Effective field theory of boson-fermion mixtures

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We construct a Galilean invariant low-energy effective field theory of boson-fermion mixtures and study bound fermion states on a vortex of boson superfluid. We derive a simple criterion to determine for which values of the fermion angular momentum $l$ there exist an infinite number of bound energy levels. We apply our formalism to two boson-fermion mixed systems: the dilute solution of $^3$He in $^4$He superfluid and the cold polarized Fermi gas on the BEC side of the “splitting point.” For the $^3$He-$^4$He mixture, we determine parameters of the effective theory from experimental data as functions of pressure. We predict that infinitely many bound $^3$He states on a superfluid vortex with $l = -2, -1, 0$ are realized in a whole range of pressure 0–20 atm, where experimental data are available. As for the cold polarized Fermi gas, while only $S$-wave ($l = 0$) and $P$-wave ($l = \pm 1$) bound fermion states are possible in the BEC limit, those with higher negative angular momentum become available as one moves away from the BEC limit.

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I. INTRODUCTION

Quantum degenerate mixtures of bosons and fermions provide interesting playgrounds where effects of quantum statistics become explicit. Several experiments on the boson-fermion mixtures have been performed, first on the classic example of the $^3$He-$^4$He mixture [1, 2] and more recently in cold atomic gases such as $^6$Li-$^7$Li [3, 4], $^6$Li-$^{23}$Na [5], $^{40}$K-$^{87}$Rb [6], and metastable triplet $^3$He-$^4$He* mixtures [7]. Molecular condensates, formed through the BCS-BEC crossover, provide another possibility for the boson-fermion mixtures [8, 9, 10, 11, 12]. Stimulated by experiments, theoretical studies of the boson-fermion mixtures predict new phenomena such as phase separation between Fermi gas and Bose-Einstein condensates [13] or sympathetic cooling of Fermi gas by the superfluidity of Bose-Einstein condensates [14].

The main goal of this paper is to investigate the bound states of fermions on a superfluid vortex. Superfluid vortices were realized by the recent experiment in cold Fermi gas throughout the BCS and BEC regimes [15]. The problem has been considered in the literature [16, 17, 18], with most of the treatments following the original approach by Caroli, de Gennes, and Matricon [19] (or the more rigorous approach based on the Bogoliubov–de Gennes equation [20]). The reason for us to revisit the problem is that the Caroli–de Gennes–Matricon formalism is expected to work well only in the BCS regime where there exist well-defined quasiparticles. Its application to the BEC regime where two fermions form a bound molecule would require an exceedingly large number of mean-field quasiparticle states for an accurate description of the strongly bound molecules. It also cannot be applied to the important case of the $^3$He-$^4$He mixture.

We are interested in the case when the minimum of the fermion dispersion curve $\varepsilon(p)$ sits at zero momentum:

$$\varepsilon(p) = \varepsilon_0 + \frac{p^2}{2m^*} + \cdots, \quad (1)$$

where $m^*$ is positive and quartic and higher order corrections are negligible. This case is opposite to the situation in the BCS theory, where the minimum is near the Fermi momentum, but it holds for the solution of $^3$He in $^4$He and also for cold atom gases sufficiently deep in the BEC regime. Our technique relies on the use of an effective field theory. The effective field theory formalism is especially powerful in strongly coupled systems where perturbative methods are not applicable. In our cases the details of the interactions become unimportant, as they are encoded into a few low-energy parameters. Thus the theory developed in this paper is universal and can be applied to many different physical systems.

In writing down the effective field theory, we put special emphasis on the Galilean invariance [21, 22, 23]. The Galilean invariance as well as the global symmetries of the system considerably restrict the possible form of the effective Lagrangian. We show that coupling constants appearing in the effective field theory can be expressed through the effective fermion mass and the derivative of fermion energy gap with the chemical potential. The latter is directly related to the Bardeen-Baym-Pines (BBP) parameter. Both the effective fermion mass and the BBP parameter have been measured by experiments for the $^3$He-$^4$He mixture.

After the construction of the effective field theory, we use it to the study of bound fermion states on a vortex of the boson superfluid. We derive a simple criterion which tells us at which values of the angular momentum $l$ there are an infinite number of bound energy levels. We will apply our effective field theory to two boson-fermion mixed systems, a dilute solution of $^3$He in $^4$He superfluid in Sec. [11] and the BEC regime of the cold polarized...
Fermi gas in Sec. I in order to predict possible angular momenta for bound fermion states.

II. DILUTE FERMIONS IN BOSON SUPERFLUID

A. Galilean invariance and effective field theory

Here we demonstrate how to construct an effective field theory of dilute fermions in a boson superfluid in accordance with the Galilean invariance and global symmetries. Since the microscopic theory is Galilean invariant, its low-energy effective Lagrangian should satisfy the following identity required by the Galilean invariance:

\[ T_{0i} = m J_i, \]  

(2)

where \( T_{0i} \) is the momentum density, \( m \) is the mass of the particle, and \( J_i \) is the particle number current \[21, 22\]. If there are more than one species of particles in the system, the right hand side of the equation is a sum over all species. This identity simply indicates that the momentum density should be equal to the mass carried by the particle number current in the Galilean invariant system. The Galilean invariance as well as the global symmetries of the system considerably restrict the possible form of the effective Lagrangian. For definiteness, we consider the particular case of the dilute solution of \(^4\)He atoms. Since \( \Psi \) is the particle number current \[21, 22\].

1. \(^4\)He superfluid phonon

First of all, let us consider an effective field theory for the pure \(^4\)He superfluid. The physical degree of freedom at long distance is the phase of the condensate \( \vartheta \), which is defined as \( \langle \Psi \rangle = |\langle \Psi | \rangle e^{i \vartheta} \) with \( \Psi \) being a field for the \(^4\)He atoms. Under the \( U(1) \) symmetry associated with the number conservation of \(^4\)He atoms, the field \( \vartheta \) transforms as \( \vartheta \rightarrow \vartheta + \chi \). Therefore the effective Lagrangian of the \(^4\)He superfluid obeying this symmetry should be a function of derivatives of the field \( \vartheta \) and \( \partial \vartheta \). We note that terms where derivatives act more than once on one field such as \( \vartheta \) or \( \partial^2 \vartheta \) are also possible. However, they are in higher orders in the power counting scheme where \( \vartheta \) and \( \partial \vartheta \) are regarded as order \( O(1) \). In consideration of the rotational symmetry, the leading order effective Lagrangian \( \mathcal{L}_4 \) should be written by a polynomial of \( \dot{\vartheta} \) and \( (\partial \vartheta)^2 \) as follows:

\[ \mathcal{L}_4 = \mathcal{L}_4(\dot{\vartheta}, (\partial \vartheta)^2). \]  

(3)

Now we impose the Galilean invariance \( T_{0i} = m_4 J_i^{(4)} \) on the effective Lagrangian, where \( m_4 \) and \( J_i^{(4)} \) are the mass and the number current of the \(^4\)He atoms. Since the momentum density and the particle number current are given by

\[ T_{0i} = \frac{\delta L_4}{\delta \vartheta}, \]  

(4)

\[ J_i^{(4)} = \frac{\delta L_4}{\delta \partial \vartheta} = 2 \partial_i \vartheta \frac{\delta L_4}{\delta (\partial \vartheta)^2}, \]  

(5)

the Galilean invariance \( T_{0i} = m_4 J_i^{(4)} \) results in

\[ \frac{\delta L_4}{\delta \vartheta} = 2 m_4 \frac{\delta L_4}{\delta (\partial \vartheta)^2}. \]  

(6)

This equality requires that the effective Lagrangian depends on the field \( \vartheta \) only through the combination \( \vartheta + (\partial \vartheta)^2 / 2 m_4 \) as follows:

\[ \mathcal{L}_4 = P \left( \vartheta - \frac{(\partial \vartheta)^2}{2 m_4} \right). \]  

(7)

Here \( P(\cdots) \) is an arbitrary polynomial, which will be identified to the pressure of the pure \(^4\)He superfluid as a function of the chemical potential later.

