Algebraic Geometry over Free Metabelian Lie Algebra II: Finite Field Case

E. Yu. Daniyarova∗ I. V. Kazachkov†
V. N. Remeslennikov†

February 2, 2008

Abstract

This paper is the second in a series of three, the aim of which is to construct algebraic geometry over a free metabelian Lie algebra $F$. For the universal closure of free metabelian Lie algebra of finite rank $r \geq 2$ over a finite field $k$ we find a convenient set of axioms in the language of Lie algebras $L$ and the language $L_{Fe}$ enriched by constants from $F$. We give a description of:

1. the structure of finitely generated algebras from the universal closure of $F_r$ in both $L$ and $L_{Fe}$;
2. the structure of irreducible algebraic sets over $F_r$ and respective coordinate algebras.

We also prove that the universal theory of a free metabelian Lie algebra over a finite field is decidable in both languages.

Contents

1 Introduction 2
2 Elements of Algebraic Geometry over Lie Algebras 3
  2.1 General Case 3
  2.2 The Case of $F_r$ 8

∗The first author is supported by the RFFI grant N02-01-00192.
†The second and the third authors are supported by the ‘Universitety Rossii’ grant.
3 Universal Axioms and the \( \Phi_r \)-Algebras

4 Theorems on Universal Closures of the Algebra \( F_r \)
   4.1 Formulation of Main Results ........................................... 21
   4.2 Proofs of the Theorems .................................................. 23

5 Irreducible Algebraic Sets over \( F_r \) and Dimension
   5.1 Classification of Irreducible Algebraic Sets over \( F_r \) .......... 24
   5.2 Dimension ................................................................. 28

1 Introduction

This paper is the second in a series of papers the main object of which is to construct algebraic geometry over free metabelian Lie algebra. In this paper we consider the free metabelian Lie algebra \( F_r \) of a finite rank \( r \geq 2 \) over a finite field \( k \). Throughout this paper we use the results, notation and definitions of the first paper of the current series [6].

The object of Section 2 which arises from papers [2] and [8] is to lay the foundations of algebraic geometry over Lie algebras. In [2] and [8] the authors conduct their arguments and prove the results in the category of groups, however the proofs are absolutely analogous for Lie algebras and, therefore, most of the results in Section 2 are omitted.

Section 3 holds main technical complications of the current paper. There we introduce two collections of seven series of universal axioms \( \Phi_r \) and \( \Phi'_r \), \( r \geq 2 \). The axioms of the collection \( \Phi_r \) are universal formulas of the standard first order language \( L \) of theory of Lie algebras over the field \( k \) and the axioms of the collection \( \Phi'_r \) are universal formulas in the enriched language \( L_{F_r} \), obtained from \( L \) by joining constants for the elements of \( F_r \). There we establish some properties of Lie algebras that satisfy either of the collections mentioned.

The key results of the current paper are formulated in Section 4 (see Theorems 4.1 - 4.5). We list the most important of these results:

- for the universal closure of the free metabelian Lie algebra of finite rank \( r \geq 2 \) over a finite field \( k \) we find a convenient set of axioms (\( \Phi_r \) and \( \Phi'_r \)) in \( L \) and \( L_{F_r} \),
- describe the structure of finitely generated algebras from \( F_r - \text{uc}1(F_r) \) and \( \text{uc}1(F_r) \),


• prove that the universal theory of the free metabelian Lie algebra over a finite field is decidable in both $L$ and $L_F$.

In Section 5 we apply theorems from Section 4 to algebraic geometry over the algebra $F_r$, $r \geq 2$ over a finite field $k$. The main results of this section are:

• given a structural description of coordinate algebras of irreducible algebraic sets over $F_r$;

• given a description of the structure of irreducible algebraic sets;

• constructed a theory of dimension in the category of algebraic sets over $F_r$.

2 Elements of Algebraic Geometry over Lie Algebras

In paper [2] the authors introduce main notions of algebraic geometry over groups. In Subsection 2.1 below we introduce main notions of algebraic geometry over Lie algebras. Following paper [2] we list several results and theorems, involving these notions. Subsection 2.2 highlights some of the aspects of algebraic geometry over the free metabelian Lie algebra $F_r$ of finite rank $r$, $r \geq 2$.

2.1 General Case

Let $A$ be a fixed Lie algebra over a field $k$.

Recall that a Lie algebra $B$ over a field $k$ is called an $A$–Lie algebra if and only if it contains a designated copy of $A$, which we shall for most part identify with $A$. A homomorphism $\varphi$ from an $A$–Lie algebra $B_1$ to an $A$–Lie algebra $B_2$ is an $A$–homomorphism of Lie algebras if it is the identity on $A$, $\varphi(a) = a$, $\forall a \in A$. Set $\text{Hom}_A(B_1, B_2)$ to be the set of all $A$–homomorphisms from $B_1$ to $B_2$. We use the symbol ‘$\cong_A$’ ($A$–isomorphism) to express that two $A$–Lie algebras are isomorphic in the category of $A$–Lie algebras.

The family of all $A$–Lie algebras together with the collection of all $A$–homomorphisms form a category in the obvious way.
Let $X = \{x_1, \ldots, x_n\}$ be a finite set. The free $A$–Lie algebra with the free base $X$

$$A [X] = A \ast F(X),$$

is the free Lie product of the free (in the category of Lie $k$-algebras) Lie algebra $F(X)$ and the algebra $A$. We think of elements of $A [X]$ as polynomials with coefficients in $A$. We use functional notation here,

$$f = f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, a_1, \ldots, a_r)$$

thereby expressing the fact that the Lie polynomial representing $f$ in $A [X]$ involves the variables $x_1, \ldots, x_n$ and, as needed, the constants $a_1, \ldots, a_r \in A$.

Using the standard argument from universal algebra one verifies that the algebra $A [X]$ is the free algebra in the category of $A$–algebras.

Let $B$ be an $A$–Lie algebra and let $S$ be a subset of $A [X]$. Then the set

$$B^n = \{(b_1, \ldots, b_n) \mid b_i \in B\}.$$

is termed the affine $n$-dimensional space over the algebra $B$.

A point $p = (b_1, \ldots, b_n) \in B^n$ such that

$$f(p) = f(b_1, \ldots, b_n, a_1, \ldots, a_r) = 0.$$

is termed a root of the polynomial $f \in A [X]$. In that case we also say that the polynomial $f$ vanishes at the point $p$.

A point $p \in B^n$ is a root or a solution of the system $S \subseteq A [X]$ if every polynomial from $S$ vanishes at $p$, i.e. if $p$ is a root of every polynomial from the system $S$.

The set

$$V_B(S) = \{p \in B^n \mid f(p) = 0 \ \forall f \in S\}$$

is termed the (affine) algebraic set over $B$ defined by the system of equations $S$.

Let $S_1$ and $S_2$ be subsets of $A [X]$. Then the systems $S_1$ and $S_2$ are termed equivalent over $B$ if $V_B(S_1) = V_B(S_2)$.

**Example 2.1 (typical examples of algebraic sets)**

1. Every element $a \in A$ forms an algebraic set, $\{a\}$: $S = \{x - a = 0\}$, $V_B(S) = \{a\}$. In this example $n = 1$ and $X = \{x\}$. 
2. The centraliser $C_B(M)$ of an arbitrary set of elements $M$ from $A$ is the algebraic set defined by the system $S = \{ x \circ m = 0 \mid m \in M \}$.

Let $Y$ be an arbitrary algebraic set ($Y = V_B(S)$) from $B^n$. The set

$$\text{Rad}_B(S) = \text{Rad}_B(Y) = \{ f \in A[X] \mid f(p) = 0 \ \forall p \in Y \}$$

is termed the radical of the set $Y$. If $Y = \emptyset$ then, by the definition, its radical is the algebra $A[X]$.

Clearly, the radical of a set is an ideal of the algebra $A[X]$. A polynomial $f \in A[X]$ is termed a consequence of a system $S \subseteq A[X]$ if

$$V(f) \supseteq V(S).$$

The radical of an algebraic set describes it uniquely, i.e. for two arbitrary algebraic sets $Y_1, Y_2 \subseteq B^n$

$$Y_1 = Y_2 \text{ if and only if } \text{Rad}_B(Y_1) = \text{Rad}_B(Y_2).$$

Let $B$ be an $A$–Lie algebra, $S$ be a subset of $A[X]$ and $Y \subseteq B^n$ be the algebraic set defined by the system $S$. Then the factor-algebra

$$\Gamma_B(Y) = \Gamma_B(S) = A[X]/_{\text{Rad}_B(Y)}$$

is termed the coordinate algebra of the algebraic set $Y$ (or of the system $S$).

Note that coordinate algebras of consistent systems of equations are $A$–Lie algebras and form a subcategory of the category of all $A$–Lie algebras.

**Lemma 2.1** For any algebraic set $V_B(S)$ there is a one-to-one correspondence between the points of $V_B(S)$ and $A$–homomorphisms from $\Gamma_B(V_B(S))$ to $B$.

**Proof.** To a point $y \in V_B(S)$ we link an $A$–homomorphism

$$\varphi \in \text{Hom}_A(A[X], B) \text{ given by } f \to f(y), \ f \in A[X].$$

This homomorphism on $A[X]$ is a correct homomorphism on the factor-algebra $\Gamma_B(S)$. Conversely, if $\varphi \in \text{Hom}_A(A[X], B)$ then the point corresponding to $\varphi$ is the following $y = (\varphi(\bar{x}_1), \ldots, \varphi(\bar{x}_n))$, where $\bar{x}_1, \ldots, \bar{x}_n$ are the images of $X = \{x_1, \ldots, x_n\}$ in the factor-algebra $\Gamma_B(V_B(S))$. Clearly, $y \in V_B(S)$. Obviously, the two maps given are mutually inverse. ■
Remark 2.1 The coordinate algebra $\Gamma(Y)$ is an $A$–Lie algebra isomorphic to the algebra of all polynomial functions from $Y$ into $B$ defined by the rule $y \in Y, \; y \mapsto f(y)$ for $f \in A[X]$.

