Comments on the paper by K. Zengel “The electromagnetic analogy of a ball on a rotating conical turntable”

Alexey V. Borisov\textsuperscript{1,\#}, Tatiana B. Ivanova\textsuperscript{1,\#},
Alexander A. Kilin\textsuperscript{1,\#}, Ivan S. Mamaev\textsuperscript{\dagger}

\textsuperscript{1} Moscow Institute of Physics and Technology (State University),
Institutskii per. 9, Dolgoprudny, 141700 Russia;
\textsuperscript{2} Udmurt State University, ul. Universitetskaya 1, Izhevsk, 426034 Russia;
\textsuperscript{3} Kalashnikov Izhevsk State Technical University,
ul. Studencheskaya 7, Izhevsk, 426069 Russia.

Abstract. This paper is concerned with the study of the rolling without slipping of a dynamically symmetric (in particular, homogeneous) heavy ball on a cone which rotates uniformly about its symmetry axis. The equations of motion of the system are obtained, partial periodic solutions are found and their stability is analyzed.

Keywords rolling, rotating cone, nonholonomic constraint, stability

Introduction

This paper studies the motion of a dynamically symmetric (in particular, homogeneous) heavy ball rolling without slipping on a cone. The cone rotates uniformly about its symmetry axis.

In Ref. \cite{5} equations of motion are presented for the system of interest, the existence of an electromagnetic analogy is shown, and a stability analysis of the periodic solutions is made. Unfortunately, the equations of motion are written incorrectly.

The main error made by the authors of Ref. \cite{5} is that they do not take into account the time dependence of the normal to the surface of rolling at the point of contact of the ball during the motion on the cone. This vector does not depend on time during the motion of the ball on the horizontal or inclined plane,
as pointed out correctly in the first part of Ref. [5]. In addition, in the equations written in Ref. [5] in spherical coordinates, the gravitational field appears with a wrong sign, which ultimately leads to erroneous conclusions.

In this work, we present equations of motion of a ball on a cone taking into account the time dependence of the normal to the surface at the point of contact. We also present results on the linear stability analysis of periodic solutions.

1 Derivation of equations of motion

Consider the rolling without slipping of a heavy completely dynamically symmetric (in particular, homogeneous) ball on the surface of a circular cone whose symmetry axis is vertical. The cone rotates about the symmetry axis with constant angular velocity $\Omega$ (Fig. 1). Let $\theta \in [0, \pi/2)$ denote the constant apex angle of the cone (measured from the vertical axis).

We will consider the motion of the ball relative to an inertial (fixed) coordinate system $Oxyz$ in which the axis $Oz$ is directed along the axis of rotation, that is, $\Omega = (0, 0, \Omega)$.

![Figure 1: A ball on a rotating cone. In the left figure, the unit vector $e_\varphi$ is directed perpendicularly to the figure plane (from the observer). The right figure shows a view from above, the vector $\Omega$ is directed perpendicularly to the figure plane (to the observer), the unit vectors $e_\theta, e_\varphi$ lie in the same plane perpendicular to the figure plane.

Since there is no slipping, the velocity of the point of contact $P$ on the ball coincides with the velocity of an analogous point on the rotating surface, that is,

$$v + R\gamma \times \omega = \Omega \times r_p, \quad r_p = r - R\gamma,$$

(1)

where $r = (x, y, z)$ is the radius vector of the center of mass of the ball, $v = \dot{r}$ and $\omega$ are, respectively, the velocity of the center of mass and the angular velocity of the ball in the coordinate system $Oxyz$, $\gamma$ is the normal at the point of contact, $R$ is the radius of the ball, and $r_p$ is the vector directed from the origin of the coordinates to the point of contact $P$. Here and in what follows, all vectors are written in boldface font.
The change in the linear and angular momenta of the ball relative to its center can be written in the form of Newton–Euler equations:

\[ m\ddot{r} = N + F, \quad I\dot{\omega} = RN \times \gamma + M, \quad (2) \]

where \( I \) is the central tensor of inertia of the ball, \( F \) is the resultant of the external active forces, \( M \) is the resultant moment of external forces relative to the center of the ball, and \( N \) is the reaction force acting on the ball at the point of contact \( P \) (in the general case it can have any direction).

