Regge pole description of scattering of scalar and electromagnetic waves by a Schwarzschild black hole

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We revisit the problem of plane monochromatic waves impinging upon a Schwarzschild black hole from complex angular momentum techniques. We focus more particularly on the differential scattering cross sections associated with scalar and electromagnetic waves. We provide an exact representation of these cross sections by replacing the discrete sum over integer values of the angular momentum which defines their partial wave expansions by a background integral in the complex angular momentum plane plus a sum over the Regge poles of the $S$-matrix. We show that, surprisingly, the background integral is numerically negligible for intermediate and high reduced frequencies (i.e., in the short-wavelength regime) and, as a consequence, that the cross sections can be reconstructed, for arbitrary scattering angles, in terms of Regge poles with a very good agreement. We show in particular that, for large values of the scattering angle, a small number of Regge poles permits us to describe the black-hole glory and that, by increasing the number of Regge poles, we can reconstruct very efficiently the differential scattering cross sections for small and intermediate scattering angles and therefore describe the orbiting oscillations. In fact, in this wavelength regime, the sum over Regge poles allows us to extract by resummation the physical information encoded in the partial wave expansion defining a differential scattering cross section and, moreover, to overcome the difficulties linked to its lack of convergence due to the long-range nature of the fields propagating on the black hole. We finally discuss the role of the background integral for low reduced frequencies (i.e., in the long-wavelength regime).

\section{I. INTRODUCTION}

Studies concerning the scattering of waves by black holes (BHs) are mainly based on partial wave expansions (see, e.g., Ref. [1]). This is due to the high degree of symmetry of the BH spacetimes usually considered and physically or astrophysically interesting. For example, the Schwarzschild BH is a static spherically symmetric solution of the vacuum Einstein’s equations while the Kerr BH is a stationary axisymmetric solution of these equations with, as a consequence, the separability of wave equations on these gravitational backgrounds (see, e.g., Ref. [2]). Even if the approach based on partial wave expansions is natural and very effective in the context of BH physics (see, e.g., Refs. [15–30]), it presents some flaws. Due to the long-range nature of the fields propagating on a BH, some partial wave expansions encountered are formally divergent (see, e.g., Ref. [1]) and, moreover, it is in general rather difficult to interpret physically the results described in terms of partial wave expansions. These problems can be overcome by using complex angular momentum (CAM) techniques (analytic continuation of partial wave expansions in the CAM plane, effective resummations involving the poles of the $S$-matrix in the CAM plane, i.e., the so-called Regge poles, and the associated residues, semiclassical interpretations of Regge pole expansions, etc.). Such techniques, which proved to be very helpful in quantum mechanics (see, e.g., Refs. [3, 4]), in electromagnetism and optics (see, e.g., Refs. [4–8]), in acoustics and seismology (see, e.g., Refs. [9, 10]) and in high energy physics (see, e.g., Refs. [11–14]) to describe and analyze resonant scattering are now also used in the context of BH physics (see, e.g., Refs. [15–30]).

In this article we revisit the problem of plane monochromatic waves impinging upon a Schwarzschild BH from CAM techniques. More precisely, we focus on the differential scattering cross sections associated with scalar and electromagnetic waves. It should be recalled that the partial wave expansions of these cross sections have been obtained a long time ago by Matzner [31] for the scalar field and by Mashhoon [32, 33] and Fabbri [34] for the electromagnetic field and that many additional works have since been done which have theoretically and numerically completed these first investigations (see, e.g., Refs. [15, 16, 35–42] for some articles directly relevant to our own study). Here, we construct an exact representation of these differential scattering cross sections by replacing the discrete sum over integer values of the angular momentum which defines their partial wave expansions by a background integral in the CAM plane plus a sum over the Regge poles of the $S$-matrix. Surprisingly, we find that the background integral is numerically negligible for intermediate and high reduced frequencies (i.e., in the short-wavelength regime) and, as a consequence, that the differential scattering cross sections can be described in terms of Regge poles with a very good agreement for.

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arbitrary scattering angles. In fact, in this wavelength regime, the sum over Regge poles allows us to extract by resummation the physical information encoded in the partial wave expansion defining a differential scattering cross section and, moreover, to overcome the difficulties linked to its lack of convergence due to the long-range nature of the fields propagating on the BH. We show in particular that, for large values of the scattering angle, i.e., in the backward direction, a small number of Regge poles permits us to describe the BH glory (see Refs. [37] and [16] for semiclassical interpretations) and that, by increasing the number of Regge poles, we can reconstruct very efficiently the differential scattering cross sections for small and intermediate scattering angles and therefore describe the “orbiting” oscillations (see Refs. [38] and [16] for semiclassical interpretations).

It is important to point out that our work extends (but also corrects) the studies of Andersson and Thylwe [15, 16] where we can find the first application of CAM techniques in BH physics. In Ref. [15], Andersson and Thylwe have considered the scattering of scalar waves by a Schwarzschild BH from a theoretical point of view and adapted the CAM formalism to this problem. They have established some properties of the Regge poles and of the S-matrix in the CAM plane. In Ref. [16], Andersson has used this formalism to interpret semiclassically the BH glory and the orbiting oscillations. He has, in particular, considered “surface waves” propagating close to the unstable circular photon (graviton) orbit at \( r = 3M \), i.e., near the so-called photon sphere, and associated them with the Regge poles. It is interesting to note that we have developed this point of view in a series of papers establishing that the complex frequencies of the weakly damped quasinormal modes (QNMs) are Breit-Wigner-type resonances generated by the surface waves previously mentioned. We have then been able to construct semiclassically the spectrum of the QNM complex frequencies from the Regge trajectories, i.e., from the curves traced out in the CAM plane by the Regge poles as a function of the frequency [17, 19, 29], establishing on a “rigorous” basis the physically intuitive interpretation of the Schwarzschild BH QNMs suggested, as early as 1972, by Goebel [43] (see Refs. [20–23] for the extension of these results to other BHs and to massive fields). Moreover, from the Regge trajectories and the residues of the greybody factors, we have described analytically the high-energy absorption cross section for a wide class of BHs endowed with a photon sphere and explained its oscillations in terms of the geometrical characteristics (orbital period and Lyapunov exponent) of the null unstable geodesics lying on the photon sphere [23–25]. All these results highlight the interpretative power of CAM techniques in BH physics.

