The gravitational-wave memory from eccentric binaries

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The nonlinear gravitational-wave memory causes a time-changing but nonoscillatory correction to the gravitational-wave polarizations. It arises from gravitational-waves that are sourced by gravitational-waves. Previous considerations of the nonlinear memory effect have focused on quasi-circular inspiralling binaries. Here I consider the nonlinear memory from Newtonian orbits with arbitrary eccentricity. Expressions for the waveform polarizations and spin-weighted spherical-harmonic modes are derived for elliptic, hyperbolic, parabolic, and radial orbits. In the hyperbolic, parabolic, and radial cases the nonlinear memory provides a 2.5 post-Newtonian (PN) correction to the leading-order waveforms. In the elliptical case the nonlinear memory corrects the waveform at leading (0PN) order. This calculation completes our knowledge of the leading-order waveform for elliptical orbits. In their past, even quasicircular binaries had large eccentricities. Because the nonlinear memory depends sensitively on the past evolution of a binary, I discuss the effect of this early-time eccentricity on the value of the late-time memory in nearly-circularized binaries. I also estimate the detectability of the linear and nonlinear memory from hyperbolic and parabolic binaries.

I. INTRODUCTION

Gravitational-waves (GWs) are usually thought of as purely oscillatory phenomena. For example, the GWs from the coalescence of compact-object binaries tend to have a characteristic structure: as a GW passes through a detector the frequency and amplitude increase, but at late times the amplitude exponentially decays to zero from its peak value. However, the GWs from a variety of sources display nonoscillatory components as well. The simplest example is the GWs produced by the scattering of two unbound masses in a hyperbolic orbit [1]. Other examples include the GW signal from the asymmetric ejection of matter [2–4] or neutrino’s [5, 6] in supernova explosions or gamma-ray-bursts [7, 8]. The GW signals from these sources all exhibit a property called GW memory. This refers to a long-timescale difference in the values of the observed metric perturbation associated with the GW:

$$\Delta h_{+, x} = \lim_{t \to +\infty} h_{+, x} - \lim_{t \to -\infty} h_{+, x},$$  (1.1)

where $h_{+, x}$ are the two GW polarizations. In a GW detector that is truly freely-falling (one that follows geodesics), a GW with memory can cause a permanent deformation in the detector (hence the term memory).

The nonoscillatory sources mentioned above are all examples of the linear memory [9–11]. In these cases the nonoscillatory component to the GW arises from nonoscillatory motions of the source that are encoded in the matter stress-energy tensor $T_{\mu \nu}$. Because unbound gravitating systems have sources that undergo nonoscillatory motions, the linear memory tends to occur in such systems. In addition to the linear memory, there is also a nonlinear memory effect [12–15]. This nonlinear memory arises from the gravitational-waves produced by gravitational-waves: it is sourced not by nonoscillatory motions encoded in $T_{\mu \nu}$ but rather in $\mu_{\nu}^{gw}$ [16, 17], which describes the stress-energy of radiated GWs. These radiated GWs are themselves always “unbound” [18] and hence produce a memory. Furthermore, since this effect originates directly from the radiated GWs and not the motion of the source, the nonlinear memory is present in all sources of GWs, including bounded systems.

Why study the nonlinear memory? One of the key goals of gravitational-wave astronomy is to experimentally probe our understanding of general relativity (GR). While solar system and binary pulsar tests can only probe GR in the weak-field regime where the theory is nearly Newtonian, observations of coalescing compact-object binaries (and especially merging binary black holes) will produce GWs whose properties will depend on the strong-field, highly-dynamical sector of GR. There are a variety of nonlinear effects which will imprint themselves on the waveforms from such systems. Of these effects the nonlinear memory is unique because (i) it describes waves produced by waves (perhaps the most nonlinear of interactions present in the theory), (ii) its nonoscillatory imprint on the GW signal is distinct from the manifestation of other nonlinear effects, and (iii) although it arises from a higher-order nonlinear interaction, it can enter the post-Newtonian (PN) expansion of the waveform amplitude at leading (0PN) order.

The nonlinear memory also has the property of being hereditary. This means that the nonlinear memory piece of the waveform amplitude at some observer’s time $t$ depends not only on the configuration of the source at the

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corresponding retarded time \( t - R \) (where \( R \) is the distance to the source), but rather on the entire past-history of the source. To illustrate this property, compare the leading-order “quadrupole” expression for the GW field,

\[
h_{jk}^{TT} = \frac{2}{\mathcal{R}} j_{jk}^{\text{TT}}(T_R),
\]

with the corresponding expression for the nonlinear memory \cite{19},

\[
h_{jk}^{TT, \text{(mem)}} = 4 \int_{-\infty}^{T_R} dt' \left[ \int dE_{gw} \frac{n_j' n_k'}{1 - n' \cdot \mathcal{N}} (dT') \right]^{TT},
\]

where \( I_j \) is the mass quadrupole moment, \( \frac{dE_{gw}}{dt'} \) is the GW energy flux, \( n_j \) is a unit radial vector, \( \mathcal{N} \) is a unit vector pointing from the source to the observer, and TT means to take the transverse-traceless projection (see also Sec. 1 of \cite{20}). In Eq. (1.2) one can clearly see that the primary GWs measured by a detector are directly determined by the retarded time configuration of the source (as encoded by its mass quadrupole). But in the expression for the memory [Eq. (1.3)], its value at time \( T_R \) depends on an integral into the infinite past.\footnote{The GW tail effect exhibits a similar property, but in that case the integrand drops off more steeply in the past than does the memory, so only the nearby past need be considered (see Sec. 4 of \cite{21} or Sec. 5.3.4 of \cite{22} for a more detailed discussion).} So to determine the value of the nonlinear memory at one instant, one needs to know the energy flux (and hence the motion of the source) at all previous times.

### A. Motivation

Previous calculations of the nonlinear memory have focused almost exclusively on quasicircular binaries. Early work focused on computing the memory during the inspiral phase for quasicircular orbits, starting with Will and Wiseman \cite{19}\footnote{Note that the expression for the memory in the text after Eq. (17) of \cite{19} is missing a factor of 2, but the curves in their Figure 1 agree with the expressions below.} (see also \cite{23}). Remarkably, they found that the nonlinear memory affects the GW polarizations at leading (0PN) order:

\[
\begin{align*}
\mathcal{h}_+ &= -2 \frac{\eta M}{\mathcal{R}} x \left[ (1 + c_0^2) \cos 2(\varphi - \Phi) + \frac{s_0^2}{96} (17 + c_0^3) \right], \\
\mathcal{h}_x &= -4 \frac{\eta M}{\mathcal{R}} x c_0^2 \sin 2(\varphi - \Phi),
\end{align*}
\]

where \( M = m_1 + m_2 \) is the binary’s total mass, \( \eta = m_1 m_2 / M^2 \) is its reduced mass ratio, \( x \equiv (M \omega) / 3 \) is the standard PN expansion variable for quasicircular binaries, \( \varphi(t) \) is the orbital phase, \( \omega = \dot{\varphi} \) is the orbital angular frequency, \( c_0 \equiv \cos \Theta, s_0 \equiv \sin \Theta \), and \( (\Theta, \Phi) \) are the angles in the source frame that point in the observer’s direction \( \mathcal{N} \). The second term in Eq. (1.4a) is the nonoscillatory memory term (for a convenient and standard choice of the polarization triad, there is no nonlinear memory contribution to the \( \times \) polarization). In Favata \cite{20} these nonlinear memory contributions were computed to 3PN order\footnote{Arun et al \cite{21} previously showed that the 0.5PN memory contribution vanishes.}, thus completing the PN expansion of the waveform amplitude consistently to 3PN order.

More recent work has investigated in detail the nonlinear memory produced by merging binary black holes. Using a simple analytic model as well as an effective-one-body \cite{24} approach, the full evolution of the memory for the inspiral, merger, and ringdown was computed in \cite{25,26} and the prospects for its detection with interferometers were examined.\footnote{Thorne \cite{18} also examined the memory’s detectability, treating it as an unmodeled burst, while Kennefick \cite{23} considered only the inspiral contribution to the memory.} Calculations of the nonlinear memory from numerical relativity were performed in \cite{27,28}, and the detectability with pulsar-timing-arrays is considered in \cite{29–31}.

The purpose of this study is to analyze the behavior of the nonlinear memory for arbitrarily eccentric binaries. There are several reasons why this generalization is worth considering. First (and most importantly) real binaries will be eccentric and not quasicircular. Even though gravitational radiation-reaction tends to circularize binaries, we expect to observe some binaries with nonnegligible ellipticity (see, e.g., Section I and Appendix A of \cite{32}), or with hyperbolic trajectories (from scattering events in clusters or galactic nuclei). However, even if one is only interested in nearly-circular binaries, the hereditary nature of the nonlinear memory makes it important to consider the eccentric case as well. Because quasicircular binaries become more eccentric in the past, this increasing eccentricity affects the orbital motion and could potentially influence the calculation of the nonlinear memory integral in Eq. (1.3). Clearly, if we hope to actually observe the nonlinear memory effect, we must also be prepared to account for binary eccentricity.

A second motivation comes simply from the desire to have a complete and consistent understanding of the waveforms produced in the general two-body problem. For example, waveform polarizations for elliptical binaries are implicit in the classic work by Peters and Mathews \cite{33} (although they focus on computing the radiated power), and is given explicitly first in Wahlquist \cite{34} and later in Refs. \cite{35–37} to leading (0PN) order. These elliptical waveforms have since been extended to 1PN order by Junker and Schäfer \cite{38} (see also \cite{39,40}), to 1.5PN order in Blanchet and Schäfer \cite{41}, and to 2PN order (neglecting tails) in Gopakumar and Iyer \cite{42}.\footnote{Additional works also consider the GW polarizations in the case} Frequency-domain waveforms are given in \cite{32,43–46}.\footnote{The GW
phasing for elliptical binaries is currently known to relative 3PN order in the conservative [47–52] and dissipative parts [38, 41, 43, 46, 53–57]. As we will see later, even the leading-order (Peters-Mathews) waveforms are incomplete because—as in the quasicircular case—the nonlinear memory modifies the polarizations at 0PN order.

In the case of hyperbolic orbits, waveform polarizations were first derived at 0PN order in Turner [1], with 1PN corrections computed in [38, 40]. In the large-eccentricity (bremstrahlung) limit, waveforms were computed to 1PN order in [53, 58] and using the “post-linear” formalism in [59, 60]. These waveforms already show a linear memory, but the nonlinear memory has only been calculated in the large-eccentricity limit [19]. Waveforms for radial orbits are also considered up to 1PN order in [53] (although the energy flux has been computed at 2PN [61] and 3PN [62] orders), but the nonlinear memory contribution in the radial case has not yet been computed.

In this paper I will attempt to complete our knowledge of the nonlinear memory for binaries with arbitrary eccentricity, including elliptical, hyperbolic, parabolic and radial orbits. Unlike in Ref. [20] where the nonlinear memory was computed to 3PN order in the quasicircular case, here I will restrict the calculation to the leading-PN-order piece of the nonlinear memory. I will also discuss how the nonlinear memory (because of its hereditary nature) is affected by the past-eccentricity of a nearly-circularized binary.

B. Summary

In Sec. II A we begin by reviewing the prescription in [20] for computing the nonlinear memory from the GW energy flux. This involves decomposing the GW polarizations into a sum of spin-weighted spherical harmonic modes \( h_{lm}^{(mem)} \), and relating the nonlinear memory modes \( h_{lm}^{(mem)} \) to “lower-order” oscillatory modes \( h_{lm}^{(p)} \) that are described at Newtonian (0PN) order. At this order these modes depend only on the mass quadrupole moment \( I_{2m} \) of the binary, for which explicit expressions are easily derived for general Newtonian binaries [Sec. II B]. In Sec. II B1 formulas for Keplerian orbits are reviewed, and general expressions for Keplerian waveforms are presented. The material in this section is concisely presented and might be of broad interest to those interested in leading-order waveforms valid for any eccentricity.

In Sections II C, II D, and II E, I specialize the nonlinear memory calculation to the cases of elliptical, hyperbolic, parabolic, and radial orbits. The primary results are explicit expressions for the nonlinear memory modes \( h_{lm}^{(mem)} \) and the corresponding polarizations \( h_{+,x} \). Aside from these explicit expressions, the following are some of the primary results of this analysis: In the case of inspiralling elliptical binaries, the nonlinear memory behaves similarly to the quasicircular case. The primary contributions come from the \( m = 0 \) modes, which have the scaling

\[
h_{l0}^{(mem), \text{ellip.}} \propto \frac{M}{R} \frac{M}{p} \mathcal{F}_{l0}(e_-, e_t),
\]

where \( p(t) \) is the semi-latus rectum of the ellipse and \( \mathcal{F}_{lm} \) is a hypergeometric function that depends on the eccentricity \( e_t(t) \) and its value \( e_- \) at some early time [see Appendix B for the exact expressions, or Eqs. (2.35) for the low-eccentricity limit]. Note that as in the quasicircular case, the nonlinear memory modifies the waveform at leading (Newtonian) order \( [M/p \sim O(x); \text{c.f. Eq.} (1.4a)] \). As discussed below, this result would hold even if we were to extend our analysis to orbits that undergo periastron advance. In the case of hyperbolic and parabolic orbits this scaling is very different:

\[
\Delta h_{lm}^{(mem), \text{hyperb.}} \propto \eta^2 \left( \frac{M}{p} \right)^{7/2} \mathcal{H}_{lm}(e_t),
\]

where \( \mathcal{H}_{lm} \) is a function of the (in this case, constant) eccentricity \( e_t \) which can be read off of Eqs. (2.51). Note that in this case all of the \((l, m)\) modes contribute to the nonlinear memory, which is a factor of \( \eta(M/p)^{5/2} \) smaller than in the elliptical case. (A similar scaling also holds in the case of radial orbits.)

