Emergence of special and doubly special relativity from superstatistics path integrals

P Jizba and F Scardigli

1 FNSPE, Czech Technical University in Prague, Břehová 7, 115 19 Praha 1, Czech Republic
2 Leung Center for Cosmology and Particle Astrophysics (LeCosPA), Department of Physics, National Taiwan University, Taipei 106, Taiwan

E-mail: p.jizba@fjfi.cvut.cz
fabio@phys.ntu.edu.tw

Abstract. Using the concept known as a superstatistics path integral we show that a Wiener process on a short spatial scale can originate a relativistic motion on scales that are larger than particle's Compton wavelength. Viewed in this way, special relativity is not a primitive concept, but rather it statistically emerges when a coarse graining average over distances of order, or longer than the Compton wavelength is taken. We also present the modifications necessary to accommodate in our scheme the doubly special relativistic dynamics. In this way, a previously unsuspected, common statistical origin of the two frameworks is revealed.

1. Introduction

Lorentz symmetry (LS) is one of the cornerstones of contemporary physics. However, recent years have witnessed an accumulation of theoretical ideas that reduce the role of Lorentz invariance from a fundamental symmetry down to an emergent phenomenon. Motivations range from the desire to establish a consistent framework for experimental tests of the LS [1, 2, 3, 4, 5, 6, 7], through models of modified gravity that could explain recent cosmological observations [8, 9, 10, 11], to new approaches to quantization of gravity [12, 13, 14, 15, 16, 17].

In this paper we wish to contribute to this debate by showing that a relativistic quantum mechanics, as formulated through path integrals (PI), bears in itself a seed of understanding how LS can be broken at short spatio-temporal scales and yet emerge as an apparently exact symmetry at large scales. Our argument is based upon a recent observation [18, 19, 20] that PI for both fermionic and bosonic relativistic particles may be interpreted (when analytically continued to imaginary times) as describing a doubly-stochastic process that operates on two vastly different spatio-temporal scales. The short spatial scale, which is much smaller than the Compton length, describes a non-relativistic Wiener process with a fluctuating Newtonian mass. This might be visualized as if the particle would be randomly propagating through a granular or “polycrystalline” medium. The large spatial scale corresponds, on the other hand, to distances that are much larger than particle’s Compton length. At such a scale the particle evolves according to usual relativistic rules, with a sharp value of the mass coinciding with the Lorentz-invariant rest mass. Particularly striking is the fact that when we average the particle’s velocity over the correlation distance (i.e., over particle’s Compton wavelength) we obtain the velocity of light \( c \). So the picture that emerges from this analysis is that the particle (with a non-zero...
mass!) propagates over the correlation distance $1/mc$ (hereafter $\hbar = 1$) with an average velocity $c$, while at larger distance scales (i.e., when a more coarse grained view is taken) the particle propagates as a relativistic particle with a sharp mass and an average velocity that is smaller than $c$. This bears a strong resemblance with Feynman’s chessboard PI for a relativistic Dirac fermion in $1 + 1$ dimensions [21]. There, an analogous situation occurs, i.e., a massive particle propagates over distances of Compton length with velocity $c$, and it is only on much larger spatial scales where the Brownian motion with a sub-luminal average velocity emerges [21, 22]. The analogy with Feynman’s chessboard PI appears also on the level of Hausdorff dimensions of representative trajectories. While below the Compton wavelength the Haussdorff dimension $d_H = 1$, which corresponds to a super-diffusive process, on scales much larger than the Compton length one has $d_H = 2$, which is the usual Brownian diffusion. In passing, we may stress that the outlined superposition of two stochastic processes with widely separated times scales fits the conceptual framework which is often referred to as a superstatistics [23].

Certainly, a single-particle relativistic quantum theory is a logically untenable concept, since a multi-particle production is allowed whenever the particle reaches the threshold energy for pair production. At the same time, the PI for a single relativistic particle is a perfectly legitimate building block in quantum field theory (QFT). Indeed, QFT can be viewed as a grand-canonical ensemble of particle histories where Feynman diagrammatic representation of quantum fields depicts directly the pictures of the world-lines in a grand-canonical ensemble. In particular, the partition function for quantized relativistic fields can be fully rephrased in terms of single-particle relativistic PI’s. This view is epitomized, e.g., in the Bern–Kosower “string-inspired” (or world-line) approach to QFT [24] or in Kleinert’s disorder field theory [25].

