NEUTRAL AND INDIFFERENCE PRICING WITH STOCHASTIC CORRELATION AND VOLATILITY

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ABSTRACT. In this paper, we consider a Wishart Affine Stochastic Correlation (WASC) model which accounts for the stochastic volatilities of the assets and for the stochastic correlations not only between the underlying assets’ returns but also between their volatilities. Under the assumptions of the model, we derive the neutral and indifference pricing for general European-style financial contracts. The paper shows that comparing to risk-neutral pricing, the utility-based pricing methods are generally feasible and avoid factitiously dealing with some risk premia corresponding to the volatilities-correlations as a consequence of the incompleteness of the market.

1. Introduction. In 1970s, Black and Scholes derived an analytic formula to pricing options, which is based on the no-arbitrage valuation theory ([3]). Since then, there have been many works to price financial contracts. In the early financial theories, the completeness of the markets is one of the most important hypotheses, with which one can eliminate randomness from the time evolution of a financial portfolio. However, risk-elimination can be achieved only for simple underlying models. In reality, eliminating all risk (such as stochastic interest rate, stochastic volatility and so on) completely is impossible in an incomplete market. Nowadays, as a pricing framework and a methodology, the utility-based pricing theory has been widely used to price the remaining risk and to find the so-called risk premium (see, for example, [5] and the references therein).

In utility-based pricing, there exist two most important frameworks: the neutral pricing and the indifference pricing. In [15], for a certain type of the investor’s risk-aversion, Kallsen introduced the neutral price of an asset (or a portfolio of assets) so that it is optimal to hold a specific investment in the market. In [14] and [7], for a given utility function, the authors proposed the indifference price of an asset so that having a specific position or not having one is indifferent from the point of view of investing in the rest of the market. Two concepts of pricing could be distinguished in the following way: the neutral price is the price at which it is optimal for the investor to make no investment (neither long nor short) into the

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asset (or the portfolio of assets), while the indifference price is the price at which it is irrelevant whether the investor make investments into the the asset (or the portfolio of assets) or not from the point of view of investing in the rest of the market ([24]). Above all, in [25], Stojanovic established a feasible and systemic method to obtain the neutral and indifference price of a European-style financial contract (or contracts per single).

In recent years, empirical evidence (see [1, 8]) has shown that the classic Black-Scholes assumption of lognormal stock diffusion with constant volatility is not consistent with the market price, which is often referred to as the volatility skew or smile. The authors of [6] and [26] show that the correlations between financial assets evolve stochastically and are far from remaining static through time. Furthermore, in [16] and [20], there is evidence which presents that the higher the market volatility is, the higher the correlations between financial assets tend to be. On the other hand, there has been tremendous growth of multi-asset financial contracts (out-performance options, for example) which exhibit sensitivity to both the volatilities and the correlations of the underlying assets. As a result, it is meaningful to price contracts corresponding to such underlying assets.

To deal with such phenomenon, Gourieroux and Sufana [11] introduced the Wishart process, which was mathematically developed in [4], to describe the stochastic volatilities of the assets and to capture the stochastic correlations not only between the underlying assets’ returns but also between their volatilities. With the Wishart process models, by means of risk-neutral pricing, the authors of [9] and [19] give the price of the ‘best of’ basket options and the outerperformance options, respectively. Both of them show the advantages of using the Wishart specification which is consistent with the empirical evidence. They use the following Wishart process (see Section 2)

\[
d\Sigma_t = (\Omega \cdot \Omega^T + M \cdot \Sigma_t + \Sigma_t \cdot M^T)dt + \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot d\overline{B}_t \cdot \sqrt{\Sigma_t},
\]

under the risk-neutral measure to picture the stochastic correlation and volatility of the assets. However, following [11], under the risk-neutral measure, (1) should be as follows:

\[
d\Sigma_t = (\Omega \cdot \Omega^T + M \cdot \Sigma_t + \Sigma_t \cdot M^T + 2\Sigma_t \cdot C_t \cdot Q^T \cdot Q + 2Q^T \cdot Q \cdot C_t^T \cdot \Sigma_t)dt
+ \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot d\overline{B}_t \cdot \sqrt{\Sigma_t},
\]

where \( C_t \) is a consequence of market incompleteness. Let \( C_t = C \) and \( M^* = M + 2Q^T \cdot Q \cdot C^T \) in (2). Then it is easy to see that (1) is a special case of (2) when \( C_t \) is constant.

Generally speaking, the Wishart specification leads to the incompleteness of the market and so we still face dealing with some coefficients which determine the risk premia corresponding to the volatilities-correlations (\( C_t \) in equation (2)) if we want to use the risk-neutral pricing. Usually, \( C_t \) is assumed to be constant ([9] and [19] are such cases), which is hard to guarantee the true property of the incomplete market. The PDE systems of neutral and indifference pricing introduced by Stojanovic [25] are proved to be efficient for general European-style financial contracts with Ito specification and avoid the assumption that \( C_t \) is constant by determining the unknown coefficient in the pricing PDE. However, to the best of our knowledge, there is no paper to study the utility-based pricing for WASC model. The main purpose of the this paper is to derive the neutral and indifference pricing for general European-style financial contracts with the WASC specification.
The rest of the paper is organized as follows. Section 2 sketches the main assumptions and results of the WASC model and of the neutral and indifference pricing in sense of Stojanovic [25]. Then we derive the neutral and indifference pricing for general European-style financial contracts with the Wishart specification in Section 3. In Section 4, we show some applications of our main results to the forward contracts. Finally, some conclusions and future problems are included in Section 5.

2. Preliminaries. In this section, we shall give some basic and important results for Wishart Affine Stochastic Correlation (WASC) model and for the neutral and indifference pricing, which can be found in [4, 9, 12, 21, 22, 23, 24, 25].

Let \((\Lambda, \mathcal{F}, \mathbb{P})\) be the probability space, where \(\mathbb{P}\) denotes real (observed) world probability measure. Let \(S_n(\mathbb{R}) (S_n^+(\mathbb{R}))\) denote the set of all real-valued \(n \times n\) symmetric (positive definite, respectively) matrices.

In probability space \((\Lambda, \mathcal{F}, \mathbb{P})\), we shall derive the following dynamic Wishart Affine Stochastic Correlation (WASC) model. Now assume that there exist \(n(n \geq 1)\) underlying assets in the market that have such an Itô dynamic equation:

\[
dS(t) = S^\ast(t) \cdot (a_S(t, \Sigma_t) - q(t, \Sigma_t))dt + S^\ast(t) \cdot \sqrt{\Sigma_t} \cdot dH(t),
\]

where the dot \(\cdot\) throughout denotes the product of two matrices, \(a_S\) is an \(n \times 1\)-matrix of predividend appreciation rates for \(S\), \(q\) is an \(n \times 1\)-matrix of corresponding dividend rates, \(\Sigma_t \in S_n^+(\mathbb{R})\) is a symmetric positive definite \(n \times n\)-matrix of volatility, \(H(t) = (H_1(t), \ldots, H_n(t))^T\) is an \(n\)-dimensional Brownian motion and \(S^\ast(t)\) is an \(n \times n\)-matrix corresponding to \(S(t)\), i.e.

\[
S^\ast(t) = \begin{pmatrix}
S_1(t) & 0 & \cdots & 0 \\
0 & S_2(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & S_n(t)
\end{pmatrix}.
\]

Following [11], the volatility matrix \(\Sigma_t\) is assumed to be the following Wishart process:

\[
d\Sigma_t = (\Omega \cdot \Omega^T + M \cdot \Sigma_t + \Sigma_t \cdot M^T)dt + \sqrt{\Sigma_t} \cdot dB_t + Q \cdot \sqrt{\Sigma_t} \cdot dB_t^T,
\]

with \(\Sigma_0 \in S_n^+(\mathbb{R})\), a.s., where \(\Omega, M, Q\) are \(n \times n\)-matrices with \(\Omega\) invertible, \(B(t)\) is a matrix of Brownian motion in \(\mathbb{R}^{n \times n}\). From [4], in order to guarantee the properties of volatility which is strictly positive and has typical mean reverting feature, the matrix \(M\) is assumed to be negative semi-definite and \(\Omega\) satisfies

\[
\Omega \cdot \Omega^T = \beta Q^T \cdot Q,
\]

where \(\beta\) is a real number and \(\beta \geq n + 1\) (and for more details one can refer to [2]) such that (4) has a unique strong solution in \(S_n^+(\mathbb{R})\) on \(t \in [0, +\infty)\). To be precise, following from [4], more complex conditions (which do not exist in [2, 9, 19]) are needed to ensure the unique strong solution in \(S_n^+(\mathbb{R})\) on \(t \in [0, +\infty)\) of (4). From Remark 4.16 in [17], we give the condition as \(Q^T \cdot Q\) and \(M\) are commute.

Moreover, following [9], the Brownian motion \(H_t\) in equation (3) is related to the Brownian matrix \(B_t\) in equation (4) in the form of

\[
H_t = B_t \cdot \rho + \sqrt{1 - \rho^2} \cdot \rho W_t,
\]

where \(W_t\) is another \(n\)-dimensional Brownian motion independent of \(B_t\) and \(\rho = (\rho_1, \ldots, \rho_n)^T\) with \(\rho_1, \ldots, \rho_n \in [-1, 1]\), which is a fixed correlation vector between the returns and the state variables.
The structure of stochastic correlation and volatility in WASC model, as well as its importance, has been discussed in details and one can refer to [9, 11, 19]. For example, the stochastic cross correlation, denoted by \(\rho_{ij}^j\), between asset \(i\) and asset \(j\) \((i, j = 1, 2, \cdots, n\) and \(i \neq j\)) is given by

\[
\rho_{ij}^j = \frac{\Sigma_{ij}(t)}{\sqrt{\Sigma_{ii}(t)\Sigma_{jj}(t)}}
\]

We note that equations (3)-(5) form the basic theory of WASC model which describes the dynamic diffusion of the price of multiple assets without involving pricing contracts. In order to price contracts by employing the WASC model, we need to introduce some concepts in connection with the simple economy and some known results for the neutral and indifference pricing as lemmas.

**Definition 2.1.** A simple economy \(\mathcal{C}\) is made up of three factors \(S(t), A(t), r(t, A(t))\) which are assumed to obey the following Itô stochastic differential equations dynamics (SDE, see [10]):

\[
\begin{align*}
\{ & dS(t) = S^*(t) \cdot (a_S(t, A(t)) - D(t, A(t))) dt + S^*(t) \cdot \sigma_S(t, A(t)) \cdot dZ(t), \\
& dA(t) = b(t, A(t)) dt + c(t, A(t)) \cdot dZ(t),
\}
\]

(6)

where

- \(S(t) = (S_1(t), S_2(t), \cdots, S_k(t))^T\) is a market that consists of a finite nonempty set of stochastic tradable financial securities;
- \(A(t) = (A_1(t), A_2(t), \cdots, A_m(t))^T\) is a finite (possible empty) stochastic dynamic factors in the market;
- \(r(t, A(t))\) is the short rate in the market;
- \(Z(t) = (Z_1(t), Z_2(t), \cdots, Z_n(t))^T\) is an \(n\)-dimensional Brownian motion;
- \(a_S(t, A(t))\) is the \(k \times 1\)-matrix of predividend appreciation rates for the tradables, and \(D(t, A(t))\) is the \(k \times 1\)-matrix of corresponding dividend rates;
- \(\sigma_S(t, A(t))\) is the \(k \times n\) stochastic volatility matrix;
- \(b(t, A(t)), c(t, A(t))\) are the \(m \times 1\)-matrix of factor drifts and \(m \times n\)-matrix of factor diffusion, respectively.

**Definition 2.2.** The state space for a simple economy \(\mathcal{C}\) consists of all possible values of the factor \(A(t)\), denoted by \(\mathcal{A}\).

