The Poisson bracket compatible with the classical reflection equation algebra

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We introduce a family of compatible Poisson brackets on the space of $2 \times 2$ polynomial matrices, which contains the reflection equation algebra bracket. Then we use it to derive a multi-Hamiltonian structure for a set of integrable systems that includes the XXX Heisenberg magnet with boundary conditions, the generalized Toda lattices and the Kowalevski top.

1 Introduction

In this paper we study a class of finite-dimensional Liouville integrable systems described by the representations of the quadratic $r$-matrix Poisson algebra:

$$\{1^T(\lambda), 2^T(\mu)\}_0 = [r(\lambda - \mu), 1^T(\lambda)2^T(\mu)]_0 + \frac{1}{2} T(\lambda)r(\lambda + \mu)T(\mu) - \frac{2}{2} T(\mu)r(\lambda + \mu)T(\lambda),$$

where $1^T(\lambda) = T(\lambda) \otimes I$, $2^T(\mu) = I \otimes T(\mu)$ and $r(\lambda, \mu)$ is a classical $r$-matrix.

The reflection equation algebra (1.1) appeared in the quantum inverse scattering method [13]. Its representations play an important role in the classification and studies of classical integrable systems (see, for instance, [4, 7, 8, 13] and references therein).

The main result of this paper is construction of the Poisson brackets $\{\ldots\}_1$ compatible with the bracket $\{\ldots\}_0$ [11] in the simplest case of the $4 \times 4$ rational $r$-matrix

$$r(\lambda - \mu) = \frac{-\eta}{\lambda - \mu} \Pi, \quad \Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \eta \in \mathbb{C},$$

and $2 \times 2$ matrix $T(\lambda)$, which depends polynomially on the parameter $\lambda$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & A(-\lambda) \end{pmatrix}, \quad \deg T(\lambda) = \begin{pmatrix} 2n + 1 & 2n + 1 \\ 2n - 1 & 2n + 1 \end{pmatrix}. \quad (1.3)$$

Coefficients of the entries

$$A(\lambda) = \alpha \lambda^{2n+1} + A_{2n} \lambda^{2n} + A_{2n-1} \lambda^{2n-1} \ldots + A_0,$$

$$B(\lambda) = \lambda^{2n+1} + B_n \lambda^{2n-1} + B_{n-1} \lambda^{2n-3} \ldots + B_1 \lambda,$$

$$C(\lambda) = C_n \lambda^{2n-1} + \ldots + C_2 \lambda^3 + C_1 \lambda,$$

are generators of the quadratic Poisson algebra [11]. The leading coefficient $\alpha$ and $2n + 1$ coefficients of the det $T(\lambda)$

$$d(\lambda) = \det T(\lambda) = Q_{2n} \lambda^{4n} + Q_{2n-1} \lambda^{4n-2} + \ldots + Q_0.$$
are Casimirs of the bracket \(^{(1.1)}\). Therefore, we have a \(4n + 1\)-dimensional space of the coefficients

\[
A_0, \ldots, A_{2n}, B_1, \ldots, B_n, \ C_1, \ldots, C_n
\]

with \(2n + 1\) Casimir operators \(Q_i\), leaving us with \(n\) degrees of freedom.

The use of the algebra \[(1.1)\] for the theory of the integrable systems is based on the following construction of commutative subalgebras \[13\ [14]\]. Let us introduce boundary matrix

\[
K(\lambda) = \begin{pmatrix}
A(\lambda) & 0 \\
C(\lambda) & D(\lambda)
\end{pmatrix}
\]

where entries \(A(\lambda), D(\lambda)\) are polynomials with numerical coefficients and \(C(\lambda)\) is arbitrary polynomial on \(\lambda\). If the polynomial

\[
\tau(\lambda) = \text{tr}K(\lambda)T(\lambda) = \sum \tau_k \lambda^k
\]

has \(n\) independent dynamical coefficients \(H_1, \ldots, H_n\) only, then

\[
\{\tau(\lambda), \tau(\mu)\}_0 = 0, \quad \Rightarrow \quad \{H_i, H_j\}_0 = 0, \quad i, j = 1, \ldots, n.
\]

These Poisson involutive integrals of motion \(H_i\) define the Liouville integrable systems, which are our generic models for the whole paper.

2 The compatible bracket

The Poisson brackets \({\ldots}_0\) and \({\ldots}_1\) are compatible if every linear combination of them is still a Poisson bracket \[3\ [11]\].

**Proposition 1** The bracket \((1.1)\) belongs to the following family of compatible Poisson brackets:

\[
\{B(\lambda), B(\mu)\}_k = 0, \quad k = 0, 1,
\]

\[
\{A(\lambda), A(\mu)\}_k = \frac{n}{\lambda + \mu} (\mu^{2k} B(\lambda)C(\mu) - \lambda^{2k} B(\mu)C(\lambda)),
\]

\[
\{C(\lambda), C(\mu)\}_k = 2\eta (\rho_k(\mu)A(\lambda) - \rho_k(\lambda)A(\mu))C(\lambda)
\]

\[
-2\eta (\rho_k(\mu)A(\lambda) - \rho_k(\lambda)A(\mu))C(\mu),
\]

\[
\{A(\lambda), B(\mu)\}_k = \frac{n}{\lambda + \mu} (\lambda^{2k} A(\lambda)B(\mu) - \mu^{2k} A(\mu)B(\lambda))
\]

\[
+ \frac{n}{\lambda + \mu} (\lambda^{2k} A(\lambda)B(\mu) + \mu^{2k} A(\mu)B(\lambda)) - 2\eta \rho_k(\lambda)B(\lambda)B(\mu),
\]

\[
\{A(\lambda), C(\mu)\}_k = - \frac{n}{\lambda + \mu} (\mu^{2k} A(\lambda)C(\mu) - \lambda^{2k} A(\mu)C(\lambda)) - 2\eta k A(\lambda)C(\mu)
\]

