Simulation of static and random errors on Grover’s search algorithm implemented in an Ising nuclear spin chain quantum computer with a few qubits

T Gorin, L Lara and G V López

Departamento de Física, Universidad de Guadalajara, Blvd. Marcelino García Barragan y Calzada Olímpica, 44840 Guadalajara, Jalisco, México
E-mail: thomas.gorin@red.cucei.udg.mx

Received 27 November 2009, in final form 10 March 2010
Published 5 April 2010
Online at stacks.iop.org/JPhysB/43/085508

Abstract

We consider Grover’s search algorithm on a model quantum computer implemented on a chain of four or five nuclear spins with first- and second-neighbour Ising interactions. Noise is introduced into the system in terms of random fluctuations of the external fields. By averaging over many repetitions of the algorithm, the output state becomes effectively a mixed state. We study its overlap with the nominal output state of the algorithm, which is called the fidelity. We analyse the behaviour of the fidelity as a function of the noise intensity for static and random noise on either the Larmor frequencies or the Rabi frequency, and we compare our results to theoretical predictions and numerical simulations which are based on more abstract quantum computer models.

1. Introduction

While in the early days of quantum computer studies the focus was on the development of algorithms as such, more recently the effects of errors and imperfections have received considerable attention [1–6]. Generically, one may distinguish different types of errors: unitary errors, noise, decoherence. Moreover, errors may be distinguished by their correlation time. This is the time scale on which the perturbation or noise fluctuates. Theoretical investigations aim at understanding and predicting the effects of these errors without having to simulate the whole quantum dynamical system exactly. Eventually this is taken to the point where the effects are considered to be independent of the actual dynamics of the quantum system. The approach usually taken uses the fact that any algorithm may be decomposed into a sequence of only a few elementary gates. Errors are then introduced on the level of these elementary gates and one studies their effects by simulating the algorithm with these modified gates.

One may then wonder what these results calculated with abstract gates can actually tell about the same algorithm if it is realized on a real quantum computer. It is unlikely that an experiment can be done in the near future to answer this question. However, we may try the next best thing. We can study the implementation of the algorithm on a microscopic model quantum computer with an idealized but still physically reasonable dynamics, and simulate that system on a normal classical computer. In contrast to the studies mentioned before, there are then physical parameters which have to be controlled in order that a certain sequence of quantum gates is performed. Furthermore, errors during the algorithm can then be related to the systems’ real imperfections, which are given from the outset and need not be introduced artificially. In this study, we choose a model for a solid state quantum computer which is made up of a chain of four or five nuclear paramagnetic spins with first- and second-nearest neighbour interactions of Ising type [7]. This model system uses radio-frequency pulses (RF-pulses) to implement quantum gates [8, 9], and the $2\pi k$-method [10] to control the non-resonant transitions in the system. We study an implementation of Grover’s quantum search algorithm, for which the effects of errors and noise have been studied extensively [1–3, 5, 11]. It is well known that
Grover’s search algorithm for searching an item in a quantum database of size $N = 2^n$, where $n$ is the number of qubits in the quantum register, provides a quadratic speed-up with respect to any classical search algorithm [12, 13]. Some applications of this algorithm can be found in [14–18], while physical implementations for two, three and four qubits have been realized or proposed in [19–23].

In a previous paper [7], we considered unitary errors, which arise from non-resonant transitions. In the present work, we study the effects of noise arising from fluctuations in the external fields; variations in the nominally constant magnetic field which result in variations of the Larmor frequencies of the nuclear spins; and variations in the amplitude of the RF-pulses, which affect the Rabi frequency. In section 2, we introduce the physical model quantum computer. In section 3, we explain how the quantum search algorithm is implemented with the help of certain radio-frequency pulses (RF-pulses). Section 4 describes the implementation of noise, by introducing small fluctuations in the Rabi frequency (Rabi-frequency noise), or the Larmor frequencies (Larmor-frequency noise). The effect of noise on the quantum computation is quantified using the fidelity, first introduced in [24] in an attempt to help of certain radio-frequency pulses (RF-pulses). Section 4 studies the effects of noise arising from fluctuations in the Rabi frequency (Rabi-frequency noise), which arise from non-resonant transitions. In the present work, the quantum system undergoes a unitary time evolution which is denoted by $R^{\text{RF}}(\varphi, \theta)$. It describes a rotation of the occupation amplitudes of the involved states by the angle $\theta = \Omega \tau$, controlled by the duration $\tau$ of the pulse.

