EXISTENCE AND EXTINCTION IN FINITE TIME FOR STRATONOVICH GRADIENT NOISE POROUS MEDIA EQUATIONS

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Abstract. We study existence and uniqueness of distributional solutions to the stochastic partial differential equation
\[ dX - (\nu \Delta X + \Delta \psi(X)) dt = \sum_{i=1}^{N} \langle b_i, \nabla X \rangle \circ d\beta_i \text{ in } [0,T] \times \Omega, \]
with \( X(0) = x(\xi) \text{ in } \Omega \) and \( X = 0 \) on \( \partial \Omega \). Moreover, we prove extinction in finite time of the solutions in the special case of fast diffusion model and of self-organized criticality model.

1. Introduction. In this work we consider stochastic porous media equations with Stratonovich gradient noise. In particular, we deal with existence and uniqueness of a solution to such kind of equations, providing also some results concerning its asymptotic behaviour. To be precise, let \( \Omega \subset \mathbb{R}^d \) be an open, bounded set with regular boundary and \( T > 0 \), then we consider the following stochastic partial differential equation (SPDE) in \( [0,T] \times \Omega \),
\[ dX(t,\xi) - (\nu \Delta X(t,\xi) + \Delta \psi(X(t,\xi))) dt = \sum_{i=1}^{N} \langle b_i(\xi), \nabla X(t,\xi) \rangle \circ d\beta_i(t), \]
where \( \circ \) denotes that the integration is intended in the Stratonovich sense, \( \nu > 0 \), \( \psi: \mathbb{R} \to 2^\mathbb{R} \) is a maximal monotone function with polynomial growth, \( b_i: \Omega \subset \mathbb{R}^d \to \mathbb{R}^d \) are \( C^2 \) functions and \( \beta = (\beta_i)_{i=1,...,N} \) is an \( N \)-dimensional Brownian motion on a given probability space.

We provide existence of a distributional solution to eq. (1), essentially studying problem (1) with \( \psi \) substituted by its Yosida approximation, \( \psi_{\lambda} \), \( \lambda > 0 \), which, as we will see, admits a solution for all \( \lambda > 0 \), and then showing that the associated sequence of solutions, namely the sequence of solutions of the Yosida approximation scheme related to eq. (1), converges to the one we are looking for, as \( \lambda \) goes to zero. This approach has been employed also in, e.g., [11] to study porous media equations with multiplicative noise.

We then treat two particular examples of eq. (1), namely fast diffusion and self-organized criticality, and prove an asymptotic result concerning the solution to the equation in those frameworks. As we shall see, the solution will be zero from a certain time on with positive probability. It is worth to mention that the method

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we have used to obtain the aforementioned results is based on the one developed in [8], where the multiplicative noise case is studied.

Perturbing a problem by a Stratonovich noise of gradient type is useful in a wide range of applications, as in the case of image processing, see [24, 28], where it has been proved that considering those kind of perturbations improves the solution obtained by the total variation regularization.

Some recent results regarding equations with Stratonovich gradient noise have been studied in the case of $p$-Laplacian and total variation flow drift. For instance the reader may refer to [6] which deals with $p$-Laplace equations in the case $p > 1$, to [15] for the case with Neumann boundary conditions, to [25] for $p$-Laplace equation and total variation flow on a $d$-dimensional torus, to [7] for a distributional solution when $b_i$ is divergence-free and to [22] for the case of total variation flow with Dirichlet boundary conditions.

See also [16, 17] for other results related to eq. (1), and more generally pathwise well-posedness and entropy solutions for nonlinear diffusion equations with nonlinear conservative noise.

1.1. Structure of the work. The work is organized as follows. We begin introducing the problem and discussing the assumptions, the definition of solution to eq. (1) and some preliminaries in Section 2. Section 3 is devoted to the construction of the approximating problem and its properties. We prove the existence and uniqueness of a solution to our problem in Section 4. Extinction in finite time for the fast diffusion model is discussed in Section 5. Self-organized criticality model is treated in Section 6. We conclude with some final remarks and considerations in Section 7.

1.2. Notation. Let $O \subset \mathbb{R}^d$ be an open and bounded set with regular boundary $\partial O$. Then we denote by $L^p(O)$, $p \in [0, \infty]$, the Banach space of all $p$-summable (equivalence classes of) functions from $O$ to $\mathbb{R}$ and by $\| \cdot \|_p$ its corresponding norm, while we indicate by $H^k(O)$, $k \in \mathbb{N}$, the Sobolev space of functions in $L^2(O)$ whose distributional derivatives of order less than $k$ belong to $L^2$. $H^1_0(O)$ is the set of $H^1$ function vanishing on $\partial O$, its corresponding norm is given by

$$
\| u \|_1 := \left( \int_{\mathbb{R}^d} |\nabla u|^2 \right)^{1/2} .
$$

$H^{-1}(O)$ is the dual of $H^1_0(O)$ and its norm is denoted by $\| \cdot \|_{-1}$, $(\cdot, \cdot)_{-1}$ being its inner product. We indicate by $(\cdot, \cdot)$, or sometimes simply by $\cdot$, the scalar product on $\mathbb{R}^N$.

2. Framework. Let $O \subset \mathbb{R}^d$ be a open and bounded set with smooth boundary $\partial O$. We aim at providing existence and uniqueness of a solution to the following nonlinear SPDE with Stratonovich gradient noise for $X : \Omega \times [0, T] \times O \to \mathbb{R}$,

$$
\begin{aligned}
&dX(t) - (\nu \Delta X(t) + \Delta \psi(X(t))) dt = \sum_{i=1}^N \langle b_i, \nabla X(t) \rangle \circ d\beta_i(t), \quad \text{in } [0, T] \times O, \\
&X(0, \xi) = x(\xi), \quad \text{in } O, \\
&X(t, \xi) = 0, \quad \text{on } [0, T] \times \partial O ,
\end{aligned}
$$

where $\nu > 0$, $b = (b_1, \ldots, b_N) : \overline{O} \to \mathbb{R}^{N \times d}$, $\psi : \mathbb{R} \to \mathbb{R}$ is a (possibly multivalued) map and $\beta = (\beta_i)_{i=1, \ldots, N}$ is an $N$-dimensional Brownian motion on a filtered
probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). For the sake of simplicity, we will often omit to write explicitly the dependence of \(X\) on \((\omega, t, \xi) \in \Omega \times [0, T] \times \mathcal{O}\) as well as the dependence of \(b\) on \(\xi\).

Equation (2) can be equivalently written as the following SPDE in the Itô sense

\[
\begin{aligned}
&dX(t) - \left( \nu \Delta X(t) + \Delta \psi(X(t)) + \frac{1}{2} \text{div}(b^* b \nabla X(t)) \right) dt = \langle b \nabla X(t), d\beta(t) \rangle, \\
&\text{in } ]0, T[ \times \mathcal{O}, \\
&X(0, \xi) = x(\xi), \\
&X(t, \xi) = 0,
\end{aligned}
\]

where \(b^*\) is the transpose of \(b\).

