NONCOMMUTATIVE GEOMETRY THROUGH MONOIDAL CATEGORIES

TOMASZ MASZCZYK‡

Abstract. After introducing a noncommutative counterpart of commutative algebraic geometry based on monoidal categories of quasi-coherent sheaves we show that various constructions in noncommutative geometry (e.g. Morita equivalences, Hopf-Galois extensions) can be given geometric meaning extending their geometric interpretations in the commutative case. On the other hand, we show that some constructions in commutative geometry (e.g. faithfully flat descent theory, principal fibrations, equivariant and infinitesimal geometry) can be interpreted as noncommutative geometric constructions applied to commutative objects. For such generalized geometry we define global invariants constructing cyclic objects from which we derive Hochschild, cyclic and periodic cyclic homology (with coefficients) in the standard way.

Contents

1. Introduction 1
2. Noncommutative schemes as monoidal categories 3
   2.1. Category of noncommutative schemes 3
   2.2. Affine morphisms and spectra 10
   2.3. Affine schemes and affine morphisms 11
   2.4. Flat covers and cospectra 12
   2.5. Sections and the sheaf condition 13
   2.6. Flat covers and Galois extensions 14
   2.7. Infinitesimals and differential operators 23
   2.8. Cyclic homology of noncommutative schemes 25
References 33

1. Introduction

Abelian categories as a replacement for spaces (schemes) can be justified by the following reconstruction theorem.

1991 Mathematics Subject Classification. 14A22, 16S38, 16W30, 16E40.
‡The author was partially supported by KBN grants 1P03A 036 26 and 115/E-343/SPB/6.PR UE/DIE 50/2005-2008.
Theorem. (P. Gabriel for noetherian schemes ([16], Ch. VI); A.L. Rosenberg in quasicompact case ([38]); and in general case ([115])) Every scheme \( X \) can be reconstructed from the abelian category \( \text{Qcoh}X \) with the distinguished object \( O_X \) uniquely up to an isomorphism of schemes.

Morphisms between schemes are encoded on the level of quasi-coherent sheaves as pairs of adjoint functors (the direct image and the inverse image as its left adjoint), in a way resembling geometric morphisms among topos [33].

The idea of a noncommutative algebraic geometry, based on abelian categories or their generalizations (triangulated categories, dg-categories and \( A_\infty \)-categories) [34, 17, 1, 45, 38, 26, 36] is derived from the following observation. The category of modules makes sense for any associative, not necessarily commutative, ring. Therefore arbitrary (with some working restrigions) abelian (or triangulated, dg, \( A_\infty \)) categories should be regarded as categories of quasi-coherent sheaves (or complexes of sheaves) on, possibly non-affine, non-commutative “schemes”. This theory develops in close relation with representation theory [22, 31].

However, in this approach one important point from commutative geometry is lost. Classical algebraic geometry is based on polynomials. They describe varieties and morphisms between them. Composition of morphisms is defined by substitution of polynomials into polynomials. The natural environment for polynomials are symmetric monoidal categories, and categories of quasi-coherent sheaves are such. Polynomial substitutions produce (co)monoidal functors between these monoidal categories. Lack of monoidal structures is the main drawback of module categories over noncommutative rings. Although one can derive from a module category its monoidal category of bimodules regarded as endofunctors [46, 13], in general there is no way to transport them along module-theoretic geometric morphisms, and if it is accidentally possible, the result is different from the result obtained for symmetric bimodules (over a commutative ring) regarded simply as modules.

One could argue that modules are important because of representation theory. But group algebras and enveloping algebras of Lie algebras are augmented algebras and modules over them can be regarded as bimodules (symmetric over a ground field) with the second side defined by means of the augmentation. Note that as such they can be used as coefficients of Hochschild (co)homology computing group and Lie algebra (co)homology.

Many natural constructions on noncommutative rings (or algebras) produce bimodules (algebras, ideals, universal differentials). Explicit natural modules for such rings, different from natural bimodules with one side forgotten, in general are not known.

The aim of the present paper is to persuade monoidal categories as models of quasicoheren sheaves on noncommutative schemes. This approach is justified by the monoidal version of the reconstruction theorem due to Balmer [8]. From this perspective, algebras and coalgebras are not primary objects but artifacts
of geometric morphisms between noncommutative schemes. Instead of thinking of classical spaces as of commutative algebras, we think of abelian symmetric monoidal categories. Since even commutative algebras admit non-symmetric bimodules, this provides some room to consider non-classical (non-local) effects even for classical spaces. We show that in this framework one can study global and infinitesimal structures of a noncommutative scheme. We compare purely geometric constructions (i.e. these which use only some geometric morphisms on the purely categorial level) and purely algebraic constructions (i.e. these which use homomorphisms of some algebraic structures).

In the global picture we prove theorem about equivalence of flat covers in the category of noncommutative affine schemes and noncommutative Galois extensions. It means that descent data or coactions, which are encoded in comodule structures, can be understood as geometric gluing or geometric quotienting by symmetries.

In the infinitesimal picture we establish a noncommutative duality between infinitesimals and differential operators (well known in the classical situation) realized by passing to the opposite category. To achieve this we prove that infinitesimals and differential operators arise as specializations of two dual categorial constructions.

Finally, we construct global invariants of our noncommutative schemes. They are Hochschild, cyclic and periodic cyclic homology derived from cyclic objects. Our construction allows to introduce coefficients into the theory, which are non-commutative analogs of sheaves with integrable connection from the theory of the DeRham cohomology. In a sense, the respective “integrability condition” in terms of some braiding is as general as possible, because it is derived from the very structure of the cyclic object, in opposite to other approaches where it is based on some ideas from category theory (comonads and distributivity laws in [2] or symmetric monoidal categories and cocartesian objects in [23]) producing some cyclic objects. We compare different types of diagrams standing behind our construction and constructions based on these categorial ideas.

The present paper is a part of some kind “noncommutative EGA in a nutshell”, tout proportion garde, whose further topics will appear in subsequent papers.

2. Noncommutative schemes as monoidal categories

2.1. Category of noncommutative schemes.

Definition. We define the category $\mathcal{S}ch$ of (noncommutative) schemes as follows. Objects of $\mathcal{S}ch$, usually denoted by $X$, are abelian monoidal categories, usually denoted by $(\text{Qcoh}(X), \otimes_X, \mathcal{O}_X)$. Morphisms $f : X \to Y$ are isoclasses of pairs $(f_*, \mathcal{O}_Y \to f_*\mathcal{O}_X)$, where $f_*$ is an additive monoidal functor $f_* : \text{Qcoh}(X) \to \text{Qcoh}(Y)$ having the left adjoint $f^*$, and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism in the category $\text{Qcoh}(Y)$, with natural composition.
**Remark.** Morphisms \( f : X \to Y \) in \( \mathcal{S}ch \) can be equivalently defined as isoclasses of pairs \((f^*, f^* \mathcal{O}_Y \to \mathcal{O}_X)\), where \( f^* \) is an additive comonoidal functor \( f^* : \text{Qcoh}(Y) \to \text{Qcoh}(X) \) having the right adjoint \( f_* \), and \( f^* \mathcal{O}_Y \to \mathcal{O}_X \) is a morphism in the category \( \text{Qcoh}(X) \), with natural composition.

**Example 1. (Commutative schemes).** With every commutative scheme \( X \) one can associate its abelian category \( \text{Qcoh}(X) \) of complexes of quasicoherent sheaves, with the distinguished structural sheaf \( \mathcal{O}_X \). The tensor product \( \otimes_X \) of \( \mathcal{O}_X \)-modules makes \( \text{Qcoh}(X) \) a monoidal category with \( \mathcal{O}_X \) as the unit object. With every morphism of commutative schemes \( f : X \to Y \) one can associate the additive monoidal (direct image) functor \( f^* : \text{Qcoh}(X) \to \text{Qcoh}(Y) \), which has the left adjoint \( f_* \) (the inverse image functor), and a morphism \( \mathcal{O}_Y \to f_* \mathcal{O}_X \) in the category \( \text{Qcoh}(Y) \).

This example has some special features. The monoidal categories \( \text{Qcoh}(X) \) are symmetric, the direct images \( f_* \mathcal{O}_X \) are commutative algebras in symmetric monoidal categories \( \text{Qcoh}(Y) \), and finally, the inverse image functor \( f^* \) is strongly comonoidal.

Instead of the category of quasicoherent sheaves one can consider the derived category of perfect complexes, with its canonical monoidal structure. The benefit from this upgrading is the reconstruction theorem of Balmer [3], which provides a construction on symmetric tensor triangulated categories with values in locally ringed spaces, functorial with respect to all tensor triangulated functors, reconstructing a topologically noetherian scheme from its derived category of perfect complexes.

**Example 2. (Finite flat correspondences of commutative schemes).** One can consider category, whose objects are commutative schemes but morphisms \( f \) from a scheme \( X \) to a scheme \( Y \) are defined as isoclasses of diagrams of the form

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & Y \\
\pi \downarrow & & \downarrow \\
X & & \\
\end{array}
\]

in the category of schemes, with \( \pi \) finite flat and \( \tilde{f} \) separable and quasi-compact (this is a technical assumption on \( \tilde{f} \) for the flat base change isomorphism). The composition of morphisms is defined by means of the following diagrams

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \xrightarrow{\tilde{g}} & Z \\
\tilde{\pi} \downarrow & \square & \downarrow & \rho \\
\tilde{X} & \xrightarrow{\tilde{f}} & Y \\
\pi \downarrow & & & \\
X & & & \\
\end{array}
\]

where \( \square \) denotes a cartesian square.
Every such a morphism \( f = (\pi, \tilde{f}) \) defines a functor
\[
(1) \quad f_* := \tilde{f}_* \pi^* : \text{Qcoh}(X) \to \text{Qcoh}(Y).
\]
Since \( \tilde{f}_* \) is monoidal and \( \pi^* \) is strongly comonoidal, \( f_* \) is monoidal as well. There is also a canonical homomorphism
\[
(2) \quad \mathcal{O}_Y \to f_* \mathcal{O}_X = \tilde{f}_* \pi^* \mathcal{O}_X = \tilde{f}_* \mathcal{O}_{\tilde{X}}.
\]
of algebras in Qcoh\((Y)\). To prove that \( f_* \) has the left adjoint it is enough to prove this property for \( \pi^* \).

**Lemma 1.** For every \( \pi : \tilde{X} \to X \) finite flat \( \pi^* = \mathcal{H}om_X((\pi_* \mathcal{O}_{\tilde{X}})^\vee, \mathcal{F})^\sim \) and has the left adjoint \((\pi_* \mathcal{O}_{\tilde{X}})^\vee \otimes_{\pi_* \mathcal{O}_{\tilde{X}}} \pi_*(-)\).

**Proof.** Notice first that being affine \( \pi \) satisfies
\[
(3) \quad \pi^* \mathcal{F} = (\pi_* \mathcal{O}_{\tilde{X}} \otimes_X \mathcal{F})^\sim.
\]
Since \( \pi_* \mathcal{O}_{\tilde{X}} \) is locally free coherent on \( X \) the latter can be rewritten as follows
\[
(4) \quad (\pi_* \mathcal{O}_{\tilde{X}} \otimes_X \mathcal{F})^\sim = \mathcal{H}om_X((\pi_* \mathcal{O}_{\tilde{X}})^\vee, \mathcal{F})^\sim.
\]
Here \((\pi_* \mathcal{O}_{\tilde{X}})^\vee\) is equipped with the canonical contragredient \( \pi_* \mathcal{O}_{\tilde{X}}\)-module structure. Using the fact that \( \pi_* \) is an equivalence between Qcoh\((\tilde{X})\) and the subcategory of Qcoh\((X)\) of \( \pi_* \mathcal{O}_{\tilde{X}}\)-modules with \( \pi_* \mathcal{O}_{\tilde{X}}\)-linear morphisms with the inverse \((-)^\sim\) and the tensor-hom adjunction we obtain the left adjoint of \( \pi^* \) of the form \((\pi_* \mathcal{O}_{\tilde{X}})^\vee \otimes_{\pi_* \mathcal{O}_{\tilde{X}}} \pi_*(-)\). \( \blacksquare \)

**Definition.** Using the fact that the \( \mathcal{O}_X\)-module \( \pi_* \mathcal{O}_{\tilde{X}} \) is locally free coherent on \( X \) we can dualize the unit \( \mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}} \) and the multiplication \( \pi_* \mathcal{O}_{\tilde{X}} \otimes_X \pi_* \mathcal{O}_{\tilde{X}} \to \pi_* \mathcal{O}_{\tilde{X}} \) to define a counital coalgebra \( \mathcal{D} := (\pi_* \mathcal{O}_{\tilde{X}})^\vee \). Then the algebra \( \pi_* \mathcal{O}_{\tilde{X}} \) itself can be regarded as a convolution algebra \( \mathcal{H}om_X(\mathcal{D}, \mathcal{O}_X) \) of the coalgebra \( \mathcal{D} \), hence \( \tilde{X} = \text{Spec}_X(\mathcal{H}om_X(\mathcal{D}, \mathcal{O}_X)) \).

