A Heuristic Subexponential Algorithm to Find Paths in Markoff Graphs Over Finite Fields

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Abstract. Charles, Goren, and Lauter [5] explained how one can construct hash functions using expander graphs in which it is hard to find paths between specified vertices. The set of solutions to the classical Markoff equation $X^2 + Y^2 + Z^2 = XYZ$ in a finite field $\mathbb{F}_q$ has a natural structure as a tri-partite graph using three non-commuting polynomial automorphisms to connect the points. These graphs conjecturally form an expander family, and Fuchs, Lauter, Litman, and Tran [8] suggested using this family of Markoff graphs in the CGL construction. In this note we show that in both a theoretical and a practical sense, assuming two randomness hypotheses, the path problem in a Markoff graph over $\mathbb{F}_q$ can be solved in subexponential time, and is more-or-less equivalent in difficulty to factoring $q-1$ and solving three discrete logarithm problems in $\mathbb{F}_q^*$. 

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1 Introduction

The classical Markoff surface is the affine surface given by the equation
\[ \mathcal{M} : X^2 + Y^2 + Z^2 = 3XYZ. \] (1)

There are three double covers \( \mathcal{M} \to \mathbb{A}^2 \) that give rise to three non-commuting involutions \( \sigma_1, \sigma_2, \sigma_3 \). A famous theorem of Markoff [12] says that every positive integer solution of (1) can be obtained from \((1,1,1)\) by applying the \( \sigma_i \) and permuting the coordinates.

In this note we consider solutions to (1) in a finite field \( \mathbb{F}_q \) of characteristic at least 5. There has recently been a lot of interest in studying the orbit structure of \( \mathcal{M}(\mathbb{F}_q) \) [2–4, 6, 7, 11]. We mention in particular a deep result of William Chen [6], which builds on work of Bourgain, Gamburd, and Sarnak [2, 3] and says that if \( q \) is a sufficiently large prime, then the reduction modulo \( q \) map

\[ \mathcal{M}(\mathbb{Z}) \to \mathcal{M}(\mathbb{F}_q) \]

is surjective.

More precisely, we consider the three non-commuting automorphisms
\[ \rho_1, \rho_2, \rho_3 : \mathcal{M} \to \mathcal{M} \]

given by the formulas
\[ \rho_1 = (X, Z, 3XZ - Y), \quad \rho_2 = (3XY - Z, Y, X), \quad \rho_3 = (Y, 3YZ - X, Z), \]

where \( \rho_1, \rho_2, \rho_3 \) are obtained from \( \sigma_1, \sigma_2, \sigma_3 \) by composing with appropriate coordinate permutations. (The \( \rho_i \) are called “rotations” in [2, 3].) Let
\[ \mathcal{R} = \langle \rho_1, \rho_2, \rho_3 \rangle \subset \text{Aut}(\mathcal{M}) \]
be the group of automorphisms generated by the \( \rho_i \), and let

\[ \mathcal{M}^*(\mathbb{F}_q) = \mathcal{M}(\mathbb{F}_q) \setminus \{(0,0,0)\}. \]

Then Chen’s proves that \( \mathcal{M}^*(\mathbb{F}_q) \) consists of a single \( \mathcal{R} \)-orbit.

We consider the Cayley graph associated to \( \mathcal{M}^*(\mathbb{F}_q) \) and \( \{\rho_1, \rho_2, \rho_3\} \), i.e., we form an undirected graph \( \overline{\mathcal{M}}(\mathbb{F}_q) \) whose vertices and edges are given by

\[ \text{Vertices}\left(\overline{\mathcal{M}}(\mathbb{F}_q)\right) = \mathcal{M}^*(\mathbb{F}_q), \]
\[ \text{Edges}\left(\overline{\mathcal{M}}(\mathbb{F}_q)\right) = \{[P, \rho_i(P)] : P \in \mathcal{M}^*(\mathbb{F}_q), \ i = 1, 2, 3\}. \]

It is conjectured in \([2,3]\) that \( \overline{\mathcal{M}}(\mathbb{F}_q) \) is a family of expander graphs; see also \([7]\).

Charles, Goren, and Lauter \([5]\) have explained how one can build cryptographic hash functions from expander graphs provided that it is hard to find paths in the graph connecting two given vertices. This led Fuchs, Lauter, Litman, and Tran \([8]\) to suggest using the Markoff graph \( \overline{\mathcal{M}}(\mathbb{F}_q) \) to construct a hash function. They prove, using the connectivity ideas from \([2,3]\), that there is a path-finding algorithm for \( \overline{\mathcal{M}}(\mathbb{F}_q) \) that runs in deterministic time \( O(q \log \log q) \), and they speculate that any path-finding algorithm in \( \overline{\mathcal{M}}(\mathbb{F}_q) \) must take time at least \( O(q) \). This leads them to suggest that “these graphs may be good candidates” for the CGL hash function construction.

Our goal in this note is to show that under some reasonable heuristic assumptions, it is possible to solve the path-finding problem in \( \overline{\mathcal{M}}(\mathbb{F}_q) \) in subexponential time on a classical computer and polynomial time on a quantum computer. More precisely, up to small polynomial-time tasks, it suffices to factor \( q - 1 \) and solve three discrete logarithm problems in \( \mathbb{F}_q^* \), as described in the following theorem.

**Theorem 1 (Markoff Path-Finding Algorithm).** We set the following notation:

\[ \text{PATH}(q) = \text{time to find a path between points in } \overline{\mathcal{M}}(\mathbb{F}_q). \]
\[ \text{DLP}(q) = \text{time to solve the DLP in } \mathbb{F}_q^*. \]
\[ \text{FACTOR}(N) = \text{time to factor } N. \]

Assume that Heuristics 2 and 3 are valid. Then with high probability,

\[ \text{PATH}(q) \leq \text{FACTOR}(q - 1) + 3 \text{DLP}(q) + C_1(\log q)^{C_2} \]

for some small constant \( C_1 \) and some small exponent \( C_2 \).

Theorem 1 is proven in Section 6.

**Remark 1.** We note that the paths constructed by our Markoff path-finder algorithm (Algorithm 1) have the following two properties that are unlikely to be present in the paths generated by the CGL graph-theoretic hash function \([5]\):

- They are quite long, in the sense that the number of \( \rho_i \) used to connect \( P \) to \( Q \) is almost certainly larger than, say, \( q^{1/2} \). 

• The path connecting $P$ to $Q$ has long stretches in which it repeatedly applies on of the $\rho_i$, e.g., it is almost certainly true that somewhere in the path there is a $\rho_k$ that is repeated at least $q^{1/2}$ times without using the other two $\rho_i$.

Thus one might still consider using the Markoff graph for a CGL hash function with the proviso that long or repetitive paths are disallowed. On the other hand, the fact that one can create paths and collisions, even of a disallowed type, may cause some disquiet, as well as making it more difficult to construct a security reduction proof.

Remark 2. We have not tried to find explicit constants in Theorem 1, but since the operations that go into that term involve things such as taking square roots in $\mathbb{F}_q$ and exponentiating elements of $\mathbb{F}_q$, taking $C_1 = 1000$ and $C_2 = 3$ should suffice, and indeed, should be a significant overestimate of the time required.

