A note on the spin connection representation of gravity

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Abstract

The formulation of gravity in 3 + 1 dimensions in which the spin connection is the basic field (\(\omega\)-frame) leads to a theory with first and second class constraints. Here, the Dirac brackets for the second class constraints are evaluated and the Dirac algebra of first class constraints is found to be the usual algebra associated to space-time reparametrizations and tangent space rotations. This establishes the classical equivalence with the vierbein approach (\(e\)-frame). The explicit form of the path integral for this theory is given and the quantum equivalence with the \(e\)-frame is also established.

1 Introduction

The standard description of the spacetime geometry in General Relativity uses the metric tensor \(g_{\mu\nu}\) as the fundamental field. In hamiltonian form, the action is a functional of the spatial metric \(h_{ij}\) and its canonical momentum \(\pi^{ij}\), as well as four Lagrange multipliers associated with spatial reparametrizations, \(N^i\) (shifts) and normal deformations, \(N^\perp\) (lapse) \([1, 2]\).

The Einstein-Hilbert action can also be written in terms of the spin connection \(\omega^{ab}_{\mu}\), the tetrad field \(e^a_\nu\) and their exterior derivatives (first order formalism) \([3, 4]\). In this form, the hamiltonian construction needed to identify the dynamical degrees of freedom is not straightforward. The torsion

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\(^1\)In this analysis the torsion tensor is assumed to vanish identically.
tensor does not vanish identically, but only as a consequence of the classical equations. Thus, it is not legitimate to eliminate the spin connection for the vierbein and one is left with a theory for 40 independent fields $\omega_i^{ab}$ (18), and $e_i^a$ (12). Also, many of the fields do not have time derivatives: out of the 40, only 12 have time derivatives in the lagrangian. This gives rise to a number of first and second class constraints.

1.1 The two frames

An additional difficulty is that there are two natural choices of coordinates and momenta, which have radically different phase space and constraint structure. Thus, two options arise, depending on whether the lagrangian involves only time derivatives of the vierbein ($e$-frame), or the spin connection ($\omega$-frame). These two choices are related by a canonical transformation (they differ by a total derivative) and therefore should be classically identical in content\footnote{It is extremely difficult to establish the quantum mechanical equivalence between canonically related formulations of a theory, and the equivalence may not even exist. It has been shown that quantum mechanics could be formulated in a way that is invariant under the simpler class of point canonical transformations \cite{5}, but a similar proof for general canonical transformations is not yet known.}. However, as the corresponding phase spaces are so radically different, proving the equivalence even at the classical level is non trivial.

To make the discussion more concrete, let us recall some facts about the first order formulation of gravity. The first order Lagrangian is the Einstein-Hilbert four-form (wedge product of forms is understood)

$$L_{E-H} = \epsilon_{abcd} R_{ab}^{e} e^{d} + dB,$$

(1)

where $R_{ab}^{e} = d\omega_{ab}^{e} + \omega_{ac}^{e} \omega^{cb}$ is the curvature two-form, $\omega$ is the spin connection one-form, $e$ is the vierbein one-form and $dB$ stands for some arbitrary boundary term. Different choices of $B(e, \omega)$ give rise to different choices of canonical coordinates (frames). Two natural choices are:

1.1.1 The $e$-frame

In the hamiltonian analysis of this action in first order form \cite{6} the spin connection splits in two pieces. One of them corresponds to the canonical
momentum of the tetrad field, and the other corresponds to a set of auxiliary variables that can be eliminated from their own equations of motion in terms of the tetrad. The resulting hamiltonian is a functional of the tetrad and its canonical momentum only, and the spin connection drops out. In this way the usual vierbein formulation of gravity is obtained [7].

1.1.2 The $\omega$-frame

Alternatively, one can start from the Einstein-Hilbert action, eliminate the tetrad field and build a hamiltonian action that depends on the spin connection and its canonical momentum only [8]. A preliminary discussion of the equivalence between the $\omega$ and $e$-frames was presented in [9]. In this letter we want address some points of the analysis in the $\omega$-frame.

