Weak convergence of path-dependent SDEs with irregular coefficients

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Abstract

In this paper we develop via Girsanov’s transformation a perturbation argument to investigate weak convergence of Euler-Maruyama (EM) scheme for path-dependent SDEs with Hölder continuous drifts. This approach is available to other scenarios, e.g., truncated EM schemes for non-degenerate SDEs with finite memory or infinite memory. Also, such trick can be applied to study weak convergence of truncated EM scheme for a range of stochastic Hamiltonian systems with irregular coefficients and with memory, which are typical degenerate dynamical systems. Moreover, the weak convergence of path-dependent SDEs under integrability condition is investigated by establishing, via the dimension-free Harnack inequality, exponential integrability of irregular drifts w.r.t. the invariant probability measure constructed explicitly in advance.

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1 Introduction

The strong/weak convergence of numerical schemes for SDEs with regular coefficients has been investigated extensively; see, e.g., [3, 10, 11, 8, 18, 20, 19, 23] and reference therein. Also, the weak convergence for SDEs with irregular terms has gained much attention; see, e.g., [7, 15] with the payoff function being smooth. For path-dependent SDEs (which, in terminology, are also named as functional SDEs or SDEs with delays), there is considerable literature on strong convergence of various numerical schemes (e.g., truncated/tamed EM scheme) under regular conditions; see, for instance, [7, 13] and references within. In contrast, weak convergence analysis of numerical algorithms for path-dependent SDEs is scarce. The path-dependent SDEs under

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irregular conditions are much more difficult than SDEs under irregular conditions. This work
deals with the weak approximation of numerical algorithms for path-dependent SDEs under
Hölder continuity condition or certain integrability condition.

As far as path-dependent SDEs are concerned, the weak convergence of numerical methods
was initiated in [?] whereas the rigorous justification of their statements was unavailable. With
regard to weak convergence of EM scheme and its variants, we refer to [3] for a class of semi-
linear path-dependent SDEs via the so-called “lift-up” approach, [6] for path-dependent SDEs
with distributed delays by means of the duality trick, and [4] for path-dependent SDEs with
point delays with the help of Malliavin calculus and the tamed Itô formula. In the references
[4] [6], as for the drift term \( b \) and the diffusion term \( \sigma \), the assumptions that \( b, \sigma \in C^\infty_b(\mathbb{R}^d) \) and
the payoff function \( f \in C^3_b(\mathbb{R}^d) \) were imposed. Subsequently, by the aid of Malliavin calculus,
[31] extended [4] [6] in a certain sense that the payoff function \( f \in \mathcal{B}_b(\mathbb{R}^d) \) while \( b, \sigma \in C^\infty_b(\mathbb{R}^d) \)
therein. It is worthy of pointing out that the approaches adopted in [4] [6] [31] are applicable
merely for path-dependent SDEs with regular coefficients. In the literature [4] [31], the tamed
Itô formula plays a crucial role in investigating weak convergence of EM scheme. Nevertheless,
the tamed Itô formula works barely for path-dependent SDEs with distributed delays or point
delays so that it seems hard to extend [4] [31] to path-dependent SDEs with general delays.

To study weak convergence of numerical schemes for path-independent SDEs with regular
coefficients, the approach on the Kolmogorov backward equation is one of the powerful methods.
However, concerning path-dependent SDEs, the Kolmogrov backward equation is in general
unavailable so that it cannot be adopted to deal with weak convergence of numerical schemes.
As we stated above, concerning path-dependent SDEs, the Malliavin calculus is the effective
tool to cope with weak convergence; see, for example, [4] [6] [31]. Whereas, a little bit strong
assumptions are imposed therein and the proof is not succinct in certain sense. Moreover,
Zvonkin’s trasformation [32] is one of the powerful tools to investigate strong convergence of
EM schemes for path-independent SDEs with singular coefficients; see, e.g., [10]. Nevertheless,
such trick no longer works for path-dependent SDEs provided the appearance of the delay terms.

In this work we aim to develop a perturbation approach to study weak convergence of
(truncated) EM scheme for path-dependent SDEs with additive noise, which allows the drift
terms to be irregular (e.g., Hölder continuous drifts and integrability drifts) and even the diffusion
coefficients to be degenerate. Elaborate estimation of the growth of a stochastic process under
Hölder continuity condition and the application of the dimension-free Harnack inequality under
integrability condition play the crucial role in current work.

The content of this paper is arranged as follows. In Section 2 we investigate weak con-
vergence of EM scheme for a class of non-degenerate SDEs with memory and reveal the weak
convergence rate; In Section 3 we apply the approach adopted in Section 2 to other scenarios,
e.g., truncated EM scheme for non-degenerate SDEs with finite memory or infinite memory;
In Section 4 we focus on weak convergence and reveal the weak convergence order of truncated EM
scheme for a range of stochastic Hamiltonian systems with singular drifts and with memory; In
the last section, we are interested in weak convergence of EM scheme for path-dependent SDEs
under integrability conditions, which allow the drift terms to be singular.

Before proceeding further, a few words about the notation are in order. Throughout this
paper, \( c > 0 \) stands for a generic constant which might change from occurrence to occurrence
and depend on the time parameters.
2 Weak Convergence: Non-degenerate Case

Let \((\mathbb{R}^d, \langle \cdot, \cdot \rangle, | \cdot |)\) be the \(d\)-dimensional Euclidean space with the inner product \(\langle \cdot, \cdot \rangle\) which induces the norm \(| \cdot |\). Let \(M^d_{\text{non}}\) be the set of all non-singular \(d \times d\)-matrices with real entries, equipped with the Hilbert-Schmidt norm \(| \cdot |_{\text{HS}}\). \(A^*\) means the transpose of the matrix \(A\). For a sub-interval \(U \subseteq \mathbb{R}\), denote \(C(U; \mathbb{R}^d)\) by the family of all continuous functions \(f : U \rightarrow \mathbb{R}^d\). Let \(\tau > 0\) be a fixed number and \(\mathcal{C} = C([-\tau, 0]; \mathbb{R}^d)\), which is endowed with the uniform norm \(|f|_\infty := \sup_{-\tau \leq \theta \leq 0} |f(\theta)|\). For \(f \in C([-\tau, \infty); \mathbb{R}^d)\) and fixed \(t \geq 0\), let \(f_t \in \mathcal{C}\) be defined by \(f_t(\theta) = f(t + \theta), \theta \in [-\tau, 0]\). In terminology, \((f_t)_{t \geq 0}\) is called the segment (or window) process corresponding to \((f(t))_{t \geq -\tau}\). For \(a \geq 0\), \([a]\) stipulates the integer part of \(a\). Let \(\mathcal{B}_h(\mathbb{R}^d)\) be the collection of all bounded measurable functions \(f : \mathbb{R}^d \rightarrow \mathbb{R}\), endowed with the norm \(|f|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)|\). Let \(\textbf{0} \in \mathbb{R}^d\) be the zero vector and \(\xi_0(\theta) \equiv 0\) for any \(\theta \in [-\tau, 0]\).

In this section, we are interested in the following path-dependent SDE

\[
\text{d}X(t) = \{b(X(t)) + Z(X_t)\} \text{d}t + \sigma \text{d}W(t), \quad t > 0, \quad X_0 = \xi \in \mathcal{C},
\]

where \(b : \mathbb{R}^d \rightarrow \mathbb{R}^d\), \(Z : \mathcal{C} \rightarrow \mathbb{R}^d\), \(\sigma \in M^d_{\text{non}}\) and \((W(t))_{t \geq 0}\) is a \(d\)-dimensional Brownian motion on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). We assume that

(A1) \(b\) is Lipschitz with the Lipschitz constant \(L_1\), i.e., \(|b(x) - b(y)| \leq L_1 |x - y|\), \(x, y \in \mathbb{R}^d\), and there exist constants \(C > 0\) and \(\beta \in \mathbb{R}\) such that

\[
2 \langle x, b(x) \rangle \leq C + \beta |x|^2, \quad x \in \mathbb{R}^d;
\]

(A2) \(Z\) is Hölder continuous with the Hölder exponent \(\alpha \in (0, 1]\) and the Hölder constant \(L_2\), i.e., \(|Z(\xi) - Z(\eta)| \leq L_2 \|\xi - \eta\|_\infty\), \(\xi, \eta \in \mathcal{C}\);

(A3) The initial value \(\xi \in \mathcal{C}\) is Lipschitz continuous with the Lipschitz constant \(L_3 > 0\), i.e., \(|\xi(t) - \xi(s)| \leq L_3 |t - s|\), \(s, t \in [-\tau, 0]\).

Under (A1) and (A2), (2.1) enjoys a unique weak solution \((X^\xi(t))_{t \geq 0}\) with the initial datum \(X_0^\xi = \xi \in \mathcal{C}\); see Lemma 2.2 below for more details. Evidently, (2.2) holds with \(\beta > 0\) whenever \(b\) obeys the global Lipschitz condition. It is worthy to emphasize that \(\beta\) in (2.2) need not to be positive, which may allow the time horizontal \(T\) to be much bigger as Lemma 2.3 below manifests. Moreover, (A3) is just imposed for the sake of continuity of the displacement of segment process. For further details, please refer to Lemma 2.4 below.

For existence and uniqueness of strong solutions to path-dependent SDEs with regular coefficients, we refer to e.g. [12, 16, 21] and references therein. Recently, path-dependant SDEs with irregular coefficients have also received much attention; see e.g. [11] on existence and uniqueness of strong solutions, [2] upon strong Feller property of the semigroup generated by the functional solution (i.e., the segment process associated with the solution process), and [25] about the regularity estimates for the density of invariant probability measures.

Let \(\delta \in (0, 1)\) be the stepsize given by \(\delta = \tau/M\) for some \(M \in \mathbb{N}\) sufficiently large. Given the stepsize \(\delta \in (0, 1)\), the continuous-time EM scheme associated with (2.1) is defined as below

\[
\text{d}X^{(\delta)}(t) = \{b(X^{(\delta)}(t)) + Z(\tilde{X}_{t/\delta}^{(\delta)})\} \text{d}t + \sigma \text{d}W(t), \quad t > 0
\]
with the initial value \( X^{(\delta)}(\theta) = X(\theta) = \xi(\theta), \theta \in [-\tau, 0] \). Herein, \( t_\delta := [t/\delta] \delta \) and, for any \( k \in \mathbb{N} \), \( \hat{X}_{k\delta}^{(\delta)} \in \mathcal{C} \) is defined by

\[
(2.4) \quad \hat{X}_{k\delta}^{(\delta)}(\theta) = \frac{\theta + (1 + i)\delta}{\delta} X^{(\delta)}((k - i)\delta) - \frac{\theta + i\delta}{\delta} X^{(\delta)}((k - i - 1)\delta)
\]

whenever \( \theta \in [-i + 1)\delta, -i\delta] \) for \( i \in \mathbb{S} := \{0, 1, \cdots, M - 1\} \), that is, the \( \mathcal{C} \)-valued process \( (\hat{X}_{k\delta}^{(\delta)})_{k \in \mathbb{N}} \) is constructed by the linear interpolations between the points on the gridpoints.

