The influence of shape on the vibratory response of variable beams

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Abstract. Pure analytical solution for variable cantilever beams subjected to free bending oscillations and inertial forces is presented in this paper. Following the Euler-Bernoulli scheme for light slender cross section beams and Henri Bouasses’s analytical procedure the in-plane linear motion differential equation is deduced. Taking into account the beam’s shape a generalized non-dimensional differential equation is derived and characterized by only four geometric parameters. Beam’s shape whose cross section area and moment of inertia variation obey a power law with respect to the longitudinal distance to the tip section of the bar and truncation are considered. The differential equation is analytically solved using the symbolic Wolfram Mathematica module for a number of typical shapes from uniform to zero area tip section with linear to cubic root variation and both rectangular and circular cross sections. The natural frequencies are determined and compared. Applications of results to high-rise building design and variable beams found in nature like tree stem are suggested.

1. Introduction

The main question that could be assessed is what shape of a variable beam is the best for a certain function or application considering its dynamic response. In the case of high-rise buildings both uniform cross section like beams and variable beams have been applied. A very important parameter of a building structure for seismic response is its natural frequency. It is of major importance to get a natural frequency as high as possible in order to avoid coupling or resonances. In the case of stem trees was found that pine, spruce and birch taper curve is clearly non-uniform non linear. What frequencies and mode shapes can be expected for these deformable solids? Let’s see how the Euler-Bernoulli scheme for slender like beams could assess for that.

Although a numerical approach based, for instance, on the finite element method can be extremely useful to solve specific problems, it is also of great importance to get an analytical solution in order to predict which parameters are essential to the variable beams behavior. Pure theoretical analysis of beams are not only possible, but well developed as well since the nineteenth century. The dynamics of a beam, characterized by its uniform cross section, such as a rectangular cross section, is widely understood. Texts books like that of Belluzzi [1] reviews and pay attention to the dynamics of variable beam. However, truncated variable beams are still hard to be studied from a merely theoretical approach. Some studies have been made using different strategies, such as Bessel functions (Julián Muñoz González [2], 2009; De Rosa and Auciello [3], 1996) and orthogonal polynomials (Caruntu
[4, 2007] among others. Truncated beams with circular cross section were also approached in [3], while many authors approached the case of truncated rectangular beam. Several papers have been published solving the problem of a certain variation for the width, such as linear (Sanger [5], 1968), power variation (Wang [6], 1967) or stepped beams (Mazzei [7], 2014). Also Stanton and Mann ([8], 2010) analytically approached discontinuous Euler-Bernoulli beams, while different boundary conditions, such as rotational springs at both ends were taken into account by Maurizi and Bellés ([9], 1993).

The main purposes for this paper are reaching both an exact and yet easy solution to vibrations of variable beams using symbolic software interface. Although complicated frequency equations will be obtained for simple cases such as a beam with linearly variable thickness, this direct method will prove exact and yet easily applicable to a large variety of beam shapes and cross section shapes. The advantages of a pure theoretical non-dimensional solution make it possible to predict the particular solution for any case - including truncated beams – and study the response of different kinds of variation, such as linear or square root, on the natural frequencies and mode shapes.

2. Theoretical Procedure
According to the scheme in figure 1 the Euler-Bernoulli equation for vibrations of slender beam of ideal length \(L\), truncated at section \(x=v\), Young modulus \(E(x)\), mass density \(\rho\), variable cross area \(S(x)\) and inertia \(I(x)\) takes the form

\[
\frac{\partial^2}{\partial x^2} \left[ E(x)I(x) \frac{\partial^2 w}{\partial x^2} (x,t) \right] + \rho S(x) \frac{\partial^2 w}{\partial t^2} (x,t) = 0
\]  

(1)

where \(w\) stands for the lateral displacement due to bending, \(t\) is the time and \(x\) the longitudinal coordinate - its origin being located at the virtual beam’s tip section of zero area, pointing towards the embedment. If a uniform medium is considered, constant \(E\) modulus will apply. The cross area \(S(x)\) and inertia \(I(x)\), considering a power law variation, can be expressed by

\[
S(x) = sx^n
\]

\[
I(x) = ix^m
\]

