Cohomological tautness for Riemannian foliations

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30 Mai 2008

To Nicolae Teleman on his 65th birthday.

Abstract

In this paper we present some new results on the tautness of Riemannian foliations in their historical context. The first part of the paper gives a short history of the problem. For a closed manifold, the tautness of a Riemannian foliation can be characterized cohomologically. We extend this cohomological characterization to a class of foliations which includes the foliated strata of any singular Riemannian foliation of a closed manifold.

Y. Carrière in his paper, cf. [5], based on his Ph.D. thesis conjectured that for Riemannian foliations of compact manifolds, the property “taut” understood as the existence of a Riemannian bundle-like metric making all leaves minimal is equivalent to the non-triviality of the top dimension basic cohomology group. The conjecture was based on the previous results of A. Haefliger, cf. [16], demonstrating that “being taut” is a transverse property and on his own research into Riemannian flows on 3-manifolds. For over a decade the conjecture was the subject of intensive study by a group of “feuilleteurs”, being finally solved by X. Masa, cf. [22], and refined by J.A. Álvarez, [1]. The best account of the development of the theory up to 1995 can be found in Ph. Tondeur’s book, cf. [43].

The case of non-compact manifolds is much more complicated as the tautness class of a Riemannian foliation cannot be defined in the standard way as in the case of closed manifolds, cf. [7]. However, for some non-compact manifolds it is possible to propose a similar characterization. In a previous paper we proved that if a foliated Riemannian manifold \((M, g, \mathcal{F})\) can be embedded as a regular stratum of a singular Riemannian foliation (SRF), then the following conditions are equivalent:

1) \(\mathcal{F}\) is taut;

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2) $\kappa = 0$, where $\kappa = [\kappa_\mu] \in H^1(M/F)$, and $\kappa_\mu$ is the mean curvature form of the bundle-like Riemannian metric $\mu$;

3) $H^0_{\kappa_\mu}(M/F) \neq 0$, where $\mu$ is a bundle-like Riemannian metric;

4) $H^n_\kappa(M/F) \neq 0$, where $n = \text{codim } F$ and the foliation is transversally oriented.

In this paper we extend this characterization to a class of non-compact foliated Riemannian manifolds which include not only regular strata of SRFs, but other strata as well (cf. Theorem 3.2, 3.3 and 3.5).

In the sequel $M$ and $N$ are connected, second countable, Hausdorff, without boundary and smooth (of class $C^\infty$) manifolds of dimension $m$. All the maps are considered smooth unless something else is indicated. We consider on $M$ a Riemannian foliation whose codimension is $n$. If $V$ is a saturated submanifold of $M$ we shall denote by $(V, F)$ the induced foliated manifold and $F|_V$ the induced Riemannian foliation.

1 An historical overview of the problem

An involutive subbundle $E$ of dimension $p$ of $TM$ is called a foliation of dimension $p$ and codimension $n = m - p$. The foliation $F$ is said to be modelled on a $n$-manifold $N_0$ if it is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_I$ modelled on $N_0$, i.e.

1. $\{U_i\}$ is an open covering of $M$,
2. $f_i: U_i \rightarrow N_0$ are submersions with connected fibres, and
3. $g_{ij}f_j = f_i$ on $U_i \cap U_j$.

The $n$-manifold $T = \bigsqcup_i T_i$, $T_i = f_i(U_i)$, is called the transverse manifold associated to the cocycle $\mathcal{U}$ and the pseudogroup $\mathcal{H}$ of local diffeomorphisms of $T$ generated by $g_{ij}$ the holonomy pseudogroup representative on $T$ (associated to the cocycle $\mathcal{U}$). $T$ is a complete transverse manifold. The equivalence class of $\mathcal{H}$ we call the holonomy pseudogroup of $F$ (or $(M, F)$). It is not difficult to check that different cocycles defining the same foliation provide us with equivalent holonomy pseudogroups, cf. [17, 18]. In general, the converse is not true. The notion of a Riemannian foliation was introduced by Bruce Reinhart in [30, 31].

A foliation $F$ on the smooth manifold $M$ is Riemannian if on $M$ there exists a bundle-like metric $\mu$ for the foliation $F$, (i.e., a geodesic perpendicular to a leaf of $F$ at a point remains perpendicular to every leaf it meets). In a local adapted chart $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ the bundle-like metric $\mu$ has a representation

$$\sum_{ij=1}^p \mu_{ij}(x, y)v_i \otimes v_j + \sum_{\alpha\beta=1}^n \mu_{\alpha\beta}(y)dy_\alpha \otimes dy_\beta$$

where $v_i$ is a 1-form annihilating the bundle $T\mathcal{F}^\perp$ and $v_i(\partial/\partial x_j) = \delta^i_j$.

Let $(M, F)$ be a Riemannian foliation with a bundle-like metric $\mu$. Then it is defined by a cocycle $\mathcal{U} = \{U_i, f_i, k_{ij}\}_I$ modelled on a Riemannian manifold $(N_0, \bar{g})$ such that

\[\text{1For the notions related with Riemannian foliations we refer the reader to [27, 43].} \]
(i) \( f_i : (U_i, \mu) \rightarrow (T_0, \bar{\mu}) \) is a Riemannian submersion with connected fibres;
(ii) \( k_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j) \) are local isometries of \((T_0, \bar{\mu});\)
(iii) \( f_j = k_{ji}f_i \) on \( f_i(U_i \cap U_j) \).

A foliation \( \mathcal{F} \) on a Riemannian manifold \((M, \mu)\) is said to be minimal if all its leaves are minimal submanifolds of \((M, \mu)\). A foliation \( \mathcal{F} \) on a manifold \( M \) is said to be taut if there exists a Riemannian metric \( \mu \) on the manifold \( M \) for which all leaves are minimal submanifolds of \((M, \mu)\).

Among other things B. Reinhart introduced and studied the basic cohomology of these foliations.

In the presence of the Riemannian metric \( \mu \), the tangent bundle \( TM \) admits an orthogonal splitting \( TM = T\mathcal{F} \oplus T\mathcal{F}^\perp \). We say that the \( k \)-form \( \alpha \) is of pure type \((r, s)\), \( r+s=k \), if for any point of \( M \) there exists an adapted chart \((x_1, \ldots, x_p, y_1, \ldots, y_n)\) such that

\[
\alpha = \sum f_{IJ} v_{i_1} \wedge \cdots \wedge v_{i_r} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_s},
\]

where \( 1 \leq i_1 < \cdots < i_r \leq p, 1 \leq j_1 < \cdots < j_s \leq n, I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_s) \).

Let us denote by \( \Omega^k(M) \) the space of \( k \)-forms \( M \), and by \( \Omega^{r,s}(M) \) the space of forms of pure type \((r, s)\). Then

\[
\Omega^k(M) = \bigoplus_{r+s=k} \Omega^{r,s}(M),
\]

for short \( \Omega^k = \bigoplus_{r+s=k} \Omega^{r,s} \).

The exterior differential \( d : \Omega^k(M) \rightarrow \Omega^{k+1}(M) \) decomposes itself into three components

\[
d = d_{\mathcal{F}} + d_T + \delta,
\]

where \( d_{\mathcal{F}} \) is of bidegree \((1,0)\) and \( d_T \) is of bidegree \((0,1)\), and \( \delta \) is of bidegree \((-1,2)\), i.e.

\[
d_{\mathcal{F}} : \Omega^{r,s} \rightarrow \Omega^{r+1,s}, \quad d_T : \Omega^{r,s} \rightarrow \Omega^{r,s+1}, \quad \text{and} \quad \delta : \Omega^{r,s} \rightarrow \Omega^{r-1,s+2}.
\]

In this work, we shall use three types of cohomologies.

(a) The basic cohomology \( H^\ast(M/\mathcal{F}) \) is the cohomology of the complex \( \Omega^\ast(M/\mathcal{F}) \) of basic forms. A differential form \( \omega \) is basic when \( i_X\omega = i_Xd\omega = 0 \) for every vector field \( X \) tangent to \( \mathcal{F} \). The complex \( \Omega^\ast(M/\mathcal{F}) \) can be identified with the complex of holonomy invariant forms on the transverse manifold \( T - \Omega^\ast_{\mathcal{H}}(T) \).

(b) The compactly supported basic cohomology \( H_c^\ast(M/\mathcal{F}) \) is the cohomology of the basic subcomplex \( \Omega_c^\ast(M/\mathcal{F}) = \{ \omega \in \Omega^\ast(M/\mathcal{F}) \mid \text{the support of } \omega \text{ is compact} \} \).

(c) The twisted basic cohomology \( H^\ast_{\kappa}(M/\mathcal{F}) \), relatively to the cycle \( \kappa \in \Omega^1(M/\mathcal{F}) \), is the cohomology of the basic complex \( \Omega^\ast(M/\mathcal{F}) \) relatively to the differential \( d_{\kappa} \omega = d\omega - \kappa \wedge \omega \). This cohomology does not depend on the choice of the cycle: we have \( H^\ast_{\kappa}(M/\mathcal{F}) \cong H^\ast_{\kappa+\delta}(M/\mathcal{F}) \) through the isomorphism: \( [\omega] \mapsto [e^\delta \omega] \).
1.1 Example (E. Ghys, [13]). Consider the unimodular matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ inducing a diffeomorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $T^3_A$ be the torus bundle over $S^1$ determined by $A$ and $\mathcal{F}$ be the flow obtained by suspending $A$. Then, the basic cohomology $H^2(T^3_A/\mathcal{F})$ is infinite dimensional as basic forms correspond to $A$-invariant forms on $T^2$ - i.e., 1-forms are of the form $f(x)dx$, thus closed, and 2-forms are of the form $u(x)dx \wedge dy$.

In [31], Reinhart claimed that the basic cohomology of a Riemannian foliated closed manifold is finite dimensional and satisfies the Poincaré duality property:

$$H^k(M/\mathcal{F}) \cong H^{n-k}(M/\mathcal{F}).$$

Soon it became apparent that the proof was not rigorous and contained some gaps. For a long time it remained an open problem. Only the beginnings of the eighties brought some striking new developments. In addition to Sullivan's characterization of taut foliations (A foliation is taut if and only if no foliation cycle is the limit of boundaries of tangent chains., cf. [41]) and H. Rummler's thesis, [35], A. Haefliger published in 1980, perhaps the most influential result of this theory, cf. [16] - the property of being "taut" is a transverse property, i.e. it depends only the properties of the holonomy pseudogroup. Then F. Kamber and Ph. Tondeur proved their correct version of the Poincaré duality property for taut Riemannian foliations on closed manifolds, cf. [20, 21]. However, it was not until Y. Carrière's thesis (1981) that a counterexample was found. The thesis presented the classification of 1-dimensional tangentially orientable Riemannian foliations (flows) on closed 3-manifolds. He found some flows which are not defined by a Killing vector field, and these flows are characterized by the property that the 2-dimensional basic cohomology is trivial, cf. [5].