To cope with the weak convergence of EM scheme (2.3) with the singular coefficient \( Y \), we demonstrate that the weak convergence rate is \( \alpha \). Hence, Theorem 2.1 improves e.g. \([4, 6, 20, 31]\) in a certain sense. Nevertheless, in the present work, we might allow the drift \( Z \) to be unbounded and even Hölder continuous and most importantly the payoff function \( f \) to be non-smooth. Hence, Theorem 2.1 improves e.g. \([4, 6, 20, 31]\) in a certain sense. Last but not least, the approached adopted to prove Theorem 2.1 is universal in a sense that it is applicable to the other scenarios as show in the Sections 3 and 4.

To this goal, we introduce the following reference SDE on \( \mathbb{R}^d \)

\[
(2.5) \quad dY(t) = b(Y(t)) dt + \sigma dW(t), \quad t > 0, \quad Y(0) = x \in \mathbb{R}^d.
\]

Under (A1), (2.5) has a unique strong solution \( (Y^\varepsilon(t))_{t \geq 0} \) with the initial value \( Y(0) = x; \) see, for example, \([16, \text{Theorem 2.1, p34}]\). Now, let’s extend \( Y^\varepsilon(t) \), solving (2.5), from \([0, \infty)\) into \([-\tau, \infty)\) in the manner below:

\[
(2.6) \quad Y^\varepsilon(t) := \xi(t) 1_{[-\tau, 0]}(t) + Y^{\varepsilon(0)}(t) 1_{[0, \infty)}(t), \quad t \in [-\tau, \infty), \quad \xi \in \mathcal{C}.
\]

Let \( (Y^\varepsilon(t))_{t \geq 0} \) be the segment process corresponding to \( (Y^\varepsilon(t))_{t \geq -\tau} \).

Our main result in this section is stated as follows, which in particular reveals the weak convergence rate of EM algorithm (2.3) associated with (2.1), which allows the drift term to be Hölder continuous.

**Theorem 2.1.** Let (A1), (A2) and (A3) hold. Then, for any \( \kappa \in (0, \alpha/2) \) and \( T > 0 \) such that

\[
(2.7) \quad 2 \|\sigma\|^2_{HS} \|\sigma^{-1}\|^2_{HS} \{(4L_1^2 + L_2^2)1_{\{\alpha = 1\}} + L_1^2 1_{\{\alpha \in (0, 1)\}}\} < e^{-(1+\beta T)/T^2},
\]

there exists a constant \( C_{1,T} > 0 \) such that

\[
(2.8) \quad |\mathbb{E}f(X(t)) - \mathbb{E}f(X^{(\delta)}(t))| \leq C_{1,T} \delta^\kappa, \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad t \in [0, T].
\]

**Remark 2.1.** For the path-independent SDE (2.1) with Hölder continuous drift, \([20]\) revealed the weak convergence order is \( \frac{\alpha}{2} \wedge \frac{1}{4} \), where \( \alpha \in (0, 1) \) is the Hölder exponent. Whereas, in Theorem 2.1, we demonstrate that the weak convergence rate is \( \alpha/2 \). So Theorem 2.1 is new even for path-independent SDEs with irregular drifts. For path-dependent SDEs with point delays or distributed delays, \([4, 6]\) investigated the weak convergence under the regular assumption \( Z \in C_b^\infty \) and with the payoff function \( f \in C_b^\delta \). Nevertheless, in the present work, we might allow the drift term to be unbounded and even Hölder continuous and most importantly the payoff function \( f \) to be non-smooth. Hence, Theorem 2.1 improves e.g. \([4, 6, 20, 31]\) in a certain sense. Before we move forward to complete the proof of Theorem 2.1, let’s prepare some warm-up lemmas. The following lemma address existence and uniqueness of weak solutions to (2.1).

**Lemma 2.2.** Under (A1) and (A2), (2.1) admits a unique weak solution.
Proof. First of all, we show existence of a weak solution to (2.1). Set
\[ R_1^\xi(t) := \exp \left( \int_0^t \sigma^{-1} Z(Y_s^\xi), dW(s) - \frac{1}{2} \int_0^t |\sigma^{-1} Z(Y_s^\xi)|^2 ds \right), \quad t \geq 0, \]
and \( dQ_1^\xi := R_1^\xi(T)dP \), where \( T > 0 \) satisfies \( \|\sigma\|^2_{\text{HS}} \|\sigma^{-1}\|^2_{\text{HS}} L_s^2 < e^{-(1+\beta T)/T^2} \) for the setup of the Hölder exponent \( \alpha = 1 \) and \( T > 0 \) is arbitrary with \( \alpha \in (0, 1) \). Moreover, let
\[ W_1^\xi(t) = W(t) - \int_0^t \sigma^{-1} Z(Y_s^\xi) ds, \quad t \geq 0. \]
According to Lemma 2.3 below, we infer that
\[ \mathbb{E} e^{\frac{1}{2} \int_0^T |\sigma^{-1} Z(Y_s^\xi)|^2 ds} < \infty, \]
that is, the Novikov condition holds true. Thus the Girsanov theorem implies that \( (W_1^\xi(t))_{t \in [0,T]} \) is a Brownian motion under the weighted probability measure \( Q_1^\xi \). Note that (2.5) can be formulated as
\[ dY^\xi(t) = \{b(Y^\xi(t)) + Z(Y^\xi(t))\} dt + \sigma dW_1^\xi(t), \quad t \in [0,T], \quad Y_0^\xi = \xi. \]
So \( (Y^\xi(t), W_1^\xi(t))_{t \in [0,T]} \) is a weak solution to (2.1) under the probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q_1^\xi) \). Analogously, we can show inductively that (2.1) admits a weak solution on \([T,2T], [2T,3T], \cdots\). Hence, (2.1) admits a global weak solution.

Now we proceed to justify uniqueness of weak solutions to (2.1). In the sequel, it is sufficient to show the weak uniqueness on the time interval \([0,T]\) since it can be done analogously on \([T,2T], [2T,3T], \cdots\). Let \( (X^{(i),\xi}(t), W^{(i)}(t))_{t \in [0,T]} \) be the weak solution to (2.1) under the probability space \( (\Omega^{(i)}, \mathcal{F}^{(i)}, (\mathcal{F}^{(i)}_t)_{t \geq 0}, \mathbb{P}^\xi_i) \), \( i = 1,2 \). In terms of [21 Proposition 2.1, p169, & Corollary, p206], it remains to show that
\[ \mathbb{E}_{\mathbb{P}^\xi_i} f(X^{(1),\xi}([0,T]),W^{(1)}([0,T])) = \mathbb{E}_{\mathbb{P}^\xi_2} f(X^{(2),\xi}([0,T]),W^{(2)}([0,T])) \]
for any \( f \in C_b(C([0,T];\mathbb{R}^d) \times C([0,T];\mathbb{R}^d);\mathbb{R}) \), where \( \mathbb{E}_{\mathbb{P}^\xi} \) means the expectation w.r.t. \( \mathbb{P}^\xi \). Whereas (2.10) can be done exactly by following the argument of [25 Theorem 2.1 (2)]. We therefore complete the proof.

The lemma below examines the exponential integrability of functionals for segment process.

Lemma 2.3. Assume that (A1) holds. Then, for any \( T > 0 \),
\[ \mathbb{E} e^{\lambda \int_0^T \|Y_1^\xi\|_{\infty}^2 dt} < \infty, \quad \lambda < \frac{e^{-(1+\beta T)/T^2}}{2 \|\sigma\|^2_{\text{HS}} T^2}. \]

Proof. Applying Jensen’s inequality and using the fact that \( \|Y_1^\xi\|_{\infty} \leq \|\xi\|_{\infty} \vee \sup_{0 \leq s \leq t} |Y^\xi(s)| \), we have
\[ \mathbb{E} e^{\lambda \int_0^T \|Y_1^\xi\|_{\infty}^2 dt} \leq \frac{1}{T} \int_0^T \mathbb{E} e^{\lambda T \|Y_1^\xi\|_{\infty}^2} dt \leq \frac{e^{\lambda T \|\xi\|_{\infty}^2}}{T} \int_0^T \mathbb{E} \left( \sup_{0 \leq s \leq t} e^{\lambda T \|Y^\xi(0)(s)\|^2} \right) dt, \quad T > 0. \]
Next, by Itô’s formula, it follows from (A1) that
\begin{equation}
\begin{split}
d(e^{-\gamma t}|Y^{(0)}(t)|^2) &= e^{-\gamma t}\{-|Y^{(0)}(t)|^2 + 2\langle Y^{(0)}(t), b(Y^{(0)}(t)) \rangle + \|\sigma\|_{\text{HS}}^2\}dt \\
&\quad + 2e^{-\gamma t}\langle \sigma Y^{(0)}(t), dW(t) \rangle \\
&\leq e^{-\gamma t}\{c - (\gamma - \beta)|Y^{(0)}(t)|^2\}dt + 2e^{-\gamma t}\langle \sigma Y^{(0)}(t), dW(t) \rangle, \quad \gamma > 0.
\end{split}
\end{equation}

Also, via Itô’s formula, we deduce from (2.13) that
\begin{equation}
\begin{split}
de\varepsilon e^{-\gamma t}|Y^{(0)}(t)|^2 &\leq -\varepsilon(\gamma - \beta - 2\|\sigma\|_{\text{HS}}^2) e^{-\gamma t}\varepsilon e^{-\gamma t}|Y^{(0)}(t)|^2 |Y^{(0)}(t)|^2 dt \\
&\quad + c e^{-\gamma t}|Y^{(0)}(t)|^2 dt \\
&\quad + 2\varepsilon e^{-\gamma t} e^{-\gamma t}\langle |Y^{(0)}(t)|^2, dW(t) \rangle, \quad \gamma > 0, \quad \varepsilon > 0,
\end{split}
\end{equation}
which implies that, for any $\gamma > \beta + 2\|\sigma\|_{\text{HS}}^2$, by Gronwall’s inequality,
\begin{equation}
E e^{-\gamma t}|Y^{(0)}(t)|^2 \leq e^{c t}\varepsilon (1 + |\xi(0)|^2),
\end{equation}
so that
\begin{equation}
\varepsilon(\gamma - \beta - 2\|\sigma\|_{\text{HS}}^2) \int_0^t e^{-\gamma s} E(e^{-\gamma s}|Y^{(0)}(s)|^2 |Y^{(0)}(s)|^2) ds \leq (1 + e^{c t})\varepsilon (1 + |\xi(0)|^2).
\end{equation}