It is easy to understand the reason for this difference in scalings. The nonlinear memory can be written as a time integral of the form

\[
h_{lm}^{(mem)} = \int_{-\infty}^{T_{RR}} dt h_{lm}^{(mem)(1)},
\]

where the time derivative \( h_{lm}^{(mem)(1)} \equiv dh_{lm}^{(mem)}/dt \) has the same leading-order scaling \( [\eta^2(M/p)^{5/2}] \) for all orbits. The difference in the scalings results from the timescale on which the integration is carried out. In the case of elliptical orbits one must integrate over the entire inspiral, which occurs on a radiation-reaction timescale \( T_{RR} \), so that

\[
\Delta h_{lm}^{(mem), \text{ellip.}} \sim h_{lm}^{(mem)(1)} T_{RR} \sim h_{lm}^{(mem)(1)} \frac{M}{\eta} \left( \frac{p}{M} \right)^4.
\]

In the hyperbolic case one integrates only over the much shorter orbital timescale \( T_{\text{orb}} \), so

\[
\Delta h_{lm}^{(mem), \text{hyperb.}} \sim h_{lm}^{(mem)(1)} T_{\text{orb}} \sim h_{lm}^{(mem)(1)} \frac{M}{\eta} \left( \frac{p}{M} \right)^{3/2}.
\]

Since

\[
\frac{T_{\text{orb}}}{T_{RR}} \sim \eta \left( \frac{M}{p} \right)^{5/2},
\]
we can see why the nonlinear memory in the hyperbolic/parabolic case enters at a much higher PN order than the elliptical/quasicircular case. Integrating over the much longer radiation-reaction time effectively allows the memory to “build-up” to a much larger value than one would naively expect from such a high-order PN effect.

One of the more important results from this study concerns the analysis of the dependence of the memory on the early-time history of the binary (Sec. III). For example, even a quasicircular binary has an eccentricity that grows in the past, eventually leading to a hyperbolic binary. To address this issue I computed the evolution of the dominant mode of the nonlinear memory waveform as a function of time for a nearly circular binary. This evolution was computed (i) assuming that the binary always remains quasicircular, and (ii) assuming that the binary’s eccentricity increases as time evolves into the past. A comparison of these two evolutions is shown in Figure 5 (where time is parameterized in terms of the changing eccentricity of the binary). Accounting for the evolving eccentricity makes a small (but non-negligible) correction to the memory. The eccentricity correction is small because even though eccentricity is increasing into the past, so is the orbital separation (or the semi-latus rectum). Since the integrand in Eq. (1.7) is weighted by a factor of \((M/p)^5\), its value drops off at larger separations; so late-time values (smaller \(p\), when \(e_i\) is also small) are weighted more heavily than the distant past (when the eccentricity is large, but \(M/p\) is very small).

Of course, it is also possible for the binary to experience a large memory jump in the distant past, for example due to its sudden formation in a capture process. Such a memory jump would in principle be observable, but only if one’s GW detector is operating when (in retarded time) that jump occurred. In general, effects of the early-time history of the binary are not observable in the memory signal if they occurred before the start of the observation. Hereditary effects are therefore only important over the observation time, and one need not worry about knowing the state of the binary prior to the start of the observation. This is illustrated with an explicit example in Sec. III and Figure 6.

Lastly, in Sec. IV I discuss how to use the waveforms for hyperbolic orbits to make rough estimates for the signal-to-noise ratio (SNR) of a memory signal from a gravitational-scattering event. Unlike the case of bound, inspiralling binaries (where the innermost stable circular orbit or the ringdown provides a natural high-frequency cutoff), the SNRs from hyperbolic binaries depend on a distance of closest approach (as well as an eccentricity parameter) and are dominated by the linear rather than the nonlinear memory. For future ground-based detectors, linear and nonlinear memory signals from the scattering of stellar-mass binaries within our galaxy are potentially detectable. Space-based detectors could see linear memory signals from the scattering of supermassive black holes (SMBHs), but detecting the nonlinear memory from these events will be more difficult. (This is in contrast to the nonlinear memory from merging SMBHs, which should be detectable to moderate redshifts [25–27, 63].)

Some useful results are presented in the Appendices. Appendix A derives general expressions for the spherical harmonic modes of the mass and current source multipole moments at Newtonian order. Appendix B shows how to express certain integrals over eccentricity in terms of hypergeometric functions. Appendix C discusses the role of wavelength averaging of the GW stress-energy tensor in the nonlinear memory calculation.

II. COMPUTING THE NONLINEAR MEMORY

We begin this section by first reviewing the decomposition of the GW polarizations in terms of multipole modes \(h_{lm}\) on a spin-weighted spherical harmonic basis (Sec. II A). The nonlinear memory pieces of these modes \([h_{lm}^{\text{mem}}]\) are related to integrals over the GW energy flux \(dE_{gw}/dt\). This is described in detail in [20], and the reader is directed there for a more detailed exposition. Since we are only interested in the leading-order memory, it is sufficient to express the energy flux in terms of the Newtonian-order non-memory modes \(h_{lm}^N\). Explicit expressions for these Newtonian-order modes are given and specialized to Keplerian orbits parameterized in terms of the true anomaly angle \(v\) (Sec. II B). These modes are then substituted into the expression for the nonlinear memory modes (Sec. II B 1). Sections II C, II D, and II E then specialize the calculation to the case of elliptical, hyperbolic, parabolic, and radial orbits, providing explicit expressions for the \(h_{lm}^{\text{mem}}\) modes and their corresponding + and \(\times\) polarizations.

A. General expressions for the waveform and nonlinear-memory modes

The GW polarizations can be decomposed as a sum over multipole modes via

\[
h_+ - ih_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_{lm}^{\text{mem}} Y_{lm}(\Theta, \Phi), \tag{2.1}
\]

where

\[
h_{lm}^{\text{mem}} = \frac{G}{\sqrt{2} R c^4} \left[ U_{lm}(T_R) - \frac{i}{6} V_{lm}(T_R) \right]. \tag{2.2}
\]

Here \(-2Y_{lm}(\Theta, \Phi)\) are the spin-weighted spherical harmonics, \((R, \Theta, \Phi)\) are the distance and angles that point from the source to the observer, and \(U_{lm}\) and \(V_{lm}\) are the spherical harmonic representations of the radiative mass and current multipole moments (see Sec. II A of [20] for details and notation). Constructing the GW polarizations requires explicit expressions for the \(-2Y_{lm}(\Theta, \Phi)\).
The general formula for the $-2Y^{lm}(\Theta, \Phi)$ can be found in Eqs. (2.13)–(2.14) of [20]. Here we will only need the following modes:

\[
\begin{align*}
-2Y^{20} &= \frac{3}{4} \sqrt{\frac{5}{6\pi}} s^2_0, \\
-2Y^{2\pm 2} &= \frac{1}{8} \sqrt{\frac{5}{\pi}} (1 \pm c_\Theta) e^{\pm 2\Phi}, \\
-2Y^{40} &= \frac{3}{8} \sqrt{\frac{5}{2\pi}} s^2_0 (7c_\Theta^2 - 1), \\
-2Y^{4\pm 2} &= \frac{3}{8} \sqrt{\frac{1}{\pi}} (1 \pm c_\Theta)^2 \left( 1 + 7c_\Theta + 7c_\Theta^2 \right) e^{\pm 2\Phi}, \\
-2Y^{4\pm 4} &= \frac{3}{16} \sqrt{\frac{7}{\pi}} s^2_0 (1 \pm c_\Theta)^2 e^{\pm 4\Phi},
\end{align*}
\]  

where we define \(c_\Theta \equiv \cos \Theta\) and \(s_\Theta \equiv \sin \Theta\).

At leading order, the radiative moments \(U_{lm}\) and \(V_{lm}\) reduce to the source moments \(I_{lm}\) and \(J_{lm}\). At higher PN orders the radiative moments are corrected by tail terms and other nonlinear couplings [see, e.g., Eqs. (2.21), (2.22), and (2.32) of [20]]. Since we are focused on only the leading-order memory contribution, we will ignore all of these higher-order terms except for the nonlinear memory contribution \(U^{(\text{mem})}\) itself. Note that there is no nonlinear memory contribution to \(V_{lm}\). (For our purposes, we can entirely ignore all current multipole moments.) This allows us to approximate the waveform modes as

\[h_{lm} \approx \frac{1}{\sqrt{2\pi}} I^{(l)}_{lm} + h^{(\text{mem})}_{lm}.\]

Here the nonlinear memory piece is given by [Eqs. (2.32) and (3.3) of [20]]

\[h^{(\text{mem})}_{lm} = \frac{16\pi}{R} \int \frac{l(l-2)!}{(l+2)!} \left[ \int dt \frac{dE_{\text{GW}}}{dt} Y^*_{lm}(\Omega) \right] I^{(l)}_{lm} + \frac{1}{16} \int_{-\infty}^{\infty} dt \int_0^{2\pi} dt \left[ h^{(l)}_{m',m''} \right],\]

where in the second line we have substituted the expansion for the energy flux \(\frac{dE_{\text{GW}}}{dt}\) in terms of the \(h_{lm}\) modes [Eq. (2.28) of [20]] and \(G_{lm}^{(l)l^*l+m'+m''}\) is an angular integral proportional to the product of three spin-weighted spherical harmonics [see Appendix A of [20]]. The angle brackets \(\langle \rangle\) mean to average over several wavelengths of the GW and arise from the averaging needed to construct a well-defined GW stress-energy tensor [16, 17] (see Appendix C for a discussion of the implications of this averaging).

Note that the nonlinear memory modes \(h^{(\text{mem})}_{lm}\) are themselves defined in terms of the full \(h_{lm}\) modes. But in practice, the “memory contribution to the memory” is negligible, so we only need substitute the non-memory modes into the right-hand-side of Eq. (2.5). Furthermore, since in our case we are only concerned with the leading-order memory, we need only substitute the leading-PN-order piece of the \(l = 2\) mode \(h_{2m}\) into the right-hand-side of Eq. (2.5).

Now we proceed to compute the angular integrals in Eq. (2.5) and explicitly express the \(h^{(\text{mem})}_{lm}\) in terms of the individual \(h_{2m}\) modes. We begin by noting that the angular integration implies certain selection rules on the maximum \(l\) for which the \(h^{(\text{mem})}_{lm}\) are nonzero [see Sec. III B of [20]]. Since our calculation is to leading-order, only the \(h_{2m}\) modes enter the right-hand-side of Eq. (2.5).

The selection rules then imply that only \(h_{2m}^{(\text{mem})}, h_{3m}^{(\text{mem})}\), and \(h_{4m}^{(\text{mem})}\) will be nonzero; all higher-\(l\) memory modes vanish (this was checked by explicit calculation).

Our results are further simplified by the fact that the nonmemory piece of the \(h_{lm}\) modes are approximated by

\[h^{(\text{mem})}_{lm} \equiv \frac{I^{(l)}_{lm}}{\sqrt{2\pi}}.\]

where the \(N\) emphasizes that our results are valid only at Newtonian order. Since \(I^{(l)}_{lm} = (-1)^m I_{l-m}\), the non-memory modes also satisfy \(h^{(\text{mem})}_{lm} = (-1)^m h^{(\text{mem})}_{l-m}\). We also note that since \(I_{lm} \propto Y^*_{lm}(\Theta, \Phi)\) [see Appendix A], and if we assume that the orbit lies in the \(x-y\) plane (so that \(\theta = \pi/2\)), then \(h^{(\text{mem})}_{2m} \propto Y^*_{\pm 1, \pm 1} \sin 2\theta = 0\). These simplifications imply

\[h^{(\text{mem})}_{20} = h^{(\text{mem})}_{20}, h^{(\text{mem})}_{2\pm 2} = h^{(\text{mem})}_{2\mp 2}, h^{(\text{mem})}_{2\pm 1} = h^{(\text{mem})}_{2\mp 1} = 0.\]

Defining \(h^{(\text{mem})}_{lm}(\Omega) = \frac{dI^{(l)}_{lm}}{dT_R}\), explicitly evaluating the angular integrals in Eq. (2.5) (with \(h_{lm} \rightarrow h^{(\text{mem})}_{lm}\) on the right-hand-side) then yields

\[h^{(\text{mem})}_{2\pm 1} = h^{(\text{mem})}_{3\pm 1} = h^{(\text{mem})}_{4\pm 1} = h^{(\text{mem})}_{4\pm 3} = 0,\]

\[h^{(\text{mem})}_{20} = \frac{R}{42} \sqrt{\frac{15}{2\pi}} \left( |h_{20}^{(N)|^2 - |h_{20}^{(N)|^2} \right),\]

\[h^{(\text{mem})}_{2\pm 2} = \frac{R}{21} \sqrt{15} \frac{3}{2\pi} \left( |h_{2\pm 2}^{(N)|^2 + |h_{2\pm 2}^{(N)|^2} \right),\]

\[h^{(\text{mem})}_{40} = \frac{R}{1260} \sqrt{\frac{5}{2\pi}} \left( |h_{40}^{(N)|^2 + |h_{40}^{(N)|^2 \right),\]

\[h^{(\text{mem})}_{4\pm 2} = \frac{R}{252} \sqrt{\frac{15}{2\pi} \left( h_{20}^{(N)|^2 + |h_{20}^{(N)|^2 \right),\]

\[h^{(\text{mem})}_{4\pm 4} = \frac{R}{504} \sqrt{\frac{14}{2\pi} \left( h_{20}^{(N)|^2 + |h_{20}^{(N)|^2 \right).\]

B. Explicit expressions for the \(h^{(N)}_{lm}\) and \(h^{(\text{mem})}_{lm}(\Omega)\) modes for Newtonian binaries

Now we write out explicit expressions for the \(h^{(N)}_{lm}\) modes. A derivation of the source mass and current mul-
tipole moments for Newtonian binaries is given in Appendix A. The result for the mass quadrupole moment is found in Eq. (A15a). The $l = 2$ moments evaluated on an equatorial orbit are

\begin{align}
I_{20}^N &= -4\sqrt{\frac{\pi}{15}} \eta Mr(t)^2, \quad (2.9a) \\
I_{2\pm 2}^N &= 2\sqrt{\frac{2\pi}{5}} \eta Mr(t)^2 e^{\mp 2i\varphi(t)}, \quad (2.9b)
\end{align}

where $M = m_1 + m_2$, $\eta = m_1 m_2 / M^2$, $r(t)$ is the relative orbital separation, and $\varphi(t)$ is the relative orbital phase.

