SUPERMARTINGALE DECOMPOSITION WITH A GENERAL INDEX SET

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Abstract. We prove results on the existence of Doléans-Dade measures and of the Doob-Meyer decomposition for supermartingales indexed by a general index set.

1. Introduction

By Doob’s theorem, supermartingales indexed by the natural numbers decompose into the difference of a uniformly integrable martingale and an increasing process. The relative ease of working with increasing processes explains the prominent role of this result in stochastic analysis and in the theory of stochastic integration. Meyer [19] then proved that, under the usual conditions, Doob’s decomposition exists for right continuous supermartingales indexed by the positive reals if and only if the class $D$ property is satisfied. Doléans-Dade [9] was the first to represent supermartingales as measures over predictable rectangles and to prove that a supermartingale is of class $D$ if and only if its Doléans-Dade measure is countably additive. This line of approach has then become dominant in the work of authors such as Föllmer [13] and Metivier and Pellaumail [18].

In this paper we consider the case of processes indexed by general index sets, illustrated in the following example.

Example 1. Let $\mathcal{C}$ be a collection of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable functions on $\Omega \times \mathbb{R}_+$ and $\mathcal{U} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ an algebra. For each $u \in \mathcal{U}$, let $\mathcal{B}_u = \sigma\{\int_v f(c)\,dt : v \in \mathcal{U}, u \subset v, c \in \mathcal{C}\}$. Assume that $\mathcal{C}$ possesses the following property: $c, c^0 \in \mathcal{C}, u \in \mathcal{U}$ and $F \in \mathcal{B}_u$ imply $c^0 + (c - c^0)1_F 1_{u^c} \in \mathcal{C}$. Let $c^*$ be a solution to the problem

\[
(1.1) \quad \sup_{c^0 \in \mathcal{C}} P \left( \int f(c)\,dt \right)
\]

If $c^0 \in \mathcal{C}$ and $V_u = \int_{u^c} f(c^0)\,dt + P\left( \int_v f(c^*)\,dt \, | \mathcal{B}_u \right)$ then $(V_u : u \in \mathcal{U})$ is a supermartingale. In fact, $v \subset u$ implies $\mathcal{B}_u \subset \mathcal{B}_v$ and $P(V_v|\mathcal{B}_u) = V_u + P\left( \int_{u^v \cap v^c} f(c^0) - f(c^*) \right)\,dt \, | \mathcal{B}_u) \leq V_u$.

We prove, in Theorem 1, a necessary and sufficient condition, the class $D_0$ property, for the existence of a Doléans-Dade measure associated to a supermartingale. Based on this, we establish, in Theorem 2, a sufficient condition, the class $D_*$ property, for the existence of a Doob-Meyer decomposition. In Corollary 1 we consider supermartingales of uniformly integrable variation. The two key properties involved in our results are the possibility of extending the original supermartingale to a larger filtration and some form of the optional sampling theorem. From a mathematical perspective we exploit extensively results from the theory of finitely additive measures.

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Among the many papers devoted to the theory of stochastic processes indexed by general sets, the ones more directly related to ours are those of Dozzi, Ivanoff and Merzbach [10], who obtain a form of the Doob Meyer decomposition, of Ivanoff and Merzbach [15], who extend such decomposition by a localization argument, and, to a much lesser extent, of De Giosa and Mininni [7]. These works, which draw in turn from an unpublished paper of Norberg [21], apply classical techniques, based on right continuity, separability and uniform integrability and, to make this possible, introduce a number of set-theoretical as well as of topological restrictions on the index set. These assumptions represent the main difference with the approach followed here, which owes, perhaps, more to the work of Mertens [17] than to that of Meyer [19].

2. Preliminaries and notation

We fix some general notation, mainly in accordance with [11]. When \( S \) is a set \( 2^S \) denotes its power set and \( 1_S \) its indicator function. If \( \Sigma \subset 2^S \), typically an algebra, the symbols \( ba(\Sigma) \) (resp. \( ca(\Sigma) \)) and \( B(\Sigma) \) designate the spaces of finitely (resp. countably) additive set functions on \( \Sigma \) and the closure of the set of \( \Sigma \) simple functions with respect to the supremum norm, respectively (however we prefer \( B(S) \) to \( B(2^S) \)). \( P(\Sigma) \) will be the subcollection of \( ba(\Sigma)_+ \), consisting of those elements \( Q \) whose restriction \( Q|\Sigma \) to \( \Sigma \) is a countably additive probability measure. Finitely additive measures will throughout be identified with the linear functional arising from the corresponding expected value so that we prefer writing \( \mu(f) \) rather than \( \int f d\mu \). We recall that if \( \Sigma \) is an algebra of subsets of \( S \) and \( \mu \in ba(\Sigma)_+ \) then there exists \( \bar{\mu} \in ba(2^S)_+ \) such that \( \bar{\mu}|\Sigma = \mu \), [2] theorem 3.2.10, p. 70).

We take two sets \( \Omega \) and \( I \) as given, put \( \bar{\Omega} \equiv \Omega \times I \) and write \( ba(\bar{\Omega}) \) and \( B(\bar{\Omega}) \) more briefly as \( ba \) and \( B \). \( 2^{\bar{\Omega}} \) is ordered by reverse inclusion that is \( s \leq t \) whenever \( t \subset s \); \( s < t \) means \( s \leq t \) and \( s \cap t^c \neq \emptyset \). \( t(i) \) denotes the \( i \)-section \( \{ \omega \in \bar{\Omega} : (\omega, i) \in t \} \) of \( t \) and \( \{ s < t \} = \bigcup_{i \in I} (s \cap t^c)(i) \) : thus \( \{ s < \emptyset \} \) is just the projection of \( s \) on \( \bar{\Omega} \). The special case where \( I = \mathbb{R}_+ \) and some probability measure \( P \) on \( \Omega \) is given will be referred to as the classical theory.

