The Error in Calculated Distance due to the propagation of Positions Error

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Abstract. It is known that distance can be measured directly using the instruments of distance measurement, or indirectly using the coordinates of two edge points using the popular Pythagorean formula. In most of the geomatics engineering studies, the error in distances is expressed by the uncertainty of the direct measured distances only, while there is not enough studies that highlight the error in a computed distance. This paper presents the error in distance in a novel formulated approach, which considers the planimetric position error in both edges of the computed distance, to determine the error in that distance. Furthermore, the research concludes that the size and direction of error ellipse at edge points and the azimuth of the distance, are the main factors that combine to define the value of error in those distances that are measured indirectly.

Keywords
Position error ellipse; Error in the calculated distance; Coefficient of correlation; Distance’s orientation.

1. Introduction
The uncertainty of the directly measured distances depends totally on the accuracy of the used instrument and the method of measurement. Whereas, the uncertainty of the distance measured indirectly (by inverse computations), is a matter of error propagation from both edges (position error of two endpoints) indeed. However, the point’s planimetric error has major attention in many surveying literature. Most of them focused on the ellipse of position error and discussed how it combined with the error ellipses in other adjoining positions through a certain geometrical network. In turn, it has been found that a little attention paid to discuss the relationship between the edge’s position error and the error in the indirectly measured distance (calculated from edge coordinates) that probably obtained due to the propagation of the position error in both edges points. Furthermore, in common surveying literature, the distance error determined by multiplying position error with some factors, which predefined according to even the size of distance, or to some applied practical experiences in different countries. The current research aims to derive a novel formula that defines the error in calculated distance based on the errors in positions of the edge points, to enrich the studies related to the assessment of map accuracy.

2. Propagation of position error of edge points
The distance between any two known points is commonly determined using the inverse computations as:
\[ D = \sqrt[2]{(E_2 - E_1)^2 + (N_2 - N_1)^2} \]  

(1)

The well-known (well-known or popular?) ‘equation (1)’ shows that the calculated distance \( D \) is a function of planimetric coordinates of the two edge points \( (1 \& 2) \), i.e., \( D = f(E_1, N_1, E_2, N_2) \). Thus, according to the general formula of error propagation [9], the precision of the distance determined indirectly using the coordinates of edge points is as follow:

\[ \sigma_D^2 = \left( \frac{\partial D}{\partial E_1} \right)^2 \sigma_{E_1}^2 + \left( \frac{\partial D}{\partial N_1} \right)^2 \sigma_{N_1}^2 + \left( \frac{\partial D}{\partial E_2} \right)^2 \sigma_{E_2}^2 + \left( \frac{\partial D}{\partial N_2} \right)^2 \sigma_{N_2}^2 + 2 \left( \frac{\partial D}{\partial E_1} \right) \left( \frac{\partial D}{\partial E_2} \right) \sigma_{E_1} \sigma_{E_2} + 2 \left( \frac{\partial D}{\partial N_1} \right) \left( \frac{\partial D}{\partial N_2} \right) \sigma_{N_1} \sigma_{N_2} \]  

(2)

Where: \( \sigma_{E_1}, \sigma_{N_1}, \sigma_{E_2}, \sigma_{N_2} \) – are the variances of easting and northing of the edge points \( (1 \& 2), \) respectively.

\( \sigma_{E_1N_1}, \sigma_{E_1N_2}, \sigma_{E_2N_1}, \sigma_{E_2N_2}, \sigma_{N_1N_2} \) – are the covariances of easting and northing of the edge points, respectively. However, ‘equation (2)’ represents the general formula of the distance’s error (measured indirectly), based on the propagation of the position error at both edge points.

