On the study of solutions for a non linear differential equation on compact Riemannian Manifolds.

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Abstract

In this paper we study the existence of solutions for a class of non-linear differential equation on compact Riemannian manifolds. We establish a lower and upper solutions’ method to show the existence of a smooth positive solution for the equation (1)

\[ \Delta u + a(x)u = f(x)F(u) + h(x)H(u), \]

where \( a, f, h \) are positive smooth functions on \( M^n \), a \( n \)-dimensional compact Riemannian manifold, and \( F, H \) are non-decreasing smooth functions on \( \mathbb{R} \). In [6] the equation (1) was studied when \( F(u) = u^{2^* - 1} \) and \( H(u) = u^q \) in the Riemannian context, i.e.,

\[ \Delta u + a(x)u = f(x)u^{2^* - 1} + h(x)u^q, \]

where \( 0 < q < 1 \). In [4] Corrêa, Gonçalves and Melo studied an equation of the type equation (2), in the Euclidean context.

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1 Introduction

The study of the theory of nonlinear differential equations on Riemannian manifolds, has began in 1960 with the so-called Yamabe problem. At a time when little was known about the methods of attacking a non-linear equation, the Yamabe problem came to light of a geometric idea and from time sealed a merger of the areas of geometry and differential equations. Let \((M, g)\) be a compact Riemannian manifold of dimension \( n, \ n \geq 3 \). Given \( \tilde{g} = u^{4/(n-2)}g \) some
conformal metrical to the metric $g$, is well known that the scalar curvatures $R$ and $\tilde{R}$ of the metrics $g$ and $\tilde{g}$, respectively, satisfy the law of transformation

$$
\Delta u + \frac{n-2}{4(n-1)} R u = \frac{n-2}{4(n-1)} \tilde{R} u^{2^*-1}
$$

where $\Delta$ denote the Laplacian operator associated to $g$.

In 1960, Yamabe [14] announced that for every compact Riemannian manifold $(M, g)$ there exist a metric $\tilde{g}$ conformal to $g$ for which $\tilde{R}$ is constant. In another words, this mean that for every compact Riemannian manifold $(M, g)$ there exist $u \in C^\infty(M)$, $u > 0$ on $M$ and $\lambda \in \mathbb{R}$ such that

$$
\Delta u + \frac{n-2}{4(n-1)} R u = \lambda u^{2^*-1}.
$$

(Y)

In 1968, Trüdinger [13] found an error in the work of Yamabe, which generated a race to solve what became known as the Yamabe problem, today it is completely positively resolved, that is, the assertion of Yamabe is true.

The main step towards the resolution of the Yamabe problem was given in 1976 by Aubin in his classic article [1]. In [1] Aubin showed that the statement was true since the manifold satisfy a condition on an invariant (called Yamabe invariant). Then he used tests functions, locally defined to show that non locally conformal flat manifolds, of dimension $n > 6$, satisfying this condition. Finally, the problem for $n \geq 3$ was completed solved by R. Schoen [10].

As previously reported, several disturbances were the Yamabe problem, all of analytical character, both in the sense of equation (with the addition of other factors) and in the sense of the operator (the Laplacian for the $p$-Laplacian), and all (at least those in this study) using the idea of estimating the functional Aubin corresponding multiple functions $u_\lambda$. We can cite some articles, such as [2], [3], [5], [6], [7] and [9].