Also given are a collection \( T \) of subsets of \( \bar{\Omega} \) containing \( \{ \bar{\Omega} \} \) and a filtration \( \mathcal{A} = (\mathcal{A}_t : t \in T) \), that is an increasing collection of algebras of subsets of \( \Omega \) satisfying:

\[
F \cap (s \cap t^c)(i) \in \mathcal{A}_s \cap \mathcal{A}_t \quad \text{and} \quad F \cap \{ s < t \} \in \mathcal{A}_s \cap \mathcal{A}_t \quad s, t \in T, \quad F \in \mathcal{A}_s, \quad i \in I
\]

Define also \( \mathcal{A} = \bigcup_{t \in T} \mathcal{A}_t \) and \( \mathcal{F} = \sigma \mathcal{A} \).

In the classical theory, \( T \) would typically be a family of stochastic intervals such as \( ]\tau, \infty[ \) or \( [\tau, \infty[ \) with \( \tau \) a stopping time; the case where each \( t \in T \) is deterministic, i.e. of the form \( t = \Omega \times J \) with \( J \subset I \), is of course a possibility. Dozzi et al. [10] take \( T \) to be a lattice of closed subsets of a (locally) compact topological space and assume that \( T \) is closed with respect to countable intersections. One should remark that in the present setting the second inclusion in (2.1) does not follow from the first one and must therefore be explicitly assumed.

Repeatedly in what follows we shall take into consideration an alternative filtration \( \tilde{\mathcal{A}} = (\mathcal{A}_u : u \in U) \) where \( U \subset 2^{\bar{\Omega}} \) is closed under union and intersection, \( T \cup \{ \emptyset \} \subset U \) and \( \mathcal{A}_t \subset \mathcal{A}_u \) for all \( t \in T \). As a matter of notation, the same symbol denoting some object defined relatively to \( \mathcal{A} \) will be used with an upper bar to designate the corresponding object defined relatively to \( \tilde{\mathcal{A}} \).

\[\text{A more explicit comparison with this approach will be made in the following sections. See in particular Remark} \]

3. Finitely Additive Supermartingales

A finitely additive process (on $\mathcal{A}$) is an element $m = (m_t : t \in T)$ of the product space $\prod_{t \in T} ba(\mathcal{A}_t)$. We shall always use the convention of letting $m_\emptyset$ be the null measure on $\mathcal{A}$. A finitely additive process $m$ is bounded if $\|m\| = \sup_{t \in T} \|m_t\| < \infty$; it is of bounded variation if

$$\|m\|_V \equiv \sup \sum_{n=1}^N \|m_{s_n} - m_{t_n}\|_{\mathcal{A}_{s_n}} < \infty$$

the supremum being over the family $\mathcal{D}$ of all finite, disjoint collections

$$d = \{s_n \cap t_n^c : n = 1, \ldots, N\} \quad \text{with} \quad s_n, t_n \in T \cup \{\emptyset\}, \ s_n \leq t_n \quad n = 1, \ldots, N$$

Processes of bounded variation are thus bounded. Our definition (3.1) slightly departs from the original one of Fisk [12] as we do not require $t_n \leq s_{n+1}$ for all $n$.

We speak of the finitely additive process $m$ as a finitely additive supermartingale if $m_t|\mathcal{A}_s \leq m_s$ for all $s, t \in T$ such that $s \leq t$. $f : \bar{\Omega} \rightarrow \mathbb{R}$ is an elementary process, $f \in \mathcal{E}$, if it may be written in the form

$$f = \sum_{n=1}^N f_n 1_{t_n} \quad \text{with} \quad f_n \in \mathcal{B}(\mathcal{A}_{t_n}), \ t_n \in T \quad n = 1, \ldots, N$$

$\mathcal{P}$ denotes the (predictable) $\sigma$ algebra generated by the elementary processes. We write $f \in \mathcal{E}^*$ if the requirement $f_n \in \mathcal{B}(\mathcal{A}_{t_n})$ in (3.3) is replaced by $f_n \in \mathcal{B}(\mathcal{P})$ for $n = 1, \ldots, N$.

A finitely additive supermartingale $m$ is strong if it is of bounded variation and

$$0 = \sum_{n=1}^N f_n 1_{t_n} \in \mathcal{E} \quad \text{imply} \quad \sum_{n=1}^N m_{t_n}(f_n) = 0$$

As $1_{s < t} = 1_{s \cap t^c} = 0$, a strong finitely additive supermartingale $m$ must satisfy $m_s(\{s < t\}^c) = m_t(\{s < t\}^c)$ for all $s \leq t$: implicit in (3.1) is thus a version of the optional sampling theorem. It is known that this theorem is far from obvious with a general index set (see [14] and [16]) and it may actually fail even with $\mathbb{R}_+$ as the index set unless the usual conditions hold (see [8] p. 393), from which our terminology is borrowed.

The assumption that a finitely additive supermartingale is strong will thus play a major role.

As argued by Dozzi et al. [10], for many a purpose the index set $T$ does not possess enough mathematical structure, a problem that induces to consider possible extensions. The following example illustrates that this may not be entirely obvious.

