Collective Modes in Two-band Superconductors.

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We analyze collective modes in two-band superconductors in the dirty limit. It is shown that these modes exist at all temperatures $T$ below $T_c$ provided the frequency of the modes is higher than the inelastic scattering rate and lower than the energy gaps $\Delta_{a,b}$. At low temperatures these modes are related to counterphase oscillations of the condensate currents in each band. The spectrum of the collective oscillations is similar to the spectrum of the Josephson "plasma" modes in a tunnel Josephson junction but the velocity of the mode propagation in the case under consideration is much lower. At higher temperatures ($\Delta_b < T < T_c$) the spectrum consists of two branches. One of them is gapless (sound-like) and the second one has a threshold that depends on coupling between the bands. We formulate the conditions under which both types of collective modes can exist. The spectrum of the collective modes can be determined by measuring the I-V characteristics of a Josephson junction in a way as it was done by Carlson and Goldman [12].

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I. INTRODUCTION

The conventional BCS theory has been developed for a single-band metal with an attractive interaction between electrons with opposite spins and momenta (s-wave, singlet pairing) [1] and describes well most low-$T_c$ superconductors. The universality of the description of the superconductors is a consequence of an assumption about a simple shape of the Fermi surface.

However, some superconducting materials have a rather complicated band structure. For example, there is a consensus that the recently discovered new superconductor MgB$_2$ ($T_c \approx 40$ K) is a two-band superconductor [2]. In contrast to conventional BCS superconductors, two-band superconductors may have two different order parameters $\Delta_{a,b}$ (we label the bands by subscripts $a$ and $b$) and an additional degree of freedom - the phase difference between the order parameters: $\varphi = \chi_a - \chi_b$. This is a new variable that has to be accounted for in the proper theory of such a superconductivity. Naturally, one can expect new phenomena in two-band superconductors related to this new degree of freedom.

One of the examples of this kind are $\varphi$-phase solitons predicted in [3]. On the basis of the Ginzburg-Landau (GL) functional it was shown that these solitons are described by the sine-Gordon equation. A Gedanken-experiment that allows the observation of these solitons was suggested in [4]. The authors considered a two-band superconductor in contact with a normal metal N. Using a version of the time-dependent GL equation, they analyzed a current through the S/N interface and showed that under certain conditions the phase solitons may be created in the superconductor. Equation describing dynamics of these solitons is similar to the one for a dissipative Josephson junction. In Ref. [8] an analogy between the Ginzburg-Landau functional for two-band superconductors and for some models in the particle physics (an extended version of Faddeev’s nonlinear $\sigma$ model) was used and, on this basis, topologically different vortices in two-gap superconductors have been predicted. In Ref.[8] the critical current in these superconductors was calculated.

Another effect that can arise in two-band superconductors is weakly damped oscillations of the phase difference $\varphi$ of the order parameters or, in other words, collective modes (CMs) related to oscillations of the phase $\varphi$ in space and time. It is known that in single-band superconductors CMs can exist only at temperatures close to the critical temperature $T_c$ (see reviews [8, 9]). In these modes the condensate current $j_S$ and the quasiparticle current $j_q$ oscillate. Because the variation of the total current density $\delta j$ should be zero due to the quasineutrality condition ($\delta j = \delta j_S + \delta j_q = 0$), the oscillations of $\delta j_S$ are accompanied by counter-phase oscillations of the quasiparticle current $\delta j_q \approx \sigma E$. The phase of the order parameter also oscillates but the amplitude of the order parameter $\Delta$ remains constant. These phase modes have an acoustic spectrum $\omega \sim kv_{cm}$ and exist in a sub-gap region ($\Delta^2/T < \omega \ll \Delta, 1/\tau$; where $\tau$ is the elastic scattering time) in impure superconductors. The CMs have been observed by Carlson and Goldman [12] and explained theoretically in [10, 11].

It is clear that CMs cannot exist in conventional superconductors at zero temperature because in this limit only one degree of freedom, namely, the condensate current $j_S$ exists. Any oscillations of the current density $j_S$ with not very high frequencies would lead to violation of the charge neutrality. In contrast, in two-band superconductors, even at zero temperature $T$ there are two degrees of freedom: the condensates currents $j_{S_{a,b}}$ in each band that can oscillate in counterphase such that the total current density remains constant: $\delta j_S(r,t) = \delta j_{S_a}(r,t) + \delta j_{S_b}(r,t) = 0$. This is
similar to what happens in layered superconductors \[13\] where the condensate currents in different layers oscillate in counterphase. The CMs in the two-band superconductors at zero (or low) temperature are related to oscillations of the phase difference $\varphi$. These CMs are similar to the Leggett mode that can be excited in superfluid $He^{3} \[14\]$. Theoretically, the Leggett-type CMs in two-band superconductors at zero temperature were studied in \[15\], and the influence of the CMs on the Josephson effect in $S_{1}/I/S_{2}$ was analyzed in \[16\] (here $S_{1,2}$ is a single- and two-band superconductor). This Leggett-type CM has a spectrum

$$\Omega^{2} = \Omega_{0}^{2} + v_{L}^{2}k^{2}, \quad (1)$$

which is typical for the Josephson tunnel junction, where $\Omega_{0}$ is a threshold frequency, $v_{L}$ is a velocity of this CM (see Sec. III). The similarity between the Josephson “plasma waves” and the Leggett-type CM in two-band superconductors is quite natural because the coupling between different bands in two-band superconductors looks like the Josephson coupling between superconductors in a Josephson tunnel junction.

According to estimates carried out in Ref. \[12\], the energy corresponding to the threshold frequency $\Omega_{0}$ in $MgB_{2}$ is higher than the smaller superconducting gap. This corresponds to a strong damping of the CMs in this two-band superconductor. Therefore it would be of interest to investigate under what conditions the gapless Carlson-Goldman CMs can propagate in two-band superconductors and this is the subject of the present paper.

The paper is organized as follows. In Sec.II we formulate a model of two-band superconductors that will be used in calculations and present microscopic equations for quasiclassical Green’s functions. These equations determine the spatial and temporal behavior of the retarded (advanced) Green’s functions $\hat{g}_{i}^{R(A)}(t,t';r)$, as well as of the Keldysh function $\hat{g}_{i}^{K}(t,t';r)$. The functions $\hat{g}_{i}^{R(A)}$ and $\hat{g}_{i}^{K}$ are matrices in the particle-hole (Gor’kov-Nambu) space. Using these equations we find a linear response $\delta g_{i}^{R(A)}$ and $\delta g_{i}^{K}$ to small perturbations of the electric field and condensate velocities in both the bands. This gives us possibility to find the spectrum of the CMs at arbitrary temperatures. Note that the spectrum of CMs cannot be obtained from the generalized GL equations used in \[4\]. In Sec.IV we analyze a method that may enable one to observe the CMs.

