Ritt’s theorem and the Heins map in hyperbolic complex manifolds

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Abstract. Let $X$ be a Kobayashi hyperbolic complex manifold, and assume that $X$ does not contain compact complex submanifolds of positive dimension (e.g., $X$ Stein). We shall prove the following generalization of Ritt’s theorem: every holomorphic self-map $f: X \to X$ such that $f(X)$ is relatively compact in $X$ has a unique fixed point $\tau(f) \in X$, which is attracting. Furthermore, we shall prove that $\tau(f)$ depends holomorphically on $f$ in a suitable sense, generalizing results by Heins, Joseph-Kwack and the second author.

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0. Introduction

The classical Wolff-Denjoy theorem (see, e.g., [A2, Theorem 1.3.9]) says that the sequence of iterates of a holomorphic self-map $f$ of the unit disk $\Delta \subset \mathbb{C}$, except when $f$ is an elliptic automorphism of $\Delta$ or the identity, converges uniformly on compact subsets to a point $\tau(f) \in \mathbb{C}$, the Wolff point of $f$. Furthermore, if $\tau(f) \in \Delta$ then it is the unique fixed point of $f$; and if $\tau(f) \in \partial \Delta$ then it is still morally fixed, in the sense that $f(\zeta)$ tends to $\tau(f)$ when $\zeta$ tends to $\tau(f)$ non-tangentially.

In 1941, Heins [H] proved that the map $\tau: \text{Hol}\,(\Delta, \Delta) \setminus \{\text{id}\} \to \mathbb{C}$, associating to every elliptic automorphism its fixed point and to any other map its Wolff point, is continuous. More than half a century later, using the first author’s version (see [A1]) of the Wolff-Denjoy theorem for strongly convex domains in $\mathbb{C}^n$, Joseph and Kwack [JK] extended Heins’ result to strongly convex domains.

In 2002, the second author started investigating further regularity properties of the Heins map. If $D$ is a bounded domain in $\mathbb{C}^n$, then $\text{Hol}(D, D)$ is a subset of the complex Banach space $H^\infty(D)^n$ of $n$-uples of bounded holomorphic functions defined on $D$; so one may ask whether the Heins map, when defined, is holomorphic on some suitable open subset of $\text{Hol}(D, D)$. And indeed, in [B] the second author proved that, when $D$ is strongly convex, the Heins map is well-defined and holomorphic on $\text{Hol}_c(D, D)$, the open subset of holomorphic self-maps of $D$ whose image is relatively compact in $D$.

The aim of this paper is to prove a similar result for the space $\text{Hol}_c(X, X)$ of the holomorphic self-maps of a Kobayashi hyperbolic Stein manifold whose image is relatively compact in $X$. First of all, we shall generalize the classical Ritt’s theorem, proving (Theorem 1.1) that every $f \in \text{Hol}_c(X, X)$ admits a unique fixed point $\tau(f) \in X$; therefore the Heins map $f \mapsto \tau(f)$ is well-defined and continuous (Lemma 2.1).

To study further regularity properties of the Heins map, one apparently needs a complex structure on $\text{Hol}_c(X, X)$. Unfortunately, we do not know whether such a structure exists in general; so we shall instead prove (Theorem 2.3) that the Heins map is holomorphic when restricted to any holomorphic family inside $\text{Hol}_c(X, X)$, a fact equivalent to $\tau$ being holomorphic with respect to any sensible complex structure on $\text{Hol}_c(X, X)$. For instance, we obtain (Corollary 2.4) that the Heins map is holomorphic on $\text{Hol}_c(D, D)$ for any bounded domain $D$ in $\mathbb{C}^n$.

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1. Ritt’s theorem

Let $X$ be a complex manifold. We shall denote by $\text{Hol}_c(X, X)$ the space of holomorphic self-maps $f: X \to X$ of $X$ such that $f(X)$ is relatively compact in $X$.

In 1920, Ritt [R] proved that if $X$ is a non-compact Riemann surface then every $f \in \text{Hol}_c(X, X)$ has a unique fixed point $z_0 \in X$. Furthermore, this fixed point is attractive in the sense that the sequence $\{f^k\}$ of iterates of $f$ converges, uniformly on compact subsets, to the constant map $z_0$. This theorem has been generalized to bounded domains in $\mathbb{C}^n$ by Wavre [W]; see also Hervé [He, p. 83]. Arguing as in [A2, Corollary 2.1.32] we shall now prove a far-reaching generalization of Ritt’s theorem:

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Theorem 1.1: Let $X$ be a hyperbolic manifold with no compact complex submanifolds of positive dimension. Then every $f \in \text{Hol}_c(X, X)$ has a unique fixed point $z_0 \in X$. Furthermore, the sequence of iterates of $f$ converges, uniformly on compact subsets, to the constant map $z_0$.

