Poisson–Lie structures on Galilei group

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Abstract
The complete list of Poisson–Lie structures on 4-d Galilei group is presented.

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I Introduction

Quantum groups have emerged in physics in connection with an attempt to understand the symmetries underlying exact solvability of certain quantum-mechanical and statistical models; they appeared to be quite powerful in this respect.

Therefore, it is natural to ask whether their range of applicability as a mathematical tool for describing physical symmetries is wider and covers, in particular, the most important case of space–time symmetries. The theory of Hopf algebras offers a variety of structures which can be viewed as deformations of classical space–time symmetry groups. For example, a number of deformed Poincare groups were considered [1]. They possess many attractive features. However, if one is going to take seriously the very idea of quantum space time symmetries, many conceptual problems arise which solution seems to be quite difficult. It is rather obvious that one should concentrate on deformations of relativistic symmetries because it is a high energy/small distance region where the deviations from the predictions of “classical” theory should occur. However, as far as the conceptual problems are concerned, the nonrelativistic region provides similar challenge. On the other hand it seems that the study of nonrelativistic deformed symmetries is slightly simpler. One is not here faced with some complications typical for relativistic quantum theory as, for example, the nonexistence of particle number conserving nontrivial interactions.

In the present paper as a first step towards the understanding of nonrelativistic quantum space–time symmetries we classify all nonequivalent Poisson–Lie structures on the four-dimensional Galilei group. The method we use (c.f. [2]) is based on solving directly the cocycle condition; it has been already used for finding Poisson–Lie structures on the two-dimensional Galilei group [3],[4].

We find families of structures which cannot be related to each other by the automorphisms of the Galilei group; contrary to the case of the Poincare group [5] many of them are not of the coboundary type.

The paper is organized as follows. In sec. II we sketch a general strategy while the results are presented in sec. III. All technical details are relegated to the number of appendices.

Let us conclude the introduction which some details concerning the Galilei group [6].

The generic element $g$ of the ten–parameter Galilei group $G$ is denoted
by

\[ g = (t, \vec{a}, \vec{v}, R) \]  

(1)

We shall denote by

\( T = \{(t, \vec{0}, \vec{0}, I)\} \)

\( S = \{(0, \vec{a}, \vec{0}, I)\} \)

\( V = \{(0, \vec{0}, \vec{v}, I)\} \)

\( \mathcal{R} = \{(0, \vec{0}, \vec{0}, R)\} \)

the subgroups of time translations, space–translations, pure Galilei transformations (boosts), rotations, respectively.

The group law is expressed by

\[
g'' = g' \cdot g = (t', \vec{a}', \vec{v}', R')(t, \vec{a}, \vec{v}, R) = (t + t', \vec{a}' + R'\vec{a} - \vec{v}'t, \vec{v}' + R'\vec{v}, R'R) \]  

(2)

The identity for the group is

\[ e = (0, \vec{0}, \vec{0}, I) \]  

(3)

and the inverse of the generic element is given by

\[ g^{-1} = (t, \vec{a}, \vec{v}, R)^{-1} = (-t, -R^{-1}(\vec{a} - t\vec{v}), -R^{-1}\vec{v}, R^{-1}) \]  

(4)

The generators \( H, \vec{P}, \vec{K} \) and \( \vec{J} \) of the Galilei Lie algebra \( \mathcal{G} \) are defined with the help of the exponential parametrization

\[ g = e^{-itH} e^{i\vec{a}\vec{P}} e^{i\vec{v}\vec{K}} e^{i\vec{j}\vec{J}} \]  

(5)

and they obey the following commutation rules (only the nonvanishing ones are written up)

\[
\begin{align*}
[J_i, J_j] &= i\varepsilon_{ijk} J_k \\
[J_i, K_j] &= i\varepsilon_{ijk} K_k \\
[J_i, P_j] &= i\varepsilon_{ijk} P_k \\
[K_i, H] &= iP_i
\end{align*}
\]  

(6)

(7)

here and in the sequel the summation over repeated indices is understood \((i, j, k = 1, 2, 3)\).
The automorphism group of $G$ consists of the inner automorphisms together with two outer ones generated by space dilations and time dilations:

\[
(t, \vec{a}, \vec{v}, R) \rightarrow (t, a\vec{a}, a\vec{v}, R) \\
(t, \vec{a}, \vec{v}, R) \rightarrow (bt, \vec{a}, b^{-1}\vec{v}, R)
\]  

(8)

In what follows we shall need the right-invariant vector fields on $G$. Denoting by $\hat{X}$ the right-invariant vector field corresponding to the element $X$ of the Lie algebra $\mathfrak{g}$ we have

\[
\hat{H} = -i \frac{\partial}{\partial t} \\
\hat{P}_i = i \frac{\partial}{\partial a_i} \\
K_i = i(t \frac{\partial}{\partial a_i} + \frac{\partial}{\partial v_i}) \\
J_i = -i \varepsilon_{ijk} a_j \frac{\partial}{\partial a_k} - i \varepsilon_{ijk} v_j \frac{\partial}{\partial v_k} - i \varepsilon_{ijk} R_{jl} \frac{\partial}{\partial R_{kl}}
\]  

(9)

II Poisson–Lie structures on Galilei group — the general strategy

Let us remind the notion of Poisson–Lie group \[7,8\]. It is a Lie group $\tilde{G}$ which has a Poisson structure $\{ , \}$ such that the multiplication map $m : \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ is a Poisson map (here $\tilde{G} \times \tilde{G}$ is given by the product Poisson structure).

Poisson–Lie structures can be described explicitly as follows. Let $\tilde{G}$ be the Lie algebra of $\tilde{G}$; denote by $\{X_i\}$ an arbitrary basis in $\tilde{G}$ and let $c^k_{ij}$ be the corresponding structure constants. One defines a mapping $\eta : \tilde{G} \rightarrow \wedge^2 \tilde{G}$, $\eta(g) \equiv \eta^{ij}(g) X_i \otimes X_j \quad \eta^{ij}(g) = -\eta^{ji}(g)$  

(10)

Let $\{X_i^R\}$ be the realization of $\tilde{G}$ in terms of right invariant vector fields on $G$. The Poisson bracket on $\tilde{G}$ given by

\[
\{\Phi, \Psi\} = \eta^{ij}(X_i^R\Phi)(X_j^R\Psi)
\]  

(11)

defines the Poisson–Lie structure on $\tilde{G}$ provided the following conditions are obeyed
(i) Poisson–Lie property (co-cycle condition)
\[ \eta(g'g) = \eta(g') + Ad(g')\eta(g) \] (12)

(ii) Jacobi identity
\[ \eta^{il}X^R_l\eta^{jk} + \eta^{kl}X^R_l\eta^{ij} + \eta^{jl}X^R_l\eta^{ki} - \\
- c^i_{lp}\eta^{il}\eta^{pk} - c^i_{lp}\eta^{kl}\eta^{pj} - c^k_{lp}\eta^{jl}\eta^{pi} = 0 \] (13)

The inverse is also true: any Poisson–Lie structure on \( \tilde{G} \) can be written in the above form. The infinitesimal analogues of Poisson–Lie groups are Lie bialgebras. For any \( X \in \tilde{G} \) define
\[ \delta(X) \equiv \frac{d}{dt}\eta(e^{itX})|_{t=0} \] (14)

Then it can be easily shown that \( \delta : \tilde{G} \to \wedge^2 \tilde{G} \) has the following properties which are the infinitesimal counterparts of (i) and (ii):

(i') co-cycle conditions
\[ \delta([X,Y]) = [X \otimes I + I \otimes X, \delta(Y)] + [\delta(X), I \otimes Y + Y \otimes I] \] (15)

(ii') co-Jacobi identity
\[ \sum_{c.p.}(\delta \otimes id) \circ \delta(X) = 0 \] (16)

where \( c.p. \) means the summation over cyclic permutation of the factors in \( \tilde{G} \otimes \tilde{G} \otimes \tilde{G} \).

