INTEGRAL COHOMOLOGY AND MIRROR SYMMETRY
FOR
CALABI-YAU 3-FOLDS

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Abstract. In this paper we compute the integral cohomology groups for all examples of Calabi-Yau 3-folds obtained from hypersurfaces in 4-dimensional Gorenstein toric Fano varieties. Among 473,800,776 families of Calabi-Yau 3-folds \( X \) corresponding to 4-dimensional reflexive polytopes there exist exactly 32 families having non-trivial torsion in \( H^i(X,\mathbb{Z}) \). We came to an interesting observation that the torsion subgroups in \( H^2 \) and \( H^3 \) are exchanged by the mirror symmetry involution.

Introduction

Let \( X \) be a smooth projective Calabi-Yau \( d \)-fold over \( \mathbb{C} \). We shall always assume that \( h^i(X,\mathcal{O}_X) = 0 \) for \( 0 < i < d \). Our main interest is the torsion in the integral cohomology ring \( H^*(X,\mathbb{Z}) \). For a finite abelian group \( G \) we denote by \( G^* \) the dual abelian group \( \text{Hom}(G,\mathbb{Q}/\mathbb{Z}) \). Using the isomorphism from the universal coefficient theorem
\[
\text{Tors}(H_i(X,\mathbb{Z})) \cong [\text{Tors}(H^{i+1}(X,\mathbb{Z}))]^* \]
and the Poincaré duality
\[
\text{Tors}(H_i(X,\mathbb{Z})) \cong \text{Tors}(H^{2d-i}(X,\mathbb{Z})) ,
\]
we obtain for \( d = 3 \):
\[
H^0(X,\mathbb{Z}) \cong H^0(X,\mathbb{Z}) \cong \mathbb{Z}, \quad H^1(X,\mathbb{Z}) = 0,
\]
\[
H^2(X,\mathbb{Z}) \cong A(X) \oplus \mathbb{Z}^a, \quad H^4(X,\mathbb{Z}) \cong B(X)^* \oplus \mathbb{Z}^a
\]
\[
H^3(X,\mathbb{Z}) \cong B(X) \oplus \mathbb{Z}^{2b+2}, \quad H^5(X,\mathbb{Z}) \cong A(X)^* ,
\]
where the finite abelian groups
\[
A(X) := \text{Tors}(H^2(X,\mathbb{Z})), \quad B(X) := \text{Tors}(H^3(X,\mathbb{Z}))
\]
determine completely the torsion in \( H^*(X,\mathbb{Z}) \). In particular, we have the isomorphisms
\[
A(X) \oplus B(X)^* \cong \text{Tors}(H^\text{even}(X,\mathbb{Z})) \cong [\text{Tors}(H^\text{odd}(X,\mathbb{Z}))]^* .
\]
It is known that mirror symmetry exchanges the integers \( a,b \) so that for a mirror Calabi-Yau 3-fold \( X^* \) one has
\[
\mathbb{Q}^{2a+2} \cong H^\text{even}(X,\mathbb{Q}) \cong H^\text{odd}(X^*,\mathbb{Q}) ,
\]
\[
\mathbb{Q}^{2b+2} \cong H^\text{odd}(X,\mathbb{Q}) \cong H^\text{even}(X^*,\mathbb{Q}) .
\]
An interesting mathematical question is to understand the behavior of the torsion in $H^\ast(X, \mathbb{Z})$, i.e. the groups $A(X)$ and $B(X)$, under the mirror symmetry. The group $B(X)$ is isomorphic to the torsion in the étale cohomology group $H^2_{\text{ét}}(X, \mathcal{O}_X^*)$ and it is called the cohomological Brauer group of $X$. By a recent result of Kresch and Vistoli [16], $B(X)$ is also isomorphic to the Brauer group of Azumaya algebras on $X$. On the other hand, there are canonical isomorphisms

$$A(X) \cong \text{Hom}(\pi_1(X), \mathbb{Q}/\mathbb{Z}) \cong \text{Tors}(\text{Pic}(X)).$$

From the physical point of view it is more natural to consider the topological $K$-groups $K^0(X)$ and $K^1(X)$ together with some natural filtrations [7, 8]. Using the Atiyah-Hirzebruch spectral sequence (see [7], 2.5), one obtains two short exact sequences

$$0 \to A(X)^* \to \text{Tors}(K^1(X)) \to B(X) \to 0,$$

$$0 \to B(X)^* \to \text{Tors}(K^0(X)) \to A(X) \to 0.$$

It is expected that the mirror symmetry exchange $K^0$ and $K^1$ so that for a mirror pair $(X, X^*)$ one has the isomorphisms

$$\text{Tors}(K^1(X)) \cong \text{Tors}(K^0(X^*)), \quad \text{Tors}(K^0(X)) \cong \text{Tors}(K^1(X^*)).$$

These isomorphisms agree with predictions of SYZ-construction and topological calculations of M. Gross [12]. The compatibility of the above isomorphisms with the natural filtrations in $K^i$ would imply the isomorphisms

$$(1) \quad A(X) \cong B(X^*), \quad B(X) \cong A(X^*).$$

The main purpose of this paper is to verify these isomorphisms for all examples of Calabi-Yau 3-folds obtained from hypersurfaces in 4-dimensional Gorenstein toric Fano varieties.

There exist 473 800 776 families of Calabi-Yau 3-folds $X$ corresponding to 4-dimensional reflexive polytopes [17, 18]. These families give rise to 30 108 different pairs of numbers $(a, b)$. We show that among 473 800 776 families of Calabi-Yau 3-folds $X$ there exist exactly 32 families having non-trivial torsion in $H^\ast(X, \mathbb{Z})$. More precisely, there are exactly 16 families of simply-connected Calabi-Yau 3-folds $X$ having non-trivial Brauer group $B(X) \cong \mathbb{Z}/p\mathbb{Z}$ ($p = 2, 3, 5$). They are mirror dual to another 16 families of Calabi-Yau 3-folds $X^*$ having trivial Brauer group and a non-trivial cyclic fundamental group of order $p = 2, 3, 5$ [15]. Thus, we come to the observation that for all families of Calabi-Yau 3-folds one has the isomorphisms (1). Although the groups $A(X)$ and $B(X)$ can be computed in a purely combinatorial way using lattice points in faces of 4-dimensional reflexive polytopes $\Delta$ (or dual reflexive polytopes $\Delta^*$) we do not see immediately the isomorphisms (1) from the combinatorial duality between $\Delta$ and $\Delta^*$. Therefore, it would
be interesting to find a general mathematical explanation for the mirror symmetry isomorphism

\[ B(X) \cong \text{Hom}(\pi_1(X^*), \mathbb{Q}/\mathbb{Z}). \]

If a finite group \( G \) acts on a smooth Calabi-Yau manifold \( V \), then there exists a conformal field theory of the orbifold \( V/G \) associated with an element \( \alpha \in H^2(G, U(1)) \) (discrete torsion). In [28] Vafa and Witten have suggested that there should be some connection between elements in the Brauer group of \( V/G \) (\( B \)-fields) and elements in \( H^2(G, U(1)) \) (discrete torsion 2-cocycles).

In this paper we show that all 16 families of Calabi-Yau 3-folds with non-trivial cyclic Brauer group are birational to Calabi-Yau orbifolds \( V/G \) (\( G \sim \mathbb{Z}/p \mathbb{Z} \oplus \mathbb{Z}/p \mathbb{Z}, \ p = 2, 3, 5 \)) and the corresponding elements in the cyclic Brauer group \( B(X) \cong H_2(G) \cong \mathbb{Z}/p \mathbb{Z} \) can be identified with the discrete torsion 2-cocycles.