In the superfluid phase, the symmetry \( \vartheta \rightarrow \vartheta + \chi \) is spontaneously broken. At finite chemical potential \( \mu_4 \), the ground state of the superfluid system corresponds to \( \vartheta = 0 \). Then we expand the field around the ground state as \( \vartheta = \vartheta_0 + \varphi \), where fluctuations of \( \varphi \) around zero corresponds to superfluid phonon excitations. Substitution of the expression \( \vartheta = \vartheta_0 + \varphi \) into Eq. \( 7 \) results in

\[ P(\mu_4 - \varphi - (\partial \varphi)^2 / 2 m_4). \]  

(8)

Now we can show that the function \( P(\cdots) \) is identical to the pressure as a function of \( \mu_4 \) at zero temperature up to an irrelevant constant. For that purpose, we calculate the number density of \(^4\)He atoms \( n_4 \) by differentiate the Lagrangian with \( \mu_4 \) and we obtain

\[ n_4(\mu_4) = \frac{\partial \mathcal{L}_4}{\partial \mu_4} = \frac{\partial P}{\partial \mu_4}. \]  

(9)

This equation implies that \( P \) is a function of the chemical potential satisfying \( P(\mu_4) = n_4(\mu_4) \), which means that \( P(\mu_4) \) is identical to the thermodynamic pressure up to an irrelevant constant. Once equation of state of the \(^4\)He superfluid \( P(\mu_4) \) is given as a function of the chemical potential, the low-energy effective field theory of superfluid phonons is simply given by replacing \( \mu_4 \) with \( \mu_4 - \varphi - (\partial \varphi)^2 / 2 m_4 \).

In order to proceed our analysis further, we expand Eq. \( 8 \) up to the second order in fields

\[ \mathcal{L}_4 \simeq P(\mu_4) - \frac{\partial P}{\partial \mu_4} \left[ \varphi + \frac{(\partial \varphi)^2}{2 m_4} \right] + \frac{1}{2} \frac{\partial^2 P}{\partial \mu_4^2} \left[ \varphi + \frac{(\partial \varphi)^2}{2 m_4} \right]^2 \]

\[ \simeq P(\mu_4) - n_4 \varphi + \frac{\partial n_4}{\partial \mu_4} \frac{\varphi^2}{2} - \frac{n_4}{m_4} \frac{m_4}{2}. \]  

(10)
The first term gives the pressure without phonon excitations. The second term is a total derivative of the field, which does not affect the equation of motion. The third and fourth terms represent the propagation of phonon with its sound velocity \[ \sqrt{\frac{n_4}{m_4} \frac{\partial n_4}{\partial t}} = \sqrt{\frac{\partial P}{m_4 \partial n_4}}. \] (11)

The higher order terms which are not shown in Eq. (10) represent self-interactions among phonons.

2. Minimal coupling between \(^4\)He and \(^3\)He

Next, we consider the coupling between the superfluid phonon \(\varphi\) and a \(^3\)He atom \(\psi\). One coupling term can be written down from the following argument. We note that there is an energy cost to introduce a single \(^3\)He atom into the \(^4\)He superfluid. This energy is some function \(\Delta(\mu_4)\) of the \(^4\)He chemical potential \(\mu_4\). However, Galileian invariance tells us that \(\mu_4\) always enters the Lagrangian in the combination \(\mu_4 - \varphi - (\partial \varphi)^2/2m_4\). Thus, the Lagrangian contains the following term:

\[ \mathcal{L}_{\text{gap}} = -\Delta \left( \mu_4 - \varphi - \frac{(\partial \varphi)^2}{2m_4} \right) \psi^\dagger \psi = -\Delta(\mu_4) \psi^\dagger \psi + \frac{\partial \Delta}{\partial \mu_4} \left[ \mu_4 - \varphi - \frac{(\partial \varphi)^2}{2m_4} \right] \psi^\dagger \psi. \] (12)

The first term represents the energy cost of introducing a \(^3\)He atom into the pure \(^4\)He superfluid. The second term proportional to \(\partial \Delta/\partial \mu_4\) gives a Galileian invariant coupling between the superfluid phonon and the \(^3\)He atom put into the \(^4\)He superfluid. However, this is not the only coupling between \(^3\)He and \(^4\)He, as we shall see below.

3. \(^3\)He kinetic term

Now, let us consider the kinetic term of the \(^3\)He field \(\psi\). The \(^3\)He atom put into the \(^4\)He superfluid has the effective mass \(m_3\), not equal to the bare mass \(m_3\) due to the strong interaction with the \(^4\)He superfluid. Thus the kinetic term of the \(^3\)He atom is

\[ \mathcal{L}_3(\psi, \psi^\dagger) = \frac{i\psi^\dagger \frac{\partial}{\partial t} \psi - |\partial \psi|^2}{2m_3^2} + \mu_3 \psi^\dagger \psi \] (13)

with \(\mu_3\) being the chemical potential for the \(^3\)He atom. However, this Lagrangian does not satisfy the Galilean invariance \(T_{0i} = m_3 J_i^{(3)}\), where \(J_i^{(3)}\) is the number current of the \(^3\)He atom. This is because the Galilean invariance condition involves the bare mass, while the kinetic term in the Lagrangian involves the effective mass.

The resolution to this apparent paradox is that there are interaction terms that contribute to both the momentum density and the particle number current in order to restore the Galilean invariance. Therefore, the Galilean invariance imposes some relationships between the interaction terms and the \(^3\)He effective mass. This situation here is reminiscent of the Fermi liquid theory, where there exists a relationship between an effective fermion mass and a Landau parameter.

One way to construct the Lagrangian that obeys the Galilean invariance is as follows. The effective mass of \(^3\)He atom \(m_3\) originates in the fact that the \(^3\)He quasiparticle in the \(^4\)He superfluid entrains superfluid \(^4\)He atoms due to the strong interaction between them. Therefore, an elementary excitation of the system, or the \(^3\)He quasiparticle \(\psi\), should be written as \(\psi = e^{i\eta \varphi} \psi_0\). Here \(\eta\) is a parameter defined by \(m_4 = m_3 + \eta m_4\), which represents a “fraction” of superfluid \(^4\)He atoms in the \(^3\)He quasiparticle. Using the \(^3\)He quasiparticle field \(\psi\) instead of the bare \(^3\)He field \(\psi\) in Eq. (13), we have

\[ \mathcal{L}_3(\psi, \psi^\dagger) = \frac{i\psi^\dagger \frac{\partial}{\partial t} \psi - |\partial \psi|^2}{2m_3^2} + \mu_3 \psi^\dagger \psi \]

\[ + \eta \frac{\partial \varphi}{2m_3^2} \left[ \eta \varphi + \frac{(\partial \varphi)^2}{2m_3^2} \right] \psi^\dagger \psi. \] (14)

We can verify that this modified Lagrangian (14) obeys the Galilean invariance. The momentum density of the system is given by

\[ T_{0i} = \frac{\delta \mathcal{L}_3}{\delta \psi} \frac{\partial}{\partial t} \psi + \frac{\delta \mathcal{L}_3}{\delta \psi^\dagger} \frac{\partial}{\partial \psi^\dagger} \psi + \frac{\delta \mathcal{L}_3}{\delta \varphi} \frac{\partial}{\partial \varphi} \psi^\dagger \]

\[ = \frac{i\psi^\dagger \frac{\partial}{\partial t} \psi}{2} - \eta \frac{\partial}{\partial \varphi} \psi^\dagger \psi. \] (15)

On the other hand, the number currents associated with the \(^3\)He atoms and \(^4\)He atoms are, respectively, given by

\[ J_i^{(3)} = \frac{\delta \mathcal{L}_3}{\delta \psi^\dagger} \frac{\partial}{\partial t} \psi - \frac{\delta \mathcal{L}_3}{\delta \psi} \frac{\partial}{\partial \psi^\dagger} \psi = \frac{i\psi^\dagger \frac{\partial}{\partial t} \psi}{2m_3^2} - \eta \frac{\partial}{\partial \varphi} \psi^\dagger \psi \]

\[ + \eta \frac{\partial \varphi}{2m_3^2} \left[ \eta \varphi + \frac{(\partial \varphi)^2}{2m_3^2} \right] \psi^\dagger \psi \] (16)

and

\[ J_i^{(4)} = \frac{\delta \mathcal{L}_3}{\delta \psi^\dagger} \frac{\partial}{\partial t} \psi - \frac{\delta \mathcal{L}_3}{\delta \psi} \frac{\partial}{\partial \psi^\dagger} \psi = \eta \frac{\partial}{\partial \varphi} \psi^\dagger \psi \]

\[ - \eta^2 \frac{\partial \varphi}{m_3^2} \psi^\dagger \psi. \] (17)

Recalling the definition of \(\eta\), the Galilean invariance \(T_{0i} = m_3 J_i^{(3)} + m_4 J_i^{(4)}\) is indeed satisfied by the Lagrangian (14). We note that the \(^3\)He self-interactions are negligible in the dilute limit of the \(^4\)He density, while in general they appear as higher order terms in the Lagrangian.