Making use of BDG’s inequality and Jensen’s inequality, we derive from (2.14) and (2.15) that
\begin{equation}
\begin{split}
E\left(\sup_{0 \leq s \leq t} e^{-\gamma s}|Y^{(0)}(s)|^2\right) &\leq e^{\gamma (1 + |\xi(0)|^2)} + c \int_0^t E e^{-\gamma s}|Y^{(0)}(s)|^2 ds \\
&\quad + 2\varepsilon E\left(\sup_{0 \leq s \leq t} \int_0^s e^{-\gamma u} e^{-\gamma s(1 + |Y^{(0)}(u)|^2)} \langle \sigma Y^{(0)}(u), dW(u) \rangle\right) \\
&\leq (1 + e^{c t})\varepsilon e^{\gamma (1 + |\xi(0)|^2)} \\
&\quad + 8\sqrt{2}\varepsilon E\left(\int_0^t e^{-2\gamma s} e^{2\varepsilon e^{-\gamma s}(1 + |Y^{(0)}(s)|^2)} |\sigma Y^{(0)}(s)|^2 ds\right)^{1/2} \\
&\leq (1 + e^{c t})\varepsilon e^{\gamma (1 + |\xi(0)|^2)} + \frac{1}{2} E\left(\sup_{0 \leq s \leq t} e^{-\gamma s(1 + |Y^{(0)}(s)|^2)}\right) \\
&\quad + 64\|\sigma\|_{\text{HS}}^2 e^{\gamma (1 + |\xi(0)|^2)} \int_0^t e^{-\gamma s} E(e^{-\gamma s(1 + |Y^{(0)}(s)|^2)} |Y^{(0)}(s)|^2) ds, \quad \gamma > \beta + 2\|\sigma\|_{\text{HS}}^2 \varepsilon.
\end{split}
\end{equation}

So plugging (2.16) back into (2.17) yields that
\begin{equation}
E\left(\sup_{0 \leq s \leq T} e^{-\gamma s}|Y^{(0)}(s)|^2\right) < \infty, \quad \gamma > \beta + 2\|\sigma\|_{\text{HS}}^2 \varepsilon.
\end{equation}

Note that
\begin{equation}
\sup_{\varepsilon > 0}(e^{-\gamma (1 + 2\|\sigma\|_{\text{HS}}^2 \varepsilon) T}) = \lambda_T := \frac{1}{2\|\sigma\|_{\text{HS}}^2 T} e^{-(\beta T + 1)}.
\end{equation}
Consequently, in (2.18), by taking $\gamma \downarrow \beta + \frac{1}{T}$, we arrive at

$$ (2.19) \quad \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\lambda_0 |Y^\xi(t)|^2} \right) < \infty, \quad \lambda_0 \in (0, \lambda_T) $$

In the end, (2.11) follows from (2.12) and (2.19) in case of $\lambda_T < \lambda_T$.

**Remark 2.2.** In terms of Lemma 2.3, (2.11) holds for small $T > 0$ provided that (2.5) is non-dissipative, i.e., $\beta \geq 0$ in (2.2). Also, (2.11) is satisfied with large $T > 0$ in case that (2.5) is dissipative, i.e., $\beta < 0$ in (2.2).

For notation brevity, we set

$$ (2.20) \quad h_\xi(t) := \sigma^{-1} \{ b(Y^\xi(t)) - b(Y^\xi(t_\delta)) - Z(\hat{Y}^\xi(t_\delta)) \}, \quad t \geq 0, \quad \xi \in \mathcal{C}, $$

where $\hat{Y}^\xi$ is defined exactly as in (2.4) with $X(\delta)$ replaced by $Y^\xi$.

The lemma below plays an important role in checking the Novikov condition so that the Girsanov theorem is applicable and investigating weak error analysis.

**Lemma 2.4.** Suppose that (A1) and (A2) hold. Then,

$$ (2.21) \quad \mathbb{E} e^{\lambda \int_0^T |\sigma^{-1}Z(Y^\xi)|^2 dt} < \infty $$

whenever $\lambda, T > 0$ such that

$$ \lambda < \frac{e^{-(1+\beta T)}}{2 \|\sigma\|_{\text{HS}}^2 \|\sigma^{-1}\|_{\text{HS}}^2 \left\{ L_2^2 1_{\{\alpha=1\}} + 0 1_{\{\alpha \in (0,1)\}} \right\} T^2}, $$

where we set $\frac{1}{0} = \infty$. Moreover,

$$ (2.22) \quad \mathbb{E} e^{\lambda \int_0^T |h_\xi(t)|^2 dt} < \infty $$

provided that $\lambda, T > 0$ such that

$$ \lambda < \frac{e^{-(1+\beta T)}}{4 \|\sigma\|_{\text{HS}}^2 \|\sigma^{-1}\|_{\text{HS}}^2 \left\{ (4L_1^2 + L_2^2) 1_{\{\alpha=1\}} + L_1^2 1_{\{\alpha \in (0,1)\}} \right\} T^2}. $$

**Proof.** From (A2), it is obvious to see that

$$ (2.23) \quad |Z(\xi)| \leq |Z(\xi_0)| + L_2 \|\xi\|_\infty, \quad \xi \in \mathcal{C}, $$

which, in addition to Young’s inequality, implies that

$$ (2.24) \quad |\sigma^{-1}Z(Y^\xi)|^2 \leq c_\varepsilon + \|\sigma^{-1}\|^2_{\text{HS}} \left\{ (1 + \varepsilon) L_2^2 1_{\{\alpha=1\}} + \varepsilon 1_{\{\alpha \in (0,1)\}} \right\} \|\xi\|^2_\infty, \quad \varepsilon > 0 $$

for some constant $c_\varepsilon > 0$. As a consequence, (2.21) holds true from (2.24) and by taking advantage of (2.11) followed by choosing $\varepsilon \in (0,1)$ sufficiently small.
By the definition of $\hat{Y}^ξ$ (see (2.1) with $X^{(δ)}$ replaced by $Y^ξ$ for more details), a straightforward calculation shows that

$$\|\hat{Y}^ξ\|_∞ = \sup_{-τ ≤ θ ≤ 0} |\hat{Y}^ξ(θ)|$$

(2.25)

$$≤ \max_{k∈\mathbb{N}} \sup_{-(k+1)δ ≤ θ ≤ -kδ} \left(\frac{θ + (1 + k)δ}{δ}|Y^ξ(t_δ - kδ)| - \frac{θ + kδ}{δ}|Y^ξ(t_δ - (k + 1)δ)|\right)$$

$$≤ \|Y^ξ\|_∞ \lor \|Y^ξ\|_{L^∞, t ≥ 0}$$

due to the fact that $(θ + (1 + k)δ)/δ - (θ + kδ)/δ = 1$. Subsequently, (2.25), together with (A1) as well as (2.24), yields that

$$|h^ξ_1(t)|^2 ≤ μ_ε + ν_ε(\|Y^ξ\|_∞ \lor \|Y^ξ\|_{L^∞, t ≥ 0}), \quad ε > 0, \quad t ≥ 0$$

for some $μ_ε > 0$ and

$$ν_ε := 2\|σ^{-1}\|_{HS}^2 \{(4L^2 + (1 + ε)L^2_1)1_{(α=1)} + L^2_1(1 + ε)1_{(α∈(0,1))}\}$$

Thereby, (2.22) follows from (2.19) and (2.26) and by noting that

$$\int_0^T e^{\lambda(\|Y^ξ\|_∞ \lor \|Y^ξ\|_{L^∞, t ≥ 0})} dt ≤ τ\|\xi\|_∞^2 + 2 \int_0^T e^{\lambda\|Y^ξ\|_∞^2} dt, \quad λ > 0,$$

where we set $ξ(θ) := ξ(-τ)$ for any $θ ∈ [-2τ, -τ]$.

Next we intend to show that the displacement of segment process is continuous in the sense of $L^p$-norm sense.

**Lemma 2.5.** Under (A1) and (A3), for any $p > 2$ and $T > 0$, there exists a constant $C_{p,T} > 0$ such that

$$\sup_{0 ≤ t ≤ T} E\|Y^ξ - \hat{Y}^ξ\|_∞^p ≤ C_{p,T} δ^{(p-2)/2}.$$

(2.27)

**Proof.** By [12] Theorem 4.4, p61], for any $p > 0$ and $T > 0$, there exists $\tilde{C}_{p,T} > 0$ such that

$$E\left(\sup_{-τ ≤ t ≤ T} |Y^ξ(t)|^p\right) ≤ \tilde{C}_{p,T}(1 + \|\xi\|_∞^p).$$

(2.28)

By utilizing Hölder’s inequality and BDG’s inequality, it follows from (A1) and (2.28) that

$$E\left(\sup_{kδ ≤ t ≤ k+2δ} |Y^ξ(t) - Y^ξ(kδ)|^p\right) ≤ c\left\{\delta^{p-1} \int_{kδ}^{(k+2)δ} E|b(Y^ξ(t))|^p dt + E\left(\sup_{0 ≤ t ≤ 2δ} |W(t)|^p\right)\right\}$$

(2.29)

$$≤ c\left\{\delta^{p-1} \int_{kδ}^{(k+2)δ} (1 + E|Y^ξ(t)|^p) dt + δ^{p/2}\right\}$$

$$≤ c\delta^{p/2}, \quad p > 2, \quad k ∈ \mathbb{N}.$$
Trivially, there exists an integer $k_0$ such that $t \in [k_0 \delta, (k_0 + 1)\delta]$. So, for any $p > 2$,

$$\mathbb{E}\|Y^\xi_t - \hat{Y}^\xi_{t_0}\|_\infty \leq M \max_{k \in \mathbb{S}} \mathbb{E} \left( \sup_{-(k+1)\delta \leq \theta \leq (k_0 - 1)\delta} |Y^\xi(t + \theta) - \hat{Y}^\xi_{k_0 \delta}(\theta)|^p \right)$$

$$\leq c M \max_{k \in \mathbb{S}} \mathbb{E} |Y^\xi((k_0 - k)\delta) - Y^\xi((k_0 - k - 1)\delta)|^p$$

$$+ c M \max_{k \in \mathbb{S}} \mathbb{E} \left( \sup_{(k_0 - k - 1)\delta \leq s \leq (k_0 - k + 1)\delta} |Y^\xi(s) - Y^\xi((k_0 - k - 1)\delta)|^p \right).$$

In case of $k \leq k_0 - 1$, we find from (2.29) that (2.27) holds. On the other hand, if $k = k_0$, from (A3), (2.29) and $M\delta = \tau$, then one gets that (2.27) holds. Moreover, for $k \geq 1 + k_0$, (2.27) is still true due to (A3). The proof is therefore complete.

