ON THE LOWER CENTRAL SERIES OF SOME VIRTUAL KNOT GROUPS

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Abstract. It is well known that for any classical knot group, the second term of the lower central series is equal to the third term of the lower central series. In this paper, we study groups of some virtual knots with small number of crossings and prove that there is a virtual knot with long lower central series which, in particular, implies that there is a virtual knot with residually nilpotent group. This gives a possibility to construct invariants of virtual knots using quotients by terms of the lower central series of knot groups. Also, we study decomposition of virtual knot groups as semi direct product and free product with amalgamation. In particular, we prove that the groups of some virtual knots are extensions of finitely generated free groups by infinite cyclic groups.

1. Introduction

Virtual links were introduced by Kauffman [11] as a generalization of classical links. Topologically, virtual links can be interpreted as isotopy classes of embeddings of classical links in thickened surfaces of higher genus. Several invariants of classical links can be extended to virtual links. Unlike classical knots, there is no natural definition of a virtual link group. Due to this, several definitions of virtual link groups can be found in the literature [2, 3, 4, 5, 8, 9, 15, 16]. For example, Kauffman, in [11], extended the idea of fundamental group of classical knots by ignoring virtual crossings. Some of these definitions use representations of the virtual braid group $VB_n$ by automorphisms of free products $F_{n,k} = F_n \ast \mathbb{Z}^k$ for some $k$.

It is well-known that the Artin braid group $B_n$ can be represented as a subgroup of $Aut(F_n)$. In [2, 13], an extension of the classical Artin representation $\varphi_A : VB_n \to Aut(F_{n+1})$ is constructed. Using the representation $\varphi_A$, for each virtual link $L$, a group $G_A(L)$ is defined. Later, in [8], a more general representation of $VB_n$ and its associated virtual knot group is defined. Recently, in [4], a new representation $\phi_M$ of the virtual braid group is constructed, which generalizes all previously mentioned representations. If $K$ is a virtual knot and $G_M(K)$ the group associated to representation $\phi_M$, then it has been shown that $G_A(K)$ is isomorphic to $G_M(K)$. A natural question is whether the same holds for virtual links with more than one component. In this paper, we show that if $L$ is the virtual Hopf link, then $G_A(L)$ is not isomorphic to $G_M(L)$.

It is well-known that the knot group for classical knots has a short lower central series, that is, the second term coincides with the third term. Therefore, factorization by the terms of lower central series cannot be used to distinguish classical knots. In [6], it was shown $G_A(K)$ of virtual trefoil knot has a homomorphism onto a nilpotent group of step 4. Also it was asked whether there is a virtual knot whose knot group is residually nilpotent. In this paper, we give positive answer...
to this question. Also, we show that there is a virtual knot whose group is not residually nilpotent and has the length of lower central series $\leq \omega^2$, where $\omega$ is the first infinite ordinal.

The paper is organized as follows. In section 2, we recall definition of the virtual braid group $VB_n$ and definitions of the virtual link groups $G_A(L)$ and $\tilde{G}_M(L)$.

In section 3, we consider virtual knots with small number of crossings from the virtual knot table [17]. Using Vogel’s algorithm [18, 19], we determine virtual braids associated to these virtual knots and compute their knot groups corresponding to representation $\varphi_A$. Further, we study quotients of virtual knot groups by terms of lower central series, and prove that all these knots are non trivial using the quotient by the fifth term of the lower central series. We show that there are virtual knots whose knot groups have long lower central series. We also study structures of these groups and prove that some of them are extensions of finitely generated free groups by infinite cyclic groups.

In section 4, we consider the virtual Hopf link $L$ and prove that $G_A(L)$ and $\tilde{G}_M(L)$ are right angled Artin groups but $G_A(L)$ is not isomorphic to $\tilde{G}_M(L)$.

2. Preliminaries and Notations

Throughout the paper, we shall denote $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$, where $x, y$ are elements of some group. If $A$ and $B$ are subgroups of a group $G$, then $[A, B]$ is a subgroup that is generated by all commutators $[a, b]$, $a \in A, b \in B$. A non trivial group $G$ is called residually nilpotent if for any $1 \neq g \in G$ there is a nilpotent group $N$ and a homomorphism $\varphi : G \to N$ such that $\varphi(g) \neq 1$.

For a group $G$, transfinite lower central series is defined as,

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_\omega(G) \geq \gamma_{\omega+1}(G) \geq \ldots,$$

where

$$\gamma_{\alpha+1}(G) = \langle [g_\alpha, g] \mid g_\alpha \in \gamma_{\alpha}(G), g \in G \rangle,$$

and if $\alpha$ is a limit ordinal, then

$$\gamma_\alpha(G) = \bigcap_{\beta < \alpha} \gamma_\beta(G).$$

In particular, $G$ is residually nilpotent if and only if

$$\gamma_\omega(G) = \bigcap_{i=1}^{\infty} \gamma_i(G) = 1.$$

The maximal $\alpha$ such that $\gamma_\alpha(G) \neq \gamma_{\alpha+1}(G)$ is called the length of the lower central series of $G$.

As we noted in the introduction, if $K$ is a classical knot, then its group $G(K)$ is not residually nilpotent since $\gamma_2(G(K)) = \gamma_3(G(K))$. 
The virtual braid group $VB_n$ is generated by the classical braid group $B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \rangle$ and the symmetric group $S_n = \langle \rho_1, \rho_2, \ldots, \rho_{n-1} \rangle$ such that the following relations hold:

$$
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & |i - j| \geq 2, \\
\rho_i^2 &= 1 & i = 1, 2, \ldots, n - 2, \\
\rho_i \rho_j &= \rho_j \rho_i & |i - j| \geq 2, \\
\rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_i & i = 1, 2, \ldots, n - 2.
\end{align*}
$$

In addition, the following mixed defining relations hold:

$$
\begin{align*}
\sigma_i \rho_j &= \rho_j \sigma_i & |i - j| \geq 2, \\
\rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_i & i = 1, 2, \ldots, n - 2.
\end{align*}
$$

Recall the two representations of the virtual braid group, which are extensions of the Artin representation of $B_n$ into $\text{Aut}(F_n)$ \cite{1}. Let $F_{n+1} = \langle x_1, x_2, \ldots, x_n, y \rangle$ be the free group of rank $n + 1$. The map

$$
\varphi_A: VB_n \to \text{Aut}(F_{n+1})
$$

defined by setting

$$
\varphi_A(\sigma_i) = \begin{cases} 
  x_i \mapsto x_i x_{i+1} x_i^{-1}, \\
  x_{i+1} \mapsto x_i, \\
  x_j \mapsto x_j & \text{for } j \neq i, i + 1, \\
  y \mapsto y.
\end{cases}
$$

and

$$
\varphi_A(\rho_i) = \begin{cases} 
  x_i \mapsto x_i^{y^{-1}}, \\
  x_{i+1} \mapsto x_i^y, \\
  x_j \mapsto x_j & \text{for } j \neq i, i + 1, \\
  y \mapsto y.
\end{cases}
$$

is a representation of $VB_n$ to $\text{Aut}(F_{n+1})$.

Next, we consider the second representation of $VB_n$. Let $F_{n,n} = F_n \ast \mathbb{Z}^n$, where $F_n = \langle y_1, y_2, \ldots, y_n \rangle$ and $\mathbb{Z}^n = \langle v_1, v_2, \ldots, v_n \rangle$ is the free abelian group of rank $n$. The map

$$
\tilde{\varphi}_M: VB_n \to \text{Aut}(F_{n,n})
$$

defined by setting

$$
\tilde{\varphi}_M(\sigma_i) = \begin{cases} 
  y_i \mapsto y_i y_{i+1} y_i^{-1}, \\
  y_{i+1} \mapsto y_i, \\
  y_j \mapsto y_j & \text{for } j \neq i, i + 1.
\end{cases}
$$

and

$$
\tilde{\varphi}_M(\rho_i) = \begin{cases} 
  y_i \mapsto y_i^{v_{i+1}^{-1}}, \\
  y_{i+1} \mapsto y_i v_i, \\
  y_j \mapsto y_j & \text{for } j \neq i, i + 1.
\end{cases}
$$
is a representation of $VB_n$ [4].

