Newton numbers and residual measures of plurisubharmonic functions

ALEXANDER RASHKOVSIIK

Abstract. We study the masses charged by \((dd^c u)^n\) at isolated singularity points of plurisubharmonic functions \(u\). It is done by means of the local indicators of plurisubharmonic functions introduced in \([15]\). As a consequence, bounds for the masses are obtained in terms of the directional Lelong numbers of \(u\), and the notion of the Newton number for a holomorphic mapping is extended to arbitrary plurisubharmonic functions. We also describe the local indicator of \(u\) as the logarithmic tangent to \(u\).

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1 Introduction

The principal information on local behaviour of a subharmonic function \(u\) in the complex plane can be obtained by studying its Riesz measure \(\mu_u\). If \(u\) has a logarithmic singularity at a point \(x\), the main term of its asymptotics near \(x\) is \(\mu_u(\{x\}) \log |z - x|\). For plurisubharmonic functions \(u\) in \(\mathbb{C}^n\), \(n > 1\), the situation is not so simple. The local properties of \(u\) are controlled by the current \(dd^c u\) (we use the notation \(d = \partial + \bar{\partial}\), \(d^c = (\partial - \bar{\partial})/2\pi i\)) which cannot charge isolated points. The trace measure \(\sigma_u = dd^c u \wedge \beta_{n-1}\) of this current is precisely the Riesz measure of \(u\); here \(\beta_p = (p!)^{-1}2^p(dd^c|z|^2)^p\) is the volume element of \(\mathbb{C}^p\). A significant role is played by the Lelong numbers \(\nu(u, x)\) of the function \(u\) at points \(x\):

\[
\nu(u, x) = \lim_{r \to 0} (\tau_{2n-2r^{2n-2}})^{-1} \sigma_u[B^{2n}(x, r)],
\]

where \(\tau_{2p}\) is the volume of the unit ball \(B^{2p}(0, 1)\) of \(\mathbb{C}^p\). If \(\nu(u, x) > 0\) then \(\nu(u, x) \log |z - x|\) gives an upper bound for \(u(z)\) near \(x\), however the difference between these two functions can be comparable to \(\log |z - x|\).

Another important object generated by the current \(dd^c u\) is the Monge-Ampère measure \((dd^c u)^n\). For the definition and basic facts on the complex
Monge-Ampère operator \((dd^c)^n\) and Lelong numbers, we refer the reader to the books [12], [14] and [8], and for more advanced results, to [2]. Here we mention that \((dd^c)^n\) cannot be defined for all plurisubharmonic functions \(u\), however if \(u \in \text{PSH}(\Omega) \cap L^\infty_{\text{loc}}(\Omega \setminus K)\) with \(K \subset \subset \Omega\), then \((dd^c)^n\) is well defined as a positive closed current of the bidimension \((0, 0)\) (or, which is the same, as a positive measure) on \(\Omega\). This measure cannot charge pluripolar subsets of \(\Omega \setminus K\), and it can have positive masses at points of \(K\), e.g. \((dd^c \log |z|)^n = \delta(0)\), the Dirac measure at 0, \(|z| = (\sum |z_j|^2)^{1/2}\). More generally, if \(f : \Omega \to \mathbb{C}^N, N \geq n\), is a holomorphic mapping with isolated zeros at \(x^{(k)} \in \Omega\) of multiplicities \(m_k\), then \((dd^c \log |f|)^n|_{x^{(k)}} = m_k \delta(x^{(k)})\).

So, the masses of \((dd^c)^n\) at isolated points of singularity of \(u\) (the residual measures of \(u\)) are of especial importance.

Let a plurisubharmonic function \(u\) belong to \(L^\infty_{\text{loc}}(\Omega \setminus \{x\})\); its residual mass at the point \(x\) will be denoted by \(\tau(u, x)\):

\[
\tau(u, x) = (dd^c u)^n|_{\{x\}}.
\]

The problem under consideration is evaluation of this value.

The following well-known relation compares \(\tau(u, x)\) with the Lelong number \(\nu(u, x)\):

\[
\tau(u, x) \geq [\nu(u, x)]^n.
\]

The equality in (1) means that, roughly speaking, the function \(u(z)\) behaves near \(x\) as \(\nu(u, x) \log |z - x|\). In many cases however relation (1) is not optimal; e.g. for

\[
u(u, 0) = k_1 k_2 > k_2^2 = [\nu(u, 0)]^2.
\]

As follows from the Comparison Theorem due to Demailly (see Theorem A below), the residual mass is determined by asymptotic behaviour of the function near its singularity, so one needs to find appropriate characteristics for the behaviour. To this end, a notion of local indicator was proposed in [15]. Note that \(\nu(u, x)\) can be calculated as

\[
\nu(u, x) = \lim_{r \to -\infty} r^{-1} \sup \{ \nu(z) : |z - x| \leq e^r \} = \lim_{r \to -\infty} r^{-1} \mathcal{M}(u, x, r),
\]

where \(\mathcal{M}(u, x, r)\) is the mean value of \(u\) over the sphere \(|z - x| = e^r\), see [4]. In [3], the refined, or directional, Lelong numbers were introduced as

\[
\nu(u, x, a) = \lim_{r \to -\infty} r^{-1} \sup \{ \nu(z) : |z_k - x_k| \leq e^{ra_k}, 1 \leq k \leq n \}
\]

\[
= \lim_{r \to -\infty} r^{-1} g(u, x, ra),
\]
\[ a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \] and \( g(u, x, b) \) is the mean value of \( u \) over the set \( \{ z : |z_k - x_k| = \exp b_k, 1 \leq k \leq n \} \). For \( x \) fixed, the collection \( \{ \nu(u, x, a) \}_{a \in \mathbb{R}_+^n} \) gives a more detailed information about the function \( u \) near \( x \) than \( \nu(u, x) \) does, so one can expect for a more precise bound for \( \tau(u, x) \) in terms of the directional Lelong numbers. It was noticed already in [5] that \( a \mapsto \nu(u, x, a) \) is a concave function on \( \mathbb{R}_+^n \). In [15], it was observed that this function produces the following plurisubharmonic function \( \Psi_{u, x} \) in the unit polydisk \( D = \{ y \in \mathbb{C}^n : |y_k| < 1, 1 \leq k \leq n \} \):

\[ \Psi_{u, x}(y) = -\nu(u, x, (-\log |y_k|)), \]

the local indicator of the function \( u \) at \( x \). It is the largest negative plurisubharmonic function in \( D \) whose directional Lelong numbers at 0 coincide with those of \( u \) at \( x \), \( (dd^c \Psi_{u, x}) = \tau(\Psi_{u, x}, 0) \delta(0) \), and finally,

\[ \tau(u, x) \geq \tau(\Psi_{u, x}, 0), \]  

so the singularity of \( u \) at \( x \) is controlled by its indicator \( \Psi_{u, x} \).

Since \( \tau(\Psi_{u, x}, 0) \geq [\nu(\Psi_{u, x}, 0)]^n = [\nu(u, x)]^n \), (4) is a refinement of (1).

