An Assessment of and Solution to the Intensity Diffusion Error Intrinsic to Short-characteristic Radiative Transfer Methods

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Abstract

Radiative transfer coupled with highly realistic simulations of the solar atmosphere is routinely used to infer the physical properties underlying solar observations. Due to its computational efficiency, the method of short-characteristics is often employed, despite it introducing numerical diffusion as an interpolation artifact. In this paper, we quantify the effect of the numerical diffusion on the spatial resolution of synthesized emergent intensity images, and derive a closed form analytical model of the diffusive error made as a function of viewing angle when using linear interpolation. We demonstrate that the consequent image degradation adversely affects the comparison between simulated data and observations away from disk center, unless the simulations are computed at much higher intrinsic resolutions than the observations. We also show that the diffusive error is readily avoided by interpolating the simulation solution on a viewing angle aligned grid prior to computing the radiative transfer. Doing this will be critical for comparisons with observations using the upcoming large aperture telescopes—the Daniel K. Inouye Solar Telescope and the European Solar Telescope.

Key words: radiative transfer – methods: numerical – Sun: photosphere

1. Introduction

Computational radiative transfer is critical to inferring the physical properties of astrophysical objects from observed spectra. Moreover, radiation is often a key energy transport mechanism and radiative transfer modeling plays an important role in hydrodynamic and magnetohydrodynamic (MHD) simulations of a wide range of phenomena from planetary and stellar atmospheres to accretion disks around compact objects. A reliable solution of the radiative transfer equation is required to both determine the radiative heating rate in the objects. A reliable solution of the radiative transfer equation is sought along rays by evaluating the formal solution

\[ I(\tau_0) = I_0(0)e^{-\tau_0} + \int_0^{\tau_0} S_0(t_0)e^{-(\tau_0-t_0)} dt_0 \]  \hspace{1cm} (1)

with

\[ \tau_0 = \int_0^L \kappa_\nu \rho ds, \]  \hspace{1cm} (2)

where \( I_0 \) is the specific intensity at frequency \( \nu \), \( \tau_0 \) is the optical depth along ray path \( ds \), \( \kappa_\nu \) is the frequency specific opacity, and \( S_0 \) is the source function (the ratio of the thermal emissivity to the opacity), which in local thermodynamic equilibrium is taken to be the Planck function (see Mihalas & Mihalas 1984 for details and Carlsson 2008 for a short review of solution methods as applied to a three-dimensional radiative transfer in stellar atmospheres of cool stars). Moments of the radiation field are then evaluated by numerical quadrature (Carlson 1963; Lathrop & Carlson 1965; Carlson 1970) using a limited number of ray directions.

Numerical integration of the formal solution (Equation (1)) along rays typically employs one of two strategies: the long-characteristic (Mihalas et al. 1978) or the short-characteristic (Kunasz & Auer 1988) method. These are illustrated in Figure 1 for radiation propagating from the bottom of the domain, where \( I_0(0) \) is specified, to a point within the domain. The long-characteristic method solves the radiative transfer equation along rays connecting downwind grid points in the domain (the points for which the specific intensity is computed directly from the transfer equation, position O in the left panel of Figure 1) to the last upwind point where the ray originates. Because the ray does not necessarily intersect the numerical grid except at the upwind point, this method requires interpolation of the plasma properties (\( \kappa_\nu, \rho, \) and \( S_0 \)) at ray intersections with the grid rows or columns when evaluating the transfer equation, as well as interpolation of the specific intensity along the computed rays to neighboring grid points, which do not lie directly on the ray paths. The long-characteristic method faces some challenges when, for large problems, domain decomposition is needed to distribute the computation among many processors (e.g., Carlsson 2008).

The short-characteristic method, on the other hand, updates the specific intensity row-by-row by solving the radiative transfer equation for each grid point along rays starting from interpolated values on the previous row (right panel in Figure 1) or column, depending on the ray direction. Sweeping the grid in this way allows for somewhat more straightforward domain decomposition, since the radiative transfer solution along the short rays depends only on the local plasma properties and the value of the specific intensity must be communicated only at the grid locations. The short-characteristic method is routinely employed in radiative MHD solvers (e.g., Vogler et al. 2005; Hayek et al. 2010; Gudiksen et al. 2011; Davis et al. 2012) and stand-alone radiative transfer solvers (e.g., Uitenbroek 2001; Criscuoli & Rast 2009; Leenaarts & Carlsson 2009; Ibgui et al. 2013;
The most significant drawback of the short-characteristic method is that, for nonvertical or nongrid-point intersecting ray directions, the specific intensity suffers diffusion because it is successively interpolated as it is propagated. In the long-characteristic method, only the plasma properties and not the specific intensity itself need to be interpolated to integrate along the ray path. The diffusive error introduced by the short-characteristic method has been known since the foundational work of Kunasz & Auer (1988), which demonstrated the occurrence of angular dispersion of the radiation for nongrid-aligned propagation directions. The error introduced decreases with increasing order of the interpolation scheme, but at the cost of ringing in regions with steep specific intensity gradients. This can lead to negative intensity values, which in turn can be mitigated by monotonic interpolation schemes (e.g., Auer & Paletou 1994; Criscuoli 2007; Hayek et al. 2010; Ibgui et al. 2013), but a consequence of these is the nonconservation of the radiative energy (Criscuoli 2007). For this reason, and for its computational efficiency, many widely used radiative transfer solvers rely on the short-characteristic method with linear interpolation despite the diffusive error introduced (e.g., RH Uitenbroek 2001, PORTA Štěpán & Trujillo Bueno 2013, RH1.5 Pereira & Uitenbroek 2015, MULTI3D Leenaarts & Carlsson 2009).

