Tropical lines on smooth tropical surfaces

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Abstract

Given a tropical line $L$ on a tropical surface $X$, we define its combinatorial position on $X$ to be a certain decorated graph, showing the relative positions of vertices of $X$ on $L$, and how the vertices of $L$ are positioned on $X$. We classify all possible combinatorial positions of a tropical line on general smooth tropical surfaces of any degree. This classification allows one to give an upper bound for the number of tropical lines on a general smooth tropical surface of degree $\geq 3$ with a given subdivision. As a concrete example, we offer a subdivision for which the associated tropical surfaces are smooth cubics with exactly 27 tropical lines in the general case, and always at least 27 tropical lines. We also give examples of smooth tropical surfaces of arbitrary degree $> 3$ containing no tropical lines.

1 Introduction

A celebrated theorem in classical geometry states that any smooth algebraic cubic surface in complex projective three-space contains exactly 27 distinct lines. This was first established in 1849 in a correspondence between Arthur Cayley and George Salmon.

Since the appearance of tropical geometry a few years ago, it has been a recurring question whether there is a tropical analogue of this result. It is a common opinion among tropical geometers that this is indeed the case. Explicit examples of smooth tropical surfaces with 27 distinct tropical lines have been found by Mikhalkin and by Gross[5].

However, nothing on the subject has been published as yet. Furthermore, it is far from obvious what the correct formulation of the tropical analogue should be. For example, in [7] we showed that there exist smooth tropical cubic surfaces containing infinitely many tropical lines.

The purpose of this paper is to give a systematic approach to the subject of tropical lines on smooth tropical surfaces of arbitrary degree. As it turns out, this allows us to give a partial answer to the above questions.

Tropical surfaces in $\mathbb{R}^3$ are unbounded polyhedral cell complexes of dimension 2 with certain properties. Most importantly, each tropical surface is dual to a regular lattice subdivision of a lattice polytope in $\mathbb{R}^3$. We say that the tropical surface is smooth of degree $\delta$, if the dual subdivision is an elementary (unimodular) triangulation of the polytope $\Gamma_\delta = \text{conv}\{(0,0,0),(\delta,0,0),(0,\delta,0),(0,0,\delta)\}$.

A large portion of our results hold only for general smooth tropical surfaces. The concept of generality used here should be noted: The parameter spaces of tropical surfaces are cones (or, more generally, fans) in some large Euclidean space. When we speak about general tropical surfaces with a given dual subdivision, we mean the surfaces corresponding to points in some open dense subset (in the Euclidean topology) of the parameter cone.

One can show that the general intersection of two tropical planes (i.e., tropical surfaces of degree 1), is an unbounded one-dimensional polyhedral cell complex, called a tropical line. Its underlying topological space is homeomorphic to the graph in Figure 4 with its 1-valent vertices removed.

The main core of this paper is an analysis of the different ways in which a tropical line $L$ can lie on a smooth tropical surface $X$. A crucial concept in our arguments is the notion of the combinatorial position of $L$ on $X$. This is a decoration of the underlying graph of $L$, displaying the relative positions of vertices of $X$ on $L$, and vertices of $L$ on $X$. (See Figure 2 for a typical example). We are able to show that on a general smooth $X$, only 17 such combinatorial positions are possible. Of these, only nine occur if $X$ has degree three.
Let $X$ be a tropical surface, and $S$ its associated subdivision. The elements of $S$ dual to the cells of $X$ intersecting $L$, form a subcomplex called a line subcomplex. In most cases, the cell structure of this subcomplex is determined by the combinatorial position of $L$. Hence, by counting subcomplexes of $S$, we obtain an upper bound for the number of tropical lines on $X$.

As an application of the above technique we provide examples of general smooth tropical surfaces of arbitrary degree greater than 3, containing no tropical lines (see Proposition 6.4). This complements a result in [7], where we found general smooth tropical surfaces of arbitrary degree containing infinitely many tropical lines.

In the final section of this paper we consider smooth tropical cubic surfaces. For the subdivision $S_{\alpha,3}$, shown in Figure 18, we prove:

**Theorem 7.1**

a) A general tropical surface with subdivision $S_{\alpha,3}$ contains exactly 27 tropical lines.

b) Any tropical surface with subdivision $S_{\alpha,3}$ contains at least 27 tropical lines.

c) There exist tropical surfaces with subdivision $S_{\alpha,3}$ containing infinitely many tropical lines.

2 Preliminaries

2.1 Convex polyhedra and polytopes

A convex polyhedron in $\mathbb{R}^n$ is the intersection of finitely many closed halfspaces. A cone is a convex polyhedron, all of whose defining hyperplanes contain the origin. A convex polytope is convex polyhedron which is bounded. Equivalently, a convex polytope can be defined as the convex hull of a finite set of points in $\mathbb{R}^n$. Throughout this paper, all polyhedra and polytopes will be assumed to be convex unless explicitly stated otherwise.

For any polyhedron $\Delta \subseteq \mathbb{R}^n$ we denote its affine hull by $\text{Aff}(\Delta)$, and its relative interior (as a subset of $\text{Aff}(\Delta)$) by $\text{int}(\Delta)$. The dimension of $\Delta$ is defined as $\dim \text{Aff}(\Delta)$. By convention, $\dim \emptyset = -1$. A face of $\Delta$ is a polyhedron of the form $\Delta \cap H$, where $H$ is a hyperplane such that $\Delta$ is entirely contained in one of the closed halfspaces defined by $H$. In particular, the empty set is considered a face of $\Delta$. Faces of dimensions 0, 1 and $n-1$ are called vertices, edges and facets of $\Delta$ respectively. If $\Delta$ is a polytope, then the vertices of $\Delta$ form the minimal set $\mathcal{A}$ such that $\Delta = \text{conv}(\mathcal{A})$.

A lattice polytope in $\mathbb{R}^n$ is a polytope of the form $\Delta = \text{conv}(\mathcal{A})$, where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n$. We say that $\Delta$ is elementary, or unimodular if it is $n$-dimensional and its volume is $\frac{1}{n!}$. It is easy to see that a necessary condition for $\Delta$ to be elementary is that it is a simplex, i.e. the convex hull of $n+1$ affinely independent points.

2.2 Polyhedral complexes and subdivisions

A (finite) polyhedral complex in $\mathbb{R}^n$ is a finite collection $X$ of convex polyhedra in $\mathbb{R}^n$ such that

- if $C \in X$, then all faces of $C$ are in $X$, and
- if $C, C' \in X$, then $C \cap C'$ is a face of both $C$ and $C'$. 


The elements of $X$ of dimension $d$ are called the $d$-cells of $X$. The dimension of $X$ itself is defined as $\max\{\dim C \mid C \in X\}$. Furthermore, if all the maximal cells (w.r.t. inclusion) have the same dimension, we say that $X$ is of pure dimension.

If all the elements of a polyhedral complex are cones, the complex is a fan.

A subdivision of a polytope $\Delta$ is a polyhedral complex $S$ such that $|S| = \Delta$, where $|S|$ denotes the union of all the elements of $S$. It follows that $S$ is of pure dimension $\dim \Delta$. If all the maximal elements of $S$ are simplices, we call $S$ a triangulation.

If $\Delta$ is a lattice polytope, we can consider lattice subdivisions of $\Delta$, i.e., subdivisions in which every element is a lattice polytope. In particular, a lattice subdivision is an elementary triangulation if all its maximal elements are elementary simplices.

2.3 Regular subdivisions and their secondary cones

Let $A \subseteq \mathbb{R}^n$ be a finite set of points, and let $\Delta = \text{conv}(A)$. For any function $\alpha : A \rightarrow \mathbb{R}$ we can consider the lifted polytope

$$\tilde{\Delta} = \text{conv}\{(v, \alpha(v)) \mid v \in A\} \subseteq \mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1}.$$ 

Projecting the top faces of $\tilde{\Delta}$ to $\mathbb{R}^n$ by forgetting the last coordinate, gives a collection of sub-polytopes of $\Delta$. They form a subdivision $S_\alpha$ of $\Delta$, called the regular (or coherent) subdivision induced by $\alpha$. The function $\alpha$ is called a lifting function associated to $S_\alpha$. Note that if $A \subseteq \mathbb{Z}^n$, then $S_\alpha$ is a lattice subdivision of $\Delta$. Most of the subdivisions we will encounter in this paper, are regular elementary triangulations, or RE-triangulations for short.

Fixing some order of the elements of $A$, there is a natural 1-1 correspondence between the set of functions $\alpha : A \rightarrow \mathbb{R}$ and $\mathbb{R}^N$, where $N = |A|$. Hence, for any regular subdivision $S$ of $\text{conv}(A)$, we can regard the set

$$K(S) := \{\alpha : A \rightarrow \mathbb{R} \mid S_\alpha = S\}$$

as a subset of $\mathbb{R}^N$. The following was observed in [2, Chapter 7]:

**Proposition 2.1.** $K(S)$ is an open cone in $\mathbb{R}^N$. If $S$ is an RE-triangulation, then $\dim K(S) = N$.

The cone $K(S)$ is called the secondary cone associated to $S$.

2.3.1 Example

For arbitrary $\delta \in \mathbb{N}$, define $A_\delta$ to be the set of lattice points contained in the simplex

$$\Gamma_\delta := \text{conv}\{(0, 0, 0), (\delta, 0, 0), (0, \delta, 0), (0, 0, \delta)\}.$$

In particular, the number of points in $A_\delta$ equals $\binom{\delta + 3}{3}$, the $(\delta - 1)'th$ tetrahedral number. Let $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the polynomial function given by

$$\alpha(x, y, z) = -2x^2 - 2y^2 - 2z^2 - xy - 2xz - 2yz.$$  

(1)

For any given $\delta$, the restriction of $\alpha$ to $A_\delta$ induces - as explained above - a regular subdivision of $\Gamma_\delta$. We denote this subdivision by $S_{\alpha, \delta}$.

**Proposition 2.2.** For any $\delta \in \mathbb{N}$, $S_{\alpha, \delta}$ is an RE-triangulation of $\Gamma_\delta$.

**Proof.** We introduce the following six families of elementary tetrahedra in $\mathbb{R}^3$: For each lattice point $P = (p, q, r) \in \mathbb{Z}^3$, let

$$T^1_P = \text{conv}\{(p, p, p), (p + 1, q, r), (p, q + 1, r), (p, q, r + 1)\}$$

$$T^2_P = \text{conv}\{(p + 1, q + 1, r), (p + 1, q, r), (p, q + 1, r), (p, q, r + 1)\}$$

$$T^3_P = \text{conv}\{(p + 1, q, r), (p + 1, q + 1, r), (p, q + 1, r), (p, q + 1, r + 1)\}$$

$$T^4_P = \text{conv}\{(p, q, q), (p + 1, q + 1, r), (p, q, q + 1, r), (p, q + 1, r + 1)\}$$

$$T^5_P = \text{conv}\{(p + 1, q + 1, r), (p + 1, q, r), (p, q + 1, r), (p, q, q + 1, r)\}$$

$$T^6_P = \text{conv}\{(p + 1, q, q), (p + 1, q + 1, r), (p, q + 1, r), (p, q + 1, q + 1, r)\}$$
The tetrahedra $T_p^1, \ldots, T_p^6$ have disjoint interiors, and they form a subdivision of the unit cube with diagonal $(p, q, r)(p + 1, q + 1, r + 1)$. (See Figure 3) In particular, the set $\{T_p^i\}_{i=1, \ldots, 6, p \in \mathbb{Z}^3}$ is a covering of $\mathbb{R}^3$.