**Definition 2.3.** A market \(S(t) = (S_1(t), S_2(t), \cdots, S_k(t))^T\) is said to be nonredundant if \(|\sigma_S \cdot \sigma_S^T| > 0\).

Throughout this paper, we always assume that the market is nonredundant.

**Definition 2.4.** A market is said to be complete if \(m = 0\) or if the following conditions are satisfied: \(m \geq 1\) and

\[
c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T = O_{m \times m},
\]

(7)

where \(I_n\) denotes the \(n\)-identity matrix and \(O_{j \times k}\) denotes the \(j \times k\) zero matrix for any integers \(j, k > 0\). Otherwise, the market is incomplete.

**Remark 1.** It follows from Lemma 2.3.1 in [25] that the completeness of the market given by Definition 2.4 is equivalent to the following condition:

\[
\text{SpanOfRows}[c] \subseteq \text{SpanOfRows}[\sigma_S]
\]

which means that “all factor-randomness is tradable”.

Theorem 2.5. If \(\sigma_S \cdot \sigma_S^T > 0\), then the market is complete.
Let $X \in \mathbb{R}$ denote the investors' wealth. Assume that the investors' preferences are quantified by a wealth utility function, denoted by $\psi(X)$, which is a continuous, strictly increasing, strictly concave and continuously differentiable function defined on its domain $\mathcal{X} = \text{dom}(\psi)$. There are two families of well-known and widely used utility functions. One family is the constant absolute risk-aversion (CARA) utility functions

$$
\psi(X) = \frac{1 - e^{\omega(1-X)}}{\omega}
$$

for $\omega > 0$ and $\mathcal{X} = (-\infty, \infty)$. And the other is the constant relative risk-aversion (CRRA) utility functions

$$
\psi(X) = \frac{X^{1-\gamma} - 1}{1-\gamma}
$$

for $\gamma > 0, \gamma \neq 1$ and $\mathcal{X} = (0, \infty)$.

Now in a simple economy, we can consider the problem of neutral and indifference pricing. In this paper we only consider the case of the single contract pricing. Let $V(t)$ be a European contract expiring at time $T \leq \infty$ (if $T = \infty$, it means the contract is perpetual) with $D$ denoting the dividend payoff per year and $V(T) = v(A(T))$ denoting the terminal payoff. Assume that $\Pi(t, X(t), A(t)) = (\Pi_1(t, X(t), A(t)), \ldots, \Pi_{k+1}(t, X(t), A(t)))^T$ denotes an investor’s (continuous) trading strategy, where $X(t)$ is the investor’s wealth at time $t$, $\Pi_l (1 \leq l \leq k)$ is the investment on $S_l$, $\Pi_{k+1}$ is the investment on $V$ and $X^{\Pi}(t)$ denotes the wealth corresponding to the trading strategy $\Pi$.

**Definition 2.5.** Giving a utility function $\psi$, a market position $\kappa(t, X, A)$, terminal payoff $v(A)$ and dividend payoff $D(t, A)$, the $\psi$-$\kappa$-neutral price $V_{\psi, \kappa}(t, X, A)$ is defined as a solution of the following equation

$$
\Pi^{*}_{\psi}(t, X, A) = (\sim, \kappa(t, X, A)V(t, X, A))^T
$$

for $t < T$, where $\sim$ is an arbitrary expression, $\Pi^{*}_{\psi}$ is the solution of following problem

$$
\sup_{\Pi} E_{t,X,A}\psi(X^{\Pi}(T)) = E_{t,X,A}\psi(X^{\Pi^{*}_{\psi}}(T)),
$$

with the terminal condition

$$
V_{\psi, \kappa}(T, X, A) = v(A).
$$

**Remark 2.** The market position or portfolio $\kappa$ mentioned in Definition 2.5 is the number of financial contracts held at time $t$. In the case $\kappa > 0$, contracts with the number $\kappa$ are bought by the investor and in the case $\kappa < 0$, contracts with the number $\kappa$ are shorted by the investor. We note that $\kappa = 0$ is also allowed, which means that the investor make no investment (neither long nor short) into the contracts at time $t$. Therefore, $\kappa$ quantifies the financial contracts to be priced.

**Definition 2.6.** Giving a utility function $\psi$, a market position $\kappa(t, X, A)$, terminal payoff $v(A)$ and dividend payoff $D(t, A)$, the $\psi$-$\kappa$-indifference price $V_{\psi, \kappa}(t, A)$ is defined as a solution of the equation

$$
\sup_{\Pi=(\Pi_1, \ldots, \Pi_{k+1})} E_{t,X,A}\psi(X^{\Pi}(T)) = \sup_{\Pi=(\Pi_1, \ldots, \Pi_{k})} E_{t,X,A}\psi(X^{\Pi}(T))
$$

for $t < T$, with the terminal condition (9).

We also need the following lemmas, whose proofs can be found in Chapter 4 of [25].
Lemma 2.7. Providing $V = V(t, A)$ (i.e. the price $V$ does not depend on the investor’s wealth $X$), for a position $\kappa = \kappa(t, X, A)$ and any utility function $\psi = \psi(X)$, the $\psi$-$\kappa$-neutral price $V_{\psi, \kappa}(t, A)$ is the solution of the following coupled pricing system of partial differential equations (PDEs):

\[
\mathcal{M}(\varphi) - \frac{1}{2} \left( \frac{\partial^2 \varphi}{\partial X^2} \right)^2 \kappa^2 (\nabla_A V)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V) = 0 \quad (11)
\]

for $t < T$, $A \in A$, with the terminal condition $\varphi(T, X, A) = \psi(X)$, and

\[
\begin{align*}
\frac{\partial V}{\partial t} + & \frac{1}{2} \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A^T V) \right] - rV \\
+ & \left( b^T - (a_S - r\tilde{r})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \right) \cdot (\nabla_A V) \\
+ & \left( \frac{\partial \varphi}{\partial X} \right)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V) \\
+ & \left( \frac{\partial^2 \varphi}{\partial X^2} \right) \kappa (\nabla_A V)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V) \\
= & -\mathcal{D}
\end{align*}
\]

for $t < T$, $A \in A$, with the terminal condition $V(T, A) = v(A)$, where $\text{Tr}$ denotes the trace of a matrix,

\[
\nabla_A = \left( \frac{\partial}{\partial A_1}, \cdots, \frac{\partial}{\partial A_m} \right)^T,
\]

$\tilde{r} = (1, 1, \cdots, 1)^T$ is a $k \times 1$-matrix, and $\mathcal{M}$ is the differential operator of Monge-Ampère type

\[
\mathcal{M}(\varphi) = \frac{\partial^2 \varphi}{\partial X^2} \cdot \frac{\partial \varphi}{\partial X} \cdot \frac{\partial \varphi}{\partial X} \cdot \frac{\partial \varphi}{\partial X} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial X^2} \cdot \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A^T \varphi) \right] - \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^2 (a_S - r\tilde{r})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot (a_S - r\tilde{r}) + \frac{\partial^2 \varphi}{\partial X^2} (b^T \cdot \nabla_A \varphi) - \frac{\partial \varphi}{\partial X} (a_S - r\tilde{r})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \cdot \left( \nabla_A \frac{\partial \varphi}{\partial X} \right) - \frac{1}{2} \left( \nabla_A \frac{\partial \varphi}{\partial X} \right)^T \cdot c \cdot \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \cdot \left( \nabla_A \frac{\partial \varphi}{\partial X} \right) + rX \frac{\partial^2 \varphi}{\partial X^2} \frac{\partial \varphi}{\partial X}.
\]

Lemma 2.8. Providing $V = V(t, A)$, for a position $\kappa = \kappa(t, X, A)$ and any utility function $\psi = \psi(X)$, the $\psi$-$\kappa$-indifference price $V_{\psi, \kappa}(t, A)$ is the solution of the following uncoupled pricing system of PDEs:

\[
\mathcal{M}(\varphi) = 0
\]

for $t < T$, $A \in A$, with the terminal condition $\varphi(T, X, A) = \psi(X)$, and

\[
\begin{align*}
\frac{\partial V}{\partial t} + & \frac{1}{2} \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A^T V) \right] - rV \\
+ & \left( b^T - (a_S - r\tilde{r})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \right) \cdot (\nabla_A V) \\
+ & \left( \nabla_A \frac{\partial \varphi}{\partial X} \right)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V)
\end{align*}
\]
\[
+ \frac{1}{2} \left( \frac{\partial^2 \varphi}{\partial X^2} \right) \kappa(\nabla_A V)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V)
= -D
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( V(T, A) = v(A) \).

**Lemma 2.9.** In the above framework, providing that the interest rate is deterministic, for a position \( \kappa = \kappa(t, A) \) and for the CARA utility function \( \psi(X) = (1 - e^{(1-X)\omega})/\omega \), the \( \omega\)-\( \kappa \)-neutral price \( V_{\omega, \kappa}(t, A) \) is the solution of the following coupled pricing system of PDEs:

\[
\frac{\partial g_\omega}{\partial t} + \frac{1}{2} Tr \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla^T_A g_\omega) \right] + \left( b^T - (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \right) \cdot (\nabla_A g_\omega) + \frac{1}{2} (\nabla_A g_\omega)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V)
= \frac{1}{2} (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot (a_S - r\bar{1})
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( g_\omega(T, A) = 0 \), and

\[
\frac{\partial V}{\partial t} + \frac{1}{2} Tr \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla^T_A V) \right] - rV + \left( b^T - (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \right) \cdot (\nabla_A V) + \frac{1}{2} (\nabla_A g_\omega)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V)
= \frac{1}{2} (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot (a_S - r\bar{1})
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( V(T, A) = v(A) \).

**Lemma 2.10.** In the above framework, providing that the interest rate is deterministic, for a position \( \kappa = \kappa(t, A) \) and for the CARA utility function \( \psi(X) = (1 - e^{(1-X)\omega})/\omega \), the \( \omega\)-\( \kappa \)-indifference price \( V_{\omega, \kappa}(t, A) \) is the solution of the following uncoupled pricing system of PDEs:

\[
\frac{\partial g_\omega}{\partial t} + \frac{1}{2} Tr \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla^T_A g_\omega) \right] + \left( b^T - (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \right) \cdot (\nabla_A g_\omega) + \frac{1}{2} (\nabla_A g_\omega)^T \cdot c \cdot (I_n - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla_A V)
= \frac{1}{2} (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot (a_S - r\bar{1})
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( g_\omega(T, A) = 0 \), and

\[
\frac{\partial V}{\partial t} + \frac{1}{2} Tr \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla^T_A V) \right] - rV + \left( b^T - (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S \cdot c^T \right) \cdot (\nabla_A V)
= \frac{1}{2} (a_S - r\bar{1})^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot (a_S - r\bar{1})
\]
\[
+ (\nabla_A g_\gamma)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
- \frac{1}{2} \int_0^T \tau \omega \cdot (\nabla_A V)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
= -D
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( V(T, A) = v(A) \).

**Lemma 2.11.** In the above framework, providing \( V = V(t, A) \), for a position \( \kappa(t, X, A) = X \kappa_0(t, A) \) and CRRA utility function \( \psi_\gamma(X) = \frac{X^{1/\gamma} - 1}{1-\gamma} \) (\( \gamma > 0 \) and \( \gamma \neq 1 \)), the \( \gamma \)-\( \kappa \)-neutral price \( V_{\gamma, \kappa}(t, A) \) is the solution of the following PDE system:

\[
\frac{\partial g_\gamma}{\partial t} + \frac{1}{2} \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A g_\gamma) \right] \\
+ \left[ b^T \cdot \frac{1}{\gamma} (a_S - r I)^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \cdot c^T \right] \cdot (\nabla_A g_\gamma) \\
+ \frac{1}{2} (\nabla_A g_\gamma)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
- \frac{\gamma}{2} \kappa_0 (\nabla_A V)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
= \frac{\gamma - 1}{2} \gamma (a_S - r I)^T \cdot (\sigma_S \cdot \sigma_S^{T^{-1}}) \cdot (a_S - r I) + 2 r \gamma 
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( g_\gamma(T, A) = 0 \), and

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A V) \right] - r V \\
+ \left[ b^T \cdot (a_S - r I)^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \cdot c^T \right] \cdot (\nabla_A V) \\
+ (\nabla_A g_\gamma)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
- \gamma \kappa_0 (\nabla_A V)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
= -D
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( V(T, A) = v(A) \).