4. Effective Lagrangian

Finally, getting Eqs. (10), (12) and (14) together, we have the low-energy effective Lagrangian of the \(^4\)He su-
perfluid phonons and $^3$He excitations as follows:
\[
\mathcal{L}_{\text{eff}} = -\frac{f_0^2}{2} \frac{\partial^2 |\psi|^2 - \frac{f_0^2}{2} (\partial \phi)^2}{2m_3^*} + \frac{i\psi^+ \partial_0 \psi}{2} - \frac{|\partial \psi|^2}{2m_3^*} + (\mu_3 - \Delta) |\psi|^2 \psi
\]
\[
+ g_1 \partial \phi \cdot \frac{i\psi^+ \partial_0 \psi}{2m_3^*} + [g_2 \phi^2 + g_3 (\partial \phi)^2] \psi^2 \psi.
\]
Here we have introduced low-energy parameters as
\[
f_1^2 = \frac{\partial \mu_4}{\partial \mu_4}, \quad f^2 = \frac{n_4}{m_4}
\]
and couplings as
\[
g_1 = \eta \frac{m_4}{m_3^*}, \quad g_2 = \frac{\partial \Delta}{\partial \mu_4} - \eta, \quad g_3 = \frac{\partial \Delta}{\partial \mu_4} - \eta^2 \frac{m_4}{m_3^*}.
\]

While there are three independent terms representing interactions between the $^4$He superfluid phonons and $^3$He excitations, their couplings are not independent. We have one constraint on the three couplings
\[
g_3 = g_2 + \frac{m_3^*}{m_4} g_1
\]
as a consequence of the Galilean invariance. Moreover $g_1$ is completely determined by the $^3$He effective mass $m_3^*$, and $g_2$ (and therefore $g_3$) is determined by $m_3^*$ and the derivative of the energy cost for introducing an $^3$He atom with the chemical potential $\Delta'(\mu_4)$. These free parameters should be either determined by experiments or computed from microscopic theories. We emphasize again that our low-energy effective Lagrangian and its consequences are applicable to any other dilute fermion excitations in a boson superfluid. Details of strong interactions among bare particles are encoded into the effective mass $m^*$ and the energy cost function $\Delta(\mu)$.

**B. Bound states on a superfluid vortex**

One of the consequences derived from the effective Lagrangian is on bound $^3$He states on a vortex of $^4$He superfluid. Let us consider a single vortex with its winding number $w$ in the $^4$He superfluid. The phase $\phi$ around the vortex is given by $\phi(t, r) = w\theta$ in cylindrical coordinates.

The equation of motion of the field $\psi$ from Eq. (18) gives a Schrödinger equation for a $^3$He atom on the vortex $\phi = w\theta$ as follows:
\[
\tilde{E}\psi(r) = -\frac{\partial^2 \psi(r)}{2m_3^*} - i g_1 w \frac{\partial \psi(r)}{\partial \theta} - \frac{g_3 w^2}{2m_4 r^2} \psi(r).
\]
$\tilde{E}$ is the energy eigenvalue of the $^3$He atom and is negative for bound states. Separation of variables leads to the equation for the radial wave function $R(r)$ as
\[
\left(2m_3^* \tilde{E} - k_z^2\right) R(r) = \left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{l^2}{r^2} + 2g_1 m_3^* w l}{m_4 r^2} - \frac{g_3 m_4^* w^2}{m_4 r^2}\right] R(r).
\]
Here $l$ is an angular momentum of the $^3$He atom and $k_z$ is its momentum along the vortex line. Introducing $E = \tilde{E} - k_z^2 / 2m_3^* < 0$ and
\[
\kappa = \sqrt{g_3 m_3^* w^2 - 2g_1 m_3^* w l - l^2},
\]
we can rewrite Eq. (25) as
\[
\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} - \left[2m_3^* |E| - \frac{\kappa^2}{r^2}\right] R = 0.
\]

Solutions of this equation are given in terms of the modified Bessel functions as $R(r) = I_{\pm \kappa}(\sqrt{2m_3^* |E|} r)$.

One should remind that our effective field theory written in terms of the phase of the condensate is valid only far away from the vortex core where the magnitude of the condensate is almost constant. Suppose the Schrödinger equation (25) is valid for $r \gtrsim r_0$ and the binding energy $|E|$ is small enough to satisfy $\sqrt{2m_3^* |E|} r_0 \ll 1$. In this instance, the solution of Eq. (25) near the point $r = r_0$ turns out to be
\[
R(r) = C \sin \left[\kappa \ln \left(\sqrt{2m_3^* |E|} r\right) - \phi\right]
\]
with $C$ and $\phi$ being arbitrary constants. This radial wave function for $r \gtrsim r_0$ should smoothly connect with the radial wave function from $r \lesssim r_0$. If the binding energy $|E|$ is small enough compared to the potential energy at $r \lesssim r_0$, the radial wave function for $r \lesssim r_0$ will not depend on $|E|$. Therefore, the solution (25) is required to be independent of $|E|$ at $r = r_0$.

The logarithmic derivative of Eq. (26) at $r = r_0$ results in
\[
\frac{R'(r_0)}{R(r_0)} = \kappa \cot \left[\kappa \ln \left(\sqrt{2m_3^* |E|} r_0\right) - \phi\right].
\]
In order to satisfy the requirement, allowed energy levels $|E|$ should be discretized as
\[
\kappa \ln \left(\sqrt{2m_3^* |E_n|} r_0\right) = D - n\pi
\]
or, equivalently,
\[
E_n = \frac{e^{2D/\kappa}}{2m_3^* r_0^2} e^{-2n\pi/\kappa}.
\]
Here $D$ is an arbitrary constant and $n$ is an arbitrary integer large enough to satisfy $|E_n| \ll 1/(2m_3^* r_0^2)$. Therefore, as long as $\kappa^2$ is positive, we have an infinite number of energy levels for bound $^3$He states. While we can not determine the absolute values of bound energy levels in our approach, their ratios are independent of unknown constants $r_0$, $D$ and are asymptotically given by
\[
\frac{E_n}{E_{n-1}} = e^{-2\pi/\kappa} \quad \text{for} \quad n \to \infty.
\]
TABLE I: Pressure dependence of measured quantities $m_3^*$ and $\alpha_0$ and calculated quantities $g_{1,2,3}$ and $l_\pm$ at zero temperature and zero $^3$He concentration. The ratio of the $^3$He effective mass to its bare mass $m_3^*(P)/m_3$ is taken from Ref. [26], which is obtained by fitting experimental data [26] to the zero concentration. The BBP parameter at zero concentration $\alpha_0(P)$ is determined up to 10 atm by the experiment in Ref. [29], while they are at 6.0% concentration and not extrapolated to the zero concentration limit. The couplings $g_{1,2,3}$ in the effective Lagrangian and $l_\pm$ in Eq. (32) are calculated from the measured values with the use of Eqs. (29) and (30).

| $P$ [atm] | $m_3^*(P)/m_3$ | $\alpha_0(P)$ | $g_1$ | $g_2$ | $g_3$ | $l_-$ | $l_+$ |
|-----------|----------------|--------------|-------|-------|-------|-------|-------|
| 0         | 2.18           | 0.288        | 0.541 | 0.403 | 0.809 | −2.34 | 0.566 |
| 5         | 2.31           | 0.242        | 0.567 | 0.260 | 0.685 | −2.45 | 0.485 |
| 10        | 2.44           | 0.218        | 0.590 | 0.138 | 0.581 | −2.57 | 0.413 |
| 15        | 2.54           | 0.172        | 0.606 | 0.0170 | 0.472 | −2.65 | 0.338 |
| 20        | 2.64           | 0.148        | 0.621 | −0.0820 | 0.384 | −2.74 | 0.278 |

The criterion for an angular momentum $l$ of $^3$He atom in which bound $^3$He states are available is simply given by

$$\kappa^2 = g_3 \frac{m_3^*}{m_4} w^2 - 2g_1 \frac{m_3^*}{m_4} w l - l^2 > 0. \quad (31)$$

Using the definitions of the couplings [20] and $\eta, m_3^* = m_3 + \eta m_4$, the criterion can be rewritten as

$$l_- < l < l_+ \quad \text{with}$$

$$l_\pm = \frac{m_3^* - m_3}{m_4} \pm \sqrt{\frac{\partial \Delta}{\partial \mu_4} m_3^*/m_4}. \quad (32)$$

An infinite number of bound energy levels with their asymptotic ratio $e^{-2\tau/\kappa}$ appear for each integer $l$ satisfying $l_- < l < l_+$. We note that binding energies themselves are not determined within the effective Lagrangian, because it cannot access the vicinity of the vortex core where the magnitude of the condensate changes.

C. Determination of parameters and results for $^3$He-$^4$He mixture

We should determine the free parameters of our theory $m_3^*$ and $\partial \Delta/\partial \mu_4$ from experiments. We adopt the $^3$He effective mass $m_3^*$ at zero temperature from Ref. [25], which is obtained by fitting experimental data [26] to the zero $^3$He concentration. The ratio to the bare $^3$He mass $m_3^*(P)/m_3$ is cited in Table I at various pressures $P = 0, 5, 10, 15,$ and 20 atm.