To prove the Proposition, it is enough to show that for any $\delta$, each maximal element of $S_{\alpha, \delta}$ equals $T_p^i$ for some $1 \leq i \leq 6$ and $P \in \mathbb{N}_0^3$. Equivalently, it suffices to prove that any tetrahedron of the form $T_p^i$ is a maximal element of $S_{\alpha, \delta}$, for all $\delta$ such that $T_p^i \subseteq \Gamma_\delta$.

Let $P = (p, q, r)$ be any point in $\mathbb{N}_0^3$, and consider the lifting (by $\alpha$) of the tetrahedron $T_p^i$. The affine hull of the resulting polytope $\tilde{T}_p^i$ is an affine hyperplane in $\mathbb{R}^4$, namely the graph of the function

$$
\beta(x, y, z) = 2p + 2q + 2r - \alpha(p, q, r) - (4p + q + 2r + 2)x - (p + 4q + 2r + 2)y - (2p + 2q + 4r + 2)z.
$$

(Proof: It is easy to check that the functions $\beta$ and $\alpha$ are equal on the vertices of $T_p^i$.)

We claim that the difference $\gamma := \beta - \alpha$ is strictly positive at all lattice points $(x, y, z) \in \mathbb{Z}^3 \setminus T_p^i$. Note that correctness of this claim implies that $\tilde{T}_p^i$ is a top facet of $\tilde{\mathcal{A}}_\delta$, and therefore that $T_p^i \in S_{\alpha, \delta}$ for all big enough $\delta$. To prove the claim, observe that the subset $\{(x, y, z) \mid \gamma(x, y, z) \leq 0\}$ is a solid ellipsoid circumscribing $T_p^i$. Translating such that $P \mapsto (0, 0, 0)$, we get the ellipsoid $Q_1$ with equation

$$
\gamma(x + p, y + q, z + r) = 0,
$$

or

$$
2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 2x - 2y - 2z = 0.
$$

The ellipsoid $Q_1$ is shown in Figure 4. It is clear that it contains no lattice points besides the vertices of $T_p^i_{(0,0,0)}$. This proves the claim.

The five remaining cases are treated similarly. More precisely, for each $i = 2, \ldots, 6$, the problem of proving that $T_p^i \in S_{\alpha, \delta}$ reduces to that of showing that a certain ellipsoid, $Q_i$, contains no lattice points outside $T_p^i_{(0,0,0)}$. This is a trivial task, once one calculates the equations of these ellipsoids:

$$
\begin{align*}
Q_2: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 3x - 3y - 3z + 1 = 0 \\
Q_3: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 4x - 3y - 4z + 2 = 0 \\
Q_4: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 3x - 4y - 4z + 2 = 0 \\
Q_5: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 4x - 4y - 5z + 3 = 0 \\
Q_6: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 5x - 6y - 5z + 5 = 0
\end{align*}
$$

We conclude that any tetrahedron of the form $T_p^i$ contained in $\Gamma_\delta$, is a maximal element of $S_{\alpha, \delta}$. Since the $T_p^i$’s are elementary, have disjoint interiors and cover $\mathbb{R}^3$, this proves that $S_{\alpha, \delta}$ is elementary. □
3 Tropical surfaces and tropical lines in $\mathbb{R}^3$

3.1 Definition of tropical surfaces

Here we go through the basic definitions and our notation concerning tropical surfaces in $\mathbb{R}^3$. Note that these concepts can be immediately generalized to hypersurfaces in $\mathbb{R}^n$. (See [3], [4], and [1]).

We work over the tropical semiring $\mathbb{R}_t := (\mathbb{R}, \max, +)$. To simplify the reading of tropical expressions, we adopt the following convention: If a (classical) expression is written in quotation marks, all arithmetic operations should be interpreted as tropical. Hence, if $x, y \in \mathbb{R}$ and $k \in \mathbb{N}_0$ we have for example 

"$x + y" = \max(x, y)$, "$xy" = x + y$ and "$x^k" = kx$.

A tropical polynomial in indeterminates $x_1, x_2, x_3$ is an expression of the form

$$f(x_1, x_2, x_3) = " \sum_{(a_1, a_2, a_3) \in \mathcal{A}} \lambda_{a_1a_2a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} "$$

(2)

where the support $\mathcal{A}$ is a finite subset of $\mathbb{Z}^3$, and the coefficients $\lambda_{a_1a_2a_3}$ are real numbers. We can write the expression for $f$ more compactly using vector notation, with $x = (x_1, x_2, x_3)$ and $a = (a_1, a_2, a_3)$, as $f(x) = " \sum_{a \in \mathcal{A}} \lambda_a x^a "$. Translating to classical arithmetic, we see that $f$ is the maximum of a finite number of affine-linear expressions. Hence, $f: \mathbb{R}^3 \to \mathbb{R}$ is a concave, piecewise-linear function. The non-linear locus of $f$, denoted $V_tr(f)$, is called the tropical surface associated to $f$. It is well known (see e.g. [3] and [4]) that $V_tr(f)$ is a connected polyhedral complex of pure dimension 2, some of whose cells are unbounded in $\mathbb{R}^3$.

Definition 3.1. Let $\delta \in \mathbb{N}$. A tropical surface of degree $\delta$ is a subset of $\mathbb{R}^3$ of the form $V_tr(f)$, where $f$ is a tropical polynomial whose support is the set $\mathcal{A}_\delta$ defined in section [2.3.1].

3.2 Duality

Many of the techniques used in this paper rest on the duality - detailed below - between cells in a tropical surface of degree $\delta$ and in its subdivision of $\Gamma_\delta$.

Let $X$ be a tropical surface of degree $\delta$. Writing $\mathcal{A}_\delta := \Gamma_\delta \cap \mathbb{Z}^3$, this means (by Definition 3.1) that $X$ is of the form $X = V_tr(f)$, for some tropical polynomial $f(x) = " \sum_{a \in \mathcal{A}_\delta} \lambda_a x^a "$. As explained in section 2.3, the function $a \mapsto \lambda_a$ induces a regular lattice subdivision of $\Gamma_\delta$. We denote this by Subdiv($f$). Any element $\Delta \in \text{Subdiv}(f)$ of dimension at least 1, corresponds in a natural way to a cell $\Delta^\vee \subseteq V_tr(f)$. Namely, if the vertices of $\Delta$ are $a_1, \ldots, a_r$, then $\Delta^\vee$ is the solution set of the equalities and inequalities

$$\lambda_{a_1} + (a_1, x) = \cdots = \lambda_{a_r} + (a_r, x) \geq \lambda_b + (b, x), \quad \text{for all } b \in \mathcal{A}_\delta \setminus \{a_1, \ldots, a_r\}.$$  

(3)

(Here, $(\cdot, \cdot)$ denotes the Euclidean inner product on $\mathbb{R}^3$.) Moreover, we have the following theorem (see [4] Proposition 3.11]):

Theorem 3.2. The association $\Delta \mapsto \Delta^\vee$ gives a one-to-one correspondence between the $k$-cells of Subdiv($f$) and the $(n-k)$-cells of $V_tr(f)$, for each $k = 1, 2, 3$. Furthermore, for any cells $\Delta, \Lambda \in \text{Subdiv}(f)$ of dimensions at least 1, we have that

i) If $\Delta$ is a face of $\Lambda$, then $\Lambda^\vee$ is a face of $\Delta^\vee$ in $V_tr(f)$.

ii) The affine-linear subspaces $\text{Aff}(\Delta)$ and $\text{Aff}(\Delta^\vee)$ are orthogonal in $\mathbb{R}^3$.

iii) $\Delta \subseteq \partial(\Gamma_\delta)$ if and only if $\Delta^\vee$ is an unbounded cell of $V_tr(f)$.

If $C$ is a cell of $V_tr(f)$, we denote its corresponding cell in Subdiv($f$) by $C^\vee$. The cells $C$ and $C^\vee$ are said to be dual to each other.

Definition 3.3. We say that $V_tr(f)$ is a smooth tropical surface if Subdiv($f$) is an elementary (unimodular) triangulation.

For example, let $f_\delta(x) = " \sum_{a \in \mathcal{A}_\delta} \alpha(a) x^a "$, where $\alpha$ is the lifting function defined in [11]. Then according to Definitions 3.1 and 3.3 the tropical surface $V_tr(f_\delta)$ is smooth of degree $\delta$. 

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3.3 Tropical lines in $\mathbb{R}^3$

Let $L$ be an unrooted tree with five edges, and six vertices, two of which are 3-valent and the rest 1-valent. We define a tropical line in $\mathbb{R}^3$ to be any realization of $L$ in $\mathbb{R}^3$ such that

- the realization is a polyhedral complex, with four unbounded rays (the 1-valent vertices of $L$ are pushed to infinity),
- the unbounded rays have direction vectors $-e_1, -e_2, -e_3, e_1 + e_2 + e_3$,
- The realization is balanced at each vertex, i.e., the primitive integer vectors in the directions of all outgoing edges adjacent to a given vertex, sum to zero.

If the bounded edge has length zero, the tropical line is called degenerate. For non-degenerate tropical lines, there are three combinatorial types of tropical lines in $\mathbb{R}^3$, as shown in Figure 5. The combinatorial types of the lines in Figure 5, from left to right, are denoted by (((12)(34))), ((13)(24)) and ((14)(23)). Each innermost pair of digits indicate the directions of two adjacent rays.

![Figure 5: The combinatorial types of tropical lines in $\mathbb{R}^3$.](image)

Remark 3.4. This definition is equivalent to the more standard algebraic definition of tropical lines in $\mathbb{R}^3$. See [Examples 2.8 and 3.8].

In classical geometry, any two distinct points lie on a unique line. When we turn to tropical lines, this is true only for generic points. In fact, for special choices of points $P$ and $Q$ there are infinitely many tropical lines passing through $P$ and $Q$. The precise statement is as follows:

Lemma 3.5. Let $P, Q \in \mathbb{R}^3$. There exist infinitely many tropical lines containing $P$ and $Q$ if and only if one of the coordinates of the vector $Q - P$ is zero, or two of them coincide. In all other cases, $P$ and $Q$ lie on a unique line.

Definition 3.6. An infinite collection of tropical lines in $\mathbb{R}^3$, is called a two-point family if there exist two points lying on all lines in the collection.

3.4 Group actions of $S_4$

The group of permutations of four elements, $S_4$, acts naturally on many of the spaces involved with tropical surfaces. Firstly, observe that $S_4$ is the symmetry group of the simplex $\Gamma_3 \subseteq \mathbb{R}^3$. This induces an action of the set of lattice points $\mathcal{A}_3 = \Gamma_3 \cap \mathbb{Z}^3$ (in fact on all of $\mathbb{Z}^3$), described explicitly as follows: Let $\sigma \in S_4$ be a permutation of four elements. For any $a = (a_1, a_2, a_3) \in \mathcal{A}_3$, let $a^\text{hom} := (a_1, a_2, a_3, \delta - a_1 - a_2 - a_3)$. We define $\sigma(a)$ to be the point in $\mathcal{A}$ whose coordinates are the first three coordinates of $\sigma(a^\text{hom})$. Obviously, this action of $S_4$ on $\mathcal{A}_3$ also induce an action of $S_4$ on the set of subdivisions of $\Gamma_3$.

