The BSSN formulation is a partially constrained evolution system

Adrian P Gentle
Department of Mathematics, University of Southern Indiana, Evansville, IN 47712
E-mail: apgentle@usi.edu

Abstract. Relativistic simulations in 3+1 dimensions typically monitor the Hamiltonian and momentum constraints during evolution, with significant violations of these constraints indicating the presence of instabilities. In this paper we rewrite the momentum constraints as first-order evolution equations, and show that the popular BSSN formulation of the Einstein equations explicitly uses the momentum constraints as evolution equations. We conjecture that this feature is a key reason for the relative success of the BSSN formulation in numerical relativity.

PACS numbers: 04.20.-q, 04.25.Dm

1. Introduction

Three decades after the earliest attempts to study black hole spacetimes numerically, the problem of simulating the collision and coalescence of two black holes remains at the forefront of numerical relativity. Only in the last year have the first stable, long-term numerical simulations of black hole collision and coalescence in three-dimensions been achieved [1, 2]. The primary focus of numerical relativity has at last shifted to the astrophysical implications of these events, with recent simulations investigating recoil in the final product of binary black hole mergers, and the extraction of gravitational wave profiles for comparison with observations [3, 4, 5].

Despite these advances, progress is still needed in understanding both the mathematical and computational formulations of the equations. The most popular formulation of the Einstein equations in numerical relativity is the BSSN system [6, 7] (but see [8] for an alternate approach). The Einstein equations are split into distinct evolution and constraint equations, with the constraints being used to construct initial data, after which they are monitored (rather than enforced) as the evolution proceeds. In addition, a variety of technical and computational techniques are employed to achieve stable evolutions, including causal differencing, artificial diffusion, grid excision, moving punctures, special shift conditions, and many others which have been tried over the last few decades.

In this paper we consider the role played by the momentum constraints in evolution, and in particular, their appearance in the BSSN formulation. It is well known that the
momentum constraints are vital to the stability of the BSSN system, where they are used to eliminate the spatial divergence of the trace-free extrinsic curvature from one of the evolution equations. This is a necessary condition for stability [7]. The Hamiltonian and momentum constraints are then monitored during evolution, and give an indication of the quality and stability of the simulation.

The close link between unstable simulations and uncontrolled violation of the constraints has led to the development of several constrained evolution algorithms [9, 10, 11, 12]. These schemes are designed to enforce the Hamiltonian and/or momentum constraints throughout the evolution, either through direct projection or gentle numerical relaxation onto the constraint surface.

We will show that the BSSN system is already a partially constrained formulation of the Einstein equations, since the evolution equations used for the conformal connection functions are the momentum constraints rewritten as first-order evolution equations. It is in this sense that the BSSN formulation is a partially constrained evolution scheme, with a form of the momentum constraints being actively enforced throughout the evolution.

We begin by briefly reviewing the BSSN formulation and its derivation, highlighting the role played by the momentum constraints. In section 3 the momentum constraints are recast as first-order evolution equations for a particular combination of the metric and its spatial derivatives. Then, in section 4, a conformal transformation on this variable demonstrates that the momentum constraints may be used to evolve the conformal connection functions. A more direct calculation demonstrates that the resulting equation is identical to the standard BSSN evolution equation for $\tilde{\Gamma}^i$. In the final section we consider the consequences of this observation.

2. The ADM and BSSN evolution equations

The Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation of the Einstein equations [6, 7] can be derived from the canonical ADM form of the equations. With the metric in the 3+1 form

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) \, dt^2 + 2\beta_i \, dt \, dx^i + g_{ij} \, dx^i dx^j,$$

(1)

the components of the three-metric $g_{ij}$ become the dynamical variables, and the remaining coordinate freedoms are expressed by the lapse function $\alpha$ and shift vector $\beta^i$. The Einstein equations consist of six second order (in both time and space), non-linear partial differential equations, together with an additional four constraints which only contain first temporal derivatives of the $g_{ij}$. By introducing the extrinsic curvature tensor $K_{ij}$ the Einstein equations can be written as a set of twelve evolution equations which are explicitly first-order in time (neglecting source terms for simplicity),

$$(\partial_t - \mathcal{L}_\beta) g_{ij} = -2\alpha K_{ij}$$

(2)

