The Gauss linking number in quantum gravity

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Abstract

We show that the exponential of the Gauss (self) linking number of a knot is a solution of the Wheeler-DeWitt equation in loop space with a cosmological constant. Using this fact, it is straightforward to prove that the second coefficient of the Jones Polynomial is a solution of the Wheeler-DeWitt equation in loop space with no cosmological constant. We perform calculations from scratch, starting from the connection representation and give details of the proof. Implications for the possibility of generation of other solutions are also discussed.

1 Introduction

The introduction of the loop representation for quantum gravity has made it possible for the first time to find solutions to the Wheeler-DeWitt equation (the quantum Hamiltonian constraint) and therefore to have possible candidates to become physical wavefunctions of the gravitational field. In the loop representation the Hamiltonian constraint has nonvanishing action only on functions of intersecting loops. It was first argued that by considering wavefunctions with support on smooth loops one could solve the constraint straightforwardly [1, 2]. However it was later realized that such solutions are associated with degenerate metrics (metrics with zero determinant) and this posed inconsistencies if one wanted to couple the theory [3]. For instance if one considered general relativity with a cosmological constant it turns out that nonintersecting loop states also solve the Wheeler-DeWitt equation for arbitrary values of the cosmological constant. This does not appear as reasonable since different values of the cosmological constant lead to widely different behaviors in general relativity.

Therefore the problem of solving the Wheeler-DeWitt equation in loop
At least a triple intersection is needed to have a state associated with a nondegenerate three dimensional metric in the loop representation. And even in this case the metric is nondegenerate only at the point of intersection.

Space is far from solved and has to be tackled by considering the action of the Hamiltonian constraint in the loop representation on states based on (at least triply-)intersecting loops, as depicted in figure 1.

Although performing these kind of direct calculations is now possible, since well defined expressions for the Hamiltonian constraint exist in terms of the loop derivative \( \frac{\delta}{\delta A_i^a} \) (see also [7]) and actually some solutions have been found with this approach \( \ref{eq:1.2} \), another line of reasoning has also proved to be useful.

This other approach is based on the fact that the exponential of the Chern-Simons term based on Ashtekar’s connection,

\[
\Psi^\text{CS}_A[A] = \exp \left( -\frac{\Lambda}{6} \int d^3 x (A_i^a \partial_i A_i^a + \frac{2}{3} A_i^a A_j^b A_k^c \varepsilon^{ijk} \varepsilon^{abc}) \right) \quad (1.1)
\]

is a solution of the Wheeler-DeWitt equation in the connection representation with a cosmological term,

\[
\hat{H} = \varepsilon^{ijk} \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b} F_{ab}^k + \frac{\Lambda}{6} \varepsilon^{ijk} \varepsilon^{abc} \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b} \frac{\delta}{\delta A_k^c} \quad (1.2)
\]

(the first term is just the usual Hamiltonian constraint with \( \Lambda = 0 \) and the second term is just \( \Lambda \det g \) where \( \det g \) is the determinant of the spatial part of the metric written in terms of triads.

To prove this fact one simply needs to notice that for the Chern-Simons state introduced above,

\[
\frac{\delta}{\delta A_i^a} \Psi^\text{CS}_A[A] = -\frac{6}{\Lambda} \varepsilon^{abc} F_{bi}^c \Psi^\text{CS}_A[A] \quad (1.3)
\]
and therefore the rightmost functional derivative in the cosmological constant term of the Hamiltonian produces a term in $F_{ab}$ that exactly cancels the contribution from the vacuum Hamiltonian constraint.

This result for the Chern-Simons state in the connection representation has an immediate counterpart in the loop representation, since the transform of the Chern-Simons state into the loop representation,

$$\Psi_{CS}^\Lambda(\gamma) = \int dA \Psi_{CS}^\Lambda[A] W_\gamma[A]$$

(1.4)

can be interpreted as the expectation value of the Wilson loop in a Chern-Simons theory,

$$\Psi_{CS}^\Lambda(\gamma) = \int dA e^{-\frac{6}{\Lambda} S_{CS}} W_\gamma[A] = \langle W_\gamma \rangle$$

(1.5)

with coupling constant $\frac{6}{\Lambda}$, and we know due to the insight of Witten that this coincides with the Kauffman bracket of the loop. Therefore the state

$$\Psi_{CS}^\Lambda(\gamma) = \text{Kauffman Bracket}_\Lambda(\gamma)$$

(1.6)

should be a solution of the Wheeler-DeWitt equation in loop space.