**II. MODEL AND BASIC EQUATIONS**

We start our discussion considering a simple model of a two-band superconductor and deriving microscopic equations for quasiclassical Green’s functions. Using these equations we can obtain equations for macroscopic quantities and calculate the spectrum of CMs. We restrict ourselves with the dirty limit assuming that the elastic impurity scattering time $\tau$ is sufficiently small: $\Delta_{a,b}\tau \ll 1$. In the equilibrium case these equations can be reduced to an Usadel-like equation used in \[17\].

The Hamiltonian of the considered two-band superconductor has the form (see for example \[18\])

$$H = H_{a} + H_{b} - \sum_{\{p,q;i,k\}} \{V_{i,k}(q)\psi_{i,p+q}^\dagger \psi_{i,p-q}^\dagger \psi_{k,-p} \psi_{k,p+q} + c.c.\}, \quad (2)$$

with

$$H_{a} = \sum_{\{p,q\}} \{\psi_{ap}^\dagger v_{a}(p-p_{a})\psi_{ap}^\dagger + \psi_{ap[p+q]}^\dagger V_{a,imp}(q) + V_{a}(q)\psi_{ap}\} \quad (3)$$

where the third term in Eq.(2) describing the electron-electron interaction leads to the superconductivity; $p,q$ are momenta (strictly speaking, we must assign to $p$ also a spin index $\sigma$ but we omit it here for the sake of brevity) and the indices $\{i,k\}$ numerate the bands $\{a,b\}$. The first two terms are one-particle Hamiltonians for each band that include the kinetic energy $s_{a,b}(p) = v_{a,b}(p - p_{a,b})$ counted from the Fermi level and the terms describing an elastic impurity scattering. Beside a short-range potential $V_{a,imp}$ due to impurities, $H_{a}$ contains also a long-range self-consistent potential $V_{a}$ due to coulomb interaction. The Hamiltonian for the $b$ band, $H_{b}$, is obtained from $H_{a}$ by replacing subindices $a \rightarrow b$. We neglect the interband impurity scattering (arguments supporting this assumption have been given in \[19\]).

The derivation of the equations for the quasiclassical Green’s functions is carried out in a standard way \[20\], \[21\]. However, the presence of the two bands makes the situation more complicated. In order to avoid unnecessary technical difficulties, we make several assumptions: a) in the BCS theory we use the mean-field approximation representing the product of four $\psi$ operators in the form $\Delta_{k}\psi_{k,-p}^\dagger \psi_{k,p+q}^\dagger + \Delta_{l}\psi_{l,-p}^\dagger \psi_{l,p+q}^\dagger$, b) we neglect a change in the single-electron spectrum due to a possible tunneling between the bands because the strong impurity scattering assumed here...
destroys such a change, c) we neglect terms corresponding to pairing of electrons from different bands \((\psi_{i}^{\dagger} - p \psi_{k,p'q}^{\dagger})\) (such a pairing was taken into account in [22] where the possibility of triplet pairing in a clean layered superconductor was analyzed). With these assumptions used also in previous works, we can derive a microscopic equation for the matrix quasiclassical Green’s functions \(\hat{g}\) in the same way as for a one-band superconductor [21]. This equation in the dirty limit \((\tau \Delta_{a,b} \ll 1)\) has the standard form

\[
- i D_{a} \nabla (\hat{g} \nabla \hat{g})_{a} + i [\hat{\tau}_{3} \partial \hat{g}_{a} / \partial t + \partial \hat{g}_{a} / \partial t' \hat{\tau}_{3}] + [\hat{\Delta}_{a}, \hat{g}_{a}] + s_{a}[\hat{\Delta}_{b}, \hat{g}_{a}] - e [V(t)\hat{g} - \hat{g} V(t')] = 0, \tag{4}
\]

where \(D_{a} = (v_{F}l)_{a}/3\) is the diffusion coefficient in the \(a\) band, \(\hat{g}_{a}(r,t,t')\) is a 4x4 matrix depending on the coordinate \(r\) and two times \(t\) and \(t'\). The elements of this matrix are the retarded (advanced) matrix Green’s functions \(\hat{g}^{R(A)}\) (elements (11) and (22)) and the matrix Keldysh function \(\hat{g}^{K}\) (element (12)). The parameter \(s_{a} = V_{ab}/V_{b}\) determines the strength of the coupling between superconducting pairing in the \(a\) and \(b\) bands (this type of pairing for the two-band superconductors was suggested earlier in [23]). The same equation as Eq. (4) is valid for the \(b\) band provided the subscripts are exchanged, \(a \sim b\).

Eq.(4) is complemented by the normalization condition

\[
\hat{g}_{a}(t,t_{1}) \circ \hat{g}_{a}(t_{1},t') = \delta(t - t') \tag{5}
\]

and the self-consistency equation

\[
\hat{\Delta}_{a,b} = \lambda_{a,b} \hat{f}_{a,b}(t,t;r) \tag{6}
\]

where \(\lambda_{a,b} = (V_{F}v)_{a,b}\) are the coupling constant and the density of states in each band. If we wrote down Eq.(1) for the retarded (advanced) Green’s functions in the Matsubara representation, we would obtain a generalized Usadel equation. Such an equation was used by Koshelev and Golubov [17]. Note that our definition of the order parameter differs from the one \((\Delta_{a})_{KG}\) used in Ref. [17] and the correspondence between both the definitions is given by \((\Delta_{a})_{KG} = \Delta_{a} + s_{a} \Delta_{b}\).

The current density \(j_{a,b}\) in each band is expressed in terms of the Keldysh matrix as

\[
j_{a,b}(t,r) = (\pi/4) \sigma_{a,b} T r \{[\hat{\tau}_{3} \hat{g}^{R}(t,t_{1};r) \nabla \hat{g}^{K}(t_{1},t;r) + \hat{g}^{K}(t,t_{1};r) \nabla \hat{g}^{A}(t_{1},t;r)]_{a,b}\} \tag{7}
\]

Our aim is to find the response \(\delta \hat{g}_{a,b}\) of the system to a perturbation of the electric potential \(V(r,t)\) and \(\nabla \chi_{a,b}\). To be more precise, we are interested in the response to perturbations of the gauge-invariant potential \(\mu_{a,b}\) and the condensate momentum \(Q_{a,b}\)

\[
\mu_{a,b} = eV + (1/2) \partial \chi_{a,b}/\partial t; \quad Q_{a,b} = (1/2)[\nabla \chi_{a,b} - (2\pi/\Phi_{0}) \mathbf{A}] \tag{8}
\]

where \(\Phi_{0} = hc/2e\) is the magnetic flux quantum. These are the responses that enter physical quantities. One more quantity \(s_{a}[\hat{\Delta}_{b}, \hat{g}_{a}]\) will also be considered as a perturbation with \(\Delta_{b}\) and \(\hat{g}_{a}\) equal to their equilibrium values.

