Sharp Bounds for the Concentration of the Resolvent
in Convex Concentration Settings

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Considering random matrix $X \in \mathcal{M}_{p,n}$ with independent columns satisfying the convex concentration properties issued from a famous theorem of Talagrand, we express the linear concentration of the resolvent $Q = (I_p - \frac{1}{n}XX^T)^{-1}$ around a classical deterministic equivalent with a good observable diameter for the nuclear norm. The general proof relies on a decomposition of the resolvent as a series of powers of $X$.

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Introduction

Initiated by Milman in the 70s [Mil71], the theory of concentration of the measure provided a wide range of concentration inequalities, mainly concerning continuous distribution (i.e. with no atoms), in particular thanks to the beautiful interpretation with the bound on the Ricci curvature [Gro79]. To give a simple fundamental example (more examples can be found in [Led05]), a random vector $Z \in \mathbb{R}^n$ having independent and Gaussian entries with unit variance satisfies for any 1-Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}$:

$$\forall t > 0 : \quad \mathbb{P} (|f(Z) - \mathbb{E}[f(Z)]| \geq t) \leq 2e^{-t^2/2}. \quad (0.1)$$

This inequality is powerful for two reasons: first, it is independent on the dimension $n$, second it concerns any 1-Lipschitz mapping $f$. It is then interesting to formalize this behavior to introduce a class of “Lipschitz concentrated random vectors” satisfying the same concentration inequality as the Gaussian distribution (in particular, all the Lipschitz transformation of a Gaussian vector). This was done in several books and papers where this approach proved its efficiency in the study of random matrices [Tao12, Nou09, LC21b]...
We want here to extend those results to a new class of concentrated vectors discovered by Talagrand in [Ta95]. Although the concentration result looks similar, its nature is quite different as it concerns bounded distributions for which classical tools of differential geometry do not operate. In a sense, it could be seen as a combinatorial result of concentration. Given a random vector $Z \in [0, 1]^n$ with independent entries, this result sets that for any $1$-Lipschitz and convex mapping $f : [0, 1] \to \mathbb{R}$:

$$
\forall t > 0 : \ P \left( |f(Z) - \mathbb{E}[f(Z)]| \geq t \right) \leq 2e^{-t^2/4}.
$$

(0.2)

We can mention here the recent results of [HT21] that extend this kind of inequalities for random vectors with independent and subgaussian entries. Adopting the terminology of [VW14,MS11,Ada11], we call those vectors convexly concentrated random vector (see Definition 1.2 below). The convexity required for the observations to concentrate makes the discussion on convexly concentrated random vector far more delicate. There is no more stability towards Lipschitz transformations and given a convexly concentrated random vector $Z$, just its affine transformations are sure to be concentrated. This issue raises naturally for one of the major objects of random matrix theory, namely the resolvent $Q_z = (zI_n - X)^{-1}$ that can provide important eigen properties on $X$. In the case of convex concentration, the concentration of the resolvent $Q_z = (zI_n - X)^{-1}$ is no more a mere consequence of a bound on its differential on $X \in \mathcal{M}_p$. Still, as first shown by [GZ00], it is possible to obtain concentration properties on the sum of Lipschitz functionals of the eigen values. Here we pursue the study, looking at linear concentration properties of $Q_z$ for which similar inequalities to (0.1) or (0.2) are only satisfied by $1$-Lipschitz and linear functionals $f$. The well known identity

$$
\frac{1}{p} \sum_{\lambda \in \text{Sp}(X)} f(\lambda) = -\frac{1}{2i\pi} \oint_\gamma f(z) \text{Tr}(Q_z^2)dz,
$$

(0.3)

is true for any analytical mapping $f$ defined on the interior of a path $\gamma \in \mathbb{C}$ containing the spectrum of $X$ (or any limit of such mappings), therefore, our results on the concentration of $Q_z^2$ concern in particular the quantities studied in [GZ00].

Although it is weaker, the class of linearly concentrated vectors behaves very well towards the dependence and the sum and allows us to obtain the concentration of the resolvent expressing it as a sum $Q_z^2 = \frac{1}{p} \sum_{i=1}^{\infty} (X/z)^i$. The linear concentration of the powers of $X$ was justified in [MS11] in the case of convexly concentrated random matrix $X$. We call this weakening of the concentration property “the degeneracy of the convex concentration through multiplication”. The linear concentration of the resolvent is though sufficient for most practical applications that rely on an estimation of the Stieltjes transform $m(z) = \frac{1}{p} \text{Tr}(Q_z^2)$.

\footnote{Lipschitz concentrated random vectors are convexly and linearly concentrated, convexly concentrated random vectors are linearly concentrated.}

\footnote{We provide an alternative proof in the appendix.}
We present below our main contribution without the useful but non-standard formalism introduced in the rest of the article. It concerns the concentration and the estimation of

$$Q^z \equiv \left( zI_p - \frac{1}{n}XX^T \right)^{-1}$$

for a random matrix $X \in \mathcal{M}_{p,n}$. Following the formalism of the random matrix theory, the computable estimation of $E[Q^z]$ will be called a “deterministic equivalent”. Its definition relies on a well known result that states that given a family of symmetric matrices $\Sigma = (\Sigma_1, \ldots, \Sigma_n) \in \mathcal{M}_{n,n}$, there exists a unique vector $\tilde{\Lambda}^z \in \mathbb{C}^n$ satisfying:

$$\forall i \in [n]: \quad [\tilde{\Lambda}^z]_i = \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^z_{\Sigma_i} \right)$$

with $\tilde{Q}^z_{\Sigma} = \left( zI_p - \frac{1}{n} \sum_{j=1}^{n} \Sigma_j \right)^{-1}$.

With those notations at hand, let us state:

**Theorem 0.1 (Concentration of the resolvent).** Considering two sequences $(p_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$, $(\sigma_n)_{n \in \mathbb{N}} \in \mathbb{R}_{+}^\mathbb{N}$ and four constants $c, C, K, \gamma > 0$, we suppose that we are given for any $n \in \mathbb{N}$ a random matrix: $X_n = (x_1^{(n)}, \ldots, x_n^{(n)}) \in \mathcal{M}_{p_n,n}$ such that

- $p_n \leq \gamma n$
- for all $n \in \mathbb{N}$, $x_1^{(n)}, \ldots, x_n^{(n)}$ are independent,
- $\sup_{n \in \mathbb{N}, j \in [n]} \left\| E \left[ x_j^{(n)} \right] \right\| \leq K \sqrt{n}$
- for any $n \in \mathbb{N}$ quasi-convex mapping $g: \mathcal{M}_{n,p_n} \to \mathbb{R}$, 1-Lipschitz for the euclidean norm:

$$\mathbb{P} \left( \left| g(X_n) - E[g(X_n)] \right| \geq t \right) \leq Ce^{-c(t/\sigma_n)^2}.$$

Then for any constant $\varepsilon > 0$, there exist two constants $c', C' > 0$ such that for all $n \in \mathbb{N}$, for any deterministic matrix $A \in \mathbb{R}^n$ such that $\|A\| \leq 1$ and for any $z \in \mathbb{C}$, such that $d(z, \text{Sp}(\frac{1}{n}XX^T)) \geq \varepsilon$:

$$\mathbb{P} \left( \left| \text{Tr} \left( \left( Q^z - \tilde{Q}^z_{\Sigma} \right) A \right) \right| \geq t \right) \leq C' e^{-c'(t/\sigma_n)^2} + C'e^{-c'n},$$

where $\Sigma_i = E[x_i x_i^T]$. Besides there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$:

$$\left\| E[Q^z] - \tilde{Q}^z_{\Sigma} \right\|_F \leq \frac{K}{\sqrt{n}}.$$

This theorem allows us to get good inferences on the eigen values distribution through the identity (0.3) and the estimation of the Stieltjes transform $g(z)$

$^c$The concentration of $(zI_p - \frac{1}{\sqrt{n}}X)^{-1}$ for a positive symmetric matrix $X \in \mathcal{M}_p$ is immediate from our approach. The estimation of its expectation is more laborious and goes beyond the scope of the paper although it can be obtained with the same tools.
\[-\frac{1}{p} \text{Tr}(Q^z)\] satisfying the concentration inequality:

\[
P\left( \left| g(z) + \frac{1}{n} \text{Tr} \left( \tilde{Q}^z \right) \right| \geq t \right) \leq Ce^{-c_n^2t^2},
\]

for two constants \(C, c > 0\) (and for \(d(z, \text{Sp}(\frac{1}{n}XX^T)) \geq O(1)\)).

When the distribution of the spectrum of \(\frac{1}{n}XX^T\) presents different bulks, this theorem also allows us to understand the eigen-spaces associated to those different bulks. Indeed, considering a path \(\gamma \in \mathbb{C}\) containing a bulk of eigen-values \(B \subset \text{Sp}(\frac{1}{n}XX^T)\), if we note \(E_B\) the associated random eigen-space and \(\Pi_B\) the orthogonal projector on \(E_B\), then for any deterministic matrix \(A \in \mathcal{M}_p\):

\[
\text{Tr}(\Pi_B A) = -\frac{1}{2i\pi} \int_{\gamma} \text{Tr}(AQ^z)dz
\]

we can estimate this projection \(\Pi_B\) defining \(E_B\) thanks to the concentration inequality\(^4\).

\[
\forall t > 0 : \quad P\left( \left| \frac{1}{\text{Rg}(\Pi)} \text{Tr}(\Pi Q^z) - \frac{1}{\text{Rg}(\Pi)} \text{Tr} \left( \Pi \tilde{Q}^z \right) \right| \geq t \right) \leq Ce^{-c \text{Rg}(\Pi)^2t^2},
\]

for some constants \(C, c > 0\) and for any projector \(\Pi\) defined on \(\mathbb{R}^p\).

The approach we present here does not only allows us to set the concentration of \(Q^z\), but also the concentration of any polynomial of finite degree taking as variable combination of \(Q^z\), \(X\) and \(X^T\). The general idea is to develop the polynomial as an infinite series of powers of \(X\) in a way that the observable diameters of the different terms of the series sum to the smallest value possible. As it is described in the proof of Proposition 4.2, the summation becomes slightly elaborate when \(z\) gets close to the spectrum.

