Difference Series Spaces and Matrix Transformations

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Abstract: This paper deals with new series space \( |C_a|_p(V) \) introduced by using Cesàro means and difference operator. It is shown that this newly defined space \( |C_a|_p(V) \) is a BK - space and has Schauder basis. Furthermore, the \( \alpha \), \( \beta \), and \( \gamma \)-duals of \( |C_a|_p(V) \) are computed and the characterizations of classes of matrix mappings from \( |C_a|_p(V) \) to \( X = \{ \ell_\infty, c, c_0 \} \) are also given.

Keywords
Difference sequence spaces, \( \alpha \)-\( \beta \) and \( \gamma \)-duals, Matrix operators, BK spaces

1. INTRODUCTION

Recently, there has been a lot of interest in studies on the sequence spaces. In the literature, the basic concept is to generate new sequence spaces by means of the matrix domain of triangles (see, [1-17]). Besides this, several authors have studied difference sequence spaces using some newly defined infinite matrices. Also, they have studied some topological properties of them, and they have given the inclusion relations and some characterizations of related matrix transformations.

Throughout this study, \( \omega \), \( \ell_\infty \), \( c \), and \( c_0 \) will be spaces of all, bounded, convergent and null sequences \( x = (x_k) \) with complex terms, respectively. Also, by \( bs \), \( cs \) and \( \ell_p \) \( (1 \leq p < \infty) \), we denote the spaces of all bounded, convergent and \( p \)-absolutely convergent series, respectively. A Banach sequence space \( X \) is called a BK-space provided each of the maps \( P_n : X \rightarrow \mathbb{C} \) defined by \( P_n(x) = x_n \ (n \geq 0) \) is continuous, where \( \mathbb{C} \) denotes the complex field.

Let \( U \) and \( V \) be two sequence spaces and \( T = (t_{nk}) \) be an infinite matrix of complex number. The matrix domain \( U_T \) is defined as

\[
U_T = \{ u \in \omega : Tu \in U \}. \tag{1}
\]

Define the set \( M(U, V) \) as

\[
M(U, V) = \{ a \in \omega : au = a_0 u_k \in V \text{ for all } u = (u_k) \in U \}. \tag{2}
\]

By the notation (2), the \( \alpha \), \( \beta \), and \( \gamma \)-duals of the space \( U \) are defined by

\[
U^\alpha = M(U, \ell_1), U^\beta = M(U, cs) \text{ and } U^\gamma = M(U, bs), \]

respectively.
Also, $T$ defines a mapping from $U$ into $V$, that is, for every $u = (u_k) \in U$, the sequence $Tu = (T_n(u))$, the $T$-transform of $u$, exists and is in $V$, where

$$T_n(u) = \sum_{k=0}^{\infty} t_{nk}u_k$$

for $n \geq 0$. $(U, V)$ denotes the class of all such matrices that maps $U$ into $V$. Thus, $T \in (U, V)$ if and only if $T_n = (t_{nk})_{k=0}^{\infty} \in U^p$ for each $n$ and $Tu \in V$ for all $u \in U$.

Throughout this study, $q$ shows the conjugate of $p$, i.e., $1/p + 1/q = 1$.

2. DIFFERENCE SERIES SPACES AND CESÁRO MEANS

The notion of difference sequence spaces has been introduced by Kizmaz [18] as follows.

$$X(\Delta) = \{ x = (x_k) : \Delta x \in X \}$$

for $X \in c_0, c, \ell_\infty$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$. After, Sarıgöl [14] has defined the sequence space

$$X(\Delta_q) = \{ x = (x_k) : \Delta_q x = (q^k (x_k - x_{k+1})) \in X, \quad q < 1 \}.$$

Later on, some new sequence spaces are defined by using the difference operator. For example, several authors including Çolak and Et [3], Orhan [19], Polat and Altay [20], Aydin and Başar [1], Başar and Altay [2], Demiriz and Çakan [4] and others have introduced and studied new sequence spaces by considering difference operators. In this section, following [1-4, 6-11, 14-16], we introduce the difference series space $|C_\alpha|_p(V)$ by using Cesáro means and difference operator and we prove that this space linearly isomorphic to space $\ell_p$, and also construct its bases.