\[
- \frac{n}{\lambda + \mu} (\mu^{2k} A(\lambda)C(\mu) + \lambda^{2k} A(\mu)C(\lambda)) + 2\eta \rho_k(\lambda)B(\lambda)C(\mu),
\]

\[
\{B(\lambda), C(\mu)\}_k = \frac{n\lambda^{2k}}{\lambda - \mu} \left( A(-\lambda)A(\mu) - A(\lambda)A(-\mu) \right)
\]

\[
+ \frac{n\lambda^{2k}}{\lambda + \mu} \left( A(\lambda)A(\mu) - A(-\lambda)A(-\mu) \right)
\]

\[
+ 2\eta (\rho_k(\mu)A(-\mu) - \rho_k(-\mu)A(\mu))B(\lambda).
\]
Here $k = 0, 1$ and $\rho_k = \left[ \frac{\lambda^{2k+1}A(\lambda)}{B(\lambda)} \right]$ is the quotient of polynomials in variable $\lambda$ over a field, such that
\[ \rho_0 = 0, \quad \text{and} \quad \rho_1 = \alpha \lambda + A_2n. \]

**Proof:** It is sufficient to check the statement on an open dense subset of the reflection equation algebra defined by the assumption that $A(\lambda)$ and $B(\lambda)$ are co-prime and all double roots of $B(\lambda)$ are distinct.

This assumption allows us to construct a separation representation for the reflection equation algebra $(1.1)$. In this special representation one has $n$ pairs of Darboux variables, $\lambda_i, \mu_i, i = 1, \ldots, n$, having the standard Poisson brackets,
\[ \{\lambda_i, \lambda_j\}_0 = \{\mu_i, \mu_j\}_0 = 0, \quad \{\lambda_i, \mu_j\}_0 = \delta_{ij}, \quad (2.2) \]
with the $\lambda$-variables being $n$ zeros of the polynomial $B(\lambda)$ and the $\mu$-variables being values of the polynomial $A(\lambda)$ at those zeros,
\[ B(\pm \lambda_i) = 0, \quad \mu_i = \eta^{-1} \ln A(\lambda_i), \quad i = 1, \ldots, n. \quad (2.3) \]

The interpolation data $(2.3)$ plus $n + 2$ identities
\[ A(\lambda_i)A(-\lambda_i) = -d(\lambda_i), \quad A(0) = \sqrt{Q_0}, \quad A = \alpha \lambda^{2n+1} + \ldots, \]
allow us to construct the needed separation representation for the whole algebra:
\[ B(\lambda) = \lambda \prod_{k=1}^{n} (\lambda^2 - \lambda_k^2), \tag{2.4} \]
\[ A(\lambda) = \left( \frac{\lambda - \eta^{-1} \ln A(\lambda)}{\prod \lambda_k^2} \right) \prod_{k=1}^{n} (\lambda^2 - \lambda_k^2) + \sum_{k=1}^{n} \left[ \frac{\lambda(\lambda - \lambda_k)e^{\eta \mu_k}}{2\lambda_k^2} + \frac{\lambda(\lambda + \lambda_k)e^{-\eta \mu_k}}{2\lambda_k^2} \right], \]
\[ C(\lambda) = \frac{A(\lambda)A(-\lambda) - d(\lambda)}{B(\lambda)}. \]

Using this representation we can easy calculate the bracket $\{.,.\} (2.1)$ in $(\lambda, \mu)$-variables
\[ \{\lambda_i, \lambda_j\}_1 = \{\mu_i, \mu_j\}_1 = 0, \quad \{\lambda_i, \mu_j\}_1 = \lambda_i^2 \delta_{ij}, \quad (2.5) \]
In order to complete the proof we have to check that brackets $(2.5)$ is compatible with the canonical brackets $(2.2)$. The compatibility of the brackets $(2.2), (2.5)$ implies the compatibility of the brackets $(1.1), (2.4)$ and vice versa. This completes the proof.

**Remark 1** The coefficients of the determinant $d(\lambda)$ $(1.5)$ are the Casimir functions for the both brackets $\{.,.\}_0$ and $\{.,.\}_1$. It means that the Poisson bracket $\{.,.\}_1$ has the same foliation by symplectic leaves as $\{.,.\}_0$.

**Proposition 2** The brackets $(2.1)$ may be rewritten in the following $r$-matrix form
\[ \frac{1}{T(\lambda)} \left[ T^{(k)}(\lambda, \mu) \right]_k = r^{(k)}_{12}(\lambda, \mu) \frac{1}{T(\mu)}T^{(k)}(\mu) - \frac{1}{T(\mu)}T^{(k)}(\lambda) r^{(k)}_{21}(\lambda, \mu) - \frac{1}{T(\lambda)} \left( \frac{1}{T(\mu)}r^{(k)}_{12}(\lambda, \mu) \right) \left( \frac{2}{T(\mu)}s^{(k)}_{12}(\lambda, \mu) \right) \left( \frac{1}{T(\lambda)} T^{(k)}(\mu) \right), \tag{2.6} \]
\[ + \frac{1}{T(\lambda)} s^{(k)}_{12}(\lambda, \mu) \frac{1}{T(\mu)} - \frac{1}{T(\mu)} s^{(k)}_{21}(\lambda, \mu) \frac{1}{T(\lambda)} T^{(k)}(\mu) \]
where