This last fact can actually be checked in a direct fashion using the expressions for the Hamiltonian constraint in loop space of reference. In order to do this we make use of the identity,

$$\text{Kauffman Bracket}_\Lambda(\gamma) = e^{\Lambda \text{Gauss}(\gamma)} \text{Jones Polynomial}_\Lambda(\gamma)$$

(1.7)

which relates the Kauffman Bracket, the Gauss (self) linking number and the Jones Polynomial. The Gauss self linking number is framing dependent, and so is the Kauffman Bracket. The Jones Polynomial however, is framing independent. This raises the issue of up to what extent statements about the Kauffman Bracket being a state of gravity are valid since one expects states of quantum gravity to be truly diffeomorphism invariant objects and framing is always dependent on an external device which should conflict with the diffeomorphism invariance of the theory. Unfortunately it is not clear at present how to settle this issue since it is tied to the regularization procedures used to define the constraints. We will return to these issues in the final discussion.

We now expand both the Jones Polynomial and the exponential of the Gauss linking number in terms of $\Lambda$ and get the expression,

$$\text{Kauffman Bracket}_\Lambda(\gamma) = 1 + \text{Gauss}(\gamma) \Lambda +$$

(1.8)

$$+ (\text{Gauss}(\gamma)^2 + a_2(\gamma)) \Lambda^2 +$$

$$+ (\text{Gauss}(\gamma)^3 + \text{Gauss}(\gamma)^2 a_2(\gamma) + a_3(\gamma)) \Lambda^3 +$$

$$+ \ldots$$
where $a_2, a_3$ are the second and third coefficient of the infinite expansion of the Jones Polynomial in terms of $\Lambda$. $a_2$ is known to coincide with the second coefficient of the Conway polynomial [7].

One could now apply the Hamiltonian constraint in the loop representation with a cosmological constant to the expansion (1.9) and one would find that certain conditions have to be satisfied if the Kauffman Bracket is to be a solution. Among them it was noticed [8] that

$$\hat{H}_0 a_2(\gamma) = 0 \quad (1.9)$$

where $\hat{H}_0$ is the Hamiltonian constraint without cosmological constant (more recently it has also been shown that $\hat{H}_0 a_3(\gamma) = 0$ [9] but we will not discuss it here).

Summarizing, the fact that the Kauffman Bracket is a solution of the Wheeler-DeWitt equation with cosmological constant seems to have as a direct consequence that the coefficients of the Jones Polynomial are solutions of the vacuum ($\Lambda = 0$) Wheeler-DeWitt equation!. This partially answers the problem of framing we pointed out above. Even if one is reluctant to accept the Kauffman Bracket as a state because of its framing dependence, it can be viewed as an intermediate step of a framing-dependent proof that the Jones Polynomial (which is a framing independent invariant) solves the Wheeler-DeWitt equation (it should be stressed that we only have evidence that the first two nontrivial coefficients are solutions).

The purpose of this paper is to present a rederivation of these facts from a different, and to our understanding simpler, perspective. We will show that the (framing-dependent) knot invariant,

$$\Psi^G_{\Lambda}(\gamma) = e^{\Lambda \text{Gauss}(\gamma)} \quad (1.10)$$

is also a solution of the Wheeler-DeWitt equation with cosmological constant. It can be viewed as an "Abelian limit" of the Kauffman Bracket (more on this in the conclusions). Given this fact, one can therefore consider their difference divided by $\Lambda^2$,

$$D_{\Lambda}(\gamma) = \frac{\text{Kauffman Bracket}_{\Lambda}(\gamma) - \Psi^G_{\Lambda}(\gamma)}{\Lambda^2} \quad (1.11)$$

which is also a solution of the Wheeler-DeWitt equation with a cosmological constant. This difference is of the form,

$$D_{\Lambda}(\gamma) = a_2(\gamma) + (a_3(\gamma) + \text{Gauss}(\gamma)^2 a_2) \Lambda + \ldots \quad (1.12)$$

Now, this difference is a state for all values of $\Lambda$, in particular, for $\Lambda = 0$. This means that $a_2(\gamma)$ should be a solution of the Wheeler-DeWitt equation. This confirms the proof given in references [4, 8].