**Lemma 2.12.** In the above framework, providing \( V = V(t, A) \), for a position \( \kappa(t, X, A) = X \kappa_0(t, A) \) and CRRA utility function \( \psi_\gamma(X) = \frac{X^{1/\gamma} - 1}{1-\gamma} \) (\( \gamma > 0 \) and \( \gamma \neq 1 \)), the \( \gamma \)-\( \kappa \)-indifference price \( V_{\gamma, \kappa}(t, A) \) is the solution of the following PDE system:

\[
\frac{\partial g_\gamma}{\partial t} + \frac{1}{2} \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A g_\gamma) \right] \\
+ \left[ b^T \cdot \frac{1}{\gamma} (a_S - r I)^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \cdot c^T \right] \cdot (\nabla_A g_\gamma) \\
+ \frac{1}{2} (\nabla_A g_\gamma)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A g_\gamma) \\
- \frac{\gamma}{2} \kappa_0 (\nabla_A V)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
= \frac{\gamma - 1}{2} \gamma (a_S - r I)^T \cdot (\sigma_S \cdot \sigma_S^{T^{-1}}) \cdot (a_S - r I) + 2 r \gamma 
\]
for \( t < T, A \in \mathcal{A} \), with the terminal condition \( g_\gamma(T, A) = 0 \), and

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \text{Tr} \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A V) \right] - r V \\
+ \left[ b^T \cdot (a_S - r I)^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \cdot c^T \right] \cdot (\nabla_A V) \\
+ (\nabla_A g_\gamma)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A g_\gamma) \\
- \gamma \kappa_0 (\nabla_A V)^T \cdot c \cdot \left( I_n - \sigma_S^T \cdot \sigma_S^{T^{-1}} \cdot \sigma_S \right) \cdot c^T \cdot (\nabla_A V) \\
= -D
\]
where 1-dimensional standard Brownian motion \( \Phi \)

**Remark 3.** In above Lemmas, the pricing systems are achieved only when it is assumed a priori that \( \varphi \) (\( g_\omega \), \( g_\gamma \), respectively) and \( V \) are sufficiently differentiable functions for \( t < T \), \( X \in \mathfrak{X} \), \( A \in \mathcal{A} \), and are continuous for \( 0 \leq t \leq T \), \( X \in \mathfrak{X} \), \( A \in \mathcal{A} \). In fact, it is not hard to see ([18]) that it suffices to assume that

\[
\varphi(t, X, A) \in C^{1,2}([0, T] \times \mathfrak{X} \times \mathcal{A}) \cap C^0([0, T] \times \mathfrak{X} \times \mathcal{A})
\]

and

\[
g_\omega(t, A), \ g_\gamma(t, A), \ V(t, A) \in C^{1,2}([0, T] \times \mathcal{A}) \cap C^0([0, T] \times \mathcal{A}),
\]

where

- \( C^0([0, T] \times \mathfrak{X} \times \mathcal{A}) \) is the space of real-valued continuous functions \( f \) on \([0, T] \times \mathfrak{X} \times \mathcal{A}; \)
- \( C^{1,2}([0, T] \times \mathfrak{X} \times \mathcal{A}) \) is the space of real-valued functions on \([0, T] \times \mathfrak{X} \times \mathcal{A} \) whose partial derivatives \( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial X}, \frac{\partial^2 f}{\partial X^2}, \frac{\partial f}{\partial A}, \frac{\partial^2 f}{\partial A \partial X}, 1 \leq i, j \leq m \), exist and are continuous on \([0, T]; \)
- \( C^0([0, T] \times \mathcal{A}) \) and \( C^{1,2}([0, T] \times \mathcal{A}) \) are defined in a similar way.

**3. Neutral and indifference pricing with stochastic correlation and volatility.** In this section, we will price the European-style financial contract with neutral and indifference pricing. Our main goal is to apply the neutral and indifference pricing method in any simple economy by constructing a simple economy whose three factors (the tradables, the dynamic factors and the short rate) come from the WASC model. Then, by applying the general neutral and indifference pricing method in any simple economy, we derive the neutral and indifference pricing with stochastic correlation and volatility in the simple economy corresponding to the WASC model.

Using (5), we can reform (3) and (4) as

\[
dS(t) = S^*(t) \cdot (aS(t, \Sigma_t) - q(t, \Sigma_t))dt + S^*(t) \cdot \sqrt{\Sigma_t} \cdot dH(t)
\]

\[
= S^*(t) \cdot (aS(t, \Sigma_t) - q(t, \Sigma_t))dt + S^*(t) \cdot \sqrt{\Sigma_t} \cdot (dB_t \cdot \rho + \sqrt{1 - \rho^2} \cdot \rho dW_t)
\]

and

\[
d \begin{pmatrix} \\
\Sigma_{1k}(t) \\
\vdots\\n\Sigma_{nk}(t) \\
\end{pmatrix} \\
= d\Sigma_t \cdot e_k
\]

\[
= (\Omega \cdot \Omega^T + M \cdot \Sigma_t + \Sigma_t \cdot M^T)dt \cdot e_k + \sqrt{\Sigma_t} \cdot dB_t \cdot Q \cdot e_k + Q^T \cdot dB_t \cdot \sqrt{\Sigma_t} \cdot e_k,
\]

where \( e_1 = (1, 0, \cdots, 0)^T, e_2 = (0, 1, \cdots, 0)^T, \cdots, e_n = (0, 0, \cdots, 1)^T \) and \( k = 1, 2, \cdots, n. \)

Since Brownian motion \( W_t \) is independent of \( B_t \), we can define a new \( n(n + 1) \)-dimensional standard Brownian motion

\[
Z(t) = (W_1(t), \cdots, W_n(t), B_{11}(t), \cdots, B_{nn}(t), \cdots, B_{n1}(t), \cdots, B_{nn}(t))^T.
\]
Define $A(t)$ in the simple economy $\mathcal{E}$ in Section 2 as

$$A(t) = (S_1(t), \ldots, S_n(t), \Sigma_{11}(t), \ldots, \Sigma_{1n}(t), \ldots, \Sigma_{nn}(t))^T. \quad (13)$$

It is easy to see that

$$\mathcal{A} = \left\{ A \in (0, \infty)^n \times (-\infty, \infty)^n : \Sigma_t \in S_n^+ (\mathbb{R}) \right\}.$$ 

Now we can reform WASC model in the simple economy $\mathcal{E}$:

$$\begin{cases}
\frac{dS(t)}{S_t} = S^*(t) \cdot (a_S(t, A(t)) - q(t, A(t))) \, dt + S^*(t) \cdot \sigma_S(t, A(t)) \cdot dZ(t), \\
\frac{dA(t)}{A(t)} = b(t, A(t)) \, dt + c(t, A(t)) \cdot dZ(t),
\end{cases} \quad (14)$$

where

$$a_S(t, A(t)) = a_S(t, \Sigma_t);$$

$$q(t, A(t)) = q(t, \Sigma_t);$$

$$\sigma_S(t, A(t)) = \left( \sqrt{1 - \rho^T \cdot \rho \Sigma_t}, \rho_1 \sqrt{\Sigma_t}, \ldots, \rho_n \sqrt{\Sigma_t} \right)_{n \times (n+1)}; \quad (15)$$

$$b(t, A(t)) = \begin{pmatrix}
S^*(t) \cdot (a_S(t, \Sigma_t) - q(t, \Sigma_t)) \\
(\Omega \cdot \Omega^T + M \cdot \Sigma_t + \Sigma_t \cdot M^T) \cdot e_1 \\
\vdots \\
(\Omega \cdot \Omega^T + M \cdot \Sigma_t + \Sigma_t \cdot M^T) \cdot e_n
\end{pmatrix}_{n(n+1) \times 1};$$

$$c(t, A(t)) = \begin{pmatrix}
S^*_t \cdot \sqrt{1 - \rho^T \cdot \rho \Sigma_t} \\
O_{n^2 \times n}
\end{pmatrix}_{n(n+1) \times n};$$

$$E_t = \begin{pmatrix}
q_{11} \\
\vdots \\
q_{nn}
\end{pmatrix}_{n \times n};$$

$$F_t = \begin{pmatrix}
q_{11} & \cdots & q_{1n} \\
\vdots & \ddots & \vdots \\
q_{n1} & \cdots & q_{nn}
\end{pmatrix};$$

and $q_{km}$ is an $n \times n$ matrix for $k, m = 1, 2, \ldots, n$, satisfying

$$\left( \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma_t} \right) \cdot e_k = \sum_{m=1}^{n} q_{km} \cdot dB_t \cdot e_m. \quad (17)$$

From (15) or (16), it is necessary to assume that $\rho^T \cdot \rho \leq 1$. It follows from (17) that

$$\sum_{k=1}^{n} e_k^T \cdot \left( \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma_t} \right) \cdot e_k = \sum_{k=1}^{n} \sum_{m=1}^{n} e_k^T \cdot q_{km} \cdot dB_t \cdot e_m. \quad (18)$$

Moreover, the fact that the above equation is equal to a number implies

$$\begin{align*}
\sum_{k=1}^{n} e_k^T \cdot \left( \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma_t} \right) \cdot e_k &= \text{Tr} \left[ \sum_{k=1}^{n} e_k^T \cdot \left( \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma_t} \right) \cdot e_k \right]
\end{align*}$$
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\[\begin{align*}
&= Tr \left[ \sum_{k=1}^{n} (e_k \cdot e_k^T) \cdot \left( \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma_t} \right) \right] \\
&= Tr \left[ \sqrt{\Sigma_t} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma_t} \right] \\
&= Tr \left[ 2Q \cdot \sqrt{\Sigma_t} \cdot dB_t \right] \\
\end{align*}\]

(19)

By making use of the arbitrariness of \(B_t\) and (18)-(20), one can obtain

\[\begin{align*}
&= Tr \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_k^T \cdot q_{km} \cdot dB_t \cdot e_m \right] \\
&= Tr \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_k^T \cdot q_{km}^* \cdot dB_t \right]. \\
\end{align*}\]

(20)

Because of the arbitrariness of \(\Sigma\), there must exist \(n \times n\) matrices \(q_{km}(k,m = 1, 2, \cdots, n)\) so that

\[q_{km}^* = q_{km} \cdot \sqrt{\Sigma}.\]

(22)

In order to use Lemmas in Section 2, we need the following lemmas.

