SHARP INEQUALITIES IN THE UNIT POLYDISC

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Abstract. Motivated by some results of Burbea we prove that if certain sharp inequality holds for functions in the unit polydisc which belong to concrete Hardy spaces, we prove that it also holds in the case of functions from arbitrary Hardy spaces. We also examine the equality case. The second half of the paper is devoted to the applications.

1. Classes of analytic functions in the unit polydisc

Denote by $U$ the unit disc in the complex plane $\mathbb{C}$, i.e., $\{z \in \mathbb{C} : |z| < 1\}$. Let $n$ be an integer $\geq 1$ and let $\mathbb{C}^n = \{z = (\ldots, z_k, \ldots) : z_k \in \mathbb{C}, \ k = 1, \ldots, n\}$ be an $n$–dimensional complex vector space. Let $U^n \subseteq \mathbb{C}^n$ be the unit polydisc and $T^n$ the torus. The measure $m_n$ is the normalized Lebesgue measure on $T^n$. If $(X, \nu)$ is a measure space and $0 < p \leq \infty$, then $L^p(X, \nu)$ stands for the Lebesgue space.

Following the Rudin monograph [27] we recall the definitions of some known function spaces in the unit polydisc. The class $N(U^n)$ contains all analytic functions $f(z)$ in the unit polydisc which satisfy the growth condition

$$\sup_{0 \leq r < 1} \int_{T^n} \log^+ |f(r\zeta)| \, dm_n(\zeta) < +\infty.$$ 

In other words, the functions $\log^+ |f_r(\zeta)|$ are required to lie in a bounded subset of the Lebesgue space $L^1(T^n, m_n)$. The function $f_r(z)$ is the dilatation $f_r(z)$, $z \in U$. Recall that $\log^+ x = \log x$ if $x > 1$, $\log x = 0$ if $x < 1$. The class $N^*(U^n)$ is the class of all $f \in N(U^n)$ for which the functions $\log^+ |f_r(\zeta)|$ form a uniformly integrable family.

We call a function $\varphi$ strongly convex if $\varphi$ is convex on $(-\infty, +\infty)$, $\varphi \geq 0$, $\varphi$ is non–decreasing, and $\varphi(t)/t \to +\infty$ as $t \to +\infty$. If $\varphi$ is a strongly convex function, define $H_{\varphi}(U^n)$ to be the class of all analytic functions $f$ in $U^n$ for which

$$\sup_{0 \leq r < 1} \int_{T^n} \varphi(\log|f(\zeta)|) \, dm_n(\zeta) < +\infty.$$ 

For $0 < p < +\infty$, $H^p(U^n)$ is $H_{\varphi}(U^n)$, with $\varphi(t) = e^{pt}$, and

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(f, r),$$ 

where

$$M_p(f, r) = \left\{ \int_{T^n} |f(r\zeta)|^p \, dm_n(\zeta) \right\}^{1/p}.$$ 

Since $|f|$ is $n$–subharmonic, $\sup_{0 \leq r < 1}$ in the expression for $\|f\|_p$ can be replaced by $\lim_{r \to 1-}$. By Theorem 3.1.2 in [27], $N^*(U^n)$ is the union of all $H_{\varphi}(U^n)$.

\textit{2010 Mathematics Subject Classification:} Primary 30H10

\textit{Key words and phrases.} Hardy spaces, logarithmically subharmonic functions, the isoperimetric inequality.

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If \( f(z) \) is any function in \( U^n \), we define \( f^*(\zeta) \) by
\[
f^*(\zeta) = \lim_{r \to 1^-} f(r\zeta)
\]
at every \( \zeta \in \mathbb{T}^n \) at which this radial limit exists. If \( f \in N(U^n) \), it is known that \( f^*(\zeta) \) exist for almost every \( \zeta \in \mathbb{T}^n \) and \( \log |f^*(\zeta)| \in L^1(\mathbb{T}^n, m_n) \). Within \( N^*(U) \), the \( H_k \)-classes are characterized by their boundary values: Suppose \( f(z) \in N^*(U^n) \) and \( \varphi \) is strongly convex, then
\[
f(z) \in H_\varphi(U^n) \iff \varphi(\log |f^*(\zeta)|) \in L^1(\mathbb{T}^n, m_n).
\]
If this is the case, then \( \int_{\mathbb{T}^n} \varphi(\log |f_r(\zeta)|) \, d\mu(\zeta) \) is increasing in \( 0 \leq r < 1 \), and
\[
\lim_{r \to 1^-} \int_{\mathbb{T}^n} \varphi(\log |f_r(\zeta)|) \, d\mu(\zeta) = \int_{\mathbb{T}^n} \varphi(\log |f^*(\zeta)|) \, d\mu(\zeta).
\]
Moreover, by Theorem 3.4.3, if \( 0 < p < +\infty \) and \( f \in H^p(U^n) \), then we have convergence in mean:
\[
\lim_{r \to 1^-} \int_{\mathbb{T}^n} |f_r(\zeta) - f^*(\zeta)|^p \, d\mu(\zeta) = 0.
\]
Every \( f(z) \in H^p(U^n) \), \( 1 \leq p < \infty \) is the Poisson integral as well as the Cauchy integral of \( f^*(\zeta) \). I.e.,
\[
f(z) = \int_{\mathbb{T}^n} K(z, \zeta) f^*(\zeta) \, d\mu(\zeta), \quad z \in U^n,
\]
here \( K(z, \zeta) \) stands for the Cauchy kernel:
\[
K(z, \zeta) = \prod_{k=1}^n (1 - z_k \bar{\zeta}_k)^{-1},
\]
for \( z = (\ldots, z_k, \ldots) \in U^n \) and \( \zeta = (\ldots, \zeta_k, \ldots) \in \mathbb{T}^n \).

2. The main result of this paper

Let \( dA \) stands for the Lebesgue measure in \( \mathbb{C} \). A weighted measure in the unit disc is a measure of the form
\[
d\mu(z) = g(z) \, dA(z), \quad g(z) > 0, \quad z \in U.
\]
A weighted measure in the polydisc \( U^n \) is a product of \( n \) weighted measures in the unit disc.

In the sequel the letter \( m \) always denotes an integer \( \geq 1 \) and \( p \) (with or without an index) any positive number. Denote by \( \mathbb{R}^m \) the set \( \{ (\ldots, x_j, \ldots) \in \mathbb{R}^m : x_j \geq 0, \ j = 1, \ldots, m \} \). Let \( \Phi : \mathbb{R}^m \to \mathbb{R}_+ \) be continuous and strictly increasing in each variable which moreover satisfies \( \Phi(x_1, \ldots, x_m) = 0 \) if \( x_j = 0 \) for some \( 1 \leq j \leq m \).

For a weighted measure in the unit disc \( \mu \) denote by \( \nu_n = \nu_{n-1} \times \mu \) a weighted measure in the unit polydisc \( U^n \). We will prove our main result under an assumption that \( \Phi \) and \( \mu \) satisfy the following condition:

\( \dagger \) There exists \( \tilde{p}_j \) \( (j = 1, \ldots, m) \) such that one can establish
\[
\Phi(|f_1|^{\tilde{p}_1}, \ldots, |f_n|^{\tilde{p}_n}) \in L^1(U^n, \nu_n)
\]
and the inequality
\[
\int_{U^n} \Phi(|f_1(z)|^{\tilde{p}_1}, \ldots, |f_n(z)|^{\tilde{p}_n}) \, d\nu_n(z) \leq \Phi(\|f_1\|^{\tilde{p}_1}, \ldots, \|f_n\|^{\tilde{p}_n})
\]
for all $f_j(z) \in H^{p_j}(\mathbb{U}^n)$ $(j = 1, \ldots, m)$, with the equality sigh if and only if either

1. $\prod_{j=1}^{m} f_j \equiv 0$ (i.e. $f_j \equiv 0$ for some $1 \leq j \leq m$) or
2. each $f_j$ $(j = 1, \ldots, m)$ is equal to $f_j(z) = \Psi^p_j(z) \not\equiv 0$, where $(\ldots, \Psi^p_j, \ldots)$ belongs to a class denoted by $\mathcal{E}(\Phi, \nu_n)$.

The class $\mathcal{E}(\Phi, \nu_n)$ is the family of extremals for the inequality (1). Note that if $C_j$ is a constant which satisfies $|C_j| = 1$ (for all $j = 1, \ldots, m$) and if $(\ldots, \Psi^p_j, \ldots) \in \mathcal{E}(\Phi, \nu_n)$, then also $(\ldots, C_j \Psi^p_j, \ldots) \in \mathcal{E}(\Phi, \nu_n)$. If $\Phi$ satisfies $\Phi(\ldots, \alpha_j x_j, \ldots) = \alpha_j \Phi(\ldots, x_j, \ldots), \alpha_j > 0, x_j \geq 0, j = 1, \ldots, m$, then the preceding holds for all $C_j \not\equiv 0$.

One of our main goals in this paper is to prove the next

**Theorem 2.1.** Assume that $\Phi$ and $\mu$ satisfy the condition (†). Let $f_j(z) \in H^{p_j}(\mathbb{U}^n), 0 < p_j < +\infty$ for all $j = 1, \ldots, m$. Then

$$
\Phi(|f_1|^{p_1}, \ldots, |f_n|^{p_n}) \in L^1(\mathbb{U}^n, \nu_n)
$$

with

$$
\int_{\mathbb{U}^n} \Phi(|f_1(z)|^{p_1}, \ldots, |f_n(z)|^{p_n}) \, d\nu_n(z) \leq \Phi(\|f_1\|^{p_1}, \ldots, \|f_n\|^{p_n}).
$$

Moreover:

1. Each extremal $\Psi^p_j, j = 1, \ldots, m$ for the inequality (1), i.e., $(\ldots, \Psi^p_j, \ldots) \in \mathcal{E}(\Phi, \nu_n)$, vanishes nowhere in $\mathbb{U}^n$, and
2. equality attains in (2) if and only if either $\prod_{j=1}^{m} f_j \equiv 0$ or each $f_j$ $(j = 1, \ldots, m)$ is of the form $f_j(z) = \Psi^p_j(z)^{\tilde{p}_j/p_j}$ for some $(\ldots, \Psi^p_j, \ldots) \in \mathcal{E}(\Phi, \nu_n)$.

