Higher order difference equations with homogeneous governing functions nonincreasing in each variable with unbounded solutions

Stevo Stević1,2*, A. El-Sayed Ahmed3, Bratislav Iričanin4,5 and Witold Kosmala6

*Correspondence: sstevic@ptt.rs
1Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia
2Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China
Full list of author information is available at the end of the article

Abstract
By using a comparison method and some difference inequalities we show that the following higher order difference equation

\[ x_{n+k} = \frac{1}{f(x_{n+k-1}, \ldots, x_n)}, \quad n \in \mathbb{N}, \]

where \( k \in \mathbb{N} \), \( f: [0, +\infty)^k \to [0, +\infty) \) is a homogeneous function of order strictly bigger than one, which is nondecreasing in each variable and satisfies some additional conditions, has unbounded solutions, presenting a large class of such equations. The class can be used as a useful counterexample in dealing with the boundedness character of solutions to some difference equations. Some analyses related to such equations and a global convergence result are also given.

MSC: 39A10; 39A22

Keywords: Difference equation; Unbounded solutions; Homogeneous function

1 Introduction
Let \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C} \) be the sets of natural, whole, real, and complex numbers respectively, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( \mathbb{R}_+ = (0, +\infty) \). If \( s, t \in \mathbb{Z} \), then we use the notation \( r = \overline{s, t} \) instead of \( s \leq r \leq t \), \( r \in \mathbb{Z} \).

Difference equations have been attracting attention of scientists for centuries. Since the time of de Moivre, many equations have been investigated so far. For some classical results see, e.g., [3, 5, 10–12, 14] and the related references therein.

1.1 First order difference equation, monotonicity, and some known facts
The general first order difference equation

\[ x_{n+1} = f(x_n), \quad n \in \mathbb{N}_0, \]

© The Author(s) 2022. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.
has been investigated for a long time, and nowadays many results on the equation and its special cases are known.

Here we recall some very basic facts on the behavior of solutions to equation (1) when the function $f$ is monotone. If the function $f$ is a self-map of an interval $I \subseteq \mathbb{R}$, then the case when $f$ is monotone is one of the basic ones. If $f$ is nondecreasing and $x_0 \leq x_1 = f(x_0)$, then the sequence $(x_n)_{n \in \mathbb{N}_0}$ is nondecreasing, whereas if $x_0 \geq x_1 = f(x_0)$ then the sequence $(x_n)_{n \in \mathbb{N}_0}$ is nonincreasing. If $f$ is additionally bounded, then the sequence is convergent (see, e.g., [1, Problem 9.34]). If the function $f$ is nonincreasing, then the sequences $(x_{2n})_{n \in \mathbb{N}_0}$ and $(x_{2n+1})_{n \in \mathbb{N}_0}$ are monotone, one of them is nondecreasing and the other is nonincreasing. If $x_0$ does not belong to the interval with the endpoints $x_1$ and $x_2$, then the sequences $(x_{2n})_{n \in \mathbb{N}_0}$ and $(x_{2n+1})_{n \in \mathbb{N}_0}$ are convergent (see, e.g., [1, Problem 9.35]). These results are some of the basic ones, and along with the additional condition on the continuity of the function $f$ they are frequently used in determining convergence of the solutions to equation (1). Many examples can be found, e.g., in [1] and [11].

We prove the statement related to the case when the function $f$ is nonincreasing, for completeness and benefit of the reader, and as a good motivation and a suggestion for part of the arguments in the rest of the paper. If

$$x_0 \leq f(x_0) = x_1,$$

then from (1), (2), and the monotonicity of $f$, we have $x_2 = f(x_1) \leq f(x_0) = x_1$.

There are two cases to be considered.

**Case 1.** If $x_0 \leq x_2 = f(f(x_0))$, then since in this case $x_0 \leq x_2 \leq x_1$, the monotonicity of $f$ implies $f(x_1) \leq f(x_2) \leq f(x_0)$, from which along with $x_0 \leq x_2$ it follows that $x_0 \leq x_2 \leq x_3 \leq x_1$. Assume that we have proved

$$x_0 \leq x_2 \leq \cdots \leq x_{2n-2} \leq x_{2n} \leq x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_3 \leq x_1$$

for some $n \in \mathbb{N}$. Then using the monotonicity of $f$, (1), (3), and $x_0 \leq x_2$, we obtain

$$x_0 \leq x_2 \leq \cdots \leq x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_3 \leq x_1,$$

from which by another application of the same procedure it follows that

$$x_0 \leq x_2 \leq \cdots \leq x_{2n+2} \leq x_{2n+3} \leq x_{2n+1} \leq \cdots \leq x_3 \leq x_1.$$

From this and by induction we have proved that (3) holds for every $n \in \mathbb{N}_0$.