Remark 3. In this article we have restricted attention to the classical Markoff equation (1), but we note that the method works, mutatis mutandis, for more general Markoff–Hurwitz type equations

$$a_1X^2 + a_2Y^2 + a_3Z^2 + b_1XY + b_2XZ + b_3YZ + c_1X + c_2Y + c_3Z + dXYZ + e = 0$$

(2)

that admit three non-commuting involutions.

We give an initial high-level description of the Markoff path-finder algorithm in Section 2 using pseudo-code (Table 1) and a picture (Table 2). A more detailed description that includes the path-finder algorithm (Algorithm 1) and its subroutines is given in Section B. The proof that the Markoff path-finder algorithm finds a path and has the indicated running time is given in Section 6. A key observation in constructing the path-finder algorithm, as already exploited in [2, 3], is to note that the action of the $\rho_i$ on appropriate fibers of $\mathcal{M} \to \mathbb{A}^1$ is described via repeated application of a linear transformation in $\text{SL}_2$. (In fancier terminology, the fibers are $\mathbb{G}_m$-torsors.) This means that if we are given two points on a fiber, then finding a power of $\rho_i$ that links the given points can be rephrased as a discrete logarithm problem, either in $\mathbb{F}_q^*$ or in the subgroup of norm 1 elements of $\mathbb{F}_q^2$.

The heuristic part of our algorithm comes from assuming that if we take a random point in $\mathcal{M}(\mathbb{F}_q)$ or on a related curve, and if we take one of the coordinates $t \in \mathbb{F}_q$ of that point, then

$$\text{Prob}(T^2 - 3tT + 1 \text{ has a root } \lambda \in \mathbb{F}_q^* \text{ that generates } \mathbb{F}_q^*) \approx \frac{1}{2} \cdot \frac{\varphi(q - 1)}{q - 1}.$$  \hspace{1cm} (3)

The factor of $\frac{1}{2}$ in (3) comes from the probability that a random quadratic polynomial has its roots $\mathbb{F}_q$, and the factor of $\frac{\varphi(q - 1)}{q - 1}$ comes from the well-known fact that there are $\varphi(q - 1)$ generators (primitive roots) in $\mathbb{F}_q^*$. We describe the heuristic assumptions more precisely in Section 5, and we do some numerical experiments to test the assumptions in Section A.
We illustrate the Markoff path-finding algorithm in Section 7 by executing it on a numerical example with \( q = 70687 \). We find paths between some randomly chosen points in \( \mathcal{M}(\mathbb{F}_q) \), and a non-trivial loop from a point back to itself. Finally, in Section 8 we briefly discuss a family of K3 surfaces that is analogous to the Markoff surface (1) and its generalizations (2) and explain how path-finding on the associated graphs can be heuristically reduced to the elliptic curve discrete logarithm problem (ECDLP).

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## 2 A High-Level Description of the Markoff Path-Finding Algorithm

For the convenience of the reader, Table 1 gives an informal description of our heuristically subexponential algorithm for finding paths in \( \mathcal{M}(\mathbb{F}_q) \). In this algorithm, we say that an element \( t \in \mathbb{F}_q^* \) is maximally elliptic if the quadratic polynomial \( T^2 - 3T + 1 \) has a root \( \lambda \in \mathbb{F}_q \) that is a primitive root, i.e., is a generator for \( \mathbb{F}_q^* \): cf. Definition 3. The algorithm is also illustrated by the picture in Table 2. We refer the reader to Section B for a more detailed description of the Markoff path-finding algorithm, and to Section 6 for a proof that the Markoff path-finding algorithm operates successfully in subexponential time.

## 3 Rotations on a Fiber and an Associated Matrix

The map \( \rho_1 \) may be written in matrix form as

\[
\rho_1(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3x - 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

(4)

Thus computing \( \rho_1^n(x, y, z) \) amounts to taking the \( n \)th power of the matrix \( \begin{pmatrix} 3x - 1 & 0 \\ 1 & 0 \end{pmatrix} \). Similar considerations apply to \( \rho_2 \) and \( \rho_3 \). This prompts the following definitions.

**Definition 1.** For \( t \in \mathbb{F}_q^* \), we set the following notation:

\[
L_t = \begin{pmatrix} 3t - 1 & 0 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{F}_q),
\]

\( \lambda_t, \lambda_t^{-1} = \text{the eigenvalues of } L_t. \)

Formula (4) tells us that if we apply iterates of \( \rho_1 \) to a point \( (x, y, z) \in \mathcal{M}^*(\mathbb{F}_q) \), then

\[
\rho_1^n(x, y, z) = (x, y_n, z_n) \quad \text{with} \quad \begin{pmatrix} y_n \\ z_n \end{pmatrix} = L_x^n \begin{pmatrix} y \\ z \end{pmatrix},
\]

(5)

and similarly for \( \rho_2 \) and \( \rho_3 \). This often allows us to find paths in fibers of \( \mathcal{M}(\mathbb{F}_q) \) by solving a classical DLP in \( \mathbb{F}_q^* \), as described in the following proposition.
• **Input:** $\mathbb{F}_q$ a finite field of characteristic at least 5

$P,Q$ points in $\mathcal{M}^*(\mathbb{F}_q)$

- Use $\rho_1$ and $\rho_3$ to randomly move $P$ in $\mathcal{M}(\mathbb{F}_q)$ until reaching a point $P'$ satisfying $y(P')$ is maximally elliptic. This gives $i_1, \ldots, i_\alpha \in \{1, 3\}$ such that

  $$P' = \rho_{i_\alpha} \circ \cdots \circ \rho_{i_1}(P).$$

- Use $\rho_1^{-1}$ and $\rho_3^{-1}$ to randomly move $Q$ in $\mathcal{M}(\mathbb{F}_q)$ until reaching a point $Q'$ satisfying $z(Q')$ is maximally elliptic. This gives $j_1, \ldots, j_\beta \in \{1, 2\}$ such that

  $$Q = \rho_{j_\beta} \circ \cdots \circ \rho_{j_1}(Q').$$

- Let $F(X,Y,Z) = X^2 + Y^2 + Z^2 - 3XYZ$. Randomly select maximally elliptic $x_0 \in \mathbb{F}_q^*$ until finding a value for which the quadratic equations

  $$F(x_0, y(P'), Z) = F(x_0, Y, z(Q')) = 0$$

  have a solution $(y_0, z_0) \in \mathbb{F}_q^2$. Set

  $$P'' \leftarrow (x_0, y(P'), z_0) \text{ and } Q'' \leftarrow (x_0, y_0, z(Q')).$$

  We note that:

  - $P''$ and $Q''$ are on the same maximally elliptic $x$-fiber,
  - $P'$ and $P''$ are on the same maximally elliptic $y$-fiber,
  - $Q'$ and $Q''$ are on the same maximally elliptic $z$-fiber.

- Find $a, b, c$ satisfying

  $$P'' = \rho_2^a(P'), \quad Q' = \rho_3^b(Q''), \quad Q'' = \rho_1^c(P').$$

  As explained in Proposition 1, this involves solving three DLPs in $\mathbb{F}_q^*$.

