Low-Temperature Series for Ising Model by Finite-Lattice Method

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We have calculated the low-temperature series for the second moment of the correlation function in $d = 3$ Ising model to order $u^{26}$ and for the free energy of Absolute Value Solid-on-Solid (ASOS) model to order $u^{23}$, using the finite-lattice method.

1. INTRODUCTION

Recently the low-temperature series of $d = 3$ Ising model or equivalently strong-coupling series of $d = 3$ $Z_2$ lattice gauge theory have been extended to higher orders using finite-lattice method. The finite-lattice method to obtain series expansion was originally developed by Neef and Enting \cite{1}. In this method the expansion series of the free energy density in the infinite volume limit to an order in the expansion-parameter is given by the appropriate linear combination of the free energies on finite-size lattices. The coefficients of the linear combination are given by Möbius inversion \cite{2}. This procedure avoids the problem involved in the graphical method, in which it is rather difficult to give the algorithm for listing all the diagrams completely that contribute to the relevant order of the series. The finite-lattice method is more effective in lower dimensions \cite{3} and it was applied intensively to two-dimensional systems \cite{4}. The Möbius inversion was introduced to the field of lattice gauge theory by Mack \cite{5} and was used to make a partial resummation of the strong-coupling expansion of the theory \cite{6,7}. Fujiwara and one of the author (H. A.) developed the finite-lattice method of strong-coupling expansion and its full resummation in lattice gauge theory using the Möbius inversion \cite{8,9}. The full-resummation method was applied to the calculation of the free energy and string tension for $SU(2)$ \cite{10} and $Z_2$ \cite{11} lattice gauge theory in three dimensions, and of the free energy for $Z_2$ \cite{12} and $SU(2)$ \cite{13} lattice gauge theory in four dimensions. The method was also applied to obtain the strong-coupling expansion series of the free energy to order $u^{20}$, string tension to order $u^{13}$ \cite{14} and the mass gap to order $u^{11}$ \cite{15} in $d = 3$ $Z_2$ lattice gauge theory, and of the free energy in $d = 4$ $Z_2$ lattice gauge theory to order $u^{11}$ \cite{16}, where $u = \tanh^2 \beta_{gauge}$ and $\beta_{gauge}$ is the inverse gauge coupling constant.

The $d = 3$ $Z_2$ lattice gauge theory is dual to the $d = 3$ Ising model and the strong-coupling series of the free energy, string tension and mass gap in the former is exactly the same as the low-temperature series of the free energy, surface tension and true inverse correlation-length in the latter, respectively, if the expansion-parameter $u = \tanh^2 \beta_{gauge}$ in the former is read as $u = \exp (-4\beta)$ and $\beta = J/kT$ in the latter. Thus the finite-lattice method of strong-coupling expansion in $d = 3$ lattice gauge theory is exactly the same as that of low-temperature expansion in $d = 3$ Ising model.

Recently the low-temperature series of $d = 3$ Ising model were extended to higher orders by the finite-lattice method using transfer matrix formalism for calculating the partition function based on building up finite-size lattices one site at a time, which was originally invented by Enting \cite{17}. The calculated quantities are the free energy to order $u^{25}$ \cite{18}, the free energy, magnetization and susceptibility to order $u^{26}$ \cite{19}, the surface tension to order $u^{17}$ \cite{20} and the true inverse correlation-length to order $u^{15}$ \cite{21}.

We should mention that Vohwinkel \cite{22} ob-
tained low-temperature Ising series for free energy, magnetization and susceptibility, which are longer than those of the reference [23] or [14] using a modification of the shadow-lattice technique. His method appears to be so powerful even in three dimensions, although it is more efficient for higher dimensional systems. We think, however, that the finite-lattice method can still be the method of choice for a range of problems in three dimensions [3].

We report here the application of the finite-lattice method to calculate the low-temperature series for the second moment of the correlation function in $d = 3$ Ising model to order $u^{20}$ [20] and for the free energy of Absolute Value Solid-on-Solid (ASOS) model to order $u^{23}$ [21].