In equilibrium the Keldysh function \(\hat{g}^{K}_{a,b}(\epsilon)\) equals

\[
\hat{g}^{K}_{a,b}(\epsilon) = (\hat{g}^{R}(\epsilon) - \hat{g}^{A}(\epsilon))_{a,b} \tanh(\epsilon \beta) \tag{9}
\]

where \(\hat{g}^{R(A)}_{a,b}(\epsilon) = [\hat{\tau}_{3} \hat{g}_{a,b}(\epsilon) + i \hat{\tau}_{2} f_{a,b}(\epsilon)]^{R(A)}, f_{a,b}(\epsilon) = \Delta_{a,b}/\xi_{a,b}^{R(A)} = (\Delta_{a,b}/\epsilon) g^{R(A)}_{a,b}(\epsilon), \xi_{a,b}^{R(A)}(\epsilon) = \sqrt{(\epsilon + i0)^{2} - \Delta^{2}_{a,b}}\) and \(\beta = (1/2T)\). As usual, the quasiclassical functions \(g\) and \(f\) stand for the normal and condensate quasiclassical Green functions and \(\tau_{3}\) are the Pauli matrices in Gorkov-Nambu space. The method of solution we will use is similar to that presented in [8].

First, we single out the phase \(\chi_{a,b}\) using the transformation \(\hat{g}_{a,b} = (\hat{U} \hat{g}_{a,b} \hat{U}^{-1})_{a,b}\), where \(\hat{U}_{a,b} = \exp(i \hat{\tau}_{3} \chi_{a,b}/2)\). Then Eq.(4) for the new matrix \(\hat{g}_{a,b}^{\text{new}}\) acquires the form (we drop the subindex new)

\[
- i D_{a} \nabla (\hat{g} \nabla \hat{g})_{a} + i [\hat{\tau}_{3} \partial \hat{g}_{a} / \partial t + \partial \hat{g}_{a} / \partial t' \hat{\tau}_{3}] + [\hat{\Delta}_{a}, \hat{g}_{a}] + s_{a}[\hat{\Delta}_{b}, \hat{g}_{a}] - [\mu(t)\hat{g} - \hat{g} \mu(t')]_{a} + D_{a} \nabla \cdot Q_{a} \hat{g}_{a}[\hat{\tau}_{3}, \hat{g}_{a}] + i D_{a} Q_{a}^{2}[\hat{\tau}_{3}, \hat{g}_{a} \hat{\tau}_{3} \hat{g}_{a}] = 0, \tag{10}
\]
After that we linearize Eq. [10] with respect to perturbations $\delta \hat{g}(t, t'; r) = \hat{g} - \hat{g}_{eq}$ and make the Fourier transformations

$$\delta \hat{g}(\epsilon, \epsilon', k) = \int dt dt' \exp(i\epsilon t - i\epsilon' t') \delta \hat{g}(t, t', k)$$  \hspace{1cm} (11)

where $\delta \hat{g}(t, t', r) \sim \delta \hat{g}(t, t', r) \exp(ikr)$. We represent the perturbations $\mu(t, r)$ and $Q(t, r)$ in the form: $\mu(t, r) \sim \mu(\Omega, k) \exp(ikr - i\Omega t);$ $Q(t, r) \sim Q(\Omega, k) \exp(ikr - i\Omega t)$.

Writing down equations for elements (11) and (22), i.e., for the matrices $\hat{g}^R$ and $\hat{g}^A$, we can obtain the expressions for the perturbations $\delta \hat{g}^R(\epsilon, \epsilon')$ of the retarded Green’s functions

$$\delta \hat{g}^R_a(\epsilon, \epsilon') = \frac{1}{M^R_a}(s_a(\hat{\Delta}_b \delta \hat{g}^R_{ac} + \mu_a(\hat{g}^R_{ae} \hat{g}^R_{ac} - 1) - iD_a kQ_a(\hat{\tau}_3 \hat{g}^R_{ac} - \hat{g}^R_{ac} \hat{\tau}_3)) \}$$  \hspace{1cm} (12)

where $M^R_a(\epsilon, \epsilon') = (\xi^R + \xi^R_a)_a + i k^2 D_a, (\xi^R)_a = \xi^R(\epsilon)$ and $\xi^R_a(\epsilon)$ is defined in Eq.[11].

The matrices of the perturbations $\delta \hat{g}^R_a(\epsilon, \epsilon')$ and $\delta \hat{g}^A_a(\epsilon, \epsilon')$ are determined by Eq.[12] after the permutation of the subscripts: $a \to b$ and $R \to A$. Eq.[12] coincides with a corresponding equation in [8] provided the limit $\Delta \tau << 1$ is taken and $s_a$ is set to zero: $s_a = 0$.

In order to find the perturbation of the Keldysh function $\delta \hat{g}_{a}(\epsilon, \epsilon') \equiv \delta \hat{g}^R_a(\epsilon, \epsilon')$, we represent $\delta \hat{g}_{a}(\epsilon, \epsilon')$ in the form of a sum of a regular $\delta \hat{g}_{reg}$ and anomalous $\delta \hat{g}_{an}$ part

$$\delta \hat{g}_{a}(\epsilon, \epsilon') = (\delta \hat{g}_{reg}(\epsilon, \epsilon') + \delta \hat{g}_{an}(\epsilon, \epsilon'))_a$$  \hspace{1cm} (13)

where $\delta \hat{g}_{reg}(\epsilon, \epsilon') = \delta \hat{g}^R(\epsilon, \epsilon') \tanh(\epsilon') - \tanh(\epsilon \beta) \delta \hat{g}^A(\epsilon, \epsilon')$ (we drop the indices $a, b$). The anomalous part is obtained in a way similar to that in [8]. It has the form

$$\langle \delta \hat{g}_{an}(\epsilon, \epsilon') \rangle_a = \frac{\tanh(\epsilon') - \tanh(\epsilon \beta)}{M_a}{\{\hat{\Delta}_b - \hat{g}^R_{ae} \hat{\Delta}_b \hat{g}^R_{ae} + \mu_a(1 - \hat{g}^R_{ae} \hat{g}^A_{ae}) - iD_a kQ_a(\hat{\tau}_3 \hat{g}^R_{ac} - \hat{g}^R_{ac} \hat{\tau}_3)}$$  \hspace{1cm} (14)

where $M_a(\epsilon, \epsilon') = (\xi^A + \xi^A_a)_a + i k^2 D_a$. The energies $\epsilon, \epsilon'$ in Eqs.[12-14] are equal to $\epsilon = \bar{\epsilon} + \Omega/2, \epsilon' = \bar{\epsilon} - \Omega/2$, where $\bar{\epsilon} = (\epsilon + \epsilon')/2$ and $\Omega$ is the frequency of oscillations.

