Positivity of Dunkl's Intertwining Operator

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Abstract

For a finite reflection group on \( \mathbb{R}^N \), the associated Dunkl operators are parametrized first-order differential-difference operators which generalize the usual partial derivatives. They generate a commutative algebra which is – under weak assumptions – intertwined with the algebra of partial differential operators by a unique linear and homogeneous isomorphism on polynomials. In this paper it is shown that for non-negative parameter values, this intertwining operator is positivity-preserving on polynomials and allows a positive integral representation on certain algebras of analytic functions. This result in particular implies that the generalized exponential kernel of the Dunkl transform is positive-definite.

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1 Introduction and results

During the last years, the theory of Dunkl operators has found a wide area of applications in mathematics and mathematical physics. Besides their use in the study of multivariable orthogonality structures with certain reflection symmetries (see e.g. [D1-2], [V1], [R], [X3]), these operators are for example closely related to certain representations of degenerate affine Hecke algebras (see [J], [O2] and, for some background, [K]). Moreover, they have been successfully involved in the description and solution of Calogero-Moser-Sutherland type quantum many body systems; among the wide literature in this context, we refer to [P], [L-V] and [B-F].

Let \( G \subset O(N, \mathbb{R}) \) be a finite reflection group on \( \mathbb{R}^N \). For \( \alpha \in \mathbb{R}^N \setminus \{0\} \), we denote by \( \sigma_\alpha \) the reflection in the hyperplane orthogonal to \( \alpha \), i.e.

\[
\sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha,
\]

where \( \langle .., . \rangle \) denotes the Euclidean scalar product on \( \mathbb{R}^N \) and \( |x| := \sqrt{\langle x, x \rangle} \). (We use the same notations for the standard Hermitian inner product and norm on \( \mathbb{C}^N \).) Let further \( R \) be the
root system of $G$, normalized such that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$, and fix a positive subsystem $R_+$ of $R$. We recall from the general theory of reflection groups (see e.g. [1]) that the set of reflections in $G$ coincides with \( \{ \sigma_\alpha, \alpha \in R_+ \} \), and that the orbits in $R$ under the natural action of $G$ correspond to the conjugacy classes of reflections in $G$. A function $k : R \to \mathbb{C}$ is called a multiplicity function on $R$, if it is $G$-invariant. We write $\text{Re} \, k \geq 0$ if $\text{Re} \, k(\alpha) \geq 0$ for all $\alpha \in R$ and $k \geq 0$ if $k(\alpha) \geq 0$ for all $\alpha \in R$.

The Dunkl operators associated with $G$ are first-order differential-difference operators on $\mathbb{R}^N$ which are parametrized by some multiplicity function $k$ on $R$. For $\xi \in \mathbb{R}^N$, the corresponding Dunkl operator $T_\xi(k)$ is given by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^N);$$

here $\partial_\xi$ denotes the directional derivative corresponding to $\xi$. As $k$ is $G$-invariant, the above definition is independent of the choice of $R_+$. In case $k = 0$, the $T_\xi(k)$ reduce to the corresponding directional derivatives. The operators $T_\xi(k)$ were introduced and first studied by Dunkl in a series of papers ([D1-4]) in connection with a generalization of the classical theory of spherical harmonics: here the uniform surface measure on the $(N - 1)$-dimensional unit sphere is modified by a weight function which is invariant under the action of a given reflection group $G$ and associated with a multiplicity function $k \geq 0$, namely

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}. \quad (1.1)$$

The most important properties of the operators $T_\xi(k)$ are as follows: Let $\Pi^N = \mathbb{C}[\mathbb{R}^N]$ denote the algebra of polynomial functions on $\mathbb{R}^N$ and $\mathcal{P}_n^N$ ($n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots \}$) the subspace of homogeneous polynomials of (total) degree $n$. Then

1. The $T_\xi(k), \xi \in \mathbb{R}^N$, generate a commuting family of linear operators on $\Pi^N$.

2. Each $T_\xi(k)$ is homogeneous of degree $-1$ on $\Pi^N$, that is, $T_\xi(k)(p) \in \mathcal{P}_{n-1}^N$ for $p \in \mathcal{P}_n^N$.

3. For all but a singular set of multiplicity functions, in particular for $k \geq 0$, there exists a unique linear isomorphism $V_k$ of $\Pi^N$ such that

$$V_k(\mathcal{P}_n^N) = \mathcal{P}_n^N, \quad V_k|_{\mathcal{P}_0^N} = \text{id} \quad \text{and} \quad T_\xi(k)V_k = V_k \partial_\xi \quad \text{for all } \xi \in \mathbb{R}^N.$$

Properties (1) and (2) were shown in [D1], while the existence of an intertwining operator according to (3) was first shown in [D2] under the assumption $k \geq 0$. An abstract and extended treatment of the above items is given in [D-J-O].

The intertwining operator $V_k$ plays a central part in Dunkl’s theory and its applications. It is in particular involved in the definition of Dunkl’s kernel $K_G(x, y)$ (see below), which generalizes the usual exponential kernel $e^{i(x,y)}$ and arises as the integral kernel of the Dunkl transform (see [D4] and [L]). An explicit form of $V_k$ is known so far only in very special cases:

1. The one-dimensional case, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$; here the multiplicity function is given by a single parameter $k \geq 0$, and the intertwining operator $V_k$ has the
integral representation (see [D3], Th. 5.1)
\[ V_k p(x) = c_k \int_{-1}^{1} p(x(t)) (1 - t)^{k-1} (1 + t)^k \, dt \quad \text{with} \quad c_k = \frac{\Gamma(k + 1/2)}{\Gamma(1/2) \Gamma(k)} \] (1.2)

2. The direct product case, associated with the reflection group \( \mathbb{Z}_2^N \) on \( \mathbb{R}^N \); here a closed form of the intertwining operator was determined in [K1].

3. The case of the symmetric group \( S_3 \) on \( \mathbb{R}^3 \), which has been studied in [D5].

In [D3], the intertwining operator \( V_k \) is, for \( k \geq 0 \), extended to a bounded linear operator on a suitably normed algebra of series of homogeneous polynomials on the unit ball. To allow a more convenient formulation of our statements, we introduce a slightly extended notation: For \( r > 0 \) let \( K_r := \{ x \in \mathbb{R}^N : |x| \leq r \} \) denote the ball of radius \( r \) and define

\[ A_r := \{ f : K_r \to \mathbb{C}, f = \sum_{n=0}^{\infty} f_n \quad \text{with} \quad f_n \in \mathcal{P}_n^N \quad \text{and} \quad \|f\|_{A_r} := \sum_{n=0}^{\infty} \|f_n\|_{\infty, K_r} < \infty \}. \] (1.3)