Since the covariance indicates the degree of dependency between the measured quantities, so it should be zeroed in cases of no correlation. Thus, in the next development, the variances of coordinates of the edge points \( (\sigma_{E_i}, \sigma_{N_i}, \sigma_{E_j}, \sigma_{N_j}) \) will be assumed equal to zero because the two points are measured independently. Then, the general formula of the calculated distance precision becomes:

\[ \sigma_D^2 = \left( \frac{\partial D}{\partial E_1} \right)^2 \sigma_{E_1}^2 + \left( \frac{\partial D}{\partial N_1} \right)^2 \sigma_{N_1}^2 + \left( \frac{\partial D}{\partial E_2} \right)^2 \sigma_{E_2}^2 + \left( \frac{\partial D}{\partial N_2} \right)^2 \sigma_{N_2}^2 + 2 \left( \frac{\partial D}{\partial E_1} \right) \left( \frac{\partial D}{\partial E_2} \right) \sigma_{E_1} \sigma_{E_2} + 2 \left( \frac{\partial D}{\partial N_1} \right) \left( \frac{\partial D}{\partial N_2} \right) \sigma_{N_1} \sigma_{N_2} \]  

(3)

Substituting the equivalent values of the partial derivatives in ‘equation (3)’ gives:

\[ \sigma_D^2 = \frac{(E_2 - E_1)^2}{b^2} (\sigma_{E_1}^2 + \sigma_{E_2}^2) + \frac{(N_2 - N_1)^2}{a^2} (\sigma_{N_1}^2 + \sigma_{N_2}^2) + \frac{2(E_2 - E_1)(N_2 - N_1)}{b^2a^2} \cdot (\sigma_{E_1N_1} + \sigma_{E_2N_2}) \]  

(4)

For practical purposes, the values \( (E_2 - E_i)/D \) and \( (N_2 - N_i)/D \) can be replaced by their trigonometric equivalents as follow:

\[ (E_2 - E_i)/D = \sin \alpha; \quad \text{and} \quad (N_2 - N_i)/D = \cos \alpha \]

Where: \( \alpha \) - represents the azimuth of the calculated distance (the distance orientation).

Thus, the ‘equation (4)’ can be rewritten as follow:

\[ \sigma_D^2 = \sin^2 \alpha (\sigma_{E_1}^2 + \sigma_{E_2}^2) + \cos^2 \alpha (\sigma_{N_1}^2 + \sigma_{N_2}^2) + 2 \sin \alpha \cos \alpha (\sigma_{E_1N_1} + \sigma_{E_2N_2}) \]  

(5)

The novel derived ‘equation (5)’ represents the general form of the equation that estimates the error in the calculated distances. Therefore, one can claim that the following factors influence the precision of the indirectly measured distance:

I- The position error at both distance’s edges \( (\sigma_{E_i} \& \sigma_{N_i}) \).
II- The azimuth of the line between the distance’s edge points \( (\alpha) \).
III- The coefficient of correlation \( (\rho_i) \) between the coordinates of the \( i^{th} \) edge point.

2.1 Effect of the position error at edge points

In order to explain how the position error at edge points affects the value of distance’s error, ‘equation (5)’ is used to calculate the error in the distance as shown in (table 1) below.

| Table 1. The distance error computations according to values of position error at edges points. | }
The relationship between the position error at edge points and the obtained distance error, illustrated graphically in ‘figure 1’, according to which one can state that there is an exact direct linear relationship between them (in case the position error increases similarly at both edges).

### Table 1

| $\sigma_{E1}$ (m) | $\sigma_{N1}$ (m) | $\sigma_{E2}$ (m) | $\sigma_{N2}$ (m) | $\alpha$ (deg.) | $\sigma_{\text{Dis}}$ (m) |
|-------------------|-------------------|-------------------|-------------------|-----------------|------------------|
| 0.015             | 0.015             | 0.015             | 0.015             | 45°             | 0.0277           |
| 0.025             | 0.025             | 0.025             | 0.025             | 45°             | 0.0461           |
| 0.035             | 0.035             | 0.035             | 0.035             | 45°             | 0.0645           |
| 0.05              | 0.05              | 0.05              | 0.05              | 45°             | 0.0922           |
| 0.075             | 0.075             | 0.075             | 0.075             | 45°             | 0.1383           |
| 0.1               | 0.1               | 0.01              | 0.01              | 45°             | 0.1844           |

*a Covariances are determined from the value of the coefficient of correlation ($\rho$ = 0.7) at both edges.*

**Figure 1.** Variation of distance error according to the variation of position error at the edges.

2.2 **Impression of the coefficient of correlation**

The ellipse of position error can take different degrees of flattening and different directions according to the nature of the correlation between the coordinates. The ‘figure 2’ below, shows how the direction of the error ellipse changes, according to change in the coefficient of correlation ($\rho$) for an assumed position error ($\sigma_{E1} = 0.063$ m; $\sigma_{N1} = 0.045$ m).

