A QUATERNIONIC PROOF OF THE
UNIVERSALITY OF SOME QUADRATIC FORMS

JESSE IRA DEUTSCH

ABSTRACT. The problem of finding all quadratic forms over \( \mathbb{Z} \) that represent each positive integer received significant attention in a paper of Ramanujan in 1917. Exactly fifty four quaternary quadratic forms of this type without cross product terms were shown to represent all positive integers. The classical case of the quadratic form that is just the sum of four squares received an alternate proof by Hurwitz using a special ring of quaternions. Here we prove that seven other quaternary quadratic forms can be shown to represent all positive integers by investigation of the corresponding quaternion rings.

1. Introduction

A generalization of the classical Four Squares Theorem concerns the representation of all positive integers by quadratic forms over \( \mathbb{Z} \). Such a quadratic form is called universal. In a 1917 paper, Ramanujan investigated the case of quaternary quadratic forms without cross product terms. It was eventually determined that there were a total of 54 such universal quaternary quadratic forms. Various techniques have been used to demonstrate the universality of these forms including the theory of ternary quadratic forms and the theory of theta functions. See Ramanujan [10] and Fine [2] for details of these demonstrations. Duke [1] is useful for background material and an alternate approach to the problem.

In 1919 Hurwitz demonstrated the classical Four Squares Theorem by means of a special ring of quaternions. The Hurwitz quaternions are the \( \mathbb{Z} \) module with generators \( \{1, i, j, \frac{1}{2}(1 + i + j + k)\} \). Note that each of these generators is a unit.

It is easy to see that this set is closed under multiplication and thus forms a non-commutative subring of the ring of all quaternions with real coefficients. Some of the key properties needed in the proof are that the Hurwitz quaternions comprise a norm Euclidean ring and that the quadratic form under consideration, \( x^2 + y^2 + z^2 + w^2 \), obeys a product law. This product law is simply an alternate statement of the fact that the norm of a product of two quaternions is the product of the norms of each quaternion. Another useful fact is that each Hurwitz quaternion has an associate with rational integer coefficients with respect to the generators \( \{1, i, j, k\} \), though this can be avoided by using a trick of Euler. For details see Hardy and Wright [3, chap. XX] and Herstein [4].
It is natural to consider what other quaternary quadratic forms on the above list of 54 can be proved universal by consideration of appropriate subrings of quaternions. We first focus our consideration on those quadratic forms which obey a product law. One way to generate a product law is to associate a quadratic form with an underlying $\mathbb{Z}$ module of quaternions closed under multiplication such that the norm of an element of the $\mathbb{Z}$ module is the quadratic form in question. For diagonal forms this property is satisfied when the product of any two non-initial coefficients is always an integer square times the third. See Lemma 1 of section 2 for details. In addition we create a second ring, related to the first, which is norm Euclidean. This second ring is actually an order in its corresponding quaternion algebra over $\mathbb{Q}$ (see Vigneras [13]). Finally we show that the product of an arbitrary element of the second ring with properly chosen units is an element of the $\mathbb{Z}$ module or a suitable submodule.

The coefficient condition is satisfied by a total of seven forms on Ramanujan’s list. Two additional forms each have coefficients off by square integer factors from one of the seven quadratic forms, and are amenable to quaternionic techniques. Using the notation $(1, a, b, c)$ for the form $x^2 + ay^2 + bz^2 + cw^2$ these nine forms are

$$(1, 1, 1, 1) \quad (1, 1, 1, 4) \quad (1, 1, 2, 2) \quad (1, 1, 2, 8)$$

$$(1, 1, 3, 3) \quad (1, 2, 2, 4) \quad (1, 2, 3, 6) \quad (1, 2, 4, 8) \quad (1, 2, 5, 10). \quad (1.1)$$

The first form, $(1, 1, 1, 1)$ corresponds to the classical Four Squares Theorem. The second form $(1, 1, 1, 4)$ can be shown universal using the same Hurwitz quaternions and a submodule of the module of integer coefficient quaternions. The forms $(1, 1, 2, 2), (1, 1, 2, 8), (1, 2, 2, 4)$ and $(1, 2, 4, 8)$ all utilize another norm Euclidean order of quaternions in their demonstration of universality.

The forms $(1, 1, 3, 3)$ and $(1, 2, 3, 6)$ each require the introduction of new norm Euclidean orders. Universality follows from an application of computer algebra utilizing the appropriate units. The last form $(1, 2, 5, 10)$ defies analysis by the above technique. The reasons why a straightforward approach will not work in this case are quickly covered.

To fix notation, we use $\mathbb{H}$ to represent the ring of all real quaternions.

$$\mathbb{H} = \{a_1 + a_2i + a_3j + a_4k \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\} \quad (1.2)$$

where $i, j, k$ are the standard noncommutative basis elements with squares equal to $-1$. We use bold type to represent quaternions, so that the typical element of $\mathbb{H}$ as listed in (1.2) can be simply written as $a$. The quaternion conjugate to $a$ is

$$\overline{a} = a_1 - a_2i - a_3j - a_4k \quad (1.3)$$

and it is well known that $a \cdot \overline{a}$ is real with zero coefficient for $i, j$ and $k$. There is a canonical norm for $\mathbb{H}$,

$$N(a) = a_1^2 + a_2^2 + a_3^2 + a_4^2 = a \cdot \overline{a}. \quad (1.4)$$

We use the standard notation $\left(\frac{m, n}{q}\right)$ for the generalized quaternion algebra over the rationals. With algebra generators denoted $\hat{i}$ and $\hat{j}$ we have $\hat{i}^2 = m$ and $\hat{j}^2 = n$ for
We define a quadratic form of four variables to be norm quaternionic if it is the norm of a \( \mathbb{Z} \) module of quaternions that is closed under quaternion multiplication. More specifically we let \( \mathbb{Z} \) module of quaternions with generators \( \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \) and \( \mathbf{g}_4 \) that is closed under multiplication. Then we say that \( f \) is norm quaternionic if

\[
f(x, y, z, w) = N(x \mathbf{g}_1 + y \mathbf{g}_2 + z \mathbf{g}_3 + w \mathbf{g}_4) \tag{2.1}
\]

The typical example is \( x^2 + y^2 + z^2 + w^2 \) which is the norm of the element \( x + y \mathbf{i} + z \mathbf{j} + w \mathbf{k} \) of the module \( \mathbb{Z}[1, \mathbf{i}, \mathbf{j}, \mathbf{k}] \). More generally it is easy to see that \( x^2 + ay^2 + bz^2 + abw^2 \) is the norm of \( x + y \sqrt{a} \mathbf{i} + z \sqrt{b} \mathbf{j} + w \sqrt{ab} \mathbf{k} \) while the elements of the module \( \mathbb{Z} \mathbf{i}, \sqrt{a} \mathbf{i}, \sqrt{b} \mathbf{j}, \sqrt{ab} \mathbf{k} \) satisfy a multiplication law, as will be seen below.

Suppose we have a norm quaternionic form \( f \) that represents each of two integers. Then taking the product of the underlying quaternions we conclude that the product of the two integers is also represented by \( f \). Specifically we have the following.

**Lemma 1.** Suppose \( a, b, c \) are positive integers such that

\[
\sqrt{ab} = k_c \sqrt{c}, \quad \sqrt{ac} = k_b \sqrt{b}, \quad \sqrt{bc} = k_a \sqrt{a} \tag{2.2}
\]

where \( k_a, k_b, \) and \( k_c \) are positive integers. Then the quadratic form \( x^2 + ay^2 + bz^2 + cw^2 \) is norm quaternionic.