The presented line of reasonings can be easily extended into various doubly special relativistic (DSR) systems. In those models a further invariant scale $\ell$, besides the speed of light $c$, is introduced, and $\ell$ is assumed typically to be of the order of the Planck length. By following the same strategy as in the special relativistic context, i.e., analyzing the structure of paths which enter the Feynman summation, one can again identify correlation lengths, canonical commutation relations and the respective Hausdorff dimensions. All of these critically depend on the DSR model at hand, and may serve to gain insight into the underlying stochastic process which is, as a rule, related by an analytic continuation with the corresponding quantum mechanical dynamics.

The structure of the paper is as follows. We start in the next section with some fundamentals of superstatistics path integrals (SPI). Section 3 is devoted to application of SPI in relativistic quantum mechanics. We restrict our presentation to bosons of zero spin that are described by the Klein-Gordon equation. Thought the Klein–Gordon particle (KGP) is not a key for the results obtained, it is instrumental in elucidating the physics behind our reasonings. In particular, we show how a transitional amplitude and ensuing partition function for the KGP can be formulated as a superposition of non-relativistic free-particle PI’s with different Newtonian masses. The concept of emergent relativity is discussed in Section 4. There we observe that the SPI for a relativistic particle allows the following probabilistic interpretation: the single-particle relativistic theory might be viewed as a single-particle non-relativistic theory (i.e. Wiener process) whose Newtonian mass $\tilde{m}$ is a fluctuating parameter, whose average approaches the true relativistic mass $m$ at times that are much larger than the Compton time. On a spatial scale greater than the particle’s Compton wave length the particle follows the standard relativistic motion with a sharp mass and a sub-luminal average velocity. In Section 5, we extend our approach to doubly special relativistic dynamics. In Section 6 we conclude the paper with some brief remarks.
2. Smearing of distributions and superstatistics path integrals

In general, whenever a conditional probability density functions (PDF’s) is formulated through PI then it satisfies the Chapman–Kolmogorov equation (CKE) for continuous Markovian processes [21, 26], i.e.

\[ P(x_b, t_b|x_a, t_a) = \int_{-\infty}^{\infty} dx \frac{P(x_b, t_b|x, t)}{P(x, t|x_a, t_a)} P(x, t|x_a, t_a). \]  

(1)

Conversely, any probability satisfying CKE possesses a PI representation [21, 27].

More often than not, one encounters in statistical physics conditional probabilities formulated as a superposition of PI’s, e.g.

\[ \tilde{P}(x_b, t_b|x_a, t_a) = \int_{0}^{\infty} dv \omega(v, t_{ba}) \int_{x(t_a)=x_a}^{x(t_b)=x_b} D\!x D\!p \ e^{\int_{t_{ba}}^{t_b} dt (ip \dot{x} - eH(p, x))}. \]  

(2)

Here \( \omega(v, t_{ba}) \) with \( t_{ba} = t_b - t_a \) is a normalized PDF. The random variable \( v \) is related to the inverse temperature, proper time, volatility, etc.

Since the Markovian behavior is a fundamental ingredient in numerous statistical models it is important to know whether \( \tilde{P}(x_b, t_b|x_a, t_a) \) can satisfy the CKE (1). The answer is affirmative provided \( \omega(v, t) \) fulfills a certain simple functional equation. Following Ref. [18] one may define a rescaled weight function

\[ w(v, t) \equiv \frac{\omega(v/t, t)}{t}, \]  

(3)

and calculate its (one-sided) Laplace transform

\[ \tilde{w}(p_v, t) \equiv \int_{0}^{\infty} dv e^{-pv} w(v, t). \]  

(4)

Then \( \tilde{P}(x_b, t_b|x_a, t_a) \) satisfies CKE only if

\[ \tilde{w}(p_v, t_1 + t_2) = \tilde{w}(p_v, t_2) \tilde{w}(p_v, t_1). \]  

(5)

Assuming continuity in \( t \), solution of (5) is unique and can be written as

\[ \tilde{w}(p_v, t) = [G(p_v)]^t = e^{-tF(p_v)}. \]  

(6)

A function \( F(p_v) \) must increase monotonically in order to allow for inverse Laplace transform, and must satisfy the condition \( F(0) = 0 \) to ensure that \( \omega \) is normalized to one. Finally the Laplace inverse of \( \tilde{w}(p_v, t) \) yields \( \omega(v, t) \).

