ON THE DRINFELD TWIST FOR QUANTUM $sl(2)$

Ludwik Dąbrowski
SISSA, Via Beirut 2-4, Trieste, Italy.
E-MAIL: DABROW@SISSA.IT

Fabrizio Nesti
SISSA, Via Beirut 2-4, Trieste, Italy.
E-MAIL: NESTI@SISSA.IT

Pasquale Siniscalco
SISSA, Via Beirut 2-4, Trieste, Italy.
E-MAIL: SINIS@SISSA.IT

Abstract

An isomorphism, up to a twist, between the quasitriangular quantum enveloping algebra $U_h(sl(2))$ and the (classical) $U(sl(2))[[h]]$ is discussed. The universal twisting element $\mathcal{F}$ is given up to the second order in the deformation parameter $h$. 

SISSA 130/96/FM
1 Introduction

In 1989 Drinfeld showed by cohomological arguments that, as a formal series in a deformation parameter \( h \), all the quantum symmetries (quasitriangular Hopf algebras) \( U_h(g) \), where \( g \) is a semisimple Lie algebra, are isomorphic with \( U(g)[[h]] \), up to a twist \( \mathcal{F} \) [1, 2]. He also posed a problem (cf. [2]) to find a concrete pair \((m, \mathcal{F})\), consisting of an isomorphism \( m \) and a universal twisting element \( \mathcal{F} \). This turns out to be a formidable task, which as far as we know, is not yet solved in general. The only case when it has been performed concerns the q-deformed Heisenberg algebra \( \mathcal{H}_q(1) \) [3]. The next important case to be investigated is the quantum deformation of \( sl(2) \) (as a matter of fact \( \mathcal{H}_q(1) \) can be obtained from it by a contraction). As far as \( U_h(sl(2)) \) is regarded, a candidate for the isomorphism \( m \) is actually known [4]. Also, a series of related particular matrix solutions for the twist element \( \mathcal{F} \) were reported, namely \( \mathcal{F} \) in the representations \( \frac{1}{2} \otimes j \), where \( \frac{1}{2} \) denotes the fundamental representation and \( j \) denotes the irreducible \((2j+1)\)-dimensional representation of \( sl(2) \) [3, 4], (see also [4]). Moreover, in [4] a sort of a ‘semi-universal’ form of \( \mathcal{F} \) has been given, i.e. the expression for \((\frac{1}{2} \otimes \text{id})(\mathcal{F})\). However, the universal element \( \mathcal{F} \) itself has not been known beyond the first order in the deformation parameter \( h \) (the first order coefficient being given by the classical \( r \)-matrix \( r \)). In this letter, we investigate and report the solution up to the second order in \( h \). In the subsequent sections we separately discuss the problem on the levels of algebra, Hopf algebra and quasitriangular Hopf algebra.

It is worth to mention that evaluating \( \mathcal{F} \) in the representation \( \rho_L \otimes \rho_L \), where \( \rho_L \) is the representation of \( sl(2) \) in terms of the left-invariant vector fields on \( SL(2) \), one obtains a quantization of the Lie-Poisson bracket on \( SL(2) \) given by \( r \) [4]. In particular, the second coefficient of \((\rho_L \otimes \rho_L)(\mathcal{F})\) provides an interesting second order (bi)differential operator on \( SL(2) \).

2 Algebra level

We start by specifying our conventions about Lie algebra \( sl(2) \). The generators are \( H, E, F \) with the commutation relations:

\[
[H, E] = E, \quad [H, F] = -F, \quad [E, F] = H.
\] (2.1)

As a consequence we have the following exchange relations between any polynomial \( \phi(H) \) in \( H \) and the powers of \( E \) and \( F \):

\[
\phi(H)E^n = E^n\phi(H + n), \quad \phi(H)F^n = F^n\phi(H - n).
\] (2.2)
The quadratic Casimir element in the universal enveloping algebra $U(sl(2))$ is

\[ I = 2EF + H(H-1) = 2FE + H(H+1) = j(j+1) \, . \] (2.3)

A possible basis for the enveloping algebra is provided by the set \{${H^l}E^mF^n$\}, but using the relations (2.3) we can pass to the basis given by \{${H^a}P^bE^c \oplus H^dI^eF^f$\}. This basis will be more suitable for our computations.

