Abstract

This paper presents the identification of heterogeneous elasticities in the Cobb-Douglas production function. The identification is constructive with closed-form formulas for the elasticity with respect to each input for each firm. We propose that the flexible input cost ratio plays the role of a control function under “non-collinear heterogeneity” between elasticities with respect to two flexible inputs. The \textit{ex ante} flexible input cost share can be used to identify the elasticities with respect to flexible inputs for each firm. The elasticities with respect to labor and capital can be subsequently identified for each firm under the timing assumption admitting the functional independence.

\textbf{Keywords:} Cobb-Douglas production function, heterogeneous elasticity, identification.

1 Introduction

Heterogeneous output elasticities and non-neutral productivity are natural features of production technologies. In addition, the heterogeneity and non-neutrality are related to a num-

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*Tong Li. Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-1819. Email: tong.li@vanderbilt.edu. Phone: (615) 322-3582

†Yuya Sasaki. Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-1819. Email: yuya.sasaki@vanderbilt.edu. Phone: (615) 343-3016.
ber of empirical questions in development economics, economic growth, industrial organization, and international trade. The econometrics literature is relatively sparse about methods of identifying production functions allowing for these empirically relevant technological features. A couple of innovations have been made relatively recently. By extending the approach of Doraszelski and Jaumandreu (2013), Doraszelski and Jaumandreu (2015) propose an empirical strategy to analyze constant elasticity of substitution (CES) production function with labor augmenting productivity, which allows for multi-dimensional heterogeneity and non-neutral productivity. By extending the method of Gandhi, Navarro, and Rivers (2017), Kasahara, Schrimpf, and Suzuki (2015) propose an empirical strategy to analyze Cobb-Douglas production function with finitely supported heterogeneous output elasticities.

We study identification of the Cobb-Douglas production function model with infinitely supported heterogeneous coefficients indexed by unobserved latent technologies. Our objective is to identify the vector of both non-additive and additive parts of the productivity for each firm, where the non-additive part consists of the output elasticity with respect to each input and the additive part is the traditional neutral productivity. We provide constructive identification with closed-form identifying formulas for each of the heterogeneous output elasticities and the additive productivity for each firm. Our constructive identification with closed-form formulas provides a transparent argument in relation to potential identification failures due to subtle yet critical issues, such as the functional dependence problem pointed out by Ackerberg, Caves, and Frazer (2015) and the instrument irrelevance problem pointed out by Gandhi, Navarro, and Rivers (2017).
2 Relation to the Literature

One of the challenges in empirical analysis of production functions is to overcome the simultaneity in the choice of input quantities by rational firms, which biases naïve estimates (Marschak and Andrews, 1944). The literature on identification of production functions has a long history (see e.g., Griliches and Mairesse, 1998; Ackerberg, Benkard, Berry, and Pakes, 2007), and remarkable progresses have been made over the past two decades. Various ideas proposed in this literature facilitate the identification result that we develop in this paper. Furthermore, this literature has discovered some subtle yet critical sources of potential identification failure, which we need to carefully take into account when we construct our identification results. As such, it is useful to discuss in detail the relations between our identification strategy and the principal ideas developed by this literature.

A family of approaches widely used in practice today to identify parameters of the Cobb-Douglas production function are based on control functions. Olley and Pakes (1996) propose to use the inverse of the reduced-form investment choice function as a control function for latent technology. Levinsohn and Petrin (2003) propose to use the inverse of the reduced-form flexible input choice function as a control function for the latent technology. See also Wooldridge (2009) for estimation of the relevant models. The main advantage of these identification strategies is that an econometrician can be agnostic on the form of the control function other than the requirement for the invertibility of the function. Like the control function literature, we employ a control function for latent technologies. However, unlike Olley and Pakes (1996) or Levinsohn and Petrin (2003), we do not directly assume an invertible mapping between an observed choice by firm and unobserved technology. Instead, we only assume for construction of a control function that the ratio of heterogeneous elasticities with respect to two flexible
inputs are not globally collinear – see Assumption 4 ahead and discussions thereafter. In other words, our approach requires “non-collinear heterogeneity” between elasticities with respect to two flexible inputs, in place of the traditional assumption of invertible mapping.

Ackerberg, Caves, and Frazer (2015) point out the so-called functional dependence problem in the control function approaches of Olley and Pakes (1996) and Levinsohn and Petrin (2003), and propose a few alternative structural assumptions to circumvent this problem. The functional dependence problem refers to the rank deficiency for identifying labor elasticity that arises because labor input that depends on the current state variables loses data variations once the state variables are fixed through the control function. Among alternative structural assumptions to avoid this problem, Ackerberg, Caves, and Frazer (2015) suggest a timing assumption where labor input is determined slightly before the current state realizes – also see Ackerberg and Hahn (2015). This structural assumption is empirically supported by Hu, Huang, and Sasaki (2017). The structural assumptions (Assumptions 1–5) invoked in the present paper are consistent with the timing assumption suggested by Ackerberg, Caves, and Frazer (2015), and relevant data generating processes can allow for the functional independence by a similar argument to those of Ackerberg, Caves, and Frazer (2015) and Kasahara, Schrimpf, and Suzuki (2015).

Gandhi, Navarro, and Rivers (2017) point out another source of identification failure in the approach of using the flexible input choice function as a control function. Namely, the Markovian model of state evolution which is commonly assumed in this literature certainly induces orthogonality restrictions, but it also nullifies the instrumental power or instrumental relevance for identification of flexible input elasticities. Noting the role of instruments from viewpoint of simultaneous equations, Doraszelski and Jaumandreu (2013, 2015) suggest to use lagged input prices as alternative instruments assumed to satisfy both the instrument indepen-
dence and instrument relevance, and thus solve this problem. In fact, for the Cobb-Douglas production functions, it is long known that the first-order conditions and the implied input cost or revenue shares inform us of input elasticities (Solow, 1957). This approach has been more recently revisited by van Biesebroeck (2003), Doraszelski and Jaumandreu (2013, 2015), Kasahara, Schrimpf, and Suzuki (2015), Gandhi, Navarro, and Rivers (2017), and Grieco, Li, and Zhang (2016) in and beyond the context of the Cobb-Douglas functions. The present paper also takes a similar approach. We argue that the ratio of flexible input costs in conjunction with the aforementioned assumption of non-collinear heterogeneity constructs a control variable for the latent technology. Furthermore, the “ex ante input cost share” defined as the share of input cost relative to the conditional expectation of output value from firm’s point of view is effective for constructive identification of the heterogeneous elasticities with respect to flexible inputs. This ex ante input cost share is also directly identifiable from data by econometricians once we construct the control variable from the flexible input costs. The idea of using input cost share to identify flexible input elasticity was hinted in Gandhi, Navarro, and Rivers (2017), and we further devise a way to extend this idea to models with non-additive productivity. The model considered by Gandhi, Navarro, and Rivers (2017) and the model considered in the present paper are complementary, in that the former is nonparametric with additive productivity while the latter is linear with non-additive productivity.

Productivity is sometimes treated as incidental parameters in panel data analysis, but the literature on production functions has often circumvented the incidental parameters problem (cf. Neyman and Scott, 1948) via inversions of maps representing choice rules of rational firms (e.g., Olley and Pakes, 1996; Levinsohn and Petrin, 2003). As already mentioned, we also circumvent this problem via a control function based on the assumption of non-collinear heterogeneity. Nonetheless, the existing methods to identify production functions still utilize
panel data to form orthogonality restrictions based on the first difference in productivity (e.g., Olley and Pakes, 1996; Levinsohn and Petrin, 2003; Wooldridge, 2009; Ackerberg, Caves, and Frazer, 2015). On the other hand, we do not form such orthogonality restrictions based on panel data, as we can explicitly identify the output elasticities with respect to flexible inputs via the aforementioned \textit{ex ante} input cost shares. This aspect of our approach is similar to that of Gandhi, Navarro, and Rivers (2017).