It is easily checked that \( A_r \) is a commutative Banach-*-algebra (with complex conjugation as involution), see Section 4. Moreover, it follows from Theorem 2.7 of [D3] that \( V_k \) extends to a continuous linear operator on \( A_r \) by \( V_k f := \sum_{n=0}^{\infty} V_k f_n \) for \( f = \sum_{n=0}^{\infty} f_n \in A_r \). Up to now, it has been an open question whether for \( k \geq 0 \) the intertwining operator \( V_k \) is always positive, i.e. \( V_k p \geq 0 \) on \( \mathbb{R}^N \) for each nonnegative polynomial \( p \in \Pi^N \). More generally, we may ask whether for every \( x \in \mathbb{R}^N \) with \( |x| \leq r \), the functional \( f \mapsto V_k f(x) \) is positive on \( A_r \). This property, which was first conjectured in [D3] (in a slightly different setting), is obvious in the above listed special cases 1 and 2 from the explicit representation of \( V_k \); in the \( S_3 \)-case however, the integral representations derived in [D5] failed to infer this result - at least for a large range of \( k \). It is the aim of this paper to prove the above conjecture for general reflection groups and nonnegative multiplicity functions. Our first central result establishes positivity of \( V_k \) on polynomials:

**1.1. Theorem.** Assume that \( k \geq 0 \) and let \( p \in \Pi^N \) with \( p(x) \geq 0 \) for all \( x \in \mathbb{R}^N \). Then also \( V_k p(x) \geq 0 \) for all \( x \in \mathbb{R}^N \).

More detailed information about \( V_k \) is then obtained by its extension to the algebras \( A_r \). This leads to the following theorem, which is the main result of this paper:

**1.2. Theorem.** Assume that \( k \geq 0 \). Then for each \( x \in \mathbb{R}^N \) there exists a unique probability measure \( \mu_x \) on the Borel-\( \sigma \)-algebra of \( \mathbb{R}^N \) such that

\[ V_k f(x) = \int_{\mathbb{R}^N} f(\mathbf{\xi}) \, d\mu_x(\mathbf{\xi}) \quad \text{for all} \quad f \in A_{|x|}. \] (1.4)

The representing measures \( \mu_x \) are compactly supported with \( \text{supp} \mu_x \subseteq \{ \mathbf{\xi} \in \mathbb{R}^N : |\mathbf{\xi}| \leq |x| \} \).

Moreover, they satisfy

\[ \mu_{r\mathbf{x}}(B) = \mu_{\mathbf{x}}(r^{-1}B), \quad \mu_{g\mathbf{x}}(B) = \mu_{\mathbf{x}}(g^{-1}(B)) \] (1.5)

for each \( r > 0 \), \( g \in G \) and each Borel set \( B \subseteq \mathbb{R}^N \).
An important consequence of Theorem 1.2 concerns the generalized exponential kernel $K_G$, which is defined by

$$K_G(x, y) := V_k(e^{\langle -y \rangle})(x) \quad (x, y \in \mathbb{R}^N),$$

see [D3]. The function $K_G$ has a holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$ and is symmetric in its arguments. We also remark that by a result of [O1], the function $x \mapsto K_G(x, y)$ may be characterized as the unique analytic solution of the system $T_\xi(k)f = \langle \xi, y \rangle f \quad (\xi \in \mathbb{R}^N)$ with $f(0) = 1$. Theorem 1.2 implies that for fixed $y \in \mathbb{R}^N$ the kernel $K_G(x, iy)$ is positive-definite as a function of $x$ on $\mathbb{R}^N$, and the same holds for the "generalized Bessel function"

$$J_G(x, iy) := \frac{1}{|G|} \sum_{g \in G} K_G(x, igy) \quad (x, y \in \mathbb{R}^N).$$

As noted in [O1], the kernel $J_G$ allows in some cases (for Weyl groups $G$ and certain discrete sets of multiplicity functions) an interpretation as the spherical function for some Euclidean symmetric space; in these cases positive-definiteness of $J_G$ is obvious. There are no similar interpretations known for the kernel $K_G$. Nevertheless, the conjecture that it should be positive-definite has been confirmed by several of its properties (see [dJ]), and in particular by the fact that $K_G(x, y) > 0$ for all $x, y \in \mathbb{R}^N$; this was proved in [R] in connection with the study of a generalized heat semigroup for Dunkl operators.

The main parts of Theorem 1.2 are obtained by a standard argumentation from Theorem 1.1. The proof of Theorem 1.1, however, is much more involved. Its crucial step is a reduction from the $N$-dimensional to a one-dimensional problem, using semigroup techniques for linear operators on spaces of polynomials. The generators of the semigroups under consideration are certain differential-difference operators whose common decisive property is that they are "degree-lowering". This setting is introduced in Section 2, together with a Hille-Yosida type theorem which characterizes positivity of such semigroups by means of their generator. Theorem 1.1 is then proved in Section 3. Section 4 is introduced with a short discussion of the algebras $A_r$ and their spectral properties, which is the basis for the subsequent proof of Theorem 1.2. In the last section we discuss some implications of our results in the theory of Dunkl operators and related applications.

### 2 Semigroups generated by degree-lowering operators on polynomials

We start with some general notations: Let $\Pi_N^+ := \{ p \in \Pi_N : p(x) \geq 0 \text{ for all } x \in \mathbb{R}^N \}$ denote the cone of nonnegative polynomials on $\mathbb{R}^N$, and $\Pi_n^+ := \bigoplus_{k=0}^{n} P_k^N (n \in \mathbb{Z}_+)$ the space of polynomials of (total) degree at most $n$. The action of a subgroup $H \subseteq O(N, \mathbb{R})$ on $\Pi_N$ will always be the natural one, given by $hp(x) := p(h^{-1}x)$ ($h \in H$, $p \in \Pi_N$). Finally, for a locally compact Hausdorff space $X$, $M_b(X)$ is the space of all regular bounded Borel measures on $X$ and $M_b^+(X)$ the subspace of those which are non-negative.

#### 2.1. Definition

A linear operator $A$ on $\Pi_N$ is called
• positive, if $Ap \in \Pi^N_+$ for each $p \in \Pi^N_+$.
• degree-lowering, if $A(\Pi^N_n) \subseteq \Pi^N_{n-1}$ for all $n \in \mathbb{Z}_+$.

Important examples of degree-lowering operators are linear operators which are homogeneous of some degree $-n$, $n \geq 1$, on $\Pi^N$. This includes in particular usual partial derivatives and Dunkl operators, as well as products and linear combinations of those. If $A$ is degree-lowering on $\Pi^N$, then for every analytic function $f : \mathbb{R} \to \mathbb{C}$ with power series $f(x) = \sum_{k=0}^{\infty} c_k x^k$, there is a linear operator $f(A)$ on $\Pi^N$ defined by the terminating series

$$f(A)p(x) := \sum_{k=0}^{\infty} c_k A^k p(x).$$

Notice that $f(A)(\Pi^N_n) \subseteq \Pi^N_n$ for each $n \in \mathbb{Z}_+$. This yields a natural restriction of $f(A)$ to a linear operator on the finite-dimensional vector space $\Pi^N_n$. In particular, the well-known product and exponential formulas for linear operators on finite-dimensional vector spaces (see, e.g. §4.7 of [Ka]) imply corresponding exponential formulas for degree-lowering operators on $\Pi^N$, where the topology may be choosen to be the one of pointwise convergence. We note two results of this type, which will be used later on:

2.2. Lemma. Let $A$ and $B$ be degree-lowering linear operators on $\Pi^N$. Then for all $p \in \Pi^N$ and $x \in \mathbb{R}^N$,

(i) $e^{A} p(x) = \lim_{n \to \infty} \left( I - \frac{A}{n} \right)^{-n} p(x)$.
(ii) $e^{A+B} p(x) = \lim_{n \to \infty} (e^{A/n} e^{B/n})^n p(x).$ (Trotter product formula).