Next, the generators $J^+, J^-, J^0$ of the q-deformed algebra obey the following commutation relations:

\[ [J^0, J^+] = J^+, \quad [J^0, J^-] = -J^-, \quad [J^+, J^-] = \frac{1}{2}[2J^0] \, . \] (2.4)

where \([x]\), the $q$-analogue of $x$, is defined as:

\[ [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \, . \] (2.5)

The ‘deforming maps’ introduced in [4], provide (cf.[10]) an isomorphism $m$ between $U_h(sl(2))$ and $U(sl(2))[h]$ which is given by mapping the generators $J^0, J^+, J^-$ to the following combinations of $H, E, F$

\[ J^0 \rightarrow H, \quad J^+ \rightarrow \phi^+ E, \quad J^- \rightarrow \phi^- F = F\phi^+ \, , \] (2.6)

where

\[ \phi^\pm = \sqrt{\frac{(j \pm H)(1 + j \mp H)}{(j \pm H)[1 + j \mp H]}} \, . \] (2.7)

We remark that (2.7) is a well defined expression, as the inverse and square root operations are admissible in the $h$-adic topology. In fact, with $q = e^h$, we can write the expansion in $h$ up to the second order as

\[ \phi^\pm = 1 + \frac{1}{12}h^2 (2I + 2H(H \mp 1) - 1) + o(h^3) \doteq 1 + h^2 \phi^+_2 + o(h^3) \, . \] (2.8)

It will be useful to mention [11], that any other isomorphism $m'$ can differ at most by a similarity via an invertible element $M \in U(sl(2))[h]$, i.e.

\[ m' = MmM^{-1} \, . \] (2.9)

We conclude this section with few remarks. Note that (2.6) is in fact valid also for the $^*$-algebras $U_h(su(2))$ and $U(su(2))[h]$, since it fulfills the relevant hermicity condition. In this respect, more general isomorphisms belonging to the one-parameter family introduced in [4] do not satisfy such a hermicity requirement. In addition, they are not suitable for our purposes since the coefficients of the expansion in $h$ are not polynomial in the generators.
3 Hopf algebra level

The enveloping algebra $U(sl(2))[[h]]$ with relations \((2.1)\) when equipped with the usual coproduct
\[
\Delta(x) = 1 \otimes x + x \otimes 1, \quad \forall x \in sl(2), \tag{3.10}
\]
becomes a Hopf algebra. In the quantum case, the coproduct in $U_h(sl(2))$ is defined as:
\[
\Delta_q(J^0) = 1 \otimes J^0 + J^0 \otimes 1,
\]
\[
\Delta_q(J^\pm) = J^\pm \otimes q^{J^0} + q^{-J^0} \otimes J^\pm. \tag{3.11}
\]
(The counit and coinverse are not needed for our purposes).

The main part of Drinfeld Theorem guarantees that these two classical and quantum coproducts are related via a twist by an invertible $F \in (U(sl(2)) \otimes U(sl(2)))[[h]]$.

More precisely, defining
\[
\tilde{\Delta}_q = (m \otimes m) \circ \Delta_q \circ m^{-1} \tag{3.12}
\]
we have
\[
\tilde{\Delta}_q(x) = F \Delta(x) F^{-1}, \quad \forall x \in U(sl(2))[[h]] \tag{3.13}
\]
It is sufficient (and necessary) to verify this equation by substituting for $x$ the image by $m$ of the generators $J^0, J^+, J^-$. We remark that there is no loss of generality in restricting ourselves to a specific isomorphism \((2.6)\). Indeed, had we used another isomorphism $m'$, it turns out from \((2.9)\) that the corresponding $F'$ would be given by $(M \otimes M)\tilde{\Delta}_q(M)F$.