Notice that we can write

\[
\text{div}(b^* b \nabla X) = \sum_{k=1}^{d} \frac{\partial}{\partial \xi_k} \left( \sum_{j=1}^{d} A_{kj}(\xi) \frac{\partial X(t, \xi)}{\partial \xi_j} \right),
\]

where

\[
A_{kj}(\xi) = \sum_{i=1}^{N} b_{ik}(\xi) b_{ij}(\xi), \quad k, j = 1, \ldots, d.
\]

Before proceeding further we provide the following result, which will be useful in the proof of the existence of a solution to eq. (3).

**Lemma 2.1.** Let \(b\) be defined as above and assume that \(b_{ij} \in C^2(\mathcal{O})\). Then there exist \(\bar{C} > 0\), depending on \(\mathcal{O}\), and \(\gamma > 0\), depending on \(b\), such that

\[
\|(-\Delta)^{-1} \text{div}(b^* b \nabla u)\|_2 \leq \bar{C} \gamma \|u\|_2, \quad \text{for every } u \in L^2(\mathcal{O}).
\]

**Proof.** Let \(z = (-\Delta)^{-1} \text{div}(b^* b \nabla u)\) in \(\mathcal{O}\), that is, equivalently,

\[
\begin{aligned}
&-\Delta z = \text{div}(b^* b \nabla u), \quad \text{in } \mathcal{O}, \\
&z = 0, \quad \text{in } \partial \mathcal{O}.
\end{aligned}
\]

Let \(f \in L^2(\mathcal{O})\) and \(v \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})\) be the solution to

\[
\begin{aligned}
&-\Delta v = f, \quad \text{in } \mathcal{O}, \\
v = 0, \quad \text{on } \partial \mathcal{O}.
\end{aligned}
\]

Multiplying the first equation in (6) by \(v\) and then integrating we have, by Green’s formula and (7),

\[
\int_{\mathcal{O}} z f \, d\xi = \int_{\mathcal{O}} \text{div}(b^* b \nabla u) v \, d\xi = \int_{\mathcal{O}} u \, \text{div}(b^* b \nabla v) \, d\xi,
\]

therefore, by (4), we get
Assumption 2.2.\ 
\[ \|z, f\|_2 \leq |u|_2 \left| \text{div}(b^* b \nabla v) \right|_2 \]
\[ \leq |u|_2 \sum_{k,j=1}^d \left( |A_{kj}|_{\infty} |D_j^2 v|_2 + |D_k A_{kj}|_{\infty} |D_j^2 v|_2 \right) \]
\[ \leq C \gamma |u|_2 \left( \|v\|_{H^2(O)} + \|v\|_{H^3(O)} \right), \]
where \( C = C(O) \) and \( \gamma = \gamma(b) = \max \{ |A_{kj}|_{\infty} + |D_k A_{kj}|_{\infty} : k, j \in \{1, \ldots, d\} \} \).

By eq. (7) we have
\[ \|v\|_{H^3(O)} + \|v\|_{H^2(O)} \leq K |f|_2, \]
for some \( K \) depending on \( O \), so that
\[ \|z, f\|_2 \leq \tilde{C} \gamma |u|_2 |f|_2, \]
for every \( f \in L^2(O) \),
where \( \tilde{C} = C K \) depends on \( O \). Hence \( |z|_2 \leq \tilde{C} \gamma |u|_2 \), for every \( u \in L^2(O) \), which concludes the proof. \( \square \)

We assume that the following hypotheses on \( \psi \), \( b \) and \( \nu \) hold.

**Assumption 2.2.** The following hypotheses hold:
(i) \( \psi : \mathbb{R} \to \mathbb{R}^d \) is maximal monotone with \( 0 \in \psi(0) \).
(ii) There exist \( C > 0 \) and \( m \geq 0 \) such that
\[ \sup \{ |\theta| : \theta \in \psi(r) \} \leq C (1 + |r|^m), \]
for every \( r \in \mathbb{R} \).
Moreover, we assume that \( m \leq d/(d - 2) \) if \( d \geq 3 \).
(iii) The functions \( A_{kj} \) defined in (5) are bounded for any \( k, j = 1, \ldots, d \).
(iv) \( b_i \in C^2(\overline{\Omega}; \mathbb{R}^d) \) for every \( i = 1, \ldots, N \) and
\[ \tilde{C} \gamma(b) + |b|_{\infty}^2 \leq 2 \nu, \]
where \( \tilde{C} \) and \( \gamma \) are as in Lemma 2.1.

Notice that condition (8) tells us that the nearer \( \nu \) is to 0, the stricter the condition on the norm of \( b \) is. In particular, the case \( \nu = 0 \) implies \( b \equiv 0 \), reducing eq. (3) to the (deterministic) PDE \( \partial_t X = \Delta \psi(X) \).

The solution we are looking for is to be intended in the following sense.

**Definition 2.3.** A solution to eq. (3) in \([0, T]\) is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted stochastic process \( X \) such that
(i) \( X \in L^2(\Omega; L^\infty([0, T]; H^{-1}(O))) \cap L^2(\Omega \times [0, T]; H^0_0(O)), \)
(ii) there exists a process \( \eta \in L^2(\Omega \times [0, T] \times O) \) such that \( \eta \in \psi(X) \) a.e., and, for every \( j \in \mathbb{N}, t \in [0, T], \mathbb{P}\)-a.s.
\[ \langle X(t), f_j \rangle_{-1} = \langle x, f_j \rangle_{-1} - \nu \int_0^t \langle X(s), f_j \rangle_2 ds - \int_0^t \langle \eta(s), f_j \rangle_2 ds \]
\[ + \frac{1}{2} \int_0^t \langle \text{div}(b^* b \nabla X), f_j \rangle_{-1} ds + \int_0^t \langle b \nabla X(s), f_j \rangle_{-1} d\beta(s), \]
where \( (f_j)_{j \in \mathbb{N}} \) is an orthonormal basis for \( -\Delta \) in \( H^{-1} \).
(iii) \( X \) is pathwise continuous from \([0, T]\) to \( H^{-1}(O) \).
A solution of this type is also referred to as distributional solution since we can equivalently write eq. (9) as

\[ X(t) = x - \nu \int_0^t \Delta X(s) \, ds - \Delta \int_0^t \eta(s) \, ds + \frac{1}{2} \int_0^t \text{div}(b^* b \nabla X(s)) \, ds + \int_0^t b \nabla X(s) \cdot d\beta(s), \]

where \( \Delta : H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O}) \) is taken in the sense of distributions on \( \mathcal{O} \).

3. The approximating problem. Under Assumption 2.2 we define, for every \( \lambda > 0 \), the resolvent and the Yosida approximation of \( \psi \),

\[ J_\lambda = (I + \lambda \psi)^{-1}, \quad \psi_\lambda = \frac{1}{\lambda}(I - J_\lambda), \]

respectively, which are known to be Lipschitz-continuous, see, e.g., [4]. We shall consider thus the following approximating problem, for \( \lambda > 0 \),

\[
\begin{cases}
  dX_\lambda - \left( \nu \Delta X_\lambda + \Delta \psi_\lambda(X_\lambda) + \frac{1}{2} \text{div}(b^* b \nabla X_\lambda) \right) dt &= (b \nabla X_\lambda, d\beta), \quad \text{in }]0,T[ \times \mathcal{O}, \\
  X_\lambda(0, \xi) &= x(\xi), \\
  X_\lambda(t, \xi) &= 0, \quad \text{on }]0,T[ \times \partial \mathcal{O}.}
\end{cases}
\]

We will need the following result.