In this article, we consider not only the case of the scalar field but, in addition, we treat the important case of the electromagnetic field which is of fundamental interest with the emergence of multimessenger astronomy as well as with the possibility to produce experimentally, in a near future, the first BH image or, more precisely, to “photograph” with the Event Horizon Telescope the shadow of the supermassive BH at the Galactic Center [44, 45]. Moreover, we correct the results numerically obtained by Andersson in Ref. [16] for the residues associated to the Regge poles (see also Ref. [18]) and we show that the background integral in the CAM plane is numerically negligible for intermediate and high reduced frequencies. It is important to recall that, in Refs. [15, 16], Andersson and Thylwe tried to extract some physical information of this background integral and that, in electromagnetism and optics [4–8] as well as in acoustics and seismology [9, 10]), it can be associated semiclassically (i.e., for high reduced frequencies) with incident and reflected rays. In the context of scattering by BHs (or, more precisely, if we focus on differential scattering cross sections in the short-wavelength regime), it plays only a minor role.

Our paper is organized as follows. In Sec. II, by means of the Sommerfeld-Watson transform [4–6] and Cauchy’s theorem, we construct exact CAM representations of the differential scattering cross sections for plane scalar and electromagnetic waves impinging upon a Schwarzschild BH from their partial wave expansions. These CAM representations are split into a background integral in the CAM plane and a sum over the Regge poles of the S-matrix involving the associated residues. In Sec. III, we obtain numerically, for various reduced frequencies, the Regge poles of the S-matrix, the associated residues and the background integral. This permits us to reconstruct, for these particular frequencies of the impinging waves, the differential scattering cross sections of the BH and to show that, in the short-wavelength regime, they can be described from the Regge pole sum alone with a very good agreement. We also discuss the role of the background integral for low reduced frequencies, i.e., in the long-wavelength regime. In the Conclusion, we summarize our main results and briefly consider possible extensions of our work. In an Appendix, we discuss the numerical evaluation of the background integrals. Due to the long-range nature of the fields propagating on a Schwarzschild BH, these integrals in the CAM plane (as the partial wave expansions) suffer of a lack of convergence. We overcome this problem, i.e., we accelerate their convergence, by extending to integrals the iterative method developed in Ref. [46] for partial wave expansions.

Throughout this article, we adopt units such that \( G = c = 1 \). We furthermore consider that the exterior of the Schwarzschild BH is defined by the line element \( ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \) where \( f(r) = 1 - 2M/r \) and \( M \) is the mass of the BH while \( t \in [-\infty, +\infty], r \in [2M, +\infty[ \), \( \theta \in [0, \pi] \) and \( \varphi \in [0, 2\pi] \) are the usual Schwarzschild coordinates. We finally assume a time dependence \( \exp(-i\omega t) \) for the plane monochromatic waves considered.
II. DIFFERENTIAL SCATTERING CROSS SECTIONS FOR SCALAR AND ELECTROMAGNETIC WAVES, THEIR CAM REPRESENTATIONS AND THEIR REGGE POLE APPROXIMATIONS

In this section, we recall the partial wave expansions of the differential scattering cross sections for plane monochromatic scalar and electromagnetic waves impinging upon a Schwarzschild BH and we construct exact CAM representations of these cross sections by means of the Sommerfeld-Watson transform [4–6] and Cauchy’s theorem. These CAM representations are split into a background integral in the CAM plane and a sum over the Regge poles of the S-matrix involving the associated residues.

A. Partial wave expansions of differential scattering cross sections

We recall that, for the scalar field, the differential scattering cross section is given by [31]

$$\frac{d\sigma}{d\Omega} = |f(\omega, \theta)|^2$$

(1)

where

$$f(\omega, \theta) = \frac{1}{2i\omega} \sum_{\ell=0}^{\infty} (2\ell + 1) |S_\ell(\omega) - 1| P_\ell(\cos \theta)$$

(2)

denotes the scattering amplitude and that, for the electromagnetic field, the differential scattering cross section can be written in the form [32, 33] (see also Refs. [34, 42])

$$\frac{d\sigma}{d\Omega} = |A(\omega, \theta)|^2$$

(3)

where the scattering amplitude is given by

$$A(\omega, \theta) = D_\theta B(\omega, \theta)$$

(4)

with

$$B(\omega, \theta) = \frac{1}{2i\omega} \sum_{\ell=1}^{\infty} \frac{(2\ell + 1)}{\ell(\ell + 1)} |S_\ell(\omega) - 1| P_\ell(\cos \theta)$$

(5)

and

$$D_\theta = -(1 + \cos \theta) \frac{d}{d\cos \theta} \left[ (1 - \cos \theta) \frac{d}{d\cos \theta} \right]$$

(6a)

$$= -\left( \frac{d^2}{d\theta^2} + \frac{1}{\sin \theta} \frac{d}{d\theta} \right).$$

(6b)

The expression (4)-(6) takes into account the two polarizations of the electromagnetic field. In Eqs. (2) and (5), the functions $P_\ell(\cos \theta)$ are the Legendre polynomials [47]. We also recall that the $S$-matrix elements $S_\ell(\omega)$ appearing in Eqs. (2) and (5) can be defined from the modes $\phi_{\omega \ell}^{in}$ solutions of the homogeneous Regge-Wheeler equation

$$\left[ \frac{d^2}{dr^2} + \omega^2 - V_{\ell}(r) \right] \phi_{\omega \ell} = 0$$

(7)

(here $r^* = r + 2M \ln[r/(2M) - 1] + \text{const}$ denotes the tortoise coordinate) where

$$V_{\ell}(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{\ell(\ell + 1)}{r^2} + (1 - s^2) \frac{2M}{r^3} \right)$$

(8)

