On the Holomorphic Factorization for Superconformal Fields

François Gieres†§
Max-Planck-Institut für Physik
Werner-Heisenberg-Institut
Föhringer Ring 6
D - 8000 - München 40

Abstract

For a generic value of the central charge, we prove the holomorphic factorization of partition functions for free superconformal fields which are defined on a compact Riemann surface without boundary. The partition functions are viewed as functionals of the Beltrami coefficients and their fermionic partners which variables parametrize superconformal classes of metrics.

† Alexander von Humboldt Fellow.
§ E-mail address: FRG@DM0MPI11.

MPI-Ph/92-78
September 1992
1 Introduction

The fundamental theorem on holomorphic factorization in string theory goes back to the work of Belavin and Knizhnik [1] (see [2] and [3] for further elaborations and references): this result concerns the factorization of partition functions for free conformal fields on a compact Riemann surface without boundary. Recently [3], this theorem was extended from the case of vanishing central charge to the general case, the partition functions being regarded as functionals on the space of Beltrami coefficients parametrizing complex structures on the Riemann surface. This extension relies on a change of the renormalization prescription which amounts to the addition of a local counterterm shifting the Weyl anomaly to the chirally split diffeomorphism anomaly. Since the supersymmetric generalization of this local counterterm was recently determined [4], we can now address the problem of holomorphic factorization (for arbitrary central charge) in superstring theory along the lines of reference [3]. For previous investigations with vanishing central charge, we refer to [2] and the references given therein.

We work in component field formalism on a compact Riemann surface $\Sigma$ without boundary and we rely on the articles [5] and [4]: in the first of these references, the classification of superconformal structures has been discussed and in the second, the local counterterm that we use in the present work was derived. Although all of these results are based on the geometry of super Riemann surfaces and although our starting equations are derived from superfield considerations, we carry out the main computation in the component field formalism, since some of the initial superspace expressions are quite complex and not very transparent.

The space of superconformal structures on the Riemann surface $\Sigma$ is parametrized by the Beltrami coefficient $\mu \equiv \mu \bar{z}$ and its fermionic partner, the Beltramino $\alpha \equiv \alpha \bar{z}^\theta$, both variables depending on a reference coordinate system $(z, \bar{z})$. The partition function $\Gamma$ is regarded as a functional on this space; it is given by the properly renormalized superdeterminant of the Laplacian associated to a system of free superconformal fields (super bc system). We are interested in proving the holomorphic factorization theorem

$$\delta_{\mu,\bar{\alpha}} \delta_{\mu,\alpha} \left[ \Gamma + \Delta \Gamma \right] = 0 . \quad (1)$$

Here, $\Delta \Gamma$ is made up of local counterterms which allow to shift the super Weyl anomaly to the chirally split form of the superdiffeomorphism anomaly [4]; $\delta_{\mu,\alpha}$ ($\delta_{\mu,\bar{\alpha}}$) represents the variation w.r.t. $\mu$ and $\alpha$ ($\bar{\mu}$ and $\bar{\alpha}$).

2 Free superconformal fields

Component field equations on the Riemann surface $\Sigma$ are best obtained from superfield expressions defined on a super Riemann surface $S \Sigma$ containing $\Sigma$. We now present the relevant superspace expressions and then project them to
component field equations. We use the notation and formalism of reference [3] with \( H_0 \) = 0 (see also appendices A and B of reference [4]).

The action for a super bc system of spin \( j \) reads

\[
S_j = \int_{\mathbf{Z}^0} d^4Z B_{1-j} D_{\theta} C_j \equiv \int_{\mathbf{Z}^0} dZ d\bar{Z} d\theta d\bar{\theta} B_{1-j} D_{\theta} C_j ,
\]

where \( B \) and \( C \) transform according to

\[
B'_{1-j}(Z', \bar{Z'}, \Theta', \bar{\Theta}') = B_{1-j}(Z, \bar{Z}, \Theta, \bar{\Theta}) (D_{\theta} \Theta')^{-(1-j)}
\]

\[
C'_{j}(Z', \bar{Z'}, \Theta', \bar{\Theta}') = C_{j}(Z, \bar{Z}, \Theta, \bar{\Theta}) (D_{\theta} \Theta')^{-j}
\]