### 2. SECOND MOMENT OF THE CORRELATION FUNCTION IN D=3 ISING MODEL

The point in the algorithm of the low-temperature expansion for the second moment $\mu_2$ is the following. We consider the partition function

$$Z(\beta, h, \eta, \gamma_1, \gamma_2, \gamma_3) = \sum_{\{s_i\}} \exp (-\mathcal{H}), \quad (1)$$

with the Hamiltonian

$$\mathcal{H} = \beta \sum_{\langle ij \rangle} s_is_j + \sum_i (h + \gamma_1 x_i + \gamma_2 y_i + \gamma_3 z_i + \eta \sigma_i^z) s_i, \quad (2)$$

for the three-dimensional lattice with a volume $V$. The second moment is given by the second derivative of the free energy density in the infinite-volume limit as

$$\mu_2 = \lim_{V \to \infty} \frac{2}{V} \left( \frac{\partial^2}{\partial \eta \partial \eta} - \frac{\partial^2}{\partial \gamma_1^2} - \frac{\partial^2}{\partial \gamma_2^2} - \frac{\partial^2}{\partial \gamma_3^2} \right) \times \ln Z(\beta, h, \eta, \gamma_1, \gamma_2, \gamma_3)|_{h=\eta=\gamma_1=\gamma_2=\gamma_3=0}. \quad (3)$$

Then the finite-lattice method can be applied to the low-temperature expansion of the free energy density, which should be calculated to the order of $u^N \eta^2$ or $u^N \gamma_i^2$ ($i = 1, 2, 3$) to obtain the second-moment series to $u^N$.

We have obtained the series for the second moment to order $u^{26}$, extending the previous result of order $u^{15}$ calculated by Tarko and Fisher [22] using the standard graphical method and of order $u^{19}$ calculated by Vohwinkel and Weisz [23] using the shadow-lattice technique. Vohwinkel and Weisz also gave an estimate of the series to order $u^{29}$. Our exact series to order $u^{19}$ coincides with their exact result and our exact coefficients from order $u^{20}$ to $u^{26}$ are consistent with their estimate within an accuracy of 1 per cent for each of the orders. It gives the low-temperature series for the second-moment correlation length squared $\Lambda_2 = \xi_2^2$ to order $u^{23}$, when combined with the known low-temperature series of the susceptibility [14,29]. This is longer by six terms than the low-temperature series for the true correlation length squared $\Lambda^2$ that was derived from the true inverse correlation-length given in Ref. [18].

An analysis of the obtained series by inhomogeneous differential approximants gives critical exponents $2\nu^* + \gamma = 2.509(38)$ for the second moment and $2\nu' = 1.247(19)$ for the correlation length squared. These are consistent with the results from high-temperature series, $\epsilon$-expansion and Monte Carlo analysis and we can conclude that the scaling relation between the high- and low-temperature exponents as $\nu = \nu'$ and $\gamma = \gamma'$ is satisfied within an accuracy of about 2 per cent. The details can be seen in the reference [20].

### 3. FREE ENERGY OF ASOS MODEL

In the low-temperature expansion of the surface tension to order $u^{17}$ [13] we found that the series coefficients change their sign at the order of $u^{13}$. The surface tension in three-dimensional Ising model or the string tension in three- or higher-dimensional lattice gauge theory suffers from the roughening transition [24,29] and it is expected to exhibit Kosterlitz-Thouless type singularity like

$$f(u) = A(u) \exp \left[-c(u_r - u)^{-1/2}\right] + B(u). \quad (4)$$

It has the essential singularity at the roughening transition point $u_r$. As was pointed out by Hasenbusch and Pinn [27], the sign-change is just the signal of the K-T type singularity in equation (4). In fact if we expand the function (4) in terms of $u$ we would obtain a series with a sign-change. The order of the sign-change depends on
Absolute Value Solid-on-Solid (ASOS) model is an approximation of the interface of \( d = 3 \) Ising model. It neglects overhangs and disconnected parts and is also expected to exhibit K-T type phase transition. The free energy of ASOS model just corresponds to the surface tension of \( d = 3 \) Ising model and is expected to behave like (1). Hasenbusch and Pinn [27] calculated the low-temperature series for the free energy of ASOS model to order \( u^{12} \) using the finite-lattice method, extending the previous work by Weeks et al [28] and found the expected sign-change at the order of \( u^{11} \).

We have extended the low-temperature series for the free energy to \( u^{23} \) using the finite-lattice method [21], which is longer by 11 terms than that by Hasenbusch and Pinn. In the longer series the sign-change is seen only at the order of \( u^{11} \) found by Hasenbusch and Pinn. We have fitted the obtained series to the Taylor expansion of the function (4) with \( A(u) = \text{constant} \) and \( B(u) = 0 \). A good fitting is obtained if we take \( u_r = 0.214 \) and \( c = 0.527 \). This fitted value of \( u_r \) should be compared with \( u_r = 0.207(9) \) from the series analysis of the surface width and \( u_r = 0.1994(1) \) from the Monte Carlo Renormalization group analysis [24]. These results confirm that the roughening phase-transition of ASOS model is of K-T type.

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