Example 2.2 (the coordinate algebra of a point) If $a \in A, \; Y = \{a\}$ then $\Gamma_A(Y) \cong A$.

We next fix an $A$–Lie algebra $B$. Let $Y \subseteq B^n$ and $Z \subseteq B^m$ be algebraic sets. Then the map

$$\psi : Y \longrightarrow Z$$

is termed a morphism from the algebraic set $Y$ to the algebraic set $Z$ if there exist $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ so that for any $(b_1, \ldots, b_n) \in Y$

$$\psi(b_1, \ldots, b_n) = (f_1(b_1, \ldots, b_n), \ldots, f_m(b_1, \ldots, b_n)) \in Z.$$  

Algebraic sets $Y$ and $Z$ are termed isomorphic if there exist morphisms:

$$\psi : Y \rightarrow Z \text{ and } \theta : Z \rightarrow Y$$

such that $\theta \psi = \text{id}_Y$ and $\psi \theta = \text{id}_Z$. We shall make use of the notation $\text{Hom}(Y, Z)$ for the set of all morphisms from $Y$ to $Z$.

The collection of all algebraic sets $V_B(S), \; S \subset A[x_1, \ldots, x_n]$ over $B^n$, where $n$ is a non-fixed positive integer form the family of objects of $\mathcal{AS}_{A,B}$. Morphisms of this category are the morphisms of algebraic sets.

Following the argument of paper [2] one can prove that the categories of coordinate algebras and algebraic sets are equivalent. We formulate this result by the means of the following two lemmas:

Lemma 2.2 Coordinate algebras define algebraic sets up to isomorphism:

$$Y \cong Y' \iff \Gamma(Y) \cong_A \Gamma(Y').$$

Lemma 2.3 There exists a one-to-one correspondence between $\text{Hom}(Y, Y')$ and $\text{Hom}_A(\Gamma(Y'), \Gamma(Y))$. Furthermore, whenever we have an embedding of algebraic sets $Y \subseteq Y'$ the correspondent map $\varphi : \Gamma(Y') \rightarrow \Gamma(Y)$ is an $A$–epimorphism of coordinate algebras. Moreover, if $Y \subset Y_1$ then the kernel $\ker \varphi \neq 0$ is non-trivial.
Example 2.3 A vector space $A$ over a field $k$ with the trivial multiplication $(\forall u, v \in A \ [u \circ v = 0])$ is a particular case of a Lie $k$-algebra. Let $A$ be a Lie $k$-algebra with the trivial multiplication and assume that $B = A$. Applying theorems of linear algebra one shows that:

1. Every consistent system of equations over $A$ is equivalent to a triangular system of equations (see [9] for definitions).

2. Every algebraic set $Y \subseteq A^n$ is isomorphic to an algebraic set of the form 
   
   $$(A, A, \ldots, A, 0, \ldots, 0), \ 0 \leq s \leq n.$$ 

3. Every coordinate algebra $\Gamma(Y)$ is $A$-isomorphic to $A \oplus \text{lin}_k \{x_1, \ldots, x_s\}$, where here $0 \leq s \leq n$, and $\text{lin}_k \{x_1, \ldots, x_s\}$ is the linear span of the elements $\{x_1, \ldots, x_s\}$ over $k$.

The union of two algebraic set is not necessarily again an algebraic set. The correspondent counterexample is easy to construct using Example 2.3. We, therefore, define a topology in $B^n$ by taking algebraic sets in $B^n$ as a sub-basis for closed sets. We term this topology the Zariski topology.

A closed set $Y$ is termed irreducible if $Y = Y_1 \cup Y_2$, where $Y_1$ and $Y_2$ are closed, implies that either $Y = Y_1$ or $Y = Y_2$.

An $A$–Lie algebra $B$ is termed $A$–equationally Noetherian if for every $n \in \mathbb{N}$ and for every system $S \subseteq A[x_1, \ldots, x_n]$ there exists a finite subsystem $S_0 \subseteq S$ such that $V_B(S) = V_B(S_0)$.

Theorem 2.1 Every closed subset $Y$ of $B^n$ over $A$–equationally Noetherian $A$–Lie algebra $B$ can be expressed as a finite union of irreducible algebraic sets:

$$Y = Y_1 \cup \ldots \cup Y_l.$$ 

These sets are usually referred to as the irreducible components of $Y$, which turn out to be unique, if for every $i, j = 1, \ldots, l; \ i \neq j \ Y_i \nsubseteq Y_j$.

The main aim of algebraic geometry over an $A$–Lie algebra $B$ is to give a description of algebraic sets over $B$ up to isomorphism. Or, which is equivalent, to give a description of coordinate algebras of algebraic sets up to $A$–isomorphism.
We next treat some of the properties of model-theoretical classes generated by an $A$–Lie algebra $B$. Our interest to the universal closure generated by $B$ is justified by the following circumstance: finitely generated Lie algebras from the universal closure $A – \text{uc1}(B)$ are exactly the coordinate algebras of irreducible algebraic sets over $B$.

Recall that the universal closure $A – \text{uc1}(B)$ generated by $B$ is the class of all $A$–Lie algebras that satisfy all the universal sentences satisfied by $B$ (for details see [6]).

**Theorem 2.2** Let $B$ be an $A$–equationally Noetherian $A$–Lie algebra. Then finitely generated $A$–Lie algebra $C$ is the coordinate algebra of an irreducible algebraic set over $B$ if and only if $C \in A – \text{uc1}(B)$.

### 2.2 The Case of $F_r$

Let $F_r$ be the free metabelian Lie $k$-algebra of the rank $r$, let $\{a_1, \ldots, a_r\}$ be its free base and let $R = k[x_1, \ldots, x_r]$ be the ring of polynomials from $r$ variables. Recall that the Fitting’s radical of the algebra $F_r$ coincides with its commutant $F_r^2$ and admits the structure of an $R$–module. Further, the multiplication by the variables $x_i$’s of the ring $R$ is interpreted as the multiplication by free generators $a_i$’s (see [6]).

In the current paper we consider so called ‘diophantine geometry’, i.e. we consider systems of equations with coefficients in $F_r$ and solutions of these systems from $F_r$. In the event that $r = 1$, $F_r$ is Abelian and this extreme case has been already considered in Example 2.3 and we, therefore, consider only non-degenerated alternative of $r \geq 2$.

One of the most important algebraic sets over the free metabelian Lie algebra $F_r$ is $\text{Fit}(F_r)$ is an algebraic set in the affine space $F_r^1$. To prove this consider an equation $(a_1a_2)x = 0$ with one indeterminant $x$, where here $a_1$ and $a_2$ are two distinct elements of the free base of $F_r$. Since $F_r$ is a $U$–algebra (see [9]), its Fitting’s radical is Abelian. Consequently every element of $\text{Fit}(F_r)$ satisfies this equation and $\text{Fit}(F_r) \subseteq V_{F_r}(\{(a_1a_2)x = 0\})$. To prove the reverse inclusion take $c \notin \text{Fit}(F_r)$. Since $\text{Fit}(F_r)$ is a torsion free module over the ring of polynomials $R$, we obtain $(a_1a_2)c \neq 0$, which implies that $\text{Fit}(F_r) = V_{F_r}(\{(a_1a_2)x = 0\})$. Below we show that the Fitting’s radical is an irreducible algebraic set and that its coordinate algebra is $F_r \oplus T_1$, where here $T_1$ is the $R$–free module of the rank $1$ (for definition of $F_r \oplus T_s$, where $T_s$ is the $R$–free module of the rank $s \geq 1$, see [6]).
Notation. By $F_{r,s}$ we denote the direct module extension of the Fitting’s radical of $F_r$ by the free $R$-module $T_s$ of the rank $s$, $F_{r,s} = F_r \oplus T_s$.

Lemma 2.4 The free metabelian Lie algebra $F_r$ is equationally Noetherian.

Proof. Let $F_r[X]$ be the free $F_r$-algebra generated by the alphabet $X = \{x_1, \ldots, x_n\}$. Let $V_{2n}$ be the verbal ideal of $F_r[X]$ that defines the variety of all metabelian Lie algebras, $V_{2n} = \langle (ab)(cd) \mid a, b, c, d \in F_r[X] \rangle$. Consider the factor-algebra of the free $F_r$–algebra by $V_{2n}$:

$$F_r[X] / V_{2n} \cong F_r \ast_{2n} F_{2n}(X),$$

where $\ast_{2n}$ stands for the free metabelian product of metabelian Lie algebras and $F_{2n}(X) = F_n$ is the free metabelian Lie algebra generated by the set $X$. Consequently, the obtained factor-algebra is isomorphic to the free algebra $F_{r+n}$. The Lie algebra $F_{r+n}$ is metabelian, thus is Noetherian (in the usual classical sense, i.e. every its ideal is finitely generated).

Fix an arbitrary system $S \subseteq F_r[X]$. To prove the lemma it suffices to find a finite subsystem $S_0 \subseteq S$ so that $V_{F_r}(S) = V_{F_r}(S_0)$. Let $\bar{S}$ be the image of $S$ in the factor-algebra $F_r \ast_{2n} F_{2n}(X)$ and let $I$ be the ideal generated by $\bar{S}$. Since $F_r \ast_{2n} F_{2n}(X) \cong F_{r+n}$ is a metabelian algebra, the ideal $I$ is finitely generated. Choose a finite subsystem $S_0 \subseteq S$ so that the set $S_0$ generates the ideal $I$. Consider the pre-images of $S$ in $F_r[X]$ and take $S_0 \subseteq S$ to be an injective subset of pre-images for the set $\bar{S}_0$. In the above notation, thanks to the choice of $S_0$, it is clear that $V_{F_r}(S) = V_{F_r}(S_0)$.