After presenting the definition and the basic properties of the convex and linear concentration (Section 1), we express the concentration of the sum of linearly concentrated random vectors (Section 2). Then we express the concentration of the entry wise product and the matricial product of convexly concentrated random vectors and matrices (Section 3). Finally we deduce the concentration of the resolvent and provide a computable deterministic equivalent (Section 4).

1. Definition and first properties

The concentration inequality (0.2) is actually also valid for quasi-convex functionals defined folowingly.

**Definition 1.1.** Given a normed vector space \((E, \| \cdot \|)\), an application \(f : E \rightarrow \mathbb{R}\) is said to be quasi-convex if for any \(t \in \mathbb{R}\), the set \(\{f \leq t\} \equiv \{x \in E \mid f(x) \leq t\}\) is convex.

\(^4\) For the concentration to be valid on all the values of the path \(\gamma\), one must be careful to consider a path staying at a distance \(O(1)\) from the bulk, that is why we only consider here multiple bulk distributions
The theory of concentration of measure becomes relevant only when dimensions get big. In the cases under study in this paper, the dimension is either given by the number of entries, either by the number of columns $n$ of random matrices - the number of rows $p$ is then understood to depend on $n$, we will sometimes note $p = p_n$. We follow then the approach with Levy families [Lé51] whose aim is to track the concentration speed through dimensionality. Therefore, we do not talk about a static concentration of a vector but about the concentration of a sequence of random vectors as seen in the definition below. In this paper, $E_n$ will either be $\mathbb{R}^n$, $\mathbb{R}^{p_n}$, $\mathcal{M}_{n}$, $\mathcal{M}_{p_n}$ or $\mathcal{M}_{p_n,n}$.

There will generally be three possibilities for the norms defining the Lipschitz character of the concentrated observations. Talagrand Theorem gives the concentration for the euclidean norm - i.e. the Frobenius norm for matrices - but we will see that some concentrations are expressed with the nuclear norm (the dual norm of the spectral norm). Given two integers $l,m \in \mathbb{N}$, the euclidean norm on $\mathbb{R}^l$ is noted $\| \cdot \|_l$, the spectral, Frobenius and nuclear norm are respectively defined for any $M \in \mathcal{M}_{l,m}$ with the expressions:

$$\| M \| = \sup_{x \in \mathbb{R}^m} \| Mx \|; \quad \| M \|_F = \sqrt{\text{Tr}(MM^T)}; \quad \| M \|_* = \text{Tr} \left( \sqrt{MM^T} \right).$$

**Definition 1.2.**

Given a sequence of normed vector spaces $(E_n, \| \cdot \|_n)_{n \geq 0}$, a sequence of random vectors $(Z_n)_{n \geq 0} \in \prod_{n \geq 0} E_n$, a sequence of positive reals $(\sigma_n)_{n \geq 0} \in \mathbb{R}_+^N$, we say that $Z = (Z_n)_{n \geq 1}$ is convexly concentrated with an observable diameter of order $O(\sigma_n)$ iff there exist two positive constants $C, c > 0$ such that $\forall n \in \mathbb{N}$ and for any 1-Lipschitz and quasi-convex function $f : E_n \to \mathbb{R}$ (for the norms $\| \cdot \|_n$),

$$\forall t > 0 : \quad \mathbb{P} \left( | f(Z_n) - \mathbb{E}[f(Z_n)] | \geq t \right) \leq Ce^{-c(t/\sigma_n)^2},$$

We write in that case $Z_n \sim_c \mathcal{E}_2(\sigma_n)$ (or more simply $Z \sim_c \mathcal{E}_2(\sigma)$).

Theorem of Talagrand then writes:

**Theorem 1.1 (Tal95).** A (sequence of) random vector $Z \in [0,1]^n$ with independent entries satisfies $Z \sim_c \mathcal{E}_2$.

Convex concentration is preserved through affine transformations (as for the class of linearly concentrated vectors). Given two vector spaces, $E$ and $F$, we note $\mathcal{A}(E,F)$ the set of affine transformation from $E$ to $F$, and given $\phi \in \mathcal{A}(E,F)$, we

In this inequality, one could have replaced the term “$\mathbb{E}[f(Z_n)]$” by “$f(Z_n'')$” (with $Z_n''$ a independent copy of $Z_n$) or by “$m_f$” (with $m_f$ a median of $f(Z_n)$). All those three definitions are equivalent.

The index 2 in “$\mathcal{E}_2$” is here a reference to the power of $t$ in the concentration bound $Ce^{-c(t/\sigma_n)^2}$, we will see some example where this exponent is 1, in particular in the Hanson-Wright Theorem 3.3 where we will let appear a notation “$\mathcal{E}_1$".
decompose \( \phi = \mathcal{L}(\phi) + \phi(0) \), where \( \mathcal{L}(\phi) \) is the linear part of \( \phi \) and \( \phi(0) \) is the translation part. When \( E = F \), \( \mathcal{A}(E, F) \) is simply noted \( \mathcal{A}(E) \).

**Proposition 1.1.** Given two normed vector spaces \( (E, \|\cdot\|) \) and \( (F, \|\cdot\|') \), a random vector \( Z \in E \) and an affine mapping \( \phi \in \mathcal{A}(E, F) \) such that \( \|\mathcal{L}(\phi)\| \leq \lambda \):

\[
Z \propto \mathcal{E}_2(\sigma) \quad \implies \quad \phi(Z) \propto \mathcal{E}_2(\lambda \sigma).
\]

We pursue our presentation with the introduction of the linear concentration. It is the "minimal" hypothesis necessary on a random vector \( X \) to be able to bound quantities of the form \( \mathbb{E}[\|X - \mathbb{E}[X]\|] \), as it has been explained in [LC19]. Here we will need its stability towards the sum when we will express \( Q^z \) as an infinite series.

**Definition 1.3 (Linearly concentrated vectors).** Given a sequence of normed vector spaces \( (E_n, \|\cdot\|)_n \), a sequence of random vectors \((Z_n)_n \in \prod_{n \geq 0} E_n\), a sequence of deterministic vectors \((\tilde{Z}_n)_n \in \prod_{n \geq 0} E_n\), a sequence of positive reals \((\sigma_n)_n \in \mathbb{R}^n_+\), \( Z_n \) is said to be linearly concentrated around the deterministic equivalent \( \tilde{Z}_n \) with an observable diameter of order \( O(\sigma_n) \) if there exist two constants \( c, C > 0 \) such that \( \forall n \in \mathbb{N} \neq 0 \) and for any unit-normed linear form \( f \in E_n' \) (\( \forall n \in \mathbb{N}, \forall x \in E \) : \( |f(x)| \leq \|x\|_n \)):

\[
\forall t > 0 : \quad \mathbb{P} \left( \left| f(Z_n) - f(\tilde{Z}_n) \right| \geq t \right) \leq Ce^{-c(t/\sigma_n)^2}.
\]

When the property holds, we write \( Z \in \tilde{Z} \pm \mathcal{E}_2(\sigma) \). If it is unnecessary to mention the deterministic equivalent, we will simply write \( Z \in \mathcal{E}_2(\sigma) \); and if we just need to control the order of the norm of the deterministic equivalent, we can write \( Z \in O(\theta) \pm \mathcal{E}_2(\sigma) \) when \( \|\tilde{Z}_n\|_n \leq O(\theta_n) \).

In the literature [BLM13], those vectors are commonly called sub-Gaussian random vectors.

The notions of linear concentration, convex concentration (and Lipschitz concentration) are equivalent for random variables and we have this important characterization with the moments:

**Proposition 1.2 ([Led05], Proposition 1.8., [LC19], Lemma 1.22.).** Given a sequence of random variables \( Z_n \in \mathbb{R} \) and a sequence of positive parameters \( \sigma_n > 0 \), we have the equivalence:

\[
Z_n \propto \mathcal{E}_2(\sigma_n) \iff Z_n \in \mathbb{E}[Z_n] \pm \mathcal{E}_2(\sigma_n)
\]

\[
\iff \exists C > 0 \ | \ \forall n, m \in \mathbb{N} : \mathbb{E}[\|Z_n - \mathbb{E}[Z_n]\|^m] \leq Cm^\frac{m}{2} \sigma_n^m
\]

\[
\iff \exists C > 0 \ | \ \forall n \in \mathbb{N}, \forall r > 0 : \mathbb{E}[\|Z_n - \mathbb{E}[Z_n]\|^r] \leq Cr^\frac{r}{2} \sigma_n^r.
\]

We end with a simple lemma that allows us to state that "every deterministic vector at a distance smaller than the observable diameter to a deterministic equivalent is also a deterministic equivalent".

**Lemma 1.1 ([LC19], Lemma 2.6.).** Given a sequence of random vectors \( Z_n \in E_n \) and two sequence of deterministic random vector \( \hat{Z}_n, \hat{Z}'_n \in E_n \), if \( \|Z_n - \hat{Z}_n\| \leq \|Z_n - \hat{Z}'_n\| \leq \]
$O(\sigma_n)$, then:

$$Z \in \tilde{Z} \pm \mathcal{E}_2(\sigma) \iff Z \in \tilde{Z}' \pm \mathcal{E}_2(\sigma).$$

2. Linear concentration through sums and integrals

Independence is known to be a key element to most of concentration inequalities. However, linear concentration behaves particularly well for the concatenation of random vectors whose dependence cannot be disentangled.

