$L^2$ boundedness of Hilbert transforms along variable flat curves

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Abstract
In this paper, the $L^2$ boundedness of the Hilbert transform along variable flat curve $(t, P(x_1)\gamma(t))$

$$H_{P,\gamma}f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - P(x_1)\gamma(t)) \frac{dt}{t}, \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

is studied, where $P$ is a real polynomial on $\mathbb{R}$. A new sufficient condition on the curve $\gamma$ is introduced.

Keywords Hilbert transform · Variable flat curve

Mathematics Subject Classification Primary 42B20; Secondary 42B25

1 Introduction

In this paper we consider the Hilbert transform along variable flat curve $(t, P(x_1)\gamma(t))$

$$H_{P,\gamma}f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - P(x_1)\gamma(t)) \frac{dt}{t}, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \quad (1.1)$$

where $P : \mathbb{R} \to \mathbb{R}$ is a real polynomial. Bennett [4] pointed out that the prime interest in the study of the operator in (1.1) is to include some curves $\gamma$ which vanish to infinite order at the origin, or with other words infinitely flat, this further implies that our curve becomes
very close to being a “line”. One will see that the curve \((2.1)\) is such kind of curve and satisfies all the conditions in Theorem 1.1. Therefore, our results satisfy Bennett’s concerns. The history of this operator in (1.1) can be traced back to the outstanding Stein conjecture. For any measurable map \(v : \mathbb{R}^2 \to \{ x \in \mathbb{R}^2 : |x| = 1 \}\) and any Schwartz function \(f\) on \(\mathbb{R}^2\), define
\[
H_{v,e} f(x) := p.v. \int_{-\varepsilon}^{\varepsilon} f(x - v(x)t) \frac{dt}{t}, \quad \forall x \in \mathbb{R}^2.
\] (1.2)

In [25], Stein conjectured that \(H_{v,e}\) maps \(L^2\) into weak \(L^2\) whenever \(v\) is a Lipschitz function with \(\|v\|_{\text{Lip}} \approx \varepsilon^{-1}\). For smooth vector field \(v\), Christ et al. [15] established the \(L^p\) boundedness of \(H_{v,e}\) under some extra curvature conditions, where \(p \in (1, \infty)\). Later, Lacey and Li [19] obtained a conditional result saying that if a suitable Kakeya maximal operator is bounded then the \(C^{1+\alpha}\)-regularity of the vector field \(v\) implies the \(L^2\) boundedness of \(H_{v,e}\), where \(\alpha > 0\). More recently, Stein and Street [26] established the \(L^p\) boundedness of \(H_{v,e}\) for real analytic vector field \(v\), where \(p \in (1, \infty)\). For more details of Stein conjecture we refer the reader to the nice memoir [19]. To connect our operator with \(H_{v,e}\) in (1.2), let us ignore the cut-off and consider
\[
H_0 f(x) := p.v. \int_{-\infty}^{\infty} f(x - v(x)t) \frac{dt}{t}, \quad \forall x \in \mathbb{R}^2.
\]

In 2006, Lacey and Li [18] considered the case that \(H_v\) applied on a dyadic piece \(P_k f\) with \(k \in \mathbb{Z}\). Here \(P_k\) denotes the Littlewood-Paley projector operator. They showed that \(H_v P_k\) maps \(L^2\) into weak \(L^2\), and \(L^p\) into \(L^p\) for any measurable vector field \(v\) with the bounds that are independent of \(k\), where \(p \in (2, \infty)\).

Now we consider the case of the one-variable vector field, i.e. \(v(x_1, x_2) := (1, u(x_1))\). Let us define
\[
H_{u,\gamma} f(x_1, x_2) := p.v. \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \frac{dt}{t}, \quad \forall (x_1, x_2) \in \mathbb{R}^2.
\] (1.3)

For \(\gamma(t) := t\), based on Lacey and Li’s works [18,19], Bateman [1] proved the single annulus \(L^p\) estimates for \(H_{u,\gamma}\) with any measurable function \(u\), where \(p \in (1, \infty)\), and in [2], Bateman and Thiele obtained that \(H_{u,\gamma}\) is bounded on \(L^p(\mathbb{R}^2)\) for any given \(p \in (2, \infty)\). Moreover, let \(\gamma\) be \(|t|^{\alpha}\) or \(\text{sgn}(t)|t|^{\alpha}\), \(\alpha > 0, \alpha \neq 1\), Guo et al. [17] obtained the \(L^p\) boundedness of \(H_{u,\gamma}\) for all \(p \in (1, \infty)\).

In this paper we consider a more special case saying that \(u\) is a polynomial, i.e. \(u := P\). Then \(H_{u,\gamma}\) in (1.3) becomes the operator \(H_{p,\gamma}\) in (1.1). In this case, our proof goes by induction on the degree of the polynomial. The start point of the induction is the \(L^p\) boundedness of the following Hilbert transform along curve \((t, \gamma(t))\)
\[
H_{\gamma} f(x_1, x_2) := p.v. \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \gamma(t)) \frac{dt}{t}, \quad \forall (x_1, x_2) \in \mathbb{R}^2.
\] (1.4)

This operator has independent interests, which is another motivation of this paper. A fundamental question here is to establish the \(L^p\) boundedness of (1.4) under some general conditions of the curve \(\gamma\). We refer to [5,9,11,14,16,20,28,29] for details. As Stein and Waigner pointed out in [27] that the curvature of the considered curve plays a crucial role in this project. In the same paper, Stein and Waigner showed that if \(\gamma\) is well-curved\(^1\) then \(H_{\gamma}\) is bounded on \(L^p(\mathbb{R}^2)\) for any given \(p \in (1, \infty)\). In [12], the well-curved condition was

\(^1\) We refer the reader to P.1240 in [27] for the definition of the well-curved curve.
replaced by \( \gamma \in C^2(0, \infty) \) is either odd or even, convex curve, \( \gamma(0) = \gamma'(0) = 0 \), and satisfies the following doubling condition:

There exists \( \lambda \in (1, \infty) \) so that \( \gamma'(\lambda t) \geq 2\gamma'(t) \) for any \( t \in (0, \infty) \). (D)

Let \( h(t) := t\gamma'(t) - \gamma(t) \). In [6] condition (D) was replaced by the following infinitesimally doubling condition:

There exists \( \varepsilon_0 \in (0, \infty) \) so that \( h'(t) \geq \varepsilon_0 \frac{h(t)}{t} \) for any \( t \in (0, \infty) \). (ID)

There are more general curves to guarantee the \( L^p \) boundedness of (1.4) for any given \( p \in (1, \infty) \), see [30]. We only need to (D) or (ID) in this paper. We can now state our main result on the boundedness of \( H_{P, \gamma} \) in (1.1).

**Theorem 1.1** Let \( P : \mathbb{R} \to \mathbb{R} \) be a real polynomial of degree \( n \), and \( \gamma \in C^2(\mathbb{R}) \) be either odd or even, convex curve on \( (0, \infty) \), and satisfying

(i) \( \gamma(0) = \gamma'(0) = 0 \),

(ii) \( \frac{\gamma''(t)}{\gamma'(t)} \) is decreasing on \( (0, \infty) \),

(iii) There exists a positive constant \( C_1 \) such that \( \frac{\gamma''(t)}{\gamma'(t)} \geq C_1 \) for any \( t \in (0, \infty) \),

(iv) \( \gamma''(t) \) is monotone on \( (0, \infty) \).

Then the Hilbert transform \( H_{P, \gamma} \) is bounded on \( L^2(\mathbb{R}^2) \) with a bound that can be taken to be independent of the coefficients of \( P \) and depends only on \( n \) and \( C_1 \).