**Corollary 1.** If \( f = (\pi, \tilde{f}) \) is a finite flat correspondence as above, then
\[
(5) \quad f_* = \tilde{f}_*(\mathcal{H}om_X(\mathcal{D}, -)^\sim)
\]
and has the left adjoint
\[
(6) \quad f^* := \mathcal{D} \otimes_{\pi_* \mathcal{O}_{\tilde{X}}} \pi_* \tilde{f}^*(-).
\]
Moreover, there is a canonical homomorphism of quasicoherent algebras on \( Y \)
\[
(7) \quad \mathcal{O}_Y \to f_* \mathcal{O}_X = \tilde{f}_* \mathcal{O}_{\tilde{X}}.
\]
Finally, we have the following lemma.

**Lemma 2.** If \( f = (\pi, \tilde{f}) \) and \( g = (\rho, \tilde{g}) \) are finite flat correspondences as above, then
\[
(8) \quad g_* f_* = (gf)_*.
\]
Proof. By the flat base change formula applied to the cartesian square in the above composition of correspondences we have

\[ \rho^* \tilde{f}_* = \tilde{f}_* \pi^* \]  

which implies that

\[ g_* f_* = g_* \tilde{g} \rho^* \tilde{f}_* \pi^* = \tilde{g} \tilde{f}_* \pi^* \pi^* = (\tilde{g} \tilde{f})_* (\pi \pi)^* = (gf)_*. \]

Now we are to prove that this composition of functors \( g_* \) and \( f_* \) can be described by means of coalgebras which define these functors as in Corollary 1. We need for that an easy base change formula with affine morphisms.

**Lemma 3.** For an arbitrary cartesian square of schemes

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\pi & \downarrow & \downarrow \rho \\
\tilde{X} & \xrightarrow{f} & Y
\end{array}
\]

with \( \rho \) affine, the natural base change transformation

\[ \tilde{f}^* \rho_* \rightarrow \pi_* \tilde{f}^* \]

is an isomorphism.

**Proof.** Since \( \tilde{f}^* \) is strong comonoidal it transforms the \( \mathcal{O}_Y \)-algebra \( \rho_* \mathcal{O}_Y \) into an \( \mathcal{O}_{\tilde{X}} \)-algebra \( \tilde{f}^* \rho_* \mathcal{O}_Y = \pi_* \mathcal{O}_{\tilde{X}} \) and every \( \rho_* \mathcal{O}_Y \)-module \( \mathcal{G} \) quasicoherent on \( Y \) into a \( \pi_* \mathcal{O}_{\tilde{X}} \)-module \( \tilde{f}^* \mathcal{G} \) quasicoherent on \( \tilde{X} \). Let us denote by \( \mathcal{G}^\sim \) and \( (\tilde{f}^* \mathcal{G})^\sim \) the corresponding quasicoherent sheaves on \( \tilde{Y} = \text{Spec}_Y(\rho_* \mathcal{O}_Y) \) and \( \tilde{X} = \text{Spec}_{\tilde{X}}(\pi_* \mathcal{O}_{\tilde{X}}) \), respectively. Then

\[ \tilde{f}^*(\mathcal{G}^\sim) = (\tilde{f}^* \mathcal{G})^\sim. \]

Applying now \( \pi_* \) and the fact that \( \mathcal{G} = \rho_*(\mathcal{G}^\sim) \) we obtain

\[ \pi_* \tilde{f}^*(\mathcal{G}^\sim) = \tilde{f}^* \rho_*(\mathcal{G}^\sim). \]

This ends the proof of the lemma, because all quasicoherent sheaves on \( \tilde{Y} = \text{Spec}_Y(\rho_* \mathcal{O}_Y) \) are of the form \( \mathcal{G}^\sim \). ■

Now we can prove that the composition of inverse image functors corresponding to finite flat correspondences can be defined in terms of coalgebras corresponding to them according to Corollary 1, without any reference to cartesian squares of schemes.

**Lemma 4.** If \( D \) and \( E \) are coalgebras defining functors \( f^* \) and \( g^* \), respectively, according to Corollary 1, the composition \( f^* g^* \) is defined by a counital coalgebra \( D \otimes_{\pi_* \mathcal{O}_{\tilde{X}}} \pi_* \tilde{f}^* E \).
Proof. First we have to prove that coalgebra structures on \( D \) and \( E \) define a coalgebra structure on \( D \otimes_{\pi_*O_X} \pi_*f^*E \). It is so because \( \pi_* \) being affine is a strong monoidal equivalence between \( \text{Qcoh}(\tilde{X}) \) and the subcategory of \( \text{Qcoh}(X) \) consisting of \( \pi_*O_X \)-modules with \( \pi_*O_X \)-linear morphisms and the tensor product over \( \pi_*O_X \). The counit is defined as the natural composite

\[
\begin{align*}
D \otimes_{\pi_*O_X} \pi_*f^*E &\rightarrow D \otimes_{\pi_*O_X} \pi_*f^*O_Y = D \otimes_{\pi_*O_X} \pi_*O_{\tilde{X}} = D \rightarrow O_X.
\end{align*}
\]

Now the composition. Below we apply the following facts:

\[
\begin{align*}
- \otimes_{\rho_*O_Y} (-) &= \text{coker}((-) \otimes Y \rho_*O_Y \otimes Y (-) \rightarrow (-) \otimes Y (-)), \\
 f^* &\text{ is strong comonoidal and right exact, } \pi_* &\text{ is strong monoidal as above and (right) exact, } \rho &\text{ is affine and we use Lemma 3.}
\end{align*}
\]

\[
\begin{align*}
(14) \quad D \otimes_{\pi_*O_X} \pi_*f^*E &\rightarrow D \otimes_{\pi_*O_X} \pi_*f^*O_Y = D \otimes_{\pi_*O_X} \pi_*O_{\tilde{X}} = D \rightarrow O_X. \\
(15) \quad (-) \otimes_{\rho_*O_Y} (-) &= \text{coker}((-) \otimes Y \rho_*O_Y \otimes Y (-) \rightarrow (-) \otimes Y (-)), \\
 f^* &\text{ is strong comonoidal and right exact, } \pi_* &\text{ is strong monoidal as above and (right) exact, } \rho &\text{ is affine and we use Lemma 3.}
\end{align*}
\]

In the next example we will recognize Corollary 1 and Lemma 4 in their noncommutative versions.

Example 3. (Noncommutative affine schemes).

Definition. We call (convolution) representation of a ring \( B \) over another ring \( A \) an arbitrary ring homomorphism \( B \rightarrow \text{Hom}_A(D, A)_A \) to the convolution ring of a given \( D \in \text{Coalg}(A) \). Two representations \( B \rightarrow \text{Hom}_A(D, A)_A \) and \( B \rightarrow \text{Hom}_A(D', A)_A \) are isomorphic if there is an isomorphism of counital coalgebras \( D \rightarrow D' \) making the diagram

\[
\begin{array}{ccc}
B & \rightarrow & \text{Hom}_A(D, A)_A \\
\| & & \uparrow \\
B & \rightarrow & \text{Hom}_A(D', A)_A
\end{array}
\]

commutative.

Example. Let \( P \) be a finitely generated projective right \( A \)-module and \( D = P^* \otimes P \) be a coalgebra in \( \text{Bimod}(A) \), with the comultiplication given by the dual basis map and the counit given by the evaluation map. Then \( \text{Hom}_A(D, A)_A = \text{End}(P)_A \). We call such a representation linear.

Definition. Now we construct a composition of convolution representations. Given two such

\[
B \rightarrow \text{Hom}_A(D, A)_A, \quad F \rightarrow \text{Hom}_B(E, B)_B.
\]
we define a third one. It is a convolution representation given as the composite of canonical ring homomorphisms

\[ E \to \text{Hom}_B(D, B) \to \text{Hom}_B(E, \text{Hom}_A(D, A))_B = \text{Hom}_A(E \otimes_{B^o \otimes B} D, A)_A, \]

where the structure of a left \((B^o \otimes B)\)-module on \(D\) comes from the \(B\)-bimodule structure defined in (22), and the structure of a coalgebra in \(\text{Bimod}(A)\) on \(E \otimes_{B^o \otimes B} D\) is defined as follows: the counits \(E \to B\) and \(D \to A\) define a counit given as the composite of canonical homomorphisms in \(\text{Bimod}(A)\)

\[ E \otimes_{B^o \otimes B} D \to B \otimes_{B^o \otimes B} D = D/[D, B] \to A, \]

the comultiplications

\[ D \to D \otimes_A D, \quad E \to E \otimes_B E, \]
\[ d \mapsto d_{(1)} \otimes d_{(2)}, \quad e \mapsto e_{(1)} \otimes e_{(2)}, \]

define a comultiplication

\[ E \otimes_{B^o \otimes B} D \to (E \otimes_{B^o \otimes B} D) \otimes_A (E \otimes_{B^o \otimes B} D), \]
\[ e \otimes d \mapsto (e_{(1)} \otimes d_{(1)}) \otimes (e_{(2)} \otimes d_{(2)}). \]

Identities have the following description

\[ \text{id}_A : A \xrightarrow{\text{id}} \text{Hom}_A(A \otimes A, A)_A. \]

In this way we obtain a category whose objects are (unital associative) rings and morphisms are isoclasses of convolution representations.

**Theorem 1.** There is a contravariant fully faithful embedding of the category of rings with isoclasses of convolution representations as morphisms into the category of noncommutative schemes.

**Proof.** To every associative ring \(A\) we assign its monoidal category \((\text{Bimod}(A), \otimes_A, A)\) of bimodules. Given a convolution representation

\[ B \to \text{Hom}_A(D, A)_A, \]
\[ b \mapsto (c \mapsto b(d)), \]

for some coalgebra \(D\) in \(\text{Bimod}(A)\) we have a canonical structure of a \(B\)-bimodule on \(D\), commuting with the original \(A\)-bimodule structure, defined as follows

\[ b \cdot d := c_{(1)} b(d_{(2)}), \quad d \cdot b := b(d_{(1)}) d_{(2)}. \]

Regarding a \(B\)-bimodule structure as the same as a left or right \(B^o \otimes B\)-module structure we obtain the following pair of adjoint functors \(f^* \dashv f_*\)

\[ f^* : \text{Bimod}(B) \to \text{Bimod}(A), \quad f^* N = N \otimes_{B^o \otimes B} D, \]
\[ f_* : \text{Bimod}(A) \to \text{Bimod}(B), \quad f_* M = \text{Hom}_A(D, M)_A. \]

For any \(A\)-bimodule \(M\) below, we will denote by \(N\) the \(B\)-bimodule \(f_* M = \text{Hom}_A(D, M)_A\) and we will regard elements of \(N\) as maps \(d \mapsto n(d)\). Then we
have the following natural transformation $f_*(-) \otimes_B f_*(-) \to f_*(\_ \otimes_A \_)$ of bifunctors $\text{Bimod}(A) \times \text{Bimod}(A) \to \text{Bimod}(B)$

\begin{equation}
\begin{aligned}
&f_*(M_1 \otimes_B f_*M_2) \to f_*(M_1 \otimes_A M_2),
&(d \mapsto n_1(d)) \otimes (d \mapsto n_2(d)) \mapsto (c \mapsto n_1(d(1)) \otimes n_2(d(2))),
\end{aligned}
\end{equation}

making $f_*$ a monoidal functor.

Note that the representation (21) gives rise automatically to a morphism $B \to f_*A$ in $\text{Bimod}(B)$. In this way a convolution representation gives rise to a morphism of noncommutative schemes.