**Degeneracies in $H_0$.** At a first glance, one would expect accidental degeneracies in the Hamiltonian $H_0$ to cause problems in the realization of any quantum algorithm, since a given microwave field could drive unwanted transitions involving degenerate states. Such degeneracies can be avoided by requiring the Larmor frequencies $\omega_k$ to increase in powers of 2: $\omega_k \propto 2^k$. Without interactions $J = J' = 0$, this produces an equidistant spectrum; for $J \ll J' \ll \omega_0$ this guarantees a spectrum without any degeneracies. Such a choice of the Larmor frequencies would however destroy the scalability of the present system as a model for quantum computation.

Fortunately, degeneracies are typically not harmful. This is due to the fact that the interaction operator couples states $|\alpha\rangle$ and $|\beta\rangle$ only if they differ by no more than one qubit. In the interaction picture, $W(t)$ becomes

$$ W_{\alpha\beta}(t) = \frac{-i\Omega}{2} \begin{cases} e^{i(\Delta t + \varphi)} : \beta - \alpha = (0, \ldots, 1, \ldots, 0) \\ e^{-i(\Delta t + \varphi)} : \beta - \alpha = (0, \ldots, 1, \ldots, 0) \\ 0 : \text{else} \end{cases}, $$

where the notation $\alpha - \beta = (0, \ldots, 1, \ldots, 0)$ means that the multi-indices $\alpha, \beta$ are equal except at one position (qubit $k$), where the qubit in the $\alpha$-state is excited and the qubit in the $\beta$-state is not. For an accidental degeneracy caused by the unfortunate choice of the Larmor frequencies, there must exist transitions such that $w_k + \mu J + v J' = w_{k'} + \mu' J + v' J'$ for some $k \neq l$. But this can be safely avoided by choosing the spacing between neighbouring Larmor frequencies much larger than the coupling constants: $|w_k - w_{k-1}| \ll J \ll J'$. Since this allows for a linear increase in the Larmor frequency as a function of the number of qubits, the scalability of the quantum computer model is not affected.

**Near-resonant approximation.** Assume that the frequency $w$ of the RF-pulse matches a particular transition so well that all other transitions can be neglected. For a rectangular RF-pulse as described above, with detuning $\Delta = w - \omega_k - \mu J - v J'$, the evolution operator in the basis of the two states involved reads

$$ U_{\text{pulse}} \equiv \begin{pmatrix} e^{i\Delta t/2} & 0 \\ 0 & e^{-ir\Delta/2} \end{pmatrix} \times \begin{pmatrix} \cos \frac{\Omega t}{2} - i\frac{\omega}{\Omega} \sin \frac{\Omega t}{2} & \frac{\omega}{\Omega} e^{i\varphi} \sin \frac{\Omega t}{2} \\ \frac{\omega}{\Omega} e^{-i\varphi} \sin \frac{\Omega t}{2} & \cos \frac{\Omega t}{2} + i\frac{\omega}{\Omega} \sin \frac{\Omega t}{2} \end{pmatrix}. $$

---

1 Note: in the standard spin algebra, we would expect $I^+ = I^x + i I^y = |1\rangle\langle 0|$ to be the raising operator.
where \( \Omega_c = \sqrt{\Delta^2 + \Delta^2} \). For \( \Delta = 0 \), we obtain the resonant transitions:

\[
R(\varphi, \theta) = \begin{pmatrix}
\cos \theta/2 & i e^{i\varphi} \sin \theta/2 \\
i e^{-i\varphi} \sin \theta/2 & \cos \theta/2
\end{pmatrix}, \quad \theta = \Omega \tau.
\]  
(8)

### 3. Implementation of Grover’s search algorithm

For our case, Grover’ search algorithm requires three qubits to prepare the ‘inquiry’ states and a single qubit (‘ancilla’) for the oracle to communicate its answer [28]. Starting from the ground state, \( |000\rangle \), a superposition state with the qubits \( k = 0, 2, 3 \) is generated by applying the corresponding Hadamard gates

\[
H(3) = H_3 H_2 H_0, \tag{9}
\]

where \( H_i \) is the Hadamard gate acting on the \( k \)th qubit. Then, one applies the Grover operator

\[
G = O_\alpha H(3) S_0 H(3) \tag{10}
\]
twice, since for \( N = 8 = 2^3 \) registers, the probability of finding the searched state is maximum for \( [\pi \sqrt{8}/4 - 1/2] \approx 2 \) applications [29]. The operator \( S_0 \) represents the phase inversion of the searched state, and the operator \( O_\alpha \) represents the oracle where \( \alpha \) is the index of the target state in decimal notation. The implementation of these operators by appropriate RF-pulses is discussed in [7]:

\[
H_0 = \prod_{\mu, \nu = -1, 1} R_{0\mu0\nu}^{0\mu} \left( \frac{\pi}{2}, \pi \right) \prod_{\mu, \nu = -1, 1} R_{0\mu0\nu}^{0\mu} \left( \frac{\pi}{2}, \pi \right)
\times \prod_{\mu, \nu = -1, 1} R_{0\mu0\nu}^{0\mu} \left( \pi, \pi \right), \tag{11}
\]