• **Output:** The list of integers $(i_1, \ldots, i_\alpha)$, $(j_1, \ldots, j_\beta)$, $(a, b, c)$ specifies the path

  $$Q = \rho_{j_1} \circ \cdots \circ \rho_{j_\beta} \circ \rho_{j_\beta}^b \circ \rho_{j_1}^a \circ \rho_{i_\alpha} \circ \cdots \circ \rho_{i_1}(P).$$

Table 1: High-level description of the Markoff path-finding algorithm
Proposition 1. Fix an $x \in \mathbb{F}_q^*$ such that an eigenvalue $\lambda_x$ of $L_x$ satisfies $\lambda_x \in \mathbb{F}_q^*$ and $\lambda_x^2 \neq 1$. Let

$$P = (x, y, z) \in M^*(\mathbb{F}_q) \quad \text{and} \quad P' = (x, y', z') \in M^*(\mathbb{F}_q)$$

be points on the $x$-fiber of $M$ such that there exists an $n \geq 0$ such that

$$P' = \rho_1^n(P).$$

Then one can compute $n$ by solving a quadratic equation in $\mathbb{F}_q$ and then solving a DLP in the group $\mathbb{F}_q^*$. (See Algorithm 2 in Table 10 for an explicit algorithm.)

Proof. The assumption that the eigenvalues $\lambda_x, \lambda_x^{-1}$ are in $\mathbb{F}_q^* \setminus \{\pm 1\}$ means that we can diagonalize $L_x$ working over $\mathbb{F}_q$. Explicitly,

$$U = \begin{pmatrix} 1 & -\lambda_x^{-1} \\ -1 & \lambda_x \end{pmatrix} \in \text{GL}_2(\mathbb{F}_q) \quad \text{satisfies} \quad U L_x U^{-1} = \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_x^{-1} \end{pmatrix}. \quad (6)$$

(Note that $U$ is invertible, since $\det(U) = \lambda_x^{-1}(\lambda_x^2 - 1)$. It follows that

$$P' = \rho_1^n(P) \iff \begin{pmatrix} y' \\ z' \end{pmatrix} = L_x^n \begin{pmatrix} y \\ z \end{pmatrix} \quad \text{from (5)},$$

$$\iff \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \lambda_x^n & 0 \\ 0 & \lambda_x^{-n} \end{pmatrix} U \begin{pmatrix} y \\ z \end{pmatrix} \quad \text{from (6)}.$$

Using the explicit value of $U$, we see that it suffices to find a value of $n$ that satisfies

$$\lambda_x^n = (y' - \lambda_x^{-1} z') \cdot (y - \lambda_x^{-1} z)^{-1},$$

where $\lambda_x, y, z, y', z' \in \mathbb{F}_q$ are known quantities. Thus it suffices to solve a DLP in $\mathbb{F}_q^*$.
Definition 2. In [2,3,8], the various \( L_t \) are separated into three cases, analogous to the classification of elements of \( \text{SL}_2(\mathbb{R}) \). We say that \( t \in \mathbb{F}_q^* \) is

- elliptic: if \( \lambda_t \in \mathbb{F}_q \setminus \{ \pm 1 \} \),
- parabolic: if \( \lambda_t = \pm 1 \),
- hyperbolic: if \( \lambda_t \in \mathbb{F}_q^* \setminus \mathbb{F}_q^2 \).

Remark 4. The characteristic polynomial of \( L_t \) is \( T^2 - 3tT + 1 \), whose discriminant is \( 9t^2 - 4 \), so we see that

- \( L_t \) is elliptic \( \iff \) \( 9t^2 - 4 \not\in \mathbb{F}_q^2 \),
- \( L_t \) is parabolic \( \iff \) \( 9t^2 - 4 = 0 \),
- \( L_t \) is hyperbolic \( \iff \) \( 9t^2 - 4 / \in \mathbb{F}_q^2 \).

Hence if \( q \) is large, then a randomly chosen \( t \in \mathbb{F}_q \) has a roughly 50% chance of being elliptic, a 50% chance of being hyperbolic, and negligible chance of being parabolic.

Definition 3. We say that \( t \in \mathbb{F}_q^* \) is maximally elliptic if any one of the following equivalent conditions is true:

- \( L_t \) is elliptic and has order \( q - 1 \) in \( \text{SL}_2(\mathbb{F}_q) \)
- An eigenvalue \( \lambda_t \) of \( L_t \) is in \( \mathbb{F}_q \) and generates the multiplicative group \( \mathbb{F}_q^* \)
- There is a generator \( \lambda \in \mathbb{F}_q^* \) such that \( t = \frac{1}{3}(\lambda + \lambda^{-1}) \).

Proposition 2. We have

\[
\# \{ t \in \mathbb{F}_q^* : t \text{ is maximally elliptic} \} = \frac{\varphi(q - 1)}{2},
\]

\[
\text{Prob}(t \text{ is maximally elliptic} \mid t \in \mathbb{F}_q^*) = \frac{1}{2} \cdot \frac{\varphi(q - 1)}{q - 1}.
\]

Proof. Let \( \text{Gen}(q) \subset \mathbb{F}_q^* \) denote the set of generators of \( \mathbb{F}_q^* \), and consider the map

\[
f : \mathbb{F}_q^* \longrightarrow \mathbb{F}_q, \quad \lambda \longrightarrow \frac{1}{3}(\lambda + \lambda^{-1}).
\]

We want to count \( \#f(\text{Gen}(q)) \). The map \( f \) is exactly 2-to-1 onto its image, since \( f(\lambda) = f(\lambda^{-1}) \), except for the two points \( f(\pm 1) = \pm \frac{2}{3} \) that have only one pre-image. The set of generators of \( \mathbb{F}_q^* \) is invariant under inversion and does not contain \( \pm 1 \), so

\[
\#f(\text{Gen}(q)) = \frac{1}{2} \# \text{Gen}(q).
\]

The group \( \mathbb{F}_q^* \) is cyclic of order \( q - 1 \), so \( \# \text{Gen}(q) = \phi(q - 1) \), which gives (7), and (8) is an immediate corollary.
Remark 5. We have focused on points of \( \mathcal{M}(\mathbb{F}_q) \) that have a coordinate that is maximally elliptic for reasons of computational and expositional simplicity. But we note that we could in also use points with a maximally hyperbolic coordinate, where \( t \in \mathbb{F}_q^* \) is said to be \textit{maximally hyperbolic} if \( L_t \) has order \( q + 1 \) in \( \text{SL}_2(\mathbb{F}_q) \), or equivalently if the roots of the polynomial \( T^2 - 3tT + 1 \) generate the subgroup of \( \mathbb{F}_q^* \) of index \( q - 1 \). Checking for maximal hyperbolicity requires a factorization of \( q + 1 \), which could be an advantage if \( q + 1 \) is easier than \( q - 1 \) to factor. And using maximal hyperbolic points would more-or-less double the probability of finding a point having a fiber on which the associated rotation acts transitively.