The diffusive errors inherent in short-characteristic solvers leads to viewing angle dependent inaccuracies in the calculated emergent intensity, whether the emergent intensity is computed directly within the MHD simulation or post-facto using a stand-alone radiative transfer solver for spectral synthesis. Moreover, it can affect the numerical solution of the radiative MHD equations through the radiative heating rate in the energy equation. Some work has been done to demonstrate and quantify the errors (e.g., Kunasz & Auer 1988; Bruls et al. 1999), but a more detailed understanding is needed. This is particularly true in light of the upcoming large aperture telescopes, such as the NSF’s Daniel K. Inouye Solar Telescope (DKIST; Elmore et al. 2014; Tritschler et al. 2016) and the planned European Solar Telescope (EST; Collados et al. 2013; Matthews et al. 2016), for which the diffraction limit is nearing the resolution of radiative MHD simulations of the solar surface (e.g., Galsgaard & Nordlund 1996; Freytag et al. 2002; Rempel 2014). In these circumstances, comparisons between observations and simulations become sensitive to the diffusive errors in the calculation of the emergent intensity.

Here we quantify the effective reduction in spatial resolution that results when employing the short-characteristic radiative transfer method to calculate the emergent intensity at inclined viewing angles. We derive a closed form analytical model for the specific intensity of a beam at each point of the grid as a function of beam angle when the beam is initiated as a delta function at the bottom of the computation domain. The emergent intensity at the top of the domain is the effective point-spread function (PSF) of the numerical scheme. We validate the model by comparing the intensity obtained by a numerical short-characteristic solution of the radiation emerging from a 3D MHD simulation snapshot with that predicted by the analytic model. Finally, we demonstrate that the diffusive error is readily avoided by interpolating the simulation atmosphere onto a viewing angle aligned grid prior to computing the radiative transfer solution. Section 2 describes the analytical model. Section 3 assesses the effect of short-characteristic intensity diffusion on the spatial resolution of the emergent intensity, and demonstrates that pre-tilting the simulation atmosphere avoids the diffusive error. Section 4 examines the effect of a higher-order interpolation on both the diffusive error and the total emergent intensity as functions of ray inclination.

2. Analytical Model of the Short-characteristic Diffusive Error

To derive an analytic expression for the effective diffusion introduced by the short-characteristic radiative transfer method, we consider a single delta-function point source of radiation at the bottom of a 3D domain and the subsequent propagation of the specific intensity through the domain along inclined ray directions (i.e., the search-beam problem, Kunasz & Auer 1988). To maintain analytic tractability, we examine only linear interpolation on a regular rectangular grid and assume that the beam propagates through a vacuum, so that only interpolation of the specific intensity (and not that of the plasma properties) contributes to diffusion effects. The emergent intensity at the top of the domain is then effectively the PSF of the short-characteristic solution. Moreover, since, as we will see in more detail, specific intensity interpolation errors are compounded with height, they dominate the error budget even in nonvacuum calculations.
Depending on the ray propagation direction, the short-characteristic method interpolates the specific intensity either on horizontal (xy) or vertical (xz or yz) planes. The full derivation of the general 3D solution for the arbitrary ray direction is given in the Appendix. Here, for simplicity of presentation, we discuss the solution for the special case, where \( \phi = 0^\circ \). This corresponds to ray propagation in the xz plane (as shown in the right panel of Figure 1). Interpolation then occurs on horizontal (in x) or vertical (in z) grid lines only, depending on the ray propagation direction \( \theta \). Note that \( \theta \), the search-beam inclination angle, is defined with respect to the horizontal so that \( \theta = 90^\circ \) for a vertical propagating ray.

When \( \phi = 0^\circ \), the 3D solution (Equations (7) and (11) in Appendices A.1 and A.2) reduces to a PSF in x only, as no diffusion occurs in the y direction for a ray confined to the xz plane. For ray angles \( 45^\circ < \theta < 90^\circ \) (interpolation on horizontal grid lines), the intensity at any grid point can be written as

\[
I_{n_x, n_z}^{\text{h}} = I_{\text{source}} \times \frac{n_z!}{n_z!(n_z - n_x)!} \left( 1 - \frac{dz}{dx \tan \theta} \right)^{n_x - n_z} \left( \frac{dz}{dx \tan \theta} \right)^{n_z},
\]

where \( I_{\text{source}} \) is the initial point-source intensity, and \( n_x \) and \( n_z \) are the integer number of grid point displacements in the ray direction from the source location. For ray angles \( 0^\circ < \theta < 45^\circ \) (interpolation on vertical grid lines), the expression becomes

\[
I_{n_x, n_z}^{\text{v}} = I_{\text{source}} \times \frac{n_x - 1!}{(n_x - n_z)!(n_x - 1)!} \left( 1 - \frac{dx \tan \theta}{dz} \right)^{n_x - n_z} \left( \frac{dx \tan \theta}{dz} \right)^{n_z}.\]

The intensity distributions (as a function of \( n_z \)) in Equations (3) and (4) represent the one-dimensional PSFs introduced by the short-characteristic method, which depends on the ray inclination angle \( \theta \) and the number of horizontal grid plane displacements \( n_z \) above the initial source height. One advantage of writing the intensity distributions in terms of discrete grid point displacements is that they become recognizable as standard and negative binomial distributions in \( n_z \) for fixed \( n_x \) with standard deviations

\[
\sigma = \sqrt{\pm n_z \frac{dz}{dx \tan \theta} \left( 1 - \frac{dz}{dx \tan \theta} \right)^{n_x - n_z} \left( \frac{dz}{dx \tan \theta} \right)^{n_z}},
\]

where the plus and minus signs apply to the horizontal (Equation (3)) and vertical (Equation (4)) grid line interpolation solutions, respectively, and \( dz/dx \) is the ratio of the grid spacing. These standard deviations are one measure of the spatial smearing or diffusion introduced by the short-characteristic method.