Let $\Sigma x_v$ be an infinite series with $n$th partial sums $(s_n)$, then the $n$th Cesáro mean $(\bar{C}, \alpha)$ of order $\alpha$ $(\alpha > -1)$ of the sequence $(s_n)$ is defined by

$$u_n^\alpha = \frac{1}{E_n^\alpha} \sum_{v=0}^{n} E_n^{\alpha-1} s_v,$$

where $E_n^0 = 1$, $E_n^{\alpha} = \left(\frac{\alpha + n}{n}\right)$, $E_n^{-n} = 0$, $n \geq 1$. The series $\Sigma x_n$ is said to be summable $|C, \alpha|_p$, $p \geq 1$, if (see [21])

$$\sum_{n=1}^{\infty} n^{p-1} |u_n^\alpha - u_{n-1}^\alpha|^p < \infty.$$

Using the method $|C, \alpha|_p$, the absolute Cesáro series space $|C_\alpha|_p$ has been defined by Sarıgöl in [16]. For any given sequence $x = (x_k) \in |C_\alpha|_p$, $H^{(p)}$ -transform of $x$ is in $\ell_p$, where the matrix $H^{(p)} = (h_{nk}^p)$ is defined by

$$h_{nk}^p = \begin{cases} \frac{E_n^{p-k}}{n!} & 1 \leq k \leq n \\ 0 & k > n. \end{cases}$$

The main purpose of this study is to define further generalization of the absolute Cesáro series space $|C_\alpha|_p(V)$ using difference operator by

$$|C_\alpha|_p(V) = \{ x = (x_k) : (\nabla x_k) \in |C_\alpha|_p \}$$

where $\nabla x_k = x_k - x_{k-1}$ for each $k \in \mathbb{N}$.

We first define the difference space $|C_\alpha|_p(V)$ by

$$|C_\alpha|_p(V) = \left\{ x = (x_k) \in \omega : \left( \sum_{n=1}^{\infty} \left| \frac{1}{n!} \sum_{v=0}^{n} E_n^{p-1} \nabla x_v \right|^p \right)^{1/p} < \infty \right\}.$$

Let us define the sequence $y = (y_n)$ as the $H^{(p)}(\nabla)$ transform of the sequence $x = (x_k)$, that is,

$$y_n = \frac{1}{n!} \sum_{v=0}^{n} E_n^{p-1} \nabla x_v$$

for each $n \in \mathbb{N}$.

Then the difference space $|C_\alpha|_p(V)$ can be redefined by all sequences whose $H^{(p)}(\nabla)$ transform is in $\ell_p$. This leads us together with (1) to the fact that

$$|C_\alpha|_p(V) = (\ell_p, H^{(p)}(\nabla)).$$

Now, we begin with following theorems which are required in the study.

**Theorem 2.1.** The difference space $|C_\alpha|_p(V)$ is a $BK$-space with the norm $\|x\|_{|C_\alpha|_p(V)} = \|H^{(p)}(\nabla) x\|_\ell_p$, that is

$$\|x\|_{|C_\alpha|_p(V)} = \left( \sum_{n=1}^{\infty} \left| H_n^{(p)}(\nabla) x \right|^p \right)^{1/p}.$$ 

**Proof.** It is known that $\ell_p$ is a $BK$ space according to usual $p$-norm, (4) holds and the matrix $H^{(p)}(\nabla)$ is a triangle. So, we deduce from Theorem 4.3.2 in [22] that space $|C_\alpha|_p(V)$ is a $BK$-space with the given norm. This concludes the proof.

**Theorem 2.2.** The difference space $|C_\alpha|_p(V)$ is linearly isomorphic to the space $\ell_p$ for $p \geq 1$, that is, $|C_\alpha|_p(V) \cong \ell_p$. 