As it is known (cf. \[9\]) a particular solution up to first order in $h$ is just $F = 1 + hr$, where
\[
r = F \otimes E - E \otimes F \tag{3.14}
\]
is the standard classical $r$-matrix. More generally and up to order two in $h$ we write
\[
F = F_0 + hF_1 + h^2F_2 + o(h^3), \tag{3.15}
\]
with $F_i$ belonging to $U(sl(2)) \otimes U(sl(2))$.

Using \((2.8)\), we obtain the following coupled system of equations to solve by recursion:
\[
[\mathcal{F}_0, \Delta H] = 0,
\]
\[
[\mathcal{F}_0, \Delta E] = 0,
\]
\[
[\mathcal{F}_0, \Delta F] = 0. \tag{3.16}
\]
\[
[\mathcal{F}_1, \Delta H] = 0,
\]
\[
[\mathcal{F}_1, \Delta E] = (E \otimes H - H \otimes E)\mathcal{F}_0,
\]
\[
[\mathcal{F}_1, \Delta F] = (F \otimes H - H \otimes F)\mathcal{F}_0. \tag{3.17}
\]
\[ [\mathcal{F}_2, \Delta H] = 0 \, , \]
\[ [\mathcal{F}_2, \Delta E] = (E \otimes H - H \otimes E)\mathcal{F}_1 - \mathcal{F}_0 \Delta \phi_2^+ \Delta E \]
\[ + \left( \frac{1}{2} E \otimes H^2 + \frac{1}{2} H^2 \otimes \phi_2^+ E + \phi_2^+ E \otimes 1 + 1 \otimes \phi_2^+ E \right) \mathcal{F}_0 \, , \]
\[ [\mathcal{F}_2, \Delta F] = (F \otimes H - H \otimes F)\mathcal{F}_1 - \mathcal{F}_0 \Delta \phi_2^+ \Delta F \]
\[ + \left( \frac{1}{2} F \otimes H^2 + \frac{1}{2} H^2 \otimes \phi_2^+ F + \phi_2^+ F \otimes 1 + 1 \otimes \phi_2^+ F \right) \mathcal{F}_0 . \]  

(3.18)

Besides \( \mathcal{F}_0 = 1 \otimes 1 \), any arbitrary polynomial \( f_0 \) in the variables \((I \otimes 1, 1 \otimes I, \Delta I)\) satisfies equations (3.16). Due to linearity of the equation we can write then:

\[ \mathcal{F}_0 = 1 \otimes 1 + f_0 . \]  

(3.19)

As regards \( \mathcal{F}_1 \), besides the solution \( \tilde{\mathcal{F}}_1 = r \) of the equations (3.17) (with \( f_0 = 0 \)), a solution for the general case is given by

\[ \mathcal{F}_1 = \tilde{\mathcal{F}}_1 (1 \otimes 1 + f_0) + f_1 , \]  

(3.20)

with \( f_1 \) being a solution of (3.16).

Similarly for \( \mathcal{F}_2 \): if one finds a particular solution \( \tilde{\mathcal{F}}_2 \) of (3.18) (with \( f_0 = f_1 = 0 \)), the most general one is given by

\[ \mathcal{F}_2 = \tilde{\mathcal{F}}_2 (1 \otimes 1 + f_0) + \tilde{\mathcal{F}}_1 f_1 + f_2 , \]  

(3.21)

with \( f_2 \) solution of (3.16).

The possibility of adding pure kernel (i.e. satisfying the homogeneous equations (3.16)) terms \( f_1 \) and \( f_2 \) comes from the fact that the last two equations for \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are linear non homogeneous, whose associated homogeneous ones are the last two equations in (3.16).