(2)  

(3)

where \(s, i, n\) and \(m\) are determined to fit a particular cross section and beam’s geometry - for example, for an ideal rectangular cross section \((v=0)\) of linear thickness variation, constant width \(b\) and maximum thickness \(h\): \(n=1\), \(s=bh/L\), \(m=3\) and \(i=bh^3/(12L^3)\), which do not depend on \(x\) nor \(t\).
We state separated functions for $x$ and $t$ for the lateral displacement

$$w(x, t) = u(x) \sin(2\pi\omega t)$$

(4)

where $u$ is a function that depends exclusively on $x$ and $\omega$ is the frequency of the free oscillations. Substituting (2-4) into (1) for a beam of constant Young modulus $E$ and assuming $\omega \neq 0$:

$$4\pi^2\omega^2 \frac{\rho_S}{Ei} u(x) x^n = \frac{d^2}{dx^2} \left[ x^m \frac{d^2 u(x)}{dx^2} \right]$$

(5)

Applying a generalized Bouasse’s manipulation [10] a change of variable $x$ to a new dimensionless variable $y=Kx$ can be performed for easiest integration and systematic solution. A like change of variables is performed in [11]. Then equation (5) reduces to:

$$u(y)y^n = \frac{d^2}{dy^2} \left[ y^m \frac{d^2 u(y)}{dy^2} \right]$$

(6)

If:

$$K = \left( \frac{4\pi^2\omega^2 \rho_S}{EI} \right)^{n-m+1}$$

(7)

$K$ has always $[L]^{-1}$ dimension, regardless of $n$ and $m$. From (2) and (3) it is easy to prove that $s$ has $[L]^{2-n}$ dimension and $i [L]^{n-4}$ dimension.

Equation (6) represents a general differential equation valid for most slender variable beams whose geometry can be expressed as (2) and (3). This includes a wide variety of possible geometries and the cases we are interested in for smooth section variations which has been revealed to be a frequent case in nature [12]. See some geometries that apply in tables 1 and 2. A general case has to be approached directly by (5) without performing the above explained change of variable.

Now it is far more interesting to solve the problem following the simplified equation (6) and applying a non-dimensional analysis. Then, the result will be based on the dimensionless frequency $\omega^*$ given by:

$$\omega^* = \frac{\omega L}{\sqrt{E/\rho}}$$

(8)

and a new dimensionless $K$: $K^* = KL$. An explicit definition for $\omega^*$ can now be obtained substituting (8) in (7):

$$\omega^* = \left[ \frac{i(K^*)^{n-m+4}}{4\pi^2Sl^{n-m+2}} \right]^{1/2}$$

(9)

Equation (6) is then solved using mathematical software symbolic interface -such as Wolfram Mathematica—. Figure 2 shows the display dump of the solution in which $x$ represents the dimensionless $y$, $C[\cdot]$ for $C_s$, BesselI, BesselJ and MeijerG are Mathematica functions that were applied by the function DSolve [13] of the Wolfram Mathematica platform[14].
The boundary conditions for the truncated cantilever beam must now be imposed: null displacement and rotation on the fixed section \( u(K*)=0; \ u'(K*)=0 \); and null bending moment and shear force on the free end section \( u''(v*K*)=0; \ u'''(v*K*)=0 \). This leads to a set of four equations with the four unknown integration constants due to (6) is a fourth order differential equation. Then, using the symbolic interface, the compatibility of the four equations is forced (the determinant of the matrix form for the set of equations for a vector with the unknowns to be zero) which leads to an algebraic equation dependent on \( K^* \) which is the frequency equation.