This work aims to work with problems related to the equation (Y), although, as we shall see, with different methods from those used by Yamabe, these results were obtained in [11], and some of them were published in [12]. The above equation was studied by Gonçalves and Alves [4], in the Euclidean space, here by using the methods in [4] we study the case in the Riemannian space.
Definition 1. We say that a function $u$ (respectively, $\overline{u}$) $\in H^2_1 \cap L^\infty$, $u \geq 0$ ($\overline{u} \geq 0$) is a lower solution (respectively, upper solution) of equation (1) if for all $\varphi \in H^2_1$, $\varphi \geq$ 0
\[ \int_M \nabla u \cdot \nabla \varphi dV + \int_M a u \varphi dV \leq \int_M f(x) F(u) \varphi dV + \int_M h(x) H(u) \varphi dV \]
(respectively,
\[ \int_M \nabla \overline{u} \cdot \nabla \varphi dV + \int_M a \overline{u} \varphi dV \geq \int_M f(x) F(\overline{u}) \varphi dV + \int_M h(x) H(\overline{u}) \varphi dV. \]

We consider the conditions:

\begin{align*}
(\alpha_1) \quad & \begin{cases}
0 \leq F(t) \leq t^{2^* - 1} \text{ and } 0 \leq H(t) \leq t^q & \text{if } t \geq 0 \\
F(t) = H(t) = 0 & \text{if } t < 0
\end{cases} \\
\text{where } & 2^* = \frac{2n}{n-2} \text{ and } 0 < q < 2^* - 1.
\end{align*}

(\alpha_2) $a > 0$, $f \geq 0$, $f \not\equiv 0$ and $h \geq 0$, $h \not\equiv 0$.

We proved, in the main result

Theorem 1. Let $(M, g)$ be a compact $n-$dimensional Riemannian manifold $n \geq 3$. Suppose that (\alpha_1) and (\alpha_2) holds. If $u, \overline{u} \in H^2_1 \cap L^\infty$ are, respectively, the lower and the upper solutions of the equation (1) with $0 \leq u \leq \overline{u}$ a.e. in $M$ and $u \not\equiv 0$, then the equation (1)
\[ \Delta u + a(x) u = f(x) F(u) + h(x) H(u) \]
admits a positive regular solution $u$, such that $\underline{u} \leq u \leq \overline{u}$.

2 An auxiliary Lemma

We will need an auxiliary lemma.

Let us consider the following equation:
\[ \Delta u + a(x) u = \Psi \text{ in } M. \] (3)
Lemma 1. Assume that $a > 0$. If $\Psi \in (H^2_1)'$, then the equation (3) has a unique solution $u \in H^2_1$. Moreover, the $T$ operator
\[
T : (H^2_1)' \longrightarrow H^2_1 \\
\Psi \longrightarrow T(\Psi) = u
\]
is continuous and non-decreasing.

Proof of the Lemma 1:
Consider the functional $I : H^2_1 \longrightarrow \mathbb{R}$ given by
\[
I(u) = \frac{1}{2} \int_M |\nabla u|^2 dV + \frac{1}{2} \int_M au^2 dV - \langle \Psi, u \rangle, \quad u \in H^2_1.
\]
Then, $I \in C^1(H^2_1, \mathbb{R})$ and
\[
\langle I'(u), v \rangle = \int_M \nabla u \nabla v dV + \int_M au v dV - \langle \Psi, v \rangle, \quad u, v \in H^2_1.
\]
Therefore, the solutions of the equation (3) are critical points of the functional $I$.

As $a > 0$ and $M$ is a compact manifold,
\[
I(u) \geq \frac{1}{2} \int_M |\nabla u|^2 dV + \frac{\min a}{2} \int_M u^2 dV - \langle \Psi, u \rangle \geq C \|u\|^2_{H^2_1} - \langle \Psi, u \rangle, \quad u \in H^2_1,
\]
where $C = \min \{1/2, (\min a)/2 \} > 0$.

Since $\langle \Psi, u \rangle \leq \|\Psi\|_{(H^2_1)'} \|u\|_{H^2_1}$, we have that
\[
I(u) \geq C \|u\|^2_{H^2_1} - \|\Psi\|_{(H^2_1)'} \|u\|_{H^2_1} \quad \forall \quad u \in H^2_1.
\]
Then, $I$ is coercive.

