Canonical tensor models with local time

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Abstract

It is an intriguing question how local time can be introduced in the emergent picture of spacetime. In this paper, this problem is discussed in the context of tensor models. To consistently incorporate local time into tensor models, a rank-three tensor model with first class constraints in Hamilton formalism is presented. In the limit of usual continuous spaces, the algebra of constraints reproduces that of general relativity in Hamilton formalism. While the momentum constraints can be realized rather easily by the symmetry of the tensor models, the form of the Hamiltonian constraints is strongly limited by the condition of the closure of the whole constraint algebra. Thus the Hamiltonian constraints have been determined on the assumption that they are local and at most cubic in canonical variables. The form of the Hamiltonian constraints has similarity with the Hamiltonian in the $c < 1$ string field theory, but it seems impossible to realize such a constraint algebras in the framework of vector or matrix models. Instead these models are rather useful as matter theories coupled with the tensor model. In this sense, a three-index tensor is the minimum-rank dynamical variable necessary to describe gravity in terms of tensor models.

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1 Introduction

Tensor models have originally been introduced \[1-3\] to describe the simplicial quantum gravity in general dimensions higher than two, with the hope to extend the success of the matrix models in the study of the two-dimensional simplicial quantum gravity. Tensor models have later been extended to describe the spin foam and loop quantum gravities by considering Lie-group valued indices \[4-6\]. These models with group indices, called group field theory \[7\], are actively studied with various interesting recent progress \[8-23\]. A key issue is the emergence of a new kind of tensor models, called colored tensor models \[9\], which have more intimate topological correspondence to simplicial manifolds than the original versions. There has also been a systematic study of tensor models in semi-classical approximations \[24-31\] under the interpretation of the rank-three tensor models as theory of dynamical fuzzy spaces \[32,33\], and the emergence of Euclidean general relativity on emergent spaces has been observed \[26,27\].

So far the study of tensor models has basically been limited to the cases of Euclidean signature. In the standard field theories on flat spaces, after setting up appropriate causal structures, the computations in Minkowski signature can be obtained from those in Euclidean signature by means of analytic continuation. However, it is generally not clear how one may extend the standard procedures such as Wick rotation to the quantum gravitational situation with fluctuating geometries. Moreover, the study of the causal dynamical triangulation \[34\] indicates that the dynamics of quantum gravity in Minkowski signature may be substantially different from that in Euclidean signature. The main purpose of this paper is to discuss how to incorporate time in the framework of tensor models.

The advent and correctness of the theory of relativity have established that time is not an absolute entity but is rather a relative quantity measured by physical phenomena dubbed as a “clock”. Since a “clock” is a local object due to the speed limit of light, the definition of time is necessarily local and is generally dependent on how the system of “clocks” is organized. The principle that physical phenomena themselves should not depend on this kind of ambiguity of defining local time provides strong constraints on possible forms of consistent theories of nature.

In the Lagrangian formalism, this ambiguity can well be incorporated by imposing the invariance of theories under the general coordinate transformations of spacetime coordinates. In the rank-three tensor models for instance, however, the only dynamical variable is a three-index tensor, \( M_{abc} (a, b, c = 1, 2, \ldots, N) \), and there are neither space nor locality built in the framework: a space and its locality are emergent phenomena \[35\]. Therefore one would have difficulties in introducing local time into the tensor models and imposing the constraints coming from the principle mentioned above. One would try to introduce time \( t \) simply as an argument of...
the tensor as $M_{abc}(t)$, and impose the invariance of the models under reparametrization $t'(t)$. However, this way of introducing time will necessarily allow an entity of a non-local global time to exist when a space is emergent. Then emergent field theories on the emergent space will not generally satisfy the above principle; emergent field theories on an emergent flat space will seriously violate Lorentz symmetry, that is rather strongly constrained experimentally and theoretically [36–38]. Another option to try would be to introduce time for each index of the tensor as $M_{a t_a, b t_b, c t_c}$. This option seems to have an appealing feature concerning locality of time. However, the contraction of indices of the tensor such as $\sum_{abc} \int dt_a dt_b dt_c M_{a t_a, b t_b, c t_c} M_{a t_a, b t_b, c t_c}$ will introduce multiple integrals over times into the Lagrangian formalism. Therefore this option seems to require extension of the Lagrangian formalism for multiple time parameters in advance, before applying it to the tensor models.

The above lack of good guiding principles for introducing time to the tensor variable would require us to abandon starting with Lagrangian formalism for the purpose. There exists another formalism of mechanics, Hamilton formalism, in which time is rather intimately related to dynamical evolution of a system. Of course, the two formulations of mechanics are equivalent (at least classically), but in the present confusing situation on local time, the latter formalism is superior to the former one, because one does not have to know in advance how time is represented in the dynamical variable. If necessary, once the Hamilton formalism of the tensor models is consistently obtained, one would also be able to obtain the corresponding Lagrangian formalism.

In Hamilton formalism of general relativity, the consistency of dynamics under ambiguous choices of local time is guaranteed by a set of first class constraints, which are the generators of the local coordinate transformations containing the time direction. In this paper, the set of constraints of general relativity will be rewritten in terms of the dynamical variables of tensor models, and a rank-three tensor model with first class constraints in Hamilton formalism will be presented. This should be equivalent to introducing local time in tensor models.

The canonical formulations of discrete models of gravity have been discussed in previous literatures [39–43]. An important difference of the present work from the previous approaches is that the spatial diffeomorphism is exactly represented by the symmetry of the tensor model. Therefore the gauge freedom of spatial diffeomorphism is exactly incorporated by first class momentum constraints of the tensor model. Then Hamiltonian constraints will be determined by the condition of the closure of the whole set of first class constraints. It is also peculiar that time is just a continuous variable unlike some previous approaches [41, 42]. Thus the standard Dirac procedure is applicable, and a rank-three tensor model with local time will be formulated in terms of the standard Hamilton formalism with a set of first class constraints.
This paper is organized as follows. In Section 2, the general relativity in Hamilton formalism is summarized for the discussions in this paper. In Section 3, the rank-three tensor models are briefly overviewed. The limit of usual continuous spaces in the tensor models is explained. In Section 4, an algebra expressed in terms of the canonical variables of a vector model is considered, and it is shown that the algebra of the first class constraints of general relativity can be reproduced in the limit of usual continuous spaces. In Section 5, however, some difficulties in the realization in Section 4 are pointed out, and the necessity of a three-index tensor is argued. In Section 6, a rank-three tensor model with first class constraints in Hamilton formalism is presented. The momentum constraints represent the kinematical symmetry of the rank-three tensor model, and contain the spatial diffeomorphism symmetry in the limit of usual continuous spaces. The Hamiltonian constraints are determined by the closure of the algebra of the whole first class constraints under the assumption that they are at most cubic and respect locality. It turns out to be necessary to break the time-reversal symmetry. In Section 7, it is shown that matter degrees of freedom coupled to the rank-three tensor model can be added to the system without destroying the algebraic structure of the first class constraints. The matter degrees of freedom can be given by any rank tensors. Section 8 is devoted to summary, discussions and future prospects.

