Handlebody-knot invariants derived from unimodular Hopf algebras

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Abstract

A handlebody-knot is a handlebody embedded in the 3-sphere. We establish a uniform method to construct invariants for handlebody-links. We introduce the category $\mathcal{T}$ of handlebody-tangles and present it by generators and relations. The result tells us that every functor on $\mathcal{T}$ that gives rise to invariants is derived from what we call a quantum-commutative quantum-symmetric algebra in the target category. The example of such algebras of our main concern is finite-dimensional unimodular Hopf algebras. We investigate how those Hopf algebras give rise to handlebody-knot invariants.

1 Introduction

A handlebody-knot is a handlebody embedded in the 3-sphere $S^3$; it is alternatively called a knotted handlebody or a spatial handlebody. A handlebody-link is a disjoint union of handlebodies embedded in $S^3$. Two handlebody-links are equivalent if there exists an isotopy of $S^3$ which takes one to the other, or equivalently if there exists an orientation-preserving self-homeomorphism of $S^3$ which sends one to the other. The aim of this paper is to establish a uniform method to construct invariants for handlebody-links.

A handlebody-knot was first introduced as a neighborhood equivalence class of a spatial graph by Suzuki [17]. Since neighborhood equivalence classes of knots coincide with ambient isotopy classes of knots, genus 1 handlebody-knots correspond to knots, which means that a handlebody-knot is a generalization of a knot. Study of knots with invariants has made great progress since the discovery of the Jones polynomial and the subsequent so-called quantum invariants; see, for example, [13]. Quantum invariants are derived from representations of quantum groups. Superiority of a quantum invariant is that it can be derived from any representation of any quantum group. A functor from the category of tangles to that of vector spaces, which is obtained via a representation of a quantum group, gives a quantum invariant of tangles, especially of links.

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Invariants of handlebody-links can be realized as those of spatial trivalent graphs which are invariant under IH-moves \cite{4}, where an IH-move is a local move on spatial trivalent graphs. When a handlebody-link $H$ is a regular neighborhood of a spatial graph $K$, we say that $H$ is represented by $K$. In this paper trivalent graphs may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. The existence of trivalent vertices distinguishes mostly handlebody-links from ordinary links, and gives us the biggest barrier when we construct functors on handlebody-tangles. In order to get over this barrier, we introduce quantum-commutative quantum-symmetric algebras (see Definition \cite{4}), and assign the multiplication mapping to a trivalent vertex. Then invariance under IH-moves follows from the associativity of multiplication. We obtain an invariant for handlebody-links with every quantum-commutative quantum-symmetric algebra. A good example of quantum-commutative quantum-symmetric algebras is given by finite-dimensional unimodular Hopf algebras, which include finite groups as the simplest example. Our invariant derived from a finite group coincides with the number of the homomorphisms from the fundamental group of the exterior of a handlebody-knot to the group.

We remark that Mizusawa and Murakami \cite{11} constructed quantum $U_q(sl_2)$ type invariants for handlebody-knots in $S^3$ via Yokota’s invariants \cite{20}.

This paper is organized as follows. In Section 2, we introduce the category $T$ of handlebody-tangles and present it by generators and relations; the result enables us to construct on $T$ the functors which gives our invariants. In Section 3, we introduce the notion of quantum-commutative quantum-symmetric algebras and see how the invariants are obtained from those algebras. In Section 4, we focus on quantum-commutative quantum-symmetric algebras in the categories of Yetter–Drinfeld modules and show that every finite-dimensional unimodular Hopf algebra $A$ is a quantum-commutative quantum-symmetric algebra in the category of Yetter–Drinfeld modules over $A$. In Section 5, we investigate the invariants derived from unimodular Hopf algebras in some algebraic and geometric situations such as the disk sum or the mirror image. In Sections 6 and 7, we give examples of invariants derived from unimodular Hopf algebras together with their data needed to compute the invariants.

2 The category of handlebody-tangles

A handlebody-tangle is a disjoint union of handlebodies embedded in a cube $I^3$ such that the intersection of the handlebodies and the boundary of $I^3$ is the union of sequences of disks in the top and bottom squares as shown in Figure \ref{fig:1}. We call the disks in the top (resp. bottom) square the top (resp. bottom) end disks of the handlebody-tangle. Two handlebody-tangles are assumed to be the same if one can be transformed into the other by an isotopy of $I^3$ preserving the order of the end disks in the top and bottom squares. We remark that a handlebody-tangle with no end disks corresponds to a handlebody-link.

We define a strict tensor category $\mathcal{T}$ of handlebody-tangles as follows. The
objects of $\mathcal{T}$ consist of finite sequences of disks. We denote by the number $n$ the sequence of $n$ disks. Then $\text{Ob}(\mathcal{T}) = \{0, 1, 2, \ldots\}$. The morphisms of $\mathcal{T}$ are handlebody-tangles. The source $s(T)$ and the target $b(T)$ of a handlebody-tangle $T$ with $m$ top end disks and $n$ bottom end disks are defined by $s(T) = n$ and $b(T) = m$. The identity morphism $\text{id}_n$ of an object $n$ is the equivalence class of the trivial handlebody-tangle with $n$ top end disks and $n$ bottom end disks, where the trivial handlebody-tangle is the direct product of disks and the interval $I$ as shown in the right picture of Figure 1. We remark that the identity morphism $\text{id}_0$ is the empty set. For handlebody-tangles $T, T'$ such that $s(T) = b(T')$, the composition $T \circ T'$ of $T$ and $T'$ is the handlebody-tangle obtained by placing $T$ on top of $T'$ and gluing the bottom end disks of $T$ and the top end disks of $T'$ as shown in the left picture of Figure 2. Then $\mathcal{T}$ is a category. We equip $\mathcal{T}$ with a tensor product as follows. For objects $m, n$ of $\mathcal{T}$, we define $m \otimes n := m + n$. For handlebody-tangles $T, T'$, the tensor product $T \otimes T'$ is the handlebody-tangle obtained by placing $T'$ to the right of $T$ as shown in the right picture of Figure 2. Then the handlebody-tangle category $\mathcal{T}$ equipped with the tensor product is a strict tensor category with the unit 0.

We give generators and relations for the strict tensor category $\mathcal{T}$. Every
morphism in $\mathcal{T}$ can be presented by the generators with the operations of composing and tensoring applied. Two morphisms are identical if and only if they, presented as above, deform to each other by using the relations. We refer the reader to [6, Chapter XII] for details of generators and relations for a strict tensor category. Let $\vert, \cap, \cup, \lambda, \gamma, X, \bar{X}$ be the handlebody-tangles depicted in Figure 3. The following proposition immediately follows from Theorem 11 in [3].

**Proposition 1.** The strict tensor category $\mathcal{T}$ is generated by the six morphisms

$\cap, \cup, \lambda, \gamma, X, \bar{X}$

and the relations

1. $(\vert \cap) \circ (\cup \vert) = \vert = (\cap \vert) \circ (\cup \vert)$,
2. $(\vert \cap) \circ (X \vert) = (\cap \vert) \circ (X \bar{X})$,
3. $(\vert \cap) \circ (X \bar{X}) = (\cap \vert) \circ (X \vert)$,
4. $(\vert \cap) \circ (\gamma \vert) = \lambda = (\cap \vert) \circ (\gamma \vert)$,
5. $(\vert \cap) \circ (X \bar{X}) \circ (\cup \vert) = \vert = (\cap \vert) \circ (\gamma \vert) \circ (\cup \vert)$,
6. $X \circ \bar{X} = \vert$,
7. $(X \vert) \circ ((X \vert) \circ (X \vert) = (X \vert) \circ (X \vert) \circ (X \vert)$,
8. $\lambda \circ X = \lambda = \lambda \circ \bar{X}$,
9. $(\lambda \vert) \circ (X \vert) \circ (X \vert) = X \circ (\lambda \vert)$,
10. $(\lambda \vert) \circ (X \vert) \circ (X \vert) = X \circ (\lambda \vert)$,
11. $\lambda \circ (\lambda \vert) = \lambda \circ (\lambda \vert)$,

where we have presented $A \otimes B$ by the juxtaposition $AB$.

We improve below the presentation of $\mathcal{T}$ above into a more economical one, which will be crucial when we prove Proposition 6.