under a superconformal change of local coordinates \((Z, \bar{Z}, \Theta, \bar{\Theta}) \rightarrow (Z', \bar{Z}', \Theta', \bar{\Theta}')\) on \( \mathbf{Z}^0 \). If we redefine the variables \( B \) and \( C \) by virtue of the integrating factor \( \Lambda \) which depends on the reference coordinates \((z, \bar{z}, \theta, \bar{\theta})\),

\[
B_{1-j}(Z, \bar{Z}, \Theta, \bar{\Theta}) \equiv \left( B_{1-j} \Lambda^{-(1-j)/2} \right) (z, \bar{z}, \theta, \bar{\theta}) , \quad C_{j}(Z, \bar{Z}, \Theta, \bar{\Theta}) \equiv \left( C_{j} \Lambda^{-j/2} \right) (z, \bar{z}, \theta, \bar{\theta}) ,
\]

we get the local action [3]

\[
S_j = \int_{\mathbf{Z}^0} d^4z \frac{1}{(k_\theta^2)^2} B_{1-j} \left[ (\bar{D} - k_\theta \bar{\theta} \bar{\partial}) - j \left( \frac{1}{2} \partial H_\theta \bar{z} - k_\theta \partial H z \right) \right] C_j .
\]

This functional of \( B \) and \( C \) does not depend on \( \Lambda \), but only on the Beltrami superfield \( H_\theta \) which contains the ordinary Beltrami coefficient \( \mu \) and its fermionic partner \( \alpha \) among its components: by substituting the \( \theta \)-expansions (of the WZ-gauge)

\[
H_\theta \equiv \bar{\theta} \mu + \theta \bar{\theta} \left[-i\alpha \right] , \quad C_{2j} = c_j + \theta [i\epsilon_{j+1}] , \quad B_{1-2j} = i\beta_{1-j} + \theta b_{1-j} ,
\]

we get the component field action

\[
S_{2j} = -\int_{\mathbf{Z}^0} d^2z \left\{ b \left[ \bar{\partial} - \mu \bar{\partial} - j (\partial \mu) \right] c + \frac{1}{2} \alpha \epsilon \right\}
\]

\[
+ \beta \left[ \bar{\partial} - \mu \bar{\partial} - (j + \frac{1}{2}) (\partial \mu) \right] \epsilon + \frac{1}{2} \alpha (\partial c) - j c (\partial \alpha) \right\} .
\]

Here, \( b, c, \beta, \epsilon \) are ordinary conformal fields and we recognize the action for an ordinary bc system of spin \( j \).

Following reference [3], we now introduce a supermetric [4]

\[
\check{\rho}_{zz} \equiv \Lambda^{-1} \Lambda^{-1} \check{\rho}_{zz} .
\]

Furthermore, for a superconformal field \( C_{j\bar{j}}(Z, \bar{Z}, \Theta, \bar{\Theta}) \) transforming as

\[
C'_{j\bar{j}} = C_{j\bar{j}} \left( D_{\theta} \Theta' \right)^{-j} \left( D_{\theta} \bar{\Theta}' \right)^{-j}
\]
and related to \( C_{j\bar{j}}(z, \bar{z}, \theta, \bar{\theta}) \) by
\[
C_{j\bar{j}} = C_{j\bar{j}} \Lambda^{-j/2} \bar{\Lambda}^{-\bar{j}/2},
\]
we define a norm
\[
\| C_{j\bar{j}} \|^2 = \int_{\mathbb{S}^4} d^4 Z \left| C_{j\bar{j}} \right|^2 \left( \partial_0 \rho_z \right)^{(1-j-\bar{j})}
\]
\[
= \int_{\mathbb{S}^4} d^4 z \frac{\sqrt{A}}{k_0} \left| C_{j\bar{j}} \right|^2 \left( \partial_{z\bar{z}} \right)^{(1-j-\bar{j})} \equiv \| C_{j\bar{j}} \|^2. \tag{9}
\]
Here, \( A \) and \( k_0 \) are polynomials in \( H_{\bar{\theta}} \) and its complex conjugate and in their derivatives. For instance, \( \| D_0 X \|^2 \) represents the standard action for a scalar superfield \( X \) for which functional the component field expression can be found in \cite{5}. For \( C_{j\bar{j}} = D_0 C_j \), it is easy to derive an explicit expression for the norm using appendix A of reference \cite{4}: from the \( \theta \)-expansions (WZ-gauge) of \( H_{\bar{\theta}} \), \( C_{2j} \) and
\[
\partial_{z\bar{z}} = \partial_0 + \theta [i \partial_1] + \bar{\theta} [-i \bar{\partial}_1] + \theta \bar{\theta} [-\frac{1}{2} \partial_1 \bar{\partial}_1 / \partial_0], \tag{10}
\]
one then finds
\[
\| D_0 C_{2j} \|^2 = \int_{\Sigma} d^2 z \frac{1}{1 - \mu \bar{\mu}} \left( \partial_0 \right)^{-j} \left| \partial - \mu \partial - j(\partial \mu)\right| c + \frac{1}{2} \alpha \epsilon \|^2. \tag{11}
\]
This expression encompasses the results of the bosonic theory \cite{3}.

