In the large-$N$ limit, we study saddle points of two SYK chains coupled by an interaction that is nonlocal in Euclidean time. We study the free model with the order of the fermionic interaction $q = 2$ analytically and also investigate the model with interaction in the case $q = 4$ numerically. We show that in both cases, there is a nontrivial phase structure with an infinite number of phases. Each phase corresponds to a saddle point in the noninteracting two-replica SYK. The nontrivial saddle points have a nonzero value of the replica-nondiagonal correlator in the sense of quasiaveraging if the coupling between replicas is turned off. The nonlocal interaction between replicas thus provides a protocol for turning the nonperturbatively subleading effects in SYK into nonequilibrium configurations that dominate at large $N$. For comparison, we also study two SYK chains with local interaction for $q = 2$ and $q = 4$. We show that the $q=2$ model has a similar phase structure, while the phase structure differs in the $q=4$ model, dual to the traversable wormhole.

Keywords: SYK model, large-$N$ limit, nonperturbative effect, replica-nondiagonal solution, quasiaverage, spontaneous symmetry breaking

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1. Introduction

The Sachdev–Ye–Kitaev (SYK) model [1]–[4] is an example of a maximally chaotic holographic quantum theory in 0+1 dimensions that is solvable at large $N$. An important feature of the SYK model is that the exact path integral can be formulated in terms of bilocal collective fields [2], [5], [6]. This framework allows performing the $1/N$ expansion using a steepest-descent method.

The theory dual to SYK contains the two-dimensional Jackiw–Teitelboim (JT) theory [7]–[12] as the gravity subsector. More correctly, the holographic correspondence dictates that the partition function of $M$ copies (replicas) of the SYK model should be equal to the partition function of the bulk theory that includes the JT gravity on constant negative curvature space–times with $M$ boundaries [13]–[18]. Therefore, we can regard the collective-replica field formulation of the SYK model as a representation of the bulk-theory path integral. From this standpoint, the saddle points of the path integral correspond to some bulk space–time geometries of constant negative curvature with $M$ boundaries, and the perturbative contributions to the $1/N$ expansion come from quantum gravity effects on a fixed geometry of a saddle point.
Recent studies of the quantum chaotic nature of SYK-type theories and random matrices [13], [14], [19]–[28] show that nonperturbative effects in the $1/N$ expansion of SYK-type models play essential role in manifestations of quantum chaos and the universality of random matrices. In particular, it was shown that the replica-nondiagonal saddle points of the path integral over the replica field are related to the ramp behavior of the spectral form factor [14]. Replica-nondiagonal structures of saddle points of SYK-like models were also studied [21], [29]–[34], in particular, in relation to the problem of spin glasses and replica symmetry breaking. From the gravity standpoint, the nonperturbative effects arising from the subleading saddle points are related to the topologies of higher genus [15], [17], [35] and to the discrete spectrum of black hole microstates; they are also important for recovering lost information [13]–[15].

We previously obtained [33] a family of exact replica-nondiagonal saddle points in the SYK model with $M$ replicas. We showed that these saddle points give nonperturbative contributions to the path integrals and these contributions are exponentially suppressed at large $N$ compared with the replica-diagonal saddle points. Here, we study the model of $M=2$ coupled SYK replicas with interaction, which makes these saddle points dominating. We treat the quadratic SYK$_2$ case analytically and the case of quartic interaction SYK$_4$ numerically. In both cases, the interaction between replicas generates a nontrivial phase structure with infinite number of phases. The order parameter for these phases is the replica-nondiagonal correlator. We show that on nontrivial phases, it acquires a nonzero value in the sense of Bogoliubov quasiaveraging, matching the corresponding subleading saddle point in the SYK with the interaction turned off.

This paper is organized as follows. In Sec. 2, we introduce the model of two SYK chains with nonlocal coupling. In Sec. 3, we study large-$N$ saddle points and the phase structure in the integrable version of the model with $q = 2$, where $q$ is the order of the fermionic interaction. In Sec. 4, we numerically study the phase structure in the $q=4$ version of the model, using the results in the preceding section for the initial approximation of the numerical solution of saddle point equations. In Sec. 5, we discuss the model with local coupling that is related to the two-dimensional traversable wormhole [18], [20], [36]. We also study saddle points in $q=2$ and $q=4$ versions of the model and compare them to our model with nonlocal interaction. In Sec. 6, we establish the connection between the interaction between SYK replicas and spontaneous symmetry breaking in the sense of Bogoliubov quasiaverages in replica-nondiagonal saddle points of the SYK model. Finally, in Sec. 7, we discuss the results and open questions.

2. The model of two nonlocally interacting SYK chains

We study the model of $2N$ Majorana fermions $\psi^\alpha$ in 0+1 dimensions. Here, the replica index is $\alpha = L, R$, and $i = 1, \ldots, N$ is the color (or lattice) index. The two replicas have disordered $q$-fermion interactions with Gaussian random couplings $j = \{j_{i_1, \ldots, i_q}\}$, equal between replicas for every realization of the randomness.

The total action for the fermions is given by

$$S[\psi, j] = S_L[\psi^L, j] + S_R[\psi^R, j] + S_{\text{int}}[\psi].$$

Here,

$$S_{\alpha}[\psi^\alpha, j] = \int_0^\beta d\tau \left( -\frac{1}{2} \sum_{i=1}^N \psi_i^\alpha \frac{d}{d\tau} \psi_i^\alpha - \frac{q/2}{q!} \sum_{i_1, \ldots, i_q=1}^N j_{i_1, \ldots, i_q} \psi_{i_1}^\alpha \psi_{i_2}^\alpha \cdots \psi_{i_q}^\alpha \right)$$

is the single-replica SYK$_q$ action. The number $q$ is the order of the fermionic interaction. Here, we consider the cases $q = 2$ and $q = 4$. 

\[1\]In this paper, we work exclusively in the Euclidean signature.
We also have the term in (2.1) with the interaction between replicas, which has the bilocal form

\[ S_{\text{int}} = \int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \sum_{i=1}^N \psi_i^L(\tau_1)\eta_{L,R}(\tau_1 - \tau_2)\psi_i^R(\tau_2). \]  

(2.3)

We choose the interaction \( \eta \) such that at a finite temperature \( T = \beta^{-1} \),

\[ \eta_{L,R}(\tau_1 - \tau_2) = \eta_{RL}(\tau_1 - \tau_2) = \frac{\nu}{\beta \sin(\pi(\tau_1 - \tau_2)/\beta)}. \]  

