) again and \( G \) acting from the left as before.

Recall the chosen representatives \( g_1, \ldots, g_r \) of \( G/H \) from Construction 4.29. We now consider the map \( k : G/H \to G//H \) given by sending \( [g_i] \) to \( g_i \), which defines an isomorphism from \( G/H \to BH \) to \( p : G//H \to BH \) in \( \text{core}_H \mathbb{F}_{BH} \). We claim that this is \( G \)-equivariant, which amounts to saying that there is for every \( g \in G \) an invertible 2-cell in \( \mathbb{F}_{BH} \) between

\[
\begin{array}{ccc}
G/H & \xrightarrow{k} & G//H \\
\downarrow & & \downarrow \circ \g \\
BH & \xrightarrow{p} & BH
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G/H & \xrightarrow{g} & G//H \\
\downarrow & & \downarrow \circ \g \\
BH & \xrightarrow{p} & BH
\end{array}
\]

The top composite \( G/H \to G//H \) on the left sends \( [g_i] \) to \( g_{g_i} = g_{g(g)(i)}h_i(g) \) while the corresponding composite in the right hand diagram sends \( [g_i] \) to \( g_{g(g)(i)} \). It is then easy to check that the maps \( h_i(g) : g_{g(g)(i)} \to g_{g_i} \) define the desired natural isomorphism. \( \triangle \)
By the claim, we are then reduced to constructing an invertible modification filling

\[
\begin{array}{ccc}
\text{Fun}^\otimes(\mathcal{F}_H, \mathcal{C}) & \xrightarrow{\text{Fun}^\otimes(\text{incl}_H^*, \mathcal{C})} & \text{Fun}^\otimes(\mathcal{F}_G, \mathcal{C}) \\
\downarrow^{\gamma_H} & & \downarrow^{\gamma_G} \\
H-C & \xrightarrow{h G} & G-C.
\end{array}
\]

By the \textbf{Cat}-enriched Yoneda Lemma it will be enough to show that for \(C = \text{core } h \mathcal{F}_{BH}\), the images of the identity under the two paths through this diagram are isomorphic in \(G-(\text{core } h \mathcal{F}_{BH})\), which amounts to the relation

\[\text{tr}_H^G(t_H) \cong \text{incl}^* \circ t_G\]

which actually holds by definition of \(\text{incl}^*\). \(\square\)

\textbf{Remark 4.33.} One can give similar explicit descriptions of the covariant functoriality of \(\text{swan}(C)\) in the fold maps \(G \amalg G \to G\) as well as the summand inclusions \(G \hookrightarrow G \amalg G\) for every finite gropoid \(G\), which then altogether yields a complete description of \(\text{swan}(C)\) on 1-cells.

\section{Global spectral Mackey functors vs. global spectra}

\subsection{Group completions.} Building on the above results, we will prove Theorem A on the comparison between global spectra and spectral Mackey functors on \(\mathcal{A}_{\text{gl}}\) in this section. This will rely on the notion of global group completions, and we begin by recalling the corresponding non-equivariant theory.

\textbf{Definition 5.1.} A \(\Gamma\)-space \(X\) is called \textit{very special} (or \textit{grouplike}) if it is special and the monoid structure on \(\pi_0(X(1^+))\) induced by the zig-zag

\[X(1^+) \times X(1^+) \xleftarrow{\sim} X(2^+) \xrightarrow{X(\mu)} X(1^+),\]

where \(\mu: 2^+ \to 1^+\) is given by \(\mu(1) = \mu(2) = 1\), is a group structure.

A map \(f\) in \((\GammaSSet_{\text{sp}})_{\infty}\) is called a \textit{very special weak equivalence} (or \textit{stable weak equivalence}), if for every very special \(T\), the restriction maps \((f,T): \text{maps}(Y,T) \to \text{maps}(X,T)\) is an isomorphism in \(\text{h} \mathcal{S}\).

\textbf{Definition 5.2.} A map \(f: X \to Y\) in \((\GammaSSet_{\text{sp}})_{\infty}\) is called a \textit{group completion} if \(Y\) is very special and \(f\) is a very special weak equivalence.

Let us recall some classical constructions of group completion maps:

\textbf{Example 5.3.} There is a functor deloop: \(\GammaSSet_{\text{sp}} \to \text{Spectra}\) that induces a Bousfield localization at the very special weak equivalences, see [BF78, Theorem 5.1]. Explicitly, \((\text{deloop}X)(A) = X(S^4)\) with the evident \(\Sigma_A\)-action, where we view \(X\) as a functor from finite pointed simplicial sets to \(\text{SSet}_{\text{sp}}\) via left Kan extension, and the structure maps are given by the assembly maps. It follows formally that for every special \(X\) the derived unit of this adjunction is a group completion of \(X\). In particular, as deloop sends special \(\Gamma\)-spaces to positive \(\Omega\)-spectra [BF78, Theorem 4.4], we see that an explicit model of the group completion is given by the adjunct \(X \to R\Omega X(S^1 \wedge -)\) of the assembly maps \(S^1 \wedge X \to X(S^1 \wedge -)\).
Example 5.4. For any special $\Gamma$-space $X$, there exists a specific map $X \to R\Omega BX$ into the derived loop space of the bar construction. This is another classical model of the group completion, see e.g. [Nik17, proof of Theorem 1] for a proof in modern language.

Next, we come to a useful criterion that allows us to detect group completions without ever having to worry about higher coherences. To this end, we first recall that for each special $\Gamma$-space $X$ the underlying space $X(1^+)$ naturally carries the structure of a homotopy commutative monoid, i.e. an object of $\operatorname{CMon}(h\mathcal{S})$, via the zig-zag (5.1).

Definition 5.5. A map $f : X \to Y$ of homotopy commutative monoids is a group completion if the induced map $H_*(f)$ on homology is a localization at the image of $\pi_0(X)$ in $H_0(X) \subset H_*(X)$. (Note that in this case $\pi_0(Y)$ is automatically a group, namely the group completion of $\pi_0(X)$ in the usual sense.)

Theorem 5.6. Let $f : X \to Y$ be a map in $(\Gamma\text{-}\mathcal{SSet},\text{special})_\infty$. Then $f$ is a group completion of $\Gamma$-spaces (in the sense of Definition 5.2) if and only if the underlying map of homotopy commutative monoids is a group completion in the sense of Definition 5.5.

Proof. The implication ‘$\Rightarrow$’ is the classical Group Completion Theorem [MS76], also see for example [Nik17, Example 2] for a proof in our language and generality.

For the other direction, we let $X \to X^+$ be any group completion; note that such a map indeed exists by Example 5.3. Then $Y$ is in particular a very special $\Gamma$-space, so $f$ factors through a map $\tilde{f} : X^+ \to Y$ by definition of the very special weak equivalences, and it suffices to show that this map defines an isomorphism in $h\mathcal{S}$. But indeed, the group completion map $X \to X^+$ induces a localization in homology at $\pi_0(X)$ by the previous direction, and so does $f$ by assumption, so that $\tilde{f}$ is a homology isomorphism. As both $X^+$ and $Y$ are in particular grouplike homotopy commutative monoids, their path components are simple, so the claim follows from Whitehead’s Theorem. □

Now we are ready to introduce the corresponding global notions:

Definition 5.7. A global $\Gamma$-space $X \in \Gamma\text{-}\mathcal{SSet}_\text{special}$ is called very special if it is special and for every finite group $H$ and some (hence any) complete $H$-set universe $U_H$, the induced monoid structure on $\pi_0(X(1^+)(U_H)^H)$ is a group structure.

A map $f : X \to Y$ in $(\Gamma\text{-}\mathcal{SSet}_\text{special})_\infty$ is called a very special weak equivalence if maps($f, T$) is an isomorphism in $h\mathcal{S}$ for each very special global $\Gamma$-space $T$.