**Proposition 2.** The strict tensor category $\mathcal{T}$ is generated by the five morphisms

$\cap, \cup, \lambda, X, \bar{X}$
and the relations (1), (5), (7), (9), (10), (11) together with
\[
\n \cap \circ (|\lambda|) = \cap \circ (|\lambda|), \tag{12}
\]
\[
\n \cap \circ X = \cap, \tag{13}
\]
\[
\n (\cap |) \circ (|X) \circ (X |) = |\cap, \tag{14}
\]
\[
\n (|\cap| \circ (X |) \circ (|X) = |\cap|, \tag{15}
\]
\[
\n \lambda \circ X = \lambda. \tag{16}
\]

**Proof.** The five morphisms generate \(\mathcal{T}\) since we have
\[
\gamma = (\lambda |) \circ (|\cup).
\]
By this equality the relation (14) turns into the relation
\[
(\lambda \cap) \circ (|\cup|) = \lambda = (\cap |) \circ (|\lambda|) \circ (|\cup) \tag{17}
\]
We see that the conditions in Proposition 2 follow from those in Proposition 1. To prove the converse it suffices to verify (2), (3), (5) and (17). This is done as follows.
\[
(|\cap| \circ (X |) = (|\cap| \circ (X |) \circ (|X) \circ (|\overline{X}) = (|\cap|) \circ (|\overline{X}), \tag{11}
\]
\[
(|\cap| \circ (X |) = (\cap |) \circ (X |) \circ (X |) \circ (|\overline{X}) = (\cap |) \circ (|X), \tag{12}
\]
\[
(|\cap| \circ (X |) \circ (|\cup) = (|\cap| \circ (\cup X)) \circ (|\overline{X}) \circ (|\overline{X}) \circ (|\cup)
\]
\[
= (|\cap| \circ (|\cup)| \circ (|\overline{X}) \circ (|\overline{X}) \circ (|\overline{X}) \circ (|\cup)
\]
\[
= (|\cap| \circ (|X) \circ (|\overline{X}) \circ (|\cup) = (|\cap| \circ (|X) \circ (|\cup)
\]
\[
(|\cap| \circ (X |) \circ (|\cup) = (|\cap| \circ (\cup X) \circ (|\overline{X}) \circ (|\overline{X}) \circ (|\cup)
\]
\[
= (|\cap| \circ (|\cup) \circ (|X) \circ (|\overline{X}) \circ (|\overline{X}) \circ (|\cup)
\]
\[
= (|\cap| \circ (|\cup) = |\lambda.
\]
\[
\lambda = \lambda \circ X \circ \overline{X} = \lambda \circ \overline{X},
\]
\[
(\lambda \cap) \circ (|\cup|) = \lambda = (\cap |) \circ (\lambda \cup) = (\cap |) \circ (|\lambda|) \circ (|\cup).\]
\]

\[\square\]

The following is now easy to see.

**Proposition 3.** The tensor category \(\mathcal{T}\) is braided with respect to the braiding depicted in Figure 4.
3 Quantum-commutative quantum-symmetric algebras

In this section, $\mathcal{B} = (\mathcal{B}, \otimes, I)$ denotes a braided tensor category unless otherwise stated. The braiding and its inverse will be denoted by, and depicted as

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V;$$

$$c^{-1}_{W,V} : W \otimes V \xrightarrow{\sim} V \otimes W;$$

where $V, W \in \text{Ob}(\mathcal{B})$. Suppose that $A$ is a (non-unital) algebra in $\mathcal{B}$. Thus, $A$ is equipped with a morphism in $\mathcal{B}$

$$m_A : A \otimes A \rightarrow A;$$

which satisfies the associativity

Here and in what follows we omit the associativity constraint in diagrams.
Definition 4. A (non-unital) algebra $A$ in $B$, equipped with two morphisms

$$\text{ev}_A : A \otimes A \to I; \quad \begin{array}{c}
\text{ev}_A
\end{array}$$

$$\text{coev}_A : I \to A \otimes A; \quad \begin{array}{c}
\text{coev}_A
\end{array}$$

is called a quantum-commutative quantum-symmetric algebra if it satisfies the selfduality (18), the Frobenius property (19), the quantum-commutativity (20) and the quantum-symmetry (21), given below.

These coincide respectively with (1), (12), (16), (13) given before, if the $\times$, $\cap$, $\cup$ and $X$ before read $m_A, \text{ev}_A, \text{coev}_A$ and $c_{A,A}$, respectively.

Proposition 5. (1) In the braided tensor category $T$ of handlebody-tangles, the object $1$ equipped with $\times$, $\cap$, $\cup$ is a quantum-commutative quantum-symmetric algebra.

(2) If $F : T \to B$ is a braided tensor functor, then the object $F(1)$, together with the morphisms which are given by $F(\times)$, $F(\cap)$, $F(\cup)$ composed with the tensor structure $I \simeq F(0), F(1) \otimes F(1) \simeq F(2)$ of $F$, forms a quantum-commutative quantum-symmetric algebra in $B$.

Proof. (1) This follows by Proposition 2.
This follows by a standard argument which depends on the fact that the notion of quantum-commutative quantum-symmetric algebras is tensor-categorical.

\begin{proposition}
Assume that \( \mathcal{B} \) is strict as a tensor category. Given a quantum-commutative quantum-symmetric algebra \( A = (A, m_A, ev_A, coev_A) \) in \( \mathcal{B} \), there uniquely exists a braided strict tensor functor \( F_A : T \to \mathcal{B} \) such that

\[
F_A(1) = A, \quad F_A(\lambda) = m_A, \quad F_A(\cap) = ev_A, \quad F_A(\cup) = coev_A.
\]

\end{proposition}

\begin{proof}
Define objects \( A^\otimes n \in \text{Ob}(\mathcal{B}) \) by

\[
A^\otimes 0 = I, \quad A^\otimes 1 = A, \quad A^\otimes n = A^\otimes (n-1) \otimes A \quad (n > 1),
\]

and define a map \( F : \text{Ob}(T) \to \text{Ob}(\mathcal{B}) \) by

\[
F(n) = A^\otimes n.
\]

Then this \( F \) strictly preserves the tensor product.

Let us see that the relations in Proposition 2 with \( X, X, \cap, \cup \) replaced by \( c_{A, A}, c^{-1}_{A, A}, ev_A, coev_A \) are satisfied. Since \( c_{A, A} \) is a braiding, the relations \( 7, 9, 10, 14 \) and \( 15 \) are satisfied; cf. Remark 7 below. The rest is satisfied by the coincidence noted below \( 18 \)–\( 21 \) and since \( c_{A, A} \) and \( c^{-1}_{A, A} \) are inverses of each other.

It follows by [6, Proposition XII.1.4] that the map \( F \) defined above gives rise uniquely to a strict tensor functor \( F_A : T \to \mathcal{B} \) which satisfies the equalities in \( 22 \) as well as \( F_A(X) = c_{A, A}, \quad F_A(X) = c^{-1}_{A, A} \). One sees easily that this \( F_A \) preserves the braiding, and indeed is unique such as described above.

\begin{remark}
Assume that \( \mathcal{B} \) is strict, but is not necessarily braided. As was essentially shown above, if we have a (non-unital) algebra \( A \) in \( \mathcal{B} \) equipped with morphisms \( c_{A, A}, c^{-1}_{A, A}, ev_A, coev_A \) which satisfy the relations in Proposition 2 except \( 11 \), there uniquely exists a strict tensor functor \( F_A : T \to \mathcal{B} \) which satisfies the equalities in \( 22 \) as well as \( F_A(X) = c_{A, A}, \quad F_A(X) = c^{-1}_{A, A} \). In fact, it is essentially from such functors that Ishihara and the first author [3] constructed invariants of handlebody-knots. However, we choose, in this paper, to work with braided tensor categories, because the algebras of our main concern sit in such a category.