The next proposition sets that the observable diameter for the $\ell_\infty$ norm remains unchanged through concatenation. Given a product $\prod_{1 \leq i \leq m} E_i$ where $(E_1, \| \cdot \|_\infty), \ldots, (E_m, \| \cdot \|_\infty)$ are $m$ normed vector spaces we define the $\ell_\infty$ norm on $E$ with the following identity:

$$(z_1, \ldots, z_m) \in E : \|(z_1, \ldots, z_m)\|_{\ell_\infty} = \sup_{1 \leq i \leq m} \|z_i\|_i.$$  \hfill (2.1)

**Proposition 2.1.** Given two sequences $m \in \mathbb{N}$ and $\sigma \in \mathbb{R}^\mathbb{N}$, a constant $q$, $m$ sequences of normed vector spaces $(E_i, \| \cdot \|_i)_{1 \leq i \leq m}$, $m$ sequences of deterministic vectors $\tilde{Z}_1 \in E_1, \ldots, \tilde{Z}_m \in E_m$, and $m$ sequences of random vectors $Z_1 \in E_1, \ldots, Z_m \in E_m$ (possibly dependent) satisfying, for any $i \in \{1, \ldots, m\}$, $Z_i \in \tilde{Z}_i \pm \mathcal{E}_2(\sigma)$, we have the concentration:

$$(Z_1, \ldots, Z_m) \in (\tilde{Z}_1, \ldots, \tilde{Z}_m) \pm \mathcal{E}_2(\sigma), \text{ in } (E, \| \cdot \|_{\ell_\infty}).$$

In other words, the linear observable diameter of $(Z_1, \ldots, Z_m)$ can not be bigger than the observable diameter of $(Z_1, \ldots, Z)$, where $Z$ is chosen as the worse possible random vector satisfying the hypotheses of $Z_1, \ldots, Z_m$.

**Remark 2.1.** Example 2.27 in [LC19] shows that this stability towards concatenation is not true for Lipschitz and convex concentration.

**Proof.** Let us consider a linear function $u : E \to \mathbb{R}$, such that

$$\|u\|_\infty \equiv \sup_{\|z\|_\infty \leq 1} |u(z)| \leq 1.$$  

Given $i \in [m]$, let us note $u_i : E_i \to \mathbb{R}$ the function defined as $u_i(z) = u((0, \ldots, 0, z, 0, \ldots, 0))$ (where $z$ is in the $i$th entry). For any $z \in E$, one can write:

$$u(z) = \sum_{i=1}^m n_i u'_i(z_i),$$

where $n_i \equiv \|u_i\| = \sup_{\|z\|_i \leq 1} u_i(z)$ and $u'_i = u_i/n_i$ ($\|u'_i\| = 1$). We have the inequality:

$$\sum_{i=1}^m n_i = \sum_{i=1}^m n_i \sup_{\|z\|_i \leq 1} u'_i(z_i) = \sup_{\|z\| \leq 1} u(z) \leq 1.$$
With this bound at hand, we plan to employ the characterization with the centered moments. Let us conclude thanks to Proposition 2.1 and the convexity of $t \mapsto t^l$, for any $l \geq 1$:

$$
\mathbb{E} \left[ |u(Z) - u(\tilde{Z})|^l \right] \leq \mathbb{E} \left[ \left( \sum_{i=1}^{m} n_i \left| u_i'(Z_i) - u_i'(\tilde{Z}_i) \right| \right)^l \right] \\
\leq \left( \sum_{i=1}^{m} n_i \right)^l \mathbb{E} \left[ \sum_{i=1}^{m} n_i \left| u_i'(Z_i) - u_i'(\tilde{Z}_i) \right|^l \right] \\
\leq \sup_{i \in [m]} \mathbb{E} \left[ \left| u_i'(Z_i) - u_i'(\tilde{Z}_i) \right|^l \right] \leq C\|z\|^l.
$$

If we want to consider the concatenation of vectors with different observable diameter, it is more convenient to look at the concentration in a space $(\prod_{i=1}^{m} E_i, \ell^r)$, for any given $r > 0$, where, for any $(z_1, \ldots, z_m) \in \prod_{i=1}^{m} E_i$:

$$
\| (z_1, \ldots, z_m) \|_{\ell^r} = \left( \sum_{i=1}^{m} \| z_i \|_{\ell^r}^r \right)^{1/r}.
$$

**Corollary 1.** Given two constants $q, r > 0$, $m \in \mathbb{N}^\ast$, $\sigma_1, \ldots, \sigma_m \in (\mathbb{R}^+_m)^m$, $m$ sequences of $(E_i, \| \cdot \|_i)_{1 \leq i \leq m}$, $m$ sequences of deterministic vectors $\tilde{Z}_1 \in E_1, \ldots, \tilde{Z}_m \in E_m$, and $m$ sequences of random vectors $Z_1 \in E_1, \ldots, Z_m \in E_m$ (possibly dependent) satisfying, for any $i \in \{1, \ldots, m\}$, $Z_i \in \tilde{Z}_i \pm \mathcal{E}_2(\sigma_i)$, we have the concentration:

$$
(Z_1, \ldots, Z_m) \in (\tilde{Z}_1, \ldots, \tilde{Z}_m) \pm \mathcal{E}_2(\| \sigma \|_r), \text{ in } (E, \| \cdot \|_{\ell^r}),
$$

**Remark 2.2.** When $E_1 = \cdots = E_m = E$ in the setting of Corollary 1 then for any vector $a = (a_1, \ldots, a_m) \in \mathbb{R}^m_+\ast$, we know that:

$$
\sum_{i=1}^{m} a_i Z_i \in \sum_{i=1}^{m} a_i \tilde{Z}_i \pm \mathcal{E}_2(\| a \|_{\ell^r} \sigma),
$$

where $|a| = (|a_1|, \ldots, |a_m|) \in \mathbb{R}^m_+\ast$.

**Proof.** We already know from Proposition 2.1 that:

$$
\left( \frac{Z_1}{\sigma_1}, \ldots, \frac{Z_m}{\sigma_m} \right) \in \left( \frac{\tilde{Z}_1}{\sigma_1}, \ldots, \frac{\tilde{Z}_m}{\sigma_m} \right) \pm \mathcal{E}_2, \text{ in } (E, \| \cdot \|_{\ell^r}).
$$

Let us then consider the linear mapping:

$$
\phi : (E, \| \cdot \|_{\ell^r}) \rightarrow (E, \| \cdot \|_{\ell^r}) \\
(z_1, \ldots, z_m) \mapsto (\sigma_1 z_1, \ldots, \sigma_m z_m),
$$

the Lipschitz character of $\phi$ is clearly $\| \sigma \|_r = (\sum_{i=1}^{m} \sigma_i^r)^{1/r}$, and we can deduce the concentration of $Z = \phi(\sigma_1 Z_1, \ldots, \sigma_m Z_m)$.
Corollary 1 is very useful to set the concentration of infinite series of concentrated random variables. This is settled thanks to an elementary result of [LC19] that sets that the observable diameter of a limit of random vectors is equal to the limit of the observable vectors. Be careful that rigorously, there are two indexes, \( n \) coming from Definition 1.3 that only describes the concentration of sequences of random vectors, and \( m \) particular to this lemma that will tend to infinity. For clarity, we do not mention the index \( n \).

**Lemma 2.1 ([LC19], Proposition 1.12.).** Given a sequence of random vectors \((Z_m)_{m \in \mathbb{N}} \in E^N\), a sequence of positive reals \((\sigma_m)_{m \in \mathbb{N}} \in \mathbb{R}^N_+\) and a sequence of deterministic vectors \((\tilde{Z}_m)_{m \in \mathbb{N}} \in E^N\) such that:

\[
Z_m \in \tilde{Z}_m \pm E_2(\sigma_m),
\]

if we assume that \((Z_m)_{m \in \mathbb{N}}\) converges in law\(^6\) when \(m\) tends to infinity to a random vector \((Z_\infty) \in E\), that \(\sigma_m \xrightarrow{n \to \infty} \sigma_\infty\) and that \(\tilde{Z}_m \xrightarrow{n \to \infty} \tilde{Z}_\infty\), then:

\[
Z_\infty \in \tilde{Z}_\infty \pm E_2(\sigma_\infty).
\]

(The result also holds for Lipschitz and convex concentration)

**Corollary 2.**

Given two constants \(q, r > 0, \sigma_1, \ldots, \sigma_m \ldots \in (\mathbb{R}^N_+)^N\), a (sequences of) normed vector spaces \((E, \| \cdot \|)\), \(\tilde{Z}_1, \ldots, \tilde{Z}_m, \ldots \in E^N\) deterministic, and \(Z_1, \ldots, Z_m, \ldots \in E^N\) random (possibly dependent) satisfying, for any \(n \in \mathbb{N}\), \(Z_m \in \tilde{Z}_m \pm E_2(\sigma_m)\). If we assume that \(Z \equiv \sum_{m \in \mathbb{N}} Z_m\) is pointwise convergent\(^7\), that \(\sum_{m \in \mathbb{N}} \tilde{Z}_m\) is well defined and that \(\sum_{m \in \mathbb{N}} \sigma_i \leq \infty\), then we have the concentration:

\[
\sum_{m \in \mathbb{N}} Z_m \in \sum_{m \in \mathbb{N}} \tilde{Z}_m \pm E_2 \left( \sum_{m \in \mathbb{N}} \sigma_m \right), \quad \text{in} \ (E, \| \cdot \|),
\]

**Proof.** We already know from Corollary 1 that for all \(m \in \mathbb{N}\):

\[
\sum_{m=1}^M Z_m \in \sum_{m=1}^M \tilde{Z}_m \pm E_2 \left( \sum_{m \in \mathbb{N}} \sigma_m \right), \quad \text{in} \ (E, \| \cdot \|).
\]

Thus in order to employ Lemma 2.1 let us note that for any bounded continuous mapping \(f : E \to \mathbb{R}\), the dominated convergence theorem allows us to set that:

\[
\mathbb{E} \left[ f \left( \sum_{m=1}^M Z_m \right) \right] \xrightarrow{M \to \infty} \mathbb{E} \left[ f \left( \sum_{m=1}^\infty Z_m \right) \right],
\]

\(^6\)For any \(n \in \mathbb{N}\), for any bounded continuous mapping \(f : \prod_{m \geq 0} E_p \to \mathbb{R}^N\):

\[
\sup_{n \in \mathbb{N}} \| \mathbb{E}[f(Z_{n,m})] - \mathbb{E}[f(Z_{n,\infty})] \| \xrightarrow{m \to \infty} 0
\]

\(^7\)For any \(w \in \Omega\), \(\sum_{m \in \mathbb{N}} \| Z_m(w) \| \leq \infty\) and we define \(Z(w) \equiv \sum_{m \in \mathbb{N}} Z_m(w)\)
thus $\left(\sum_{m=1}^{M} Z_m\right)_{N \in \mathbb{N}}$ converges in law to $\sum_{m=1}^{\infty} Z_m$, which allows us to set the result of the corollary.