Equation (11) shows how the dimensionless frequency \( \omega^* \) depends linearly on the slenderness ratio \( h/L \) of the beam. For \( h/L=0 \), \( \omega^*=\omega=0 \).
Let us consider another important variable beam which is that of circular cross section cantilevered beam whose diameter $d$, increases by the power law $d = d(x/L)^a$ to get the maximum diameter $d$ in the fixed end where $a$ is the exponent. Now we get from (2) and (3):

\[
s = \frac{\pi d^2}{4L^2 a}; \quad i = \frac{\pi d^4}{64L^4 a}; \quad n = 2a; \quad m = 4a
\] (12)

Substituting upon (9):

\[
\omega^* = \frac{(K^*)^{2-a}}{8\pi} \cdot \frac{d}{L}
\] (13)

Equation (13) shows how the dimensionless frequency $\omega^*$ depends again linearly on the slenderness ratio $d/L$ of the beam. Equations (11) and (13) can be synthesized and simplified setting the slope $C$ with respect to the slenderness ratio, named frequency coefficient and defined by:

\[
\omega^* = C \cdot \frac{n}{L} \quad \text{for rectangular cross section where} \quad C = \frac{(K^*)^{2-a}}{4\pi\sqrt{3}}
\] (14)

\[
\omega^* = C \cdot \frac{d}{L} \quad \text{for circular cross section where} \quad C = \frac{(K^*)^{2-a}}{8\pi}
\]

The same process can be followed with different cases. As long as equations (11) and (13) are valid, the problem reduces to determine the corresponding solution values $K_j^*$ ($j=1,2,3,...$) from the frequency equation, that is, its roots. These are correspondent to the natural frequency $\omega_1$ ($j=1$) and its harmonics $\omega_j$ ($j=2,3,...$) of the beam.

3. Results

Tables 1 and 2 show the first three frequency coefficients $C_1$, $C_2$ and $C_3$ and the volume ratio for the volume of the variable beam to that for the equivalent uniform beam $V_v/V_u$ for a number of variable beams, plus the uniform rectangular and circular cases.

| Beam type                  | Drawing | $a$ | $n$ | $m$ | $V_v/V_u$ | $C_1$  | $C_2$  | $C_3$  |
|----------------------------|---------|-----|-----|-----|-----------|--------|--------|--------|
| Rectangular, uniform       |         | 0   | 0   | 0   | 1         | 0.1615 | 1.0124 | 2.8346 |
| Rectangular, cubic root variation |       | 1/3 | 1/3 | 3   | 3/4       | 0.1963 | 0.9279 | 2.3148 |
| Rectangular, square root   |         | 1/2 | 1/2 | 3/2 | 2/3       | 0.2113 | 0.8774 | 2.0678 |
| Rectangular, 1.8 root      |         | 1/1.8 | 1/1.8 | 3/1.8 | 0.6429   | 0.2159 | 0.8595 | 1.9874 |
| Rectangular, 1.6 root      |         | 1/1.6 | 1/1.6 | 3/1.6 | 0.6154   | 0.2214 | 0.8363 | 1.8882 |
| Rectangular, 1.4 root      |         | 1/1.4 | 1/1.4 | 3/1.4 | 0.5833   | 0.2279 | 0.8055 | 1.7630 |
| Rectangular, 1.2 root      |         | 1/1.2 | 1/1.2 | 3/1.2 | 0.5455   | 0.2356 | 0.7624 | 1.5999 |
| Rectangular, linear beam   |         | 1    | 1   | 3   | 0.5       | 0.2442 | 0.6987 | 1.3792 |
Table 2. First three natural frequency coefficients and volume ratio of variable circular-section cantilever beams.

| Beam type                     | Drawing | $a$ | $n$ | $m$ | $V_v/V_u$ | $C_1$ | $C_2$ | $C_3$ |
|-------------------------------|---------|-----|-----|-----|-----------|-------|-------|-------|
| Circular, uniform             |         | 0   | 0   | 0   | 1         | 0.1399| 0.8767| 2.4549|
| Circular, cubic root variation|         | 1/3 | 2/3 | 4/3 | 0.6       | 0.2146| 0.9059| 2.1607|
| Circular, square root         |         | 1/2 | 1   | 2   | 1/2       | 0.2506| 0.9040| 2.008 |
| Circular, 1.8 root            |         | 1/1.8| 2/1.8| 4/1.8| 0.4737   | 0.2623| 0.9011| 1.9565|
| Circular, 1.6 root            |         | 1/1.6| 2/1.6| 4/1.6| 0.4445   | 0.2766| 0.8961| 1.8914|
| Circular, 1.4 root            |         | 1/1.4| 2/1.4| 4/1.4| 0.4118   | 0.2945| 0.8872| 1.8069|
| Circular, 1.2 root            |         | 1/1.2| 2/1.2| 4/1.2| 0.3750   | 0.3174| 0.8713| 1.6927|
| Circular, linear beam         |         | 1    | 2   | 4   | 1/3       | 0.3469| 0.8414| 1.5300|