It is easy to see that there are no algebraic curves in \( X \) representing the torsion classes in \( H_2(X, \mathbb{Z}) \). Aspinwall and Morrison suggested in [2] that the instanton expansion of the Yukawa coupling for a Calabi-Yau 3-fold with the Brauer group \( B(X) \cong \mathbb{Z}/p \mathbb{Z} \) should be weighted by powers of \( p \)-th roots of unity. It would be interesting to verify this prediction for the above 16 toric families of Calabi-Yau 3-folds with non-trivial Brauer group.

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1. The fundamental group of toric hypersurfaces

First we recall a combinatorial computation of the fundamental group of a toric variety [9]. Let \( M \cong \mathbb{Z}^d \) be a lattice of rank \( d \) and \( N := \text{Hom}(M, \mathbb{Z}) \) the dual lattice. We denote by \( \langle *, * \rangle \) the canonical pairing \( M \times N \to \mathbb{Z} \). For an algebraic torus \( T^d := \text{Spec} \mathbb{C}[M] \cong (\mathbb{C}^*)^d \) there are canonical isomorphisms

\[ \pi_1^{\text{top}}(T^d) \cong N \cong H_1(T^d, \mathbb{Z}), \quad M \cong H^1(T^d, \mathbb{Z}). \]

If \( \mathbb{P}_\Sigma \) is a smooth partial toric compactification of \( T^d \) defined by a fan \( \Sigma \subset N_\mathbb{R} = N \otimes \mathbb{R} \), then \( \mathbb{P}_\Sigma \setminus T^d \) is a union of divisors \( D_1, \ldots, D_n \) which 1-to-1 correspond to primitive lattice vectors \( e_1, \ldots, e_n \in N \) generating 1-dimensional cones in \( \Sigma \). If \( D^*_i \) is a dense torus orbit in \( D_i \), then we have \( \mathbb{P}_i := T^d \cup D^*_i \cong \mathbb{C} \times T^{d-1} \). The embedding \( T^d \hookrightarrow \mathbb{P}_i \) defines the homomorphism of the fundamental groups

\[ \rho_i : N = \pi_1^{\text{top}}(T^d) \to \pi_1^{\text{top}}(\mathbb{P}_i) \cong N/\mathbb{Z}e_i. \]
Combining all the homomorphisms $\rho_i$ ($i = 1, \ldots, n$), we obtain the isomorphisms
\[ \pi_1^{\text{top}}(\mathbb{P}_\Sigma) \cong \pi_1^{\text{top}}(\mathbb{P}_{\Sigma(1)}) \cong N/\sum_{i=1}^{n} \mathbb{Z}e_i, \]
where $\mathbb{P}_{\Sigma(1)}$ is a toric variety associated to the subfan $\Sigma^{(1)} \subset \Sigma$ of all cones of dimension $\leq 1$ in $\Sigma$. In particular, we have
\[ H_1(\mathbb{P}_\Sigma, \mathbb{Z}) \cong H_1(\mathbb{P}_{\Sigma(1)}, \mathbb{Z}) \cong N/\sum_{i=1}^{n} \mathbb{Z}e_i. \]

In order to compute the fundamental group of hypersurfaces in toric varieties one applies the following result of Oka [22]:

**Theorem 1.1.** Let $\Delta \subset M_\mathbb{R}$ be a lattice polytope of dimension $d$,
\[ f(t) = \sum_{m \in \Delta\cap M} c_m t^m \in \mathbb{C}[M] \cong \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \]
be a $\Delta$-nondegenerate Laurent polynomial, and $W \subset \mathbb{T}^d$ the nondegenerate affine hypersurface defined by the equation $f(t) = 0$. Then the embedding $j : W \hookrightarrow \mathbb{T}^d$ is a homotopy $(d - 1)$-equivalence, i.e. the homomorphisms of the homotopy groups
\[ j_* : \pi_i(W) \to \pi_i(\mathbb{T}^d) \]
is bijective for $i < d - 1$ and surjective for $i = d - 1$.

By Whitehead’s theorem, we get:

**Corollary 1.2.** The homomorphism
\[ j_* : H_i(W, \mathbb{Z}) \to H_i(\mathbb{T}^d, \mathbb{Z}) \]
is bijective for $i < d - 1$ and surjective for $i = d - 1$.

Since $H_i(\mathbb{T}^d, \mathbb{Z})$ is a free abelian group of rank $d \choose i$, using the universal coefficients theorem, we also obtain:

**Corollary 1.3.** For any abelian group $A$, the homomorphism
\[ j^* : H^i(\mathbb{T}^d, A) \cong \Lambda^i M \otimes_{\mathbb{Z}} A \to H^i(W, A) \]
is bijective for $i < d - 1$ and injective for $i = d - 1$.

Let $\Sigma := \Sigma(\Delta) \subset N_\mathbb{R}$ be the normal fan of the polytope $\Delta \subset M_\mathbb{R}$. The cones $\sigma = \sigma(\theta) \in \Sigma(\Delta)$ 1-to-1 correspond to faces $\theta \subset \Delta$ in the following way:
\[ \sigma(\theta) = \{ y \in N_\mathbb{R} : \min_{x \in \Delta} \langle x, y \rangle = \langle x, z \rangle \ \forall z \in \theta \}, \]
i.e. $\sigma(\theta)$ consists of those $y \in N_\mathbb{R}$ for which the minimum of the linear function $\langle *, y \rangle$ on $\Delta$ is attained at any point of $\theta$. Sometimes we denote the corresponding projective toric variety $\mathbb{P}_\Sigma = \mathbb{P}_{\Sigma(\Delta)}$ simply by $\mathbb{P}_\Delta$. Let $\overline{W} \subset \mathbb{P}_\Sigma$ be the Zariski closure of $W$ in $\mathbb{P}_\Sigma$. If $\Sigma^{(1)} \subset \Sigma$ is the subfan of all cones of dimension $\leq 1$ in $\Sigma$, then $\mathbb{P}_{\Sigma^{(1)}}$ and $\overline{W}^{(1)} := \mathbb{P}_{\Sigma^{(1)}} \cap \overline{W}_f$ are smooth quasi-projective varieties.
**Theorem 1.4.** Let $d \geq 3$. Then the embedding $j : \overline{W}^{(1)} \hookrightarrow \mathbb{P}_{\Sigma(1)}$ induces the isomorphism

$$j_* : \pi_1(\overline{W}^{(1)}) \cong \pi_1(\mathbb{P}_{\Sigma(1)}) \cong N/\sum_{i=1}^{n} \mathbb{Z}e_i,$$

where $e_1, \ldots, e_n$ are primitive generators of 1-dimensional cones in $\Sigma^{(1)}$.

**Proof.** Let $\mathbb{P}_{\Sigma(1)} \setminus \mathbb{T}^d = D_1^\circ \cup \cdots \cup D_n^\circ \ (D_i^\circ \cong \mathbb{T}^{d-1})$. Using the isomorphisms $\pi_1(W) \cong \pi_1(\mathbb{T}^{d}) \cong N$ and the surjectivity of $\pi_1(W) \to \pi_1(\overline{W}^{(1)})$, we obtain that

$$j_* : \pi_1(\overline{W}^{(1)}) \to \pi_1(\mathbb{P}_{\Sigma(1)})$$

is surjective. Therefore the universal covering $\overline{U}^{(1)}$ of $\overline{W}^{(1)}$ is induced by some unramified covering $U$ of $\mathbb{T}^d$.

