A Brief History of Algebra with a Focus on the Distributive Law and Semiring Theory

Peyman Nasehpour
Department of Engineering Science
Golpayegan University of Technology
Golpayegan, Isfahan Province
IRAN
nasehpour@gut.ac.ir, nasehpour@gmail.com

August 1, 2018

Abstract

In this note, we investigate the history of algebra briefly. We particularly focus on the history of rings, semirings, and the distributive law.

“I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history.”
- J. W. L. Glaisher (1848–1928), English mathematician

“One can conceive history as an argument without end.”
- Pieter Catharinus Arie Geijl (1887–1966), Dutch historian

1 Etymology of Algebra

The word “algebra” is derived from the Arabic word Al-Jabr, and this comes from the treatise written in 820 by the medieval Persian mathematician [44], Muhammad ibn Musa al-Khwarizmi, entitled, in Arabic “Kitāb al-mukhtaṣar fī hisab al-ğabr wa-l-muqābala”, which can be translated as “The Compendious Book on Calculation by Completion and Balancing” [48].

1This is a translation of Geyl’s sentence in his book with the title “Napoleon: Voor en Tegen in de Franse Geschiedschrijving”, published in 1946. The original sentence in Dutch [21] is: “Men kan de geschiedschrijving opvatten als een discussie zonder eind.”

2Key words and phrases. History of Algebra, Distributive Law, Semiring Theory

2010 Mathematics Subject Classification. 01A05, 97A30
2 A Brief History of Algebra

One may consider the history of algebra to have two main stages, i.e., “classical algebra” that mostly was devoted to solving the (polynomial) equations and “abstract algebra”, also called “modern algebra” that is all about the study of algebraic structures, something that today algebraists do.

2.1 Classical Algebra

Classical algebra can be divided into three sub-stages:

1. In the very early stages of algebra, concepts of algebra were geometric, proposed by the Babylonians apparently and developed by the Greeks and later revived by Persian mathematician Omar Khayyam (1048–1131) since the main purpose of Khayyam in this matter was to solve the cubic equations with intersecting conics [15, p. 64].

2. The main step of passing from this substage to the substage of equation-solving was taken by Persian mathematician al-Khwarizmi (c.780–c. 850) in his treatise Al-Jabr that it gives a detailed account of solving polynomials up to the second degree.

3. While the mathematical notion of a function was implicit in trigonometric tables, apparently the idea of a function was proposed by the Persian mathematician Sharaf al-Din al-Tusi (died 1213/4), though his approach was not very explicit, perhaps because of this point that dealing with functions without symbols is very difficult. Anyhow algebra did not decisively move to the dynamic function substage until the German mathematician Gottfried Leibniz (1646–1716) [33].

One may also view the history of the development of classical algebra in another perspective namely passing from rhetorical substage to full symbolic substage. According to this view, the history of algebra is divided into three substages through the development of symbolism: 1. Rhetorical algebra, 2. Syncopated algebra, and 3. Symbolic algebra [46].

1. In rhetorical algebra, all equations are written in full sentences, but in symbolic algebra, full symbolism is used, the method that we do today. So far as we know, rhetorical algebra was first developed by the ancient Babylonians and was developed up to 16th century.

2. Syncopated algebra used some symbolism but not as full as symbolic algebra. It is said that syncopated algebraic expression first appeared in the book Arithmetica by the ancient Greek mathematician Diophantus (born sometime between AD 201 and 215 and passed away sometime between AD 285 and 299) and was continued in the book Brahma Sphuta Siddhanta by the ancient Indian mathematician Brahmagupta (598–c.670 CE).

3 Also, see The Evolution of Algebra, Science, 18(452) (1891), 183–187. Retrieved from http://www.jstor.org/stable/1766702.
3. The full symbolism can be seen in the works of the French mathematician, René Descartes (1596–1650), though early steps of symbolic algebra was taken by Moroccan mathematician Ibn al-Bannā al-Marakūshī (1256–c.1321) and Andalusian mathematician Abū al-Hasan al-Qalasādī (1412–1486) [41, p. 162].