Therefore we see that by noticing that the Gauss linking number is a state with cosmological constant, it is easy to prove that the second coeff-
ficient of the infinite expansion of the Jones Polynomial (which coincides with the second coefficient of the Conway Polynomial) is a solution of the Wheeler-DeWitt equation with \( \Lambda = 0 \).

The rest of this paper will be devoted to a detailed proof that the exponential of the Gauss linking number solves the Wheeler-DeWitt equation with cosmological constant. To this aim we will derive expressions for the Hamiltonian constraint with cosmological constant in the loop representation. We will perform the calculation explicitly for the case of a triply-self intersecting loop, the more interesting case for gravity purposes (it should be noticed that all the arguments presented above were independent of the number and order of intersections of the loops, we just present the explicit proof for a triple intersection since in three spatial dimensions it represents the most generic type of intersection).

Apart from presenting this new state, we think the calculations exhibited in this paper should help the reader get into the details of how these calculations are performed and make an intuitive contact between the expressions in the connection and the loop representation.

In section 2 we derive the expression of the Hamiltonian constraint (with a cosmological constant) in the loop representation for a triply intersecting loop in terms of the loop derivative. In section 3 we write an explicit analytic expression for the Gauss linking number and prove that it is a state of the theory. We end in section 4 with a discussion of the results.

### 2 The Wheeler-DeWitt equation in terms of loops

Here we derive the explicit form in the loop representation of the Hamiltonian constraint with a cosmological constant. The derivation proceeds along the following lines. Suppose one wants to define the action of an operator \( \hat{O}_L \) on a wavefunction in the loop representation \( \Psi(\gamma) \). Applying the transform,

\[
\hat{O}_L \Psi(\gamma) \equiv \int dA \hat{O}_L W_\gamma[A] \Psi[A]
\]

(2.1)

the operator \( \hat{O} \) in the right member acts on the loop dependence of the Wilson loop. On the other hand, this definition should agree with,

\[
\hat{O}_L = \int dAW_\gamma[A] \hat{O}_C \Psi[A]
\]

(2.2)

where \( \hat{O}_C \) is the connection representation version of the operator in question. Therefore, it is clear that,

\[
\hat{O}_L W_\gamma[A] \equiv \hat{O}^\dagger_C W_\gamma[A]
\]

(2.3)

where \( ^\dagger \) means the adjoint operator with respect to the measure of integration \( dA \). If one assumes that the measure is trivial, the only effect of taking the adjoint is to reverse the factor ordering of the operators.
Concretely, in the case of the Hamiltonian constraint (without cosmological constant)

\[ \hat{H}_C = \epsilon^{ijk} \frac{\delta}{\delta A^i_a} \frac{\delta}{\delta A^j_b} F^{k}_{ab} \]  

(2.4)

and therefore

\[ \hat{H}_L W_\gamma[A] \equiv \epsilon^{ijk} F^k_{ab} \frac{\delta}{\delta A^i_a} \frac{\delta}{\delta A^j_b} W_\gamma[A]. \]  

(2.5)

We now need to compute this quantity explicitly. For that we need the expression of the functional derivative of the Wilson loop with respect to the connection,

\[ \frac{\delta}{\delta A^i_a(x)} W_\gamma[A] = \oint dy^a \delta(y - x) \text{Tr}[\text{Pexp}(\int_o^y dz^b A_b) \tau^i \text{Pexp}(\int_y^o dz^b A_b)] \]  

(2.6)

where \( o \) is the basepoint of the loop. The Wilson loop is therefore "broken" at the point of action of the functional derivative and a Pauli matrix (\( \tau^i \)) is inserted. It is evident that with this action of the functional derivative the Hamiltonian operator is not well defined. We need to regularize it,

\[ \hat{H}_L W_\gamma[A] = \lim_{\epsilon \to 0} f_\epsilon(x - z) \epsilon^{ijk} F^k_{ab} \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(z)} W_\gamma[A]. \]  