Claim 1 The functional $I$ is l.s.s.c. (lower sequentially semi continuous), namely, for all $u_i \rightharpoonup u$ in $H^2_1$ implies that $I(u) \leq \lim \inf_{i \rightarrow \infty} I(u_i)$.

Indeed, suppose that $u_i \rightharpoonup u$ in $H^2_1$.

Since the embedded $H^2_1 \hookrightarrow L^2$ is compact and $H^2_1$ is reflexive,
\[
u_i \longrightarrow u \text{ in } L^2,
\]
and
An auxiliary Lemma

\[ \|u\|_{H^1_1}^2 \leq \lim_{i \to \infty} \inf \|u_i\|_{H^1_1}^2. \]

With this,
\[ \int_M |\nabla u|^2 dV \leq \lim_{i \to \infty} \inf \int_M |\nabla u_i|^2 dV, \]
and
\[ \int_M au^2 dV = \lim_{i \to \infty} \int_M a(u_i)^2 dV. \]

Thus,
\[ I(u) = \frac{1}{2} \int_M |\nabla u|^2 dV + \frac{1}{2} \int_M au^2 dV - \langle \Psi, u \rangle \]
\[ \leq \frac{1}{2} \lim_{i \to \infty} \inf \int_M |\nabla u_i|^2 dV + \frac{1}{2} \lim_{i \to \infty} \int_M a(u_i)^2 dV - \lim_{i \to \infty} \langle \Psi, u_i \rangle \]
\[ = \lim_{i \to \infty} \inf I(u_i) \]
what prove the Claim 1.

As \( H^2_1 \) is a reflexive space, (see [8]) there is \( u \in H^2_1 \), such that
\[ I(u) = \min_{v \in H^2_1} I(v) \]
and, consequently, \( u \) is a solution of the equation (3).

Claim 2 The uniqueness of solution for the equation \( (3) \) holds.

Let us suppose that \( u_1 \) and \( u_2 \) are solutions of the equation \( (3) \). Then, \( \forall \varphi \in H^2_1 \), we have that
\[ \int_M \nabla u_1 \nabla \varphi dV + \int_M au_1 \varphi dV = \langle \Psi, \varphi \rangle, \] \( (4) \)
\[ \int_M \nabla u_2 \nabla \varphi dV + \int_M au_2 \varphi dV = \langle \Psi, \varphi \rangle. \] \( (5) \)
Taking \( \varphi = u_1 - u_2 \) and considering the difference \( (4) \) and \( (5) \) we get
\[ \int_M |\nabla(u_1 - u_2)|^2 dV + \int_M a(u_1 - u_2)^2 dV = 0. \]
Thus, as \( a > 0 \), then \( u_1 = u_2 \) in \( H^2_1 \). What give us the proof of Claim 2.
\( \square \)
Claim 3 The operator $T$ is continuous.

Indeed, let $\{\Psi_i\}$ and $\Psi \in (H^2_1)'$ such that

$$\Psi_i \rightarrow \Psi \text{ in } (H^2_1)' .$$

Taking

$$u_i = T(\Psi_i) \text{ and } u = T(\Psi),$$

obtain $\forall \varphi \in H^2_1$,

$$\int_M \nabla u_i \nabla \varphi dV + \int_M a u_i \varphi dV = \langle \Psi_i, \varphi \rangle, \quad (6)$$

$$\int_M \nabla u \nabla \varphi dV + \int_M a u \varphi dV = \langle \Psi, \varphi \rangle. \quad (7)$$

Substituting $\varphi_i = u_i - u \in H^2_1$ in (6) and (7) we get that

$$\int_M \nabla u_i (\nabla u_i - \nabla u) dV + \int_M a u_i (u_i - u) dV = \langle \Psi_i, u_i - u \rangle, \quad (8)$$

$$\int_M \nabla u (\nabla u_i - \nabla u) dV + \int_M a u (u_i - u) dV = \langle \Psi, u_i - u \rangle. \quad (9)$$