Secondly, $S_4$ acts on the set of tropical surfaces of degree $\delta$. Let $X = V_{tr}(f)$, where $f(x) = \sum_{a \in \mathcal{A}_3} \lambda_a x^a$. For a given $\sigma \in S_4$, we define $\sigma(X)$ to be the surface $V_{tr}(\sigma(f))$, where $\sigma(f) = \sum_{a \in \mathcal{A}_3} \lambda_a x^{\sigma(a)}$. Clearly, $\sigma(X)$ is still of degree $\delta$, and the resulting action is compatible with the action of $S_4$ on the subdivisions of $\Gamma_3$. In other words, $\text{Subdiv}_{\sigma(X)} = \sigma(\text{Subdiv}_X)$.

4 Properties of tropical lines on tropical surfaces

4.1 Notation

The topic of this paper is to study tropical lines contained in tropical surfaces. It is important to note that 'containment' here is meant purely set-theoretically. For notational convenience, we fix the following:
The symbols $X$ and $L$ will always refer to the underlying point set in $\mathbb{R}^3$ of a tropical surface of degree $\delta$, and a tropical line, respectively. The associated polyhedral cell complexes are denoted by $\text{Complex}(X)$ and $\text{Complex}(L)$ respectively. Hence in particular, the statement $L \subseteq X$ means that $L$ is contained in $X$ as subsets of $\mathbb{R}^3$. There is a natural map

$$c_X : X \rightarrow \text{Complex}(X),$$

(4)

taking a point $p \in X$ to the minimal cell of $X$ containing $p$.

Furthermore, for the remaining part of the paper we fix the following notation:

- For $\delta \in \mathbb{N}$, $\Gamma_\delta$ is the simplex with vertices $(0,0,0), (\delta,0,0), (0,0,\delta), (0,\delta,0), (0,0,\delta)$, and $A_\delta := \Gamma_\delta \cap \mathbb{Z}^3$.
- The vectors $-e_1, -e_2, -e_3$ and $e_1 + e_2 + e_3$ are denoted $\omega_1, \ldots, \omega_4$. The coordinate variables in $\mathbb{R}^3$ are $x_1, x_2, x_3$, and we set $x_4 := \delta - x_1 - x_2 - x_3$.
- If $\ell$ is the equation of a plane in $\mathbb{R}^3$, then $P_{\ell}$ denotes this plane.
- For a fixed $\delta \in \mathbb{N}$, $F_i$ is the facet of $\Gamma_\delta$ with outer normal vector $\omega_i, i = 1, \ldots, 4$. (Note that $F_i$ is contained in the plane $P_{x_i=0}$.) Moreover, for distinct $i, j \in \{1, \ldots, 4\}$ we set $F_{ij} := F_i \cap F_j$.
- If $X$ is a tropical surface of degree $\delta$, then we set $\text{Subdiv}_X := \text{Subdiv}(f)$, where $f$ is any tropical polynomial with support $A_\delta$, such that $X = V_{tr}(f)$. (It is easy to see that $\text{Subdiv}(f)$ is the same for all such $f$, so $\text{Subdiv}_X$ is well defined.)
- Let $S$ be an RE-triangulation of $\Gamma_\delta$ for some $\delta \in \mathbb{N}$, and $K(S)$ is its secondary cone. If $\alpha$ is a point in $K(S)$, then $X_\alpha$ is the associated tropical surface. More precisely, $X_\alpha := V_{tr}(f)$, where

$$f(x) = \sum_{a \in A_\delta} \alpha(a)x^a.$$

4.2 Trespassing line segments on $X$

In [7] we introduced the notion of trespassing line segments on $X$. If $\ell \subseteq X$ is any ray or line segment, we say that $\ell$ is trespassing on $X$ if there exist distinct cells $C, C' \subseteq X$ such that

$$\dim(\text{int}(C) \cap \ell) = \dim(\text{int}(C') \cap \ell) = 1.$$  

(5)

Alternatively, $\ell$ is trespassing on $X$ if it is not contained in the closure of any single cell of $X$. For smooth $X$, trespassing can happen in one way only, as shown in [7] Lemma 6.2):

**Lemma 4.1.** Let $\ell \subseteq X$, where $\ell$ is a line segment and $X$ is smooth. If $C, C' \subseteq X$ are any cells satisfying [5] and such that $\ell \subseteq C \cup C'$, then $C$ and $C'$ are maximal cells of $X$ whose intersection is a vertex $V$ of $X$.

An immediate consequence of this is that $C^V$ and $(C')^V$ are opposite edges of the tetrahedron $V^V$. The following converse to Lemma 4.1 is straightforward:

**Lemma 4.2.** Let $\Lambda$ and $\Lambda'$ be opposite edges of a tetrahedron $\Delta \in \text{Subdiv}_X$. Then there is a trespassing line segment on $X$ passing through the vertex $\Delta^V \in X$, and which is orthogonal to both $\Lambda$ and $\Lambda'$.

We recall one more result from [7], again valid for smooth $X$ (cf. [7] Lemma 6.4)):

**Lemma 4.3.** Suppose $L \subseteq X$ is non-degenerate, and that the vertex $v$ of $L$ lies in the interior of a 1-cell $E$ of $X$. Then $L \cap E = \{v\}$, and the three edges of $L$ adjacent to $v$ start off in different 2-cells of $X$ adjacent to $E$.

The last statement of Lemma 4.3 can be reformulated "dually" as follows: Suppose $\omega_i$ and $\omega_j$ are the direction vectors of the unbounded edges of $L$ emanating from $v$. Then the edges of the triangle $E^\omega$ are orthogonal to $\omega_i$, $\omega_j$ and $\omega_i + \omega_j$ respectively.
4.3 The combinatorial position of \( L \subseteq X \)

Consider as before a tropical line \( L \) on a tropical surface \( X \). We next describe a way of displaying the essential information of how \( L \) lies on \( X \).

For any tropical line \( L \subseteq \mathbb{R}^3 \), the underlying graph of \( L \) is one of the two shown in Figure 6. A decoration of either of these consists of a finite number of dots (possibly none) on each edge, and at each vertex either a dot, a vertical line segment, or nothing. Note that we consider the graphs without metrics, so moving an edge-dot along its edge does not change the decoration. Also, two decorations \( \mathcal{C} \) and \( \mathcal{C}' \) (of the same graph) are said to be equal if there is an automorphism of the graph taking \( \mathcal{C} \) to \( \mathcal{C}' \). See Figure 7 for examples.

**Definition 4.4.** Let \( X \) be a smooth tropical surface, and let \( L \subseteq X \) be any tropical line. The combinatorial position of \( L \) on \( X \) is the following decoration of the underlying graph of \( L \):

- If an edge of \( L \) passes through \( k \) vertices of \( X \), the corresponding edge of the underlying graph has \( k \) dots.
- For each vertex \( v \) of \( L \), the corresponding vertex of the graph has a dot if \( \dim c_X(v) = 0 \), and a vertical line segment if \( \dim c_X(v) = 1 \).

4.4 Line subcomplexes of \( \text{Subdiv}_X \)

Recall that for any tropical surface \( X \), we have the natural map \( c_X : X \to \text{Complex}(X) \), taking a point \( p \in X \) to the minimal cell of \( X \) containing \( p \). Combining \( c_X \) with dualization, we get the map

\[
e^X : X \to \text{Subdiv}_X \quad p \mapsto c_X(p)^\vee.
\]

(6)

If \( Y \subseteq X \) is any subset, we set

\[
e^X(Y) := \bigcup_{y \in Y} e^X(y).
\]

Note that if \( Y \) is connected, then \( e^X(Y) \) is a connected subcomplex of \( \text{Subdiv}_X \).

**Definition 4.5.** Let \( S \) be a regular subdivision of \( \Gamma_\delta \). A subcomplex \( R \subseteq S \) is called a line subcomplex if there exists a tropical surface \( X \) and a tropical line \( L \subseteq X \) such that \( \text{Subdiv}_X = S \), and \( e^X(L) = R \).

Conversely, suppose \( R \subseteq S \) is a line subcomplex. Then if \( X' \) is any tropical surface with \( \text{Subdiv}_{X'} = S \), we say that \( R \) is realized on \( X' \) if there is a tropical line \( L \subseteq X' \) such that \( e^{X'}(L) = R \).

Because tropical lines in \( \mathbb{R}^3 \) are unbounded, any line subcomplex in \( \text{Subdiv}_X \) contain cells dual to unbounded cells of \( X \). Recall from Theorem 3.2c) that such cells of \( \text{Subdiv}_X \) are precisely those lying in the boundary of \( \Gamma_\delta \). Taking this a bit further, we define subpolytopes with exits in \( \Gamma_\delta \), introduced in [7]:

**Definition 4.6.** Let \( \Delta \) be a lattice polytope (of dimension 1, 2 or 3) contained in \( \Gamma_\delta \). We say that \( \Delta \) has an exit in the direction of \( \omega_i \) if at least one edge of \( \Delta \) lies in \( F_i \). If \( \Delta \) has exits in the directions of \( k \) of the \( \omega_i \)’s, we say that \( \Delta \) has \( k \) exits.

The relevance of this definition should be clear from the following observation: Let \( C \) be any cell of \( X \), and let \( p \in C \) be an arbitrary point. Then \( C \) contains the ray with starting point \( p \) and direction \( \omega_i \), if and only if \( C^\vee \) has an exit in direction \( \omega_i \).

When \( X \) is smooth, the cell structure of a line subcomplex \( e^X(L) \) is in many cases uniquely determined by the combinatorial position of \( L \) on \( X \). Moreover, using Lemma 4.1 and Lemma 4.3, we can often
describe explicitly the exits required of the edges of $c^X_L$. For example, the two rightmost combinatorial positions in Figure 7 imply the same cell structure of $c^X_L$, but with different exit properties (see Figure 8).

**Remark 4.7.** A line subcomplex will often have more exits than those required by the combinatorial position. Hence it is usually more difficult to reverse the process described in the last paragraph, i.e., to determine the combinatorial position of $L$ on $X$, given a line subcomplex $c^X_L \subseteq \text{Subdiv} X$.

We conclude this section by mentioning one case where the cell structure of $c^X_L$ is not determined by the combinatorial position of $L$ on $X$. Namely, when both vertices of $L$ are vertices of $X$, and the middle edge of $L$ is not trespassing. In this case, the middle edge of $L$ may or may not be an edge of $X$, giving different structures of $c^X_L$. (See Figure 9.) The two tetrahedra $P^\gamma$ and $Q^\gamma$ have a common facet if $PQ$ is an edge of $X$ (case i), but only an edge in common otherwise (i.e., if $PQ$ goes across a 2-cell of $X$). Note that if the middle edge were trespassing, there would be no ambiguity: By Lemma 4.1, no point of $PQ$ could then be in the interior of a 1-cell of $X$.

### 4.5 Deformations and specializations

Let $S$ be a given subdivision of $\Gamma_δ$, and let $K = K(S)$ be the corresponding secondary cone. We define the incidence $X_S \subseteq K \times \mathbb{R}^3$ by

$$X_S := \{(\alpha, x) \mid x \in X_\alpha\} \subseteq K \times \mathbb{R}^3.$$ 

Using the Euclidean metric on both $K$ and $\mathbb{R}^3$, we give $X_S$ the topology induced by the product topology on $K \times \mathbb{R}^3$. This makes the projections on $K$ and $\mathbb{R}^3$, denoted by $p_1$ and $p_2$ respectively, continuous. Note that for any $\alpha \in K$, $p_2(p_1^{-1}(\alpha))$ is the tropical surface $X_\alpha$.

**Definition 4.8.** A family of tropical lines associated to $S$ is a subset $L \subseteq X_S$ satisfying the following conditions:

- For any $\alpha \in K$, $p_2(p_1^{-1}(\alpha) \cap L)$ is a tropical line $L_\alpha \subseteq X_\alpha$.
- The projections from $L$ to $K$ and $\mathbb{R}^3$ are continuous.