**Proof.** We wish to show that the underlying module \( \mathbb{Z}[1, \sqrt{a} \mathbf{i}, \sqrt{b} \mathbf{j}, \sqrt{c} \mathbf{k}] \) is closed under quaternion multiplication, but this is a simple exercise in algebra since

\[
\begin{align*}
(x_1 + y_1 \sqrt{a} \mathbf{i} + z_1 \sqrt{b} \mathbf{j} + w_1 \sqrt{c} \mathbf{k}) \cdot (x_2 + y_2 \sqrt{a} \mathbf{i} + z_2 \sqrt{b} \mathbf{j} + w_2 \sqrt{c} \mathbf{k}) &= (x_1 x_2 - a y_1 y_2 - b z_1 z_2 - c w_1 w_2) \\
&+ \sqrt{a} \mathbf{i} (x_1 y_2 + x_2 y_1 - k_a w_1 z_2 + k_a w_2 z_1) \\
&+ \sqrt{b} \mathbf{j} (x_1 z_2 + x_2 z_1 + k_b w_1 y_2 - k_b w_2 y_1) \\
&+ \sqrt{c} \mathbf{k} (x_1 w_2 + x_2 w_1 + k_c y_1 z_2 - k_c y_2 z_1). \tag{2.3}
\end{align*}
\]

Of the forms listed in (1.1) all are norm quaternionic except \( (1, 1, 1, 4) \) and \( (1, 1, 2, 8) \).
3. Orders, Norm Euclidean property

For our purposes, an order of the rational quaternion algebra \( \left( \frac{m, n}{\mathbb{Q}} \right) \), or simply an order, is a finitely generated \( \mathbb{Z} \) module \( H \) contained in the algebra such that \( H \) is a ring with unity and

\[
\mathbb{Q} \otimes_{\mathbb{Z}} H \simeq \left( \frac{m, n}{\mathbb{Q}} \right).
\]  

(3.1)

This last condition means that the whole quaternion algebra is spanned when the scalars on the module \( H \) are extended from \( \mathbb{Z} \) to \( \mathbb{Q} \). For the general definition and properties of orders of quaternion algebras see Vignéras [13, pp. 19-21].

It is easy to observe that the intersection of an order of a rational quaternion algebra and the real numbers is \( \mathbb{Z} \). This intersection is a finitely generated \( \mathbb{Z} \) module, a ring with unity, and is contained in the rationals. If the intersection contained any rational number outside of \( \mathbb{Z} \) it would also contain rationals with arbitrarily large denominators when written in lowest terms. This contradicts being a finitely generated \( \mathbb{Z} \) module.

It is also the case that the norm map on an order of a rational quaternion algebra takes values in \( \mathbb{Z} \). This is implicit in the equivalent definitions for an order given in Vignéras [13, p. 20]. It is also not difficult to prove directly.

**Lemma 2.** The norm of any element of an order of a rational quaternion algebra is a rational integer.

**Proof.** Let \( H \) be a \( \mathbb{Z} \) order of a rational quaternion algebra, and let \( a \in H \). By Reiner [11, p. 3] \( a \) is integral over \( \mathbb{Z} \). It follows from Reiner [11, p. 6] that the minimal polynomial over \( \mathbb{Q} \) of \( a \) is in \( \mathbb{Z}[x] \).

Note that \( a \) satisfies a quadratic polynomial over \( \mathbb{Q} \), namely,

\[
x^2 - (a + \overline{a})x + a\overline{a} = 0.
\]  

(3.2)

If this polynomial factors, since \( H \) has no zero divisors it follows that \( a \) must be a rational number. By a comment just before the Lemma, it follows that \( a \in \mathbb{Z} \) so that the norm of \( a \) is also in \( \mathbb{Z} \).

If the polynomial above does not factor, it is a monic polynomial of smallest degree satisfied by \( a \) over \( \mathbb{Q} \), hence it is the minimal polynomial for \( a \) and the coefficients must be in \( \mathbb{Z} \). \( \square \)

We say that a ring \( R \) of quaternions is norm Euclidean if there is a \( \delta > 0 \) such that \( \forall q \in \mathbb{H} \) \( \exists a \in R \) with \( N(q - a) \leq \delta < 1 \). An associate of \( r \in R \) is an element of the form \( \varepsilon r \) or \( r \varepsilon \) where \( \varepsilon \) is a unit. The next results follow using the same proofs as in the case of Hurwitz quaternions or in the classical case of the rational integers. Thus we shall only mention the results, and refer to Hardy and Wright [3] for details.

**Lemma 3.** Let \( R \) be a norm Euclidean ring of quaternions and \( a, b \in R \). Suppose \( b \neq 0 \) the zero quaternion. Then there exists \( g, h \in R \) such that

\[
a = gb + h, \quad N(h) < N(b)
\]  

(3.3)

We now need the concept of greatest common divisor. The noncommutativity of \( \mathbb{H} \) must be taken into account.
Definition 4. For any two nonzero elements \(a, b\) of a subring \(R\) of \(\mathbb{H}\), \(d\) is a right greatest common divisor if it divides each of \(a\) and \(b\) on the right, and for any \(e \in R\) that divides these two elements on the right, \(e\) divides \(d\) in the right. In notation

\[
\begin{align*}
(i) \quad & \exists c_1, c_2 \in R \ni a = c_1 d, \ b = c_2 d \\
(ii) \quad & \exists e \in R \ni a = e f_1, \ b = e f_2 \implies \exists g \in R \ni d = g e.
\end{align*}
\]

Lemma 5. The right greatest common divisor is unique up to a factor invertible in \(R\).

Now we show criteria under which a right greatest common divisor must exist.

Lemma 6. Let \(R\) be a ring of quaternions, \(R \subset \mathbb{H}\). Suppose that \(R\) is norm Euclidean and that the norm on \(\mathbb{H}\), \(N\), maps \(R\) into the nonnegative rational integers. Let \(a\) and \(b\) be any nonzero elements of \(R\). Then a right greatest common divisor, \(d\), of \(a\) and \(b\) exists in \(R\) and can be written as a right linear combination of \(a\) and \(b\), i.e.

\[
\exists r, s \in R \ni d = ra + sb
\]

Some information is known about norm Euclidean orders. For orders of a ring whose field of fractions is a number field, the norm Euclidean property implies that the order is maximal (see Vignéras [13, p. 91]). In the case of an algebra “totalement définie” over \(\mathbb{Q}\) the norm Euclidean property implies that the reduced discriminant is 2, 3 or 5 (see Vignéras [13, p. 156]). Criteria exist concerning the Euclidean property for functions other than the norm. For more details on this aspect of the theory, see R. Markanda and V. Albis-Gonzales [8].

4. The form \((1, 1, 1, 4)\)

We start with the ring of Hurwitz quaternions \(H_{1,1,1} = \mathbb{Z}[i,j,\frac{1}{2}(1+i+j+k)]\) and recall from the demonstration of the classical Four Squares Theorem that for every positive integer \(n\) there exists a quaternion \(q \in H_{1,1,1}\) such that \(N(q) = n\). Also every element of \(H_{1,1,1}\) has an associate in \(\mathbb{Z}[1, i, j, k]\). See Hardy and Wright [3], Herstein [4] or Hurwitz [6] for details. An analogue of these results suffices for a demonstration of the universality of the form \((1, 1, 1, 4)\) despite the fact that this form is not itself norm quaternionic. We set \(h = \frac{1}{2}(1+i+j+k)\) for brevity.

Lemma 7. Every element of \(H_{1,1,1}\) has an associate in the \(\mathbb{Z}\)-module \(\mathbb{Z}[1, i, j, 2k]\).