Remark 2.2 The argument of Lemma 2.4 holds for an arbitrary finitely generated metabelian Lie algebra $A$, i.e. every finitely generated metabelian Lie algebra is equationally Noetherian.

Let $\mathfrak{V}$ be an arbitrary variety of Lie algebras and let $A \in \mathfrak{V}$. We shall make use of the following notation. Let $F_{\mathfrak{V}}(X)$ be the free Lie algebra with the free base $X$ in the variety $\mathfrak{V}$ and denote by $A_{\mathfrak{V}}[X] = A \ast_{\mathfrak{V}} F_{\mathfrak{V}}(X)$ the $\mathfrak{V}$-free product of $A$ and $F_{\mathfrak{V}}(X)$ (for definitions see [1]). Suppose that $\varphi$ is the canonical homomorphism from the free $A$–algebra $A[X]$ to $A_{\mathfrak{V}}[X]$. Then for every system of equations $S \subseteq A[X]$ holds $\ker \varphi \subseteq \text{Rad}_A(S)$. Which implies that

$$\Gamma(S) = A[X] / \text{Rad}_A(S) \cong_A A_{\mathfrak{V}}[X] / \text{Rad}_{\mathfrak{V}}(S), \quad \text{Rad}_A(S) = \text{Rad}_A(S).$$
Where here $\text{Rad}_A(S)$ denotes the image of $\text{Rad}_A(S)$ in $A_{\mathfrak{M}}[X]$ under $\varphi$. The definition of $\text{Rad}_A(S)$ coincides with the one of $\text{Rad}_A(S)$ with all the preliminary notions given with respect to the algebra $A_{\mathfrak{M}}[X]$

**Remark 2.3** Let $\mathfrak{M}$ be the variety of all metabelian Lie algebras and let $A \in \mathfrak{M}$. From the above discussion follows that all arguments for the radicals of systems of equations over $A$ and respective coordinate algebras can be performed in the metabelian Lie algebra $A_{\mathfrak{M}}[X] = A *_{\mathfrak{M}} F[X]$.

Let $S$ be a system of equations over $F_r$. On behalf of Remark 2.3 we may assume that $S \subseteq (F_r)_{\mathfrak{M}}[X] = F_r *_{\mathfrak{M}} F_r[X]$. Denote by $I_X = \langle X \rangle$ the ideal of $(F_r)_{\mathfrak{M}}[X]$, generated by the alphabet $X$. Consider an equation $f(x_1, \ldots, x_n) \in S$. We next write $f$ as a sum of homogeneous (by the variables $x_1, \ldots, x_n$) monomials:

$$f = c + x_1h_1 + \cdots + x_nh_n + g(x_1, \ldots, x_n),$$

where here $c \in F_r$, $h_i \in R$ are polynomials, $i = 1, \ldots, n$, $g(x_1, \ldots, x_n) \in I_X^2$. Note that the expression $x_ih_i$ is not uniquely defined but the choice of presentations of $h_1, \ldots, h_n$ is not significant, see [6]. Since any solution of the system $S$ is a sum of the linear part and the part from $\text{Fit}(F_r)$, to solve the system $S$ it is convenient to write the variables $x_i$, $i = 1, \ldots, n$ as a sum of two variables:

$$x_i = z_i + y_i; \quad y_i \in \text{Fit}(F_r), \quad z_i = \alpha_{i1}a_1 + \cdots + \alpha_{ir}a_r, \quad \alpha_{ij} \in k.$$

This increases the number of variables to $nr + n$: the variables $\alpha_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, r$ are valued in $k$ and the variables $y_1, \ldots, y_n$ in $\text{Fit}(F_r)$.

However, this increase of the number of variables simplifies the system $S$. Indeed, substituting the variables $x_i$ in the form $z_i + y_i$ into the equation $f = 0$ parts the system into two. We separately equate to zero the linear part and the part from $\text{Fit}(F_r)$. The first system is a regular linear system of equations, which is solved using the methods of linear algebra. If the correspondent system of equations over $k$ is inconsistent, then so is the initial system $S$ over $F_r$. Suppose that the linear system of equations is consistent. To every its solution $\alpha_{ij}'$, $i = 1, \ldots, n$, $j = 1, \ldots, r$ we associate secondary module system of equations over $\text{Fit}(F_r)$ with coefficients in $R$. 

10
3 Universal Axioms and the $\Phi_r$-Algebras

The above notation apply. In this section we formulate two collections of seven series of universal axioms $\Phi_r$ and $\Phi'_r$ in the languages $L$ and $L_{F_r}$. In the next section we prove that $\Phi_r$ and $\Phi'_r$ axiomatise the universal classes $\text{uc}1(F_r)$ and $F_r - \text{uc}1(F_r)$. Most of these formulas are the formulas of the first order language $L$. Consequently they are the formulas of both $\Phi_r$ and $\Phi'_r$. We, therefore, write these series simultaneously, pointing the differences between $\Phi_r$ and $\Phi'_r$.

Since the algebra $F_r$ is metabelian we write the metabelian identity

$$\Phi_1 : \forall x_1, x_2, x_3, x_4 \ (x_1 x_2)(x_3 x_4) = 0.$$ 

On account of Lemma 3.3 in [6] the algebra $F_r$ satisfies two following axioms

$$\Phi_2 : \forall x \forall y \ xyx = 0 \land xyy = 0 \rightarrow xy = 0.$$  

$$\Phi_3 : \forall x \forall y \forall z \ x \neq 0 \land xy = 0 \land xz = 0 \rightarrow yz = 0.$$ 

The universal formula $\Phi_3$ is called the CT-axiom (commutative transitivity axiom).

Let $\mathcal{N}_2$ be the quasi variety of all Lie algebras defined by $\Phi_1$ and $\Phi_2$ and let $\mathcal{N}_3$ be the universal class axiomatised by $\Phi_1$, $\Phi_2$ and $\Phi_3$.

**Lemma 3.1** Let $B \in \mathcal{N}_2$. Then $B$ is a metabelian Lie algebra such that $\text{Fit}(B)$ and every nilpotent subalgebra of $B$ are Abelian.

**Proof.** By axiom $\Phi_1$ the algebra $B$ is metabelian. Let $C$ be a nilpotent subalgebra of $B$ and $c_1, c_2 \in C$. Suppose that $c_1 \circ c_2 \neq 0$. Then on account of Lemma 2.2 from [6] there exists a two-generated nilpotent subalgebra $D = \langle d_1, d_2 \rangle$ of class two in $C$. It is essentially immediate that $d_1 d_2 d_1 = d_1 d_2 d_2 = 0$, while $d_1 d_2 \neq 0$, deriving a contradiction to $\Phi_2$. Finally, recall that if every nilpotent subalgebra of $B$ is Abelian then $\text{Fit}(B)$ is Abelian (see Lemma 2.4 in [6]).

**Lemma 3.2** Let $B \in \mathcal{N}_3$ and let $a \in B \setminus \text{Fit}(B)$. Then for an arbitrary non-zero element $b$ from $\text{Fit}(B)$ holds $ab \neq 0$. 

11
**Proof.** Assume the converse: \( ab = 0 \) for a non-zero element \( b \in \text{Fit}(B) \). For \( \text{Fit}(B) \) is Abelian, \( bd = 0 \) for any \( d \in \text{Fit}(B) \). By the CT-axiom it follows that \( ad = 0 \), i.e. \( a \) commutes with elements from \( \text{Fit}(B) \). Consequently, see Lemma 2.4 in [6], \( a \in \text{Fit}(B) \). This contradicts the assumption of the lemma. \( \square \)

We next introduce universal formula \( \text{Fit}(x) \) of the language \( L \) with one variable \( x \). The formula \( \text{Fit}(x) \) defines the Fitting’s radical

\[
\text{Fit}(x) \equiv (\forall y \ xyx = 0).
\]  

(1)

The analogue of Formula (1) in the language \( L_F \) is

\[
\text{Fit'}(x) \equiv (\land_{i} (xa_{i}x = 0)).
\]  

(2)

**Lemma 3.3** Let \( B \in \mathfrak{N}_3 \). Then the truth domain of Formula (1) is \( \text{Fit}(B) \). In the event that \( B \) is an \( F_r \)-algebra the truth domain of \( \text{Fit}(x) \) is also \( \text{Fit}(B) \).

**Proof.** According to Lemma 3.1 the Fitting’s radical \( \text{Fit}(B) \) is Abelian. It, therefore, is contained in the truth domain of the Formula (1) (Formula 2, respectively). Conversely, if \( b \in B \) satisfies Formula (1) then the ideal \( I = \langle b \rangle \) is Abelian and consequently \( b \in \text{Fit}(B) \).

For the case of \( F_r \)-algebras, we note that if \( b \notin \text{Fit}(B) \) then, by axiom \( \Phi_3 \), for an element \( a_{i} \) from the free base of \( F_r \) holds \( ba_{i} \neq 0 \). Lemma 3.2 therefore implies that \( ba_{i}b \neq 0 \). \( \square \)

**Nota Bene** We next restrict ourselves to the case of a finite field \( k \). In which case the vector space \( F/F_\text{Fit}(F_r) \) is finite and its dimension over \( k \) is \( r \).