**Example 2.** Let $\bar{m} = (\bar{m}_u : u \in U)$ be a finitely additive supermartingale on $\bar{\mathcal{A}}$ and define the semi-algebra

$$U(d) = \{u \cap v^c : u, v \in U, \ u \leq v\}$$

For each $u \cap v^c \in U(d)$ one may be tempted to extend $\bar{m}$ to $U(d)$ by letting

$$\bar{m}_{u \cap v^c} = (\bar{m}_u - \bar{m}_v)|\mathcal{A}_{u^c} \in ba(\mathcal{A}_{u^c})$$

Such an extension however need not be well defined if $\bar{m}$ is not strong. Consider in fact $u_1, v_1, u_2, v_2 \in U$ such that $u_1 \leq v_1 \leq u_2 \leq v_2$ and $\bar{m}_{u_1}(\Omega) > \bar{m}_{v_1}(\Omega) > \bar{m}_{u_2}(\Omega) = \bar{m}_{v_2}(\Omega)$. If $u_1 \cap v_1^c = u_2 \cap v_2^c$ then (3.6) implies $(\bar{m}_{u_1} - \bar{m}_{v_1})|\mathcal{A}_{u_1} = (\bar{m}_{u_2} - \bar{m}_{v_2})|\mathcal{A}_{u_1} = 0$ which contradicts $(\bar{m}_{u_1} - \bar{m}_{v_1})(\Omega) > 0$. In the case of classical processes, when each $\bar{m}_u \in ba(\mathcal{A}_{u^c})$ is replaced by a corresponding $\sigma\mathcal{A}_{u^c}$ measurable random quantity $X_u$, assume that $X_{u_2} = X_{v_2}$. Then (3.6) implies $X_{u_1} = X_{v_1}$ which is contradictory if $X_{v_1}$ fails to be $\sigma\mathcal{A}_{u_1}$ measurable.
Remark 1. In many papers (see, e.g., [7, p. 74], [10, proposition 2.1] or [15, p. 85]) the extension \(3.6\) of a process indexed by a lattice of sets to its generated semi-ring is claimed to exist by virtue of [21, proposition 2.3, p. 9]. This claim is however not correct for all lattices and all processes, as Example 2 shows. In fact in the paper by Norberg, the index set consists of all lower sets \(f \equiv \{g \in \mathcal{L} : g \leq f\}\) of elements \(f\) of some lattice \(\mathcal{L}\) satisfying the property

\[
(3.8) \quad f \leq g \quad \text{and} \quad g \leq f \quad \text{imply} \quad f = g, \quad f, g \in \mathcal{L}
\]

In this setting if \(f_i, g_i \in \mathcal{L}\) and \(g_i \leq f_i\) for \(i = 1, 2\) then \(f_1 \cap (\downarrow g_1)^c = f_2 \cap (\downarrow g_2)^c\) if and only if either \(f_i = g_i\) for \(i = 1, 2\) or \(f_1 = f_2\) and \(g_1 = g_2\). Thus Example 2 does not apply.

Implicit in the above remarks is the importance of the set-theoretic properties of the index set. This is specially true for what concerns the representation of elementary processes. Let, for example, \(f = \sum_{k=1}^{K} f_k 1_{u_k^c} \in \tilde{\mathcal{D}}\). Denote by \(\{\pi_1, \ldots, \pi_N\}\) the collection of non empty subsets of \(\{1, \ldots, K\}\) and, for \(n = 1, \ldots, N\), define (with the convention \(\bigcup\emptyset = \emptyset\))

\[
u_n = \bigcap_{k \in \pi_n} u_k^c, \quad v_n = u_n \cap \bigcup_{j \notin \pi_n} u_j^c \quad \text{and} \quad f_n = \sum_{k \in \pi_n} f_k
\]

Then

\[
(3.9) \quad f = \sum_{n=1}^{N} f_n 1_{u_n \cap v_n^c} \quad \text{where} \quad f_n \in \mathcal{I}(\mathcal{A}_{u_n}) \quad n = 1, \ldots, N \quad \text{and} \quad \{u_n \cap v_n^c : n = 1, \ldots, N\} \in \tilde{\mathcal{D}}
\]

(see \(3.2\) for the definition of \(\tilde{\mathcal{D}}\)). We will henceforth refer to \((3.8)\) as the canonical representation of \(f\).

Another noteworthy implication of the set theoretic properties of \(U\) is that the collection \(\tilde{\mathcal{D}}\) defined as in \(3.2\) is a directed set relatively to refinement, that is if we write \(\delta' \geq \delta\) whenever \(\delta, \delta' \in \tilde{\mathcal{D}}\) and each \(u \cap v^c \in \delta\) may be written as \(\bigcup_{n=1}^{N} u_n \cap v_n^c\) with \(u_n \cap v_n^c \in \delta' n = 1, \ldots, N\).

Eventually we have the following (with \(f^+ = f \lor 0\)):

Lemma 1. Let \(\tilde{m}\) be a positive, strong, finitely additive supermartingale on \(\tilde{\mathcal{A}}\). Then

\[
(3.10) \quad \sum_{n=1}^{N} \tilde{m}_{u_n^c}(f_n') \leq \|\tilde{m}\|_V \|f^+\| \quad f = \sum_{n=1}^{N} f_n' 1_{u_n^c} \in \tilde{\mathcal{D}}
\]