3. Preliminaries for the proof of Theorem 2.1

3.1. The case $n = 1$ of our Theorem 2.1 is straightforward to obtain and we consider it here. We will prove

**Theorem 3.1.** Assume that $\Phi$ and $\mu$ satisfy the condition (†). Let $f_j(z) \in H^{p_j}, 0 < p_j < +\infty$ for all $j = 1, \ldots, m$. Then

$$
\Phi(|f_1|^{p_1}, \ldots, |f_n|^{p_n}) \in L^1(\mathbb{U}, \mu)
$$

with

$$
\int_{\mathbb{U}} \Phi(|f_1(z)|^{p_1}, \ldots, |f_n(z)|^{p_n}) \, d\mu(z) \leq \Phi(\|f_1\|^{p_1}, \ldots, \|f_n\|^{p_n}).
$$

Moreover:

1. Every $\Psi^p_j (j = 1, \ldots, m)$, where $(\ldots, \Psi^p_j, \ldots) \in \mathcal{E}(\Phi, \mu)$, annihilates nowhere in the unit disc.
2. Equality attains in (3) if and only if either $\prod_{j=1}^{m} f_j \equiv 0$ or each $f_j$ $(j = 1, \ldots, m)$ is of the form $f_j(z) = \Psi^p_j(z)^{\tilde{p}_j/p_j}$ for some $(\ldots, \Psi^p_j, \ldots) \in \mathcal{E}(\Phi, \mu)$.

**Proof.** Without lost of generality, suppose that $f_j \not\equiv 0$ for all $j = 1, \ldots, m$. By the Riesz theorem it is possible to obtain the factorization $f_j(z) = B_j(z) h_j(z)$, where $B_j$ is the Blaschke product for $f_j$. Recall that one takes $|B_j| = 1$, if $f_j$ is zero-free. Since $h_j$ does not vanish in the unit disc, it is possible to obtain a branch $\tilde{h}_j(z) = h_j(z)^{\tilde{p}_j/p_j}$ there. Since $|B_j(z)| \leq 1$ everywhere in the disc, we have

$$
|f_j(z)| \leq |h_j(z)|, \quad z \in \mathbb{U}.
$$
Since \(|B_j(\zeta)| = 1\) for almost every \(\zeta \in \mathbb{T}\), it follows \(|h_j(\zeta)| = |f_j(\zeta)|\) a.e. on \(\mathbb{T}\). Thus
\[
(5) \quad \|\tilde{h}_j\|_{\tilde{p}_j} = \|f_j\|_{p_j},
\]
which implies \(\tilde{h}_j \in H^{\tilde{p}_j}\).

In view of (4), since \(\Phi\) is increasing in each variable, we have
\[
(6) \quad \int_{\mathbb{U}} \Phi(\ldots, |f_j(z)|^{p_j}, \ldots) \, d\mu(z) \leq \int_{\mathbb{U}} \Phi(\ldots, |\tilde{h}_j(z)|^{\tilde{p}_j}, \ldots) \, d\mu(z).
\]
Regarding (5), we first obtain \(\Phi(\ldots, |\tilde{h}_j|^{\tilde{p}_j}, \ldots) \in L^1(\mathbb{U}, \mu)\). This means that both integrals in (6) are finite, hence \(\Phi(\ldots, |f_j|^{p_j}, \ldots) \in L^1(\mathbb{U}, \mu)\). Further,
\[
(7) \quad \int_{\mathbb{U}} \Phi(\ldots, |\tilde{h}_j(z)|^{\tilde{p}_j}, \ldots) \, d\mu(z) \leq \Phi(\ldots, \|\tilde{h}_j\|_{p_j}^{p_j}, \ldots).
\]
Regarding (5), the inequality of this theorem follows from the relations (6) and (7).

Let us consider now the second half of this theorem. If equality attains in (3), then equality must hold in (7) and (6). Applying the equality statement of (7), we infer that equality holds in (5) if and only if \(\tilde{h}_j(z) = \Psi_j^1(z), \ j = 1, \ldots, m\) for some \((\ldots, \Psi_j^1, \ldots) \in E(\Phi, \mu)\). It follows that each \(\Psi_j^1\) is zero–free. Now, equality holds in (6) if and only if \(|B_j(z)| \equiv 1\) for all \(j = 1, \ldots, m\). This means that
\[
(\setminus) \quad f_j(z) = C_j \Psi_j^1(z)^{\tilde{p}_j/p_j} = \{C_j \Psi_j^1(z)\}^{\tilde{p}_j/p_j},
\]
where \(|C_j| = |\tilde{C}_j| = 1\) are new constants (for all \(j = 1, \ldots, m\)). \(\Box\)

**Remark 3.1.** Due to the non–existence of a direct analogue of the Riesz factorization theorem for Hardy spaces (and for the Nevanlinna space) on the unit polydisc, a proof of our theorem is somewhat harder for \(n > 1\). However, it is possible to give a proof using a factorization theorem, but with some constrains. This will be shown at the end of the fourth section.

3.2. In our complete proof of Theorem 2.1 the main role play logarithmically subharmonic functions of the Hardy classes (classes \(PL_p\) for positive \(p\) introduced in the sequel). Recall, a function \(U\) is logarithmically subharmonic in a domain \(D\) if \(U \equiv 0\), or it is possible to represent it in the form \(U(z) = e^{u(z)}, \ z \in D\), where \(u(z)\) is a subharmonic function in \(D\). In the next section we will use the following two lemmas concerning (logarithmically) subharmonic functions. Their proofs may be found at the beginning of the Ronkin monograph [25]. Actually, regarding the following remark, it is enough to consider both lemmas for subharmonic functions:

A function \(U(x, y)\) is logarithmically subharmonic in \(D\) if and only if \(e^{\alpha x + \beta y}U(x, y)\) is subharmonic in \(D\) for every choice of real numbers \(\alpha\) and \(\beta\).

**Lemma 3.1** (cf. [25]). Let \(X\) be a non–empty set and let \(\{U_x, \ x \in X\}\) be a family of (logarithmically) subharmonic functions in a domain \(D\). Then
\[
u(z) = \sup_{x \in X} U_x(z)
\]
is also (logarithmically) subharmonic in \(D\) if it is upper semi–continuous in this domain.
Lemma 3.2 (cf. [25]). Let $U(z,x)$ be upper semi–continuous in $D \times X$, where $D$ is a domain and $X$ is a topological space. Moreover, let $\mu$ be a finite measure on $X$. Then

$$U(z) = \int_X U(z,x) d\mu(x)$$

is (logarithmically) subharmonic in $D$, if $U_x$ is (logarithmically) subharmonic in $D$ for a.e. $x \in X$.

Introduce now the Hardy class $PL_p$, $0 < p < +\infty$ of logarithmically subharmonic functions in the unit disc which plays main role in this paper. The class $PL_p$ contains all continuous logarithmically subharmonic functions $U$ in the unit disc that satisfy: $M_p(U,r) = 0$ for $r < 1$. Since $U_p$ is also (logarithmically) subharmonic in the unit disc in the definition, instead of finiteness of supremum, we could ask for the existence of boundary value $\lim_{r \to 1^-} M_p(U,r)$.

It is known that every $U \in PL_p$ has the radial boundary values at almost every point $\zeta \in \mathbb{T}$. The boundary value will be denoted (when exist) $U^*(\zeta)$, or simply as $U(\zeta)$. It may be proved that $U(\zeta) \in L_p(\mathbb{T},m_1)$, log $U(\zeta) \in L^1(\mathbb{T},m_1)$, and one can prove the convergence in mean. One can introduce a norm by

$$\|U\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} U(e^{i\theta})^p d\theta \right\}^{1/p}.$$ 

We call $\| \cdot \|_p$ the norm of $U$ (although it is not in the strong sense). For these classes of subharmonic functions we refer to the work of Privalov [22]. See also Yamashita’s work [31] for more deeply consideration.

We establish now the following useful extension of the case $n = 1$ of our main theorem.

Theorem 3.2. Assume (†) holds for $\Phi$ and $\mu$. Let $U_j(z) \in PL_1$ ($j = 1, \ldots, m$). Then

$$\Phi(U_1, \ldots, U_n) \in L^1(\mathbb{U}, \mu)$$

with

$$\int_{\mathbb{U}} \Phi(U_1(z), \ldots, U_n(z)) dm(z) \leq \Phi(\|U_1\|_1, \ldots, \|U_n\|_1).$$

Equality attains if and only if either there exist $1 \leq j_0 \leq m$ such that $U_{j_0} \equiv 0$ or each $U_j$ ($j = 1, \ldots, m$) is of the form $U_j(z) = |\Psi_j(z)|^{p_j}$ for some $\Psi_j, \ldots \in E(\Phi, \mu)$.

The following simple lemma will be useful in the proof.

Lemma 3.3. For every $U(z) \in PL_p$ there exist $f(z) \in H^p$ such that $U(z) \leq |f(z)|$ for $z \in \mathbb{U}$ and $U(\zeta) = |f(\zeta)|$ for a.e. $\zeta \in \mathbb{T}$.

Proof. Let $U(z) \in PL_p$ and suppose w.l.g. that $U \not\equiv 0$. Since $U(\zeta) \in L^p(\mathbb{T},m_1)$ and log $U(\zeta) \in L^1(\mathbb{T},m_1)$, consider the outer function for the class $H^p$:

$$f(z) = \exp \left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log U(\zeta) d\mu_1(\zeta) \right\}.$$ 

It is clear that $|f(\zeta)| = U(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Since log $U(z)$ is subharmonic in the unit disc, we have

$$\log U(z) \leq \frac{1}{2\pi} \int_0^{2\pi} P(r,\theta - t) \log U(e^{it}) dt = \log |f(z)|, \quad z = re^{i\theta}.$$ 

It follows $U(z) \leq |f(z)|$ for $z \in \mathbb{U}$. \hfill $\square$
Proof of Theorem 3.2. W.l.o.g. assume that \( U_j \neq 0 \) for all \( j = 1, \ldots, m \). According to Lemma 3.3 there exist (an outer function) \( f \in H^1 \) such that
(8) \[ U_j(z) \leq |f_j(z)|, \quad z \in \mathbb{U} \]
and
(9) \[ U_j(\zeta) = |f_j(\zeta)| \quad \text{for a.e. } \zeta \in \mathbb{T} \]
In (†) take above functions \( f_j \) \((j = 1, \ldots, m)\). Using (8) and (9), and monotonicity of \( \Phi \) we obtain
(10) \[
\int_{\mathbb{U}} \Phi(\ldots, U_j(z), \ldots) \, d\mu(z) \leq \int_{\mathbb{U}} \Phi(\ldots, |f_j(z)|, \ldots) \, d\mu(z)
\leq \Phi(\ldots, \|f_j\|_1, \ldots) = \Phi(\ldots, \|U_j\|_1, \ldots),
\]
what proofs the inequality we need.

Equality holds at the second place of (10) (regarding the equality statement of (†)) if and only if each \( f_j \) \((j = 1, \ldots, m)\) is of the form \( f_j = \Psi_j \) for some \((\ldots, \Psi_j, \ldots) \in \mathcal{E}(\Phi, \mu)\). In this case \(|f_j(z)|\) does not vanish anywhere in the unit disc. In view of (8), it follows that equality occurs at the first place of (10) if and only if \( U_j(z) = |f_j(z)| \) for all \( z \in \mathbb{U} \) and \( j = 1, \ldots, m \). All together, equality attains at both places if and only if
\[ U_j(z) = |f_j(z)| = |\Psi_j(z)|, \quad z \in \mathbb{U} \]
for all \( j = 1, \ldots, m \). \( \square \)

Remark 3.2. Since
\[ \{ U = |f|^p : f \in H^p \} \subseteq \text{PL}_1, \]
note that Theorem 3.2 may be seen as a generalization of our theorem for \( n = 1 \). The equality statement of Theorem 3.1 could be derived from the equality statement of Theorem 3.2 having on the mind the fact: if \( \varphi \) and \( \psi \) are analytic functions in the unit disc and if \(|\varphi(z)| = |\psi(z)|\) for all \( z \in \mathbb{U} \), then \( \varphi = \alpha \psi \) for some constant \( \alpha \) of the unit modulus.