(here $s = 0$ corresponds to the scalar field and $s = 1$ to the electromagnetic field) which have a purely ingoing behavior at the event horizon $r = 2M$ (i.e., for $r_+ \to -\infty$)

$$\phi_{\omega \ell}^{in}(r) \sim e^{-i\omega r}$$

(9a)

and, at spatial infinity $r \to +\infty$ (i.e., for $r_+ \to +\infty$), an asymptotic behavior of the form

$$\phi_{\omega \ell}^{in}(r) \sim A_\ell^{-}(\omega)e^{-i\omega r} + A_\ell^{+}(\omega)e^{+i\omega r}.$$  

(9b)

In this last equation, the coefficients $A_\ell^{(-)}(\omega)$ and $A_\ell^{(+)}(\omega)$ are complex amplitudes and we have

$$S_{\ell}(\omega) = e^{i(\ell + 1)\pi} \frac{A_\ell^{(+)}(\omega)}{A_\ell^{(-)}(\omega)}.$$  

(10)

B. CAM representation of the scattering amplitude for scalar waves

By means of the Sommerfeld-Watson transformation [4–6] which permits us to write

$$\sum_{\ell=0}^{+\infty} (-1)^\ell F(\ell) = \frac{i}{\pi} \int_{C} d\lambda F(\lambda - 1/2) \frac{\lambda}{\cos(\pi \lambda)}$$

(11)

for a function $F$ without any singularities on the real $\lambda$ axis, we can replace in Eq. (2) the discrete sum over the ordinary angular momentum $\ell$ by a contour integral in the complex $\lambda$ plane (i.e., in the complex $\ell$ plane with $\lambda = \ell + 1/2$). By noting that $P_\ell(\cos \theta) = (-1)^\ell P_\ell(-\cos \theta)$, we obtain

$$f(\omega, \theta) = \frac{1}{2i\omega} \int_{C} d\lambda \frac{\lambda}{\cos(\pi \lambda)} \times [S_{\lambda-1/2}(\omega) - 1] P_{\lambda-1/2}(-\cos \theta).$$

(12)

In Eqs. (11) and (12), the integration contour encircles counterclockwise the positive real axis of the complex $\lambda$ plane, i.e., we take $C = \{ \lambda \in +\infty + i\epsilon, +i\epsilon \cup [-i\epsilon, -i\epsilon] \cup [-i\epsilon, +\infty - i\epsilon] \cup [0, +\infty - i\epsilon] \}$ with $\epsilon \to 0_+$. We can recover (2) from (12) by using Cauchy’s residue theorem and by noting that the poles of the integrand in (12) that are enclosed into $C$ are the zeros of $\cos(\pi \lambda)$, i.e., the semi-integers $\lambda = \ell + 1/2$ with $\ell \in \mathbb{N}$. It should be recalled that, in Eq. (12), the Legendre function of first kind $P_{\lambda-1/2}(z)$ denotes the analytic extension of the Legendre polynomials $P_\ell(\theta)$. It is defined in terms of hypergeometric functions by [47]

$$P_{\lambda-1/2}(z) = F[1/2 - \lambda, 1/2 + \lambda; 1; (1 - z)/2].$$

(13)
In Eq. (12), $S_{\lambda-1/2}(\omega)$ denotes “the” analytic extension of $S_\ell(\omega)$. It is given by [see Eq. (10)]

$$S_{\lambda-1/2}(\omega) = e^{i(\lambda+1/2)\pi} \frac{A^{(+)}_{\lambda-1/2}(\omega)}{A^{(-)}_{\lambda-1/2}(\omega)}$$

(14)

where the complex amplitudes $A^{(+)}_{\lambda-1/2}(\omega)$ and $A^{(-)}_{\lambda-1/2}(\omega)$ are defined from the analytic extensions of the modes $\phi^{in}_{\omega,\lambda-1/2}$, i.e., from the function $\phi^{in}_{\omega,\lambda-1/2}$ solution of the problem (7)-(9) where we now replace $\ell$ by $\lambda - 1/2$. With the deformation of the contour $C$ in mind, it is important to note the symmetry property

$$e^{i\pi \lambda} S_{-\lambda-1/2}(\omega) = e^{-i\pi \lambda} S_{\lambda-1/2}(\omega)$$

(15)

of the $S$-matrix which can be easily obtained from its definition (see also Ref. [15]). It is also important to note that the poles of $S_{\lambda-1/2}(\omega)$ in the complex $\lambda$ plane (i.e., the Regge poles) lie in the first and third quadrants of this plane symmetrically distributed with respect to the origin $O$. The poles lying in the first quadrant can be defined as the zeros $\lambda_n(\omega)$ with $n = 1, 2, 3, \ldots$ of the coefficient $A^{(-)}_{\lambda-1/2}(\omega)$ [see Eq. (14)]. They therefore satisfy

$$A^{(-)}_{\lambda_n(\omega)-1/2}(\omega) = 0.$$  

(16)

We now deform the contour $C$ in Eq. (12) in order to collect, by using Cauchy’s residue theorem, the contributions of the Regge poles lying in the first quadrant of the CAM plane (for more details, see, e.g., Ref. [4]). This must be done very carefully and, in particular, we must deal with the contributions coming from the quarter circles at infinity with great care. We first note that

$$\frac{\lambda}{\cos(\pi \lambda)} [S_{\lambda-1/2}(\omega) - 1] P_{-\lambda-1/2}(-\cos \theta)$$

(17)

vanishes rapidly for $|\lambda| \to +\infty$ and $\text{Im} \lambda > 0$, that

$$\frac{\lambda}{\cos(\pi \lambda)} P_{-\lambda-1/2}(-\cos \theta)$$

(18)

vanishes rapidly for $|\lambda| \to +\infty$ and $\text{Im} \lambda < 0$, and that

$$\frac{\lambda}{\cos(\pi \lambda)} S_{\lambda-1/2}(\omega) P_{-\lambda-1/2}(-\cos \theta)$$