**Lemma 3.1.** We have, for every \( r \in \mathbb{R} \) and \( \lambda > 0 \),

\[ |\psi_\lambda(r)| \leq C(1 + |r|^m). \]

**Proof.** It holds, for every \( r \in \mathbb{R} \),

\[ |\psi_\lambda(r)| \leq \sup \{|\theta| : \theta \in \psi(J_\lambda(r))\} \leq C(1 + |J_\lambda(r)|^m) \leq C(1 + |r|^m). \]

Moreover, one can also see that \( r \mapsto \psi_\lambda(r) + \nu r \) is strictly monotonically increasing, bounded by \( C(1 + |r|^m) \) and \( (\psi_\lambda(r) + \nu r)r \geq \nu |r|^2 \) for all \( r \in \mathbb{R} \).

We have, thus, the following existence result, which is a consequence of Krylov and Rozovskii Theorem, see [19] or the more recent book [21].

**Proposition 1.** Suppose Assumption 2.2 holds. Then eq. (10) admits a unique variational solution.

We conclude this section with the following result.

**Lemma 3.2.** Let \( X_\lambda \) be a solution to eq. (10). Then there exist \( C_1, C_2 > 0 \), depending on \( \nu \) and on the \( L^2 \)-norm of the initial condition \( x \) but not on the parameter \( \lambda \), such that

\[ \mathbb{E} |X_\lambda(t)|^2 + C_1 \mathbb{E} \int_0^t |\nabla X_\lambda(s)|^2 \, ds \leq C_2. \]

**Proof.** Let \( X_\lambda \) be the solution to eq. (10), then by Itô’s formula in \( L^2 \) we have

\[
  d|X_\lambda(t)|^2 = 2\langle X_\lambda(t), \nu \Delta X_\lambda(t) + \Delta \psi_\lambda(X_\lambda(t)) \rangle dt + 2 \sum_{i=1}^N \langle X_\lambda(t), b_i \cdot \nabla X_\lambda(t) \, d\beta_i(t) \rangle dt.
\]
Theorem 4.1. If Assumption 2.2 holds and then, taking the expectations, the maximal monotonicity of which can be written as

\[ |X_\lambda(t)|^2 - 2 \int_0^t \langle X_\lambda(s), \nu \Delta X_\lambda(s) + \Delta \psi_\lambda(X_\lambda(s)) \rangle_2 \, ds = \]

\[ = 2 \sum_{i=1}^N \int_0^t \langle X_\lambda(s), b_i \cdot \nabla X_\lambda(s) \, d\beta_i(s) \rangle_2, \]

which gives

\[ |X_\lambda(t)|^2 + 2\nu \int_0^t |\nabla X_\lambda(s)|_2^2 \, ds + 2 \int_0^t \int_\Omega \psi_\lambda'(X_\lambda(s)) |\nabla X_\lambda(s)|^2 \, d\xi \, ds = \]

\[ = |x|^2 + 2 \sum_{i=1}^N \int_0^t \langle X_\lambda(s), b_i \cdot \nabla X_\lambda(s) \, d\beta_i(s) \rangle_2. \]

Then, taking the expectations,

\[ E |X_\lambda(t)|^2 + 2\nu E \int_0^t |\nabla X_\lambda(s)|_2^2 \, ds + 2 E \int_0^t \int_\Omega \psi_\lambda'(X_\lambda(s)) |\nabla X_\lambda(s)|^2 \, d\xi \, ds = |x|^2. \]

The maximal monotonicity of \( r \mapsto \psi(r) \) yields the result. \( \Box \)

4. Existence and uniqueness of the solution. This section is devoted to the proof of the main result of the work, namely the existence and uniqueness of a solution to eq. (3).

Theorem 4.1. If Assumption 2.2 holds and \( x \in L^2(\Omega) \), then eq. (3) admits a unique solution in the sense of Definition 2.3.

Proof. As concerns existence, we now prove that the sequence of solutions to eq. (10), \( (X_\lambda)_{\lambda>0} \), is a Cauchy sequence in \( L^2(\Omega; L^\infty([0,T]; H^{-1}(\Omega))) \) as \( \lambda \to 0 \). Consider \( X_\lambda \) and \( X_\mu \), with \( \lambda, \mu > 0 \), then by Itô’s formula in \( H^{-1} \) we get

\[ \|X_\lambda(t) - X_\mu(t)\|_{-1}^2 = 2\nu \int_0^t \langle \Delta(X_\lambda(s) - X_\mu(s)), X_\lambda(s) - X_\mu(s) \rangle_{-1} \, ds \]

\[ + 2 \int_0^t \langle \Delta \psi_\lambda(X_\lambda(s)) - \Delta \psi_\mu(X_\mu(s)), X_\lambda(s) - X_\mu(s) \rangle_{-1} \, ds \]

\[ + \int_0^t \langle \text{div}(b^* b \nabla(X_\lambda(s) - X_\mu(s))), X_\lambda(s) - X_\mu(s) \rangle_{-1} \, ds \]

\[ + \int_0^t \|b \nabla(X_\lambda(s) - X_\mu(s))\|_{-1}^2 \, ds \]

\[ + 2 \sum_{i=1}^N \int_0^t \langle X_\lambda(s) - X_\mu(s), b_i \cdot \nabla(X_\lambda(s) - X_\mu(s)) \, d\beta_i(s) \rangle_{-1}, \]

which can be written as

\[ \|X_\lambda(t) - X_\mu(t)\|_{-1}^2 = -2\nu \int_0^t |X_\lambda(s) - X_\mu(s)|_2^2 \, ds \]

\[ - 2 \int_0^t \langle \psi_\lambda(X_\lambda(s)) - \psi_\mu(X_\mu(s)), X_\lambda(s) - X_\mu(s) \rangle_2 \, ds \]

\[ + \int_0^t \langle \text{div}(b^* b \nabla(X_\lambda(s) - X_\mu(s))), X_\lambda(s) - X_\mu(s) \rangle_{-1} \, ds \]

(11)
We shall now provide some estimates of the right-hand side terms in (11). Therefore we divide the remaining part of the proof in different steps.

