ADJOINT RINGS ARE FINITELY GENERATED

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ABSTRACT. This paper proves finite generation of the log canonical ring without Mori theory.

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1. INTRODUCTION

The main goal of this paper is to prove the following theorem while avoiding techniques of the Minimal Model Program.

Theorem 1.1. Let $(X,\Delta)$ be a projective klt pair. Then the log canonical ring $R(X,K_X + \Delta)$ is finitely generated.

Let me sketch the strategy for the proof of finite generation in this paper and present difficulties that arise on the way. The natural idea is to pick a smooth divisor $S$ on $X$ and to restrict the algebra to it. If we are very lucky, the restricted algebra will be finitely generated and we might hope that the generators lift to generators on $X$. There are several issues with this approach.

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This paper was written while I was a PhD student at the University of Cambridge, a research visitor at the Max-Planck-Institut für Mathematik and a postdoc at the Institut Fourier.
First, to obtain something meaningful on $S$, we require $S$ to be a log canonical centre of some pair $(X, \Delta')$ such that the rings $R(X, K_X + \Delta)$ and $R(X, K_X + \Delta')$ share a common truncation.

Second, even if the restricted algebra were finitely generated, the same might not be obvious for the kernel of the restriction map. So far this seems to have been the greatest conceptual issue in attempts to prove the finite generation by the plan just outlined.

Third, the natural strategy is to use the Hacon-Mckernan extension theorem, and hence we must be able to ensure that $S$ does not belong to the stable base locus of $K_X + \Delta'$. The idea to resolve the kernel issue is to view $R(X, K_X + \Delta)$ as a subalgebra of a much bigger algebra containing generators of the kernel by construction. The new algebra is graded by a monoid whose rank corresponds roughly to the number of components of $\Delta$ and of an effective divisor $D \sim Q K_X + \Delta$. A basic example which models the general lines of the proof in $§10$ is presented in Lemma $A.2$.

It is natural to try and restrict to a component of $\Delta$, the issue of course being that $(X, \Delta)$ does not have log canonical centres. Therefore I allow restrictions to components of some effective divisor $D \sim Q K_X + \Delta$, and a tie-breaking-like technique allows me to create log canonical centres. Algebras encountered this way are, in effect, plt algebras, and their restriction is handled in $§7$. This is technically the most involved part of the proof.

Since the algebras we consider are of higher rank, not all divisors will have the same log canonical centres. I therefore restrict to available centres, and lift generators from algebras that live on different divisors. Since the restrictions will also be algebras of higher rank, the induction process must start from them. The contents of this paper can be summarised in the following result.

**Theorem 1.2.** Let $X$ be a projective variety, and let $D_i = k_i(K_X + \Delta_i + A) \in \text{Div}(X)$, where $A$ is an ample $Q$-divisor and $(X, \Delta_i + A)$ is a klt pair for $i = 1, \ldots, \ell$. Then the adjoint ring $R(X; D_1, \ldots, D_\ell)$ is finitely generated.

Theorem $1.1$ is a corollary to the previous theorem. Techniques of the MMP were used to prove Theorem $1.1$ in the seminal paper $[BCHM06]$. A proof of finite generation of the canonical ring of general type by analytic methods is announced in $[Siu06]$.

In the following result I recall some of the well-known consequences of Theorems $1.1$ and $1.2$; further discussion is in the appendix.

**Corollary 1.3.** The following holds.

1. Klt flips exist.
2. Canonical models of klt pairs of log general type exist.
3. Log Fano klt pairs are Mori dream spaces.

In the appendix I give a very short history of Mori theory, and also outline a new approach which aims to turn the conventional thinking about classification on its head. Finite generation comes at the beginning of the theory and all main results of the Minimal Model Program should be derived from it. In light of this new viewpoint, it is my hope...
that the techniques of this paper could be adapted to handle finite generation in the case of log canonical singularities and the Abundance Conjecture.

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2. Notation and conventions

Unless stated otherwise, varieties in this paper are projective and normal over \( \mathbb{C} \). However, all results hold when \( X \) is, instead of being projective, assumed to be projective over an affine variety \( Z \). The group of Weil, respectively Cartier, divisors on a variety \( X \) is denoted by \( \text{WDiv}(X) \), respectively \( \text{Div}(X) \). Subscripts denote the rings in which the coefficients are taken. For a divisor \( D \), \( [D] \) denotes its class in \( N^1(X) \).

We say an ample \( \mathbb{Q} \)-divisor \( A \) on a variety \( X \) is general if there is a sufficiently divisible positive integer \( k \) such that \( kA \) is very ample and \( kA \) is a general section of \( |kA| \). In particular we can assume that for some \( k \gg 0 \), \( kA \) is a smooth divisor on \( X \). In practice, we fix \( k \) in advance, and generality is most often needed to ensure that \( A \) does not make singularities of pairs worse.

For any two formal sums of prime divisors \( P = \sum p_iE_i \) and \( Q = \sum q_iE_i \) on \( X \), set
\[
P \wedge Q = \sum \min\{p_i, q_i\}E_i.
\]

For definitions and basic properties of multiplier ideals used in this paper see [HM08]. The sets of non-negative (respectively non-positive) rational and real numbers are denoted by \( \mathbb{Q}^+ \) and \( \mathbb{R}^+ \) (respectively \( \mathbb{Q}^- \) and \( \mathbb{R}^- \)), and similarly for \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{R}_{> 0} \). For two subsets of \( A \) and \( B \) of a vector space \( V \), \( A + B \) denotes their Minkowski sum, i.e. the set \( \{a + b : a \in A, b \in B\} \).

b-Divisors. I use basic properties of b-divisors, see [Cor07]. The cone of mobile b-divisors on \( X \) is denoted by \( \text{Mob}(X) \).

Definition 2.1. Let \( (X, \Delta) \) be a log pair. For a model \( f : Y \to X \) we can write uniquely
\[
K_Y + B_Y = f^*(K_X + \Delta) + E_Y,
\]
where \( B_Y \) and \( E_Y \) are effective with no common components, and \( E_Y \) is \( f \)-exceptional. The boundary b-divisor \( \mathbf{B}(X, \Delta) \) is defined by \( \mathbf{B}(X, \Delta)_Y = B_Y \) for every model \( Y \to X \).

Lemma 2.2. If \( (X, \Delta) \) is a log pair, then the b-divisor \( \mathbf{B}(X, \Delta) \) is well-defined.
Lemma 2.3. Let $\sum_R$ Convex geometry. Proof. generated submonoid of $\sum_R$. Let $h: Y' \to Y$. Pushing forward $K_{Y'} + B_{Y'} = g^*(K_X + \Delta) + E_{Y'}$ via $h_*$ yields

$$K_Y + h_*B_{Y'} = f^*(K_X + \Delta) + h_*E_{Y'},$$

and thus $h_*B_{Y'} = B_Y$ since $h_*B_{Y'}$ and $h_*E_{Y'}$ have no common components. \hfill \Box

If $\{D\}$ denotes the fractional part of a divisor $D$, we have:

**Lemma 2.3.** Let $(X, \Delta)$ be a log canonical pair. There exists a log resolution $Y \to X$ such that the components of $\{B(X, \Delta)Y\}$ are disjoint.

**Proof.** See [KM98, 2.36] or [HM05, 6.7]. \hfill \Box

**Convex geometry.** If $\mathcal{I} = \sum N e_i$ is a submonoid of $\mathbb{N}^n$, I denote $\mathcal{I}_Q = \sum Q_+ e_i$ and $\mathcal{I}_R = \sum \mathbb{R}_+ e_i$. A monoid $\mathcal{I} \subset \mathbb{N}^n$ is saturated if $\mathcal{I} = \mathcal{I}_R \cap \mathbb{N}^n$.

If $\mathcal{I} = \sum_{i=1}^n N e_i$ and $\kappa_1, \ldots, \kappa_n$ are positive integers, the submonoid $\sum_{i=1}^n N \kappa_i e_i$ is called a truncation of $\mathcal{I}$. If $\kappa_1 = \cdots = \kappa_n = \kappa$, I denote $\mathcal{I}(\kappa) = \sum_{i=1}^n \mathbb{N} \kappa_i e_i$, and this truncation does not depend on a choice of generators of $\mathcal{I}$.

A submonoid $\mathcal{I} = \sum N e_i$ of $\mathbb{N}^n$ (respectively a cone $\mathcal{C} = \sum \mathbb{R}_+ e_i$ in $\mathbb{R}^n$) is called simplicial if its generators $e_i$ are linearly independent in $\mathbb{R}^n$, and the $e_i$ form a basis of $\mathcal{I}$ (respectively $\mathcal{C}$).

I often use without explicit mention that if $f: \mathcal{M} \to \mathcal{I}$ is an additive surjective map between finitely generated saturated monoids, and if $\mathcal{C}$ is a rational polyhedral cone in $\mathcal{I}_R$, then $f^{-1}(\mathcal{I} \cap \mathcal{C}) = \mathcal{M} \cap f^{-1}(\mathcal{C})$. In particular, if $\mathcal{M}$ and $\mathcal{I}$ are saturated, the inverse image of a saturated finitely generated submonoid of $\mathcal{I}$ is a saturated finitely generated submonoid of $\mathcal{M}$.

For a polytope $\mathcal{P} \subset \mathbb{R}^n$, I denote $\mathcal{P}_Q = \mathcal{P} \cap \mathbb{Q}^n$. A polytope is rational if it is the convex hull of finitely many rational points. The dimension of a polytope $\mathcal{P}$, denoted $\dim \mathcal{P}$, is the dimension of the smallest affine space containing $\mathcal{P}$.

If $\mathcal{B} \subset \mathbb{R}^n$ is a convex set, then $\mathbb{R}_+ \mathcal{B}$ denotes the set $\{rb : r \in \mathbb{R}_+, b \in \mathcal{B}\}$. In particular, if $\mathcal{B}$ is a rational polytope, $\mathbb{R}_+ \mathcal{B}$ is a rational polyhedral cone.

**Remark 2.4.** The following will be used in the proof of Theorem 7.9. Assume that $\mathcal{P} \subset \mathbb{R}^n$ is an $n$-dimensional polytope and let $\{\mathcal{H}_\alpha\}$ be a collection of countably many affine hyperplanes in $\mathbb{R}^n$. Then $\mathcal{P} \not\subseteq \bigcup_\alpha \mathcal{H}_\alpha$. To see this, I argue by induction on $n$. The statement obviously stands for $n = 1$, so assume that $n > 1$. Fix a point $p$ in the interior of $\mathcal{P}$, and assume $\mathcal{P} \subset \bigcup_\alpha \mathcal{H}_\alpha$. Since the number of affine hyperplanes passing through $p$ is uncountable, there is an affine hyperplane $\mathcal{H} \ni p$ different from all $\mathcal{H}_\alpha$. Now $\mathcal{P} \cap \mathcal{H} \subset \bigcup_\alpha (\mathcal{H}_\alpha \cap \mathcal{H})$, but this is a contradiction since $\{\mathcal{H}_\alpha \cap \mathcal{H}\}$ is at most countable collection of hyperplanes in $\mathcal{H}$.

Let $\mathcal{C} \subset \mathbb{R}^n$ be a rational polyhedral cone and $V$ an $\mathbb{R}$-vector space with an ordering. A function $f: \mathcal{C} \to V$ is: positively homogeneous if $f(\lambda x) = \lambda f(x)$ for $x \in \mathcal{C}, \lambda \in \mathbb{R}_+$, and superlinear if $\lambda f(x) + \mu f(y) \leq f(\lambda x + \mu y)$ for $x, y \in \mathcal{C}, \lambda, \mu \in \mathbb{R}_+$. It is piecewise linear
if there is a finite polyhedral decomposition $\mathcal{C} = \bigcup \mathcal{C}_i$ such that $f|_{\mathcal{C}_i}$ is linear for every $i$; additionally, if each $\mathcal{C}_i$ is a rational cone, it is \textit{rationally piecewise linear}. Similarly for \textit{sublinear, superadditive, subadditive, (rationally) piecewise linear}. Assume furthermore that $f$ is linear and $\dim \mathcal{C} = n$. The \textit{linear extension of $f$ to $\mathbb{R}^n$} is the unique linear function $\ell : \mathbb{R}^n \to V$ such that $\ell|_{\mathcal{C}} = f$.

Unless otherwise stated, cones considered in this paper do not contain lines, and the \textit{relative interior} of a cone $\mathcal{C} = \sum \mathbb{R}_+ e_i \subset \mathbb{R}^n$, denoted by $\text{relint}\mathcal{C}$, is the origin union the topological interior of $\mathcal{C}$ in the space $\sum \mathbb{R} e_i$; if $\dim \mathcal{C} = n$, we call it the \textit{interior} of $\mathcal{C}$ and denote it by $\text{int}\mathcal{C}$. The boundary of a closed set $\mathcal{D}$ is denoted by $\partial \mathcal{D}$. If a norm $\| \cdot \|$ on $\mathbb{R}^n$ is given, then for $x \in \mathbb{R}^n$ and for any $r > 0$, the closed ball of radius $r$ with centre at $x$ is denoted by $B(x, r)$. Unless otherwise stated, the norm considered is always the sup-norm $\| \cdot \|_{\infty}$, and note that then $B(x, r)$ is a hypercube in the Euclidean norm.

I will need the following lemma in the proof of Theorem 9.1.

\textbf{Lemma 2.5.} Let $\mathcal{D} \subset \mathbb{R}^\ell$ be a closed cone, let $z_i \in \mathbb{R}^\ell \setminus \mathcal{D}$ be linearly independent points, and denote $\mathcal{Z} = \sum \mathbb{R}_+ z_i$ and $\mathcal{C} = \mathcal{Z} + \mathcal{D}$. Assume that $x_m \in \mathcal{C}$ is a sequence converging to $x_\infty = \sum \alpha_i z_i$ with $\alpha_i > 0$ for all $i$, and that $x_m \neq x_\infty$ for all $m$. Assume further that if $x_\infty = z + d$ with $z \in \mathcal{Z}$ and $d \in \mathcal{D}$, then $d = 0$, and in particular $x_\infty \notin \mathcal{D}$. Then for every $m_0 \in \mathbb{N}$ there exist $m \geq m_0$ and $x_m \in \mathcal{C}$ such that $x_m \in (x_\infty, x_m')$.

\textbf{Proof.} Let $H \ni x_\infty$ be an affine hyperplane such that $\mathcal{Z}_H = \mathcal{Z} \cap H$ is a polytope. For every $m \gg 0$, the intersection of $\mathbb{R}^n_+ x_m$ and $H$ is a point, and denote it by $y_m$. If there is $y'_m \in \mathcal{C}$ such that $y_m \in (x_\infty, y'_m)$, then it is easy to see that there is $x'_m \in \mathbb{R}^n_+ y'_m$ such that $x_m \in (x_\infty, x'_m)$. Therefore, replacing $x_m$ by $y_m$ and passing to a subsequence, I can assume that $x_m \in H$ for all $m$.

Write $x_m = s'_m + d'_m$ for every $m$, where $s'_m \in \mathcal{Z}$ and $d'_m \in \mathcal{D}$; note that $s'_m \neq 0$ for $m \gg 0$ as $x_\infty \notin \mathcal{D}$. Since $\mathbb{R}^n_+ s'_m$ intersects $H$, then $\mathbb{R}^n_+ d'_m$ also intersects $H$, and denote the intersection points by $s_m$ and $d_m$, respectively. Setting $\alpha_m = \frac{\|x_m - d_m\|}{\|s_m - d_m\|}$, we have

\begin{equation}
    x_m = \alpha_m s_m + (1 - \alpha_m) d_m.
\end{equation}

Observe that $\|s_m - x_m\|$ is bounded from above for $m \gg 0$ since $\mathcal{Z}_H$ is compact, and that $\|x_m - d_m\|$ is bounded from below for $m \gg 0$ as $x_m \notin \mathcal{D}$ and $\mathcal{D}$ is closed. Therefore $\frac{1}{\alpha_m} = 1 + \frac{\|x_m - d_m\|}{\|x_m - s_m\|}$ is bounded from above, and thus $\alpha_m$ is bounded away from zero as $m \to \infty$. By passing to a subsequence we can assume that $\lim_{m \to \infty} \alpha_m = \alpha_\infty > 0$, and that $\lim_{m \to \infty} s_m = s_\infty \in \mathcal{Z}_H$ since $\mathcal{Z}_H$ is compact. Therefore, $d_\infty = \lim_{m \to \infty} d_m$ exists in $\mathcal{D}$, and $x_\infty = \alpha_\infty s_\infty + (1 - \alpha_\infty) d_\infty$. But then $(1 - \alpha_\infty)d_\infty = 0$ by assumptions of the lemma.

Thus $\lim_{m \to \infty} \alpha_m s_m = x_\infty$. If $x_\infty = \alpha_m s_m$ for some $m$, then by (1) we have $x_m \in (x_\infty, x'_m)$, where $x'_m = (1 - \alpha_m)d_m$. Therefore, I can assume that $x_m \neq \alpha_m s_m$ for all $m$. Since $x_\infty \in \text{relint}\mathcal{Z}$ by assumption and $\alpha_m s_m \in \mathcal{Z}$, for $m \gg 0$ there exist $\hat{s}_m \in \mathcal{Z}$ and $t_m \in (0, 1)$ such that $\alpha_m s_m = t_m x_\infty + (1 - t_m) \hat{s}_m$. Let $\hat{d}_m = \frac{1}{1 - t_m} d_m \in \mathcal{D}$ and set $x'_m = \hat{s}_m + \hat{d}_m$. Then it is easy to check that $x_m = t_m x_\infty + (1 - t_m) x'_m$, and we are done.
Remark 2.6. The following situation will appear in the proof of Theorem 9.1. Let $\mathcal{H}_i$ be finitely many half-spaces of $\mathbb{R}^\ell$ bounded by affine hyperplanes $\mathcal{H}_i$, and let $\mathcal{D} = \bigcap_i \mathcal{H}_i$. Let $x_m \in \mathbb{R}^\ell \setminus \mathcal{D}$ be a convergent sequence of points and fix $z \in \mathbb{R}^\ell$. Assume that for each $m \in \mathbb{N} \cup \{\infty\}$ there exists a point $y_m \in (x_m + \mathbb{R}_- z) \cap \partial \mathcal{D}$ closest to $x_m$, and that $x_\infty = \lim_{m \to \infty} x_m \in \mathcal{H}_i$ for all $i$. Then I claim that a subsequence of $y_m$ converges to $x_\infty$. To see this, by passing to a subsequence I can assume that $y_m \in \mathcal{H}_{i_0}$ for all $m$ and for a fixed $i_0$. If $\mathbb{R}z \cap \mathcal{H}_{i_0} = \emptyset$, then $x_m \in \mathcal{H}_{i_0}$ for all $m$, and by replacing $\mathbb{R}^\ell$ by $\mathcal{H}_{i_0}$ we can finish by induction on $\ell$. If $\mathbb{R}z \cap \mathcal{H}_{i_0} \neq \emptyset$, then $\{y_m\} = (x_m + \mathbb{R}_- z) \cap \mathcal{H}_{i_0}$, and it is easy to see that $\lim_{m \to \infty} y_m = x_\infty$.

Asymptotic invariants. The standard references on asymptotic invariants arising from linear series are [Nak04, ELM+06].

Definition 2.7. Let $X$ be a variety and $D \in \text{WDiv}(X)_{\mathbb{R}}$. For $k \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, define
$$|D|_k = \{C \in \text{WDiv}(X)_k : C \geq 0, C \sim_k D\}.$$ If $T$ is a prime divisor on $X$ such that $T \not\subseteq \text{Fix}(D)$, then $|D|_T$ denotes the image of the linear system $|D|$ under restriction to $T$. The stable base locus of $D$ is $B(D) = \bigcap_{C \in |D|_{\mathbb{R}}} \text{Supp} \ C$ if $|D|_{\mathbb{R}} \neq \emptyset$, otherwise we set $B(D) = X$. The diminished base locus is $B_-(D) = \bigcup_{\ell > 0} B(D + \varepsilon A)$ for an ample divisor $A$; this does not depend on a choice of $A$. In particular, $B_-(D) \subset B(D)$.

We denote $\text{WDiv}(X)^{\kappa \geq 0} = \{D \in \text{WDiv}(X) : |D|_{\mathbb{R}} \neq \emptyset\}$, and similarly for $\text{Div}(X)^{\kappa \geq 0}$ and for versions of these sets with subscripts $\mathbb{Q}$ and $\mathbb{R}$. Observe that when $D \in \text{WDiv}(X)$, the condition $|D|_{\mathbb{R}} \neq \emptyset$ is equivalent to $\kappa(X, D) \geq 0$ by Lemma 2.12 below, where $\kappa$ is the Iitaka dimension.

It is elementary that $B(D_1 + D_2) \subset B(D_1) \cup B(D_2)$ for $D_1, D_2 \in \text{WDiv}(X)_{\mathbb{R}}$. In other words, the set $\{D \in \text{WDiv}(X)_{\mathbb{R}} : x \not\in B(D)\}$ is convex for every point $x \in X$. By [BCHM06, 3.5.3], $B(D) = \bigcap_{C \in |D|_{\mathbb{Q}}} \text{Supp} \ C$ when $D$ is a $\mathbb{Q}$-divisor, which is the standard definition of the stable base locus.

Definition 2.8. Let $Z$ be a closed subvariety of a smooth variety $X$ and let $D \in \text{Div}(X)^{\kappa \geq 0}_{\mathbb{R}}$. The asymptotic order of vanishing of $D$ along $Z$ is
$$\text{ord}_Z \|D\| = \inf \{\text{mult}_Z C : C \in |D|_{\mathbb{R}}\}.$$ Remark 2.9. In the case of rational divisors, the infimum above can be taken over rational divisors, see Lemma 2.12 below. More generally, one can consider any discrete valuation $\nu$ of $k(X)$ and define
$$\nu \|D\| = \inf \{\nu(C) : C \in |D|_{\mathbb{Q}}\}$$ for an effective $\mathbb{Q}$-divisor $D$. Then [ELM+06] shows that $\nu \|D_1\| = \nu \|D_2\|$ if $D_1$ and $D_2$ are numerically equivalent big divisors, and that $\nu$ extends to a sublinear function on $\text{Big}(X)_{\mathbb{R}}$. When $E$ is a prime divisor on a birational model over $X$, I write $\text{ord}_E \| \ |$ for the corresponding geometric valuation.
Remark 2.10. Nakayama [Nak04] defines a function $\sigma_Z : \text{Big}(X) \to \mathbb{R}_+$ by
\[
\sigma_Z(D) = \liminf_{\varepsilon \to 0} \|D + \varepsilon A\|
\]
for any ample $\mathbb{R}$-divisor $A$, and shows that it agrees with $\text{ord}_Z \| \cdot \|$ on big classes. Then we define the formal sum $N_\sigma \|D\| = \sum \sigma_Z(D) \cdot Z$ over all prime divisors $Z$ on $X$. Analytic properties of these invariants were studied in [Bou04].

We now define the restricted version of the invariant introduced.

Definition 2.11. Let $S$ be a smooth divisor on a smooth variety $X$ and let $D \in \text{Div}(X)^{\geq 0}_{\mathbb{R}}$ be such that $S \not\subset \mathcal{B}(D)$. Let $P$ be a closed subvariety of $S$. The restricted asymptotic order of vanishing of $D$ along $P$ is
\[
\text{ord}_P \|D\|_S = \inf \{ \text{mult}_P C|_S : C \in |D|_{\mathbb{R}}, S \not\subset \text{Supp}C \}.
\]

Lemma 2.12. Let $X$ be a smooth variety, $D \in \text{Div}(X)^{\geq 0}_{\mathbb{Q}}$ and let $D' \geq 0$ be an $\mathbb{R}$-divisor such that $D \sim \mathbb{R} D'$. Then for every $\varepsilon > 0$ there is a $\mathbb{Q}$-divisor $D'' \geq 0$ such that $D \sim \mathbb{Q} D''$. If $A$ is an ample $\mathbb{R}$-divisor such that $D \sim \mathbb{R} A$, then for every closed subvariety $P \subset S$ we have
\[
\text{ord}_P \|D\|_S = \inf \{ \text{mult}_P C|_S : C \in |D|_{\mathbb{Q}}, S \not\subset \text{Supp}C \}.
\]

Proof. Let $D' = D + \sum_{j=1}^p r_i(f_i)$ for $r_i \in \mathbb{R}$ and $f_i \in k(X)$. Let $F_1, \ldots, F_N$ be the components of $D$ and of all $(f_i)$, and assume that $\text{mult}_{F_j} D' = 0$ for $j \leq \ell$ and $\text{mult}_{F_j} D' > 0$ for $j > \ell$. Let $(f_i) = \sum_{j=1}^N \phi_i, F_j$ for all $i$, and $D = \sum_{j=1}^N \delta_j F_j$. Then we have $\delta_j + \sum_{j=1}^p \delta_j r_i = 0$ for $j = 1, \ldots, \ell$. Let $\mathcal{H} \subset \mathbb{R}^p$ be the space of solutions of the system $\sum_{j=1}^p \phi_{ij} x_i = -\delta_j$ for $j = 1, \ldots, \ell$. Then $\mathcal{H}$ is a rational affine subspace and $(r_1, \ldots, r_p) \in \mathcal{H}$, thus for $0 < \eta < 1$ there is a rational point $(s_1, \ldots, s_p) \in \mathcal{H}$ with $|s_i - r_i| < \eta$ for all $i$. Therefore for $\eta$ sufficiently small, setting $D'' = D + \sum_{i=1}^p s_i(f_i)$ we have the desired properties. \square

Remark 2.13. Let $\mathcal{B}_S(X) \subset \text{Big}(X)$ be the set of classes of divisors $D$ such that $S \not\subset \mathcal{B}_-(D)$. Similarly as in Remark 2.10 [Hac08] introduces the function $\sigma_P \| \cdot \|_S : \mathcal{B}_S(X) \to \mathbb{R}_+$ by
\[
\sigma_P \|D\|_S = \liminf_{\varepsilon \to 0} \|D + \varepsilon A\|_S
\]
for any ample $\mathbb{R}$-divisor $A$. Then one can define a formal sum $N_\sigma \|D\|_S = \sum \sigma P \|D\|_S \cdot P$ over all prime divisors $P$ on $S$. If $S \not\subset \mathcal{B}(D)$, then $\lim_{\varepsilon \to 0} \text{ord}_P \|D + \varepsilon A\|_S = \text{ord}_P \|D + \varepsilon_0 A\|_S$ for every $\varepsilon_0 > 0$ and for any ample divisor $A$ on $X$ similarly as in [Nak04] 2.1.1.

In this paper I need a few basic properties cf. [Hac08].

Lemma 2.14. Let $S$ be a smooth divisor on a smooth projective variety $X$ and let $P$ be a closed subvariety of $S$.