Every Poisson–Lie structure on \( \tilde{G} \) defines the bialgebra structure on \( \tilde{G} \). The inverse is true provided \( \tilde{G} \) is connected and simply connected. Two Poisson–Lie structures on \( \tilde{G} \) will be called equivalent if there exists an automorphism of \( \tilde{G} \) which is a Poisson map.

The main aim of the present paper is to classify, up to equivalence, all Poisson–Lie structures on the four-dimensional Galilei group \( G \). We adopt the following, rather straightforward, strategy.

First, we write out \( \eta \) in the form
\[ \eta(g) = \Psi_i(g)H \wedge J_i + \Phi_i(g)H \wedge P_i + \Gamma_i(g)H \wedge K_i \\
+ \Lambda_i(g)\varepsilon_{ijk}P_j \wedge P_k + \Upsilon_{ij}(g)P_i \wedge K_j \\
+ \Sigma_{ij}(g)P_i \wedge J_j + \Xi_i(g)\varepsilon_{ijk}K_i \wedge K_k + \\
+ \Omega_{ij}(g)K_i \wedge J_j + \Pi_i(g)\varepsilon_{ijk}J_i \wedge J_k \] (17)
where \( g \equiv (t, \vec{a}, \vec{v}, R) \) is an arbitrary element of the Galilei group \( G \).

Inserting the expansion (17) into the one-cocycle condition (12) one obtains the following set of functional equations for the coefficients \( \Psi_i, \Phi_i, \) etc.

\[
\begin{align*}
\Psi_i(gg') &= \Psi_i(g) + R_{il}\Psi_l(g') \\
\Phi_i(gg') &= \Phi_i(g) + R_{il}\Phi_l(g') + \varepsilon_{ink}(v_nt - a_n)R_{kl}\Psi_l(g') - tR_{il}\Gamma_l(g') \\
\Gamma_i(gg') &= \Gamma_i(g) + R_{il}\Gamma_l(g') - \varepsilon_{ink}v_nR_{kl}\Psi_l(g') \\
\Lambda_i(gg') &= \Lambda_i(g) + R_{il}\Lambda_l(g') - \frac{1}{2}\varepsilon_{imn}v_mR_{nl}\Phi_l(g') + \frac{1}{2}[(\varepsilon^2t - a_m)v_m]R_{nl}\Psi_l(g') + \\
&\quad + \frac{1}{2}t\varepsilon_{imn}v_mR_{nl}\Gamma_l(g') - \frac{1}{2}tR_{il}\varepsilon_{imn}\Upsilon_{mn}(g') + \\
&\quad + \frac{1}{2}[(tv_i - a_i)\delta_{mp} - R_{im}R_{np}(tv_n - a_n)](\Sigma_{pm}(g')) \\
&\quad - \frac{1}{2}t\Omega_{pm}(g')) + t^2R_{il}\Xi_l(g') + \\
&\quad + (t^2v_pv_i - v_pa_i) - v_i\delta_{mp} + a_p a_i)R_{pm}\Pi_m(g') \\
\Upsilon_{ij}(gg') &= \Upsilon_{ij}(g) + R_{im}R_{jn}\Upsilon_{mn}(g') + v_i\varepsilon_{jnk}v_nR_{kl}\Psi_l(g') \\
&\quad - v_iR_{jl}\Gamma_l(g') - \varepsilon_{jnl}v_nR_{ip}R_{lk}\Sigma_{pk} \\
&\quad - 2t\varepsilon_{ijkl}\Xi_l(g') + \\
&\quad + [\varepsilon_{iln}(a_n - v_n)t]R_{jp} + \varepsilon_{jnl}v_nR_{ip}R_{lk}\Omega_{pk}(g') \\
&\quad + 2[\varepsilon_{ijn}(a_p - v_p)t]R_{pm} - \varepsilon_{ijk}v_nR_{lm}\Pi_m(g') (18) \\
\Sigma_{ij}(gg') &= \Sigma_{ij}(g) + R_{ip}R_{jk}\Sigma_{pk}(g') - v_iR_{jl}\Psi_l(g') \\
&\quad - tR_{ip}R_{jk}\Omega_{pk}(g') + \\
&\quad + 2[(a_j - tv_j)R_{jm} - (a_l - tv_l)R_{lm}\delta_{ij}\Pi_m(g') \\
\Xi_{i}(gg') &= \Xi_{i}(g) + R_{in}\Xi_n(g') - \frac{1}{2}(v_i\delta_{pm} - R_{im}R_{np}v_n)\Omega_{pm}(g') + \\
&\quad + v_iv_pR_{pm}\Pi_m(g') \\
\Omega_{ij}(gg') &= \Omega_{ij}(g) + R_{ip}R_{jk}\Omega_{pk}(g') + 2(R_{im}v_j - R_{lm}v_l\delta_{ij})\Pi_m(g') \\
\Pi_{i}(gg') &= \Pi_{i}(g) + R_{im}\Pi_m(g') \\
\end{align*}
\]

In spite of their complicated structure they can be solved in the following way (cf.[2],[3],[4]). One decomposes the general element \( g = (t, \vec{a}, \vec{v}, R) \) into
the product of four elements belonging to the subgroups of time–and–space–translations, boosts and rotations, (see sec.I).

\[(t, \vec{a}, \vec{v}, R) = (t, \vec{0}, \vec{0}, I) \cdot (0, \vec{a}, 0, I) \cdot (0, \vec{0}, \vec{v}, I) \cdot (0, \vec{0}, \vec{0}, R)\]  (19)

According to the condition (i) one can successively calculate \(\eta(g)\) using the above decomposition provided the form of \(\eta\) for all four subgroups is known. In order to find the latter we specify eqs.(18) to those subgroups. The resulting equations can be easily solved; this is done in Appendix A. However, in obtaining the final form of \(\eta\) we apply eq.(19) with some definite order of multiplication (for example using \((t, \vec{a}, \vec{v}, R) = (t, \vec{0}, \vec{0}, I)\cdot\cdot\cdot((0, \vec{a}, 0, I)((0, 0, \vec{v}, I)(0, \vec{0}, \vec{0}, R))))\) so there could be further constraints on the parameters entering \(\eta\) following from associativity. Therefore, we reintert \(\eta\) into eq.(12) with arbitrary \(g\) and \(g'\) to find all missed relations between parameters. In this way we produce the general solution to eq.(12) described in Appendix B (see eq.(B.1)).

There remains to solve (ii) which imposes additional relations between coefficients of \(\eta\). It is very tedious to try to solve eq.(13) directly so we adopt a different method. From our general form of \(\eta\) we calculate \(\delta\) and impose (ii') which, in this context, is equivalent to (ii). On the other hand it is well known that (ii') can be restated as the condition that the dual map \(\delta^*\) defines a Lie algebra structure on \(G^*\). Therefore, we first calculate \(\delta\) and the commutators on \(G^*\) resulting from it and then we solve the Jacobi identities. This is still a complicated problem but it can be simplified by using the boost and translation automorphisms of the Galilei group/algebra. Once this is done there remains only to use the residual automorphisms to put our solutions in canonical position. The more detailed discussion is given in Appendix C.