There were some results in this topic; see, for example, [7,8,23,24]. The condition (i) in Theorem 1.1 shows that the curve \( \gamma \) has a certain smoothness at the origin. Except for this condition, the conditions (ii),(iii) and (iv) in Theorem 1.1 are used to describe the curvature of the curve \( \gamma \). In [10], Carbery et al. set up the \( L^p \) boundedness of \( H_{P, \gamma} \) with \( P(x_1) := x_1 \) for any given \( p \in (1, \infty) \), where the curvature conditions are as follows:

\[ \lambda(t) := \frac{t\gamma''(t)}{\gamma'(t)} \text{ is decreasing on } (0, \infty) \text{ and has a positive bounded from below.} \quad \text{(CWW)} \]

Under the same conditions, Bennett [4] established the \( L^2 \) boundedness of \( H_{P, \gamma} \) for general polynomial \( P \). More recently, Chen and Zhu [13] obtained the \( L^2 \) boundedness of \( H_{P, \gamma} \) by asking the curvature conditions as

\[ \left( \frac{\gamma''}{\gamma'} \right)'(t) \leq -\frac{\lambda}{t^2} \text{ for any } t \in (0, \infty) \text{ and some positive constant } \lambda. \quad \text{(CZ)} \]

The most significant difference in the conditions imposed on curve \( \gamma \) between the current paper and Bennett [4] is that Theorem 1.1 requires \( \frac{\gamma''(t)}{\gamma'(t)} \) is decreasing on \( (0, \infty) \) rather that \( \frac{t\gamma''(t)}{\gamma'(t)} \) is decreasing on \( (0, \infty) \). This is essentially because, in Bennett [4], many of estimates are established from the inequality that

\[ \left| \frac{\lambda'(t)}{t} - \frac{\lambda(t)}{t^2} \right| \geq \left| \frac{\lambda(t)}{t^2} \right|, \quad \forall t \in [1, 2], \]

such as [4, P.4883]. But in this paper, we do not need this inequality. On the other hand, Bennett [3] obtained the \( L^2 \) boundedness and weak type \((1,1)\) boundeness for

\[ \text{p.v.} \int_{-\infty}^{\infty} e^{-i\lambda x y(x-y)} f(y) \, dy, \quad \forall x \in \mathbb{R}, \]

by asking the condition that \( \frac{\gamma''(t)}{\gamma'(t)} \) is decreasing on \( (0, \infty) \) and some other conditions, which further implies that the \( L^2 \) boundedness of \( H_{P, \gamma} \) at the case that \( P(x_1) := x_1 \). Therefore, it
is natural to want to obtain the $L^2$ boundedness of $H_{P, \gamma}$ for all polynomial $P$ under the main condition that $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing on $(0, \infty)$.

This paper is organized as following. In Sect. 2, we want to make clear that our curvature conditions are not stronger than (CWW) or (CZ). Indeed, the conditions (ii) and (iii) in Theorem 1.1 are implied by (CWW) and (CZ). Our condition (iv) in Theorem 1.1 is not too strong. We will give an example which does not satisfy (CWW) or (CZ) but verifies our conditions. Section 3 contains some primary lemmas which will be used in the proof of the main result. In Sect. 4, we will give the proof of Theorem 1.1.

Throughout this paper, we always use $C$ to denote a positive constant, independent of the main parameters involved, but whose value may differ from line to line. We use $C_{(n)}$ to denote a positive constant depending on the indicated parameters $n$, and also whose value may differ from line to line. The positive constants with subscripts, such as $C_1$, do not change in different occurrences. For two real functions $f$ and $g$, if $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; if $f \lesssim g \lesssim f$, we then write $f \approx g$.

## 2 Curve $\gamma$

Since $\gamma$ is convex on $(0, \infty)$ and belongs to $C^2(0, \infty)$, thus

$$\gamma''(t) \geq 0, \quad \forall \ t \in (0, \infty).$$

It means that $\gamma'(t)$ is increasing on $(0, \infty)$. This, combined with the fact that $\gamma'(0) = 0$, further implies that

$$\gamma'(t) \geq 0, \quad \forall \ t \in (0, \infty).$$

The homogeneous curve $\gamma(t) := t^\alpha, \ t \in (0, \infty)$ with $\alpha > 1$ satisfies all the conditions in Theorem 1.1, for $t \in (-\infty, 0]$, $\gamma$ is given by its even or odd property. And the homogeneous curve is the model curve. Follows are some other curves $\gamma$ satisfy our conditions in Theorem 1.1. we here only write the part $t \in [0, \infty)$, and define $\gamma(t) := \pm \gamma(-t)$ for $t \in (-\infty, 0]$.

(i) for any $t \in [0, \infty)$, $\gamma(t) := t^2 \log(1 + t)$,

(ii) for any $t \in [0, \infty)$, $\gamma(t) := t^2 e^{-t}$,

(iii) for any $t \in [0, \infty)$, $\gamma(t) := \int_0^t \tau^\alpha \log(1 + \tau) \, d\tau, \ \alpha \in [1, \infty)$,

(iv) for any $t \in [0, \infty)$, $\gamma(t) := \int_0^t \tau^\alpha e^{-\frac{1}{2} \tau} \, d\tau, \ \alpha \in [1, \infty)$,

(v) for any $t \in [0, \infty)$, $\gamma(t) := \int_0^t \tau^\alpha \arctan \tau \, d\tau, \ \alpha \in [1, \infty)$.

It is easy to see that (CWW) implies conditions (ii) and (iii). It is also clear that (CZ) implies condition (ii). Actually in [13], instead of (CZ), Chen and Zhu used a weaker condition

$$\frac{\gamma''(t)}{\gamma'(t)} - \frac{\gamma''(s)}{\gamma'(s)} \geq \frac{\lambda(s-t)^M}{(s+t)^{M+1}}, \quad \forall \ 0 < t < s, \quad \text{(wCZ)}$$

for some positive constants $\lambda$ and $M$, which is also stronger than the condition that $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing on $(0, \infty)$. Meanwhile, condition (CZ) also implies condition (iii). To see this, let us denote

$$F(t) := \frac{\gamma''(t)}{\gamma'(t)} - \frac{\lambda}{t}, \quad \forall \ t \in (0, \infty).$$
Since we know that $\gamma''(t) \geq 0$ for any $t \in (0, \infty)$ and $-\frac{\lambda}{t} \to 0$ as $t \to \infty$. We have $\lim_{t \to \infty} F(t) \geq 0$. By (CZ), we have

$$F'(t) = \left(\frac{\gamma''}{\gamma'}\right)'(t) + \frac{\lambda}{t^2} \leq 0, \quad \forall \ t \in (0, \infty).$$

Thus we have

$$F(t) \geq 0 \iff \frac{t\gamma''(t)}{\gamma'(t)} \geq \lambda, \quad \forall \ t \in (0, \infty).$$

The condition (iv) has already appeared in [21]. There, Nagel and Winger proved that under conditions (i),(iii) and (iv), $H_p$ in (1.4) is bounded on $L^p(\mathbb{R}^2)$ for $p$ is very close to 2. Actually, if $\gamma''(t)$ is increasing on $(0, \infty)$, then (iv) implies (iii). But if $\gamma''(t)$ is decreasing on $(0, \infty)$, then all the conditions in Theorem 1.1 are independent from each other. For given $x, y \in (0, \infty)$, let $0 < z - y < z - x < \infty$, we set

$$\varphi(z) := \gamma''(z - x) - \gamma''(z - y).$$

Condition (iv) is used to guarantee that the sign of $\varphi$ changes only a finite number of times on $(0, \infty)$ and the number does not depend on $x, y, z$. There are many different ways to form up this condition, and here we use condition (iv) in the proof of Theorem 1.1.