1. Let $D \in \text{Div}(X)^{\geq 0}_{\mathbb{R}}$ be such that $S \not\subset \mathcal{B}(D)$. If $A$ is an ample $\mathbb{R}$-divisor on $X$, then $\text{ord}_P \|D + A\|_S \leq \text{ord}_P \|D\|_S$, and in particular $\sigma_P \|D\|_S \leq \text{ord}_P \|D\|_S$. 


(2) If $D \in \mathcal{B}(X)$, and if $A_m$ is a sequence of ample $\mathbb{R}$-divisors on $X$ such that
\[
\lim_{m \to \infty} \|A_m\| = 0,
\]
then
\[
\lim_{m \to \infty} \text{ord}_p \|D + A_m\|_S = \sigma_p \|D\|_S.
\]

(3) If $D, E \in \mathcal{B}(X)$, then
\[
\lim_{\varepsilon \to 0} \sigma_p \|(1 - \varepsilon)D + \varepsilon E\|_S = \sigma_p \|D\|_S.
\]

(4) Let $D$ be a pseudo-effective $\mathbb{Q}$-divisor on $X$ such that $\sigma_p \|D\|_S = 0$. If $A$ is an
ample $\mathbb{Q}$-divisor on $X$, then there is $l \in \mathbb{Z}_{>0}$ such that $\text{mult}_p \text{Fix} |l(D + A)|_S = 0$.

Proof. Statement (1) is trivial. The proof of (2) is standard: fix an ample divisor $A$ on $X$, and let $0 < \varepsilon \ll 1$. For $m \gg 0$ the divisor $\varepsilon A - A_m$ is ample, and so by (1) we have
\[
\text{ord}_p \|D + \varepsilon A\|_S = \text{ord}_p \|D + A_m + (\varepsilon A - A_m)\|_S \leq \text{ord}_p \|D + A_m\|_S.
\]

Letting $m \to \infty$, and then $\varepsilon \downarrow 0$ we obtain
\[
\sigma_p \|D\|_S \leq \lim_{m \to \infty} \text{ord}_p \|D + A_m\|_S,
\]
and similarly for the opposite inequality.

For (3), let $A$ be an ample $\mathbb{Q}$-divisor such that $E - D + A$ is ample. Then by convexity,
\[
\sigma_p \|D\|_S = \lim_{\varepsilon \to 0} \sigma_p \|D + \varepsilon (E - D + A)\|_S \leq \lim_{\varepsilon \to 0} \sigma_p \|D + \varepsilon (E - D)\|_S
\]
\[
\leq \lim_{\varepsilon \to 0} \left( (1 - \varepsilon) \sigma_p \|D\| + \varepsilon \sigma_p \|E\|_S \right) = \sigma_p \|D\|_S,
\]
thus the desired equality follows.

Finally, for (4), set $n = \dim X$, let $H$ be a very ample divisor on $X$, and fix a positive
integer $l$ such that $l(D + A)$ is Cartier and $H^l = \frac{l}{2}A - (K_X + S) - (n + 1)H$ is ample. Since
\[
\text{ord}_p \|D + \frac{1}{2}A\|_S \leq \sigma_p \|D\|_S = 0
\]
by (1), there exists a $\mathbb{Q}$-divisor $\Delta \sim Q D + \frac{1}{2}A$ such that
$S \not\subset \text{Supp} \Delta$ and $\text{mult}_p \Delta\|_S < 1/l$. Since $l(D + A)|_S \sim Q K_S + l\Delta|_S + H^l|_S + (n + 1)H|_S$, by
Nadel vanishing we have
\[
H^l(S, \mathcal{J}_{l\Delta|_S}((l(D + A) + mh)) = 0
\]
for $m \geq -n$, and so the sheaf $\mathcal{J}_{l\Delta|_S}(l(D + A))$ is globally generated by \cite[5.7]{HM08} and
its sections lift to $H^0(X, l(D + A))$ by \cite[4.4(3)]{HM08}. Since $\text{mult}_p (l\Delta|_S) < 1$, the pair
$(S, l\Delta|_S)$ is klt around the generic point $\eta$ of $P$. Therefore the sheaf $\mathcal{J}_{l\Delta|_S}$ is trivial at $\eta$, and so $\text{mult}_p \text{Fix} |l(D + A)|_S = 0$.

Remark 2.15. Analogously to Lemma \cite[2.14]{HM08}, one can prove that if $D$ is a pseudo-effective
$\mathbb{R}$-divisor such that $\sigma_Z \|D\| = 0$ for a closed subvariety of $Z$ of $X$, then $Z \not\subset \mathcal{B}(D)$. In particular, if $D|_Z$ is defined, then $D|_Z - N_D \|D\|_Z$ is pseudo-effective. Further, let $f : Y \to X$ be a log resolution and denote $Z' = f^{-1}_*Z$. Then I claim $\sigma_{Z'} \|f^*D\| = 0$. To that end, we have first that $Z \not\subset \mathcal{B}(D + \varepsilon A)$ for an ample divisor $A$ and for any $\varepsilon > 0$. 

\[\square\]
Therefore $Z' \not\subset B(f^*D + \varepsilon f^*A)$, and thus $\sigma_{Z'}(f^*D + \varepsilon f^*A) \leq \text{ord}_{Z'}(f^*D + \varepsilon f^*A) = 0$. But then

$$\sigma_{Z'}(f^*D) = \lim_{\varepsilon \to 0} \sigma_{Z'}(f^*D + \varepsilon f^*A) = 0$$

by [Nak04, 2.1.4(2)].

**Convex sets in** $\text{WDiv}(X)_\mathbb{R}$. Let $X$ be a variety and let $V$ be a finite dimensional affine subspace of $\text{WDiv}(X)_\mathbb{R}$. Fix an ample $\mathbb{Q}$-divisor $A$ and a prime divisor $G$ on $X$, and define

$$\mathcal{L}_V = \{ \Phi \in V : K_X + \Phi \text{ is log canonical} \},$$

$$\mathcal{E}_{V,A} = \{ \Phi \in \mathcal{L}_V : K_X + \Phi + A \text{ is pseudo-effective} \},$$

$$\mathcal{R}_{V,A}^G = \{ \Phi \in \mathcal{L}_V : G \not\subset B(K_X + \Phi + A) \},$$

$$\mathcal{R}_{V,A}^{G^=} = \{ \Phi \in \mathcal{L}_V : \text{mult}_G \Phi = 1, G \not\subset B(K_X + \Phi + A) \}.$$

If $V$ is a rational affine subspace, the set $\mathcal{L}_V$ is a rational polytope by [BCHM06, 3.7.2]. Similarly as in Lemma 5.8 below, one can prove that Theorem 1.2 implies that then also $\mathcal{E}_{V,A}, \mathcal{R}_{V,A}^G$ and $\mathcal{R}_{V,A}^{G^=}$ are rational polytopes.

**Remark 2.16.** Assume the notation as above. In the proofs of Theorems 8.1 and 9.1 we will have the following situation. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be properties of divisor classes in $N^1(X)$; namely, assume $\mathcal{P}_1$ is the property that the class is pseudo-effective, and $\mathcal{P}_2$ that $\sigma_G(\psi) = 0$ for a class $\psi \in \text{Big}(X)$. Denote $\mathcal{P}_{V,A}^1 = \{ \Phi \in \mathcal{L}_V : K_X + \Phi + A \text{ has } \mathcal{P}_1 \}$, $\mathcal{P}_{V,A}^2 = \{ \Phi \in \mathcal{L}_V : \text{mult}_G \Phi = 1, K_X + \Phi + A \text{ has } \mathcal{P}_2 \}$ and $\mathcal{C}^i = \mathbb{R}_+(K_X + A + \mathcal{P}_{V,A}^i) \subset \text{Div}(X)_\mathbb{R}$. Assume that we know that $\mathcal{P}_{V,A}^i$ are closed convex sets, and in particular that $\mathcal{C}^i$ are closed cones, and that we need to prove that $\mathcal{C}^i$ are polyhedral.

The strategy is as follows. Fix $i$ and assume the contrary, i.e. that $\mathcal{C}^i$ has infinitely many extremal rays. Then there are distinct divisors $\Delta_m \in \mathcal{P}_{V,A}^i$ for $m \in \mathbb{N} \cup \{ \infty \}$ such that the rays $\mathbb{R}_+ \Gamma_m$ are extremal in $\mathcal{C}^i$ and $\lim_{m \to \infty} \Delta_m = \Delta_\infty$, where $\Gamma_m = K_X + \Delta_m + A$. I achieve contradiction by showing that for some $m \gg 0$ there is a point $\Gamma'_m \in \mathcal{C}^i$ such that $\Gamma_m \in (\Gamma_\infty, \Gamma'_m)$, so that the ray $\mathbb{R}_+ \Gamma_m$ cannot be extremal in $\mathcal{C}^i$.

I make the following observations. Let $W$ be the vector space spanned by the components of $K_X, A$ and by the prime divisors in $V$. I claim that we can assume that $[\Gamma_m] \neq [\Gamma_\infty]$ for all $m \gg 0$. Assuming the contrary and passing to a subsequence, we have $[\Gamma_m] = [\Gamma_\infty]$ for all $m$, and let $\phi : W \to N^1(X)$ be the map sending a divisor to its numerical class. Then since $\phi^{-1}([\Gamma_\infty])$ is an affine subspace of $W$, there is a divisor $\Phi_m \in \phi^{-1}([\Gamma_\infty])$ such that $\Gamma_m \in (\Gamma_\infty, \Phi_m)$, and note that $\Phi_m$ has $\mathcal{P}^i$ since $[\Phi_m] = [\Gamma_\infty]$. Since $\mathbb{R}_+(K_X + A + \mathcal{L}_V)$ is a rational polyhedral cone, for $m \gg 0$ we have

$$[\Gamma_\infty, \Gamma_m] \subset (\mathbb{R}_+ + \mathbb{R}_+(\Gamma_m - \Gamma_\infty)) \cap \mathbb{R}_+(K_X + A + \mathcal{L}_V),$$

so in particular there exists a divisor $\Gamma'_m \in (\Gamma_m, \Phi_m) \cap \mathbb{R}_+(K_X + A + \mathcal{L}_V)$ which has $\mathcal{P}^i$ since $[\Gamma'_m] = [\Gamma_\infty]$. Also $\text{mult}_G \Gamma'_m = \text{mult}_G \Gamma_\infty$, so $\Gamma'_m \in \mathcal{C}^i$ and $\mathbb{R}_+ \Gamma_m \subset \text{relint}(\mathbb{R}_+ \Gamma_\infty + \mathbb{R}_+ \Gamma_m)$, which implies that $\mathbb{R}_+ \Gamma_m$ is not an extremal ray of $\mathcal{C}^i$. 


Further, I claim that in order to achieve contradiction, it is enough to prove that for 
$m \gg 0$, there is a class $\hat{\Phi}_m \in N^1(X)$ which has $\mathcal{H}^i$ and a real number $0 < t < 1$ such that 
$[\Gamma_m] = t[\Upsilon_\infty] + (1 - t)\hat{\Phi}_m$. To see this, let $\Phi_m = \frac{1}{1 - t}(\Gamma_m - t\Upsilon_\infty)$. Then $[\Phi_m] = \hat{\Phi}_m$, and thus $\Phi_m$ has $\mathcal{H}^i$. Since $\Gamma_m \in (\Upsilon_\infty, \Phi_m)$ and $\mathcal{H}_{V,A}^i$ is convex, we finish the proof of the claim as above.

Therefore, I am allowed to, and will without explicit mention in the proofs of Theorems 8.1 and 9.1, increase $V$ and consider divisors up to $\mathbb{R}$-linear equivalence, since this does not change their numerical classes.

3. OUTLINE OF THE INDUCTION

As part of the induction, I will prove the following three theorems.

**Theorem A.** Let $X$ be a smooth projective variety, and let $D_i = k_i(K_X + \Delta_i + A) \in \text{Div}(X)$, where $A$ is an ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a log smooth log canonical pair with $|D_i| \neq \emptyset$ for $i = 1, \ldots, \ell$. Then the adjoint ring $R(X; D_1, \ldots, D_\ell)$ is finitely generated.

**Theorem B.** Let $X$ be a smooth projective variety, $B$ a simple normal crossings divisor and $A$ a general ample $\mathbb{Q}$-divisor on $X$. Let $V \subset \text{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of $B$. Then for any component $G$ of $B$, the set $\mathcal{R}^{G = 1}_{V,A}$ is a rational polytope, and we have

$$ \mathcal{R}^{G = 1}_{V,A} = \{ \Phi \in \mathcal{L}_V : \text{mult}_G \Phi = 1, \sigma_G(K_X + \Phi + A) = 0 \}.$$ 

**Theorem C.** Let $X$ be a smooth projective variety, $B$ a simple normal crossings divisor and $A$ a general ample $\mathbb{Q}$-divisor on $X$. Let $V \subset \text{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of $B$. Then the set $\mathcal{E}_{V,A}$ is a rational polytope, and we have

$$ \mathcal{E}_{V,A} = \{ \Phi \in \mathcal{L}_V : |K_X + \Phi + A|_{\mathbb{R}} \neq 0 \}.$$ 

Let me give an outline of the paper, where e.g. “Theorem $A_n$” stands for “Theorem A in dimension $n$.”

Sections 4 and 5 develop tools to deal with algebras of higher rank and to test whether functions are piecewise linear. Section 6 contains results from Diophantine approximation which will be necessary in Sections 7, 8 and 9.

In 8 I prove that Theorems $A_{n-1}$ and $C_{n-1}$ imply Theorem $B_n$, and this part of the proof uses techniques from 7. In 9 I show how Theorems $A_{n-1}$, $B_n$ and $C_{n-1}$ imply Theorem $C_n$. Finally, Sections 7 and 10 contain the proof that Theorems $A_{n-1}$, $B_n$ and $C_{n-1}$ imply Theorem $A_n$. Section 7 is technically the most difficult part of the proof, whereas 10 contains the main new idea on which the whole paper is based.

At the end of this section, let me sketch the proofs of Theorems A, B and C when $X$ is a curve of genus $g$. Since by Riemann-Roch the condition that a divisor $E$ on $X$ is pseudo-effective is equivalent to $\deg E \geq 0$, and this condition is linear on the coefficients, this proves Theorem C. For Theorem A, when $g \geq 1$ every divisor $D_i$ is ample, and when $g = 0$, since $\deg D_i \geq 0$ we have that $D_i$ is basepoint free, so the statement follows from [HK00].
2.8] Furthermore, this shows that every divisor of the form \( K_X + \Phi + A \) is semiample, so \( \mathcal{R}_{V,A}^G = \mathcal{E}_{V,A} \) and Theorem B follows.

4. Convex geometry

Results of this section will be used in the rest of the paper to study relations between superadditive and superlinear functions, and to test their piecewise linearity. The following proposition can be found in [HUL93] and I add the proof for completeness.

**Proposition 4.1.** Let \( \mathcal{C} \subset \mathbb{R}^n \) be a cone and \( f: \mathcal{C} \to \mathbb{R} \) a concave function. Then \( f \) is locally Lipschitz continuous on the topological interior of \( \mathcal{C} \) with respect to any norm \( || \cdot || \) on \( \mathbb{R}^n \).

In particular, if \( \mathcal{C} \) is rational polyhedral and \( g: \mathcal{C}_Q \to \mathbb{Q} \) is a superadditive map which satisfies \( g(\lambda x) = \lambda g(x) \) for all \( x \in \mathcal{C}_Q, \lambda \in \mathbb{Q}_+ \), then \( g \) extends in a unique way to a superlinear function on \( \mathcal{C} \).

**Proof.** Since \( f \) is locally Lipschitz if and only if \( -f \) is locally Lipschitz, we can assume \( f \) is convex. Fix \( 0 \neq x = (x_1, \ldots, x_n) \in \text{int} \mathcal{C} \), and let \( \Delta = \{(y_1, \ldots, y_n) \in \mathbb{R}_+^n : \sum y_i \leq 1 \} \). It is easy to check that translations of the domain do not affect the result, so we may assume \( x \in \text{int} \Delta \subset \text{int} \mathcal{C} \).

First, let us prove that \( f \) is locally bounded from above around \( x \). Let \( e_i \) be the standard basis vectors of \( \mathbb{R}^n \) and set \( M = \max \{ f(0), f(e_1), \ldots, f(e_n) \} \). If \( y = (y_1, \ldots, y_n) \in \Delta \) and \( y_0 = 1 - \sum y_i \geq 0 \), then

\[
    f(y) = f\left( \sum y_i e_i + y_0 \cdot 0 \right) \leq \sum y_i f(e_i) + y_0 f(0) \leq M.
\]

Now choose \( \delta \) such that \( B(x, 2\delta) \subset \Delta \). Again by translating the domain and composing \( f \) with a linear function we may assume that \( x = 0 \) and \( f(0) = 0 \). Then for all \( y \in B(0, 2\delta) \) we have

\[
    -f(y) = -f(y) + 2f(0) \leq -f(y) + (f(y) + f(-y)) = f(-y) \leq M,
\]

so \( |f| \leq M \) on \( B(0, 2\delta) \).

Set \( L = 2M/\delta \). Fix \( u, v \in B(0, \delta) \), and set \( \alpha = \frac{1}{\delta} \|v - u\| \) and \( w = v + \frac{1}{\alpha}(v - u) \in B(0, 2\delta) \), so that \( v = \frac{\alpha}{\alpha + 1} w + \frac{1}{\alpha + 1} u \). Then by convexity,

\[
    f(v) - f(u) \leq \frac{\alpha}{\alpha + 1} f(w) + \frac{1}{\alpha + 1} f(u) - f(u)
    = \frac{\alpha}{\alpha + 1} (f(w) - f(u)) \leq 2M\alpha = L \|v - u\|,
\]

and similarly \( f(u) - f(v) \leq L \|u - v\| \), which proves the first claim.

For the second one, observe that the sup-norm \( \| \cdot \|_\infty \) takes values in \( \mathbb{Q} \) on \( \mathcal{C}_Q \). The proof above applied to the interior of \( \mathcal{C} \) and to the relative interiors of the faces of \( \mathcal{C} \) shows that \( g \) is locally Lipschitz, and the claim follows.

The following result is classically referred to as Gordan’s lemma, and I often use it without explicit mention.
Lemma 4.2. Let $\mathcal{I} \subset \mathbb{N}^r$ be a finitely generated monoid and let $\mathcal{C} \subset \mathbb{R}^r$ be a rational polyhedral cone. Then the monoid $\mathcal{I} \cap \mathcal{C}$ is finitely generated.

Proof. Assume first that $\dim \mathcal{C} = r$. Let $\ell_1, \ldots, \ell_m$ be linear functions on $\mathbb{R}^r$ with integral coefficients such that $\mathcal{C} = \bigcap_{i=1}^m \{ z \in \mathbb{R}^r : \ell_i(z) \geq 0 \}$, and define $\mathcal{I}_0 = \mathcal{I}$ and $\mathcal{I}_i = \mathcal{I}_{i-1} \cap \{ z \in \mathbb{R}^r : \ell_i(z) \geq 0 \}$ for $i = 1, \ldots, m$; observe that $\mathcal{I} \cap \mathcal{C} = \mathcal{I}_m$. Assuming by induction that $\mathcal{I}_{i-1}$ is finitely generated, by [Swa92, Theorem 4.4] we have that $\mathcal{I}_i$ is finitely generated.

Now assume $\dim \mathcal{C} < r$ and let $\mathcal{H}$ be a rational hyperplane containing $\mathcal{C}$. Let $\ell$ be a linear function with integral coefficients such that $\mathcal{H} = \ker(\ell)$. The monoid $\mathcal{I} \cap \mathcal{H}$ is finitely generated by the first part of the proof applied to the functions $\ell$ and $-\ell$. Now we conclude by induction on $r$. $\square$

The next lemma will turn out to be indispensable and it shows that it is enough to check additivity of a superadditive map at one point only.

Lemma 4.3. Let $\mathcal{I} = \sum \mathbb{N} e_i$ be a finitely generated monoid and let $f : \mathcal{I} \rightarrow G$ be a superadditive map to an ordered monoid $G$ (respectively let $f : \mathcal{I}_{\mathbb{R}} \rightarrow V$ be a superlinear map to a cone $V$ with an ordering). Assume that there is a point $s_0 = \sum s_i e_i \in \mathcal{I}$ with all $s_i > 0$, such that $f(s_0) = \sum s_i f(e_i)$ and $f(\kappa s_0) = \kappa f(s_0)$ for every positive integer $\kappa$ (respectively there is a point $s_0 = \sum s_i e_i \in \mathcal{I}_{\mathbb{R}}$ with all $s_i > 0$ such that $f(s_0) = \sum s_i f(e_i)$). Then $f$ is additive (respectively linear).

Proof. I prove the lemma when $f$ is superadditive, the other claim is proved analogously. For $p = \sum p_i e_i \in \mathcal{I}$, choose $\kappa_0 \in \mathbb{Z}_{>0}$ so that $\kappa_0 s_i \geq p_i$ for all $i$. Then

$$\sum \kappa_0 s_i f(e_i) = \kappa_0 f(s_0) = f(\kappa_0 s_0) \geq f(p) + \sum f((\kappa_0 s_i - p_i) e_i) \geq \sum p_i f(e_i) + \sum (\kappa_0 s_i - p_i) f(e_i) = \sum \kappa_0 s_i f(e_i).$$

Therefore all inequalities are equalities and $f(p) = \sum p_i f(e_i)$. $\square$

Now we are ready to prove the main result of this section, which will be crucial.

Theorem 4.4. Let $f$ be a superlinear function on a polyhedral cone $\mathcal{C} \subset \mathbb{R}^{r+1}$ with $\dim \mathcal{C} = r + 1$, such that for every 2-plane $H \subset \mathbb{R}^{r+1}$ the function $f|_{\mathcal{C} \cap \mathcal{C}}$ is piecewise linear. Then $f$ is piecewise linear.

Proof. I prove the lemma by induction on $r$. In the proof, $\| \cdot \|$ denotes the standard Euclidean norm and $S^r \subset \mathbb{R}^{r+1}$ is the unit sphere.

Step 1. Fix a ray $R \subset \mathcal{C}$. In this step I prove there is a collection of $(r+1)$-dimensional polyhedral cones $\{ \mathcal{C}_\alpha \}_{\alpha \in \mathcal{I}_R}$ with $\mathcal{C}_\alpha \subset \mathcal{C}$, such that

(i) for every $c \in \mathcal{C} \setminus R$ there is $\alpha \in \mathcal{I}_R$ such that $R \subset \mathcal{C}_\alpha \cap (R + \mathbb{R}_+ c)$,

(ii) for every $\alpha \in \mathcal{I}_R$ the map $f|_{\mathcal{C}_\alpha}$ is linear,

(iii) for every two distinct $\alpha, \beta \in \mathcal{I}_R$ the linear extensions of $f|_{\mathcal{C}_\alpha}$ and $f|_{\mathcal{C}_\beta}$ to $\mathbb{R}^{r+1}$ are different.
Fix \( c \in C \setminus R \), and choose a hyperplane \( \mathcal{H}_r \supset R + \mathbb{R}+c \). By induction there is an \( r \)-dimensional polyhedral cone \( \mathcal{C}_r = \sum_{i=1}^{r+1} \mathbb{R}+e_i \subset \mathcal{H}_r \cap C \) such that \( R \not\subset \mathcal{C}_r \supset (R + \mathbb{R}+c) \) and \( f|_{\mathcal{C}_r} \) is linear. Then \( f(e_0) = \sum_{i=1}^{r+1} f(e_i) \), where \( e_0 = \sum_{i=1}^{r+1} e_i \), and let \( \mathcal{P} \) be a 2-plane such that \( \mathcal{P} \cap \mathcal{H}_r = \mathbb{R}+e_0 \). By assumption, there is a point \( e_{r+1} \in (\mathcal{P} \cap C) \setminus \mathbb{R}+e_0 \) such that \( f|_{\mathbb{R}+e_0 + \mathbb{R}+e_{r+1}} \) is linear, and in particular \( f(e_0 + e_{r+1}) = f(e_0) + f(e_{r+1}) \). Setting \( \mathcal{C}_{r+1} = \sum_{i=1}^{r+1} \mathbb{R}+e_i \), we have

\[
\sum_{i=1}^{r+1} e_i = f(e_0 + e_{r+1}) = f(e_0) + f(e_{r+1}) = \sum_{i=1}^{r+1} f(e_i),
\]

so the map \( f|_{\mathcal{C}_{r+1}} \) is linear by Lemma 4.3. Let \( \ell \) be the linear extension of \( f|_{\mathcal{C}_{r+1}} \) to \( \mathbb{R}^{r+1} \), and set \( C_c = \{ z \in C : f(z) = \ell(z) \} \). I claim \( C_c \) is a closed cone. To that end, if \( u,v \in C_c \), then there are real numbers \( u_i,v_i \) such that \( u = \sum_{i=1}^{r+1} u_i e_i \) and \( v = \sum_{i=1}^{r+1} v_i e_i \), and set \( e = \sum_{i=1}^{r+1} (1 + |u_i| + |v_i|) e_i \). Note that \( e \) and \( e + u + v \) belong to \( C_{r+1} \subset C_c \), thus

\[
f(e + u + v) = \ell(e + u + v) = \ell(e) + \ell(u) + \ell(v) = f(e) + f(u) + f(v),
\]

so \( f \) is linear on the cone \( C_{r+1} + \mathbb{R}+u + \mathbb{R}+v \) by Lemma 4.3. In particular \( f(u + v) = f(u) + f(v) = \ell(u) + \ell(v) = \ell(u + v) \), hence \( C_c \) is a cone. Denote by \( \mathcal{Q} \) the closure of \( C_c \), and fix \( q \in \mathcal{Q} \). Then for every \( p \in C_c \), the function \( f|_{\mathbb{R}+p+\mathbb{R}+q} \) is piecewise linear by assumption, and in particular continuous. Since \( \mathbb{R}+p + \mathbb{R}+0q \subset \text{int} \mathcal{Q} \subset C_c \), and \( f \) and \( \ell \) agree on \( C_c \), this implies that \( f \) is linear on \( \mathbb{R}+p + \mathbb{R}+q \), so \( C_c \) is closed. Now, by varying \( c \in C \setminus R \) we obtain the desired collection of cones.

**Step 2.** In this step I prove that \( \mathcal{I} \) is a finite set for every ray \( R \subset C \).

Fix \( R \), and arguing by contradiction assume that \( \mathcal{I} \) is infinite; we can assume that \( \mathbb{N} \subset \mathcal{I} \). For each \( n \in \mathbb{N} \) choose \( x_n \in \text{int} C \setminus R \), and denote \( \mathcal{H}_n = (R + \mathbb{R}+x_n) \cup (-R + \mathbb{R}+x_n) \). Let \( R_n \subset \mathcal{H}_n \) be the unique ray orthogonal to \( R \) and let \( S' \cap R_n = \{ Q_n \} \). By passing to a subsequence, we can assume that points \( Q_n \) converge to \( Q_\infty \in S' \), and set \( \mathcal{H}_\infty = (R + \mathbb{R}+Q_\infty) \cup (-R + \mathbb{R}+Q_\infty) \).