\[
\begin{align*}
  r_{12}^{(k)}(\lambda, \mu) &= -\eta \begin{pmatrix}
    \lambda^{2k+1} - 2k^2 - 1 & 0 & 0 & 0 \\
    0 & 0 & \lambda^{2k} - 1 & 0 \\
    0 & \lambda^{2k} & 0 & 0 \\
    0 & 2\rho_k(\lambda) - 2\rho_k(\mu) & \lambda^{2k+1} - 2k^2 + 1 & 0
  \end{pmatrix}, \\
  r_{21}^{(k)}(\lambda, \mu) &= -\eta \begin{pmatrix}
    \lambda^{2k+1} - 2k^2 - 1 & 0 & 0 & 0 \\
    0 & -2\rho_k(\lambda) & 0 & \lambda^{2k} - 1 \\
    2\rho_k(\mu) & 0 & \lambda^{2k} & 0 \\
    0 & \lambda^{2k+1} - 2k^2 + 1 & 0 & 0
  \end{pmatrix}, \\
  s_{12}^{(k)}(\lambda, \mu) &= -\eta \begin{pmatrix}
    \lambda^{2k+1} - 2k^2 - 1 & 0 & 0 & 0 \\
    0 & -2\rho_k(\lambda) & 0 & \lambda^{2k} - 1 \\
    0 & \lambda^{2k} & 0 & 0 \\
    0 & 0 & \lambda^{2k+1} - 2k^2 + 1 & 0
  \end{pmatrix}, \\
  s_{21}^{(k)}(\lambda, \mu) &= \eta^2 \begin{pmatrix}
    \lambda^{2k+1} - 2k^2 - 1 & 0 & 0 & 0 \\
    0 & -2\rho_k(\lambda) & 0 & \lambda^{2k} - 1 \\
    0 & \lambda^{2k} & 0 & 0 \\
    0 & 0 & \lambda^{2k+1} - 2k^2 + 1 & 0
  \end{pmatrix}
\end{align*}
\]

and

\[
s_{21}^{(k)}(\lambda, \mu) = \Pi s_{12}^{(k)}(\mu, \lambda)\Pi.
\]

The proof consists of the straightforward calculations.

Using the separated representation (2.7) we can rewrite the higher order Poisson brackets

\[
\{\lambda_i, \lambda_j\}_k = \{\mu_i, \mu_j\}_k = 0, \quad \{\lambda_i, \mu_j\}_k = \lambda_i^{2k} \delta_{ij}, \quad k = 0, \ldots, n
\]

(2.8)

at the r-matrix form (2.6). As a result we obtain a family of the Poisson brackets compatible with the bracket (1.1). For the Sklyanin r-matrix algebra such family of the brackets has been constructed in [17].

**Proposition 3** Integrals of motion \(H_i\) from \(\text{tr} K(\lambda) T(\lambda)\) are in the bi-involution

\[
\{H_i, H_j\}_0 = \{H_i, H_j\}_1 = 0,
\]

with respect to the brackets (2.7) or (2.6).

**Proof:** According to [13] variables \(\lambda_i, \mu_i\) (2.4) are the separated coordinates for the coefficients \(H_i\) of the polynomial \(\tau = \text{tr} K(\lambda) T(\lambda)\) and the separated relations look like

\[
\text{tr} K(\lambda) T(\lambda) = \sum \eta_k \lambda_i^k = A(\lambda_i) e^{\eta_i} + D(\lambda_i) e^{-\eta_i}, \quad i = 1, \ldots, n.
\]

Remind that \(A(\lambda), D(\lambda)\) are numerical polynomials and among all the coefficients \(\tau_k\) we have only \(n\) integrals of motion \(H_i\).

On the other hand from (2.5) follows that \(\lambda_i, \mu_i\) are the Darboux-Nijenhuis variables for the brackets \(\{\ldots\}_0, 1\) (2.4). So, integrals of motion \(H_i\) are in the bi-involution with respect to the brackets (2.7) or (2.6) according to the Theorem 3.2 from [3]. This completes the proof.

Summing up, we have proved a bi-involution of the integrals of motion \(H_i\) using the Darboux-Nijenhuis variables \(\lambda_i, \mu_i\) and the separation representation (2.4) for the reflection equation algebra.

### 3 The Heisenberg magnet

Another important representations of the quadratic Poisson algebra with the generators \(A_i, B_i, C_i\) comes as a consequence of the co-multiplication property of the reflection equation algebra (1.4). Essentially, it means that the matrix \(T(\lambda)\) (1.3) can be factorized into a product of elementary matrices, each containing only one degree of freedom [13]. In this picture, our main
model turns out to be an \( n \)-site Heisenberg magnet with boundary conditions, which is an integrable lattice of \( n \) \( \text{sl}(2) \) spins with nearest neighbor interaction.

According to [13] the \( 2 \times 2 \) Lax matrix for the generalized Heisenberg magnet acquires the form

\[
T = K_+(\lambda)T(\lambda), \quad \text{where} \quad K_+ = \begin{pmatrix} b_1 \lambda + b_0 & 0 \\ -b_1 \lambda + b_0 \end{pmatrix},
\]

and matrix

\[
T(\lambda) = \left( \prod_{m=1}^{n} L_m(\lambda) \right) K_-(-\lambda) \left( \prod_{m=1}^{n} L_m(-\lambda) \right)^{-1}, \quad \text{with}
\]

\[
L_m(\lambda) = \begin{pmatrix} \lambda - s_{3}^{(m)} & s_{1}^{(m)} + is_{2}^{(m)} \\ s_{1}^{(m)} - is_{2}^{(m)} & \lambda + s_{3}^{(m)} \end{pmatrix}, \quad K_-(\lambda) = \begin{pmatrix} a_1 \lambda + a_0 & \lambda \\ 0 & -a_1 \lambda + a_0 \end{pmatrix},
\]

satisfies to the reflection equation algebra at \( \alpha = a_1 \) and \( \eta = -i \). Here \( a_0, b_0, c_m \) are arbitrary numbers, \( i^2 = -1 \) and \( s_{3}^{(m)} \) are dynamical variables on the direct sum of \( \text{sl}(2) \)

\[
\left\{ e_i^{(m)}, e_j^{(m)} \right\}_1 = \varepsilon_{ijk} s_k^{(m)},
\]

where \( \varepsilon_{ijk} \) is the totally skew-symmetric tensor.