**Figure 2.** The direction of error ellipse according to change in the coefficient of correlation ($\rho$).
Therefore, to explain the effect of the correlation between the position’s coordinates, two assumed scenarios will be discussed as follow:

2.2.1 Not correlated coordinates at the edges. The first scenario, where the coordinates of edge points are not correlated (i.e., \( \rho_i = 0 \)), the covariances are equal to zero (i.e., \( \sigma_{E_i N_i} = \sigma_{E_i N_2} = 0 \)). Thus, ‘equation (5)’ shall take the following special form for the case above:

\[
\sigma_D^2 = \sin^2 \alpha (\sigma_{E_1}^2 + \sigma_{E_2}^2) + \cos^2 \alpha (\sigma_{N_1}^2 + \sigma_{N_2}^2)
\]  

(6)

The ‘equation (6)’ has been used to calculate the value (\( \sigma_{Dis} \)) shown in (table 2) to figure out the behavior of the distance error when there is no correlation between the coordinates at edge points.

| \( \sigma_{E_1} \) (m) | \( \sigma_{N_1} \) (m) | \( \sigma_{E_2} \) (m) | \( \sigma_{N_2} \) (m) | \( \alpha \) (deg.) | \( \sigma_{Dis} \) (m) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.05            | 0.05            | 0.03            | 0.03            | 45°             | 0.0583          |
| 0.05            | 0.03            | 0.03            | 0.05            | 45°             | 0.0583          |
| 0.03            | 0.05            | 0.05            | 0.03            | 45°             | 0.0583          |
| 0.03            | 0.05            | 0.05            | 0.03            | 135°            | 0.0583          |
| 0.05            | 0.05            | 0.03            | 0.03            | 135°            | 0.0583          |
| 0.05            | 0.05            | 0.03            | 0.03            | 0°              | 0.0583          |
| 0.05            | 0.05            | 0.03            | 0.03            | 90°             | 0.0583          |
| 0.04            | 0.06            | 0.03            | 0.07            | 45°             | 0.0742          |
| 0.05            | 0.06            | 0.08            | 0.07            | 45°             | 0.0933          |

The obtained results in (table 2) show that when \( \rho_i = 0 \) and \( \sigma_{E_i} = \sigma_{N_i} \) at both edges, the value of distance’s error does not change despite changing the distance’s azimuth (\( \alpha \)). That is because the ellipse of position error conserves its original shape in the absence of correlation. Meanwhile, the distance error (\( \sigma_{Dis} \)) increases with decrease in the precision of the edge points.

2.2.2 Correlated coordinates at the edges. The correlation between any two variables varies according to the nature of dependency one to another. Thus, the sign and the value of the coefficient of correlation of the position coordinates will define the shape and the direction of position error ellipse. The other scenario highlights the impact of coordinate’s correlation on the value of distance’s error (\( \sigma_{Dis} \)). For that purpose, calculations based on ‘equation (5)’ are done for different values of the coefficient of correlation. Table 3 shows the results of this scenario.

| \( \sigma_{E_1} = \sigma_{N_1} \) (m) | \( \sigma_{E_2} = \sigma_{N_2} \) (m) | \( \rho_1 \) | \( \rho_2 \) | Az. Ell.1 (deg.) | Az. Ell.2 (deg.) | \( \alpha \) (deg.) | \( \sigma_{Dis} \) (m) |
|-----------------|-----------------|---------|---------|-----------------|-----------------|-----------------|-----------------|
| 0.05            | 0.03            | -1      | -1      | 315°            | 315°            | 45°             | 0.00            |
| 0.05            | 0.03            | 1       | 1       | 45°             | 45°             | 45°             | 0.083           |
| 0.05            | 0.03            | 1       | -1      | 45°             | 315°            | 45°             | 0.070           |
| 0.05            | 0.03            | -1      | 1       | 315°            | 45°             | 45°             | 0.042           |
| 0.05            | 0.03            | 0.5     | 0.5     | 45°             | 45°             | 45°             | 0.071           |
| 0.05            | 0.03            | -0.5    | -0.5    | 315°            | 315°            | 45°             | 0.041           |

