Hawking radiation in a $d$-dimensional static spherically-symmetric black Hole surrounded by quintessence

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Abstract

We present a solution of Einstein equations with quintessential matter surrounding a $d$-dimensional black hole, whose asymptotic structures are determined by the state of the quintessential matter. We examine the thermodynamics of this black hole and find that the mass of the black hole depends on the equation of state of the quintessence, while the first law is universal. Investigating the Hawking radiation in this black hole background, we observe that the Hawking radiation dominates on the brane in the low-energy regime. For different asymptotic structures caused by the equation of state of the quintessential matter surrounding the black hole, we learn that the influences by the state parameter of the quintessence on Hawking radiation are different.

PACS numbers: 04.30.Nk, 04.70.Bw
I. INTRODUCTION

String theory predicts the existence of extra dimensions. This inspired a lot of interest to study whether extra dimensions can be observed, which can present the signature of string and the correctness of string theory. A great deal of effort has been expanded for the detection of extra dimensions. One among them is the study of perturbations around braneworld black holes. It has been argued that the extra dimension can imprint in the wave dynamics in the branworld black hole background [1, 2, 3, 4]. Another chief possibility to observe the extra dimension is the spectrum of Hawking radiation which is expected to be detected in particle accelerator experiments [5]-[17]. Recently through the study of Hawking radiation from squashed Kaluza-Klein black holes [14, 15, 16], it was argued that the luminosity of Hawking radiation can tell us the size of the extra dimension which opens a window to observe extra dimensions.

Recent astronomical observations strongly suggest that our universe is currently undergoing a phase of accelerated expansion, likely driven by some exotic component called dark energy. Despite the mounting observational evidence, the nature and origin of dark energy remains elusive and it has become a source of vivid debate. Quintessence is one candidate for the dark energy, which has negative pressure. If the quintessence exists everywhere in the universe, it can cause the universe to accelerate. Besides with the quintessence around a black hole, the black hole spacetime will be deformed. The Einstein equations for the static spherically-symmetric quintessence surrounding a black hole in four dimensions were studied in [18]. It was found that the condition of additivity and linearity in the energy-momentum tensor allows one to get correct limit to the known solutions for the electromagnetic static field and for the case of cosmological constant.

In this work we first extend [18] to the solution of Einstein equations with quintessential matter surrounding a d-dimensional black hole by assuming that quintessence is not only on the brane but full in the bulk. We get a new d-dimensional black hole, whose asymptotic structures are determined by the state of the quintessential matter surrounding the black hole. We examine the thermodynamics of this black hole and find that the mass of the black hole depends on the equation of state of the quintessence, while the first law keeps the same form independent of the dimensions and the state of quintessence. Investigating Hawking radiation in this black hole background, we observe that Hawking radiation dominates on the brane. For different asymptotic structures caused by the equation of state of the quintessential matter surrounding the black hole, we learn that the influences by the state parameter of the quintessence on Hawking radiation are different.
The signature of the dimension in Hawking radiation is also presented.

II. \textit{d-DIMENSIONAL STATIC SPHERICALLY-SYMMETRIC BLACK HOLES SURROUNDED BY QUINTESSENCE}

We study the Einstein equation for a static spherically-symmetric black hole surrounded by quintessence in \textit{d} dimensions. The \textit{d}-dimensional static black hole is described by

\[ ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2d\theta_1^2 - r^2\sin^2\theta_1d\theta_2^2 - \cdots - r^2\sin^2\theta_1\cdots\sin^2\theta_{d-3}d\theta_{d-2}^2, \tag{1} \]

where \( \nu \) and \( \lambda \) are functions of radial coordinate \( r \). The energy-momentum tensor of the quintessence in the static spherically symmetric state can be written as \[ T_{tt} = A(r), \quad T_{ij} = 0, \quad T_{ij} = C(r)r_ir_j + B(r)\delta^j_i. \tag{2} \]

After averaging over the angles of isotropic state we get \[ \langle T_{ij} \rangle = D(r)\delta^j_i, \quad D(r) = -\frac{1}{d-1}C(r)r^2 + B(r). \tag{3} \]

For quintessence, we have \[ D(r) = -\omega_qA(r). \tag{4} \]

Thus, in terms of density \( A(r) \), we can get the expression of \( D(r) \) for fixed state parameter \( \omega_q \). As in Ref. \[18\], the appropriate constant coefficient \( C(r)/B(r) \) is defined by the condition of additivity and linearity.

The Einstein equations of metric (1) have the form

\[ 2T_{tt} = \frac{d-2}{2} \left[ -e^{-\lambda}\left(\frac{d-3}{r^2} - \frac{\lambda'}{r} \right) + \frac{d-3}{r^2} \right], \tag{5} \]

\[ 2T_{rr} = \frac{d-2}{2} \left[ -e^{-\lambda}\left(\frac{d-3}{r^2} + \frac{\nu'}{r} \right) + \frac{d-3}{r^2} \right], \tag{6} \]

\[ 2T_{\theta_1\theta_2} = 2T_{\theta_2\theta_1} = \cdots 2T_{\theta_{d-2}\theta_{d-2}} \]

\[ = -\frac{e^{-\lambda}}{2} \left[ \nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{r} + \frac{(d-3)(\nu' - \lambda')}{r^2} + \frac{(d-3)(d-4)}{2r^2} \right] + \frac{(d-3)(d-4)}{2r^2}, \tag{7} \]

where the prime denotes the derivative with respect to \( r \).