2.2 Modern Algebra

The transition of algebra from the “classical” to the “modern” form occurred in about the middle of 19th century when mathematicians noticed that classical tools are not enough to solve their problems. During the time of classical algebra in the Renaissance, Italian mathematicians Scipione del Ferro (1465–1526) and Niccolò Tartaglia (1499/1500–1557) found the solution of the equations of degree 3 [29, p. 10] and another Italian mathematician Lodovico Ferrari (1522–1565) solved equations of degree 4, but, then, it was the Italian mathematician and philosopher Paolo Ruffini (1765–1822) and later the Norwegian mathematician Niels Henrik Abel (1802–1829) who used abstract algebra techniques to show that equations of degree 5 and of higher than degree 5 are not always solvable using radicals (known as Abel-Ruffini theorem [47, §13]). Finally, the French mathematician Évariste Galois (1811–1832) used group theory techniques to give a criterion deciding if an equation is solvable using radicals. We also need to add that the Italian mathematician Gerolamo Cardano (1501–1576), who is definitely one of the most influential mathematicians of the Renaissance, received a method of solving cubic equation from Tartaglia and promised not to publish it but he did. Since Cardano was the first to publish the explicit formula for solving the cubic equations, this is most probably why this formula is called Cardano’s formula. At the end, though the British mathematician Arthur Cayley (1821–1895) was the first who gave the abstract definition of a finite group [12], the English mathematician George Boole (1815–1864), in his book Mathematical Analysis of Logic (1847), was most probably the first who formulated an example of a non-numerical algebra, a formal system, which can be investigated without explicit resource to their intended interpretations [18, p. 1]. It is also good to mention that the first statement of the modern definition of an abstract group was given by the German mathematician Walther Franz Anton von Dyck (1856–1934) [56].

3 A Brief History of the Distributive Law

An algebraic structure on a set (called underlying set or carrier set) is essentially a collection of finitary operations on it [14, p. 41, 48]. Since ring-like structures have two binary operations, often called addition and multiplication, with multiplication distributing over addition and the algebraic structure “semiring” is one of them, we continue this note by discussing the history of the distributive law in mathematics briefly.

The distributive law, in mathematics, is the law relating the operations of multiplication and addition, stated symbolically, \( a(b + c) = ab + ac \). Ancient Greeks were aware of this law. The first six books of Elements presented the rules and techniques of plane geometry. Book I included theorems about congruent triangles, constructions using a ruler and compass, and the proof of the Pythagorean theorem about the
lengths of the sides of a right triangle. Book II presented geometric versions of the distributive law \(a(b + c + d) = ab + ac + ad\) and formulas about squares, such as \((a + b)^2 = a^2 + 2ab + b^2\) and \(a^2 - b^2 = (a + b)(a - b)\) [6, p. 32].

Apart from arithmetic, the distributive law had been noticed, years before the birth of abstract algebra, by the inventors of symbolic methods in the calculus. While the first use of the name distributive operation is generally credited to the French mathematician François-Joseph Servois (1768–1847) [45], the Scottish mathematician Duncan Farquharson Gregory (1813–1844), who wrote a paper in 1839 entitled *On the Real Nature of Symbolical Algebra*, brought out clearly the commutative and distributive laws [11, p. 331].

Boole in his book entitled *Mathematical Analysis of Logic* (1847) mentioned this law by giving the name distributive to it and since he was, most probably, the first who formulated an example of a non-numerical algebra, one may consider him the first mathematician who used this law in abstract algebra [4]. Some years later, Cayley, in a paper entitled *A Memoir on the Theory of Matrices* (1858), showed that multiplication of matrices is associative and distributes over their finite addition [15].

The distributive law appeared naturally in ring-like structures as well. Though the first axiomatic definition of a ring was given by the German mathematician Abraham Halevi (Adolf) Fraenkel (1891–1965) and he did mention the phrase “the distributive law” (in German “die distributiven Gesetze”), but his axioms were stricter than those in the modern definition [20, p. 11]. Actually, the German mathematician Emmy Noether (1882–1935) who proposed the first axiomatic modern definition of (commutative) rings in her paper entitled “Idealtheorie in Ringbereichen”, did also mention the phrase “Dem distributiven Gesetz” among others such as “Dem assoziativen Gesetz” and “Dem kommutativen Gesetz” [42, p. 29].