Note also that we can prove Theorem 3.2 using only (†). That proof is of the same length and it is of some interest to observe that we have a proof of our theorem in the case \( n = 1 \) which does not use the Riesz factorization theorem for Hardy spaces of the unit disc.

4. A THEOREM ON RESTRICTED ANALYTIC FUNCTIONS

Beside Theorem 3.1 and Theorem 3.2 for a complete proof of Theorem 2.1 we will need some additional results which are of interest on its own right.

4.1. We collect some facts concerning the Calderón–Zygmund theorem on iterated limits. We follow mostly the work of Davis [5].

If \( f \in N(\mathbb{U}^n) \) (or if \( f \in H_\sigma(\mathbb{U}^n) \)) and \( z = (z_{j_1}, \ldots, z_{j_k}) \in \mathbb{U}^k \) \((1 \leq k \leq n-1, j_1 < \cdots < j_k)\), then for the restricted function \( f_z = f_{z_{j_1} \cdots z_{j_k}} \) holds \( f_z \in N(\mathbb{U}^k) \) (i.e., \( f_z \in H_\sigma(\mathbb{U}^k) \)). This is seen most easily by using the \( n \)–harmonic majorant for \( \log^+ |f| \) or \( \varphi(\log |f|) \).

If \( f \in N(\mathbb{U}^n) \) and \( \zeta_1 \in \mathbb{T} \), let
\[
f_{\zeta_1}(z_2, \ldots, z_n) = \lim_{r \to 1^-} f(r\zeta_1, z_2, \ldots, z_n),
\]
whenever this limit defines an analytic function on \( U^{n-1} \). Zygmund in [32] proved: if \( f \in N(U^n) \), then for almost all \( \zeta_1 \in T \), \( f_{\zeta_1} \in N(U^{n-1}) \). For \( \zeta_1 \in T \) satisfying the previous condition, we may then consider

\[
f_{\zeta_1 \zeta_2}(z_3, \ldots, z_n) = \lim_{r \to 1^+} f_{\zeta_1}(r \zeta_2, z_3, \ldots, z_n) \in N(U^{n-2}),
\]

for almost all \( \zeta_2 \in T \). Continuing in this manner we may then consider

\[
f_{\zeta_1 \cdots \zeta_n} = \lim_{r \to 1^+} f_{\zeta_1 \cdots \zeta_{n-1}}(r \zeta_n),
\]

whenever this limit exists. In [3] it is shown that if \( f \in N^+(U^n) \), then the iterated limits of \( f \) are almost everywhere independent of the order of iteration. In fact, the iterated limit and the radial limit are equal almost everywhere. A similar theorem hold for \( H_{\phi} \) (as a consequence). Also in [32] Zygmund proved the preceding, but for \( f \in N_{n-1}(U^n) \supseteq N^+(U^n) \). Calderón and Zygmund posed the question whether \( N_{n-1}(U^n) \) may be replaced by \( N(U^n) \). The result of Davis gives a partial answer.

In the sequel we will need only the following proposition. A proof follows from above discussion.

**Proposition 4.1.** Let \( f(z) = f(z_1, \ldots, z_n) \in H_{\phi}(U^n) \). Then for an integer \( 1 \leq k \leq n - 1 \) and mutually disjoint integers \( \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\} \) we have

1. \( f_{j_1 \cdots j_k} \in H_{\phi}(U^{n-k}) \) for all \( (z_{j_1}, \ldots, z_{j_k}) \in U^k \);
2. \( f_{\zeta_{j_1} \cdots \zeta_{j_k}} \in H_{\phi}(U^{n-k}) \) for almost every \( (\zeta_{j_1}, \ldots, \zeta_{j_k}) \in T^k \).

Particularly, the iterated boundary function \( f_{\zeta_{j_1} \cdots \zeta_{j_k}} \) is the same as the radial boundary function \( f_{j_1 \cdots j_k}^* \) for almost every \( (\zeta_{j_1}, \ldots, \zeta_{j_k}) \in T^k \).

**Remark 4.1.** It follows that the function \( f_{\zeta_{j_1} \cdots \zeta_{j_k}} \) does not depend on the order of iteration; thus, we may assume \( 1 < j_1 < \cdots < j_k \leq n \).

4.2. We introduce \( PL_p(U^n) \) (\( 0 < p < \infty \)) as the class of all \( n \)-logarithmically subharmonic functions in \( U^n \) such that

\[
\|U\|_p = \sup_{0 < r < 1} M_p(U, r) = \sup_{0 < r < 1} \left\{ \int_{U^n} U(r\zeta)^p \, dm_n(\zeta) \right\}^{1/p}.
\]

It is known that every \( U \in PL_p(U^n) \) has the radial limit a.e., i.e., there exist

\[
U(\zeta) = \lim_{r \to 1^+} U(r\zeta) \quad \text{for a.e. } \zeta \in T^n.
\]

Obviously, \( H_p(U^n) \subseteq PL_p(U^n) \).

This place is devoted to the proof of the following

**Theorem 4.1.** For \( f(z) = f(z_1, \ldots, z_n) \in H_p(U^n) \) and integers \( 1 \leq j_1 < \cdots < j_k \leq n \), where \( 1 \leq k < n \) is also an integer, the function

\[
U(z_{j_1}, \ldots, z_{j_k}) = \|f_{j_1 \cdots j_k}\|_p^p
\]

is well defined in \( U^k \). Moreover, \( U \in PL_1(U^k) \) and \( PL_1 \)-norm of \( U \) is given by

\[
\|U\|_1 = \|f\|_{p, *}^p.
\]

For a proof of Theorem 4.1 we need the auxiliary results that are formulated below.
**Lemma 4.1** (cf. [29]). For $z \in \mathbb{U}^n$ holds the (optimal) estimate of the modulus

$$|F(z)|^p \leq \frac{1}{(1 - |z_1|^2) \cdots (1 - |z_n|^2)} \|F\|_p,$$

of $F(z) = F(z_1, \ldots, z_n) \in H^p(\mathbb{U}^n)$ ($0 < p < \infty$). The equality sign occurs if and only if

$$F(z) = \lambda K_w(z)^{2/p},$$

where $w \in \mathbb{U}^n$ and $\lambda$ is any constant. Here, $K(z, w) = \prod_{j=1}^n (1 - z_j \overline{w}_j)^{-1}$ stands for the Cauchy–Szegő kernel for the unit polydisc.

**Remark 4.2.** A similar statement to Lemma 4.1 may be found in Vukotić work [29] for functions which belong to weighed Bergman spaces in the unit ball in $\mathbb{C}^n$. However, the same idea may be applied to derive the preceding one result.

In what follows we will use several times the next lemma which follows immediately from the Lebesgue Dominant Convergence Theorem. If $(X, \mu)$ is a measurable space and $\Lambda$ a non–empty set of indexes, we say that $G$ is dominant for the family $\{F_\alpha : \alpha \in \Lambda\}$ of real–valued measurable functions in $X$ if $F_\alpha(z) \leq G(z)$ for almost every $z \in X$ and for all $\alpha \in \Lambda$.

**Lemma 4.2.** Let $D$ be a domain, $(X, \mu)$ a measurable space, and let $F(z, w)$ be defined in $D \times X$ such that $F_z = F(z, \cdot) \in L^1(X, \mu)$ for all $z \in D$ and $F_w = F(\cdot, w) \in C(D)$ for almost all $w \in X$. If for any compact set $K \subseteq D$ there exist $G \in L^1(X, \mu)$ dominant for the family

$$\{F_z : z \in K\},$$

then

$$z \mapsto \int_X F(z, w)d\mu(w)$$

is continuous in $D$.

**Proof of Theorem 4.1** For simplicity, we will assume $j_1 = n - k + 1$ and $j_k = n$.

In view of the first part of Corollary 4.1, the value of

$$U(z_{n-k+1}, \ldots, z_n) = \|f_{z_{n-k+1} \ldots z_n}\|_p^p$$

is finite for all $(z_{n-k+1}, \ldots, z_n) \in \mathbb{U}^k$.

We have used the identity

$$f_{z_{n-k+1} \ldots z_n}(\zeta_1, \ldots, \zeta_{n-k}) = f_{\zeta_1 \ldots \zeta_{n-k}}(z_{n-k+1}, \ldots, z_n)$$

for all $(z_{n-k+1}, \ldots, z_n) \in \mathbb{U}^k$ and almost all $(\zeta_1, \ldots, \zeta_{n-k}) \in T^{n-k}$. Let us say that on the left side of the preceding relation we mean the radial boundary value of $f_{z_{n-k+1} \ldots z_n}$ in $(\zeta_1, \ldots, \zeta_{n-k}) \in T$, on the right side we have the value of radial boundary function $f_{\zeta_1 \ldots \zeta_{n-k}}$ in $(z_{n-k+1}, \ldots, z_n) \in \mathbb{U}^k$. In both cases it is about boundary value

$$\lim_{r \to 1^-} f(r\zeta_1, \ldots, r\zeta_{n-k}, z_{n-k+1}, \ldots, z_n).$$

Denote $z' = (z_1, \ldots, z_{n-k})$ and $z'' = (z_{n-k+1}, \ldots, z_n)$; similarly introduce $\zeta' = (\zeta_1, \ldots, \zeta_{n-k})$ and $\zeta'' = (\zeta_{n-k+1}, \ldots, \zeta_n)$. 

For all $0 \leq r < 1$ let
\[ \hat{U}_r(z'') = \int_{\mathbb{T}^k} |f_{rG}(z'')|^p dm_k(\zeta'), \quad z'' \in \mathbb{U}^k. \]

Note that $\hat{U}_r(z'')$ is not the $r$–dilation of $U(z'')$.

Regarding Lemma \[\ref{lem:3.2}\] the function $\hat{U}_r (0 \leq r < 1)$ is $k$– logarithmically subharmonic in the polydisc $\mathbb{U}^k$, since the same holds for
\[ \mathbb{U}^k \ni z'' \mapsto |f(r\zeta', z'')|^p, \]
for all $\zeta' = (\zeta_1, \ldots, \zeta_{n-k}) \in \mathbb{T}^{n-k}$. The convergence
\[ \hat{U}_r(z'') \to U(z''), \quad r \to 1^- \]
is nondecreasing in $r$, if $z'' = (z_{n-k+1}, \ldots, z_n) \in \mathbb{U}^k$ is fixed, since
\[ \hat{U}_r(z'') = M^p_p(f_{z''}, r) \]
and $f_{z''} \in H^p(\mathbb{U}^{n-k})$. It follows
\[ U(z'') = \sup_{0 \leq r < 1} \hat{U}_r(z'') \]
for all $z'' = (z_{n-k+1}, \ldots, z_n) \in \mathbb{U}^k$.

