Braneworld Kaluza-Klein Corrections in a Nutshell

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Evaluating Kaluza-Klein (KK) corrections is indispensable to test the braneworld scenario. In this paper, we propose a novel symmetry approach to a 4-dimensional effective action with KK corrections for the Randall-Sundrum two-brane system. The result can be used to assess the validity of the low energy approximation. Also, our result provides the basis for predicting CMB spectrum with KK corrections and the study of the transition from black strings to black holes.

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I. INTRODUCTION

It is generally believed that the singularity problem of the cosmology can be resolved in the context of the superstring theory. It seems that the most clear prediction of the superstring theory is the existence of extra-dimensions. This apparently contradicts our experience. Fortunately, the superstring theory itself provides a mechanism to hide extra-dimensions, which is the so-called braneworld scenario where the standard matter lives on the brane, while only the gravity can feel the bulk space-time. This scenario has been realized by Randall and Sundrum in a two-brane model [1]. Needless to say, it is important to test this new picture by the cosmological observations in the context of this model.

We need some efficient calculational method to give quantitative predictions to be compared with cosmological observations. As the observable quantities are usually represented by the 4-dimensional language, it would be advantageous if we could find purely 4-dimensional description of the braneworld which includes the enough information of the bulk geometry, i.e. KK effects. The purpose of this paper is to derive the 4-dimensional effective action with KK effects for the two-brane system. For this purpose, we propose a novel symmetry approach that utilize the conformal symmetry as a principle to determine the effective action. Our new method gives not only a simple re-derivation of known results [2, 3], but also a new result, i.e. the effective action with KK corrections.

The organization of this paper is as follows. In sec.II, we explain our method and re-derive known results. In sec.III, we derive a new result, i.e. the KK corrected effective action. In the final section, we summarize our results and discuss possible applications and extension of our results. Throughout this paper, we take the unit $8\pi G = 1$.

II. SYMMETRY APPROACH

In this paper, for simplicity, we concentrate on the vacuum two-brane system. Let us start with the 5-dimensional action for this system

$$S[\gamma_{AB}, g_{\mu\nu}, h_{\mu\nu}]$$

where $\gamma_{AB}$, $g_{\mu\nu}$ and $h_{\mu\nu}$ are the 5-dimensional bulk metric, the induced metric on the positive and the negative tension branes, respectively. Here, $A, B$ and $\mu, \nu$ label the 5-dimensional and 4-dimensional coordinates, respectively. The variation with respect to $\gamma_{AB}$ gives the bulk Einstein equations and the variation with respect to $g_{\mu\nu}$ and $h_{\mu\nu}$ yields the junction conditions. Now, suppose to solve the bulk equations of motion and the junction condition on the negative tension brane, then formally we get the relation

$$\gamma_{AB} = \gamma_{AB}[g_{\mu\nu}], \quad h_{\mu\nu} = h_{\mu\nu}[g_{\mu\nu}] \, .$$

By substituting relations $\Box$ into the original action, in principle, the 4-dimensional effective action can be obtained as

$$S_{\text{eff}} = S[\gamma_{AB}[g_{\mu\nu}], g_{\mu\nu}, h_{\mu\nu}[g_{\mu\nu}]] \, .$$

In practice, however, the above calculation is not feasible. Therefore, in the following, we will give a method to obtain the effective action without doing the above calculation. Our method combines the gradient expansion approach and the geometric approach.

The gradient expansion approach can be used at low energy. In this case, it is legitimate to assume that the action can be expanded by the local terms with increasing orders of derivatives if one includes all of the relevant degrees of freedom $\Box$. In the two-brane system, the relevant degrees of freedom are nothing but the metric and the radion which can be seen from the linear analysis $\Box$. Hence, we assume the general local action constructed from the metric $g_{\mu\nu}$ and the radion $\Psi$ as an
ansatz. Therefore, we can write the action as

\[ S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right] \]

\[ + \int d^4x \sqrt{-g} \left[ A(\Psi) (\nabla^\mu \Psi \nabla_\mu \Psi)^2 + B(\Psi) (\Box \Psi)^2 \right. \]