\[
H_2 = \prod_{\mu, \nu = -2, 0, 2} R_{2\mu2\nu}^{2\mu} \left( \pi, \pi \right) \prod_{\mu, \nu = -2, 0, 2} R_{2\mu2\nu}^{2\mu} \left( \pi, \pi \right)
\times \prod_{\mu, \nu = -1, 1} R_{2\mu2\nu}^{2\mu} \left( \pi, \pi \right), \tag{12}
\]

\[
H_3 = \prod_{\mu, \nu = -1, 1} R_{1\mu1\nu}^{1\mu} \left( \pi, \pi \right) \prod_{\mu, \nu = -1, 1} R_{1\mu1\nu}^{1\mu} \left( \pi, \pi \right)
\times \prod_{\mu, \nu = -1, 1} R_{1\mu1\nu}^{1\mu} \left( \pi, \pi \right), \tag{13}
\]

\[
S_0 = R_{2\mu2\mu}^{\mu\mu}(-1, 0, 2\pi) R_{1\mu1\mu}^{\mu\mu}(-1, 0, 2\pi) R_{0\mu0\mu}^{\mu\mu}(0, 2\pi)
\times R_{1\mu1\mu}^{\mu\mu}(-1, 0, 2\pi) R_{2\mu2\mu}^{\mu\mu}(0, 2\pi), \tag{14}
\]

\[
O_0 = R_{2\mu2\mu}^{\mu\mu}(0, 2\pi), \quad O_\alpha = R_{1\mu1\mu}^{\mu\mu}(-1, 0, 2\pi), \tag{15}
\]

\[
O_3 = R_{1\mu1\mu}^{\mu\mu}(0, 2\pi), \quad O_{13} = R_{1\mu1\mu}^{\mu\mu}(-1, 0, 2\pi). \tag{16}
\]

We need 70 pulses to realize the operator \( H(3) \), and 146 pulses to realize the Grover operator, which yields a total of 362 pulses to implement the search algorithm. The duration of the RF-pulses depends on the rotation angle \( \theta \) and on the Rabi frequency: \( \tau_{\text{pulse}} = \theta / \Omega \). Below, we will measure time in units of \( \tau_{\text{ph}} = \pi / (2\Omega) \), the duration of a \( \pi/2 \)-pulse.

### 4. Implementation of noise

The Larmor frequencies \( \omega_k \) for \( k = 0, 1, 2, 3 \) depend on a static magnetic field in the z-direction with a strong field gradient: \( \omega_k = \gamma B_k \), where \( B_k \) is the strength of the static magnetic field at the location of the \( k \)th paramagnetic nucleus of spin one-half. The Rabi frequency \( \Omega \) is determined by the amplitude of the radio-frequency (RF) pulses which is assumed to be constant during a pulse: \( \Omega = \gamma B_1 \), where \( \gamma \) is the gyromagnetic ratio and \( B_1 \) is the amplitude of the RF-pulse. The environment or imperfections in the static magnetic field and/or the RF-pulses produce noise which, in turn, leads to variations in the above parameters. To study the effect of these variations, we consider random detunings of the form

\[
\omega_k = \omega_{k0} + \epsilon_L \xi, \tag{17}
\]

and

\[
\Omega = \Omega_0 + \epsilon_R \xi, \tag{18}
\]

where \( \omega_{k0} \) for \( k = 0, 1, 2, 3 \) and \( \Omega_0 \) are the Larmor and Rabi frequencies without noise, respectively, and \( \xi \) is a random Gaussian variable, centered at zero, with unit variance. The parameters \( \epsilon_L \) and \( \epsilon_R \) determine the amplitude of the noise.

The two types of noise produce different types of errors as can be seen from the evolution operators in the resonant approximation, equations (7) and (8). In the case of noise on the Rabi frequency, the evolution operator remains of the form (8); only the angle \( \theta \) suffers a change:

\[
\theta \to \theta' = \theta + \Delta_R, \quad \Delta_R = \epsilon_R \xi \tau, \tag{19}
\]

where \( \tau \) is the duration of the RF-pulse. In the case of noise on the Larmor frequencies, the evolution operator gets distorted in a more complicated way: with \( \theta = \Omega \tau \) and \( \Delta_L = \epsilon_L \xi \tau \) we find

\[
R(\varphi, \theta) \to R'(\varphi, \theta) = \begin{pmatrix} e^{i\Delta_L/2} & 0 \\ 0 & e^{-i\Delta_L/2} \end{pmatrix}
\times \begin{pmatrix} \cos \theta/2 - \frac{\Delta_L}{2 \omega} \sin \theta/2 & i e^{i\varphi} \sin \theta/2 \\ i e^{-i\varphi} \sin \theta/2 \cos \theta/2 + \frac{\Delta_L}{2 \omega} \sin \theta/2 \end{pmatrix}, \tag{20}
\]

up to second order in \( \Delta_L = \epsilon_L \xi \tau \). In both cases, the amplitude of the noise must be compared to the smallest energy (frequency) scale in the system, which is \( J \) the coupling between second nearest neighbours.