1.2 Flows. Let us begin with Carrière's example. Let $A$ be a matrix of $SL_2(\mathbb{Z})$ with trace greater than 2 with two different eigenvalues $\lambda$ and $\frac{1}{\lambda}$ with two corresponding eigenvectors $v_1$ and $v_2$, respectively. By $T^3_A$ we denote the 3-manifold obtained by suspending $A$, i.e. it is a $T^2$-fibre bundle over $S^1$. In fact, it is obtained as the quotient space of $T^2 \times \mathbb{R}$ by the equivalence relation generated by the identification of $(m, t)$ with $(A(m), t + 1)$. The lines parallel to eigenvectors $v_1$ and $v_2$ define $A$-invariant foliations (flows) $\Phi_1$ and $\Phi_2$, respectively, on $T^2$. They, in turn, induce flows on $T^3_A$ which we denote by the same letters. Each flow is dense in the tori which form the fibres of our 3-manifold $T^3_A$. One can show, cf. [5], that the flow $\Phi_2$ on $T^3_A$ is a transversally Lie modelled on the affine group $GA$ of the real line.

As the flow is transversally Lie, there exists a developing mapping $D: \mathbb{R}^3 \rightarrow GA$ ($\mathbb{R}^3$ is the universal covering of $T^3_A$) such that the fibers of $D$ are the leaves of the lifted flow. Moreover, there exists a homomorphism of groups $h: \pi_1(T^3_A) \rightarrow GA$ and its image is called the holonomy group $\Gamma$ of the foliation. Global basic forms on $(T^3_A, \Phi_2)$ correspond to $\Gamma$-invariant forms on $GA$, thus to $K = \Gamma$-invariant forms, where $K$ is the closure of the group $\Gamma$ in $GA$. Therefore the basic cohomology of the foliated manifold $(T^3_A, \Phi_2)$ is isomorphic the cohomology of the complex of $K$-invariant forms on $GA$. If we identify the group $GA$ with the group $\mathbb{R}^2$ with the product given by the formula $(t, s)(t', s') = (t + t', \lambda s' + s)$, then the group $K$ can be identified with the group $\{(n, s) : n \in \mathbb{Z}, s \in \mathbb{R}\}$.

These considerations permit us to show that $H^2(T^3_A/\Phi_2) = 0$. To prove that fact we have to show that any $K$-left invariant 2-form on $GA$ is exact. The 1-forms $\alpha = dt$ and $\beta = \frac{ds}{s}$ are left-invariant. A smooth function if $K$ invariant if it does not depend on the variable $s$ and $f(t) = f(t + 1)$ for any real number $t$. Hence a one form $\omega$ is $K$-invariant iff $\omega = f\alpha + g\beta$ and both functions $f$ and $g$ are $K$-invariant. A $K$-invariant 2-form $\Omega$ can be written as $\Omega = h\alpha \wedge \beta$ where $h$
is a $K$-invariant function. We have to demonstrate that for any $K$-invariant function $h$ there exist $K$-invariant functions $f$ and $g$ such that $d(f\alpha + g\beta) = h\alpha \wedge \beta$, or equivalently $g'(t) + g(t)\log \lambda = h(t)$ for any real number $t$.

If we assume that $g(t) = \lambda^{-t}g_1(t)$, then we have to find a function $g_1$ such that $g_1'(t)\lambda^{-t} = h(t)$. But such a function is given by integration:

$$g_1(t) = c + \int_0^t \lambda^x h(x) dx$$

where $c$ is a real constant. And thus

$$g(t) = \lambda^{-t}(c + \int_0^t \lambda^x h(x) dx)$$

To get the invariance condition $g(t) = g(t+1)$ we need $c = \frac{1}{\lambda-1} \int_0^1 \lambda^x h(x) dx$ which is always possible as $\lambda \neq 1$.

Moreover, it is not difficult to show that the flow $\Phi_2$ is not isometric.

This example should be seen in the light of the following proposition, cf. [43, Proposition 6.6].

**Proposition 1.2.1** Let $F$ be a flow defined by a nonsingular vector field $V$ (with the normalized vector field $W = \frac{1}{|V|}V$) on a Riemannian manifold $(M, \mu)$. Then the following conditions are equivalent:

(i) all leaves of $F$ are minimal submanifolds of $(M, \mu)$, i.e. the foliation is minimal;

(ii) the orbits of $V$ are geodesics;

(iii) $\theta(W)\chi_\mu = 0$;

(iv) $\nabla_W W = 0$, where $\nabla$ is the Levi-Civita connection of $(M, \mu)$.

The combined effort of H. Gluck, D. Sullivan, cf. [14, 41], can be summarized in the following theorem, cf. [43, Proposition 6.7]. The equivalence of the fifth condition is due to Y. Carrièere, cf. [5].

**Theorem 1.2.2** Let $F$ be a flow given by the nonsingular vector field $V$ on an $m$-manifold $M$. Then the following conditions are equivalent:

(i) there exists a Riemannian metric on $M$ making the orbits of $V$ geodesics and $V$ of unit length;

(ii) there exists a 1-form $\chi \in \Omega^1(M)$ such that $\chi(V) = 1$ and $\theta(V)\chi = 0$;

(iii) there exists a 1-form $\chi \in \Omega^1(M)$ such that $\chi(V) = 1$ and $i_V d\chi = 0$;

(iv) there exists an $(m-1)$-plane subbundle $E \subset TM$, complementary the flow such that $[V, X]$ is a section of $E$ for any section $X$ of $E$;

(v) there exists a Riemannian metric on $M$ for which $V$ is a Killing vector field.
1.3 Tautness and basic cohomology. In addition to all these deep results a simple remark seems to point to a close relation between tautness and basic cohomology, cf. [43, Theorem 4.32].

Theorem 1.3.1 Let $\mathcal{F}$ be a transversally oriented Riemannian foliation of codimension $n$ of a Riemannian manifold $(M, \mu)$ whose leaves are minimal. Then the basic cohomology class of the transverse volume $\nu$ form is non-trivial.

Proof. Assume that there exists a basic form $\alpha$ such that $d\alpha = \nu$. Let $\chi_\mu$ be the volume form along the leaves of the foliation. Then

$$d(\alpha \wedge \chi_\mu) = d\alpha \wedge \chi_\mu + (-1)^{n-1} \alpha \wedge d\chi_\mu$$

The minimality assumption implies that the form $d\chi_\mu$ is of degree $(p-1, 2)$, so as the form $\alpha$ is of degree $(0, n-1)$ the form $\alpha \wedge d\chi_\mu$ vanishes. Therefore the form $\nu \wedge \chi_\mu$ which is a volume form of the manifold $M$ is exact, a contradiction.

Moreover, in [21], F. Kamber and Ph. Tondeur proved that the basic cohomology of a taut Riemannian foliation of a closed manifold is finite dimensional and satisfies the PD property. The above results and Haefliger’s theorem, [16], which assured that the existence of a Riemannian metric making all leaves minimal is a transverse property made possible in 1982 the formulation of the following conjecture by Y. Carrièere first expressed for flows:

Conjecture Let $\mathcal{F}$ be a Riemannian foliation of a closed Riemannian manifold $(M, \mu)$. Then there exists a (bundle-like) Riemannian metric making all leaves minimal (i.e., the foliation is taut) iff the top dimensional basic cohomology is non-trivial.

For flows the conjecture was solved by P. Molino and V. Sergiescu in 1985, cf. [29]:

Theorem 1.3.2 Let $\mathcal{F}$ be a Riemannian flow on a closed oriented $m$-manifold $M$. Then there exists a Riemannian metric for which $\mathcal{F}$ is an isometric flow iff the top dimensional basic cohomology is non-trivial.

However, at that time the solution of the conjecture was far away.

First, G. Hector and his students A. El Kacimi, and V. Sergiescu proved that the basic cohomology of a Riemannian closed foliated manifold $(M, g, \mathcal{F})$ is finite dimensional, cf. [11], then they developed the Hodge theory, first studied by F. Kamber and Ph. Tondeur in [20, 21], for basic forms and showed that the basic cohomology has the PD property iff the top dimensional basic cohomology is non-trivial, cf. [10].

Theorem 1.3.3 Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold $M$. Then the following two conditions are equivalent:

(i) $H^n(M/\mathcal{F}) \neq 0$;

(ii) the basic cohomology $H^*(M/\mathcal{F})$ satisfies the Poincaré duality property.

This theorem together with Kamber-Tondeur’s result mentioned at the beginning of the subsection strongly hinted that Carrièere’s intuition was correct. Finally in 1991, X. Masa, [22], showed the tautness is equivalent to the non-triviality of the top-dimensional basic cohomology, solving the conjecture positively.
Theorem 1.3.4 Let $\mathcal{F}$ be a transversally oriented Riemannian foliation of a closed manifold $M$. Then there exists a Riemannian metric on $M$ for which all leaves are minimal iff the top-dimensional basic cohomology $H^n(M/\mathcal{F})$ is non-trivial.

To complete the story of basic cohomology let us mention that in 1993 A. El Kacimi and M. Nicolau proved that the basic cohomology of a closed Riemannian foliated manifold is a topological invariant, cf. [12].

1.4 Mean curvature form In the story of the proof of the tautness conjecture a certain 1-form turned out to be of great importance.

For a foliation $\mathcal{F}$ of a Riemannian manifold $(M, \mu)$, we define the shape operator $W$ of the leaves using the natural splitting of the tangent bundle

$$TM = T\mathcal{F} \oplus T\mathcal{F}^\perp.$$ 

In fact, for any section $Y$ of $T\mathcal{F}^\perp$ and any tangent vector field $X$, we have

$$W(Y)(X) = -\pi^\perp(\nabla_X Y),$$

where $\pi^\perp : TM \to T\mathcal{F}^\perp$ is the orthogonal projection.

The trace of $W$ is linear in $Y$, so it defines a section of $T\mathcal{F}^\perp$. We extend it to a global 1-form $\kappa_\mu$ on $M$:

$$\kappa_\mu(X) = \begin{cases} \text{trace } W(X) & \text{if } X \in T\mathcal{F}^\perp \\ 0 & \text{if } X \in T\mathcal{F}. \end{cases}$$

The 1-form is called the mean curvature 1-form of $\mathcal{F}$ on the Riemannian manifold $(M, \mu)$.

If $f : (M_1, \mathcal{F}_1) \to (M, \mathcal{F})$ is a foliated imbedding between two Riemannian manifolds with $f(M_1)$ saturated in $M$ and $\dim \mathcal{F}_1 = \dim \mathcal{F}$, then

(1) $f^*\mu$ is a bundle-like metric on $(M_1, \mathcal{F}_1)$ and $f^*\kappa_\mu = \kappa_{f^*\mu}$.

So,

(2) if $U$ is an open subset of $M$ then $(\kappa_\mu)|_U = \kappa_{\mu|_U}$.

This form is of particular interest. In [20], the authors proved that if the form $\kappa_\mu$ is basic, then it is closed. So it defines a 1-basic cohomology class $[\kappa_\mu]$ which proved to be of importance in the study of taut foliations as, cf. [20],

Proposition 1.4.1 Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold $M$ with a bundle-like metric $\mu$ for which $\kappa_\mu$ is basic and $[\kappa_\mu] = 0$. Then bundle-like metric $\mu$ can be modified along the leaves to a bundle-like metric $\mu'$ for which all leaves of $\mathcal{F}$ are minimal.

