ESTIMATION OF TURBULENT DIFFUSIVITY WITH DIRECT NUMERICAL SIMULATION OF STELLAR CONVECTION

H. Hotta, Y. Iida, and T. Yokoyama

Department of Earth and Planetary Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan; hotta.h@eps.s.u-tokyo.ac.jp

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ABSTRACT

We investigate the value of horizontal turbulent diffusivity $\eta$ by numerical calculation of thermal convection. In this study, we introduce a new method whereby the turbulent diffusivity is estimated by monitoring the time development of the passive scalar, which is initially distributed in a given Gaussian function with a spatial scale $d_0$. Our conclusions are as follows: (1) assuming the relation $\eta = L_c v_{\text{rms}}/3$, where $v_{\text{rms}}$ is the root-mean-square (rms) velocity, the characteristic length $L_c$ is restricted by the shortest one among the pressure (density) scale height and the region depth. (2) The value of turbulent diffusivity becomes greater with the larger initial distribution scale $d_0$. (3) The approximation of turbulent diffusion holds better when the ratio of the initial distribution scale $d_0$ to the characteristic length $L_c$ is larger.

Key words: stars: interiors – Sun: dynamo – Sun: interior

Online-only material: color figures

1. INTRODUCTION

Turbulent diffusivity has been an important concept for the mean-field modeling of the interior convection and dynamo of the Sun and stars (see the review by Miesch 2005). It is a substantial factor for the transport of the angular momentum and magnetic field. While the non-turbulent molecular diffusivities are much smaller, i.e., molecular viscosity is $\nu \sim 1$ cm$^2$ s$^{-1}$ and molecular magnetic diffusivity is $\eta \sim 10^4$ cm$^2$ s$^{-1}$ in the solar convection zone, the random advective motion of gases in turbulence is considered to behave as a strong diffusion. The specific value is unknown but previous studies suggest that the turbulent diffusivity depends on the resolved scale, i.e., the value becomes smaller with higher resolution. Abramenko et al. (2011) also found this type of dependency through observation of bright points.

Käpylä et al. (2009) report that the value of turbulent diffusivity is proportional to the square of vertical velocity and is approximately proportional to the wavelength of the test field. Cameron et al. (2011) investigate the value of turbulent diffusivity with a realistic radiative MHD simulation and estimate the value of turbulent diffusivity from the decreasing rate of the total magnetic flux. Yousef et al. (2003) estimate the turbulent magnetic diffusivity and kinetic viscosity in a forced isotropic turbulence using a similar way, i.e., from the decay rate of the magnetic field and the velocity field. Rüdiger et al. (2011) use the cross helicity to estimate the turbulent magnetic diffusivity in a stratified medium with forced turbulence. Rüdiger et al. (2012) extend this method to numerical calculation of thermal convection and observation of the Sun.

In this study, we introduce a new method to estimate the value of turbulent diffusivity. We investigate the development of a passive scalar whose initial condition is the Gaussian function. The method is found to be well suited for a Gaussian function at each time point and its peak density and spatial extent give us necessary information on the scalar’s kinematics. A detailed explanation of the method is given in Section 2.2. The specific aims of this study are: (1) estimation of turbulent diffusivity of thermal convection with different sizes of the simulation box, (2) investigation into the validity of approximation of turbulent diffusion in thermal convection, and (3) investigation into the dependence of turbulent diffusivity and the validity of approximation on the initial distribution scale.

2. MODEL

2.1. Equations

The three-dimensional hydrodynamic equation of continuity, equation of motion, equation of energy, and equation of state are solved in Cartesian coordinates $(x, y, z)$, where $x$ and $y$ denote the horizontal directions and $z$ denotes the vertical direction. The formulations are almost the same as those used by Hotta et al. (2012). Equations are expressed as

$$\frac{\partial \rho_1}{\partial t} = - \nabla \cdot [(\rho_0 + \rho_1) \mathbf{v}],$$  