**Case 2.** If $x_2 \leq x_0$, then since $x_2 \leq x_0 \leq x_1$, the monotonicity of $f$ implies $f(x_1) \leq f(x_0) \leq f(x_2)$, from which we have $x_2 \leq x_0 \leq x_1 \leq x_3$. Using the monotonicity of $f$, (1), (2), the fact that $x_2 \leq x_0$, and the method of induction, we get

$$x_{2n} \leq x_{2n-2} \leq \cdots \leq x_2 \leq x_0 \leq x_1 \leq x_3 \leq \cdots \leq x_{2n-1} \leq x_{2n+1}, \quad n \in \mathbb{N}_0.$$

The case when, instead of (2), it is assumed that $x_1 \leq x_0$ is treated similarly. From this and the monotonicity of $f$, we have $x_1 = f(x_0) \leq f(x_1) = x_2$. 


Case 3. If \( x_1 \leq x_2 \leq x_0 \), then as above we obtain

\[
x_1 \leq x_3 \leq \cdots \leq x_{2n-1} \leq x_{2n+1} \leq x_{2n} \leq x_{2n-2} \leq \cdots \leq x_2 \leq x_0, \quad n \in \mathbb{N}_0.
\]

Case 4. If \( x_1 \leq x_0 \leq x_2 \), then as above we obtain

\[
x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_3 \leq x_1 \leq x_0 \leq x_2 \leq \cdots \leq x_{2n-2} \leq x_{2n}, \quad n \in \mathbb{N}_0.
\]

This simple and well-known analysis shows that for each solution \((x_n)_{n\in\mathbb{N}_0}\) to equation (1) its subsequences \((x_{2n})_{n\in\mathbb{N}_0}\) and \((x_{2n+1})_{n\in\mathbb{N}_0}\) are monotone, as claimed. In the first and the third case from a well-known theorem it follows that the subsequences are convergent. Recall also that in this case \(g = f \circ f\) is a nondecreasing function, from which the monotonicity and convergence results can be concluded from the ones in the case when \(f\) is nondecreasing by noting that \(x_{2n+2} = g(x_{2n})\) and \(x_{2n+3} = g(x_{2n+1}), n \in \mathbb{N}_0\).

To say more about the long-term behavior of solutions to equation (1), some additional conditions on function \(f\) should be posed.

### 1.2 A previous claim

Recent literature shows frequent applications of known global convergence results. The following statement was formulated in [13].

**Theorem 1** Assume that \(f\) has nonpositive partial derivatives and is homogeneous with degree \(s\). Then the equation

\[
x_{n+1} = f(x_{n-k}, x_{n-m}), \quad n \in \mathbb{N}_0
\]

has a unique positive equilibrium \(x^*\), and every solution to equation (4) converges to \(x^*\).

To prove the statement in Theorem 1, paper [13] quotes Theorem 1.4.7 in [9] which deals with equation (4), but only when \(k = 0\) and \(m = 1\), which means that the result cannot be applied for other values of \(k\) and \(m\). Beside this, the proof of the theorem only checks the fact that from the associated two-dimensional algebraic system

\[
l = f(L, L), \quad L = f(l, l),
\]

it follows that \(l = L\). But since under the conditions in Theorem 1 system (5) is

\[
l = Lf(1, 1), \quad L = f(1, 1),
\]

it is immediately obtained \(l = L\), when \(f(1, 1) \neq 0\). It is claimed therein that this finishes the proof of Theorem 1.

Quite recently in [28] we have shown that the claim in Theorem 1 is not correct by presenting a counterexample in the class of difference equations with interlacing indices. The class of equations seems quite suitable for providing some counterexamples in the theory of difference equations (see, e.g., recent paper [4]).
1.3 Our aim
We show that there is a related global convergence result which can be applied for all values of $k, m \in \mathbb{N}_0$, but under some additional conditions. We also show that there is a large class of governing functions $f$ satisfying the conditions in the formulation of Theorem 1, such that the corresponding difference equations have solutions which are unbounded, showing that the statement in Theorem 1 is not correct for the large class of equations.

2 Preliminary analysis
In this section we conduct some analyses of difference equations whose governing functions are homogeneous and nonincreasing in all variables.

2.1 An instructive example, a product-type difference equation
Now we give a simple, but instructive, example which contains some ideas which are useful for the study.

Example 1 Consider the following difference equation:

$$x_{n+1} = \frac{a}{x_n^\alpha}, \quad n \in \mathbb{N}_0, \quad (6)$$

where $a, \alpha \in \mathbb{R}_+$. 

First note that in this case $f(x) = \frac{a}{x^\alpha}$, which is a continuous and decreasing function on $\mathbb{R}_+$, and maps it onto itself.

Note also that by using the change of variables $x_n = a^{\frac{1}{\alpha}} y_n$, $n \in \mathbb{N}_0$, equation (6) is transformed to the following one:

$$y_{n+1} = \frac{1}{a^{\alpha}}, \quad n \in \mathbb{N}_0.$$

Hence, we may assume that $a = 1$.

