NON-$G$-COMPLETELY REDUCIBLE SUBGROUPS OF THE EXCEPTIONAL
ALGEBRAIC GROUPS

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Abstract. Let $G$ be an exceptional algebraic group defined over an algebraically closed field $k$ of characteristic $p > 0$ and let $H$ be a subgroup of $G$. Then following Serre we say $H$ is $G$-completely reducible or $G$-cr if, whenever $H$ is contained in a parabolic subgroup $P$ of $G$, then $H$ is in a Levi subgroup of that parabolic. Building on work of Liebeck and Seitz, we find all triples $(X, G, p)$ such that there exists a closed, connected, simple non-$G$-cr subgroup $H \leq G$ with root system $X$.

1. Introduction

Let $G$ be an algebraic group defined over an algebraically closed field $k$ of characteristic $p > 0$ and let $H$ be a subgroup of $G$. Then following Serre [Ser98] we say $H$ is $G$-completely reducible or $G$-cr if, whenever $H$ is contained in a parabolic subgroup $P$ of $G$, then $H$ is in a Levi subgroup of that parabolic. This is a natural generalisation of the notion of a group acting completely reducibly on a module $V$: if we set $G = GL(V)$ then saying $H$ is $G$-completely reducible is precisely the same as saying that $H$ acts semisimply on $V$.

This notion is important in unifying some other pre-existing notions and results. For instance, in [BMR05], it was shown that a subgroup $H$ is $G$-cr if and only if it satisfied Richardson’s notion of being strongly reductive in $G$. It also allows one to state some previous results due to Liebeck–Seitz and Liebeck–Saxl–Testerman on the subgroup structure of the exceptional algebraic groups in a particularly satisfying form:

Assume $G$ is simple of one of the five exceptional types and let $X$ be a simple root system. The result [LS96, Theorem 1] asserts a number $N(X, G)$ such that if $H$ is closed, connected and simple, with root system $X$, then $H$ is $G$-cr whenever the characteristic $p$ of $k$ is bigger than $N(X, G)$. In particular if $p$ is bigger than 7 then they show that all closed, connected, reductive subgroups of $G$ are $G$-cr. There is some overlap in that paper with the contemporaneous work of [LST96]. If $H$ is a simple subgroup of rank greater than half the rank of $G$, then [Theorem 1, ibid.] finds all conjugacy classes of simple subgroups of $G$, the proofs indicate where these conjugacy classes are $G$-completely reducible. With essentially one class of exceptions, all subgroups, including the non-$G$-cr subgroups, can be located in ‘nice’ so-called subsystem subgroups of $G$. We shall mention these in greater detail later.

More recently, [Ste10a] and [Ste12] find all conjugacy classes of simple subgroups of exceptional groups of types $G_2$ and $F_4$. One consequence of this is to show that the numbers $N(X, G)$ found above can be made strict. (One need only change $N(A_1, G_2)$ from 3 to 2.) The main purpose of this article is to make all the $N(X, G)$ strict. That is, for each of the five types of exceptional algebraic group $G$, for each prime $p = \text{char } k$ and for each simple root system $X$, we give in a table of Theorem 1 an example $H = E(X, G, p)$ of a connected, closed, simple non-$G$-cr subgroup $H$ with

\footnote{(thus, $H$ is in some parabolic $P$, but in no Levi subgroup $L$ of $P$)}
root system $X$, precisely when this is possible. In other words we classify the triples $(X, G, p)$ where there exists a connected, closed, simple non-$G$-cr subgroup $H$ with root system $X$. Moreover, in all but one case (where $(X, G, p) = (G_2, E_7, 7)$), we can locate $E(X, G, p)$ in a subsystem subgroup.

Our main theorem can thus be viewed as the best possible improvement of the result [LS96, Theorem 1], in the spirit of that result. Before we state our main theorem in full, we need a definition: A subsystem subgroup of $G$ is a simple, closed, connected subgroup $Y$ which is normalised by a maximal torus $T$ of $G$. Let $\Phi$ be the root system of $G$ corresponding to a choice of Borel subgroup $B \supseteq T$ and for $\alpha \in \Phi$, let $U_{\alpha}$ denote the $T$-root subgroup corresponding to $\alpha$. Then $Y = \langle U_{\alpha} | \alpha \in \Phi \rangle$ where either $\Phi$ is a closed subsystem of $\Phi$ or $(\Phi, p)$ is $(B_n, 2)$, $(C_n, 2)$, $(F_4, 2)$ or $(G_2, 3)$ and $\Phi$ lies in the dual of a closed subsystem. The subsystem subgroups of $G$ are easily determined by the Borel–de Siebenthal algorithm. Most of our examples $H = E(X, G, p)$ are described in terms of an embedding of $H$ into a subsystem subgroup $M$. Here we describe $M$ just by giving its root system.

**Theorem 1.** Let $G$ be an exceptional algebraic group defined over an algebraically closed field $k$ of characteristic $p > 0$. Suppose there exists a non-$G$-cr closed, connected, simple subgroup $H$ of $G$ with root system $X$. Then $(X, G, p)$ has an entry in Table 1.

Conversely, for each $(X, G, p)$ given in Table 1, the last column guarantees an example of a closed, connected, simple, non-$G$-cr subgroup $E(X, G, p)$ with root system $X$.

In particular we can improve on [LS96, Theorem 1]. In the table in Corollary 2 we have struck out the primes which were used in the hypotheses in [loc. cit.]. This is done partly to show where we have made improvements but mainly to facilitate reading the proof of the first part of Theorem 1.

**Corollary 2.** Let $G$ be an exceptional algebraic group over a field $k$ of characteristic $p$. Let $X$ be a simple root system and let $N(X, G)$ be a list of primes defined by the table below. Suppose $H$ is a closed, connected, reductive subgroup of $G$ with root system having simple components $X_1, \ldots, X_n$. Then if $p \not\in \bigcup_i N(X_i, G)$, $H$ is $G$-cr.

| $X$ | $G = E_8$ | $E_7$ | $E_6$ | $F_4$ | $G_2$ |
|-----|-----------|-------|-------|-------|-------|
| $A_1$ | $\leq 7$ | $\leq 7$ | $\leq 5$ | $\leq 3$ | $\beta_2$ |
| $A_2$ | $\beta$ 3 2 | $\beta$ 3 2 | $3 2$ | $3 2$ | $2$ |
| $B_2$ | $5 \beta$ 2 | $2 \beta$ | $2 \beta$ | $2$ | $2$ |
| $G_2$ | $7 \beta$ 3 2 | $7 \beta$ 3 2 | $\beta$ 2 | $2$ | $2$ |
| $A_3$ | $2$ | $2$ | $2$ | $2$ | $2$ |
| $B_3$ | $2$ | $2$ | $2$ | $2$ | $2$ |
| $C_3$ | $3$ | $2$ | $2$ | $2$ | $2$ |
| $B_4$ | $2$ | $2$ | $2$ | $2$ | $2$ |
| $C_4, D_4$ | $2$ | $2$ | $2$ | $2$ | $2$ |

Using the above description of $N(X, G)$ one also gets generalisations to each of the other results [LS96 Theorems 2–8], by replacing the hypothesis ‘$p > N(X, G)$’ by ‘$p \not\in N(X, G)$’.