Pick \( x \in R \setminus \{0\} \). By assumption, there is a point \( y \in \mathcal{H}_\infty \setminus R \) such that \( f|_{\mathbb{R}+R+y} \) is linear, and in particular \( f(x+y) = f(x) + f(y) \). If \( \mathcal{H}' \) is a hyperplane such that \( \mathcal{H}' \cap (\mathbb{R}+\mathbb{R}y) = \mathbb{R}+(x+y) \) and \( \mathcal{H}' \cap C \) is an \( r \)-dimensional cone, by induction there are finitely many \( r \)-dimensional polyhedral cones \( \mathcal{Q}_i \subset \mathcal{H}' \cap C \) containing \( x+y \) such that the map \( f|_{\mathcal{Q}_i} \) is linear for every \( i \), and for every \( c \in (\mathcal{H}' \cap C) \setminus (\mathbb{R}+x+y) \) we have \( \mathbb{R}+x+y \subset \bigcup_i \mathcal{Q}_i \cap (\mathbb{R}+x+y) + \mathbb{R}+c \). If \( g_{ij} \) are finitely many generators of \( \mathcal{Q}_i \), then

\[
f(\sum_j g_{ij} + x+y) = \sum_j f(g_{ij}) + f(x+y) = \sum f(g_{ij}) + f(x) + f(y),
\]

so \( f \) is linear on the cone \( \mathcal{Q}_i \) by Lemma 4.3.

I claim that for every \( c \in C \setminus (\mathbb{R}+x+y) \) we have \( \mathbb{R}+x+y \subset \bigcup_i \mathcal{Q}_i \cap (\mathbb{R}+x+y) + \mathbb{R}+c \), and in particular there exists \( 0 < \varepsilon < 1 \) such that \( \mathbb{R}+B(x+y,\varepsilon) \cap C = \mathbb{R}+B(x+y,\varepsilon) \cap \bigcup_i \mathcal{Q}_i \). To that end, if \( c \in \mathcal{H} \), then the claim follows by assumption on the cones \( \mathcal{Q}_i \). Otherwise, let \( \{ t \} = \mathcal{H} \cap (x,y) \), let \( c_m \in (c,t) \) be a sequence converging to \( t \), and without loss of generality assume that \( y \) and all \( c_m \) are on the same side of \( \mathcal{H} \). If \( \{ z_m \} = \mathcal{H} \cap \ldots \)
\((x, c_m)\), then \(c_m - z_m = \alpha_m(z_m - x)\) and \(t - x = \beta(y - t)\) for some \(\alpha_m, \beta \in \mathbb{R}_{>0}\), and thus \(c_m = (1 - \alpha_m \beta)(t + \frac{\alpha_m + 1}{1 - \alpha_m \beta}(z_m - t)) + \alpha_m \beta y\). We have \(z_m \to t\) and \(\alpha_m \to 0\) when \(m \to \infty\), hence \(t + \frac{\alpha_m + 1}{1 - \alpha_m \beta}(z_m - t) \in \bigcup_i \bar{\mathcal{D}}_i\) for \(m \gg 0\) by assumption, so \(c_m \in \bigcup_i \bar{\mathcal{D}}_i\), proving the claim.

Since \(\lim_{n \to \infty} Q_n = Q_\infty\), we obtain \(\mathcal{H}_n \cap \text{int} B(x + y, \varepsilon) \neq \emptyset\) for \(n \gg 0\), and therefore, by the claim above and by passing to a subsequence, there is an index \(i_0\) such that \(\tilde{\mathcal{D}}_{i_0}\) intersects all \(\mathcal{H}_n \setminus \mathcal{R}\). In particular, since \((x, x_n) \subset \text{int} \mathcal{C}_n\) by the choice of \(x_n\), we have \(\tilde{\mathcal{D}}_{i_0} \cap \text{int} \mathcal{C}_n \neq \emptyset\), and therefore \(\tilde{\mathcal{D}}_{i_0} \cap \mathcal{C}_n\) is an \((r + 1)\)-dimensional cone for all \(n\). Thus the linear extensions of all \(f_{|\mathcal{C}_n}\) to \(\mathbb{R}^{r+1}\) are the same since they coincide with the linear extension of \(f_{|\tilde{\mathcal{D}}_{i_0}}\), a contradiction.

Step 3. Therefore, for every ray \(R \subset \mathcal{C}\) the map \(f_{|\mathcal{C}_i}\) is linear for \(i \in \mathcal{I}_R\), and there is small ball \(B_R\) centred at \(R \cap s'\) such that \(B_R \cap s' \cap \mathcal{C} = B_R \cap s' \cap \bigcup_{i \in \mathcal{I}_R} \mathcal{C}_i\). There are finitely many open sets \(B_R\) which cover the compact set \(s' \cap \mathcal{C}\), and therefore we can choose finitely many cones \(\mathcal{C}_i\) with \(\mathcal{C} = \bigcup_i \mathcal{C}_i\). Note that by the construction in Step 1, the linear extensions of \(f_{|\mathcal{C}_i}\) to \(\mathbb{R}^{r+1}\) are pairwise different.

It remains to show that all \(\mathcal{C}_i\) are polyhedral cones. Assume that \(\mathcal{C}_{i_0}\) is not polyhedral for some \(i_0\), and let \(L_n\) be its distinct extremal rays for \(n \in \mathbb{N}\). If infinitely many \(L_n\) do not belong to any other cone \(\mathcal{C}_i\), then passing to a subsequence I can assume that they belong to a face of \(\mathcal{C}\), and we derive contradiction by induction. Therefore, I can assume that for every \(n \gg 0\) there is an index \(i_n \neq i_0\) such that \(L_n \subset \mathcal{C}_{i_n}\). Passing to a subsequence, there is an index \(j_0 \neq i_0\) such that \(L_n \subset \mathcal{C}_{i_0} \cap \mathcal{C}_{j_0}\) for all \(n\). As before, we can assume that there does not exist a hyperplane containing infinitely many \(L_n\), so there are finitely many indices \(k\) such that \(\text{dim}(\sum L_k) = r + 1\). Thus the linear extensions of \(f_{|\mathcal{C}_{i_0}}\) and \(f_{|\mathcal{C}_{j_0}}\) to \(\mathbb{R}^{r+1}\) are the same since they coincide with the linear extension of \(f_{|\sum L_k}\), a contradiction. \(\square\)

5. Higher rank algebras

**Definition 5.1.** Let \(X\) be a variety, \(\mathcal{I} \subset \mathbb{N}'\) a finitely generated monoid, let \(\mu: \mathcal{I} \to \text{WDiv}(X)^{\kappa \geq 0}\) be an additive map and \(\text{Mob}_\mu: \mathcal{I} \to \text{Mob}(X)\) the subadditive map defined by \(\text{Mob}_\mu(s) = \text{Mob}(\mu(s))\) for every \(s \in \mathcal{I}\). Then

\[ R(X, \mu(\mathcal{I})) = \bigoplus_{s \in \mathcal{I}} H^0(X, \mathcal{O}_X(\mu(s))) \]

is the divisorial \(\mathcal{I}\)-graded algebra associated to \(\mu\). The \(b\)-divisorial \(\mathcal{I}\)-graded algebra associated to \(\mu\) is

\[ R(X, \text{Mob}_\mu(\mathcal{I})) = \bigoplus_{s \in \mathcal{I}} H^0(X, \mathcal{O}_X(\text{Mob}_\mu(s))) , \]

and we obviously have \(R(X, \text{Mob}_\mu(\mathcal{I})) \simeq R(X, \mu(\mathcal{I}))\). If \(e_1, \ldots, e_\ell\) are generators of \(\mathcal{I}\) and if \(\mu(e_i) = k_i(K_X + \Delta_i)\), where \(\Delta_i\) is an effective \(\mathbb{Q}\)-divisor for every \(i\), the algebra \(R(X, \mu(\mathcal{I}))\) is the adjoint ring associated to \(\mu\).
Remark 5.2. When \( \mathcal{I} = \bigoplus_{i=1}^f \mathbb{N}e_i \) is a simplicial cone, the algebra \( R(X, \mu(\mathcal{I})) \) is denoted also by \( R(X; \mu(e_1), \ldots, \mu(e_f)) \). If \( \mathcal{I}' \) is a finitely generated submonoid of \( \mathcal{I} \), \( R(X, \mu(\mathcal{I}')) \) is used to denote \( R(X; \mu(\mathcal{I}')) \). If \( \mathcal{I} \) is a submonoid of \( \text{WDiv}(X)^{\mathbb{K} \geq 0} \) and \( t: \mathcal{I} \to \mathcal{I} \) is the identity map, \( R(X, \mathcal{I}) \) is used to denote \( R(X, t(\mathcal{I})) \).

Remark 5.3. Algebras considered in this paper are algebras of sections when varieties are smooth. I will occasionally, and without explicit mention, view them as algebras of rational functions, in particular to be able to write \( H^0(X, D) \simeq H^0(X, \text{Mob}(D)) \subset k(X) \).

Assume now that \( X \) is smooth, \( D \in \text{Div}(X) \) and that \( \Gamma \) is a prime divisor on \( X \). If \( \sigma_{\Gamma} \) is the global section of \( \mathcal{O}_X(\Gamma) \) such that \( \text{div} \sigma_{\Gamma} = \Gamma \), from the exact sequence

\[
0 \to H^0(X, \mathcal{O}_X(D - \Gamma)) \xrightarrow{\sigma_{\Gamma}} H^0(X, \mathcal{O}_X(D)) \xrightarrow{\rho_{D, \Gamma}} H^0(\Gamma, \mathcal{O}_{\Gamma}(D))
\]

we define \( \text{res}_{\Gamma} H^0(X, \mathcal{O}_X(D)) = \text{Im}(\rho_{D, \Gamma}) \). For \( \sigma \in H^0(X, \mathcal{O}_X(D)) \), denote \( \sigma_{|\Gamma} = \rho_{D, \Gamma}(\sigma) \).

Observe that

\[
(2) \quad \ker(\rho_{D, \Gamma}) = H^0(X, \mathcal{O}_X(D - \Gamma)) \cdot \sigma_{\Gamma},
\]

and that \( \text{res}_{\Gamma} H^0(X, \mathcal{O}_X(D)) = 0 \) if \( \Gamma \subset \text{Bs} |D| \). If \( D \sim D' \) is such that the restriction \( D'|_{\Gamma} \) is defined, then

\[
\text{res}_{\Gamma} H^0(X, \mathcal{O}_X(D)) \simeq \text{res}_{\Gamma} H^0(X, \mathcal{O}_X(D')) \subset H^0(\Gamma, \mathcal{O}_{\Gamma}(D'|_{\Gamma})).
\]

The restriction of \( R(X, \mu(\mathcal{I})) \) to \( \Gamma \) is defined as

\[
\text{res}_{\Gamma} R(X, \mu(\mathcal{I})) = \bigoplus_{s \in \mathcal{I}} \text{res}_{\Gamma} H^0(X, \mathcal{O}_X(\mu(s))).
\]

This is an \( \mathcal{I} \)-graded, not necessarily divisorial algebra.

The following lemma summarises basic properties of higher rank finite generation.

Lemma 5.4. Let \( \mathcal{I} \subset \mathbb{N}^n \) be a finitely generated monoid and let \( R = \bigoplus_{s \in \mathcal{I}} R_s \) be an \( \mathcal{I} \)-graded algebra.

1. Let \( \mathcal{I}' \) be a truncation of \( \mathcal{I} \). If the \( \mathcal{I}' \)-graded algebra \( R' = \bigoplus_{s \in \mathcal{I}'} R_s \) is finitely generated over \( R_0 \), then \( R \) is finitely generated over \( R_0 \).

2. Assume furthermore that \( \mathcal{I} \) is saturated and let \( \mathcal{I}'' \subset \mathcal{I} \) be a finitely generated saturated submonoid. If \( R \) is finitely generated over \( R_0 \), then the \( \mathcal{I}'' \)-graded algebra \( R'' = \bigoplus_{s \in \mathcal{I}''} R_s \) is finitely generated over \( R_0 \).

3. Let \( X \) be a variety and let \( \mu: \mathcal{I} \to \text{WDiv}(X)^{\mathbb{K} \geq 0} \) be an additive map. If there exists a rational polyhedral subdivision \( \mathcal{I}_R = \bigcup_{i=1}^k \Delta_i \) such that \( \text{Mob}_{\mu|\Delta_i} \) is additive up to truncation for each \( i \), then \( R(X, \mu(\mathcal{I})) \) is finitely generated.

Proof. If \( \mathcal{I} = \sum_{i=1}^n \mathbb{N}e_i \) and \( \mathcal{I}' = \sum_{i=1}^n \mathbb{N}e_i \), then for any \( f \in R \) we have \( f^{k_1 \cdots k_n} \in R' \), so \( R \) is an integral extension of \( R' \), and (1) follows by the theorem of Emmy Noether on finiteness of integral closure.

Claim (2) is [ELM+06, 4.8].
For (3), denote \( m = \text{Mob}_\mu \) and \( \mathcal{I}_i = \Delta_i \cap \mathcal{I} \). By Lemma 4.2, choose a set of generators \( \{e_{ij} : j = 1, \ldots, k_i\} \) of \( \mathcal{I}_i \), and let \( \kappa \) be a positive integer such that \( m \) is additive on each \( \mathcal{I}_i^{(\kappa)} \). By (1), it is enough to show that the algebra \( R(X, m(\mathcal{I}^{(\kappa)})) \) is finitely generated. To that end, let \( Y \to X \) be a model such that \( m(\kappa e_{ij}) \) descend to \( Y \) for all \( i, j \). Then it is easy to see that \( m(s) \) descends to \( Y \) for every \( s \in \bigcup_i \mathcal{I}_i^{(\kappa)} \), and thus \( R(X, m(\mathcal{I}_i^{(\kappa)})) \simeq \bigoplus_{s \in \mathcal{I}_i^{(\kappa)}} H^0(Y, m(s)Y) \) for every \( i \). Since \( R(Y; m_Y(e_{i1}), \ldots, m_Y(e_{ik})) \) is finitely generated by [HK00, 2.8], so is the algebra \( R(X, m(\mathcal{I}_i^{(\kappa)})) \) by projection. Since \( \mathcal{I}_i^{(\kappa)} = \Delta_i \cap \mathcal{I}^{(\kappa)} \), the union of sets of generators of all \( R(X, m(\mathcal{I}^{(\kappa)})) \) spans \( R(X, m(\mathcal{I}^{(\kappa)})) \). □

I will need the next result in the proof of Proposition 5.7, and in §7.

**Lemma 5.5.** Let \( X \) be a variety, let \( \mathcal{I} \subset \mathbb{N}^r \) be a finitely generated monoid and let \( f : \mathcal{I} \to G \) be a superadditive map to a monoid \( G \) which is a subset of \( \text{WDiv}(X) \) or \( \text{Mob}(X) \), such that for every \( s \in \mathcal{I} \) the map \( f|_{\mathbb{N}s} \) is additive up to truncation.

Then there is a unique superlinear function \( f^\sharp : \mathcal{I}_\mathbb{R} \to G_\mathbb{R} \) such that for every \( s \in \mathcal{I} \) there exists \( \lambda_s \in \mathbb{Z}_{>0} \) with \( f(\lambda_s s) = f^\sharp(\lambda_s s) \). Furthermore, if \( \mathcal{C} \subset \mathbb{R}^r \) is a rational polyhedral cone, then \( f^\sharp|_{\mathcal{C} \cap \mathcal{I}} \) is additive up to truncation if and only if \( f^\sharp|_{\mathcal{C}} \) is linear.

In particular, if \( G = \text{Div}(X) \) and \( f = \text{Mob}_\mu \), where \( \mu : \mathcal{I} \to \text{Div}(X) \) is an additive map, then

\[
(3) \quad f^\sharp(s) = \overline{\mu(s)} - \sum E \|\mu(s)\| E,
\]

where the sum runs over all geometric valuations \( E \) on \( X \).

**Proof.** The construction will show that \( f^\sharp \) is the unique function with the stated properties.

To start, fix a point \( s \in \mathcal{I}_\mathbb{Q} \), choose \( \kappa \in \mathbb{Z}_{>0} \) so that \( \kappa s \in \mathcal{I} \) and \( f|_{\mathbb{N}\kappa s} \) is additive, and set

\[
f^\sharp(s) = f(\kappa s) / \kappa.
\]

This is well-defined: if \( \kappa' \) is another positive integer such that \( \kappa' s \in \mathcal{I} \) and \( f|_{\mathbb{N}\kappa' s} \) is additive, then \( \kappa f(\kappa' s) = f(\kappa \kappa' s) = \kappa' f(\kappa s) \), so \( f(\kappa s) / \kappa = f(\kappa' s) / \kappa' \).

Now fix \( \xi \in \mathbb{Q}_+ \), and let \( \lambda \) be a positive integer such that \( \lambda \xi \in \mathbb{N} \), \( \lambda \xi s \in \mathcal{I} \) and \( f|_{\mathbb{N}\lambda \xi s} \) is additive. Then

\[
f^\sharp(\lambda \xi s) = f(\lambda \xi s) / \lambda = \xi f(\lambda \xi s) / \lambda \xi = \xi f^\sharp(s),
\]

so \( f^\sharp \) is positively homogeneous with respect to rational scalars. Further, let \( s_1, s_2 \in \mathcal{I}_\mathbb{Q} \) and \( \kappa \in \mathbb{Z}_{>0} \) be such that \( f(\kappa s_1) = f^\sharp(\kappa s_1) \), \( f(\kappa s_2) = f^\sharp(\kappa s_2) \) and \( f(\kappa(\kappa s_1 + s_2)) = f^\sharp(\kappa(\kappa s_1 + s_2)) \). By superadditivity of \( f \) we have \( f(\kappa s_1) + f(\kappa s_2) \leq f(\kappa(\kappa s_1 + s_2)) \), so dividing this by \( \kappa \) we obtain superadditivity of \( f^\sharp \).

Let \( E \) be any divisor on \( X \), respectively any geometric valuation \( E \) over \( X \), when \( G \subset \text{WDiv}(X) \), respectively \( G \subset \text{Mob}(X) \). Consider the function \( f_E^\sharp \) given by \( f_E^\sharp(s) = \text{mult}_E f^\sharp(s) \). Proposition 4.7, applied to each \( f_E^\sharp \), shows that \( f^\sharp \) extends to a superlinear function on \( \mathcal{I}_\mathbb{R} \). Now (3) is just a restatement of the definition of \( f^\sharp \) when \( f = \text{Mob}_\mu \).
As for the statement on cones, necessity is clear. Now assume $f^i|_{\mathcal{I}}$ is linear and let $e_i$ be finitely many generators of $\mathcal{G} \cap \mathcal{I}$, cf. Lemma 4.2. Let $s_0 = \sum e_i$, and let $\mu$ be a positive integer such that $f(\mu s_0) = f^i(\mu s_0)$ and $f(\mu e_i) = f^i(\mu e_i)$ for all $i$. Then from $f^i(s_0) = \sum f^i(e_i)$ we obtain $f(\mu s_0) = \sum f(\mu e_i)$, and Lemma 4.3 implies that $f^i$ is additive on $(\mathcal{G} \cap \mathcal{I})(\mu)$.

**Definition 5.6.** In the context of Lemma 5.5, $f^i$ is called the straightening of $f$.

**Proposition 5.7.** Let $X$ be a variety, $\mathcal{I} \subset \mathbb{N}^r$ a finitely generated saturated monoid and $\mu \colon \mathcal{I} \to \text{WDiv}(X)^{\mathbb{K} \geq 0}$ an additive map. Let $\mathcal{L}$ be a finitely generated submonoid of $\mathcal{I}$ and assume $R(X, \mu(\mathcal{L}))$ is finitely generated. Then $R(X, \mu(\mathcal{L}))$ is finitely generated. Moreover, the map $m = \text{Mob}_\mu|_{\mathcal{L}}$ is rationally piecewise additive up to truncation. In particular, there is a positive integer $p$ such that $\text{Mob}_\mu(ips) = i\text{Mob}_\mu(ps)$ for every $i \in \mathbb{N}$ and every $s \in \mathcal{L}$.

**Proof.** Denote $\mathcal{M} = \mathcal{L}_\mathbb{R} \cap \mathbb{N}^r$. By Lemma 5.4(2), $R(X, \mu(\mathcal{M}))$ is finitely generated, and by the proof of [ELM+06, 4.1], there is a finite rational polyhedral subdivision $\mathcal{M} = \bigcup \Delta_i$ such that for every geometric valuation $E$ on $X$, the map $\text{ord}_E \rho_{\mathcal{M}}$ is linear on $\Delta_i$ for every $i$. Since for every saturated rank 1 submonoid $\mathcal{R} \subset \mathcal{M}$ the algebra $R(X, \mu(\mathcal{R}))$ is finitely generated by Lemma 5.4(2), the map $m|_{\mathcal{R}}$ is additive up to truncation by [Cor07, 2.3.53], and thus there is the well-defined straightening $m^i \colon \mathcal{L}_\mathbb{R} \to \text{Mob}(X)_\mathbb{R}$ since $\mathcal{M} = \mathcal{L}_\mathbb{R}$. Then equation 3 implies that $m^i|_{\Delta_i}$ is linear for every $i$, hence by Lemma 5.5 the map $\mathbf{m}$ is rationally piecewise additive up to truncation. Therefore $R(X, \mu(\mathcal{L}))$ is finitely generated by Lemma 5.4(3).

The following lemma shows that finite generation implies certain boundedness on the convex geometry of boundaries.

**Lemma 5.8.** Let $X$ be a smooth variety of dimension $n$, let $B$ be a simple normal crossings divisor and let $A$ be a general ample $\mathbb{Q}$-divisor on $X$. Let $V \subset \text{Div}(X)_\mathbb{R}$ be the vector space spanned by the components of $B$. Assume Theorems $A_n$ and $C_n$.

Then for each prime divisor $G$ on $X$, the set $\mathcal{B}_V^G$ is a rational polytope. Furthermore, there exists a positive integer $r$ such that:

1. for each prime divisor $G$ on $X$, for every $\Phi \in (\mathcal{B}_V^G)_\mathbb{Q}$, and for every positive integer $k$ such that $k(K_X + \Phi + A)/r$ is Cartier, we have $G \not\subset \text{Fix} |k(K_X + \Phi + A)|$

2. for every $\Phi \in (\mathcal{B}_V^A)_\mathbb{Q}$, and for every positive integer $k$ such that $k(K_X + \Phi + A)/r$ is Cartier, we have $|k(K_X + \Phi + A)| \neq \emptyset$.

**Proof.** Let $K_X$ be a divisor such that $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\text{Supp} A \not\subset \text{Supp} K_X$, and let $\Lambda \subset \text{Div}(X)$ be the monoid spanned by components of $K_X, B$ and $A$. Let $G$ be a prime divisor on $X$. By Theorem $C_n$ the set $\mathcal{B}_V^A$ is a rational polytope, and let $D_1, \ldots, D_k$ be generators of the finitely generated monoid $\mathcal{G} = \mathbb{R}_+ (K_X + A + \mathcal{B}_V^A) \cap \Lambda$, cf. Lemma 4.2. Since every $D_i$ is proportional to an adjoint bundle, by Theorem $A_n$ and by Lemma 5.4(1) the ring $R(X; D_1, \ldots, D_k)$ is finitely generated, and thus so is the algebra $R(X, \mathcal{G})$ by projection.
By Proposition 5.7 the map \( \text{Mob}_t |_{G \cap A(0)} \) is rationally piecewise additive for some positive integer \( r \), where \( t : \Lambda \to \Lambda \) is the identity map. Now (2) is straightforward.

Furthermore, the set \( \mathcal{O} = \{ Y \in C_\mathbb{R} : \text{ord}_G \| Y \| = 0 \} \) is a rational polyhedral cone by the proof of [ELM+06, 4.1], and \( \mathbb{R}_+ (K_X + A + B_{VA}) \subset \mathcal{O} \). Since for every \( Y \in \mathcal{O} \) we have \( G \not\subset B(Y) \) by Theorem A, and by [Cor07] 2.3.53, this implies \( \mathcal{O} \subset \mathbb{R}_+ (K_X + A + B_{VA}) \) as extremal rays of \( \mathcal{O} \) are rational. Therefore \( B_{VA} \) is a rational polytope, and now (1) follows similarly as above. \( \square \)

6. Diophantine approximation

I need a few results from Diophantine approximation theory.

**Lemma 6.1.** Let \( \Lambda \subset \mathbb{R}^n \) be a lattice spanned by rational vectors, and let \( V = \Lambda \otimes \mathbb{Z} \mathbb{R} \). Fix \( v \in V \) and denote \( X = \mathbb{N} v + \Lambda \). Then the closure of \( X \) is symmetric with respect to the origin. Moreover, if \( \pi : V \to V / \Lambda \) is the quotient map, then the closure of \( \pi(X) \) is a finite disjoint union of connected components. If \( v \) is not contained in any proper rational affine subspace of \( V \), then \( X \) is dense in \( V \).

**Proof.** I am closely following the proof of [BCHM06, 3.7.6]. Let \( G \) be the closure of \( \pi(X) \). Since \( G \) is infinite and \( V / \Lambda \) is compact, \( G \) has an accumulation point. It then follows that zero is also an accumulation point and that \( G \) is a closed subgroup. The connected component \( G_0 \) of the identity in \( G \) is a Lie subgroup of \( V / \Lambda \) and so by [Bum04, Theorem 15.2], \( G_0 \) is a torus. Thus \( G_0 = V_0 / \Lambda_0 \), where \( V_0 = \Lambda_0 \otimes \mathbb{Z} \mathbb{R} \) is a rational subspace of \( V \). Since \( G / G_0 \) is discrete and compact, it is finite, and it is straightforward that \( X \) is symmetric with respect to the origin. Therefore a translate of \( v \) by a rational vector is contained in \( V_0 \), and so if \( v \) is not contained in any proper rational affine subspace of \( V \), then \( V_0 = V \). \( \square \)

**Definition 6.2.** Let \( x \in \mathbb{R}^n \), \( \varepsilon \in \mathbb{R}_{>0} \) and \( k \in \mathbb{Z}_{>0} \). We say that \( (x_i, k, k_i, r_i) \in \mathbb{Q}^n \times \mathbb{Z}_{>0}^n \times \mathbb{R}_{>0} \) uniformly approximate \( x \) with error \( \varepsilon \), for \( i = 1, \ldots, p \), if

1. \( k_i x_i / k \) is integral for every \( i \),
2. \( \| x - x_i \| < \varepsilon / k_i \) for every \( i \),
3. \( x = \sum r_i x_i \) and \( \sum r_i = 1 \).

The next result is [BCHM06, 3.7.7].