The appropriate general expression of the energy-momentum tensor of quintessence in the \textit{d}–dimensional spherically-symmetric spacetime is given by

\[ T_{tt} = \rho_q(r), \quad T_{ij} = \rho_q(r)\alpha \left\{ -1 + (d-1)B(r)\frac{r_ir_j}{r_n^nr_m^r} + B\delta^j_i \right\}. \tag{8} \]
This leads the spatial part of the energy-momentum tensor in proportional to the time component with the arbitrary parameter \( B(r) \) which depends on the internal structure of quintessence. After taking isotropic average over the angles, we obtain

\[
\langle r_j r^j \rangle = \frac{1}{d-1} r^n r^m \delta_{ij},
\]

(9)

\[
\langle T^i_j \rangle = -\rho_q(r) \frac{\alpha}{d-1} \delta^i_j = -p_q \delta^i_j.
\]

(10)

From the state equation \( p_q = \omega_q \rho_q \), it is easy to see

\[
\omega_q = \frac{\alpha}{d-1}.
\]

(11)

For quintessence, we have \(-1 < \omega_q < 0\) and \(-(d-1) < \alpha < 0\).

As in Ref. [18], we can define a principle of additivity and linearity by the equality

\[
T^t_t = T^r_r \implies \lambda + \nu = 0.
\]

(12)

And then substituting

\[
\lambda = -\ln f,
\]

(13)

we can obtain the linear differential equations in \( f \)

\[
T^t_t = T^r_r = -\frac{d-2}{4r^2} [rf' + (d-3)(f-1)],
\]

(14)

\[
T^\theta_\theta^1 = T^\theta_\theta^2 = \cdots T^\theta_\theta^{d-2} = \frac{1}{4r^2} [r^2f'' + 2(d-3)rf' + (d-4)(d-3)(f-1)].
\]

(15)

From equations (8) and (14), we can fix the free parameter \( B \) in the energy-momentum tensor for the matter

\[
B = -\frac{(d-1)\omega_q + 1}{(d-1)(d-2)\omega_q}.
\]

(16)

Thus the energy-momentum tensor (8) has the form

\[
T^t_t = T^r_r = \rho_q,
\]

(17)

\[
T^\theta_\theta^1 = T^\theta_\theta^2 = \cdots T^\theta_\theta^{d-2} = -\frac{1}{d-2} \rho_q [(d-1)\omega_q + 1].
\]

(18)

Making use of equations (14)-(15) and (17)-(18), we obtain a differential equation for \( f \)

\[
r^2f'' + [(d-1)\omega_q + 2d-5]rf' + (d-3)[(d-1)\omega_q + d-3](f-1) = 0.
\]

(19)

The general solution of the above equation has the form

\[
f = 1 - \frac{rg}{r^{d-3}} + \frac{c_1}{r^{(d-1)\omega_q + d-3}}.
\]

(20)
where \( r_g \) and \( c_1 \) are normalization factors. When \( c_1 = 0 \), the function \( f \) describes the usual \( d \)-dimensional Schwarzschild black hole. Moreover, we also note that in the case \( \omega_q = 0 \), the second and the third term in \( f \) have the same order of \( r \).

The energy density \( \rho_q \) for quintessence can be described by

\[
\rho_q = \frac{c_1 \omega_q (d-1)(d-2)}{4r^{(d-1)(\omega_q+1)}} \tag{21}
\]

which should be positive. Since \( \omega_q \leq 0 \), it requires the normalization constant \( c_1 \) for quintessence to be negative. If we take \( r_g = 2M \) and \( c = -c_1 \), the metric of the \( d \)-dimensional spherically symmetric black hole surrounded by quintessence reads

\[
ds^2 = \left[ 1 - \frac{2M}{r^{d-3}} - \frac{c}{r^{(d-1)\omega_q+d-3}} \right] dt^2 - \left[ 1 - \frac{2M}{r^{d-3}} - \frac{c}{r^{(d-1)\omega_q+d-3}} \right]^{-1} dr^2 - r^2 d\Omega_{d-2}. \tag{22}
\]

This spacetime depends not only on the dimension \( d \), but also on the state parameter \( \omega_q \) of quintessence. When \( d = 4 \), our result reduces to that obtained in [18]. In the limit \( \omega_q = -1 \), the metric (22) becomes

\[
ds^2 = \left[ 1 - \frac{2M}{r^{d-3}} - cr^2 \right] dt^2 - \left[ 1 - \frac{2M}{r^{d-3}} - cr^2 \right]^{-1} dr^2 - r^2 d\Omega_{d-2}, \tag{23}
\]

which reduces to the \( d \)-dimensional Schwarzschild de Sitter black hole. We also note that the metric (22) can reduce to the \( d \)-dimensional Reissner-Nordström black hole if we take

\[
\omega_q = \frac{d - 3}{d - 1}. \tag{24}
\]

This implies that the state parameter \( \omega_q \) of the electromagnetic field is a function of the dimension \( d \) of the spacetime, so that we might fix the number of extra dimensions of the spacetime by measuring the relation between the pressure \( p_q \) and the energy density \( \rho_q \).

III. THERMODYNAMICS OF THE \( d \)-DIMENSIONAL STATIC SPHERICALLY SYMMETRIC BLACK HOLE SURROUNDED BY QUINTESSENCE

We now study the thermodynamical property at the black hole event horizon in the background (22).