While the literature on identification of production functions often considers the CES productions functions (including the Cobb-Douglas and translog approximation cases) with additive latent technologies, a departure from Hicks-neutral productivity allows for answering many important economic questions as emphasized in the introduction. Doraszelski and Jaumandreu (2015) extend the identification strategy of Doraszelski and Jaumandreu (2013) to the framework of CES production function with labor-augmenting technologies. The present paper shares similar motivations to that of the preceding work by Doraszelski and Jaumandreu (2015), but in different and complementary directions. The labor-augmented CES production function in the Cobb-Douglas limit case entails neutral productivity, and hence the present paper focusing on non-neutral productivity in the Cobb-Douglas production function attempts to complement the CES framework of Doraszelski and Jaumandreu (2015). Cobb-Douglas production functions with non-additive heterogeneity are studied in Kasahara, Schrimpf, and Suzuki (2015) and Balat, Brambilla, and Sasaki (2016). Kasahara, Schrimpf, and Suzuki (2015) treat heterogeneous productivity via a mixture of the models of Gandhi, Navarro, and Rivers (2017), and propose to identify the mixture components. On the other hand, our framework allows for infinitely supported coefficients and our method constructs identifying formulas for each coefficient for each firm. For an application to international trade, Balat, Brambilla, and Sasaki (2016) consider infinitely supported coefficients in the Cobb-Douglas production function, where...
they almost directly assume identification for the moment restrictions in a similar manner to Ackerberg, Caves, and Frazer (2015), based on a multi-dimensional invertibility assumption for the reduced-form flexible input choice. On the other hand, the present paper develops the identification strategy instead of assuming the identification, and complements Balat, Brambilla, and Sasaki (2016) by formally establishing the identification result for a closely related model. We take advantage of the first-order conditions, instead of relying on the invertibility assumption, for the purpose of unambiguous identification of output elasticities with respect to flexible inputs as emphasized earlier.

3 The Model and Notations

Consider the gross-output production function in logarithm:

\[ y_t = \Psi (l_t, k_t, m^1_t, m^2_t, \omega_t) + \eta_t \quad \text{E[}\eta_t\text{]} = 0, \quad (3.1) \]

where \( y_t \) is the logarithm of output produced, \( l_t \) is the logarithm of labor input, \( k_t \) is the logarithm of capital, \( m^1_t \) is the logarithm of a flexible input such as materials, \( m^2_t \) is the logarithm of another flexible input such electricity, \( \omega_t \) is an index of latent technology, and \( \eta_t \) is an idiosyncratic productivity shock. The Cobb-Douglas production function takes the form

\[ \Psi (l_t, k_t, m^1_t, m^2_t, \omega_t) = \beta_l(\omega_t)l_t + \beta_k(\omega_t)k_t + \beta_{m^1}(\omega_t)m^1_t + \beta_{m^2}(\omega_t)m^2_t + \beta_0(\omega_t) \quad (3.2) \]

with heterogeneous coefficients \((\beta_l(\omega_t), \beta_k(\omega_t), \beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t), \beta_0(\omega_t))\) that are indexed by the latent technology \( \omega_t \). The latent technology \( \omega_t \) affects the additive productivity \( \beta_0(\omega_t) \) and the elasticities \( \beta_l(\omega_t), \beta_k(\omega_t), \beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t) \) in non-parametric and non-linear ways. Econometricians may not know the functional forms of \( \beta_l(\cdot), \beta_k(\cdot), \beta_{m^1}(\cdot), \beta_{m^2}(\cdot), \) or \( \beta_0(\cdot) \).

Let \( p^y_t \) denote the unit price of the output faced by firm \( j \) at time period \( t \). Let \( p^{m_1}_t \) and \( p^{m_2}_t \)
denote the unit prices of the two types \((k = 1 \text{ and } 2)\) of the flexible input, respectively. (We
remark that these prices need not be observed in data for our iden-tification argument. We only
require to observe the output value, \(p_y^t \exp(y_t)\), and input costs, \(p_{m1}^t \exp(m_{1t})\) and \(p_{m2}^t \exp(m_{2t})\),
for our identification results.) With these notations, we make the following assumption on
flexible input choice by firms.

**Assumption 1** (Flexible Input Choice). A firm at time \(t\) with the state variables \((l_t, k_t, \omega_t)\)
chooses the flexible input vector \((m_{1t}, m_{2t})\) by the optimization problem

\[
\max_{(m_1, m_2) \in \mathbb{R}_+^2} p_y^t \exp\left(\Psi\left(l_t, k_t, m_{1t}, m_{2t}, \omega_t\right)\right) E[\exp(\eta_t)] - p_{m1}^t \exp\left(m_{1t}\right) - p_{m2}^t \exp\left(m_{2t}\right),
\]

where \(p_{m1}^t > 0\) and \(p_{m2}^t > 0\) almost surely.

This assumption consists of two parts. First, each firm makes the choice of the flexible
input vector \((m_{1t}, m_{2t})\) by the expected profit maximization against unforeseen shocks \(\eta_t\) given
the state variables \((l_t, k_t, \omega_t)\). Second, firms almost surely face strictly positive flexible input
prices. Furthermore, in order to guarantee the existence of these flexible input solutions, we
make the following assumption of diminishing returns with respect to flexible input.

**Assumption 2** (Finite Solution). \(\beta_{m1}(\omega_t) + \beta_{m2}(\omega_t) < 1\) almost surely.

Note that this assumption only requires diminishing returns with respect to the subvec-tor \((m_{1t}, m_{2t})\) of only flexible inputs, and not necessarily with respect to the entire vector
\((l_t, k_t, m_{1t}, m_{2t})\) of production factors. Assumptions \(\Box\) and \(\square\) stated below concern about the
heterogeneous elasticity functions \((\beta_l(\cdot), \beta_k(\cdot), \beta_{m1}(\cdot), \beta_{m2}(\cdot), \beta_0(\cdot))\), and involve requirements
for a number of functions to be measurable.

**Assumption 3** (Measurability). \(\beta_l(\cdot), \beta_k(\cdot), \beta_{m1}(\cdot), \beta_{m2}(\cdot)\) and \(\beta_0(\cdot)\) are measurable functions.
Assumption 3 is satisfied if, for example, $\beta_l(\cdot)$, $\beta_k(\cdot)$, $\beta_{m^1}(\cdot)$, $\beta_{m^2}(\cdot)$ and $\beta_0(\cdot)$ are continuous functions of the latent technology $\omega_t$. In particular, for the additive-productivity models considered in the literature (e.g., Olley and Pakes, 1996; Levinsohn and Petrin, 2003; Wooldridge, 2009), this assumption is trivially satisfied as $\beta_l(\cdot)$, $\beta_k(\cdot)$, $\beta_{m^1}(\cdot)$, and $\beta_{m^2}(\cdot)$, being constant functions (i.e., $\beta_l(\omega_t) \equiv \beta_l$, $\beta_k(\omega_t) \equiv \beta_k$, $\beta_{m^1}(\omega_t) \equiv \beta_{m^1}$, and $\beta_{m^2}(\omega_t) \equiv \beta_{m^2}$) are continuous, and $\beta_0(\cdot)$ being the identity function (i.e., $\beta_0(\omega_t) \equiv \omega_t$) is continuous as well.

Assumption 4 (Non-Collinear Heterogeneity). The function $\omega \mapsto \beta_{m^1}(\omega)/\beta_{m^2}(\omega)$ is measurable and invertible with measurable inverse.

Assumption 4 is satisfied if the rate at which the latent technology $\omega_t$ increases the output elasticity $\beta_{m^1}(\omega_t)$ with respect to $m^1$ is strictly higher or strictly lower than the rate at which the latent technology $\omega_t$ increases the output elasticity $\beta_{m^2}(\omega_t)$ with respect to $m^2$. Figure 1 provides a geometric illustration of Assumption 4. The bold solid curves indicate the technological paths $\omega_t \mapsto (\beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t))$. The dashed rays from the origin indicate linear paths. The left column, (a) and (b), of the figure illustrates cases that satisfy Assumption 4. In these graphs, the ratio $\beta_{m^1}(\omega)/\beta_{m^2}(\omega)$ is associated with a unique value of $\omega$ provided an injective technology $\omega_t \mapsto (\beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t))$. Assumption 4 is a nonparametric shape restriction, as opposed to a parametric functional restriction, and hence both a simple linear case (a) and a nonlinear case (b) satisfy this assumption. The right column, (a’), and (b’), of the figure illustrates cases that violate Assumption 4. In these graphs, the ratio $\beta_{m^1}(\omega)/\beta_{m^2}(\omega)$ is not associated with a unique value of $\omega$ even if the technology $\omega_t \mapsto (\beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t))$ is injective.