(2.4)

At zero temperature, we have

\[ \eta_{L,R}(\tau_1 - \tau_2) = \eta_{RL}(\tau_1 - \tau_2) = \frac{\nu}{\pi(\tau_1 - \tau_2)}. \]  

(2.5)

We choose the interaction of this form because in the frequency space in both cases (2.4) and (2.5), it has the simple form

\[ \eta_{L,R}(\omega) = \eta_{RL}(\omega) = i\nu \text{sgn}(\omega). \]  

(2.6)

Indeed,

\[ \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega \ \text{sgn}(\omega)e^{-i\omega\tau} = \frac{1}{\pi\tau}, \]  

(2.7)

where the integral is understood in the sense of principal value. At a finite temperature, the frequency is quantized as

\[ \omega_n = \frac{2\pi}{\beta} \left( n + \frac{1}{2} \right), \quad n \in \mathbb{Z}, \]  

(2.8)

and the Fourier transform in this case is

\[ \frac{i}{\beta} \sum_{n=-\infty}^{+\infty} \text{sgn}(\omega_n)e^{-i\omega_n\tau} = \frac{1}{\beta \sin(\pi\tau/\beta)}. \]  

(2.9)

We note that we must also understand integral (2.3) in the regularized sense (principal value). The constant \( \nu \) has the dimension of energy and determines the strength of the interaction between replicas. We note that for \( q = 2 \), action (2.3) looks similar to the single-replica nonlocal kinetic term considered in [37]. On the other hand, in previous works where coupled replicas were considered [14], [18], [20], [36], [38], the interaction was local.

The partition function is given by

\[ Z_4(\beta) = \int D\psi e^{-S[\psi,j]}. \]  

(2.10)

Performing the standard procedure of averaging over the Gaussian disorder and integrating over the fermions (see [4] for the detailed derivation), we obtain the path integral over the bilocal collective replica fields \( G_{\alpha\beta}(\tau_1, \tau_2) \) and \( \Sigma_{\alpha\beta}(\tau_1, \tau_2) \)

\[ Z(\eta) = \int DG \ D\Sigma e^{-NI[G, \Sigma; \eta]}, \]  

(2.11)

where

\[ I[G, \Sigma; \eta] = -\log \text{Pf}[\delta_{\alpha\beta}\partial_{\tau} - \hat{\Sigma}_{\alpha\beta}] + \frac{1}{2} \int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \left( \Sigma_{\alpha\beta}(\tau_1, \tau_2)G_{\alpha\beta}(\tau_1, \tau_2) - \frac{J^2}{q} G_{\alpha\beta}(\tau_1, \tau_2)^q \right) - \frac{1}{2} \int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \eta_{\alpha\beta}(\tau_1, \tau_2)G_{\alpha\beta}(\tau_1, \tau_2). \]  

(2.12)
We assume that the bilocal fields satisfy the antisymmetry condition
\[ G_{\alpha\beta}(\tau_1, \tau_2) = -G_{\beta\alpha}(\tau_2, \tau_1), \quad \Sigma_{\alpha\beta}(\tau_1, \tau_2) = -\Sigma_{\beta\alpha}(\tau_2, \tau_1). \] (2.13)

The interaction between replicas of form (2.3), which is bilinear in fermions, is equivalent to including the source term for the field \( G \). More specifically,
\[ \eta(\tau_1, \tau_2) = \begin{pmatrix} 0 & \zeta(\tau_1 - \tau_2) \\ \zeta(\tau_1 - \tau_2) & 0 \end{pmatrix}, \] (2.14)

where the function \( \zeta \) has form (2.6) in the frequency space and (2.4) or (2.5) in the coordinate space. We note that this is a symmetric matrix in the replica space.

If we assume that \( \nu = \text{const} \), then the Fourier transform to the temporal representation is given by formula (2.5) at zero temperature and (2.4) at a finite temperature. Both those expressions define integral kernels of antisymmetric operators in the space of functions of time. The combined symmetry properties of the source agree with the general antisymmetry condition
\[ \eta_{LR}(\tau_1, \tau_2) = -\eta_{RL}(\tau_2, \tau_1). \] (2.15)

We are interested in studying saddle points of path integral (2.11). The saddle point equations are
\[ \partial_\tau G_{\alpha\gamma}(\tau, \tau'') - \int d\tau' G_{\alpha\beta}(\tau, \tau') \Sigma_{\beta\gamma}(\tau', \tau'') = \delta_{\alpha\gamma} \delta(\tau - \tau''), \] (2.16)
\[ \Sigma_{\alpha\beta}(\tau, \tau') = J^2 G_{\alpha\beta}(\tau, \tau')^{q-1} + \eta_{\alpha\beta}(\tau, \tau'). \] (2.17)

The replica-symmetric form of the source with (2.17) taken into account also prescribes a replica-symmetric form of the solutions:
\[ G_{LL} = G_{RR} = G_0, \quad G_{LR} = G_{RL} = G_1, \] (2.18)

where \( G_0 \) and \( G_1 \) are both assumed to be odd functions in the frequency and time representations and to be antiperiodic in time. In [33], the solutions constructed in the noninteracting-replica case were obtained under the same assumptions.

In concluding this section, we make some remarks for future reference.

**Remark 1.** One more possibility to satisfy condition (2.15) is to consider the case where \( \eta \) is antisymmetric in the replica space and symmetric in the coordinate space. This case is widely considered in literature, in particular, in the context of traversable wormholes and the chaotic thermofield-double dynamics [14], [18], [17], [36] and \( O(N) \) symmetry breaking [38]. We consider it separately in Sec. 5.

**Remark 2.** An important assumption about the source \( \eta \) is that it depends only on difference of times. We thus focus only on solutions that are translation invariant under simultaneous translation in both replicas, breaking a part of the total translational symmetry. We discuss this in Sec. 6.

