Assouad type dimensions of parabolic Julia sets

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Abstract

We prove that the Assouad dimension of a parabolic Julia set is \( \max\{1, h\} \) where \( h \) is the Hausdorff dimension of the Julia set. Since \( h \) may be strictly less than 1, this provides examples where the Assouad and Hausdorff dimension are distinct. The box and packing dimensions of the Julia set are known to coincide with \( h \) and, moreover, \( h \) can be characterised by a topological pressure function. This distinctive behaviour of the Assouad dimension invites further analysis of the Assouad type dimensions, including the Assouad and lower spectra. We compute all of the Assouad type dimensions for parabolic Julia sets and the associated \( h \)-conformal measure. Further, we show that if a Julia set has a Cremer point, then the Assouad dimension is 2.

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1 Introduction

The Julia set of a rational map of the extended complex plane is typically a beautiful fractal with intricate geometric properties. Our main case of interest is when the rational map has a rationally indifferent periodic point (a parabolic point) but the Julia set contains no critical points (that is, we consider parabolic Julia sets). The Hausdorff, box and packing dimensions of a parabolic Julia set are known to coincide and are given by the smallest zero of the topological pressure, which we denote by \( h \), see [8]. The Assouad dimension is a notion of growing relevance in fractal geometry and dimension theory of dynamical systems and up until now the Assouad dimension of Julia sets has not been considered. Our main result is that, unlike the dimensions discussed above, the Assouad dimension of a parabolic Julia set is \textit{not} necessarily equal to \( h \). It is instead given by \( \max\{1, h\} \) and, since \( h \) may be strictly less than 1, this provides examples where the Assouad and Hausdorff dimension are distinct. Moreover, it is known that \( h < 2 \) and so the Assouad dimension is also strictly less than 2.

Therefore, our result can be viewed as a refinement of the (known) result that parabolic Julia sets are porous, see [17, Theorem 1.4]. On the other hand, if the Julia set contains an \textit{irrationally} indifferent fixed point (a Cremer point), then we show that its Assouad dimension is 2.

The distinctive behaviour of the Assouad dimension of parabolic Julia sets invites further analysis of the Assouad type dimensions. We derive formulae for the Assouad and lower dimension as well as the
Assouad and lower spectra, both of the Julia set itself and the associated $h$-conformal measure. These results shed some new light on the ‘Sullivan dictionary’ in the context of dimension theory, see \[15\]. The Assouad and lower spectra were introduced in \[16\] and provide an ‘interpolation’ between the box dimension and the Assouad and lower dimensions, respectively. The motivation for the introduction of these ‘dimension spectra’ was to gain a more nuanced understanding of fractal sets than that provided by the dimensions considered in isolation. This is already proving a fruitful programme with applications emerging in a variety of settings including to problems in harmonic analysis, see work of Anderson, Hughes, Roos and Seeger \[2, 27\] which uses the Assouad spectrum to study spherical maximal functions.

Our proofs use a variety of techniques. We take some inspiration from the paper \[13\] which gave the Assouad dimension of Kleinian limit sets with parabolic points. That said, parallels in the strategy only go so far, partly due to the lack of understanding of the ‘hidden 3-dimensional geometry’ of Julia sets, see \[22, 24\]. We also take inspiration from the papers \[9, 28, 29\] where ideas from Diophantine approximation are applied in the context of conformal dynamics. We also require a quantitative version of the Leau-Fatou flower theorem which describes the geometry of the Julia set near parabolic points. Interestingly, we only require the flower theorem to study the lower dimension and spectrum, emphasising that the lower dimension, although the natural dual to the Assouad dimension, does not always yield to dual arguments.

For notational convenience throughout, we write $A \lesssim B$ if there exists a constant $C \geq 1$ such that $A \leq CB$, and $A \gtrsim B$ if $B \lesssim A$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. The constant $C$ is allowed to depend on parameters fixed in the hypotheses of the theorems presented, but not on parameters introduced in the proofs.

## 2 Definitions and Background

### 2.1 Dimensions of sets and measures

We recall the key notions from fractal geometry and dimension theory which we will use throughout the paper. For a more in-depth treatment see the books \[4, 11\] for background on Hausdorff and box dimensions, and \[14\] for Assouad type dimensions. Julia sets will be subsets of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. However, by a standard reduction we will assume that the Julia sets are bounded subsets of the complex plane $\mathbb{C}$, which we identify with $\mathbb{R}^2$. Therefore, it is convenient to recall dimension theory in Euclidean space only.

Let $F \subseteq \mathbb{R}^d$. Perhaps the most commonly used notion of fractal dimension is the Hausdorff dimension. We write $\dim_H F$, $\overline{\dim}_B F$ and $\underline{\dim}_B F$ for the Hausdorff, upper and lower box dimensions of $F$, respectively, but refer the reader to \[4, 11\] for the precise definition since we do not use it directly. We write $\dim_B F$ for the box dimension of $F$ when it exists. We write

$$|F| = \sup_{x, y \in F} |x - y| \in [0, \infty]$$

to denote the diameter of $F$. Given $r > 0$, we write $N_r(F)$ for the smallest number of balls of radius $r$ required to cover $F$. We write $M_r(F)$ to denote the largest cardinality of a packing of $F$ by balls of radius $r$ centred in $F$. In what follows, it is easy to see that replacing $N_r(F)$ by $M_r(F)$ yields an equivalent definition and so we sometimes switch between minimal coverings and maximal packings in our arguments. This is standard in dimension theory.
The Assouad dimension of $F \subseteq \mathbb{R}^d$ is defined by

$$\dim_A F = \inf \left\{ s \geq 0 : \exists C > 0 : \forall 0 < r < R : \forall x \in F : N_r(B(x, R) \cap F) \leq C \left( \frac{R}{r} \right)^s \right\}.$$  

Similarly, the lower dimension of $F$ is defined by

$$\dim_L F = \sup \left\{ s \geq 0 : \exists C > 0 : \forall 0 < r < R \leq |F| : \forall x \in F : N_r(B(x, R) \cap F) \geq C \left( \frac{R}{r} \right)^s \right\}$$

provided $|F| > 0$ and otherwise it is 0. Importantly, for compact $F$ we have

$$\dim_L F \leq \dim_H F \leq \dim_B F \leq \dim_A F.$$  

The Assouad and lower spectrum, introduced in [16], interpolate between the box dimensions and the Assouad and lower dimensions in a meaningful way. They provide a parametrised family of dimensions by fixing the relationship between the two scales $r < R$ used to define Assouad and lower dimension. Studying the dependence on the parameter within this family thus yields finer and more nuanced information about the local structure of the set. For example, one may understand which scales ‘witness’ the behaviour described by the Assouad and lower dimensions. For $\theta \in (0,1)$, the Assouad spectrum of $F$ is given by

$$\dim_A^\theta F = \inf \left\{ s \geq 0 : \exists C > 0 : \forall 0 < r < 1 : \forall x \in F : N_r(B(x, r^\theta) \cap F) \leq C \left( \frac{r^\theta}{r} \right)^s \right\}$$

and the lower spectrum of $F$ by

$$\dim_L^\theta F = \sup \left\{ s \geq 0 : \exists C > 0 : \forall 0 < r < 1 : \forall x \in F : N_r(B(x, r^\theta) \cap F) \geq C \left( \frac{r^\theta}{r} \right)^s \right\}.$$  

See [14] for more background and basic properties of the Assouad and lower spectra. It was shown in [16] that for a bounded set $F \subseteq \mathbb{R}^d$, we have

$$\dim_B F \leq \dim_A^\theta F \leq \min \left\{ \dim_A F, \frac{\dim_B F}{1 - \theta} \right\} \quad \text{(2.1)}$$

In particular, $\dim_A^\theta F \to \dim_B F$ as $\theta \to 0$. Whilst the analogous statement does not hold for the lower spectrum in general, it was proved in [14, Theorem 6.3.1] that $\dim_L^\theta F \to \dim_B F$ as $\theta \to 0$ provided $F$ satisfies a strong form of dynamical invariance. Whilst the fractals we study are not quite covered by this result, we shall see that this interpolation holds nevertheless. The limits $\lim_{\theta \to 1} \dim_A^\theta F$ and $\lim_{\theta \to 1} \dim_L^\theta F$ are known to exist in general but may not be the Assouad and lower dimensions, respectively.

There is an analogous dimension theory of measures, and the interplay between the dimension theory of fractals and the measures they support is fundamental to fractal geometry, especially in the dimension theory of dynamical systems. Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^d$. The Assouad dimension of $\mu$ with support given by $F$ is defined by

$$\dim_A \mu = \inf \left\{ s \geq 0 : \exists C > 0 : \forall 0 < r < R < |F| : \forall x \in F : \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \left( \frac{R}{r} \right)^s \right\}.$$
and, provided $|F| > 0$, the lower dimension of $\mu$ is given by

$$\dim_L \mu = \sup \left\{ s \geq 0 \mid \exists C > 0 : \forall 0 < r < |F| : \forall x \in F : \frac{\mu(B(x,R))}{\mu(B(x,r))} \geq C \left( \frac{R}{r} \right)^s \right\}$$

and otherwise it is 0. By convention we assume that $\inf \emptyset = \infty$. The Assouad and lower dimensions of measures were introduced in [18], where they were referred to as the upper and lower regularity dimensions, respectively. It is well known (see [14, Lemma 4.1.2]) that, for a Borel probability measure $\mu$ with support $F \subseteq \mathbb{R}^d$,

$$\dim_L \mu \leq \dim_L F \leq \dim_A F \leq \dim_A \mu.$$

For $\theta \in (0,1)$, the Assouad spectrum of $\mu$ with support given by $F$ is defined by

$$\dim^\theta_A \mu = \inf \left\{ s \geq 0 \mid \exists C > 0 : \forall 0 < r < |F| : \forall x \in F : \frac{\mu(B(x,r^\theta))}{\mu(B(x,r))} \leq C \left( \frac{r^\theta}{r} \right)^s \right\}$$

and, provided $|F| > 0$, the lower spectrum of $\mu$ is given by

$$\dim^\theta_L \mu = \sup \left\{ s \geq 0 \mid \exists C > 0 : \forall 0 < r < |F| : \forall x \in F : \frac{\mu(B(x,r^\theta))}{\mu(B(x,r))} \geq C \left( \frac{r^\theta}{r} \right)^s \right\}$$

and otherwise it is 0. It is known (see [12] for example) that

$$\dim_L \mu \leq \dim^\theta_L \mu \leq \dim^\theta_A \mu \leq \dim_A \mu$$

and, if $\mu$ is fully supported on a closed set $F$, then

$$\dim^\theta_L \mu \leq \dim^\theta_F \mu \leq \dim^\theta_A F \leq \dim^\theta_A \mu.$$