The intensity distributions and their standard deviations are plotted in Figure 2 as a function of \( \theta \) for the case of a square grid \((dx = dz)\). In the plots, \( n_x \) and \( \sigma \) have been scaled by the sine of the inclination angle to account for foreshortening when the emergent intensity is viewed at inclined angles. This allows for direct comparison with simulations (Section 3). For illustrative purposes, the distributions (Figure 2(a)) and standard deviations (thick dashed curve in Figure 2(b)) are shown at multiple ray angles for fixed \( n_z = 65 \) (corresponding to the depth of the mean vertical ray \( \tau = 1 \) surface in the MHD simulation described in Section 3). The short-characteristic ray direction is grid aligned at \( \theta = 90^\circ \) and \( 45^\circ \), and the specific intensity distribution collapses to a delta function at those angles (no diffusive error). While \( \theta = 0^\circ \) is grid aligned as well, that case is pathological because a strictly horizontal ray never reaches the upper boundary. Moreover, when interpolation occurs on vertical grid lines, as it does for \( 0^\circ < \theta < 45^\circ \), an increasing number of interpolations are required (to cross an equal number of grid plane lines, \( n_z \)) with decreasing inclination angle. Thus the diffusive error increases monotonically as \( \theta \to 0^\circ \). For horizontal grid interpolation (\( 45^\circ < \theta < 90^\circ \)), the maximum diffusive error \( \sigma \) occurs at \( 67.5^\circ \), half-way between the grid-aligned directions.

The specific intensity distributions given by the analytic solutions of Equations (3) and (4) (more generally by Equations (7) and (11) in Appendices A.1 and A.2) and plotted in Figure 2(a) represent the PSFs of the short-characteristic solution for any ray propagation angle \( \theta \) through a vacuum domain. The width of the distribution \( \sigma \) (Equation (5) and Figure 2(b) thick dashed curve) captures the effective image smearing introduced when computing the emergent intensity. It depends explicitly on \( n_z \), the number of interpolations above the source point. Since the model assumes ray propagation through a vacuum, the analytic solution captures image degradation in a realistic 3D solution due to radiation propagation above the optically thick to optically thin transition. Below the \( \tau = 1 \) transition, the medium is optically thick and the diffusive error of the short-characteristic method, while it may contribute to the accuracy with which the divergence of the radiative flux is determined in the radiative MHD simulation, does not contribute significantly to degradation of the emergent intensity. Thus the specific intensity error made depends on the depth within the domain from which the radiation escapes. Since the optical depth surfaces are not aligned with the numerical grid, that depth depends not only on the simulation solution, but on the viewing angle and the wavelength of the radiation. This must be accounted for when evaluating the short-characteristic error in the emergent intensity from a simulation solution.

### 3. Diffusive Error in the Emergent Intensity from 3D MHD Simulations

To assess the amount of image degradation in the emergent intensity images computed from 3D MHD simulations, we examine a snapshot from a MURaM (Vögler et al. 2005) simulation of a \( 6 \times 6 \times 1.2 \) Mm region of solar granulation including magnetic field generation by small-scale dynamo processes (Rempel 2014). The solution has 8 km grid spacing in both the vertical and horizontal directions, the latter comparable to high resolution images that will be forthcoming with the DKIST. It thus provides a good test bed for future comparisons between simulations and observations. The synthesized emergent intensity was computed using the RH radiative transfer solver (Uitenbroek 2001) for viewing angles between \( \mu \) of 0.2 and 1 (\( \mu = \sin \theta \), with \( \theta \) as defined previously) and at \( \lambda = 500 \) nm. For comparison with Section 2, we confine the viewing angles to the xz plane (\( \phi = 0^\circ \)).

We perform the radiative transfer in two ways. In the first, we solve the radiative transfer equation using the standard
short-characteristics scheme, linearly interpolating the specific intensity along the inclined rays. The resulting emergent intensity images for \( \lambda = 500 \) nm and viewing angles \( \mu = 0.20, 0.49, 0.82, \) and 0.99 are shown along the top row of Figure 3. The effect of numerical diffusion in the direction of inclination is visually apparent in the images. In the second, we compute the radiative transfer on the same atmosphere by pre-tilting the atmosphere to the required viewing angle, interpolating the atmospheric properties along the viewing angles, and solving the transfer equation, using the same short-characteristic scheme, along the now grid-aligned rays. We perform the interpolation such that \( dz \) in the ray-aligned atmosphere remains unchanged from its value in the original atmosphere. The interpolation is therefore not restricted to horizontal or vertical planes of the atmosphere, and a bilinear interpolation scheme is used. This method allows for calculation of the emergent intensity at highly inclined angles without causing very sparse grids that would result from interpolation restricted to grid planes. In this way, we avoid the compounded specific intensity interpolation error of the short-characteristic method, though the plasma properties are still interpolated as required for all radiative transfer solvers. Similar approaches to avoid diffusion error in the emergent intensity have been used in other works (e.g., Koesterke et al. 2008; Hayek et al. 2011; Amarsi et al. 2016). The important point is that the pre-tilting procedure eliminates the compounded specific intensity interpolation error without adding additional sources of error. The bottom row of Figure 3 shows the emergent intensity that results when the pre-tilted scheme is employed. It is visually apparent that the images retain higher spatial resolution than those obtained with the standard short-characteristic method.