Throughout, we shall use the facts that \(S^*, \Sigma\) are symmetric matrices and that for arbitrary matrices \(X, Y\), if the notations are meaningful, then

\[Tr[X \cdot Y] = Tr[Y \cdot X], \quad Tr[X] = Tr[X^T].\]

We shall assume \(F = F(t, S, \Sigma)\) and \(G = G(t, S, \Sigma)\) are two arbitrary sufficiently smooth functions and use the following notations:

\[\nabla \Sigma = \begin{pmatrix}
\frac{\partial}{\partial \Sigma_{11}} & \cdots & \frac{\partial}{\partial \Sigma_{1n}} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial \Sigma_{n1}} & \cdots & \frac{\partial}{\partial \Sigma_{nn}}
\end{pmatrix}, \quad \nabla S = \begin{pmatrix}
\frac{\partial}{\partial S_1} \\
\vdots \\
\frac{\partial}{\partial S_n}
\end{pmatrix}.
\]

Lemma 3.1. For any function \(F = F(t, S, \Sigma)\), we have the following result:

\[\begin{align*}
\frac{1}{2} Tr[c \cdot c^T \cdot (\nabla_A \cdot \nabla_A^T F)] &= 2Tr[\Sigma \cdot \nabla \Sigma \cdot (Q^T \cdot Q) \cdot \nabla \Sigma] F + 2[q^T \cdot Q \cdot \nabla S \cdot \Sigma^* \cdot \nabla S] F \\
&\quad + \frac{1}{2}[\nabla S^T \cdot (S^* \cdot \Sigma) \cdot \nabla S] F.
\end{align*}\]

(23)

Proof. It follows from (13) that

\[\nabla_A = \begin{pmatrix}
\nabla S \\
\nabla \Sigma \cdot e_1 \\
\vdots \\
\nabla \Sigma \cdot e_n
\end{pmatrix}.
\]
By direct computation, we have
\[ Tr \left[ c \cdot c^T \cdot (\nabla_A \cdot \nabla_A T) \right] F \]
\[ = Tr \left[ (\nabla_S \cdot \nabla_S^T) \cdot (S^* \cdot \Sigma \cdot S^*) \right] F + 2Tr \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} \rho_m \nabla_S \cdot e_k^T \cdot \nabla_S \cdot q_{km} \cdot \Sigma \cdot q_{lm} \right] F \]
\[ + Tr \left[ \nabla_S \cdot e_k^T \cdot \nabla_S \cdot \Sigma \cdot q_{km} \cdot \Sigma \cdot q_{lm} \right] F \]
\[ = Tr \left[ \nabla_S \cdot \Sigma \cdot q_{km} \cdot \Sigma \cdot q_{lm} \right] F \]
\[ = Tr \left[ \nabla_S \cdot \Sigma \cdot q_{km} \cdot \Sigma \cdot q_{lm} \right] F. \tag{24} \]

and
\[ Tr \left[ (\nabla_S \cdot \nabla_S^T) \cdot (S^* \cdot \Sigma \cdot S^*) \right] F \]
\[ = Tr \left[ \nabla_S \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla_S \right] F \]
\[ = \left[ \nabla_S \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla_S \right] F. \tag{25} \]

Now let us consider (17). With (17) and (22), one has
\[ \sum_{k=1}^{n} e_k^T \cdot \nabla_S \cdot \left( \sqrt{\Sigma} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma} \right) \cdot e_k \]
\[ = \sum_{k=1}^{n} e_k^T \cdot \nabla_S \cdot q_{km} \cdot dB_t \cdot e_m. \tag{26} \]

Moreover, the fact that the above equation is equal to a number implies that
\[ \sum_{k=1}^{n} e_k^T \cdot \nabla_S \cdot \left( \sqrt{\Sigma} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma} \right) \cdot e_k \]
\[ = Tr \left[ \sum_{k=1}^{n} e_k^T \cdot \nabla_S \cdot \left( \sqrt{\Sigma} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma} \right) \cdot e_k \right] \]
\[ = Tr \left[ \nabla_S \cdot \left( \sqrt{\Sigma} \cdot dB_t \cdot Q + Q^T \cdot dB_t^T \cdot \sqrt{\Sigma} \right) \right] \]
\[ = Tr \left[ 2Q \cdot \nabla_S \cdot \sqrt{\Sigma} \cdot dB_t \right]. \tag{27} \]

and
\[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_k^T \cdot \nabla_S \cdot q_{km} \cdot \sqrt{\Sigma} \cdot dB_t \cdot e_m \]
\[ = Tr \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_k^T \cdot \nabla_S \cdot q_{km} \cdot \sqrt{\Sigma} \cdot dB_t \cdot e_m \right] \]
\[ = Tr \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_m \cdot e_k^T \cdot \nabla_S \cdot q_{km} \cdot \sqrt{\Sigma} \cdot dB_t \right]. \tag{28} \]

By making use of the arbitrariness of \( B_t \) and (26)-(28), one can obtain
\[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_m \cdot e_k^T \cdot \nabla_S \cdot q_{km} = 2Q \cdot \nabla_S. \tag{29} \]
As a result of (29), we can get
\[
\sum_{k=1}^{n} \sum_{m=1}^{n} \rho_m e_k^T \cdot \nabla \Sigma \cdot q_{km} = \sum_{k=1}^{n} \sum_{m=1}^{n} \rho^T \cdot e_m \cdot e_k^T \cdot \nabla \Sigma \cdot q_{km} = 2 \rho^T \cdot \nabla \Sigma \cdot q_{km} \quad (30)
\]
and
\[
4 \text{Tr} \left[ Q \cdot \nabla \Sigma \cdot \Sigma \cdot \nabla \Sigma \cdot Q^T \right] = \text{Tr} \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} e_m \cdot e_k^T \cdot \nabla \Sigma \cdot q_{km} \cdot \Sigma \cdot q_{km}^T \cdot \nabla \Sigma \cdot e_l \right] = \text{Tr} \left[ \sum_{k=1}^{n} \sum_{m=1}^{n} \nabla \Sigma \cdot e_l \cdot e_k^T \cdot \nabla \Sigma \cdot q_{km} \cdot \Sigma \cdot q_{lm}^T \cdot \nabla \Sigma \cdot e_l \right]. \quad (31)
\]
Finally, by substituting (25), (30) and (31) into (24), we prove (23). □

**Lemma 3.2.** For any function \( F = F(t, S, \Sigma) \), the following equality holds:
\[
\left( \sigma_S \cdot \sigma_S^T \right)^{-1} \cdot \sigma_S \cdot c^T \cdot \nabla_A F = S^* \cdot \nabla_S F + 2 \nabla \Sigma \cdot (Q^T \cdot \rho). \quad (32)
\]

**Proof.** By directly computing, one has
\[
\sigma_S \cdot \sigma_S^T = \left( \sqrt{1 - \rho^T \cdot \rho \sqrt{\Sigma} \cdot \rho_1 \sqrt{\Sigma} \cdot \cdots \cdot \rho_n \sqrt{\Sigma}} \right)^T \cdot \left( \sqrt{1 - \rho^T \cdot \rho \sqrt{\Sigma} \cdot \rho_1 \sqrt{\Sigma} \cdot \cdots \cdot \rho_n \sqrt{\Sigma}} \right) = \Sigma
\]
and
\[
\sigma_S \cdot c^T = \left( \sqrt{1 - \rho^T \cdot \rho \sqrt{\Sigma} \cdot \rho_1 \sqrt{\Sigma} \cdot \cdots \cdot \rho_2 \sqrt{\Sigma}} \right) \cdot \left( \begin{array}{cccc}
\sqrt{1 - \rho^T \cdot \rho \sqrt{\Sigma} \cdot S^*} & 0 & \cdots & 0 \\
\rho_1 \sqrt{\Sigma} \cdot S^* & \sqrt{\Sigma} \cdot q_{11}^T & \cdots & \sqrt{\Sigma} \cdot q_{1n}^T \\
\vdots & \vdots & \ddots & \vdots \\
\rho_n \sqrt{\Sigma} \cdot S^* & \sqrt{\Sigma} \cdot q_{1n}^T & \cdots & \sqrt{\Sigma} \cdot q_{nn}^T \\
\end{array} \right) = \left( \Sigma \cdot S^*, \sum_{m=1}^{n} \rho_m \Sigma \cdot q_{1m}^T, \cdots, \sum_{m=1}^{n} \rho_m \Sigma \cdot q_{nm}^T \right).
\]
Thus, we have
\[
\left( \sigma_S \cdot \sigma_S^T \right)^{-1} \cdot \sigma_S \cdot c^T \cdot (\nabla_A F)
\]
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\[ \begin{align*}
&= \Sigma^{-1} \cdot \left( \Sigma \cdot S^* + \sum_{m=1}^{n} \rho_m \Sigma \cdot q_{1m}^T, \cdots, \sum_{m=1}^{n} \rho_m \Sigma \cdot q_{nm}^T \right) \cdot \left( \begin{array}{c}
\nabla S \\
\nabla \Sigma \cdot e_1 \\
\vdots \\
\nabla \Sigma \cdot e_n
\end{array} \right) F \\
&= S^* \cdot \nabla S F + \sum_{k=1}^{n} \sum_{m=1}^{n} (\rho_m q_{km}^T \cdot \nabla \Sigma \cdot e_k) F \\
&= S^* \cdot \nabla S F + 2 \nabla \Sigma F \cdot (Q^T \cdot \rho)
\end{align*} \]

where the last equality comes from (30). This completes the proof. \( \square \)

**Lemma 3.3.** For any function \( F = F(t, S, \Sigma) \), the following equality holds:
\[ b^T \cdot \nabla A F = (a_S - q)^T \cdot S^* \cdot \nabla S F + Tr \left( [\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T] \cdot \nabla \Sigma F \right) \] (33)

**Proof.** By direct computation, it is easy to see that
\[ b^T \cdot \nabla A F = (a_S - q)^T \cdot S^* \cdot \nabla S F + \sum_{m=1}^{n} e_m^T \cdot (\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T) \cdot \nabla \Sigma F \cdot e_m \]
\[ = (a_S - q)^T \cdot S^* \cdot \nabla S F + \sum_{m=1}^{n} Tr \left( (e_m^T \cdot e_m) \cdot (\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T) \cdot \nabla \Sigma F \right) \]
\[ = (a_S - q)^T \cdot S^* \cdot \nabla S F + Tr \left( [\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T] \cdot \nabla \Sigma F \right), \]
which proves that the equality is true. \( \square \)

**Lemma 3.4.** For any functions \( F = F(t, S, \Sigma) \) and \( G = G(t, S, \Sigma) \), the following equality holds:
\[ (\nabla A F)^T \cdot c \cdot (I_{n(n+1)} - \sigma_S^T \cdot (\sigma_S \cdot \sigma^T_S)^{-1} \cdot \sigma_S) \cdot c^T \cdot (\nabla A G) = 4 Tr \left( \nabla \Sigma G \cdot (Q^T \cdot Q) \cdot \nabla \Sigma F \right) - 4 (\rho^T \cdot Q) \cdot \nabla \Sigma F \cdot \Sigma \cdot \nabla \Sigma G \cdot (Q^T \cdot \rho) \] (34)

**Proof.** Since
\[ \begin{align*}
(\nabla A F)^T \cdot c \cdot \sigma_S^T \cdot (\sigma_S \cdot \sigma^T_S)^{-1} \cdot \sigma_S \cdot c^T \cdot (\nabla A G) &= Tr \left( (\nabla A F)^T \cdot c \cdot \sigma_S^T \cdot (\sigma_S \cdot \sigma^T_S)^{-1} \cdot \sigma_S \cdot c^T \cdot (\nabla A G) \right) \\
&= Tr \left( (\nabla A G) \cdot (\nabla A F)^T \cdot c \cdot \sigma_S^T \cdot (\sigma_S \cdot \sigma^T_S)^{-1} \cdot \sigma_S \cdot c^T \right),
\end{align*} \]
by a similar way in Lemma 3.1, one can prove the following equation
\[ \begin{align*}
&= \frac{1}{2} (\nabla A F)^T \cdot c \cdot \sigma_S^T \cdot (\sigma_S \cdot \sigma^T_S)^{-1} \cdot \sigma_S \cdot c^T \cdot (\nabla A G) \\
&= \frac{1}{2} \nabla \Sigma F \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla \Sigma G + 2 (\rho^T \cdot Q) \cdot \nabla \Sigma F \cdot \Sigma \cdot \nabla \Sigma G \cdot (Q^T \cdot \rho) \\
&\quad + 2 (\rho^T \cdot Q) \cdot \nabla \Sigma F \cdot \Sigma \cdot S^* \cdot \nabla \Sigma G.
\end{align*} \] (35)

Thus, Lemma 3.1 and (35) imply that the equality is true. \( \square \)

Before we give the way to price, we shall show an important property of the market in the simple economy \( \mathcal{C} \).