Proof: Since $[\kappa_\mu] = 0$, there exists a smooth basic function $f$ on $(M/\mathcal{F})$ such that $\kappa_\mu = df$. Put $\lambda = e^f$ and modify the metric $\mu$ as follows

$$\mu' = \lambda^2 \mu_{\mathcal{F}} \oplus \mu^\perp.$$
where $p$ is the dimension of leaves, $\mu_F$ and $\mu^\perp$ is the Riemannian metric induced on leaves of $\mathcal{F}$ and the orthogonal subbundle, respectively. The splitting is the splitting defined by the metric $\mu$. The mean curvature form $\kappa_\mu'$ is equal to $\kappa_\mu - d\log \lambda = 0$.

Let $\mathcal{F}$ be a tangentially oriented foliation. We define the characteristic form $\chi_\mu$, a $p$-form, as follows:

for any $p$-tuple $(Y_1, ..., Y_p)$ of vectors of $T_xM$

$$\chi_\mu(Y_1, ..., Y_p) = \det(\mu(Y_i, E_j))_{ij}$$

where $i, j = 1, ..., p$ and $E_1, ..., E_p$ is an oriented orthonormal frame of $T_x\mathcal{F}$.

There is a close relation between the characteristic form and the mean curvature form. Namely, cf. [35],

**Theorem 1.4.2** Let $\mathcal{F}$ be a tangentially oriented foliation of a Riemannian manifold $(M, \mu)$, $\chi_\mu$ its characteristic form and $\kappa_\mu$ its mean curvature form. Then, for any vector field $Y$ orthogonal to the foliation, we have:

$$\theta(Y)\chi_\mu = -\kappa_\mu(Y)\chi_\mu + \beta$$

where $\beta$ is a $p$-form of type $(p - 1, 1)$.

As a corollary we get the following:

**Corollary 1.4.3** A tangentially oriented foliation $\mathcal{F}$ is taut iff for any vector field $Y$ orthogonal to the foliation the form $\theta(Y)\chi_\mu$ is of type $(p - 1, 1)$, which is equivalent to the condition that $d\chi_\mu$ is of type $(p - 1, 2)$, that is for any vector $Y$ and any vectors $(Y_1, ..., Y_p)$ tangent to the foliation

$$d\chi_\mu(Y, Y_1, ..., Y_p) = 0.$$  

The research into the tautness conjecture concentrated on the study of the basic cohomology and the mean curvature form.

The following theorem, cf. [20, 21], gave further evidence that the tautness, the mean curvature class and the PD property for basic cohomology are linked in some way. And the result of A. El Kacimi and G. Hector suggested that the non-vanishing of the top dimensional basic cohomology can be related to the tautness of the foliation, i.e. the vanishing of the mean curvature form.

**Theorem 1.4.4** Let $\mathcal{F}$ be a transversally oriented Riemannian foliation foliation on a closed oriented manifold $M$. Let $g$ be a bundle-like metric with basic mean curvature form. Then the pairing $\alpha \otimes \beta \rightarrow \int_M \alpha \wedge \beta \wedge \chi_\mu$ induces a non-degenerate pairing

$$H^\tau(M/\mathcal{F}) \otimes H^{n-\tau}_{\kappa_\mu}(M/\mathcal{F}) \rightarrow \mathbb{R}$$

of finite-dimensional vector spaces.
In the development of the theory Álvarez López’s paper [1] of 1992 proved to be of great interest. In the paper, Álvarez demonstrates that the space of smooth forms $\Omega(M)$ on a foliated closed Riemannian manifold $(M, \mu, \mathcal{F})$ can be decomposed as the direct sum of $\Omega(M/\mathcal{F})$ of basic forms and its orthogonal complement $\Omega(M/\mathcal{F})^\perp$. Therefore the mean curvature form $\kappa_\mu$ of $(M, \mu, \mathcal{F})$ can be decomposed into the basic component $\kappa_{\mu,b}$ and the orthogonal one. The 1-form $\kappa_{\mu,b}$ is closed and it defines the 1-basic cohomology class $\kappa = [\kappa_{\mu,b}]$, which does not depend on $\mu$. Moreover, Álvarez proves that any form cohomologous to $\kappa_{\mu,b}$ (in the complex of basic forms) can be realized as the basic component of the mean curvature form of some bundle-like metric of $\mathcal{F}$ with the same transverse Riemannian metric. Additionally, one can verify that changing the orthogonal complement of $\mathcal{F}$ does not change the form $\kappa_{\mu,b}$.

As an application, the assumption of the orientability of $M$ in the original formulation of Theorem 1.3.4 is removed.

For some time the condition that the mean curvature form is basic seemed to be a major obstacle to the existence of such a Riemannian metric. But at last in 1995 D. Domínguez published his theorem stating that, [8, 9],

**Theorem 1.4.5** Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold $M$. Then there exists a bundle-like metric for $\mathcal{F}$ for which the mean curvature form is basic.

These Riemannian metrics are very important in the remaining part of the paper. Therefore, a bundle-like metric for which the mean curvature form is basic we call a $D$-metric.

This Theorem together with Proposition 1.4.1 ensures that the “taut” Riemannian metric can be chosen to be a $D$-metric. In the sequel we shall use the following fact:

(3) if $U$ is a saturated open subset of $M$ such that $\mu|_U$ is a $D$-metric then $\kappa_{\mu,b}|_U = \kappa_{\mu|_U}$.

The final characterization of taut Riemannian foliations of closed manifolds can be summarized in the following theorem, cf. [43, 7.56]:

**Theorem 1.4.6** Let $\mathcal{F}$ be a transversally oriented Riemannian foliation foliation on a closed oriented Riemannian manifold $(M, \mu)$. Then $H^n(\mathcal{F}) \cong \mathbb{R}$ and the following conditions are equivalent:

(i) $H^n(\mathcal{F}) \cong \mathbb{R}$,
(ii) $\mathcal{F}$ is taut;
(iii) $\kappa = 0$;
(iv) $H^0(\mathcal{F}) = \mathbb{R}$.

Moreover, then the basic cohomology of the foliated manifold $(M/\mathcal{F})$ has the Poincaré duality property.

1.5 Open manifolds The theory has not been well-developed for open manifolds. We have a fine and very general version of Poincaré duality theorem published by V. Sergiescu in 1985, cf. [38]. Then in 1997, Cairns and Escobales presented a very interesting example, cf. [7], of a Riemannian foliation on an open manifold for which the mean curvature form is basic but not closed.
1.5.1 The SRFs. A singular Riemannian foliation \(^2\) (SRF for short) on a connected manifold \(X\) is a partition \(\mathcal{K}\) by connected immersed submanifolds, called *leaves*, verifying the following properties:

I- The module of smooth vector fields tangent to the leaves is transitive on each leaf.

II- There exists a Riemannian metric \(\nu\) on \(N\), called *adapted metric*, such that each geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

The first condition implies that \((X, \mathcal{K})\) is a singular foliation in the sense of [40] and [42]. Notice that the restriction of \(\mathcal{K}\) to a saturated open subset produces a SRF. Each (regular) Riemannian foliation (RF in short) is a SRF, but the first interesting examples are the following:

- The orbits of the action by isometries of a Lie group.

- The closures of the leaves of a regular Riemannian foliation.

1.5.2 Stratification. Classifying the points of \(X\) by the dimension of the leaves one gets a stratification \(S_{\mathcal{K}}\) of \(X\) whose elements are called strata. The restriction of \(\mathcal{K}\) to a stratum \(S\) is the RF \(\mathcal{K}_S\). The strata are ordered by: \(S_1 \preceq S_2\) \(\iff\) \(S_1 \subset \overline{S_2}\). The minimal (resp. maximal) strata are the closed strata (resp. open strata). We shall denote by \(S_{\min}\) the union of the closed strata. Since \(X\) is connected, there is just one open stratum, denoted \(\mathcal{R}_{\mathcal{K}}\). It is a dense subset. This is the *regular stratum*, the other strata are the *singular strata*.

The depth of \(S_{\mathcal{K}}\), written depth \(S_{\mathcal{K}}\), is defined to be the largest \(i\) for which there exists a chain of strata \(S_0 \prec S_1 \prec \cdots \prec S_i\). So, depth \(S_{\mathcal{K}} = 0\) if and only if the foliation \(\mathcal{K}\) is regular. The *depth* of a stratum \(S \in \mathcal{H}_{i}\), written depth \(\mathcal{H}_{i} S\), is defined to be the largest \(i\) for which there exists a chain of strata \(S_0 \prec S_1 \prec \cdots \prec S_i = S\).

The basic cohomology of such foliations on closed manifolds is finite dimensional and it is a topological invariant, cf. [44]. However, as far as the tautness property is concerned the situation is totally different.

1.5.3 Example. Let us consider the isometric action \(\Phi: \mathbb{R} \times S^{2d+2} \to S^{2d+2}\) given by the formula

\[
\Phi(t, (z_0, \ldots, z_d, x)) = (e^{a_0 \pi it} \cdot z_0, \ldots, e^{a_d \pi it} \cdot z_d, x),
\]

with \((a_0, \ldots, a_d) \neq (0, \ldots, 0)\). Here, \(S^{2d+2} = \{(z_0, \ldots, z_d, x) \in \mathbb{C}^{d+1} \times \mathbb{R} \mid |z_0|^2 + \cdots + |z_d|^2 + x^2 = 1\}\). There are two singular strata: the north pole \(S_1 = (0, \ldots, 0, 1)\) and the south pole \(S_2 = (0, \ldots, -1)\). The regular stratum is \(S^{2d+1} \setminus [-1, 1]\). Let \(r\) be the variable of \([-1, 1]\). The basic cohomology \(H^*(S^{2d+2}/\mathcal{F})\) of the foliation is

\[
\begin{array}{cccccccc}
  i = 0 & i = 1 & i = 2 & i = 3 & i = 4 & i = 5 & \cdots & i = 2d & i = 2d + 1 \\
 1 & 0 & 0 & |dr \wedge e| & 0 & |dr \wedge e^2| & \cdots & 0 & |dr \wedge e^d|
\end{array}
\]

where \(e \in \Omega^2(S^{2d+2}/\mathcal{F})\) is an Euler form.

The top dimensional basic cohomology group is isomorphic to \(\mathbb{R}\), but this cohomology does not have the Poincaré duality property in spite of the fact that the flow is isometric. And, of course, the foliation is not minimal for any adapted (bundle-like) Riemannian metric.

\(^2\)For the notions related with singular Riemannian foliations we refer the reader to [2, 26, 27, 28].
Moreover, in [24], the authors proved that a singular foliation on a closed manifold admitting an adapted Riemannian metric for which all leaves are minimal must be regular. These facts have led us to study closer singular Riemannian foliations. We have introduced basic intersection cohomology in view to recover some kind of Poincaré duality, cf. [36, 37, 33]. We hope that soon we will complete our task and demonstrate the perverse version of the Poincaré duality property for basic intersection cohomology for singular Riemannian foliations of closed manifolds. In his thesis, [32], written under the supervision of M. Saralegi and M. Macho, J.I. Royo Prieto demonstrated, among other results on singular Riemannian flows, the Poincaré duality for basic intersection cohomology and the singular version of the Molino-Sergiescu theorem. Inspired by these results, we have started to investigate the possible generalizations to the SRF case and at the same time we have found that our research gives some interesting insights into the problem on non-compact manifolds, cf. [33, 34]. The second part of this work is concerned with this problem.