Having determined the perturbations $\delta \hat{g}_{a}(\epsilon, \epsilon') \equiv \delta \hat{g}^R(\epsilon, \epsilon')$ and $\delta \hat{g}_{a}(\epsilon, \epsilon')$, we can readily derive equations for such macroscopic quantities as $\mu_{a,b}(t, r), \delta \hat{g}_{a,b}(t, r)$, etc., in each band and obtain the spectrum of the CMs.

### III. MACROSCOPIC QUANTITIES. SPECTRUM OF OSCILLATIONS.

As follows from its definition, the condensate momentum $Q_{a,b}$ obeys the equations

$$\partial Q_{a,b}/\partial t = eE + \nabla \mu_{a,b}$$  \hspace{1cm} (15)

In order to obtain an equation for $\mu_{a,b}$, we can use the self-consistency equation [9] written for the phases $\chi_{a,b}$. This means that terms proportional to $\hat{\tau}_1$ in Eq.[9] should be equal to zero. The variation of the current density is found from Eq.[11].

We consider first the case of low temperatures: $T << \Delta_{a,b}$.

a) $T << \Delta_{a,b}$. In this case the main contribution is due to the regular part $\delta \hat{g}_{reg}(\epsilon, \epsilon')$. The anomalous part gives small corrections of the order $\Omega/\Delta_{a,b}$ because we assume that $\Omega/\Delta_{a,b} << 1$.

Let us obtain an equation for $\mu_{a,b}$ using Eq.[9]. Calculating the contribution from the regular part and setting $\epsilon = i\omega + \Omega/2$ and $\epsilon' = i\omega - \Omega/2$ we can transform the integration over $\bar{\epsilon}$ into a sum over the Matsubara frequencies:

$$\int d\bar{\epsilon} \delta \hat{g}^R(\epsilon, \epsilon') \tanh(\epsilon') - \tanh(\epsilon \beta) \delta \hat{g}^A(\epsilon, \epsilon') = (2\pi i)(2T) \sum_\omega \delta \hat{g}^R(\epsilon, \epsilon')$$

Substituting the perturbations $\delta \hat{g}$ into Eq. [10] we get
where \( \Omega^2 = 2(V_{ab}/V_b \nu)\Delta_a \Delta_b, \bar{\nu}_a = \nu_a/(\nu_a + \nu_b) \) is the normalized density-of-states in the \( a \) band, \( \nu = \nu_a + \nu_b \), and \( \varphi = \chi_a - \chi_b \). The same equation with the interchange of indices \( a \leftrightarrow b \) is valid for the \( b \) band (one has to keep in mind that the sign in front of \( \sin \varphi \) changes as a result of this interchange).

Eq. (16) is the continuity equation for the charge of Cooper pairs \( q_a \sim \nu_a \mu_a \) in the \( a \) band because the third term is proportional to the divergence of the condensate current \( j_a \). The first term in Eq. (16) describes a Josephson-like coupling between the bands and may be considered as a drain (\( \varphi > 0 \)) or source (\( \varphi < 0 \)) of Cooper pairs in the \( a \) band. Eq. (16) is valid for frequencies exceeding the effective relaxation time \( \tau_{imb} \) for a charge imbalance, \( \Omega >> 1/\tau_{imb} \).

This relaxation time is determined, in particular, by the electron-phonon and electron-electron inelastic scattering. If charge imbalance relaxation processes are taken into account, the term \( \partial \mu_a/\partial t \) in Eq. (16) should be replaced by \( (\partial/\partial t + \gamma) \), where \( \gamma^{-1} = \tau_{imb} \).

At low temperatures the current density \( j_a \) coincides in the main approximation with the condensate current and equals

\[
j_a = \pi \sigma_a \Delta_a \mathbf{Q}_a/e \quad (17)
\]

This expression and Eqs. (15), (16) describe the system at low temperatures. One should add also the charge neutrality condition

\[
\delta j = \delta (j_a + j_b) = 0. \quad (18)
\]

Then, we can exclude all the variables except \( \varphi \) and obtain the equation for the phase difference

\[
\Omega_0^2 \sin \varphi + (\partial/\partial t + \gamma) \partial \varphi/\partial t - v_{cm}^2 \nabla^2 \varphi = 0 \quad (19)
\]

where \( \Omega_0^2 = 2\bar{\nu}_a^2/\bar{\nu}_b, \gamma = 1/\tau_{imb} \) is a damping rate and

\[
v_{cm} = [\pi \Delta_a \Delta_b D_a D_b]^{1/2} \quad (20)
\]

is velocity of the CMs or the limiting velocity of phase solitons in the two band superconductors.

Eq. (19) is similar to the sine-Gordon equation for a tunnel Josephson junction but the velocity \( v_{cm} \) is much smaller than the corresponding velocity of the Swihart waves in a Josephson junction. By the order of magnitude the velocity \( v_{cm} \) is equal to the velocity \( \sim (D/\Delta) \) of the Carlson-Goldman CM in ordinary superconductors. However, in contrast to the Carlson-Goldman CMs in single-band superconductors that are weakly damped only near \( T_c \), the CM described by Eq. (19) exist at low temperatures. In these modes, condensate in the \( a \) band oscillates with respect to the condensate in the \( b \) band and these oscillations are accompanied by oscillations of the phase difference \( \varphi \). The spectrum of the small amplitude oscillations is given by

\[
\Omega^2 = \Omega_0^2 + k^2 v_{cm}^2. \quad (21)
\]

The threshold frequency \( \Omega_0 \) is analogous to the Josephson “plasma frequency”. As we assume that \( \Omega \) is less than \( \Delta_{a,b} \), our consideration is valid provided \( V_{ab}(\nu_a + \nu_b)/2V_b \nu_a \nu_b < \Delta_b/\Delta_a \). This type of CMs that is analogous to the Leggett mode in \( Hc^3 \) was studied theoretically in Refs. [12], [16]. The dependence of \( \Omega(k) \) is depicted in Fig. 1a for certain parameters of the model.