With the previous Lemmas in hand, we are now in the position to complete the

**Proof of Theorem 2.1**

Let

$$(2.30) \quad W^\xi(t) = W(t) + \int_0^t h^\xi_1(s) ds, \quad t \geq 0,$$

where $h^\xi_1$ was introduced in (2.20). Define

$$R^\xi_2(t) = \exp \left( - \int_0^t \langle h^\xi_1(s), dW(s) \rangle - \frac{1}{2} \int_0^t \| h^\xi_1(s) \|^2 ds \right), \quad t \geq 0$$

and $dQ^\xi_2 = R^\xi_2(T)d\mathbb{P}$, where $T > 0$ such that (2.27). Due to (2.27) and (2.22), the Girsanov theorem implies that $(W^\xi_2(t))_{t \in [0,T]}$ is a Brownian motion under the probability measure $Q^\xi_2$. Thus, (2.3) can be rewritten in the following form

$$(2.31) \quad dY^\xi(t) = \{b(Y^\xi(t)) + Z(Y^\xi(t))\} dt + \sigma dW^\xi_2(t), \quad t > 0$$

with the initial value $Y^\xi(\theta) = \xi(\theta), \theta \in [-\tau,0]$ so that $(Y^\xi(t), W^\xi_2(t))_{t \in [0,T]}$ is a weak solution to (2.3) under $Q^\xi_2$. Obviously, (2.3) has a unique strong solution so as to the weak solution is unique. Since, by (2.7) and (2.21), $(Y^\xi(t), W^\xi_1(t))_{t \in [0,T]}$ is a weak solution to (2.1) under $Q^\xi_1$ and $(Y^\xi(t), W^\xi_2(t))_{t \in [0,T]}$ is a weak solution to (2.3) under $Q^\xi_2$, we deduce from the weak uniqueness
due to Lemma 2.2 and Hölder’s inequality that
\[
\begin{align*}
|\mathbb{E} f(X(t)) - \mathbb{E} f(X(t))| &= |\mathbb{E}_{\xi_1} f(Y_1(t)) - \mathbb{E}_{\xi_2} f(Y_2(t))| \\
&= |\mathbb{E}((R_1^1(T) - R_1^2(T)) f(Y_1(t)))| \\
&\leq ||f||_\infty \mathbb{E}|R_1^1(T) - R_1^2(T)| \\
&\leq ||f||_\infty \mathbb{E}\left( (R_1^1(T) + R_2^2(T)) \left( \int_0^t |\sigma^{-1}Z(Y_1^\xi(s)) + h_1^\xi(s), dW(s)| \right) + \frac{1}{2} \int_0^t |h_1^\xi(s)|^2 - |\sigma^{-1}Z(Y_1^\xi(s)|^2|ds) \right) \\
&= ||f||_\infty \mathbb{E}\left( (R_1^1(T))^q + (R_2^2(T))^q \right) \\
&\times \left\{ \left( \mathbb{E}\left( \left( \int_0^t |\sigma^{-1}Z(Y_1^\xi(s)) + h_1^\xi(s), dW(s)| \right)^p \right) \right)^{1/p} + \frac{1}{2} \int_0^t \mathbb{E}|h_1^\xi(s)|^2 - |\sigma^{-1}Z(Y_1^\xi(s)|^2|ds \right\} \\
&=: ||f||_\infty \Gamma(T) \{ \Theta_1(t) + \Theta_2(t) \}, \quad t \in [0, T]
\end{align*}
\]
for \(1/p + 1/q = 1, p, q > 1\), where in the second inequality we utilized the fundamental inequality:
\[
|e^x - e^y| \leq (e^x + e^y)|x - y|, \quad x, y \in \mathbb{R},
\]
and, in the last two procedure, employed the Minkowski inequality. Let
\[
M_1(t) = \int_0^t |\sigma^{-1}Z(Y_1^\xi(s), dW(s)) \quad \text{and} \quad M_2(t) = -\int_0^t h_1^\xi(s), dW(s), \quad t \geq 0.
\]
For any \(q > 1\), using Hölder’s inequality and the fact that \(e^{2qM_i(t) - 2q^2(M_i(t))}, i = 1, 2, \) is an exponential martingale leads to
\[
\mathbb{E}(R_1^1(T))^q + \mathbb{E}(R_2^2(T))^q \\
= \mathbb{E}e^{qM_1(t) - \frac{q^2}{2}(M_1(T))} + \mathbb{E}e^{qM_2(t) - \frac{q^2}{2}(M_2(T))} \\
\leq (\mathbb{E}e^{(2q^2-q)(M_1(T))})^{1/2} + (\mathbb{E}e^{(2q^2-q)(M_2(T))})^{1/2} \\
\leq 2\left(\mathbb{E}\exp\left((2q^2-q)\int_0^T |\sigma^{-1}Z(Y_1^\xi(s)|^2) ds\right)\right)^{1/2} + \left(\mathbb{E}\exp\left((2q^2-q)\int_0^T |h_1^\xi(s)|^2 ds\right)\right)^{1/2}.
\]
Whence, by taking \(q \downarrow 1\) and exploiting (2.27), (2.21), and (2.22), one has, for some \(\tilde{C}_{q,T} > 0\),
\[
(2.33) \quad \Gamma(T) \leq \tilde{C}_{q,T}.
\]
In view of (A1) and (A2), in addition to \(|Y^\xi(t) - Y^\xi(t_s)| \leq \|Y^\xi - \hat{Y}^\xi\|_\infty\), it holds that
\[
\begin{align*}
|\sigma^{-1}Z(Y_1^\xi) + h_1^\xi(t)| &\leq c \{ |b(Y^\xi(t)) - b(Y^\xi(t_s))| + |Z(Y_1^\xi) - \hat{Y}_1^\xi| \} \\
&\leq c \left\{ L_1|Y_1^\xi(t) - Y^\xi(t_s)| + L_2\|Y_1^\xi - \hat{Y}_1^\xi\|_\infty \right\} \\
&\leq c \{ \|Y_1^\xi - \hat{Y}_1^\xi\|_\infty + \|Y_1^\xi - \hat{Y}_1^\xi\|_\infty \}.
\end{align*}
\]
\[
(2.34)
\]
This, besides BDG’s inequality followed by Hölder’s inequality, yields that
\[
\Theta_1(t) \leq c \left( \int_0^t \mathbb{E}[\sigma^{-1}Z(Y^\xi_s) + h^\xi_1(s)]p \right)^{1/p} ds
\]
\[
\leq c \left( \int_0^t \{\mathbb{E}[|Y^\xi_s - \hat{Y}^\xi_{s_1}|p] + \mathbb{E}[|Y^\xi_s - \hat{Y}^\xi_{s_1}|p] \} ds \right)^{1/p}
\leq c \delta^{{\alpha \gamma - 1 \over 2}}, \quad p > 2/\alpha,
\]
where we utilized (2.27) in the last display. On the other hand, applying Hölder’s inequality and combining (A1) with (A2) and (2.34) enables us to obtain that
\[
\Theta_2(t) \leq \frac{1}{2} \int_0^t \{(\mathbb{E}[h^\xi_1(s) - \sigma^{-1}Z(Y^\xi_s)^{p/(p-1)}] - \mathbb{E}\sigma^{-1}Z(Y^\xi_s) + h^\xi_1(s)]p \}^{1/p} ds
\leq c \int_0^t \{(1 + \mathbb{E}[|Y^\xi_s|p] + \mathbb{E}[|\hat{Y}^\xi_{s_1}|p] + \mathbb{E}[|Y^\xi_s - \hat{Y}^\xi_{s_1}|p])^{1/p} ds
\leq c \int_0^t \{\mathbb{E}[|Y^\xi_s - \hat{Y}^\xi_{s_1}|p] + \mathbb{E}[|Y^\xi_s - \hat{Y}^\xi_{s_1}|p] \}^{1/p} ds
\leq c \delta^{{\alpha \gamma - 1 \over 2}}, \quad p > 2/\alpha,
\]
where we used (2.25) and (2.28) in the penultimate procedure and exploited (2.27) in the last step. Consequently, substituting (2.33), (2.35) and (2.36) into (2.32) and taking \( p > 2/\alpha \) sufficiently large (so that \( q \downarrow 1 \)) yields the assertions (2.3).

3 Extensions to Other Scenarios

In this section, we intend to extend the approach to derive Theorem 2.1 and investigate the weak convergence of other kind of numerical schemes for path-dependent SDEs with irregular coefficients.

3.1 Extension to Truncated EM Scheme

In this subsection we are still interested in (2.1). Rather than the EM scheme (2.3), we introduce the following truncated EM scheme associated with (2.1)
\[
\text{d}X^{(\delta)}(t) = \{b(X^{(\delta)}(t)) + Z(\hat{X}^{(\delta)}))\}dt + \sigma\text{d}W(t), \quad t > 0
\]
with the initial value \( X^{(\delta)}(\theta) = X(\theta) = \xi(\theta), \theta \in [-\tau, 0] \), where \( \hat{X}^{(\delta)} \in \mathcal{C} \) is defined in the way
\[
\hat{X}^{(\delta)}(\theta) := X^{(\delta)}((t + \theta) \wedge t_0), \theta \in [-\tau, 0].
\]

As for the truncated EM scheme (3.1), the main result in this subsection is stated as below.

**Theorem 3.1.** Let (A1) and (A2) hold. Then, for any \( T > 0 \) such that
\[
2\|\sigma\|^2_{\text{HS}} \|\sigma^{-1}\|^2_{\text{HS}}\{(4L_1^2 + L_2^2)\mathbf{1}_{\{\alpha=1\}} + L_1^2\mathbf{1}_{\{\alpha=0,1\}}\} \leq e^{-(1+\beta T)/T^2},
\]
there exists a constant \( C_{2,T} > 0 \) such that
\[
\|\mathbb{E}f(X(t)) - \mathbb{E}f(X^{(\delta)}(t))\| \leq C_{2,T} \delta^\alpha/2, \quad f \in \mathcal{B}_b(\mathbb{R}^d), \ t \in [0,T].
\]

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Proof. Herein we just list some dissimilarities since the argument of Theorem 3.1 is parallel to that of Theorem 2.1. Set
\[ h_2^\xi(t) := \sigma^{-1}\{b(Y^\xi(t)) - b(Y^\xi(t_\delta)) - Z(\hat{Y}_t^\xi)\}, \quad t \geq 0, \quad \xi \in \mathcal{C} \]
with
\[ \hat{Y}_t^\xi(\theta) = Y^\xi((t + \theta) \wedge t_\delta), \quad \theta \in [-\tau, 0]. \]
It is easy to see that
\[ \|\hat{Y}_t^\xi\|_\infty = \sup_{t-\tau \leq s \leq t} |Y^\xi(s \wedge t_\delta)| \leq \|Y_t^\xi\|_\infty. \]
So Lemma 2.4 still holds with \( h_1^\xi \) replaced by \( h_2^\xi \) by virtue of Lemma 2.3. On the other hand, by (A1) and (2.28), we infer from Hölder’s inequality and BDG’s inequality that
\[
\mathbb{E}\|Y_t^\xi - \hat{Y}_t^\xi\|_\infty^p = \mathbb{E}\left( \sup_{t-\tau \leq s \leq t} |Y^\xi(s) - Y^\xi(s \wedge t_\delta)|^p \right) \\
= \mathbb{E}\left( \sup_{t-\tau \leq s \leq t} |Y^\xi(s) - Y^\xi(t_\delta)|^p 1_{\{s \geq t_\delta\}} \right) \\
= \mathbb{E}\left( \sup_{t-\tau \leq s \leq t} \left| \int_{t_\delta}^s b(Y^\xi(u))du + \int_{t_\delta}^s \sigma dW(s) \right|^p 1_{\{s \geq t_\delta\}} \right) \\
\leq c \left\{ \delta^{p-1} \int_{t_\delta}^t |b(Y^\xi(u))|^p du + \mathbb{E}\left( \sup_{t_\delta \leq s \leq t} \left| \int_{t_\delta}^s \sigma dW(s) \right|^p \right) \right\} \\
\leq c \delta^{p/2}, \quad p \geq 1.
\]
Having Lemma 2.4 with writing \( h_2^\xi \) in lieu of \( h_1^\xi \) and (3.3) in hand, the proof of Theorem 3.1 is therefore complete by inspecting the argument of Theorem 2.1.