Proof. Write a typical \(a \in H_{1,1,1}\) in the form

\[
a = 4(c_1 \cdot 1 + c_2 i + c_3 j + c_4 h) + d_1 \cdot 1 + d_2 i + d_3 j + d_4 h \quad (4.1)
\]

where \(c_1, \ldots, c_4\) are rational integers and \(d_1, \ldots, d_4 \in \{0, 1, 2, 3\}\). Let \(d = d_1 + d_2 i + d_3 j + d_4 h\) and define \(c\) analogously. Then \(c \in H_{1,1,1}\). Computation shows there exists a unit \(u \in H_{1,1,1}\) for which \(d \cdot u \in \mathbb{Z}[1, i, j, 2k]\). See Table I for a sample of these calculations. Since \(c \cdot u\) is an element of the ring \(H_{1,1,1}\) clearly \(4c \cdot u\) is an element of \(\mathbb{Z}[1, i, j, 2k]\). Hence

\[
a u = 4cu + du \quad (4.2)
\]
Table I. Some associates of elements of $H_{1,1,1}$ that are members of $\mathbb{Z}[1, i, j, 2k]$

| Quaternion $q$ | Unit $u$ | Associate $q \cdot u$ |
|---------------|----------|---------------------|
| $3 + 2j + 2h$ | $2h - j - i - 1$ | $4k - j + 3i - 1$ |
| $3 + 2j + 3h$ | $h$       | $2k + 4j + 4i - 1$ |
| $3 + 3j$      | $1$       | $3j + 3$            |
| $3 + 3j + h$  | $h - 1$   | $3i - 4$            |
| $3 + 3j + 2h$ | $i$       | $-4k + j + 4i - 1$  |
| $3 + 3j + 3h$ | $h - 1$   | $3i - 6$            |
| $3 + i$       | $1$       | $i + 3$             |
| $3 + i + h$   | $h - 1$   | $2k + j + i - 3$    |
| $3 + i + 2h$  | $j$       | $2k + 4j - i - 1$   |
| $3 + i + 3h$  | $h - 1$   | $2k + j + i - 5$    |
| $3 + i + j$   | $1$       | $j + i + 3$         |

is also an element of $\mathbb{Z}[1, i, j, 2k]$. □

Theorem 8. The form $(1, 1, 1, 4)$ is universal.

Proof. Fix any positive integer $n$. By the proof of the classical Four Squares Theorem via quaternions, there exists a $q \in H_{1,1,1}$ for which $N(q) = n$ (see Hardy and Wright [3]). By the previous Lemma there exists a unit $u \in H_{1,1,1}$ such that $qu \in \mathbb{Z}[1, i, j, 2k]$. Write

$$qu = x + yi + zj + w2k$$

with $x, y, z, w$ rational integers. Then

$$n = N(q) = N(qu) = x^2 + y^2 + z^2 + 4w^2.$$ □

5. The Forms $(1, 1, 2, 2), (1, 1, 2, 8), (1, 2, 2, 4), (1, 2, 4, 8)$

We note that $(1, 1, 2, 8), (1, 2, 2, 4)$ and $(1, 2, 4, 8)$ differ from the first form listed above by square factors in the coefficients. We call $(1, 1, 2, 2)$ the base form and construct the $\mathbb{Z}$ module and quaternion ring for this form. The non-base forms will then have their universality demonstrated in a manner similar to the derivation of the universality of $(1, 1, 1, 4)$ from the Hurwitz quaternions.

Since clearly

$$N\left(x + yi + z\sqrt{2}j + w\sqrt{2}k\right) = x^2 + y^2 + 2z^2 + 2w^2.$$ (5.1)
A QUATERNIONIC PROOF OF THE UNIVERSALITY OF SOME QUADRATIC FORMS

Table II. Multiplication Table for $H_{1,2,2}$

|   | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|-------|-------|-------|-------|
| $v_1$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
| $v_2$ | $v_2$ | $-v_1$ | $v_4 - v_1$ | $-v_3 + v_2$ |
| $v_3$ | $v_3$ | $-v_4 + v_2$ | $v_3 - v_1$ | $v_2$ |
| $v_4$ | $v_4$ | $v_3 - v_1$ | $v_4 + v_3 - v_2 - v_1$ | $v_4 - v_1$ |

the $\mathbb{Z}$ module associated with this form is $\mathbb{Z}[1, i, \sqrt{2}j, \sqrt{2}k]$. We look for units with low denominators analogous to Hurwitz’s special quaternion $\frac{1}{2}(1 + i + j + k)$. Choosing a set of these which are closed under multiplication we have

$$v_1 = 1, \ v_2 = i, \ v_3 = \frac{1}{2} \left(1 + i + \sqrt{2}j \right), \ v_4 = \frac{1}{2} \left(1 + i + \sqrt{2}k \right). \quad (5.2)$$

We set $H_{1,2,2} = \mathbb{Z}[v_1, v_2, v_3, v_4]$. We note that $H_{1,2,2} \cap \mathbb{R} = \mathbb{Z}$ and the canonical norm maps $H_{1,2,2}$ into the nonnegative integers. It is easy to see that this module is closed under quaternion conjugation.

**Lemma 9.** The $\mathbb{Z}$ module $H_{1,2,2}$ is closed under multiplication and is thus an order in the quaternion algebra $\left(\frac{-1}{-1}\right)$.

*Proof.* Table II, generated by computer algebra, shows that $H_{1,2,2}$ is closed under multiplication. It is easy to see that rational linear combinations of the generators span all of the associated quaternion algebra. □

**Lemma 10.** For any rational prime $p \neq 2$ there exist rational integers $a$ and $b$ such that $2a^2 + 2b^2 + 1 \equiv 0 \pmod{p}$.

*Proof.* Since $p \neq 2$ the sets $\{2a^2\}$ and $\{-1 - 2b^2\}$ each contain $1 + \frac{p-1}{2}$ different elements as $a$ and $b$ run through all the integers modulo $p$. This is because exactly half the integers from 1 to $p - 1$ are squares modulo $p$. If these two sets had no intersection modulo $p$ we would have a total of $2 \cdot \frac{p-1}{2} = p + 1$ different elements modulo $p$, a contradiction. Hence there exists $a, b \in \mathbb{Z}$ such that $2a^2 \equiv -1 - 2b^2 \pmod{p}$. □

An important step in the proof of the Four Squares Theorem by Hurwitz quaternions is the result that any such quaternion has an associate with integer coordinates with respect to the basis $1, i, j, k$ (see Hardy and Wright [3, chap. XX]). The analogous result for $H_{1,2,2}$ can be obtained by first listing the units in the ring.

**Lemma 11.** The units of $H_{1,2,2}$ are

$$\pm \left\{1, \ i, \ \frac{1}{2} \pm \frac{1}{2}i \pm \frac{\sqrt{2}}{2}j, \ \frac{1}{2} \pm \frac{1}{2}i \pm \frac{\sqrt{2}}{2}k, \ \frac{\sqrt{2}}{2}j \pm \frac{\sqrt{2}}{2}k \right\} \quad (5.3)$$
or in terms of basis elements
\[ \pm \left\{ v_1, v_2, v_3, v_4 - v_1, v_3 - v_2, v_3 - v_2 - v_1, v_4 - v_1, v_4 - v_2, v_4 - v_2 - v_1 \right\}. \quad (5.4) \]

Proof. To find all the units of \( H_{1,2,2} \) write a typical element of the ring in the form
\[ a_1 + a_2i + a_3\sqrt[4]{2}j + a_4\sqrt[4]{2}k. \] The coefficients \( a_1, \ldots, a_4 \) are rational numbers with denominators 1 or 2. To be a unit, the sum \( a_1^2 + a_2^2 + 2a_3^2 + 2a_4^2 \) must equal one. Thus the absolute values of \( a_3 \) and \( a_4 \) are less than or equal to \( \sqrt{2}/2 \) while the absolute values of \( a_1 \) and \( a_2 \) are less than or equal to one. \( a_3 \) and \( a_4 \) can only take the values 0 or \( \pm \frac{1}{2} \) while \( a_1 \) and \( a_2 \) can take on the additional values of \( \pm 1 \).