(19)

diverging for $|\lambda| \to +\infty$ and $\text{Im} \lambda < 0$, it is more convenient to replace it by

$$\frac{\lambda}{\cos(\pi \lambda)} [S_{\lambda-1/2}(\omega) e^{-i\pi(\lambda-1/2)} P_{-\lambda-1/2}(+\cos \theta)$$

$$+ \frac{1}{2} \lambda S_{\lambda-1/2}(\omega) Q_{\lambda-1/2}(\cos \theta + i0)]$$

(20)

where the first term vanishes rapidly for $|\lambda| \to +\infty$ and $\text{Im} \lambda < 0$. In Eq. (20) we have introduced the Legendre function of the second kind $Q_{\lambda-1/2}(z)$ which satisfies [47]

$$Q_{\lambda-1/2}(z + i0) = \frac{\pi}{2 \cos(\pi \lambda)} \left[ P_{\lambda-1/2}(-z)$$

$$- e^{-i\pi(\lambda-1/2)} P_{\lambda-1/2}(+z) \right].$$

(21)

Then, by using (15) and the relation [47]

$$P_{-\lambda-1/2}(z) = P_{\lambda-1/2}(z)$$

(22)

we finally obtain

$$f(\omega, \theta) = f^b(\omega, \theta) + f^{RP}(\omega, \theta)$$

(23)

where

$$f^b(\omega, \theta) = f^{b, re}(\omega, \theta) + f^{b, im}(\omega, \theta)$$

(24a)

with

$$f^{b, re}(\omega, \theta) = \frac{1}{\pi \omega} \int_{C_-} d\lambda \lambda S_{\lambda-1/2}(\omega) Q_{\lambda-1/2}(\cos \theta + i0)$$

(24b)

and

$$f^{b, im}(\omega, \theta) = \frac{1}{\pi \omega} \int_{+\infty}^{0} d\lambda \lambda S_{\lambda-1/2}(\omega) Q_{\lambda-1/2}(\cos \theta + i0)$$

(25)

is a background integral contribution [here we have $C_- = [0, -i\epsilon] \cup [-i\epsilon, +\infty - i\epsilon]$ with $\epsilon \to 0$, and we can note that the path of integration defining the background integral (24a) is a continuous one running down first the positive imaginary axis and then running along $C_-$, i.e., slightly below the positive real axis] and where

$$f^{RP}(\omega, \theta) = -\frac{i\pi}{\omega} \sum_{n=1}^{+\infty} \lambda_n(\omega) r_n(\omega)$$

$$\times P_{\lambda_n(\omega)-1/2}(-\cos \theta)$$

(25)

is a sum over the Regge poles lying in the first quadrant of the CAM plane. In Eq. (25) we have introduced the residue of the matrix $S_{\lambda-1/2}(\omega)$ at the pole $\lambda = \lambda_n(\omega)$ defined by

$$r_n(\omega) = e^{i\pi |\lambda_n(\omega)+1/2|} \left[ \frac{A^{+}_{\lambda_n(\omega)-1/2}(\omega)}{2 \pi A^{(-)}_{\lambda_n(\omega)-1/2}(\omega)} \right]_{\lambda=\lambda_n(\omega)}.$$  

(26)

Of course, Eqs. (23), (24) and (25) provide an exact representation of the scattering amplitude $f(\omega, \theta)$ for the scalar field equivalent to the initial partial wave expansion (2). From this CAM representation, we can extract the contribution $f^{RP}(\omega, \theta)$ given by (25) which, as a sum over Regge poles, is only an approximation of $f(\omega, \theta)$, and which provides us with an approximation of the differential scattering cross section (1).

C. CAM representation of the scattering amplitude for electromagnetic waves

In order to obtain the CAM representation of the scattering amplitudes (4) and (5), we use the Sommerfeld-Watson transformation in the form

$$\sum_{\ell=1}^{+\infty} (-1)^\ell F(\ell) = \frac{i}{2} \int_{C'} d\lambda \frac{F(\lambda-1/2)}{\cos(\pi \lambda)}.$$  

(27)
Here $C^\prime = +\infty + i \epsilon, 1 + i \epsilon \cup [1 + i \epsilon, 1 - i \epsilon] \cup [1 - i \epsilon, +\infty - i \epsilon]$ with $\epsilon \to 0^+$ because we must take into account that the sum over $\ell$ defining the scattering amplitude (5) begins at $\ell = 1$. We then obtain

$$B(\omega, \theta) = \frac{1}{2\omega} \int_{C^\prime} d\lambda \frac{\lambda}{(\lambda^2 - 1/4) \cos(\pi \lambda)} \times \left[ S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(-\cos \theta).$$  \hspace{1cm} (28)

By now noting that

$$D_\theta \ln \left[ \frac{1}{2}(1 - \cos \theta) \right] = 0 \hspace{1cm} (31)$$

we can finally write

$$B(\omega, \theta) = \frac{1}{2\omega} \int_{C^\prime} d\lambda \frac{\lambda}{(\lambda^2 - 1/4) \cos(\pi \lambda)} \times \left[ S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(-\cos \theta).$$  \hspace{1cm} (32)

Here we have dropped terms which do not contribute to the scattering amplitude $A(\omega, \theta)$.

We now deform the contour $C$ in Eq. (32) in order to collect, by using Cauchy’s residue theorem, the Regge pole contributions. This is achieved by following, *mutatis mutandis*, the approach of Sec. II B. We obtain

$$A(\omega, \theta) = A_b(\omega, \theta) + A^{RP}(\omega, \theta)\hspace{1cm} (33)$$

where

$$A_b(\omega, \theta) = A_{b,Re}(\omega, \theta) + A_{b,Im}(\omega, \theta)\hspace{1cm} (34a)$$

with

$$A_{b,Re}(\omega, \theta) = \frac{1}{\pi \omega} \int_{C_{-\infty}} d\lambda \frac{\lambda}{\lambda^2 - 1/4} S_{\lambda-1/2}(\omega) \times D_\theta Q_{\lambda-1/2}(\cos \theta + i0)\hspace{1cm} (34b)$$