**Lemma 3.4** Let \( k \) be a finite field and \( n \in \mathbb{N}, n \leq r \). Then existential formula

\[
\varphi(x_1, \ldots, x_n) \equiv (\land_{(\alpha_1, \ldots, \alpha_n) \neq 0} \neg \text{Fit}(\alpha_1x_1 + \cdots + \alpha_nx_n)).
\]

of the language \( L \) is true on the elements \( \{b_1, \ldots, b_n\} \) of \( F_r \) if and only if \( b_1, \ldots, b_n \) are linearly independent modulo \( \text{Fit}(F_r) \).
Proof. First we note that, since \( k \) is finite, there exist only a finite number of \( n \)-tuples \( \alpha_1, \ldots, \alpha_n \in k \). In what it follows that \( \varphi(x_1, \ldots, x_n) \) is a formula of the language \( L \). Now, since \( \text{Fit}(x) \) is a universal formula, the negation \( \neg \text{Fit}(x) \) is an existential formula and so is the formula \( \varphi(x_1, \ldots, x_n) \).

Let \( \{ b_1, \ldots, b_n \} \) be a system of elements from the truth domain of the formula \( \varphi \). By Lemma 3.3 this system is linearly independent modulo \( \text{Fit}(F_r) \). The converse is obviously also true.

Resulting from Lemma 3.3 the truth domain of \( \varphi(x_1, \ldots, x_n) \) in the algebra \( B \in \mathfrak{N}_3 \) is the set of all linearly independent modulo \( \text{Fit}(B) \) \( n \)-tuples, \( n \leq r \). Therefore, the formula \( \varphi \) formalises the notion of linear independence modulo the Fitting’s radical for a tuple of elements, provided that the field \( k \) is finite. The formula \( \varphi \) is very convenient to use in the language \( L \). In the language \( L_F \), there is another, more simple way to test whether the elements \( b_1, \ldots, b_n \) are linearly independent modulo \( \text{Fit}(B) \).

Lemma 3.5 Let \( B \) be an \( F_r \)-Lie algebra, \( B \in \mathfrak{N}_3 \) and let \( c_1, \ldots, c_n, n \leq r \) be linearly independent elements modulo the Fitting’s radical of the designated copy of \( F_r \). Then the elements \( c_1, \ldots, c_n \) are linearly independent modulo \( \text{Fit}(B) \).

Proof. Suppose that a non-trivial linear combination \( c = \alpha_1 c_1 + \cdots + \alpha_n c_n, \alpha_i \in k \) lies in \( \text{Fit}(B) \). Since \( \text{Fit}(B) \) is Abelian, \( a_1 a_2 c = 0 \). This derives a contradiction, for \( F_r \) is a \( U \)-algebra.

With the help of the formula \( \varphi \) we next write the dimension axiom

\[
\Phi_4 : \forall x_1, \ldots, x_{r+1} \neg \varphi(x_1, \ldots, x_{r+1}).
\]

For \( \varphi \) is an existential formula, the formula \( \neg \varphi \) is a universal formula, thus so is the formula \( \Phi_4 \). This axiom postulates that the dimension of the factor-space \( B/\text{Fit}(B) \) is lower than or equals \( r \), provided that \( B \in \mathfrak{N}_3 \).

Recall that \( \text{Fit}(F_r) \) allows the structure of a module over the ring of polynomials \( R = k[x_1, \ldots, x_r] \). The series of axioms \( \Phi_5, \Phi'_5, \Phi_6, \Phi_7 \) and \( \Phi'_7 \) express module properties of \( \text{Fit}(F_r) \). We use module notation here, i.e. the multiplication of elements of an algebra on elements from \( R \). By this notation (we refer to [6] for details) we mean that the polynomial \( f(x_1, \ldots, x_n), n \leq r \) rewrites into the signature of metabelian Lie algebras.

The Fitting’s radical of the free metabelian Lie algebra is a torsion free module over the ring \( R \). Consequently we write the following infinite series
of axioms. For every non-zero polynomial \( f \in k[x_1, \ldots, x_n] \), \( n \leq r \) write

\[
\Phi_5 : \forall z_1, z_2 \forall x_1, \ldots, x_n \quad (z_1 z_2 \cdot f(x_1, \ldots, x_n) = 0 \land z_1 z_2 \neq 0) \rightarrow \\
\rightarrow (\neg \varphi(x_1, \ldots, x_n)).
\]

Since \( \varphi(x_1, \ldots, x_n) \) is a \( \exists \)-formula the formula \( \Phi_5 \) is a \( \forall \)-formula.

For the collection \( \Phi'_r \), i.e. for axioms in the language \( L_{F_r} \), this fact can be expressed in a more simple way,

\[
\Phi'_5 : \forall z_1, z_2 \quad (z_1 z_2 \cdot f(a_1, \ldots, a_r) = 0 \rightarrow z_1 z_2 = 0).
\]

Here \( f \) is a non-zero polynomial from \( k[x_1, \ldots, x_r] \).

The main advantage of this formula is that it does not involve the formula \( \varphi \), which implies that the restriction on the cardinality of the field \( k \) is not significant.

Let \( \mathfrak{N}_5 \) and \( \mathfrak{N}'_5 \) correspondingly be the universal classes generated by the series of axioms \( \Phi_1 - \Phi_5 \) and \( \Phi_1 - \Phi_4, \Phi'_5 \).

**Lemma 3.6** The class \( \mathfrak{N}_5 \) is the class of all \( U \)-algebras \( B \) such that \( \dim B/\text{Fit}(B) \leq r \).

The class \( \mathfrak{N}'_5 \) is the class of all \( F_r \)-\( U \)-algebras \( B \) such that \( \dim B/\text{Fit}(B) = r \).

**Proof.** Let \( B \) be a non-abelian Lie algebra from the class \( \mathfrak{N}_5 \). Then, according to Lemma 3.1, \( \text{Fit}(B) \) is an Abelian ideal. Assume that \( \dim B/\text{Fit}(B) = n \). The inequality \( n \leq r \) follows immediately from Lemma 3.2 and axiom \( \Phi_4 \). In the event that \( B \) is an \( F_r \)-algebra \( n \) equals \( r \) (see Lemma 3.5).

We next show that \( B \) is a \( U \)-algebra. The Fitting’s radical \( \text{Fit}(B) \) admits a structure of a module over the ring \( k[x_1, \ldots, x_n] \). We show that \( \text{Fit}(B) \) is torsion-free. Take an element \( 0 \neq b \in \text{Fit}(B) \) and a non-zero polynomial \( f(x_1, \ldots, x_n) \). By Lemma 3.2, \( ba \neq 0 \) for any \( a \in B \setminus \text{Fit}(B) \). By axiom \( \Phi_4 \), \( (ba) \cdot f(x_1, \ldots, x_n) \neq 0 \). Since \( b \in \text{Fit}(B) \), the product \( ba \) can be written as: \( ba = b \cdot g(x_1, \ldots, x_n) \), where \( g(x_1, \ldots, x_n) \) is a linear non-zero polynomial. In what follows that \( b \cdot f(x_1, \ldots, x_n) \neq 0 \), that \( \text{Fit}(B) \) is a torsion-free module, and therefore \( B \) is a \( U \)-algebra.

Conversely, let \( B \) be a metabelian \( U \)-algebra and let \( \dim B/\text{Fit}(B) = n \), \( n \leq r \). We show that \( B \) lies in \( \mathfrak{N}_5 \). From elementary properties of metabelian \( U \)-algebras (see Theorem 3.4 [6]) it follows that \( B \) lies in \( \mathfrak{N}_3 \). From Lemmas 3.4 and 3.5 it is immediate that \( B \) lies in \( \mathfrak{N}_5 \) (\( \mathfrak{N}'_5 \)). \( \blacksquare \)
According to Corollary 2.4 in [6], every n-tuple of linearly independent modulo $\text{Fit}(F_r)$ elements $\{b_1, \ldots, b_n\}$, $n \leq r$ of $F_r$ freely generates the free metabelian Lie algebra of the rank $n$. In the event that $k$ is finite, the statement of Corollary 2.4 for algebras from $\mathfrak{N}_5$ and $\mathfrak{N}'_5$ can be written by the means of universal formulas.

For every non-zero Lie polynomial $l(a_1, \ldots, a_n)$, $n \leq r$ of the letters $a_1, \ldots, a_r$ from the free base of $F_r$ write

$$\Phi_6 : \forall x_1, \ldots, x_n \ varphi(x_1, \ldots, x_n) \rightarrow (l(x_1, \ldots, x_n) \neq 0).$$

Since $\varphi(x_1, \ldots, x_n)$ is a $\exists$-formula the formula $\Phi_6$ is a $\forall$-formula.

Denote by $\mathfrak{N}_6$ and $\mathfrak{N}'_6$, correspondingly, the universal classes generated by the series of axioms $\Phi_1 - \Phi_6$ and $\Phi_1 - \Phi_4, \Phi'_5, \Phi_6$.

Lemma 3.7 Let $B = F_n \oplus M$ be the direct module extension of $F_n$ (see Section 4.3 in [6]), where here $M$ is a torsion free module over $k[x_1, \ldots, x_n]$, $n \leq r$. Then $B \in \mathfrak{N}_6$ and, in the event that $n = r$, $B \in \mathfrak{N}'_6$.

Proof. Note that, according to Lemma 3.6, the free metabelian Lie algebra $F_n$ of the rank $n \leq r$ lies in $\mathfrak{N}_5$. Therefore, it is clear that $F_n \in \mathfrak{N}_6$. Since $\text{uc1}(F_n) = \text{uc1}(B)$ (see Proposition 4.4 in [6]), the algebra $B$ also lies in $\mathfrak{N}_6$.