**Step 1.** Estimate of \( \int_0^t (\text{div}(b^* b \nabla (X_\lambda(s) - X_\mu(s))), X_\lambda(s) - X_\mu(s)) \) ds.  
By Cauchy-Schwartz inequality we have

\[
\left| (\text{div}(b^* b \nabla (X_\lambda - X_\mu)), X_\lambda - X_\mu) \right|_1 
\leq \left| (\nabla \cdot (b^* b \nabla (X_\lambda - X_\mu)), X_\lambda - X_\mu) \right|_2 
\leq \left| (-\nabla)^{-1} \text{div}(b^* b \nabla (X_\lambda - X_\mu)) \right|_2 \| X_\lambda - X_\mu \|_2.
\]

Exploiting now Lemma 2.1 with \( u = X_\lambda - X_\mu \) we get

\[
\left| (-\nabla)^{-1} \text{div}(b^* b \nabla (X_\lambda - X_\mu)) \right|_2 \leq \tilde{C}_\gamma \| X_\lambda - X_\mu \|_2,
\]

which yields

\[
\|(\text{div}(b^* b \nabla (X_\lambda - X_\mu)), X_\lambda - X_\mu)\|_1 \leq \tilde{C}_\gamma \| X_\lambda - X_\mu \|_2^2. \tag{12}
\]

**Step 2.** Estimate of \( \int_0^t \| b \nabla (X_\lambda(s) - X_\mu(s)) \|_{-1}^2 ds \).
The integrand can be equivalently written as

\[
\| b \nabla (X_\lambda - X_\mu) \|_{-1}^2 = \left\| (b_j \cdot \nabla (X_\lambda - X_\mu))_{j=1,...,N} \right\|_{-1}^2.
\]

In particular, we can rewrite, for all \( j = 1, \ldots, N \),

\[
b_j \cdot \nabla (X_\lambda - X_\mu) = \text{div}(b_j (X_\lambda - X_\mu)) - \text{div}(b_j)(X_\lambda - X_\mu),
\]

so that

\[
\| b \nabla (X_\lambda - X_\mu) \|_{-1}^2 = \| (\text{div}(b_j (X_\lambda - X_\mu)) - \text{div}(b_j)(X_\lambda - X_\mu))_{j=1,...,N} \|_{-1}^2.
\]

Hence we get

\[
\| b \nabla (X_\lambda - X_\mu) \|_{-1}^2 \leq |b|_\infty^2 \| X_\lambda - X_\mu \|_2^2 + |(\text{div} b)_{j} \|_\infty \| X_\lambda - X_\mu \|_{-1}^2. \tag{13}
\]

**Step 3.** Estimate of \(-2 \int_0^t \langle \psi_\lambda(X_\lambda(s)) - \psi_\mu(X_\mu(s), X_\lambda(s) - X_\mu(s) \rangle ds \).  
We have, for every \( \lambda, \mu > 0 \),

\[
-2 \int_0^t \langle \psi_\lambda(X_\lambda(s)) - \psi_\mu(X_\mu(s), X_\lambda(s) - X_\mu(s) \rangle ds \leq K(\lambda + \mu). \tag{14}
\]

Indeed, we can rewrite

\[
X_\lambda - X_\mu = X_\lambda - J_\lambda(X_\lambda) + J_\lambda(X_\lambda) - J_\mu(X_\mu) + J_\mu(X_\mu) - X_\mu = \lambda \psi_\lambda(X_\lambda) + J_\lambda(X_\lambda) - J_\mu(X_\mu) - \mu \psi_\mu(X_\mu).
\]

Thus, since we have \( \langle J_\lambda(X_\lambda) - J_\mu(X_\mu), \psi_\lambda(X_\lambda) - \psi_\mu(X_\mu) \rangle \geq 0 \) (see, e.g., [4, Prop. 2.3]),

\[
\int_0^t \langle X_\lambda - X_\mu, \psi_\lambda(X_\lambda) - \psi_\mu(X_\mu) \rangle ds \geq \lambda \int_0^t |\psi_\lambda(X_\lambda)|_2^2 ds
\]

\[
+ \mu \int_0^t |\psi_\mu(X_\mu)|_2^2 ds - (\lambda + \mu) \int_0^t |\psi_\lambda(X_\lambda)|_2 |\psi_\lambda(X_\lambda)|_2 ds.
\]
By Cauchy’s inequality with epsilon we have

\[(\lambda + \mu) |\psi_\lambda(X_\lambda)|_2 |\psi_\lambda(X_\mu)|_2 \leq \lambda \left( |\psi_\lambda(X_\lambda)|_2^2 + \frac{1}{4} |\psi_\lambda(X_\mu)|_2^2 \right) + \mu \left( |\psi_\mu(X_\mu)|_2^2 + \frac{1}{4} |\psi_\lambda(X_\lambda)|_2^2 \right),\]

and so,

\[
\int_0^t \langle X_\lambda - X_\mu, \psi_\lambda(X_\lambda) - \psi_\mu(X_\mu) \rangle_2 ds \geq - \frac{1}{4} \int_0^t (\lambda |\psi_\lambda(X_\lambda)|_2^2 + \mu |\psi_\mu(X_\mu)|_2^2) ds.
\]

Recalling Assumption 2.2(ii) and exploiting Lemma 3.1, Lemma 3.2 and Sobolev embedding Theorem, we have

\[
\mathbb{E} \int_0^t |\psi_\lambda(X_\lambda)|_2^2 ds \leq C_1 \mathbb{E} \int_0^t |X_\lambda|^{2m} d\xi ds \leq C_2,
\]

so that

\[
\mathbb{E} \int_0^t \langle X_\lambda - X_\mu, \psi_\lambda(X_\lambda) - \psi_\mu(X_\mu) \rangle_2 ds \geq - \frac{tC_2}{4} (\lambda + \mu) \geq - \frac{TC_2}{4} (\lambda + \mu),
\]

which implies the claim, for every \( \mu, \lambda > 0 \).

**Step 4.** Proof that \( (X_\lambda)_\lambda \) is a Cauchy sequence in \( L^2(\Omega; L^\infty([0,T]; H^{-1}(\mathcal{O}))) \), as \( \lambda \to 0 \).

Consider eq. (11), then we can exploit the previous steps of the proof to get, by eqs. (12) and (13),

\[
\|X_\lambda(t) - X_\mu(t)\|_2^2 + 2\nu \int_0^t |X_\lambda(s) - X_\mu(s)|_2^2 ds \leq 0
\]

\[
\leq - 2 \int_0^t \langle \psi_\lambda(X_\lambda(s)) - \psi_\mu(X_\mu(s)), X_\lambda(s) - X_\mu(s) \rangle_2 ds
\]

\[
+ (\tilde{C}\gamma + |b|_\infty^2) \int_0^t |X_\lambda(s) - X_\mu(s)|_2^2 ds
\]

\[
+ |(\text{div } b_j)_j|_\infty^2 \int_0^t \|X_\lambda(s) - X_\mu(s)\|_{-1}^2 ds
\]

\[
+ 2 \sum_{i=1}^N \int_0^t \langle X_\lambda(s) - X_\mu(s), b_i \cdot \nabla(X_\lambda(s) - X_\mu(s)) \rangle d\beta_i(s)_{-1}.
\]

Recalling (8), and by Burkholder-Davis-Gundy inequality, we have

\[
\mathbb{E} \sup_{s \in [0,t]} \|X_\lambda(s) - X_\mu(s)\|_2^2 \leq - 2 \mathbb{E} \int_0^t \langle \psi_\lambda(X_\lambda(s)) - \psi_\mu(X_\mu(s)), X_\lambda(s) - X_\mu(s) \rangle_2 ds
\]

\[
+ |(\text{div } b_j)_j|_\infty^2 \mathbb{E} \int_0^t \|X_\lambda(s) - X_\mu(s)\|_{-1}^2 ds
\]

\[
+ C_3 \mathbb{E} \int_0^t \|X_\lambda(s) - X_\mu(s)\|_2^2 ds.
\]

Now estimate (14) gives

\[
\mathbb{E} \sup_{s \in [0,t]} \|X_\lambda(s) - X_\mu(s)\|_{-1}^2 \leq K(\lambda + \mu) + C_4 \mathbb{E} \int_0^t \|X_\lambda(s) - X_\mu(s)\|_{-1}^2 ds,
\]
where \( C_4 = |(\text{div} b_j)_{j\in\mathbb{N}}|^2 + C_3 \). Applying Gronwall’s inequality we get the result.