Proof. Fix \(f = \sum_{n=1}^{N} f_n' 1_{u_n^c} \in \tilde{\mathcal{D}}\) and let \(\sum_{k=1}^{K} f_k 1_{u_k^c} \in \tilde{\mathcal{D}}\) be its canonical representation as in \((3.8)\). For \(k = 1, \ldots, K\) let \(\mu_{f,k} \in ba_+\) be an extension of \(\tilde{m}_{u_k} - \tilde{m}_{v_k} \mid_\mathcal{A}_{u_k} \) to \(2^\Omega\). Put \(\mu_f = \sum_{k=1}^{K} \mu_{f,k}\): the inequality \(|\mu_f| \leq \|\tilde{m}\|_V\) is obvious. Thus

\[
\sum_{n=1}^{N} \tilde{m}_{u_n^c}(f_n') = \sum_{k=1}^{K} (\tilde{m}_{u_k} - \tilde{m}_{v_k})(f_k 1_{\{u_k < v_k\}}) = \sum_{k=1}^{K} \mu_{f,k}(f_k 1_{\{u_k < v_k\}}) \leq \mu_f \left(\sup_{i \in I} f(i)\right)
\]

the first equality being a consequence of the fact that \(\tilde{m}\) is strong. \(\Box\)

Lemma 11 legitimates a special interest for conditions that permit the construction of an extension of our original finitely additive supermartingale \(m\), that is of a finitely additive process \(\tilde{m}\) defined on \(\tilde{\mathcal{A}}\) such that \(\tilde{m}_t \mid_\mathcal{A}_t = m_t\) for all \(t \in T\) (in symbols \(\tilde{m} \mid_\mathcal{A} = m\)). It turns out that the problem of extending \(m\) is related to the time honoured question of whether finitely additive supermartingales may be represented as measures on \(\tilde{\Omega}\), i.e. the existence of Doléans-Dade measures.

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2The proof of [21, lemma 2.2] uses property \((3.7)\) without explicitly assuming it.
Theorem 1. Let $m$ be a finitely additive process on $\mathbb{A}$. The following are equivalent:

(i) $m$ is of class $D_0$, that is

$$\sup \left\{ \sum_{n=1}^{N} m_{t_n}(f_n) : 1 \geq \sum_{n=1}^{N} f_n 1_{t_n} \in \mathcal{E} \right\} < \infty$$

(ii) $m$ admits a Doléans-Dade measure, that is an element of the set

$$\mathcal{M}(m) = \{ \lambda \in ba_+ : \lambda(F1_t) = m_t(F), \ F \in \mathcal{A}_t, \ t \in T \}$$

(iii) there exists a positive, strong, finitely additive supermartingale $\bar{m}$ on $\mathbb{A}$ such that $\bar{m}|\mathbb{A} = m$.

Proof. For $t \in T$, let $\mathcal{L}_t = \{ f1_t : f \in \mathcal{B}(\mathcal{A}_t) \}$ and define $\phi_t : \mathcal{L}_t \rightarrow \mathbb{R}$ implicitly as $\phi_t(f1_t) = m_t(f)$. Then $\mathcal{L}_t$ is a linear subspace of $\mathcal{B}$ and $\phi_t$ a linear functional on it. Condition (i) corresponds to requiring that the collection $(\phi_t : t \in T)$ is coherent in the sense of [5, Corollary 1]; thus (i) is equivalent to (ii). For $\lambda \in \mathcal{M}(m)$, define $\bar{m} \in \prod_{u \in U} ba(\mathcal{A}_u)$ implicitly as

$$\bar{m}_u(F) = \lambda(F1_u) \quad u \in U, \ F \in \mathcal{A}_u$$

It is clear that $\bar{m}$ is a positive, strong, finitely additive supermartingale and that $\bar{m}|\mathbb{A} = m$. The implication (iii)$\Rightarrow$(i) follows from Lemma [1].

A Doléans-Dade measure is usually defined to be countably additive relatively to $\mathcal{P}$ (see [9], [13] and [18]). We believe that this additional property is not really essential to obtain several interesting results, such as the Doob Meyer decomposition. The advantage of taking $\mathcal{M}(m)$ to be a subset of $ba$, implicit in Theorem [1], is that the existence of a measure so defined turns out to be a property independent of the given filtration. It should be mentioned that a strong finitely additive supermartingale need not be of class $D_0$, if the index set is not closed with respect to union and intersection.

The next task is to establish a version of the Doob Meyer decomposition. A first step in this direction is made by remarking that, if $\lambda \in \mathcal{M}(m)$ and $H \subset \bar{\Omega}$, we may define $\lambda_H \in ba(\Omega)$ implicitly as

$$\lambda_H(F) = \lambda((F \times I) \cap H) \quad F \subset \Omega$$

Then $(\lambda_u : u \in U)$ is an increasing family in $ba(\Omega)$; moreover

$$m_t = \lambda_{\bar{\Omega}} |\mathcal{A}_t - \lambda_u |\mathcal{A}_t \quad t \in T$$

i.e. $m$ decomposes into the difference of a finitely additive martingale and a finitely additive increasing process, a result first obtained by Armstrong [1] (see also [4, corollary 1, p. 591]).

4. Classical Supermartingales

A particularly interesting special case is that of classical supermartingales that we treat, in accordance with [4], without the assumption of a given probability measure. In order to avoid additional notation we assume in what follows (and without loss of generality) that $\mathcal{A}_u$ is a $\sigma$ algebra for each $u \in U$.