Quantifying this difference, and its agreement with the model of Section 2, requires that \( n_z \) be determined for use in Equations (3) and (4). As discussed previously, \( n_z \) is the number of horizontal grid plane displacements above the source height. This is the vertical displacement between the height from which the radiation escapes and the observer’s position (approximately the number of horizontal grid levels crossed along the ray between the mean \( \tau = 1 \) surface and the top of the domain). Since we are examining the emergent intensity at \( \lambda = 500 \) nm, we measure, for select viewing angles, the depth of the mean \( \tau_{500} = 1 \) surface in the simulation. We determine that \( n_z = 53–66 \) for lines of sight between \( 10^\circ \) and vertical, and plot those values as filled circles in Figure 2(b). For other angles, we use a logarithmic fit to the measured points (dotted gray curve in Figure 2(b)), and use these to calculate \( \sigma \) from Equations (3) and (4). This is plotted as a solid black curve in Figure 2(b). A comparison between the \( \sigma \) values computed from the 3D MHD snapshot, for which \( n_z \) is a function of the inclination angles (solid black curve), and those found for the case of a ray propagating in vacuum (Section 2) for which \( n_z \) is not a function of \( \theta \) (dashed black curve), shows the dependence of the smearing on the formation height of radiation. In this example, for which we have chosen the value \( n_z \) in the vacuum calculation to coincide with the depth of the \( \tau_{500} = 1 \) surface in the vertical direction, the amounts of smearing differs significantly for angles smaller than \( \theta \lesssim 30^\circ \).

We compare the effective smearing from the analytic model directly with the measured reduction in spatial resolution induced by the standard short-characteristic method in the images themselves. In this way, we can demonstrate agreement with the model of Section 2, and show that the error can be eliminated in real model analysis by pre-tilting and interpolating before computing the radiative transfer. To make this comparison, we examine the amplitude spectra of the two image sets. Since the error occurs only in the inclined direction of the viewing angle prescribed, we compute the average 1D spatial amplitude spectra along the direction of the tilt (x-dimension). We then convolve the rows of the pre-tilted image with the generic analytic solution (Equations (3) or (4), and use a least-squares measure of the difference between the average of those convolutions and the average 1D spectrum of the standard short-characteristic image to determine the standard deviation of the best-fit binomial convolution kernel.

![Figure 2](image-url)

**Figure 2.** In (a), intensity distributions from Equations (3) and (4) with \( n_z = 65 \). In (b), the effective image smearing computed as the standard deviation of the model (Equation (3)) for \( n_z = 65 \) (dashed black curve) and for \( n_z \) computed using the mean \( \tau_{500} = 1 \) formation depths in the MHD snapshot when accounting for viewing angle (solid black curve). Both \( n_z \) and \( \sigma \) are scaled by \( \sin \theta \) to account for apparent foreshortening caused by viewing the emergent intensity at inclined angles. The mean \( \tau_{500} = 1 \) formation depths were explicitly computed for viewing angles plotted with filled gray circles and determined by a logarithmic fit otherwise (dotted gray curve). The gray band indicates the \( \sigma \) range for \( n_z \) between the mean \( \tau_{500} = 0.5 \) and 1.5 formation depths. Pink diamonds represent the standard deviation of the binomial convolution kernel applied to the pre-tilt amplitude spectrum that best fits the standard short-characteristic spectrum. The solid blue curve and accompanying gray band indicates the standard deviation for the same mean \( \tau_{500} \) formation depths when employing cubic monotonic interpolation in the short-characteristic solution.
Examples of the average 1D spectra used in this procedure are plotted in Figure 4, and the estimates of the standard deviation $\sigma$ of the binomial convolution kernel are shown in Figure 2(b) with pink diamonds. The error in those measures, estimated as the uncertainty in the fit between the spectra, is smaller than the symbol size, and the values agree with the image degradation caused by inclined ray interpolation in the short-characteristic method. The agreement demonstrates that the model correctly captures the image degradation induced by the short-characteristic computation of the emergent intensity, and that the degradation can be avoided by pre-tilting the solution before computing the short-characteristic radiative transfer.