2 The first class constraints from general relativity

The ADM formulation \[44\] of general relativity leads to the following algebra of first class constraints \[45, 46\],

\[
\{\mathcal{H}(x), \mathcal{H}(x')\} = \varepsilon \left( \mathcal{H}^i(x)\delta_i(x, x') - \mathcal{H}^i(x')\delta_i(x, x) \right),
\]

\[
\{\mathcal{H}_i(x), \mathcal{H}(x')\} = \mathcal{H}(x)\delta_i(x, x'),
\]

\[
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\delta_j(x, x') + \mathcal{H}_j(x)\delta_i(x, x'),
\]

where \{ , \} denotes Poisson bracket, and \(\delta_i(x, x')\) denotes the derivative of the delta function with respect to \(x^i\). The signature \(\epsilon\) takes \(\epsilon = 1\) and \(\epsilon = -1\) for Minkowski and Euclidean signatures, respectively. \(\mathcal{H}(x)\) and \(\mathcal{H}_i(x)\) are the super-Hamiltonian and the super-momentums, respectively, and they are explicitly given by

\[
\mathcal{H}(x) = \frac{16\pi G}{\sqrt{|g|}} \left( \pi_{ij}\pi^{ij} - \frac{1}{2} \left( \pi_i \right)^2 \right) - \frac{\sqrt{|g|}}{16\pi G} \left( R^{(3)} - 2\Lambda \right),
\]

\[
\mathcal{H}_i(x) = -2D_j\pi_i^j,
\]

*The trivial ones with the conjugate momentums \(\pi_N, \pi_{Ni}\) of the lapse and shift variables are omitted.*
where $\pi_{ij}(x)$ is the conjugate momentum to the spatial geometry $g_{ij}(x)$, and $G$ and $\Lambda$ are the gravitational and cosmological constants, respectively. The algebra (1), (2) and (3) can be equivalently expressed as

$$\{H(v), H(w)\} = \epsilon D(v^i \partial^i w - w^i \partial^i v),$$  \hspace{1cm} (6)

$$\{D(v^i), H(w)\} = H(v^i \partial_i w),$$  \hspace{1cm} (7)

$$\{D(v^i), D(w^i)\} = D(v^j \partial_j w^i - w^j \partial_j v^i),$$  \hspace{1cm} (8)

where $H(v)$ and $D(w^i)$ are defined by

$$H(v) \equiv \int dx \, v(x) \mathcal{H}(x),$$  \hspace{1cm} (9)

$$D(v^i) \equiv \int dx \, v^i(x) \mathcal{H}_i(x),$$  \hspace{1cm} (10)

with $v(x)$ and $v^i(x)$ independent of the canonical variables $g_{ij}(x)$ and $\pi_{ij}(x)$.

The fundamental roles of the first class constraint algebra (1), (2) and (3) in geometrodynamics have been discussed in [47]. An important feature is that the right-hand side of (1) contains the inverse spatial metric $g^{ij}(x)$ to raise the index of $H^i(x)$ to $H_i(x)$. Therefore the constraint algebra has structure functions depending on the canonical variable, and is not a Lie algebra with constant structure constants. To reconcile the constraint algebra with the spacetime diffeomorphism, the on-shell conditions $\mathcal{H} = \mathcal{H}_i = 0$ must be imposed. In addition, under some (reasonable) assumptions, the form of the super-Hamiltonian has been shown to be given uniquely by the form (4). It is also argued that the constraint algebra does not change by adding matters with non-derivative couplings with gravity.

3 A brief overview of tensor models

This section will give a brief overview of the rank-three tensor models (with no time) to prepare for the discussions in the following sections. As explained in Section 1, tensor models have some variations with distinct interpretations. This paper deals with the rank-three tensor models, which have a three-index tensor as their only dynamical variable. The rank-three tensor models can be regarded as theory of dynamical fuzzy spaces [32,33]. This interpretation of the tensor models is especially convenient to understand the semi-classical behavior of the tensor models: classical solutions are regarded as background fuzzy spaces which approximate continuum spaces, and the perturbations around the classical solutions are interpreted by effective field theories on the spaces. In fact, various classical solutions and the perturbations around them are studied to show the phenomena of emergent Euclidean general relativity.
on emergent spaces [24–31]. It is especially noteworthy that, as theory of dynamical fuzzy spaces, the rank-three tensor models can deal with any dimensional spaces, unlike the original proposals [1–3], in which the ranks of tensors are related with dimensions.

Let me denote the three-index dynamical tensor by $M_{abc}$. The tensor is assumed to satisfy the generalized Hermiticity condition [1–3],

\[ M_{abc} = M_{bca} = M_{cab} = M_{a\star bc} = M_{b\star ac} = M_{c\star ab}, \]  

where * denotes complex conjugation, and the indices run as $a, b, c = 1, 2, \ldots, N$. Because of the generalized Hermiticity condition (11), the symmetry which can be associated to the rank-three tensor models is the orthogonal group $O(N)$,

\[ M'_{abc} = O_a^\prime O_b^\prime O_c^\prime M_{a'b'c'}, \quad O_a^b \in O(N), \]  

instead of a unitary group of an hermitian matrix model.

While a continuous manifold can be described by a coordinate system, a fuzzy space is defined by an algebra of functions on it. The algebra of functions $\phi_a$ ($a = 1, 2, \ldots, N$) can be characterized by its structure constants $C_{abc}$ as

\[ \phi_a \phi_b = C_{abc} \phi_c. \]  

While a usual continuous space can be characterized by a commutative and associative algebra of functions, a noncommutative space for instance can be characterized by a noncommutative associative algebra. One may even consider a nonassociative algebra to define a nonassociative space [48–51].

To relate fuzzy spaces to the configurations of the rank-three tensor models, let me introduce an inner product [32],

\[ \langle \phi_a | \phi_b \rangle = \delta_{ab}, \]  

which is assumed to be bi-linear. Now let me identify the structure constants with the dynamical variable of the tensor model,

\[ C_{abc} = M_{abc}. \]  

This physically means that the rank-three tensor models are interpreted as theory describing the dynamics of fuzzy spaces. The identification (15) and the generalized Hermiticity condition (11) imply the following cyclicity property on the algebraic structure,

\[ \langle \phi_a | \phi_b | \phi_c \rangle = \langle \phi_c | \phi_b | \phi_a \rangle = \langle \phi_b | \phi_c | \phi_a \rangle, \]

\[ \text{†Repeated indices are assumed to be summed over throughout this paper, unless otherwise stated.} \]
and also the properties on complex conjugation,

$$\phi_a = \phi_a^*, \quad (17)$$

$$\phi_a \phi_b = \phi_b \phi_a. \quad (18)$$

These properties (16), (17) and (18) characterize the fuzzy spaces which can be associated with the configurations of the rank-three tensor models. The fuzzy spaces with these properties have various interesting properties concerning symmetries, uncertainties, and reduction procedures [32, 52–54].

In the analysis of emergent Euclidean general relativity from the rank-three tensor models [24–29], the following particular form of $M_{abc}$ with Gaussian functions has extremely been useful. The indices of functions $\phi_a$ are assumed to be given by the coordinates of a usual continuous $D$-dimensional space,

$$a = x = (x^1, x^2, \ldots, x^D), \quad x^i \in \mathbb{R},$$

and $M_{x_1x_2x_3}$ is assumed to be given by the following Gaussian form,

$$M^G_{x_1x_2x_3} = B g(x_1)^{1/4} g(x_2)^{1/4} g(x_3)^{1/4} \exp \left[-\beta (d(x_1, x_2)^2 + d(x_2, x_3)^2 + d(x_3, x_1)^2)\right], \quad (20)$$

where $B$ and $\beta$ are positive constants. The fuzziness of the spaces is characterized by the parameter $\beta$. Here a metric tensor field $g_{ij}(x)$ is assumed to exist on the $D$-dimensional space, $g(x) = \text{Det}[g_{ij}(x)]$, and $d(x_1, x_2)$ denotes the geodesic distance between two points $x_1$ and $x_2$. The form of (20) respects the diffeomorphism symmetry, as $g(x)^{1/4}$ guarantees the diffeomorphism invariance of an index contraction: $M^G_{xx_1x_2} M^G_{xx_3x_4} = \int dx \sqrt{g(x)} \cdots$.