*Az. Ell.1 & Az. Ell.2: The azimuth of ellipse semi-major axis at edge point 1 & 2, respectively.*
The results in (table 3) affirm that the distance’s error varies according to change in the coefficient of correlation without changing the value of distance’s azimuth (α). Furthermore, the distance’s error might be maximum or exactly zero, when there is a perfect correlation between the coordinates at both edges ($\rho_1 = \rho_2 = \pm 1$). The maximum value is obtained, if the azimuth of distance is identical to the azimuth of the ellipse’s semi-major axis at both edges, whereas zero value is obtained, when the distance’s azimuth is perpendicular to the azimuth of the semi-major axis of the ellipse at both edge points.

2.3 The effect of the azimuth of the calculated distance
To understand the effect of the distance’s orientation on the value of distance’s error ($\sigma_{\text{Dis}}$), different scenarios were created based on the assumed combination of the relevant factors, the azimuth (α) and the coordinate’s coefficient correlation (ρ).

| $\sigma_{E1}$ (m) | $\sigma_{N1}$ (m) | $\sigma_{E2}$ (m) | $\sigma_{N2}$ (m) | $\rho_1 = \rho_2$ | Ell. Az. at 1 & 2 (deg.) | α (deg.) | $\sigma_{\text{Dis}}$ (m) |
|------------------|------------------|------------------|------------------|-----------------|-------------------------|---------|-----------------|
| 0.05             | 0.05             | 0.03             | 0.03             | -1              | 135°                     | 45°     | 0.00            |
| 0.05             | 0.05             | 0.03             | 0.03             | -1              | 135°                     | 135°    | 0.083           |
| 0.05             | 0.05             | 0.03             | 0.03             | +1              | 45°                     | 45°     | 0.083           |
| 0.05             | 0.05             | 0.03             | 0.03             | +1              | 45°                     | 135°    | 0.00            |
| 0.05             | 0.05             | 0.03             | 0.03             | +1              | 45°                     | 0°      | 0.058           |
| 0.05             | 0.05             | 0.03             | 0.03             | +1              | 45°                     | 90°     | 0.026           |
| 0.05             | 0.05             | 0.03             | 0.03             | -1              | 315°                    | 45°     | 0.00            |
| 0.05             | 0.05             | 0.03             | 0.03             | +1              | 225°                    | 45°     | 0.083           |
| 0.05             | 0.05             | 0.03             | 0.03             | -1              | 315°                    | 135°    | 0.083           |

The results shown in (table 4), explain the relationship between the direction of error ellipse at edge points (which follow the sign of correlation) and the azimuth of calculated distance (which defined by the geometrical relation between the edge points). The obtained results illustrate the effect of this relationship on the value of distance error ($\sigma_{\text{Dis}}$).

The obtained data in (table 4) strengthen the idea of a coherent relationship between the azimuth of calculated distance (α), and the direction of error ellipses at edge points (Ell. Az.). Therefore, it is worth mentioning that the obtained results in (tables 3 & 4) are identical. The value of ($\sigma_{\text{Dis}}$) can become zero, when the direction of error ellipse (Ell. Az. at edges points) is perpendicular to the azimuth of the calculated distance (α) with the existence of a perfect correlation. The combination of these two factors directly affects the value of the distance’s error, indeed.

3. The relationship between the distance’s error and the ellipse of position error
The results in (tables 3 & 4) illustrate the possible combination of the ellipse direction at edge points (Ell. Az.) and the distance’s azimuth (α). However, in case of a perfect correlation, the semi-minor axis of error ellipse will be equal to zero, and the error ellipse becomes just a line with a length equal to the semi-major axis.

Therefore, the value of ($\sigma_{\text{Dis}}$) becomes maximum when the error ellipse at both edges coincide with the distance’s azimuth with a perfect correlation. Thus, the distance’s error in such a special case can be formulated as follow:

$$\sigma_{\text{Dis}} (\text{max.}) = \sqrt{(\text{maj}_1)^2 + (\text{maj}_2)^2}$$  \hspace{1cm} (7)
Where: \( \text{maj}_i \) - is the semi-major axis of error ellipse at the \( i^{th} \) edge.