Note the difference between Assumption 4 and the invertibility assumptions used in the nonparametric control function approaches (e.g., Olley and Pakes, 1996; Levinsohn and Petrin, 2003), where an invertible mapping between the technology $\omega_t$ and an observable, such as...
Figure 1: Illustrations of Assumption 4. The bold solid curves indicate the paths representing the technology $\omega_t \mapsto (\beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t))$. The dashed rays from the origin indicate linear paths. The left column, (a) and (b), of the figure illustrates cases that satisfy Assumption 4. In these graphs, the ratio $\beta_{m^1}(\omega)/\beta_{m^2}(\omega)$ is associated with a unique value of $\omega$ provided an injective technology $\omega_t \mapsto (\beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t))$. The right column, (a’) and (b’), of the figure illustrates cases that violate Assumption 4. In these graphs, the ratio $\beta_{m^1}(\omega)/\beta_{m^2}(\omega)$ is not associated with a unique value of $\omega$ even if the technology $\omega_t \mapsto (\beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t))$ is injective.
investment choice or flexible input choice, is assumed to exist. Assumption \[1\] does not assume such an inversion between the technology and observed choices by firms. Assumption \[4\] only requires that the latent technology \(\omega_t\) has a one-to-one relation with the ratio \(\beta_{m1}(\omega_t)/\beta_{m2}(\omega_t)\) of the elasticities with respect to two flexible inputs. One restrictive feature of this non-collinearity assumption is that it rules out constant coefficients \((\beta_{m1}(\omega_t) \equiv \beta_{m1} \text{ and } \beta_{m2}(\omega_t) \equiv \beta_{m2})\) for the two flexible inputs.

Finally, we state the following independence assumption for the idiosyncratic shock \(\eta_t\), which is standard in the literature. It requires that the idiosyncratic shock \(\eta_t\) is unknown by a firm at the time when it makes input decisions for production to take place in period \(t\).

**Assumption 5 (Independence).** \((l_t, k_t, \omega_t, p_{m1}^t, p_{m2}^t) \perp \eta_t\).

We introduce the flexible input cost ratio defined by

\[
r_{t}^{1,2} = \frac{p_{m1}^t \exp(m_1^t)}{p_{m2}^t \exp(m_2^t)}.
\]  

(3.3)

We argue in Lemmas \[1\] and \[2\] below in the identification section that this ratio plays an important role as a control variable for the latent productivity \(\omega_t\) under Assumptions \[1\] and \[4\].

We emphasize that the input prices, \(p_{m1}^t\) and \(p_{m2}^t\), need not be observed in data. It suffices to observe the input costs, \(p_{m1}^t \exp(m_1^t)\) and \(p_{m2}^t \exp(m_2^t)\).

We also introduce the \textit{ex ante} input cost share of \(m^t\) for each \(\iota \in \{1, 2\}\), defined by

\[
s_t^{\iota} = \frac{p_{m\iota}^t \exp(m^t)}{\mathbb{E}[p_{y}^t \exp(y_t) | l_t, k_t, m_1^t, m_2^t, r_{t}^{1,2}]}.
\]

(3.4)

This is an \textit{ex ante} share because the output is given by the conditional expectation given the information \((l_t, k_t, m_1^t, m_2^t, r_{t}^{1,2})\) observable from the viewpoint of the econometrician (as well as the firm) before the idiosyncratic shock \(\eta_t\) is realized. The prices, \(p_{m1}^t\), \(p_{m2}^t\) or \(p_{y}^t\), need not be observed in data. It suffices to observe the input costs, \(p_{m1}^t \exp(m_1^t)\) and \(p_{m2}^t \exp(m_2^t)\), and the
output value, $p_t^y \exp(y_t)$. We argue in Lemma 3 below in the identification section that this 
ex ante cost share can be used to identify the coefficients, $\beta_{m_1}(\omega_t)$ and $\beta_{m_2}(\omega_t)$, of the flexible inputs, $m_1$ and $m_2$, under Assumptions 1, 2, 3, 4, and 5.

4 An Overview

4.1 An Overview of the Construction of a Control Variable

It is perhaps intuitive and is also formally shown in Lemma 1 below that Assumption 1 (flexible input choice) implies the equality $r_{1,2}^t = \beta_{m_1}(\omega_t)/\beta_{m_2}(\omega_t)$ between the flexible input cost ratio and the ratio of the output elasticities with respect to the two flexible inputs. By this equality, econometricians may use the flexible input cost ratio $r_{1,2}^t$ in order to control for the latent technology $\omega_t$ under Assumption 4. Figure 2 graphically describes how this is possible. The base figure used here is copied from panel (b) of Figure 1, which illustrates a case where Assumption 4 is satisfied. Figure 2 shows that firms with the flexible input cost ratio $r_{1,2}^t = 0.8, 1.0, and 1.2$ are associated with the latent technological levels of $\omega_t = \omega', \omega'', and \omega'''$, respectively. In this manner, Assumption 4 implies that controlling for $r_{1,2}^t$ is equivalent to controlling for $w_t$. This intuitive illustration also indicates how powerful and important Assumption 4 is for our identification strategy.

4.2 An Overview of the Closed-Form Identifying Formulas

In this section, we present a brief overview of all the closed-form identifying formulas which we formally derive in Section 5. For the model (3.1)–(3.2) equipped with Assumptions 1, 2, 3, 4, and 5, the heterogeneous coefficients are identified for each firm residing in the “locality of identification” (to be formally defined in Section 5.1) with the following closed-form formulas.
Figure 2: Illustrations of the construction of the control variable $r_t^{1,2}$ for the latent technology $\omega_t$ under Assumption 4. Firms with the flexible input cost ratio $r_t^{1,2} = 0.8$, 1.0, and 1.2 are associated with the latent technological levels of $\omega_t = \omega', \omega''$, and $\omega'''$, respectively.
The coefficients of flexible inputs, \( m^1 \) and \( m^2 \), are identified in closed form by

\[
\beta_{m^1}(\omega_t) = s^1_t \quad \text{and} \quad \beta_{m^2}(\omega_t) = s^2_t.
\]  

(4.1)

The coefficients of labor \( l \) and capital \( k \) are identified in turn in closed form by

\[
\beta_l(\omega_t) = \frac{\partial}{\partial l} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t \mid l_t = l, k_t, r^{1,2}_t] \bigg|_{l=l_t} \quad \text{and} \quad \beta_k(\omega_t) = \frac{\partial}{\partial k} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t \mid l_t, k_t = k, r^{1,2}_t] \bigg|_{k=k_t}.
\]  

(4.2)

Finally, the additive productivity is identified in closed form by

\[
\beta_0(\omega_t) = E[y_t - \beta_l(\omega_t)l_t - \beta_k(\omega_t)k_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t \mid l_t, k_t, r^{1,2}_t].
\]  

(4.3)

The next section presents a formal argument to derive these closed-form identifying formulas.

5 Main Results

5.1 Locality of Identification

A potential obstacle to identification of production function models is that functional dependence of input choices on the latent technology may induce rank deficiency and thus a failure in identification (Ackerberg, Caves, and Frazer, 2015). A few structural models to eliminate this functional dependence problem are proposed by Ackerberg, Caves, and Frazer (2015). Among those structural models, Hu, Huang, and Sasaki (2017) empirically support the second model of Ackerberg, Caves, and Frazer (2015) that labor input \( l_t \) as well as capital \( k_t \) are chosen earlier than period \( t \). This structural assumption is also consistent with our Assumption \( \Pi \) where only \( m^1_t \) and \( m^2_t \) are treated as flexible inputs. If \( l_t \) and \( k_t \) are chosen prior to realization of \( \omega_t \), then
stochastic evolution of $\omega_t$ will allow for elimination of functional dependence in the sense that it allows for a non-degenerate conditional distribution of $l_t$ given $(k_t, \omega_t)$ and a non-degenerate conditional distribution of $k_t$ given $(l_t, \omega_t)$. As we discuss below after Lemma 4, this non-degeneracy is crucial for identification of $\beta_l(\cdot)$ and $\beta_k(\cdot)$ in the relevant locality. In light of the timing assumption with stochastic evolution of $\omega_t$, we assume that $l_t$, $k_t$, and $\omega_t$ are continuous random variables throughout this paper, and thus the location $(l_t, k_t, \omega_t) = (l^*, k^*, \omega^*)$ defined below enables the local identification.