The Gaussian configuration (20) is merely an idealized working hypothesis which singles out the modes corresponding to those of general relativity. This hypothesis has very well explained the qualitative features of the results of the numerical analysis [25–29], which have shown the emergence of Euclidean general relativity on emergent spaces. The detailed values of $M_{abc}$ may be different from that, because the actual degrees of freedom of the rank-three tensor models are discrete and finite, while they are continuous and infinite for the continuum index (19). Also the simple Gaussian damping form is idealizing $M_{x_1x_2x_3}$ which is locally distributed with respect to the relative locations of $x_i$. Thus the Gaussian configuration (20) should be regarded as an infrared effective idealized description, which would be obtained from a coarse-graining procedure [24].

The algebra (13) with (15) and (20),

$$\phi_{x_1} \phi_{x_2} = M^G_{x_1x_2x_3} \phi_{x_3}, \quad (21)$$
defines a fuzzy space of the kind satisfying (16), (17) and (18). Intuitively, the function $\phi_x$ represents a fuzzy point at $x$, which has fuzziness with a length scale $\sim 1/\sqrt{\beta}$. For the flat case $g_{ij}(x) = \delta_{ij}$, the Gaussian configuration (20) becomes

$$M_{x_1x_2x_3}^{\text{flat}} = B \exp \left[ -\beta \left( (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \right) \right].$$

(22)

This configuration indeed respects the Poincare symmetry and can be considered to represent a flat $D$-dimensional fuzzy space [51]. In the limit $\beta \to \infty$ with an appropriate normalization $B$, the fuzzy space algebra approaches

$$\phi_{x_1}\phi_{x_2} = \delta^D(x_1 - x_2) \phi_{x_1} \text{ in } \beta \to \infty.$$ 

(23)

This is the limit to the usual continuous spaces with no fuzziness. Indeed the algebra (23) is commutative and associative. In the rest of this paper, this limit will be denoted by the pointwise limit, and will be used to reproduce the constraint algebra of general relativity in Hamilton formalism from tensor models.

In fact, in the subsequent discussions, the Gaussian form (20) or (22) is not essential. It is merely a representative of the configurations of $M_{abc}$, which have relatively local distributions when the $O(N)$ gauge symmetry (12) is appropriately fixed, and have a parameter that can be tuned to take the pointwise limit (23). Another important thing is that one cannot take the pointwise limit as a starting point. As will be seen, one has to start with a finite fuzziness and then take the pointwise limit. For simplicity, I will only use the flat expression (22) in the subsequent computations, but it should be straightforward to extend to the general cases by using the diffeomorphism invariant expression (20).

## 4 Realization of constraints by a vector model

In this section, I will try to incorporate the constraint algebra of general relativity (1), (2) and (3) in the framework of vector models in Hamilton formalism. The discussions will proceed almost well, but some difficulties, which will be discussed in Section 5, will arise in a final step. Matrix models will also be abandoned due to the same difficulties.

The degrees of freedom of a vector model in Hamilton formalism are assumed to be given by $M_a$ ($a = 1, 2, \ldots, N$) and their conjugate momentums $\pi_a$. They are assumed to satisfy the canonical relation of Poisson bracket,

$$\{M_a, \pi_b\} = \delta_{ab}.$$ 

(24)
Let me define
\[ C_{(ab)}^V \equiv \frac{1}{2} (\pi_a \pi_b + \epsilon'M_a M_b) , \tag{25} \]
\[ C_{[ab]}^V \equiv \frac{1}{2} (\pi_a M_b - \pi_b M_a) , \tag{26} \]
where \( \epsilon' = \pm 1 \) is a signature, and its relation with \( \epsilon \) in (11) or (6) will be given later. The \( C^V \)'s satisfy \( C_{(ab)}^V = C_{(ba)}^V \) and \( C_{[ab]}^V = -C_{[ba]}^V \), respectively. As will be explained in detail below, (25) and (26) mimic the super-Hamiltonian and super-momentum of general relativity, respectively.

From (24), (25) and (26), one can straightforwardly obtain the following algebraic relations,
\[ \{ H_V(v^S), H_V(w^S) \} = -\epsilon'D_V([v^S, w^S]), \tag{27} \]
\[ \{ D_V(v^A), H_V(w^S) \} = H_V([v^A, w^S]), \tag{28} \]
\[ \{ D_V(v^A), D_V(w^A) \} = D_V([v^A, w^A]) , \tag{29} \]
which look similar to (6), (7) and (8) of general relativity. Here \( H_V \) and \( D_V \) are defined by
\[ H_V(v^S) \equiv v^S_{ab} C_{(ab)}^V , \tag{30} \]
\[ D_V(v^A) \equiv v^A_{ab} C_{[ab]}^V , \tag{31} \]
and the upper indices \( S \) and \( A \) indicate the symmetric properties of the matrices, \( v^S_{ab} = v^S_{ba} \) and \( v^A_{ab} = -v^A_{ab} \), respectively. The \( v, w \)'s are assumed to be independent of the canonical variables. The square bracket \([v, w]\) denotes the commutator of matrices, \([v, w]_{ab} \equiv v_{ac} w_{cb} - w_{ac} v_{cb}\). Thus \( C_{(ab)}^V \) and \( C_{[ab]}^V \) form a Lie algebra under the Poisson bracket. Especially, due to (29) and the anti-symmetry of \( v^A_{ab}, C_{[ab]}^V \) form the Lie algebra \( so(N) \) of the orthogonal group, which is the kinematical symmetry of the vector model.

In the following, I will show how the constraint algebra of general relativity (6), (7) and (8) can be reproduced from the algebra of a vector model (27), (28) and (29). Let me assume that a situation similar to the Gaussian configurations in Section 3 is occurring in the vector model. Then the indices are assumed to take the coordinates of a continuous \( D \)-dimensional space as in (19).

Let me first discuss (29). In (29), the computation of the Poisson bracket has been reduced to a commutator of matrices. Let me consider a (infinite-dimensional) matrix in the form,
\[ v^A_{xy} = \frac{1}{2}(v^i(x) + v^i(y))\delta_i(x, y) , \tag{32} \]
which is anti-symmetric $v^A_{xy} = -v^A_{yx}$. Here $v^i(x)$ are assumed to be arbitrary smooth functions on the $D$-dimensional space. Then the commutator between two such matrices is given by

$$[v^A, w^A]_{xy} = \frac{1}{4} \int dz \left( v^i(x) + v^i(z) \right) \delta_i(x, z) \left( w^j(z) + w^j(y) \right) \delta_j(z, y) - (x \leftrightarrow y).$$

(33)

Because of the derivatives of the delta functions, the computation of the right-hand side of (33) tends to become cumbersome. It is much more straightforward and easier to do the computation by considering a test function $f(y)$. By multiplying the right-hand side of (33) with $f(y)$ and performing the partial integrations over $y$ and $z$, one obtains

$$\int dz dy f(y) \left[ (v^i(x) + v^i(z)) \delta_i(x, z) \left( w^j(z) + w^j(y) \right) \delta_j(z, y) - (x \leftrightarrow y) \right]$$

$$= 2v^i(x)(\partial_i \partial_j w^j(x)) f(x) + 4v^i(x)(\partial_i w^j(x)) \partial_j f(x) - (v \leftrightarrow w).$$

(34)

Comparing the last expression of (34) with the right-hand side of

$$\int dy \left( v^i(x) + v^i(y) \right) \delta_i(x, y) f(y) = (\partial_i v^i(x)) f(x) + 2v^i(x) \partial_i f(x),$$

(35)

one obtains

$$[v^A, w^A]_{xy} = \frac{1}{2} \left( [v, w]^i(x) + [v, w]^i(y) \right) \delta_i(x, y),$$

(36)

where

$$[v, w]^i(x) \equiv v^j(x) \partial_j w^i(x) - w^j(x) \partial_j v^i(x).$$

(37)

Comparing (36) and (37) with (32), one concludes that (29) with (32) exactly reproduces (8). Thus $D_V(V^A)$ is the analogue to the generators of spatial diffeomorphism in the general relativity.