Once the above conditions are satisfied, then \( \tilde{P}(x_b, t_b|x_a, t_a) \) possesses a PI representation on its own. The new Hamiltonian is given by the relation \( \tilde{H}(p, x) = F(H(p, x)) \). Further discussion which involves also the issue of operator ordering was pursued in Ref. [18].

3. Superstatistics and spinless relativistic particle

As reported in Refs. [18, 19], the Newton–Wigner propagator for a spinless relativistic particle, i.e. [28, 29]

\[ P(x_b, t_b|x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} D\!x \frac{D\!p}{(2\pi)^D} \exp \left\{ \int_{t_a}^{t_b} dt \left[ i p \cdot \dot{x} - c \sqrt{p^2 + m^2c^2} \right] \right\}, \]  

(7)
can be considered as a superposition of non-relativistic free-particle PI’s provided one chooses the generating function $G(p) = e^{-a\sqrt{p^2 + m^2c^2}}$ with $a \in \mathbb{R}^+$. In such a case one gets

$$P(x_b, t_b | x_a, t_a) = \int_0^\infty dv \omega(v, t_b) \int_{x(t_a) = x_a}^{x(t_b) = x_b} Dx \frac{Dp}{(2\pi)^D} \exp \left\{ \int_{t_a}^{t_b} d\tau \left[ ip \cdot \dot{x} - v(p^2 c^2 + m^2c^4) \right] \right\},$$  \hspace{1cm} \text{(8)}

with $\omega(v, t)$ representing the Weibull distribution of order 1. Weibull’s PDF of order $a$ is defined as [30]

$$\omega(v, a, t) = a \exp \left( -\frac{a^2 t/4v}{2\sqrt{\pi}v^3/t} \right).$$  \hspace{1cm} \text{(9)}

Note, however, that neither (7) nor (8) are genuine propagators for Klein–Gordon equation. As stressed first by Stuckelberg [31, 32], the true relativistic propagator must include also the negative energy spectrum, reflecting the existence of antiparticles.

The resolution of this problem can be readily found when the Klein–Gordon particle is reformulated in the so-called Feshbach–Villars (FV) representation, i.e. [19]

$$i\partial_t \Psi = \hat{H}_{\text{FV}}(p) \Psi,$$

$$\hat{H}_{\text{FV}}(p) = (\sigma_3 + i\sigma_2) \frac{p^2}{2m} + \sigma_3 mc^2,$$  \hspace{1cm} \text{(10)}

where $\Psi$ is a two component wave function. The two components are related to opposite parity states — fact that is automatically fulfilled by Dirac bispinors in case of Dirac’s equation. FV representation together with ensuing superstatistics PI was already thoroughly discussed in Ref. [19] and we shall refrain from going to further details here. Let us only mention that in order to handle the full PI representation of the Klein–Gordon particle it will suffice to discuss the PI relation (8) alone. The latter serves as a building block of the $2 \times 2$ matrix structure of the FV propagator from which the KG propagator can be easily reconstructed.

### 4. Emergent Special Relativity

Let us now consider in Eq. (8) the change of variable $vc^2 \leftrightarrow 1/2\tilde{m}$, then the RHS can be rewritten in the form

$$\int_{x(0) = x'}^{x(t)} Dx \frac{Dp}{(2\pi)^D} \exp \left\{ \int_0^t d\tau \left[ ip \cdot \dot{x} - c\sqrt{p^2 + m^2c^2} \right] \right\} = \int_0^{\infty} d\tilde{m} f_{\frac{p}{2}}(\tilde{m}, tc^2, tc^2m^2) \int_{x(0) = x'}^{x(t)} Dx \frac{Dp}{(2\pi)^D} \exp \left\{ \int_0^t d\tau \left[ ip \cdot \dot{x} - \frac{p^2}{2\tilde{m}} - mc^2 \right] \right\},$$  \hspace{1cm} \text{(11)}

