A SYMPLECTIC PROOF OF THE HORN INEQUALITIES

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ABSTRACT. In this paper, we give a symplectic proof of the Horn inequalities on eigenvalues of a sum of two Hermitian matrices with given spectra. Our method is a combination of tropical calculus for matrix eigenvalues, combinatorics of planar networks, and estimates for the Liouville volume. As a corollary, we give a tropical description of the Duistermaat–Heckman measure on the Horn polytope.

1. Introduction

1.1. The Horn problem. Fix a positive integer $n$, and let $\mathcal{H}$ be the set of Hermitian matrices of size $n$. For $K \in \mathcal{H}$, denote by $\lambda(K) = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ the set of eigenvalues of $K$ listed in decreasing order, and introduce the map $l : \mathcal{H} \to \mathbb{R}^n$ defined by the equalities

$$l_1(K) = \lambda_1, \ l_2(K) = \lambda_1 + \lambda_2, \ldots, \ l_n(K) = \lambda_1 + \cdots + \lambda_n = \text{Tr}(K).$$

We will call the set

$$\mathcal{C}_{\text{Horn}} = \{(a, b, c) \in \mathbb{R}^{3n}; \exists (K_1, K_2) \in \mathcal{H}^2 : l(K_1) = a, l(K_2) = b, l(K_1 + K_2) = c\}$$

(1)

the Horn cone.

Clearly, $\mathcal{C}_{\text{Horn}}$ is a closed subset of the hyperplane

$$\{(a, b, c) \in \mathbb{R}^{3n}; a_n + b_n = c_n\} \subset \mathbb{R}^{3n},$$

and $\tau \mathcal{C}_{\text{Horn}} = \mathcal{C}_{\text{Horn}}$ for any $\tau > 0$.

The problem of determining this cone, known as the Horn problem, has a long history (see [7] for details). The first conjectural description was given by Horn [9] in 1962; it presents $\mathcal{C}_{\text{Horn}}$ as the set of solutions of a complicated, recursively defined list of linear inequalities. This description, in particular, implies that $\mathcal{C}_{\text{Horn}}$ is a closed convex cone. Later, a natural explanation for this fact was found in terms of convexity properties of moment maps in symplectic geometry.

In 1999, Knutson and Tao came up with the following much simpler, albeit implicit description of $\mathcal{C}_{\text{Horn}}$ (see [11], [4]). Consider the regular triangulation of order $n$ of an equilateral triangle. The triangle is divided into $n^2$ small
triangles. Two adjacent triangles form a rhombus, which can be of one of the three types shown in Figure 1.

![Figure 1: The triangular tableau with three types of rhombi.](image)

We will call the assignment of a real number to each of the nodes of the triangulation a tableau. Denoting by $\nabla$ the set of nodes of the triangulation, we can identify the space of tableaux with $\mathbb{R}^\nabla$.

Let $l^k_i$ be the number at the $i$th node in the $k$th row of the triangulation, $0 \leq i \leq k \leq n$. Then each rhombus gives rise to an inequality: the sum of the two numbers assigned to the endpoints of the short diagonal is greater than or equal to the sum of the two numbers assigned to the endpoints of the long diagonal. A tableau is called a hive if it satisfies all the inequalities, i.e., if for $0 < i \leq k < n$,

\begin{align}
  l^k_{i+1} + l^k_{i-1} &\geq l^k_{i-1} + l^k_i, \\
  l^k_{i+1} + l^k_{i} &\geq l^k_{i+1} + l^k_{i-1}, \\
  l^k_{i} + l^k_{i-1} &\geq l^k_{i+1} + l^k_{i-1}.
\end{align}

(2)

Clearly, the set of hives $\mathcal{C}_3$, defined by the three sets of inequalities (2), is a closed cone in $\mathbb{R}^\nabla$. Now consider the boundary map: $\partial : \mathbb{R}^\nabla \rightarrow \mathbb{R}^{3n}$ given by

$a_i = l^n_i, \quad b_i = l^n_{n-i} - l^n_n, \quad c_i = l^n_{n-i}, \quad 1 \leq i \leq n$

Denote by $\mathcal{C}_{KT} = \partial(\mathcal{C}_3)$ the polyhedral cone obtained as the image of $\mathcal{C}_3$ along this map.

**Theorem 1** (Knutson–Tao). The Horn cone coincides with the Knutson–Tao cone: $\mathcal{C}_{Horn} = \mathcal{C}_{KT}$.

Speyer [13] gave another proof of this theorem using Viro’s patchworking and Vinnikov curves. The purpose of the present article is to provide a proof based on a combination of ideas from tropical and symplectic geometry.
1.2. **The multiplicative problem.** There is a similar multiplicative problem defined for the group $\mathcal{B}$ of complex upper-triangular matrices of size $n$ with positive entries on the diagonal.

For a matrix $A \in \mathcal{B}$, the *singular values* are defined as the eigenvalues $\lambda_i(AA^*)$, $i = 1, \ldots, n$, of the matrix $AA^*$, which are positive real numbers in this case. The map

$$l^\mathcal{B} : \mathcal{B} \to \mathbb{R}^n, \quad l^\mathcal{B}_i(A) = \frac{1}{2} \sum_{k=1}^{i} \log \lambda_k(AA^*), \quad i = 1, \ldots, n$$

is intertwined with the map $l : \mathcal{H} \to \mathbb{R}^n$ by the diffeomorphism between $\mathcal{H}$ and $\mathcal{B}$ given by $\exp(2K) = AA^*$.

We can also define the multiplicative analog of the Horn cone:

$$C_\mathcal{B} = \{ (r, s, t) \in \mathbb{R}^{3n}; \exists (A, C) \in \mathcal{B} \times \mathcal{B}; l^\mathcal{B}(A) = r, l^\mathcal{B}(C) = s, l^\mathcal{B}(AC) = t \}.$$

The following surprising result (also known as the Thompson Conjecture) was proved by Klyachko [10]:

**Theorem 2 (Klyachko).** The set $C_\mathcal{B}$ coincides with the Horn cone:

$$C_\mathcal{B} = C_{\text{Horn}}.$$

In particular, this implies that $C_\mathcal{B}$ is a polyhedral cone.

1.3. **Planar networks.** We define one more subset of $\mathbb{R}^{3n}$, this time using the theory of planar networks.

Recall that a planar network is the following data:

- a finite oriented planar graph $\Gamma$ with vertex set $V_\Gamma$ and edge set $E_\Gamma$,
- an embedding of $\Gamma$ into the strip $\{ x_0 \leq x \leq x_1 \} \subset \mathbb{R}^2$ such that the image of each edge is a segment of a straight line, which is not parallel to the $y$-axis. This condition allows us to define an orientation of $\Gamma$: we orient each edge in the positive direction along the $x$-axis.

The vertices on the line $\{ x = x_0 \}$ are called *sources* and the vertices on the line $\{ x = x_1 \}$ are called *sinks* of $\Gamma$. A planar network with $n$ sources and $n$ sinks is called a planar network of rank $n$. Without loss of generality, we can assume that the set of $y$-coordinates of the sources and sinks is the set of the first $n$ integers $\{1, 2, \ldots, n\}$.

A $k$-path in $\Gamma$ is a collection of $k$ vertex-disjoint oriented paths connecting $k$ sources with $k$ sinks. The set of $k$-paths in $\Gamma$ is denoted by $P_k \Gamma$. For $I, J \subset \{1, \ldots, n\}$, two subsets of cardinality $k$, we denote by $P_k \Gamma(I, J)$ the set of $k$-paths with the sources corresponding to $I$ and sinks corresponding to $J$. 

Let $Q$ be an abelian semigroup with unit, and let $W(\Gamma, Q)$ be the set of weightings of $\Gamma$ with values in this semigroup: $W(\Gamma, Q) = Q^{\mathcal{E}\Gamma}$. For $w \in W(\Gamma, Q)$ and a collection of edges $\alpha \subset E\Gamma$, we set

$$w(\alpha) = \prod_{e \in \alpha} w(e).$$

If $\alpha = \emptyset$, then we set $w(\alpha) = 1_Q$.