In view of Lemma \[\ref{lem:3.4}\] to show that $U$ is continuous $k$–logarithmically subharmonic in $\mathbb{U}^k$, i.e., that belongs to the class $PL(\mathbb{U}^k)$, it remains to prove that $U$ is continuous in the same domain.

To realize that, we will use Lemma \[\ref{lem:4.2}\]. Let $K$ be any compact subset of $\mathbb{U}^k$. We will show that there exists an integrable dominant function for the family
\[ \{|f_{z''}^*(\zeta')|^p : z'' = (z_{n-k+1}, \ldots, z_n) \in K, \zeta' = (\zeta_1, \ldots, \zeta_{n-k}) \in \mathbb{T}^{n-k} \}. \]

There exists a constant $C = C(K)$ such that
\[ \frac{1}{(1 - |z_{n-k+1}|^2) \cdots (1 - |z_n|^2)} \leq C \]
for $z'' = (z_{n-k+1}, \ldots, z_n) \in K$. Since $f_{\zeta_1 \ldots \zeta_{n-k}}^* \in H^p(\mathbb{U}^k)$ for almost every $(\zeta_1, \ldots, \zeta_{n-k}) \in \mathbb{T}^{n-k}$ (see the second part of Corollary \[\ref{cor:4.1}\]), according to \[\ref{cor:4.1}\] we find
\[ |f_{\zeta_1 \ldots \zeta_{n-k}}^*(z_{n-k+1}, \ldots, z_n)|^p \leq \frac{\|f_{\zeta_1 \ldots \zeta_{n-k}}^*\|_p^p}{(1 - |z_{n-k+1}|^2) \cdots (1 - |z_n|^2)}, \]
for $(z_{n-k+1}, \ldots, z_n) \in \mathbb{U}^k$. The growth estimate \[\ref{eq:12}\] gives the dominant function. Denote
\[ G(\zeta_1, \ldots, \zeta_{n-k}) = C \|f_{\zeta_1 \ldots \zeta_{n-k}}^*\|_p^p \]
\[ = C \int_{\mathbb{T}^n} |f_{\zeta_1 \ldots \zeta_{n-k}}^*(\zeta_{n-k+1}, \ldots, \zeta_n)|^p dm_k(\zeta_{n-k+1}, \ldots, \zeta_n), \]
the function defined almost everywhere on the torus $\mathbb{T}^{n-k}$. The function $G$ is integrable, since applying the Fubini theorem and Theorem \[\ref{thm:4.1}\] we find
\[ \int_{\mathbb{T}^{n-k}} G(\zeta_1, \ldots, \zeta_{n-k}) = C \int_{\mathbb{T}^n} |f^*(\zeta_1, \ldots, \zeta_n)|^p dm_n(\zeta_1, \ldots, \zeta_n) = C\|f\|_p^p < \infty. \]
Using \[\ref{eq:12}\] and \[\ref{eq:11}\], it follows
\[ |f_{z_{n-k+1} \ldots z_n}^*(\zeta_1, \ldots, \zeta_{n-k})|^p \leq G(\zeta_1, \ldots, \zeta_{n-k}), \]
for almost every \((\zeta_1, \ldots, \zeta_{n-k}) \in \mathbb{T}^{n-k}\) and all \((z_{n-k+1}, \ldots, z_n) \in K\). Since the function \(|f_{\zeta}^*(z'')|^p\) is continuous (for a.e. \(\zeta' = (\zeta_1, \ldots, \zeta_{n-k}) \in \mathbb{T}^{n-k}\), as a modulus of analytic function, what follows from Corollary 4.1), applying Lemma 4.2 we obtain that

\[
U(z_{n-k+1}, \ldots, z_n) = \int_{\mathbb{T}^{n-k}} |f_{\zeta}^*|_{z_{n-k+1} \ldots z_n} (\zeta_1, \ldots, \zeta_{n-k})|^p dm_{n-k}(\zeta_1, \ldots, \zeta_{n-k})
\]

is also continuous in \(U_k\).

In order to finish the proof of this lemma, we have to show that \(U \in PL_1(U_k)\) and \(\|U\|_1 = \|f\|_p^p\). First of all, for \(0 \leq r < 1\) we have

\[
M_1(U, r) = \int_{\mathbb{T}^k} \|f_{\zeta''}^*\|_p^p dm_k(\zeta'') = \int_{\mathbb{T}^k} \left\{ \int_{\mathbb{T}^{n-k}} |f_{\zeta''}^*|^{p} dm_{n-k}(\zeta') \right\} dm_k(\zeta'') = \int_{\mathbb{T}^{n-k}} |f_{\zeta''}^*|_{\zeta'}^{p} dm_{n-k}(\zeta') \leq \int_{\mathbb{T}^{n-k}} \|f_{\zeta''}^*\|_p^p dm_{n-k}(\zeta') = \|f\|_p^p < \infty.
\]

It follows

\[
\|U\|_1 = \sup_{0 \leq r < 1} M_1(U, r) \leq \|f\|_p^p.
\]

Thus, \(U \in PL_1(U_k)\).

To show the reverse inequality, it is enough to apply the Fatou lemma. Since \(U \in PL_1\), the radial boundary value \(U(\zeta'')\) exists in almost all points \(\zeta'' = (\zeta_{n-k+1}, \ldots, \zeta_n) \in \mathbb{T}^k\) and holds

\[
U(\zeta_{n-k+1}, \ldots, \zeta_n) = \lim_{r \to 1^{-}} U(r\zeta_{n-k+1}, \ldots, r\zeta_n) = \lim_{r \to 1^{-}} \|f(r\zeta_{n-k+1}, \ldots, r\zeta_n)\|_p^p \geq \|f_{\zeta_{n-k+1} \ldots \zeta_n}^*\|_p^p.
\]

Now we obtain

\[
\|U\|_1 = \int_{\mathbb{T}^k} U(\zeta'') dm_k(\zeta'') \geq \int_{\mathbb{T}^k} \|f_{\zeta''}^*\|_p^p dm_k(\zeta'') = \|f\|_p^p.
\]

what finishes this proof.

\[\square\]

5. Proof of Theorem 2.1

Here we will propose a proof of the main theorem. We use the induction on the dimension parameter \(n\).

Proof of Theorem 2.1 For \(n = 1\) we have already proved the theorem (Theorem 3.1).

Assume now that this theorem is valid for \(n-1\), where \(n \geq 2\). We are going to prove it for \(n\). Let \(f_j(z) = f_j(z', z_n) \in H^p_j(U^n)\) for all \(j = 1, \ldots, m\). We
have denoted \( z' = (z_1, \ldots, z_{n-1}) \). Since \( d\nu_n(z) = d\nu_{n-1}(z') \times d\mu(z_n) \), applying the Fubini theorem we obtain

\[
\int_{U^n} \Phi(\ldots, |f_j(z)|^{p_j}, \ldots) \, d\nu_n(z)
\]

\[
= \int_{U^n} \left\{ \int_{U^{n-1}} \Phi(\ldots, |f_j(z')|^{p_j}, \ldots) \, d\nu_{n-1}(z') \right\} \, d\mu(z_n)
\]

\[
\leq \int_{U^n} \Phi(\ldots, \|f_j(z')\|^{p_j}, \ldots) \, d\mu(z_n) \leq \Phi(\ldots, \|f_j\|^{p_j}, \ldots),
\]

what proves the inequality in our theorem.

More precisely, since \( f_j^z \in H_{p_j}(\mathbb{U}^{n-1}) \) for every fixed \( z_n \in \mathbb{U} \), applying the induction hypothesis, we obtain the first inequality above:

\[
\int_{U^{n-1}} \Phi(\ldots, |f_j^z(z')|^{p_j}, \ldots) \, d\nu_{n-1}(z') \leq \Phi(\ldots, \|f_j^z\|^{p_j}, \ldots).
\]

In view of Theorem 4.1, let \( U_j \in PL_1 \) (\( j = 1, \ldots, m \)) be defined in the unit disc in the following way

\[
U_j(z_n) = \|f_j^z\|^{p_j}, \quad z_n \in \mathbb{U}.
\]

By the same theorem we have

\[
\|U_j\|_1 = \|f_j\|^{p_j}
\]

for all \( j = 1, \ldots, m \). The second inequality

\[
\int_{U^n} \Phi(\ldots, \|f_j^z\|^{p_j}, \ldots) \, d\mu(z_n) = \int_{U^n} \Phi(\ldots, U_j(z_n), \ldots) \, d\mu(z_n)
\]

\[
\leq \Phi(\ldots, \|U_j\|_1, \ldots) = \Phi(\ldots, \|f_j\|^{p_j}, \ldots)
\]

follows from the logarithmically subharmonic version of the case \( n = 1 \) of our theorem (Theorem 3.2).

Regarding Theorem 3.2 it is not hard to see that the main inequality holds if we take \( U_j \in PL_1(\mathbb{U}^n) \) instead of \( f_j \in H^1(\mathbb{U}^n) \) (for all \( j = 1, \ldots, m \)) with \( PL_1(\mathbb{U}^n) \)–norms of functions \( U_j \) (\( j = 1, \ldots, m \)) on the right side. This is important to note for the rest of this proof which is devoted to the extremal functions. To establish this version of the main inequality, it is enough to prove it for dilatations \( (U_j)_r, \ 0 < r < 1 \). One can prove this as we have just done for analytic functions (even easily, since we have continuous functions in a velocity of the unit polydisc).

Letting \( r \to 1 \) we obtain the desire inequality since the \( PL_1(\mathbb{U}^n) \)–norm of a dilation converge to \( PL_1 \)–norm of the same function. This approach, however, does not give all extremal functions for the main inequality in the subharmonic case neither in the analytic case.

