G-Gaussian Processes under Sublinear Expectations and $q$-Brownian Motion in Quantum Mechanics

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Abstract

We provide a general approach to construct a stochastic process with a given consistent family of finite dimensional distributions under a non-linear expectation space. We use this approach to construct a generalized Gaussian process under a sublinear expectation and a $q$-Brownian motion, under a complex-valued linear expectation, with which a new type of Feynman-Kac formula can be derived to represent the solution of a Schrödinger equation.

1 Introduction

In this paper we are concerned with constructions of some typical stochastic processes under a generalized notion of expectation. We have already introduced, in [Peng2004-2010], $G$-Markovian processes, $G$-Brownian motions and the related stochastic calculus under nonlinear expectation space (see also [Hu-Peng2010] for $G$-Lévy processes). The main idea is to directly use a nonlinear (sublinear) expectation to value the random variables depending on the paths of the corresponding stochastic process. For example, a generalized Brownian motion $(B_t)_{t \geq 0}$, called $G$-Brownian motion, is merely a process with continuous paths under a specifically designed nonlinear expectation so that each increments of $(B_t)_{t \geq 0}$ is stable and independent of its historical path. It was proved that each of its increments is $G$-normally distributed random variables. On the other hand, such type of $G$-normal distributions can be also obtained as the limit in law of a sum of an i.i.d. sequence of random variables.

In this paper we will construct two types of stochastic processes. The first one is a Gaussian process under a sublinear expectation. All finite dimensional distributions of this process are $G$-normally distributed. When we deduce to a classical framework the corresponding expectation is linear and thus the $G$-normal distributions become the classical normal distributions. In this case this process is noting but a classical Gaussian. Quite different from a classical situation, in general, a $G$-Brownian
motion \((B_t)_{t \geq 0}\) is not a Gaussian process although each increment of a \(G\)-Brownian motion is normally distributed.

The second type of stochastic process is constructed under a complex-valued linear expectation in the place of a real valued expectation. Under such \(\mathbb{C}\)-valued expectation we can define a new type of random variable which is \(q\)-normally distribution and then a \(q\)-Brownian motion, where \(q\) stands for “quantum”. Under this framework the solution of the well-known Schrödinger equation can be expressed by a new formula of Feynman-Kac type. We then get a clear path picture via this \(q\)-Brownian motion.

The paper is organized as follows. In section 2 we present some preliminary notations and results of a general framework of nonlinear expectations and nonlinear expectation spaces, including nonlinear distribution, independence, \(G\)-normal distribution. In Section 3 we give a general construction of a stochastic process with a given consistent family of finite dimensional distributions. It is in fact a generalization of the well-know approach of Kolmogorov’s consistency under a nonlinear expectation. We also discuss how to find the upper expectation of a family of probability measures for some given sublinear expectation and under what condition we can obtain the continuity of stochastic process. In Section 4 the construction of a \(G\)-Gaussian process are introduced. In the last section we introduce a complex valued linear expectation under which the \(q\)-normal random variable and the \(q\)-Brownian motion are defined. We also provide an Appendix for the convenience to read the paper.

2 Preliminaries: sublinear expectation

A sublinear expectation is also called an upper expectation. It is frequently applied to situations when the probability models themselves have uncertainty. In this section, we present the basic notion of sublinear expectations and the corresponding sublinear expectation spaces.

2.1 Nonlinear expectation

Let \(\Omega\) be a given set and let \(\mathcal{H}\) be a linear space of real valued functions defined on \(\Omega\) containing constants. In this paper, we often suppose that \(|X| \in \mathcal{H}\) if \(X \in \mathcal{H}\). The space \(\mathcal{H}\) is also called the space of random variables.

Definition 2.1 A Sublinear expectation \(\mathbb{E}\) is a functional \(\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}\) satisfying

(i) **Monotonicity:**

\[
\mathbb{E}[X] \geq \mathbb{E}[Y] \quad \text{if} \quad X \geq Y.
\]

(ii) **Constant preserving:**

\[
\mathbb{E}[c] = c \quad \text{for} \quad c \in \mathbb{R}.
\]

(iii) **Sub-additivity:** For each \(X,Y \in \mathcal{H}\),

\[
\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y].
\]
(iv) **Positive homogeneity:**
\[ E[\lambda X] = \lambda E[X] \quad \text{for} \ \lambda \geq 0. \]

The triple \((\Omega, \mathcal{H}, E)\) is called a **sublinear expectation space.** If only (i) and (ii) are satisfied, \(E\) is called a **nonlinear expectation** and the triple \((\Omega, \mathcal{H}, E)\) is called a **nonlinear expectation space.**

**Remark 2.2** In more general situation the above \(E\) may be \(\mathbb{R}^k\)-valued, namely, the functions of \(\mathcal{H}\) are \(\mathbb{R}^k\)-valued and \(E\) maps \(\mathcal{H}\) to \(\mathbb{R}^k\). For linear situation we usually only assume \(E[c] = c\), for \(c \in \mathbb{R}^d\) and \(E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]\), for \(\alpha, \beta \in \mathbb{R}\) and \(X, Y \in \mathcal{H}\). \(E\) is called an \(\mathbb{R}^d\)-valued linear expectation, or simply \(\mathbb{R}^d\)-expectation. A type of \(C\)-valued linear expectation will be discussed in the Section 5.

**Definition 2.3** Let \(E_1\) and \(E_2\) be two nonlinear expectations defined on \((\Omega, \mathcal{H})\). \(E_1\) is said to be **dominated** by \(E_2\) if
\[ E_1[X] - E_1[Y] \leq E_2[X - Y] \quad \text{for} \ X, Y \in \mathcal{H}. \]

**Remark 2.4** If the inequality in (iii) becomes equality, then \(E\) is a linear expectation, i.e., \(E\) is a linear functional satisfying (i) and (ii).

**Remark 2.5** (iii)+(iv) is called **sublinearity.** This sublinearity implies (v) **Convexity:**
\[ E[\alpha X + (1 - \alpha)Y] \leq \alpha E[X] + (1 - \alpha)E[Y] \quad \text{for} \ \alpha \in [0, 1]. \]

If a nonlinear expectation \(E\) satisfies convexity, we call it a **convex expectation.**

The properties (ii)+(iii) implies (vi) **Constant translatability:**
\[ E[X + c] = E[X] + c \quad \text{for} \ c \in \mathbb{R}. \]

In fact, we have
\[ E[X] + c = E[X] - E[-c] \leq E[X + c] \leq E[X] + E[c] = E[X] + c. \]

For property (iv), an equivalence form is
\[ E[\lambda X] = \lambda^+ E[X] + \lambda^- E[-X] \quad \text{for} \ \lambda \in \mathbb{R}. \]

In this paper, we mainly consider the following type of nonlinear expectation spaces \((\Omega, \mathcal{H}, E)\): if \(X_1, \ldots, X_n \in \mathcal{H}\) then \(\varphi(X_1, \ldots, X_n) \in \mathcal{H}\) for each \(\varphi \in C_{\text{Lip}}(\mathbb{R}^n)\) where \(C_{\text{Lip}}(\mathbb{R}^n)\) denotes the linear space of functions \(\varphi\) satisfying
\[ |\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \quad \text{for} \ x, y \in \mathbb{R}^n, \]
where some \(C > 0, m \in \mathbb{N}\) depending on \(\varphi\).

In this case \(X = (X_1, \ldots, X_n)\) is called an \(n\)-dimensional random vector, denoted by \(X \in \mathcal{H}^n.\)
Remark 2.6 It is clear that if $X \in \mathcal{H}$ then $|X|, X^m \in \mathcal{H}$. More generally, $\varphi(X)\psi(Y) \in \mathcal{H}$ if $X, Y \in \mathcal{H}$ and $\varphi, \psi \in C_{Lip}(\mathbb{R})$. In particular, if $X \in \mathcal{H}$ then $\mathbb{E}[|X|^n] < \infty$ for each $n \in \mathbb{N}$.

Here we use $C_{Lip}(\mathbb{R}^n)$ in our framework only for some convenience of techniques. In fact our essential requirement is that $\mathcal{H}$ contains all constants and, moreover, $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. In general, $C_{Lip}(\mathbb{R}^n)$ can be replaced by any one of the following spaces of functions defined on $\mathbb{R}^n$:

- $L^\infty(\mathbb{R}^n)$: the space of bounded Borel-measurable functions;
- $C_b(\mathbb{R}^n)$: the space of bounded and continuous functions;
- $C^k_b(\mathbb{R}^n)$: the space of bounded and $k$-time continuously differentiable functions with bounded derivatives of all orders less than or equal to $k$;
- $C_{unif}(\mathbb{R}^n)$: the space of bounded and uniformly continuous functions;
- $C_{b,Lip}(\mathbb{R}^n)$: the space of bounded and Lipschitz continuous functions;
- $L^0(\mathbb{R}^n)$: the space of Borel measurable functions.

2.2 Representation of a sublinear expectation

A sublinear expectation can be expressed as a supremum of linear expectations.

Theorem 2.7 Let $\mathbb{E}$ be a functional defined on a linear space $\mathcal{H}$ satisfying sub-additivity and positive homogeneity. Then there exists a family of linear functionals $\{E_\theta : \theta \in \Theta\}$ defined on $\mathcal{H}$ such that

$$E[X] = \sup_{\theta \in \Theta} E_\theta[X] \quad \text{for } X \in \mathcal{H}$$

and, for each $X \in \mathcal{H}$, there exists $\theta_X \in \Theta$ such that $E[X] = E_{\theta_X}[X]$.

Furthermore, if $\mathbb{E}$ is a sublinear expectation, then the corresponding $E_\theta$ is a linear expectation.