**Example 1.** When $Q = U(1)$ is the group of unitary complex numbers, then we will write $\Phi(\Gamma) = W(\Gamma, U(1))$. For a weighting $\phi \in \Phi(\Gamma)$, we have

$$\phi(\alpha) = \prod_{e \in \alpha} \phi(e).$$

**Example 2.** Consider the tropical semigroup $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with group law given by addition: $(x, y) \mapsto x + y$. Then for $w \in W(\Gamma, \mathbb{T})$ we have

$$w(\alpha) = \sum_{e \in \alpha} w(e).$$

1.4. **Correspondence map.** Let $\Gamma$ be a planar network of rank $n$ and $w \in W(\Gamma, Q)$ a weighting of $\Gamma$ with values in a commutative semiring $Q$. To this pair, we can associate an $n$-by-$n$ matrix with matrix elements in $Q$:

$$M_{i,j}(\Gamma; w) = \sum_{\alpha \in P_1\Gamma(i,j)} w(\alpha) = \sum_{\alpha \in P_1(i,j)} \prod_{e \in \alpha} w(e).$$

In case the set of paths $P_1\Gamma(i,j)$ is empty, we set $M_{i,j}(\Gamma; w)$ equal to the additive unit (zero) of $Q$.

Let $\Gamma_1$ and $\Gamma_2$ be two rank-$n$ planar networks and let $\Gamma = \Gamma_1 \circ \Gamma_2$ be their concatenation, i.e. $\Gamma$ is a union of $\Gamma_1$ and $\Gamma_2$ with sinks of $\Gamma_1$ identified with sources of $\Gamma_2$. Then a pair of weightings $w_1 \in W(\Gamma_1, Q)$, $w_2 \in W(\Gamma_2, Q)$ gives rise to the weighting $w = w_1 \circ w_2 \in W(\Gamma, Q)$, where $w(e) = w_1(e)$ if $e \in ET_1$ and $w(e) = w_2(e)$ if $e \in ET_2$.

Under the correspondence map, the concatenation of planar networks corresponds to matrix multiplication:

$$M(\Gamma_1 \circ \Gamma_2; w_1 \circ w_2) = M(\Gamma_1; w_1) \cdot M(\Gamma_2; w_2).$$

If $Q$ is a commutative ring, then we can define the minors $M_{I,J}(\Gamma; w)$ of the matrix $M(\Gamma; w)$, where $I, J \subset \{1, \ldots, n\}$ with $|I| = |J| = k$, but this definition does not work for a semiring, since it involves signs. The Lindström Lemma asserts that these minors can be expressed in terms of multi-paths in $\Gamma$ as follows:

$$M_{I,J}(\Gamma; w) = \sum_{\alpha \in P\Gamma(I,J)} w(\alpha).$$

Note that the right hand side is well-defined even if $Q$ is only a semiring.
For $k = 1, \ldots, n$, we introduce the functions
\[
m_k(\Gamma, w) = \sum_{I, J; |I| = |J| = k} M_{I,J}(\Gamma; w).
\]
If it is clear which planar network is used, we omit $\Gamma$ and use the shorthand notation $m_k(w)$. When we want to emphasize the semiring in which $m_k$ takes values, we include it in the notation $m_Q^T(w)$.

**Example 3.** Let $T = \mathbb{R} \cup \{-\infty\}$ be the tropical semiring, with addition given by $(x, y) \mapsto \max(x, y)$ and with multiplication $(x, y) \mapsto x + y$. The tropical weights are then defined by the formula (5). The functions $m^T_k(\Gamma, w)$ take the form
\[
(8) \quad m^T_k(\Gamma, w) = \max\{w(\alpha) | \alpha \in P_k \Gamma\}.
\]
In the case when $P_k \Gamma$ is empty, we set $m^T_k(\Gamma, w) = -\infty$ for all weightings $w$.

Later on, we will see that the functions $m^T_k$ provide a “tropical counterpart” of the sums of singular values $l^B_k$. In view of this analogy, we can introduce “tropical singular values” as
\[
\lambda^T_i(w) = m^T_i(w) - m^T_{i-1}(w),
\]
for $i \geq 2$, and $\lambda^T_1(w) = m^T_1(w)$. One can show that $\lambda^T_i \geq \lambda^T_{i+1}$ for all $i = 1, \ldots, n - 1$ (the proof is similar to that of Theorem 2 in [3]).

**Example 4.** Let $Q = T \times U(1)$. The map $(u, \phi) \mapsto \exp(u)\phi$ from $Q$ to $\mathbb{C}$ is a homomorphism for the product. We will use the correspondence map to define the composition
\[
W(\Gamma, T \times U(1)) \to W(\Gamma, \mathbb{C}) \to \text{Mat}_n(\mathbb{C}).
\]
The result is given by the formula
\[
M_{i,j}(\Gamma; u, \phi) = \sum_{\alpha \in P_{i,j} \Gamma} \exp(u(\alpha))\phi(\alpha),
\]
where $u$ is a weighting with values in $\mathbb{T}$ and $\phi$ is a weighting with values in $U(1)$.

1.5. **Results: comparison of different cones.** Let $\Gamma_0$ be the planar network of rank $n$ shown in Figure 2.

Note that the matrices defined by the correspondence map $M(\Gamma_0; w)$ are upper-triangular.

Inspired by the analogy with the multiplicative problem for $B$ (cf. (7)), we can define the following tropical cone:
\[
C_T = \{(r, s, t) \in \mathbb{T}^3n; \exists (w_1, w_2) \in W(\Gamma_0, \mathbb{T})^2; m^T(w_1) = r, m^T(w_2) = s, m^T(w_1 \circ w_2) = t\}.
\]
Figure 2: The planar network $\Gamma_0$.

Note that the set of multi-paths $P_k \Gamma_0$ is nonempty for every $k$, and hence, we can consider the “real” part of this cone:

\[(10)\]

\[C^0_T = C_T \cap \mathbb{R}^{3n}.\]

In [3], we proved the following theorem, which may be thought of as the solution of the tropical Horn problem.

**Theorem 3.** $C^0_T = C_{KT}$.

The main result of this paper is as follows:

**Theorem 4.** $C^0_T = C_B$.

In combination with Klyachko’s theorem (Theorem [2]), this result implies $C^0_T = C_{Horn}$. Together with Theorem [3] this gives a new proof of the Knutson–Tao theorem (Theorem [1]).

1.6. The structure of the paper. Our purpose in Section [2] is to study the relation between $C^0_T$ and $C_B$ via the correspondence map (6). First, in Proposition [2] we show that away from a small set of tropical weights, tropical singular values approximate the corresponding ordinary singular values exponentially well.

A refinement of this statement is Proposition [5] where we show that this approximation is valid on a large part of $B$, where we measure size in terms of the image with respect to the Gelfand–Zeitlin map.

The main result of the section is Proposition [6]. It states that the singular values of the matrices $A$, $C$, and $AC$, for $(A, C) \in B \times B$, are exponentially close the corresponding tropical singular values except for a small part of $B \times B$. This is sufficient to prove the inclusion $C^0_T \subset C_B$.

There is a canonical Poisson structure on the space of Hermitian matrices $H$, whose symplectic leaves $H_r$ are Hermitian matrices with fixed eigenvalues $r \in \mathbb{R}^n$, and it induces a Liouville measure $\mu_r$ on $H_r$. Similarly, there is a canonical Poisson structure on the group $B$ with symplectic leaves the upper-triangular matrices with fixed singular values $\exp(r)$. The corresponding Liouville measure is denoted $\mu^B_r$. 
In Section 3, we first recall the fact that these measures are compatible with the corresponding Gelfand–Zeitlin maps (Theorem 6 and Section 3.3) in the sense that the pushforwards of the measures of $\mu_r$ and $\mu_r^B$ onto the Gelfand–Zeitlin polytope are equal to the Lebesgue measure. This is a corollary of the complete integrability of the Gelfand–Zeitlin system.

According to Klyachko’s theorem, the images of the map $(K, L) \to l(K + L)$ on $H_r \times H_s$, and the map $(A, C) \to l_B(AC)$ on $B_r \times B_s$ coincide. This image is a polytope, that we denote by $\Pi_{r,s} \subset \mathbb{R}^n$. Theorem 7 is a refinement of this theorem: it states that the pushforward measures on $\Pi_{r,s} \subset \mathbb{R}^n$ of the measures $\mu_r \times \mu_s$ and $\mu_r^B \times \mu_s^B$ along these maps coincide; we denote this measure by $\mu_{r,s}$. The proof of this theorem (given in Appendix) uses the theory of Poisson–Lie groups.

Combining these two pieces of information about pushforward measures with our tropical analysis from Section 2 we present our final argument in the proof of Theorem 8. Here we consider the hypothetical exceptional part of $\Pi_{r,s}$ which does not lie in $C^0_T$, and prove that the measure of this part is zero.