Definition 5.8. A map $f : X \to Y$ in $(\Gamma\text{-}\mathcal{SSet}_\text{special})_\infty$ is called a global group completion if $Y$ is very special and $f$ is a very special weak equivalence.

Example 5.9. Analogously to the non-equivariant situation, global $\Gamma$-spaces come with a homotopical delooping functor $E^\otimes$ to global spectra; explicitly, $E^\otimes X(A) = X(S^A)(A)$ for any global $\Gamma$-space $X$, with structure maps given by assembly and the functoriality of $X$ in $I$. The induced functor on quasi-categories admits a right adjoint $R\Phi^\otimes$, and the resulting adjunction is a Bousfield localization at the very special weak equivalences, see [Len20a, Theorem 3.4.21]. In particular we see that for special $X$ the unit $X \to R\Phi^\otimes E^\otimes X$ is a global group completion.
Before we can state our criterion for global group completions, we have to explain how the homotopy category of a semi-additive category is enriched in CMon(h\(\mathcal{S}\)). While we could of course do this via the machinery of Section 3, this is easy enough to make explicit here:

Remark 5.10. A classical observation in algebra says that every semiadditive category \(\mathcal{C}\) has a unique enrichment in commutative monoids, and that any direct sum-preserving functor (as well as any natural transformation of such) is automatically enriched. Concretely, any \(Y \in \mathcal{C}\) inherits a commutative monoid structure from the composition \(Y \times Y \cong Y \amalg Y \to Y\) of the canonical isomorphism with the fold map and dually every \(X\) inherits a commutative comonoid structure \(X \to X \amalg X\) from the diagonal; this induces two compatible commutative monoid structures on \(\text{Hom}(X,Y)\) which then agree by the Eckmann-Hilton argument.

This immediately generalizes to the case that \(\mathcal{C}\) itself is enriched in a closed symmetric monoidal category \(V\), as long as the coproducts and products above are \(V\)-(co)products. In particular, we can apply this to the \(h\mathcal{S}\)-enriched homotopy category of any semiadditive quasi-category, for example the homotopy categories of global spectra (Theorem 1.13) or global \(\Gamma\)-spaces (Theorem 4.9).

Note that the above indeed agrees with the commutative monoid structures transferred from \(y\text{SADD}\) by abstract nonsense, using once more that the forgetful functor \(\text{Fun}(\mathcal{C}, \text{CMon}(h\mathcal{S})) \to \text{Fun}(\mathcal{C}, h\mathcal{S})\) is fully faithful [GGN15, Corollary 2.5-(iii)].

Example 5.11. For a global \(\Gamma\)-space \(X\), the commutative monoid structure in the homotopy category is obtained similarly to (5.1) from the zig-zag

\[
X \times X \xleftarrow{X(\mu \wedge -)} X(2^+ \wedge -) \xrightarrow{X(\mu \wedge -)} X;
\]

indeed, it suffices to show that after restricting along \(X \vee X \to X \times X\) this agrees with the fold map, which is a trivial computation.

Theorem 5.12. Let \(f: X \to Y\) be a map in \((\Gamma_{\mathcal{I}}\text{-SSet}_{\text{special}}^{\ast})^\infty\) such that \(Y\) is very special. Then the following are equivalent:

1. \(f\) is a global group completion.
2. The map \(E^\otimes(f)\) is an equivalence in \(\text{Spectra}_{\text{global}}^\infty\).
3. For every finite group \(H\) and some (hence any) complete \(H\)-set universe \(U_H\), \(f(U_H)^H: X(U_H)^H \to Y(U_H)^H\) is a group completion of \(\Gamma\)-spaces.
4. For every finite group \(H\) and every finite faithful \(H\)-set \(A\), the induced map

\[
\text{maps}(\Gamma(1^+, -) \wedge I(A, -)/H^+, X) \to \text{maps}(\Gamma(1^+, -) \wedge I(A, -)/H^+, Y)
\]

is a group completion of homotopy commutative monoids (with respect to the above monoid structures).

Proof. The equivalence (1) ⇔ (2) follows from the fact that \(E^\otimes\) inverts precisely the very special weak equivalences [Len20a, Theorem 3.4.21].

By Theorem 5.6 \(X(U_H)^H \to Y(U_H)^H\) is a group completion of \(\Gamma\)-spaces if and only if its underlying map of homotopy commutative monoids is a group completion. Thus, (3) ⇔ (4) follows as \(\Gamma(1^+, -) \wedge I(A, -)/H\) corepresents the functor sending a special \(\Gamma\)-space \(Z\) to \(Z(U_H)^H\) (from the homotopy category of global \(\Gamma\)-spaces with respect to the special weak equivalences to the category of homotopy commutative monoids); namely, these are equivalent as functors to \(h\mathcal{S}\), and
by semiadditivity they then also have to be equivalent as functors to $\text{CMon}(h\mathcal{S})$ \cite[Corollary 2.5-(iii)]{GGN15}.

We will now prove $(\ref{item:adjunction-isomorphism}) \Rightarrow (\ref{item:maps})$; the implication $(\ref{item:maps}) \Rightarrow (\ref{item:adjunction-isomorphism})$ will then follow formally by a similar argument as in Theorem 5.6. To this end, we may assume without loss of generality that $f$ is represented by an actual map in $\Gamma\mathcal{Z}\text{-SSet}_{\ast,\text{special}}$ that we denote by $f$ again. Then the spaces $X(U_H)(S^A)$ for varying finite sets $A$ together with the assembly maps form a positive $H$-$\Omega$-spectrum, while the spaces $Y(U_H)(S^A)$ likewise assemble into an actual $H$-$\Omega$-spectrum, and the maps $f(U_H)(S^A) : X(U_H)(S^A) \to Y(U_H)(S^A)$ define an $H$-weak equivalence, see \cite[Lemma 3.4.19]{Len20a}. Thus, the right hand vertical and the bottom horizontal map in

$$
\begin{array}{ccc}
X(U_H)(S^0)^H & \xrightarrow{f(U_H)(S^0)^H} & Y(U_H)(S^0)^H \\
\sigma \downarrow & & \downarrow \tilde{\sigma} \\
R\Omega^1 X(U_H)(S^1)^H & \xrightarrow{R\Omega^1 f(U_H)(S^1)^H} & R\Omega^1 Y(U_H)(S^1)^H
\end{array}
$$

are isomorphisms in $h\mathcal{S}$. On the other hand, with respect to the induced homotopy commutative monoid structures, the left hand vertical map is a group completion (Example 5.3 together with Theorem 5.6) and hence so is the top horizontal map. The claim follows from another application of Theorem 5.6. \hfill $\square$

**Corollary 5.13.** Let $X, Y \in (\Gamma\mathcal{Z}\text{-SSet}_{\ast,\text{special}})^{\infty}$ with $X \simeq \Gamma(1^+, -) \wedge \mathcal{I}(A, -)/H$ for some finite group $H$ and some finite faithful $H$-set $A$. Then $\mathcal{E}^\otimes$ induces a group completion of homotopy commutative monoids

$$
(\ref{eq:maps}) \quad \text{maps}(X, Y) \to \text{maps}(\mathcal{E}^\otimes X, \mathcal{E}^\otimes Y).
$$

**Proof.** The adjunction isomorphism $\text{maps}(\mathcal{E}^\otimes X, \mathcal{E}^\otimes Y) \cong \text{maps}(X, R\mathcal{F}^\otimes \mathcal{E}^\otimes Y)$ in $h\mathcal{S}$ is actually a map of homotopy commutative monoids as it can be obtained as a composition of applying the additive functor $R\mathcal{F}^\otimes$ and restricting along the unit. But by naturality the resulting composition $\text{maps}(X, Y) \to \text{maps}(X, R\mathcal{F}^\otimes \mathcal{E}^\otimes Y)$ is simply postcomposition with the unit $\eta$ of $Y$. As $\eta$ is a group completion (Example 5.3), the claim follows immediately from the previous theorem. \hfill $\square$

As an upshot of the above discussion we can now prove a variant of the global Barratt-Priddy-Quillen Theorem characterizing the effect of *global algebraic K-theory* on certain hom-spaces. For this we first recall:

**Definition 5.14.** We define $K_{gl}$ as the composition

$$
\text{SymMonCat}^0 \xrightarrow{\Gamma_{gl}} \Gamma \text{-EAct-SSet}_{\ast,\text{special}} \xrightarrow{(-)_{\mathcal{E}^\otimes}} \Gamma \text{-Z-SSet}_{\ast,\text{special}} \xrightarrow{\mathcal{E}^\otimes} \text{Spectra}_{\text{global}}
$$

of homotopical functors, and call it the *global algebraic K-theory functor*. By Remark 4.17 this extends essentially uniquely to a functor $N_{\Delta}(\text{SymMonCat}_{(2,1)}) \to \text{Spectra}_{\text{global}}$ that we again denote by $K_{gl}$.