The concentration of infinite series directly implies the concentration of resolvents and other related operators (like $(I_n - X/\sqrt{p})^{-1}X^k$ for instance).

**Corollary 3.** Given a (sequence of) vector space $(E, \| \cdot \|)$, let $\phi \in \mathcal{A}(E)$ be a (sequence of) random affine mapping such that there exists a constant $\varepsilon > 0$ satisfying $\|L(\phi)\| \leq 1 - \varepsilon$ and a (sequence of) integers $\sigma > 0$ satisfying for all (sequence of) integer $k$:

$$L(\phi)^k(\phi(0)) \in \mathcal{E}_2(\sigma(1 - \varepsilon)^k) \quad \text{in} \quad (E, \| \cdot \|)$$

Then the random equation

$$Y = \phi(Y)$$

admits a unique solution $Y = (Id_E - L(\phi))^{-1}\phi(0)$ satisfying the linear concentration:

$$Y \in \mathcal{E}_2(\sigma).$$

In practical examples, $\|L(\phi)\|$ is rarely bounded by $1 - \varepsilon$ for all drawings of $\phi$ and to obtain the concentration of $L(\phi)^k$ with an observable diameter of order $\sigma(1 - \varepsilon)^k$, one needs to place oneself on an event $\mathcal{A}_\phi$ satisfying $\mathcal{A}_\phi \subset \{\|L(\phi)\| \leq 1 - \varepsilon\}$. Then, thanks to a simple adaptation of Lemma 4.2 below to the case of linear concentration, we have the concentration $(Y \mid \mathcal{A}_\phi) \in \mathcal{E}_2(\sigma)$. When $\mathbb{E}[\|L(\phi)\|] \leq 1 - 2\varepsilon$ for $\varepsilon \geq O(1)$ and $\phi$ is sufficiently concentrated, it is generally possible to chose an event $\mathcal{A}_\phi$ of overwhelming probability.

As it will be seen in Subsection 4, this corollary finds its relevancy under convex concentration hypotheses, where the linear concentration seems to be the best concentration property to obtain on the resolvent $Q = (zI_p - \frac{1}{n}XX^T)^{-1}$.

**Proof.** By contractivity of $\phi$, $Y$ is well defined and expresses:

$$Y = (Id_E - L(\phi))^{-1}\phi(0) = \sum_{k=0}^{\infty} L(\phi)^k\phi(0).$$

One can then conclude with Corollary 2 that $Y \in \mathcal{E}_2(\sigma/\varepsilon) = \mathcal{E}_2(\sigma)$.

In order to satisfy the hypothesis of Corollary 3 but also for independent interest, we are now going to express the concentration of the product of convexly concentrated random matrices.
3. Degeneracy of convex concentration through product

Given two convexly concentrated random vectors \(X, Y \in E\) satisfying \(X, Y \propto c E_2(\sigma)\), the convex concentration of the couple \((X, Y) \propto c E_2(\sigma)\) is ensured if:

1. \(X\) and \(Y\) are independent
2. \((X, Y) = u(Z)\) with \(u\) affine and \(Z \propto c E_2(\sigma)\).

We can then in particular state the concentration of \(X + Y\) as it is a linear transformation of \((X, Y)\). For the product it is not as simple as for the Lipschitz concentration, let us first consider the particular case of the entry-wise product in \(E^p = \mathbb{R}^p\).

Since this result is not important for the rest of the paper, we left its proof in \[Appendix A\].

Theorem 3.1. Given a (sequence of) integer \(m \in \mathbb{N}\) and a (sequence of) positive number \(\sigma > 0\) such that \(m \leq O(p)\), a (sequence of) \(m\) random vectors \(X_1, \ldots, X_m \in \mathbb{R}^p\), if we suppose that \(X \equiv (X_1, \ldots, X_m) \propto c E_2(\sigma)\) in \((\mathbb{R}^p)^m, \|\cdot\|_{\ell^\infty})\), and that there exists a (sequence of) positive numbers \(\kappa > 0\) such that \(\forall i \in [m]: \|X_i\|_{\ell^\infty} \leq \kappa\), then:

\[X_1 \odot \cdots \odot X_m \in E_2((2e\kappa)^m \sigma)\] in \((\mathbb{R}^p, \|\cdot\|)\).

And if \(X_1 = \cdots = X_m = X\), the constant \(2e\) is no more needed and we get the concentration \(X^{\odot m} \in E_2(\kappa^{m-1}\sigma)\).

Remark 3.1. If we replace the strong assumption \(\forall i \in [m]: \|X_i\|_{\ell^\infty} \leq \kappa\), with the bound \(\sup_{1 \leq i \leq m} \|\mathbb{E}[X_i]\|_{\ell^\infty} \leq O((\log(p))^{1/3})\) we can still deduce a similar result to [LC21, Example 4.], stating the existence of a constant \(\kappa \leq O(1)\) such that:

\[X_1 \odot \cdots \odot X_m \in E_2\left((\kappa\sigma)^m (\log(p))^{(m-1)/3}\right) + E_{q/m}(\kappa\sigma)^m\] in \((\mathbb{R}^p, \|\cdot\|)\).

The result of concentration of a product of matrices convexly concentrated was already proven in [MS11] but since their formulation is slightly different, we reprove in \[Appendix A\] the following result with the formulation required for the study of the resolvent.

Theorem 3.2 ([MS11, Theorem 1]).

Let us consider three sequences \(m \in \mathbb{N}\) and \(\sigma, \kappa \in \mathbb{R}_+\), and a sequence of \(m\) random random matrices \(X_1 \in M_{n_0, n_1}, \ldots, X_m \in M_{n_m, n_m}, \) satisfying:

\((X_1, \ldots, X_m) \propto c E_2(\sigma)\) in \((M_{n_0, n_1} \times \cdots \times M_{n_m, n_m}, \|\cdot\|_F)\),

\(^{1}\)The norm \(\|\cdot\|_F\) is defined on \(M_{n_0, n_1} \times \cdots \times M_{n_m, n_m}\) by the identity:

\[\|(M_1, \ldots, M_p)\|_F = \sqrt{\|M_1\|_F^2 + \cdots + \|M_p\|_F^2}\]
In the particular case where \( X_1 = \cdots = X_n \equiv X \), it is sufficient to assume that \( X \sim \mathcal{E}_2(\sigma) \) in \((\mathcal{M}_n, \| \cdot \|_F)\). If there exist a sequence of positive values \( \kappa > 0 \) such that \( \forall i \in [m], \|X_i\| \leq \kappa \), then the product is concentrated for the nuclear norm:

\[
X_1 \cdots X_m \in \mathcal{E}_2 \left( \frac{K^m \sigma \sqrt{n_0 + \cdots + n_m}}{\kappa} \right)
\]

\[
in (\mathcal{M}_{n_0, n_m}, \| \cdot \|_*),
\]

where, for any \( M \in \mathcal{M}_{n_0, n_m} \), \( \|M\|_* = \text{Tr}(\sqrt{MM^T}) \) (it is the dual norm of the spectral norm).

Remark 3.2. The hypothesis \( \|X\| \leq \kappa \) might look quite strong, however in classical settings where \( X \sim \mathcal{E}_2 \) and \( \|E[X]\| \lesssim O(\sqrt{n}) \) it has been shown that there exist three constants \( C, c, K > 0 \) such that \( \mathbb{P}(\|X\| \geq K \sqrt{n}) \leq C e^{-cn} \). Placing ourselves on the event \( \mathcal{A} = \{\|X\| \leq K \sqrt{n}\} \), we can then show from Lemma 4.2 below that:

\[
((X/\sqrt{n})^m \mid A) \in \mathcal{E}_2 \left( \frac{K^{m-1} }{\sqrt{n}} \right)
\]

\[
\text{and } \mathbb{P}(A^c) \leq C e^{-cn},
\]

(here \( \sigma = 1/\sqrt{n} \) and \( \kappa = K \)). The same inferences hold for the concentration of \( (XX^T/(n+p))^m \).

We end this section on the concentration of the product of convexly concentrated random vectors with the Hanson-Wright Theorem that will find some use of the estimation of \( E[Q^2] \). This result was first proven in [Ada15], an alternative proof with our notations is provided in [LC21a, Proposition 8].

Theorem 3.3 ([Ada15]). Given two random matrices \( X, Y \in \mathcal{M}_{p,n} \) such that \( (X, Y) \sim \mathcal{E}_2 \) and \( \|E[X]\|_F, \|E[Y]\|_F \leq O(1) \), for any \( A \in \mathcal{M}_p \):

\[
Y^TAX \in \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|).
\]

4. Concentration of the resolvent of the sample covariance matrix of convexly concentrated data

4.1. Assumptions on \( X \) and “concentration zone” of the resolvent

Given \( n \) data \( x_1, \ldots, x_n \in \mathbb{R}^p \), to study the eigen behavior of the sample (non centered) covariance matrix \( \frac{1}{n}XX^T \), where \( X = (x_1, \ldots, x_n) \in \mathcal{M}_{p,n} \), one classically studies the resolvent \( Q^2 = (zI_p - \frac{1}{n}XX^T)^{-1} \) for the values of \( z \) where it is defined. Let us note the \( p \) eigen values of \( \frac{1}{n}XX^T \): \( \lambda_i = \sigma_i(\frac{1}{n}XX^T) \), for \( i \in [p] \) (then \( \lambda_1 \geq \cdots \geq \lambda_n \)), then the spectral distribution of \( \frac{1}{n}XX^T \):

\[
\mu = \frac{1}{p} \sum_{i=1}^{p} \delta_i
\]

\(^1\)Be careful that \( X \sim \mathcal{E}_2(\sigma) \) does not imply that \( (X_1, \ldots, X_n) \sim \mathcal{E}_2(\sigma) \), it is only true when \( (\mathcal{M}_n)^m \) is endowed with the norm \( \| \cdot \|_{F,cw} \), satisfying for any \( M = (M_1, \ldots, M_m) \in (\mathcal{M}_n)^m \),

\[
\|M\|_{F,cw} = \sup_{1 \leq i \leq m} \|M_i\|_F
\]

\(^2\)This paper only studies the Lipschitz concentration case, however, since quadratic forms are convex, the arguments stays the same with convex concentration hypotheses.
has for Stieltjes transform \( g : z \mapsto \frac{1}{z} \text{Tr}(Q^2) \).