Suppose that \( \mathcal{B} \) is braided. To modify the proposition above when \( \mathcal{B} \) is not necessarily strict, let us say that a tensor functor \( F : T \to \mathcal{B} \) is almost strict, if it satisfies the following three conditions: (i) \( F(0) = I \), (ii) if we set \( A = F(1) \), then for each \( n > 1 \), \( F(n) = A^\otimes n \), where \( A^\otimes n \) is defined by \( 23 \), and (iii) the tensor structure \( \varphi_0, \varphi_2 \) of \( F \) is as follows,

\begin{enumerate}
\item \( \varphi_0 : I \xrightarrow{\sim} F(0) \) is the identity \( \text{id}_I \);
\end{enumerate}
(b) for every $n, m > 0$, $\varphi_2(n, m) : F(n) \otimes F(m) \xrightarrow{\sim} F(n + m)$ is canonical in the sense that it is built from the associativity constraint $a_{A,A,A} : (A \otimes A) \otimes A \xrightarrow{\sim} A \otimes (A \otimes A)$ and the identity $id_A$ by composing and tensoring; such an isomorphism is unique by MacLane’s coherence theorem.

One sees from (a) that if $n = 0$ or $m = 0$, then $\varphi_2(n, m)$ must coincide with the left or right unit constraint $l_{A^{\otimes n}} : I \otimes A^{\otimes m} \xrightarrow{\sim} A^{\otimes n}$, $r_{A^{\otimes n}} : A^{\otimes n} \otimes I \xrightarrow{\sim} A^{\otimes n}$.

**Proposition 8.** Given a quantum-commutative quantum-symmetric algebra $A = (A, m_A, ev_A, coev_A)$ in $\mathcal{B}$, there uniquely exists a braided, almost strict tensor functor $F_A : \mathcal{T} \rightarrow \mathcal{B}$ which satisfies the equalities given in (22).

**Proof.** For a tensor category $\mathcal{B}$ in general, a strict tensor category $\mathcal{B}^{str}$ together with a strict tensor equivalence $G : \mathcal{B} \rightarrow \mathcal{B}^{str}$ is constructed in [6 Sect. XI5]. Suppose we are in our special situation. It follows by the construction of [6 Sect. XI5] that for every $n, m > 0$, the unique canonical isomorphism $A^{\otimes n} \otimes A^{\otimes m} \xrightarrow{\sim} A^{\otimes (n+m)}$ is sent by $G$ to the identity on $G(A)^{\otimes (n+m)}$. We can apply Remark [7] to $(G(A), G(m_A), G(ev_A), G(coev_A))$ in $\mathcal{B}^{str}$. The result is translated via $G$ so that the same result as in the remark holds true with “a strict tensor functor $F_A$” replaced with “an almost strict tensor functor $F_A$.” This implies the desired result.

**Corollary 9.** Given a quantum-commutative quantum-symmetric algebra $A$ in a braided tensor category, $F_A(H)$ gives an invariant for handlebody-links $H$, which has values in the endomorphism monoid $\text{End}(I)$.

**Proof.** This is a direct consequence of Proposition 8.

## 4 Quantum-commutative quantum-symmetric algebras in Yetter–Drinfeld modules

We will show that every finite-dimensional unimodular Hopf algebra over a field, regarded as an algebra in a braided tensor category of Yetter–Drinfeld modules, is a quantum-commutative quantum-symmetric algebra, which is unital.

In what follows we work over a fixed base field $k$; the tensor products $\otimes$ denote those for vector spaces over $k$. Let $A$ be a Hopf algebra with the coproduct $\Delta : A \rightarrow A \otimes A$, the counit $\varepsilon : A \rightarrow k$ and the antipode $S : A \rightarrow A$. For $\Delta$, we will use the following variant of the Sweedler notation [18 Sect. 1.2]:

$$\Delta(a) = a_{(1)} \otimes a_{(2)}, \quad \Delta(a_{(1)}) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} = a_{(1)} \otimes \Delta(a_{(2)}).$$

For our purpose we may and we do assume that $A$ is finite-dimensional. Then $A$ includes the one-dimensional subspaces

$$I_l(A) := \{ \lambda \in A | a \lambda = \varepsilon(a) \lambda, \ a \in A \}$$

resp., $I_r(A) := \{ \lambda \in A | \lambda a = \varepsilon(a) \lambda, \ a \in A \}$

consisting of all left and resp., right integrals; see [18 Corollary 5.1.6]. It possibly happen that $I_l(A) \neq I_r(A)$. 

Definition 10 ([12 Definition 2.1.1]). $A$ is said to be unimodular, if $I_l(A) = I_r(A)$.

The assumption $\dim A < \infty$ ensures that the antipode $S$ is bijective; see [18 Corollary 5.1.6].

Given a left $A$-comodule $V$, we will write its structure, say $\rho : V \to A \otimes V$, explicitly so that

$$\rho(v) = v_{(-1)} \otimes v_{(0)}; \text{ cf. } [18 \text{ Sect. 2.0}].$$

Let $A^\text{YD}$ denote the $k$-linear abelian, braided tensor category of Yetter–Drinfeld modules over $A$; see [12, Definition 10.6.10]. Such a module is by definition a left $A$-module $V$ given a left $A$-comodule structure $\rho : V \to A \otimes V$ which satisfies

$$\rho(av) = a_{(1)}v_{(-1)}S(a_{(3)}) \otimes a_{(2)}v_{(0)}, \quad a \in A, v \in V.$$

Morphisms in $A^\text{YD}$ are $A$-linear and $A$-colinear maps. In $A^\text{YD}$, the tensor product, the unit object (that is $k$), and the associativity and unit constraints are the obvious ones, being the same as those for left (co)modules. The braiding is defined by

$$c_{V, W} : V \otimes W \xrightarrow{\cong} W \otimes V, \quad c_{V, W}(v \otimes w) = v_{(-1)}w \otimes v_{(0)},$$

whose inverse is given by

$$c_{V, W}^{-1}(w \otimes v) = v_{(0)} \otimes S^{-1}(v_{(-1)})w.$$

As is well known, the braided tensor category $A^\text{YD}$ thus defined is naturally identified with that of left modules over the quantum double $D(A)$; see [12 Sect. 10.6] or [6 Sect. IX.5]. Though the latter category might be more familiar, the former is more suitable for our purpose. In the following section we will treat with the quantum double $D(kG)$ of a finite group algebra $kG$, as an example of the present $A$.

We regard $A$ as a left $A$-module with respect to the conjugate action $\triangleright$ defined by

$$a \triangleright b = a_{(1)}bS(a_{(2)}), \quad a, b \in A.$$

We regard $A$ as a left $A$-comodule with respect to the coproduct $\Delta : A \to A \otimes A$.

Lemma 11. We have $(A, \triangleright, \Delta) \in A^\text{YD}$. Moreover, this $A$, equipped with the original algebra structure, turns into a unital algebra in $A^\text{YD}$ which satisfies the quantum-commutativity (20).

Proof. This is directly verified; the quantum-commutativity follows from $(a_{(1)} \triangleright b)a_{(2)} = ab$. □

Continue to suppose that $A$ is a finite-dimensional Hopf algebra. The dual vector space $A^* = \text{Hom}_k(A, k)$ of $A$ forms naturally a Hopf algebra (see [18 Sect. 6.2]), so that we have $I_l(A^*), I_r(A^*)$. We will use the following well-known fact; see Proposition 1 (e) and Corollary 1 of [13], for example.
Proposition 12. Let $\lambda$ be a non-zero left or right integral in $A^*$.

(1) If $\Lambda$ is a left (resp., right) integral in $A$, then $S^{\pm 1}(\Lambda)$ is a right (resp., left) integral in $A$, and $\lambda(\Lambda) = \lambda(S^{\pm 1}(\Lambda))$.

(2) There exists uniquely a left or right integral $\Lambda$ in $A$ such that $\lambda(\Lambda) = 1$.

It follows that the evaluation map $I_l(A^*) \otimes I_l(A) \to k$ and the analogous ones are all linear isomorphisms.

Choose $0 \neq \lambda \in I_l(A^*)$, and define a bilinear form on $A$ by

$$\langle \cdot, \cdot \rangle_\lambda : A \times A \to k, \quad \langle a, b \rangle_\lambda = \lambda(ab).$$

The following is well known; see [12, Theorem 2.1.3].

Proposition 13. This bilinear form is non-degenerate.