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\(^2\) An equivalent way to think about \( \eta \) (up to a straightforward redefinition of field variables) is as a shift of the \( \Sigma \) field [30], [31].
3. Quadratic case

3.1. Equations of motion. We start the study of saddle points and phase structure with the case $q = 2$, which corresponds to the integrable variant of the SYK model or random mass fermions. We seek solutions of Eqs. (2.16) and (2.17). With the ansatz for $G$ in form (2.18), saddle point equation (2.17) in the frequency space has the form

$$
\Sigma_0(\omega) = J^2 G_0(\omega), \quad \Sigma_1(\omega) = J^2 G_1(\omega) + i\nu \text{sgn}(\omega).
$$

(3.1)

Substituting this in Eq. (2.16), we obtain a system of two algebraic equations for two unknowns:

$$
-i\omega G_0(\omega) - J^2 G_0(\omega) G_0(\omega) - (J^2 G_1(\omega) + i\nu \text{sgn}(\omega)) G_1(\omega) = 1,
$$

$$
-i\omega G_1(\omega) - J^2 G_0(\omega) G_1(\omega) - (J^2 G_1(\omega) + i\nu \text{sgn}(\omega)) G_0(\omega) = 0.
$$

(3.2)

This form of the equations holds in both the zero and finite temperature cases for $q = 2$. In the case $\nu = 0$, we recover the equations for the decoupled replicas for $M = 2$, which were solved in [33].

Before proceeding to discuss the solutions, we note that a peculiar feature of the $q=2$ case is that the source $\eta$ in form (2.6) respects the conformal symmetry in the IR limit and transforms with the fixed conformal dimension $\Delta = 1/2$. The same form of the source breaks the conformal symmetry in the IR limit in the cases where $q > 2$.

3.2. Solutions. We now discuss the solutions. We focus on the case of a finite temperature. In this case, the frequencies are quantized according to Matsubara rule (2.8). Equations (3.2) for a general value of $\nu$ admit four solutions for every Matsubara frequency. They are given by the formulas

$$
G_0^{(1)}(\omega_n) = -\frac{i}{2J^2} \left[ \omega_n - \frac{\text{sgn}(\omega_n)}{\sqrt{2}} A_1 \right],
$$

(3.3a)

$$
G_1^{(1)}(\omega_n) = \frac{i}{8J^2\nu\omega_n} \left( \sqrt{2} A_1 (4J^2 + \omega_n^2 - B) - \nu^2 (4|\omega_n| - \sqrt{2} A_1) \right),
$$

(3.3b)

$$
G_0^{(2)}(\omega_n) = -\frac{i}{2J^2} \left[ \omega_n + \frac{\text{sgn}(\omega_n)}{\sqrt{2}} A_1 \right],
$$

(3.3c)

$$
G_1^{(2)}(\omega_n) = \frac{i}{8J^2\nu\omega_n} \left( -\sqrt{2} A_1 (4J^2 + \omega_n^2 - B) - \nu^2 (4|\omega_n| + \sqrt{2} A_1) \right),
$$

(3.3d)

$$
G_0^{(3)}(\omega_n) = -\frac{i}{2J^2} \left[ \omega_n + \frac{\text{sgn}(\omega_n)}{\sqrt{2}} A_2 \right],
$$

(3.3e)

$$
G_1^{(3)}(\omega_n) = \frac{i}{8J^2\nu\omega_n} \left( -\sqrt{2} A_2 (4J^2 + \omega_n^2 + B) - \nu^2 (4|\omega_n| + \sqrt{2} A_2) \right),
$$

(3.3f)

$$
G_0^{(4)}(\omega_n) = -\frac{i}{2J^2} \left[ \omega_n - \frac{\text{sgn}(\omega_n)}{\sqrt{2}} A_2 \right],
$$

(3.3g)

$$
G_1^{(4)}(\omega_n) = \frac{i}{8J^2\nu\omega_n} \left( \sqrt{2} A_2 (4J^2 + \omega_n^2 + B) - \nu^2 (4|\omega_n| - \sqrt{2} A_2) \right).
$$

(3.3h)

Here, we introduce the auxiliary notation

$$
A_1 = \sqrt{4J^2 + \omega_n^2 + \nu^2 + B}, \quad A_2 = \sqrt{4J^2 + \omega_n^2 + \nu^2 - B}, \quad B = \sqrt{16J^2\omega_n^2 + (4J^2 + \nu^2 - \omega_n^2)^2}.
$$

(3.4)
We expand these solutions (3.3) in small $\nu$. Taking leading and subleading terms into account, we obtain

\begin{align}
G_0^{(1)}(\omega_n) &= -\frac{i\omega_n + i\text{sgn}(\omega_n)\sqrt{4J^2 + \omega_n^2}}{2J^2} + \frac{i\text{sgn}(\omega_n)}{(4J^2 + \omega_n^2)^{3/2}}\nu^2 + O(\nu^4), \\
G_1^{(1)}(\omega_n) &= \frac{i\omega_n - i\text{sgn}(\omega_n)\sqrt{4J^2 + \omega_n^2}}{2J^2} - \frac{i\omega_n}{(4J^2 + \omega_n^2)^{3/2}}\nu^2 + O(\nu^4), \\
G_0^{(2)}(\omega_n) &= -\frac{i\omega_n - i\text{sgn}(\omega_n)\sqrt{4J^2 + \omega_n^2}}{2J^2} - \frac{i\text{sgn}(\omega_n)}{(4J^2 + \omega_n^2)^{3/2}}\nu^2 + O(\nu^4), \\
G_1^{(2)}(\omega_n) &= \frac{i\omega_n - i\text{sgn}(\omega_n)\sqrt{4J^2 + \omega_n^2}}{2J^2} + \frac{i\omega_n}{(4J^2 + \omega_n^2)^{3/2}}\nu^2 + O(\nu^4), \\
G_0^{(3)}(\omega_n) &= -\frac{i\omega_n}{2J^2} - \frac{i\omega_n}{2J^2\sqrt{4J^2 + \omega_n^2}}\nu + O(\nu^3), \\
G_1^{(3)}(\omega_n) &= -i\text{sgn}(\omega_n)\frac{\sqrt{4J^2 + \omega_n^2}}{2J^2} - \frac{i\text{sgn}(\omega_n)}{2J^2}\nu + O(\nu^3), \\
G_0^{(4)}(\omega_n) &= -\frac{i\omega_n}{2J^2} + \frac{i\omega_n}{2J^2\sqrt{4J^2 + \omega_n^2}}\nu + O(\nu^3), \\
G_1^{(4)}(\omega_n) &= i\text{sgn}(\omega_n)\frac{\sqrt{4J^2 + \omega_n^2}}{2J^2} - \frac{i\text{sgn}(\omega_n)}{2J^2}\nu + O(\nu^3).
\end{align}

Comparing the obtained formulas with the results in Sec. 3.1 in [33] in the decoupled case $\nu = 0$, we see that in the limit $\nu \to 0$, the first and second solutions become replica-diagonal solutions while the third and fourth solutions become replica-nondiagonal solutions. It is also important that for the first and second solutions, the subleading powers in $\nu$ do not change the leading UV asymptotic behavior as $\omega_n \to \infty$.