Now we consider the limit of temperatures close to \( T_c \).

b) \( \Delta_{a,b} << T_c \). The calculations in this case become more cumbersome because the anomalous Green’s function \( \delta g_{an,a}(\epsilon, \epsilon') \), Eq. (14), also gives an essential contribution. In this case the currents \( j_{a,b} \) are equal to

\[
\mathbf{j}_{a,b} = \sigma_{a,b}[(\pi \Delta_{a,b}^2/2T)\mathbf{Q}_{a,b}/e + \mathbf{E}] \quad (22)
\]

where the first term is the supercurrent and the second term is the quasiparticle current \( j_q = \sigma E \). In the main approximation this current originates from the anomalous part \( \delta g_{an,a}(\epsilon, \epsilon') \), Eq. (14).
FIG. 1: a) Calculated CM spectrum for the case of low temperatures \((T << \Delta_{a,b})\). There is a gap \(\Omega_0\) in the CM spectrum. \((k_o = \Omega_0/v_{CM})\) b) The CM spectrum for high temperatures \((T >> \Delta_{a,b})\). There are two branches of the CMs: the sound-like mode and the mode with a gap (analogous to the Leggett mode). The sound-like mode exists for frequencies greater than inverse energy relaxation time: \(\Omega > \gamma\).

The equation for \(\mu_a\) acquires the form

\[
\tilde{\epsilon}_0^2 \sin \varphi + (\nu p)_a(\partial \mu / \partial t + \gamma)\mu_a - (\bar{\nu} p)_a v_a^2 \nabla Q_a = 0
\]

where \(\tilde{\epsilon}_0^2 = (4T/\pi \Delta)\epsilon_0^2\), \(\Delta = \Delta_a + \Delta_b\), \(p_o = \bar{\nu}_a \Delta_a/(\Delta_a + \Delta_b)\) and \(v_a = \sqrt{2D_a \Delta_a}\) is the velocity of the Carlson-Goldman mode in the \(a\)-band. In the limit \(\{\gamma, \Delta_{a,b}^2/T\} < \Omega < \Delta_{a,b}\) the equation for \(Q_{a,b}\) is reduced in the main approximation to \(\partial Q_{a,b}/\partial t \approx \nabla \mu_{a,b}\). The spectrum of the CMs consists of two branches determined by the roots of equation

\[
(\Omega^2 - k^2 v_a^2)(\Omega^2 - k^2 v_b^2) = 2\tilde{\epsilon}_0^2[(\Omega^2 - k^2 v_a^2)/p_b + (\Omega^2 - k^2 v_b^2)/p_a)]
\]

In the long-wave limit these branches are described by the expressions

\[
\Omega_1^2 = 2\epsilon_0^2(1/p_a + 1/p_b) + k^2(v_a^2 p_b + v_b^2 p_a)/(p_a + p_b)
\]

and

\[
\Omega_2^2 = k^2(v_a^2 p_a + v_b^2 p_b)/(p_a + p_b)
\]

Therefore one branch of the CMs has a sound-like spectrum \(\Omega_2\) and another one a Josephson-like spectrum \(\Omega_1\). If \(\nu_a >> \nu_b\) and \(\Delta_a >> \Delta_b\) (this limit corresponds to the two-band superconductor \(MgB_2\), one has \(\Omega_2^2 = k^2 v_a^2\), that is, the low-frequency mode coincides with the Carlson-Goldman mode in the band with a higher gap. This low frequency mode may be excited in such two-band superconductor as \(MgB_2\). One can show that if \(\hbar \Omega < \Delta_{a,b}^2/T\), then the sound-like branch of the CMs is strongly damped. Therefore at low temperatures, the “soft” mode can hardly exist. Below we clarify this point in more detail. The form of the spectrum at high temperatures is shown in Fig.1b. Consider now the case of intermediate temperatures.

c) \(\Delta_b < T < \Delta_a\).

The analysis given above shows that the equation for the potential \(\mu\) can be written in limiting cases as

\[
\mp \epsilon_0^2 \sin \varphi - (\bar{\nu} r)_{a,b} [\partial \mu / \partial t - \bar{\nu}^2]_{a,b} \nabla Q_{a,b} = 0
\]

where the coefficients \(r_{a,b}\) and velocities \(\bar{\nu}^2_{a,b}\) are equal to

\[
r_{a,b} = \begin{cases} 1, & \pi \Delta_{a,b}/4T, \\ \pi/2, & T << \Delta_{a,b} \\ 1, & T >> \Delta_{a,b} \end{cases}
\]

\(\bar{\nu}^2_{a,b} = v^2_{a,b} \begin{cases} \pi^2, & T << \Delta_{a,b} \\ 1, & T >> \Delta_{a,b} \end{cases}\)
FIG. 2: CM spectrum at intermediate temperatures ($\Delta_b < T < \Delta_a$) for two cases: $\Omega >> \Omega_{a,b}$ (a) and $\Omega << \Omega_{a,b}$ (b). There are two branches of the CM spectrum in the both limiting cases: the sound-like mode and the mode with a gap. The frequency and wave vector of CMs are normalized to $\Omega^*$ and $k^*$ equal to: a) $\Omega^* = E_1(T = 8\Delta_b)$, $k^* = E_1/v_1$ and b) $\Omega^* = E_1(T = 2\Delta_b)$, $k^* = E_1/v_1$. With increasing temperature the two modes are getting closer to each other.

$$v^2_{a,b} = \sqrt{2(D\Delta)_{a,b}}$$ The current density in each band is given by

$$j_{a,b} = (\sigma\rho)_{a,b} \Delta_{a,b} Q_{a,b} + \tilde{\sigma}_{a,b} E$$

(29)

with limiting values of conductivities

$$(\sigma\rho)_{a,b} = \sigma_{a,b} \left\{ \begin{array}{ll} \pi, & \pi \Delta_{a,b}/2T, \\ \pi, & \pi \Delta_{a,b}/2T \end{array} \right.$$

$$\tilde{\sigma}_{a,b} = \sigma_{a,b} \left\{ \begin{array}{ll} \exp(-\Delta_{a,b}/T), & T << \Delta_{a,b} \\ 1, & T >> \Delta_{a,b} \end{array} \right.$$

(30)

Qualitatively, dynamics of the CMs at any temperatures is described by Eqs. (15,18,27,29). One can exclude the variables $Q_{a,b}$ and $E$ and obtain the equations

$$i\Omega \tilde{\nu}_a \tilde{\nu}_b (\tilde{\nu}_a A_b - \tilde{\nu}_b A_a) = 2\xi_0^2 (\tilde{\nu}_a A_b + \tilde{\nu}_b A_a - \tilde{\nu}_a B_b - \tilde{\nu}_b B_a)$$

(33)

In the limiting cases $T << \Delta_{a,b}$ and $\Delta_{a,b} << T$ we obtain the formulas for the CM spectrum given above (see Eqs. (21,25,26)). The "soft", sound-like mode, which exists in the temperature range $\Delta_{a,b} << T << T_c$, is of a special interest because another mode with a threshold frequency of the order of $\epsilon_0/\hbar$ may be strongly damped if $\epsilon_0 > \Delta_b$ (this case seems to correspond to $MgB_2$ [15]).

