Eight-dimensional SU(3)-manifolds of cohomogeneity one

ANDREA GAMBIOLI

Abstract

In this paper, we classify 8-dimensional manifolds $M$ admitting an $SU(3)$ action of cohomogeneity one such that (i) $M$ is simply connected and the orbit space $M/G$ is isomorphic to $[0, 1]$, and (ii) $M/G \cong S^1$ and the principal orbits are simply connected. We discuss applications to the study of the group manifold $SU(3)$ and to 8-dimensional quaternion-Kähler spaces.

MSC classification: 57S25; 22E46, 57S15, 53C30, 53C26, 58D05.

1 Introduction

Let $M$ be a differentiable manifold, and $G$ a compact semisimple group acting smoothly on $M$. Then $M$ is said to be a cohomogeneity-one $G$-space if the principal orbits are codimension-one submanifolds. A result due to Mostert [31] asserts that the quotient space $M/G$ is isomorphic to $[0, 1]$ or to $S^1$ if $M$ is compact, to $[0, 1)$ or $\mathbb{R}$ if $M$ is non-compact. In the case of the interval $[0, 1]$, there are precisely two singular orbits corresponding to the endpoints.

Manifolds with a cohomogeneity-one group action have been increasingly studied in recent years. This is mainly due to the fact that many problems concerning the existence of $G$-invariant structures on them can be reduced to ODE’s, which are sometimes straightforward to handle. As typical examples, we cite [9], [13], [17], in which such techniques were used to construct Einstein metrics and examples of metrics with exceptional holonomy.

More recently, cohomogeneity-one quaternion-Kähler and hyperkähler manifolds were classified in [15], [16]. General criteria for the classification of cohomogeneity-one manifolds were also developed in [1], [2], and used to partially classify such manifolds with $\chi(M) > 0$ (and a corresponding family of quaternion-Kähler manifolds) in [4]. Cohomogeneity-one $SU(3)$ manifolds of dimension 7 are the subject of [32].

In this paper, we shall focus on 8-dimensional simply-connected smooth manifolds admitting an action of $SU(3)$ of cohomogeneity one. The interest
in these manifolds arises from the following considerations. Firstly, the 8-dimensional quaternion-Kähler (QK) spaces

\[ \mathbb{H}P^2, \quad \text{Gr}_2(\mathbb{C}^4) \cong \text{Gr}_4(\mathbb{R}^6), \quad G_2/\text{SO}(4), \]  

remarkably all admit an SU(3)-action of cohomogeneity one. (See [38], [3], [34] for the theory of such Wolf spaces.) In [20], the author studied the moment mapping \( \mu \) of a QK space into the Grassmannian \( \tilde{\text{Gr}}_3(\mathfrak{g}) \) of oriented 3-planes in the Lie algebra \( \mathfrak{g} \). Whilst \( \mu \) is a branched covering of \( G_2/\text{SO}(4) \) onto its image in \( \tilde{\text{Gr}}_3(\mathfrak{su}(3)) \) [28], we shall point out that the first two spaces in (1) give rise to 7-dimensional images.

Another observation is that 8 is precisely the dimension of the Lie group SU(3) itself, and it is natural to ask whether there is a cohomogeneity-one action of SU(3) on itself. Whilst the Adjoint action has cohomogeneity-two, a positive answer to the question comes from a modification called the \( \sigma \)-action (see [14], [24] and [29]). For the case of SU(3), this coincides with the more elementary consimilarity action, studied independently in the theory of matrices [26]. In any case, the tangent space at a generic point of an 8-dimensional Riemannian manifold with an isometric SU(3)-action of cohomogeneity-one can be naturally identified with the Lie algebra \( \mathfrak{su}(3) \).

Such considerations suggest the importance of setting these four examples in a wider context, in order to understand more deeply their common structure. Although SU(3) does not admit a global QK structure, we show that it has features in common with (1) that allow it to be regarded as an “honorary Wolf space”. For example, SU(3) minus a 5-sphere is SU(3)-diffeomorphic to \( G_2/\text{SO}(4) \) minus a complex projective plane \( \mathbb{C}P^2 \), and we explain that open dense sets of both SU(3) and \( \mathbb{H}P^2 \) are the total spaces of \( S^1 \) bundles over the vector bundle \( \Lambda^2 \mathbb{C}P^2 \). The manifold SU(3) admits an invariant hypercomplex structure [27], and a PSU(3) structure in the sense of [25]. The theory also has links with Spin(7) structures [22].

In the present article, we classify compact 8-dimensional differentiable manifolds \( M \) admitting a cohomogeneity-one SU(3) action such that the quotient space \( M/\text{SU}(3) \) is \([0, 1]\). In this case, the generic orbit has type \( SU(3)/H \) where the connected component at the identity is \( S^1 \), and there are precisely two singular orbits \( M_1, M_2 \) of type \( SU(3)/K_i \), \( i = 1, 2 \), satisfying the relations \( SU(3) \supset K_i \supset H \) (we refer the reader to [11] for this basic theory). We also give a partial classification of the case \( M/G \cong S^1 \), where in almost all cases, \( M \) turns out to be a product of \( S^1 \) with an Aloff-Wallach space, which is the principal orbit.