Remark 2.8 It is important to observe that the above linear expectation $E_\theta$ is only assumed to be finitely additive. But we can apply the well-known Daniell-Stone Theorem to prove that there is a unique $\sigma$-additive probability measure $P_\theta$ on $(\Omega, \sigma(\mathcal{H}))$ such that

$$E_\theta[X] = \int_\Omega X(\omega) dP_\theta, \quad X \in \mathcal{H}.$$ 

The corresponding model uncertainty of probabilities is the subset $\{P_\theta : \theta \in \Theta\}$, and the corresponding uncertainty of distributions for an $n$-dimensional random vector $X$ in $\mathcal{H}$ is $\{F_X(\theta, A) := P_\theta(X \in A) : A \in \mathcal{B}(\mathbb{R}^n)\}$. 

4
2.3 Distributions, independence and product spaces

We now give the notion of distributions of random variables under non-linear expectations.

Let \( X = (X_1, \cdots, X_n) \) be a given \( n \)-dimensional random vector on a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). We define a functional on \( C_{l.Lip}(\mathbb{R}^n) \) by

\[
F^X[\varphi] := \mathbb{E}[\varphi(X)] : \varphi \in C_{l.Lip}(\mathbb{R}^n) \rightarrow \mathbb{R}.
\]

The triple \((\mathbb{R}^n, C_{l.Lip}(\mathbb{R}^n), F^X)\) forms a nonlinear expectation space. \( F^X \) is called the distribution of \( X \) under \( \mathbb{E} \). In the \( \sigma \)-additive situation (see Remark 2.8), we have the following form:

\[
F_X[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}^n} \varphi(x) F_X(\theta, dx).
\]

**Definition 2.9** Let \( X_1 \) and \( X_2 \) be two \( n \)-dimensional random vectors defined on nonlinear expectation spaces \((\Omega_1, \mathcal{H}_1, \mathbb{E}_1)\) and \((\Omega_2, \mathcal{H}_2, \mathbb{E}_2)\), respectively. They are called identically distributed, denoted by \( X_1 \overset{d}{=} X_2 \) or \( F^{X_1} = F^{X_2} \), if

\[
\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)] \quad \text{for } \varphi \in C_{l.Lip}(\mathbb{R}^n).
\]

It is clear that \( X_1 \overset{d}{=} X_2 \) if and only if their distributions coincide. We say that the distribution of \( X_1 \) is stronger than that of \( X_2 \) if \( \mathbb{E}_1[\varphi(X_1)] \geq \mathbb{E}_2[\varphi(X_2)] \), for each \( \varphi \in C_{l.Lip}(\mathbb{R}^n) \).

**Remark 2.10** In the case of sublinear expectations, \( X_1 \overset{d}{=} X_2 \) implies that the uncertainty subsets of distributions of \( X_1 \) and \( X_2 \) are the same, e.g., in the framework of Remark 2.8,

\[
\{F_{X_1}(\theta_1, \cdot) : \theta_1 \in \Theta_1\} = \{F_{X_2}(\theta_2, \cdot) : \theta_2 \in \Theta_2\}.
\]

Similarly if the distribution of \( X_1 \) is stronger than that of \( X_2 \), then

\[
\{F_{X_1}(\theta_1, \cdot) : \theta_1 \in \Theta_1\} \supset \{F_{X_2}(\theta_2, \cdot) : \theta_2 \in \Theta_2\}.
\]

The distribution of \( X \in \mathcal{H} \) has the following four typical parameters:

\[
\bar{\mu} := \mathbb{E}[X], \quad \underline{\mu} := -\mathbb{E}[-X], \quad \sigma^2 := \mathbb{E}[X^2], \quad \underline{\sigma}^2 := -\mathbb{E}[-X^2].
\]

The intervals \([\underline{\mu}, \bar{\mu}]\) and \([\underline{\sigma}^2, \bar{\sigma}^2]\) characterize the mean-uncertainty and the variance-uncertainty of \( X \) respectively.

The following property is very useful in our sublinear expectation theory.

**Proposition 2.11** Let \((\Omega, \mathcal{H}, \mathbb{E})\) be a sublinear expectation space and \( X, Y \) be two random variables such that \( \mathbb{E}[Y] = -\mathbb{E}[-Y] \), i.e., \( Y \) has no mean-uncertainty. Then we have

\[
\mathbb{E}[X + \alpha Y] = \mathbb{E}[X] + \alpha \mathbb{E}[Y] \quad \text{for } \alpha \in \mathbb{R}.
\]

In particular, if \( \mathbb{E}[Y] = \mathbb{E}[-Y] = 0 \), then \( \mathbb{E}[X + \alpha Y] = \mathbb{E}[X] \).
Definition 2.12 A sequence of n-dimensional random vectors \( \{ \eta_i \}_{i=1}^{\infty} \) defined on a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is said to converge in distribution (or converge in law) under \( \mathbb{E} \) if for each \( \varphi \in C_{b,Lip}(\mathbb{R}^n) \), the sequence \( \{ \mathbb{E}[\varphi(\eta_i)] \}_{i=1}^{\infty} \) converges.

The following result is easy to check.

Proposition 2.13 Let \( \{ \eta_i \}_{i=1}^{\infty} \) converges in law in the above sense. Then the mapping \( F[\cdot] : C_{b,Lip}(\mathbb{R}^n) \rightarrow \mathbb{R} \) defined by

\[ F[\varphi] := \lim_{i \rightarrow \infty} \mathbb{E}[\varphi(\eta_i)] \quad \text{for} \quad \varphi \in C_{b,Lip}(\mathbb{R}^n) \]

is a nonlinear expectation defined on \((\mathbb{R}^n, C_{b,Lip}(\mathbb{R}^n))\). If \( \mathbb{E} \) is sublinear (resp. linear), then \( F \) is also sublinear (resp. linear).

The following notion of independence plays a key role in the sublinear expectation theory.

Definition 2.14 In a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), a random vector \( Y \in \mathcal{H}^n \) is said to be independent of another random vector \( X \in \mathcal{H}^m \) under \( \mathbb{E}[\cdot] \) if for each test function \( \varphi \in C_{l,Lip}(\mathbb{R}^{m+n}) \) we have

\[ \mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\varphi(x,Y)]_{x=X}. \]

Remark 2.15 In a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), \( Y \) is independent of \( X \) means that the uncertainty of distributions \( \{ F_Y(\theta, \cdot) : \theta \in \Theta \} \) of \( Y \) does not change after the realization of \( X = x \). In other words, the “conditional sublinear expectation” of \( Y \) with respect to \( X \) is \( \mathbb{E}[\varphi(x,Y)]_{x=X} \). In the case of linear expectation, this notion of independence is just the classical one.

Remark 2.16 It is important to note that under a sublinear expectation the condition “\( Y \) is independent from \( X \)” does not imply automatically that “\( X \) is independent from \( Y \)”.

The independence property of two random vectors \( X, Y \) involves only the “joint distribution” of \((X,Y)\). The following result tells us how to construct random vectors with given “marginal distributions” and with a specific direction of independence.

Definition 2.17 Let \((\Omega_i, \mathcal{H}_i, \mathbb{E}_i)_{i \in I}\) be nonlinear expectation spaces indexed by \( I \). We denote

\[ \prod_{i \in I} \Omega_i = \{ (\omega_i : i \in I) : \omega_i \in \Omega_i, \ i \in I \} \]

\[ \bigotimes_{i \in I} \mathcal{H}_i := \{ Y(\omega_1, \ldots, \omega_n) = \varphi(X_{11}(\omega_1), \ldots, X_{1n}(\omega_n)) : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2, \]

\[ X_{ik} \in \mathcal{H}_{ik}^{d_{ik}}, \ k = 1, \ldots, n, \ \varphi \in C_{l,Lip}(\mathbb{R}^{d_1 + \cdots + d_n}) \}, \]

We denote by \( \Omega := \prod_{i \in I} \Omega_i \) and \( \mathcal{H} := \bigotimes_{i \in I} \mathcal{H}_i \). It is clear that \( \mathcal{H} \) forms a vector lattice on \( \Omega \). Let \( \mathbb{E} \) be a nonlinear expectation defined on \( \mathcal{H} \). If for each \( i \in I \) and \( X_i \in \mathcal{H}_i^{d_i} \) we always have \( \mathbb{E}[\varphi(X_i)] = \mathbb{E}_i[\varphi(X_i)] \), then we say that the margin of \( \mathbb{E} \) coincides with \( \mathbb{E}_i \).

Remark 2.18 In the last section, the above notion of independence is extended to an \( \mathbb{C} \)-valued linear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\).
2.4 Completion of a sublinear expectation space

Let \((\Omega, \mathcal{H}, \mathbb{E})\) be a sublinear expectation space. We have the following useful inequalities.

We first give the following well-known inequalities.

**Lemma 2.19** For \(r > 0\) and \(1 < p, q < \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\), we have

\[
|a + b|^p \leq \max\{1, 2^{r-1}\}(|a|^r + |b|^r) \quad \text{for } a, b \in \mathbb{R},
\]

\[
|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.
\]

**Proposition 2.20** For each \(X, Y \in \mathcal{H}\), we have

\[
\mathbb{E}[|X + Y|^r] \leq 2^{r-1}(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r]),
\]

\[
\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{1/p} \cdot (\mathbb{E}[|Y|^q])^{1/q},
\]

\[
(\mathbb{E}[|X + Y|^p])^{1/p} \leq (\mathbb{E}[|X|^p])^{1/p} + (\mathbb{E}[|Y|^p])^{1/p},
\]

where \(r \geq 1\) and \(1 < p, q < \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\).

