The influence of dipole-dipole interaction on entanglement of two superconducting qubits in the framework of double Jaynes-Cummins model

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The influence of dipole-dipole interaction on entanglement of two superconducting qubits in the framework of double Jaynes-Cummins model

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Abstract. In this paper we investigated the entanglement dynamics between two superconducting qubits interacting with two microwave modes of independent coplanar cavities taking into account the direct dipole-dipole interaction. The model with different qubit-field couplings and detunings is under consideration. Using the dressed-states representation technique we derived the exact solution for considered model with qubits prepared initially in entangled states and vacuum field modes. We have carried out the dependence of the atom-atom entanglement on the strength of the dipole-dipole interaction and other parameters of the considered system such as different coupling constants and detunings. The results showed that the presence of a sufficiently large dipole-dipole interaction leads to stabilization of entanglement.

1. Introduction
Coupling distant qubits is an important goal for quantum information and for its potential applications. This kind of coupling needs the study of interaction between qubits and photons, which has been widely studied in a cavity quantum electrodynamics (QED) [1]. The Jaynes–Cummings model (JCM) is the simplest possible physical model in cavity QED that describes the interaction of a natural or artificial two-level atom (qubit) with a single-mode cavity, and has been used to understand a wide variety of phenomena in quantum optics and condensed matter systems, such as trapped ions, quantum dots, spins, superconducting circuits, optical and microwave cavity QED [1]. In order to explore a wider range of phenomena caused by the interaction of the qubits with the quantum fields in resonators the numerous generalizations of the JCM have been investigated in recent years. Eberly and co-authors [2] have proposed the so-called double JCM (DJCM), consisting of two two-level atoms and two resonator modes, provided that each atom interacts only with one field of the resonator, and investigated the entanglement dynamics of this model. Recently, the DJCM have been extensively investigated. Hu et al. [3, 4] have studied the effect of the Stark shift and detuning and different couplings on entanglement in the framework of DJCM. Du et al. [5] have investigated the sudden birth of entanglement between two initially separate atoms interacting with two entangled photons in a DJCM, and have discussed the influences of different atomic initial states on entanglement among atoms. Xie and Fang [6] have considered the entanglement dynamics of intensity-dependent coupling DJCM with two different initial light fields, which one is in the squeezed vacuum state and another is in the coherent state. Liao et al. [7] have investigated the dynamics of entanglement between two atoms in a two-photon DJCM model when the cavity field is previously in a maximally entangled state. Xie and Fang [8] have examined the entanglement dynamics of the atoms in the DJCM with the
Kerr medium. The entanglement dynamics of a two-photon DJCM with Kerr-like medium have also been studied by Quyang et al. [9]. Vieira et al. [10] have reported on the geometric character of the entanglement dynamics of two pairs of qubits evolving according to the DJCM. Baghshahi et al. [11] have studied the interaction between two two-level atoms and two coupled modes of a quantized radiation field in the form of parametric frequency converter. Zhou and Fang [12] have investigated the analytical solution and entanglement swapping of a DJCM in non-Markovian environments. Zhu et al. [13] have studied the dynamics of quantum entanglement of DJCM model analytically in the framework of corrections to the rotating-wave approximations.

In all previous studies, the direct dipole-dipole interaction between the qubits doesn’t takes into account. But for superconducting Josephson qubits the effective dipole-dipole interaction constant may exceed the coupling constant between the qubit and cavity field (see references in [14]). In this paper, we consider a non-resonant DJCM taking into account the direct dipole-dipole interaction between qubits. We investigate the entanglement between qubits for Bell’s initial qubits states and vacuum resonators states, and discuss dependence of the entanglement on the parameters of the considered system, such as different intensity of dipole interaction, coupling constants and the detunings between the atomic transition frequency and the cavity field frequencies.

2. The model and its time-dependent dynamics

We consider two identical superconducting qubits labelled $A$ and $B$, and two cavity modes of coplanar or $LC$ circuits labelled “$a$” and “$b$”. Qubit $A$ not-resonantly interacts with a single-mode cavity field “$a$”, and qubit $B$ not-resonantly interacts with a single-mode cavity field “$b$”. Due to the randomness of the qubits positions in the cavity, it is very difficult to control the couplings between different atom-cavity systems to be the same. Therefore the coupling constants between the atoms and cavities are assumed to be unequal. For superconducting qubits interacting with microwave coplanar resonators or $LC$ superconducting circuits the intensity of effective dipole-dipole interaction can be compared with the atom-cavity coupling constant. In this case the dipole-dipole interaction should be included in the model Hamiltonian. Therefore, Hamiltonian for the system under the rotating wave approximation can be written as