(1)
\[ \frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\nabla p_1}{\rho_0} - \frac{\rho_1}{\rho_0} \mathbf{g}_z + \frac{1}{\rho_0} \nabla \cdot \Pi, \]

\[ \frac{\partial s_1}{\partial t} = - (\mathbf{v} \cdot \nabla)(s_0 + s_1) + \frac{1}{\rho_0 T_0} \nabla \cdot (K\rho_0 T_0 \nabla s_1) \]

\[ + \frac{\gamma - 1}{\rho_0} (\Pi \cdot \nabla) \mathbf{v}, \]

\[ p_1 = p_0 \left( \frac{\gamma p_1}{\rho_0} + s_1 \right), \]

where \( \rho_0(z), p_0(z), T_0(z), \) and \( s_0(z) \) denote the time-independent, plane-parallel reference density, pressure, temperature, and entropy, respectively, and \( \mathbf{e}_z \) denotes the unit vector along the \( z \)-direction. \( \gamma \) is the ratio of specific heats, with the value for an ideal gas being \( \gamma = 5/3 \). \( \rho_1, p_1, \) and \( s_1 \) denote the fluctuations of density, pressure, and entropy from reference atmosphere, respectively. Note that the entropy is normalized by specific heat capacity at constant volume \( c_v \) and the Prandtl number \( Pr \equiv v/c_P \), the value for an ideal gas being \( Pr = 0.72 \).

\[ \delta \equiv \frac{\delta z}{\delta r} \equiv \frac{\delta}{\delta r}, \]

and \( v \) and \( K \) denote the viscosity and thermal diffusivity, respectively. \( v \) and \( K \) are assumed to be constant throughout the simulation domain.

We assume an adiabatically stratified polytrope for the reference atmosphere except for entropy:

\[ \rho_0(z) = \rho_r \left[ 1 - \frac{z}{(m + 1)H_r} \right]^m, \]

\[ p_0(z) = \rho_r \left[ 1 - \frac{z}{(m + 1)H_r} \right]^{m+1}, \]

\[ T_0(z) = T_r \left[ 1 - \frac{z}{(m + 1)H_r} \right], \]

\[ H_0(z) = \frac{\rho_0}{\rho_0 g}, \]

where \( \rho_r, p_r, T_r, \) and \( H_r \) denote the values of \( \rho_0, p_0, T_0, H_0 \) (the pressure scale height) at the bottom boundary \( z = 0 \). The profile of the reference entropy \( s_0(z) \) is defined with a steady state solution of the thermal diffusion equation \( \nabla \cdot (K\rho_0 T_0 \nabla s_0) = 0 \) with constant \( K \):

\[ \frac{ds_0}{dz} = -\frac{\gamma \delta(z)}{H_0(z)}, \]

\[ \delta(z) = \delta \equiv \frac{\rho_r}{\rho_0 g}(z), \]

where \( \delta \) is the non-dimensional superadiabaticity and \( \delta \) is the value of \( \delta \) at \( z = 0 \). In spite of a non-zero value of superadiabaticity, the adiabatic stratification is acceptable due to the small value of superadiabaticity. The strength of the diffusion coefficients \( \nu \) and \( K \) are expressed with the following non-dimensional parameters: the Reynolds number \( Re \equiv \nu c z / H_r \), and the Prandtl number \( Pr \equiv \nu / K \), where the velocity scale \( \nu_c \equiv (8\delta g H_r)^{1/2} \). In all cases of this study, the parameters are set as \( Re = 300, Pr = 1, \delta = 1 \times 10^{-2} \). We calculate three cases with different box sizes (see Table 1). The horizontal size is the same in all calculations, i.e., \( L_x = L_y = L = 26.16H_r \), and the number of grids in the \( x, y \) directions are set as \( N_x = N_y = 1152 \). We adopt three different vertical sizes of box, \( L_z = 2.18H_r, 1.635H_r, \) and \( 1.09H_r \) for cases 1, 2, and 3, respectively. The number of grids in these cases are set as \( N_z = 96, 72, \) and 48, respectively. The Rayleigh number, which is defined as

\[ R_a \equiv \frac{g H_r^2}{\nu K} \frac{\Delta s}{L_c^2}, \]

in these cases are estimated to be \( 1.3 \times 10^5, 3.4 \times 10^4 \), and \( 1.7 \times 10^4 \), respectively, where \( \Delta s \) denotes the difference of entropy between the top and the bottom boundaries. The calculation domain is \(-L/2 < x < L/2, -L/2 < y < L/2, 0 < z < L_z\). The boundary conditions and the numerical method are the same as those used by Hotta et al. (2012). The boundary condition for the \( x, y \) directions is periodic for all variables, and the stress-free and impenetrative boundary conditions are adopted and the entropy is fixed, i.e., \( s_1 = 0 \) at \( z = 0 \) and \( L_z \).