If \( a_3 \) and \( a_4 \) are both zero we have the units of the Gaussian integers \( \pm 1, \pm i \). If \( a_3 \) is zero and \( a_4 \) is not zero then we have \( a_1^2 + a_2^2 = \frac{1}{2} \) which forces both \( a_1 \) and \( a_2 \) to be nonzero. The only solution with half integers is \( a_1 = \pm \frac{1}{2} \) and \( a_2 = \pm \frac{1}{2} \). The situation where \( a_4 \) is zero and \( a_3 \) is not zero is completely analogous to the previous case. Finally if \( a_3 \) and \( a_4 \) are both nonzero then their squares are \( \frac{1}{4} \) and this forces \( a_1 = a_2 = 0 \). The units in this case correspond to \( a_3 = \pm \frac{1}{2} \) and \( a_4 = \pm \frac{1}{2} \). \( \square \)

Lemma 12. Every element of \( H_{1,2,2} \) has an associate in \( \mathbb{Z}[1, i, \sqrt{2}j, \sqrt{2}k] \).

Proof. Write \( a = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \) in the form
\[ a = 2(c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4) + d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4 \quad (5.5) \]
where \( d_1, \ldots, d_4 \in \{0, 1\} \). Let \( c = \sum_{t=1}^{4} c_t v_t \) and \( d = \sum_{t=1}^{4} d_t v_t \). Take the unit \( u \) from Table III such that \( d u \in \mathbb{Z}[1, i, \sqrt{2}j, \sqrt{2}k] \). Then
\[ a u = 2 c u + d u = 2 \hat{c} + d u \quad (5.6) \]
where \( \hat{c} \in H_{1,2,2} \) as \( H_{1,2,2} \) is a ring. Since twice any element of \( H_{1,2,2} \) must be in \( \mathbb{Z}[1, i, \sqrt{2}j, \sqrt{2}k] \) and \( d u \) is also in this module, we find that \( a u \) is in the module, the result wanted. \( \square \)

Any number of the form \( u_1 a u_2 \) with \( u_1 \) and \( u_2 \) units will be called a two sided associate of \( a \).

Lemma 13. Every element of \( H_{1,2,2} \) has a two sided associate in each of the modules
\[ \mathbb{Z}[1, i, \sqrt{2}j, 2 \sqrt{2}k], \quad \mathbb{Z}[1, 2i, \sqrt{2}j, \sqrt{2}k], \quad \mathbb{Z}[1, 2i, \sqrt{2}j, 2 \sqrt{2}k]. \quad (5.7) \]

Proof. The proof is similar to the previous demonstration. Write \( a = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \) in the form
\[ a = 4(c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4) + d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4 \quad (5.8) \]
where \( d_1, \ldots, d_4 \in \{0, 1, 2, 3\} \). Let \( c = \sum_{t=1}^{4} c_t v_t \) and \( d = \sum_{t=1}^{4} d_t v_t \). Computation shows that there exists units \( u_1 \) and \( u_2 \) in \( H_{1,2,2} \) such that \( u_1 d u_2 \in \mathbb{Z}[1, i, \sqrt{2}j, 2 \sqrt{2}k] \). We have
\[ u_1 a u_2 = 4 u_1 c u_2 + u_1 d u_2 = 4 \hat{c} + u_1 d u_2 \quad (5.9) \]
where \( \hat{c} \in H_{1,2,2} \) as \( H_{1,2,2} \) is a ring. Since four times any element of \( H_{1,2,2} \) must be in \( \mathbb{Z}[2, 2i, \sqrt{2}j, 2 \sqrt{2}k] \) we find \( u_1 a u_2 \) is in \( \mathbb{Z}[1, i, \sqrt{2}j, 2 \sqrt{2}k] \).

A similar argument holds in the other two cases of the Lemma. \( \square \)


Table III. Associates of elements of $H_{1,2,2}$ that are members of $\mathbb{Z}[1,i,\sqrt{2}j,\sqrt{2}k]$

| Quaternion $q$ | Unit $u$ | Associate $q \cdot u$ |
|---------------|----------|------------------------|
| $v_1$         | $v_1$    | $1$                    |
| $v_2$         | $v_2$    | $-1$                   |
| $v_3$         | $v_4$    | $i$                    |
| $v_4$         | $v_3 - v_2 - v_1$ | $-i$       |
| $v_1 + v_2$   | $v_1$    | $1 + i$                |
| $v_1 + v_3$   | $v_3$    | $i + \sqrt{2}j$       |
| $v_1 + v_4$   | $v_4$    | $i + \sqrt{2}k$       |
| $v_2 + v_3$   | $v_4 - v_1$ | $-1 - \sqrt{2}j$     |
| $v_2 + v_4$   | $v_3 - v_2$ | $1 + \sqrt{2}k$       |
| $v_3 + v_4$   | $v_4 - v_3$ | $i - \sqrt{2}j$       |
| $v_1 + v_2 + v_3$ | $v_4 - v_2 - v_1$ | $-i - \sqrt{2}j + \sqrt{2}k$ |
| $v_1 + v_2 + v_4$ | $v_3$ | $i + \sqrt{2}j + \sqrt{2}k$ |
| $v_1 + v_3 + v_4$ | $v_4$ | $2i + \sqrt{2}k$       |
| $v_2 + v_3 + v_4$ | $v_3$ | $-1 + i + \sqrt{2}j + \sqrt{2}k$ |
| $v_1 + v_2 + v_3 + v_4$ | $v_4 + v_3 - v_2 - v_1$ | $-1 + 2\sqrt{2}k$ |

**Lemma 14.** $H_{1,2,2}$ is norm Euclidean.

**Proof.** Fix any $q \in \mathbb{H}$. We may choose $a_4v_4$ such that $a_4 \in \mathbb{Z}$ and the absolute value of the coefficient of $k$ of $q - a_4v_4$ is less than or equal to $\sqrt{2}/4$. Similarly we find $a_3v_3$ with $a_3 \in \mathbb{Z}$ such that $q - a_3v_3 - a_4v_4$ additionally has the absolute value of the coefficient of $j$ less than or equal to $\sqrt{2}/4$.

Then we need only subtract off a further $a_1 + a_2i$ with rational integral $a_1$, $a_2$ to force the coefficients of $i$ and $1$ to be less than $1/2$ in absolute value. Consequently

$$N(q - a_1v_1 - a_2v_2 - a_3v_3 - a_4v_4) \leq 2 \left(\frac{1}{2}\right)^2 + 2 \left(\frac{\sqrt{2}}{4}\right)^2 = \frac{3}{4}. \quad (5.10)$$

Thus the norm Euclidean property holds. □

**Theorem 15.** Every rational prime $p$ can be represented by the form $(1, 1, 2, 2)$.

**Proof.** If $p = 2$ this is trivial. Otherwise, by Lemma 10 there exists $a, b \in \mathbb{Z}$ such that $2a^2 + 2b^2 + 1 \equiv 0 \pmod{p}$. We may choose $|a|, |b| \leq (p - 1)/2$. Thus

$$2a^2 + 2b^2 + 1 \leq 2 \left(\frac{p-1}{4}\right)^2 + 2 \left(\frac{p-1}{4}\right)^2 + 1 = (p-1)^2 + 1 = p^2 - 2p + 2 < p^2 \quad (5.11)$$

for all $p > 1$. Hence $2a^2 + 2b^2 + 1 = pr$ where $0 < r < p$. Set $a = 1-a\sqrt{2}j + b\sqrt{2}k$
and note $N(a) = 2a^2 + 2b^2 + 1$. Calculate
\[
2b v_4 - 2a v_3 + v_1 - (b - a)(v_1 + v_2)
= b (1 + i + \sqrt{2} k) - a (1 + i + \sqrt{2} j) + 1 - (b - a) (1 + i) 
= b\sqrt{2}k - a\sqrt{2}j + 1 = a.
\] (5.12)

So $a \in H_{1,2,2}$. Since $H_{1,2,2}$ is norm Euclidean, right greatest common divisors exist. Suppose the right greatest common divisor of $a$ and $p$ is a unit $u$. Then there exists $s, t \in H_{1,2,2}$ such that
\[
u = sa + tp.
\] (5.13)

Taking conjugates we find
\[
u^* = \overline{sa + tp}
\] (5.14)
so by multiplying the previous two equations we have for some $q \in H_{1,2,2}$
\[
u^* = u^* = sa\overline{s} + tp
\] (5.15)
but $H_{1,2,2} \cap \mathbb{R} = \mathbb{Z}$, a contradiction. Hence the right greatest common divisor of $a$ and $p$ is not a unit. Call this right greatest common divisor $d$. We may write
\[
u = f d, \quad p = g d, \quad N(d) > 1
\] (5.16)
for some $f, g \in H_{1,2,2}$. But then $p^2 = N(p) = N(g)N(d)$ so $N(d)$ must divide $p^2$. Also $pr = N(a) = N(f)N(d)$ so $N(d)$ divides $pr$. Thus $N(d)$ divides the greatest common divisor of $p^2$ and $pr$ in the ring $\mathbb{Z}$, which is simply $p$. Since $N(d)$ cannot equal one, it must equal $p$.