Next we compute time derivatives of these mass moments. We eliminate second derivatives using the Newtonian equations of motion:

\begin{align}
\dot{r} &= r \varphi^2 - \frac{M}{r^2}, \quad (2.10a) \\
\dot{\varphi} &= -\frac{2r\dot{r}}{r}. \quad (2.10b)
\end{align}

The resulting derivatives of the moments are

\begin{align}
I_{20}^{(1)} &= -8\sqrt{\frac{\pi}{15}} \eta Mr \dot{r}, \quad (2.11a) \\
I_{2\pm 2}^{(1)} &= 4\sqrt{\frac{2\pi}{5}} \eta Mr (\dot{r} \mp ir\dot{\varphi}) e^{\mp 2i\varphi}, \quad (2.11b) \\
I_{20}^{(2)} &= -8\sqrt{\frac{\pi}{15}} \eta M \left( \dot{r}^2 + r^2 \dot{\varphi}^2 - \frac{M}{r} \right), \quad (2.11c) \\
I_{2\pm 2}^{(2)} &= 4\sqrt{\frac{2\pi}{5}} \eta Me^{\mp 2i\varphi} \left( \dot{r}^2 - r^2 \dot{\varphi}^2 - \frac{M}{r} \mp 2ir\ddot{r} \right), \quad (2.11d) \\
I_{20}^{(3)} &= 8\sqrt{\frac{\pi}{15}} \left( \frac{M}{r} \right)^2 \dot{r}, \quad (2.11e) \\
I_{2\pm 2}^{(3)} &= -4\sqrt{\frac{2\pi}{5}} \eta \left( \frac{M}{r} \right)^2 e^{\mp 2i\varphi} (\dot{r} \mp 4ir\ddot{r}). \quad (2.11f)
\end{align}

These expressions are valid for a general orbit that satisfies the Newtonian equations of motion (2.10). The explicit Newtonian-order polarizations are found from substituting Eqs. (2.6) and (2.3) into Eq. (2.1) and summing only over the $l = 2$ modes.\(^6\)

\begin{align}
\dot{h}_N &= \frac{\eta M}{R} \left\{ (1 + c_{\Theta}) \left[ (\dot{r}^2 - r^2 \dot{\varphi}^2 - \frac{M}{r}) \cos 2(\varphi - \Phi) \\
&- 2\dot{r}r\dot{\varphi} \sin 2(\varphi - \Phi) \right] - 2s_{\Theta}^2 \left( \dot{r}^2 + r^2 \dot{\varphi}^2 - \frac{M}{r} \right) \right\}, \quad (2.12a)
\end{align}

Substituting $\dot{h}_{2m}^N = I_{2m}^{(3)} / (R\sqrt{2})$ and Eqs. (2.11) into Eqs. (2.8) then gives

\begin{align}
\dot{h}_{20}^{(\text{mem})(1)} &= \frac{16}{21R} \sqrt{\frac{2\pi}{15}} \eta^2 \left\{ \left( \frac{M}{r} \right)^4 (\dot{r}^2 + 24r^2 \dot{\varphi}^2) \right\}, \quad (2.13a) \\
\dot{h}_{2\pm 2}^{(\text{mem})(1)} &= -\frac{16}{21R} \sqrt{\frac{2\pi}{5}} \eta^2 \left\{ \left( \frac{M}{r} \right)^4 \dot{r} e^{\mp 2i\varphi} (\dot{r} \mp 4ir\dot{\varphi}) \right\}, \quad (2.13b) \\
\dot{h}_{20}^{(\text{mem})(1)} &= \frac{2}{315R} \sqrt{\frac{2\pi}{5}} \eta^2 \left\{ \left( \frac{M}{r} \right)^4 (3\dot{r}^2 + 16r^2 \dot{\varphi}^2) \right\}, \quad (2.13c) \\
\dot{h}_{2\pm 2}^{(\text{mem})(1)} &= -\frac{4}{315R} \sqrt{\pi} \eta^2 \left\{ \left( \frac{M}{r} \right)^4 \dot{r} e^{\mp 2i\varphi} (\dot{r} \mp 4ir\dot{\varphi}) \right\}, \quad (2.13d) \\
\dot{h}_{2\pm 4}^{(\text{mem})(1)} &= \frac{2}{45R} \sqrt{\pi} \eta^2 \left\{ \left( \frac{M}{r} \right)^4 e^{4i\varphi} (\dot{r} \mp 4ir\dot{\varphi}) \right\}, \quad (2.13e)
\end{align}

for the nonzero memory modes.

1. Formulas for Keplerian orbits

Now we wish to specialize these expressions to Keplerian orbits. The time evolution of the orbital separation $r$ and phase $\varphi$ is parameterized in terms of the true anomaly $v$,

\begin{align}
\dot{r} &= \frac{p}{1 + e\cos v}, \quad (2.14a) \\
v &= \varphi - \omega, \quad (2.14b) \\
\dot{\varphi} &= \dot{v} = \sqrt{\frac{pM}{r^3}}. \quad (2.14c)
\end{align}

where, for planar orbits, only three orbital elements are needed to parameterize the orbit: the semi-latus rectum $p$, the eccentricity\(^7\) $e_\ell$, and the argument of pericenter $\omega$.

---

\(^6\) These formula agree with Eqs. (6) of [50] if we choose $\Phi = \pm \pi / 2$ and change the overall sign on both polarizations. This difference arises from the choice of the polarization tensors (see Sec. II A of [20] for the convention used here).

\(^7\) Throughout this article we denote the eccentricity by $e_\ell$ to avoid confusion with the mathematical constant $e$ and to emphasize that the eccentricity can evolve with time. This choice is not meant to suggest an identification of our eccentricity parameter with the “time eccentricity” used in the quasi-Keplerian formalism that describes PN elliptical orbits (e.g., [50] and references therein). In that formalism three eccentricity parameters, $e_t$, $e_r$, and $e_\varphi$, are introduced; but at Newtonian order these three eccentricities are equivalent and we can identify either of them with the $e_\ell$ used here.
FIG. 1. (color online). Notation and parameters for describing elliptical and hyperbolic orbits. The left figure shows a particle moving on an ellipse at a distance $r$ from the focus and making an angle $v$ (the true anomaly) with respect to the pericenter. The ellipse has an eccentricity $e$ and a size described by either its semimajor axis $a$, focus-pericenter distance $r_p$, or semi-latus rectum $p$. Also indicated is the eccentric anomaly $u$, which is the angle from the pericenter to the projection of the particle’s position on the circle that circumscribes the ellipse. The right figure shows a particle moving on a hyperbolic orbit, where we additionally indicate the asymptotes of the hyperbola (dashed lines), the particle’s impact parameter $b$, and the scattering angle $\Theta$. The argument of pericenter $\omega$. Figure 1 illustrates the meaning of the various orbital parameters introduced throughout this article. Time is determined by integrating Eq. (2.14c):

$$t - t_0 = \sqrt{\frac{p^3}{M}} \int \frac{dv}{(1 + e \cos v)^2}. \quad (2.15)$$

The following derivatives follow from Eqs. (2.14):

$$\dot{r} = e \sqrt{\frac{M}{p}} \sin v, \quad (2.16a)$$

$$\ddot{r} = \frac{M}{p^2} (p - r), \quad (2.16b)$$

$$\dot{\varphi} = -\frac{2M}{r^3} e \sin v. \quad (2.16c)$$

Note also that the instantaneous orbital velocity is given by

$$V^2 = \dot{r}^2 + r^2 \dot{\varphi}^2 = \frac{M}{r} (1 + e^2 + 2e \cos v), \quad (2.17)$$

the orbital energy per reduced mass is

$$\frac{E}{\mu} = \frac{V^2}{2} - \frac{M}{r} = -\frac{1}{2} \frac{M}{p} (1 - e^2), \quad (2.18)$$

and the semi-latus rectum is related to the semi-major axis $a$ by

$$p = a |1 - e^2| \quad (2.19)$$

and to the pericenter distance $r_p$ by

$$p = r_p (1 + e_t). \quad (2.20)$$

Applying Eqs. (2.14) and (2.16), Eqs. (2.11) simplify to

$$I_{20}^{(2)} = -8 \sqrt{\frac{\pi}{15}} \frac{M^2}{p} e_t (e_t + \cos v), \quad (2.21a)$$

$$I_{2 \pm 2}^{(2)} = -4 \sqrt{\frac{2\pi}{5}} \frac{M^2}{p} e^{\mp 2i\varphi} \times \left[1 - e_t^2 + (1 + e_t \cos v)(1 + 2e_t e^{\pm iv})\right], \quad (2.21b)$$

$$I_{20}^{(3)} = 8 \sqrt{\frac{\pi}{15}} \left(\frac{M}{p}\right)^{5/2} e_t \sin v (1 + e_t \cos v)^2, \quad (2.22a)$$

$$I_{2 \pm 2}^{(3)} = -4 \sqrt{\frac{2\pi}{5}} \frac{M}{p} e^{\mp 2i\varphi} (1 + e_t \cos v)^2 \times [e_t \sin v \mp 4i(1 + e_t \cos v)]. \quad (2.22b)$$

The modes that appear in the waveform polarizations [Eqs. (2.21)] are plotted in Figure 2 for hyperbolic and elliptic orbits. To produce these plots the differential
equation for the true anomaly \( [(2.14c)] \) was solved numerically.\(^8\) Note that the hyperbolic waveforms show a linear memory in the \((2, \pm 2)\) mode. This is discussed in more detail in Sec. II.D.

To compute the polarizations in terms of the true anomaly, one substitutes the following expressions into Eqs. (2.12):

\[
\dot{r} = \frac{M}{p} (1 + e_t \cos v) e_t \sin v, \tag{2.23a}
\]

\[
\dot{r}^2 + r^2 \dot{\phi}^2 - \frac{M}{r} = \frac{M}{p} e_t (e_t \cos v + v), \tag{2.23b}
\]

\[
\dot{r}^2 - r^2 \dot{\phi}^2 - \frac{M}{r} = -\frac{M}{p} (2 + 3e_t \cos v + e_t^2 \cos 2v). \tag{2.23c}
\]

Likewise, we can use the above expressions for \( r, \dot{r}, \) and \( \dot{\phi} \) to write Eqs. (2.13) in terms of \( e_t, p, \) and \( v \):

\[
h_{20}^{(\text{mem})} = \frac{32}{21R} \sqrt{\frac{\pi}{30}} \eta^2 \left\langle \left( \frac{M}{p} \right)^5 (1 + e_t \cos v)^4 \times (24 + e_t^2 + 48e_t \cos v + 23e_t^2 \cos^2 v) \right\rangle, \tag{2.24a}
\]

\[
h_{2\pm 2}^{(\text{mem})} = -\frac{16}{21R} \sqrt{\frac{\pi}{5}} \eta^2 \left\langle \left( \frac{M}{p} \right)^5 e^{\mp 2i\phi} e_t \sin v \times (1 + e_t \cos v)^4 \right\rangle, \tag{2.24b}
\]

\[
h_{40}^{(\text{mem})} = \frac{2}{315R} \sqrt{\frac{2\pi}{5}} \eta^2 \left\langle \left( \frac{M}{p} \right)^5 (1 + e_t \cos v)^4 \times (16 + 3e_t^2 + 32e_t \cos v + 13e_t^2 \cos^2 v) \right\rangle, \tag{2.24c}
\]

\[
h_{4\pm 2}^{(\text{mem})} = -\frac{4\sqrt{\pi}}{315R} \eta^2 \left\langle \left( \frac{M}{p} \right)^5 e^{\mp 4i\phi} e_t \sin v \times (1 + e_t \cos v)^4 \right\rangle, \tag{2.24d}
\]

\[
h_{4\pm 4}^{(\text{mem})} = -\frac{2}{45R} \sqrt{\frac{\pi}{7}} \eta^2 \left\langle \left( \frac{M}{p} \right)^5 e^{\mp 4i\phi} (1 + e_t \cos v)^4 \times [16 - e_t^2 + 32e_t \cos v + 17e_t^2 \cos^2 v \pm 8ie_t \sin v (1 + e_t \cos v)] \right\rangle, \tag{2.24e}
\]

where the angle brackets again arise from the averaging inherent in the definition of the GW energy flux.

---

\(^8\) These waveforms could also be produced by using a representation in terms of the eccentric anomaly \( u \). In this case the problem of solving for the time involves finding a root of the so-called Kepler equation (rather than solving a differential equation), but a separate set of equations must be used to treat elliptic, hyperbolic, and parabolic orbits (see, e.g., Ch. 6 of [64]).
C. Elliptical orbits

For bound, eccentric orbits \(0 \leq e_t < 1\) the wavelength averaging in Eqs. (2.24) is accomplished by explicitly averaging over an orbital period \(P_{\text{orb}}\). For any function \(F(t)\) this orbit-averaging is defined by

\[
\langle F(t) \rangle = \frac{1}{P_{\text{orb}}} \int_0^{P_{\text{orb}}} dt F(t) = \frac{(1 - e_t^2)^{3/2}}{2\pi} \int_0^{2\pi} dv \frac{F(v)}{(1 + e_t \cos v)^2},
\]

where

\[
P_{\text{orb}} = \frac{2\pi}{(1 - e_t^2)^{3/2}} \sqrt{\frac{p^3}{M}}
\]

follows from Eq. (2.15). Averaging Eqs. (2.24) then yields

\[
h_{20}^{\text{(mem)(1)}}(t) = \frac{256}{7R} \sqrt{\frac{\pi}{30}} \eta \left( \frac{M}{p} \right)^5 (1 - e_t^2)^{3/2} \times \left( 1 + 145 \frac{e_t^2}{48} + 73 \frac{\eta}{192} e_t^4 \right),
\]

\[
h_{2 \pm 2}^{\text{(mem)(1)}}(t) = \frac{52}{21R} \sqrt{\frac{\pi}{5}} \eta \left( \frac{M}{p} \right)^5 e_t^2 (1 - e_t^2)^{3/2} \times \left( 1 + \frac{2}{13} e_t^2 \right) e^{\pm 2i\omega},
\]

\[
h_{40}^{\text{(mem)(1)}}(t) = \frac{64}{315R} \sqrt{\frac{\pi}{10}} \eta \left( \frac{M}{p} \right)^5 (1 - e_t^2)^{3/2} \times \left( 1 + \frac{99}{32} e_t^2 + \frac{51}{128} e_t^4 \right),
\]

\[
h_{4 \pm 2}^{\text{(mem)(1)}}(t) = \frac{13}{315R} \sqrt{\pi} \eta \left( \frac{M}{p} \right)^5 e_t^2 (1 - e_t^2)^{3/2} \times \left( 1 + \frac{2}{13} e_t^2 \right) e^{\pm 2i\omega},
\]

\[
h_{4 \pm 4}^{\text{(mem)(1)}}(t) = -\frac{5}{72R} \sqrt{\frac{\pi}{7}} \eta \left( \frac{M}{p} \right)^5 e_t^4 (1 - e_t^2)^{3/2} e^{\pm 4i\omega}.
\]

This averaging has essentially removed the high frequency (\(\sim 1/P_{\text{orb}}\)) structure from the memory waveform. Appendix C explicitly shows that this averaging procedure has only a very small effect on the memory.