(2.7)

where \( f_\epsilon(x - z) \to \delta(x - z) \) when \( \epsilon \to 0 \). Caution should be exercised, since such point-splitting breaks the gauge invariance of the Hamiltonian. There are a number of ways of fixing this situation in the language of loops. One of them is to define the Hamiltonian inserting pieces of holonomies connecting the points \( x \) and \( z \) between the functional derivatives to produce a gauge invariant quantity [1]. Here we will only study the operator in the limit in which the regulator is removed, therefore we will not be concerned with these issues. A proper calculation would require their careful study.

It is immediate from the above definitions that the Hamiltonian constraint vanishes in any regular point of the loop, since it yields a term \( dy^a dy^b F^i_{ab} \) which vanishes due to the antisymmetry of \( F^i_{ab} \) and the symmetry of \( dy^a dy^b \) at points where the loop is smooth. However, at intersections there can be nontrivial contributions. Here we compute the contribution at a point of triple self-intersection,

\[ \epsilon^{ijk} F^k_{ab}(x) \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(z)} W_{\gamma_1 \circ \gamma_2 \circ \gamma_3}[A] = \epsilon^{ijk} F^k_{ab}(x) \times \]  

(2.8)

\[ \times (\gamma_1 \gamma_2 W_{\gamma_1 \gamma_2 \gamma_3}[A] + \gamma_2 \gamma_3^b W_{\gamma_2 \gamma_3 \gamma_1}[A] + \gamma_1 \gamma_3^b W_{\gamma_1 \gamma_3 \gamma_2}[A]) \]

where we have denoted \( \gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3 \) where \( \gamma_i \) are the "petals" of the loop as indicated in figure 1. By \( W_{\gamma_1 \gamma_2 \gamma_3}[A] \) we really mean take the holonomy from the basepoint along \( \gamma_1 \) up to just before the intersection.
point, insert a Pauli matrix, continue along $\gamma_1$ to the intersection, continue along $\gamma_2$ and just before the intersection insert another Pauli matrix, continue to the intersection and complete the loop along $\gamma_3$. One could pick "after" the intersection instead of "before" to include the Pauli matrices and it would make no difference since we are concentrating in the limit in which the regulator is removed in which the insertions are done at the intersection. By $\dot{\gamma}^1_a$ we mean the tangent to the petal number 1 just before the intersection (where the Pauli matrix was inserted). This is just a shorthand for expressions like

$$\oint dy^a \delta(x - y)$$

when the point $x$ is close to the intersection, so strictly speaking $\dot{\gamma}^1_a$ really is a distribution that is nonvanishing only at the point of intersection.

One now uses the following identity for traces of $SU(2)$ matrices,

$$\epsilon^{ijk} W_{\alpha \tau^j} [A] W_{\beta \tau^k} [A] = \frac{1}{2} (W_{\alpha \tau^j} [A] W_{\beta \tau^k} [A] - W_{\alpha \tau^k} [A] W_{\beta \tau^j} [A])$$

which is a natural generalization to the case of loops with insertions of the $SU(2)$ Mandelstam identities,

$$W_{\alpha} [A] W_{\beta} [A] = W_{\alpha \beta} [A] + W_{\alpha \bar{\beta}} [A]$$

where $\bar{\beta}$ means the loop opposite to $\beta$. The result of the application of these identities to the expression of the Hamiltonian is,

$$\hat{H} W_{\gamma \tau} [A] = \frac{1}{2} F_{ab} \left( \dot{\gamma}_1 \dot{\gamma}_2 W_{\gamma_2 \gamma_3 \gamma_1 \tau} [A] - \dot{\gamma}_1 \dot{\gamma}_3 W_{\gamma_1 \gamma_2 \gamma_3 \tau} [A] + \dot{\gamma}_2 \dot{\gamma}_3 W_{\gamma_3 \gamma_1 \gamma_2 \tau} [A] \right).$$