Substituting matrix \( T(\lambda) \) into the brackets \( \{,\} \) one gets the overdetermined system of algebraic equations on the elements of the Poisson bivector \( P_1 \) associated with the Poisson brackets \( \{,\} \):

\[
\{ f(z), g(z) \}_1 = (df, P_1 dg) = \sum_{i,k} P_i^{jk}(z) \frac{\partial f(z)}{\partial z_i} \frac{\partial g(z)}{\partial z_k},
\]

where \( z = (z_1, \ldots, z_m) \) are coordinates on the Poisson manifold. In our case \( z \) consist of coordinates \( s_i^{(m)} \) on \( n \) copies of \( \text{sl}(2) \). Solving this system of equations we obtain the second cubic bracket \( \{,\}_1 \) compatible with \( \{,\} \). As an example the local brackets look like

\[
\left\{ s_i^{(m)}, s_j^{(m)} \right\}_1 = \varepsilon_{ijk} s_k^{(m)} \left( (s_3^{(m)} + c_m)^2 - 2a_1(s_1^{(m)} + is_2^{(m)})(s_3^{(m)} + c_m) - 2a_0(s_1^{(m)} + is_2^{(m)}) \right).
\]

For the sake of brevity we omit the explicit form of the nonlocal cubic brackets \( \left\{ e_i^{(m)}, e_j^{(m)} \right\}_1 \) in this paper.

## 4 The generalized Toda lattices

The Toda lattices appear as another specialization of our basic model. Let us consider generalized open Toda lattices with the Hamiltonians

\[
H_g = \sum_{i=1}^{n} p_i^2 + 2 \sum_{i=1}^{n-1} \epsilon^{q_{i-1}q_i} + V_g(q), \quad \text{where}
\]

\[
\begin{cases}
V \theta = 2a_0 \epsilon^\theta_n \\
V \varphi = a_1^2 \epsilon^{2\theta_n} \\
V \phi = 2a_2 \epsilon^{\theta_{n-1}+\theta_n}
\end{cases}
\]

These Toda lattices are associated with the root systems \( B_n, C_n \) and \( D_n \).

According to [13, 14] the \( 2 \times 2 \) Lax matrix for the generalized open Toda lattice acquires the form

\[
T_{\text{open}} = K_+(\lambda)T(\lambda), \quad \text{where} \quad K_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
and matrix
\[
T(\lambda) = \left( \prod_{k=1}^{n} L_k(\lambda) \right) K_-(\lambda) \left( \prod_{k=1}^{n} L_k(-\lambda) \right)^{-1}, \tag{4.3}
\]
with
\[
L_i = \begin{pmatrix} \lambda - p_i & -e^{q_i} \\ e^{-q_i} & 0 \end{pmatrix}, \quad K_-(\lambda) = \begin{pmatrix} 2a_2\lambda^2 - ia_1\lambda + a_0 & (4a_2e^{\eta_n} + 1)\lambda \\ 0 & 2a_2\lambda^2 + ia_1\lambda + a_0 \end{pmatrix},
\]
satisfies to the reflection equation algebra at \(\alpha = 0\) and \(\eta = 1\). Here \(p_i, q_i\) are dynamical variables and \(a_k\) are parameters.

The polynomial
\[
\text{tr} T_{\text{open}} = \lambda^{2n+1} + \sum_{i=1}^{n} H_i \lambda^{2(n-i)}
\]
is a generating function of independent integrals of motion \(H_i\). The first integral \(H_1\) coincides with one of the the Hamiltonians \((4.1)\), but for the Toda lattices of \(C_n\) and \(D_n\) type we have to change variables
\[
p_n \to p_n - ia_1e^{q_n}. \tag{4.4}
\]
and
\[
p_n \to -p_n \cosh(q_n) + 1, \quad q_n \to -\ln(-2a_2 \cosh(q_n + \ln(-a_2)) + 1). \tag{4.5}
\]
respectively.

Substituting matrix \(T(\lambda)\) \((4.3)\) into the brackets \(\{\ldots\}_k\) \((2.1)\) one can rewrite the Poisson brackets \(\{\ldots\}_k\) in \((p, q)\) variables. Of course, at \(k = 0\) we obtain canonical bracket
\[
\{q_i, q_j\}_0 = \{p_i, p_j\}_0 = 0, \quad \{q_i, p_j\}_0 = \delta_{ij}.
\]
For the Toda lattice associated with \(B\) \(C_n\) root system after canonical transformation \((4.3)\) one get the following non-zero brackets
\[
i < j \quad \{q_i, q_j\}_1 = 2p_i,
\]
\[
i = 1, \ldots, n - 1 \quad \{p_i, p_{i+1}\}_1 = -(p_i + p_{i+1})e^{q_{i} - q_{i+1}}, \quad \{q_{i+1}, p_i\}_1 = e^{q_{i+1} - q_{i}},
\]
\[
i = 1, \ldots, n - 2 \quad \{q_i, p_{i+1}\}_1 = 2e^{q_{i+1} - q_i} - e^{q_{i+1} - q_i + 1},
\]
\[
\{p_n, q_i\}_1 = 2e^{q_{n-1} - q_n} - 2a_0 e^{q_n} - 2a_1^2 e^{2q_n},
\]
\[
i = 3, \ldots, n - 1 \quad \{p_i, q_j\}_1 = 2e^{q_{i+1} - q_i} - 2e^{q_{i+1} - q_i + 1}, \quad 1 \leq j \leq i - 2,
\]
and
\[
\{q_n, p_n\}_1 = p_n^2 + 2a_0 e^{q_n} + a_1^2 e^{2q_n}, \quad \{q_{n-1}, p_n\}_1 = 2a_0 e^{q_n} + 2a_1^2 e^{2q_n} - e^{q_{n-1} - q_n}.
\]
For the Toda lattice associated with \(D_n\) root system after canonical transformation \((4.3)\) one
gets the following non-zero brackets