The main tool used to obtain these bounds is the Comparison Theorem due to Demailly. To formulate it we give the following

**Definition 1.** A \( q \)-tuple of plurisubharmonic functions \( u_1, \ldots, u_q \) is said to be in general position if their unboundedness loci \( A_1, \ldots, A_q \) satisfy the following condition: for all choices of indices \( j_1 < \ldots < j_k, k \leq q \), the \((2q - 2k + 1)\)-dimensional Hausdorff measure of \( A_{j_1} \cap \ldots \cap A_{j_k} \) equals zero.

**Theorem A** (Comparison Theorem, [2], Th. 5.9). Let \( n \)-tuples of plurisubharmonic functions \( u_1, \ldots, u_n \) and \( v_1, \ldots, v_n \) be in general position

\[ dd^c u_1 \wedge \ldots \wedge dd^c u_n \vert_{\{x\}} \geq \prod_j \nu(u_j, x, a) \frac{a_1 \ldots a_n}{a_1 \ldots a_n} \forall a \in \mathbb{R}_+^n. \]
on a neighbourhood of a point \( x \in \mathbb{C}^n \). Suppose that \( u_j(x) = -\infty, \ 1 \leq j \leq n \), and
\[
\limsup_{z \to x} \frac{v_j(z)}{u_j(z)} = l_j < \infty.
\]
Then
\[
dd^c v_1 \wedge \ldots \wedge \dd^c v_n |_{\{x\}} \leq l_1 \ldots l_n \dd^c u_1 \wedge \ldots \wedge \dd^c u_n |_{\{x\}}.
\]

We also obtain a geometric interpretation for the value \( N(u, x) \) (Theorem [7]). Let \( \Theta_{u,x} \) be the set of points \( b \in \mathbb{R}^n_+ \) such that \( \nu(u, x, a) \geq \langle b, a \rangle \) for some \( a \in \mathbb{R}^n_+ \), then
\[
\tau(u, x) \geq N(u, x) = n! Vol(\Theta_{u,x}).
\]

In many cases the volume of \( \Theta_{u,x} \) can be easily calculated, so (1) gives an effective formula for \( N(u, x) \).

To illustrate these results, consider functions \( u = \log |f|, \ f = (f_1, \ldots, f_n) \) being an equidimensional holomorphic mapping with an isolated zero at a point \( x \). It is probably the only class of functions whose residual measures were studied in details before. In this case, \( \tau(u, x) \) equals \( m \), the multiplicity of \( f \) at \( x \), and
\[
\nu(\log |f|, x, a) = I(f, x, a) := \inf \{ \langle a, p \rangle : p \in \omega_x \}
\]
where
\[
\omega_x = \{ p \in \mathbb{Z}^n_+ : \sum_j |\frac{\partial^p f_j}{\partial z^p}(x)| \neq 0 \}
\]
(see [13]). For polynomials \( F : \mathbb{C}^n \to \mathbb{C} \), the value \( I(F, x, a) \) is a known object (the index of \( F \) at \( x \) with respect to the weight \( a \)) used in number theory (see e.g. [11]).

Relation (1) gives us \( m = \tau(\log |f|, x) \geq N(\log |f|, x) \). In general, the value \( N(\log |f|, x) \) is not comparable to \( m_1 \ldots m_n \) with \( m_j \) the multiplicity of the function \( f_j \): for \( f(z) = (z_1^2 + z_2, z_2) \) and \( x = 0, m_1m_2 = 1 < 2 = N(\log |f|, x) = m \) while for \( f(z) = (z_1^2 + z_2, z_2^3) \), \( N(\log |f|, x) = 2 < 3 = m_1m_2 < 6 = m \). A more sharp bound for \( m \) can be obtained by (3) with \( u_j = \log |f_j|, \ 1 \leq j \leq n \). In this case, the left-hand side of (3) equals \( m \), and its right-hand side with \( a_1 = \ldots = a_n = m_1 \ldots m_n \). For the both above examples of the mapping \( f \), the supremum of the right-hand side of (3) over \( a \in \mathbb{R}^n_+ \) equals \( m \). For \( a_1, \ldots, a_n \) rational, relation (3) is a known bound for \( m \) via the multiplicities of weighted homogeneous initial Taylor polynomials of \( f_j \) with respect to the weights \( (a_1, \ldots, a_n) \) ([11], Th. 22.7).

Recall that the convex hull \( \Gamma_+(f, x) \) of the set \( \bigcup_p \{ p + \mathbb{R}^n_+ \} \), \( p \in \omega_x \) is called the Newton polyhedron of \( (f_1, \ldots, f_n) \) at \( x \), the union \( \Gamma(f, x) \) of the
compact faces of the boundary of $\Gamma_+(f, x)$ is called the Newton boundary of $(f_1, \ldots, f_n)$ at $x$, and the value $N_{f,x} = n! \text{Vol}(\Gamma_-(f, x))$ with $\Gamma_-(f, x) = \{ t \in \Gamma(f, x), \ 0 \leq \lambda \leq 1 \}$ is called the Newton number of $(f_1, \ldots, f_n)$ at $x$ (see [4], [1]). The relation

\[ m \geq N_{f,x} \]

was established by A.G. Kouchnirenko [3] (see also [1], Th. 22.8). Since $\Theta_{\log |f|, x} = \Gamma_-(f, x)$, (3) is a particular case of the relation (3). It is the reason to call $N(u, x)$ the Newton number of $u$ at $x$.

These observations show that the technique of plurisubharmonic functions (and local indicators in particular) is quite a powerful tool to produce, in a unified and simple way, sharp bounds for the multiplicities of holomorphic mappings.

Finally, we obtain a description for the indicator $\Psi_{u,x}(z)$ as the weak limit of the functions $m^{-1} u(x_1 + z_1^m, \ldots, x_n + z_n^m)$ as $m \to \infty$ (Theorem 8), so $\Psi_{u,x}$ can be viewed as the tangent (in the logarithmic coordinates) for the function $u$ at $x$. Using this approach we obtain a sufficient condition, in terms of $C_{n-1}$-capacity, for the residual mass $\tau(u, x)$ to coincide with the Newton number of $u$ at $x$ (Theorem 9).

## 2 Indicators and their masses

We will use the following notations. For a domain $\Omega$ of $\mathbb{C}^n$, $PSH(\Omega)$ will denote the class of all plurisubharmonic functions on $\Omega$, $PSH_-(\Omega)$ the subclass of the nonpositive functions, and $PSH(\Omega, x) = PSH(\Omega) \cap L^\infty_{loc}(\Omega \setminus \{x\})$ with $x \in \Omega$.

Let $D = \{ z \in \mathbb{C}^n : |z_k| < 1, 1 \leq k \leq n \}$ be the unit polydisk, $D^* = \{ z \in D : z_1 \cdot \ldots \cdot z_n \neq 0 \}$, $\mathbb{R}_+^n = \{ t \in \mathbb{R}^n : \pm t_k > 0 \}$. By $\text{CNVI}_-(\mathbb{R}^n)$ we denote the collection of all nonpositive convex functions on $\mathbb{R}^n$ increasing in each variable $t_k$. The mapping $\text{Log} : D^* \to \mathbb{R}^n$ is defined as $\text{Log}(z) = (\log |z_1|, \ldots, \log |z_n|)$, and $\text{Exp} : \mathbb{R}^n \to D^*$ is given by $\text{Exp}(t) = (\exp t_1, \ldots, \exp t_n)$.