We write the mass \( E \) of a \( d \)-dimensional black hole as a product

\[
E = F(d)M, \tag{25}
\]

where \( F(d) \) is a function of dimension \( d \). We will see that the first law of thermodynamics at the black hole event horizon does not depend on this function. The entropy \( S \), mass \( E \) and Hawking temperature \( T \) of the
black hole (22) can be described by

\[ S = \frac{A_h}{4} = \frac{(d-1)\pi^{(d-1)/2}}{4 \Gamma\left(\frac{d+1}{2}\right)} r_h^{d-2} = r_h^{d-2} \frac{G(d)}{G(d-1)} \]

\[ E = \frac{F(d)}{2} (G(d)S)^{\frac{d-3}{d-2}} - \frac{F(d)c}{2} (G(d)S)^{-\frac{\omega_q(d-1)}{d-2}}, \]

\[ T = \frac{F(d)G(d)}{2(d-2)} \left[ (d-3)(G(d)S)^{\frac{d-3}{d-2}} + \omega_q(c - 1)(G(d)S)^{-\frac{\omega_q(d-1)+d-2}{d-2}} \right], \]

respectively. As did in [19], we treat the constant \( c \) as a variable, and have the generalized force

\[ \Theta_c = \left( \frac{\partial E}{\partial c} \right)_S = -\frac{F(d)}{2} (G(d)S)^{-\frac{\omega_q(d-1)}{d-2}}. \]

We find that the first law takes the form

\[ \frac{d-3}{d-2} E = TS + \frac{\omega_q(d-1) + d-3}{d-2} \Theta_c c. \]

It is clear that the mass depends on the state parameter \( \omega_q \) of quintessence. In the limit \( \omega_q \to -1 \), the second term in the right-side of equation (30) becomes \(-\frac{2}{d-2} \Theta_c c\). Setting \( c = \frac{1}{l^2} \), where \( l^2 \) is defined as the cosmological constant \( \Lambda \) through \( l^2 = \frac{(d-1)(d-2)}{2\Lambda} \), we have \( \Theta_l = -\frac{2}{d-2} \Theta_c \) and \(-\frac{2}{d-2} \Theta_c c = \frac{1}{l^2} \Theta_l l \). Then the equation (30) reduces to the first law in the \( d \)-dimensional de Sitter black hole spacetimes [19]. In the case \( \omega_q = \frac{d-3}{d-1} \), setting \( c = -Q^2 \), we can obtain \( \Theta_Q = -2Q\Theta_c \), and \( \frac{2(d-3)}{d-2} \Theta_c c = \frac{d-3}{d-2} \Theta_Q Q \). The first law returns to that in the \( d \)-dimensional Reissner-Nordström black hole [20].

From equations (26), (27) and (28), it is easy to obtain that

\[ T = \left( \frac{\partial E}{\partial S} \right)_c. \]

Combining it with equation (29), we have

\[ dE = \left( \frac{\partial E}{\partial S} \right)_c dS + \left( \frac{\partial E}{\partial c} \right)_S d c = T dS + \Theta_c d c, \]

which is the differential form of the first law of thermodynamics in the background of (22). Obviously, in the case of a \( d \)-dimensional static black hole surrounded by spherically-symmetric quintessence, the differential form of the first law does not depend on the state parameter \( \omega_q \).

### IV. GREYBODY FACTOR FOR A \( d \)-DIMENSIONAL STATIC SPHERICALLY-SYMMETRIC BLACK HOLE SURROUNDED BY QUINTESSENCE

In this section, we study the greybody factors for the emission of scalar field in the low energy limit on the brane and into the bulk from the \( d \)-dimensional static spherically-symmetric black hole surrounded
by quintessence \cite{22}. The greybody factors of the scalar field in the $d$-dimensional Schwarzschild and Schwarzschild-dS blackholes have been investigated \cite{21, 22}.

The equation of motion for a massless scalar particle propagating in the curved spacetime is described by

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu}) \Phi(t, r, \Omega) = 0,$$

where $\Phi(t, r, \Omega)$ denotes the scalar field. Separating $\Phi(t, r, \Omega) = e^{-i\omega t} \Psi_{\text{bulk}}(r) Y_{lm}(\Omega)$, we can obtain the radial equation for the scalar field propagating into the bulk

$$\frac{1}{r^{d-2}} \frac{d}{dr} \left[ r^{d-2} f \frac{d\Psi_{\text{bulk}}(r)}{dr} \right] + \left[ \frac{\omega^2}{f} - \frac{l(l + d - 3)}{r^2} \right] \Psi_{\text{bulk}}(r) = 0,$$

with $f = 1 - \frac{2M}{r} - \frac{c}{r^{d-1} \omega_q}$. Similarly, we can also obtain the radial equation for scalar field propagating on the brane

$$\frac{1}{r^2} \frac{d}{dr} \left[ r^2 f \frac{d\Psi_{\text{brane}}(r)}{dr} \right] + \left[ \frac{\omega^2}{f} - \frac{l(l + 1)}{r^2} \right] \Psi_{\text{brane}}(r) = 0.$$