Next we need to compute the time integrals of the above expressions. In the circular case treated in [20], this was accomplished by changing variables to \(x = (M\omega)^{2/3}\). In the eccentric case, both the eccentricity \(e_t\) and the semi-latus rectum \(p\) vary with time, but \(p\) is easily expressed in terms of \(e_t\). We therefore change variables from time \(t\) to eccentricity, and integrate from some early-time value of the eccentricity \(e_-\) to its value at some later time \(e_+ = e_t(t)\):

\[
h_{lm}^{\text{(mem)(1)}}(t) = \int_{e_-}^{e_+} \frac{h_{lm}^{\text{(mem)(1)}}(\tilde{t})}{de_t/\tilde{t}} \, d\tilde{t}.
\]

In computing the above integral we will need to make use of the following equations for the evolution of \(p\) and \(e_t\) [these are easily derived from the results of [50, 65], along with the Newtonian-order relations in Eqs. (2.26) and (2.19)]:

\[
\frac{dp}{dt} = -\frac{8}{5} \eta \left( \frac{M}{p} \right)^3 (1 - e_t^2)^{3/2} (8 + 7 e_t^2),
\]

\[
\frac{de_t}{dt} = -\frac{\eta}{15M} \left( \frac{M}{p} \right)^4 e_t (1 - e_t^2)^{3/2} (304 + 121 e_t^2).
\]

Dividing the first equation by the second yields

\[
\frac{dp}{de_t} = 24 \frac{p}{e_t} (8 + 7 e_t^2),
\]

which is easily solved to give

\[
p(e_t) = \frac{p_0}{C_0} e_t^{12/19} (304 + 121 e_t^2)^{270/2299},
\]

where

\[
C_0 = e_0^{12/19} (304 + 121 e_0^2)^{270/2299},
\]

and \(p_0\) is the value of \(p\) at some arbitrary reference time when \(e_t = e_0\). Since both \(p\) and \(e_t\) evolve with time, this relation allows us to eliminate the time-dependent \(p(t)\) terms in Eq. (2.28) and instead express the integrand entirely in terms of the evolving eccentricity \(e_t(t)\) (which is our new integration variable) and the constants \(p_0\) and \(e_0\). The integrand in Eq. (2.28) for the relevant modes then becomes:

\[
\frac{dh_{20}^{\text{(mem)}}}{de_t} = -\frac{2}{7} \frac{\sqrt{\frac{10\pi \eta M^2}{3}}}{R_{p_0}} \frac{C_0}{e_t^{31/19}} (192 + 580 e_t^2 + 73 e_t^4)\]

\[
\frac{dh_{2 \pm 2}^{\text{(mem)}}}{de_t} = -\frac{4\sqrt{\frac{\pi \eta M^2}{7}}}{R_{p_0}} \frac{C_0}{e_t^{31/19}} (13 + 2 e_t^2) e^{\pm 2i\omega},
\]

\[
\frac{dh_{40}^{\text{(mem)}}}{de_t} = -\frac{1}{42} \frac{\sqrt{\pi \eta M^2}}{R_{p_0}} \frac{C_0}{e_t^{31/19}} (128 + 396 e_t^2 + 51 e_t^4)\]

\[
\frac{dh_{4 \pm 2}^{\text{(mem)}}}{de_t} = -\frac{\sqrt{\pi \eta M^2}}{21} \frac{C_0}{R_{p_0}} (304 + 121 e_t^2)^{3169/2299} e^{\pm 2i\omega}.
\]
\[
\frac{dh_{4\pm 4}^{(\text{mem})}}{de_t} = \frac{25}{24\sqrt{\frac{3\pi}{7}} R_p \eta M^2} \frac{C_0 e_t^{45/19}}{304 + 121 e_t^2} 3169/2299 e^{\mp 4i\pi}. \tag{2.33e}
\]

These expressions can be analytically integrated and the result expressed in terms of hypergeometric functions. Details of this are given in Appendix B, where we show that all of the above modes can be expressed in the form
\[
h_{lm}^{(\text{mem})} = A_{lm} C_0 (e_0) e^{\mp im\pi} \left[ F_{lm}(e_t) - F_{lm}(e_-) \right], \tag{2.34}
\]
where \(A_{lm}\) are constants that can be read off of Eqs. (B1), and \(F_{lm}\) is a sum of hypergeometric functions given in Eq. (B3). Note that in computing the integral over \(e_t\) we have chosen the integration constant such that the memory vanishes at an early-time eccentricity value of \(e_-\). We have also ignored periastron precession (choosing \(\omega\) to be fixed), but the relaxation of this assumption will be discussed below.

In the limit of small \(e_t\), we can easily evaluate the integrals of Eqs. (2.33):
\[
h_{20}^{(\text{mem})} = \frac{2}{7} \sqrt{\frac{10\pi}{3}} R_p \eta M^2 \left[ \left( \frac{e_0}{e_t} \right)^{12/19} - \left( \frac{e_0}{e_-} \right)^{12/19} \right], \tag{2.35a}
\]
\[
h_{2\pm 2}^{(\text{mem})} = -\frac{\sqrt{5\pi}}{56} \frac{\eta M^2}{R_p} e^{\mp 2i\pi} e_0^2 \left( \frac{e_t}{e_0} \right)^{26/19} - \left( \frac{e_-}{e_0} \right)^{26/19}, \tag{2.35b}
\]
\[
h_{40}^{(\text{mem})} = \frac{1}{63} \sqrt{\frac{10\pi}{3}} R_p \eta M^2 \left[ \left( \frac{e_0}{e_t} \right)^{12/19} - \left( \frac{e_0}{e_-} \right)^{12/19} \right], \tag{2.35c}
\]
\[
h_{4\pm 2}^{(\text{mem})} = -\frac{\sqrt{5\pi}}{672} \frac{\eta M^2}{R_p} e^{\mp 2i\pi} e_0^2 \left( \frac{e_t}{e_0} \right)^{26/19} - \left( \frac{e_-}{e_0} \right)^{26/19}, \tag{2.35d}
\]
\[
h_{4\pm 4}^{(\text{mem})} = \frac{25\sqrt{\pi}}{172032} R_p \eta M^2 e^{\mp 4i\pi} e_0^4 \left[ \left( \frac{e_t}{e_0} \right)^{64/19} - \left( \frac{e_-}{e_0} \right)^{64/19} \right], \tag{2.35e}
\]
where we have used \(C_0 \approx e_0^{12/19} 304^{7870/2299}\), and in the second line of each equation we have reexpressed the result in terms of the time-dependent \(p(t) = p_0(e_t/e_0)^{12/19}\) using Eq. (2.31).

In the zero-eccentricity limit, \(h_{2\pm 2}^{(\text{mem})}, h_{4\pm 2}^{(\text{mem})}\), and \(h_{4\pm 4}^{(\text{mem})}\) vanish and \(h_{20}^{(\text{mem})}\) and \(h_{40}^{(\text{mem})}\) reduce to the circular-orbit values found in Eqs. (4.1)–(4.3) of [20]. In the eccentric case, the \(m \neq 0\) modes do not contribute to the memory for several reasons: first, they are suppressed by factors of \(e_0^m\) or \(e_0^{-m}\), and their numerical coefficients tend to be much smaller than the coefficient of \(h_{20}^{(\text{mem})}\). More importantly, the factor of \(e^{\mp im\pi}\) is not actually constant as we have assumed so far. Post-Newtonian corrections result in periastron precession, which causes \(\omega\) to vary with time [or with changing \(e_t(t)\)]. For example, the rate of periastron advance is
\[
\dot{\omega} = \frac{2\pi}{T_{\text{orb}}} = 3 \frac{M}{p} \left( 1 - e_t^2 \right)^{3/2}, \tag{2.36}
\]
for \(k = 3M/p\) at leading PN order. Using Eqs. (2.29b) and (2.31), the pericenter angle can be obtained as a function of eccentricity by solving
\[
\frac{d\bar{\omega}}{de_t} = -\frac{45}{\eta} \left( p_0/M \right)^{3/2} C_0^{-3/2} e_t^{-1/19} \frac{(304 + 121 e_t^2)^{994/2299}}{192(1 + e_t)^2}. \tag{2.37}
\]
For arbitrary (bound) eccentricities, this equation can be integrated using Eq. (B2). For small eccentricity this equation has the solution
\[
\bar{\omega} = \bar{\omega}_0 + \frac{5}{32}\eta \left( p_0/M \right)^{3/2} \left( 1 - \frac{e_t}{e_0} \right)^{18/19}, \tag{2.38}
\]
where \(\bar{\omega}_0\) is the value of \(\bar{\omega}\) when \(e_t = e_0\). As the eccentricity varies from \(e_0\) to 0, the pericenter angle changes by \(\Delta \bar{\omega} \sim (1/\eta)(p_0/M)^{3/2}\); this leads to oscillations in Eqs. (2.33) that, upon integration, cause the \(m \neq 0\) terms to be further suppressed. Hence, as in the quasicircular case, only the \(m = 0\) terms contribute to a secularly increasing memory effect.

In the left plot of Figure 3 we plot the \(h_{20}\) and \(h_{40}\) modes as a function of \(e_t\). This is obtained from both the full analytic solution for the mode evolution [Eq. (2.34), solid lines] and the low-eccentricity limit [Eqs. (2.35), dashed lines], choosing \(e_- = 1\) in both cases. Note that the \(h_{40}\) mode is much smaller than the \(h_{20}\) mode.

It is also convenient to express the above results in terms of the pericenter distance \(r_p\) rather than \(p\). The time evolution of \(r_p\) is found from differentiating Eq. (2.20) and using Eqs. (2.29):
\[
\frac{dr_p}{dt} = -\frac{\eta}{15} \left( \frac{M}{r_p} \right)^3 (1 - e_t)^{3/2} \left( 192 - 112e_t + 168e_t^2 + 47e_t^3 \right). \tag{2.39}
\]
The evolution with eccentricity \(r_p(e_t)\) is easily found from Eq. (2.31),
\[
r_p = \frac{r_0}{C_0} (1 + e_t) \left( 304 + 121 e_t^2 \right)^{870/2299}. \tag{2.40a}
\]
FIG. 3. (color online) The left plot shows the \( h_{20} \) and \( h_{40} \) memory modes as a function of eccentricity \( e_t \) for two values of the reference eccentricity \( e_0 \) and for \( e_- = 1 \). The dashed lines show the small-eccentricity approximation [Eqs. (2.35)]. The right plot shows the evolution of the \( h_{20}^{(\text{mem})} \) mode for various values of the indicated reference eccentricity \( e_0 \) (every curve passes through the point \( e_t = e_0 \) when \( r_p = 20M \)). The integration is terminated at the last-stable-orbit (at which point the curves flatten); in a real merger the memory would continue to evolve past this point, eventually saturating at a different value.

\[
C_0' = \frac{e_0^{12/19}}{(1 + e_0)} (304 + 121e_0^2)^{870/2299}. 
\tag{2.40b}
\]

This allows Eqs. (2.27), (2.33), and (2.35) to be expressed in terms of \( r_p \) or \( r_0 \).

The right plot of Figure 3 attempts to further illustrate the dependence of the \( h_{20}^{(\text{mem})} \) memory mode for different eccentricities. In place of a time variable \( t \), we can parameterize the temporal evolution in terms of the eccentricity \( e_t \). This is because, at Newtonian order in the conservative dynamics, an inspiralling eccentric binary passes through every value of \( e_t \in (0, 1) \) at some point in its evolution (provided we neglect the details of the binary’s formation or its interactions with the external universe). So to distinguish one eccentric binary from another, we need to specify the value of the eccentricity at some fiducial orbital separation. The different curves in the right-plot of Figure 3 are parameterized by the value of the eccentricity \( e_t = e_0 \) when the binary passes through a pericenter distance of \( r_p = 20M \). The curves are obtained from the analytic solution for the \((0, 0)\) mode in Eq. (2.34), choosing \( e_- = 1 \). The \((0, 0)\) mode is allowed to grow until the last-stable-orbit (LSO) is reached, corresponding to the condition \( p \equiv r_p(1 + e_t) = 6 + 2e_t \) [60] [the critical value of \( e_t = e_\text{LSO} \) is determined by combining this condition with Eq. (2.40)]. This plot shows that the more eccentric binaries reach the LSO while they are still mildly eccentric and at slightly smaller values of the memory. (However, GWs radiated during the merger and ringdown cause the memory to grow past its LSO value.)

The polarizations for the nonlinear memory waves for bound, eccentric orbits can be simply computed by summing the \( m = 0 \) modes in Eq. (2.1):

\[
h_{+}^{(\text{mem})} = \frac{1}{8} \sqrt{\frac{30}{\pi}} s_0^2 \left[ h_{20}^{(\text{mem})} + \frac{\sqrt{3}}{2} h_{40}^{(\text{mem})} (7e_0^2 - 1) \right], 
\tag{2.41a}
\]

\[
h_{x}^{(\text{mem})} = 0. 
\tag{2.41b}
\]

For arbitrary eccentricities, Eq. (2.34) for the \( l = 2 \) and \( l = 4 \) modes must be substituted into the above equation. In the small-eccentricity case Eqs. (2.35) yield

\[
h_{+}^{(\text{mem})} = \frac{\eta}{48 R_p} s_0^2 (17 + c_3^2) \left[ \left( \frac{e_0}{e_t} \right)^{12/19} - \left( \frac{e_0}{e_-} \right)^{12/19} \right], 
\tag{2.42}
\]

Note that in the circular limit these agree with Eqs. (4.4)–(4.6a) of [20].

\[\text{D. Hyperbolic and parabolic orbits}\]

To treat the case of hyperbolic and parabolic orbits, we ignore the possibility of periastron advance and we fix the periastron direction to lie along the +\( x \) axis. In this case the reduced mass particle swings around the origin in a counter-clockwise sense, entering at very early times along the asymptote at \( \phi = \Phi_- = \pi - \arccos(-1/e) \), and exiting at very late times along the asymptote at \( \phi = \Phi_+ = \arccos(-1/e) \) (see Figure 1). The corresponding scattering angle \( \Theta_s \) is given by

\[
\Theta_s = 2\arccos(-1/e_t) - \pi. 
\tag{2.43}
\]
For hyperbolic orbits it is also useful to define two additional parameters that can be used in place of \( p \) or \( e_t \). The asymptotic velocity is

\[
V_\infty^2 = \frac{2E}{\mu} = \frac{M}{p}(e_t^2 - 1).
\]  \hfill (2.44)

The impact parameter \( b \), defined to be the perpendicular distance from the center of mass \( M \) to the ingoing or outgoing asymptote of the hyperbola, is found to be

\[
b = \frac{p}{\sqrt{e_t^2 - 1}}. \]  \hfill (2.45)

The equations below can alternatively be expressed in terms of \( V_\infty \) or \( b \) using the above relations.