$$H = H_0 + H_1,$$

$$H_0 = \hbar \omega_0 (\sigma^x_a/2 + \sigma^y_a/2 + a^+ a + b^+ b),$$

$$H_1 = \hbar \Delta_a a^+ a + \hbar \Delta_b b^+ b + \hbar \gamma_a (\sigma^x_a + a^+ a^x_a) + \hbar \gamma_b (\sigma^x_b + b^+ b^x_b) + \hbar J (\sigma^x_a \sigma^x_b + \sigma^x_b \sigma^x_a),$$

where $(1/2)\sigma^x_i$ is the inversion operator for the $i$th qubit ($i = A, B$), $\sigma^+ = |+\rangle\langle -|$, and $\sigma^- = |\rangle\langle +|$. $a^+$ and $a$ are the creation and the annihilation operators of a cavity mode photon (or plasmon for $LC$-resonator) of the cavity mode “$a$”, $b^+$ and $b$ are the creation and the annihilation operators of photons of the cavity mode $b$, $\hbar \omega_0$ is the superconducting (SC) gap energy, $\gamma_a \equiv \gamma$ is the coupling constant between qubit $A$ and the cavity field “$a$” and $\gamma_b$ is the coupling constant between qubit $A$ and the cavity field “$b$”. $\Delta_a$ and $\Delta_b$ are the detunings for mode “$a$” and “$b$”. $J$ is the coupling constant of the dipole interaction between the qubits $A$ and $B$.

Firstly we take two qubits initially in the Bell-like pure state of the following form

$$|\Psi(0)\rangle_A = \cos \theta |+, -\rangle + \sin \theta |-, +\rangle,$$

where $0 \leq \theta \leq \pi$, and the cavity fields are in a vacuum states $|0, 0\rangle$. Then the full initial state for considered model is

$$|\Psi(0)\rangle = (\cos \theta |+, -\rangle + \sin \theta |-, +\rangle) \otimes |0, 0\rangle.$$  

To obtain the time-dependent wave function of considered model we used the so-called dressed states or eigenvectors of the Hamiltonian $H_1$ as in [15, 16].
|\Phi_t\rangle = \xi_t(0,1,0)+X_{12}|X_{12}|X_{13}|X_{14}|X_{14})(i=1,2,3,4),

where

\xi_t = \frac{1}{\sqrt{|X_{11}|^2 + |X_{12}|^2 + |X_{13}|^2 + |X_{14}|^2}},

X_{11} = \frac{1}{\alpha}(-a^2 + g^2 \lambda_1 - \lambda_1^2), \quad X_{12} = \frac{g}{\delta_1 - \lambda_1}, \quad X_{13} = \frac{g^2}{\delta_1 - \lambda_1} + \lambda_1, \quad X_{14} = 1,

X_{11} = \frac{1}{\alpha}(-a^2 + g^2 \lambda_2 - \lambda_2^2), \quad X_{12} = \frac{g}{\delta_2 - \lambda_2}, \quad X_{13} = \frac{g^2}{\delta_2 - \lambda_2} + \lambda_2, \quad X_{14} = 1,

X_{13} = \frac{1}{\alpha}(-a^2 + g^2 \lambda_3 - \lambda_3^2), \quad X_{12} = \frac{g}{\delta_3 - \lambda_3}, \quad X_{13} = \frac{g^2}{\delta_3 - \lambda_3} + \lambda_3, \quad X_{14} = 1,

X_{14} = \frac{1}{\alpha}(-a^2 + g^2 \lambda_4 - \lambda_4^2), \quad X_{12} = \frac{g}{\delta_4 - \lambda_4}, \quad X_{13} = \frac{g^2}{\delta_4 - \lambda_4} + \lambda_4, \quad X_{14} = 1.

The corresponding eigenvalues are

E_1 = \frac{h}{12} \left( 6 \Delta_a - \sqrt{6(T_1 + T_1)} \right), \quad E_2 = \frac{h}{12} \left( 6 \Delta_b - \sqrt{6(T_2 + T_2)} \right),

E_3 = \frac{h}{12} \left( 6 \Delta_c + \sqrt{6(T_3 + T_3)} \right), \quad E_4 = \frac{h}{12} \left( 6 \Delta_d + \sqrt{6(T_3 + T_3)} \right).