Using standard arguments from symplectic geometry, this quickly leads to the proof of our main theorem (Theorem 9). We conclude the paper with an interesting corollary: we provide a tropical description of $\mu_{r,s}$ in Theorem 10.

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# Index of notations

| $\mathcal{H}$ | the set of Hermitian matrices |
|--------------|-------------------------------|
| $\mathcal{B}$ | the set of upper-triangular matrices with positive diagonal entries |
| $\mathcal{C}_{\text{Horn}}$ | Horn cone |
| $\mathcal{C}_{\text{KT}}$ | Knutson–Tao cone defined in terms of hives |
| $\mathcal{C}_B$ | the multiplicative Horn cone |
| $\mathcal{C}_T$ | the tropical Horn cone |
| $P_k\Gamma(I, J)$ | the set of $k$-paths in $\Gamma$ connecting sources with labels from $I$ and sinks with labels from $J$ |
| $W(\Gamma, Q)$ | weightings of network $\Gamma$ with values in the semigroup $Q$ |
| $\overline{W}(\Gamma, Q)$ | subset of $W(\Gamma, Q)$ with vanishing weightings on all horizontal edges except those adjacent to sinks |
| $\Phi(\Gamma)$ | weightings of network $\Gamma$ with values in $U(1)$ |
| $\overline{\Phi}(\Gamma)$ | weightings from $\Phi(\Gamma)$ with all horizontal edges having weight equal to 1 |
| $l(K)$ | vector of sums of eigenvalues of $K$ |
| $l^B(A)$ | multiplicative analog of $l$ |
| $M(\Gamma; w)$ | correspondence map |
| $m_k^T(\Gamma, w)$ | maximal value of the weighting $w$ on $k$-paths in $\Gamma$ |
| $L_H$ | the Gelfand–Zeitlin map $\mathcal{H} \to \mathbb{R}^\Gamma$ recording the values of $l$ on principal submatrices |
| $L_B$ | the map $\mathcal{B} \to \mathbb{R}^\Gamma$ with components given by values of $l^B$ on principal submatrices |
| $L_T$ | the map $W(\Gamma, T) \to T^\Gamma$ with components defined by values of $m^T\Gamma$ on subnetworks |
| $W_\delta(\Gamma, \mathbb{R})$ | subset of points of $W(\Gamma, \mathbb{R})$ with distance $> \delta$ from certain critical hyperplanes |
| $\Delta_{\text{GZ}}$ | Gelfand–Zeitlin cone defined by interlacing inequalities |
| $\Delta_0$ | cone isomorphic to $\Delta_{\text{GZ}}$ via $L_T$ |
In this section, we establish tropical approximation estimates for singular values of matrices defined by planar networks. These estimates imply the inclusion of cones $C^o_T \subset C_B$.

2. Preliminaries. We begin by recalling some standard facts about interlacing inequalities and Gelfand–Zeiltin completely integrable systems.

2.1.1. The Gelfand–Zeiltin system and interlacing inequalities. For a given $n$, let $S^\n$ be the set of maps from the vertices of the triangular tableau of size $n$ to the set $S$. For instance, $\mathbb{R}^\n$ is the set of triangular tableaux of size $n$ filled with real numbers $l_{ik}$ for $0 \leq i \leq k \leq n$.

We define the Gelfand–Zeiltin map $L_H : H \to \mathbb{R}^\n$ as follows. For $A \in H$, we assign $l_i(A(k))$ to the $i$th node of the $k$th row for $i > 0$, where $A(k)$ is the principal $k$-by-$k$ submatrix of $A$; we also set the first element in each row to zero.

The basic result is that, for any $A \in H$, the tableau $L_H(A)$ lies in the Gelfand–Zeiltin cone $\Delta_{GZ} \subset \mathbb{R}^\n$ defined by the system

\begin{align}
    l_k^0 &= 0 \\
    l_k^{i+1} + l_{i-1}^k &\geq l_{i-1}^{k+1} + l_i^k, \\
    l_k^{i+1} + l_i^k &\geq l_{i+1}^{k+1} + l_{i-1}^k.
\end{align}

These inequalities are also called the interlacing inequalities, since the numbers $\lambda_i^k = l_i^k - l_{i-1}^k$ (corresponding to the eigenvalues of Hermitian matrices and their principal submatrices) satisfy the inequalities

$\lambda_i^k \geq \lambda_{i-1}^{k-1} \geq \lambda_{i+1}^k$.

Remark 1. Note that, somewhat surprisingly, the inequalities (11) are part of the Knutson–Tao inequalities (2).

Example 5. For the case of $n = 2$, the interlacing inequalities read as follows:

\[ l_2^1 \geq l_1^1 \quad \text{and} \quad l_2^2 + l_1^1 \geq l_2^1. \]

2.1.2. Tropical Gelfand–Zeiltin map. Let $\Gamma$ be a planar network of rank $n$ and let $Q = T$ the tropical semiring. We denote by $\Gamma^{(k)}$ the maximal subgraph of $\Gamma$ that does not contain the sinks or sources with $y$-coordinates above the line $\{ y = k \}$. A weighting $w \in W(\Gamma, T)$ induces weightings on $\Gamma^{(k)}$ for all $k$, which, by abuse of notations, we also denote by $w$. For each $k = 1, \ldots, n$, consider the collection of functions $m_i^k(w) = m_i^T(\Gamma^{(k)}, w)$, for $i = 1, \ldots, k$, and place the corresponding values in the $k$-th row of the triangular tableau (see Figure [I]). We will call the resulting map $L_T : W(\Gamma, T) \to T^\n$ the tropical Gelfand–Zeiltin map.
Theorem 5 (3 Theorem 2). For any planar network $\Gamma$ of rank $n$ and any weighting $w \in W(\Gamma, T)$, the components $m^T_i(\Gamma^{(k)}, w)$ satisfy the interlacing inequalities (11).

Let $\Gamma_0$ be the planar network shown in Figure 2. Consider the subset $W(\Gamma_0, T) \subset W(\Gamma_0, T)$ which consists of weightings $w$ vanishing on all horizontal edges with the exception of those which end on a sink. Note that the number of edges carrying non-vanishing weights is then exactly $N = n(n + 1)/2$, which coincides with the number of entries in the triangular tableaux and with the number of functions $m^k_i$. The following result is proved in [3]:

Proposition 1. There exists a cone $\Delta_0 \subset W(\Gamma_0, T)$ such that the restriction of $L_T$ to $\Delta_0$ is an isomorphism to $\Delta_{GZ}$.

Proposition 1, with some abuse of notation, allows us to define the bijective inverse map, $L^{-1}_T : \Delta_{GZ} \to \Delta_0$.

2.2. Tropical estimates for a single matrix. Let $\Gamma$ be a planar network of rank $n$. It will be convenient to work with the subset $W(\Gamma, \mathbb{R}) \subset W(\Gamma, T)$ of real weightings of $\Gamma$, considering the reals $\mathbb{R} = T \setminus \{-\infty\}$ as a subset of the tropical numbers. Naturally, the “multiplication” in this situation is the addition of real numbers, thus, for example, for $w \in W(\Gamma, \mathbb{R})$, we have

$$w(\alpha) = \sum_{e \in \alpha} w(e).$$

For $\delta > 0$, denote by $W_\delta(\Gamma, \mathbb{R})$ the subset of weightings $w \in W(\Gamma, \mathbb{R})$ such that

- for any two distinct subsets $\alpha, \beta \subset ET$, we have

$$|w(\alpha) - w(\beta)| > \delta.$$  

- in all interlacing inequalities (11) for $m^k_i(w)$, the left hand side is greater than the right hand side by at least $\delta$ (cf. Theorem 5):

$$l^k_0 = 0 \quad \text{for all } k,$$

$$l^{k+1}_i + l^k_{i-1} > l^{k+1}_{i-1} + l^k_i + \delta,$$

$$l^{k+1}_i + l^k_i > l^{k+1}_{i+1} + l^k_{i-1} + \delta.$$  

Note that the second condition implies, in particular, that we have the gap inequality $\lambda^T_i(\Gamma^{(k)}) - \lambda^T_{i+1}(\Gamma^{(k)}) > \delta$ for the tropical eigenvalues of the principal subnetworks of $\Gamma$.