Considering (8) - (9) we obtain

$$\int_M |\nabla (u_i - u)|^2 dV \leq \int_M |\nabla \Psi|^2 dV + \int_M a |u_i - u|^2 dV = \langle \Psi_i - \Psi, u_i - u \rangle \leq \|\Psi_i - \Psi\|_{(H^2_1)'} \|u_i - u\|_{H^2_1}. \quad (10)$$

As

$$\int_M |\nabla (u_i - u)|^2 dV \leq \int_M |\nabla u|^2 dV + \int_M a |u_i - u|^2 dV \leq C \|u_i - u\|_{H^2_1}^2,$$

where

$$C = \min \{1, \min a\} > 0,$$

we obtain

$$\|u_i - u\|_{H^2_1} \leq \frac{1}{C} \|\Psi_i - \Psi\|_{(H^2_1)'} \rightarrow 0 .$$

Therefore,

$$u_i \rightarrow u \text{ in } H^2_1,$$
namely, $T$ is continuous. Proving the Claim 3. \hfill \Box

Claim 4 The operator $T$ is non-decreasing.

Let $\Psi_1, \Psi_2 \in (H^2_1)'$ such that $\Psi_1 \leq \Psi_2$ in the sense of that

$$\langle \Psi_1, \varphi \rangle \leq \langle \Psi_2, \varphi \rangle, \quad \forall \varphi \in H^2_1, \varphi \geq 0.$$  

Taking $u_1 = T(\Psi_1)$ and $u_2 = T(\Psi_2) \in H^2_1$, we want to show that $u_1 \leq u_2$ a.e. in $M$. Indeed, as for all $\varphi \in H^2_1, \varphi \geq 0$,

$$\int_M \nabla u_1 \nabla \varphi dV + \int_M a u_1 \varphi dV = \langle \Psi_1, \varphi \rangle \leq \langle \Psi_2, \varphi \rangle = \int_M \nabla u_2 \nabla \varphi dV + \int_M a u_2 \varphi dV.$$

It follows by the Weak Comparison Principle that

$u_1 \leq u_2$ a.e. in $M$. \hfill \Box

The proof of Lemma 1 follows immediately using Claims 1-4. \hfill ■

3 The Proof of the Theorem 1

Now we will proof the Main Result.

Consider the “interval”

$$[u, \overline{u}] = \{ v \in H^2_1; u(x) \leq v(x) \leq \overline{u}(x) \text{ a.e. in } M \}$$

with the topology of the a.e. convergence, consider

$$S : [u, \overline{u}] \longrightarrow (H^2_1)'$$

by

$$\langle S(v), \varphi \rangle = \int_M f(x)F(v)\varphi dV + \int_M h(x)H(v)\varphi dV, \quad v \in [u, \overline{u}], \varphi \in H^2_1.$$
Claim 5 $S$ is a continuous, non-decreasing and bounded operator.

Proof of Claim 5:

(i) $S$ is bounded.

Indeed, if $v \in [u, \overline{u}]$ and $\varphi \in H^2_1$, we have that

$$|\langle S(v), \varphi \rangle| \leq \int_M f(x)F(u)\varphi dV + \int_M h(x)H(u)\varphi dV$$

$$\leq \|fF(u)\|_2 \|\varphi\|_2 + \|hH(u)\|_2 \|\varphi\|_2$$

$$= \left(\|fF(u)\|_2 + \|hH(u)\|_2\right) \|\varphi\|_2$$

where in the last inequality we use the H"older’s inequality.

As $H^2_1 \hookrightarrow L^2$, we obtain that

$$|\langle S(v), \varphi \rangle| \leq \left(\|fF(u)\|_2 + \|hH(u)\|_2\right) C \|\varphi\|_{H^2_1} = A \|\varphi\|_{H^2_1},$$

where $A = C \left(\|fF(u)\|_2 + \|hH(u)\|_2\right) > 0$.