Each degree-lowering operator $A$ on $\Pi^N$ generates a semigroup $(e^{tA})_{t \geq 0}$ of linear operators on $\Pi^N$ and, in fact, on each of the $\Pi^N_n$. The generator $A$ is uniquely determined from the semigroup by

$$Ap(x) = \lim_{t \downarrow 0} t^{-1} (e^{tA} - I) p(x) \quad \text{for all } p \in \Pi^N.$$

The following key-result characterizes positive semigroups generated by degree-lowering operators; it is an adaption of a well-known Hille-Yosida type characterization theorem for Feller-Markov semigroups on $C(K)$, $K$ a compact Hausdorff space (see, e.g. §II.4 of [G-S]):

2.3. Theorem. Let $A$ be a degree-lowering linear operator on $\Pi^N$. Then these are equivalent:

1. $e^{tA}$ is positive on $\Pi^N$ for all $t \geq 0$.
2. $A$ satisfies the “positive minimum principle”

(M) For every $p \in \Pi^N_+$ and $x_0 \in \mathbb{R}^N$, $p(x_0) = 0$ implies $A p(x_0) \geq 0$.

Proof. (1) $\Rightarrow$ (2): Let $p \in \Pi^N_+$ with $p(x_0) = 0$. Then

$$Ap(x_0) = \lim_{t \downarrow 0} \frac{e^{tA} p(x_0) - p(x_0)}{t} = \lim_{t \downarrow 0} \frac{1}{t} e^{tA} p(x_0) \geq 0.$$
(2) \Rightarrow (1): Notice first that for each \( \lambda \neq 0 \), the operator \( \lambda I - A \) is bijective on \( \Pi^N \). In fact, \( \lambda I - A \) is injective on \( \Pi^N \), because otherwise there would exist some \( p \in \Pi^N \), \( p \neq 0 \), with \( Ap = \lambda p \), in contradiction to the degree-lowering character of \( A \). As \( (\lambda I - A)(\Pi^N) \subseteq \Pi^N \), this already proves bijectivity of \( \lambda I - A \) on each \( \Pi^N \), hence on \( \Pi^N \) as well. We next claim that for every \( \lambda > 0 \) the resolvent operator \( R(\lambda; A) := (\lambda I - A)^{-1} \) is positive on \( \Pi^N \). For this, let \( p \in \Pi^N \) and \( q := R(\lambda; A)p \). If \( p \) is constant, then \( q = \frac{1}{\lambda} p \geq 0 \). We may therefore restrict to the case that the total degree \( n \) of \( p \) (which must be even) is greater than 0. Suppose first that \( p(x) \geq c|x|^n \) for all \( x \in \mathbb{R}^N \), with some constant \( c > 0 \). As \( A \) lowers the degree, we may write \( q = \frac{1}{\lambda} p + r \) with a polynomial \( r \) of total degree less than \( n \). Hence \( \lim_{|x| \to \infty} q(x) = \infty \), which shows that \( q \) takes an absolute minimum, let us say in \( x_0 \in \mathbb{R}^N \). Put \( \tilde{q}(x) := q(x) - q(x_0) \). Then \( \tilde{q} \in \Pi^N \) with \( \tilde{q}(x_0) = 0 \), and property (M) assures that \( A\tilde{q}(x_0) = A\tilde{q}(x_0) \geq 0 \). For \( \lambda > 0 \) and \( x \in \mathbb{R}^N \) we therefore obtain

\[
\lambda q(x) - \lambda q(x_0) = \lambda(\lambda I - A)q(x_0) + A(q(x) - q(x_0)) \geq p(x_0) - p(x_0) = 0.
\]

If \( p \in \Pi^N_+ \) is arbitrary, then consider the polynomials \( p_\epsilon(x) := p(x) + \epsilon |x|^n \) for \( \epsilon > 0 \), where \( n \) is the degree of \( p \). As \( A \) is degree-lowering, and by the above result, we obtain

\[
R(\lambda; A) p(x) = \lim_{\epsilon \to 0} R(\lambda; A) p_\epsilon(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N.
\]

This proves the stated positivity of \( R(\lambda; A) \) for \( \lambda > 0 \). Now let \( p \in \Pi^N_+ \) and \( \lambda > 0 \). Then according to Lemma 2.2.(i),

\[
e^{tA} p(x) = \lim_{n \to \infty} \left( I - \frac{tA}{n} \right)^{-n} p(x) = \lim_{n \to \infty} \left( \frac{n}{t} R \left( \frac{n}{t}; A \right) \right)^n p(x) \geq 0
\]

for all \( x \in \mathbb{R}^N \). This finishes the proof. \( \square \)

3 \quad \textbf{Positivity of } V_k \textbf{ on polynomials}

This section is devoted to the proof of Theorem 1.1. The outline of this proof is as follows: In a first step, we consider the (one-dimensional) differential-difference operators

\[
\Lambda_s := e^{-sD^2} \delta e^{sD^2}, \quad s \geq 0
\]

on \( \Pi^1 \). Here \( D \) denotes the usual first derivative, i.e. \( Dp(x) = p'(x) \) for \( x \in \mathbb{R} \), and \( \delta \) is the linear operator on \( \Pi^1 \) given by

\[
\delta p(x) = \frac{p'(x)}{x} - \frac{p(x) - p(-x)}{2x^2} = \frac{1}{2} \int_{-1}^1 (D^2 p)(tx)(1 + t) \, dt.
\]

This operator is related to the Dunkl operator \( T(k) \) associated with the reflection group \( \mathbb{Z}_2 \) on \( \mathbb{R} \) and the multiplicity parameter \( k \geq 0 \) by

\[
T(k)^2 = D^2 + 2k\delta.
\]

As both \( D^2 \) and \( \delta \) are homogeneous of degree \(-2\) on \( \Pi^1 \), the operators \( \Lambda_s \) are well-defined and degree-lowering on \( \Pi^1 \). We shall prove that they have the following decisive property:
3.1. Proposition. The operators $\Lambda_s$, $s \geq 0$, satisfy the positive minimum principle (M) on $\Pi^1$.

We next turn to the general $N$-dimensional setting: Here $G$ is an arbitrary finite reflection group on $\mathbb{R}^N$ with multiplicity function $k \geq 0$. We consider the generalized Laplacian associated with $G$ and $k$, which is defined by

$$\Delta_k := \sum_{i=1}^{N} T^2_{\xi_i}$$

with an arbitrary orthonormal basis $(\xi_1, \ldots, \xi_n)$ of $\mathbb{R}^N$ (see [D1]). It is homogeneous of degree $-2$ on $\Pi^N$ and (with our convention $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R_+$) given explicitly by

$$\Delta_k = \Delta + 2 \sum_{\alpha \in R_+} k(\alpha) \delta_{\alpha} \quad \text{with} \quad \delta_{\alpha} f(x) = \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_{\alpha} x)}{\langle \alpha, x \rangle^2};$$

(3.2)

here $\Delta$ and $\nabla$ denote the usual Laplacian and gradient respectively. Theorem 2.3 is the key to infer from the one-dimensional setting of Proposition 3.1 to a general multi-variable extension:

3.2. Proposition. Let $L_k := \Delta_k - \Delta$. Then for $k \geq 0$, the operators

$$e^{-s\Delta} e^{s L_k} e^{s \Delta} \quad (s, t \geq 0)$$

are positive on $\Pi^N$.