**Theorem 3.5.** In the simple economy \( \mathcal{C} \), assume that \( \Omega \) is invertible (as described in Section 2). If \( n = 1 \) and \( \rho = 1 \), then the market is complete. Otherwise, the market is incomplete.
Proof. It is easy to check that when \( n = 1 \) and \( \rho = 1 \), (7) is fulfilled and so the market is complete. Now we argue by contradiction, which yields that the market is complete and that \( n = 1, \rho = 1 \) do not hold. Then from Definition 2.4, we have

\[
c \cdot (I_{n(n+1)} - \sigma_S^T \cdot (\sigma_S \cdot \sigma_S^T)^{-1} \cdot \sigma_S) \cdot c^T = O_{n(n+1) \times n(n+1)}.
\]

Following Lemma 3.4, it is obvious to see that

\[
\text{Tr} \left[ \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot Q) \cdot \nabla_\Sigma G \right] - (\rho^T \cdot Q) \cdot \nabla_\Sigma F \cdot \Sigma \cdot \nabla_\Sigma G \cdot (Q^T \cdot \rho) = 0.
\]

On the other hand, we have

\[
\text{Tr} \left[ \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot Q) \cdot \nabla_\Sigma G \right] - (\rho^T \cdot Q) \cdot \nabla_\Sigma F \cdot \Sigma \cdot \nabla_\Sigma G \cdot (Q^T \cdot \rho)
= \text{Tr} \left[ \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot Q) - \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot \rho) \cdot (Q^T \cdot Q) \right]
= \text{Tr} \left[ \nabla_\Sigma G \cdot \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot (I_n - \rho \cdot \rho^T)) \cdot Q \right].
\]

Because of the arbitrariness of \( \Sigma, F \) and \( G \), we must have

\[
Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q = O_{n \times n}.
\]

Then we obtain

\[
I_n - \rho \cdot \rho^T = O_{n \times n}, \quad (36)
\]

because \( \Omega \cdot \Omega^T = \beta Q^T \cdot Q \) implies that \( Q \) is invertible if \( \Omega \) is invertible.

However, it is easy to see that there is not such a \( \rho \) satisfying (36) except for the case \( n = 1 \) and \( \rho = 1 \). As a result, the assumption of completeness is not true, which completes the proof.

Now we turn to consider the neutral and indifference pricing with stochastic correlation and volatility. With the aid of above lemmas in this section and lemmas in Section 2, the following pricing theorems are easy to prove. Therefore, we shall prove the first theorem, and then just state the rest theorems and omit their proofs.

Let us define some infinitesimal generators of the couple \((S, \Sigma)\):

\[
\mathcal{L}_\Sigma = 2 \text{Tr}[\Sigma \cdot \nabla_\Sigma \cdot (Q^T \cdot Q) \cdot \nabla_\Sigma] + \text{Tr}[(\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T) \cdot \nabla_\Sigma],
\]

\[
\mathcal{L}_S = \frac{1}{2} \left[ \nabla_S \cdot (S^* \cdot \Sigma) \cdot \nabla_S \right] + (r^T - q)^T \cdot S^* \cdot \nabla_S,
\]

\[
\mathcal{L}_{(S,\Sigma)} = 2[\rho^T \cdot Q \cdot \nabla_\Sigma \cdot \Sigma \cdot S^* \cdot \nabla_S],
\]

\[
\mathcal{L}_{S,\Sigma} = \mathcal{L}_S + \mathcal{L}_\Sigma + \mathcal{L}_{(S,\Sigma)},
\]

\[
\mathcal{L}_{\Sigma^2}(F,G) = \text{Tr}\left[\nabla_\Sigma G \cdot \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot (I_n - \rho \cdot \rho^T)) \cdot Q \right],
\]

where:

- \( \mathcal{L}_\Sigma \) represents the infinitesimal generator for the Wishart process \( \Sigma \), which has been discussed by Bru in [4];
- \( \mathcal{L}_S \) denotes the infinitesimal generator for assets’ price \( S \);
- \( \mathcal{L}_{(S,\Sigma)} \) denotes the cross infinitesimal generator for assets’ price and volatility;
- \( \mathcal{L}_{\Sigma^2}(F,G) \) signifies the influence of the incompleteness of the market.

**Remark 4.** It is not difficult to check that \( \mathcal{L}_{\Sigma^2}(F,G) \) does not depend on the order of \( F \) and \( G \). Indeed, with the symmetry of \( \nabla_\Sigma G, \nabla_\Sigma F \) and \( \Sigma \), it suffices to check that

\[
\mathcal{L}_{\Sigma^2}(G,F) = \text{Tr}\left[\nabla_\Sigma F \cdot \Sigma \cdot \nabla_\Sigma G \cdot (Q^T \cdot (I_n - \rho \cdot \rho^T)) \cdot Q \right]
= \text{Tr}\left[\nabla_\Sigma F \cdot \Sigma \cdot \nabla_\Sigma G \cdot (Q^T \cdot (I_n - \rho \cdot \rho^T)) \cdot Q \right]^T
= \text{Tr}\left[Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot \nabla_\Sigma G \cdot \Sigma \cdot \nabla_\Sigma F \right]
= \text{Tr}\left[\nabla_\Sigma G \cdot \Sigma \cdot \nabla_\Sigma F \cdot (Q^T \cdot (I_n - \rho \cdot \rho^T)) \cdot Q \right]
= \mathcal{L}_{\Sigma^2}(F,G).
\]
Theorem 3.6. Providing \( V = V(t, S, \Sigma) \), under aforementioned assumptions, for a position \( \kappa = \kappa(t, X, S, \Sigma) \) and any utility function \( \psi = \psi(X) \), the \( \psi, \kappa \)-neutral price \( \bar{V}_{\psi, \kappa}(t, S, \Sigma) \) is the solution of the following coupled pricing system of PDEs:

\[
\mathcal{M}(\varphi) - 2 \left( \frac{\partial^2 \varphi}{\partial X^2} \right)^2 \kappa^2 \mathcal{L}_{\Sigma_2}(V, V) = 0, \tag{37}
\]

for \( t < T \), \( A \in \mathcal{A} \), with the terminal condition \( \varphi(T, X, S, \Sigma) = \psi(X) \), and

\[
\begin{align*}
\frac{\partial V}{\partial t} + \mathcal{L}_{\Sigma, \Sigma} V &- 2(a_S - r)T \cdot \nabla_{\Sigma} V \cdot (Q_T \cdot \rho) \\
+ 4 \left( 1 / \frac{\partial \varphi}{\partial X} \right) \mathcal{L}_{\Sigma_2}(\frac{\partial \varphi}{\partial X}, V) + 4 \left( \frac{\partial^2 \varphi}{\partial X^2} / \frac{\partial \varphi}{\partial X} \right) \kappa \mathcal{L}_{\Sigma_2}(V, V) - r V \\
&= -D,
\end{align*}
\]

for \( t < T \), \( A \in \mathcal{A} \), with the terminal condition \( V(T, S, \Sigma) = \nu(S, \Sigma) \), where operator \( \mathcal{M} \) is the differential operator,

\[
\mathcal{M}(\varphi) = \frac{\partial^2 \varphi}{\partial X^2} \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial X^2} \left( \mathcal{L}_{\Sigma, \Sigma} \varphi + (a_S - r)T \cdot S^* \cdot \nabla_{\Sigma} \varphi \right) \\
- \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^2 (a_S - r)T \cdot \Sigma^{-1} \cdot (a_S - r)I \\
- \frac{\partial \varphi}{\partial X} (a_S - r)T \cdot \left( S^* \cdot \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) + 2 \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) \cdot (Q_T \cdot \rho) \right) \\
- \frac{1}{2} \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) \cdot (Q_T \cdot \rho) \\
- 2(\rho T \cdot Q) \cdot \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) \cdot \Sigma \cdot \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) + r X \frac{\partial^2 \varphi}{\partial X^2} \frac{\partial \varphi}{\partial X}. \tag{39}
\]

Proof. By using Lemma 2.7 and Lemma 3.1-3.4, (11) and (12) turn out to be

\[
\mathcal{M}(\varphi) - 2 \left( \frac{\partial^2 \varphi}{\partial X^2} \right)^2 \kappa^2 \left( Tr[\Sigma \cdot \nabla_\Sigma V \cdot (Q_T \cdot Q) \cdot \nabla_\Sigma V] \\
- (\rho T \cdot Q) \cdot \nabla_\Sigma V \cdot \Sigma \cdot \nabla_\Sigma V \cdot (Q_T \cdot \rho) \right) = 0
\]

and

\[
\begin{align*}
\frac{\partial V}{\partial t} + 2 Tr[\Sigma \cdot \nabla_\Sigma \cdot (Q_T \cdot Q) \cdot \nabla_\Sigma V] &+ 2(\rho T \cdot Q) \cdot \nabla_\Sigma \cdot \Sigma \cdot S^* \cdot \nabla_\Sigma V \\
+ \frac{1}{2} \left( \nabla_{\Sigma} \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla_\Sigma V \right) + (rT - q)T \cdot S^* \cdot \nabla_\Sigma V - rV \\
+ Tr[(\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T) \cdot \nabla_\Sigma V] - 2(a_S - r)T \cdot \nabla_\Sigma V \cdot (Q_T \cdot \rho) \\
+ 4 \left( 1 / \frac{\partial \varphi}{\partial X} \right) \left( Tr[\Sigma \cdot \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) \cdot (Q_T \cdot Q) \cdot \nabla_\Sigma V] \\
- (\rho T \cdot Q) \cdot \left( \nabla_{\Sigma} \frac{\partial \varphi}{\partial X} \right) \cdot \Sigma \cdot \nabla_\Sigma V \cdot (Q_T \cdot \rho) \right) + 4 \left( \frac{\partial^2 \varphi}{\partial X^2} / \frac{\partial \varphi}{\partial X} \right) \kappa \\
\times \left( Tr[\Sigma \cdot \nabla_\Sigma V \cdot (Q_T \cdot Q) \cdot \nabla_\Sigma V] - (\rho T \cdot Q) \cdot \nabla_\Sigma V \cdot \Sigma \cdot \nabla_\Sigma V \cdot (Q_T \cdot \rho) \right) \\
&= -D,
\end{align*}
\]
where the operator $\mathcal{M}$ is the differential operator,
\[
\mathcal{M}(\varphi) = \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial^2 \varphi}{\partial X^2} \left( 2T r\Sigma \cdot \nabla \Sigma \cdot (Q^T \cdot Q) \cdot \nabla \Sigma \right) \varphi \\
+ 2\rho^T \cdot Q \cdot \nabla \Sigma \cdot \Sigma \cdot S \cdot \nabla S \varphi + \frac{1}{2} \left[ \nabla \Sigma^T \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla S \right] \varphi \\
- \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right)^2 \left( a_s - r \bar{I} \right)^T \Sigma^{-1} \cdot \left( a_s - r \bar{I} \right) + \frac{\partial^2 \varphi}{\partial X^2} \left( (a_s - q) \right)^T \cdot S^* \cdot \nabla S \varphi \\
+ Tr \left( \Sigma \cdot \Sigma \cdot \Sigma \cdot M \cdot \Sigma + M^T \right) \cdot \nabla \Sigma \varphi \}
\]

- $\frac{\partial \varphi}{\partial X}(a_s - r)^T \left( S^* \cdot \left( \nabla_a \frac{\partial \varphi}{\partial X} \right) + 2 \left( \nabla_a \frac{\partial \varphi}{\partial X} \right) \cdot (Q^T \cdot \rho) \right)$

- $\frac{1}{2} \left( \nabla_a \frac{\partial \varphi}{\partial X} \right) \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \left( \nabla_a \frac{\partial \varphi}{\partial X} \right) + 2 \left( \rho^T \cdot Q \right) \cdot \left( \nabla_a \frac{\partial \varphi}{\partial X} \right) + rX \frac{\partial^2 \varphi}{\partial X^2} \frac{\partial \varphi}{\partial X}$.