In classical knot theory, the Alexander theorem states that for each oriented link $L$, there exists a braid whose closure is equivalent to $L$. An analogous result holds for virtual links and virtual braids [12]. Let $L$ be a virtual link which is equivalent to closure of a virtual braid $\beta \in VB_n$. Suppose that we have a representation $\varphi : VB_n \rightarrow \text{Aut}(K)$ of the virtual braid group into the automorphism group of a group $K = \langle k_1, k_2, \ldots, k_m \mid R \rangle$, where $R$ is the set of defining relations. Then to each $\beta \in VB_n$, we assign the group

$$G_\varphi(\beta) = \langle k_1, k_2, \ldots, k_m \mid R, k_i = \varphi(\beta)(k_i) \text{ for } i = 1, 2, \ldots, m \rangle.$$ 

Using the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, we have $\beta = \sigma_1^{-1} \sigma_2 \rho_1 \rho_2$. For simplicity, we denote $G_\varphi(\beta)$ by $G(\beta)$. Then the group $G_\varphi(\beta)$ has a presentation

$$G_\varphi(\beta) = \langle x, y, z \mid z^{-xyx} = z[xz, y][xz, y]^y, z = [y^{-1}, x^{-1}] \rangle.$$

3. Computations for virtual knots with small number of crossings

In this section, we study some virtual knots with small number of crossings from the knot table [17]. The virtual trefoil knot was already studied in [6, 7]. Using Vogel’s algorithm [18, 19], we present virtual knots as a closure of some braids. Using the representation $\varphi_A$ we compute their associated virtual knot groups and study structures of these groups.

3.1. Virtual knot $K_1$ (The knot 4.33 in [17]).

It is not difficult to prove that $K_1$ is the closure of the braid $\beta = \sigma_2(\sigma_1 \sigma_2 \sigma_1)^{-1} \rho_1 \rho_2 \in VB_3$. Using the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, we have $\beta = \sigma_1^{-1} \sigma_2 \rho_1 \rho_2$. For simplicity, we denote $G(\beta)$ by $G(\beta)$. Then the group

$$G(\beta) = G(K_1) = \langle x_1, x_2, x_3, y \mid x_i = x_i \beta, i = 1, 2, 3 \rangle,$$ 

where $x_i \beta = \varphi_A(\beta)$, has a presentation

$$G(K_1) = \langle x, y, z \mid z^{-xyx} = z[xz, y][xz, y]^y, z = [y^{-1}, x^{-1}] \rangle.$$

We remove the generator $z = [y^{-1}, x^{-1}]$ using the second relation of $G(K_1)$. Transforming the first relation we have:

$$z^{-xyx} = z[xz, y][xz, y]^y,$$
$$[xyx]^{-1}z^{-1}(xyx) = z(z^{-1}x^{-1}y^{-1}xzy)y^{-1}(z^{-1}x^{-1}y^{-1}xzy)y,$$
$$[xyx]^{-1}z^{-1}(xyx) = x^{-1}y^{-1}y^{-1}xzy^2,$$
$$x^{-1}z^{-1}(xyx) = y^{-1}xzy^2,$$
$$x^{-1}(xyx^{-1}y^{-1})(xyx) = y^{-1}x(yxy^{-1}x^{-1})y^2,$$
$$x^{-1}y^{-1}xy = y^{-2}x(yxy^{-1}x^{-1})y^2x^{-1},$$
$$[x, y] = x^{y^{-1}x^{-1}y^2}x^{-1},$$
$$[x, y] = x^{xy^{-1}x^{-1}y^2}x^{-1},$$
$$[x, y] = x^{[x^{-1}, y]y}x^{-1},$$
$$[x, y] = [[x^{-1}, y]y, x^{-1}],$$
$$[x, y] = [[x^{-1}, y], x^{-1}]y[y, x^{-1}],$$
$$[x, y][x^{-1}, y] = [x^{-1}, y, x^{-1}]y.$$ 

Putting $t = x^{-1}$, we obtain

$$[t^{-1}, y][t, y] = [t, y, t]^y.$$ 

So we have,

$$[y, t][t^{-1}, y] = [t, y, t]^y,$$
$$([t, y]^{-1})[t^{-1}, y] = [t, y, t]^y,$$
$$[t^{-1}, [t, y]] = [t, y, t]^y,$$
$$[t, y, t]^{-1} = [t, y, t]^y.$$ 

Hence, relation has the form

$$[t, y, t, yt] = 1$$

and we get

$$G(K_1) = \langle x, y \parallel [x^{-1}, y, x^{-1}, yx^{-1}] = 1 \rangle.$$ 

**Theorem 3.1.** (1) For the knot $K_1$, we have

$$G(K_1)/\gamma_4G(K_1) \cong F_2/\gamma_4F_2,$$
$$\gamma_4G(K_1)/\gamma_5G(K_1) \cong \mathbb{Z}^2,$$
$$G(K_1)/\gamma_5G(K_1) \not\cong F_2/\gamma_5F_2.$$ 

(2) The group $G(K_1)$ is an extension of free group of rank 3 by infinite cyclic group.

**Proof.** (1) We see that the left hand side of the defining relation of $G(K_1)$ lies in $\gamma_4F_2$, where $F_2 = \langle x, y \rangle$. Hence, $G(K_1)/\gamma_4G(K_1) \cong F_2/\gamma_4F_2$. 

(2) The group $G(K_1)$ is an extension of free group of rank 3 by infinite cyclic group.
The quotient $\gamma_4 G(K_1) / \gamma_5 G(K_1)$ is generated by the basis commutators of weight 4. In the quotient $G(K_1) / \gamma_5 G(K_1)$, we have

$$1 = [x^{-1}, y, x^{-1}, yxy^{-1}] \equiv [x^{-1}, y, x^{-1}, y][x^{-1}, y, x^{-1}, x^{-1}] \equiv [x, y, x, y][x, y, x, x]^{-1}.$$ 

Therefore, the defining relation in $G(K_1) / \gamma_5 G(K_1)$ is of the form

$$[x, y, x, y] \equiv [x, y, y, x].$$

The commutators $[x, y, y, x], [x, y, x, x], [x, y, y, y]$ are the basic commutators of weight 4, i.e. $\gamma_4 F_2 / \gamma_5 F_2 \cong \mathbb{Z}^3$, whereas from the defining relation of $G(K_1)$, it follows that $\gamma_4 G(K_1) / \gamma_5 G(K_1) \cong \mathbb{Z}^2$.