The statement of Theorem 1.1 will then finally be reduced to the following consequence of Proposition 3.2:

3.3. Corollary. The operator $e^{-\Delta/2} e^{\Delta/2}$ is positive on $\Pi^N$.

Proof. Applying Trotter’s product formula of Lemma 2.2, we obtain

$$e^{-\Delta/2} e^{\Delta/2} p(x) = e^{-\Delta/2} e^{\Delta/2 + L_k/2} p(x) = \lim_{n \to \infty} e^{-\Delta/2} \left( e^{\Delta/2n} e^{L_k/2n} \right)^n p(x)$$

$$= \lim_{n \to \infty} \prod_{j=1}^{n} \left( e^{-(1-j/n)\Delta/2} e^{L_k/2n} e^{(1-j/n)\Delta/2} \right) p(x) \quad (p \in \Pi^N, x \in \mathbb{R}^N).$$

By Proposition 3.2, each of the $n$ factors in the above product is a positive operator on $\Pi^N$. Hence $e^{-\Delta/2} e^{\Delta/2}$ is also positive on $\Pi^N$. \[\square\]

We now turn to the proof of Proposition 3.1. We start with two elementary auxiliary results:

3.4. Lemma. For each $p \in \Pi^1$ and $c \in \mathbb{R}$,

$$e^{cD^2} (xp(x)) = xe^{cD^2} p(x) + 2c e^{cD^2} p'(x).$$

Proof. Power series expansion of $e^{cD^2}$ yields

$$e^{cD^2} (xp(x)) = \sum_{n=0}^{\infty} \frac{c^n}{n!} D^{2n} (xp(x)) = xp(x) + \sum_{n=1}^{\infty} \frac{c^n}{n!} (xD^{2n} p(x) + 2nD^{2n-1} p(x))$$

$$= xe^{cD^2} p(x) + 2c \sum_{n=1}^{\infty} \frac{c^{n-1}}{(n-1)!} D^{2n-1} p(x) = xe^{cD^2} p(x) + 2c e^{cD^2} p'(x).$$

\[\square\]
3.5. Lemma. Let \( p \in \Pi_{2n+1} \), \( n \in \mathbb{Z}_+ \), be an odd polynomial. Then the differential equation

\[
y' - xy = p \quad (c > 0)
\]

has exactly one polynomial solution (which belongs to \( \Pi_{2n} \)), namely

\[
y_p(x) = \frac{1}{c} e^{x^2/2c} \int_{-\infty}^{x} e^{-t^2/2c} p(t) \, dt.
\]

Proof. The general solution of (3.4) is

\[
y(x) = a e^{x^2/2c} + \frac{1}{c} e^{x^2/2c} \int_{-\infty}^{x} e^{-t^2/2c} p(t) \, dt, \quad a \in \mathbb{R}.
\]

It therefore remains to prove that

\[
x \mapsto e^{x^2/2c} \int_{-\infty}^{x} e^{-t^2/2c} p(t) \, dt
\]

is a polynomial. We use induction by \( n \): For \( n = 0 \), the statement is obvious. For \( n \geq 1 \), write \( p(x) = -e^{-1}x r(x) \) with \( r \in \Pi_{2n} \). Partial integration then yields

\[
\int_{-\infty}^{x} e^{-t^2/2c} p(t) \, dt = -\frac{1}{c} \int_{-\infty}^{x} t e^{-t^2/2c} r(t) \, dt = e^{-x^2/2c} r(x) - \int_{-\infty}^{x} e^{-t^2/2c} r'(t) \, dt.
\]

By our induction hypothesis, this equals \( e^{-x^2/2c}(r(x) - \tilde{r}(x)) \) with some polynomial \( \tilde{r} \in \Pi_{2n-2} \). This finishes the proof.

Proof of Proposition 3.1. 1. The case \( s = 0 \) is easy and may be treated separately: Let \( p \in \Pi_1 \) with \( p(x_0) = 0 \). Then \( p'(x_0) = 0 \) and \( p''(x_0) \geq 0 \). Thus if \( x_0 \neq 0 \), then \( \delta p(x_0) = p(-x_0)/(2x_0^2) \geq 0 \). In case \( x_0 = 0 \), it is seen from the integral representation (3.1) that \( \delta p(0) = p''(0) \geq 0 \). From now on, we may therefore assume that \( s > 0 \).

2. We first derive an explicit representation of the operator \( \Lambda_s \) (\( s > 0 \)), which allows to check property (M) easily: We claim that

\[
\Lambda_s p(x) = -\frac{1}{2s} p(x) - \frac{1}{4s^2} e^{x^2/4s} \left( \int_{-\infty}^{x} g_{p,x}(t) \, dt - \int_{-x}^{\infty} g_{p,x}(t) \, dt \right) \quad \text{for } p \in \Pi_1,
\]

with \( g_{p,x}(t) = e^{-t^2/4s} (t+x) p(t) \).

This may of course be verified by a (tedious) direct computation of \( \Lambda_s(x^k) \), \( k \in \mathbb{Z}_+ \), and an explicit evaluation of the corresponding integrals on the right side by series expansions of the involved exponentials. We prefer, however, to give a more instructive proof:

Note first that the operators \( D^2 \) and \( \delta \) map even polynomials to even ones and odd polynomials to odd ones again, and that

\[
\delta p(x) = \begin{cases} 
\frac{1}{x} p'(x) & \text{if } p \text{ is even}, \\
\left( \frac{1}{x} p(x) \right)' & \text{if } p \text{ is odd}.
\end{cases}
\]

Now fix \( s > 0 \) and suppose that \( p \in \Pi_1 \) is even. Then the polynomials \( e^{sD^2} p \) and \( q := \Lambda_s p \) are also even, and we obtain the following equivalences:

\[
q = \Lambda_s p \iff \delta(e^{sD^2} p) = e^{sD^2} q \iff p'(x) = e^{-sD^2} (x e^{sD^2} q)(x).
\]
By use of Lemma 3.4, this becomes

\[ p'(x) = xq(x) - 2sq'(x), \] (3.6)

which is a differential equation of type (3.3) for \( q \). Lemma 3.5, together with a further partial integration, now implies that

\[ \Lambda_s p(x) = -\frac{1}{2s} e^{x^2/4s} \int_{-\infty}^{x} e^{-t^2/4s} p'(t) dt \]
\[ = -\frac{1}{2s} p(x) - \frac{1}{4s^2} e^{x^2/4s} \int_{-\infty}^{x} e^{-t^2/4s} tp(t) dt \quad (p \text{ even}). \] (3.7)