**Lemma 6.3.** Let \( x \in \mathbb{R}^n \) and let \( W \) be the smallest rational affine space containing \( x \). Fix a positive integer \( k \) and a positive real number \( \varepsilon \). Then there are finitely many \( (x_i, k, k_i, r_i) \in (W \cap \mathbb{Q}^n) \times \mathbb{Z}_{>0}^n \times \mathbb{R}_{>0} \) which uniformly approximate \( x \) with error \( \varepsilon \).

I will need a refinement of this lemma when the approximation is not necessarily happening in the smallest rational affine space containing a point.

**Lemma 6.4.** Let \( x \in \mathbb{R}^n \) and let \( W \) be the smallest rational affine space containing \( x \). Let \( 0 < \varepsilon, \eta \ll 1 \) be rational numbers, \( k \) a positive integer, and assume that there are
Suppose points \( x_1 \in \mathbb{Q}^n \) and \( k_1 \in \mathbb{Z}_{>0} \) such that \( \|x - x_1\| < \varepsilon/k_1 \) and \( k_1 x_1/k \) is integral. Then there are finitely many \( x_i \in \mathbb{Q}^n \) and \( k_i \in \mathbb{Z}_{>0} \) for \( i \geq 2 \), and positive real numbers \( r_i \) for \( i \geq 1 \), such that \((x_i, k_i, r_i)\) uniformly approximate \( x \) with error \( \varepsilon \). Furthermore, we can assume that \( x_i \in W \) for \( i \geq 3 \), and we can write

\[
x = \frac{k_1}{k_1 + k_2} x_1 + \frac{k_2}{k_1 + k_2} x_2 + \xi,
\]

with \( \|\xi\| < \eta/(k_1 + k_2) \).

**Proof.** Rescaling by \( k \), we can assume that \( k = 1 \). Let \( \pi: \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \) be the quotient map and let \( G \) be the closure of the set \( \pi(\mathbb{N} x + \mathbb{Z}^n) \). Then by Lemma 6.1 we have \( \pi(-k_1 x) \in G \) and there is \( k_2 \in \mathbb{N} \) such that \( \pi(k_2 x) \) is in the connected component of \( \pi(-k_1 x) \) in \( G \) and \( \|k_2 x - y\| < \eta \) for some \( y \in \mathbb{R}^n \) with \( \pi(y) = \pi(-k_1 x) \). Thus there is a point \( x_2 \in \mathbb{Q}^n \) such that \( k_2 x_2 \in \mathbb{Z}^n \), \( \|k_2 x - k_2 x_2\| < \varepsilon \) and the open segment \((x_1, x_2)\) intersects \( W \) at a point \( u \).

By Lemma 6.3 there exist \((x_1, 1, k_i, p_i) \in (W \cap \mathbb{Q}^n) \times \mathbb{Z}_{>0} \times \mathbb{R}_{>0} \) for \( i \geq 3 \) which uniformly approximate \( x \) with error \( \varepsilon \). In particular, \( x \) is in the interior of the rational polytope with vertices \( x_i \) for \( i \geq 3 \), so there exists a point \( v = \sum_{i \geq 3} q_i x_i \) with \( q_i > 0 \) and \( \sum q_i = 1 \), such that \( x \in (u, v) \). Let \( \alpha, \beta \in (0, 1) \) be such that \( u = \alpha x_1 + (1 - \alpha) x_2 \) and \( x = \beta u + (1 - \beta) v \), and set \( r_1 = \alpha \beta \), \( r_2 = (1 - \alpha) \beta \), and \( r_i = (1 - \beta) q_i \) for \( i \geq 3 \). Then \((x_i, k_i, r_i)\) uniformly approximate \( x \) with error \( \varepsilon \).

Finally, observe that the vector \( y/k_2 - x_2 \) is parallel to the vector \( x - x_1 \) and \( \|y - k_2 x_2\| = \|k_1 x - k_1 x_1\| \). Denote \( z = x - y/k_2 \). Then

\[
\frac{x - x_1}{(x_2 + z) - x} = \frac{x - x_1}{x_2 - y/k_2} = \frac{k_2}{k_1},
\]

so

\[
x = \frac{k_1}{k_1 + k_2} x_1 + \frac{k_2}{k_1 + k_2} (x_2 + z) = \frac{k_1}{k_1 + k_2} x_1 + \frac{k_2}{k_1 + k_2} x_2 + \xi,
\]

where \( \|\xi\| = \|k_2 z/(k_1 + k_2)\| < \eta/(k_1 + k_2) \). \(\square\)

**Remark 6.5.** Assuming notation from the previous proof, the connected components of \( G \) are precisely the connected components of the closure of the set \( \pi(\bigcup_{k>0} kW) \). Therefore \( y/k_2 \in W \).

**Remark 6.6.** Suppose points \((y_i, k_i, r_i)\) uniformly approximate \( x \in \mathbb{R}^k \) with error \( \varepsilon \), in the sup-norm. Let \( x_j \) denote the \( j \)-th coordinate of \( x \), and similarly for other vectors. I claim that by choosing \( \varepsilon \ll 1 \) and \( k_i \gg 0 \) we have \( y_{ip} \geq y_{iq} \) whenever \( x_p \geq x_q \). To that end, if \( x_p = x_q \), then by triangle inequality \( |k_i(y_{ip} - y_{iq})| \leq |k_i(x_p - y_{ip})| + |k_i(x_q - y_{iq})| < 2\varepsilon \), so \( y_{ip} = y_{iq} \) since \( k_i y_{ip} \) and \( k_i y_{iq} \) are integers. If \( x_p > x_q \), then since \( |x_p - y_{ip}| + |x_q - y_{iq}| < 2\varepsilon/k_i < x_p - x_q \), we must have \( y_{ip} > y_{iq} \), and the claim follows.

## 7. Restricting PLT Algebras

In this section I establish one of the technically most difficult steps in the scheme of the proof, that Theorems A\(_{n-1}\), B\(_n\) and C\(_{n-1}\) imply Theorem A\(_n\). Crucial techniques will be those developed in [HM08] and in Sections 4 and 5.
The key result is the following Hacon-McKernan extension theorem [HM08, 6.3], whose proof relies on deep techniques initiated by [Siu98].

**Theorem 7.1.** Let \((X, \Delta = S + A + B)\) be a projective plt pair such that \(S = |\Delta|\) is irreducible, \(\Delta \in \text{WDiv}(X)\), \((X, S)\) is log smooth, \(A\) is a general ample \(\mathbb{Q}\)-divisor and \((S, \Omega + A|_{S})\) is canonical, where \(\Omega = (\Delta - S)|_{S}\). Assume \(S \not\subset \mathcal{B}(K_{X} + \Delta)\), and let

\[
F = \liminf_{m \to \infty} \frac{1}{m} \text{Fix} |m(K_{X} + \Delta)|_{S}.
\]

If \(\varepsilon > 0\) is any rational number such that \(\varepsilon (K_{X} + \Delta) + A\) is ample and if \(\Phi\) is any \(\mathbb{Q}\)-divisor on \(S\) and \(k > 0\) is any integer such that both \(k\Delta\) and \(k\Phi\) are Cartier, and \(\Omega \wedge (1 - \frac{\varepsilon}{k})F \leq \Phi \leq \Omega\), then

\[
|k(K_{S} + \Omega - \Phi)| + k\Phi \subset |k(K_{X} + \Delta)|_{S}.
\]

The immediate consequence is:

**Corollary 7.2.** Let \((X, \Delta = S + A + B)\) be a projective plt pair such that \(S = |\Delta|\) is irreducible, \(\Delta \in \text{WDiv}(X)\), \((X, S)\) is log smooth, \(A\) is a general ample \(\mathbb{Q}\)-divisor and \((S, \Omega + A|_{S})\) is canonical, where \(\Omega = (\Delta - S)|_{S}\). Assume \(S \not\subset \mathcal{B}(K_{X} + \Delta)\), and let \(\Phi_{m} = \Omega \wedge \frac{1}{m} \text{Fix} |m(K_{X} + \Delta)|_{S}\) for every \(m\) such that \(m\Delta\) is Cartier. Then

\[
|m(K_{S} + \Omega - \Phi_{m})| + m\Phi_{m} = |m(K_{X} + \Delta)|_{S}.
\]

The following result will be used several times to test inclusions of linear series. It is extracted and copied almost verbatim from [Hac08], and Step 2 of the proof below first appeared in [Tak06]. Similar techniques in the analytic setting appeared in [Pâu08].

**Proposition 7.3.** Let \((X, \Delta = S + A + B)\) be a projective plt pair such that \(S = |\Delta|\) is irreducible, \(\Delta \in \text{WDiv}(X)\), \((X, S)\) is log smooth, \(A\) is a general ample \(\mathbb{Q}\)-divisor and \((S, \Omega + A|_{S})\) is canonical, where \(\Omega = (\Delta - S)|_{S}\). Let \(0 \leq \Theta \leq \Omega\) be a \(\mathbb{Q}\)-divisor on \(S\), let \(k\) be a positive integer such that \(k\Delta\) and \(k\Theta\) are integral, and denote \(A' = A/k\). Assume that \(S \not\subset \mathcal{B}(K_{X} + \Delta + A')\) and that for any \(l > 0\) sufficiently divisible we have

\[
(4) \quad \Omega \wedge \frac{1}{l} \text{Fix} |l(K_{X} + \Delta + A')|_{S} \leq \Omega - \Theta.
\]

Then

\[
|k(K_{S} + \Theta)| + k(\Omega - \Theta) \subset |k(K_{X} + \Delta)|_{S}.
\]

**Proof.** Step 1. We first prove that there exists an effective divisor \(H\) on \(X\) not containing \(S\) such that for all sufficiently divisible \(m\) we have

\[
(5) \quad |m(K_{S} + \Theta)| + m(\Omega - \Theta) + (mA' + H)|_{S} \subset |m(K_{X} + \Delta) + mA'|_{S}.
\]

Taking \(l\) as in (4) sufficiently divisible, we can assume \(S \not\subset \text{Bs}|l(K_{X} + \Delta + A')|\). Let \(f: Y \to X\) be a log resolution of \((X, \Delta + A')\) and of \(|l(K_{X} + \Delta + A')|\). Denote \(\Gamma = \mathcal{B}(X, \Delta + A')_{Y}\) and \(E = K_{Y} + \Gamma - f^{\ast}(K_{X} + \Delta + A')\), and define

\[
\Xi = \Gamma - \Gamma \wedge \frac{1}{l} \text{Fix} |l(K_{Y} + \Gamma)|.
\]
Then \(l(K_Y + Ξ)\) is Cartier, \(\text{Fix} \ l(K_Y + Ξ) \cap Ξ = 0\) and \(\text{Mob}(l(K_Y + Ξ))\) is free. Since \(\text{Fix} \ l(K_Y + Ξ) \cap Ξ\) has simple normal crossings support, it follows that \(B(K_Y + Ξ)\) contains no log canonical centres of \((Y, [Ξ])\). Denote \(T = f_s^{-1} S, Γ_T = (Γ - T)|_T\) and \(Ξ_T = (Ξ - T)|_T\), let \(l\) be any positive integer divisible by \(l\) and consider a section

\[
σ ∈ H^0(T, Ω_T(m(K_T + Ξ_T))) = H^0(T, J_{|m(K_T + Ξ_T)|(m(K_T + Ξ_T))}).
\]

By [HM08 5.3], there is an ample divisor \(H'\) on \(Y\) such that if \(τ ∈ H^0(T, Ω_T(H'))\), then

\[
σ · τ ∈ \text{Im} (H^0(Y, Ω_Y(m(K_Y + Ξ) + H')) → H^0(T, Ω_T(m(K_Y + Ξ) + H')))\).
\]

Therefore

\[
|Ω_T| + m(Γ_T - Ξ_T) + H'_T ⊂ m(K_Y + Γ) + H'|_T.
\]

We claim that

\[
Ω + A'_{|S} ≥ (f_{|T}|)_* Ξ_T ≥ Θ + A'_{|S}.
\]

Assuming the claim, as \((S, Ω + A'_{|S})\) is canonical, we have

\[
|Ω_T| + m((f_{|T}|)_* Ξ_T - Θ) ⊂ |m(K_S + (f_{|T}|)_* Ξ_T)| = (f_{|T}|)_* |m(K_T + Ξ_T)|.
\]

Pushing forward the inclusion \((6)\), we obtain \((5)\) for \(H = f_* H'\).

Now we prove the claim. Since \(Ξ_T ⊂ Γ_T\) and \((f_{|T}|)_* Γ_T = Ω + A'_{|S}\), the first inequality in \((7)\) follows. In order to prove the second inequality, let \(P\) be any prime divisor on \(S\) and let \(P' = (f_{|T}|)^{-1}_* P\). Assume that \(P ⊂ \text{Supp} Ω\), and thus \(P' ⊂ \text{Supp} Γ_T\). Then there is a component \(Q\) of the support of \(Γ\) such that

\[
\text{mult}_{P'} \text{Fix} |l(K_Y + Γ)|_T = \text{mult}_Q \text{Fix} |l(K_Y + Γ)| \quad \text{and} \quad \text{mult}_{P'} Γ_T = \text{mult}_Q Γ,
\]

and thus

\[
\text{mult}_{P'} Ξ_T = \text{mult}_{P'} Γ_T - \min \{\text{mult}_{P'} Γ_T, \text{mult}_{P'} \frac{1}{l} \text{Fix} |l(K_Y + Γ)|_T\}.
\]

Notice that \(\text{mult}_{P'} Γ_T = \text{mult}_P (Ω + A'_{|S})\) and since \(E|_T\) is exceptional, we have

\[
\text{mult}_{P'} \text{Fix} |l(K_Y + Γ)|_T = \text{mult}_P \text{Fix} |l(K_X + Δ + A')|_S.
\]

Therefore

\[
(f_{|T}|)_* Ξ_T = Ω + A'_{|S} - Ω ∧ \frac{1}{l} \text{Fix} |l(K_X + Δ + A')|_S.
\]

The inequality now follows from \((4)\).

**Step 2.** Let \(m \gg 0\) be as in Step 1 and divisible by \(k\), and such that \(A' - \frac{k - 1}{m} H\) is ample and \((S, Ω + \frac{k - 1}{m} H|_S)\) is klt, which is possible since \((S, Ω)\) is canonical. In particular, \(\text{mult}_P \frac{1}{m} H|_S = \text{mult}_S\).

By Step 1, for any \(Σ ∈ |k(K_S + Θ)|\) there is a divisor \(G ∈ |m(K_X + Δ) + mA' + H|\) such that \(G|_S = \frac{m}{k} Σ + m(Ω - Θ) + (mA' + H)|_S\). Set \(Δ = \frac{k - 1}{m} (G + Δ - S - A)\), and observe that

\[
A|_S - (Σ + k(Ω - Θ)) = \frac{k - 1}{m} (G|_S + Ω - A|_S - (Σ + k(Ω - Θ)) \leq Ω + \frac{k - 1}{m} H|_S.
\]
Therefore,

\[ \mathcal{J}_{\sum + k(\Omega - \Theta)} \subset \mathcal{J}_{\Lambda|S} \]

by [HM08 4.3(3)]. Since \( k(K_X + \Delta) \sim_\mathbb{Q} K_X + S + \Lambda + (A' - \frac{k-1}{m} H) \), the homomorphism

\[ H^0(X, \mathcal{J}_{S\Lambda}(k(K_X + \Delta))) \to H^0(S, \mathcal{J}_{\Lambda|S}(k(K_X + \Delta))) \]

is surjective by [HM08 4.4(3)]. This together with (8) implies

\[ \sum + k(\Omega - \Theta) \in |k(K_X + \Delta)|_S, \]

which finishes the proof. \( \square \)

The main result of this section is the following.

**Theorem 7.4.** Let \( X \) be a smooth variety of dimension \( n \), \( S \) a smooth prime divisor and \( A \) a general ample \( \mathbb{Q} \)-divisor on \( X \). For \( i = 1, \ldots, \ell \), let \( D_i = k_i(K_X + \Delta_i) \in \text{Div}(X) \), where \((X, \Delta_i = S + B_i + A)\) is a log smooth plt pair with \(|\Delta_i| = S\) and \(|D_i| \neq 0\). Assume Theorems \( A_{n-1}, B_n \) and \( C_{n-1} \). Then the algebra \( \text{res}_S R(X; D_1, \ldots, D_\ell) \) is finitely generated.

**Proof.** Step 1. I first show that we can assume \( S \notin \text{Fix} |D_i| \) for all \( i \).

To that end, let \( K_X \) be a divisor with \( \mathcal{O}_X(K_X) \cong \omega_K \) and \( \text{Supp} A \supset \text{Supp} K_X \), and let \( \Lambda \) be the monoid in \( \text{Div}(X) \) generated by the components of \( K_X \) and \( \sum \Delta_i \). Denote \( \mathcal{C}_S = \{ P \in \Lambda_{\mathbb{R}} : S \notin \mathcal{B}(P) \} \). By Theorem \( B_n \), the set \( \mathcal{A} = \sum_{i=1}^\ell \mathbb{R} \cdot D_i \cap \mathcal{C}_S \) is a rational polyhedral cone.

The monoid \( \sum_{i=1}^\ell \mathbb{R} \cdot D_i \cap \Lambda \) is finitely generated by Lemma 4.2 and let \( D_{\ell+1}, \ldots, D_q \) be its generators. Let \( e_i \) be the standard generators of \( \mathbb{R}^q \). If \( \mu : \bigoplus_{i=1}^\ell \mathbb{N} e_i \to \text{Div}(X) \) denotes the additive map given by \( \mu(e_i) = D_i \), then \( \mathcal{A} = \mu^{-1}((\mathcal{A} \cap \Lambda) \cap \bigoplus_{i=1}^\ell \mathbb{N} e_i) \) is a finitely generated monoid, and let \( h_1, \ldots, h_m \) be generators of \( \mathcal{A} \). Observe that \( \mu(h_i) \) is a multiple of an adjoint bundle for every \( i \), and that \( R(X, \mu(\bigoplus_{i=1}^\ell \mathbb{N} e_i)) = R(X; D_1, \ldots, D_\ell) \).

The algebra \( \text{res}_S R(X, \mu(\bigoplus_{i=1}^\ell \mathbb{N} e_i)) \) is finitely generated if and only if \( \text{res}_S R(X, \mu(\mathcal{A})) \) is, since \( \text{res}_S H^0(X, \mu(s)) = 0 \) for every \( s \in \bigoplus_{i=1}^\ell \mathbb{N} e_i \setminus \mathcal{A} \). Then it is enough to prove that the restricted algebra \( \text{res}_S R(X; \mu(h_1), \ldots, \mu(h_m)) \) is finitely generated, as we have the natural projection

\[ \text{res}_S R(X; \mu(h_1), \ldots, \mu(h_m)) \to \text{res}_S R(X, \mu(\mathcal{A})). \]

By passing to a truncation, cf. Lemma 5.4(1), I can assume further that \( S \notin \text{Fix} |\mu(h_i)| \) for \( i = 1, \ldots, m \). Now by replacing \( \mathcal{A} \) by \( \bigoplus_{i=1}^m \mathbb{N} \mu(h_i) \), I assume \( \mathcal{A} = \bigoplus_{i=1}^m \mathbb{N} e_i \) and \( \mu(e_i) = D_i \) for every \( i \).

**Step 2.** For \( s = \sum_{i=1}^\ell t_i e_i \in \mathcal{A}_{\mathbb{R}} \), denote

\[ t_s = \sum_{i=1}^\ell t_i k_i, \quad \Delta_s = \sum_{i=1}^\ell \frac{t_i k_i}{t_s} \Delta_i, \quad \text{and} \quad \Omega_s = (\Delta_s - S)|_S, \]
and observe that then
\[ R(X;D_1,\ldots,D_\ell) = \bigoplus_{s \in \mathcal{S}} H^0(X, t_s(K_X + \Delta_s)). \]

In this step I show that we can assume that the pair \((S,\Omega_s + A|_S)\) is terminal for every \(s \in \mathcal{S}\).

Let \(\sum F_k = \bigcup_i \text{Supp} B_i\), and denote \(B_i = B(X, \Delta_i)\) and \(B = B(X, S + \nu \sum F_k + A)\), where \(\nu = \max_{i,k} \{\text{mult}_{F_k} B_i\}\). By Lemma 2.3 there is a log resolution \(f : Y \rightarrow X\) such that the components of \(\{B_Y\}\) do not intersect, and denote \(D_i' = k_i(K_Y + B_i)|_{Y}\). Observe that
\[ (9) \quad R(X;D_1,\ldots,D_\ell) \simeq R(Y;D_1',\ldots,D_\ell'). \]

Since \(B_i \leq \nu \sum F_k\), by comparing discrepancies we see that the components of \(\{B_Y\}\) do not intersect and notice that \(f^*A = f^{-1}_a A \leq B\) since \(A\) is general. Denote \(\Delta'_s = \sum_{i=1}^{\ell} \frac{l_{i,s}}{t_s} B_{i Y}\). Let \(H\) be a small effective \(f\)-exceptional \(\mathbb{Q}\)-divisor such that \(f^*A - H\) is ample and let \(A' \rightsquigarrow_{\mathbb{Q}} f^*A - H\) be a general ample \(\mathbb{Q}\)-divisor. Let \(T = f^{-1}_a S\), \(\Psi_s = \Delta'_s - f^*A - T + H \geq 0\) and \(\Omega'_s = (\Psi_s + A')|_T\). Then the pair \((T,\Omega'_s + A'|_T)\) is terminal and
\[ K_Y + T + \Psi_s + A' \rightsquigarrow_{\mathbb{Q}} K_Y + \Delta'_s. \]

Now replace \(X\) by \(Y\), \(S\) by \(T\), \(\Delta_s\) by \(T + \Psi_s + A'\) and \(\Omega_s\) by \(\Omega'_s\).

**Step 3.** Write
\[ \text{res}_S R(X;D_1,\ldots,D_\ell) = \bigoplus_{s \in \mathcal{S}} \mathcal{R}_s. \]

Then, denoting \(\theta_s = \Omega_s - \Omega_s \wedge \frac{1}{t_s} \text{Fix} |t_s(K_X + \Delta_s)|_S\), we have
\[ (10) \quad \mathcal{R}_s = H^0(S, t_s(K_S + \theta_s)) \]
by Corollary 7.2. Let \(m : \mathcal{S} \rightarrow \text{Div}(S)\) be the map given by \(m(s) = \text{Mob}(t_s(K_S + \theta_s))\). Since \(\mathcal{R}_{s_1} \mathcal{R}_{s_2} \subset \mathcal{R}_{s_1 + s_2}\) for all \(s_1, s_2 \in \mathcal{S}\), \(m\) is superadditive, cf. [Cor07, 2.3.34].

For \(s \in \mathcal{S}\), set \(\Theta_s = \limsup_{m \rightarrow \infty} \theta_{m,s}\). Then similarly as in the proof of [HM08, 7.1], by Theorem 7.1 and Lemma 5.8 we obtain that \(\Theta_s\) is rational and
\[ (11) \quad \bigoplus_{p \in \mathbb{N}} \mathcal{R}_{p\ell_s} \simeq R(S, \ell_s t_s(K_S + \Theta_s)), \]
where \(\ell_s\) is a positive integer such that \(\ell_s \Delta_s\) and \(\ell_s \Theta_s\) are Cartier. By Theorem \(A_{n-1}\), the algebra \(R(S, \ell_st_s(K_S + \Theta_s))\) is finitely generated, and since \(\mathcal{R}_{p\ell_s} = H^0(S, m(p\ell_s))\), the map \(m|_{p\ell_s}\) is additive up to truncation by [Cor07, 2.3.53]. Therefore, there is a well-defined straightening \(m^* : \mathcal{S} \rightarrow \text{Div}(S)|_\mathcal{S}\) by Lemma 5.5.

Define the maps \(\Theta, \lambda : \mathcal{S} \rightarrow \text{Div}(S)|_\mathcal{S}\) by
\[ \Theta(s) = \Theta_s, \quad \lambda(s) = t_s(K_S + \Theta_s) \]
for \(s \in \mathcal{S}\). Note that, by definition of \(\theta_s\) and by (10), for every component \(G\) of \(\theta_s\) we have \(G \notin \text{Fix} |t_s(K_S + \theta_s)|\), and so \(\text{mult}_{G}(t_s(K_S + \theta_s)) = \text{mult}_{G} m(s)\). Therefore, by the
construction of $m^s$ from the proof of Lemma 5.5, \( \text{mult}_G(t_s(K_S + \Theta_s)) = \text{mult}_G m^s(s) \) for every component $G$ of $\Theta_s$, and thus $\Theta$ and $\lambda$ extend to $S_R$.

I claim that there exists a finite rational polyhedral subdivision $S_R = \bigcup C_i$ such that $\lambda$ is linear on each $C_i$. Grant this for the moment. By Lemma 4.2, let $s_1', \ldots, s_r'$ be generators of $C_i = C_i \cap C_i$, and let $\kappa$ be a sufficiently divisible positive integer such that $\lambda(\kappa s_i') = \lambda(\kappa s_j') \in \text{Div}(S)$ for all $i$ and $j$. Then the ring $R(S; \lambda(\kappa s_i'), \ldots, \lambda(\kappa s_j'))$ is finitely generated by Theorem $A_{n-1}$, and so is $\bigoplus_{s \in C_i} \mathcal{R}_S$ by projection. Thus the algebra $\bigoplus_{s \in C_i} \mathcal{R}_S$ is finitely generated by Lemma 5.8, and so is $\text{res}_S R(X; D_1, \ldots, D_\ell)$ by putting all those generators together.

The claim stated above is Theorem 7.9, and this is proved in the remainder of this section.

\[ \square \]

**Remark 7.5.** Note that for every $s \in S$ we have $\Theta_s - A_s \in S_{V_{S,A_s}}$ for every component $G$ of $\Theta_s$, since $\Theta_{ms} - A_s \in S_{V_{S,A_s}}$ for every component $G$ of $\Theta_s$, and each $S_{V_{S,A_s}}$ is a rational polytope by Lemma 5.8 and in particular closed.

**Notation 7.6.** With notation from the previous proof, for $s \in S_R$ I usually denote $\Theta(s)$ and $\lambda(s)$ by $\Theta_s$ and $\lambda_s$, respectively. Denote $\Pi = \{s \in S_R : t_s = 1\}$; this is a rational polytope in $\mathbb{R}^\ell$. Let $\Delta: \mathbb{R} \Pi \to \text{Div}(X)_{\mathbb{R}}$ be the linear map given by $\Delta(q_1) = \Delta_q$ for linearly independent points $q_1, \ldots, q_\ell \in \Pi$, and then extended linearly. This is well defined since $S_R$ is an affine map on $\Pi$. Similarly, observe that, since the function $\text{ord}_P \| \cdot \|_S$ is convex for every $P$, the set $\{s \in \Pi : \Theta_s > 0\}$ is convex and $\Theta$ is concave on it. Let $L$ denote the norm of the linear map $\Delta$, i.e. the smallest global Lipschitz constant of $\Delta$. Denote by $V_{S} \subset \text{Div}(S)_{\mathbb{R}}$ the vector space spanned by the components of $\bigcup_{s \in S_R} \text{Supp}(\Theta_s - A_s)$. For a prime divisor $P$ on $S$, let $\lambda_P: S_R \to \mathbb{R}$ be the function given by $\lambda_P(s) = \text{mult}_P \lambda(s)$, and similarly for $\Theta_P$.