Proof. We show that the existence of a linear bijection between the spaces $|C_{α}|_{1p}(V)$ and $ℓ_p$. Consider the transformation $H^{(p)}(V) : |C_{α}|_{1p}(V) → ℓ_p$ such that $H^{(p)}(V)(x) = y$ defined by (3). The linearity of $H^{(p)}(V)$ is clear and also it is seen that $x = θ$ whenever $H^{(p)}(V)(x) = θ$. So, $H^{(p)}(V)$ is injective. Furthermore, let $y ∈ ℓ_p$ and we define a sequence $x = (x_n)$ by

$$x_n = \sum_{j=1}^{n} \sum_{τ=j}^{n} E_{r-j}^{α-1}E_{l}^{α} \frac{1}{r} j^{1/p} y_j$$

and so

$$\|x\|_{|C_{α}|_{1p}(V)} = \left\|H^{(p)}(V)(x)\right\|_{ℓ_p} = \left(\sum_{n=1}^{∞} \left|H^{(p)}_{n}(V)(x)\right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=1}^{∞} \frac{1}{n^p} E_{n}^{α-1}v \sum_{l=1}^{n} E_{r-l}^{α}v x_l\right)^{\frac{1}{p}}$$

$$= \|y\|_{ℓ_p}.$$ 

Therefore, $H^{(p)}(V)$ is norm preserving and $x ∈ |C_{α}|_{1p}(V)$ for all $y ∈ ℓ_p$, namely, $H^{(p)}(V)$ is surjective. Consequently, $H^{(p)}(V)$ is a linear bijection, the fact that $|C_{α}|_{1p}(V) ≅ ℓ_p$ which concludes the proof.

Now, we determine the Schauder basis of the space $|C_{α}|_{1p}(V)$.

A sequence $(b_n)$ is called a Schauder basis (or briefly basis) of a normed sequence space $X$, if for each $x ∈ X$, there exists a unique sequence $(α_n)$ of scalars such that

$$\lim_{m→∞} \left\|x - \sum_{k=0}^{m} α_k b_k\right\|_X = 0$$

and in this case, we write $x = \sum_{k=0}^{∞} α_k b_k$.

Since $|C_{α}|_{1p}(V) ≅ ℓ_p$, the Schauder basis of the new space $|C_{α}|_{1p}(V)$ is the inverse image of the basis $(e^{(n)})_{k=0}^{∞}$ of the space $ℓ_p$, where $e^{(n)}(n = 0, 1, ...)$ is the sequence with $e^{(n)}_n = 1$, $e^{(n)}_v = 0 (v ≠ n)$ for all $n ≥ 0$.

So, we have the following theorem without proof.

**Theorem 2.3.** Let $α_k = (H^{(p)}(V)(x))_k$, for all $k ∈ ℤ$. Define the sequence $τ^{(j)} = (τ^{(j)}_n)$ as

$$τ^{(j)}_n = \begin{cases} 1/p \sum_{r=j}^{n} E_{r-j}^{α-1}E_{l}^{α} \frac{1}{r} j^{1/p} y_j, & 1 ≤ j ≤ n \\ 0, & j > n. \end{cases}$$

The sequence $τ^{(j)}$ is a basis for the space $|C_{α}|_{1p}(V)$ and any $x ∈ |C_{α}|_{1p}(V)$ has a unique representation of the form

$$x = \sum_{j=1}^{∞} α_j τ^{(j)}.$$ 

### 3. Dual spaces and matrix transformations

We devote the last section of the paper to determine the $α$, $β$ and $γ$-duals of spaces $|C_{α}|_{1p}(V)$ and to give characterizations of certain matrix classes concerning the spaces $|C_{α}|_{1p}(V)$.

We continue with quoting following lemmas due to Stieglitz and Tietz [23], Sarıgöl [24] and Maddox [25] for our main results.

**Lemma 3.1** [23]. The following statements hold:

- a) $T = (t_{nk}) ∈ (ℓ_{1}, c)$ if and only if

$$\lim_{n→∞} t_{nk}$$ exists for each $k ∈ ℤ$ (6)

and

$$\sup_{n,k} |t_{nk}| < ∞.$$

- b) Let $1 < p < ∞$. Then, $T = (t_{nk}) ∈ (ℓ_{p}, c)$ if and only if (6) holds, and

$$\sup_{n} \sum_{k=0}^{∞} |t_{nk}|^q < ∞.$$ (8)

- c) $T = (t_{nk}) ∈ (ℓ_{1}, ℓ_p)$ if and only if (7) holds.