1.1.3 The Ashtekar approach

The alternative approach to canonical gravity proposed by Ashtekar [10] in the past decade is yet another canonically equivalent description of General Relativity. The Ashtekar frame is obtained through a complex canonical transformation from the $e$-frame [11]. It has been often discussed whether Ashtekar’s theory is quantum mechanically equivalent to standard metric gravity and the answer still seems uncertain and possibly irrelevant. As we show here, the $\omega$ and $e$-frames are not only equivalent classically through a real canonical transformation but, if there were a quantum description for either one, it would be equivalent to the quantum description for the other.

2 First order formalism (in the $\omega$ frame)

As shown in [8, 9], dropping the boundary term in (1), the first order action for gravity in 3+1 dimensions can also be written as

$$I[w, e] = \int (\dot{\omega}_k^{ab} e_{abcd} \epsilon_i^{ijk} e_j^d + \omega_0^{ab} J_{ab} + e_0^a P_a),$$  \(2\)

\footnote{Our conventions are that $\epsilon^{ijk} = \pm 1, 0$ is a tensor density of weight 1 (i.e., it transforms like a tensor of third rank times $\sqrt{g}$). Hence, $\epsilon_{ijk} = g_{il} g_{jm} g_{kn} \epsilon^{lmn}$ is also a tensor density of weight 1, but it takes values $\pm g, 0$.}
where $J_{ab} = \epsilon_{abcd}e^i^j_kT_{ij}^c e_d^k = D_k(\epsilon_{abcd}e^i^j_k e^e_d^k)$, $P_a = (\epsilon_{abcd}e^i^j_k P^b_j e^e_d^k)$. If $e_0^a$ is decomposed as $e_0^a = N \eta^a + N^i e_i^a$, where $\eta^a$ is the normal to spacelike surfaces, $\eta_a e_i^a = 0$, the action can be written in terms of the $\omega^{ab}_i$ and its canonically conjugate momentum $P^k_{ab} = \epsilon_{abcd}e^i^j_k e^e_d^c$, in the form

$$I = \int (\dot{\omega}^{ab}_k P^k_{ab} - \omega^{ab}_0 J_{ab} + NH_\perp + N^i H_i + \mu_{ij} \phi^{ij}),$$

where

$$H_\perp = g^{-1/2}P^k_{ab} P^{ij}_b R_{ij}^a,$$

$$H_i = P^j_{ab} R^{ij}_b - \omega^{ab}_i J_{ab},$$

$$J_{ab} = D_i P^i_{ab},$$

and

$$\phi^{ij} = \epsilon_{abcd} P^a_{ab} P^b_{cd}.$$

Here $\omega^{ab}_0$, $N$, and $N^i$ are Lagrange multipliers corresponding to the constraints $J_{ab} = H_\perp = H_i = 0$, and $g = det(g_{ij})$, $g_{ij} \equiv e_i^a e_j^a$.

The presence of the constraint $\phi^{ij} = 0$ deserves some discussion. The substitution of $P^k_{ab}$ for $e_i^a e_j^i e^d_k$ conceals the fact that there are only 12 independent fields ($e_i^a$) and not 18 ($P^k_{ab}$) in the phase space. The elimination of the 6 spurious fields is enforced by the 6 conditions $\phi^{ij} = 0$. The Jacobian of the transformation $e_i^a \rightarrow P^i_{ab}$ is $\Omega^{ab}_{cd} = 2\epsilon_{abcd}e^i^j_k e^e_d^k$, which has maximum rank (twelve) on configurations for which the local orthonormal frames $e_i^a$ are generic, that is, they span a 3-dimensional volume (see below).

Once the second class constraints have been eliminated, $H_i$ and $J_{ab}$ become the generators of spatial diffeomorphism and local rotations, respectively.

Preservation in time of the constraint $\phi^{ij} = 0$ implies a new constraint

$$\chi^{kl}(x, y) = \{\phi^{kl}(x), H_\perp(y)\} = g^{-1/2}D_i (P^{(k}_{ec} P^l_{ab} P^{ci}_{f} \epsilon^{abfe} \delta(x, y),$$

where the parentheses indicate symmetrization in $k, l$. Preservation of $\chi^{kl} = 0$ in turn, implies

$$N\{H_\perp, \chi^{kl}\} + \mu_{mn}\{\phi^{mn}, \chi^{kl}\} = 0.$$
These are equations for the Lagrange multipliers, which can be solved for $\mu$ because the constraints $\phi^{mn} = \chi^{kl} = 0$ obey a second class algebra,

$$
\begin{align*}
\{ \phi^{mn}(x), \phi^{kl}(y) \} &= 0 \\
\{ \chi^{mn}(x), \chi^{kl}(y) \} &\neq 0 \\
\{ \chi^{ij}(x), \phi^{mn}(y) \} &= g^{-1/2} B^{ijmn}(x, y) \\
&= g^{-1/2}(G^{ijmn}(\tilde{g}(x)) + G^{ijmn}(\phi(x))) \delta(x - y).
\end{align*}
$$