Unfortunately not every finitely generated algebra from $\mathfrak{N}_6$ has the form $F_n \oplus M$. On the other hand, all finitely generated algebras from $\text{uc1}(F_r)$ and $F_r - \text{uc1}(F_r)$ have this form (see Theorems 4.1 and 4.2). The point is that every Lie algebra from $\mathfrak{N}_6$ (correspondingly, $\mathfrak{N}'_6$) is obtained from $F_n$, $n \leq r$ (correspondingly from $F_r$) by the means of an extension of its Fitting’s radical. But in general, this extension is not the direct module extension, i.e. the Fitting’s radical of the algebra from $\mathfrak{N}_6$ (or from $\mathfrak{N}'_6$) is not the direct sum of a new module and the initial Fitting’s radical. To narrow the classes $\mathfrak{N}_6$ and $\mathfrak{N}'_6$ we need to write the final the most sophisticated series of axioms $\Phi_7$ and $\Phi'_7$.

We first introduce higher-dimensional analogues of Formulas (1) and (2)

$$\text{Fit}(y_1, \ldots, y_l; x_1, \ldots, x_n) \equiv \bigwedge_{i=1}^{n} (y_1 x_i y_1 = 0) \land \ldots \land \bigwedge_{i=1}^{n} (y_l x_i y_l = 0), \quad (3)$$

$$\text{Fit}'(y_1, \ldots, y_l) \equiv \text{Fit}'(y_1) \land \ldots \land \text{Fit}'(y_l). \quad (4)$$
Formula (I) defines the subset

$$\operatorname{Fit}(B) \times \cdots \times \operatorname{Fit}(B) \subseteq B \times \cdots \times B,$$

provided that $B$ is an $F_r$–Lie algebra from $\mathfrak{N}_3$.

Let $B$ be an algebra from $\mathfrak{N}_6$ and let $\{b_1, \ldots, b_n\}$ be a linearly independent modulo $\operatorname{Fit}(B)$ set of elements from $B$, $n \leq r$, $n \geq 2$. The truth domain of the formula $\operatorname{Fit}(y_1, \ldots, y_l; b_1, \ldots, b_n)$ is $\operatorname{Fit}(B) \times \cdots \times \operatorname{Fit}(B)$. In the event that $n = 1$ the same holds, provided that $\dim B/\operatorname{Fit}(B) = 1$.

Before we introduce the final series of axioms we need to explain the syntax of these formulas. We begin with the series of axioms $\Phi^7$ in the language $L_{F_r}$. Let $S$ be a fixed finite system of module equations with variables $y_1, \ldots, y_l$ over the module $\operatorname{Fit}(F_r)$. Every equation from $S$ has the form

$$h = y_1 f_1(\bar{x}) + \cdots + y_l f_l(\bar{x}) - c = 0, \quad c = c(a_1, \ldots, a_r) \in \operatorname{Fit}(F_r),$$

where here $\bar{x} = \{x_1, \ldots, x_r\}$ is a vector of variables and $f_1, \ldots, f_l \in R = k[\bar{x}]$. Suppose that $S$ is inconsistent over $\operatorname{Fit}(F_r)$. This fact can be easily written in the signature of a module. The system $S$ gives rise to a system of equations $S_1$ over $F_r$. Replace every module equation $h_i = 0$ from $S$ by the equation $h'_i = 0$, $i = 1, \ldots, m$ in the signature of $L_{F_r}$ (see [9]). This results a system of equations $S_1$ over $F_r$. By every inconsistent module system of equations $S$ write

$$\Phi^7 : \quad \psi'_S \equiv \forall y_1, \ldots, y_l \quad \operatorname{Fit}'(y_1, \ldots, y_l) \rightarrow \bigvee_{i=1}^m h'_i(y_1, \ldots, y_l) \neq 0.$$

Notice that the restriction on the cardinality of the field $k$ is not used.

Denote by $\mathfrak{V}_7'$ the universal class axiomatised by $\Phi_1 - \Phi_4, \Phi'_5, \Phi_6, \Phi'_7$.

**Lemma 3.8** Let $B$ be a finitely generated $F_r$–algebra from $\mathfrak{N}_7'$. Then $B$ is $F_r$–isomorphic to the algebra $F_r \oplus M$ for some finitely generated torsion free module $M$ over $R$. 

16
Proof. Suppose that the elements $a_1, \ldots, a_r$ generate the designated copy of $F_r$ in $B$. By Lemma 3.5 these elements are linearly independent modulo $\text{Fit}(B)$. For $B$ is a finitely generated algebra from $\mathcal{N}_r$, from Lemma 3.6 we conclude that $\text{Fit}(B)$ is a finitely generated torsion-free module over the ring $R$ (see Lemma 2.6 in [6]). It, therefore, suffices to prove that the submodule $\text{Fit}(F_r)$ of the module $\text{Fit}(B)$ is a direct summand.

For the sake of brevity, set $P = \text{Fit}(B)$, $N = \text{Fit}(F_r)$. We prove that the factor-module $P/N$ is torsion-free. Assume the converse, then there exists an element $d \in \text{Fit}(F_r)$ and a non-zero polynomial $f(x_1, \ldots, x_r)$ so that the equation

$$y \cdot f(x_1, \ldots, x_r) = d$$

is compatible over $\text{Fit}(B)$ (let $y_1 \in \text{Fit}(B)$ be its solution) but incompatible over $\text{Fit}(F_r)$. Assume that, on the contrary, there exists a solution $y_2 \in \text{Fit}(F_r)$. In what follows that $(y_1 - y_2) \cdot f(x_1, \ldots, x_r) = 0$, deriving a contradiction with the fact that $P$ is torsion-free.

Equation (5) gives rise to an incompatible module system of equations $S_0$ over $\text{Fit}(F_r)$. In what it follows that the axiom $\psi'_{S_0}$ from the series $\Phi'7$ is true in $B$, which is obviously false. Further, if $P/N$ is a free module then $P = N \oplus M$ and $B = F_r \oplus M$. So we assume that $P/N$ is not a free module and that the images of the elements $m_1, \ldots, m_l \in P$ are its generators. Let $S$ be a finite system of module relations for the generators $\bar{m}_1, \ldots, \bar{m}_l$ of the module $P/N$. If in the left-hand sides of the relations we replace the generators by the variables $y_i$’s and in the right-hand side we write the values of the left-hand side relations with $y_i$’s valued as $m_i$’s, we obtain a finite system $S_1$ of module equations over $N$. The system $S_1$ has a solution $\{m_1, \ldots, m_l\}$ in $P$. Consequently, by the axioms of the series $\Phi'7$, the system $S_1$ has a solution $\{c_1, \ldots, c_l\}$ in $\text{Fit}(F_r)$. Denote by $M$ the submodule of the module $P$ generated by the elements $m_i - c_i = m_i'$, $i = 1, \ldots, l$. Clearly, $M$ is isomorphic to the module $P/N$, and consequently $P = N \oplus M$. In what follows that $B = F_r \oplus M$ (see Lemma 4.8 [6]).

Now we turn to the collection $\Phi_r$ in the language $L$. Let $B \in \mathfrak{N}_0$ and let $C$ be its subalgebra generated by the elements $c_1, \ldots, c_n$, $n \leq r$. Assume that $c_1, \ldots, c_n$ are linearly independent modulo $\text{Fit}(B)$. Then $C$ is isomorphic to $F_n$. Although, even in the event that $B = F_r$ the subalgebra $C \subset F_r$ does not yield to the decomposition $F_r \oplus M$. To avoid this problem we use the notion of $\Delta$-localisation of $F_r$ (see Section 4.2 in [6]).

The basic idea of the formula $\Phi'7$ is to write it in such a way that an
analogue of Lemma 3.8 holds for ∆-local Lie algebras from \( \mathfrak{H}_7 \).

We next introduce some auxiliary notation. Let \( S \) be a finite system of module equations with variables \( y_1, \ldots, y_l \) over \( \text{Fit}(F_n) \), \( n \leq r \) and let \( f(x_1, \ldots, x_n) \) be a polynomial from \( R \setminus \Delta \), where here \( \Delta = \langle x_1, \ldots, x_r \rangle \). Denote by \( UD(f) \) the collection of all unitary divisors of \( f \), \( UD(f) \subseteq R \setminus \Delta \). For each \( f \) and each \( \alpha \in UD(f) \times \cdots \times UD(f) \), \( \alpha = (d_1, \ldots, d_l) \) define a system of equations \( S_{f,\alpha} \). The system \( S_{f,\alpha} \) is obtained from \( S \) by multiplying the equations from \( S \) by the polynomial \( d = d_1 \cdots d_l \) and dividing the coefficient of the term \( y_i \) by \( d_i \).

**Lemma 3.9** Let \( B \) be an algebra from \( \mathfrak{H}_6 \). In the above notation, the system \( S \) is consistent over \( \text{Fit}_\Delta(B) \) if and only if there exist \( f(x_1, \ldots, x_n) \in R \setminus \Delta \) and \( \alpha \in UD(f) \times \cdots \times UD(f) \) such that the system \( S_{f,\alpha} \) is consistent over \( \text{Fit}(B) \).

**Proof.** The proof is straightforward. \[\Box\]

**Remark 3.1** If \( S \) is inconsistent over \( \text{Fit}_\Delta(B) \), then for all \( f \) and \( \alpha \) the system \( S_{f,\alpha} \) is inconsistent over \( \text{Fit}_\Delta(B) \).