**Definition 4.9.** A deformation of $L \subseteq X_\alpha$ is a family $L$ of tropical lines associated to $S$, such that

- $p_1(L)$ contains $\alpha$, and is homeomorphic to an interval,
- for any two points $\beta \neq \gamma$ in $p_1(L)$, we have $X_\beta \neq X_\gamma$.

Figure 9: A combinatorial position giving two possible line subcomplexes, depending on whether $PQ$ is a 1-cell of $X$ (case i) of not (case i).
Note that a deformation of $L \subseteq X$ can be thought of as a map $t \mapsto (L_t, X_t)$, where $t$ runs through some interval $I \subseteq \mathbb{R}$ containing 0, and where $(L, X) = (L_0, X_0)$. In particular, 0 can be an endpoint of $I$, as in $I = [0, 1)$.

**Definition 4.10.** Let $L$ be a tropical line with combinatorial position $C$ on $X$. We say that $L$ **deforms into combinatorial position** $C'$, if there exist a deformation $t \mapsto (L_t, X_t)$ of $L \subseteq X$ such that for all $t \in I \setminus \{0\}$, the combinatorial position of $L_t$ on $X_t$ is $C'$.

The following lemma gives a simple property of deformations, namely that one cannot deform a tropical line away from a vertex through which it is trespassing.

**Lemma 4.11.** Suppose $L \subseteq X$ has a trespassing edge $\ell$, passing through the vertex $\Delta^\vee \subseteq X$, where $\Delta$ is a tetrahedron in $\text{Subdiv}_X$. For any deformation $t \mapsto (L_t, X_t)$, $t \in I$ of $L \subseteq X$, let $\ell_t$ be the edge of $L_t$ parallel to $\ell$. Then for $t$ small enough, $\ell_t$ is trespassing through $\Delta^\vee \subseteq X_t$.

**Proof.** Since $\ell$ is trespassing through $\Delta^\vee \subseteq X$, Lemma 4.1 gives that $\dim(\ell \cap \text{int}(\Delta^\vee)) = \dim(\ell \cap \text{int}(\Delta^\vee)) = 1$, for some pair of opposite edges $\Lambda, \Lambda'$ of $\Delta$. By continuity of the deformation, this implies that $\dim(\ell_t \cap \text{int}(\Delta^\vee)) = \dim(\ell_t \cap \text{int}(\Delta^\vee)) = 1$ for small enough $t$. Hence $\ell_t$ is trespassing through $\Delta^\vee$. \[\square\]

**Remark 4.12.** Note that the proof of Lemma 4.11 rests on Lemma 4.1 which requires $X$ to be smooth. In fact, it is not hard to produce examples of non-smooth $X$ where one cannot deform away from trespassing vertices.

Related to deformations is the concept of specialization:

**Definition 4.13.** Let $(L_t, X_t)$ be a deformation of $L_0 \subseteq X_0$, where $t \in [0, 1]$. We say that $L_0 \subseteq X_0$ **specializes to** $L_1 \subseteq X_1$ if the combinatorial position of $L_t \subseteq X_t$ is constant for all $t \in [0, 1)$ but differs for $t = 1$.

## 5 Classification of tropical lines on general smooth tropical surfaces

Let $\delta \in \mathbb{N}$ be fixed, and let $S$ be an RE-triangulation of $\Gamma_\delta$. By Proposition 2.1, the secondary cone $K(S)$ has dimension $N$ in $\mathbb{R}^N$, where $N = |A_\delta| = (\delta + 3)^3$. Recall that each $\alpha \in K(S)$ corresponds to a smooth tropical surface $X_\alpha$ with subdivision $S$.

**Definition 5.1.** We say that a property $\Pi$ holds for **general tropical surfaces with subdivision** $S$ if $\Pi$ holds for $X_\alpha$ for every $\alpha$ in some open, dense subset of $K(S)$.

More generally, $\Pi$ holds for **general smooth tropical surfaces of degree** $\delta$ if $\Pi$ holds for general tropical surface with subdivision $S$, for all RE-triangulations $S$ of $\Gamma_\delta$.

Finally, $\Pi$ holds for **general smooth tropical surfaces** if $\Pi$ holds for general smooth tropical surfaces of degree $\delta$, for all $\delta \in \mathbb{N}$.

The next lemma gives an important example of a property held by general smooth tropical surfaces (in all degrees).

**Lemma 5.2.** A general smooth $X$ contains no doubly trespassing line segments.

**Proof.** Let $X = V_t(f)$ be a tropical surface, given by a tropical polynomial $f = \sum \lambda_\alpha x^\alpha$, and suppose $\ell \subseteq X$ is a line segment containing two vertices of $X$, say $P$ and $Q$, in its relative interior. We will show that this condition implies that there is a linear relation between the coefficients $\lambda_\alpha$.

The situation is shown in Figure 10. From Lemma 4.11 it follows that $\text{Subdiv}_X$ contains three 1-cells $AB, CD, EF$ such that $\ell \subseteq (AB)^\vee \cup (CD)^\vee \cup (EF)^\vee$, and such that $P^\vee = ABCD$ and $Q^\vee = CDEF$. Obviously, a necessary condition for this to happen is that $\ell$ is parallel to both vectors products $\vec{AB} \times \vec{CD}$ and $\vec{CD} \times \vec{EF}$. 

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Proof. Let \( \ell \) be an arbitrary smooth combinatorial type of degree \( \delta \), corresponding to some elementary subdivision \( S \) of \( \Gamma_\delta \). In \( S \), we look for all pairs of tetrahedra with a common edge, and such that (with the notation of Figure 10) \( AB \times CD \parallel CD \times EF \). Clearly there are at most finitely many such pairs. As seen above, each such pair gives rise to a hyperplane section of \( \Delta \), and any surface containing a doubly trespassing line, corresponds to a point on one of these hyperplanes. This proves the lemma.

The above lemma greatly limits the number of ways in which a tropical line can lie on a general smooth tropical surface. In particular, the lemma says that for general \( X \), each of the five edges of \( L \subseteq X \) contains at most one vertex of \( X \) in its relative interior. An immediate implication of this is the following interesting result: There exists a finite list of combinatorial positions, such for a general smooth \( X \) (of any degree), the combinatorial position of any tropical line on \( X \) is in the list.

However, there are some combinatorial positions of \( L \) that do not occur on general \( X \), but which are not excluded by Lemma 5.2. Many of these can be identified using the lemma to follow.

By a 3-star on \( X \) we will mean the union of 3 line segments, pairwise non-parallel, with a common endpoint, all of which is pointwise contained in \( X \). If \( Y \) is a 3-star on \( X \) with center vertex \( v \), we say that \( Y \) is special if the number of trespassing edges of \( Y \) is exactly \( \dim c_X(v) + 1 \). Combinatorially, this gives three possibilities, shown in Figure 11.

**Lemma 5.3.** A general smooth \( X \) contains no special 3-stars.

**Proof.** Let \( X = V_{\text{tr}}(f) \) be smooth of some arbitrary degree \( \delta \), where \( f = \sum \lambda_a x^a \). For any 3-star \( Y \subseteq X \), we can consider the 3-star subcomplex \( c_X^Y(Y) \) in \( \text{Subdiv } X \). (Cf. the definition of line subcomplexes

![Figure 10: A doubly trespassing line, and the dual configuration.](image)

![Figure 11: Special 3-stars on X, where dim c_X(v) equals i) 2, ii) 1, and iii) 0.](image)
in section 4.4.) For the three special 3-stars in Figure 11, the structures of the corresponding 3-star subcomplexes are depicted in Figure 12. We claim that given any special 3-star subcomplex $R \subseteq \text{Subdiv}_X$, it is realized as a 3-star on $X$ only if the coefficients $\lambda_a$ satisfy a linear condition, dependent of $R$. To show this, the idea is to find in each case the equations $v$ must satisfy and arrange them in a matrix form, similar to (8). For example, in case i), we see that $v$ lies on each of the planes spanned by $(AB)^\vee$, $(CD)^\vee$, $(EF)^\vee$, and $(GH)^\vee$. Writing out the corresponding equations, we obtain

$$
\begin{bmatrix}
\vec{AB} \lambda_B - \lambda_A \\
\vec{CD} \lambda_D - \lambda_C \\
\vec{EF} \lambda_F - \lambda_E \\
\vec{GH} \lambda_H - \lambda_G \\
\end{bmatrix} \begin{bmatrix} v_1 \\
1 \\
\end{bmatrix} = 0.
$$

Observe that the leftmost matrix in (9) is a 4 x 4-matrix; let us call it $M$. Since the null-space of $M$ is non-trivial (it contains $(v, 1)^T$), we must have $\det M = 0$, giving a linear relation in the $\lambda$’s. Note that this would reduce to an empty condition (i.e. 0 = 0) if rank $M \leq 2$, but it is easy to see that in our case $AB, CD, EF, GH$ span all of $R^3$, so rank $M = 3$. This proves the claim in case i).

The cases ii) and iii) are done in the same way, but with the matrix $M$ exchanged with

$$
\begin{bmatrix}
\vec{AB} \lambda_B - \lambda_A \\
\vec{AC} \lambda_C - \lambda_A \\
\vec{DE} \lambda_E - \lambda_D \\
\vec{FG} \lambda_G - \lambda_F \\
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
\vec{AB} \lambda_B - \lambda_A \\
\vec{AC} \lambda_C - \lambda_A \\
\vec{AD} \lambda_D - \lambda_A \\
\vec{EF} \lambda_F - \lambda_E \\
\end{bmatrix}.
$$

It is now straightforward to show that a general smooth $X$ contains no special 3-stars. Indeed, let $S$ be any elementary subdivision of $\Gamma_3$, and $\alpha$ a point in the parameter cone $K(S)$. Then as we have seen, any 3-star subcomplex in $S$ like those in Figure 12 can be realized on $X_\alpha$ only if $\alpha$ lies on a certain hyperplane. Moreover, $S$ contains at most finitely many of the 3-star subcomplexes in Figure 12. Hence for any $\alpha$ in the complement of a finite union of hyperplanes, $X_\alpha$ contains no special 3-stars.

**Corollary 5.4.** On a general smooth tropical surface $X$, any vertex $v$ of a tropical line $L \subseteq X$ satisfies

$$\sharp(\text{trespassing edges of } L \text{ adjacent to } v) \leq \dim_{c_X}(v).$$

**Proof.** It is easy to see that if $v$ is any vertex of a tropical line $L \subseteq X$, for which (10) is not satisfied, then $v$ is the center vertex of a special 3-star. The result therefore follows from Lemma 5.3.

By the help of Corollary 5.3, it is straightforward to construct a list containing all possible combinatorial positions of a non-degenerate tropical line on a general smooth tropical surface. The result is shown in Table 1.

Note that we do not claim that all the entries of Table 1 actually occur on general smooth surfaces. For starters, the following combinatorial positions are clearly impossible on any tropical surface:

In each case, the middle segment of $L$ is contained in a 2-cell of $X$. But the part of $L$ contained in this cell spans $R^3$, which is a contradiction.

Furthermore, we have the following lemma:
Table 1: Combinatorial positions of tropical lines on a general smooth $X$. Gray=nonexistent or non-general; dash=non-compatible 3-stars.
Lemma 5.5.  

a) A general smooth $X$ has no tropical lines with the combinatorial positions shown in Figure 13.

b) A general smooth $X$ has no tropical line such that i) its combinatorial position is the one in Figure 14 and ii) its middle segment goes across a 2-cell of $X$.