**Proof of piecewise linearity.** To finish the proof of Theorem 7.4, it remains to prove that the map $\lambda$ is rationally piecewise linear. I first briefly sketch the strategy of the proof of this fact, which occupies the rest of this section.

Until the end of the section I fix a prime divisor $Z$ on $S$, and the goal is to prove that $\lambda_Z$ is rationally piecewise linear – it is clear that then $\lambda$ is rationally piecewise linear by taking a subdivision of the cone $S_R$ that works for all prime divisors. By suitably replacing $S_R$ and $\lambda_Z$, I can assume that $\lambda_Z$ is a superlinear map, see the proof of Theorem 7.9, and also that $\Theta_s - A_s \in S_{V_{S,A_s}}$ for every $s \in S_R$. In order to prove that $\lambda_Z$ is piecewise linear, it is enough to show that $\lambda_Z|_{S_R \cap H}$ is piecewise linear for every $2$-plane $H \subset \mathbb{R}^\ell$ by Theorem 4.4, and the first step is Theorem 7.7(1), which claims that $\lambda|_{S_R \cap H}$ is continuous.

The method of the proof is as follows: starting from a point $s \in S_R$ and a $2$-plane $H \ni s$, I approximate $(s, \Theta_s) \in S_R \times \text{Div}(S)_{\mathbb{R}}$ by points $(t_s, \Theta_s') \in S_R \times \text{Div}(S)_{\mathbb{R}}$ such that $\mathbb{R}^\ell \ni s \subset C_{s,H} \cap H$, where $C_{s,H} = \sum \mathbb{R}^\ell t_s$. Furthermore, if the approximation is sufficiently good, I can assume that $\Theta_s' \in S_{V_{S,A_s}}$ by Theorems $A_{n-1}$ and $C_{n-1}$. Then there are suitable inclusions of linear series which force $\lambda_Z$ to be convex on $C_{s,H}$. However, since $\lambda_Z$ is
concave, this implies it is linear on \( C_{s,H} \), and thus on \( \mathcal{H} \cap H \) by an easy compactness argument. The fact that \( \lambda_Z \) is rationally piecewise linear then follows easily, and this is done in Step 3 of the proof of Theorem 7.9.

**Theorem 7.7.** Fix \( s \in \Pi \), let \( U \subseteq \mathbb{R}^\ell \) be the smallest rational affine space containing \( s \) and let \( P \) be a prime divisor on \( S \). Then:

1. for any \( t \in \Pi \) we have \( \lim_{\varepsilon \to 0} \Theta_{(1-\varepsilon)s+\varepsilon t} = \Theta_s \),
2. if \( \Theta_P(s) > 0 \), then the map \( \lambda_P \) is linear in a neighbourhood of \( s \) contained in \( U \).

**Proof.** First note that \( U \cap \mathcal{H} \subseteq \Pi \), and let \( r \) be a positive integer as in Lemma 5.8 with respect to the vector space \( V_S \) and the ample divisor \( A_{tS} \).

Note that in order to prove the claim (1), it is enough to show that for every \( u \in \mathcal{H} \),

\[
\Theta_u = \Theta_u^\sigma,
\]

where \( \Theta_u^\sigma = \Omega_u - \Omega_u \wedge N_\sigma ||K_X + \Delta_u||_S \), cf. Remark 2.13 since then

\[
\lim_{\varepsilon \to 0} \Theta_{(1-\varepsilon)s+\varepsilon t} = \lim_{\varepsilon \to 0} \Theta_{(1-\varepsilon)s+\varepsilon t}^\sigma = \Theta_s^\sigma = \Theta_s
\]

by Lemma 2.14(3). Therefore I concentrate on proving (12) and the claim (2). Without loss of generality I assume \( u = s \). In Step 1 I am closely following [Hac08].

**Step 1.** Let \( 0 < \phi < 1 \) be the smallest positive coefficient of \( \Omega_s - \Theta_s^\sigma \) if it exists, and set \( \phi = 1 \) otherwise. Let \( W \subseteq \text{Div}(S)_{\mathbb{R}} \) be the smallest rational affine space containing \( \Theta_s^\sigma \). Let \( 0 < \eta \ll 1 \) be a rational number such that \( (L+1)\eta(K_X + \Delta') + \frac{1}{2}A \) and \( \Delta' - \Delta_s + \frac{1}{2}A \) are ample divisors whenever \( \Delta' \leq B(\Delta_s, L\eta) \), cf. Notation 7.6.

By Lemma 6.3 there exist rational points \((t_i, \Theta_{t_i}^\sigma) \in U \times W \), integers \( p_{t_i} \gg 0 \) and \( r_{t_i} \in \mathbb{R}_{>0} \) such that \((t_i, \Theta_{t_i}^\sigma, r_{t_i}, p_{t_i}) \) uniformly approximate \((s, \Theta_s^\sigma) \in U \times W \) with error \( \phi \eta \). Note that then \((\Omega_{t_i}, \Theta_{t_i}^\sigma, r_{t_i}, p_{t_i}) \) uniformly approximate \((\Omega_s, \Theta_s^\sigma) \in \text{Div}(S)_{\mathbb{R}} \times W \) with error max\{\( \phi \eta, L\phi \eta \), and thus \( \Theta_{t_i}^\sigma \leq \Omega_{t_i} \), by Remark 6.6. Furthermore, for every prime divisor \( P \) on \( S \) we have

\[
(1 - \frac{(L+1)\eta}{p_{t_i}}) \text{mult}_P(\Omega_s - \Theta_s^\sigma) \leq \text{mult}_P(\Omega_{t_i} - \Theta_{t_i}^\sigma).
\]

To see this, note that (13) is trivial when \( \text{mult}_P(\Omega_s - \Theta_s^\sigma) = 0 \). Therefore I can assume that \( 0 < \phi < 1 \) and \( \text{mult}_P(\Omega_s - \Theta_s^\sigma) \geq \phi \). Since \( ||\Omega_s - \Omega_{t_i}|| < L\phi \eta / p_{t_i} \) and \( ||\Theta_{t_i}^\sigma - \Theta_s^\sigma|| < \phi \eta / p_{t_i} \), by triangle inequality we have

\[
\text{mult}_P(\Omega_s - \Theta_s^\sigma) \leq \text{mult}_P(\Omega_{t_i} - \Theta_{t_i}^\sigma) + ||\Omega_s - \Omega_{t_i}|| + ||\Theta_{t_i}^\sigma - \Theta_s^\sigma||
\leq \text{mult}_P(\Omega_{t_i} - \Theta_{t_i}^\sigma) + \frac{(L+1)\eta}{p_{t_i}} \leq \text{mult}_P(\Omega_{t_i} - \Theta_{t_i}^\sigma) + \frac{(L+1)\eta}{p_{t_i}} \text{mult}_P(\Omega_s - \Theta_s^\sigma),
\]

and (13) follows.

I claim that

\[
|p_{t_i}(K_S + \Theta_{t_i}^\sigma)| + p_{t_i}(\Omega_{t_i} - \Theta_{t_i}^\sigma) \subseteq |p_{t_i}(K_S + \Delta_{t_i})|_S
\]
for every $i$. To that end, set $A_i = A/p_i$, and recall that $S \not\subseteq \mathcal{B}(K_X + \Delta_u)$ for every $u \in \mathcal{S}$ by Step 1 of the proof of Theorem 7.4. Therefore $S \not\subseteq \mathcal{B}(K_X + \Delta_i)$ for every $i$ since $t_i \in V$, $p_i \gg 0$ and $\mathcal{S}$ is a rational polyhedral cone, and so $S \not\subseteq \mathcal{B}(K_X + \Delta_i + A_i)$.

Thus, to prove (14), by Proposition 7.3 it is enough to show that for any component $P \subset \text{Supp} \Omega_s$, and for any $l > 0$ sufficiently divisible,

$$\text{mult}_P(\Omega_i \wedge \frac{1}{l} \text{Fix} |l(K_X + \Delta_i + A_i)|_S) \leq \text{mult}_P(\Omega_i - \Theta_i'),$$

and so by (13) it suffices to prove that

(15) $$\text{mult}_P(\Omega_i \wedge \frac{1}{l} \text{Fix} |l(K_X + \Delta_i + A_i)|_S) \leq (1 - \frac{(L+1)\eta}{p_i}) \text{mult}_P(\Omega_s - \Theta_s'),$$

Let $\delta > (L+1)\eta/p_i$ be a rational number such that $\delta(K_X + \Delta_i) + \frac{1}{2}A_i$ is ample. Since

$K_X + \Delta_i + A_i = (1 - \delta)(K_X + \Delta_i + \frac{1}{2}A_i) + (\delta(K_X + \Delta_i) + \frac{1}{2}A_i)$,

and $\text{ord}_P |K_X + \Delta_i + \frac{1}{2}A_i|_S = \sigma_P |K_X + \Delta_i + \frac{1}{2}A_i|_S$ by Remark 2.13 we have

$$\text{ord}_P |K_X + \Delta_i + A_i|_S \leq (1 - \delta)\sigma_P |K_X + \Delta_i + \frac{1}{2}A_i|_S,$$

and thus

(16) $$\text{mult}_P \frac{1}{l} \text{Fix} |l(K_X + \Delta_i + A_i)|_S \leq (1 - \frac{(L+1)\eta}{p_i}) \sigma_P |K_X + \Delta_i + \frac{1}{2}A_i|_S$$

for $l$ sufficiently divisible, cf. Lemma 2.14(4). The divisor $\Delta_i - \delta + \frac{1}{2}A_i$ is ample by the choice of $\delta$, so

$$\sigma_P |K_X + \Delta_i + \frac{1}{2}A_i|_S = \sigma_P |K_X + \Delta_i + (\Delta_i - \delta + \frac{1}{2}A_i)|_S \leq \sigma_P |K_X + \Delta_i|_S.$$ 

This together with (16) gives (15).

Step 2. Let $H$ be a general ample $\mathbb{Q}$-divisor, and let $A_m$ be ample divisors with $\text{Supp} A_m \subset \text{Supp}(\Delta_s - S - A + H)$ such that $\Delta_s + A_m$ are $\mathbb{Q}$-divisors and $\lim_{m \to \infty} ||A_m|| = 0$. Denote $\Delta_m = \Delta_s + A_m$, $\Omega_m = (\Delta_m - S)|_S$ and

$$\Theta_m^\sigma = \Omega_m - \Omega_m \wedge N_\sigma |K_X + \Delta_m|_S. $$

Observe that $\Theta_m^\sigma = \lim_{m \to \infty} \Theta_m^\sigma$ by Lemma 2.14(2), and that

$$N_\sigma |K_X + \Delta_m|_S = \sum \text{ord}_P |K_X + \Delta_m|_S \cdot P$$

for all prime divisors $P$ on $S$ and for all $m$, cf. Remark 2.13. Thus by Remark 7.5 $\Theta_m^\sigma - A_s \in \mathcal{R}_{V_{S,H}A_S}$ for all $m$ and for every component $G$ of $\Theta_m^\sigma$, where $V_{S,H} = V_S + \mathbb{R}H|_S$.

Since $\mathcal{R}_{V_{S,H}A_S}$ is a rational polytope by Lemma 5.8 and in particular is closed, this yields $\Theta_s^\sigma - A_s \in \mathcal{R}_{V_{S,H}A_S}$ for every component $G$ of $\Theta_s^\sigma$. Since $W$ is the smallest rational affine space containing $\Theta_s^\sigma$ and $p_i \gg 0$, we have $\Theta_i' - A_s \in \mathcal{R}_{V_{S,H}A_S}$ for every $i$. Now since $p_i \Theta_i'/r$ is Cartier, we have $G \not\subseteq \text{Fix} |p_i(K_S + \Theta_i')|$ by Lemma 5.8(1). In particular, then (14) implies

$$\Omega_i - \Theta_i' \geq \Omega_i \wedge \frac{1}{p_i} \text{Fix} |p_i(K_X + \Delta_i)|_S,$$
and since by definition $\Omega_i \wedge \frac{1}{p_i} \text{Fix } |p_i(K_X + \Delta_i)|_S \geq \Omega_i - \Theta_i$, we obtain

$$\Theta_i \geq \Theta'_i.$$  

To prove (12), since $\Theta'_s \geq \Theta_s$ by Lemma 2.14(1), it is enough to show that $\text{mult}_P \Theta'_s \leq \text{mult}_P \Theta_s$ for every prime divisor $P$ on $S$. If $\text{mult}_P \Theta'_s = 0$, then immediately $\text{mult}_P \Theta_s = 0$, and we are done. If $\text{mult}_P \Theta'_s > 0$, then $\text{mult}_P \Theta'_i > 0$ for all $i$, and thus $\text{mult}_P \Theta_i > 0$ by (17). In particular, $\text{mult}_P \Theta_i = \text{mult}_P \Omega_i - \text{ord}_P \|K_X + \Delta_i\|_S$. Since $\Omega_s = \sum r_i \Omega_i$, and

$$\text{ord}_P \|K_X + \Delta_i\|_S = \text{ord}_P \|\sum r_i(K_X + \Delta_i)\|_S \leq \sum r_i \text{ord}_P \|K_X + \Delta_i\|_S$$

by convexity, using (17) we have

$$\text{mult}_P \Theta_s \geq \text{mult}_P \Omega_s - \text{ord}_P \|K_X + \Delta_s\|_S \geq \sum r_i (\text{mult}_P \Omega_i - \text{ord}_P \|K_X + \Delta_i\|_S)$$

$$= \sum r_i \text{mult}_P \Theta_i \geq \sum r_i \text{mult}_P \Theta'_i = \text{mult}_P \Theta'_s \geq \text{mult}_P \Theta_s.$$

Therefore all inequalities are equalities, so this proves (12), and also the claim (2), since then $\lambda_P$ is linear on the cone $\sum \mathbb{R}^+ t_i$ by Lemma 4.3.

Next I prove that, under certain conditions, $\lambda_Z|_{\mathcal{R} \cap \mathcal{H}}$ is piecewise linear for every 2-plane $H \subset \mathbb{R}^\ell$.

**Theorem 7.8.** Assume that the map $\lambda_Z$ is superlinear and that $\Theta_Z(w) > 0$ for all $w \in \mathcal{R}$.

Fix distinct points $s, u \in \Pi$. Then there exists $t \in (s, u)$ such that the map $\lambda_Z|_{\mathcal{R} \cap \mathcal{H}}$ is linear. In particular, for every 2-plane $H \subset \mathbb{R}^\ell$, the map $\lambda_Z|_{\mathcal{R} \cap \mathcal{H}}$ is piecewise linear.

**Proof.** In Step 1 I prove the first claim in the case $s \in \Pi_{\mathbb{Q}}$, and in Step 2 when $s \not\in \Pi_{\mathbb{Q}}$. Then this is put together in Step 3 to prove the second claim.

Let $r$ be a positive integer as in Lemma 5.8 with respect to the vector space $V_s$ and the ample divisor $A|_S$.

**Step 1.** Assume $s \in \Pi_{\mathbb{Q}}$. Let $W$ be the smallest rational affine subspace containing $s$ and $u$, and note that $W \cap \mathcal{R} \subset \Pi$.

Let $\mathcal{P}$ be the set of all prime divisors $P$ on $S$ such that $\text{mult}_P(\Omega_s - \Theta_s) > 0$. If $\mathcal{P} \neq \emptyset$, by Theorem 7.7(1) there is a positive number $\varepsilon \ll 1$ such that

$$\phi = \min \{\text{mult}_P(\Omega_v - \Theta_v) : P \in \mathcal{P}, v \in [s, u] \cap B(s, \varepsilon)\} > 0,$$

and set $\phi = 1$ if $\mathcal{P} = \emptyset$. We can further assume that $\varepsilon$ is small enough so that $(L + 1)\varepsilon(K_X + \Delta') + \frac{1}{r}A$ and $\Delta' - \Delta_s + \frac{1}{r}A$ are ample divisors whenever $\Delta' \in B(\Delta_s, 2L\varepsilon)$.

Let $p_s$ be a positive integer such that $p_s\Delta_s/r$ and $p_s\Theta_s/r$ are integral, and

$$\|p_s(K_X + \Delta_s)\|_S = \|p_s(\Omega_s - \Theta_s)\|_S,$$

cf. relation (11) in Step 3 of the proof of Theorem 7.4. Pick $t \in (s, u)$ such that the smallest rational affine subspace containing $t$ is precisely $W$, $\|s - t\| < \phi\varepsilon/p_s$, and $\|\Theta_s - \Theta_t\| < \phi\varepsilon/p_s$, which is possible by Theorem 7.7(1). Denote by $V \subset \text{Div}(S)_\mathbb{R}$ the smallest rational affine space containing $\Theta_s$ and $\Theta_t$. 

Pick $0 < \eta < 1$. Then by Lemma 6.4 there exist rational points $(t_i, \Theta_i') \in W \times V$, integers $p_i \gg 0$, and $r_i \in \mathbb{R}_{>0}$ for $i = 1, \ldots, w$ such that:

1. $(t_i, \Theta_i', r, p_i, r_i)$ uniformly approximate $(t, \Theta_i) \in W \times V$ with error $\phi \varepsilon$, where $t_1 = s,$ $\Theta_i' = \Theta_i, \quad p_1 = p,$
2. $(t_i, \Theta_i')$ belong to the smallest rational affine space containing $(t, \Theta_i)$ for $i = 3, \ldots, w,$
3. $t = \frac{p_1}{p_1 + p_2}t_1 + \frac{p_2}{p_1 + p_2}t + \tau$, where $\|\tau\| \leq \frac{\eta}{p_1 + p_2},$
4. $\Theta_i = \frac{p_1}{p_1 + p_2}\Theta_i' + \frac{p_2}{p_1 + p_2}\Phi_i + \Phi$, where $\|\Phi\| < \frac{\eta}{p_1 + p_2}$.

Note that $t_i \in \Pi$ since $W$ is the smallest rational affine space containing $t$ and $p_i \gg 0$, thus all divisors above are well defined. By applying the map $\Delta$ from Notation 7.6 to the condition (3), we get

$$\Delta_i = \frac{p_1}{p_1 + p_2}\Delta_1 + \frac{p_2}{p_1 + p_2}\Delta_2 + \Psi,$$

where $\|\Psi\| < \frac{L \eta}{p_1 + p_2}$.

Now that $\Theta_i' \leq \Omega_i$ for all $i$ by Remark 6.6. Furthermore, for every prime divisor $P$ on $S$ we have

$$(1 - \frac{(L+1)\varepsilon}{p_1}) \mu_p(\Omega - \Theta) \leq \mu_p(\Omega - \Theta).$$

To prove this, note that (19) is trivial when $\mu_p(\Omega - \Theta) = 0$. Thus I can assume $\mu_p(\Omega - \Theta) > 0$, and then, by the choice of $\eta$,

$$\mu_p \left( \Omega - \Theta - \frac{p_1 + p_2}{p_1}\left(\Psi - \Phi\right) \right) > 0,$$

since conditions (4) and (5) give $\left\| \frac{p_1 + p_2}{p_1}\left(\Psi - \Phi\right) \right\| < \frac{(L+1)\eta}{p_1}$. If $P \notin \mathcal{P}$, then $\mu_p(\Omega - \Theta) = 0$, and (20) together with conditions (4) and (5) gives

$$\mu_p(\Omega - \Theta) \leq \mu_p \left( \Omega - \Theta - \frac{p_1 + p_2}{p_1}\left(\Omega - \Theta - \frac{p_1 + p_2}{p_1}\left(\Psi - \Phi\right) \right) \right) = \mu_p(\Omega - \Theta'),$$

which implies (19). If $P \in \mathcal{P}$, then $\mu_p(\Omega - \Theta) \geq \phi$, and (19) follows similarly as (13) in Step 1 of the proof of Theorem 7.7.

I claim that

$$\left| p_i(K_S + \Theta_i') \right| + p_i(\Omega_i - \Theta_i') \subset \left| p_i(K_X + \Delta_i) \right|_S$$

for all $i$. Granting this for the moment, note that $\mathcal{B}_{V_S,A_S}$ is a rational polytope by Lemma 5.8, and $\Theta_p - A_{|S|} \subset \mathcal{B}_{V_S,A_S}$ for every $p \in \Pi$ by Remark 7.5. Therefore when $\varepsilon \ll 1$, as in Step 2 of the proof of Theorem 7.7 we have that $\lambda_2$ is linear on the cone $\sum_{i=1}^w \mathbb{R}_{>0}t_i$, and in particular on the cone $\mathbb{R}_{>0}s + \mathbb{R}_{>0}t$, so we are done.

To prove the claim, note that (21) follows from (18) for $i = 1$, and for $i = 3, \ldots, w$ it is proved as (14) in Step 1 of the proof of Theorem 7.7. For $i = 2$, by Proposition 7.3 it is enough to show that for a prime divisor $P$ and for $l > 0$ sufficiently divisible we have

$$\mu_p(\Omega - \Theta) \leq \mu_p(\Omega - \Theta').$$
where \( A_{t_2} = A/p_{t_2} \), and so by (19) it suffices to prove that
\[
\text{mult}(\Omega_{t_2} \wedge 1/\ell) \text{Fix}|(K_X + \Delta_{t_2} + A_{t_2})|s) \leq \left(1 - \frac{(L+1)\varepsilon}{p_{t_2}}\right) \text{mult}(\Omega_r - \Theta_r).
\]
But this is proved similarly as (15) in Step 1 of the proof of Theorem 7.7.

**Step 2.** Assume in this step that \( s \not\in \Pi \). By Theorem 7.7(2) there is a cone \( C = \sum \mathbb{R}_+ g_i \) with finitely many \( g_i \in \mathcal{S}_\mathbb{Q} \), such that \( s = \sum \alpha_i g_i \) with all \( \alpha_i > 0 \), and \( \lambda_Z|_C \) is linear. Then \( \lambda_Z(g) = \sum \lambda_Z(g_i) \), where \( g = \sum g_i \in \mathcal{C}_\mathbb{Q} \). By Step 1 there is a point \( v = \alpha g + (1 - \alpha)u \) with \( 0 < \alpha < 1 \) such that the map \( \lambda_Z|_{\mathbb{R}_+ g + \mathbb{R}_+ v} \) is linear, and in particular \( \lambda_Z(g + v) = \lambda_Z(g) + \lambda_Z(v) \).

Finally, let \( H \) be any 2-plane in \( \mathbb{R}^\ell \). By Steps 1 and 2, for every \( s \in \Pi \cap H \) there is a positive number \( \varepsilon_s \) such that \( \lambda_Z\mid_{\mathbb{R}_+ (\Pi \cap H \cap B(s, \varepsilon_s))} \) is piecewise linear. By compactness, there are finitely many points \( s_i \in \Pi \cap H \) such that \( \Pi \cap H \subset \bigcup_i B(s_i, \varepsilon_{s_i}) \), and thus \( \lambda_Z|_{\mathbb{R}^\ell \cap H} \) is piecewise linear.

Finally, we have

**Theorem 7.9.** For every prime divisor \( Z \) on \( S \), the map \( \lambda_Z \) is rationally piecewise linear. Therefore, \( \lambda \) is rationally piecewise linear.

**Proof.** *Step 1.* Let \( \sum G_i = \bigcup_{s \in \mathcal{S}} \text{Supp}(\Delta_s - S - A) \), and set \( v = \max_{s \in \mathcal{S}}\{\text{mult}G_i, \Delta_s\} \) < 1. Let \( 0 < \eta < 1 - v \) be a rational number such that \( A - \eta \sum G_i \) is ample, and let \( A \sim_{\mathbb{Q}} A - \eta \sum G_i \) be a general ample \( \mathbb{Q} \)-divisor. Denote \( \tilde{\Delta}_s = \Delta_s - A + \eta \sum G_i + \tilde{A} \geq 0 \) for every \( s \in \mathcal{S} \), and observe that \( \tilde{\Delta}_s \sim_{\mathbb{Q}} \Delta_s \), \( \tilde{\Delta}_s = S \) and \( (S, \tilde{\Delta}_s) = (\Delta_s - S)|_S \) is terminal since \( \eta < 1 \).

Fix a sufficiently divisible positive integer \( k \) such that \( k\text{Fix}_t(K_X + \tilde{\Delta}_s) \in \text{Div}(X) \) for all \( s \in \mathcal{S} \), and define
\[
\tilde{\Theta}_s = \limsup_{m \to \infty} (\tilde{\Theta}_s - \tilde{\Theta}_s \wedge 1/mk\text{Fix}_m k\text{Fix}_t(K_X + \tilde{\Delta}_s)|_S).
\]
Then as in Step 3 of the proof of Theorem 7.4 and in Notation 7.6 we have associated maps \( \tilde{\Theta}, \tilde{\lambda}, \lambda : \mathcal{S}_R \to \text{Div}(S)|_R \) and \( \tilde{\Theta}_Z, \tilde{\lambda}_Z : \mathcal{S}_R \to \mathbb{R} \). Let \( \mathcal{L}_Z \) and \( \tilde{\mathcal{L}}_Z \) be the closures of sets \( \{s \in \mathcal{S}_R : \tilde{\Theta}(s) > 0\} \) and \( \{s \in \mathcal{S}_R : \tilde{\Theta}(s) > 0\} \), respectively; they are closed cones. By construction, \( \text{ord}_Z(\lambda(s))/\text{Fix}_t = \text{ord}_Z(\tilde{\lambda}(s))/\text{Fix}_t \), and thus \( \tilde{\Theta}(s) = \Theta(s) + \eta \) for every \( s \in \mathcal{L}_Z \). In particular, \( \mathcal{L}_Z \) is the closure of the set \( \{s \in \mathcal{S}_R : \tilde{\Theta}(s) > \eta\} \), and \( \mathcal{L}_Z \subset \tilde{\mathcal{L}}_Z \). Therefore, for every \( s \in \mathcal{L}_Z \) there is a sequence \( s_m \) such that \( \lim_{m \to \infty} s_m = s \) and \( \tilde{\Theta}(s_m) > \eta \).
thus similarly as in [Nak04, 2.1.4], we have $\tilde{\Theta}(s) \geq \limsup_{m \to \infty} \tilde{\Theta}(s_m) \geq \eta$.

**Step 2.** If $L_Z = \emptyset$, then $\lambda_Z$ is trivially a linear map, so until the end of the proof I assume $L_Z \neq \emptyset$. In this step I prove that there is a rational polyhedral cone $M_Z$ such that $L_Z \subset M_Z \subset \hat{L}_Z$.