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = \alpha \left( R_{ij} - 2K_{kl} \, K'_{lj} + KK_{ij} \right) - \nabla_i \nabla_j \alpha,$$

(3)
where $\mathcal{L}_\beta$ is the Lie derivative along $\beta^i$ and $K = g^{ij}K_{ij}$. The constraint equations are

$$R + (K)^2 - K_{ij}K^{ij} = 2\rho, \quad (4)$$
$$\nabla_jK^{ij} - g^{ij}\nabla_jK = S^i, \quad (5)$$

and are known as the Hamiltonian and momentum constraints, respectively. These are mathematically conserved by the evolution equations, in the sense that if they are satisfied on some spacelike hypersurface then solutions of the evolution equations will continue to satisfy the constraints for all time.

This 3+1 form of the Einstein equations is known as the ADM system, and was the basis of initial efforts to construct numerical solutions. Until quite recently all attempts to evolve black holes in three dimensions were unsuccessful (see [13] for a review), with the impending failure of the code signposted by rapid growth in violations of the constraints. This led many investigators to propose and test new formulations of the equations, among the most popular being the so-called BSSN system developed by Shibata and Nakamura [6], and later popularized by Baumgarte and Shapiro [7].

Their approach splits the extrinsic curvature tensor into its trace ($K$) and trace-free ($\tilde{A}^{ij}$) parts, and then performs a conformal decomposition of the three-geometry, with $g_{ij} = e^{4\phi}\tilde{g}_{ij}$ and $A_{ij} = e^{4\phi}\tilde{A}_{ij}$. Then

$$K_{ij} = e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{g}_{ij}K\right), \quad (6)$$

and for consistency we require that $\det(\tilde{g}_{ij}) = 1$ and $\tilde{A}^i_i = \tilde{g}^{ij}\tilde{A}_{ij} = 0$.

The ADM evolution equations (2) and (3) are transformed to evolve $\tilde{g}_{ij}$ and $\tilde{A}_{ij}$,

$$\left(\partial_t - \mathcal{L}_\beta\right)\tilde{g}_{ij} = -2\alpha\tilde{A}_{ij} \quad (7)$$
$$\left(\partial_t - \mathcal{L}_\beta\right)\tilde{A}_{ij} = e^{-4\phi}\left[\alpha R_{ij} - \nabla_i\nabla_j\alpha\right]_{\text{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) \quad (8)$$

where “TF” denotes the trace-free portion of the bracketed expression. Similarly, the ADM equations lead to

$$\left(\partial_t - \mathcal{L}_\beta\right)\phi = -\frac{1}{6}\alpha K \quad (9)$$
$$\left(\partial_t - \mathcal{L}_\beta\right)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \nabla_i\nabla^i\alpha \quad (10)$$

which are used to evolve $\phi$ and $K$.

The BSSN approach also introduces the conformal connection functions

$$\bar{\Gamma}^i = -\partial_j\tilde{g}^{ij}, \quad (11)$$

for which evolution equations can be obtained by differentiating with respect to the time coordinate and commuting the partial derivatives. This yields

$$\partial_t\bar{\Gamma}^i = -\partial_j\left(2\alpha\tilde{A}^{ij} + \mathcal{L}_\beta\tilde{g}^{ij}\right) \quad (12)$$

where equation (7) has been used to eliminate explicit time derivatives of the conformal three-metric. The Lie derivative is given by

$$\mathcal{L}_\beta\tilde{g}^{ij} = \beta^k\partial_k\tilde{g}^{ij} - 2\tilde{g}^{k(i}\partial_k\beta^{j)} + \frac{2}{3}\tilde{g}^{ij}\partial_k\beta^k, \quad (13)$$
since $\tilde{g}^{ij}$ is a tensor density of weight $\frac{2}{3}$.