**Remark 5.15.** Schwede’s original construction \cite[discussion below Proposition 11.9]{Sch19b} used the language of parsummable categories; for the equivalence to the above definition, we refer the reader to \cite[Theorem 4.1.33]{Len20a}.

**Theorem 5.16.** Let $H$ be a finite group and $C$ a symmetric monoidal groupoid. Then

$$
K_{gl} : \text{maps}(\mathcal{F}_H, C) \to \text{maps}(K_{gl} \mathcal{F}_H, K_{gl} C)
$$
is a group completion of homotopy commutative monoids.

Proof. By construction of $K_{gl}$ the map in question factors as the composition

$$\maps(\mathfrak{S}_H, \mathcal{C}) \xrightarrow{\Gamma_{gl}} \maps(\Gamma_{gl}\mathfrak{S}_H, \Gamma_{gl}\mathcal{C}) \xrightarrow{(-)[\omega^*]} \maps(\Gamma_{gl}\mathfrak{S}_H[\omega^*], \Gamma_{gl}\mathcal{C}[\omega^*]) \xrightarrow{E} \maps(K_{gl}\mathfrak{S}_H, K_{gl}\mathcal{C})$$

of maps of homotopy commutative monoids. Of these the first is an equivalence by Proposition 4.21, and so is the second one by Theorem 4.7. Finally, Theorem 4.18 together with another application of Theorem 4.7 provides us with an equivalence

$$\Gamma_{gl}(\mathfrak{S}_H)[\omega^*] \simeq \Gamma(1^+, -) \wedge I(H, -)/H_+, \text{ so the final map is a group completion by Corollary 5.13, finishing the proof of the theorem.} \qed$$

5.2. The comparison. We now have everything in place to prove the following precise version of Theorem A from the introduction:

**Theorem 5.17.** The composition

$$\text{mack: } \text{Spectra}_{\infty}^{\text{global}} \xrightarrow{y^{\text{ST}}} \text{Fun}^\oplus((\text{Spectra}_{\infty}^{\text{global}})^{\text{op}}, \mathcal{F}p) \xrightarrow{(K_{gl} \circ \text{core} \circ \Psi)^*} \text{Fun}^\oplus(A_{gl}, \mathcal{F}p)$$

is an equivalence.

Proof. The usual global Barratt Priddy Quillen Theorem [Len20a, Theorem 4.2.21] shows that the global spectra $K_{gl}(\mathfrak{S}_H)$ for (representatives of isomorphism classes of) finite groups $H$ form a set of compact generators of $\text{Spectra}_{\infty}^{\text{global}},$ also see [Sch19b, Theorem 9.7]. Thus, if we write $\mathcal{A}_{gl}$ for the full subcategory of $\text{Spectra}_{\infty}^{\text{global}}$ spanned by the finite products of the global spectra of the form $K_{gl}(\mathfrak{S}_H)$, then Theorem 3.1 shows that $y^{\text{ST}}$ induces an equivalence $\text{Spectra}_{\infty}^{\text{global}} \simeq \text{Fun}^\oplus(\mathcal{A}_{gl}, \mathcal{F}p)$.

Next, we observe that Theorem 5.16 shows that $K_{gl}$ restricts to a local group completion $A_{gl} \to A_{gl}$, i.e. it is essentially surjective and it induces group completions (of $\Gamma$-spaces or equivalently of homotopy commutative monoids) on morphism spaces. On the other hand, Theorem 2.26 shows that core $\circ \Psi$ defines an equivalence between $A_{gl}^{\text{op}}$ and $A_{gl}$. Altogether we therefore get a local group completion $A_{gl} \to A_{gl}^{\text{op}}$, and as $\mathcal{F}p$ is additive, we conclude that $(K_{gl} \circ \text{core} \circ \Psi)^*: \text{Fun}^\oplus(\mathcal{A}_{gl}^{\text{op}}, \mathcal{F}p) \to \text{Fun}^\oplus(A_{gl}, \mathcal{F}p)$ is an equivalence as desired. \qed

6. Unravelling the Mackey functor structure

In this section we will give a more intrinsic description of the spectral Mackey functor $\text{mack}(X)$ associated to a global spectrum $X$ in terms of fixed point spectra together with transfers and restrictions.

6.1. Restrictions and transfers between fixed point spectra. As a first step, we have to recall some general facts about fixed point spectra and equivariant homotopy groups; we will be somewhat terse here and refer the reader to [Hau17, Hau19] for additional background.

**Construction 6.1.** Let $G$ be a finite group. We consider the the composition

$$\xymatrix{ \text{Spectra}_{\text{global}} \ar[r]^{\Omega^G \text{sh}^G} & G^{\text{Spectra}} \ar[r]^{(-)^G} & \text{Spectra}_{\text{non-equivariant}} }$$

where $\text{sh}^G X = X(G \wr (-))$ with the evident structure maps and functoriality, and $G$ acts on $\Omega^G \text{sh}^G X$ via its left action on itself.
We now observe that $F^G \colon \text{Spectra}_G \to \mathcal{S}$ as the ‘the’ right derived functor of $\Omega G$; more precisely, we pick a fibrant replacement $\iota : \text{id} \Rightarrow P$ and we define $F^G$ as the functor induced by the composition $(-)^G \circ \Omega G \sh P$. If $X$ is a global spectrum, we call $F^G X$ the (genuine) $G$-fixed point spectrum.

**Lemma 6.2.** The functor $F^G$ is corepresented (in the sense of Definition 3.14) by $(\Sigma^*_+ I(G,-)/G, \tau_G)$ where $\tau_G \in \pi_0 R\Omega^\infty F^G \Sigma^*_+ I(G,-)/G$ is given by the class $\pi_0 R\Omega^\infty (i)[S^G \land [\text{id}_G] : S^G \Rightarrow S^G \land I(G,-)/G]$. 

Here we write $R\Omega^\infty$ instead of the notation $\Omega^\infty$ employed before to stress that in the model of symmetric spectra this is not simply given by sending a spectrum $X$ to $\text{colim}_A \Omega^A X(A)$.