The present study was already lead in previous papers in the case of Lipschitz concentration of \( X \) \cite{LC21b}, or in the case of convex concentration of \( X \) but with negative \( z \) \cite{LC19}. The goal of this section, is mainly to present the consequences of Theorem 3.2 and adapt the recent results of \cite{LC21b} on the case of convex concentration. We adopt here classical hypotheses and assume a convex concentration for \( X = (x_1, \ldots, x_n) \).

**Assumption 4.1 (Convergence scheme).** \( p = O(n) \).

**Assumption 4.2 (Independence).** \( x_1, \ldots, x_n \) are independent.

**Assumption 4.3 (Concentration).** \( X \preceq_c \mathcal{E}_2 \).

**Assumption 4.4 (Bounding condition).** \( \sup_{i \in [n]} \|E[x_i]\| \leq O(1) \).

When \( n \) gets big, \( \mu \) distributes along a finite number of bulks. To describe them, let us consider a positive parameter, \( \varepsilon > 0 \), that could be chosen arbitrarily small (it will though be chosen independent with \( n \) in most practical cases) and introduce as in \cite{LC21b} the sets:

\[
\mathcal{S} = \{ \lambda_i \}_{i \in [p]} \quad \tilde{\mathcal{S}} = \{ E[\lambda_i] \}_{i \in [p]} \quad \tilde{\mathcal{S}}^\varepsilon = \{ x \in \mathbb{R}, \exists i \in [n], |x - \lambda_i| \leq \varepsilon \}
\]

One can show that \( \nu = \sup \tilde{\mathcal{S}} = E[\lambda_1] \leq O(1) \) and introducing the event:

\[
\mathcal{A}_\varepsilon = \left\{ \forall i \in [p], \sigma_i \left( \frac{1}{n} XX^T \right) \in \tilde{\mathcal{S}}^\varepsilon \right\},
\]

the concentration of \( \sigma(X)/\sqrt{n} - \nu \in \mathbb{E}[\sigma(X)] \pm \mathcal{E}_2(1/\sqrt{n}) \), allows us to set \( \mathcal{E}_2 \).

**Lemma 4.1 (\cite{LC21b}, Lemma 3.).** There exist two constants \( C, c > 0 \) such that

\[
\mathbb{P}( \mathcal{A}^c ) \leq Ce^{-cn^2}.
\]

The following lemma allows us to conduct the concentration study on the highly probable event \( \mathcal{A}_\varepsilon \) (when \( \varepsilon \geq O(1) \)).

**Lemma 4.2.** Given a (sequence of) positive numbers \( \sigma > 0 \), a (sequence of) random vector \( Z \in E \) satisfying \( Z \preceq_c \mathcal{E}_2(\sigma) \), and a (sequence of) convex subsets \( A \subset E \), if there exists a constant \( K > 0 \) such that \( \mathbb{P}(Z \in A) \geq K \) then:

\[
(Z | Z \in A) \preceq_c \mathcal{E}_2(\sigma).
\]

\(^1\)As already done in \cite{LC19} (but with real negative \( z \)), one can obtain the same conclusion assuming that there are a finite number of classes for the distribution of the columns \( x_1, \ldots, x_n \) and that \( \sup_{i \in [n]} \|E[x_i]\| \leq O(\sqrt{n}) \).

\(^2\)In \cite{LC21b}, the proof is conducted for Lipschitz concentration hypotheses on \( X \). However, since only the linear concentration of \( \sigma(X) \) is needed, the justification are the same in a context of convex concentration thanks to Theorem \cite[Appendix A.1]{LC21b}:

\[
\forall t > 0 : \mathbb{P}( |f(Z) - E[f(Z)] | \geq t | Z \in A) \leq Ce^{-(t/c)^2},
\]

and similar concentration occur around any median of \( f(Z) \) or any independent copy of \( Z \) (under \( \{ Z \in A \} \)).
Proof. The proof is the same as the one provided in LC21b, Lemma 2, except that this time, one needs the additional argument that since \( S = \{ f \leq m_f \} \) (for \( m_f \), a median of \( f \)) is convex, the mappings \( z \mapsto d(z, S) \) and \( z \mapsto -d(z, S) \) are both quasi-convex thanks to the triangular inequality.

We can deduce from Lemma 4.2 that for all \( \varepsilon \geq O(1) \), \( (X | A_\varepsilon) \propto \mathcal{E}_2 \), and the random matrix \( (X | A_\varepsilon) \) is far easier to control because \( \| (X | A_\varepsilon) \| \leq \nu + \frac{\varepsilon}{2} \) (we recall that \( \nu \equiv \mathbb{E}[\lambda_1] \)).

### 4.2. Concentration of the resolvent

Placing ourselves under the event \( A_\varepsilon \), let us first show that the resolvent \( Q^z \equiv (zI_p - \frac{1}{n} XX^T)^{-1} \) is concentrated if \( z \) has a big enough modulus. Be careful that the following concentration is expressed for the nuclear norm (for any deterministic matrix \( A \in \mathcal{M}_p \) such that \( \| A \| \leq O(1) \), \( \text{Tr}(AQ^z) \in \mathcal{E}_2 \)). All the following results are provided under Assumptions 4.1-4.4. The next proposition is just provided as a first direct application of Theorem 3.2 and Corollary 2, a stronger result is provided in Proposition 4.2.

**Proposition 4.1.** Given two parameters \( \varepsilon > 0 \) and \( z \in \mathbb{C} \) such that \( |z| \geq \nu + \varepsilon \):

\[
(Q^z | A_\varepsilon) \in \mathcal{E}_2 \left( \frac{4}{\varepsilon} (\nu + \varepsilon) \right)
\]

in \( (\mathcal{M}_p, \| \cdot \|_*) \).

**Proof.** We know from Lemma 4.2 that \( (X | A_\varepsilon) \propto \mathcal{E}_2 \) and from Theorem 3.2 that (here \( \kappa = \nu + \frac{\varepsilon}{2} \leq O(1) \), \( \sigma = 1/\sqrt{n} \) and \( p = O(n) \)):

\[
\text{Under } A_\varepsilon: \quad \left( \frac{1}{n} XX^T \right)^m \in \mathcal{E}_2 \left( \left( \nu + \frac{\varepsilon}{2} \right)^m \sqrt{m} \right)
\]

in \( (\mathcal{M}_p, \| \cdot \|_*) \).

Let us then note that \( \left( \nu + \frac{\varepsilon}{2} \right)^m \sqrt{m} = O \left( \left( \nu + \frac{3\varepsilon}{4} \right)^m \right) \) and for \( z \in \mathbb{C} \) satisfying our hypotheses: \( (\nu + \frac{3\varepsilon}{4})/|z| \leq 1 - \frac{\varepsilon}{2(\nu + \varepsilon)} \). We can then deduce from Corollary 2 that under \( A_\varepsilon \):

\[
Q^z = \frac{1}{z} \left( I_p - \frac{1}{2n} XX^T \right)^{-1} = \frac{1}{z} \sum_{i=1}^{\infty} \left( \frac{1}{2n} XX^T \right)^i \in \mathcal{E}_2 \left( \frac{4}{\varepsilon} (\nu + \varepsilon) \right).
\]

Let us now try to study the concentration of \( Q^z \) when \( z \) gets close to the spectrum, for that we now require \( \varepsilon > 0 \) to be a constant (\( \varepsilon \geq O(1) \)).

**Proposition 4.2.** Given \( \varepsilon \geq O(1) \), for all \( z \in \mathbb{C} \setminus \mathcal{S}_\varepsilon \):

\[
(Q^z | A_\varepsilon) \in \mathcal{E}_2 \quad \text{in } (\mathcal{M}_p, \| \cdot \|_*),
\]

and we recall that there exist two constants \( C, c > 0 \) such that \( \mathbb{P}(A_\varepsilon^c) \leq Ce^{-cn} \).

**Proof.** Proposition 4.1 already set the result for \( |z| \geq \nu + \varepsilon \equiv \rho \), therefore, let us now suppose that \( |z| \leq \rho \).
With the notation $|Q^z|^2 = \left(\Im(z)^2 + (\Re(z) - \frac{1}{n}XX^T)^2\right)^{-1}$, let us decompose:

$$Q^z = \left(\Re(z) - \frac{1}{n}XX^T\right)|Q^z|^2 - \Im(z)|Q^z|^2.$$  \hfill (4.1)

We can then deduce the linear concentration of $|Q^z|^2$ with the same justifications as previously thanks to the Taylor decomposition:

$$|Q^z|^2 = \frac{1}{\rho^2} \sum_{m=0}^{\infty} 1 - \left(\frac{\Im(z)}{\rho^2} - (\Re(z) - \frac{1}{n}XX^T)^2\right)^m.$$  

Indeed, $\|\Re(z)I_p - \frac{1}{n}XX^T\| \leq d(\Re(z), S)$ and $d(z, S)^2 = \Im(z)^2 + d(\Re(z), S)^2 \leq \rho$

thus:

$$\left\|1 - \frac{\Im(z)^2}{\rho^2} - \frac{1}{\rho^2} (\Re(z)I_p - \frac{1}{n}XX^T)^2\right\| \leq 1 - \frac{d(z, S)^2}{\rho^2} \leq 1 - \frac{\varepsilon^2}{\rho^2} < 1.$$  

We therefore deduce from (4.1) that:

$$(Q^z | \mathcal{A}_z) \in \mathcal{E}_2 \left(\frac{2}{\varepsilon} \left(\Im(z) + |\Re(z)| + \nu + \frac{\varepsilon}{2}\right)\right) = \mathcal{E}_2. \quad \square$$

For the sake of completeness, we left in the appendix an alternative laborious proof (but somehow more direct) already presented in [LC19].