The solutions of the full SYK$_2$ model are constructed by independently choosing any of the four roots for each Matsubara frequency. If we choose the first root for every Matsubara frequency, then we obtain the standard saddle point [3], [13]. As $\nu \to 0$, it tends to the saddle point that dominates saddle points determined by other solutions [33]. To obtain a UV-finite action, we must consider the solutions for which the second, third, or fourth roots are chosen for only a finite number of Matsubara modes and the UV-asymptotic behavior is determined exclusively by the first root.

### 3.3. On-shell action and phase structure

We now discuss the contributions of saddle points to path integral (2.11). To study the dominance of the saddle points, we consider the action density $\rho$ defined by the formula

$$I[G, \Sigma]_{\text{on-shell}} = \sum_n \rho(\omega_n, J, \nu).$$

The explicit expression for $\rho$ is obtained from (2.12) by setting $q = 2$, using Eq. (2.17), and passing to the frequency space. The action density is written as

$$\rho = -1 + \frac{J^2}{2}m,$$

where we have the contribution from the Pfaffian term

$$1 = \log\left(1 + \frac{J^2G_0(\omega_n)}{i\omega_n}\right)^2 - \left(\frac{J^2G_1(\omega_n) + i\nu\text{sgn}(\omega_n)}{i\omega_n}\right)^2.$$

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Fig. 1. Action density (3.10) as a function of the number \( n \) of Matsubara frequencies for four solutions with \( J = 1 \) and \( T = 0.1 \) for different values of \( \nu \): (a) \( \nu = 0 \), (b) \( \nu = \pi T \simeq 0.314 \), and (c) \( \nu = 2 \).

and the polynomial part

\[ m = |G_0(\omega_n)|^2 + |G_1(\omega_n)|^2. \]

(3.9)

For every Matsubara mode, we have four different branches of solutions given by Eqs. (3.3). Every solution contributes to the action density, which is denoted by

\[ \rho^{(j)}(\omega_n, J, \nu) = -1^{(j)} + \frac{J^2}{2}m^{(j)}, \quad j = 1, 2, 3, 4. \]

(3.10)

We present plots of \( \rho^{(j)} \) as function of \( n \) for different values of the coupling \( \nu \) in Fig. 1. For the decoupled case \( \nu = 0 \), as shown in [33], the first solution, which corresponds to the standard replica-diagonal saddle, has the lowest action and hence the lowest action density for every Matsubara mode (see Fig. 1a). The second solution, which is also replica-diagonal, has the highest action. Meanwhile, the third and fourth solutions have the same action. As \( \nu \) increases, the action density of the third solution increases, while the action density of the fourth solution decreases. As shown in Fig. 1b, at a certain value of \( \nu \), we have \( \rho^{(1)}(\pm \omega_0) = \rho^{(4)}(\pm \omega_0) \). This signifies the first phase transition: after that point, the solution with \( G(\pm \omega_0) = G^{(4)}(\pm \omega_0) \) dominates the solution involving the \( G^{(1)} \) root for all Matsubara modes. The value of the action on the dominant saddle point develops a discontinuity of the first derivative with respect to temperature, which indicates a first-order phase transition. As \( \nu \) increases further, the curve \( \rho^{(1)} \) goes down further, and more phase transitions occur. In Fig. 1, the dominant saddle point has \( G(\pm \omega_0) = G^{(4)}(\pm \omega_0) \) and \( G(\pm \omega_1) = G^{(4)}(\pm \omega_1) \) (the other Matsubara modes are still defined by \( G^{(1)} \)).

In summary, when the coupling increases or, equivalently, the temperature is lowered, we encounter an infinite number of phase transitions. Each phase transition consecutively replaces the root \( G^{(1)} \) with \( G^{(4)} \) for the Matsubara modes starting from \( \pm \omega_0 \) on the dominant saddle point. We introduce some terminology for these phases.
Fig. 2. Phase diagram of two nonlocally coupled SYK2 chains in the ($\nu, T$) plane, as defined by Eq. (3.12): the numbers label the nontrivial phases.

- The phase where the dominant saddle point has $G(\omega_n) = G^{(1)}(\omega_n)$ for all $n$ is called the paramagnetic phase.
- We label a nonparamagnetic phase with the bold integer $n_0$ if we have $G(\pm \omega_n) = G^{(4)}(\pm \omega_n)$ for $n < n_0$ and $G(\omega_n) = G^{(1)}(\omega_n)$ for the other Matsubara modes. For example, phase 0 is defined by $G(\pm \omega_0) = G^{(4)}(\pm \omega_0)$ and $G(\omega_n) = G^{(1)}(\omega_n)$ for the other $n$.

We consider the transition from phase $n - 1$ to phase $n$. We can find the critical curve for such a phase transition from the equation

$$\rho^{(4)}(\omega_n(T), J, \nu) = \rho^{(1)}(\omega_n(T), J, \nu)$$

for any $n$. Performing some straightforward but tedious algebra, we can verify that the solution of this equation depends linearly on $n$:

$$\nu_{cr} = 2\pi T_{cr} \left( n + \frac{1}{2} \right).$$

We note that there is no dependence on $J$. The corresponding phase diagram in the plane of the replica coupling $\nu$ and the temperature $T$ is shown in Fig. 2. We make some comments on it.

- At any given temperature (coupling), we can probe an infinite number of phases by increasing the coupling (decreasing the temperature).
- At any given temperature, by decreasing the coupling quasistatically, we always reach the paramagnetic phase.