In the next place, let me discuss (28). Consider

$$w^S_{xy} = c(\beta) w \left( \frac{x + y}{2} \right) \exp(-\beta(x - y)^2),$$

(38)

where $(x - y)^2 = (x - y)^i(x - y)^i$. The (38) satisfies the symmetry $w^S_{xy} = w^S_{yx}$. The function $w(x)$ is assumed to be an arbitrary smooth function on the $D$-dimensional space and its argument in (38) takes the middle point between $x$ and $y$. The coefficient $c(\beta)$ depending on $\beta$ will be determined later.

The Gaussian form in (38) follows the Gaussian configurations in Section 3, and, at the final step of the following computations, the pointwise limit $\beta \to \infty$ will be taken to compare with the constraint algebra of general relativity. This Gaussian form is considered just because
of its simplicity. In fact, the following discussions do not depend on the details of the form. What is needed is that $w_{xy}^S$ has distributions within finite ranges of relative distances between $x$ and $y$, and one can finally take a smooth pointwise limit.

The computation of the Poisson bracket (28) has been reduced to the commutator of the matrices (32) and (38), which is given by

$$[v^A, w^S]_{xy} = \frac{c(\beta)}{2} \int dz (v^i(x) + v^i(z)) \delta_i(x, z) w \left( \frac{z + y}{2} \right) \exp(-\beta(z - y)^2) + (x \leftrightarrow y)$$

$$= \frac{c(\beta)}{2} \left[ \partial_i v^i(x) w \left( \frac{x + y}{2} \right) \exp(-\beta(x - y)^2) + 2v^i(x) \partial_i^x \left( w \left( \frac{x + y}{2} \right) \exp(-\beta(x - y)^2) \right) \right] + (x \leftrightarrow y). \quad (39)$$

The last expression is rather confusing, because the derivative with respect to $x$ in the last line produces a factor $\beta$, which makes the pointwise limit difficult to handle. It is again much easier to do the computations by considering a test function $f(y)$. By multiplying the right-hand side of (39) with $f(y)$ and integrating over $y$, one obtains, in the leading order of $1/\beta$,

$$\int dy f(y) \frac{c(\beta)}{2} \left[ \partial_i v^i(x) w \left( \frac{x + y}{2} \right) \exp(-\beta(x - y)^2) + 2v^i(x) \partial_i^x \left( w \left( \frac{x + y}{2} \right) \exp(-\beta(x - y)^2) \right) \right] + (x \leftrightarrow y)$$

$$= c(\beta) \left( \int dz \exp(-\beta z^2) \right) \left[ v^i(x) \partial_i w(x) + O(\beta^{-1}) \right] f(x). \quad (40)$$

This concludes

$$[v^A, w^S]_{xy} = c(\beta) v^i \partial_i w \left( \frac{x + y}{2} \right) \exp(-\beta(x - y)^2) \quad (41)$$

in the leading order of $1/\beta$. Thus, by comparing the right-hand side of (41) with (38), (28) with (32) and (38) exactly reproduces (7) in the pointwise limit $\beta \to \infty$.

Finally let me discuss (27). What should be computed is the commutator between the matrices in the form (38):

$$[v^S, w^S]_{xy} = c(\beta)^2 \int dz \left[ v \left( \frac{x + z}{2} \right) w \left( \frac{z + y}{2} \right) - (x \leftrightarrow y) \right] \exp(-\beta(x - z)^2 - \beta(z - y)^2). \quad (42)$$

To systematically do the computation, let me again consider a test function $f(y)$ and evaluate

$$c(\beta)^2 \int dy dz f(y) \left[ v \left( \frac{x + z}{2} \right) w \left( \frac{z + y}{2} \right) - (x \leftrightarrow y) \right] \exp(-\beta(x - z)^2 - \beta(z - y)^2). \quad (43)$$
By a change of variables, \( y = a + b + x, \ z = b + x \), the integration becomes
\[
c(\beta)^2 \int dadb f(a + b + x) [v(x + b/2)w(a/2 + b + x) - (v \leftrightarrow w)] \exp(-\beta a^2 - \beta b^2). \tag{44}
\]
Because of the Gaussian damping factor for \( a \) and \( b \), the large \( \beta \) limit of this integral can well be evaluated after Taylor expanding in \( a \) and \( b \) the integrand other than the exponential. Then one obtains
\[
c(\beta)^2 \beta^{-D-1} \left( \frac{1}{D} \int dzz^2 \exp(-z^2) \right) \left( \int dz \exp(-z^2) \right) \times \left[ \partial_i f(x) (v(x)\partial_i w(x) - w(x)\partial_i v(x)) + \frac{1}{2} f(x) (v(x)\partial_i \partial_i w(x) - w(x)\partial_i \partial_i v(x)) \right]. \tag{45}
\]
Comparing with (35), this concludes
\[
[v^S, w^S]_{xy} = c_1 c(\beta)^2 \beta^{-D-1} \frac{1}{2} (l_i(x) + l_i(y)) \delta_i(x, y), \tag{46}
\]
where \( c_1 \) is a numerical factor, and
\[
l_i(x) = v(x)\partial_i w(x) - w(x)\partial_i v(x). \tag{47}
\]

The ugly contraction of the indices in (46) comes from the simplified assumption \( z^2 = z_i z^i \) in (38). If a general metric is assumed as \( z^2 = g_{ij} z^i z^j \), one will have
\[
\int da \sqrt{g} a^i a^j \exp(-\beta a^2) \sim \beta^{-\frac{D}{2}-1} g^{ij}, \tag{48}
\]
and (46) will contain \( g^{ij} \). It would be straightforward to do the computations in a full covariant fashion following the expressions in Section 3.

Thus, by taking
\[
c(\beta) = \beta^{D+1} c_1^{-\frac{1}{2}}, \tag{49}
\]
\[
\epsilon' = -\epsilon, \tag{50}
\]
one obtains (6) of general relativity from (27).

An important fact in the above computations is that the finiteness of the range \(|x - y|^2 \lesssim 1/\beta \) of the distribution in (38) plays an essential role in the derivation of (46), even though the pointwise limit \( \beta \to \infty \) is finally taken. This is indicated by the extra factor \( \beta^{-1} \) in (46). This is also the reason why the final expression depends on the inverse metric \( g^{ij} \), which appears also in the constraint algebra (6) of general relativity. If the matrix (38) were assumed to have a full diagonal expression like \( \delta(x, y) \) for instance, (6) would not be reproduced.
5 Difficulties and a solution

From the discussions in Section 4, the super-Hamiltonians in the vector model should be given by $C^{V}_{ab}$ in (25). Then the generators of an infinitesimal local “time” translation will be given by $\delta t_{(ab)}C^{V}_{(ab)}$. However, in the regime discussed in Section 3 the indices label each “point” in a space, and the “time” is generally highly non-local, since $x$ and $y$ in $\delta t_{(xy)}$ can freely take any values. Therefore this “time” is very different from the conventional notion of time in physics. One may instead consider a “time” in a diagonal form $\delta t_{ab} \sim \delta t_{a}\delta_{ab}$. But, as discussed in the last paragraph in Section 3 such a full diagonal form cannot correctly reproduce the constraint algebra of general relativity.