Here $t_{ba} = t$, and

$$f_{\frac{p}{2}}(z, a, b) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} 2^{p-1} e^{-(az+b)/\tilde{m}},$$  \hspace{1cm} \text{(12)}

is the generalized inverse Gaussian distribution [33] with $K_p$ being the modified Bessel function of the second kind with index $p$. The structure of (11) hints that $\tilde{m}$ can be interpreted as a Newtonian mass which takes on continuous values distributed according to $f_{\frac{p}{2}}(\tilde{m}, tc^2, tc^2m^2)$ with $\langle \tilde{m} \rangle = m + 1/ct^2$ and $\text{var}(\tilde{m}) = m/ct^2 + 2/t^2c^4$. As a result one may view a single-particle
relativistic theory as a single-particle non-relativistic theory where the particle’s Newtonian mass $\bar{m}$ represents a fluctuating parameter which approaches on average the Einstein rest mass $m$ in the large $t$ limit. We stress that the time $t$ in question should be understood as a time after which the position measurement is made. In particular, during this period the system is unperturbed. In this respect the smearing distribution $f_{\frac{1}{2}}(\bar{m}, t\bar{c}^2, t\bar{c}^2m^2)$ represents a temporal coarse-grained distribution for a Newtonian mass. So, the longer the time between measurements, the poorer the resolution of mass fluctuations. One can thus justly expect that in the long run all mass fluctuations will be washed out and only a sharp time-independent effective mass will be perceived. The form of $\langle \bar{m} \rangle$ identifies the time scale at which this happens with $t \sim 1/mc^2$. The latter is nothing but the Compton time $t_C$.

The PI (11) can be identified with a PI for a relativistic particle in, the so-called, Polyakov’s gauge [34]. So, the form of the smearing function $f_{\frac{1}{2}}$ naturally fixes the gauge, which in this case turns out to be to Polyakov’s gauge. With the help of Ward identities it is easy to show [20] that the ensuing canonical commutation relation (CCR) reads

$$\{\hat{x}_j, \hat{p}_i\} = i \left( \delta_{ij} + \frac{\hat{p}_i \hat{p}_j}{m^2 c^2} \right).$$  \hspace{1cm} (13)$$

This is the correct special-relativistic CCR corresponding to quantized Dirac brackets.

Referring to identity (11) the fluctuations of the Newtonian mass can be portrayed as originating from particle’s evolution in an “inhomogeneous” or a “polycrystalline” medium. Granularity, as known, for instance, from solid-state systems, typically leads to corrections in the local dispersion relation [35] and hence to alterations in the local effective mass. The following picture thus emerges: on the short-distance scale, a single particle can be envisaged essentially Brownian, the local probability density matrix (PDM) conditioned on some fixed grain endowing it with a mass $\bar{m}$. This fast-time process has a time scale $\sim 1/mc^2$. An averaged value of the time scale can be computed with the help of the smearing distribution $f_{\frac{1}{2}}(\bar{m}, tc^2, tc^2m^2)$, which gives a transient temporal scale $(1/mc^2) = 1/mc^2$. The latter coincides with particle’s Compton time $t_C$. At time scales much longer than $t_C$, the probability that the particle encounters a grain which endows it with a mass $\bar{m}$ is $f_{\frac{1}{2}}(\bar{m}, tc^2, tc^2m^2)$. Because the fast-time scale motion is essentially Brownian, the local probability density matrix (PDM) conditioned on some fixed $\bar{m}$ in a given grain is Gaussian, i.e.

$$\hat{\rho}(p, t|\bar{m}) = \left( \frac{t}{\bar{m}2\pi} \right)^{3/2} \exp \left[ -\frac{t \hat{p}^2}{2\bar{m}} \right].$$  \hspace{1cm} (14)$$

As the particle moves through a “grainy environment” the Newtonian mass $\bar{m}$ fluctuates and the corresponding joint PDM will be $\hat{\rho}(p, t; \bar{m}) = f_{\frac{1}{2}}(\bar{m}, tc^2, tc^2m^2) \hat{\rho}(p, t|\bar{m})$. The marginal PDM describing the mass-averaged (i.e. long-term) behavior is then

$$\hat{\rho}(p, t) = \int_0^\infty d\bar{m} f_{\frac{1}{2}}(\bar{m}, tc^2, tc^2m^2) \hat{\rho}(p, t|\bar{m}).$$  \hspace{1cm} (15)$$

The matrix elements of $\hat{\rho}(p, t)$ in the $x$-basis are then described by the PI from Eq. (11).