\[
\begin{align*}
&i < j \quad \{q_i, q_j\}_1 = 2p_i, \\
&i = 1, \ldots, n - 2 \quad \{p_i, p_{i+1}\}_1 = -(p_i + p_{i+1}) e^{q_i - q_{i+1}}, \quad \{q_{i+1}, p_i\}_1 = e^{q_i - q_{i+1}}, \\
&\{q_i, p_i\}_1 = p_i^2 + 2e^{q_i - q_{i+1}}, \\
&\{p_n, q_i\}_1 = 2e^{q_{n-1} - q_n - 2a_2^2e^{q_{n-1} - q_n}}, \\
&i = 1, \ldots, n - 3 \quad \{p_{n-1}, q_i\}_1 = 2e^{q_{n-2} - q_{n-1} - 2e^{q_{n-1} - q_n} - 2a_2^2e^{q_{n-1} - q_n}}, \\
&i = 3, \ldots, n - 1 \quad \{p_i, q_j\}_1 = 2e^{q_i - q_{i+1} - 2e^{q_{i+1} - q_j}}, \quad 1 \leq j \leq i - 2,
\end{align*}
\]

and

\[
\begin{align*}
&\{q_n, p_n\}_1 = p_n^2, \quad \{q_{n-2}, p_{n-1}\}_1 = -2e^{q_{n-2} - q_{n-1}} + 2e^{q_{n-1} - q_n} + 2a_2^2e^{q_{n-1} - q_n}, \\
&\{q_{n-1}, p_{n-1}\}_1 = p_{n-1}^2 - 1 + 2e^{q_{n-1} - q_n} + 2a_2^2e^{q_{n-1} - q_n}, \\
&\{q_{n-2}, p_{n-2}\}_1 = -2e^{q_{n-2} - q_{n-1}} + 2e^{q_{n-1} - q_n} + 2a_2^2e^{q_{n-1} - q_n}, \\
&\{p_{n-1}, p_{n-1}\}_1 = -(p_{n-1} + p_n)e^{q_{n-1} - q_n} + (p_{n-1} - p_n)a_2^2e^{q_{n-1} - q_n}.
\end{align*}
\]

So, using bracket (2.1) at \(k = 1\) we recovered all the known second brackets for the \(BC_n\) and \(D_n\) Toda lattices [2].

**Remark 2** According to [9, 14] if \(a_k \neq 0\) then after more complicated canonical transformation of \(p_n\) and \(q_n\) we can describe generalized Toda lattice with the following potential

\[
V_g = 2a_2^2e^{q_{n-1} - q_n} + \frac{a_1}{\sinh^2 q_n} + \frac{2a_0}{\sinh^2 (q_n/2)},
\]

which was discovered by Inozemtsev [6]. Of course, the bracket (2.1) gives rise the second Poisson structure for this system.

**Remark 3** According to [7, 9, 13, 14] if we multiply matrix \(T(\lambda)\) [13] on the matrix

\[
K_+(\lambda) = \begin{pmatrix}
2b_2\lambda^2 - ib_1\lambda + b_0 & 0 \\
4b_2e^{q_1} + 1 & 2b_2\lambda^2 + ib_1\lambda + b_0
\end{pmatrix}
\]

the \(\tau(\lambda) = \text{tr} K_+(\lambda)T(\lambda)\) are generating function of integrals of motion for the periodic Toda lattices associated with all the classical root systems. The Darboux-Nijenhuis variables \(\lambda_i, \mu_i\) [2.1] are the separated coordinates for these integrals and, therefore, integrals of motion for the periodic Toda lattices are in bi-involution with respect to the same brackets (2.1) or (4.6-4.7).

All the details may be found in [18].

## 5 The Kowalevski top on \(so^*(4)\)

Let us consider the Kowalevski top on \(so^*(4)\) with the Hamilton function

\[
H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2bx_1,
\]

and the second integral of motion

\[
H_2 = (J_1^2 + 2bx_+ - \mathcal{P}b^2)(J_2^2 + 2bx_- - \mathcal{P}b^2),
\]

\[
J_\pm = J_1 \pm iJ_2, \quad x_\pm = x_1 \pm ix_2, \quad (i^2 = -1),
\]

\[
(5.2)
\]

\[
(5.1)
\]
which are Poisson commuting

\[ \{H_1, H_2\}_0 = 0 \]  

(5.3)

with respect to the following Poisson brackets

\[ \{J_i, J_k\}_0 = \varepsilon_{ikl} J_l, \quad \{J_i, x_k\}_0 = \varepsilon_{ikl} x_l, \quad \{x_i, x_k\}_0 = -\mathcal{P} \varepsilon_{ikl} J_l. \]  

(5.4)

Here \( \varepsilon_{ikl} \) is the completely antisymmetric tensor, \( \mathcal{P} \) is a complex (or real) parameter, \( x = (x_1, x_2, x_3) \) and \( J = (J_1, J_2, J_3) \) are coordinates on the Poisson manifold \( \text{so}^*(4) \). The Casimirs of the bracket (5.4) have the form

\[ C_1 = x_1 J_1 + x_2 J_2 + x_3 J_3, \quad C_2 = x_1^2 + x_2^2 + x_3^2 - \mathcal{P}(J_1^2 + J_2^2 + J_3^2). \]  

(5.5)

According to [10] the 2 × 2 Lax matrix for the Kowalevski top acquires the form

\[ \mathcal{T} = K_+(\lambda) T(\lambda), \quad \text{where} \quad K_+ = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \]  