Adopting the tortoise coordinate $x = \int \frac{dr}{f}$, radial equations (34) and (35) can be further written as

$$\left[ \frac{d^2}{dx^2} + \omega^2 - V_{\text{bulk}}(r) \right] \left[ r^{d-2} \Psi_{\text{bulk}}(r) \right] = 0,$$

and

$$\left[ \frac{d^2}{dx^2} + \omega^2 - V_{\text{brane}}(r) \right] \left[ r \Psi_{\text{brane}}(r) \right] = 0,$$

with the effective potentials

$$V_{\text{bulk}}(r) = f \left[ \frac{(d - 2)(d - 4)f}{4r^2} + \frac{(d - 2)f'}{2r} + \frac{l(l + d - 3)}{r^2} \right],$$

and

$$V_{\text{brane}}(r) = f \left[ \frac{f'}{r} + \frac{l(l + 1)}{r^2} \right].$$

Here we only consider the greybody factor for the mode $l = 0$ which dominates in the low-energy regime $\omega \ll T_H$ and $\omega R H \ll 1$.

From the expression of $f$, we find that the spacetime \cite{22} is asymptotically flat if $0 > \omega_q > -(d - 3)/(d - 1)$ and is asymptotically dS-like when $-1 \leq \omega_q < -(d - 3)/(d - 1)$. As in \cite{21}, we write $f = f_a(r) + f_h(r)$. The function $f_a(r)$ is the asymptotic part of $f$ and the function $f_h(r)$ contains physics which is specific for the black hole. We can define the asymptotic region to be $f_a(r) \gg f_h(r)$. 
Near the black hole horizon \( r \sim R_H \), taking into account the ingoing boundary condition, we obtain the solution of the radial equations (34) and (35) in the same form

\[
\Psi(r)_{rH} = A_I e^{i\omega x}. \tag{40}
\]

Near the black hole horizon \( x \sim \frac{1}{2\kappa_H} \log \frac{r-R_H}{R_H} \), the solution (40) can be written as

\[
\Psi(r)_{rH} = A_I \left[ 1 + \frac{i\omega}{2\kappa_H} \log \frac{r-R_H}{R_H} \right]. \tag{41}
\]

In the intermediate region where the effective potentials \( V_{bulk}(r) \gg \omega^2 \) and \( V_{brane}(r) \gg \omega^2 \), the radial equations (34) and (35) can be reduced to

\[
\frac{1}{r^{d-2}} \frac{d}{dr} \left[ r^{d-2} f \frac{d\Psi_{bulk}(r)}{dr} \right] = 0, \tag{42}
\]

and

\[
\frac{1}{r^2} \frac{d}{dr} \left[ r^2 f \frac{d\Psi_{brane}(r)}{dr} \right] = 0, \tag{43}
\]

respectively. The general solutions for equation (42) and (43) can be got

\[
\Psi_{bulk}(r) = A_{II} + B_{II} G(r), \quad \text{and} \quad \Psi_{brane}(r) = A_{II} + B'_{II} G'(r), \tag{44}
\]

where

\[
G(r) = \int_{\infty}^{r} \frac{dr}{r^{d-2} f}, \quad \text{and} \quad G'(r) = \int_{\infty}^{r} \frac{dr}{r^2 f}. \tag{45}
\]

For \( r \sim R_H \), we have

\[
G(r) = \frac{1}{2R_H^{d-2}\kappa_H} \log (r-R_H), \quad \text{and} \quad G'(r) = \frac{1}{2R_H^{d-2}\kappa_H} \log (r-R_H). \tag{46}
\]

Matching this solution to the wave-function (41) of the near black hole horizon region, we obtain

\[
A_{II} = A_I, \quad B_{II} = i\omega R_H^{d-2} A_I, \quad B'_{II} = i\omega R_H^2 A_I. \tag{47}
\]

The asymptotic expressions of wave functions (41) in the limit \( r \gg R_H \) with \( V(r) \gg \omega^2 \) read

\[
\Psi_{bulk}(r) = A_I \left( 1 + i\omega R_H^{d-2} \int_{\infty}^{r} \frac{dr}{r^{d-2} f_a(r)} \right), \quad \Psi_{brane}(r) = A_I \left( 1 + i\omega R_H^2 \int_{\infty}^{r} \frac{dr}{r^2 f_a(r)} \right). \tag{48}
\]

We shall use these expressions to match the general solution for the scalar wave equation in the asymptotic region.
Until now we just concentrate on the black hole, in the following we will do the matching in the asymptotic region for the case of asymptotically flat spacetime and asymptotically dS spacetime, respectively.

Let us first consider the asymptotically flat spacetime where \( 0 > \omega_q > -(d-3)/(d-1) \). In this case, we take \( f_a(r) = 1 \).

The general solutions of the wave equation (34) and (35) in asymptotically flat spacetime are given by

\[
\Psi_{\text{bulk}}(r) = \rho^{3-d/2} \left[ C_1 H^{(1)}_{\frac{d-3}{2}}(\rho) + C_2 H^{(2)}_{\frac{d-3}{2}}(\rho) \right], \\
\Psi_{\text{brane}}(r) = \rho^{-1/2} \left[ C_1 H^{(1)}_{\frac{d}{2}}(\rho) + C_2 H^{(2)}_{\frac{d}{2}}(\rho) \right],
\]