Now we proceed to exhibit the aforementioned particular solutions \( \tilde{\mathcal{F}}_i \) of this set of equations. In \( U(sl(2)) \otimes U(sl(2)) \) we use the basis

\[ \{ H^{a_1} I^{b_1} E^{c_1} \oplus H^{a_2} I^{b_2} E^{c_2} \} \otimes \{ H^{a_2} I^{b_2} E^{c_2} \oplus H^{a_2} I^{b_2} E^{c_2} \} . \]

In order to simplify the notation, for any \( x \in U(sl(2)) \) we set \( x_1 = x \otimes 1, x_2 = 1 \otimes x \). From \([\mathcal{F}_i, \Delta H] = 0\), for all \( i \), it is easily seen that any \( \mathcal{F}_i \) is of the form \( \mathcal{F}_i = a_d E_i^d F_i^d + b_d E_i^d F_i^d \), where \( a_d \) and \( b_d \) are polynomials in \( H_1, H_2, I_1, I_2 \).

We’ve already mentioned that \( \tilde{\mathcal{F}}_0 = 1 \) is a solution for equations (3.16).

Next we pass to the first order term. For simplicity we drop the index \( i = 1 \) in the following formulae and define

\[ \delta_1(a_k) = a_k(H_1, H_2, I_1, I_2) - a_k(H_1 - 1, H_2, I_1, I_2) , \]
\[ \delta_2(a_k) = a_k(H_1, H_2, I_1, I_2) - a_k(H_1, H_2 - 1, I_1, I_2) , \]
and similarly for \( b_k \). The equations (3.17) give the following system of coupled partial difference equations for the coefficients \( a_i \) and \( b_i \):

\[
\begin{align*}
\delta_1(a_{n-1}) &= -\frac{1}{2}(I_2 + H_2 - H_2^2)\delta_2(a_n) + (nH_2 + \frac{n^2-n}{2})a_n, \\
\delta_2(a_{n-1}) &= -\frac{1}{2}(I_1 - H_1 - H_1^2)\delta_1(a_n(H_1+1, H_2-1)) + (nH_1 - \frac{n^2-n}{2})a_n(H_1, H_2-1), \\
\delta_1(b_{n-1}) &= -\frac{1}{2}(I_2 - H_2 - H_2^2)\delta_2(b_n(H_1-1, H_2+1)) + (nH_2 - \frac{n^2-n}{2})b_n(H_1-1, H_2), \\
\delta_2(b_{n-1}) &= -\frac{1}{2}(I_1 + H_1 - H_1^2)\delta_1(b_n) + (nH_1 + \frac{n^2-n}{2})b_n. \\
\end{align*}
\]

for any \( n \geq 2 \), whereas for \( n = 1 \) we have:

\[
\begin{align*}
\delta_1(a_0 + b_0) &= -\frac{1}{2}(I_2 + H_2 - H_2^2)\delta_2(a_1) + H_2a_1 + H_2, \\
\delta_1(a_0 + b_0) &= -\frac{1}{2}(I_2 - H_2 - H_2^2)\delta_2(b_1(H_1-1, H_2+1)) + H_2b_1(H_1-1, H_2) - H_2, \\
\delta_2(a_0 + b_0) &= -\frac{1}{2}(I_1 - H_1 - H_1^2)\delta_1(a_1(H_1+1, H_2-1)) + H_1a_1(H_1, H_2-1) + H_1, \\
\delta_2(a_0 + b_0) &= -\frac{1}{2}(I_1 + H_1 - H_1^2)\delta_1(b_1) + H_1b_1 - H_1. \\
\end{align*}
\]

In order to find a particular solution of this system of equations, one can fix a couple \( \{N, K\} \) such that \( a_n = b_k = 0, \forall n \geq N \) and \( \forall k \geq K \), in order to set the maximum degree for the polynomials in \( E_1^+F_2^+ \) and \( E_2^+F_1^+ \), and then solve recursively the equations for the lower degree terms by partial finite integration.

By making a minimal choice, putting \( a_n = b_n = 0 \), for any \( n \geq 2 \), we recover the solution:

\[
\tilde{\mathcal{F}}_1 = r, \\
\]

with \( r \) given by (3.14). Consistently with what we explained in the previous section, had we decided to fix our cut-off at higher degree terms we would have adjoined to \( \tilde{\mathcal{F}}_1 \) some \( f_1 \) solution of the pure kernel part.