Choose bases $(\alpha_i)$, $(\beta_i)$ of $A$ which are dual to each other with respect to $\langle \cdot, \cdot \rangle_\lambda$, so that $\langle \alpha_i, \beta_j \rangle_\lambda = \delta_{ij}$. Set

$$U_\lambda = \sum_i \beta_i \otimes \alpha_i \in A \otimes A.$$ 

This element is characterized by the property that

$$\sum_i \beta_i \langle \alpha_i, a \rangle_\lambda = a \quad \text{or} \quad \sum_i \langle a, \beta_i \rangle_\lambda \alpha_i = a \quad (25)$$

for all $a \in A$.

Lemma 14. Let $0 \neq \lambda \in I_l(A^*)$ as above.

(1) Let $\Lambda$ be a unique right integral in $A$ such that $\lambda(\Lambda) = 1$. Then

$$U_\lambda = S(\Lambda(1)) \otimes \Lambda(2).$$

(2) Let $\Lambda$ be a unique left integral in $A$ such that $\lambda(\Lambda) = 1$. Then

$$U_\lambda = \Lambda(2) \otimes S^{-1}(\Lambda(1)).$$

Proof. (1) Since $\Lambda \in I_r(A)$, we have

$$\Lambda(1)S^{-1}(a) \otimes \Lambda(2) = \Lambda(1) \otimes \Lambda(2)a, \quad a \in A. \quad (26)$$

To see that the element $S(\Lambda(1)) \otimes \Lambda(2)$ satisfies the first equation of (25), we have to show that for any $a \in A$, $f \in A^*$,

$$f(S(\Lambda(1))) \lambda(\Lambda(2)a) = f(a).$$

From (26) and $\lambda \in I_l(A^*)$, we see that this left-hand side equals

$$S^*(f \leftarrow a)(\Lambda(1)) \lambda(\Lambda(2)) = S^*(f \leftarrow a)(1) \lambda(\Lambda) = f(a),$$

as desired, where $f \leftarrow a$ is defined by $(f \leftarrow a)(b) = f(ab)$, $b \in A$. 

11
(2) If \( \Lambda \in I_l(A) \) with \( \lambda(\Lambda) = 1 \), then \( S^{-1}(\Lambda) \in I_l(A) \) with \( \lambda(S^{-1}(\Lambda)) = 1 \), by Proposition 12 (1). Part 1 applied to \( S^{-1}(\Lambda) \) shows Part 2.

To continue our construction we define linear maps,

- \( \text{ev}_A : A \otimes A \to k \), \( \text{ev}_A(a \otimes b) = \langle a, b \rangle^\lambda \),
- \( \text{coev}_A : k \to A \otimes A \), \( \text{coev}_A(1) = U^\lambda \).

Let \( m_A : A \otimes A \to A \) denote the product on \( A \). Then we have

\[
\text{ev}_A = \lambda \circ m_A. \tag{27}
\]

Obviously, \( \text{ev}_A, \text{coev}_A \) defined above satisfy the selfduality \( \text{(18)} \) and the Frobenius property \( \text{(19)} \).

**Proposition 15.** Assume that \( A \) is unimodular.

1. \( \text{ev}_A \) and \( \text{coev}_A \) defined above are both morphisms in \( \mathcal{YD}_A \).

2. The object \( A = (A, \triangleright, \Delta) \) in \( \mathcal{YD}_A \), equipped with \( m_A, \text{ev}_A, \text{coev}_A \), is a quantum-commutative quantum-symmetric algebra in \( \mathcal{YD}_A \).

**Proof.** (1) First, we show, without the unimodularity assumption, that \( \text{ev}_A \) and \( \text{coev}_A \) are \( A \)-colinear. Since \( \lambda \), regarded as a linear map \( A \to k \), is left \( A \)-colinear, it follows by (27) that \( \text{ev}_A \) is \( A \)-colinear, since \( m_A \) is obviously \( A \)-colinear. For \( \text{coev}_A \), let us use the expression of \( U^\lambda \) given by Lemma 14 (1). Then the \( A \)-colinearity of \( \text{coev}_A \) follows since we see

\[
S(\Lambda(1)(1) \otimes S(\Lambda(1)(2) \otimes (\Lambda(2)(1) \otimes (\Lambda(2)(2) = 1 \otimes U^\lambda.
\]

To show the remaining \( A \)-linearity, assume that \( A \) is unimodular. Let \( (A^{op})^{cop} \) denote the Hopf algebra \( A \) with the opposite product and co-product; it has the same antipode as \( A \), and our \( \lambda \) is a right integral in its dual Hopf algebra. Apply the equality (a) of \( \text{[14, Theorem 3]} \) to this \( (A^{op})^{cop} \). Since the unimodularity assumption implies that the \( \alpha \) in that equality equals \( \varepsilon \), it follows that

\[
\lambda(ab) = \lambda(b S^2(a)), \quad a, b \in A. \tag{28}
\]

Since the product \( m_A \) is obviously \( A \)-linear, it follows by (27) that in order to prove the \( A \)-linearity of \( \text{ev}_A \), it suffices to see that \( \lambda : A \to k \) is \( A \)-linear. In fact, this holds true, since we see from (28) that for \( a, b \in A \),

\[
\lambda(a(1)b S(a(2))) = \lambda(b S(a(2)) S^2(a(1)))) = \varepsilon(a) \lambda(b).
\]

The \( A \)-linearity of \( \text{coev}_A \) will follow if one sees, using the same expression of \( U^\lambda \) as above, that for every \( a \in A \),

\[
a(1) \triangleright S(\Lambda(1)) \otimes a(2) \triangleright \Lambda(2) = \varepsilon(a) U^\lambda.
\]
Use \( (26) \) and the analogous equation

\[
a\Lambda(1) \otimes \Lambda(2) = \Lambda(1) \otimes S^{-1}(a)\Lambda(2),
\]

which holds since \( \Lambda \in I_l(A) \). Then we see that the left-hand side of the desired equation equals

\[
a_{(1)}S(\Lambda_{(1)})S(a_{(2)}) \otimes a_{(3)}\Lambda_{(2)}S(a_{(4)}) = S(\Lambda_{(1)}) \otimes a_{(3)}S^{-1}(a_{(2)})\Lambda_{(2)}a_{(1)}S(a_{(4)}),
\]

which is seen to equal the right-hand side.

(2) It remains to verify the quantum-symmetry. By \( (27) \), this desired property follows from the quantum-commutativity which was verified by Lemma \( 11 \).

**Remark 16.** The construction above is generalized as follows. Suppose that \( A, L \) are finite-dimensional Hopf algebras, and that \( B \) is an \((L, A)\)-biGalois object, that is, an \((L, A)\)-bicomodule algebra which is a Galois extension \([12, \text{Definition 8.1.1}]\) over \( k \) on both sides. Choose \( 0 \neq \lambda \in I_l(A^*) \), and define \( \langle b, c \rangle_\lambda = b_{(0)}c_{(0)}\lambda(b_{(1)}c_{(1)}) \) for \( b, c \in B \), where \( b \mapsto b_{(0)} \otimes b_{(1)} \) denotes the right \( A \)-comodule structure on \( B \). Then one can prove that this last defines indeed a bilinear form \( \langle \cdot, \cdot \rangle_\lambda : B \times B \to k \) which is non-degenerate. By the same procedure as above, we see that if \( A \) is unimodular, then \( B \) turns into a quantum-commutative quantum-symmetric algebra in \( LYD \), where the left \( L \)-module structure on \( B \) is given by the so-called Miyashita-Ulbrich action. This construction applied to \( A \), which is regarded naturally as an \((A, A)\)-biGalois object, produces the quantum-commutative quantum-symmetric algebra in \( AYD \) given by the last proposition.

However, we have a natural equivalence \( AYD \simeq LYD \) of braided tensor categories (see \([10, \text{Proposition 5.1}]\), for example), under which \( A \) and \( B \) correspond to each other, so that the associated braided tensor functors \( F_A, F_B \) are identified via the equivalence. Therefore, we may restrict ourselves to Hopf algebras, without working with biGalois objects.