### 4.3. Computable deterministic equivalent

We are going to look for a deterministic equivalent of $Q$. We mainly follow the lines of [LC21b], we thus allow ourselves to present the justifications rather succinctly. Although Proposition 4.2 gives us a concentration of $Q^z$ in nuclear norm, we will provide a deterministic equivalent for the Frobenius norm with a better observable diameter. For any $z \in \mathbb{C} \setminus \mathcal{S}^z$, let us introduce $\tilde{\Lambda}^z = (\Tr(\Sigma_i E[Q^z]))_{i \in [n]}$ and recall that for any $\delta \in \mathbb{C}^n$, we note $\tilde{Q}_\delta^z = (zI_p - \frac{1}{n}\sum_{i=1}^n \frac{\Sigma_i}{1-\delta_i})^z$. We have the following first approximation to $E[Q^z]$:

**Proposition 4.3.** For any $z \in \mathbb{C} \setminus \mathcal{S}^z$:

$$\|\tilde{Q}_\delta^z\| \leq O(1) \quad \text{and} \quad \|E[Q^z] - \tilde{Q}_\delta^z\|_F \leq O \left(\frac{1}{\sqrt{n}}\right).$$

To prove this proposition, we will play on the dependence of $Q^z$ towards $x_i$ with the notation $X_{-i} \equiv (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in \mathcal{M}_{p,n}$ and:

$$Q_{-i}^z \equiv \left(zI_p - \frac{1}{n}X_{-i}X_{-i}^T\right)^{-1}.$$  

To link $Q^z$ to $Q_{-i}^z$, we will extensively use a direct application of the Schur identity:

$$Q^zx_i = \frac{Q_{-i}^zx_i}{1 - \frac{1}{n}x_i^TQ_{-i}^zx_i}. \quad (4.2)$$
Proof. All the estimations hold under $\mathcal{A}_e$, therefore the expectation should also be taken under $\mathcal{A}_e$ to be fully rigorous. Note that if $Q_{-i}$ and $x_i$ are independent on the whole universe, they are no more independent under $\mathcal{A}_e$. However, since the probability of $\mathcal{A}_e$ is overwhelming, the correction terms are negligible, we thus allow ourselves to abusively expel from this proof the independence and approximation issues related to $\mathcal{A}_e$, a rigorous justification is provided in [LC21b].

Let us bound for any deterministic matrix $A \in \mathcal{M}_p$ such that $\|A\|_F \leq 1$:

$$\left| \text{Tr} \left( A \left( \mathbb{E}[Q^-] - \tilde{Q}_{\Lambda}^x \right) \right) \right| \leq \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \text{Tr} \left( A \left( Q^-_{-i} \left( \frac{\Sigma_i}{1+\bar{\Lambda}^i} - x_i x_i^T \right) \tilde{Q}_{\Lambda}^x \right) \right) \right] \right| .$$

We can then develop with (4.2):

$$\left| \text{Tr} \left( A \mathbb{E}[Q^-] - \tilde{Q}_{\Lambda}^x \right) \right| \leq \frac{1}{n} \sum_{i=1}^{n} \left| \text{Tr} \left( A \mathbb{E} [Q^- - Q^-_{-i}] \Sigma_i \tilde{Q}_{\Lambda}^x \right) \right|$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \text{Tr} \left( A \left( Q^-_{-i} \left( \frac{\Sigma_i}{1+\bar{\Lambda}^i} - \frac{x_i x_i^T}{1+\frac{1}{n} x_i^T Q_{-i} x_i} \right) \tilde{Q}_{\Lambda}^x \right) \right) \right] \right|$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E} \left[ x_i^T \tilde{Q}_{\Lambda}^x A Q^-_{x_i} \left( \frac{1}{n} x_i^T Q_{-i} x_i - \bar{\Lambda}^i \right) \right] \right| + O \left( \frac{\|	ilde{Q}_{\Lambda}^x\|}{\sqrt{n}} \right),$$

thanks to Lemma [4.5] and the independence between $Q_{-i}$ and $x_i$. We can then bound thanks to Hölder inequality and Lemma [4.6] below:

$$\left| \mathbb{E} \left[ x_i^T \tilde{Q}_{\Lambda}^x A Q^-_{x_i} \left( \frac{1}{n} x_i^T Q_{-i} x_i - \bar{\Lambda}^i \right) \right] \right|$$

$$= \left| \mathbb{E} \left[ x_i^T \tilde{Q}_{\Lambda}^x A Q^-_{x_i} - \mathbb{E} \left[ x_i^T \tilde{Q}_{\Lambda}^x A Q^-_{x_i} \right] \left( \frac{1}{n} x_i^T Q_{-i} x_i - \mathbb{E} \left[ \frac{1}{n} x_i^T Q_{-i} x_i \right] \right) \right] \right|$$

$$\leq \sqrt{\mathbb{E} \left[ \left( x_i^T A Q_{\Lambda}^x_{-i} x_i - \mathbb{E} \left[ x_i^T A Q_{\Lambda}^x_{-i} x_i \right] \right)^2 \right]} + \mathbb{E} \left[ \left( \frac{1}{1-\frac{1}{n} x_i^T Q_{-i} x_i} - \mathbb{E} \left[ \frac{1}{1-\frac{1}{n} x_i^T Q_{-i} x_i} \right] \right)^2 \right]^{1/2}$$

$$\leq O \left( \frac{1}{\sqrt{n}} \right) \left( \mathbb{E} \left[ \left( \frac{1}{1-\frac{1}{n} x_i^T Q_{-i} x_i} - \mathbb{E} \left[ \frac{1}{1-\frac{1}{n} x_i^T Q_{-i} x_i} \right] \right)^2 \right] \right)^{1/2}$$

indeed since we know that $\left| \frac{1}{1-\frac{1}{n} x_i^T Q_{-i} x_i} \right| \leq O(1)$ from Lemma [4.3] $\frac{1}{1-\frac{1}{n} x_i^T Q_{-i} x_i}$ is a $O(1)$-Lipschitz transformation of $\frac{1}{n} x_i^T Q_{-i} x_i$, therefore, it follows the same concen-
tration inequality (with a variance of order $O(1/n)$). Since this inequality is true for any $A \in \mathcal{M}_p$, we can bound:

$$\left\| \hat{Q}_{\lambda^*} \right\| \leq \left\| \hat{Q}_{\lambda^*} - E[Q^+] \right\|_F + \|E[Q^+]\| \leq O\left( \frac{\left\| \hat{Q}_{\lambda^*} \right\|}{\sqrt{n}} \right) + O(1),$$

which directly implies that $\left\| \hat{Q}_{\lambda^*} \right\| \leq O(1)$ and $\left\| E[Q^+] - \hat{Q}_{\lambda^*} \right\|_F \leq O(1/\sqrt{n})$. 

**Lemma 4.3** ([LC21b], Lemmas 4., 8.). $\forall z \in \bar{S}^c$, under $\mathcal{A}^c$:

$$\|Q^z\| \leq \frac{2}{\epsilon} \quad \text{and} \quad \sup_{i \in [n]} \left| \frac{1}{1 - 1/n^2 Q_{-i}x_i} \right| \leq O(1)$$

**Lemma 4.4.** For any $z \in \mathbb{C} \setminus \bar{S}^c$, any $i \in [n]$ and any $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:

$$(u^T Q^z_{-i}x_i \mid \mathcal{A}_c), (u^T Q^z_{-i}x_i \mid \mathcal{A}_c) \in O(1) \pm \mathcal{E}_2.$$

**Proof.** We do not care about the independence issues brought by $\mathcal{A}_c$. Let us simply bound for any $t > 0$ and under $\mathcal{A}_c$:

$$P\left( |u^T Q^z_{-i}x_i - E[Q^z_{-i}x_i]| \geq t \right) \leq P\left( |u^T Q^z_{-i}(x_i - \mu_i)| \geq \frac{t}{2} \right) + P\left( |u^T (Q^z_{-i} - E[Q^z_{-i}]) \mu_i| \geq \frac{t}{2} \right) \leq E\left[ Ce^{-c'nt^2/\|Q_{-i}\|^2} \right] + Cc' e^{-c'nt^2} \leq 2C e^{-c'nt^2},$$

for some constants $C, c, c' > 0$. Besides, we can bound:

$$|E[Q^z_{-i}x_i]| = |u^T E[Q^z_{-i}] \mu_i| \leq O(1),$$

thanks to Lemma 4.3 and Assumption 4.4.

The concentration of $u^T Q^z x_i$ is a consequence of the concentration $QX \in \mathcal{E}_2$ that can be shown thanks to Corollary 2 as in the proof of Proposition 4.2. We are then left to bounding $E[u^T Q^z x_i]$. For this purpose, let us write:

$$|E[u^T Q^z x_i]| = |E[u^T Q^z_{-i}x_i] - E\left[ (u^T Q^z_{-i}x_i) \left( \frac{1}{n} x_i^T Q^z x_i \right) \right]| \leq O(1) + O\left( \sqrt{E\left[ (u^T Q^z_{-i}x_i)^2 \right] E\left[ (\frac{1}{n} x_i^T Q^z x_i)^2 \right]} \right) \leq O(1),$$

thanks to Cauchy-Schwarz inequality Lemma 4.3 and the bound on $x_i$, valid under $\mathcal{A}_c$.