There exists also another more general and serious difficulty. Under the Poisson bracket, the constraints (25) and (26) form a Lie algebra with constant structure constants. This means that the future time evolution is completely determined by the action of the Lie group element parameterized by the “time”. As reviewed in Section 2 this is substantially different from the gravitational case, in which the constraint algebra has structure functions depending on the metric. One cannot expect non-trivial dynamics to occur from the constraint algebra of the vector model.

These problems cannot be solved by considering a matrix model. One would try a set of constraints,

\begin{align}
C^{M}_{(ab)} &= \pi_{ac}\pi_{bc} + \epsilon^M_{ac}M_{bc}, \\
C^{M}_{[ab]} &= \pi_{ac}M_{bc} - M_{ac}\pi_{bc}.
\end{align}

(51)  
(52)

But the problems above appear in the same as in the vector model.

A solution can be given, if there exists a three-index tensor $M_{abc}$. By using $M_{abc}$, the two indices of $C^{V}_{(ab)}$ can be contracted as

$$C^{V}_{a} \equiv M_{abc}C^{V}_{(ab)}.$$  
(53)

Then the infinitesimal time can have only one index as $\delta t_{a}C^{V}_{a}$. Moreover, in the regime explained in Section 3 $\delta t_{x}M_{xyz}$ will provide a non-diagonal distribution of a finite range for $y, z$, that is necessary to reproduce the constraint algebra of general relativity in the pointwise limit $\beta \to \infty$, as is discussed in the last paragraph of Section 4. And also, as will be seen in Section 6 the constraint algebra does not have structure constants, but they rather depend on $M_{abc}$ and its conjugate.

In principle, the indices of the super-momentums of the vector model $C^{V}_{[ab]}$ can also be contracted by $M_{abc}$. However, this is not a valid option, because, as explained in Section 3
the tensor models have the orthogonal group symmetry, and $C^V_{[ab]}$ are the generators of this kinematical symmetry. If they were contracted, the number of constraints would be reduced, and the symmetry could not be fully incorporated by the constraints.

Another important thing is that $M_{abc}$ must be a dynamical variable. If not, the Poisson bracket between the super-momentums $C^V_{[ab]}$ and $C^V_a$ would not close:

$$\{ v^A_{ab} C^V_{[ab]}, v^c M_{cde} C^V_{(de)} \} = v^A_{ab} v^c M_{cde} \{ C^V_{[ab]}, C^V_{(de)} \}$$

$$= \{ v^A, vM \}_{ab} C^V_{(ab)},$$  \hfill (54)

$$= [v^A, vM]_{ab} C^V_{(ab)},$$  \hfill (55)

where $vM$ denotes a matrix $vM_{ab} \equiv v^c M_{cab}$. The expression in the last line does not in general have the form of linear combinations of (53). Therefore one has to include $M_{abc}$ as a canonical variable, and make it transform appropriately under the Poisson bracket with the super-momentums.

The discussions in this section can be summarized as follows. To correctly reproduce the constraint algebra of general relativity and the usual notion of time, it is necessary to include a three-index tensor $M_{abc}$ as a dynamical variable. In this sense, $M_{abc}$ is the dynamical variable corresponding to gravity. The vector and matrix variables may be incorporated consistently in the constraint algebra, but they are not necessary. Instead they can rather be regarded as some matter degrees of freedom, which can be added consistently, as will be discussed in Section 7.

6 Realization by a rank-three tensor model

The discussions in Section 5 imply that the pure gravitational system should be obtained from a rank-three tensor model which has a three-index tensor as its only dynamical variable. The canonical variables are assumed to be given by $M_{abc}$ and $\pi_{abc}$. They are assumed to satisfy the generalized Hermiticity condition (11) and its conjugate correspondence,

$$\pi_{abc} = \pi_{cba} = \pi_{cab} = \pi^*_{bac} = \pi^*_{acb} = \pi^*_{cba},$$  \hfill (56)

respectively. The Poisson bracket between the canonical variables is assumed to be given by

$$\{M_{abc}, \pi_{def}\} = \delta_{abc,def} \equiv \delta_{ad} \delta_{bc} \delta_{cf} + \delta_{ac} \delta_{bf} \delta_{cd} + \delta_{af} \delta_{bd} \delta_{ce}.$$  \hfill (57)

Let me consider

$$C_{(ab)} \equiv \frac{1}{2} (\pi_{acd} \pi_{bde} - \epsilon M_{acd} M_{bde}),$$  \hfill (58)
\[ C_{(ab)} \equiv \frac{1}{2} (\bar{\pi}_{acd} M_{bcd} - M_{acd} \bar{\pi}_{bcd}), \]  

(59)

where the relation \( \epsilon' = -\epsilon \) in (50) has already been used. Let me define

\[ H'(v^S) \equiv v^S_{ab} C_{(ab)}, \]  

(60)

\[ D(v^A) \equiv v^A_{ab} C_{[ab]}, \]  

(61)

where \( v^S_{ab} = v^S_{ba} \) and \( v^A_{ab} = -v^A_{ba} \), and they are assumed to be independent of the canonical variables. Then \( H' \) and \( D \) satisfy

\[ \{ H'(v^S), H'(w^S) \} = \epsilon D([v^S, w^S]), \]  

(62)

\[ \{ D(v^A), H'(w^S) \} = H'([v^A, w^S]), \]  

(63)

\[ \{ D(v^A), D(w^A) \} = D([v^A, w^A]), \]  

(64)

which are actually the same as (27), (28) and (29) of the vector model.

The discussions in Section 4 for a vector model uses only the fact that the algebra of constraints under Poisson bracket can be written as commutators of matrices as in (27), (28) and (29). Since this is the same for the tensor model as in (62), (63) and (64), the constraint algebra of general relativity can be reproduced in the same way as in the vector model.

As discussed in Section 5, let me contract the indices of \( C_{(ab)} \) with \( M_{abc} \) to construct the possible super-Hamiltonians defined by

\[ \bar{C}_a \equiv M_{abc} (C_{(bc)} + \lambda \delta_{bc}), \]  

(65)

where I have included a new term with a coefficient \( \lambda \). This term is meaningful because it adds a non-constant term to \( \bar{C}_a \). The term is consistent with the kinematical symmetry of the tensor models, since \( \delta_{ab} \) is an invariant of \( O(N) \).

Now let me define

\[ \bar{H}(v) \equiv v_a \bar{C}_a, \]  

(66)

for an infinitesimal parameter \( v_a \) independent of the canonical variables. Because of the \( O(N) \) invariant form of (66), it is obvious that

\[ \{ D(v^A), \bar{H}(w) \} = \bar{H}(v^A w), \]  

(67)

where \( v^A w_a \equiv v^A_{ab} w_b \), and therefore the Poisson bracket between \( D \) and \( \bar{H} \) closes.