### 4. Extension to Higher-order Interpolation

As discussed in Section 1, the interpolation error introduced by the short-characteristics method can be reduced by employing a higher-order monotonic interpolation scheme (e.g., Kunasz & Auer 1988; Hayek et al. 2010). To assess the effectiveness of this approach, we revisit the search-beam problem, this time employing cubic monotonic interpolation (Fritsch & Carlson 1980) in the short-characteristic solution. Unlike the linear interpolation solution, we were unable to derive an analytical solution for the intensity on the grid, and so instead propagated the search beam through the vacuum domain numerically. For direct comparison with the linear interpolation results, as applied to the MHD simulations, the short-characteristic solution was iterated through $n_z$ heights on a uniform grid $dx = dz$, where $n_z$ was taken to be the depth of the $\tau$ surfaces measured as a function of angle in the previous section. The width of the emergent intensity distribution was determined by a skew-normal fit, and the resulting values are shown as the solid blue curve in Figure 2(b). Numerical diffusion is significantly reduced (by a factor of up to about three) by employing the higher-order interpolation scheme, but because of the monotonicity constraint needed to avoid negative specific intensity values, energy is not conserved. The integrated intensity at the top of the domain (normalized by the source value) as a function of beam angle is plotted for both linear and cubic monotonic interpolation in Figure 5. While the total intensity is conserved by linear interpolation for all inclination angles (black triangles), cubic monotonic interpolation (blue circles) conserves intensity only at $45^\circ$, $90^\circ$, and near $62^\circ$. For all other beam inclinations, the total emergent intensity is overestimated.
Figure 4. Average (over image rows) intensity amplitude spectrum for $\mu = 0.49$ after employing the standard (solid curve) and pre-tilted (dotted curve) short-characteristic methods. Dotted-dashed curve shows the averaged spectra of the pre-tilted image rows convolved with a binomial convolution kernel with a width that yields the best-fit to the standard spectrum. Spectra are normalized to share the same integrated amplitude.

Figure 5. Normalized integrated intensity for linear (black triangles) and cubic monotonic (blue circles) interpolation. Note how the integrated intensity is conserved using linear interpolation, but not using cubic monotonic interpolation.

5. Discussion and Conclusion

We have quantified the effect of interpolation errors inherent in the short-characteristic radiative transfer method, deriving an analytical model of the diffusion of a point source through a grid at arbitrary viewing angle and comparing that with the spatial resolution degradation of the emergent intensity calculated using a 3D MHD snapshot. We have shown that, due to the compounding nature of the error, intensity interpolation accounts for most, if not all, of the reduction in resolution introduced by the short-characteristic method. Interpolation of the plasma properties along the ray path, common to both short- and long-characteristic methods, likely contributes negligibly to resolution reduction. While higher-order interpolation schemes reduce the diffusion error in the short-characteristic method, they do not conserve energy. We have demonstrated that some problems can be circumvented by pre-tilting the computational domain to a ray-aligned grid and performing the short-characteristic radiative transfer vertically through this domain. This requires only that the interpolation of the atmospheric properties along the ray directions occurs prior to the ray propagation, rather than after it.

Without such pre-tilting, the short-characteristic diffusion error is now known analytically for linear interpolation, the emergent intensity could simply be deconvolved to obtain the fully resolved image. This is true in the case of local thermodynamic equilibrium where the specific intensity along a ray depends only on the local plasma properties, but is likely much more difficult for nonlocal thermodynamic equilibrium solutions for which the radiation field is iteratively solved and the error may propagate nonlinearly through the domain. Employing higher-order interpolation schemes yields an improvement (to 28 km resolution for the cubic monotonic interpolation in the example above) at the expense of specific intensity nonconservation. This nonconservation of energy does not affect the emergent intensity image contrast at any given viewing angle, but does affect the relative intensity at differing angles, again likely posing difficulties for nonlocal thermodynamic equilibrium solutions.

Similarly, since some radiative MHD solvers use short-characteristic radiative transfer in combination with discrete angle-weighted quadrature schemes to evaluate the divergence of the radiative flux in the solution of the energy equation, angle dependent specific intensity errors may introduce computational artifacts in the radiative MHD solutions themselves. The effects of numerical diffusion on the moments of the radiative transfer equation were discussed in Bruls et al. (1999) for the case of specific triangular grids. Similarly, angle dependent artifacts have been noted in solutions for the photon density in cosmological solutions for optically thin radiative transfer (Finlator et al. 2009).

A detailed analysis of how short-characteristic error propagates through the quadrature schemes employed by MHD solvers in the optically thick to optically thin transition of the solar photosphere is warranted, and the subject of future work.

Finally, we reiterate that the diffusive errors introduced by the short-characteristic method can be avoided by interpolating the atmosphere onto a ray-aligned grid before computing the radiative transfer. Since the total number of interpolations incurred by pre-tilting is less than that for the standard short-characteristic method (only the plasma properties not the specific intensity values must be interpolated if pre-tilting is employed), the pre-tilting method may save computational time in the solution of the radiative transfer equation. However, if the specific intensity is needed on the original grid, for quadrature calculation of the flux divergence, for example, interpolation of the specific intensity back to the original grid is also required, making the total interpolation cost the same, though careful comparison between methods on multiprocessor hardware must be made. The clear advantage of pre-tilting compared to the standard short-characteristic method is that the interpolation of the specific intensity back to the original grid does not compound the error. Other solutions to the numerical diffusion problem have been explored. Long-characteristic solvers...
are routinely employed (e.g., Bifrost Gudiksen et al. 2011; COSBOLD, Freytag et al. 2012; Stagger, Magic et al. 2013; StellarBox, Wray et al. 2015), and these fundamentally avoid the intensity diffusion error. Hybrid radiative transfer schemes using adaptive mesh refinement have also been developed (Rijkhorst et al. 2006) and help mitigate the error. Which approach proves most accurate and computationally efficient is still an open question, but pre-tilting is simple, highly effective, and can perhaps be seamlessly integrated into existing radiative MHD solvers.