and

$$A_{b,Im}(\omega, \theta) = \frac{1}{\pi \omega} \int_{+\infty}^{0} d\lambda \frac{\lambda}{\lambda^2 - 1/4} S_{\lambda-1/2}(\omega) \times D_\theta Q_{\lambda-1/2}(\cos \theta + i0)\hspace{1cm} (34c)$$

with the collect of Regge poles in mind, the contour $C^\prime$ is not as convenient as the contour $C$ used in Sec. II B. However, it is possible to move $C^\prime$ to the left so that it coincides with $C$ but we then introduce a spurious double pole at $\lambda = 1/2$ (i.e., at $\ell = 0$) corresponding to the term $1/[(\lambda - 1/2) \cos(\pi \lambda)]$. It is necessary to remove the associated residue contribution and we obtain

$$B(\omega, \theta) = \frac{1}{2\omega} \int_{C} d\lambda \frac{\lambda}{(\lambda^2 - 1/4) \cos(\pi \lambda)} \times \left[ S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(-\cos \theta)$$

$$-2i\pi \lim_{\lambda \to 1/2} \frac{d}{d\lambda} \left[ (\lambda - 1/2)^2 \times \frac{\lambda}{(\lambda^2 - 1/4) \cos(\pi \lambda)} \times \left[ S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(-\cos \theta) \right].$$  \hspace{1cm} (29)

By using the definition (13) and by noting that, for the electromagnetic field, we have formally $S_0(\omega) = 1$ [see Eqs. (7)-(9)], we can write

$$B(\omega, \theta) = \frac{1}{2\omega} \int_{C} d\lambda \frac{\lambda}{(\lambda^2 - 1/4) \cos(\pi \lambda)} \times \left[ S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(-\cos \theta)$$

$$- i \frac{\pi}{2\omega} \ln \left[ \frac{1}{2}(1 - \cos \theta) \right] + \text{terms independant of } \theta.\hspace{1cm} (30)$$

is a background integral contribution and where

$$A^{RP}(\omega, \theta) = -i \frac{\pi}{\omega} \sum_{n=1}^{+\infty} \frac{\lambda_n(\omega) r_n(\omega)}{[\lambda_n(\omega)^2 - 1/4] \cos[\pi \lambda_n(\omega)]} \times D_\theta P_{\lambda_n(\omega)-1/2}(-\cos \theta)\hspace{1cm} (35)$$

is a sum over the Regge poles lying in the first quadrant of the CAM plane.

III. RECONSTRUCTION OF DIFFERENTIAL SCATTERING CROSS SECTIONS FROM REGGE POLE SUMS

In this section, we compare numerically the partial wave expansions of the differential scattering cross sections with their CAM representations or, more precisely, with their Regge pole approximations in order to highlight the benefits of working with Regge pole sums.

A. Computational methods

In order to construct numerically the scattering amplitudes (2) and (4), the background integrals (24b), (24c), (34b) and (34c) as well as the Regge pole sums (25) and (35):

(1) We have to solve the problem (7)-(9) permitting us to obtain the function $\phi^b_{\ell,\ell}(r)$, the coefficients $A_{\ell,\ell}^{(-)}(\omega)$ and $A_{\ell,\ell}^{(+)}(\omega)$ and the $S$-matrix elements $S_{\ell}(\omega)$. This must be achieved (i) for $\ell \in \mathbb{N}$ and $\omega > 0$ as well as (ii) for $\ell = \lambda - 1/2 \in \mathbb{C}$ and $\omega > 0$.\hspace{1cm}
We have to determine for $\omega > 0$ the Regge poles $\lambda_n(\omega)$, i.e., the solutions of (16) and to obtain the corresponding residues (26).

All these numerical results can be obtained by using, mutatis mutandis, the methods that have permitted us to provide in Ref. [29] a description of gravitational radiation from BHs based on CAM techniques (see, in particular, Secs. IIIIB and IVA of this previous paper). It is moreover important to note that, due to the long-range nature of the fields propagating on a Schwarzschild BH, the scattering amplitudes (2) and (4) and the background integrals (24b) and (34b) suffer of a lack of convergence (this is not the case for the background integrals (24c) and (34c) because their integrands vanish exponentially as $\lambda \to +i\infty$). In the Appendix, we explain how to overcome this problem, i.e., how to accelerate their convergence by employing an iterative method, the number of iterations being chosen to obtain stable numerical results. It should be noted that we have performed all the numerical calculations by using Mathematica [48].

B. Results and comments

In Figs. 1-6, we focus on the scalar field and we compare the differential scattering cross section (1) constructed from the partial wave expansion (2) with its Regge pole approximation obtained from the Regge pole sum (25). In Figs. 7-12 we focus on the electromagnetic field and we compare the differential scattering cross section (3) constructed from the partial wave expansion (4)-(6) with its Regge pole approximation obtained from the Regge pole sum (35). The comparisons are achieved for the reduced frequencies $2M\omega = 0.1, 0.3, 0.6, 1, 3$ and 6 and, for these frequencies, we have displayed the lowest Regge poles and the associated residues in Table I (for the scalar field) and in Table II (for the electromagnetic field). The higher Regge poles and their residues that have been necessary to obtain some of the results displayed in Figs. 1-12 are available upon request from the authors.

In Figs. 3-6 and 9-12, we display the results obtained for intermediate and high reduced frequencies (here, we consider the reduced frequencies $2M\omega = 0.1$ and 0.3). We can observe that, in the “short”-wavelength regime, the Regge pole approximations (25) and (35) alone do not permit us to reconstruct the differential scattering cross sections but that this can be achieved by taking into account the background integral contributions (24) and (34). Cally obtain these contributions and observe that they are completely negligible for intermediate and large scattering angles. It seems they begin to play a role only for small angles. In Table III, we have considered, for the electromagnetic field at $2M\omega = 1$, the various contributions to the CAM representation (33). We can see that, for the scattering angles $\theta = 15^\circ$ and $\theta = 20^\circ$, the background integrals, although not completely negligible, play a minor role. It would be interesting to check if this remains valid even for scattering angles $\theta \ll 1/(2M\omega)$ but, due to numerical instabilities when $\theta \to 0$, we are not currently able to provide such a result.