In particular, for \(1 \leq p < p'\), we have \((\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^{p'})^{1/p'}\).

**Proof.** The inequality (3) follows from (1).

For the case \(\mathbb{E}[|X|^p] \cdot \mathbb{E}[|Y|^q] > 0\), we set

\[
\xi = \frac{X}{(\mathbb{E}[|X|^p])^{1/p}}, \quad \eta = \frac{Y}{(\mathbb{E}[|Y|^q])^{1/q}}.
\]

By (2) we have

\[
\mathbb{E}[|\xi \eta|] \leq \mathbb{E}[\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}] \leq \mathbb{E}[\frac{|\xi|^p}{p}] + \mathbb{E}[\frac{|\eta|^q}{q}]
\]

\[
= \frac{1}{p} + \frac{1}{q} = 1.
\]

Thus (4) follows.

For the case \(\mathbb{E}[|X|^p] \cdot \mathbb{E}[|Y|^q] = 0\), we consider \(\mathbb{E}[|X|^p] + \varepsilon\) and \(\mathbb{E}[|Y|^q] + \varepsilon\) for \(\varepsilon > 0\). Applying the above method and letting \(\varepsilon \to 0\), we get (4).

We now prove (5). We only consider the case \(\mathbb{E}[|X + Y|^p] > 0\).

\[
\mathbb{E}[|X + Y|^p] = \mathbb{E}[|X + Y|] \cdot |X + Y|^{p-1}
\]

\[
\leq \mathbb{E}[|X|] \cdot |X + Y|^{p-1} + \mathbb{E}[|Y|] \cdot |X + Y|^{p-1}
\]

\[
\leq (\mathbb{E}[|X|^p])^{1/p} \cdot (\mathbb{E}[|X + Y|^{(p-1)q}])^{1/q}
\]

\[
+ (\mathbb{E}[|Y|^p])^{1/p} \cdot (\mathbb{E}[|X + Y|^{(p-1)q}])^{1/q}.
\]

Since \((p-1)q = p\), we have (5).

By (1), it is easy to deduce that \((\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^{p'})^{1/p'}\) for \(1 \leq p < p'\).

For each fixed \(p \geq 1\), we observe that \(\mathcal{H}_0^p = \{ X \in \mathcal{H}, \mathbb{E}[|X|^p] = 0\}\) is a linear subspace of \(\mathcal{H}\). Taking \(\mathcal{H}_0^p\) as our null space, we introduce
the quotient space $\mathcal{H}/\mathcal{H}^0_p$. Observing that, for every $\{X\} \in \mathcal{H}/\mathcal{H}^0_p$ with a representation $X \in \mathcal{H}$, we can define an expectation $E[\{X\}] := \mathbb{E}[X]$ which is still a sublinear expectation. We set $\|X\|_p := \left(\mathbb{E}[|X|^p]\right)^{\frac{1}{p}}$. By Proposition 2.20, it is easy to check that $\|\cdot\|_p$ forms a Banach norm on $\mathcal{H}/\mathcal{H}^0_p$. We extend $\mathcal{H}/\mathcal{H}^0_p$ to its completion $\hat{\mathcal{H}}_p$ under this norm, then $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ is a Banach space. In particular, when $p = 1$, we denote it by $(\hat{\mathcal{H}}, \|\cdot\|)$. For each $X \in \mathcal{H}$, the mappings
\[
X^+(\omega) : \mathcal{H} \to \mathcal{H} \quad \text{and} \quad X^- (\omega) : \mathcal{H} \to \mathcal{H}
\]
satisfy
\[
|X^+ - Y^+| \leq |X - Y| \quad \text{and} \quad |X^- - Y^-| = |(-X)^+ - (-Y)^+| \leq |X - Y|.
\]
Thus they are both contraction mappings under $\|\cdot\|_p$ and can be continuously extended to the Banach space $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$.

**Definition 2.21** An element $X$ in $(\hat{\mathcal{H}}, \|\cdot\|)$ is said to be nonnegative, or $X \geq 0$, $0 \leq X$, if $X = X^+$. We also denote by $X \geq Y$, or $Y \leq X$, if $X - Y \geq 0$.

It is easy to check that $X \geq Y$ and $Y \geq X$ imply $X = Y$ on $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$.

For each $X, Y \in \mathcal{H}$, note that
\[
|E[X] - E[Y]| \leq \|X - Y\| \leq \|[X - Y]\|_p.
\]
Thus the sublinear expectation $E[\cdot]$ can be continuously extended to $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ on which it is still a sublinear expectation.

Let $(\Omega, \mathcal{H}, \mathbb{E}_1)$ be a nonlinear expectation space. $\mathbb{E}_1$ is said to be dominated by $\mathbb{E}$ if
\[
\mathbb{E}_1[X] - \mathbb{E}_1[Y] \leq \mathbb{E}[X - Y] \quad \text{for } X, Y \in \mathcal{H}.
\]
From this we can easily deduce that $|\mathbb{E}_1[X] - \mathbb{E}_1[Y]| \leq \|X - Y\|_1$, thus the nonlinear expectation $\mathbb{E}_1[\cdot]$ can be continuously extended to $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ on which it is still a nonlinear expectation.

**Remark 2.22** It is important to note that $X_1, \ldots, X_n \in \hat{\mathcal{H}}$ does not imply $\varphi (X_1, \ldots, X_n) \in \hat{\mathcal{H}}$ for each $\varphi \in C_{1, \text{Lip}}(\mathbb{R}^n)$. Thus, when we talk about the notions of distributions, independence and product spaces on $(\Omega, \hat{\mathcal{H}}, \mathbb{E})$, the space $C_{1, \text{Lip}}(\mathbb{R}^n)$ is replaced by $C_{b, \text{Lip}}(\mathbb{R}^n)$ unless otherwise stated.

### 2.5 $G$-normal distributions

A well-known characterization for a zero-mean $d$-dimensional normally distributed random variable $X$ is
\[
aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X \quad \text{for } a, b \geq 0,
\]
where $\bar{X}$ is an independent copy of $X$. The covariance matrix $\Sigma$ is defined by $\Sigma = E[XX^T]$. We now consider the so called $G$-normal distribution in probability model uncertainty situation.
Definition 2.23 (G-normal distribution) A d-dimensional random vector \( X = (X_1, \cdots, X_d) \) on a sublinear expectation space \((\Omega, \mathcal{H}, E)\) is called (centralized) G-normal distributed if
\[
aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X \quad \text{for } a, b \geq 0,
\]
where \( \bar{X} \) is an independent copy of \( X \).

Remark 2.24 Noting that \( E[X + \bar{X}] = 2E[X] \) and \( E[X + \bar{X}] = \sqrt{2}E[X] = \sqrt{2}E[X] \), we then have \( E[X] = 0 \). Similarly, we can prove that \( E[-X] = 0 \). Namely, \( X \) has no mean-uncertainty.

The following property is easy to prove by the definition.

Proposition 2.25 Let \( X \) be G-normal distributed. Then for each \( A \in \mathbb{R}^{m \times d} \), \( AX \) is also G-normal distributed. In particular, for each \( a \in \mathbb{R}^d \), \( \langle a, X \rangle \) is a 1-dimensional G-normal distributed random variable.

We denote by \( S(d) \) the collection of all \( d \times d \) symmetric matrices. Let \( X \) be a \( d \)-dimensional G-normal distributed random vector in \((\Omega, \mathcal{H}, E)\). The following function is very important to characterize their distributions:
\[
G(A) := \mathbb{E}\left[ \frac{1}{2} \langle AX, X \rangle \right], \quad A \in S(d).
\]
It is easy to check that \( G \) is a sublinear function monotonic in \( A \in S(d) \) in the following sense: for each \( A, \bar{A} \in S(d) \)
\[
\begin{align*}
G(A + \bar{A}) & \leq G(A) + G(\bar{A}), \\
G(\lambda A) & = \lambda G(A), \quad \forall \lambda \geq 0, \\
G(A) & \geq G(\bar{A}), \quad \text{if } A \succeq \bar{A}.
\end{align*}
\]
Clearly, \( G \) is also a continuous function. By Theorem 2.7, there exists a bounded and closed subset \( \Gamma \subset \mathbb{R}^{d \times d} \) such that
\[
G(A) = \sup_{Q \in \Gamma} \frac{1}{2} \text{tr}[AQ^T] \quad \text{for } A \in S(d).
\]
The following copy result can be found in [Peng2010].

Proposition 2.26 Let \( G : S(d) \to \mathbb{R} \) be a given sublinear and continuous function, monotonic in \( A \in S(d) \) in the sense of (9). Then there exists a G-normal distributed \( d \)-dimensional random vector \( X \) on some sublinear expectation space \((\Omega, \mathcal{H}, E)\). satisfying (7). Moreover, if both \( X \) and \( Y \) are G-normal distributed with the same function \( G \), namely
\[
E[\langle AX, X \rangle] = E[\langle AY, Y \rangle] = 2G(A), \quad \forall A \in S(d),
\]
then \( X \overset{d}{=} Y \).