## 4 A Brief History of Semirings

The most familiar examples for semirings in classical algebra are the semiring of non-negative integers or the semiring of non-negative real numbers. The first examples of semirings in modern algebra appeared in the works of the German mathematician Richard Dedekind (1831–1916) [16], when he worked on the algebra of the ideals of rings [25]. In point of fact, it was Dedekind who proposed the concept of ideals in his earlier works on number theory, as a generalization of the concept of “ideal numbers” developed by the German mathematician Ernst Kummer (1810–1893) [37]. Others such as the English mathematician Francis Sowerby Macaulay (1862–1937) and the German mathematicians Emanuel Lasker (1868–1941), Emmy Noether (1882–1935), Wolfgang Krull (1899–1971) and Paul Lorenzen (1915–1994) also studied the algebra of ideals of rings [25, 38, 40, 42]. Semirings appeared implicitly in the works of the German mathematician David Hilbert (1862–1943) and the American mathematician Edward Vermilye Huntington (1874–1952) in connection with the axiomatization of the natural and nonnegative rational numbers [25, 31, 32]. But, then, it was the American mathematician Harry Schultz Vandiver (1882–1973) who used the term “semi-ring” in his 1934 paper entitled “Note on a simple type of algebra in which cancellation law of addition does not hold” for introducing an algebraic structure with two operations
of addition and multiplication such that multiplication distributes on addition, while
cancellation law of addition does not hold [49]. The foundations of algebraic theory
for semirings were laid by Samuel Bourne and others in the 1950’s [1, p. 6]. For
example, the concept of ideals for semirings was introduced by Samuel Bourne [5].
In the years between 1939 and 1956, V andiver published at least six more papers on
semirings [50–55], but it seems he was not successful to draw the attention of mathe-
maticians to consider semirings as an independent algebraic structure that is worth to
be developed [27]. In fact, semirings, as most of the other concepts in mathematics,
were not developed as an exercise for generalization, only for the sake of generaliza-
tion! Actually, in the late 1960s, semirings were considered as a more serious topic
by researchers when real applications were found for them. The Polish-born American
mathematician Samuel Eilenberg (1913–1998) and a couple of other mathematicians
started developing formal languages and automata theory systematically [19], which
they have strong connections to semirings. Since then many mathematicians and com-
puter scientists have broadened the theory of semirings and related structures [23].
Definitely, the reference books on semirings and other ring-like algebraic structures
including the books [1, 3, 17, 24–26, 30, 36, 43] have helped to the popularity of these
rather new algebraic structures. Today many journals specializing in algebra have edi-
tors who are responsible for semirings. Semirings not only have significant applications
in different fields such as automata theory in theoretical computer science, (combinato-
rial) optimization theory, and generalized fuzzy computation, but are fairly interesting
generalizations of two broadly studied algebraic structures, i.e., rings and bounded dis-
tributive lattices [23, 25]. The number of publications in the field of semiring theory, the
beauty of the semirings and their broad applications in different areas of science should
convince us that today semiring theory is an established one and its development, even
in pure mathematics, is valuable and important.

5 References for further studies

In order to have a bit more of “an argument without end”, the reader may like to refer
the books [6–10, 57] on the general history of mathematics and the books [2, 34, 48] on
the history of algebra.

Acknowledgements

The author’s main interests in algebra are in commutative algebra and semiring theory.
It is a pleasure to thank both Professor Dara Moazzami and Professor Winfried Bruns
to help and encourage the author in order to work on these fields of algebra. The
author is supported in part by the Department of Engineering Science at Golpayegan
University of Technology and his special thanks go to the Department for providing all
the necessary facilities available to him for successfully conducting this research.
References

[1] J. Ahsan, J. N. Mordeson and M. Shabir, Fuzzy Semirings with Applications to Automata Theory, Springer, Berlin, 2012.