In Figs. 1, 2, 7 and 8, we focus on the results obtained for low reduced frequencies (here, we consider the reduced frequencies $2M\omega = 0.1$ and 0.3). We can observe that, in the long-wavelength regime, the Regge pole approximations (25) and (35) alone do not permit us to reconstruct the differential scattering cross sections but that this can be achieved by taking into account the background integral contributions (24) and (34).
| $n$ | $\omega$ | $\lambda_n(\omega)$ | $r_n(\omega)$ |
|-----|--------|-------------------|-------------|
| 1   | 0.1    | 0.299705 + 0.532035i | 0.110788 − 0.088226i |
| 0.3 | 0.768109 + 0.531109i | 0.229999 − 0.159051i |
| 0.6 | 1.543385 + 0.511888i | 0.391142 − 0.295794i |
| 1   | 2.586845 + 0.507436i | 0.527572 − 0.019732i |
| 3   | 7.79018 + 0.500553i | −0.136670 + 0.901866i |
| 6   | 15.586384 + 0.500139i | −0.594258 − 1.144512i |
| 2   | 0.1    | 0.495889 + 1.176250i | 0.099116 − 0.087234i |
| 0.3 | 0.976977 + 1.364123i | 0.119559 − 0.228260i |
| 0.6 | 1.695146 + 1.448184i | 0.181312 − 0.487058i |
| 1   | 2.68353 + 1.479587i | 0.447998 − 0.888968i |
| 3   | 7.82561 + 1.497769i | 4.906565 + 1.422003i |
| 6   | 15.604187 + 1.494447i | −12.410929 + 5.305460i |
| 3   | 0.1    | 0.650192 + 1.778513i | 0.087664 − 0.087739i |
| 0.3 | 1.164769 + 2.098578i | 0.056734 − 0.256550i |
| 0.6 | 1.87896 + 2.298202i | −0.078148 − 0.523737i |
| 1   | 2.843095 + 2.391435i | −0.357084 − 1.174098i |
| 3   | 7.893083 + 2.484084i | 7.751902 − 10.715411i |
| 6   | 15.639426 + 2.495896i | 16.785993 + 69.494274i |
| 4   | 0.1    | 0.788665 + 2.349868i | 0.078994 − 0.087791i |
| 0.3 | 1.339372 + 2.783917i | 0.016812 − 0.232342i |
| 0.6 | 0.206530 + 3.068702i | −0.247707 − 0.466596i |
| 1   | 3.018970 + 3.248630i | −0.987253 − 0.882511i |
| 3   | 7.990003 + 3.453496i | −11.471634 − 23.211359i |
| 6   | 15.691407 + 4.477447i | 256.923913 + 5.083667i |
| 5   | 0.1    | 0.917952 + 2.899702i | 0.074199 − 0.087641i |
| 0.3 | 1.504128 + 3.437384i | −0.011069 − 0.224974i |
| 0.6 | 2.248056 + 3.809074i | −0.348103 − 0.384562i |
| 1   | 3.203185 + 4.065913i | −0.357084 − 1.174098i |
| 3   | 8.109029 + 4.405790i | 7.751902 − 10.715411i |
| 6   | 15.759163 + 4.473447i | 16.785993 + 69.494274i |
| 6   | 0.1    | 1.039655 + 3.433425i | 0.069763 − 0.087418i |
| 0.3 | 1.661166 + 4.067794i | −0.031853 − 0.216897i |
| 0.6 | 2.425852 + 4.520191i | −0.410128 − 0.305258i |
| 1   | 3.386161 + 4.845810i | −1.384667 − 0.289334i |
| 3   | 8.245770 + 5.337166i | −54.31516 + 35.68649i |
| 6   | 15.841536 + 5.451746i | −114.818563 − 114.173130i |
| 7   | 0.1    | 1.156528 + 3.954439i | 0.066156 − 0.087107i |
| 0.3 | 1.811951 + 4.680400i | −0.048060 − 0.208882i |
| 0.6 | 2.598670 + 5.208747i | −0.446908 − 0.227138i |
| 1   | 3.569316 + 5.602025i | −1.328185 + 0.409929i |
| 3   | 8.390600 + 6.248658i | −12.517767 − 72.751371i |
| 6   | 15.937250 + 6.421641i | −3071.847422 + 904.874699i |
| 8   | 0.1    | 1.269207 + 4.465047i | 0.061333 − 0.086918i |
| 0.3 | 1.95732 + 5.278644i | −0.061116 − 0.201188i |
| 0.6 | 2.766835 + 5.879184i | −0.467343 − 0.159566i |
| 1   | 3.750196 + 6.337011i | −1.193204 − 0.709887i |
| 3   | 8.556293 + 7.141111i | 41.280909 + 78.754435i |
| 6   | 16.044990 + 7.382406i | −1576.690323 + 5414.570984i |
| 9   | 0.1    | 1.378403 + 4.966897i | 0.060539 − 0.086669i |
| 0.3 | 2.098952 + 5.864393i | −0.071892 − 0.193894i |
| 0.6 | 2.930753 + 6.534615i | −0.476862 − 0.099550i |
| 1   | 3.928352 + 7.054311i | −1.017928 − 0.937403i |
| 3   | 8.723992 + 8.015714i | 89.972545 + 49.380751i |
| 6   | 16.163453 + 8.333555i | 5509.049968 + 7209.599940i |
FIG. 1. The scalar cross section of a Schwarzschild BH for $2M\omega = 0.1$, its Regge pole approximation and the background integral contribution.

FIG. 2. The scalar cross section of a Schwarzschild BH for $2M\omega = 0.3$, its Regge pole approximation and the background integral contribution.
FIG. 3. The scalar cross section of a Schwarzschild BH for $2M\omega = 0.6$ and its Regge pole approximation.

FIG. 4. The scalar cross section of a Schwarzschild BH for $2M\omega = 1$ and its Regge pole approximation.
FIG. 5. The scalar cross section of a Schwarzschild BH for $2M\omega = 3$ and its Regge pole approximation.