We present a central limit theorem in the framework of sublinear expectation (see [Peng2007], [Peng2009] or [Peng2010, ThmII.3.3]).
Theorem 2.27 Let \( \{X_i\}_{i=1}^{\infty} \) be a sequence of \( \mathbb{R}^d \) valued random vectors in a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). We assume that \( \{X_i\}_{i=1}^{\infty} \) is an i.i.d. sequence, i.e., for each \( i = 1, 2, \ldots, X_{i+1} \overset{d}{=} X_1 \) and it is independent of \((X_1, \cdots, X_i)\). We assume furthermore that \( \mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0 \). Then the sequence \( \{\bar{S}_n\}_{n=1}^{\infty} \) defined by
\[
\bar{S}_n = (X_1 + \cdots + X_n)/\sqrt{n}
\]
converges in law to \( X \):
\[
\lim_{n \to \infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(X)], \quad \varphi \in C_b(\mathbb{R}^d),
\]
where \( X \) is a \( d \)-dimensional \( G \)-normally distributed random variable with
\[
G(A) = \frac{1}{2} \mathbb{E}[\langle AX_1, X_1 \rangle].
\]

3 Construction of stochastic processes with a given family of finite dimensional distributions

Definition 3.1 Let \((\Omega, \mathcal{H}, \mathbb{E})\) be a nonlinear expectation space. \((X_t)_{t \geq 0}\) is called a \( d \)-dimensional stochastic process if for each \( t \geq 0 \), \( X_t \) is a \( d \)-dimensional random vector in \( \mathcal{H} \).

A typical example of such type of stochastic processes defined on a space of nonlinear expectation is the so-called \( G \)-Brownian motion.

Definition 3.2 A \( d \)-dimensional process \((B_t)_{t \geq 0}\) on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is called a \( G \)-Brownian motion if the following properties are satisfied:

(i) \( B_0(\omega) = 0 \);

(ii) For each \( t, s \geq 0 \), the increment \( B_{t+s} - B_t \overset{d}{=} B_s \) and is independent of \((B_{t_1}, B_{t_2}, \cdots, B_{t_n})\), for each \( n \in \mathbb{N} \) and \( t_1, \cdots, t_n \in [0, t] \).

(iii) \( \lim_{t \to 0} \mathbb{E}[|B_t|^3]/t = 0 \).

The following theorem gives a characterization of \( G \)-Brownian motion.

Theorem 3.3 Let \((B_t)_{t \geq 0}\) be a \( d \)-dimensional \( G \)-Brownian motion defined on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), such that \( \mathbb{E}[B_t] = \mathbb{E}[-B_t] = 0 \), \( t \geq 0 \). Then, for each \( t > 0 \), \( B_t \) is normally distributed, namely,
\[
aB_t + b(B_{2t} - B_t) \overset{d}{=} \sqrt{a^2 + b^2} B_t, \quad a, b \geq 0.
\]

Using a generalization of Kolmogorov approach, a new type of Markovian processes was introduced in [Peng2004] and then \( G \)-Brownian motion in [Peng2007].

In this section we will use this approach to give a general construction of stochastic processes so that it can be also applied to construct a new type of Gaussian processes under a sublinear expectation.

We first notice that, just as in the classical situation, one can define the family of finite dimensional distributions for a given stochastic process \( X = (X_t)_{t \geq 0} \). We denote
\[ \mathcal{T} := \{ \ul{t} = (t_1, \ldots, t_m) : \forall m \in \mathbb{N}, t_i \in [0, \infty), t_i \neq t_j, 0 \leq i,j \leq m, i \neq j \}. \]

**Definition 3.4** If for each \( \ul{t} = (t_1, \ldots, t_m) \in \mathcal{T} \), \( F_{\ul{t}} \) is a nonlinear expectation defined on \((\mathbb{R}^{m \times d}, C_{l.lip}(\mathbb{R}^{m \times d}))\) then we call \((F_{\ul{t}})_{\ul{t} \in \mathcal{T}}\) a family of finite dimensional distributions on \( \mathbb{R}^d \).

**Definition 3.5** Let \((X_t)_{t \geq 0}\) be an \( \mathbb{R}^d \)-valued stochastic process defined in a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). For each \( \ul{t} = (t_1, \cdots, t_n) \in \mathcal{T} \) the random variable \( X_{\ul{t}} = (X_{t_1}, \ldots, X_{t_n}) \) induces a distribution \( F_{\ul{t}} X = \mathbb{E}[\varphi(X_{\ul{t}})] \), \( \varphi \in C_{l.lip}(\mathbb{R}^{n \times d}) \). We call \((F_{\ul{t}} X)_{\ul{t} \in \mathcal{T}}\) the family of finite dimensional distributions corresponding to \((X_t)_{t \geq 0}\).

**Definition 3.6** Let \((X_t)_{t \geq 0}\) and \((\tilde{X}_t)_{t \geq 0}\) be \( d \)-dimensional stochastic processes defined respectively in nonlinear expectation spaces \((\Omega, \mathcal{H}, \mathbb{E})\) and \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})\). \( X \) and \( \tilde{X} \) are said to be identically distributed if their families of finite dimensional distributions coincide from each other:

\[ F_{\ul{t}} X = F_{\ul{t}} \tilde{X}, \quad \forall \ul{t} \in \mathcal{T}. \]

For a given stochastic process, the family of its finite dimensional distributions satisfies the following properties of consistency:

**Definition 3.7** A family of finite dimensional distributions \( \{ F_{\ul{t}}[\varphi] \}_{\ul{t} \in \mathcal{T}} \) on \( \mathbb{R}^d \) is said to be consistent if, for each \( \ul{t} = (t_1, \cdots, t_n) \in \mathcal{T} \), we have

(i) \[ F_{\ul{t}}[\varphi] = F_{\ul{t}}[\varphi], \quad \varphi \in C_{l.lip}(\mathbb{R}^{n \times d}). \]

Here, on the right hand side, \( \varphi \) is considered as a function defined on \( \mathbb{R}^{n \times d} \times \mathbb{R}^d \) which does not depend on the last coordinate.

(ii) For each permutation \( \sigma \) of \( (1, 2, \cdots, n) \)

\[ F_{\sigma(t_1), \cdots, t_n}[\varphi] = F_{t_1, \cdots, t_n}[\varphi_{\sigma}] \]

where

\[ \varphi_{\sigma}(x_1, \cdots, x_n) = \varphi(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), \quad x_i \in \mathbb{R}^d, \quad i = 1, \cdots, n. \]

It is clear that the finite dimensional distributions of the process \( X \) is consistent. Inversely, we have:

**Theorem 3.8** Let \((F_{\ul{t}})_{\ul{t} \in \mathcal{T}}\) be a family of consistent nonlinear distributions. Then there exists a \( d \)-dimensional stochastic process \((X_t)_{t \geq 0}\) defined on a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) such that the family of finite dimensional distributions of \( X \) coincides with \((F_{\ul{t}})_{\ul{t} \in \mathcal{T}}\). \( \mathbb{E} \) can be a sublinear (resp. linear) expectation if the distributions in \((F_{\ul{t}})_{\ul{t} \in \mathcal{T}}\) are all sublinear (resp. linear).

**Proof.** Let \( \Omega = (\mathbb{R}^d)^{[0, \infty)} \) denote the space of all \( \mathbb{R}^d \)-valued functions \((\omega_t)_{t \in \mathbb{R}^+}\). We denote by \( \tilde{\mathcal{X}}(\omega) = \omega_t, \ t \in [0, \infty), \ \omega \in \Omega \), the corresponding canonical process. The space of Lipschitzian cylinder functions on \( \Omega \) is denoted by

\[ \mathcal{H} = L_{l.lip}(\Omega) := \{ \varphi(X_{t_1}, \cdots, X_{t_n}), \ul{t} = (t_1, \cdots, t_n) \in \mathcal{T}, \forall \varphi \in C_{l.lip}(\mathbb{R}^{d \times n}) \}. \]
It is clear that $L_{ip} (\Omega)$ is a vector lattice. For each $\xi \in L_{ip}(\Omega)$ of the form $\xi(\omega) = \varphi(X_{t_1}(\omega), \ldots, X_{t_n}(\omega))$, we set

$$E[\varphi(\xi)] = F_{t_1, \ldots, t_n}[\varphi].$$

It is clear that the mapping $E : L_{ip}(\Omega) \rightarrow \mathbb{R}$ forms a consistently defined nonlinear expectation on $(\Omega, L_{ip}(\Omega))$ and the family of the finite dimensional distributions of $X$ is $(F_{t})_{t \in T}$. Consequently, $E$ is a sublinear expectation.

We see that with this construction $E$ is sublinear (resp. linear) expectation if the distributions in $(F_{t})_{t \in T}$ are all sublinear (resp. linear).

**Remark 3.9** In the proof of Theorem 3.8 we can also use $\Omega = C([0, \infty); \mathbb{R}^d)$ in the place of $\Omega = (\mathbb{R}^d)^{[0, \infty)}$. But we will need the later one in the proof of Lemma 3.12. In many situations, similar to the classical situation, we need to introduce the natural capacity $\hat{c}$ associated to $E$ and use the related “c-‘quasi sure’ analysis” to study the continuity of the process $X$. We refer to [Denis-Hu-Peng2010] or [Hu-Peng2010] for the proof of the continuity of a G-Brownian motion.

**Remark 3.10** Definitions 2.4-2.7 as well as Theorem 3.8 can be extended to construct a $\mathbb{R}^d$-valued nonlinear expectation. We will see in Section 6 a typical $C$-valued expectation.

We will prove that when $(F_{t})_{t \in T}$ is sublinear, then the corresponding sublinear expectation $E$ constructed above is an upper expectation of a family of probability measures on $(\Omega, \mathcal{F})$. We need the following lemmas:

**Lemma 3.11** Let $\xi \in \mathcal{H}^m$ be a given random vector in a linear expectation space $(\Omega, \mathcal{H}, E)$ such that $\varphi(\xi) \in \mathcal{H}$ for each $\varphi \in C_{l.\text{lip}}(\mathbb{R}^m)$. Then there exists a unique probability measure $Q$ on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ such that $E_Q[\varphi] = E[\varphi(\xi)]$, $\varphi \in C_{l.\text{lip}}(\mathbb{R}^m)$.