(10)

where $G^{ijmn}(A)$ is the inverse supermetric for a symmetric matrix $A^{ij}$.

$$
G^{ijkl}(A) = 2A^{ij} A^{kl} - A^{il} A^{kj} - A^{ik} A^{lj}.
$$

(11)

The precise form of $\{ \chi, \chi \}$ is not essential as we will see below. The matrix $B^{ijmn}(x, y)$ has a formal inverse $B^{-1}_{ijmn}$ given by the series

$$
B = G(g) - G(g) G^{-1}(\phi) G(g) + G(g) G^{-1}(\phi) G(g) G^{-1}(\phi) G(g) + ... 
$$

(12)

where we have defined $G \sim G_{ijkl}$, $B \sim B_{ijkl}$, $G^{-1} \sim G^{ijkl}$, etc.

Obviously $B^{ijmn}$ coincides with the supermetric $G^{ijmn}$ on the constraint surface $\phi = 0, \chi = 0$. In this way we can solve (9) for $\mu_{ij}$,

$$
\mu_{ij} = \frac{1}{2} B^{ijmn} N \{ H_{\perp} , \chi^{mn} \}.
$$

(13)

Thus no new constraints appear from the preservation in time of $\chi$. The initial $H_{\perp}$ has nonvanishing Poisson brackets with $\phi$ or $\chi$, but the modified one

$$
\tilde{H}_{\perp} = H_{\perp} - \frac{1}{N} \mu_{ij} \phi^{ij},
$$

(14)

do es.

### 3 Dirac brackets

The second class constraints can be eliminated through Dirac bracket defined by

$$
\{ U, V \}^* = \{ U, V \} - \{ U, \varphi^\alpha \} C_{\alpha \beta} \{ \varphi^\beta, V \},
$$

(15)

\footnote{Here, the metric is not defined yet, $\tilde{g}^{ij}$ is just a shorthand for $P_{ab}^{k} P_{bik}$ which will eventually be related to the canonical metric through $\tilde{g}^{ij} = -8g_{ij}$.}
where $C_{\alpha\beta}$ is the inverse of the Dirac matrix $C^{\alpha\beta} = \{\phi^\alpha, \phi^\beta\}$, where $\phi^\alpha$ denote generic second class constraints. In our case, the Dirac matrix

$$C^{\alpha\beta}(x, y) = \begin{pmatrix} \{\chi^{ij}(x), \chi^{mn}(y)\} & \{\phi^{ij}(x), \phi^{mn}(y)\} \\ \{\phi^{ij}(x), \chi^{mn}(y)\} & \{\phi^{ij}(x), \phi^{mn}(y)\} \end{pmatrix},$$  

has the form $\begin{pmatrix} A & B^{-1} \\ -B^{-1} & 0 \end{pmatrix}$, because $\{\phi^{ij}, \phi^{mn}\} = 0$. Its inverse takes the form

$$C^{-1}_{\alpha\beta} = \begin{pmatrix} 0 & -B \\ B & BAB \end{pmatrix},$$

or

$$C^{-1}_{\alpha\beta} = \begin{pmatrix} 0 & -B_{ijmn} \\ B_{ijmn} & \chi^{pq} \chi^{kl} B_{klmn} \end{pmatrix}. \quad (17)$$