On the other hand, the derivative of the $^3$He energy cost with the $^4$He chemical potential $\partial \Delta/\partial \mu_4$ is related to the relative fractional molar volume of $^3$He in a $^3$He-$^4$He solution, or the Bardeen-Baym-Pines (BBP) parameter [27], at zero concentration $\alpha_0$ by

$$\frac{\partial \Delta}{\partial \mu_4} = \alpha_0(P) + 1. \quad (33)$$

$\alpha_0$ has been determined as a function of the pressure $P$ up to about 10 atm by the experiment [28] and is shown in Table II. The values for 15, 20 atm in Table II are quoted from Ref. [30], while they are at 6.0% concentration and not extrapolated to the zero concentration limit.

Then, $l_\pm$ in Eq. (32) as well as the couplings $g_{1,2,3}$ in our effective Lagrangian [20] are determined from experimental values as functions of the pressure, which are listed in Table I. We have put $w = 1$ because only $|w| = 1$ vortex is energetically stable. The results show that bound $^3$He states appear for each $l = 0, 1, 2$ channels in a whole range of pressure, 0–20 atm, where experimental data are available. Note that because parity is broken by the vortex, the $l = 1$ and $l = 2$ states are not bound while the $l = -1$ and $l = -2$ states are bound. The asymptotic ratios of energy levels [31] of bound $^3$He states for each possible angular momentum are shown at $P = 0, 5, 10, 15,$ and 20 atm in Table III. We conclude that those values are in principle measurable by accurate experiments.

TABLE II: Asymptotic ratios of energy levels $E_{\infty}/E_{\infty-1} = e^{-2\pi/\kappa}$ of bound $^3$He states for each possible angular momentum at pressures $P = 0, 5, 10, 15,$ and 20 atm.

| $P$ [atm] | $l = 0$ | $l = -1$ | $l = -2$ |
|-----------|---------|---------|---------|
| 0         | 4.24 $\times 10^{-3}$ | 1.30 $\times 10^{-2}$ | 1.15 $\times 10^{-3}$ |
| 5         | 3.13 $\times 10^{-3}$ | 1.38 $\times 10^{-2}$ | 2.62 $\times 10^{-3}$ |
| 10        | 2.25 $\times 10^{-3}$ | 1.48 $\times 10^{-2}$ | 4.48 $\times 10^{-3}$ |
| 15        | 1.32 $\times 10^{-3}$ | 1.46 $\times 10^{-2}$ | 6.11 $\times 10^{-3}$ |
| 20        | 7.42 $\times 10^{-4}$ | 1.47 $\times 10^{-2}$ | 7.85 $\times 10^{-3}$ |

III. COLD FERMI GAS

The effective field theory described above makes no assumption about the nature of the bosonic ($\varphi$) and fermionic ($\psi$) degrees of freedom. This means we can use it for a two-component Fermi gas when $\varphi$ is the phase of the Cooper pair and $\psi$ represents the fermion quasi-particles of a chosen component (say, the spin-up component if the two components correspond to different spins). The fact that the Cooper pair is made up from the two
fermions, one of which is the $\psi$ fermion, is of no importance from the point of view of an effective field theory.

Let us consider the cold two-component Fermi gas with a scattering length $a$, which has been recently archived by experiments using the technique of Feshbach resonance \[8, 11, 12\]. Its ground state is found to be the superfluid in a whole range of $a$ via usual BCS mechanism for $a < 0$ (BCS regime) or Bose-Einstein condensation of tightly bound Cooper pairs (molecules) for $a > 0$ (BEC regime) \[31, 32, 33\].

One can have in mind the situation of a slightly unequal number densities in the two fermion components. Then, the ground state would be a homogeneous mixture of superfluid Cooper pairs and extra dilute fermions carrying the single component. Since the minimum of the fermion dispersion curve sits at zero momentum sufficiently deep in the BEC regime (corresponding to phase III in Ref. \[34\]), such a system turns out to be within the scope of our low-energy effective field theory for boson-fermion mixtures. As in the case for $^3$He-$^4$He mixture, we refer to the inequality $n a^3 \rightarrow 0$.

In our subsequent discussion, we refer to the inequality in the number densities as “polarization” according to the terminology of Ref. \[34\] in the case of spin-$\frac{1}{2}$ fermions. Also, we shall use the term “BEC regime” for the case $a > 0$, and “BEC limit” for the limit $na^3 \rightarrow +0$. We introduce the notation $\mu = (\mu_\uparrow + \mu_\downarrow)/2$ and $H = (\mu_\uparrow - \mu_\downarrow)/2$, with $\mu_\uparrow$ and $\mu_\downarrow$ being chemical potentials of each component of fermions for later use.

### A. BEC limit

#### 1. Microscopic description of the system

It would be instructive to start with the microscopic description of the system in the BEC limit $na^3 \ll 1$, where $n$ is the fermion number density without polarization. Since the system is dilute in this limit, the bound molecules can be regarded as pointlike bosons. Therefore, the dynamics of the molecules could be described by the following local Lagrangian \[35\]:

$$\mathcal{L}_B = \frac{i \psi_\uparrow^\dagger \partial_\uparrow \psi_\uparrow}{2} - \frac{\partial_\uparrow \psi_\uparrow^2}{4m} + \mu_\uparrow \psi_\uparrow^\dagger \psi_\uparrow - \frac{g_b}{2} (\psi_\uparrow^\dagger \psi_\uparrow)^2. \quad (34)$$

Here the field $\Psi$ represents the superfluid molecule which has its mass $2m$ with $m$ being the fermion mass. $\mu_b = 2\mu + E_0$ is the chemical potential of molecule with $E_0$ being its binding energy. The coupling of its self-interaction $g_b$ is characterized in terms of the two-body scattering length between molecules by $g_b = 2\pi a_b/m$.

If we introduce the chemical potential for the polarization $H$ larger than the energy gap of a single fermion $|H| \gtrsim E_0/2$, extra fermions carrying one sign of spin will be created on the top of the BEC ground state. The Lagrangian describing such fermions and their interaction with the superfluid molecules will be given by

$$\mathcal{L}_F = \frac{i \psi_\uparrow^\dagger \partial_\uparrow \psi_\uparrow}{2} - \frac{\partial_\uparrow \psi_\uparrow^2}{2m} + (\mu + |H|) \psi_\uparrow^\dagger \psi_\uparrow - g_b \psi_\uparrow^\dagger \psi_\uparrow^\dagger \psi_\uparrow^\dagger \psi_\uparrow. \quad (35)$$

With $\psi_\uparrow$ or $\psi_\downarrow$ depending on the sign of $H$. Hereafter $H > 0$ is assumed to be positive without losing generality. The coupling of the interaction between the extra fermion and the bound molecule $g_b$ is characterized in terms of their two-body scattering length by $g_b = 3\pi a_b/m$. Self-interactions among fermions are negligible in the dilute limit of extra fermions. We should note here that the mass of the extra fermion is provided by its bare mass $m$, because interaction effects become infinitely small in the BEC limit $na^3 \rightarrow 0$.

The equation of motion from Eq. (34) leads to the Gross-Pitaevskii equation

$$(i \partial_\uparrow + \mu_b) \Psi(t, r) = \frac{\partial_\uparrow^2 \Psi(t, r)}{4m} + g_b |\Psi(t, r)|^2 \Psi(t, r). \quad (36)$$

In order to consider a single vortex in the molecular superfluid, we set $\Psi(t, r) = \sqrt{n_b} h(r)e^{iw_\theta}$ in cylindrical coordinates. $n_b = n/2 = \mu_b/g_b$ is the density of molecules far away from the vortex core and $w$ is a winding number of the vortex. $h(r)$ is some function of the radius $r$ from the vortex core, which satisfies the boundary conditions

$$h(r \rightarrow 0) \rightarrow 0 \quad \text{and} \quad h(r \rightarrow \infty) \rightarrow 1. \quad (37)$$

Substituting $\Psi(t, r)$ into Eq. (36), we obtain an equation for $h(x)$,

$$\frac{\partial^2 h}{\partial x^2} + \frac{1}{x} \frac{\partial h}{\partial x} - w^2 \frac{h}{x^2} + h - h^3 = 0, \quad (38)$$

where $x = r/\xi$ is the dimensionless radius normalized by the healing length of the molecular superfluid $\xi = 1/\sqrt{4\pi n_b g_b}$.
This equation should be solved under the boundary conditions. Asymptotic forms of $h(x)$ at $x \to 0$ and $x \to \infty$ are easily read from Eq. (38) as follows:

$$h(x) \propto x^w \quad \text{for} \quad x \to 0, \quad (39)$$

$$h(x) = 1 - \frac{w^2}{2x^2} \quad \text{for} \quad x \to \infty. \quad (40)$$

The solutions of Eq. (38), connecting these two limits can be obtained numerically, which are shown in Fig. 1 for $w = 1, 2, \text{and} 3$.