**Proof.** As $\Omega^\infty \circ (-)^G \circ \Omega G \sh P : \text{Spectra} \to \text{SSet}$ is just the (simplicially enriched) functor $\epsilon_G$ sending $X$ to $(\Omega^G X(G))^G$, it suffices to show that we have for every a fibrant global spectrum $X$ an isomorphism

$$\text{maps}(\Sigma^*_+ I(G,-)/G, X) \xrightarrow{\sim} \text{maps}\left(\left(\Omega^G (S^G \land I(G,-)/G_+)\right)^G, (\Omega^G X(G))^G\right)$$

in $\mathcal{S}$. This follows at once from the fact that $\Sigma^*_+ I(G,-)/G$ is cofibrant and represents $\epsilon_G$ in the simplicially enriched sense via evaluation at $S^G \land [\text{id}_G]$ as a consequence of the simplicially enriched Yoneda Lemma. \qed

**Construction 6.3.** Let $G$ be a finite group and fix once and for all a complete $G$-set universe $U_G$. The *naive zeroth $G$-equivariant homotopy group* of $X$ is defined as the colimit

$$\pi^{G, \text{naive}}_0(X) := \text{colim}_{A \subset U_G} \lim_{\text{finite } G\text{-set}} [\text{colim}_{A \in U_G} [S^A, X(A)]]_G$$

where $[\cdot, \cdot]^G_*$ denotes the set of $G$-equivariant based homotopy classes. The structure maps of the colimit are induced by the structure maps of $X$, and the group structure comes from the pinch map on $S^A$ for $A^G \neq \emptyset$. Alternatively, $\pi^{G, \text{naive}}_0$ preserves finite products by direct inspection, so there is a unique way to define the group structures such that the above becomes a functor into $\text{Ab}$. Beware that the naive homotopy groups are not invariant under $G$-equivariant weak equivalences in general, but only for so-called semistable $G$-spectra [Hau17, Corollary 3.37], which includes all ‘eventual $G$-$\Omega$-spectra’, i.e. all $G$-spectra for which there exists a finite $G$-set $A_0$ such that the derived adjoint structure maps $X(B) \to R\Omega^G X(A \amalg B)$ are $G$-equivariant weak equivalences for all finite $G$-sets $A, B$ with $A_0 \subset A$.

If now $X$ is a global spectrum, then we define $\pi^{G, \text{naive}}_0(X)$ as the zeroth naive $G$-equivariant homotopy group of $X$ equipped with the trivial $G$-action. Moreover, using the fibrant replacement $\iota : \text{id} \Rightarrow P$ from the previous construction, we define the *true zeroth $G$-equivariant homotopy group* $\pi^{G, \text{true}}_0(X)$ as $\pi^{G, \text{naive}}_0(PX)$. 
As global Ω-spectra are eventual $G$-spectra, $\pi_0^{G,\text{true}}$ is indeed invariant under global weak equivalences. Moreover, $\iota$ induces a natural group homomorphism $\pi_0^{G,\text{naïve}}(X) \to \pi_0^{G,\text{true}}(X)$ for every global spectrum $X$, that we again denote by $\iota$. Obviously, $\iota$ is an isomorphism whenever $X$ is a global $\Omega$-spectrum.

Lemma 6.4. Let $G$ be a finite group and fix a $G$-equivariant injection $\kappa_G : G \to \mathcal{U}_G$. We write $\tau'_G$ for the image of the class of

$$\left| S^{\kappa_G(G)} \right| \xrightarrow{\sim} \left| S^{\kappa_G(G)} \wedge I(G, \kappa_G(G))/G_+ \right| = \left| (\Sigma_+ I(G,-))/G_+ \right|$$

under $\iota$. Then $\pi_0^{G,\text{true}}$ is corepresented by $(\Sigma_+ I(A,-), \tau'_G)$.

Proof. It suffices to prove this as functors to $\text{Set}$ as both $\pi_0^{G,\text{true}}$ and the functor corepresented by $\Sigma_+ I(G,-)$ are additive. This is then a similar computation to the proof of Lemma 6.2 also see [Len20a, Example 3.2.5]. □

In what follows we will suppress the chosen injection $\kappa_G$ from the notation and instead simply pretend $G$ is literally embedded as a $G$-subset of $\mathcal{U}_G$.

Corollary 6.5. We have a natural isomorphism $\psi : \pi_0^{G,\text{true}}(X) \cong \pi_0 R\Omega^\infty F^G(X)$ for every global spectrum $X$, which is uniquely characterized by the condition that it sends $\tau'_G \in \pi_0^{G,\text{true}}(\Sigma_+ I(G,-)/G_+)$ to $\tau_G \in \pi_0 R\Omega^\infty F^G(\Sigma_+ I(G,-)/G_+)$.

Proof. This is immediate from Lemma 6.4 together with Lemma 2.24 □

Below we will use these results to define natural maps between genuine fixed point spectra. Namely, if $T : \textbf{Spectra}^{\infty}_{\text{global}} \to \mathcal{F} p$ is any functor, then the spectral Yoneda Lemma (Lemma 5.17) implies that a natural transformation $F^G \Rightarrow T$ is characterized uniquely up to (non-canonical) homotopy by a natural transformation $\pi_0 R\Omega^\infty F^G \Rightarrow \pi_0 R\Omega^\infty T$, which we then may identify with a natural transformation from $\pi_0^{G,\text{true}}$ by the previous corollary. In particular, if $T = F^H$, then any natural transformation $\pi_0^{G,\text{true}} \Rightarrow \pi_0^{H,\text{true}}$ gives rise to a natural transformation $F^G \Rightarrow F^H$, unique up to non-canonical homotopy.

Construction 6.6. Let $\varphi : H \to G$ be any homomorphism of finite groups. We recall the construction of the natural restriction map $\varphi^* : \pi_0^{G,\text{true}} \Rightarrow \pi_0^{H,\text{true}}$ from [Hau19, 4.4]. For this we first will construct an analogous map on naïve homotopy groups, which will depend on the choice of an $H$-equivariant injection $\lambda_\varphi : \varphi^* \mathcal{U}_G \to \mathcal{U}_H$.

Namely, if $X$ is any global spectrum, then we send an element represented by a $G$-equivariant map $f : |S^A| \to |X(A)|$ to the element represented by the $H$-equivariant map

$$|S^{\lambda_\varphi(\varphi^*A)}| \cong |S^{\varphi^*A}| \xrightarrow{\varphi^* f} |X(\varphi^*A)| \cong |X(\lambda_\varphi(\varphi^*A))|$$

where the unlabelled isomorphisms are induced by $\lambda_\varphi$. Applying this to our chosen fibrant replacements then yields a natural transformation $\varphi^* : \pi_0^{G,\text{true}} \Rightarrow \pi_0^{H,\text{true}}$; moreover, while the natural transformation between naïve homotopy groups depended on the choice of $\lambda_\varphi$, the natural transformation of true homotopy groups is actually independent of this choice [Hau19, 4.6].

We moreover also write $\varphi^* : F^G \Rightarrow F^H$ for the natural transformation corresponding to $\varphi^* : \pi_0^{G} \Rightarrow \pi_0^{H}$, which is characterized uniquely up to non-canonical homotopy by the relation $\pi_0 R\Omega^\infty (\varphi^*) \tau_G = \psi(\varphi^* \tau'_G)$ where $\psi$ is the isomorphism from Corollary 6.5.
Construction 6.7. Let $G$ be a finite group and let $H \subset G$ be a subgroup. For any pointed $H$-set $X$, we write $\Psi$ for the map $G_+ \wedge H X \to X$ projecting onto the preferred coset $[1]$ of $G/H$, i.e. the $H$-equivariant based map

$$[g,x] \mapsto \begin{cases} g.x & \text{if } g \in H \\ * & \text{otherwise.} \end{cases}$$

If now $Y$ is any $G$-spectrum, we will also write $\Psi: G_+ \wedge Y \to Y$ for the map of $H$-spectra obtained by applying the above levelwise in the simplicial and spectral directions. The Wirthmüller map is then defined as the composite

$$\text{Wirth}_H^*: \pi_0^G,\text{naive}(Y) \xrightarrow{\text{incl}^* \pi_0^H,\text{naive}(Y)} \pi_0^H,\text{naive}(G_+ \wedge H Y),$$

and one can show that this map is an isomorphism (without any cofibrancy or semistability assumptions on $Y$), see e.g. [Hau17, Proposition 3.7]. We then define the transfer from $H$ to $G$ as the composition

$$\text{tr}_H^*: \pi_0^H,\text{naive}(Y) \xrightarrow{\eta} \pi_0^H,\text{naive}(G_+ \wedge H Y) \xrightarrow{(\text{Wirth}_H^*)^{-1}} \pi_0^G,\text{naive}(Y).$$

In particular, if $X$ is any global spectrum, we can apply this to $Y$ equipped with trivial $G$-action, yielding a natural transformation $\text{tr}_H^*: \pi_0^H,\text{naive} \Rightarrow \pi_0^G,\text{naive}$ of functors $\text{Spectra}_{\infty,\text{global}} \to \text{Ab}$, and applying this to our chosen fibrant replacements then yields a natural transformation $\text{tr}_H^*: \pi_0^H,\text{true} \Rightarrow \pi_0^G,\text{true}$.