Hence, we obtain

$$G(K_1) / \gamma_5 G(K_1) \not\cong F_2 / \gamma_5 F_2.$$ 

(2) Rewriting the relation of $G(K_1)$, we have

$$x^{-1} y^{-1} xy = y^{-2} xyxy^{-1} x^{-1} y^2 x^{-1}.$$ 

Putting $y_k = x^{-k} y x^k, \quad k \in \mathbb{Z},$

the relation is equivalent to

$$y_1^{-1} y_0 = y_0^{-2} y_1 y_2^{-1} y_1^{-1}.$$ 

On shifting the indexes by two, we get

$$y_3^{-1} y_2 = y_2^{-2} y_1 y_0^{-1} y_1^{-1},$$

which implies

$$y_3 = y_2 y_1^{-2} y_0 y_1^{-1} y_2^{-2}.$$ 

Therefore, $G(K_1)$ has the presentation

$$G(K_1) = \langle x, y_0, y_1, y_2 \parallel y_0^x = y_1, y_1^x = y_2, y_2^x = y_2 y_1^{-2} y_0 y_1^{-1} y_2^{-2}, y_2^{-1} = y_1, y_1^{-1} = y_0, y_0^{-1} = y_0^2 y_1^{-2} y_2 y_1^{-2} y_0 \rangle.$$ 

So we have,

$$G(K_1) = F_3 \rtimes \mathbb{Z},$$

where $F_3 = \langle y_0, y_1, y_2 \rangle$ is the free group of rank 3 and $\mathbb{Z} = \langle x \rangle$.

Conjugation of $F_3$ by $x$ induces a linear transformation on the quotient $F_3 / F_3'$, which has the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$ 

Since all the eigenvalues of $A$ are equal to 1, we have $(A - E)^3 = 0$. Using [14, Lemma 2] we have the following:
Corollary 3.2. The group $G(K_1)$ is residually nilpotent.

3.2. Virtual knot $K_2$ (The knot 4.21 in [17]).

$K_2$ is equivalent to the closure of the braid $\beta = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\rho_1\rho_2 \in VB_3$ and has group

$$G(K_2) = \langle x_1, x_2, x_3, y \mid x_i = x_i\beta, i = 1, 2, 3 \rangle,$$

where the relations are of the form

$$
\begin{align*}
x_1^y &= x_3^{(x_1x_2)^{-1}}, \\
x_2^y &= x_1^{x_2}, \\
x_3^{-2} &= x_3^{(x_1x_2)^{-1}}(x_1x_2x_3^{-1})x_2^{-1}x_1^{-1}.
\end{align*}
$$

Considering the third relation, we get

$$
x_3^{-2} = x_3^{(x_1x_2)^{-1}}x_2x_3^{-1}(x_1x_2)^{-1},
$$

which implies

$$
x_3^{-2} = x_1^yx_2x_1^{-y}.
$$

Rewriting the second relation, we obtain

$$
x_1 = x_2^{yx_2^{-1}}.
$$

Hence, we have

$$
x_3^{-2} = x_2^{yx_2^{-1}}x_2x_2^{-y}x_2^{-1}y \iff x_3 = \left( x_2^{yx_2^{-1}}x_2x_2^{-y}x_2^{-1}y \right)^{y^2}.
$$
Substituting the expressions for $x_1$ and $x_3$ in the first relation, we obtain

$$x_1^y = (x_1 x_2) x_3 (x_1 x_2)^{-1},$$
$$x_2^{y x_1} y = (x_2^{y x_1} x_2) x_3 (x_2^{y x_1} x_2)^{-1},$$
$$x_2^{y x_1} y = (x_2 y^{-1} x_2 y) x_3 (x_2 y^{-1} x_2 y)^{-1},$$
$$x_2^{y x_1} x_2 y^{-1} x_2 y = x_3,$$
$$x_2^{y x_1} x_2 y^{-1} x_2 y = \left( x_2^{y x_1} y x_2 \right)^{y^2},$$
$$x_2^{y x_1} x_2 y^{-1} x_2 y = x_2^{y x_1} y x_2^{-1} y, $$
$$x_2^{y x_1} x_2 y^{-1} x_2 y = x_2^{y x_1} y x_2^{-1} y,$$
$$[x_2^{-1} x_2 y^{-1} x_2^{y x_1} y, x_2] = 1.$$

If we denote $x_2 = x$, then the group $G(K_2)$ has the following presentation

$$G(K_2) = \langle x, y \parallel [x y^{-1} x y^{-1}, x x y^{-1}, x] = 1 \rangle.$$

Simplifying the relation and using the commutator identities, we get

$$x^2 [x, y^{-1} x y^{-1}] [x, y^{-1} x y^{-1}, x] [x, y x^{-1} y], x] = 1,$$
$$[x, y^{-1} x y^{-1}] [x, y^{-1} x y^{-1}, x] [x, y x^{-1} y], x] = 1,$$
$$[x, y^{-1} x y^{-1}] [x, y^{-1} x y^{-1}, x] [x, y x^{-1} y], x] = 1.$$

Performing the transformations modulo $\gamma_2 G(K_2)$, we obtain

$$[x, x^{-1} x y^{-1}, x] [x, y^{-1} x y^{-1}, x, x] [x, y^{-1} y, x] \equiv 1,$$
$$[x, x^{-1} x y^{-1}, x] [x, y^{-1} x y^{-1}, x, x] [x, y^{-1} y, x] \equiv 1,$$
$$[[y^{-1} x y^{-1}, x]^{-1}, x] [y^{-1} y, x]^{-1}, x] \equiv [x, y, x, x]^2,$$
$$[x, [y^{-1} x y^{-1}, x]] [x, [y^{-1} y, x]] \equiv [x, y, x, x]^2,$$
$$[y^{-1} x y^{-1}, x, x]^{-1} [y^{-1} y, x, y^{-1}] \equiv [x, y, x, x]^2,$$
$$[x^{-1} y^{-1} x y^{-1}, x, x]^{-1} [y^{-1} y, x, y^{-1}] \equiv [x, y, x, x]^2,$$
$$[[x, y] y^{-2}, x, x]^{-1} [[x^{-1}, y^{-1}] y^{-2}, x, x]^{-1} \equiv [x, y, x, x]^2,$$
$$[x, y, x, x]^{-1} [y^{-2}, x, x]^{-1} [x^{-1}, y^{-1}, x, x]^{-1} [y^{-2}, x, x]^{-1} \equiv [x, y, x, x]^2,$$
$$[x, y, x, x]^{-1} [y^{-2}, x, x]^{-1} \equiv [x, y, x, x]^2.$$

Further, we have

$$[y^{-2}, x, x] = [[x, y] y^{-2}, x] \equiv [x, y, x, y^{-2}, y, x] \equiv [x, y, x, y^{-2}, x] \equiv [x, y, x, y^{-2}, x]^{-1} \equiv [x, y, x, y^{-2}, x]^{-1} \equiv [x, y, x, y^{-2}, x]^{-1},$$
$$\equiv [x, y, x, x]^{-1} [y^{-2}, x, x]^{-1} \equiv [x, y, x, x]^{-1} [y^{-2}, x, x]^{-1} \equiv [x, y, x, x]^{-1} [y^{-2}, x, x]^{-1}.$$
Therefore, in the quotient $G(K_2)/\gamma_5G(K_2)$ the following unique relation holds

$$([x, y, y, x]^{-1}[x, y, x, x])^4 \equiv 1.$$ 

The commutators $[x, y, y, x], [x, y, x, x], [x, y, y, y]$ are the basic commutators of weight 4, hence we have proved the first part of the following theorem:

**Theorem 3.3.** (1) The first five terms of lower central series of the group $G(K_2)$ are different from each other. Moreover, we have

$$G(K_2)/\gamma_4G(K_2) \cong F_2/\gamma_4F_2,$$

$$\gamma_4G(K_2)/\gamma_5G(K_2) \cong \mathbb{Z}^2 \times \mathbb{Z}_4,$$

$$G(K_2)/\gamma_5G(K_2) \not\cong F_2/\gamma_5F_2.$$ 

(2) The group $G(K_2)$ is an extension of free group of rank 5 by infinite cyclic group.