At the end, having the form of \(\eta\) (eq.(17)) and the form of the right-invariant vector fields (eq.(9)), using eq.(11) and taking into account that for all families of solutions \(\Pi_i(g) \equiv 0\) (see eq.(B.1) and sec.III) one can easily calculate the following fundamental Poisson–Lie brackets.

\[
\begin{align*}
\{R_{ab}, R_{cd}\} & = 0 \\
\{v_a, R_{bc}\} & = \varepsilon_{bji} R_{lc} \Omega_{aj} \\
\{a_a, R_{bc}\} & = \varepsilon_{bji} R_{lc} (\Sigma_{aj} + t \Omega_{aj}) \\
\{t, R_{bc}\} & = -\varepsilon_{bji} R_{lc} \Psi_j \\
\{t, v_a\} & = \Gamma_a - \varepsilon_{ajl} v_l \Psi_j
\end{align*}
\]  (20)
\{v_a, v_b\} = -2\varepsilon_{abj}\Xi_j + \varepsilon_{bjl}v_l\Omega_{aj} - \varepsilon_{ajl}v_l\Omega_{bj}
\{a_a, v_b\} = -\Upsilon_{ab} + \varepsilon_{bjl}v_l\Sigma_{aj} - 2t\varepsilon_{abj}\Xi_j + t\varepsilon_{bjl}v_l\Omega_{aj} - \varepsilon_{ajl}v_l\Omega_{bj}
\{a_a, a_b\} = -2\varepsilon_{abj}\Lambda_j + \varepsilon_{bjl}v_l\Sigma_{aj} - \varepsilon_{ajl}v_l\Omega_{bj}
\{t, a_a\} = -\varepsilon_{ina}\alpha_n\Psi_i + \Phi_a + \Gamma_a

III  Poisson–Lie structures on Galilei group
— the results

By applying the procedure outlined above we solve the relevant Jacobi identities for dual algebra (making use of boost and translation automorphisms) and we arrive at the following families of Poisson–Lie structures (for all cases \(\vec{n} = 0\)).

I.  \(\vec{\alpha}\)–arbitrary, \(\beta \neq 0\), \(\vec{\gamma} = 0\), \(\vec{\phi} = 0\), \(v = 0\), \(\vec{\zeta} = 0\), \(\theta = 0\), \(\rho = 0\), \(\sigma_{ij} = 0\), \(\chi_{ij} = 0\), \(\omega_{ij} = 0\)

free parameters: \(\vec{\alpha}, \beta \neq 0\)

II. \(\vec{\alpha} \neq 0\), \(\beta = 0\), \(\vec{\gamma} = 0\), \(\vec{\phi} = F\vec{\alpha}\), \(\vec{\lambda} = L\vec{\alpha}\), \(v\)–arbitrary, \(\vec{\zeta} = 0\), \(\theta = 0\), \(\rho = 0\), \(\sigma_{ij} = 0\), \(\omega_{ij} = W(\alpha^2\delta_{ij} - \alpha_i\alpha_j)\), \(\chi_{ij} = B\alpha_i\alpha_j - \frac{1}{3}\alpha^2\delta_{ij}\) + \(2Wv\varepsilon_{ijk}\alpha_k\)

free parameters: \(F, L, v, W \neq 0, B\)

III. \(\vec{\alpha} = 0\), \(\beta = 0\), \(\vec{\gamma} = 0\), \(\vec{\phi} = F\vec{\mu}\), \(\vec{\lambda} = L\vec{\mu}\), \(v = 0\), \(\vec{\zeta} = 0\), \(\theta = 0\), \(\rho = 0\), \(\sigma_{ij} = 0\), \(\omega_{ij} = W(\delta_{ij} - \mu_i\mu_j)\), \(\chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k\)

free parameters: \(F \neq 0, L, B, C, W \neq 0, \vec{\mu}, ||\vec{\mu}|| = 1\)

IV. \(\vec{\alpha} = 0\), \(\beta = 0\), \(\vec{\gamma} = 0\), \(\vec{\phi} = F\vec{\alpha}\), \(\vec{\lambda} = L\vec{\alpha}\), \(v = 0\), \(\vec{\zeta} = X\vec{\mu}\), \(\theta\)–arbitrary, \(\rho = 0\), \(\sigma_{ij} = 0\), \(\omega_{ij} = W(\delta_{ij} - \mu_i\mu_j)\), \(\chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k\)

free parameters: \(L, X, \theta, W \neq 0, B, C, \vec{\mu}, ||\vec{\mu}|| = 1\)

V. \(\vec{\alpha} \neq 0\), \(\beta = 0\), \(\vec{\gamma} = 0\), \(\vec{\phi} = F\vec{\alpha}\), \(\vec{\lambda} = L\vec{\alpha}\), \(v\)–arbitrary, \(\vec{\zeta} = 0\), \(\theta = 0\), \(\rho = 0\), \(\sigma_{ij} = 0\), \(\omega_{ij} = 0\), \(\chi_{ij} = B(\alpha_i\alpha_j - \frac{1}{3}\alpha^2\delta_{ij})\)

free parameters: \(\vec{\alpha} \neq 0, F, L, v, B\)
VI. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\xi} = 0$ $\theta$-arbitrary, $\rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, \vec{\lambda}$ and $\chi_{ij}$-arbitrary except that $\varepsilon_{abc}\chi_{ab} = 0$
free parameters: $\vec{\lambda}, \chi_{ab}, \theta$

VII. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \phi_i = F\varepsilon_{imn}\chi_{mn}, \vec{\lambda} = 0, \varepsilon = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, \chi_{ij}$-arbitrary except that $\varepsilon_{imn}\chi_{mn} \neq 0$
free parameters: $F \neq 0, \chi_{mn}$

VIII. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \phi_i \neq F\varepsilon_{imn}\chi_{mn}$ and $\vec{\phi} \neq 0, \vec{\lambda} = L\vec{\phi}, \varepsilon = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, \chi_{ij}$-arbitrary
free parameters: $\vec{\phi} \neq 0, L, \chi_{ij}$

IX. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = \theta = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, \chi_{ij}$-arbitrary except that $\varepsilon_{abc}\chi_{ab}\xi_c = 0$
free parameters: $\vec{\lambda}, \vec{\xi} \neq 0, \theta, \chi_{ij}$

X. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = \theta = 0, \rho = -\frac{1}{3}S, \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0, \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k$
free parameters: $S \neq 0, B, \theta, \mu, ||\vec{\mu}|| = 1$

XI. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = L\vec{\mu}, \varepsilon = 0, \vec{\xi} = 0, \theta = 0, \rho = -\frac{1}{3}S, \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0, \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k$
free parameters: $S \neq 0, B, C \neq 0, \theta, \mu, ||\vec{\mu}|| = 1$

XII. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = \theta = 0, \rho = -\frac{1}{3}S, \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0, \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k$
free parameters: $S \neq 0, v = 0, B, C, \mu, ||\vec{\mu}|| = 1$

XIII. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = \mu, \varepsilon = 0, \vec{\xi} = 0, \theta = 0, \rho = -\frac{1}{3}S, \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0, \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k$
free parameters: $S \neq 0, v = 0, B, C, \mu, ||\vec{\mu}|| = 1$

XIV. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = L\vec{\mu}, \varepsilon = 0, \vec{\xi} = X\vec{\mu}, \theta$-arbitrary, $\rho = -\frac{1}{3}S, \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0, \chi_{ij} = \mu, ||\vec{\mu}|| = 1$
\[ B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) \]
free parameters: \( S \neq 0 \), \( X \neq 0 \), \( L \), \( \theta \), \( B \), \( \vec{\mu} \), \( ||\vec{\mu}|| = 1 \)

**XV.** \( \vec{\alpha} = 0 \), \( \beta = 0 \), \( \vec{\gamma} = 0 \), \( \vec{\phi} = F\vec{\mu} \), \( \vec{\lambda} = 0 \), \( v \neq 0 \), \( \vec{\xi} = 0 \), \( \theta = 0 \), \( \rho = -\frac{1}{3}S \), \( \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) \), \( \omega_{ij} = 0 \), \( \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k \)
free parameters: \( S \neq 0 \), \( F \neq 0 \), \( v \neq 0 \), \( B \), \( C \), \( \vec{\mu} \), \( ||\vec{\mu}|| = 1 \)