(49)

where \( \rho = r \omega, \ H^{(1)}_{\nu}(\rho) = J_{\nu}(\rho) + i N_{\nu}(\rho) \) and \( H^{(2)}_{\nu}(\rho) = J_{\nu}(\rho) - i N_{\nu}(\rho) \) are the Hankel functions defined by the Bessel functions \( J_{\nu}(\rho) \) and \( N_{\nu}(\rho) \). In the limit \( \rho \ll 1 \), we have

\[
\Psi_{\text{bulk}}(r) \sim \frac{C_1 + C_2}{\Gamma(d-1)2\frac{d-3}{2}} - i(C_1 - C_2)\frac{\Gamma(d-3)2\frac{d-3}{2}}{2\pi\rho^{d-2}}, \\
\Psi_{\text{brane}}(r) \sim \frac{\sqrt{2}(C_1 + C_2)}{\sqrt{\pi}} - i\frac{\sqrt{2}(C_1 - C_2)}{\sqrt{\pi}\rho^{d-2}}.
\]

(50)

| \( \omega_q \) | \( R_H \) | \( R_H^{d-2} \) |
|---|---|---|
| -0.1 | 2.0122 | 2.0151 |
| -0.2 | 1.4183 | 1.4189 |
| -0.3 | 1.2623 | 1.2626 |
| -0.4 | 1.1909 | 1.1910 |

TABLE I: The changes of \( R_H \) and \( R_H^{d-2} \) with different state parameter \( \omega_q \) and dimension numbers \( d \) in the asymptotically flat case. Here \( M = 1 \) and \( c = 0.01 \).

Matching the wave-function (50) in the asymptotic region to that in the intermediate region, we get the relationship between the coefficients \( C_1 \) and \( C_2 \)

\[
C_1 + C_2 = \Gamma\left(\frac{d-1}{2}\right)2\frac{d-3}{2} \pi^{d-2} A_I, \quad C_1 - C_2 = \frac{\pi \omega^{d-2} R_H^{d-2}}{(d-3)\Gamma\left(\frac{d-3}{2}\right)2\frac{d-3}{2}} A_I,
\]

(51)
in the bulk and

\[
C_1 + C_2 = \sqrt{\frac{\pi}{2}} A_I, \quad C_1 - C_2 = \sqrt{\frac{\pi}{2}} \omega^{d-2} R_H^2 A_I,
\]

(52)
on the brane. From the definition of greybody factor in the low energy limit \( \omega R_H \ll 1 \),

\[
\gamma(\omega) = 1 - \frac{|C_2|^2}{|C_1|^2} \simeq 4 \frac{C_1 - C_2}{C_1 + C_2},
\]

(53)
we obtain the greybody factor

$$\gamma(\omega) = \frac{4\pi\omega^{d-2}R_H^{d-2}}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)^2},$$

(54)

in the bulk and

$$\gamma(\omega) = 4\omega^2R_H^2,$$

(55)
on the brane. Obviously, the greybody factors in the bulk and on the brane depend on the black hole horizon radius. The changes of $R_H$ and $R_H^{d-2}$ with the state parameter $\omega_q$ and dimension $d$ are listed in table (I) and the factor $2^{d-2}\Gamma\left(\frac{d-1}{2}\right)^2$ in (54) is only a monotonically increased function of the dimension numbers $d$, thus the greybody factors (54) and (55) increase with the increase of the absolute value of $\omega_q$ and decrease with the increase of $d$.

In the $d$ dimensional black hole spacetime, the luminosity of the black hole Hawking radiation for the mode $l = 0$ in the bulk and on the brane is given by

$$L_{\text{bulk}} = \int_0^\infty d\omega \frac{\omega^{d-1}R_H^{d-2}}{2^{d-3}\Gamma\left(\frac{d-1}{2}\right)^2} \frac{1}{e^{\omega/T_H}-1},$$

(56)

$$L_{\text{brane}} = \int_0^\infty d\omega \frac{4\omega^3R_H^2}{2\pi e^{\omega/T_H}-1}.$$

(57)
The integral expressions above are just for the sake of completeness by writing the integral range from 0 to infinity. However, as our analysis has focused only in the low-energy regime of the spectrum, an upper cutoff will be imposed on the energy parameter such that the low-energy conditions $\omega \ll T_H$ and $\omega R \ll 1$ are satisfied. The values derived for the luminosities of the black hole on the brane and in the bulk will therefore be based on the lower part of the spectrum and modifications may appear when the high-energy part of the spectrum is included in the calculation. The Hawking temperature $T_H$ of the black hole in the

| $\omega_q$ | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ |
|------------|---------|---------|---------|---------|
| $-0.1$     | 0.03947 | 0.11208 | 0.18895 | 0.26707 |
| $-0.2$     | 0.03931 | 0.11187 | 0.18869 | 0.26676 |
| $-0.3$     | 0.03909 | 0.11161 | 0.18837 | 0.26639 |
| $-0.4$     |         | 0.11127 | 0.18798 | 0.26569 |
| $-0.5$     |         | 0.11086 | 0.18752 | 0.26595 |
| $-0.6$     |         | 0.11086 | 0.18696 | 0.26648 |

TABLE II: The change of $T_H$ with different state parameter $\omega_q$ and dimension numbers $d$ in the asymptotically flat case. Here $M = 1$ and $c = 0.01$.

asymptotically flat spacetime is listed in table (II). It is shown that $T_H$ increases with of the dimension number.
$d$ and decreases with the increase of the absolute value of $\omega_q$. Table (III) tells us that both the luminosity of Hawking radiation in the bulk and on the brane decrease with the increase of the absolute $\omega_q$ and increase with the dimension number $d$. We observe that Hawking radiation dominates on the brane and the ratio $L_{brane}/L_{bulk}$ increases with the magnitude of $\omega_q$ and dimension $d$.