A function $u$ on $D^*$ is called $n$-circled if

\[ u(z) = u(|z_1|, \ldots, |z_n|), \]

i.e. if $\text{Log}^* \text{Exp}^* u = u$. Any $n$-circled function $u \in PSH_-(D^*)$ has a unique extension to the whole polydisk $D$ keeping the property (9). The class of such functions will be denoted by $PSH_\circ(D)$. The cones $\text{CNVI}_-(\mathbb{R}^n_+)$ and $PSH_\circ(D)$ are isomorphic: $u \in PSH_\circ(D) \iff \text{Exp}^* u \in \text{CNVI}_-(\mathbb{R}^n_-)$, $h \in \text{CNVI}_-(\mathbb{R}^n_-) \iff \text{Log}^* h \in PSH_\circ(D)$.  

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**Definition 2**. A function $\Psi \in \mathcal{PSH}^c(D)$ is called an *indicator* if its convex image $\text{Exp}^*\Psi$ satisfies

$$\text{Exp}^*\Psi(ct) = c\text{Exp}^*\Psi(t) \quad \forall c > 0, \forall t \in \mathbb{R}^n. \quad (10)$$

The collection of all indicators will be denoted by $I$. It is a convex subcone of $\mathcal{PSH}^c(D)$, closed in $D'$ (or equivalently, in $L^1_{\text{loc}}(D)$). Besides, if $\Psi_1, \Psi_2 \in I$ then $\sup\{\Psi_1, \Psi_2\} \in I$, too.

Every indicator is locally bounded in $D^*$. In what follows we will often consider indicators locally bounded in $D \setminus \{0\}$; the class of such indicators will be denoted by $I_0$: $I_0 = I \cap \mathcal{PSH}(D,0)$.

An example of indicators can be given by the functions $\varphi_a(z) = \sup_k a_k \log |z_k|$, $a_k \geq 0$. If all $a_k > 0$, then $\varphi_a \in I_0$.

**Proposition 1** Let $\Psi \in I_0$, $\Psi \not\equiv 0$. Then

(a) there exist $\nu_1, \ldots, \nu_n > 0$ such that

$$\Psi(z) \geq \varphi_{\nu}(z) \quad \forall z \in D; \quad (11)$$

(b) $\Psi \in C(\overline{D} \setminus \{0\})$, $\Psi|_{\partial D} = 0$;

(c) the directional Lelong numbers $\nu(\Psi, 0, a)$ of $\Psi$ at the origin with respect to $a \in \mathbb{R}^n_+$ (3) are

$$\nu(\Psi, 0, a) = -\Psi(\text{Exp}(-a)), \quad (12)$$

and its Lelong number $\nu(\Psi, 0) = -\Psi(e^{-1}, \ldots, e^{-1})$;

(d) $(dd^c\Psi)^n = 0$ on $D \setminus \{0\}$.

**Proof.** Let $\Psi_k(z_k)$ denote the restriction of the indicator $\Psi(z)$ to the disk $D^{(k)} = \{ z \in D : z_j = 0 \ \forall j \neq k \}$. By monotonicity of $\text{Exp}^*\Psi$, $\Psi(z) \geq \Psi_k(z_k)$. Since $\Psi_k$ is a nonzero indicator in the disk $D^{(k)} \subset \mathbb{C}$, $\Psi_k(z_k) = \nu_k \log |z_k|$ with some $\nu_k > 0$, and (a) follows.

As $\text{Exp}^*\Psi \in C(\mathbb{R}^n_+)$, $\Psi \in C(D^*)$. Its continuity in $D \setminus \{0\}$ can be shown by induction in $n$. For $n = 1$ it is obvious, so assuming it for $n \leq l$, consider any point $z^0 \neq 0$ with $z^0_j = 0$. Let $z^* \to z^0$, then the points $\tilde{z}^*$ with $\tilde{z}^*_j = 0$ and $\tilde{z}^*_m = z^*_m$, $m \neq j$, also tend to $z^0$, and by the induction hypothesis, $\Psi(\tilde{z}^*) \to \Psi(z^0) = \Psi(z^0)$. So, $\liminf_{s \to \infty} \Psi(z^s) \geq \lim_{s \to \infty} \Psi(\tilde{z}^s) = \Psi(z^0)$, i.e.
Ψ is lower semicontinuous and hence continuous at $z^0$. Continuity of $Ψ$ up to $\partial D$ and the boundary condition follow from (11).

Equality (12) is an immediate consequence of the definition of the directional Lelong numbers (3) and the homogeneity condition (10). The relation $\nu(u, x) = \nu(u, x, (1, \ldots, 1))$ [5] gives us the desired expression for $\nu(Ψ, 0)$.

Finally, statement (d) follows from the homogeneity condition (10), see [15], Proposition 4.

For functions $Ψ \in I_0$, the complex Monge-Ampère operator $(dd^cΨ)^n$ is well defined and gives a nonnegative measure on $D$. By Proposition 1,

$$(dd^cΨ)^n = \tau(Ψ) \delta(0)$$

with some constant $\tau(Ψ) \geq 0$ which is strictly positive unless $Ψ \equiv 0$. In this section, we will study the value $\tau(Ψ)$.

An upper bound for $\tau(Ψ)$ is given by

**Proposition 2** For $Ψ \in I_0$,

$$(13) \quad \tau(Ψ) \leq \nu_1 \ldots \nu_n$$

with $ν_1, \ldots, ν_n$ the same as in Proposition [1], (a).

**Proof.** The function $ϕ_ν(z) \in I_0$, and (11) implies

$$\limsup_{z \to 0} Ψ(z) \leq 1,$$

so (13) follows by Theorem A.

To obtain a lower bound for $\tau(Ψ)$, we need a relation between $Ψ(z)$ and $Ψ(z^0)$ for $z, z^0 \in D$. Denote

$$Φ(z, z^0) = \sup_k \frac{\log |z_k|}{|\log |z_k^0||}, \quad z \in D, \ z^0 \in D^*.$$ 

Being considered as a function of $z$ with $z^0$ fixed, $Φ(z, z^0) \in I_0$.

**Proposition 3** For any $Ψ \in I$, $Ψ(z) \leq |Ψ(z^0)|Φ(z, z^0)$ $∀z \in D, \ z^0 \in D^*$.

**Proof.** For a fixed $z^0 \in D^*$ and $t^0 = Log(z^0)$, define $u = |Ψ(z^0)|^{-1}Exp^*Ψ$ and $v = Exp^*Φ = sup_k t_k/|t^0_k|$. It suffices to establish the inequality $u(t) \leq v(t)$ for all $t \in R^n$ with $t^0_k < t_k < 0$, $1 \leq k \leq n$. Given such a $t$, denote $λ_0 = [1 + v(t)]^{-1}$. Since $\{t^0 + λ(t - t^0) : 0 \leq λ \leq λ_0\} \subset R^d$, the functions
$u_t(\lambda) := u(t^0 + \lambda(t - t^0))$ and $v_t(\lambda) := v(t^0 + \lambda(t - t^0))$ are well defined on $[0, \lambda_0]$. Furthermore, $u_t$ is convex and $v_t$ is linear there, $u_t(0) = v_t(0) = -1$, $u_t(\lambda_0) \leq v_t(\lambda_0) = 0$. It implies $u_t(\lambda) \leq v_t(\lambda)$ for all $\lambda \in [0, \lambda_0]$. In particular, as $\lambda_0 > 1$, $u(t) = u_t(1) \leq v_t(1) = v(t)$, that completes the proof.