The upper box dimension of $\mu$ with support given by $F$ is defined by

$$\overline{\dim}_B \mu = \inf \left\{ s \mid \exists C > 0 : \forall 0 < r < |F| : \forall x \in F : \mu(B(x,r)) \geq C r^s \right\}$$

and the lower box dimension of $\mu$ is given by

$$\underline{\dim}_B \mu = \inf \left\{ s \mid \exists C > 0 : \forall r_0 > 0 : \exists 0 < r < r_0 : \forall x \in F : \mu(B(x,r)) \geq C r^s \right\}.$$

If $\overline{\dim}_B \mu = \underline{\dim}_B \mu$, then we refer to the common value as the box dimension of $\mu$, denoted by $\dim_B \mu$. These definitions of the box dimension of a measure were introduced only recently in [12] where it was also shown that, for $\theta \in (0,1)$,

$$\overline{\dim}_B \mu \leq \dim^\theta_A \mu \leq \min \left\{ \dim_A \mu, \frac{\overline{\dim}_B \mu}{1 - \theta} \right\}$$

and if the support of $\mu$ is a compact set $F$, then

$$\overline{\dim}_B F \leq \overline{\dim}_B \mu.$$
2.2 Rational maps and Julia sets

Let \( T : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map, and write \( J(T) \) to denote the Julia set of \( T \), which is equal to the closure of the repelling periodic points of \( T \). The Julia set is closed and \( T \)-invariant. We may assume that \( J(T) \) is a compact subset of \( \mathbb{C} \) by a standard reduction. If this is not the case, then simply conjugate a point \( z \notin J(T) \) to \( \infty \) via a Möbius inversion and then the closedness of the resulting Julia set ensures it lies in a bounded region of \( \mathbb{C} \). This is essentially just choosing a different point on the Riemann sphere to represent the point at infinity. For a more detailed discussion on the dynamics of rational maps, see [3, 25].

A periodic point \( \xi \in \hat{\mathbb{C}} \) with period \( p \) is said to be rationally indifferent if \( (T^p)'(\xi) = e^{2\pi i q} \) for some \( q \in \mathbb{Q} \). We say that \( T \) and \( J(T) \) are parabolic if \( J(T) \) contains no critical points of \( T \), but contains at least one rationally indifferent point. This is our main case of interest and we assume throughout that \( J(T) \) is parabolic unless we explicitly state otherwise. We write \( \Omega \) to denote the finite set of parabolic points of \( T \) and let

\[
\Omega_0 = \{ \xi \in \Omega \mid T(\xi) = \xi, \ T'(\xi) = 1 \}.
\]

As \( J(T^n) = J(T) \) for every \( n \in \mathbb{N} \), we may assume without loss of generality that \( \Omega = \Omega_0 \).

It was proven in [8] that \( h = \dim_H J(T) \) is given by the smallest zero of the function \( t \mapsto P(T, -t \log|T'|) \) where \( P \) is the topological pressure. A similar result for hyperbolic Julia sets with ‘smallest’ replaced with ‘only’ is often referred to as the Bowen-Manning-McCluskey formula, see [3, 21].

We recall, see [9, 28], that for each \( \omega \in \Omega \), we can find a ball \( U_\omega = B(\omega, r_\omega) \) with sufficiently small radius such that on \( B(\omega, r_\omega) \), there exists a unique holomorphic inverse branch \( T_\omega^{-1} \) of \( T \) such that \( T_\omega^{-1}(\omega) = \omega \). For a parabolic point \( \omega \in \Omega \), the Taylor series of \( T \) about \( \omega \) is of the form

\[
z + a(z - \omega)^{p(\omega) + 1} + \cdots
\]

for some \( a \neq 0 \). We call \( p(\omega) \in \mathbb{N} \) the petal number of \( \omega \), and we write

\[
p_{\text{min}} = \min\{p(\omega) \mid \omega \in \Omega\}
\]

\[
p_{\text{max}} = \max\{p(\omega) \mid \omega \in \Omega\}.
\]

It was proven in [1] that \( h > p_{\text{max}}/(1 + p_{\text{max}}) \). We define the set of pre-parabolic points \( J_p(T) \) by

\[
J_p(T) = \bigcup_{k=0}^{\infty} T^{-k}(\Omega).
\]

It was proven in [8] that there exists a constant \( C > 0 \) such that to each \( \xi \in J(T) \setminus J_p(T) \), we can associate a unique maximal sequence of integers \( n_j(\xi) \) such that for each \( j \in \mathbb{N} \), the inverse branches \( T_\xi^{-n_j(\xi)} \) are well defined on \( B(T^{n_j(\xi)}(\xi), C) \). We call \( J_r(T) = J(T) \setminus J_p(T) \) the radial Julia set. Following [28, 29], we define

\[
r_j(\xi) = |(T^{n_j(\xi)})'(\xi)|^{-1}
\]

and call the sequence \( (r_j(\xi))_{j \in \mathbb{N}} \) the hyperbolic zoom at \( \xi \). Similarly, for each \( \xi \in J_p(T) \), we can associate its terminating hyperbolic zoom \( (r_j(\xi))_{j \in \{1, \ldots, t\}} \). We also require the concept of a canonical ball, see [29]. Let \( \omega \in \Omega \), and let \( I(\omega) = T^{-1}(\omega) \setminus \{\omega\} \). Then for each integer \( n \geq 0 \), we define the canonical radius \( r_\xi \) at \( \xi \in T^{-n}(I(\omega)) \) by

\[
r_\xi = |(T^n)'(\xi)|^{-1}
\]
and we call $B(ξ, r_ξ)$ the canonical ball. We will use the fact that $r_ξ ≈ r_ℓ$, where $r_ℓ$ is the last element in the terminating hyperbolic zoom at $ξ$.

Due to work from Aaronson, Denker and Urbanski [1, 7, 8] it is known that given a parabolic rational map $T$, there exists a unique $h$-conformal measure $m$ supported on $J(T)$, i.e. $m$ is a probability measure such that for each Borel set $F ⊂ J(T)$ on which $T$ is injective,

$$m(T(F)) = \int_F |T'(ξ)|^h dm(ξ).$$

In [9], it was shown that $m$ has Hausdorff dimension $h$, and also that the box and packing dimensions of $J(T)$ are equal to $h$. It also follows from, for example, [29] that $m$ is exact dimensional and therefore the packing and entropy dimensions are also given by $h$. We derive formulae for the Assouad and lower dimensions and spectra of $J(T)$ and $m$ in Theorems 3.1 and 3.2. It was shown in [28] that $m$ has an associated global measure formula which we will make use of throughout.

**Theorem 2.1** (Global Measure Formula). Let $T$ be a parabolic rational map with Julia set $J(T)$ of Hausdorff dimension $h$. Let $m$ denote the associated $h$-conformal measure supported on $J(T)$. Then there exists a function $φ : J(T) × ℝ^+ → ℝ^+$ such that for all $ξ ∈ J(T)$ and $0 < r < |J(T)|$, we have

$$m(B(ξ, r)) ≈ r^h φ(ξ, r).$$

The values of $φ$ are determined as follows:

i) Suppose $ξ ∈ J_r(T)$ has associated optimal sequence $(n_j(ξ))_{j∈ℕ}$ and hyperboliczooms $(r_j(ξ))_{j∈ℕ}$ and $r$ is such that $r_{j+1}(ξ) ≤ r < r_j(ξ)$ for some $j ∈ ℕ$ and $T^k(ξ) ∈ U_ω$ for all $n_j(ξ) < k < n_{j+1}(ξ)$ and for some $ω ∈ Ω$. Then

$$φ(ξ, r) ≈ \begin{cases} \left( \frac{r}{r_j(ξ)} \right)^{(h-1)p(ω)} & r > r_j(ξ) \left( \frac{r_{j+1}(ξ)}{r_j(ξ)} \right) \left( \frac{1}{1+π(ω)} \right) \\ \left( \frac{r_j(ξ)}{r} \right)^{h-1} & r ≤ r_j(ξ) \left( \frac{r_{j+1}(ξ)}{r_j(ξ)} \right) \left( \frac{1}{1+π(ω)} \right) \end{cases}.$$  

ii) Suppose $ξ ∈ J_p(T)$ has associated terminating optimal sequence $(n_j(ξ))_{j=1,...,l}$ and hyperbolic zooms $(r_j(ξ))_{j=1,...,l}$. Suppose $T^{n_l}(ξ) = ω$ for some $ω ∈ Ω$. If $r > r_l(ξ)$, the values of $φ$ are determined as in the radial case, and if $r ≤ r_l(ξ)$, then

$$φ(ξ, r) ≈ \left( \frac{r}{r_l(ξ)} \right)^{(h-1)p(ω)}.$$  

If $J(T)$ contains no parabolic points nor critical points, then it is hyperbolic and

$$\dim_A J(T) = \dim_L J(T) = \dim_A m = \dim_L m = \dim_B m = h$$

and

$$\dim^θ_A J(T) = \dim^θ_A m = \dim^θ_L J(T) = \dim^θ_L m = h$$

for all $θ ∈ (0, 1)$. A final case of interest is when $J(T)$ contains an irrationally indifferent periodic point (a Cremer point). In this case the Jacobian derivative of $T$ at the Cremer point is an irrational rotation and $T$ is not linearisable in a neighbourhood of the Cremer point. Julia sets with Cremer points are notoriously difficult to study and it is conjectured that they should have Hausdorff dimension 2 (even positive Lebesgue measure). See [6] for more discussion of this problem and some partial results.
3 Results

Let $T$ be a parabolic rational map and $m$ the associated $h$-conformal measure.