In principle this completes the system of evolution equations, since the conformal connection coefficients could be evolved using equation (12). However, it is found that the system is only more stable than the original ADM formulation if the spatial divergence of $\tilde{A}^{ij}$ is eliminated from equation (12). This is achieved using the momentum constraints [7]. Under the conformal transformation described above, the momentum constraints take the form

$$\partial_j \tilde{A}^{ij} + 6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{g}^{ij} \partial_j K + \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} = e^{4\phi} S^i,$$

where $S^i$ is a source term. Using these to eliminate the spatial divergence of $\tilde{A}^{ij}$ from (12) gives

$$\partial_t \tilde{\Gamma}^i_j = -2 \tilde{A}^{ij} \partial_j \alpha + 2\alpha \left( 6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{g}^{ij} \partial_j K + \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} - \tilde{S}^i \right),$$

The standard BSSN evolution equations consist of (7)-(10) together with (15), and are used to evolve the BSSN variables $\tilde{g}^{ij}$, $\tilde{A}^{ij}$, $\phi$, $K$ and $\tilde{\Gamma}^i_j$. This system has been found to be relatively stable, and is basis for most of the successful recent binary black hole calculations [1, 2, 5].

3. The momentum constraints as first-order evolution equations

The momentum constraints (5) are a set of three differential constraints relating the three-metric and extrinsic curvature. Since the extrinsic curvature is itself related to the time development of the three-metric through its definition, equation (2), the momentum constraints can be viewed as equations involving the time derivative of $g_{ij}$. In this section we explicitly rewrite the momentum constraints as first-order evolution equations for a function of the three-metric and its spatial derivatives.

The momentum constraints with source terms, equation (5), can be written as

$$\partial_j K^{ij} + \Gamma^j_{jk} K^{ik} + \Gamma^i_{jk} K^{jk} - g^{ij} \partial_j K = S^i,$$

where $\Gamma^i_{jk}$ are the Christoffel symbols and $S^i$ represents the sources. Using the definition of extrinsic curvature, equation (2), the first term in the constraints becomes

$$\partial_j K^{ij} = \partial_j \left( \frac{1}{2\alpha} \left( \partial_t g^{ij} - \mathcal{L}_\beta g^{ij} \right) \right),$$

and by commuting the spatial and temporal partial derivatives of $g_{ij}$ we find that

$$\partial_j K^{ij} = \frac{1}{2\alpha} \left( \partial_t \left( \partial_j g^{ij} \right) - 2K^{ij} \partial_j \alpha - \partial_j \left( \mathcal{L}_\beta g^{ij} \right) \right).$$

Applying a similar procedure to the gradient of $K$, we have

$$g^{ij} \partial_j K = \frac{1}{2\alpha} \left( -\partial_t \left( g^{ij} g^{ab} \partial_j g_{ab} \right) - 2K g^{ij} \partial_j \alpha + g^{ab} K^{ij} \partial_i g_{ab} \\
+ g^{ij} g^{ab} \partial_j \left( \mathcal{L}_\beta g_{ab} \right) + g^{ij} \partial_j g_{ab} \mathcal{L}_\beta g^{ab} + g^{ab} \partial_j g_{ab} \mathcal{L}_\beta g^{ij} \right).$$
Combining these results we can write the momentum constraints (16) as
\[ \partial_t \left( \partial_j g^{ij} + g^{ij} \partial_j \ln g \right) = 2 \left( K^{ij} - g^{ij} K \right) \partial_j \alpha 
+ 2 \alpha \left( K^{ij} \partial_j \ln g + S^i - \Gamma^j_{jk} K^{ik} - \Gamma^i_{jk} K^{jk} \right) 
+ \partial_j \left( L^j_{\beta} g^{ij} \right) + g^{ij} \partial_j \left( g^{ab} L_{\beta} g_{ab} \right) + (\partial_j \ln g) \mathcal{L}_\beta g^{ij}, \]
where we have used \( g^{ab} \partial_j g_{ab} = \partial_j \ln g \) and \( g = \det(g_{ij}). \)