Suppose that \( S \) is inconsistent over the Fitting’s radical \( \text{Fit}_\Delta(F_n) \) of ∆-local Lie algebra \( (F_n)\Delta \) system of \( m \) module equations over \( \text{Fit}(F_n) \). For every \( n \in \mathbb{N} \), \( n \leq r \) and a system \( S \) we write

\[
\Phi_7 : \psi_{n,S} \equiv \forall x_1, \ldots, x_n \forall y_1, \ldots, y_l \quad \varphi(x_1, \ldots, x_n) \land \text{Fit}(y_1, \ldots, y_l) \rightarrow \nabla_{i=1}^m h_i(y_1, \ldots, y_l; x_1, \ldots, x_n) \neq 0.
\]

The Lie polynomials \( h_i, i = 1, \ldots, m \) are obtained from the system \( S \). Consider the \( i \)-th polynomial from \( S \). It has the form

\[
h'_i = y_1 f_1(x_1, \ldots, x_n) + \cdots + y_l f_l(x_1, \ldots, x_n) - c = 0,
\]

\[
f_i \in R, \quad c = c(a_1, \ldots, a_n) \in \text{Fit}(F_n).
\]

Its interpretation in the signature of Lie algebras (see [6]) with every occurrence of \( a_j \) in \( c(a_1, \ldots, a_n) \) replaced by \( x_j, j = 1, \ldots, n \) results the polynomial \( h_i \).

**Lemma 3.10** The axioms of the series \( \Phi_7 \) are true in the free metabelian Lie algebra \( F_r \).
Lemma 3.11 If an algebra $F_r$ is incompatible over $\text{Fit}(F_n)$, $n \leq r$. In which case it is clear that $S$ is incompatible over $\text{Fit}(F_r)$. Take a tuple of arbitrary linearly independent modulo $\text{Fit}(F_r)$ elements $c_1, \ldots, c_r \in F_r$. Denote by $C$ the subalgebra of $F_r$ generated by these elements. The subalgebra $C$ is isomorphic to the algebra $F_r$. Therefore, the system $S$ is incompatible over $\text{Fit}(C)$ and thus $h_i(b_1, \ldots, b_i; c_1, \ldots, c_n) \neq 0$, $i = 1, \ldots, m$ for any $b_1, \ldots, b_i \in \text{Fit}(C)$. But if we treat $C$ as a set it might not coincide with $F_r$. However we have $C_\Delta = (F_r)_\Delta$ and $\text{Fit}(C_\Delta) = \text{Fit}(F_r)$ (see Proposition 4.2 [3]). We, therefore, obtain that the correspondent formula $\psi_{n,S}$ is true in the algebra $F_r$.

Denote by $\mathcal{N}_7$ the universal class axiomatised by $\Phi_1 - \Phi_7$.

Lemma 3.11 If an algebra $B$ lies in $\mathcal{N}_7$ then its $\Delta$-localisation $B_\Delta$ lies in $\mathcal{N}_7$ and has the form $B_\Delta = (F_n)_\Delta \oplus M_\Delta$ for some $n \leq r$ and some finitely generated torsion free module $M$ over the ring $k[x_1, \ldots, x_n]$. Furthermore, there exists an integer $s \in \mathbb{N}$ such that the algebra $B$ is a subalgebra of $F_{r,s}$.

Proof. Let $\dim B_{\text{Fit}(B)} = n$. By Lemma 3.6 $n \leq r$ and $B$ is a $U$-algebra. By the axioms of the series $\Phi_6$, a tuple of linearly independent modulo $\text{Fit}(B)$ elements $b_1, \ldots, b_n$ generates a subalgebra of $B$ isomorphic to $F_n$.

The fact that $B_\Delta \in \mathcal{N}_7$ is implied by the coincidence of universal closures $\text{uc}l(B) = \text{uc}l(B_\Delta)$, which is true for an arbitrary $U$-algebra $B$ (see Proposition 4.1 [6]).

We next show that if a finite system of module equations $S$ has a solution in $\text{Fit}_\Delta(B)$, then it has a solution in $\text{Fit}_\Delta(F_n)$. If $S$ is compatible over $\text{Fit}_\Delta(B)$ then by Lemma 3.9 for some $f(x_1, \ldots, x_n) \in R \setminus \Delta$ and some $\alpha \in D(f) \times \cdots \times D(f)$ the system $S_{f,\alpha}$ is compatible over $\text{Fit}(B)$. From the series of axioms $\Phi_7$ it, therefore, follows that $S_{f,\alpha}$ has a solution in $\text{Fit}_\Delta(F_n)$ and consequently, on account of Lemma 3.9 the system $S$ has a solution in $\text{Fit}_\Delta(F_n)$. To prove that $B_\Delta$ has the form $(F_n)_\Delta \oplus M_\Delta$ it suffices to conduct an argument analogous to the one of Lemma 3.3. We leave this to the reader.

Since $(F_n)_\Delta \oplus M_\Delta = (F_n \oplus M)_\Delta$ (see Lemma 4.5 in [6]) and since $B$ embeds into $B_\Delta$, the algebra $B$ is a finitely generated subalgebra of a $\Delta$-local algebra $(F_n \oplus M)_\Delta$. In what follows that $B$ embeds into the algebra $F_n \oplus M$ (see Lemma 4.1 in [6]). The module $M$ embeds into the free module $T_s$ of the rank $s$ over $R$ (see [3], [7]). This embedding gives rise to an embedding of the algebra $F_n \oplus M$ into the algebra $F_{n,s}$ (see Lemma 4.8 [6]), which is
a subalgebra of $F_{r,s}$. We, therefore, have proven that $B$ embeds into the algebra $F_{r,s}$. ■

Denote, correspondingly, by $\Phi_r$ and by $\Phi'_r$ the universal classes axiomatised by $\Phi_1 - \Phi_7$ and by $\Phi_1 - \Phi_4, \Phi'_5, \Phi_6, \Phi'_7$ respectively.

**Definition 3.1** The algebras from $\Phi_r$ and $\Phi'_r$ are termed, correspondingly, $\Phi_r$-algebras and $\Phi'_r$-algebras.

**Lemma 3.12** For every $r \in \mathbb{N}$ and every $n \leq r$ and $m > r$ the algebra $F_n$ is a $\Phi_r$-algebra, while $F_m$ is not a $\Phi_r$-algebra. The algebra $F_r$ lies in $\Phi'_r$ and $F_r \notin \Phi'_n$, provided that $n \neq r$.

**Proof.** From the axioms $\Phi_r$ and the properties of $F_r$ proved above, the algebra $F_n$ is a $\Phi_r$-algebra. However, the algebra, $F_m$, $m > r$ does not lie in $\Phi_r$, for the dimension axiom is false if $m > r$. ■

**Corollary 3.1** The following sequence of strict inclusions holds

$$\Phi_1 \subsetneq \Phi_2 \subsetneq \ldots \subsetneq \Phi_r \subsetneq \ldots$$

The classes $\Phi'_{r_1}$ and $\Phi'_{r_2}$ are disjoint.

**Corollary 3.2** Let $A$ be a $\Phi_r$-algebra and let $\dim_{F_{\text{fit}}(A)} = n < r$. Then $A$ is a $\Phi_n$-algebra but not a $\Phi_m$-algebra, where here $m < n$.

Finally, we emphasise the aspects which impose the restriction on the cardinality of the field $k$.

First of all, the formula $\varphi$ is finite only because the ground field $k$ is finite. Recall that the formula $\varphi$ is written in the language $L$ and formalises the notion of linear independence modulo the Fitting’s radical. The formula $\varphi$ is used in all the axioms of the series $\Phi_4 - \Phi_7$.

In the enriched language $L_{F_r}$ the formula $\varphi$ is almost unnecessary. It is involved in neither $\Phi'_5$ nor $\Phi'_7$ and is only used in the dimension axiom $\Phi_4$ and in the series $\Phi_6$. But the axioms of $\Phi_6$ can be excluded from the collection $\Phi'_r$. The series $\Phi_6$ postulates that every set of linearly independent modulo the Fitting’s radical elements $\{b_1, \ldots, b_n\}$, $n \leq r$ freely generates an algebra isomorphic to $F_n$. This property is important only in the proofs of Lemmas [3.8] and [3.11] which require only the existence of such $n$-tuples.
Since every $F_r$-algebra contains a designated copy of $F_r$, which possesses such an $n$-tuple, the series $\Phi_6$ can be omitted in the case of $L_{F_r}$.

However, the axiom $\Phi_4$ is significant. We can not write this axiom without the use of the formula $\varphi$ and can not exclude $\Phi_4$ from $\Phi'$. In the event that the field $k$ is finite the dimension axiom is true in $F_r$ and is true in every algebra from $F_r - \text{uc1}(F_r)$. But in the event that the main field is infinite the algebra $F_r, r \geq 2$ is discriminated by $F_2$. Which essentially implies that $F_r - \text{uc1}(F_r)$ contains algebras $A$ of ‘unlimited’ dimension of $A/F_{\text{Fit}}(A)$.

4 Theorems on Universal Closures of the Algebra $F_r$

In this section we formulate and prove several theorems on universal classes $U_r = \text{uc1}(F_r)$ and $U'_r = F_r - \text{uc1}(F_r)$. The main results here are

- The set of axioms $\Phi_r$ ($\Phi'_r$) axiomatise the universal closure of the free metabelian Lie algebra of finite rank $r \geq 2$ over a finite field $k$,

- Given a description of the structure of finitely generated algebras from $F_r - \text{uc1}(F_r)$ and $\text{uc1}(F_r)$,

- Investigated the structure of irreducible algebraic sets over $F_r$ and the structure of correspondent coordinate algebras,

- Proved that the universal theory of the free metabelian Lie algebra is decidable in both $L$ and $L_F$.

4.1 Formulation of Main Results

Theorem 4.1 Let $A$ be an arbitrary finitely generated metabelian Lie algebra over a finite field $k$. Then the following conditions are equivalent

- $A \in \text{uc1}(F_r)$;

- $A$ is a $\Phi_r$-algebra;

- there exists $s \in \mathbb{N}$ such that $A$ is a subalgebra of $F_{r,s}$.