**Proof.** (2) Consider the relation of $G(K_2)$:

$$[x^{y^{-1}xy^{-1}y^{-1}y^{-1}}, x] = 1$$

Transforming it, we obtain

$$x^{y^{-1}xy^{-1}y^{-1}y^{-1}}x = x^{y^{-1}xy^{-1}y^{-1}y^{-1}},$$

$$x^{-3}(x^{y^{-1}xy^{-1}y^{-1}y^{-1}}x^{y^{-1}y^{-1}y^{-1}}x) = x^{-2}(x^{y^{-1}xy^{-1}y^{-1}y^{-1}}x^{y^{-1}y^{-1}y^{-1}}).$$

Putting

$$y_k = x^{-k}yx^k, \quad k \in \mathbb{Z}.$$ 

The above relation has the form

$$y_3y_4y_3^{-1}y_2^{-2}y_1^{-1}y_0y_1 = y_3y_4y_3^{-1}y_2^{-2}y_1^{-1}y_0y_1.$$ 

Shifting the indexes by 1, we get

$$y_4y_5y_4^{-1}y_3^{-2}y_2^{-1}y_1y_2 = y_3y_4y_3^{-1}y_2^{-2}y_1^{-1}y_0y_1.$$ 

Therefore, we have

$$y_5 = y_4^{-1}y_3y_4y_3^{-1}y_2^{-2}y_1^{-1}y_0y_1y_2^{-1}y_1^{-1}y_2y_3^{-2}y_4.$$ 

Hence, $G(K_2)$ in the generators $x, y_0, y_1, y_2, y_3, y_4$ has the following presentation:

$$G(K_2) = \langle x, y_0, y_1, y_2, y_3, y_4 \parallel y_0^2 = y_1, y_2^2 = y_3, y_3^2 = y_4, y_4^2 = y_3y_4y_3^{-1}y_2^{-2}y_1^{-1}y_0y_1y_2^{-1}y_1^{-1}y_2y_3^{-2}y_4, y_4^{-1} = y_3, y_3^{-1} = y_2, y_2^{-1} = y_1, y_1^{-1} = y_0, y_0^{-1} = y_0y_1^2y_2y_3^{-1}y_2^{-1}y_3y_4y_3^{-1}y_2^{-2}y_1^{-1}y_0y_1y_0^{-1} \rangle.$$ 

Therefore, we obtain

$$G(K_2) = F_5 \ltimes \mathbb{Z},$$

where $F_5 = \langle y_0, y_1, y_2, y_3, y_4 \rangle$ is the free group of rank 5 and $\mathbb{Z} = \langle x \rangle.$
In the quotient $F_5/F'_5$ the matrix

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & -1 & -2 & 2 & 1
\end{pmatrix}$$

corresponds to conjugation by element $x$. Since the characteristic polynomial of $A$ is

$$\chi(\lambda) = \lambda^5 - \lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda - 1 = (\lambda - 1)^3(\lambda + 1)^2,$$

using [14, Lemma 2], we obtain the following result:

**Corollary 3.4.** The length of the lower central series of $G(K_2)$ is not less than $\omega$ and not more than $\omega^2$.

**Proof.** It follows from the fact that $\gamma_\omega G(K_2) \subseteq F'_5$ and $\gamma_\omega^2 G(K_2) = 1$. \qed

**Question 3.5.** Is it true that the length of the lower central series of $G(K_2)$ is equal to $\omega^2$?

### 3.3. Virtual knot $K_3$(The knot 3.5 in [17]).

The braid $\beta = \rho_1\sigma_1^{-2}\rho_1\sigma_1^{-1} \in VB_2$ is equivalent to closure of knot $K_3$. It has the group

$$G(\beta) = \langle x_1, x_2, y \parallel x_i = x_1^\beta, i = 1, 2 \rangle,$$

where the relations are of the form

$$x_1 = x_2^{-1}(x_2^{-1}x_1x_2)^{-y^2}x_2(x_2^{-1}x_1x_2)y^{-2}x_2,$$

$$x_2 = x_2^{-y^2}x_1x_2^{-y^2}.$$  

Transforming the first relation, we get

$$x_1^y = (x_2^{-y^2}x_2^{-1}x_1^{-1}x_2x_2^y)(x_2^{-1}x_1x_2x_2^y).$$

Using the second relation, we have

$$x_1^y = x_2^{-1}x_2^y x_2,$$

which implies

$$x_1 = x_2^{-y^2}x_2x_2^y.$$

Substituting the second relation in the first one, we obtain

$$x_2 = x_2^{-y^2}x_2^{-1}x_2^{-y^2}x_2x_2^y.$$
We put $x_2 = x$ and we see that $G(K_3)$ has the following presentation
\[ G(K_3) = \langle x, y \mid x = x^{-y^2}x^{-x^{-y^2}xx^{-2}xy^2} \rangle. \]

**Theorem 3.6.** (1) For the group $G(K_3)$ we have,
\[ G(K_3)/\gamma_4G(K_3) \cong F_2/\gamma_4F_2, \]
\[ \gamma_4G(K_3)/\gamma_5G(K_3) \cong \mathbb{Z}^2 \times \mathbb{Z}_4, \]
\[ G(K_3)/\gamma_5G(K_3) \not\cong F_2/\gamma_5F_2. \]

(2) $G(K_3)$ is an extension of infinite free product with amalgamation by infinite cyclic group.

**Proof.** (1) Transforming the relation, we have
\[ [x^{y^2}xx^2, x] = 1, \]
\[ [x, y^{-2}[x, y^2], x] = 1, \]
\[ [x^3[x, y^{-2}][x, y^2], x] = 1, \]
\[ [[x, y^{-2}][x, y^2], x] = 1, \]
\[ [y^2, xx^2, x] = 1, \]
\[ [y^2, x][y^2, x][y^{-2}, x^2][x, y^2], x] = 1. \]

Performing the transformations modulo $\gamma_5G(K_3)$, we get
\[ [y^2, x, y^{-2}, x][x, y^{-2}, x, x] \equiv 1, \]
\[ [x, y, y][x, x, x]^{-4} \equiv 1. \]

So, by modulo $\gamma_5G(K_3)$, the relation has the form
\[ ([x, y, x][x, y, x]^{-1})^4 \equiv 1. \]

So, we get
\[ G(K_3)/\gamma_4G(K_3) \cong F_2/\gamma_4F_2, \]
\[ \gamma_4G(K_3)/\gamma_5G(K_3) \cong \mathbb{Z}^2 \times \mathbb{Z}_4, \]
\[ G(K_3)/\gamma_5G(K_3) \not\cong F_2/\gamma_5F_2. \]

(2) We consider the relation of $G(K_3)$
\[ x = x^{-y^2}x^{-x^{-y^2}xx^{-2}xy^2}, \]

transferring it, we get
\[ x^{y^2}xx^2 = x^{x^{-y^2}xx^{-2}xy^2}. \]

Putting
\[ x_k = y^{-k}yx^k, \quad k \in \mathbb{Z} \]

and rewriting the relation, we have
\[ x_2x_2 = x_0x_2x_0, \]
which implies
\[ [x_{-2}, x_0] = [x_0^{-1} x_2^{-1}]. \]
Let
\[ H_3 = \langle x_k, k \in \mathbb{Z} \mid [x_k, x_{k+2}] = [x_{k+2}^{-1}, x_{k+4}], k \in \mathbb{Z} \rangle, \]
then
\[ G(K_3) = H_3 \times \mathbb{Z}, \]
where \( \mathbb{Z} = \langle y \rangle \) and \( x_k^y = x_{k+1}, k \in \mathbb{Z} \).
For any \( k \in \mathbb{Z} \), let
\[ A_k = \langle x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, [x_k, x_{k+2}] = [x_{k+2}^{-1}, x_{k+4}] \rangle, \]
and
\[ B_k = \langle x_k, x_{k+1}, x_{k+2}, x_{k+3} \rangle. \]
then,
\[ H_3 = \cdots * A_k * A_{k+1} * A_{k+2} * A_{k+3} * \cdots \]
is the infinite free product with amalgamation. \( B_k \cong F_4 \), where \( F_4 \) is the free group of rank 4, \( k \in \mathbb{Z} \).