**Lemma 4.5.** Under $\mathcal{A}_c$, for any $z \in \mathbb{C} \setminus \bar{S}^c$ and any $i \in [n]$: $\|E[q^z - Q^z_{-i}]\| \leq O\left( \frac{1}{n} \right).$
Proof. For any $u \in \mathbb{R}^p$, we can bound thanks to Lemma 4.4

$$|u^T \mathbb{E}[Q^z - Q^z_{-1}]u| \leq \frac{1}{n} |\mathbb{E}[u^T Q^z x_i x_i^T Q^z_{-1}u]|$$

$$\leq \frac{1}{n} \sqrt{\mathbb{E}[(u^T Q^z x_i)^2] \mathbb{E}[(x_i^T Q^z_{-1}u)^2]} \leq O\left(\frac{1}{n}\right). \quad \blacksquare$$

Lemma 4.6. For any $z \in \mathbb{C} \setminus \tilde{S}_z$ deterministic matrix $A \in \mathcal{M}_p$:

$$(x_i^T A Q^z_{-1} x_i \mid A \in \mathcal{T}) \in \text{Tr}(\Sigma_i A \mathbb{E}[Q^z]) \pm \mathcal{E}_2(||A||_F) + \mathcal{E}_1(||A||).$$

Proof. Once again, without referring to $A \in \mathcal{T}$, we assume that $||X|| \leq O(1)$ and $||Q^z|| \leq O(1)$. Given $i \in [n]$, since we know from Lemma 4.5 that $||\mathbb{E}[Q^z - Q^z_{-1}]|| \leq O(1/\sqrt{n})$, we want to bound:

$$|x_i^T A Q^z_{-1} x_i - \text{Tr}(\Sigma_i A \mathbb{E}[Q^z_{-1}])| \leq |x_i^T A Q^z_{-1} x_i - \text{Tr}(\Sigma_i A Q^z_{-1})| + |\text{Tr}(\Sigma_i A(Q^z_{-1} - \mathbb{E}[Q^z_{-1}]))|.$$

Now we know that, for $X \in \mathcal{M}_p$, fixed, we can bound thanks to Theorem 3.3

$$\mathbb{P}\left(|x_i^T A Q^z_{-1} x_i - \text{Tr}(\Sigma_i A \mathbb{E}[Q^z_{-1}])| \geq t\right) \leq \mathbb{E}\left[C e^{-c/(||Q^z_{-1}||_F)^2 + C e^{-c||A||}}\right]$$

$$\leq C e^{-c t^2/||A||^2} + C e^{-c t/||A||},$$

for some constants $C, c, c' > 0$, thanks to Lemma 4.6.

Besides, we know from Proposition 1.2 and Lemma 1.1 that $Q^z_{-1} \in \mathbb{E}[Q^z] \pm \mathcal{E}_2(1/\sqrt{n})$ in $(\mathcal{M}_p, ||\cdot||_F)$, which allows us to bound:

$$\mathbb{P}\left(|\text{Tr}(\Sigma_i A Q^z_{-1}) - \text{Tr}(\Sigma_i A \mathbb{E}[Q^z])| \geq t\right) \leq C e^{-c t^2/||A||^2},$$

for some constants $C, c > 0$, since $||\Sigma_i|| \leq O(1)$. Putting the two concentration inequalities together, we obtain the result of the lemma. \hfill \blacksquare

Theorem 0.1 is then a consequence of the following proposition proven in [LC21b] (once Proposition 4.3 is proven, the convex concentration particularities do not intervene anymore). Recall that $\tilde{A}^z \in \mathbb{C}^n$ is defined as the unique solution to the equation:

$$\forall i \in [n] : \tilde{A}^z_i = \frac{1}{n} \text{Tr}\left(\Sigma_i \tilde{Q}_{\tilde{A}^z_i}\right),$$

where $\tilde{Q}_{\tilde{A}^z_i} = (z I_p - \frac{1}{n} \sum_{i=1}^n \Sigma_i \tilde{A}^z_i)$.

**Proposition 4.4.** For all $z \in \mathbb{C} \setminus \tilde{S}_z$:

$$\left\|\mathbb{E}[Q] - \tilde{Q}_{\tilde{A}^z}\right\|_{F} \leq O\left(\frac{1}{\sqrt{n}}\right).$$
Appendix A. Proofs of the concentration of products of convexly
concentrated random vectors and of convexly
concentrated random matrices

We will use several time the following elementary result:

Lemma Appendix A.1. Given a convex mapping $f : \mathbb{R} \to \mathbb{R}$, and a vector $a \in \mathbb{R}^n$, the mapping $F : \mathbb{R}^p \ni (z_1, \ldots, z_p) \mapsto \sum_{i=1}^p a_i f(z_i) \in \mathbb{R}$ is convex (so in particular quasi-convex).

To efficiently manage the concentration rate when multiplying a large number of random vectors, we will also need:

Lemma Appendix A.2. Given $m$ commutative or non commutative variables $a_1, \ldots, a_m$ of a given algebra, we have the identity:

$$\sum_{\sigma \in \mathfrak{S}_m} a_{\sigma(1)} \cdots a_{\sigma(m)} = (-1)^m \sum_{I \subseteq [m]} (-1)^{|I|} \left( \sum_{i \in I} a_i \right)^m,$$

where $|I|$ is the cardinality of $I$.

Proof. The idea is to inverse the identity:

$$(a_1 + \cdots + a_m)^m = \sum_{J \subseteq I} \sum_{\{i_1, \ldots, i_m\} = J} a_{i_1} \cdots a_{i_m},$$

thanks to the Rota formula (see [Rol06]) that sets for any mappings $f, g$ defined on the set subsets of $\mathbb{N}$ and having values in a commutative group (for the sum):

$$\forall I \subseteq \mathbb{N}, f(I) = \sum_{J \subseteq I} g(J) \iff \forall I \subseteq \mathbb{N}, g(I) = \sum_{J \subseteq I} \mu_{\mathcal{P}(\mathbb{N})}(J, I) f(J),$$

where $\mu_{\mathcal{P}(\mathbb{N})}(J, I) = (-1)^{|I \setminus J|}$ is an analog of the Mo"ebius function for the order relation induced by the inclusions in $\mathcal{P}(\mathbb{N})$. In our case, for any $J \subseteq [m]$, if we set:

$$f(J) = \left( \sum_{i \in J} a_i \right)^m \quad \text{and} \quad g(J) = \sum_{\{i_1, \ldots, i_m\} = J} a_{i_1} \cdots a_{i_m},$$

we see that for any $I \subseteq [m]$, $f(I) = \sum_{J \subseteq I} g(J)$, therefore taking the Rota formula in the case $I = [m]$, we obtain the result of the Lemma (in that case, $\mu_{\mathcal{P}(\mathbb{N})}(J, I) = (-1)^{m-|I|}$ and $\sum_{\{i_1, \ldots, i_m\} = J} a_{i_1} \cdots a_{i_m} = \sum_{\sigma \in \mathfrak{S}_m} a_{\sigma(1)} \cdots a_{\sigma(m)}$).

Proof. [Proof of Theorem 3.1] Let us first assume that all the $X_i$ are equal to a vector $Z \in \mathbb{R}^p$. Considering $a = (a_1, \ldots, a_p) \in \mathbb{R}^p$, we want to show the concentration of $a^T Z \circ m = \sum_{i=1}^p a_i z_i^m$ where $z_1, \ldots, z_p$ are the entries of $Z$.

The mapping $p_m : x \mapsto x^m$ is not quasi-convex when $m$ is odd, therefore, in that case we decompose it into the difference of two convex mappings $p_m(z) = p_m^+(z) - p_m^-(z)$ where:

$$p_m^+ : z \mapsto \max(z^m, 0) \quad \text{and} \quad p_m^- : z \mapsto -\min(z^m, 0), \quad (A.1)$$

where $p_m^+$ is the positive part of $p_m$ and $p_m^-$ is the negative part. The mapping $p_m$ is then the difference of two convex mappings $p_m = p_m^+ - p_m^-$.
(say that, if $m$ is even, then we set $p_m^+ = p_m$ and $p_m^- : z \mapsto 0$). For the same reasons, we decompose $\phi^+_a : z \mapsto a^T p_m^+(z)$ and $\phi^-_a : z \mapsto a^T p_m^-(z)$ into:

$$\phi^+_a = \phi^+_{|a|} - \phi^+_{|a|-a}$$
$$\phi^-_a = \phi^-_{|a|} - \phi^-_{|a|-a}$$

(for $|a| = (|a_i|)_{1 \leq i \leq p}$), so that:

$$a^T Z^\otimes m = \phi^+_{|a|}(Z) - \phi^+_{|a|-a}(Z) - \phi^-_{|a|}(Z) + \phi^-_{|a|-a}(Z)$$

becomes a combination of quasi-convex functionals of $Z$ and the same holds for $\phi^+(z)$ for $|z| = (|z_i|)_{1 \leq i \leq p}$.

Let us give some definitions. For the concentration of the matrix product, we introduce a new notion of transversal convex concentration, namely the

Definition Appendix A.2.

Let us exploit Proposition 2.1 to obtain:

$$\left( \sum_{i \in I} X_i \right)^\otimes m \in \mathcal{E}_2(m^m \kappa^{-1} \sigma)$$

(note that $\#\{I \subset [m]\} = 2^m$) Thus summing the $2^m$ concentration inequalities, we can conclude from Equation (A.2), and the Stirling formula $\frac{m^n}{m!} = \frac{e^n}{\sqrt{2\pi n}} + O(1)$ that:

$$X_1 \otimes \cdots \otimes X_m \in \mathcal{E}_2((2\kappa)^{m-1} \sigma)$$

For the concentration of the matrix product, we introduce a new notion of concentration, namely the transversal convex concentration. Let us give some definitions.

Definition Appendix A.1.

Given a sequence of normed vector spaces $(E_n, \| \cdot \|_n)_{n \geq 0}$, a sequence of groups $(G_n)_{n \geq 0}$, each $G_n$ (for $n \in \mathbb{N}$) acting on $E_n$, a sequence of random vectors $(Z_n)_{n \geq 0} \in \prod_{n \geq 0} E_n$, a sequence of positive reals $(\sigma_n)_{n \geq 0} \in \mathbb{R}_{+}^n$, we say that $Z = (Z_n)_{n \geq 0}$ is convexly concentrated transversally to the action of $G$ with an observable diameter of order $\sigma$ and we note $Z \sim_T \mathcal{E}_2(\sigma)$ if there exist two constants $C, c \leq O(1)$
such that $\forall n \in \mathbb{N}$ and for any 1-Lipschitz, quasi-convex and $G$-invariant function $f : E_n \to \mathbb{R}$, $\forall t > 0$:

$$
\mathbb{P} \left( |f(Z_n) - \mathbb{E}[f(Z_n)]| \geq t \right) \leq C e^{-c(t/\sigma_n)^2}.
$$

**Remark Appendix A.1.** Given a normed vector space $(E, \|\cdot\|)$, a group $G$ acting on $E$ and a random vector $Z \in E$, we have the implication chain:

$$
Z \propto \mathcal{E}_2(\sigma) \implies Z \propto_c \mathcal{E}_2(\sigma) \implies Z \propto_T^G \mathcal{E}_2(\sigma).
$$

Considering the actions:

1. $S_n$ on $\mathbb{R}^p$ where for $\sigma \in S_n$ and $x \in \mathbb{R}^p$, $\sigma \cdot x = (x_{\sigma(i)})_{1 \leq i \leq p}$,
2. $O_{p,n} \equiv O_p \times O_n$ on $\mathcal{M}_{p,n}$ where for $(P, Q) \in O_{p,n}$ and $M \in \mathcal{M}_{p,n}$, $(P, Q) \cdot M = PMQ$,

the convex concentration in $\mathcal{M}_{p,n}$ transversally to $O_{p,n}$ can be expressed as a concentration on $\mathbb{R}^p$ transversally to $S_n$ thanks to the introduction the mapping $\sigma$ providing to any matrix the ordered sequence of its singular values:

$$
\sigma : \mathcal{M}_{p,n} \to \mathbb{R}^d_+ \quad \text{with} \quad d = \min(p, n) \quad M \mapsto (\sigma_1(M), \ldots, \sigma_d(M)).
$$

(there exists $(P, Q) \in O_{p,n}$ such that $M = P\Sigma(M)Q$, where $\Sigma \in \mathcal{M}_{p,n}$ has $\sigma_1(M) \geq \cdots \geq \sigma_d(M)$ on the diagonal).