Besides the amplitude also the frequencies of the noise fluctuations are important. It makes a difference whether the variation of the Larmor frequencies (Rabi frequency) occurs on a time scale which is of the order of the duration of the whole algorithm (low-frequency noise) or of the duration of a single RF-pulse. To investigate this issue, we consider two types of noise (errors): static noise, where the respective frequencies are detuned for each realization, but kept fixed during the whole algorithm; random noise, where the respective frequencies are detuned anew for each RF-pulse, the latter simulating noise which fluctuates on a time scale of the order of the inverse Rabi frequency.
Fidelity. In order to measure the effect of noise on the realization of Grover’s quantum search algorithm, we use the fidelity (or quantum Loschmidt echo) [24–26]. We perform averages over n_{rep} repetitions (where n_{rep} is of the order of 100) of the algorithm. This effectively turns the final state of the quantum computer into a mixed state ϱ_{real}. The fidelity is computed by projecting ϱ_{real} on the outcome of the ideal realization of the algorithm:

\[ F_{\text{end}} = \langle \Psi_{\text{ideal}} | \rho_{\text{real}} | \Psi_{\text{ideal}} \rangle, \]
\[ \rho_{\text{real}} = \frac{1}{n} \sum_{r=1}^{n} | \Psi_{\text{real}}^{(r)} \rangle \langle \Psi_{\text{real}}^{(r)} |, \]  

where | \Psi_{\text{real}}^{(r)} \rangle are the final states obtained from the algorithm taking into account the random detuning of the respective frequencies.

When analysing the final fidelity, there are two options: we can compare the final state after the evolution in the presence of noise and unwanted transitions with the final state obtained from an error-free execution of the quantum protocol, or we can compare the final state with the desired target state. These two options are not the same because the Grover protocol arrives at the desired target state only with a final probability, which is close to but still different from 1 [29]. Both options yield very similar results for the behaviour of fidelity. For the results presented in this paper, we chose the first option.

5. Numerical method and results

The evolution of the quantum state in the qubit register is calculated by solving the time-dependent Schrödinger equation numerically in the interaction picture. In view of the rapidly oscillating interaction matrix, any sophisticated adaptive step size routine is of limited use. We therefore apply a simple fourth-order Runge–Kutta algorithm to solve the ordinary first-order differential equation. The step size Δt was fixed according to the largest Larmor frequency w_{n} which is the largest frequency in the problem: Δt = π/(2w_{n}).

The computation time depends mainly on the time needed to evaluate the right-hand side of the differential equation. For each of the N = 2^n complex amplitudes one has to calculate a sum over the n non-zero elements of the interaction matrix. Therefore, the time needed to advance the solution by one time step scales as nN = 2^{n+1} with the number of qubits. Finally, for a larger database with more qubits the Grover algorithm requires a longer pulse sequence. Roughly, the length of the pulse sequence scales with √{N} as explained in [28]. The total computation time therefore scales as 2^{3n/2} with the number of qubits.

The simulations presented in this work are performed on standard single processor machines, where a single run of the Grover algorithm with four or five qubits takes approximately 4 min of computing time.

Choice of the model parameters. The following parameters are all frequencies (see equation (2)) and given in units of 2π × MHz. Note however that the model depends only on their relative values, and not on the frequency unit itself.

On the basis of the discussion of accidental degeneracies in section 2, we expect that the Larmor frequencies may be chosen to increase linearly with the number of qubits. Thus, we chose

\[ w_{0} = 50, \quad w_{1} = 200, \quad w_{2} = 350, \quad w_{3} = 500 \]  

for the calculations with four qubits and in addition w_{4} = 650 for the five-qubit calculations. Before settling on these values we compared the errors in different schemes (exponential increase, avoiding any degeneracies in H_{0}, quadratic increase and the present linear increase). We found consistently that the different schemes only lead to appreciable errors if at least one of the differences w_{k} − w_{j} approaches the value of one of the Ising coupling constants J or J’.