**Theorem 3.1.** Let $\theta \in (0, 1)$. Then

i) $\dim_B m = \max\{h, h + (h - 1)p_{\max}\}$.

ii) $\dim_A m = \max\{1, h + (h - 1)p_{\max}\}$.

iii) $\dim_L m = \min\{1, h + (h - 1)p_{\max}\}$.

iv) If $h < 1$, then

$$\dim^\theta_A m = h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h)$$

and if $h \geq 1$, then $\dim^\theta_A m = h + (h - 1)p_{\max}$.

v) If $h < 1$, then $\dim^\theta_L m = h + (h - 1)p_{\max}$ and if $h \geq 1$, then

$$\dim^\theta_L m = h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h).$$

We prove Theorem 3.1 in Sections 4.2 - 4.6. Next we turn our attention to the Julia set $J(T)$.

**Theorem 3.2.** Let $\theta \in (0, 1)$. Then

i) $\dim_A J(T) = \max\{1, h\}$.

ii) $\dim_L J(T) = \min\{1, h\}$.

iii) If $h < 1$, then

$$\dim^\theta_A J(T) = h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h)$$

and if $h \geq 1$, then $\dim^\theta_A J(T) = h$.

iv) If $h < 1$, then $\dim^\theta_L J(T) = h$ and if $h \geq 1$, then

$$\dim^\theta_L J(T) = h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h).$$

We prove Theorem 3.2 in Sections 4.7 - 4.10. The case where the rational map is not parabolic (or hyperbolic) remains an interesting open programme. For example, we do not know how to compute the Assouad dimension when the rational map has a Herman ring or a Siegel disk. McMullen [23] constructed examples of quadratic polynomials with Siegel disks whose Julia set is porous. In particular, the Assouad (and Hausdorff) dimension of the Julia set is strictly less than 2 by [20, Theorem 5.2], where it was proved that porous sets in $\mathbb{R}^d$ must have Assouad dimension strictly less than $d$.

We obtain the following modest result for Julia sets with Cremer points.

**Theorem 3.3.** If $T$ is a rational map and $J(T)$ contains a Cremer point, then $\dim_A J(T) = 2$.

We prove Theorem 3.3 in Section 4.11.

4 Proofs

4.1 Preliminaries

The following lemma will be used to count canonical balls of certain sizes.
Lemma 4.1. Let $\xi \in J(T)$ and $R > r > 0$. For $R$ sufficiently small, we have
\[
\sum_{c(\omega) \in I \cap B(\xi, R)} r_{c(\omega)}^{h} \lesssim \log(R/r) \ m(B(\xi, R))
\]
where $I := \bigcup_{n \geq 0} T^{-n}(I(\omega))$ for some $\omega \in \Omega$.

Proof. By [29, Theorem 3.1], there exists a constant $\kappa > 0$ dependant only on $T$ such that for each $\omega \in \Omega$ and for sufficiently small $r > 0$, we have
\[
J(T) \subseteq \bigcup_{c(\omega) \in I \cap B(\xi, R)} B(c(\omega), \kappa r_{c(\omega)}^{p(\omega)/(1+p(\omega))} r^{1/(1+p(\omega))})
\]
with multiplicity $\lesssim 1$. In particular, for $R > r > 0$ with $R$ sufficiently small, the set
\[
\bigcup_{c(\omega) \in I \cap B(\xi, R)} B(c(\omega), \kappa r_{c(\omega)}^{p(\omega)/(1+p(\omega))} r^{1/(1+p(\omega))})
\]
has multiplicity $\lesssim 1$ and is contained in the ball $B(\xi, (\kappa + 1)R)$. Therefore, we have
\[
m(B(\xi, (\kappa + 1)R)) \gtrsim \sum_{c(\omega) \in I \cap B(\xi, R)} m\left(B(c(\omega), \kappa r_{c(\omega)}^{p(\omega)/(1+p(\omega))} r^{1/(1+p(\omega))})\right)
\]
\[
\gtrsim \sum_{c(\omega) \in I \cap B(\xi, R)} \left(r_{c(\omega)}^{p(\omega)/(1+p(\omega))} r^{1/(1+p(\omega))}\right)^{h} \left(r_{c(\omega)}^{p(\omega)/(1+p(\omega))} r^{1/(1+p(\omega))}\right)^{(h-1)p(\omega)}
\]
\[
\gtrsim r^{h} \sum_{c(\omega) \in I \cap B(\xi, R)} \left(\frac{r_{c(\omega)}}{r}\right)^{p(\omega)/(1+p(\omega))} \gtrsim r^{h} \sum_{c(\omega) \in I \cap B(\xi, R)} 1 \tag{4.1}
\]
where the second inequality is an application of Theorem 2.1 and the third uses the fact that $r_{c(\omega)} \approx r_{I}$. Therefore
\[
\sum_{c(\omega) \in I \cap B(\xi, R)} r_{c(\omega)}^{h} \lesssim \sum_{m \in \mathbb{Z}[0, \log(R/r)]} \sum_{c(\omega) \in I \cap B(\xi, R)} r_{c(\omega)}^{h} e^{mh}
\]
\[
\lesssim \sum_{m \in \mathbb{Z}[0, \log(R/r)]} \sum_{c(\omega) \in I \cap B(\xi, R)} r_{c(\omega)}^{h} e^{mh}
\]
\[
\lesssim \sum_{m \in \mathbb{Z}[0, \log(R/r)]} r_{c(\omega)}^{h} e^{mh} \sum_{c(\omega) \in I \cap B(\xi, R)} \frac{1}{e^{m^{2}}} \lesssim \sum_{m \in \mathbb{Z}[0, \log(R/r)]} r_{c(\omega)}^{h} e^{mh} \left(r_{e^{m}}^{-h} m(B(\xi, (\kappa + 1)R))\right) \quad \text{by } (4.1)
\]
\[
\lesssim \log(R/r) \ m(B(\xi, R))
\]
where the last inequality uses the fact that $m$ is a doubling measure. \hfill \Box
We also require the following key lemma.

**Lemma 4.2.** Let $\xi \in J_r(T)$ and $R > 0$ be sufficiently small. Then there exists some $c(\omega) \in J_p(T)$ and some constant $C \geq 1$ such that

$$B(\xi, R) \subset B(c(\omega), C\phi(\xi, R)^{(h-1)p(\omega)}r_{c(\omega)})$$

**Proof.** It was shown in [9, Section 5] that

$$B(\xi, R) \subset T_{\xi}^{-n}(B(\omega, C\phi(\xi, R)^{(h-1)p(\omega)}r_{\omega}))$$

for some $\omega \in \Omega$ and uniform $C \geq 1$ where $T_{\xi}^{-n}$ is an appropriately chosen holomorphic inverse branch of $T^n$. By definition $T_{\xi}^{-n}(\omega) \in J_p(T)$, and then the result is an immediate consequence of the fact that $(T_{\xi}^{-n})'(\omega)r_{\omega} \approx r_{c(\omega)}$, using the Koebe distortion theorem and the fact that $r_{\omega} \approx 1$ as $\Omega$ is a finite set (see [9, 29]).

### 4.2 The box dimension of $m$

#### 4.2.1 Upper bound

We show $\overline{\dim}_B m \leq \max\{h, h + (h-1)p_{\max}\}$. Note that when $h \geq 1$, we have

$$\overline{\dim}_B m \leq \dim_A m \leq h + (h-1)p_{\max}$$

(see Section 4.3) so we assume $h < 1$. Then note that for any $\xi \in J(T)$ and $r < |J(T)|$

$$m(B(\xi, r)) \approx r^h \phi(\xi, r) \geq r^h$$

which proves $\overline{\dim}_B m \leq h$, as required.

#### 4.2.2 Lower bound

We show $\underline{\dim}_B m \geq \max\{h, h + (h-1)p_{\max}\}$. Suppose $h \geq 1$. Let $\xi \in J_p(T)$ with associated terminating optimal sequence $(n_j(\xi))_{j=1,...,l}$ and hyperbolic zooms $(r_j(\xi))_{j=1,...,l}$ such that $T^{n_j(\xi)}(\xi) = \omega$ for some $\omega \in \Omega$ with $p(\omega) = p_{\max}$. Then using Theorem 2.1 we have for all sufficiently small $r > 0$

$$m(B(\xi, r)) \lesssim r^{h+(h-1)p_{\max}}$$

which proves $\underline{\dim}_B m \geq h + (h-1)p_{\max}$, as required.

If $h < 1$, then we have $\underline{\dim}_B m \geq \underline{\dim}_B J(T) = h$, as required.

### 4.3 The Assouad dimension of $m$

The lower bound will follow from our lower bound for the Assouad spectrum of $m$, see Section 4.5. Therefore we only need to prove the upper bound.

We show $\dim_A m \leq \max\{1, h + (h-1)p_{\max}\}$.

We only argue the case where $\xi \in J_r(T)$, and note that the case when $\xi \in J_p(T)$ follows similarly. We make extensive use of Theorem 2.1 throughout.
Suppose $\xi \in J_r(T)$, with associated optimal sequence $(n_j(\xi))_{j \in \mathbb{N}}$ and hyperbolic zooms $(r_j(\xi))_{j \in \mathbb{N}}$. Suppose that $r_{j+1}(\xi) \leq r < R < r_j(\xi)$ and that $T^{r_j(\xi)+1}(\xi) \in U_\omega = B(\omega, r)$ for some $\omega \in \Omega$, and let $r_m = r_j(\xi) (r_{j+1}(\xi)/r_j(\xi))^{r_j(\xi)/r_m(\xi)}$.