Let $m$ be a positive finitely additive supermartingale on $\mathbb{A}$ and define

$$\mathcal{M}^{uc} = \{ \lambda \in ba_+ : \lambda|\mathcal{F} \in ca(\mathcal{F}) \} \quad \text{and} \quad \mathcal{M}^{uc}(m) = \mathcal{M}^{uc} \cap \mathcal{M}(m)$$

By a terminological curiosum, a finitely additive supermartingale as in Theorem [1](iii) is called a weak supermartingale in [10, definition 2.1, p. 503].
The property \( \mathcal{S}^{uc} (m) \neq \emptyset \) is actually necessary and sufficient for a finitely additive supermartingale indexed by \( \mathbb{R}_+ \) to admit a Doob Meyer decomposition, \cite{4} theorem 4, p. 597. This conclusion extends to the case of a linearly ordered index set but needs to be strengthened considerably in the more general case. It suffices, however, to get a primitive version of this fundamental result. Let in fact \( \lambda \in \mathcal{H}^{uc} (m) \) and fix \( P \in \mathbb{P}(\mathcal{F}) \) such that \( \Lambda_\Omega |\mathcal{F} \ll P |\mathcal{F} \) (in symbols \( P \in \mathbb{P}(\mathcal{F}, \lambda) \)). Then, letting \( \tilde{M} \), \( \tilde{A}_u \), \( \tilde{X}_u \in L(P) \) be the Radon Nikodym derivatives of \( \Lambda_\Omega |\mathcal{F} \), \( \lambda_u |\mathcal{F} \) and \( \lambda |\mathcal{F} \) with respect to \( P |\mathcal{F} \) we get from \( \ref{3.13} \):

\[
(4.2) \quad \tilde{m}_u (F) = P (\tilde{X}_u 1_F) = P ((\tilde{M} - \tilde{A}_u) 1_F) \quad u \in U, \; F \in \mathcal{F}_u
\]

where \( \tilde{m} \) is the extension of \( m \) to \( \tilde{A} \) mentioned in Theorem \( \ref{1}(iii) \). The crucial step is thus obtaining a version of \( \ref{4.2} \) in which \( \tilde{A} \) is predictable in some due sense.

In the classical theory, where \( P \in \mathbb{P}(\mathcal{F}) \) is given, there is a strict relationship between the property \( \mathcal{S}^{uc} \neq \emptyset \) and the existence of a Doléans-Dade measure which is countably additive in restriction to the predictable \( \sigma \) algebra. In fact ordinary properties of the predictable projection guarantee that if \( \lambda \in ba \) that vanishes on \( P \) negligible sets and is countably additive in restriction to \( \mathcal{F} \) then it has an extension which is countably additive on \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \) and thus on \( \mathcal{F} \). Conversely, each \( \lambda \in \mathcal{S}^{uc} \) is a Daniell integral in restriction to any vector sublattice \( \mathcal{L} \) of \( \mathcal{B} \) with the property that \( \sup_{i \in I} f(i) \) is \( \mathcal{F} \) measurable for each \( f \in \mathcal{L} \) and that each sequence \( (f_n)_{n \in \mathbb{N}} \) in \( \mathcal{L} \) with \( f_n \downarrow 0 \) converges to 0 uniformly in \( i \in I \). With \( I = \mathbb{R}_+ \) a simple variant of Dini’s theorem shows that processes with finite range and upperly semicontinuous paths possess this property, a fact used by Dellacherie and Meyer \cite{8} theorem 2, p. 184] to prove, after Mertens \cite{17}, a version of the Doob Meyer decomposition that does not require the usual conditions. Under the current assumptions, however, this last argument does not hold without introducing additional topological properties; on the other hand, the extension of the notion of predictable projection to our setting, that in Dozzi et al. \cite{10}, A3, p. 516] is by assumption, is not obvious.

For given \( P \in \mathbb{P}(\mathcal{F}) \) and \( d \in \mathcal{D} \) define the following elementary process (up to a \( P \) equivalence class)

\[
(4.3) \quad \mathcal{P}_P (b) = \sum_{u \cap w \in c} P \left( \inf_{i \in w} b(i) \right| \mathcal{F}_u) 1_{u \cap w} = b \in \mathcal{B}
\]

where the conditional expectation appearing in \( \ref{4.3} \) is defined as in \cite{4} Theorem 1, p. 588]. Of course if \( b \in \mathcal{E} \) and \( d \in \mathcal{D} \) is large enough then \( \mathcal{P}_P (b) = b \).

**Lemma 2.** If \( \lambda \in \mathcal{H}^{uc} \) and \( P \in \mathbb{P}(\mathcal{F}, \lambda) \) then there is \( \lambda^P \in ba_+ \) such that \( \lambda^P \) vanishes on \( P \) null sets and \( \lambda^P (f) = \lim_{d \in \mathcal{D}} \lambda^P (\mathcal{P}_P (f)) \) \( f \in \mathcal{E}^* \), \( g \in \mathcal{E} \).

If \( \lambda^P | \mathcal{D} \in ca(\mathcal{D}) \) then there exists \( \mathcal{P}_P : \mathcal{B} \rightarrow L^\infty (\lambda | \mathcal{D}) \) such that

\[
(4.5) \quad \mathcal{P}_P (g) = g \quad \text{and} \quad \lambda^P (f) = \lambda^P (\mathcal{P}_P (f)) \quad f \in \mathcal{E}^*, \; g \in \mathcal{E}
\]

**Proof.** Consider the functional \( \gamma : \mathcal{B} \rightarrow \mathbb{R} \) defined as \( \gamma (f) = \lim_{d \in \mathcal{D}} \lambda (\mathcal{P}_P (f)) \) for any \( f \in \mathcal{B} \). Then, \( \gamma \) is a concave integral in the sense of \cite{5} definition 1, it is linear on \( \mathcal{E}^* \) and such that \( \gamma = \lambda \) in restriction to \( \mathcal{E} \); moreover, \( \gamma (b) = 0 \) for all \( b \) in the linear space \( \mathcal{L} = \{ g \in \mathcal{B} : P (|g| > \eta) = 0 \text{ for all } \eta > 0 \} \). Given that \( \mathcal{L} \) is a linear space, by \cite{5} lemma 2, p. 4] there exists \( \lambda^P \in ba_+ \) such that

\[
\lambda^P (g) = 0 \quad \text{and} \quad \lambda^P (f) \geq \gamma (f) \quad g \in \mathcal{L}, \; f \in \mathcal{B}
\]