As before this corresponds to a natural transformation $\text{tr}^H_G: F^H \Rightarrow F^G$ of functors $\text{Spectra}_{\infty,\text{global}} \to \mathcal{F}p$, unique up to non-canonical homotopy.

6.2. Restrictions and transfers for global $\Gamma$-spaces. We will now relate the above homotopy theoretic constructions to more combinatorial constructions on the level of special global $\Gamma$-spaces. For this we begin by introducing the respective functoriality following [Sch19b] Construction 6.9 and Construction A.2).

The restrictions actually already exist on the level of global spaces:

Construction 6.8. Let $\varphi: H \to G$ be any homomorphism of finite groups, and let $\lambda_\varphi: \varphi^*\omega^G \to \omega^H$ be any $H$-equivariant injection (where $H$ acts on $\omega^H$ in the evident way, and similarly for $G$); this indeed exists as the source is countable and the target is a complete $H$-set universe by [Sch19b, Proposition 2.19]. We then define for each $X \in E(M)\text{-SSet}$ the restriction

$$\varphi^*: \pi_0(X[\omega^G]^G) \to \pi_0(X[\omega^H]^H)$$

as the map induced by the composition

$$X[\omega^G]^G \subset X[\omega^G]^{\varphi(H)} = X[\varphi^*\omega^G]^{\varphi(H)} \xrightarrow{X[\lambda_\varphi]} X[\omega^H]^H.$$  

Note that if $\lambda_\varphi$ is any other $H$-equivariant injection $\varphi^*\omega^G \to \omega^H$, then the maps $X[\lambda_\varphi]$ and $X[\lambda'_\varphi]$ are $H$-equivariantly homotopic via $X[\lambda_\varphi,\lambda'_\varphi]$, so $\varphi^*$ is independent of the choice of $\lambda_\varphi$. Moreover, $\varphi^*$ is clearly natural.

Remark 6.9. Below we will apply to the special case that $X = \text{NFun}(EM,\mathcal{C})$ for some category $\mathcal{C}$ with trivial $EM$-action. In this case, the above obviously agrees up to conjugation with the isomorphisms induced by the chosen bijections $\omega^H \cong \omega$ and $\omega^G \cong \omega$ with the map

$$\pi_0(\text{NFun}(\text{EInj}(\omega^G,\mathcal{C}),\mathcal{C})^G) \to \pi_0(\text{NFun}(\text{EInj}(\omega^H,\omega),\mathcal{C})^H)$$

given by restricting along $\text{EInj}(\lambda_\varphi,\omega)$. 
Construction 6.10. Let $G$ be a finite group and let $H \subset G$ be a subgroup. For every special global $\Gamma$-space $X$ we define the transfer

$$\text{tr}^H_G : \pi_0(X[\omega^H]^H) \to \pi_0(X[\omega^G]^G)$$

as follows: we have a zig-zag

$$X(1^+)[\omega^H]^H \xrightarrow{X(1^+)[i_1]} X(1^+)[\omega^G]^H \xrightarrow{p_{[1]}} X(G/H_\omega)[\omega^G]^G \xrightarrow{p_{[\omega_0]}} X(1^+)[\omega^G]^G$$

where $i_1$ again denotes the ‘extension by zero’ map $\omega^H \to \omega^G$, $p_{[1]}$ is the characteristic map of the coset $[1] \in G/H$, and $\nabla : G/H_\omega \to 1^+$ is the fold map sending every coset to $1$. An easy computation using the specialness of $X$ shows that $p_{[1]}$ is an equivalence, so the above zig-zag induces a map $\text{tr}^H_G : \pi_0(X[\omega^H]^H) \to \pi_0(X[\omega^G]^G)$.

On the other hand, we can relate the homotopy groups of $X(1^+)$ to the ones of the global spectrum $E^\infty X[\omega^*]$ as follows:

Construction 6.11. Let $X \in \mathbf{GEMSet}^\text{special}$ be a special global $\Gamma$-space, and let $G$ be any finite group. Following [Sch19b, Construction 6.19] we define a map $\beta : \pi_0(X(1^+)[\omega^G]^G) \to \pi_0^G(\mathcal{E}^\otimes (X[\omega^*]))$ as follows: a $G$-fixed $0$-simplex $x$ of $X(1^+)[\omega^G]$ is sent to the image under $\iota$ of

$$[S^G \xrightarrow{\omega^G \wedge x} S^G \wedge X(1^+)[\omega^G] \xrightarrow{\text{assembly}} X(S^G)[\omega^G]].$$

Here we have again secretly identified $G$ with a subset of $U_G$.

Remark 6.12. One can show that $\beta$ is a group completion in the usual sense [Sch19b, Proposition A.3], but we will not need this below.

Proposition 6.13. Let $X$ be a special global $\Gamma$-space.

1. Let $\phi : H \to G$ be a homomorphism of finite groups. Then $\beta^G = \beta^H \phi$ as maps $\pi_0(X(1^+)[\omega^G]^G) \to \pi_0^G(\mathcal{E}^\otimes (X[\omega^*]))$.

2. Let $G$ be any finite group and let $H \subset G$ be a subgroup. Then $\beta \text{tr}^H_G = \text{tr}^H_G \beta$ as maps $\pi_0(X(1^+)[\omega^H]^H) \to \pi_0^G(\mathcal{E}^\otimes (X(1^+)[\omega^G]^G))$.

Proof. If $X$ is the special global $\Gamma$-space associated to a parsummable category, this is [Sch19b, Theorem 6.21] and its proof. The general case now follows from the fact that every special global $\Gamma$-space is $G$-globally level weakly equivalent to the special global $\Gamma$-space associated to a parsummable category [Len20a, Theorem 4.3.3].

6.3. Restrictions and transfers in mack. We are now finally ready to make precise in which sense $\text{mack}(X)$ is given for a global spectrum $X$ by genuine fixed points together with restrictions and transfers between them. For this we begin by constructing the equivalences $\text{mack}(X)(BG) \simeq F^G X$:

Construction 6.14. Let $T_G$ be any right $G$-invariant functor with $T_G(j(g), 1) = g : 1 \to 1$ for all $g \in G$, where $j : G \to \mathcal{M}$ is the injective homomorphism induced from our chosen bijection $\omega^G \cong \omega$. In [Len20a, Theorem 4.4.21] we showed that the map $\Sigma^*_G T(G, -)/G \to K_{gl}(\mathfrak{F}_{gl})$ classifying $\mathfrak{u}_G := \beta(T_G)$ is an equivalence. We moreover define $\mathfrak{l}_G \in \pi_0 R\Omega^\infty F^G K_{gl}(\mathfrak{F}_{gl})$ as the image of $\tau_G$ under this equivalence.

Construction 6.15. We recall the equivalence $\mathfrak{F}_{gl} \to \text{core} hF^G_{gl}$ from Construction 2.22 which in particular induces an equivalence on $K_{gl}$. We write $\tau_G$ and $\mathfrak{l}_G^*$ for the images of the universal classes $\mathfrak{l}_G$ and $\tau_G$ under this equivalence. It follows again
formally that $(\mathbf{K}_{gl}(\text{core } h\mathbb{F}_{BG}), \tilde{\tau}_G)$ corepresents $F^G$ while $(\mathbf{K}_{gl}(\text{core } h\mathbb{F}_{BG}), \tilde{\tau}'_G)$ corepresents $\omega_0^{G,\text{true}}$.