FIG. 6. The scalar cross section of a Schwarzschild BH for $2M\omega = 6$ and its Regge pole approximation.
FIG. 7. The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 0.1$, its Regge pole approximation and the background integral contribution.

FIG. 8. The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 0.3$, its Regge pole approximation and the background integral contribution.
FIG. 9. The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 0.6$ and its Regge pole approximation.

FIG. 10. The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 1$ and its Regge pole approximation.
FIG. 11. The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 3$ and its Regge pole approximation.

FIG. 12. The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 6$ and its Regge pole approximation.
TABLE III: We compare, for the electromagnetic field at $\omega = 1 \ (2M = 1)$, the exact value of the scattering amplitude (4) with the sum over Regge poles (35) and we highlight the minor role of the background integral contributions (34b) and (34a).

| Electromagnetic field at $\omega = 1 \ (2M = 1)$ | $\theta = 15^\circ$ | $\theta = 20^\circ$ |
|-----------------------------------------------|------------------|------------------|
| $|A(\omega, \theta)|^2$ (n=1, 90) 1023.0681 | 335.2717 | 1025.5432 |
| $|A^{RP}(\omega, \theta)|^2$ (n=1, 60) 1025.4767 | 335.8305 | 1023.0754 |
| $|A^{RP}(\omega, \theta)|^2$ (n=1, 90) 1025.4767 | 335.8343 | 1023.0754 |
| $|A^{RP}(\omega, \theta)|^2$ (n=1, 90) 1023.0762 | 335.2718 | 1023.0762 |

IV. CONCLUSION

In this article, we have considered the scattering of scalar and electromagnetic waves by a Schwarzschild BH by focusing on the associated differential scattering cross sections and we have shown that, for intermediate and high reduced frequencies, these cross sections can be reconstructed in terms of Regge poles with great precision. This is really surprising and certainly due to the fact that BHs are very particular physical objects. Indeed, in quantum mechanics [3, 4], electromagnetism and optics [4–8] and acoustics [9], background integral contributions are never negligible (this remains true regardless of the frequency) and, as a consequence, a Regge pole sum alone does not permit us to reproduce a differential scattering cross section. In the context of scattering of scalar and electromagnetic waves by a Schwarzschild BH, we have observed that it is necessary to take into account background integral contributions only for low reduced frequencies.

In the short-wavelength regime, from the Regge pole sum alone, we have also been able to describe, with a very good agreement, the BH glory occurring in the backward direction as well as the orbiting oscillations appearing on the differential scattering cross sections for small and intermediate scattering angles. Moreover, it is important to note that working with Regge pole sums has permitted us to overcome the difficulties linked to the lack of convergence of the partial wave expansions defining the cross sections which are due to the long-range nature of the fields propagating on a Schwarzschild background.

Here, it is important to recall that the CAM approach to scattering is usually combined with asymptotic methods in order to provide semiclassical interpretations of the phenomenons observed. This is the case in quantum mechanics [3, 4], in electromagnetism and optics [4–8] as well as in acoustics and seismology [9, 10] where the association “surface wave-Regge pole” is of fundamental interest. This is also the case in BH physics where the Regge poles can be associated with surface waves “trapped” on the BH photon spheres (see, e.g., Refs. [16, 17, 19, 21, 22]). This point of view, which is valid in the high-frequency limit and has permitted us to understand the existence of the weakly damped QNMs as well as to describe analytically the high-energy absorption cross section for a wide class of BHs endowed with a photon sphere [23–25], is not adopted in the present article. The CAM approach provides an exact representation of partial wave expansions valid for arbitrary frequencies and we have based our study on this point of view (see also our recent article [29] where we have previously adhered to such a point of view). In other words, we have here chosen to discard that part of the CAM machinery involving asymptotic methods. As a consequence, we have renounced to provide physical interpretations of the results obtained with, in return, impressive agreements between exact calculations and Regge pole approximations.

We hope in next works to extend our study to scattering of gravitational waves by a Schwarzschild BH as well as to scattering of waves by a Kerr BH (see Refs. [40, 41, 49–52] for articles concerning these two topics and which could serve as departure points for such works).

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Appendix: Iterative method to accelerate the convergence of the background integrals

Due to the long-range nature of the fields propagating on a Schwarzschild BH, the background integrals (24b) and (34b) suffer of a lack of convergence. In this Appendix we explain how to overcome this problem, i.e., how to accelerate their convergence. In fact, the same problem occurs for the partial wave expansions (2), (5) and (4) (see, e.g., Refs. [38] where the case of the scalar field is discussed). For these partial wave expansions, the convergence can be obtained by employing an iterative method introduced a long time ago in the context of Coulomb scattering by Yennie, Ravenhall and Wilson [46] (see also Refs. [41, 42] where this method is used in the context of scattering by BHs and see, e.g., Ref. [38] for another approach based on the truncation of partial wave expansions and on the matching of the Schwarzschild S-matrix elements $S(\omega)$ with the Newtonian ones). In this Appendix, we briefly recall this iterative method because we use it in Sec. III in order to accelerate the convergence of the partial wave expansions (2) and (5) and we generalize it in order to deal with the background integrals (24b) and (34b).
1. Acceleration of the convergence of partial wave expansions

In order to improve the convergence of the partial wave expansion

\[ \alpha(\theta) = \sum_{\ell=0}^{\infty} a_{\ell}\cos^{\ell}(\theta) \]  

(A.1)

[see Eqs. (2) and (5)] we introduce the associated "reduced" series [46]

\[ \tilde{\alpha}(n)(\theta) = \sum_{\ell=0}^{\infty} \tilde{a}_{\ell}(n)\cos^{\ell}(\theta) \]  

(A.2)

defined by

\[ \alpha(\theta) = (1 - \cos^{n}\theta) - n\tilde{a}(\theta). \]  

(A.3)