**XVI.** \( \vec{\alpha} = 0 \), \( \beta = 0 \), \( \vec{\gamma} = 0 \), \( \vec{\phi} = F\vec{\mu} \), \( \vec{\lambda} = L\vec{\mu} \), \( v = 0 \), \( \vec{\xi} = 0 \), \( \theta = 0 \), \( \rho = -\frac{1}{3}S \), \( \sigma_{ij} = S(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) \), \( \omega_{ij} = 0 \), \( \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) \)
free parameters: \( S \neq 0 \), \( F \neq 0 \), \( B \), \( L \), \( \vec{\mu} \), \( ||\vec{\mu}|| = 1 \)

**XVII.** \( \vec{\alpha} = 0 \), \( \beta = 0 \), \( \vec{\gamma} = 0 \), \( \vec{\phi} = F\vec{\mu} \), \( \vec{\lambda} = L\vec{\gamma} \), \( v = 0 \), \( \vec{\xi} = X\vec{\gamma} \), \( \theta = 0 \)
\( \rho = 0 \), \( \sigma_{ij} = -\varepsilon_{ijk}\gamma_k \), \( \omega_{ij} = 0 \), \( \chi_{ij} = 0 \)
free parameters: \( X \), \( L \).

Let us note that all Poisson–Lie structures with \( \beta = 0 \), \( v = 0 \) and \( \theta = 0 \) are coboundary and the corresponding r–matrix reads

\[
\begin{align*}
    r &= i\phi_k H \wedge P_k + i\gamma_k H \wedge K_k + i\lambda_k H \wedge J_k + \\
    &+ i\varepsilon_{ijk}\lambda_k P_i \wedge P_j + i(\sigma_{ij} - \rho\delta_{ij})P_i \wedge J_j + \\
    &+ i\chi_{ij} P_i \wedge K_j - i(2\omega_{ij} - \omega_{mn}\delta_{ij})J_i \wedge K_j + \\
    &+ i\varepsilon_{ijk}\xi_k K_i \wedge K_j
\end{align*}
\]

Now there remains only to classify the orbits under the action of residual automorphisms corresponding to the rotations and scaling and put our structure in the canonical form. This is a straightforward although very tedious task. The result can be summarized as follows. There are 69 families of inequivalent Poisson–Lie structures which have been grouped for convenience into eight groups. Each group is described by an appropriate tables which are given below. They provide the main result of our paper.

Let us note that for all groups \( \vec{n} = 0 \). In the last column (labelled by \#) we indicate the number of essential parameters.

I. \( \rho = 0 \), \( \sigma_{ij} = 0 \), \( \omega_{ij} = 0 \), \( \chi_{ij} = 0 \), \( \vec{\gamma} = 0 \)
\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
N & \(\tilde{\alpha}\) & \(\beta\) & \(\tilde{\phi}\) & \(\tilde{\lambda}\) & \(v\) & \(\tilde{\xi}\) & \(\theta\) & \# \\
\hline
1 & (0, 0, \alpha) & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & (0, 0, 1) & 0 & (0, 0, 1) & (0, 0, L) & v & 0 & 0 & 2 \\
3 & (0, 0, 1) & 0 & 0 & (0, 0, \pm 1) & v & 0 & 0 & 1 \\
4 & (0, 0, 1) & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & (0, 0, 1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & (0, 0, 1) & (0, 0, \pm 1) & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & (0, 0, 1) & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & (0, 0, 1) & 0 & 1 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & \lambda_1 = 0 & 0 & (0, 0, 1) & \theta & 2 \\
10 & 0 & 0 & 0 & (0, 0, 1) & 0 & 0 & \pm 1 & 0 \\
11 & 0 & 0 & 0 & (0, 0, 1) & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & \theta & 1 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\end{table}

II. \(\beta = 0, \tilde{\gamma} = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = W(\delta_{ij} - \delta_{i3}\delta_{j3}), \chi_{ij} = B(\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ij3}\)

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
N & \(\tilde{\alpha}\) & \(\tilde{\phi}\) & \(\tilde{\lambda}\) & \(v\) & \(\tilde{\xi}\) & \(\theta\) & \(B\) & \(C\) & \(W\) & \# \\
\hline
15 & (0, 0, 1) & (0, 0, F) & (0, 0, L) & v & 0 & 0 & B & 2v & 1 & 4 \\
16 & (0, 0, 1) & (0, 0, F) & (0, 0, L) & v & 0 & 0 & 1 & 0 & 0 & 3 \\
17 & 0 & (0, 0, \pm 1) & (0, 0, L) & 0 & 0 & 0 & B & C & 1 & 3 \\
18 & 0 & \tilde{\phi} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\
19 & 0 & 0 & (0, 0, L) & 0 & (0, 0, X) & \theta & 2B^2 + 6C^2 = 3 & 1 & 4 \\
20 & 0 & 0 & (0, 0, \pm 1) & 0 & (0, 0, X) & \theta & 0 & 0 & 1 & 2 \\
21 & 0 & 0 & 0 & 0 & (0, 0, X) & \theta & 0 & 0 & 1 & 2 \\
\hline
\end{tabular}
\end{table}

III. \(\tilde{\alpha} = 0, \beta = 0, \tilde{\gamma} = 0, \rho = -\frac{1}{3}, \sigma_{ij} = (\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0, \chi_{ij} = B(\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ij3}\)
IV. $\vec{\alpha} = 0, \beta = 0, \phi = 0, v = 0, \theta = 0, \rho = 0, \omega_{ij} = 0, \chi_{ij} = 0, \sigma_{ij} = -\varepsilon_{ij3}$

| N | $\vec{\phi}$ | $\vec{\lambda}$ | $\vec{v}$ | $\vec{\xi}$ | $\vec{\theta}$ | $B$ | $C$ | # |
|---|---|---|---|---|---|---|---|---|
| 22 | (0,0,1) | (0,0,$L$) | 0 | 0 | 0 | $B$ | 0 | 2 |
| 23 | (0,0,1) | 0 | $v \neq 0$ | 0 | 0 | $B$ | 0 | 2 |
| 24 | (0,0,1) | 0 | $v$ | 0 | 0 | $B$ | $C \neq 0$ | 3 |
| 25 | 0 | (0,0,$L$) | 0 | (0,0,$\pm 1$) | $\theta$ | $B$ | 0 | 3 |
| 26 | 0 | (0,0,$L$) | 0 | 0 | $\pm 1$ | $B$ | 0 | 2 |
| 27 | 0 | (0,0,$L$) | 0 | 0 | 0 | 1 | 0 | 1 |
| 28 | 0 | (0,0,$L$) | 0 | 0 | 0 | 0 | 0 | 1 |
| 29 | 0 | 0 | 0 | 1 | 0 | 0 | $B$ | $C$ | 2 |
| 30 | 0 | 0 | 0 | 0 | $\theta$ | $2B^2 + 6C^2 = 3$ | $C \neq 0$ | 2 |

V. $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, v = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, tr\chi = 0, \sum_{ij} \chi_{ij}^2 = 1, \chi_{ij} = diag(\chi_{11}, \chi_{22}, \chi_{33}), \chi_{11} \neq \chi_{22} \neq \chi_{33}$

| N | $\vec{\gamma}$ | $\vec{\lambda}$ | $\vec{\xi}$ | # |
|---|---|---|---|---|
| 31 | (0,0,1) | (0,0,$L$) | (0,0,$\pm 1$) | 1 |
| 32 | (0,0,1) | (0,0,$L$) | 0 | 1 |