Remark 6.16. Under the isomorphism $\text{Fun}(EM, 3_G)[\omega^G] \cong \text{Fun}(E\text{Inj}(\omega^G, \omega), 3_G)$ induced by our chosen bijection $\rho_G: \omega^G \cong \omega$ the functor $T_G$ corresponds to a functor $\tilde{T}_G: E\text{Inj}(\omega^G, \omega) \to 3_G$, which is uniquely described up to $G$-equivariant isomorphism by the fact that it is $G$-invariant and that $\tilde{T}_G(\rho_G(1)), \rho_G(1)) = 1 \to 1$.

Now the map $g \mapsto \rho_G(g) \in \text{Inj}(\omega^G, \omega)$ is right $G$-equivariant with respect to the usual (left) $G$-action on $\omega^G$; if $k$ is any other right $G$-equivariant map $G \to \text{Inj}(\omega^G, \omega)$, then the induced maps $EG \to E\text{Inj}(\omega^G, \omega)$ are right $G$-equivariantly homotopic. Thus, a $G$-invariant functor $T: E\text{Inj}(\omega^G, \omega) \to 3_G$ is isomorphic to $\tilde{T}_G$ if and only if its restriction along some (hence any) such injection $k$ is isomorphic to the functor $(g, 1) \mapsto g: 1 \to 1$.

We can now state our compatibility result:

**Theorem 6.17.**

1. Let $\varphi: H \to G$ be a homomorphism of finite groups. Then the diagram

\[
\begin{array}{ccc}
\mathbf{y}^{\text{ST}}(\mathbf{K}_{gl}(\text{core } h\mathbb{F}_{BG})) & \xrightarrow{\mathbf{y}^{\text{ST}}(\mathbf{K}_{gl}(\text{incl}^\tau))} & \mathbf{y}^{\text{ST}}(\mathbf{K}_{gl}(\text{core } h\mathbb{F}_{BH})) \\
\; \downarrow \mathbf{F}^G & & \; \downarrow \mathbf{F}^H \\
\mathbf{F}^H & \xrightarrow{\varphi^*} & \mathbf{F}^G
\end{array}
\]

of natural transformations of functors $\mathbf{Spectra}^\infty_{\text{global}} \to \mathcal{I}p$ commutes up to (non-canonical) homotopy.

2. Let $G$ be a finite group, let $H \subset G$ be a subgroup, and write incl: $H \hookrightarrow G$ for the inclusion. Then the diagram

\[
\begin{array}{ccc}
\mathbf{y}^{\text{ST}}(\mathbf{K}_{gl}(\text{core } h\mathbb{F}_{BH})) & \xrightarrow{\mathbf{y}^{\text{ST}}(\mathbf{K}_{gl}(\text{incl}^\tau))} & \mathbf{y}^{\text{ST}}(\mathbf{K}_{gl}(\text{core } h\mathbb{F}_{BG})) \\
\; \downarrow \text{tr}^H_G & & \; \downarrow \tilde{\tau}_G \\
\mathbf{F}^H & \xrightarrow{\text{tr}^H_G} & \mathbf{F}^G
\end{array}
\]

of natural transformations of functors $\mathbf{Spectra}^\infty_{\text{global}} \to \mathcal{I}p$ commutes up to (non-canonical) homotopy.

Put differently, under the chosen identifications the restrictions in $\text{mack}(X)$ (i.e. the functoriality in the contravariant legs of $A^{\text{gl}}$) are given by the usual homotopy theoretic restrictions, and analogously for the transfers.

The proof will rely on two separate computations:

**Proposition 6.18.** Let $\varphi: H \to G$ be a homomorphism of finite groups and let $\varphi_1: 3_H \to 3_G$ be the essentially unique strong symmetric monoidal functor with $\varphi_1(1: 1 \to 1) = (\varphi(1: 1 \to 1))$. Then the relation

$$\varphi^*[T_G] = [\varphi_1 \circ T_H]$$

holds in $\pi_0(\mathbf{NFun}(EM, 3_G)[\omega^H^H])$.

**Proof.** It suffices to prove the analogous relation in $\pi_0(\mathbf{NFun}(\text{Inj}(\omega^H, \omega), 3_G)^H)$ (with $\tilde{T}_G$ and $\tilde{T}_H$ in place of $T_G$ and $T_H$, respectively). For this we make some very specific choices: we begin by picking any right $G$-equivariant injection $\ell: G \to
Inj(ω^G, ω) and a left H-equivariant injection λ_ϕ: ϕ^*ω^G → ω^H. Next, we pick an injection u ∈ Inj(ω^H, ω) such that uλ_ϕ /∈ ℓ(G); this is certainly possible as the right hand side is finite. This then yields a right H-equivariant injection k: H → Inj(ω^H, ω) via k(h) = uh. Finally, we pick a G-equivariant map r: Inj(ω^G, G) → G sending both ℓ(1) and k(1)λ_ϕ to 1; this is possible since Inj(ω^G, ω) is free and since ℓ(1) and k(1)λ_ϕ = uλ_ϕ belong to distinct orbits by construction.

The claim now amounts to showing that ϕ_1 ○ T_H and T_G(− ○ λ_ϕ) are right H-equivariantly isomorphic after restricting along Ek: EH → EInj(ω^H, ω). To this end we agree to take T_G(v, u) = (r(v)r(u)^{-1}: 1 → 1) (which has the desired properties by construction of r) and we let T_H be any right H-invariant injection EInj(ω^H, ω) → F_H with T_H(k(h), 1) = (h: 1 → 1). We claim that in this case we even have an equality of functors

ϕ_1 ○ T_H ○ Ek = T_G(− ○ λ_ϕ) ○ Ek.

Indeed, if we plug in the edge (h, 1) for h ∈ H then the left hand side evaluates to ϕ(h): 1 → 1 while the right hand side evaluates to T_G(k(h)λ_ϕ, k(1)λ_ϕ) = r(k(h)λ_ϕ)r(k(1)λ_ϕ)^{-1}: 1 → 1. It will therefore suffice to show that

r(k(h)λ_ϕ) = ϕ(h)

for every h ∈ H. Indeed, for h = 1 this holds by construction of r. For varying r both sides are right H-equivariant maps H → G (with H acting on G via ϕ), which proves the desired relation and hence the proposition.

Let G be a finite group, let H be a subgroup, and let incl: H → G be the inclusion. For the next statement we recall the symmetric monoidal functor incl*: ˜g_H → ˜g_H from the proof of Proposition 4.32 classifying the object r with G action induced by the injective homomorphism Ϣ: G → Σ_r \ H associated to a choice of a system of coset representatives g_1, ..., g_r of G/H.

Proposition 6.19. In the above situation, the relation

tr^H_G[T_H] = [incl* ○ T_G]

holds in π_0(NFun(EM, ˜g_H)[ω^G]^G).

Proof. The claim is clear if H = G. Thus, we may assume without loss of generality that r ≥ 2.

Both sides are represented by right G-invariant functors EM → ˜g_H, and it will suffice to show that their restrictions along ℓ: EG → EM agree, where ℓ: G → M is the injective homomorphism induced by the G-action on ω^G via the chosen bijection ω^G ≅ ω. Plugging in the definitions, we see that the restriction of the right hand side is represented by the functor (g, 1) → (ι(g): r → r).