The Dirac bracket for two arbitrary functionals $U, V$ is given by:

$$\{U, V\}^* = \{U, V\} - \{U, \chi\} B\{\phi, V\} - \{U, \phi\} B\{\chi, V\} - \{U, \phi\} B\{\phi, \phi\} B\{\phi, V\},$$

where sum and integration over discrete and continuous indices is assumed. It can be shown that when $U$ and $V$ belong to the set $\{\tilde{H}, H_i, J_{ab}\}$, the second term of the right hand side of (18) vanishes on the constraint surface $\phi = 0, \chi = 0$. In particular, direct substitution in (18) yields

$$\{\tilde{H}, \tilde{H}\}^* = \{H, H\}^* = \{H, H\}.$$

and using the results of [8], we finally have

$$\{H, H\}^* \sim g^{ij} H_j \partial_i \delta(x, y).$$

In the same way, the complete Dirac algebra can be shown to be given by

$$\{H[N], H[M]\}^* = \int [(\partial_i N) M - (\partial_i M) N] g^{ij}(P) H_j,$$

$$\{H[N], H^i\}^* = \int (M^i \partial_i N - N \partial_i M^i) H,$$

$$\{H^i[N], H^j\}^* = \int (N^i \partial_i M^m - M^i \partial_i N^m) H_m,$$

$$\{J[N^{ab}], J[M^{cd}]\}^* = \int J[(M \times N)^{ab}],$$

6
Thus, the Dirac algebra reduces to a direct sum of the usual algebra of
spacetime diffeomorphism plus tangent space rotations.

Note that when $P^{b}_{ab}$ is replaced by its expression in terms of the tetrad,
the $\phi_{ij}$ constraints vanish identically, but the secondary constraints $\chi_{ij}$ do not. In the vierbein frame it can also be shown that prior to eliminating
the auxiliary variables, apart from $J_{ab}, H_{\perp}, H_{i}$ the constraints

$$\gamma_{ij} = E^i_a (\epsilon^{mnj}) T^a_{mn} = 0,$$

are found, where $T^a_{ij}$ are the spatial components of the torsion tensor, and
$E^i_a \equiv e_{aj} g^{ij}$. Equation (27) is one of the field equations, from which the auxiliary variables $\lambda_{ij}$ can be eliminated. Replacing $P^i_{ab} = \epsilon^{ijk} \epsilon_{abcd} e_j e_k$ in the
definition (8), $\chi_{ij}$ can be identified with $\gamma_{ij}$ in the e-frame.

The algebra (21–26) is the same as the one found in the vielbein-frame
once the constraint $\gamma_{ij}$ is strongly set equal to zero. The two frames can be
compared and contrasted in the following table:

|                     | e-frame                                      | $\omega$-frame                             |
|---------------------|----------------------------------------------|--------------------------------------------|
| dynamical variables | $e^a_i, \pi^i_a = \epsilon^{ijk} \epsilon_{abcd} e_j e_k \omega_k$ | $\omega^a_i, P^i_{ab} = \epsilon^{ijk} \epsilon_{abcd} e_j e_k$ |
| (q,p)               | (12) , (12)                                  | (18) , (18)                                |
| First class constraint | $H_{\perp}, H_{i}, J_{ab}$                   | $H_{\perp}, H_{i}, J_{ab}$                 |
| second class constraint | ———                                          | $\chi_{ij}, \phi^{kl}$                     |
| prop. degrees of freedom | 12 - 4 - 6 = 2                               | 18 - 10 - $\frac{1}{2}$ 12 = 2            |
(Here \(\gamma^{ij}\) has been eliminated in the \(e\)-frame). The number of propagating degrees of freedom is

\[ g = c - f - \frac{1}{2}s, \]

where \(c\) is the number of coordinates, \(f\) the number of first class constraints and \(s\) the number of second class constraints.

4 Path integral

We now consider the path integral for this system. As shown in [14], the path integral for a system with second class constraints \(\chi, \phi\) has a measure proportional to

\[ \delta(\chi)\delta(\phi)\sqrt{\det C^{\alpha\beta}}, \]

where \(C^{\alpha\beta}\) is the Dirac matrix. In our case, \(C^{\alpha\beta}\) as given in (16) and (10), yields

\[ \sqrt{\det C^{\alpha\beta}} = \det(G^{ijmn}(\tilde{g}) + G^{ijmn}(\phi)). \]

The delta functions in (28) restrict the integration to the constraint surface \(\phi = 0, \chi = 0\), so path integral reads

\[ Z = \int [D\omega_{ab}] [D\omega_{ik}] [D\omega_{ik}] \delta(\phi^{ij}) \delta(\chi^{mn}) \det M_{\alpha\beta} \delta(H_{\perp}) \delta(H_{i}) \delta(J_{ab}) \exp \frac{i}{\hbar} S \]

with

\[ S = \int \omega_{ab}^{ik} P_{ik}^{k}, \]

and \(M_{\alpha\beta}\) is the matrix of Poisson brackets

\[ M_{\alpha\beta} = \{ F_{\alpha}, \varphi_{\beta} \}^{*}, \]

where \(F_{\alpha}\) are gauge condition for the first class constraint set \(\varphi_{\beta} = \{ H_{\perp}, H_{i}, J_{ab} \}\).