- d) Let $1 < p < ∞$. Then, $T = (t_{nk}) ∈ (ℓ_{p}, ℓ_{∞})$ holds.

- e) $T = (t_{nk}) ∈ (ℓ_{1}, c_0)$ holds, and

$$\lim_{n→∞} t_{nk} = 0,$$ for each $k ∈ ℤ$. (9)

- f) Let $1 < p < ∞$. Then, $T = (t_{nk}) ∈ (ℓ_{p}, c_0)$ holds and

$$\sup_{n} \left(\sum_{k=0}^{∞} |t_{nk}|^q\right) < ∞.$$ (10)

**Lemma 3.2** [24]. Let $1 < p < ∞$. Then, $T = (t_{nk}) ∈ (ℓ_{p}, ℓ_1)$ if and only if

$$\sum_{k=0}^{∞} \left(\sum_{n=0}^{∞} |t_{nk}|^q\right) < ∞.$$ (11)

**Lemma 3.3** [25]. Let $1 ≤ p < ∞$. Then, $T = (t_{nk}) ∈ (ℓ_{p}, ℓ_p)$ if and only if
We now give details about duals of the spaces $|C_{a}\|_{p}(\mathcal{V})$.

**Theorem 3.4.** Let define the sets $A_1$ and $A_2$ as follows.

$$A_1 = \left\{ a = (a_n) \in \omega : \sup_{j} \sum_{n\geq j} \left( \sum_{r=j}^{\infty} \frac{a_n E_{r-j}^{-1} E_j^{a}}{r} \right)^{j/p} < \infty \right\}$$

and

$$A_2 = \left\{ a = (a_n) \in \omega : \sup_{j} \sum_{n\geq j} \left( \sum_{r=j}^{\infty} \frac{a_n E_{r-j}^{-1} E_j^{a}}{r} \right)^{j/p} \left( \sum_{r=j}^{\infty} \frac{a_n E_{r-j}^{-1} E_j^{a}}{r} \right)^{j/p} < \infty \right\}.$$ 

Then, the $\alpha$-dual of the spaces $|C_{a}\|_{p}(\mathcal{V})$ for $p > 1$ and $|C_{a}\|_{1}(\mathcal{V})$ are given by

$$\{ |C_{a}\|_{p}(\mathcal{V}) \}^{\alpha} = A_1$$

and

$$\{ |C_{a}\|_{1}(\mathcal{V}) \}^{\alpha} = A_2,$$

respectively.

**Proof.** Let $a = (a_n) \in w$ and $p > 1$. Then, we write

$$a_n x_n = \sum_{j=1}^{n} \sum_{r=j}^{n} \frac{E_{r-j}^{-1} E_j^{a}}{r} y_j \left( f_{i_{nj}}^{p} \right)$$

where the matrix $F^p = (f_{i_{nj}}^{p})$ is defined via the sequence $a = (a_n)$ by

$$f_{i_{nj}}^{p} = \left( \sum_{r=j}^{n} \frac{a_n E_{r-j}^{-1} E_j^{a}}{r} \right)^{j/p}, 1 \leq j \leq n$$

and $0, j > n$.

Therefore, we deduce that $ax = (a_n x_n) \in \ell_1$ whenever $x \in |C_{a}\|_{p}(\mathcal{V})$ if and only if $F^p y \in \ell_1$ whenever $y \in \ell_p$.

Thus, we have $\{ |C_{a}\|_{p}(\mathcal{V}) \}^{\alpha} = A_1$.

Using Lemma 3.3 instead of Lemma 3.2, the proof can be completed in a similar way.