Proof: Since $X$ is hyperbolic, by [A3] the space $\text{Hol}(X, X)$ of holomorphic self-maps of $X$ is relatively compact in the space $C^0(X, X^*)$ of continuous maps of $X$ into the one-point compactification $X^* = X \cup \{\infty\}$, endowed with the compact-open topology. If $f \in \text{Hol}_c(X, X)$, this implies that the sequence of iterates of $f$ is relatively compact in $\text{Hol}(X, X)$, because $f(X) \subset X$.

Let then $\{f^k\}$ be a subsequence of $\{f^\nu\}$ converging to $h_0 \in \text{Hol}(X, X)$. We can also assume that $p_\nu = h_{\nu+1} - h_\nu$ and $q_\nu = p_\nu - h_\nu$, tend to $+\infty$ as $\nu \to +\infty$, and that there are $\rho_0, g_0 \in \text{Hol}(X, X)$ such that $f^\nu \to \rho_0$ and $f^\nu \to g_0$ in $\text{Hol}(X, X)$. Then it is easy to see that

$$h_0 \circ \rho_0 = h_0 = \rho_0 \circ h_0 \quad \text{and} \quad g_0 \circ h_0 = \rho_0 = h_0 \circ g_0,$$

and so

$$\rho_0^2 = \rho_0 \circ \rho_0 = g_0 \circ h_0 \circ \rho_0 = g_0 \circ h_0 = \rho_0.$$

Thus $\rho_0$ is a holomorphic retraction, whose image is contained in the closure of $f(X)$, which is compact. This means (see Rossi [Ro] and Cartan [C]) that $\rho_0(X)$ is a compact connected complex submanifold of $X$, i.e., a point $z_0 \in X$. Therefore $\rho_0 \equiv z_0$ and $z_0$ is a fixed point of $f$, since $f$ clearly commutes with $\rho_0$.

We are left to proving that $f^k \to z_0$, which implies in particular that $z_0$ is the only fixed point of $f$. Since $\{f^k\}$ is relatively compact in $\text{Hol}(X, X)$, it suffices to show that $z_0$ is the unique limit point of any converging subsequence of $\{f^k\}$. So let $\{f^k_\nu\}$ be a subsequence converging toward a map $h \in \text{Hol}(X, X)$. Arguing as before we find a holomorphic retraction $\rho \in \text{Hol}(X, X)$ such that $h = \rho \circ h$. Furthermore, $\rho$ must again be constant; but since it is obtained as a limit of a subsequence of iterates of $f$, it must commute with $\rho_0$, and this is possible if and only if $\rho \equiv z_0$. But then $h = \rho \circ h \equiv z_0$ too, and we are done. \qed

In particular this theorem holds for hyperbolic Stein manifolds, because a Stein manifold has no compact complex submanifolds of positive dimension.

Remark 1.1: If $f^k \to z_0$, then the spectral radius of $df_{z_0}$ is strictly less than one. Indeed, if $df_{z_0}$ had an eigenvalue $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$, then $d(f^k)_{z_0}$ would have $\lambda^k$ as eigenvalue, and $\lambda^k \neq 0$ whereas $d(f^k)_{z_0} \to O$.

2. The Heins map

Let $X$ be a hyperbolic manifold with no compact complex submanifolds of positive dimension. The Heins map of $X$ is the map $\tau: \text{Hol}_c(X, X) \to X$ that associates to any $f \in \text{Hol}_c(X, X)$ its unique fixed point $\tau(f) \in X$, whose existence is proved in Theorem 1.1.

The first observation is that the Heins map is continuous:

Lemma 2.1: Let $X$ be a hyperbolic manifold with no compact complex submanifolds of positive dimension. Then the Heins map $\tau: \text{Hol}_c(X, X) \to X$ is continuous.

Proof: Let $\{f_k\} \subset \text{Hol}_c(X, X)$ be a sequence converging toward a map $f \in \text{Hol}_c(X, X)$; we must show that $\tau(f_k) \to \tau(f) \in X$.

First of all, we claim that the set $\{\tau(f_k)\}$ is relatively compact in $X$. Assume that this is not true; then, up to passing to a subsequence, we can assume that the sequence $\{\tau(f_k)\}$ eventually leaves any compact subset of $X$. Now, the set $f(X)$ is relatively compact in $X$; we can then find an open set $D$ in $X$ such that $f(X) \subset D \subset X$.