I first show that for every point $x \in \Pi \cap L_Z$ there is a neighbourhood $U \subset \mathbb{R}^\ell$ of $x$, in the sup-norm, such that $U \cap \mathcal{S}_R \subset \hat{L}_Z$. To that end, recall that $\mathcal{S}_R = \sum \mathbb{R}_+ e_i$, and choose points $x_i \in \mathbb{R}_+ e_i \setminus \{x\}$. Since $\tilde{\Theta}(x) > 0$, by Theorem [7.7.1] there exists a point $y_i \in (x, x_i)$ such that $\tilde{\Theta}(y_i) > 0$ for each $i$. Therefore $\bigcup \mathbb{R}_+ y_i \subset \hat{L}_Z$, and it is sufficient to take any neighbourhood $U$ of $x$ such that $U \cap \mathcal{S}_R \subset \sum \mathbb{R}_+ y_i$.

By compactness, there is a rational number $0 < \xi < 1$ and finitely many rational points $z_1, \ldots, z_p \in \Pi \cap L_Z$ such that $L_Z \subset \bigcup (\mathbb{R}_+ B(z_i, \xi)) \cap \mathcal{S}_R \subset \hat{L}_Z$. The convex hull $B$ of $\bigcup B(z_i, \xi)$ is a rational polytope, and define $M_Z = \mathbb{R}_+ B \cap \mathcal{S}_R$.

**Step 3.** Note that, by construction, $\tilde{\Theta}(s) > 0$ for all $s \in M_Z$, and that the map $\tilde{\lambda}_Z|_M$ is superlinear, cf. the argument in Notation 7.6.

I claim that it is enough to prove that $\tilde{\lambda}_Z|_M$ is rationally piecewise linear. To that end, since $L_Z$ is the closure of the set $\{s \in \mathcal{S}_R : \tilde{\Theta}(s) \geq \eta\}$ and $\eta \in \mathbb{Q}$, we have that then $L_Z$ is a rational polyhedral cone, and thus the map $\tilde{\lambda}_Z|_M$ is rationally piecewise linear.

Therefore so is $\lambda_Z$, since $\tilde{\Theta}(s) = \Theta(s) + \eta$ for every $s \in L_Z$, and this proves the claim.

By Lemma [4.2], there are finitely many generators $g_i$ of $M_Z \cap \mathcal{S}_R$, and let $\varphi : \bigoplus i \mathbb{N} g_i \rightarrow M_Z \cap \mathcal{S}_R$ be the projection map. Replacing $\mathcal{S}_R$ by $\bigoplus i \mathbb{N} g_i$, $\lambda_Z$ by $\lambda_Z \circ \varphi$ and $\Theta$ by $\tilde{\Theta} \circ \varphi$, I can assume that $\lambda_Z$ is a superlinear function on $\mathcal{S}_R$ and $\Theta_Z(s) > 0$ for all $s \in \mathcal{S}_R$.

By Theorem [7.8] for any 2-plane $H \subset \mathbb{R}^\ell$ the map $\lambda_Z|_{\mathcal{S}_R \cap H}$ is piecewise linear, and thus $\lambda_Z$ is piecewise linear by Theorem [4.3].

Finally, to prove that $\lambda_Z$ is rationally piecewise linear, let $\mathcal{S}_R = \bigcup C_m$ be a finite polyhedral decomposition such that the maps $\lambda_Z|_{C_m}$ are linear, and their linear extensions to $\mathbb{R}^\ell$ are pairwise different. Let $\mathcal{F}$ be a common $(\ell - 1)$-dimensional face of cones $C_i$ and $C_j$, and assume $\mathcal{F}$ does not belong to a rational hyperplane. Let $\mathcal{H}$ be the smallest affine space containing $\mathcal{F}_\Pi = \mathcal{F} \cap \Pi$, and note that $\mathcal{H}$ is not rational and dim $\mathcal{H} = \ell - 2$. If for every $f \in \mathcal{F}_\Pi$ there existed a rational affine space $\mathcal{H}_f \ni f$ of dimension $\ell - 2$, this would contradict Remark 2.4 since countably many $\mathcal{H} \cap \mathcal{F}$ would cover $\mathcal{F}_\Pi \subset \mathcal{H}$.

Therefore, there is a point $s \in \mathcal{F}_\Pi$ and an $\ell$-dimensional cone $C_s$ such that $s \in \text{int} C_s$ and the map $\tilde{\lambda}_Z|_{C_s}$ is linear, by Theorem [7.7.2]. But then the cones $C_s \cap C_i$ and $C_s \cap C_j$ are $\ell$-dimensional and linear extensions of $\tilde{\lambda}_Z|_{C_s}$ and $\tilde{\lambda}_Z|_{C_i}$ coincide since they are equal to the linear extension of $\tilde{\lambda}_Z|_{C_s}$, a contradiction. Thus all $(\ell - 1)$-dimensional faces of the cones $C_m$ belong to rational $(\ell - 1)$-planes, so $C_m$ are rational cones.

8. Stable base loci

**Theorem 8.1.** Theorems $A_{n-1}$ and $C_{n-1}$ imply Theorem $B_n$. 

Proof. Let $X$ be a smooth variety as in Theorem B, let $K_X$ be a divisor with $\mathcal{O}_X(K_X) \cong \omega_X$ and $A \not\subset \text{Supp } K_X$, and denote $\mathcal{C} = \mathbb{R}_+(K_X + A + \mathcal{B}^{G=1}_{V,A})$. It suffices to prove that the cone $\mathcal{C}$ is rational polyhedral.

Step 1. Denote

$$\mathcal{D}_{V,A}^{G=1} = \{ \phi \in \mathcal{L}_V : \text{mult}_G \phi = 1, \sigma_G\|K_X + \phi + A\| = 0 \}.$$

This is a convex set, and it is also closed: if $D_m \in K_X + A + \mathcal{D}_{V,A}^{G=1}$ is a sequence such that $\lim_{m \to \infty} D_m = D \in K_X + A + \mathcal{L}_V$, then $\sigma_G\|D\| \leq \liminf_{m \to \infty} \sigma_G\|D_m\| = 0$ by [Nak04, 2.14], so $D \in K_X + A + \mathcal{D}_{V,A}^{G=1}$.

In this step I show that $\mathcal{D}_{V,A}^{G=1} = \mathcal{B}_{V,A}^{G=1}$, and also that $\mathcal{C}$ is a rational cone, i.e. that its extremal rays are rational. Note that $\mathcal{B}_{V,A}^{G=1} \subset \mathcal{D}_{V,A}^{G=1}$ is trivial by Lemma [2.14], so I concentrate on proving the reverse inclusion.

Let $\Delta \in \mathcal{L}_V + A$ be a divisor such that $\text{mult}_G \Delta = 1$ and $\sigma_G\|K_X + \Delta\| = 0$. I first claim that we can assume that $(X, \Delta)$ is plt, $|\Delta| = G$, and $(X, \Omega + A|G)$ is terminal, where $\Omega = (\Delta - G)|G$.

To that end, let $\mathcal{P}$ be the set of prime divisors $F \neq G$ with $\text{mult}_F \Delta = 1$, and choose $0 < \eta \ll 1$ such that $A + \mathcal{P}$ is ample, where $\mathcal{P} = \eta \sum_{F \in \mathcal{P}} F$. Replacing $A$ by a general ample $\mathcal{P}$-linearly equivalent to $A + \mathcal{P}$ and $\Delta$ by $\Delta - \mathcal{P}$, we can assume that $(X, \Delta)$ is plt and $|\Delta| = G$. Let $f : Y \to X$ be a log resolution such that the components of $\{ B(X, \Delta)|_Y \}$ are disjoint as in Lemma [2.3], and in particular $(Y, (\Delta' - G')|_{G'})$ is terminal, where $G' = f_*^{-1}G$ and $\Delta' = B(X, \Delta)|_Y$. Note that $f^*A = f_*^{-1}A \leq \Delta'$ since $A$ is general, let $H$ be a small effective $f$-exceptional divisor such that $f^*A - H$ is ample, and let $A' \sim_{\mathcal{Q}} f^*A - H$ be a general ample $\mathcal{Q}$-divisor. Let $V'$ be the vector space spanned by proper transforms of elements of $V$ and by exceptional divisors. Then $\Delta' \in \mathcal{D}_{V',A'}^{G=1} + A'$ by Remark [2.15], so it is enough to show that $\Delta' \in \mathcal{B}_{V',A'}^{G=1} + A'$ and that the cone $\mathbb{R}_+(K_Y + A' + \mathcal{B}_{V',A'}^{G=1})$ is rational locally around $K_Y + A'$. Replacing $X$ by $Y$, $G$ by $G'$, $\Delta$ by $\Delta' - f^*A + H + A'$ and $V$ by $V'$ proves the claim.

Since $\sigma_G\|K_X + \Delta\| = 0$, by Remark [2.15] the formal sum $N_\sigma\|K_X + \Delta\|_G$ is well-defined and $K_G + \Theta$ is pseudo-effective, where $\Theta = \Omega - \Omega \wedge N_\sigma\|K_X + \Delta\|_G$. Let $\phi < 1$ be the smallest positive coefficient of $\Omega - \Theta$ if it exists, and set $\phi = 1$ otherwise. Denote by $V_G \subset \text{Div}(G)_\mathbb{R}$ the vector space spanned by components of divisors in $\{ F|_G : F \in V, G \not\subset \text{Supp } F \}$. Let $r$ be a positive integer as in Lemma [5.8] with respect to $V_G$ and $A|G$, and let $W \subset \text{Div}(X)_\mathbb{R}$ and $U \subset \text{Div}(G)_\mathbb{R}$ be the smallest rational affine subspaces containing $\Delta$ and $\Theta$, respectively. Choose $\varepsilon > 0$ such that $\varepsilon(K_X + \hat{\Delta}) + \frac{1}{2}A$ and $\hat{\Delta} - \Delta + \frac{1}{2}A$ are ample divisors whenever $\hat{\Delta} \in B(\Delta, \varepsilon)$.

Then by Lemma [6.3] there exist rational points $(\Delta_i, \Theta_i) \in W \times U$, integers $k_i \gg 0$, and $r_i \in \mathbb{R}_{>0}$ such that $(\Delta_i, \Theta_i, r_i, k_i, r_i)$ uniformly approximate $(\Delta, \Theta) \in W \times U$ with error $\phi \varepsilon/2$. Note that then, for each $i$, $(X, \Delta_i)$ is plt, $(G, \Omega_i + A_i|G)$ is terminal with $\Omega_i = (\Delta_i - G)|G$, and $\Theta_i \leq \Omega_i$ by Remark [6.6].
Since $\sigma_G\Vert K_X + \Delta \Vert = 0$ we have $G \not\subset B(K_X + \Delta + \frac{1}{2}A_i)$ by Remark 2.15 and since $\Delta - \Delta_i + \frac{1}{2}A_i$ is ample, it follows that $G \not\subset B(K_X + \Delta + A_i)$. Therefore, similarly as in Step 1 of the proof of Theorem 7.7.

(22) $|k_i(K G + \Theta_i)| + k_i(\Omega_i - \Theta_i) \subset |k_i(K_X + \Delta_i)|_G$.

In particular, since $U$ is the smallest rational affine space containing $\Theta$, $k_i \gg 0$ and $\Theta_{V,G,A_i}$ is a rational polytope by Theorem C, we have $\Theta_i - A_i|G \in \Theta_{V,G,A_i}$, and Lemma 5.8 (2) yields $|k_i(K G + \Theta_i)| \neq 0$. Thus (22) implies that there is an effective divisor $D_i \in |k_i(K_X + \Delta_i)|$ with $G \not\subset \text{Supp}D_i$. But then $K_X + \Delta \sim_{\mathbb{R}} \sum m_iD_i$ and $G \not\subset B(K_X + \Delta)$, so $\Delta \in \mathcal{R}_{V,A}G^{-1} + A$, as desired.

**Step 2.** It remains to prove that $C$ is polyhedral. To that end, I will prove it has only finitely many extremal rays.

Assume that there are distinct rational divisors $\Delta_m \in \mathcal{R}_{V,A}G^{-1} + A$ for $m \in \mathbb{N} \cup \{\infty\}$ such that the rays $\mathbb{R} \cdot Y_m$ are extremal in $C$ and $\lim_{m \to \infty} \Delta_m = \Delta_{\infty}$, where $\Sigma_m = K_X + \Delta_m$. As explained in Remark 2.16 I achieve contradiction by showing that for some $m \gg 0$ there is a point $Y_m \in \mathcal{C}$ such that $Y_m \in (Y_\infty, Y_m')$.

I claim that we can assume that $(X, \Delta_m)$ is plt, $|\Delta_m| = G$, and each pair $(G, \Omega_m + A |G)$ is canonical for $m \in \mathbb{N} \cup \{\infty\}$, where $\Omega_m = (\Delta_m - G)|G$. To that end, by passing to a subsequence, as in Step 1 we can assume that $(X, \Delta_m)$ is plt and $|\Delta_m| = G$ for each $m$. Let $f: Y \to X$ be a log resolution such that the components of $\{B(X, \Delta_{\infty})Y\}$ are disjoint as in Lemma 2.3 and in particular $(Y, (B(X, \Delta_m)Y - G')|G)$ is terminal for $m \gg 0$, where $G' = f_*^{-1}G$. Let $H$, $A'$ and $V'$ be as in Step 1, and denote $\Delta'_m = B(X, \Delta_m)Y - f^*A + H + A'$. Now, if for every $m \gg 0$ there is a divisor $\Delta_m \in \mathcal{R}_{V',A'}G^{-1} + A'$ such that $K_Y + \Delta'_m \in (K_Y + \Delta_{\infty}, K_Y + \tilde{\Delta}_m)$, then $f_*\tilde{\Delta}_m \in \mathcal{R}_{V,A}G^{-1} + A$ and $[\Sigma_m] \in ([Y_\infty], [K_X + f_*\tilde{\Delta}_m])$ as $\Sigma_m \sim_{\mathbb{Q}} K_X + f_*\Delta'_m$ for all $m$. Therefore, since $\sigma_G\Vert K_X + f_*\tilde{\Delta}_m\Vert = 0$ by Step 1, the ray $\mathbb{R} \cdot Y_m$ is not extremal in $C$, as explained in Remark 2.16. Replacing $X$ by $Y$, $G$ by $G'$ and $\Delta_m$ by $\Delta'_m$ proves the claim.

Let $\Theta_m = \Omega_m - \Omega_m \wedge N_G\Vert Y_m\Vert_G$, and note that $\Theta_m = \Omega_m - \Omega_m \wedge (\sum \text{ord} P \Vert Y_m\Vert_G \cdot P)$ by the relation (12) in Theorem 7.7. By Step 3 of the proof of Theorem 7.4 each $\Theta_m$ is a rational divisor, and $\Theta_{\infty} \geq \limsup_{m \to \infty} \Theta_m$ as in the proof of [Nak04, 2.1.4]. By passing to a subsequence, we can assume that there is a divisor $\Theta^0_{\infty}$ such that $\lim_{m \to \infty} \Theta_m = \Theta^0_{\infty} \leq \Theta_{\infty}$. If we define $V_G$ as in Step 1, then $\Theta_{V_G,A_i}|G$ is a rational polytope by Theorem C, and thus

(23) $\Theta^0_{\infty} - A_i|G \in \Theta_{V_G,A_i}|G$

since $\Theta_m - A_i|G \in \Theta_{V_G,A_i}|G$ for all $m$.

Let $\mathcal{P}$ be the set of all prime divisors $P$ on $S$ such that $\text{mult}_P(\Omega_{\infty} - \Theta^0_{\infty}) > 0$. If $\mathcal{P} \neq \emptyset$, by passing to a subsequence we can assume that

$$\phi = \min\{\text{mult}_P(\Omega_m - \Theta_m) : P \in \mathcal{P}, m \in \mathbb{N} \cup \{\infty\}\} > 0,$$
and set $\phi = 1$ if $\mathcal{P} = \emptyset$. Let $r$ be a positive integer as in Lemma 5.8 with respect to $V_G$ and $A_G$, and let $U^0$ be the smallest rational affine space containing $\Theta_0$.

Let $0 < \varepsilon \ll 1$ be a rational number such that $\varepsilon (K_X + \tilde{\Delta}) + \frac{1}{2}A$ and $\tilde{\Delta} - \Delta_\infty + \frac{1}{2}A$ are ample divisors whenever $\tilde{\Delta} \in B(\Delta_\infty, 2\varepsilon)$. Let $q$ be a positive integer such that $q\Delta_\infty / r$ is integral. By Lemma 6.3 there exist a $\mathbb{Q}$-divisor $\tilde{\Theta} \in U^0$ and a positive integer $k_\infty \gg 0$ such that $\|\tilde{\Theta} - \Theta_0\| < \phi \varepsilon / 2k_\infty$ and $k_\infty \tilde{\Theta} / q$ is integral; in particular $k_\infty \Delta_\infty / r$ and $k_\infty \tilde{\Theta} / r$ are integral, and $\tilde{\Theta} - A_G \in \delta_{V_G A_G}$ by (23) since $\delta_{V_G A_G}$ is a rational polytope by Theorem $C_{n-1}$. By passing to a subsequence again, we can assume that $\|\Delta_\infty - \Delta_m\| < \phi \varepsilon / 2k_\infty$ and $\|\tilde{\Theta} - \Theta_m\| < \phi \varepsilon / 2k_\infty$ for all $m$.

Then by Lemma 6.4, for every $m \in \mathbb{N}$ there is a point $(\Delta_m', \Theta_m') \in \operatorname{Div}(X)_\mathbb{Q} \times \operatorname{Div}(G)_\mathbb{Q}$ and a positive integer $k_m \gg 0$ such that:

1. $\Delta_m = \frac{k_\infty}{k_\infty + k_m} \Delta_\infty + \frac{k_m}{k_\infty + k_m} \Delta_m'$ and $\Theta_m = \frac{k_m}{k_\infty + k_m} \tilde{\Theta} + \frac{k_m}{k_\infty + k_m} \Theta_m'$,
2. $k_m \Delta_m'/r$ is integral and $\|\Delta_m - \Delta_m'\| < \phi \varepsilon / 2k_m$,
3. $k_m \Theta_m'/r$ is integral and $\|\Theta_m - \Theta_m'\| < \phi \varepsilon / 2k_m$.

Denote $\Omega_m' = (\Delta_m' - G)|_G$, and note that $\Theta_m' \leq \Omega_m'$ by Remark 6.6. Furthermore, since $\delta_{V_G A_G}$ is a rational polytope, for $m \gg 0$ we have

$$[\tilde{\Theta}, \Theta_m] \subseteq ([\tilde{\Theta} + \mathbb{R}_+ (\Theta_m - \tilde{\Theta})) \cap \delta_{V_G A_G},$$

so in particular $\Theta_m' \in \delta_{V_G A_G}$ since $k_m \gg 0$. I claim that

$$\|k_m(K_G + \Theta_m')| + k_m(\Omega_m' - \Theta_m') \| \leq \|k_m(K_X + \Delta_m')|_G.$$  

Grant the claim for the moment. Then $|k_m(K_G + \Theta_m')| \neq \emptyset$ by Lemma 5.8(2), and thus $G \not\subset B(K_X + \Delta_m')$ by (24). But then by the condition (1) above, the ray $\mathbb{R}_+ \tilde{\gamma}_m$ is not extremal in $\mathcal{E}'$, a contradiction.

Now I prove the claim. By Proposition 7.3 it is enough to show that for a prime divisor $P$ on $S$ and for $l > 0$ sufficiently divisible we have

$$\operatorname{mult}_P(\Omega_m' \wedge \frac{1}{l} \operatorname{Fix} |l(K_X + \Delta_m' + A_m)|_S) \leq \operatorname{mult}_P(\Omega_m' - \Theta_m'),$$

where $A_m = A / k_m$. First I show

$$\|1 - \frac{\varepsilon}{k_m}\| \operatorname{mult}_P(\Omega_m' \wedge \frac{1}{l} \operatorname{Fix} |l(K_X + \Delta_m' + A_m)|_S) \leq \operatorname{mult}_P(\Omega_m' - \Theta_m').$$

To see this, note that (25) is trivial when $\operatorname{mult}_P(\Omega_m - \Theta_m) = 0$, so I assume $\operatorname{mult}_P(\Omega_m - \Theta_m) > 0$. If $P \not\in \mathcal{P}$, then $\operatorname{mult}_P(\Theta_0) = \operatorname{mult}_P \Omega_\infty \in \mathbb{Q}$, so in particular $\operatorname{mult}_P \tilde{\Theta} = \operatorname{mult}_P \Omega_\infty$ as $\tilde{\Theta} \in U^0$. Therefore, by condition (1) above,

$$\operatorname{mult}_P(\Omega_m - \Theta_m) \leq \frac{k_m + k_m}{k_m} \operatorname{mult}_P(\Omega_m - \Theta_m') = \operatorname{mult}_P(\Omega_m' - \Theta_m'),$$

which implies (26). If $P \in \mathcal{P}$, then $\operatorname{mult}_P(\Omega_m - \Theta_m) \geq \phi$, and (26) follows similarly as (13) in Step 1 of the proof of Theorem 7.7.

Therefore, by (25) and (26) it suffices to prove that

$$\operatorname{mult}_P(\Omega_m' \wedge \frac{1}{l} \operatorname{Fix} |l(K_X + \Delta_m' + A_m)|_S) \leq (1 - \frac{\varepsilon}{k_m}) \operatorname{mult}_P(\Omega_m - \Theta_m).$$
But this is proved similarly as (15) in Step 1 of the proof of Theorem 7.7 and we are done. □

9. PSEUDO-EFFECTIVITY AND NON-VANISHING

In this section I prove the following.

**Theorem 9.1.** Theorems $A_{n-1}$, $B_n$ and $C_{n-1}$ imply Theorem $C_n$.

I first make a few remarks that will be used in the proof.

**Remark 9.2.** Let $D \leq 0$ be a divisor on a smooth variety $X$. I claim that then $D$ is pseudo-effective if and only if $D = 0$. To that end, if $A$ is an ample divisor, then $D + \varepsilon A$ is big for every $\varepsilon > 0$. In particular, $|D + \varepsilon A|_\RR \neq \emptyset$, and thus $\deg(D + \varepsilon A) \geq 0$. But then letting $\varepsilon \downarrow 0$ implies $\deg D \geq 0$, so $D = 0$.

**Remark 9.3.** With notation from Theorem B, let $0 < \xi \ll 1$ be a rational number such that $A - \Xi$ is ample for all $\Xi \in V$ with $\|\Xi\| \leq \xi$, let $\mathcal{L}_V, \xi$ be the $\xi$-neighbourhood of $\mathcal{L}_V$ in the sup-norm, and set

$$
\mathcal{B}^{G=1}_{V,A,\xi} = \{ \Phi \in \mathcal{L}_V, \xi : \text{mult}_G \Phi = 1, G \not\subset B(K_X + \Phi + A) \}.
$$

Then I claim Theorem B implies that $\mathcal{B}^{G=1}_{V,A,\xi}$ is a rational polytope. To that end, fix $\Phi \in \mathcal{B}^{G=1}_{V,A,\xi}$. Let $\mathcal{Z}$ be the set of all prime divisors $Z \in V \setminus \{G\}$ such that $\text{mult}_Z \Phi \geq 1$, and let $A' \sim_{\QQ} A + \xi \sum_{Z \in \mathcal{Z}} Z$ be a general ample $\QQ$-divisor. Then for every $\Phi' \in \mathcal{B}^{G=1}_{V,A,\xi}$ with $\|\Phi - \Phi'\| < \xi$, we have $\Phi' - \xi \sum_{Z \in \mathcal{Z}} Z \in \mathcal{B}^{G=1}_{V,A'}$ since $\xi \ll 1$. As $\mathcal{B}^{G=1}_{V,A'}$ is a rational polytope by Theorem B, this implies that $\mathcal{B}^{G=1}_{V,A,\xi}$ is locally a rational polytope around $\Phi$, and the claim follows by compactness of $\mathcal{B}^{G=1}_{V,A,\xi}$.

**Proof of Theorem 9.1.** Let $X$ be a smooth variety and $B$ a divisor on $X$ as in Theorem $C_n$. Fix a divisor $K_X$ such that $\mathcal{E}_X(K_X) \simeq \mathcal{O}_X$ and $A \not\subset \text{Supp} K_X$. It suffices to prove that the cone $\mathcal{C} = \RR^+ (K_X + A + \mathcal{E}_V) \subset \text{Div}(X)_\RR$ is rational polyhedral. Observe that $\mathcal{C}$ is closed since $\mathcal{E}_{V,A}$ is.

**Step 1.** Fix $\Delta \in \mathcal{E}_{V,A}$. I first show that there exists an effective divisor $D \in \text{Div}(X)_\RR$ such that $K_X + \Delta \equiv D$. This was proved essentially in [Hac08], and I will sketch the proof here for completeness.

First I claim that we can assume $(X, \Delta)$ is klt. To see this, let $\mathcal{G}$ be the set of all prime divisors $G$ with $\text{mult}_G \Delta = 1$ and choose $0 < \eta \ll 1$ such that $A + \eta \sum_{G \in \mathcal{G}} G$ is ample. Let $A' \sim_{\QQ} A + \eta \sum_{G \in \mathcal{G}} G$ be a general ample $\QQ$-divisor and set $\Delta' = \Delta - \eta \sum_{G \in \mathcal{G}} G + A'$. Then $K_X + \Delta \sim_{\QQ} K_X' + \Delta'$ and $(X, \Delta')$ is klt, so replace $\Delta$ by $\Delta'$ and $A$ by $A'$.

Now, if $v(X, D) = 0$, cf. Definition A.4, then the result follows from [BCHM06, 3.3.2]. If $v(X, D) > 0$, then by [BCHM06, 6.2] we can assume that $(X, \Delta)$ is plt, $A$ is a general ample $\QQ$-divisor, $|\Delta| = S$, $(S, (\Delta - S)|_S)$ is canonical, and $\sigma_S|K_X + \Delta|_\RR = 0$. But now as in Step 1 of the proof of Theorem 8.1 we have $|K_X + \Delta|_\RR \neq \emptyset$.
Step 2. In this step we assume further that $\Delta \in \text{Div}(X)_\mathbb{Q}$, and prove that $|K_X + \Delta|_\mathbb{Q} \neq \emptyset$. This argument uses Shokurov’s trick from his proof of the classical Non-vanishing theorem, and I will present an algebraic proof following the analytic version from [Pău08].

By Step 1, $K_X + \Delta \equiv D$ for some effective $\mathbb{R}$-divisor $D$, and write $\Delta = \Phi + A$. Let $W \subset \text{Div}(X)_\mathbb{R}$ be the vector space spanned by the components of $K_X$, $A$, and by the prime divisors in $V$, and let $\phi: W \to N^1(X)$ be the linear map sending a divisor to its numerical class. Since $\phi^{-1}([K_X + \Delta])$ is a rational affine subspace of $W$, we can assume that $D$ is an effective $\mathbb{Q}$-divisor.