We complete the section with the presentation of the BIC for the above example, in which the PD property can be easily seen.

If we consider the BIC of our example the picture changes. The following table presents the BIC $IH^\ast_p(S^{k=2d+2}/\mathcal{F})$ for the constant perversities:

| $i$  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | $k-2$ | $k-1$ |
|------|----|----|----|----|----|----|----|----|-------|-------|
| $\tilde{p} < 0$ | $[d]$ | 0  | $[e \wedge dr]$ | 0  | $[e^2 \wedge dr]$ | 0  | $[e^3 \wedge dr]$ | $\cdots$ | 0  | $[e^d \wedge dr]$ |
| $\tilde{p} = 0, 1$ | 1  | 0  | 0  | $[e \wedge dr]$ | 0  | $[e^2 \wedge dr]$ | 0  | $[e^3 \wedge dr]$ | $\cdots$ | 0  | $[e^d \wedge dr]$ |
| $\tilde{p} = 2, 3$ | 0  | 0  | $[e]$ | 0  | 0  | $[e^2 \wedge dr]$ | 0  | $[e^3 \wedge dr]$ | $\cdots$ | 0  | $[e^d \wedge dr]$ |
| $\tilde{p} = 4, 5$ | 1  | 0  | $[e]$ | 0  | $[e^2]$ | 0  | 0  | $[e^3 \wedge dr]$ | $\cdots$ | 0  | $[e^d \wedge dr]$ |
| $\tilde{p} = k-4, k-3$ | 0  | $[e]$ | 0  | $[e^2]$ | 0  | $[e^3]$ | 0  | $\cdots$ | 0  | $[e^d \wedge dr]$ |
| $\tilde{p} \geq k-2$ | 1  | 0  | $[e]$ | 0  | $[e^2]$ | 0  | $[e^3]$ | 0  | $\cdots$ | $[e^d]$ | 0  |

We notice that the top dimensional basic cohomology group is isomorphic either to 0 or $\mathbb{R}$. These cohomology groups are finite dimensional. We recover the Poincaré duality in the perverse sense:

$$IH_p^\ast(S^k/\mathcal{F}) \cong IH_q^{k-1-\ast}(S^k/\mathcal{F})$$

for two complementary perversities: $\tilde{p} + \tilde{q} = \tilde{t} = k-3$.

**Theorem 1.5.4** Let $M$ be a connected closed manifold endowed with an SRF $\mathcal{F}$. If $\ell = \text{codim}_M \mathcal{F}$ and $\tilde{p}$ a perversity on $(M/\mathcal{F})$, then

$$IH^\ast_p(M/\mathcal{F}) = 0 \text{ or } \mathbb{R}.$$ 

**Corollary 1.5.5** Let $M$ be a connected compact manifold endowed with an SRF $\mathcal{F}$. Let us suppose that $\mathcal{F}$ is transversally orientable. Consider $\tilde{p}$ a perversity on $(M, \mathcal{F})$ with $\tilde{p} \leq \tilde{t}$. If $\ell = \text{codim}_M \mathcal{F}$, then the two following statements are equivalent:

1. The foliation $\mathcal{F}_R$ is taut, where $R$ is the regular stratum of $(M, \mathcal{F})$;
(2) The cohomology group $IH^\ell_{\bar{p}}(M/\mathcal{F})$ is $\mathbb{R}$.

Otherwise, $IH^\ell_{\bar{p}}(M/\mathcal{F}) = 0$.

The BIC of a conical foliation $\mathcal{F}$ defined on $M$ by an isometric action of an abelian Lie group on an oriented manifold $M$ verifies the Poincaré duality:

\begin{equation}
IH^\ell_{\bar{p}}(M/\mathcal{F}) \cong IH^{\ell-\ast}_{\bar{q},c}(M/\mathcal{F}).
\end{equation}

Here $\ell = \text{codim}_M \mathcal{F}$ and the two perversities $\bar{p}$ and $\bar{q}$ are complementary.

Note: Due to the limited space we could dedicate to this overview of the problem we have not mentioned many partial results, (e.g. [1, 19, 18]), and some reviews papers (e.g. [6, 39]).

2 Geometrical preliminaries.

We present in this section the kind of foliations we are going to use in this work: the CERFs. A CERF is essentially a Riemannian foliation defined on a non-compact manifold which is imbeddable in a closed manifold in a nice way.

2.1 The CERFs. We shall consider in this work a particular case of Riemannian foliations defined on non-compact manifolds. They have an outside compact manifold (zipper) and an inside compact submanifold (reppiz). Consider a manifold $M$ endowed with a Riemannian foliation $\mathcal{F}$.

A zipper of $\mathcal{F}$ is a closed manifold $N$ endowed with a (regular) Riemannian foliation $\mathcal{H}$ verifying the following properties:

(a) The manifold $M$ is a saturated open subset of $N$ and $\mathcal{H}_M = \mathcal{F}$.

The open subset $M$ is also $\mathcal{F}$-saturated. Thus, the closure $\overline{L}$ of a leaf $L \in \mathcal{F}$ is compact.

A reppiz of $\mathcal{F}$ is a saturated open subset $U$ of $M$ verifying the following properties:

(b) the closure $\overline{U}$ (in $M$) is compact.

(c) the inclusion $U \hookrightarrow M$ induces the isomorphism $H^\ast(U/\mathcal{F}) \cong H^\ast(M/\mathcal{F})$.

It is not true that any saturated open subset of $M$ is a reppiz. Just consider $M = S^1$ endowed with the pointwise foliation and take $U = S^1 \setminus \{(\cos(2\pi/n), \sin(2\pi/n)) / n \in \mathbb{N}\setminus\{0\}\}$.

We say that $\mathcal{F}$ is a Compactly Embeddable Riemannian Foliation (or CERF)$^3$ if $(M, \mathcal{F})$ possesses a zipper and a reppiz. When $M$ is closed, then $(M, \mathcal{F})$ is clearly a CERF, being $M$ itself a zipper and a reppiz. Neither the zipper nor the reppiz are unique.

The main example of a CERF is given by the strata of a singular Riemannian foliation defined on a closed manifold. This family will be treated in the next Section. The interior of a Riemannian foliation defined on a manifold with boundary is a CERF when the foliation is tangent to the boundary; we can consider the double of the manifold as a zipper. When the foliation is transverse to the boundary then the foliation is not a CERF.

We present now some geometrical tools we shall use for the study of a SRF $(X, \mathcal{K})$.

---

$^3$The definition of CERF given in [34] is more restrictive: see Proposition 2.4.
2.2 Tubular neighborhood. A singular stratum $S \in S_X$ is a proper submanifold of the Riemannian manifold $(X, \nu)$. So, it possesses a tubular neighborhood $(T_S, \tau_S, S)$. Recall that associated with this neighborhood there are the following smooth maps:

+ \text{The radius map } \rho_S: T_S \to [0,1] \text{ defined fiberwise by } z \mapsto |z|. \text{ Each } t \neq 0 \text{ is a regular value of the } \rho_S. \text{ The pre-image } \rho_S^{-1}(0) \text{ is } S.

+ \text{The contraction } H_S: T_S \times [0,1] \to T_S \text{ defined fiberwise by } (z,r) \mapsto r \cdot z. \text{ The restriction } (H_S)_t: T_S \to T_S \text{ is an imbedding for each } t \neq 0 \text{ and } (H_S)_0 \equiv \tau_S.

These maps verify $\rho_S(r \cdot z) = r \rho_S(z)$. This tubular neighborhood can be chosen verifying the two following important properties (cf. [27]):

(a) Each $(\rho_S^{-1}(t), K)$ is a SRF, and
(b) Each $(H_S)_t: (T_S, F) \to (T_S, F)$ is a foliated map.

We shall say that $(T_S, \tau_S, S)$ is a foliated tubular neighborhood of $S$.

The hypersurface $D_S = \rho_S^{-1}(1/2)$ is the core of the tubular neighborhood. We have the equality $\text{depth } S_{K_{D_S}} = \text{depth } S_{K_{T_S}} - 1$. Notice that the map

$$(5) \quad \mathcal{L}_S: (D_S \times [0,1], K \times I) \to ((T_S \setminus S), K),$$

defined by $\mathcal{L}_S(z,t) = H_S(z,2t)$, is a foliated diffeomorphism.

A family of foliated tubular neighborhoods $\{T_S \mid S \in S^\infty_X\}$ is a foliated Thom-Mather system of $(N, H)$ if the following conditions are verified.

(TM1) For each pair of singular strata $S, S'$ we have

$$T_S \cap T_{S'} \neq \emptyset \iff S \preceq S' \text{ or } S' \preceq S.$$ 

Let us suppose that $S' \prec S$. The two other conditions are:

(TM2) $T_S \cap T_{S'} = \tau_S^{-1}(T_{S'} \cap S)$.

(TM3) $\rho_{S'} = \rho_{S'} \circ \tau_S$ on $T_S \cap T_{S'}$.

We have seen in [34] that each closed manifold endowed with a SRF possesses a foliated Thom-Mather system. We fix for the sequel of this work a such foliated Thom-Mather system.

2.3 Blow up. Molino’s blow up of a SRF produces a new SRF of the same generic dimension but with smaller depth (see [27] and also [37],[34]). The main idea is to replace each point of the closed strata by its link (a sphere).

In fact, given a SRF $(X, K)$ with depth $S_K > 0$, there exists another SRF $(\tilde{X}, \tilde{K})$ and a continuous map $\mathcal{L}: \tilde{X} \to X$, called blow up of $(X, K)$, verifying:

- $\text{depth } S_{K_{\mathcal{L}}} = \text{depth } S_K - 1$.
- there exists a commutative diagram

\[ \mathcal{L}^{-1}(X \setminus S_{\min}) \xrightarrow{f_0} (X \setminus S_{\min}) \times \{-1, 1\} \]

\[ \mathcal{L} \]

\[ X \setminus S_{\min} \]

projection

where \( f_0 : (\mathcal{L}^{-1}(X \setminus S_{\min}), \mathcal{K}) \rightarrow (X \setminus S_{\min} \times \{-1, 1\}, \mathcal{K} \times \mathcal{I}) \) is a foliated diffeomorphism. Here, \( \mathcal{I} \) denotes the foliation by points.

- for each minimal (closed) stratum \( S_c \), there exists a commutative diagram

\[ \mathcal{L}^{-1}(T_{S_c}) \xrightarrow{f_{S_c}} D_{S_c} \times [-1, 1] \]

\[ \mathcal{L} \]

\[ T_{S_c} \]

\[ \mathcal{L}_{S_c} \]

where \( f_{S_c} : (\mathcal{L}^{-1}(T_{S_c}), \mathcal{K}) \rightarrow (D_{S_c} \times [-1, 1], \mathcal{K} \times \mathcal{I}) \) is a foliated diffeomorphism and the map \( \mathcal{L}_{S_c} \) is defined by \( \mathcal{L}_{S_c}(z, t) = H_{S_c}(z, 2|t|) \). Notice that \( f_{S_c} : (\mathcal{L}^{-1}(S_c), \mathcal{K}) \rightarrow (D_{S_c} \times \{0\}, \mathcal{K} \times \mathcal{I}) \) is also a foliated diffeomorphism.