Consider now a morphism of bimodule categories in the category of noncommutative schemes given by a comonoidal functor $f^* : \text{Bimod}(B) \to \text{Bimod}(A)$ and a homomorphism $f^*B \to A$ in $\text{Bimod}(A)$. Since $B \otimes B$ is a coalgebra in $\text{Bimod}(B)$, with the counit $B \otimes B \to B$

\begin{equation}
\begin{aligned}
&B \otimes B \to B \\
&b_1 \otimes b_2 \to b_1b_2,
\end{aligned}
\end{equation}

and the comultiplication

\begin{equation}
\begin{aligned}
&B \otimes B \to (B \otimes B) \otimes_B (B \otimes B) \\
&b_1 \otimes b_2 \to (b_1 \otimes 1) \otimes (1 \otimes b_2),
\end{aligned}
\end{equation}

$D := f^*(B \otimes B)$ is a coalgebra in $\text{Bimod}(A)$ as well. Let us consider the composite of the following canonical maps, using the morphism $f^*B \to A$ in $\text{Bimod}(A)$

\begin{equation}
\begin{aligned}
&B = \text{Hom}_B(B \otimes B, B)_B \to \text{Hom}_A(f^*(B \otimes B), f^*(B))_A \\
&\quad \to \text{Hom}_A(f^*(B \otimes B), A)_A = \text{Hom}_A(D, A)_A.
\end{aligned}
\end{equation}

Since $f^*$ is comonoidal, the above composite is a homomorphism of rings, where $\text{Hom}_A(D, A)_A$ is equipped with the canonical convolution product. One can check easily that the functor $f^*$ is an image under our assignment of this convolution representation of $B$ over $A$. ■

**Definition.** We call the essential image of the above embedding *category of affine noncommutative schemes*. Let $A$ be an associative ring. Then we define the object $X := \text{Spec}(A)$ of $\mathfrak{Sch}$ as follows: $\text{Qcoh}(X) := \text{Bimod}(A)$, $\otimes_X := \otimes_A$, $\mathcal{O}_X := A$. We call such an affine noncommutative scheme *noncommutative spectrum*. In particular, we have a distinguished affine scheme $S := \text{Spec}(\mathbb{Z})$, where $\text{Qcoh}(S) := \text{Ab} = \text{Bimod}(\mathbb{Z})$, $\otimes_S := \otimes = \otimes_{\mathbb{Z}}$, $\mathcal{O}_S := \mathbb{Z}$.

2.1.1. **Morita invariance of the spectrum.** The following fact is a corollary of the above structural theorem. Essentially, it is a monoidal enhancement of the well known Morita invariance of bimodule categories.

**Theorem 2.** A Morita equivalence between associative rings $A$ and $B$ induces an isomorphism between affine noncommutative schemes $\text{Spec}(A)$ and $\text{Spec}(B)$. 
Proof. A Morita equivalence of $A$ and $B$ can be described as a representation which is an isomorphism of rings

$$B \to \text{Hom}_A(P^* \otimes P, A)_A = \text{End}(P)_A,$$

where $P$ is a finitely generated projective generator in the category $\text{Mod}-A$ of right $A$-modules. This defines a morphism of affine schemes $\text{Spec}(A) \to \text{Spec}(B)$. By the Morita theory $P^*$ is a finitely generated projective generator in $\text{Mod}-B$ and the homomorphism of rings

$$A \to \text{Hom}_A(P^* \otimes P, A)_A = \text{End}(P)_B,$$

is an isomorphism. This defines a morphism $\text{Spec}(A) \leftarrow \text{Spec}(B)$ in the opposite direction in the same way as (29) defined $f$. It is inverse to $f$ since by the Morita theory

$$P^* \otimes_B P \cong A \quad (\text{resp. } P \otimes_A P^* \cong B)$$

in $\text{Bimod}(A)$ (resp. in $\text{Bimod}(B)$), hence composites of these two morphisms are represented by coalgebras

$$(P \otimes P^*) \otimes_{B^\circ \otimes B} (P^* \otimes P) = (P^* \otimes_B P) \otimes (P^* \otimes_B P) \cong A \otimes A
\text{ (resp. } (P^* \otimes P) \otimes_{A^\circ \otimes A} (P \otimes P^*) = (P \otimes_A P^*) \otimes (P \otimes_A P^*) \cong B \otimes B)$$

in $\text{Bimod}(A)$ (resp. in $\text{Bimod}(B)$) isomorphic to coalgebras representing identity morphisms. ■

The problem of description of monoidal equivalences of categories of bimodules was first considered by Takeuchi \[43, 44\] under the name of $\sqrt{\text{Morita}}$-equivalences. Since any strong monoidal equivalence admits the left adjoint (which is necessarily strong comonoidal) $\sqrt{\text{Morita}}$-equivalences are isomorphism in the category of affine noncommutative schemes, and all such isomorphism are of that kind.

2.2. Affine morphisms and spectra.

Definition. For every $A \in \text{Alg}(X)$ and $\mathcal{F}_1, \mathcal{F}_2 \in \text{Bimod}_X(A)$ we define the tensor product $\mathcal{F}_1 \otimes_A \mathcal{F}_2$, as usual, as the cokernel of the canonical pair of morphisms

$$\mathcal{F}_1 \otimes_X A \otimes_X \mathcal{F}_2 \Rightarrow \mathcal{F}_1 \otimes_X \mathcal{F}_2.$$

Definition. We call a morphism $f : X \to Y$ affine if the functor $f_*$ is faithful, exact, and the natural transformation of bifunctors

$$f_*(-) \otimes_{f_* \circ X} f_*(-) \Rightarrow f_*(- \otimes_X -)$$

is an isomorphism.
**Definition.** For every $A \in \text{Alg}(X)$ such that the category $(\text{Bimod}_X(A), \otimes_A, A)$ is abelian monoidal we define the following noncommutative scheme

$$\text{Spec}_X(A) := (\text{Bimod}_X(A), \otimes_A, A)$$

and a canonical morphism $\text{Spec}_X(A) \to X$ whose direct image functor is forgetting of the $A$-bimodule structure.

**Remark.** We have a canonical isomorphism of noncommutative schemes $X \to \text{Spec}_X(O_X)$.

**Proposition 1.** Given an affine morphism $f : X \to Y$ the category $(\text{Bimod}_Y(f_*(O_X)), \otimes_{f_*(O_X)}, f_*(O_X))$ is abelian monoidal and we have the following canonical decomposition

$$X \xrightarrow{f} Y, \quad \text{Spec}_Y(f_*(O_X))$$

where the south-east arrow is an isomorphism.

**Proof.** Since $f_*$ is monoidal it admits the following canonical decomposition

$$\text{Qcoh}(X) \xrightarrow{f_*} \text{Qcoh}(Y)$$

where the right hand side arrow is the forgetting of the $f_*O_X$-bimodule structure.

Since $f_*$ is monoidal we have, for every two $\mathcal{F}_1, \mathcal{F}_2 \in \text{Qcoh}(X)$, the following decomposition

$$f_*(\mathcal{F}_1 \otimes_X \mathcal{F}_2) \xrightarrow{f_*} f_*(\mathcal{F}_1 \otimes f_*O_X \mathcal{F}_2).$$

Since every equivalence of categories has the left adjoint (equal to the inverse) and forgetting the bimodule structure has the left adjoint, $f_*$ has the left adjoint as well. Since $f_*$ being faithful exact induces by the Barr-Beck theorem an equivalence $\text{Qcoh}(X) \to \text{Bimod}_Y(f_*O_X)$ the latter category is abelian. $(\text{Bimod}_Y(f_*O_X), \otimes_{f_*O_X}, f_*O_X)$ is a monoidal category and the equivalence is strong monoidal by **[3]** applied to the latter decomposition. ■

2.3. **Affine schemes and affine morphisms.** Exactly as in the commutative algebraic geometry we have the following proposition.

**Proposition 2.** A noncommutative scheme is affine iff it admits an affine morphism to $\text{Spec}(\mathbb{Z})$. 
Proof. If $X = \text{Spec}(A)$ then the unique ring homomorphism $\mathbb{Z} \to A$ defines a morphism $X \to \text{Spec}(\mathbb{Z})$. Since algebras over $\mathbb{Z}$ are simply rings, then by Proposition 1 every affine morphism $X \to \text{Spec}(\mathbb{Z})$ defines an isomorphism $X \xrightarrow{\cong} \text{Spec}_{\text{Spec}(\mathbb{Z})}(f_*\mathcal{O}_X) = \text{Spec}(f_*\mathcal{O}_X)$. ■

Remark. Although every affine scheme $\text{Spec}(A)$ admits a morphism $\text{Spec}(A) \to \text{Spec}(\mathbb{Z})$ corresponding to the unique ring homomorphism $\mathbb{Z} \to A$, $\text{Spec}(\mathbb{Z})$ is not a final object in the category of noncommutative affine schemes because of possible non-identical morphisms $\text{Spec}(\mathbb{Z}) \to \text{Spec}(\mathbb{Z})$ corresponding to arbitrary coalgebras $D$ defined over $\mathbb{Z}$. This resembles the situation in homotopy theory of topological $G$-spaces, where the one-point space is not a final object and there are spaces leaving under it, e.g. classifying spaces. In this analogy coalgebras play the role of group actions. More precise relation between coalgebras and group actions (or Hopf algebra coactions) needs some regularity conditions discussed in the next section.

2.4. Flat covers and cospectra.

Definition. For every $C \in \text{Coalg}(X)$ and $\mathcal{F}_1, \mathcal{F}_2 \in \text{Bicomod}_X(C)$ we define the cotensor product $\mathcal{F}_1 \boxtimes^C \mathcal{F}_2$, as usual, as the kernel of the canonical pair of morphisms

\[
\mathcal{F}_1 \otimes_X \mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_1 \otimes_X C \otimes_X \mathcal{F}_2.
\]

Definition. We call a morphism $f : X \to Y$ flat if the functor $f^*$ is exact and the natural transformation of bifunctors

\[
f^*(-) \boxtimes_{f^*\mathcal{O}_Y} f^*(-) \leftrightarrow f^*(- \otimes_Y -)
\]

is an isomorphism, and cover if $f^*$ is faithful.

Definition. For every $C \in \text{Coalg}(X)$ such that the category $(\text{Bicomod}_X(C), \boxtimes^C, C)$ is abelian monoidal we define the following noncommutative scheme

\[
\text{Cospec}_X(C) := (\text{Bicomod}_X(C), \boxtimes^C, C).
\]

and a canonical morphism $X \to \text{Cospec}_X(C)$, whose inverse image functor is forgetting of the $C$-bicomodule structure.

Remark. We have a canonical isomorphism of noncommutative schemes

\[
\text{Cospec}_X(\mathcal{O}_X) \to X.
\]

Proposition 3. Given a flat cover $f : X \to Y$ the category $(\text{Bicomod}_X(f^*\mathcal{O}_Y), \boxtimes^{f^*\mathcal{O}_Y}, f^*\mathcal{O}_Y)$ is abelian monoidal and we have the following
canonical decomposition

\[
\begin{array}{c}
X \\ \downarrow \cospec_X(f^*\mathcal{O}_Y) \\
\end{array} \xrightarrow{f} \begin{array}{c}
Y \\ \nearrow \end{array}
\]

where the north-east arrow is an isomorphism.

**Proof.** Since \( f^* \) is comonoidal it admits the following canonical decomposition

\[
\begin{array}{c}
\text{Qcoh}(X) \\ \downarrow \text{Bicomod}_X(f^*\mathcal{O}_Y) \\
\end{array} \xleftarrow{f^*} \begin{array}{c}
\text{Qcoh}(Y) \\ \nearrow \end{array}
\]

where the north-west arrow is forgetting of the \( f^*\mathcal{O}_Y \)-bicomodule structure.

Since \( f^* \) is comonoidal we have, for every two \( G_1, G_2 \in \text{Qcoh}(Y) \), the following decomposition

\[
f^*G_1 \otimes_X f^*G_2 \leftarrow \begin{array}{c}
\text{Qcoh}(Y) \\ \downarrow \end{array} \xrightarrow{f^*} \begin{array}{c}
f^*(G_1 \otimes_Y G_2). \\
\end{array}
\]

Since every equivalence of categories has the right adjoint (equal to the inverse) and forgetting the bicomodule structure has the right adjoint, \( f^* \) has the right adjoint as well. Since \( f^* \) being faithful exact induces by the Barr-Beck theorem an equivalence \( \text{Qcoh}(Y) \leftarrow \text{Bicomod}_X(f^*\mathcal{O}_Y) \) the latter category is abelian. (\( \text{Bicomod}_X(f^*\mathcal{O}_Y), \square f^*\mathcal{O}_Y, f^*\mathcal{O}_Y \)) is a monoidal category and the equivalence is strong comonoidal by (38) applied to the latter decomposition. \( \blacksquare \)

2.5. **Sections and the sheaf condition.** Consider a commutative diagram

\[
\begin{array}{c}
U \\
\downarrow \alpha \\
S \\
\end{array} \xrightarrow{\pi} \begin{array}{c}
X \\ \swarrow \beta \\
\end{array}
\]

in the category of noncommutative schemes over \( S := \text{Spec}(\mathbb{Z}) \) with \( \pi \) flat. Let \( \mathcal{F} \in \text{Qcoh}(X) \).