**Theorem 3.5.** Let define the sets $A_3$, $A_4$ and $A_5$ by

$$A_3 \left\{ a = (a_n) \in \omega : \sup_{m} \sum_{n \geq j}^{m} \sum_{r=j}^{m} a_n \frac{E_{r-j}^{-1} E_j^{a}}{r} y_j^{q} \right\} < \infty,$$

and

$$A_4 \left\{ a = (a_n) \in \omega : \sup_{m} \sum_{n \geq j}^{m} a_n \frac{E_{r-j}^{-1} E_j^{a}}{r} y_j^{q} \right\} < \infty,$$

respectively. Then, the $\beta$-dual of the spaces $|C_{a}\|_{p}(\mathcal{V})$ for $p > 1$ and $|C_{a}\|_{1}(\mathcal{V})$ are given by

$$\{ |C_{a}\|_{p}(\mathcal{V}) \}^{\beta} = A_3 \cap A_4$$

and

$$\{ |C_{a}\|_{1}(\mathcal{V}) \}^{\beta} = A_4 \cap A_5$$

respectively.

**Proof.** Let $a = (a_n) \in w$ and $p > 1$. Then, we consider the following equation.

$$\sum_{n=1}^{m} a_n x_n = \sum_{n=1}^{m} a_n \sum_{j=1}^{n} \sum_{r=j}^{n} \frac{E_{r-j}^{-1} E_j^{a}}{r} y_j^{q}$$

$$= \sum_{j=1}^{m} \sum_{r=j}^{m} a_n \frac{E_{r-j}^{-1} E_j^{a}}{r} y_j^{q}$$

$$= \sum_{j=1}^{m} b_{nj} y_j = (By)_m.$$
where the matrix \( B = (b_{mj}) \) is defined via the sequence \( a = (a_n) \) by

\[
b_{mj} = \left( j^{1/p} E_j^a \right)^{m} a_n \sum_{n=j}^{\infty} \frac{E_{r-j}^{-a-1}}{r}, 1 \leq j \leq m, 0, j > m.
\]

Therefore, we deduce that \( ax = (a_n x_n) \in cs \) whenever \( x \in |C_{a}p|p(\mathbb{V}) \) and if only if \( By \in c \) whenever \( y \in \ell_p \), which implies that \( a \in \{|C_{a}p|p(\mathbb{V})\}^\beta \) if and only if \( B \in (\ell_p, c) \), by part b- of Lemma 3.1, we obtain that \( a \in \{|C_{a}p|p(\mathbb{V})\}^\beta \) if and only if

\[
\sup_n \frac{1}{1/p} \sum_{j=1}^{m} \left( j^{1/p} E_j^a \right)^{m} a_n \sum_{n=j}^{\infty} \frac{E_{r-j}^{-a-1}}{r} < \infty
\]

and

\[
\lim_{m \to \infty} \sum_{n=j}^{\infty} a_n \frac{E_{r-j}^{-a-1}}{r} \text{ exists for each } j \in \mathbb{N}.
\]

Thus, we have \( \{|C_{a}p|p(\mathbb{V})\}^\beta = A_3 \cap A_5 \).

Using part a-) instead of part b-) of Lemma 3.1, the proof can be completed in a similar way.

Since the proof is similar to the previous one, we give following theorem without proof.

**Theorem 3.6.** Let define the sets \( A_3 \) and \( A_4 \) by (10) and (11), respectively. The \( \gamma \)-dual of the spaces \(|C_{a}p|p(\mathbb{V})\) for \( p > 1 \) and \(|C_{a}1|p(\mathbb{V})\) are given by

\[
\{|C_{a}p|p(\mathbb{V})\}^\gamma = A_3
\]

and

\[
\{|C_{a}1|p(\mathbb{V})\}^\gamma = A_4,
\]

respectively.

Now, we characterize matrix transformations from \(|C_{a}p|p(\mathbb{V})\) to \( \ell_{\infty}, c, c_0 \). Let us define the matrix \( B^{(p)} = (b_{mj}^{(p)}) \) via an infinite matrix \( T = (t_{nk}) \) by

\[
b_{mj}^{(p)} = \sum_{k=j}^{\infty} t_{nk} \sum_{r=j}^{\infty} \frac{E_{r-j}^{-a-1} E_j^{a}}{r}, 1 \leq j \leq m, 0, j > m.
\]

We may begin with characterization of matrix classes \(|C_{a}1|p(\mathbb{V}), X\), where \( X = \{\ell_{\infty}, c, c_0\}\).