(Vb) $\chi_{11} = \chi_{22} \neq \chi_{33}$

| N | $\vec{\phi}$ | $\vec{\lambda}$ | $\vec{\xi}$ | $\vec{\theta}$ | # |
|---|---|---|---|---|---|
| 33 | 0 | $||\vec{\lambda}|| = 1$ | 0 | $\theta$ | 4 |
| 34 | 0 | $\vec{\lambda}$ | $||\vec{\xi}|| = 1$ | $\theta$ | 7 |
| 35 | $||\vec{\phi}|| = 1$ | $\vec{\lambda} = L\vec{\phi}$ | 0 | $\theta$ | 4 |

| N | $\vec{\phi}$ | $\vec{\lambda}$ | $\vec{\xi}$ | $\vec{\theta}$ | # |
|---|---|---|---|---|---|
| 36 | 0 | $\chi_{1} = 0$ | 0 | $\theta$ | 2 |
| 37 | 0 | $\chi_{2}^2 + \chi_{3}^2 = 1$ | $\chi_{1}$ | $\chi_{3}$ | $\xi_{1} = 0$ | $\theta$ | 5 |
| 38 | $\phi_{1} = 0$ | $\phi_{2}^2 + \phi_{3}^2 = 1$ | $\chi_{2}^2$ | $\phi_{1}$ | $\phi_{3}^2 = 1$ | $\chi_{1}$ | $\phi_{1}$ | $\phi_{3}^2 = 1$ | $\phi_{1}$ | $\phi_{3}^2 = 1$ | 2 |
\[ VI. \vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, v = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, tr\chi = 0, \sum_{ij}^{3} \chi_{ij}^2 = 1, \chi_{32} = -\chi_{23} \neq 0, \chi_{12} = \chi_{21}, \chi_{13} = \chi_{32} \]

(VIa) \[ \chi_{22} \neq \chi_{33} \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
N & \phi & \lambda & \xi & \theta & \# \\
\hline
39 & 0 & \lambda_1 = 0 & 0 & \theta & 6 \\
40 & 0 & \lambda_2^2 + \lambda_3^2 = 1 & \xi_1 = 0 & \theta & 9 \\
41 & ||\vec{\phi}|| = 1 & \vec{\lambda} = L\vec{\phi} & \xi_2 + \xi_3^2 = 1 & 0 & 7 \\
\hline
\end{array}
\]

(VIb) \[ \chi_{22} = \chi_{33} \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
N & \phi & \lambda & \xi & \theta & \# \\
\hline
42 & 0 & \lambda_1 = \lambda_2 = 0 & 0 & \theta & 4 \\
43 & 0 & \lambda_2 = \pm 1 & \xi_1 = \xi_2 = 0 & \theta & 7 \\
44 & (F, 0, 1) & \vec{\lambda} = L\vec{\phi} & \xi_3 = \pm 1 & 0 & 5 \\
\hline
\end{array}
\]

\[ VII. \vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\lambda} = 0, \vec{\xi} = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, tr\chi = 0, \sum_{ij}^{3} \chi_{ij}^2 = 1, \]

(VIIa) \[ \chi_{11} \neq \chi_{22} \neq \chi_{33}, \chi_{13} = -\chi_{31}, \chi_{23} = -\chi_{32}, \chi_{12} = -\chi_{21} \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
N & \phi & v & \theta & \# \\
\hline
45 & 0 & 0 & \pm 1, 0 & 4 \\
46 & \vec{\phi} & 1 & 0 & 7 \\
\hline
\end{array}
\]

(VIIb) \[ \chi_{11} = \chi_{22} \neq \chi_{33}, \chi_{13} = -\chi_{31}, \chi_{23} = \chi_{32} = 0, \chi_{12} = -\chi_{21} \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
N & \phi & v & \theta & \# \\
\hline
47 & 0 & 0 & \pm 1, 0 & 2 \\
48 & \vec{\phi} & 1 & 0 & 5 \\
\hline
\end{array}
\]

(VIIc) \[ \chi_{11} = \chi_{22}, \chi_{13} = \chi_{31} = \chi_{23} = \chi_{32} = 0, \chi_{12} = -\chi_{21} \]
\[
\begin{array}{c|cccc}
N & \phi & v & \theta & \# \\
\hline
49 & 0 & 0 & \pm 1, 0 & 1 \\
50 & (0, \phi_2, \phi_3) & 1 & 0 & 3 \\
\end{array}
\]

VIII. \( \bar{\alpha} = 0, \beta = 0, \bar{\gamma} = 0, \bar{\phi} = (0, 0, 1), \bar{\lambda} = 0, v = 0, \bar{\xi} = 0, \theta = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, tr\chi = 0, \sum_{ij} \chi_{ij}^2 = 1, \chi_{12} = -\chi_{21} = \frac{1}{F} \)

(VIIIa) \( \chi_{11} \neq \chi_{22}, \chi_{13} = \chi_{31}, \chi_{23} = \chi_{32} \)

(VIIIb) \( \chi_{11} = \chi_{22}, \chi_{13} = \chi_{31} = 0, \chi_{23} = \chi_{32} \)

Now, inserting the appropriate values of the parameters (listed above) to eq. (B.1) and using eq. (20) one can easily calculate the fundamental Poisson brackets for all nonequivalent structures.

IV Summary

We have obtained all Lie–Poisson structures on the four-dimensional Galilei group and classified them up to the equivalence implied by group automorphisms. The resulting set of structures appears to be quite rich; in particular, it includes many non–coboundary structures, to be contrasted with the Poincare group case [5]. In spite of that, part of them can surely be obtained from those on the Poincare group by a contraction procedure.

The next step to be done is to quantize the Lie-Poisson structures. In general, the consistent quantization is not an easy task. However, the preliminary study already done by us shows, that most of the cases described here are quantization friendly.

V Appendix A: The cocycle condition for subgroups

Let us specify the equations (18) for the subgroups of rotations, boosts, space and time translations. They read, respectively:

\[
\begin{align*}
\Psi_i(\mathcal{R}R') &= \Psi_i(\mathcal{R}) + R_i\Psi_l(\mathcal{R}') \\
\Phi_i(\mathcal{R}R') &= \Phi_i(\mathcal{R}) + R_i\Phi_l(\mathcal{R}')
\end{align*}
\]
\[ \Gamma_i(RR') = \Gamma_i(R) + R_{ij} \Gamma_i(R') \]
\[ \Lambda_i(RR') = \Lambda_i(R) + R_{ij} \Lambda_i(R') \]
\[ \Upsilon_{ij}(RR') = \Upsilon_{ij}(R) + R_{im} R_{jl} \Upsilon_{ml}(R') \]
\[ \Sigma_{ij}(RR') = \Sigma_{ij}(R) + R_{im} R_{jl} \Sigma_{ml}(R') \]
\[ \Xi_i(RR') = \Xi_i(R) + R_{im} \Xi_m(R') \]
\[ \Omega_{ij}(RR') = \Omega_{ij}(R) + R_{im} R_{jl} \Omega_{ml}(R') \]
\[ \Pi_i(RR') = \Pi_i(R) + R_{ij} \Pi_i(R') \]

\[ \Psi_i(\vec{v} + \vec{v}') = \Psi_i(\vec{v}) + \Psi_i(\vec{v}') \]
\[ \Phi_i(\vec{v} + \vec{v}') = \Phi_i(\vec{v}) + \Phi_i(\vec{v}') \]
\[ \Gamma_i(\vec{v} + \vec{v}') = \Gamma_i(\vec{v}) + \Gamma_i(\vec{v}') - \varepsilon_{ink} v_n \Psi_k(\vec{v}') \]
\[ \Lambda_i(\vec{v} + \vec{v}') = \Lambda_i(\vec{v}) + \Lambda_i(\vec{v}') - \frac{1}{2} \varepsilon_{imm} v_m \Phi_n(\vec{v}') \]