We may also observe that the coarse-grained velocity over the correlation time $t = 1/mc^2$ equals the speed of light $c$. In fact

$$\langle |v| \rangle_{t=1/mc^2} = \frac{\langle |p| \rangle}{\langle \bar{m} \rangle}_{t=1/mc^2} = c.$$  \hspace{1cm} (16)$$
Figure 1. The roughness of the representative trajectories in the relativistic path integral (11) depends on a spatial/temporal scale. On a fine scale (A), where $t \ll t_C$ a particle can be considered as propagating with a sharp Newtonian mass $\tilde{m}$ in a single grain with a Brownian motion controlled by the Hamiltonian $\frac{p^2}{2\tilde{m}}$. On the intermediate scale of order $\lambda_C$ the particle propagates with an average velocity equal to the speed of light $c$. On a coarser scale (B) the particle appears to follow a Brownian process with a sharp Lorentz invariant mass $m$, and the particle’s net velocity is then less than $c$.

So on a short-time scale of order $\lambda_C$ the Klein–Gordon particle propagates with an averaged velocity which is the speed of light $c$. But if one checks the particle’s position at widely separated intervals (much larger than $\lambda_C$), then many directional reversals along a typical PI trajectory will take place, and the particle’s net velocity will be then $c$, as it should be for a massive particle (see Fig. 1). In addition, the time-compounded smearing distribution tends for large times rapidly to the delta-function distribution $\delta(\tilde{m} - m)$ thanks to the central limit theorem. This means that the particle acquires a sharp mass equal to Einstein’s Lorentz invariant mass, and the process (not being hindered by fluctuating masses) turns out to be purely Brownian.

There are a number of additional theories that are simple to construct. For example, all previous remarks extend straightaway to Dirac’s Hamiltonian [19]

$$H_{DA}^{AV} = c\gamma_0 \gamma \cdot (p - eA/c) + \gamma_0(mc^2 + V) + eA_0,$$

and to the Feshbach–Villars Hamiltonian [19]

$$H_{FV}^{AV} = (\sigma_3 + i\sigma_2) \frac{1}{2\tilde{m}} (p - eA/c)^2 + \sigma_3(mc^2 + V) + eA_0.$$

For example, in the case when $V = 0$, $A_x = -By$ ($B_z \equiv B$), and $A_y = A_z = 0$, then the PI for Dirac’s Hamiltonian yields the “fast scale” Hamiltonian [see Ref. [19] for details]

$$H_{SP} = \frac{1}{2\tilde{m}} \left[ (p_x + \frac{e}{c} By)^2 + p_y^2 + p_z^2 \right] - \mu_B B \sigma_3.$$  

This is the Schrödinger–Pauli Hamiltonian with $\mu_B = e\hbar/2\tilde{m}$ representing the Bohr magneton. The corresponding grain distribution is again the inverse Gauss distribution. Analogous reasonings can be carried on also for charged spin-0 particles, such as, e.g., $\pi^\pm$ mesons.
5. Emergent Doubly Special Relativity

Doubly Special Relativity (DSR) stands among the prominent ideas introduced in physics during the last decade. At the same time it is also one of the most controversial ideas. Many foundational issues about DSR, such as the multi-particle sector of the theory (the so called soccer-ball-problem) or a lack of consistent formulation of the DSR in position space are still being debated [36]. In a nutshell, DSR is a theory which coherently tries to implement a second invariant, besides the speed of light, into the transformations among inertial frames. This new invariant it is usually assumed to be an observer-independent length-scale — the Planck length $\ell_p$, or its inverse, i.e., the Planck energy $E_p = c\ell_p^{-1}$. Thus, it is not so surprising that the relations mainly studied are those between DSR and various quantum gravity models [37, 38, 39]. In a particularly suggestive approach [40], DSR has been presented as the low energy limit of Quantum Gravity. Connections between DSR and other theories (non commutative geometry, AdS space-time, etc.) have also been recently investigated [41].

To extend our reasonings to DSR, we start by considering the modified invariant, or deformed dispersion relation,

$$\frac{\eta^{ab}p_a p_b}{(1 - \ell_p p_0)^2} = m^2 c^2,$$

proposed by Magueijo and Smolin [42, 43]. Here $m$ plays the role of the DSR invariant mass. Assuming a metric signature $(+,-,-,-)$, we can solve (20) with respect to $p_0$, which essentially coincides with the physical Hamiltonian $\hat{H} = cp_0$, the generator of the temporal translations with respect to the coordinate time $t$. Our starting Hamiltonian is therefore