Where \( \lim \) denotes the Banach limit.
Consequently if \( g \in \bar{\mathcal{E}} \) and \( f \in \bar{\mathcal{E}}^* \), then \( fg \in \bar{\mathcal{E}}^* \) and thus

\[
\lambda^P(fg) = \gamma(fg) = \text{LIM}_{d \in \mathcal{D}} \lambda(\mathcal{P}^d_p(fg)) = \text{LIM}_{d \in \mathcal{D}} \lambda(\mathcal{P}^d_p(f)g) = \text{LIM}_{d \in \mathcal{D}} \lambda^P(\mathcal{P}^d_p(f)g)
\]

The last claim follows by taking \( \mathcal{P}^d_p(f) = \lambda^P(f|\mathcal{P}) \).

When \( \lambda \in \mathcal{M}^{uc} \) and \( P \in \mathbb{P}(\mathcal{F}, \lambda) \) then \( \lambda^P \in ba_{+} \), defined as in Lemma \([2]\) will be referred to as the \( P \) compensator of \( \lambda \), disregarding non uniqueness. Remark that if \( \lambda \in \mathcal{M}^{uc}(m) \) for some finitely additive supermartingale \( m \), then its \( P \) compensator \( \lambda^P \) is itself a Doléans-Dade measure for \( m \), i.e. \( \lambda^P \in \mathcal{M}(m) \). It is, however, not possible to conclude that \( \lambda^P \in \mathcal{M}^{uc}(m) \) in the general case, i.e. that

\[
\mathcal{M}^*(m) = \{ \lambda^P \in \mathcal{M}^{uc}(m) : \lambda \in \mathcal{M}^{uc}(m) \} \neq \emptyset
\]

We shall refer to \((4.6)\) by saying that \( m \) is of class \( D_{*} \).

When \( P \in \mathbb{P}(\mathcal{F}) \), \( (\bar{A}_u : u \in U) \) is a \( P \) increasing process if \( P(0 = \bar{A}_0 \leq \bar{A}_u \leq \bar{A}_v) = 1 \) for all \( u, v \in U \) with \( u \leq v \). \( (\bar{B}_u : u \in U) \) is then a modification of \( \bar{A} \) if \( P(\bar{A}_u = \bar{B}_u) = 1 \) for all \( u \in U \).

**Theorem 2.** Let \( \bar{m} \) be a finitely additive supermartingale of class \( D_{*} \) on \( \mathbb{A} \). Then for some \( P \in \mathbb{P}(\mathcal{F}) \) there exists one and only one (up to modification) way of writing

\[
(4.7) \quad \bar{m}_u(F) = P((M - \bar{A}_u)1_F) \quad u \in U, \ F \in \mathcal{A}_u
\]

where \( M \in L(P) \) and \( \bar{A} \) is an increasing process, adapted to \( \mathbb{A} \) and such that

\[
(4.8) \quad P \int fd\bar{A} = \text{LIM}_{d \in \mathcal{D}} P \int_{\mathcal{E}^*} \mathcal{P}^d_p(f)d\bar{A} \quad f \in \bar{\mathcal{E}}^*
\]

**Proof.** Let \( \lambda \in \mathcal{M}^{uc}(m) \), \( P \in \mathbb{P}(\mathcal{F}, \lambda) \) and \( \lambda^P \in \mathcal{M}^*(m) \). Define \( M \) and \( \bar{A}'_u \) to be the Radon Nikodym derivatives with respect to \( P|\mathcal{F} \) of \( \lambda^P_{|\mathcal{F}} \) and \( \lambda^P_{|\mathcal{F}} \). \((4.7)\) is thus a version of \((4.2)\). Clearly,

\[
(4.9) \quad P \int fd\bar{A}' = P \sum_{u \cap v \in e} f_u(\bar{A}'_u - \bar{A}'_v) = \lambda^P(f) \quad \text{for all} \quad f = \sum_{u \cap v \in e} f_u 1_{u \cap v} \in \bar{\mathcal{E}}^*
\]

\((4.4)\) then implies that \((4.8)\) holds for \( \bar{A}' \) and its modifications, among which there exists one which is adapted. In fact, if \( b \in \mathcal{B}(\mathcal{F}) \), \( \bar{u} \in U \) and \( P(b|\mathcal{A}_u) = 0 \), then, choosing \( d \in \mathcal{D} \) finer than \( \{\bar{w} \} \)

\[
\mathcal{P}^d_p(b1_{\bar{u}^c}) = \sum_{\{u \cap w \in e : u \cap w \subset \bar{u} \}} P(b|\mathcal{A}_u)1_{u \cap w} = \sum_{\{u \cap w \in e : u \cap w \subset \bar{w} \}} P(b1_{\{u \leq \bar{u} \}}|\mathcal{A}_u)1_{u \cap w} = 0
\]

a conclusion following from \((2.1)\) and the fact that \( P(b1_{\{u \leq \bar{u} \}}|\mathcal{A}_u) = P(P(b|\mathcal{A}_u)1_{\{u \leq \bar{u} \}}|\mathcal{A}_u) \). We conclude that \( \lambda^P_{|\mathcal{F}}(b) = \lambda^P(\bar{A}_{\bar{u}} - \bar{A}_u) = 0 \). Let \( \bar{A}_u = P(\bar{A}'_u|\mathcal{A}_u) \) and \( F \in \mathcal{F} \). Then,

\[
(4.10) \quad P(\bar{A}'_u1_F) = \lambda^P_{|\mathcal{F}}(F) = \lambda^P_{|\mathcal{F}}(P(F|\mathcal{A}_u)) = P(\bar{A}'_uP(F|\mathcal{A}_u)) = P(\bar{A}_u1_F)
\]