If $r > r_m$, then

$$\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \left( \frac{R/r_j(\xi)}{r_j(\xi)/r} \right)^{(h-1)p(\omega)} \leq \left( \frac{R}{r} \right)^{h+(h-1)p_{\text{max}}}.$$ 

If $R < r_m$, then

$$\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \left( \frac{r_j(\xi)}{r_{j+1}(\xi)} \right)^{(h-1)p(\omega)} = \left( \frac{R}{r} \right)^{h+(h-1)p(\omega)}.$$ 

If $r \leq r_m \leq R$, then

$$\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \left( \frac{R/r_j(\xi)}{r_{j+1}(\xi)/r} \right)^{(h-1)p(\omega)} = \frac{R^{h+(h-1)p(\omega)}}{r_j(\xi)^{(h-1)p(\omega)} r_{j+1}(\xi)^{(h-1)}}.$$ (4.2)

If we assume that $h \geq 1$, then note that

$$\left( \frac{R}{r} \right)^{h+(h-1)p(\omega)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \frac{r_j(\xi)^{(h-1)(1+p(\omega))}}{r_{j+1}(\xi)^{(h-1)+1(1+p(\omega))}}$$

is maximised when $r = r_m$. Therefore

$$\left( \frac{R}{r} \right)^{h+(h-1)p(\omega)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \leq \frac{r_j(\xi)^{(h-1)(1+p(\omega))}}{r_{j+1}(\xi)^{(h-1)+1(1+p(\omega))}} = 1$$

which proves that

$$\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^{h+(h-1)p(\omega)} \leq \left( \frac{R}{r} \right)^{h+(h-1)p_{\text{max}}}.$$ 

Similarly, if $h < 1$, then by (4.2)

$$\left( \frac{R}{r} \right)^{h+(h-1)p(\omega)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \frac{R^{(h-1)(1+p(\omega))}}{r_j(\xi)^{(h-1)p(\omega)} r_{j+1}(\xi)^{(h-1)}}$$

is maximised when $R = r_m$. Therefore, as above

$$\left( \frac{R}{r} \right)^{h+(h-1)p(\omega)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \leq 1$$

which proves

$$\frac{m(B(\xi, R))}{m(B(\xi, r))} \leq \left( \frac{R}{r} \right).$$ 

This covers all cases when $r_{j+1}(\xi) \leq r < R < r_j(\xi)$.

Now, we consider the cases when $r_{j+1}(\xi) \leq R < r_j(\xi)$, $r_{l+1}(\xi) \leq r < r_l(\xi)$, with $l > j$, $T^{r_j(\xi)+1}(\xi) \in U_{\omega_1}$ and $T^{r_{l+1}(\xi)+1}(\xi) \in U_{\omega_2}$ for some $\omega_1, \omega_2 \in \Omega$. 

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Let \( r_m = r_j(\xi) (r_{j+1}(\xi)/r_j(\xi))^{\frac{1}{p_1-1}} \) and \( r_n = r_l(\xi) (r_{l+1}(\xi)/r_l(\xi))^{\frac{1}{p_1-1}} \).

**Case 1:** \( R > r_m, r > r_n \).

We have

\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \frac{(R/r_j(\xi))^{(h-1)p(\omega_1)}}{(r/r_l(\xi))^{(h-1)p(\omega_2)}}. \tag{4.3}
\]

If \( h < 1 \), then

\[
\left( \frac{r}{R} \right) \frac{m(B(\xi, R))}{m(B(\xi, r))} \lesssim \frac{R^{(h-1)(1+p(\omega_1))}}{r^{(h-1)p(\omega_1)}} \lesssim \frac{r_j^{(h-1)(1+p(\omega_1))}}{r^{(h-1)p(\omega_1)}} \cdot \frac{r_j^{(h-1)(1+p(\omega_1))}}{r_j^{(h-1)p(\omega_1)}} = 1,
\]

and if \( h \geq 1 \), then by (4.3)

\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \lesssim \left( \frac{R}{r} \right)^h (r/l(\xi))^{(h-1)p(\omega_2)} \lesssim \left( \frac{R}{r} \right)^{h+(h-1)p(\omega_2)}
\]

using \( r_l(\xi) < R \).

**Case 2:** \( R \leq r_m, r \leq r_n \).

We have

\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \frac{(r_{j+1}(\xi)/r_j^{h-1})}{(r_{l+1}(\xi)/r_l^{h-1})}. \tag{4.4}
\]

If \( h < 1 \), then

\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right) \frac{r_{j+1}(\xi)}{r_{l+1}(\xi)} \lesssim \left( \frac{R}{r} \right)
\]

and if \( h \geq 1 \), then by (4.4)

\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \lesssim \frac{R^h}{r^{r_{l+1}(\xi)^{h-1}}}
\]

and therefore

\[
\left( \frac{r}{R} \right)^{h+(h-1)p(\omega_2)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \lesssim \frac{R^{(1-h)p(\omega_2)}}{r^{(1-h)(1+p(\omega_2))}} \frac{r_{l+1}(\xi)^{(1-h)p(\omega_2)}}{r_{l+1}(\xi)^{(1-h)p(\omega_2)}} = 1.
\]

**Case 3:** \( R > r_m, r \leq r_n \).

We have

\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \frac{(R/r_j(\xi))^{(h-1)p(\omega_1)}}{(r_l(\xi)/r_l(\xi))^{(h-1)p(\omega_2)}} = \frac{R^{h+(h-1)p(\omega_1)}}{rr_j^{(h-1)p(\omega_1)}r_{l+1}(\xi)^{h-1}}. \tag{4.5}
\]
If \( h < 1 \), then
\[
\left( \frac{r}{R} \right) \frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \frac{R^{(h-1)(1+p(\omega_1))}}{r_j(\xi)^{(h-1)p(\omega_1)} r_{l+1}(\xi)^{h-1}} \leq \frac{r_j(\xi)^{(h-1)(1+p(\omega_1))}}{r_j(\xi)^{(h-1)p(\omega_1)} r_{j+1}(\xi)^{h-1}} = 1.
\]

If \( h \geq 1 \) and \( p(\omega_1) \geq p(\omega_2) \), then by (4.5)
\[
\left( \frac{r}{R} \right)^{h+(h-1)p(\omega_1)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \frac{r^{(h-1)(1+p(\omega_1))}}{r_j(\xi)^{(h-1)p(\omega_1)} r_{l+1}(\xi)^{h-1}} \leq \frac{r_j(\xi)^{(h-1)(1+p(\omega_1))}}{r_j(\xi)^{(h-1)p(\omega_1)} r_{l+1}(\xi)^{h-1}} = \left( \frac{r_l(\xi)}{r_{l+1}(\xi)} \right)^{(h-1)(1+\frac{p(\omega_1)}{1+p(\omega_2)})} \leq 1.
\]

and if \( h \geq 1 \) and \( p(\omega_1) < p(\omega_2) \), then by (4.5)
\[
\left( \frac{r}{R} \right)^{h+(h-1)p(\omega_2)} \frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \frac{r^{(h-1)(1+p(\omega_2))} R^{(h-1)(p(\omega_1)-p(\omega_2))}}{r_j(\xi)^{(h-1)p(\omega_1)} r_{l+1}(\xi)^{h-1}} \leq \frac{r_j(\xi)^{(h-1)(1+p(\omega_1))} R^{(h-1)(1+p(\omega_1))}}{r_j(\xi)^{(h-1)p(\omega_1)} r_{l+1}(\xi)^{h-1}} = \left( \frac{R}{r_l(\xi)} \right)^{(h-1)(p(\omega_1)-p(\omega_2))} \leq 1.
\]

Case 4: \( R \leq r_m, r > r_n \).

This gives
\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \left( \frac{r_{j+1}(\xi)}{R} \right)^{(h-1)} = 1. \tag{4.6}
\]

If \( h < 1 \), then
\[
\left( \frac{r}{R} \right) \frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \frac{r^{(1-h)(1+p(\omega_1))}}{r_{j+1}(\xi)^{1-h} r_{l}(\xi)^{(1-h)p(\omega_2)}} \leq \frac{r_j(\xi)^{(1-h)(1+p(\omega_2))}}{r_j(\xi)^{(1-h)p(\omega_2)} r_{l}(\xi)^{(1-h)p(\omega_2)}} = 1
\]

and if \( h \geq 1 \), then by (4.6)
\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \approx \left( \frac{R}{r} \right)^h \frac{r_l(\xi)^{(h-1)p(\omega_2)}}{r_l(\xi)^{(h-1)p(\omega_2)}} \leq \left( \frac{R}{r} \right)^{h+(h-1)p(\omega_2)}.
\]

In all possible cases, we have
\[
\frac{m(B(\xi, R))}{m(B(\xi, r))} \leq \left( \frac{R}{r} \right)^\max\{1,h+(h-1)p(\omega_1)\} \leq \left( \frac{R}{r} \right)^\max\{1,h+(h-1)p_{\text{max}}\}
\]

which proves the desired upper bound.

4.4 The lower dimension of \( m \)

The upper bound will follow from our upper bound for the lower spectrum of \( m \), see Section 4.6.

Therefore we only need to prove the lower bound. However, the lower bound for \( \text{dim}_m m \) can be proved by a completely analogous argument to that given for the upper bound for \( \text{dim}_A m \) in the previous section, and so we leave it for the reader. In particular, the roles of \( h < 1 \) and \( h \geq 1 \) are reversed which reverses many of the inequalities.
4.5 The Assouad spectrum of $m$

4.5.1 When $h < 1$

The lower bound here follows from the lower bound for the Assouad spectrum of $J(T)$, see Section 4.9.1. Therefore it remains to prove the upper bound.

We show

$$\dim^\theta_A m \leq h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h).$$

The case when $\theta \geq 1/(1 + p_{\max})$ follows easily, as

$$\dim^\theta_A m \leq \dim_A m \leq 1,$$

so we assume $\theta < 1/(1 + p_{\max})$. Let $\xi \in J_r(T)$, and assume that $r_{j+1}(\xi) \leq r^\theta < r_j(\xi)$, $r_{l+1}(\xi) \leq r < r_l(\xi)$, with $l \geq j$, $T^{n_j+1}(\xi) \in U_{\omega_1}$ and $T^{n_l+1}(\xi) \in U_{\omega_2}$ for some $\omega_1, \omega_2 \in \Omega$. The case $\xi \in J_p(T)$ is similar and omitted. Let $r_m = r_j(\xi) (r_{j+1}(\xi)/r_j(\xi))^{1/(p(\omega_1))}$. If $r^\theta > r_m$, then by Theorem 2.1

$$\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \lesssim \left( \frac{r^\theta}{r} \right)^h \left( \frac{r_{j+1}(\xi)}{r^\theta} \right)^{(h-1)p(\omega_1)} \leq \left( \frac{r^\theta}{r} \right)^h r_j(\xi)^{(1-h)p(\omega_1)} \lesssim \left( \frac{r^\theta}{r} \right)^h \frac{\theta p_{\max}}{1 - \theta} (1 - h)$$

and if $r^\theta \leq r_m$, then by Theorem 2.1

$$\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \lesssim \left( \frac{r^\theta}{r} \right)^h \left( \frac{r_{j+1}(\xi)}{r^\theta} \right)^{-1} \leq \left( \frac{r^\theta}{r} \right)^h r_j(\xi)^{(1-h)p(\omega_1)} \lesssim \left( \frac{r^\theta}{r} \right)^h \frac{\theta p_{\max}}{1 - \theta} (1 - h).$$

In either case, we have

$$\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \lesssim \left( \frac{r^\theta}{r} \right)^h \frac{\theta p_{\max}}{1 - \theta} (1 - h)$$

which proves

$$\dim^\theta_A m \leq h + \frac{\theta p_{\max}}{1 - \theta} (1 - h)$$

as required.