Thus, the second half of this proof is devoted to the extremal functions for the main inequality. However, we will make a digression in order to show that the function

\[
F(z_n) = \int_{U^{n-1}} \Phi(\ldots, |f_j^z(z')|^{p_j}, \ldots) \, d\nu_{n-1}(z'),
\]

which appears at the beginning of this proof, under the integral sign of the relation (14) is continuous at every point \( z_n \in \mathbb{U} \). In view of Lemma 4.2 it is enough to
show that for any compact set \( K \subseteq U \) there exist \( G(z') \in L^1(U^{n-1}, \nu_{n-1}) \) dominant for the family
\[
\{ \tilde{F}_n(z') = \Phi(\ldots, |f_j(z', z_n)|^{p_j}, \ldots) : z_n \in K \}.
\]
To finish this, let a constant \( C = C(K) \) be chosen in the such way that
\[
\frac{1}{1 - |z_n|^2} \leq C, \quad z_n \in K.
\]
Since \( f_j(z') \in H^{p_j} \) for every fixed \( z' \in U^{n-1} \) by Lemma \( \text{L.1} \) we obtain the estimate
\[
|f_j(z_n)|^{p_j} \leq \frac{1}{1 - |z_n|^2} \parallel f_j \parallel_{p_j}^{p_j}, \quad z_n \in U.
\]
Thus, for fixed \( z_n \in K \) have
\[
\tilde{F}_n(z') = \Phi(\ldots, |f_j(z', z_n)|^{p_j}, \ldots) \leq \Phi(\ldots, C\tilde{U}_j(z'), \ldots),
\]
where we have denoted
\[
\tilde{U}_j(z') = \| f_j(z) \|_{p_j}, \quad z' \in U^{n-1}
\]
for each \( j = 1, 2, \ldots, m \); see also Theorem \( \text{L.1} \) Thus,
\[
\tilde{F}_n(z') \leq G(z'),
\]
for
\[
G(z') = \Phi(\ldots, C\tilde{U}_j(z'), \ldots), \quad z' \in U^{n-1}.
\]
It remains to show that \( G \in L^1(U^{n-1}, \nu_{n-1}) \); since \( \tilde{U}_j \in \text{PL}_1(U^{n-1}) \) for all \( j = 1, \ldots, m \), according to the main inequality for the class \( \text{PL}_1(U^{n-1}) \), we obtain
\[
\int_{U^{n-1}} G(z') d\nu_{n-1}(z') = \int_{U^{n-1}} \Phi(\ldots, C\tilde{U}_j(z'), \ldots) d\nu_{n-1}(z')
\]
\[
\leq \Phi(\ldots, C\|\tilde{U}_j\|_{1, \ldots}) < \infty.
\]
Let us now prove the second half of our theorem – the equality statement.

Equality attains in the main inequality if and only if equality holds at both places in \( \text{L.3} \). If \( f_j \equiv 0 \) for some \( 1 \leq j \leq m \), then equality obvious holds at both places in \( \text{L.3} \). In the sequel we assume this is not the case and will prove first that if equality attains in the main inequality, then each \( f_j \) \( (j = 1, \ldots, m) \) does not vanish in \( \tilde{U}^n \). After that we will use the equality statement of \( (\dagger) \) in order to derive the equality statement in general.

Since \( f_j \not\equiv 0 \), in view of \( \text{L.5} \) we have \( U_j \not\equiv 0 \) for all \( j = 1, \ldots, m \). Thus, equality holds at the second place in \( \text{L.3} \) if and only if
\[
U_j(z_n) = |\Psi_j(z_n)|, \quad z_n \in U
\]
for all \( j = 1, \ldots, m \) (see the equality statement of Theorem \( \text{L.2} \)).

This means that each \( U_j \) does not vanish in the unit disc. Note now that the first inequality in \( \text{L.3} \) may be rewritten in the following form
\[
\int_{\tilde{U}} F(z_n) d\nu_{n-1}(z_n) \leq \int_{\tilde{U}} \Phi(\ldots, U_j(z_n), \ldots) d\nu_{n-1}(z_n).
\]
Since \( F(z_n) \) and \( \Phi(\ldots, U_j(z_n), \ldots) \) are continuous in all points \( z_n \in \tilde{U} \), equality occurs in the preceding inequality if and only if
\[
F(z_n) = \Phi(\ldots, U_j(z_n), \ldots) \quad \text{for all} \quad z_n \in \tilde{U}.
This means that equality holds in (14) also for all \( z_n \in \mathbb{U} \), what is possible (in view of the equality statement of the induction hypothesis) if and only if
\[
f_j^{z_n}(z') = \Psi_j^{n-1}(z'), \quad z' \in \mathbb{U}^{n-1}
\]
for all \( j = 1, \ldots, m \); here \( \Psi_j^{n-1} \in \mathcal{E}_j(\Phi, \nu_{n-1}) \) for \( z_n \in \mathbb{U} \) and \( j = 1, \ldots, m \). Namely, since each \( U_j(z_n) \) \( (j = 1, \ldots, m) \) does not vanish in the unit disc, it is not possible to exist \( 1 \leq j \leq m \) and \( z_n \in \mathbb{U} \) such that \( f_j^{z_n} = 0 \) (see (15)).

All together we have proved that if equality holds in the main inequality, then each \( f_j (j = 1, \ldots, m) \) does not vanish in \( \mathbb{U}^n \). Thus, we can take some branch \( f_j^{p_j/\hat{p}_j} \in H^{\hat{p}_j}(\mathbb{U}^n) \) for all \( j = 1, \ldots, m \). Using now the equality statement of (\textit{1}) for \( f_j^{p_j/\hat{p}_j} \), \( j = 1, \ldots, m \), we find that equality holds if and only if each \( f_j (j = 1, \ldots, m) \) is of the form \( f_j^{p_j/\hat{p}_j}(z) = \Psi_j^p(z) \), or what is the same
\[
f_j(z) = \Psi_j^p(z)^{\hat{p}_j/p_j}, \quad z \in \mathbb{U}^n
\]
for some and some \( \Psi_j^p \in \mathcal{E}(\Phi, \nu_n) \). This finishes the proof of the equality statement of Theorem 2.1. \( \square \)

**Remark 5.1.** An inner function in \( \mathbb{U}^n \) is a function \( G \in H^\infty(\mathbb{U}^n) \) whose radial boundary values satisfy \( |G^*(\zeta)| = 1 \) a.e. on \( T^n \). An inner function \( G \) in \( \mathbb{U}^n \) is said to be good if \( U[G] \equiv 0 \). Here, \( U[G] \) stands for the least \( n \)-harmonic majorant of \( \log |G| \) in \( \mathbb{U}^n \). One can prove that an inner function \( G \) is good if and only if \( U[G](0) = 0 \). In the unit disc, the good inner functions in \( \mathbb{U} \) are precisely the Blaschke products.

As is well known, in one variable, there corresponds to every \( f \in H^p(\mathbb{U}) \) a Blaschke product \( B \) such that \( h = f/B \) has no zeros in \( \mathbb{U} \), \( h \in H^p(\mathbb{U}) \), and even \( \|h\|_p = \|f\|_p \). If \( U \) is replaced by \( \mathbb{U}^n \), where \( n > 1 \), one might expect that the role of the Blaschke products is taken over by the good inner functions. This is true, but only to a certain extent: the analogues of the one-variable theory hold for exactly those \( f \in H^p(\mathbb{U}^n) \) for which the least \( n \)-harmonic majorant \( U[f] \) of \( \log |f| \) is the real part of an analytic function – the class \( \text{RP}(\mathbb{U}^n) \). We have

**Theorem 5.1** (cf. [27]). Assume \( f \in N(\mathbb{U}^n) \). Then

1. If \( U[f] \) is not in \( \text{RP}(\mathbb{U}^n) \) then no good inner function has the same zeros as \( f \);
2. If \( U[f] \in \text{RP}(\mathbb{U}^n) \) then there is a good inner function (unique up to a multiplicative constant) with the same zeros as \( f \).

Let \( G \) be a good inner function, \( h \) an analytic in \( \mathbb{U}^n \), and \( f = G \cdot h \). Then \( h \in N(\mathbb{U}^n) \).

Moreover,

1. If \( f \in N^*(\mathbb{U}^n) \), then \( h \in N^*(\mathbb{U}^n) \).
2. If \( 0 < p < \infty \) and \( f \in H^p(\mathbb{U}^n) \), then \( h \in H^p(\mathbb{U}^n) \), and \( \|h\|_p = \|f\|_p \).

We will sketch now a simple but incomplete proof of Theorem 2.1 based on the preceding theorem. This proof is motivated by the method which is almost standard in the theory of Hardy spaces (of the unit disc). However, as we have seen, this approach gives a complete proof in the one-dimensional case of our main theorem, what is Theorem 3.1. Let \( f_j \in H^p(\mathbb{U}^n) \), and without lost of generality, suppose that \( f_j \neq 0 \) for all \( j = 1, \ldots, m \). Assume, moreover, that each \( \log |f_j| \) has a \( n \)-harmonic majorant which belongs to \( \text{RP}(\mathbb{U}^n) \). With the preceding additional assumption, it is possible to obtain the factorization \( f_j = G_j \cdot h_j \), where \( G_j \) is a
good inner function on the unit polydisc with same zeroes as $f_j$. Recall, we take $|G_j| \equiv 1$, if $f_j$ is zero–free. Now the proof goes in the same way as in the case $n = 1$.

6. AN APPLICATION: AN ISOPERIMETRIC TYPE INEQUALITY

Let $D$ be a simply–connected domain in the plain such that $\partial D$ is a rectifiable curve. Then the area of $D$ and the length of $\partial D$ satisfy

$$4\pi \cdot \text{Area}(D) \leq \text{Length}(\partial D)^2,$$

with equality if and only if $D$ is a disc. This inequality is known as the classical isoperimetric inequality. For a broader discussion on (15), various connections with some known analytic inequalities, we refer to the survey article [21] of Osserman.

6.1. We say some words on the Carleman approach in the proving the isoperimetric inequality. In [7] Carleman deduced [13] using the theory of functions of a complex variable. Strebel [28, Theorem 19.9, pp. 96–98] modified the original Carleman argument in order to establish the following isoperimetric type inequality (a variation of Carleman’s): If $f(z)$ belongs to Hardy space $H^p$, where $p$ is any positive number, then it belongs also to the Bergman space $L^2_a$, and

$$4\pi \int_U |f(z)|^{2p} dA(z) \leq \left\{ \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right\}^{2p}.$$

Equality holds in (19) if and only if

$$f(z) = \frac{\lambda}{(1 - zw)^{2/p}}$$

for some $w \in \mathbb{U}$ and constant $\lambda$. A proof of (19) and the corresponding equality statement is also exposed in [30], where it is observed that the original Carleman approach leads to a simple proof of the result of Hardy and Littlewood on bound-
edness of the inclusion operator from $H^p$ into the Bergman space $L^2_a$ (a similar approach is given much more earlier by Mateljević [17]). Thus, there exist a short proof of (15) and the corresponding equality statement in the Hardy space theory. This proof includes the classical results of Riemann, Carathéodory and Smirnov for conformal mapping, and the modified Carleman inequality (19). For this approach see [30].

The isoperimetric inequality holds for simply–connected domains on a surface of the non–positive Guass curvature; see [2] and [3]. Of, course, in that case, the relation (15) should be understood in the context of the intrinsic geometry of a surface. Historically, the first result concerning the classical isoperimetric inequality for surfaces of variable Gaussian curvature is due also to Carleman for minimal surfaces [7]. His original proof of the isoperimetric inequality for simply–connected domains bounded by an analytic curve is based on the Weierstrass–Enneper parametrization of minimal surfaces and the following inequality for analytic function which is a variation of (19), as may be showed: Let $f_1, f_2 \in H^1$, then

$$4\pi \cdot \int_U |f_1(z)|^2 |f_2(z)|^2 dA(z) \leq \int_0^{2\pi} |f_1(e^{i\theta})|^2 d\theta \cdot \int_0^{2\pi} |f_2(e^{i\theta})|^2 d\theta.$$
with equality if and only if either \( f_1 f_2 \equiv 0 \) or
\[
f_1(z) = \frac{C_1}{(1 - zw)^2}, \quad f_2(z) = \frac{C_2}{(1 - zw)^2}
\]
for some \( w \in \mathbb{U} \) and some nonzero constants \( C_1 \) and \( C_2 \).