(5.6)

and matrix \( T(\lambda) \) is the following representation of the reflection equation algebra at \( n = 2, \alpha = 0 \) and \( \eta = 2i \):

\[
\begin{align*}
A_4 &= -\frac{1}{2} \left( X^2 + J_2^2 - \mathcal{P} b^2 / 2 \right), \quad A_3 = -\frac{i}{2} \left( X^3 + (J_3^2 + x_1 b - \mathcal{P} b^2) X + x_2 J_3 \right), \\
A_2 &= \frac{J_2^2}{2} \left( X^2 + J_3^2 \right) + b x_2 J_3 X - x_1 J_3^2 + b^2 \left( J_3^2 - x_3^2 + C_2 \right) / 2, \quad A_0 = -b^2 C_2^2 / 4, \\
A_1 &= \frac{i}{2} \left( J_2^2 X^3 + 2 b x_2 J_3 X^2 + (J_3^2 (J_2^2 - x_1 b)) + b^2 (x_2^2 - x_3^2 - \mathcal{P} J_3^2) \\ &\quad + b J_1 (x_1 J_1 + x_2 J_2) ) X + b x_2 J_3 (J_2^2 + J_3^2 - x_1 b) + J_1 J_2 J_3 (b x_1 - \mathcal{P} b^2) \right), \\
B_3 &= X^2 - J_2^2 + 2 b x_1 - \mathcal{P} b^2, \\
B_1 &= -J_2 X^2 - 2b (x_2 J_3 - x_3 J_2) X - 2 b C_1 J_1 - b^2 (x_2^2 + x_3^2 - \mathcal{P} J_1^2), \\
C_3 &= \frac{1}{4} \left( X^2 + J_3^2 \right) \left( X^2 + J_3^2 - \mathcal{P} b^2 \right), \\
C_1 &= -\frac{1}{4} \left( J_2^2 X^4 + 2 b x_2 J_3 X^3 + (2 J_3^2 (J_2^2 - b x_1 + \mathcal{P} b^2 / 2) + C_2 b^2 - b^2 (x_3^2 + x_1^2)) X^2 \\ &\quad + 2 b x_2 J_3 (J_3^2 - b x_1) X + J_3^4 (J_2^2 - 2 b x_1) - b^2 J_3^2 (x_3^2 - x_1^2 + \mathcal{P} (J_1^2 + J_3^2)) + \mathcal{P} b^4 x_3^2 \right), \\
\end{align*}
\]

(5.7)

where

\[
X = \frac{J_1 J_3 + b x_3}{J_2}, \quad J^2 = J_1^2 + J_2^2 + J_3^2. \]  

(5.8)

In contrast with [10] we use the transposed matrices \( K \) and \( T \) in (5.6) and solving the resulting system of algebraic equations one gets the following Poisson
Here \( \lambda \) theory. Remind, that according to [3] the bi-involutivity of integrals of motion

On the other hand we can get these Darboux-Nijenhuis variables using the control matrix

\[
\{J_1, J_2\} = -J_3 X^2 - b(x_3 J_1 + x_2 X),
\]
\[
\{J_1, J_3\} = J_2 X^2 - \frac{b(x_3 J_2 - x_2 J_3)X}{J_2} + \frac{bx_2 (2J_1 J_2 + bx_2)}{J_2},
\]
\[
\{J_2, J_3\} = -J_1 X^2 - b(x_1 J_1 + x_3 J_3) - \frac{bx_2 (J_1^2 - J_3^2 + bx_1)}{J_2},
\]
\[
\{x_1, J_1\} = -\frac{2Q_1 (b(x_2 J_2 + x_3 J_3) + (J_2^2 + J_3^2) J_1)}{J_2},
\]
\[
\{x_1, J_2\} = -x_3 X^2 + bP J_2 X + \frac{Q_1 (2J_2^2 - H_1) - bx_2 x_3 J_3}{J_2},
\]
\[
\{x_1, J_3\} = x_2 X^2 - bP (J_1 J_2 + bx_2) + \frac{Q_1 (J_1 X + J_2 J_3) + bx_2^2 J_1}{J_2},
\]
\[
\{x_2, J_1\} = -x_3 J_2^2 - \frac{(bx_2 + J_3 X + J_1 J_2) ((bP - 2x_1) J_3 + x_3 J_1)}{J_2},
\]
\[
\{x_2, J_2\} = -\frac{Q_2 (H_1 - J_2^2 + b^2) + bJ_3 (P J_3^2 + x_3^2)}{J_2}
\]
\[
\{x_2, J_3\} = bP (J_1^2 + J_2^2) + \frac{Q_2 (J_2 J_3 + J_1 X) + bP (bP - x_1) (bx_2 + J_3) J_1 + bx_2 x_3 J_3}{J_2}
\]
\[
\{x_3, J_1\} = x_2 J_1^2 + \frac{(b(x_2 J_2 + x_3 J_3) + (J_2^2 + J_3^2) J_1) \left( bP - 2x_1 + \frac{x_2 J_2}{J_2} \right)}{J_2}
\]
\[
\{x_3, J_2\} = x_1 X^2 - bP (bx_1 + J_2^2 - J_3^2) + \frac{Q_3 (2J_1^2 - H_1) + bx_2 (x_1 J_1 + x_3 J_3)}{J_2}
\]
\[
\{x_3, J_3\} = \frac{(Q_3 - bP J_2) (J_2 J_3 + J_1 X)}{J_2} - \frac{bx_2 (x_1 X + x_3 J_3)}{J_2}
\]

and

\[
\{x_1, x_2\} = (Z_1 x_3 - Z_2 J_3) - bP J_1 Q_1 \frac{x_2 J_2}{J_2}
\]
\[
\{x_1, x_3\} = -(Z_1 x_2 - Z_2 J_2) + bP X Q_1 \frac{x_2 J_2}{J_2}
\]
\[
\{x_2, x_3\} = (Z_1 x_1 - Z_2 J_1) + bP (x_1 J_1 - x_3 J_3) - \frac{bP x_2 (J_1^2 - J_3^2)}{J_2}
\]

Here

\[
Q = x \wedge J = [x_2 J_3 - x_3 J_2, x_3 J_1 - x_1 J_3, x_1 J_2 - J_1 x_2]
\]

and

\[
Z_1 = x_3 J_3 - x_2 J_2 - 2x_1 J_1 + \frac{x_2 (J_1^2 - 2J_3^2)}{J_2} + \frac{(J_3 (bP - 2x_1) + x_3 J_1) X}{J_2}
\]
\[
Z_2 = b^2 P^2 + (H_1 - J_3^2) P.
\]

Functions \( C_{1,2} \) are Casimirs with respect to the both Poisson structures \( \{.,.\}_{0,1} \) \( (5.4), (5.9) \).