Remark 3.1. In terms of Theorems 2.1 and 3.1, we conclude that the truncated EM scheme (3.1) enjoys a better weak convergence rate than the EM scheme (2.3). On the other hand, with regard to the truncated EM scheme, we drop the assumption (A3) in Theorem 3.1. Furthermore, we point out that the EM scheme (2.3) established via interpolation works merely for path-dependent SDEs with finite memory. While the truncated EM scheme (3.1) is available for path-dependent SDEs with infinite memory as the following subsection demonstrates.

3.2 Extension to path-dependent SDEs with infinite memory

As we depicted in Remark 3.1, one of the advantages of the truncated EM scheme (3.1) is that it is applicable to path-dependent SDEs with infinite memory. To proceed, let’s introduce some additional notation. For a fixed number \( r \in (0, \infty) \), let
\[ \mathcal{C}_r = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^d) : \|\phi\|_r := \sup_{-\infty < \theta < 0} (e^{r\theta} |\phi(\theta)|) < \infty \right\}, \]
which is a Polish space under the metric induced by \( \| \cdot \|_r \).

In this subsection, we focus on the following path-dependent SDE with infinite memory
\[
dX(t) = \{b(X(t)) + Z(X_t)\}dt + \sigma dW(t), \quad t > 0, \quad X_0 = \xi \in \mathcal{C}_r,
\]
in which
(A2') $Z : \mathcal{C}_r \to \mathbb{R}^d$ is Hölder continuous, i.e., there exist $\alpha \in (0, 1]$ and $L_4 > 0$ such that
\[
|Z(\xi) - Z(\eta)| \leq L_4|\xi - \eta|^\alpha, \quad \xi, \eta \in \mathcal{C}_r,
\]
and the other quantities are stipulated exactly as in (3.1). Similar to (3.1), we define the truncated EM scheme associated with (3.4) by
\[
dX^\delta(t) = \{b(X^\delta(t)) + Z(X_t^\delta)\} dt + \sigma dW(t), \quad t > 0
\]
with the initial datum $X^\delta(\theta) = X(\theta) = \xi(\theta), \theta \in (-\infty, 0]$, in which $X_t^\delta \in \mathcal{C}_r$ is designed by
\[
\hat{X}_t^\delta(\theta) := X^\delta((t + \theta) \wedge t_\delta), \theta \in (-\infty, 0].
\]

The main result in this subsection is presented as follows.

**Theorem 3.2.** Assume assumptions of Theorem 3.1 hold with (A2) replaced by (A2'). Then, there exists a constant $C_{3,T} > 0$ such that
\[
|E_f(X(t)) - E_f(X^\delta(t))| \leq C_{3,T} \delta^{\alpha/2}, \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad t \in [0, T]
\]
provided that the stepsize $\delta \in (0, 1)$ is sufficiently small.

**Proof.** Since
\[
\|Y_\xi^\delta\|_r \leq \|\xi\|_r + \sup_{0 \leq s \leq t} |Y_\xi(s)|,
\]
Lemma 2.2 still holds with $\|\cdot\|_\infty$ replaced by $\|\cdot\|_r$. Also, (2.21) holds under the assumptions (A1) and (A2') so that (3.4) has a unique weak solution by following the argument of Lemma 2.2.

Let
\[
h^\delta_3(t) = \sigma^{-1} \{b(Y_\xi^\delta(t)) - b(Y_\xi^\delta(t_\delta)) - Z(Y_\xi^\delta(t))\}, \quad t \geq 0, \quad \xi \in \mathcal{C}_r,
\]
where
\[
\hat{Y}_t^\xi(\theta) := Y_\xi((t + \theta) \wedge t_\delta), \quad \theta \in (-\infty, 0].
\]

Clearly, we have
\[
\|\hat{Y}_t^\xi\|_r = e^{-\delta t} \sup_{-\infty < s \leq t} (e^{\delta s} |Y^\xi(s)|1_{\{s \leq t_\delta\}}) + e^{-\delta t} \sup_{-\infty < s \leq t} (e^{\delta s} |Y^\xi(t_\delta)|1_{\{t_\delta \leq s\}})
\]
\[
\leq e^{-\delta t} \sup_{-\infty < s \leq t} (e^{\delta s} |Y^\xi(s)|1_{\{s \leq t_\delta\}}) + e^{\delta t} e^{-\delta t} \sup_{-\infty < s \leq t} (e^{\delta s} |Y^\xi(t_\delta)|1_{\{t_\delta \leq s\}})
\]
\[
\leq e^{\delta t} \|Y^\xi\|_r.
\]
So (2.22) with writing $h^\delta_3(t)$ instead of $h^\xi(t)$ remains true whenever the stepsize $\delta \in (0, 1)$ is sufficiently small. Moreover, by virtue of (A1), (2.28), Hölder’s inequality as well as BDG’s inequality, it follows that
\[
E\|Y^\xi_t - \hat{Y}_t^\xi\|^p = e^{-\delta t} E\left(\sup_{-\infty < s \leq t} (e^{\delta s} |Y^\xi(s) - Y^\xi(s \wedge t_\delta)|^p)\right)
\]
\[
\leq E\left(\sup_{t_\delta < s \leq t} \left(\left\|b(Y^\xi(0)(s))ds + \sigma(W(s) - W(t_\delta))\right\|^p\right)\right)
\]
\[
\leq e^{\delta p/2}, \quad p \geq 2.
\]
Afterwards, carrying out a similar argument to derive Theorem 2.1, we obtain the desired assertion (3.6).
Remark 3.2. To the best of knowledge, Theorem 3.2 is the first result upon weak convergence for path-dependent SDEs with infinite memory and irregular drifts. For path-dependent SDEs with finite memory, Theorems 2.1 and 3.1 shows that the weak convergence order can be achieved for any \( \delta \in (0, 1) \). However, concerning path-dependent SDEs with infinite memory, the weak convergence rate can only be available whenever the stepsize \( \delta \in (0, 1) \) is sufficiently small. This illustrates one of the essential features between SDEs with finite memory and SDEs infinite memory. Moreover, Theorem 3.2 further shows the superiority of the truncated EM scheme (3.1) with contrast to the EM scheme established by interpolations at discrete-time points.

4 Weak Convergence: Degenerate Case

In the previous sections, we investigate weak convergence of EM schemes and its variants for non-degenerate path-dependent SDEs with Hölder continuous drifts. In this section, we are still interested in the same topic but concerned with a class of degenerate SDE on \( \mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d \)

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{dX(t)}{dt} = \{X(t) + Y(t)\}dt \\
\frac{dY(t)}{dt} = \{b(X(t), Y(t)) + Z(X_t, Y_t)\}dt + \sigma dW(t), \\
\end{array}
\right. \quad t \geq 0
\end{aligned}
\]

with the initial datum \( (X_0, Y_0) = (\xi, \eta) \in \mathcal{C}^2 \), where \( b : \mathbb{R}^{2d} \to \mathbb{R}^d \), \( Z : \mathcal{C}^2 \to \mathbb{R}^d \), \( \sigma \in \mathbb{M}^d_{\text{non}} \), and \( (W(t))_{t \geq 0} \) is a \( d \)-dimensional Brownian motion on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).

(4.1) is the so-called stochastic Hamiltonian systems, which has been investigated considerably in \([14, 22, 26, 29, 31]\), to name a few.

Throughout this section, we assume that

\((H1)\) \( b \) is Lipschitz continuous, that is, there exists \( K_1 > 0 \) such that

\[
b(x_1, y_1) - b(x_2, y_2) \leq K_1(|x_1 - x_2| + |y_1 - y_2|), \quad \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^{2d}
\]

and there exist \( \alpha, \beta, \lambda, C > 0 \) and \( \gamma \in (-\alpha \beta, \alpha \beta) \) such that

\[
\langle \alpha x + \gamma y, x + y \rangle + \langle \beta y + \gamma x, b(x, y) \rangle \leq C - \lambda(|x|^2 + |y|^2), \quad \forall (x, y) \in \mathbb{R}^{2d}.
\]

\((H2)\) \( Z \) is Hölder continuous, i.e., there exist \( \alpha \in (0, 1] \) and \( K_2 > 0 \) such that

\[
|Z(\xi, \eta_1) - Z(\xi, \eta_2)| \leq K_2(\|\xi_1 - \xi_2\|_\infty + |\eta_1 - \eta_2|) \leq K_2(\|\xi_1 - \xi_2\|_\infty + |\eta_1 - \eta_2|_\infty), \quad (\xi, \eta_1, (\xi_2, \eta_2) \in \mathcal{C}^2.
\]

By carrying out a similar argument to derive Lemma 2.2 and taking advantage of Lemma 4.1 below, (4.1) has a unique weak solution under \((H1)\) and \((H2)\). With the assumption (4.2), the following reference SDE

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{dU(t)}{dt} = \{U(t) + V(t)\}dt \\
\frac{dV(t)}{dt} = b(U(t), V(t))dt + \sigma dW(t), \\
\end{array}
\right. \quad t \geq 0
\end{aligned}
\]

with the initial data \((U(0), V(0)) = (u, v) \in \mathbb{R}^{2d}\) is wellposed. To emphasize the initial value \((u, v) \in \mathbb{R}^{2d}\), we shall write \((U^u, v(t), V^u, v(t))\) instead of \((U(t), V(t))\). Analogously to (4.2), we can extend respectively \(U(t)\) and \(V(t)\) in the following way:

\[
U^{\xi, \eta}(t) = \xi(t)1_{[-\tau, 0)}(t) + U^{\xi(0), \eta(0)}(t)1_{[0, \infty)}(t), \quad t \in [-\tau, \infty), \quad (\xi, \eta) \in \mathcal{C}^2
\]

\[
V^{\xi, \eta}(t) = \eta(t)1_{[-\tau, 0)}(t) + V^{\xi(0), \eta(0)}(t)1_{[0, \infty)}(t), \quad t \in [-\tau, \infty), \quad (\xi, \eta) \in \mathcal{C}^2
\]
and
\[ V^{\xi, \eta}(t) = \eta(t)1_{[-\tau, 0]}(t) + V^{0, \eta}(t)1_{[0, \infty)}(t), \quad t \in [-\tau, \infty), \quad (\xi, \eta) \in \mathcal{C}^2. \]