In what follows, we will write all variables and equations in dimensionless form. To do so, we take the radius of the ball \( R \) as a unit of distance, the quantity \( \sqrt{R/g} \) as a unit of time, and the mass of the ball \( m \) as a unit of mass, that is, we make the following change of variables:

\[ \frac{r}{R} \rightarrow r, \quad t\sqrt{\frac{g}{R}} \rightarrow t, \quad \omega\sqrt{\frac{R}{g}} \rightarrow \omega, \quad \frac{F}{mg} \rightarrow F, \quad \frac{M}{mgR} \rightarrow M. \quad (3) \]

Eliminating the reaction force from the second of Eqs. (2) and performing a vector product by \( \gamma \), we obtain

\[ \gamma \times \dot{\omega} = \frac{1}{k}(\ddot{r} - \gamma(\gamma, \ddot{r}) - F + \gamma(\gamma, F) + \gamma \times M), \quad (4) \]

where \( k = I/(mR^2) \); the parameter \( k \leq 1 \), \( k = 0 \) for a material point, \( k = 2/5 \) for a homogeneous ball.

The vector product on the left-hand side of (4) can be expressed from the derivative of the constraint equation (1) with respect to time. Taking Eq. (3) into account, we obtain

\[ \gamma \times \dot{\omega} = -\ddot{r} + \Omega \times \dot{r} + \dot{\gamma} \times (\Omega - \omega). \quad (5) \]

Equating the right-hand sides of Eqs. (4) and (5), we obtain an equation governing the evolution of the vector \( r \):

\[ (k + 1)\ddot{r} - \gamma(\gamma, \ddot{r}) = F - \gamma(\gamma, F) + k\Omega \times \dot{r} + M \times \gamma + k\dot{\gamma} \times (\Omega - \omega). \quad (6) \]

**Remark 1.** The last term in Eq. (6), which was not taken into account by the authors of Ref., [5] vanishes as the ball moves on the plane (\( \gamma = \text{const} \)). In the general case, in particular, when the ball rolls on the surface of the cone, \( \dot{\gamma} \neq 0 \).

The vectors \( r \) and \( \gamma \) in Eq. (6) are dependent. If the surface on which the center of the ball moves is defined by the equation

\[ \Phi(r) = 0, \]

then the normal vector to this surface turns out to be collinear to the normal vector to the surface on which the ball rolls, and is given by the Gauss map

\[ \gamma = -\frac{\nabla\Phi(r)}{||\nabla\Phi(r)||}. \quad (7) \]
In the case of motion of the ball on the internal surface of the cone, the coordinates of the center of the ball are related to each other by
\[ \Phi(r) = z^2 \sin^2 \theta - (x^2 + y^2) \cos^2 \theta = 0. \] (8)

This gives us the normal to the surface in the form
\[ \gamma_1 = -\frac{x \cos \theta}{\sqrt{x^2 + y^2}}, \quad \gamma_2 = -\frac{y \cos \theta}{\sqrt{x^2 + y^2}}, \quad \gamma_3 = \sin \theta. \]

Equation (6) contains the angular velocity \( \omega \). Its projections, except for the projection onto the direction of the vector \( \gamma \), can be obtained from the constraint equation (1).

In order to obtain the missing equation for the projection of the angular velocity onto the direction of the vector \( \gamma \), we perform a scalar product of the second of Eqs. (2) by \( \gamma \) and obtain
\[ k(\dot{\omega}, \gamma) = (M, \gamma), \]
whence
\[ (\omega, \gamma) = (\omega, \dot{\gamma}) + \frac{1}{k} (M, \gamma). \] (9)

Thus, equations (1), (6)–(9) with restriction to the surface (8) with (3) and (7) taken into account form a complete closed system of differential equations governing the motion of a homogeneous ball on the surface of the cone.

**Remark 2.** Since the authors of Ref. [5] have lost a term proportional to \( \dot{\gamma} \), their equations of motion do not contain the projection of the angular velocity onto the direction \( \gamma \). Equation (9) is not presented and not taken into account in Ref. [5]. When it is taken into account, the equations of motion cannot be written in a natural Hamiltonian form.

In this paper, we shall assume that the ball is acted upon by the gravity force, and the resultant moment of external forces relative to the center of the ball is zero \( M = 0 \), a nonholonomic system, without friction).

### 1.1 Equations of motion in spherical coordinates

In this section, as in Ref., [5] we make use of spherical coordinates related to the Cartesian coordinates by
\[ x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta. \] (10)

We write the projections of Eqs. (1), (6)–(9) using Eq. (3) onto the spherical basis \( e_r, e_\theta, e_\varphi \) (see Fig. 1).