### 2.2. Method for Estimation of Turbulent Diffusivity

In this study, we calculate the evolution of passive scalar to estimate the value of turbulent diffusivity. Along with Equations (1)–(4), we simultaneously solve the advection equation of the passive scalar as

\[ \frac{\partial Q}{\partial t} = -\nabla \cdot (Q\mathbf{v}), \]

where \( Q \) is passive scalar density. Although in Equation (13), the diffusion term does not appear explicitly, we use tiny artificial viscosity on the passive scalar, a technique which is adopted in (Rempel et al. 2009). Its initial condition is set as

\[ Q(x, y, z, t = 0) = \exp \left( -\frac{x^2 + y^2}{a_0^2} \right). \]

We adopt three initial conditions, i.e., \( d_0 = 2.5H_r, 5.0H_r, \) and \( 7.5H_r \) for each of the different depth settings (cases 1–3); hence, the total number of cases is nine. In the initial condition the passive scalar does not depend on \( z \), since we focus on the turbulent diffusion in the horizontal direction. Since the transport of the passive scalar is assumed to be approximated by a diffusion process with constant diffusivity \( \eta \), then its density should obey the two-dimensional diffusion equation as

\[ \frac{\partial Q}{\partial t} = \eta \left( \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} \right). \]

When the calculation domain is infinite, the analytical solution of Equation (15) with the initial condition of Equation (14) is expressed as

\[ Q = \left( \frac{d_0}{a_0} \right)^2 \exp \left( -\frac{x^2 + y^2}{a_0^2} \right), \]
where, \( d^2 = 4\eta t + d_0^2 \). In this study we adopt periodic boundary conditions; thus, the analytic solution is given by the periodic superposition of the above formula and can be expressed as

\[
Q = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left( \frac{d_0}{d} \right)^2 \exp \left[ -\frac{(x - iL)^2 + (y - jL)^2}{d^2} \right].
\]  (17)

When the width of the Gaussian function is narrower than the box size \( d < L \), the analytical solution in the range, \(-L/2 < x < L/2\) and \(-L/2 < y < L/2\), can be approximated as

\[
Q \sim \sum_{i=-1}^{1} \sum_{j=-1}^{1} \left( \frac{d_0}{d} \right)^2 \exp \left[ -\frac{(x - iL)^2 + (y - jL)^2}{d^2} \right].
\]  (18)

We estimate the value of turbulent diffusivity by the following steps.

1. The advection Equation (13) is calculated with the obtained velocity of thermal convection.

2. The obtained passive scalar in each step is vertically averaged as

\[
\tilde{Q} = \frac{1}{L_z} \int_0^{L_z} Q \, dz.
\]  (19)

Note that by using this method, we will obtain an averaged turbulent diffusivity along the \( z \)-direction.

3. The result of averaging, i.e., Equation (19), is fitted with Equation (18). Note that the fitting has only one parameter \( d(t) \), and this parameter has information on both the height and the width of the Gaussian function.

4. According to the analytical relation, \( d^2 = 4\eta t + d_0^2 \), we obtain the value of turbulent diffusivity from the slope of \( d^2(t) \).

3. RESULTS AND DISCUSSION

Figure 1 shows the results of our hydrodynamic calculation. The three panels in the left, middle, and right columns show the contours of entropy in cases 1, 2, and 3, respectively. Due to the large Rayleigh number, the velocity with a large box size is high (see the fourth row in Table 1). A detailed investigation of
cell size distribution will be reported in our forthcoming paper (Y. Iida et al. 2012, in preparation).