4.5.2 When $h \geq 1$

We show $\dim^\theta_A m = h + (h - 1)p_{\max}$. This follows easily, since

$$h + (h - 1)p_{\max} = \dim_B m \leq \dim^\theta_A m \leq \dim_A m \leq h + (h - 1)p_{\max}.$$
so we need only prove the upper bound. To do this, let \( \xi \in J_p(T) \) with associated terminating optimal sequence \((n_j(\xi))_{j=1,...,l} \) and hyperbolic zooms \((r_j(\xi))_{j=1,...,l}\) such that \( T^{n_l}(\xi) = \omega \) for some \( \omega \in \Omega \) with \( p(\omega) = p_{\text{max}} \). Then using Theorem 2.1 we have for all sufficiently small \( r > 0 \)

\[
\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \leq \left( \frac{r^\theta}{r} \right)^{h} \left( \frac{r^\theta}{r_j(\xi)} \right)^{(h-1)p(\omega)} = \left( \frac{r^\theta}{r} \right)^{h+(h-1)p_{\text{max}}}
\]

which proves \( \dim_1^\theta m \leq h + (h-1)p_{\text{max}} \), as required.

4.6.2 When \( h \geq 1 \)

The upper bound here follows from the upper bound for the lower spectrum of \( J(T) \), see Section 4.10.1 Therefore it remains to prove the lower bound.

We show

\[
\dim_1^\theta m \geq h + \min \left\{ 1, \frac{\theta p_{\text{max}}}{1 - \theta} \right\} (1 - h).
\]

The case when \( \theta \geq 1/(1 + p_{\text{max}}) \) follows easily, as \( \dim_1^\theta m \geq \dim_1 m \geq 1 \), so we assume \( \theta < 1/(1 + p_{\text{max}}) \). Let \( \xi \in J_r(T) \), and assume that \( r_{j+1}(\xi) < r_j(\xi), \theta r_{j+1}(\xi) < r < r_{j}(\xi) \), with \( l \geq j \), \( T^{n_{j+1}}(\xi) \in U_{\omega_1} \) and \( T^{n_{j}+1}(\xi) \in U_{\omega_2} \) for some \( \omega_1, \omega_2 \in \Omega \). The case \( \xi \in J_p(T) \) is similar and omitted. Let \( r_m = r_j(\xi)(r_{j+1}(\xi)/r_j(\xi))^{1/(1+p(\omega_1))} \). If \( r^\theta > r_m \), then by Theorem 2.1

\[
\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \geq \left( \frac{r^\theta}{r} \right)^{h} \left( \frac{r^\theta}{r_j(\xi)} \right)^{(h-1)p(\omega)} \geq \left( \frac{r^\theta}{r} \right)^{h} r^\theta(1-h)p(\omega_1) r_j(\xi)^{(1-h)p(\omega)} \geq \left( \frac{r^\theta}{r} \right)^{h + \frac{\theta p_{\text{max}}}{1 - \theta} (1-h)}
\]

and if \( r^\theta \leq r_m \), Theorem 2.1 gives

\[
\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \geq \left( \frac{r^\theta}{r} \right)^{h} \left( \frac{r^\theta}{r_j(\xi)} \right)^{h-1} \geq \left( \frac{r^\theta}{r} \right)^{h} \left( \frac{r^\theta}{r_j(\xi)} \right)^{(h-1)(1+p(\omega_1))} r_j(\xi)^{(1-h)p(\omega_1)} \geq \left( \frac{r^\theta}{r} \right)^{h + \frac{\theta p_{\text{max}}}{1 - \theta} (1-h)}.
\]

In either case, we have

\[
\frac{m(B(\xi, r^\theta))}{m(B(\xi, r))} \geq \left( \frac{r^\theta}{r} \right)^{h + \frac{\theta p_{\text{max}}}{1 - \theta} (1-h)}
\]

which proves

\[
\dim_1^\theta m \geq h + \frac{\theta p_{\text{max}}}{1 - \theta} (1-h)
\]

as required.

4.7 The Assouad dimension of \( J(T) \)

The lower bound will follow from our lower bound for the Assouad spectrum of \( J(T) \), see Section 4.9. Therefore we only need to prove the upper bound.

We show

\[
\dim_A J(T) \leq \max \{ 1, h \}.
\]
Note that when \( h \leq 1 \), we have \( \dim_A J(T) \leq \dim_A m \leq 1 \), so throughout we assume that \( h > 1 \).

Let \( \xi \in J(T) \), \( \varepsilon > 0 \), and \( R > r > 0 \) with \( R/r \geq \max\{e^{\varepsilon^{-1}}, 10\} \). Let \( \{B(x_i, r)\}_{i \in X} \) be a centred \( r \)-packing of \( B(\xi, R) \cap J(T) \) of maximal cardinality. We assume for convenience that each \( x_i \in J_r(T) \), which we may do since \( J_r(T) \) is dense in \( J(T) \). This is not really necessary but allows for efficient application of Lemma 4.2. Each \( x_i \) has a particular \( \omega = \omega(i) \in \Omega \) associated with it, coming from the global measure formula for \( m(B(x_i, r)) \). In particular, \( x_i \) belongs to an associated canonical ball \( B(c(\omega), r_{c(\omega)}) \). Decompose \( X \) as

\[
X = X_0 \cup X_1 \cup \bigcup_{n=2}^{\infty} X_n
\]

where

\[
X_0 = \{ i \in X \mid x_i \in B(c(\omega), r_{c(\omega)}) \text{ with } r_{c(\omega)} \geq 5R \}
\]

\[
X_1 = \{ i \in X \setminus X_0 \mid \phi(x_i, r) \geq (r/R)^{\varepsilon} \}
\]

and

\[
X_n = \{ i \in X \mid (X_0 \cup X_1) \mid e^{-n} \leq \phi(x_i, r) < e^{-(n-1)} \}.
\]

To study those \( i \in X_0 \), we decompose \( X_0 \) further as

\[
X_0 = X_0^0 \cup \bigcup_{n=1}^{\infty} X_0^n
\]

where

\[
X_0^0 = \{ i \in X_0 \mid \phi(x_i, r) \geq \phi(\xi, R) \}
\]

\[
X_0^n = \{ i \in X_0 \mid e^{-n} \phi(\xi, R) \leq \phi(x_i, r) < e^{-(n-1)} \phi(\xi, R) \}.
\]

If \( i \in X_0^0 \), then by Theorem 2.1

\[
R^h \phi(\xi, R) \gtrsim m(B(\xi, R)) \gtrsim m\left( \bigcup_{i \in X_0^0} B(x_i, r) \right) \gtrsim \min_{i \in X_0^0} |X_0^0|^r \phi(x_i, r) \geq |X_0^0|^r \phi(\xi, R)
\]

which implies that

\[
|X_0^0| \lesssim r^{-h}.
\]

Turning our attention to \( X_0^n \), for \( c(\omega) \in J_p(T) \), write \( X_0^n(c(\omega)) \) to denote the set of all \( i \in X_0^n \) which are associated with \( c(\omega) \) (that is, \( \phi(x_i, r) \) is defined via \( c(\omega) \) in the context of Theorem 2.1). In particular, Lemma 4.2 ensures that

\[
B(x_i, r) \subseteq B(c(\omega), C(\phi(\xi, R) e^{-(n-1)} \frac{1}{(n-1)p(\omega)} r_{c(\omega)})
\]

for all \( i \in X_0^n(c(\omega)) \), where \( C \geq 1 \) is the constant coming from Lemma 4.2. Temporarily fix \( c(\omega) \in J_p(T) \) such that \( X_0^n(c(\omega)) \neq \emptyset \). We have

\[
|c(\omega) - z| \approx \phi(z, \rho \frac{1}{(n-1)p(\omega)} r_{c(\omega)})
\]

for all \( z \in J(T) \) and \( \rho > 0 \) such that \( \phi(z, \rho) \) is defined via \( c(\omega) \) in the context of Theorem 2.1. This is proved in [9]. Specifically, [9, Equations (5.3) and (5.5)] give

\[
\phi(z, \rho \frac{1}{(n-1)p(\omega)}) \approx |T^n(z) - \omega|
\]
where \( n \in \mathbb{N} \) is such that \( T^n(c(\omega)) = \omega \) and \(|(T^n)'(c(\omega))| \approx r_{c(\omega)}^{-1} \) (we note that the \( \phi \) used in [9] is different than the notation we are using, so we have translated the equation into our notation which is consistent with [28, 29]). Then, applying the Koebe Distortion Theorem,

\[
|T^n(z) - \omega| \approx r_{c(\omega)}^{-1}|z - c(\omega)|
\]

establishing (4.7).