## 5 Invariants derived from unimodular Hopf algebras

Let \( A \) be a finite-dimensional unimodular Hopf algebra, and choose \( 0 \neq \lambda \in I_l(A^*) \). By Proposition \( 15 \) \( A \) turns into a quantum-commutative quantum-symmetric algebra \( A \) in \( AYD \). By Corollary \( 9 \) this \( A \) gives an invariant \( F_A(H) \) for handlebody-links \( H \), which has values in \( k \) since the endomorphism ring \( \text{End}(k) \) in \( AYD \) coincides with \( k \). The invariant \( F_A(H) \) depends on choice of \( \lambda \), but we will not indicate it within the notation except in the following.
Remark 17. Let us write here $F_{A,\lambda}(H)$ for $F_A(H)$, indicating $\lambda$. Let $0 \neq c \in k$. If we replace $\lambda$ with $c\lambda$, then $\text{ev}_A$ (resp., $\text{coev}_A$) is replaced by its scalar multiple by $c$ (resp., by $c^{-1}$). Therefore, we have

$$F_{A,c\lambda}(H) = c^{##\cap-##\cup}F_{A,\lambda}(H),$$

where $##\cap$, $##\cup$ respectively denote the numbers of $\cap$, $\cup$ in $H$.

Here is the simplest example of computations.

Example 18. Let $O$ be the trivial genus 1 handlebody-knot, which is represented by the trivial knot. Then,

$$F_A(O) = \text{Trace} S^2,$$

the trace of the linear endomorphism $S^2 = S \circ S$ of $A$. In particular, $F_A(O) = (\dim A)1$, if $S$ is an involution, that is, $S^2 = \text{id}_A$. To prove the formula above, we use the expression of $U_\lambda$ given by Lemma 14 (1). Then it follows by Eq. (3) of [7] that

$$F_A(O) = \lambda(S(A(1))A(2)) = \lambda(1)\varepsilon(A) = \text{Trace} S^2.$$

We should carefully choose $A$ for the invariant, as is seen from the following proposition, whose proof will be postponed for a moment.

Proposition 19. If $A$ is not cosemisimple, then $F_A(H) = 0$ for any handlebody-link $H$.

We say that a finite-dimensional Hopf algebra $A$ is cosemisimple if $A^*$ is semisimple as an algebra. We recall the following fundamental results on (co)semisimplicity; see [18, Theorem 5.1.8] for (1), [7, Theorem 4], [8, Theorem 3.3] for (2), and [2, Corollary 3.2] for (3).

Theorem 20. Let $A$ be a finite-dimensional Hopf algebra.

1. (Sweedler) The following are equivalent:
   
   (a) $A$ is cosemisimple;
   
   (b) There exists a left or right integral $\lambda$ in $A^*$ such that $\lambda(1) = 1$;
   
   (c) There exists a left and right integral $\lambda$ in $A^*$ such that $\lambda(1) = 1$.

   In particular, if $A$ is cosemisimple, then $A^*$ is unimodular.

2. (Larson–Radford) Assume $\text{char } k = 0$. Then the following are equivalent:
   
   (a) $A$ is cosemisimple;
   
   (d) $A$ is semisimple;
   
   (e) The antipode $S$ is an involution.

3. (Etingof–Gelaki) Assume $\text{char } k > 0$. Then the following are equivalent:
(f) $A$ is semisimple and cosemisimple;

(g) $S$ is an involution, and $\text{char } k$ does not divide $\dim A$.

We remark that $\lambda$ such as in (b), (c) of Part 1 above is unique. As the dual result of Part 1, a finite-dimensional semisimple Hopf algebra is unimodular. It follows from Part 2 that if $\text{char } k = 0$, then a finite-dimensional cosemisimple Hopf algebra is necessarily unimodular. Whether the same statement holds true in positive characteristic seems an open problem.

Our proof of Proposition 19 is based on the following fact.

**Lemma 21.** Let $A$ be a finite-dimensional Hopf algebra. The vector space $\mathcal{A}(k, A)$ of all morphisms $\phi : k \to A = (A, \nabla, \Delta)$ in $\mathcal{A}(k, A)$ is isomorphic, via $\phi \mapsto \phi(1)$, to the sub-vector space $k1$ in $A$.

**Proof.** Note that given an object $V$ in $\mathcal{A}(k, A)$, the vector space $\mathcal{A}(k, V)$ of all morphisms $\phi : k \to V$ is isomorphic, via $\phi \mapsto \phi(1)$, to the sub-vector space of $V$ which consists of the elements $v$ such that $av = \varepsilon(a)v$, $a \in A$ and $\rho(v) = 1 \otimes v$. (29)

Suppose $V = A$. Then the second condition of (29) is equivalent to $v \in k1$, which implies the first condition. This proves the lemma. □

For the following proof and for later use, let $H$ be a handlebody-link. Choose arbitrarily one from the top handlebody-tangles $\cap$ in $H$, and replace it by $\lambda$. Let $H^\wedge$ denote the resulting handlebody-tangle; see Figure 5. We thus have $s(H^\wedge) = 0$, $b(H^\wedge) = 1$. We call $H^\wedge$ a handlebody-tangle horned to $H$; this varies according to choice of the top $\cap$.

**Proof of Proposition 19.** Let $H$, $H^\wedge$ be as above. Lemma 21 shows that $F_A(H^\wedge)$ has values in $k1 (\subset A)$. Since $I_1(A^*)$ is spanned by $\lambda$, Theorem 20 (1) shows that the non-cosemisimplicity assumption is equivalent to the condition that $\lambda$ vanishes on $k1$, which implies that $F_A(H) = \lambda \circ F_A(H^\wedge)(1) = 0$, since $F_A(\cap) = ev_A = \lambda \circ m_A$ by (27). □

By modifying $F_A(H)$, we wish to obtain some meaningful invariant of handlebody-links $H$, when $A$ is not necessarily cosemisimple. Let $A$ be a finite-dimensional unimodular Hopf algebra, and choose $0 \neq \lambda \in I_1(A^*)$. Let $Z(A)$ denote the center of $A$. Assume that

$$\lambda(z) = \lambda(S(z)), \quad z \in Z(A). \quad (30)$$
This assumption is independent of choice of $\lambda$, and is satisfied if $A^*$ is unimodular, since then $\lambda = \lambda \circ S$, as is seen from Proposition 12. In particular, it is satisfied if $A$ is cosemisimple; see Theorem 20 (1). There are known examples of finite-dimensional cocommutative Hopf algebras which are not unimodular; see [12, p.238], for example. Their dual Hopf algebras are examples of finite-dimensional unimodular Hopf algebras which do not satisfy (30).

Definition 22. Let $A, \lambda$ be as above. For a handlebody-link $H$, we define a scalar $v_A(H)$ in $k$ by
\[ v_A(H) = \varepsilon \circ F_A(H^\wedge)(1), \]
where $H^\wedge$ is a handlebody-tangle horned to $H$. Notice from Lemma 21 that $v_A(H) = F_A(H^\wedge)$ in $k$.

We have to show that the value $F_A(H^\wedge)(1)$ is independent of choice of the top $\cap$ to be replaced by $\wedge$. This will be proved below Lemma 26.

Remark 23. (1) Suppose that $A$ is cosemisimple. Then by Theorem 20 (1), $\lambda$ can be chosen so that $\lambda(1) = 1$. In this case we have $v_A(H) = F_A(H)$ for every handlebody-link $H$, since
\[ v_A(H) = \varepsilon \circ F_A(H^\wedge)(1) = \lambda \circ F_A(H^\wedge)(1) = F_A(H). \]

(2) If $A$ is not cosemisimple, we do not have any canonical choice of $\lambda$ as above. We see from Remark 17 that if $\lambda$ is replaced by $c\lambda$ with $0 \neq c \in k$, then $v_A(H)$ changes by the scalar multiple by $c^{\#(\cap - \#A - 1)}$.

Convention 24. Taking Part 1 above into account, we will hereafter choose $\lambda$ so that $\lambda(1) = 1$ if $A$ is cosemisimple.

Lemma 25. Let $A$ be a finite-dimensional Hopf algebra. Then the vector space $A_YD(k, A \otimes A)$ of all morphisms $\phi : k \to A \otimes A$ in $A_YD$, where $A \otimes A$ is the tensor product of two copies of $A = (A, \triangledown, \Delta)$, is isomorphic, via $\phi \mapsto \phi(1)$, to the sub-vector space of $A \otimes A$ consisting of the elements $S(z_{(1)}) \otimes z_{(2)}$, where $z$ is an arbitrary element in the center $Z(A)$ of $A$.

