The fundamental group of the harmonic archipelago

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Abstract

The harmonic archipelago $HA$ is obtained by attaching a large pinched annulus to every pair of consecutive loops of the Hawaiian earring. We clarify $\pi_1(HA)$ as a quotient of the Hawaiian earring group, provide a precise description of the kernel, show that both $\pi_1(HA)$ and the kernel are uncountable, and that $\pi_1(HA)$ has the indiscrete topology.

1 Introduction

This note serves to clarify certain properties of $\pi_1(HA)$ and $\pi_1(HE)$, the topological fundamental groups respectively of the harmonic archipelago and the Hawaiian earring.

The Hawaiian earring $HE$ is the union of a null sequence of simple closed curves meeting in a common point.

Introduced by Bogley and Sieradski, the harmonic archipelago $HA$, is the space obtained by attaching large pinched annuli, one for each pair of consecutive loops, to the Hawaiian earring $HE$. In [2] Bogley and Sieradski construct a comprehensive theory which provides a useful framework for investigating the fundamental groups of locally complicated spaces such as $HE$ and $HA$. Various properties of $\pi_1(HA)$, such as its uncountability, are uncovered in [2].

In [1], Biss also uses $HE$ and $HA$ as motivating examples, and proves some nice general results on topological fundamental groups and their generalized covering spaces.

However, an oversight in [1] (see Remark [2], leads to a false description of $\pi_1(HA)$ and its false generalization Theorem 8.1. There is also a typographical error in the description of ker$(j^*)$, the kernel of the epimorphism $j^* : \pi_1(HE) \to \pi_1(HA)$, induced by inclusion $j : HE \hookrightarrow HA$. 

1
After adjusting for this, there remains arguably room for further discussion regarding which elements of $\pi_1(HE)$ belong to $\ker(j^*)$.

For example it follows from the investigations of Morgan/Morrison $[6]$, DeSmit $[4]$, and Cannon/Conner $[3]$ that elements of $\pi_1(HE)$ can be seen as “transfinite words” over an infinite alphabet $\{x_1, x_2, \ldots\}$ with each letter appearing finitely many times.

As described in $[1]$, $\ker(j^*)$ is generated by the relations $x_i = x_j$ for all $i$ and $j$. Does this mean two transfinite words over $\{x_1, x_2, \ldots\}$ are equivalent in $\pi_1(HE)$ if and only if one can be transformed into the other by finitely many substitutions and finitely many cancellations of consecutive letters? No, for example $x_1x_2^{-1}(x_3x_4\ldots)x_1x_2^{-1}(\ldots x_4^{-1}x_3^{-1})$ is trivial in $\pi_1(HE)$, but cannot be transformed into the trivial word with finitely many such operations.

Are two transfinite words over $\{x_1, x_2, \ldots\}$ equivalent in $\pi_1(HE)$ if one can be transformed into the other after “infinitely many substitutions”? No, for then the essential element $(x_1x_2^{-1}x_3x_4^{-1}\ldots)$ could be transformed into the inessential $x_1x_1^{-1}x_2x_2^{-1}\ldots$.

We provide a precise description of $\ker(j^*)$ in Theorem $[7]$ and prove as Corollary $[8]$ that $\ker(j^*)$ is uncountable. Corollaries $[9]$ and $[10]$ provide proofs of results also indicated in $[2]$ and $[1]$: $\pi_1(HE)$ is uncountable and, despite its large cardinality, $\pi_1(HE)$ has the indiscrete topology.

2 Definitions

For $n \in \{1, 2, 3, \ldots\}$ let $X_n \subset \mathbb{R}^2$ denote the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$. Let $Y_n = \bigcup_{i=n}^{\infty} X_i$. Thus $Y_n$ is the Hawaiian earring determined by the loops $X_n, X_{n+1}, \ldots$.

Let $A_n \subset \mathbb{R}^2$ denote the closed pinched annulus bounded by $X_n \cup X_{n+1}$. Let $Y^n = Y_1 \cup A_1 \cup \ldots \cup A_n$. Endow $Y^n$ with the subspace topology inherited from $\mathbb{R}^2$.