We have $\tau(f_k) \notin \overline{D}$ eventually; therefore for $k$ large enough we can find $R_k > 0$ such that

$$B(\tau(f_k), R_k) \cap D = \emptyset \quad \text{and} \quad B(\tau(f_k), R_k) \cap \partial D \neq \emptyset,$$

where $B(z, R)$ is the ball of center $z \in X$ and radius $R > 0$ with respect to the Kobayashi distance of $X$. Choose $z_k \in B(\tau(f_k), R_k) \cap \partial D$ for every $k$ large enough; since $\partial D$ is compact, up to a subsequence we can assume that $z_k \to z_0 \in \partial D$. In particular, then, $f_k(z_k) \to f(z_0) \in f(X) \subset D$. But, on the other hand, we
have \( f_k(z_k) \in B(\tau(f_k), R_k) \subset X \setminus D \) for all \( k \) large enough, because \( \tau(f_k) \) is fixed by \( f_k \) and the Kobayashi distance is contracted by holomorphic maps; therefore \( f(z_0) \in X \setminus D \), contradiction.

So \( \{ \tau(f_k) \} \) is relatively compact in \( X \); to prove that \( \tau(f_k) \to \tau(f) \) it suffices to show that \( \tau(f) \) is the unique limit point of the sequence \( \{ \tau(f_k) \} \). But indeed if \( \tau(f_k) \to x \in X \) we have

\[
\begin{align*}
  f(x) &= \lim_{\nu \to +\infty} f_{k_\nu}(\tau(f_{k_\nu})) = \lim_{\nu \to +\infty} \tau(f_{k_\nu}) = x;
\end{align*}
\]

but \( \tau(f) \) is the only fixed point of \( f \), and we are done. \( \square \)

As stated in the introduction, our aim is to prove that the Heins map is holomorphic in a suitable sense. Since we do not know how to define a holomorphic structure on \( \text{Hol}_c(X, X) \) for general manifolds, we shall prove another result which is equivalent to the holomorphy of \( \tau \) in any reasonable setting (see for instance Corollary 2.4 below). We shall need the following lemma:

**Lemma 2.2:** Let \( P \subset \mathbb{C}^n \) be a polydisk centered in \( p_0 \in \mathbb{C}^n \), and \( h: P \to \mathbb{C}^n \) a holomorphic map. Then there is a holomorphic map \( A: \mathbb{R} \to M(n, \mathbb{C}) \), where \( M(n, \mathbb{C}) \) is the space of \( n \times n \) complex matrices, satisfying the following properties:

(i) \( h(z) - h(p_0) = A(z) \cdot (z - p_0) \) for all \( z \in P \);

(ii) \( A(p_0) = dh_{p_0} \);

(iii) for every polydisk \( P_1 \subset\subset P \) centered at \( p_0 \) there is a constant \( C(P_1) > 0 \) such that \( ||A||_{P_1} \leq C(P_1) ||h||_{P_1} \).

**Proof:** We can write

\[
  h(z) - h(p_0) = \int_0^1 \frac{\partial}{\partial t} h(z_0 + t(z - p_0)) \, dt = \sum_{j=1}^n (z_j - p_j^0) \int_0^1 \frac{\partial h}{\partial z_j}(z_0 + t(z - p_0)) \, dt.
\]

Therefore taking

\[
  A_j(z) = \int_0^1 \frac{\partial h}{\partial z_j}(z_0 + t(z - p_0)) \, dt
\]

the matrix \( A = (A_j) \) clearly satisfies (i) and (ii), and (iii) follows from the Cauchy estimates. \( \square \)

**Theorem 2.3:** Let \( X \) be a hyperbolic manifold with no compact complex submanifolds of positive dimension, \( Y \) another complex manifold, and \( F: Y \times X \to X \) a holomorphic map so that \( f_y = F(y, \cdot) \in \text{Hol}_c(X, X) \) for every \( y \in Y \). Then the map \( \tau_F: Y \to X \) given by \( \tau_F(y) = \tau(f_y) \) is holomorphic. Furthermore, for every \( y_0 \in Y \) the differential of \( \tau_F \) at \( y_0 \) is given by

\[
  d(\tau_F)_{y_0} = (id - d(f_{y_0})_{\tau(f_{y_0})})^{-1} \circ dF_{(y_0, \tau(f_{y_0}))} (\cdot, O).
\]

Notice that, by Remark 1.1, \( id - d(f_{y_0})_{\tau(f_{y_0})} \) is invertible.

**Proof:** Without loss of generality, we can assume that \( Y \) is a ball \( B^m \subset \mathbb{C}^m \) centered at \( y_0 \). Set \( p_0 = \tau(f_{y_0}) \), and let \( P_0 \subset X \) be the domain of a polydisk chart centered at \( p_0 \). Since \( f_{y_0}(p_0) = p_0 \), we can find a polydisk \( P_1 \subset P_0 \) centered at \( p_0 \) such that \( f_{y_0}(P_1) \subset P_0 \). Furthermore, by Lemma 2.1 there is also a \( \delta > 0 \) such that \( ||y - y_0|| < \delta \) implies \( \tau(f_y) \in P_1 \) and \( f_y(P_1) \subset P_0 \). This means that as soon as \( y \) is close enough to \( y_0 \) we can work inside \( P_0 \) and assume, without loss of generality, that \( X \) is contained in some \( \mathbb{C}^n \).