On the other hand, we would then need to solve the DLP in \( \mathbb{F}_q^* \), which is more difficult than in \( \mathbb{F}_q \), although still subexponential. More precisely, we would need to solve the DLP in the subgroup of \( \mathbb{F}_q^* \) of order \( q + 1 \). In any case, the algorithm still requires solving three DLPs.

### 4 Checking If \( t \in \mathbb{F}_q^* \) Is Maximally Elliptic

We analyze the running time of Algorithm 3 in Table 11, which checks whether a given \( t \in \mathbb{F}_q^* \) is maximally elliptic, i.e., whether the matrix \( L_t = \begin{pmatrix} 3t & -1 \\ 1 & 0 \end{pmatrix} \) has order \( q - 1 \) in \( \text{SL}_2(\mathbb{F}_q) \). Algorithm 3 is then invoked in Steps 5, 13, and 22 of the Markoff path-finder algorithm (Algorithm 1 in Table 9).

The first step in Algorithm 3 is to find a root \( \lambda \in \mathbb{F}_q \) of the quadratic equation

\[
T^2 - 3tT + 1 = 0,
\]

or to show that there are no roots. Quadratic reciprocity provides a very fast way to check if there is a root, and there are practical polynomial-time algorithms for taking square roots modulo primes.

Assuming that the equation (9) has a root \( \lambda \in \mathbb{F}_q \), it remains to check whether \( \lambda \) generates \( \mathbb{F}_q^* \), i.e., whether \( \lambda \) is a primitive root. The most straightforward way to check this is to first factor \( q - 1 \),

\[
q - 1 = \prod_{i=1}^{r} p_i^{e_i},
\]

which need only be done once, and then use the elementary fact:

\[
\lambda \text{ is a primitive root } \iff \lambda^{(q-1)/p_i} \neq 1 \text{ for all } 1 \leq i \leq r.
\]

Hence once \( q - 1 \) has been factored, we have:

\[
\left( \text{time to check if } t \in \mathbb{F}_q \text{ is maximally elliptic} \right) = \left( \text{time to compute a square root in } \mathbb{F}_q \right) + \left( \text{time to compute } r \text{ exponentiations in } \mathbb{F}_q \right).
\]

Since taking square roots and doing exponentiations take practical polynomial time, and since \( r < \log_2(q) \), the time to check if an element of \( \mathbb{F}_q^* \) is maximally elliptic is negligible.
Remark 6. We note that the factorization of \( q - 1 \) is used to make it easy to check if an element of \( \mathbb{F}_q^* \) is a primitive root. However, rather than completely factoring \( q - 1 \), we could instead use Lenstra’s elliptic curve factorization algorithm to find all moderately small prime factors. This is very efficient, since the running time for Lenstra’s algorithm to factor an integer \( N \) depends on the size of the smallest prime factor of \( N \). We can then use that to create a probabilistic primitive root algorithm that has a high success rate. Thus for example, if \( q \approx 2^{4000} \) and we use Lenstra’s algorithm to find all primes \( p < 2^{100} \) that divide \( q - 1 \), we can consider the algorithm

\[
\lambda \text{ is probably a primitive root } \iff \lambda^{(q-1)/p} \neq 1 \text{ for all } p \mid q - 1, \ p < 2^{100}.
\]  

(10)

The probability that (10) misidentifies an element of \( \mathbb{F}_q^* \) as a primitive root when \( q \approx 2^{4000} \) is less than

\[
1 - \prod_{p \mid q-1, \ p > 2^{100}} \left( 1 - \frac{1}{p} \right) < 1 - \left( 1 - \frac{1}{2^{100}} \right)^{40} \approx \frac{1}{2^{94}},
\]

so the probability is negligible. And even if (10) returns a false positive, the path-finding algorithm can simply restart. Finally, we note the since factor ing \( q - 1 \) and solving the DLP in \( \mathbb{F}_q^* \) are of roughly the same order of difficulty using the best known algorithms, the saving in only partially factoring \( q - 1 \) is minimal at best.

5 Heuristic Assumptions

In this section we describe the heuristic assumptions that we will use in our Markoff path-finding algorithm. They say roughly that if we choose a \( t \in \mathbb{F}_q^* \) that is the coordinate of a random point in \( \mathcal{M}(\mathbb{F}_q) \) or a random point on a certain curve, then the probability that \( t \) is maximally elliptic is roughly the same as if \( t \) were chosen randomly in \( \mathbb{F}_q^* \). We refer the reader to Section A for data that supports Heuristics 2 and 3.

Heuristic Assumption 2 Let \( (x_0, y_0, z_0) \in \mathcal{M}(\mathbb{F}_q) \). For \( n \geq 0 \), randomly choose \( i_1, i_2, \ldots, i_n \in \{1, 3\} \) and set

\[
(x_n, y_n, z_n) \leftarrow \rho_{i_n} \circ \rho_{i_{n-1}} \circ \cdots \circ \rho_{i_2} \circ \rho_{i_1}(x_0, y_0, z_0).
\]

Then

\[
\text{Prob}(y_n \text{ is maximally elliptic}) \approx \frac{1}{2} \frac{\varphi(q-1)}{q-1}.
\]  

(11)

Proof (Justification). As we randomly use \( \rho_1 \) and \( \rho_3 \) to “rotate” on the \( x \)-fiber and the \( z \)-fiber, it is reasonable to view \( y_0, y_1, y_2, \ldots \) as being independent random elements of \( \mathbb{F}_q^* \), at least insofar as to whether they are maximally elliptic, which recall means that an associated quadratic polynomial has a root that generates \( \mathbb{F}_q^* \). Proposition 2 says that a random element of \( \mathbb{F}_q^* \) has probability \( \varphi(q-1)/2(q-1) \) of being maximally elliptic, which gives the desired justification.
Heuristic Assumption 3 Let
\[ F(X, Y, Z) = X^2 + Y^2 + Z^2 - 3XYZ, \]
\[ \text{let } a, b \in \mathbb{F}_q^*, \text{ and let } C_{a,b} \subset \mathbb{A}^3 \text{ be the affine curve} \]
\[ C_{a,b} : F(X, a, Z) = F(X, Y, b) = 0. \]

Then
\[ \operatorname{Prob}_{t \in \mathbb{F}_q^*} \left( t \text{ is maximally elliptic and is the } x\text{-coordinate of a point in } C_{a,b}(\mathbb{F}_q) \right) \approx \frac{1}{8} \cdot \frac{\varphi(q-1)}{q}. \] (12)

Proof (Justification). We note that \( t \in \mathbb{F}_q^* \) is the \( x \)-coordinate of a point in \( C_{a,b}(\mathbb{F}_q) \) if and only if the quadratic equations
\[ F(t, a, Z) = 0 \quad \text{and} \quad F(t, Y, b) = 0 \]
have roots in \( \mathbb{F}_q \), so if and only if
\[ 9a^2t^2 - 4(t^2 + a^2) \in \mathbb{F}_q^2 \quad \text{and} \quad 9b^2t^2 - 4(t^2 + b^2) \in \mathbb{F}_q^2. \]

Hence we expect that
\[ \operatorname{Prob}_{t \in \mathbb{F}_q^*} \left( t \text{ is the } x\text{-coordinate of a point in } C_{a,b}(\mathbb{F}_q) \right) \approx \frac{1}{4}, \]
since about half the elements of \( \mathbb{F}_q^* \) are squares. (If \( a^2 = b^2 \), the probability increases to \( \frac{1}{2} \), which helps the attacker.)