Consequently, we choose J = 10 (one order of magnitude smaller than the spacing between the Larmor frequencies) and J’ = 0.4 (one order of magnitude smaller than J). In this way, we ensure that the dominant source for errors is near-resonant transitions with detunings of the order of J’. Finally, we choose the Rabi frequency as Ω_{s} = 0.1008 according to the 2πk-condition, such that near-resonant transitions due to the second-neighbour coupling are efficiently suppressed.

5.1. Larmor frequency noise

To start our study, we analyse single runs without any averaging. We compare calculations with and without noise, as well as exact calculations and calculations which apply the near-resonant approximation. We use a quantum register of 3+1 qubits to simulate Grover’s search algorithm. The qubits are arranged according to the string xx.x, where x represents a data qubit, while . represents the auxiliary qubit which is used to implement the oracle and the conditional reflection. The algorithm consists of an initial sequence of Hadamard gates H^{(1)} followed by two Grover steps.

In the simulation for figure 1 the algorithm searches for the state 0 (string 0000). On the graph, the dark grey bars indicate the implementation of the quantum oracle (narrow bar), and the conditional reflection, respectively. The Grover step starts with the quantum oracle, so the figure shows the preparation of the superposition state and two subsequent Grover steps. The figure compares exact simulations (solid lines) and simulations with the near-resonant approximation (dashed lines) in the presence (green)/absence (blue) of static Larmor frequency noise. Even without noise, the fidelity drops with more or less constant rate from F(0) = 1 to F_{end} ≈ 0.92. This reminds us that the implementation of the quantum gates via RF-pulses as such already introduces errors. In this case exact simulation and the near-resonant approximation agree very well, though the fidelity is generally a bit larger for the latter. The near-resonant approximation neglects the far resonant transitions, where the frequency mismatch is of the order of the Larmor frequencies. The agreement between the exact calculation and the near-resonant approximation is completely lost as soon as...
the noise (here it is static noise) is switched on (green solid and dashed lines). The solid line shows the exact calculation for the noise amplitude $\epsilon_L = 0.02$ which only initially follows the curve for the fidelity decay without noise. More or less in the middle of the algorithm, the fidelity curve drops rather abruptly to a level of $F(t) \approx 0.7$, where it saturates. In contrast to that, the near-resonant approximation stays close to the results without noise. Apparently, the Larmor frequency noise introduces additional errors primarily via far resonant and/or indirect transitions, which are neglected in the near-resonant approximation.

Figure 2 shows the average fidelity as described by equation (21) for static and random noise on the Larmor frequencies. The simulations are averaged over an ensemble of $n_{\text{rep}}$ independent repetitions of the algorithm. In the case of the exact numerical simulations, $n_{\text{rep}} = 50$ (static noise) and 25 (random noise), respectively. In the case of the near-resonant approximations, $n_{\text{rep}} = 1000$ in both cases. This figure should illustrate several points: first, there are large variations in the behaviour of the fidelity among individual realizations of the algorithm. This can be deduced by comparison with figure 1 which shows an individual realization where the fidelity curve behaves very differently. Second, the simulations reveal that on average, the fidelity decays particularly quickly during the phase inversion gates. Even the two oracles which require only a small fraction of the whole processing time, make a considerable contribution to the overall fidelity decay. Third, we find that the fidelity decay is stronger in the case of random noise. If we compare the fidelity curve for random (red solid line) and static (green solid line) noise, we find that it is during the Hadamard gates where the random noise wins over the static noise. This unusual behaviour is further discussed below, where we consider the fidelity of the whole quantum evolution as a function of the noise amplitude. Finally, figure 2 again demonstrates the failure of the near-resonant approximation in the presence of static or random Larmor frequency noise.

**Static versus random Larmor frequency noise.** Here, we analyse the average fidelity for the whole algorithm as a function of the noise amplitude. Averages are now taken over ensembles which contain typically about $n_{\text{rep}} = 100$ repetitions. To estimate the statistical error on these averages, we used a numerical procedure\(^2\) which divides each set of $n_{\text{rep}}$ data values into $p = 10$ subsets of equal size. It then estimates the variance of the $n_{\text{rep}} = 100$ average on the basis of the calculated variance of the $p$ separate averages of each subset. It does so assuming that the normalized variance is inversely proportional to the number of data points.

Figure 3 shows the average fidelity $F_{\text{end}}$ of the whole algorithm as a function of $\epsilon_L$, the amplitude of the Larmor frequency noise. We compare the behaviour of the fidelity for static noise (panel (a)) and random noise (panel (b)). In both cases, we use different target states ($0, 4$ and 27) and change the size of the quantum register (four and five qubits). The error bars show the statistical error which has been estimated as described above. It depends on the number of realizations which have been taken into account when calculating the average fidelity; in both cases, we used values for $n_{\text{rep}}$ between 25 and 100. The solid lines, which serve to guide the eye, show best fits to the data using the model function:

$$f(\epsilon) = f_{\text{bas}} + (1 - f_{\text{bas}}) \exp\left[-(\epsilon/\epsilon_L)^2 - \epsilon/\epsilon_L\right].$$

We choose this particular form because it allows us to obtain a quantitative estimate of the saturation level at large $\epsilon_L$.