Since \((X_\lambda)_\lambda\) is Cauchy in \( L^2(\Omega; C([0, T]; H^{-1}(\mathcal{O}))) \) and by eq. (15), we know that, as \( \lambda \to 0 \),

\[
X_\lambda \to X, \quad \text{in } L^2(\Omega; C([0, T]; H^{-1}(\mathcal{O}))),
\]

\[
X_\lambda \to X, \quad \text{in } L^2(\Omega \times [0, T] \times \mathcal{O}),
\]

\[
\psi_\lambda(X_\lambda) \rightharpoonup \eta, \quad \text{weakly in } L^2(\Omega \times [0, T] \times \mathcal{O}),
\]

and \( \eta \in \psi(X) \) by the maximal monotonicity of \( y \mapsto \psi(y) \) in \( L^2(\Omega \times [0, T] \times \mathcal{O}) \), indeed \( \psi_\lambda(X_\lambda) = \psi((1 + \lambda \psi)^{-1} X_\lambda) \) and \((1 + \lambda \psi)^{-1} X_\lambda \to X \) in \( L^2(\Omega \times [0, T] \times \mathcal{O}) \) as \( \lambda \to 0^+ \). On the other hand \( \psi_\lambda(X_\lambda) = \psi((1 + \lambda \psi)^{-1} X_\lambda) \rightharpoonup \eta \), and so \( \eta \in \psi(X) \) a.e. on \((\Omega \times [0, T] \times \mathcal{O})\), which proves the existence of a solution.

Step 5. Uniqueness.

Consider two solution \( X \) and \( Y \) of eq. (3), then, by Itô’s formula in \( H^{-1} \) and exploiting the estimates in the existence proof, we have

\[
\|X(t) - Y(t)\|_{-1}^2 + 2\nu \int_0^t |X(s) - Y(s)|_2^2 \, ds \leq
\]

\[
\leq -2 \int_0^t \langle \psi(X(s)) - \psi(Y(s)), X(s) - Y(s) \rangle_{2} \, ds
+ (\dot{C}_\gamma + |b|_{\infty}^2) \int_0^t |X(s) - Y(s)|_2^2 \, ds
+ |(\text{div} b_j)_{j} |_{\infty}^2 \int_0^t \|X(s) - Y(s)\|_{-1}^2 \, ds
+ 2 \sum_{i=1}^N \int_0^t \langle X(s) - Y(s), b_i \cdot \nabla(X(s) - Y(s)) \rangle \, d\beta_i(s)_{-1}.
\]

Now, taking expectation and recalling the monotonicity of \( \psi \), we obtain

\[
\mathbb{E} \|X(t) - Y(t)\|_{-1}^2 + \left(2\nu - (\dot{C}_\gamma + |b|_{\infty}^2)\right) \mathbb{E} \int_0^t |X(s) - Y(s)|_2^2 \, ds \leq
\]

\[
\leq |(\text{div} b_j)_{j} |_{\infty}^2 \mathbb{E} \int_0^t \|X(s) - Y(s)\|_{-1}^2 \, ds.
\]

Hence, by (8) we have

\[
\mathbb{E} \|X(t) - Y(t)\|_{-1}^2 \leq |(\text{div} b_j)_{j} |_{\infty}^2 \int_0^t \mathbb{E} \|X(s) - Y(s)\|_{-1}^2 \, ds,
\]

and Gronwall’s inequality yields the result. \( \square \)

5. Extinction in finite time for the fast diffusion model. Porous media equations of the type

\[
\frac{\partial u}{\partial t} - \Delta \psi(u) = f
\]

were first used to describe the dynamics of the flow in a porous medium, see, e.g., [20, 23]. Indeed, the standard model of diffusion of a gas through a porous media is that where

\[
\psi(r) = \rho |r|^{m-1} r, \quad \text{for every } r \in \mathbb{R},
\]
with \( \rho > 0 \) and \( m > 1 \), which is the so-called slow diffusion model. More generally, one can consider the case of a continuous monotone function satisfying
\[
\rho |r|^{m+1} \leq r \psi(r) \leq a_1 |r|^{q+1} + a_2 r, \quad \text{for every } r \in \mathbb{R},
\]
for \( q > m > 1 \), and \( \rho, a_1 > 0 \).

The case \( m \in [0, 1[ \), which we are concerned here, is that of the fast diffusion model. This model is relevant in the description of plasma physics, the kinetic theory of gas or fluid transportation in porous media, as suggested in, e.g., [13, 14].

The reader is referred to [26, 27] for a complete treatment of porous media equations.

A general feature of the fast diffusion case is that it models diffusion processes with a fast speed of mass transportation and this is one reason why the process terminates within finite time with positive probability. This is, in fact, what we are going to show in this section for the Stratonovich gradient noise case. The result has been proved for the case of linear multiplicative noise, see [8, 10]. The approach used in the following is the same as the ones used in those works and in [11, Ch. 3.7].

So, from now on, we will focus on the fast diffusion model and we will work under the following conditions.

**Assumption 5.1.** The following hypotheses hold:

(i) \( \psi \) is as in the fast diffusion model, i.e.,
\[
\psi(r) = \rho |r|^{m-1} r, \quad \text{for every } r \in \mathbb{R}, \text{ with } m \in ]0, 1[, \rho > 0. \tag{16}
\]

(ii) Assumption 2.2 (iii)–(iv) hold.

(iii) We have
\[
1 \leq d < \frac{2(1 + m)}{1 - m}. \tag{17}
\]

**Remark 1.** The function \( \psi \) defined in (16) satisfies (i)–(ii) in Assumption 2.2 and so eq. (3) admits a unique solution, according to Theorem 4.1.

The meaning of assumption (17) will be clear after the following lemma, which gives us an estimate on \( \|X_\lambda(t)\|^{-m}_{-1} \), where \( X_\lambda \) is the solution to the approximating problem (10) for \( \lambda > 0 \).