The Poisson bracket between two \( \bar{H} \)'s is given by

\[ \{ \bar{H}(v), \bar{H}(w) \} = \{ \bar{C}_{ab}, \bar{C}_{cd} \} \bar{v}_{ab} \bar{w}_{cd} + \{ \bar{v}_{ab}, \bar{C}_{cd} \} \bar{w}_{cd} \bar{C}_{ab} + \{ \bar{C}_{ab}, \bar{w}_{cd} \} \bar{v}_{ab} \bar{C}_{cd} + \{ \bar{v}_{ab}, \bar{w}_{cd} \} \bar{C}_{ab} \bar{C}_{cd}, \]  

(68)
where I have defined
\[
\bar{C}_{ab} \equiv C_{(ab)} + \lambda \delta_{ab},
\]
\[
\bar{v}_{ab} \equiv v_c M_{(ab)c},
\]
\[
\bar{w}_{ab} \equiv w_c M_{(ab)c},
\]
for notational simplicity. Here \(M_{(ab)c}\) (and \(\pi_{(ab)c}\)) is the symmetrization defined by
\[
M_{(ab)c} \equiv \frac{1}{2} (M_{abc} + M_{bac}),
\]
\[
\pi_{(ab)c} \equiv \frac{1}{2} (\pi_{abc} + \pi_{bac}).
\]
Note that, thanks to (11) and (56), \(M_{(ab)c}\) and \(\pi_{(ab)c}\) are totally symmetric with respect to their indices. The computation of the first term in (68) is similar to (62). The last term in (68) trivially vanishes. The sum of the second and the third terms can be computed as
\[
\frac{1}{2} v_e \{ M_{(ab)e}, \pi_{efg} \pi_{dgf} - \epsilon_{efg} M_{dfg} \} w_h M_{(cd)h} \bar{C}_{ab} - (v \leftrightarrow w)
\]
\[
= v_e \pi_{efg} \delta_{(ab)e,dfg} w_h M_{(cd)h} \bar{C}_{ab} - (v \leftrightarrow w)
\]
\[
= 2 (v_a w_b - w_a v_b) \bar{C}_{cd} \pi_{(ac)e} M_{(bd)e}. \quad (74)
\]
Thus the result of (68) is obtained as
\[
\{ \bar{H}(v), \bar{H}(w) \} = \epsilon D([\bar{v}, \bar{w}]) + 2 (v_a w_b - w_a v_b) \bar{C}_{cd} \pi_{(ac)e} M_{(bd)e}. \quad (75)
\]
The first term of the right-hand side is what is desired. The matrices \(\bar{v}\) and \(\bar{w}\) are distributed by \(M_{abc}\) as in (70) and (71). Therefore, this term exactly reproduces the constraint algebra of general relativity in the pointwise limit \(\beta \to \infty\), as has been discussed in Section 4. However, the second term is proportional to neither \(C_{[ab]}\) nor \(\bar{C}_a\). Therefore \(\bar{C}_a\) are not appropriate as super-Hamiltonians, since the constraint algebra does not close.

To cancel the second term in (75), let me first proceed under the assumption that the super-Hamiltonians be invariant under the time-reversal transformation \(\pi_{abc} \to -\pi_{abc}\). I also assume that the terms are at most cubic and local: each term must be connected, not like \(M_{abb}M_{cde}M_{cde}\) for instance. Then the possible cubic terms other than those in (65) can be listed as
\[
C_a^{(1)} \equiv \pi_{abc} \pi_{bde} M_{cde},
\]
\[
C_a^{(2)} \equiv M_{abc} \pi_{bcd} \pi_{dee},
\]
\[
C_a^{(3)} \equiv \pi_{abc} M_{bcd} \pi_{dee},
\]
(76)
(77)
(78)
\[ \mathcal{C}_a^{(4)} \equiv \pi_{abc} \pi_{bcd} M_{dec}, \]  
\[ \mathcal{C}_a^{(5)} \equiv M_{abc} M_{bcd} M_{dec}, \]

where I have not cared the order of the indices, since this is not essential in the following discussions. It can easily be shown that \( \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(5)} \) are not useful to cancel the second term in (75). When \( \mathcal{C}^{(1)} \) is added to (65), the Poisson bracket will have additional contributions, \{\( \mathcal{C}^{(1)}, \mathcal{C}^{(1)} \)\} and \{\( \mathcal{C}^{(1)}, \bar{C} \)\}. From explicit computations, \{\( \mathcal{C}^{(1)}, \mathcal{C}^{(1)} \)\} is proportional to super-momentums, and might be allowed. However, \{\( \mathcal{C}^{(1)}, \bar{C} \)\} contains terms

\[ v_a w_b \{\mathcal{C}^{(1)}_a, \bar{C}_b\} \sim d_1 v_a w_c M_{def} \pi_{acd} M_{fij} M_{cji} + d_2 v_a w_b M_{def} M_{gbc} \pi_{acd} + \cdots, \]  

where \( d_1, d_2 \) are some integers. The first term in (81) can be used to cancel part of the unwanted term in (75), but the second term in (81) is problematic. This term is not proportional to the super-momentums, and it can also be shown that this term cannot be canceled by adding \( \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(5)} \). Thus it is not possible to construct super-Hamiltonians that contain (65), are cubic at highest and local, and respect the time-reversal symmetry.

A simple solution can be found, if one relaxes the time-reversal symmetry, while the other assumptions are kept intact. Let me consider

\[ \mathcal{C}_a \equiv (M_{abc} + \epsilon'' \pi_{abc})(\mathcal{C}_{(bc)} + \lambda \delta_{bc}), \] 

where \( \epsilon'' \) is a parameter which will be determined in the following. This will become the final form of super-Hamiltonians. Define

\[ H(v) = v_a \mathcal{C}_a \] 

for an infinitesimal parameter \( v_a \) independent of the canonical variables. Because of the \( O(N) \) invariant form of (83), one obtains

\[ \{D(v^A), H(w)\} = H(v^A w), \] 

where \( v^A w_a \equiv v^A_{ab} w_b \).

The Poisson bracket between \( H \)’s is given by

\[ \{H(v), H(w)\} = \{\bar{C}_{ab}, \bar{C}_{cd}\} v'_a w'_c + \{v'_a, \bar{C}_{cd}\} w'_c \bar{C}_{ab} + \{\bar{C}_{ab}, w'_d\} v'_a \bar{C}_{cd} + \{v'_a, w'_d\} \bar{C}_{ab} \bar{C}_{cd}, \] 

where

\[ v'_a \equiv v_c (M_{(ab)c} + \epsilon'' \pi_{(ab)c}), \]
\[ w'_{ab} \equiv w_c (M_{(ab)c} + \epsilon'' \pi_{(ab)c}). \] (87)

Computations similar to the case of (68) result in

\[ \{ H(v), H(w) \} = \epsilon D([v', w']) + 2 (1 - \epsilon (\epsilon'')^2) (v_a w_b - w_a v_b) \bar{\mathcal{C}}_{cd} \pi_{(ac)e} M_{(bd)e}. \] (88)

For the closure of the constraint algebra, \( \epsilon'' \) is determined as

\[ \epsilon'' = \pm 1/\sqrt{\epsilon}. \] (89)

Then

\[ \{ H(v), H(w) \} = \epsilon D([v', w']). \] (90)

The commutator in (90) is of the matrices (86) and (87). This contains not only \( M_{abc} \) but also \( \pi_{abc} \). But, as is stressed in Section 4, the constraint algebra of general relativity can be reproduced without the details of the distribution of these matrices. One would be able to expect that, in the regime discussed in Section 3, not only \( M_{abc} \) but also the conjugate momenta \( \pi_{abc} \) have relatively local distributions which become pointwise in the limit \( \beta \to \infty \). Therefore the addition of \( \pi_{abc} \) will not affect the derivation of the constraint algebra of general relativity in the pointwise limit.