\[ + C(\Psi) \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi + D(\Psi) R \Box \Psi \]

\[ + E(\Psi) R^\mu \nabla_\mu \Psi + F(\Psi) R^\mu \nabla_\mu \Psi \nabla_\nu \Psi + G(\Psi) R^2 \]

\[ + H(\Psi) R^\mu R_\mu + I(\Psi) R^\mu \nabla_\mu \lambda \nabla_\nu \lambda + \cdots \] , \hspace{1cm} (4)

where \( \nabla_\mu \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \). Here, we have listed up all of the possible local terms which have derivatives up to fourth-order. This series will continue infinitely. This can be regarded as the generalization of the scalar-tensor theory including the higher derivative terms. We have the freedom to redefine the scalar field \( \Psi \). In fact, we have used this freedom to fix the functional form of the coefficient of \( R \).

However, we can not determine other coefficient functions without any information about the bulk geometry. The geometric approach yields, instead of the action, the conformal symmetry \( \Box \). First, we must find \( \Lambda(\Psi) \). There are two equations \( (12) \) and \( (13) \) to be compatible, \( \Lambda \) and \( \omega \) must satisfy

\[ \Box \Psi = -\frac{\omega}{3\Psi} \nabla^\mu \nabla_\mu \Psi - \frac{4}{3}(\Lambda - \Psi) \] . (12)

This is the equation for the radion \( \Psi \). Hence, we also have the equation for \( \Psi \) from the action as

\[ \Box \Psi = \left( \frac{1}{2\Psi} - \frac{\omega'}{2\omega} \right) \nabla^\alpha \Psi \nabla_\alpha \Psi - \frac{\Psi}{2\omega} R + \frac{\Psi}{\omega} \Lambda' \] , (13)

where the prime denotes the derivative with respect to \( \Psi \). In order for these two Eqs. (12) and (13) to be compatible, \( \Lambda \) and \( \omega \) must satisfy

\[ -\frac{\omega}{3\Psi} = \frac{1}{2\Psi} - \frac{\omega'}{2\omega} \] , \hspace{1cm} (14)

\[ \frac{4}{3}(\Lambda - \Psi) = \frac{\Psi}{\omega} (2\Lambda - \Lambda') \] , (15)

where we used \( R = 4\lambda \) which comes from the trace part of Eq. (7). Eqs. (14) and (15) can be integrated as

\[ \Lambda(\Psi) = \lambda + \lambda \beta (1 - \Psi)^2 \] , \hspace{1cm} (16)

where the constant of integration \( \beta \) represents the ratio of the cosmological constant on the negative tension geometry. Notice that the property (8) implies the conformal invariance of this effective matter. Clearly, both approaches should agree to each other. Hence, the radion must play a role of the conformally invariant matter \( E_{\mu\nu} \). This requirement gives a stringent constraint on the action, more precisely, the conformal symmetry (8) determines radion dependent coefficients in the action (4).

Let us illustrate our method using the following truncated action

\[ S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right] \] (10)

which is nothing but the scalar-tensor theory with coupling function \( \omega(\Psi) \) and the potential function \( \Lambda(\Psi) \). Note that this is the most general local action which contains up to the second order derivatives and has the general coordinate invariance. It should be stressed that the scalar-tensor theory is, in general, not related to the braneworld. However, we know a special type of scalar-tensor theory corresponds to the low energy braneworld [2,3]. Here, we will present a simple derivation of this known fact. First, we must find \( E_{\mu\nu} \). The above action gives the equations of motion for the metric as

\[ G_{\mu\nu} = -\frac{\lambda}{\Psi} \nabla_\mu \nabla_\nu \Psi - E_{\mu\nu} \] , (7)

\[ G_{\mu\nu} = \frac{\Lambda}{\Psi} g_{\mu\nu} + \frac{1}{\Psi} (\nabla_\mu \nabla_\nu \Psi - g_{\mu\nu} \Box \Psi) \]

\[ + \frac{\omega}{\Psi^2} \left( \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Psi \nabla_\alpha \Psi \right) \] (11)