\(^2\) The numerical procedure has been developed for the analysis of correlated data where the division into smaller subgroups is important. Here, the data are of course uncorrelated and the variance of the $n_{\text{rep}} = 100$ average could have been estimated as well from the variance of the individual data points.
of the Hilbert space.

We find that the variation of the fidelity \( F \) we find for static noise, we find a roughly exponential decay and a slower exponential decay, which is more corrupting than static noise. This will be further discussed in the conclusions.

### 5.2. Rabi-frequency noise

In this section we study the effects of amplitude noise on the Rabi-frequency. As in the previous case, we consider static and random noise. In the case of static noise, the fidelity depends on only one random variable, the detuning \( \epsilon_1 \), and the fidelity decay is closer to an exponential \( \epsilon_1 \approx 0.023 \). For static noise, this number is equal to the number of qubits, while for random noise it has to be multiplied by the number of RF-pulses. Hence, for random noise we have several hundreds of random numbers involved.

Overall, the different target states and the small change in the size of the quantum register have only a minor effect on the behaviour of the fidelity. For static noise, we find a roughly exponential decay and a saturation value for large \( \epsilon_L \), of approximately \( f_{\text{bas}} \approx 0.18 \). At \( \epsilon_L \approx 0.023 \), \( F_{\text{end}} \) has dropped by one-half. By contrast, we find for random noise that the fidelity decay is more similar to a Gaussian, the saturation value is much lower \( (f_{\text{bas}} \approx 0.08) \), and already at \( \epsilon_L \approx 0.013 \), \( F_{\text{end}} \) has dropped by one-half. Overall, we can say that for Larmor frequency noise, random noise is more corrupting than static noise. This will be further discussed in the conclusions.

Figure 3. Exact numerical simulations for the fidelity \( F_{\text{end}} \) for static (a) and random (b) Larmor frequency noise: target state 0 (red points ‘+’), target state 8 (green points ‘x’), calculated with a four-qubit register. Target state 27 (blue points ‘*’), with five qubits. The error bars show the statistical error, estimated as explained in the text. The solid lines of the corresponding colour show best fits with equation (23) to the numerical data.
Hence, to some extent even the resonant approximation could reproduce some of the fidelity loss.

Figure 5 shows the average fidelity for static (red line) with error bars: exact numerical calculation; green line with error bars: near-resonant approximation) and dynamic (blue line with error bars: exact numerical calculation; pink line with error bars: near-resonant approximation) Rabi frequency noise. The error bars represent the estimate of the statistical error as explained earlier. The exact calculations are performed with \( n_{\text{rep}} = 100 \) repetitions, and the near-resonant approximations with \( n_{\text{rep}} = 1000 \). Note that the statistical uncertainty is quite large for static noise whereas it is very small for dynamic noise. We made the same observation in the case of static and random Larmor frequency noise. The explanation we found there probably also applies here.

When comparing exact simulations and the near-resonant approximation, we find relatively good agreement. Here, in figure 5 we included error bars to find out whether there are significant differences between both calculations. In the case of random noise (red points, solid and dashed lines) this is clearly not the case. In the case of static noise, the near-resonant approximation separates from the exact result more and more towards the end of the algorithm. Nevertheless, the deviations remain within the limits of the error also there. In contrast to the case of Larmor frequency noise, the fidelity does not show any particular behaviour during the phase inversion gates (indicated as dark grey vertical bars). Also, here the fidelity decays faster for static noise than for random noise. This is, what one would normally expect since in the case of random noise subsequent detunings may have different signs and thereby some tendency to compensate each other (see also [30]).

**Static versus random Rabi frequency noise.** From the previous results for the time dependence of the fidelity during the Grover algorithm, we saw that the near-resonant approximation works very well in the case of Rabi frequency noise. Since it provides an enormous speed-up of the simulations, we will use it almost exclusively in the present section.