The stratification induced by \( \mathcal{K} \) can be described as follows. For each non minimal stratum \( S \in \mathcal{S}_K \) there exists a unique stratum \( S^e \in \mathcal{S}_{\mathcal{K}} \) with \( \mathcal{L}^{-1}(S) \subset S^e \), and we have

\[ \mathcal{S}_{\mathcal{K}} = \{ S^e / S \in \mathcal{S}_K \text{ and } S \cap S_{\min} = \emptyset \} \]

In fact,

\[
\begin{align*}
& f_0 \left( S^e \cap \mathcal{L}^{-1}(X \setminus S_{\min}) \right) = S \times \{-1, 1\} \text{ and } \\
& f_{S_c} \left( S^e \cap \mathcal{L}^{-1}(T_{S_c}) \right) = (S \cap D_{S_c}) \times [-1, 1] \text{ if } S_c \text{ is a closed stratum with } S_c \preceq S.
\end{align*}
\]

The CERFs and the SRFs are related by the following result.

**Proposition 2.4** Let \( X \) be a closed manifold endowed with a SRF \( \mathcal{K} \). For any stratum \( S \) of \( \mathcal{S}_K \) the foliation \( \mathcal{K}_S \) is a CERF.

**Proof.** When \( S \) is a closed stratum it suffices to take the zipper \((S, \mathcal{K})\) and the reppiz \( S \). Consider now the case where \( S \) is not closed (minimal). We proceed in two steps.

A zipper for \((S, \mathcal{K})\). Proceeding by induction on depth \( \mathcal{S}_K \) we know that there exists a zipper \((N, \mathcal{H})\) of \((S^e, \mathcal{K})\). Since the map \( \xi : (S, \mathcal{K}) \rightarrow (S^e, \mathcal{K}) \), defined by \( x \mapsto f_0^{-1}(x, 1) \), is an open foliated imbedding we can identify \((S, \mathcal{K})\) with its (open) image \((\xi(S), \mathcal{K})\). So, the foliated manifold \((N, \mathcal{H})\) is a zipper of \((S, \mathcal{K})\).

A reppiz for \((S, \mathcal{K})\). For each \( i \in \{0, \ldots, s - 1\} \), where \( s = \text{depth} \mathcal{H} S \), we denote by :

\[ \Sigma_i = \cup \{ S' \in \mathcal{S}_H \mid \text{ depth} \mathcal{H} S' \leq i \} \]
$T_i$ the union of the disjoint tubular neighborhoods $\{T_{s_i} \cap T_{s_i}' \subset \Sigma_i \setminus \Sigma_{i-1}\}$,
- $\rho_i : T_i \to [0,1]$ its radius function, and
- $D_i = \rho_i^{-1}(0)$ the core of $T_i$.

The family $\{S \cap T_0, S \setminus \rho_0^{-1}([0,7/8])\}$ is a saturated open covering of $S$. The inclusion $I : ((S \cap T_0) \setminus \rho_0^{-1}([0,7/8]), K) \hookrightarrow (S \cap T_0, K)$ induces an isomorphism for the basic cohomology. This comes from the fact that the inclusion $I$ is foliated diffeomorphic to the inclusion $J : ((S \cap D_0) \times [7/8,1], \mathcal{K} \times \mathcal{I}) \hookrightarrow ((S \cap D_0) \times [0,1], \mathcal{K} \times \mathcal{I})$ (cf. (5) and $S \cap \Sigma_0 = \emptyset$). From the Mayer-Vietoris sequence (see for example [34]) we conclude that the inclusion $S \setminus \rho_0^{-1}([0,7/8]) \hookrightarrow S$ induces the isomorphism $H^*(S/K) \cong H^*((S \setminus \rho_0^{-1}([0,7/8]))/K)$. The family $\{T_{s'} \setminus \rho_{s'}^{-1}([0,7/8]) \mid s' \in S, \text{depth } s' > 0\}$ is a foliated Thom-Mather system of $(S \setminus \rho_0^{-1}([0,7/8]), \mathcal{H})$ (cf. [34, (1.6)]). The same previous argument applied to the stratum $S \setminus \rho_0^{-1}([0,7/8])$ gives $H^*((S \setminus \rho_0^{-1}([0,7/8]))/K) \cong H^*((S \setminus \rho_0^{-1}([0,7/8])) \setminus \rho_1^{-1}([0,7/8]))/K)$. This procedure leads us to

$$H^*(S/K) \cong H^*((S \setminus \rho_0^{-1}([0,7/8]))/K) \cong H^*((S \setminus (\rho_0^{-1}([0,7/8]) \cup \rho_1^{-1}([0,7/8])))/K) \cong \cdots \cong H^*((S \setminus \rho_0^{-1}([0,7/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,7/8])))/K).$$

Take $U = S \setminus (\rho_0^{-1}([0,7/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,7/8]))$, which is an open saturated subset included on $S$. By construction, the inclusion $U \hookrightarrow S$ induces the isomorphism $H^*(S/K) \cong H^*(U/K)$. This gives (a).

Consider $K = S \setminus (\rho_0^{-1}([0,1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,1/8]))$, which is a subset of $S$ containing $U$. We compute its closure in $S$:

$$\overline{K} = \overline{S \setminus (\rho_0^{-1}([0,1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,1/8]))} = \overline{S \setminus (\rho_0^{-1}([0,1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,1/8])))} \subset S \setminus (\rho_0^{-1}([0,1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,1/8])) = S \setminus \rho_0^{-1}([0,1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,1/8])) \setminus \rho_0^{-1}([0,1/8]),$$

since $\overline{S \setminus \rho_0^{-1}([0,1/8])} \subset S \setminus \rho_0^{-1}([0,1/8]) \cup \cdots \cup \rho_{s-1}^{-1}([0,1/8])$. This implies that $K$ is a closed subset of $S$ and therefore compact. This gives (b).

### 2.5 Basic cohomology

As in the regular case, the basic cohomology $H^*(X/K)$ is the cohomology of the complex $\Omega^*(X/K)$ of basic forms (cf. [44]). A differential form $\omega$ is basic when $i_X \omega = i_X d\omega = 0$ for every vector field $X$ tangent to $F$.

Associated to a covering $\{U, V\}$ of $X$ by saturated open subsets we have the Mayer-Vietoris short exact sequence

$$0 \to (\Omega^*(X/K), d) \to (\Omega^*(U/K), d) \oplus (\Omega^*(V/K), d) \to (\Omega^*((U \cap V)/K), d) \to 0,$$

where the maps are defined by restriction (the same proof of [34] for the regular case works).
3 Tautness in the non-compact case.

We prove in this section that the previous cohomological characterizations of the tautness of a RF $\mathcal{F}$ are still valid when the manifold is non-compact but the foliation $\mathcal{F}$ is a CERF.

We fix for the rest of this section a CERF $\mathcal{F}$ defined on a manifold $M$. We also fix a zipper $(N, \mathcal{H})$ and a reppiz $U$.

3.1 Tautness class of $\mathcal{F}$. Since $N$ is compact we get from Theorem 1.4.5 that $M$ possesses a D-metric $\mu$. The tautness class of $(M, \mathcal{F})$ is the class $\kappa = [\kappa_\mu] \in H^1(M/\mathcal{F})$ (cf. page 9). This class is well defined since:

**Proposition 3.1.1** Two D-metrics on $(M, \mathcal{F})$ define the same tautness class.

**Proof.** Fix a zipper $(N, \mathcal{H})$ and a reppiz $U$. Since $N$ is compact then the tautness class $\kappa_N$ is well defined. Let $\mu$ be a D-metric on $M$. The key point of the proof is to relate the class $[\kappa_\mu]$ with $\kappa_N$.

From 2.1 (a) and (b), we have that $\{M, N\setminus \overline{U}\}$ is a saturated open covering of $M$. It possesses a subordinated partition of the unity $\{f, g\}$ made up with basic functions (cf. [34]). Consider $\nu$ a D-metric on $N$, which always exists since $N$ is compact. So, the metric

$$\lambda = f\mu + (1-f)\nu$$

is a bundle-like metric on $N$ with $\lambda|_U = \mu|_U$, which is a D-metric. This gives

$$\kappa_{\lambda,b}|_U = \kappa_{\mu}|_U = \kappa_{\nu}|_U$$

(cf. (3) and (2)). Denote by $I: U \to M$ and $J: U \to N$ the natural inclusions. We have

$$I^*[\kappa_\mu] = [\kappa_\mu]|_U = [\kappa_{\lambda,b}] = J^*[\kappa_{\lambda,b}] = J^*\kappa_N.$$  

Consider $\mu'$ another D-metric on $M$. The above equality gives $I^*[\kappa_\mu] = I^*[\kappa_{\mu'}]$. From 2.1 (c) we get $[\kappa_\mu] = [\kappa_{\mu'}]$. 

The first characterization of the tautness is the following.

**Theorem 3.2** Let $M$ be a manifold endowed with a CERF $\mathcal{F}$. Then the following two statements are equivalent:

(a) The foliation $\mathcal{F}$ is taut.

(b) The tautness class $\kappa \in H^1(M/\mathcal{F})$ vanishes.

**Proof.** We prove the two implications.

(a) $\Rightarrow$ (b). There exists a D-metric $\mu$ on $M$ with $\kappa_\mu = 0$. Then $\kappa = [\kappa_\mu] = 0$.

(b) $\Rightarrow$ (a). See [43, Proposition 7.6]$^4$.

The second characterization of the tautness we use the twisted basic cohomology $H^*_\kappa(M/\mathcal{F})$, where $\mu$ is a D-metric. Notice that this cohomology does not depend on the choice of the D-metric (cf. Proposition 3.1.1 and (c) of page 3).

$^4$At the beginning of Chapter 7 of [43] it is said the the manifold $M$ must be compact. In fact, this condition is not necessary on the proof of Proposition 7.6.
Theorem 3.3 Let $M$ be a manifold endowed with a CERF $\mathcal{F}$. Consider $\mu$ a D-metric on $M$. Then, the following two statements are equivalent:

(a) The foliation $\mathcal{F}$ is taut.

(b) The cohomology group $H^0_{\kappa_{\mu}}(M/\mathcal{F})$ is $\mathbb{R}$.

Otherwise, $H^0_{\kappa_{\mu}}(M/\mathcal{F}) = 0$.

Proof. We proceed in two steps.

$(a) \Rightarrow (b)$. If $\mathcal{F}$ is taut then $\kappa = [\kappa_{\mu}] = 0$. So, $H^0_{\kappa_{\mu}}(M/\mathcal{F}) \cong H^0(M/\mathcal{F}) = \mathbb{R}$.