Computing the transfer on the left hand side directly from the definition would require us to make the higher terms of Γ_g ˜g_H precise and become quite cumbersome. Fortunately, there is a way around this: we write k: H → M for the injective homomorphism (with universal image) associated to action on ω induced by the action on ω^G, we fix a free H-orbit A ⊂ k^*ω, and we consider the commutative diagram
\[
\begin{align*}
\Gamma_{\text{gl}} \mathcal{F}_H(G/H) [\omega^G]^H & \xrightarrow{\sim} \Gamma_{\text{gl}} \mathcal{F}_H(G/H) [\omega^G]^G \\
\Gamma_{\text{gl}} \mathcal{F}_H(G/H) [\omega^G]^H & \xrightarrow{\sim} \Gamma_{\text{gl}} \mathcal{F}_H(G/H) [\omega^G]^G \\
N_F \mathcal{F}_H(G/H) [\omega^G]^H & \xrightarrow{\sim} N_F \mathcal{F}_H(G/H) [\omega^G]^G
\end{align*}
\]

where the unlabelled weak equivalences on the left are again induced by the extension by zero map \(\omega^H \to \omega^G\) while the vertical weak equivalences are as in the proof of Theorem 4.18. Again, we will only need to know what they do in degree 1, where the lower arrow is simply the unit. Unlike in the proof of Theorem 4.18, we will however have to make the functor \(j : \mathcal{F}_H \to \mathcal{F}_H = \bigsqcup_{n \geq 0} E\text{Inn}(n \times A, \omega)/\Sigma_n \wr H\) explicit this time, see [Len20a, Construction 4.2.18]: to do this, we first pick an injection \(f^{(n)} : n \times A \to \omega\) for every \(n \geq 0\); we agree at this point that \(f^{(1)}\) is the inclusion of \(A\), while we will at a later point make a concrete choice of \(f^{(n)}\). Then the equivalence \(j : \mathcal{F}_H \to \mathcal{F}_H\) is simply given by sending \((\sigma; h_1, \ldots, h_n) : n \to n\) to \([f^{(n)}(\sigma; h_1, \ldots, h_n), f^{(n)}]\), where we let \(\Sigma_n \wr H\) act on \(n \times A\) as usual.

Now we consider the vertex \(T_H\) of \(N_F \mathcal{F}_H = N_F \mathcal{F}_H(G/H) [\omega^G]^H\) given by the inclusion \(A \hookrightarrow \omega\). Then \(\eta(T_H)\) and \(j \circ T_H\) agree after restricting along \(E\ell\), so they define isomorphic objects in \(\Gamma_{\text{gl}} \mathcal{F}_H(1^+) [\omega^G]^H\). We will now compute a representative of transfer of \(T_H\), and then show that its image in \(\Gamma_{\text{gl}} \mathcal{F}_H(1^+) [\omega^G]^H\) is isomorphic to the image of \(\text{Incl}^* \circ T_G\), which will then complete the proof of the proposition.

The computation of the transfer can be found in [Sch19b, Construction 6.9 and Theorem 6.21] (for arbitrary parsummable categories); we recall the relevant details: we fix a \(G\)-equivariant injection \(\psi : G \times H \to \omega^G\), and we set \(\psi_i := \psi|_{g_i \cdot -}\); for later use we remark that the \(G\)-equivariance of \(\psi\) then translates to the condition

\[(g - )\psi_i = \psi_{\sigma(g)i}(h_i(g) - )\]

where we again write \(\iota(g) := (\sigma(g); h_1(g), \ldots, h_n(g))\).

By loc. cit., the transfer \(\text{tr}^H_T\) of the map \(\psi(\psi)\) is then represented by the object

\[\sum_{i=1}^r F_\psi(\psi_i)(T_H)\]

If we write \(\varphi_i := \rho_G \psi_i \rho_H^{-1}\), then this is by definition the class in \(E\text{Inn}(r \times A, \omega)/\Sigma_r \wr H\) of the map \(\varphi : (i, a) \to \varphi_i(a)\). Moreover, the relation \(n \geq 0\) means that \(\varphi\) is \(G\)-equivariant when we let \(G\) act on \(r \times A\) via \(\iota : G \to \Sigma_r \wr H\) and on \(\omega\) via the chosen bijection \(\rho_G : \omega^G \cong \omega\), i.e. via \(\ell : G \to M\). This is in fact all we will need to know below.

Now is the time to make the choice of the injection \(f^{(r)} : r \times A \to \omega\) from the construction of \(j : \mathcal{F}_H(1^+) \to \mathcal{F}_H\) explicit: namely, we simply take \(f^{(r)} = \varphi\). To prove now that \(\text{Incl}^* \circ T_G\) and \(\varphi\) have isomorphic images in \(\text{Fun}(E\mathcal{M}, \mathcal{F}_H)[\omega^G]^G\), it will be enough to show that their images are \(G\)-equivariantly isomorphic after restricting along \(E\ell\). However, plugging in the definitions we see that

\[j(\text{Incl}^* \circ T_G)(g, 1) = [\varphi \cdot \iota(g), \varphi] = [\ell(g) \varphi, \varphi]\]

(where the second equation uses \(G\)-equivariance of \(\varphi\), which is actually even equal to \(\eta(\varphi)(\ell(g), \ell(1))\)).
Proof of Theorem 6.17. For the first statement, all functors in question are exact, so the spectral Yoneda Lemma (Lemma 3.17) implies that it is enough to show that the two paths through
\[
\begin{align*}
&\mathbf{K}_{\text{gl core } hF_{BG}}, \mathbf{K}_{\text{gl core } hF_{BG}} \\
&\xrightarrow{[\phi, \mathbf{K}_{\text{gl core } hF_{BG}}]} \mathbf{K}_{\text{gl core } hF_{BH}}, \mathbf{K}_{\text{gl core } hF_{BG}} \\
&\xrightarrow{\tau'_{G}} \pi_{0} R\Omega^{\infty} F^{G}(\mathbf{K}_{\text{gl core } hF_{BG}}) \\
&\xrightarrow{\varphi^{*}} \pi_{0} R\Omega^{\infty} F^{H}(\mathbf{K}_{\text{gl core } hF_{BG}})
\end{align*}
\]
send the class of the identity to the same element in \(\pi_{0} R\Omega^{\infty} F^{H} X\), where the brackets denote the hom sets in the global stable homotopy category. By definition of \(\varphi^{*}\) and the universal classes \(\hat{\tau}'_{G}, \hat{\tau}'_{H}\) it is enough to instead prove this for the square
\[
\begin{align*}
&\mathbf{K}_{\text{gl core } hF_{BG}}, \mathbf{K}_{\text{gl core } hF_{BG}} \\
&\xrightarrow{[\phi, \mathbf{K}_{\text{gl core } hF_{BG}}]} \mathbf{K}_{\text{gl core } hF_{BH}}, \mathbf{K}_{\text{gl core } hF_{BG}} \\
&\xrightarrow{(\hat{\tau}'_{G})^{*}} \pi_{0}^{G, \text{true}}(\mathbf{K}_{\text{gl core } hF_{BG}}) \\
&\xrightarrow{\varphi^{*}} \pi_{0}^{H, \text{true}}(\mathbf{K}_{\text{gl core } hF_{BG}})
\end{align*}
\]
Recall now the strong symmetric monoidal functor \(\phi^{!}: \mathcal{H}_{G} \rightarrow \mathcal{H}_{G}\) from Proposition 6.18 sending \(h: 1 \rightarrow 1\) to \(\phi(h): 1 \rightarrow 1\). Chasing through the universal \(H\)-object shows that
\[
\mathcal{H}_{G} \xrightarrow{\phi^{!}} \mathcal{H}_{G}
\]
commutes up to symmetric monoidal isomorphism, where the unlabelled vertical arrows are the equivalences from Construction 2.22. By definition of \(\hat{\tau}'_{G}\) we are therefore altogether reduced to showing that the two paths through
\[
\begin{align*}
&\mathbf{K}_{\text{gl core } hF_{BG}}, \mathbf{K}_{\text{gl core } hF_{BG}} \\
&\xrightarrow{[\phi, \mathbf{K}_{\text{gl core } hF_{BG}}]} \mathbf{K}_{\text{gl core } hF_{H}}, \mathbf{K}_{\text{gl core } hF_{BG}} \\
&\xrightarrow{(\hat{\tau}'_{G})^{*}} \pi_{0}^{G, \text{true}}(\mathbf{K}_{\text{gl core } hF_{BG}}) \\
&\xrightarrow{\varphi^{*}} \pi_{0}^{H, \text{true}}(\mathbf{K}_{\text{gl core } hF_{BG}})
\end{align*}
\]
send the identity of \(\mathcal{H}_{G}\) to the same class, which amounts to the relation
\[
\varphi^{*}(\hat{\tau}'_{G}) = \mathbf{K}_{\text{gl core } hF_{G}}^{\phi}(\varphi_{G})^{*} \hat{\tau}'_{H}
\]
in \(\pi_{0}^{H, \text{true}}(\mathbf{K}_{\text{gl core } hF_{G}})\). Using that \(\hat{\tau}'_{G} = \beta[T_{G}]\) and \(\hat{\tau}'_{H} = \beta[T_{H}]\) by definition, this then finally follows from Proposition 6.18 together with Proposition 6.13.