In order to prove that $j_*$ is an isomorphism it remains to show that the image of every element $e_i \in \pi_1(W) \cong N \ (i = 1, \ldots, n)$ is zero in $\pi_1(\overline{W}^{(1)})$. Let $\mathbb{P}_i := \mathbb{T}^d \cup D_i^\circ$ and $\overline{W}_i^{(1)} := \overline{W}^{(1)} \cap \mathbb{P}_i$. Using open inclusions $W \subset \overline{W}_i^{(1)} \subset \overline{W}^{(1)}$, it is enough to prove that the image of $e_i \in \pi_1(W)$ is zero already in $\pi_1(\overline{W}_i^{(1)})$ so that one obtains $\pi_1(\overline{W}_i^{(1)}) \cong N/\mathbb{Z}e_i$. We can consider $\overline{W}_i^{(1)}$ as an affine hypersurface in $\mathbb{P}_i \cong \mathbb{T}^{d-1} \times \mathbb{C} \cong \text{Spec} \mathbb{C}[t_1^{\pm 1}, \ldots, t_{d-1}^{\pm 1}, t_d]$ where the divisor $D_i^\circ \subset \mathbb{P}_i$ is defined by $t_d = 0$. Moreover, the element $e_j \in \pi_1(\mathbb{T}^d) = N$ is represented by a small 1-cycle $\gamma_i \subset \mathbb{C}^*$ around 0 in $\mathbb{C}$ which can be contracted to 0 in $\mathbb{T}^{d-1} \times \mathbb{C}$. The transversality of $\overline{W}_i^{(1)}$ and $D_i^\circ$ implies that the 1-cycle $\gamma_i$ can be choosen inside $W_f$ so that it will be contractible in $\overline{W}_i^{(1)}$. Therefore all elements $e_1, \ldots, e_n$ are in the kernel of the homomorphism $\pi_1(W) \to \pi_1(\overline{W}^{(1)})$. We remark that the isomorphism $\pi_1(\overline{W}_i^{(1)}) \cong \pi_1(\mathbb{P}_i)$ can be interpreted also as a Lefschetz-type statement for a nondegenerate hypersurface $\overline{W}_i^{(1)}$ in $\mathbb{T}^{d-1} \times \mathbb{C}$. \hfill $\square$

Since $e_1, \ldots, e_n$ generate $N$ over $\mathbb{Q}$, we obtain:

**Corollary 1.5.** The fundamental group of the quasi-projective hypersurface $\overline{W}^{(1)}$ is a finite abelian group. In particular, $\pi_1(\overline{W}^{(1)})$ coincides with the algebraic fundamental group of $\overline{W}^{(1)}$ and the universal covering of $\overline{W}^{(1)}$ is a toric quasi-projective hypersurface $\tilde{W}^{(1)}$ in the finite universal covering of $\mathbb{P}_{\Sigma(1)} \to \mathbb{P}_{\Sigma(1)}$ determined by the finite index sublattice $N' = \sum_{i=1}^{n} \mathbb{Z}e_i \subset N$.

**Theorem 1.6.** Assume that $d \geq 3$. Let $\mathbb{P}_{\widetilde{\Sigma}}$ be a smooth $d$-dimensional projective toric variety defined by a fan $\widetilde{\Sigma} \subset N_{\mathbb{R}}$ which is a subdivision of $\Sigma = \Sigma(\Delta)$. We denote by $\Gamma$ the set of primitive generators $v$ of 1-dimensional cones in $\widetilde{\Sigma}^{(1)}$ such that the minimum of $\langle *, v \rangle$ on $\Delta$ is attained on a face $\theta \subset \Delta$ of dimension $\geq 1$. Define the sublattice $N_{\Delta}^{(1)} \subset N$ to be generated by all $v \in \Gamma$. If $\tilde{W}$ is the closure of $W$ in $\mathbb{P}_{\widetilde{\Sigma}}$, then the fundamental
group \( \pi_1(\hat{W}) \) is isomorphic to the cyclic group \( N/N^{(1)}_\Delta \).

**Proof.** Let \( \mathcal{E} := \{e_1, \ldots, e_n\} \) be the set of primitive generators of 1-dimensional cones in \( \Sigma^{(1)} \). Since every linear function \( \langle *, e_i \rangle \) attains its minimum of \( \Delta \) on a \( (d - 1) \)-dimensional face of \( \Delta \), we have \( \mathcal{E} \subseteq \Gamma \) and we can write \( \Gamma = \{e_1, \ldots, e_n, \ldots, e_{n+k}\} \). Moreover, \( \hat{W}^{(1)} \) can be considered as a dense open subset of \( \hat{W} \). Therefore, we have a surjective homomorphism

\[
\psi : \pi_1(\hat{W}^{(1)}) \to \pi_1(\hat{W})
\]

and the dual injective homomorphism

\[
\psi^* : \text{Hom}(\pi_1(\hat{W}, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\pi_1(\hat{W}^{(1)}), \mathbb{Q}/\mathbb{Z})).
\]

Since \( \hat{W} \setminus \hat{W}^{(1)} = Z_1 \cup \cdots Z_k \), where

\[
Z_i = D_{n+i} \cap \hat{W}, \ i = 1, \ldots, k,
\]

we obtain that an element \( g \in \text{Hom}(\pi_1(\hat{W}^{(1)}), \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(N/ \sum_{i=1}^n Ze_i, \mathbb{Q}/\mathbb{Z}) \) belongs to \( \text{Im} \psi^* \) if and only if the cyclic Galois covering of \( \hat{W}^{(1)} \) corresponding to \( g \) is unramified along all divisors \( Z_1, \cdots, Z_k \). Using the same arguments as in the proof of [3], we obtain that the latter holds exactly when \( g(e_{n+1}) = \cdots = g(e_{n+k}) = 0 \). Therefore,

\[
\text{Im} \psi \cong \text{Hom}(N/ \sum_{i=1}^{n+k} Ze_i, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(N/N^{(1)}_\Delta, \mathbb{Q}/\mathbb{Z})
\]

and by duality \( \pi_1(\hat{W}) \cong N/N^{(1)}_\Delta \). The group \( N/N^{(1)}_\Delta \) is cyclic, because the smoothness of \( \hat{W} \) implies that for any \( (d - 1) \)-dimensional cone \( \sigma = \sigma(\theta) \in \Sigma \) the subset \( \sigma \cap \Gamma \) generate a direct summand of the rank \( d - 1 \) of the lattice \( N \) (all \( (d - 1) \)-dimensional cones in \( \hat{\Sigma} \) which subdivide \( \sigma \) are generated by part of a \( \mathbb{Z} \)-basis of \( N \)). \( \square \)

**Corollary 1.7.** For any lattice point \( v \in N \), we denote by

\[
M_v := \{x \in M : \langle x, v \rangle = 0\}
\]

the orthogonal complement to \( v \) in \( M \). Then we have the exact sequence

\[
\bigoplus_{v \in \Gamma} \Lambda^{d-1}M_v \to \Lambda^{d-1}M \to H^{2d-3}_{\text{tor}}(\hat{W}, \mathbb{Z}) \to 0.
\]

**Proof.** The exactness of

\[
\bigoplus_{v \in \Gamma} \Lambda^{d-1}M_v \to \Lambda^{d-1}M \to H^{2d-3}_{\text{tor}}(\hat{W}, \mathbb{Z}) \to 0
\]

follows from the exactness of

\[
\bigoplus_{v \in \Gamma} \mathbb{Z}[v] \to N \to \pi_1(\hat{W}) \to 0
\]
together with the canonical isomorphisms
\[ \pi_1(\hat{W}) \cong H^{2d-3}_{\text{tor}}(\hat{W}, \mathbb{Z}), \quad \Lambda^{d-1}M_v \cong \mathbb{Z}, \quad \Lambda^{d-1}M \cong N \]

after choosing an isomorphism (orientation) \( \Lambda^d \cong \mathbb{Z} \).

\[ \blacksquare \]

Remark 1.8. Theorem 1.6 can be considered as a special case of a more general result of Oka (see [24], Chapter V, §5).