Now by inserting the infinitesimal generators $\mathcal{L}_\Sigma$, $\mathcal{L}_S$, $\mathcal{L}_{(S,\Sigma)}$, $\mathcal{L}_{S,\Sigma}$ and $\mathcal{L}_{\Sigma^2}$, we just obtain the desired results (37)-(39).

**Theorem 3.7.** Providing $V = V(t, S, \Sigma)$, under aforementioned assumptions, for a position $\kappa = \kappa(t, X, S, \Sigma)$ and any utility function $\psi = \psi(X)$, the $\psi$-indifference price $V_{\psi,\kappa}(t, S, \Sigma)$ is the solution of the following uncoupled pricing system of PDEs:
\[
\mathcal{M}(\varphi) = 0,
\]
for $t < T$, $A \in A$, with the terminal condition $\varphi(T, X, S, \Sigma) = \psi(X)$, and
\[
\frac{\partial V}{\partial t} + \mathcal{L}_{S,\Sigma} V - 2(a_s - r \bar{I})^T \cdot \nabla \Sigma \cdot (Q^T \cdot \rho) \\
+ 4 \left( \frac{1}{2} \left( \frac{\partial \varphi}{\partial X} \right) \mathcal{L}_{\Sigma^2} \frac{\partial \varphi}{\partial X}, V \right) + 2 \left( \frac{\partial^2 \varphi}{\partial X^2} \right) \kappa \mathcal{L}_{\Sigma^2} (V, V) - rV = -D,
\]
for $t < T$, $A \in A$, with the terminal condition $V(T, S, \Sigma) = v(S, \Sigma)$.

**Theorem 3.8.** In the above framework, providing that the interest rate is deterministic, for a position $\kappa = \kappa(t, S, \Sigma)$ and for the CARA utility function $\psi(X) = (1 - e^{(1-X)\omega})/\omega$, the $\psi$-neutral price $V_{\omega,\kappa}(t, S, \Sigma)$ is the solution of the following coupled pricing system of PDEs:
\[
\frac{\partial g_{\omega}}{\partial t} + \mathcal{L}_{S,\Sigma} g_{\omega} - 2(a_s - r \bar{I})^T \cdot \nabla \Sigma \cdot g_{\omega} \cdot (Q^T \cdot \rho) \\
+ 2 \mathcal{L}_{\Sigma^2} (g_{\omega} \cdot g_{\omega}) - 2e^2 \int_{t}^{T} r(\tau) d\tau \omega^2 \kappa^2 \mathcal{L}_{\Sigma^2} (V, V) \\
= \frac{1}{2} \left( a_s - r \bar{I} \right)^T \Sigma^{-1} \cdot \left( a_s - r \bar{I} \right),
\]
for $t < T$, $A \in A$, with the terminal condition $g_{\omega}(T, S, \Sigma) = 0$, and
\[
\frac{\partial V}{\partial t} + \mathcal{L}_{S,\Sigma} V - 2(a_s - r \bar{I})^T \cdot \nabla \Sigma \cdot (Q^T \cdot \rho) \\
+ 4 \mathcal{L}_{\Sigma^2} (g_{\omega}, V) - 4 e^2 \int_{t}^{T} r(\tau) d\tau \omega \kappa \mathcal{L}_{\Sigma^2} (V, V) - rV
\]
for $t < T$, $A \in A$, with the terminal condition

$$V(T, S, \Sigma) = v(S, \Sigma).$$

**Theorem 3.9.** In the above framework, providing that the interest rate is deterministic, for a position $\kappa = \kappa(t, S, \Sigma)$ and for the CARA utility function $\psi(X) = \frac{1 - e^{(1 - \kappa)X}}{\omega}$, the $\psi$-$\kappa$-indifference price $V_{\omega, \kappa}(t, S, \Sigma)$ is the solution of the following uncoupled pricing system of PDEs:

$$\frac{\partial g_\omega}{\partial t} + \mathcal{L}_{S, \Sigma} g_\omega - 2(a_S - r \bar{I})^T \cdot \nabla_{S} g_\omega \cdot (Q^T \cdot \rho) + 2\mathcal{L}_{\Sigma^2}(g_\omega, g_\omega)$$

$$= \frac{1}{2} (a_S - r \bar{I})^T \cdot \Sigma^{-1} \cdot (a_S - r \bar{I}),$$

for $t < T$, $A \in A$, with the terminal condition $g_\omega(T, S, \Sigma) = 0$, and

$$\frac{\partial V}{\partial t} + \mathcal{L}_{S, \Sigma} V - 2(a_S - r \bar{I})^T \cdot \nabla_{S} V \cdot (Q^T \cdot \rho)$$

$$+ 4\mathcal{L}_{\Sigma^2}(g_\omega, V) - 2e^{\gamma t} r(t)dr \omega \kappa \mathcal{L}_{\Sigma^2}(V, V) - r V$$

$$= -D,$$

for $t < T$, $A \in A$, with the terminal condition $V(T, S, \Sigma) = v(S, \Sigma)$.

**Theorem 3.10.** In the above framework, providing $V = V(t, S, \Sigma)$, for a position $\kappa(t, X, S, \Sigma) = X \kappa_0(t, S, \Sigma)$ and CRRA utility function $\psi_\gamma(X) = \frac{X^{1-\gamma}}{1-\gamma}$ ($\gamma > 0$ and $\gamma \neq 1$), the $\gamma$-$\kappa$-neutral price $V_{\gamma, \kappa}(t, S, \Sigma)$ is the solution of the following PDE system:

$$\frac{\partial g_\gamma}{\partial t} + \mathcal{L}_{S, \Sigma} g_\gamma + \frac{1}{\gamma} (a_S - r \bar{I})^T \cdot S^* \cdot \nabla_S g_\gamma$$

$$- \frac{2}{\gamma^2} (a_S - r \bar{I})^T \cdot \nabla_{S} g_\gamma \cdot (Q^T \cdot \rho) + 2\mathcal{L}_{\Sigma^2}(g_\gamma, g_\gamma)$$

$$+ \frac{2}{\gamma} (\rho^T \cdot Q) \cdot \nabla_{S} g_\gamma \cdot \Sigma \cdot \nabla_{S} g_\gamma \cdot (Q^T \cdot \rho) + \frac{2}{\gamma^2} (\rho^T \cdot Q) \cdot \nabla_{S} g_\gamma \cdot \Sigma \cdot S^* \cdot \nabla_{S} g_\gamma$$

$$+ \frac{1}{\gamma^2} \nabla^T_S g_\gamma \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla_S g_\gamma - 2\gamma(\gamma - 1)\kappa_0^2 \mathcal{L}_{\Sigma^2}(V, V)$$

$$= \frac{\gamma - 1}{\gamma}(a_S - r \bar{I})^T \cdot \Sigma^{-1} \cdot (a_S - r \bar{I}) + 2r\gamma,$$

for $t < T$, $A \in A$, with the terminal condition $g_\gamma(T, S, \Sigma) = 0$, and

$$\frac{\partial V}{\partial t} + \mathcal{L}_{S, \Sigma} V - 2(a_S - r \bar{I})^T \cdot \nabla_{S} V \cdot (Q^T \cdot \rho)$$

$$+ 4\mathcal{L}_{\Sigma^2}(g_\gamma, V) - 4\gamma \kappa_0 \mathcal{L}_{\Sigma^2}(V, V) - r V$$

$$= -D,$$

for $t < T$, $A \in A$, with the terminal condition $V(T, S, \Sigma) = v(S, \Sigma)$.

**Theorem 3.11.** In the above framework, providing $V = V(t, S, \Sigma)$, for a position $\kappa(t, X, S, \Sigma) = X \kappa_0(t, S, \Sigma)$ and CRRA utility function $\psi_\gamma(X) = \frac{X^{1-\gamma}}{1-\gamma}$ ($\gamma > 0$ and $\gamma \neq 1$), the $\gamma$-$\kappa$-indifference price $V_{\gamma, \kappa}(t, S, \Sigma)$ is the solution of the following PDE system:

$$\frac{\partial g_\gamma}{\partial t} + \mathcal{L}_{S, \Sigma} g_\gamma + \frac{1}{\gamma} (a_S - r \bar{I})^T \cdot S^* \cdot \nabla_S g_\gamma$$
\[-2 \frac{\gamma - 1}{\gamma} (a_s - r\bar{1})^T \cdot \nabla g_{\gamma} \cdot (Q^T \cdot \rho) + 2 \mathcal{L}_{g_{\gamma}}(g_{\gamma}, g_{\gamma}) + \frac{2}{\gamma} (\rho^T \cdot Q) \cdot \nabla g_{\gamma} \cdot \Sigma \cdot \nabla g_{\gamma} \cdot (Q^T \cdot \rho) + \frac{2}{\gamma} (\rho^T \cdot Q) \cdot \nabla g_{\gamma} \cdot \Sigma \cdot S^* \cdot \nabla g_{\gamma} + \frac{1}{2\gamma} \nabla g_{\gamma} \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla g_{\gamma} \]

\[= \frac{1}{2} \frac{\gamma - 1}{\gamma} ((a_s - r\bar{1})^T \cdot \Sigma^{-1} \cdot (a_s - r\bar{1}) + 2r\gamma),\]

for \(t < T, \quad A \in A,\) with the terminal condition \(g_{\gamma}(T, S, \Sigma) = 0,\) and

\[\frac{\partial V}{\partial t} + \mathcal{L}_{g_{\gamma}}(g_{\gamma}, g_{\gamma}) - 2(a_s - r\bar{1})^T \cdot \nabla g_{\gamma} \cdot (Q^T \cdot \rho) + 2 \mathcal{L}_{g_{\gamma}}(g_{\gamma}, g_{\gamma}) + \frac{1}{2} (a_s - r\bar{1})^T \cdot \Sigma^{-1} \cdot (a_s - r\bar{1}),\]

for \(t < T, \quad A \in A,\) with the terminal condition \(g_{\omega}(T, S, \Sigma) = 0,\) and

\[\frac{\partial V}{\partial t} + \mathcal{L}_{g_{\omega}}(g_{\omega}, g_{\omega}) - 2(a_s - r\bar{1})^T \cdot \nabla g_{\omega} \cdot (Q^T \cdot \rho) + 4 \mathcal{L}_{g_{\omega}}(g_{\omega}, V) - rV = -D,\]

for \(t < T, \quad A \in A,\) with the terminal condition \(V(T, S, \Sigma) = v(S, \Sigma).\)

**Corollary 1.** In the above framework, providing that the interest rate is deterministic, for a position \(\kappa = \kappa(t, S, \Sigma)\) and for the CARA utility function \(\psi(X) = (1 - e^{(1-X)\omega})/\omega,\) if \(\kappa = 0,\) then the CARA neutral price is equal to the CARA indifference price, denoted by \(V_{\omega, \kappa=0},\) which is the solution of the following uncoupled pricing system of PDEs:

\[\frac{\partial g_{\omega}}{\partial t} + \mathcal{L}_{g_{\omega}}(g_{\omega}, g_{\omega}) - 2(a_s - r\bar{1})^T \cdot \nabla g_{\omega} \cdot (Q^T \cdot \rho) + 2 \mathcal{L}_{g_{\omega}}(g_{\omega}, g_{\omega}) + \frac{1}{2} (a_s - r\bar{1})^T \cdot \Sigma^{-1} \cdot (a_s - r\bar{1})\]

for \(t < T, \quad A \in A,\) with the terminal condition \(g_{\omega}(T, S, \Sigma) = 0,\) and