As regards \( \mathcal{F}_2 \), the structure of the equations for \( a_l \) and \( b_l \) remains unchanged for \( n \geq 3 \), whereas for \( n = \{2, 1\} \) some extra term appear, due to \( \phi_2^+ \) and \( \phi_2^- \).

We skip the explicit (and lengthy) form of them, and we just give the expression for a particular solution:

\[
\tilde{\mathcal{F}}_2 = \frac{1}{2}(I \otimes H^2 + H^2 \otimes I) + \frac{1}{3}(E \otimes HF - HE \otimes F + HF \otimes E - F \otimes HE) \\
+ \frac{1}{6}H \otimes H(1 - 3P) - \frac{11}{24}P + \frac{1}{2}((1 + P)^2 - 1 - 2I \otimes I), \\
\]

where

\[
P = 2(E \otimes F + F \otimes E + H \otimes H)
\]

is the Cartan-Killing metric.

Applying representations of \( sl(2) \) we can obtain explicit matrix expressions for \( \tilde{\mathcal{F}} \). It
turns out that our particular solution $\tilde{F}$, when composed with $\frac{1}{2} \otimes \text{id}$, reproduces the semi-universal solution presented in [3] in terms of $2 \times 2$ matrices with coefficients in $U(sl(2))[\hbar]$ (up to the second order in $\hbar$). Thus, as a consequence it also coincides with the matrix solutions in the representations $\frac{1}{2} \otimes j$.

We remark that in the literature one may find often other properties of the twisting element $F$. For instance, $F$ may be supposed to satisfy the

i) ‘normalization’ condition

$$(\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1,$$

sometimes also expressed as $F(x, 0) = F(0, y) = 1$. With the standard definition of counit $\varepsilon$ this implies $F_0 = 1$, i.e. $f_0 = 0$.

ii) unitarity condition $\sigma(F)F = 1$. In our case $F$ fulfills this condition in the particular representation $\frac{1}{2} \otimes \frac{1}{2}$, but not in general.

iii) condition $(F \otimes \text{id})(\Delta \otimes \text{id})F = (\text{id} \otimes F)(\text{id} \otimes \Delta)F$. In our case $F$ does not fulfill it, not even in a representation (except the trivial one). We remark that this condition is a stronger requirement with respect to the coassociativity of the twisted coproduct, which in our case follows directly from the definition.

## 4 Quasitriangular Hopf algebra level

From the Drinfeld theorem, the quantum universal R-matrix

$$R_q = q^{2j_0 \otimes j^0} \sum_{n=0}^{\infty} q^{-n(n-1)/2} \frac{2^n(1-q^{-2})^n}{[n]!} (q^{j_0} J^+ \otimes q^{-j_0} J^-)^n,$$

and the undeformed universal R-matrix, though not the simple $1 \otimes 1$ but rather,

$$R = q^P,$$

with $P$ given by (3.26), should be related by the isomorphism up to a twist. Thus, setting

$$\tilde{R}_q \doteq (m \otimes m)(R_q),$$

$F$ is supposed to verify the equation

$$\tilde{R}_q F = \sigma(F) R,$$

where $\sigma$ is the flip operator and $[n]! \doteq [n][n-1] \ldots 1$.

We have the following expansions

$$R = 1 + \hbar R^{(1)} + \hbar^2 R^{(2)} + o(\hbar^3)
= 1 + \hbar(2E \otimes F + 2F \otimes E + 2H \otimes H)
+ \hbar^2(-H \otimes H - 2E \otimes F - 2F \otimes E + 2E^2 \otimes F^2 + 2F^2 \otimes E^2
- 2E \otimes HF - 2HF \otimes E + 2F \otimes HE + 2HE \otimes F + 4HE \otimes HF + 4HF \otimes HE
+ 3H^2 \otimes H^2 + I \otimes I - I \otimes H^2 - H^2 \otimes I),$$
\[ \widetilde{R}_q = 1 + hR_q^{(1)} + h^2 R_q^{(2)} + o(h^3) \]
\[ = 1 + h(4E \otimes F + 2H \otimes H) + h^2(2H^2 \otimes H^2 - 4E \otimes F - 4E \otimes HF + 8E^2 \otimes F^2 + 4HE \otimes F + 8HE \otimes HF) . \] (4.33)

At the zero-order in \( h \), choosing \( f_0 = 0 \), (4.31) is identically satisfied \((1 = 1)\).