The complement to the set $W_\delta(\Gamma, \mathbb{R})$ is contained in the $\delta$-neighborhood of a finite number of hyperplanes defined by the equations $w(\alpha) = w(\beta)$ and by the equations resulting from the interlacing inequalities.
Recall the definition of the correspondence map (9). The tropical approximation estimate is described by the following proposition:

**Proposition 2.** Let $\Gamma$ be a planar network of rank $n$, fix $\delta > 0$, and let $w \in W_\delta(\Gamma, \mathbb{R})$. Then there is a constant $c$ depending only on $\Gamma$, such that for $\tau \geq 1$ and for any $\phi \in \Phi(\Gamma) = W(\Gamma, U(1))$, the inequalities

$$
\left| \frac{1}{\tau} \log B_i(M(\Gamma; \tau w, \phi)) - m_i^\tau(\Gamma, w) \right| < c \cdot e^{-\tau \delta}, \quad i = 1, \ldots, n,
$$

hold.

**Proof.** Let $\sigma_i(A)$ be the elementary symmetric functions of the singular values of $A$:

$$
1 + \sum_{i=1}^n \sigma_i(A) q^i = \prod_{i=1}^n (1 + q \lambda_i(AA^*)).
$$

The determinantal expansion for $AA^*$ with $A = M(\Gamma; \tau w, \phi)$ gives the formula

$$
\sigma_i(M(\Gamma; \tau w, \phi)) = \sum_{|I| = i} \left| \sum_{\alpha \in P_i(I, J)} \phi(\alpha) \exp(\tau w(\alpha)) \right|^2.
$$

Isolating the dominant term $\exp(2\tau m_i^\tau (\Gamma, w))$ of the sum, and using condition (12) of $w \in W_\delta(\Gamma, \mathbb{R})$, we obtain the estimate

$$
\left| \sigma_i(M(\Gamma; \tau w, \phi))/\exp(2\tau m_i^\tau (\Gamma, w)) \right| \leq 1 + c_1 e^{-\tau \delta}
$$

for some constant $c_1$. This implies

$$
|\log \sigma_i(M(\Gamma; \tau w, \phi)) - 2\tau m_i^\tau (\Gamma, w)| \leq c_1 e^{-\tau \delta}.
$$

A simple calculation using the second condition (13) shows that $\sigma_i(M(\Gamma; \tau w, \phi))$ may be replaced by the dominant term given by the product of the top $k$ singular values:

$$
|\log \sigma_i(M(\Gamma; \tau w, \phi)) - \log \prod_{j=1}^{i} \lambda_j(M(\Gamma; \tau w, \phi)M(\Gamma; \tau w, \phi)^*)| < c_2 e^{-\tau \delta}.
$$

Now, combining (15) and (16), and using notation (3), we obtain the desired inequality (14) for $\tau \geq 1$. \(\square\)

**Example 6.** Consider the case of $\Gamma_0$ and $n = 2$. To simplify things, we choose $\phi(e) = 1$ for all the edges of the network.

Then we have

$$
m_1^\tau(w) = \max(x, y, z), \quad m_2^\tau(w) = x + z.$$

The planar network $\Gamma$ for $n = 2$.

The correspondence map gives the matrix

$$M(\Gamma; \tau w) = \begin{pmatrix} e^{\tau x} & e^{\tau y} \\ 0 & e^{\tau z} \end{pmatrix}.$$

For its singular values, we have

$$l_{B1}(M(\Gamma; \tau w)) = \frac{1}{2} \log \left( \frac{1}{2} \left( U + \sqrt{U^2 - 4V} \right) \right), \quad l_{B2}(M(\Gamma; \tau w)) = \tau(x + y),$$

where $U = e^{2\tau x} + e^{2\tau y} + e^{2\tau z}$ and $V = e^{2\tau(x+z)}$. Clearly, $l_{B2}(M(\Gamma; \tau w)) = \tau m^T_2(w)$, and it is easy to verify that

$$\lim_{\tau \to +\infty} \frac{1}{\tau} l_{B1}(M(\Gamma; \tau w)) = \max(x,y,z) = m^T_1(w).$$

Proposition 2 has the following corollary:

**Corollary 3.** Let $\Gamma$ be a planar network of rank $n$. Then the convex polyhedral cone $L^T(W(\Gamma, \mathbb{R}))$ is contained in the smallest closed cone containing the image of $W(\Gamma, \mathbb{R})$ under the map $L^B \circ M$.

**Proof.** Indeed, (14) implies that the distance of any element of the interior of the cone $L^T(W_\delta(\Gamma, \mathbb{R}))$ from the image of $W(\Gamma, \mathbb{R})$ under the map $\frac{1}{\tau}L^B \circ M$ is exponentially small as $\tau \to \infty$. But any point of $L^T(W(\Gamma, \mathbb{R}))$ may be approximated by a point in $L^T(W_\delta(\Gamma, \mathbb{R}))$ for $\delta$ small enough. \qed

2.3. Preimages under the Gelfand–Zeitlin map. Following Flaschka–Ratiu [5], we introduce a notion of the Gelfand–Zeitlin map for upper triangular matrices. By definition, the components of the map $L_B : B \to \mathbb{R}^\nabla$ are the values $l^B_i(A^{(k)})$, where $A^{(k)}$ is the principal $k$-by-$k$ submatrix of $A$.

The following statement is a standard fact from linear algebra:

**Proposition 4.** The image of the map $L_B$ is the cone $\Delta_{GZ}$, and the fibers of $L_B : B \to \mathbb{R}^\nabla$ are topological tori of various dimensions (or empty). In particular, for an interior point $\Xi \in \Delta_{GZ}$ in the Gelfand–Zeitlin cone, the fiber of $L_B$ is a torus is of the maximal possible dimension $n(n - 1)/2$.

For the convenience of the reader, we sketch a proof of this proposition. Let $A$ be in $B_{k+1}$, and set $a = AA^*$ to be the corresponding positive definite Hermitian matrix. Denote the eigenvalues of $a$ by $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_k$. By
conjugating the matrix $a$ with an element of $h \in U(k) \subset U(k+1)$, we can bring it to the form

$$nah^{-1} = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,k} \\ a_{1,0} & \mu_1 & 0 & \cdots & 0 \\ a_{2,0} & 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,0} & 0 & 0 & \cdots & \mu_k \end{pmatrix}.$$  

Then the condition that $\lambda_i$’s are zeros of the characteristic polynomial of $a$ gives rise to a system for linear equations on $a_{0,0}, |a_{0,1}|^2, \ldots, |a_{0,k}|^2$, which admits a unique solution. Hence, the set of matrices $a$ of the form (17) with given eigenvalues is a torus of dimension at most $k$. When the eigenvalues $\lambda_i$ and $\mu_j$ are all distinct, we have $|a_{0,i}| \neq 0$ for all $i$. In this case, the torus is parametrized by the angles $\phi_i = \text{Arg}(a_{0,i})$, and hence it is of dimension exactly equal to $k$.

Applying this procedure to the natural chain of projections $B_n \to B_{n-1} \to \ldots \to B_1$, we see that that the fibers of $L_B$ are tori of dimension at most $1 + \cdots + (n-1) = n(n-1)/2$. When all eigenvalues of the principal submatrices of $a$ are distinct (i.e. $L_B(A)$ is in the interior of the Gelfand–Zeiltin cone), then the dimension of the fiber is exactly $n(n-1)/2$. □

The following proposition describes the preimages of the map $L_B$ for sufficiently large values of the scaling parameter $\tau$. Introduce the notation $\Delta^\circ(\delta) = W_\delta(\Gamma_0, \mathbb{R}) \cap \Delta_0$, $\Delta_{GZ}(\delta) = L_T(\Delta^\circ(\delta))$, and let $\overline{\Phi} \subset W(\Gamma_0, U(1))$ stand for the weightings of $\Gamma_0$ with values in $U(1)$ taking value 1 on all horizontal edges.

**Proposition 5.** For every $\delta > 0$, there exists $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ the following statement holds: for $\Xi \in \Delta_{GZ}(\delta)$ and $A \in L_B^{-1}(\tau\Xi)$, there exist $w \in \Delta^\circ(\delta/2)$ and $\phi \in \overline{\Phi}$ such that $A = M(\Gamma; \tau w, \phi)$.