\( \bar{A} \) is thus an adapted modification of \( \bar{A}' \) and therefore itself an increasing process meeting \((4.8)\). Suppose that \( P(N|\mathcal{A}_u) - \bar{B}_u \) is another decomposition such as \((4.7)\). Then if \( F \in \mathcal{A}_u \) and \( d \in \mathcal{D} \)

\[
P \int \mathcal{P}^d_p(F1_u)d\bar{A} = -P \int \mathcal{P}^d_p(F1_u)d\bar{X} = P \int \mathcal{P}^d_p(F1_u)d\bar{B}
\]

and, if both \( \bar{A} \) and \( \bar{B} \) meet \((4.8)\), \( P(\bar{A}_u1_F) = P(\bar{B}_u1_F) \).
Remark that Theorem 2 is actually weaker than the classical Doob Meyer decomposition first of all because the class $D_*$ property is only a sufficient condition but need not be necessary. Second, we established uniqueness only up to a modification rather than indistinguishability, a circumstance which is almost unavoidable in the absence of separability of the index set and of right continuity of the process.

It should also be remarked that it may not be possible to establish the above decomposition for the original index set $T$ because the increasing process $\tilde{A}$ may not be adapted to the original filtration $\mathcal{A}$. On the other hand the class $D_*$ property is a global property and is thus preserved under any enlargement of the filtration. It appears therefore that the decomposition of Doob Meyer depends more on the structure of the index set than on the filtration.

A less general decomposition is based on the following, further uniform integrability condition for processes.

**Definition 1.** A stochastic process $\bar{Y}$ on $\mathcal{A}$ is said to be of $P$ uniformly integrable variation for some $P \in \mathbb{P}(\mathcal{F})$ if the collection $\mathcal{V}(\bar{Y}) = \{\sum_{u \cap \nu^c \in d} \bar{Y}_u - P(\bar{Y}_v|\mathcal{A}_u) : d \in \mathcal{D}\}$ is uniformly $P$ integrable.

**Corollary 1.** Let $P \in \mathbb{P}(\mathcal{F})$ and $\bar{X}$ be a positive, strong, $P$ supermartingale on $\mathcal{A}$. Then $\bar{X}$ is of $P$ uniformly integrable variation if and only if it decomposes as in (4.7) with the increasing process $\tilde{A}$ being, in addition, of $P$ uniformly integrable variation.

**Proof.** If (4.7) holds then $\mathcal{V}(\bar{X}) = \mathcal{V}(\tilde{A})$ so that $\bar{X}$ is of $P$ uniformly integrable variation if and only if so is $\tilde{A}$. Assume then that $\bar{X}$ is of $P$ uniformly integrable variation and let $\bar{m}$ be the finitely additive supermartingale associated to $\bar{X}$. Choose $\lambda \in \mathcal{M}^uc(\bar{m})$ and let $\bar{X}$ be its $P$ compensator.

If $F \in \mathcal{F}$ then

$$\lambda(\mathcal{P}^d(F)) = \sum_{u \cap \nu^c \in d} P(F|\mathcal{A}_u)(X_u - X_v) = P\left(1_F \sum_{u \cap \nu^c \in d} X_u - P(X_v|\mathcal{A}_u)\right) \leq \sup_{V \in \mathcal{D}} P(V^{1_F})$$

Thus, $\lambda^d_{\bar{X}} \mathcal{F} \ll P|\mathcal{F}$ and $\bar{m}$ is thus of class $D_*$: (4.7) follows from Theorem 2.

The characterisation provided in Corollary 1 is less satisfactory then it may appear at first sight. In fact the property involved is significantly stronger than what considered in the classical setting. In fact even increasing processes may fail to be of uniformly integrable variation.

The special case of a linearly ordered index set is eventually considered.

**Corollary 2.** Let $m$ be a finitely additive supermartingale on $\mathcal{A}$ and assume that $U$ is linearly ordered. Then the following are equivalent:

(i) $\mathcal{M}^uc(m) \neq \emptyset$;

(ii) $m$ is of class $D_*$;

(iii) there exists $P \in \mathbb{P}(\mathcal{F})$ and a strong $P$ supermartingale $\bar{X}$ of uniformly integrable variation that meet (4.2).