It is reasonable to view the \( x \)-coordinates of the points in \( C_{a,b}(\mathbb{F}_q) \) as independent random elements of \( \mathbb{F}_q \). Proposition 2 says that a random element of \( \mathbb{F}_q^* \) has probability \( \frac{\varphi(q-1)}{2(q-1)} \) of being maximally elliptic, so multiplying this by the probability \( \frac{1}{4} \) that \( t \) is the \( x \)-coordinate of a point in \( C_{a,b}(\mathbb{F}_q) \) yields (12). We also note that Weil’s estimate says that
\[ \#C_{a,b}(\mathbb{F}_q) = q + O(\sqrt{q}). \]

so when \( q \) is large, the set \( C_{a,b}(\mathbb{F}_q) \) is also large.

6 The Markoff Path-Finder Algorithm

We give the proof of our main result (Theorem 1), which we restate for the convenience of the reader

Theorem 4 (Theorem 1). We set the following notation:
\[ \operatorname{PATH}(q) = \text{time to find a path between points in } \overline{M}(\mathbb{F}_q), \]
\[ \operatorname{DLP}(q) = \text{time to solve the DLP in } \mathbb{F}_q^*. \]
\[ \operatorname{FACTOR}(N) = \text{time to factor } N. \]
Assume that Heuristics 2 and 3 are valid. Then with high probability, the Markoff path-finder Algorithm described in detail as Algorithm 1 in Table 9 will find a path between randomly given points in the graph $\overline{M}(\mathbb{F}_q)$ in time

$$\text{PATH}(q) \leq \text{FACTOR}(q - 1) + 3\text{DLP}(q) + C_1(\log q)^{C_2} \quad (13)$$

for some small constant $C_1$ and some small exponent $C_2$.

**Proof.** The Markoff path-finder algorithm (Algorithm 1 in Table 9) terminates with a list of positive integers $(i_1, \ldots, i_{\alpha}), (j_1, \ldots, j_{\beta}), (a, b, c)$ satisfying

- $P' = \rho_{i_\alpha} \circ \rho_{i_{\alpha-1}} \circ \cdots \circ \rho_{i_2} \circ \rho_{i_1}(P)$, Steps 3–10
- $Q = \rho_{j_1} \circ \rho_{j_2} \circ \cdots \circ \rho_{j_{\beta-1}} \circ \rho_{j_\beta}(Q')$, Steps 11–18
- $P'' = \rho_{a}^2(P'), Q' = \rho_{b}^3(Q''), Q'' = \rho_{c}^1(P'')$, Steps 26–30

We use these to compute

$$Q = \rho_{j_1} \circ \cdots \circ \rho_{j_{\beta}}(Q')$$
$$= \rho_{j_1} \circ \cdots \circ \rho_{j_{\beta}} \circ \rho_{b}^3(Q'')$$
$$= \rho_{j_1} \circ \cdots \circ \rho_{j_{\beta}} \circ \rho_{b}^3 \circ \rho_{c}^1(P'')$$
$$= \rho_{j_1} \circ \cdots \circ \rho_{j_{\beta}} \circ \rho_{b}^3 \circ \rho_{c}^1 \circ \rho_{a}^2 \circ \rho_{i_{\alpha}} \circ \cdots \circ \rho_{i_1}(P).$$

Hence Algorithm 1 gives the a path in $\overline{M}(\mathbb{F}_q)$ from $P$ to $Q$.

We next consider the running time of each step of the algorithm. In Step 2 we factor the integer $q - 1$. Once this is done, the time to check whether an element $t \in \mathbb{F}_q$ is maximally elliptic is negligible; see Section 4.

In Steps 3–10 and Steps 11–18, we randomly move a point around $\overline{M}(\mathbb{F}_q)$ and check whether one of its coordinates is maximally elliptic. Heuristic 2 says that each of these loops needs to look at an average of $2(q - 1)/\varphi(q - 1)$ points before terminating, and as already noted, checking maximal ellipticity takes negligible time once we have factored $q - 1$. Similarly, Heuristic 3 says that the loop in Steps 22 is executed an average of $8(q - 1)/\varphi(q - 1)$ times, with the maximal ellipticity and the square root computations taking negligible time. Hence Steps 3–25 take average time $12(q - 1)/\varphi(q - 1)$ multiplied by some small power of $\log(q)$. Finally, we use the classical estimate [10, Sections 18.4 and 22.9]

$$\frac{\varphi(N)}{N} \geq \frac{C_3}{\log \log N}$$

(where $C_3$ is not particularly small) to conclude that Steps 3–25 take a negligible amount of time.
Steps 26–30 use the MarkoffDLP algorithm three times, and the MarkoffDLP algorithm (Algorithm 2 in Table 10) requires taking a square root in $\mathbb{F}_q^*$ (negligible time) and computing a discrete logarithm in $\mathbb{F}_q^*$. Hence the time to execute Steps 26–30 is essentially the time it takes to compute three DLPs in $\mathbb{F}_q^*$.

Adding these time estimates yields (13), which completes the proof that the Markoff path-finding algorithm terminates in the specified time.

7 The Markoff Path Finder Algorithm in Action: An Example

We illustrate the Markoff path-finder algorithm (Algorithm 1 in Table 9) by computing a numerical example. We take $q = 70687$, $q - 1 = 2 \cdot 3^3 \cdot 7 \cdot 11 \cdot 17$, $P = (45506, 13064, 18) \in M(\mathbb{F}_q)$, $Q = (11229, 5772, 56858) \in M(\mathbb{F}_q)$.

We use a simplified version of the algorithm in which $i_\alpha = 1$ for all $\alpha$ and $j_\beta = 1$ for all $\beta$, since in practice this almost always works. Thus Steps 3–10 say to apply $\rho_1$ to $P$ until the $y$-coordinate is maximally elliptic. We do a similar computation in Steps 11–18, except now we apply iterates of $\rho_1^{-1}$ to $Q$ and stop when we reach an iterate whose $z$-coordinate is maximally elliptic. Tables 3 and 4 show our computations. They list the iterates and indicate whether the appropriate coordinate is elliptic or hyperbolic, and if it is elliptic, it lists $o(\lambda)$, the order of an eigenvalue of the matrix in $\mathbb{F}_q$. We find that $y(\rho_1^2(P))$ and $z(\rho_1^{-15}(Q))$ are maximally elliptic, so the output from Steps 3–18 are

$$\alpha = 2, \quad i_1 = i_2 = 1, \quad P' = \rho_1^2(P) = (45506, 40902, 10340),$$

$$\beta = 15, \quad j_1 = \cdots = j_{15} = 1, \quad Q' = \rho_1^{-15}(Q) = (11229, 2424, 19535).$$

| $i$ | $P_i = (45506, 13064, 18)$ | $y_i = 13064$ elliptic, $o(\lambda) = 1683$ |
|-----|--------------------------|---------------------------------|
| $i = 1$ | $\rho_1(P) = (45506, 18, 40902)$ | $y = 18$ elliptic, $o(\lambda) = 4158$ |
| $i = 2$ | $\rho_1^2(P) = (45506, 40902, 10340)$ | $y = 40902$ elliptic, $o(\lambda) = 70686$ |