Consider now the function

$$P(z) = -\prod_{1 \leq k \leq n} |\log |z_k||^{1/n} \in I.$$

**Theorem 1** For any $\Psi \in I_0$,

$$\tau(\Psi) \geq \frac{\Psi(z^0)}{P(z^0)} \quad \forall z^0 \in D^*.$$  

**Proof.** By Proposition 3,

$$\frac{\Psi(z)}{\Phi(z, z^0)} \leq |\Psi(z^0)| \quad \forall z \in D, \ z^0 \in D^*.$$  

By Theorem A,

$$\langle dd^c \Psi \rangle^\star \leq |\Psi(z^0)| \langle dd^c \Phi(z, z^0) \rangle^\star,$$

and the statement follows from the fact that

$$\langle dd^c \Phi(z, z^0) \rangle^\star = \prod_{1 \leq k \leq n} |\log |z^0_k||^{-1} = |P(z^0)|^{-n}.$$  

**Remarks.** 1. One can consider the value

$$A_\Psi = \sup_{z \in D} \frac{\Psi(z)}{P(z)};$$

by Theorem 4,

$$\tau(\Psi) \geq A_\Psi.$$  

2. Let $I_{0,M} = \{ \Psi \in I_0 : \tau(\Psi) \leq M \}, \ M > 0$. Then (14) gives the lower bound for the class $I_{0,M}$:

$$\Psi(z) \geq M^{1/n} P(z) \quad \forall z \in D, \ \forall \Psi \in I_{0,M}.$$  

Let now $\Psi_1, \ldots, \Psi_n \in I$ be in general position in the sense of Definition 1. Then the current $\wedge_k dd^c \Psi_k$ is well defined, as well as $(dd^c \Psi)^\star$ with $\Psi = \sup_k \Psi_k$. Moreover, we have
Proposition 4 If $\Psi_1, \ldots, \Psi_n \in I$ are in general position, then
\begin{equation}
\bigwedge_k dd^c \Psi_k = 0 \text{ on } D \setminus \{0\}.
\end{equation}

Proof. For $\Psi_1, \ldots, \Psi_n \in I_0$, the statement follows from Proposition (d), and the polarization formula
\begin{equation}
\bigwedge_k dd^c \Psi_k = \frac{(-1)^n}{n!} \sum_{j=1}^{n} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq n} \left( dd^c \sum_{k=1}^{j} \Psi_{j_k} \right)^n.
\end{equation}

When the only condition on $\{\Psi_k\}$ is to be in general position, we can replace $\Psi_k(z)$ with $\Psi_{k,N}(z) = \sup \{\Psi_k(z), N \sup_j \log |z_j| \} \in I_0$ for which $\bigwedge_k dd^c \Psi_{k,N} = 0$ on $D \setminus \{0\}$. Since $\Psi_{k,N} \rightarrow \Psi_k$ as $N \rightarrow \infty$, it gives us (17).

The mass of $\bigwedge_k dd^c \Psi_k$ will be denoted by $\tau(\Psi_1, \ldots, \Psi_n)$.

Theorem 2 Let $\Psi_1, \ldots, \Psi_n \in I$ be in general position, $\Psi = \sup_k \Psi_k$. Then
\begin{enumerate}[(a)]
\item $\tau(\Psi) \leq \tau(\Psi_1, \ldots, \Psi_n)$;
\item $\tau(\Psi_1, \ldots, \Psi_n) \geq |P(z^0)|^{-n} \prod_k |\Psi_k(z^0)| \quad \forall z^0 \in D^*$.
\end{enumerate}

Proof. Since
\[
\frac{\Psi(z)}{\Psi_k(z)} \leq 1 \quad \forall z \neq 0,
\]
statement (a) follows from Theorem A.

Statement (b) results from Proposition (a) exactly like the statement of Theorem A.

3 Geometric interpretation

In this section we study the masses $\tau(\Psi)$ of indicators $\Psi \in I_0$ by means of their convex images $Exp^* \Psi \in CNVI_-(\mathbb{R}^n)$.

Let $V \in PSH^c(rD) \cap C^2(rD)$, $r < 1$, and $v = Exp^* V \in CNVI_-(((\mathbb{R}_- + \log r)^n)$. Since
\[
\frac{\partial^2 V(z)}{\partial z_j \partial \bar{z}_k} = \frac{1}{4\bar{z}_j \bar{z}_k} \frac{\partial^2 v(t)}{\partial t_j \partial t_k} \bigg|_{t = \log(z)}, \quad z \in rD^*;
\]
\[
\det \left( \frac{\partial^2 V(z)}{\partial z_j \partial \bar{z}_k} \right) = 4^{-n} |z_1 \ldots z_n|^{-2} \det \left( \frac{\partial^2 v(t)}{\partial t_j \partial t_k} \right) \bigg|_{t = \log(z)}.
\]
By setting $z_j = \exp\{t_j + i\theta_j\}$, $0 \leq \theta \leq 2\pi$, we get $eta_n(z) = |z_1 \ldots z_n|^2 dt d\theta$, so

$$(19) (dd^c V)^n = n! \left( \frac{2\pi}{n} \right)^n \det \left( \frac{\partial^2 V}{\partial z_j \partial \bar{z}_k} \right) \beta_n = n! (2\pi)^{-n} \det \left( \frac{\partial^2 v}{\partial t_j \partial \bar{t}_k} \right) dt d\theta.$$  

Every function $U \in PSH_c(D) \cap L^\infty(D)$ is the limit of a decreasing sequence of functions $U_l \in PSH_c(E) \cap C^2(E)$ on an $n$-circled domain $E \subset D$, and by the convergence theorem for the complex Monge-Ampère operators,

$$(20) (dd^c U_l)^n|_E \rightarrow (dd^c U)^n|_E.$$  

On the other hand, for $u_l = \exp^* U_l$ and $u = \exp^* U$,

$$(21) \det \frac{\partial^2 u_l}{\partial t_j \partial \bar{t}_k} dt \bigg|_{Log(D^* \cap E)} \rightarrow \mathcal{MA}[u]|_{Log(D^* \cap E)},$$  

the real Monge-Ampère operator of $U$.

Since $(dd^c U_l)^n$ and $(dd^c U)^n$ cannot charge pluripolar sets, $(19)$ with $V = U_l$ and $(20), (21)$ imply

$$(dd^c U)^n(E) = n! (2\pi)^{-n} \mathcal{MA}[u] d\theta (Log(E) \times [0, 2\pi]^n)$$  

for any $n$-circled Borel set $E \in D$, i.e.

$$(22) (dd^c U)^n(E) = n! \mathcal{MA}[u](Log(E)).$$  

This relation allows us to calculate $\tau(\Psi)$ by using the technique of real Monge-Ampère operators in $\mathbb{R}^n$ (see [16]).