Let \( U^{\xi, \eta}_t \) and \( V^{\xi, \eta}_t \) be the segment process associated with \( U^{\xi, \eta}(t) \) and \( V^{\xi, \eta}(t) \), respectively. Next, the truncated EM scheme corresponding to (4.1) is given by
\[
\begin{aligned}
&dX^{(\delta)}(t) = \{X^{(\delta)}(t) + Y^{(\delta)}(t)\}dt \\
&dY^{(\delta)}(t) = \{b(X^{(\delta)}(t\delta), Y^{(\delta)}(t\delta)) + Z(\widetilde{X}^{(\delta)}_t, \widetilde{Y}^{(\delta)}_t)\}dt + \sigma dW(t)
\end{aligned}
\]

with the initial value \((X^{(\delta)}(0), Y^{(\delta)}(0)) = (X(0), Y(0)) = (\xi(0), \eta(0)) \in \mathbb{R}^{2d}, \theta \in [-\tau, 0]\), where
\[
\widetilde{X}^{(\delta)}_t(\theta) := X^{(\delta)}((t + \theta) \land t\delta) \quad \text{and} \quad \widetilde{Y}^{(\delta)}_t(\theta) := Y^{(\delta)}((t + \theta) \land t\delta), \quad \theta \in [-\tau, 0].
\]

Observe that
\[
dX^{(\delta)}(t) = \{X^{(\delta)}(t) + (b(X^{(\delta)}(0), Y^{(\delta)}(0))t + Z(\widetilde{X}^{(\delta)}_t, \widetilde{Y}^{(\delta)}_t)) + \sigma W(t)\}dt, \quad t \in [0, \delta]
\]

where, for any \( \theta \in [-\tau, 0] \),
\[
\Lambda(t) := \int_0^t Z(\widetilde{X}^{(\delta)}_s, \widetilde{Y}^{(\delta)}_s)ds \quad \text{with} \quad \widetilde{X}^{(\delta)}_t(\theta) = X((t + \theta) \land 0), \quad \widetilde{Y}^{(\delta)}_t(\theta) = Y((t + \theta) \land 0).
\]

Thus, \((X^{(\delta)}(t))_{t \in [0, \delta]}\) can be obtained explicitly via the variation-of-constants formula. Inductively, \(X^{(\delta)}(t)\) enjoys explicit formula.

In the sequel, for \( \alpha, \beta, \gamma \) such that (4.3), consider the following Lyapunov function
\[
\mathbb{W}(x, y) := \frac{\alpha}{2}|x|^2 + \frac{\beta}{2}|y|^2 + \gamma \langle x, y \rangle, \quad x, y \in \mathbb{R}^d.
\]

For \( \gamma \in (-\alpha \beta, \alpha \beta) \), it is easy to see that
\[
\kappa_2(|x|^2 + |y|^2) \leq \mathbb{W}(x, y) \leq \kappa_1(|x|^2 + |y|^2), \quad x, y \in \mathbb{R}^d,
\]

in which \( \kappa_1 := (1 + \alpha)(1 + \beta)/2 \) and
\[
\kappa_2 := \frac{1}{2} \left\{ (\alpha - \frac{1}{2}(\alpha/|\gamma| + |\gamma|/\beta)) \wedge \left( \beta - \frac{2|\gamma|}{\alpha/|\gamma| + |\gamma|/\beta} \right) \right\}.
\]

The main result in this section is presented as follows.

**Theorem 4.1.** Assume (H1) and (H2) hold. Then, for any \( T > 0 \) such that
\[
2\kappa_3 \|\sigma\|_{\text{HS}}^2 (\sigma^{-1})_2 \left\{ (4K_1^2 + K_2^2)1_{\{\alpha = 1\}} + 2K_1^21_{\{\alpha \in (0, 1)\}} \right\} T^2 < \kappa_2 e^{\lambda \kappa_2 T - 1}
\]

there exists \( C_{4, T} > 0 \) such that
\[
|E f(X(t), Y(t)) - E f(X^{(\delta)}(t), Y^{(\delta)}(t))| \leq C_{4, T}\delta^\alpha/2, \quad f \in \mathcal{B}_b(\mathbb{R}^{2d}), \quad t \in [0, T].
\]

**Remark 4.1.** The dissipative condition (4.3) is imposed to guarantee that the time horizontal \( T > 0 \) in Theorem 4.1 is large in certain situation. Nevertheless, in case of \( \lambda < 0 \), (4.7) remains true but for small time horizontal. Moreover, we can also investigate weak convergence of EM scheme via interpolation for (4.1) but with an additional assumption put on the initial value. Also we point out that, whenever the numerical scheme of the second component is established by interpolation, the algorithm for the first component is much more explicit compared with the truncated EM scheme.
The proof of Theorem 4.1 is based on several lemmas below. The following lemma shows exponential integrability of segment process.

**Lemma 4.2.** Let (4.3) hold. Then, for any $T > 0$,

$$
\mathbb{E} \exp \left( \lambda \int_0^T (\|U_t^\xi\|_\infty^2 + \|V_t^\xi\|_\infty^2) dt \right) < \infty, \quad \lambda < \frac{\kappa_2 e^{\lambda \kappa_2 T - 1}}{\kappa_3 \|\sigma\|^2_{\text{HS}} T^2}
$$

with $\kappa_3 := \gamma^2 \vee \beta^2$.

**Proof.** For notation simplicity, in what follows we write $U(t)$ and $V(t)$ in lieu of $U^\xi(t)$ and $V^\eta(t)$, respectively. By a close inspection of the proof for Lemma 2.3, to verify (2.11) it is sufficient to show that, for any $\varepsilon > 0$ and $\gamma > -\lambda \kappa_2 + \kappa_3 \|\sigma\|^2_{\text{HS}} \varepsilon$,

$$
\mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\varepsilon \kappa_2 e^{-\gamma t} (|U(t)|^2 + |V(t)|^2)} \right) < \infty,
$$

where $\kappa_2$ was given in (4.6) and $\kappa_3 := \gamma^2 \vee \beta^2$. By the Itô formula, it follows from (4.3) and (4.5) that

$$
d(e^{-\gamma t} \mathbb{W}(U(t), V(t)))
$$

$$
e^{-\gamma t} \left\{ -\gamma \mathbb{W}(U(t), V(t)) + \langle \alpha U(t) + \gamma V(t), U(t) + V(t) \rangle 
$$

$$
+ \langle \gamma U(t) + \beta V(t), b(U(t), V(t)) \rangle + (C + \|\sigma\|^2_{\text{HS}}/2) \right\} dt + e^{-\gamma t} \langle \sigma^* (\gamma U(t) + \beta V(t)), dW(t) \rangle 
$$

$$
\leq e^{-\gamma t} \left\{ - (\gamma + \lambda \kappa_2) \mathbb{W}(U(t), V(t)) + (C + \|\sigma\|^2_{\text{HS}}/2) \right\} dt + e^{-\gamma t} \langle \sigma^* (\gamma U(t) + \beta V(t)), dW(t) \rangle.
$$

This implies via Itô’s formula that

$$
d e^{\varepsilon e^{-\gamma t} \mathbb{W}(U(t), V(t))} \leq -\varepsilon (\gamma + \lambda \kappa_2 - \kappa_3 \|\sigma\|^2_{\text{HS}} \varepsilon) e^{\varepsilon e^{-\gamma t} \mathbb{W}(U(t), V(t))} \mathbb{W}(U(t), V(t)) dt
$$

$$
+ c_\varepsilon e^{\varepsilon e^{-\gamma t} \mathbb{W}(U(t), V(t))} dt + e^{\varepsilon e^{-\gamma t} \mathbb{W}(U(t), V(t))} \langle \sigma^* (\gamma U(t) + \beta V(t)), dW(t) \rangle, \quad \varepsilon > 0
$$

for some constant $c_\varepsilon > 0$. For any $\gamma > -\lambda \kappa_2 + \kappa_3 \|\sigma\|^2_{\text{HS}} \varepsilon$, Gronwall’s inequality, in addition to (4.3), yields that

$$
\mathbb{E} e^{\varepsilon e^{-\gamma t} \mathbb{W}(U(t), V(t))} \leq e^{c_\varepsilon t} e^{\varepsilon \kappa_1 (|\xi(0)|^2 + |\eta(0)|^2)},
$$

which, together with (4.10), leads further to

$$
\varepsilon (\gamma + \lambda \kappa_2 - \kappa_3 \|\sigma\|^2_{\text{HS}} \varepsilon) \int_0^t e^{-\gamma s} \mathbb{E} e^{s e^{-\gamma s} \mathbb{W}(U(s), V(s))} \mathbb{W}(U(s), V(s)) ds
$$

$$
\leq (1 + e^{c_\varepsilon t}) e^{\varepsilon \kappa_1 (|\xi(0)|^2 + |\eta(0)|^2)}.
$$

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Subsequently, by means of BDG’s inequality, we derive from (4.5) and (4.12) that

\[
\mathbb{E}\left(\sup_{0 \leq s \leq t} \int_0^s e^{-\gamma u}e^{\gamma u} \mathbb{E}(U(u), V(u)) \sigma^*(\gamma U(u) + \beta V(u), dW(u))\right) \\
\leq 4\sqrt{2}\mathbb{E}\left(\int_0^t e^{-\gamma s}e^{\gamma s} \mathbb{E}(U(s), V(s)) \sigma^*(\gamma U(s) + \beta V(s))^2 ds\right)^{1/2} \\
(4.13) \leq \frac{1}{2}\mathbb{E}\left(\sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E}(U(s), V(s)) + c \int_0^t e^{-\gamma s} \mathbb{E} e^{\gamma s} \mathbb{E}(U(s), V(s)) (|U(s)|^2 + |V(s)|^2) ds\right) \\
\leq \frac{1}{2}\mathbb{E}\left(\sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E}(U(t), V(t)) + c \int_0^t e^{-\gamma s} \mathbb{E} e^{\gamma s} \mathbb{E}(U(s), V(s)) \mathbb{E}(U(s), V(s)) ds\right) \\
\leq \frac{1}{2}\mathbb{E}\left(\sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E}(U(t), V(t)) + c (1 + e^{\gamma t}) e^{\gamma t} (|\xi(0)|^2 + |\eta(0)|^2)\right).
\]

With (4.10)-(4.13) in hand, we thus arrive at

\[
\mathbb{E}\left(\sup_{0 \leq s \leq t} e^{\gamma s} \mathbb{E}(U(t), V(t))\right) < \infty.
\]

This, combining with (4.5), yields (4.9). \(\square\)

For notation brevity, we set

\[
k^\xi,\eta(t) := \sigma^{-1} \{b(U^\xi,\eta(t), V^\xi,\eta(t)) - b(U^\xi,\eta(t_2), V^\xi,\eta(t_2)) - Z(U^\xi,\eta, \nu^\xi,\eta)\}.
\]

**Lemma 4.3.** Let (H1) and (H2) hold. Then,

\[
\mathbb{E} e^{\lambda} \int_0^T |\sigma^{-1}Z(U^\xi,\eta, V^\xi,\eta)|^2 dt < \infty
\]

for any \(\lambda, T > 0\) such that

\[
\lambda < \frac{\kappa_2 e^{\lambda \kappa_2 T - 1}}{2 \kappa_3 \|\sigma\|^2_{H^2} \sigma^{-1}_H \{K^2 \mathbb{1}_{\{\alpha = 1\}} + 0 \mathbb{1}_{\{\alpha \in (0, 1)\}}\} T^2}.
\]