**Definition 1 (Locality of Identification).** The point $(l_t, k_t, \omega_t) = (l^*, k^*, \omega^*)$ is called the locality of identification if there are real numbers $\underline{f}, \overline{f} \in (0, \infty)$ such that $\underline{f} < f_{l_t|k_t,\omega_t}(l, k, \omega) < \overline{f}$ for all $(l, k, \omega)$ in a neighborhood of $(l^*, k^*, \omega^*)$.

**Proposition 1 (Functional Independence).** If $(l^*, k^*, \omega^*)$ is the locality of identification according to Definition 4, then: (i) the conditional density function $f_{l_t|k_t,\omega_t}(\cdot | k^*, \omega^*)$ exists and is positive in a neighborhood of $l = l^*$; and (ii) the conditional density function $f_{k_t|l_t,\omega_t}(\cdot | l^*, \omega^*)$ exists and is positive in a neighborhood of $k = k^*$.

**Proof.** By Definition 4 there is $\varepsilon \in (0, \infty)$ such that $\underline{f} < f_{l_t,k_t,\omega_t}(l, k, t) < \overline{f}$ for all $(l, k, t)$ in an $\varepsilon$-ball of $(l^*, k^*, \omega^*)$. Thus, $f_{k_t,\omega_t}(k^*, \omega^*) > 2\varepsilon \underline{f} > 0$. Therefore, the conditional density function $f_{l_t|k_t,\omega_t}(\cdot | k^*, \omega^*) = f_{l_t,k_t,\omega_t}(\cdot, k^*, \omega^*)/f_{k_t,\omega_t}(k^*, \omega^*)$ exists, and is bounded by $\overline{f}/f_{k_t,\omega_t}(k^*, \omega^*)$ in the $\varepsilon$-ball of $l = l^*$. This proves part (i). A proof of part (ii) similar by exchanging the roles of $l$ and $k$. \hfill $\Box$

### 5.2 Control Variable

We construct a control variable via the first-order condition explicitly exploiting the structural information. The flexible input choice rule in Assumption 4 yields the following restrictions as
the first-order condition.

\[ p_t^y / \beta_{m^1}(\omega_t) \exp \left( \Psi \left( l_t, k_t, m^1_t, m^2_t, \omega_t \right) \right) E \left[ \exp(\eta_t) \right] = p_t^{m^1} \exp(m^1) \]  

(5.1)

for each \( t \in \{1, 2\} \). Furthermore, Assumption 2 guarantees that the solution to the flexible input choice problem exists, and is explicitly given by

\[
m^1_t = \ln p_t^y + \frac{1 - \beta m^2(\omega_t)}{1 - \beta m^1(\omega_t) - \beta m^2(\omega_t)} \beta m^1(\omega_t) \ln \frac{\beta m^1(\omega_t)}{p_t^{m^1}} + (1 - \beta m(\omega_t)) \ln \frac{\beta m^2(\omega_t)}{p_t^{m^2}} \]

(5.2)

\[
m^2_t = \ln p_t^y + \frac{1 - \beta m^1(\omega_t)}{1 - \beta m^1(\omega_t) - \beta m^2(\omega_t)} \beta m^1(\omega_t) \ln \frac{\beta m^1(\omega_t)}{p_t^{m^1}} + (1 - \beta m^2(\omega_t)) \ln \frac{\beta m^2(\omega_t)}{p_t^{m^2}} \]

(5.3)

These two equations under Assumption 3 imply that \((m^1_t, m^2_t)\) is a measurable function of the state variables \((l_t, k_t, \omega_t, p_t^{m^1}, p_t^{m^2})\). With these solutions to the static optimization problem as auxiliary tools, we now proceed with the construction of a control variable for the unobserved technology \(\omega_t\). The next lemma shows that the flexible input cost ratio \(r^{1,2}_t\) defined in (3.3) identifies the ratio of the heterogeneous coefficients of two flexible inputs.

**Lemma 1** (Identification of the Ratio \(\beta_{m^1}(\omega_t)/\beta_{m^2}(\omega_t)\)). If Assumption 1 is satisfied, then

\[
\frac{\beta m^1(\omega_t)}{\beta m^2(\omega_t)} = r^{1,2}_t.
\]

(5.4)

**Proof.** Assumption 1 yields the first-order condition (5.1). Taking the ratio of this first-order condition for \( t = 1 \) over the first-order condition for \( t = 2 \) yields (5.4). \( \square \)

From Assumption 1 and (5.4) that holds under Assumption 1, we can see that the flexible input cost ratio \(r^{1,2}_t\) defined in (3.3) can be used as a control variable for the unobserved latent productivity \(\omega_t\). Specifically, we state the following lemma.
Lemma 2 (Control Variable). If Assumptions 1 and 4 satisfied, then there exists a measurable invertible function $\phi$ with a measurable inverse $\phi^{-1}$ such that

$$\omega_t = \phi(r_t^{1,2}). \quad (5.5)$$

Proof. The claim follows from Lemma 1 and Assumption 4.

In light of this lemma, we can interpret the flexible input cost ratio $r_t^{1,2}$ as a normalized observable measure of the unobserved latent productivity $\omega_t$ through the normalizing transformation $\phi$. With this interpretation, the identification of $\beta_l(\cdot)$, $\beta_k(\cdot)$, $\beta_{m1}(\cdot)$, $\beta_{m2}(\cdot)$, and $\beta_0(\cdot)$ may be achieved by the identification of $\beta_l \circ \phi(\cdot)$, $\beta_k \circ \phi(\cdot)$, $\beta_{m1} \circ \phi(\cdot)$, $\beta_{m2} \circ \phi(\cdot)$, and $\beta_0 \circ \phi(\cdot)$, respectively, which take the normalized observable measure $r_t^{1,2}$ of the unobserved productivity $\omega_t$. The closed-form identification results stated as Lemmas 3, 4, and 5 in Section 5.3 indeed consist of identifying formulas in the forms of these function compositions.

5.3 Identification

5.3.1 Coefficients of Flexible Inputs

Recall the ex ante input cost shares, $s^1_t$ and $s^2_t$, of $m^1$ and $m^2$, respectively, defined in (3.4). These ex ante input cost shares are not directly observable from data, but are directly identifiable from data. The following lemma shows that the marginal product $\beta_{m^i}(\omega_t)$ of flexible input $m^i$ can be identified by this ex ante input cost share for each $i \in \{1, 2\}$.

Lemma 3 (Identification of $\beta_{m1}(\cdot)$ and $\beta_{m2}(\cdot)$). If Assumptions 1, 2, 3, 4, and 5 are satisfied, then

$$\beta_{m1}(\omega_t) = \beta_{m1}(\phi(r_t^{1,2})) = s^1_t \quad \text{and}$$

$$\beta_{m2}(\omega_t) = \beta_{m2}(\phi(r_t^{1,2})) = s^2_t \quad (5.6)$$
hold.

This lemma establishes the identifying formulas (4.1) presented in Section 4.

Proof. The proof of this lemma consists of four steps. First, note that Lemma 2 under Assumptions 1 and 4 implies the equivalence between the two sigma algebras:

\[ \sigma(l_t, k_t, m_1^t, m_2^t, r_1^{1.2}) = \sigma(l_t, k_t, m_1^t, m_2^t, \omega_t). \] (5.7)

Second, Assumptions 1 and 2 yield (5.2) and (5.3), which in turn imply under Assumption 3 that \((m_1^t, m_2^t)\) is a measurable function of \((l_t, k_t, \omega_t, p_{m1}^t, p_{m2}^t)\). In light of this, applying Theorem 2.1.6 of Durrett (2010) to Assumption 5 yields

\[ (l_t, k_t, m_1^t, m_2^t, \omega_t) \perp \eta_t. \] (5.8)

Third, we obtain the following chain of equalities.

\[
\begin{align*}
E[\exp(y_t)|l_t, k_t, m_1^t, m_2^t, r_1^{1.2}] &= E[\exp(y_t)|l_t, k_t, m_1^t, m_2^t] \\
&= E[\exp(\Psi(l_t, k_t, m_1^t, m_2^t, \omega_t) + \eta_t)|l_t, k_t, m_1^t, m_2^t] \\
&= \exp(\Psi(l_t, k_t, m_1^t, m_2^t)) E[\exp(\eta_t)|l_t, k_t, m_1^t, m_2^t] \\
&= \exp(\Psi(l_t, k_t, m_1^t, m_2^t)) E[\exp(\eta_t)]
\end{align*}
\]

where the first equality is due to (5.7), the second equality follows from substitution of the gross-output production function (3.1), the third equality follows from the property of the conditional expectation that \(E[\Psi_1(X_1)\Psi_2(X_2)|X_1] = \Psi_1(X_1) E[\Psi_2(X_2)|X_1]\), and the fourth equality follows from (5.8).