In a similar way, we calculate \( q = \Lambda_s p \) for odd \( p \in \Pi^1 \): In this case, \( e^{sD^2} p \) and \( q = \Lambda_s p \) are odd as well, and we have the equivalence

\[ q = \Lambda_s p \iff \frac{d}{dx} \left( \frac{1}{x} e^{sD^2} p(x) \right) = e^{sD^2} q(x). \]

Hence there exists a constant \( c_1 \in \mathbb{R} \) such that

\[ e^{sD^2} p(x) = (c_1 + h(x)), \quad \text{with} \ h(x) = \int_0^x e^{sD^2} q(t) dt. \]

Applying Lemma 3.4 again, we obtain

\[ p(x) = c_1 e^{-sD^2}(x) + xe^{-sD^2} h(x) - 2s e^{-sD^2} h'(x) = c_1 x + xe^{-sD^2} h(x) - 2sq(x). \] (3.8)

In order to determine \( e^{-sD^2} h \), note that

\[ \frac{d}{dx} \left( e^{-sD^2} h(x) \right) = e^{-sD^2} h'(x) = q(x). \]

Consequently, there exists a constant \( c_2 \in \mathbb{R} \) such that

\[ e^{-sD^2} h(x) = c_2 + \int_0^x q(t) dt. \] (3.9)

Now write \( p(x) = x P(x) \) and \( q(x) = x Q(x) \) with even \( P, Q \in \Pi^1 \). Then by (3.8) and (3.9),

\[ P(x) = c_1 + c_2 + \int_0^x tQ(t) dt - 2sQ(x), \]

and therefore

\[ P'(x) = xQ(x) - 2sQ'(x). \]

This is exactly the same differential equation as we had in the even case before, and the transfer of (3.7) gives

\[ \Lambda_s p(x) = -\frac{1}{2s} p(x) - \frac{1}{4s^2} e^{x^2/4s} x \int_{-\infty}^{x} e^{-t^2/4s} p(t) dt \quad (p \text{ odd}). \] (3.10)

If finally \( p \in \Pi^1 \) is arbitrary, then write \( p = p_e + p_o \) with even part \( p_e(x) = (p(x) + p(-x))/2 \) and odd part \( p_o(x) = (p(x) - p(-x))/x \). The combination of (3.7) for \( p_e \) with (3.10) for \( p_o \) then leads to

\[ \Lambda_s p(x) = -\frac{1}{2s} p(x) - \frac{1}{4s^2} e^{x^2/4s} \int_{-\infty}^{x} e^{-t^2/4s} \left( \frac{t + x}{2} p(t) + \frac{t - x}{2} p(-t) \right) dt, \]
3. In order to prove that $\Lambda_s$ satisfies the positive minimum principle (M), define

$$F_p(x) := \int_{-\infty}^{x} g_{p,x}(t) \, dt - \int_{x}^{\infty} g_{p,x}(t) \, dt,$$

for $p \in \Pi^1$ and $x \in \mathbb{R}$.

Now let $p \in \Pi_+^1$ with $p(x_0) = 0$. Then in view of (3.5),

$$\Lambda_s p(x_0) = -\frac{1}{4s^2} e^{x_0^2/4s} F_p(x_0),$$

and it remains to check that $F_p(x_0) \leq 0$. For this, we rewrite $F_p$ as

$$F_p(x) = \int_{-|x|}^{-\infty} g_{p,x}(t) \, dt - \int_{\infty}^{\infty} |x| g_{p,x}(t) \, dt.$$

As $p$ is nonnegative, the sign of $g_{p,x}(t)$ coincides with the sign of $(x + t)$ for all $x, t \in \mathbb{R}$. This shows that in fact, $F_p(x) \leq 0$ for all $x \in \mathbb{R}$, which completes the proof.

**Proof of Proposition 3.2.** For fixed $s \geq 0$, the operators $(e^{-s\Delta} e^{t L_k e^{s\Delta}})_{t \geq 0}$ form a semigroup on $\Pi^N$ with generator $e^{-s\Delta} L_k e^{s\Delta}$. According to Theorem 2.3, it therefore suffices to prove that this generator satisfies the positive minimum principle (M) on $\Pi^N$. With the notation of (3.2), we have

$$L_k = 2 \sum_{\alpha \in R_+} k(\alpha) \delta_\alpha \quad \text{with} \quad k(\alpha) \geq 0 \quad \text{for all} \quad \alpha \in R_+.$$

It is therefore enough to make sure that each of the operators

$$\rho^s_\alpha := e^{-s\Delta} \delta_\alpha e^{s\Delta} \quad (\alpha \in R_+)$$

satisfies (M). (Here the assumption $k \geq 0$ is crucial!) Now fix $\alpha \in R_+$. An easy calculation shows that $\delta_\alpha$ and hence also $\rho^s_\alpha$ is rotation-equivariant, i.e.

$$g \circ \rho^s_\alpha \circ g^{-1} = \rho^s_{g(\alpha)} \quad \text{for} \quad g \in SO(N, \mathbb{R}).$$

We may therefore assume that $\alpha = \sqrt{2} e_1 = (\sqrt{2}, 0, \ldots, 0)$. As $\delta_{\sqrt{2} e_1}$ obviously commutes with each of the partial derivatives $\partial_2, \ldots, \partial_N$ on $\mathbb{R}^N$, we obtain

$$\rho^s_{\sqrt{2} e_1} = e^{-s\partial_1^2} \delta_{\sqrt{2} e_1} e^{s\partial_1^2}.$$

But this operator acts on the first variable only, namely via $\Lambda_s$:

$$\rho^s_{\sqrt{2} e_1} p(x_1, \ldots, x_N) = \Lambda_s p_{x_2, \ldots, x_N}(x_1), \quad \text{where} \quad p_{x_2, \ldots, x_N}(x_1) := p(x_1, x_2, \ldots, x_N) \quad \text{for} \quad p \in \Pi^N.$$

The assertion now follows from Proposition 3.1. \[\square\]

In order to complete the proof of Theorem 1.1, we employ the following bilinear form on $\Pi^N$ associated with $G$ and $k$, which was introduced in [D3] (for a further discussion, see also [D-J-O]):

$$[p, q]_k := (p(T_k) q)(0) \quad \text{for} \quad p, q \in \Pi^N;$$
here \( p(T_k) \) is the differential-difference operator which is obtained from \( p(x) \) by replacing each \( x_i \) by the corresponding Dunkl operator \( T_{e_i}(k) \). The case \( k = 0 \) will be distinguished by the notation \( p(\partial) \). Notice that \([p,q]_k = 0 \) for \( p \in \mathcal{P}_n^N \) and \( q \in \mathcal{P}_m^N \) with \( n \neq m \). It was shown in [D3] that for \( k \geq 0 \) and for all \( p,q \in \Pi^N \),

\[
[p,q]_k = c_k \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x) e^{-\Delta_k/2} q(x) e^{-|x|^2/2} w_k(x) dx, \tag{3.11}
\]

where \( w_k \) is the weight function defined in (1.1) and \( c_k := \left( \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x) dx \right)^{-1} \). We remark that (3.11) can also be proved in a completely independent way by using certain biorthogonal polynomial systems (Appell characters and cocharacters) in \( L^2(\mathbb{R}^N, e^{-|x|^2/2} w_k(x) dx) \); see [R-V1].