The paper is organized as follows. In Section 2, we describe our approach to the classification, along with some results concerning connected principal stabilizers and the sphere-transitive representations of \( U(2) \) and \( T^2 \); in the latter case are also discussed non-connected pricipal stabilizers, which can appear only in presence of this type of singular stabilizer. In Section 3, we
carry out the classification distinguishing two possible situations: the case in which both singular stabilizers are connected (Theorem 3.1), and that in which at least one is not connected (Proposition 3.5). Moreover, we discuss the case in which $M/G \cong S^1$ and the principal orbits are simply connected (Theorem 3.2).

In Section 4, we shall identify some of the manifolds obtained during the classification, and discuss more extensively the consimilarity action of $SU(3)$ on itself. Afterwards, in Section 5 we apply ideas behind the classification results to discuss the QK moment mappings induced on $\mathbb{H}P^2$ and $Gr_2(\mathbb{C}^4)$ under the action of $SU(3)$, and relate these 8-dimensional manifolds with examples of 7-dimensional $SU(3)$-manifolds via circle actions.

2 Preliminary results

In general, for arbitrary $G$-manifolds $M$ with orbit space isomorphic to $[0, 1]$ there are two singular orbits $M_1$, $M_2$ and a normal (or slice) representation for each of them; let $V$ denote such representation at a point $x$ of a singular orbit $M_i$; then the bundle obtained as the twisted product

$$G \times_{K_i} V$$

is $G$-equivariantly isomorphic to a tube around $M_i$. If we consider the corresponding disk bundle $D_i$, we can describe $M$ as

$$M = M_\phi = D_1 \cup_\phi D_2,$$

where

$$\phi : \partial D_1 \longrightarrow \partial D_2$$

is a $G$-equivariant diffeomorphism identifying the points of the two boundaries. The latter are precisely the principal orbits: $\partial D_i \cong G/H$, where $H$ is the principal stabilizer.

In [37], Uchida used this approach in order to classify cohomology complex projective spaces with a cohomogeneity-one action. We cite his useful sufficient conditions to decide if the manifolds obtained using different maps $\phi$ are isomorphic as $G$-spaces (see [37] Lemma 5.3.1): let $\phi, \psi : \partial D_1 \rightarrow \partial D_2$ be $G$-equivariant maps as in (2); then $M_\phi$ and $M_\psi$ are $G$-equivariantly diffeomorphic if one of the following conditions are satisfied:

1. $\phi$ and $\psi$ are $G$-diffeotopic, or
2. $\psi \circ \phi^{-1}$ can be extended to a $G$-equivariant diffeomorphism of $D_1$ on itself, or
3. $\phi \circ \psi^{-1}$ can be extended to a $G$-equivariant diffeomorphism of $D_2$ on itself.
Our problem can therefore be reduced to classifying automorphisms of the generic orbit $SU(3)/U(1)$ up to these conditions.

One can obtain $G$-equivariant automorphisms of homogeneous spaces $G/H$ as follows: let $a \in N(H)$; then the map $\phi^a$ given by

$$\phi^a(gH) = ga^{-1}H$$

is well defined and commutes with the left multiplication for elements $g \in G$.

It can be shown that all $G$-equivariant automorphisms of $G/H$ have this form (see [11, Chap I, Th. 4.2]); we have therefore the identification

$$\text{Aut}_G(G/H) \cong \frac{N(H)}{H}.$$  

Let us discuss in some detail the case that two $SU(3)$-spaces obtained by distinct gluing maps are isomorphic (as $SU(3)$-spaces). In general if $M_\phi$ and $M_\psi$ are two such manifolds, then an equivariant morphism $\Phi : M_\phi \to M_\psi$ would restrict on the two tubular neighborhoods to a couple of equivariant morphisms $\phi^a$ and $\phi^b$, as described in (3), which make the following diagram commutative:

$$\begin{array}{ccc}
G/K_1 & \overset{p_1}{\leftarrow} & G/H \overset{\psi}{\leftarrow} G/H \overset{p_2}{\rightarrow} G/K_2 \\
\phi^a & \uparrow & \phi^a \\
G/K_1 & \overset{p_1}{\leftarrow} & G/H \overset{\phi^b}{\rightarrow} G/H \overset{p_2}{\rightarrow} G/K_2
\end{array}$$

In this case, $a \in N(H) \cap N(K_1)$ and $b \in N(H) \cap N(K_2)$.