\[ \Upsilon_{ij}(\vec{v} + \vec{v}') = \Upsilon_{ij}(\vec{v}) + \varepsilon_{jnk} v_n \Psi_k(\vec{v}') v_i - \Gamma_j(\vec{v}') v_i - \varepsilon_{jnk} v_n \Sigma_{ik}(\vec{v}') \]
\[ \Sigma_{ij}(\vec{v} + \vec{v}') = \Sigma_{ij}(\vec{v}) + \Sigma_{ij}(\vec{v}') - v_i \Psi_j(\vec{v}') \]
\[ \Xi_i(\vec{v} + \vec{v}') = \Xi_i(\vec{v}) + \Xi_i(\vec{v}') - \frac{1}{2} (v_i \Omega_{mm}(\vec{v}') - v_j \Omega_{mm}(\vec{v}') \Pi_{im}(\vec{v}') v_i \]
\[ \Omega_{ij}(\vec{v} + \vec{v}') = \Omega_{ij}(\vec{v}) + \Omega_{ij}(\vec{v}') + 2 \Pi_i(\vec{v}') v_j - 2 \Pi_m(\vec{v}') v_m \delta_{ij} \]
\[ \Pi_i(\vec{v} + \vec{v}') = \Pi_i(\vec{v}) + \Pi_i(\vec{v}') \]

\[ \Psi_i(\vec{a} + \vec{a}') = \Psi_i(\vec{a}) + \Psi_i(\vec{a}') \]
\[ \Phi_i(\vec{a} + \vec{a}') = \Phi_i(\vec{a}) + \Phi_i(\vec{a}') - \varepsilon_{ink} a_n \Psi_k(\vec{a}') \]
\[ \Gamma_i(\vec{a} + \vec{a}') = \Gamma_i(\vec{a}) + \Gamma_i(\vec{a}') \]
\[ \Lambda_i(\vec{a} + \vec{a}') = \Lambda_i(\vec{a}) + \Lambda_i(\vec{a}') + \Pi_m(\vec{a}') a_m a_i - \frac{1}{2} (a_i \delta_{lm} - a_l \delta_{im}) \Sigma_{lm}(\vec{a}') \]
\[ \Upsilon_{ij}(\vec{a} + \vec{a}') = \Upsilon_{ij}(\vec{a}) + \Upsilon_{ij}(\vec{a}') + \varepsilon_{jnk} a_n \Omega_{jk}(\vec{a}') \]
\[ \Sigma_{ij}(\vec{a} + \vec{a}') = \Sigma_{ij}(\vec{a}) + \Sigma_{ij}(\vec{a}') + 2 (\Pi_i(\vec{a}') a_j - \Pi_l(\vec{a}') a_l \delta_{ij}) \]
\[ \Xi_i(\vec{a} + \vec{a}') = \Xi_i(a) + \Xi_i(a) \]
\[ \Omega_{ij}(\vec{a} + \vec{a}') = \Omega_{ij}(\vec{a}) + \Omega_{ij}(\vec{a}') \]
\[ \Pi_i(\vec{a} + \vec{a}') = \Pi_i(\vec{a}) + \Pi_i(\vec{a}') \]
\[ \begin{align*}
\Psi_i(t + t') &= \Psi_i(t) + \Psi_i(t') \\
\Phi_i(t + t') &= \Phi_i(t) + \Phi_i(t') - t\Gamma_i(t') \\
\Gamma_i(t + t') &= \Gamma_i(t) + \Gamma_i(t') \\
\Lambda_i(t + t') &= \Lambda_i(t) - \Lambda_i(t') - \frac{1}{2} t\varepsilon_{imn} \Upsilon_{mn}(t') + t^2 \Xi_i(t') \\
\Upsilon_{ij}(t + t') &= \Upsilon_{ij}(t) + \Upsilon_{ij}(t') - 2t\varepsilon_{ijl} \Xi_l(t') \\
\Xi_i(t + t') &= \Xi_i(t) + \Xi_i(t') - t\Omega_i(t') \\
\Omega_{ij}(t + t') &= \Omega_{ij}(t) + \Omega_{ij}(t') \\
\Pi_i(t + t') &= \Pi_i(t) + \Pi_i(t')
\end{align*} \] (A.4)

Note that all eqs. (A.1) have the same structure:
\[ T_{i_1...i_k}(RR') = T_{i_1...i_k}(R) + R_{i_1j_1} \cdots R_{i_kj_k} T_{j_1...j_k}(R') \] (A.5)

They can be solved by integrating over \( R' \) with respect to the Haar measure on \( SO(3) \):
\[ T_{i_1...i_k}(R) = (R_{i_1j_1} \cdots R_{i_kj_k} - \delta_{i_1j_1} \cdots \delta_{i_kj_k}) c_{j_1...j_k} \] (A.6)

where \( c_{j_1...j_k} \) are constants. This result agrees with general theorem that all cocycles on semisimple groups are coboundaries. On the other hand it follows immediately from eqs. (A.2–A.4) that all functions entering there are polynomials in the relevant parameters. This allows us to write out explicitly the general solutions.

\[ \begin{align*}
\Psi_i(\vec{v}) &= 0 \\
\Phi_i(\vec{v}) &= 0 \\
\Gamma_i(\vec{v}) &= a_{ij} v_j \\
\Lambda_i(\vec{v}) &= b_{ij} v_j \\
\Upsilon_{ij}(\vec{v}) &= c_{ijk} v_k - \frac{1}{2}(\delta_{ik} a_{jl} + \delta_{jl} a_{ik} - a_{ij} \delta_{ki} + \frac{1}{2}\varepsilon_{jkl}\varepsilon_{imn} a_{nm}) v_k v_l \\
\Sigma_{ij}(\vec{v}) &= \left( \frac{1}{2}\delta_{jk} \varepsilon_{imn} a_{nm} - \varepsilon_{ijn} a_{kn} \right) v_k \\
\Xi_i(\vec{v}) &= d_{ik} v_k + \left( \frac{1}{4}e_{ijk} - \frac{1}{8}(e_{jik} \delta_{ik} + e_{ikj}) \right) v_j v_k \\
\Omega_{ij}(\vec{v}) &= (e_{ijk} + \frac{1}{2}(e_{k} \delta_{ji} - e_{i} \delta_{jk})) v_k \\
\Pi_i(\vec{v}) &= 0
\end{align*} \] (A.7)
where \( e_{ijk} = e_{kji}, \ e_{iik} = 0; \)

\[
\begin{align*}
\Psi_i(\vec{a}) &= 0 \\
\Phi_i(\vec{a}) &= f_{ij} a_j \\
\Gamma_i(\vec{a}) &= g_{ij} a_j \\
\Lambda_i(\vec{a}) &= h_{ijk} a_j + \frac{1}{4} (h_{jik} - \frac{1}{2} (h_j \delta_{ik} + h_k \delta_{ij})) a_j a_k \\
\Upsilon_{ij}(\vec{a}) &= k_{ijk} a_k \\
\Sigma_{ij}(\vec{a}) &= (h_{ijk} + \frac{1}{2} (h_k \delta_{ij} - h_i \delta_{jk})) a_k \\
\Xi_i(\vec{a}) &= l_{ij} a_j \\
\Omega_{ij}(\vec{a}) &= 0 \\
\Pi_i(\vec{a}) &= 0,
\end{align*}
\]

where \( h_{ijk} = h_{kji}, \ h_{iik} = 0; \)

\[
\begin{align*}
\Psi(t) &= p_i t \\
\Phi_i(t) &= r_i t - \frac{1}{2} s_i t^2 \\
\Gamma_i(t) &= s_i t \\
\Lambda_i(t) &= u_i t - \frac{1}{4} \varepsilon_{imn} x_{mn} t^2 + \frac{1}{3} w_i t^3 \\
\Upsilon_{ij}(t) &= x_{ij} t - \varepsilon_{ijk} w_k t^2 \\
\Sigma_{ij}(t) &= y_{ij} t - \frac{1}{2} z_{ij} t^2 \\
\Xi_i(t) &= w_i t \\
\Omega_{ij}(t) &= z_{ij} t \\
\Pi_i(t) &= m_i t.
\end{align*}
\]