This expression can be further rearranged making use of the loop derivative. The loop derivative $\Delta_{\tau} \left( \pi^x \right)$ is the differentiation operator that appears in loop space when one considers two loops to be "close" if they differ by an infinitesimal loop appended through a path $\pi^x$ going from the basepoint to a point of the manifold $x$ as shown in figure 2. Its definition is

$$\Psi(\pi^x \delta \gamma \pi^0_\gamma) = (1 + \sigma^{ab} \Delta_{\tau^a} (\pi^x_\gamma)) \Psi(\gamma)$$

where $\sigma^{ab}$ is the element area of the infinitesimal loop $\delta \gamma$ and by $\pi^x_\gamma \delta \gamma \pi^0_\gamma$ we mean the loop obtained by traversing the path $\pi$ from the basepoint to $x$, the infinitesimal loop $\delta \gamma$, the path $\pi$ from $x$ to the basepoint and then the loop $\gamma$.

We will not discuss all its properties here. The only one we need is that the loop derivative of a Wilson loop taken with a path along the loop is given by,

$$\Delta_{\tau} (\gamma^x_\alpha) W_{\gamma} [A] = \text{Tr}[F_{ab}(x)\text{Pexp}(\oint dy^a A_c)]$$

which reflects the intuitive notion that a holonomy of an infinitesimal loop
Fig. 2. The loop defining the loop derivative

is related with the field tensor. Therefore we can write expressions like,

\[ F_{ab}^i W_{\gamma_2 \gamma_3 \gamma_1 \tau}^\tau [A] \]  \hspace{1cm} (2.14)

as,

\[ \Delta_{ab}(\gamma_1) W_{\gamma_2 \gamma_3 \gamma_1}^\tau [A] \]  \hspace{1cm} (2.15)

and the final expression for the Hamiltonian constraint in the loop representation can therefore be read off as follows,

\[ \hat{H} \Psi(\gamma) = \frac{1}{2} (\xi_{a 1}^{a 2} \Delta_{ab}(\gamma_1) \Psi(\bar{\gamma}_2 \gamma_3 \gamma_1) + \gamma_{a 1}^{a 2} \Delta_{ab}(\gamma_3) \Psi(\bar{\gamma}_1 \gamma_2 \gamma_3) + \xi_{a 1}^{a 2} \Delta_{ab}(\gamma_3) \Psi(\bar{\gamma}_3 \gamma_1 \gamma_2)) \]  \hspace{1cm} (2.16)

This expression could be obtained by particularizing that of reference \[4\] to the case of a triple self-intersecting loop and rearranging terms a bit using the Mandelstam identities. However we thought that a direct derivation for this particular case would be useful for pedagogical purposes.

We now have to find the loop representation form of the operator corresponding to the determinant of the metric in order to represent the second term in (1.2). We proceed in a similar fashion, first computing the action of the operator in the connection representation,

\[ \epsilon^{ijk} \epsilon_{abc} \delta_{\delta A_1}^{\delta A_2} \delta_{\delta A_3}^{\delta A_4} W_{\gamma}^\gamma [A] = \epsilon_{abc} \epsilon^{ijk} \gamma_{a 1}^{a 2} \gamma_{b 2}^{b 3} \gamma_{c 3}^{c 4} (W_{\gamma_1 \gamma_2 \gamma_3 \gamma_4}^\tau [A]) \]  \hspace{1cm} (2.18)

This latter expression can be rearranged with the following identity between
holonomies with insertions of Pauli matrices,

$$\epsilon^{ijk} W_{\gamma_1 \gamma_2 \gamma_3} [A] = \frac{1}{4} W_{\gamma_1 \gamma_2 \gamma_3} [A] + W_{\gamma_2 \gamma_3 \gamma_1} [A]. \quad (2.19)$$

It is therefore immediate to find the expression of the determinant of the metric in the loop representation,

$$\det q\Psi(\gamma) = -\frac{1}{4} \epsilon_{abc} \gamma^a_1 \gamma^b_2 \gamma^c_3 \left( \Psi(\gamma_1 \gamma_2 \gamma_3) + \Psi(\gamma_2 \gamma_1 \gamma_3) + \Psi(\gamma_2 \gamma_3 \gamma_1) \right). \quad (2.20)$$

With these elements we are in a position to perform the main calculation of this paper, to show that the exponential of the Gauss self linking number of a loop is a solution of the Hamiltonian constraint with a cosmological constant.