**Definition.** We define an abelian group of sections of \( \mathcal{F} \) over \( U \) as follows

(41) \( \mathcal{F}(U) := \text{Hom}_{\text{cospec}_X(\pi^*\mathcal{O}_X)}(\alpha^*\mathcal{O}_S, \pi^*\mathcal{F}). \)

**Proposition 4.** \( \mathcal{O}_X(-) \) is a contravariant functor from the category of \( S \)-schemes flat over \( X \) to the category of rings. For any \( \mathcal{F} \in \text{Qcoh}(X) \) sections \( \mathcal{F}(-) \) form an \( \mathcal{O}_X(-) \)-bimodule and the assignment \( \mathcal{F} \mapsto \mathcal{F}(-) \) is a monoidal functor \( \text{Qcoh}(X) \rightarrow \text{Bimod}(\mathcal{O}_X(-)) \).
Proof. Since $\beta^*O_S \in \text{Coalg}(X)$ and $\pi$ is flat the inverse image $\alpha^*O_S = \pi^*\beta^*O_S$ is a coalgebra in the monoidal category $(\text{Bicomod}(\pi^*O_X), \square^{\pi^*O_X}, \pi^*O_X)$. Then the following canonical composite

\begin{equation}
\text{Hom}_U(\pi^*O_X, \pi^*O_X)^{\pi^*O_X} \otimes \text{Hom}_U(\pi^*O_X, \pi^*O_X)^{\pi^*O_X} \\
\downarrow \\
\text{Hom}_U(\pi^*O_X, \pi^*O_X)^{\pi^*O_X,\pi^*O_X,\square^{\pi^*O_X,\pi^*O_X}}^{\pi^*O_X}
\end{equation}

defines a ring structure on sections $O_X(U) = \text{Hom}_{\text{Cospec}_X}(\pi^*O_X, \pi^*O_X) = \text{Hom}_U(\pi^*O_X, \pi^*O_X)^{\pi^*O_X}$ with the unit being the image of the identity in $\text{Hom}_U(\pi^*O_X, \pi^*O_X)^{\pi^*O_X}$ under the canonical map induced by the morphism $\alpha^*O_S = \pi^*\beta^*O_S \rightarrow \pi^*O_X$. Replacing in (42) one copy of $O_X$ in the covariant argument of Hom by $F$ we obtain an $O_X(U)$-bimodule structure on $F(U)$. Similarly, replacing in (42) two copies of $O_X$ by $F_1$, $F_2$ and inverting the isomorphism

\begin{equation}
\pi^*(F_1)\square^{\pi^*O_X}\pi^*(F_2) \leftrightarrow \pi^*(F_1 \otimes_X F_2)
\end{equation}

we obtain

\begin{equation}
F_1(U) \otimes F_2(U) \rightarrow (F_1 \otimes_X F_2)(U).
\end{equation}

One can check that the above map factorizes canonically through

\begin{equation}
F_1(U) \otimes_{O_X(U)} F_2(U) \rightarrow (F_1 \otimes_X F_2)(U).
\end{equation}

It is clear that all these constructions are functorial in $U$. ■

**Proposition 5.** If $\pi$ is a flat cover then $F(U) = F(X)$. In particular, the functor of global sections is independent of the choice of a flat cover.

**Proof.** By Proposition 2 we have

$F(U) = \text{Hom}_{\text{Cospec}_X}(\pi^*\beta^*O_S, \pi^*F) = \text{Hom}_X(\beta^*O_S, F) = F(X)$. ■

**Proposition 6.** $F(X) = \beta_*F$.

**Proof.** By the following isomorphism of functors $\text{Hom}_S(O_S, -) = id_{\text{Ab}}$, for $S = \text{Spec}(Z)$, we have

$F(X) = \text{Hom}_X(\beta^*O_S, F) = \text{Hom}_S(O_S, \beta_*F) = \beta_*F$. ■

2.6. Flat covers and Galois extensions. Let $A$ be a ring and $D \in \text{Coalg}(A)$.
Definition. Let $R$ be a ring. A morphism $\varphi : M_0 \to M_1$ in Bimod$(R)$ is called pure if for all $U \in \text{Mod}_R$, $V \in \text{Mod}$ the induced sequence
\[(46) \quad 0 \to U \otimes_R \ker(\varphi) \otimes_R V \to U \otimes_R M_0 \otimes_R V \to U \otimes_R M_1 \otimes_R V\]
is exact.

The following lemma is a bimodule version of the purity criterion of compatibility of the tensor and cotensor products \[42, 41\].

Lemma 5. Assume that there is given $C \in \text{Coalg}(A)$ and a left $R$-module right $C$-comodule $M_1$ and a right $R$-module left $C$-comodule $M_2$. Then

1) for every $U \in \text{Mod}_R$ and every $V \in \text{Mod}$ there exists a canonical, natural in $U$ and $V$, homomorphism of abelian groups
\[(47) \quad U \otimes_R (M_1 \square_A M_2) \otimes_R V \to (U \otimes_R M_1) \square_A (M_2 \otimes_R V)\]

2) the canonical morphism in Bimod$(R)$ defining the cotensor product
\[(48) \quad M_1 \otimes_A M_2 \to M_1 \otimes_A C \otimes_A M_2\]
is pure iff all homomorphisms \([47]\) are isomorphisms.

Proof. We have a canonical exact sequence in Bimod$(R)$
\[(49) \quad 0 \to M_1 \square_A M_2 \to M_1 \otimes_A M_2 \to M_1 \otimes_A C \otimes_A M_2.\]

Tensoring it from both sides by $U$ and $V$ we obtain a complex fitting into the following diagram with the exact bottom row and the left hand side arrow in the bottom row injective
\[
\begin{array}{cccccc}
U \otimes_R (M_1 \square_A M_2) \otimes_R V & \to & U \otimes_R (M_1 \otimes_A M_2) \otimes_R V & \to & U \otimes_R (M_1 \otimes_A C \otimes_A M_2) \otimes_R V \\
\downarrow & & \downarrow \cong & & \downarrow \cong \\
(U \otimes_R M_1) \square_A (M_2 \otimes_R V) & \to & (U \otimes_R M_1) \otimes_A (M_2 \otimes_R V) & \to & (U \otimes_R M_1) \otimes_A C \otimes_A (M_2 \otimes_R V)
\end{array}
\]
This defines the vertical left hand side arrow and implies that the upper row is exact and the left hand side arrow in the upper row is injective iff the vertical left hand side arrow is an isomorphism. ■

Definition. If there is given a morphism $D \to C$ in Coalg$(A)$ such that $\square_A^C$ is associative (with the unit $C$) then $D$ becomes a coalgebra in the monoidal category (Bicomod$(A,C), \square_A^C, C$) and the canonical composite
\[(50) \quad \text{Hom}^C_A(D,C)^C_A \otimes \text{Hom}^C_A(D,C)^C_A \to \text{Hom}^C_A(D \square_A^C D, C \square_A^C C)^C_A \to \text{Hom}^C_A(D,C)^C_A,\]
where the right hand arrow uses the comultiplication $D \to D \square_A^C D$ and the isomorphism $C \xrightarrow{\cong} C \square_A^C C$, defines a subring structure on $\text{Hom}^C_A(D,C)^C_A \subset \text{Hom}_A(D,A)^C_A$, which we call subring of invariants.
Definition. Assume there is given a representation \( B \to \text{Hom}_A(D, A) \) factoring through the subring of invariants \( \text{Hom}_A^C(D, C) \). It is an exercise in Sweedler’s notation to prove using (22) that

- \( D/[D, B] \) inherits a structure of a coalgebra in \( \text{Bimod}(A) \) from \( D \),
- the homomorphism \( D \to C \) in \( \text{Coalg}(A) \) factorizes in \( \text{Coalg}(A) \) canonically through

\[
D/[D, B] \to C, \tag{51}
\]
- there are structures of \( (B^o \otimes B) \)-bimodules on \( B \otimes D \) and \( D \square_A^C D \) defined as follows

\[
(b^o \otimes b'') \cdot (b \otimes d) := bb' \otimes b'' \cdot d,
(b \otimes d) \cdot (b^o \otimes b'') := b'b \otimes d \cdot b'',
(b^o \otimes b'') \cdot (d^o \otimes d'') := d' \otimes b'' \cdot d'' \cdot b',
(d^o \otimes d'') \cdot (b^o \otimes b'') := b' \cdot d' \cdot b'' \otimes d'',
\]

and there is a canonical morphism in \( \text{Bimod}(B^o \otimes B) \)

\[
B \otimes D \to D \square_A^C D,
\]
defined as follows

\[
b \otimes d \mapsto d_{(1)} b(d_{(2)}) \otimes d_{(3)} = d_{(1)} \otimes b(d_{(2)}) d_{(3)}. \tag{54}
\]
- for every \( M \in \text{Bimod}(A) \) there is a canonical morphism in \( \text{Bimod}(A) \) natural in \( M \)

\[
\text{Hom}_A(D, M)_A \otimes_{B^o \otimes B} D \to D/[D, B] \otimes_A M \otimes_A D/[D, B],
\]
defined as follows

\[
\mu \otimes d \mapsto [d_{(1)}] \otimes \mu(d_{(2)}) \otimes [d_{(3)}], \tag{55}
\]
where \([d] := d + [D, B]\).

Definition. We call a representation \( B \to \text{Hom}_A(D, A) \) noncommutative Galois ring extension if there is given a morphism \( D \to C \) in \( \text{Coalg}(A) \) such that

- (regularity) the category \( (\text{Bicomod}_A^C(C), \square_A^C, C) \) is abelian monoidal,
- (purity) the canonical morphism in \( \text{Bimod}(B^o \otimes B) \)

\[
D \otimes_A D \to D \otimes_A C \otimes_A D
\]
defining the cotensor product is pure,
- (invariants) \( B \) is mapped isomorphically onto the subring of invariants

\[
\text{Hom}_A^C(D, C) \subset \text{Hom}_A(D, A)_A,
\]
and the canonical morphism in \( \text{Bimod}(B^o \otimes B) \)

\[
B \otimes D \to D \square_A^C D,
\]
is an isomorphism,
• (faithful flatness) \( D \) is faithfully flat as a left \( B^o \otimes B \)-module,
• (freeness) the canonical morphism
  \[
  D / [D, B] \to C
  \]
in \( \text{Coalg}(A) \) is an isomorphism,
• (comonad) the natural transformation of functors
  \[
  \text{Hom}_A(D, -) \otimes_{B^o \otimes B} D \to C \otimes_A - \otimes_A C
  \]
is an isomorphism.

2.6.1. Comparison with Galois comodules. Let \( A \) be a ring, \( C \in \text{Coalg}(A) \) and \( P \) be a finitely generated projective right \( A \)-module. Assume now in addition that \( P \) is a right comodule over \( C \). Define \( B := \text{End}(P)^C_A \). Then \( P \) is called Galois comodule if the natural transformation of functors \( \text{Mod}_A \to \text{Comod}_A \)
  \[
  \text{Hom}(P, -)_A \otimes_B P \to - \otimes_A C
  \]
is an isomorphism. For the complicated history of this simple definition we refer the reader to [48] (where the assumption of being finitely generated projective is dropped). Under some assumptions ([48] Theorem 5.7. Equivalences.) the functor
  \[
  \text{Hom}(P, -)^C_A : \text{Comod}_A^C \to \text{Mod}_B
  \]
is an equivalence with the inverse \( - \otimes_B P \).

Galois comodules and their predecessors (Galois corings, Galois extensions, coalgebra-Galois extensions, Hopf-Galois extensions, [3, 4, 5, 6, 7, 14, 40, 47]) found an interesting interpretation in terms of descent theory, theory of invariants, associated vector bundles etc. in the realm of noncommutative geometry modeled on one-sided (co)modules [5]. Starting from slightly different assumptions, we realize a similar program in our noncommutative geometry modeled on monoidal categories. The main difference consists in the role of the monoidal structure. In the first approach associativity of the cotensor product fails in general ([21]) and can be achieved only for the price of painful and difficult to examine assumptions ([18]). In our approach cotensor products arising from our geometric flat covers are associative almost by definition, so we put this associativity as a necessary regularity condition in our definition of a noncommutative Galois ring extension. In the purely commutative case of schemes with trivial group scheme action, viewed as symmetric monoidal categories, the inverse image of the monoidal unit \( f^*\mathcal{O}_Y = \mathcal{O}_X \) is a monoidal unit again, so the associativity condition is void. In the sequel of this paper we will present the derived version of our monoidal noncommutative geometry, where this restriction disappears. In general, it is not easy to analyse relations between different assumptions in one-sided and monoidal noncommutative geometries. The following proposition can be regarded as a transition of the border line between these two approaches.
Proposition 7. Let $C \in \text{Coalg}(A)$ be a coalgebra over a ring $A$ such that $(\text{Bicomod}_A(C), \square^C_A, C)$ is an abelian monoidal category. Let $P$ be a right $C$-comodule which is finitely generated and projective as a right $A$-module and define a subring $B := \text{End}(P)^C_A \subset \text{End}(P)_A$. Assume the following conditions are fulfilled:

1) the functor $P^* \otimes_B - \otimes_B P : \text{Bimod}(B) \to \text{Bimod}(A)$ is faithful exact,

2) the first differential in the Amitsur complex

\begin{equation}
\text{End}(P)_A \to \text{End}(P)_A \otimes_B \text{End}(P)_A,
\end{equation}

$e \mapsto 1 \otimes e - e \otimes 1$,

is pure in $\text{Bimod}(B)$,

3) the canonical morphism in $\text{Coalg}(A)$

\begin{equation}
P^* \otimes_B P \to C,
\end{equation}

$p^* \otimes p \mapsto p^*(p_{(0)}) \cdot p_{(1)},$

is an isomorphism.