Here we use the following notations

T_1 = \sqrt{4C + 6\Delta^2 + 2\sqrt{3}S_1 + 6S_3}, \quad T_2 = 8C + 12\Delta^2 - 2\sqrt{3}S_1 - 6S_3, \quad T_3 = 12\sqrt{6} \left( B - \Delta^2 - \Delta, C \right) / S_1,

S_1 = 27B^2 - 18BC\Delta_a - 2C^2 + 108DA^2 + 72C, \quad S_2 = \sqrt{S_1 + S_1^2 - 4\chi^2}, \quad S_3 = \sqrt{2\chi / 3S_2},

A = \sqrt{a^2 + g^2 + 1}, \quad B = 2A^2 - \delta_1 - g^2 \delta_2, \quad C = A^2 - \Delta^2, \quad D = g^2 - a^2 \Delta^2,

\chi = C^2 + 6B\Delta_a + 12D, \quad \Delta_a = (\delta_1 + \delta_2) / 2, \quad \Delta_c = \sqrt{\delta_1 \delta_2},

where \delta_1 = \Delta_a / \gamma_a, \quad \delta_2 = \Delta_b / \gamma_a, \quad g = \gamma_b / \gamma_a, \quad \alpha = J / \gamma_a.

To obtain the time dependent wave function one can use the formula \Phi(t) = e^{-i\theta / \hbar} |\Phi(0)\rangle. Using the eigenvalues and eigenvectors of Hamiltonian (1) and the initial state (2) we can derive

|\Psi(t)\rangle = C^{(1)}_{11}(t)|X_{11}|X_{12}|X_{13}|X_{14}|X_{14},

where

C^{(1)}_{11} = \cos \theta Z_{11} + \sin \theta Z_{12}, \quad C^{(1)}_{12} = \cos \theta Z_{11} + \sin \theta Z_{12},

C^{(1)}_{31} = \cos \theta Z_{31} + \sin \theta Z_{32}, \quad C^{(1)}_{32} = \cos \theta Z_{31} + \sin \theta Z_{32},

Z_{11} = e^{-i\hbar / \theta} \xi_1 Y_{11} + e^{-i\hbar / \theta} \xi_2 Y_{21} + e^{-i\hbar / \theta} \xi_3 Y_{31} + e^{-i\hbar / \theta} \xi_1 Y_{11} + e^{-i\hbar / \theta} \xi_4 Y_{41},

Z_{12} = e^{-i\hbar / \theta} \xi_1 Y_{11} + e^{-i\hbar / \theta} \xi_2 Y_{21} + e^{-i\hbar / \theta} \xi_3 Y_{31} + e^{-i\hbar / \theta} \xi_4 Y_{41},
Here $\theta = \varphi^*/X_{ji}$.

Now we will consider another type of Bell-like pure initial state of two qubits $|\psi(0)\rangle = \cos |\theta\rangle + \sin |\theta\rangle$. Then for vacuum cavity field full initial state of the system is

$$|\Psi(0)\rangle = (\cos |\theta\rangle + \sin |\theta\rangle) \otimes |0,0\rangle.$$ (3)

For initial state (3) the time-dependent wave function can be written in the form

$$|\Psi(t)\rangle = C_1^{(2)}(t) |+,+,0,0\rangle + C_2^{(2)}(t) |+,-,0,1\rangle + C_3^{(2)}(t) |-,+,1,0\rangle + C_4^{(2)}(t) |+,+,1,0\rangle +$$

$$+ C_5^{(2)}(t) |-,+,0,1\rangle + C_6^{(2)}(t) |-,+,2,0\rangle + C_7^{(2)}(t) |-,-,0,2\rangle + C_8^{(2)}(t) |-,-,1,1\rangle + C_9^{(2)}(t) |-,+,0,0\rangle.$$

For considered initial state the analytical solution of dynamical equations is a very difficult problem, so for we confine ourselves to obtain numerical solution.

3. Entanglement calculations

For two-qubit system described by the reduced density operator $\rho_A(t)$, a measure of entanglement or negativity can be defined in terms of the negative eigenvalues $\mu$ of partial transpose of the reduced atomic density matrix ($\rho_A^T$) [17, 18]: $\varepsilon = -2 \sum \mu$. The partial transpose of the reduced atomic density matrix for considered initial state (2) is

\[\rho_A^T(t) = \begin{pmatrix}
0 & 0 & 0 & H(t)^\dagger \\
V(t) & 0 & 0 & 0 \\
0 & W(t) & 0 & 0 \\
H(t) & 0 & 0 & R(t)
\end{pmatrix}.\] (4)

The elements of matrix (4) are

$V(t) = |C_1^{(1)}(t)|^2$, $W(t) = |C_3^{(1)}(t)|^2$, $R(t) = |C_1^{(1)}(t)|^2 + |C_2^{(1)}(t)|^2$, $H(t) = C_4^{(1)}(t)C_4(t)^\dagger$.