Finally, taking the ratio of (5.1) to (5.9) yields

\[ \beta_{m^\iota}(\omega_t) = \frac{p_{m^\iota}^t \exp(m^\iota)}{p_{m}^t E[\exp(y_t)|l_t, k_t, m_1^t, m_2^t, r_1^{1.2}]} \]

for each \(\iota \in \{1, 2\}\). By the definition of \(s_1^t\) given in (3.4), this proves (5.6). \(\square\)

The referenced theorem says that \(X \perp \perp Y\) implies \(f_1(X) \perp \perp f_2(X)\) for any measurable functions \(f_1\) and \(f_2\).
5.3.2 Coefficients of Labor and Capital Inputs

We now introduce the short-hand notation,
\[ \tilde{y}_t := y_t - s_1^1 m_1^1 - s_2^2 m_2^2, \]
which can be interpreted as the output net of the flexible input contributions by Lemma 3 under Assumptions 1, 2, 3, 4, and 5. Thus, we can rewrite the gross-output production function (3.1)–(3.2) into the net-output production function
\[ \tilde{y}_t = \beta_l(\omega_t) l_t + \beta_k(\omega_t) k_t + \beta_0(\omega_t) + \eta_t. \] (5.10)

It remains to identify the remaining heterogeneous coefficient functions \( \beta_l(\cdot), \beta_k(\cdot), \) and \( \beta_0(\cdot). \)

The next lemma proposes the identification of \( \beta_l(\cdot) \) and \( \beta_k(\cdot). \)

Lemma 4 (Identification of \( \beta_l(\cdot) \) and \( \beta_k(\cdot) \)). If Assumptions 1, 2, 3, 4, and 5 are satisfied, then
\[ \beta_l(\omega_t) = \beta_l(\phi(r_t^{1,2})) = \frac{\partial}{\partial l} E[\tilde{y}_t | l_t = l, k_t, r_t^{1,2}] \bigg|_{l=l_t} \quad \text{and} \]
\[ \beta_k(\omega_t) = \beta_k(\phi(r_t^{1,2})) = \frac{\partial}{\partial k} E[\tilde{y}_t | l_t, k_t = k, r_t^{1,2}] \bigg|_{k=k_t} \] (5.11)
hold.

This lemma establishes the identifying formulas (4.2) presented in Section 4.

Proof. The proof of this lemma consists of four steps. First, note that Assumption 4 together with Lemma 1 under Assumption 1 implies the equivalence between the two sigma algebras:
\[ \sigma(l_t, k_t, r_t^{1,2}) = \sigma(l_t, k_t, \omega_t). \] (5.12)

Second, applying the decomposition property of the semi-graphoid axiom (Pearl, 2000, pp. 11) to Assumption 3 yields \( (l_t, k_t, \omega_t) \perp \eta_t. \) This independence and the restriction \( E[\eta_t] \) in the
gross-output production function model (3.1) together yield

\[ E[\eta_t | l_t, k_t, \omega_t] = 0. \quad (5.13) \]

Third, we obtain the following chain of equalities.

\[
\begin{align*}
E[\tilde{y}_t | l_t, k_t, r_t^{1,2}] &= E[\tilde{y}_t | l_t, k_t, \omega_t] \\
&= E[\beta_l(\omega_t)l_t + \beta_k(\omega_t)k_t + \beta_0(\omega_t) + \eta_t | l_t, k_t, \omega_t] \\
&= \beta_l(\omega_t)l_t + \beta_k(\omega_t)k_t + \beta_0(\omega_t) + E[\eta_t | l_t, k_t, \omega_t] \\
&= \beta_l(\omega_t)l_t + \beta_k(\omega_t)k_t + \beta_0(\omega_t) 
\end{align*}
\]

(5.14)

where the first equality is due to (5.12), the second equality follows from a substitution of the net-output production function (5.10) which is valid by Lemma 3 under Assumptions 1, 2, 3, 4, and 5, the third equality follows from the information \( \sigma(l_t, k_t, \omega_t) \) on which the conditional expectation is taken, and the fourth equality follows from (5.13).

Fourth, substituting (5.5) under Assumptions 1 and 4 in (5.14), we obtain

\[ E[\tilde{y}_t | l_t = l, k_t = k, r_t^{1,2} = r] = \beta_l(\phi(r))l + \beta_k(\phi(r))k + \beta_0(\phi(r)). \]

The map \((l, k, r) \mapsto E[\tilde{y}_t | l_t = l, k_t = k, r_t^{1,2} = r] \) is thus an affine function of \((l, k)\), and thus is differentiable in \((l, k)\) in particular. Differentiating both sides of this equation with respect to \( l \) and \( k \) at \((l_t, k_t, r_t^{1,2})\) yields

\[
\frac{\partial}{\partial l} E[\tilde{y}_t | l_t = l, k_t = k, r_t^{1,2} = r] \bigg|_{(l, k, r) = (l_t, k_t, r_t^{1,2})} = \beta_l(\phi(r_t^{1,2})) = \beta_l(\omega_t) \quad \text{and}
\]

\[
\frac{\partial}{\partial k} E[\tilde{y}_t | l_t = l, k_t = k, r_t^{1,2} = r] \bigg|_{(l, k, r) = (l_t, k_t, r_t^{1,2})} = \beta_k(\phi(r_t^{1,2})) = \beta_k(\omega_t),
\]

respectively. This shows (5.11).

Lemma 4 paves the way for identification of \( \beta_l(\omega_t) \) and \( \beta_k(\omega_t) \), but it in fact does not guarantee the identification by itself. To make sense of (5.11) as identifying formulas, \( l_t \) should
be functionally independent of \((k_t, r_t^{1,2})\), and, similarly, \(k_t\) should be functionally independent of \((l_t, r_t^{1,2})\) in the language of Ackerberg, Caves, and Frazer (2015). Proposition 4 shows that the locality of identification introduced in Definition 4 satisfies the functional independence. Therefore, \((5.11)\) can be interpreted as the identifying formulas at such localities.

5.3.3 Additive Technology

We now introduce the further short-hand notation,

\[
\tilde{y}_t := \bar{y}_t - \frac{\partial}{\partial l} E[\tilde{y}_t | l_t = l, k_t, r_t^{1,2}] \bigg|_{l_t = l_t} - \frac{\partial}{\partial k} E[\tilde{y}_t | l_t, k_t = k, r_t^{1,2}] \bigg|_{k_t = k_t},
\]

(5.15)

which can be interpreted as the residual of the production function \((3.1)\) by Lemma 3 and 4 under Assumptions 1, 2, 3, 4, and 5. With this new notation, therefore, we can rewrite the net-output production function \((5.10)\) into

\[
\tilde{y}_t = \beta_0(\omega_t) + \eta_t.
\]

The final step is to identify the additive productivity \(\beta_0(\omega_t)\). The next lemma provides the identification of \(\beta_0(\omega_t)\).

**Lemma 5** (Identification of \(\beta_0(\cdot)\)). If Assumptions 1, 2, 3, 4, and 5 are satisfied, then

\[
\beta_0(\omega_t) = \beta_0(\phi(r_t^{1,2})) = E[\tilde{y}_t | l_t, k_t, r_t^{1,2}] \tag{5.16}
\]

holds.

This lemma establishes the identifying formula \((4.3)\) presented in Section 4.

**Proof.** Equation \((5.14)\) in the proof of Lemma 4 can be rewritten as

\[
E[\tilde{y}_t - \beta_l(\omega_t) l_t - \beta_k(\omega_t) k_t, r_t^{1,2}] = \beta_0(\omega_t).
\]

Substituting the definition \((5.15)\) of \(\tilde{y}_t\) together with \((5.11)\) of Lemma 4 in the above equation proves the corollary. \(\Box\)
5.4 Summary of the Main Results

Summarizing the identification steps stated as Proposition 1 and Lemmas 1, 2, 3, 4, and 5, we obtain the following theorem.