**Proof.** Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $C_{l.\text{lip}}(\mathbb{R}^m)$ satisfying $\varphi_n \downarrow 0$. For each $N > 0$, it is clear that

$$\varphi_n(x) \leq k^n_N + \varphi_1(x)I_{\{|x| > N\}} \leq k^n_N + \frac{\varphi_1(x)|x|}{N}$$

for each $x \in \mathbb{R}^d \times \mathbb{R}^m$,

where $k^n_N = \max_{|x| \leq N} \varphi_n(x)$. Noting that $\varphi_1(x)|x| \in C_{l.\text{lip}}(\mathbb{R}^m)$, we have

$$E[\varphi_n(\xi)] \leq k^n_N + \frac{1}{N} E[\varphi_1(\xi)|\xi|].$$

It follows from $\varphi_n \downarrow 0$ that $k^n_N \downarrow 0$. Thus we have $\lim_{N \rightarrow \infty} E[\varphi_n(\xi)] \leq \frac{1}{N} E[\varphi_1(\xi)|\xi|]$. Since $N$ can be arbitrarily large, we get $E[\varphi_n(\xi)] \downarrow 0$. Consequently, $E[\varphi_n(\xi)] \downarrow 0$.

It follows from Daniell-Stone’s theorem that there exists a unique probability measure $Q$ on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ such that $E_Q[\varphi] = E[\varphi(X)]$, for each $\varphi \in C_{l.\text{lip}}(\mathbb{R}^m)$. ■

**Lemma 3.12** Let $E$ be a sublinear expectation on $(\Omega, L_{ip}(\Omega))$ and $E$ be a (finitely additive) linear expectation on $(\Omega, L_{ip}(\Omega))$ which is dominated by $E$. Then there exists a unique probability measure $Q$ on $(\Omega, \sigma(L_{ip}(\Omega))$ such that $E[X] = E_Q[\xi]$ for each $\xi \in L_{ip}(\Omega)$. ■
Proof. For each fixed \( t = (t_1, \ldots, t_m) \in \mathcal{T} \), we denote \( X = (X_{t_1}, \ldots, X_{t_m}) \).
by Lemma 3.11 there exists a unique probability measure \( Q_t \) on \((\mathbb{R}^{d \times m}, \mathcal{B}(\mathbb{R}^{d \times m}))\)
such that \( \mathbb{E}^Q_t[\varphi] = \mathbb{E}[\varphi(X_{t_1}, \ldots, X_{t_m})] \) for each \( \varphi \in \mathcal{C}_{Lip}(\mathbb{R}^{d \times m}) \).
Thus, corresponding to \((X_{t_1}, \ldots, X_{t_m}), t = (t_1, \ldots, t_m) \in \mathcal{T}\), we get a family
of finite dimensional distributions \( \{Q_t : t \in \mathcal{T}\} \).
It is easy to check that \( \{Q_t : t \in \mathcal{T}\} \) is consistent. Then by Kolmogorov’s consistent the-
orem, there exists a probability measure \( Q \) on \((\Omega, \sigma(Lip(\Omega)))\) such that
\( \{Q_t : t \in \mathcal{T}\} \) is the finite dimensional distributions of \( Q \). Now assume
that there exists another probability measure \( \bar{Q} \) satisfying the condition,
by Daniell-Stone’s theorem, \( Q \) and \( \bar{Q} \) have the same finite-dimensional
distributions. Then by monotone class theorem, \( Q = \bar{Q} \). The proof is
complete. \( \square \)

Lemma 3.13 There exists a family of probability measures \( \mathcal{P}_e \) on \((\Omega, \sigma(\Omega))\)
such that
\( \mathbb{E}^\mathcal{P}_e[X] = \max_{Q \in \mathcal{P}_e} \mathbb{E}^Q[X] \), for \( X \in Lip(\Omega) \).
Proof. By the representation theorem of sublinear expectation and Lemma 3.12, it is easy to get the result. \( \square \)

For this \( \mathcal{P}_e \), we define the associated capacity:
\[ \bar{c}(A) := \sup_{Q \in \mathcal{P}_e} Q(A), \quad A \in \mathcal{B}(\Omega), \]
and the upper expectation for each \( \mathcal{B}(\Omega) \)-measurable real function \( X \) which makes the following definition meaningful:
\[ \mathbb{E}^\mathcal{P}_e[X] := \sup_{Q \in \mathcal{P}_e} \mathbb{E}^Q[X]. \]

4 Gaussian processes in a sublinear ex-
pectation space

In this section we generalize the notion of Gaussian processes to the situa-
tion in sublinear expectation space.

Definition 4.1 An \( \mathbb{R}^d \)-valued stochastic process \( X = (X_t)_{t \geq 0} \) defined in
a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is called a Gaussian process if for
each \( t = (t_1, \ldots, t_n) \in \mathcal{T}, X_t = (X_{t_1}, \ldots, X_{t_n}) \) is an \( \mathbb{R}^{n \times d} \)-valued nor-
mally distributed random variable.

In this section we are only concerned with the processes satisfying
\( \mathbb{E}[X_t] = \mathbb{E}[-X_t] = 0 \).

Definition 4.2 Let \((X_t)_{t \geq 0}\) be a given \( d \)-dimensional stochastic process.
We denote, for each \( t = (t_1, \ldots, t_n) \in \mathcal{T}, \)
\[ G^X_t(A) := \frac{1}{2} \mathbb{E}[\langle AX_t, X_t \rangle] : A \in \mathcal{S}_{n \times d}. \]
We called \( \{G^X_t\}_{t \in \mathcal{T}} \) the family of 2nd moments of the process \( X \). It is
clear that for each \( t = (t_1, \ldots, t_n) \in \mathcal{T}, G^X_t : \mathcal{S}_{n \times d} \rightarrow \mathbb{R} \) is a sublinear
and monotone function.
Definition 4.3 A family $G_{t_1,\cdots,t_n} : \mathbb{S}_{n \times d} \mapsto \mathbb{R}$, $\mathcal{L} = (t_1, \cdots, t_n) \in \mathcal{T}$ of sublinear and monotone functions is called consistent if it satisfies, for each $\mathcal{L} = (t_1, \cdots, t_n) \in \mathcal{T}$ and $t_{n+1} \geq 0$,

(i) $G_{t_1,\cdots,t_{n+1}}(\mathcal{A}) = G_{\mathcal{A}}(A)$, for each $A \in \mathbb{S}_{n \times d}$ where

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_{(n+1) \times d}$$

(ii) $G_{t_2(t_1),\cdots,t_{(n)}(n)}(A) = G_{\mathcal{L}}(\sigma(A))$, where, for each $A \in \mathbb{S}_{n \times d}$ with the form $A = [A_{ij}]_{j=1}^n$, $A_{ij} \in \mathbb{R}^{d \times d}$, $\sigma(A)$ is defined by $\sigma(A) = [A_{\sigma(i)\sigma(j)}]_{j=1}^n$.

It is clear that the family $(G^X_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}}$ of 2nd moments of the process $X$ is consistent. Inversely, we have:

Proposition 4.4 Let a family of sublinear monotone functions $\{G_{\mathcal{L}}\}_{\mathcal{L} \in \mathcal{T}}$ be consistent. Then for each $\mathcal{L} = (t_1, \cdots, t_n) \in \mathcal{T}$, there is a unique $n \times d$-dimensional normal distribution $\mathbb{F}_{\mathcal{L}}$ defined on $(\mathbb{R}^{n \times d}, \mathcal{L}_{\mathcal{L}}(\mathbb{R}^{n \times d}))$. Moreover the family of finite distributions $\{\mathbb{F}_{\mathcal{L}}\}_{\mathcal{L} \in \mathcal{T}}$ is consistent. Consequently there exists a $d$-dimensional Gaussian process $(X_t)_{t \geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ such that

$$(G^X_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}} = (G_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}}, \quad (\mathbb{F}^X_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}} = (\mathbb{F}_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}}$$

for each $\mathcal{L} = (t_1, \cdots, t_n) \in \mathcal{T}$, the random vector $X_{\mathcal{L}} = (X_{t_1}, \cdots, X_{t_n})$ is $G$-normal distributed with $G = G_{\mathcal{L}}$.

Example 4.5 Let $(B_t)_{t \geq 0}$ be a $d$-dimensional $G$-Brownian motion in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ with

$$G(A) = \frac{1}{2} \mathbb{E}([AB_1, B_1])$$

For each $\mathcal{L} = (t_1, \cdots, t_n) \in \mathcal{T}$, we set $B_{\mathcal{L}} = (B_{t_1}, \cdots, B_{t_n})$ and

$$G^B_{\mathcal{L}}(A) := \frac{1}{2} \mathbb{E}([AB_{\mathcal{L}}, B_{\mathcal{L}}]) : A \in \mathbb{S}_{n \times d} \mapsto \mathbb{R}.$$ $$\{G^B_{\mathcal{L}}\}_{\mathcal{L} \in \mathcal{T}}$$ is the family of 2nd moments of $(B)_t_{t \geq 0}$ and thus satisfying the above consistency. We then can construct a $G$-Gaussian process $(X_t)_{t \geq 0}$ such that, $G^B_{\mathcal{L}} = G^X_{\mathcal{L}}$ for each $\mathcal{L} \in \mathcal{T}$. But, in general, their family of finite dimensional distributions $(\mathbb{F}^B_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}}$ and $(\mathbb{F}^X_{\mathcal{L}})_{\mathcal{L} \in \mathcal{T}}$ are not the same.