Proof. Set $V = A \otimes A$. Give to the same vector space $A \otimes A$, an alternative structure of a Yetter–Derinfeld module by defining
\[ a(b \otimes c) := a(1)bS(a(3)) \otimes a(2)cS(a(3)), \quad \rho(b \otimes c) = b(1) \otimes (b(2) \otimes c), \]
where $a \in A$, $b \otimes c \in A \otimes A$. Let $V'$ denote the thus defined object. We see that $b \otimes c \mapsto bc(1) \otimes c(2)$ gives an isomorphism $V \xrightarrow{\sim} V'$ in $A_YD$, whose inverse is given by $b \otimes c \mapsto bS(c(1)) \otimes c(2)$. As is easily seen, the elements $1 \otimes z, z \in Z(A)$ are precisely those elements in $V'$ which satisfies the conditions (29). It follows that the elements $S(z_{(1)}) \otimes z_{(2)}, z \in Z(A)$ are precisely those which satisfies the same conditions. The proof of Lemma 24 shows the desired result.

For the rest of this section, let $A$ be a finite-dimensional unimodular Hopf algebra, and choose $0 \neq \lambda \in I_l(A^*)$ (so that $\lambda(1) = 1$ if $A$ is cosemisimple).
Lemma 26. Assume (30). For any handlebody-tangle $T$ such that $s(T) = 0, b(T) = 2$, we have
\[(\varepsilon \otimes \lambda) \circ F_A(T) = (\lambda \otimes \varepsilon) \circ F_A(T).\]

Proof. By Lemma 25 for a morphism $F_A(T) : k \to A \otimes A$ in $\mathcal{A}YD$, we have $F_A(T)(1) = S(z(1)) \otimes z(2)$ for some $z \in Z(A)$. The desired result will follow if we see that $\lambda(S(z(1)))z(2) = S(z(1))\lambda(z(2))$, or equivalently,
\[\lambda(S(z(1)))f(z(2)) = f(S(z(1)))\lambda(z(2)), \quad f \in A^*.\]
Since $\lambda \in I_l(A^*)$ and $\lambda \circ S = S^*(\lambda) \in I_r(A^*)$, we see that the assumption (30) ensures this last desired condition.

The desired independency of the value $v_A(H)$ follows since Lemmas 21 and 26 show $\tau_0 : k \to k$.

Proposition 27. Assume (30). Given two handlebody-links $H_i (\subset B_i), i = 1, 2$ contained in disjoint balls $B_i$, let $H_1 \# H_2$ denote the handlebody-link obtained by attaching them by a 1-handle. Then we have
\[v_A(H_1 \# H_2) = v_A(H_1) v_A(H_2).\]

Proof. For $i = 1, 2$, let $H_i^{\alpha}$ be a handlebody-tangle horned to $H_i$. Then $H_1^{\alpha} H_2^{\alpha} = \cap \circ (H_1^{\alpha} \otimes H_2^{\alpha})$. Since $\varepsilon : A \to k$ is an algebra map, we have
\[v_A(H_1 \# H_2) = \varepsilon \circ m_A(F_A(H_1^{\alpha}))(1) \otimes F_A(H_2^{\alpha})(1) = v_A(H_1) v_A(H_2).\]

Given a handlebody-link (or more generally, a handlebody-tangle) $H$, let $H^*$ denote its mirror image. Let us evaluate $v_A(H^*)$. Let $A^o p$ denote the Hopf algebra $A$ with the opposite product; it has $S^{-1}$ as its antipode. We can and do choose the same $\lambda$ as the original one as a non-zero left integral in $(A^o p)^*$. Proposition 28. For a handlebody-link $H$, we have
\[v_A(H^*) = v_{A^o p}(H).\]

We prove this in a generalized situation. Let $\tau_1 : A \xrightarrow{\cong} A^o p$, $\tau_1(a) = a^o p$ denote the canonical linear isomorphism, so that $a^o p b^o p = (ba)^o p$, where $a, b \in A$. Let $\tau_0 : k \to k$ be the identity map. For $n > 1$, let $\tau_n : A^o p \otimes \ldots \otimes (A^o p)^{\otimes n}$ be the linear isomorphism defined by
\[\tau_n(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = a_n^o p \otimes \ldots \otimes a_2^o p \otimes a_1^o p, \quad a_i \in A.\]
Proposition 29. Let $T$ be a handlebody-tangle such that $s(T) = m$, $b(T) = n$. Then we have
\[ \tau_n \circ F_A(T^*) = F_{A^{op}}(T) \circ \tau_m. \]

Proof. By Proposition 2, we may suppose that $T$ is one of the five tangles listed there.

Suppose $T = X$. Then $T^* = \overline{X}$. An element $a \otimes b$ in $A \otimes A$ is sent by $\tau_2 \circ F_A(X)$ to $(S^{-1}(b)(2)ab(1))^{op} \otimes b^{op}(3)$, while it is sent by $F_{A^{op}}(X) \circ \tau_2$ to $b^{op}(1)g^{op}S^{-1}(b)(2) \otimes b^{op}(3)$. Obviously, the two results coincide.

Suppose $T = \cup$. Then $T^* = \cup$. If $(\alpha_i)$, $(\beta_i)$ are the dual bases with respect to $\langle \cdot, \cdot \rangle : A \times A \to k$, that is, $\lambda(\alpha_i \beta_j) = \delta_{ij}$, then $(\beta^{op}_i)$, $(\alpha^{op}_i)$ are the dual bases with respect to $\langle \cdot, \cdot \rangle : A^{op} \times A^{op} \to k$. This implies that $\tau_2 \circ F_A(\cup)(1) = F_{A^{op}}(\cup) \circ \tau_0(1)$.

Similarly, the desired results follow in the remaining three cases. \qed

Since $\varepsilon \circ \tau_1 = \varepsilon$, Proposition 28 follows from Proposition 29 in the special situation when $m = 0, n = 1$.

6 First examples of unimodular Hopf algebras

We raise below three examples of finite-dimensional unimodular Hopf algebras $A$, giving their data needed to compute the invariants $v_A(H)$. The duals $A^*$ are all unimodular, and the antipodes $S$ of $A$ are involutions. It follows by Theorem 20 that if $\text{char } k \nmid \text{dim } A$, then $A$ is semisimple and cosemisimple. We choose two-sided integrals $\lambda$ in $A^*$ and $\Lambda$ in $A$ such that $\lambda(1) = 1$.

Example 30. Let $A = kG$ be the group algebra, where $G$ is a finite group. The Hopf algebra structure is given by
\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}
\]
where $g \in G$. We have
\[
c_{A,A}(g \otimes h) = ghg^{-1} \otimes g, \quad \lambda(g) = \delta_{1,g}, \quad \Lambda = \sum_{g \in G} g,
\]
\[
ev_A(g \otimes h) = \delta_{1,gh}, \quad \text{coev}_A(1) = \sum_{g \in G} g \otimes g^{-1},
\]
where $g, h \in G$.

Note $\Lambda(1) = 1$ and that this $A$ is cosemisimple. By Maschke’s Theorem, $A$ is semisimple if and only if $\text{char } k$ does not divide the order $|G|$ of $G$.