Case $\alpha = 1$. Assume that $\alpha = 1$. Then equation (6) becomes

$$x_{n+1} = \frac{1}{x_n}, \quad n \in \mathbb{N}_0. \quad (7)$$

Let $x_0 \in \mathbb{R}_+$. Then, by using (7) and a simple inductive argument, we have

$$x_{2n} = x_0 \quad \text{and} \quad x_{2n+1} = \frac{1}{x_0}, \quad n \in \mathbb{N}_0.$$

If $x_0 \in [1, \infty)$, then we have $x_1 = \frac{1}{x_0} \leq x_0 = x_2$. So, this situation corresponds to Case 3, and also Case 4 above. Moreover, if $x_0 \in (1, \infty)$, then the subsequences $(x_{2n})_{n \in \mathbb{N}_0}$ and $(x_{2n+1})_{n \in \mathbb{N}_0}$ are convergent, but the whole sequence is not. If $x_0 = 1$, then $x_n = 1$ for every $n \in \mathbb{N}_0$, and then the sequence is convergent.

If $x_0 \in (0, 1]$, then we have $x_2 = x_0 \leq \frac{1}{x_0} = x_1$. So, this situation corresponds to Case 1 and also Case 2 above. Moreover, if $x_0 \in (0, 1)$, then the subsequences $(x_{2n})_{n \in \mathbb{N}_0}$ and $(x_{2n+1})_{n \in \mathbb{N}_0}$ are convergent, but the sequence $(x_n)_{n \in \mathbb{N}_0}$ is not.
Now note that by using (6) with \( a = 1 \) twice, we obtain
\[
x_{2n+2} = x_{2n}^2 \quad \text{and} \quad x_{2n+3} = x_{2n+1}^2, \quad n \in \mathbb{N}_0.
\] (8)
From (8) it is easily obtained
\[
x_{2n} = x_0^{2n} \quad \text{and} \quad x_{2n+1} = x_1^{2n}, \quad n \in \mathbb{N}_0.
\] (9)
There are two cases to be considered.

Case \( \alpha \in (0, 1) \). In this case the sequences \( \alpha^{2n} \) and \( \alpha^{2n+1} \) decreasingly converge to zero, and consequently
\[
\lim_{n \to +\infty} x_{2n} = 1 \quad \text{and} \quad \lim_{n \to +\infty} x_{2n+1} = 1.
\]
Moreover, if \( x_0 \in (0, 1) \), then \( x_{2n} \) increases to one, whereas \( x_{2n+1} \) decreases to one. Since in this case \( x_0 < x_2 = \alpha^{2} < x_1 = \frac{1}{x_0} \), it corresponds to Case 1 above. If \( x_0 > 1 \), then \( x_{2n} \) decreases to one, whereas \( x_{2n+1} \) increases to one. Since in this case \( x_1 = \frac{1}{x_0} < x_2 = \alpha^{2} < x_0 \), we have the situation as in Case 3 above.

Case \( \alpha \in (1, \infty) \). In this case the sequences \( \alpha^{2n} \) and \( \alpha^{2n+1} \) increasingly tend to \( +\infty \), and consequently, if \( x_0 \in (0, 1) \), by letting \( n \to +\infty \) in (9) we get
\[
\lim_{n \to +\infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n \to +\infty} x_{2n+1} = +\infty,
\]
x_{2n} is decreasing and \( x_{2n+1} \) is increasing. Since \( x_2 = \alpha^{2} < x_0 < \frac{1}{x_0} = x_1 \), we have the situation in Case 2. If \( x_0 > 1 \), then
\[
\lim_{n \to +\infty} x_{2n} = +\infty \quad \text{and} \quad \lim_{n \to +\infty} x_{2n+1} = 0,
\]
x_{2n} is increasing and \( x_{2n+1} \) is decreasing. Since \( x_1 = \frac{1}{x_0} < x_0 < \alpha^{2} = x_2 \), we have the situation in Case 4.

Remark 1 Equation (6) is one of the simplest product-type difference equations which is solvable in closed form. The solvability is essentially what enables the simple analysis given in Example 1. For some more complex solvable product-type difference equations and systems see, for example, [24, 29] and the related references therein. Some classical results on solvability can be found, e.g., in [1, 3, 5, 8, 10–12, 14], whereas some other recent ones can be found, e.g., in [2, 20, 23, 25] (see also the related references therein).

Remark 2 Note that the function \( f(x) = \frac{1}{x^\alpha} \) in Example 1 is homogeneous with degree \(-\alpha\). Recall also that it is decreasing. But, as we have seen in the analysis in the example, if \( \alpha > 1 \), then there are unbounded solutions to equation (6), although
\[
l = \frac{1}{L^\alpha} \quad \text{and} \quad L = \frac{1}{l^\alpha}
\]
implies \( l = L \) in this case.
This homogeneous function of one variable strikingly suggests that similar situation should appear also in the case of homogeneous functions of several variables. This example along with the construction of difference equations with interlacing indices (for some examples of the equations see, e.g. [26, 27]) was used for constructing the counterexample in [28].