It allows us to obtain the recursion operator \( N \) on the symplectic leaves of \( so^* (4) \) and the corresponding Darboux-Nijenhuis variables \( \lambda_{1,2} \).

On the other hand we can get these Darboux-Nijenhuis variables using the control matrix theory. Remind, that according to [3] the bi-involutivity of integrals of motion

\[
\{H_1, H_2\} = \{H_1, H_2\}_0 = 0
\]

(5.10)
is equivalent to the existence of the non-degenerate control matrix \( F \), such that

\[
P_1 dH_i = P_0 \sum_{j=1}^{2} F_{ij} dH_j, \quad i = 1, 2. \tag{5.11}
\]

Here \( P_{0,1} \) are the Poisson bivectors associated with the brackets \( \{ , \} _{0,1} \). In our case \( F \) looks like

\[
F = \begin{pmatrix}
\frac{H_1 + b^2 \mathcal{P}}{2} - X^2 - J_3^2 & \frac{1}{4} \\
Z_3 & \frac{H_1 + b^2 \mathcal{P}}{2}
\end{pmatrix},
\]

where

\[
Z_3 = H_2 - 2(b(b \mathcal{P} - 2x_1) - J_1^2)X^2 + 8bx_2J_3X + (2bJ_3^2 - 2x_3)b^2 + 4bQ_2J_3 + 2(J_2^2 + 2J_1^2)J_3^2.
\]

Then we can prove that entry \( B(\lambda) \) of the matrix \( T(\lambda) \) coincides with the characteristic polynomial of \( F \) and, therefore, with the minimal characteristic polynomial of the recursion operator \( N \) on the symplectic leaves of \( \mathfrak{s}o^*(4) \).

Summing up, we have proved that roots \( \lambda_{1,2} \) of the polynomial \( B(\lambda) \) are the Darboux-Nijenhuis coordinates with respect to the Poisson structures \( \{ \lambda \} _{4} \) and \( \{ \lambda \} _{10} \). At \( \mathcal{P} = 0 \) these variables \( \lambda_{1,2} \) coincide with the well-known Kowalevski variables \( \{ \lambda \} _{10} \).

6 The Kowalevski-Goryachev-Chaplygin gyrostat.

Let us consider the Kowalevski-Goryachev-Chaplygin gyrostat with the following Hamilton function

\[
H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2\rho J_3 + c_1x_1 + c_2x_2 + c_3(x_1^2 - x_2^2) + c_4x_1x_2 + \frac{\delta}{x_3}, \tag{6.1}
\]

\[
c_1, c_2, c_3, c_4, \rho, \delta \in \mathbb{R}.
\]

Here \( x = (x_1, x_2, x_3) \) and \( J = (J_1, J_2, J_3) \) are coordinates on the dual Lie algebra \( \mathfrak{e}^* (3) \) with the Lie-Poisson brackets

\[
\{ J_1 , J_2 \}_0 = \varepsilon_{ijk} J_k, \quad \{ J_1 , J_3 \}_0 = \varepsilon_{ijk} x_k, \quad \{ x_1 , x_2 \}_0 = 0, \tag{6.2}
\]

and with the following Casimirs

\[
C_1 = x_1J_1 + x_2J_2 + x_3J_3, \quad C_2 = x_1^2 + x_2^2 + x_3^2. \tag{6.3}
\]

The Hamilton function \( \{ \lambda \} _{6} \) determines dynamical system on \( \mathfrak{e}^* (3) \), which is an integrable by Liouville at \( C_1 = 0 \) only.

Let us start with the 2 × 2 Lax matrix for the symmetric Neumann system

\[
L(\lambda) = \begin{pmatrix}
\lambda^2 - 2J_3\lambda - J_1^2 - J_2^2 - \frac{\delta}{x_3} & \lambda(ix_1 + x_2) - x_3(iJ_1 + J_2) \\
\lambda(ix_1 - x_2) - x_3(iJ_1 - J_2) & x_3^2
\end{pmatrix}, \tag{6.4}
\]

which is a representation of the Sklyanin algebra on the subset of \( \mathfrak{e}^* (3) \) defined by \( C_1 = 0 \). Using a family of the Poisson brackets compatible with the Sklyanin algebra \( \{ \lambda \} _{17} \) we can get compatible bi-hamiltonian structures on \( \mathfrak{e}^* (3) \) associated with the matrix \( L(\lambda) \). All the detail may be found in \( \{ \lambda \} _{3} \) and \( \{ \lambda \} _{17} \).
According to [8, 15], the Lax matrix for the Kowalevski-Goryachev-Chaplygin gyrostat acquires the form

$$T(\lambda) = K_+(\lambda)T(\lambda),$$

where

$$T(\lambda) = L(\lambda - \rho) K_- (\lambda) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) L^T(-\lambda - \rho) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad \rho \in \mathbb{R}$$

(6.5)

is the representation of the reflection equation algebra (1.1) at \(n = 2, \alpha = 0\) and \(\eta = 2i\). Here \(L(\lambda)\) is given by (6.1) and

$$K_- = \left( \begin{array}{cc} a_1 \lambda + a_0 & \lambda \\ 0 & -a_1 \lambda + a_0 \end{array} \right), \quad K_+ = \left( \begin{array}{cc} b_1 \lambda + b_0 & 0 \\ \lambda & -b_1 \lambda + b_0 \end{array} \right)$$

(6.6)

are numerical matrices depending on arbitrary parameters \(a_{0,1}\) and \(b_{0,1}\).