2. Notation

When discussing roots or weights, we use the Bourbaki conventions [Bou82 VI. Planches I-IX]. We use a lot of representation theory for algebraic groups whose notation we have taken largely consistent with [Jan03]. For an algebraic group $G$, recall that a $G$-module is a comodule for the Hopf algebra $k[G]$; in particular every $G$-module is a $kG$-module. Let $B$ be a Borel subgroup of
each of the subgroups of $A_2$:

- $E_5$:
  - $E \leftrightarrow A_5^\sharp_2$; $x \mapsto (x,x)$
  - $E \leftrightarrow A_5^\sharp_3$; $V_6 \rightarrow E = V(20) = 10^{[4]}/01$
  - $E \leftrightarrow A_5^\sharp_2$; $x \mapsto (x,x)$

- $E_6$:
  - $E \leftrightarrow A_6^\sharp_2$; $(V_3, V_3) \rightarrow E = (2,2)$
  - $E \leftrightarrow A_6^\sharp_3$; $V_{10} \rightarrow E = (2,2)$

- $E_7$:
  - $E \leftrightarrow A_7^\sharp_2$; $x \mapsto (x,x)$
  - $E \leftrightarrow A_7^\sharp_3$; $V_8 \rightarrow E = W(7) = 1^{[4]}/5$
  - $E \leftrightarrow A_7^\sharp_2$; $x \mapsto (x,x)$

- $E_8$:
  - $E \leftrightarrow A_8^\sharp_2$; $V_9 \rightarrow E = (20) = 00/00/00$

| $G$ | $X$ | $p$ | Example $E = E(X, G, p)$ |
|-----|-----|-----|--------------------------|
| $G_2$ | $A_1$ | 2 | $E \leftrightarrow A_1^\sharp_1$; $x \mapsto (x,x)$ |
| $E_8$ | $A_3$ | 2 | $E \leftrightarrow A_3^\sharp_2$; $(V_3, V_3) \rightarrow E = (2,2)$ |
| $E_8$ | $B_2$ | 2 | $E_2 \leq D_3$ |
| $E_8$ | $B_3$ | 2 | $E \leq D_4$ |
| $E_8$ | $B_4$ | 2 | $E \leq D_4$ |

Table 1. Simple non-$G$-cr subgroups of type $X$ in the exceptional groups

a reductive algebraic group $G$, containing a maximal torus $T$ of $G$. Recall that for each dominant weight $\lambda \in X^+(T)$ for $G$, the space $H^0(\lambda) := H^0(G/B, \lambda) = \text{Ind}^G_B(\lambda)$ is a $G$-module with highest weight $\lambda$ and with socle $\text{Soc}^G_B(\lambda) = L(\lambda)$, the irreducible $G$-module of highest weight $\lambda$. The Weyl module of highest weight $\lambda$ is $V(\lambda) \cong H^0(-w_0\lambda)^*$ where $w_0$ is the longest element in the Weyl group. We identify $X(T)$ with $\mathbb{Z}^r$ for $r$ the rank of $G$ and for $\lambda \in X^+(T)$, write $\lambda = (a_1, a_2, \ldots, a_r) = a_1\omega_1 + \cdots + a_r\omega_r$ where $\omega_i$ are the fundamental dominant weights; a $\mathbb{Z}_{\geq 0}$-basis of $X(T)^+$. Put also $L(\lambda) = L(a_1, a_2, \ldots, a_r)$. When $0 \leq a_i < p$ for all $i$, we say that $\lambda$ is a restricted weight and we write $\lambda \in X^1_1(T)$. Recall that any module $V$ has a Frobenius twist $V^{[n]}$ induced by raising entries of matrices in $GL(V)$ to the $p^n$th power. Steinberg’s tensor product theorem states that $L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_n)^{[n]}$ where $\lambda_i \in X^1_1(T)$ and $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^n\lambda_n$ is the $p$-adic expansion of $\lambda \in \mathbb{Z}_r^+$. We refer to $\lambda_0$ as the restricted part of $\lambda$.

The right derived functors of $\text{Hom}(V, \ast)$ are denoted by $\text{Ext}^i_G(V, \ast)$ and when $V = k$, the trivial $G$-module, we have the identity $\text{Ext}^i_G(k, \ast) = H^i(G, \ast)$ giving the Hochschild cohomology groups.

We recall some standard modules; when $G$ is classical, there is a ‘natural module’ which we refer to by $V_{\text{nat}}$; or $V_m$ where $m$ is the dimension of $V_{\text{nat}}$. It is always the Weyl module $V(\omega_1)$, which
is irreducible unless \( p = 2 \) and \( G \) is of type \( B_n \); in the latter case it has a 1-dimensional radical. Certain properties of these modules is described in [Jan03 8.21]. Of importance to us is the fact that when \( G = SL_n \), \( F_r(L(\omega_1)) = L(\omega_r) \) for \( r \leq n - 1 \). We use this fact without further reference.

Recall that \( F_4 \) has a 26-dimensional Weyl module which we denote \( V_{26} \). When \( p \neq 3 \), \( V_{26} \) is the irreducible representation of high weight \( 0001 = \omega_4 \). When \( p = 3 \), \( V_{26} \) has a one-dimensional radical, with a 25-dimensional irreducible quotient of high weight \( 0001 \). \( E_6 \) (resp. \( E_7 \), \( E_8 \)) has a module of dimension 27 (resp. 56, 248) of high weight \( \omega_1 \) (resp. \( \omega_7 \), \( \omega_8 \)) which is irreducible in all characteristics. We refer to this module as \( V_{27} \) (resp. \( V_{56} \), \( \text{Lie}(E_8) \)).

We will often want to consider restrictions of simple \( G \)-modules to reductive subgroups \( H \) of \( G \). Where we write \( V_1|V_2|\ldots|V_n \) we list the composition factors \( V_i \) of an \( H \)-module. For a direct sum of \( H \)-modules, we write \( V_1 + V_2 \). Where a module is uniserial, we will write \( V_1/\ldots/V_n \) to indicate the socle and radical series: here the head is \( V_1 \) and the socle \( V_n \). On rare occasions we use \( V/W \) to indicate a quotient. It will be clear from the context which is being discussed.

Recall also the notion of a tilting module as one having a filtration by modules \( V(\mu) \) for various \( \mu \) and also a filtration by modules \( H(\mu) \) for various \( \mu \) (equiv. dual Weyl modules). Let us record in a lemma some key properties of tilting modules which we use:

**Lemma 2.1.**

(i) For each \( \lambda \in X(T)^+ \) there is a unique indecomposable tilting module \( T(\lambda) \) of high weight \( \lambda \);

(ii) A direct summand of a tilting module is a tilting module;

(iii) The tensor product of two tilting modules is a tilting module;

(iv) \( \text{Ext}^1_G(T(\lambda), T(\mu)) = 0 \); in particular \( H^1(G, T(\lambda)) = 0 \).