Take $D := P^* \otimes P$ and the canonical morphism $D \to C$ in $\text{Coalg}(A)$ defined as in (59). Then there is a canonical ring isomorphism $\text{End}(P)_A \cong \text{Hom}_A(D, A)_A$ such that $(B \to \text{Hom}_A(D, A)_A, D \to C)$ is a noncommutative Galois ring extension.

Proof. Since $P$ is finitely generated projective there is a canonical isomorphism of rings

\begin{equation}
\text{End}(P)_A \cong \text{Hom}_A(P^* \otimes P, A)_A,
\end{equation}

$e \mapsto (p^* \otimes p \mapsto p^*(e(p)))$,

with the inverse defined by means of the dual basis $((p_i)_{i \in I}, (p^*_i)_{i \in I})$, $p^*_i(p_j) = \delta_{ij}$, as follows

\begin{equation}
(p \mapsto \sum_{i \in I} p_i \cdot h(p^*_i \otimes p)) \leftrightarrow h,
\end{equation}

For a right $C$-comodule $P$, with the comultiplication

\begin{equation}
P \to P \otimes_A C,
\end{equation}

$p \mapsto p_{(0)} \otimes p_{(1)},$

which is finitely generated and projective as a right $A$-module, the canonical left $C$-comodule structure on the dual $P^* := \text{Hom}(P, A)_A$ can be defined by means of the dual basis as follows

\begin{equation}
P^* \to C \otimes_A P^*,
\end{equation}
\[(64) \quad p^* \mapsto p_{(-1)}^* \otimes p_{(0)}^* := \sum_{i \in I} p^*(p_i(0)) \cdot p_i(1) \otimes p_i^*.\]

Then the canonical isomorphism \[(60)\] induces an isomorphism
\[(65) \quad \text{End}(P) \overset{\sim}{\to} \text{Hom}^C_A(P^* \otimes P, A)_A^C,\]
which, by the definition of \(B\), proves the first part of the invariants condition.

By the canonical isomorphism
\[(66) \quad P \otimes_A P^* \to \text{End}(P)_A,\]
with the inverse (dual basis)
\[(67) \quad \sum_{i \in I} e(p_i) \otimes p_i^* \leftrightarrow e,\]
and the isomorphism \[(59)\] the first differential in the Amitsur complex is isomorphic to the map defining the cotensor product \[P \square_A C \otimes_A P^*,\]
\[(68) \quad p \otimes p^* \mapsto (p \mapsto p \cdot p^*(p')),\]
By purity of \[(68)\] and Lemma \[5\], the homomorphism
\[(69) \quad B \otimes D \to D \square_A C D,\]
can be rewritten as follows
\[(70) \quad B \otimes P^* \otimes P \to (P^* \otimes P) \square_A C (P^* \otimes P) \cong P^* \otimes (P \square_A P^*) \otimes P.\]
The isomorphism \[(66)\] induces an isomorphism
\[(71) \quad P \square_A P^* \to \text{End}(P)_A^C,\]
hence \[(70)\] can be rewritten as
\[(72) \quad B \otimes P^* \otimes P \to P^* \otimes \text{End}(P)_A^C \otimes P \cong \text{End}(P)_A^C \otimes P^* \otimes P\]
which is induced by the homomorphism \(B \to \text{End}(P)_A^C\). If the latter homomorphism is an isomorphism, \[(69)\] is an isomorphism as well, which proves the second part of the invariants condition.

Again by purity of \[(68)\] for all \(U, V \in \text{Bimod}(B)\) the canonical sequence
\[
\begin{align*}
0 & \quad (P^* \otimes_B U) \otimes_B (P \square_A P^*) \otimes_B (V \otimes_B P) \\
& \downarrow \\
& (P^* \otimes_B U) \otimes_B (P \otimes_A P^*) \otimes_B (V \otimes_B P) \\
& \downarrow \\
& (P^* \otimes_B U) \otimes_B (P \otimes_A C \otimes_A P^*) \otimes_B (V \otimes_B P),
\end{align*}
\]
is exact. It can be rewritten as
\[
0 \to U \otimes_{B^* \otimes B} (P^* \otimes (P \boxtimes^C_A P^*) \otimes P) \otimes_{B^* \otimes B} V \\
\downarrow \\
U \otimes_{B^* \otimes B} (P^* \otimes (P \otimes A^* C \otimes_A P^*) \otimes P) \otimes_{B^* \otimes B} V \\
\downarrow \\
U \otimes_{B^* \otimes B} (P^* \otimes (P \otimes A^* C \otimes_A P^*) \otimes P) \otimes_{B^* \otimes B} V,
\]
and further, using purity of (68) together with Lemma 5, and finally the definition of \( D := P^* \otimes P \), it can be rewritten as
\[
0 \to U \otimes_{B^* \otimes B} (D \boxtimes^C_A D) \otimes_{B^* \otimes B} V \\
\downarrow \\
U \otimes_{B^* \otimes B} (D \otimes_A D) \otimes_{B^* \otimes B} V \\
\downarrow \\
U \otimes_{B^* \otimes B} (D \otimes_A C \otimes_A D) \otimes_{B^* \otimes B} V,
\]
which proves purity of
\[
D \otimes_A D \to D \otimes_A C \otimes_A D, \tag{73}
\]
i.e. the purity condition.

The identification
\[
(P^* \otimes P)/[P^* \otimes P, B] = P^* \otimes_B P \tag{74}
\]
and the assumption 3) prove the freeness condition.

By duality between right and left finitely generated projective modules we have
\[
\text{Hom}_A(P^* \otimes P, -)_{A} \otimes_{B^* \otimes B} (P^* \otimes P) = (P^* \otimes_B P) \otimes_A - \otimes_A (P^* \otimes_B P), \tag{75}
\]
which proves the comonad condition. \( \blacksquare \)

**Remark.** If \( K \) is a commutative ring, \( H \) a Hopf algebra over \( K \), \( A \) a right comodule algebra over \( H \), then \( C := A \otimes_K H \in \text{Coalg}(A) \), \( P := A \in \text{Comod}^C_A \), \( B := \text{Hom}_A^C(D, C)^C_A = A^\text{co } H \), and the canonical map
\[
A \otimes_B A \to A \otimes_K H \tag{76}
\]
form a Hopf-Galois context. One can also take instead of a Hopf algebra an arbitrary (symmetric) coalgebra over \( K \) \([3, 6]\) and form a coalgebra-Galois context.

If \( S \) is a commutative affine scheme, \( G \) a commutative group scheme flat affine over \( S \), acting freely \( U \times_S G \to U \) on a commutative scheme \( U \) flat affine over \( S \) with a good quotient \( X = U/\!/G \), then \( K := \mathcal{O}(S) \), \( H := \mathcal{O}(G) \), \( A := \mathcal{O}(U) \), and \( B = \mathcal{O}(X) = \mathcal{O}(U/\!/G) = A^G \) form a Hopf-Galois context.
In particular, if $K$ is a field, $G$ a finite group of automorphisms of a finite field extension $K \subset A$ then the Hopf-Galois context with $H := \text{Map}(G, K)$ is equivalent to the classical $G$-Galois field extension context
\begin{equation}
K \xrightarrow{\cong} A^G \iff A \otimes_K A \xrightarrow{\cong} \text{Map}(G, A).
\end{equation}

**Remark.** The purity assumption in Proposition 7 is satisfied if the Amitsur complex admits a homotopy contracting it to $B$, i.e. if $B$ is a direct summand in the $B$-bimodule $\text{End}(P)_A$. By the *invariants* condition the latter property is equivalent to the existence of a generalized Reynolds operator, i.e. to generalized linear reductivity of the coaction of $C$.

2.6.2. Flat covers and Galois extensions. The next theorem is the main result of this paper. In particular, it means that various global constructions in commutative geometry related to group actions and gluing (invariants, descent theory) can be viewed as natural constructions in noncommutative geometry.

**Theorem 3.** There is one-to-one correspondence between flat covers in the category of noncommutative affine schemes and noncommutative Galois ring extensions.

**Proof.** (Flat covers $\leadsto$ Galois extensions) Consider a flat cover $f : \text{Spec}(A) \to \text{Spec}(B)$.

The multiplication morphism $B \otimes B \to B$ in $\text{Coalg}(B)$ induces a homomorphism in $\text{Coalg}(A)$
\begin{equation}
D := f^*(B \otimes B) \to C := f^*B.
\end{equation}

By Proposition 2 and the flat cover condition the category $\left(\text{Bicomod}_A(C), \square^C_A, C\right)$ is abelian monoidal (regularity condition).

Since $f$ is a flat cover the canonical composite
\begin{equation}
B = \text{Hom}_B(B \otimes B, B)_B \xrightarrow{f^*} \text{Hom}_A(f^*(B \otimes B), f^*B)_A \xrightarrow{f^*B} = \text{Hom}_A(D, C)_A^C.
\end{equation}
is, by Proposition 2, an isomorphism of rings (first part of the *invariants* condition). The isomorphism
\begin{equation}
f^*((B \otimes B) \otimes_B (B \otimes B)) \xrightarrow{\cong} f^*(B \otimes B) \square_A f^*(B \otimes B)
\end{equation}
can be rewritten, using $f^*(-) = (-)_{B^\otimes B}D$, as
\begin{equation}
B \otimes D \xrightarrow{\cong} D \square_A^C D
\end{equation}
(second part of the *invariants* condition).

By (81) we have, for $R := B^\otimes B$ and all $N_1, N_2 \in \text{Bimod}(B) \simeq R\text{Mod} \simeq \text{Mod}_R$, the canonical isomorphism
\begin{equation}
(N_1 \otimes_B N_2) \otimes_R D = N_1 \otimes_R (B \otimes D) \otimes_R N_2 \xrightarrow{\cong} N_1 \otimes_R (D \square_A^C D) \otimes_R N_2,
\end{equation}
so the canonical map (see Lemma 5)

\[ N_1 \otimes R (D \Box_A D) \otimes_R N_2 \to (N_1 \otimes_R D) \Box_A (D \otimes_R N_2) \]

is isomorphic to the canonical homomorphism

\[ f^*(N_1 \otimes_B N_2) \to f^*N_1 \Box^B f^*N_2, \]

which is an isomorphism by flatness of \( f \). This together with Lemma 5 proves the purity condition.

For any \( N \in \text{Bimod}(B) \) we define

\[ M := f^*N \in \text{Bicomod}_A(f^*B) = \text{Bicomod}_A(C). \]

As in (79) we obtain the following isomorphism of \( B \)-bimodules, defined as the canonical composite

\[ N = \text{Hom}_B(B \otimes B, N)_B \xrightarrow{f^*} \text{Hom}_A^{f^*B}(f^*(B \otimes B), f^*N)_{f^*B} = \text{Hom}_A^C(D, M)_A, \]

where the structure of a \( B \)-bimodule on the right hand side is induced by (79) and the canonical structure of \( \text{Hom}_A^C(D, C)_A \)-bimodule is given as the pair of canonical composites using the comultiplication \( \Delta \). We use (85) in the following canonical composite

\[ \text{Hom}_B(B, N)_B \xrightarrow{\text{Hom}_B(B, \text{Hom}_A^C(D, M)_A)_B} \text{Hom}_A^C(B \otimes_{B^o \otimes B} D, M)_A = \text{Hom}_A^C(D/[D, B], M)_A. \]

On the other hand, we have the following composite similar to a part of (85)

\[ \text{Hom}_B(B, N)_B \xrightarrow{f^*} \text{Hom}_A^{f^*B}(f^*B, f^*N)_{f^*B} = \text{Hom}_A^C(C, M)_A; \]

Since (86) and (87) are natural isomorphisms and the functor \( f^* : \text{Bimod}(B) \to \text{Bicomod}_A(f^*B) = \text{Bicomod}_A(C) \) is essentially surjective on objects the canonical morphism \( D/[D, B] \to C \) in \( \text{Coalg}(A) \) is an isomorphism (freeness condition).