Choose an associate of $d$ that is in the module $\mathbb{Z}[1, i, \sqrt{2} j, \sqrt{2} k]$. Denote this associate $\hat{d}$. Let
\[
\hat{d} = a_1 + a_2i + a_3\sqrt{2}j + a_4\sqrt{2}k, \quad a_1, a_2, a_3, a_4 \in \mathbb{Z}.
\] (5.17)

Then
\[
p = N(d) = N(\hat{d}) = a_1^2 + a_2^2 + 2a_3^2 + 2a_4^2.
\] (5.18)

**Theorem 16.** Every positive integer can be represented by the form $(1, 1, 2, 2)$.

**Proof.** This follows immediately as the quadratic form in question is norm quaternionic and also represents the integer one. □
Theorem 17. Every positive integer can be represented by each of the forms 
\((1, 1, 2, 8), (1, 2, 2, 4), (1, 2, 4, 8)\).

Proof. The proof for the case \((1, 1, 2, 2)\) showed that for each positive integer \(n\) there exists \(q \in H_{1,2,2}\) such that \(N(q) = n\). For the first quadratic form of the Theorem, choose a two sided associate of \(q\) that is in the module \(\mathbb{Z}[1, i, \sqrt{2}j, 2\sqrt{2}k]\). Then there exist units \(u_1, u_2 \in H_{1,2,2}\) such that \(u_1 q u_2\) is in this module. It follows that
\[
N(x + y\sqrt{2}i + z\sqrt{3}j + w\sqrt{6}k) = x^2 + 2y^2 + 3z^2 + 6w^2
\]
for integers \(x, y, z, w\). Similar argumentation demonstrates the Theorem in the case of the other two quadratic forms.

Alternatively, one may proceed from the representation of rational primes \(p\) to universality directly in the case of the norm Euclidean forms \((1, 2, 2, 4)\) and \((1, 2, 4, 8)\). By the argument in the previous paragraph, each rational prime and 1 is represented by each of the above two norm Euclidean forms. Using the product law for norm Euclidean forms, universality follows immediately. \(\square\)

6. The form \((1, 2, 3, 6)\)

The proof follows closely that of the form \((1, 1, 2, 2)\). Considering the identity
\[
N(x + y\sqrt{2}i + z\sqrt{3}j + w\sqrt{6}k) = x^2 + 2y^2 + 3z^2 + 6w^2.
\]
the \(\mathbb{Z}\) module associated with this form is \(\mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k]\). Again we look for likely units and find
\[
w_1 = 1, \quad w_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}j, \quad w_3 = \frac{1}{2} + \frac{\sqrt{3}}{6}j + \frac{\sqrt{6}}{3}k, \quad w_4 = \frac{\sqrt{2}}{2}i + \frac{\sqrt{6}}{3}j + \frac{\sqrt{6}}{6}k.
\]

We set \(H_{2,3,6} = \mathbb{Z}[w_1, w_2, w_3, w_4]\).

Lemma 18. The \(\mathbb{Z}\) module \(H_{2,3,6}\) is closed under multiplication and is thus an order in the quaternion algebra \((-\frac{2}{3}, -\frac{3}{q})\).

Proof. Table IV, generated by computer algebra, demonstrates closure under multiplication. The proof follows as in Lemma 9 above. \(\square\)

It follows that \(H_{2,3,6} \cap \mathbb{R} = \mathbb{Z}\) while the canonical norm maps \(H_{2,3,6}\) into the non-negative integers. It is also easy to see that this module is closed under quaternion conjugation.

Lemma 19. For any rational prime \(p\) there exist rational integers \(a\) and \(b\) such that \(a^2 + 2b^2 = 3 \equiv 0 \pmod{p}\).

Proof. If \(p = 2\) this is trivial. Suppose \(p \neq 2\). The result follows from the same line of reasoning as in Lemma 10 using the sets \(\{a^2\}\) and \(\{-3 - 2b^2\}\) instead. \(\square\)
Write a typical element of $\mathbb{Z}$ statement suffices for our purposes.

Lemma 20. The units of $H_{2,3,6}$ are

$$\pm \left\{ \begin{array}{c}
w_1, w_2, w_3, w_4, w_4 - w_3, w_3 - w_2, \\
w_2 - w_1, w_3 - w_1, w_4 - w_2, \\
w_4 - w_2 + w_1, w_4 - w_3 - w_2 + w_1, w_4 - w_3 + w_1 \end{array} \right\}. \quad (6.3)$$

Proof. Write a typical element of $H_{2,3,6}$ in the form $w = a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4$ with $a_1, \ldots, a_4 \in \mathbb{Z}$. The coefficient of $i$ in $w$ is $a_4\sqrt{2}$. To be a unit the sums of the squares of the $i, j, k$ and real coefficients must equal one. Thus $\frac{1}{7}a_4^2 \leq 1$. This forces $|a_4| \leq 1$.

For the $k$ coefficient of $w$ we have $a_4\sqrt{2} + a_3\sqrt{6} = (a_4 + 2a_3)\frac{\sqrt{6}}{2}$, the square of which must be less than or equal to one. Taking square roots, we have $|a_4 + 2a_3| \leq \sqrt{6}$ so

$$2|a_3| - |a_4| \leq |a_4 + 2a_3| \leq \sqrt{6}$$

$$\implies |a_3| \leq \frac{1}{2}(\sqrt{6} + 1) = 1.72\ldots \quad (6.4)$$

which implies that $|a_3| \leq 1$. Similar reasoning with the $j$ coefficient yields $|a_2| \leq 2$ while working with the real coefficient gives $|a_1| \leq 2$.

A computer scan was run using a computer algebra system to find all the units, which are listed in (6.3). \qed

Computation implies that not every element of $H_{2,3,6}$ has an associate which is an element of the module $\mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k]$. However, the following weaker statement suffices for our purposes.

Lemma 21. Every element of $H_{2,3,6}$ has a two sided associate in the module $\mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k]$.

Proof. Write $a = a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4$ in the form

$$a = 6(c_1w_1 + c_2w_2 + c_3w_3 + c_4w_4) + d_1w_1 + d_2w_2 + d_3w_3 + d_4w_4 \quad (6.5)$$

where $d_1, \ldots, d_4 \in \{0, 1, 2, 3, 4, 5\}$. Let $c = \sum_{t=1}^{4}c_tw_t$ and $d = \sum_{t=1}^{4}d_tw_t$.