Because of the interaction between the molecule and the fermion, the vortex structure in the molecular superfluid acts as a potential for the fermion. If the interaction between the molecule and the fermion is repulsive ($g_{bf} > 0$), the fermion will be attracted to the vortex core because there are less molecules. Due to this effective attraction, bound fermion states on the vortex turn out to be possible. The equation of motion from Eq. (35) with the solution of Eq. (36) gives the Schrödinger equation for the fermion under the vortex potential

$$\tilde{E}\psi(r) = -\frac{\partial^2\psi(r)}{2m} + g_{bf}n_a \{h(r)^2 - 1\} \psi(r). \quad (41)$$

$\tilde{E}$ is the energy eigenvalue of the fermion and is negative for bound states.

Separation of variables and use of the normalized radial $x = r/\xi$ lead to the equation for the radial wave function $R(x)$ as follows:

$$\epsilon R(x) = \left[ -\frac{\partial^2}{\partial x^2} - \frac{1}{x} \frac{\partial}{\partial x} + \frac{\kappa^2}{x^2} + \gamma \{t^2 - 1\} \right] R(x). \quad (42)$$

Here $l$ is an angular momentum of the fermion and we have defined a dimensionless energy eigenvalue $\epsilon = (2m\tilde{E} - k_z^2)\epsilon^2 < 0$ and a coupling ratio $\gamma = g_{bf}/2g_b$. $k_z$ is a momentum of the fermion along the vortex line. Note that Eq. (42) is invariant under $l \to -l$. This invariance cannot be exact because the vortex breaks parity. It is only an approximate property of the deep BEC limit, in which the fermion field feels only the magnitude of the condensate but not its phase [Eq. (11)].

### TABLE III: Bound energy levels $\epsilon_n$ and their ratios $\epsilon_n/\epsilon_{n-1}$ in the S-wave channel ($l = 0$) for $\gamma = 1.47$.

| $n$ | $\epsilon_n$ | $\epsilon_n/\epsilon_{n-1}$ |
|-----|---------------|-----------------------------|
| 0   | $-4.93 \times 10^{-1}$ | —                          |
| 1   | $-3.39 \times 10^{-3}$  | 6.88 \times 10^{-3}         |
| 2   | $-1.91 \times 10^{-5}$  | 5.64 \times 10^{-3}         |
| 3   | $-1.08 \times 10^{-7}$  | 5.65 \times 10^{-3}         |
| 4   | $-6.10 \times 10^{-10}$ | 5.65 \times 10^{-3}         |
| 5   | $-3.45 \times 10^{-12}$ | 5.65 \times 10^{-3}         |
| 6   | $-1.95 \times 10^{-14}$ | 5.65 \times 10^{-3}         |
| $\infty$ | —                          | 5.65 \times 10^{-3}         |

### TABLE IV: Bound energy levels $\epsilon_n$ and their ratios $\epsilon_n/\epsilon_{n-1}$ in the P-wave channel ($l = \pm 1$) for $\gamma = 1.47$.

| $n$ | $\epsilon_n$ | $\epsilon_n/\epsilon_{n-1}$ |
|-----|---------------|-----------------------------|
| 0   | $-1.57 \times 10^{-4}$ | —                          |
| 1   | $-1.55 \times 10^{-6}$  | 9.86 \times 10^{-5}         |
| 2   | $-1.68 \times 10^{-10}$ | 1.09 \times 10^{-4}         |
| 3   | $-1.83 \times 10^{-14}$ | 1.09 \times 10^{-4}         |
| $\infty$ | —                          | 1.09 \times 10^{-4}         |

2. **Bound fermion states on a superfluid vortex**

The Schrödinger equation with the asymptotic behavior of the vortex potential predicts an infinity number of energy levels for bound fermion states on a superfluid vortex at least in the S-wave channel. We can rewrite Eq. (12) far away from the vortex core $(x \gg 1)$ as

$$\frac{\partial^2 R}{\partial x^2} + \frac{1}{x} \frac{\partial R}{\partial x} - \left[ |\epsilon - \kappa^2|/x^2 \right] R = 0 \quad (43)$$

with $\kappa = \sqrt{\gamma w^2 - l^2}$. This equation has the same form as Eq. (26) so that the discussion in Sec. II B is applicable here. Consequently, the criterion for the angular momentum of the fermion in which bound fermion states are available is given by

$$\kappa^2 = \gamma w^2 - l^2 > 0. \quad (44)$$

An infinite number of energy levels for the bound fermion states appear with each integer $l$ satisfying $l^2 < \gamma w^2$ and their ratios are asymptotically given by

$$\frac{\epsilon_n}{\epsilon_{n-1}} = e^{-2\pi/\kappa} \quad \text{for} \quad n \to \infty. \quad (45)$$

Whether we have bound fermion states with a certain $l$ depends only on the value of the coupling ratio $\gamma = g_{bf}/2g_b = 3a_{bf}/4a_b > 0$. In particular, one can conclude that they are always possible for the S-wave channel ($l = 0$) in the BEC limit.

Now we restrict ourselves to the energetically stable vortex $w = 1$. Since the system is weakly coupled in the BEC limit $na^3 \to 0$, the scattering length between a molecule and a molecule or fermion is calculable as a function of the fermion scattering length $a$ and results in $a_{bf} = 0.60 - 0.75a$, or $a_{bf} = 1.79a$ or $a_{bf} = 4 \times 10^{-3}$ for the S-wave channel $l = \pm 1$ as well as in the S-wave channel. Hereafter we adopt the value $a_{bf} = 0.60a$ which has been confirmed by a quantum Monte Carlo simulation and experiments. Then, the asymptotic ratios of bound energy levels $\epsilon_{n}/\epsilon_{n-1}$ are given by $5.65 \times 10^{-3}$ for the S-wave channel and $1.09 \times 10^{-4}$ for the P-wave channel.

The absolute values of binding energies should be determined by solving the Schrödinger equation numerically. Results on bound energy levels for $\epsilon_n \leq 10^{-15}$
FIG. 2: (Color online) Solutions of Eq. (42) in the $S$-wave channel ($l = 0$) for $\gamma = 1.47$. Left panel: Bound energy levels $\epsilon_n$ for $n = 0, 1, \ldots, 6$ (dotted lines) and the potential energy $\gamma \{ h(x)^2 - 1 \}$ (solid curve) are shown as functions of $x = r / \xi$. The normalized wave function of the ground state ($n = 0$) is also shown by the dashed curve. Right panel: $|\epsilon_n|$ (dotted lines) are shown in the log scale.

FIG. 3: (Color online) Solutions of Eq. (42) in the $P$-wave channel ($l = \pm 1$) for $\gamma = 1.47$. Left panel: Bound energy levels $\epsilon_n$ for $n = 0, 1, 2, 3$ (dotted lines) and the potential energy $1/x^2 + \gamma \{ h(x)^2 - 1 \}$ (solid curve) are shown as functions of $x = r / \xi$. The normalized wave function of the ground state ($n = 0$) is also shown by the dashed curve. Right panel: $|\epsilon_n|$ (dotted lines) are shown in the log scale.

and their ratios $\epsilon_n / \epsilon_{n-1}$ are listed in Table III for the $S$-wave channel and in Table IV for the $P$-wave channel. The ratios of bound energy levels $\epsilon_n / \epsilon_{n-1}$ rapidly converge to their asymptotic values from $n \gtrsim 2$ for both the $S$- and $P$-wave channels. Even in the first ratios $\epsilon_1 / \epsilon_0$, their deviations from the asymptotic values are only 22\% for the $S$-wave channel and 9.1\% for the $P$-wave channel.

The bound energy levels (dotted lines) are shown in Fig. 2 for the $S$-wave channel and in Fig. 3 for the $P$-wave channel as well as the potential energies $l^2 / x^2 + \gamma \{ h(x)^2 - 1 \}$ (solid curves) as functions of $x = r / \xi$. While the binding energies of excited states ($n \geq 1$) seems degenerated into zero in the linear scale (left panels) for each channels, they are allocated with equal intervals $2\pi / (\kappa \ln 10)$ in the log scale (right panels) as indicated by Eq. (45). The wave functions of the ground state ($n = 0$) normalized as

$$\int_0^\infty dx x R(x)^2 = 1$$

(46)

are also shown in each figure (dashed curves). In the $S$-wave channel, the ground state wave function is well localized around the vortex core $r / \xi \lesssim 5$ and its binding energy is comparable to the bottom of potential energy $|\epsilon_0| / \gamma = 0.334$. Thus, we conclude that those bound fermion states in the BEC limit could be measurable by future experiments at least in the ground state of the $S$-wave channel.