The super Laplacian \( \Delta_{2j} \) (acting on \( C_{2j} \)) associated to the supermetric \( \partial_{z\bar{z}} \) is now defined by
\[
\| D_0 C_{2j} \|^2 = \langle C_{2j} \mid -\Delta_{2j} C_{2j} \rangle \tag{12}
\]
and the partition function \( \Gamma \) is given by the superdeterminant of \( \Delta_{2j} \):
\[
\Gamma \equiv \Gamma[\partial_0; \mu, \bar{\mu}, \alpha, \bar{\alpha}] - \Gamma[\partial_0; 0, 0, 0, 0] \tag{13}
\]
\[
\Gamma[\partial_0; \mu, \bar{\mu}, \alpha, \bar{\alpha}] \equiv -\frac{1}{2} \ln \text{sdet} (-\Delta_{2j}).
\]
The superdeterminant is assumed to be renormalized by the \( \zeta \)-function and to involve a mass term accounting for global zero modes \cite{3}.

3 The local counterterm \( \Delta \Gamma \)

By virtue of eq.\((11)\), the supermetric \( \partial_{z\bar{z}} \) has bosonic (fermionic) components \( \partial_0 \) (\( \partial_1 \)). These variables give rise to a non-holomorphic superaffine connection with components \cite{4}
\[
\Gamma^\alpha_z = \partial \ln \partial_0, \quad \bar{\eta}_\theta = \partial_1 / \partial_0. \tag{14}
\]
The corresponding 'field strengths' define a non-holomorphic superprojective connection,

\[ \dot{r}_{zz} = \left( \partial \dot{\Gamma}_z - \frac{1}{2} \dot{\Gamma}_z^2 \right) - \frac{1}{2} \dot{\eta}_\theta \partial \dot{\eta}_\theta, \quad \dot{\chi}_{z\theta} = \left( \partial - \frac{1}{2} \dot{\Gamma}_z \right) \dot{\eta}_\theta, \]  

while the 'field strength' of \( \mu \) is given by the supercovariant derivative

\[ \mathcal{D}_\mu = \left( \partial + \dot{\Gamma}_z \right) \mu + \frac{1}{2} \dot{\eta}_\theta \alpha. \]  

In order to simplify the notation in the following, we will suppress the indices on all component fields.

The local counterterm relating the super Weyl and the chirally split superdiffeomorphism anomalies involves the local functional [4]

\[ \Delta \Gamma = \Gamma_{II} + \Gamma_{III} \]

\[ \Gamma_{II} = k \int_{\Sigma} d^2z \left\{ \mu \left( r - \dot{r} \right) - \alpha \left( \chi - \dot{\chi} \right) \right\} + \text{c.c.} \]

\[ \Gamma_{III} = -k \int_{\Sigma} d^2z \frac{1}{1 - \mu \bar{\mu}} \left\{ \left( \mathcal{D}_\mu \right) \left( \mathcal{D}_{\bar{\mu}} \right) - \frac{1}{2} \bar{\mu} \left( \mathcal{D}_\mu \right)^2 - \frac{1}{2} \mu \left( \mathcal{D}_{\bar{\mu}} \right)^2 \right\}. \]

Here, \( r \equiv r_{zz}(z) \) and \( \chi \equiv \chi_{z\theta}(z) \) denote, respectively, the bosonic and fermionic components of a holomorphic superprojective connection and

\[ k = \frac{Q_j}{12\pi} \quad \text{with} \quad Q_j = 6j(j - 1) + 1 \]

represents the central charge of our model.