Another useful identity for \([\ldots,\ldots]_k \) is

\[
[V_k p,q]_k = [p,q]_0 \quad \text{for all } p,q \in \Pi^N. \tag{3.12}
\]

In fact, for \( p,q \in \mathcal{P}_n^N \) with \( n \in \mathbb{Z}_+ \), one obtains

\[
[V_k p,q]_k = [q,V_k p]_k = q(T_k)(V_k p) = V_k(q(\partial)p) = q(\partial)(p) = [p,q]_0;
\]

here the characterizing properties of \( V_k \) and the fact that \( q(\partial)p \) is a constant have been used. For general \( p,q \in \Pi^N \), (3.12) then follows from the orthogonality of the spaces \( \mathcal{P}_n^N \), \( n \in \mathbb{Z}_+ \), with respect to both scalar products.

Finally, we shall need the following positivity criterion for polynomials:

**3.6. Lemma.** Let \( \alpha > 0 \) and suppose that \( h \in C_b(\mathbb{R}^N) \) satisfies

\[
\int_{\mathbb{R}^N} h(x) p(x) e^{-\alpha|x|^2} w_k(x) dx \geq 0 \quad \text{for all } p \in \Pi^N_+. \tag{3.13}
\]

Then \( h(x) \geq 0 \) for all \( x \in \mathbb{R}^N \).

**Proof.** For abbreviation, put

\[
dm_k(x) := e^{-\alpha|x|^2} w_k(x) dx \in M_b^+(\mathbb{R}^N)
\]

1. We shall use that \( \Pi^N \) is dense in \( L^2(\mathbb{R}^N, dm_k) \). This is proved (with \( \alpha = 1/2 \)) in Theorem 2.5 of [D4] by referring to a well-known Theorem of Hamburger for one-dimensional distributions, but it can also be seen directly as follows: Suppose that \( \Pi^N \) is not dense in \( L^2(\mathbb{R}^N, dm_k) \). Then there exists some \( f \in L^2(\mathbb{R}^N, dm_k) \), \( f \neq 0 \), with \( \int_{\mathbb{R}^N} f p dm_k = 0 \) for all \( p \in \Pi^N \). Now consider the measure \( \nu := f m_k \in M_b(\mathbb{R}^N) \) and its (classical) Fourier-Stieltjes transform

\[
\widehat{\nu}(\lambda) = \int_{\mathbb{R}^N} e^{-i\langle \lambda,x \rangle} d\nu(x) = \int_{\mathbb{R}^N} f(x) e^{-i\langle \lambda,x \rangle} dm_k(x).
\]

As \( x \mapsto e^{||\lambda||x} \) belongs to \( L^2(\mathbb{R}^N, dm_k) \) for all \( \lambda \in \mathbb{R}^N \), the dominated convergence theorem yields

\[
\widehat{\nu}(\lambda) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^N} f(x) \langle \lambda,x \rangle^n dm_k(x) = 0.
\]
By injectivity of the Fourier-Stieltjes transform on $M_b(\mathbb{R}^N)$, it follows that $\nu = 0$ and hence $f = 0$ a.e., a contradiction.

2. Now assume that $h \in C_b(\mathbb{R}^N)$ satisfies (3.13). In order to prove $h \geq 0$, it suffices to check that

$$\int_{\mathbb{R}^N} fh \, dm_k \geq 0 \quad \text{for all } f \in C_c^+(\mathbb{R}^N).$$

(3.14)

For this, let $f \in C_c^+(\mathbb{R}^N)$ and $\epsilon > 0$. By density of $\Pi^N$ in $L^2(\mathbb{R}^N, dm_k)$ there exists some $p = p_\epsilon \in \Pi^N$ with $\|\sqrt{f} - p\|_{2,m_k} < \epsilon$. With $M := \|h\|_{\infty,\mathbb{R}^N}$ it follows that

$$\left| \int_{\mathbb{R}^N} fh \, dm_k - \int_{\mathbb{R}^N} p^2h \, dm_k \right| \leq M \int_{\mathbb{R}^N} |f - p^2| \, dm_k \leq M \epsilon \cdot 2 \|\sqrt{f}\|_{2,m_k} = M \epsilon \cdot \sqrt{2} \|f\|_{2,m_k} \epsilon,$$

which tends to 0 with $\epsilon \to 0$. This proves (3.14) and yields the assertion.

The proof of Theorem 1.1 is now easily accomplished:

**Proof of Theorem 1.1.** Combining formulas (3.12) and (3.11), we obtain for all $p, q \in \Pi^N$ the identity

$$c_k \int_{\mathbb{R}^N} e^{-\Delta k/2}(V_k p)(x) e^{-\Delta k/2} q(x) e^{-|x|^2/2} w_k(x) \, dx = c_0 \int_{\mathbb{R}^N} e^{-\Delta/2} p(x) e^{-\Delta/2} q(x) e^{-|x|^2/2} \, dx.$$

As $e^{-\Delta k/2}(V_k p) = V_k(e^{-\Delta/2} p)$, and as we may also replace $p$ by $e^{\Delta/2} p$ and $q$ by $e^{\Delta k/2} q$ in the above identity, it follows that for all $p, q \in \Pi^N$

$$c_k \int_{\mathbb{R}^N} V_k p(x) q(x) e^{-|x|^2/2} w_k(x) \, dx = c_0 \int_{\mathbb{R}^N} p(x) e^{-\Delta/2} e^{\Delta k/2} q(x) e^{-|x|^2/2} \, dx.$$

Corollary 3.3 now implies that

$$\int_{\mathbb{R}^N} V_k p(x) q(x) e^{-|x|^2/2} w_k(x) \, dx \geq 0 \quad \text{for all } p, q \in \Pi^N_+.$$

For fixed $p \in \Pi^N_+$ we may therefore apply Lemma 3.6 with, let us say, $\alpha = 1/4$, to the function $h(x) := e^{-|x|^2/4} V_k p(x) \in C_b(\mathbb{R}^N)$. This shows that $V_k p(x) \geq 0$ for all $x \in \mathbb{R}^N$ and yields the assertion.

4 Proof of the main result

We start this section with a short discussion of the algebras $A_r$ ($r > 0$) introduced in [3]. We should first point out that these are complex algebras, whereas in [2] only series of real-valued polynomials are considered. It is easily checked that $A_r$ is a subalgebra of the space of functions which are continuous on the ball $K_r$ and real analytic in its interior: in fact, for real-valued $p \in \mathcal{P}^N_n$ and $i = 1, \ldots, N$ the inequality $|\partial_i p|_{\infty,K_1} \leq n \|p\|_{\infty,K_1}$ holds as a consequence of the Van der Corput-Schaake inequality, see [3]; this allows to differentiate $f = \sum_{n=0}^{\infty} f_n \in A_r$ termwise and arbitrary often. The topology of $A_r$ is stronger than the topology induced by the uniform norm on $K_r$. Notice also that $A_r$ is not closed with respect to $\|\cdot\|_{\infty,K_r}$ and that $A_r \subseteq A_s$ with $\|\cdot\|_{A_r} \geq \|\cdot\|_{A_s}$ for $s \leq r$. The following observation is straightforward:
4.1. **Lemma.** \((A_r, \|\cdot\|_{A_r})\) is a commutative Banach-\(*\)-algebra with the pointwise multiplication of functions, complex conjugation as involution, and with unit 1.