Suppose \( i \in X_0^N(c(\omega)) \) for some large \( N \), which implies \( \phi(x_i, r) \leq e^{-(N-1)}\phi(\xi, R) \). Recall that \( r_{c(\omega)} \geq 5R \). Let \( j \in \mathbb{N} \) be such that \( n_j + 1 = n_{j+1}(x_i) \geq n_j(x_i) = n_j \) and

\[
|(T^{n_j+1})'(x_i)|^{-1} = r_{j+1}(x_i) \leq r < r_j(x_i)
\]

and \( T^k(x_i) \in U_\omega \) for all \( n_j < k < n_{j+1} \). For \( n \) such that \( T^n(c(\omega)) = \omega \) and \(|(T^n)'(c(\omega))| \approx r_{c(\omega)}^{-1} \), we have \( T^n(\xi) \in U_\omega \). Note that

\[
|(T^{n_j+1})'(x_i)|^{-1} = r_{j+1}(x_i) \leq r < R \leq r_{c(\omega)}/5 \approx |(T^n)'(c(\omega))|^{-1},
\]

and we have \( T^k(\xi) \in U_\omega \) for all \( n_j \leq n \leq k \leq l \leq n_{j+1} \) for some \( l \) satisfying \(|(T^l)'(\xi)|^{-1} \geq R \). It follows that \( \phi(\xi, aR) \) is also defined via \( c(\omega) \) in the context of Theorem 2.1 for some \( a \approx 1 \). Then by (4.7)

\[
|x_i - c(\omega)| \lesssim \phi(x_i, r)^{-1} r_{c(\omega)} \lesssim e^{-(N-1)\phi(\xi, R)^{1/(n-1)\phi(\xi, R)}} r_{c(\omega)}
\]

and therefore (see Figure 1)

\[
R \geq |\xi - c(\omega)| - |x_i - c(\omega)| \gtrsim \phi(\xi, R)^{1/(n-1)\phi(\xi, R)} r_{c(\omega)}
\]

for \( N \) chosen large enough depending only on various implicit constants. We fix such \( N \) in the following discussion.

\[
\begin{align*}
\xi & \gtrsim \phi(\xi, R)^{1/(n-1)\phi(\xi, R)} r_{c(\omega)} \\
x_i & \lesssim e^{-(N-1)\phi(\xi, R)^{1/(n-1)\phi(\xi, R)}} r_{c(\omega)}
\end{align*}
\]

Figure 1: Bounding \( R \) from below.
We may assume $X_0^n(c(\omega)) \neq \emptyset$ for some $n > N$, since otherwise $\phi(x_i, r) \succ \phi(\xi, R)$ for all $i \in X_0$ and the argument bounding $|X_0^n|$ also applies to bound $|X_0|$.

By Theorem 2.1

$$m \left( \bigcup_{i \in X_0^n(c(\omega))} B(x_i, r) \right) \lesssim m \left( B(c(\omega), C(\phi(\xi, R)e^{-(n-1)} \frac{1}{N^p} r_c(\omega)}) \right) \lesssim (e^{-n} \phi(\xi, R))^{\frac{h}{(n-1)p(\omega)}} r_{c(\omega)}^h e^{-n} \phi(\xi, R).$$

In the other direction, as $\{x_i\}_{i \in X_0^n(c(\omega))}$ is an $r$-packing,

$$m \left( \bigcup_{i \in X_0^n(c(\omega))} B(x_i, r) \right) \geq \sum_{i \in X_0^n(c(\omega))} m(B(x_i, r)) \gtrsim |X_0^n(c(\omega))| r^h e^{-n} \phi(\xi, R)$$

and so

$$|X_0^n(c(\omega))| \lesssim (e^{-n} \phi(\xi, R))^{\frac{h}{(n-1)p(\omega)}} r_{c(\omega)}^h r^{-h} \lesssim e^{\frac{-nh}{(n-1)p_{\max}}} \left( \frac{R}{r} \right)^h$$

by (4.8). The number of distinct squeezed canonical balls giving rise to non-empty $X_0^n(c(\omega))$ with $n \geq N$ is $\lesssim 1$ since $\phi(\xi, aR)$ is defined via $c(\omega)$ for some $a \approx 1$ and any such $c(\omega)$ (see the argument leading up to (4.7)). Therefore

$$|X_0^n| \lesssim e^{\frac{-nh}{(n-1)p_{\max}}} \left( \frac{R}{r} \right)^h.$$

Pulling these estimates together, we get

$$|X_0| = |X_0^0| + \sum_{n=1}^{\infty} |X_0^n| \lesssim \left( \frac{R}{r} \right)^h \sum_{n=1}^{\infty} e^{\frac{-nh}{(n-1)p_{\max}}} \left( \frac{R}{r} \right)^h \lesssim \left( \frac{R}{r} \right)^h. \tag{4.9}$$

If $i \in X_1$, then

$$R^h \phi(\xi, R) \gtrsim m(B(\xi, R)) \gtrsim \min_{i \in X_1} |X_1^r| r^h \phi(x_i, r) \gtrsim |X_1| r^h \left( \frac{r}{R} \right)^\varepsilon$$

which proves

$$|X_1| \lesssim \left( \frac{R}{r} \right)^{h+\varepsilon}. \tag{4.10}$$

Finally, we turn our attention to $X_n$. If $i \in X_n$ for $n \geq 2$, then $\phi(x_i, r) < e^{-(n-1)}$, and therefore by Lemma 1.2 the ball $B(x_i, r)$ is contained in the squeezed canonical ball

$$B(c(\omega), C e^{\frac{-(n-1)}{p(\omega)}} r_c(\omega))$$

for some $c(\omega) \in J_p(T)$. Therefore, $r/C \leq r_{c(\omega)} < 5R < 6CR$ and, noting that $h > 1$,

$$|c(\omega) - \xi| \leq |c(\omega) - x_i| + |x_i - \xi| \leq Cr_{c(\omega)} + R \leq 5CR + R \leq 6CR$$

and so $c(\omega) \in B(\xi, 6CR)$. For $p \in \{p_{\min}, \ldots, p_{\max}\}$, let

$$X_n^p = \{ i \in X_n \mid p(\omega) = p \}$$
and let

\[ I_p = \bigcup_{\omega \in \Omega \atop p(\omega) = p} I \]

where \( I \) is defined in the same way as in Lemma 4.1. Then we have

\[
m \left( \bigcup_{i \in X_h^p} B(x_i, r) \right) \leq m \left( \bigcup_{c(\omega) \in I_p \cap B(\xi, 6CR) \atop 6CR > r_{c(\omega)} \geq r/C} B(c(\omega), Ce^{-\frac{1}{h}r} r_{c(\omega)}) \right) \leq \sum_{c(\omega) \in I_p \cap B(\xi, 6CR) \atop 6CR > r_{c(\omega)} \geq r/C} m \left( B(c(\omega), Ce^{-\frac{1}{h}r} r_{c(\omega)}) \right) \]

\[
\lesssim \sum_{c(\omega) \in I_p \cap B(\xi, 6CR) \atop 6CR > r_{c(\omega)} \geq r/C} e^{-\frac{1}{h}r} r_{c(\omega)}^{h} e^{-\frac{1}{h}r} \quad \text{by Theorem 2.1 (ii)}
\]

\[
\lesssim e^{-\frac{1}{h}r} \log(R/r) + \log(6C^2) m(B(\xi, R)) \quad \text{by Lemma 4.1}
\]

\[
\lesssim e^{-\frac{1}{h}r} \log(R/r) R^{h} \phi(\xi, R)
\]

\[
\lesssim e^{-\frac{1}{h}r} \log(R/r) R^{h} e^{-\frac{1}{h}r} R^{h}.
\]

The final line uses the estimate \((r/R)^{\xi} > e^{-\frac{1}{h}}\) which holds whenever \(X_n\) is non-empty. On the other hand, by Theorem 2.1

\[
m \left( \bigcup_{i \in X_h^p} B(x_i, r) \right) \geq \sum_{i \in X_h^p} m(B(x_i, r)) \gtrsim |X_n^p| R^{h} e^{-\frac{1}{h}}.
\]

Therefore

\[
|X_n^p| \lesssim e^{-\frac{1}{h}n} \left( \frac{R}{r} \right)^{h} \lesssim e^{-\frac{1}{h}n e^{-\frac{\frac{1}{h}R}{\max(\frac{1}{h}, \min(\frac{1}{h}))}}} \left( \frac{R}{r} \right)^{h}
\]

which gives

\[
|X_n| \leq \sum_{p = p_{\min}}^{p_{\max}} |X_n^p| \lesssim e^{-\frac{1}{h}n} \left( \frac{R}{r} \right)^{h}
\]

(4.11)

Combining (4.9), (4.10) and (4.11), we have

\[
|X| = |X_0| + |X_1| + \sum_{n=2}^{\infty} |X_n| \lesssim \left( \frac{R}{r} \right)^{h} + \left( \frac{R}{r} \right)^{h+\varepsilon} + \sum_{n=2}^{\infty} e^{-\frac{1}{h}n} \left( \frac{R}{r} \right)^{h}
\]

\[
\lesssim \left( \frac{R}{r} \right)^{h+\varepsilon} + e^{-1} \left( \frac{R}{r} \right)^{h}
\]

which proves that \(\dim_{\mathcal{A}} J(T) \leq h + \varepsilon\), and letting \(\varepsilon \to 0\) proves that \(\dim_{\mathcal{A}} J(T) \leq h\), as required.
4.8 The lower dimension of $J(T)$

The upper bound will follow from our upper bound for the lower spectrum of $J(T)$, see Section 4.10. Therefore we only need to prove the lower bound.

We show

$$\dim_{\text{l}} J(T) \geq \min\{1, h\}.$$  

Note that when $h < 1$, we have $\dim_{\text{l}} J(T) \geq \dim_{\text{l}} m \geq 1$ so we may assume throughout that $h < 1$.

Let $\xi \in J(T)$, and $R > r > 0$ with $R/r \geq 10$. Let $\{B(y_i, r)\}_{i \in \mathcal{Y}}$ be a centred $r$-covering of $B(\xi, R) \cap J(T)$ of minimal cardinality. We assume for convenience that each $y_i \in J_r(T)$, which we may do since $J_r(T)$ is dense in $J(T)$. Each $y_i$ has a particular $\omega = \omega(i) \in \Omega$ associated with it, coming from the global measure formula for $m(B(x_i, r))$. In particular, $y_i$ belongs to an associated canonical ball $B(c(\omega), r_{c(\omega)})$.

Decompose $Y$ as $Y = Y_0 \cup Y_1$ where

$$Y_0 = \{ i \in Y \mid y_i \in B(c(\omega), r_{c(\omega)}) \text{ with } r_{c(\omega)} \geq 5R \}$$

$$Y_1 = Y \setminus Y_0.$$  

As $\{B(y_i, r)\}_{i \in \mathcal{Y}}$ is a covering of $B(\xi, R) \cap J(T)$, we have

$$m(B(\xi, R)) \leq m(\cup_{i \in \mathcal{Y}} B(y_i, r)) = m(\cup_{i \in Y_0} B(y_i, r)) + m(\cup_{i \in Y_1} B(y_i, r))$$

and therefore one of the terms in (4.12) must be at least $(m(\xi, R))/2$.

Suppose that the term involving $Y_0$ is at least $(m(\xi, R))/2$. Then we write

$$Y_0^0 = \{ i \in Y_0 \mid \phi(y_i, r) \leq K\phi(\xi, R) \}$$

where $K > 0$ is a constant chosen according to the following lemma.