2.2 An example with a heuristic asymptotic approach

Here we consider a difference equation heuristically. The first higher order difference equation with non-interlacing indices, such that the governing function satisfies the conditions in the formulation of Theorem 1 and is related to the one-dimensional equation (6) with $a = 1$ that came to our mind, is the following second order one:

$$x_{n+2} = \frac{2}{x_{n+1} + x_n^2}, \quad n \in \mathbb{N},$$

with $x_1, x_2 \in \mathbb{R}_+$. Consider the equation with

$$x_1 = x_2 = \varepsilon,$$

where

$$\varepsilon \in \left(0, \frac{1}{\sqrt{2}}\right)$$

is very small.

From (10) and (11) we have

$$x_3 = \frac{2}{x_2^2 + x_1} = \frac{1}{\varepsilon^2}.$$  \hspace{1cm} (13)

From (10), (11), (13) and by the Taylor formula with Peano’s remainder, we have

$$x_4 = \frac{2}{x_3^2 + x_2} = \frac{2\varepsilon^4}{1 + \varepsilon^6} = 2\varepsilon^2 (1 + O(\varepsilon^{2+2^3})).$$  \hspace{1cm} (14)

From (10), (13), (14) and by the Taylor formula with Peano’s remainder, we have

$$x_5 = \frac{2}{x_4^2 + x_3} = \frac{2\varepsilon^4}{1 + 2\varepsilon^{2+2^3} + o(\varepsilon^{12})} = 2\varepsilon^2 (1 + O(2^2\varepsilon^{2+2^3})).$$  \hspace{1cm} (15)

From (10), (14), (15) as above, we have

$$x_6 = \frac{2}{x_5^2 + x_4} = \frac{1}{2^2 \varepsilon^4 (1 + O(\varepsilon^{2+2^3}))} = \frac{1}{2^2 \varepsilon^2 (1 + O(\varepsilon^{2+2^3}))}.$$  \hspace{1cm} (16)

From (10), (15), (16), we have

$$x_7 = \frac{2}{x_6^2 + x_5} = \frac{21+2^2 \varepsilon^{2+2^4}}{1 + O(\varepsilon^{2+2^3})} = 21 \varepsilon^2 (1 + O(\varepsilon^{2+2^3})).$$  \hspace{1cm} (17)
From (10), (16), (17), we have
\[ x_8 = \frac{2}{x_7^2 + x_6^2} = \frac{2^{1+2^2} \varepsilon^{2+4}}{1 + O(\varepsilon^{2+2})} = 2^{1+2^2}\varepsilon^{2^4} \left(1 + O(\varepsilon^{2^2})\right). \] (18)

From (10), (17), (18), we have
\[ x_9 = \frac{2}{x_8^2 + x_7^2} = \frac{1}{2^{2+2^2}\varepsilon^{2^5} \left(1 + O(\varepsilon^{2^2})\right)} = \frac{1}{2^{2+2^2}\varepsilon^{2^5}} \left(1 + O(\varepsilon^{2^2})\right). \] (19)

Formulas (13)–(19) suggest that the following relations hold:
\[ x_{3n} = \frac{1}{2^{2+2^2+\cdots+2^{(n-1)}} e^{2^{2n-1}} \left(1 + O(\varepsilon^{2^2})\right)} = \frac{1 + O(\varepsilon^{2^2})}{2^{2^{2n-1}-2} \varepsilon^{2^{2n-1}}} = \frac{\sqrt[3]{4} \left(1 + O(\varepsilon^{2^2})\right)}{\sqrt[3]{2^2} e^{2^{2(n-1)}} \left(1 + O(\varepsilon^{2^2})\right)}, \] (20)
\[ x_{3n+1} = 2^{1+2^2+\cdots+2^{(n-1)}} e^{2^{2n}} \left(1 + O(\varepsilon^{2^2})\right) = \frac{\sqrt[3]{2^2} \varepsilon^{2^{2n}} \left(1 + O(\varepsilon^{2^2})\right)}{\sqrt[3]{2} e^{2^{2n}}} = \left(\frac{\varepsilon^{2^{2n}}}{\sqrt[3]{2}}\right) \left(1 + O(\varepsilon^{2^2})\right), \] (21)
\[ x_{3n+2} = 2^{1+2^2+\cdots+2^{(n-1)}} e^{2^{2n}} \left(1 + O(\varepsilon^{2^2})\right) = \frac{\sqrt[3]{2^2} \varepsilon^{2^{2n}} \left(1 + O(\varepsilon^{2^2})\right)}{\sqrt[3]{2} e^{2^{2n}}} = \left(\frac{\varepsilon^{2^{2n}}}{\sqrt[3]{2}}\right) \left(1 + O(\varepsilon^{2^2})\right) \] (22)
for \( n \leq n_0 \) for some large but fixed \( n_0 \).