**Theorem 1** (Identification). Suppose that Assumptions 1, 2, 3, 4, and 5 are satisfied for the model (3.1)–(3.2). The parameter vector \((\beta_l(\omega_t), \beta_k(\omega_t), \beta_{m_1}(\omega_t), \beta_{m_2}(\omega_t), \beta_0(\omega_t))\) with the latent productivity \(\omega_t\) is identified if there exists \((l_t, k_t)\) such that \((l_t, k_t, \omega_t)\) is at a locality of identification (Definition 1).

For a summary of all the closed-form identifying formulas, we refer the readers the overview in Section 4.2. In this paper, we focus on the identification problem, and leave aside methods of estimation. Since what we identify are functions, \(\beta_l \circ \phi(\cdot), \beta_k \circ \phi(\cdot), \beta_{m_1} \circ \phi(\cdot), \beta_{m_2} \circ \phi(\cdot), \) and \(\beta_0 \circ \phi(\cdot)\), explicitly expressed in terms of nonparametric conditional expectation functions and their derivatives, one may use analog nonparametric estimation methods (e.g., Chen, 2007) to obtain function estimates \(\hat{\beta}_l \circ \phi(\cdot), \hat{\beta}_k \circ \phi(\cdot), \hat{\beta}_{m_1} \circ \phi(\cdot), \hat{\beta}_{m_2} \circ \phi(\cdot), \) and \(\hat{\beta}_0 \circ \phi(\cdot)\). The estimates of heterogeneous coefficients are then computed by \(\hat{\beta}_l(\omega_t) = \hat{\beta}_l \circ \phi(r_t^{1,2}), \hat{\beta}_k(\omega_t) = \hat{\beta}_k \circ \phi(r_t^{1,2}), \hat{\beta}_{m_1}(\omega_t) = \hat{\beta}_{m_1} \circ \phi(r_t^{1,2}), \) \(\hat{\beta}_{m_2}(\omega_t) = \hat{\beta}_{m_2} \circ \phi(r_t^{1,2}), \) and \(\hat{\beta}_0(\omega_t) = \hat{\beta}_0 \circ \phi(r_t^{1,2}).\)

6 Alternative Models

In the baseline model, we considered a parsimonious form that consists of only the two observed state variables \((l_t, k_t)\) and the two observed flexible input variables \((m_{1t}, m_{2t})\). In this section, we remark that it is possible to augment the vectors of the observed state variables \((l_t, k_t)\) and/or the observed flexible input variables \((m_{1t}, m_{2t})\) – see Sections 6.1, 6.2. On the other hand, it is also possible to reduce the model with just one \(m_t\) variable, provided that \(l_t\) satisfies the timing assumption as a flexible input – see Section 6.3.
6.1 More State Variables

The baseline model treats the labor input \( l_t \) only of a single type. In applications, however, researchers often distinguish skilled labor input \( l_t^s \) and unskilled labor input \( l_t^u \) (e.g., Levinsohn and Petrin, 2003). To accommodate this distinction, we can augment the gross-output production function in the logarithm (3.1) as

\[
y_t = \Psi \left( l_t^s, l_t^u, k_t, m_{t1}^1, m_{t2}^2, \omega_t \right) + \eta_t \quad E[\eta_t] = 0, \tag{6.1}
\]

where \( l_t^s \) is the logarithm of skilled labor input, \( l_t^u \) is the logarithm of unskilled labor input, and all the other variables are the same as in the baseline model. Accordingly, the Cobb-Douglas form (3.2) is augmented as

\[
\Psi \left( l_t^s, l_t^u, k, m_{t1}^1, m_{t2}^2, \omega_t \right) = \beta_{ls}(\omega_t) l_t^s + \beta_{lu}(\omega_t) l_t^u + \beta_k(\omega_t) k_t + \beta_{m1}(\omega_t) m_{t1}^1 + \beta_{m2}(\omega_t) m_{t2}^2 + \beta_0(\omega_t) \tag{6.2}
\]

with heterogeneous coefficients \((\beta_{ls}(\omega_t), \beta_{lu}(\omega_t), \beta_k(\omega_t), \beta_{m1}(\omega_t), \beta_{m2}(\omega_t), \beta_0(\omega_t))\). With the following modifications of Assumptions 1, 3, and 5 adapted to the current augmented model, we can construct the identification results in similar lines of argument to those we had for the baseline model.

**Assumption 1** (Flexible Input Choice). A firm at time \( t \) with the state variables \((l_t^s, l_t^u, k_t, \omega_t)\) chooses the flexible input vector \((m_{t1}^1, m_{t2}^2)\) by the optimization problem

\[
\max_{(m_{t1}^1, m_{t2}^2) \in \mathbb{R}_+^2} p_t^u \exp \left( \Psi \left( l_t^s, l_t^u, k_t, m_{t1}^1, m_{t2}^2, \omega_t \right) \right) \quad E[\exp (\eta_t)] - p_t^{m1} \exp \left( m_{t1}^1 \right) - p_t^{m2} \exp \left( m_{t2}^2 \right),
\]

where \( p_t^{m1} > 0 \) and \( p_t^{m2} > 0 \) almost surely.

**Assumption 3** (Measurability). \( \beta_{ls}(\cdot), \beta_{lu}(\cdot), \beta_k(\cdot), \beta_{m1}(\cdot), \beta_{m2}(\cdot) \) and \( \beta_0(\cdot) \) are measurable functions.

**Assumption 5** (Independence). \((l_t^s, l_t^u, k_t, \omega_t, p_t^{m1}, p_t^{m2}) \perp \eta_t\).
Assumption 1 proposes that both \( \ell^s_t \) and \( \ell^u_t \) are predetermined, which is empirically supported by Hu, Huang, and Sasaki (2017). Assumptions 3 and 5 are straightforward extensions of Assumptions 3 and 5 respectively, suitable for the current model (6.1)–(6.2). With the following modification to the definition of the locality of identification adapted to the current augmented model, we state as a theorem the extended identification result.

**Definition** 1 (Locality of Identification). The point \((\ell^s_t, \ell^u_t, k_t, \omega_t) = (\ell^{s\ast}, \ell^{u\ast}, k^\ast, \omega^\ast)\) is called the locality of identification if there are real numbers \( \underline{f}, \overline{f} \in (0, \infty) \) such that \( \underline{f} < f_{\ell^s_t, \ell^u_t, k_t, \omega_t}(\ell^s, \ell^u, k, \omega) < \overline{f} \) for all \((\ell^s, \ell^u, k, \omega)\) in a neighborhood of \((\ell^{s\ast}, \ell^{u\ast}, k^\ast, \omega^\ast)\).

**Theorem 2** (Identification). Suppose that Assumptions 1, 2, 3, 4, and 5 are satisfied for the model (6.1)–(6.2). The parameter vector \((\beta_{l^s}(\omega_t), \beta_{l^u}(\omega_t), \beta_k(\omega_t), \beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t), \beta_0(\omega_t))\) with the latent productivity \(\omega_t\) is identified if there exists \((\ell^s_t, \ell^u_t, k_t)\) such that \((\ell^s_t, \ell^u_t, k_t, \omega_t)\) is at a locality of identification (Definition 1).

The coefficients of flexible inputs, \(m^1\) and \(m^2\), are identified in closed form by

\[
\beta_{m^1}(\omega_t) = s^1_t = \frac{p^{m_1}_1 \exp(m^1)}{E[p_t^y \exp(y_t) | \ell^s_t, \ell^u_t, k_t, m^1_t, m^2_t, r_t^{1\cdot}]} \quad \text{and} \quad \\
\beta_{m^2}(\omega_t) = s^2_t = \frac{p^{m^2}_2 \exp(m^2)}{E[p_t^y \exp(y_t) | \ell^s_t, \ell^u_t, k_t, m^1_t, m^2_t, r_t^{1\cdot}]} 
\]

The coefficients of skilled labor \(l^s\), unskilled labor \(l^u\), and capital \(k\) are identified in turn in closed form by

\[
\beta_{l^s}(\omega_t) = \left. \frac{\partial}{\partial l^s} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t | \ell^s_t = l^s, \ell^u_t, k_t, r_t^{1\cdot}] \right|_{l^s = l^s_t}
\]

\[
\beta_{l^u}(\omega_t) = \left. \frac{\partial}{\partial l^u} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t | \ell^s_t, \ell^u_t = l, k_t, r_t^{1\cdot}] \right|_{l^u = l^u_t} \quad \text{and} \quad \\
\beta_k(\omega_t) = \left. \frac{\partial}{\partial k} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t | \ell^s_t, \ell^u_t, k_t = k, r_t^{1\cdot}] \right|_{k = k_t}.
\]
The additive productivity is identified in closed form by

\[ \beta_0(\omega_t) = E[\gamma_t - \beta_l(\omega_t)l_t^\omega - \beta_k(\omega_t)k_t - \beta_{m^1}(\omega_t)m_t^1 - \beta_{m^2}(\omega_t)m_t^2 | l_t, k_t, r_t^{1.2}] \].