**Lemma 4.3.** We may choose $K > 0$ independently of $R$ and $r$ sufficiently large such that

$$\frac{m(\cup_{i \in Y_0 \setminus Y_0^0} B(y_i, r))}{m(B(\xi, R))} \leq \frac{1}{100}.$$  

**Proof.** Write $Y(c(\omega))$ to denote the set of all $i \in Y_0 \setminus Y_0^0$ such that

$$B(y_i, r) \subseteq B(c(\omega), C\phi(y_i, r)(\frac{1}{(n-1)p(\omega)} r_{c(\omega)}) \subseteq B(c(\omega), C(K\phi(\xi, R))((\frac{1}{(n-1)p(\omega)} r_{c(\omega)})$$

using the definition of $Y_0 \setminus Y_0^0$ and where $C \geq 1$ is the constant from Lemma 4.2. By Lemma 4.2 all $i \in Y_0 \setminus Y_0^0$ belong to some $Y(c(\omega))$. Consider non-empty $Y(c(\omega))$. Since $r_{c(\omega)} \geq 5R$ we may follow the proof of (4.8) to show that, provided $Y(c(\omega)) \neq \emptyset$, $\phi(\xi, aR)$ is defined via $c(\omega)$ in the context of Theorem 2.1 for some $\alpha \approx 1$ and that $K$ can be chosen large enough such that $|c(\omega) - \xi| \leq R$. Then, since $m$ is doubling, by Theorem 2.1

$$R^h \phi(\xi, R) \approx m(B(\xi, R)) \approx m(B(c(\omega), R)) \approx R^h \phi(c(\omega), R) \approx R^h \left( \frac{R}{r_{c(\omega)}} \right)^{(\frac{1}{n-1})p(\omega)}$$

and so

$$r_{c(\omega)} \phi(\xi, R)^{(\frac{1}{n-1})p(\omega)} \approx R.$$  

(4.15)
Applying Theorem 2.1 and (4.14),
\[
\frac{m \left( \bigcup_{i \in Y(y, c)} B(y, r) \right)}{m(B(x, R))} \leq \left( \frac{K \phi(\xi, R)}{\phi(\xi, R)} \right)^{1/(n-1)p(\omega)} \frac{r^h}{c(\omega)}
\]
\[
= K^{1+ \frac{h}{(n-1)p(\omega)}} \left( \frac{\phi(\xi, R)}{R} \right)^{\frac{h}{p(\omega)}} \lesssim K^{1+ \frac{h}{(n-1)p(\omega)}}
\]
by (4.15). Note that the number of distinct squeezed canonical balls giving rise to non-empty \( Y(c, \omega) \) is \( \lesssim 1 \). This is because \( \phi(\xi, aR) \) is defined via \( c(\omega) \) for some \( a \approx 1 \) and any such \( c(\omega) \). Using the general bound \( h > p_{\text{max}}/(1 + p_{\text{max}}) \), we see \( 1 + h/(h - 1)p(\omega) \) < 0, and therefore we may choose \( K \) large enough to ensure \( (4.13) \).

Applying (4.13), we have
\[
m \left( \bigcup_{i \in Y_0^\omega} B(y, r) \right) \approx m \left( \bigcup_{i \in Y_0} B(y, r) \right) \geq m(B(x, R))/2
\]
which gives
\[
R^h \phi(x, R) \lesssim m(B(x, R)) \lesssim m \left( \bigcup_{i \in Y_0^\omega} B(y, r) \right) \lesssim |Y_0^\omega| R^h \phi(x, R)
\]
where the last inequality uses the definition of \( Y_0^\omega \). Therefore
\[
|Y_0| \geq |Y_0^\omega| \gtrsim \left( \frac{R}{r} \right)^h.
\]
(4.16)

Now, suppose that the second term of (4.12) is at least \( m(B(x, R))/2 \). Let \( \varepsilon > 0 \) and write
\[
Y_1 = \left\{ i \in Y_1 \mid \phi(y, r) \leq \left( \frac{R}{r} \right)^{\varepsilon} \right\}.
\]
If \( i \in Y_1 \setminus Y_0^\omega \), then this implies that \( \phi(y, r) > (R/r)^{\varepsilon} \), and therefore by Lemma 4.2 the ball \( B(y, r) \) is contained in the squeezed canonical ball
\[
B \left( c(\omega), C \left( \frac{R}{r} \right)^{\frac{\varepsilon}{(n-1)p(\omega)}} r(\omega) \right)
\]
for some \( c(\omega) \in J_p(T) \). Therefore, recalling the definition of \( Y_1 \), \( r/C \leq r(\omega) < 5R < 6CR \) and, using \( h < 1 \),
\[
|c(\omega) - \xi| \leq |c(\omega) - y| + |y - \xi| \leq C r(\omega) + R \leq 5CR + R \leq 6CR
\]
and so \( c(\omega) \in B(\xi, 6CR) \). Therefore
\[
m \left( \bigcup_{i \in Y_1 \setminus Y_0^\omega} B(y, r) \right) \lesssim \sum_{c(\omega) \in J_p(T) \cap B(\xi, 6CR)} m \left( B(c(\omega), C \left( \frac{R}{r} \right)^{\frac{\varepsilon}{(n-1)p(\omega)}} r(\omega) \right)\]
\[
\lesssim \sum_{c(\omega) \in J_p(T) \cap B(\xi, 6CR)} \left( \frac{R}{r} \right)^{\frac{\varepsilon}{(n-1)p(\omega)}} r(\omega)^h \phi \left( c(\omega), \left( \frac{R}{r} \right)^{\frac{\varepsilon}{(n-1)p(\omega)}} r(\omega) \right)
\]

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\[ \sum_{c(\omega) \in J_p(T) \cap B(\xi, 6 C R)} m(\xi, R) \leq (\log(R/r) + \log(6C^2)) \left( \frac{R}{r} \right)^{\varepsilon h p_{\max} + 2} \]

where the last inequality uses Lemma 4.1 and the fact that \( m \) is doubling. Note that the exponent of the term involving \( R/r \) is negative, recalling that \( h > p_{\max}/(1 + p_{\max}) \), so for sufficiently large \( R/r \), balls with centres in \( Y_1 \setminus Y_1^0 \) cannot carry a fixed proportion of \( m(B(\xi, R)) \), and so

\[ m \left( \bigcup_{i \in Y_1^0} B(y_i, r) \right) \approx m \left( \bigcup_{i \in Y_1} B(y_i, r) \right) \geq m(B(\xi, R))/2. \]

Therefore, we have

\[ R^h \phi(\xi, R) \lesssim m(B(\xi, R)) \approx m \left( \bigcup_{i \in Y_1^0} B(y_i, r) \right) \lesssim |Y_1^0| |r^h \left( \frac{R}{r} \right)^{\varepsilon}. \]

where the last inequality uses the definition of \( Y_1^0 \). Therefore

\[ |Y_1| \approx |Y_1^0| \gtrsim \left( \frac{R}{r} \right)^{h-\varepsilon} \phi(\xi, R) \gtrsim \left( \frac{R}{r} \right)^{h-\varepsilon}. \]  \hspace{1cm} (4.17)

We have proven that at least one of (4.16) and (4.17) must hold, and therefore

\[ |Y| = |Y_0| + |Y_1| \gtrsim \left( \frac{R}{r} \right)^{h-\varepsilon} \]

which proves that \( \dim_L J(T) \geq h - \varepsilon \), and letting \( \varepsilon \to 0 \) proves the desired result.

### 4.9 The Assouad spectrum of \( J(T) \)

#### 4.9.1 When \( h < 1 \)

We show

\[ \dim^\theta_A J(T) = h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h). \]

The upper bound follows from

\[ \dim^\theta_A J(T) \leq \dim^\theta_A m \leq h + \min \left\{ 1, \frac{\theta p_{\max}}{1 - \theta} \right\} (1 - h) \]

and for the lower bound, we can apply the following proposition (see [14, Theorem 3.4.8]).

**Proposition 4.4.** Let \( F \subseteq \mathbb{R}^n \) and suppose that

\[ \rho = \inf \{ \theta \in (0, 1) \mid \dim^\theta_A F = \dim_A F \} \]

exists and \( \rho \in (0, 1) \) and \( \dim_L F = \dim_B F \). Then for \( \theta \in (0, \rho) \),

\[ \dim^\theta_A F \geq \dim_B F + \frac{(1 - \rho)\theta}{(1 - \theta)\rho} (\dim_A F - \dim_B F). \]
As $h < 1$, we have $\dim L J(T) = \dim B J(T) = h$ and the upper bound for $\dim^0 A J(T)$ shows that $\rho \geq 1/(1 + p_{\text{max}})$, so we need only show that $\rho \leq 1(1 + p_{\text{max}})$. To do this, let $\omega \in \Omega$ such that $p(\omega) = p_{\text{max}}$, and recall that there exists some sufficiently small neighbourhood $U_{\omega} = B(\omega, r_{\omega})$ such that there exists a unique holomorphic inverse branch $T^{-1}_\omega$ of $T$ on $U_{\omega}$ such that $T^{-1}_\omega(\omega) = \omega$. Using $[10, \text{Lemma 1}]$, we have that for $\xi \in U_{\omega} \cap (J(T) \setminus \{\omega\})$ and $n \in \mathbb{N}$,

$$|T^{-n}_\omega(\xi) - \omega| \approx n^{-1} p_{\text{max}}$$

which, using $T$-invariance of $J(T)$ and bi-Lipschitz stability of the Assouad spectrum, implies that

$$\dim^0 A J(T) \geq \dim^0 A \{n^{-1/p_{\text{max}}}: n \in \mathbb{N}\} = \min \left\{ 1, \frac{p_{\text{max}}}{(1 + p_{\text{max}})(1 - \theta)} \right\}$$

by $[10, \text{Corollary 6.4}]$. This is enough to ensure $\rho \leq 1/(1 + p_{\text{max}})$. Therefore, by Proposition 4.4 for $\theta \in (0, 1/(1 + p_{\text{max}}))$ we have

$$\dim^0 A J(T) \geq h + \frac{(1 - 1/(1 + p_{\text{max}}))\theta}{(1 - \theta)/(1 + p_{\text{max}})(1 - h)} = h + \frac{\theta p_{\text{max}}}{1 - \theta}(1 - h)$$

as required.