5 The \(\omega-e\) transformation

Consider now the following transformation, which maps the 18 coordinates \(\omega_{ab}^{ik}\) and their 18 canonically conjugate momenta \(P_{ab}^{k}\) into 12 \(e_{i}^{a}\)’s, 12 \(\pi_{a}^{i}\)’s, 6
auxiliary variables $\lambda_{mn}$ and $6 \rho_{mn}$ ($\lambda_{mn}$ and $\rho_{mn}$ are symmetric and m,n take the values 1,2,3.)

$$\omega_k^{ab} = \Theta_k^{ab} c j c + U_k^{mn} \lambda_{mn}$$  \hfill (33)

$$P_k^{ab} = \frac{1}{2} \Omega_k^{ab} c j c + V_k^{ab} \rho_{mn}.$$  \hfill (34)

Here $\Theta$ and $\Omega$ are rectangular matrices,

$$\Theta_i^{ab} c j = \frac{1}{8 \sqrt{g}} [\epsilon_i^{[a} \eta_{b]} c_j - \epsilon_i^{[a} \epsilon_{j]} \eta_{c}^c - 2 \epsilon_i^{[a} \eta_{b]} \epsilon_{c]}^c]$$.  \hfill (35)

$$\Omega^i_j = 2 \epsilon_{abcd} e^d_k \epsilon_{ijk},$$  \hfill (36)

where the square brackets indicate antisymmetrization. $U$ and $V$ are null vectors for $\Omega$ and $\Theta$, given by

$$U_k^{ab} mn = \frac{1}{2} \delta_i^{(m} \epsilon_{n)kl} e_k^a e_l^b,$$  \hfill (37)

$$V_k^{ab} mn = \frac{1}{g} E_a^r E_b^s \delta_{rs(m} \delta_{n)}^i.$$  \hfill (38)

These objects satisfy the following relations

$$\Omega_k^{ab} c j \Theta_k^{cd} m = \delta_d^c \delta_j^i,$$  \hfill (39)

$$\Omega_k^{ab} c j U_k^{mn} = 0,$$  \hfill (40)

$$\Theta_k^{ab} c j V_k^{mn} = 0,$$  \hfill (41)

$$U_k^{ab} mn V_k^{ab} pq = \delta_{(mn)}^{(pq)}.$$  \hfill (42)

One can think of $\Theta$ and $\Omega$ as a collection of twelve vectors –labeled by the indices ($^a_i$) and ($^i_a$) respectively–, in an 18-dimensional vector space with components ($^{ab}_i$) and ($^i_{ab}$), respectively. By the same token, $U$ and $V$ are six vectors (labeled by the index (mn)) in an 18-dimensional vector space.

In this sense the properties (39,...42) are nothing but orthogonality relations among the vectors $\Theta$, $\Omega$, $U$ and $V$. These relations imply the following completeness relation

$$\Theta_i^{ab} e \Omega_{cd}^i l + U_i^{ab} mn V_{cd}^j mn = \delta_{[cd]}^{[ab]} \delta_{j]}^i,$$  \hfill (43)
which will be used in what follows. In this way, the 18 vectors $\Theta_{ij}^{ab}$, $U_{i}^{ab}$ are a basis of the space of contravariant vectors $L_{i}^{ab}$, while $\Omega_{ij}^{ab}$, $V_{ij}^{ab}$ are a basis for the dual (covariant vectors, $L_{i}^{ab}$). Thus, the field transformations (33), (34) correspond to the expansions of $\omega_{i}^{ab}$ and $P_{i}^{ab}$ in the contravariant and covariant bases, respectively.