Now we give an example that is not obeying the condition (CWW) or (CZ) but it does satisfy our conditions (i),(ii),(iii) and (iv). Thus the conditions in Theorem 1.1 are not trivial. Let

$$\gamma(t) := \int_0^t e^{\tau} e^{-\frac{1}{\tau}} \, d\tau, \quad \forall \ t \in [0, \infty), \quad (2.1)$$

and

$$\gamma(t) := \pm \gamma(-t), \quad \forall \ t \in (-\infty, 0].$$

We calculate

$$\gamma'(t) = e^t e^{-\frac{1}{t}}, \quad \forall \ t \in [0, \infty),$$

$$\gamma''(t) = e^t e^{-\frac{1}{t}} + e^t e^{-\frac{1}{t}} t^{-2} \geq 0, \quad \forall \ t \in [0, \infty),$$

$$\gamma'''(t) = e^t e^{-\frac{1}{t}} \left(1 + 2t^{-2} - 2t^{-3} + t^{-4}\right) = e^t e^{-\frac{1}{t}} \frac{t^4 - 2(t - \frac{1}{2})^2 + \frac{1}{4}}{t^4} \geq 0, \quad \forall \ t \in [0, \infty).$$

So we have $\gamma \in C^2(\mathbb{R})$ is either odd or even, convex curve on $(0, \infty)$, and $\gamma''(t)$ is increasing on $(0, \infty)$, and

$$\gamma(0) = \gamma'(0) = 0,$$

$$\frac{\gamma''(t)}{\gamma'(t)} = 1 + \frac{1}{t^2}, \quad \forall \ t \in [0, \infty).$$

Therefore, $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing on $(0, \infty)$ and

$$\frac{t\gamma''(t)}{\gamma'(t)} = t + \frac{1}{t} \geq 2, \quad \forall \ t \in [0, \infty).$$

Then, curve $\gamma$ satisfies the conditions (i), (ii), (iii) and (iv) in Theorem 1.1.
We now show that this curve does not satisfy (CWW). For
\[ \lambda(t) = \frac{ty''(t)}{y'(t)} = t + \frac{1}{t}, \quad \forall \, t \in (0, \infty). \]

It is easy to get that
\[ \lambda'(t) = 1 - \frac{1}{t^2}, \quad \forall \, t \in (0, \infty). \]
Thus, \( \lambda(t) \) is not decreasing on \( (0, \infty) \).

For condition (CZ), we have
\[ \left( \frac{y''}{y'} \right)'(t) = -2t^{-3}, \quad \forall \, t \in (0, \infty). \]
Thus, there is no positive number \( \lambda \) such that (CZ) is true.

Even for the weaker condition (wCZ). Let \( s := 2t \), if \( y \) satisfies (wCZ), then
\[ \frac{1}{t^2} - \frac{1}{4t^2} \geq \frac{\lambda t^M}{(3t)^{M+1}}, \quad \forall \, t \in (0, \infty), \]
thus
\[ \frac{3}{4t} \geq \frac{\lambda}{3^{M+1}}, \quad \forall \, t \in (0, \infty). \]
But such positive constants \( \lambda \) and \( M \) can not exist as \( t \to \infty \).

Next, we try to give a geometric explanation of the conditions (ii) and (iii). Condition (ii) is equivalent to the condition that \( \frac{y''(t)}{y'(t)} \) is increasing on \( (0, \infty) \). See Fig. 1, point \( A \) is the intersection between \( x \)-axis and the tangent line to the curve \((x, y) = (t, y'(t))\), and point \( B(0, 0) \) is the intersection between \( x \)-axis and the vertical line to the curve \((x, y) = (t, y'(t))\), then the formula \( \frac{y'(t)}{y''(t)} \) is the distance between points \( A \) and \( B \). Condition (ii) said that the distance between points \( A \) and \( B \) will grow large as \( t \) goes to right. But condition (iii) said that the distance is always less than \( \frac{C_1}{L} \), where \( C_1 \) is the constant in condition (iii). If we simply set \( C_1 \geq 1 \) then it means that \( y' \) is also convex on \( (0, \infty) \).

Actually, if \( y' \) is also convex on \( (0, \infty) \), then we have that \( y''(t) \) is increasing on \( (0, \infty) \). Thus condition (iii) can also be true with \( C_1 := 1 \). Thus we have the following corollary:

**Corollary 2.1** Let \( P : \mathbb{R} \to \mathbb{R} \) be a real polynomial of degree \( n \), and \( y \in C^2(\mathbb{R}) \) be either odd or even, convex curve on \( (0, \infty) \), and satisfying

1. \( y(0) = y'(0) = 0 \),
2. \( y''(0) \) is decreasing on \( (0, \infty) \),
3. \( y'(t) \) is also convex on \( (0, \infty) \).

Then the Hilbert transform \( H_{P, y} \) is bounded on \( L^2(\mathbb{R}^2) \) with a bound that can be taken to be independent of the coefficients of \( P \) and depends only on \( n \).

### 3 Some Lemmas

In this section, we collect several lemmas which will be used in the proof of Theorem 1.1.

**Lemma 3.1** The conditions in Theorem 1.1 imply the doubling condition (D) and infinitesimally doubling condition (ID).
Proof For doubling condition (D), notice that $\gamma(0) = \gamma'(0) = 0$, $\gamma'(t)$ is increasing on $(0, \infty)$, and there exists a positive constant $C_1$ such that $\frac{\gamma''(t)}{\gamma'(t)} \geq C_1$ for any $t \in (0, \infty)$. Let $\lambda := e^{2C_1}$, then

$$\gamma'\left(\lambda t\right) = \int_0^{\lambda t} \gamma''(\tau) d\tau \geq \int_t^{\lambda t} \gamma''(\tau) d\tau \geq C_1 \int_t^{\lambda t} \frac{\gamma'(\tau)}{\tau} d\tau \geq C_1 \gamma'(t) \log \lambda = 2\gamma'(t), \quad \forall t \in (0, \infty).$$

To verify the infinitesimally doubling condition (ID), let $\epsilon_0 := C_1$, then

$$\frac{h'(t)}{\gamma'(t)} = \frac{t\gamma''(t)}{\gamma'(t)} \geq C_1 \geq C_1 \frac{t\gamma'(t) - \gamma(t)}{t\gamma'(t)} = C_1 \frac{h(t)}{t\gamma'(t)}, \quad \forall t \in (0, \infty),$$

thus

$$h'(t) \geq \epsilon_0 \frac{h(t)}{t}, \quad \forall t \in (0, \infty).$$

\[ \square \]

Lemma 3.2 Let $\gamma$ be the same as in Theorems 1.1. Let $x, y \in \mathbb{R}$ be fixed. If $1 \leq z - y < z - x \leq 2$, then $\Upsilon(z) := \frac{\gamma'(x) - \gamma'(y)}{\gamma'(x) - \gamma'(y)}$ is increasing on this domain, and $\Upsilon(z) \leq \frac{2}{C_1}$ uniformly in $\omega$ and $k$ on this domain, where $\omega \in \mathbb{R}$, $\omega > 0$ and $k \in \mathbb{Z}, k \geq 0$.

Proof Let $g(s, t) := \frac{\gamma'(s) - \gamma'(t)}{\gamma'(s) - \gamma'(t)}$ for any $s > t > 0$. By generalized mean value theorem and notice that $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing on $(0, \infty)$, it is easy to see that $\frac{\partial g}{\partial s} \geq 0$ and $\frac{\partial g}{\partial t} \geq 0$. Thus, for fixed $x, y \in \mathbb{R}$, for any $z$ satisfies $1 \leq z - y < z - x \leq 2$, we have $\Upsilon(z) = \frac{1}{\omega2^k} g(\omega2^k(z - x), \omega2^k(z - y))$. Therefore, we get

$$\Upsilon'(z) = \frac{\partial g}{\partial s}(\omega2^k(z - x), \omega2^k(z - y)) + \frac{\partial g}{\partial t}(\omega2^k(z - x), \omega2^k(z - y)) \geq 0.$$ 

Thus, $\Upsilon$ is increasing uniformly in $\omega$ and $k$ on this domain.
By the generalised mean value theorem, and the fact that \( \frac{\gamma''(t)}{\gamma'(t)} \geq C_1 \) for any \( t \in (0, \infty) \), we obtain
\[
\gamma(z) = \frac{\gamma'(\omega 2^k \theta_1)}{\omega 2^k \gamma''(\omega 2^k \theta_1)} = \theta_1 \frac{\gamma'(\omega 2^k \theta_1)}{\omega 2^k \theta_1 \gamma''(\omega 2^k \theta_1)} \leq \frac{2}{C_1}
\]
on this domain, where \( 1 \leq z - y \leq \theta_1 \leq z - x \leq 2 \).