Computation shows that for every possible value of $d$ there exist units $u_1, u_2$ in...
A QUATERNIONIC PROOF OF THE UNIVERSALITY OF SOME QUADRATIC FORMS

Table V. Some two sided associates of elements of $H_{2,3,6}$ that are members of $\mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k]$

| Unit $u_1$ | Quaternion $q$ | Unit $u_2$ | $u_1 \cdot q \cdot u_2$ |
|------------|----------------|------------|------------------------|
| $w_3$      | $2w_2 + w_3 + 4w_4$ | $w_4 - w_2$ | $-2\sqrt{6}k + \sqrt{2}i + 3$ |
| $w_3 - w_4$ | $2w_2 + w_3 + 5w_4$ | $w_3 - w_1$ | $\sqrt{6}k + 4\sqrt{2}i - 3$ |
| $w_1$      | $2w_2 + 2w_3$ | $w_3$ | $\sqrt{6}k + \sqrt{3}j + \sqrt{2}i - 1$ |
| $w_3$      | $2w_2 + 2w_3 + w_4$ | $w_2$ | $-2\sqrt{2}i - 3$ |
| $w_1$      | $2w_2 + 2w_3 + 2w_4$ | $w_1$ | $\sqrt{6}k + 2\sqrt{3}j + \sqrt{2}i + 2$ |
| :         | :              | :         | :                      |
| $w_1$      | $w_1 + w_2 + 3w_4$ | $w_3$ | $\sqrt{6}k + 2\sqrt{2}i - 1$ |
| $w_3 - w_4$ | $w_1 + w_2 + 4w_4$ | $w_1$ | $2\sqrt{3}j - 2\sqrt{2}i + 3$ |
| $w_2$      | $w_1 + w_2 + 5w_4$ | $w_4 - w_1$ | $-3\sqrt{3}j + \sqrt{2}i + 2$ |
| $w_1$      | $w_1 + w_2 + w_3$ | $w_4$ | $\sqrt{3}j + \sqrt{2}i - 1$ |
| $w_1$      | $w_1 + w_2 + w_3 + w_4$ | $w_4$ | $\sqrt{3}j + \sqrt{2}i - 2$ |
| :         | :              | :         | :                      |

$H_{2,3,6}$ such that $u_1 d u_2$ is an element of $\mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k]$. Table V contains a section of the relevant computer generated output.

Consequently

$$u_1 a u_2 = 6u_1 c u_2 + u_1 d u_2 = 6\hat{c} + u_1 d u_2 \quad (6.6)$$

where $\hat{c} \in H_{2,3,6}$ as $H_{2,3,6}$ is a ring. Since six times any element of $H_{2,3,6}$ must be in $\mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k]$ and $u_1 d u_2$ is also in this module, we find that $u_1 a u_2$ is in the module, the result wanted. □

Lemma 22. $H_{2,3,6}$ is norm Euclidean.

Proof. Fix any $q \in \mathbb{H}$. We may choose $a_4 w_4$ such that $a_4 \in \mathbb{Z}$ and the absolute value of the coefficient of $i$ of $q - a_4 w_4$ is less than or equal to $\sqrt{2}/4$. Similarly we find $a_3 w_3$ with $a_3 \in \mathbb{Z}$ such that $q - a_3 w_3 - a_4 w_4$ additionally has the absolute value of the coefficient of $k$ less than or equal to $\sqrt{6}/6$.

Now we find an integer multiple of $w_2$ so that the $j$ coefficient of $q - a_2 w_2 - a_3 w_3 - a_4 w_4$ is additionally less than $\sqrt{3}/4$ in absolute value. We need only adjust the above difference by a rational integer $a_1 w_1$ to force the real part less than one half in absolute value. All together we have

$$N\left(q - \sum_{n=1}^{4} a_n w_n\right) \leq \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{\sqrt{6}}{6}\right)^2 = \frac{35}{48}. \quad (6.7)$$

Thus the norm Euclidean property holds. □
Theorem 23. Every rational prime \( p \) can be represented by the form \((1, 2, 3, 6)\).

Proof. For \( p = 2 \) this is trivial. For odd primes \( p \), by Lemma 19 there exists \( a, b \in \mathbb{Z} \) such that \( a^2 + 2b^2 + 3 \equiv 0 \pmod{p} \). By reasoning similar to the previous case we show that there exist integers \( a \) and \( b \) for which \( a^2 + 2b^2 + 3 = pr \) where \( 0 < r < p \).

Setting \( a = a - b\sqrt{2}i + \sqrt{3}j \) it follows that \( N(a) = a^2 + 2b^2 + 3 = pr \).

Some algebra shows
\[
(a - b - 1)w_1 + (b + 2)w_2 + b w_3 - 2bw_4 = a - b\sqrt{2}i + \sqrt{3}j.
\] (6.8)

Thus \( a \in H_{2,3,6} \) and the proof proceeds as in the previous case until the issue of associates arises. As before we find \( d \in H_{2,3,6} \) such that \( N(d) = p \).

Choose units \( u_1 \) and \( u_2 \) such that \( u_1 du_2 \) is in the module \( \mathbb{Z}[1, \sqrt{2}i, \sqrt{3}j, \sqrt{6}k] \). Let
\[
u_1 du_2 = a_1 + a_2\sqrt{2}i + a_3\sqrt{3}j + a_4\sqrt{6}k, \quad a_1, a_2, a_3, a_4 \in \mathbb{Z}.
\] (6.9)

Then
\[
p = N(d) = N(u_1 d u_2) = a_1^2 + 2a_2^2 + 3a_3^2 + 6a_4^2.
\] (6.10)

Theorem 24. Every positive integer can be represented by the form \((1, 2, 3, 6)\).

Proof. Again, this follows immediately as the quadratic form in question is norm quaternionic and also represents the integer one. \( \square \)

7. The form \((1, 1, 3, 3)\)

The demonstration of the universal property of the form \((1, 1, 3, 3)\) follows the previous cases closely with one small difficulty. Two sided units only reduce matters so far, and we need an extension of a trick of Euler to transform a representation of \( 2n \) to one of \( n \). Set
\[
x_1 = 1, \quad x_2 = \frac{1}{2}i - \frac{\sqrt{3}}{2}k, \quad x_3 = \frac{1}{4}i + \frac{\sqrt{3}}{4}j - \frac{\sqrt{3}}{4}k, \quad x_4 = \frac{1}{2} + \frac{3}{4}i + \frac{\sqrt{3}}{4}k
\] (7.1)

and let \( H_{1,3,3} = \mathbb{Z}[x_1, x_2, x_3, x_4] \).

Lemma 25. The \( \mathbb{Z} \) module \( H_{1,3,3} \) is closed under multiplication and is thus an order in the quaternion algebra \( \left(\frac{-1, -3}{q}\right) \).

Proof. Table VI, generated by computer algebra, demonstrates closure under multiplication. Consideration of rational linear combinations of the basis elements completes the proof. \( \square \)
Table VI. Multiplication Table for $H_{1,3,3}$

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| $x_2$ | $x_2$ | $-x_1$ | $x_4 - x_1$ | $-x_3 + x_2$ |
| $x_3$ | $x_3$ | $-x_4$ | $-x_1$ | $x_2$ |
| $x_4$ | $x_4$ | $x_3$ | $x_3 - x_2$ | $x_4 - x_1$ |

Just as for the previously considered $\mathbb{Z}$ modules $H_{a,b,c}$ we note that $H_{1,3,3} \cap \mathbb{R} = \mathbb{Z}$. Further the canonical norm on quaternions maps $H_{1,3,3}$ into the nonnegative rational integers.

**Lemma 26.** The units in $H_{1,3,3}$ are

$$\pm \{x_1, x_2, x_3, x_4 - x_1, x_3 - x_2\}$$  \hspace{2cm} (7.2)

**Proof.** We wish to solve

$$1 = N(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_1 a_4 + a_2 a_3$$  \hspace{2cm} (7.3)

or equivalently

$$2 = 2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_4^2 + 2a_1 a_4 + 2a_2 a_3$$

$$= (a_1 + a_4)^2 + a_2^2 + (a_1 + a_3)^2 + a_2^2 + a_3^2.$$  \hspace{2cm} (7.4)

We must find those integers $a_1$ and $a_4$ such that $(a_1 + a_4)^2 + a_2^2 + a_3^2 \leq 2$ and similarly for $a_2$ and $a_3$. This yields the list

$$(a_1, a_4) \in \{(1, -1), (0, -1), (0, 1), (1, 0), (0, 0), (-1, 0), (-1, 1)\}$$  \hspace{2cm} (7.5)

and the same for the pair $(a_2, a_3)$. Putting together those four tuples for which the expression in (7.4) adds up to two yields the following as the coefficients of all possible units.

$$(a_1, a_2, a_3, a_4) = \pm \left\{ (1, 0, 0, -1), (1, 0, 0, 0), (0, 0, 0, 1) \right\}$$  \hspace{2cm} (7.6)

which are the units in (7.2). \hspace{1cm} $\Box$

**Lemma 27.** For every element $a$ in $H_{1,3,3}$ there exists two units $u_1, u_2 \in H_{1,3,3}$ such that $u_1 2a u_2$ is an element of $\mathbb{Z}[1, i, \sqrt{3}j, \sqrt{3}k]$.