4. Quartic case

4.1. Numerical solution of saddle-point equations. Having studied the analytically tractable case where $q = 2$, we now proceed to study saddle points in the interacting case where $q = 4$. Specifically, we solve saddle point equations (2.16) and (2.17) with source (2.14) numerically, using replica-symmetric

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3The paramagnetic phase here corresponds to $-1$. 1592
ansatz (2.18). Under these assumptions, the saddle point equations are

\[-i\omega_n G_0(\omega_n) - G_0(\omega_n) \Sigma_0(\omega_n) - G_1(\omega_n) \Sigma_1(\omega_n) = 1,\]
\[i\omega_n G_1(\omega_n) + G_1(\omega_n) \Sigma_0(\omega_n) + G_0(\omega_n) \Sigma_1(\omega_n) = 0,\]  
(4.1)

\[
\Sigma_0(\tau, \tau') = J^2 G_0(\tau, \tau')^{q-1}, \quad \Sigma_1(\tau, \tau') = J^2 G_1(\tau, \tau')^{q-1} + \zeta(\tau - \tau').
\]

We solve the equations at a finite temperature, and \(\zeta\) is therefore given by (2.4) in the coordinate space. To construct numerical solutions, we use the approach described in Sec. 4.1 in [33] generalized to the case of interacting replicas. The fixed parameters of the system are \(J, \nu\), and the temperature \(T = \beta^{-1}\). The initial condition for the iteration procedure is given by a solution of the \(q=2\) version of the model. The resulting solution is obtained by iterating the equations of motion until convergence with the desired accuracy is achieved.

As in the case with decoupled replicas [33], for each \(q=2\) solution, there exists a solution in the \(q=4\) model. It is observed that for the studied solutions, the saddle-point dominance hierarchy is very similar to the \(q=2\) case. Different phases correspond to solutions obtained by iterating from the dominant solutions in the \(q=2\) model. In other words, the competition happens between the solutions obtained from \(q=2\) trial functions \(G_{\alpha\beta}(\omega_n)\) containing different numbers of Matsubara modes with the fourth root (3.3g) and (3.3h).

We use the same terminology for phases as in the \(q=2\) case.

4.2. Phase structure. To study the phase structure of the model, we compute action (2.12) on numerical solutions of Eqs. (4.1). We use the equations of motion, subtract the free part from the logarithm, and assume that fields depend only on \(u = \tau_1 - \tau_2\). We then obtain

\[I|_{\text{on-shell}} = -\frac{1}{2} \sum_n \log \det \left[ \delta_{\alpha\beta} + \frac{\Sigma_{\alpha\beta}(\omega_n)}{i\omega_n} \right] + \left( 1 - \frac{1}{q} \right) J^2 \int_0^\beta du \sum_{\alpha,\beta} G_{\alpha\beta}(u)^q |_{\text{on-shell}}.\]  
(4.2)

Constructing the numerical solutions of saddle point equations and computing the on-shell action, we find that the phase structure of the \(q=4\) model is similar to that of the \(q=2\) model: as the temperature decreases at a fixed coupling, the system undergoes an infinite number of phase transitions. Solutions corresponding to different phases are shown in Figs. 3 and 4. In Fig. 3, the coupling between replicas is weak compared with the SYK self-interaction of fermions, \(\nu = 0.1 < J = 1\). The behavior of solutions in the nonparamagnetic phases have many similarities to the replica-nondiagonal solutions of decoupled replicas in [33]. The effect of the coupling \(\nu\) reduces to shifting the long-time region by a constant and introducing extra local extremums in the short-time region. In Fig. 4, the coupling between replicas is strong compared with the SYK self-interaction of fermions, \(\nu = 5 > J = 1\). In this case, the oscillatory behavior dominates the regular SYK4-like dynamics.

To study the phase transitions, we define the annealed free energy as \(F = -T \log Z(\eta)\), where \(T = \beta^{-1}\). On the dominant saddle point in the leading order of large \(N\), we express it in terms of the on-shell action by

\[F = TN \left( -\log 2 + I|_{\text{on-shell}} \right).\]  
(4.3)

Here, \(\log 2\) is subtracted for the entropy to match the free result \(N \log 2\) for \(J = \nu = 0\).

We plot the free energy near first three critical points in Fig. 5. Specifically, we show the following phase transitions (in order of increasing temperature):

- Figs. 5a and 5d: phase 1 $\rightarrow$ phase 2,
- Figs. 5b and 5e: phase 0 $\rightarrow$ phase 1,
- Figs. 5c and 5f: paramagnetic phase $\rightarrow$ phase 0.
First, we confirm that these are first-order phase transitions. Second, we see that although the paramagnetic phase has approximately linear behavior of the free energy, which is characteristic for SYK models [3], [18], [20], [38], the nontrivial phases 0, 1, 2, … have significantly nonlinear behavior of the free energy.
Fig. 5. Free energy during phase transitions in the $q=4$ model: the bold curve is the free energy of the paramagnetic phase, the thin solid curve is the free energy of phase 0, the dashed curve is the free energy of phase 1, and the dotted curve is the free energy of phase 2. Plots (a)–(c) correspond to $\nu = 0.1$, plots (d)–(f) correspond to $\nu = 5$, and $J = 1$ in all plots. For comparison, the vertical dashed line shows the corresponding critical point of the $q=2$ model determined by Eq. (3.12).

energy. For a weak coupling $\nu$ (see Fig. 3), the total free energy decreases monotonically, as shown in Figs. 5a–5c. But for a strong coupling between replicas (see Fig. 4), the free energy is nonmonotonic and increases towards each critical point, as shown in Figs. 5d–5f.

One point that seems common for both strong and weak replica interaction is that the entropy decreases with temperature on the nontrivial phases 0, 1, 2, ... That means that the heat capacity $C = -T \partial^2 F / \partial T^2$ is negative for these phases. Moreover, in the case $\nu = 5$, the increase of the free energy indicates negative entropy. This might suggest that we should perhaps restrict the allowed values of $\nu$ such that the entropy remains positive, but we note that the heat capacity on nontrivial phases is negative for all values of $\nu$.

For comparison, we also show the critical points of the $q=2$ model, determined by phase diagram equation (3.12), by vertical dashed lines in Fig. 5. This shows that the effect of the $q=4$ self-interaction is that the critical temperatures decrease compared with the $q=2$ case, and the difference is largest for the highest-temperature critical point of the transition between the paramagnetic phase and phase 0.