3 The Gauss (self) linking number as a solution

In order to be able to apply the expressions we derived in the previous section for the constraints to the Gauss self linking number we need an expression for it in terms of which it is possible to compute the loop derivative. This is furnished by the well known integral expression,

$$\text{Gauss}(\gamma) = \frac{1}{4\pi} \oint_{\gamma} dx^a \oint_{\gamma} dy^b \epsilon_{abc} \frac{(x - y)^c}{|x - y|^3} \quad (3.1)$$

where $|x - y|$ is the distance between $x$ and $y$ with a fiducial metric. This formula is most well known when the two loop integrals are computed along different loops. In that case the formula gives 1 if the loops are linked or 0 if the are not. In the present case we are considering the expression of the linking of a curve with itself. This is in general not well defined without the introduction of a framing [7].

We will rewrite the above expression in a more convenient fashion,

$$\text{Gauss}(\gamma) = \int d^3x \int d^3y X^a(x, \gamma) X^b(y, \gamma) g_{ab}(x, y) \quad (3.2)$$

where the vector densities $X$ are defined as,

$$X^a(x, \gamma) = \oint_{\gamma} dz^a \delta(z - x) \quad (3.3)$$

and the quantity $g_{ab}(x, y)$ is the propagator of a Chern-Simons theory [7],

$$g_{ab}(x, y) = \epsilon_{abc} \frac{(x - y)^c}{|x - y|^3}. \quad (3.4)$$

For calculational convenience it is useful to introduce the notation,

$$\text{Gauss}(\gamma) = X^{ax}(\gamma) X^{by}(\gamma) g_{ax by} \quad (3.5)$$

where we have promoted the point dependence in $x, y$ to a "continuous index" and assumed a "generalized Einstein convention" which means sum
The loop derivative that appears in the definition of the Hamiltonian is evaluated along a path that follows the loop over repeated indices $a$, $b$ and integrate over the three manifold for repeated continuous indices. This notation is also faithful to the fact that the index $a$ behaves as a vector density index at the point $x$, that is, it is natural to pair $a$ and $x$ together.

The only dependence on the loop of the Gauss self linking number is through the $X'$s, so we just need to compute the action of the loop derivative on one of them to be in a position to perform the calculation straightforwardly. In order to do this we apply the definition of loop derivative, that is, we consider the change in the $X$ when one appends an infinitesimal loop to the loop $\gamma$ as illustrated in figure 3. We partition the integral in a portion going from the basepoint to the point $z$ where we append the infinitesimal loop, which we characterize as four segments along the integral curves of two vector fields $u^a$ and $v^b$ of associated lengths $\epsilon_1$ and $\epsilon_2$ and then we continue from there back to the basepoint along the loop. Therefore,

$$
(1 + \sigma^{ab}_k \Delta_{ab}(\gamma_z^x) X^{ax}_x(\gamma)) \equiv \int_{\gamma_z^x} dy^a \delta(x - y) + \epsilon_1 u^a \delta(x - z) + \\
+ \epsilon_2 v^b (1 + u^c \partial_c) \delta(x - z) - \epsilon_1 u^a (1 + (u^c + v^c) \partial_c) \delta(x - z) - \\
- \epsilon_2 v^a (1 + v^c \partial_c) \delta(x - z) + \int_{\gamma_z^x} dy^a \delta(x - y).
$$

The last and first term combine to give back $X^{ax}(\gamma)$ and therefore one can read off the action of the loop derivative from the other terms. Rearranging one gets,

$$
\Delta_{ab}(\gamma_z^x) X^{ax}(\gamma) = \partial_a \delta_x^c (x - z) \quad (3.7)
$$

where the notation $\delta_{5}^{x}(x - z)$ stands for $\delta_{5}^{x}(x - z)$, the product of the
Kronecker and Dirac deltas. This is really all we need to compute the action of the Hamiltonian. We therefore now consider the action of the vacuum (Λ = 0) part of the Hamiltonian on the exponential of the Gauss Linking number. The action of the loop derivative is,

\[ \Delta_{ab}(\gamma^x) \exp \left( X^{cy}(\gamma)X^{dz}(\gamma)g_{cy}dz \right) = (3.8) \]

\[ 2\partial_i\delta_{ij}(x - y)X^{dz}(\gamma)g_{ cy}dz \exp \left( X^{cw}(\gamma)X^{fu'}(\gamma)g_{cw'fu} \right) \]