As shown in the appendix, using (33), (34) the path integral (30) can now be written in terms of the coordinates of the $e$-frame as

$$Z = \int [De_{i}] [D\pi_{i}^{a}] \det M_{\alpha\beta} \delta(H) \delta(H) \delta(J_{ab}) \exp \frac{i}{\hbar} S,$$

which is the path integral one would write in the $e$-frame. This shows the equivalence between quantum theories one would obtain in the two frames. The different constraints $H_{\perp}$, $H_{i}$, and $J_{ab}$ can be written explicitly in terms of $e$-frame variables, as

$$\frac{1}{2} H_{\perp} = \eta^{a} \partial_{i} \pi_{a}^{i} - \frac{1}{2} E_{ij}^{a} \partial_{i} e_{k}^{d} \eta^{b} \pi_{b}^{j} - G_{ab}^{\perp} \pi_{a}^{i} \pi_{b}^{j} - g^{3/2} G^{mnpq} \lambda_{mn}^{0} \lambda_{pq}^{0},$$

$$N_{m} H_{m} = N_{m} \left[ \frac{1}{2} (g^{-1} E_{ij}^{a} \partial_{i} e_{k}^{a} \epsilon_{ij}^{ab} - E_{ij}^{a} \partial_{i} e_{k}^{a} \epsilon_{ij}^{ab}) + \eta^{a} \partial_{i} \pi_{a}^{i} + G_{ij}^{ab} \pi_{a}^{i} \pi_{b}^{j} \right] + \frac{1}{2} N_{m} (e_{i}^{a} e_{j}^{b} J_{ab} \epsilon_{ij}^{ab}) \lambda_{mn}^{0},$$

$$J_{ab} = 2 \epsilon_{abcd} \partial_{i}^{c} \epsilon_{k}^{d},$$

where

$$\lambda_{pq}^{0} = \frac{1}{2g} C_{pqmn} E_{a}^{(m} \partial_{i} e_{j}^{a \epsilon_{ij}^{ab})},$$

$$G_{ab}^{\perp} = \frac{1}{16 \sqrt{g}} [e_{i}^{a} e_{j}^{b} - 2 e_{i}^{a} e_{j}^{b} - g_{ij} \eta^{a} \eta^{b}],$$

and

$$G_{ij}^{ab} = \frac{1}{16 \sqrt{g}} [g_{ij} \eta^{a} e_{m}^{b} + 2 g_{im} (e_{j}^{a} \eta^{b} - e_{j}^{b} \eta^{a})].$$

Finally, the kinetic term $P_{ab} \dot{\omega}_{i}^{ab}$ in the action $S$ reduces via the $e$-ω transformation to the usual $e$-frame kinetic term $\pi_{a}^{i} e_{i}^{a}$. This completes the classical and quantum equivalence between the $\omega$ and $e$ frames.

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Appendix: Equivalence of the measure in the $\omega$ and $e$ frames

Using (33,34), the path integral in the $\omega$ frame given in (30) reads

\[
Z = \int \frac{\mathcal{D}e_a}{\mathcal{D}\pi_i} \frac{\mathcal{D}\lambda_{mn}}{\mathcal{D}\rho_{mn}} \frac{J}{\det M_{\alpha\beta}} \delta(H_\perp) \delta(H_i) \delta(J_{ab}) \exp \frac{\bar{h}S}{\hbar} \tag{51}
\]

where $J$ is the determinant of the Jacobian matrix of the transformation $(\omega, P) \rightarrow (e, \pi, \lambda, \rho)$.

The different constraints must be written in terms of the new variables. Consider first $\phi^{ij} = \epsilon^{abcd} P_{ia} P_{jb}$. Using (34), $\phi^{ij}$ it is easily shown that

\[
\phi^{ij} = 32 \rho^{ij},
\]

so that $\delta(\phi^{ij}) = \delta(32 \rho^{ij})$. In the same way, substituting (34) in (8), the constraints $\chi^{ij}$ become

\[
\chi^{ij} = \frac{g^{1/2}}{2} E^{(i} e^{j)mn} T^{a}_{mn},
\]

which are recognized as the second class constraints in the $e$-frame (27). The $\chi$ constraints can be rewritten substituting $\omega$ from (33) in $T^{a}_{ij}(\omega, e)$ in the form

\[
\chi^{ij} = -2g^{3/2} G^{ijmn}(g)(\lambda_{mn} - \lambda_{0mn}^{0}),
\]

where $\lambda_{mn}$ are given by (48), then

\[
\delta(\chi^{ij}) = \delta(-2g^{3/2} G^{ijmn}(g)(\lambda_{mn} - \lambda_{0mn}^{0})) \tag{55}
\]

\[
= \frac{\delta(\lambda_{mn} - \lambda_{0mn}^{0})}{\det(-2g^{3/2} G^{ijmn}(g))}.
\]