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Appendix

The analytical form of the diffusion error intrinsic to the short-characteristic method is derived here assuming a single delta-function point source of intensity at the bottom of the 3D domain. The ray propagates through vacuum so there is no need to interpolate the plasma properties. To maintain analytical tractability, we assume linear interpolation and a regular spatial grid with sampling \( dx \) by \( dy \) and \( dz \) in the horizontal and vertical directions respectively. In the short-characteristic method, rays are propagated differently depending on whether the upwind intensity is interpolated in a vertical or horizontal plane. We present the two cases separately. For the ray orientations shown in Figure 6, the intensity for both cases is updated from left to right and then from front to back. The solution is applicable to other ray orientations as the indices used represent the number of grid points away from the \( I_{\text{source}} \) location (e.g., \( I_{000} \) represents the source location and \( I_{n_n_n} \) represents the point \( n_x, n_y, \) and \( n_z \) grid steps from \( I_{\text{source}} \) in the \( x, y, \) and \( z \) directions). The domain is taken to have sufficient horizontal extent such that the fully dispersed beam intersects the upper domain boundary before exiting the sides. If this is not the case, horizontal periodicity must be imposed on the solution.

### A.1. Solution for Ray Directions for which Interpolation Occurs on Horizontal Planes

The ray propagation for this subset of angles is shown on the left of Figure 6. In 3D geometry, the boundary condition \( I_{\text{source}} \) on the upwind \( xy \)-plane results in four intensity values at the downwind grid points that are linearly interpolated as

\[
I_{001} = I_{\text{source}} \left[ 1 - \frac{dz \cos \phi}{dx \tan \theta} \right] \left[ 1 - \frac{dz \sin \phi}{dy \tan \theta} \right],
\]

\[
I_{011} = I_{\text{source}} \left[ \frac{dz \cos \phi}{dx \tan \theta} \right] \left[ \frac{dz \sin \phi}{dy \tan \theta} \right],
\]

\[
I_{111} = I_{\text{source}} \left[ \frac{dz \cos \phi}{dx \tan \theta} \right] \left[ \frac{dz \sin \phi}{dy \tan \theta} \right].
\]

and

\[
I_{111} = I_{\text{source}} \left[ \frac{dz \cos \phi}{dx \tan \theta} \right] \left[ \frac{dz \sin \phi}{dy \tan \theta} \right].
\]

Repeating the interpolation at each subsequent plane (and thus compounding the interpolation error) yields the emergent intensity for point \( I_{n_n_n} \):

\[
I_{n_n_n} = I_{\text{source}} \times I_{n_x} \times I_{n_y} \times I_{n_z},
\]

with

\[
I_{n_x} = \frac{n_x!}{(n_x)!_!(n_z - n_x)!} \left[ 1 - \frac{dz \cos \phi}{dx \tan \theta} \right]^{n_z - n_x} \left( \frac{dz \cos \phi}{dx \tan \theta} \right)^{n_x},
\]

and

\[
I_{n_z} = \frac{n_z!}{(n_z)!_!(n_y - n_z)!} \left[ 1 - \frac{dz \sin \phi}{dy \tan \theta} \right]^{n_y - n_z} \left( \frac{dz \sin \phi}{dy \tan \theta} \right)^{n_z}.
\]

\[
I_{n_n_n} \text{ and } I_{n_n_n} \text{ are binomial distributions with variances}
\]

\[
\sigma^2 = \sqrt{nz \frac{dz \cos \phi}{dx \tan \theta} \left[ 1 - \frac{dz \cos \phi}{dx \tan \theta} \right]}.
\]
and

\[ \sigma_y = \sqrt{n_z \frac{d\sigma_y}{dy \tan \theta} \left( 1 - \frac{d\sigma_y}{dy \tan \theta} \right)} \] \hspace{1cm} (8)

For interpolation in horizontal planes, the two-dimensional PSF (Equation (7)) has widths (Equation (8)) that are independent functions, which depend on the propagation angle and the number of horizontal planes \( n_z \) through which the beam has propagated.

It is straightforward to show that source intensity is conserved with height. At any given height \( n_z = N_z \) the total intensity is shown to be

\[ I_{\text{total}} = \sum_{n_y=0}^{N_y} \sum_{n_z=0}^{N_z} I_{n_y, n_z, n_z} = I_{\text{source}} \left( 1 - \frac{d\sigma_y}{dy \tan \theta} \right)^{N_y} \left( 1 - \frac{d\sigma_y}{dy \tan \theta} \right)^{N_z} \times \left[ \sum_{n_z=0}^{N_z} \frac{N_z!}{(N_z - n_z)!} \left( \frac{d\sigma_y}{dy \tan \theta} \right)^{n_z} \right] \times \left[ \sum_{n_y=0}^{N_y} \frac{N_y!}{(N_y - n_y)!} \left( \frac{d\sigma_y}{dy \tan \theta} \right)^{n_y} \right] = I_{\text{source}}, \] \hspace{1cm} (9)

where we have used the binomial series expansion \( (1 + x)^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^k \) with \( \alpha = N_z \).

The PSFs for three viewing angles are shown in Figure 7 after being scaled to account for apparent foreshortening (x and y in Figure 7 have been defined so that the images display the intensity as it would be observed from an inclined angle, not the solutions to Equation (7) on the xy grid plane).

A.2. Solution for Ray Directions for which Interpolation Occurs on Vertical Planes

Using the same method as in the previous subsection, we derive the beam diffusion for when the incident angle is such that the interpolation occurs in xz or yz planes. Here we explicitly present the solution for interpolation in yz planes, as illustrated by the right-hand panel of Figure 6. As indicated below, the solution can be readily generalized for the alternative case (interpolation in xz planes).