Since we still have

$$\mathbb{E}[|X_t - X_s|^4] = \mathbb{E}[|B_t - B_s|^4] \leq d|t - s|^2,$$

We then can apply the same arguments as in the case of $G$-Brownian motion to prove that there exists a weakly compact family $\mathcal{F}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$, where $\Omega = C([0, \infty); \mathbb{R}^d)$ equipped with the usual local uniform convergence topology, such that, the canonical process $\hat{B}_t(\omega) = \hat{\omega}_t, t \geq 0$ is a Gaussian process such that

$$G^B_{\mathcal{L}} = G^B_{\mathcal{L}} : \mathcal{L} \in \mathcal{T}.$$ 

Readers who are interested in the details can see our appendix.
Definition 4.6 Let \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) be two stochastic processes in a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). They are called identically distributed if \(\mathbb{F}_t^X = \mathbb{F}_t^Y\), for each \(t \in T\). \((Y_t)_{t \geq 0}\) is said to be distributionally independent of another process \((Z_t)_{t \geq 0}\) if, for each \(t = (t_1, \cdots, t_n) \in T\), \((Y_{t_1}, \cdots, Y_{t_n})\) is independent of \((Z_{t_1}, \cdots, Z_{t_n})\).

Definition 4.7 A sequence of \(d\)-dimensional stochastic processes \(\{(X_t)_{t \geq 0}\}_{i=1}^\infty\) in a nonlinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is said to be convergence in finite dimensional distributions if for each \(t = (t_1, \cdots, t_n) \in T\) and for each \(\varphi \in C_b(L^1(\mathbb{R}^d))\), the limit \(\lim_{n \to \infty} \mathbb{E}[\varphi(X_{t_1}, \cdots, X_{t_n})]\) exists.

Theorem 4.8 Let \(\{(X_t^i)_{t \geq 0}\}_{i=1}^\infty\) be a sequence of \(d\)-dimensional stochastic processes in a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) such that, for each \(i = 1, 2, \cdots\),

(i) \((X_t^i)_{t \geq 0}\) and \((X_t^{i+1})_{t \geq 0}\) are identically distributed;

(ii) \((X_t^{i+1})_{t \geq 0}\) is distributionally independent of \((X_t^1, \cdots, X_t^{i+1})_{t \geq 0}\);

(iii) \(\mathbb{E}[X_t^1] = \mathbb{E}[-X_t^1] \equiv 0\), \(t \geq 0\).

Then the sum

\[
Z_t^{(N)} := \frac{1}{\sqrt{N}} \sum_{i=1}^N X_t^i, \quad t \geq 0,
\]

converges in finite dimensional distributions to a Gaussian process \((Z_t)_{t \geq 0}\) under a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). The family of the 2nd moments of \((X_t^1)_{t \geq 0}\) and that of \((Z_t)_{t \geq 0}\) are the same, namely \(\mathbb{E}[Z_t] = \mathbb{E}[-Z_t] \equiv 0\) and \(G_t^{X_1} = G_t^Z\), for each \(t \in T\).

Remark 4.9 It is worth to stress that, even in the case where the above \(\{(X_t^i)_{t \geq 0}\}_{i=1}^\infty\) is a sequence of \(G\)-Brownian motions, the limit \((Z_t)_{t \geq 0}\) is a Gaussian process but it may not be a \(G\)-Brownian motion.

Proof. The proof is simply from the central limit theorem, i.e., Theorem 3.3. Indeed, for each fixed \(t = (t_1, \cdots, t_n) \in T\), \((X_t^i)_{i=1}^\infty := (X_{t_1}, \cdots, X_{t_n})_{i=1}^\infty\) is a sequence of \(\mathbb{R}^d\)-valued random vectors which is i.i.d. in the sense that \(X_{t_1}^i \overset{d}{=} X_{t_2}^i\) and \(X_{t_i}^{i+1} \overset{d}{=} X_{t_i}^1, \cdots, X_{t_i}^i\), for \(i = 1, 2, \cdots\). We then can apply the central limit theorem under the sublinear expectation \(\mathbb{E}\) to prove that \(Z_t^{(N)} := \frac{1}{\sqrt{N}} \sum_{i=1}^N X_t^i\) converges in law to an \(\mathbb{R}^d\)-valued random vector \(Z_t\), of which the distribution denoted by \(\mathbb{F}_t^Z\) is \(G\)-normal. Moreover the family \(\{(Z_t^0)_{t \in T}\}\) is consistent. It follows from Theorem 3.3 that there exists a \(d\)-dimensional stochastic process \((Z_t)_{t \geq 0}\) in some sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) such that \((\mathbb{F}_t^Z)_{t \in T} = (\mathbb{F}_t^N)_{t \in T}\). Thus \((Z_t)_{t \geq 0}\) is a Gaussian process.

5 \(q\)-normal distribution and \(q\)-Brownian motion in quantum mechanics

The approach to construct stochastic processes such as \(G\)-Brownian motion, \(G\)-Gaussian processes as well as some other typical stochastic processes in a nonlinear expectation, e.g. Lévy processes and Markovian processes, can be also applied to construct some new stochastic processes in an \(\mathbb{R}^n\)-valued expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). As a very typical example
we explain how to construct a $q$-Brownian motion under a $\mathbb{C}$-valued linear expectation space.

Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of complex valued functions defined on $\Omega$ such that $c \in \mathcal{H}$ for each complex constant $c$. The space $\mathcal{H}$ is the random space in our consideration.

**Definition 5.1** A $\mathbb{C}$-valued **linear expectation** $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{C}$ satisfying

(i) **Constant preserving:**

\[ \mathbb{E}[c] = c \text{ for } c \in \mathbb{C}. \]

(ii) **Linearity:** For each $X, Y \in \mathcal{H}$,

\[ \mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y], \quad \alpha, \beta \in \mathbb{C} \]

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a $\mathbb{C}$-valued linear expectation space.

Let $X_1$ and $X_2$ be two $\mathbb{C}^m$-valued random vectors defined on a $\mathbb{C}$-valued linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if, for each function $\varphi$ defined on $\mathbb{C}^m$ such that $\varphi(X_1) \in \mathcal{H}$ (resp. $\varphi(X_2) \in \mathcal{H}$) implies $\varphi(X_2) \in \mathcal{H}$ (resp. $\varphi(X_1) \in \mathcal{H}$) and

\[ \mathbb{E}[\varphi(X_1)] = \mathbb{E}[\varphi(X_2)]. \]

**Definition 5.2** In a $\mathbb{C}$-valued linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, for two random vectors $X \in \mathcal{H}^m$ and $Y \in \mathcal{H}^n$, $Y$ is said to be independent of $X$ under $\mathbb{E}$ if we have

\[ \mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}], \]

for each function $\varphi$ on $\mathbb{C}^{m+n}$ such that the above operations of expectations are meaningful. $Y$ is said to be an independent copy of $X$ if moreover $Y \overset{d}{=} X$.

We refer to [Peng2010-chI] for the product space method to construct independent random variables with specific distributions.

**Definition 5.3** An $\mathbb{R}^d$-valued valued random vector $X = (X_1, \cdots, X_d)$ on a linear valuation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a standard $q$-normal distributed if its components are independent from each others with $\varphi(X) \in \mathcal{H}$ and $\mathbb{E}[X_k^2] = -i$ ($i$ stands for the imaginary number) and

\[ aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X \quad \text{for } a, b \in \mathbb{R}, \quad (11) \]

where $\bar{X}$ is an independent copy of $X$. Here $h(\mathbb{R}^d)$ is the space of complex valued functions on $\mathbb{R}^d$ spanned by polynomials of $(x_1, \cdots, x_d)$ and all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\partial_{x}^{\nu} \varphi \in L^2(\mathbb{R}^d)$, $n = 0, 1, 2, \cdots$.

**Remark 5.4** Noting that $\mathbb{E}[X_k + \bar{X}_k] = 2\mathbb{E}[X_k]$ and $\mathbb{E}[X_k + \bar{X}_k] = \mathbb{E}[\sqrt{2}X_k] = \sqrt{2}\mathbb{E}[X_k]$, we then have $\mathbb{E}[X_k] = 0, k = 1, \cdots, d$. 

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Just like in the case of normal distribution we can also define, for a
fixed complex valued function \( \varphi \in h(C^d) \),
\[
w(t, x) := E[\varphi(x + \sqrt{t}X)].
\]
From the definition we have
\[
w(t, x) = E[\varphi(x + \sqrt{\delta}X + \sqrt{t}X)]
= E[w(t, x + \sqrt{\delta}X)].
\]

Similar to the situation of \( G \)-normal distributions, the function \( w \) solves the following free Schrödinger equation (for \( d = 1 \)):
\[
\partial_t w(t, x) = \frac{i}{2} \Delta w(t, x).
\]

We can use the same method as in [Peng2010] to prove the existence
of such type of \( q \)-normal distributed random variable \( X \), using classical results of PDE.

**Definition 5.5** We call \( (X_t(\omega))_{t \geq 0} \) a complex-valued stochastic process
define in a linear expectation space \( (\Omega, \mathcal{H}, E) \) if \( X_t \in \mathcal{H} \) for each \( t \geq 0 \).

**Definition 5.6** A stochastic process \( (B_t)_{t \geq 0} \) define in a \( C \)-valued linear
valuation space \( (\Omega, \mathcal{H}, E) \) is call a \( q \)-Brownian motion if it satisfies: for
each \( t, s \geq 0 \),
(i) \( B_{t+s} - B_s \overset{d}{=} B_t \) and \( B_{t+s} - B_s \) is independent of \( B_1, \ldots, B_n \), \( t_i \leq s \), \( i = 1, 2, \ldots \);
(ii) \( E[B_t] \equiv 0 \) and \( E[B_t^2] = -it \).