Finally, we have the following two natural composites which are isomorphisms for every \( L \in \text{Bimod}(A) \) and \( M, N \) as above

\[ \text{Hom}_B(N, f_*L)_B \xrightarrow{f^*} \text{Hom}_A^{f^*B}(f^*N, f_*f_*L)_{f^*B} = \text{Hom}_A^C(M, \text{Hom}_A(D, L)_A \otimes_{B^o \otimes B} D)_A, \]
Hom\_B(N, f\_\ast L\_B) = Hom\_A(f\_\ast N, L\_A) = Hom\_A(M, L\_A) = Hom\_A(M, C \otimes\_A L \otimes\_A C\_A).

Similarly, since (88) and (89) are natural isomorphisms and the functor f\_\ast : \text{Bimod}(B) \to \text{Bicomod}_A(f\_\ast B) = \text{Bicomod}_A(C) is essentially surjective on objects we obtain a natural isomorphism of functors \text{Bimod}(A) \to \text{Bicomod}_A(C) (comonad condition)

\[ (90) \quad \text{Hom}_A(D, -) \otimes\_B \otimes\_D D \to C \otimes\_A - \otimes\_A C. \]

(\textit{Galois extensions }\rightsquigarrow \textit{Flat covers} ) Let us consider a noncommutative Galois extension \( (B \to \text{Spec}(A) \to \text{Spec}(D, A) \to C). \)

By Theorem 1 it determines a morphism \( f : \text{Spec}(A) \to \text{Spec}(B), \) where \( f\_\ast = \text{Hom}_A(D, -)_A, f\_\ast = - \otimes\_B \otimes\_D D. \) By the freeness condition and the first part of the invariants condition we have the canonical isomorphism in Coalg(A)

\[ (91) \quad f\_\ast B = B \otimes\_B \otimes\_D D = D/[D, B] \cong C. \]

By (91) and the comonad condition we have an isomorphism of functors

\[ (92) \quad f\_\ast f\_\ast \to C \otimes\_A - \otimes\_A C = f\_\ast B \otimes\_A - \otimes\_A f\_\ast B. \]

In fact, it is an isomorphism of comonads on \text{Bimod}(A). Therefore the category of comodules over the comonad \( f\_\ast f\_\ast \) is equivalent to \text{Bicomod}_A(f\_\ast B). By the faithful flatness condition \( f\_\ast \) is faithful and exact. Therefore by the Barr-Beck theorem and the comonad condition \( f\_\ast \) induces an equivalence \text{Bimod}(B) \to \text{Bicomod}_A(f\_\ast B). By the regularity condition the latter category is monoidal with respect to the cotensor product over \( C = f\_\ast B \) with the monoidal unit \( C \) and by Proposition 2 this equivalence induced by \( f\_\ast \) is (lax) comonoidal. By the second part of the invariants condition and next by the purity condition and Lemma 5 we obtain isomorphisms in the following canonical decomposition of the canonical homomorphism (93) (see also (82))

\[ (93) \quad f\_\ast(N_1 \otimes\_B N_2) = (N_1 \otimes\_B N_2) \otimes\_R D = N_1 \otimes\_R (B \otimes\_D) \otimes\_R N_2 \cong N_1 \otimes\_R (D \square\_A D) \otimes\_R N_2 \cong (N_1 \otimes\_R D) \square\_A (D \otimes\_R N_2) = f\_\ast N_1 \square\_B f\_\ast N_2. \]

This implies that the above equivalence is strong comonoidal, hence \( f \) is a flat cover.

2.7. \textbf{Infinitesimals and differential operators.} In this section we show how to define and study infinitesimal structure of noncommutative schemes. First, we fix the terminology related to towers and filtrations in abelian categories.

With every tower descending from \( \mathcal{F} \)

\[ \mathcal{F} = \mathcal{F}^0 \to \cdots \to \mathcal{F}^p \to \mathcal{F}^{p+1} \to \cdots \to 0 \]
we associate an increasing filtration in $\mathcal{F}$

$$0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_p \hookrightarrow \mathcal{F}_{p+1} \hookrightarrow \cdots \hookrightarrow \mathcal{F},$$

taking $\mathcal{F}_p := \ker(\mathcal{F} \to \mathcal{F}_{p+1})$.

In the dual manner, with every decreasing filtration in $\mathcal{G}$

$$\mathcal{G} = \mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_p \leftarrow \mathcal{G}_{p+1} \leftarrow \cdots \leftarrow 0$$

we associate a tower ascending to $\mathcal{G}$

$$0 \leftarrow \mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_p \leftarrow \mathcal{G}_{p+1} \leftarrow \cdots \leftarrow \mathcal{G},$$

taking $\mathcal{G}_p := \mathrm{coker}(\mathcal{G} \leftarrow \mathcal{G}_{p+1})$.

Let us consider now a morphism of noncommutative schemes $f : X \to Y$. For every $\mathcal{F} \in \text{Qcoh}(X)$ we define by induction a tower descending from $\mathcal{F}$

$$f^* f_* \mathcal{F}^p \to \mathcal{F}^p \to \mathcal{F}_{p+1} \to 0,$$

inducing an increasing filtration $\mathcal{F}_p$ on $\mathcal{F}$, and for every $\mathcal{G} \in \text{Qcoh}(Y)$ a decreasing filtration in $\mathcal{G}$

$$f_* f^* \mathcal{G}^p \leftarrow \mathcal{G}^p \leftarrow \mathcal{G}_{p+1} \leftarrow 0,$$

inducing a tower $\mathcal{G}_p$ ascending to $\mathcal{G}$.

Finally, assuming that $\text{Qcoh}(X)$ is cocomplete and $\text{Qcoh}(Y)$ is complete, we can define $\mathcal{F}_f := \text{colim}_p \mathcal{F}_p$ and $\mathcal{G}^f := \text{lim}_p \mathcal{G}_p$.

In the next proposition we will denote by $\text{Diff}^{A/K}(M, N)$ differential operators in the sense of Lunts - Rosenberg [28], which agree with the definition of Grothendieck [19] if $A$ is commutative.

**Proposition 8.** Let $K$ be a commutative ring, $A$ a $K$-algebra, $\text{Qcoh}(X)$ consists of $A$-bimodules symmetric over $K$, $\text{Qcoh}(Y)$ consists of symmetric $K$-bimodules and $f^* = A \otimes_K (-)$. Let $\mathcal{F} := \text{Hom}_K(M, N)$ for some $M, N \in_A \text{Mod}$. Then $\mathcal{F}_f = \text{Diff}^{A/K}(M, N)$.

**Proof.** By the definition of the increasing filtration $\mathcal{F}_p$ the exact sequence (94) is isomorphic to

$$f^* f_* (\mathcal{F} / \mathcal{F}_{p-1}) \to \mathcal{F} / \mathcal{F}_{p-1} \to \mathcal{F} / \mathcal{F}_p \to 0.$$

By exactness in the second term this implies the canonical isomorphism

$$\mathcal{F}_p / \mathcal{F}_{p-1} = \text{im}(f^* f_* (\mathcal{F} / \mathcal{F}_{p-1}) \to \mathcal{F} / \mathcal{F}_{p-1}).$$

Since $f^* = A \otimes_K (-)$ has as the right adjoint $f_* = \text{Hom}_A(A, -)_A = Z_A(-)$, the center of an $A$-bimodule, the functor $\text{im}(f^* f_* (\mathcal{F} / \mathcal{F}_{p-1}) \to (-))$ is nothing but the functor of the sub-$A$-bimodule generated by the center of a given $A$-bimodule. Applying this fact to (97) we obtain the inductive definition of the $p$-th differential part of an $A$-bimodule $\mathcal{F}$, symmetric over $K$, according to [28]. In the special case of $\mathcal{F} = \text{Hom}_K(M, N)$ one obtains $\mathcal{F}_p = \text{Diff}^{A/K}_p(M, N)$, i.e. differential operators of order $\leq p$. ■
Remark. In general, as in [28], provided only $f^*$ is strongly comonoidal, the increasing filtration $F_p$ in $F$ is monoidal, which means that we have natural transformations

\[(98) \quad F_p' \otimes_X F_p'' \to (F'_p \otimes_X F''_p)_{p'+p''}.\]

This implies that for every additive category enriched in $\text{Qcoh}(X)$ (e.g. enriched in $\text{Bimod}(A)$ as in [24]) a generalized differential part is well defined, generalizing the category with differential operators as morphisms.

**Proposition 9.** Let $A$ be a commutative ring, $I$ an ideal in $A$, $\text{Qcoh}(X)$ and $\text{Qcoh}(Y)$ consist of symmetric bimodules over $A/I$ and $A$, respectively, and $f_*$ is the base forgetting from $A/I$ to $A$. Then $G^f = \hat{G}_I$ ($I$-adic completion).

**Proof.** Since the forgetting functor $f_*$ has as the left adjoint $f^* \mathcal{G} = A/I \otimes_A \mathcal{G} = \mathcal{G}/IG$ the exact sequence (95) is isomorphic to

\[(99) \quad \mathcal{G}^p/IG^p \leftarrow \mathcal{G}^p \leftarrow \mathcal{G}^{p+1} \leftarrow 0,\]

which implies that $\mathcal{G}^{p+1} = I\mathcal{G}^p$, hence by induction $\mathcal{G}^p = I^p \mathcal{G}$. Therefore by the definition of the tower $\mathcal{G}_p$ ascending to $\mathcal{G}$ we have $\mathcal{G}_p = \mathcal{G}/I^p \mathcal{G}$. Finally, passing to the limit we obtain the $I$-adic completion. $\blacksquare$

### 2.8. Cyclic homology of noncommutative schemes.

To define cyclic homology of noncommutative schemes we need some additional structure which cannot be derived from the plain abelian monoidal structure. We derive the axioms of this additional structure from the canonical structures of the category of affine noncommutative schemes. First we observe that in the case of $X = \text{Spec}(A)$ we have a functor

\[(100) \quad \text{Tr}_X := (-) \otimes_{A^\circ \otimes A} A\]

\[(101) \quad \text{Tr}_X : \text{Qcoh}(X) \to \text{Ab}\]

satisfying the following flip symmetry property

\[(102) \quad \text{Tr}_X(F_1 \otimes_X F_2) \overset{\cong}{\to} \text{Tr}_X(F_2 \otimes_X F_1).\]

For any geometric morphism $X = \text{Spec}(A) \to \text{Spec}(B) = Y$, which is equivalent to a representation

\[(103) \quad B \to \text{Hom}_A(D, A)_A\]

we have the following functors

\[(104) \quad f_* = \text{Hom}_A(D, -)_A, \quad f^* = (-)_{B^\circ \otimes B} D\]
By the tensor-hom adjunction they form an adjoint pair $f^* \dashv f_*$, i.e. there is an isomorphism of bifunctors (first hom adjunction axiom)

\[(105) \quad \text{Hom}_X(f^* \mathcal{G}, \mathcal{F}) = \text{Hom}_Y(\mathcal{G}, f_* \mathcal{F}).\]

Since $f_*$ is a monoidal functor $f^*$ is a comonoidal one. This means that $f_*$ transforms algebras (monoids in an abelian monoidal category) into algebras, while $f^*$ transforms coalgebras (comonoids in an abelian monoidal category) into coalgebras. In particular $f_* \mathcal{O}_X$ is always an algebra while $f^* \mathcal{O}_Y$ is always a coalgebra. Structural morphisms $\mathcal{O}_Y \to f_* \mathcal{O}_X$ and equivalent to it $f^* \mathcal{O}_Y \to f_* \mathcal{O}_X$ are the unit and the counit for $f_* \mathcal{O}_X$ and $f^* \mathcal{O}_Y$.