5. Local replica coupling

We present an analogous study of two coupled SYK models related to the eternal traversable wormhole [18], [20], [36], [39]. In terms of bilocal replica field action (2.12), we obtain this model if we assume
that the source $\eta_{\alpha\beta}$ has the replica-antisymmetric form
\[
\hat{\eta}(\tau_1 - \tau_2) = \begin{pmatrix}
0 & \zeta(\tau_1 - \tau_2) \\
-\zeta(\tau_1 - \tau_2) & 0
\end{pmatrix},
\]
(5.1)
where
\[
\zeta(\tau) = i\mu\delta(\tau).
\]
(5.2)
In the momentum space, we simply have $\zeta(\omega_n) = i\mu$. This results in a local coupling of traversable wormhole models [36], [18] of the form
\[
H_{\text{int}} = -i\mu \sum_{i} \psi_i^L(\tau)\psi_i^R(\tau).
\]
(5.3)

We expect that the replica-antisymmetric source supports a solution
\[
G_{LL}(\tau_1, \tau_2) = G_{RR}(\tau_1, \tau_2) = G_0(\tau_1, \tau_2),
\]
\[
G_{LR}(\tau_1, \tau_2) = -G_{RL}(\tau_1, \tau_2) = G_1(\tau_1, \tau_2).
\]
(5.4)
The dynamical variables are $G_0$ and $G_1$. If we take antisymmetry condition (2.13) into account, then ansatz (5.4) implies that $G_0$ must be an odd function in the frequency space and $G_1$ must be an even function.

5.1. The case $q = 2$. As in the case of the replica-symmetric source, we can also solve the model in the $q=2$ case. With ansatz (5.4) for $G$, saddle point equations (2.16) and (2.17) have the forms
\[
\Sigma_0(\omega_n) = J^2G_0(\omega), \quad \Sigma_1(\omega) = -\Sigma_{RL}(\omega) = J^2G_1(\omega) + i\mu,
\]
(5.5)
and we have the equations in the frequency space
\[
-i\omega G_0(\omega) - J^2G_0(\omega)G_0(\omega) + (J^2G_1(\omega) + i\mu)G_1(\omega) = 1,
\]
\[
-i\omega G_1(\omega) - J^2G_0(\omega)G_1(\omega) - (J^2G_1(\omega) + i\mu)G_0(\omega) = 0.
\]
(5.6)
We note that the antisymmetric matrices do not form a closed algebra under matrix multiplication, unlike the symmetric matrices. Therefore, for an arbitrary replica number $M$, there is no guarantee for a consistent system of equations that would admit a nontrivial replica-nondiagonal solution. Fortunately, this is the case for $M = 2$.

5.1.1. Solutions. The solutions with a general value of $\mu$ are given by
\[
G_0^{(1)}(\omega_n) = \frac{i}{2J^2} \left[ \omega_n - \frac{\text{sgn}(\omega_n)}{\sqrt{2}} C_1 \right],
\]
(5.7a)
\[
G_1^{(1)}(\omega_n) = \frac{i}{8J^2\mu\omega_n} \left[ -\sqrt{2}C_1(4J^2 + \omega_n^2 - D) - \mu^2(4|\omega_n| - \sqrt{2}C_1) \right],
\]
(5.7b)
\[
G_0^{(2)}(\omega_n) = \frac{i}{2J^2} \left[ \omega_n + \frac{\text{sgn}(\omega_n)}{\sqrt{2}} C_1 \right],
\]
(5.7c)

---

4 We note that the sign of $\mu$ in our case is opposite to that in [18]. While the sign is significant for the gravity analysis [18], [39] and for exact diagonalization studies [18], [20], it is unimportant for solving the saddle point equations.

5 These assumptions about the symmetry properties of the functions correspond to the properties of correlators in the thermofield double and other studies of coupled SYK models [14], [18], [20], [38].
third and fourth solutions have a nonzero limit for $G$, density $\rho$.

Analogously to the analysis in Sec. 3.2, we can also expand these solutions in powers of $\mu$.

In Fig. 6b, the critical point is shown where solutions are always subleading. The saddle points with the third and fourth $\mu$ for the other Matsubara modes. We also note that the coupling $\mu$ does not lift the degeneracy between the third and fourth solutions, unlike in the nonlocal case. The saddle points with the third and fourth solutions are always subleading.

The critical curves are defined by the equations

$$\rho^{(2)}(\omega_n(T), J, \mu) = \rho^{(1)}(\omega_n(T), J, \mu)$$

for any $n$. This equation does not have an analytic solution, but it can be solved numerically to obtain the phase diagram. In Fig. 7, we show three critical curves with the highest temperatures. As in the
Fig. 6. Action densities (3.10) as functions of Matsubara frequency label $n$ on 4 roots for different values of $\mu$. Here $J = 1$ and $T = 0.1$ for different values of $\mu$: (a) $\mu = 0$, (b) $\mu = 1.76$, and (c) $\mu = 2.8$.

Fig. 7. Phase diagram for replica-antisymmetric solutions with local coupling $\mu$ in the $(\mu, T)$ plane, as defined by the equation (5.10). The numbers label the nontrivial phases. Here $J = 1$. Only the critical curves which correspond to the three highest temperature phase transitions are shown.

In the symmetric nonlocal case, we have an infinite number of phases. The distinctive feature here is the presence of a gap in $\mu$: at a sufficiently small (but nonzero) coupling $\mu < \mu_0$, there are no phase transitions, and the paramagnetic phase is unique.

5.2. The case $q = 4$: Traversable wormhole dual. The exact saddle point equations of the system with source (5.1) in the case $q = 4$ was studied numerically in [18], [20]. The main phenomenon that was discovered is the presence of a phase transition dual to the Hawking–Page phase transition in the bulk. We also applied our approach of numerical integration explained in [33] and outlined in Sec. 4 to this model. We summarize our findings:

- At temperatures significantly higher than the Hawking–Page temperature, there exist new solutions
that can be obtained by choosing second, third, or fourth solutions (5.7c)-(5.7h) in the $q=2$ trial function. Not all such solutions that we find have an action higher than that of the standard saddle point.

- The Maldacena–Qi hysteresis [18] of the free energy is reproduced by our approach, as shown by the bold curve in Fig. 8. In terms of the trial functions, the $q=4$ solutions on the horizontal branch of the hysteresis are obtained from $q=2$ solutions with a sufficiently large number of second roots among the Matsubara modes (in our case, 14). The solutions on the linearly decaying branch are obtained from the trial function with the first root for all Matsubara modes.

- We also found that at low temperatures, the saddle points with $G^{(2)}$ solutions in the trial function do not exist. They appear around the Hawking–Page transition and have a higher free energy, which moreover increases with temperature. In Fig. 8, the thin curve is the free energy of a $q=4$ saddle point obtained from the solution with $G_{\alpha\beta}(\pm \omega_0) = G^{(2)}_{\alpha\beta}(\pm \omega_0)$, which corresponds to phase 0 in the $q=2$ model.