We remark that if $\text{char } k = 0$, then the invariant $v_A(H)$ coincides with the number of the homomorphisms from the fundamental group of the exterior of a handlebody-knot $H$ to the group $G$. For, when we regard the value of the invariant as a state sum, each state corresponds to the $G$-coloring of the diagram.
Example 31. Let $A = D(kG)$ be the quantum double of $kG$, where $G$ is a finite group. Note that the dual Hopf algebra $(kG)^\ast$ of $kG$ is spanned by those orthogonal idempotents $e_g$, $g \in G$, which are defined by $e_g(h) = \delta_{g,h}$, where $g, h \in G$. As a coalgebra, $A = kG \otimes (kG)^\ast$, and so

$$\Delta(a \otimes e_g) = \sum_{h \in G} (a \otimes h) \otimes (a \otimes h^{-1}g), \quad \varepsilon(a \otimes e_g) = \delta_1, g,$$

where $a, g \in G$. The product and the antipode on $A$ are given by

$$(a \otimes e_g)(b \otimes e_h) = \delta_{g,hb^{-1}}ab \otimes e_h,$$

$$S(a \otimes e_g) = (1 \otimes e_g^{-1})(a^{-1} \otimes 1) = a^{-1} \otimes e_{ag^{-1}a^{-1}},$$

where $a, b, g, h \in G$. The unit equals $1 \otimes 1$. The remaining data are given by

$$c_{A,A}((a \otimes e_g) \otimes (b \otimes e_h)) = (aba^{-1} \otimes e_{aha^{-1}}) \otimes (a \otimes e_{hbb^{-1}b^{-1}g}),$$

$$\lambda(a \otimes e_g) = \delta_{1,a}, \quad \Lambda = \sum_{a \in G} a \otimes e_1,$$

$$\text{ev}_A((a \otimes e_g) \otimes (b \otimes e_h)) = \delta_{1,a}\delta_{g,hb^{-1}},$$

$$\text{coev}_A(1 \otimes 1) = \sum_{a \in G} (a \otimes e_g) \otimes (a^{-1} \otimes e_{aga^{-1}}).$$

Note $\lambda(1) = |G|$. This $A$ is semisimple if and only if it is cosemisimple if and only if char $k \nmid |G|$. If these equivalent conditions hold, we should replace the integrals $\lambda, \Lambda$ above with $|G|^{-1}\lambda, |G|\Lambda$, respectively.

Example 32. Assume that the characteristic char $k$ of $k$ is not 2. Fix an integer $m > 2$. Let $A = B_{3m}$ be the Hopf algebra as defined by [9, Definition 3.3(2)]. As an algebra this is generated by three elements, $a, t, z$, and is defined by the relations

$$a^2 = t^2 = 1, \quad ta = at, \quad za = az, \quad z^m = a, \quad zt = tz^{-1}.$$ 

Here we have re-chosen the generators $s_{\pm 1}$ given in [9, Definition 3.3(2)] so that $t = s_+, z = s_+s_-$, as in [9, Page 203, line -3]. Note $z^{-1} = az^{m-1}$. Set $e_0 = (1/2)(1 + a)$, $e_1 = (1/2)(1 - a)$; these are central idempotents in $A$ such that $e_0e_1 = 0$, $e_0 + e_1 = 1$. The structure on $A$ is given by

$$\Delta(a) = a \otimes a, \quad \varepsilon(a) = 1, \quad S(a) = a,$$

$$\Delta(t) = t \otimes e_0t + tz \otimes e_1t, \quad \varepsilon(t) = 1, \quad S(t) = t(e_0 + e_1z),$$

$$\Delta(z) = z \otimes e_0z + z^{-1} \otimes e_1z, \quad \varepsilon(z) = 1, \quad S(z) = e_0z^{-1} + e_1z.$$

This $A$ has $(a^i t^j z^k)_{0 \leq i, j, k \leq m}$ as a basis, so that dim $A = 4m$. Note that $(e_i t^j z^k)_{0 \leq i, j, k \leq m}$ is another basis of $A$. Let $0 \leq i, j, p, q < 2$, $0 \leq k, r < m$. Set

$$d(i, j, k, p, q, r) = (-1)^j \{r - (-1)^i p(2k - j)q\} + jq.$$
Then we have
\[ c_{A,A}(e_i t^j z^k \otimes e_p t^q z^r) = e_p t^q z^{d(i,j,k,p,q,r)} \otimes e_i t^j z^k. \]

Note that if \( q = 0 \) in particular, then
\[ c_{A,A}(e_i t^j z^k \otimes e_p t^q z^r) = e_p z^{(-1)^i r} \otimes e_i t^j z^k. \]

The remaining data are given by
\[
\lambda(a^i t^j z^k) = \delta(i,j,k),
\Lambda = (1 + a)(1 + t)(1 + z + \cdots + z^{m-2} + az^{m-1}),
\]
\[
ev_A(a^i t^j z^k \otimes a^p t^q z^r) = \delta_{(i,j,k),(p,q,r)}(\delta_{(j,k),(0,0)} + \delta_{j,1}) + \delta_{(i,j,k),(1-p,q,m-r)}\delta_{j,0},
\]
\[
\text{coev}_A(1) = \sum_{0 \leq i < 2} a^i \otimes a^i + \sum_{0 \leq i < 2, 0 \leq k < m} a^i t^k z^k \otimes a^i t^k z^k + \sum_{0 \leq i < 2} a^i z^k \otimes a^{i+1} z^{m-k}.\]

It is easy to represent these data with respect to the other basis \((e_i t^j z^k)_{0 \leq i,j < 2, 0 \leq k < m}\).

Note \( \lambda(1) = 1 \), and that this \( A \) is cosemisimple. It is known that \( A \) is semisimple if and only if \( \text{char } k \nmid 2m \). Moreover, if \( k \) contains a primitive \( 4m \)-th root of 1, then \( A \) is selfdual, that is, \( A \simeq A^* \) as Hopf algebras.

Table 7 lists the invariant \( v_A(H) \) for the handlebody-knots \( 0_1, \ldots, 6_{16} \) in the table given in [5] when \( m = 3, \ldots, 7 \).

### 7 The invariants derived from the finite quantum group \( \overline{U}_q \)

Recall from [6, Sect.VI.5] the finite quantum group \( \overline{U}_q \) associated to \( \text{sl}_2 \). Let \( q \in k \setminus \{ \pm 1 \} \) be a root of 1, and let \( e \) (\( > 1 \)) denote the order of \( q^2 \). As an algebra, \( \overline{U}_q \) is generated by \( K, E \) and \( F \), and is defined by the relations
\[
KE = q^2 KF, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},
\]
\[
K^e = 1, \quad E^e = F^e = 0.
\]

This \( \overline{U}_q \) is a Hopf algebra with respect to the structure
\[
\Delta(K) = K \otimes K, \quad \varepsilon(K) = 1, \quad S(K) = K^{-1},
\]
\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \varepsilon(E) = 0, \quad S(E) = -EK^{-1},
\]
\[
\Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \varepsilon(F) = 0, \quad S(F) = -KF.
\]

To apply results of Radford [13], it is convenient to replace the generators above with
\[
a = K, \quad x = \frac{1}{q - q^{-1}} FK, \quad y = E.
\]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
   & $m = 3$ & $m = 4$ & $m = 5$ & $m = 6$ & $m = 7$ \\
\hline
0_i & 144 & 256 & 400 & 576 & 784 \\
4_i & 216 & 256 & 400 & 864 & 784 \\
5_i & 144 & 256 & 400 & 576 & 784 \\
5_2 & 216 & 256 & 400 & 864 & 784 \\
5_3 & 144 & 256 & 400 & 576 & 784 \\
5_4 & 144 & 256 & 400 & 576 & 784 \\
6_1 & 144 & 256 & 400 & 576 & 784 \\
6_2 & 144 & 256 & 400 & 576 & 784 \\
6_3 & 144 & 256 & 400 & 576 & 784 \\
6_4 & 144 & 256 & 400 & 576 & 784 \\
6_5 & 144 & 256 & 400 & 576 & 784 \\
6_6 & 144 & 256 & 400 & 576 & 784 \\
6_7 & 144 & 256 & 800 & 576 & 784 \\
6_8 & 144 & 256 & 400 & 576 & 784 \\
6_9 & 216 & 256 & 400 & 864 & 784 \\
6_{10} & 144 & 256 & 400 & 576 & 784 \\
6_{11} & 144 & 256 & 400 & 576 & 784 \\
6_{12} & 144 & 256 & 800 & 576 & 784 \\
6_{13} & 216 & 256 & 400 & 864 & 784 \\
6_{14} & 288 & 256 & 400 & 1152 & 784 \\
6_{15} & 288 & 256 & 400 & 1152 & 784 \\
6_{16} & 144 & 256 & 400 & 576 & 784 \\
\hline
\end{tabular}
\caption{$\mathcal{B}_{4m}$}
\end{table}
Then the defining relations turn into

\[ xa = q^2ax, \quad ya = q^{-2}ay, \quad xy - q^{-2}yx = a^2 - 1, \quad a^c = 1, \quad x^e = y^e = 0. \]

We have

\[ \Delta(x) = 1 \otimes x + x \otimes a, \quad \varepsilon(x) = 0, \quad S(x) = -xa^{-1}. \]

The Hopf algebra \( \mathbb{U}_q \) thus presented coincides with Radford’s \( \mathbb{U}_{(N, \nu, \omega)} \) in the special situation when \( N = e, \nu = 1 \) and \( \omega = q^2 \); see [15, Sect.5.2]. We remark that the \( q \) in [15] should read our \( q^2 \). The first three parts of the following proposition are proved in Propositions 10, 11 of [15].