The momentum constraints can thus be viewed as the natural evolution equations for the subsidiary variable
\[ \Lambda^i = \partial_j g^{ij} + g^{ij} \partial_j \ln g, \]
the first term of which is suggestive of the conformal connection coefficients defined by equation (11). The momentum constraints written in their most general form as evolution equations are then
\[ \partial_t \Lambda^i = 2 \left( K^{ij} - g^{ij} K \right) \partial_j \alpha 
+ 2 \alpha \left( K^{ij} \partial_j \ln g + S^i - \Gamma^j_{jk} K^{ik} - \Gamma^i_{jk} K^{jk} \right) 
+ \partial_j \left( L^j_{\beta} g^{ij} \right) + g^{ij} \partial_j \left( g^{ab} L_{\beta} g_{ab} \right) + (\partial_j \ln g) \mathcal{L}_\beta g^{ij}, \quad (17) \]
where \( \Lambda^i \) is a subsidiary variable since it is not truly independent, but rather a function of the metric and its spatial derivatives. We conjecture that using the momentum constraints to evolve \( \Lambda^i \) in the ADM formulation, alongside the evolution of \( g_{ij} \) and \( K_{ij} \), would enforce the consistency of the constrained evolution system more thoroughly than evolving the three-metric and extrinsic curvature alone.

Finally, we note that equation (17) is essentially the same form of the momentum constraint used in the first-order flux-conservative formulation of Bona and Masso [14]. The Bona-Masso approach introduces a new variable \( V_i \) which is evolved using the momentum constraint, and in our notation \( V_i = \frac{1}{2} g_{ij} \Lambda^j \).

4. Momentum constraints in the BSSN formulation

In the previous section we demonstrated that the momentum constraints can be used to evolve the subsidiary variable \( \Lambda^i \), which is itself a function of the metric and its spatial derivatives. In this section we show that under a conformal transformation \( \Lambda^i \) is closely related to the BSSN variable \( \tilde{\Gamma}^i \). Guided by this insight, we will derive the evolution equation for \( \tilde{\Gamma}^i \) directly from the conformally decomposed momentum constraints.

Beginning with the variable \( \Lambda^i \) introduced above, the conformal decomposition used in section 2 gives
\[ \Lambda^i = e^{-4\phi} \left( \partial_j \tilde{g}^{ij} - 8 \tilde{g}^{ij} \partial_j \phi \right), \]
which suggests that we define \( \Lambda^i = e^{-4\phi} \tilde{\Lambda}^i \), so that
\[ \tilde{\Lambda}^i = -\tilde{\Gamma}^i - 8 \tilde{g}^{ij} \partial_j \phi. \]
It is clear that \( \tilde{\Lambda}^i \) splits into the BSSN conformal connection coefficient and the gradient of the conformal factor.
It follows from the results of the previous section that the momentum constraint can be used to evolve $\tilde{\Gamma}^i$. By calculating the time derivative of $\Lambda^i$ we find that

$$\partial_t \Lambda^i = e^{-4\phi} \left( \partial_t \tilde{\Gamma}^i - 4 \partial_t \left( \tilde{g}^{ij} \partial_i \phi \right) + 2 \left( \partial_i \tilde{g}^{ij} + 8 \tilde{g}^{ij} \partial_i \phi \right) \partial_t \phi \right),$$

and eliminating the $\partial_t \phi$ and $\partial_t \tilde{g}^{ij}$ terms, using equations (7) and (9), relates the time evolution of $\Lambda^i$ directly to the evolution of $\tilde{\Gamma}^i$. Substituting this into the momentum constraints in the form of equation (17) will then give an evolution equation for $\tilde{\Gamma}^i$.