The second statement follows analogously from Proposition 6.19 together with the fact that
\[
\hat{\mathcal{H}}_{G} \xrightarrow{\text{incl}^{*}} \hat{\mathcal{H}}_{H}
\]
(with the vertical equivalences as before) commutes up to symmetric monoidal isomorphism, as verified in the proof of Proposition 4.32. \(\square\)
7. Global algebraic $K$-theory vs. Swan $K$-theory

We can now finally prove Theorem 7.1 relating the spectral Mackey functors underlying global algebraic $K$-theory to certain categorically defined spectral Mackey functors. With all of the above results at hand, this will be surprisingly formal:

**Theorem 7.1.** The diagram

$$
\begin{array}{ccc}
N_{\Delta}(\text{SymMonCat}_{(2,1)}) & \xrightarrow{\text{K}_\text{gl}} & \text{Spectra}^\infty_{\text{global}} \\
\text{swan} & & \downarrow \text{mack} \\
\text{Fun}^\oplus(\text{A}^\text{gl}, \text{SymMonCat}^\infty_{\text{w.e.}}) & \xrightarrow{\text{Fun}^\oplus(\text{A}^\text{gl}, \text{K})} & \text{Fun}^\oplus(\text{A}^\text{gl}, \mathscr{S}^\text{p})
\end{array}
$$

commutes up to preferred natural equivalence.

Together with the explicit descriptions of Mackey functor structures in Subsections 4.4 and 6.3.3 this in particular makes precise that the homotopy theoretic restrictions and transfers on the fixed points of global algebraic $K$-theory come from the categorically defined restrictions and transfers on the corresponding categories of $G$-objects.

**Proof.** We first observe that both paths through this diagram descend to cocontinuous functors from $\text{SymMonCat}^\infty_{\text{global}}$: for the upper path this follows from Theorem 4.23 together with [Len20a, Theorem 3.4.21]. On the other hand, the equivalences $\text{swan}(C)(G) \simeq G-C$ show that the lower path inverts global weak equivalences, and by Theorem 4.26 the induced functor $\text{SymMonCat}^\infty_{\text{global}} \rightarrow \text{Fun}^\oplus(\text{A}^\text{gl}, \text{SymMonCat}^\infty_{\text{w.e.}})$ is an equivalence while $\text{Fun}^\oplus(\text{A}^\text{gl}, \text{K})$ is a left adjoint functor by Mandell’s result which we recalled in Proposition 4.24.

By Proposition 4.27 we are now reduced to showing that the restrictions to $\text{A}^\text{gl}$ are equivalent. On the other hand, both paths actually factor through $\text{Fun}^\oplus(\text{A}^\text{gl}, \mathscr{S}^\text{p}^\geq_0) \simeq \text{Fun}^\oplus(\text{A}^\text{gl}, \text{CGrp}(\mathscr{S})) \subset \text{Fun}^\oplus(\text{A}^\text{gl}, \text{CMon}(\mathscr{S})) \simeq \text{Fun}^\times(\text{A}^\text{gl}, \mathscr{S})$ (where $\mathscr{S}^\text{p}^\geq_0 \subset \mathscr{S}^\text{p}$ again denotes the full subcategory of connective spectra), so we are reduced to showing an equivalence of functors $\text{A}^\text{gl} \rightarrow \text{Fun}^\times(\text{A}^\text{gl}, \mathscr{S})$. Using that both paths through the diagram by construction factor through restriction along core $c\Psi: \text{A}^\text{gl} \rightarrow \text{A}^\text{op}_{\text{gl}}$ we are then finally reduced to showing that

$$
\begin{array}{ccc}
\text{A}^\text{gl} & \xrightarrow{\text{K}_\text{gl}} & \text{Spectra}^\infty_{\text{global}} \\
\downarrow & & \downarrow y \\
\text{Fun}^\oplus(\text{A}^\text{op}_{\text{gl}}, \text{SymMonCat}^\infty_{\text{w.e.}}) & \xrightarrow{\text{Fun}^\times(\text{A}^\text{op}_{\text{gl}}, \mathscr{S})} & \text{Fun}^\times(\text{A}^\text{op}_{\text{gl}}, \mathscr{S})
\end{array}
$$

commutes up to equivalence: here the unlabelled arrow on the upper left is the composite from the construction of $\text{swan}$ (Construction 4.25). To this end we will compare both paths through this diagram to the Yoneda embedding of $\text{A}^\text{gl}$. More precisely, we will construct natural transformations from the Yoneda embedding to the two composites, and show that they are pointwise group completions of homotopy commutative monoids (with respect to the usual homotopy commutative...
monoid structures), hence group completions of coherently commutative monoids. It will then follow formally that there is an essentially unique natural equivalence filling the above diagram that is compatible with the comparison map from the Yoneda embedding, which will then complete the proof of the theorem.

For the upper path, we can simply take the natural transformation induced by $K_{gl}$ itself, which is pointwise a group completion by Theorem 5.16.

For the lower path we first factor $K: \text{SymMonCat}_{w.e.}^{\infty} \to \mathcal{F} p^{\geq 0}$ up to equivalence as the composite

$$\text{SymMonCat}_{w.e.}^{\infty} \xrightarrow{\mathbb{N}} \text{CMon}(\mathcal{S}) \xrightarrow{\text{group complete}} \text{CGrp}(\mathcal{S}) \xrightarrow{\cong} \mathcal{F} p^{\geq 0}$$

for $\mathbb{N}$ the essentially unique lift of $N \circ \text{forget}: \text{SymMonCat}_{w.e.}^{\infty} \to \text{Cat}_{w.e.}^{\infty} \to \mathcal{S}$. As the composite

$$\mathcal{A}_{gl} \to \text{Fun}^\oplus(\mathcal{A}_{gl}^{op}, \text{SymMonCat}_{w.e.}^{\infty}) \xrightarrow{\text{forget}} \text{Fun}^\times(\mathcal{A}_{gl}^{op}, \text{Cat}_{w.e.}^{\infty}) \xrightarrow{\mathbb{N}} \text{Fun}^\times(\mathcal{A}_{gl}^{op}, \mathcal{S})$$

is equivalent to the Yoneda embedding (see the claim in the proof of Theorem 4.26), we conclude that the lower left composite is equivalent to

$$\mathcal{A}_{gl} \xrightarrow{y_{\text{SADD}}} \text{Fun}^\oplus(\mathcal{A}_{gl}^{op}, \text{CMon}(\mathcal{S})) \xrightarrow{\text{group completion}} \text{Fun}^\oplus(\mathcal{A}_{gl}^{op}, \text{CGrp}(\mathcal{S})) \xrightarrow{\text{forget}} \text{Fun}^\times(\mathcal{A}_{gl}^{op}, \mathcal{S}).$$

Now the unit of the adjunction $\text{CMon}(\mathcal{S}) \dashv \text{CGrp}(\mathcal{S})$ induces a natural transformation from $y \circ \text{forget} \circ y_{\text{SADD}}$ to the above composite, and this is a pointwise group completion by design. Altogether this finishes the proof of the theorem. □

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