### 6.2 More Flexible Input Variables

The baseline model includes only two flexible inputs, \( m_t^1 \) and \( m_t^2 \). In applications, however, researchers often use more types of flexible inputs, such as materials, electricity, and fuels (e.g., Levinsohn and Petrin, 2003). To accommodate such applications with three flexible inputs, for example, we can augment the gross-output production function in the logarithm (3.1) as

\[ y_t = \Psi(l_t, k_t, m_t^1, m_t^2, m_t^3, \omega_t) + \eta_t \quad E[\eta_t] = 0, \]  

(6.3)

where \( m_t^3 \) is the logarithm of the third flexible input and all the other variables are the same as in the baseline model. Accordingly, the Cobb-Douglas form (3.2) is augmented as

\[ \Psi(l_t, k, m_t^1, m_t^2, m_t^3, \omega_t) = \beta_l(\omega_t)l_t + \beta_k(\omega_t)k_t + \beta_{m^1}(\omega_t)m_t^1 + \beta_{m^2}(\omega_t)m_t^2 + \beta_{m^3}(\omega_t)m_t^3 + \beta_0(\omega_t) \]  

(6.4)

with heterogeneous coefficients \((\beta_l(\omega_t), \beta_k(\omega_t), \beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t), \beta_{m^3}(\omega_t), \beta_0(\omega_t))\). With the following modifications of Assumptions 1, 2, 3, 4, and 5 adapted to the current augmented model, we can construct the identification results in similar lines of argument to those we had for the baseline model.

**Assumption 1′′ (Flexible Input Choice).** A firm at time \( t \) with the state variables \((l_t, k_t, \omega_t)\) chooses the flexible input vector \((m_t^1, m_t^2, m_t^3)\) by the optimization problem

\[
\max_{(m^1, m^2, m^3) \in \mathbb{R}_+^3} p_t^{m^1} \exp \left( \Psi(l_t, k_t, m_t^1, m_t^2, m_t^3, \omega_t) \right) E[\exp(\eta_t)] \\
- p_t^{m^1} \exp(m^1) - p_t^{m^2} \exp(m^2) - p_t^{m^3} \exp(m^3),
\]

where \( p_t^{m^1} > 0, p_t^{m^2} > 0, \) and \( p_t^{m^3} > 0 \) almost surely.
Assumption 2' (Finite Solution). $\beta_{m^1}(\omega_t) + \beta_{m^2}(\omega_t) + \beta_{m^3}(\omega_t) < 1$ almost surely.

Assumption 3' (Measurability). $\beta_l(\cdot)$, $\beta_k(\cdot)$, $\beta_{m^1}(\cdot)$, $\beta_{m^2}(\cdot)$, $\beta_{m^3}(\cdot)$, and $\beta_0(\cdot)$ are measurable functions.

Assumption 4' (Non-Collinear Heterogeneity). One of the functions, $\omega \mapsto \beta_{m^1}(\omega) / \beta_{m^2}(\omega)$, $\omega \mapsto \beta_{m^2}(\omega) / \beta_{m^3}(\omega)$, $\omega \mapsto \beta_{m^3}(\omega) / \beta_{m^1}(\omega)$, or $\omega \mapsto (\beta_{m^1}(\omega) / \beta_{m^2}(\omega), \beta_{m^2}(\omega) / \beta_{m^3}(\omega))$, is measurable and invertible with measurable inverse.

Assumption 5' (Independence). $(l_t, k_t, \omega_t, p_{m^1}^t, p_{m^2}^t, p_{m^3}^t) \perp \eta_t$.

Assumption 1' formally requires that $m^1_t$, $m^2_t$, and $m^3_t$ are the three flexible inputs while the others are state variables. Assumption 2' requires diminishing returns with respect to the three flexible inputs, but not necessarily with respect to all the production factors. Assumptions 3' and 5' are straightforward modifications of Assumptions 3 and 5 respectively, suitable for the current model (6.3)–(6.4). Assumption 4' is a less straightforward modification of Assumption 4 and merits some discussions concerning its implication for the latent technology. If the assumption holds for one of the first three maps, namely $\omega \mapsto \beta_{m^1}(\omega) / \beta_{m^2}(\omega)$, $\omega \mapsto \beta_{m^2}(\omega) / \beta_{m^3}(\omega)$, or $\omega \mapsto \beta_{m^3}(\omega) / \beta_{m^1}(\omega)$, then this assumption still requires the latent technology $\omega_t$ to be one-dimensional, and the interpretation of Assumption 4' is analogous to that of Assumption 4. On the other hand, if the assumption holds for the last map, namely $\omega \mapsto (\beta_{m^1}(\omega) / \beta_{m^2}(\omega), \beta_{m^2}(\omega) / \beta_{m^3}(\omega))$, then this assumption requires the latent technology $\omega_t$ to be two-dimensional. In this case, similar lines of arguments to those for the baseline model yield the vector of input cost ratios $(r_{t}^{1,2}, r_{t}^{2,3}) = (\beta_{m^1}(\omega) / \beta_{m^2}(\omega), \beta_{m^2}(\omega) / \beta_{m^3}(\omega))$ as a control variable for the two-dimensional technology $\omega_t$, where $r_{t}^{2,3}$ is defined by $r_{t}^{2,3} = p_{t}^{m^2} \exp(m_{t}^{2}) / p_{t}^{m^3} \exp(m_{t}^{3})$ analogously to (3.3), and the identifying formulas will thus entail controlling for these two ratios.

We state as a theorem the extended identification result based on these modified assumptions.
Theorem 3 (Identification). Suppose that Assumptions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ are satisfied for the model (6.3)–(6.4). The parameter vector $(\beta_l(\omega_t), \beta_k(\omega_t), \beta_{m^1}(\omega_t), \beta_{m^2}(\omega_t), \beta_{m^3}(\omega_t), \beta_0(\omega_t))$ with the latent productivity $\omega_t$ is identified if there exists $(l^*_t, l^*_t, k_t)$ such that $(l^*_t, l^*_t, k_t, \omega_t)$ is at a locality of identification (Definition 1).

The coefficients of flexible inputs, $m^1$ and $m^2$, are identified in closed form by

$$
\beta_{m^1}(\omega_t) = s^1_t = \frac{p_{l^*_t}^{m^1}\exp(m^1)}{E[p_l^t\exp(y_t)\mid l_t, k_t, m^1_t, m^2_t, m^3_t, r_{l^*_t}^{1,2}, r_{l^*_t}^{2,3}]},
$$

$$
\beta_{m^2}(\omega_t) = s^2_t = \frac{p_{l^*_t}^{m^2}\exp(m^2)}{E[p_l^t\exp(y_t)\mid l_t, k_t, m^1_t, m^2_t, m^3_t, r_{l^*_t}^{1,2}, r_{l^*_t}^{2,3}]},
$$

and

$$
\beta_{m^3}(\omega_t) = s^3_t = \frac{p_{l^*_t}^{m^3}\exp(m^3)}{E[p_l^t\exp(y_t)\mid l_t, k_t, m^1_t, m^2_t, m^3_t, r_{l^*_t}^{1,2}, r_{l^*_t}^{2,3}]}
$$

The coefficients of labor $l$ and capital $k$ are identified in turn in closed form by

$$
\beta_l(\omega_t) = \left. \frac{\partial}{\partial l} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t - \beta_{m^3}(\omega_t)m^3_t \mid l_t = l, k_t, r_{l^*_t}^{1,2}, r_{l^*_t}^{2,3}] \right|_{l=l_t},
$$

and

$$
\beta_k(\omega_t) = \left. \frac{\partial}{\partial k} E[y_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t - \beta_{m^3}(\omega_t)m^3_t \mid l_t, k_t = k, r_{l^*_t}^{1,2}, r_{l^*_t}^{2,3}] \right|_{k=k_t}.
$$