Figure 6 shows the average fidelity \( F(t) \) for the complete algorithm as a function of the amplitude \( \epsilon_R \) of the Rabi frequency noise. In panel (a), we consider static noise, and in panel (b), random noise. We find a striking difference in the functional dependence of the fidelity between the two cases. For random noise (panel (b)) we find that our generic model function from the previous section, equation (23), works quite well. As mentioned, this model function simply interpolates between Gaussian and exponential decay allowing at the same time for a finite saturation value at large noise amplitudes. However, the fidelity decay in panel (a) is rather algebraic, so there is no way for the function of equation (23) to describe the data. As it turned out, a rather simple modification of equation (23) does the job:

\[
    g(\epsilon) = g_{\text{bas}} + \frac{1 - g_{\text{bas}}}{(\epsilon/\epsilon_0)^2 + \epsilon/\epsilon_1 + 1}. \tag{24}
\]

For all cases studied in panel (a) of figure 6 it provides excellent best fits. For \( \epsilon_1 = \infty \) and \( g_{\text{bas}} = 0 \) we would recover the theoretical prediction for the fidelity of a quantum chaotic system in the presence of noise [6] (this point will be discussed in more detail in the conclusions, section 6). The deviations at small noise amplitudes must be expected, since the fidelity is also affected by unitary errors. Due to these errors, the fidelity is smaller than 1, even if no noise is present (see figures 1 and 4).

We may again ask whether static or random noise is more corrupting to the fidelity of the algorithm. Here, the situation is not as clear. While in panel (b), random noise, the three different cases shown are all very close together, this is not so in panel (a), static noise, due to the quite different decay rates.
Still we may say that at small noise amplitudes $F_{\text{end}}$ decays faster for static noise, while at larger noise amplitudes $F_{\text{end}}$ decays faster for random noise. Also, $F_{\text{end}}$ saturates at a much lower level for random noise. Thus, while the situation for weak noise or large fidelity is as expected, the situation gets reversed for larger noise. Note however that the five-qubit case in panel (a) shows the fastest decay at the beginning, and in general we may expect that for larger quantum registers and more Grover steps, the noise amplitudes must be reduced. That would mean that in those cases the weak noise regime became more relevant where the static errors dominate.

### 6. Conclusion

We studied the effect of noise on the performance of Grover’s search algorithm implemented on a nuclear spin quantum computer with four and five spins coupled via first- and second-neighbour Ising interactions. Starting from the ground state, the algorithm attempts to build up the target state in the quantum register, using the information obtained from inquiries of a quantum oracle. We used the fidelity to quantify the effect of noise on the final state, i.e., the result of the algorithm. We considered different types of noise: (i) noise affecting the Larmor frequencies which would be due to fluctuations in the static magnetic field, and (ii) noise affecting the Rabi frequency, which would be due to fluctuations in the intensity of the radio-frequency pulses. We simulate the noise by randomly detuning the respective frequencies from their nominal values. For both cases (i) and (ii) we considered static and random noise. In the case of random noise, the respective frequencies are randomly detuned for each RF-pulse independently. In the case of static noise, the random detuning is kept fixed during the whole algorithm. As in a real quantum experiment, the algorithm is repeated many times averaging the final state over the repetitions. In this way, we obtain a mixed state which is compared to the nominal final state in terms of fidelity.

In what follows we discuss our results. While the fidelity as a function of time shows no simple behaviour, the final fidelity as a function of the noise amplitude is surprisingly regular. It is a smooth and monotonously decaying function. This is demonstrated with appropriate fits to two essentially phenomenological model functions. The first one interpolates between a Gaussian and an exponential decay, while the second one describes an algebraic decay. The Gaussian-exponential model works well except for the case of static Rabi frequency noise which is well described by the algebraic model. This algebraic decay can be explained theoretically, if one assumes chaotic dynamics perturbed by static noise as it has been discussed in [6]. Unfortunately, one would then have to explain why in the case of static Larmor frequency noise, the decay is not algebraic.

In the case of Rabi frequency noise, the near-resonant approximation works very well, and we use it to calculate the overall fidelity and its dependence on the noise amplitude. By contrast, for Larmor frequency noise, we need exact simulations which are extremely time consuming. In part, this may be due to the required higher noise amplitudes, which are roughly twice as high as in the case of Rabi frequency noise.

Interestingly, we find that in the case of Larmor frequency noise, random errors are more corrupting than static errors. The same situation also holds in the case of Rabi frequency noise, if the noise is sufficiently strong ($F_{\text{end}} < 0.5$). Only for weak Rabi frequency noise (where $F_{\text{end}} > 0.5$) do we find the opposite behaviour. Normally, one would expect that random errors following one another during a quantum protocol tend to compensate each other, so that their net corrupting effect is reduced. Why would this mechanism not be effective in our case? Possibly this can be explained with the fact that the Grover algorithm implements a rotation in the two-dimensional subspace, spanned by the initial superposition and the target state. In this situation, errors may have two effects: they may drive the evolution out of the subspace or lead to errors in the rotation angles of subsequent rotations towards the target state. If errors mainly drive the system out of the subspace, random errors are as bad as static errors.

![Figure 6](image_url)
Since the subspace is so small, there is only little chance that a subsequent error undoes the effects of the previous one.