3.4. Virtual knot \( K_4 \) (The knot 3.7 in [17]).

\[ K_4 \] is closure of a braid \( \beta = \rho_1 \sigma_1^{-2} \rho_1 \sigma_1 \in VB_2 \) and we have,
\[ G(K_4) = \langle x_1, x_2, y \mid x_i = x_i \beta, i = 1, 2 \rangle, \]
where the relations are of the form
\[
\begin{align*}
x_1 &= (x_1 x_2 x_1^{-1})^{-1} x_1^{-y^2} (x_1 x_2 x_1^{-1}) x_1^{y^2} (x_1 x_2 x_1^{-1}), \\
x_2 &= (x_1 x_2 x_1^{-1})^{-y^2} x_1 (x_1 x_2 x_1^{-1}) y^2.
\end{align*}
\]
Transforming it, we obtain
\[
\begin{align*}
x_1 &= (x_1 x_2 x_1^{-1})^{-1} x_1^{-y^2} (x_1 x_2 x_1^{-1}) x_1^{y^2} (x_1 x_2 x_1^{-1}), \\
x_2^{y^2} &= (x_1 x_2 x_1^{-1})^{-1} x_1^{y^2} (x_1 x_2 x_1^{-1}),
\end{align*}
\]
\[
\begin{align*}
x_1 &= x_2^{-y^2} x_1^{y^2} (x_1 x_2 x_1^{-1}), \\
x_2^{y^2} &= (x_1 x_2 x_1^{-1})^{-1} x_1^{y^2} (x_1 x_2 x_1^{-1}),
\end{align*}
\]
\[
\begin{align*}
x_1 x_2^{y^2} &= (x_1 x_2 x_1^{-1})^{y^2} x_1, \\
x_1 x_2^{y^2} &= x_2^{y^2} x_1^{y^2} x_1^{y^2},
\end{align*}
\]
The quotient

The following holds for the group

Theorem 3.7. The following holds for the group $G(K_4)$:

(1) The quotient $G(K_4)/\langle y \rangle^{G(K_4)}$ of $G(K_4)$ by the normal closure of $y$ in $G(K_4)$ is isomorphic to

$$\langle x, z \mid z^3 = 1, z^x = z^{-1} \rangle \ast \mathbb{Z}_2.$$  

In particular, it contains a subgroup that is isomorphic to the free product $\mathbb{Z}_3 \ast \mathbb{Z}_2$.

(2)

$$G(K_4)/\gamma_4G(K_4) \cong F_2/\gamma_4F_2,$$

$$\gamma_4G(K_4)/\gamma_5G(K_4) \cong \mathbb{Z} \times \mathbb{Z}_4,$$

$$G(K_4)/\gamma_5G(K_4) \not\cong F_2/\gamma_5F_2.$$  

Proof. (1) We consider the second relation and we obtain

$$x_2^{-1}x_1 = \left[ x_1^{-1}, x_2^{-y^2} \right]^{y^2} \iff (x_2^{-1}x_1)^{y^{-2}} = \left[ x_1^{-1}, x_2^{-y^2} \right]$$

$$\iff (x_2^{-1}x_1)^{y^{-2}} = \left[ x_2^{-y^2}, x_1 \right]^{x_1^{-1}} \iff \left[ x_2^{-y^2}, x_1^{-1} \right] = (x_2^{-1}x_1)^{-1}.$$  

Using the first relation

$$\left[ x_2^{-1}x_1 \right]^{-1} = x_1x_2^{-1} = (x_2^{-1}x_1)^{-1},$$  

we get

$$\left[ x_2^{-1}x_1, y^{-2}x_1^2 \right] = 1.$$  

Denote $x_1 = x, y = y, x_2^{-1}x_1 = z$, then $x_2^{-1} = zx_1^{-1}$ and the relations are of the form

$$\left[ z, y^{-2}x_2 \right] = 1, \quad xzx^{-1} = [(zx^{-1})y^{-2}, x].$$  

In the quotient $G(K_4)/\langle y \rangle^{G(K_4)}$, relations are of the form

$$\left[ z, x^2 \right] \equiv 1, \quad xzx^{-1} \equiv [zx^{-1}, x].$$  

The second relation is equivalent to relation $z^x = z^2$ and from the first relation it follows that

$$z^3 = 1.$$  

So,

$$G(K_4)/\langle y \rangle^{G(K_4)} \cong \langle x, z \mid z^3 = 1, z^x = z^{-1} \rangle.$$  

Therefore, we obtain

$$G(K_4)/\langle y \rangle^{G(K_4)} \cong \langle x, z \mid z^3 = 1, z^x = z^{-1} \rangle \ast \mathbb{Z}_2.$$
(2) Considering the relation \( x_2^{-1} x_1 = [x_1^{-y^2}, x_2^{-1}] \) and transforming it, we get
\[
x_1 = x_2[x_1^{-y^2}, x_2^{-1}] = x_2[y^{-2} x_1^{-1} y^2, x_2^{-1}] = x_2[[y^2, x_1] x_1^{-1}, x_2^{-1}] =
\]
\[
= x_2[y^2, x_1, x_2^{-1}] x_1^{-1} [x_1^{-1}, x_2^{-1}] = x_2[y^2, x_1, x_2^{-1}] [y^2, x_1, x_2^{-1}, x_1^{-1}] [x_1^{-1}, x_2^{-1}] =
\]
\[
= x_2[y^2, x_1, x_2^{-1}] [y^2, x_1, x_2^{-1}, x_1^{-1}] [x_2^{-1}, x_1] x_1^{-1} =
\]
\[
= x_2[y^2, x_1, x_2^{-1}] [y^2, x_1, x_2^{-1}, x_1^{-1}] [x_2^{-1}, x_1] [x_2^{-1}, x_1, x_1^{-1}],
\]
which implies
\[
x_1 = x_2[y^2, x_1, x_2^{-1}] [y^2, x_1, x_2^{-1}, x_1^{-1}] [x_2^{-1}, x_1] [x_2^{-1}, x_1, x_1^{-1}].
\]
Performing the transformations modulo \( \gamma_4 G(K_4) \), we get
\[
x_1 \equiv x_2[y, x_2, x_2]^{-2} [x_2^{-1}, x_1] [x_2, x_1, x_1] \equiv x_2[y, x_2, x_2]^{-2} [x_2^{-1}, x_1] \equiv
\]
\[
\equiv x_2[y, x_2, x_2]^{-2} [x_2^{-1}, x_2^{-1} x_2^{-1}, x_1] \equiv x_2[y, x_2, x_2]^{-2} [x_2^{-1}, x_2^{-1}, x_1] \equiv x_2[y, x_2, x_2]^{-2}.
\]
Hence, the relation \( x_2^{-1} x_1 = [x_1^{-y^2}, x_2^{-1}] \) has the form
\[
x_1 \equiv x_2[y, x_2, x_2]^{-2} (\text{mod } \gamma_4 G(K_4)).
\]
Further, in the quotient \( G(K_4) / \gamma_4 G(K_4) \), the second relation \( x_1 = [x_2^{-y^2}, x_1] x_2 \) has the form
\[
x_2[y, x_2, x_2]^{-2} \equiv [x_2^{-y^2}, x_2] x_2,
\]
\[
x_2[y, x_2, x_2]^{-2} \equiv [[y^{-2}, x_2] x_2^{-1}, x_2] x_2,
\]
\[
x_2[y, x_2, x_2]^{-2} \equiv [y^{-2}, x_2] x_2^{-2} x_2 \equiv 1.
\]
So, we have
\[
G(K_4) / \gamma_4 G(K_4) \cong \langle x_1, x_2, y \mid x_1 = x_2[y, x_2, x_2]^{-2} \rangle \cong F_2 / \gamma_4 F_2.
\]
Let us now consider the following relation
\[
[x_2^{-1} x_1, y^{-2} x_1] = 1,
\]
then transforming it modulo \( \gamma_5 G(K_4) \) and using the relation
\[
x_1 \equiv x_2[y, x_2, x_2]^{-2} (\text{mod } \gamma_4 G(K_4)),
\]
we get
\[
[x_2^{-1} x_2[y, x_2, x_2]^{-2}, y^{-2} x_2[y, x_2, x_2]^{-4}] \equiv 1 (\text{mod } \gamma_5 G(K_4)),
\]
\[
[y, x_2, x_2, y]^4[y, x_2, x_2, x_2]^{-4} \equiv 1 (\text{mod } \gamma_5 G(K_4)),
\]
\[
([y, x_2, x_2, y][y, x_2, x_2, x_2]^{-1})^4 \equiv 1 (\text{mod } \gamma_5 G(K_4)).
\]
Hence, we finally have
\[
G(K_4) / \gamma_4 G(K_4) \cong F_2 / \gamma_4 F_2,
\]
\[
\gamma_4 G(K_4) / \gamma_5 G(K_4) \cong \mathbb{Z}^2 \times \mathbb{Z}_4,
\]
\[
G(K_4) / \gamma_5 G(K_4) \not\cong F_2 / \gamma_5 F_2.
\]
3.5. Virtual knot $K_5$ (The knot 4.43 in [17]).