The spectrum of this and other (plasma-like) mode may be obtained from Eq. (33). Consider different limits of high and low frequencies $\Omega$ with respect to the characteristic frequencies $\Omega_{a,b}$. These frequencies in the considered temperature range ($\Delta_b < T < \Delta_a$) are equal to: $\Omega_a \approx \pi(\sigma_a/\sigma_b)\Delta_a$, $\Omega_b \approx \pi\Delta_b^2/2T$.

C1) Assume that the frequency of oscillations is low

$$\Omega << \Omega_{a,b}$$

(34)
In the considered temperature range this condition implies that \( \Omega << \Delta_a^2/T << \Delta_b << \Delta_a \). Then the functions \( A_{a,b} \) and \( B_{a,b} \) acquire the form

\[
A_{a,b} \approx (r_{a,b}/i\Omega)[\Omega^2 - (k\tilde{v}_{a,b})^2\frac{\Omega_{b,a}}{\Omega_+}]
\]

\[
B_{a,b} \approx -(r_{b,a}/i\Omega)(k\tilde{v}_{b,a})^2\frac{\Omega_{a,b}}{\Omega_+}
\]

Substituting these expressions for \( A_{a,b} \) and \( B_{a,b} \), we obtain the spectrum of oscillations

\[
\Omega^2 = 2\epsilon_1^2 + k^2v_1^2
\]

where \( \epsilon_1^2 = \epsilon_0^2[(\tilde{\nu}_a r_a)^{-1} + (\tilde{\nu}_b r_b)^{-1}], v_1^2 = (\tilde{\nu}_a^2\Omega_b + \tilde{\nu}_b^2\Omega_a)/\Omega_+ \), \( r_a \approx 1 \), \( r_b \approx \pi\Delta_a^2/4T \). We see that in this case the spectrum has a threshold frequency \( \sim (8/\pi)\epsilon_0^2 T/(\Delta_b \tilde{\nu}_b) \). The sound-like mode is strongly damped in this limit.

c2) For large frequencies satisfying the condition

\[
\Omega >> \Omega_{a,b}
\]

we can write the functions \( A_{a,b} \) and \( B_{a,b} \) in the form

\[
A_{a,b} \approx (r_{a,b}/i\Omega)[\Omega^2 - (k\tilde{v}_{a,b})^2]; \quad B_{a,b} \approx 0
\]

The roots of Eq.\ref{roots} are

\[
\Omega_{1,2}^2 = 2\epsilon_1^2 + \frac{1}{2}k^2(\tilde{v}_a^2 + \tilde{v}_b^2) \pm \left\{ \epsilon_1^4 + \frac{1}{4}k^4(\tilde{v}_a^2 - \tilde{v}_b^2) + \epsilon_1^2k^2(\tilde{v}_a^2 + \tilde{v}_b^2) - 2\epsilon_1^2k^2\left(\frac{\tilde{v}_a^2}{(\tilde{\nu}_a r_a)} + \frac{\tilde{v}_b^2}{(\tilde{\nu}_b r_b)}\right)\right\}^{1/2}
\]

In the long-wave limit we obtain for \( \Omega_1^2 \) and \( \Omega_2^2 \)

\[
\Omega_1^2 = 2\epsilon_1^2 + k^2v_2^2; \quad \Omega_2^2 = k^2v_3^2
\]

where \( v_2^2 = (\tilde{v}_a^2(\tilde{\nu}_b r_b) + \tilde{v}_b^2(\tilde{\nu}_a r_a)/(\tilde{\nu}_a r_a + \tilde{\nu}_b r_b)), \) \( v_3^2 = (\tilde{v}_a^2(\tilde{\nu}_b r_b) + \tilde{v}_b^2(\tilde{\nu}_a r_a)/(\tilde{\nu}_a r_a + \tilde{\nu}_b r_b)) \). Thus, we have a "hard" mode \( (\Omega_1) \) and a "soft" mode \( (\Omega_2) \). However, the considered limit of high frequencies corresponds to a hypothetical case which is hardly realizable in experiments. We assumed that the frequency \( \Omega \) is higher than \( \Omega_{a,b} \), but lower than \( \Delta_{a,b} \).

This means, in particular, that the inequality \( \Omega_a < \Delta_b \) should be satisfied. This inequality can be presented in the form: \( \sigma_a/\sigma_b << (\Delta_b/\pi \Delta_a) \); that is, the conductivity of the \( a \) band with the higher energy gap \( \Delta_a \) should be much less than the conductivity of the \( b \) band.

In Fig.2 we plot the dispersion curves \( \Omega(k) \) for different temperatures determined by Eq.\ref{dispersion}.

**IV. OBSERVATION OF COLLECTIVE MODES IN A JOSEPHSON JUNCTION**

In this section we analyze an experimental method that allows one to observe the CMs and to determine the spectrum of these modes. The idea to identify the CMs in a superconductor has been suggested by Carlson and Goldman \[8, 9, 12\] and also used in Ref.\ref{10}.

According to this idea one has to measure the I-V characteristics of a tunnel Josephson junction in the presence of a bias voltage \( V_B \) across the junction and of a weak external magnetic field \( H \). As in Ref.\ref{10}, we consider a tunnel Josephson junction \( S_1/1/S_2 \) consisting of two superconductors one of which is a two-band superconductor \( (S_2) \) and another one \( (S_1) \) is a single band superconductor. However, unlike Ref.\ref{10}, we take into account a magnetic field, which allows one to determine the spatial dispersion of the CM spectrum and use the microscopic approach to derive necessary equations. As in Ref.\ref{10}, we calculate the Josephson current \( j_J \) taking into account terms of high order.
in transparency $|T_k|^2$. As is well known, in the main approximation the Josephson current at a finite voltage $V_B$ and in the presence of a magnetic field $H$ has the form of a traveling wave (see, for example, [26, 27, 28])

$$j_I(x,t) = j_J \sin(\Omega_V t - K_H x).$$

(42)

where $\Omega_V = 2eV_B/h$ and $K_H$ is given by Eq. (12). This current is injected into the superconductor $S_2$ and modifies Eqs. (23) for the gauge-invariant potentials $\mu_{a,b}$ describing the conservation of charge of the Cooper pairs in each band. We show that, as in the Carlson and Goldman experiment [12], a resonance occurs if the frequency $\Omega$ and the wave vector $k$ coincide with $\Omega_V$ and $K_H$ respectively. As was noted in Ref. [10], this resonance is analogous to the Fiske resonance in a Josephson tunnel junction when the velocity of the traveling wave $|42, 43|$ $\Omega_V/K_H$ coincides with velocity of the Josephson "plasma" waves.