For the above geometric morphism between affine noncommutative schemes we have also another pair of functors

\[(106) \quad f! := D \otimes A \circ \otimes A (\mathcal{F}), \quad f! := \text{Hom}_B(D, -)_B.\]

By the tensor-hom adjunction they form an adjoint pair $f_! \dashv f^!$, i.e. there is an isomorphism of bifunctors (second hom adjunction axiom)

\[(107) \quad \text{Hom}_Y(f_! \mathcal{F}, \mathcal{G}) = \text{Hom}_X(\mathcal{F}, f^! \mathcal{G}).\]

**Lemma 6.** The pair of functors $(f_!, f^!)$ admits two natural transformations (projection axiom)

\[(108) \quad f_! \mathcal{F} \otimes_Y \mathcal{G} \to f_!(\mathcal{F} \otimes_X f^* \mathcal{G})\]

\[(109) \quad \mathcal{G} \otimes_Y f_! \mathcal{F} \to f_!(f^* \mathcal{G} \otimes_X \mathcal{F})\]

**Proof.** Using the equivalence of bimodules and left or right modules over the enveloping ring it is enough to construct only the last transformation, the previous one we obtain in an analogical way, changing the role of left and right modules.

\[(110) \quad \mathcal{G} \otimes_Y f_! \mathcal{F} = \mathcal{G} \otimes_B (D \otimes_{A^e \otimes A} \mathcal{F})\]

\[(111) \quad = ((B \otimes B) \otimes_B \mathcal{G}) \otimes_{B^e \otimes B} D \otimes_{A^e \otimes A} \mathcal{F}\]

\[(112) \quad = (f^*((B \otimes B) \otimes_B \mathcal{G}) \otimes_{A^e \otimes A} \mathcal{F}) \otimes_{A^e \otimes A} A\]

\[(113) \quad \to (f^*(B \otimes B) \otimes_A f^* \mathcal{G} \otimes_A \mathcal{F}) \otimes_{A^e \otimes A} A\]

\[(114) \quad = (D \otimes_A f^* \mathcal{G} \otimes_A \mathcal{F}) \otimes_{A^e \otimes A} A\]

\[(115) \quad = D \otimes_{A^e \otimes A} (f^* \mathcal{G} \otimes_A \mathcal{F})\]

\[(116) \quad = f_!(f^* \mathcal{G} \otimes_X \mathcal{F}).\]

\[\blacksquare\]

**Lemma 7.** There exists a natural isomorphism of bifunctors (trace adjunction axiom)

\[(117) \quad \text{Tr}_X(\mathcal{F} \otimes_X f^* \mathcal{G}) \cong \text{Tr}_Y(f_! \mathcal{F} \otimes_Y \mathcal{G})\]
Proof. Using the flip transformations under $\text{Tr}_X$ and $\text{Tr}_Y$ it is enough to construct the flip-equivalent transformation.

\begin{align}
(118) \quad \text{Tr}_X(f^*G \otimes_X F) &= ((G \otimes_{B^o \otimes B} D) \otimes_A F) \otimes_{A^o \otimes A} A \nonumber \\
(119) &= G \otimes_{B^o \otimes B} D \otimes_{A^o \otimes A} F \nonumber \\
(120) &= (G \otimes_B (D \otimes_{A^o \otimes A} F)) \otimes_{B^o \otimes B} B \nonumber \\
(121) &= \text{Tr}_Y(G \otimes_Y f^*F). \nonumber
\end{align}

Definition. Let $p : X \to S$ be a morphism of noncommutative schemes. A system of coefficients (relative to $S$) is an object $M \in \text{Qcoh}(X)$ equipped with a braiding

$$\beta : p^*\mathcal{O}_S \otimes_X M \to M \otimes_X p^*\mathcal{O}_S,$$

such that the following diagrams commute (where $C := p^*\mathcal{O}_S \in \text{Coalg}(X)$, $\otimes := \otimes_X$, transpositions of $M$ and copies of $C$ are obtained via $\beta$ and $\beta^{-1}$, and $\Delta_i, \varepsilon_i$ come from $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathcal{O}_X$ applied to the $i$-th factor.)

Diagram I.

$$
\begin{array}{c}
\begin{aligned} 
C \otimes C \otimes M & \to M \otimes C \otimes C \\
\Delta_2 \downarrow & \downarrow \Delta_2 \\
C \otimes C \otimes C \otimes M & \to M \otimes C \otimes C \otimes C
\end{aligned}
\end{array}
$$

Diagram II.

$$
\begin{array}{c}
\begin{aligned} 
M \otimes C \xrightarrow{\Delta_2} M \otimes C \otimes C & \to C \otimes M \otimes C \xrightarrow{\Delta_1} C \otimes C \otimes M \otimes C \\
\downarrow & \downarrow \\
C \otimes M \xrightarrow{\Delta_1} C \otimes C \otimes M & \to C \otimes M \otimes C \xrightarrow{\Delta_1} C \otimes M \otimes C \otimes C
\end{aligned}
\end{array}
$$

Diagram III.

$$
\begin{array}{c}
\begin{aligned} 
C \otimes M \otimes C \xrightarrow{\Delta_1} C \otimes C \otimes M \otimes C & \to C \otimes M \otimes C \otimes C \xrightarrow{\varepsilon_3} C \otimes M \otimes C \\
\downarrow & \downarrow \\
M \otimes C \otimes C \xrightarrow{\varepsilon_3} M \otimes C & \to C \otimes M \xrightarrow{\Delta_1} C \otimes C \otimes M
\end{aligned}
\end{array}
$$

Diagram IV.

$$
\begin{array}{c}
\begin{aligned} 
C \otimes M \xrightarrow{\Delta_1} \quad & \to M \otimes C \\
\Delta_1 \downarrow & \uparrow \varepsilon_2 \\
C \otimes C \otimes M & \to M \otimes C \otimes C
\end{aligned}
\end{array}
$$
Diagram V.

\[
\begin{array}{ccc}
C \otimes C \otimes M & \rightarrow & C \otimes M \otimes C \\
\varepsilon_2 \downarrow & & \downarrow \varepsilon_3 \\
C \otimes M & \rightarrow & C \otimes M \\
\downarrow & & \downarrow \\
M \otimes C & \rightarrow & M \otimes C \\
\varepsilon_2 \downarrow & & \varepsilon_2 \\
& & M
\end{array}
\]

Diagram VI.

\[
\begin{array}{ccc}
C \otimes C \otimes M & \rightarrow & M \otimes C \otimes C \\
\varepsilon_2 \downarrow & & \downarrow \varepsilon_2 \\
C \otimes M & \rightarrow & M \otimes C
\end{array}
\]

**Definition.** \(M\) is a cyclic system of coefficients relative to \(S\) if for all \(p, q > 0\) the following diagrams commute (braiding-trace compatibility):

\[
\begin{array}{ccc}
\text{Tr}_X(C^{\otimes p} \otimes M \otimes C^{\otimes q}) & \rightarrow & \text{Tr}_X(C^{\otimes p+1} \otimes M \otimes C^{\otimes q-1}) \\
\downarrow & & \downarrow \\
\text{Tr}_X(C^{\otimes p-1} \otimes M \otimes C^{\otimes q+1}) & \rightarrow & \text{Tr}_X(C^{\otimes p} \otimes M \otimes C^{\otimes q}),
\end{array}
\]

\[
\begin{array}{ccc}
\text{Tr}_X(C^{\otimes p} \otimes M) & \rightarrow & \text{Tr}_X(M \otimes C^{\otimes p}) \\
\downarrow & & \downarrow \\
\text{Tr}_X(C^{\otimes p-1} \otimes M \otimes C) & \rightarrow & \text{Tr}_X(C^{\otimes p} \otimes M),
\end{array}
\]

where vertical arrows are induced by braiding transpositions \(C \otimes M \rightarrow M \otimes C\) or their inverses, and horizontal ones are induced by natural flip isomorphisms of \(\text{Tr}_X\).

**Example.** \(C = p^* \mathcal{O}_S\) itself is a (trivial) cyclic system of coefficients relative to \(S\) with the identity braiding \(\beta : C \otimes C \rightarrow C \otimes C\).

**Definition.** A cyclic object in an abelian category consists of a collection of morphisms \(\partial_i : C_n \rightarrow C_{n-1}, s_i : C_n \rightarrow C_{n+1}, i = 0, \ldots, n,\) and \(t_n : C_n \rightarrow C_n,\)
satisfying

\[ \partial_i \partial_j = \partial_{j-1} \partial_i, \quad i < j, \]
\[ s_i s_j = s_{j+1} s_i, \quad i \leq j, \]

\[ \partial_i s_j = \begin{cases} 
  s_{j-1} \partial_i, & i < j, \\
  \text{id}, & i = j, j+1, \\
  s_j \partial_{i-1}, & i > j+1,
\end{cases} \]

\[ \partial_i t_n = t_{n-1} \partial_{i-1}, \quad i = 1, \ldots, n, \]
\[ \partial_0 t_n = \partial_n, \]
\[ s_i t_n = t_{n+1} s_{i-1}, \quad i = 1, \ldots, n, \]
\[ s_0 t_n = t_{n+1}^2 s_n, \]
\[ t_{n+1}^n = 1. \]

**Definition.** For every cyclic system $M$ of coefficients relative to $S$ we define

\[ C_{X/S}(M)_n := \text{Tr}_X(M \otimes_X (p^* \mathcal{O}_S)^{\otimes X^n}), \]

and (in the element-wise convention!)

\[ \partial_i : C_{X/S}(M)_n \to C_{X/S}(M)_{n-1}, \]

\[ \partial_0 \text{Tr}_X(m, c_1, \ldots, c_n) := \text{Tr}_X(m \varepsilon(c_1), c_2, \ldots, c_n) \]
\[ \partial_i \text{Tr}_X(m, c_1, \ldots, c_n) := \text{Tr}_X(m, c_1, \ldots, c_{i-1} \varepsilon(c_{i+1}), \ldots, c_n), \quad i = 1, \ldots, n-1, \]
\[ \partial_n \text{Tr}_X(m, c_1, \ldots, c_n) := \text{Tr}_X((1 \otimes \varepsilon) \beta(c_n, m), c_1, \ldots, c_{n-1}) \]

\[ s_i : C_{X/S}(M)_n \to C_{X/S}(M)_{n+1}, \]

\[ s_i \text{Tr}_X(m, c_1, \ldots, c_n) := \text{Tr}_X(m, c_1, \ldots, \Delta(c_{i+1}), \ldots, c_n), \quad i = 0, \ldots, n-1, \]
\[ s_n \text{Tr}_X(m, c_1, \ldots, c_n) := \sum \text{Tr}_X(\beta(c_{i+1}', m'), c_2, \ldots, c_n, c_{i+1}'(1)), \]

where

\[ \beta^{-1}(m, c) = \sum c' \otimes m', \quad \Delta(c) = \sum c_{(1)} \otimes c_{(2)}, \]

\[ t_n : C_{X/S}(M)_n \to C_{X/S}(M)_n, \]

\[ t_n \text{Tr}_X(m, c_1, \ldots, c_n) := \text{Tr}_X(\beta(c_n, m), c_1, \ldots, c_{n-1}). \]
**Definition.** We say that $\text{Tr}_X$ is *faithful relative to $S$* if for every $n > 0$ the functor $C_{X/S}(-)_n : \text{Qcoh}(X) \to \text{Ab}$ is faithful.

**Theorem 4.** For every cyclic system $M$ of coefficients relative to $S$ the system $(C_{X/S}(M)_\bullet, \partial_\bullet, s_\bullet, t_\bullet)$ is a cyclic object in the category of abelian groups. Assume that $\text{Tr}_X$ is faithful relative to $S$. Then for every $M$ with a braiding with respect to $p^*\mathcal{O}_S$ compatible with $\text{Tr}_X$ the para-cyclic relations between $(\partial_\bullet, s_\bullet, t_\bullet)$ defined as above define on $M$ a structure of a cyclic system of coefficients relative to $S$.

**Proof.** Diagram V implies $\partial_{n-1}\partial_n = \partial_{n-1}\partial_{n-1}$. Diagram II implies $s_0s_n = s_{n+1}s_0$. Coassociativity of $\Delta : C \to C \otimes C$ implies $s_is_i = s_{i+1}s_i$ for $i = 0, \ldots, n-1$. Coassociativity and invertibility of braiding imply $s_ns_n = s_{n+1}s_n$. Diagram III implies $\partial_0s_n = s_{n-1}\partial_0$. The left counit property $(\varepsilon \otimes \text{id})\Delta = \text{id}$ implies $\partial_is_i = \text{id}$ and $\partial_{i+1}s_i = \text{id}$ for $i = 0, \ldots, n-1$. Diagram IV implies $\partial_{n+1}s_n = \text{id}$. Diagram VI implies $\partial_0t_n = t_{n-1}\partial_{n-1}$. Diagram I implies $s_0t_n = t^2_{n+1}s_n$. The trace flip-braiding compatibility implies $t^2_{n+1} = \text{id}$. All other cyclic object relations are fulfilled automatically by the definition of $(\partial_\bullet, s_\bullet, t_\bullet)$.

If $\text{Tr}_X$ is faithful relative to $S$, all implications between commutativity of diagrams I-VI and cyclic object relations become equivalences.

We denote the respective Hochschild, cyclic and periodic cyclic homology by $HH_{X/S}(M)_\bullet$, $HC_{X/S}(M)_\bullet$ and $HP_{X/S}(M)_\bullet$, respectively.