Proof.  
a) The idea is basically the same in the proofs of Lemma 5.2 and Lemma 5.3. Each case implies some linear relation between the coefficients $\lambda_a$ of the polynomial defining $X$. We sketch a typical argument: Suppose $L$ has the leftmost combinatorial position, and observe that the line subcomplex of $L$ is then homeomorphic to case i) of Figure 12. Let $v_1$ be the vertex of $L$ which is also a vertex of $X$; assume this is dual to the tetrahedron $ABGH$. Then, by the definition of duality, $v_1$ is uniquely determined by $\lambda_A, \lambda_B, \lambda_G$ and $\lambda_H$. (In fact, the coordinates of $v_1$ are linear forms in these $\lambda$’s.) Similarly, the other vertex of $L$, $v_2$, is determined by $\lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_E$ and $\lambda_F$ (this corresponds to solving the equation (9), but with the last row removed.) Finally, since $v_1v_2$ is the middle segment of a tropical line, it has a prescribed direction. This forces a linear relation between the coefficients. The remaining three cases of the above claim are done similarly.

b) This combinatorial position was discussed in the last paragraph of section 4.4. If the middle segment of $L$ goes across a 2-cell of $X$, the line subcomplex is homeomorphic to case ii) of Figure 9. In this case the argument sketched in a) applies again: Each vertex of $L$ is determined by the $\lambda$’s, and the direction vector of the middle segment implies a linear relation between these.

Note that this argument does not apply if the middle segment of $L$ is a 1-cell of $X$: In this case the direction of the middle segment is encoded in the line subcomplex as a normal vector of the common facet of the two tetrahedra (cf. case i) of Figure 9).

5.1 The classification theorem

In the last section we identified 10 entries of Table 1 that were either impossible on any tropical surface $X$, or non-general, meaning that they do not occur on general smooth $X$. In this section we analyze the remaining 17 combinatorial positions. The main result is the following classification:

Theorem 5.6. For a general smooth tropical surface $X$, the combinatorial position of any non-degenerate tropical line on $X$ is one of the 17 listed in Table 2. Moreover, we have:

a) 14 of the combinatorial positions occur only of surfaces of a particular degree:

- 1A and 1B occur only on surfaces of degree 1.
- 2A, 2B, 2C, 2D, 2E and 2F occur only on surfaces of degree 2.
- 3A, 3B, 3C, 3D, 3E and 3F occur only on surfaces of degree 3.

b) 3 of the combinatorial positions occur on surfaces of arbitrary degree. More precisely:

- 3G and 3H occur on surfaces of any degree $\delta \geq 2$.
- 3I occur on surfaces of any degree $\delta \geq 1$.

The 17 combinatorial positions mentioned in Theorem 5.6 are called general combinatorial positions.

Proof. The proof of Theorem 5.6 will occupy most of this section. To avoid repeating ourselves too much, we start by giving some auxiliary observations about tropical half lines on $X$, which will apply frequently. A tropical half line in $\mathbb{R}^3$ is the remaining part of a non-degenerate tropical line, after removing two adjacent rays. Figure 15 shows tropical half lines on $X$ in different positions.
| Only deg $X = 1$ | 1A | 1B |
|------------------|----|----|
| 2A               | ![Diagram](image1) | ![Diagram](image2) |
| Only deg $X = 2$ | 2A | 2B | 2C |
| 2D               | ![Diagram](image3) | ![Diagram](image4) | ![Diagram](image5) |
| Only deg $X = 3$ | 3A | 3B | 3C |
| 3D               | ![Diagram](image6) | ![Diagram](image7) | ![Diagram](image8) |
| Any deg $X \geq 2$ | 3G | 3H |
| Any deg $X \geq 1$ | 3I |

Table 2: The 17 combinatorial positions of non-degenerate lines on general smooth surfaces.

For a tropical half line $H$, let $H^b$ be its bounded segment. Note that if $H^b \subseteq X$ is non-trespassing, then there is a unique cell of $X$, denoted $C^b$, containing $H^b$. The following lemma gives information on the position of the dual cell $(C^b)^\vee \subseteq \text{Subdiv}_X$. As always, $X$ is assumed to be smooth of some fixed degree $\delta$.

**Lemma 5.7.** Let $H \subseteq X$ be a tropical half line with unbounded rays in the directions $\omega_i$ and $\omega_j$, such that $H^b$ is non-trespassing and contained in a cell $C^b$ of $X$. Then $(C^b)^\vee$ is contained in a plane with equation $x_i + x_j = K$ for some non-negative integer $K$.

**Proof.** Recall that any vector contained in $C^b$ is orthogonal to $(C^b)^\vee$. In the case where $i, j \neq 4$, this immediately proves the assertion, since by assumption $C^b$ contains the vector $-\omega_i - \omega_j = e_i + e_j$. For the remaining case, suppose $j = 4$, and let $(i, i', i'')$ be any permutation of $(1, 2, 3)$. Then $C^b$ contains the vector $\omega_i + \omega_4 = e_i + e_4$, so $(C^b)^\vee$ lies in a plane with equation $x_i + x_{i''} = \text{constant}$. This is equivalent to the statement in the lemma, since $x_i + x_4 = K \iff x_i + x_{i''} = \delta - K$.

**Lemma 5.8.** Let $H$ be as in Lemma 5.7, and suppose in addition that $\dim c_X(v) \geq k$, where $v$ is the vertex of $H$, and $k$ is the number of unbounded rays of $H$ which are trespassing on $X$. If either

- $\dim c_X(v) > 0$,
- $\dim c_X(v) = 0$ and $\dim C^b = 1$,

then $(C^b)^\vee$ lies in the plane with equation $x_i + x_j = \dim C^b - \dim c_X(v) + k$. 

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Remark 5.9. Lemma 5.8 implies in particular that in the cases mentioned, the integer $k$ in Lemma 5.7 is dependent only of $\dim C^b$, $\dim c_X(v)$ and $k$. As we will see, this does no longer hold if $\dim c_X(v) = 0$ and $\dim C^b = 2$ (the only case not covered by the lemma), and this fact is what allows the positions $3G$ and $3H$ to occur on surfaces of arbitrarily high degree.

Proof. By symmetry we can assume that $i = 1$ and $j = 2$, so by Lemma 5.7 $(C^b)^\vee$ lies in a plane given by $x_1 + x_2 = K$ for some $K$. Suppose first $\dim c_X(v) > 0$. It is easy to see that it suffices to consider the five cases shown in Figure 15. Note that in all these cases, $\dim C^b = 2$.

In case i), $\dim c_X(v) = 1$ and $k = 0$, so we must show that $K = 2 + 0 - 1 = 1$. We see that the triangle $E^\vee$ has one edge on $F_1$, another edge on $F_2$, while its last edge is $(C^b)^\vee$. Hence the vertices of $E^\vee$ are of the form $(0, 0, a)$, $(0, K, b)$ and $(K, 0, c)$, where $a, b, c \in \mathbb{N}_0$. Let $P = (p, q, r)$ be the fourth vertex of a tetrahedron in Subdiv$_X$ having $E^\vee$ as a facet. A standard calculation shows that the volume of this tetrahedron is divisible by $K$. Hence unimodularity of Subdiv$_X$ implies $K = 1$, as wanted.

In ii), we must show that $K = 2$. Here, $E^\vee$ has one edge in $F_1$, another in the plane $x_2 = 1$, and the third is $(C^b)^\vee$. Thus the vertices of $E^\vee$ are $(0, 1, a), (0, K, b), (K - 1, 1, c)$ for some $a, b, c \in \mathbb{N}_0$. A volume calculation shows that any integral tetrahedron having $E^\vee$ as facet, has a volume divisible by $K - 1$. Thus $K = 2$.

In iii) we must show $K = 0$. It is clear that $(C^b)^\vee$ lies in both facets $F_1$ and $F_2$, and therefore in the edge $F_{12}$ of $\Gamma_\delta$. Since $F_{12}$ is contained in the plane $x_1 + x_2 = 0$, we are done.

In iv) we find similarly that $(C^b)^\vee$ lies in the intersection of $F_1$ (where $x_1 = 0$) and the plane given by $x_2 = 1$. In particular, $(C^b)^\vee$ lies in the plane where $x_1 + x_2 = 1$, as claimed in the lemma.

Finally, in case v) $(C^b)^\vee$ lies in the intersection of the planes with equations $x_1 = 1$ and $x_2 = 1$. In particular, this means $x_1 + x_2 = 2$, which is again what we needed to prove.

It remains to treat the case where $\dim c_X(v) = 0$ and $\dim C^b = 1$. In other words, $v$ is a vertex of $X$ and $H^b$ is contained in an edge $C^b$ of $X$. Dually, $(C^b)^\vee$ is a triangle $ABC \in$ Subdiv$_X$, and $v^\vee$ is a tetrahedron having $(C^b)^\vee$ as a facet, i.e. $v^\vee = ABCD$ for some integral point $D$. By Lemma 5.7 $ABC$ lies in a plane with equation $x_1 + x_2 = K$, and by obvious volume considerations, $D$ must then lie in a plane given by $x_1 + x_2 = K \pm 1$. We have to prove that $K = 1$.

Observe that $ABCD$ has exits in both directions $\omega_1$ and $\omega_2$, since $H$ has no trespassing rays (recall the assumption $k \leq \dim c_X(v)$). Now, if the triangle $ABC$ has exits in neither of the two directions, then we must have $D \in F_{12}$, implying $K = 1$. If $ABC$ has an exit in exactly one of the two directions, say $\omega_1$, then we must have (possibly after renaming) that $AB \subseteq F_1$ and $CD \subseteq F_2$. Writing out what this means for the coordinates of $A$, $B$, $C$ and $D$, a volume calculation of $ABCD$ shows that unimodularity again implies $K = 1$. Note that $ABC$ (being a non-degenerate triangle contained in the plane of the form $x_1 + x_2 = K$) cannot have exits in both directions $\omega_1$ and $\omega_2$. Hence we have covered all cases.

This concludes the proof of the lemma.

After this preparatory work, we turn to the proof of part a) of Theorem 5.6.

Proof of Theorem 5.6(a). We examine the combinatorial positions individually:

- Position 1A: Suppose $L \subseteq X$, where $X$ has degree $\delta$, and $L$ has combinatorial position 3A on $X$. We can assume w.l.o.g. that $L$ has combinatorial type $((12)(34))$, so the situation is as follows:

```
  2
 / \
1   4
  \
  3
```

Regard $L$ as the union of two tropical half lines on $X$, sharing the same bounded segment. Let $C^b$ be the 2-cell of $X$ containing this bounded segment. Applying Lemma 5.8 to the half line on the left (i.e. the
Figure 16: Line subcomplexes in $\Gamma_4$ associated to tropical lines in position $3G$, $3H$ and $3I$, respectively.

one with rays 1 and 2) it follows that $(C^h)^\vee$ lies in the plane with equation $x_1 + x_2 = 1$. On the other hand, the same lemma applied to the other half line implies that $(C^h)^\vee$ lies in the plane with equation $x_3 + x_4 = 0$, i.e. $x_1 + x_2 = \delta$. We conclude that $\delta = 1$.

- An analogue argument works in all cases mentioned in part a) where the middle segment of $L$ is not trespassing on $X$, i.e. positions $2A$, $2B$, $2D$, $2F$, $3A$, $3D$ and $3E$. Note in particular that in case $2F$, we have $\dim C^b = 1$ by Lemma 5.5b), so Lemma 5.8 applies.