The additive productivity is identified in closed form by

$$
\beta_0(\omega_t) = E[y_t - \beta_l(\omega_t)l_t - \beta_k(\omega_t)k_t - \beta_{m^1}(\omega_t)m^1_t - \beta_{m^2}(\omega_t)m^2_t - \beta_{m^3}(\omega_t)m^3_t \mid l_t, k_t, r_{l^*_t}^{1,2}, r_{l^*_t}^{2,3}].
$$

6.3 Single $m_t$ Variable

The baseline model includes two flexible inputs, $m^1_t$ and $m^2_t$. Researchers sometimes include only one type of inputs, such as materials, other than labor and capital. We may accommodate such a reduced model at the cost of an alternative timing assumption for labor input, namely concurrent choice of labor input. Write a parsimonious version of the gross-output production function in the logarithm (3.1) as

$$
y_t = \Psi(l_t, k_t, m_t, \omega_t) + \eta_t \quad E[\eta_t] = 0, \quad (6.5)
$$
Accordingly, the Cobb-Douglas form (3.2) is reduced as
\[
\Psi (l_t, k_t, m_t, \omega_t) = \beta_l(\omega_t)l_t + \beta_k(\omega_t)k_t + \beta_m(\omega_t)m_t + \beta_0(\omega_t)
\] (6.6)

with heterogeneous coefficients \((\beta_l(\omega_t), \beta_k(\omega_t), \beta_m(\omega_t), \beta_0(\omega_t))\). With the following modifications of Assumptions 1, 2, 3, 4, and 5 adapted to the current augmented model, we can construct the identification results in similar lines of argument to those we had for the baseline model.

**Assumption 1'' (Flexible Input Choice).** A firm at time \(t\) with the state variables \((k_t, \omega_t)\) chooses the flexible input vector \((l_t, m_t)\) by the optimization problem
\[
\max_{(l, m) \in \mathbb{R}_+^2} p^l_t \exp (\Psi (l, k_t, m, \omega_t)) E[\exp (\eta_t)] - p^l_t \exp (l) - p^m_t \exp (m),
\]
where \(p^l_t > 0\) and \(p^m_t > 0\) almost surely.

**Assumption 2'' (Finite Solution).** \(\beta_l(\omega_t) + \beta_m(\omega_t)< 1\) almost surely.

**Assumption 3'' (Measurability).** \(\beta_l(\cdot), \beta_k(\cdot), \beta_m(\cdot),\) and \(\beta_0(\cdot)\) are measurable functions.

**Assumption 4'' (Non-Collinear Heterogeneity).** The function \(\omega \mapsto \beta_l(\omega)/\beta_m(\omega)\) is measurable and invertible with measurable inverse.

**Assumption 5'' (Independence).** \((l_t, k_t, \omega_t, p^l_t, p^m_t) \perp \eta_t\).

Assumption 1'' asserts that, unlike the baseline model, the labor input \(l_t\) is treated as a flexible input along with the materials \(m_t\), as opposed to a predetermined quantity. This will not incur the functional independence problem (Ackerberg, Caves, and Frazer, 2015) because the output elasticity with respect to labor in this case is unambiguously identified via the first-order condition just like the output elasticity with respect to materials. Accordingly, the
locality of identification defined for the current model below does not require data variations in \( l_t \) given the state variables fixed. The identification argument in the current reduced model relies on the non-collinear heterogeneity in the ratio of the elasticity with respect to labor to the elasticity with respect to materials. With the following modification to the definition of the locality of identification adapted to the current augmented model, we state as a theorem the identification result.

**Definition 1** (Locality of Identification). The point \((k_t, \omega_t) = (k^*, \omega^*)\) is called the locality of identification if there are real numbers \( f, \tilde{f} \in (0, \infty) \) such that \( f < f_{k_t, \omega_t}(k, \omega) < \tilde{f} \) for all \((k, \omega)\) in a neighborhood of \((k^*, \omega^*)\).

**Theorem 4** (Identification). Suppose that Assumptions 1, 2, 3, 4, and 5 are satisfied for the model (6.5)–(6.6). The parameter vector \((\beta_l(\omega_t), \beta_k(\omega_t), \beta_m(\omega_t), \beta_0(\omega_t))\) with the latent productivity \(\omega_t\) is identified if there exists \(k_t\) such that \((k_t, \omega_t)\) is at a locality of identification (Definition 1').

With the flexible input cost ratio modified as

\[
\frac{r_{l,m}^t}{p_t^l \exp(l_t)} = \frac{p_t^l \exp(l_t)}{p_t^m \exp(m_t)},
\]

the coefficients of flexible inputs, \(l\) and \(m\), are identified in closed form by

\[
\beta_l(\omega_t) = s_t^l = \frac{p_t^l \exp(l)}{E[p_t^y \exp(y_t)|l_t, k_t, m_t, r_{l,m}^t]} \quad \text{and} \quad \beta_m(\omega_t) = s_t^m = \frac{p_t^m \exp(m)}{E[p_t^y \exp(y_t)|l_t, k_t, m_t, r_{l,m}^t]}
\]

The coefficient capital \(k\) is identified in turn in closed form by

\[
\beta_k(\omega_t) = \left. \frac{\partial}{\partial k} E[y_t - \beta_l(\omega_t)l_t - \beta_m(\omega_t)m_t \mid k_t = k, r_{l,m}^t] \right|_{k=k_t}.
\]

The additive productivity is identified in closed form by

\[
\beta_0(\omega_t) = E[y_t - \beta_l(\omega_t)l_t - \beta_k(\omega_t)k_t - \beta_m(\omega_t)m_t \mid k_t, r_{l,m}^t].
\]
7 Summary and Discussions

In this paper, we develop the identification of heterogeneous elasticities in the Cobb-Douglas production function. The identification is constructively achieved with closed-form formulas for the output elasticity with respect to each input, as well as the additive productivity, for each firm. The flexible input cost ratio plays the role of a control function under the assumption of non-collinear heterogeneity between elasticities with respect to two flexible inputs. The *ex ante* flexible input cost share is shown to be useful to identify the elasticities with respect to flexible inputs for each firm. The elasticities with respect to labor and capital can be identified for each firm under the timing assumption admitting the functional independence. Extended identification results are provided for three alternative models that are frequently used in empirical analysis.

In light of the fact that conventional identification strategies for production functions use panel data, it is unusual for our identification strategy not to rely on panel data. Note that the existing papers use panel data to form orthogonality restrictions to estimate input coefficients. Our explicit identification of the flexible input coefficients via the first-order conditions does not involve any panel structure. This feature entails a couple of limitations which are the costs that we pay for not relying on panel data and for our ability to identify heterogeneous elasticities. While these limitations are shared by other recent papers that also use the first-order conditions for identification, we discuss them below and propose a scope of future research.

The first limitation of our identification method is the requirement for input and output prices, which are not always available in empirical data. In certain types of production analysis, however, the prices are normalized to one and thus are assumed to be known. For example, many important papers, including Levinsohn and Petrin (2003), in estimation of production
functions use a data set that is based on the census for plants collected by Chile’s Instituto Nacional de Estadística (cf. Lui, 1991). For this data set, the output (exp(y_t)) is the gross revenue deflated to real Chilean pesos in a baseline year. Flexible inputs include measures of materials (exp(m^1_t)), electricity (exp(m^2_t)), and fuels (exp(m^3_t)) all measured in terms of pecuniary values deflated to real Chilean pesos in a baseline year. Since they are measured in terms of values as opposed to quantities, the researchers effectively set \( p^y_t = p^{m^1}_t = p^{m^2}_t = p^{m^3}_t = 1 \) in their analysis. For this data set, therefore, our first limitation is not binding.

The second limitation of our identification approach is the assumption of price-taking firms in both the output market and the flexible input markets. This assumption rules out market power that is relevant to answering some important policy questions in industry studies and international trade (e.g., De Loecker and Warzynski, 2012; De Loecker, Goldberg, Khandelwal, and Pavcnik, 2016). We leave identification strategies under market power for future research.
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