Matrix (4) has only one eigenvalue, which may take a negative value. As a result, the negativity can be written as

$$\varepsilon(t) = \sqrt{R(t)^2 + 4|H(t)|^2} - R(t).$$

The partial transpose of the reduced atomic density matrix $\rho_A^T$ for initial state (3) is
\[
\rho_A(t)_{ij} = \begin{pmatrix}
U_1(t) & 0 & 0 & H_2(t)^* \\
0 & V_1(t) & H_1(t)^* & 0 \\
0 & H_1(t) & W_1(t) & 0 \\
H_2(t) & 0 & 0 & R_1(t)
\end{pmatrix},
\]

where \( U_1(t) = |C_1^{(2)}|^2 \), \( H_1(t) = C_4^{(2)}C_9^{(2)*} \), \( H_2(t) = C_5^{(2)}C_8^{(2)*} + C_4^{(2)}C_3^{(2)*} \), \( V_1(t) = |C_1^{(2)}|^2 + |C_2^{(2)}|^2 \), \( W_1(t) = |C_3^{(2)}|^2 + |C_5^{(2)}|^2 \), \( R_1(t) = |C_6^{(2)}|^2 + |C_7^{(2)}|^2 + |C_8^{(2)}|^2 + |C_9^{(2)}|^2 \).

Matrix (5) has two eigenvalues, which may take a negative value. Then, the negativity can be written as a superposition of two terms. At the same time, each term contributes to the total amount, as long as it takes a positive value. As a result the negativity is

\[
\varepsilon(t) = \sqrt{(U_1(t) - R_1(t))^2 + 4|H_2(t)|^2 - U_1(t) - R_1(t) + \sqrt{(V_1(t) - W_1(t))^2 + 4|H_1(t)|^2 - V_1(t) - W_1(t)}}.
\]

The results of entanglement parameter calculations for Bell’s type initial atomic state \(|\psi_1\rangle = 1/\sqrt{2}(|+,-\rangle + |-,+-\rangle)\) are shown in Figures 1 and 2 and these for initial atomic state \(|\psi_2\rangle = 1/\sqrt{2}(|+,-\rangle + |-,+-\rangle)\) are displayed in Figures 3 and 4. Figure 1 shows that in the case of exact resonance, the dependence of the negativity evolves periodically between 0 and 1. In this case the inclusion of the dipole-dipole interaction leads to a stabilization of entanglement behaviour. Figure 2 show the effect of dipole-dipole interaction on negativity for non-resonant interaction. If qubits A and B interact with a single-mode cavity fields via not-zero detuning the presence of dipole-dipole interaction with intermediate strength leads to increasing of the amplitudes of the negativity oscillations. But for large values of dipole-dipole interaction strength one can see the stabilization of entanglement oscillations as in the case of exact resonance for both equal and unequal qubit-field couplings. Figure 3 shows the time dependence of negativity for initial qubits state (3) and different strength of dipole-dipole interaction in the case of exact resonance. This is different from the results obtained for the previous case. The dipole-dipole interaction in the present case does not lead to stabilization of the entanglement, but has only an effect on the periods and amplitudes of the oscillations of entanglement. However, for non resonant interaction between qubits and fields the influence of dipole-dipole interaction on the entanglement is similar to the previous case as one can see from Figure 4. For considered state we show the entanglement behaviour for different detunings and couplings. For sufficiently large values of the dipole-dipole interaction strength we have to deal with the stabilization of entanglement.

**Figure 1.** Negativity vs. \(\gamma t\) for initial state (2) with \(\theta = \pi / 4\) and \(\delta_a = \delta_b = 0\), \(\gamma_b = \gamma_a\). Parameter \(\alpha = 0\) (dotted), \(\alpha = 3\) (dashed) and \(\alpha = 10\) (solid).

**Figure 2.** Negativity vs. \(\gamma t\) for initial state (2) with \(\theta = \pi / 4\) and \(\delta_b = 5, \delta_a = 3\), \(\gamma_b = \gamma_a\). Parameter \(\alpha = 0\) (dotted), \(\alpha = 3\) (dashed) and \(\alpha = 10\) (solid).
4. Conclusions

In this paper, we investigate the entanglement between two superconducting qubits in the framework of a DJCM with different coupling constants and detunings between the atomic transition frequencies and the cavity fields frequencies taking into account the direct dipole-dipole interaction, and discuss dependence of the atom-atom entanglement on strength of the dipole-dipole interaction and other parameters of the considered system such as the different coupling constants and the detunings. The results show that these parameters have great impact on the amplitude and the period of the atom-atom entanglement evolution. In addition, the presence of sufficiently large dipole-dipole interaction leads to stabilization of entanglement for all Bell-types initial qubits states and different couplings and detunings.

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