Closure of the braid $\beta = \sigma_2\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\rho_1 \in VB_3$ represents the virtual knot $K_5$. Its group is

$$G(K_5) = \langle x_1, x_2, x_3, y \parallel x_i = x_i\beta, i = 1, 2, 3\rangle,$$

where the relations are of the form

$$x_3 = x_1^{yx_1},$$

$$x_2 = (x_1x_2x_3x_1^{-1}x_2^{-1}x_1^{-1})^y,$$

$$x_2 = x_3^{x_1^{-1}}.$$

Excluding the generator $x_3$ gives

$$x_2 = (x_1x_2x_1^{-1}y^{-1}x_1yx_1^{-1}y^{-1}x_1^{-1}yx_1^{-1}x_1^{-1})^y,$$

$$x_2 = x_1^{yx_1yx_1^{-1}yx_1^{-1}}.$$

Excluding $x_2$ from the right side of the first relation, we get

$$x_2 = x_1^{y^{-1}x_1^{-1}yx_1^{-1}x_1^{-1}y^2}.$$

Denote $x_1 = x$, we see that $G(K_5)$ has the presentation

$$G(K_5) = \langle x, y \parallel x^{y^{-1}x^{-1}yx^{-1}x^{-1}y^2} = x^{yx^{-1}x^{-1}y} \rangle.$$

**Theorem 3.8.** The following holds for $G(K_5)$:

1. The group $G(K_5)$ is isomorphic to $G(K_5) = H_5 \rtimes \mathbb{Z}$, where

$$H_5 = \cdots * B_{k-1} A_k A_{k+1} * B_{k+1} A_{k+2} * B_{k+2} * \cdots$$

is the amalgamated free product with

$$A_k = \langle x_k, x_{k+1}, x_{k+2} \parallel [x_{k+1}x_k^{-1}x_{k+2}x_k^{-1}x_{k+2}^{-1}x_{k+1}^{-1}x_k^{-1}] = 1 \rangle,$$

and $B_k$ is free group of rank 2, $k \in \mathbb{Z}$. 

(2) The quotient of \( G(K_5) \) by the terms of lower central series gives

\[
G(K_5)/\gamma_4 G(K_5) \cong F_2/\gamma_4 F_2, \\
\gamma_4 G(K_5)/\gamma_5 G(K_5) \cong \mathbb{Z}^2 \times \mathbb{Z}_2, \\
G(K_5)/\gamma_5 G(K_5) \not\cong F_2/\gamma_5 F_2.
\]

Proof.

(1) We consider the relation

\[ x^{y^{-1}x^{-1}yxy^{-1}x^{-1}y^2} = x^{yxy^{-1}x^{-1}yx}. \]

Transforming it, we obtain

\[ x^{y^{-1}x^{-1}yxy^{-1}x^{-1}y^2x^{-1}y^{-1}y} = x, \]

which implies

\[ [x, y^{-1}x^{-1}yxy^{-1}x^{-1}y^2x^{-1}y^{-1}yx^{-1}y^{-1}] = 1. \]

If we introduce the notation

\[ x_k = y^{-k}xy^k, \quad k \in \mathbb{Z}, \]

then we can rewrite the above relation as

\[ [x_0, x_1^{-1}x_1^{-1}x_1^{-1}x_1^{-1}] = 1. \]

Let

\[ H_5 = \langle x_k, \ k \in \mathbb{Z} \mid [x_{k+1}, x_{k+2}^{-1}x_k x_{k+2} x_k^{-1} x_{k+1} x_k^{-1}] = 1, \ k \in \mathbb{Z} \rangle. \]

Then

\[ G(K_5) = H_5 \rtimes \mathbb{Z}, \]

where \( \mathbb{Z} = \langle y \rangle \) and \( x_k^y = x_{k+1}, \ k \in \mathbb{Z}. \) Further, for every \( k \in \mathbb{Z} \) let

\[ A_k = \langle x_k, \ x_{k+1}, \ x_{k+2} \mid [x_{k+1}, x_{k+2}^{-1}x_k x_{k+2} x_k^{-1} x_{k+1} x_k^{-1}] = 1 \rangle, \]

and

\[ B_k = \langle x_{k+1}, \ x_{k+2} \rangle. \]

Then

\[ H_5 = \cdots * B_{k-1} A_k B_k A_{k+1} B_{k+1} A_{k+2} * \cdots \]

is an infinite free product with amalgamation. Note that \( B_k \cong F_2 \) is a free group of rank 2, \( k \in \mathbb{Z}. \)

(2) Transforming the relation, we get

\[
\begin{align*}
x^{y^{-1}x^{-1}yxy^{-1}x^{-1}y^2} &= x^{yxy^{-1}x^{-1}yx}, \\
x^{y^{-1}x^{-1}yxy^{-1}x^{-1}yx} &= x^{yxy^{-1}x^{-1}yxy^{-1}x}, \\
x^{[y, x]^2} &= x^{-2}yx[y, x]y^{-1}x, \\
[y, x, x][y, x, x][y, x][y, x] &= [x^{-2}yx[y, x]y^{-1}x, x], \\
[y, x, x][y, x, x][y, x][y, x][y, x] &= [[x, y^{-1}]^2[y, x][y, x, y^{-1}x], x].
\end{align*}
\]
Performing the transformations modulo $\gamma_5 G(K_5)$, we obtain

\[
\begin{align*}
[y, x, x]^2 & \equiv [[x, y^{-1}] [x, y^{-1}, x] [y, x] [y, x, y^{-1} x], x], \\
[y, x, x]^2 & \equiv [[x, y^{-1}] [y, x] [x] [x, y, x]^{-1} [y, x, y, x]^{-1} [y, x, x], x], \\
[y, x, x]^2 & \equiv [[y, x] [y, x, x]^{-1} [y, x] [y, x, x]^{-1} [y, x, x], x], \\
[y, x, x]^2 & \equiv [[y, x, x, x] [y, x, x]^{-1} [y, x, x, x]^{-1} [y, x, x, x], x], \\
1 & \equiv [y, x, x, x]^{-2} [y, x, x, x]^{-2}.
\end{align*}
\]

Therefore, in the quotient $G(K_5)/\gamma_5 G(K_5)$, the relation is of the form

\[
([x, y, y, x] [x, y, x, x]^{-1})^2 = 1.
\]

Since the commutators $[x, y, y, x]$, $[x, y, x, x]$, $[x, y, y, y]$ are the basic commutators of weight 4, we have

\[
\begin{align*}
G(K_5)/\gamma_4 G(K_5) & \cong F_2/\gamma_4 F_2, \\
\gamma_4 G(K_5)/\gamma_5 G(K_5) & \cong \mathbb{Z}^2 \times \mathbb{Z}_2, \\
G(K_5)/\gamma_5 G(K_5) & \cong F_2/\gamma_5 F_2.
\end{align*}
\]

3.6. **Virtual knot $K_6$ (The knot 4.100 in [17])**.