At the order one we have the following equation:
\[ \sigma(f_1) - f_1 = R_q^{(1)} - R^{(1)} - \left( \sigma(\widetilde{F}_1) - \widetilde{F}_1 \right) . \] (4.34)

It comes from direct computations that the right-hand-side is zero, which implies that \( f_1 \) must be symmetric.

At the second order we obtain
\[ \sigma(f_2) - f_2 = \widetilde{F}_2 - \sigma(\widetilde{F}_2) - \sigma(\widetilde{F}_1)R^{(1)} + R_q^{(1)} \widetilde{F}_1 + R_q^{(2)} - R^{(2)} . \] (4.35)

Again the right-hand-side is zero, and hence also \( f_2 \) must be symmetric.

Since, in particular, \( f_1 \) and \( f_2 \) can be equal to zero, we have that our particular solution \( \widetilde{F} \) satisfies (4.31).

5 Conclusions

In accordance with the theorem of Drinfeld, we have exhibited an isomorphism from \( U_h(sl(2)) \) to \( U(sl(2))[[h]] \) and (up to the second order in \( h \)) a class of universal twisting elements \( F \in (U(sl(2)) \otimes U(sl(2)))[[h]] \). Such \( F \) perform a gauge transformation (twist) from the ordinary coproduct and from the universal R-matrix \( R = q^P \) in \( U(sl(2))[h] \) to their quantum counterparts in \( U_h(sl(2)) \).

We have identified a particular universal element \( \widetilde{F} \) in this class which, after applying the representation \( \frac{1}{2} \) to its first leg, coincides with the ‘semi-universal’ solution in [8] (up to the second order in \( h \)). Consequently, it also coincides with the known matrix solutions in the representations \( \frac{1}{2} \otimes j \).

The computation of the higher order terms, with the help of ‘Mathematica’, is in progress.
References

[1] V.G. Drinfeld “Quasi-Hopf Algebras and Knizhnik-Zamolodchikov equations” Res. Rep. Phys., 1989, Springer

[2] V.G. Drinfeld “Quasi-Hopf Algebras” Leningrad Math. J. 1990 1 (6) 1419–1457

[3] M. Bonechi, R. Giachetti, E. Sorace & M. Tarlini Commun. Math. Phys. 1995 169 (243) 627–634

[4] T.L. Curtright & C.K. Zachos “Deforming maps for quantum algebras” Phys. Lett. B 1990 3 (243) 237–244

[5] T.L. Curtright “Deformations, Coproducts, and $U$” in Quantum Groups T.L. Curtright, D.B. Fairlie & C.K. Zachos eds World Scientific 1991

[6] C.K. Zachos “Quantum Deformations” in Quantum Groups T.L. Curtight, D.B. Fairlie & C.K. Zachos eds. World Scientific 1991

[7] R.A. Engeldinger “On the Drinfeld-Kohno Equivalence of groups and Quantum Groups” Prep. LMU-TPW 95-13 (q-alg/9509001)

[8] T.L. Curtright, G.I. Ghandour, C.K. Zachos “Quantum algebra deforming maps, Clebsch-Gordan coefficients, coproducts, R and U matrices” J. Math. Phys. 1991 32 (3) 676–688

[9] L.A. Takhtajan “Lectures on Quantum Groups” in Introduction to quantum group and integrable massive models of quantum field theory M. Ge & B. Zhao eds. World Scientific 1989

[10] L. Dąbrowski “Drinfeld twisting and nonstandard quantum groups” in Proc. 10th Naz.Conv.Gen.Rel., Bardonecchia 1992; 661-665, World Scientific

[11] C. Kassel Quantum Groups Springer-Verlag 1995