- Position 1B: Again we assume that $L$ has combinatorial type $((12)(34))$: $V_2 4 1 3$

This time we regard $L$ as the union of two tropical half lines on $X$, intersecting in the point $V$ only. Let $C^1_b$ be the 2-cell containing the bounded segment of the half line with rays 1 and 2, and similarly $C^2_b$ the 2-cell containing the bounded segment of the other half line. Now we apply Lemma 5.8 twice, to find that $(C^1_b)^\vee$ and $(C^2_b)^\vee$ lie in the planes with equations $x_1 + x_2 = 0$, and $x_1 + x_2 = \delta$ respectively. But $(C^1_b)^\vee$ and $(C^2_b)^\vee$ are opposite edges of the unimodular tetrahedron $V^\vee$. This is only possibly if $\delta = 1$.

- An analogue argument works in all cases mentioned in part a) where the middle segment is trespassing, i.e. the positions $2C$, $2E$, $3B$, $3C$ and $3F$.

In Table 3 we have summarized the cell structures of the line subcomplexes associated to the combinatorial positions $3A, \ldots, 3I$, including the additional information provided by Lemma 5.8.

We make the following definition:

**Definition 5.10.** A lattice complex contained in $\Gamma_\delta$ is said to be of type $3A$ (resp. $3B, \ldots$) if it has the same cell structure as the line subcomplex given in Table 3 for position $3A$ (resp. $3B, \ldots$), and it meets the associated conditions given in Table 3.

It should be noted that the conditions given in Table 3 are only necessary conditions: If $\mathcal{S}$ is an RE-triangulation of $\Gamma_\delta$, then a subcomplex of $\mathcal{S}$ of a given type is not necessarily a line subcomplex in $\mathcal{S}$.

We apply this terminology in the proof of the remaining part of Theorem 5.6:

**Proof of Theorem 5.6b.** As usual, let $\delta$ be the degree of $X$.

- Position 3G: Assume $\delta \geq 2$ (this is clearly necessary for $L \subseteq X$ to have combinatorial position 3G), and consider the complex $\mathcal{R}_G$ with maximal elements $\Delta_1 = ABCD$ and $\Delta_2 = CDEF$, where $A = (0, 0, 1), \quad B = (0, 1, 1), \quad C = (\delta - 1, 0, 0), \quad D = (\delta - 2, 1, 0), \quad E = (\delta - 1, 1, 0), \quad F = (\delta - 1, 0, 1)$.

The leftmost picture of Figure 16 shows $\mathcal{R}_G$ for $\delta = 4$; it is clear that $\mathcal{R}_G$ is of type 3G. We claim that if $\text{Subdiv}_X$ contains $\mathcal{R}_G$, then $X$ contains a line with combinatorial position 3G. Indeed, this can be checked directly by examining the shape of the 2-cell of $X$ dual to the edge $CD$. Furthermore, by applying the techniques described in [7, Section 4], one can construct RE-triangulations of $\Gamma_\delta$ containing $\mathcal{R}_G$, for all $\delta \geq 2$. This proves the assertion in Theorem 5.6b) concerning position 3G.
| Combinatorial position of $L \subseteq X$ | Structure of line subcomplex $c_X^\nu(L) \subseteq \text{Subdiv}_X$ | Necessary conditions |
|----------------------------------------|-------------------------------------------------|---------------------|
| 3A                                    | ![Diagram]                                     | Exits: $AB \subseteq F_i$, $BD \subseteq F_j$, $AC \subseteq F_k$, $EF \subseteq F_l$. $AD \subseteq P_{x_i+x_j=1}$, $CD \subseteq P_{x_i+x_j=1}$. |
| 3B                                    | ![Diagram]                                     | Exits: $AB \subseteq F_i$, $AC \subseteq F_j$, $DF \subseteq F_k$, $EF \subseteq F_l$. $BC \subseteq P_{x_i+x_j=1}$, $ED \subseteq P_{x_i+x_j=2}$. |
| 3C                                    | ![Diagram]                                     | Exits: $AB \subseteq F_i$, $AC \subseteq F_j$, $DE \subseteq F_k$, $FG \subseteq F_l$. $AD \subseteq P_{x_i+x_j=1}$, $DE \subseteq P_{x_i+x_j=2} \cap F_k$. |
| 3D                                    | ![Diagram]                                     | Exits: $CE \subseteq F_i$, $AB \subseteq F_j$, $DE \subseteq F_k$, $FG \subseteq F_l$. $CD \subseteq P_{x_i+x_j=1}$, $DE \subseteq P_{x_i+x_j=2} \cap F_k$. |
| 3E                                    | ![Diagram]                                     | Exits: $AB \subseteq F_i$, $AC \subseteq F_j$, $DE \subseteq F_k$, $FG \subseteq F_l$. $BC \subseteq P_{x_i+x_j=1} \cap P_{x_i}$. |
| 3F                                    | ![Diagram]                                     | Exits: $CD \subseteq F_i$, $AB \subseteq F_j$, $EF \subseteq F_k$, $GH \subseteq F_l$. $CD \subseteq P_{x_i+x_j=1} \cap F_i$, $EF \subseteq P_{x_i+x_j=2} \cap F_k$. |
| 3G                                    | ![Diagram]                                     | Exits: $CD \subseteq F_k$, $EF \subseteq F_i$, $ABCD$ has edges also in $F_i$ and $F_j$. $CD \subseteq P_{x_i=1} \cap F_k$. |
| 3H                                    | ![Diagram]                                     | Exits: $CE \subseteq F_k$, $DE \subseteq F_i$, $ABCD$ has edges also in $F_i$ and $F_j$. $CD \subseteq P_{x_i+x_j=1}$. |
| 3I                                    | ![Diagram]                                     | Exits: $CD \subseteq F_k \cap F_i$, $ABCD$ has edges also in $F_i$ and $F_j$. (In particular, $ABCD$ has four exits.) |

Table 3: The cell structures of all line subcomplexes in $\text{Subdiv}_X$ for general smooth $X$ of degree $\delta \geq 3$. For positions $3A - 3F$ we have $\delta = 3$. The bold lines indicate edges with required exits.
• Position 3H: Let $\mathcal{R}_H$ be the complex with maximal elements the tetrahedron $ABCD$ and the triangle $CDE$, where

$$A = (0, 0, 1), \quad B = (0, 1, 2), \quad C = (\delta - 1, 0, 0), \quad D = (\delta - 2, 1, 1), \quad E = (\delta - 1, 1, 0).$$

(See Figure 16 middle picture.) Then $\mathcal{R}_H$ is of type $3H$. Suppose $S$ is an RE-triangulation of $\Gamma_\delta$ containing $\mathcal{R}_H$. By examining the shape of the 2-cell $(CD)\vee$, one can see that $\mathcal{R}_H$ is a line subcomplex in $S$, realizable on $X_\alpha$ for all $\alpha$ in some open, full-dimensional cone in $K(S)$. Finally, as above, one can construct RE-triangulations of $\Gamma_\delta$ containing $\mathcal{R}_H$, for all $\delta \geq 2$.

• Position 3I: Consider the tetrahedron $\Delta$ with vertices $(0, 0, 0), (0, 0, 1), (1, 0, \delta - 1)$ and $(\delta - 1, 1, 0)$, shown to the left in Figure 16 for $\delta = 4$. Clearly, $\Delta$ is a complex of type $3I$. In the proof of [7] Theorem 9.2 we showed that if $\Delta \in \text{Subdiv}_X$, then $X$ has a tropical line with combinatorial position $3I$. Moreover, we showed that for all $\delta \in \mathbb{N}$ there exists an RE-triangulation of $\Gamma_\delta$ which contains $\Delta$.

This concludes the proof of Theorem 5.6.

We conclude this section with a result important for the next section.

**Proposition 5.11.** Let $X$ be a smooth tropical surface of degree at least 3.

a) If $X$ contains a tropical line with combinatorial position $3I$, then $X$ contains a two-point family of tropical lines.

b) Suppose $L \subseteq X$ has general combinatorial position other than $3I$. If $L' \subseteq X$ is any tropical line, we have

$$c_X^L(L') = c_X^L(L) \implies L' = L.$$

Alternatively, b) can be formulated as follows: Let $\mathcal{R}$ be a subcomplex of $\text{Subdiv}_X$ of type either $3A, 3B, 3C, 3D, 3E, 3F, 3G$ or $3H$. Then there is either none or exactly one tropical line on $X$ with line subcomplex $\mathcal{R}$.

**Proof.** a) This follows from the convexity of the cells of $X$: Suppose $L \subseteq X$ has combinatorial position $3I$, with vertices $v_1$ and $v_2$, where $C := c_X(v_2)$ has dimension 2. If $\vec{u}$ is the vector from $v_1$ to $v_2$, then convexity of $C$ implies that the tropical line $L_t$ with vertices $v_1$ and $v_1 := v_1 + t\vec{u}$ lies on $X$ for all $t \geq 0$. Let $V$ be the common vertex of $L$ and $X$, and let $C$ be the cell of $X$ holding the other vertex of $L$.

b) This is a consequence of [7] Proposition 8.3], which states that if $\deg X \geq 3$, then any $L \subseteq X$ not belonging to a two-point family on $X$ is uniquely determined by its set of data, $D_X(L)$, introduced in [7]. The main difference between $c_X^L(L)$ and $D_X(L)$ is that the latter includes the combinatorial type of $L$. However, one can check that if $L \subseteq X$ has any general combinatorial position (and $\deg X \geq 3$), then its combinatorial type - and thus $D_X(L)$ - is uniquely determined by $c_X^L(L)$ and $X$.

To prove the lemma, we argue as follows: Let $L$ be as in the statement, and suppose $c_X^L(L') = c_X^L(L)$ for some $L' \subseteq X$. This implies that $D_X(L) = D_X(L')$, and thus, by [7] Proposition 8.3, that either $L = L'$, or that $L$ belong to a two-point family on $X$. It is straightforward to check that the latter possibility cannot happen (see the proof [7] Proposition 8.3 for details).

### 6 Counting tropical lines on general smooth tropical surfaces

The classification in the last section can be used to count tropical lines on smooth tropical surfaces. More precisely, let $X$ be any smooth tropical surface of degree $\geq 3$. First, check whether $\text{Subdiv}_X$ contains a subcomplex of type $3I$, i.e., a tetrahedron with four exits. If it does, then by Proposition 5.11 a) $X$ contains infinitely many tropical lines.

Suppose $\text{Subdiv}_X$ contains no subcomplexes of type $3I$. Then Proposition 5.11 b) implies that for any general combinatorial position $\mathcal{P}$, there is an injection of sets

$$\{\text{tropical lines on } X \text{ with combinatorial position } \mathcal{P}\} \xrightarrow{c_X^L} \{\text{subcomplexes of } \text{Subdiv}_X \text{ of type } \mathcal{P}\}.$$  

Since on a general smooth $X$, every tropical line has general combinatorial position (Theorem 5.6), we have:
Figure 17: A quartic subdivision (left) whose general tropical surface (right) has no tropical lines.

**Proposition 6.1.** Let $S$ be an RE-triangulation of $\Gamma_\delta$ without subcomplexes of type $3I$, where $\delta \geq 3$. If $X$ is a general smooth tropical surface with subdivision $S$, then

$$\sharp\{\text{tropical lines on } X\} \leq \sharp\{\text{subcomplexes of } S \text{ of general type}\}.$$ 

**Remark 6.2.** Proposition 6.1 gives an computationally accessible upper bound for the number of tropical lines on a general tropical surface with given subdivision. Namely, if $S$ is a subdivision of $\Gamma_\delta$, its subcomplexes of general type can be found in the following easily programmable way: For each type, identify all subcomplexes in $S$ with the cell structure associated to that type, as given in Table 3. Thereafter, check which of these satisfy the associated conditions (in the rightmost column of Table 3).