The Hamilton function \(H_1\) from

$$\text{tr} \ T(\lambda) = \lambda^6 - 2\tilde{H}_1 \lambda^4 + \tilde{H}_2 \lambda^2 + 2a_0b_0(\rho^2\mathcal{C}_2 - \delta)$$

(6.7)

coincides with the previous function \(H\) (6.1) after canonical transformation of variables

$$J \rightarrow J + Ux, \quad U = \left( \begin{array}{ccc} 0 & 0 & i\beta_+ \\ 0 & 0 & \beta_- \end{array} \right), \quad \beta_\pm = \frac{a_1 \pm b_1}{2}$$

and exchange of parameters

$$a_1^2 = \left( c_3 + \frac{ic_4}{2} \right), \quad b_1^2 = \left( c_3 - \frac{ic_4}{2} \right), \quad a_0 = \frac{ic_1 - c_2}{2}, \quad b_0 = \frac{ic_1 + c_2}{2}.$$  

Substituting matrix \(T(\lambda)\) (6.5) into the brackets \(\{ , \}_k\) (2.1) one get the second cubic bracket \(\{ , \}_1\) compatible with bracket \(\{ , \}_2\). For brevity we present these brackets at \(c_3 = c_4 = \rho = \delta = 0\) only, i.e. for the Kowalevski top:

\[
\begin{align*}
\{J_1, J_2\}_1 &= -(J_1(J_1 - iJ_2) + 2a_0x_3)(J_1 + iJ_2) \\
\{J_1, J_3\}_1 &= -(J_1^2 + J_2^2 + 2J_3^2 - 2a_0(x_1 - ix_2))J_2 \\
\{J_2, J_3\}_1 &= (J_1^2 + J_2^2 + 2J_3^2 - 2a_0(x_1 - ix_2))J_1 \\
\{x_1, J_1\}_1 &= (2J_2^2 - 2a_0x_2)x_3 - 2x_2(J_1 + iJ_2)J_3 \\
\{x_1, J_2\}_1 &= -i(J_1^2 - 2iJ_1J_2 + J_2^2 - 2ia_0x_2)x_3 + 2ix_2(J_1 + iJ_2)J_3 \\
\{x_1, J_3\}_1 &= -(J_1^2 + J_2^2 + 4J_3^2 - 2a_0(x_1 + ix_2))x_2 + 2x_3J_2J_3 \\
\{x_2, J_1\}_1 &= (J_1^2 + 2iJ_1J_2 + J_2^2 - 2a_0x_1)x_3 + 2x_1(J_1 + iJ_2)J_3, \\
\{x_2, J_2\}_1 &= 2i(J_1^2 + a_0x_1)x_3 - 2ix_1(J_1 + iJ_2)J_3, \\
\{x_2, J_3\}_1 &= (J_1^2 + J_2^2 + 4J_3^2 - 2a_0(x_1 + ix_2))x_1 - 2x_3J_1J_3, \\
\{x_3, J_1\}_1 &= x_2(J_1^2 + 2iJ_1J_2 + J_2^2 - 2a_0x_1)x_2 - 2ix_1J_2^2, \\
\{x_3, J_2\}_1 &= -x_1(J_1^2 - 2iJ_1J_2 + J_2^2) - 2ix_2J_1^2, \\
\{x_3, J_3\}_1 &= 2(x_2J_1 - x_1J_2)J_3
\end{align*}
\]

and

$$\{x_i, x_j\}_1 = 2i\epsilon_{ijk}(x_1J_2 - x_2J_1 - ix_3J_3) x_k.$$
At $C_1 = 0$ integrals of motion $\tilde{H}_{1,2}$ (6.7) are in the bi-involution with respect to the compatible brackets $\{.,\}_0,1$ and satisfy to the following relations

$$P_1d\tilde{H}_i = P_0 \sum_{j=1}^{2} F_{ij} d\tilde{H}_j, \quad i = 1, 2,$$

(6.9)

where $P_{0,1}$ are the Poisson bivectors associated with the brackets $\{.,\}_{0,1}$. and

$$F = \begin{pmatrix}
2J_1^2 + 2J_2^2 + 4J_3^2 - 2a_0(x_1 + ix_2) & 1 \\
2a_0(ix_2 J_1 - ix_1 J_2 + x_3 J_3)(J_1 + iJ_2) & -(J_1^2 + J_2^2)^2
\end{pmatrix}. $$

Of course, entry $B(\lambda)$ of the matrix $T(\lambda)$ (6.5) coincides with the characteristic polynomial of $F$ and, therefore, with the minimal characteristic polynomial of the recursion operator $N$ on the symplectic leaves of $e^*(3)$.

Summing up, we have proved that at $C_1 = 0$ the Kowalevski top has the polynomial brackets (6.8) in additional to the rational brackets (5.9) considered above. These different Poisson structures are related with the different separated variables, which give rise to the different representations of the reflection equation algebra.

**Remark 4** There is another bi-hamiltonian structure for the Kowalevski top on $e^*(3)$, associated with linear $r$-matrix algebra [12]. According to [16] the corresponding Poisson tensor is the rational function $\hat{P}_1 = C_1^{-1}P_{pol}$, where $C_1$ is the Casimir function (6.3) and $P_{pol}$ is a cubic polynomial in variables $(x, J)$.

The Casimir $C_2$ does not the Casimir function for the second Poisson bivector $\hat{P}_1$ from [12, 16], in contrast with the second Poisson bivectors associated with the considered above brackets (5.9) and (6.8).

## 7 Conclusion

We present a family of compatible Poisson brackets (2.1), (2.6), that includes the reflection equation algebra. The application of the $r$-matrix formalism is extremely useful here resulting in drastic reduction of the calculations for a whole set of integrable systems.

For the rational $4 \times 4$ matrix $r(\lambda - \mu)$ (1.2) the similar construction has been applied to other $r$-matrix algebras in [17, 19]. It will be interesting to construct the similar families of compatible Poisson brackets associated with the $4 \times 4$ matrices $r(\lambda - \mu)$, which are trigonometric and elliptic functions on spectral parameter.

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