**Lemma 3.3** [4, Lemma 13] Let \( P \) be a real monic polynomial of degree \( n \), \( n > 0 \), and \( U \) be the union of the set of roots of \( P' \) and of \( P'' \) over \( \mathbb{R} \). There exists \( C > 0 \) depending only on \( n \), such that if \( \text{dist}(z, U) > \delta \), then \( |P'(z)| \geq C \delta^{n-1} \) for any \( \delta > 0 \).

**Lemma 3.4** Let \( P \) be a real monic polynomial of degree \( n \), \( n > 0 \), and
\[
E_k := \left\{ x \in \mathbb{R} : \left| \frac{\bar{P}_k(x)}{\bar{P}_k(x)} \right| \leq \frac{4}{C_1} \text{ and } \left( \frac{\bar{P}_k}{\bar{P}_k} \right)'(x) \leq \frac{1}{8n} \right\},
\]
where \( \bar{P}_k(x) := 2^{-nk}P(2^k x), k \geq 0, k \in \mathbb{Z} \). Then for any \( \alpha \in (0, 1) \), we have \( \Sigma_{k \geq 0}|E_k|^\alpha \leq C \), where \( C \) only depends on \( n \) and \( C_1 \).

**Proof** We denote the roots of \( P \) as \( \{v_j\}_{j=1}^m \subset \mathbb{R} \) and \( \{\beta_j\}_{j=1}^{n'} \cup \{\bar{\beta}_j\}_{j=1}^{n''} \subset \mathbb{C} \setminus \mathbb{R} \), where \( 0 \leq m \leq n \), \( n' := \frac{n-m}{2} \) and \( \beta_j := a_j + ib_j \). Then
\[
\bar{P}_k(x) = \prod_{j=1}^m (x - 2^{-k} v_j) \prod_{j=1}^{n'} (x - 2^{-k} \beta_j) \prod_{j=1}^{n''} (x - 2^{-k} \bar{\beta}_j).
\]
Thus
\[
\left( \frac{\bar{P}_k}{\bar{P}_k} \right)'(x) = -\sum_{j=1}^m \frac{1}{(x - 2^{-k} v_j)^2} - 2 \sum_{j=1}^{n'} \frac{(x - 2^{-k} a_j)^2 - (2^{-k} b_j)^2}{(x - 2^{-k} a_j)^2 + (2^{-k} b_j)^2}.
\]
For any \( l \in \mathbb{Z} \), we set
\[
E_{k,l} := \left\{ x \in \mathbb{R} : 2^{l-1} \frac{4}{C_1} \leq \left| \frac{\bar{P}_k(x)}{\bar{P}_k(x)} \right| \leq 2^l \frac{4}{C_1} \text{ and } \left( \frac{\bar{P}_k}{\bar{P}_k} \right)'(x) \leq \frac{1}{8n} \right\}.
\]
Hence,
\[
|E_k| = \sum_{l=-\infty}^{0} |E_{k,l}|. \tag{3.1}
\]
Notice that \( \left( \frac{\bar{P}_k}{\bar{P}_k} \right)'(x) = -\left( \frac{\bar{P}_k}{\bar{P}_k} \right)^2 \left( \frac{\bar{P}_k}{\bar{P}_k} \right)'(x) \), for any \( x \in E_{k,l} \), by definition
\[
\left( \frac{\bar{P}_k}{\bar{P}_k} \right)^2 \left( \frac{\bar{P}_k}{\bar{P}_k} \right)'(x) = \sum_{j=1}^m \frac{1}{(x - 2^{-k} v_j)^2} + 2 \sum_{j=1}^{n'} \frac{(x - 2^{-k} a_j)^2 - (2^{-k} b_j)^2}{(x - 2^{-k} a_j)^2 + (2^{-k} b_j)^2} \leq \frac{1}{8n} \tag{3.2}
\]
and
\[ \left| \sum_{j=1}^{m} \frac{1}{x - 2^{-k}v_j} + 2 \sum_{j=1}^{n'} \frac{x - 2^{-k}a_j}{(x - 2^{-k}a_j)^2 + (2^{-k}b_j)^2} \right| \geq C_1 \frac{4}{2^l}. \] (3.3)

From which we obtain
\[ \sum_{j=1}^{m} \frac{1}{|x - 2^{-k}v_j|} + 2 \sum_{j=1}^{n'} \frac{|x - 2^{-k}a_j|}{(x - 2^{-k}a_j)^2 + (2^{-k}b_j)^2} \geq \frac{C_1}{4 \cdot 2^l}. \] (3.4)

By Hölder’s inequality, (3.4) implies
\[ \sum_{j=1}^{m} \frac{1}{(x - 2^{-k}v_j)^2} + 2 \sum_{j=1}^{n'} \frac{(x - 2^{-k}a_j)^2}{(x - 2^{-k}a_j)^2 + (2^{-k}b_j)^2} \geq \frac{1}{n} \left( \frac{C_1}{4 \cdot 2^l} \right)^2. \] (3.5)

From (3.2), we have
\[ \sum_{j=1}^{m} \frac{1}{(x - 2^{-k}v_j)^2} + 2 \sum_{j=1}^{n'} \frac{(x - 2^{-k}a_j)^2 - (2^{-k}b_j)^2}{(x - 2^{-k}a_j)^2 + (2^{-k}b_j)^2} \leq \frac{1}{8n} \left( \frac{C_1}{2 \cdot 2^l} \right)^2. \] (3.6)

Combining (3.5) and (3.6), we have
\[ \sum_{j=1}^{n'} \frac{(2^{-k}b_j)^2}{((x - 2^{-k}a_j)^2 + (2^{-k}b_j)^2)^2} \geq \frac{1}{2} \left( \frac{1}{n} \left( \frac{C_1}{4 \cdot 2^l} \right)^2 - \frac{1}{8n} \left( \frac{C_1}{2 \cdot 2^l} \right)^2 \right) = \frac{1}{64n} \left( \frac{C_1}{2^l} \right)^2. \] (3.7)

By Minkowski’s inequality, it gives
\[ \sum_{j=1}^{n'} \frac{|2^{-k}b_j|}{(x - 2^{-k}a_j)^2 + (2^{-k}b_j)^2} \geq \frac{1}{8n^{1/2}} \frac{C_1}{2^l}. \] (3.8)

By pigeonholing, there exists \( 1 \leq j_0 \leq n' \) such that
\[ \frac{|2^{-k}b_{j_0}|}{(x - 2^{-k}a_{j_0})^2 + (2^{-k}b_{j_0})^2} \geq \frac{1}{8n' \cdot 2^{l/2}} \frac{C_1}{2^l}. \] (3.9)

It is then not hard to check that
\[ E_{k,l} \subset \bigcup_{j=1}^{n'} E_{k,l,j}, \]
where
\[ E_{k,l,j} := \left\{ x \in \mathbb{R} : \frac{8n' \cdot 2^{l/2}}{C_1} \text{ and } \frac{|2^{-k}b_{j_0}|}{(x - 2^{-k}a_{j_0})^2 + (2^{-k}b_{j_0})^2} \leq \left( \frac{8n' \cdot 2^{l/2}}{C_1} \right)^{1/2} \frac{1}{2^{l/2}} \right\}. \] (3.10)
Since $E_{k,l,j} = \emptyset$ if $|2^{-k}b_j| \geq \frac{8n'\frac{1}{2}l}{c_1}$, and $|E_{k,l,j}| \leq \left(\frac{8n'\frac{1}{2}l}{c_1}\right)^{\frac{1}{2}} |2^{-k}b_j|^\frac{1}{2}$ if $|2^{-k}b_j| \leq \frac{8n'\frac{1}{2}l}{c_1}$, thus

$$\sum_{k \geq 0} |E_{k,l,j}|^\alpha \lesssim 2^{\alpha l}$$

uniformly in $j$ and $l$. Hence, for any $\alpha \in (0, 1)$,

$$\sum_{k \geq 0} |E_k|^\alpha \leq \sum_{k \geq 0} \left(\sum_{l=-\infty}^{0} \sum_{j=1}^{n'} |E_{k,l,j}|\right)^\alpha \leq \sum_{l=-\infty}^{0} \sum_{j=1}^{n} \sum_{k \geq 0} |E_{k,l,j}|^\alpha \lesssim \sum_{l=-\infty}^{0} 2^{\alpha l} \lesssim 1.$$ 

So far we have finished the proof of Lemma 3.4. \qed

It is easy to see that

**Lemma 3.5** Let $f \in C(\mathbb{R})$, suppose that the sign of $f'$ changes $m$ times on $\mathbb{R}$ and there exists a positive constant $C$ such that $|f(z)| \leq C$ for any $z \in \mathbb{R}$. Then $\int_{-\infty}^{\infty} |f'(z)| \, dz \leq 2(m+1)C$.