In analogous to \( G \)-Brownian motions we can the construction a \( q \)-
Brownian motion as follows. Let \( \Omega = (\mathbb{R}^d)^{[0, \infty)} \) be the space of all \( d \)-
dimensional complexed valued process and let \( B_t(\omega) = \omega_t, t \geq 0 \), be the
canonical process. We define
\[
\mathcal{H} = \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : \xi = (t_1, \ldots, t_n) \in \mathcal{T}, \varphi \in h(\mathbb{R}^{n \times d}) \}.
\]
It then remains to construct consistently a \( C \)-valued linear expectation of
\( E \) on \( (\Omega, \mathcal{H}) \) under which the canonical process \((B_t)_{t \geq 0}\) is a \( q \)-Brownian motion.
To this end we are given a sequence of standard \( q \)-normally
distributed random variables \( \{X_i\}_{i=1}^n \) of a \( C \)-valued expectation space
\( (\Omega, \mathcal{H}, E) \) such that \( X_{i+1} \) is independent of \( (X_1, \ldots, X_i) \) for \( i = 1, 2, \ldots \).
For each \( \xi = (t_1, \ldots, t_n) \in \mathcal{T} \) with \( t_1 \leq t_2 \leq \cdots \leq t_n \), we set, for each
\( \xi \in \mathcal{H} \) of the form \( \xi(\omega) = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) \), we set
\[
E[\xi] = E[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})]
:= E[\varphi(\sqrt{t_1}X_1, \sqrt{t_2 - t_1}X_2, \ldots, \sqrt{t_n - t_{n-1}}X_n)].
\]
We see that \( E : \mathcal{H} \rightarrow C \) consistently defines a \( C \)-valued linear expectation
under which \( (B_t)_{t \geq 0} \) becomes a \( q \)-Brownian motion.
We can also check that \( w(t, x) := E[\varphi(x + B_t)], t \geq 0, x \in \mathbb{R}^d \) solves the following free Schrödinger equation
\[
\partial_t w(t, x) = \frac{i}{2} \Delta w(t, x), \quad w|_{t=0} = \varphi.
\]
The situation with potential $V(x)$ corresponds to:

$$w(t, x) = \mathbb{E}[\varphi(x + B_t \exp \int_0^t V(x + B_s) ds)], \ t \geq 0, x \in \mathbb{R}^d.$$  

This forms a new type of Feynman-Kac formula to give a path-representation of the solution of a Schrödinger equation.

6 Appendix

6.1 Appendix A: Parabolic PDE associated with $G$-normal distributions

The distribution of a $G$-normally distributed random vector $X$ is characterized by the following parabolic partial differential equation (PDE for short) defined on $[0, \infty) \times \mathbb{R}^d$:

$$\partial_t u - G(D^2 u) = 0,$$

with Cauchy condition $u|_{t=0} = \varphi$, where $G : \mathbb{S}(d) \to \mathbb{R}$ is defined by (5) and $D^2 u = (\partial^2_{i,j} u)^d_{i,j=1}$, $Du = (\partial_i u)^d_{i=1}$. The PDE (12) is called a $G$-heat equation.

**Proposition 6.1** Let $X \in \mathcal{N}^d$ be normally distributed, i.e., (7) holds. Given a function $\varphi \in C_{Lip} (\mathbb{R}^d)$, we define

$$u(t, x) := \mathbb{E}[\varphi(x + \sqrt{s}X)], \ (t, x) \in [0, \infty) \times \mathbb{R}^d.$$  

Then we have

$$u(t + s, x) = \mathbb{E}[u(t, x + \sqrt{s}X)], \ s \geq 0. \quad (13)$$

We also have the estimates: for each $T > 0$, there exist constants $C, k > 0$ such that, for all $t, s \in [0, T]$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$,

$$|u(t, x) - u(t, \bar{x})| \leq C(1 + |x|^k + |\bar{x}|^k)(|x - \bar{x}|) \quad (14)$$

and

$$|u(t, x) - u(t + s, x)| \leq C(1 + |x|^k)(s + |s|^{1/2}). \quad (15)$$

Moreover, $u$ is the unique viscosity solution, continuous in the sense of (14) and (15), of the PDE (12).

**Corollary 6.2** If both $X$ and $\bar{X}$ satisfy (7) with the same $G$, i.e.,

$$G(A) := \mathbb{E}[\frac{1}{2} \langle AX, X \rangle] = \mathbb{E}[\frac{1}{2} \langle A\bar{X}, \bar{X} \rangle] \quad \text{for } A \in \mathbb{S}(d),$$

then $X \overset{d}{=} \bar{X}$. In particular, $X \overset{d}{=} -X$.

**Example 6.3** Let $X$ be $G$-normal distributed. The distribution of $X$ is characterized by

$$u(t, x) = \mathbb{E}[\varphi(x + \sqrt{t}X)], \ \varphi \in C_{Lip} (\mathbb{R}^d).$$
In particular, \( E[\varphi(X)] = u(1,0) \), where \( u \) is the solution of the following parabolic PDE defined on \([0, \infty) \times \mathbb{R}^d \):

\[
\partial_t u - G(D^2 u) = 0, \quad u|_{t=0} = \varphi, \tag{16}
\]

where \( G = G_X(A) : \mathbb{S}(d) \to \mathbb{R} \) is defined by

\[
G(A) := \frac{1}{2} E[\langle AX, X \rangle], \quad A \in \mathbb{S}(d).
\]

The parabolic PDE (16) is called a \( G \)-heat equation.

It is easy to check that \( G \) is a sublinear function defined on \( \mathbb{S}(d) \). By Theorem 2.7 there exists a bounded, convex and closed subset \( \Theta \subset \mathbb{S}(d) \) such that

\[
\frac{1}{2} E[\langle AX, X \rangle] = G(A) = \frac{1}{2} \sup_{Q \in \Theta} \text{tr}[AQ], \quad A \in \mathbb{S}(d).
\]

Since \( G(A) \) is monotonic: \( G(A_1) \geq G(A_2) \), for \( A_1 \geq A_2 \), it follows that \( \Theta \subset \mathbb{S}_+(d) = \{\theta \in \mathbb{S}(d) : \theta \geq 0\} = \{BB^T : B \in \mathbb{R}^{d \times d}\} \), where \( \mathbb{R}^{d \times d} \) is the set of all \( d \times d \) matrices. If \( \Theta \) is a singleton: \( \Theta = \{Q\} \), then \( X \) is classical zero-mean normal distributed with covariance \( Q \). In general, \( \Theta \) characterizes the covariance uncertainty of \( X \).

When \( d = 1 \), we have \( X \dist N\left(\{0\} \times [\sigma^2, \bar{\sigma}^2]\right) \) (We also denoted by \( X \dist N(0, [\bar{\sigma}^2, \sigma^2]) \)), where \( \bar{\sigma}^2 = E[X^2] \) and \( \sigma^2 = -E[-X^2] \). The corresponding \( G \)-heat equation is

\[
\partial_t u - \frac{1}{2}(\bar{\sigma}^2(\partial^2_{xx} u)^+ - \bar{\sigma}^2(\partial^2_{xx} u)^-) = 0, \quad u|_{t=0} = \varphi.
\]

For the case \( \sigma^2 > 0 \), this equation is also called the Barenblatt equation.

In the following two typical situations, the calculation of \( E[\varphi(X)] \) is very easy:

- For each convex function \( \varphi \), we have

\[
E[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\sigma^2 y) \exp\left(-\frac{y^2}{2}\right) dy.
\]

Indeed, for each fixed \( t \geq 0 \), it is easy to check that the function \( u(t, x) := E[\varphi(x + \sqrt{t}X)] \) is convex in \( x \):

\[
u(t, \alpha x + (1 - \alpha) y) = E[\varphi(\alpha x + (1 - \alpha)y + \sqrt{t}X)] \leq \alpha E[\varphi(x + \sqrt{t}X)] + (1 - \alpha)E[\varphi(x + \sqrt{t}X)] = \alpha u(t, x) + (1 - \alpha)u(t, x).
\]

It follows that \((\partial^2_{xx} u)^- = 0\) and thus the above \( G \)-heat equation becomes

\[
\partial_t u = \frac{\sigma^2}{2} \partial^2_{xx} u, \quad u|_{t=0} = \varphi.
\]
For each concave function $\varphi$, we have

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{y^2}{2}\right) dy.$$ 

In particular,

$$\mathbb{E}[X] = \mathbb{E}[-X] = 0, \quad \mathbb{E}[X^2] = \sigma^2, \quad -\mathbb{E}[-X^2] = \sigma^2.$$

6.2 Appendix B: Kolmogorov’s criterion in the situation of sublinear expectation

Definition 6.4 Let $I$ be a set of indices, $(X_t)_{t \in I}$ and $(Y_t)_{t \in I}$ be two processes indexed by $I$. We say that $Y$ is a modification of $X$ if for all $t \in I$, $X_t = Y_t$ q.s.

We now give a Kolmogorov criterion for a process indexed by $[0,1]^d$.