In the spherical coordinates, in view of Eq. (10), the surface equation (8) and the vector \( \gamma \) (7) can be reduced to the following very simple form:
\[ \theta = \text{const}, \quad \gamma = (0, -1, 0). \] (11)
Further, we write out the projections of the vectors appearing in Eqs. (1), (6), (9), and their time derivatives onto the basis $e_r, e_\theta, e_\varphi$, taking into account that $\dot{\theta} = \ddot{\theta} = 0$ (for details on the rules of differentiation of vectors in spherical coordinates, see, e. g., Ref. [6]):

\[
\begin{align*}
\mathbf{r} &= (r, 0, 0), \quad \mathbf{\omega} = (\omega_r, \omega_\theta, \omega_\varphi), \\
\dot{\mathbf{r}} &= (\dot{r}, 0, r \dot{\varphi} \sin \theta), \quad \dot{\mathbf{\omega}} = (0, 0, -\dot{\varphi} \cos \theta), \\
\mathbf{\omega} &= (\dot{\omega}_r - \dot{\varphi} \omega_\varphi \sin \theta, \dot{\omega}_\theta - \dot{\varphi} \omega_\varphi \cos \theta, \dot{\omega}_\varphi + \dot{\varphi} (\omega_\theta \cos \theta + \omega_r \sin \theta)), \\
\dot{\mathbf{\omega}} &= (\dot{r} - r \dot{\varphi}^2 \sin^2 \theta, -r \dot{\varphi}^2 \cos \theta \sin \theta, (r \dot{\varphi} + 2\dot{r} \dot{\varphi}) \sin \theta), \\
\mathbf{F} &= (-\cos \theta, \sin \theta, 0), \quad \Omega = \Omega(\cos \theta, -\sin \theta, 0).
\end{align*}
\]

Using Eqs. (11) and (12), the nontrivial projections of Eqs. (1), (6) and (9) onto the basis $e_r, e_\theta, e_\varphi$ can be represented in the form of four differential first-order equations

\[
\begin{align*}
\dot{r} &= V_r, \\
\dot{\omega}_\theta &= V_r V_\varphi \cos \theta, \\
\dot{V}_r &= r V_\varphi^2 \sin^2 \theta - \frac{k V_\varphi}{1 + k} (\Omega \sin \theta (r \sin \theta + \cos \theta) + \omega_\theta \cos \theta) - \frac{\cos \theta}{1 + k}, \\
\dot{V}_\varphi &= \frac{\Omega k}{r^2} (\frac{\Omega k}{1 + k} - 2V_\varphi),
\end{align*}
\]

and two algebraic equations

\[
\begin{align*}
\omega_r &= \Omega(\cos \theta + r \sin \theta) - r V_\varphi \sin \theta, \quad \omega_\varphi = V_r, \\
\end{align*}
\]

from which we have excluded the equation governing the evolution of $\varphi$:

\[
\dot{\varphi} = V_\varphi.
\]

The equation for $\dot{\varphi}$ (15) decouples from the general system, since the variable $\varphi$ does not appear explicitly in the system (13). This is due to the invariance of the initial system under rotations about the vertical axis.

In addition, the variables $\omega_r, \omega_\varphi$ do not appear explicitly in the resulting differential equations either and can be calculated immediately from Eq. (14) after integrating the system (13).

Thus, the system (13) is closed relative to the unknowns $(r, \omega_\theta, V_r, V_\varphi)$ and completely defines the dynamics of the ball on the cone. In order to restore the motion of the center of mass of the ball in space and to construct its trajectory in the fixed coordinate system $Oxyz$, it is necessary to add the quadrature (15) to the system (13) and to calculate the dependence of the Cartesian coordinates $(x, y, z)$ on time by using their relation to the spherical coordinates (10).
2 Periodic solutions and analysis of their stability

In this section we investigate the circular orbits considered in Ref. [5]. They correspond to motion of the center of mass of the ball on a cone in the horizontal plane with some constant frequency. In this section we show that they are fixed points of the reduced system (13), and analyze their stability.

1. In the case of the circular periodic motion of the center of mass, the following relations must be satisfied:

\[ r = r_0 = \text{const}, \quad V_r = 0, \quad V_\phi = \omega_0 = \text{const}. \]

\[ \dot{V}_r = 0, \quad \dot{V}_\phi = 0. \]

Using Eq. (16), we obtain from the equation for \( \dot{\omega}_\theta \) (13)

\[ \dot{\omega}_\theta = 0, \quad \omega_\theta = \Omega_\theta = \text{const}. \]

Consequently, in the case of motion in circular orbits according to Eqs. (16), (17) and the first of Eqs. (13), the derivatives \( \dot{r}, \dot{\omega}_\theta, \dot{V}_r, \dot{V}_\phi \) vanish, which corresponds to definition of the fixed points of the reduced system (13) or periodic motions of the complete system (13), (15).