Away from the BEC limit, the fermion mass changes to the effective mass $m^*$, which is larger than the bare mass $m^* > m$. Accordingly, the coupling ratio in the Schrödinger equation (42) changes to the effective
one \( \gamma^* = (m^*/m)\gamma \). Because \( \gamma^* > \gamma \), bound fermion states with higher angular momenta are expected to appear as one departs from the BEC limit. However, if one is away from the BEC limit \( na^3 \gg 1 \), molecules overlap each other and then the microscopic description with local interactions Eqs. (54) and (55) will no longer be valid. In order to treat the problem away from the BEC limit, the effective field theory in terms of low-energy excitations is efficient.

3. Connection to effective field theory

Let us see how the effective field theory in terms of low-energy excitations, superfluid phonons and extra fermions, emerges from the microscopic Lagrangians (54) and (55). For that purpose, we parametrize the field of bound molecule as \( \Psi(t, r) = \sqrt{n_b(t, r)} e^{2i\varphi(t, r)} \) and assume that the magnitude \( n_b(t, r) \) is slowly varying in time and space, which is satisfied far away from the vortex core. Then, Eqs. (54) and (55) can be rewritten by

\[
\mathcal{L}_B + \mathcal{L}_F = n_B \left[ \mu_\mu - 2\dot{\varphi} - \left( \frac{\partial \varphi}{m} - g_{bd} \psi^\dagger \psi \right) \right] - \frac{g_{b^2}}{2} n_B^2 + \frac{i\psi^\dagger \partial_t \psi}{2} - \frac{\left| \partial \varphi \right|^2}{2m} + (\mu + H) \psi^\dagger \psi. \tag{47}
\]

Since now the density of bound molecules \( n_B \) is a variational parameter, it should be determined by minimizing the action and results in

\[
g_{b^2} n_B = \mu_\mu - 2\dot{\varphi} - \left( \frac{\partial \varphi}{m} - g_{bd} \psi^\dagger \psi \right). \tag{48}
\]

Substitution of this expression into Eq. (47) leads to an effective theory in terms of the superfluid phonon \( \varphi \) and the fermion excitation \( \psi \) as follows:

\[
\mathcal{L}_B + \mathcal{L}_F \simeq P(\mu) + \frac{f_1^2}{2} \varphi^2 - \frac{f_2^2}{2} \left( \frac{\partial \varphi}{m} \right)^2 + \frac{i\psi^\dagger \partial_t \psi}{2} - \frac{\left| \partial \varphi \right|^2}{2m} + (\mu + H) \psi^\dagger \psi \tag{49}
\]

\[-\Delta(\mu) \psi^\dagger \psi + \frac{\partial \Delta}{\partial \mu} \left[ \phi + \frac{\left( \frac{\partial \varphi}{m} \right)^2}{2m} \right] \psi^\dagger \psi.
\]

We have dropped a total derivative term and higher order terms in \( \varphi \). Here \( P(\mu) = \mu_\mu^2/2g_{b^2} \) is identical to the pressure of the molecular superfluid up to an irrelevant constant, and \( \Delta(\mu) = g_{bd} n_B/m \) is the energy cost to introduce a single fermion into the superfluid originating in its interaction with the bound molecules. The low-energy parameters in the Lagrangian turn out to be given by

\[
f_1^2 = \frac{\partial n}{\partial \mu}, \quad f_2^2 = \frac{n}{m} \quad \text{with} \quad n = \frac{\partial P}{\partial \mu}. \tag{50}
\]

This is the manifestation of general properties of Sec. II A in the molecular superfluid case. The coupling between the superfluid phonons and extra fermions is also given by the derivative of the fermion energy cost \( \Delta'(\mu) = 2g_{bd}/g_{b^2} \) as in Eq. (42).

B. Away from the BEC limit

1. Effective field theory

As mentioned above, the problem of writing down an effective field theory that couples the superfluid phonons \( \varphi \) and the extra fermions \( \psi \) is identical to the same problem in the \(^3\)He-\(^4\)He mixture. This is because their symmetries and the pattern of symmetry breaking in the two cases are the same. Let us elaborate on this point in more detail.

In the \(^3\)He-\(^4\)He mixed system, there are two conserved charges: the number of \(^4\)He atoms \( N_4 \) and the number of \(^3\)He atoms \( N_3 \). The first charge is spontaneously broken by the \(^4\)He superfluid ground state. The order parameter carries a unit \( N_4 \) charge, but is neutral with respect to the \( N_3 \) charge. The \(^3\)He atoms carry only the \( N_3 \) charge which is unbroken, while being neutral with respect to the \(^4\)He charge.

In the slightly polarized fermion system, the number of spin-up fermions \( N_\uparrow \) and the number of spin-down fermions \( N_\downarrow \) are conserved separately. We suppose all extra fermions have spin up. It is, however, more convenient to use another basis, consisting of \( N_\uparrow \) and the total polarization \( Y = N_\uparrow - N_\downarrow \). The Cooper pairs carry unit \( N_4 \) charge and its condensation spontaneously breaks this symmetry. The extra fermions carry a unit \( Y \) charge but are neutral with respect to \( N_4 \). Thus, \( N_4 \) is equivalent to \( N_4 \) in the \(^3\)He-\(^4\)He mixture, and \( Y = N_\uparrow - N_\downarrow \) is equivalent to \( N_3 \).

It is now possible to write down the effective Lagrangian for the case at hand by direct analogy with the case for \(^3\)He-\(^4\)He mixture. We simply need to make the following replacement for the particle masses and chemical potentials:

\[
m_4 \to 2m, \quad m_3 \to m, \quad \mu_4 \to 2\mu, \quad \mu_3 \to \mu + H. \tag{52}
\]

The equations in Eq. (55) follow from the requirement that \( \mu_4 n_4 + \mu_3 n_3 = \mu_\uparrow n_\uparrow + \mu_\downarrow n_\downarrow \) when \( n_4 \to n_\downarrow \) and \( n_3 \to n_\uparrow - n_\downarrow \). For convenience, we shall also make some rescalings so as to absorb extra factors 2 appearing in the equations:

\[
\varphi \to 2\varphi, \quad 2n \to n. \tag{53}
\]

As a result, the Galilean invariant effective Lagrangian \( \mathcal{L}_{\text{eff}} \) becomes

\[
\mathcal{L}_{\text{eff}} = \frac{f_1^2}{2} \varphi^2 - \frac{f_2^2}{2} \left( \frac{\partial \varphi}{m} \right)^2 + \frac{i\psi^\dagger \partial_t \psi}{2} - \frac{\left| \partial \varphi \right|^2}{2m^*} + (\mu + H - \Delta) \psi^\dagger \psi \tag{54}
\]

\[+ g_{1} \varphi \cdot \frac{i\psi^\dagger \partial \psi}{2m} + \left[ g_{2} \varphi + g_{3} \frac{\left( \partial \varphi \right)^2}{2m} \right] \psi^\dagger \psi. \]

Here \( m^* \) is the effective fermion mass and the couplings
in the effective Lagrangian are given by

\[ g_1 = \frac{m}{m^*}, \quad g_2 = \frac{\partial \Delta}{\partial \mu} - \eta, \quad g_3 = \frac{\partial \Delta}{\partial \mu} - \eta^2 \frac{m}{m^*}, \quad (55) \]

with \( \eta \) being defined by \( m^* = (1 + \eta)m \). The universal relation among these couplings

\[ g_3 = g_2 + g_1 \quad (56) \]

is a consequence of the Galilean invariance. In particular, \( g_1 = 0 \) and \( g_2 = g_3 = \partial \Delta / \partial \mu = 4\gamma \) in the BEC limit \( m^* \to m \), which reproduce the result of Eq. (19).

2. Bound fermion states on a superfluid vortex

Let us consider the bound fermion states on a superfluid vortex with its winding number \( w \). The discussion is parallel to that in Sec. II B. What we have to be careful here is that the phase \( \varphi \) of the condensate around the vortex is given by \( \varphi(t, r) = w\theta/2 \) in cylindrical coordinates. Extra one-half compared to the \( ^3\text{He}-^4\text{He} \) mixture case is because \( \varphi \) is normalized to be half of the phase of the Cooper pair \( \Psi = |\Psi|e^{i\varphi} \) in Eq. (55) for the fermion superfluid case. The equation of motion for the fermion field \( \psi \) from Eq. (51) gives a Schrödinger equation as follows:

\[ \hat{E}\psi(r) = -\frac{\partial^2 \psi(r)}{2m^*} - \frac{ig_1w}{2m^*} \frac{\partial \psi(r)}{\partial \theta} - \frac{g_3w^2}{8mr^2}\psi(r). \quad (57) \]

The third term corresponds to the \( 1/\mu^2 \) vortex potential far away from the vortex core in Eq. (11) in the BEC limit. On the other hand, the second term, which was absent in the BEC limit, is required by the Galilean invariance when \( m^* \neq m \). This term represents that the fermion with angular momentum in the opposite direction to the vortex’s one is energetically favored, because it decreases the total angular momentum of the system.