It is important to note that the waveforms from hyperbolic orbits already contain a linear memory [1]. For example, consider Eqs. (2.21a) and (2.21b) for \( e_t > 1 \) and \( \varphi = v \) varying between \( v_- \) and \( v_+ \). This difference in the orbital phase angle does not affect the \( I_{2m}^{(2)} \) mode (since \( \cos v \) is even), but it does affect the imaginary part of \( I_{2m}^{(2)} \) (which is odd), leading to a memory in that mode (see Figure 2). More explicitly, the linear memory jump between late and early times for a hyperbolic orbit is found from the difference \( I_{2m}^{(2)}(v_+) - I_{2m}^{(2)}(v_-) \), yielding [see also Eqs. (10) of [63]]

\[
\Delta h_{20}^{(\text{lin. mem})} = 0, \]  \hfill (2.46a)

\[
\Delta h_{2\pm 2}^{(\text{lin. mem})} = \pm i16 \sqrt{\frac{\pi}{5}} \sqrt{\frac{M}{p}} \frac{M}{e_t} \frac{(e_t^2 - 1)^{3/2}}{e_t^2}. \]  \hfill (2.46b)

The corresponding memory jump in the polarizations is

\[
\Delta h_{+}^{(\text{lin. mem})} = -4i \frac{M}{p} \frac{M}{\sqrt{e_t^2}} \frac{(e_t^2 - 1)^{3/2}}{e_t^2} \frac{1 + e_t^2}{\cos 2\Phi}, \]  \hfill (2.47a)

\[
\Delta h_{\times}^{(\text{lin. mem})} = -\frac{8\eta}{\sqrt{\pi}} \sqrt{\frac{2}{5}} \sqrt{\frac{e_t^2}{R}} \frac{M}{p} \frac{(e_t^2 - 1)^{3/2}}{e_t^2} \frac{1}{c_\phi} \cos 2\Phi. \]  \hfill (2.47b)

Note that for large \( e_t \),

\[
\frac{M}{p} \frac{(e_t^2 - 1)^{3/2}}{e_t^2} \approx \frac{M}{b}, \]  \hfill (2.48)

and Eqs. (2.47) agree with Eqs. (15) of [19]. Note also that for parabolic orbits \((e_t = 1)\), the linear memory vanishes. This is because the asymptotic incoming and outgoing directions of the orbit are now the same \((v_- = -\pi, v_+ = +\pi)\).

To compute the nonlinear memory we proceed from Eqs. (2.24). Since unbound orbits are no longer periodic, there is no averaging over an orbital period. Instead we directly perform the time integrals over Eq. (2.24), changing variables to the true anomaly using Eq. (2.14c):

\[
h_{lm}^{(\text{mem})}(v) = \int_{v_-}^{v_+} \frac{h_{lm}^{(\text{mem})(1)}}{v} \, dv. \]  \hfill (2.49)

The integrand is a sum over powers of sines and cosines or their products, and is easily evaluated for any limit of integration. For simplicity (and because periastron passage happens relatively quickly for hyperbolic orbits), we will focus on computing only the overall memory jump \( \Delta h_{lm}^{(\text{mem})} \), rather than the evolution of the memory with time:

\[
\Delta h_{lm}^{(\text{mem})} = \int_{v_-}^{v_+} \frac{h_{lm}^{(\text{mem})(1)}}{v} \, dv. \]  \hfill (2.50)

The resulting memory modes for any \( e_t \geq 1 \) are

\[
\Delta h_{20}^{(\text{mem})} = \frac{8}{63} \sqrt{\frac{\pi}{30}} \eta^2 \frac{M}{R} \frac{M}{p} \frac{1}{e_t^2} \left[ 3(73e_t^4 + 580e_t^2 + 192)(\pi - \arccos e_t^{-1}) + (1333e_t^2 + 1202) \sqrt{e_t^2 - 1} \right], \]  \hfill (2.51a)

\[
\Delta h_{2\pm 2}^{(\text{mem})} = \frac{8}{63} \sqrt{\frac{\pi}{30}} \eta^2 \frac{M}{R} \frac{M}{p} \frac{1}{e_t^2} \left[ 3e_t^2(2e_t^2 + 13)(\pi - \arccos e_t^{-1}) + (34e_t^2 + 13 - 2/e_t^2) \sqrt{e_t^2 - 1} \right], \]  \hfill (2.51b)

\[
\Delta h_{40}^{(\text{mem})} = \frac{1}{945} \sqrt{\frac{\pi}{10}} \eta^2 \frac{M}{R} \frac{M}{p} \frac{1}{e_t^2} \left[ 3(51e_t^4 + 396e_t^2 + 128)(\pi - \arccos e_t^{-1}) + (919e_t^2 + 806) \sqrt{e_t^2 - 1} \right], \]  \hfill (2.51c)

\[
\Delta h_{4\pm 2}^{(\text{mem})} = \frac{2}{945} \sqrt{\frac{\pi}{10}} \eta^2 \frac{M}{R} \frac{M}{p} \frac{1}{e_t^2} \left[ 3e_t^4(2e_t^2 + 13)(\pi - \arccos e_t^{-1}) + (34e_t^2 + 13 - 2/e_t^2) \sqrt{e_t^2 - 1} \right], \]  \hfill (2.51d)

\[
\Delta h_{4\pm 4}^{(\text{mem})} = \frac{1}{2700} \sqrt{\frac{\pi}{7}} \eta^2 \frac{M}{R} \frac{M}{p} \frac{1}{e_t^2} \left[ 375e_t^4(\pi - \arccos e_t^{-1}) + (1001e_t^2 - 1178 + 728/e_t^2 - 176/e_t^4) \sqrt{e_t^2 - 1} \right]. \]  \hfill (2.51e)

---

\(^9\) In this case we have \( V_\infty^2 \approx e_t^2(M/p) \) and \( b \approx p/e_t \), or alternatively, \( e_t \approx V_\infty^2(b/M) \) and \( p \approx V_\infty^2 (b^2/M) \). The \( e_t \gg 1 \) limit then corresponds to the bremsstrahlung (small-angle scattering) limit, \( V_\infty^2 \gg M/b \). Note that the scattering angle for \( e_t \gg 1 \) is \( \Theta_s \approx 2/e_t \approx (2/V_\infty^2)(M/b) \ll 1 \).
Note that there is a memory contribution of order $\eta^2 (M/p)^{5/2}$ in each mode. Note also that each mode is real-valued. These modes are plotted in Figure 4. The resulting polarizations are

$$
\Delta h_{+}^{(\text{mem})} = \frac{\eta^2 M}{960 R} \left( \frac{M}{p} \right)^{7/2} \left\{ c_{\Theta} (c_{\Theta}^2 - 1) \left[ 50c_{\Theta}^4(\pi - \arccos c_{\Theta}) + \sqrt{c_{\Theta}^2 - 1} \left( 2002c_{\Theta}^2 - 2356 + \frac{1456}{c_{\Theta}^2} - \frac{352}{c_{\Theta}^4} \right) \right] \cos 4\Phi \\
+ \frac{32}{3} (3 + 2c_{\Theta}^2 + c_{\Theta}^4) \left[ 3c_{\Theta}^2(2c_{\Theta}^2 + 13)(\pi - \arccos c_{\Theta}) + \sqrt{c_{\Theta}^2 - 1} \left( 34c_{\Theta}^2 + 13 - \frac{2}{c_{\Theta}^2} \right) \right] \cos 2\Phi \\
+ 4(\pi - \arccos c_{\Theta}) \left[ (827c_{\Theta}^4 + 6572c_{\Theta}^2 + 2176) - (776c_{\Theta}^4 + 6176c_{\Theta}^2 + 2048)c_{\Theta}^2 - (51c_{\Theta}^4 + 396c_{\Theta}^2 + 128)c_{\Theta}^4 \right] \\
+ \frac{4}{3} \sqrt{c_{\Theta}^2 - 1} \left[ 15103c_{\Theta}^2 + 13622 - (14184c_{\Theta}^2 + 12816)c_{\Theta}^2 - (919c_{\Theta}^2 + 806)c_{\Theta}^4 \right] \right\} (2.52a)
$$

$$
\Delta h_{x}^{(\text{mem})} = \frac{\eta^2 M}{90 R} \left( \frac{M}{p} \right)^{7/2} c_{\Theta} \left\{ (a_{\Theta}^2 - 6) \left[ c_{\Theta}^2(6c_{\Theta}^2 + 39)(\pi - \arccos c_{\Theta}) + \sqrt{c_{\Theta}^2 - 1} \left( 34c_{\Theta}^2 + 13 - \frac{2}{c_{\Theta}^2} \right) \right] \sin 2\Phi \\
+ \frac{c_{\Theta}^2}{40} \left[ 375c_{\Theta}^4(\pi - \arccos c_{\Theta}) + \sqrt{c_{\Theta}^2 - 1} \left( 1001c_{\Theta}^2 - 1178 + \frac{728}{c_{\Theta}^2} - 176 \right) \right] \sin 4\Phi \right\}. (2.52b)
$$

For parabolic orbits ($e_{t} = 1$), Eqs. (2.51) simplify to:

$$
\Delta h_{20,e_{t}=1}^{(\text{mem})} = \frac{676\pi\sqrt{30\pi}}{63} \eta^2 M \left( \frac{M}{p} \right)^{7/2}, \quad (2.53a)
$$

$$
\Delta h_{2\pm 2,e_{t}=1}^{(\text{mem})} = \frac{3\sqrt{6}}{169} \Delta h_{20,e_{t}=1}^{(\text{mem})}, \quad (2.53b)
$$

$$
\Delta h_{40,e_{t}=1}^{(\text{mem})} = \frac{23\sqrt{3}}{4056} \Delta h_{20,e_{t}=1}^{(\text{mem})}, \quad (2.53c)
$$

$$
\Delta h_{4\pm 2,e_{t}=1}^{(\text{mem})} = \frac{3\sqrt{6}}{3380} \Delta h_{20,e_{t}=1}^{(\text{mem})}, \quad (2.53d)
$$

$$
\Delta h_{4\pm 4,e_{t}=1}^{(\text{mem})} = -\frac{\sqrt{210}}{16224} \Delta h_{4\pm 2,e_{t}=1}^{(\text{mem})}, \quad (2.53e)
$$

and the polarizations are

$$
\Delta h_{+,e_{t}=1}^{(\text{mem})} = \frac{\pi\eta^2 M}{4} \left( \frac{M}{p} \right)^{7/2} \left\{ (3 + 2c_{\Theta}^2 + c_{\Theta}^4) \cos 2\Phi \\
+ \frac{5}{48} c_{\Theta}^2 \left[ 766 + 46c_{\Theta}^2 - (1 + c_{\Theta}^2) \cos 4\Phi \right] \right\}. (2.54a)
$$

$$
\Delta h_{x,e_{t}=1}^{(\text{mem})} = \frac{\pi\eta^2 M}{4} \left( \frac{M}{p} \right)^{7/2} c_{\Theta} \times \left[ \frac{5}{24} c_{\Theta}^2 \sin 4\Phi - (5 + c_{\Theta}^2) \sin 2\Phi \right]. (2.54b)
$$

Unlike in the linear-memory case, the nonlinear memory for parabolic orbits is nonzero. Even though parabolic orbits are marginally bound, the radiated GWs are unbound and hence contribute to the nonlinear memory.

We can also examine the $e_{t} \gg 1$ limit. In this case it is easy to extract the large-$e_{t}$ behavior from Eqs. (2.51):

$$
\Delta h_{20,e_{t}>1}^{(\text{mem})} = \frac{292\pi}{21} \sqrt{\frac{\pi}{30}} \eta^2 M \left( \frac{M}{p} \right)^{7/2} e_{t}^4, \quad (2.55a)
$$

$$
\Delta h_{2\pm 2,e_{t}>1}^{(\text{mem})} = \frac{2\sqrt{6}}{73} \Delta h_{20,e_{t}>1}^{(\text{mem})}, \quad (2.55b)
$$

$$
\Delta h_{40,e_{t}>1}^{(\text{mem})} = \frac{17\sqrt{3}}{2920} \Delta h_{20,e_{t}>1}^{(\text{mem})}, \quad (2.55c)
$$

$$
\Delta h_{4\pm 2,e_{t}>1}^{(\text{mem})} = \frac{\sqrt{30}}{2190} \Delta h_{20,e_{t}>1}^{(\text{mem})}, \quad (2.55d)
$$

$$
\Delta h_{4\pm 4,e_{t}>1}^{(\text{mem})} = -\frac{5\sqrt{210}}{7008} \Delta h_{20,e_{t}>1}^{(\text{mem})}. \quad (2.55e)
$$

FIG. 4. (color online) The $\Delta h_{lm}^{(\text{mem})}$ modes for hyperbolic orbits from Eqs. (2.51). For $e_{t} = 1$ these reduce to Eqs. (2.53), while for $e_{t} \gg 1$ they asymptote to Eqs. (2.55).
Using
\[
\left(\frac{M}{p}\right)^{7/2} \epsilon_t^4 \approx \left(\frac{M}{b}\right)^3 V_\infty,
\]
the polarizations are given by
\[
\Delta h^{(\text{mem})}_{+,e_t\gg 1} = \frac{\pi}{960} \eta^2 \frac{M}{R} \left(\frac{M}{b}\right)^3 V_\infty
\]
\[
\times \left[192 \cos 2\Phi + s_0^2 (1756 - 128 \cos 2\Phi - 50 \cos 4\Phi)
\right. \\
- \left. s_0^2 (102 - 32 \cos 2\Phi - 25 \cos 4\Phi)\right],
\]
(2.57a)
\[
\Delta h^{(\text{mem})}_{\times,e_t\gg 1} = -\frac{\pi}{480} \eta^2 \frac{M}{R} \left(\frac{M}{b}\right)^3 V_\infty \epsilon_0
\]
\[
\times \left[96 \sin 2\Phi - s_0^2 (16 \sin 2\Phi + 25 \sin 4\Phi)\right].
\]
(2.57b)
This agrees exactly with Eq. (16) of [19], providing further confirmation of the correctness of the above results.