**Proof.** For (i), see [Don93 1.1(i)]; (ii) is clear by projecting a filtration to a direct summand and using the fact that Weyl modules and induced modules are indecomposable; (iii) is [Don93 1.2]; (iv) follows from [Jan03 II.4.13 (2)].

As we are considering very low weight representations in general, it is possible to spot that a module is a \( T(\lambda) \); for instance when \( p = 2 \), the natural Weyl module for \( B_n \) has a 1-dimensional radical, so its structure is \( W(\lambda_1) = L(\lambda_1)/k \). It is then the case that giving the Loewy series for a module \( k/L(\lambda_1)/k \) uniquely characterises it as a tilting module \( T(\lambda_1) \).

Recall that a parabolic subgroup \( P \) of \( G \) has a Levi decomposition,

\[
1 \to Q \to P \to L \to 1
\]

where \( Q \) is the unipotent radical of the \( P \). Recall also \( L = L'Z(L) \) with \( L' \) being semisimple.

### 3. Outline

Theorem 1 has two facets. The first proves that if \( p \notin N(X, G) \) for \( N(X, G) \) as defined in Corollary 2, then \( X \) is \( G \)-cr. The second proves the existence of the examples given in Table 1 and proves that they are non-\( G \)-cr.

The proof of the first part runs along the same lines as that of [LS96 Theorem 1]; Assume \( H \) is a closed, connected, simple non-\( G \)-cr subgroup of \( G \). Then \( H \) is a subgroup of \( P = LQ \); let \( \tilde{H} \) be its image in \( L' \). Almost all the time, \( H \cap Q = \{1\} \) as group-schemes and so we have \( HQ = \tilde{H}Q \) and \( H \) is a complement to \( Q \) in \( HQ \). Then the possibilities for \( H \) are parameterised by \( H^1(H, Q) \);
in fact, in any case, the possibilities for $H$ are parameterised by $H^1(\tilde{H}, Q^{[1]})$. This is the content of [Ste10b, Lemma 3.6.1].

From [ABS90], $Q$ has a filtration $Q = Q_1 \geq Q_2 \geq Q_3 \ldots$ with successive quotients being known (usually semisimple) $L$-modules. So if we have $H^1(\tilde{H}, (Q_i/Q_{i+1})^{[1]}) = 0$ for each $i$, then (by [4.4(ii)]) $H^1(\tilde{H}, Q^{[1]}) = 0$ and $H$ is conjugate to $\tilde{H}$.

Now, for an exceptional algebraic group $G$ over $k$ of characteristic $p$ and a simple root system $X$ we consider possible embeddings $\tilde{H} \leq L'$ where $\tilde{H}$ is an $L'$-irreducible subgroup (which can be determined using [4.8] and/or by working down through [4.9]). The composition factors $V$ of the restrictions of the $L$-modules $Q_i/Q_{i+1}$ are investigated, and then conditions for the vanishing of $H^1(\tilde{H}, V)$ found, for all relevant $V$. (Usually the dimensions of the composition factors are too small to admit non-vanishing of $H^1(\tilde{H}, V)$.)

With essentially one exception, one can reduce to the case where $V$ is of the form $L(\lambda) \otimes L(\mu)^{[1]}$ with $L(\lambda)$ non-trivial and restricted. There are any number of computer programs one can use to calculate the values of $H^1(X, V)$ where $\mu$ is 0. Since the possible dimension of $V$ is limited to a subset of roots of $G$, this process is finite.

For the proof of the second part of Theorem 1, we must show that for each of the remaining cases (where some composition factor $V$ of $Q^{[1]}$ has $H^1(\tilde{H}, V) \neq 0$), we exhibit a non-$G$-cr subgroup $H$ with the required root system over the required characteristic. In almost all cases we can give an example in a classical subgroup of $G$. Here it is easy to see when it is in a parabolic subgroup using [4.8]. In two cases this is not possible, yet we can assert the existence of such a group using a cohomological argument.

4. Preliminaries

One needs to be careful about the notion of complements in semidirect products of algebraic groups. These are treated systematically in [McN10]. We recall some of the main facts.

**Definition 4.1** (cf. [McN10 4.3.1]). Let $G = H \ltimes Q$ be a semidirect product of algebraic groups as in [Jan03 I.2.6].

A closed subgroup $H'$ of $G$ is a complement to $Q$ if it satisfies the following equivalent conditions:

(i) Multiplication is an isomorphism $H' \ltimes Q \rightarrow G$.
(ii) $\pi_{H'} : H' \rightarrow H$ is an isomorphism of algebraic groups
(iii) As group-schemes, $H'Q = G$ and $H' \cap Q = \{1\}$.
(iv) For the groups of $k$-points, one has $H'(k)Q(k) = G(k)$, $H'(k) \cap Q(k) = \{1\}$ and $\text{Lie}(H') \cap \text{Lie}(Q) = 0$.

**Remark 4.2.** See [Ste10b §3.2] for a discussion. Note that [LS96] uses item (iv) above as its definition of a complement, without the last condition on Lie algebras.

**Definition 4.3.** A rational map $\gamma : H \rightarrow Q$ is a 1-cocycle if $\gamma(nm) = \gamma(n)^m\gamma(m)$ for each $n, m \in N(k)$. We write $Z^1(H, Q)$ for the set of 1-cocycles.

We say $\gamma \sim \delta$ if there is an element $q \in Q(k)$ with $q^{-h}\gamma(h)q = \delta(h)$ for each $h \in H(k)$. We write $H^1(H, Q)$ for the set of equivalence classes of 1-cocycles $Z^1(H, Q)/\sim$.

\[\text{We use the data on Frank Lübeck's website which accompanies Lüb01.}\]
Lemma 4.4.  
(i) The set of 1-cocycles $Z(H,Q)$ is in bijection with the set of complements to $Q$ in $HQ$. Two cocycles are equivalent if the corresponding complements are conjugate by an element of $H(k)$.
(ii) Suppose $H$ is a closed, connected, reductive subgroup of a parabolic subgroup $P = LQ$ of $G$ and denote by $H$ the subgroup of $L$ given by the image of $H$ under the quotient map $\pi : P \to L$.

Then as abstract groups $H(k)$ is a complement to $Q(k)$ in $\bar{H}(k)Q(k)$; and either (1) $H$ is a complement to $Q$ in $\bar{H}Q$; or (2)
(a) $p = 2$;
(b) There exists a component $SO_{2n+1}$ of the semisimple group $H/Z(H)^0$;
(c) the image of this component in $H/Z(H)^0$ is isomorphic to $Sp_{2n}$; and
(d) the natural module for $Sp_{2n}$ appears in a filtration of $Q$ by $H$-modules.