Corollary 4.1 The universal closure $\text{uc1}(F_r)$ of the free metabelian Lie algebra $F_r$ is axiomatised by $\Phi_r$. 
Theorem 4.2 Let $A$ be an arbitrary finitely generated metabelian $F_r$–Lie algebra over a finite field $k$. Then the following conditions are equivalent

- $A \in F_r - \text{ucl}(F_r)$;
- $A$ is a $\Phi'_r$-algebra;
- $A$ is $F_r$–isomorphic to the algebra $F_r \oplus M$, where here $M$ is a torsion free module over the ring of polynomials $R = k[x_1, \ldots, x_r]$;

Corollary 4.2 The universal closure $F_r - \text{ucl}(F_r)$ of the free metabelian Lie algebra $F_r$ is axiomatised by $\Phi'_r$.

Recall that the fact that the field $k$ is finite, is significant for the axiom $\Phi_4$, for series of axioms $\Phi_5$, $\Phi_6$, $\Phi_7$.

Two next theorems treat the decidability of universal theory of the algebra $F_r$ in the languages $L$ and $L_{F_r}$.

Theorem 4.3 Axioms $\Phi_r$ form a recursive set and the universal theory in the language $L$ of the algebra $F_r$ over a finite field $k$ is decidable.

Theorem 4.4 Axioms $\Phi'_r$ form a recursive set and the universal theory in the language $L_{F_r}$ of the algebra $F_r$ (treated as a $F_r$–algebra) over a finite field $k$ is decidable.

Theorem 4.5 Compatibility problem for system of equations over the free metabelian Lie algebra $F_r$ is decidable.

This result contrasts with a result of V.A. Roman’kov on the compatibility problem over some metabelian algebraic systems. In [10] he proves that this problem is undecidable for free metabelian groups of a large enough rank. The argument of [10] holds for free metabelian Lie rings and for free metabelian Lie algebras, provided that the compatibility problem for the ground field is undecidable.
4.2 Proofs of the Theorems

Proof of Theorem 4.1

1 → 2 On behalf of Lemma 3.12, \( F_r \) is a \( \Phi_r \)-algebra. Consequently, every (unnecessarily finitely generated) Lie algebra from \( \text{uc1}(F_r) \) is a \( \Phi_r \)-algebra.

2 → 3 On account of Lemma 3.11, every finitely generated Lie algebra from \( \Phi_r \) is a subalgebra of \( F_{r,s} \) for some \( s \in \mathbb{N} \).

3 → 1 Finally, from Proposition 4.4 in [6], follows that \( F_{r,s} \in \text{uc1}(F_r) \). ■

Proof of Corollary 4.1

In the proof of Theorem 4.1 above we have already mentioned that every Lie algebra from \( \text{uc1}(F_r) \) is a \( \Phi_r \)-algebra. We, therefore, are to show the converse. Let \( B \) be an arbitrary \( \Phi_r \)-algebra. Since the formulas of the collection \( \Phi_r \) are universal, we conclude that every finitely generated subalgebra \( A \) of \( B \) is a \( \Phi_r \)-algebra and consequently lies in \( \text{uc1}(F_r) \). Therefore, \( B \in \text{uc1}(F_r) \).

The proofs of Theorem 4.2 and Corollary 4.2 are analogous to the ones of Theorem 4.1 and Corollary 4.1.

Remark 4.1 In [6] the authors show (see Lemma 4.8) that for any torsion free \( R \)-module \( M \) the algebra \( F_r \oplus M F_r \)-embeds into the algebra \( F_{r,s} \) for some \( s \in \mathbb{N} \). In what follows that \( \Phi_r \)-algebras can be treated as \( F_r \)-subalgebras of \( F_{r,s} \).

Proof of Theorem 4.3

The statements of the theorem regard a universal class of a single object. From general model-theoretical facts follows that it suffices to prove the first statement only.

All the series of axioms \( \Phi_1 - \Phi_6 \) are obviously recursive, provided that the field \( k \) is finite. We now treat the series \( \Phi_7 \). The axioms of this series are enumerated by the set of finite systems of equations over the module \( \text{Fit}(F_r) \) which are inconsistent over \( \text{Fit}_\Delta(F_r) \). According to [11], the compatibility problem for systems of equations over finitely generated modules over Noetherian commutative rings is decidable. Since \( \text{Fit}_\Delta(F_r) \) is a finitely generated module over Noetherian commutative ring \( R_\Delta \) and since the set of all finite systems of equations over \( \text{Fit}(F_r) \) is recursive (in the event that the ground field \( k \) is finite), the statement follows.

The proof of Theorem 4.4 is analogous and, therefore, omitted.

Proof of Theorem 4.5
In the end of Subsection 2.2 we have pointed out that to solve a system of equations over $F_r$ it is convenient to part it into two. The first one is a linear system of equations over $k$. Every solution of this system gives rise to a module system of equations over $\text{Fit}(F_r)$. In the event that the field $k$ is finite these problems are algorithmically soluble.

5  Irreducible Algebraic Sets over $F_r$ and Dimension

In Section 2 we have introduced the category of algebraic sets over an arbitrary Lie algebra and the category of coordinate algebras. We show there that these two categories are equivalent. In particular, the classification of coordinate algebras gives us a classification of algebraic sets. Furthermore, the algebraic set is defined by its coordinate algebra up to isomorphism in the respective category. In Subsection 5.1 below we give a classification of irreducible algebraic sets over $F_r$, using the classification of its coordinate algebras and in Subsection 5.2 following the custom of classical algebraic geometry, we introduce its counterpart – the definition of a dimension of an algebraic set. In Theorem 5.2 we show how one can find the dimension of an arbitrary irreducible algebraic set over $F_r$.

5.1  Classification of Irreducible Algebraic Sets over $F_r$

Throughout this subsection we use the notation $a_1, \ldots, a_r$ for the free base of the free metabelian Lie algebra $F_r$, $r \geq 2$. Theorem 2.2 states that the collection of all coordinate algebras of irreducible algebraic sets over $F_r$ coincides with the family of all finitely generated algebras from $F_r - \text{uc1}(F_r)$. Theorem 4.2 gives a structural description of such algebras. The following proposition combines these results.

\textbf{Proposition 5.1 (on irreducible coordinate algebras over $F_r$)}

Let $\Gamma$ be a $F_r$–Lie algebra. Then $\Gamma$ is the coordinate algebra of an irreducible algebraic set over $F_r$ if and only if $\Gamma$ is $F_r$–isomorphic to $F_r \oplus M$, where here $M$ is a torsion free module over $k[x_1, \ldots, x_r]$.

Using Proposition 5.1 we next approach the problem of classification of irreducible algebraic sets over $F_r$.
We shall make use of the following notation. Let $R = k[x_1, \ldots, x_r]$ and let $M$ be a finitely generated torsion-free module over $R$. Let $\text{Hom}_R(M, \text{Fit}(F_r))$ be the set of all $R$-homomorphisms from $M$ to $\text{Fit}(F_r)$ treated as a module over $R$. This set of homomorphisms can be treated from another point of view. Fix a system of generators $\{m_1, \ldots, m_n\}$ of a module $M$. Defining an $R$-homomorphism is equivalent to defining the images of the elements $m_1, \ldots, m_n$. This defines an embedding

$$\alpha : \text{Hom}_R(M, \text{Fit}(F_r)) \rightarrow \text{Fit}(F_r) \times \cdots \times \text{Fit}(F_r),$$

where here $\phi \in \text{Hom}_R(M, \text{Fit}(F_r))$. Every relation imposed on the $n$-tuple $\{m_1, \ldots, m_n\}$ is a relation between $\phi(m_1), \ldots, \phi(m_n)$. We, therefore, identify the set $\text{Hom}_R(M, \text{Fit}(F_r))$ with its image $\alpha(\text{Hom}_R(M, \text{Fit}(F_r)))$ in $\text{Fit}(F_r) \times \cdots \times \text{Fit}(F_r)$.

**Lemma 5.1** In this notation, the following chain of one-to-one correspondences holds

$$\text{Hom}_R(M, \text{Fit}(F_r)) \leftrightarrow \text{Hom}_{F_r}(F_r \oplus M, F_r) \leftrightarrow Y,$$

where here $Y$ is an irreducible algebraic set over $F_r$ such that $\Gamma(Y) = F_r \oplus M$.

**Proof.** The result follows directly from Lemma 2.1 and Lemma 4.6 in [6].

**Theorem 5.1 (on irreducible algebraic sets over $F_r$)**

Every irreducible algebraic set over $F_r$ is, up to isomorphism, either

- a point or
- $\text{Hom}_R(M, \text{Fit}(F_r))$ for some finitely generated torsion free module $M$ over the ring $R$.

Conversely, any set of the above form is algebraic.
Proof. Consider an arbitrary irreducible algebraic set $Y$ over the algebra $F_r$. By Proposition 5.1 its coordinate algebra $F_r$ is isomorphic to the algebra of the form $F_r \oplus M$, where $M$ is a finitely generated torsion-free module over the ring $R$.

If $M = 0$, i.e. the coordinate algebra is $F_r$, in which case there exists the only $F_r$–homomorphism $\varphi$ from $\Gamma(Y)$ into $F_r$. Set $\varphi(x_i) = b_i$, $i = 1, \ldots, n$. Then, by Lemma 2.1 $Y = \{(b_1, \ldots, b_l)\} \cong \{0\}$ is a point. Conversely, every point is an algebraic set, obviously an irreducible one. The corresponding coordinate algebra is isomorphic to $F_r$ (see Examples 2.1 and 2.2).