The upper bound given in Proposition 6.1 is not sharp in general. However, in concrete examples, it is often fairly easy to improve the inequality, or even find the exact number of tropical lines. We give a detailed example of this in section 7, where we analyze the subdivision $S_{\alpha,3}$. However, we first look at tropical surfaces without any tropical lines.

### 6.1 Tropical surfaces with no tropical lines

In classical geometry, it is well known that a general smooth algebraic surface of degree higher than 3 in projective three-space contains no lines. (See [6, p. 28] for an early reference.)

As shown in [7], this statement fails to hold for tropical surfaces. To restate the result precisely, recall our notion of generality for smooth tropical surfaces of degree $\delta$: For a general such surface to have a certain property, we require that for each RE-triangulation $S$ of $\Gamma_\delta$ there is an open dense subset $U \subseteq K(S)$ such that for all $\alpha \in U$, $X_\alpha$ has the property. In [7, Theorem 9.2] we showed that

**Theorem 6.3.** For any $\delta \in \mathbb{N}$ there exists an RE-triangulation $S$ of $\Gamma_\delta$ such that $X_\alpha$ contains infinitely many tropical lines for all $\alpha \in K(S)$.

We will now prove a theorem to the converse effect: That there exist RE-triangulations of $\Gamma_\delta$ for arbitrary $\delta \geq 4$, for which a general surface contains no tropical lines.

Recall the RE-triangulation $S_{\alpha,\delta}$ of $\Gamma_\delta$, defined in section 2.3.4, as the subdivision induced by the lifting function $\alpha(a,b,c) = -2a^2 - 2b^2 - 2c^2 - ab - 2ac - 2bc$. (Figure 17 shows $S_{\alpha,4}$ and one of its associated tropical surfaces.)

**Proposition 6.4.** Let $\delta \geq 4$. A general tropical surface with subdivision $S_{\alpha,\delta}$ contains no tropical lines.

**Proof.** Let $X$ be a general tropical surface with subdivision $S_{\alpha,\delta}$. Since we assume $\deg X \geq 4$, Theorem 5.6 guarantees that any tropical line on $X$ has combinatorial position either $3G$, $3H$, or $3I$. Hence to prove the proposition it is enough to show that $S_{\alpha,\delta}$ has no subcomplexes of types $3G$, $3H$, $3I$. Combining the description of the maximal elements of $S_{\alpha,\delta}$ (Section 2.3.4) with the information given in Table 3, this is a simple exercise.
7 Tropical lines on smooth tropical cubic surfaces.

We start this section by giving two conjectures concerning tropical lines on smooth tropical cubic (i.e. degree 3) surfaces. Subsequently, we examine one specific subdivision, and show that the first conjecture hold for the tropical cubics associated to this subdivision.

**Conjecture 1.** A general smooth tropical cubic surface contains exactly 27 tropical lines.

Extending to all smooth tropical cubics, we conjecture the following:

**Conjecture 2.** For a smooth tropical surface $X$, let $f$ be the number of two-point families on $X$, and $i$ the number of tropical lines on $X$ not part of any two-point family on $X$. Then we have

$$f + i = 27.$$

In particular, $X$ contains either 27 or infinitely many tropical lines.

7.1 An example

In this section we will analyze the RE-triangulation $S_{\alpha,3}$ (shown in Figure 18), which was defined in section 2.3.1. The aim is to prove the following theorem:

**Theorem 7.1.**

a) A general tropical surface with subdivision $S_{\alpha,3}$ contains exactly 27 tropical lines.

b) Any tropical surface with subdivision $S_{\alpha,3}$ contains at least 27 tropical lines.

c) There exist tropical surfaces with subdivision $S_{\alpha,3}$ containing infinitely many tropical lines.

We will show this through a series of lemmas, looking at how many lines $X$ has in the different general combinatorial positions. We will frequently use that $S_{\alpha,3}$ is invariant under the subgroup $G \subseteq S_4$ generated by the three involutions (12), (34) and (13)(24). In particular, $|G| = 8$.

Some local notation used in this section: The elements of $A_3$ will be denoted $A_{000}, A_{100}, \ldots, A_{003}$, where the indices indicates the coordinates of the lattice points. Furthermore, $X$ is assumed to be a tropical surface with subdivision $S_{\alpha,3}$. Thus $X$ corresponds to a point $(\lambda_{000}, \lambda_{100}, \ldots, \lambda_{003})$ in the secondary cone $K(S_{\alpha,3})$, where the ordering is chosen such that $\lambda_{ijk}$ is the lifting value of $A_{ijk}$. In other words, $X = V_{tr}(f)$, where

$$f(x_1, x_2, x_3) = \sum_{(i,j,k) \in A_3} \lambda_{ijk} x_1^i x_2^j x_3^k.$$

**Lemma 7.2.** $X$ has no tropical lines in either of the combinatorial positions $3C$, $3G$, $3H$ or $3I$.

Proof. It is enough to observe that $S_{\alpha,3}$ has no subcomplexes of types $3C$, $3G$, $3H$ or $3I$. This is a straight-forward (although somewhat tedious if done by hand) check, using the cell structures given in Table 3.

\[\square\]
Lemma 7.3.  

a) A general $X$ has exactly 12 tropical lines with combinatorial position $3A$ or $3D$.

b) Exactly 4 of the tropical lines in a) specialize to a two-point family.

c) Any $X$ has exactly 12 tropical lines which deforms into combinatorial position $3A$ or $3D$. Neither of these deforms into any other general combinatorial position.

Proof. a) Consider the three subcomplexes $\mathcal{R}_{3A}, \mathcal{R}_{3D}, \mathcal{R}'_{3D} \subseteq S_{\alpha,3}$ shown in Figure 19. In $S_{\alpha,3}$ we find a total of eight subcomplexes of type $3A$; these are all equivalent modulo $G$ to $\mathcal{R}_{3A}$. Furthermore, there are 12 subcomplexes of type $3D$. Of these, eight are equivalent to $\mathcal{R}_{3D}$, while the remaining four are equivalent to $\mathcal{R}'_{3D}$.

Let $h_1 = \lambda_{210} + \lambda_{002} - \lambda_{201} - \lambda_{011}$, and $h_2 = 2\lambda_{210} - 2\lambda_{120} + \lambda_{020} - \lambda_{200}$. We claim that:

i) $h_1 > 0 \iff \mathcal{R}_{3A}$ can be realized uniquely on $X$ as a tropical line with combinatorial position $3A$,

ii) $h_1 < 0 \iff \mathcal{R}_{3D}$ can be realized uniquely on $X$ as a tropical line with combinatorial position $3D$,

iii) $h_2 \neq 0 \iff \mathcal{R}'_{3D}$ can be realized uniquely on $X$ as a tropical line with combinatorial position $3D$.

To prove claims i) and ii) we sketch the 2-cells of $X$ dual to the three edges $A_{101}A_{111}$, $A_{101}A_{201}$ and $A_{101}A_{210}$ (see Figure 20). In the figure, $P$ and $Q$ are the points dual to the tetrahedra $A_{002}A_{101}A_{111}A_{011}$ and $A_{101}A_{111}A_{201}A_{210}$ respectively. By Lemma 4.2, $X$ contains a line segment trespassing through $P$, with direction vector $\omega_1$. This segment can be extended uniquely to a ray $\ell_1 \subseteq X$, starting somewhere on the polygonal arc $QQ''$.

![Figure 20: If the ray $\ell_1$ meets the interior of the segment $QQ''$ (resp. the interior of $QQ'$), it can be extended uniquely to a tropical line on $X$ with combinatorial position $3A$ (resp. $3D$).]
In particular, observe that \((\mathbb{Q}^2)^{\omega_4}\) and \((\mathbb{R})^{\omega_3}\) have combinatorial type \(((23)(14))\), with one vertex in each of \(\text{int}(\mathbb{Q}^2)^{\omega_4}\) and \(\text{int}(\mathbb{R})^{\omega_3}\). Clearly, \(L\) has combinatorial position \(3A\), and \(c_3(L) = R_{3A}\), so claim i) is proved.

Similarly, if \(h_1 < 0\), then \(\ell_1\) starts in \(\text{int}(\mathbb{Q}^2)^{\omega_4}\). From the facts that \((\mathbb{Q}^2)^{\omega_4} = A_{101}A_{111}A_{201}\) has an exit in the direction \(\omega_4\), and that \((\mathbb{R})^{\omega_3}\) has exits in both directions \(\omega_2\) and \(\omega_3\), hence it is evident from Figure 20 that if \(h_1 > 0\), then \(\ell_1\) can be extended uniquely to a tropical line \(\mathbb{T}\) of combinatorial type \(((23)(14))\), with one vertex on each of \(\text{int}(\mathbb{Q}^2)^{\omega_4}\) and \(\text{int}(\mathbb{R})^{\omega_3}\). Clearly, \(L\) has combinatorial position \(3A\), and \(c_3(L) = R_{3A}\), so claim ii) is proved.

For claim iii) we refer to Figure 21 showing the 2-cells dual to the edges \(A_{110}A_{210}\) and \(A_{110}A_{120}\). If the side lengths \(a + b \neq c + d\) then \(X\) contains a unique tropical line \(L\) containing the vertices \(S, T \in X\): If \(a + b < c + d\), then \(L\) has combinatorial type \(((13)(24))\) and one vertex in \(\text{int}((A_{110}A_{120}))^{\omega_4}\). If \(a + b > c + d\) then \(L\) has combinatorial type \(((14)(23))\) and one vertex in \(\text{int}((A_{110}A_{120}))^{\omega_3}\). In both cases, \(L\) has one vertex in \(\text{int}(\mathbb{Q}^2)^{\omega_4}\) and the other in the interior of the 2-cell \((A_{110}A_{120})\). The combinatorial position of this line is \(3D\), and the associated line subcomplex in \(\text{Subdiv}_X\) is precisely \(R_{3D}\). Therefore, calculating the vertex coordinates, one finds that \(a + b - c - d = h_2\). This proves claim iii).


de
d

Observe that claim i) remains valid if we exchange \(h_1\) and \(R_{3A}\) by \(\sigma(h_1)\) and \(\sigma(R_{3A})\), where \(\sigma\) is any element of \(G \subseteq S_4\), and similarly for the claims ii) and iii). From this we conclude two things. Firstly, if \(\alpha\) lies away from the hyperplanes given by \(\sigma(h_1) = 0\), for all \(\sigma \in G\), then the 16 subcomplexes in the orbits of \(R_{3A}\) and \(R_{3D}\) give rise to exactly 8 tropical lines on \(X_0\). Secondly, if \(\alpha\) lies away from the hyperplanes \(\sigma(h_2) = 0\), for all \(\sigma \in G\), then the four subcomplexes in the orbit if \(R_{3D}\) give rise to exactly four tropical lines on \(X_0\). Hence, a general \(X\) with subdivision \(S_{0,3}\) has exactly \(8 + 4 = 12\) tropical lines with combinatorial position either \(3A\) or \(3D\).

b) Let us first analyze the cases \(h_1 = 0\) and \(h_2 = 0\). If \(h_1 = 0\), then \(X\) contains the tropical line with vertices \(Q\) and \(R\) (see Figure 20). It has non-general combinatorial position, and it does not belong to any two-point family on \(X\).