Let $\Psi \in I$. Consider the set

$$B_\Psi = \{ a \in \mathbb{R}_+^n : \langle a, t \rangle \leq \exp^* \Psi(t) \ \forall t \in \mathbb{R}_+^n \}$$  

and define

$$\Theta_\Psi = \mathbb{R}_+^n \setminus B_\Psi.$$  

Clearly, the set $B_\Psi$ is convex, so $\exp^* \Psi$ is the restriction of its support function to $\mathbb{R}^n$. If $\Psi \in I_0$, the set $\Theta_\Psi$ is bounded. Indeed, $a \in \Theta_\Psi$ if and only if $\langle a, t^0 \rangle \geq \exp^* \Psi(t^0)$ for some $t^0 \in \mathbb{R}_+^n$, that implies $|a_j| \leq |\exp^* \Psi(t^0)/t^0_j| \forall j$. By Proposition [4], (a), $|\exp^* \Psi(t^0)| \leq \nu_j |t_j|$ and therefore $|a_j| \leq \nu_j \forall j$.

Given a set $F \in \mathbb{R}^n$, we denote its Euclidean volume by $Vol(F)$.

**Theorem 3** $\forall \Psi \in I_0$,  

$$(23) \tau(\Psi) = n! Vol(\Theta_\Psi).$$
Proof. Denote \(U(z) = \sup\{\Psi(z), -1\} \in PSH_{c}(D) \cap C(D), \ u = \text{Exp}^{*} U \in CNVI_{-}(\mathbb{R}^{n})\). Since \(U(z) = \Psi(z)\) near \(\partial D\),

\[
\tau(\Psi) = \int_{D} (dd^{c} U)^{n}.
\]

Furthermore, as \((dd^{c} U)^{n} = 0\) outside the set \(E = \{z \in D : \Psi(z) = -1\}\),

\[
\tau(\Psi) = \int_{E} (dd^{c} U)^{n}.
\]

In view of (22),

\[
\int_{E} (dd^{c} U)^{n} = n! \int_{\text{Log}(E)} \mathcal{M}A[u].
\]

As was shown in [16], for any convex function \(v\) in a domain \(\Omega \subset \mathbb{R}^{n}\),

\[
\int_{F} \mathcal{M}A[v] = \text{Vol}(\omega(F, v)) \quad \forall F \subset \Omega,
\]

where

\[
\omega(F, v) = \bigcup_{v^{0} \in F} \{a \in \mathbb{R}^{n} : v(t) \geq v(t^{0}) + \langle a, t - t^{0} \rangle \forall t \in \Omega\}
\]

is the gradient image of the set \(F\) for the surface \(\{y = v(x), x \in \Omega\}\).

We claim that

\[
\Theta_{\Psi} = \omega(\text{Log}(E), u).
\]

Observe that

\[
\Theta_{\Psi} = \{a \in \mathbb{R}^{n}_{+} : \sup_{\psi(t) = -1} \langle a, t \rangle \geq -1\}
\]

where \(\psi = \text{Exp}^{*} \Psi\).

If \(a \in \omega(\text{Log}(E), u)\), then for some \(t^{0} \in \mathbb{R}^{n}\) with \(\psi(t^{0}) = 1\) we have \(\langle a, t^{0} \rangle \geq \langle a, t \rangle\) for all \(t \in \mathbb{R}^{n}\) such that \(\psi(t) < -1\). Taking here \(t_{j} \to -\infty\) we get \(a_{j} \geq 0\), i.e. \(a \in \mathbb{R}^{n}_{+}\). Besides, \(\langle a, t^{0} \rangle \geq \langle a, t \rangle - 1 - \psi(t)\) for all \(t \in \mathbb{R}^{n}\) with \(\psi(t) > -1\), and applying this for \(t \to 0\) we derive \(\langle a, t^{0} \rangle \geq -1\).

Therefore, \(a \in \Theta_{\Psi}\) and \(\Theta_{\Psi} \supset \omega(\text{Log}(E), u)\).

Now we prove the converse inclusion. If \(a \in \Theta_{\Psi} \cap \mathbb{R}^{n}_{+}\), then

\[
\sup\{\langle a, t^{0} \rangle : t^{0} \in \text{Log}(E)\} \geq -1.
\]

Let \(t\) be such that \(\psi(t) = -\delta > -1\), then \(t/\delta \in \text{Log}(E)\) and thus

\[
\langle a, t \rangle - 1 - \psi(t) = \delta(a, t/\delta) - 1 + \delta \leq \delta \sup_{t^{0} \in \text{Log}(E)} \langle a, t^{0} \rangle - 1 + \delta \leq \sup_{t^{0} \in \text{Log}(E)} \langle a, t^{0} \rangle = \sup_{z^{0} \in E} \langle a, \text{Log}(z^{0}) \rangle.
\]
Since $E$ is compact, the latter supremum is attained at some point $z^0$. Furthermore, $z^0 \in E \cap D^*$ because $a_k \neq 0$, $1 \leq k \leq n$. Hence $\sup_{t \in \log(E)} (a, t^0) = \langle a, t^0 \rangle$ with $t^0 = \log(z^0) \in \mathbb{R}^n_+$, so that $a \in \omega(\log(E), u)$ and $\Theta_\Psi \cap \mathbb{R}^n_+ \subset \omega(\log(E), u)$. Since $\omega(\log(E), u)$ is closed, this implies $\Theta_\Psi = \omega(\log(E), u)$, and (27) follows.

Now relation (23) is a consequence of (24)–(27). The theorem is proved.

Note that the value $\tau(\Psi_1, \ldots, \Psi_n)$ also can be expressed in geometric terms. Namely, if $\Psi_1, \ldots, \Psi_n \in I_0$, the polarization formula (18) gives us, by Theorem 3,

$$\tau(\Psi_1, \ldots, \Psi_n) = (-1)^n \sum_{j=1}^n (-1)^j \sum_{1 \leq i_1 < \ldots < i_j \leq n} \text{Vol}(\Theta_{\sum_k \Psi_{i_k}}).$$

We can also give an interpretation for the bound (16). Write $A_\Psi$ from (15) as

$$A_\Psi = \sup_{a \in \mathbb{R}^n_+} \frac{|\psi(-a)|^n}{a_1 \ldots a_n} = \sup_{a \in \mathbb{R}^n_+} |\psi(-a/a_1) \ldots \psi(-a/a_n)|,$$

$\psi = \text{Exp}^* \Psi$. For any $a \in \mathbb{R}^n_+$, the point $a^{(j)}$ whose $j$th coordinate equals $|\psi(-a/a_j)|$ and the others are zero, has the property $\langle a^{(j)}, -a \rangle = \psi(-a)$. This remains true for every convex combination $\sum \rho_j a^{(j)}$ of the points $a^{(j)}$, and thus $r \sum \rho_j a^{(j)} \in \Theta_\Psi$ with any $r \in [0, 1]$. Since $$(n!)^{-1} |\psi(-a/a_1) \ldots \psi(-a/a_n)|$$ is the volume of the simplex generated by the points $0, a^{(1)}, \ldots, a^{(n)}$, we see from (28) that $(n!)^{-1} A_\Psi$ is the supremum of the volumes of all simplices contained in $\Theta_\Psi$.

Besides, $(n!)^{-1} [\nu(\Psi, 0)]^n$ is the volume of the simplex

$$\{a \in \mathbb{R}^n_+ : \langle a, (1, \ldots, 1) \rangle \leq \nu(\Psi, 0) \} \subset \Theta_\Psi.$$ 
It is a geometric description for the "standard" bound $\tau(\Psi) \geq [\nu(\Psi, 0)]^n$.