$$\hat{H} = \left(1 - \frac{m^2 c^2 \ell}{1 - m^2 c^2 \ell^2}\right)^{-1},$$

which we take as the transformed Hamiltonian $\hat{H}(p,x) = F(H(p,x))$ entering the PI representation of $\hat{P}(x_b, t_b|x_a, t_a)$, see Eq. (6) and the comments below [see also Ref. [18]]. The same line of reasonings that brought us to (11) will give us [20]

$$\int_{x(0) = x'}^x Dx \frac{Dp}{(2\pi)^D} \exp \left\{ \int_0^t d\tau \left[ i p \cdot \dot{x} + \frac{1}{2} \left( \frac{m^2 c^2 \ell}{1 - m^2 c^2 \ell^2}\right) \right] \right\},$$

where $E_0 = mc^2/(1 + m c \ell)$ is the particle’s rest energy and $\lambda = 1/(1 - m^2 c^2 \ell^2)$ is the deformation parameter. From the form of smearing distribution (12) one can easily drawn that $\langle \tilde{m} \rangle = m + 1/(tc^2 \lambda)$ and $\var(\tilde{m}) = m/tc^2 \lambda + 2/t^2 c^4 \lambda^2$.

The superstatistics identity (22) allows to deduce the CCR via the standard PI analysis [20, 21]. In the PI literature the CCR are directly related to the degree of roughness (described through Hausdorff dimension $d_H$ or Hurst exponent $h$) of typical PI paths [21, 44]. For instance, the usual non-relativistic canonical relation $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$ results from the fact that, for a typical path occurring in non-relativistic PI’s have $d_H = 2$ and $h = 1/2$. In fact, in non-relativistic quantum mechanics all local potentials fall into the same universality class (as for their scaling behavior) as the free system [44]. The latter might be viewed as a PI justification of the universal form of non-relativistic CCR’s.

With the help of Ward identities the PI identity (22) implies the commutators [20]

$$[\hat{x}_i, \hat{p}_j]_{DSR} = i \left( \delta_{ij} + \frac{c^2 - m^2 c^2}{c^2 m^2 c^2} \hat{p}_i \hat{p}_j \right).$$
Here $\kappa = 1/\ell$. The CCR (23) coincides with the Snyder version of the deformed CCR associated to the dispersion relation (20) [cf. Refs. [45, 46, 47, 48]]. The same commutator appears also in works of Ghosh [47] and Mignemi [49].

The minimal length interval $\ell$ is typically set to be the Planck length $\ell_p$, or more generally the Compton length $\lambda_C$ (which reduces to the Planck length for a Planck mass). For definiteness we will in the following identify $\ell$ with $\ell_p$. In this connection, note that when $mc \to \kappa$, i.e., when $m$ coincides with the Planck mass, then the CCR (23) becomes non-relativistic. This can also alternatively seen from (22), where for $m \to M_p$ the deformation parameter $\lambda \to \infty$, and the smearing distribution $f_\tau(\tilde{m}, tc^2\lambda, tc^2m^2\lambda) \to \delta(m - \tilde{m})$, which yields the usual PI for a Wiener process.

For the sake of completeness, we should also note that commutator (23) does not coincide with the corresponding commutator from the Magueijo–Smolin paper [43], although it comes from the same dispersion relation (20). Moreover, the commutators (23) does not enjoy an important property of Magueijo–Smolin’s commutator. In particular, when the energy of the boosted system approaches the Planck energy, then Magueijo–Smolin’s commutator reads

$$\left[ \hat{x}_j, \hat{p}_i \right]_{\text{MS}} = i \delta_{ij} \left( 1 - \frac{E}{E_p} \right).$$

So, at the Planck scale the CCR (24) predicts possibility of a classical (non-quantal) world. On the other hand, when the energy of the particle under boost approaches the Planck value, our commutator (23) becomes

$$\left[ \hat{x}_j, \hat{p}_i \right]_{\text{DSR}} = i \delta_{ij}.$$

The qualitative difference in behavior of both CCR can be traced back to the fact that commutators in (doubly-)special relativity depend on two things; First, the fundamental commutators are essentially the Dirac brackets of the canonical variables. The explicit definition of the Dirac brackets depends on the choice of a gauge (gauge fixing condition), which for relativistic systems corresponds to choice of a specific physical time. So the commutation relations are generally gauge fixing dependent in both SR and DSR systems. Second, the fundamental commutator $[\hat{x}_j, \hat{p}_i]$ depends (through the Jacobi identities) on the whole symplectic structure of the system (and therefore also on the commutator $[\hat{x}_j, \hat{x}_i]$, for example). These are not specified by a particular DSR model, but they have to be chosen aside. Of course, one obtains different theories for different choices of $[\hat{x}_j, \hat{x}_i]$.