It is then possible to obtain the exact values - according to Euler-Bernoulli theory - for every $\omega_j^*$ only by multiplying the corresponding $C_j$ - table 1 or 2 - by the slender ratio of the sample $h/L$ or $d/L$ according to (14). For instance, a rectangular linearly variable beam, with $h/L=0.1$ gives the results shown in table 3 for the non-dimensional frequencies $\omega_*$. Variable rectangular beam frequencies $f_i$ can also be expressed in terms of $f_{ui}$, the $i$-th frequency for the rectangular uniform beam. It can be derived from (8) that the ratio $f_i/f_{ui}$ -also shown in Table 3- is coincident with the ratio $C_i/C_{ui}$ where $C_{ui}$ is the slope $C$ for the $i$-th frequency rectangular uniform beam -provided that both beams are the same length and material-. Results shown in table 1 and 2 have been validated using the approximate method of Rayleigh-Ritz and FEM SolidWorks simulation tool [16] and those shown in bold by comparison with specific exact solutions in the literature [1, 10, 15].

Table 3. First three natural frequencies of a rectangular, linear beam

| Beam type            | $h/L$ | $\omega_*^1$ | $\omega_*^2$ | $\omega_*^3$ | $f_i/f_{ui}$ | $f_i/f_{u2}$ | $f_i/f_{u3}$ |
|----------------------|-------|-------------|-------------|-------------|--------------|--------------|--------------|
| Rectangular, linear  | 0.1   | 0.0244      | 0.0699      | 0.1379      | 1.5117       | 0.6902       | 0.4866       |

Now, once the frequencies are known, their associated mode shapes can be calculated by substituting the value of any of the frequencies $\omega_*^j$, in the set of boundary conditions equations, the four constants appearing on the solution for (6) are solved, and, unmaking the change of variables, the full solution for every mode shape is obtained. As an example, a linear rectangular beam vibrating in the horizontal plane can be analyzed and compared with results given in [10] (chapter II, page 123). Bouasse obtains 0.3290 for the node of the second harmonic - as position from the origin divided by length - and 0.2136 and 0.5495 for both nodes corresponding to the third harmonic. Plotting the full solution, as shown in figure 4, for each mode following the steps above, 0.3287 is obtained as the node’s position for the second mode, while 0.2133 and 0.5490 stand for the third mode.
4. Conclusions
A non-dimensional generalized differential equation and boundary conditions were derived from classical Euler-Bernoulli equation for free vibrations of variable-shape cantilever beams by applying a generalized Bouasse change of variable with certain limitations for the cross section area and inertia power laws. It was found that the fundamental frequencies for tip ended rectangular and circular cross section variable beams (14) are linearly dependent on the slenderness ratio ($h/L$ or $d/L$) whose slope $C(14)$ depends on the thickness or diameter power law exponent $a$. The equations are solved using the Wolfram Mathematic symbolic interface. Exact frequencies for rectangular/circular cross section of a varying thickness/ diameter cantilever beams are determined for a number of different power laws. It was found that the fundamental frequency of a linear, square root or cubic root shaped rectangular/circular beam is significantly higher than that of a similar uniform beam although its volume is lower than that of the uniform one. The second and third frequencies are lower. Finally, by viewing (5) for the rectangular cross section values shown in (10), it can be easily concluded that the depth $b$-constant along the beam’s span- has no influence on the natural frequencies.

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