[2] H.-W. Alten, A. Djafari Naini, B. Eick, M. Folkerts, H. Schlosser, K.-H. Schlote, H. Wesemüller-Kock and H. Wußing, 4000 Jahre Algebra: Geschichte - Kulturen - Menschen, Springer Spektrum, Berlin, 2014.

[3] S. Bistarelli, Semirings for Soft Constraint Solving and Programming, Springer-Verlag, Berlin, 2004.

[4] G. Boole, The Mathematical Analysis of Logic, Being an Essay towards a Calculus of Deductive Reasoning, Cambridge: Macmillan, Barclay, & Macmilan, London: George Bell, 1847.

[5] S. Bourne, The Jacobson radical of a semiring, Proc. Nat. Acad. Sci. 37 (1951), 163–170.

[6] M. J. Bradley, The Birth of Mathematics: Ancient Times to 1300 Vol. 1. Chelsea House Infobase Publishing, New York, 2006.

[7] M. J. Bradley, The Age of Genius: 1300 to 1800, Vol. 2. Chelsea House Infobase Publishing, New York, 2006.

[8] M. J. Bradley, The Foundations of Mathematics: 1800 to 1900, Vol. 3. Chelsea House Infobase Publishing, New York, 2006.

[9] M. J. Bradley, Modern Mathematics: 1900 to 1950, Vol. 4. Chelsea House Infobase Publishing, New York, 2006.

[10] M. J. Bradley, Mathematics Frontiers: 1950 to the Present, Vol. 5. Chelsea House Infobase Publishing, New York, 2006.

[11] F. Cajori, A History of Mathematics, The Macmillan Company, London, 1909.

[12] A. Cayley, VII. On the theory of groups, as depending on the symbolic equation θn = 1, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 7(42) (1854), 40–47.

[13] A. Cayley, A Memoir on the Theory of Matrices, Philos. Trans. R. Soc. Lond., 148 (1858), 17–37.

[14] P. M. Cohn, Universal Algebra, D. Reidel Publishing Company, Dordrecht, 1981.

[15] R. Cooke, Classical Algebra, Its Nature, Origins, and Uses, John Wiley and Sons, Inc. Publication, Hoboken, 2008.

[16] R. Dedekind, Über die Theorie der ganzen algebraischen Zahlen, Supplement XI to P.G. Lejeune Dirichlet: Vorlesung über Zahlentheorie 4 Aufl., Druck und Verlag, Braunschweig, 1894.
[17] M. Droste, W. Kuich and H. Vogler, *Handbook of Weighted Automata*, EATCS Monographs in Theoretical Computer Science, Springer, Berlin, 2009.

[18] J. M. Dunn and G. M. Hardegree, *Algebraic methods in philosophical logic*, Oxford University Press, New York, 2001.

[19] S. Eilenberg, *Automata, Languages, and Machines*, Vol. A., Academic Press, New York, 1974.

[20] A. A. Fraenkel, *Über die Teiler der Null und die Zerlegung von Ringen*, dissertation, Marburg University, 1914.

[21] P. C. A. Geyl, *Napoleon: Voor en Tegen in de Franse Geschiedschrijving*, Oostenhoek’s Uitgevers Mij. N.V., Utrecht, 1946.

[22] W. J. Gilbert and S. A. Vanstone, *Classical Algebra*, 3rd ed., Waterloo Mathematics Foundation, Waterloo, 1993.

[23] K. Głazek, *A guide to the literature on semirings and their applications in mathematics and information sciences*, Kluwer, Dordrecht, 2002.

[24] J. S. Golan, *Power Algebras over Semirings, with Applications in Mathematics and Computer Science*, Kluwer, Dordrecht, 1999.

[25] J. S. Golan, *Semirings and Their Applications*, Kluwer, Dordrecht, 1999.

[26] J. S. Golan, *Semirings and Affine Equations over Them: Theory and Applications*, Kluwer, Dordrecht, 2003.

[27] J. S. Golan, *Some recent applications of semiring theory*, International Conference on Algebra in Memory of Kostia Beider at National Cheng Kung University, Tainan, 2005.

[28] M. Gondran, M. Minoux, *Graphs, Diods and Semirings*, Springer, New York, 2008.