However, since the calculation errors are accumulated from one step to another, we will not conduct further our analysis in the direction nor try to prove (20)–(22), but will simply leave it as a heuristic asymptotic analysis.

Note that if (20)–(22) were true, then by using assumption (12), i.e., \( \varepsilon \sqrt[3]{2} \in (0, 1) \), in (20)–(22), it would follow that
\[ \lim_{n \to \infty} x_{3n} = +\infty, \quad \lim_{n \to \infty} x_{3n+1} = \lim_{n \to \infty} x_{3n+2} = 0, \]
which would show that the solution to equation (10) satisfying (11) is unbounded.

The heuristic proof suggests that another method should be employed. Since we do not have exact applicable formulas, one of the ideas is to compare some of the solutions to the equation with solutions to another equation for which it is possible to find the solutions in closed form. This idea will be used in the next section.

Remark 3 If \( x_1 = x_2 = 0 \), then the solution to equation (10) is not well defined. If \( x_1 \neq 0 \) or \( x_2 \neq 0 \), then \( x_3 > 0 \), and a simple inductive argument shows that \( x_n > 0 \) for every \( n \in \mathbb{N}_0 \). Hence, all solutions except the one obtained for \( x_1 = x_2 = 0 \) are well defined.

Remark 4 The only real equilibrium of equation (10) is \( x^* = 1 \). Hence, the equation has a bounded solution
\[ x_n \equiv 1, \quad n \in \mathbb{N}, \]
which is, of course, convergent. The solution is obtained for \( x_1 = x_2 = 1 \).

Assume that \( x_{n_0 + 1} = x_{n_0 + 2} = 1 \) for some \( n_0 \geq 3 \). Then we have

\[
1 = x_{n_0 + 2} = \frac{2}{x_{n_0 + 1}^2 + x_{n_0}^2} = \frac{2}{1 + x_{n_0}^2},
\]

from which it follows that \( x_{n_0} = 1 \) (by Remark 3, \( x_n > 0 \), \( n \geq 3 \) for each solution to equation (10)). By a simple inductive argument, we get \( x_{n_0} = 1 \) for \( 3 \leq n \leq n_0 + 2 \). If \( x_3 = x_4 = 1 \), then \( x_2^2 = 1 \) is similarly obtained, that is, \( |x_2| = 1 \), from which in the same way \( |x_1| = 1 \) is obtained. Hence there are four solutions which are eventually equal to one, namely those satisfying the conditions

\[
|x_1| = |x_2| = 1.
\]

3 Main results

Here we prove the main results in this paper, which give some answers to the questions posed in the introduction and show that the statement in Theorem 1 is not true by presenting a large class of difference equations possessing unbounded solutions.

The following result on global convergence of solutions to a difference equation is folklore and one of many existing in the literature (see, e.g., [6, 7, 9, 15, 16, 18, 21]). The proof is standard and essentially given in [9, Theorem A.0.8]. We present it for the completeness and the benefit of the reader.

**Theorem 2** Let \( k \in \mathbb{N} \) and \( f : [a, b]^k \to [a, b] \) be a continuous function which is nonincreasing in each variable, and such that from

\[
f(l, \ldots, l) = L \quad \text{and} \quad f(L, \ldots, L) = l,
\]

where \( l, L \in [a, b] \), it follows that \( l = L \).

Then the following difference equation

\[
x_{n+k} = f(x_{n+k-1}, \ldots, x_n), \quad n \in \mathbb{N},
\]

has a unique equilibrium \( x^* \in [a, b] \) and every solution to (24) converges to \( x^* \).

**Proof** Let \( g(x) = f(x, \ldots, x) - x \). Since \( g(a) \geq a \) and \( g(b) \leq b \), the continuity of \( g \) implies that there is \( x^* \in [a, b] \) such that \( g(x^*) = 0 \). Assume that there is \( y^* \in [a, b] \), \( x^* \neq y^* \) such that \( g(y^*) = 0 \). We may assume that \( x^* > y^* \), since the other case is dual. Then from the relation \( f(x^*, \ldots, x^*) - f(y^*, \ldots, y^*) = x^* - y^* \) and monotonicity of \( f \) in each variable we have

\[
0 < x^* - y^* = f(x^*, \ldots, x^*) - f(y^*, \ldots, y^*) \leq 0,
\]

which is a contradiction. Hence, it must be \( x^* = y^* \), proving the uniqueness.