4 Singularities of plurisubharmonic functions

Let $u$ be a plurisubharmonic function in a domain $\Omega \subset \mathbb{C}^n$, and $\nu(u, x, a)$ be its directional Lelong number (3) at $x \in \Omega$ with respect to $a \in \mathbb{R}^n_+$. Fix a point $x$. As is known [3], the function $a \mapsto \nu(u, x, a)$ is a concave function on $\mathbb{R}^n_+$. So, the function

$$\psi_{u,x}(t) := -\nu(u, x, -t), \quad t \in \mathbb{R}^n_-,$$
belongs to $CNVI_-(\mathbb{R}^n_+)$ and thus

$$
\Psi_{u,x} := \log^* \psi_{u,x} \in PSH^c_-(D).
$$

Moreover, due to the positive homogeneity of $\nu(u, x, a)$ in $a$, $\Psi_{u,x} \in I$. The function $\Psi_{u,x}$ was introduced in [15] as (local) indicator of $u$ at $x$. According to (3),

$$
\Psi_{u,x}(z) = \lim_{R \to +\infty} R^{-1} \sup \{ u(y) : |y_k - x_k| \leq |z_k|^R, \ 1 \leq k \leq n \} = \lim_{R \to +\infty} \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} u(x_k + |z_k|^R e^{i\theta_k}) d\theta_1 \ldots d\theta_n.
$$

Clearly, $\Psi_{u,x} \equiv 0$ if and only if $\nu(u, x) = 0$. It is easy to see that $\Psi(\Phi, 0) = \Phi \forall \Phi \in I$. In particular,

$$
(29) \quad \nu(u, x, a) = \nu(\Psi_{u,x}, 0, a) = -\Psi_{u,x}(Exp(-a)) \ \forall a \in \mathbb{R}^n_+.
$$

So, the results of the previous sections can be applied to study the directional Lelong numbers of arbitrary plurisubharmonic functions.

**Proposition 5** (cf. [7], Pr. 5.3) For any $u \in PSH(\Omega)$,

$$
\nu(u, x, a) \geq \nu(u, x, b) \sup_k \frac{a_k}{b_k} \ \forall x \in \Omega, \ \forall a, b \in \mathbb{R}^n_+.
$$

**Proof.** In view of (29), the relation follows from Proposition 4.

Given $r \in \mathbb{R}^n_+$ and $z \in \mathbb{C}^n$, we denote $r^{-1} = (r_1^{-1}, \ldots, r_n^{-1})$ and $r \cdot z = (r_1 z_1, \ldots, r_n z_n)$.

**Proposition 6** ([15]). If $u \in PSH(\Omega)$ then

$$
(30) \quad u(z) \leq \Psi_{u,x}(r^{-1} \cdot z) + \sup \{ u(y) : y \in D_r(x) \}
$$

for all $z \in D_r(x) = \{ y : |y_k - x_k| \leq r_k, \ 1 \leq k \leq n \} \subset \subset \Omega$.

**Proof.** Let us assume for simplicity $x = 0$, $D_r(0) = D_r$. Consider the function $v(z) = u(r \cdot z) - \sup \{ u(y) : y \in D_r \} \in PSH_-(D)$. The function $g_v(R, t) := \sup \{ v(z) : |z_k| \leq \exp\{R r_k\}, \ 1 \leq k \leq n \}$ is convex in $R > 0$ and $t \in \mathbb{R}^n$, so for $R \to \infty$

$$
(31) \quad \frac{g_v(R, t) - g_v(R_1, t)}{R - R_1} > \psi_{v,0}(t),
$$

13
\[ \psi_{v,0} = \text{Exp}^* \Psi_{v,0}. \]

For \( R = 1, R_1 \to 0 \), (31) gives us \( g_v(1, t) \leq \psi_{v,0}(t) \) and thus (30). The proposition is proved.

Let \( \Omega_k(x) \) be the connected component of the set \( \Omega \cap \{ z \in \mathbb{C}^n : z_j = x_j \ \forall j \neq k \} \) containing the point \( x \). If for some \( x \in \Omega, u|_{\Omega_k(x)} \neq -\infty \ \forall k \), then \( \Psi_{u,x} \in I_0 \). For example, it is the case for \( u \in \text{PSH}(\Omega, x) \).

If \( u \in \text{PSH}(\Omega, x) \), the measure \((dd^c u)^n\) is defined on \( \Omega \). Its residual mass at \( x \) will be denoted by \( \tau(u,x) \):

\[ \tau(u,x) = (dd^c u)^n|_{\{x\}}. \]

Besides, the indicator \( \Psi_{u,x} \in I_0 \). Denote \( N(u,x) = \tau(\Psi_{u,x}) \).

**Proposition 7** (\[15\], Th. 1). If \( u \in \text{PSH}(\Omega, x) \), then \( \tau(u, x) \geq N(u, x) \).

**Proof.** Inequality (30) implies

\[ \limsup_{z \to x} \frac{\Psi_{u,x}(r^{-1} \cdot (z - x))}{u(z)} \leq 1, \]

and since

\[ \lim_{y \to 0} \frac{\Psi_{u,x}(r^{-1} \cdot y))}{\Psi_{u,x}(y)} = 1 \ \forall r \in \mathbb{R}^n_+, \]

the statement follows from Theorem A.

So, to estimate \( \tau(u, x) \) we may apply the bounds for \( \tau(\Psi_{u,x}) \) from the previous section.

**Theorem 4** If \( u \in \text{PSH}(\Omega, x) \), then

\[ \tau(u,x) \geq \left[ \mu(u,x,a) \right]^n \quad \forall a \in \mathbb{R}^n_+; \]

in other words, \( \tau(u, x) \geq A_{u,x} \) where \( A_{u,x} = A_{\Psi_{u,x}} \) is defined by (15).

**Proof.** The result follows from Theorem 4 and Proposition 7.

Let now \( u_1, \ldots, u_n \in \text{PSH}(\Omega) \) be in general position in the sense of Definition 1. Then the current \( \wedge_k dd^c u_k \) is defined on \( \Omega \) (\[4\], Th. 2.5); denote its residual mass at a point \( x \) by \( \tau(u_1, \ldots, u_n; x) \). Besides, the \( n \)-tuple of the indicators \( \Psi_{u_k,x} \) is in general position, too, that implies \( \wedge_k dd^c \Psi_{u_k,x} = \tau(\Psi_{u_1,x}, \ldots, \Psi_{u_n,x}) \delta(0) \) (Proposition 4). In view of Theorem A and Proposition 3 we have
Theorem 5 $\tau(u_1, \ldots, u_n; x) \geq \tau(\Psi u_1, \ldots, \Psi u_n, x)$.

Now Theorems 2 and 5 give us

Theorem 6

$$\tau(u_1, \ldots, u_n; x) \geq \prod_j \nu(u_j, x, a) \quad \forall a \in \mathbb{R}^n_+.$$ \hfill (32)

Remark. For $a_1 = \ldots = a_n = 1$, inequality (32) is proved in [2], Cor. 5.10.