This knot is equivalent to the closure of the braid $\beta = \sigma_1^4 \rho_1 \in VB_2$ and has the group $G(K_6) = \langle x_1, x_2, y \mid x_i = x_i \beta, i = 1, 2 \rangle$, where the relations have the form

\[
\begin{align*}
x_1^{\sigma_1^4 \rho_1} & = x_1, & x_2^{\sigma_1^4 \rho_1} & = x_2.
\end{align*}
\]

Transforming the relations, we have

\[
\begin{align*}
(x_1 x_2 x_1^{-1})^{\sigma_1^4 \rho_1} & = x_1, & x_1^{\sigma_1^2 \rho_1} & = x_2, \\
x_2 x_2^{\sigma_1^2 \rho_1} x_2^{-1} & = x_1, & x_1^{\sigma_1^2 \rho_1} & = x_2, \\
x_1^{\sigma_1^2 \rho_1} & = x_2^{-1} x_1 x_2, & (x_1 x_2 x_1^{-1})^{\sigma_1^2 \rho_1} & = x_2, \\
x_1^{\sigma_1^2 \rho_1} & = x_2^{-1} x_1 x_2, & x_2^{\sigma_1^2 \rho_1} x_2^{-1} & = x_1 x_2 = x_2,
\end{align*}
\]
By modulo \( x \)

Further using (1) and transforming the commutator

Let us transform the relation

Proof.

So, the group \( G(K_6) \) has the presentation

with 3 generators and 2 defining relations.

**Theorem 3.9.** The first five terms of lower central series of the group \( G(K_6) \) are different from each other. Moreover, we have

\[
G(K_6)/\gamma_4 G(K_6) \cong F_2/\gamma_4 F_2,
\]

\[
\gamma_4 G(K_6)/\gamma_5 G(K_6) \cong \mathbb{Z}^2 \times \mathbb{Z}_2,
\]

\[
G(K_6)/\gamma_5 G(K_6) \not\cong F_2/\gamma_5 F_2.
\]

**Proof.** Let us transform the relation

by modulo \( \gamma_4 G(K_6) \). We get

by modulo \( \gamma_4 G(K_6) \), the relation \( x_2^{y^{-1}} = x_1^{(x_1 x_2)^2} \) has the form

Transforming the relation

by modulo \( \gamma_4 G(K_6) \), we get

Hence, by modulo \( \gamma_4 G(K_6) \), the relation \( x_1^y = x_2^{(x_1 x_2)^2} \) has the form

Further using (1) and transforming the commutator \([x_1, x_2] (\text{mod } \gamma_4 G(K_6))\), we obtain

\[
[x_1, x_2] = [x_1, x_1 [x_1, y] [x_1, x_2]^2] \equiv [x_1, [x_1, y] [x_1, x_2]^2] \equiv [x_1, [x_1, y] [x_1, x_2]^2] = \ldots
\]
\[ \equiv [x_1, [x_1, y]][x_1, [x_1, x_2]]^2 \equiv [x_1, [x_1, y]][x_1, [x_1, x_1]]^2 \equiv [y, x_1, x_1]. \]

So, we have

\[ [x_1, x_2] \equiv [y, x_1, x_1] \pmod{\gamma_4 G(K_6)}. \]

Hence, from the relation (1) we have

\[ x_2 \equiv x_1[x_1, y][y, x_1, x_1]^2 \pmod{\gamma_4 G(K_6)}. \]

Similarly, from the relation (2) we have

\[ x_2 \equiv x_1[x_1, y][y, x_1, x_1]^2 \pmod{\gamma_4 G(K_6)}. \]

Therefore,

\[ G(K_6)/\gamma_4 G(K_6) \cong \langle x_1, x_2, y \mid x_2 = x_1[x_1, y][y, x_1, x_1]^2, \gamma_4 G(K_6) \rangle, \]

\[ \cong \langle x_1, y \mid \gamma_4 G(K_6) \rangle, \]

\[ \cong F_2/\gamma_4 F_2. \]

Let us now show that

\[ G(K_6)/\gamma_5 G(K_6) \ncong F_2/\gamma_5 F_2. \]

From the relations

\[ x_1^y = x_2(x_1 x_2)^2, \quad x_2^{y^{-1}} = x_1(x_1 x_2)^2, \]

it follows that

\[ ([x_1 x_2]^2 y(x_1 x_2)^2, x_1] = [y, x_1]. \]

We transform the left side of the above relation modulo \( \gamma_5 G(K_6) \) by using the relation

\[ x_2 \equiv x_1[x_1, y][y, x_1, x_1]^2 \pmod{\gamma_4 G(K_6)}. \]

(3)

We have

\[ [(x_1 x_2)^2 y(x_1 x_2)^2, x_1] \equiv [(x_1 x_2)^2 y, x_1]((x_1 x_2)^2, x_1), \]

\[ \equiv [(x_1 x_2)^2 y, x_1][((x_1 x_2)^2 y, x_1, (x_1 x_2)^2)][(x_1 x_2)^2, x_1], \]

\[ \equiv [(x_1 x_2)^2, x_1]^y[y, x_1][((x_1 x_2)^2 y, x_1, (x_1 x_2)^2)][(x_1 x_2)^2, x_1], \]

\[ \equiv [(x_1 x_2)^2, x_1][((x_1 x_2)^2, x_1, y][y, x_1][((x_1 x_2)^2 y, x_1, (x_1 x_2)^2)][(x_1 x_2)^2, x_1], \]

\[ \equiv [(x_1 x_2)^2, x_1][(x_1 x_2)^2, x_1, y][y, x_1], \]

\[ [((x_1 x_2)^2, x_1][(x_1 x_2)^2, x_1, y][y, x_1], (x_1 x_2)^2][((x_1 x_2)^2, x_1]. \]
Using the commutator identities and (3), we transform \([\sigma_1 \sigma_2 \sigma_1] \sigma_1 \) in the quotient \(G(K_6) \gamma_5 G(K_6)\) and we get

\[
\begin{align*}
[\sigma_1 \sigma_2 \sigma_1, \sigma_1] &= [\sigma_1 \sigma_2 \sigma_1, \sigma_1] \sigma_1, \\
&= [\sigma_1, \sigma_2 \sigma_1] \sigma_1 \sigma_2 \sigma_1, \\
&= [\sigma_1, \sigma_1] \sigma_2 \sigma_1, \\
&= [\sigma_1, \sigma_1] [\sigma_1 \sigma_2 \sigma_1, \sigma_1], \\
&= [\sigma_1, \sigma_1] [\sigma_1 \sigma_1 \sigma_2, \sigma_1] \\
&\equiv [\sigma_1, \sigma_1] [\sigma_1 \sigma_1 \sigma_2, \sigma_1].
\end{align*}
\]