Let \( l_1 = a, L_1 = b \),

\[
L_{n+1} = f(l_n, \ldots, l_n) \quad \text{and} \quad l_{n+1} = f(L_n, \ldots, L_n), \quad n \in \mathbb{N}.
\]
Then from (25) and by using induction it is routinely obtained
\[ l_1 \leq l_2 \leq \cdots \leq l_{n-1} \leq l_n \leq L_{n-1} \leq \cdots \leq L_2 \leq L_1 \]
(26)
for every \( n \in \mathbb{N} \), and for each solution \((x_n)_{n\in\mathbb{N}}\) to (24) we have
\[ l_n \leq x_j \leq l_n, \quad \text{for } j \geq k(n-1)+1. \]
(27)
From (26) we have
\[ \hat{l} = \lim_{n \to +\infty} l_n \quad \text{and} \quad \hat{L} = \lim_{n \to +\infty} L_n \]
for some \( \hat{l}, \hat{L} \in [a, b] \). By letting \( n \to +\infty \) in (25) we get (23) with \( l = \hat{l} \) and \( L = \hat{L} \), which implies \( l = \hat{l} \). From this and by letting \( n \to +\infty \) in (27) we get
\[ \lim_{n \to +\infty} x_n = \hat{l} = \hat{L} = x^*, \]
finishing the proof. \( \square \)

Remark 5 Theorem 2 is the result which can be applied in the case when the function \( f \) is nonincreasing in each variable, whether or not the function \( f(t_1, \ldots, t_k) \) depends on all the variables. It could have been applied in the proof of Theorem 3.3 in [13], but only if all its conditions are verified, which was not the case therein.

The following theorem is the main result in the paper. It shows that there is a large class of functions satisfying the conditions in Theorem 1, such that the corresponding difference equations have solutions which are unbounded, showing that the claim in Theorem 1 is not correct. In the proof of the theorem we use a comparison argument. For some related comparison arguments see, e.g., [17, 19, 22].

**Theorem 3** Consider the difference equation
\[ x_{n+k} = \frac{1}{f(x_{n+k-1}, \ldots, x_n)}, \quad n \in \mathbb{N}, \]
(28)
with \( x_j \in \mathbb{R}_+, j = 1, k \), where the function \( f : [0, +\infty)^k \to [0, +\infty) \) is homogeneous of order \( \alpha > 1 \), that is, \( f(\lambda t_1, \ldots, \lambda t_k) = \lambda^\alpha f(t_1, \ldots, t_k) \) for every \( \lambda \in [0, +\infty) \), nondecreasing in each variable, \( f(1, \ldots, 1) = 1 \), and that
\[ q := \min \{ f(1, 0, \ldots, 0), f(0, 1, 0, \ldots, 0), \ldots, f(0, \ldots, 0, 1) \} > 0. \]

Then every solution to equation (28) such that
\[ 0 < \max_{j=1, k} x_j < q^{\frac{1}{\alpha+1}} \]
(29)
is unbounded.
Remark 6 If $f(1, \ldots, 1) \neq 1$, then by using the change of variables
\[ x_n = \frac{y_n}{(f(1, \ldots, 1))^{1/(1+\alpha)}}, \quad n \in \mathbb{N}, \]
the sequence $(y_n)_{n \in \mathbb{N}}$ satisfies the equation
\[ y_{n+k} = \frac{f(1, \ldots, 1)}{f(y_{n+k-1}, \ldots, y_n)}, \quad n \in \mathbb{N}, \]
and the function
\[ \tilde{f}(t_1, \ldots, t_k) := \frac{f(t_1, \ldots, t_k)}{f(1, \ldots, 1)} \]
satisfies the condition $\tilde{f}(1, \ldots, 1) = 1$. Hence, we may assume $f(1, \ldots, 1) = 1$.

Proof of Theorem 3 Let
\[ m_0 := \max_{j \in \mathbb{Z}} x_j > 0. \]
Then, from (28) and the conditions of the theorem, we have
\[ x_{k+1} = \frac{1}{f(x_k, \ldots, x_k)} \geq \frac{1}{f(m_0, \ldots, m_0)} = \frac{1}{m_0^\alpha} = \frac{1}{m_0^\alpha}. \tag{30} \]
Let
\[ M_1 := \frac{1}{m_0^\alpha}. \tag{31} \]
Then, from (28), (30), and the conditions of the theorem, we have
\[ x_{k+2} = \frac{1}{f(x_{k+1}, \ldots, x_2)} \leq \frac{1}{f(M_1, 0, \ldots, 0)} = \frac{1}{M_1^\alpha f(1, 0, \ldots, 0)} \leq \frac{1}{q M_1^\alpha}, \]
\[ x_{k+3} = \frac{1}{f(x_{k+2}, x_{k+1}, \ldots, x_3)} \leq \frac{1}{f(0, M_1, 0, \ldots, 0)} = \frac{1}{M_1^\alpha f(0, 1, 0, \ldots, 0)} \leq \frac{1}{q M_1^\alpha}, \]
\[ \vdots \leq \frac{1}{q M_1^\alpha}. \tag{32} \]
Let
\[ m_1 := \frac{1}{q M_1^\alpha}. \]
Then, from (28), (32), and the conditions of the theorem, we have
\[ x_{2k+2} = \frac{1}{f(x_{2k+1}, \ldots, x_{k+2})} \geq \frac{1}{f(m_1, \ldots, m_1)} = \frac{1}{m_1^\alpha} = \frac{1}{m_1^\alpha}. \]
Let \((M_n)_{n \in \mathbb{N}}\) and \((m_n)_{n \in \mathbb{N}_0}\) be sequences defined as follows:

\[
M_{n+1} = \frac{1}{m_n}, \quad m_{n+1} = \frac{1}{qM_n^{\alpha}}, \quad n \in \mathbb{N}_0. \tag{33}
\]