Hence, $S$ is bounded in $[u, \overline{u}]$.

(ii) $S$ is non-decreasing.

Indeed, if $u_1, u_2 \in [u, \overline{u}]$ are such that $u_1 \leq u_2$ a.e., it follows that, by the fact that $F$ and $H$ are non-decreasing and by $(\alpha_2)$, that for all $\varphi \in H^2_1$, $\varphi \geq 0$,

$$\langle S(u_1), \varphi \rangle = \int_M f(x)F(u_1)\varphi dV + \int_M h(x)H(u_1)\varphi dV$$

$$\leq \int_M f(x)F(u_2)\varphi dV + \int_M h(x)H(u_2)\varphi dV$$

$$= \langle S(u_2), \varphi \rangle.$$

(iii) $S$ is continuous.

Let $(v_i)$ and $v \in [u, \overline{u}]$ such that $v_i \rightarrow v$ a.e. in $M$.

We observe that $fF(v_i) + hH(v_i) \rightarrow fF(v) + hH(v)$ a.e. in $M$ and

$$|fF(v_i) + hH(v_i)|^2 \leq |fF(u) + hH(u)|^2. \quad (10)$$
On the other hand, for $\varphi \in H^2_1$

$$\| \langle S(v_i) - S(v), \varphi \rangle \| \leq \int_M \| (fF(v_i) + hH(v_i)) - (fF(v) + hH(v)) \| \varphi |dV$$

$$\leq \| (fF(v_i) + hH(v_i)) - (fF(v) + hH(v)) \|_2 \| \varphi \|_2$$

$$\leq C \| (fF(v_i) + hH(v_i)) - (fF(v) + hH(v)) \|_2 \| \varphi \|_{H^2_1}$$

where in the two last inequalities we used the Hölder's inequality and the Sobolev's embedded $H^2_1 \hookrightarrow L^2$, respectively.

Then, by equation (10) we can apply the Theorem of Dominated Convergence of Lebesgue to conclude that

$$\| S(v_i) - S(v) \|_{(H^2_1)'^*} = o(1).$$

Therefore, $S$ is continuous in $[\underline{u}, \overline{u}]$ what proves the Claim 5. \hfill $\square$

Consider, now, $J : [\underline{u}, \overline{u}] \rightarrow H^2_1$ given by $J = T \circ S$. Namely, for all $v \in [\underline{u}, \overline{u}]$, $u = J(v)$ is the unique solution of equation

$$\Delta u + au = fF(v) + hH(v).$$

Taking

$$u_1 = J(\underline{u}), \quad u_1 = J(\overline{u}), \quad u_{i+1} = J(u_i) \text{ and } u^{i+1} = J(u^i), \quad i \geq 1,$$

we obtain

$$\Delta u_1 + au_1 = fF(u) + hH(u) \geq \Delta u + au$$

and by the Weak Comparison Principle $u_1 \geq u$ a.e.

Analogously,

$$\Delta u_2 + au_2 = fF(u_1) + hH(u_1) \geq fF(u) + hH(u) = \Delta u_1 + au_1$$

and, thus, $u_2 \geq u_1$ a.e.

Considering this process, successively, we obtain that

$$\underline{u} \leq u_1 \leq u_2 \leq \ldots \leq u_i \leq \ldots$$
With the same argument we conclude that

\[ \ldots \leq u^i \leq \ldots \leq u^2 \leq u^1 \leq \overline{u}. \]

On the other hand,

\[ \Delta u^1 + au^1 = fF(\overline{u}) + hH(\overline{u}) \geq fF(u) + hH(u) = \Delta u_1 + au_1 \]

what give us \( u^1 \geq u_1 \geq u \).

Analogously,

\[ \Delta u^2 + au^2 = fF(u^1) + hH(u^1) \geq fF(u_1) + hH(u_1) = \Delta u_2 + au_2 \]

then \( u^2 \geq u_2 \).