**4 Proof of the main result**

In this section, we devote to the proof of Theorem 1.1. By Fourier transform and Plancherel’s formula, see [22], we have

$$\|H_{P,\gamma}\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \leq \sup_{u \in \mathbb{R}} \|S_u\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})},$$

where

$$S_u f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{-iuP(x)\gamma(y)} f(x - y) \, \frac{dy}{y}, \quad \forall x \in \mathbb{R}. \quad (4.1)$$

Here $P$ is a polynomial of degree $n$. Thus, the remainder of the proof is devoted to the proof of the following Proposition:

**Proposition 4.1** Let $S_u$ be defined as in (4.1), we have

$$\|S_u\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C. \quad (4.2)$$

Here $C$ is independent of the coefficients of polynomial $P$ and $u$, it is depends only on $n$ and $C_1$.

**Proof** As in [4,10,13], we will proceed by induction on the degree of the polynomial $P$. The start point is the case that $n = 0$. The $L^2$ boundedness of $S_u$ then can be converted to the $L^2$ boundedness of the following directional Hilbert transform $H_{\lambda,\gamma}$ along curve $\gamma$ defined for a fixed direction $(1, \lambda)$ for any $\lambda \in \mathbb{R}$ as

$$H_{\lambda,\gamma} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \lambda \gamma(t)) \, \frac{dt}{t}, \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$ 

By scaling on the second variable we know that the $L^2$ boundedness of $H_{\lambda,\gamma}$, with $\lambda \neq 0$, is the same as the Hilbert transform (1.4). Since the $L^2$ boundedness of $H_{0,\gamma}$ is trivial, thus we could obtain an uniform $L^2$ estimate for $S_u$ if we obtain the $L^2$ boundedness of $H_{\gamma}$. As
we pointed out in Lemma 3.1, the conditions in Theorem 1.1 imply the doubling condition (D) and infinitesimally doubling condition (ID), by applying the result in [6,12], we obtain the result for the case that \(n = 0\).

We now consider the case that \(n > 0\). Suppose \(s\) is the coefficient of the term of the highest order in \(P\), since \(\gamma(0) = 0\) and \(\gamma(t)\) is increasing on \((0, \infty)\), we can take a positive constant \(\omega\), unless \(\gamma' \equiv 0\) which is trivial, such that

\[
| -us|\alpha^n \gamma(\alpha) = 1. \tag{4.3}
\]

Thus

\[
S_u f(\omega x) = p.v. \int_{-\infty}^{\infty} e^{-i\omega P(\omega x)\gamma(\omega x - y)} f(y) \frac{dy}{\omega x - y} = p.v. \int_{-\infty}^{\infty} e^{-i\omega \gamma(\omega) P(\omega) \frac{\gamma(\omega x - y)}{\gamma(\omega)}} f(\omega y) \frac{dy}{x - y}.
\]

We define

\[
S f(x) := p.v. \int_{-\infty}^{\infty} e^{i\omega x (x-y)} f(y) \frac{dy}{x - y}, \tag{4.4}
\]

where \(P\) is a monic polynomial, \(\check{\gamma}(t) := \frac{\gamma(\omega t)}{\gamma(\omega)}\). By scaling, we need then to set up

\[
\|S\|_{L^2(\mathbb{R})} \lesssim 1. \tag{4.5}
\]

Suppose that \(S\) is bounded on \(L^2(\mathbb{R})\) for all polynomials \(P\) of degree less than \(n\), with a bound that is independent of the coefficients of \(P\). We decompose

\[
S f(x) = \int |x-y| \leq 1 e^{i\omega x (x-y)} f(y) \frac{dy}{x - y} + \sum_{k \geq 0} \int_{2^k \leq |x-y| \leq 2^{k+1}} e^{i\omega x (x-y)} f(y) \frac{dy}{x - y} \equiv S^1 f(x) + \sum_{k \geq 0} S_k f(x). \tag{4.6}
\]

The first part is easy. It is exactly as the local part in [4], where the author asked for \(\check{\gamma}\) is convex on \((0, \infty)\), \(\check{\gamma}(0) = 0\), \(\check{\gamma}(1) = 1\), and the inductive hypothesis. Thus

\[
\|S^1\|_{L^2(\mathbb{R})} \lesssim 1
\]

with a bound that is independent of the coefficients of \(P\).

For \(k \geq 0\), we set

\[
\tilde{S}_k f(x) := \int_{1 \leq |x-y| \leq 2} e^{i2^{nk} \frac{\gamma(\omega x)}{\gamma(\omega)}} P_k(x) \frac{f(y)}{x - y} dy,
\]

where \(P_k(x) := 2^{-nk} P(2^k x)\) is a real monic polynomial. \(\tilde{S}_k\) is a scaling of \(S_k\) and thus share the same \(L^p\) norm for any given \(p \in (1, \infty)\). Since \(\gamma\) is either even or odd, we here only consider the operator

\[
\tilde{S}_k f(x) := \int_{1 \leq |x-y| \leq 2} e^{i2^{nk} \frac{\gamma(\omega x)}{\gamma(\omega)}} P_k(x) \frac{f(y)}{x - y} dy.
\]

To distinct the critical points of the phase function, let

\[
E_k := \left\{ x \in \mathbb{R} : \left| \frac{P_k'(x)}{P_k(x)} \right| \leq \frac{4}{C_1} \text{ and } \frac{P_k(x)}{P_k'}(x) \leq \frac{1}{8n} \right\}.
\]
By this set, we decompose \( \tilde{S}_k \) further as \( \tilde{S}_{ka} \) where the phase function has critical points, and \( \tilde{S}_{kb} \) where the phase function has not critical points.

\[
\tilde{S}_{ka} f(x) := \chi_{E_k}(x) \int_{1 \leq x - y \leq 2} e^{i 2^{nk} \gamma(2^k)} \frac{\tilde{P}_k(x) z(2^k(x - y))}{\gamma(2^k)} f(y) \frac{dy}{x - y}
\]

and

\[
\tilde{S}_{kb} f(x) := \chi_{E_k}(x) \int_{1 \leq x - y \leq 2} e^{i 2^{nk} \gamma(2^k)} \frac{\tilde{P}_k(x) z(2^k(x - y))}{\gamma(2^k)} f(y) \frac{dy}{x - y}.
\]

For \( \tilde{S}_{ka} \), we use the character that the set \( E_k \) is not very large. It is easy to see that

\[
\| \tilde{S}_{ka} f \|_{L^1(\mathbb{R})} \leq |E_k| \| f \|_{L^1(\mathbb{R})}
\]

and

\[
\| \tilde{S}_{ka} f \|_{L^\infty(\mathbb{R})} \lesssim \| f \|_{L^\infty(\mathbb{R})}.
\]

By interpolation,

\[
\| \tilde{S}_{ka} f \|_{L^p(\mathbb{R})} \lesssim |E_k|^{\frac{1}{p}} \| f \|_{L^p(\mathbb{R})}.
\]

Thus by Lemma 3.4,

\[
\sum_{k \geq 0} \| \tilde{S}_{ka} \|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})} \lesssim \sum_{k \geq 0} |E_k|^{\frac{1}{p}} \lesssim 1. \tag{4.7}
\]

For \( \tilde{S}_{kb} \), the bad news is that the size of \( E_k^b \) is large, but the good news is that there is no critical points of the phase function. Thus we run a \( TT^* \) argument. It is easy to see that

\[
\| \tilde{S}_{kb} \|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \| \tilde{S}_{kb} \tilde{S}_{kb} \|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}^{\frac{1}{2}} \tag{4.8}
\]