The constants \( r_0, \omega_0, \Omega_\theta \) parameterize the circular orbits under consideration. Substituting Eqs. (16) and (17) into the third of Eqs. (13), we obtain an equation that relates these parameters:

\[ -r_0 \sin^2 \theta \omega_0^2 + \frac{k \omega_0}{1 + k} (\Omega \sin \theta (r_0 \sin \theta + \cos \theta) + \Omega_\theta \cos \theta) + \frac{\cos \theta}{1 + k} = 0. \]

(18)

Thus, only two parameters are independent. Following Ref. [5] we choose \( r_0 \) and \( \omega_\theta \) as independent parameters. As for the third parameter \( \omega_0 \), we find it from Eq. (18), which is quadratic in \( \omega_0 \). In the general case there are two roots:

\[ \omega_{01}(r_0, \Omega_\theta) = B(r_0, \Omega_\theta) + \sqrt{B(r_0, \Omega_\theta)^2 + \frac{4 \cos \theta}{r_0 (1 + k) \sin^2 \theta}}, \]

\[ \omega_{02}(r_0, \Omega_\theta) = B(r_0, \Omega_\theta) - \sqrt{B(r_0, \Omega_\theta)^2 + \frac{4 \cos \theta}{r_0 (1 + k) \sin^2 \theta}}, \]

where, to abbreviate the formula, we have introduced \( B(r_0, \omega_\theta) \):

\[ B(r_0, \omega_\theta) = \frac{k (\Omega \sin \theta (r_0 \sin \theta + \cos \theta) + \Omega_\theta \cos \theta)}{(1 + k) r_0 \sin^2 \theta}. \]

For \( r_0 > 0, \theta \in (0, \pi/2) \), the radical expression (19) is nonnegative. Thus, the following proposition holds.
Proposition 1. There exist two two-parameter families of periodic solutions to Eqs. (13) and (15):

1. \( r = r_0, \quad V_r = 0, \quad \omega_\theta = \Omega_\theta, \quad V_\phi = \omega_01(r_0, \Omega_\theta), \)

2. \( r = r_0, \quad V_r = 0, \quad \omega_\theta = \Omega_\theta, \quad V_\phi = \omega_02(r_0, \Omega_\theta), \)

where \( r_0, \Omega_\theta \) are some constants, parameters of the family.

Thus, for any fixed values \((r_0, \Omega_\theta)\) there exist two different frequencies of stationary motion. In addition, according to Eq. (19), these frequencies correspond to motion of the center of mass of the ball in a circle in opposite directions.

Remark 3. It follows from Eq. (19) that there exists no minimal radius \( r_0 \), as obtained in Refs. [5, 4] starting from which there exist periodic solutions (16).

2. Let us analyze the linear stability of the periodic solution (20). For this we represent the system (13) as

\[
\dot{z} = F(z),
\]

where \( z = (r, \omega_\theta, V_r, V_\phi) \) is the vector of the variables and \( F(z) \) is the vector whose components are functions of the variables \( z \) and of the system parameters.

Next, we expand \( F(z) \) near the partial solution (20). For convenience of calculations we choose \( r_0 \) and \( \omega_0 \) as independent parameters. In this case, \( \Omega_\theta \) can be uniquely expressed from Eq. (18).

We obtain the following system of differential equations:

\[
\dot{\tilde{z}} = L\tilde{z}, \quad L = \left. \frac{\partial F}{\partial z} \right|_{z = z_0},
\]

where \( L \) is the linearization matrix, \( \tilde{z} = z - z_0, \ z_0 = (r_0, \Omega_\theta, 0, \omega_0) \) is the vector whose components correspond to Eq. (20).