In case (2), $H$ corresponds to a cocycle $\gamma \in Z^1(\bar{H},Q^{[1]})$ such that $[\gamma]$ has no preimage in $H^1(\bar{H},Q)$ under the inclusion $H^1(\bar{H},Q) \to H^1(\bar{H},Q^{[1]})$.

Thus there is a bijection between the set of conjugacy classes of closed, connected, reductive subgroups $H$ of $HQ$ and the set $H^1(\bar{H},Q^{[1]})$.

(iii) In a filtration of a unipotent algebraic $H$-group $Q$ by $H$-modules (such as that given by [5.1]) if each composition factor $V$ satisfies $H^1(H,V) = 0$ then $H^1(H,Q) = 0$.

Proof. (i) is [Ste10b, Lemma 3.2.2]; (ii) is [Ste10b, Lemma 3.6.1]. For (iii), such a filtration is ‘sectioned’ in the sense of [Ste10b, Definition 3.2.7] using [Ste10b, Lemma 3.2.8]. Now one uses the exact sequence of non-abelian cohomology in [Ste10b, 2.1(i)] inductively. (See the discussion in [Ste10b, §3.2] on the validity of this sequence for rational cohomology.)

In almost all cases the cohomology group $H^1(G,V)$ for a semisimple algebraic group $G$ satisfies $H^1(G,V) \cong H^1(G,V^{[1]})$. This fact allows us to reduce our considerations to simple modules with non-trivial restricted parts.

Lemma 4.5. Let $G$ be a simple algebraic group and $V$ a simple $G$-module. Then $H^1(G,V) \cong H^1(G,V^{[1]})$ unless $G$ is $Sp_{2n}$ and $V$ is its $2n$-dimensional natural module.

Moreover $H^1(G,V^{[1]})$ is isomorphic to its generic cohomology $H^1_{gen}(G,V)$.

Proof. See [Jan03, II.12.2, Remark] and [CPSvdK77, 7.1].

There are many papers finding the values $\text{Ext}_H^n(L,M)$ with $H$ of low rank and $L,M$ simple. Taking $L = k$, the trivial module, one gets the following result, where we have included more data than necessary for our purposes for completion’s sake.

Lemma 4.6. Let $V$ be a simple module for a simple algebraic group $H$ where $H$ is one of $SL_2$, $SL_3$, $Sp_4$ over an algebraically closed field of any characteristic $p$; $G_2$ for $p = 2, 3$ or $p \geq 13$; or $SL_4$, $Sp_6$ or $Sp_8$ when $p = 2$. Then $H^1(H,V)$ is at most one-dimensional, and is non-zero if and only if $V$ is a Frobenius twist of one of the modules in the following table.

In the table we also give some useful dimension data, often in only specific characteristic.

Proof. These are special cases from [Cli79] for $SL_2$, [Yeh82, 4.2.2] for $SL_3$, [Ye90] for $Sp_4$, $p \geq 3$, [LY93] for $G_2$ ($p \geq 13$), [Sin94, Proposition 2.2] for $Sp_4$ ($p = 2$), [Sin94, Proposition 3.4] for $G_2$.
Lemma 4.7. Let $G = G_2$ over a field of characteristic 5 and let $L$ be a simple module for $G$ with $H^1(G, L) \neq 0$. Then $\dim L > 56$.

Proof. One reduces to the case where the restricted part of $L$ is non-trivial using [4.30] so we may assume $L = M$. Start with the case that $M$ is restricted. One can use the data from [Lüb01] to establish that all Weyl modules of dimension less than 97 are irreducible. But then $H^1(G, L(\lambda)) \cong H^0(G, H^0(\lambda)/\text{Soc}_G(H^0(\lambda))) = 0$.

If $M$ is not restricted, then it is $M = M_1 \otimes M_2^{[1]}$ for $M_1$ restricted and $M_2$ non-trivial. The lowest dimension $M_1$ and $M_2$ can have is 7, the next is 14, but $14 \times 7 > 56$, so we conclude $M_1 = L(1,0)$ and $M_2 = L(1,0)^{[r]}$. Now by [LS96] 1.15] (or the linkage principle), one gets $H^1(G, M) = 0$. $\square$

The next lemma is useful for establishing $L'$-irreducible embeddings $\bar{H} \leq L'$ when $L'$ and also for deciding when a subgroup $H$ is in a parabolic of a classical subgroup $M$ of $G$.

Lemma 4.8 ([LS96] p32-33). Let $G$ be a simple algebraic group of classical type, with natural module $V = V_G(\lambda_1)$, and let $H$ be a $G$-irreducible subgroup of $G$.

(i) If $G = A_n$, then $H$ acts irreducibly on $V$.
(ii) If $G = B_n$, $C_n$, or $D_n$ with $p \neq 2$, then $V \downarrow H = V_1 \perp \cdots \perp V_k$ with the $V_i$ all non-degenerate, irreducible, and inequivalent as $X$-modules.
(iii) If $G = D_n$ and $p = 2$, then $V \downarrow H = V_1 \perp \cdots \perp V_k$ with the $V_i$ all non-degenerate, $V_2 \downarrow H, \ldots, V_k \downarrow H$, irreducible and inequivalent, and if $V_1 \neq 0$, $H$ acting on $V_1$ as a $B_{m-1}$-irreducible subgroup where $\dim V_1 = 2m$.

On a couple of occasions we need to know the reductive maximal subgroups of $E_6$ and $E_7$. 

\begin{tabular}{|c|c|c|c|}
\hline
$H$ & $p$ & $L$ & $\dim L$ \\
\hline
$SL_2$ & any & $(p-2) \otimes L(1)^{[1]}$ & $2p-2$ \\
\hline
$SL_3$ & $p \geq 3$ & $(p-2,0,0)$ & $(p-1)^2 - 1$ \\
 & $p = 2$ & $(p-2,1) \otimes L(1)^{[1]}$ & $54$ for $p = 5$ \\
 & & $(1,0) \otimes L(1)^{[1]}$ & $9$ \\
 & & $(0,1) \otimes L(1)^{[1]}$ & $9$ \\
$Sp_4$ & $p \geq 5$ & $(2,2,0)$ & $125$ for $p = 3$ \\
 & $p = 2$ & $(2,1) \otimes L(1)^{[1]}$ & $49$ \\
 & & $(1,0)^{[1]}$ & $49$ \\
 & & $(0,1)$ & $84$ \\
$G_2$ & $p \geq 13$ & $(p-5,0)$ & $49$ \\
 & & $(p-2,1) \otimes L(1)^{[1]}$ & $49$ \\
 & & $(4,0,0,0)$ & $6$ \\
 & & $(0,1) \otimes L(1)^{[1]}$ & $84$ \\
 & $p = 3$ & $(3,0,0,0,0)$ & $49$ \\
 & & $(3,0,0,0,0)$ & $49$ \\
 & & $(1,0,0,0,0)$ & $84$ \\
 & $p = 2$ & $(0,0,0,0,0)$ & $84$ \\
$SL_4$ & $p = 2$ & $(1,0,0,0)$ & $14$ \\
 & & $(0,1,0)$ & $48$ \\
 & & $(0,0,1)$ & $84$ \\
\hline
$Sp_6$ & $p = 2$ & $(1,0,0,0)^{[1]}$ & $6$ \\
 & & $(1,0,1)$ & $26$ \\
 & & $(1,0,0,1)$ & $246$ \\
 & & $(1,0,1,0)$ & $6396$ \\
 & & $(1,0,0,1)$ & $6396$ \\
$Sp_8$ & $p = 2$ & $(1,0,0,0,0)^{[1]}$ & $8$ \\
 & & $(1,0,1,0)$ & $26$ \\
\end{tabular}
Lemma 4.9 (c.f. [LS04, Theorem 1]). Let $G$ be an exceptional group not of type $E_8$ and let $M$ be a closed, connected, reductive maximal subgroup of $G$ without factors of $A_1$ or connected centre. Then $M$ is in the following list