Corollary 1.9. Let \( \hat{W} \) be a nondegenerate \((d - 1)\)-dimensional Calabi-Yau hypersurface in the Gorenstein toric Fano variety \( \mathbb{P}_\Delta \) associated with a \( d \)-dimensional reflexive polytope \( \Delta \subset M_\mathbb{R} \). Assume \( d \geq 3 \) and that there exists a smooth projective crepant resolution \( \hat{W} \) of singularities in \( W \) induced by a convex (coherent) triangulation of the dual reflexive polytope \( \Delta^* \subset \mathbb{N}_\mathbb{R} \). We denote by \( N'_\Delta \) the sublattice in \( N \) generated by all lattice points in \( \Delta^* \cap N \) which belong to faces of codimension > 1 of \( \Delta^* \). Then the fundamental group of \( \hat{W} \) is isomorphic to the cyclic group
\[ N/N'_\Delta. \]

Proof. The statement follows immediately from 1.6 because for a crepant desingularization \( \hat{W} \to W \) the set of primitive generators \( \Gamma \) is exactly the set of all lattice points in faces of codimension < 1 of the dual reflexive polytope \( \Delta^* \) (see [3]).

\[ \blacksquare \]

Remark 1.10. Recently Haase and Nill proved that for a reflexive polytope \( \Delta^* \) of arbitrary dimension \( d > 2 \) the sublattice in \( N \) generated by all lattice points in \( \Delta^* \cap N \) coincides with \( N'_\Delta \) [14]. The proof of this result uses the fact that the interior lattice points in codimension-1 faces of \( \Delta^* \) are Demazure roots for the automorphism group of the corresponding Gorenstein toric Fano variety \( \mathbb{P}_{\Delta^*} \) [20]. In particular, this result easily implies that \( N = N'_\Delta \) for all reflexive polyhedra \( \Delta^* \) of dimension \( d = 3 \). The last statement follows also from 1.9 and from the fact that every smooth \( K3 \)-surface is simply connected.

Remark 1.11. It is known that for \( d = 4 \) there always exists a smooth projective crepant resolution \( \hat{W} \) of singularities in \( W \) induced by a convex (coherent) triangulation of the dual reflexive polytope \( \Delta^* \subset \mathbb{N}_\mathbb{R} \). For \( d \geq 5 \) such a resolution does not exist in general. However, one can use \( N/N'_\Delta \) as a candidate for a stringy fundamental group of a singular Calabi-Yau variety \( W \) or its maximal partial crepant desingularization \( \hat{W} \) (similar to stringy Hodge numbers [3]). It would be interesting to know whether for \( d \geq 5 \) the dual group \( (N/N'_\Delta)^* \) coincides with the torsion subgroup in the Picard group of the Deligne-Mumford Calabi-Yau stack associated with the \( V \)-manifold \( \hat{W} \) (see [6, 25, 26]).
2. The list of non-simply connected Calabi-Yau 3-folds

Using the Calabi-Yau data [18] and the program package PALP [19] one can check all 473 800 776 reflexive polytopes $\Delta$ and find that there exist exactly 16 examples of reflexive polytopes $\Delta_i$ ($1 \leq i \leq 16$) such that $|N/N'_\Delta_i| > 1$. In the latter case, $|N/N'_\Delta_i|$ is always a prime number $p = 2, 3, 5$.

The most known example of a non-simply connected Calabi-Yau 3-fold obtained from a hypersurface in a Gorenstein toric variety $\mathbb{P}_\Delta$ is a free $\mu_5$-quotient of a smooth quintic 3-fold in $\mathbb{P}^4$ obtained as follows. Consider the action of the cyclic group $\mu_5 = \langle \zeta \rangle$ by $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ on $\mathbb{P}^4$ and take $\mathbb{P}_{\Delta_1} := \mathbb{P}^4/\mu_5$. The Gorenstein toric Fano variety $\mathbb{P}_{\Delta_1}$ contains exactly 5 isolated singular points. A non-degenerate hypersurface $\mathbb{W}_1 \subset \mathbb{P}_{\Delta_1}$ is smooth, i.e. $\mathbb{W}_1 = \hat{\mathbb{W}}_1$, it has the Hodge numbers $h^{1,1}(\mathbb{W}_1) = 1$, $h^{2,1}(\mathbb{W}_1) = 21$, and $\pi_1(\mathbb{W}_1) \cong \mathbb{Z}/5\mathbb{Z}$. This is the single example in the case $|N/N'_\Delta_1| = 5$.

Similarly one obtains a free $\mu_3$-quotient $\mathbb{W}_2$ of a smooth hypersurface of bidegree $(3, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ using the action of the cyclic group $\mu_3 = \langle \zeta \rangle$ on $\mathbb{P}^2 \times \mathbb{P}^2$ by $(1, \zeta, \zeta^2)$ on each $\mathbb{P}^2$. The toric variety $\mathbb{P}_{\Delta_2} := \mathbb{P}^2 \times \mathbb{P}^2/\mu_3$ has 9 isolated singularities. A non-degenerate hypersurface $\mathbb{W}_2 = \hat{\mathbb{W}}_2 \subset \mathbb{P}_{\Delta_2}$ is smooth, it has the Hodge numbers $h^{1,1}(\mathbb{W}_2) = 2$, $h^{2,1}(\mathbb{W}_2) = 29$, and $\pi_1(\mathbb{W}_2) \cong \mathbb{Z}/3\mathbb{Z}$.

We remark that the dual reflexive polytopes $\Delta_1^*, \ldots, \Delta_{16}^*$ have at most 8 vertices. $\Delta_{14}^*$ is the single dual reflexive polytope with 8 vertices. The corresponding Gorenstein toric Fano variety $\mathbb{P}_{\Delta_{14}}$ is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1/\mu_2$ where the cyclic group $\mu_2 = \langle \zeta \rangle$ acts by $(1, \zeta)$ on each $\mathbb{P}^1$.

We collected the information about all 16 reflexive polytopes $\Delta_1, \ldots, \Delta_{16}$ in the table below. In this table $\mathcal{P}_\Delta = |\Delta \cap M|$, $\mathcal{V}_\Delta = |\text{vert}(\Delta)|$, $\mathcal{P}_\Delta^* = |\Delta^* \cap N|$, and $\mathcal{V}_\Delta^* = |\text{vert}(\Delta^*)|$ denote numbers of points and vertices, respectively. Since any reflexive polytope $\Delta_i \subset N_{\mathbb{R}}$ is uniquely determined by its dual polytope $\Delta_i^* \subset N_{\mathbb{R}}$, we prefer to use the small number of vertices $v_1, \ldots, v_{\mathcal{V}_\Delta^*}$ in $\Delta_i^*$ in order to describe $\Delta_i^*$ by $\mathcal{V}_\Delta^* - 4$ independent linear relations among them.