The characteristic equation \( \det(L - \lambda E) = 0 \) (with \( \lambda \) being its roots) is

\[
\lambda^2 \left( \lambda^2 + \frac{(k + \sin^2 \theta)\omega_0^2}{1 + k} + 2 \cos \theta (\omega_0 - \omega_0^*) \right) = 0,
\]

where, to abbreviate the formula, we have introduced the notation

\[
\omega_0^* = \frac{k\Omega}{2(1 + k)}.
\]

Two zero roots of the characteristic equation correspond to the parameters \( r_0, \omega_0 \) of the family (20). The nonzero roots of Eq. (21) can be imaginary or real (of different signs) depending on the parameters \( r_0 \) and \( \omega_0 \) and the value of the angular velocity of rotation of the cone \( \Omega \). But since the system (13) is invariant under the substitution

\[
(\Omega, \dot{\phi}, \omega_\theta) \rightarrow (-\Omega, -\dot{\phi}, -\omega_\theta),
\]

7
we shall consider, without loss of generality, only the values $\Omega \geq 0$. The analysis of the system for $\Omega < 0$ will be identical up to a change of sign of the variables $\dot{\varphi}$ and $\omega_\theta$.

We enumerate all possible cases.

1. When $\Omega \neq 0$, the nonzero roots of Eq. (21) are imaginary, the family of periodic solutions (20) is linearly stable and of center type in the cases
   
   (a) $\omega_0 > \omega_0^*$ and $\omega_0 < 0$ for any $r_0$,
   (b) $0 < \omega_0 < \omega_0^*$ and $r_0 > \frac{2 \cos \theta (\omega_0^* - \omega_0)}{\omega_0^2 (k + \sin^2 \theta)}$.

2. When $\Omega = 0$, the nonzero roots of Eq. (21) are imaginary, the family of periodic solutions (20) is linearly stable and of center type for any $\omega_0 \neq 0$ and $r_0$. When $\omega_0 = 0$, according to Eq. (18), there exist no periodic solutions to Eqs. (13) and (15).

3. In other cases, the family of periodic solutions (20) is linearly unstable and of saddle type. For the sake of illustration, the instability region (20) is shown in grey in Fig. 2 on the plane $(\omega_0, r_0)$.

![Figure 2: Instability regions of the periodic solution (20) on the plane $(r_0, \omega_0)$ are shown in grey and constructed for the parameters $k = 2/5, \Omega = 0.31, \theta = 89\pi/180$ (for a ball of radius 19 mm the corresponding dimensional angular velocity is 7 rad/s)](image)

3 Discussion

Our main goal in writing this paper was to eliminate inaccuracies in the analysis made in Ref. [5] of the nonholonomic rolling of a ball on a cone. In this note, we have considered only a partial solution corresponding to the periodic rolling of the center of mass of the ball in a horizontal circle on a cone, and presented a criterion for stability of this solution which determines the relation of the system parameters.

The analysis of motion of the homogeneous ball on the cone is not restricted to investigating a partial periodic solution. The problem that remains open is that of exploring the rolling of the ball on the cone depending on initial conditions in the general case. In addition, given the results of Refs. [2, 3] the system can be reduced to quadratures.
Another open problem is to examine the rolling of the ball with friction. In the recent paper [1], in the case of motion of a homogeneous ball on a plane, a good agreement was shown between the experimental trajectory and the theoretical trajectory obtained by adding the moment of rolling friction which is proportional to the angular velocity of the ball. Using this friction model, it was shown that all trajectories asymptotically tend to an untwisting spiral. For motion on the cone there may exist initial conditions under which the center of mass of the ball will fall to the center of the cone.

References

[1] A. V. Borisov, T. B. Ivanova, Y. L. Karavaev, and I. S. Mamaev, “Theoretical and experimental investigations of the rolling of a ball on a rotating plane (turntable),” Eur. J. Phys. 39 (6), 065001, 13 (2018).

[2] A. V. Borisov, I. S. Mamaev, and A. A. Kilin, “The rolling motion of a ball on a surface: New integrals and hierarchy of dynamics” Regul. Chaotic Dyn. 7 (2), 201–219 (2002).

[3] A. V. Borisov, I. S. Mamaev, and I. A. Bizyaev, “The Jacobi Integral in Nonholonomic Mechanics,” Regul. Chaotic Dyn. 20 (3), 383–400 (2015).

[4] K. Weltner, “Movement of spheres on rotating discs: a new method to Measure coefficients of rolling friction by the central drift,” Mech. Res. Commun. 10 (4), 223–232 (1983).

[5] K. Zengel, “The electromagnetic analogy of a ball on a rotating conical turntable,” Am. J. Phys. 85 (12), 901–907 (2017).

[6] O. M. O’Reilly, Intermediate dynamics for engineers: a unified treatment of Newton-Euler and Lagrangian mechanics, (Cambridge University Press, New York, 2008).