VI Appendix B: The general solution to the cocycle condition

According to the procedure outlined in sec. I, we use the cocycle condition(i) together with the decomposition (19) and the expressions written out in Appendix A to produce the Ansatz for \( \eta(g) \). Inserting it back into (18) we
find the general solution for $\eta$ of the form.

\[
\Psi_i(g) = (R_{ij} - \delta_{ij}) \alpha_j \\
\Phi_i(g) = (R_{ij} - \delta_{ij}) \phi_j + \beta(a_i - v_i t) - \gamma_j R_{ij} t + \varepsilon_{ijk} \alpha_i R_{jl}(a_k - v_k t) \\
\Gamma_i(g) = (R_{ij} - \delta_{ij}) \gamma_j + \beta v_i + \varepsilon_{ijk} \alpha_i R_{jl} v_k \\
\Lambda_i(g) = (R_{ij} - \delta_{ij}) \lambda_j + (\rho - \frac{1}{2} \sigma_{nn})(a_i - v_i t) + \frac{1}{2} \beta \varepsilon_{ijk} a_j v_k
\]

B.1

\[
\Upsilon_{ij}(g) = (R_{ik} R_{jl} - \delta_{ik} \delta_{jl}) \chi_{kl} + \varepsilon_{ijl} R_{ik} R_{jl} v_k t - \frac{1}{2} \delta_{ij} \beta \bar{v}^2 + \\
+ \varepsilon_{jkl} \alpha_n R_{ln} v_k v_l - \gamma_k R_{jk} v_i - 2 \varepsilon_{ijkl} \xi_l R_{kl} t \\
- \varepsilon_{jkl} \sigma_n R_{ln} R_{ik} v_k + \rho \varepsilon_{ijk} v_k + \omega_{nn} \varepsilon_{ijk} a_k \\
+ 2 \omega_{ns} (\varepsilon_{jkl} R_{js} R_{ln} a_k - \varepsilon_{ijl} R_{nk} R_{ln} v_k t) + 2 n_l (R_{jl} v_k - R_{ml} v_m \delta_{ik}) \varepsilon_{kin} a_n + \\
- 2 n_k v_m R_{nk} \varepsilon_{nij} + \omega_{lk} R_{nk} R_{il} v_n t^2
\]

\[
\Sigma_{ij}(g) = (R_{ik} R_{jl} - \delta_{ik} \delta_{jl}) \sigma_{kl} - \beta \varepsilon_{ijk} v_k - \alpha_k R_{jk} v_i \\
- 2 \omega_{ik} R_{ik} R_{jl} t + \omega_{nn} \delta_{ij} t + \\
+ 2 n_k (R_{ik} a_j - R_{mk} a_m \delta_{ij} - R_{ik} v_j + R_{mk} v_m \delta_{ij})
\]

\[
\Xi_i(g) = (R_{ij} - \delta_{ij}) \xi_j + \omega_{jl} R_{ij} R_{kl} v_k + n_k R_{mk} v_m v_i \\
\Omega_{ij}(g) = 2 (R_{ik} R_{jl} - \delta_{ik} \delta_{jl}) \omega_{lk} + 2 n_k (R_{ik} v_j - R_{mk} v_m \delta_{ij}) \\
\Pi_i(g) = (R_{ij} - \delta_{ij}) n_j
\]

VII Appendix C: Jacobi identities

Using the general form of $\eta$ described in Appendix B we find from eq.(14):

\[
\delta(H) = \gamma_i H \wedge P_i + \frac{1}{2} (\chi_{kj} - \chi_{jk}) P_j \wedge P_k + (2 \varepsilon_{ijk} \xi_k - \theta \delta_{ij}) P_i \wedge K_j
\]
Let \( \tilde{X}_i \) denote a basis in \( \mathcal{G}^* \) defined by \( \langle \tilde{X}_i, X_j \rangle = \delta_{ij} \). Then \( \delta \) imposes the following commutator structure in \( \mathcal{G}^* \).

\[
\begin{align*}
[\tilde{H}, \tilde{J}] &= \varepsilon_{ikl}\alpha_l \tilde{J}_i \\
[\tilde{H}, \tilde{P}_k] &= \gamma_k \tilde{H} + (\beta \delta_{ik} + \varepsilon_{ikl}\alpha_l) \tilde{P}_i + \varepsilon_{ikl}\phi_l \tilde{J}_i \\
[\tilde{H}, \tilde{K}_k] &= (\beta \delta_{ik} + \varepsilon_{ikl}\alpha_l) \tilde{K}_i + \varepsilon_{ikl}\gamma_l \tilde{J}_i \\
[\tilde{K}_k, \tilde{J}_i] &= 2(n_k \delta_{li} - n_i \delta_{kl}) \tilde{K}_i + 2(\varepsilon_{ikn}\omega_{ln} + \varepsilon_{iln}\omega_{nk}) \tilde{J}_i \\
[\tilde{P}_l, \tilde{P}_m] &= (\chi_{lm} - \chi_{ml}) \tilde{H} + 2\varepsilon_{klm}[\rho \delta_{ki} + \frac{1}{2}(\sigma_{ik} - \sigma_{mn} \delta_{ik})] \tilde{P}_i + 2(\varepsilon_{ilm} + \frac{1}{2}(\phi_l \delta_{im} - \phi_m \delta_{li})) \tilde{K}_i + 2(\lambda_l \delta_{im} - \lambda_m \delta_{ik}) \tilde{J}_i \\
[\tilde{P}_k, \tilde{K}_i] &= (2\xi_{kn} \varepsilon_{nkl} - \theta \delta_{ik}) \tilde{H} + (2\dot{\varepsilon}_{nk} \omega_{nl} - \omega_{nn} \dot{\varepsilon}_{nkl}) \tilde{P}_i + (\rho \varepsilon_{kli} - \varepsilon_{lin} \sigma_{kn} - \delta_{kl} \gamma_l) \tilde{K}_i + (\varepsilon_{inkn} \chi_{nl} + \varepsilon_{iln} \chi_{kn}) \tilde{J}_i \\
[\tilde{P}_k, \tilde{J}_i] &= (2\dot{\varepsilon}_{nk} \omega_{nl} - \omega_{nn} \dot{\varepsilon}_{nkl}) \tilde{H} + 2(n_k \delta_{li} - n_i \delta_{kl}) \tilde{P}_i - (\beta \varepsilon_{nkl} + \alpha \delta_{kl}) \tilde{K}_i + (\varepsilon_{inkn} \sigma_{nl} + \varepsilon_{iln} \sigma_{kn}) \tilde{J}_i \\
[\tilde{K}_m, \tilde{K}_n] &= 2\varepsilon_{kmn} \omega_{kl} \tilde{K}_i + 2(\xi_m \delta_{ni} - \xi_n \delta_{mi}) \tilde{J}_i \\
[\tilde{J}_k, \tilde{J}_l] &= 2(n_k \delta_{li} - n_l \delta_{ki}) \tilde{J}_i
\end{align*}
\]
Now we have to solve the Jacobi identities for the structure described in (C.2). This is still a complicated task so we apply a mixed procedure consisting in solving part of Jacobi identities directly and applying the group of automorphisms to simplify the remaining ones.