**Lemma 5.2.** Suppose Assumption 5.1 holds. Then there exists \( K_m > 0 \) such that, for every \( \lambda > 0 \) and \( 0 \leq r \leq t \),
\[
\|X_\lambda(t)\|^{-m}_{-1} + (1 - m)\rho \int_r^t e^{K_m(t-s)} \|X_\lambda(s)\|^{-m-1}_{-1} |X_\lambda(s)|^{m+1}_{m+1} \mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0}(s) \, ds \\
\leq e^{K_m(r-t)} \|X_\lambda(r)\|^{-m}_{-1} + (1 - m) \\
\int_r^t e^{K_m(t-s)} \|X_\lambda(s)\|^{-m-1}_{-1} \langle X_\lambda(s), b \nabla X_\lambda(s) \cdot \left( \mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0}(s) \, d\beta(s) \right) \rangle_{-1}.
\]

**Proof.** In order to get the estimate on \( \|X_\lambda(t)\|^{-m}_{-1} \) we start estimating \( \phi_\varepsilon(X_\lambda(t)) \), where, for any \( \varepsilon > 0 \),
\[
\phi_\varepsilon(y) \doteq \left( \|y\|_{-1}^2 + \varepsilon^2 \right)^{-\frac{1+m}{2}}.
\]
Notice that

\[
D\phi_{\varepsilon}(y) = (1-m) \left(\|y\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} y,
\]

\[
D^2\phi_{\varepsilon}(y) = (1-m) \left(\|y\|_{-1}^2 + \varepsilon\right)^{\frac{1+m}{2}} - (1-m^2) \left(\|y\|_{-1}^2 + \varepsilon^2\right)^{-\frac{3+m}{2}} y \otimes y.
\]

By Itô’s formula we have

\[
d\phi_{\varepsilon}(X_{\lambda}(t)) - (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), \nu \Delta X_{\lambda}(t) + \Delta \psi_{\lambda}(X_{\lambda}(t)) \rangle_{-1} dt
\]

\[
= (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), \frac{1}{2} \text{div}(b^* \cdot b \nabla X_{\lambda}(t)) \rangle_{-1} dt
\]

\[
+ \frac{1}{2} (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon\right)^{\frac{1+m}{2}} \|b \nabla X_{\lambda}(t)\|_{-1}^2 dt
\]

\[
- \frac{1}{2} (1-m^2) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{3+m}{2}} \|X_{\lambda}(t)\|_{-1} \|b \nabla X_{\lambda}(t)\|_{-1} dt
\]

\[
+ (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), b \nabla X_{\lambda}(t) \cdot d\beta(t) \rangle_{-1},
\]

which can be rewritten as

\[
d\phi_{\varepsilon}(X_{\lambda}(t)) + (1-m) \nu \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_{\lambda}(t)|_{2}^2 dt
\]

\[
+ (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), \psi_{\lambda}(X_{\lambda}(t)) \rangle_{2} dt
\]

\[
= \frac{1}{2} (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), \text{div}(b^* \cdot b \nabla X_{\lambda}(t)) \rangle_{-1} dt
\]

\[
+ \frac{1}{2} (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon\right)^{\frac{1+m}{2}} \|b \nabla X_{\lambda}(t)\|_{-1}^2 dt
\]

\[
- \frac{1}{2} (1-m^2) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{3+m}{2}} \|X_{\lambda}(t)\|_{-1} \|b \nabla X_{\lambda}(t)\|_{-1} dt
\]

\[
+ (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), b \nabla X_{\lambda}(t) \cdot d\beta(t) \rangle_{-1}.
\]

Now noticing that \(\langle X_{\lambda}, \psi_{\lambda}(X_{\lambda}) \rangle_{2} = \rho |X_{\lambda}|_{1+m} \) and estimating the terms on the right-hand side as we did in the proof of Theorem 4.1, we get

\[
d\phi_{\varepsilon}(X_{\lambda}(t)) + (1-m) \nu \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_{\lambda}(t)|_{2}^2 dt
\]

\[
+ (1-m) \rho \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_{\lambda}(t)|_{1+m} dt
\]

\[
\leq \frac{1}{2} (1-m) \tilde{C} \gamma \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_{\lambda}(t)|_{2}^2 dt
\]

\[
+ \frac{1}{2} (1-m) |b_{\infty}| \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_{\lambda}(t)|_{2}^2 dt
\]

\[
+ \frac{1}{2} (1-m) |(\text{div} b_{\infty})_{\infty}| \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \|X_{\lambda}(t)\|_{-1} dt
\]

\[
+ (1-m) \left(\|X_{\lambda}(t)\|_{-1}^2 + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_{\lambda}(t), b \nabla X_{\lambda}(t) \cdot d\beta(t) \rangle_{-1}.
\]
Let $C_1 = \nu - \frac{c_\gamma + |b|^2}{2}$ and $C_2 = \|(\text{div } b)\|_\infty^2$, then integrating with respect to time from $r$ to $t$, we have

\[
\|X_\lambda(t)\|_{-1}^{-m} + (1 - m)C_1 \int_r^t \left(\|X_\lambda(s)\|_{-1}^{-m} + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_\lambda(s)|_2^2 ds \\
+ (1 - m)\rho \int_r^t \left(\|X_\lambda(s)\|_{-1}^{-m} + \varepsilon^2\right)^{-\frac{1+m}{2}} |X_\lambda(s)|_{1+m}^2 ds \\
\leq \|X_\lambda(r)\|_{-1}^{-m} + C_2(1 - m) \int_r^t \left(\|X_\lambda(s)\|_{-1}^{-m} + \varepsilon^2\right)^{-\frac{1+m}{2}} \|X_\lambda(s)\|_{-1}^2 ds \\
+ (1 - m) \int_r^t \left(\|X_\lambda(s)\|_{-1}^{-m} + \varepsilon^2\right)^{-\frac{1+m}{2}} \langle X_\lambda(s), b \nabla X_\lambda(s) \cdot d\beta(s) \rangle_{-1}.
\]

Taking $\varepsilon \to 0$ yields

\[
\|X_\lambda(t)\|_{-1}^{-m} + (1 - m)C_1 \int_r^t \|X_\lambda(s)\|_{-1}^{-m} |X_\lambda(s)|_2^2 \mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0} ds \\
+ (1 - m)\rho \int_r^t \|X_\lambda(s)\|_{-1}^{-m} |X_\lambda(s)|_{1+m}^2 \mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0} ds \\
\leq \|X_\lambda(r)\|_{-1}^{-m} + C_2(1 - m) \int_r^t \|X_\lambda(s)\|_{-1}^{-m} ds \\
+ (1 - m) \int_r^t \|X_\lambda(s)\|_{-1}^{-m} \langle X_\lambda(s), b \nabla X_\lambda(s) \cdot \left(\mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0} d\beta(s)\right) \rangle_{-1}.
\]

Set now $K_m = C_2(1 - m)/2$. By the stochastic Gronwall's lemma we have

\[
\|X_\lambda(t)\|_{-1}^{-m} + (1 - m)C_1 \int_r^t e^{K_m(t-s)} \|X_\lambda(s)\|_{-1}^{-m} |X_\lambda(s)|_2^2 \mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0} ds \\
+ (1 - m)\rho \int_r^t e^{K_m(t-s)} \|X_\lambda(s)\|_{-1}^{-m} |X_\lambda(s)|_{1+m}^2 \mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0} ds \\
\leq e^{K_m(t-r)} \|X_\lambda(r)\|_{-1}^{-m} + (1 - m) \int_r^t e^{K_m(t-s)} \\
\|X_\lambda(s)\|_{-1}^{-m} \langle X_\lambda(s), b \nabla X_\lambda(s) \cdot \left(\mathbb{1}_{\|X_\lambda(s)\|_{-1} > 0} d\beta(s)\right) \rangle_{-1}.
\]

Recalling that $C_1 > 0$ because of (8), we get (18).