Note also the different scalings between the linear [Eq. (2.46)] and nonlinear memories in the \(e_t \gg 1\) limit:
\[
\Delta t^{\text{lin,mem}}_{+,e_t\gg 1} \propto \frac{M M}{\eta R b},
\]
(2.58)
\[
\Delta t^{\text{nonlin,mem}}_{\times,e_t\gg 1} \propto \frac{\eta^2 M}{R} \left(\frac{M}{b}\right)^3 V_\infty
\]
(2.59)
This indicates that the nonlinear memory for high-velocity gravitational scattering is typically much smaller than the linear memory (see also Sec. IV). However, for bound eccentric (and circular) orbits, the linear memory vanishes (but see Sec. V B of [20] for a caveat), while the nonlinear memory is \(\propto \eta\). This is essentially due to differences in the integration time over which the nonlinear memory builds up. For unbound orbits the nonlinear memory scales with the orbital time
\[
h^{(\text{mem}),\text{unbound}}_{lm} \sim h^{(\text{mem})}_{lm} \left(\frac{p}{M}\right)^{3/2},
\]
while for bound orbits it scales with the reaction time
\[
h^{(\text{mem}),\text{bound}}_{lm} \sim h^{(\text{mem})}_{lm} \left(\frac{p}{M}\right)^4.
\]
(2.60)

\section{E. Radial orbits}

Next we consider radial orbits corresponding to the head-on collision or separation of two masses. In this case the equations of motion and conserved energy yield
\[
\dot{\varphi} = \ddot{\varphi} = 0, \quad \dot{r} = -\frac{M}{r^2},
\]
(2.62a)
\[
\dot{E} = \frac{E}{\mu} = \frac{\dot{r}^2}{2} - \frac{M}{r}.
\]
(2.62b)
The multipole modes in Eqs. (2.11) easily simplify in the radial case (where we can choose \(\varphi = \text{const} = 0\), and the leading-order waveform polarizations become
\[
h^N_+ = \frac{\eta M}{R} \left(\frac{\dot{r}^2 - \frac{M}{r}}{\mu}\right) \left(1 + c_0^2 \cos 2\Phi - s_0^2\right)
\]
(2.63a)
\[
h^N_\times = -\frac{2\eta M}{R} \left(\frac{\dot{r}^2 - \frac{M}{r}}{\mu}\right) \epsilon_0 \sin 2\Theta.
\]
(2.63b)
If the relative radial velocity approaches \(v_\infty\) at infinite separation, \(\dot{r}^2 - M/r \to v_\infty^2 + M/r\). Radial waveforms can therefore show a linear memory effect that depends on \(v_\infty\) and the initial and final values of \(M/r\).

To compute the nonlinear memory we simplify Eqs. (2.13) (again choosing \(\varphi = 0\)). We easily see that all of the leading-order memory modes have the form
\[
h^{(\text{mem})(1)} \propto \eta \frac{M}{r} \dot{r}^2.
\]
(2.64)
Converting the time-integral to a radial integral and using Eq. (2.62b),
\[
\dot{r} = \pm \sqrt{2 E + M/r},
\]
(2.65)
(where the “+” sign refers to radial expansion and the “−” sign refers to radial infall), the \(h^{(\text{mem})}_{lm}\) modes can be expressed as
\[
h^{(\text{mem})}_{lm} = \pm \sqrt{2 C_{lm}} \eta \frac{M}{r} \int_{r_-}^{r_+} \left(\frac{M}{r}\right)^4 \sqrt{E + M/r} \, dr,
\]
(2.66)
where \(r_\pm\) refers to the value of \(r\) at late or early times. The \(C_{lm}\) are constants that can be read off of Eqs. (2.13). Evaluating the integral yields
\[
h^{(\text{mem})}_{lm} = \pm \sqrt{\frac{2}{105}} C_{lm} \eta \frac{M}{r} \left(\frac{E + M/r}{r}\right)^{3/2}
\]
\[
\times \left[8 E^2 - 12 E M/r + 15 \left(\frac{M}{r}\right)^2\right]_{r_-}^{r_+}.
\]
(2.67)

For case of radial infall from rest at infinity, the above simplifies to
\[
h^{(\text{mem})}_{lm} = \pm \sqrt{\frac{2}{7}} C_{lm} \eta \frac{M}{r} \left[\frac{M}{r(t)}\right]^{7/2},
\]
(2.68)
where \(r_+ \to r(t)\). For ejection with an asymptotic velocity of \(v_\infty\), \(E \to v_\infty^2/2\) at late times when \(r_+ \to \infty\), while at early times (when only a single bound object exists) \(E \to -M/r_-\). The resulting nonlinear memory shift in this case evaluates to
\[
\Delta h^{(\text{mem})}_{lm} = -\frac{2}{105} C_{lm} \eta \frac{M}{r} v_\infty^7,
\]
(2.69)
where the dependence on $r_-$ cancels. Note that in both cases the nonlinear memory is a relative 2.5PN correction to the Newtonian waveform. In all cases, the waveform polarizations for radial orbits are given explicitly by

$$h_+^{(\text{mem})} = \hat{h}^{(\text{mem})} \left[ \frac{c_{\text{eff}}}{60} (11 + 3c_{\text{eff}}^2) - \frac{1}{15} (3 + 2c_{\text{eff}} + c_{\text{eff}}^4) \cos 2\Phi + \frac{1}{60} (1 - c_{\text{eff}}^4) \cos 4\Phi \right],$$

$$h_\times^{(\text{mem})} = \hat{h}^{(\text{mem})} \left[ \frac{c_{\text{eff}}}{15} (5 + c_{\text{eff}}^2) \sin 2\Phi - \frac{1}{30} s_{\text{eff}}^2 c_{\text{eff}} \sin 4\Phi \right],$$

(2.70a)

(2.70b)

where $\hat{h}^{(\text{mem})}$ is given by Eqs. (2.67), (2.68), or (2.67) with $\hat{h}^{(\text{mem})} = h_{lm}^{(\text{mem})} / C_{lm}$.

III. SENSITIVITY OF THE MEMORY TO THE EARLY-TIME HISTORY OF A BINARY

In this section we wish to evaluate the degree to which the nonlinear memory from a quasicircular inspiralling binary is sensitive to its deviations from circularity. These deviations arise from the binary’s initial eccentricity, which gets damped by radiation reaction. To perform this evaluation, we compare two model waveforms for the evolution of the $h_{20}^{(\text{mem})}$ mode. (For simplicity and because they tend to be much smaller, I will neglect the other memory modes.) In the first model we consider the $h_{20}^{(\text{mem,ellip})}$ mode for an elliptical binary described via Eq. (2.34), with $e_- = 0.99$ and $e_0 = 0.01$ at a pericenter distance of $r_0 = p_0 / (1 + e_0) = 6M$. This mode is plotted as the blue (solid) line in Figure 5. We will also need to model how the eccentricity evolves with time. To do this I evolve Eqs. (2.29b) and (2.39), but I change to a new time variable $T = -t$ so that I can more easily evolve the system “backwards” in time starting from the initial conditions $r_p(T = 0) = r_0 = 6M$ and $e_r(T = 0) = e_0 = 0.01$. For an equal-mass binary, I find that the “early-time” eccentricity $e_- = 0.99$ is reached at a time $T_-/M \approx 2.031 \times 10^8$. This mode is compared with a purely quasicircular model for the $h_{20}^{(\text{mem})}$ mode which is given by

$$h_{20}^{(\text{mem,circ})} = \frac{2}{7} \sqrt{\frac{10\pi}{3}} \frac{M}{M} \left( \frac{r}{r} - \frac{M}{M} \right),$$

(3.1)

where

$$r(T) = r_0 \left( 1 + \frac{T}{\tau_{rr}} \right)^{1/4}, \quad \tau_{rr} = \frac{5}{256} \frac{M}{\eta} \left( \frac{r_0}{M} \right)^4,$$

(3.2)

and $r_- = r(T_-) \approx 225.8M$. This model forces both the quasicircular mode $h_{20}^{(\text{mem,circ})}$ and the elliptical mode $h_{20}^{(\text{mem,ellip})}$ to vanish at the same time ($T = T_-$), which can be considered the start of the observation. It also ensures that both orbits have a pericenter separation $r_p = r_0 = 6M$ at time $T = 0$. The two modes are plotted in Figure 5, where the value of time for both modes is parameterized in terms of the eccentricity of the elliptical mode. This figure indicates that the quasicircular model provides a moderately accurate representation of the true evolution of the memory mode (which accounts for the orbit’s past eccentricity). At the end of the evolution ($T = 0$, $r_p = 6M$), the two modes have a fractional error of $\approx 1.5\%$.

To better quantify the degree to which the two modes “overlap,” I have computed the following normalized inner product:

$$\mathcal{O} = \frac{\int_0^{T_-} h_1(t) h_2(t) \, dt}{\sqrt{\int_0^{T_-} h_1^2(t) \, dt \left[ \int_0^{T_-} h_2^2(t) \, dt \right]}}.$$

(3.3)

where $h_1 = h_{20}^{(\text{mem,circ})}$ and $h_2 = h_{20}^{(\text{mem,ellip})}$. This is equivalent to the commonly computed overlap between two GW signals, but here assuming white noise. For values of $e_0 = 0.01$ or 0.001, I find the value $\mathcal{O} \approx 0.976$; this decreases slightly to 0.975 for $e_0 = 0.1$. Although I have not considered a realistic noise model, this calculation suggests that ignoring the effects of past eccentricity in quasicircular binaries is a reasonable approximation and
is not likely to result in significant reduction in the signal-to-noise ratio.

Now let us consider the memory that results over the entire lifetime of a binary system, including its initial formation. As one would intuitively expect, any bound eccentric binary experiencing gravitational radiation-reaction evolves to larger eccentricities (and larger orbital separations) in the past until $e_i > 1$ and the binary becomes unbound. This was proved rigorously by Walker and Will [67]. Equivalently, for certain choices of its initial orbital parameters, a hyperbolic binary can lose energy from gravitational-wave emission and become bound. The waveform for such a scenario can be approxi-
mately modeled using Eqs. (2.21a) and (2.21b) combined with a prescription for the instantaneous evolution of the orbital elements (see, e.g., [67]). If we choose our time and angular coordinates so that capture happens at periastron, a schematic description of the waveform modes from such a captured binary would look like the left plot of Figure 2 for $t \equiv (t/M)(M/p)^{3/2} < 0$ smoothly matched onto the right plot of Figure 2 for $t > 0$. (Note that the different modes and their slopes in that figure have the correct signs at $t = 0$ to allow for such a matching.) After capture, such a binary would circularize and eventually merge, with the waveforms evolving in the standard way for $t > 0$. Of course, this description is somewhat idealized. In the real world other interactions (e.g., tidal dissipation, three-body interactions, gas drag, dynamical friction) are more likely to result in binary capture (although gravitational radiation losses could play an important role in very dense stellar systems such as globular clusters or galactic nuclei). However, for the purpose of considering the size of the memory jump over very long timescales, let us assume that at some early time the binary is an unbound, hyperbolic orbit, while at some later time it is a bound, eccentric binary that circularizes and merges. For such a binary the total memory jump is roughly given by Eq. (1.1) with

$$\lim_{t \to -\infty} h_{+,(x)} \sim \frac{\eta M^2}{R p_i} \left( e_i^2 - 1 \right)^{3/2},$$  \hspace{1cm} (3.4a)

$$\lim_{t \to \infty} h_{+,(x)} \sim \frac{\Delta E}{R},$$  \hspace{1cm} (3.4b)

where $p_i$ and $e_i$ are the semi-latus rectum and eccentricity prior to capture, and $\Delta E$ is the energy radiated in GWs throughout the inspiral, merger, and ringdown. This suggests that large memory jumps can result not only from the nonlinear memory (which grows most rapidly during the final phases of coalescence), but also through the linear memory associated with binary capture.

A more relevant issue is the observability of some signature of the formation or early-time state of the binary. Clearly, if one’s GW detector is operating when the binary capture process occurs (in retarded time), then the signature of the capture, including the resulting memory, will be seen in the detector’s output (provided the detector is sensitive to low-frequency effects like the memory). However, what if the capture process (and the associated passing GWs) occurred long before the start of the observation period? Does the capture process still leave an “imprint” on the waves observed at later times? Intuitively one expects the answer to this question to be “no.” This is indeed correct as can be seen with the following argument.

where $\Delta E = \int_0^t \frac{dE}{d\tau} \ d\tau$. An expression for $\Delta E/\tau$ can be derived by considering the Lagrange planetary equation (cf. Danby [64]) for an osculating Keplerian ellipse under the action of the 2.5PN radiation reaction force. [See also the last equation in [67] or Eq. (2.14) of [64].]

However, note that the nonlinear memory is only proportional to the radiated energy at leading-order in an $(l, m)$ mode expansion of the energy flux [26].
Consider a simplified GW detector consisting of two particles floating in space separated by a distance $L$. Placing the first particle at the origin of its own proper reference frame, the position $x^j$ of the second particle relative to the first is given by the equation of motion (Ch. 35.5 of [17])

$$\ddot{x}^j = \frac{1}{2} \hat{h}^{TT}_{jk} x^k,$$  

(3.6)

where overdots here refer to the derivative of the proper time at the first particle, and $\hat{h}^{TT}_{jk}$ is the metric perturbation in transverse-traceless gauge. We can choose to orient our two particles and the resulting coordinate system such that their motion along their direction of separation $\hat{x}$ is given by

$$\ddot{x}(t) = \frac{1}{2} \hat{h}_+(t) x(t).$$  

(3.7)

For very small displacements, $x(t) \approx L + \delta x(t)$, and the equation for the difference in the particles’ relative separation simplifies to

$$\delta x(t) = \frac{L}{2} \hat{h}_+(t).$$  

(3.8)

Now we consider two scenarios: in the first we assume that our detector has been freely-floating for all times, so it observes the entire build-up of the memory. In the distant past, we assume that our memory signal approaches the value $h^{(-\infty)}$ and its derivative vanishes, $\dot{h}^{(\text{mem})}_+ (\infty) \to 0$. In this case Eq. (3.8) has the solution

$$\delta x(t) = \frac{L}{2} \left[ h^{(\text{mem})}_+(t) - h^{(-\infty)}_+ \right].$$  

(3.9)

Already we see in this case that the value of the GW field in the asymptotic past $[h^{(-\infty)}]$ is not observable; instead only the difference between that asymptotic value and the current value (at time $t$) is observable. However, since the detector has been operating for arbitrarily long times, the measured value of the memory retains any imprint of the past evolution of the binary [e.g., the value of the nonlinear memory at time $t$ would depend on the motion of the source all the way to $t \to -\infty$, but not on the value of $h^{(-\infty)}$].