The metric $\tilde{g}^{ij} = P_{ai}^{i} P_{bj}^{j}$ becomes, after using (34), $\tilde{g}^{ij} = -8g^{ij}$, so $G^{ijmn}(\tilde{g}) = 64g^{2} G^{ijmn}(g)$.
Thus, the path integral reads, up to a normalization constant,

\[ Z[e, \pi, \lambda, \rho] = \int [De][D\pi][D\lambda_{mn}][D\rho_{mn}] \, J \, \delta(\rho_{mn}) \delta(\lambda_{mn} - \lambda_{mn}^0) \]

\[ \times \det M_{\alpha\beta} \, \delta(H_\perp) \delta(H_i) \delta(J_{ab}) \, \exp \frac{i}{\hbar} S. \]  

(56)

Integrating over \( \lambda \) and \( \rho \) one obtains

\[ Z[e, \pi] = \int [De][D\pi] \, J_0 \, \det M_{\alpha\beta} \, \delta(H_\perp) \delta(H_i) \delta(J_{ab}) \, \exp \frac{i}{\hbar} S, \]  

(57)

where \( J_0 \) is the Jacobian evaluated at \( \lambda = \lambda_0 \) and \( \rho = 0 \). Now we will show that this Jacobian is one, that is, the measure is invariant under the transformation (33), (34). In what follows we denote de collective indeces \((ab) \rightarrow A, (i) \rightarrow a \) and \((mn) \rightarrow \alpha\). Then, varying the fields in the transformation (33, 34) yields

\[ \delta\omega^A = \frac{\partial \Theta^{Ab}}{\partial e^a} \pi_b \, \delta e^a + \frac{\partial U^{Ab}}{\partial e^a} \lambda_b \, \delta e^a + \Theta^{Aa} \delta \pi_a + U^{Aa} \delta \lambda_a, \]

(58)

\[ \delta P_A = \omega_A a \delta e^a + \frac{\partial V_{A\beta}}{\partial e^a} \rho^\beta \, \delta e^a + 0 \, \delta \pi_a + V_{Aa} \, \delta \rho^a, \]

(59)

so that the Jacobian matrix is given by

\[
J = \begin{bmatrix}
\frac{\partial \Theta^{Ab}}{\partial e^a} \pi_b + \frac{\partial U^{Ab}}{\partial e^a} \lambda_b & 0 & \Theta^{Aa} & U^{Aa} \\
\omega_{Aa} & \omega_{Aa} & V_{Aa} & 0 & 0 \\
\end{bmatrix},
\]

(60)

which has the block form \( J = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \), so that \( det(J) = det(C) \, det(B) = det(C)det(B^t) = det(CB^t) \). In our case

\[
CB^t = [(\omega_{Aa} + \frac{\partial V_{A\alpha}}{\partial e^a} \rho^\beta) \, V_{Aa}] \begin{bmatrix} \Theta^{Ba} & U^{Ba} \\
\end{bmatrix} = \omega_{Aa} \Theta^{Ba} + V_{Aa} U^{Ba} + \frac{\partial V_{A\alpha}}{\partial e^a} \rho^\beta \Theta^{Ba}.
\]

(61)

The first two terms reproduce exactly the completneness relation (43), so the Jacobian is

\[ J = det(\delta_A + \frac{\partial V_{A\alpha}}{\partial e^a} \rho^\beta \Theta^{Ba}). \]

(62)

Finally, evaluating the Jacobian on the constraint surface \( \rho = 0, \lambda = \lambda_0 \), one finds \( J|_{\rho=0} = det(\delta_A^B) = 1 \), and the path integral can be finally written as

\[ Z = \int [De][D\pi] \, \det M_{\alpha\beta} \, \delta(H_\perp) \delta(H_i) \delta(J_{ab}) \, \exp \frac{i}{\hbar} S, \]

(63)

which is the expected expression for the path integral in the \( e \)-frame.