This shows, somewhat indirectly, that the momentum constraints can be used to evolve the conformal connection coefficients $\tilde{\Gamma}^i$. It is more enlightening, however, if we begin with the conformal form of the momentum constraints,

$$\partial_j \tilde{A}^{ij} + 6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{g}^{ij} \partial_j k + \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} = e^{4\phi} S^i \quad (18)$$

and perform a calculation analogous to that of the previous section.

It follows directly from the definition of $\tilde{A}^{ij}$, equation (7), that

$$\tilde{A}^{ij} = \frac{1}{2\alpha} \left( \partial_t \tilde{g}^{ij} - \mathcal{L}_{\beta} \tilde{g}^{ij} \right).$$

This allows us to re-express the first term in the momentum constraints as

$$\partial_j \tilde{A}^{ij} = \frac{1}{2\alpha} \left( \partial_j \partial_t \tilde{g}^{ij} - \partial_j \left( \mathcal{L}_{\beta} \tilde{g}^{ij} \right) \right) - \frac{1}{\alpha} \tilde{A}^{ij} \partial_j \alpha,$$

and commuting the partial derivatives of $\tilde{g}^{ij}$ and using the definition of $\tilde{\Gamma}^i$ we can write

$$\partial_j \tilde{A}^{ij} = -\frac{1}{2\alpha} \left( \partial_t \tilde{\Gamma}^i + 2 \tilde{A}^{ij} \partial_j \alpha + \partial_j \left( \mathcal{L}_{\beta} \tilde{g}^{ij} \right) \right).$$

Using this result to replace the divergence of $\tilde{A}^{ij}$ in the conformal momentum constraints, equation (18), we find

$$-\frac{1}{2\alpha} \left( \partial_t \tilde{\Gamma}^i + 2 \tilde{A}^{ij} \partial_j \alpha + \partial_j \left( \mathcal{L}_{\beta} \tilde{g}^{ij} \right) \right) + 6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{g}^{ij} \partial_j k + \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} = e^{4\phi} S^i.$$

It is important to note that these are precisely the momentum constraints; all that we have done is replace $\tilde{A}^{ij}$ with its definition in terms of $\tilde{g}^{ij}$. Finally, expanding the Lie derivative and rearranging yields the momentum constraints in a new form:

$$\partial_t \tilde{\Gamma}^i = -2 \tilde{A}^{ij} \partial_j \alpha + 2\alpha \left( 6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{g}^{ij} \partial_j k + \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} - \tilde{S}^i \right)$$

$$- \partial_j \left( \beta^k \partial_k \tilde{g}^{ij} - 2 \tilde{g}^{k(i} \partial_k \beta^{j)} + \frac{2}{3} \tilde{g}^{ij} \partial_k \beta^k \right).$$

This is identical to equation (15), the standard equation used to evolve the conformal connection coefficients in the BSSN formulation. From this we conclude that the usual statement that the BSSN formulation is stable when the spatial divergence of $\tilde{A}^{ij}$ is eliminated from the evolution equation for $\tilde{\Gamma}^i$ [7] is equivalent to the statement that the BSSN system is stable when the momentum constraints themselves are used to evolve the conformal connection coefficients.
5. Conclusions

We have shown that the BSSN formulation utilizes the momentum constraints as evolution equations for the conformal connection functions $\tilde{\Gamma}^i$. It follows that the momentum constraints should be satisfied to computational accuracy throughout the evolution, although in practice violations of the “definitional constraints”

$$\tilde{\Gamma}^i + \partial_j \tilde{g}^{ij} = 0, \quad \det(\tilde{g}_{ij}) = 1, \quad \text{and} \quad \tilde{g}^{ij} \tilde{A}_{ij} = 0,$$

(19)
could lead to errors when the momentum constraints (18) are calculated during evolution.