Write \( p_y = \tau(f_y) \in P_1 \), and define \( h_y: \overline{P_1} \to \mathbb{C}^n \) by \( h_y = f_y - f_{y_0} \). We have

\[
  p_y - p_0 = f_{y_0}(p_y) - f_{y_0}(p_0) + h_y(p_y);
\]

therefore Lemma 2.2 applied to \( f_{y_0} \) yields a matrix \( A(y) \), depending continuously on \( y \) by Lemma 2.1, such that

\[
  p_y - p_0 = A(y) \cdot (p_y - p_0) + h_y(p_y).
\]

Since \( A(y) \to d(f_{y_0})_{p_0} \) as \( y \to y_0 \), for \( y \) close to \( y_0 \) \( id - A(y) \) is invertible, and so we can write

\[
  p_y - p_0 = (id - A(y))^{-1} \cdot h_y(p_y).
\]

(2.1)
Now, we have
\[ dF_{(y_0, \tau(f_{y_0}))}(\cdot, O) = \text{Jac}_y(f_y(p_0))(y_0), \]
where $\text{Jac}_y$ is the Jacobian matrix computed with respect to the $y$ variables; in particular,
\[ h_y(p_0) - dF_{(y_0, \tau(f_{y_0}))}(y - y_0, O) = o(\|y - y_0\|). \]
This means that to show that $\tau_F$ is holomorphic and $d\tau_F$ has the claimed expression it suffices to show that
\[ \lim_{y \to y_0} \left| \frac{\tau_F(y) - \tau_F(y_0) - (\text{id} - d(f_{y_0})_{p_0})^{-1} \cdot h_y(p_0)}{\|y - y_0\|} \right| = 0, \]
which is equivalent to proving that
\[ \lim_{y \to y_0} \left| \frac{(\text{id} - d(f_{y_0})_{p_0}) \cdot (p_y - p_0) - h_y(p_0)}{\|y - y_0\|} \right| = 0. \quad (2.2) \]
Now, (2.1) yields
\[ \left\| \frac{(\text{id} - d(f_{y_0})_{p_0}) \cdot (p_y - p_0) - h_y(p_0)}{\|y - y_0\|} \right\| = \left\| \frac{(\text{id} - A(y)) \cdot (p_y - p_0) - h_y(p_0) + (A(y) - d(f_{y_0})_{p_0}) \cdot (p_y - p_0)}{\|y - y_0\|} \right\| \leq \frac{\|h_y(p_y) - h_y(p_0)\|}{\|y - y_0\|} + \|A(y) - d(f_{y_0})_{p_0}\| \frac{\|p_y - p_0\|}{\|y - y_0\|}. \quad (2.3) \]
Since $h_y(z)$ is holomorphic both in $y$ and in $z$, we have
\[ h_y(z) - h_y(z_1) = O(\|y - y_1\|, \|z - z_1\|); \]
in particular,
\[ h_y(z) = h_y(z) - h_{y_0}(z) = O(\|y - y_0\|) \quad (2.4) \]
uniformly on $P_1$. So (2.1) implies that $p_y - p_0 = O(\|y - y_0\|)$, and thus the second summand in (2.3) tends to zero as $y \to y_0$.

Finally, if we apply Lemma 2.2 to $h_y$ we get a matrix $B(y)$ and a constant $C > 0$ such that
\[ \|h_y(p_y) - h_y(p_0)\| \leq \|B(y)\| \cdot \|p_y - p_0\| \leq C \|h_y\| \|p_y - p_0\| \]
when $y$ is close enough to $y_0$, where $P_2 \subset P_1$ is a fixed polydisk centered at $p_0$. But then (2.4) yields
\[ \|h_y(p_y) - h_y(p_0)\| = O(\|y - y_0\|^2), \]
and so (2.2) is proved. \[ \square \]

If $X$ is a bounded domain in $C^n$, then $\text{Hol}_c(X, X)$ is an open subset of $H^\infty(X)^n$, the complex Banach space of $n$-uples of bounded holomorphic functions defined on $X$. Therefore in this case $\text{Hol}_c(X, X)$ has a natural complex structure, and we obtain the following generalization of the main result in [B]:

**Corollary 2.4:** Let $D \subset \subset \mathbb{C}^n$ be a bounded domain. Then the Heins map $\tau : \text{Hol}_c(D, D) \to D$ is holomorphic.

**Proof:** It follows from Theorem 2.3 and [FV, Theorem II.3.10]. \[ \square \]
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