**Proof.** Write $a = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$ in the form

$$a = 2(c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4) + d_1 x_1 + d_2 x_2 + d_3 x_3 + d_4 x_4$$  \hspace{2cm} (7.7)

where $d_1, \ldots, d_4 \in \{0, 1\}$ and $c_1, \ldots, c_4$ are rational integers. Let $c = \sum_{t=1}^{4} c_t x_t$ and $d = \sum_{t=1}^{4} d_t x_t$. Computation shows that for every possible value of $d$ there exist units $u_1, u_2$ in $H_{1,3,3}$ such that $u_1 2d u_2$ is an element of $\mathbb{Z}[1, i, \sqrt{3}j, \sqrt{3}k]$. We consider $u_1 2d u_2$ and the demonstration concludes as in the previous cases. \hspace{1cm} $\Box$

We now proceed to an extension of a trick of Euler that was used in demonstrating the classical Four Squares Theorem.
Lemma 28. Suppose $n$ is an integer for which $2n$ can be represented by the quadratic form $(1, 1, 3, 3)$. Then $n$ can also be represented by this form.

Proof. Start with $2n = x^2 + y^2 + 3z^2 + 3w^2$ with $x, y, z, w \in \mathbb{Z}$. Consider this modulo 2.

$$0 \equiv x^2 + y^2 + 3z^2 + 3w^2 \equiv x + y + z + w \pmod{2}. \quad (7.8)$$

If $x$ and $y$ have the same parity, then so do $z$ and $w$. We can directly use Euler’s trick. Since $(x+y)/2, (x-y)/2, (z+w)/2$ and $(z-w)/2$ are integers we compute

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + \left(\frac{z+w}{2}\right)^2 + \left(\frac{z-w}{2}\right)^2 = \frac{1}{2}2n = n. \quad (7.9)$$

The other case is if $x$ and $y$ do not have the same parity. With no loss of generality, assume $x$ is even and $y$ is odd. Then $1 \equiv z + w \pmod{2}$ so $z$ and $w$ must have different parities too. Again with no loss of generality suppose $z$ is odd and $w$ is even. Consider $y^2 + 3z^2$ where $y$ and $z$ are both odd. Set

$$y_1 = \frac{y + 3z}{2}, \quad z_1 = \frac{y - z}{2}, \quad y_2 = \frac{y - 3z}{2}, \quad z_2 = \frac{y + z}{2}. \quad (7.10)$$

Then

$$y_1^2 + 3z_1^2 = \frac{1}{4} (y^2 + 6yz + 9z^2 + 3(y^2 - 2yz + z^2))$$

$$= \frac{1}{4} (4y^2 + 12z^2) = y^2 + 3z^2. \quad (7.11)$$

Similarly $y_2^2 + 3z_2^2 = y^2 + 3z^2$.

Since $y$ and $z$ are odd they must be congruent to 1 or 3 modulo 4. If $y \equiv z \pmod{4}$ then $(y - z)/2$ is even and so is $(y + 3z)/2$. Hence $y_1$ and $z_1$ are even, but

$$x^2 + y^2 + 3z^2 + 3w^2 = x^2 + y_1^2 + 3z_1^2 + 3w^2 = 2n \quad (7.12)$$

and we are in the case where $x$ and $y_1$ have the same parity. Thus $n$ is represented by $(1, 1, 3, 3)$ using the arguments of the same parity case above.

If $y \not\equiv z \pmod{4}$ then $y + z \equiv 0 \pmod{4}$ so $(y + z)/2$ and $(y - 3z)/2$ are even. Thus $y_2$ and $z_2$ are even and

$$x^2 + y^2 + 3z^2 + 3w^2 = x^2 + y_2^2 + 3z_2^2 + 3w^2 = 2n. \quad (7.13)$$

We are in the same parity case and a representation of $n$ by $(1, 1, 3, 3)$ is derived as above. $\square$

Lemma 29. $H_{1,3,3}$ is norm Euclidean.

Proof. Fix any quaternion $q \in \mathbb{H}$. We first find an integer $a_3$ such that $q - a_3x_3$ has $j$ component less than or equal to $\sqrt{3}/4$ in absolute value. We must now consider the $i$ and $k$ components simultaneously. Let $x_2^i$ and $x_3^i$ be the projections of $x_2$ and $x_3$ onto the $i$ - $k$ plane,

$$x_2^i = \frac{1}{2}i - \frac{\sqrt{3}}{2}k, \quad x_3^i = \frac{3}{4}i + \frac{\sqrt{3}}{4}k. \quad (7.14)$$
We wish to approximate an arbitrary point in the $i$- $k$ plane, $ai + bk$, by an integral linear combination of the above two projections. Set $m$ equal to the nearest integer to $(a - \sqrt{3}b)/2$ and $n$ equal to the nearest integer to $(3a + \sqrt{3}b)/3$. Thus

$$\left| m - \frac{a - \sqrt{3}b}{2} \right| \leq \frac{1}{2}, \quad \left| n - \frac{3a + \sqrt{3}b}{3} \right| \leq \frac{1}{2}. \quad (7.15)$$

Set $-c$ equal to the expression inside the absolute value signs on the left, while $-d$ is set equal to the corresponding term on the right. Note

$$ai + bk - mx_2 - nx_4 = \left( a - \frac{1}{2}m - \frac{3}{4}n \right)i + \left( b + \frac{\sqrt{3}}{2}m - \frac{\sqrt{3}}{4}n \right)k \quad (7.16)$$

which implies

$$||ai + bk - mx_2 - nx_4||^2 = \left( a - \frac{1}{2}m - \frac{3}{4}n \right)^2 + \left( b + \frac{\sqrt{3}}{2}m - \frac{\sqrt{3}}{4}n \right)^2 \quad (7.17)$$

as $i$ and $k$ are orthogonal. We have the further equations

$$a - \frac{1}{2}m - \frac{3}{4}n = \left( \frac{a - \sqrt{3}b}{2} - m \right) \cdot \frac{1}{2} + \left( \frac{3a + \sqrt{3}b}{3} - n \right) \cdot \frac{3}{4}$$

$$b + \frac{\sqrt{3}}{2}m - \frac{\sqrt{3}}{4}n = \left( \frac{a - \sqrt{3}b}{2} - m \right) \cdot \left( -\frac{\sqrt{3}}{2} \right) + \left( \frac{3a + \sqrt{3}b}{3} - n \right) \cdot \frac{\sqrt{3}}{4} \quad (7.18)$$

Then

$$||ai + bk - mx_2 - nx_4||^2 = \left( \frac{1}{2}c + \frac{3}{4}d \right)^2 + \left( -\frac{\sqrt{3}}{2}c + \frac{\sqrt{3}}{4}d \right)^2 \quad (7.19)$$

since $c$ and $d$ are bounded by $1/2$ in absolute value.