**Proof.** To show completeness, let \((f^m)_{m \in \mathbb{Z}_+}\) be a Cauchy sequence in \(A_r\). Then for \(\epsilon > 0\) there exists an index \(m(\epsilon) \in \mathbb{Z}_+\) such that
\[
\sum_{n=0}^{\infty} \|f^m_n - f^{m'}_n\|_{\infty, K_r} < \epsilon \quad \text{for } m, m' > m(\epsilon).
\] (4.1)
In particular, for each degree \(n\) the homogeneous parts \((f^m_n)_{m \in \mathbb{Z}_+}\) converge uniformly on \(K_r\), and hence within \(P^N_r\) to some \(g_n \in P^N_r\). It further follows from (4.1) that
\[
\sum_{n=0}^{\infty} \|g_n - f^m_n\|_{\infty, K_r} < \epsilon \quad \text{for } m > m(\epsilon).
\]
Therefore \(g := \sum_{n=0}^{\infty} g_n\) belongs to \(A_r\) with \(\|g - f^m\|_{A_r} \to 0\) for \(m \to \infty\). It is also easily checked by a Cauchy-product argument that \(A_r\) is an algebra with \(\|fg\|_{A_r} \leq \|f\|_{A_r} \cdot \|g\|_{A_r}\) for all \(f, g \in A_r\). The rest is clear. \(\square\)

We next determine the symmetric spectrum of \(A_r\), i.e. the subspace of the spectrum \(\Delta(A_r)\) given by
\[
\Delta_S(A_r) := \{ \varphi \in \Delta(A_r) : \varphi(\overline{f}) = \overline{\varphi(f)} \quad \text{for all } f \in A_r\}.
\]
As usual, \(\Delta_S(A_r)\) is equipped with the Gelfand-topology. For \(x \in K_r\) the evaluation homomorphism at \(x\) is defined by \(\varphi_x : A_r \to \mathbb{C}, \varphi_x(f) := f(x)\).

4.2. **Lemma.** \(\Delta_S(A_r) = \{ \varphi_x : x \in K_r \}\), and the mapping \(x \mapsto \varphi_x\) is a homeomorphism from \(K_r\) onto \(\Delta_S(A_r)\).

**Proof.** It is obvious that \(\varphi_x\) belongs to \(\Delta_S(A_r)\) for each \(x \in K_r\), with \(\varphi_x \neq \varphi_y\) for \(x \neq y\), and that the mapping \(x \mapsto \varphi_x\) is continuous on \(K_r\). It remains to show that each \(\varphi \in \Delta_S(A_r)\) is of the form \(\varphi_x\) with some \(x \in K_r\). To this end, put \(\lambda_i := \varphi(x_i)\) for \(i = 1, \ldots, N\). By symmetry of \(\varphi\) we have \(\lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N\). Moreover,
\[
|\lambda|^2 = \varphi(|x|^2) \leq \|x|^2\|_{A_r} = r^2.
\]
This shows that \(\lambda \in K_r\). By definition of \(\lambda\), the identity \(p(\lambda) = \varphi(p)\) holds for all polynomials \(p \in \Pi^N\). The assertion now follows from the density of \(\Pi^N\) in \((A_r, \|\cdot\|_{A_r})\). \(\square\)

**Proof of Theorem 1.2.** Fix \(x \in \mathbb{R}^N\) and put \(r = |x|\). Then the mapping
\[
\Phi_x : f \mapsto V_k f(x)
\]
is a bounded linear functional on \(A_r\), and Theorem 1.1 implies that it is positive on the dense subalgebra \(\Pi^N\) of \(A_r\), i.e. \(\Phi_x(|p|^2) \geq 0\) for all \(p \in \Pi^N\). Consequently, \(\Phi_x\) is a positive functional on the whole Banach-\(*\)-algebra \(A_r\). Now, by a well-known Bochner-type representation theorem
for positive functionals on commutative Banach-∗-algebras (see e.g. Theorem 21.2 of [F-D]), there
exists a unique measure \( \nu_x \) such that
\[
\Phi_x(f) = \int_{\Delta_S(A_r)} \hat{f}(\varphi) \, d\nu_x(\varphi) \quad \text{for all } f \in A_r,
\]
with \( \hat{f} \) the Gelfand transform of \( f \). Denote by \( \mu_x \) the image measure of \( \nu_x \) under the homeo-
morphism \( \Delta_S(A_r) \to K_r, \varphi_x \to x \). Equation (4.2) then becomes
\[
V_k f(x) = \int_{\{||\xi|| \leq |x|\}} f(\xi) \, d\mu_x(\xi) \quad \text{for all } f \in A_{|x|}.
\]
The normalization \( V_k 1 = 1 \) implies that \( \mu_x \) is a probability measure on \( \{\xi \in \mathbb{R}^N : ||\xi|| \leq |x|\} \).

5 Some consequences and applications

In this final section, we discuss only a short selection of implications which arise from the positivity
of Dunkl’s intertwining operator. We expect that several more useful applications can be found,
and it would of course also be of interest to have an explicit form for \( V_k \) for larger classes of
reflection groups. In the sequel, it is always assumed that \( k \geq 0 \). The most prominent consequence
of Theorem 1.2, as already mentioned in the introduction, is positive-definiteness of Dunkl’s
generalized exponential kernel. Up to now, this has been known only in the special cases where
positivity of \( V_k \) is visible from an explicit integral representation. In particular, for the reflection
group \( G = \mathbb{Z}_2 \) on \( \mathbb{R} \) and multiplicity parameter \( k \geq 0 \), formula (1.2) shows that
\[
K_G(x, iy) = c_k \int_{-1}^{1} e^{ity} (1 - t)^{k-1}(1 + t)^k \, dt = e^{ity} \text{I}_1 F_1(k, 2k + 1, -2ity).
\]
The following general result is an immediate consequence of Theorem 1.2 with \( f(x) = e^{(x,z)} \) and
Bochner’s theorem:

5.1. Proposition. For each \( z \in \mathbb{C}^N \), the function \( x \mapsto K_G(x, z) \) has the Bochner-type represen-
tation
\[
K_G(x, z) = \int_{\mathbb{R}^N} e^{\xi \cdot z} \, d\mu_x(\xi); \quad (5.1)
\]
here the \( \mu_x \) are the representing measures from Theorem 1.2. In particular, \( K_G(x, y) > 0 \) for all
\( x, y \in \mathbb{R}^N \), and for each \( y \in \mathbb{R}^N \) the function \( x \mapsto K_G(x, iy) \) is positive-definite on \( \mathbb{R}^N \).