Let $m$ be a positive integer such that $m\Delta$ and $mD$ are integral. By Nadel vanishing

$$H^i(X, \mathcal{J}_{(m-1)D} + \Phi(m(K_X + \Delta))) = 0 \quad \text{and} \quad H^i(X, \mathcal{J}_{(m-1)D} + \Phi(mD)) = 0$$

for $i > 0$, and since the Euler characteristic is a numerical invariant,

$$h^0(X, \mathcal{J}_{(m-1)D} + \Phi(m(K_X + \Delta))) = h^0(X, \mathcal{J}_{(m-1)D} + \Phi(mD)).$$

Let $\sigma \in H^0(X, mD)$ be the section with $\text{div} \sigma = mD$. Since

$$((m-1)D + \Phi) - mD \leq \Phi,$$

by [HM08 4.3(3)] we have $\mathcal{J}_{mD} \subset \mathcal{J}_{(m-1)D} + \Phi$, and thus

$$\sigma \in H^0(X, \mathcal{J}_{(m-1)D} + \Phi(mD)).$$

Therefore (27) implies $h^0(X, m(K_X + \Delta)) > 0$, and we are done.

Step 3. In this step I prove that $\mathcal{C}$ is a rational cone, and that

$$\mathcal{E}_{V,A} = \{ \Phi \in \mathcal{L}_V : |K_X + \Phi + A|_\mathbb{R} \neq \emptyset \}.$$

Fix $\Delta \in \mathcal{E}_{V,A} + A$. By Step 1 there is an effective $\mathbb{R}$-divisor $D$ such that $K_X + \Delta \equiv D$. Write $\Delta = A + \sum \delta_i F_i$ with $\delta_i \in [0, 1]$, and $D = \sum f_i F_i$, where we can assume $F_i \neq \text{Supp} A$ for all $i$ since $A$ is general.

I claim that we can assume that $\sum F_i$ has simple normal crossings. To that end, let $f: Y \to X$ be a log resolution of $(X, \sum F_i)$, and denote $G = f_*^{-1}G$ and $\Delta' = \mathcal{B}(X, \Delta)$. Note that $f^*A = f_*^{-1}A \leq \Delta'$ since $A$ is general, let $H$ be a small effective $f$-exceptional divisor such that $f^*A - H$ is ample, and let $A' \sim_\mathbb{Q} f^*A - H$ be a general ample $\mathbb{Q}$-divisor. Let $V'$ be the vector space spanned by proper transforms of elements of $V$ and by exceptional divisors. Then $K_Y + \Delta' \equiv f^*D + E$, where $E = K_Y + \Delta' - f^*(K_X + \Delta)$ is effective and $f$-exceptional, and $\Delta' \in \mathcal{E}_{V',A'} + A'$, so it is enough to show that the cone $\mathbb{R}_+(K_Y + \Delta' + \mathcal{E}_{V',A'})$ is rational locally around $K_Y + \Delta'$. Replacing $X$ by $Y$, $G$ by $G'$, $\Delta$ by $\Delta' - f^*A + H + A'$, and $V'$ by $V'$ proves the claim.

Define $W$ and $\phi$ as in Step 2, and let $0 < \varepsilon \ll 1$ be a rational number such that $A + \Phi$ is ample for any divisor $\Phi \in W$ with $|\Phi| \leq \varepsilon$. Choose $0 \leq f_i' \leq f_i$ be rational numbers such that $f_i - f_i' < \varepsilon$. Then

$$K_X + \Delta' \equiv \sum f_i' F_i,$$
where $\Delta' = \Delta - \sum(f_i - f'_i)F_i$. Since $\mathcal{P} = \phi^{-1}(\sum f'_i[F_i])$ is a rational affine subspace of $W$, there are rational divisors $\Delta_j \in V + A$ such that $\|\Delta' - \Delta_j\| < \varepsilon$, $K_X + \Delta_j \in \mathcal{P}$ and $K_X + \Delta' = \sum \rho_j(K_X + \Delta_j)$ for some positive numbers $\rho_j$ with $\sum \rho_j = 1$. Setting $\Phi_j = \sum \max\{0, \text{mult}\_F \Delta_j - \varepsilon\}F_j$, the divisor $\Delta_j - \Phi_j$ is ample since $\|\Delta_j - A - \Phi_j\| \leq \varepsilon$, and let $A' \sim Q \Delta_j - \Phi_j$ be a general ample $Q$-divisor. Therefore each $K_X + \Delta_j \sim Q K_X + \Phi_j + A'$ is a rational pseudo-effective divisor, and since $(X, \Phi_j + A')$ is klt, it is $Q$-linearly equivalent to an effective divisor by Step 2. For each $j$, denote $\mathcal{B}_j = [\sum \text{mult}\_F \Delta_j, 1]F_i$, and let $\mathcal{B}$ be the convex hull of $\bigcup \mathcal{B}_j$; observe that $\mathcal{B}$ is a rational polytope. Then, since $V \subset W$,

$$K_X + \Delta \in (K_X + A + \mathcal{B}) \cap (K_X + A + \mathcal{L}_V).$$

Therefore, (28) shows that $K_X + \Delta$ is $\mathbb{R}$-linearly equivalent to an effective divisor, and that $\mathcal{C}$ is locally rational around every $K_X + \Delta$, and thus it is a rational cone.

**Step 4.** It remains to prove that the cone $\mathcal{C}$ is polyhedral, i.e. that it has finitely many extremal rays. Let $G_1, \ldots, G_N$ be prime divisors on $X$ such that $\text{Supp} K_X \cup \text{Supp} B \subset \sum G_i$.

Assume that $\mathcal{C}$ has infinitely many extremal rays. Thus, since $\mathcal{C}$ is a rational cone, there are distinct rational divisors $\Delta_m \in \mathcal{B}_{VA} + A$ for $m \in \mathbb{N} \cup \{\infty\}$ such that the rays $\mathbb{R} \Delta_m$ are extremal in $\mathcal{C}$ and $\lim_{m \to \infty} \Delta_m = \Delta_\infty$, where $\Delta_\infty = K_X + \Delta_m$. As explained in Remark 2.16. I achieve contradiction by showing that for some $m \gg 0$ there is a point $\Gamma_m \in \mathcal{C}$ such that $\Delta_m = (\Gamma_\infty, \Gamma_m)$.

By Step 2, there is an effective divisor $D_\infty$ such that $\Delta_\infty \sim Q D_\infty$. By possibly adding components, cf. Remark 2.16. I can assume that $\text{Supp} D_\infty \subset \sum G_j$ and that $V = \sum \mathbb{R} G_j$.

Similarly as in Step 1, and possibly by passing to a subsequence, we can assume that $(X, \Delta_m)$ is klt for all $m$, and by taking a log resolution as in Step 3, we can assume further that $\sum G_j$ has simple normal crossings.

**Step 5.** If $D_\infty = 0$, then for all $m \in \mathbb{N}$ the class $[\Gamma_m]$ belongs to the segment $([\Gamma_\infty], [2 \Gamma_m])$, and $2 \Gamma_m$ is pseudo-effective, so we derive contradiction as in Remark 2.16.

Thus, until the end of the proof I assume that $D_\infty \neq 0$, and write $D_\infty = \sum d_j G_j$. Assume that $\text{Supp} N_\sigma[\Gamma_\infty] = \text{Supp} D_\infty$. Then $N_\sigma[\Gamma_\infty] = N_\sigma[D_\infty] = D_\infty$ by Nakayama’s Lemma 2.16.1, and $[G_j]$ are linearly independent in $\text{N}^1(X)$. Let $\mathcal{E} \subset \text{N}^1(X)$ denote the pseudo-effective cone. Similarly as in the proof of [Bou04 3.19], we have $\mathcal{E} = \sum \mathbb{R} [G_j] + \bigcap_j \mathcal{E}_{G_j}$, where $\mathcal{E}_{G_j} = \{ \Xi \in \mathcal{E} : \sigma G_j \| \Xi \| = 0 \}$ is a closed cone for every $j$. I claim that if

$$[D_\infty] = \sum d_j^j[G_j] + \Phi$$

with $d_j^j \geq 0$ and $\Phi \in \bigcap_j \mathcal{E}_{G_j}$, then $\Phi = 0$. To that end, denote $\alpha_j = d_j - d_j^j$ and let $J = \{ j : \alpha_j > 0 \}$. Then (29) gives

$$\sum_{j \in J} \alpha_j [G_j] = - \sum_{j \notin J} \alpha_j [G_j] + \Phi.$$
Assume that there exists $i_0 \in J$. Then, by [Nak04, 2.1.6] again,
\[
0 < \alpha_{i_0} = \sigma_{G_{i_0}} \| \sum_{j \in J} \alpha_j G_j \| = \sigma_{G_{i_0}} \| - \sum_{j \notin J} \alpha_j [G_j] + \Phi \| \leq \sigma_{G_{i_0}} \| - \sum_{j \notin J} \alpha_j G_j \| + \sigma_{G_{i_0}} \| \Phi \| = 0,
\]
a contradiction. Therefore $J = \emptyset$ and $\Phi = \sum_{j \notin J} \alpha_j [G_j]$, thus $\Phi = 0$ by Remark 9.2.

Therefore, by Lemma 2.5 applied to the cone $\mathcal{C}$ and to the sequence $\{Y_m\}$, there exists $\Phi_m \in \mathcal{C}$ such that $[Y_m] \in ([1_\infty], \Phi_m)$, a contradiction by Remark 2.16.

**Step 6.** Therefore, from now on I assume that $\text{Supp} \, N_0 \| Y_\infty \| \neq \text{Supp} \, D_\infty$, and in particular, there is an index $j_0$ such that $d_{j_0} > 0$ and $\sigma_{G_{j_0}} \| Y_\infty \| = 0$. For $m \in \mathbb{N} \cup \{\infty\}$, write $\Delta_m = A + \sum \delta^m_j G_j$ with $\delta^m_j \in [0, 1)$. Then from $Y_\infty \sim_Q D_\infty$ we have
\[
K_X + A \sim_Q \sum f_j G_j
\]
with $f_j = d_j - \delta^m_j$. In this step I prove that there exist pseudo-effective divisors $\Sigma_m$, for $m \in \mathbb{N} \cup \{\infty\}$, with the following properties:

(i) $\lim_{m \to \infty} \Sigma_m = \Sigma_\infty$,

(ii) $\Sigma_\infty \in \sum f_j G_j + \mathbb{R}_{V,A}^{G_{k_0}}$ for some $k_0$ with $\text{mult}_{G_{k_0}} \Sigma_\infty > 0$,

(iii) if for $m \gg 0$ there exists a pseudo-effective divisor $\Sigma'_m$ such that $\Sigma_m \in (\Sigma_\infty, \Sigma'_m)$, then $\mathbb{R}_+ \Sigma_m$ is not an extremal ray of $\mathcal{C}$.

For each $t \in \mathbb{R}_+$, let $\Delta'_t = \Delta_\infty + tD_\infty$ and $\Theta'_t = \Delta'_t - \Delta'_t \cap \text{Supp} \, [\sigma_N] (\tau + 1) Y_\infty$. Note that $K_X + \Delta'_t \sim_Q (\tau + 1) Y_\infty$, $\Theta'_t$ is a continuous function in $t$, and $\text{mult}_{G_{j_0}} \Theta'_t = \delta^m_{j_0} + td_{j_0}$ for all $t$. Therefore, since $(X, \Theta'_t)$ is klt, there exists $t_0 \in \mathbb{R}_+ > 0$ such that
\[
t_0 = \sup \{ t \in \mathbb{R}_+: (X, \Theta'_t) \text{ is log canonical} \}.
\]

By construction, there is $k_0$ with $d_{k_0} > 0$ such that $G_{k_0}$ is a log canonical centre of $(X, \Theta'_t)$ and $\sigma_{G_{k_0}} \| K_X + \Theta'_t \| = 0$, thus by Theorem B, we have
\[
(30) \quad \Theta'_0 - A \in \mathbb{R}_{V,A}^{G_{k_0}}.
\]

Define $D_m = \sum (f_j + \delta^m_j) G_j$ and $\Xi_m = (t_0 + 1) D_m$ for $m \in \mathbb{N} \cup \{\infty\}$, and observe that
\[
(31) \quad (t_0 + 1) Y_m \sim_Q \Xi_m = \sum f_j G_j + \Delta_m - A + t_0 D_m \sim_Q K_X + \Delta_m + t_0 D_m
\]
and $\lim_{m \to \infty} \Xi_m = \Xi_\infty$. Denote $\Lambda_m = (\Delta_\infty + t_0 D_\infty) \cap \text{Supp} \, [\sigma_N] \Xi_\infty$ and
\[
\Lambda_m = (\Delta_m + t_0 D_m) \cap \sum_{Z \subset \text{Supp} \, \Lambda_\infty} \sigma_Z \Xi_m
\]
for $m \in \mathbb{N}$. Note that $0 \leq \Lambda_m \leq \text{Supp} \, [\sigma_N] \Xi_m$ for $m \gg 0$, and therefore $\Xi_m - \Lambda_m$ is pseudo-effective. Similarly as in [Nak04, 2.1.4] we have $\Lambda_\infty \leq \liminf_{m \to \infty} \Lambda_m$, and in particular, $\text{Supp} \Lambda_m = \text{Supp} \Lambda_\infty$ for $m \gg 0$. Therefore, there exists a sequence of rational numbers $\epsilon_m \uparrow 1$ such that $\Lambda_m \geq \epsilon_m \Lambda_\infty$, and set $\epsilon_\infty = 1$. 


Now define \( \Sigma_m = \Xi_m - \epsilon_m \Lambda_\infty \) for \( m \in \mathbb{N} \cup \{ \infty \} \), and note that \( \Sigma_m \geq \Xi_m - \Lambda_m \) are pseudo-effective divisors satisfying (1). Also, note that \( \Sigma_\infty = \sum f_j G_j + \Theta_\infty^{00} - A \), and

\[
\text{mult}_{G_{k_0}} \Sigma_\infty = f_{k_0} + 1 \geq f_{k_0} + \delta_{k_0} = d_{k_0} > 0,
\]

so this together with (30) gives (ii).

In order to show (iii), let \( 0 < \alpha_m < 1 \) be such that \( \Sigma_m = \alpha_m \Sigma_\infty + (1 - \alpha_m) \Sigma'_m \). Since every point on the segment \( [\Sigma_m, \Sigma'_m] \) is pseudo-effective, we can assume \( \alpha_m \ll 1 \). Then setting \( \Upsilon'_m = \Sigma'_m + \frac{\epsilon_m - \alpha_m}{1 - \alpha_m} \Lambda_\infty \), we have \( \Sigma_m = \alpha_m \Sigma_\infty + (1 - \alpha_m) \Upsilon'_m \), and this together with (31) gives \( \Upsilon_m = \alpha_m \Upsilon_\infty + (1 - \alpha_m) \Upsilon'_m \), so \( R_+ \Upsilon_m \) is not an extremal ray of \( \mathcal{C} \) by Remark 2.16.

**Step 7.** Let \( 0 < \xi \ll 1 \) be a rational number such that \( A - \Xi \) is ample for all \( \Xi \in V \) with \( \| \Xi \| \leq \xi \), and let \( L_{V, \xi} \) be the \( \xi \)-neighbourhood of \( L_V \) in the sup-norm. With notation from Remark 2.3, set

\[
D_\xi = \mathbb{R}_+ (\Sigma f_j G_j + B_{G A, \xi}^{V=1}) \subset V.
\]

By Remark 2.3, \( D_\xi \) is a rational polyhedral cone. Note that \( \{ \Theta_\infty^{00} - A + \Xi : 0 \leq \Xi \in V, \| \Xi \| \leq \xi, \text{mult}_{G_{k_0}} \Xi = 0 \} \subset B_{G A, \xi}^{V=1} \), so dim \( D_\xi \) = dim \( V \) and \( \Sigma_\infty \in D_\xi \). If \( \Sigma_\infty \in \text{int} D_\xi \), then it is obvious that for \( m > 0 \) there exists \( \Sigma'_m \in D_\xi \) such that \( \Sigma_m \in (\Sigma_\infty, \Sigma'_m) \), which is a contradiction by (iii) above.

Otherwise, let \( H_i \), for \( i = 1, \ldots, \ell \), be the supporting hyperplanes of codimension 1 faces of the cone \( D_\xi \), which contain \( \Sigma_\infty \), where \( \ell \leq \text{dim} V - 1 \). Let \( \mathcal{H}_i \) be the half-spaces determined by \( H_i \) which contain \( D_\xi \), and denote \( \mathcal{D} = \bigcap \mathcal{H}_i \). If \( \Sigma_m \in \mathcal{D} \) for infinitely many \( m \), then for some \( m > 0 \) there exists \( \Sigma'_m \in D_\xi \) such that \( \Sigma_m \in (\Sigma_\infty, \Sigma'_m) \) since \( D_\xi \) is polyhedral, a contradiction again.

Therefore, by passing to a subsequence, I can assume that \( \Sigma_m \notin \mathcal{D} \) for all \( m \). For each \( m \in \mathbb{N} \), denote \( \Gamma_m = \Sigma_m - \alpha_m \Xi_m \cdot G_{k_0} \). I claim that \( \mathcal{D} \cap (\mathbb{R}_{\geq0} \Gamma_m + \mathbb{R}_{>0} \Sigma_\infty) \neq \emptyset \) for each \( m \in \mathbb{N} \), and in particular \( \mathcal{H}_i \cap (\mathbb{R}_{>0} \Gamma_m + \mathbb{R}_{>0} \Sigma_\infty) \neq \emptyset \) for all \( i \). Granting the claim, let me show how it yields contradiction.

Since \( \Sigma_\infty \in \mathcal{H}_i \) for every \( i \), and the cone \( \mathbb{R}_+ \Gamma_m + \mathbb{R}_+ \Sigma_\infty \) is convex, the claim implies \( \Gamma_m \in \mathcal{H}_i \), and thus \( \Gamma_m \in \mathcal{D} \). Since \( \Sigma_m \notin \mathcal{D} \), segments \( [\Gamma_m, \Sigma_m] \) intersect \( \partial \mathcal{D} \), and in particular there exists a point \( P_m \in (\Sigma_m + \mathbb{R}_- G) \cap \partial \mathcal{D} \), closest to \( \Sigma_m \). By passing to a subsequence, we have \( \lim_{m \to \infty} P_m = \Sigma_\infty \) by Remark 2.6 and thus for every \( m > 0 \) there exists a codimension 1 face of \( D_\xi \) that contains \( \Sigma_\infty \) and \( P_m \). Since \( D_\xi \) is polyhedral, for \( m > 0 \) there are points \( Q_m \in D_\xi \) such that \( P_m = \mu_m Q_m + (1 - \mu_m) \Sigma_\infty \) for some \( 0 < \mu_m < 1 \). Set \( \Sigma'_m = Q_m + \frac{1}{\mu_m} (\Sigma_m - P_m) \), and note that \( \Sigma'_m \geq Q_m \) is pseudo-effective. Then \( \Sigma_m = \mu_m \Sigma_m + (1 - \mu_m) \Sigma_\infty \), and this is a contradiction by (iii) above.

Finally, let me prove the claim stated above. Observe that for every \( \Psi \in \mathbb{R}_+ \Gamma_m + \mathbb{R}_+ \Upsilon_\infty \) we have \( \sigma_{G_{k_0}} \ll \Psi \ll 0 \). Therefore, as \( \Sigma_\infty \in \sum f_j G_j + B_{G A, \xi}^{V=1} \), it is enough to find \( \Pi_m \in (\mathbb{R}_{\geq0} \Gamma_m + \mathbb{R}_{>0} \Sigma_\infty) \cap \mathcal{B}(\Sigma_\infty, \xi) \) such that \( \text{mult}_{G_{k_0}} \Pi_m = \text{mult}_{G_{k_0}} \Sigma_\infty \). Write \( \Gamma_m = \sum \gamma_m G_j \geq 0 \) and \( \Sigma_\infty = \sum \sigma_j G_j \), where \( \sigma_{k_0} > 0 \) by the condition (ii) above. If \( \gamma_{m, k_0} \neq 0 \),
choose 0 < β_m < 1 so that \( (1 - β_m)|σ_kσ_jγ_m,i - σ_jγ_m,k_0| < ξ |γ_m,k_0| \) for all \( j \), and set \( α_m = (1 - β_m)σ_kσ_jγ_m,k_0 \). If \( γ_m,k_0 = 0 \), let \( β_m = 1 \), and pick \( α_m > 0 \) so that \( |α_mγ_m,i| < ξ \) for all \( j \).

Then it is easy to check that \( \Pi_m = α_mΓ_m + β_mΣ∞ \) is the desired one.

**Remark 9.4.** If \((X, Δ)\) is a klt pair such that \( Δ \) is big, the existence of an effective divisor \( D ∈ \text{Div}(X)_{\mathbb{R}} \) such that \( K_X + Δ ∼ D \) was proved in [P˘ aú08] with analytic tools.

10. Finite generation

**Theorem 10.1.** Theorems \( A_{n-1}, B_n \) and \( C_{n-1} \) imply Theorem \( A_n \).

**Proof.** Let \( F_1, \ldots, F_N \) be prime divisors on \( X \) such that \( Δ_i = \sum_j δ_{ij}F_j \) with \( δ_{ij} ∈ [0, 1] \), and \( K_X + Δ_i + A ∼_Q \sum_j γ_{ij}F_j ≥ 0 \).

**Step 1.** I first show that we can assume \( A \) is a general ample \( \mathbb{Q} \)-divisor, all pairs \((X, Δ_i + A)\) are klt, and the divisor \( \sum F_i \) has simple normal crossings.

Fix an integer \( p \gg 0 \) such that \( Δ_i + pA \) is ample for every \( i \), and let \( A_i ∼_Q \frac{1}{p+1}Δ_i + \frac{1}{p+1}A \) be general ample \( \mathbb{Q} \)-divisors. Set \( Δ'_i = \frac{p}{p+1}Δ_i + A_i \). Then the pairs \((X, Δ'_i + A')\) are klt and \( K_X + Δ_i + A ∼_Q K_X + Δ'_i + A' \).

Let \( g : Y → X \) be a log resolution of the pair \((X, Δ'_i + A')\), denote \( B_i = B(X, Δ'_i + A') \) for all \( i \), and note that \( g^*A' = g_{-1}^*A' ≤ B_i \) since \( A' \) is general. Let \( H \) be a small effective \( g \)-exceptional \( \mathbb{Q} \)-divisor such that \( g^*A' - H \) is ample, and let \( A_Y ∼_Q g^*A' - H \) be a general ample \( \mathbb{Q} \)-divisor. Denote \( Δ_i,Y = B_i - g^*A' + H ≥ 0 \), and note that the divisor \( E_i = K_Y + B_i - g^*(K_X + Δ'_i + A') \) is effective and \( g \)-exceptional for every \( i \). Then

\[
K_Y + Δ_i,Y + A_Y ∼_Q K_Y + B_i,Y ∼_Q g^*(K_X + Δ_i + A) + E_i ∼_Q g^*(\sum γ_{ij}F_j) + E_i,
\]

and \( g^*(\sum γ_{ij}F_j) + E_i \) has simple normal crossings support. Choose \( q ∈ \mathbb{Z}_{>0} \) such that \( D'_i = qk_i(K_Y + Δ_i,Y + A_Y) \) is Cartier for every \( i \), and \( D'_i ∼_Q gg^*D_i + qk_iE_i \).

Then \( R(X; gD_1, \ldots, gD_{ℓ}) ∼_Q R(Y; D'_1, \ldots, D'_{ℓ}) \), and by Lemma[5.41] it suffices to prove that \( R(Y; D'_1, \ldots, D'_{ℓ}) \) is finitely generated. Now replace \( X \) by \( Y \), \( A \) by \( A_Y \) and \( Δ_i \) by \( Δ_i,Y \).

**Step 2.** Denote \( \mathcal{T} = \{(t_1, \ldots, t_{ℓ}) : t_i ≥ 0, \sum t_i = 1\} ⊂ \mathbb{R}^{ℓ} \) and \( f_{ij} = γ_{ij} - δ_{ij} \), and note that \( f_{ij} > 1 \). For each \( τ = (t_1, \ldots, t_{ℓ}) ∈ \mathcal{T} \), set

\[
δ_{τj} = \sum_t t_i δ_{ij} \quad \text{and} \quad f_{τj} = \sum_t t_i f_{ij},
\]

and observe that

\[
K_X + A ∼_\mathbb{R} \sum f_{τj}F_j.
\]

Let \( Λ = \bigoplus_j NF_j ⊂ \text{Div}(X) \), and denote \( \mathcal{B}_τ = \sum_j[f_{τj} + δ_{τj}, f_{τj} + 1]F_j ⊂ Λ_\mathbb{R} \) and \( \mathcal{B} = \bigcup_{τ ∈ \mathcal{T}} \mathcal{B}_τ \). Since every point in \( \mathcal{B} \) is a barycentric combination of the vertices of \( \mathcal{B}_e \), where \( e_i \) are the standard basis vectors of \( \mathbb{R}^{ℓ} \), \( \mathcal{B} \) is a rational polytope, and thus \( C = \mathbb{R}_+\mathcal{B} \) is a rational polyhedral cone.

For every \( j = 1, \ldots, N \), let

\[
\mathcal{F}_{τj} = (f_{τj} + 1)F_j + \sum_{k ≠ j}[f_{τk} + δ_{τk}, f_{τk} + 1]F_k,
\]
and set \( \mathcal{P}_j = \bigcup_{\tau \in \mathcal{T}} \mathcal{P}_{\tau j} \), which is a rational polytope similarly as above. Then \( \mathcal{C}_j = \mathbb{R}_+ \mathcal{P}_j \) is a rational polyhedral cone, and I claim that \( \mathcal{C} = \bigcup_j \mathcal{C}_j \). To see this, fix \( s \in \mathcal{C} \setminus \{0\} \). Then there exists \( \tau \in \mathcal{T} \) such that \( s \in \mathbb{R}_+ \mathcal{P}_{\tau} \), hence \( s = r_s \sum_j (f_{\tau j} + b_{\tau j})F_j \) for some \( r_s \in \mathbb{R}_{>0} \), \( b_{\tau j} \in [\delta_{\tau j}, 1] \). Setting
\[
\tau = \max_j \left\{ \frac{f_{\tau j} + b_{\tau j}}{f_{\tau j} + 1} \right\}, \quad \text{and} \quad b'_{\tau j} = -\frac{f_{\tau j} + b_{\tau j}}{r_{\tau}},
\]
we have
\[
s = r_s r_{\tau} \sum_j (f_{\tau j} + b'_{\tau j})F_j.
\]
Note that \( r_{\tau} \in (0, 1] \), \( b'_{\tau j} \in [\delta_{\tau j}, 1] \) for all \( j \), and there exists \( j_0 \) such that \( b'_{\tau j_0} = 1 \). Therefore \( s \in \mathbb{R}_+ \mathcal{P}_{\tau j_0} \subset \mathcal{C}_{j_0} \), and the claim is proved.