**Theorem 3.7.** Consider the matrix \( B^{(p)} = (b_{nk}^{(p)}) \) as in (12) with \( p = 1 \). Then, i-)

\[
T = (t_{nk}) \in (|C_{a}1|p|V), \ell_{\infty}) \text{ if and only if}
\]

\[
\lim_{m \to \infty} \sum_{k=1}^{m} t_{nk} \sum_{r=j}^{\infty} \frac{E_{r-j}^{-a-1} E_j^{a}}{r} \text{ exists for all } n, j \in \mathbb{N},
\]

(ii-)

\[
\sup_{n,k} |b_{nk}(1)| < \infty.
\]

(iii-)

\[
\lim_{n \to \infty} b_{nk}(1) \text{ exists for each } k \in \mathbb{N}.
\]

**Proof.** i-) \( T = (t_{nk}) \in (|C_{a}1|p|V), \ell_{\infty}) \text{ iff } T x \text{ exists and is in } \ell_{\infty} \text{ for all } x \in |C_{a}1|p(\mathbb{V}) \). Then \( (t_{nk})_{k=1}^{\infty} \in (|C_{a}1|p|V)|^{\beta} \) and so the conditions (13) and (14) hold.

Moreover, the series \( \Sigma_{k} t_{nk} x_{k} \) converges uniformly in \( n \) and so

\[
\lim_{n \to \infty} \Sigma_{k=0}^{\infty} \lim_{n \to \infty} \Sigma_{k=0}^{\infty} t_{nk} x_{k} = 0, \text{ for each } k \in \mathbb{N}.
\]

To prove sufficiency and necessity of (15), let \( x \in |C_{a}1|p|V| \) be given and consider the operator \( H^{(1)}(\mathbb{V}) : |C_{a}1|p|V| \to \ell_{1} \), defined by (3) with \( p = 1 \). Further, \( x \in |C_{a}1|p|V| \text{ iff } y = H^{(1)}(\mathbb{V})(x) \in \ell_{1} \), and also by (5), let us consider the equality

\[
\sum_{k=1}^{m} t_{nk} x_k = \sum_{k=1}^{m} \sum_{j=1}^{r} t_{nk} \sum_{r=j}^{\infty} \frac{E_{r-j}^{-a-1} E_j^{a}}{r} y_j
\]

\[
= \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{r=j}^{\infty} \frac{E_{r-j}^{-a-1} E_j^{a}}{r} y_j
\]

\[
= \sum_{j=1}^{m} \psi_{(m/j)}^{(n)} y_j;
\]

where

\[
\psi_{(m/j)}^{(n)} = \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{r=j}^{\infty} \frac{E_{r-j}^{-a-1} E_j^{a}}{r} y_j, 1 \leq j \leq m
\]

\[
0, j > m.
\]

Then, since \( y \in \ell_{1} \) and \( \psi^{(n)} = (\psi_{(m/j)}^{(n)}) \in (\ell_{1}, c), \psi^{(n)} \) exists and so the series \( \Sigma_{j} \psi_{(m/j)}^{(n)} y_j \) converges uniformly
for every \( n \in \mathbb{N} \). Hence, by (16), this yields us under the assumption that as \( m \to \infty \) in (17),

\[
T_n(x) = \sum_{j=1}^{\infty} \left( \lim_{m \to \infty} \psi_m^{(n)}(y) \right) = \sum_{j=1}^{\infty} b_{kj}^{(n)}y_j = B_n^{(n)}(y),
\]

where \( b_{kj}^{(n)} = \lim_{m \to \infty} \psi_m^{(n)} \). This means that \( T \in \ell_\infty \) whenever \( x \in \mathbb{C}_{a1}(\mathbb{V}) \) if and only if \( B^{(n)} \) in \( \ell_\infty \) whenever \( y \in \ell_\infty \). Therefore, it follows from part c) of Lemma 3.1 that \( B^{(n)} \in (\ell_\infty, \ell_\infty) \) iff (15) is satisfied, and this step completes the proof of the part i).

Since ii-) and iii-) are proved easily as in i-) using parts a), e-) instead of part c-) of Lemma 3.1, so we omit the detail.

Now, we prove the following result on matrix transformations.