Table 3: $\rho_1$ iterates of $P$ until reaching a maximally elliptic $y$-fiber

In Steps 19–25 we randomly choose $x \in \mathbb{F}_q^*$ and check whether $x$ is maximally elliptic and whether the quadratic equations

$$F(x, 40902, Z) = 0 \quad \text{and} \quad F(x, Y, 19535) = 0$$

have a solutions $y, z \in \mathbb{F}_q$. It took 5 tries, as listed in Table 5. So the output from Steps 19–25 consists of the two points

$$P'' = (29896, 40902, 935) \quad \text{and} \quad Q'' = (29896, 595, 19535).$$

| $j$ | $Q = (11229, 5772, 56858)$ | $z = 56858$ | hyperbolic |
|-----|--------------------------|-------------|------------|
| 0   | $\rho_1^{-1}(Q) = (11229, 65943, 5772)$ | $z = 5772$ | elliptic, $\alpha(\lambda) = 35343$ |
| 1   | $\rho_1^{-1}(Q) = (11229, 6407, 65943)$ | $z = 65943$ | elliptic, $\alpha(\lambda) = 10098$ |
| 2   | $\rho_1^{-1}(Q) = (11229, 29942, 6407)$ | $z = 6407$ | hyperbolic |
| 3   | $\rho_1^{-1}(Q) = (11229, 16944, 29942)$ | $z = 29942$ | hyperbolic |
| 4   | $\rho_1^{-1}(Q) = (11229, 35748, 16944)$ | $z = 16944$ | hyperbolic |
| 5   | $\rho_1^{-1}(Q) = (11229, 23200, 35748)$ | $z = 35748$ | hyperbolic |
| 6   | $\rho_1^{-1}(Q) = (11229, 66363, 23200)$ | $z = 23200$ | hyperbolic |
| 7   | $\rho_1^{-1}(Q) = (11229, 21119, 66363)$ | $z = 66363$ | hyperbolic |
| 8   | $\rho_1^{-1}(Q) = (11229, 46109, 21119)$ | $z = 21119$ | hyperbolic |
| 9   | $\rho_1^{-1}(Q) = (11229, 47313, 46109)$ | $z = 46109$ | elliptic, $\alpha(\lambda) = 594$ |
| 10  | $\rho_1^{-1}(Q) = (11229, 7133, 47313)$ | $z = 47313$ | hyperbolic |
| 11  | $\rho_1^{-1}(Q) = (11229, 47632, 7133)$ | $z = 7133$ | elliptic, $\alpha(\lambda) = 5049$ |
| 12  | $\rho_1^{-1}(Q) = (11229, 47838, 47632)$ | $z = 47632$ | elliptic, $\alpha(\lambda) = 10098$ |
| 13  | $\rho_1^{-1}(Q) = (11229, 19535, 47838)$ | $z = 47838$ | elliptic, $\alpha(\lambda) = 7854$ |
| 14  | $\rho_1^{-1}(Q) = (11229, 2424, 19535)$ | $z = 19535$ | elliptic, $\alpha(\lambda) = 70686$ |

Table 4: $\rho_1^{-1}$ iterates of $Q$ until reaching a maximally elliptic $z$-fiber

| $x$ | $F(x, 40902, Z)$ | $F(x, Y, 19535)$ |
|-----|----------------|----------------|
| 29628 | irreducible | irreducible |
| 19562 | irreducible | $(Y - 42621)(Y - 57310)$ |
| 43036 | irreducible | irreducible |
| 6057 | $(Z - 27506)(Z - 70305)$ | irreducible |
| 29896 | $(Z - 935)(Z - 45089)$ | $(Y - 595)(Y - 6503)$ |

Table 5: Finding a point on $F(x, 40902, Z) = F(x, Y, 19535) = 0$

In Steps 26–30 we find a path on the $y$-fiber from $P'$ to $P''$, a path on the $z$-fiber from $Q''$ to $P'$, and a path on the $x$-fiber from $Q'$ to $Q''$. This is done using the Markoff DLP Algorithm (Algorithm 2 in Table 10), which uses Proposition 1 to convert the path problem in a maximal elliptic fiber into a classical discrete logarithm problem in $\mathbb{F}_q$. Implementing this algorithm, we find that

$$P'' = \rho_2^{26986}(P'), \quad Q' = \rho_3^{30287}(Q''), \quad Q'' = \rho_1^{65193}(P'').$$

Finally, the algorithm outputs

$$(1, 1), (1, 1, \ldots, 1), (26986, 30287, 65193),$$
where the second item is a 15-tuple. We check that this gives a path from $P$ to $Q$ by computing

$$P = (45506, 13064, 18)$$

$$\rho_1^2(P) = (45506, 40902, 10340)$$

$$\rho_2^{26986} \circ \rho_1^2(P) = (29896, 40902, 935)$$

$$\rho_1^{30287} \circ \rho_2^{26986} \circ \rho_1^2(P) = (29896, 595, 19535)$$

$$\rho_3^{65193} \circ \rho_1^{30287} \circ \rho_2^{26986} \circ \rho_1^2(P) = (11229, 2424, 19535)$$

$$\rho_1^{15} \circ \rho_3^{65193} \circ \rho_1^{30287} \circ \rho_2^{26986} \circ \rho_1^2(P) = (11229, 5772, 56858) = Q.$$

If we run the algorithm a second time, the randomness in Steps 19–25 means that we are likely to obtain a different path. (And if we hadn’t simplified the choices of the $i_\alpha$ and $j_\beta$, that randomness would also lead to different paths.) For example, using the same $(q, P, Q)$ as input and running the algorithm again, we obtained the output

$$(a, c, b) = (26703, 52102, 29583),$$

which gives the path

$$Q = \rho_1^{15} \circ \rho_3^{29583} \circ \rho_1^{52102} \circ \rho_2^{26703} \circ \rho_1^2(P).$$

We also note that we can combine a path from $P$ to $Q$ with a path from $Q$ to $P$ to find a non-trivial loop that starts and returns to $P$, since it is unlikely that the two paths will be exact inverses of one another. Indeed, running the algorithm to find a path from $Q$ to $P$, we found

$$(1, 1, 1), (1, 1, \ldots, 1), (389, 14491, 39906),$$

which gives the path

$$P = \rho_1^{11} \circ \rho_3^{39906} \circ \rho_1^{14491} \circ \rho_2^{389} \circ \rho_1^3(Q).$$

Combining this with the first path from $P$ to $Q$ that we found earlier gives the loop

$$P = \rho_1^{11} \circ \rho_3^{39906} \circ \rho_1^{14491} \circ \rho_2^{389} \circ \rho_1^{18} \circ \rho_3^{65193} \circ \rho_1^{30287} \circ \rho_2^{26986} \circ \rho_1^2(P)$$

where we have combined the middle $\rho_1^3 \circ \rho_1^{15}$ into a single $\rho_1^{15}$. 

