Moutard type transformation for matrix generalized analytic functions and gauge transformations

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There has recently been considerable progress in the theory of Darboux–Moutard type transformations for two-dimensional linear differential systems with applications to geometry, spectral theory, and soliton equations (see [1]–[4], for instance). In the present note we derive such a transformation for the matrix generalized function system

\[ \partial_\tau \Psi + A \Psi + B \overline{\Psi} = 0, \]

where \( \partial_\tau = \partial/\partial_\tau \), and the coefficients \( A \) and \( B \) and the solutions \( \Psi \) are \( N \times N \) matrix functions defined on an open simply connected domain \( D \) in \( \mathbb{C} \). In particular, this generalizes the transformation for \( N = 1 \) found in [4] with \( A = 0 \). In addition, we show that the Moutard type transformation for a system of the form (1) with \( B = 0 \) is equivalent to a gauge transformation for the connection \( \nabla_\tau = \partial_\tau + A \). In turn, our studies show that the Moutard type transformation for (1)) with \( A = 0 \) can be treated as a proper analogue of the indicated gauge transformation.

As for \( N = 1 \), the system (1) can be reduced to the system

\[ \partial_\tau \Psi + B \overline{\Psi} = 0, \]

that is, to (1) with \( A = 0 \), by the gauge transformation

\[ \Psi \rightarrow \tilde{\Psi} = g^{-1} \Psi, \quad B \rightarrow \tilde{B} = g^{-1} B \overline{g}, \quad \partial_\tau g + A g = 0, \quad \det g \neq 0. \]

We say that the system

\[ \partial_\tau \Psi^+ - \overline{\Psi}^+ B = 0 \]

is conjugate to (2) (see [5] for a similar definition for \( N = 1 \)).

We have the following result.

**Theorem 1.** The systems (2) and (3) are covariant, that is, are mapped into systems of the same type, with respect to the Moutard type transformation

\[ \Psi \rightarrow \tilde{\Psi} = \Psi - F \omega_{F,F^+}^{-1} \omega_{\Psi,F^+}, \quad \Psi^+ \rightarrow \tilde{\Psi}^+ = \Psi^+ - \omega_{\Psi^+,\Psi} \omega_{F,F^+}^{-1} F^+, \]

\[ B \rightarrow \tilde{B} = B + F \omega_{F,F^+}^{-1} F^+, \]

where \( F \) and \( F^+ \) are arbitrary fixed solutions of (2) and (3), respectively,

\[ \partial_\tau \omega_{\Psi^+,\Psi} = \Phi^+ \overline{\Phi}, \quad \text{Re} \omega_{\Psi^+,\Psi} = 0 \]

for \( \Phi \) and \( \Phi^+ \) satisfying equations (2) and (3), and \( \det \omega_{F,F^+} \neq 0 \).

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To find $\omega_{\Phi, \Phi^+}$ satisfying (5) we use also the fact that $\partial_z \omega_{\Phi, \Phi^+} = -\overline{\Phi}^+ \Phi$. Moreover, our definition of $\omega_{\Phi, \Phi^+}$ is self-consistent up to a purely imaginary matrix integration constant in view of the identity $\partial_z \Phi^+ \Phi = -\partial_z \Phi^+ \Phi$. The last equality follows from the systems (2) and (3) for $\Phi$ and $\Phi^+$, respectively. Recall that the domain $D$ is simply connected.

For given $\omega_{F, F^+}, \omega_{\Phi, F^+}$, and $\omega_{F, \Phi^+}$, Theorem 1 is proved by straightforward computations.

In addition, for the system

$$
\partial_z \Psi + A \Psi = 0,
$$

that is, for (1) with $B = 0$, the following result holds.

**Proposition 1.** The system (6) is covariant under the following Moutard type transformation:

$$
\Psi \rightarrow \tilde{\Psi} = \Psi - F \tilde{\omega}_{F, F^+}^{-1} \omega_{\Phi, F^+}, \quad A \rightarrow \tilde{A} = A + F \tilde{\omega}_{F, F^+}^{-1} F^+,
$$

where $F$ is an arbitrary fixed solution of (6), $F^+$ is an arbitrary fixed matrix function,

$$
\partial_z \tilde{\omega}_{F, F^+} = F^+ \Phi
$$

for any matrix function $\Phi$, and $\det \tilde{\omega}_{F, F^+} \neq 0$.

Equations (7) and (8) are analogues of (4) and (5). However, in contrast to (5) we do not require the matrix functions $\tilde{\omega}_{F, F^+}$ to be purely imaginary. Equation (8) is solvable for $\tilde{\omega}_{F, F^+}$, and Proposition 1 is proved by straightforward computations.

**Remark.** Let $A$, $\tilde{A}$, $\Psi$, $F$, $F^+$, and $\tilde{\omega}_{\Phi, F^+}$ be the same as in Proposition 1. Let $g = 1 - F \tilde{\omega}_{F, F^+}^{-1} \Lambda$, with $\Lambda \Psi = \Lambda A + F^+$. Then $\partial_z (g \Psi) + \tilde{A}(g \Psi) = 0$. This is proved by straightforward computations, and it shows that for invertible $g$ the transformation $A \rightarrow \tilde{A}$ reduces to a gauge transformation.

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