For $\xi \in \Delta_0$ such that $L_T(\xi) = \Xi \in \Delta_{GZ}(\delta)$, the point $\tau\Xi$ has the unique preimage $\tau\xi \in \Delta_0$ under the map $L_T$. Proposition 5 states that for $U_\varepsilon$ an $\varepsilon$-neighborhood of $\tau\xi$, the image of $U_\varepsilon \times \overline{\Phi}$ under the correspondence map $M$ contains the full preimage $L_B^{-1}(\tau\Xi) \subset B$.

$$\Delta_0 \times \overline{\Phi} \xrightarrow{M} B \\
\Delta_0 \xrightarrow{L_T} \Delta_{GZ} \xrightarrow{L_B} B$$

**Proof.** Fix $\phi \in \overline{\Phi}$, and consider the map

$$f_\phi : \Delta_0 \to \Delta_0, \quad f_\phi(w) = L_T^{-1} \circ L_B \circ M(\Gamma; w, \phi).$$
Fix $\varepsilon > 0$. According to Proposition 2 for some $\tau_0$ and $c$, as long as $w \in W_{\delta/2}(\Gamma_0, \mathbb{R})$ and $\tau > \tau_0$ we have

$$\left| \frac{1}{\tau} L_B \circ M(\Gamma; \tau w, \phi) - L_T(w) \right| \leq c_1 e^{-\tau \delta/2}.$$ 

This inequality can be rewritten as

$$\left| \frac{1}{\tau} L_T(f_{\phi}(\tau w)) - L_T(w) \right| \leq c_1 \cdot e^{-\tau \delta/2}. \tag{18}$$

Since the restriction of $L_T$ to $\Delta_0$ is a non-degenerate linear map, this implies

$$\left| \frac{1}{\tau} f_{\phi}(\tau w) - w \right| \leq c_2 \cdot e^{-\tau \delta/2}.$$

Recall the following standard homological argument: let $S$ be a convex subset of $\mathbb{R}^N$, and $g : S \to \mathbb{R}^N$ be a continuous map satisfying $|g(w) - w| < \varepsilon$; then the image $g(S)$ contains all points of $S$ that are at least at the distance $\varepsilon$ from its boundary:

$$g(S) \supset \{s \in S; \ d(s, \partial S) > \varepsilon \}.$$

Now let $S = \Delta_0(\delta/2)$, $g : w \mapsto \frac{1}{\tau} f_{\phi}(\tau w)$ and $\varepsilon = \delta/2$. The argument above shows that for large enough $\tau$ the image of the set $\Delta_0(\delta/2)$ under the map $w \mapsto \frac{1}{\tau} f_{\phi}(\tau w)$ will contain $\Delta_0(\delta)$.

Hence, for every $\phi$, we constructed a point $A_{\phi} = M(\Gamma; \tau w, \phi)$ in the preimage $L_B^{-1}(\tau \Xi)$. By equation (18), we have

$$|\Xi - L_T(w)| \leq c_1 \exp(-\delta \tau/2)$$

and

$$|\xi - w| \leq c_2 \exp(-\delta \tau/2).$$

By choosing $\tau$ sufficiently large, we can make sure that $|\xi - w| \leq \varepsilon$. Then

$$M((\Delta_0(\delta/2)\setminus U_{\varepsilon}) \times \overline{\Phi}) \cap L_B^{-1}(\tau \Xi) = \emptyset.$$ 

We would like to show that

$$L_B^{-1}(\tau \Xi) \subset M(U_{\varepsilon} \times \overline{\Phi}).$$

By Proposition 4 the preimage $L_B^{-1}(\tau \Xi)$ is connected. The argument above shows that

$$L_B^{-1}(\tau \Xi) \cap M(\tau \Delta_0(\delta/2) \times \overline{\Phi})$$

is a nonempty connected component of $L_B^{-1}(\tau \Xi)$. This means that the entire set $L_B^{-1}(\tau \Xi)$ is contained in $M(\tau \Delta_0(\delta/2) \times \overline{\Phi})$. 

$\Box$
2.4. **Tropical analysis of the Horn problem.** Now we can pass to our main focus, the Horn problem. The tropical analysis is quite similar to that of the previous section.

First, we need to define an appropriate analog of the set $W_\delta$. Let $\hat{W}_\delta \subset W(\Gamma_0 \circ \Gamma_0, \mathbb{R})$ be defined by the following conditions:

- for any two distinct subsets $\alpha, \beta \subset \Gamma_0 \circ \Gamma_0$, we have $|\alpha(w_1 \circ w_2) - \beta(w_1 \circ w_2)| > \delta$,
- $L_T(w_1), L_T(w_2), L_T(w_1 \circ w_2) \in \Delta_G\delta(\delta)$.

We will also need the corresponding image set $\Sigma(\delta) = L_T^{x^2 \hat{W}_\delta} \subset \mathbb{R}^\n \times \mathbb{R}^\n$.

**Example 7.** Consider the case of $n = 2$. The cone $\Delta_G\times \Delta_G$ is defined by the inequalities $r_2 - r_1 \leq l \leq r_1$ and $s_2 - s_1 \leq m \leq s_1$. Among others, we have the following inequalities defining $\Sigma(\delta)$:

$$(r_2 - r_1) + \delta < l, \ l < r_1 - \delta, \ (s_2 - s_1) + \delta < m, \ m < s_1 - \delta,$$

$$(r_2 + s_1) - (l + m + r_1) > \delta.$$ 

Note that the first four inequalities state that the point is inside $\Delta_G\delta(\delta) \times \Delta_G\delta(\delta)$ whereas the last inequality involves both copies of the cone $\Delta_G\delta$ at the same time.

We also introduce the tropical and the usual Horn maps:

$$H_T(w_1, w_2) = (m^T(\Gamma_0, w_1), m^T(\Gamma_0, w_2), m^T(\Gamma_0 \circ \Gamma_0, w_1 \circ w_2)),$$

$$H_B(A_1, A_2) = (\ell^B(A_1), \ell^B(A_2), \ell^B(A_1 A_2)).$$

With these preparations, we can formulate the tropical estimate for the Horn problem as follows:

**Proposition 6.** For every $\delta > 0$, there exist $\tau_0 > 0$ and a constant $c$ such that for every $\tau \geq \tau_0$ the following statement holds: for $(\Xi_1, \Xi_2) \in \Sigma(\delta)$ and $A_1 \in L_B^{-1}(\tau \Xi_1)$, $A_2 \in L_B^{-1}(\tau \Xi_2)$, there exist $w_1 \circ w_2 \in \hat{W}_{\delta/2}$, and $\phi_1, \phi_2 \in \hat{W}(\Gamma_0, U(1))$ such that $M(\Gamma; \tau w_1, \phi_1) = A_1$, $M(\Gamma; \tau w_2, \phi_2) = A_2$, and

$$\left| \frac{1}{\tau} H_B(A_1, A_2) - H_T(w_1, w_2) \right| \leq ce^{-\delta \tau}. \quad (19)$$

**Proof.** Indeed, using Proposition 5 we can conclude that there are weights $w_1, w_2 \in \hat{W}(\Gamma_0, \mathbb{R})$ and $\phi_1, \phi_2 \in \hat{\Phi}(\Gamma_0)$ such that $M(\Gamma; \tau w_1, \phi_1) = A_1$ and $M(\Gamma; \tau w_2, \phi_2) = A_2$ with

$$\left| \frac{1}{\tau} L_B(A_1) - L_T(w_1) \right| < ce^{-\delta \tau}, \quad \left| \frac{1}{\tau} L_B(A_2) - L_T(w_2) \right| < ce^{-\delta \tau}.$$ 

Since $(\Xi_1, \Xi_2) \in \Sigma(\delta)$, for $\tau$ sufficiently large we have $w_1 \circ w_2 \in \hat{W}_{\delta/2}$. 

Then the equality
\[ M(\Gamma; \tau w_1, \phi_1) \cdot M(\Gamma; \tau w_2, \phi_2) = M(\Gamma; \Gamma; \tau w_1 \circ w_2, \phi_1 \circ \phi_2). \]
together with Proposition 2 yields
\[ \left| \frac{1}{\tau} L_B(A_1 \cdot A_2) - L_T(w_1 \circ w_2) \right| < e^{-\delta \tau}, \]
which clearly implies (19). \(\square\)

**Corollary 7.** \(C_\tau^0 \subset C_B\)

Indeed, using Proposition 6 we can prove that the smallest closed cone containing \(C_B\) contains \(C_\tau^0\) exactly in the same way as we deduced Corollary 3 from Proposition 2. Now, it follows from Klyachko’s theorem that \(C_B\) is itself a closed cone, and hence we can conclude that \(C_\tau^0 \subset C_B\).