| $G$  | Subsystem $M$                                      | Non-subsystem $M$ |
|------|----------------------------------------------------|-------------------|
| $G_2$ | $A_2$, $A_2$ ($p = 3$)                            | $G_2$ ($p = 7$)   |
| $F_4$ | $B_3, C_4$ ($p = 2$), $D_4$, $D_4$, $A_2A_2$ ($p = 2$), $A_2A_2$ | $A_2$ ($p = 2$), $F_4$, $A_2G_2$ |
| $E_6$ | $A_2^3$                                            | $C_4$ ($p = 2$), $G_2$ ($p = 7$), $A_2$ ($p = 2$), $F_4$, $A_2G_2$. |
| $E_7$ | $A_7$, $A_2A_5$                                    | $A_2$ ($p = 5$), $G_2C_3$ |

5. Proof of Theorem 1

In [Ste10a] and [Ste12] we find all semisimple non-$G$-cr subgroups of $G$ where $G$ is $G_2$ and $F_4$ respectively. So the result follows for these cases. It remains to deal with the cases $G = E_6$, $E_7$ and $E_8$. We start by honing the Liebeck and Seitz result to show that if $H$ is a closed, connected, simple subgroup of $G$ with root system $X$ and $p$ is not in our list $N(X, G)$ then $H$ is $G$-cr. Then we check that the examples given in Table 1 are indeed non-$G$-cr.

A filtration for unipotent radicals of parabolics by $L$-modules is given in [ABS90]; to find the isomorphism types of the composition factors is a simple calculation using the root system of $G$.

Summarising the results for our situation, we get:

Lemma 5.1 ( [LS96, 3.1]). Let $G = E_6$, $E_7$ or $E_8$ and let $P = LQ$ be a parabolic subgroup of $G$. The $L'$-composition factors within $Q$ have the structure of high weight modules for $L'$. If $L_0$ is a simple factor of $L'$, then the possible high weights $\lambda$ of non-trivial $L_0$-composition factors and their dimensions are as follows:

(i) $L_0 = A_n$: $\lambda = \lambda_j$ or $\lambda_{n+1-j}$ $(j = 1, 2, 3)$, dimensions $\binom{n+1}{j}$;

(ii) $L_0 = D_n$: $\lambda = \lambda_1$, $\lambda_{n-1}$ or $\lambda_n$, dimensions $2n$, $2^{n-1}$ and $2^{n-1}$, resp.;

(iii) $L_0 = E_6$: $\lambda = \lambda_1$ or $\lambda_6$, dimension 27 each;

(iv) $L_0 = E_7$: $\lambda = \lambda_7$, dimension 56.

Corollary 5.2. With the hypotheses of the lemma, let $V$ be a $L'$-composition factor of $Q$ and suppose $L'$ does not contain a component of type $A_1$. Then either $\dim V \leq 60$ or $G = E_8$, $L' = D_7$ and $V$ is a spin module for $L'$ of dimension 64.

If $G = E_7$, $\dim V \leq 35$; if $G = E_6$, $\dim V \leq 20$.

Proof. If $L'$ is itself simple, this follows from the lemma. Also, if $G = E_6$ or $E_7$ then the number of positive roots is less than 56, so the result is clear. So we may assume $G = E_8$. The possibilities for $L$ are $A_2A_2$, $A_2A_3$, $A_2A_4$, $A_3A_3$, $A_3A_4$, $A_2D_4$ and $A_2D_5$. Since $V$ is simple, it must be a tensor product of simple modules for the two factors, with the simple modules occurring in the lemma. One checks that the highest dimension possible for this is when $L = A_3A_4$, $V = L(\lambda_2) \otimes L(\lambda_2)$ with $\dim V = 6 \times 10 = 60$.

For the second part, if $G = E_7$ and $L'$ is simple this follows from Lemma 5.1, the largest case occurring when $L' = A_6$. If $L'$ is not simple, then it is $A_4A_2$, $A_3A_2$ or $A_2A_2$. Then the largest possible dimension comes from the first option and is at most $10 \times 3 = 30 \leq 35$-dimensional. □
$p \notin N(X, G)$ implies that $H$ is $G$-cr. Since we are building on [LS96 Theorem 1], we need only deal with the struck out numbers in the table in Corollary 2.

Proof of the first statement of Theorem 1:

Looking for a contradiction, we will assume $H$ is non-$G$-cr; then we can make the following assumption, using $L.3$

We have $H \leq P = LQ$ with $\tilde{H}$ being $L$-ir, and either (i) $H$ is a complement to $Q$ in $\tilde{H}Q$ and there exists a composition factor $V$ of $Q$ with $H^1(\tilde{H}, V) \neq 0$; or (ii) $p = 2$, $H = SO_{2n}$, $\tilde{H} = Sp_{2n}$ and $V = L(\omega_1)$ appears as a composition factor of $Q$.

The cases to consider are

$$(X, G, p) \in \{(B_2, \bullet, 3), (G_2, \bullet, 5), (G_2, E_6, 3), (A_2, \bullet, 5), (A_3, E_6, 2), (A_3, E_7, 2), (B_4, E_6, 2), (B_4, E_7, 2), (D_4, E_6, 2), (C_3, \bullet, 2), (C_4, \bullet, 2)\},$$

where $\bullet$ can be replaced by $E_6$, $E_7$ or $E_8$.

By Corollary 5.2 the largest possibility for the dimension of $V$ occurs when $G = E_8$, $L' = D_7$ and $V$ has dimension 64. By 1.6 there is no such $V$ when $H = G_2$ and $p = 5$. This rules out $(G_2, \bullet, 5)$.