The kernel of \( \tilde{S}_{kb} \) can be written as

\[
L_k(x, y) := \int_{1 \leq z - x, z - y \leq 2, z \notin E_k} e^{i 2^{nk} \gamma(2^k) \frac{\tilde{P}_k(z)(\gamma(2^k(z - x)) - \gamma(2^k(z - y)))}{\gamma(2^k)}} (z - x)(z - y) dz.
\]

By symmetry, it suffices to consider the kernel

\[
\mathbb{L}_k(x, y) := \int_{1 \leq z - x, z - y \leq 2, z \notin E_k} e^{i 2^{nk} \gamma(2^k) \frac{\tilde{P}_k(z)(\gamma(2^k(z - x)) - \gamma(2^k(z - y)))}{\gamma(2^k)}} (z - x)(z - y) dz.
\]

For fixed \( x, y \in \mathbb{R} \), and for any \( z \in \mathbb{R} \) satisfies \( 1 \leq z - y < z - x \leq 2 \), let

\[
\phi(z) := 2^{nk} \frac{\gamma(2^k)}{\gamma(\omega)} \left( \frac{\gamma(2^k(z - x))}{\gamma(2^k)} - \frac{\gamma(2^k(z - y))}{\gamma(2^k)} \right),
\]

\[
\psi(z) := \phi(z) \tilde{P}_k(z),
\]

and for any \( r \in \mathbb{R} \) satisfies \( 1 \leq r - y < r - x \leq 2 \), let

\[
J' := \int_{(y + 1, r) \setminus E_k} e^{i \psi(z)} dz. \tag{4.9}
\]
For (4.9), we collect two estimates in Proposition 4.2 which we will give the proof later. We notice that \( \frac{d}{dz} \frac{1}{(z-x)(z-y)} < 0 \) for any \( z \) satisfies \( 1 \leq z - y < z - x \leq 2 \) and \( E_k \) is made up by \( C(n) \) intervals. If we have the estimate (4.17), then

\[
\mathbb{I}_k(x, y) = \int_{1 \leq z - y < z - x \leq 2, z \notin E_k} \frac{d}{dz} \left( \frac{1}{(z-x)(z-y)} \right) \, dz \\
= \left[ \frac{J_z}{(z-x)(z-y)} \right]_{1 \leq z - y < z - x \leq 2, z \notin E_k} - \int_{1 \leq z - y < z - x \leq 2, z \notin E_k} J_z \frac{d}{dz} \left( \frac{1}{(z-x)(z-y)} \right) \, dz \\
\lesssim \left( \sup_{1 \leq z - y < z - x \leq 2} |J_z| \right) \frac{1}{|x-y|} \\
+ \left( \sup_{1 \leq z - y < z - x \leq 2} |J_z| \right) \int_{1 \leq z - y < z - x \leq 2, z \notin E_k} \frac{d}{dz} \left( \frac{1}{(z-x)(z-y)} \right) \, dz \\
\approx \left( \sup_{1 \leq z - y < z - x \leq 2} |J_z| \right) \frac{1}{|x-y|} \\
+ \left( \sup_{1 \leq z - y < z - x \leq 2} |J_z| \right) \int_{1 \leq z - y < z - x \leq 2, z \notin E_k} \frac{d}{dz} \left( \frac{1}{(z-x)(z-y)} \right) \, dz \\
\lesssim \left( \sup_{1 \leq z - y < z - x \leq 2} |J_z| \right) \frac{1}{|x-y|} \lesssim \left( \frac{1}{|x-y|} \right)^{\frac{n+1}{n}} 2^{-k}. \tag{4.10}
\]

Meanwhile, we have the following trivial estimate,

\[
|\mathbb{I}_k(x, y)| \leq \int_{1 \leq z - y < z - x \leq 2, z \notin E_k} \frac{1}{|(z-x)(z-y)|} \, dz \lesssim 1. \tag{4.11}
\]

Therefore,

\[
|\mathbb{I}_k(x, y)| \lesssim \min \left\{ \left( \frac{1}{|x-y|} \right)^{\frac{n+1}{n}} 2^{-k}, 1 \right\} \lesssim \left( \frac{1}{|x-y|} \right)^{\frac{1}{2}} 2^{-\frac{n}{2(n+1)} k}. \tag{4.12}
\]

From (4.11) and (4.12),

\[
\int_{-\infty}^{\infty} |\mathbb{I}_k(x, y)| \, dx \lesssim \int_{|x-y| \leq 1} \left( \frac{1}{|x-y|} \right)^{\frac{1}{2}} 2^{-\frac{n}{2(n+1)} k} \, dx \\
+ \int_{|x-y| \geq 1} \left( \frac{1}{|x-y|} \right)^{\frac{n+1}{n}} 2^{-k} \, dx \lesssim 2^{-\frac{n}{2(n+1)} k}. \tag{4.13}
\]

If we have the estimate (4.18), by the same argument as (4.10) we obtain

\[
\mathbb{I}_k(x, y) \lesssim \left( \frac{1}{|x-y|} \right)^{\frac{n+2}{n+1}} 2^{-\frac{n}{n+1} k}. \tag{4.14}
\]

Combine (4.11), which leads to

\[
|\mathbb{I}_k(x, y)| \lesssim \min \left\{ \left( \frac{1}{|x-y|} \right)^{\frac{n+2}{n+1}} 2^{-\frac{n}{n+1} k}, 1 \right\} \lesssim \left( \frac{1}{|x-y|} \right)^{\frac{1}{2}} 2^{-\frac{n}{2(n+2)} k}. \tag{4.15}
\]

From (4.14) and (4.15),

\[
\int_{-\infty}^{\infty} |\mathbb{I}_k(x, y)| \, dx \lesssim \int_{|x-y| \leq 1} \left( \frac{1}{|x-y|} \right)^{\frac{1}{2}} 2^{-\frac{n}{2(n+2)} k} \, dx
\]
\[
+ \int_{|x-y| \geq 1} \left( \frac{1}{|x-y|} \right)^{\frac{n+2}{n+1}} 2^{-\frac{n}{n+1} k} \, dx \lesssim 2^{-\frac{n}{n+2} k}. \tag{4.16}
\]

Both (4.13) and (4.16) lead us to
\[
\int_{-\infty}^{\infty} |L_k(x, y)| \, dx \lesssim 2^{-\frac{n}{n+1} k}.
\]

And it is easy to check that
\[
\int_{-\infty}^{\infty} |L_k(x, y)| \, dy \lesssim 2^{-\frac{n}{n+1} k}.
\]

By interpolation and (4.8), we have
\[
\sum_{k \geq 0} \| \tilde{S}_{kb} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim \sum_{k \geq 0} 2^{-\frac{n}{n+2} k} \lesssim 1.
\]

This finishes the proof of Proposition 4.1. \(\square\)

**Proposition 4.2** For fixed \(x < y\) and for any \(r \in \mathbb{R}\) satisfies \(1 \leq r - y < r - x \leq 2\), and \(z \notin E_k\), \(J'\) is defined in (4.10). We have the following estimates:

1. If \(\left| \frac{\tilde{P}_k(z)}{P_k'(z)} \right| > \frac{4}{C_1}\), then
   \[
   |J'| \leq C \left( \frac{1}{2^{nk}|x-y|} \right)^{\frac{1}{n}}. \tag{4.17}
   \]

2. If \(\left( \frac{\tilde{P}_k(z)}{P_k'(z)} \right)'(z) > \frac{1}{8n}\), then
   \[
   |J'| \leq C \left( \frac{1}{2^{nk}|x-y|} \right)^{\frac{1}{n+1}}. \tag{4.18}
   \]

Here the bound \(C\) is independent of the coefficients of polynomial \(P\) and depends only on \(n\) and \(C_1\).