For the underlying set let $HA = \bigcup_{n=1}^{\infty} Y^n$. However we define the topology of $HA$ such that $Y^n$ inherits the usual topology but such that 1) There exists a sequence $z_n \in \text{int}(A_n)$ such that $\{z_1, z_2, \ldots\}$ has no subsequential limit and 2) If $p \in HA$ is a subsequential limit of the sequence $y_1, y_2, \ldots$, and if $y_n \in \text{int}(A_n)$ for all $n$, then $p = (0, 0)$.

Let $G_n = \pi_1(Y_n, (0,0))$.

Let $F_{N,n}$ denote the free group on the letters $\{x_N, x_{N+1}, \ldots x_n\}$ coupled with the symbol $1$ denoting the trivial element.

Let $\phi_{N,n} : F_{N,n+1} \to F_{N,n}$ denote the homomorphism such that $\phi_{N,n}(x_i) = x_i$ if $N \leq i \leq n$ and $\phi_{N,n}(x_{n+1}) = 1$.

For $N \geq 1$ let $G^N$ denote the inverse limit of free groups determined by $F_{N,N} \leftarrow F_{N,N+1} \ldots$ under the bonding maps $\phi_{N,N}$.

Formally elements of $F_{N,n}$ are equivalence classes of words under
the obvious cancellations, and the group operation is catcatanation. However each element of \( F_{N,n} \) has a unique representative with a minimal number of nontrivial letters. Consequently each element of \( G^N \) is uniquely determined by a **canonical sequence** \( w_N, w_{N+1}, ... \) such that \( w_n \in F_{N,n} \) and \( w_n \) is a maximally reduced word in \( F_{N,n} \).

3 \( \pi_1(HA) \)

It is a nontrivial fact ([3], [4], [6]) that \( \pi_1(Y_1) \) injects naturally into the inverse limit of free groups.

**Remark 1** Given \( n \geq N \geq 1 \) let \( Z_{N,n} = \bigcup_{i=N}^{n} X_{i,n} \). Let \( r_{N,n} : Y_N \to Z_{N,n} \) denote the retraction collapsing \( X_i \) to \( (0,0) \) for \( i > n \). Since \( r_{N,n}(r_{N,n+1}) = r_{N,n} \), the maps \( r_{N,n} \) induce a homomorphism \( \psi_N : \pi_1(Y_N,(0,0)) \to \lim_{\to} \pi_1(Z_{N,n},(0,0)) \). It is shown in [5] and [6] that \( \psi_N \) is one to one. Moreover \( G_N = \text{im}(\psi_N) \) consists of all canonical sequences \( (w_N, w_{N+1}, ...) \) such that for each \( i \) there exists \( M_i \) such that for all \( n \geq N \), \( x_i \) appears at most \( M_i \) times in \( w_n \).

It is falsely asserted in [6] that \( \psi_1 : \pi_1(HE) \to \lim_{\to} F_{1,n} \) is an isomorphism. Consequently Theorem 8.1 of [6] is false.

**Remark 2** The canonical monomorphism \( \psi_1 : \pi_1(Y_1,(0,0)) \hookrightarrow G^1 \) is **not** surjective. By compactness of \([0,1]\), a given path in \( Y_1 \) can traverse each loop only finitely many times. Thus the element

\[
(1, x_1 x_2 x_1^{-1} x_2^{-1}, x_1 x_2 x_1^{-1} x_2^{-1} x_3 x_4 x_1^{-1} x_3^{-1}, ...) \in \lim_{\to} F_{1,n}
\]

has no preimage in the Hawaiian earring group.

**Remark 3** \( X_n \cup X_{n+1} \) is a strong deformation retract of \( A_n \setminus \{z_n\} \). Thus \( Y^n \) is a strong deformation retract of \( HA \setminus \{z_{n+1}, z_{n+2}, ...\} \).

**Lemma 4** Suppose \( f : S^1 \to Y_1 \) is any map. Then \( f \) is inessential in \( HA \) if and only if there exists \( N \) such that \( f \) is inessential in \( Y^N \).