Suppose $H$ is of type $B_2$ and $p = 3$. Since $\tilde{H}$ is $D_7$-irreducible, it must have act on the natural module $V_{14}$ for $L'$ as specified in 1.8. Checking [Lib01], one finds the simple untwisted representations of dimension no more than 14 are $L(0, 1)$, $L(1, 0)$, $L(0, 2)$, $L(2, 0)$ with dimensions 4, 5, 10 and 14, respectively. But $L(0, 1)$ is the natural representation for $Sp_4$, thus carries a symplectic structure, which cannot be non-degenerate. Hence $V_{14} \downarrow H = L(2, 0)$; moreover, as $L(2, 0)$ is an irreducible Weyl module when $p = 3$, the embedding $H \hookrightarrow L'$ can be seen as the reduction mod $p$ of an embedding $H_Z \hookrightarrow L'_{Z}$. Now [LS96 Proposition 2.12] gives that $V_z \downarrow \tilde{H}_Z$ is the irreducible Weyl module $V(1, 3)$. Using [Lib01] one can calculate the composition factors of a reduction mod 3 of this module; one sees that $V \downarrow H$ has composition factors $L(1, 3)|0|L(2, 1)|0$. Since none of these modules appears in 1.6, this rules out $(X, G, p) = (B_2, \bullet, 3)$.

By 5.2 the largest possibility for the dimension $V$ when $G = E_7$ is 35; when $G = E_6$ it is 16.

Then dimension considerations using 1.6 and 4.7 also rule out $(X, G, p) = (A_2, E_7, 5)$ and $(G_2, E_6, 3)$, respectively.

For $(A_2, E_8, 5)$, the fact that $V$ has dimension at least 54 forces $L' = E_7$, $D_7$ or $A_7$ but simple $E_7$- and $D_7$-modules are self-dual, so the possibilities for $V$ coming from 1.6 are discounted as they are not self-dual. Thus we may assume that $L' = A_7$ and $V = L(\omega_3) = L^3(\omega_1)$. Since $H$ is $L'$-ir, $L'$ must act irreducibly on the natural $8$-dimensional module $V_8$ for $L'$. A check of [Lib01] forces $V_8|L' = L(1, 1)$. But $L^3(1, 1)$ has highest weights $(2, 2)$ and $(0, 3)$. But the weights appearing in 1.6 are all higher than these (in the dominance order). This rules out $(A_2, E_8, 5)$.

Consider next the case $(X, G, p) = (A_3, E_6, 2)$. By 5.2 we have dim $V \leq 20$ so 1.6 shows that $V$ must be 14-dimensional; this forces $L' = D_5$ or $A_5$. Examining low dimensional representations for $A_3$, it is easy to see using 1.8 that there is no $D_5$-irreducible embedding $H \hookrightarrow D_5$, so we must have $H \hookrightarrow L' = A_5$ by $V_{16}|H = L(0, 1, 0)$. Here, $Q$ has factors $L(\lambda_3) = L^3(V_6)$ and a trivial module. Now $L(0, 1, 0)$ has weights $\pm(0, 1, 0), \pm(1, 0, -1), \pm(-1, 1, 0)$, so $L^3(0, 1, 0)$ has dominant weights $(0, 0, 2), (2, 0, 0)$ and $(0, 1, 0)$. These do not appear in 1.6. Thus $H^1(H, L^3(0, 1, 0)) = 0$ and this case is ruled out.
Let \((X, G, p) = (A_3, E_7, 2)\). Again \(V\) is at least 14-dimensional. So \(L' = A_5, A_6, D_5, D_6\) or \(E_6\). Using [4.8] and [4.9] for \(L' = D_5\) and \(E_6\) respectively, one finds there are no \(L'\)-irreducible subgroups of type \(A_3\). Thus \(L'\) is \(A_5\) or \(D_6\); a similar analysis to the case \((A_3, E_6, 2)\) rules out the former as an option. So we have \(\bar{H} \leq A_3^2 \leq L' = D_6\) by \(V_{12} \downarrow \bar{H} = L(0,1,0) + L(0,1,0)^r\). Now \(Q\) has \(L'\)-composition factors \(k\) and \(L(\omega_6)\), a spin module. We wish to calculate \(L(\omega_6) \downarrow \bar{H}\). Since \(\bar{H} \leq A_3^2\) it is instructive to work out \(L(\omega_6)\) restricted to one of these factors. Using [LS96] 2.6 and 2.7 this is \(L(1,0,0)^4 + L(0,0,1)^4\). Thus we must have \(L(\omega_6) \downarrow A_3^2 = (L(1,0,0), L(1,0,0)) + (L(0,0,1), L(0,0,1))\) so that \(L(\omega_6) \downarrow \bar{H} = L(1,0,0) \otimes L(1,0,0)^r[1] + L(0,0,1) \otimes L(0,0,1)^r[1]\) which now implies \(H^1(\bar{H}, Q) = 0\).

In case \((B_4, E_6, 2)\) we must have \(\bar{H} \leq D_5\), with \(Q\) a spin module for \(L'\). But then \(Q \downarrow \bar{H} = V \cong L(0001)\) using [LS96] 2.7 is a spin module for \(H\) with \(V(0001) = L(0001)\). So \(H^1(B_4, V) = 0\) and this case is ruled out.

Lastly take case \((X, G, p) = (C_3, \bullet, 2)\) of type \(C_3\). We need an \(L'\)-ir embedding of \(\bar{H}\) in \(L'\) and an \(H\)-composition factor \(V\) of \(Q\) with \(H^1(H, V) \neq 0\). We will see this is impossible. As above, if \(G = E_6\), \(L'\) has to be type \(A_5\), with \(Q\) having \(L'\)-composition factors \(k\) and \(L(0,0,1,0,0) = \sqrt[3]{2} L(1,0,0,0)\).

Hence \(Q\) has \(\bar{H}\) composition factors which are \(k\) or in \(\sqrt[3]{2} L(1,0,0)\) which has composition factors \(L(0,0,1,0)^r[2]\). Since these do not appear in [4.6] this case is ruled out. Similarly if \(G = E_7\) or \(E_8\) we must still have \(L' = A_5\) and we must also consider the restrictions of \(L(0,1,0,0)\) and its dual, \(L(0,0,0,1,0)\) to \(\bar{H}\). These are \(\sqrt[2]{2} L(1,0,0) \cong \sqrt[4]{2} L(0,0,1)\) which also contain no composition factors with non-trivial \(H^1\).

Since there are no embeddings of a subgroup of type \(C_4\) into any proper Levi of \(E_6\), this case is ruled out too.

This completes the proof of the first statement of Theorem 1.

\(p \in N(X, G)\) implies the existence of a non-\(G\)-cr subgroup \(H\) with root system \(X\). The examples when \(G = G_2\) and \(F_4\) were shown already in [Ste10a Theorem 1] and [Ste12 Theorem 1(A)(B)] to be non-\(G\)-cr, so we need only deal with the cases \(G = E_6, E_7\) and \(E_8\).