$(b) \Rightarrow (a)$. If $H^0_{\kappa_{\mu}}(M/\mathcal{F}) \neq 0$ then there exists $0 \neq f \in \Omega^0(M/\mathcal{F})$ with $df = f\kappa_{\mu}$. The set $Z(f) = f^{-1}(0)$ is clearly a closed subset of $M$. Let us see that it is also an open subset. Take $x \in Z(f)$ and consider a contractible open subset $V \subset M$ containing $x$. So, there exists a smooth map $g: V \to \mathbb{R}$ with $\kappa_{\mu} = dg$ on $U$. The calculation

$$d(fe^{-g}) = e^{-g}df - fe^{-g}dg = e^{-g}f\kappa_{\mu} - e^{-g}f\kappa_{\mu} = 0$$

shows that $fe^{-g}$ is constant on $V$. Since $x \in Z(f)$ then $f \equiv 0$ on $V$ and therefore $x \in V \subset Z(f)$. We have proved that $Z(f)$ is an open subset.

As $M$ is connected, we have $Z(f) = \emptyset$. From $d(\log |f|) = \frac{1}{f}df = \kappa_{\mu}$ we conclude that $\kappa = 0$.

The foliation $\mathcal{F}$ is taut.

Notice that we have also proved: $H^0_{\kappa_{\mu}}(M/\mathcal{F}) \neq 0 \Rightarrow H^0_{\kappa_{\mu}}(M/\mathcal{F}) = \mathbb{R}$.

3.3.1 Remark. To prove $(b) \Rightarrow (a)$ we do not need $\mathcal{F}$ to be a CERF but just the existence of a D-metric $\mu$ (see [43, Proposition 7.6]).

For the third characterization we need to extend the basic Poincaré duality to the non-compact case. We find in [38] another version of this Poincaré duality using the cohomological orientation sheaf instead of the twisted basic cohomology we use here. Also compare with [43, Proposition 7.54].

Theorem 3.4 Let $M$ be a manifold endowed with a transversally oriented RF $\mathcal{F}$ possessing a zipper. Consider $\mu$ a D-metric on $M$. If $n = \text{codim } \mathcal{F}$, then

$$H^c_*(M/\mathcal{F}) \cong H^{n-*}_{\kappa_{\mu}}(M/\mathcal{F}).$$

Proof. See Appendix I.

The third characterization of the tautness is the following. Compare with [43, Proposition 7.56].

Theorem 3.5 Let $M$ be a manifold endowed with a CERF $\mathcal{F}$. Let us suppose that $\mathcal{F}$ is transversally oriented. If $n = \text{codim } \mathcal{F}$, then the two following statements are equivalent:

(a) The foliation $\mathcal{F}$ is taut.

(b) The cohomology group $H^n_c(M/\mathcal{F})$ is $\mathbb{R}$.

Otherwise, $H^n_c(M/\mathcal{F}) = 0$. 
Proof. It suffices to apply Theorem 3.4 and Theorem 3.3.

As a direct application, we extend the scope of a well known result for closed manifolds (cf. [1, Corollary 6.6]) to arbitrary CERFs.

Corollary 3.6 Any codimension one CERF is taut.

Proof. Let \( F \) be a codimension one CERF defined on a manifold \( M \). Without loss of generality we can suppose that \( F \) is transversally oriented (cf. [1, Lemma 6.3]). By reductio ad absurdum, let’s suppose that \( F \) is not taut. Then, by Theorem 3.3, we get \( H^0_{\kappa \mu}(M/F) = 0 \), and by Theorem 3.2, we get \( \kappa \neq 0 \), thus \( H^1(M/F) \neq 0 \). Now, from Remark 4.11 (b) we get \( H^0_{\kappa \mu}(M/F) \neq 0 \). Theorem 3.4 yields \( H^1_{\kappa}(M/F) \neq 0 \), a contradiction.

\( \blacklozenge \)

4 Appendix.

This appendix is devoted to the proof the Theorem 3.4. We distinguish two cases following the orientability of \( M \). Beforehand, we introduce two technical tools.

4.1 Bredon’s Trick. The Mayer-Vietoris sequence allows us to make computations when the manifold is covered by a finite suitable covering. The passage from the finite case to the general case may be done using an adapted version of the Bredon’s trick of [3, page 289] we present now.

Let \( X \) be a paracompact topological space and let \( \{U_\alpha\} \) be an open covering, closed for finite intersections. Suppose that \( Q(U) \) is a statement about open subsets of \( X \), satisfying the following three properties:

1. (BT1) \( Q(U_\alpha) \) is true for each \( \alpha \);
2. (BT2) \( Q(U), Q(V) \) and \( Q(U \cap V) \Rightarrow Q(U \cup V) \), where \( U \) and \( V \) are open subsets of \( X \);
3. (BT3) \( Q(U_i) \Rightarrow Q\left( \bigcup U_i \right) \), where \( \{U_i\} \) is an arbitrary disjoint family of open subsets of \( X \).

Then \( Q(X) \) is true.

4.2 Mayer-Vietoris. Associated to a covering \( \{U, V\} \) of \( M \) by saturated open subsets we have the Mayer-Vietoris exact short sequence

\[
0 \to (\Omega^*(M/F), d) \to (\Omega^*(U/F), d) \oplus (\Omega^*(V/F), d) \to (\Omega^*((U \cap V)/F), d) \to 0,
\]

where the maps are defined by restriction (see for example [34]). In the compact support context we have the Mayer-Vietoris sequence

\[
0 \to (\Omega^*_c((U \cap V)/F), d) \to (\Omega^*_c(U/F), d) \oplus (\Omega^*_c(V/F), d) \to (\Omega^*_c(M/F), d) \to 0,
\]

where the maps are defined by extension (see for example [34]). Finally, for the twisted basic cohomology, we have the Mayer-Vietoris sequence

\[
0 \to (\Omega^*(M/F), d_{\kappa \mu}) \to (\Omega^*(U/F), d_{\kappa \mu}) \oplus (\Omega^*(V/F), d_{\kappa \mu}) \to (\Omega^*((U \cap V)/F), d_{\kappa \mu}) \to 0,
\]

where the maps are defined by restriction.
4.3 Integration. In order to define the duality operator, we fix

(a) an oriented manifold \( M \),

(b) a transversally oriented RF \( \mathcal{F} \) (TORF for short) on \( M \), and

(c) a \( D \)-metric \( \mu \) on \((M, \mathcal{F})\).

We shall say that \((M, \mathcal{F}, \mu)\) is a \( D \)-triple. The associated tangent volume form is \( \chi_\mu \) (it exists since \( \mathcal{F} \) is also oriented). With all these ingredients we define the morphism

\[ \int_M : H^*(M/\mathcal{F}) \longrightarrow \text{Hom} \left( H^{n-*}_\mu (M/\mathcal{F}); \mathbb{R} \right) = \left( H^{n-*}_\mu (M/\mathcal{F}) \right)^* \]

by \( \int_M ([\alpha])([\beta]) = \int_M \alpha \wedge \beta \wedge \chi_\mu \). Here \( n = \text{codim}_M \mathcal{F} \). This operator is well defined since \( M \) is oriented and we have the Rummler formula

\[ i_{Y_1} \cdots i_{Y_r} d\chi_\mu + \chi_\mu (Y_1, \ldots, Y_r) \cdot \kappa_\mu = 0 \]

when \( \{Y_1, \ldots, Y_r\} \) are vector fields tangent to \( T\mathcal{F} \) and \( r = \dim \mathcal{F} \) (see [43]). We prove in this section that the operator \( \int_M \) is an isomorphism.

Before proving the general case first we consider some particular cases.

**Lemma 4.4** Suppose that the \( D \)-triple \((M, \mathcal{F}, \mu)\) is \((E \times \mathbb{R}, \mathcal{E} \times \mathcal{I}, \mu)\) where \( E \) is a closed manifold and the leaves of \( \mathcal{E} \) are dense. Then the operator \( \int_M \) is an isomorphism.

**Proof.** For the proof of Lemma we proceed in several steps. We shall use the following notation. Given a differential form (or Riemannian metric) \( \omega \) on \( E \times \mathbb{R} \) we shall denote by \( \omega(t) \) the restriction \( I_t^* \omega \), where \( I_t : E \rightarrow E \times \mathbb{R} \) is defined by \( I_t(x) = (x, t) \) for each \( t \in \mathbb{R} \).

**First Step.** The cohomology \( H^*_c((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \). Consider \( f : \mathbb{R} \rightarrow [0, 1] \) a smooth function with compact support such that \( \int_\mathbb{R} f dt = 1 \). We know that the correspondence \( [\gamma] \mapsto [f \gamma \wedge dt] \) establishes an isomorphism between \( H^*_c(E/\mathcal{E}) \) and \( H^{*+1}_c((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \) (cf. [34]). In fact, this isomorphism does not depend on choice of \( f \).

**Second Step.** The cohomology \( H^*_\mu((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \). Notice that the metric \( \mu(0) \) is a \( D \)-metric (cf. (1)). So, we have two \( D \)-metrics on \( E \times \mathbb{R} \): \( \mu \) and \( \mu(0) + dt^2 \) with \( \kappa_\mu(0) + dt^2 = \kappa_\mu(0) \). Since \( \mathcal{E} \times \mathcal{I} \) is a CERF (it suffices to take \( E \times [-1, 1] \) as a repzip and \( (E \times \mathbb{S}^1, \mathcal{E} \times \mathcal{I}) \) as a zipper) then there exists a function \( g \in \Omega^0((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \) with \( \kappa_\mu = \kappa_\mu(0) + dg \) (cf. Proposition 3.1.1). Since the leaves of \( \mathcal{E} \) are dense on \( N \) then the (basic) function \( g \) is a smooth function on \( \mathbb{R} \).

We know (see (c) of page 3) that the assignment \( [\omega] \mapsto [e^g \omega] \) establishes an isomorphism between \( H^*_\mu(0)((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \) and \( H^*_\mu((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \). The usual techniques give that the assignment \( [\omega] \mapsto [\omega] \) establishes an isomorphism between \( H^*_\mu(0)(E/\mathcal{E}) \) and \( H^*_\mu((E \times \mathbb{R})/\mathcal{E} \times \mathcal{I}) \).

**Last Step.** Notice that \((E, \mathcal{E}, \mu(0))\) is a \( D \)-triple. Since \( E \) is compact, the morphism

\[ \int_E : H^*(E/\mathcal{E}) \longrightarrow \left( H^{n-1-*}_\mu(0)(E/\mathcal{E}) \right)^* \]
defined by \( \int_E ([\gamma])([\zeta]) = \int_E \gamma \wedge \zeta \wedge \chi_\mu(0) \), is an isomorphism (see [21]). Following the previous steps it suffices to show that the morphism

\[ \int_E^*: H^*(E/\mathcal{E}) \longrightarrow \left(H_{\tau(0)}^{n-1-*}(E/\mathcal{E})\right)^* , \]

defined by \( \int_E^* ([\gamma])([\zeta]) = \int_{E \times \mathbb{R}} f e^g \gamma \wedge dt \wedge \zeta \wedge \chi_\mu \), is an isomorphism. Let us see that.

- \( \int_E^* \) is a monomorphism.