### 4.9.2 When $h \geq 1$

We show $\dim^0 A J(T) = h$. This follows easily, since $h = \dim B J(T) \leq \dim^0 A J(T) \leq \dim A J(T) = h$.

### 4.10 The lower spectrum of $J(T)$

#### 4.10.1 When $h < 1$

We show $\dim^0 L J(T) = h$. This follows easily, since $h = \dim L J(T) \leq \dim^0 L J(T) \leq \dim B J(T) = h$.

#### 4.10.2 When $h \geq 1$

We show

$$\dim^0 L J(T) = h + \min \left\{ 1, \frac{\theta p_{\text{max}}}{1 - \theta} \right\} (1 - h).$$

Note that we have

$$\dim^0 L J(T) \geq \dim^0 L m \geq h + \min \left\{ 1, \frac{\theta p_{\text{max}}}{1 - \theta} \right\} (1 - h)$$

and so it suffices to prove the upper bound. To do this, we require the following technical lemma, which is a quantitative version of the Leau-Fatou flower theorem (see $[19, 325–363]$ and $[25]$). This was not known to us initially, but seems to be standard in the complex dynamics community. We thank Davoud Cheraghi for explaining this version to us. We note that the non-quantitative version, e.g. that stated in $[1, 9]$, is enough to bound the lower dimension from above, but not the lower spectrum.

**Lemma 4.5.** Let $\omega \in \Omega$ be a parabolic fixed point with petal number $p(\omega)$. Then there exists a constant $C > 0$ such that for all sufficiently small $r > 0$, $B(\omega, r) \cap J(T)$ is contained in a $Cr^{1+p(\omega)}$-neighbourhood of the set of $p(\omega)$ lines emanating from $\omega$ in the repelling directions.
Proof. We only sketch the proof. We may assume via standard reductions that $\omega = 0$ and that the repelling directions are $e^{n2\pi i/p(\omega)}$ for $n = 0, 1, \ldots, p(\omega) - 1$. By the (non-quantitative) Leau-Fatou flower theorem, $B(0, r) \cap J(T)$ is contained in a cuspidal neighbourhood of the repelling directions. Apply the coordinate transformation $z \mapsto 1/z^{p(\omega)}$ which sends the fixed point to infinity and the repelling directions to 1. The linearisation of the conjugated map at infinity is a (real) translation and this linearisation can be used to show that the Julia set is contained in a half-infinite horizontal strip of bounded height. The pre-image of this strip under the coordinate transformation consists of cuspidal neighbourhoods of the $p(\omega)^{th}$ roots of unity, and an easy calculation gives the desired result. \(\square\)

We can use our work on the lower spectrum of $m$ to show that the exponent used in Lemma 4.5 is sharp. Again, this seems to be well-known in the complex dynamics community (even a stronger form of sharpness than we give) but we mention it since we provide a new approach.

**Corollary 4.6.** In the case where $\omega \in \Omega$ is of maximal rank, the expression $Cr^{1+p(\omega)}$ in Lemma 4.5 cannot be replaced by $Cr^{1+p(\omega)+\varepsilon}$ for any $\varepsilon > 0$.

**Proof.** Suppose that such an $\varepsilon$-improvement was possible for some $\omega \in \Omega$ of maximal rank. Then, taking efficient $r$-covers of the improved cuspidal neighbourhood of $B(\omega, r^{1+p_{\text{max}}+\varepsilon}) \cap J(T)$, we would obtain $\dim_{r, \theta}J(T) \leq 1$ for all $\theta > 1/(1 + p_{\text{max}} + \varepsilon)$. This contradicts the lower bound for the lower spectrum of $m$, proved in Section 4.6.2. \(\square\)

We can now prove the upper bound. Note that when $\theta \geq 1/(1 + p_{\text{max}})$, we immediately have $\dim_{l, \theta}J(T) \leq 1$ by Lemma 4.5 so we assume that $\theta \in (0, 1/(1 + p_{\text{max}}))$. Let $\omega \in \Omega$ be such that $p(\omega) = p_{\text{max}}$, and let $r > 0$ be sufficiently small. Then we can estimate $N_r(B(\omega, r^\theta) \cap J(T))$ by first covering $B(\omega, r^\theta) \cap J(T)$ with balls of radius $r^{\theta(1+p_{\text{max}})}$, and then covering each of those balls with balls of radius $r$. Using the fact that $\dim_{A}J(T) = h$, we have

\[
N_r(B(\omega, r^\theta) \cap J(T)) \leq N_r\theta(1+p_{\text{max}})(B(\omega, r^\theta) \cap J(T)) \left(\frac{r^{\theta(1+p_{\text{max}})}}{r}\right)^h
\]

\[
\leq \frac{r^\theta}{\theta(1+p_{\text{max}})} \left(\frac{r^{\theta(1+p_{\text{max}})}}{r}\right)^h
\]

\[
= r^{-\theta p_{\text{max}} + h(\theta(1+p_{\text{max}}) - 1)} = \left(r^{\theta - 1}\right)^{h + \frac{\theta p_{\text{max}}}{\theta - 1}(h-1)}
\]

which proves that $\dim_{l, \theta}J(T) \leq h + \frac{\theta p_{\text{max}}}{\theta - 1}(1 - h)$ as required.

### 4.11 Proof of Theorem 3.3

We assume without loss of generality that 0 is the Cremer fixed point. Then

\[
T(z) = e^{2\pi i \alpha}z + O(z^2)
\]

in a neighbourhood of 0 for some $\alpha \notin \mathbb{Q}$. Without loss of generality we may assume (4.18) holds in $B(0, 1)$. It follows that for sufficiently small $z$ and $n \in \mathbb{N}$ such that \{z, T(z),\ldots, T^{n-1}(z)\} $\subseteq B(0, 1)$

\[
T^n(z) = e^{2\pi i n \alpha}z + 5^n O(z^2)
\]
where the implicit constants are the same as in (4.18), in particular, independent of \( n \) and \( z \). This can be proved by a simple induction. Let \( C \geq 1 \) denote the implicit constant from (4.18) and suppose \( |z| \leq 1/C \) and that \( \{ z, T(z), \ldots, T^{n-1}(z) \} \subseteq B(0, 1) \). Assume (4.19) has been verified for \( T^{n-1} \) (Inductive Hypothesis).

Then
\[
|T^n(z) - e^{2\pi in\alpha}z| \leq |T^{n-1}(T(z)) - e^{2\pi i(n-1)\alpha}T(z)| + |e^{2\pi i(n-1)\alpha}T(z) - e^{2\pi in\alpha}z|
\]
\[
\leq 5^nC|T(z)|^2 + |T(z) - e^{2\pi in\alpha}z| \quad \text{(by Inductive Hypothesis)}
\]
\[
\leq 5^nC \left( |z| + C|z|^2 \right)^2 + C|z|^2 \quad \text{(by (4.18))}
\]
\[
= C|z|^2 \left( 5^{n-1}(1+2C|z|+C|z|^2)+1 \right)
\]
\[
\leq 5^nC|z|^2 \quad \text{(using } C|z| \leq 1 \text{)}
\]
proving (4.19). For \( \delta \in (0, 1) \), we say a set \( A \subseteq \mathbb{R}^d \) is \( \delta \)-dense in a set \( B \subseteq \mathbb{R}^d \) if \( \sup_{b \in B} \inf_{a \in A} |a-b| \leq \delta \) and let
\[
M(\alpha, \delta) = \min \left\{ m : \{ 0, \alpha \text{(mod } 1), \ldots, m\alpha \text{(mod } 1) \} \text{ is } \delta \text{-dense in } [0,1) \right\}.
\]
Note that \( M(\alpha, \delta) \) is finite for all \( \delta \in (0,1) \) since \( \alpha \notin \mathbb{Q} \) and that \( M(\alpha, \delta) \to \infty \) as \( \delta \to 0 \).

To prove that \( \dim_A J(T) = 2 \), we show there exist \( 0 < r < R \) with \( r/R \) arbitrarily small such that \( J(T) \) is \( (cr) \)-dense in \( B(0, R) \) for some uniform constant \( c \geq 1 \). Let \( R \in (0, 1) \) be small and choose \( r \) depending on \( R \) such that \( r/R \to 0 \) sufficiently slowly as \( R \to 0 \) to ensure
\[
5^{M(\alpha, r/R)}R^2 \leq r.
\]
(4.20)
Note that this forces \( r/R^2 \to \infty \). Let \( y \in B(0, R) \) be arbitrary and choose \( z \in J(T) \) with \( |z| = |y| \). We can choose \( z \) in this way for sufficiently small \( R \) using Pérez-Marco’s celebrated result that Cremer fixed points are contained in a non-trivial connected component of the Julia set \( [26, \text{ Theorem } 1] \). By definition we can find \( n \in \mathbb{Z} \) with \( 1 \leq n \leq M(\alpha, r/|y|) \leq M(\alpha, r/R) \) such that
\[
|\arg(z) + 2\pi n\alpha)\mod 2\pi - \arg(y)| \leq 2\pi r/|y|.
\]
(4.21)
Using (4.20),
\[
|z| + 5^{M(\alpha, r/R)}C|z|^2 \leq R + 5^{M(\alpha, r/R)}CR^2 \leq R + Cr \leq 1
\]
for sufficiently small \( R \) and so \( \{ z, T(z), \ldots, T^{n-1}(z) \} \subseteq B(0, 1) \) by iteratively applying (4.19). Therefore, by (4.19), (4.20), and (4.21)
\[
|\arg(T^n(z)) - \arg(y)| \leq |\arg(z) + 2\pi n\alpha)\mod 2\pi - \arg(y)| + 5^n|z|^2 \lesssim r/|y| + 5^{M(\alpha, r/R)}R^2 \lesssim r/|y|.
\]
(4.22)
Moreover, by (4.19) and (4.20),
\[
|T^n(z) - |y|| = ||T^n(z)| - |z|| \lesssim 5^n|z|^2 \lesssim 5^{M(\alpha, r/R)}R^2 \leq r.
\]
(4.23)
Together (4.22) and (4.23) yield
\[
|T^n(z) - y| \lesssim r
\]
and since the Julia set is \( T \)-invariant the result follows.

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