**Proof of the second part of Theorem 1:** The subgroups listed in Table 1 are non-\(G\)-cr:

The proof of many of these cases is similar. Let \(H = E(X, G, p)\) for one of the examples in Table 1. We locate \(H\) within a parabolic subgroup of \(G\) and establish the embedding \(\bar{H} \leq L\). Next we take a low dimensional (faithful) \(G\)-module \(V\) and calculate the restriction to \(H\) and \(\bar{H}\) of this \(G\)-module; in all cases under consideration these will be non-isomorphic. Thus we can conclude that since \(V \downarrow H \not\cong V \downarrow H\), \(H\) is not even \(GL(V)\)-conjugate to \(\bar{H}\), let alone \(G\)-conjugate to \(\bar{H}\). Further, in all the cases under consideration we will conclude that if \(H\) is non-\(F_r\)-cr, then it is also non-\(E_r\)-cr for \(6 \leq r \leq 8\) using the embeddings \(F_4 \leq E_6 \leq E_7 \leq E_8\); unfortunately we seem to need to do this mostly case by case.

We will now give a few examples.

\(H = E(E_6, A_1, 2)\)

Here \(H\) is a subgroup of type \(A_1\) in a subsystem \(A_1^2\) given by \(A_1 \hookrightarrow A_1^2\) by \(x \mapsto (x, x)\). From [LS04 Table 10.1] we have \(V_{27} \downarrow F_4 = V_{26} + k\). Now from [Ste12 5.1] we have \(V_{26} \downarrow A_1^2 = \ldots\)

\(\ldots\)This statement is made without loss of generality: one can embed with graph automorphisms to have dual versions of these modules.
$L(1, 1) + k^6 + L(1, 0)^4 + L(0, 1)^4$, so $V_{27} \downarrow H = L(1) \otimes L(1) + L(1)^8 + k^7 = T(2) + L(1)^8 + k^7$. In [Ste12] it is shown that $H$ is in a long $A_1$-parabolic of $F_4$, hence $H$ is in an $A_1$-parabolic of $E_6$ (so that $L'$ of type $A_1$). But $V_{27} \downarrow L' = L(1)^9 + k^9$ and so $H$ is not $GL(V_{27})$-conjugate to (a subgroup of) $L'$, let alone $E_6$-conjugate. Now $V_{27} \downarrow L' \cong V_{27}^* \downarrow L'$ and $V_{27} \downarrow H \cong V_{27}^* \downarrow H$. Since the $E_7$-module $V_{56}$ has $V_{56} \downarrow E_6 = V_{27} \oplus V_{27}^* + k^2$ we see $H$ is also non-$E_7$-cr.

To show it is also non-$E_8$-cr, note that $L(E_8) \downarrow E_7 = L(E_7) + L(T_1) + L(Q)^2$ where $Q$ is the unipotent radical of an $E_7$-parabolic of $E_8$, with $L(Q) \downarrow E_7 = V_{56} + k$. Thus $L(E_8) \downarrow H$ contains at least two submodules isomorphic to $T'(2)$ (contained in the two $V_{56}$). On the other hand $L(E_8) \downarrow L' = L(A_1 T_1) + k^6 + M^2$ where $M$ is the restriction to $L'$ of the Lie algebra of the unipotent radical of an $A_1$-parabolic. Using [5.1] $M$ has composition factors with high weights 1 or 0, which must be semisimple since $\text{Ext}^1_{A_1}(L(1), L(1)) = \text{Ext}^1_{A_1}(L(1), L(0)) = 0$. In particular, while the direct summand $L(A_1 T_1)$ is an indecomposable module $T'(2)$ for $L'$, it is the only one in $L(E_8)$; for $H$ there are at least two such (in $L(Q)$). Thus $H$ is also non-$E_8$-cr.

$(G, X, p) = (E_6, A_2, 3)$

Let $\tau$ denote a graph automorphism of $G$ with induced action on the Dynkin diagram for $G$. If $G_\tau$ denotes the fixed points of $\tau$ in $G$, we have $G_\tau \cong F_4$ such that the root groups corresponding to simple short roots are contained in the subsystem (of type $A_2 A_2$) determined by the nodes in the Dynkin diagram of $G$ on which $\tau$ acts non-trivially. Thus $H$ is contained in $A_2 A_2 \leq F_4$ by $x \mapsto (x, x)$. It is shown in [Ste12, 4.4.1, 4.4.2] that this subgroup is in a $B_3$-parabolic of $F_4$ with $V_7 \downarrow H = A_2(11)$.

In [Ste12, 5.1] the restrictions of the $F_4$-module $V_{26} = V(0001) \cong 0001/0000$ to $H$ and $\tilde{H}$ is calculated. Using this together with $V_{27} \downarrow F_4 = T(0001) = 0000/0000$ we see that $V_{27} \downarrow \tilde{H}$ cannot be the same as $V_{27} \downarrow H$: the former is an extension by the trivial module of $V_{26} \downarrow \tilde{H} = 11^3 + 005$ where the resulting module is self-dual, so must be $11^3 + 00^5$ whereas the latter is forced to be $T(11)^3$. By a similar argument as before, we also get that this subgroup is non-$E_7$-cr and non-$E_8$-cr.

We give an example of a subgroup not arising from a non-$F_4$-cr subgroup (these being found in [Ste12]):

$(G, X, p) = (E_6, A_1, 5)$.

The module $T(8) = L(0)/L(8)/L(0)$ is a direct summand of the 25-dimensional module $L(4) \otimes L(4) = T(4) \otimes T(4)$ by [2.1]. The two tensor factors here admit orthogonal forms, so the tensor product does too. Hence we get a subgroup of type $SL_2$ in $GL_{25}$ which is actually contained in $SO_{25}$. Indeed as the 10-dimensional direct factor $T(8)$ is the unique such, the duality must preserve this factor. Hence we get an $A_1 \leq SO_{10} \times SO_{15}$ and so projecting to the first orthogonal group, we get $H \leq SO_{10}$ with $V_{10}|H = T(8)$.

Now, by [4.8] we have that this subgroup is in a parabolic of $SO_{10}$. Considering dimensions of composition factors of Levi subgroups of $D_5$ acting on the natural module shows that $H$ must in fact be in a $D_4$-parabolic of $D_5$ with $\tilde{H}$ being $D_4$-irreducible and $V_8 \downarrow \tilde{H} = L(8)$. By e.g. [Ste12, 5.1] we can calculate $V_{27} \downarrow D_4 = L(\omega_1) + L(\omega_3) + L(\omega_4) + k^3$. We wish to restrict this further to get $V_{27}|\tilde{H}$ and $V_{27}|H$. Note that since $L(8) \cong L(3) \otimes L(1)^4$, we have $\tilde{H} \leq S_{\omega_4} \times S_{\omega_3} \leq D_4$. Let $\tilde{H}'$ (resp. $\tilde{H}''$) denote the projection of the $\tilde{H}$ in the first (resp. second) factor. Taking a graph automorphism, we can consider $SL_4$ as type $D_3$ corresponding to nodes 2, 3 and 4 of the Dynkin diagram. Then we have $L_{D_4}(\omega_1)|SL_4 = L(010) + k^2$, thus $L_{D_4}(\omega_1)|\tilde{H}' = \Lambda^2(L(3)) + k^2$.
if \( L(4) + k^3 \), with \( L_{D_4}(\omega_1)|\tilde{H} = L(4) + L(1)[k] \) or \( L_{D_4}(\omega_1)|\tilde{H} = L(4) + L(2)[k] \). As \( \tilde{H} \) is \( D_4 \)-irr, it must be the latter, since \( L(1)[k] \) carries a symplectic form. Also from \([LS96, 2.7]\) one sees that \( L_{D_4}(\omega_3) \downarrow SL_4 \cong L_{D_4}(\omega_4) \downarrow SL_4 = L(100) + L(001) \) and so \( L_{D_4}(\omega_3) \cong L_{D_4}(\omega_4)|\tilde{H}' = L(3)^2 \). Thus \( L_{D_4}(\omega_3) \cong L_{D_4}(\omega_4)|\tilde{H} = (L(3) \otimes L(1)[k]) = L(8) \).