In the Euclidean case \( \epsilon = -1 \), (89) implies that \( \epsilon'' \) must take an imaginary value \( \epsilon'' = \pm i \). Then an infinitesimal time-like shift will generate

\[ \delta M_{abc} = \{ M_{abc}, H(v) \} = \pm iv_a \bar{\mathcal{C}}_{bc} + \cdots. \] (91)

This direct appearance of the imaginary unit \( i \) will generally violate the generalized Hermiticity condition (11). Therefore the Euclidean case is not consistent.

7 Coupling with matters

Although the vector model considered in Section 4 is not suited for describing gravity, it may be added as a matter. Because of the kinematical character, the total super-momentums are simply given by the sum of those of the three-tensor (59) and the vector (26) as

\[ C_{total}^{[ab]} \equiv C_{[ab]} + C_{V[ab]}, \] (92)

which are obviously the generators of the \( O(N) \) transformation. The total super-Hamiltonians would be defined by a summation,

\[ C_{a}^{total} \equiv C_a + (M_{abc} + \epsilon'' \pi_{abc}) C_{(bc)}^V, \] (93)
where $C_a$ and $C^V_{(ab)}$ are the super-Hamiltonians of the three-tensor (82) and the vector (25), respectively. Here the coupling between gravity and matter is realized by the contraction of $C^V_{(ab)}$ with $M_{abc} + \epsilon'' \pi_{abc}$. To check whether this actually produces the desired result, let me compute the Poisson bracket,

$$\{H^{total}(v), H^{total}(w)\} = v_a w_b \left( \{C_a, C_b\} + \{C_a, M_{bcd} + \epsilon'' \pi_{bcd}\} C^V_{(cd)} + \{M_{acd} + \epsilon'' \pi_{acd}, C_b\} C^V_{(cd)} + \{M_{acd} + \epsilon'' \pi_{acd}, M_{bef} + \epsilon'' \pi_{bef}\} \{C^V_{(cd)}, C^V_{(ef)}\} \right),$$

(94)

where

$$H^{total}(v) = v_a C^a_{total}.$$  

(95)

The first and the last terms in (94) produce the desired form, and the fourth term obviously vanishes. The computation of the sum of the second and the third terms is essentially the same as (85), and one obtains

$$\{H^{total}(v), H^{total}(w)\} = \epsilon D^{total}([v', w']),$$

(96)

if the condition (89) is satisfied, where

$$D^{total}(v^A) = v^A_{ab} C^a_{[ab]},$$

(97)

and $v'$ and $w'$ are defined in (86) and (87), respectively. Because of the apparent $O(N)$ invariant forms, it is straightforward to derive

$$\{D^{total}(v^A), H^{total}(w)\} = H^{total}(v^A w).$$

(98)

It is obvious that one can add various matters in similar manners. It will be straightforward to extend (24), (25), (26), (92) and (93) to such general cases. Any rank of tensors and statistics, bosonic or fermionic, will be allowed.

8 Summary, discussions and future prospects

In this paper, I have discussed how local time can be introduced in tensor models. Since it was not clear how to formulate such tensor models in the Lagrangian formalism, a rank-three tensor model with first class constraints in Hamilton formalism has been presented. The discussions have made it clear that a three-index tensor has a prominent feature necessary for the purpose. The other rank tensors can be added as matter sectors which has coupling with the three-index tensor. In this sense, the rank-three tensor model, which contains a
three-index tensor as its only dynamical variable, can be regarded as the gravitational sector which can universally couple with matters.

The momentum constraints have straightforwardly been constructed so that they incorporate the kinematical symmetry of the rank-three tensor models. Then the consistency of the local time evolution requires Hamiltonian constraints and the momentum constraints to compose a closed first class constraint algebra. This closure condition gives strong limitations on the possible forms of super-Hamiltonians, and they have been determined on the assumptions that they are local and cubic at highest. They have turned out to contain terms which break the time-reversal symmetry. It has also been shown that the constraint algebra reproduces the first class constraint algebra of general relativity in the limit of usual continuous spaces. The first class constraint algebra closes exactly without any approximations, and this is a good feature which marks distinction from the previous approaches [39–43] to discrete gravity in the canonical formalism.

The most important implication of this paper is that it is in principle possible for the rank-three tensor model to have the first class constraint algebra which reproduces that of general relativity in the limit of usual continuous spaces. However, the limiting procedure is rather formal assuming the regime discussed in Section 3. Therefore it will be necessary to investigate whether the limit can be generated by the actual dynamics of the rank-three tensor model. Not only the constraint algebra, but it would also be necessary to check whether the equations of motion of general relativity can directly be reproduced in the continuum limit from the super-Hamiltonians. Especially, the fate of the terms with the breaking of the time-reversal symmetry has to be studied. If the breaking effect has turned out to be too large to be allowed by experiments/observations, the super-Hamiltonians must be reconsidered under relaxed assumptions. By canonical transformations of the variables, it is also possible to change the property of the super-Hamiltonians under the time-reversal transformation.

Regardless of whether such a regime in Section 3 actually exists in the dynamics, the canonical rank-three tensor model of this paper has a significance of its own as a model of emergent space and gravity with consistently incorporated local time. The form of the super-Hamiltonians has similarity with the Hamiltonian of the $c < 1$ string field theory [55–57], which has applications to random surfaces. Since the $c < 1$ string field theory may be regarded as (1+1)-dimensional gravity with matters, the rank-three tensor model of this paper may have some connections to it. The connections may provide an interesting testing ground as well as some hints for the dynamics of the tensor model.
References

[1] J. Ambjorn, B. Durhuus, and T. Jonsson, “Three-dimensional simplicial quantum gravity and generalized matrix models,” *Mod.Phys.Lett.* **A6** (1991) 1133–1146.

[2] N. Sasakura, “Tensor model for gravity and orientability of manifold,” *Mod.Phys.Lett.* **A6** (1991) 2613–2624.

[3] N. Godfrey and M. Gross, “Simplicial quantum gravity in more than two-dimensions,” *Phys.Rev.* **D43** (1991) 1749–1753.

[4] D. Boulatov, “A Model of three-dimensional lattice gravity,” *Mod.Phys.Lett.* **A7** (1992) 1629–1646, [hep-th/9202074].

[5] H. Ooguri, “Topological lattice models in four-dimensions,” *Mod.Phys.Lett.* **A7** (1992) 2799–2810, [hep-th/9205090].

[6] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli, “Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space,” *Nucl.Phys.* **B574** (2000) 785–806, [hep-th/9907154].

[7] D. Oriti, “The microscopic dynamics of quantum space as a group field theory,” [1110.5606].

[8] R. Gurau, “The Double Scaling Limit in Arbitrary Dimensions: A Toy Model,” [1110.2460].

[9] R. Gurau and J. P. Ryan, “Colored Tensor Models - a review,” [1109.4812].

[10] V. Bonzom, R. Gurau, and V. Rivasseau, “The Ising Model on Random Lattices in Arbitrary Dimensions,” [1108.6269].

[11] D. Benedetti and R. Gurau, “Phase Transition in Dually Weighted Colored Tensor Models,” [1108.5389].

[12] A. Baratin and D. Oriti, “Quantum simplicial geometry in the group field theory formalism: reconsidering the Barrett-Crane model,” [1108.1178].

[13] R. Gurau, “A generalization of the Virasoro algebra to arbitrary dimensions,” *Nucl.Phys.* **B852** (2011) 592–614, [1105.6072].

[14] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, “Critical behavior of colored tensor models in the large N limit,” *Nucl.Phys.* **B853** (2011) 174–195, [1105.3122].
[15] E. R. Livine, D. Oriti, and J. P. Ryan, “Effective Hamiltonian Constraint from Group Field Theory,” [1104.5509].