Therefore, we conclude that the phase diagram of the traversable wormhole model with $q = 4$ differs essentially from its $q = 2$ counterpart because of the Hawking–Page transition, which does not occur in the $q=2$ case. This is a key distinction between the traversable wormhole model and our model with a nonlocal symmetric coupling. More specifically, the model with interaction (2.4) promotes the subleading saddle points of the two decoupled SYK replicas into the nontrivial phase structure, while the model dual to the traversable wormhole cannot do this because of the Hawking–Page transition.

### 6. Symmetry breaking and quasiaveraging

This section is devoted to discussing patterns of continuous symmetry breaking in two SYK replicas with and without interaction.

#### 6.1. Spontaneous symmetry breaking in two decoupled replicas

If the interaction is turned off, $\eta = 0$, then path integral (2.11) at a finite temperature is invariant under time translations in each replica $U(1) \times U(1)$. Replica-diagonal saddle points preserve this symmetry, but replica-nondiagonal saddle
points, generally speaking, spontaneously break this symmetry. In particular, the saddle points for which $G_{\alpha\beta}$ depend only on the difference of times spontaneously break the full time-translation symmetry as $U(1) \times U(1) \to U(1)$. This has important consequences. In particular, it was shown in [14] that this symmetry breaking in the spectral form factor explains the ramp behavior.

Because of the spontaneous symmetry breaking, we can generate other replica-nondiagonal solutions by acting with a broken generator. The broken symmetry acts on the components of a solution for the field $G(\tau_1 - \tau_2)$ as

$$G_{LL}(\tau) \to G_{LL}(\tau), \quad G_{RR}(\tau) \to G_{RR}(\tau), \quad (6.1)$$

$$G_{LR}(\tau) \to G_{LR}(\tau + \alpha), \quad G_{RL}(\tau) \to G_{RL}(\tau - \alpha), \quad (6.2)$$

In the coordinate space and as

$$G_{LL}(\omega) \to G_{LL}(\omega), \quad G_{RR}(\omega) \to G_{RR}(\omega), \quad (6.3)$$

$$G_{LR}(\omega) \to e^{-i\omega\alpha}G_{LR}(\omega), \quad G_{RL}(\omega) \to e^{+i\omega\alpha}G_{RL}(\omega) \quad (6.4)$$

in the frequency space. It can be verified that for generic $q$, on-shell action (4.2) is invariant under these transformations if $\eta = 0$. We now make a few remarks.

**Remark 3.** Because the replica-nondiagonal solutions are subleading at large $N$ in the decoupled case, we here use the term “spontaneous symmetry breaking” in a generalized sense. It does not imply the existence of an ordered phase with spontaneously broken symmetry in the thermodynamic limit $N \to \infty$, but also takes subleading saddle points into account.

**Remark 4.** An important particular transformation of form (6.2) is realized by setting $\alpha = \beta/2$. It can be used to map replica-symmetric solutions in the SYK with $M=2$ decoupled replicas to replica-antisymmetric solutions related to thermofield double correlators [33]. This simple map works only for $M=2$ replicas, but for general $M$, we can in principle generate other solutions, starting from the replica-symmetric solutions or solutions with a Parisi pattern of replica symmetry breaking.

**Remark 5.** For $q = 2$, the Matsubara modes are decoupled. Therefore, symmetry (6.4) can be treated as a gauge symmetry with $\alpha = \alpha(\omega_n)$.

### 6.2. Quasiaveraging.

The interaction between the replicas works as a source term in the bilocal replica field action. As mentioned at the end of Sec. 2, it converts the spontaneous breaking of the time translation symmetry into an explicit one. In other words, the source $\eta$ lifts the degeneracy between the replica-nondiagonal solutions of the decoupled system. In Secs. 3 and 4, we showed that the source of the particular form (2.4) generates a nontrivial phase structure from the replica-nondiagonal solutions with the lowest free energy after the split of the degeneracy. To work toward the physical implications of these effects, it is important to define a quantity that is sensitive to the lifting of the degeneracy.

In the SYK with two decoupled replicas, the exact two-point correlator is defined as

$$G_{\alpha\beta}(\tau_1, \tau_2) = \frac{\delta}{\delta \eta_{\alpha\beta}(\tau_1, \tau_2)} \log \mathcal{Z}(\eta) \bigg|_{\eta=0}. \quad (6.5)$$

We are specifically interested in the case where $\alpha \neq \beta$. In terms of path integral (2.11), we have

$$G_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{\mathcal{Z}(0)} \int DG D\Sigma e^{-NI[G,\Sigma;\eta]}G_{\alpha\beta}(\tau_1, \tau_2) \big|_{\eta=0}. \quad (6.6)$$
We want to study this quantity in the thermodynamic limit $N \to \infty$, but we have the spontaneous symmetry breaking, which is made explicit by the source $\eta$. Therefore, we must especially attend to the order of taking the limits $N \to \infty$ and $\eta \to 0$ [40].

We can define the usual quantum average in the thermodynamic limit

$$\langle G_L R(T_1, T_2) \rangle = \lim_{N \to \infty} \frac{1}{Z(0)} \int DG D\Sigma e^{-NI[G, \Sigma; \eta]} G_L R(T_1, T_2) \bigg|_{\eta=0}. \quad (6.7)$$

To compute this quantity, we first set $\eta = 0$ and then evaluate the path integral by the saddle point. The dominant saddle point for $\eta = 0$ is the standard replica-diagonal solution, which preserves the time translation symmetry. Therefore, the result is $\langle G_L R \rangle = 0$.

The more suitable quantity for probing the replica-nondiagonal structure is the Bogoliubov quasiaverage [40], [31]. In our case, we define it as

$$\prec G_L R(T_1, T_2) \succ = \lim_{\eta \to 0} \lim_{N \to \infty} \frac{1}{Z(0)} \int DG D\Sigma e^{-NI[G, \Sigma; \eta]} G_L R(T_1, T_2). \quad (6.8)$$

To compute this quantity, we must evaluate the path integral in the saddle point approximation at some nonzero $\eta$ and then take the limit $\eta \to 0$. This quantity is multivalued: its value depends on the initial value of $\eta$ because it determines the dominant saddle point. This allows this quantity to probe the replica-nondiagonal structure.