For the next step, we need an additional assumption concerning $d$ and $m$. In particular, we would like to have a constant $C_m \geq 0$ such that

\[
\|y\|_{-1}^{-m} |y|_{1+m}^2 \geq C_m, \quad \text{for every } y \in H^{-1},
\]

which is equivalent to having

\[
L^{1+m}(O) \subset H^{-1}(O), \quad \text{with continuous embedding.}
\]

By duality, this is equivalent to

\[
H^d_0(O) \subset L^{\frac{1+m}{m}}(O), \quad \text{with continuous embedding},
\]

hence, by Sobolev embedding Theorem, we have that (19) holds provided

\[
\frac{m}{m+1} > \frac{1}{2} - \frac{1}{d},
\]

which explains the meaning of hypothesis (17).

We are now ready to prove our extinction in finite time result.
Theorem 5.3. Suppose Assumption 5.1 holds. Let $X$ be a solution to eq. (3) and set $	au_x = \inf\{t > 0 : X(t) = 0\}$, then $X(t, \omega) = 0$ for every $t > \tau_x(\omega)$. Moreover, the extinction probability is finite and

$$\mathbb{P}(\tau_x > t) \leq \frac{K_m \|x\|_{-1}^{1-m}}{\rho C_m (1-m) (1-e^{-K_m t})}.$$  \hspace{1cm} (20)

Proof. By (18), taking into account (19), we have for every $t \geq r \geq 0$

\[ \|X_x(t)\|_{-1}^{1-m} + C_m (1-m) \rho \int_r^t e^{K_m(t-s)} \mathbb{E} \|X_x(s)\|_{-1} > 0 \, ds \leq e^{K_m(t-r)} \|X_x(r)\|_{-1}^{1-m} + (1-m) \int_r^t e^{K_m(t-s)} \|X_x(s)\|_{-1}^{m-1} \langle X_x(s), b\nabla X_x(s) \cdot (\mathbb{E} \|X_x(s)\|_{-1} > 0 \, d\beta(s)) \rangle_{-1}. \]

Let $\lambda \to 0$ to find

\[ \|X(x)\|_{-1}^{1-m} + C_m (1-m) \rho \int_r^t e^{K_m(t-s)} \mathbb{E} \|X_x(s)\|_{-1} > 0 \, ds \leq e^{K_m(t-r)} \|X_x(r)\|_{-1}^{1-m} + (1-m) \int_r^t e^{K_m(t-s)} \|X_x(s)\|_{-1}^{m-1} \langle X_x(s), b\nabla X_x(s) \cdot (\mathbb{E} \|X_x(s)\|_{-1} > 0 \, d\beta(s)) \rangle_{-1}. \]

which can be equivalently written as

\[ e^{-K_m t} \|X_x(t)\|_{-1}^{1-m} + C_m (1-m) \rho \int_r^t e^{-K_m s} \mathbb{E} \|X_x(s)\|_{-1} > 0 \, ds \leq e^{-K_m r} \|X_x(r)\|_{-1}^{1-m} + (1-m) \int_r^t e^{-K_m s} \|X_x(s)\|_{-1}^{m-1} \langle X_x(s), b\nabla X_x(s) \cdot (\mathbb{E} \|X_x(s)\|_{-1} > 0 \, d\beta(s)) \rangle_{-1}. \]

Recalling that $K_m = C_2(1-m)/2$ and defining $Y(t) = e^{-C_2 t/2} X(t)$, this proves that the process $\|Y(t)\|_{-1}^{1-m}$ is a nonnegative supermartingale, i.e.,

\[ \mathbb{E} \|Y(t)\|_{-1}^{1-m} | \mathcal{F}_r \| \leq \|Y(r)\|_{-1}^{1-m}, \quad \text{for every } t \geq r. \]

This implies, for any couple of stopping times $\tau_1$ and $\tau_2$, that

\[ \tau_1 > \tau_2 \Rightarrow \|Y(\tau_1)\|_{-1} \leq \|Y(\tau_2)\|_{-1}, \]

and, in particular, for any $t > \tau_x = \inf\{t > 0 : X(t) = 0\}$, we have

\[ \|Y(t)\|_{-1} \leq \|Y(\tau_x)\|_{-1} = 0, \]

that is,

\[ \|X(t)\|_{-1} = \|X(\tau_x)\|_{-1} = 0, \quad \mathbb{P}\text{-a.s.} \]

Now set $r = 0$ in (21) and take the expectation to get

\[ \mathbb{E} \|Y(t)\|_{-1}^{1-m} + C_m (1-m) \rho \int_0^t e^{-K_m s} \mathbb{P}(\tau_x > s) \, ds \leq \|x\|_{-1}^{1-m}, \]

which gives

\[ \mathbb{E} \|Y(t)\|_{-1}^{1-m} + C_m (1-m) \rho \int_0^t e^{-K_m s} \mathbb{P}(\tau_x > t) \, ds \leq \|x\|_{-1}^{1-m}, \]

and (20) follows. \hfill \Box
6. Self-organized criticality model. In this section we will introduce the self-organized criticality (SOC) model, which is a special case of porous media equation with
\[ \psi(r) = \rho \text{sign} r + \phi(r), \quad \text{for all } r \in \mathbb{R}, \]
where \( \rho > 0, \phi \) a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) and
\[ \text{sign} r = \begin{cases} \frac{r}{|r|}, & \text{for } r \neq 0, \\ \{r \in \mathbb{R}; |r| \leq 1\}, & \text{for } r = 0. \end{cases} \]
Such a choice of \( \psi \) represents the so-called sand-pile model or Bak-Tang-Wiesenfeld model. The deterministic version of the sand-pile model was first introduced in [1, 2], while its stochastic counterpart has been studied, e.g., in [5, 9, 12].

In the following we formalize the deterministic model referring to the method treated in [3]. Let \( O \) be an \( N \times N \) discrete region of points, we label each of those points with an integer index \( i \in \{1, \ldots, N^2\} \). We associate a height, \( X_i(t) \), to every index \( i \) at a certain time \( t \). Now, we can select, randomly, a site \( i \) and increase \( X_i(t) \) by 1, leaving the other sites unchanged. A toppling event occurs if the height at a site exceed of a given critical value \( X_c \). A site whose height is greater than \( X_c \) is called activated site.

If \( X_i(t) > X_c \), then
\[ X_j(t+1) = X_j(t) - Z_{ij}, \quad j = 1, \ldots, N^2, \tag{22} \]
where \( Z = (Z_{ij})_{ij} \) is a \( N^2 \times N^2 \) matrix such that
\[ Z_{ij} = \begin{cases} 4, & \text{if } i = j, \\ -1, & \text{if } i \text{ and } j \text{ nearest neighbours}, \\ 0, & \text{otherwise}. \end{cases} \]

Consider \( X(t) = (X_i(t))_i \), then, the dynamic of the system can be written, starting from equation (22), as
\[ X(t+1) = X(t) - Z f(t), \tag{23} \]
where \( f(t) = (f_i(t))_i = (H(X_i(t) - X_c))_i \) and \( H \) is the Heaviside function defined as
\[ H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \]