6.2. Some generalizations of Carleman’s inequality \(^{(19)}\) are exposed here. We recall the result which belongs to Mateljević and Pavlović \(^{(19)}\) as well as Burbea \(^{(6)}\) (obtained in different ways and motivated by different reasons). It is about the following sharp inequality
\[
(21) \quad \frac{m-1}{\pi} \int_{\mathbb{U}} |f(z)|^{mp} (1 - |z|^2)^{m-2} dA(z) \leq \|f\|_{mp}^p.
\]
Here, \( f \in H^p \), \( 0 < p < +\infty \) and \( m \geq 2 \) is an integer. Extremal functions are the same which appear in the particular case \( m = 2 \) of this inequality, when we have the modern version of the Carleman inequality. A function \( f \in H^p \) is extremal for \((21)\) if and only if \( f \) is of the form
\[
f(z) = \frac{\lambda}{(1 - wz)^{2/p}}
\]
for some \( |w| < 1 \) and a constant \( \lambda \).

In \(^{(6)}\), which is mostly based on his earlier work \(^{(4)}\), Burbea recognized that it is possible to derive \((21)\) with a little effort from his earlier results. Inequality \((21)\), and all extremal functions as well, may be seen as a direct corollary of

**Theorem 6.1 (cf. \(^{(6)}\)).** Let \( m \geq 2 \) be an integer and \( f_j(z) \in H^{p_j} \), \( 0 < p_j < +\infty \) for all \( j = 1, 2, \ldots, m \). Then
\[
\prod_{j=1}^{m} |f_j|^{p_j} \in L_{m-2}^1
\]
and
\[
\int_{\mathbb{U}} \left\{ \prod_{j=1}^{m} |f_j(z)|^{p_j} \right\} da_{m-2}(z) \leq \prod_{j=1}^{m} \|f_j\|_{p_j}^{p_j}
\]
with equality if and only if either \( \prod_{j=1}^{m} f_j \equiv 0 \) or each \( f_j \ (j = 1, 2, \ldots, m) \) is of the form
\[
f_j(z) = c_j K_w(z)^{2/p_j}
\]
for some (common) \( w \in \mathbb{U} \) and \( c_j \neq 0 \). Here, \( K(z, w) = (1 - zw)^{-1} \) stands for the Cauchy–Szegő kernel for the unit disc.

We should say why the reproducing kernel appears in above formulation. Namely, the case \( p_j = 2 \) for all \( j = 1, 2, \ldots, m \), which we call the Hilbert case, of the preceding theorem may be derived from a very general consideration involving the theory of reproducing kernels (see Theorem \(^{(6,2)}\)). This case is crucial in proving Theorem \(^{(6,4)}\) the general case then follows in a usual way in the Hardy space theory, using the Riesz factorization for Hardy spaces (as in Theorem \(^{(3,1)}\)). This approach was also used in Burbea’s paper \(^{(6)}\). In the third section we presented a proof without the factorization theorem, moreover, one can establish Burbea’s inequality for functions in the class \( PL_1 \) (see Theorem \(^{(3,2)}\)).

It is of some interest to note that Theorem \(^{(6,1)}\) is, actually, equivalent to the inequality \((21)\) with the corresponding equality statement, as observed and showed
to us by Professor Pavlović. In detail, for all \( j = 1, 2, \ldots, m \) let \( f_j \) be as in the theorem. As we know, it is enough to consider the Hilbet case. Moreover, we may assume, without lost of generality, that each \( f_j(z) \) \( (j = 1, 2, \ldots, m) \) is zero–free in the disc \( U \). Denote
\[
g(z) = \{ f_1(z) f_2(z) \cdots f_m(z) \}^{1/m}
\]
(we take some brunch). Applying the Hölder inequality, we obtain
\[
\| g \|_2^m = \left\| g^m \right\|_2 = \left\| \prod_{j=1}^{m} f_j \right\|_{2/m} \leq \prod_{j=1}^{m} \| f_j \|_2 < \infty.
\]
Thus \( g \in H^2 \). Using \((21)\) for \( p = 2 \) and \( f = g \), then the preceding inequality, we infer
\[
\int_{U} \left\{ \prod_{j=1}^{m} |f_j(z)|^2 \right\} \, da_{m-2}(z) = \int_{U} |g(z)|^{2m} \, da_{m-2}(z)
\leq \| g \|_2^{2m} \leq \prod_{j=1}^{m} \| f_j \|_2^2.
\]
If equality holds at each place above, then from the first inequality, we conclude that \( g = \lambda K_w \) for some \( w \in U \) and a non–zero constant \( \lambda \) (according to the equality statement for \((21)\)). Regarding the equality case for the Hölder inequality, and the relation between \( g \) and \( f_j \), it follows easy that equality holds at the second place if and only if \( f_j = c_j K_w \), where \( c_j \) is also a non–zero constant, for all \( j = 1, 2, \ldots, m \). Thus, we have derived Theorem 6.1 from inequality \((21)\) and the corresponding equality statement.

6.3. We recall here a version of the Hilbert case of Carleman’s inequality in a very general context – as an inequality between analytic functions on any complete Reinhardt domain. This result is proved by Burbea, and we give a somewhat extended version of the result. In the Appendix we simplify the proof of Burbea.

A domain \( D \subseteq \mathbb{C}^n \) is a complete Reinhardt if \( z \in D \) implies \( z \cdot w \in D \) for all \( w \in \overline{U}^n \). We have denoted by \( \cdot \) the following operation on \( \mathbb{C}^n \)
\[
z \cdot \overline{w} = (z_1 \overline{w}_1, \ldots, z_n \overline{w}_n)
\]
where
\[
\overline{w} = (\overline{w}_1, \ldots, \overline{w}_n).
\]
is the conjugate vector for \( w \). Here \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \) are the coordinate representations of \( z \) and \( w \) in the standard base of \( \mathbb{C}^n \) (considered as a vector space over the field \( \mathbb{C} \)). It is clear that a complete Reinhardt domain is a star–shaped domain which contains \( 0 \in \mathbb{C}^n \). One of the well known facts in the theory of analytic functions of several complex variables is that any analytic function \( f \) in such one domain \( D \) has the unique power series expansion
\[
f(z) = \sum_{\alpha} a_{\alpha} z^\alpha, \quad z \in D.
\]
The convergence is uniform (and absolute) on compact subsets of \( D \). In other words \( H(D) \) is a closure (in the locally uniform topology of \( D \)) of the linear space spanned
by \( \{ z^\alpha : \alpha \in \mathbb{Z}_+^n \} \). Moreover, for coefficients in expansion (22) we have

\[
a_\alpha = a_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!}, \quad \alpha \in \mathbb{Z}_+^n.
\]

Until the rest of this paper \( D \) always stands for a complete Reinhardt domain.

Introduce the following two classes of analytic functions in \( D \), the class \( P(D) \) and \( P_\infty(D) \subseteq P(D) \). The first one contains all \( \phi \) with an expansion

\[
\phi(z) = \sum_{\alpha} c_\alpha z^\alpha, \quad z \in \Omega,
\]

where the series is (absolutely) uniformly convergent on compact subsets of \( D \) and

\[
c_\alpha > 0 \quad \text{for all} \quad \alpha \in \mathbb{Z}_+^n.
\]

The subclass \( P_\infty(D) \) is defined in the following way: \( \phi \in P_\infty(D) \) if \( \phi \in P(D) \) and \( \phi(z \cdot \overline{z}) = \infty \) for all \( z \in \partial D \).

Denote

\[
K_\phi(z, w) = \phi(z \cdot \overline{w}), \quad z, w \in D.
\]

It is easy to check that \( K_\phi(z, w) \) is a sesqui–analytic positively defined kernel on \( D \times D \). From the theory of reproducing kernels [1] it follows that there exist a unique reproducing kernel Hilbert space (RKHS) for which \( K_\phi \) is a reproducing kernel. Since the kernel is locally bounded on \( D \times D \), we have that \( H_{K_\phi} \) contains analytic functions. One can prove that the space \( H_{K_\phi} \) coincides with the following one

\[
H_\phi = \{ f \in H(D) : \| f \|_\phi < \infty \}.
\]

For \( f \in H(D) \) with the expansion \( f(z) = \sum_{\alpha} a_\alpha z^\alpha, \quad z \in D \) we have denoted

\[
\| f \|_\phi^2 = \sum_{\alpha} c_\alpha^{-1} |a_\alpha|^2,
\]

the norm generated by the inner product

\[
\langle f, g \rangle_\phi = \sum_{\alpha} c_\alpha^{-1} a_\alpha \overline{b_\alpha},
\]

where \( g(z) = \sum_{\alpha} b_\alpha z^\alpha, \quad z \in D \). Moreover, the inner product (and the corresponding norm) of \( H_{K_\phi} \) and \( H_\phi \) are the same.

Theorem 6.2, which we prove in the Appendix, establishes a general result giving a sharp relationship between the reproducing kernel spaces of analytic functions of several complex variables defined on a Reinhardt domain and the reproducing kernel space determined by the product of their reproducing kernels. Observe firstly that \( \phi_j \in P(D) \) implies \( \prod_{j=1}^m \phi_j \in P(D) \).

**Theorem 6.2** (cf. [6]). Let \( D \) be an arbitrary Reinhardt domain in \( \mathbb{C}^n \) and let \( m \geq 2 \) be an integer. For all \( j = 1, 2, \ldots, m \) let \( \phi_j \in P(D) \), let \( K_{\phi_j}(z, w) \) be the corresponding reproducing kernel for \( H_{\phi_j} \), and \( f_j \in H_{\phi_j} \). Then

\[
\prod_{j=1}^m f_j \in H_{\phi_1 \phi_2 \cdots \phi_m}.
\]
with
\[ \prod_{j=1}^{m} f_j \leq \prod_{j=1}^{m} \|f_j\|_{\phi_j}. \]

Equality attains if and only if either \( \prod_{j=1}^{m} f_j \equiv 0 \) or each \( f_j \) \((j = 1, 2, \ldots, m)\) is of the form
\[ f_j = C_j K_{\phi_j}^w \]
for some (common) \( w \in \mathbb{C}^n \), which satisfies \( \phi_j(w \cdot \bar{w}) < +\infty \) for all \( j = 1, 2, \ldots, m \), and for some nonzero \( C_j \). If there exists \( 1 \leq j \leq m \) such that \( \phi_j \in P_\infty(D) \), then the preceding condition for \( w \) may be replaced with \( w \in D \).