| $L_{bulk}$ ($10^{-4}$) | $L_{brane}$ ($10^{-4}$) | $L_{brane}/L_{bulk}$ |
|---------------------|---------------------|---------------------|
| $\omega_q = -0.1$ | $\omega_q = -0.2$ | $\omega_q = -0.3$ | $\omega_q = -0.1$ | $\omega_q = -0.2$ | $\omega_q = -0.3$ |
| $d = 4$ | 4.06400 | 4.0087 | 3.9330 | -- | -- | -- |
| $d = 5$ | 31.405 | 31.145 | 30.818 | 131.27 | 130.39 | 129.26 |
| $d = 6$ | 99.774 | 99.036 | 98.135 | 839.67 | 835.39 | 830.17 |
| $d = 7$ | 263.32 | 261.38 | 259.06 | 2982.9 | 2969.9 | 2954.4 |

TABLE III: The changes of $L_{bulk}$, $L_{brane}$ and $L_{brane}/L_{bulk}$ with different state parameter $\omega_q$ and dimension numbers $d$ in the asymptotically flat case. Here $M = 1$ and $c = 0.01$.

Now we start to consider asymptotically dS like spacetime with $-1 \leq \omega_q < -(d - 3)/(d - 1)$. The function $f_a(r)$ is now given by

$$f_a(r) = 1 - \frac{c}{r^{\omega_q(d-1)+d-3}}. \quad (58)$$

The metric (22) now has a cosmological-like horizon located at $r = r_c = c^{1/[(\omega_q(d-1)+d-3)]}$. Assuming that $r_c \gg R_H$, we have that $f_a(r) \ll f_a(r)$ for $r \gg R_H$ and the $f_a(r)$ contribution to $f(r)$ is negligible. This allows us to define an intermediate region, $R_H < r < r_c$, in between the near horizon and the asymptotic region. Thus, for $r \gg R_H$, $r/r_c \ll 1$ and $r \omega \ll 1$, the wave functions (48) have the form

$$\Psi_{bulk}(r) = A_I \left(1 - i \frac{\omega R_H^2 r^{d-2}}{(d-3)r^{d-3}}\right), \quad \Psi_{brane}(r) = A_I \left(1 - i \frac{\omega R_H^2}{r}\right). \quad (59)$$

In the asymptotic region, $r \gg R_H$, we define the coordinate

$$z = \left(\frac{r}{r_c}\right)^{-n}, \quad (60)$$

with $n = \omega_q(d-1)+d-3$, which is negative because that in this asymptotically dS-like spacetime $\omega_q < -(d - 3)/(d - 1)$. Then radial equations (34) and (35) can be approximated as

$$(1 - z)^{n+1} \frac{d^2 P_{bulk}}{dz^2} - \left[n + \frac{2n+1}{n} \frac{dP_{bulk}}{dz} + \left(\frac{\omega^2 r_c^2}{n^2(1-z)} - \frac{d-2}{4n^2(z(d-2n-4))}\right)\right]P_{bulk} = 0, \quad (61)$$

and

$$(1 - z)^{n+1} \frac{d^2 P_{brane}}{dz^2} - \left[n + \frac{2n+1}{n} \frac{dP_{brane}}{dz} + \left(\frac{\omega^2 r_c^2}{n^2(1-z)} - \frac{1}{n}\right)\right]P_{brane} = 0, \quad (62)$$
respectively. Here $P_{\text{bulk}} = r^{(d-2)/2} \Psi_{\text{bulk}}(r)$ and $P_{\text{brane}} = r \Psi_{\text{brane}}(r)$.

The general solution to the equation (61) is

$$P_{\text{bulk}} = C_1 z^{-\frac{d-2}{2n}} \left(1 - z\right)^{\frac{i\omega r_c}{n}} 2F_1\left[\left(\frac{i\omega r_c}{n}, \frac{3 - d + n + i\omega r_c}{n}\right), 1 + \frac{i\omega r_c}{n}; 1 - z\right]$$

$$+ \ C_2 z^{\frac{d-4}{2n}} \left(1 - z\right)^{\frac{i\omega r_c}{n}} 2F_1\left[\left(\frac{\omega r_c}{n}, \frac{d - 3 + n + i\omega r_c}{n}\right), 1 + \frac{i\omega r_c}{n}; 1 - z\right],$$

where $2F_1[a, b; c; z]$ is the standard hypergeometric function. Since, $n < 0$, for $z \rightarrow 0$, or $r/r_c \ll 1$, we have

$$\Psi_{\text{bulk}}(r) = C_1 r_c^{\frac{d-4}{2n}} + \frac{C_2 r_c^{\frac{d-4}{2n}}}{r^{d-3}}.$$ (64)

Matching this wave function to the behavior (59) in the intermediate region, we can fix the coefficients $C_1$ and $C_2$

$$C_1 = r_c^{-\frac{d-4}{2n}} A_I; \quad C_2 = -i r_c^{\frac{d-4}{2n}} \frac{\omega P_{\text{brane}}^{d-2}}{(d-3)} A_I.$$ (65)