**Corollary 2.** In the above framework, providing \(V = V(t, S, \Sigma),\) for a position \(\kappa(t, X, S, \Sigma) = X\kappa_0(t, S, \Sigma)\) and CRRA utility function \(\psi_{\gamma}(X) = \frac{X^{1-\gamma} - 1}{1-\gamma} (\gamma > 0\) and \(\gamma \neq 1),\) if \(\kappa = 0,\) then the CRRA neutral price is equal to the CRRA indifference price, denoted by \(V_{\gamma, \kappa=0},\) which is the solution of the following uncoupled pricing system of PDEs:

\[\frac{\partial g_{\gamma}}{\partial t} + \mathcal{L}_{g_{\gamma}}(g_{\gamma}, g_{\gamma}) + \frac{1}{\gamma} (a_s - r\bar{1})^T \cdot S^* \cdot \nabla g_{\gamma} - 2 \frac{\gamma - 1}{\gamma} (a_s - r\bar{1})^T \cdot \nabla g_{\gamma} \cdot (Q^T \cdot \rho) + 2 \mathcal{L}_{g_{\gamma}}(g_{\gamma}, g_{\gamma}) + \frac{2}{\gamma} (\rho^T \cdot Q) \cdot \nabla g_{\gamma} \cdot \Sigma \cdot \nabla g_{\gamma} \cdot (Q^T \cdot \rho) + \frac{2}{\gamma} (\rho^T \cdot Q) \cdot \nabla g_{\gamma} \cdot \Sigma \cdot S^* \cdot \nabla g_{\gamma} + \frac{1}{2\gamma} \nabla g_{\gamma} \cdot (S^* \cdot \Sigma \cdot S^*) \cdot \nabla g_{\gamma} \]

\[= \frac{1}{2} \frac{\gamma - 1}{\gamma} ((a_s - r\bar{1})^T \cdot \Sigma^{-1} \cdot (a_s - r\bar{1}) + 2r\gamma),\]
for \( t < T, \ A \in \mathcal{A}, \) with the terminal condition \( g_\gamma(T, S, \Sigma) = 0, \) and
\[
\frac{\partial V}{\partial t} + \mathcal{L}_{S, \Sigma} V - 2(a_S - r) f^T \cdot \nabla V \cdot (Q^T \cdot \rho) + 4\mathcal{L}_{\Sigma^2}(g_\gamma, V) - r V = -D,
\]
for \( t < T, \ A \in \mathcal{A}, \) with the terminal condition \( V(T, S, \Sigma) = v(S, \Sigma). \)

**Proof.** Let \( \kappa = 0 \) in Theorems 3.10 and 3.11, and the result is obvious. \( \square \)

**Remark 5.** Following [25] (see Remark 3.4.1), the first PDEs of pricing systems in all above theorems and corollaries hold in their domains (\( \varphi \) in \( \mathcal{X} \times \mathcal{A}, \) and \( g_\omega, g_\gamma \) in \( \mathcal{A}, \) specifically) and no more boundary conditions except for the conditions at \( t = T \) are needed, nor can we impose any. On the other hand, for the second PDEs, we may just need the conditions in all above theorems and corollaries to price the contracts. However, we can give more boundary conditions which depend on the terminal condition \( v(S, \Sigma) \) to satisfy the conditions in Remark 3 (One can see an example in [13]). For example, let’s consider two contracts \( V_1 \) and \( V_2 \) with the same underlying assets \( S_1, S_2 \) and expiring time \( T, \) but their terminal payoffs are \( V_1(T, S, \Sigma) = (S_1 T - S_2 T - K)^+ \) and \( V_2(T, S, \Sigma) = (S_2 T - S_1 T - K)^+, \) respectively. The appropriate boundary condition for \( S_1 = \infty \) are \( \frac{\partial V_1}{\partial S_1}(t, \infty, S_2, \Sigma) = 1 \) and \( V_2(t, \infty, S_2, \Sigma) = 0, \) respectively.

4. Application to forward contracts. In this section, we shall apply the neutral and indifference pricing with stochastic correlation and volatility to general forward contracts. For the general case, we can convert the pricing to some matrix Riccati equations and use the numerical method to solve the equations.

Firstly, we recall that a forward contract is an agreement between two parties that the holder agrees to buy an asset (or assets) from the writer at a delivery time in the future for a predetermined delivery price. It is obvious that a forward contract is typically European-style. If the holder want to buy more than one asset from the writer, it is practicable to write a forward contract for each asset when these assets are independent. However, it is impossible to assume that the assets in our market are independent from each other and as a result, it is much more reasonable to treat these assets as a multi-asset underlying and use the WASC model to price the forward contract.

Providing that \( V \) is the value of a forward contract delivering at time \( T < \infty, \) with the delivery price \( K \in \mathbb{R}, \) underlying assets \( S = (S_1, S_2, \cdots, S_n)^T (n \geq 1), \) \( D = 0 \) and the terminal payoff \( v(T) = \alpha^T \cdot S_T + Tr[\theta \cdot \Sigma_T] - K, \) where \( \alpha \in \mathbb{R}^n \) is a given \( n \times 1 \)-matrix, and \( \theta \in S_n(\mathbb{R}) \) is a given symmetric \( n \times n \)-matrix.

**Remark 6.** It is worth mentioning that we do not assume that \( \alpha > 0 \) (i.e. \( \alpha_i > 0 \) for \( 1 \leq i \leq n \)). When \( \alpha_i = 0 \) for some \( i, \) it means that the holder do not buy any asset \( i. \) When \( \alpha_i < 0 \) for some \( i, \) it means that the holder shall sell the asset \( i \) to the writer. Moreover, we note that \( Tr[\theta \cdot \Sigma_T] \) is a generalization for a forward contract for instantaneous variance \( \Sigma, \) which was also discussed by Stojeanovic in [25]. Indeed, if the price of a forward contract is based on the Black-Scholes’ lognormal stochastic diffusion with constant volatility, then the investor does not need to consider the risk coming from the uncertainty of the volatility. However, concerned with the stochastic volatility financial model considered in this paper, the price of the contract exhibits sensitivity to both the volatilities and the correlations of the underlying assets and so the investor could not ignore the risk caused by the uncertainty of the stochastic volatility. Thus, it is reasonable
to consider $Tr[\theta \cdot \Sigma_T]$ corresponding to the factor of volatility matrix $\Sigma_T$ in the terminal payoff function.

For the simplest case $n = 1$, the WASC model in our paper is the same as in [25], and the typical and classical case is the Heston’s Model in [13]. For the application of neutral and indifference pricing to forward contracts, one can refer to [25] (P129, Section 4.13).

In the following part, we shall talk about the solutions to the above pricing systems for the general case which is, of course, suitable for the case $n = 1$.

Now assume that $r(t, \Sigma) = r$, $a_S = r \bar{I} + \Sigma \cdot \vartheta$, $q(t, \Sigma) = q = (q_1, q_2, \ldots, q_n)^T$ ($r > 0, q_i \in \mathbb{R}(1 \leq i \leq n)$ are constant numbers and \( \vartheta \) is an $n \times 1$-matrix). The assumption $a_S = r \bar{I} + \Sigma \cdot \vartheta$ implies a constant market price \( \vartheta \) of stochastic volatility-correlation risk, which is also used in [13] for the Heston’s model and in [2] for optimal portfolios model. Under such assumptions, we seek for the solution of the CRRA neutral pricing system (40)-(41), in the form $\{g_\gamma(t, S, \Sigma), V(t, S, \Sigma)\}$:

$$g_\gamma(t, S, \Sigma) = E_\gamma(t) + Tr[F_\gamma(t) \cdot \Sigma],$$

$$V(t, S, \Sigma) = E(t) + Tr[F(t) \cdot \Sigma] + L^T(t) \cdot S,$$  

with the boundary conditions:

$$E_\gamma(T) = 0, \quad F_\gamma(T) = 0; \quad E(T) = -K, \quad F(T) = \theta, \quad L(T) = \alpha;$$

where $E, F \in \mathbb{R}$ are real numbers, $F, \theta, L \in \mathbb{S}_n(\mathbb{R})$ are symmetric $n \times n$ matrices and $L, L \in \mathbb{R}^{n \times 1}$ are $n \times 1$ matrices.

By plugging (42)-(43) into (40)-(41), we arrive at:

$$\frac{dE_\gamma}{dt} + Tr \left[ \frac{dF_\gamma}{dt} \cdot \Sigma \right] + Tr[\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T) \cdot F_\gamma]$$

$$- 2\gamma - 1 \frac{1}{\gamma} \theta^T \cdot \Sigma \cdot F_\gamma \cdot Q^T \cdot \rho + 2Tr \left[ F_\gamma \cdot \Sigma \cdot F_\gamma \cdot Q^T \cdot (I_n - \gamma \frac{1}{\gamma} \rho \cdot \rho^T) \cdot Q \right]$$

$$- 2\gamma(\gamma - 1)\kappa_0^2 Tr[F \cdot \Sigma \cdot F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q]$$

$$= \frac{1}{2} \gamma - 1 \frac{1}{\gamma} \theta^T \cdot \Sigma \cdot \vartheta + (\gamma - 1)r$$  

and

$$\frac{dE}{dt} + Tr \left[ \frac{dF}{dt} \cdot \Sigma \right] + \frac{dL^T}{dt} \cdot S + (r \bar{I} - q)^T \cdot S^* \cdot L$$

$$+ Tr[\Omega \cdot \Omega^T + M \cdot \Sigma + \Sigma \cdot M^T) \cdot F] - 2\theta^T \cdot \Sigma \cdot F \cdot Q^T \cdot \rho$$

$$+ 4Tr[F \gamma \cdot \Sigma + F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q]$$

$$- 4\gamma \kappa_0 Tr[F \cdot \Sigma \cdot F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q]$$

$$- rE - Tr[r F \cdot \Sigma] - rL^T \cdot S = 0.$$  

Making use of the arbitrariness of $S$ and $\Sigma$, we obtain a following system of ordinary differential equations (ODEs):

$$\frac{dE_\gamma}{dt} + Tr[\Omega \cdot \Omega^T \cdot F_\gamma] = (\gamma - 1)r,$$
\[
\frac{dF_t}{dt} + F_t \cdot (M - \frac{\gamma - 1}{\gamma} Q^T \cdot \rho \cdot \vartheta^T) + (M - \frac{\gamma - 1}{\gamma} Q^T \cdot \rho \cdot \vartheta^T) \cdot F_t \\
+ 2F_t \cdot Q^T \cdot (I_n - \frac{\gamma - 1}{\gamma} \rho \cdot \rho^T) \cdot Q \cdot F_t \\
- 2\gamma(\gamma - 1)\kappa_0 F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F \\
= \frac{1}{2} \frac{\gamma - 1}{\gamma} \rho \cdot \vartheta^T, \quad (49)
\]

\[
\frac{dE}{dt} + Tr[\Omega \cdot \Omega^T \cdot F] - rE = 0, \quad (50)
\]

\[
\frac{dF}{dt} + F \cdot (M - Q^T \cdot \rho \cdot \vartheta^T) + (M - Q^T \cdot \rho \cdot \vartheta^T) \cdot F - rF \\
+ 2F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F + 2F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F_t \\
- 4\gamma\kappa_0 F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F \\
= 0, \quad (51)
\]

for \(0 \leq t < T\), together with the boundary conditions (44)-(45).

With the boundary conditions (45), it is easy to solve (52) and

\[
L_i(t) = \alpha_i e^{-q_i(T-t)}(1 \leq i \leq n), \quad (53)
\]

for \(0 \leq t \leq T\).

Moreover, from (50) and the boundary conditions (45) for \(E(t)\), we can reform \(E(t)\) as the following equation:

\[
E(t) = -e^{-r(T-t)}K + \int_t^T e^{-r(\tau-t)}Tr[\Omega \cdot \Omega^T \cdot F(\tau)]d\tau. \quad (54)
\]

Summarizing, we obtain the following proposition.