Figure 2 shows the contour of the passive scalar whose Gaussian function width at the initial condition is \(d_0=2.5H_r\). We can see that the passive scalar is diffused with turbulent convection. The dependences of \(d^2\) on time are provided in Figure 3, and are shown to be almost linear. This shows the validity of the diffusive description for the turbulent transport by the convective motion. The estimated turbulent diffusivity is shown in Figure 4(a). It is derived through linear fittings to the curves in Figure 3 in the range of \(0 < t < t_{\text{max}}\), where \(t_{\text{max}}\) is chosen so as to reduce the fitting error; it is given in Table 1.

The scaling behavior of the obtained diffusion is studied by changing the depth of the simulation box in cases 1, 2, and 3. In Figure 4(a), the blue, green, and red lines show the values of turbulent diffusivity with \(d_0=7.5H_r\), \(5.0H_r\), and \(2.5H_r\) respectively. The value of turbulent diffusivity scales with the size of the box, which is discussed in the next paragraph. The value of turbulent diffusivity also scales with the initial width of the Gaussian function \(d_0\). Using a wider Gaussian function makes the larger size of the convection cell work more efficiently and generates a larger value of turbulent diffusivity.

In the mean-field model, it is thought that the coefficient of turbulent diffusion can be expressed as \(\eta = \frac{L_c v_{\text{rms}}}{3}\), where \(L_c\) is the characteristic length scale of turbulence and \(v_{\text{rms}}\) is the root-mean-square (rms) velocity. The value of turbulent diffusivity is obtained in this study, and we can estimate the value of \(L_c\) based on the mean-field model, i.e., \(L_c = 3\eta / v_{\text{rms}}\). The estimated horizontal rms velocities and characteristic length scale are shown in Figures 4(b) and (c), respectively. We discuss...
the dependence of $L_c$ on the box size with $d_0 = 5.0H_r$ and $7.5H_r$. With a smaller box, i.e., $1.635H_r$ (case 2) and $1.09H_r$ (case 3), the characteristic lengths are almost the same as the sizes of the boxes (the size of the box is indicated by the dashed line). It is natural that the largest cell size is determined by the size of the box and that the largest cell is most effective for advecting the passive scalar. Although we expected that the characteristic length of case 1 would also be the same as $L_c$, the obtained characteristic length scale was smaller than $L_c$ even with $d_0 = 5.0H_r$ and $7.5H_r$. A possible reason for this result is that the characteristic length $L_c$ is restricted by the convection cell, which is also limited vertically by the pressure scale height ($H_r$) or the density scale height ($\gamma H_r \sim 1.67H_r$). Although $L_c$ should be evaluated in the horizontal scale, the mixture of the passive scalar may occur at approximately the same distance with the vertical scale. It should also be noted that when the narrowest Gaussian function, i.e., $d_0 = 2.5H_r$ (red line), is used, the characteristic lengths are restricted by the width of the Gaussian function in cases 1 and 2.

Next, we discuss the validity of the approximation of turbulent diffusivity quantitatively. We calculate the estimated error of the linear fitting of $d^2$ as

$$\sigma = \sqrt{\frac{1}{N-2} \sum_{n=1}^{N} \left( \frac{d^2(t_n) - d^2(t_{n0})}{d^2(t_{n0})} \right)^2}, \quad (20)$$

where $N$ is the number of data points along the time, $d^2(t_n)$ is the $n$th estimated result, and $d^2(t_{n0})$ is the $n$th result of the fitted line. In Figure 4(d), we found a dependence of $\sigma$ on $d_0$, i.e., $\sigma$ is larger with narrower $d_0$. Although the qualitative relation is not clear, it indicates that the estimated error $\sigma$ tends to become smaller with a larger ratio $d_0/L_c$ of the initial width of Gaussian function to the characteristic length ($L_c$).