In the second scenario, let us suppose that the memory signal has started arriving, but our detector is rigidly fixed in position until some time $t_0$ when we allow our particles to be free-floating. In this case, Eq. (3.8) has the solution

$$\ddot{x}(t) = \frac{L}{2} \left[ h^{(\text{mem})}_+(t) - h^{(\text{mem})}_+(t_0) - \dot{h}^{(\text{mem})}_+(t_0)(t - t_0) \right].$$  

(3.10)

Here (somewhat obviously) we see that the memory loses its dependence on times before $t_0$. We also see that a linear drift (proportional to the slope of the waveform at $t_0$) also develops. This drift arises from the initial impulse the detector receives from the passing wave when it is released.

As a simple analytic example, consider a schematic model for a nonlinear memory waveform given by the arctangent function:

$$h^{(\text{mem})}_+ = \frac{h^{(+\infty)} - h^{(-\infty)}}{\pi} \arctan \left( \frac{t}{\tau} \right) + \frac{h^{(+\infty)} + h^{(-\infty)}}{2},$$  

(3.11)

where $h^{(\pm\infty)}$ are the asymptotic values of the memory and $\tau$ is the characteristic rise time of the memory. The second time-derivative of this function is

$$\ddot{h}^{(\text{mem})}_+ = - \frac{2[h^{(+\infty)} - h^{(-\infty)}]t\tau}{\pi(t^2 + \tau^2)^2}.$$  

(3.12)

These functions and the resulting differential displacements are plotted in Figure 6 for the two scenarios mentioned above. A similar model could be based on the hyperbolic tangent function,

$$h^{(\text{mem})}_+ = \frac{h^{(+\infty)} - h^{(-\infty)}}{2} \tanh \left( \frac{t}{\tau} \right) + \frac{h^{(+\infty)} + h^{(-\infty)}}{2},$$  

(3.13)

which approaches its asymptotes more quickly and has a second derivative given by

$$\ddot{h}^{(\text{mem})}_+ = - \frac{[h^{(+\infty)} - h^{(-\infty)}] \sinh(t/\tau)}{\tau^2 \cosh^3(t/\tau)}.$$  

(3.14)

IV. ESTIMATING THE SIGNAL-TO-NOISE RATIO OF MEMORY JUMPS

Here I provide some simple formulas for estimating the detectability of the memory. For the case of merging quasicircular binaries, detectability estimates are presented in [26] and will be discussed in more detail in [27]. For elliptical binaries, we have seen that the behavior quite similarly to the quasicircular case, so the estimates of detectability would be little changed. Here I will on the linear and nonlinear memory for hyperbolic and parabolic orbits.

We begin by defining the angle-averaged square of the signal-to-noise ratio as

$$\langle \rho^2 \rangle = \int_0^\infty \frac{h_n^2(f) df}{S_n(f)} f,$$  

(4.1)

where the average is over all sky positions, source orientations, and polarization angles [see, e.g., Eqs. (2.33)–(2.36) of [68]]. Here $h_n(f) = \sqrt{\alpha f S_n(f)}$ is the sky-averaged rms noise amplitude per logarithmic frequency interval. The factor $\alpha$ is $5$ for orthogonal arm detectors like LIGO and $20/3 = 5/\sin^2(60^\circ)$ for

\[\text{footnote: This situation is equivalent to solving Eq. (3.8) with the right-hand-side multiplied by a Heaviside function $\Theta(t - t_0)$.}\]
equilateral triangles like LISA or the Einstein Telescope [69]. The characteristic amplitude is given by

\[ h_c(f) = 2f⟨|\hat{h}_+(f)|^2 + |\hat{h}_x(f)|^2⟩^{1/2}. \]  

(4.2)

where a tilde denotes a Fourier transform. If we approximate the memory as a step-function, then its Fourier transform is given by\(^\dagger\)

\[ |\hat{h}_{+x}(f)| = \frac{\Delta h_{+x}}{2\pi f}. \]  

(4.3)

However, a real memory signal has some finite rise time \(\tau\) which imposes a high-frequency cutoff at \(f_c \sim 1/\tau\) in the Fourier transform.\(^\ddagger\) We can therefore approximate the characteristic strain by

\[ h_c = \frac{1}{\pi}⟨|\Delta h_+|^2 + |\Delta h_x|^2⟩^{1/2} \Theta(f_c - f). \]  

(4.4)

The SNR then becomes

\[ (\rho^2)^{1/2} = \frac{\hat{h}_c}{\mathcal{N}}, \]  

(4.5)

where we define \(\hat{h}_c\) to be Eq. (4.4) without the Heaviside factor and

\[ \mathcal{N} = \left(\int_0^{f_c} \frac{df}{fh_n^2}\right)^{-1/2}. \]  

(4.6)

\(^\dagger\) For cosmological sources, one must replace \(f \rightarrow (1 + z)f\) and \(R \rightarrow D_L(z)/(1 + z)\) in this expression, where \(z\) is the redshift and \(D_L\) is the luminosity distance.

\(^\ddagger\) This follows from the Fourier transform of the Heaviside function,

\[ \int_{-\infty}^{\infty} \hat{H}(\pm t)e^{\pm i\omega t}dt = \frac{\delta(f)}{2} \mp \frac{i}{2\pi f}. \]

The step-function approximation is equivalent to the zero-frequency limit (ZFL) discussed in [6, 70-72]. In that case one approximates the Fourier transform of the time-derivative of a signal \(h(t)\) near \(f \approx 0\) via

\[ \hat{h}(f) = \int_{-\infty}^{\infty} \hat{h}(t)e^{i\omega t}dt ≈ \int_{-\infty}^{\infty} \hat{h}(t)dt \]

\[ = h(+\infty) - h(-\infty) \equiv \Delta h, \]

and uses the usual relation for the Fourier transform of a derivative, \(\hat{h}(f) = (-2\pi i f)\hat{h}(f)\) to arrive at

\[ \hat{h}_{ZFL}(f) = \frac{\Delta h}{2\pi f}. \]

\(^\triangleright\) For an explicit example of this, consider the Fourier transform of the signal in Eq. (3.13) for \(f > 0\) [73]:

\[ \hat{h}_{+}^{\mathrm{mem}} = \frac{[h(+\infty) - h(-\infty)]}{i\tau c \text{ch}(\sigma^2 f)} \]

\[ = \frac{2}{2\pi f} \left\{ 1 - \frac{\pi^4}{6} (\tau f)^2 + O((\tau f)^4) \right\}. \]

Here one can directly see a sharp cutoff in the ZFL value of the Fourier transform when \(f \sim 1/\tau\).

\(^\dagger\) If we define the rise time by taking the integral in Eq. (2.15) over \(\nu \in [-\Theta_s/2, \Theta_s/2]\), then we find that \(\kappa \leq 4/3\) and asymptotically approaches \(2/\epsilon_s^2\) for \(\epsilon_s \gg 1\).
\[ \hat{h}_{c,e>1}^{(\text{lin. mem})} = \frac{8}{\pi} \sqrt{\frac{2}{5}} \frac{M M}{R r_p}, \quad (4.8b) \]
\[ \hat{h}_{c,e>1}^{(\text{mem})} = \frac{\sqrt{21075910}}{3600} \eta^2 \frac{M}{R} \left( \frac{M}{r_p} \right)^3 V_\infty, \quad (4.8c) \]

where the second and third equations show the linear and nonlinear memory for hyperbolic orbits in the large-eccentricity limit (for which the memory is largest). Note that in the hyperbolic case, the nonlinear memory is smaller than the linear memory by a factor 0.79\(\eta(M/r_p)^2V_\infty\) (in practice this amounts to a factor \(\gg 4\) orders of magnitude). Plugging in some numbers yields

\[ \hat{h}_{c,e=0}^{(\text{mem})} = 2.2 \times 10^{-22} \left( \frac{\eta}{0.25} \right)^2 \left( \frac{M/10M_\odot}{R/10\text{kpc}} \right) \left( \frac{20M}{r_p} \right)^{7/2} \]
\[ = 1.7 \times 10^{-29} \left( \frac{\eta}{10^{-5}} \right)^2 \left( \frac{M/10^6M_\odot}{R/20\text{Mpc}} \right) \left( \frac{20M}{r_p} \right)^{7/2} \]
\[ = 2.2 \times 10^{-22} \left( \frac{\eta}{0.25} \right)^2 \left( \frac{M/10^6M_\odot}{R/1\text{Gpc}} \right) \left( \frac{20M}{r_p} \right)^{7/2} \quad (4.9) \]

\[ \hat{h}_{c,e>1}^{(\text{lin. mem})} = 9.6 \times 10^{-19} \left( \frac{\eta}{0.25} \right) \left( \frac{M/10M_\odot}{R/10\text{kpc}} \right) \left( \frac{20M}{r_p} \right) \]
\[ = 1.9 \times 10^{-21} \left( \frac{\eta}{10^{-5}} \right) \left( \frac{M/10^6M_\odot}{R/20\text{Mpc}} \right) \left( \frac{20M}{r_p} \right) \]
\[ = 9.6 \times 10^{-19} \left( \frac{\eta}{0.25} \right) \left( \frac{M/10^6M_\odot}{R/1\text{Gpc}} \right) \left( \frac{20M}{r_p} \right). \quad (4.10) \]

For cosmological distances we should take \(M/R \rightarrow M_L/D_L\) in the above expressions, where \(M_L = (1 + z)M\) is the redshifted mass. These rough estimates indicate that GW bursts with linear memory should be easily detectable with second-generation ground-based detectors and future space-based detectors. The nonlinear memory from GW bursts from unbound (or marginally) bound binaries will be more difficult to detect and will likely require third-generation detectors. Current and near-term PTAs are not sufficiently sensitive to detect memory bursts of the types considered here.

V. CONCLUSIONS

To briefly summarize, I have here generalized previous computations of the nonlinear gravitational-wave memory effect to the case of binaries with arbitrary eccentricity. In the case of hyperbolic, parabolic, and radial orbits, the nonlinear memory is a 2.5PN correction to the waveform. In the case of elliptical binaries, the nonlinear memory affects the waveform at leading order (just as in the quasicircular case). The resulting expressions for the waveforms complete the leading-order expressions first derived by Peters and Mathews [33]. In addition, I have also investigated the sensitivity of the nonlinear memory to the early-time history of the binary. In the case of quasicircular binaries that were initially eccentric, the early-time eccentricity provides only a small correction to the memory. In addition, any contributions to the memory made outside of the observation time are undetectable. I have also showed how one can make quick calculations of the signal-to-noise ratio for memory bursts from unbound orbits.

There are a variety of areas in which this study could be extended. Here I have only restricted to the leading-order nonlinear memory corrections. For hyperbolic, parabolic, and radial orbits the waveforms are only known to 1PN order, so there is little motivation to compute higher-order corrections to the leading-order nonlinear memory terms (which enter at 2.5PN order). However, in the elliptical case the oscillatory waveform polarizations are known to 2PN order, so 2PN order corrections to the leading-order nonlinear memory terms would be needed to have complete 2PN order eccentric waveforms. In addition, the effects of spinning binary components on the nonlinear memory have not yet been computed. This calculation is in progress in the case of quasicircular binaries and will be reported elsewhere. The case of computing the nonlinear memory for eccentric, spinning binaries will be left for future work. If future observations detect the nonlinear memory, more accurate waveform models than are currently available may be needed to properly extract all of the information that such signals may contain. It would also be interesting to investigate the relative sizes of the linear and nonlinear memory in the case of ultrarelativistic collisions and scatterings [81–86]. These situations should show a very large memory effect.