3. Poisson Geometry and Duistermaat–Heckman measures

3.1. **The Gelfand–Zeitlin system on Hermitian matrices.** Recall that the set of Hermitian matrices \(H\) can be naturally identified with the dual of the Lie algebra \(u(n)\) by means of the nondegenerate pairing
\[ \langle a, \xi \rangle = \text{Im } \text{Tr}(a \xi), \]
where \(a \in H\) and \(\xi \in u(n)\) (viewed as a skew-Hermitian matrix). Since \(H \cong u^*(n)\), it carries a linear Kirillov–Kostant–Souriau (KKS) Poisson bracket \(\pi_H\). Symplectic leaves under this bracket are formed by matrices with fixed eigenvalues:
\[ \mathcal{H}_r = \{ a \in \mathcal{H}; l(a) = r \}, \text{ where } r \in \mathbb{R}^n. \]

**Example 8.** Consider the case of \(n = 2\). The space of Hermitian 2-by-2 matrices is isomorphic to \(\mathbb{R}^4\),
\[ (x, y, z, t) \mapsto \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix}. \]
Under the KKS bracket, \(t\) is a Casimir function (i.e., belongs to the Poisson center), and brackets of the other variables take the form
\[ \{x, y\} = z, \{y, z\} = x, \{z, x\} = y. \]
The symplectic leaves are either points (if \(x = y = z = 0\)) or 2-spheres:
\[ \mathcal{H}_{(r_1, r_2)} = \left\{ \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix}; r_1 = t + \sqrt{x^2 + y^2 + z^2}, r_2 = 2t \right\}. \]
Recall the definition of the Gelfand–Zeitlin map \(L_H : \mathcal{H} \to \mathbb{R}^\nabla\) from Section 2.1.1. The following theorem is due to Guillemin and Sternberg [8]:
Theorem 6. The map $L_H : (\mathcal{H}, \pi_{\text{KKS}}) \to (\mathbb{R}^N, 0)$ is a Poisson map. Its components $l_i = l_i^n$ are Casimir functions. For $k < n$, the functions $l_i^k$ generate a densely defined action of a torus of dimension $n(n - 1)/2$.

Over each symplectic leaf $\mathcal{H}_r$ with $r \in \mathbb{R}^n$, the map $L_H$ defines a completely integrable system in the sense of Liouville–Arnold, i.e.

$$\{l_i^k, l_j^m\} = 0,$$

and the number of independent functions is equal to $\dim \mathcal{H}_r/2$.

For generic $r$, the symplectic form on $\mathcal{H}_r$ is given by

$$(20) \quad \omega_r = \sum_{k=1}^{n-1} \sum_{i=1}^k dl_i^k \wedge d\phi_i^k,$$

where $\phi_i^{(k)}$'s are some linear combinations (with integer coefficients) of the angles defining the torus action. In these coordinates, the Liouville volume form is expressed as

$$\Lambda_r = \prod_{k=1}^{n-1} \prod_{i=1}^k dl_i^k \wedge d\phi_i^k.$$

Denote by $P_r = L_H(\mathcal{H}_r)$ the image of $\mathcal{H}_r$ under the map $L_H$. This is a convex polytope defined by the interlacing inequalities and it carries a natural measure $(L_H)_* \Lambda_r$ which is equal to the Lebesgue measure on $P_r$:

$$(L_H)_* \Lambda_r = \chi_{P_r} \prod_{k=1}^{n-1} \prod_{i=1}^k dl_i^k.$$

Here $\chi_{P_r}$ is the characteristic function of $P_r$. Sometimes it is more convenient to consider the normalized measure

$$\mu_r = \frac{1}{\text{vol}(P_r)} (L_H)_* \Lambda_r.$$

Let $\tau \in \mathbb{R}_+$, and denote by $R_\tau : \mathbb{R}^n \to \mathbb{R}^n$ the dilation map $R_\tau : r \to \tau r$. Then we have $P_{\tau r} = \tau P_r$ and

$$\mu_{\tau r} = (R_\tau)_* \mu_r.$$

Example 9. For the case of $n = 2$, the components of the map $L_H$ are as follows

$$l_1^2 = t + \sqrt{x^2 + y^2 + z^2}, \quad l_2^2 = 2t, \quad l_1^1 = t - z.$$

Restricting to the symplectic leaf $\mathcal{H}_r$, we fix $l_1^2 = r_1$ and $l_2^2 = r_2$. The polytope $P_r$ is the closed interval $l_1^1 \in [r_2 - r_1, r_1]$. The corresponding normalized measure is

$$\mu_r = \frac{1}{2r_1 - r_2} \chi_{[r_2 - r_1, r_1]} dl_1^1.$$

It is invariant under scaling $r_i \mapsto \tau r_i$, $l_1^1 \mapsto \tau l_1^1$. 
3.2. The Duistermaat–Heckman measure for the Horn problem.
For \( r, s \in \mathbb{R}^n \), the symplectic manifold \( \mathcal{H}_r \times \mathcal{H}_s \) carries the diagonal Hamiltonian action of \( U(n) \) with moment map \( \Psi_H : (a, b) \rightarrow a + b \). By Kirwan’s Convexity Theorem, the image \( \Pi_{r,s} = l \circ \Psi_H (\mathcal{H}_r \times \mathcal{H}_s) \) is a convex polytope (the Horn polytope). We clearly have
\[
\Pi_{r,s} = \{ t \in \mathbb{R}^n ; (r,s,t) \in \mathcal{C}_H \}.
\]
The push-forward of the Liouville measure \( \Delta_{r,s} = (l \circ \Phi)_* (\Lambda_r \times \Lambda_s) \) is the Duistermaat–Heckman measure on \( \Pi_{r,s} \). By the Duistermaat–Heckman Theorem, \( \Delta_{r,s} \) is absolutely continuous with respect to the Lebesgue measure on \( \Pi_{r,s} \), and the corresponding Radon–Nykodim derivative is a piece-wise polynomial function, which is strictly positive on the interior of \( \Pi_{r,s} \).

We can again define the normalized measure
\[
\mu_{r,s} = \frac{1}{\text{vol}(\mathcal{P}_r) \text{vol}(\mathcal{P}_s)} \Delta_{r,s},
\]
with the obvious scaling property:
\[
\Pi_{\tau r, \tau s} = \tau \Pi_{r,s}, \quad \mu_{\tau r, \tau s} = (\tau r)_* \mu_{r,s}.
\]

Example 10. Let \( n = 2 \) and choose \( r_1 = r, r_2 = 0, s_1 = s, \) and \( s_2 = 0 \). Then it is easy to check that the Horn polytope is of the form
\[
\Pi_{r,s} = \{(t,0) \in \mathbb{R}^2 ; |r - s| \leq t \leq r + s\}.
\]
It is equipped with the normalized measure
\[
\mu_{r,s} = \frac{1}{2rs} \chi_{[|r-s|,r+s]} t dt.
\]
This measure is invariant under scaling \( r \mapsto \tau r, s \mapsto \tau s, t \mapsto \tau t \).

3.3. The Gelfand–Zeitlin system on the group \( \mathcal{B} = U^*(n) \). The group \( \mathcal{B} \) carries a natural action of the group \( U(n) \). Recall that the Iwasawa decomposition
\[
g = Au, \quad \text{for } g \in \text{Gl}(n, \mathbb{C}), A \in \mathcal{B}, \text{ and } u \in U(n),
\]
gives rise to the identification
\[
\mathcal{B} \cong \text{GL}(n, \mathbb{C})/U(n)
\]
of the group \( \mathcal{B} \) with a homogeneous space. This presentation defines a natural action of \( \text{GL}(n, \mathbb{C}) \) on \( \mathcal{B} \) by multiplication on the left. The restriction of this action to the subgroup \( U(n) \) is called the dressing action. For \( x \in U(n) \) we denote by \( \xi_x \) the corresponding fundamental vector field on \( \mathcal{B} \).

The group \( \mathcal{B} \) has a canonical multiplicative Lu–Weinstein Poisson structure \( \pi_\mathcal{B} \), that is defined as follows. Let \( dAA^{-1} \) be the right-invariant
Maurer–Cartan 1-form on $\mathcal{B}$ with values in the Lie algebra $\text{Lie}(\mathcal{B})$. There is a canonical pairing between $\text{Lie}(\mathcal{B})$ and $\mathfrak{u}(n)$ given by

$$\langle \xi, x \rangle = \text{Im} \text{Tr}(\xi x).$$

The bivector $\pi_\mathcal{B}$ is the unique bivector on $\mathcal{B}$ such that

$$\pi_\mathcal{B}(\langle dA A^{-1}, x \rangle, \cdot) = \xi_x.$$

Note that for $x$ a diagonal skew-Hermitian matrix, we have

$$\langle dA A^{-1}, x \rangle = d\langle \log(A_d), x \rangle,$$

where $A = A_d N(A)$ with $A_d$ a diagonal matrix and $N(A)$ a unipotent upper-triangular matrix. Hence, the action of the Cartan subgroup of $U(n)$ consisting of unitary diagonal matrices is Hamiltonian with the moment map $\Psi(A) = \log(A_d)$.