Consider \( [\gamma] \in H^*(E/\mathcal{E}) \) with \( \int_E^* ([\gamma]) = 0 \). We have \( \int_{\mathbb{R}} f(t)e^g(t) \left( \int_E \gamma \wedge \zeta \wedge \chi_\mu(t) \right) dt = 0 \) for each \([\zeta] \in H_{\tau(0)}^{n-1-*}(E/\mathcal{E})\) and each smooth function \( f: \mathbb{R} \rightarrow [0,1] \) with compact support and \( \int_{\mathbb{R}} f = 1 \). So, \( \int_E \gamma \wedge \zeta \wedge \chi_\mu(t) dt = 0 \) for each \([\zeta] \in H_{\tau(0)}^{n-1-*}(E/\mathcal{E})\) and each \( t \in \mathbb{R} \). We get \( \int_E ([\gamma])([\zeta]) = 0 \) for each \([\zeta] \in H_{\tau(0)}^{n-1-*}(E/\mathcal{E})\). Since \( \int_E^* \) is an isomorphism then \([\gamma] = 0\).

- \( \int_E^* \) is an epimorphism. From (7) we know that \( \dim H^*(E/\mathcal{E}) = \dim \left(H_{\tau(0)}^{n-1-*}(E/\mathcal{E})\right)^* \), which is finite since \( E \) is compact (see [11]). This gives that the monomorphism \( \int_E^* \) is also an epimorphism.

**Lemma 4.5** Let \((M, \mu, \mathcal{F})\) be a \( D \)-triple. Suppose that \((M, \mathcal{F})\) possesses a transversally parallelizable zipper. Then \( \int_M^* \) is an isomorphism.

**Proof.** Denote by \((N, \mathcal{H})\) the transversally parallelizable zipper. Since \( \mathcal{H} \) is transversally parallelizable, then there exists a fiber bundle \( \pi: N \rightarrow B \) whose fibers are the closures of the leaves of \( \mathcal{H} \). Since \( M \) is saturated for the leaves of \( \mathcal{H}_M \) then it is also saturated for the closures of these leaves. We get an open subset \( V_M \subset B \) with \( M = \pi^{-1}(V_M) \).

Fix \( V \) an open subset of \( V_M \). The foliation \( \mathcal{H}_{\pi^{-1}(V)} \) admits the zipper \((N, \mathcal{H})\). Notice that \( \pi^{-1}(V) \) is oriented and \( \mathcal{H}_{\pi^{-1}(V)} \) is a TORF. The triple \((\pi^{-1}(V), \mathcal{H}_{\pi^{-1}(V)}, \mu_{\pi^{-1}(V)})\) is a \( D \)-triple. So, the operator \( \int_{\pi^{-1}(V)} \) is well defined. We prove that this operator is non-degenerate. This will end the proof by taking \( V = V_M \).

Let \((E, \mathcal{E})\) be a generic fiber of \( \pi \). The manifold \( E \) is closed and the leaves of \( \mathcal{E} \) are dense in \( E \). We know that the fibration \( \pi: \pi^{-1}(V) \rightarrow V \) has a foliated atlas \( \mathcal{A} = \{ \varphi: (\pi^{-1}(U), \mathcal{H}) \rightarrow (U \times E, \mathcal{T} \times \mathcal{E}) \} \). We can suppose that the covering \( U = \{ U \mid \exists (U, \mathcal{H}) \in \mathcal{A} \} \):

- is a good covering of \( V \): if \( U_1, \ldots, U_k \in U \) then the intersection \( V = U_1 \cap \cdots \cap U_k \) is diffeomorphic to \( \mathbb{R}^{\dim B} \) (cf. [4]), and
- is closed for finite intersections.

We consider the statement \( Q(U): \)

“The integration operator \( \int_{\pi^{-1}(U)} \) is an isomorphism.”

where \( U \subset V \) is an open subset. Following the Bredon’s trick, it suffices to prove (BT1), (BT2) and (BT3) relatively to the covering \( \mathcal{U} \).

+ (BT1). It follows directly from the Lemma 4.4.

+ (BT2). The integration operator \( \int \) commutes with the restriction and inclusion operators. It suffices to apply the Five’s Lemma to the Mayer-Vietoris sequences of 4.2.

+ (BT3). Straightforward.

\[4.6 \text{ Frame bundle.} \]

Let \((M, \mu, \mathcal{F})\) be a \( D \)-triple possessing a zipper \((N, \mathcal{H})\). Consider \( p: \tilde{N} \to N \) the bundle of transverse oriented orthonormal frames of \( N \) (cf. [25]). It is an \( SO(n) \)-principal bundle. The canonical lift \( \tilde{\mathcal{H}} \) of \( \mathcal{H} \) is a transversally parallelizable foliation on the closed manifold \( \tilde{N} \) with \( \text{codim} \; \tilde{N} \mathcal{H} = n + \dim SO(n) \). The restriction bundle morphism \( p_*: T\tilde{\mathcal{H}} \to T\mathcal{H} \) is an isomorphism. We can lift \((M, \mathcal{F}, \mu)\) as follows.

- Lifting \( \mathcal{F} \). Since \( M \) is a saturated open subset of \((N, \mathcal{H})\) then \( \tilde{M} = p^{-1}(M) \) is a saturated open subset of \((\tilde{N}, \tilde{\mathcal{H}})\). The foliation \( \tilde{\mathcal{F}} = \tilde{\mathcal{H}}_{\tilde{M}} \) is transversally parallelizable (and then a TORF) and the manifold \( \tilde{M} \) is oriented since \( p: \tilde{M} \to M \) is a \( SO(n) \)-bundle and \( M \) is oriented. The foliation \( \tilde{\mathcal{F}} \) possesses \((\tilde{N}, \tilde{\mathcal{H}})\) as a zipper.

- Lifting \( \mu \). Consider the decomposition \( \mu = \mu_1 + \mu_2 \) relatively to the orthogonal decomposition \( TM = T\mathcal{F} \oplus (T\mathcal{F})^\perp \). Since the restriction bundle morphism \( p_*: T\tilde{\mathcal{F}} \to T\mathcal{F} \) is an isomorphism then we have the decomposition \( T\tilde{M} = T\tilde{\mathcal{F}} \oplus p_*^{-1}(T\mathcal{F})^\perp \). Moreover, since \((\tilde{M}, \tilde{\mathcal{F}})\) is a Riemannian foliated manifold (TP in fact) then there exists a Riemannian metric \( \nu_2 \) on \( p_*^{-1}(T\mathcal{F})^\perp \) such that the Riemannian metric \( \nu = p^*\mu_1 + \nu_2 \) is a bundle-like metric on \((\tilde{M}, \tilde{\mathcal{F}})\). Then, the associated volume forms verify:

\[
\chi_\nu = p^*\chi_\mu.
\]

Rummler’s formula (6) gives

\[
(8) \quad \kappa_\nu = p^*\kappa_\mu.
\]

We conclude that \((\tilde{M}, \tilde{\mathcal{F}}, \nu)\) is a \( D \)-triple possessing a transversally parallelizable zipper. From the Lemma 4.5 we know that the integration operator \( \int_{\tilde{M}}: H^*_i(\tilde{M}/\tilde{\mathcal{F}}) \to \left(H^{n+\ell-\ast}_{\nu_2}((\tilde{M}/\tilde{\mathcal{F}}))^\ast\right) \) is an isomorphism. Here \( \ell = \dim SO(n) \). We shall prove Theorem 3.4 relating \((M, \mathcal{F}, \mu)\) with \((\tilde{M}, \tilde{\mathcal{F}}, \nu)\) using two spectral sequences.
4.7 A spectral sequence. Consider the usual filtration
\[ F^p \Omega^{p+q}_c(\widetilde{M}/\widetilde{\mathcal{F}}) = \{ \omega \in \Omega^{p+q}_c(\widetilde{M}/\widetilde{\mathcal{F}}) \mid i_{x_{u_0}} \cdots i_{x_{u_q}} \omega = 0 \text{ for each } \{u_0, \ldots, u_q\} \subset \mathfrak{g} \}, \]
where \( X_u \in \mathfrak{X}(\widetilde{M}) \) is determined by the element \( u \in \mathfrak{g} \), the Lie algebra of \( SO(n) \). It induces a filtration in the differential complex \( I^*K = (\Omega^*_c(\widetilde{M}/\widetilde{\mathcal{F}}))^{SO(n)} \) by
\[ F^p I^pK^{p+q} = F^p \Omega^{p+q}_c(\widetilde{M}/\widetilde{\mathcal{F}}), \]
leading us to a first quadrant spectral sequence \( I^pE^r_{p,q} \) which verifies

(a) \( I^pE^0_{r,q} \Rightarrow H^p(\widetilde{M}/\widetilde{\mathcal{F}}) \), and

(b) \( I^pE^1_{r,q} \Rightarrow H^p(\mathcal{M}/\mathcal{F}) \otimes H^q(SO(n)) \).

Let us see that. The inclusion \( I^pK^* \hookrightarrow \Omega^*_c(\widetilde{M}/\widetilde{\mathcal{F}}) \) induces an isomorphism in cohomology. This is a standard argument based on the fact that \( SO(n) \) is a connected compact Lie group (cf. [15, Theorem I,Ch.IV,vol.II]). This gives (a).

We denote by \( \gamma_u = i_{x_u}^* \) the associated fundamental differential form. Notice that the assignment \( \alpha \otimes u \mapsto \alpha \wedge \gamma_u \) induces the identification
\[ \bigoplus_{p+q=r} \left( F^p \Omega^p_c(\widetilde{M}/\widetilde{\mathcal{F}}) \otimes \bigwedge^q \mathfrak{g} \right) = \Omega^r_c(\widetilde{M}/\widetilde{\mathcal{F}}). \]
So, we get \( I^pE^0_{r,q} \cong \left( F^p \Omega^p_c(\widetilde{M}/\widetilde{\mathcal{F}}) \otimes \bigwedge^q \mathfrak{g} \right)^{SO(n)} \). A straightforward calculation gives (see also [15, (9.2),vol.III]) that \( d_0 = - \text{Identity} \otimes \delta \), where \( d_0 \) is the 0-differential of the spectral sequence and \( \delta \) is the differential of \( \bigwedge^* \mathfrak{g} \). This gives \( I^pE^1_{r,q} \cong \left( F^p \Omega^p_c(\widetilde{M}/\widetilde{\mathcal{F}}) \right)^{SO(n)} \otimes H^q(SO(n)) \) (cf. [15, 5.28 and 5.12, vol.III]). On the other hand, we have
\[ \left( F^p \Omega^p_c(\widetilde{M}/\widetilde{\mathcal{F}}) \right)^{SO(n)} = \{ \omega \in \Omega^p_c(\widetilde{M}/\widetilde{\mathcal{F}}) \mid i_{x_u} \omega = L_{x_u} \omega = 0 \text{ for each } u \in \mathfrak{g} \} \cong p^* \Omega^p_c(M/\mathcal{F}) \]
and then \( I^pE^1_{r,q} \cong \Omega^r_c(M/\mathcal{F}) \otimes H^q(SO(n)) \). Since \( d_1 \), the 1-differential of the spectral sequence, becomes \( \delta \otimes \text{Identity} \) then we conclude \( I^pE^2_{r,q} \cong H^r_c(M/\mathcal{F}) \otimes H^q(SO(n)) \). This gives (b).