Hence, we obtain

\[
\begin{align*}
[\sigma_1 \sigma_2 \sigma_1, \sigma_1]^2 &= [\sigma_1, \sigma_1] [\sigma_1 \sigma_1 \sigma_2, \sigma_1]^2 [\sigma_1, \sigma_1] [\sigma_1 \sigma_1 \sigma_2, \sigma_1], \\
&\equiv [\sigma_1, \sigma_1] [\sigma_1 \sigma_1 \sigma_2, \sigma_1]^2 [\sigma_1, \sigma_1] [\sigma_1 \sigma_1 \sigma_2, \sigma_1].
\end{align*}
\]

Therefore, the relation

\[
[\sigma_1 \sigma_2 \sigma_1, \sigma_1] = [\sigma_1, \sigma_1]
\]

has the form

\[
[\sigma_1, \sigma_1, \sigma_1]^{-2} [\sigma_1, \sigma_1, \sigma_1]^6 \equiv 1 \mod \gamma_5 G(K_6).
\]

So, we finally have

\[
\begin{align*}
G(K_6) / \gamma_4 G(K_6) &\cong F_2 / \gamma_4 F_2, \\
\gamma_4 G(K_6) / \gamma_5 G(K_6) &\cong \mathbb{Z}^2 \times \mathbb{Z}_2, \\
G(K_6) / \gamma_5 G(K_6) &\not\cong F_2 / \gamma_5 F_2.
\end{align*}
\]
3.7. Virtual figure eight knot $K_7$ (The knot 3.2 in [17]).

This knot is the closure of $\beta = \sigma_2^{-1}\sigma_1\rho_1 \in VB_2$ and the virtual knot group has the presentation

$$G(K_7) = \langle x, y \parallel x[y^{-1}, x] = x[y, x^{-1}] \rangle.$$ 

**Theorem 3.10.** The following holds for group $G(K_7)$:

1. The quotient of $G(K_7)$ by the terms of lower central series gives
   
   $$G(K_7)/\gamma_4G(K_7) \cong F_2/\gamma_4F_2,$$
   $$\gamma_4G(K_7)/\gamma_5G(K_7) \cong \mathbb{Z}^2,$$
   $$G(K_7)/\gamma_5G(K_7) \not\cong F_2/\gamma_5F_2.$$

2. The group $G(K_7)$ is the semi direct product $H_7 \triangleleft \mathbb{Z}$, where
   
   $$H_7 = \langle x_k, k \in \mathbb{Z} \parallel [x_k, x_k^{-1}x_k^2x_{k+1}] = 1, k \in \mathbb{Z} \rangle$$

is the amalgamated free product with $A_k \cong F_2, k \in \mathbb{Z}$.

**Proof.**

1. Let us rewrite the relation of $G(K_7)$ in the form
   
   $$[[y^{-1}, x][x^{-1}, y], x] = 1.$$ 

We transform the word $[y^{-1}, x][x^{-1}, y] \pmod{\gamma_4G(K_7)}$ and we get

$$[y^{-1}, x][x^{-1}, y] = [x, y][y^{-1}[y, x]x^{-1},$$

$$= [x, y][x, y^{-1}][y, x][y, x, x^{-1}],$$

$$\equiv [x, y][y^{-1}[y, x, x^{-1}] (\text{mod } \gamma_4G(K_7)).$$

Therefore, we get

$$[[y^{-1}, x][x^{-1}, y], x] \equiv [x, y, x][y, x, x]^{-1}[x, y, x, x] (\text{mod } \gamma_5G(K_7)).$$
The commutators \([x, y, y, x], [x, y, x, x], [x, y, y, y]\) are the basic commutators of weight 4. So, we obtain
\[
G(K_7)/\gamma_4 G(K_7) \cong F_2/\gamma_4 F_2, \\
\gamma_4 G(K_7)/\gamma_5 G(K_7) \cong \mathbb{Z}^2, \\
G(K_7)/\gamma_5 G(K_7) \not\cong F_2/\gamma_5 F_2.
\]

(2) We rewrite the defining relation
\[
x[y^{-1}, x] = x[y x^{-1}]
\]
in the form
\[
x^{-y^{-1}} = x^{-y x^{-1}}.
\]
Let us denote
\[
x_k = y^{-k} x y^k, \quad k \in \mathbb{Z},
\]
then the relation is equivalent to the following set of relations:
\[
x^{-1} x_k = x_k^{-1} x_{k+1}^{-1}, \quad k \in \mathbb{Z}
\]
or
\[
[x_k, x_{k-1}^{-1} x_k^2 x_{k+1}] = 1, \quad k \in \mathbb{Z}.
\]
Putting
\[
H_7 = \langle x_k, k \in \mathbb{Z} \mid [x_k, x_{k-1}^{-1} x_k^2 x_{k+1}] = 1, k \in \mathbb{Z} \rangle,
\]
we get
\[
G(K_7) = H_7 \rtimes \mathbb{Z},
\]
where \(\mathbb{Z} = \langle y \rangle\) and \(x_k^y = x_{k+1}, k \in \mathbb{Z}\). For any \(k \in \mathbb{Z}\), let
\[
A_k = \langle x_k, x_{k+1} \rangle.
\]
Then the relation
\[
[x_k, x_{k-1}^{-1} x_k^2 x_{k+1}] = 1
\]
is equivalent to
\[
x_k^{-2} x_{k-1} x_k x_{k+1}^{-1} = x_{k+1} x_k x_{k+1}^{-1} x_k^{-2}.
\]
Let us denote
\[
u_k = x_k^{-2} x_{k-1} x_k x_{k+1}^{-1}, \quad v_k = x_{k+1} x_k x_{k+1}^{-1} x_k^{-2}, \quad k \in \mathbb{Z},
\]
then we have
\[
H_7 = \cdots * A_{k-1}^{(u_k=v_k)} * A_k * \cdots ,
\]
an infinite free product with amalgamation, where \(A_k \cong F_2\) is the free group of rank 2, \(k \in \mathbb{Z} \).

\(\Box\)
4. Groups $G_A$ and $G_{\tilde{M}}$ for the virtual Hopf link

**Proposition 4.1.** The group $G_A(L)$ and $G_{\tilde{M}}(L)$ of virtual Hopf link $L$ are right angled Artin groups and $G_A(L)$ is not isomorphic to $G_{\tilde{M}}(L)$.

**Proof.** It is easy to see that $L$ is equivalent to closure of braid $\beta = \sigma_1^{-1}\rho_1 \in VB_2$. We have

$$G_A(\beta) = \langle x_1, x_2, y || [y, x_1] = [x_1, x_2] = 1 \rangle$$

which is isomorphic to the direct product $\mathbb{Z} \times F_2$, where $\mathbb{Z} = \langle x_1 \rangle$ and $F_2 = \langle x_2, y \rangle$. On the other hand, we have

$$G_{\tilde{M}}(K) = \langle y_1, y_2, v_1, v_2 || [y_1, v_2] = [y_1, y_2] = [v_1, v_2] = 1 \rangle$$

which is isomorphic to the quotient of the free product $\mathbb{Z}^2 * \mathbb{Z}^2$ by the normal closure of the commutator $[y_1, v_2]$, where the first factor $\mathbb{Z}^2 = \langle y_1, y_2 \rangle$ and the second factor $\mathbb{Z}^2 = \langle v_1, v_2 \rangle$.

Hence, we have

$$G_A(\beta) / \gamma_2 G_A(\beta) \cong \mathbb{Z}^3,$$

and

$$G_{\tilde{M}}(\beta) / \gamma_2 G_{\tilde{M}}(\beta) \cong \mathbb{Z}^4,$$

which implies

$$G_A(L) \not\cong G_{\tilde{M}}(L).$$

□

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