4. SUMMARY

We investigated the value of horizontal turbulent diffusivity $\eta$ by a numerical calculation of thermal convection. In this study, we have introduced a new method, whereby the turbulent diffusivity is estimated by monitoring the time development of the passive scalar, which is initially distributed in a given Gaussian function with a spatial scale $d_0$. Our conclusions are as follows: (1) assuming the relation $\eta = L_c v_{\text{rms}}/3$ where $v_{\text{rms}}$ is the rms velocity, the characteristic length $L_c$ is restricted by the shortest one among the pressure (density) scale height and the region depth. (2) The value of turbulent diffusivity becomes larger with a larger initial distribution scale $d_0$. (3) The approximation of turbulent diffusion holds better when the ratio of the initial distribution scale $d_0$ to the characteristic length $L_c$ is larger.

Conclusion (2) is consistent with the results of observational study (Chae et al. 2008; Abramenko et al. 2011) and a previous
In this study, we do not estimate the correlation length directly from the thermal convection. This will be achieved in our future work with an auto-detection technique and our characteristic length ($L_c$) will be compared with directly estimated correlation length. We now assume that our characteristic length is an average of correlation length at each height (Y. Iida et al. 2012, in preparation). The turbulent diffusion in the horizontal directions is estimated in this work, but such estimations are also important in the vertical directions for addressing the solar dynamo problem from the viewpoint of the transport of magnetic flux from the surface to the bottom of the convection zone. Such a study will be conducted in the future.

Although turbulent diffusivity averaged in the whole box is estimated in this study, the dependence of this estimation on the height is important. There are, however, two reasons why it is difficult to estimate this dependence with our method. First, in our calculations the integrated passive scalar density is not conserved at each height. Second, we found that it is difficult to estimate the diffusivity separately for each horizontal plane only by solving Equation (13) two-dimensionally in the $x-y$ plane because the results show that the passive scalar density is strongly concentrated in the boundaries of the convection cells. Such a spatially intermittent structure is inappropriate for obtaining a statistical property like the turbulent diffusivity. These difficulties will necessitate some substantial improvements in our method. We are also interested in the effect of feedback from the magnetic field to the convection because of its influence on the turbulent diffusivity (e.g., Yousef et al. 2003; Rüdiger et al. 2011).

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REFERENCES

Abramenko, V. I., Carbone, V., Yurchyshyn, V., et al. 2011, ApJ, 743, 133
Brandenburg, A. 2005, Astron. Nachr., 326, 787
Brandenburg, A. 2008, Astron. Nachr., 329, 725
Cameron, R., Vogler, A., & Schüssler, M. 2011, A&A, 533, A86
Chae, J., Litvinenko, Y. E., & Sakurai, T. 2008, ApJ, 683, 1153
Choudhuri, A. R., Chatterjee, P., & Jiang, J. 2007, Phys. Rev. Lett., 98, 131103
Dikpati, M., & Charbonneau, P. 1999, ApJ, 518, 508
Dikpati, M., & Gilman, P. A. 2006, ApJ, 649, 498
Hotta, H., Rempel, M., Yokoyama, T., Iida, Y., & Fan, Y. 2012, A&A, 539, A30
Hotta, H., & Yokoyama, T. 2010a, ApJ, 709, 1009
Hotta, H., & Yokoyama, T. 2010b, ApJ, 714, L508
Hotta, H., & Yokoyama, T. 2011, ApJ, 740, 12
Käpylä, P. J., Korpi, M. J., & Brandenburg, A. 2009, A&A, 500, 633
Miesch, M. S. 2005, Living Rev. Sol. Phys., 2, 1
Rempel, M. 2007, ApJ, 655, 651
Rempel, M., Schüssler, M., & Knöllker, M. 2009, ApJ, 691, 640
Rüdiger, G., Kitchatinov, L. L., & Brandenburg, A. 2011, Sol. Phys., 269, 3
Rüdiger, G., Kueker, M., & Schnerr, R. S. 2012, arXiv:1202.1429
Schrinner, M., Rädler, K.-H., Schmitt, D., Rheinhardt, M., & Christensen, U. 2005, Astron. Nachr., 326, 245
Wang, Y., Nash, A. G., & Sheeley, N. R., Jr. 1989, ApJ, 347, 529
Yeates, A. R., Nandy, D., & Mackay, D. H. 2008, ApJ, 673, 544
Yousef, T. A., Brandenburg, A., & Rüdiger, G. 2003, A&A, 411, 321