[29] H. Grant, I. Kleiner, *Turning Points in the History of Mathematics*, Birkhäuser, New York, 2015.

[30] U. Hebisch and H. J. Weinert, *Semirings, Algebraic Theory and Applications in Computer Science*, World Scientific, Singapore, 1998.

[31] D. Hilbert, *Über den Zahlbegriff*, Jber. Deutsch. Math.-Verein. 8 (1899), 180–184.

[32] E. V. Huntington, *Complete sets of postulates for the theories of positive integral and positive rational numbers*, Trans. Amer. Math. Soc. 3 (1902), 280–284.

[33] V. J. Katz and B. Barton, *Stages in the history of algebra with implications for teaching*, Educational Studies in Mathematics 66 (2007), 185–201.

[34] I. Kleiner, *A History of Abstract Algebra*, Birkhäuser, Boston, 2007.
[35] W. Krull, *Axiomatische Begründung der Algemeinen Idealtheorie*, Sitz. phys. med. Soc. Erlangen **56** (1924), 47–63.

[36] W. Kuich and A. Salomaa, *Semirings, Automata, Languages*, Springer-Verlag, Berlin, 1986.

[37] E. E. Kummer, *Über die Zerlegung der aus Wurzeln der Einheit gebildeten komplexen Zahlen in ihre Primfaktoren*, Jour. für Math. (Crelle) **35** (1847) 327–367.

[38] E. Lasker, *Zur Theorie der Moduln und Ideale*, Math. Ann. **60** (1905), 19–116.

[39] P. Lorenzen, *Abstrakte Begründung der multiplikativen Idealtheorie*, Math. Z., **45** (1939), 533–553.

[40] F. S. Macaulay, *Algebraic Theory of Modular Systems*, Cambridge University Press, Cambridge, 1916.

[41] C. B. Boyer and U. C. Merzbach, *A History of Mathematics*, 3rd edn, John Wiley and Sons Inc., Hoboken, 2011.

[42] E. Noether, *Idealtheorie in Ringbereichen*, Mathematische Annalen **83**(1–2) (1921), 24–66.

[43] G. Pilz, *Near-Rings, The Theory and Its Applications*, Revised edition, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1983.

[44] G. Saliba, *Science and medicine*, Iranian Studies, **31**(3–4) (1998), 681–690.

[45] F. J. Servois, *Essai sur un nouveau d'exposition des principes du calcul différentiel*, Ann. Math. Pures Appl. **5** (1814), 93–140.

[46] L. Stalling, *A brief history of algebraic notation*, School Science and Mathematics, **100**(5), 230–235.

[47] J.-P. Tignol, *Galois' Theory of Algebraic Equations*, 2nd edn., World Scientific, Singapore, 2016.

[48] B. L. van der Waerden, *A History of Algebra: From al-Khwārizmi to Emmy Noether*, Springer, Berlin, 1985.

[49] H. S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. Vol. **40** (1934), 914–920.

[50] H. S. Vandiver, *On some simple types of semi-rings*, Am. Math. Mon. **46** (1939), 22–26.

[51] H. S. Vandiver, *On the imbedding of one semi-group in another, with application to semi-rings*, American Journal of Mathematics, **62**(1) (1940), 72–78.

[52] H. S. Vandiver, *A development of associative algebra and an algebraic theory of numbers. I*, Math. Mag. **25** (1952), 233–250.
[53] H. S. Vandiver, *A development of associative algebra and an algebraic theory of numbers. II*, Math. Mag. 27 (1954), 1–18.

[54] H. S. Vandiver and M. W. Weaver, *A development of associative algebra and an algebraic theory of numbers. III*, Math. Mag. 29 (1955), 135–151.

[55] H. S. Vandiver and M. W. Weaver, *A development of associative algebra and an algebraic theory of numbers. IV*, Math. Mag. 30 (1956), 1–8.

[56] W. von Dyck, *Gruppentheoretische Studien*, Mathematische Annalen 20(1) (1882), 1–44.

[57] B. Wardhaugh, *How to Read Mathematical History*, Princeton University Press, Princeton, 2010.