[16] S. Carrozza and D. Oriti, “Bounding bubbles: the vertex representation of 3d Group Field Theory and the suppression of pseudo-manifolds,” [1104.55158].

[17] R. Gurau, “The complete 1/N expansion of colored tensor models in arbitrary dimension,” [1102.5759].

[18] A. Baratin, F. Girelli, and D. Oriti, “Diffeomorphisms in group field theories,” *Phys.Rev.* D83 (2011) 104051, [1101.0590].

[19] R. Gurau, “The 1/N expansion of colored tensor models,” *Annales Henri Poincare* 12 (2011) 829–847, [1011.2725].

[20] J. Ben Geloun, R. Gurau, and V. Rivasseau, “EPRL/FK Group Field Theory,” *Europhys.Lett.* 92 (2010) 60008, [1008.0354].

[21] R. Gurau, “Lost in Translation: Topological Singularities in Group Field Theory,” *Class.Quant.Grav.* 27 (2010) 235023, [1006.0714].

[22] R. Gurau, “Topological Graph Polynomials in Colored Group Field Theory,” *Annales Henri Poincare* 11 (2010) 565–584, [0911.1945].

[23] R. Gurau, “Colored Group Field Theory,” *Commun.Math.Phys.* 304 (2011) 69–93, [0907.2582].

[24] N. Sasakura, “A Renormalization procedure for tensor models and scalar-tensor theories of gravity,” *Int.J.Mod.Phys.* A25 (2010) 4475–4492, [1005.3088].

[25] N. Sasakura, “Emergent general relativity in the tensor models possessing Gaussian classical solutions,” *AIP Conf.Proc.* 1243 (2010) 76–86, [0911.1170].

[26] N. Sasakura, “Gauge fixing in the tensor model and emergence of local gauge symmetries,” *Prog.Theor.Phys.* 122 (2009) 309–322, [0904.0046].

[27] N. Sasakura, “Emergent general relativity on fuzzy spaces from tensor models,” *Prog.Theor.Phys.* 119 (2008) 1029–1040, [0803.1717].

[28] N. Sasakura, “The Lowest modes around Gaussian solutions of tensor models and the general relativity,” *Int.J.Mod.Phys.* A23 (2008) 3863–3890, [0710.0696].
[29] N. Sasakura, “The Fluctuation spectra around a Gaussian classical solution of a tensor model and the general relativity,” *Int.J.Mod.Phys.* A23 (2008) 693–718, 0706.1618.

[30] N. Sasakura, “Tensor model and dynamical generation of commutative nonassociative fuzzy spaces,” *Class.Quant.Grav.* 23 (2006) 5397–5416, hep-th/0606066.

[31] N. Sasakura, “An Invariant approach to dynamical fuzzy spaces with a three-index variable: Euclidean models,” hep-th/0511154.

[32] N. Sasakura, “Tensor models and 3-ary algebras,” *J.Math.Phys.* 52 (2011) 103510, 1104.1463.

[33] N. Sasakura, “An Invariant approach to dynamical fuzzy spaces with a three-index variable,” *Mod.Phys.Lett.* A21 (2006) 1017–1028, hep-th/0506192.

[34] J. Ambjorn, J. Jurkiewicz, and R. Loll, “Causal Dynamical Triangulations and the Quest for Quantum Gravity,” 1004.0352.

[35] L. Sindoni, “Emergent models for gravity: An Overview,” 1110.0686.

[36] J. Polchinski, “Comment on [arXiv:1106.1417] 'Small Lorentz violations in quantum gravity: do they lead to unacceptably large effects?,” 1106.6346.

[37] R. Gambini, S. Rastgoo, and J. Pullin, “Small Lorentz violations in quantum gravity: do they lead to unacceptably large effects?,” *Class. Quant. Grav.* 28 (2011) 155005, 1106.1417.

[38] J. Collins, A. Perez, D. Sudarsky, L. Urrutia, and H. Vucetich, “Lorentz invariance and quantum gravity: an additional fine-tuning problem?,” *Phys.Rev.Lett.* 93 (2004) 191301, gr-qc/0403053.

[39] T. Piran and R. M. Williams, “A (3+1) formulation of Regge calculus,” *Phys.Rev.* D33 (1986) 1622.

[40] J. Friedman and I. Jack, “(3+1) Regge calculus with conserved momentum and Hamilton constraints,” *J.Math.Phys.* 27 (1986) 2973–2986.

[41] R. Gambini and J. Pullin, “Consistent discretization and canonical classical and quantum Regge calculus,” *Int.J.Mod.Phys.* D15 (2006) 1699–1706, gr-qc/0511096.

[42] B. Dittrich and P. A. Hoehn, “Canonical simplicial gravity,” 1108.1974.
[43] B. Bahr, R. Gambini, and J. Pullin, “Discretisations, constraints and diffeomorphisms in quantum gravity,” 1111.1879.

[44] R. L. Arnowitt, S. Deser, and C. W. Misner, “The Dynamics of general relativity,” gr-qc/0405109 Gravitation: an introduction to current research, Louis Witten ed. (Wilew 1962), chapter 7, pp 227-265.

[45] B. S. DeWitt, “Quantum Theory of Gravity. 1. The Canonical Theory,” Phys.Rev. 160 (1967) 1113–1148.

[46] J. Ehlers and H. Friedrich, eds., Canonical Gravity: From Classical to Quantum, vol. 434 of Lecture Notes in Physics, Berlin Springer Verlag, 1994.

[47] S. Hojman, K. Kuchar, and C. Teitelboim, “Geometrodynamics Regained,” Annals Phys. 96 (1976) 88–135.

[48] S. Ramgoolam, “On spherical harmonics for fuzzy spheres in diverse dimensions,” Nucl.Phys. B610 (2001) 461–488, hep-th/0105006.

[49] S. Ramgoolam, “Towards gauge theory for a class of commutative and nonassociative fuzzy spaces,” JHEP 0403 (2004) 034, hep-th/0310153.

[50] P. de Medeiros and S. Ramgoolam, “Non-associative gauge theory and higher spin interactions,” JHEP 0503 (2005) 072, hep-th/0412027.

[51] Y. Sasai and N. Sasakura, “One-loop unitarity of scalar field theories on Poincare invariant commutative nonassociative spacetimes,” JHEP 0609 (2006) 046, hep-th/0604194.

[52] N. Sasakura, “Tensor models and hierarchy of n-ary algebras,” Int.J.Mod.Phys. A26 (2011) 3249–3258, 1104.5312.

[53] N. Sasakura, “Super tensor models, super fuzzy spaces and super n-ary transformations,” Int.J.Mod.Phys. A26 (2011) 4203–4216, 1106.0379.

[54] N. Sasakura, “Fuzzy spaces from tensor models, cyclicity condition, and n-ary algebras,” (http://www.physics.ntua.gr/corfu2011/Talks/sasakura@yukawa.kyoto-u.ac.jp_01.pdf). Talk at “Workshop on Noncommutative Field Theory and Gravity”, Sep.7-11, Corfu, Greece.

[55] N. Ishibashi and H. Kawai, “String field theory of noncritical strings,” Phys.Lett. B314 (1993) 190–196, hep-th/9307045.
[56] M. Ikehara, N. Ishibashi, H. Kawai, T. Mogami, R. Nakayama, et al., “String field
theory in the temporal gauge,” *Phys.Rev.* **D50** (1994) 7467–7478. [hep-th/9406207]

[57] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra, and S. Zohren, “A String Field Theory
based on Causal Dynamical Triangulations,” *JHEP* **0805** (2008) 032, [0802.0719]