In most short-characteristic schemes, the first grid point value is solved employing a long-characteristic ray, and we assume the that is true here for the first upwind grid point to avoid complication and without loss of exactness. Unlike for interpolation on horizontal planes, for which only four grid points on the first horizontal plane downwind of \( I_{\text{source}} \) are illuminated, interpolation on vertical planes populates the entire horizontal plane downwind of the source. This is because each subsequent downwind interpolation, as one sweeps left to right, relies on the previous grid point solution. Thus sweeping from left to right each successive row front to back (in the direction of ray propagation) yields the following.

\begin{align*}
I_{010} &= I_{\text{source}} \left( 1 - \frac{d\theta}{d\tan \theta} \right) \left( \frac{d\sigma_y}{d\cos \phi} \right) \left( \frac{d\sigma_x}{d\cos \phi} \right), \\
I_{011} &= I_{\text{source}} \left( 1 - \frac{d\theta}{d\tan \theta} \right) \left( \frac{d\sigma_y}{d\cos \phi} \right) \left( \frac{d\sigma_x}{d\cos \phi} \right), \\
I_{110} &= I_{\text{source}} \left( 1 - \frac{d\theta}{d\tan \theta} \right)^2 \left( \frac{d\sigma_y}{d\cos \phi} \right)^2 \left( \frac{d\sigma_x}{d\cos \phi} \right), \\
I_{111} &= I_{\text{source}} \left( 1 - \frac{d\theta}{d\tan \theta} \right)^3 \left( \frac{d\sigma_y}{d\cos \phi} \right)^3 \left( \frac{d\sigma_x}{d\cos \phi} \right), \\
I_{000} &= \ldots, \\
I_{001} &= 0, \\
I_{012} &= I_{\text{source}} \left( \frac{d\sigma_x}{d\cos \phi} \right) \left( \frac{d\sigma_y}{d\cos \phi} \right), \\
I_{121} &= 2I_{\text{source}} \left( 1 - \frac{d\theta}{d\tan \theta} \right) \left( \frac{d\sigma_y}{d\cos \phi} \right) \left( \frac{d\sigma_x}{d\cos \phi} \right), \\
I_{311} &= \ldots.
\end{align*}
\hspace{1cm} (10)
Continuing this procedure and recognizing the negative binomial distribution with \( n_x > n_y \) yields

\[
I_{n_x n_y n_z} = I_{\text{source}} \times I_{n_x n_z} \times I_{n_y n_z},
\]

with

\[
I_{n_x n_z} = \frac{n_x - 1}{(n_x - n_z)!} \left(1 - \frac{dx \tan \theta}{dz \cos \phi}\right)^{n_x - n_z} \left(\frac{dx \tan \theta}{dz \cos \phi}\right)^{n_z},
\]

and

\[
I_{n_y n_z} = \frac{n_y!}{n_y!(n_x - n_y)!} \left(1 - \frac{dy \sin \phi}{dz \cos \phi}\right)^{n_y - n_z} \left(\frac{dy \sin \phi}{dz \cos \phi}\right)^{n_z}. \tag{11}
\]

Note that \( I_{n_x n_y n_z} \) is a negative binomial distribution, while the \( I_{n_z n_y} \) is a regular binomial distribution. \( I_{n_x n_z} \), as previously for interpolation on horizontal planes, describes diffusion in \( n_z \) and depends on the number of \( n_z \) levels through which the ray has passed. \( I_{n_y n_z} \), on the other hand, depends on \( n_x \) and \( n_z \) as diffusion in the two horizontal directions are no longer independent. The PSFs derived from Equation (11) are shown in Figure 8 for three inclination angles. They have been scaled to account for apparent foreshortening to show appearance of the PSF at inclined viewing angles. Note that the distributions are no longer aligned with the \( x \) and \( y \) axes. This is a consequence of the mixing introduced by the \( I_{n_x n_z} \) term. Since the spread in \( I_{n_x n_z} \) depends on \( n_x \), the distribution in Figure 8(c) is asymmetric, with larger broadening for larger \( n_x \) values. The variances associated with these distributions are

\[
\sigma_x = \sqrt{n_x \frac{dz \cos \phi}{dx \tan \theta} \left(\frac{dz \cos \phi}{dx \tan \theta} - 1\right)},
\]

and

\[
\sigma_y = \sqrt{n_y \frac{dy \sin \phi}{dy \cos \phi} \left(1 - \frac{dy \sin \phi}{dz \cos \phi}\right)}, \tag{12}
\]

with the variance in the \( y \) direction reflecting the asymmetry.

The derivation above is valid for ray angles for which the interpolation occurs in the \( xz \) plane. For angles by which the interpolation occurs in the \( yz \) plane, the solution is given by Equations (11) and (12) with \( x \) and \( y \) and \( \sin \phi \) and \( \cos \phi \) interchanged.