In the case of Larmor frequency noise we further find that random noise strongly affects the Hadamard gates while static noise does not. This goes so far that for static noise the fidelity loss is mainly due to the phase inversion gates which occupy only a small fraction of the total execution time. This scenario is in agreement with findings of Long et al in [1], where they study random and systematic errors in the Grover algorithm on the basis of abstract gate operations. While they do not find a general dominance of random errors, they do find such a dominance for the Hadamard gates. Note that the systematic errors used in [1] are fixed detunings and thereby different from our static errors.

Using the near-resonant approximation, the numerical studies on Rabi frequency noise can be performed very efficiently. This would allow us to consider the Grover search algorithm in a larger quantum register with more qubits. There are however two difficulties to overcome: on the one hand, we would need longer and more complex pulse sequences for the implementation of the oracle, which include qubit swapping. On the other hand, the whole algorithm would require a longer pulse sequence. In that case, the unitary errors alone would already reduce the fidelity $F_{\text{end}}$ so much that the effects of additional noise could not be analysed. It would therefore be necessary to apply additional error suppression schemes, such as the ones explored in [32]).

A very important question of principle is whether sufficiently weak noise would eventually allow us to apply the near-resonant approximation even in the case of Larmor frequency noise. As the number of qubits is increased, the pulse sequence becomes larger and the interesting noise amplitudes become smaller. If the near-resonant approximation were then applicable, it would allow us to study the Grover algorithm with much larger databases without sacrificing our model’s main advantage as an accurate simulation of a potentially realizable quantum computer.

References

[1] Long G L, Li Y S, Zhang W L and Tu Ch C 2000 Phys. Rev. A 61 042305

[2] Ellinas D and Konstadakis C 2001 arXiv:quant-ph/0110010v1
[3] Song P H and Kim I 2003 Eur. Phys. J. D 23 299
[4] Zhirov O V and Shepelyansky D L 2006 Eur. Phys. J. D 38 405
[5] Salas P J 2008 Eur. Phys. J. D 46 365
[6] Sokolov V V, Zhirov O V and Kharkov Y A 2009 Europhys. Lett. 88 60002
[7] López G V, Gorin T and Lara I 2008 J. Phys. B: At. Mol. Opt. Phys. 41 055504
[8] Berman G P, Doolen D D, López G V and Tsifrinovich V I 2000 Phys. Rev. A 61 062305
[9] López G V and Lara I 2006 J. Phys. B: At. Mol. Opt. Phys. 39 3897
[10] Berman G P, Kamenev D I, Doolen D D, López G V and Tsifrinovich V I 2002 Contemp. Math. 305 13
[11] Pablo-Norman B and Ruiz-Altaba M 1999 Phys. Rev. A 61 012301
[12] Grover L K 1997 Phys. Rev. Lett. 79 325
[13] Grover L K 1998 Phys. Rev. Lett. 80 4329
[14] Terhal B M and Smolin J A 1998 Phys. Rev. A 58 1822
[15] Brassard G, Hoyer P and Tapp A 1998 Automata Language and Programming vol 1443 ed K G Larsen, S Skyum and G Winskel (Berlin: Springer)
[16] Cerf N J, Grover L K and Williams C P 2000 Phys. Rev. A 61 032303
[17] Gingrich R M, Williams C P and Cerf N J 2000 Phys. Rev. A 61 052313
[18] Carlini A and Hosoya A 2001 Phys. Lett. A 280 114
[19] Yang W L, Chen C Y and Feng M 2007 Phys. Rev. A 76 054301
[20] Chuang I L, Gershenfeld N and Kubinec M 1998 Phys. Rev. Lett. 80 3408
[21] Jones J A Mosca M and Hansen R H 1998 Nature 393 344
[22] Kwiat P G, Mitchel J R, Schwindt P D D and White A G 2000 J. Mod. Opt. 47 257
[23] Long G L et al 2001 Phys. Lett. A 286 121
[24] Peres A 1984 Phys. Rev. A 30 1610
[25] Gorin T, Prosen T, Seligman T H and Žnidarič M 2006 Phys. Rep. 435 33
[26] Jacqoud Ph and Petitjean C 2009 Adv. Phys. 58 67
[27] López G V, Gorin T and Lara I 2008 Int. J. Theo. Phys. 47 1641
[28] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[29] Long G L 2001 Phys. Rev. A 64 022307
[30] Frahm K M, Fleckinger R and Shepelyansky D L 2004 Eur. Phys. J. D 29 139
[31] Gorin T, Prosen T and Seligman T H 2004 New J. Phys. 6 20
[32] Berman G P, Kamenev D I, Kassman R B, Pineda C and Tsifrinovich V I 2002 arXiv:quant-ph/0212070v1