Furthermore,

\[
\mathbb{E} e^{\lambda} \int_0^T |k^\xi,\eta(t)|^2 dt < \infty, \quad \lambda < \frac{\kappa_2 e^{\lambda \kappa_2 T - 1}}{4 \kappa_3 \|\sigma\|^2_{H^2} \sigma^{-1}_H \|\sigma^{-1}_H \|_{H^2} \{4K^2 + K^2 \mathbb{1}_{\{\alpha = 1\}} + 2K^2 \mathbb{1}_{\{\alpha \in (0, 1)\}}\} T^2}.
\]

for any \(\lambda, T > 0\) such that

\[
\lambda < \frac{\kappa_2 e^{\lambda \kappa_2 T - 1}}{4 \kappa_3 \|\sigma\|^2_{H^2} \sigma^{-1}_H \|\sigma^{-1}_H \|_{H^2} \{(4K^2 + K^2 \mathbb{1}_{\{\alpha = 1\}} + 2K^2 \mathbb{1}_{\{\alpha \in (0, 1)\}}\} T^2}.
\]

**Proof.** From (A2), it holds that there exists some constant \(c_\varepsilon > 0\) such that, for any \(\varepsilon > 0\),

\[
(4.16) \quad |\sigma^{-1}Z(U^\xi,\eta, V^\xi,\eta)|^2 \leq c_\varepsilon + \{2K^2 \|\sigma^{-1}_H \|_{H^2} (1 + \varepsilon) \mathbb{1}_{\{\alpha = 1\}} + \varepsilon \mathbb{1}_{\{\alpha \in (0, 1)\}}\} (\|U^\xi,\eta\|^2_{\infty} + \|V^\xi,\eta\|^2_{\infty}).
\]

Henceforth, (4.13) follows from (4.16) and Lemma 4.2.
Next, with the aid of (4.12) and (H2) and due to the facts that $\|\hat{U}_t^\xi,\eta\|_\infty \leq \|U_t^\xi,\eta\|_\infty$ and $\|\hat{V}_t^\xi,\eta\|_\infty \leq \|V_t^\xi,\eta\|_\infty$, it follows that
\begin{equation}
|h_t^\xi,\eta(t)|^2 \leq c_\varepsilon + 4 \|\sigma^{-1}\|_{\text{HS}}(4K_1^2 + K_2^2(1 + \varepsilon))\{\|U_t^\xi,\eta\|^2_\infty + \|V_t^\xi,\eta\|^2_\infty\}
\end{equation}
for some $c_\varepsilon > 0$ and
$$\nu_\varepsilon := 4 \|\sigma^{-1}\|_{\text{HS}}\{(4K_1^2 + K_2^2(1 + \varepsilon))\{\alpha\alpha(0,1)\} + 2K_1^2\{\alpha\alpha(0,1)\}\}$$
Thus, by virtue of (4.17) and Lemma 4.2 (4.15) holds true.

Hereinafter, we proceed to finish the

**Proof of Theorem 4.1.** Under the assumption (H1), it is standard to show that
$$\mathbb{E}\left(\sup_{-\tau \leq t \leq T} (|U_t^\xi,\eta(t)|^p + |V_t^\xi,\eta(t)|^p)\right) \leq C_{p,T}(\|\xi\|_\infty^p + \|\eta\|_\infty^p).$$
This, combining Hölder’s inequality with BDG’s inequality, leads to
\begin{equation}
\sup_{0 \leq t \leq T} \mathbb{E}\|U_t^\xi,\eta - \hat{U}_t^\xi,\eta\|_\infty^p + \mathbb{E}\|V_t^\xi,\eta - \hat{V}_t^\xi,\eta\|_\infty^p \leq c \delta^\alpha/2.
\end{equation}
Thus, mimicking the argument of Theorem 2.1 we obtain the desired assertion from (4.18) and Lemma 4.3.

5 Weak Convergence: Integrability Conditions

In the previous sections, we investigated weak convergence of EM schemes for path-dependent SDEs, where the irregular drifts are at most linear growth. In this section we still focus on the topic upon weak convergence but for path-dependent SDEs under integrability conditions, which might allow that the irregular drifts need not to be linear growth.

We start with some additional notation. Denote $C^2(\mathbb{R}^d)$ by the set of all continuously twice differentiable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $C_0^\infty(\mathbb{R}^d)$ by the family of arbitrarily often differentiable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with compact support. Let $\nabla$ and $\nabla^2$ mean the gradient operator and the Hessian operator, respectively. Let $\mathcal{P}(\mathbb{R}^d)$ stand for the collection of all probability measures on $\mathbb{R}^d$. For $\sigma \in M_{\text{non}}^d$ and $V \in C^2(\mathbb{R}^d)$ with $e^{-V} \in L^1(dx)$ and $\mu_0(dx) := C_V e^{-V(x)}dx \in \mathcal{P}(\mathbb{R}^d)$, where $C_V$ is the normalization, set $Z_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by
\begin{equation}
Z_0(x) := -(\sigma\sigma^*)\nabla V(x), \quad x \in \mathbb{R}^d.
\end{equation}
Thus, by the integration by parts formula, the operator
$$\mathcal{L}_0f(x) := \frac{1}{2}\text{tr}((\sigma\sigma^*)\nabla^2 f)(x) + \langle Z_0(x), \nabla f(x) \rangle, \quad x \in \mathbb{R}^d, \ f \in C_0^\infty(\mathbb{R}^d)$$
is symmetric on $L^2(\mu_0)$, i.e., for any $f, g \in C_0^\infty(\mathbb{R}^d),$
$$\mathcal{E}_0(f, g) := \langle f, \mathcal{L}_0 g \rangle_{L^2(\mu_0)} = \langle g, \mathcal{L}_0 f \rangle_{L^2(\mu_0)} = -\langle \sigma^* \nabla f, \sigma^* \nabla g \rangle_{L^2(\mu_0)}.$$
Let $H_{\sigma}^{1,2}$ be the completion of $C_0^\infty(\mathbb{R}^d)$ under the Sobolev norm
\[ \|f\|_{H_{\sigma}^{1,2}} := (\mu_0(|f|^2 + |\sigma^* f|^2))^{1/2}. \]

Then, $(\mathcal{S}_0, H_{\sigma}^{1,2})$ is a symmetric Dirichlet form on $L^2(\mu_0)$ and the associated Markov process can be constructed as the solution to the following reference SDE
\begin{equation}
(5.2) \quad dY(t) = Z_0(Y(t))dt + \sigma dW(t), \quad t > 0, \quad Y(0) = x,
\end{equation}
where $W(t)$ is a $d$-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t \geq 0}$. Assume that

(C1) $Z_0 : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous, i.e., there exist an $L_0$ such that
\[ |Z_0(x) - Z_0(y)| \leq L_0|x - y|, \quad x, y \in \mathbb{R}^d, \]
and there exists constants $C > 0$ and $\beta \in \mathbb{R}$ such that
\[ 2\langle x, Z_0(x) \rangle \leq C + \beta|x|^2, \quad x \in \mathbb{R}^d. \]

Under (C1), $(5.2)$ has a unique solution $(Y^x(t))_{t \geq 0}$ with the initial value $Y^x(0) = x$. Observe that $\mu_0$ is the invariant probability measure of the Markov semigroup $P_t f(x) := \mathbb{E}(Y^x(t))$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, the set of all bounded measurable functions on $\mathbb{R}^d$.

In this section, we consider the following path-dependent SDE
\begin{equation}
(5.3) \quad dX(t) = \left\{ Z_0(X(t)) + \int_{-\tau}^{0} Z(X(t + \theta))\rho(d\theta) \right\} dt + \sigma dW(t), \quad t \geq 0, \quad X_0 = \xi,
\end{equation}
where $\rho(\cdot)$ is a probability measure on $[-\tau, 0]$. Under the assumption $\mu_0$ below, $(5.3)$ admits a unique weak solution by following exactly the argument of Lemma 2.2. The EM scheme associated with $(5.3)$ is given by
\begin{equation}
(5.4) \quad dX^{(\delta)}(t) = \left\{ Z_0(X^{(\delta)}(t)) + \int_{-\tau}^{0} Z(\hat{X}_t^{(\delta)}(\theta))\rho(d\theta) \right\} dt + \sigma dW(t)
\end{equation}
with the initial value $X^{(\delta)}(\theta) = X(\theta) = \xi(\theta), \theta \in [-\tau, 0]$, where
\[ \hat{X}_t^{(\delta)}(\theta) := X^{(\delta)}((t + \theta) \wedge t_\delta), \quad \theta \in [-\tau, -0]. \]

Analogously, we define
\[ \hat{Y}_t^{\xi}(\theta) = Y^{\xi}((t + \theta) \wedge t_\delta), \quad \theta \in [-\tau, -0], \]
where $Y^{\xi}$ was extended as in $(2.6)$. Moreover, we set
\[ h_1^{\xi}(t) := \sigma^{-1} \left\{ Z_0(Y^{\xi}(t)) - Z_0(Y^{\xi}(t_\delta)) - \int_{-\tau}^{0} Z(\hat{Y}_t^{\xi}(\theta))\rho(d\theta) \right\}. \]

One of our main result in this section is as follows, which reveals the weak convergence order of EM scheme for path-dependent SDEs under an integrability condition.
\textbf{Theorem 2.1.} With (5.11) and (5.19) in hand, the proof of Theorem 5.1 can be done by following the line of (5.10) for any \( T > |(5.7) | \), inequality and taking advantage of (2.28), (3.3), and (5.6) enables us to obtain that \( C \).