One defect of this approach is that \( E_{\mu\nu} \) is not determined except for the following property

\[ E^\mu_\mu = 0 . \] (8)

For the isotropic homogeneous universe, Eq. (8) has sufficient information to deduce the cosmological evolution equation

\[ H^2 = \frac{\lambda}{3} + C \] , \hspace{1cm} (9)

where \( C \) is the constant of integration. This effect of the bulk acts as radiation fluid, hence it is called as dark radiation. For general spacetimes, however, this traceless condition is not sufficient to determine the evolution of the braneworld.

In the former approach, we have introduced the radion explicitly. While the radion never appears in the latter approach, instead \( E_{\mu\nu} \) is induced as the effective energy-momentum tensor reflecting the effects of the bulk
brane to that on the positive tension brane. Here, one of the constants of integration is absorbed by rescaling of $\Psi$. In doing so, we have assumed the constant of integration is positive. We can also describe the negative tension brane to that on the positive tension brane. Here, one of the coefficients in the action (4) represent the effects of KK-modes. In the case of the single-brane model, this contribution is non-local. While, in the two-brane model, as we have shown explicitly in (2), the KK-effects can be represented by the local terms if we introduce the radion field.

Thus, we get the effective action

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Psi R - \frac{3}{4(1-\Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda \beta (1-\Psi)^2 \right].$$

(17)

Surprisingly, this completely agrees with the previous result. Surprisingly enough, our simple symmetry principle $E^\mu_\mu = 0$ has determined the action completely.

As we have shown in (3), if $\beta < -1$ there exists a static deSitter two-brane solution which turns out to be unstable. In particular, two inflating branes can collide at $\Psi = 0$. This process is completely smooth for the observer on the brane. This fact led us to the born-again scenario. The similar process occurs also in the ekpyrotic (cyclic) model where the moduli approximation is used. It can be shown that the moduli approximation is nothing but the lowest order truncation of the low energy gradient expansion method developed by us. Hence, it is of great interest to see the leading order corrections due to KK modes to this process.

### III. KK CORRECTIONS

Let us extend the result in the previous section to the higher order case. We have already determined the functions $\Lambda(\Psi)$ and $\omega(\Psi)$. From the linear analysis, the action in the previous section is known to come only from zero modes. Hence, one can expect the other coefficients in the action (4) represent the effects of KK-modes. In the case of the single-brane model, this contribution is non-local. While, in the two-brane model, as we have shown explicitly in (2), the KK-effects can be represented by the local terms if we introduce the radion field.

Now we impose the conformal symmetry on the fourth order derivative terms in the action (4) as we did in the previous section. Starting from the action (4), one can read off the equation for the metric and hence $E^\mu_\mu$ can be identified. The compatibility condition between $E^\mu_\mu = 0$ and the equation for the radion $\Psi$ leads to

$$C' + 3E'' + \frac{3}{2} F''$$

(18)

$$2B' - 2C = 0$$

(19)

$$2B = (1-\Psi)[6B' - 2(1-\Psi)B'' - C + (1-\Psi)C']$$

(20)

$$3(1-\Psi)A' = C'$$

(21)

$$3(1-\Psi)(2C' - 8A) = 2C + 12E' + 5F'$$

(22)

$$3(1-\Psi)(3B' - 2C) = 2B + 3D' + F$$

$$4(1-\Psi)B' = 2B + 6D' + 6E + 3F$$

$$2(1-\Psi)B = 3D$$

$$D = 2(1-\Psi)B$$

(23)

(24)

(25)

(26)

(27)

(28)

(29)

These equations seem to be over constrained. Nevertheless, one can find solutions consistently. From Eqs. (26) and (29), we see $F = 0$. Eqs. (23) and (24) can be solved as

$$E = \frac{1}{3}(1-\Psi)C, \quad D = \frac{2}{3}(1-\Psi)B.$$ (30)