**Proof** We recall that for fixed \(x, y \in \mathbb{R}\), and for any \(z \in \mathbb{R}\) satisfies \(1 \leq z - y < z - x \leq 2\),
\[
\phi(z) = 2^{nk} \frac{\gamma'(\omega 2^k)}{\gamma(\omega)} \left( \frac{\gamma'(\omega 2^k (z-x))}{\gamma(\omega 2^k)} - \frac{\gamma'(\omega 2^k (z-y))}{\gamma(\omega 2^k)} \right),
\]
\[
\psi(z) = \phi(z) \tilde{P}_k(z).
\]

It is easy to obtain
\[
\phi'(z) = 2^{nk} \frac{\gamma'(\omega 2^k)}{\gamma(\omega)} \left( \frac{\omega 2^k \gamma'(\omega 2^k (z-x))}{\gamma(\omega 2^k)} - \frac{\omega 2^k \gamma'(\omega 2^k (z-y))}{\gamma(\omega 2^k)} \right)
\]
and
\[
\psi'(z) = \phi(z) \tilde{P}_k'(z) + \phi'(z) \tilde{P}_k(z).
\]

It is natural that the critical points of the phase function in \(J'\) will be the obstacle to obtain the estimates (4.17) and (4.18). For this reason, we introduce the following set \(\Delta\). In this set there is no critical points of the phase function and its extra set is not too large. Let \(U\) be the
union of the set of roots of $\tilde{P}_k'$ and of $\tilde{P}_k''$ over $\mathbb{R}$. For fixed $x, y \in \mathbb{R}$, for any $r \in \mathbb{R}$ satisfies $1 \leq r - y < r - x \leq 2$ and any $\delta > 0$, let

\[ \Delta := \{ z \in (y + 1, r) : z \notin E_k \text{ and } \operatorname{dist}(z, U) > \delta \} \]

Then

\[ |(y + 1, r) \setminus E_k \setminus \Delta| \leq C(n) \delta \]

and

\[
|J_{\Delta}^r| = \left| \int_{(y + 1, r) \setminus E_k} e^{i \psi(z)} \, dz \right| \\
= \left| \int_{\Delta} e^{i \psi(z)} \, dz + \int_{(y + 1, r) \setminus E_k} e^{i \psi(z)} \, dz \right| \leq \left| \int_{\Delta} e^{i \psi(z)} \, dz \right| + \left| \int_{(y + 1, r) \setminus E_k} e^{i \psi(z)} \, dz \right|. \tag{4.19}
\]

The last integral then is controlled by $C(n) \delta$. It suffices to estimate

\[ \left| \int_{\Delta} e^{i \psi(z)} \, dz \right|. \]

**Case 1** $\left| \frac{\tilde{P}_k(z)}{P_k'(z)} \right| > \frac{4}{C_1}$. For $z \in \Delta$, we can write

\[
\frac{\psi'(z)}{\tilde{P}_k'(z) \phi'(z)} = \frac{\gamma(\omega 2^k(z - x)) - \gamma(\omega 2^k(z - y))}{\omega 2^k \gamma'(\omega 2^k(z - x)) - \omega 2^k \gamma'(\omega 2^k(z - y))} + \frac{\tilde{P}_k(z)}{P_k'(z)}. \tag{4.20}
\]

By Lemma 3.2, we get

\[
\left| \frac{\psi'(z)}{\tilde{P}_k'(z) \phi'(z)} \right| \geq \left| \frac{\tilde{P}_k(z)}{P_k'(z)} \right| - \left| \frac{\gamma(\omega 2^k(z - x)) - \gamma(\omega 2^k(z - y))}{\omega 2^k \gamma'(\omega 2^k(z - x)) - \omega 2^k \gamma'(\omega 2^k(z - y))} \right| \geq \frac{4}{C_1} - \frac{2}{C_1} = \frac{2}{C_1}.
\]

On the other hand, by the generalised mean value theorem, $\gamma(t)$ and $\gamma'(t)$ are increasing on $(0, \infty)$, which yields $\frac{\gamma'(t)}{\gamma(t)} \geq 1$ for any $t \in (0, \infty)$, and

\[
|\phi'(z)| = 2^{nk} \frac{\gamma'(\omega 2^k)}{\gamma(\omega)} \left( \frac{\omega 2^k \gamma'(\omega 2^k(z - x)) - \omega 2^k \gamma'(\omega 2^k(z - y))}{\gamma(\omega 2^k)} \right) \geq 2^{nk} \frac{\gamma'(\omega 2^k)}{\gamma(\omega)} \left( \frac{\omega 2^k \gamma'(\omega 2^k(z - x)) - \omega 2^k \gamma'(\omega 2^k(z - y))}{\gamma(\omega 2^k)} \right) \geq 2^{nk} |x - y| \geq 2^{nk} |x - y|. \tag{4.21}
\]

where $1 \leq z - y \leq \theta_2 \leq z - x \leq 2$. Lemma 3.3 leads to

\[ |\psi'(z)| \geq 2^{nk} |x - y| \delta^{n-1}. \]

Then

\[
\int_{\Delta} e^{i \psi(z)} \, dz = \int_{\Delta} \frac{d}{dz} \left( \frac{e^{i \psi(z)}}{\tilde{P}_k'(z) \phi'(z)} \right) \frac{\tilde{P}_k'(z) \phi'(z)}{i \psi'(z)} \, dz.
\]
From (4.21), it follows that

\[ \frac{P_k^1(z)\phi'(z)}{P_k(z)\phi'(z)} \approx \frac{1}{\psi'(z)} \]

\[
\int_{\Delta} e^{i\psi(z)} \left( \frac{P_k^1(z)\phi'(z)}{i\psi'(z)} \right)^2 \, dz =: J_1 + J_2.
\]

And

\[
J_1 = \int_{\Delta} \frac{e^{i\psi(z)}}{P_k(z)\phi'(z)} \, dz
\]

\[
= \left[ \frac{e^{i\psi(z)}}{i\psi'(z)} \right]_{d\Delta} - \int_{\Delta} \frac{e^{i\psi(z)}}{P_k(z)\phi'(z)} \, dz =: J_{1a} + J_{1b}.
\]

From (4.21), it follows that

\[ |J_{1a}| \lesssim \frac{1}{|x-y|} \delta^{1-n} 2^{-nk}. \]

We have

\[ \left| \frac{\psi'(z)}{P_k(z)\phi'(z)} \right| = \left| \frac{\gamma(\omega 2^k(z-x)) - \gamma(\omega 2^k(z-y))}{\omega 2^k \gamma'(\omega 2^k(z-x)) - \omega 2^k \gamma'(\omega 2^k(z-y))} \right| \]

\[
\geq 1 - \left| \frac{\gamma(\omega 2^k(z-x)) - \gamma(\omega 2^k(z-y))}{\omega 2^k \gamma'(\omega 2^k(z-x)) - \omega 2^k \gamma'(\omega 2^k(z-y))} \frac{\tilde{P}_k^1(z)}{\tilde{P}_k(z)} \right|
\]

\[
\geq 1 - \frac{2}{C_1} \frac{C_1}{4} = \frac{1}{2}.
\]

From \( \frac{d}{dz} \frac{\tilde{P}_k(z)}{\tilde{P}_k^1(z)} = \frac{\tilde{P}_k(z)\tilde{P}_k^1(z) - \tilde{P}_k^1(z)\tilde{P}_k(z)}{\tilde{P}_k(z)\tilde{P}_k^1(z)} \), it implies \( \frac{d}{dz} \frac{\tilde{P}_k(z)}{\tilde{P}_k^1(z)} \) has at most 2\((n-1)\) roots. Combine Lemma 3.5,

\[
|J_{1b}| \leq \int_{\Delta} \left| \frac{e^{i\psi(z)}}{\tilde{P}_k(z)\phi'(z)} \right| \left| \frac{\tilde{P}_k^1(z)\phi'(z)}{i\psi'(z)} \right| \, dz
\]

\[
\lesssim \frac{1}{|x-y|} 2^{-nk} \delta^{1-n} \int_{\Delta} \left| \frac{d}{dz} \left( \frac{\tilde{P}_k^1(z)\phi'(z)}{\psi'(z)} \right) \right| \, dz
\]

\[
\lesssim \frac{1}{|x-y|} 2^{-nk} \delta^{1-n} \int_{\Delta} \left( \frac{\tilde{P}_k^1(z)\phi'(z)}{\psi'(z)} \right)^2 \left| \frac{d}{dz} \left( \frac{\tilde{P}_k(z)}{\psi'(z)} \right) \right| \, dz
\]

\[
\lesssim \frac{1}{|x-y|} 2^{-nk} \delta^{1-n} \left( \int_{\Delta} \left| \frac{d}{dz} \left( \frac{\tilde{P}_k(z)}{\psi'(z)} \right) \right| \, dz + \int_{\Delta} \left( \frac{\tilde{P}_k(z)\phi'(z)}{\psi'(z)} \right)^2 \left| \frac{d}{dz} \tilde{P}_k(z) \right| \, dz \right)
\]

\[
\lesssim \frac{1}{|x-y|} 2^{-nk} \delta^{1-n} \left( 1 + \int_{\Delta} \left( \frac{\tilde{P}_k(z)\phi'(z)}{\psi'(z)} \right)^2 \left| \frac{d}{dz} \tilde{P}_k(z) \right| \, dz \right)
\]

\[
\lesssim \frac{1}{|x-y|} 2^{-nk} \delta^{1-n} \left( 1 + \int_{\Delta} \left| \frac{\tilde{P}_k(z)}{\tilde{P}_k(z)} \right| \, dz \right) \lesssim \frac{1}{|x-y|} 2^{-nk} \delta^{1-n}.
\]