**Example.** Let $K \to A$ be a central ring homomorphism from a commutative ring $K$, regarded as a geometric morphism $p : \text{Spec}(A) = X \to S = \text{Spec}(K)$ (both spectra noncommutative). Then $C = p^*\mathcal{O}_S = A \otimes_K A$. We have also $\text{Tr}_X F := A \otimes_{A^\otimes A} F$ and we can take $M := C$. Then we obtain the classical cyclic object of a $K$-algebra $A$.

2.8.1. **Functoriality of cyclic objects.** Assume now that we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow p & & \downarrow q \\
S & & \\
\end{array}
$$

in the category of noncommutative schemes over a noncommutative scheme $S$ equipped with trace functors and with the geometric morphism $f^* \dashv f_*$ completed by a compatible pair $f_! \dashv f^!$. Assume that $M$ and $N$ are cyclic systems of coefficients relative to $S$ on $X$ and $Y$, respectively. Assume that we have a
morphism \( N \to f_!M \) making the canonical diagram
\[
\begin{array}{c}
q^* \mathcal{O}_S \otimes_Y N \to N \otimes_Y q^* \mathcal{O}_S \\
f_!(p^* \mathcal{O}_S \otimes_X M) \to f_!(M \otimes_X p^* \mathcal{O}_S)
\end{array}
\]
commutative.
Then it induces a morphism of cyclic objects
\[
(C_{X/S}(M), \partial_\bullet, s_\bullet, t_\bullet) \leftarrow (C_{Y/S}(N), \partial_\bullet, s_\bullet, t_\bullet).
\]
and hence the morphisms of homologies
\[
\begin{align*}
HH_{X/S}(M) & \leftarrow (HH_{Y/S}(N), \\
HC_{X/S}(M) & \leftarrow (HC_{Y/S}(N), \\
HP_{X/S}(M) & \leftarrow (HP_{Y/S}(N).
\end{align*}
\]

2.8.2. **Comparison with other constructions.** In a recent paper [2] authors consider a construction of cyclic objects based on comonads and distributivity laws. Down to the earth, restricted to the context of our construction, their structure is based on the following two diagrams

**Diagram I’**
\[
\begin{array}{c}
C \otimes M \to M \otimes C \\
\Delta_1 \downarrow \downarrow \Delta_2 \\
C \otimes C \otimes M \to M \otimes C \otimes C
\end{array}
\]

**Diagram V’**
\[
\begin{array}{c}
C \otimes M \to M \otimes C \\
\varepsilon_1 \downarrow \downarrow \varepsilon_2 \\
M.
\end{array}
\]

The first difference between this cyclic object and our consists in the number of copies of a coalgebra \( C \) in every degree in the complex. In degree \( n \) we see in their complex \( n+1 \) copies while in our we have only \( n \) of them. This suggests that these two constructions have different flavor. Indeed, the classical (commutative) object corresponding to our construction is the DeRham cohomology with values in a module with an integrable connection. In the case of the Sweedler coalgebra \( C = A \otimes_K A \) in Bimod(\( A \)) for a commutative \( K \)-algebra \( A \) over a commutative ring \( K \) we think of the bimodule \( M \) as of a noncommutative analog of a sheaf supported on the first infinitesimal neighborhood of the diagonal in the cartesian square of a scheme over \( K \). It contains the information about its restriction to the diagonal together with an infinitesimal variation of this restriction encoded in the braiding \( \beta \), i. e. a connection in the Grothendieck approach. Our diagrams I-VI together with the trace-braiding compatibility form a noncommutative analog
of the property of being supported on the first infinitesimal neighborhood and integrability of the connection.

The conditions on coefficients $M$ in the construction of [2] is a generalization of the stable anti-Yetter-Drinfeld condition from Hopf-cyclic homology with coefficients (in the dual approach of Jara-Stefan). This condition means that $M$ is regarded as a noncommutative analog of a stable equivariant sheaf on the group of symmetries acted by itself via conjugations. This means that it depends only on symmetries of the space and not on the space itself. Therefore it requires only diagrams I' and VI', which play the role of our diagrams I and VI. Moreover, the passage to the cyclic object consists there in quotienting the paracyclic object by the relation forcing the cyclic relation $t_{n+1} = \text{id}$, a la Kaygun. It is an analog of dividing the DeRham complex tensored by a module with a non-integrable connection by the image of the curvature to obtain a complex, and hence does not correspond to integrability of the connection.

In another recent paper [23] the author uses a formalism of cartesian objects in symmetric monoidal categories. In the simplest case, an algebra $A$ over a commutative ring $K$ is considered, and we can pass to our context taking the respective Sweedler construction, i.e.

$$C = A \otimes_K A.$$ 

The author assumes that there is given a twist

$$A \otimes_K M \rightarrow M \otimes_K A$$

in the category of bimodules over the algebra $A \otimes_K A$, satisfying some cocycle condition. It can be compared with our braiding, taking into account the fact that it can be written as

$$C \otimes_A M = A \otimes_K M \rightarrow M \otimes_K A = M \otimes_A C.$$ 

The respective commutative diagram encoding this cocycle condition is of the form

$$A \otimes_K A \otimes_K M \rightarrow A \otimes_K M \otimes_K A \leftarrow M \otimes_K A \otimes_K A$$

where the south-east arrow is defined as the transposition of the first and the third factor. It is clear that one has to use the symmetry of the monoidal category to do this. Also the algebra structure on tensor powers of $A$ over $K$, e.g. on $A \otimes_K A$, needs this symmetry. Therefore this construction makes no sense over a noncommutative base ring $K$. Moreover, expressing the latter diagram in terms of the Sweedler construction one gets

$$C \otimes_A C \otimes_A M \rightarrow C \otimes_A M \otimes_A C \leftarrow M \otimes_A C \otimes_A C$$
but the south-east arrow cannot be defined purily in terms of the category of $A$-bimodules, without referring to the special structure of $C$.

**References**

[1] Artin, M.; Zhang, J. J.: *Noncommutative projective schemes*, Adv. Math. 109 (1994), no. 2, pp. 228-287.

[2] Böhm, G.; Štefan, D.: *(Co)cyclic (co)homology of bialgebroids: An approach via (co)monads*. [arXiv:0705.3190v1 [math.KT]] 22 May 2007

[3] Brzeziński, T.: *On modules associated to coalgebra-Galois extensions*. J. Algebra 215 (1999), 290–317.

[4] Brzeziński, T.: *The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*. Alg. Rep. Theory 5 (2002), 389–410.

[5] Brzeziński, T.: *Galois comodules*. J. Algebra 290 (2005), 503–537.

[6] Brzeziński, T.; Hajac, P.M.: *Coalgebra extensions and algebra cocorrections of Galois type*. Comm. Algebra 27 (1999), 1347–1367.

[7] Brzeziński, T.; Wisbauer, R.: *Galois comodules*. London Math. Soc. LNS. 309, Cambridge University Press, 2003.

[8] Balmer, P.: *The spectrum of prime ideals in tensor differential graded categories*. J. Reine Angew. Math. 588 (2005), 149–168.

[9] Bergman, G. M.; Hausknecht, A. O.: *Cogroups and co-rings in categories of associative rings*, Mathematical Surveys and Monographs, vol. 45, American Mathematical Society, Providence, RI, 1996

[10] Caenepeel, S.: *Galois corings from the descent theory point of view*. Galois theory, Hopf algebras, and semiabelian categories, 163–186, Fields Inst. Commun., 43, Amer. Math. Soc., Providence, RI, 2004.

[11] Caenepeel, S.; De Groot, E.; Vercruysse, J.: *Galois theory for comatrix corings: Descent theory, Morita theory, Frobenius and separability properties*. Trans. Amer. Math. Soc., (to appear), arXiv:math.RA/0406436 (2004).

[12] Cartier, P.: *A mad day’s work: from Grothendieck to Connes and Kontsevich - the evolution of concepts of space and symmetry*. Bull. AMS 38 (2001), no. 4, 389–408.

[13] Eilenberg, S.: *Abstract description of some basic functors*, J. Indian Math. Soc. (N.S.) 24 (1960), 231-234.

[14] El Kaoutit, L.; Gómez-Torrecillas, J.: *Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings*. Math. Z. 244 (2003), 887–906.

[15] Freyd, P.: *Algebra valued functors in general and tensor products in particular*. Colloquium Mathematicum (Wrocław), 14 (1966), 89–106.

[16] Gabriel, P.: *Des catégories abéliennes*. Bull. Soc. Math. France 90 (1962), 323-448.

[17] Golan, J. S.; Raynaud, J.; van Oystaeyen, F.: *Sheaves over the spectra of certain noncommutative rings*, Comm. Alg. 4(5), (1976), pp. 491-502.

[18] Gómez-Torrecillas, J.: *Separable functors in corings*. Int. J. Math. Math. Sci. 30 (2002), no. 4, 203–225.

[19] Grothendieck, A.: *Éléments de géométrie algébrique. Étude locale de schémas*. Publ. Math. IHES 32 (1967).

[20] Grothendieck, A. et al.: *Rêvêtements étale et group fondamental*. Séminaire de Géometrie Algébrique du Bois Marie 1960-1961 (SGA 1), LNM 224, Springer, 1971.

[21] Gruenberg, L.; Paré, R.: *Families parametrized by coalgebras*. J. Algebra, 107 (1987), 316-375.

[22] Hodges, T. J.; Smith, S. P.: *Sheaves of noncommutative algebras and the Beilinson- Bernstein equivalence of categories*, Proc. AMS, 92, N. 3, 1985.
[23] Kaledin, D.: *Cyclic homology with coefficients*, arXiv:math.KT/0702068v1 3 Feb 2007
[24] Kelly, G. M.: *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Notes Series 64 (Cambridge University Press 1982).
[25] Kontsevich, M.: *Triangulated categories and geometry*, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellâiche, J.-F. Dat, I. Marin, G. Racinet and H. Randriamiholona, 1998.
[26] Kontsevich, M., Rosenberg, A. L.: *Noncommutative smooth spaces*. The Gelfand Math. Seminars, 1996–1999, pp. 85–108, Birkhäuser, 2000.
[27] Kontsevich, M., Rosenberg, A. L.: *Noncommutative spaces*, MPI 2004-35; *Noncommutative spaces and flat descent*, MPI 2004-36; *Noncommutative stacks*, MPI 2004-37; preprints, Bonn 2004.
[28] Lunts, V., Rosenberg, A. L.: *Differential operators on noncommutative rings*, Sel. Math., New Ser. 3 (1997), 335-359.
[29] Lunts, V., Rosenberg, A. L.: *Differential calculus in noncommutative algebraic geometry I*, Preprint MPI 96-53.
[30] Lunts, V., Rosenberg, A. L.: *Differential calculus in noncommutative algebraic geometry II*, Preprint MPI 96-76.
[31] Mac Lane, S.: *Natural associativity and commutativity*, Rice Univ. Stud. 49 (1963), 28–46.
[32] Mac Lane, S., Moerdijk, I.: *Sheaves in geometry and logic*, Springer 1992.
[33] Murdoch, D. C.; van Oystaeyen, F.: *Noncommutative localization and sheaves*, J. Alg. 38 (1975), pp. 500-515.
[34] Nuss, P.: *Noncommutative descent and non-abelian cohomology*. Algebra, 3. J. Math. Sci. 82 (1996), no. 6, 3824-3831.
[35] Orlov, D. O.: *Quasi-coherent sheaves in commutative and non-commutative geometry*, Izvestiya: Math. 67:3 (2003), pp. 119-138.
[36] Phung Ho Hai: *An embedding theorem for abelian monoidal categories*, Compositio Math., 132(2) (2002), 27-48.
[37] Schauenburg, P.; Schneider, H.-J.: *On generalized Hopf Galois extensions*, arXiv:math.QA/0405184 (2004).
[38] Schneider, H.-J.: *Principal homogeneous spaces for arbitrary Hopf algebras*, Israel J. Math., 72 (1-2) (1990), 167-195. arXiv:math.QA/0405184 (2004).
[39] Takeuchi, M.: *A note on geometrically reductive groups*, J. Fac. Sci., Univ. Tokyo, Sect. 1, 20 (3) (1973), 384-396.
[40] Takeuchi, M.: *Introduction to \sqrt{Morita theory}*, Proceedings of the 17th symposium on ring theory (Tsukuba, 1984), 78-86, Okayama Univ., Okayama, 1984.
[41] Verschoren, A.: *Sheaves and localization*, J. Algebra 182 (1996), no. 2, pp. 341-346.
[42] Watts, C. E.: *Intrinsic characterization of some additive functors*, Proc. AMS 11 (1960), pp. 1-8.
NONCOMMUTATIVE GEOMETRY THROUGH MONOIDAL CATEGORIES

Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8
00–956 Warszawa, Poland,

Institute of Mathematics, University of Warsaw, Banacha 2
02–097 Warszawa, Poland

E-mail address: maszczyk@mimuw.edu.pl