4.8 Another spectral sequence. Consider the usual filtration
\[ F^p \Omega^{p+q}(\widetilde{M}/\widetilde{\mathcal{F}}) = \{ \omega \in \Omega^{p+q}(\widetilde{M}/\widetilde{\mathcal{F}}) \mid i_{x_{u_0}} \cdots i_{x_{u_q}} \omega = 0 \text{ for each } \{u_0, \ldots, u_q\} \subset \mathfrak{g} \}. \]
The complex \( I^*K^* = \left( \Omega^{n+\ell-*}(\widetilde{M}/\widetilde{\mathcal{F}}) \right)^{SO(n)} \) admits \( \nabla_{\kappa_u} \), the dual of \( d_{\kappa_u} \), as a differential since (8). Consider the filtration
\[ F^p I^*K^{p+q} = \left\{ L \in I^*K^{p+q} \mid L \equiv 0 \text{ on } F^{n-p+1} \Omega^{n+\ell-(p+q)}(\widetilde{M}/\widetilde{\mathcal{F}}) \right\}. \]
A straightforward calculation gives that \( F^{p+1} I^*K^* \subset F^p I^*K^* \) and \( \nabla_{\kappa_u} (F^p I^*K^{p+q}) \subset F^p I^*K^{p+q+1} \). Thus, it induces a first quadrant spectral sequence \( I^pE^p_{r,q} \) verifying

\(^{5}\)This is the spectral sequence of [15, 9.1,Ch.IX,vol.III].

\(^{6}\)This is the spectral sequence of [15, Ch.IX,9.1,vol.III] associated to \( \left( \Omega^*_{\kappa_u}(\widetilde{M}/\widetilde{\mathcal{F}}) \right)^* \) which is not a differential graded algebra.
(a) \( I I E_r^{p,q} \Rightarrow \left( H^{n+\ell-(p+q)}(\widetilde{M}/\widetilde{F}) \right)^* \).

(b) \( I I E_2^{p,q} \cong \left( H^{n-p}(M/F) \right)^* \otimes \left( H^{\ell-q}(SO(n)) \right)^* \).

Let us see that. As in 4.7 (a), the inclusion \( (\Omega^* (\widetilde{M}/\widetilde{F}))^{SO(n)} \hookrightarrow \Omega^* (\widetilde{M}/\widetilde{F}) \) induces an isomorphism in the corresponding twisted basic cohomology. This yields (a).

The analogous identification to (9) is here

\[ \bigoplus_{p+q=\ast} \left( F^p \Omega^p (\widetilde{M}/\widetilde{F}) \otimes \wedge^q g \right) = \Omega^* (\widetilde{M}/\widetilde{F}). \]

So, we get \( I I E_0^{p,q} \cong \left( \left( F^{n-p} \Omega^{n-p} (\widetilde{M}/\widetilde{F}) \otimes \wedge^{\ell-q} g \right)^{SO(n)} \right)^* \). A straightforward calculation shows that the 0-differential of the spectral sequence is the dual of \( \text{Identity} \otimes \delta \). This gives \( I E_1^{p,q} \cong \left( \left( F^{n-p} \Omega^{n-p} (\widetilde{M}/\widetilde{F}) \right)^{SO(n)} \right)^* \otimes \left( H^{\ell-q}(SO(n)) \right)^* \). On the other hand, we have

\[ \left( F^{n-p} \Omega^{n-p} (\widetilde{M}/\widetilde{F}) \right)^{SO(n)} = \{ \omega \in \Omega^{n-p} (\widetilde{M}/\widetilde{F}) \mid i_{x_u} \omega = L_{x_u} \omega = 0 \text{ for each } u \in g \} = p^* \Omega^{n-p} (M/F) \]

and then \( I E_1^{p,q} \cong \left( \Omega^{n-p}(M/F) \right)^* \otimes \left( H^{\ell-q}(SO(n)) \right)^* \). Since the 1-differential of the spectral sequence becomes the dual of \( d_{\nu} \otimes \text{Identity} \) (cf. (8)) then we conclude \( I E_2^{p,q} \cong \left( H^{n-p}_\nu (M/F) \right)^* \otimes \left( H^{\ell-q}(SO(n)) \right)^* \).

**Proposition 4.9** Let \((M, \mu, F)\) be a D-triple. Suppose that \(F\) possesses a zipper, then the operator \( \int_M \) is an isomorphism.

**Proof.** Consider the differential operator

\[ \Delta: \left( (\Omega^*_\nu (\widetilde{M}/\widetilde{F}))^{SO(n)} ; d \right) \longrightarrow \left( \left( \Omega^{n+\ell-\ast} (\widetilde{M}/\widetilde{F}) \right)^{SO(n)} ; \nabla_\nu \right) \]

defined by \( \Delta(\omega)(\eta) = \int_{\overline{M}} \omega \wedge \eta \wedge \chi_\nu \) (cf. 4.3). By degree reasons, it preserves the involved filtrations, that is, we have \( \Delta \left( F^p, K^{p+q} \right) \subset F^p I I K^{p+q} \). It induces the following morphisms:

+ [At \( \infty \)-level] \( \int_{\overline{M}} : H^{p+q}_c(\widetilde{M}/\widetilde{F}) \longrightarrow \left( H^{n+\ell-(p+q)}(\widetilde{M}/\widetilde{F}) \right)^* \).

+ [At 2-level] \( \int_M \otimes \int_{SO(n)} : H^p_c(M/F) \otimes H^q(SO(n)) \longrightarrow \left( H^{n-p}_\nu (M/F) \right)^* \otimes \left( H^{\ell-q}(SO(n)) \right)^* \).

As the operators \( \int_{\overline{M}} \) and \( \int_{SO(n)} \) are isomorphisms (cf. Lemma 4.5), Zeeman’s comparison theorem yields that \( \int_M \) is an isomorphism (see for example [23]).
- - - Non-orientable case - - -

For the non-orientable case it suffices to consider the orientation covering in order to apply the previous results.

**Proposition 4.10** Let $\mathcal{F}$ be a TORF defined on a manifold $M$. Suppose that $\mathcal{F}$ possesses a zipper. Consider $\mu$ a $D$-metric on $M$. If $n = \text{codim} \mathcal{F}$, then

$$H_{c}^{*}(M/\mathcal{F}) \cong H_{\mu}^{n-*}(M/\mathcal{F}).$$

**Proof.** Let us suppose that $M$ is not orientable. We fix a a zipper $(N, \mathcal{H})$ of $\mathcal{F}$. We consider $\mu$ a $D$-metric on $(M, \mathcal{F})$.

Consider $\varnothing: \tilde{N} \rightarrow N$ the two-fold orientation covering of $N$. It is an oriented closed manifold. Denote by $\tilde{\mathcal{H}}$ the lifted foliation, which is a Riemannian one. In fact, there exists a smooth foliated action $\Phi: \mathbb{Z}_{2} \times (\tilde{N}, \tilde{\mathcal{H}}) \rightarrow (\tilde{N}, \tilde{\mathcal{H}})$ such that $\varnothing$ is $\mathbb{Z}_{2}$-invariant and $\tilde{N}/\mathbb{Z}_{2} = N$. Put $\varnothing: (\tilde{N}, \tilde{\mathcal{H}}) \rightarrow (\tilde{N}, \tilde{\mathcal{H}})$ the foliated diffeomorphism generating this action.

The restriction $\varnothing: \varnothing^{-1}(M) \rightarrow M$ is the the two-fold orientation covering of $M$. The manifold $\tilde{M} = \varnothing^{-1}(M)$ is oriented and $\tilde{\mathcal{H}}$-saturated. The diffeomorphism $\varnothing: \tilde{M} \rightarrow M$ preserves the foliation $\tilde{\mathcal{F}} = \mathcal{H}|_{\varnothing^{-1}(M)}$ and the $D$-metric $\tilde{\mu} = \varnothing^* \mu$.

Since the foliation $\mathcal{F}$ is transversally oriented then the foliation $\tilde{\mathcal{F}}$ is also transversally oriented. Moreover, the diffeomorphism $\varnothing: \tilde{M} \rightarrow M$ preserves the transversally orientation of $\tilde{\mathcal{F}}$. It does not preserve the orientation of $\tilde{M}$ since $M$ is a not orientable manifold. We get that $\varnothing$ does not preserve the tangential orientation of $\tilde{\mathcal{F}}$, this gives:

$$b^* \chi_{\tilde{\mu}} = -\chi_{\mu}.$$  

The foliated manifold $\tilde{\mathcal{F}}$ is a TORF on an oriented manifold $\tilde{M}$ with $(\tilde{N}, \tilde{\mathcal{H}})$ as a zipper. So, $(\tilde{M}, \tilde{\mathcal{F}}, \tilde{\mu})$ is a $D$-triple. From Theorem 3.4 we get that $\int_{\tilde{M}}$ induces an isomorphism

$$H_{c}^{*}(\tilde{M}/\tilde{\mathcal{F}}) \cong H_{\tilde{\mu}}^{n-*}(\tilde{M}/\tilde{\mathcal{F}}),$$

where $n = \text{codim} \tilde{\mathcal{F}} = \text{codim} M \mathcal{F}$.

On the other hand, the map $\varnothing$ induces the isomorphisms $H_{c}^{*}(M/\mathcal{F}) \cong (H_{c}^{*}(\tilde{M}/\tilde{\mathcal{F}}))_{\mathbb{Z}_{2}}$ and $H_{\mu}^{*}(M/\mathcal{F}) \cong \left(H_{\tilde{\mu}}^{*}(\tilde{M}/\tilde{\mathcal{F}})\right)_{\mathbb{Z}_{2}}$ since $b^* \kappa_{\tilde{\mu}} = \kappa_{\mu}$. From (11) it suffices to prove that $\int_{\tilde{M}}$ is $\mathbb{Z}_{2}$-invariant. This comes from the equality

$$\int_{\tilde{M}} b^* \alpha \wedge b^* \beta \wedge \chi_{\tilde{\mu}} \equiv \int_{\tilde{M}} b^* \alpha \wedge b^* \beta \wedge \chi_{\tilde{\mu}} \not \equiv \int_{\tilde{M}} b^* \alpha \wedge b^* \beta \wedge \chi_{\tilde{\mu}},$$

where $\alpha \in \Omega_{c}^{*}(\tilde{M}/\tilde{\mathcal{F}})$ and $\beta \in \Omega^{n-*}(\tilde{M}/\tilde{\mathcal{F}})$.

\begin{itemize}
  \item[(a)] The above proof gives also that the pairing $I: H_{c}^{*}(M/\mathcal{F}) \oplus H_{\mu}^{n-*}(M/\mathcal{F}) \rightarrow \mathbb{R}$ defined by $I(\alpha, \beta) = \int_{\tilde{M}} \varnothing^* \alpha \wedge \varnothing^* \beta \wedge \chi_{\mu}$, is non-degenerate.

  \item[(b)] Under the assumptions of Theorem 3.4 we also have

  $$H_{c}^{*}(M/\mathcal{F}) \cong H_{\mu, c}^{n-*}(M/\mathcal{F}),$$

where the twisted cohomology is with compact supports.
\end{itemize}
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