6. Concluding remarks

In this paper we have provided the picture wherein both SR and DSR systems can be seen as statistically arising from underlying non-relativistic Wiener process. Such an emergence results from superposition of two stochastic processes. On a short spatial scale (much shorter than particle’s Compton wavelength) the particle moves according to a Brownian, non-relativistic, motion. Its Newtonian mass fluctuates according to an inverse Gaussian distribution. The time-compounded smearing distribution tends, however, rapidly to the delta-function distribution due to the central limit theorem. This happens at the time scale of the order of Compton time, at which the relative mass fluctuation is of order unity. The Compton length also represents the critical length scale at which the Feynman–Hibbs scaling relation between $\Delta x$ and $\Delta t$ changes its critical exponent (Hurst exponent) from 1 to $1/2$. The Compton time and ensuing Compton length can be thus viewed as correlation time and correlation length, respectively. The averaged (or coarse-grained) velocity over the correlation time is the light velocity $c$. On a time scale much larger than the Compton time, the particle then behaves as
a relativistic particle with a sharp mass equal to the Lorentz invariant mass. In this case the particle moves with a net velocity which is less than $c$. Here the reader may notice a close analogy with the Feynman chessboard PI. In contrast to the chessboard PI, our approach is not confined to only 1 + 1 dimensional Dirac fermions.

The presented concept of statistical emergence, which is shared both by SR and DSR, can offer a new valid insight into the Planck-scale structure of space-time. The existence of a discrete polycrystalline substrate might be welcomed in various quantum gravity constructions. In fact, it has been speculated for long time that quantum gravity may lead to a discrete structure of space and time which can cure classical singularities. This idea has been embodied, in particular, in Loop Quantum Cosmology [50, 51]. A similar proposal was put forward in Ref. [52] in connection with the space-time foam. It should be stressed that many condensed matter systems show that a discrete sub-structure might lead to a genuinely relativistic dynamics at low energies, without any internal inconsistency [53]. A paradigmatic example of this is graphene, where an effective theory emerges in which conducting electrons behave, at low temperatures, as massless relativistic Dirac fermions with a “light speed” equal to the Fermi velocity.

Finally, it is also hoped that the essence of our results will continue to hold in curved spacetimes. This could be an important step in addressing the issue of quantum gravity. In this connection we may notice a conceptual similarity with the Hoňava–Lifshitz gravity theory, where, as in our case, space and time are not equivalent at the fundamental level, hence the theory is intrinsically non-relativistic at Planck energies. The relativistic concept of time together with its Lorentz invariance emerges only at distances much larger than Planck distance.

Acknowledgments

A particular thank goes to S. Mignemi for his enlightening emails on the DSR commutators and DSR symplectic structure. We are grateful also for comments from H. Kleinert, C. Schubert, F. Bastianelli, and L.S. Schulman which have helped us to understand better the ideas proposed in this paper. P.J. is supported by the Ministry of Education of the Czech Republic under the Grant No. CFRJS 1507001. F.S. is supported by Taiwan National Science Council under Project No. NSC 97-2112-M-002-026-MY3.