Otherwise $M \neq 0$. On account of Lemma 5.1 it suffices to show that the set $\text{Hom}_R(M, \text{Fit}(F_r))$ is an algebraic set over $F_r$ and that its coordinate algebra is $F_r$–isomorphic to the algebra $F_r \oplus M$.

Let $M = \langle m_1, \ldots, m_n| s_1, \ldots, s_l \rangle$ be a presentation of $M$. The system of module equations $S = \{s_1 = 0, \ldots, s_l = 0\}$ gives rise to a system of equations $S_1$ over $F_r$ (see [6]). Set $S' = S_1 \cup \{a_1a_2x_i = 0, i = 1, \ldots, n\}$.

By Lemma 3.2 $V_{F_r}(S') \subseteq \text{Fit}^n(F_r)$. Consequently, $V_{F_r}(S') = \text{Hom}_R(M, \text{Fit}(F_r))$. It follows, therefore, that the set $\text{Hom}_R(M, \text{Fit}(F_r))$ is algebraic.

We show next that $\Gamma(S') \cong_{F_r} F_r \oplus M$. Set $\theta$ to be the following $F_r$–homomorphism:

$$\theta : (F_r)[X] \rightarrow F_r \oplus M, \quad \theta(x_i) = m_i, \quad i = 1, \ldots, n, \quad \theta(a) = a, \quad a \in F_r,$$

and show that $\ker \theta = \text{Rad}(S')$. Here $(F_r)[X]$ is the free $F_r$–algebra generated by the alphabet $X$ in the variety of all metabelian Lie algebras $\mathcal{M}$ and the radical $\text{Rad}(S')$ is the radical in $\mathcal{M}$. As mentioned in Section 2 the definition of a coordinate algebra $\Gamma(S')$ carries over to the variety $\mathcal{M}$.

Finally, we show that $\ker \theta = \text{Rad}(S')$. Take a polynomial $f \in \ker \theta$ and write it in the following form

$$f = c + x_1h_1 + \cdots + x_nh_n + g(x_1, \ldots, x_n), \quad c \in F_r, \quad h_i \in R, \quad g(x_1, \ldots, x_n) \in I_X^2,$$

(see Section 2). Since $F_r \oplus M$ is the direct module extension of $F_r$ by the module $M$, we have $c = 0$. Next, for

$$g(m_1, \ldots, m_n) = 0 \text{ and } f(m_1, \ldots, m_n) = 0,$$

26
we obtain the following relation in the module $M$:

$$m_1 h_1 + \cdots + m_n h_n = 0.$$ 

Now, it is essentially immediate that for any point 

$$(b_1, \ldots, b_n) \in V_{F_r}(S')$$

holds:

$$b_1 h_1 + \cdots + b_n h_n = 0,$$

i.e.

$$x_1 h_1 + \cdots + x_n h_n \in \text{Rad}(S').$$

Therefore, $f \in \text{Rad}(S').$

Suppose next that $f \in \text{Rad}(S')$ and show that $f \in \ker \theta$. Since $(0, \ldots, 0) \in V_{F_r}(S')$, we have $c = 0$. Moreover,

$$g(m_1, \ldots, m_n) = 0.$$ 

Verify that

$$m_1 h_1 + \cdots + m_n h_n = 0.$$ 

Assume the converse, then there exists an $R$-homomorphism $\phi \in \text{Hom}_R(M, \text{Fit}(F_r))$ so that

$$\phi(m_1 h_1 + \cdots + m_n h_n) \neq 0$$

(see Lemma 4.7 in [6]). This derives a contradiction, for $f \in \text{Rad}(S')$. ■

**Remark 5.1** The system $S'$, associated to $F_r \oplus M$ depends on the choice of a presentation of the module $M$. We term such systems canonical for the algebra $F_r \oplus M$.

**Corollary 5.1** In the one-dimensional affine space $F^1_r$, $n = 1$ every irreducible algebraic set is, up to isomorphism, either

- a point or
- $\text{Fit}(F_r)$.

27
Proof. The coordinate algebra of an irreducible algebraic set is $F_r$-isomorphic to the algebra $F_r \oplus M$. If $M = 0$ then the algebra set is a point. If $M \neq 0$, since $M$ is one-generated, the module is the free module $T_1$. In which case $\text{Hom}_R(T_1, \text{Fit}(F_r))$ is isomorphic to $\text{Fit}(F_r)$. ■

5.2 Dimension

Definition 5.1 Let $Y$ be an irreducible algebraic set. As is the custom in algebraic geometry, a maximum of all integers $m$ such that there exists a chain of irreducible algebraic sets

$$Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_m.$$ 

is termed the dimension of $Y$ and is denoted by $\dim(Y)$.

Definition 5.2 Let $Y = Y_1 \cup \ldots \cup Y_l$ be an expression of an algebraic set $Y$ (not necessarily irreducible) as a finite union of irreducible algebraic sets (see Theorem 2.1). We define the dimension of $Y$ (and denote it by $\dim(Y)$) to be the maximum of the dimensions of all irreducible components.

Let $Y$ be an irreducible algebraic set over $F_r$. Let $\Gamma(Y) \cong_{F_r} F_r \oplus M$ for some finitely generated torsion free module $M$ over $R = k[x_1, \ldots, x_r]$. Theorem 5.2 shows that $\dim(Y)$ is uniquely defined by the module $M$. Recall that the rank $r(M)$ of the module $M$ over the ring $R$ is a maximum of cardinalities of linearly independent over $R$ sets of elements from $M$. By the definition we set $r(\Gamma(Y)) = r(M)$.

Theorem 5.2 For an irreducible algebraic set $Y$ over $F_r$ holds

$$\dim(Y) = r(\Gamma(Y)) = r(M).$$

Proof. Let

$$Y_m \subsetneq \ldots \subsetneq Y_1 \subsetneq Y_0 = Y$$

be a strictly descending chain of irreducible algebraic sets. Obviously $Y_m$ is a point, $\dim(Y_m) = 0$, $r(\Gamma(Y_m)) = 0$. By Lemma 2.3 the inclusion $Y_{i+1} \subsetneq Y_i$ induces an $F_r$-epimorphism of respective coordinate algebras $\varphi : F_r \oplus M_i \to F_r \oplus M_{i+1}$, moreover $\ker \varphi \neq 0$. According to Lemma 4.8 in [6], $\varphi$ induces an $R$-epimorphism $\phi : M_i \to M_{i+1}$, where $\ker \phi \neq 0$. Consequently, $r(M_i) > r(M_{i+1})$. In what follows that $\dim(Y) \leq r(\Gamma(Y))$. 28
To prove the converse inequality suppose that \( r(\Gamma(Y)) = r(M) = n \). Let \( N \) be the isolated submodule generated by a nonzero element \( 0 \neq m \in M \), and let \( \phi \) be the canonical \( R \)-epimorphism from \( M \) onto \( M_1 = M/N \). Then \( M_1 \) is a torsion free module over \( R \) and \( r(M_1) = n - 1 \). Applying the reverse of the above argument we conclude that \( \dim(Y) \geq n \).

References

[1] Yu. A. Bahturin, "Identities in Lie Algebras", M. Nauka, 1985, in Russian.

[2] G. Baumslag, A. G. Myasnikov, V. N. Remeslennikov, "Algebraic geometry over groups I. Algebraic sets and Ideal Theory", Journal of Algebra, 1999, Vol. 219, pp. 16-79.

[3] N. Bourbaki, "Elements of mathematics. Commutative algebra", Hermann, Paris; Addison-Wesley Publishing Co, 1972.

[4] C. C. Chang, H. J. Keisler, "Model Theory", Studies in Logic and the Foundations of Mathematics, 1973.

[5] E. Yu. Daniyarova, I. V. Kazatchkov, V. N. Remeslennikov, "Algebraic geometry over free metabelian Lie algebra I. \( U \)-algebras and \( A \)-modules", Omsk: OmGAU, 2001, Preprint no. 34, 25 pp.

[6] E. Yu. Daniyarova, I. V. Kazatchkov, V. N. Remeslennikov, "Algebraic Geometry over Metabelian Lie Algebras I. \( U \)-Algebras and Universal Classes", Fundamental and Applied Mathematics, to appear in 2004.

[7] S. Lang, "Algebra. Revised third edition", Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.

[8] A. G. Myasnikov, V. N. Remeslennikov, "Algebraic geometry over groups II. Logical Foundations", Journal of Algebra, 2000, Vol. 234, pp. 225-276.

[9] A. I. Malcev, "Basic Linear Algebra", 3rd edition, Moscow: Nauka, 1970, in Russian.
[10] V. A. Roman’kov, ”On equations in free metabelian groups”, Siberian Math. Jnl, 1979, v. 20, no. 3, pp. 671-673.

[11] Seidenberg A. ”Constructions in algebra”, Trans. Amer. Math. Soc. V. 197 (1974). p. 273-313.
Daniyarova Evelina Yur’evna,
644043, Russia, Omsk, Spartakovskaya st. 13-8,
tel. +7 3812 232239,
e-mail: evelina_om@mail333.com
Omsk Branch of Institute of Mathematics
(Siberian branch of Russian Academy of Science)

Kazatchkov Itia Vladimirovich,
644046, Russia, Omsk, Pushkin st. 136-22,
tel. +7 3812 312315,
e-mail: kazatchkov@mail333.com
Omsk Branch of Institute of Mathematics
(Siberian branch of Russian Academy of Science)

Remeslennikov Vladimir Nikanorovich,
644099, Russia, Omsk, Ordjonikidze st. 13-202,
tel. +7 3812 240914,
e-mail: remesl@iitam.omsk.net.ru
Omsk Branch of Institute of Mathematics
(Siberian branch of Russian Academy of Science)