As in Appendix A.1, the specific intensity is conserved. Explicitly, as before,

\[
I_{\text{total}} = \sum_{n_x=N_z}^{\infty} \sum_{n_y=0}^{n_x} I_{n_x n_y n_z}
\]

\[
= I_{\text{source}} \left(\frac{dx \tan \theta}{dz \cos \phi}\right)^{N_z} \left[\sum_{n_y=0}^{n_x} \frac{n_y!}{n_y!(n_x - n_y)!} \left(1 - \frac{dx \sin \phi}{dz \cos \phi}\right)^{n_y - n_z} \left(\frac{dx \sin \phi}{dz \cos \phi}\right)^{n_z}\right]
\]

\[
= I_{\text{source}} \left(\frac{dx \tan \theta}{dz \cos \phi}\right)^{N_z} \left[\sum_{n_y=0}^{n_x} \frac{n_y!}{n_y!(n_x - n_y)!} \left(1 - \frac{dx \sin \phi}{dz \cos \phi}\right)^{n_y - n_z} \left(\frac{dx \sin \phi}{dz \cos \phi}\right)^{n_z}\right]
\]

\[
= I_{\text{source}} \left(\frac{dx \tan \theta}{dz \cos \phi}\right)^{N_z} \left[\sum_{n_y=0}^{n_x} \frac{n_y!}{n_y!(n_x - n_y)!} \left(1 - \frac{dx \sin \phi}{dz \cos \phi}\right)^{n_y - n_z} \left(\frac{dx \sin \phi}{dz \cos \phi}\right)^{n_z}\right]
\]

\[
= I_{\text{source}}, \tag{13}
\]

where we have used the binomial and negative binomial series expansions \((1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\) with \( \alpha = n_y \) and \(1/(1 - x)^\beta = \sum_{n=0}^{\infty} \binom{n+\beta}{n} x^n\) with \( \beta = N_z - 1 \) and \( n = n_x - N_z \).

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**References**

Amarsi, A. M., Asplund, M., Collet, R., & Leenaarts, J. 2016, MNRAS, 455, 3735

Auer, L. H., & Paletou, F. 1994, A&A, 285, 675
Bruls, J. H. M. J., Vollmöller, P., & Schüssler, M. 1999, A&A, 348, 233
Carlson, B. 1965, Discrete Ordinates Angular Quadrature of the Neutron Transport Equation, Tech. Rep. LA-3186, Los Alamos National Laboratory, Los Alamos, NM
Carlsson, M. 2008, PhST, T133, 014012
Collados, M., Bettonvil, F., Cavaller, L., et al. 2013, MmSAI, 84, 379
Criscioli, S. 2007, PhD thesis, Univ. Rome Tor Vergata doi: 10.5281/zenodo.845496
Criscioli, S., & Rast, M. P. 2009, A&A, 495, 621
Davis, S. W., Stone, J. M., & Jiang, Y.-F. 2012, ApJS, 199, 9
Elmore, D. F., Rimmelte, T., Casini, R., et al. 2014, Proc. SPIE, 9147, 914707
Finlator, K., Özel, F., & Davé, R. 2009, MNRAS, 393, 1090
Freytag, B., Steffen, M., & Dorch, B. 2002, JCoPh, 231, 919
Fritsch, F., & Carlson, R. 1980, SJNA, 17, 238
Galsgaard, K., & Nordlund, k. 1996, JGR, 101, 13445
Gudiksen, B. V., Carlsson, M., Hansteen, V. H., et al. 2011, A&A, 531, A154
Hayek, W., Asplund, M., Carlsson, M., et al. 2010, A&A, 517, A49
Hayek, W., Asplund, M., Collet, R., & Nordlund, A. 2011, A&A, 529, A158
Ibgui, L., Hubeny, I., Lanz, T., & Stehlé, C. 2013, A&A, 549, A126
Koesterke, L., Allende Prieto, C., & Lambert, D. L. 2008, ApJ, 680, 764
Kunasz, P., & Auer, L. H. 1988, JQSRT, 39, 67
Lathrop, K., & Carlson, B. 1965, Discrete Ordinates Angular Quadrature of the Neutron Transport Equation, Tech. Rep. LA-3186, Los Alamos National Laboratory, Los Alamos, NM
Leenaarts, J., & Carlsson, M. 2009, in ASP Conf. Ser. 415, The Second Hinode Science Meeting: Beyond Discovery-Toward Understanding, ed. B. Lites et al. (San Francisco, CA: ASP), 87
Magic, Z., Collet, R., Asplund, M., et al. 2013, A&A, 557, A26
Matthews, S. A., Collados, M., Mathioudakis, M., & Erdelyi, R. 2016, Proc. SPIE, 9908, 990809
Mihalas, D., Auer, L. H., & Mihalas, B. R. 1978, ApJ, 220, 1001
Mihalas, D., & Mihalas, B. W. 1984, Foundations of Radiation Hydrodynamics (New York: Oxford Univ. Press)
Pereira, T. M. D., & Uitenbroek, H. 2015, A&A, 574, A3
Rempel, M. 2014, ApJ, 789, 132
Rijkhorst, E.-J., Plewa, T., Dubey, A., & Mellema, G. 2006, A&A, 452, 907
Štěpán, J., & Trujillo Bueno, J. 2013, A&A, 557, A143
Tritschler, A., Rimmelte, T. R., Berukoff, S., et al. 2016, AN, 337, 1064
Uitenbroek, H. 2001, ApJ, 557, 389
Vögler, A., Shelyag, S., Schüssler, M., et al. 2005, A&A, 429, 335
Wray, A. A., Bensassi, K., Kitashvili, I. N., Mansour, N. N., & Kosovichev, A. G. 2015, arXiv:1507.07999
Zhu, Y., Narayan, R., Sadowski, A., & Psaltis, D. 2015, MNRAS, 451, 1661