**Example 11.** For $n = 2$, we have a parametrization

$$A = \left( \begin{array}{cc} y & z \\ 0 & y^{-1} \end{array} \right),$$

with $y \in \mathbb{R}_+$ and $z \in \mathbb{C}$. The Poisson brackets read

$$\{y, z\}_B = \frac{i}{2} yz, \quad \{y, \overline{z}\}_B = -\frac{i}{2} y\overline{z}, \quad \{z, \overline{z}\}_B = i(z^2 - \overline{z}^2).$$

The dressing action of the diagonal circle $\text{diag}(\exp(i\theta), \exp(-i\theta))$ is given by

$$y \mapsto y, \quad z \mapsto z \exp(2i\theta).$$

The moment map for this action is $\Psi(A) = \log(y)$.

Symplectic leaves of the Poisson structure $\pi_\mathcal{B}$ are orbits of the dressing action and at the same time fibers of the map $l_\mathcal{B}$. For $r \in \mathbb{R}^n$, we denote by $\mathcal{B}_r = (l_\mathcal{B})^{-1}(r)$ the corresponding symplectic leaf. The leaf $\mathcal{B}_r$ consists of matrices $A \in \mathcal{B}$ such that the eigenvalues of $AA^*$ are given by $(\exp(r_1), \ldots, \exp(r_n))$.

Similarly to the case of Hermitian matrices, the map $L_\mathcal{B}$ defines a completely integrable system on each leaf $\mathcal{B}_r$. The image $L_\mathcal{B}(\mathcal{B}_r)$ is the same polytope $P_r$ defined by the interlacing inequalities. Moreover, in action-angle variables $(l^B_k, \phi^B_k)$ the symplectic form is again described by equation (20) and the induced normalized measure on $P_r$ is again the measure $\mu$ (see [5] for details).

### 3.4. Duistermaat–Heckman measure for the multiplicative problem.

For a pair of symplectic leaves $\mathcal{B}_r$ and $\mathcal{B}_s$, the space $\mathcal{B}_r \times \mathcal{B}_s$ carries the dressing action of $U(n)$ defined by $g : (A, B) \mapsto (A', B')$, where

$$gA = A'g', \quad g'B = B'g''$$

with $g, g'$, and $g''$ in $U(n)$ and $A, A', B, B' \in \mathcal{B}$. This action has a moment map in the sense of Lu, $\Psi(A, B) = AB$. By the Klyachko Theorem, the composition map $l_\mathcal{B} \circ \Psi$ sends $\mathcal{B}_r \times \mathcal{B}_s$ to the same polytope $\Pi_{r,s} \subset \mathbb{R}^n$ as in the case of Hermitian matrices. Denote the normalized push-forward measure on $\Pi_{r,s}$ by $\mu^{B}_{r,s}$.
Theorem 7.
\[ \mu^B_{r,s} = \mu_{r,s}. \]

We give a proof of this theorem in Appendix.

An important corollary of this Theorem is the scaling invariance of \( \mu^B_{r,s} \):
\[ \mu^B_{\tau r, \tau s} = (R_\tau)_* \mu^B_{r,s}, \]
which follows from the analogous (obvious) property \( \mu_{r,s} \).

4. Comparison of the multiplicative and tropical Horn problems

In this section, we put all the elements of our argument together, and prove the equivalence of the multiplicative and tropical Horn problems.

For \( r, s \in \mathbb{R}^n \), define the polytope
\[ \Pi^T_{r,s} = \{ t \in \mathbb{R}^n; (r, s, t) \in C^0_T \}. \]

Since \( C^0_T \subset C_B = C_{\text{Horn}} \), we have \( \Pi^T_{r,s} \subset \Pi_{r,s} \). We introduce the exceptional set
\[ X_{r,s} = \Pi_{r,s} \setminus \Pi^T_{r,s}. \]

Clearly, showing that \( C^0_T = C_B \) is equivalent to showing that \( X_{r,s} \) is empty.

Theorem 8.
\[ \mu_{r,s}(X_{r,s}) = 0. \]

Proof. Choose \( \delta > 0, \varepsilon > 0 \) and \( \tau > 0 \) such that \( \tau > \tau_0 \) with \( \tau_0 \) defined in Proposition and \( c \exp(-\tau \delta) < \varepsilon \), where \( c \) is the maximum of the constants corresponding to the graphs \( \Gamma_0 \) and \( \Gamma_0 \circ \Gamma_0 \).

Let
\[ (u, v) \in (P_r \times P_s) \setminus \Sigma(\delta) \]
and consider a pair \( (A, B) \in B_{\tau r} \times B_{\tau s} \) such that \( L_B(A) = \tau u, L_B(B) = \tau v. \)

Then, by Proposition there exist weights \( w_1, w_2 \in W(\Gamma_0, \mathbb{R}) \) and \( \phi_1, \phi_2 \in \widehat{V}(\Gamma_0) \subset \widehat{W}(\Gamma_0, U(1)) \) such that \( A = M(\Gamma; \tau w_1, \phi_1) \) and \( B = M(\Gamma; \tau w_2, \phi_2) \) with
\[ |L_T(w_1) - u| < \varepsilon, \quad |L_T(w_2) - v| < \varepsilon, \]
and
\[ \left| \frac{1}{\tau} L_B(AB) - L_T(w_1 \circ w_2) \right| < \varepsilon. \]

Denote by \( w_u = L_T^{-1}(u) \) and \( w_v = L_T^{-1}(v) \) the preimages of \( u \) and \( v \) under the map \( L_T \). Since \( L_T \) is a non-degenerate linear map, the inequalities imply \( |w_u - w_1| \leq c_1 \varepsilon, \quad |w_v - w_2| \leq c_1 \varepsilon \) and
\[ |L_T(w_u \circ w_v) - L_T(w_1 \circ w_2)| < c_2 \varepsilon, \]
where \(c_i\)'s are appropriately chosen constants. Then, combining this with inequality (22), we obtain
\[
\left| \frac{1}{\tau} L_B(AB) - L_T(w_u \circ w_v) \right| < c_3 \varepsilon,
\]
and, as a consequence,
\[
\left| \frac{1}{\tau} l^B(AB) - m^T(w_u \circ w_v) \right| < c_3 \varepsilon.
\]
Hence,
\[
l^B \left( (L_B \times L_B)^{-1}(P_{\tau r} \times P_{\tau s}) \backslash \Sigma(\tau \delta) \right) \subset U_{\tau \varepsilon}(\Pi_{\tau r,\tau s}^T),
\]
where \(U_{\tau \varepsilon}(\Pi_{\tau r,\tau s}^T)\) is the \(\tau \varepsilon\) neighborhood of the polytope \(\Pi_{\tau r,\tau s}^T\). This implies that
\[
\mu_{\tau r,\tau s}(X_{\tau r,\tau s}) \leq c_4 \delta + c_5 \varepsilon,
\]
where \(c_4\) is the total volume of intersections of the hyperplanes defining \(\Sigma(\delta)\) with \(P_r \times P_s\) and \(c_5\) is the total volume of the boundary of \(\Pi_{\tau r,s}^T\).