Thereby, the error in the calculated distance (\( \sigma_{\text{Dis}} \)) becomes zero when the azimuth of ellipse’s major axes at both edges offsets perpendicular to the azimuth of the distance (table 5).

**Table 5.** Distance azimuth error and point error ellipse azimuth.

| \( \rho_1 \) | \( \sigma_{E1} = \frac{\sigma_{N1}}{\rho_1} \) | \( \sigma_{E2} = \frac{\sigma_{N2}}{\rho_1} \) | \( \text{Maj}_1 \) (m) | Maj_1 Az. (deg.) | \( \rho_2 \) | \( \text{Maj}_2 \) (m) | Maj_2 Az. (deg.) | \( \alpha \) (deg.) | \( \sigma_{\text{Dis}} \) (m) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.05 | 0.03 | 0.071 | 45° | 1 | 0.042 | 45° | 45° | 0.083 |
| 1 | 0.05 | 0.03 | 0.071 | 45° | 1 | 0.042 | 45° | 135° | 0.00 |
| -1 | 0.05 | 0.03 | 0.071 | 315° | -1 | 0.042 | 315° | 45° | 0.00 |
| -1 | 0.05 | 0.03 | 0.071 | 315° | -1 | 0.042 | 315° | 135° | 0.083 |
| 1 | 0.05 | 0.03 | 0.071 | 45° | -1 | 0.042 | 315° | 45° | 0.071 * |
| -1 | 0.05 | 0.03 | 0.071 | 315° | 1 | 0.042 | 45° | 45° | 0.042 * |

\(^{\text{a}} \) (\( * \)) – indicates calculation by ‘equation (8)’.

The obtained results in (table 5) show that ‘equation (7)’ can be applied just for the following special cases:

i. **When \( \sigma_{Ei} = \sigma_{Ni} \) with a perfect coordinates correlation (\( \rho_i = \pm 1 \))**; and

ii. **When both ellipses at edges points have the same direction.**

Hence, if the directions of ellipses at edges are perpendicular to each other, and (\( \alpha \)) is parallel to one of them, which represents another special case, then the equation of distance’s error becomes as follow:

\[
\sigma_{\text{Dis}} = 2\left( (\text{min}_1)^2 + (\text{maj}_2)^2 \right) \]

Or

\[
\sigma_{\text{Dis}} = 2\left( (\text{maj}_1)^2 + (\text{min}_2)^2 \right) \]

Where: \( \text{min}_i \) - is the semi-minor axis of error ellipse at the \( i^{th} \) edge point.

Practically, the coordinates of edge’s point mostly are not perfectly correlated in fact (i.e., \( \rho < \pm 1 \)). Thus, it should be concluded that ‘equations (8 & 7)’ cannot be used as a general formula. Therefore, according to the obtained results in (table 5), it should be inferred that it is extremely difficult figuring out an exact relationship between the distance’s error and the ellipses of error at edge points through a certain formula, to be applied for all cases. However, to prove this reasoning, ‘figure 3’ shows the variation of the value (\( \sigma_{\text{Dis}} \)), due to the variation of distance’s azimuth (\( \alpha \)) within just one quarter (\( 0° - 90° \)) for fixed values of (\( \sigma_{Ei}, \sigma_{Ni}, \& \rho \)).
4. Conclusions

1- The distance that is measured indirectly (calculated mathematically) has an error; its value depends on the following factors:
   - The size of position error at edge points.
   - The existing correlation between the coordinates of edge points.
   - The azimuth (direction) of the computed distance.

2- To define the value of error in the calculated distance, a worked out ‘equation (5)’ is obtained based on consideration the position accuracy at both edge points and the orientation of the distance.

3- The value of distance’s error increases as the position error at edge points increases unless some special cases as shown through the results in table (4).

4- The value of the distance’s error is significantly affected by the correlation that exists between the coordinates of points at the edges of distance.

5- The value of error in the computed distance can be zero or maximum when the coordinates at both edges, have a perfect correlation. It depends on the relationship between the azimuth of distance and the sign of the perfect correlation (see table (3)).

6- The research strongly recommends the use of ‘equation (5)’ for determining the value of the error in the calculated distances, especially in cases of assessment formap accuracy, and for estimating the precision of the baselines measured indirectly as well.

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