Hence an integer linear combination of $x_2$ and $x_4$ can be chosen such that the sums of squares of the $i$ and $k$ components of $q - ax_3 - mx_2 - nx_4$ is bounded by $7/16$. The $j$ component is unchanged by this transformation of $q$. We need only choose integral $a_1$ for which $q - ax_3 - mx_2 - nx_4 - a_1x_1$ has real coefficient less than or equal to $1/2$. Then

$$||q - ax_3 - mx_2 - nx_4 - a_1x_1||^2 \leq \left( \frac{1}{2} \right)^2 + \left( \frac{\sqrt{3}}{4} \right)^2 + \frac{7}{16} = \frac{7}{8}. \quad (7.20)$$

Hence $H_{1,3,3}$ is norm Euclidean. □

**Lemma 30.** For any rational prime $p$ there exist rational integers $a$ and $b$ such that $a^2 + b^2 + 3 \equiv 0 \pmod{p}$.

**Proof.** The demonstration is analogous to similar Lemmas above. □
Lemma 31. \(H_{1,3,3}\) contains the elements \(1, 2i, \sqrt{3}j, i+\sqrt{3}k\).

Proof. \(x_1 = 1 \in H_{1,3,3}\). Also \(x_2 + 2x_4 = 1 + 2i\) hence \(2i \in H_{1,3,3}\). Then \(2x_3 - x_2 = \sqrt{3}j \in H_{1,3,3}\). Finally \(2x_4 - x_2 = 1 + i +\sqrt{3}k\) so \(i + \sqrt{3}k \in H_{1,3,3}\). \(\Box\)

Theorem 32. Every rational prime can be represented by the form \((1, 1, 3, 3)\).

Proof. The proof has only a few differences from previous demonstrations of analogous Theorems. By similar reasoning to the above mentioned Theorems we find that there exist integers \(a\) and \(b\) such that \(a^2 + b^2 + 3 = pr\) for integer \(r, 0 < r < p\).

To find an element \(u \in H_{1,3,3}\) with norm equal to \(a^2 + b^2 + 3\) we consider two cases. If \(b\) is even we may use \(u = a + (b/2) \cdot 2i + \sqrt{3}j\). This is in \(H_{1,3,3}\) by the previous Lemma. If \(b\) is odd then write \(b = 2s + 1\) and let
\[
u = a + s \cdot 2i + (i + \sqrt{3}k) \in H_{1,3,3}.
\]
Then \(u = a + bi + \sqrt{3}k\) so the norm of \(u\) is again \(a^2 + b^2 + 3\).

The remainder of the proof is similar with the note that it easy to see \(H_{1,3,3}\) is closed under conjugation. We end up with an element \(d \in H_{1,3,3}\) such that \(N(d) = p\). Since our results only guarantee that a two sided associate of \(2d\) is in the ring \(\mathbb{Z}[1, i, \sqrt{3}k, \sqrt{3}k]\) we find integers \(a_1, a_2, a_3, a_4\) such that
\[
4p = 4N(d) = N(2d) = a_1^2 + a_2^2 + 3a_3^2 + 3a_4^2. \tag{7.22}
\]
By Lemma 28, the form \((1, 1, 3, 3)\) represents \(2p\), and hence \(p\).

Corollary 33. The quadratic form \((1, 1, 3, 3)\) is universal.

Proof. This form represents one, every prime, and is norm quaternionic.

8. The form \((1, 2, 5, 10)\)

The attempt to demonstrate the universality of this quadratic form via quaternion runs into difficulties. When looking for a basis of units, we previously used quaternions with real part 0, \(\frac{1}{2}\) and 1. Experience shows other real parts lead to dramatically increasing denominators upon repeated multiplication. At least one of the unit generators in the above cases had real part zero. It is not difficult to show that the unit quaternions analogous to those considered for the previous quadratic forms cannot have real part zero.

Theorem 34. There is no unit quaternion of the form
\[
\frac{1}{c} \left(b \sqrt{2}i + c \sqrt{5}j + d \sqrt{10}k\right) \tag{8.1}
\]
with \(b, c, d, e \in \mathbb{Z}\).

Proof. For the norm to be one we must have \(e^2 = 2b^2 + 5c^2 + 10d^2\). With no loss of generality we may choose \(e, b, c, d\) to be positive. Consider the solution with the smallest positive value of \(e\). Then \(e^2 \equiv 2b^2 \pmod{5}\). If \(b \not\equiv 0 \pmod{5}\) then 2 is a quadratic residue modulo 5. This is a contradiction, so 5 must divide \(b\). Consequently 5 also divides \(e\).
Let $e = 5e_1$ and $b = 5b_1$. Thus $e_1$ and $b_1$ are elements of $\mathbb{Z}$ and we have

\begin{align*}
25e_1^2 &= 2 \cdot 25b_1^2 + 5c^2 + 10d^2 \\
5c_1^2 &= 10b_1^2 + c^2 + 2d^2
\end{align*}

(8.2)

Modulo 5 this becomes $0 \equiv c^2 + 2d^2 \pmod{5}$. If $d$ is not divisible by 5 then $-2$ is a quadratic residue of 5, a contradiction. Hence $d$ and thus $c$ are divisible by 5.

Let $c = 5c_1$ and $d = 5d_1$. Then $c_1$ and $d_1$ are in $\mathbb{Z}$ and

\begin{align*}
5e_1^2 &= 10b_1^2 + 25c_1^2 + 2 \cdot 25d_1^2 \\
c_1^2 &= 2b_1^2 + 5c_1^2 + 10d_1^2
\end{align*}

(8.3)

which contradicts the minimality of the choice of $e$. $\square$

9. The Computation

PUNIMAX, a computer algebra system related to MAXIMA, was used on a LINUX partition of two personal computers. One computer was a Pentium 133 with 32 megabytes of RAM and the other was a Pentium 300 with 64 megabytes of RAM. Both machines ran Linux 2.0.35. The computation of the two sided products of units for the quadratic form $(1, 2, 3, 6)$ required a total of an hour of machine time on the faster computer. A UNIX port of SNOBOL was used to change MAXIMA output into a TeX friendly form. In addition Tables II and IV were checked by manual calculations. On this basis, Table III and the cases listed in Table V were also verified manually. Table VI was verified by an elementary lex and yacc based quaternion calculator written by the author.

10. Acknowledgments

The author would like to thank A. Jarvis and N. Dummigan of Sheffield University for assistance in tracking down one of the references and explaining some current results. The author would like to thank B. Haible, the maintainer of PUNIMAX, for permitting its free use for academic purposes [5]. The author would like to thank Harvey Cohn for many encouraging email communications. The author expresses his appreciation to the referee for very useful comments and suggestions.

References

1. W. Duke, Some Old Problems and New Results about Quadratic Forms, Notices of the AMS 44 (1997), no. 2, 190–196.
2. N. Fine, Basic Hypergeometric Series and Applications, American Mathematical Society, Providence, Rhode Island, 1988.
3. G. Hardy and E. Wright, An Introduction to the Theory of Numbers, (fourth ed.), Oxford University Press, London, 1971.
4. I. Herstein, Topics in Algebra, John Wiley and Sons, New York, 1975.
5. B. Haible, Private communication (1997).
6. A. Hurwitz, Vorlesungen über die Zahlentheorie der Quaternionen, Julius Springer, Berlin, 1919.
7. D. Marcus, Number Fields, Springer–Verlag, New York, 1977.
8. R. Markanda and V. Albis-Gonzales, Euclidean Algorithm in principal arithmetic algebras, Tamkang J. Math. 15 (1984), 193–196.
9. R. Pierce, Associative Algebras, Springer–Verlag, New York, 1982.
10. S. Ramanujan, *On the expression of a number in the form* \( ax^2 + by^2 + cz^2 + du^2 \), *Proc. Cambridge Phil. Soc.*, 19 (1917), no. 1, 11–21.

11. I. Reiner, *Maximal Orders*, Academic Press, London, 1975.

12. W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, Berlin, 1985.

13. M.-F. Vigneras, *Arithmétique des Algèbres de Quaternions*, Springer-Verlag, Berlin, 1980.

Mathematics Department, University of Botswana, Private Bag 0022, Gaborone, Botswana

E-mail address: deutschj@mopipi.ub.bw