Substituting these into Eqs. (18) - (22) and Eqs. (27) and (28), we have

$$3(1-\Psi)A' = C'$$

(31)

$$3(1-\Psi)(2C' - 8A) = 2C + 12E' + 5F'$$

(32)

$$4(1-\Psi)A = C$$

(33)

$$B' - 2C = 0$$

(34)

$$3(1-\Psi)A = C$$

(35)

$$2B = (1-\Psi)[6B' - 2(1-\Psi)B'' - C + (1-\Psi)C']$$

(36)

$$3B = (1-\Psi)[2B' - C].$$

(37)

Combining Eqs. (34) and (35), we obtain

$$B = \frac{\ell^2}{(1-\Psi)^2},$$

(38)

where $\ell^2$ is the constant of integration representing the curvature scale of the bulk. Eqs. (35) and (33) give

$$C = \frac{\ell^2}{(1-\Psi)^2}, \quad A = \frac{\ell^2}{4(1-\Psi)^2}.$$ (39)

The rest of Eqs. (31), (32), (36) and (37) are identically satisfied. The coefficients $G, H$ and $I$ must be constants $g, h$ and $i$. Because of the existence of the Gauss-Bonnet topological term, we can put $i = 0$ without losing the generality. The constants $g$ and $h$ can be interpreted as the variety of the effects of the bulk gravitational waves.

Thus, we find the 4-dimensional effective action with KK corrections as

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Psi R - \frac{3}{4(1-\Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda \beta (1-\Psi)^2 \right] + \ell^2 \int d^4x \sqrt{-g} \left[ \frac{1}{4(1-\Psi)^2} (\nabla^\mu \Psi \nabla_\mu \Psi)^2 + \frac{1}{(1-\Psi)^2} \nabla^\mu \Psi \nabla_\mu \Psi \square \Psi + \frac{2}{3(1-\Psi)} R \square \Psi + \frac{1}{3(1-\Psi)^2} R \nabla^\mu \Psi \nabla_\mu \Psi + g R^2 + h R^\mu_\nu R^\nu_\mu \right].$$

(40)
It should be noted that this action becomes non-local after integrating out the radion field. This fits the fact that KK effects are non-local usually. In principle, we can continue this calculation to any order of derivatives. There are many possible applications using our effective action which will be published in separate papers. Here, we only mention the collision of two branes at low energy. In the ekpyrotic (cyclic) model and born-again model, the cosmological evolution was completely smooth at the collision time in the Jordan frame which is physical in the braneworld picture. Here, we can say this is true even if we take into account KK corrections. This is seen from the fact the action is regular at the collision point \( \Psi = 0 \).

IV. CONCLUSION

We have established a novel symmetry approach to a 4-dimensional effective action with KK corrections. This is done by combining the low energy expansion of the action and the geometric approach. Our result supports the smoothness of the collision process of two branes advocated in the ekpyrotic (cyclic) model and born-again model. Our result can be used to assess the validity of the low energy approximation. It also has a potential to make concrete predictions to be compared with observations.

As to the cosmological applications, it is important to recognize that our action can describe the inflation. Cosmological perturbations are now ready to be studied. In fact, our result provides the basis of the prediction of CMB spectrum with KK corrections.

The black hole solutions with KK corrections are also interesting subjects. If we truncate the system at the lowest order, the static solution is Schwarzschild black hole with a trivial radion which corresponds to the black string in the bulk. The Gregory-Laflamme instability occurs when the wavelength of KK modes exceeds the gravitational length of the black hole. Clearly, the lightest KK mode is important and this mode is already included in our action, hence it would be interesting to investigate if the Gregory-Laflamme instability occurs or not within our theory.

As an extension of our analysis, we can incorporate the matter fields both on the brane and in the bulk. As to the matter on the brane, at the lowest order, we obtain the same functional for \( \omega \) and the coupling the radion with the matter is that previously obtained. Inclusion of KK-effects is also possible if we take the scalar field as the matter. It is rather straightforward to incorporate bulk fields into our scheme.

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