Then, there exists \( C_{5,T} > 0 \) such that

\[ (5.7) \quad |E f(X^\xi(t)) - E f(X^{(\delta)}(t))| \leq C_{1,T} \delta^\alpha, \quad \xi \in \mathcal{C}, \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad t \in [0,T] \]

for any \( T > 0 \) such that

\[ (5.8) \quad 1 < \frac{\kappa}{2(2 \vee d) ||\sigma^{-1}||_{HS}^2 T^2} \wedge \frac{e^{-(1+\beta)T}}{32 ||\sigma||_{HS}^2 ||\sigma^{-1}||_{HS}^2 T^2} \wedge \frac{\kappa}{(1 \vee \frac{d}{2}) T}. \]

\textbf{Proof.} From \( \xi \in \mathcal{C} \) and (5.6), we infer from Lemma 5.2 below that (5.8) is available so that

\[ (5.9) \quad \mathbb{E} e^{(1+\varepsilon) \int_0^T \int_{-\tau}^0 Z(Y^\xi(t+\theta)) \rho(d\theta) d^2 dt + \mathbb{E} e^{(1+\varepsilon) \int_0^T |\mathbf{h}_\xi(t)|^2 dt} < \infty \]

for some \( \varepsilon \in (0,1) \) sufficiently small and \( T > 0 \) such that (5.8). Next, exploiting H"older's inequality and taking advantage of (2.28), (3.3), and (5.6) enables us to obtain that

\[ (5.10) \quad \int_{-\tau}^0 \mathbb{E} |Z(Y^\xi(t+\theta)) - Z(\hat{Y}_\xi(t))|^p \rho(d\theta) \leq \int_{-\tau}^0 \mathbb{E} \left( \sup_{t_3 \leq s \leq t} |Z(Y^\xi(s)) - Z(Y^\xi(t_3))|^p I_{\{t_3 + \theta \geq s\}} \right) \rho(d\theta) \leq c E \left( \sup_{t_3 \leq s \leq t} (1 + |Z(Y^\xi(s))|^p + |Y^\xi(t_3)|^p) |Y^\xi(s) - Y^\xi(t_3)|^{p\alpha} \right) \leq c \delta^{p\alpha/2}. \]

With (5.11) and (5.19) in hand, the proof of Theorem 5.1 can be done by following the line of Theorem 2.1. \qed

\textbf{Remark 5.1.} The integrability condition (5.5) is explicit and verifiable since the density of \( \mu_0 \) is explicitly given. If \( \mu_0 \) is a Gaussian measure (e.g., \( V(x) = c|x|^2 \) for some constant \( c > 0 \)) and \( Z : \mathbb{R}^d \to \mathbb{R}^d \) is H"older continuous with the H"older exponent \( \alpha \in (0,1) \), then (5.5) holds definitely for any \( \kappa > 0 \). Moreover, the linear growth of \( Z \) imposed in Lemma 2.3 is an essential ingredient, whereas the integrability condition (5.5) does not impose any growth condition on \( Z \) and even allows \( Z \) to be singular at certain setup, e.g., \( Z(x) = (\log \frac{1}{|x|^\alpha}) I_{\{|x| \leq 1\}} + x I_{\{|x| > 1\}}, \quad x \in \mathbb{R}, \) for some \( \alpha \in (0,1) \).

Via the dimension-free Harnack inequality (see e.g. [27, 28]), we can establish the following exponential integrability under an integrability condition, which is an essential ingredient in analyzing weak convergence.
Lemma 5.2. Assume that (C1) and (5.5) hold. Then, for any $\lambda, T > 0$ such that
\begin{equation}
\mathbb{E} e^{\lambda \int_0^T \gamma |Z(Y^{\varepsilon(t+\theta)})|d\theta} < \infty, \quad \lambda < \frac{\kappa}{(1 + \frac{d}{2})T}
\end{equation}
and
\begin{equation}
\mathbb{E} e^{\lambda \int_0^T |\xi(t)|^2 dt} < \infty
\end{equation}
whenever $\lambda, T > 0$ such that
\[ \lambda < \frac{\kappa}{2(2 + d) ||\sigma||^{-2}_{L^2}} \wedge \frac{e^{-(1 + \beta) T}}{32 ||\sigma||^{-2}_{L^2} ||\sigma^{-1}||^{-2}_{L^2} R^2}. \]

Proof. By Hölder’s inequality and Jensen’s inequality, it follows that
\begin{equation}
\mathbb{E} e^{\lambda \int_0^T \gamma |Z(Y^{\varepsilon(t+\theta)})|d\theta} \leq \mathbb{E} e^{\lambda \int_0^T |Z(Y^{\varepsilon(t+\theta)})|^2 d\theta} \leq \frac{1}{T} \int_0^T \mathbb{E} e^{\lambda T |Z(Y^{\varepsilon(t+\theta)})|^2} \rho(d\theta) dt
\end{equation}
\begin{equation}
\leq \frac{1}{T} \left\{ \int_0^T e^{\lambda T |Z(\xi(\theta))|^2} d\theta + \int_0^T \mathbb{E} e^{\lambda T |Z(Y^{\varepsilon(0)}(t))|^2} dt \right\}.
\end{equation}
If, for any $\gamma > 0$ and $p > (1 + d/2)$ with $p \gamma < \kappa$, there exists a continuous positive function $x \mapsto \Lambda_p(x)$ such that
\begin{equation}
\mathbb{E} e^{\gamma |\xi(X^{\varepsilon(t)})|^2} \leq \Lambda_p(x)(1 - e^{-L_0 t})^{-d/2p}(e^{\gamma |\xi(\cdot)|^2})^{1/p},
\end{equation}
then (5.11) holds true due to the facts that $1 - e^{-L_0 t} \sim L_0 t$ as $t \to 0$ and
\[ \int_0^t s^{-d/2p} ds < \infty, \quad p > \frac{d}{2}. \]
In what follows, it remains to verify that (5.14) holds. According to [27, Theorem 3.2] (see also [28, Theorem 1.1]), the following dimension-free Harnack inequality
\begin{equation}
(P_t f(x))^p \leq P_t f^p(x) \exp \left( - \frac{p L_0 |x - y|^2}{2(p - 1)(1 - e^{-L_0 t})} \right), \quad x, y \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d), \quad p > 1
\end{equation}
holds. For any $n, \gamma > 0$ and $p > (1 + d/2)$ with $p \gamma < \kappa$, applying the Harnack inequality (5.15) to the function $\mathbb{R}^d \ni x \mapsto e^{\gamma |\xi(x)|^2} \wedge n \in B_b(\mathbb{R}^d)$ yields that
\[ \exp \left( - \frac{p L_0 |x - y|^2}{2(p - 1)(1 - e^{-L_0 t})} \right) \left( \mathbb{E} (e^{\gamma |Y^{\varepsilon(t)}|^2} \wedge n) \right)^p \leq \mathbb{E} (e^{p \gamma |Z(Y^{\varepsilon(t)})|^2} \wedge n^p), \quad x, y \in \mathbb{R}^d. \]
Thereby, integrating w.r.t. $\mu_0(\text{d}y)$ on both sides and taking the invariance of $\mu_0$ and (5.5) into consideration leads to
\[ \begin{aligned}
\exp \left( - \frac{p L_0}{2(p - 1)} \right) \int_{|x - y|^2 \leq 1 - e^{-L_0 t}} \mu_0(\text{d}y) (\mathbb{E} e^{\gamma |Z(Y^{\varepsilon(t)})|^2} \wedge n)^p \\
\leq \int_{\mathbb{R}^d} \mathbb{E} (e^{p \gamma |Z(Y^{\varepsilon(t)})|^2} \wedge n^p) \mu_0(\text{d}y) \\
\leq \mu_0(e^{p \gamma |Z(\cdot)|^2} \wedge n^p) \leq \mu_0(e^{\gamma |Z(\cdot)|^2}) < \infty, \quad x \in \mathbb{R}^d, \quad p \gamma < \kappa.
\end{aligned} \]
So, by the dominated convergence theorem, we arrive at

\[
(5.16) \quad \left( \int_{|x-y|^2 \leq 1 - e^{-L_0 t}} \mu_0(dy) \right)^{1/p} \mathbb{E} e^{|Z(Y^x(t))|^2} \leq (\mu_0(e^{|Z(.)|^2}))^{1/p} \exp \left( \frac{L_0}{2(p-1)} \right).
\]

Next, from \( \mu_0(dy) = e^{-V(y)}dy \) and Taylor’s expansion, we deduce that

\[
\int_{|x-y|^2 \leq 1 - e^{-L_0 t}} \mu_0(dy) = \int_{|x-y|^2 \leq 1 - e^{-L_0 t}} e^{-V(y)}dy \geq e^{-V(x)} \int_{|z|^2 \leq 1 - e^{-L_0 t}} e^{-\int_0^1 \nabla V(x + \theta z) \cdot dz} \, dz \geq e^{-V(x)} \inf_{|y| \leq 1 + |x|} e^{-|\nabla V(y)|} \int_{|z|^2 \leq 1 - e^{-L_0 t}} e^{-|z| \cdot dz} \geq \frac{\pi^{d/2}}{\Gamma(1 + d/2)} e^{-(1+V(x))} \inf_{|y| \leq 1 + |x|} e^{-|\nabla V(y)|} (1 - e^{-L_0 t})^{d/2},
\]

where \( \Gamma(\cdot) \) is the Gamma function. Thence, inserting (5.17) back into (5.16) gives (5.14).

A direct calculation shows from (C1) and Hölder’s inequality that

\[
|h_1^e(t)|^2 \leq 2\|\sigma^{-1}\|_{\text{HS}}^2 \left\{ 2L_0^2(|Y^\xi(t)|^2 + |Y^\xi(t_0)|^2) + \int_{-\tau}^0 |Z(\hat{Y}_t^\xi(\theta))|^2 \rho(d\theta) \right\}.
\]

Thus, Hölder’s inequality implies that

\[
\mathbb{E} e^\lambda \int_0^T |h_1^e(t)|^2 dt \leq \left( \mathbb{E} e^{16\lambda L_0^2\|\sigma^{-1}\|_{\text{HS}}^2} \int_0^T |Y^\xi(t)|^2 dt \right)^{1/2} \left( \mathbb{E} e^{4\lambda L_0^2\|\sigma^{-1}\|_{\text{HS}}^2} \int_0^T e^{(1+V(\xi(\theta)))} \rho(d\theta) dt \right)^{1/2}
\]

\[
=: \sqrt{I_1(T)} \times \sqrt{I_2(T)}.
\]

On one hand, in view of (2.11), it holds that

\[
(5.18) \quad I_1(T) < \infty, \quad \lambda < \frac{e^{-(1+\beta T)}}{32 \|\sigma^{-1}\|_{\text{HS}}^2 \|\sigma^{-1}\|_{\text{HS}}^2 L_0^2 T^2}.
\]

On the other hand, the Hölder inequality and the Jensen inequality shows that for any \( \lambda > 0 \),

\[
\mathbb{E} e^\lambda \int_{-\tau}^T e^{\int_0^T \mathbb{E} \lambda T |Z(\hat{Y}_t^\xi(\theta))|^2 d\theta} \rho(d\theta) dt \leq \frac{1}{T} \int_{-\tau}^T \int_0^T e^{\lambda T |Z(\hat{Y}_t^\xi(\theta))|^2} d\theta \rho(d\theta) \\
= \frac{1}{T} \int_{-\tau}^T \int_0^T \left\{ e^{\lambda T |Z(Y^\xi(t + \theta))|^2} 1_{t + \theta \leq t_0} + e^{\lambda T |Z(Y^\xi(t_0))|^2} 1_{t + \theta > t_0} \right\} d\theta \rho(d\theta) \\
\leq \frac{1}{T} \left\{ \int_{-\tau}^T e^{\lambda T |Z(\xi(\theta))|^2} d\theta + e^{\lambda T |Z(\xi(0))|^2} + \int_0^T \mathbb{E} e^{\lambda T |Z(Y^\xi(t))|^2} dt + \int_0^T \mathbb{E} e^{\lambda T |Z(Y^\xi(t_0))|^2} dt \right\}
\]

so that, by virtue of (5.14),

\[
(5.19) \quad I_2(T) < \infty, \quad \lambda < \frac{\kappa}{2(2\sqrt{d} \|\sigma^{-1}\|_{\text{HS}}^2 T^2)}.
\]

Thus, (5.12) follows (5.18) as well as (5.19) immediately.
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