**Proposition 33.**  
(1) \( (a^ix^jy^k)_{0 \leq i,j,k < e} \) is a basis of \( \mathbb{U}_q \), so that \( \dim \mathbb{U}_q = e^3 \).

(2) \( \mathbb{U}_q \) is unimodular, and

\[ \Lambda = \left( \sum_{i=0}^{e-1} a^i \right) x^{e-1}y^{e-1} \]

is a non-zero two-sided integral in \( \mathbb{U}_q \).

(3) The elements \( \lambda, \lambda' \) of \( \mathbb{U}_q^e \) defined by

\[ \lambda(a^ix^jy^k) = \delta_{(i,j,k),(0,e-1,e-1)}, \quad \lambda'(a^ix^jy^k) = \delta_{(i,j,k),(2,e-1,e-1)}, \]

where \( 0 \leq i, j, k < e \), are the left and the right integrals, respectively, in \( \mathbb{U}_q \) such that \( \lambda(\Lambda) = 1 = \lambda'(\Lambda) \). It follows that \( \mathbb{U}_q \) is not cosemisimple.

(4) \( \mathbb{U}_q \) satisfies the assumption of Lemma [26].

**Proof.** Let us prove Part 4. Since \( \lambda(\Lambda) = \lambda'(\Lambda) \) by Part 3, we see from Proposition [12] that \( \lambda \circ S = \lambda' \). Let \( z \in Z(\mathbb{U}) \). Since \( z \) commutes with \( a \), we have \( z = \sum_{i,j=0}^{e-1} c_{ij}a^i x^j y^k \) with \( c_{ij} \in k \). To prove \( \lambda(z) = \lambda'(z) \), we wish to show that \( c_{0,e-1} = c_{2,e-1} \). The formula given in [15, p.256, lines 2–3] tells us that for each \( 0 < j < e \),

\[ yx^j = q^{-2j}x^j y + (j)_{q^2}a^2x^{j-1} - q^{-2(j-1)}(j)_{q^2}x^{j-1}, \]

(31)

where \( (j)_{q^2} = \sum_{i=0}^{j-1} q^{2i} \). This implies that the term \( a^2x^{e-2}y^{e-1} \) in \( yz \) arises from the product of \( y \) with the terms \( a^2x^{e-2}y^{e-2}, x^{e-1}y^{e-1}, a^2x^{e-1}y^{e-1} \) in \( z \). It follows that the coefficient of \( a^2x^{e-2}y^{e-1} \) in \( yz \) equals

\[ c_{2,e-2}q^{-2e} + c_{0,e-1}(e-1)_{q^2} - c_{2,e-1}q^{-2e}(e-1)_{q^2} = c_{2,e-2} + (c_{0,e-1} - c_{2,e-1})(e-1)_{q^2}, \]

while the same coefficient in \( zy \) equals \( c_{2,e-2} \). This proves \( c_{0,e-1} = c_{2,e-1} \), as desired.

\[ \square \]
In what follows we suppose that the base field $k$ is the field $\mathbb{C}$ of complex numbers. Hence, $q^{-1}$ equals the complex conjugate $\bar{q}$ of $q$. We re-choose $\Lambda, \lambda$ given in Proposition 33 (2), (3) so that the derived invariants behave preferably with mirror images. For $q$ as above, we define complex numbers $c_q, \epsilon_q$ by

$$c_q = \bar{q}^2(q - \bar{q})e^{-1}, \quad \epsilon_q = q^e.$$  

Note that $\epsilon_q = 1$ if the order $\text{ord}_q$ of $q$ is odd, and $\epsilon_q = -1$ if $\text{ord}_q$ is even, and so that $\epsilon_q = \epsilon_{\bar{q}}$. We define

$$\Lambda_q = c_q\Lambda, \quad \lambda_q = c_{\bar{q}}^{-1}\lambda.$$  

One sees that $\Lambda_q$ is a two-sided integral in $U_q$, and $\lambda_q \in I_l(U_q^*)$ with $\lambda_q(\Lambda_q) = 1$.

**Lemma 34.** We have

$$\Lambda_q = \epsilon_q^{-1} \left( \sum_{i=0}^{e-1} K^i \right) F^{e-1} E^{e-1}.$$  

**Proof.** This follows since one computes

$$\left( \sum_{i=0}^{e-1} K^i \right) F^{e-1} E^{e-1} = (q - \bar{q})^{e-1} \left( \sum_{i=0}^{e-1} a^i \right) (xa^{-1})^{e-1} y^{e-1}$$

$$= q^{-2(e-1)}(q - \bar{q})^{e-1}\Lambda = q^{-e(e-1)}q^{-2}(q - \bar{q})^{e-1}\Lambda = \epsilon_q^{-1} c_q \Lambda = \epsilon_q^{-1} \Lambda_q.$$  

\[ \square \]

**Lemma 35.** $K \mapsto K^{op}, E \mapsto E^{op}$ and $F \mapsto F^{op}$ give a Hopf algebra isomorphism $U_q \xrightarrow{\sim} U_q^{op}$, under which $\Lambda_q \mapsto \Lambda_q^{op}$.

**Proof.** It is well-known that the correspondences above gives a Hopf algebra isomorphism. To see that $\Lambda_q \mapsto \Lambda_q^{op}$, it suffices to prove that

$$E^{e-1} F^{e-1} \left( \sum_{i=0}^{e-1} K^i \right) = \left( \sum_{i=0}^{e-1} K^i \right) F^{e-1} E^{e-1}$$

in $U_q$, since $\epsilon_q = \epsilon_{\bar{q}}$. By (31) for $j = e - 1$, we see

$$y^{e-1} x^{e-1} \left( \sum_{i=0}^{e-1} a^i \right) = \bar{q}^2 \left( \sum_{i=0}^{e-1} a^i \right) x^{e-1} y^{e-1}.$$  

By multiplying $a^{-(e-1)} = a$ from the right, it follows that

$$y^{e-1} (xa^{-1})^{e-1} \left( \sum_{i=0}^{e-1} a^i \right) = \left( \sum_{i=0}^{e-1} a^i \right) (xa^{-1})^{e-1} y^{e-1},$$

which implies the desired equality.  

\[ \square \]
By Proposition 33 (4), \( \overline{U}_q \) together with \( \lambda_q \) defines the invariant \( v_{\overline{U}_q}(H) \) for each handlebody-knot \( H \). Let us write simply \( v_q(H) \) for this.

**Proposition 36.** Given a handlebody-knot \( H \), the invariant \( v_q(H^*) \) of the mirror image \( H^* \) of \( H \) equals the complex conjugate \( v_q(H) \), that is, \( v_q(H^*) = \overline{v_q(H)} \).

**Proof.** By Proposition 32, the composite of the isomorphism in Lemma 25 with \( \lambda_q \) coincides with \( \lambda_{\overline{U}_q} \). Then Proposition 28 shows \( v_q(H^*) = v_q(H) \). It remains to prove \( v_q(H^*) = \overline{v_q(H)} \). This equality holds since the Hopf algebra \( \overline{U}_q \) and the linear map \( \lambda_{\overline{U}_q} : U_q \rightarrow \mathbb{C} \) are the base extensions of \( U_q, \lambda_q \), respectively, along the complex conjugation \( \mathbb{C} \rightarrow \mathbb{C} \).

**Remark 37.** Let \( q = e^{2\pi \sqrt{-1}/n} \). For \( n \leq 4 \) we have checked by computer calculation that the invariant does not detect the handlebody-knots \( 0_1, \ldots, 6_{16} \) given in [5]. For \( n > 4 \) the calculation takes so far too long time for us to see whether the invariant is non-trivial.

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