In order to find the behavior of the wave function for $z \rightarrow 1$, we change the argument of the hypergeometric function of the solution (63) from $z$ to $1 - z$ and find that it has the form

$$P_{\text{bulk}} = C_1 b_{11} z^{-\frac{d-2}{2n}} \left(1 - z\right)^{\frac{i\omega r_c}{n}} 2F_1\left[\left(\frac{i\omega r_c}{n}, \frac{3 - d + n + i\omega r_c}{n}\right), 1 + \frac{i\omega r_c}{n}; 1 - z\right]$$

$$+ \ C_1 b_{12} z^{-\frac{d-2}{2n}} \left(1 - z\right)^{\frac{i\omega r_c}{n}} 2F_1\left[\left(\frac{\omega r_c}{n}, \frac{d - 3 + i\omega r_c}{n}\right), 1 + \frac{i\omega r_c}{n}; 1 - z\right]$$

$$+ \ C_1 b_{21} z^{-\frac{d-2}{2n}} \left(1 - z\right)^{\frac{i\omega r_c}{n}} 2F_1\left[\left(\frac{i\omega r_c}{n}, \frac{3 - d + i\omega r_c}{n}\right), 1 + \frac{i\omega r_c}{n}; 1 - z\right]$$

$$+ \ C_1 b_{22} z^{-\frac{d-2}{2n}} \left(1 - z\right)^{\frac{i\omega r_c}{n}} 2F_1\left[\left(\frac{\omega r_c}{n}, \frac{d - 3 + \omega r_c}{n}\right), 1 + \frac{i\omega r_c}{n}; 1 - z\right],$$ (66)

with

$$b_{11} = \frac{\Gamma\left(\frac{3-d+n}{n}\right)\Gamma\left(-\frac{i\omega r_c}{n}\right)}{\Gamma\left(\frac{3-d+n-i\omega r_c}{n}\right)\Gamma\left(-\frac{i\omega r_c}{n}\right)}, \quad b_{12} = \frac{\Gamma\left(\frac{d-3+n}{n}\right)\Gamma\left(-\frac{i\omega r_c}{n}\right)}{\Gamma\left(\frac{d-3+n-i\omega r_c}{n}\right)\Gamma\left(-\frac{i\omega r_c}{n}\right)}. \quad (67)$$

$$b_{21} = \frac{\Gamma\left(\frac{3-d+n}{n}\right)\Gamma\left(\frac{i\omega r_c}{n}\right)}{\Gamma\left(\frac{i\omega r_c}{n}\right)\Gamma\left(\frac{3-d+n+i\omega r_c}{n}\right)}, \quad b_{22} = \frac{\Gamma\left(\frac{d-3+n}{n}\right)\Gamma\left(\frac{i\omega r_c}{n}\right)}{\Gamma\left(\frac{d-3+n+i\omega r_c}{n}\right)\Gamma\left(\frac{i\omega r_c}{n}\right)}. \quad (68)$$

Thus, in the limit $z \rightarrow 1$, the wave function becomes

$$\Psi_{\text{bulk}}(r) = C_1 r_c^{\frac{2-d}{2n}} e^{-\frac{i\omega r_c}{n}} e^{i\omega x} + C_2 r_c^{\frac{2-d}{2n}} e^{-\frac{i\omega r_c}{n}} e^{-i\omega x}, \quad (69)$$

where $\delta = - \frac{\Gamma(1-1/n)}{\Gamma(-1/n)} \left[\text{EulerGamma} + \text{PolyGamma}(0, -1/n)\right]$. The relations between $C_1$, $C_2$ and $C_1$, $C_2$ can be expressed as

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (70)$$
Then the greybody factor is given by
\[ 
\gamma(\omega)_{\text{bulk}} = 1 - \frac{|\tilde{C}_2|^2}{|\tilde{C}_1|^2} = \left(1 - \frac{b_{11} b_{22} - b_{12} b_{21} C_2^2}{b_{11} b_{21} C_1}\right)^2
\]
\[ = 4h(\omega r_c)\left(\frac{R_H}{r_c}\right)^{d-2}, \tag{71} \]
where the function \( h(\omega r_c) \) is defined by
\[ h(\omega r_c) = \frac{1}{4|b_{11}|^2}. \tag{72} \]
Similarly, the greybody factor for the scalar emission on the brane can be given by
\[ 
\gamma(\omega)_{\text{brane}} = \frac{1}{|b'_{11}|^2} \left(\frac{R_H}{r_c}\right)^2, \tag{73} \]
with
\[ b'_{11} = \frac{\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(-\frac{2\omega r_c}{n}\right)}{\Gamma\left(n - 1 - i\omega r_c\right)\Gamma\left(-\frac{i\omega r_c}{n}\right)}. \tag{74} \]
The greybody factors \( \gamma(\omega)_{\text{bulk}} \) and \( \gamma(\omega)_{\text{brane}} \) depend on \( \omega \) and the ratio \( R_H/r_c \). The changes of \( R_H \) and \( r_c \) in the case \( \omega_q(d - 1) + d - 3 < 0 \) with \( \omega_q \) and \( d \) are listed in Table (IV). One can find that the ratio \( R_H/r_c \) increases with the increase of the absolute value of \( \omega_q \) and decreases with the increase of the dimension \( d \). In the low energy limit \( \omega r_c < 1 \), we have the quantities \( |b_{11}|^2 \sim 1/4 \) and \( |b'_{11}|^2 \sim 1/4 \), and then the greybody factors \( \gamma(\omega)_{\text{bulk}} \) and \( \gamma(\omega)_{\text{brane}} \) also increase with \( \omega_q \) and decrease with \( d \).

| \( R_H/r_c \) | \( R_H \) | \( r_c \) |
| --- | --- | --- |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |

| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |
| \( \omega_q = -0.8 \) | \( \omega_q = -0.9 \) | \( \omega_q = -1 \) |

TABLE IV: The changes of \( R_H \), \( r_c \) and \( R_H/r_c \) with different state parameter \( \omega_q \) and dimension numbers \( d \) in the asymptotically dS-like case. Here \( M = 1 \) and \( c = 0.01 \).