Separation of variables leads to the equation for the radial wave function \( R(r) \) as follows:

\[ 2m^* \hat{E} - k_z^2 \] \[ R(r) \]

\[ = \left( -\frac{\partial^2}{\partial r^2} - \frac{\partial}{r} \frac{\partial}{\partial r} + \frac{l^2}{r^2} + g_1 \frac{m^* w l}{m r^2} - g_3 \frac{m^* w^2}{4mr^2} \right) R(r). \quad (58) \]

Here \( l \) is an angular momentum of the fermion and \( k_z \) is its momentum along the vortex line. As we have proved in Sec. II B whether we have bound fermion states for a certain \( l \) is determined only by the sign of the coefficient of \( 1/r^2 \). They are possible for angular momenta \( l \) which satisfy the criterion

\[ \kappa^2 = g_3 \frac{m^*}{4m} w^2 - g_1 \frac{m^*}{m} w l - l^2 > 0. \quad (59) \]

Using the definitions of the couplings (55) and \( \eta, m^* = (1 + \eta)m \), the criterion can be rewritten as \( l_- < l < l_+ \)

with

\[ \frac{l_+}{w} = \frac{1}{2} \sqrt{\frac{m^*}{m}} \pm \frac{1}{2} \sqrt{\frac{\partial \Delta}{\partial \mu}} \frac{m^*}{m}. \quad (60) \]

An infinite number of bound energy levels with their asymptotic ratio \( e^{-2\pi / \kappa} \) appear for each integer \( l \) satisfying \( l_- < l < l_+ \).

3. Discussion on parameters and conjectures

We have two free parameters which cannot be determined within our effective field theory, the effective fermion mass \( m^*/m \) and the derivative of fermion energy cost \( \Delta'(/\mu) \) in the superfluid. These two parameters are functions of the fermion scattering length \( a \) and the fermion chemical potential \( \mu \) or density \( n \). They should be measurable by experiments in the same way as \( m_3^* \) and \( a_0 = \Delta'(/\mu_3) - 1 \) in the \( ^3\text{He}-^4\text{He} \) mixtures. While such measurements are not achieved yet, we still know some general properties of those quantities.

As we have discussed, the effective fermion mass \( m^*/(\mu, a) \) coincides with its bare mass \( m \) in the BEC limit \( na^3 \ll 1 \). On the other hand, it has been argued that \( m^*/(\mu, a) \) becomes infinite at the so-called splitting point (SP), located on the BEC half \( (a > 0) \) of the crossover diagram [34]. Therefore, it is natural to assume that \( m^*/(\mu, a)/m \) is an increasing function of \( na^3 \) from unity (the BEC limit) to infinity (the SP limit), as \( na^3 \) increases from zero to some critical value.

As for the derivative of fermion energy cost \( \Delta'(/\mu, a) \), it is presumably positive in the BEC regime. This can be seen by writing the derivative of fermion energy cost as

\[ \frac{\partial \Delta(/\mu, a)}{\partial \mu} = \frac{\partial n}{\partial \mu} \frac{\partial \Delta(n, a)}{\partial n}. \quad (61) \]

Thermodynamic stability implies \( \partial n / \partial \mu > 0 \). The energy cost \( \Delta \) to introduce a fermion into the superfluid originates in the interaction of the fermion with bound molecules in the BEC regime. Their interaction is considered to be repulsive because the Pauli principle between the extra fermion and a fermion in the bound molecule with the same sign of spin acts as an effective repulsion. Thus, the fermion energy cost will be positive in the superfluid and an increasing function of the density \( n \), which results in \( \partial \Delta / \partial n > 0 \). In fact, this is the case in the BEC limit where \( \Delta = g_{bd} n^2/2 \) with the repulsive coupling \( g_{bd} = 1.179 a > 0 \). It is also true at the unitarity limit \( (na^3 = \infty) \) where \( \Delta \sim n^{2/3} \). Accordingly, we conclude that \( \Delta'(/\mu) \) is positive in the BEC regime.

With the use of those properties on the parameters, we can derive some interesting predictions from the formula (60). Let us restrict ourselves to the energetically stable vortex \( w = 1 \). Because of \( l_+ = \sqrt{7} = 1.21 \) in the BEC limit, the \( S \)-wave \( (l = 0) \) and \( P \)-wave \( (l = \pm 1) \) bound fermion states are possible. On the other hand, \( l_+ \) goes
FIG. 4: (Color online) $l_+$ and $l_-$ in Eq. (60) as functions of $m^*/m$ for $\partial \Delta / \partial \mu = 4 \gamma$. $m^*/m = 1$ corresponds to the BEC limit, while $m^*/m \to \infty$ corresponds to the splitting point in Ref. [34]. The curves are shown for $m^*/m \geq 0$.

TABLE V: Estimated values of the fermion effective mass where bound fermion states appear $m^*/m|_{l_-=l}$ and disappear $m^*/m|_{l_+=l}$ for several angular momenta $l$.

| $l$ | $m^*/m|_{l_-=l}$ | $l$ | $m^*/m|_{l_+=l}$ |
|-----|-----------------|-----|-----------------|
| $-2$ | $1.77$          | $1$  | $3.62$          |
| $-3$ | $2.88$          | $0$  | $7.77$          |
| $-4$ | $4.09$          | $-1$ | $11.1$          |
| $-5$ | $5.37$          | $-2$ | $17.0$          |

to negative infinity in the SP limit $l_+ \to -\infty$, because $m^*/m \to \infty$ and $\partial \Delta / \partial \mu$ should be finite without phase transitions. Therefore, $l_-$ must get across integers less than $-1$ where new bound energy levels appear, and $l_+$ must get across integers not greater than $1$ where existing bound energy levels disappear between the BEC and SP limits. In particular, an infinite number of negative and large angular momenta become ready for the bound fermion states in the vicinity of the SP limit.

For illustration, let us employ a somewhat unjustified ansatz, $\Delta'(\mu, a) = 4 \gamma$, which is an extrapolation from the weak coupling BEC limit. Using the value $\gamma = 1.47$, $l_\pm$ in Eq. (60) are evaluated as functions of $m^*/m$ in Fig. 4. Bound fermion states are possible with angular momenta $l$ which are integers between $l_-$ and $l_+$. The term proportional to $g_1$ in Eq. (54) suppresses bound fermions with positive angular momenta, while it enhances bound fermions which have negative angular momenta in the opposite direction to the vortex’s one. Fig. 4 shows that bound fermion states with positive angular momentum disappear away from the BEC limit, while those with arbitrary higher negative angular momenta become possible as one approaches to the SP limit $m^*/m \to \infty$.

The fermion effective masses where bound fermion states appear $m^*/m|_{l_-=l}$ or disappear $m^*/m|_{l_+=l}$ are estimated for the first four angular momenta $l$ in Table V. Especially, the $S$-wave bound fermion states which have the deepest binding energy in the BEC limit are found to disappear when $g_3$ changes its sign at $m^*/m = 7.77$. We emphasize that quantitative results in Fig. 4 and Table V may not be reliable due to the ansatz we employed.

Finally, it would be interesting to compare our results in the polarized Fermi gas with those in the unpolarized Fermi gas from the Bogoliubov–de Gennes approach [17, 18]. The results in Ref. [18] show that bound fermion states on a vortex are possible only for $l = 0, -1$ at $na^3 = 1/\pi a^2$ in the BEC regime, while bound fermion states become possible for all $l \leq 0$ in the unitarity limit $na^3 = \infty$. The tendency that possible angular momenta for bound states increase as one gets away from the BEC limit is consistent with what we found, however, we predict $l = 1$ bound fermion states as well as $l = 0, -1$ in the BEC limit. How these two results match with each other as a function of the polarization will be an interesting future problem.

IV. SUMMARY AND CONCLUSIONS

In this paper, we constructed low-energy effective field theories for dilute fermion excitations in superfluids with emphasis on the Galilean invariance [Eq. (18) for boson superfluids or Eq. (54) for molecular superfluids]. The Galilean invariance as well as the global symmetries of the system considerably restrict the possible form of the effective Lagrangian. We showed three terms representing interactions between the fermion excitations $\psi$ and superfluid phonons $\varphi$ appear to the lowest nontrivial order of the fields. The couplings for the three interaction terms are not independent as a consequence of the Galilean invariance. They are written by the effective fermion mass $m^*/m$ and the derivative of the fermion energy cost with the chemical potential $\partial \Delta / \partial \mu$ [Eq. (21) or (59)], which are measurable quantities by experiments.