We demonstrate how this approach works on the $q=2$ version of model (2.1). We first take a finite value of $\nu = \nu_0$ such that the system is in phase 0, i.e., $G_1(\pm \omega_0) = G_1^{(4)}(\pm \omega_0)$ (see (3.3)). In this case,

$$\lim_{N \to \infty} \frac{1}{Z(0)} \int DG D\Sigma e^{-NI[G, \Sigma; \eta]} G_L R(\omega_0) = G_1^{(4)}(\omega_0). \quad (6.9)$$

We must now take the limit $\nu \to 0$. For this, we use expansion (3.5):

$$G_1|_{\nu=0}(\omega) = i \text{sgn}(\omega) \sqrt{4J^2 + \omega^2} \frac{\sqrt{4J^2 + \omega^2}}{2J^2}. \quad (6.10)$$

As a result, this gives the nonzero quasiaverage

$$\prec G_L R(\omega_0) \succ = i \frac{\sqrt{4J^2 + \omega_0^2}}{2J^2}, \quad (6.11)$$

and $\prec G_L R(\omega_n) \succ = 0$ for all $n \neq 0$. The value of the quasiaverage $\prec G_L R(T_1, T_2) \succ$ matches the value of the replica-nondiagonal component of $G_{\alpha \beta}$ on the replica-nondiagonal saddle point of the SYK model with decoupled replicas, which can be found analytically for $q = 2$ and numerically for $q = 4$ [33]. We can say that the property of model (2.1) is that the quasiaverages remember about its phase structure after $\nu$ is turned off. We note that this is not the case in the model with local coupling (5.1): in both the $q=2$ and $q=4$ versions of the model, the dominant phases have $G_1|_{\nu=0} = 0$.

Finally, we note that quasiaverages were originally proposed by Bogoliubov as a probe of ordered phases with spontaneously broken symmetry [40]. Our considerations in [33] and here show that the two-replica SYK is different in this regard: there are no phases with spontaneously broken symmetry, but there is still spontaneous symmetry breaking in the sense of nonzero quasiaverages.
7. Discussion

We have studied the model of two SYK replicas with nonlocal interaction (2.3). We showed that in both the \( q = 2 \) and \( q = 4 \) cases, this model exhibits a nontrivial phase structure with an infinite number of phases. The coupling of form (2.4) is unique because every nontrivial phase corresponds to a replica-nondiagonal saddle point in the SYK model with two decoupled replicas. This map is made explicit by consideration of quasiaverages, which correspond to the breaking of the time translation symmetry.

We now summarize our main results.

1. We solved the saddle point equations of two nonlocally coupled SYK\(_2\) chains. We derived phase diagram (3.12) (see Fig. 2). We showed that the nontrivial phases correspond to replica-nondiagonal saddle points of the two-replica SYK\(_2\) model with the coupling turned off.

2. We found numerical solutions of the saddle point equations of two nonlocally coupled SYK\(_4\) chains (see Figs. 3 and 4). We showed that this model has a nontrivial phase structure similar to the \( q = 2 \) counterpart. We studied the behavior of the annealed free energy (see Fig. 5).

3. We solved the saddle point equations of two locally coupled SYK\(_2\) chains. We also obtained the phase diagram for this model (see Fig. 7). In this case, all phases including the nontrivial phases correspond to replica-diagonal saddle points in the two-replica SYK\(_2\) model with the coupling turned off.

4. We numerically studied the saddle point structure of the model of two locally coupled SYK\(_4\) chains, which is dual to the traversable wormhole. We found that the phase structure differs essentially from the \( q = 2 \) case because of effects related to the Hawking–Page phase transition. We reproduced the Maldacena–Qi hysteresis of the free energy by a different approach and showed that other saddle points have a higher free energy than the Hawking–Page saddle points.

5. We discussed the connection between the model of two nonlocally coupled SYK chains with a spontaneous breaking of the time translation symmetry by replica-nondiagonal solutions in the two-replica SYK model. From analytic arguments in the \( q = 2 \) model and numerical solutions in the \( q = 4 \) model, we see that the thermodynamic limit \( N \to \infty \) does not commute with the limit \( \nu \to 0 \) of zero interaction between replicas. Therefore, the replica-nondiagonal correlator is nonzero in the sense of quasiaverages. This establishes the spontaneous symmetry breaking in the two-replica SYK model in the sense of nonzero quasiaverages.

The connection between the phases of model (2.1) and the quasiaverages in the decoupled SYK replicas gives a basis for proposing this model as a tool for probing the nonperturbative effects in two decoupled SYK replicas. This can be done by performing the following operations. First, we prepare an equilibrium configuration of model (2.1) by tuning the coupling \( \nu \) and temperature \( T \) such that the system is in phase \( n_0 \). Second, we instantly turn off the coupling \( \nu \) between the replicas. After such a quantum quench, we have a two-replica SYK model in a nonequilibrium configuration with a nonzero value of the replica-nondiagonal correlator. This nonequilibrium configuration is described by a subleading saddle point. It can be expected that this configuration continues for a time \( \Delta t \sim \beta/N\delta \), where \( \delta \) is the difference between the on-shell action of the replica-nondiagonal and the standard replica-diagonal saddles. Until this configuration decays, we can expect that the physics of the system at large \( N \) is determined by the replica-nondiagonal saddle point, which is usually a nonperturbatively small effect in the equilibrium state. Hence, the model of
two nonlocally coupled SYK chains (2.1) provides a protocol of a quantum quench, which can turn a small nonperturbative effect in a regular SYK model into a nonequilibrium configuration dominating the physics for a time period of the order 1/N. The computation of real-time nonequilibrium correlators in such a quench scenario is an interesting problem for future work. The obstacle in this problem, which might make the physical realization of such a quench protocol difficult, is the fact that the initial state is prepared as a phase with negative heat capacity (see Sec. 4.2). The negative heat capacity might suggest that model (2.1) as it is could not be realized physically, but we can probably overcome this difficulty by making the source $\nu$ a dynamical variable with some sufficiently slow dynamics.

The other pressing open question is to discover the precise role of the replica-nondiagonal saddle points and of coupling (2.3) in the dual gravity description. We expect that the replica-nondiagonal saddle points in the SYK model with decoupled replicas obtained in [33] describe the contributions of nontrivial topologies to the path integral of the UV completion of the JT gravity,\(^6\) which were discussed in [15]. It was shown in [36], [18] that the model with a local interaction defined by (5.1) is suitable as a holographic dual to the traversable wormhole. From our study of the model with nonlocal interaction (2.4), we can say that the holographic dual differs from a wormhole. It would be interesting to check whether our model is perhaps more related instead to some D-brane configurations in the conjectured theory of the JT string. The negative heat capacity can also motivate considering a holographic dual model where some form of the Hawking evaporation happens.

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