### 8 Markoff-Type K3 Surfaces and the ECDLP

In this section we briefly discuss K3 surfaces that are analogous to the Markoff surface. These surfaces, which were dubbed tri-involutive K3 (TIK3) surfaces in [9], are surfaces

$$W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$
given by the vanishing of a \((2, 2, 2)\) form. With appropriate non-deneracy conditions, the three double covers \(W \to \mathbb{P}^1 \times \mathbb{P}^1\) give three non-commuting involutions \(\sigma_1, \sigma_2, \sigma_3 \in \text{Aut}(W)\). If the \((2, 2, 2)\)-form is symmetric, then \(W\) also admits coordinate permutation automorphisms, in which case we can define the analogues of the rotations \(\rho_1, \rho_2, \rho_3 \in \text{Aut}(W)\), and \(W\) resembles even more closely the Markoff surface. Fuchs, Litman, Tran, and the present author studied the orbit structure of \(W(\mathbb{F}_q)\) for various groups of automorphism. In view of [8], one might consider using the graph structure on \(W(\mathbb{F}_q)\) induced by \(\{\sigma_1, \sigma_2, \sigma_3\}\) or \(\{\rho_1, \rho_2, \rho_3\}\) to implement the CLG [5] hash function algorithm. However, the three fibrations \(W \to \mathbb{P}^1\) have genus 1 fibers, the Jacobians of these fibrations elliptic surfaces of rank at least 1, and the action of the automorphisms on fibers can be described in terms of translation by a section to the elliptic surface. See for example [1], where this geometry is explained and explicit formulas are provided.

Thus the Markoff path-finder algorithm, with suitable tweaks, yields K3 path-finder algorithm whose running time is determined primarily by how long it takes to solve three instances of the elliptic curve discrete logarithm problem. Thus on a classical computer, the algorithm currently takes exponential time to find paths in \(W(\mathbb{F}_q)\), but that is reduced to polynomial time on a quantum computer. However, since the algorithm can look at many elliptic curves lying in the fibration \(W(\mathbb{F}_q) \to \mathbb{P}^1(\mathbb{F}_q)\), it may well be possible to find one whose order is fairly smooth, in which case the ECDLP becomes easier to solve. We have not pursued this further, but it might be interesting to see if under reasonable heuristic assumptions, one can solve the path-finding problem in \(W(\mathbb{F}_q)\) in subexponential time on a classical computer.

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## A Computations to Check Heuristics 2 and 3

**Remark 7 (Testing Heuristic 2).** Choose a random point $P$ in $\mathcal{M}(\mathbb{F}_q)$ and randomly apply $\rho_1$ or $\rho_3$ until the $y$-coordinate of the resulting point is maximally elliptic. For each prime in Table 6, we compute the average value of $n$ for $10^5$ randomly chosen points. We compare this with the theoretical value $2(q-1)/\varphi(q-1)$, which is the theoretical expected number of trials to find a maximally elliptic element in $\mathbb{F}_q^\ast$.

**Remark 8.** We note that the experimental values in Table 6 are somewhat larger than expected, especially when $q-1$ is quite smooth. We are not sure what is causing the discrepancy. It is probably not due to $\mathcal{M}(\mathbb{F}_q)$ having a fewer than expected number of points with a maximally elliptic coordinate, since we checked in experimentally in Table 7 that the proportion of points $P \in \mathcal{M}(\mathbb{F}_p)$ such that $x_P$ is maximally elliptic is almost identical to the proportion of elements of $\mathbb{F}_q^\ast$ that are maximally elliptic. In any case, the experimental numbers are small enough that even for $q$ of cryptographic size, the number of iterations of Steps 3–10 and Steps 11–18 in the Markoff path-finder algorithm (Algorithm 1 in Table 9) will be practical.

| $q$    | $q - 1$     | $2 \cdot \frac{q-1}{\varphi(q-1)}$ | Heuristic 2 Experiment |
|--------|-------------|-----------------------------------|-------------------------|
| 17389  | $2^5 \cdot 3^2 \cdot 7$ | 7.318                             | 8.919                   |
| 48611  | $2 \cdot 5 \cdot 4861$    | 5.001                             | 5.200                   |
| 55163  | $2 \cdot 27581$           | 4.000                             | 3.634                   |
| 70687  | $2 \cdot 3^3 \cdot 7 \cdot 11 \cdot 17$ | 8.181                             | 10.459                  |
| 104729 | $2^3 \cdot 13 \cdot 19 \cdot 63$ | 4.662                             | 4.613                   |
| 200560490131 | $2 \cdot 3 \cdot 5 \cdots 29 \cdot 31$ | 13.085                           | 19.361                  |

Table 6: Experiments to test Heuristic 2 (100000 samples)

**Remark 9 (Testing Heuristic 3).** For each prime in Table 8, we chose $10^5$ random values $t, a, b \in \mathbb{F}_q$ and checked to see if $t$ was both maximally elliptic and had the property that there are $y, z \in \mathbb{F}_q$ satisfying $F(t, a, z) = F(t, y, b) = 0$. We computed the proportion of such $t$ values and compared it with the theoretical
Table 7: Experiments to test the size of $\mathcal{M}(\mathbb{F}_q)^{\text{gen}}$, which is the number of $P \in \mathcal{M}(\mathbb{F}_q)$ such that $x_P$ is maximally elliptic in $\mathbb{F}_q^*$. (100000 samples)

| $q$ | $\#\mathcal{M}(\mathbb{F}_q)/q^2$ | $\#\mathcal{M}(\mathbb{F}_q)^{\text{gen}}/q^2$ | $\varphi(q-1)/(2(q-1))$ |
|-----|---------------------------------|---------------------------------|-----------------|
| 647 | 1.000000                        | 0.22699                        | 0.22291         |
| 757 | 1.00794                         | 0.14298                        | 0.14286         |
| 863 | 1.000000                        | 0.24930                        | 0.24942         |
| 983 | 1.000000                        | 0.25034                        | 0.24949         |
| 1091| 1.000000                        | 0.19853                        | 0.19817         |
| 1213| 1.00495                         | 0.16514                        | 0.16502         |
| 1307| 1.000000                        | 0.25003                        | 0.24962         |

Table 8: Experiments to test Heuristic 3 (100000 Samples)

| $q$ | $q - 1$ | $1/8 \cdot \varphi(q - 1)/(q - 1)$ | Heuristic 3 Experiment |
|-----|---------|-----------------------------------|------------------------|
| 17389| $2^2 \cdot 3^2 \cdot 7 \cdot 23$ | 0.0342                   | 0.0343                 |
| 48611| $2 \cdot 5 \cdot 4861$         | 0.0500                   | 0.0498                 |
| 55163| $2 \cdot 27581$                | 0.0625                   | 0.0629                 |
| 70687| $2 \cdot 3^3 \cdot 7 \cdot 11 \cdot 17$ | 0.0306                   | 0.0314                 |
| 104729| $2^4 \cdot 13 \cdot 19 \cdot 53$ | 0.0536                   | 0.0550                 |
| 200560490131| $2 \cdot 3 \cdot 5 \cdots 29 \cdot 31$ | 0.0191                   | 0.0192                 |

B The Markoff Path-Finder Algorithm and Subroutines

The following algorithms are described in Tables 9–11 in this section.