**Proof.** Suppose \( f \) is inessential in \( HA \). Let \( F : D^2 \to HA \) be a continuous extension of \( f \). Since \( \text{im}(F) \) is compact, there exists \( N \) such that \( z_n \not\in \text{im}(F) \) whenever \( n > N \). Thus \( \text{im}(F) \subset HA \setminus \{z_{N+1}, z_{N+2}, ...\} \). By remark \( \Box \) \( Y^N \) is a strong deformation retract of \( HA \setminus \{Z_{N+1}, Z_{N+2}, ...\} \). Thus \( f \) is inessential in \( Y^N \). Conversely if there exists \( N \) such that \( f \) is inessential in \( Y^N \) then \( f \) is inessential in \( HA \) since \( Y^N \subset HA \). ■

Notice \( Y_N \) is a strong deformation retract of \( Y^{N-1} \) under a homotopy \( R_{N,t} : Y^{N-1} \to Y_N \) collapsing \( A_1 \cup ... A_{N-1} \) onto the simple closed curve.
$X_N$. In particular given any loop $f : S^1 \to Y_1$ we may canonically deform $f$ in $Y^{N-1}$ to a loop $g$ such that $im(g) \subset Y_N$. Appealing to Remark 1 we may identify $\pi_1(Y_i, (0,0))$ with $G_i$. Thus the composition $Y_1 \hookrightarrow Y^{N-1} \to Y_N$ induces a homomorphism $q_N^* : G_1 \to G_N$. Combining these observations we obtain the following:

**Lemma 5** $q_N^*(w_1, w_2, ...) = (v_N, v_{N+1}, \ldots)$ if and only the following property is satisfied for each $n \geq N$: For each $i \leq N$ replace each occurrence of $x_i$ in $w_n$ with $x_N$, creating a word $w_n'$ on the letters $\{x_N, \ldots x_n\}$. Then the word $w_n$ is equivalent to $v_n$ in the free group $F_{N,n}$.

Lemma 5 is also handled in [1] and [2]. Our proof is similar to an argument that the 2 sphere is simply connected.

**Lemma 6** Let $j : Y_1 \hookrightarrow HA$ denote the inclusion map. Then the induced homomorphism $j^* : \pi_1(Y_1, (0,0)) \to \pi_1(HA, (0,0))$ is surjective.

**Proof.** Suppose $f : S^1 \to HA$ is any map such that $f(1) = (0,0)$. Let $J$ be a (nonempty) component of $f^{-1}(HA \setminus Y_1)$. Note $HA \setminus Y_1$ is the union of pairwise disjoint connected open sets $int(A_1) \cup int(A_2), \ldots$ Thus there exists $i$ such that $f(J) \subset int(A_i)$ and $f(\partial J) \subset \partial A_i$. If $z_i \in im(f)$ replace $f_J$ by a path homotopic small perturbation $\hat{f}_{J}$ such that $z_i \notin im(\hat{f}_{J})$. By uniform continuity of $f$ finitely many such surgeries are required. Note $f$ and $f^*$ are homotopic in $HA$ and $im(f^*) \subset HA \setminus \{z_1, z_2, \ldots\}$.

By remark 3 $Y_1$ is a strong deformation retract of $HA \setminus \{z_1, z_2, \ldots\}$ and hence there exists $f^* : S^1 \to Y_1$ such that $j^*[f^*] = [f^*] = [f]$.

Since $j^*$ is a surjection $\pi_1(HA, (0,0))$ is isomorphic to the quotient group $\pi_1(Y_1, (0,0))/ker j^*$. Combining Lemmas 4 and 5 we obtain the following characterization of ker $j^*$.

**Theorem 7** Let $j^* : G_1 \to \pi_1(HA, (0,0))$ denote the epimorphism induced by inclusion $j : Y_1 \hookrightarrow HA$. Let $K = ker j^*$. Then $(w_1, w_2, \ldots) \in K$ if and only if there exists $N$ such that the following holds: Suppose for each $i \leq N$ and for each $n \geq N$ each occurrence of $x_i$ in $w_n$ is replaced by $x_N$ creating a (nonreduced) word $v_n$ on the letters $\{x_N, x_{N+1}, \ldots x_n\}$. Then $v_n$ can be reduced to the trivial element of $F_{N,n}$.

**Corollary 8** ker$(j^*)$ is uncountable.

**Proof.** There exist uncountably many distinct permutations of the set $\{3, 4, 5, 6, \ldots\}$. Moreover, the loops in $Y_3$ determined by distinct permutations of $x_3, x_4, \ldots$ determine distinct elements of $\pi_1(Y_3, (0,0))$. Thus
$G_3$ is uncountable. Each $w \in G_3$ determines a homotopically distinct loop in $Y_1$ corresponding to $x_1x_2^{-1}wx_1x_2^{-1}w^{-1}$. By Theorem 7 this element is in $\ker(j^*)$. $\blacksquare$

Corollary 9 is also treated in [2].