**Theorem 3.8.** Let \( 1 < p < \infty \) and define the matrix \( B^{(p)} = (b_{nk}^{(p)}) \) as in (12). Then,

i-) \( T = (t_{nk}) \in (|C_a|_p(\mathbb{V}), \ell_\infty) \) if and only if (13) holds, and

\[
\sup_m \left\{ \sum_{j=1}^{m} \left| \sum_{k=1}^{m} t_{nk} \frac{E_{r-j}^{r-1}E_r a}{r} j^{1/p} \right|^q \right\} < \infty, \tag{18}
\]

for all \( n \geq 1 \),

ii-) \( T = (t_{nk}) \in (|C_a|_p(\mathbb{V}), c) \) if and only if (13), (18), (19) hold, and

\[
\lim_{n \to \infty} \psi_{m}^{(n)}(y) \text{ exists for each } k \in \mathbb{N}. \tag{19}
\]

iii-) \( T = (t_{nk}) \in (|C_a|_p(\mathbb{V}), c_0) \) if and only if (13), (18), (19) hold, and

\[
\lim_{n \to \infty} b_{nk}^{(p)} = 0, \text{ for each } k \in \mathbb{N}. \tag{20}
\]

**Proof.**

i-) Given \( T = (t_{nk}) \in (|C_a|_p(\mathbb{V}), \ell_\infty) \). Then, equivalently, \( T \) exists and is in \( \ell_\infty \) for all \( x \in |C_a|_p(\mathbb{V}) \). Then \( (t_{nk})_{k=1}^{m} \in (|C_a|_p(\mathbb{V}) \) and so the conditions (13) and (18) hold. Moreover, the series \( \sum_k t_{nk}x_k \) converges uniformly in \( n \) and so (16) holds.

To prove necessity and sufficiency of (19), consider the operator \( H^{(p)}(\mathbb{V}) : |C_a|_p(\mathbb{V}) \to \ell_p \) defined by (3) and let \( x \in |C_a|_p(\mathbb{V}) \) be given. Then \( x \in |C_a|_p(\mathbb{V}) \) iff \( y = H^{(p)}(\mathbb{V})(x) \in \ell_p \). Let us now consider the following equality derived by using the relation (5),

\[
\sum_{k=1}^{m} t_{nk}x_k = \sum_{j=1}^{\infty} \left( \lim_{m \to \infty} \psi_m^{(n)}(y) \right) = \sum_{j=1}^{\infty} b_{kj}^{(n)}y_j = B_n^{(n)}(y),
\]

where

\[
\psi_{m}^{(n)}(y) = \begin{cases} \sum_{k=1}^{m} t_{nk} \frac{E_{r-j}^{r-1}E_r a}{r} j^{1/p} y_j, & 1 \leq j \leq m, \\ 0, & j > m. \end{cases}
\]

Then, since \( y \in \ell_p \) and \( \psi_{m}^{(n)}(y) = (\psi_{m}^{(n)}) \in (\ell_p, c) \), \( \psi_{m}^{(n)} \) exists and so the series \( \sum_j \psi_{m}^{(n)}(y_j) \) converges uniformly for every \( n \in \mathbb{N} \). Therefore, if we pass to the limit in (20) as \( m \to \infty \), then we obtain by (16) that

\[
\sum_{j=1}^{\infty} b_{kj}^{(n)}y_j = B_n^{(n)}(y),
\]

where \( b_{kj}^{(n)} = \lim_{m \to \infty} \psi_{m}^{(n)} \), \( n \geq 1 \). Thus, we deduce that \( T \in \ell_\infty \), whenever \( x \in |C_a|_p(\mathbb{V}) \) if and only if \( B^{(p)} \) in \( \ell_\infty \) whenever \( y \in \ell_p \), which implies that \( B^{(p)} \in (\ell_p, \ell_\infty) \), and so it follows from part d-) of Lemma 3.1 that \( B^{(p)} \in (\ell_p, \ell_\infty) \) iff (19) is satisfied. This completes the proof of part i-) of the theorem.

Since parts ii-) and iii-) can be proved by using the similar way of that used in the proof of part i-) taking account of parts b-) and f-) instead of part d-) of Lemma 3.1, respectively; we leave the details to the reader.

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