5.2. Corollary. For each \( y \in \mathbb{R}^N \), the generalized Bessel function \( x \mapsto J_G(x, iy) \) is positive-
definite on \( \mathbb{R}^N \).
We mention that for the group \( G = S_3 \) this corollary follows from the integral representations in \( D_3 \). The following useful estimates of \( K_G \) are immediate from \( [l,1] \); they partially sharpen those of \( [l,1] \). (Notice that our proof of Theorem 1.2 did not involve any results from \( [l,1] \).)

**5.3. Corollary.** Let \( \nu \in \mathbb{Z}_+^N \) and \( |\nu| = \nu_1 + \ldots + \nu_N \). Then for all \( x \in \mathbb{R}^N \) and \( z \in \mathbb{C}^N \),

\[
|\partial_x^\nu K_G(x,z)| \leq |x|^{|\nu|} e^{(\gamma| Re z|}$.

(5.2)

In particular, \(|K_G(x, iy)| \leq 1\) for all \( x, y \in \mathbb{R}^N \).

From the integral representation (5.1) we also obtain further knowledge about the support of the representing measures \( \mu_x \):

**5.4. Corollary.** The measures \( \mu_x \), \( x \in \mathbb{R}^N \), satisfy

(i) \( \text{supp} \mu_x \) is contained in \( \text{co} \{ gx, g \in G \} \), the convex hull of the orbit of \( x \) under \( G \).

(ii) \( \text{supp} \mu_x \cap \{ gx, g \in G \} \neq \emptyset \).

**Proof.** (i) follows from Corollary 3.3 of \( [l,1] \). For the proof of (ii) it is therefore enough to show that

\[
\text{supp} \mu_x \cap \{ \xi \in \mathbb{R}^N : |\xi| = |x| \} \neq \emptyset.

Suppose in the contrary that \( \text{supp} \mu_x \cap \{ \xi \in \mathbb{R}^N : |\xi| = |x| \} = \emptyset \) for some \( x \in \mathbb{R}^N \). Then there exists a constant \( \sigma \in (0,1] \) such that \( \text{supp} \mu_x \subseteq \{ \xi \in \mathbb{R}^N : |\xi| \leq \sigma |x| \} \). This leads to the estimation

\[
K_G(x, y) = \int_{\{ |\xi| \leq \sigma |x| \}} e^{\langle \xi, y \rangle} d\mu_x(\xi) \leq e^{|x||y|}
\]

(5.3)

for all \( y \in \mathbb{R}^N \). On the other hand, Theorem 3.2 of \( D_3 \) with \( z = 0 \) says that

\[
c_k \int_{\mathbb{R}^N} K_G(x, y) e^{-|x|^2 + |y|^2}/2 w_k(y) dy = 1.
\]

In view of (5.3), and as \(|w_k(y)| \leq 2^\gamma |y|^{2\gamma} \) with \( \gamma := \sum_{\alpha \in R_+} k(\alpha) \), it follows that

\[
1 \leq 2^\gamma c_k \int_{\mathbb{R}^N} e^{-|x|^2 + |y|^2}/2 e^{\sigma|y| |y|^{2\gamma}} dy
\]

\[
= d_k \int_0^\infty e^{-r^2} e^{(\sigma-1)r^2} r^{2\gamma + N - 1} dr
\]

\[
\leq d_k \int_{-\infty}^\infty e^{-r^2/2} e^{(\sigma-1)(|x| + |y|)|x|} (|x| + |y|)^{2\gamma + N - 1} dr,
\]

(5.4)

with some constant \( d_k > 0 \) (which is independent of \( x \)). The integrand in (5.4) is obviously majorized by \( C \cdot e^{-r^2/2} \) with a constant \( C \) independent of \( r \) and \( x \), and converges pointwise to \( e^{-r^2/2} \) with \( |x| \to \infty \). The dominated convergence theorem now implies that the integral (5.4) tends to 0 with \( |x| \to \infty \), a contradiction.

\( \square \)
We give two further applications:

1. **Summability of orthogonal series in generalized harmonics.** The study of generalized spherical harmonics associated with a finite reflection group and a multiplicity function $k \geq 0$ was one of the starting points of Dunkl’s theory in [D3] and has been extended in [X2] and [X3]. Many results for classical spherical harmonics carry over to these spherical $k$-harmonics, where harmonicity is now meant with respect to $\Delta_k$. In particular, there is a natural decomposition of $P^N_n|_{S^{N-1}}$ into subspaces of $k$-spherical harmonics, which are orthogonal in $L^2(S^{N-1}, w_k(x)dx)$. In [X2], Cesàro summability of generalized Fourier expansions with respect to an orthonormal basis of spherical $k$-harmonics is studied. Recall that a sequence $\{s_n\}_{n \in \mathbb{Z}^+}$ is called Cesàro summable of order $\delta$ to $s$, for short, $(C, \delta)$-summable to $s$, if

$$\frac{1}{n+\delta} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k \to s \quad \text{with} \quad n \to \infty.$$ 

The following result is proven in [X2] under the requirement that the intertwining operator $V_k$ is positive on $\Pi^N$; Theorem 1.1 now assures its validity for all $k \geq 0$:

**5.5. Theorem.** Let $f : S^{N-1} \to \mathbb{C}$ be continuous, and let $\{s_n\}$ denote the sequence of partial sums in the expansion of $f$ as a Fourier series with respect to a fixed orthonormal basis of spherical $k$-harmonics. Then $\{s_n\}$ is uniformly $(C, \delta)$-summable over $S^{N-1}$, provided $\delta > \gamma + N/2 - 1$ with $\gamma = \sum_{\alpha \in R_+} k(\alpha)$.

2. **Generalized moment functions.** Recently, in [R-V2] a concept of Markov kernels and Markov processes which are homogeneous with respect to a given Dunkl transform has been developed. In this context, generalized moment functions on $\mathbb{R}^N$ provide a useful tool. They generalize the classical monomial moment functions $m_\nu(x) = x^\nu$, $\nu \in \mathbb{Z}_+^N$ and are defined as the unique analytic coefficients in the expansion

$$K_G(x, y) = \sum_{\nu \in \mathbb{Z}_+^N} \frac{m_{k, \nu}(x)}{\nu!} y^\nu \quad (x \in \mathbb{R}^N, y \in \mathbb{C}^N).$$

From the definition of $K_G$ it follows that

$$m_{k, \nu}(x) = V_k(x^\nu) \quad \text{for} \quad \nu \in \mathbb{Z}_+^N,$$

and Theorem 1.2 in particular implies the following useful relations for the generalized moment functions, which are obvious only in the classical case (again, we assume $k \geq 0$):

$$|m_{k, \nu}(x)| \leq |x|^{|\nu|} \quad \text{and} \quad 0 \leq m_{k, \nu}(x)^2 \leq m_{k, 2\nu}(x) \quad \text{for all} \quad x \in \mathbb{R}^N, \nu \in \mathbb{Z}_+^N.$$

The first inequality is clear from the support properties of the measures $\mu_x$ while the second one follows from Jensen’s inequality. Among the applications of these moments, we mention the construction of martingales from Dunkl-type Markov processes; for details, we refer to [R-V2].

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