In order to generalize Eqs. (23) to the case of a Josephson junction, we assume that there are no

The modified equation can be solved in the same way as it was done before. For example, the retarded function $\hat{g}_R$ is given again by Eq. (12) with an additional term

$$i\hat{E}_{Ja}(\hat{g}_R - \hat{g}_R \hat{g}_S \hat{g}_R)$$

(45)

where the energy $\hat{E}_{Ja} = D_a/(2R_B \sigma)_a d$ is related to the interface transparency and to the Josephson coupling energy, $d$ is the thickness of the two-band superconductor. This term results in a corresponding modification of Eq. (27), that acquires the form

$$\mp e^2 \sin \varphi - (\hat{v} s)_{a,b} \partial \mu_{a,b}/\partial t + (\hat{v} D/\sigma)_{a,b}(\nabla j_{a,b} - (j_{a,b}/d) \sin(\chi_{a,b} - \chi_S)) = 0$$

(46)

where the current $j_{a,b}$ at low temperatures is given by Eq. (17) and the Josephson critical current $J_{a,b}$ is equal to $J_{a,b} = (\Delta/eR_B)_{a,b}$, where the parameter $\Delta$ equals

$$\Delta_a = 2\pi T \sum_{\omega=0} \frac{\Delta}{\sqrt{\omega^2 + \Delta^2}} \frac{\Delta_s}{\sqrt{\omega^2 + \Delta_s^2}}$$

(47)

At low temperatures $\Delta_a = \Delta_s \ln(c_1 \Delta_a/\Delta_s)$ if $\Delta_s < \Delta_a$ and $\Delta_a = \Delta_a \ln(c_1 \Delta_s/\Delta_a)$ if $\Delta_s \gg \Delta_a$, where $c_1 = 2 \int_0^\infty dx x \ln(2x)/(x^2 + 1)^{3/2} \approx 2.8$.

The phases $\chi_{a,b}$ may be represented in the form

$$\chi_{a,b} = \bar{\chi} \pm \varphi/2$$

(48)

where $\bar{\chi} = (\chi_a + \chi_b)/2$.

We assume that there are no $\varphi$-solitons in the system (according to Ref. [4] special conditions are needed to create such solitons) so that in the main approximation one has: $\chi_a = \chi_b$, $\bar{\chi} - \chi_S = 2eV_B t/h - K_H x$ (see Appendix). Substituting these expressions into Eq. (46), we obtain two equations
FIG. 3: Corrections to the I-V characteristics of a Josephson tunnel junction due to CMs in the two-band superconductor. The Josephson junction consists of a one-band and two-band superconductors. At voltages \( V \) satisfying the equation \( \Omega V^2 = \Omega_0^2 + (K_H v_{cm})^2 \) a spike arises on the I-V characteristics. The position of the spike depends on an applied magnetic field \( H \).

The current is normalized to the value: \( j_0 = \gamma (j_{Ja} - j_{Jb}) (E^2_{Ja}/\nu_a - E^2_{Jb}/\nu_b) \).

\[
-j_0^2 \sin \varphi - \bar{\nu}_a \partial \mu_a / \partial t + (\bar{\nu}_a) \nabla Q_a = E^2_{Ja} \sin (\Omega_V t - K_H x),
\]

\[
-j_0^2 \sin \varphi - \bar{\nu}_b \partial \mu_b / \partial t + (\bar{\nu}_b) \nabla Q_b = E^2_{Jb} \sin (\Omega_V t - K_H x),
\]

where \( \alpha_a = \pi (\sigma \Delta_a) / e \) and \( E^2_{Ja} = 2 \bar{\Delta}_a \bar{\nu}_a \tilde{E}_{Ja} \). Taking into account Eqs. (49) for \( Q_{a,b} \), we can find the phase perturbation \( \varphi \) due to the "external forces" \( E^2_{Ja} \sin (\Omega_V t - K_H x) \)

\[
\varphi = 2 (E^2_{Ja}/\nu_a - E^2_{Jb}/\nu_b) Im (N^{-1} \exp (i (\Omega_V t - K_H x)))
\]

where \( N = \Omega_0^2 - \Omega_V (\Omega_V + i \gamma) + (K_H v_{cm})^2 \). The phase perturbation \( \varphi \) leads to a change of the dc Josephson current

\[
\langle j_J \rangle = (j_{Ja} \sin (\Omega_V t - K_H x) + \delta (\bar{\chi} - \chi_S) + \varphi/2) + j_{Jb} \sin (\Omega_V t - K_H x) + \delta (\bar{\chi} - \chi_S) - \varphi/2\rangle
\]

\( \delta \bar{\chi} \) is the perturbation of the phase difference between the total phase \( \bar{\chi} \) of the two-band superconductor and the phase \( \chi_S \) of the single band superconductor. The angle brackets mean the averaging in space and time. Then we expand the currents \( J_{Ja,Jb} \) with respect to \( \delta (\bar{\chi} - \chi_S) \) and \( \varphi/2 \) and take into account Eq. (51). The correction to the current \( j_J \) due to the perturbation \( \delta (\bar{\chi} - \chi_S) \) is omitted because it is not related to the CMs (this correction is real Fiske steps [20, 27, 28]) and corresponds to much larger voltages (see Appendix). The correction to the current \( j_J \) due to \( \varphi \) is

\[
\langle \delta j_J \rangle = -(j_{Ja} - j_{Jb}) (E^2_{Ja}/\nu_a - E^2_{Jb}/\nu_b) \frac{\gamma \Omega_V}{|N|^2}
\]

where \( |N|^2 = (\Omega_0^2 - \Omega_V^2 + (K_H v_{cm})^2)^2 + (\gamma \Omega_V)^2 \).

We see that if the applied voltage satisfies the condition \( \Omega_V^2 = \Omega_0^2 + (K_H v_{cm})^2 \), a spike arises on the I-V characteristics the position of which is shifted with varying magnetic field \( H \). These spikes for different \( H \) are shown in Fig.3.

V. CONCLUSIONS

In summary, using a simple model we have studied the CMs in two-band superconductors and shown that weakly damped CMs exist in these superconductors at all temperatures below \( T_c \) in the frequency range: \( \gamma < \Omega < \Delta_{a,b} \).
where $\gamma$ is an effective charge imbalance relaxation rate. At low temperatures these modes have a spectrum similar to the spectrum of the Josephson “plasma modes” in tunnel Josephson junctions. However, the velocity of the CM in two-band superconductors by the order of magnitude is $\sim \sqrt{D \Delta}$, i.e., much smaller than the velocity of plasma modes in the Josephson junction.

At high temperatures ($T >> \Delta_{a,b}$), the CM spectrum consists of two branches. One branch is analogous to the “plasma modes” in a Josephson junction but propagates much slower than the Swihart waves. Another branch is similar to the Carlson-Goldman mode in one-band ordinary superconductors, i.e., it has a sound-like spectrum. In the case of intermediate temperatures ($\Delta_b < T < \Delta_a$) the gapless (sound-like) mode has a low damping if strict conditions are satisfied.