Algorithm 1 - MarkoffPathFinder: Returns a path in $\overline{\mathcal{M}}(\mathbb{F}_q)$ from $P$ to $Q$.

Algorithm 2 - MarkoffDLP: Returns an integer $n \geq 0$ so that $P = \rho^n_k(G)$ in $\mathcal{M}(\mathbb{F}_q)$.

Algorithm 3 - MaximalEllipticQ: Returns true if $t$ is maximal elliptic in $\mathbb{F}_q^*$, i.e., if the matrix $\begin{pmatrix} 3t & -1 \\ 1 & 0 \end{pmatrix}$ has order $q - 1$ in $\text{SL}_2(\mathbb{F}_q)$; otherwise returns false. It assumes that a factorization of $q - 1$ is known; but see Remark 6.
Algorithm 1 MarkoffPathFinder

**Input:** $q, P, Q$ with $P, Q \in \mathcal{M}(\mathbb{F}_q)$

1: **comment:** Use a factorization algorithm to factor $q - 1$ and store it so that it is accessible by subroutines.
2: PrimeFactorList ← \{primes that divide $q - 1$\}
3: **comment:** Randomly move $P$ using $\rho_1$ and $\rho_3$ until the $y$-coordinate is maximally elliptic
4: $P' \leftarrow P, \alpha \leftarrow 0$
5: while MaximalEllipticQ($q, y(P')$) = false do
6: \hspace{1em} $\alpha \leftarrow \alpha + 1$
7: \hspace{1em} $i_{\alpha} \in \{1, 3\}$
8: \hspace{1em} $P' \leftarrow \rho_{i_{\alpha}}(P')$
9: end while
10: **comment:** $P' = \rho_{i_{\alpha}} \circ \rho_{i_{\alpha-1}} \circ \cdots \circ \rho_2 \circ \rho_1 (P)$
11: **comment:** Randomly move $Q$ using $\rho_1^{-1}$ and $\rho_2^{-1}$ until the $z$-coordinate is maximally elliptic
12: $Q' \leftarrow Q, \beta \leftarrow 0$
13: while MaximalEllipticQ($q, z(Q')$) = false do
14: \hspace{1em} $\beta \leftarrow \beta + 1$
15: \hspace{1em} $j_{\beta} \in \{1, 2\}$
16: \hspace{1em} $Q' \leftarrow \rho_{j_{\beta}}^{-1}(Q')$
17: end while
18: **comment:** $Q = \rho_{j_1} \circ \rho_{j_2} \circ \cdots \circ \rho_{j_{\beta-1}} \circ \rho_{j_{\beta}}(Q')$
19: **comment:** Find random points with the same maximally elliptic $x$-coordinate that can be used to connect $P'$ to $Q'$
20: repeat
21: \hspace{1em} $x \in \mathbb{F}_q^*$
22: until MaximalEllipticQ($q, x$) = true and $F(x, y(P'), Z)$ has a root $z \in \mathbb{F}_q$ and $F(x, Y, z(Q'))$ has a root $y \in \mathbb{F}_q$
23: $P'' \leftarrow (x, y(P'), z)$
24: $Q'' \leftarrow (x, y, z(Q'))$
25: **comment:** $P''$ and $Q''$ are on the same maximally elliptic $x$-fiber.
\hspace{1em} $P'$ and $P''$ are on the same maximally elliptic $y$-fiber.
\hspace{1em} $Q'$ and $Q''$ are on the same maximally elliptic $z$-fiber.
26: **comment:** Find paths $P'' \rightarrow P$ and $P' \rightarrow Q''$ and $Q'' \rightarrow P''$
27: $a \leftarrow \text{MarkoffDLP}(q, P', P'', 2)$
28: $b \leftarrow \text{MarkoffDLP}(q, Q'', P', 3)$
29: $c \leftarrow \text{MarkoffDLP}(q, P'', Q'', 1)$
30: **comment:** $P'' = \rho_a^2(P'), Q' = \rho_b^3(Q''), Q'' = \rho_c^3(P'')$

**Output:** $(i_1, \ldots, i_{\alpha}), (j_1, \ldots, j_{\beta}), (a, b, c)$

Table 9: Returns a path in $\overline{\mathcal{M}(\mathbb{F}_q)}$ from $P$ to $Q$
Algorithm 2 MarkoffDLP

Input: $q, P, Q$ $k$ with $P, Q \in \mathcal{M}(\mathbb{F}_q)$ and $k \in \{1, 2, 3\}$ and the $k$th coordinate of $G$ maximally elliptic

1: comment: if $k = 2$ (y-fiber) or $k = 3$ (z-fiber), swap coordinates to use the $x$-fiber
2: if $k = 2$ then
3: $P \leftarrow (y_P, z_P, x_P)$
4: $G \leftarrow (y_G, z_G, x_G)$
5: else if $k = 3$ then
6: $P \leftarrow (z_P, x_P, y_P)$
7: $G \leftarrow (z_G, x_G, y_G)$
8: end if
9: comment: Note that if $x_P$ is not maximally elliptic, then the algorithm may fail.
10: $\lambda \leftarrow (3x_P + \sqrt{9x_P^2 - 4})/2$ in $\mathbb{F}_q$
11: $b \leftarrow (y_G - z_G/\lambda)/(y_P - z_P/\lambda)$
12: comment: Use a DLP algorithm to find $n$ so that $\lambda^n = b$ in $\mathbb{F}_q^*$.
13: $result = \text{ClassicalDLP}(q, \lambda, b)$

Output: $result$

Table 10: Returns an integer $n \geq 0$ so that $P = \rho_k^n(G)$ in $\mathcal{M}(\mathbb{F}_q)$. See Proposition 1 for an explanation of why this algorithm works.

Algorithm 3 MaximalEllipticQ

Input: $q, t$
1: $result \leftarrow \text{false}$
2: if $9t^2 - 4$ is a square in $\mathbb{F}_q^*$ then
3: $\lambda \leftarrow (3t + \sqrt{9t^2 - 4})/2$ in $\mathbb{F}_q$
4: $result \leftarrow \text{true}$
5: for $p \in \text{PrimeFactorList}$ do
6: if $\lambda^{(q-1)/p} = 1$ in $\mathbb{F}_q^*$ then
7: $result \leftarrow \text{false}$
8: end if
9: end for
10: end if

Output: $result$

Table 11: Check whether $t$ maximal elliptic, or equivalently, whether the matrix $L_t \leftarrow \begin{pmatrix} 3t & -1 \\ 1 & 0 \end{pmatrix}$ has order $q - 1$ in $\text{SL}_2(\mathbb{F}_q)$, or equivalently, whether $L_t$ has an eigenvalue in $\mathbb{F}_q^*$ that is a primitive root.