**Remark 6.1.** The previous theorem is formulated in Burbea’s paper [6]. In a particular case a proof may be found in his earlier papers [4] and [5]. In the Appendix we will simplify the proof from the last paper, we have also noted that the same proof may be adapted for analytic functions defined in any Reinhardt domain (not only the unit ball and the whole space \( \mathbb{C}^n \) as considered in [4] and in the other Burbea papers).

6.4. Theorem 6.2 is a result with interesting applications which arise by choosing for \( \phi_j \) \((j = 1, \ldots, m)\) concrete functions in \( P(D) \) (or \( P_\infty(D) \)). Since our interest here is on the analytic functions in polydiscs, we will derive only one application of it. That is Corollary 6.1 for generalized Hardy spaces on the unit polydisc. For various interesting applications to other HSRK of holomorphic functions, such as generalized Fisher spaces, we refer to the Burbea papers. This subsection contains a new version of Carleman’s inequality for analytic functions which belong to Hardy spaces on the unit polydisc with not necessary Hilbert structure (that is \( H^p(U^n) \) for any positive \( p \)).

Recall the definition of generalized Hardy spaces of the unit polydisc. First of all, for any number \( q \) the shifted factorial (the Pochhammer symbol) is
\[ (q)_n = \begin{cases} q(q + 1) \cdots (q + n - 1), & \text{if } n > 1, \\ 1, & \text{if } n = 0, \end{cases} \]
where \( n \) is an integer. One extends this definition as follows: For \( q = (q_1, \ldots, q_n) \in \mathbb{C}^n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \) denote
\[ (q)_\alpha = \prod_{j=1}^{n} (q_j)_{\alpha_j}. \]

For \( q = (q_1, \ldots, q_n) > 0 \) (which means \( q_j > 0 \) for each \( j = 1, 2, \ldots, n \)) introduce the function
\[ \phi_q(z) = \prod_{j=1}^{n} (1 - z_j)^{-q_j}, \]
where \( z = (z_1, \ldots, z_n) \) is a point of the unit polydisc. It is easy to check that
\[ a_\alpha(\phi_q) = \frac{(q)_\alpha}{\alpha!}, \quad \alpha \in \mathbb{Z}_+^n. \]

Recall that \( a_\alpha : H(D) \rightarrow \mathbb{C} \) stands for the functional which gives \( \alpha \)-coefficient in the power series expansion of \( f \in H(D) \). Thus, \( \phi_q \in P_\infty(U^n) \).

The \( q \)-Hardy space of the unit polydisc is the RKHS generated by the kernel \( K_{\phi_q} \). This is the space \( \mathcal{H}_{\phi_q} \) by the notation of the preceding subsection. However,
in the sequel, this space we will denote simply by $H_q(U^n)$. The corresponding norm and the reproducing kernel we denote by $\| \cdot \|_q$ and $K_q$, respectively. Thus,

$$K_q(z, w) = \phi_q(z \cdot \overline{w}) = \prod_{j=1}^n (1 - z_j \overline{w_j})^{-q_j}$$

for $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in U^n$. Explicitly, the norm in $H_q(U^n)$ is given by

$$\|f\|_q^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{\alpha!}{(q|\alpha|)} |a_\alpha|^2,$$

where $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbb{Z}_+^n$ for $f \in H(U^n)$.

The family of all $q$–Hardy spaces ($q > 0$) we call the generalized Hardy spaces.

For $q \in (1, \ldots, 1)$ the square of the norm $\| \cdot \|_q$ has the integral representation

$$\|f\|_q^2 = \int |f|^2 \, da_{q-2}, \quad f \in H_q(U^n),$$

where

$$da_{q-2} = \prod_{k=1}^n da_{q_k-2}(z_k) = \prod_{k=1}^n \frac{q_k - 1}{\pi} (1 - |z_k|^2)^{q_k-2} \, dA(z_k)$$

is the normalized weighted area measure on the unit polydisc for $q = (q_1, \ldots, q_n) > 1$ (which means $q_j > 1$ for each $j = 1, \ldots, n$); for $q = 1$ ($q_j = 1$ for all $j = 1, \ldots, n$) it is convenient to set $da_q = dm_n$. In (25) we assume integration over $U$ if $q > 1$ and over $T^n$ if $q = 1$. In the last case, the object of integration is the radial boundary function for $f$.

Since reproducing kernel in the unique way determinate the Hilbert spaces, it follows that $H_1(U^n)$ is the Hardy space $H^2(U^n)$ and $H_q(U^n)$, $q > 1$ is the weighted Bergman space $L_{a,q-2}^2(U^n)$. For $0 < q < 1$ the generalized Hardy space $H_q(U^n)$ is the Bergman–Selberg space on the unit polydisc.

In the sequel, we will use the following abbreviations: $(q, \ldots, q)$ we denote shortly by $q$. Thus, the Bergman space $L^p_{a,q}(U^n)$, the corresponding measure $da_{(q, \ldots, q)}$ and the norm $\| \cdot \|_{p,q}$ is denoted by $L^p_{a,q}(U^n)$, $da_q$ and $\| \cdot \|_{p,q}$, respectively.

Theorem 6.2 has the following reformulation for generalized Hardy spaces on the unit polydisc.

**Corollary 6.1** (cf. [5]). Let $0 < q_j < +\infty$ and $f_j \in H_q(U^n)$ for all $j = 1, 2, \ldots, m$. Denote $q = \sum_{j=1}^m q_j$. Then

$$\prod_{j=1}^m f_j \in H_q(U^n)$$

and

$$\left\| \prod_{j=1}^m f_j \right\|_q \leq \prod_{j=1}^n \|f_j\|_{q_j}.$$

The equality sigh attains if and only if either $\prod_{j=1}^m f_j \equiv 0$ or each $f_j$ ($j = 1, 2, \ldots, m$) is of the form

$$f_j = C_j K_{q_j}^w$$

for some (common) $w \in U^n$ and a constant $C_j \neq 0$. 
Proof. The proof is straightforward from Theorem 6.2. It is enough to note that
\[ \prod_{j=1}^{m} \phi_{q_j} = \phi_q. \]
Thus,
\[ \| \cdot \|_{\phi_{q_1}\phi_{q_2}\cdots\phi_{q_m}} = \| \cdot \|_{\phi_q}. \]
We may use the inequality in Theorem 6.2 which gives the desired inequality.
Since each \( \phi_j \) \((j = 1, 2, \ldots, m)\) belongs to the class \( \mathcal{P}_\infty(U^n) \), it follows \( w \in U^n \). \( \square \)

Remark 6.2. The case of the unit disc of this theorem is considered in [4]. Moreover, in this paper appears an inequality for generalized Fisher spaces of holomorphic functions in the entire plane.

Our goal is to derive the next theorem proved by the author in the case of bidisc. This theorem generalizes Burbea’s Theorem 6.1.

Corollary 6.2 (cf. [16]). Let \( m \geq 2 \) be an integer and \( f_j(z) \in H^{p_j}(U^n) \), \( 0 < p_j < +\infty \) for all \( j = 1, 2, \ldots, m \). Then

\[ \prod_{j=1}^{m} |f_j|^{p_j} \in L_{m-2}(U^n) \]

with

\[ \int_{U^n} \left\{ \prod_{j=1}^{m} |f_j(z)|^{p_j} \right\} \, da_{m-2}(z) \leq \prod_{j=1}^{m} \| f_j \|_{p_j}^{p_j}. \]

Equality attains if and only if either \( \prod_{j=1}^{m} f_j \equiv 0 \) or each \( f_j \) \((j = 1, 2, \ldots, m)\) is of the form

\[ f_j(z) = c_j K_w(z)^{2/p_j}, \]

for some (common) \( w \in U^n \) and a non-zero \( c_j \). Here, \( K(z, w) = \prod_{j=1}^{m} (1 - z_j \overline{w}_j)^{-1} \) denotes the Cauchy–Szegö kernel for the unit polydisc.

Proof. The general case of this corollary follows from our Theorem 2.1, where we have to set \( \Phi : \mathbb{R}^m \rightarrow \mathbb{R}_+ \) given by

\[ \Phi(x_1, \ldots, x_m) = \prod_{j=1}^{m} x_j \]

and

\[ d\mu(z) = \frac{m-1}{\pi} (1 - |z|^2)^{m-2} dA(z). \]

The condition \((\dagger)\) is satisfied for \( \tilde{p}_j = 2 \) for all \( j = 1, 2, \ldots, m \) and the family of extremals is

\[ \mathcal{E}(\Phi, \nu_n) = \{(C_1 K_w(z), \ldots, C_m K_w(z)) : w \in U^n, C_j \neq 0, j = 1, 2, \ldots, m\}, \]

according to Corollary 6.1 (for \( q_j = 1, j = 1, 2, \ldots, m \)). Recall that \( H_1(U^n) = H^2(U^n) \) is the Hardy space and \( H_m(U^n) = L_a^{2, m-2}(U^n) \) is the weighted Bergman space. \( \square \)
7. Appendix: A proof of Theorem 6.2

A proof of Theorem 6.2 follows immediately by induction, from the next theorem. The result may be derived by applying the original Carleman idea [7]. That idea is incorporated in the following result, which should be compared with the original Carleman inequality.

**Theorem 7.1** (cf. [4, 5, 6]). Let \( D \) be a Reinhardt domain in \( \mathbb{C}^n \). Let \( \phi, \psi \in \mathcal{P}(D) \) and \( f \in \mathcal{H}_\phi, g \in \mathcal{H}_\psi \). Then

\[
f \in \mathcal{H}_\phi, g \in \mathcal{H}_\psi
\]

and

\[
\|fg\|_{\phi \psi} \leq \|f\|_{\phi} \|g\|_{\psi}.
\]

Equality holds if and only if either \( fg \equiv 0 \) (i.e., \( f \equiv 0 \) or \( g \equiv 0 \)) or \( f \) and \( g \) have the following form

\[
f = C_1 K^\zeta_{\phi}, \quad g = C_2 K^\zeta_{\psi}
\]

for some (common) \( \zeta \in \mathbb{C}^n \) which satisfies \( \phi(\zeta \cdot \overline{\zeta}) < \infty \), and some nonzero constants \( C_1, C_2 \). Here, \( K_\phi(z, \zeta) = \phi(z \cdot \overline{\zeta}) \) and \( K_\psi(z, \zeta) = \psi(z \cdot \overline{\zeta}) \) are reproducing kernels for \( \mathcal{H}_\phi \) and \( \mathcal{H}_\psi \), respectively.

If \( \phi \in \mathcal{P}_\infty(D) \) or \( \psi \in \mathcal{P}_\infty(D) \), then above condition for \( \zeta \) may be replaced with \( \zeta \in D \).

The next simple lemma we will use in our proof of Theorem 7.1.