The lattice $N$ is generated by the vertices $v_1, \ldots, v_{\mathcal{V}_\Delta^*}$, and by an additional vector $v$ expressed as a rational linear combination of the vertices $\{v_i\}$. The sublattice $N'_{\Delta_i}$ is generated by $v_1, \ldots, v_{\mathcal{V}_\Delta^*}$ except for 4 cases: $i = 4, 9, 15, 16$. In the latter cases $N'_{\Delta_i}$ is generated by $v_1, \ldots, v_{\mathcal{V}_\Delta^*}$ and by the vector $2v$. 


| $n^2$ | $|\pi_1|$ | $P_{\Delta} \cdot V_{\Delta}$ | $P_{\Delta^*} \cdot V_{\Delta^*}$ | $h^{1,1}$ | $h^{2,1}$ | $\chi(W)$ | Vertices $\{v_i\}$ of $\Delta^*$, $N := \mathbb{Z}v + \sum v_i$ |
|-------|-------|----------------|----------------|--------|--------|--------|--------------------------------------------------|
| 1     | 5     | 26 5           | 6 5            | 1 21   | -40    | $v_1 + v_2 + v_3 + v_4 + v_5 = 0$    | $v = \frac{1}{4}(v_2 + 2v_3 + 3v_4 + 4v_5)$ |
| 2     | 3     | 34 9           | 7 6            | 2 29   | -54    | $v_1 + v_2 + v_3 = v_4 + v_5 + v_6 = 0$ | $v = \frac{1}{4}(v_2 + 2v_3 + v_5 + 2v_6)$ |
| 3     | 3     | 49 5           | 7 5            | 2 38   | -72    | $3v_1 + 3v_2 + v_3 + v_4 + v_5 = 0$   | $v = \frac{1}{3}(v_1 + 2v_2 + v_3 + 2v_4)$ |
| 4     | 2     | 53 5           | 9 5            | 3 43   | -80    | $4v_1 + v_2 + v_3 + v_4 + v_5 = 0$    | $v = \frac{1}{4}(2v_1 + v_2 + 2v_3 + 3v_4)$ |
| 5     | 2     | 77 7           | 9 6            | 3 59   | -112   | $4v_1 + 2v_2 + v_3 + v_4 = 0$         | $v = \frac{1}{2}(v_1 + v_2 + v_3 + v_4)$ |
| 6     | 2     | 77 9           | 9 7            | 3 59   | -112   | $2v_1 + v_2 = 2v_1 + v_3 + v_4 + v_5 = 0$ | $v = \frac{1}{2}(v_1 + v_2 + v_3 + v_5)$ |
| 7     | 2     | 101 5          | 9 5            | 3 75   | -144   | $8v_1 + 4v_2 + 2v_3 + v_4 + v_5 = 0$   | $v = \frac{1}{8}(v_1 + v_2 + v_3 + v_4)$ |
| 8     | 2     | 101 6          | 9 6            | 3 75   | -144   | $4v_1 + 2v_2 + v_3 + v_4 = 0$         | $v = \frac{1}{2}(v_1 + v_2 + v_4 + v_5)$ |
| 9     | 2     | 29 8           | 9 6            | 4 28   | -48    | $v_1 + v_2 + v_3 + v_4 = v_5 + v_6 = 0$ | $v = \frac{1}{2}(v_2 + 2v_3 + 3v_4 + 2v_6)$ |
| 10    | 2     | 53 8           | 9 6            | 4 44   | -80    | $4v_1 + 2v_2 + v_3 + v_4 + v_5 = v_6 = 0$ | $v = \frac{1}{4}(v_1 + v_2 + v_3 + v_5)$ |
| 11    | 2     | 53 10          | 9 7            | 4 44   | -80    | $2v_1 + v_2 + v_3 = 0$                | $v = \frac{1}{2}(v_1 + v_2 + v_4 + v_5)$ |
| 12    | 2     | 41 9           | 9 6            | 4 36   | -64    | $2v_1 + v_2 + v_3 = 0$                | $v = \frac{1}{2}(v_1 + v_2 + v_4 + v_5)$ |
| 13    | 2     | 41 12          | 9 7            | 4 36   | -64    | $2v_1 + v_2 + v_3 = 0$                | $v = \frac{1}{2}(v_1 + v_2 + v_4 + v_5)$ |
| 14    | 2     | 41 16          | 9 8            | 4 36   | -64    | $v_1 + v_2 = v_3 + v_4 = 0$           | $v = \frac{1}{2}(v_1 + v_3 + v_5 + v_7)$ |
| 15    | 2     | 29 5           | 9 5            | 5 29   | -48    | $2v_1 + 2v_2 + 2v_3 + v_4 + v_5 = 0$  | $v = \frac{1}{4}(v_1 + 2v_2 + 3v_3 + 2v_5)$ |
| 16    | 2     | 29 6           | 9 6            | 5 29   | -48    | $v_1 + v_2 + v_3 + v_4 = 0$           | $v = \frac{1}{2}(v_1 + 3v_2 + 2v_4 + 2v_6)$ |
3. The Brauer group of toric hypersurfaces

Let $X$ be a smooth quasi-projective variety over $\mathbb{C}$. The cohomological Brauer group $B'(X)$ of $X$ is defined as the torsion subgroup in $H^2_{\text{et}}(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf of units in $\mathcal{O}_X$. If $B(X)$ is the Brauer group of sheaves of Azumaya algebras on $X$, then there exist a canonical injective homomorphism $B(X) \to B'(X)$. Recently Kresch and Vistoli proved that this homomorphism is in fact an isomorphism [16].

Let us summarize some well-known results about the Brauer group.

Theorem 3.1. The subgroup $B(X)_r \subset B(X)$ of elements $x \in B(X)$ with $rx = 0$ can be included into the short exact sequence

$$0 \to \text{Pic}(X)/r\text{Pic}(X) \to H^2_{\text{et}}(X, \mu_r) \to B(X)_r \to 0,$$

where $\mu_r$ denotes the subsheaf on $r$-th roots of unity in $\mathcal{O}_X^*$. More generally, there is a short exact sequence

$$0 \to \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H^2_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \to B(X) \to 0,$$

where $H^2_{\text{et}}(X, \mathbb{Q}/\mathbb{Z})$ coincides with the singular cohomology group $H^2(X, \mathbb{Q}/\mathbb{Z})$ in the usual topology of $X$ as a manifold over $\mathbb{C}$.

Proof. See [13] II, Theorem 3.1. □

Corollary 3.2. If $X$ is a smooth projective variety such that $\text{Pic}(X) \cong H^2(X, \mathbb{Z})$, then one has the isomorphism

$$B(X) \cong \text{Tors}(H^3(X, \mathbb{Z})).$$

In particular, the last statement holds for smooth Calabi-Yau $d$-folds ($d \geq 3$).

Proof. Follows from the universal coefficients theorem. □

Theorem 3.3. Let $X$ be a smooth quasi-projective variety and $Y \subset X$ a closed subvariety. Then the natural homomorphism $B(X) \to B(X \setminus Y)$ is always injective. This homomorphism is bijective if $Y$ has codimension $\geq 2$. Moreover, if $Y$ is a union of closed irreducible subvarieties $Y_1, \ldots, Y_n$ of codimension 1, then one has the exact sequence

$$0 \to B(X) \to B(X \setminus Y) \to \bigoplus_{i=1}^n H^1_{\text{et}}(Y_i, \mathbb{Q}/\mathbb{Z}),$$

where $H^1_{\text{et}}(Y_i, \mathbb{Q}/\mathbb{Z})$ coincides with the singular cohomology group $H^1(Y_i, \mathbb{Q}/\mathbb{Z})$ in the usual topology if $Y_i$ is smooth.