Therefore, we first give the action of automorphism group on parameters of $\delta$. Let us start with inner automorphisms. They read

$$
\begin{align*}
\alpha'_k &= \alpha_i \quad \gamma'_k &= \gamma_k - \beta v_k + \varepsilon_{kil} v_l \alpha_l \\
\beta' &= \beta \quad \lambda'_k &= \lambda_k - \nu v_k - \frac{1}{2} \varepsilon_{kil} \phi_i v_{\alpha} \\
\phi' &= \phi \quad \xi'_k &= \xi_k - \omega_{kn} v_n - v_k (v_n n_n) \\
\theta' &= \theta \quad \rho' &= \rho - \frac{1}{3} \alpha_n v_n \\
v' &= v \quad \omega'_{ij} &= \omega_{ij} + v_i n_j + (v_n n_n) \delta_{ij} \\
\vec{n}' &= \vec{n} \quad \sigma'_{ia} &= \sigma_{ia} + \beta \varepsilon_{ial} v_l + \alpha_a v_i - \frac{1}{3} \alpha_n v_n \delta_{ai} \\
\lambda'_{ab} &= \lambda_{ab} + v_a \gamma_b + \varepsilon_{bmn} \alpha_m v_a - \frac{1}{2} \varepsilon_{abk} \sigma_{nk} v_n \\
\rho' &= \rho - \frac{4}{3} a_k n_k \quad \sigma'_{ab} &= \sigma_{ab} + \frac{2}{3} (a_k n_k) \delta_{ab} - 2 a_n a_b \\
\chi'_{ab} &= \chi_{ab} + \varepsilon_{a} \omega_{nk} a_{k} + \varepsilon_{amk} \omega_{mb} a_k + \\
&\quad + \varepsilon_{bmk} \omega_{ma} a_k + \frac{2}{3} \varepsilon_{mnk} \omega_{mn} a_k \delta_{ab} \\
\omega'_{ab} &= \omega_{ab}
\end{align*}
$$

for boosts,

$$
\begin{align*}
\vec{v}' &= \vec{n}, \quad \vec{\alpha}' &= \alpha, \quad \vec{\beta}' &= \beta \\
v' &= v, \quad \vec{\xi}' &= \vec{\xi}, \quad \theta' &= \theta
\end{align*}
$$

(C.3)

(C.4)
for space translations and
\[\vec{n}' = \vec{n}, \quad \vec{\alpha}' = \vec{\alpha}, \quad \beta' = \beta, \quad \vec{\gamma}' = \vec{\gamma}, \quad v' = v, \quad \vec{\xi}' = \vec{\xi}, \quad \theta' = \theta, \quad \omega'_{ij} = \omega_{ij}\]
\[\bar{\phi}' = \bar{\phi} + t\vec{\gamma}, \quad \bar{\chi}' = \bar{\chi} - \frac{1}{2} t\varepsilon_{imn}\chi_{nm} + t^2 \xi_i\]
\[\rho' = \rho + \frac{1}{3} t\omega_{nn}, \quad \sigma'_{ab} = \sigma_{ab} + 2 t\omega_{ba} - \frac{2}{3} t\omega_{nn}\delta_{ab}\]
\[\chi'_{ab} = \chi_{ab} + 2 t\varepsilon_{abk}\xi_k\]
for time translations, respectively.

Under the rotations the parameters transform as tensors of appropriate rank.

Besides, there are two outer automorphisms, which correspond to rescaling of space and time unit, \((\vec{a} \rightarrow a\vec{a}, t \rightarrow bt)\). They read:
\[\vec{n}' = \vec{n}, \quad \vec{\alpha}' = \vec{\alpha}, \quad \beta' = \beta, \quad \vec{\gamma}' = \vec{\gamma}, \quad v' = v, \quad \vec{\xi}' = \vec{\xi}, \quad \theta' = \theta, \quad \omega'_{ij} = \omega_{ij}\]
\[\bar{\phi}' = \bar{\phi} + t\vec{\gamma}, \quad \bar{\chi}' = \bar{\chi} - \frac{1}{2} t\varepsilon_{imn}\chi_{nm} + t^2 \xi_i\]
\[\rho' = \rho + \frac{1}{3} t\omega_{nn}, \quad \sigma'_{ab} = \sigma_{ab} + 2 t\omega_{ba} - \frac{2}{3} t\omega_{nn}\delta_{ab}\]
\[\chi'_{ab} = \chi_{ab} + 2 t\varepsilon_{abk}\xi_k\]
\[\bar{\mu}' = \bar{\mu} + t(\vec{\alpha} \times \vec{\xi})\]
\[\bar{\lambda}' = \bar{\lambda} - \frac{1}{2} t\varepsilon_{imn}\lambda_{nm} + t^2 \xi_i\]
\[\rho' = \rho + t\omega_{nn}, \quad \sigma'_{ab} = \sigma_{ab} + 2 t\omega_{ba} - 2 t\omega_{nn}\delta_{ab}\]
\[\chi'_{ab} = \chi_{ab} + 2 t\varepsilon_{abk}\xi_k\]

We shall not enter into all details. Let us rather give a sketch of the procedure.

By solving Jacobi identities for the subalgebra generated by \(\vec{H}, \vec{J}_l, \vec{K}_n\) we find the following six families of constraints on parameters \(\vec{n}, \beta, \vec{\alpha}, \omega_{ij}, \vec{\gamma}, \) and \(\vec{\xi}\).

a) \(\vec{n} = 0, \vec{\alpha}\)–arbitrary, \(\beta \neq 0, \vec{\gamma}\)–arbitrary, \(\vec{\xi} = 0, \omega_{ij} = 0\)
b) \(\vec{n} = 0, \vec{\alpha} \neq 0, \beta = 0, \vec{\gamma}\)–arbitrary, \(\vec{\xi} = \xi\vec{\alpha} + W(\vec{\alpha} \times \vec{\gamma}), \omega_{ij} = W(\vec{\alpha}^2\delta_{ij} - \alpha_i\alpha_j), W \neq 0\)
c) \(\vec{n} = 0, \vec{\alpha} = 0, \beta = 0, \vec{\gamma}\)–arbitrary, \(\omega_{ij} = W(\delta_{ij} - \mu_i\mu_j) + V\varepsilon_{ijk}\mu_k, ||\vec{\mu}|| = 1\)
\[\vec{\gamma} = \begin{cases} \gamma\vec{\mu} & \text{if } |W| + |V| \neq 0 \\ \text{arbitrary} & \text{if } W = V = 0 \end{cases}\]
d) \(\vec{n} = 0, \vec{\alpha} \neq 0, \beta = 0, \omega_{ij} = W\delta_{ij}, \vec{\gamma} = \frac{1}{W}(\vec{\alpha} \times \vec{\xi}), W \neq 0, \vec{\xi}\)–arbitrary
e) $\vec{n} = 0, \vec{\alpha} \neq 0, \beta = 0, \omega_{ij} = 0, \vec{\xi} \parallel \vec{\alpha}, \vec{\gamma} \perp \vec{\alpha}$

f) $\vec{n} = 0, \vec{\alpha} = 0, \beta = 0, \omega_{ij} = W \delta_{ij}, \vec{\gamma} = 0, \vec{\xi}$–arbitrary

Now we have to solve the remaining Jacobi identities. First of all let us note that there are ambiguities in determining the matrices $\sigma_{ij}$ and $\chi_{ij}$. Namely, both can be redefined by adding the arbitrary multiplies of unit matrix. In order to remove this ambiguity we put $tr \sigma = tr \chi = 0$. Now we use the automorphisms generated by boosts, space and time translations to simplify the Jacobi identities. For example in the cases (a),(d) and (e) we may use the boost to put $\vec{\gamma} = 0$ from the very beginning. On the other hand in the case (b) by solving Jacobi identity for $\tilde{H}, \tilde{P}_i, \tilde{P}_k$ we find $\vec{\gamma} \vec{\alpha} = 0$ and again the boost can be used to put $\vec{\gamma} = 0$. Following this way we obtain the eighteen families of solutions described in sec.[I].

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