Now we must integrate by parts. Using the fact that \( \partial_iX^{ax}(\gamma) = 0 \) and the definition of \( g_{cy}dz \), we get

\[ \Delta_{ab}(\gamma^x) \exp \left( X^{cy}(\gamma)X^{dz}(\gamma)g_{cy}dz \right) = (3.9) \]

\[ 2\epsilon_{abc}(X^{ce}(\gamma_1) + X^{ce}(\gamma_2) + X^{ce}(\gamma_3)) \exp \left( X^{cy}(\gamma)X^{dz}(\gamma)g_{cy}dz \right) \]

Therefore the action of the Hamiltonian constraint on the Gauss linking number is,

\[ \hat{H}e^{\Lambda Gauss} = \epsilon_{abc}\gamma_1^{\alpha}\gamma_2^{\beta}\gamma_3^{\gamma}e^{\Lambda Gauss} \quad (3.10) \]

where we again have replaced the distributional tangents at the point of intersection by an expression only involving the tangents. The expression is only formal since in order to do this a divergent factor should be kept in front. We assume such factors coming from all terms to be similar and therefore ignore them.

It is straightforward now to check that applying the determinant of the metric one the Gauss linking number one gets a contribution exactly equal and opposite by inspection from expression (2.20). This concludes the main proof of this paper.

4 Discussion

We showed that the exponential of the Gauss (self) linking number is a solution of the Hamiltonian constraint of quantum gravity with a cosmological constant. This naturally can be viewed as the "Abelian" limit of the solution given by the Kauffman bracket.

What about the issue of regularization? The proof we presented is only valid in the limit where \( \epsilon \to 0 \), that is, when the regulator is removed. If one does not take the limit the various terms do not cancel. However, the expression for the Hamiltonian constraint we introduced is also only valid when the regulator is removed. A regularized form of the Hamiltonian constraint in the loop representation is more complicated than the expression we presented. If one is to point split, infinitesimal segments of loop should be used to connect the split points to preserve gauge invariance and a more careful calculation would be in order.
What does all this tell us about the physical relevance of the solutions? The situation is remarkably similar to the one present in the loop representation of the free Maxwell field \([11]\). In that case, as here, there are two terms in the Hamiltonian that need to be regularized in a different way (in the case of gravity, the determinant of the metric requires splitting three points whereas the Hamiltonian only needs two). As a consequence of this, it is not surprising that the wavefunctions that solve the constraint have some regularization dependence. In the case of the Maxwell field the vacuum in the loop representation needs to be regularized. In fact, its form is exactly the same as that of the exponential of the Gauss linking number if one replaces the propagator of the Chern-Simons theory present in the latter by the propagator of the Maxwell field. This similarity is remarkable. The problematic is therefore the same, the wavefunctions inherit regularization dependence since the regulator does not appear as an overall factor of the wave equation.

How could these regularization ambiguities be cured? In the Maxwell case they are solved by considering an "extended" loop representation in which one allows the quantities \(X^{ax}\) to become smooth vector densities on the manifold without reference to any particular loop \([12]\). In the gravitational case such construction is being actively pursued \([13]\), although it is more complicated. It is in this context that the present solutions really make sense. If one allows the \(X's\) to become smooth functions the framing problem disappears and one is left with a solution that is a function of vector fields and only reduces to the Gauss linking number in a very special (singular) limit. It has been proved that the extension of the Kauffman Bracket and Jones polynomials to the case of smooth density fields are solutions of the extended constraints. A similar proof goes through for the extended Gauss linking number. In the extended representation, there are additional multivector densities needed in the representation. The "Abelian" limit of the Kauffman bracket (the Gauss linking number) appears as the restriction of the "extended" Kauffman bracket to the case in which higher order multivector densities vanish. It would be interesting to study if such a limit could be pursued in a systematic way order by order. It would certainly provide new insights into how to construct nonperturbative quantum states of the gravitational field.

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