Proof. See [13] III, Corollary 6.2. □

The Brauer group of a $d$-dimensional algebraic torus $\mathbb{T}^d$ over $\mathbb{C}$ was computed e.g. by Magid in [21].
Theorem 3.4. Using the canonical isomorphism between the cohomology ring $H^*(\mathbb{T}^d, \mathbb{Z})$ and the exterior algebra $\Lambda^* M$ of the $\mathbb{Z}$-module $M$, one has the following canonical isomorphisms

$$B(\mathbb{T}^d) \cong H^2(\mathbb{T}^d, \mathbb{Q}/\mathbb{Z}) \cong \Lambda^2 M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong \text{Hom}(\Lambda^2 N, \mathbb{Q}/\mathbb{Z}).$$

The elements in the Brauer group $B(\mathbb{T}^d)$ can be explicitly described as follows. Take an integer $r > 1$ and two characters $\alpha, \beta \in M \subset O^*_{\mathbb{T}^d}$ of the algebraic torus $\mathbb{T}^d$. The symbol algebra $(\alpha, \beta)_r$ is generated by two elements $x, y$ with relations

$$x^r = \alpha, y^r = \beta, xy = \zeta yx,$$

where $\zeta$ is a primitive $r$-the root of unity. If $r = 2$, then $(\alpha, \beta)_2$ is the classical quaternionic algebra associated with $\alpha, \beta \in O^*_{\mathbb{T}^d}$.

Let $B(\mathbb{T}^d)_r$ be $r$-torsion elements in $B(\mathbb{T}^d)$, then $B(\mathbb{T}^d)_r$ is a free $\mathbb{Z}/r\mathbb{Z}$ module with the basis

$$(m_i, m_j)_r, \ 1 \leq i < j \leq d,$$

where $m_1, \ldots, m_d$ is a $\mathbb{Z}$-basis of the lattice $M$, i.e.

$$B(\mathbb{T}^d)_r = \Lambda^2(M) \otimes \mathbb{Z}/r\mathbb{Z}.$$\[Theorem 3.5. Let $\Delta \subset M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope and let $W \subset \mathbb{T}^d$ be a nondegenerate affine hypersurface defined by the equation

$$f(t) = \sum_{m \in \Delta \cap M} c_m t^m = 0.$$\]

Then for $d \geq 4$ the natural homomorphism

$$j^*: B(\mathbb{T}^d) \to B(W)$$

defined by the embedding $j : W \hookrightarrow \mathbb{T}^d$ is an isomorphism.

Proof. By 3.3, we obtain the commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Pic}(\mathbb{T}^d) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \\
\downarrow j^*_p & & \downarrow j^*_p \\
0 & \longrightarrow & \text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \\
\downarrow j^*_u & & \downarrow j^*_u \\
0 & \longrightarrow & H^2(W, \mathbb{Q}/\mathbb{Z}) \\
\downarrow j^* & & \downarrow j^* \\
0 & \longrightarrow & B(W) \\
\end{array}$$

By 1.3, $j^*_u$ is an isomorphism. Moreover, $\text{Pic}(\mathbb{T}^d) = 0$ since $\mathbb{T}^d$ is a Zariski open subset of $\mathbb{C}^d$. Therefore it suffices to prove that $\text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ (since by surjectivity of $\text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ would imply that $\text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$). Using toric resolution of singularities, one can compactify $W$ to a smooth projective semiample hypersurface $\tilde{W}$ in some smooth projective toric variety $\tilde{P}$ such that $Z := \tilde{W} \setminus W$ is a union of smooth irreducible divisors. By the vanishing theorem for semiample line bundles on toric varieties, we obtain that

$$h^1(\tilde{W}, \mathcal{O}_{\tilde{W}}) = \cdots = h^{d-2}(\tilde{W}, \mathcal{O}_{\tilde{W}}) = 0.$$
In particular, \( \text{Pic}(\hat{W}) \cong H^2(\hat{W}, \mathbb{Z}) \). Therefore,
\[
\dim_{\mathbb{Q}} \text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q} = \dim_{\mathbb{C}} H^2(\hat{W}, \mathbb{C}) = b_2(\hat{W}) = b_{2d-4}(\hat{W}).
\]

Let \( Z_1, \ldots, Z_k \) be irreducible components of \( Z = \hat{W} \setminus W \). Using the exact sequence
\[
\bigoplus_{i=1}^{k} \mathbb{Q}[Z_i] \to \text{Pic}(\hat{W}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0,
\]

it remains to show that the classes \([Z_1], \ldots, [Z_k] \in \text{Pic}(\hat{W})\) generate \( \text{Pic}(\hat{W}) \otimes_{\mathbb{Z}} \mathbb{Q} \). The latter follows from the Poincaré duality together with the long exact sequence for cohomology with compact supports
\[
\cdots \to H_c^{2d-4}(W, \mathbb{Q}) \to H_c^{2d-4}(\hat{W}, \mathbb{Q}) \to H_c^{2d-4}(Z, \mathbb{Q}) \to \cdots,
\]
where \( H_c^{2d-4}(Z, \mathbb{Q}) \cong \mathbb{Q}^k \), \( H_c^{2d-4}(\hat{W}, \mathbb{Q}) \cong \text{Hom}(\text{Pic}(\hat{W}), \mathbb{Q}) \), because the Hodge component of the Hodge type \((d-2, d-2)\) in \( H_c^{2d-4}(W, \mathbb{Q}) \) is trivial (see [10]).

Let us recall the computation of the Brauer group of a smooth (quasi-projective) toric variety \( \mathbb{P}_\Sigma \) obtained by Demeyer and Ford [11]. By 3.3, \( B(\mathbb{P}_\Sigma) \cong B(\mathbb{P}_{\Sigma^{(1)}}) \) where \( \Sigma^{(1)} \) is the subfan of all cones of dimension \( \leq 1 \) in \( \Sigma \). So we can assume without loss of generality that \( \Sigma = \Sigma^{(1)} \) and \( \mathbb{P}_\Sigma \) is a toric compactification of \( \mathbb{T}^d \) by divisors \( D_1^\circ, \ldots, D_n^\circ \) \( (D_i^\circ \cong \mathbb{T}^{d-1}) \) corresponding to lattice vectors \( e_1, \ldots, e_n \in N \). By 3.3 we have the exact sequence
\[
0 \to B(\mathbb{P}_\Sigma) \to B(\mathbb{T}^d) \to \bigoplus_{i=1}^{n} H^1(D_i^\circ, \mathbb{Q}/\mathbb{Z}).
\]

For any \( i = 1, \ldots, n \) we have the canonical isomorphisms
\[
H^1(D_i^\circ, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(N/\mathbb{Z}e_i, \mathbb{Q}/\mathbb{Z}) \cong M_{e_i} \otimes \mathbb{Q}/\mathbb{Z},
\]
where
\[
M_{e_i} := \{ m \in M : \langle x, e_i \rangle = 0 \}.
\]

Therefore, one obtains
\[
B(\mathbb{T}^d) \supset B(\mathbb{P}_\Sigma) = \bigcap_{i=1}^{n} \text{Ker} \varphi_i,
\]
where \( \varphi_i \) denotes the ramification map
\[
\varphi_i : B(\mathbb{T}^d) \to H^1(D_i^\circ, \mathbb{Q}/\mathbb{Z})
\]
which associates to a given symbol algebra \((\alpha, \beta)\), its ramification along the divisor \( D_i^\circ \subset \mathbb{P}_\Sigma \), i.e. a cyclic Galois covering of \( D_i^\circ \) of degree \( r \).
Theorem 3.6. Using the canonical isomorphisms

\[ B(T^d) \cong \text{Hom}(\Lambda^2 N, \mathbb{Q}/\mathbb{Z}), \quad H^1(D^\circ, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(N/\mathbb{Z}e_i, \mathbb{Q}/\mathbb{Z}), \]

one obtains the ramification map \( \varphi_i \) from the homomorphism

\[ \psi_i : N/\mathbb{Z}e_i \to \Lambda^2 N \]
\[ y \mapsto y \wedge e_i, \quad \forall y \in N \]

by applying the functor \( \text{Hom}(\ast, \mathbb{Q}/\mathbb{Z}) \). In particular, the image of \( \psi_i \) is the sublattice

\[ \text{Im} \, \psi_i = N \wedge e_i \subset \Lambda^2 N. \]