4 Variation of \( \Delta \Gamma \) and \( \Gamma \)

Since \( \mu \) and \( \alpha \) parametrize an infinite dimensional space \( \mathcal{M} \), it is preferable to replace the variation \( \delta_{\mu,\alpha} \) by \( dt = dt \partial_t \) where \( t \) is a local coordinate labeling a finite dimensional analytic family \( (\mu_t, \alpha_t) \in \mathcal{M}_t \subset \mathcal{M} \). From the expression (17), we then conclude that

\[ d_i d_t \Delta \Gamma = k \int_{\Sigma_t} d^2z \frac{1}{1 - \mu \bar{\mu}} \left[ (\partial + g) d_i \mu + \frac{1}{2} \dot{\eta} d_i \alpha \right] \left[ (\bar{\partial} + \bar{g}) d_i \bar{\mu} + \frac{1}{2} \bar{\eta} d_i \bar{\alpha} \right], \]

where

\[ g \equiv \dot{\Gamma} - \frac{1}{1 - \mu \bar{\mu}} \left[ \mathcal{D}_{\bar{\mu}} - \bar{\mu} \mathcal{D}_{\mu} \right] \]

and where \( \Sigma_t \) corresponds to a fibre of the bundle \( \Sigma \times \mathcal{M}_t \).
The variation of $\Gamma$ is given by the supersymmetric extension of the index theorem for families of elliptic operators (see [3] and references therein). The local bosonic version of this theorem reads
\begin{equation}
\int_{\Sigma_t} d^2 Z \; d\bar{t} \; dt \; | \partial_t \partial_{\bar{Z}} \ln \hat{\rho}_{ZZ} (Z, \bar{Z}) |^2 \end{equation}
and its manifestly supersymmetric generalization is given by
\begin{equation}
\int_{\Sigma_t} d^4 Z \; d\bar{t} \; dt \; | \partial_t D\bar{\Theta} \ln \hat{\rho}_{Z\bar{Z}} |^2 .
\end{equation}
Here, $\hat{\rho}_{Z\bar{Z}} (Z, \bar{Z}, \Theta, \bar{\Theta})$ is the supermetric introduced in section 2. An explicit expression for $D\bar{\Theta} \ln \hat{\rho}_{Z\bar{Z}}$ in terms of the superfields $H_{\Theta}$ and $\Lambda$ is given by the (complex conjugate of) eq.(30) of reference [4]. This expression is quite complicated, but by going over to the reference coordinates $(z, \bar{z}, \theta, \bar{\theta})$ in the integral (21) and projecting to components, we find the simple result
\begin{equation}
k \int_{\Sigma_t} d^4 Z \; d\bar{t} \; dt \; | \partial_t D\bar{\Theta} \ln \hat{\rho}_{Z\bar{Z}} |^2 = -k \int_{\Sigma_t} (\partial + g) d t \mu + 1 \eta d t \alpha |^2 .
\end{equation}
Obviously, the right hand sides of eqs.(18) and (22) coincide with each other up to an overall sign. Combining these equations with the relation (21), we get the final result
\begin{equation}
d_t d_t [ \Gamma + \Delta\Gamma ] = 0 ,
\end{equation}
i.e. the holomorphic factorization of $\Gamma + \Delta\Gamma$.

5 Conclusion

For a generic value of the central charge, we have shown that the holomorphic splitting property in superstring theory can be recovered on the space of superconformal structures (parametrized by $\mu$ and $\alpha$) by the inclusion of the local counterterm $\Delta\Gamma$.

The holomorphic factorization for ordinary free conformal fields has recently been generalized by including a coupling to Yang-Mills connections on the Riemann surface [5]. By virtue of the results derived in the present work, it should be possible to treat the coupling of superconformal and super Yang-Mills fields along the lines of reference [3].

Acknowledgements

It is a pleasure to thank M.Knecht and J.P.Ader for a critical reading of the manuscript.
References

[1] A.A. Belavin and V.G. Knizhnik, *Phys. Lett.* B168 (1986) 201;
    A.A. Belavin and V.G. Knizhnik, *Sov. Phys. JETP* 64 (1986) 214.

[2] E.D’Hoker and D.H. Phong, *Rev. Mod. Phys.* 60 (1988) 917.

[3] M. Knecht, S. Lazzarini and R. Stora, *Phys. Lett.* B262 (1991) 25.

[4] J.-P. Ader, F. Gieres and Y. Noirot, “Relating Weyl and diffeomorphism anomalies on super Riemann surfaces”, preprint MPI-Ph/92-38, to appear in *Class. Quantum Grav.*

[5] F. Delduc and F. Gieres, *Class. Quantum Grav.* 7 (1990) 1907.

[6] M. Knecht, S. Lazzarini and R. Stora, *Phys. Lett.* B273 (1991) 63.