From \( \frac{d}{dz} \frac{1}{\tilde{P}_k(z)} = -\frac{\tilde{P}_k(z)}{\tilde{P}_k(z)\tilde{P}_k^1(z)} \), this implies \( \frac{d}{dz} \frac{1}{\tilde{P}_k(z)} \) has at most \( n - 2 \) roots. From \( \frac{d}{dz} \frac{1}{\phi(z)} = -\frac{\phi''(z)}{\phi(z)\phi'(z)} \) and \( \phi''(z) = 2^n \frac{(\omega 2^k)^2}{\gamma(\omega)} \left( \gamma''(\omega 2^k(z-x)) - \gamma''(\omega 2^k(z-y)) \right) \), noticing \( \gamma'' \) is
monotone on \((0, \infty)\), we may get that the sign of \(\frac{d}{dz} \frac{1}{\phi'(z)}\) does not change on this domain. From Lemma 2.5 and together with \(|\frac{1}{P_k'(z)}| \lesssim \delta^{1-n}\) and \(|\frac{1}{\phi'(z)}| \lesssim \frac{1}{|x-y|^{2-nk}}\) enable us to obtain

\[
|J_2| \leq \int_{\Delta} \left| e^{i\psi(z)} \left( \frac{\delta_k'(z)\phi'(z)}{(P_k'(z))^2} \cdot \delta_k'(z)\phi'(z) \right) \right| \, dz 
\]

\[
\lesssim \int_{\Delta} \left| \frac{(P_k'(z))^2\phi'(z)}{(P_k'(z))^2} \right| \, dz 
\]

\[
\lesssim \int_{\Delta} \left| \frac{\delta_k'(z)\phi'(z)}{(P_k'(z))^2\phi'(z)} \right| \, dz + \int_{\Delta} \left| \frac{\phi''(z)}{(P_k'(z))^2\phi'(z)} \right| \, dz 
\]

\[
\lesssim \frac{1}{|x-y|^{2-nk}} \int_{\Delta} \left| \frac{d}{dz} \left( \frac{1}{P_k'(z)} \right) \right| \, dz + \delta^{1-n} \int_{\Delta} \left| \frac{d}{dz} \left( \frac{1}{\phi'(z)} \right) \right| \, dz \lesssim \frac{1}{|x-y|^{\delta^{1-n}2^{-nk}}}. 
\]

Therefore,

\[
\left| \int_{\Delta} e^{i\psi(z)} \, dz \right| \leq |J_{1a}| + |J_{1b}| + |J_2| \lesssim \frac{1}{|x-y|^{\delta^{1-n}2^{-nk}}}. \tag{4.22} \]

Let \(\frac{1}{|x-y|^{\delta^{1-n}2^{-nk}}} \approx C(n)\delta\), we then obtain (4.17).

**Case 2** \(\left( \frac{\delta_k'(z)}{P_k'(z)\phi'(z)} \right)'(z) > \frac{1}{8n}\). We denote \(\Delta := \bigcup_{m=1}^{C(n)} (a_m, b_m)\). On each \((a_m, b_m)\), by Lemma 3.2, the derivative of \(\delta_k'(z)/P_k'(z)\phi'(z)\) is greater than \(\frac{1}{8n}\). Then \(\psi'(z)/P_k'(z)\phi'(z)\) is strictly increasing on this domain. If it has one (and only one) zero in this interval, we denote it as \(z_m \in (a_m, b_m)\). By mean value theorem,

\[
\left| \frac{\psi'(z)}{P_k'(z)\phi'(z)} \right| \geq \frac{1}{8n} |z - z_m|. \tag{4.23} 
\]

If \(\frac{\psi'(z)}{P_k'(z)\phi'(z)}\) has no zero in \((a_m, b_m)\), we consider two cases: If \(\frac{\psi'(z)}{P_k'(z)\phi'(z)} > 0\) on \((a_m, b_m)\), we take \(z_m\) as the intersection between \(z\)-axis and the tangent line to the function \(\frac{\psi'(z)}{P_k'(z)\phi'(z)}\) at \(z = a_m\); if \(\frac{\psi'(z)}{P_k'(z)\phi'(z)} < 0\) on \((a_m, b_m)\), we take \(z_m\) as the intersection between \(z\)-axis and the tangent line to the function \(\frac{\psi'(z)}{P_k'(z)\phi'(z)}\) at \(z = b_m\). Then, we obtain a series \(\{z_m\}_{m=1}^{C(n)}\) such that, for each \(m \in [1, C(n)]\) and \(m \in \mathbb{Z}\), we have (4.23) is true.

Let \(B_\delta := \{z \in \Delta : \text{dist} \left( z, \bigcup_{m=1}^{C(n)} \{z_m\} \right) \leq \delta \}\) and \(D := \Delta \setminus B_\delta\). We know that \(|B_\delta| \leq C(n)\delta\) and \(D\) consists of \(C(n)\) intervals. Then we focus our goal to estimate

\[
\left| \int_{D} e^{i\psi(z)} \, dz \right|. 
\]

For \(z \in D\), as in Case 1,

\[
|\psi'(z)| \gtrsim 2^{nk}|x - y|^{\delta^n}. \tag{4.24} 
\]

Replacing the \(\Delta\) in (4.22) by \(D\) and running the same argument, we can see that \(J_{1a} \lesssim \frac{1}{|x-y|^{2-nk}\delta^{-n}}\) and \(J_2 \lesssim \frac{1}{|x-y|^{2-nk}\delta^{-n}}\). For \(J_{1b}\), noticing \(\frac{\psi'(z)}{P_k'(z)\phi'(z)}\) is increasing on \(D\), we
have
\[
|J_{1b}| \leq \int_D \left| \frac{e^{i\psi(z)}}{\tilde{P}'_k(z)\phi'(z)} \right| \frac{d}{dz} \left( \frac{\tilde{P}'_k(z)\phi'(z)}{i\psi'(z)} \right) dz
\]
\[
\lesssim \frac{1}{|x-y|^{2-nk\delta^{1-n}}} \int_D \left| \frac{d}{dz} \left( \frac{\tilde{P}'_k(z)\phi'(z)}{\psi'(z)} \right) \right| dz
\]
\[
\approx \frac{1}{|x-y|^{2-nk\delta^{1-n}}} \left| \int_D \frac{d}{dz} \left( \frac{\tilde{P}'_k(z)\phi'(z)}{\psi'(z)} \right) dz \right| \lesssim \frac{1}{|x-y|^{2-nk\delta^{1-n}}}.
\]
Therefore,
\[
\left| \int_D e^{i\psi(z)} dz \right| \leq |J_{1a}| + |J_{1b}| + |J_2| \lesssim \frac{1}{|x-y|^{n-2n}}. \tag{4.25}
\]
As in (4.19), \(|J^r|\) can be controlled by
\[
|J^r| \lesssim \left( \frac{1}{2^{nk|x-y|}} \right)^{1/4}. \tag{4.26}
\]
This is (4.18), we finish the proof of Proposition 4.2. \qed

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