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Appendix A: DERIVATION OF THE LEADING-ORDER MASS AND CURRENT SOURCE MOMENTS FOR A GENERAL TWO-BODY SYSTEM

The purpose of this appendix is to derive expressions for the mass and current multipole moments in the form of \((l,m)\) modes that are valid for any two-body orbit at Newtonian order. At leading-order we are only concerned about the so-called source moments which are de-
fined in terms of integrals over a stress-energy pseudo-tensor. General expressions (valid for any PN order) for the mass and current symmetric-trace-free (STF) source multipoles, $I_L$ and $J_L$, can be found in Eq. (85) of [87]. These STF tensors with $L$ indices (where $L = a_1a_2 \cdots a_l$) can be difficult to work with, and for some calculations it is more convenient to instead use the “scalarized” versions of these moments, $I_{lm}$ and $J_{lm}$. These “scalar” multipoles are simply the coefficients of the expansion of the STF mass and current multipoles on the basis of the STF spherical harmonics $Y_L^{lm}$ [these are defined in Eq. (2.12) of [88] and are related to the standard scalar spherical harmonics via Eq. (A8) below]. The STF moments and their $(l, m)$ modes are related by the following formulas [Eqs. (4.6) and (4.7) of [88]]:

$$I_L = \frac{l}{4} \sqrt{\frac{2l(l-1)}{(l+1)(l+2)}} \sum_{m=-l}^{l} I_{lm} Y_L^{lm},$$  \hspace{1cm} (A1a)

$$J_L = -\frac{(l+1)!}{8l} \sqrt{\frac{2l(l-1)}{(l+1)(l+2)}} \sum_{m=-l}^{l} J_{lm} Y_L^{lm},$$  \hspace{1cm} (A1b)

$$I_{lm} = A_l I_L Y_L^{lm},$$  \hspace{1cm} (A2a)

$$J_{lm} = B_l J_L Y_L^{lm},$$  \hspace{1cm} (A2b)

$$A_l = \frac{16\pi}{(2l+1)!!} \sqrt{\frac{(l+1)(l+2)}{2l(l-1)}},$$  \hspace{1cm} (A3a)

$$B_l = -\frac{32\pi l}{(2l+1)!!} \sqrt{\frac{(l+2)}{2l(l+1)(l-1)}}.$$  \hspace{1cm} (A3b)

Now we specialize the general form for STF mass and current moments in Eq. (85) of [87] to the Newtonian-order moments for general orbits (but arbitrary $l$-value). This derivation could be easily extended to the 1PN-order moments (see Kidder [89]). The Newtonian-order source mass and current multipole moments for a system of $N$ (nonspinning) point masses is

$$I_L^N = \sum_{A=1}^{N} m_A y_A^{<L>},$$  \hspace{1cm} (A4a)

$$J_L^N = \sum_{A=1}^{N} m_A \varepsilon^{abc} y_A^{<L-1>} y^b_A y^c_A,$$  \hspace{1cm} (A4b)

where $A$ labels the body, the multi-index $L$ refers to a product of $l$ vectors (e.g., $y^L_A = y^a_1 y^b_1 \cdots y^l_1$), $\varepsilon^{abc}$ is the Levi-Civita tensor, and the angled brackets $<>$ mean to take the STF projection on the enclosed indices. We now specialize to a 2-body system with masses $m_1$ and $m_2$, total mass $M = m_1 + m_2$, and reduced mass ratio $\eta = m_1 m_2/M^2$. We transform to the center-of-mass frame using

$$\vec{y}_1 = \frac{m_2}{M} \vec{x},$$  \hspace{1cm} (A5a)

$$\vec{y}_2 = \frac{m_1}{M} \vec{x},$$  \hspace{1cm} (A5b)

where $\vec{x} = \vec{y}_1 - \vec{y}_2 = r \vec{n}$ has length $r$ and $\vec{n}$ points from $m_2$ to $m_1$. We also define the individual and relative velocity vectors via $\vec{v}_A = \vec{y}_A$ and $\vec{v} = \vec{x}$.

Substituting the above relations into Eqs. (A4) gives [Eqs. (5.21) and (5.22) of [90]]:

$$I_L^N = \eta M s_1(\eta) x_{<L>},$$  \hspace{1cm} (A6a)

$$J_L^N = \eta M s_{l+1}(\eta) \varepsilon^{abc} y_A^{<L-1> \times c} v^b,$$  \hspace{1cm} (A6b)

where $s_l(\eta) = X_l^2 + (-1)^l X_{l-1}^2$, and we define $X_1 = \frac{m_1}{M} = \frac{1}{2}(1+\Delta)$, $X_2 = \frac{m_2}{M} = \frac{1}{2}(1-\Delta)$, and $\Delta = \frac{m_1 - m_2}{M}$ ($\pm$ sign depends on one’s convention for which mass is larger). To compute the “scalar” multipoles defined in Eq. (A2) we need to contract Eqs. (A6) with $Y_L^{lm}$*. Using the relationship between the “scalar” and STF spherical harmonics,

$$Y^{lm} = Y_L^{lm} n_L = Y_L^{lm} n_{<L>},$$  \hspace{1cm} (A8)

the Newtonian “scalar” mass multipole equivalent to (A6a) is easily seen to be

$$I_N^{lm} = A_l \eta M s_1(\eta) r^{l} \varepsilon_{<} Y^{lm}(\theta, \phi).$$  \hspace{1cm} (A9)

To derive the Newtonian “scalar” current multipole moment we use the definition of the magnetic-type “pure-spin” vector spherical harmonics [Eqs. (2.18b) and (2.23b) of [88]]:

$$Y_b^{lm} = \sqrt{\frac{l}{l+1}} \varepsilon_{bad} n_a Y^{lm}_{l-1 \times c} Y^{<l-1>},$$  \hspace{1cm} (A10)

$$= \frac{1}{\sqrt{l(l+1)}} \varepsilon_{bad} n_a \nabla_d Y^{lm}_{l-1}. $$  \hspace{1cm} (A11)

Combining this equation with Eqs. (A2b) and (A6b) yields the Newtonian scalar current multipole for general orbits:

$$J_N^{lm} = B_l \eta M s_{l+1}(\eta) r^{l} \varepsilon_{<<} \vec{Y}^{<lm}_{l-1}(\theta, \phi).$$  \hspace{1cm} (A12)

In Eqs. (A9) and (A12) the moments are given as functions of time by solving the equations of motion to determine the spherical coordinates of the relative separation vector $\vec{x}$: $r(t), \theta(t)$, and $\phi(t)$. If we restrict ourselves to orbits in the $x$–$y$ plane, we can further simplify the multipole moments by using

$$\vec{x} \times \vec{v} = r^2 \vec{\omega},$$  \hspace{1cm} (A13)

$$\vec{c}_z \times \vec{\nabla} Y_{lm} = -\frac{\sin \theta}{r} \frac{\partial Y_{lm}}{\partial \theta}.$$  \hspace{1cm} (A14)

The resulting Newtonian-order “scalar” multipole moments for general orbits restricted to the $x$–$y$ plane are

$$I_N^{lm} = A_l \eta M s_1(\eta) r^{l} \varepsilon_{<<} \vec{Y}^{<lm}_{l-1}(\theta, \phi),$$  \hspace{1cm} (A15a)

$$J_N^{lm} = -B_l \eta M s_{l+1}(\eta) r^{l+1} \phi \frac{\partial Y^{<lm}_{l-1}(\theta, \phi)}{\partial \phi}.$$  \hspace{1cm} (A15b)
Appendix B: NONLINEAR MEMORY INTEGRAL FOR ELLIPTICAL ORBITS IN TERMS OF HYPERGEOMETRIC FUNCTIONS

In this appendix we show how to derive explicit expressions for the integrals of Eqs. (2.33), which we rewrite as:

\begin{align}
\label{eq:B1a}
h_{20}^{(\text{mem})} &= \left[ -\frac{384}{7(304)^b} \sqrt{\frac{10\pi \eta M^2}{3 R_p}} \right] C_0(e_0) \left[ \int dt \frac{e_t^{1/19} (1 + 145 e_t^4)}{e_t^{31/19} (1 + \frac{121}{304} e_t^2)^b} \right] + K_{20}, \\
\label{eq:B1b}
h_{2\pm 2}^{(\text{mem})} &= \left[ -\frac{52}{7} \frac{\sqrt{5\pi} \eta M^2}{R_p} \right] C_0(e_0) e^{2\pi i \omega} \left[ \int dt \frac{e_t^{1/19} (1 + 145 e_t^4)}{e_t^{31/19} (1 + \frac{121}{304} e_t^2)^b} \right] + K_{2\pm 2}, \\
\label{eq:B1c}
h_{40}^{(\text{mem})} &= \left[ -\frac{64}{21(304)^b} \sqrt{\frac{\pi \eta M^2}{10 R_p}} \right] C_0(e_0) \left[ \int dt \frac{e_t^{19/19} (1 + 145 e_t^4)}{e_t^{31/19} (1 + \frac{121}{304} e_t^2)^b} \right] + K_{40}, \\
\label{eq:B1d}
h_{4\pm 2}^{(\text{mem})} &= \left[ -\frac{13}{21} \frac{\sqrt{\pi} \eta M^2}{R_p} \right] C_0(e_0) e^{2\pi i \omega} \left[ \int dt \frac{e_t^{19/19} (1 + 145 e_t^4)}{e_t^{31/19} (1 + \frac{121}{304} e_t^2)^b} \right] + K_{4\pm 2}, \\
\label{eq:B1e}
h_{4\pm 4}^{(\text{mem})} &= \left[ -\frac{25}{24(304)^b} \sqrt{\frac{\pi \eta M^2}{7 R_p}} \right] C_0(e_0) e^{4\pi i \omega} \left[ \int dt \frac{e_t^{5/19} (1 + 145 e_t^4)}{e_t^{31/19} (1 + \frac{121}{304} e_t^2)^b} \right] + K_{4\pm 4},
\end{align}

where \( b = \frac{3169}{2799} \), the \( K_{lm} \) are integration constants, and we refer to the constants in square brackets as \( A_{lm} \) below and in the main text. We now note that all of the indefinite integrals in square brackets can be expressed in terms of combinations of the following integral \([91]\):

\[
\int \frac{x^a}{(1 + cx)^b} dx = \frac{x^{a+1}}{a+1} \frac{1}{2} \text{F}_1(b, a + 1; a + 2; -cx),
\]

where \( \text{F}_1 \) is the hypergeometric function \([92]\). Any of the integrals in Eqs. (B1) can then be computed from:

\[
F_{lm}(e_t) \equiv \int dt \frac{e_t^{\alpha_{lm}} (1 + e_t^{c_{lm}} e_t^{d_{lm}})}{(1 + \beta e_t^2)^b} = e_t^{\alpha_{lm}} \left[ \frac{1}{\alpha_{lm} + 1} \frac{1}{2} \text{F}_1(b, \frac{\alpha_{lm} + 1}{2}, \frac{\alpha_{lm} + 3}{2}, -\beta e_t^2) + \frac{c_{lm} e_t^2}{\alpha_{lm} + 3} \frac{1}{2} \text{F}_1(b, \frac{\alpha_{lm} + 3}{2}, \frac{\alpha_{lm} + 5}{2}, -\beta e_t^2) + \frac{d_{lm} e_t^4}{\alpha_{lm} + 5} \frac{1}{2} \text{F}_1(b, \frac{\alpha_{lm} + 5}{2}, \frac{\alpha_{lm} + 7}{2}, -\beta e_t^2) \right],
\]

where the constants \( \alpha_{lm}, c_{lm}, \) and \( d_{lm} \) are easily read off of Eqs. (B1), and \( \beta = \frac{121}{304} \). The integration constants \( K_{lm} \) are then determined by the requirement that the nonlinear memory vanish at early times when the eccentricity \( e_t = e_- \). The final result for the \( h_{lm}^{(\text{mem})} \) modes is then given by

\[
h_{lm}^{(\text{mem})} = A_{lm} C_0(e_0) e^{\pi i m \pi} \left[ F_{lm}(e_t) - F_{lm}(e_-) \right].
\]

Appendix C: THE ROLE OF AVERAGING THE GRAVITATIONAL-WAVE STRESS-ENERGY TENSOR IN NONLINEAR MEMORY CALCULATIONS

In the definition of the nonlinear memory in Eq. (2.5), an explicit averaging over several wavelengths appears in the gravitational-wave energy flux \( d\Phi_{\text{gw}} \). This is consistent with the standard derivation in which averaging is necessary to obtain a well-defined GW stress-energy tensor \([16, 17]\). However, in the derivations of the nonlinear memory in \([14, 15]\), this wavelength averaging does not explicitly appear. The purpose of this appendix is to investigate (in the context of eccentric binaries) how the nonlinear memory calculation depends on whether the wavelength averaging is performed or not. The short answer to this question is that the nonlinear memory does not depend on this averaging, aside from very small amplitude oscillations at the orbital period that are superimposed on the memory when averaging is not per-
The reason why the memory is relatively insensitive to the averaging procedure is simple: the time integral that explicitly appears in the nonlinear memory calculation essentially already “averages” over the integrand [cf. Eq. (2.5)]. So by performing also the wavelength averaging $\langle \rangle$ of the integrand, one is effectively “averaging” twice. However, note that the wavelength averaging significantly simplifies the integrand, allowing for an analytic calculation.\footnote{Note also that in the circular case, this wavelength averaging issue does not arise. The integrand of those modes that contribute to the nonlinear memory are constant on an orbital timescale and are unaffected by averaging. This can be seen explicitly by comparing Eqs. (2.24) and (2.27) in the $e_t = 0$ case.}

To investigate this issue in more detail we can explicitly compute the $h^{(1)}_{20}$ nonlinear memory mode with and without wavelength averaging. In both cases we first solve for the evolution of $e_t(t), p(t),$ and $v(t)$ (the true anomaly) by numerically integrating Eqs. (2.29) and (2.14c). We assume an equal-mass binary and initial conditions $e_t(0) = 0.7, \ p(0) = 30M$, and $v(0) = 0$. We then substitute the result into the integrand of the time integral for the memory: Eq. (2.24a) in the non-averaged case (ignoring the $\langle \rangle$), and Eq. (2.27a) in the averaged case. The resulting integrands are plotted in Figure 7. There we see that if we do not perform any wavelength averaging, the integrand retains oscillations at the orbital period; these oscillations are smoothed-over by the wavelength averaging.

We then numerically integrate both the averaged and un-averaged integrands, starting from the condition $h^{(1)}_{20}(0) = 0$. The result is plotted in left-half of Figure 8. There we see that performing the wavelength averaging has had very little effect on the resulting memory. The two curves lie nearly on top of each other. As stated above, this agreement is simply due to the fact that the time integration of the un-averaged $h^{(1)}_{20}$ essentially acts as an averaging procedure. The only difference is a very small remnant oscillation about the wavelength-averaged curve. At the last-stable-orbit and for a large range of eccentricity, the two curves agree to $\sim 1\%$.  

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7}
\caption{(color online). Effect of wavelength averaging on nonlinear memory integrand. The solid (black) curve shows the time evolution (parameterized in terms of decreasing nonlinear memory integrand. The solid (black) curve shows the time evolution (parameterized in terms of decreasing $h_{20}(t)$ of the integrand $h^{(1)}_{20}$ computed without wavelength averaging via Eq. (2.24a) as described in the text. The dashed (red) curve shows the time evolution of $h^{(1)}_{20}$ computed with wavelength averaging via Eq. (2.27a). The averaging procedure removes the short timescale oscillations from the integrand. The insets zoom in on the low and high eccentricity regions.}
\end{figure}
FIG. 8. (color online). Effect of wavelength averaging on the nonlinear memory mode. The left plot shows the $h_{20}^{(mem)}$ nonlinear memory mode (the time integral of Figure 7). The solid (black) curve is without wavelength averaging; the dashed (red) curve is with averaging. The curves lie nearly on top of each other, aside from small amplitude oscillations that remain when one integrates the un-averaged integrand. The inset zooms in on the low eccentricity region. The integration is terminated at the last-stable-orbit. The right plot shows the absolute error (top panel) and the fractional error (bottom panel) between the two curves.

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