The spectrum of the CMs considered above can be determined experimentally by measuring the I-V characteristics in a Josephson junction. Such a method has been applied by Carlson and Goldman [12] for measuring the CM spectrum in a conventional superconductor. They observed a small peak on the I-V curve the position of which depends on a weak applied magnetic field. In this case the wave vector $k$ is proportional to $H$ and the frequency of phase oscillations $\Omega$ is related to the applied voltage: $\Omega = (2e/\hbar) V_B$. The measured temperature dependence of the spectrum of the CMs in two-band superconductors would allow one to elucidate the nature of superconductivity in such two-band superconductors as $MgB_2$.

Note that an evidence in favour of the existence of a Leggett-type collective mode in $MgB_2$ was obtained in a recent work, where the point-contact spectroscopy was used [29].

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VII. APPENDIX

Here we derive the relation between the phase difference $\theta$ and the applied voltage $V$ as well as the magnetic field $H$ in the main approximation, i.e., in the absence of the Josephson coupling. In addition we present the derivation of Eq.(44). For simplicity we restrict ourselves with the case of low temperatures ($T << \Delta_{a,b}$). The results for higher temperatures are qualitatively the same.

In the absence of the Josephson coupling the right-hand side of Eqs.(40) is equal to zero. Summing up these equations, we obtain the equation of the charge conservation

$$e \nu_{a,b} \partial_t \mu_{a,b} / \partial t = (\nu D/\sigma)_{a,b} \nabla j_{a,b}$$

(54)

where $j_{a,b} = \alpha_{a,b} (\partial_x \chi_{a,b} - 2\pi A_x/\Phi_0)/2$; $\Phi_0 = hc/2e$ is the magnetic flux quantum. The coefficients $\alpha_{a,b}$ are related to London penetration depth $\lambda_L$: $\alpha_{a,b} = c\Phi_0/(4\pi^2 \lambda_L^2)$. At low temperatures these coefficients are equal to $\alpha_{a,b} = (\sigma \Delta)_{a,b}/e$. Summing up these equations, we obtain the equations of the charge conservation

$$e \partial (\nu_{a} \mu_{a} + \nu_{b} \mu_{b}) / \partial t = \nabla [\alpha_{a} Q_{a} + \alpha_{b} Q_{b}]$$

(55)

where $Q_{a,b} = (\partial_x \chi_{a,b} - 2\pi A_x/\Phi_0)/2$ is the condensate momentum. Differentiating this equation on time and taking into account Eq.(14) and the Poisson equation

$$\nabla E = 4\pi \rho = 4\pi \nu e (\nu_{a} \mu_{a} + \nu_{b} \mu_{b})$$

(56)

we obtain the equation

$$v^2 \partial^2 (\nu_{a} \mu_{a} + \nu_{b} \mu_{b}) / \partial t^2 = -k_{TF}^2 (\nu_{a} \mu_{a} + \nu_{b} \mu_{b}) + \nabla^2 [\delta_{a} \mu_{a} + \delta_{b} \mu_{b}]$$

(57)
where \( v^2 = (\pi/2)[(D\nu\Delta)_{\alpha}+(D\nu\Delta)_{\beta}] \), \( \nu = \nu_a + \nu_b \), and \( \tilde{\delta}_{a,b} = (\sigma\Delta)_{a,b}/((\sigma\Delta)_{a}+(\sigma\Delta)_{b}) \). The Thomas-Fermi screening length \( k_{TF}^{-1} \) is of the order of interatomic spacing, i.e., much shorter than characteristic lengths of the problem: \( k_{TF}^{-1} \gg v/\Delta > v/\Omega_V \approx K_H \). This means that the first term on the right-hand side in Eq. (57) is much larger than other terms. Therefore we obtain

\[
\tilde{\nu}_a \mu_a + \tilde{\nu}_b \mu_b = 0
\]  

(58)

For the single band superconductor we have

\[
\mu_S = 0
\]  

(59)

Writing the potentials \( \mu_{a,b} \) in the form \( \mu_{a,b} = \partial(\chi \pm \varphi/2)/\partial t + eV \), we obtain from Eqs. (58, 59)

\[
\partial \theta/\partial t = 2eV - \frac{1}{2}(\tilde{\nu}_a - \tilde{\nu}_b)\partial \varphi/\partial t
\]  

(60)

where \( \theta = \bar{\chi} - \chi_S \), \( V_B = V - V_S \) is the applied voltage. Eq. (60) generalizes the Josephson relation to the case of a junction with a two-band superconductor.

If in the ground state there are no \( \varphi \)-solitons (\( \varphi = 0 \)) and the external magnetic field \( H_e \) is applied in the \( y \)-direction, one obtains from Eq. (60)

\[
\theta(x,t) = \Omega_V t - K_H x
\]  

(61)

where \( \Omega_V = 2eV_B/\hbar \) and the wave vector \( K_H \) is determined by the equation [27]

\[
K_H = (2\pi H_e/\Phi_0)[\lambda \tanh(d/\lambda) + \lambda_S \tanh(d_S/\lambda_S)]
\]  

(62)

where \( d_S \) is the thickness of the single band superconductor, \( \lambda = \lambda_{La} + \lambda_{Lb} \).

We turn now to finding solutions for Eqs. (49-50). We assume that the phase perturbations are small (\( \chi_{a,b}, \varphi << 1 \)) and seek for a solution in the form

\[
\varphi(x,t) = Im \varphi \Omega \exp(\Omega_V t - K_H x)
\]  

(63)

Summing up Eqs. (49, 51), we obtain

\[
- \partial(\tilde{\nu}_a \mu_a + \tilde{\nu}_b \mu_b)/\partial t + \partial[\alpha_a Q_a + \alpha_b Q_b]/\partial x = E^2_a + E^2_b
\]  

(64)

The first term on the left is small due to the quasineutrality condition. Therefore we obtain from Eq. (64)

\[
Q_b = -\frac{(\sigma\Delta)_{a}}{(\sigma\Delta)_{b}} Q_a + (E^2_a + E^2_b)/(iK_H v^2_b)\tilde{\nu}_b
\]  

(65)

From the definition of \( Q_{a,b} \) we have

\[
Q_a - Q_b = \frac{1}{2} \partial \varphi/\partial x
\]  

(66)

Finally we divide Eqs. (49, 50) by \( \tilde{\nu}_{a,b} \) and subtract from each other. We get

\[
2\nu^2(\tilde{\nu}_a^{-1} + \tilde{\nu}_b^{-1})\varphi - \frac{1}{2} \partial^2 \varphi/\partial t^2 + 2\nu^2 \tilde{\nu}_a \partial Q_a/\partial x = 2E^2_a
\]  

(67)

Using Eq. (66), we substitute \( Q_a \) into Eq. (67) and come to Eq. (51) for \( \varphi \).

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