We conjecture that the BSSN formulation shows superior stability properties to the ADM system because errors in the definitional constraints do not correspond directly to spurious sources of momentum being pumped into the simulation. Although the Hamiltonian and momentum constraints are mathematically conserved by the ADM evolution equations, small computational errors prevent the strict consistency of the system. The particular advantage of the BSSN system is that the conformal connection functions provide a way of more strictly and actively enforcing the consistency of the equations, provided the violation of the Hamiltonian constraint is not too great.

We expect that in practice any violation observed when calculating the momentum constraints (18) during evolution arises from inconsistencies introduced by violations of the new definitional constraints (19). It is possible that additional stability in the system may be gained by including the addition terms in the evolution equation for $\tilde{\Gamma}^i$ which “drop out”, but may in practice be non-zero. For example, terms containing

$$\tilde{g}^{ab} \partial_j \tilde{g}_{ab} \equiv \partial_j \ln \tilde{g}$$

are nominally zero because $\tilde{g} = \det(\tilde{g}_{ij}) = 1$, but in practice we expect small violations in one of more of the constraints (19).

Our result is consistent with recent work by Marronetti [11,12], where a constraint relaxation method is applied to the BSSN formulation. Although weak enforcement of the Hamiltonian constraint was found to provide significant improvement in the results [11], there was little change when a similar approach was applied to the momentum constraints [12]. As we have seen, the momentum constraints are already being directly enforced in the BSSN approach. It is likely that the small improvements seen in Marronetti’s momentum relaxation scheme were due to improved (albeit indirect) enforcement of the remaining constraints $\tilde{\Gamma}^i + \partial_j \tilde{g}^{ij} = 0$ and $\det(\tilde{g}_{ij}) = 1$.

We have shown that the BSSN formulation ensures that the momentum constraints are enforced throughout the evolution. The generalized harmonic approach of Pretorius [8] is also, by construction, constraint damping. Since all of the successful codes currently used in numerical relativity are based on one of these formulations, it appears that active constraint enforcement is a vital ingredient for the success of any new formulation of the Einstein equations designed for use in numerical relativity.
Acknowledgments

The author is grateful to the Lilly Endowment and the Pott Foundation for support through their Summer Research Fellowship programs, and to the School of Mathematical Sciences at Monash University for its hospitality while some of this work was undertaken. The author is indebted to many colleagues for insightful comments and conversations, including Pablo Laguna, Leo Brewin, Tony Lun, and Robert Bartnik.

References

[1] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, *Phys. Rev. Lett.*, 96, 111102 (2006), gr-qc/0511103.
[2] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, *Phys. Rev. Lett.*, 96, 111101 (2006), gr-qc/0511048.
[3] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, J. R. van Meter, and M. C. Miller, *Ap. J. Lett.*, 653, L93 (2006), astro-ph/0603204.
[4] J. A. Gonzalez, M. D. Hannam, U. Sperhake, B. Brugmann, S. Husa, gr-qc/0702052 (2007).
[5] F. Herrmann, I. Hinder, D. Shoemaker, and P. Laguna, gr-qc/07062541 (2007).
[6] M. Shibata, T. Nakamura, *Phys. Rev.*, D52 5428, 1995.
[7] T. W. Baumgarte, S. L. Shapiro, *Phys. Rev.*, D59 024007, 1998.
[8] F. Pretorius, *Class. Quant. Grav.*, 23, (2006) S529-S552.
[9] S. Bonazzola, E. Gourgoulhon, P. Grandcl and J. Novak1, *Phys.Rev.* D70 (2004) 104007.
[10] M. Anderson, R. A. Matzner, *Found. Phys.*, 35, (2005) No. 9.
[11] P. Marronetti, *Class. Quant. Grav.*, 22, (2005) 2433.
[12] P. Marronetti, *Class. Quant. Grav.*, 23, (2006) 2681.
[13] L. Lehner, *Class. Quant. Grav.*, 18 (2001) R25-R86.
[14] A. Arbona, C. Bona, J. Masso and J. Stela, *Phys.Rev.* D60 (1999) 104014.