Assume that for some \(l \in \mathbb{N}\) we have proved that

\[
x_{(k+1)|l+j} \leq m_l, \quad j = 1, k, \tag{34}
\]

\[
x_{(k+1)|l+1} \geq M_{l+1}. \tag{35}
\]

Then from (28), (35), and the conditions of the theorem we have

\[
x_{(k+1)|l+1} = \frac{1}{f(x_{(k+1)|l+1}, \ldots, x_{(k+1)|l+2})} \leq \frac{1}{f(M_{l+1}, 0, \ldots, 0)} \leq \frac{1}{qM_{l+1}^{\alpha}} = m_{l+1},
\]

\[
: \tag{36}
\]

\[
x_{(k+1)|l+k} = \frac{1}{f(x_{(k+1)|l+k-1}, \ldots, x_{(k+1)|l+1})} \leq \frac{1}{f(0, \ldots, 0, M_{l+1})} \leq \frac{1}{qM_{l+1}^{\alpha}} = m_{l+1}.
\]

Then from (28), (36), and the conditions of the theorem we have

\[
x_{(k+1)|l+2} = \frac{1}{f(x_{(k+1)|l+1}, \ldots, x_{(k+1)|l+2})} \geq \frac{1}{f(m_{l+1}, \ldots, m_{l+1})} \leq \frac{1}{m_{l+1}^{\alpha}f(1, \ldots, 1)} = \frac{1}{m_{l+1}^{\alpha}} = M_{l+2}. \tag{37}
\]

From this and by induction, we see that the inequalities in (34) and (35) hold for every \(l \in \mathbb{N}_0\).

From the equations in (33) we have

\[
M_n = \frac{1}{m_{n-1}^{\alpha}} \geq (qM_1^{\alpha})^a = q^a M_{n-1}^{\alpha^2}, \quad n \geq 2. \tag{38}
\]

Iterating equation (38) yields

\[
M_n = q^a M_{n-1}^{\alpha^2} = q^a (q^a M_{n-2}^{\alpha^2})^{\alpha^2} = q^{a(1+\alpha^2)} M_{n-2}^{(\alpha^2)^2}, \quad n \geq 3.
\]

By a simple inductive argument we have

\[
M_n = q^{a(1+\alpha^2+\cdots+\alpha^{2n-4})} M_1^{2n-2}, \quad n \in \mathbb{N},
\]

from which it follows that

\[
M_n = q^{\frac{2n-2-1}{\alpha^2}} M_1^{2n-2}, \quad n \in \mathbb{N}. \tag{39}
\]
From (39) together with (31) we have

\[ M_n = q^{2^{n-1} - \frac{a}{a^2 - 1}} m_0^{2^{n-1}} = \left( \frac{q^{\frac{1}{a^2-1}}}{m_0} \right)^{2^{n-1}} q^{-\frac{a}{a^2-1}}, \quad n \in \mathbb{N}. \quad (40) \]

Employing (40) in the second equation in (33) we get

\[ m_n = \frac{1}{q M_n^2} = \left( \frac{q^{\frac{1}{a^2-1}}}{m_0} \right)^{-2^{n}} q^{\frac{1}{a^2-1}}. \quad (41) \]

Let \( m_0 \in (0, q^{\frac{1}{a^2-1}}) \). Letting \( n \to +\infty \) in (40) and (41) and using the fact \( q^{\frac{1}{a^2-1}} / m_0 > 1 \) and the assumption \( a > 1 \), we obtain

\[ \lim_{n \to +\infty} m_n = 0 \quad (42) \]

and

\[ \lim_{n \to +\infty} M_n = +\infty. \quad (43) \]

From (34), (35), (42), and (43), we see that for each solution to equation (28) satisfying condition (29) we have

\[ \lim_{l \to +\infty} x_{(k+1)l^j} = 0, \quad j = 1, k, \]

and

\[ \lim_{l \to +\infty} x_{(k+1)l} = +\infty. \quad (44) \]

The relation in (44) shows that each solution with such chosen initial values is unbounded, finishing the proof of the theorem. \( \square \)

Acknowledgements
The paper was made during the investigation supported by the Ministry of Education, Science, and Technological Development of Serbia, contract no. 451-03-68/2022-14/200103. A. El-Sayed Ahmed would like to thank Taif University Researchers supporting Project number (TURSP-2020/159), Taif University – Saudi Arabia.