References

[1] Wolf P et al. 2006 Phys. Rev. Lett. 96 060801
[2] Cane F et al. 2004 Phys. Rev. Lett. 93 230801
[3] Stanwix P L et al. 2006 Phys. Rev. D 74 081101
[4] Gabrielse G et al. 1999 Phys. Rev. Lett. 82 3198
[5] BaBar collaboration, Aubert B et al. 2004 Phys. Rev. Lett. 92 142002
[6] FOCUS collaboration, Link J et al. 2003 Phys. Lett. B 556 7
[7] Heckel B et al. 2006 Phys. Rev. Lett. 97 021603
[8] The Pierre Auger Collaboration, Abreu P et al. 2010 Phys. Lett. B 685 239
[9] The Pierre Auger Collaboration, Abraham J 2007 et al. Science 318 938
[10] Webb J K et al. 2001 Phys. Rev. Lett. 87 091301
[11] Kostelecky V A and Mewes M 2004 Phys. Rev. D 70 031102
[12] Katori V, Kostelecky V A and Taylor R 2006 Phys. Rev. D 74 105009
[13] Damour T and Polyakov A M 1994 Nucl. Phys. B 423 532
[14] Gambini R and Pullin J 1999 Phys. Rev. D 59 124021
[15] Kostelecky V A and Samuel S 1989 Phys. Rev. D 39 683
[16] Kleinert H 2008 Multivalued Fields in Condensed Matter, Electrodynamics, and Gravitation (Singapore: World Scientific)
[17] Jizba P, Kleinert H and Scardigli F 2010 Phys. Rev. D 81 084030
[18] Jizba P and Kleinert H 2008 Phys. Rev. E 78 031122
[19] Jizba P and Kleinert H 2010 Phys. Rev. D 82 085016
[20] Jizba P and Scardigli F 2011 (Preprint hep-th/1106.5913)
[21] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[22] Gaveau B, Jacobson T, Kac M and Schulman L S 1984 *Phys. Rev. Lett.* **53** 419
  Jacobson T and Schulman L S 1984 *J. Phys. A* **17** 375
[23] Beck C 2001 *Phys. Rev. Lett.* **87** 180601
  Beck C 2004 *Physica D* **193** 195
[24] Bern Z and Kosower D A 1991 *Phys. Rev. Lett.* **66** 1669
  Bern Z and Kosower D A 1992 *Nucl. Phys. B* **379** 451
  Schubert C 2001 *Phys. Rep.* **355** 73
[25] Kleinert H 1989 *Gauge Fields in Condensed Matter, Vol. I Superflow and Vortex Lines* (Singapore: World Scientific)
[26] Kleinert H 2009 *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets* (Singapore: World Scientific)
[27] Kac M 1959 *Probability and related Topics in Physical Sciences* (New York: Interscience)
[28] Newton T and Wigner E 1949 *Rev. Mod. Phys.* **21** 400
[29] Hartle J B and Kuchar K V 1986 *Phys. Rev. D* **34** 2323
[30] Weibull W 1951 *J. Appl. Mech.-Trans. ASME* **18** 293
[31] Stöckelberg E C G 1941 *Helvetica Physica Acta* **14** 588
[32] Stöckelberg E C G 1942 *Helvetica Physica Acta* **15** 23
[33] Feller W 1966 *An Introduction to Probability Theory and its Applications, Vol. II* (London: John Wiley)
[34] Polyakov A M 1987 *Gauge Fields and Strings* (New York: Harwood)
[35] see, e.g., Johnson E A and MacKinnon A 1993 *J. Phys.: Condens. Matter* **5** 5859
[36] Kowalski-Glikman J 2005 *Introduction to Doubly Special Relativity*, Lecture Notes in Physics **669** Springer 131 - 159 (Preprint hep-th/0405273)
[37] Smolin L (Preprint hep-th/1007.0718)
  Smolin L (Preprint hep-th/0808.3765)
[38] Girelli F, Livine E R and Oriti D 2010 *Phys. Rev. D* **81** 024015
[39] Rovelli C and Speziale S 2003 *Phys. Rev. D* **67** 064019
[40] Girelli F, Livine E R and Oriti D 2005 *Nucl. Phys. B* **708** 411
[41] Girelli F and Livine E Preprint hep-th/0708.3813
[42] Magueijo J and Smolin L *Phys. Rev. Lett.* **88** 190403
  Mignemi S 2003 *Phys. Lett. A* **316** 173
[43] Magueijo J and Smolin L 2003 *Phys. Rev. D* **67** 044017
[44] Kröger H 2000 *Physics Reports* **323** 81
[45] Snyder H S 1947 *Phys. Rev.* **71** 38
[46] Ghosh S 2006 *Phys. Rev. D* **74** 084019
[47] Ghosh S 2007 *Phys. Lett. B* **648** 262
[48] Banerjee R, Kulkarni S and Samanta S 2006 *JHEP* **05** 077
[49] Mignemi S 2003 *Phys. Rev. D* **68** 065029
[50] Ashtekar A 2009 *Gen. Rel. Grav.* **41** 707
[51] Bojowald M 2009 *Class. Quant. Grav.* **26** 075020
[52] Vilenkin A 1985 *Nucl. Phys. B* **252** 141
[53] Volovik G E 2003 *The Universe in a Helium Droplet* (Oxford: Clarendon Press)