It should be noted that, by re-expressing the series (A.1) in the form (A.3), we isolate the pathological behavior of the partial wave expansions (2) and (5) occurring for \( \theta \to 0 \). By using now the recursion relation [47]

\[ (\ell + 1)P_{\ell+1}(\cos \theta) - (2\ell + 1)\cos \theta P_{\ell}(\cos \theta) + \ell P_{\ell-1}(\cos \theta) = 0 \]  

(A.4)

in the form

\[ (1 - \cos \theta)P_{\ell}(\cos \theta) = P_{\ell}(\cos \theta) - \frac{\ell + 1}{2\ell + 1}P_{\ell+1}(\cos \theta) - \frac{\ell}{2\ell + 1}P_{\ell-1}(\cos \theta) \]  

(A.5)

we can show from (A.1)-(A.3) that the coefficients \( \tilde{a}_{\ell}(n) \)

can be expressed in terms of the coefficients \( \tilde{a}_{\ell}(n-1) \). We have, for \( \ell \in \mathbb{N} \),

\[ \tilde{a}_{\ell}(n) = \tilde{a}_{\ell}(n-1) - \frac{\ell + 1}{2\ell + 1}\tilde{a}_{\ell+1}(n-1) - \frac{\ell}{2\ell - 1}\tilde{a}_{\ell-1}(n-1) \]  

(A.6)

(\text{here we take } \tilde{a}_{-1}(n-1) = 0). As noted in Ref. [46], we have for large values of \( \ell \),

\[ \tilde{a}_{\ell}(n) = \mathcal{O}\left(\tilde{a}_{\ell-1}(n-1)/\ell^{2}\right). \]  

(A.7)

This explicitly shows that using the reduced series (A.2) greatly improves the convergence of the initial partial wave expansion (A.1).

2. Acceleration of the convergence of background integrals

In order to improve the convergence of the background integral

\[ \alpha(\theta) = \int_{0}^{\infty} d\lambda a(\lambda) Q_{\lambda - 1/2}(\cos \theta + i0) \]  

(A.8)

[see Eqs. (24b) and (34b)], we first split it in the form

\[ \alpha(\theta) = \int_{0}^{\lambda_{0}} d\lambda a(\lambda) Q_{\lambda - 1/2}(\cos \theta + i0) + \int_{\lambda_{0}}^{\infty} d\lambda a(\lambda) Q_{\lambda - 1/2}(\cos \theta + i0) \]  

(A.9)

where

\[ \alpha_{\lambda_{0}}(\theta) = \int_{\lambda_{0}}^{\infty} d\lambda a(\lambda) Q_{\lambda - 1/2}(\cos \theta + i0). \]  

(A.10)

The choice of the truncation parameter \( \lambda_{0} \) will be discussed later. We then introduce the "reduced" integrals \( \tilde{\alpha}_{\lambda_{0}}(\theta) \) defined by

\[ \tilde{\alpha}_{\lambda_{0}}(\theta) = (1 - \cos \theta)^{-n}\tilde{\alpha}_{\lambda_{0}}(\theta). \]  

(A.11)

By using the relation

\[ (1 - \cos \theta)Q_{\lambda - 1/2}(\cos \theta + i0) = Q_{\lambda - 1/2}(\cos \theta + i0) \]

\[ - \left( \frac{\lambda + 1/2}{2\lambda} \right) Q_{\lambda + 1/2}(\cos \theta + i0) \]

\[ - \left( \frac{\lambda - 1/2}{2\lambda} \right) Q_{\lambda - 3/2}(\cos \theta + i0) \]  

(A.12)

which is a consequence of the definition (21) and of the relation [47]

\[ (\nu + 1)P_{\nu+1}(\cos \theta) - (2\nu + 1)(\nu + 1)\cos \theta P_{\nu}(\cos \theta) + \nu P_{\nu-1}(\cos \theta) = 0, \]  

(A.13)

we can show that these reduced integrals can be written in the form

\[ \tilde{\alpha}_{\lambda_{0}}(\theta) = \int_{\lambda_{0}}^{\infty} d\lambda \tilde{a}(\lambda) Q_{\lambda - 1/2}(\cos \theta + i0) + \tilde{R}(\theta) \]  

(A.14)

where \( \tilde{R}(\theta) \) is an integral over a finite integration domain which can be expressed in terms of the integral \( \tilde{R}(\lambda - 1/2) \) and where the function \( \tilde{a}(\lambda) \) can be expressed in terms of the function \( \tilde{a}(\lambda) \). We have

\[ \tilde{a}(\lambda) = \tilde{a}(\lambda - 1) - \left( \frac{\lambda + 1/2}{2(\lambda + 1)} \right) \tilde{a}(\lambda - 1) \]

\[ + \left( \frac{\lambda - 1/2}{2(\lambda - 1)} \right) \tilde{a}(\lambda - 1) \]  

(A.15)

and

\[ \tilde{R}(\theta) = (1 - \cos \theta)\tilde{R}(\lambda - 1/2) \]

\[ + \int_{\lambda_{0} - 1}^{\lambda_{0}} d\lambda \left[ \left( \frac{\lambda + 1/2}{2\lambda} \right) \tilde{a}(\lambda) Q_{\lambda + 1/2}(\cos \theta + i0) - \left( \frac{\lambda + 1/2}{2(\lambda + 1)} \right) \tilde{a}(\lambda + 1) Q_{\lambda - 1/2}(\cos \theta + i0) \right]. \]  

(A.16)

From the relation (A.15) we can see that, for large values of \( \lambda \),

\[ \tilde{a}(\lambda) = \mathcal{O}\left(\tilde{a}(\lambda - 1)/\lambda^{2}\right). \]  

(A.17)
This explicitly shows that using the reduced integral (A.14) greatly improves the convergence of the initial background integral (A.8). As far as the choice of the truncation parameter $\lambda_0$ is concerned, it should be noted that it depends on the number $n$ of iterations performed.

Indeed, due to the shift of the variable $\lambda$ induced by the relation (A.15), we can observe that $\overline{\alpha}^{(n)}(\lambda)$ is not defined for $\lambda \in [0,n]$ and, in order to have the integrals (A.16) and (A.14) well defined, it is necessary to take $\lambda_0 \geq n$. 

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