**Proof.** If $U$ is linearly ordered then each $d \in \mathcal{D}$ may be taken to be of the form $\{u_n \cap v_n^c : u_n \leq u_{n+1} : n = 1, \ldots, N\}$. Assume (i) and choose $\lambda \in \mathcal{M}^uc$ and $P \in \mathbb{P}(\mathcal{F}, \lambda)$. If $F \in \mathcal{F}$ and $d \in \mathcal{D}$ then, letting $M_\delta^* (F) = \sup_{1 \leq n \leq N} P(F|\mathcal{A}_{u_n})$ we have

$$(4.11) \quad \lambda(\mathcal{P}^d_\delta(F)) \leq \eta\|\lambda\| + \lambda(\mathcal{P}^d_\delta(F) > \eta) \leq \eta\|\lambda\| + \lambda^d_{\bar{X}}(M_\delta^*(F) > \eta)$$

Choose, by absolute continuity, $\delta$ such that $P(F) < \delta$ implies $\lambda^d_{\bar{X}}(F) < \eta$. By Doob maximal inequality if $P(F) < \eta\delta/3$ then $P(M_\delta^*(F) > \eta) \leq \delta$ and thus $\lambda(\mathcal{P}^d_\delta(F)) \leq \eta(\|\lambda\| + 1)$. But then $\lambda^d_{\bar{X}}(F) \leq \eta(\|\lambda\| + 1)$ and
thus $\lambda^P_0|\mathcal{F} \ll P|\mathcal{F}$, proving the implication (i)$\rightarrow$(ii). Assume (ii) and consider $\lambda \in \mathcal{M}^{uc}(m)$, $P \in \mathbb{P}(\mathcal{F},\lambda)$ and $\lambda^P \in \mathcal{M}^{uc}(m)$. Let $\tilde{X}$ be as in (iii) and thus a strong $P$ supermartingale. Then,

$$P \left( F \sum_{u \cap w \in d} X_u - P(X_v|\mathcal{A}_u) \right) = P \sum_{u \cap w \in d} P(F|\mathcal{A}_u)(X_u - X_v) = \lambda(\mathcal{D}_P^d(F))$$

which implies, together with (4.11), that $\tilde{X}$ is of uniformly integrable variation. Let $P$ and $\tilde{X}$ be as in (iii) and let $\tilde{m}$ be the generated finitely additive supermartingale on $\tilde{\Omega}$. Given that $\tilde{m}$ is strong it is of class $D_0$, by Theorem [11].

The restriction $\lambda|\mathcal{P}$ of $\lambda \in \mathcal{M}(\tilde{m})$ to $\mathcal{P}$ admits an extension to $\tilde{\mathcal{P}}^*$ (still denoted by $\lambda$), defined by letting:

$$\lambda(f) = \lim_{d \in \tilde{\mathcal{D}}} \lambda(\mathcal{D}_P^d(f))$$

Given that $\tilde{X}$ is of uniformly integrable variation and in view of (4.12) we conclude that $\lambda \in \mathcal{M}^{uc}(\tilde{m})$. \(\square\)

**References**

[1] T. E. Armstrong (1983), *Finitely Additive F-processes*, Trans. Amer. Math. Soc., **279**, 271-295.

[2] K. P. S. Bhaskara Rao, M. Bhaskara Rao (1983), *Theory of Charges*, Academic Press, London.

[3] G. Cassese (2007), *Decomposition of Supermartingales Indexed by a Linearly Ordered Set*, Stat. Prob. Lett., **77**, 795-802.

[4] G. Cassese (2008), *Finitely Additive Supermartingales*, J. Theor. Prob., **18**, 23-54.

[5] G. Cassese (2009), *Sure Wins, Separating Probabilities and the Representation of Linear Functionals*, J. Math. Anal. Appl., forthcoming.

[6] Y. S. Chow (1960), *Martingales in a σ-Finite Measure Space Indexed by Directed Sets*, Trans. Amer. Math. Soc., **97**, 254-285.

[7] De Giosa, R. Mininni (1995) *On the Doléans Function of Set-indexed Submartingales*, Stat. Prob. Lett., **24**, 71-75.

[8] J. Dellacherie, P. A. Meyer (1982), *Probabilities and Potential B*, North-Holland, Amsterdam.

[9] C. Doléans-Dade (1968), *Existence du Processus Croissant Naturel Associé à un Potentiel de la Classe (D)*, Z. Wahrsch. Verw. Geb., **9**, 309-314.

[10] M. Dozzi, B. G. Ivanoff, E. Merzbach (1994), *Doob-Meyer Decomposition for Set-indexed Submartingales*, J. Theor. Prob., **7**, 499-525.

[11] N. Dunford, J. Schwartz (1988), *Linear Operators. Part I*, Wiley, New York.

[12] D. Fisk (1965), *Quasi-martingales*, Trans. Amer. Math. Soc., **120**, 369-389.

[13] H. Föllmer (1973), *On the Representation of Semimartingales*, Ann. Prob., **5**, 580-589.

[14] H. E. Hürzeler (1985), *The Optional Sampling Theorem for Processes Indexed by a Partially Ordered Set*, Ann. Prob., **13**, 1224-1235.

[15] B. G. Ivanoff, E. Merzbach (1995), *Stopping and Set-indexed Local Martingales*, Stochastic Process. Appl., **57**, 83-98.

[16] T. G. Kurtz (1980), *The Optional Sampling Theorem for Martingales Indexed by Directed Sets*, Ann. Prob., **8**, 675-681.

[17] J. F. Mertens (1972), *Processus Stochastiques Généraux et Surmartingales*, Z. Wahrsch. Verw. Geb., **22**, 45-68.

[18] M. Metivier, J. P. Pellaumail (1975), *On Doléans-Föllmer Measure for Quasi-Martingales*, Illinois J. Math., **19**, 491-504.

[19] P. A. Meyer (1962), *A Decomposition Theorem for Supermartingales*, Illinois J. Math., **6**, 193-205.

[20] P. A. Meyer (1963), *Decomposition of Supermartingales: The Uniqueness Theorem*, Illinois J. Math., **7**, 1-17.

[21] T. Norberg (1989), *Stochastic Integration on Lattices*, Tech. Report, Chalmers Univ. of Tech., Univ. of Göteborg.

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