By combination of Proposition 7 and Theorem 3 we get

Theorem 7 For $u \in PSH(\Omega, x)$,

$$\tau(u, x) \geq N(u, x) = n! V(\Theta_{u,x})$$ \hfill (33)

with

$$\Theta_{u,x} = \{ b \in \mathbb{R}^n_+ : \sup_{a \in \mathbb{Z}^n_+} [\nu(u, x, a) - \langle b, a \rangle] \geq 0 \}.$$ 

Remark on holomorphic mappings. Let $f = (f_1, \ldots, f_n)$ be a holomorphic mapping of a neighbourhood $\Omega$ of the origin into $\mathbb{C}^n$, $f(0) = 0$ be its isolated zero. Then in a subdomain $\Omega' \subset \Omega$ the zero sets $A_j$ of the functions $f_j$ satisfy the conditions

$$A_1 \cap \ldots \cap A_n \cap \Omega' = \{0\}, \quad \text{codim } A_{j_1} \cap \ldots \cap A_{j_k} \cap \Omega' \geq k$$

for all choices of indices $j_1 < \ldots < j_k$, $k \leq n$. Denote $u = \log |f|$, $u_j = \log |f_j|$. Then, as is known, $\tau(u, 0) = \tau(u_1, \ldots, u_n; 0) = m_f$, the multiplicity of $f$ at 0. For $a = (1, \ldots, 1)$, $\nu(u_j, 0, a)$ equals $m_j$, the multiplicity of the function $f_j$ at 0. Therefore, (32) with $a = (1, \ldots, 1)$ gives us the standard bound $m_f \geq m_1 \ldots m_n$.

For $a_j$ rational, (32) is the known estimate of $m_f$ via the multiplicities of weighted homogeneous initial Taylor polynomials for $f_j$ (see e.g. [1], Th. 22.7). Indeed, due to the positive homogeneity of the directional Lelong numbers, we can take $a_j \in \mathbb{Z}^n_+$. Then by (7), $\nu(u_j, 0, a)$ is equal to the multiplicity of the function $f^{(a)}_j(z) = f_j(z^a)$.

We would also like to mention that (32) gives a lower bound for the Milnor number $\mu(F, 0)$ of a singular point 0 of a holomorphic function $F$ (i.e. for the multiplicity of the isolated zero of the mapping $f = \text{grad } F$.)
at 0) in terms of the indices \( I(F,0,a) \) of \( F \). Since \( I(\partial F/\partial z_k,0,a) \geq I(F,0,a) - a_k \),
\[
\mu(F,0) \geq \prod_{1 \leq k \leq n} \left( \frac{I(F,0,a)}{a_k} - 1 \right).
\]

Finally, as follows from (7), the set \( \mathbb{R}^n_+ \setminus \Theta_{u,0} \) is the Newton polyhedron for the system \((f_1,\ldots,f_n)\) at 0 (see Introduction). Therefore, \( n! V(\Theta_{u,0}) \) is the Newton number of \((f_1,\ldots,f_n)\) at 0, and (33) becomes the bound for \( m_f \) due to A.G. Kouchnirenko (see \cite{1}, Th. 22.8). So, for any plurisubharmonic function \( u \), we will call the value \( N(u,x) \) the Newton number of \( u \) at \( x \).

## 5 Indicators as logarithmic tangents

Let \( u \in PSH(\Omega,0) \), \( u(0) = -\infty \). We will consider the following problem: under what conditions on \( u \), its residual measure equals its Newton number? Of course, the relation
\[
\exists \lim_{z \to 0} \frac{u(z)}{\Psi_{u,0}(z)} = 1
\]

is sufficient, however it seems to be too restrictive. On the other hand, as the example \( u(z) = \log(|z_1 + z_2|^2 + |z_2|^4) \) shows, the condition
\[
\lim_{\lambda \to 0} \frac{u(\lambda z)}{\Psi_{u,0}(\lambda z)} = 1 \quad \forall z \in \mathbb{C}^n \setminus \{0\}
\]
does not guarantee the equality \( \tau(u,0) = N(u,0) \).

To weaken (34) we first give another description for the local indicators. In \cite{6}, a compact family of plurisubharmonic functions
\[
u_r(z) = u(rz) - \sup\{u(y) : |y| < r\}, r > 0
\]
was considered and the limit sets, as \( r \to 0 \), of such families were described. In particular, the limit set need not consist of a single function, so a plurisubharmonic function can have several (and thus infinitely many) tangents. Here we consider another family generated by a plurisubharmonic function \( u \).

Given \( m \in \mathbb{N} \) and \( z \in \mathbb{C}^n \), denote \( z^m = (z_1^m,\ldots,z_n^m) \) and set
\[
T_m u(z) = m^{-1} u(z^m).
\]
Clearly, \( T_m u \in PSH(\Omega \cap D) \) and \( T_m u \in PSH_-(D_r) \) for any \( r \in \mathbb{R}_+^n \cap D^* \) (i.e. \( 0 < r_k < 1 \)) for all \( m \geq m_0(r) \).
Proposition 8 The family \( \{T_m u\}_{m \geq m_0(r)} \) is compact in \( L^1_{loc}(D_r) \).

Proof. Let \( M(v, \rho) \) denote the mean value of a function \( v \) over the set \( \{ z : |z_k| = \rho_k, \ 1 \leq k \leq n \} \), \( 0 < \rho_k \leq r_k \), then \( M(T_m u, \rho) = m^{-1}M(u, \rho^m) \).

The relation
\[
M^{-1}M(u, \rho^m) \geq \Psi_{u,0}(\rho) \text{ as } m \to \infty
\]
implies \( M(T_m u, \rho) \geq M(T_{m_0} u, \rho) \). Since \( T_m u \leq 0 \) in \( D_r \), it proves the compactness.

Theorem 8
(a) \( T_m u \to \Psi_{u,0} \) in \( L^1_{loc}(D) \);

(b) if \( u \in PSH(\Omega, 0) \) then \( (dd^c T_m u)^n \to \tau(u, 0) \delta(0) \).

Proof. Let \( g \) be a partial limit of the sequence \( T_m u \), that is \( T_m u \to g \) as \( s \to \infty \) for some sequence \( m_k \). For the function \( v(z) = \sup \{ u(y) : |y_k| \leq |z_k|, \ 1 \leq k \leq n \} \) and any \( r \in \mathbb{R}^n_+ \cap D^* \) we have by (30)
\[
T_m u(z) \leq (T_m v)(z) \leq \Psi_{u,0}(r^{-1} \cdot z)
\]
and thus
\[
g(z) \leq \Psi_{u,0}(z) \forall z \in D.
\]

On the other hand, the convergence of \( T_m u \) to \( g \) in \( L^1 \) implies \( M(T_m u, r) \to M(g, r) \) (\[3\], Prop. 4.1.10). By (32), \( M(T_m u, r) \to \Psi_{u,0}(r) \), so \( M(g, r) = \Psi_{u,0}(r) \) for every \( r \in \mathbb{R}^n_+ \cap D^* \). Being compared with (36) it gives us \( g \equiv \Psi_{u,0} \), and the statement \( (a) \) follows.

To prove \( (b) \) we observe that for each \( \alpha \in (0, 1) \)
\[
\int_{\alpha D}(dd^c T_m u)^n = \int_{\alpha^m D}(dd^c u)^n \to \tau(u, 0)
\]
as \( m \to \infty \), and for \( 0 < \alpha < \beta < 1 \)
\[
\lim_{m \to \infty} \int_{\beta D \setminus \alpha D}(dd^c T_m u)^n = \lim_{m \to \infty} \left[ \int_{\beta^m D}(dd^c u)^n - \int_{\alpha^m D}(dd^c u)^n \right] = 0.
\]
The theorem is proved.