Considering this process, successively, we obtain that

\[ u \leq u_1 \leq u_2 \leq \ldots \leq u_i \leq \ldots \leq u^i \leq \ldots \leq u^2 \leq u^1 \leq \overline{u}. \]

Then, \( u_i \rightarrow u_* \) and \( u^i \rightarrow u^* \) a.e. when \( i \rightarrow \infty \) and \( u_*, u^* \in [u, \overline{u}] \) with \( u_* \leq u^* \) a.e.

As \( u_{i+1} = J(u_i) \rightarrow J(u_*) \) and \( u^{i+1} = J(u^i) \rightarrow J(u^*) \) when \( i \rightarrow \infty \), by continuity of \( J \), we conclude that \( u_*, u^* \in H^2_1 \) with \( J(u_*) = u_* \) and \( J(u^*) = u^* \), namely, \( u \leq u_* \leq u^* \leq \overline{u} \) are weak solutions of equation

\[ \Delta u + au = fF(u) + hH(u). \]

By using \((\alpha_1)\) we can use a Regularity Theorem to show that \( u_*, u^* \in C^\infty(M) \) and by the Strong Maximum Principle \( u_*, u^* > 0 \) in \( M \).

References

[1] T. AUBIN, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*. Journal de Mathématiques Pure et Appliquées, vol. 55(1976), p. 269-296.
[2] J.G. AZORERO & I.P. ALONSO, Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues. Communications in Partial Differential Equations, vol. 12(1987), p. 1389-1430.

[3] H. BRÉZIS & L. NIRENBERG, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Communications on Pure and Applied Mathematics, vol. 36(1983), p. 437-477.

[4] F.J. CORRÊA, J.V. GONCALVES & A.L. MELO, On positive radial solutions of quasilinear elliptic equations. Nonlinear Analysis, vol. 52(2003), p. 681-701.

[5] F. DEMEGEL & E. HEBEY, On some nonlinear equations involving the p-Laplacian with critical Sobolev growth. Advances in Differential Equations, vol. 3(1998), n. 4, p. 533-574.

[6] Z. DJADLI, Nonlinear elliptic equations with critical Sobolev exponent on compact riemannian manifolds. Calculus of Variations and Partial Differential Equations, vol. 8(1999), p. 293-326.

[7] O. DRUET, Generalized scalar curvature type equations on compact riemannian manifolds. Proceedings of the Royal Society of Edinburgh, vol. 130 A(2000), p. 767-788.

[8] O. KAVIAN, Introduction à la théorie des points critiques aux applications aux problèmes elliptiques. Mathématiques et Applications, 13 (1993), Springer-Verlag, Paris.

[9] O.H. MIYAGAKI, On a class of semilinear elliptic problems in $\mathbb{R}^n$ with critical growth. Nonlinear Analysis, Theory, Methods and Applications, vol. 29 (1997), n. 7, p. 773-781.

[10] R. SCHOEN, Conformal deformation of a riemannian metric to constant scalar curvature. Journal of Differential Geometry, vol. 20(1984), p. 479-495.
[11] C.R. SILVA, *Algumas Equações Diferenciais Não Lineares em Variedades Riemannianas Compactas, UnB. thesis (2004)*

[12] C.R. SILVA, *On the study of Existence of solutions for a class of equations with critical Sobolev exponent on compact Riemannian Manifolds*. Mat. Contemporânea, vol. *43*(2014), p. 223-246.

[13] N.S. TRUDINGER, *Remarks concerning the conformal deformation of riemannian structures on compact manifolds*. Ann. Scuola Norm. Sup. Pisa, vol. *3*(1968), n. 22, p. 265-274.

[14] H. YAMABE, *On a deformation of riemannian structures on compact manifolds*. Osaka Math. J., vol. *12*(1960), p. 21-37.

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