Corollary 10 $\pi_1(HA, (0, 0))$ is uncountable.

Proof. Consider the set $A$ of functions from $\{1, 3, 5, \ldots\} \to \{0, 1\}$. Each element $f \in A$ determines a permutation $\tau_f : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}$ as follows: If $f(2n - 1) = 0$ then $\tau_f$ fixes $2n - 1$ and $2n$. Otherwise $\tau_f$ swaps $2n - 1$ and $2n$. The permutation $\tau_f$ determines a loop in $Y_1$ by the following recipe. Travel clockwise once around $X_{\tau_f(1)}$, then travel clockwise once around $X_{\tau_f(2)}$, etc... This corresponds to the transfinite word $\tau_f(1)\tau_f(2)\ldots \in \pi_1(Y_1)$.

Note $A$ is uncountable. Declare two elements $\{f, g\} \subset A$ equivalent if $f$ and $g$ agree except on a finite set. Each equivalence class in $A$ has countably many elements and hence $B$, the set of equivalence classes of $A$, is uncountable. If $[f]$ and $[g]$ are distinct elements of $B$ then by Lemma 5 for each $N$, $\tau_f$ and $\tau_g$ fail to be equivalent in $Y^N$. Thus by Lemma 4 $\tau_f$ and $\tau_g$ determine distinct elements of $\pi_1(HA, (0, 0))$. Thus, since $B$ is uncountable, $\pi_1(HA, (0, 0))$ is uncountable. $\blacksquare$

It shown in [11] (and generalized in [13]) that the quotient space consisting of path components of based loops in a topological space $X$ is a topological group. Despite a large number of elements, $\pi_1(HA, (0, 0))$ has the indiscrete topology. Corollary 10 is also argued in [11].

Corollary 10 The topological group $\pi_1(HA, (0, 0))$ has exactly one nonempty open subset.

Proof. Let $L = \{f : [0, 1] \to HA | f(0) = f(1) = (0, 0)\}$ with the uniform topology. By definition $\pi_1(HA, (0, 0))$ is the collection of path components of $L$ with the quotient topology. Suppose $B \subset \pi_1(HA, (0, 0))$, suppose $B \neq \emptyset$ and suppose $[f] \in B$.

Since $im(f)$ is compact and $\{z_1, z_2, \ldots\}$ is not compact, there exists $N$ such that $im(f) \subset HA \setminus \{z_{N+1}, z_{N+2}, \ldots\}$. For $M > N$ there is a strong deformation retraction from $HA \setminus \{z_{M+1}, z_{M+2}, \ldots\}$ onto $Y^M$. Thus for large $M$ we may deform $f$ in $HA$ to a nearby map $f_M$ such that $im(f_M) \subset Y^M$. Let $A_M = f_M^{-1}(Y^M \setminus (X_{M+1} \cup X_{M+2} \ldots))$. Redefine $f_M$ over $A_M$ to be the constant $(0, 0)$ creating a map $f_M^*$. Note $im(f_M^*) \subset A_1 \cup \ldots A_{M-1}$. Notice $A_1 \cup \ldots A_{M-1}$ has the homotopy type of $S^1$. By wrapping around $X_M$ as many times as necessary, we may extend $f_M^*$ to a map of $f_M^*[0, 1 + \frac{1}{M}] \to HA$ such that $f_M^*$ is an inessential loop and $f(t) \in X_M$ for $t \in [1, \frac{1}{M}]$. Reparameterize $f_M^*$ linearly to create a
map \( f^*_M : [0,1] \to HA \). Notice \( f^*_M \to f \) uniformly. Thus the path component of the trivial loop is dense in the space of based loops of \( HA \). Thus, if \( B \) is open then \( B \) contains the trivial element.

Suppose on the other hand that \( B \) is closed. Choose \( N \) such that \( im(f) \subset Y^n \) whenever \( n \geq N \). Notice \( Y_{n+1} \) is a strong deformation retract of \( Y^n \). Thus \( [f] \) contains representatives whose images have arbitrarily small diameter. Thus \( B \) contains the trivial element.

Thus the trivial element of \( \pi_1(HA(0,0)) \) belongs to each nonempty open set and each nonempty closed set. Hence \( \pi_1(HA(0,0)) \) has only one nonempty open subset.

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