Proof. These statements are contained in [11] Lemmas 1.5-1.7. \( \square \)

Let us define the sublattice \( N' := \sum_{i=1}^n \mathbb{Z}e_i \subset N \) and write

\[ N/N' \cong \mathbb{Z}/c_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/c_d\mathbb{Z} \]

where \( c_1, \ldots, c_d \) non-negative integers such that \( c_1 | c_2 | \cdots | c_d \). We can choose a \( \mathbb{Z} \)-basis \( y_1, \ldots, y_d \) of \( N \) such that \( \{c_i y_i\}_{c_i \neq 0} \) is a \( \mathbb{Z} \)-basis of \( N' \). Consider another sublattice

\[ N \wedge N' = \sum_{i=1}^n N \wedge e_i = \sum_{i=1}^n \text{Im} \, \psi_i \subset \Lambda^2 N. \]

Then the elements \( \{c_i y_i \wedge y_j\}_{c_i \neq 0, i < j} \) form a \( \mathbb{Z} \)-basis of \( N \wedge N' \). Therefore,

\[ \Lambda^2 N/(N \wedge N') \cong \bigoplus_{i<j} \mathbb{Z}/c_i\mathbb{Z}. \]

Thus, one obtains (see [11], Theorem 1.1):

Theorem 3.7. In the above notations, the Brauer group of a nonsingular toric variety \( \mathbb{P}_\Sigma \) is isomorphic to

\[ \bigoplus_{i<j} \text{Hom}(\mathbb{Z}/c_i\mathbb{Z}, \mathbb{Q}/\mathbb{Z}). \]

In particular, if \( N' \subset N \) is a sublattice of finite index, then \( \mathbb{P}_\Sigma \) has a finite Brauer group

\[ B(\mathbb{P}_\Sigma) \cong \text{Hom}(\Lambda^2 N/(N \wedge N'), \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{i<j} (\mathbb{Z}/c_i\mathbb{Z})^*. \]

Let \( \Delta \subset M_\mathbb{R} \) be a \( d \)-dimensional lattice polytope and let \( W \subset T^d \) be a nondegenerate affine hypersurface defined by the equation

\[ f(t) = \sum_{m \in \Delta \cap M} c_m t^m = 0. \]

Denote by \( \overline{W} \) the closure of \( W \) in the projective toric variety \( \mathbb{P}_\Delta \) and by \( \widehat{W} \) a smooth projective desingularization of \( \overline{W} \) defined by the fan \( \Sigma \subset N_\mathbb{R} \) which is a subdivision of the normal fan to \( \Delta \). We denote by \( N^{(2)}_\Delta \) the sublattice in \( N \) generated by all lattice points \( v \in N \) such that the minimum of the
linear function \( \langle *, v \rangle \) on \( \Delta \) is attained on a face \( \theta \subset \Delta \) of dimension \( \geq 2 \).

Our main result in this section is the following:

**Theorem 3.8.** In the above notation, for \( d \geq 4 \) the Brauer group \( B(\hat{W}) \) of the toric hypersurface \( \hat{W} \) is a cyclic group isomorphic to

\[
\text{Hom}(\Lambda^2 N/(N \wedge N^{(2)}_\Delta), \mathbb{Q}/\mathbb{Z}).
\]

**Proof.** Let \( e_1, \ldots, e_n \) be primitive lattice generators of 1-dimensional cones in \( \Sigma \) such that the minimum of the linear function \( \langle *, v \rangle \) on \( \Delta \) is attained on a face \( \theta \subset \Delta \) of dimension \( \geq 1 \). Denote by \( \Sigma' \subset \Sigma^{(1)} \) the subfan of \( \Sigma \) consisting of 0 and all 1-dimensional cones \( \mathbb{R}_{\geq 0} e_1, \ldots, \mathbb{R}_{\geq 0} e_n \). We set \( \hat{W}^{(1)} = \hat{W} \cap \mathbb{P}_{\Sigma'} \). Since the complement to \( \hat{W}^{(1)} \) in \( \hat{W} \) has codimension \( \geq 2 \), by 3.3 it suffices to compute the Brauer group \( B(\hat{W}^{(1)}) \). By 3.3 we have the commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & B(\mathbb{P}_{\Sigma'}) & \longrightarrow & B(T^d) & \longrightarrow & \bigoplus_{i=1}^n H^1(D^\circ_i, \mathbb{Q}/\mathbb{Z}) \\
& & \alpha & \downarrow & \beta & \downarrow & \\
0 & \longrightarrow & B(\hat{W}^{(1)}) & \longrightarrow & B(W) & \longrightarrow & \bigoplus_{i=1}^n H^1(Z^\circ_i, \mathbb{Q}/\mathbb{Z})
\end{array}
\]

By 3.5 \( \beta \) is an isomorphism. In order to understand the homomorphisms \( \alpha \) and \( \gamma \) we divide the set of lattice vectors \( \mathcal{E} = \{e_1, \ldots, e_n\} \) into a union of two disjoint subsets \( \mathcal{E}_1 \cup \mathcal{E}_2 \). The set \( \mathcal{E}_2 \) consists of all \( e_i \in \mathcal{E} \) such that the minimum of \( \langle *, e_i \rangle \) on \( \Delta \) is attained on a face \( \theta \subset \Delta \) of dimension \( \geq 2 \) and we define \( \mathcal{E}_1 := \mathcal{E} \setminus \mathcal{E}_2 \).

For any \( e_i \in \mathcal{E} \) we denote by \( \theta_i \) the maximal subface of \( \Delta \) on which the minimum of \( \langle *, e_i \rangle \) on \( \Delta \) is attained.

If \( e_i \in \mathcal{E}_2 \), then \( d_i := \dim \theta_i \geq 2 \) and the divisor \( Z^\circ_i \) is isomorphic to the product \( V_i \times T^{d-d_i-1} \), where \( V_i \) is a \( \theta_i \)-nondegenerate affine hypersurface in a torus \( T^d \). By 3.8 the ramification homomorphism

\[
\phi_i : B(W) \rightarrow H^1(Z^\circ_i, \mathbb{Q}/\mathbb{Z})
\]

can be identified with the composition of the ramification homomorphism

\[
\varphi_i : B(T^d) \rightarrow H^1(D^\circ_i, \mathbb{Q}/\mathbb{Z})
\]

with the monomorphism

\[
H^1(D^\circ_i, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(Z^\circ_i, \mathbb{Q}/\mathbb{Z}).
\]

Thus, we obtain

\[
B := \bigcap_{e_i \in \mathcal{E}_2} \ker \varphi_i = \bigcap_{e_i \in \mathcal{E}_2} \ker \phi_i.
\]

Using 3.6 we can describe the dual group \( B^* \) as cokernel of the homomorphisms

\[
\psi_i : N/\mathbb{Z} e_i \rightarrow \Lambda^2 N, \ e_i \in \mathcal{E}_2,
\]
i.e. we have
\[ \sum_{e_i \in E} \text{Im} \psi_i = N \wedge N^{(2)}_\Delta \quad \text{and} \quad B^* = N/(N \wedge N^{(2)}_\Delta). \]

Therefore, it remains to prove that
\[ B = \bigcap_{e_i \in E} \ker \phi_i = B(\hat{W}^{(1)}), \]
i.e., that for any \( e_i \in E \) the image \( \text{Im} \phi_i = \text{Im} \gamma \circ \psi_i \) is contained in \( N \wedge N^{(2)}_\Delta \). Now we consider two cases for \( e_i \in E \):

Case 1: Let \( e_i \in N^{(2)}_\Delta \). Then the image of \( \phi_i \) is contained in \( N \wedge e_i \subset N \wedge N^{(2)}_\Delta \).