The luminosity of Hawking radiation for the mode \( l = 0 \) in the bulk and on the brane can be given by
\[ 
L_{\text{bulk}} = \int_0^\infty \frac{d\omega}{e^\omega/T_H - 1} \gamma(\omega)_{\text{bulk}}, \tag{75} \]
\[ L_{\text{brane}} = \int_0^\infty \frac{d\omega}{2\pi} e^\omega/T_H - 1 \gamma(\omega)_{\text{brane}}. \tag{76} \]
As did in (56)(57), here we also focus only in the low-energy regime of the spectrum and impose an upper cutoff on the energy parameter such that the low-energy conditions $\omega \ll T_H$ and $\omega R \ll 1$ are satisfied. The values obtained for the luminosities of the black hole on the brane and in the bulk are based on the lower part of the spectrum. When the high energy part spectrum is included, the results may significantly change.

$$\omega_q = -0.4 \quad \omega_q = -0.5 \quad \omega_q = -0.6 \quad \omega_q = -0.7 \quad \omega_q = -0.8 \quad \omega_q = -0.9 \quad \omega_q = -1.0$$

| $d$ | $L_{\text{bulk}} \times 10^{-9}$ | $L_{\text{brane}} \times 10^{-9}$ | $L_{\text{brane}}/L_{\text{bulk}}$ |
|-----|-------------------------------|-------------------------------|-------------------------------|
| 4   | 0.9045                        | 2.9143                        | 6.4797                        |
| 5   | 0.0361                        | 0.6386                        | 3.5966                        |
| 6   | $9.2 \times 10^{-5}$         | 0.0425                        | 0.9108                        |
| 7   | $5.5 \times 10^{-9}$         | 0.0013                        | 0.1728                        |

Table V: The change of $T_H$ with different state parameter $\omega_q$ and dimensional numbers $d$ in the asymptotically dS-like case. Here $M = 1$ and $c = 0.01$.

Although the black hole Hawking temperature $T_H$ (which is listed in table V) decreases with the increase of the absolute value of $\omega_q$, table (VI) tells us that the luminosity of the black hole Hawking radiation increases with the increase of the magnitude of $\omega_q$. This is different from that in the asymptotically flat case with $\omega_q(d - 1) + d - 3 > 0$. Moreover comparing with the asymptotically flat case, although the ratio $L_{\text{brane}}/L_{\text{bulk}}$ tells us that the black hole Hawking radiation still dominates on the brane in the dS like spacetime, its dependence on $|\omega_q|$ is different in the dS like situation from that in the asymptotically flat spacetime. These differences can be understood from the behavior of the ratio $R_H/r_c \ll 1$, which increases with $|\omega_q|$. This means that when $|\omega_q|$ becomes bigger, the black hole horizon and the cosmological horizon will come closer, so that Hawking radiation on the black hole event horizon will be enhanced by the contribution from Hawking radiation from the cosmological horizon.
In this paper, we obtain an exact solution of Einstein equations for the static spherically-symmetric quintessential matter surrounding a black hole in d-dimensional spacetimes. For different state parameters $\omega_q$ of quintessence, our solution can lead to different limits, such as the Schwarzschild, Reissner-Nordström and de Sitter black holes in $d$-dimensions. We study the thermodynamics in this d-dimensional black hole spacetime and find that the first law is universal for arbitrary state parameter $\omega_q$ of the quintessence.

We investigate the greybody factors and Hawking radiations of a scalar field in the bulk and on the brane, in the low-energy regime, in this $d$-dimensional black hole surrounded by quintessence. We observe that Hawking radiation dominates on the brane. For the case $0 > \omega_q > -(d - 3)/(d - 1)$, the black hole is asymptotically flat, the luminosity of Hawking radiation both in the bulk and on the brane decreases with the increase of $|\omega_q|$. But for the case $-(d - 3)/(d - 1) > \omega_q > -1$, the black hole is in the asymptotically dS spacetime, Hawking radiation increases with the magnitude of $|\omega_q|$. The difference can be attributed to the different asymptotic structures of the spacetimes. In the asymptotic dS spacetime, besides the black hole event horizon, there also exists the cosmological horizon. When the absolute value of $\omega_q$ becomes bigger, these two horizons come closer. The contribution of Hawking radiation from the cosmological horizon enhances the Hawking radiation near the black hole event horizon.

Acknowledgments

This work was partially supported by NNSF of China, Shanghai Education Commission and Shanghai Science and Technology Commission. R. K. Su’s work was partially supported by the National Basic Research Project of China. S. B. Chen’s work was partially supported by the China Postdoctoral Science Foundation under Grant No. 20070410685, the Scientific Research Fund of Hunan Provincial Education Department Grant No. 07B043, the National Basic Research Program of China under Grant No. 2003CB716300 and the construct program of key disciplines in Hunan Province.

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