Noticing that the matrix \( Z \) is a discretized version of the Laplace operator \( \Delta \), we can claim that equation (23) is the discrete version of the following partial differential equation for \( X: [0, +\infty[ \times O \to \mathbb{R}, \)
\[ \frac{\partial X}{\partial t}(t) = \Delta H(X(t) - X_c), \quad (t, \xi) \in [0, +\infty[ \times O, \tag{24} \]
where \( O \subset \mathbb{R}^2 \) is a continuous spatial domain, \( \Delta \) is the 2-dimensional Laplace operator and \( H \) is the Heaviside function. More generally, one can consider \( O \subset \mathbb{R}^d \), \( d = 1, 2, 3 \), and replace \( H \) by a continuous function with jump at 0. One has to associate to equation (24) an initial value condition
\[ X(0, \xi) = X_0(\xi), \quad \xi \in O, \]
with \( X_0: O \to \mathbb{R} \) representing the initial configuration of the system, and boundary conditions on \( \partial O \), a common one being the Dirichlet condition
\[ X(t, \xi) = 0, \quad (t, \xi) \in [0, +\infty[ \times \partial O. \]
We want now to treat eq. (3) in the particular case of self-organized criticality. If we consider
\[ \psi(r) = \rho \text{sign}(r), \quad r \in \mathbb{R}, \]
where \( \rho > 0 \), then Theorem 4.1 still applies, since \( \psi \) satisfies the hypotheses therein.

Theorem 5.3 holds with the same proof in the case \( m = 0 \), which is exactly the case of self-organized criticality, since
\[ \psi(r) = \rho |r|^{m-1} r = \rho |r|^m \text{sign}(r). \]
However, under this assumption, condition (17) imposes \( d = 1 \).

7. Concluding remarks. Theorem 4.1 provides a nice existence result for eq. (2), however, as we pointed out in Section 2, hypothesis (8) is quite restrictive, since it imposes in particular that \( \nu > 0 \) to avoid the loss of the noise term. It could be interesting to see if it is possible to gain existence even with \( \nu = 0 \), but keeping the noise.

As regards the asymptotic behaviour of solutions, Theorem 5.3 ensures extinction in finite time of the solution to the fast diffusion model, while for the SOC model, in Section 6, we only have the result for \( d = 1 \). One may wonder what happens in the case \( d \geq 2 \), does extinction in finite time phenomenon still take place? Some asymptotic results for the case of SOC in stochastic porous media equations of the type
\[ dX - \Delta \psi(X) \, dt = \sigma(X) \, dW, \]
have been provided by V. Barbu, G. Da Prato, and M. R"ockner in [11, Ch. 3.8] as well as B. Gess in [18], the latter guaranteeing, under some suitable assumptions, the extinction in finite time of solutions also for \( d > 1 \). However, in the case of Stratonovich gradient noise, what happens for \( d > 1 \) is still to be proved, up to our knowledge, and it could be the next step to be tackled in future works.

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REFERENCES
[1] P. Bak, C. Tang and K. Wiesenfeld, Self-organized criticality: An explanation of the 1/f noise, Phys. Rev. Lett., 59 (1987), 381–384.
[2] P. Bak, C. Tang and K. Wiesenfeld, Self-organized criticality, Phys. Rev. A, 38 (1988), 364–374.
[3] P. Bantay and M. Janosi, Self-organization and anomalous diffusion, Physica A: Stat. Mech. Appl., 185 (1992), 11–18.
[4] V. Barbu, Nonlinear Differential Equations of Monotone Types in Banach Spaces, Monographs in Mathematics, Springer-Verlag New York, 2010.
[5] V. Barbu, Self-organized criticality and convergence to equilibrium of solutions to nonlinear diffusion equations, Ann. Rev. Control, 34 (2010), 52–61.
[6] V. Barbu, Z. Brzeźniak, E. Hausenblas and L. Tubaro, Existence and convergence results for infinite dimensional nonlinear stochastic equations with multiplicative noise, Stochastic Process. Appl., 123 (2013), 934–951.
[7] V. Barbu, Z. Brzeźniak and L. Tubaro, Stochastic nonlinear parabolic equations with Stratonovich gradient noise, Appl. Math. Optim., 78 (2018), 361–377.
[8] V. Barbu, G. Da Prato and M. R"ockner, Finite time extinction for solutions to fast diffusion stochastic porous media equations, C. R. Acad. Sci. Paris, Ser. I, 347 (2009), 81–84.
[9] V. Barbu, G. Da Prato and M. R"ockner, Stochastic porous media equations and self-organized criticality, Commun. Math. Phys., 285 (2009), 901–923.
V. Barbu, G. Da Prato and M. Röckner, Finite time extinction of solutions to fast diffusion equations driven by linear multiplicative noise, *J. Math. Anal. Appl.*, 389 (2012), 147–164.

V. Barbu, G. Da Prato and M. Röckner, *Stochastic Porous Media Equations*, vol. 2163 of Lecture notes in Mathematics, Springer International Publishing, 2016.

V. Barbu and M. Röckner, Stochastic porous media equations and self-organized criticality: convergence to the critical state in all dimensions, *Commun. Math. Phys.*, 311 (2012), 539–555.

J. Berryman and C. Holland, Nonlinear diffusion problems arising in plasma physics, *Phys. Rev. Lett.*, 40 (1978), 1720–1722.

J. Berryman and C. Holland, Asymptotic behavior of the nonlinear diffusion equation $n_t = (n^{-1} n_x)_x$, *J. Math. Phys.*, 23 (1982), 983–987.

I. Ciotir and J. Tölle, Nonlinear stochastic partial differential equations with singular diffusivity and gradient Stratonovich noise, *J. Funct. Anal.*, 271 (2016), 1764–1792.

K. Dareiotis and B. Gess, Nonlinear diffusion equations with nonlinear gradient noise, preprint. arXiv:1811.08356

B. Fehrman and B. Gess, Well-posedness of stochastic porous media equations with nonlinear, conservative noise, preprint. arXiv:1712.05775

B. Gess, Finite time extinction for stochastic sign fast diffusion and self-organized criticality, *Commun. Math. Phys.*, 335 (2015), 309–344.

N. Krylov and B. Rozovskii, Stochastic evolution equations, *J. Math. Sci.*, 16 (1981), 1233–1277.

L. Leibenzon, The motion of a gas in a porous medium, in *Complete Works*, vol. 2, Acad. Sciences URSS, 1930.

W. Liu and M. Röckner, *Stochastic Partial Differential Equations: An Introduction*, Universitext, Springer International Publishing, 2015.

I. Munteanu and M. Röckner, Total variation flow perturbed by gradient linear multiplicative noise, *Infin. Dimens. Anal. Quantum Probab. Rel. Top.*, 21 (2018), 1850003, 28pp.

M. Muskat, The flow of homogeneous fluids through porous media, *Soil Science*, 46 (1938), 169.

B. Sixou, L. Wang and F. Peyrin, Stochastic diffusion equation with singular diffusivity and gradient-dependent noise in binary tomography, *J. Phys.: Conf. Ser.*, 542 (2014), 012001.

J. Tölle, Stochastic evolution equations with singular drift and gradient noise via curvature and commutation conditions, preprint. arXiv:1803.07005v3

J. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations: Equations of Porous Medium Type*, Oxford University Press, 2006.

J. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford University Press, 2007.

L. Wang, B. Sixou and F. Peyrin, Filtered stochastic optimization for binary tomography, in *2015 IEEE 12th ISBI*, 2015, 1604–1607.

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