Next, suppose \(h_2 = 0\). In this case \(a + b = c + d\) (cf. Figure 21), and the lines through \(S\) and \(T\) with direction vectors \(e_2\) and \(e_1\) respectively, meet in the point \(v := S + (0, a + b, 0) = T + (c + d, 0, 0)\) on the 1-cell dual to the triangle \(A_{210}A_{120}A_{110}\). Since this triangle has exits in both directions \(\omega_3\) and \(\omega_4\), it follows that \(X\) contains the degenerate tropical line with vertex \(v\). In fact, it is easy to see that for all \(t \geq 0\), the tropical line with vertices \(v\) and \(v + t(e_1 + e_2)\) lies on \(X\). Hence \(X\) contains the complete two-point family of tropical lines passing through \(S\) and \(T\).

Now for the specializations. As seen in a) the 12 tropical lines in question come in two groups, 8 associated to \(R_{3A}\) or \(R_{3D}\), and 4 associated to \(R_{3D}'\). Suppose \(X\) is general, and that \(L \subseteq X\) is in the first group. We can assume that \(c_3(L) = \mathbb{Q}^2\) or \(R_{3D}\). Any perturbation of \(X\) which keeps
Figure 22: The tetrahedron $T$ common to all subcomplexes of type $3B$.

$h_1 \neq 0$, induces a deformation of $L \subseteq X$ that preserves the combinatorial position of $L$. Hence to obtain a specialization of $L$, we must let $h_1 \to 0$. As observed above, this results in a specialization of $L$ to an isolated tropical line.

Next, let (on a general $X$) $L$ be in the last group, i.e., we can assume $L$ to be the realization of $R_{3D}^t$. Choose any perturbation of a second vertex of $X$ such that $h_2 \to 0$. As shown above, this will induce a specialization of $L$ to a degenerate tropical line which belongs to a two-point family on $X$.

c) On general $X$, the 12 tropical lines are of course those found in a); these clearly satisfy the requirements. If $X$ is non-general, then either $\sigma(h_1) = 0$ or $\sigma(h_2) = 0$ for some $\sigma \in G$. Suppose the former. It is enough to consider the case $h_1 = 0$, in which $X$ contains the tropical line $L_0$ with vertices $Q$ and $R$. As seen in b), $L_0$ is the unique specialization of any realization of $R_{3A}$ or $R_{3D}$. In particular, it deforms into combinatorial positions $3A$ and $3D$.

We claim that $L_0$ cannot be deformed into any combinatorial position other than $3A$ and $3D$. Let $X_0 := X$, and consider any deformation $t \to (L_t, X_t)$ of $L_0 \subseteq X_0$ into some general combinatorial position $C$. For each $t$, let $P_t \in X_t$ be the vertex corresponding to $P \in X_0$. Then we know, by Lemma 4.11 that the $\omega_1$-ray of $L_t$ is trespassing through $P_t$ for each $t$. But this, together with the assumption that the combinatorial position $\sigma$ of $L_t$ is general, implies that $C$ equals either $3A$ or $3D$. (This follows from our discussion in a), in particular Figure 21.)

Finally, suppose $\sigma(h_2) = 0$; as before it is enough to consider the case $h_2 = 0$. Then $X$ contains the two-point family of tropical lines passing through $S$ and $T$ (cf. Figure 21). Let $L_{\text{deg}}$ be the degenerate member of this two-point family. Clearly, $L_{\text{deg}}$ deforms into combinatorial position $3D$ (it is the unique specialization of any realization of $R_{3D}^t$), and it is easy to see that it does not deform into any other general combinatorial position. As for the non-degenerate tropical lines in the two-point family, none of them have general combinatorial position, nor can any of them be deformed into any general combinatorial position.

We conclude that the 12 tropical lines found in a) all have unique (and distinct) specializations, which satisfy the requirements given in the lemma. This completes the proof. \hfill $\Box$

Lemma 7.4.  a) A general $X$ has exactly 3 tropical lines with combinatorial position $3B$.

b) Each of the tropical lines in a) specialize to a two-point family.

c) Any $X$ has exactly 3 tropical lines which deforms into combinatorial position $3B$. Neither of these deforms into any other general combinatorial position.

Proof. a) There are 12 subcomplexes of type $3B$ in $S_{\alpha,3}$. All of these contain the tetrahedron $T$, shown in Figure 22 with vertices $A_{110}$, $A_{101}$, $A_{011}$ and $A_{111}$. Using Lemma 4.2 we see that the dual vertex $T' \in X$ allows trespassing line segments in three directions simultaneously: $e_1 + e_2$, $e_1 + e_3$ and $e_2 + e_3$. Drawing the shapes of the 2-cells adjacent to $T'$, one sees immediately that each of these trespassing line segments can be extended to a tropical line on $X$. For general $X$, each extension is unique on $X$, and the three resulting tropical lines all have combinatorial position $3B$.

b) Non-generality in this case means that at least one of the three trespassing line segments in a) meets a second vertex of $X$. One can check that this always allows for a second trespassing, resulting in a
two-point family on \(X\). It is even possible for the lines segment to meet a third vertex of \(X\), giving rise to a 2-dimensional two-point family on \(X\). Any of the tropical lines in a) specializes to both a 1-dimensional and 2-dimensional family obtained in this way.

c) For any of the two-point families described in b), none of its members has general combinatorial position. Using arguments similar to those in the proof of Lemma 7.3, it is not hard to show that there is exactly one tropical line in the family that can be deformed into some general combinatorial position, which must be 3B. The truth of the statement follows from this.

**Lemma 7.5.**

a) A general \(X\) has exactly 4 tropical lines with combinatorial position 3E.

b) Each of the tropical lines in b) specialize to a two-point family.

c) Any \(X\) has exactly 4 tropical lines which deforms into combinatorial position 3E. Neither of these deforms into any other general combinatorial position.

**Proof.** There are 8 subcomplexes of type 3E in \(S_{\alpha,3}\), all equivalent modulo \(G\). These 8 can be divided into 4 pairs, such that the subcomplexes in each pair contain the same tetrahedra. One of these pairs, \(R_{3E}\) and \(R'_{3E}\), is shown in Figure 23.

We claim that a general \(X\) contains exactly one tropical line \(L\) with line subcomplex either \(R_{3E}\) or \(R'_{3E}\). To prove this, we refer to Figure 24, which shows the 2-cell dual to \(A_{111}A_{101}\). The cell is a parallel hexagon whose edge directions are given in the figure. The vertices \(P\) and \(Q\) are the duals of the tetrahedra \(A_{111}A_{101}A_{210}\) and \(A_{111}A_{101}A_{011}A_{001}\), allowing (by Lemma 4.2) trespassing in directions \(\omega_3\) and \(\omega_1\) respectively. Let \(L\) be the tropical line with vertices \(v_1 = P + (a, 0, 0)\) and \(v_2 = v_1 + (\min(b, c), 0, \min(b, c))\). Observe that \(v_2\) lies either on the edge \(RR'\) (if \(c \leq b\)) or on the edge \(RR''\) (if \(b \leq c\)). Hence, since both \((RR')^e = A_{111}A_{101}A_{201}\) and \((RR'')^e = A_{111}A_{101}A_{102}\) has exits in directions \(\omega_2\) and \(\omega_4\), this ensures that \(L \subset X\). For a general \(X\), we can assume \(b \neq c\). If \(c < b\) (as shown in Figure 24), we have \(v_2 \in \int(\RR''\)), giving \(c_X^e(L) = R_{3E}\). If \(b < c\), then \(v_2 \in \int(RR')\), and \(c_X^e(L) = R'_{3E}\). In either case it is clear that \(L\) is the only tropical line on \(X\) passing through \(P\) and \(Q\). This proves the claim.

**Figure 24:** A tropical line on \(X\) with combinatorial position 3E.
Finally, the same argument applies to the three other pairs of subcomplexes, giving a total of 4 lines with combinatorial position 3E.

Parts b) and c) are proved in a similar fashion as in the corresponding parts of Lemma 7.3.

**Lemma 7.6.** Any X has exactly 8 tropical lines with combinatorial position 3F. Neither of these specialize into any other combinatorial position.

**Proof.** Modulo $G$, the only subcomplexes of type 3F in $S_{α,3}$ are $R_{3F}$ and $R'_{3F}$, shown in Figure 25. Both have orbits of length 4 under the action of $G$.

It is not hard to see that any $X$ contains exactly one tropical line with line subcomplex $R_{3F}$. Indeed, Figure 26 shows how to construct such a tropical line. For uniqueness we can e.g. apply Lemma 3.5. Denoting the side lengths by $a, b, c, d$, as indicated, we find that $PR = (0, a, a) + (0, b, 0) + (-c, 0, 0) + (0, d, d) = (-c, a + b + d, a + d)$. Since $a, b, c, d$ are strictly positive, the lemma implies that there is a unique tropical line through $P$ and $R$, and, a fortiori, that there is a unique line on $X$ with associated line subcomplex $R_{3F}$.

The same argument applies to the subcomplexes in the orbit of $R_{3F}$. Similarly, by studying the 2-cells dual to $A_{111}A_{102}$ and $A_{101}A_{201}$, one can show that $X$ always contains exactly one tropical line with $R'_{3F}$ as its line subcomplex. Hence we have a total of 8 tropical lines with combinatorial position 3F.

The lemmas 7.2 through 7.6 provide everything needed to prove Theorem 7.1.

**Proof of Theorem 7.1.** a) To sum up, we have on a general $X$, 12 tropical lines with combinatorial position 3A or 3D, 3 with 3B, 4 with 3E, 8 with 3F and none with 3C, 3G, 3H or 3I. No tropical line can have more than one combinatorial position on $X$, hence the total number of lines is exactly $12 + 3 + 4 + 8 = 27$.

b) Part c) of the lemmas 7.3 through 7.5 and Lemma 7.6 identifies, on any $X$, four sets of tropical lines. Moreover, it follows from the same results that these four sets are mutually disjoint, and contains altogether 27 tropical lines.

c) As shown in part b) of the lemmas 7.3 through 7.5 there exist tropical surfaces $X$ with subdivision $S_{α,3}$, which contains one or more two-point families of tropical lines. In particular, such $X$ has infinitely many tropical lines.

**Figure 26: A tropical line on $X$ with combinatorial position 3F.**


7.2 Comments

In principle, Conjecture 1 could be proved by subjecting every RE-subdivision of \( \Gamma_3 \) (up to the action of \( S_4 \)) to an analysis similar to that in the proof of Theorem 7.1. It is not known to the author how many such subdivisions exist. Using computer-randomized lifting functions we generated 1500 RE-subdivisions of \( \Gamma_3 \), but the actual number is presumably a lot larger. For each of the 1500 subdivisions, we calculated the total number of subcomplexes of types 3A, \ldots , 3I. This number was never below 27. (For the subdivision \( S_{\alpha,3} \) examined in section 7.1 the corresponding number is 48.)

Going through the proofs of part b) of lemmas 7.3 - 7.6 we see a clear pattern: Let \( L \) be any of the 27 tropical lines on a general \( X \) with subdivision \( S_{\alpha,3} \). When we pass to a non-general surface, then \( L \) specializes to either a unique tropical line, or a unique two-point family. This is almost enough to prove Conjecture 2 for the subdivision \( S_{\alpha,3} \). To complete the proof, one has to show in addition that it is impossible for any \( X \) to have an isolated tropical line which is not the specialization of any of the 27 general lines. It seems probable that this could be tackled by a case study of combinatorial positions, but we leave this for future research.

Finally we are tempted to pose the following question, on whose answer we dare not speculate:

**Question.** Does there exist an RE-triangulation of \( \Gamma_3 \), for which any associated tropical surface contains exactly 27 tropical lines?

Acknowledgements. I would like to thank my supervisor Kristan Ranestad for many valuable suggestions and corrections, and for encouragement during the writing of this paper.

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