Case 2: Let \( e_i \not\in N^{(2)}_\Delta \). Then the image of \( \phi_i \) is a sublattice \( N_i \wedge e_i \), where \( N_i \) is a direct summand of \( N \) of rank \( d-1 \) containing \( e_i \) (\( N_i \) is a sublattice of rank \( d-1 \) which is orthogonal to the 1-dimensional face \( \theta_i \)). The smoothness of \( \hat{W} \) implies that both lattices \( N_i \) and \( N^{(2)}_\Delta \) contain a direct summand of \( N \) of rank \( d-2 \). Therefore, \( N_i/N_i \cap N^{(2)}_\Delta \) is a cyclic group and we have
\[ N_i = \mathbb{Z}v + N_i \cap N^{(2)}_\Delta \]
for some lattice vector \( v \in N_i \) which proportional to \( e_i \). By \( \nu \wedge e_i = 0 \), we have
\[ N_i \wedge e_i \subset (\mathbb{Z}v + N_i \cap N^{(2)}_\Delta) \wedge e_i \subset N^{(2)}_\Delta \wedge e_i \subset N \wedge N^{(2)}_\Delta. \]

Thus, all divisors \( Z_i (e_i \in E_1) \) do not define non-trivial ramification conditions on the subgroup \( B \) of the Brauer group \( B(W) \) and
\[ B \cong B(\hat{W}^{(1)}) \cong B(\hat{W}) \cong \text{Hom}(\Lambda^2 N/(N \wedge N^{(2)}_\Delta), \mathbb{Q}/\mathbb{Z}). \]

\[ \square \]

**Corollary 3.9.** Let \( \hat{W} \) be a smooth projective Calabi-Yau \( d \)-fold obtained as a crepant desingularization of a nondegenerate projective hypersurface \( \overline{W} \subset \mathbb{P}_\Delta \) corresponding to a reflexive polytope \( \Delta \) of dimension \( d \geq 4 \). Then the Brauer group \( B(\hat{W}) \) is a cyclic group isomorphic to
\[ \text{Hom}(\Lambda^2 N/(N \wedge N^{(2)}_\Delta), \mathbb{Q}/\mathbb{Z}), \]
where \( N''_\Delta. \subset N \) is the sublattice generated by all lattice points in \( N \cap \Delta^* \) which are contained in faces of \( \Delta^* \) of codimension \( > 2 \).

**Proof.** The statement follows immediately from 3.8 and from the easy combinatorial equality \( N^{(2)}_\Delta = N''_\Delta \) using the duality between faces of dual reflexive polytopes \( \Delta \subset M_\mathbb{R} \) and \( \Delta^* \subset N_\mathbb{R} \). \( \square \)
4. The Brauer group of Calabi-Yau 3-folds

Let $\Delta$ be a 4-dimensional reflexive polytope. Then

$$N''_{\Delta^*} := \sum_{v \in \theta^* \cap N, \dim \theta^* = 1} Zv \subset N$$

is a sublattice of finite index. We write

$$N/N''_{\Delta^*} = \mathbb{Z}/c_1\mathbb{Z} \oplus \mathbb{Z}/c_2\mathbb{Z} \oplus \mathbb{Z}/c_3\mathbb{Z} \oplus \mathbb{Z}/c_4\mathbb{Z},$$

where $c_1 | c_2 | c_3 | c_4$. In this case $c_1 = c_2 = 1$ and

$$B(\hat{W}) \cong \text{Hom}(\Lambda^2 N/(N \wedge N''_{\Delta^*}), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/c_3\mathbb{Z}.$$

Using the Calabi-Yau date [18], one can check all 473,800,776 reflexive polytopes $\Delta$ and find exactly 16 reflexive polytopes of dimension 4 with $c_3 > 1$. For each of these 16 polytopes, one obtains

$$N/N''_{\Delta^*} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, \quad p = 2, 3, 5.$$  

So we have again 16 families with the cyclic Brauer group $\mathbb{Z}/p\mathbb{Z}$. The main observation is that these 16 families are exactly the mirrors of 16 non-simply connected families of Calabi-Yau 3-folds with the cyclic fundamental group $\mathbb{Z}/p\mathbb{Z}$ ($c = 2, 3, 5$) (see Section 2). Therefore, we can exchange the lattices $N \leftrightarrow M$ and polytopes $\Delta^* \leftrightarrow \Delta$ so that for all 4-dimensional reflexive polytopes we have the isomorphisms

$$\Lambda^2 M/(M \wedge M''_{\Delta^*}) \cong N/N''_{\Delta^*} \quad \text{and} \quad \Lambda^2 N/(N \wedge N''_{\Delta^*}) \cong M/M''_{\Delta^*}$$

Unfortunately we do not have any natural combinatorial explanation of the isomorphisms (2).

The sublattice $N''_{\Delta^*} \subset N$ defines a (ramified) Galois covering $\hat{W}'' \to \hat{W}$ with the Galois group $G = N/N''_{\Delta^*} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. If we consider $G$ as a 2-dimensional vector space over the finite field $\mathbb{F}_p$, then the Brauer group $B(\hat{W})$ is isomorphic to $\text{Hom}(\Lambda^2 G, \mathbb{Q}/\mathbb{Z})$, i.e., to a 1-dimensional $\mathbb{F}_p$-space. This allows to look at $B(\hat{W})$ as the discrete torsion group $H^2(G, U(1))$ for the orbifold $\hat{W}''/G$.

We finish this section by describing a typical example $\hat{W}_1^*$ of a Calabi-Yau 3-fold from our list (the mirror of $\hat{W}_1$ from Section 2) with Brauer group $B(\hat{W}_1^*) \cong \mathbb{Z}/5\mathbb{Z}$. This example gives a 1-parameter family of Calabi-Yau 3-folds $\hat{W}_1^*$ having a close connection to the counterexample to the global Torelli considered by Aspinwall-Morrison and B. Szendrői [1, 27].

Consider the action of the group $G := \mu_5 \times \mu_5$ on $\mathbb{P}^4$ by

$$\zeta_1 \to (1, \zeta_1, \zeta_1^2, \zeta_1^3, \zeta_1^4)$$

$$\zeta_2 \to (1, \zeta_2, \zeta_2^3, \zeta_2, 1).$$

This action is not free and the quotient \( \mathbb{P}_{\Delta_1^*} := \mathbb{P}^4/G \) has cyclic singularities of type \( \frac{1}{5}(3, 1, 1) \) along 10 curves. Let \( \hat{W}_1^* \) be the crepant desingularization of a generic Calabi-Yau hypersurface \( \overline{W}_1^* \subset \mathbb{P}_{\Delta_1^*} \). Then

\[
\pi_1(\hat{W}_1^*) = 0 = B(\hat{W}_1^*) = \Lambda^2 G \cong \mu_5 \cong \pi_1(\hat{W}_1).
\]

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