Funding
Taif University, project TURSP-2020/159.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
SS initiated the investigation, proposed some preliminary ideas, and conducted some detailed investigations. AEA, BI, and WK analyzed the proposed ideas, made some calculations, and gave many comments. All authors read and approved the final manuscript.
References
1. Bashmakov, M.I., Bekker, B.M., Gol’hovoi, VM.: Zadachi po Matematike. Algebra and Analiz. Nauka, Moskva (1982) (in Russian)
2. Berg, L., Stević, S.: On some systems of difference equations. Appl. Math. Comput. 218, 1713–1718 (2011)
3. Fort, T.: Finite Differences and Difference Equations in the Real Domain. Oxford University Press, London (1948)
4. Jekl, J.: Special cases of critical linear difference equations. Electron. J. Qual. Theory Differ. Equ. 2021, Article ID 79 (2021)
5. Jordan, C.: Calculus of Finite Differences. Chelsea, New York (1956)
6. Kent, C.M.: Convergence of solutions in a nonhyperbolic case. Nonlinear Anal. 47(7), 4651–4665 (2001)
7. Kosmala, W.A., Teixeira, C.: More on the difference equation \(y_{n+1} = \frac{p + y_{n-1}}{q + y_{n-1}}\). Appl. Anal. 81(1), 143–151 (2002)
8. Krechmar, VA.: A Problem Book in Algebra. Mir, Moscow (1974)
9. Kulenović, M.R.S., Ladas, G.: Dynamics of Second Order Rational Difference Equations. Chapman & Hall, Boca Raton (2002)
10. Milne-Thomson, L.M.: The Calculus of Finite Differences. Macmillan & Co., London (1933)
11. Mitrinović, D.S., Adamović, D.D.: Nizovii Redovi/Sequences and Series. Naučna Knjiga, Beograd (1980) (in Serbian)
12. Mitrinović, D.S., Kečkić, J.D.: Metodi Izračunavanja Konačnih Zbirova/Methods for Calculating Finite Sums. Naučna Knjiga, Beograd (1984) (in Serbian)
13. Moaaz, O.: Dynamics of difference equation \(x_{n+1} = f(x_{n}, x_{n-k})\). Adv. Differ. Equ. 2018, Article ID 447 (2018)
14. Nörlund, N.E.: Vorlesungen über Differenzenrechnung. Springer, Berlin (1924) (in German)
15. Papaschinopoulos, G., Schinas, C.J.: On a system of two nonlinear difference equations. J. Math. Anal. Appl. 219(2), 415–426 (1998)
16. Papaschinopoulos, G., Schinas, C.J.: Oscillation and asymptotic stability of two systems of difference equations of rational form. J. Differ. Equ. Appl. 7, 601–617 (2001)
17. Papaschinopoulos, G., Schinas, C.J.: On a \((k+1)\)-th order difference equation with a coefficient of period \(k+1\). J. Differ. Equ. Appl. 11(3), 215–225 (2005)
18. Papaschinopoulos, G., Schinas, C.J., Stefanidou, G.: Two modifications of the Beverton–Holt equation. Int. J. Difference Equ. 4(1), 115–136 (2009)
19. Papaschinopoulos, G., Stefanidou, G.: Trichotomy of a system of two difference equations. J. Math. Anal. Appl. 289, 216–230 (2004)
20. Papaschinopoulos, G., Stefanidou, G.: Asymptotic behavior of the solutions of a class of rational difference equations. Int. J. Difference Equ. 5(2), 233–249 (2010)
21. Stević, S.: A global convergence result with applications to periodic solutions. Indian J. Pure Appl. Math. 33(1), 45–53 (2002)
22. Stević, S.: On the recursive sequence \(x_{n+1} = (\alpha + \beta x_{n})/(1 + g(x_{n}))\). Indian J. Pure Appl. Math. 33(12), 1767–1774 (2002)
23. Stević, S.: Solvable subclasses of a class of nonlinear second-order difference equations. Adv. Nonlinear Anal. 5(2), 147–165 (2016)
24. Stević, S.: Solvable product-type system of difference equations whose associated polynomial is of the fourth order. Electron. J. Qual. Theory Differ. Equ. 2017, Article ID 13 (2017)
25. Stević, S.: New class of practically solvable systems of difference equations of hyperbolic-cotangent-type. Electron. J. Qual. Theory Differ. Equ. 2020, Article ID 89 (2020)
26. Stević, S., Ahmed, A.E., Kosmala, W., Smarda, Z.: Note on a difference equation and some of its relatives. Math. Methods Appl. Sci. 44, 10053–10061 (2021)
27. Stević, S., Diblik, J., Irčanin, B., Smarda, Z.: On some solvable difference equations and systems of difference equations. Abstr. Appl. Anal. 2012, Article ID 541761 (2012)
28. Stević, S., Irčanin, B., Kosmala, W., Smarda, Z.: Note on difference equations with the right-hand side function nonincreasing in each variable. J. Inequal. Appl. 2022, Article ID 25 (2022)
29. Stević, S., Irčanin, B., Smarda, Z.: Solvability of a close to symmetric system of difference equations. Electron. J. Differ. Equ. 2016, Article ID 159 (2016)