**Step 3.** In this step I prove that for each \( j \), the restricted algebra \( \text{res}_{F_j} R(X; \mathcal{C}_j \cap \Lambda) \) is finitely generated.

Fix \( 1 \leq j_0 \leq N \). By Lemma 4.2, pick finitely many generators \( h_1, \ldots, h_m \) of \( \mathcal{C}_{j_0} \cap \Lambda \). Similarly as in Step 1 of the proof of Theorem 7.4, it is enough to prove that the restricted algebra \( \text{res}_{F_{j_0}} R(X; h_1, \ldots, h_m) \) is finitely generated.

By definition of \( \mathcal{C}_{j_0} \), for every \( h_w \) there exist \( r_w \in \mathbb{Q}_+ \), \( \tau = (t_1, \ldots, t_i) \in \mathcal{T}_Q \), and \( b_{\tau j}^w \in [\delta_{\tau j}, 1] \) for \( j \neq j_0 \), such that \( h_w = r_w ((f_{\tau j_0} + 1) F_{j_0} + \sum_{j \neq j_0} (f_{\tau j} + b_{\tau j}^w)F_j) \). Denote \( \Phi' = \sum_{j \neq j_0} b_{\tau j}^w F_j \). Fix an integer \( p_{j_0} \gg 0 \) such that \( \Phi' + p_{j_0} A \) is ample for every \( w = 1, \ldots, m \), and let \( A_w \sim Q \frac{1}{p_{j_0} + 1} \Phi' + \frac{p_{j_0}}{p_{j_0} + 1} A \) and \( H \sim Q \frac{1}{p_{j_0} + 1} A \) be general ample \( Q \)-divisors. Set \( \Phi_w = \frac{p_{j_0}}{p_{j_0} + 1} \Phi' + A_w \). Then by (32),
\[
h_w \sim Q r_w (K_X + F_{j_0} + \Phi_w + H),
\]
and note that \( (X, F_{j_0} + \Phi_w + H) \) is a log smooth plt pair with \([F_{j_0} + \Phi_w + H] = F_{j_0}\) for every \( w \). Furthermore, we have
\[
h_w \geq r_w \sum_j (f_{\tau j} + \delta_{\tau j})F_j = r_w \sum_j t_j \sum_j (f_{ij} + \delta_{ij})F_j = r_w \sum_i t_i \sum_j y_j F_j \geq 0,
\]
so \([K_X + F_{j_0} + \Phi_w + H] = 0 \). Choose \( q_{j_0} \in \mathbb{Z}_{>0} \) such that \( q_{j_0} h_w \sim H_w \) for all \( w \), where \( H_w = q_{j_0} r_w (K_X + F_{j_0} + \Phi_w + H) \). Then
\[
\text{res}_{F_{j_0}} R(X; q_{j_0} h_1, \ldots, q_{j_0} h_m) \simeq \text{res}_{F_{j_0}} R(X; H_1, \ldots, H_m),
\]
and this last algebra is finitely generated by Theorem 7.4. Thus \( \text{res}_{F_{j_0}} R(X; h_1, \ldots, h_m) \) is finitely generated by Lemma 5.4.

**Step 4.** Let \( \sigma_j \in H^0(X, F_j) \) be the section such that \( \text{div} \sigma_j = F_j \) for each \( j \). Consider the \( \Lambda \)-graded algebra \( \mathfrak{R} = \bigoplus_{s \in \Lambda} \mathfrak{R}_s \subset R(X; F_1, \ldots, F_N) \) such that every element of \( \mathfrak{R} \) is a polynomial in elements of \( R(X; \mathcal{C} \cap \Lambda) \) and in \( \sigma_1, \ldots, \sigma_N \). Note that \( \mathfrak{R}_s = H^0(X, s) \) for every \( s \in \mathcal{C} \cap \Lambda \). In this step I show that the algebra \( \mathfrak{R} \) is finitely generated.

Let \( V = \sum_j \mathbb{R} F_j \simeq \mathbb{R}^N \), and let \( \| \cdot \| \) be the Euclidean norm on \( V \). Since the polytopes \( \mathcal{P}_j \subset V \) are compact, there is a positive constant \( C \) such that \( \mathcal{P}_j \subset B(0, C) \) for all \( j \). Let
\( \deg: \Lambda \to \mathbb{N} \) be the function given by \( \deg(\sum_j \alpha_j F_j) = \sum_j \deg \alpha_j \), and for a section \( \sigma \in \mathcal{R} \), set \( \deg \sigma = \deg s \). For every \( \mu \in \mathbb{N} \), denote \( \Lambda_{\leq \mu} = \{ s \in \Lambda : \deg s \leq \mu \} \), and \( \mathcal{R}_{\leq \mu} = \bigoplus_{s \in \Lambda_{\leq \mu}} \mathcal{R}_s \).

By Step 3, for each \( j \) there exists a finite set \( \mathcal{H}_j \subset R(X, \mathcal{C} \cap \Lambda) \) such that \( \text{res}_{F_j} R(X, \mathcal{C} \cap \Lambda) \) is generated by the set \( \{ \sigma|\mathcal{F}_j : \sigma \in \mathcal{H}_j \} \). Let \( M \) be a sufficiently large positive integer such that \( \mathcal{H}_j \subset \mathcal{R}_{\leq M} \) for all \( j \), and \( M \geq CN^{1/2} \max_{i,j} \left\{ \frac{1}{1 - \delta_{ij}} \right\} \). By H"{o}lder's inequality we have \( \|s\| \geq N^{-1/2} \deg s \) for all \( s \in \Lambda \), and thus

\[
(33) \quad \|s\|/C \geq \max_{i,j} \left\{ \frac{1}{1 - \delta_{ij}} \right\}
\]

for all \( s \in \Lambda \setminus \Lambda_{\leq M} \). Let \( \mathcal{H} \) be a finite subset of \( \mathcal{R} \) such that \( \{ \sigma_1, \ldots, \sigma_N \} \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_N \subset \mathcal{H} \), and that \( \mathcal{H} \) is a set of generators of the finite dimensional vector space \( \mathcal{R}_{\leq M} \). Let \( \mathbb{C}[\mathcal{H}] \) be the ring consisting of polynomials in the elements of \( \mathcal{H} \), and observe that trivially \( \mathbb{C}[\mathcal{H}] \subset \mathcal{R} \).

I claim that \( \mathcal{R} = \mathbb{C}[\mathcal{H}] \), and the proof is by induction on \( \deg \chi \), where \( \chi \in \mathcal{R} \).

Fix \( \chi \in \mathcal{R} \). By definition of \( \mathcal{R} \), write \( \chi = \sum_i \sigma_i^{\lambda_{i,j}} \cdots \sigma_N^{\lambda_{N,j}} \chi_i \), where \( \chi_i \in R(X, \mathcal{C} \cap \Lambda) \), and note that \( \deg \chi_i \leq \deg \chi \). Then it is enough to show that \( \chi_i \in \mathbb{C}[\mathcal{H}] \). By replacing \( \chi \) by \( \chi_i \), I assume that \( \chi \in H^0(X, c) \), where \( c \in \mathcal{C} \cap \Lambda \). If \( \deg \chi \leq M \), then \( \chi \in \mathbb{C}[\mathcal{H}] \) by definition of \( \mathcal{H} \).

Now assume \( \deg \chi > M \). By Step 2 there exists \( j_0 \) such that \( c \in \mathcal{C}_{j_0} \cap \Lambda \), and thus, by definition of \( \mathcal{H} \), there are \( \theta_1, \ldots, \theta_c \in \mathcal{H} \) and a polynomial \( \phi \in \mathbb{C}[X_1, \ldots, X_c] \) such that \( \chi|\mathcal{F}_{j_0} = \phi(\theta_1|\mathcal{F}_{j_0}, \ldots, \theta_c|\mathcal{F}_{j_0}) \). Therefore,

\[
\chi - \phi(\theta_1, \ldots, \theta_c) = \sigma_{j_0} \cdot \chi'
\]

for some \( \chi' \in H^0(X, c - F_{j_0}) \) by the relation (2) in Remark 5.3. Since \( \deg \chi' < \deg \chi \), it is enough to prove that \( \chi' \in \mathcal{R} \), since then \( \chi' \in \mathbb{C}[\mathcal{H}] \) by induction, and so \( \chi = \sigma_{j_0} \cdot \chi' + \phi(\theta_1, \ldots, \theta_c) \in \mathbb{C}[\mathcal{H}] \).

To that end, since \( c \in \mathcal{C}_{j_0} \cap \Lambda \), there exist \( \tau \in \mathbb{P}_{\mathbb{Q}}, r_c \in \mathbb{Q}_+, \) and \( b_{\tau_j} \in [\delta_{\tau_j}, 1] \) for \( j \neq j_0 \), such that \( c = r_c \tau_{j_0} \), where \( \tau_{j_0} = (f_{\tau_{j_0}} + 1)F_{j_0} + \sum_{j \neq j_0} (f_{\tau_j} + b_{\tau_j})F_j \in \mathcal{F}_{j_0} \). Then

\[
c - F_{j_0} = r_c ((f_{\tau_{j_0}} + \frac{r_{c-1}}{r_c})F_{j_0} + \sum_{j \neq j_0} (f_{\tau_j} + b_{\tau_j})F_j),
\]

and observe that \( r_c = \|c\|/\|c\tau_{j_0}\| \geq \max_{i,j} \left\{ \frac{1}{1 - \delta_{ij}} \right\} \) by (33) since \( \|c\tau_{j_0}\| \leq C \) by definition of \( C \). In particular \( r_{c-1} \geq \delta_{c_{j_0}} \), and therefore \( c - F_{j_0} \in \mathbb{R} + \mathcal{F}_c \cap \Lambda \subset \mathcal{C} \cap \Lambda \). Thus \( \chi' \in R(X, \mathcal{C} \cap \Lambda) \subset \mathcal{R} \), and we are done.

**Step 5.** Finally, in this step I derive that \( R(X;D_1, \ldots, D_\ell) \) is finitely generated.

To that end, choose \( r \in \mathbb{Z}_{>0} \) such that \( rD_i \sim \omega_i \) for \( i = 1, \ldots, \ell \), where \( \omega_i = rk_i \sum_j \gamma_{ij}F_j \). Set \( \mathcal{G} = \sum_{i=1}^\ell \mathbb{R}_+ \omega_i \cap \Lambda \) and note that \( \mathcal{G}_\mathcal{R} \subset \mathcal{C} \). Since \( \mathcal{R} \) is finitely generated by Step 4,
the algebra $R(X, \mathcal{C} \cap \Lambda)$ is finitely generated by Lemma 5.4(2), and therefore by Proposition 5.7 there is a finite rational polyhedral subdivision $\mathcal{G}_\mathbb{R} = \bigcup_k \mathcal{G}_k$ such that the map $\text{Mob}_\pi|_{\mathcal{G}_\mathbb{R} \cap \Lambda}$ is additive up to truncation for every $k$, where $\pi : \Lambda \to \Lambda$ is the identity map.

By Lemma 4.2 there are finitely many elements $\omega_{l+1}, \ldots, \omega_q \in \mathcal{G}$ that generate $\mathcal{G}$, and denote by $\pi : \bigoplus_{i=1}^q \mathbb{N} \omega_i \to \mathcal{G}$ the natural projection. Then the map $\text{Mob}_\pi|_{\pi^{-1}(\mathcal{G}_\mathbb{R} \cap \Lambda)}$ is additive up to truncation for every $k$, and thus the algebra $R(X, \pi(\bigoplus_{i=1}^q \mathbb{N} \omega_i))$ is finitely generated by Lemma 5.4(3). Since $\bigoplus_{i=1}^q \mathbb{N} \omega_i$ is a saturated submonoid of $\bigoplus_{i=1}^q \mathbb{N} \omega_i$, the algebra $R(X, \pi(\bigoplus_{i=1}^q \mathbb{N} \omega_i)) \simeq R(X; rD_1, \ldots, rD_t)$ is finitely generated by Lemma 5.4(2), and finally $R(X; D_1, \ldots, D_t)$ is finitely generated by Lemma 5.4(1). 

Finally, we have:

**Proof of Theorem 1.2.** Similarly as in Step 1 of the proof of Theorem 10.1 by passing to a log resolution $f : Y \to X$ of $(X, \sum \Delta_i)$, I can assume that $A$ is a general ample $\mathbb{Q}$-divisor and $(X, \Delta + A)$ is log smooth for every $i$.

Let $K_X$ be a divisor with $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\text{Supp} A \not\subset \text{Supp} K_X$, let $V \subset \text{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of $\sum \Delta_i$, and let $\Lambda \subset \text{Div}(X)$ be the monoid spanned by the components of $K_X$, $\sum \Delta_i$ and $A$. The set $\mathcal{C} = \sum \mathbb{R} A_i \subset \Lambda_{\mathbb{R}}$ is a rational polyhedral cone. Similarly as in Step 5 of the proof of Theorem 10.1 it is enough to prove that the algebra $R(X, C \cap \Lambda)$ is finitely generated. By Theorem C the set $\mathcal{O}_{V, \Lambda}$ is a rational polytope, and denote $\mathcal{D} = \mathbb{R} (K_X + A + \mathcal{O}_{V, \Lambda}) \cap C \subset \Lambda_{\mathbb{R}}$. Then the algebra $R(X, C \cap \Lambda)$ is finitely generated if and only if the algebra $R(X, \mathcal{D} \cap \Lambda)$ is finitely generated. Let $H_1, \ldots, H_m$ be generators of the monoid $\mathcal{D} \cap \Lambda$. Then it suffices to prove that the ring $R(X; H_1, \ldots, H_m)$ is finitely generated, and this follows from Theorem A. 

**Proof of Theorem 1.1.** By [FM00, 5.2] and by induction on $\dim X$, we may assume $K_X + \Delta$ is big. Write $K_X + \Delta \sim \mathbb{Q} A + B$ with $A$ ample and $B$ effective. Let $\varepsilon$ be a small positive rational number and set $\Delta' = (\Delta + \varepsilon B) + \varepsilon A$. Then $K_X + \Delta' \sim \mathbb{Q} (\varepsilon + 1)(K_X + \Delta)$, thus $R(X, K_X + \Delta)$ and $R(X, K_X + \Delta')$ have isomorphic truncations, so the result follows from Theorem 1.2. 

**Proof of Corollary 1.3.** Theorem 1.1 implies the claim (1) by [Fuj09a, 3.9], and (2) by [Rei80, 1.2(II)]. The claim (3) follows by Theorem 1.2 and [HK00, 2.9].
For many years the guiding philosophy of the Minimal Model Program was to prove finite generation of the canonical ring as a standard consequence of the theory, namely as a corollary to the existence of minimal models and the Abundance Conjecture. Efforts in this direction culminated in [BCHM06], which derived the finite generation in the case of klt singularities from the existence of minimal models for varieties of log general type. However, passing to the case of log canonical singularities, as well as trying to prove the Abundance Conjecture, although seemingly slight generalisations, seem to be substantially harder problems where different techniques and methods are welcome, if not needed. The aim of the new approach is to invert the conventional logic of the theory, where finite generation is not at the end, but at the beginning of the process, and the standard theorems and conjectures of Mori theory are derived as consequences. I hope the results of this paper give substantial ground to such claims.

There are many contributors to the initial development of Mori theory, Mori, Reid, Kawamata, Shokurov, Kollár, Corti to name a few. In the MMP one starts with a \( \mathbb{Q} \)-factorial log canonical pair \((X, \Delta)\), and then constructs a birational map \( \phi : X \to Y \) such that the pair \((Y, \phi_* \Delta)\) has exceptionally nice properties. Namely we expect that in the case of log canonical singularities, there is the following dichotomy:

1. if \( \kappa(X, K_X + \Delta) \geq 0 \), then \( K_Y + \phi_* \Delta \) is nef (\( Y \) is a minimal model),
2. if \( \kappa(X, K_X + \Delta) = -\infty \), then there is a contraction \( Y \to Z \) such that \( \dim Z < \dim Y \) and \( -(K_Y + \phi_* \Delta) \) is ample over \( Z \) (\( Y \) is a Mori fibre space).

If \( Y \) is a Mori fibre space, then it is known that \( \kappa(X, K_X + \Delta) = -\infty \) and \( X \) is uniruled. The reverse implication is much harder to prove. The greatest contributions in that direction are [BDPP04], which proves that if \( X \) is smooth and \( K_X \) is not pseudo-effective, then \( X \) is uniruled, and [BCHM06], which proves that if \( K_X + \Delta \) is klt and not pseudo-effective, then there is a map to \( Y \) as in (2) above.

The classical strategy is as follows: if \( K_X + \Delta \) is not nef, then by the Cone theorem (known for log canonical pairs by the work of Ambro and Fujino, see [Amb03]) there is a \((K_X + \Delta)\)-negative extremal ray \( R \) of \( \text{NE}(X) \), and by the contraction theorem there is a morphism \( \pi : X \to W \) which contracts curves whose classes belong to \( R \), and only them. If \( \dim W < \dim X \), then we are done. Otherwise \( \pi \) is birational, and there are two cases. If \( \text{codim}_X \text{Exc} \pi = 1 \), then \( \pi \) is a divisorial contraction, \( W \) is \( \mathbb{Q} \)-factorial and \( \rho(X/W) = 1 \), and we continue the process starting from the pair \((W, \pi_* \Delta)\). If \( \text{codim}_X \text{Exc} \pi \geq 2 \), then \( \pi \) is a flipping contraction, \( \rho(X/W) = 1 \), but \( K_W + \pi_* \Delta \) is no longer \( \mathbb{Q} \)-Cartier. In order to proceed, one needs to construct the flip of \( \pi \), namely a birational map \( \pi^+ : X^+ \to W \) such that \( X^+ \) is \( \mathbb{Q} \)-factorial, \( \rho(X^+/W) = 1 \) and \( K_{X^+} + \phi_* \Delta \) is ample over \( W \), where \( \phi : X \to X^+ \) is the birational map which completes the diagram. Continuing the procedure, one hopes that it ends in finitely many steps.

Therefore there are two conjectures that immediately arise in the theory: existence and termination of flips. Existence of the flip of a flipping contraction \( \pi : X \to W \) is known to
be equivalent to the finite generation of the \textit{relative canonical algebra}

\[ R(X/W, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X([m(K_X + \Delta)]), \]

and the flip is then given by $X^+ = \text{Proj}_W R(X/W, K_X + \Delta)$. The termination of flips is related to conjectures about the behaviour of the coefficients in the divisor $\Delta$, but I do not discuss it here.

Since the paper [Zar62], one of the central questions in higher dimensional birational geometry is the following:

\textbf{Conjecture A.1.} \emph{Let $(X, \Delta)$ be a projective log canonical pair. Then the canonical ring $R(X, K_X + \Delta)$ is finitely generated.}

Finite generation implies existence of flips [Fuj09a, 3.9]; moreover, one only needs to assume finite generation for pairs $(X, \Delta)$ with $K_X + \Delta$ big.

The proof of the finite generation in the case of klt singularities along the lines of the classical philosophy in [BCHM06] is as follows: by [FM00, 5.2] one can assume that $K_X + \Delta$ is big. Then by applying carefully chosen flipping contractions, one proves that the corresponding flips exist and terminate (termination with scaling), and since the process preserves the canonical ring, the finite generation follows from the basepoint free theorem.

Now consider a flipping contraction $\pi: (X, \Delta) \to W$ with additional properties that $(X, \Delta)$ is a plt pair such that $S = [\Delta]$ is an irreducible divisor which is negative over $Z$. This contraction is called \textit{pl flipping}, and the corresponding flip is the pl flip. Following the work of Shokurov, one of the steps in the proof in [BCHM06] is showing that pl flips exist, and the starting point is Lemma A.2 below. Note that in the context of pl flips, the issues which occur in the problem of global finite generation outlined in the introduction to this paper do not exist. I give a slightly modified proof of the lemma below than the one present elsewhere in the literature in order to stress the following point: I do not calculate the kernel of the restriction map, but rather chase the generators. This reflects the basic principle: if our algebra is large enough so that it contains the equation of the divisor we are restricting to, then it is automatically finitely generated assuming the restriction to the divisor is. This is one of the main ideas guiding the proof in §10.

\textbf{Lemma A.2.} \emph{Let $(X, \Delta)$ be a plt pair of dimension $n$, where $S = [\Delta]$ is a prime divisor, and let $f: X \to Z$ be a pl flipping contraction with $Z$ affine. Then $R(X/Z, K_X + \Delta)$ is finitely generated if and only if $\text{res}_S R(X/Z, K_X + \Delta)$ is finitely generated.}

\textbf{Proof.} We will concentrate on sufficiency, since necessity is obvious.

Numerical and linear equivalence over $Z$ coincide by the basepoint free theorem. Since $\rho(X/Z) = 1$, and both $S$ and $K_X + \Delta$ are $f$-negative, there exists a positive rational number $r$ such that $S \sim_{Q,f} r(K_X + \Delta)$. By considering an open cover of $Z$ we can assume that $S - r(K_X + \Delta)$ is $\mathbb{Q}$-linearly equivalent to a pullback of a principal divisor.

Therefore $S \sim_{\mathbb{Q}} r(K_X + \Delta)$, and since then $R(X,S)$ and $R(X, K_X + \Delta)$ have isomorphic truncations, it is enough to prove that $R(X,S)$ is finitely generated. As a truncation of
res₅R(X, S) is isomorphic to a truncation of res₅R(X, K_X + Δ), we have that res₅R(X, S) is finitely generated. Let σ₅ ∈ H₀(X, S) be a section such that div σ₅ = S and let H be a finite set of generators of the finite dimensional vector space ǳ=1 res₅H₀(X, iS), for some d, such that the set {hᵢS : h ∈ H} generates res₅R(X, S). Then it is easy to see that H ∪ {σ₅} is a set of generators of R(X, S), since ker(ρₖS, S) = H₀(X, (k − 1)S)·σ₅ for all k, in the notation of Remark 5.3.

One of the crucial unsolved problems in higher dimensional geometry is the following Abundance Conjecture.

**Conjecture A.3.** Let (X, Δ) be a projective log canonical pair such that K_X + Δ is nef. Then K_X + Δ is semiample.

Until the end of the appendix I discuss this conjecture more thoroughly. There are, to my knowledge, two different ways to approach this problem.

The first approach is close to the classical strategy, and goes back to [Kaw85b]. First let us recall the following definition from [Nak04]; the corresponding analytic version can be found in [Pau08].

**Definition A.4.** Let X be a projective variety. For D ∈ Big(X) denote

σ(D, A) = sup {k ∈ N : liminfₘ→∞ m⁻¹h₀(X, |mD| + A) > 0}.

Then the numerical dimension of D is

ν(X, D) = sup{σ(D, A) : A is ample}.

We know that ν(X, D) = 0 if and only if D ≡ Nₐ||D||, and that ν(X, D) is the standard numerical dimension when D is nef by [Nak04, 6.2.8]. It is well known that abundance holds when ν(X, K_X + Δ) is equal to 0 or dim X by [Kaw85a, 8.2], and when ν(X, K_X + Δ) = κ(X, K_X + Δ) by [Kaw85b, 6.1], cf. [Fuj09].

**Theorem A.5.** Let (X, Δ) be a projective klt pair of dimension n such that K_X + Δ is nef. Assume that ν(Y, K_Y + Δ_Y) > 0 implies κ(Y, K_Y + Δ_Y) > 0 for any klt pair (Y, Δ_Y) of dimension at most n. Then K_X + Δ is semiample.

**Proof.** Let (S, Δ₅) be a Q-factorial (n − 1)-dimensional klt pair with κ(S, K_S + Δ₅) = 0. Then ν(S, K_S + Δ₅) = 0 by the assumption in dimension n − 1, and thus K_S + Δ₅ ≡ Nₐ||K_S + Δ₅||. By [Dru09, 3.4] a minimal model of (S, Δ₅) exists. Now the result follows along the lines of [Kaw85b, 7.3]. □

The assumption in the theorem can be seen as a stronger version of non-vanishing.

Now I present a different approach, where one derives abundance from the finite generation. It is a result of J. M°Kernan and C. Hacon, and I am grateful to them for allowing me to include it here.

**Theorem A.6.** Assume that for every (n + 1)-dimensional projective log canonical pair (X, Δ) with K_X + Δ nef and big, the canonical ring R(X, K_X + Δ) is finitely generated. Then abundance holds for klt pairs in dimension n.
Proof. Let \((Y, \Phi)\) be an \(n\)-dimensional projective klt pair such that \(K_Y + \Phi\) is nef, and let \(Y \subset \mathbb{P}^N\) be some projectively normal embedding. Let \(X_0\) be the cone over it, let \(X = \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(1))\) be the blowup of \(X_0\) at the origin, and let \(H' \subset \mathbb{P}^N\) be a sufficiently ample divisor which does not contain the origin. Let \(\Delta\) and \(H\) be the proper transforms in \(X\) of \(\Phi\) and \(H'\), respectively, and let \(E \subset X\) be the exceptional divisor.

Then by inversion of adjunction the pair \((X, \Upsilon = E + \Delta + H)\) is log canonical, and of log general type since \(H'\) is ample enough. We have \(Y \simeq E\), and this isomorphism maps \(K_Y + \Phi\) to \(K_E + \Delta|_E\). The divisor \(K_X + \Upsilon\) is also nef: since \((K_X + E + \Delta)|_E\) is identified with \(K_Y + \Phi\), this deals with curves lying in \(E\) by nefness, and for those curves which are not in \(E\), the ampleness of \(H\) away from \(E\) ensures that the intersection product with \(K_X + \Upsilon\) is positive. By assumption, the algebra \(R(X, K_X + \Upsilon)\) is finitely generated, therefore \(K_X + \Upsilon\) is semiample by [Laz04, 2.3.15], and thus so is \(K_E + \Delta|_E = (K_X + \Upsilon)|_E\). □

Finally a note about the general alternative philosophy. Since [HK00] it has become clear that adjoint rings encode many important geometric information about the variety. In particular, by Corollary 1.3(3) and [HK00, 1.11], all the main theorems and conjectures of Mori theory hold on \(X\), such as the Cone and Contraction theorems, existence and termination of flips, abundance. In particular, the following conjecture applied to Mori dream regions [HK00, 2.12, 2.13] seems to encode the whole Mori theory.

**Conjecture A.7.** Let \(X\) be a projective variety, and let \(D_i = k_i(K_X + \Delta_i) \in \text{Div}(X)\), where \((X, \Delta_i)\) is a log canonical pair for \(i = 1, \ldots, \ell\). Then the adjoint ring \(R(X; D_1, \ldots, D_\ell)\) is finitely generated.
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