Theorem 6.5 Let $p > 0$ and $(X_t)_{t \in [0,1]^d}$ be a process such that for all $t \in [0,1]^d$, $X_t$ belongs to $L^p$. Assume that there exist positive constants $c$ and $\epsilon$ such that

$$\overline{\mathbb{E}}\left[|X_t - X_s|^p\right] \leq c|t - s|^{d+\epsilon}, \quad s,t \in [0,1]^d$$

Then $X = (X_t)_{t \in [0,1]^d}$ admits a modification $(\tilde{X}_t)_{t \in [0,1]^d}$ such that

$$\overline{\mathbb{E}}\left[\left(\sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^{\alpha}}\right)^p\right] < \infty,$$

for each $\alpha \in [0, \epsilon/p)$. As a consequence, paths of $\tilde{X}$ are quasi-surely H"older continuous of order $\alpha$ for each $\alpha < \epsilon/p$. Moreover, if $X_t \in L^p$ for each $t$, then we also have $\tilde{X}_t \in L^p$.

Lemma 6.6 Let $p > 0$. Assume that there exist positive constants $c$ and $\epsilon$ such that

$$\mathbb{F}_{t,s}[\varphi_p] \leq c|t - s|^{1+\epsilon}, \quad s,t \geq 0$$

where $\varphi_p(x_1,x_2) = |x_1 - x_2|^p$. Then for the stochastic process $X = (X_t)_{t \geq 0}$, there exists a continuous modification $\tilde{X} = \{\tilde{X}_t : t \in [0,\infty]\}$ of $X$ (i.e. $\tilde{c}(\{\tilde{X}_t \neq X_t\}) = 0$, for each $t \geq 0$).

Proof. We have $\overline{\mathbb{E}} = \mathbb{E}$ on $L_{ip}(\Omega)$. On the other hand, we have

$$\overline{\mathbb{E}}[|\tilde{X}_t - \tilde{X}_s|^p] = \mathbb{E}[|\tilde{X}_t - \tilde{X}_s|^p] = c|t - s|^{1+\epsilon}, \quad \forall s,t \in [0,\infty),$$

where $d$ is a constant depending only on $G$. By Theorem 6.5, there exists a continuous modification $\tilde{B}$ of $X$. Since $\tilde{c}(\{\tilde{X}_0 \neq 0\}) = 0$, we can set $\tilde{B}_0 = 0$. The proof is complete. \qed
6.3 Appendix C: Tightness of $\mathcal{P}_e$

For each $Q \in \mathcal{P}_e$, let $Q \circ \bar{B}^{-1}$ denote the probability measure on $(\Omega, \mathcal{B}(\Omega))$ induced by $\bar{B}$ with respect to $Q$. We denote $\mathcal{P}_1 = \{Q \circ \bar{B}^{-1} : Q \in \mathcal{P}_e\}$.

By Lemma 6.6 we get
\[
\mathbb{E}[|X_t - X_s|^p] = c|t - s|^{1+\varepsilon}, \forall s, t \in [0, \infty).
\]

Applying the well-known result of moment criterion for tightness of the above Kolmogorov-Chentsov’s type, we conclude that $\mathcal{P}_1$ is tight. We denote
\[
\mathcal{P}_1 = \{Q \circ \bar{B}^{-1} : Q \in \mathcal{P}_e\}.
\]

By Lemma 6.6, we get
\[
\tilde{E}\left[|X_t - X_s|^p\right] = c|t - s|^{1+\varepsilon}, \forall s, t \in [0, \infty).
\]

Applying the well-known result of moment criterion for tightness of the above Kolmogorov-Chentsov’s type, we conclude that $\mathcal{P}_1$ is tight. We denote
\[
\mathcal{P}_1 = \{Q \circ \bar{B}^{-1} : Q \in \mathcal{P}_e\}.
\]

Now, we give the representation of $G$-expectation.

**Theorem 6.7** For each continuous monotonic and sublinear function $G : \mathbb{S}(\mathbb{D}) \rightarrow \mathbb{R}$, let $\hat{E}_G$ be the corresponding $G$-expectation on $(\Omega, \mathcal{B}(\Omega))$. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that
\[
\hat{E}_G[X] = \max_{P \in \mathcal{P}} E_P[X], \quad \forall X \in \mathcal{L}_1(\Omega).
\]

**Proof.** By Lemma 6.13 and Lemma 6.6 we have
\[
\hat{E}_G[X] = \max_{P \in \mathcal{P}_1} E_P[X], \quad \forall X \in \mathcal{L}_1(\Omega).
\]

For each $X \in \mathcal{L}_1(\Omega)$, by Lemma 3.11 we get $\hat{E}_G[|X - (X \wedge N) \vee (-N)|] \downarrow 0$ as $N \to \infty$. Noting also that $\mathcal{P} = \overline{\mathcal{P}_1}$, then by the definition of weak convergence, we get the result. □

6.4 Appendix D: $G$-Capacity and paths of $G$-Brownian motion

According to Theorem 6.7, we obtain a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ to represent the sublinear expectation $\mathbb{E}[\cdot]$. For this $\mathcal{P}$, we define the associated $G$-capacity:
\[
\hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega),
\]

and upper expectation for each $X \in L^0(\Omega)$ which makes the following definition meaningful,
\[
\hat{E}[X] := \sup_{P \in \mathcal{P}} E_P[X].
\]

We have $\hat{E} = \hat{E}_G$ on $\mathcal{L}_1(\Omega)$. Thus the $\hat{E}_G[\cdot]$-completion and the $\hat{E}[\cdot]$-completion of $\mathcal{L}_1(\Omega)$ are the same.

For each $T > 0$, we also denote by $\Omega_T = C^d_0([0, T])$ equipped with the distance
\[
\rho(\omega^1, \omega^2) = \|\omega^1 - \omega^2\|_{C^d([0, T])} := \max_{0 \leq t \leq T} |\omega^1_t - \omega^2_t|.
\]

We now prove that $L^1_G(\Omega) = L^1$. First, we need the following classical approximation Lemma (see e.g. the well-known approximation of Barlow and Perkins for SDE and Lepeltier and San Martin to prove the existence of BSDE with continuous coefficients).
Proposition 6.8 For each $X \in C_b(\Omega)$ and $\varepsilon > 0$, there exists a $Y \in L_{tp}(\Omega)$ such that $\mathbb{E}[|Y - X|] \leq \varepsilon$.

Proof. We denote $M = \sup_{\omega \in \Omega} |X(\omega)|$. We can find $\mu > 0$, $T > 0$ and $\bar{X} \in C_b(\Omega_T)$ such that $\mathbb{E}[|X - \bar{X}|] < \varepsilon/3$, $\sup_{\omega \in \Omega} |\bar{X}(\omega)| \leq M$ and

$$|\bar{X}(\omega) - \bar{X}(\omega')| \leq \mu \|\omega - \omega'\|_{C^0_b([0,T])}, \quad \forall \omega, \omega' \in \Omega.$$

Now for each positive integer $n$, we introduce a mapping $\omega^{(n)}(\omega) : \Omega \mapsto \Omega$:

$$\omega^{(n)}(\omega)(t) = \sum_{k=0}^{n-1} \frac{1_{[t_k^{n-1}, t_k^n]}(t)}{t_k^{n} - t_k^{n-1}} \left(\left(\frac{t_k^n - t}{t_k^n - t_k^{n-1}}\right) \omega(t) + \left(\frac{t - t_k^{n-1}}{t_k^n - t_k^{n-1}}\right) \omega(t_k^n)\right) + 1_{(t, \infty)}(t)\omega(t),$$

where $t_k^n = \frac{t_k}{n}$, $k = 0, 1, \ldots, n$. We set $\bar{X}^{(n)}(\omega) := \bar{X}(\omega^{(n)}(\omega))$, then

$$|\bar{X}^{(n)}(\omega) - \bar{X}^{(n)}(\omega')| \leq \mu \sup_{t \in [0, T]} |\omega^{(n)}(\omega)(t) - \omega^{(n)}(\omega')(t)|$$

$$= \mu \sup_{k \in \{0, \ldots, n\}} |\omega(t_k^n) - \omega'(t_k^n)|.$$

We now choose a compact subset $K \subset \Omega$ such that $\mathbb{E}[1_K] \leq \varepsilon / 6M$. Since $\sup_{\omega \in K} \sup_{t \in [0, T]} |\omega(t) - \omega^{(n)}(\omega)(t)| \to 0$, as $n \to \infty$, we then can choose a sufficiently large $n_0$ such that

$$\sup_{\omega \in K} |\bar{X}(\omega) - \bar{X}^{(n_0)}(\omega)| = \sup_{\omega \in K} |\bar{X}(\omega) - \bar{X}(\omega^{(n_0)}(\omega))|$$

$$\leq \mu \sup_{\omega \in K} \sup_{t \in [0, T]} |\omega(t) - \omega^{(n_0)}(\omega)(t)|$$

$$< \varepsilon / 3.$$

Set $Y := \bar{X}^{(n_0)}$, it follows that

$$\mathbb{E}[|X - Y|] \leq \mathbb{E}[|X - \bar{X}|] + \mathbb{E}[|\bar{X} - \bar{X}^{(n_0)}|]$$

$$\leq \mathbb{E}[|X - \bar{X}|] + \mathbb{E}[1_K |X - \bar{X}^{(n_0)}|] + 2M \mathbb{E}[1_K]$$

$$< \varepsilon.$$

The proof is complete. ■

By Proposition 6.8 we can easily get $L^1_G(\Omega) = \mathbb{L}^1_G$. Furthermore, we can get $L^p_G(\Omega) = L^p_G$, $\forall p > 0$.

Thus, we obtain a pathwise description of $L^p_G(\Omega)$ for each $p > 0$:

$L^p_G(\Omega) = \{X \in L^p(\Omega) : X$ is quasi-continuous and $\lim_{n \to \infty} \mathbb{E}[|X|^p I_{|X| > n}] = 0\}$.

Furthermore, $\mathbb{E}[X] = \mathbb{E}[X]$, for each $X \in L^1_G(\Omega)$. 

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