**Lemma 7.1.** Let \( (\lambda_\gamma) \) be any sequence of complex numbers indexed by multi–indexes \( \gamma \in \mathbb{Z}_+^n \) which satisfies

\[
\lambda_\alpha +_\beta = C \lambda_\alpha \lambda_\beta
\]

for all \( \alpha, \beta \in \mathbb{Z}_+^n \) and some constant \( C \neq 0 \). Then the sequence if of the form

\[
\lambda_\gamma = C^{-|\gamma|} \zeta_\gamma \quad (\gamma \in \mathbb{Z}_+^n).
\]

Here \( \zeta \in \mathbb{C}^n \) is an arbitrary point.

**Proof.** Let \( \gamma = (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_n) \in \mathbb{Z}_+^n \). If \( \gamma_j \neq 0 \) for some \( 1 \leq j \leq n \), then

\[
\lambda_\gamma = \lambda_{\gamma - e_j}(C \lambda_{e_j}).
\]

Using the previous relation and induction on \( |\gamma| \), we immediately obtain

\[
\lambda_\gamma = \lambda_0 \prod_{j=1}^{n} \zeta_j^{\gamma_j} = \lambda_0 \zeta_\gamma,
\]

where we have denoted

\[
\zeta_j = C \lambda_{e_j}
\]

(for all \( j = 1, 2, \ldots, n \)) and

\[
\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n).
\]

Since \( \lambda_0 = C \lambda_0^2 \), it follows \( \lambda_0 = 0 \) or \( \lambda_0 = C^{-1} \). In both cases we have the representation of the sequence \( (\lambda_\gamma) \) as stated in the lemma. \( \Box \)
Proof of Theorem 7.1. If
\[ \phi(z) = \sum_{\alpha \in \mathbb{Z}_n^+} c_{\alpha} z^{\alpha}, \quad \psi(z) = \sum_{\beta \in \mathbb{Z}_n^+} d_{\beta} z^{\beta}, \]
then
\[ \phi(z)\psi(z) = \sum_{\gamma \in \mathbb{Z}_n^+} M_{\gamma} z^{\gamma}, \quad z \in D. \]
Here
\[ M_{\gamma} = \sum_{\alpha + \beta = \gamma} c_{\alpha} d_{\beta}, \quad \gamma \in \mathbb{Z}_n^+. \]
The last summation is over all pairs \((\alpha, \beta)\) satisfying \(\alpha + \beta = \gamma\).
Similarly, if
\[ f(z) = \sum_{\alpha \in \mathbb{Z}_n^+} a_{\alpha} z^{\alpha}, \quad g(z) = \sum_{\beta \in \mathbb{Z}_n^+} b_{\beta} z^{\beta}, \]
then
\[ f(z)g(z) = \sum_{\gamma \in \mathbb{Z}_n^+} A_{\gamma} z^{\gamma}, \quad z \in D, \]
where
\[ A_{\gamma} = \sum_{\alpha + \beta = \gamma} a_{\alpha} b_{\beta}, \quad \gamma \in \mathbb{Z}_n^+. \]
Thus
\[ \|f\|_{\phi}^2 = \sum_{\alpha \in \mathbb{Z}_n^+} c_{\alpha}^{-1} |a_{\alpha}|^2, \quad \|g\|_{\psi}^2 = \sum_{\beta \in \mathbb{Z}_n^+} d_{\beta}^{-1} |b_{\beta}|^2 \]
and
\[ \|fg\|_{\phi\psi}^2 = \sum_{\gamma \in \mathbb{Z}_n^+} M_{\gamma}^{-1} |A_{\gamma}|^2. \]
In view of coefficient representation we immediately see that the statement of this theorem (inequality
\[ \|fg\|_{\phi\psi} \leq \|f\|_{\phi} \|g\|_{\psi} \]
and all extremal functions) is equivalent to the next statement concerning only the coefficients:

There holds
\[ \sum_{\gamma \in \mathbb{Z}_n^+} M_{\gamma}^{-1} |A_{\gamma}|^2 \leq \left\{ \sum_{\alpha \in \mathbb{Z}_n^+} c_{\alpha}^{-1} |a_{\alpha}|^2 \right\} \left\{ \sum_{\beta \in \mathbb{Z}_n^+} d_{\beta}^{-1} |b_{\beta}|^2 \right\} \]
with equality if and only if one of the following two independent conditions is satisfied:

1. \(a_{\alpha} = 0\) for all \(\alpha \in \mathbb{Z}_n^+\) or \(b_{\beta} = 0\) for all \(\beta \in \mathbb{Z}_n^+\) (regarding the series expansion of \(f\) and \(g\) this is equivalent to \(f \equiv 0\) or \(g \equiv 0\));
2. \(a_{\alpha} = C_1 c_{\alpha} \zeta^\alpha\), \(\alpha \in \mathbb{Z}_n^+\) and \(b_{\beta} = C_2 d_{\beta} \zeta^\beta\), \(\beta \in \mathbb{Z}_n^+\), where \(C_1\) and \(C_2\) are some nonzero constants and \(\zeta \in \mathbb{C}^n\) (in other words \(f = C_1 K^\zeta_\phi\) and \(g = C_2 K^\zeta_\psi\)).
Let us proof the relation (20). By the Cauchy–Schwarz inequality we have

\[ |A_\gamma|^2 = \left| \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right|^2 \leq \left( \sum_{\alpha + \beta = \gamma} \frac{|a_\alpha|^2}{c_\alpha} d_\beta \right)^{1/2} \left( \sum_{\alpha + \beta = \gamma} c_\alpha d_\beta \right)^{1/2} \]

for all \( \gamma \in \mathbb{Z}^n_+ \). We have just proved

(27) \[ M^{-1}_\gamma |A_\gamma|^2 \leq \sum_{\alpha + \beta = \gamma} c_\alpha^{-1} |a_\alpha|^2 d_\beta^{-1} |b_\beta|^2 \quad (\gamma \in \mathbb{Z}^n_+) \]

Taking the sum of (27) over all \( \gamma \in \mathbb{Z}^n_+ \) yields (26):

\[ \sum_{\gamma \in \mathbb{Z}^n_+} M^{-1}_\gamma |A_\gamma|^2 \leq \sum_{\gamma \in \mathbb{Z}^n_+} \sum_{\alpha + \beta = \gamma} \frac{|a_\alpha|^2}{c_\alpha} d_\beta^{-1} |b_\beta|^2 \]

\[ \leq \left\{ \sum_{\alpha \in \mathbb{Z}^n_+} c_\alpha^{-1} |a_\alpha|^2 \right\} \left\{ \sum_{\beta \in \mathbb{Z}^n_+} d_\beta^{-1} |b_\beta|^2 \right\}, \]

what proves the inequality.

The rest of this proof is devoted to the equality statement and extremal functions. Note first that if \( 0 \leq A_\alpha \leq B_\alpha \) for all \( \alpha \) then \( \sum_\alpha A_\alpha \leq \sum_\alpha B_\alpha \). If moreover there exist \( a_0 \) such that \( A_{\alpha_0} < B_{\alpha_0} \), then we have the strict inequality \( \sum_\alpha A_\alpha < \sum_\alpha B_\alpha \).

In view of this fact, equality attains in (26) if and only if equality holds in (27) for all \( \gamma \in \mathbb{Z}^n_+ \). This is equivalent to the existence of \( \lambda_\gamma \in \mathbb{C}, \gamma \in \mathbb{Z}^n_+ \) such that

(28) \[ a_\alpha b_\beta = \lambda_\gamma c_\alpha d_\beta \]

for all \( \alpha, \beta \in \mathbb{Z}^n_+ \) satisfying \( \alpha + \beta = \gamma \). Let us take \( \beta = 0 \) and \( \gamma = \alpha \), then \( \alpha = 0 \) and \( \gamma = \beta \) in (25). We obtain

(29) \[ a_\alpha b_0 = \lambda_\alpha c_\alpha d_0, \quad a_0 b_\beta = \lambda_\beta c_0 d_\beta, \quad \alpha, \beta \in \mathbb{Z}^n_+ \]

We will consider two disjoint cases \( a_0 b_0 = 0 \) and \( a_0 b_0 \neq 0 \).

If \( a_0 b_0 = 0 \), then from (29) we have \( \lambda_\gamma = 0 \) for all \( \gamma \in \mathbb{Z}^n_+ \). For example, let \( a_0 = 0 \). If \( b_\beta = 0 \) for all \( \beta \), then the first item of the modified equality statement holds. If there exist \( b_{\beta_0} \neq 0 \), in (28) take \( \beta = \beta_0 \), we obtain \( a_\alpha b_{\beta_0} = 0, \quad \alpha \in \mathbb{Z}^n_+ \)

and thus \( a_\alpha = 0 \) for all \( \alpha \in \mathbb{Z}^n_+ \). It follows that the first item also holds. Now let \( a_0 \neq 0 \) and \( b_0 \neq 0 \). According to (29),

\[ a_\alpha = (b_0^{-1} c_0) \lambda_\alpha c_\alpha, \quad \alpha \in \mathbb{Z}^n_+, \quad b_\beta = (a_0^{-1} c_0) \lambda_\beta d_\beta, \quad \beta \in \mathbb{Z}^n_+. \]

If the previous two relations substitute in (28), we obtain

\[ \lambda_{\alpha + \beta} = C \lambda_\alpha \lambda_\beta, \quad \alpha, \beta \in \mathbb{Z}^n_+, \]

where we denote \( C = (a_0^{-1} c_0 b_0^{-1} d_0) \). Regarding Lemma 17.1 we obtain

\[ \lambda_\gamma = C^{-1} \zeta^\gamma, \quad \gamma \in \mathbb{Z}^n_+ \]

for some \( \zeta \in \mathbb{C}^n \). It follows

\[ a_\alpha = C_1 c_\alpha \zeta^\alpha, \quad \alpha \in \mathbb{Z}^n_+, \quad b_\beta = C_2 d_\beta \zeta^\beta, \quad \beta \in \mathbb{Z}^n_+, \]
where
\[ C_1 = a_0 c_0^{-1} \neq 0, \quad C_2 = b_0 d_0^{-1} \neq 0. \]

We have proved the second item.

Finally, if \( \phi \in \mathcal{P}_\infty \) or if \( \psi \in \mathcal{P}_\infty \), then \( \zeta \in D \). Namely, if \( f \) and \( g \) are extremal for the inequality, then
\[
\|f\|_\phi^2 = |C_1|^2 \phi(\zeta \cdot \overline{\zeta}), \quad \|f\|_\psi^2 = |C_2|^2 \psi(\zeta \cdot \overline{\zeta})
\]
(recall, \( \|K_\phi\|_\phi = \phi(\zeta \cdot \overline{\zeta})^{1/2} \), \( \|K_\psi\|_\psi = \psi(\zeta \cdot \overline{\zeta})^{1/2} \)). Since the domain of convergence of \( \phi(\zeta \cdot \overline{\zeta}) \) and \( \psi(\zeta \cdot \overline{\zeta}) \) is exactly \( D \), it follows \( \zeta \in D \). \( \square \)

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