Recall that \(\mu_{\tau r,\tau s}(X_{\tau r,\tau s}) = \mu_{r,s}(X_{r,s})\) which implies
\[
\mu_{r,s}(X_{r,s}) \leq c_4 \delta + c_5 \varepsilon.
\]
Since the constants \(\delta\) and \(\varepsilon\) were chosen arbitrarily, we can conclude that \(\mu_{r,s}(X_{r,s}) = 0\), as required. \(\square\)

**Theorem 9.** \(X_{r,s} = \emptyset\) and \(\Pi_{r,s} = \Pi_{r,s}^T\).

**Proof.** The polytope \(\Pi_{r,s}\) is the image of the symplectic manifold \(\mathcal{H}_r \times \mathcal{H}_s\) under the composition of the moment map \((a,b) \rightarrow a + b\) with the map \(l\). By the Duistermaat–Heckman theorem, the induced measure \(\mu_{r,s}\) on \(\Pi_{r,s}\) is piece-wise polynomial and non-vanishing on its interior. Since both \(\Pi_{r,s}\) and \(\Pi_{r,s}^T\) are closed polytopes, their difference \(X_{r,s}\) is either empty or contains points of the interior of \(\Pi_{r,s}\). The vanishing of the measure \(\mu_{\tau r,\tau s}(X_{\tau r,\tau s}) = 0\) implies that \(X_{r,s}\) contains no points in the interior of \(\Pi_{r,s}\). Then it must be empty, as required. \(\square\)

This construction has the following interesting corollary:

**Theorem 10.** Let \(\kappa : P_r \times P_s \rightarrow \Pi_{r,s}\) be the map defined by
\[
\kappa(u,v) = m^T(L^{-1}_r(u) \circ L^{-1}_s(v)).
\]
Then
\[
\kappa_*(\mu_r \times \mu_s) = \mu_{r,s}.
\]

**Proof.** The measure \(\mu_{r,s}\) is piece-wise polynomial by the Duistermaat–Heckman Theorem. The measure
\[
\mu_{r,s}^T = \kappa_*(\mu_r \times \mu_s)
\]
is the image of the Lebesgue measure on \(P_r \times P_s\) under a piece-wise linear map. Hence, \(\mu_{r,s}^T\) is also piece-wise polynomial, and to establish the equality
\[ \mu_{r,s}^{\top} = \mu_{r,s}, \] it is enough to compare them on open balls. Let \( B \subset \Pi_{r,s} \) be an open ball. Then, similar to the proof of the previous theorem, we have an estimate
\[ |\mu_{r,s}^{\top}(B) - \mu_{r,s}(B)| < c_4 \delta + c_6 \varepsilon, \]
where \( c_4 \) (as before) is the total volume of intersections of the hyperplanes defining \( \Sigma \) with \( P_r \times P_s \), and \( c_6 \) is the volume of the boundary of \( B \). As the constants \( \delta \) and \( \varepsilon \) can be chosen arbitrarily, we conclude that \( \mu_{r,s}^{\top}(B) = \mu_{r,s}(B) \). Hence, \( \mu_{r,s}^{\top} = \mu_{r,s} \), as required.

\[ \square \]

**Example 12.** Let \( n = 2 \) and \( r = (r, 0), s = (s, 0) \). Then, \( P_r = [-r, r], P_s = [-s, s] \). The map \( \kappa \) is of the form
\[ \kappa : (x, y) \mapsto \max(r - y, x + s). \]

It is easy to see that \( \text{im}(\kappa) = [\|r - s\|, r + s] = \Pi_{r,s} \). For \( t \in [\|r - s\|, r + s] \), and its preimage is a union of two segments:
\[ \kappa^{-1}(t) = [-r, t - s] \times \{r - t\} \cup \{t - s\} \times [r - t, s]. \]

The induced measure on \( \Pi_{r,s} \) is given by
\[
\kappa_* \left( \frac{1}{2r} \chi_{[-r,r]} dx \times \frac{1}{2s} \chi_{[-s,s]} dy \right) =
\frac{1}{4rs} \chi_{[-s,r+s]} ((t - s + r) + (s - r + t)) dt = \frac{1}{2rs} \chi_{[-s,r+s]} t dt,
\]
which coincides with the measure \( \mu_{r,s} \).
Appendix: proof of Theorem 7

In order to prove this theorem, we need to use several facts from symplectic geometry. First, we recall the Linearization Lemma of [1].

Theorem 11 (Linearization Lemma). Let $G$ be a compact connected semisimple Lie group, and $\mathfrak{g}$ its Lie algebra. Then there exist a 2-form $\nu \in \Omega^2(\mathfrak{g}^*)$ and a map $u : \mathfrak{g}^* \to G$ such that for any symplectic manifold $(M, \omega)$ endowed with a Poisson $G$-action and a map $\Psi : M \to \mathfrak{g}^*$, which is a moment map in the sense of Lu, the triple $(M, \varpi = \omega - \psi^*\nu, \psi = \text{Exp}^{-1} \circ \Psi)$ is a Hamiltonian $G$-space with the moment map $\psi$.

In addition, in this case, the map $u_M : M \to M$, $u_M(m) = u(\psi(m)) \cdot m$ is a symplectomorphism between $(M, \varpi)$ and $(M, \omega)$.

In particular, linearizations of dressing orbits on $G^*$ are coadjoint orbits in $\mathfrak{g}^*$. For example, let $G = U(n)$ with the standard Poisson structure; then $G^* = \mathcal{B}$ with the Lu–Weinstein Poisson structure. The identification of $\mathcal{B}$ with $\mathcal{H}$ via the equation $AA^* = \exp(2K)$ induces an isomorphism of $G$-spaces $\mathcal{B}_r \cong \mathcal{H}_r$ between dressing and coadjoint orbits. Moreover, the linearization of $\mathcal{B}_r$ with the Lu–Weinstein Poisson structure is exactly $\mathcal{H}_r$ with the KKS Poisson structure.

For a $G$-Hamiltonian space $(M, \varpi, \psi)$, we can define the projection map onto the positive Weyl chamber $\sigma = \pi \circ \psi : M \to W_+$ and the corresponding Duistermaat–Heckman measure $\text{DH}_0(M, \varpi, \psi) = \sigma_* \varpi^N/N!$, where $N = \text{dim} M/2$. Similarly, we define the measure $\text{DH}(M, \omega, \Psi) = \sigma_* \omega^N/N!$.

Lemma 8. In the setup of Theorem 11, we have $\text{DH}(M, \omega, \Psi) = \text{DH}_0(M, \varpi, \psi)$.

Proof. According to Theorem 11, there is a map $u_M : M \to M$ such that $u_M^*\omega = \varpi$. This implies $(u_M)_*\varpi^N = \omega^N$, and therefore we have

$$\text{DH}(M, \omega, \Psi) = \sigma_* \left( \frac{\varpi^N}{N!} \right) = \sigma_* \circ (u_M)_* \left( \frac{\varpi^N}{N!} \right) = \pi_* \circ \psi_* \circ (u_M)_* \left( \frac{\varpi^N}{N!} \right) = \pi_* \circ (\text{Ad}_u^*)_* \circ \psi_* \left( \frac{\varpi^N}{N!} \right) = \text{DH}_0(M, \varpi, \psi).$$

Here we used that $\pi \circ \text{Ad}_g^* = \pi$ for any $g \in G$. \qed

Next, recall Theorem 3.4 of [2].

Theorem 12 (“Linearization commutes with products”). Let $(M_i, \omega_i, \Psi_i)$, $i = 1, 2$, be two Hamiltonian $G$-spaces endowed with moment maps in the sense of Lu, and let $(M_i, \varpi_i, \psi_i)$, $i = 1, 2$, be the corresponding linearizations (cf. Theorem 11). Then the space $(M_1 \times M_2, \omega = \omega_1 + \omega_2, \Psi(m_1, m_2) = \Psi_1(m_1)\Psi_2(m_2))$ carries a Poisson $G$-action for which $\Psi$ is the moment map in the sense of Lu, and the linearization of this space is $G$-equivariantly symplectomorphic to the $G$-Hamiltonian space $(M_1 \times M_2, \varpi = \varpi_1 + \varpi_2, \psi = \psi_1 + \psi_2)$. \hfill \qed
ψ_1 + ψ_2). Moreover, the symplectomorphism ξ : M_1 × M_2 → M_1 × M_2 intertwines the moment maps Exp^{-1} ◦ Ψ and ψ.

Since the G-equivariant symplectomorphism ξ intertwines the moment maps, the Duistermaat–Heckman measures of the corresponding G-Hamiltonian spaces coincide, and we obtain the following corollary of Lemma 8 and Theorem 12:

**Corollary 9.** In the setup of Theorem 12, we have

$$DH(M_1 \times M_2, ω, Ψ) = DH_0(M_1 \times M_2, ω, ψ).$$

We are now ready to prove Theorem 7.

**Proof of Theorem 7.** Let $G = U(n), G^* = B$ and $M_1 = B_r ≅ H_r, M_2 = B_s ≅ H_s$. Then, $DH_0(M_1 \times M_2, ω, ψ) = µ_{r,s}$ and $DH(M_1 \times M_2, ω, Ψ) = µ_{r,s}^B$, and, the equality $µ_{r,s}^B = µ_{r,s}$ follows from Corollary 9. □

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