**Proposition 1.** In the above framework, assume that \(r(t, \Sigma) = r\) and \(a_S = r\bar{\Gamma} + \Sigma \cdot \vartheta\) with

\[
q(t, \Sigma) = q = (q_1, q_2, \cdots, q_n)^T.
\]

Then for a position \(\kappa(t, X, S, \Sigma) = X\kappa_0\) and the CRRA utility function

\[
\psi_\gamma(X) = \frac{X^{1-\gamma} - 1}{1 - \gamma} \quad (\gamma > 0, \ \gamma \neq 1),
\]

the CRRA neutral value of the forward contract \(V(t, S, \Sigma)\) with the general payoff

\[
v(T) = \alpha^T \cdot S_T + Tr[\vartheta \cdot \Sigma_T] - K
\]

is given in the following form

\[
V(t, S_t, \Sigma_t) = E(t) + Tr[F(t) \cdot \Sigma_t] + L^T(t) \cdot S_t \\
= -e^{-r(T-t)}K + \int_t^T e^{-r(\tau-t)}Tr[\Omega \cdot \Omega^T \cdot F(\tau)]d\tau \\
+ Tr[F(t) \cdot \Sigma_t] + \sum_{i=1}^n \alpha_i S_i^T e^{-q_i(T-t)} \quad (55)
\]
for \( t < T \), where \( F(t) \) is given by the coupled ODE systems (49) and (51). Consequently, the associating forward price, denoted by \( \hat{F}(t, S_t, \Sigma_t) \), is given by

\[
\hat{F}(t, S_t, \Sigma_t) = \int_t^T e^{-r(\tau-T)}Tr[\Omega \cdot \Omega^T \cdot F(\tau)]d\tau \\
+ e^{r(T-t)}Tr[F(t) \cdot \Sigma_t] + \sum_{i=1}^n \alpha_i S_t^i e^{r-q_i(T-t)} 
\]

(56)

for \( t < T \), where \( F(t) \) is also given by the coupled ODE systems (49) and (51).

Proof. (55) follows from (43), (53) and (54). Solving \( V(t, S_t, \Sigma_t) = 0 \) in (55) for \( K \), we get (56), the associating forward price. \( \square \)

Similarly, we can get the result for the indifference pricing in Theorem 3.11, which is the following proposition.

**Proposition 2.** Providing that the conditions in the proposition 1 hold true, the CRRA indifference value of the forward contract \( V(t, S_t, \Sigma_t) \) with the general payoff

\[
v(T) = \alpha^T \cdot S_T + Tr[\theta \cdot \Sigma_T] - K
\]

is given in the following form

\[
V(t, S_t, \Sigma_t) = E(t) + Tr[F(t) \cdot \Sigma_t] + L^T(t) \cdot S_t \\
= -e^{-r(T-t)}K + \int_t^T e^{-r(\tau-t)}Tr[\Omega \cdot \Omega^T \cdot F(\tau)]d\tau \\
+ Tr[F(t) \cdot \Sigma_t] + \sum_{i=1}^n \alpha_i S_t^i e^{-q_i(T-t)} 
\]

(57)

for \( t < T \). Consequently, the associating forward price, denoted by \( \hat{F}(t, S_t, \Sigma_t) \), is given by

\[
\hat{F}(t, S_t, \Sigma_t) = \int_t^T e^{-r(\tau-T)}Tr[\Omega \cdot \Omega^T \cdot F(\tau)]d\tau \\
+ e^{r(T-t)}Tr[F(t) \cdot \Sigma_t] + \sum_{i=1}^n \alpha_i S_t^i e^{r-q_i(T-t)} 
\]

(58)

for \( t < T \). \( F(t) \) in both (57) and (58) is given by the uncoupled ODE system:

\[
\frac{dF}{dt} + F \gamma \cdot (M - \gamma - \frac{1}{\gamma} Q^T \cdot \rho \cdot \vartheta^T) + (M - \gamma - \frac{1}{\gamma} Q^T \cdot \rho \cdot \vartheta^T) \cdot F \gamma \\
+ 2F \gamma \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F \gamma \\
= \frac{1}{2} \gamma - \frac{1}{\gamma} \vartheta \cdot \vartheta^T, 
\]

and

\[
\frac{dF}{dt} + F \cdot (M - Q^T \cdot \rho \cdot \vartheta^T) + (M - Q^T \cdot \rho \cdot \vartheta^T) \cdot F - rF \\
+ 2F \gamma \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F + 2F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F \gamma \\
- 2\gamma \kappa_0 F \cdot Q^T \cdot (I_n - \rho \cdot \rho^T) \cdot Q \cdot F \\
= 0,
\]
for \( t < T \), with the boundary condition

\[
F_\gamma(T) = 0, F(T) = \theta.
\]

**Remark 7.** Let \( n = 1 \) and \( \vartheta = 0 \). If we take \( \alpha = 1, \theta = 0 \) (i.e. \( v(T) = S_T - K \)), then it is not hard to see that the forward contract price \( \hat{F}_\gamma = e^{(r-q)(T-t)}S_t \), which is equal to the price in usual case. If we take \( \alpha = 0, \theta = 1 \) (i.e. \( v(T) = \Sigma_T - K \)), then we shall see that the price is the same as the result in [25] (see Section 4.13).

In order to visualize how the factors (such as the CRRA risk-aversion parameter \( \gamma \), the position \( \kappa \) and so on) in the market affect the price of the forward contract, we consider a series of numerical analyses of CRRA utility. We take the parameters from [16], which are given by

\[
M = \begin{pmatrix} -2.5 & -1.5 \\ -1.5 & -2.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.21 & -0.14 \\ 0.14 & 0.21 \end{pmatrix}, \quad \rho = \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix},
\]

\[
\Sigma_0 = \begin{pmatrix} 0.09 & -0.036 \\ -0.036 & 0.09 \end{pmatrix}, \quad \beta = 7.14286.
\]

And other parameters are assumed by

\[
\vartheta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

and

\[
r = 0.03, \quad q_1 = q_2 = 0.
\]

In the following numerical computations, we use the Runge-Kutta method and the Simpson’s rule to solve the system of ODEs (48)-(51) and (56), respectively.

Let \( T = 1, \gamma = 2 \) and \( \kappa_0 = 5 \). If \( n = 1 \), then we write two forward contracts for \( S_1 \) and \( S_2 \) with the payoffs \( v_1(T) = S_1(T) + \Sigma_{11}(T) - K_1 \) and \( v_2(T) = S_2(T) + \Sigma_{22}(T) - K_2 \), respectively. If \( n = 2 \), then we write a forward contract for \( S = (S_1, S_2)^T \) under the WASC model with the payoff

\[
v(T) = \alpha^T \cdot S_T + Tr[\theta \cdot \Sigma_T] - K_1 - K_2.
\]

The numerical results are shown in Figures 1 and 2, respectively. It is obvious that \( v(T) = v_1(T) + v_2(T) \). Figures 1 and 2 imply that the forward price of latter is not equal to the addition of the former, as \( 2.2684 > 2 \times 1.0900 \). The essential difference is that the correlation between the assets is considered in the WASC model which leads to the higher risk.

Letting \( \gamma = 0.5 \) (resp. \( \gamma = 2 \)), the delivery time \( T \) range from 0.5 to 3 with step length \( \Delta T = 0.5 \) and the position \( \kappa_0 \) range from \(-5 \) to 5 with step length \( \Delta \kappa_0 = 0.05 \), we get Figure 3 (resp. Figure 4).

From Figures 3 and 4, we can see that the forward price grows in pace with the delivery time \( T \), which is an empirical phenomenon.

For the position \( \kappa = X_\kappa_0 \), it is the number of contracts. We can take \( \kappa_0 < 0, \kappa_0 = 0 \) or \( \kappa_0 > 0 \), which stand for the short position, no position or the long position, respectively. For a given CRRA risk-aversion parameter \( \gamma \), the Figures 3 and 4 show that the forward prices decrease with \( \kappa_0 \) increasing and that the forward price with the long position is lower than the price with no position and the latter is lower than the price with the short position.

Putting \( T = 1 \) and \( \kappa_0 = -5 \) (resp. \( \kappa_0 = 5, \kappa_0 = -0.01, \kappa_0 = 0 \)) and letting the risk-aversion \( \gamma \) range from 0.01 to 0.99 and from 1.01 to 2 with step length \( \Delta \gamma = 0.01 \), we obtain Figure 5 (resp. Figure 6, Figure 7, Figure 8).
Figure 1. Solutions to $F_\gamma$ and $F$ ($n=2$)

Figure 2. Solutions to $F_\gamma$ and $F$ ($n=1$)
Figure 3. Price under different delivery times and positions ($\gamma = 0.5$)

Figure 4. Price under different delivery times and positions ($\gamma = 2$)
Figure 5. Price under different risk-aversion parameters ($\kappa_0 = -5, T = 1, \gamma \neq 1$)

Figure 6. Price under different risk-aversion parameters ($\kappa_0 = 5, T = 1, \gamma \neq 1$)
Figure 7. Price under different risk-aversion parameters ($\kappa_0 = -0.01, T = 1, \gamma \neq 1$)

Figure 8. Price under different risk-aversion parameters ($\kappa_0 = 0, T = 1, \gamma \neq 1$)
Letting $T = 1$, $\gamma$ range from 0.4 to 2.0 with step length $\Delta \gamma = 0.4$ and the position $\kappa_0$ range from $-5$ to 5 with step length $\Delta \kappa_0 = 0.05$, we get Figure 9.

**Figure 9.** Price under different risk-aversion parameters and positions ($T = 1$)

As to the CRRA risk-aversion parameter $\gamma$, empirically, the higher the risk-aversion is, the lower the market price should be. However, from Figures 5-9, its influence on the forward price depends on the position in our paper. In fact, the forward contract is written for the holder to avoid the fluctuation risk of the assets’ prices in the future so that the higher the position, the more risk averse the holder is (which is just like what we showed before in Figures 3 and 4). As a result, the total risk aversion in our model relies on both the CRRA risk-aversion parameter $\gamma$ and the position $\kappa_0$.

Roughly speaking, from the Figure 9, the forward price varies in such an approximate way under the influence of $\gamma$ and $\kappa_0$: if $\kappa_0 < 0$ ($\kappa_0 = -5$ in Figure 5, for example), the higher the risk-aversion $\gamma$, the higher the price tends to be, and if $\kappa_0 > 0$ ($\kappa_0 = 5$ in the Figure 6, for example), the higher the risk-aversion $\gamma$, the lower the price tends to be. If $\kappa_0 < 0$ is around 0 ($\kappa_0 = -0.01$ in Figure 7, for example), the influence of $\kappa_0$ on the forward price is in contrast with the influence of $\gamma$ so that the case is much more complex and relies on the choice of $\gamma$.

For no position, the CRRA parameter $\gamma$ becomes the only factor to determine the holder’s risk aversion and Figure 8 shows that the higher the risk-aversion $\gamma$, the lower the price with no position tends to be, which is the same as the empirical phenomenon.

Moreover, it is easy to see from Figure 9 that the lower the position $|\kappa_0|$, the smaller the change of the forward price caused by risk-aversion parameter $\gamma$ tends to be.
Remark 8. If $\theta = 0$, owing to the linear property of the terminal payoff, then it is easy to check that $F(t) = 0$ for $0 \leq t \leq T$, which shows that
\[
\hat{F}(t, S_t, \Sigma_t) = \sum_{i=1}^{2} \alpha_i S_i^t e^{(T-t)}
\]
and that there is no difference between writing a forward contract in WASC model and writing two forward contracts in usual case (if they have the same total delivery price like before). However, it is not always true if $\theta \neq 0$, which we have checked before.

5. Conclusions. Under the Wishart Affine Stochastic Correlation (WASC) model for the multiple underlying assets, the paper derives two general utility-based pricing for European-style financial contracts: the neutral and indifference pricing. Then we give the particular cases so that the pricing methods are more widely practical, where the utility functions are constant absolute risk-aversion (CARA) and constant relative risk-aversion (CRRA) utility function. The paper shows that comparing to risk-neutral pricing, the neutral and indifference pricing methods are generally feasible and avoid factitiously dealing with some risk premia corresponding to the volatilities-correlations as a consequence of the incompleteness of the market.

However, the neutral and indifference pricing consists two complex coupled (or uncoupled) systems of PDEs, which are usually hard to solve. Hence, looking for some efficient analytic or numerical methods to solve the systems of PDEs is an important and crucial work in the future.

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