So, Theorem 8 shows us that \( \tau(u, 0) = N(u, 0) \) if and only if \( (dd^c T_m u)^n \to (dd^c \Psi_{u,0})^n \). And now we are going to find conditions for this convergence.

Recall the definition of the inner \( C_{n-1} \)-capacity introduced in (17): for any Borel subset \( E \) of a domain \( \omega \),
\[
C_{n-1}(E, \omega) = \sup \{ \int_E (dd^c v)^{n-1} \wedge \beta_1 : v \in PSH(\omega), \ 0 < v < 1 \}.
\]
It was shown in [17] that convergence of uniformly bounded plurisubharmonic functions \( v_j \) to \( v \) in \( C_{n-1} \)-capacity implies \((dd^c v_j)^n \to (dd^c v)^n\). In our situation, neither \( T_m u \) nor \( \Psi_{u,0} \) are bounded, so we will modify the construction from [17].

Set
\[
E(u, m, \delta) = \{ z \in D \setminus \{0\} : \frac{T_m u(z)}{\Psi_{u,0}(z)} > 1 + \delta \}, \quad m \in \mathbb{N}, \ \delta > 0.
\]

**Theorem 9** Let \( u \in PSH(\Omega, 0), \rho \in (0, 1/4), N > 0, \) and a sequence \( m_s \in \mathbb{N} \) be such that

1) \( u(z) > -Nm_s \) on a neighbourhood of the sphere \( \partial B_{\rho m_s} \), \( \forall s \);

2) \( \lim_{s \to \infty} C_{n-1}(B_{\rho} \cap E(u, m_s, \delta), D) = 0 \quad \forall \delta > 0. \)

Then \( (dd^c T_m u)^n \to (dd^c \Psi_{u,0})^n \) on \( D \).

**Proof.** Without loss of generality we can take \( u \in PSH_-(D, 0) \). Consider the functions \( v_s(z) = \max \{ T_{m_s} u(z), -N \} \) and \( v = \max \{ \Psi_{u,0}(z), -N \} \). We have \( v_s = T_{m_s} u \) and \( v = \Psi_{u,0} \) on a neighbourhood of \( \partial B_{\rho} \), \( v_s = v = -N \) on a neighbourhood of \( 0, v_s \leq v \) on \( B_{\rho} \), and \( v_s \geq (1 + \delta)v \) on \( B_{\rho} \setminus E(u, m_s, \delta) \).

We will prove the relations
\[
(dd^c v_s)^k \wedge (dd^c v)^l \to (dd^c v)^{k+l}
\]
for \( k = 1, \ldots, n, \ l = 0, \ldots, n - k \). As a consequence, it will give us the statement of the theorem. Indeed, by Theorem 8
\[
\int_{B_{\rho}} (dd^c v_s)^n = \int_{B_{\rho}} (dd^c T_{m_s} u)^n \to \tau(u, 0)
\]
while
\[
\int_{B_{\rho}} (dd^c v)^n = \int_{B_{\rho}} (dd^c \Psi_{u,0})^n = N(u, 0),
\]
and \((37)\) with \( k = n \) provides the coincidence of the right-hand sides of these relations and thus the convergence of \((dd^c T_m u)^n\) to \((dd^c \Psi_{u,0})^n\).

We prove \((37)\) by induction in \( k \). Let \( k = 1, 0 \leq l \leq n - 1, \delta > 0 \). For any test form \( \phi \in \mathcal{D}_{n-l-1,n-l-1}(B_{\rho}) \),
\[
\left| \int dd^c v_s \wedge (dd^c v)^l \wedge \phi - \int (dd^c v)^{l+1} \wedge \phi \right| = \left| \int (v - v_s)(dd^c v)^l \wedge dd^c \phi \right|
\]
\[ \leq C_\phi \int_{B_\rho} (v - v_s)(dd^c v)^l \land \beta_{n-l} \]

\[ = C_\phi \left[ \int_{B_\rho \setminus E_{s,\delta}} + \int_{B_\rho \cap E_{s,\delta}} \right] (v - v_s)(dd^c v)^l \land \beta_{n-l} \]

\[ = C_\phi [I_1(s, \delta) + I_2(s, \delta)], \]

where, for brevity, \( E_{s,\delta} = E(u, m_s, \delta) \).

We have

\[ I_1(s, \delta) \leq \delta \int_{B_\rho} |v|(dd^c v)^l \land \beta_{n-l} \leq C\delta \]

with a constant \( C \) independent of \( s \), and

\[ I_2(s, \delta) \leq N \int_{B_\rho \cap E_{s,\delta}} (dd^c v)^l \land \beta_{n-l} \leq C(N, \rho, l) \cdot C_{n-1}(B_\rho \cap E_{s,\delta}, D) \to 0. \]

Since \( \delta > 0 \) is arbitrary, it proves \((37)\) for \( k = 1 \).

Let us now have got \((37)\) for \( k = j \) and \( 0 \leq l \leq n - j \). For \( \phi \in \mathcal{D}_{n-1-j,n-l}(B_\rho) \),

\[ \int (dd^c v_s)^{j+1} \land (dd^c v)^l \land \phi = \int (dd^c v_s)^j \land (dd^c v)^{l+1} \land \phi \]

\[ + \int [(dd^c v_s)^{j+1} \land (dd^c v)^l - (dd^c v_s)^j \land (dd^c v)^{l+1}] \land \phi. \]

The first integral in the right-hand side converges to \( \int (dd^c v)^{l+j+1} \land \phi \) by the induction assumption. The second integral can be estimated similarly to the case \( k = 1 \):

\[ \left| \int [(dd^c v_s)^{j+1} \land (dd^c v)^l - (dd^c v_s)^j \land (dd^c v)^{l+1}] \land \phi \right| \leq C_\phi \left[ \int_{B_\rho \setminus E_{s,\delta}} + \int_{B_\rho \cap E_{s,\delta}} \right] (v - v_s)(dd^c v_s)^j(dd^c v)^l \land \beta_{n-j-l} \]

\[ = C_\phi [I_3(s, \delta) + I_4(s, \delta)]. \]

Since \((dd^c v_s)^j \land (dd^c v)^l \to (dd^c v)^{j+1}\),

\[ \int (dd^c v_s)^j(dd^c v)^l \land \beta_{n-j-l} \leq C \quad \forall s \]

and

\[ I_3(s, \delta) \leq \delta \int_{B_\rho} |v|(dd^c v_s)^j(dd^c v)^l \land \beta_{n-j-l} \leq CN\delta. \]

Similarly, \( I_4(s, \delta) \leq N \int_{B_\rho \cap E_{s,\delta}} (dd^c v_s)^j(dd^c v)^l \land \beta_{n-j-l} \leq C(N, \rho, j, l) \cdot C_{n-1}(B_\rho \cap E_{s,\delta}, D) \to 0, \)

and \((37)\) is proved.
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Mathematical Division, Institute for Low Temperature Physics
47 Lenin Ave., Kharkov 310164, Ukraine

E-mail: rashkovskii@ilt.kharkov.ua, rashkows@ilt.kharkov.ua