Finally we conclude that \( V_{27} \downarrow \tilde{H} = L(8)^2 + L(4) + L(2)[k] + 0^2 \). In particular, \( \tilde{H} \) acts semisimply. On the other hand \( V_{27} \downarrow D_5 = L(\omega_1) + L(\omega_5) + k \) where \( L(\omega_5) \). But \( H \) does not act semisimply on \( V_{10} \). So \( \tilde{H} \) is not \( GL(V_{27}) \)-conjugate to \( H \), so neither is it \( E_6 \)-conjugate to \( H \).

The remaining cases where \( X = A_1 \) are similar.

Let us now vouch for the existence of the subgroup asserted in case \((G, X, p) = (E_8, C_3, 3)\).

First observe that since the natural module \( L(100) \) for \( Sp_6 \) admits a symplectic form, the tensor square \( M = L(100) \otimes L(100) \) admits an orthogonal form, with composition factors \( L(200)|L(010)|L(000)^2 \). Since \( L(100) \) is a tilting module, so is \( M \); and since \( L(200) = V(200) = T(200) \), while \([Lub01]\) gives \( V(010) = L(010)|L(000) \) we must have \( M \cong L(200)+T(010) \). Duality preserves these factors, so the 15-dimensional module \( T(010) \) is orthogonal for \( Sp_6 \). Thus we have a subgroup \( Sp_6 \leq SO_{15} \leq SO_{16} \) obviously in a \( D_7 \)-parabolic of this \( D_8 \).

\[ H = E(C_4, E_7, 2) \]

is discussed in \([LST96, 2.7]\). Proof: there it is shown to be in an \( E_6 \)-parabolic and not conjugate to its image \( \tilde{H} \cong C_4 \leq F_4 \leq E_6 = L' \). We need to show that this subgroup is also non-\( E_8 \)-cr. For this, restriction of \( L(E_8) \) to an \( E_6 \) Levi gives \( L(E_8)|E_6 \cong L(E_6T_2) + L(Q) + L(Q^-) \), with \( L(Q) \) having composition factors \( k, L(\omega_1) \) or \( L(\omega_6) \) by \([5.4]\). We have \( L(\omega_6)|\tilde{H} = L(\omega_1)|\tilde{H} = L(0100) + k \), and \( L(E_6T_2) \cong L(E_6) + k^2 \) has dimension 80. On the other hand, \( L(E_8)|E_7 = L(E_7T_1) + L(R) + L(R^-) \) for \( R \) the unipotent radical of an \( E_7 \)-parabolic. By \([5.4]\) \( L(R)|E_7 = V_{56} + k \). But \( V_{56}|A_7 = L(\lambda_2) + L(\lambda_6) \) from \([LS96, 2.7]\). Thus \( V_{56}|H = \lambda^2(1000) + (\lambda^2(1000))^* = T(0100)^2 \)

In particular there are 4 direct factors in \( L(E_8)|H \) which are isomorphic to the 28-dimensional module \( T(0100) \). However we found above that there are none in the submodule \( (L(Q) + L(Q^-))|\tilde{H} \) of \( L(E_8)|\tilde{H} \), so if \( H \) were conjugate to \( \tilde{H} \), one would have to find these 4 direct factors \( T(0100) \) inside \( L(E_6T_1) \); but the dimension of the latter is \( 79 < 4 \times 28 = 112 \).

There is one further case where we could not give a nice embedding as we have done above. Let \( H = (E_7, G_2, 7) \).

We first indicate how to see the existence of this subgroup then show that it cannot have any proper reductive overgroup. By \([LS04]\), when \( p = 7 \), \( F_4 \) has a maximal subgroup of type \( G_2 \). Set \( \tilde{H} \) to be this subgroup and regard \( \tilde{H} \) as subgroup of a Levi subgroup of an \( E_6 \)-parabolic; note that \( \tilde{H} \) is \( E_6 \)-irreducible. By \([4.9]\) one has \( V_{27}|\tilde{H} = L(20) + k \). Now, using \([Lub01]\), one has, when \( p = 7 \) that \( V(20) \) is uniserial with composition factors \( 20|00 \). Thus \( H^3(\tilde{H}, L(20)) = H^0(\tilde{H}, H^0(20)/L(20)) = k \). Now

\[ \text{One way to see this is to note that } T = L(1000) \otimes L(1000) \text{ is a tilting module, whose character can be decomposed to yield composition factors } L(2000)|L(0100)^2|L(0000)^4. \text{ Now, one can use Doty’s Weyl group package for GAP to see that } V_{C_4}(2000) \text{ is uniserial with successive factors } L(2000)|L(0000)|L(0100)|L(0000) \text{ and } V_{C_4}(0100) \text{ is uniserial with successive factors } L(0100)|L(0000). \text{ Thus } T(2000) \text{ is uniserial with successive factors } L(2000)|L(0100)|L(0000)|L(2000)|L(0000)|L(0100)|L(0000)|L(0000) \text{ (it is clear that it has both a Weyl- and dual Weyl-filtration). So } T = T(2000) \text{ and indecomposable. But } \lambda^2(1000) \text{ is a submodule of } T; \text{ dimension considerations imply that it consists of the last three factors. But } T(0100) = L(0000)|L(0100)|L(0000) \text{ so the claim follows.} \]

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Q/L' \cong V_{27} or V_{27}^* so one has \( H^1(\hat{H}, Q) = k \). Now by [Ste12, 3.2.15] it follows that there is a non-\( G \)-cr subgroup \( H \), which is a complement to \( Q \) in \( HQ \).

Suppose \( H \) had a proper reductive overgroup in \( G \). Then by [9] it would have to lie in a subsystem subgroup of type \( A_7 \). Also it cannot lie in any parabolic subgroup of \( A_7 \) since then \( \hat{H} \) would not be \( E_6 \)-irreducible. Checking [Lüb01] one sees that there are no irreducible 8-dimensional representations of \( H \cong G_2 \). This is a contradiction. Thus \( H \) has no proper reductive overgroup in \( G \) as required.

The remaining cases are all similar and easier. This completes the proof of Theorem 1.

Acknowledgements. We would like to thank Martin Liebeck for helpful comments and corrections on a previous version of the paper.

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