We consider that the effective Lagrangian obtained here is valid as long as the following conditions are satisfied. (i) The low-energy dynamics of superfluids is dominated by the excitations of superfluid phonons or, in other words, the variations in the magnitude of the condensate are negligible. (ii) The fermion’s dispersion is given by the quadratic power of its momentum with a positive coefficient as in Eq. (1). Higher order corrections to the dispersion are negligible in the low-energy dynamics of fermions. (iii) Fermions are dilute or weakly coupled so that their self-interactions are negligible.

With the use of the effective field theory, we studied bound fermion states on a single vortex of the superfluid. We derived a simple criterion for an angular momentum $l$ of fermion in which bound fermion states are available [Eq. (31) or (55)]. An infinite number of bound energy levels appear for angular momenta satisfying the criterion and their ratios are asymptotically given by Eq. (30).

We applied our effective field theory to two boson-fermion mixed systems, a dilute solution of $^3$He in $^4$He superfluid and the BEC regime of a cold polarized Fermi...
gas. For the $^3\text{He}-^4\text{He}$ mixture, we determined parameters of the effective theory from experimental data as functions of pressure. As a result, we predict that bound $^3\text{He}$ states with $l = -2, -1, 0$ will be realized on a vortex of the $^4\text{He}$ superfluid in a whole range of pressure, 0–20 atm, where experimental data are available. Asymptotic ratios of bound energy levels are calculated in Table III for each angular momentum. Those properties should be in principle confirmed by future accurate experiments.

As for the cold polarized Fermi gas, we determined parameters of the effective field theory from the microscopic description in the BEC limit. As a consequence, $S$-wave ($l = 0$) and $P$-wave ($l = \pm 1$) bound fermion states turned out to be realized in the BEC limit. Since the fermion mass compared to its bare mass $m^* / m$ could change from unity (the BEC limit) to infinity (the SP limit) in the BEC regime, bound fermion states with arbitrary negative angular momentum will become available away from the BEC limit. Especially, the bound fermion states with $l = 0, \pm 1$, which are realized in the BEC limit, are shown to disappear as one set away from the BEC limit.

While we have concentrated on the study of bound fermion states on a superfluid vortex in this paper, our effective field theory will be useful to study other problems in the boson-fermion mixtures. Also, our effective field theory is universal to any other boson-fermion mixed systems. Those investigations should be performed in future works.

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APPENDIX: THE BARDEEN-BAYM-PINES PARAMETER

Here let us remind of the relation between the energy cost $\Delta$ to introduce a $^3\text{He}$ quasiparticle into the $^4\text{He}$ superfluid in Eq. (12) and the relative fractional molar volume of a dilute solution of $^3\text{He}$ in $^4\text{He}$, or the Bardeen-Baym-Pines (BBP) parameter, $\alpha$. $\alpha$ is a directly measurable quantity by experiments. It is sufficient for the present purpose to limit ourselves at zero temperature for simplicity.

BBP parameter $\alpha(x, P)$ at the $^3\text{He}$ concentration $x$ and the pressure $P$ is defined through the following equation:

$$ V_{^3\text{He}}(x, P) = V_4(P) \left[ 1 + x \alpha(x, P) \right], \tag{A.1} $$

where $V_{^3\text{He}}$ and $V_4$ are the molar volume of the dilute solution of $^3\text{He}$ in $^4\text{He}$ and the pure $^4\text{He}$, respectively. Total number of $^3\text{He}$ and $^4\text{He}$ atoms $N$ is fixed here. Since we have interest in the dilute limit of $\alpha$, let us rewrite Eq. (A.1) as

$$ \alpha_0(P) \equiv \lim_{x \to 0} \frac{V_{^3\text{He}}(x, P) - V_4(P)}{x V_4(P)}, \tag{A.2} $$

and calculate $V_{^3\text{He}}$ in the leading order of $x$.

The chemical potential of the mixture is defined as

$$ \mu(x, P) = x \mu_3(x, P) + (1 - x) \mu_4(x, P) \tag{A.3} $$

with $\mu_3$ and $\mu_4$ being the chemical potentials of the $^3\text{He}$ and $^4\text{He}$ atoms. We can calculate the molar volume of the mixture by differentiating the mixture chemical potential with the pressure as

$$ V_{^3\text{He}}(x, P) = A \frac{\partial \mu}{\partial P} \bigg|_x, \tag{A.4} $$

where $A$ is the Avogadro’s constant. With the use of the Gibbs–Duhem relation

$$ x \frac{\partial \mu_3}{\partial x} \bigg|_P + (1 - x) \frac{\partial \mu_4}{\partial x} \bigg|_P = 0, \tag{A.5} $$

Eq. (A.4) is rewritten as

$$ \mu(x, P) = (1 - x) \mu_4(0, P) + x \mu_3(x, P) - (1 - x) \int_0^x dx' \frac{x'}{1 - x'} \frac{\partial \mu_3(x', P)}{\partial x'}. \tag{A.6} $$

The energy of the $^3\text{He}$ quasiparticle excitation $E(k)$ can be expressed with its effective mass $m_3^*(P)$ as

$$ E(k) = \Delta(P) + \frac{k^2}{2m_3^*(P)} + x E_{\text{int}}(x, P, k). \tag{A.7} $$

The first term $\Delta$ is the energy cost to introduce a single $^3\text{He}$ quasiparticle to the pure $^4\text{He}$ superfluid, and the third term proportional to $x$ represents the contribution of $^3\text{He} - ^3\text{He}$ interaction to the excitation energy. Since the third term gives higher order corrections of the order of $O(x^2)$ to $\alpha(x, P)$, we neglect it hereafter. Therefore, the $^3\text{He}$ chemical potential in the mixture is of the form:

$$ \mu_3(x, P) = \Delta(P) + \mu_3(x, P). \tag{A.8} $$

$\mu_F$ is the chemical potential of a free Fermi gas of mass $m_F^*$ and density $xN$, which is simply given, at zero temperature, by

$$ \mu_F(x, P) = \frac{(3\pi^2N)^{2/3}}{2m_F^*(P)} x^{2/3} = \mu_F(1, P) x^{2/3}. \tag{A.9} $$

Substituting Eqs. (A.8) and (A.9) into Eq. (A.6), we obtain

$$ \mu(x, P) = (1 - x) \mu_4(0, P) + x \Delta(P) + \frac{3}{5} x^{5/3} \mu_F(1, P) + O(x^2). \tag{A.10} $$
The differentiation of this equation with $P$ leads to the expression of the molar volume of the mixture as follows:

$$V_{34}(x, P) = V_4(P) - xV_4(P) + xA\frac{\partial \Delta(P)}{\partial P} + O(x^{5/3}),$$  \hspace{1cm} (A.11)

where $V_4(P) = A\partial\mu_4(0, P)/\partial P$. Consequently, BBP parameter at zero $^3$He concentration is given by

$$\alpha_0(P) = \frac{A}{V_4(P)}\frac{\partial \Delta(P)}{\partial P} - 1 = \frac{\partial \Delta(P)}{\partial \mu_4(0, P)} - 1.$$  \hspace{1cm} (A.12)

$\alpha_0$ has been determined as a function of the pressure $P$ by the experiment \[29\] up to about 10 atm, and found to be fitted very well by the following polynomial [Eq. (60) in \[29\]]:

$$\alpha_0(P) = \sum_{i=0}^{4} a_i P^i.$$  \hspace{1cm} (A.13)

with

$$a_0 = 2.88069661, \quad a_1 = -1.26990768 \times 10^{-2} \text{ atm}^{-1},$$

$$a_2 = 9.57212464 \times 10^{-4} \text{ atm}^{-2}, \quad (A.14)$$

$$a_3 = -6.1337394 \times 10^{-5} \text{ atm}^{-3}, \quad a_4 = 2.26173741 \times 10^{-6} \text{ atm}^{-4}.$$  \hspace{1cm}

The unit is converted from “kgf/cm²” in \[29\] to “atm” here. The values of $\alpha_0(P)$ for $P = 0, 5$ and 10 atm are shown in Table I.

Finally, we note that $\Delta$ and $\mu_4$ in Eq. \[A.12\] are identical to $\Delta$ and $\mu_4$ introduced in Eq. \[12\] of the text. Therefore, the coupling between the $^3$He quasiparticle and the $^4$He superfluid phonons $\partial \Delta/\partial \mu$ is related with the BBP parameter at zero concentration $\alpha_0$ by $\partial \Delta/\partial \mu = \alpha_0 + 1$.

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