**Theorem Appendix A.1 ([Led05], Corollary 8.23. [LC19], Theorem 2.44).**

Given a random matrix $Z \in \mathcal{M}_{p,n}$:

$$
Z \propto_{O_{p,n}}^T \mathcal{E}_2(\sigma) \iff \sigma(Z) \propto_{S_n}^T \mathcal{E}_2(\sigma)
$$

(where the concentrations inequalities are implicitly expressed for euclidean norms: $\|\cdot\|_F$ on $\mathcal{M}_{p,n}$ and $\|\cdot\|$ on $\mathbb{R}^d$).

**Proof.** [Proof of Theorem 3.2] Let us start to study the case where $X_1 = \cdots = X_m \equiv X \in \mathcal{M}_n$ and $X \propto \mathcal{E}_2$ in $(\mathcal{M}_n, \|\cdot\|_F)$. We know from Theorem Appendix A.1 that:

$$
\sigma(X) \propto_{S_n}^T \mathcal{E}_2,
$$

and therefore, as a $\sqrt{n}$-Lipschitz linear observation of $\sigma(X)^{\otimes m} \in \mathcal{E}_2 (\kappa^{m-1} \sigma)$ (see Theorem 3.1), $\text{Tr}(X^m)$ follows the concentration:

$$
\text{Tr}(X^m) = \sum_{i=1}^{m} \sigma_i(X)^m \in \mathcal{E}_2 \left( \sqrt{n} \kappa^{m-1} \sigma \right).
$$

Now, we consider the general setting where we are given $m$ matrices $X_1, \ldots, X_m$, a deterministic matrix $A \in \mathcal{M}_{n_a,n_0}$ satisfying $\|A\| \leq 1$, and we want to show the

\[\text{For any } g \in G \text{ and } x \in E, f(x) = f(g \cdot x)\]

\[\text{Once again, we point out that one could have replaced here } \mathbb{E}[f(Z_n)] \text{ by } f(Z'_n) \text{ or } mf.\]
concentration of \( tr(AX_1, \ldots, X_m) \). First note that we stay in the hypotheses of the theorem if we replace \( X_1 \) with \( AX_1 \), we are thus left to show the concentration of \( \text{Tr}(X_1 \cdot \cdots \cdot X_m) \). We can not employ again Lemma [Appendix A.2](#) without a strong hypothesis of commutativity on the matrices \( X_1, \ldots, X_n \). Indeed, one could not have gone further than a concentration on the whole term \( \sum_{\sigma \in \mathcal{S}_n} \text{Tr}(X_{\sigma(1)} \cdots X_{\sigma(m)}) \).

However, we can still introduce the random matrix

\[
Y = \begin{pmatrix}
0 & X_{m-1} & \cdots & \cdots & X_1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
X_m & & \cdots & \cdots & 0
\end{pmatrix}
\]

then

\[
Y^m = \begin{pmatrix}
0 & X_1^m & \cdots & \cdots & X_m^m \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
X_1 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\]

where for \( i, j \in \{2, \ldots, m - 1\} \), \( X_i^j = X_iX_{i+1} \cdots X_mX_1 \cdots X_j \). Since \( Y \in M_{n_0+\cdots+n_m} \) satisfies \( Y \propto \mathcal{E}_2(\sigma) \) and \( \|Y\| \leq \kappa \), the first part of the proof provides the concentration \( Y^m \in \mathcal{E}_2(\kappa^{m-1}d\sqrt{n_0 + \cdots + n_m}) \) in \( (\mathcal{M}_n, \|\|_*) \) which directly implies the concentration of \( X_1^m = X_1 \cdots X_m \).

### Appendix B. Alternative proof of Proposition [B.2](#)

We are going to show the concentration of the real part and the imaginary part of \( Q^z \), where:

\[
\Re(Q^z) = Q^z \left( \Re(z)I_p - \frac{1}{n}XX^T \right) \bar{Q}^z = (\Re(z) - z)|Q^z|^2 + \bar{Q}^z
\]

\[
\Im(Q^z) = \Im(z)|Q^z|^2
\]

Since it is harder, we will only prove the linear concentration of \( |Q^z|^2 = (\Re(z)^2 + (\Re(z) - \frac{1}{n}XX^T)^2)^{-1} \). For that we are going to decompose, for any matrix \( A \in \mathcal{M}_p \) with unit spectral norm, the random variable \( \text{Tr}(A|Q^z|^2) \) as the sum of convex and \( O(1/\sqrt{n}) \)-Lipschitz mappings of \( X \). Let us introduce the two mappings, \( \psi : \mathcal{M}_p \to \mathcal{M}_p \) and \( \phi : \mathcal{M}_{p,n} \to \mathcal{M}_p \), defined for any \( M \in \mathcal{M}_p \) and \( B \in \mathcal{M}_{p,n} \) with:

\[
\psi(M) = (\Re(z)^2 + M)^{-1} \quad \phi(B) = \Re(z)^2 - \frac{2\Re(z)}{n}BB^T + \frac{1}{n^2}BB^TB^TB.
\]

We then have the identity \( \text{Tr}(AQ^z) = \text{Tr}(A\psi \circ \phi(X)) \).

We then then look at the second derivative of \( \psi \circ \phi \) to prove convex properties on \( \text{Tr}(A\psi \circ \phi) \). Given \( H \in \mathcal{M}_p \), let us compute:

\[
d\psi|_M \cdot H = -\phi(M)H\phi(M) \quad d^2\psi|_M \cdot (H, H) = 2\phi(M)H\phi(M)H\phi(M),
\]

and given \( K \in \mathcal{M}_{p,n} \):

\[
d\phi|_B \cdot K = -\frac{2\Re(z)}{n}L(B, K) + \frac{1}{n^2}P(K, B)
\]

\[
d^2\phi|_B \cdot (K, K) = -\frac{2\Re(z)}{n}KK^T + \frac{2}{n^2}P_2(K, B),
\]

where:
\begin{itemize}
  \item $L(B, K) = BK^T + KBT$
  \item $P(B, K) = KB^TBB^T + BKTBT + BB^TKB^T + BTKB^T$
  \item $P_2(B, K) = KK^TBB^T + BKTKB^T + BKB^T + BK^T + BB^T + BB^T$.
\end{itemize}

First we deduce from the expression of the first derivative and thanks to Lemma \ref{gromov} that, on $X(A)$, $\text{Tr}(A \psi \circ \phi)$ is a $O(||A||_F/\sqrt{n}) = O(1)$-Lipschitz transformation of $X$ (for the Frobenius norm).

Second, choosing $M = \phi(B)$:
\[
d^2 \psi \circ \phi|_B \cdot (K, K) = d^2 \psi|_{MM} \cdot (d\phi_B \cdot K, d\phi_B \cdot K) = 2\phi(M) (d\phi_B \cdot K) \phi(M) (d\phi_B \cdot K) \phi(M) + \frac{2\Re(z)}{n} \phi(M) K^TK^T \phi(M) - \frac{2}{n^2} \phi(M) P_2(K, B) \phi(M).
\]

In this identity the only term raising an issue is $\frac{2}{n^2} \phi(M) P_2(K, B) \phi(M)$ because $P_2(K, B)$ is not nonnegative symmetric. We can however still bound:
\[
\frac{12}{n^2} \text{Tr}(A \phi(M) P_2(K, B) \phi(M)) \leq \frac{12}{n^2} ||A||_F ||\phi(M)||^2 ||B||^2 ||K||^2 \leq O \left( \frac{1}{n} \text{Tr}(KKT^T) \right),
\]

for $B \in X(AQ)$ (in particular $||B|| \leq O(\sqrt{n})$ and $||\phi(M)|| \leq O(1)$). Now, if we note $h : \mathcal{M}_{p,n} \to \mathbb{R}$ defined for any $B \in \mathcal{M}_{p,n}$ as $h(B) = \frac{1}{n} \text{Tr}(BB^T)$, we see that $\frac{1}{n} \text{Tr}(KKT^T) = d^2 h(B) \cdot (K, K)$ is a quadratic functional on $K$, $h$ is thus convex. It is besides $O(1)$-Lipschitz on $X(AQ)$ (for the Frobenius norm). Assuming in a first time that $A$ is nonnegative symmetric and choosing a constant $C \leq O(1)$ sufficiently big, we show that $B \mapsto \text{Tr}(A \psi \circ \phi(B)) + Ch(B)$ is convex and $O(1)$-Lipschitz on $X(AQ)$ like $Ch$. We have thus the concentration:
\[
\left( \text{Tr}(A|Q|^2) \mid A \right) \in \mathcal{E}_2.
\]

Now, given a general matrix $A \in \mathcal{M}_p$, we decompose $A = A_+ - A_- + A_0$ where $A_+$ and $A_-$ are nonnegative symmetric and $A_0$ is anti-symmetric, in that case $\text{Tr}(A|Q|^2) = \text{Tr}(A_+|Q|^2) - \text{Tr}(A_-|Q|^2)$, and we can conclude the same way. That eventually gives us the concentration of the proposition.

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