In general we cannot expect to have the same map $\phi^a$ in the first two columns of the diagram (similarly for $\phi^b$); instead, for instance, we will have $\phi^a$ and $\phi^b$ repectively at $G/K_1$ and at $G/H$. Nevertheless, the map $\Phi$ is always diffeotopic (through $SU(3)$-invariant maps) to a map $\Phi'$ for which $a$ and $b$ are constant in the respective tubular neighborhoods. The homotopy between $\Phi$ and $\Phi'$ can be described as follows: the map $\Phi$ is identified on each tubular neighborhood by a continuous function $\epsilon : [0, \frac{1}{2}] \to N(H)$, so that $\Phi = \phi^{\epsilon(t)}$; we can define

$$\eta(t, s) := \epsilon((1-s)t),$$

and $\phi^{\eta(t,s)}$ is the required homotopy. We also observe that, for instance, $a \in N(H) \cap N(K_1)$ in general, because the map $\epsilon$ is continuous and $N(H)$ is a closed subgroup.

In the sequel, we shall use the following notation to identify the most commonly used homogeneous spaces:
\( S := \frac{SU(3)}{SU(2)} \), the 5-sphere,

\( \mathbb{P} := \frac{SU(3)}{S(U(2) \times U(1))} \), the complex projective plane \( \mathbb{CP}^2 \),

\( L := \frac{SU(3)}{SO(3)} \), the set of special Lagrangian subspaces in \( \mathbb{C}^3 \),

\( A := \frac{SU(3)}{U(1)} \), any Aloff-Wallach type space,

\( F := \frac{SU(3)}{T^2} \), the complex flag manifold.

We shall actually use \( A \) to stand for any homogeneous space of the form \( SU(3)/U(1) \), even though the terminology “Aloff-Wallach” usually excludes one case (we shall be more precise in the next section). The Lagrangian interpretation of \( L \) can be found in [23], and is important for making more explicit some of our constructions, such as finding geodesics from one singular orbit to another.

### 2.1 Connected principal stabilizers

Principal stabilizers \( H \) in our case are 1-dimensional subgroups of \( SU(3) \), such that \( H^0 = U(1) \). The case in which \( H^0 = H \) will be particularly significant, so we will dedicate the first part of this section to it. We begin by defining circle subgroups of \( SU(3) \). Let \( k, l \) be integers, and let \( U_{k,l} \) denote the subgroup (isomorphic to \( U(1) \)) of \( SU(3) \) consisting of matrices

\[
\begin{pmatrix}
e^{kit} & 0 & 0 \\
0 & e^{lit} & 0 \\
0 & 0 & e^{-(k+l)it}
\end{pmatrix}.
\]

We shall denote the coset space \( SU(3)/U_{k,l} \) by \( \mathbb{A}_{k,l} \). Since \( U_{k,l} \) is unchanged when any common factor of \( k, l \) is removed, we may assume that they are coprime. The space \( \mathbb{A}_{k,l} \) is called an *Aloff-Wallach space* provided \( kl(k+l) \neq 0 \), since the pairs equivalent to \((1, -1)\) are excluded for geometrical reasons (they do not satisfy the conditions that guarantee the existence of homogeneous positively-curved metrics, see [5]). In our analysis, the subgroups \( U_{1,-1} \) will however play important roles.

Denote the 1-dimensional subalgebra of \( \mathfrak{su}(3) \) corresponding to \( U_{k,l} \) by \( \mathfrak{u}_{k,l} \). We consider the pair of orthogonal subalgebras \( \mathfrak{u}_{1,-1}, \mathfrak{u}_{1,1} \) generated by the respective elements

\[
\mathfrak{u} = \begin{pmatrix}
i & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad \mathfrak{v} = \begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -2i
\end{pmatrix}
\]
that together span a Cartan subalgebra $t$. It can be shown that $u$ is a regular element, so it belongs to a unique Cartan subalgebra $t \subset \mathfrak{su}(3)$, namely that consisting of diagonal elements; if $\alpha, \beta, \alpha + \beta$ denote the roots in $t_C$, we have that $u$ corresponds to $\alpha$, and we can identify
\[
\text{span}\{\alpha\} = -i\ u_{1,-1} \quad \text{span}\{\beta\} = -i\ u_{1,0} \quad \text{span}\{\alpha + \beta\} = -i\ u_{0,1}.
\]

On the other hand, $v$ is a singular element and is contained in three independent Cartan subalgebras $t, t_1, t_2$; the 1-dimensional orthogonal complements $v^+, v_1^+, v_2^+$ then span the subalgebra $\mathfrak{su}(2)$ corresponding to the root $\alpha$. Each root has an orthogonal singular hyperplane, which in our notation correspond to $u_{1,1}, u_{-2,1}$ and $u_{1,-2}$.

The first step to obtain our classification is that of determining the possible gluing maps between two principal orbits. In the case that the principal stabilizer is connected, this corresponds to identify the group $N(U(1))$: this depends from the way $U(1)$ is immersed in $SU(3)$, up to conjugacy. The subgroups $U_{1,-1}$ and $U_{1,1}$ represent distinguished cases, in this sense.

**Lemma 2.1.** The normalizer of $U_{k,l}$ in $SU(3)$ is given by

\[
N(U_{k,l}) = \begin{cases} 
T^2 \cup \tau T^2 & \text{if } (k, l) = (1, -1) \\
S(U(2) \times U(1)) & \text{if } (k, l) = (1, 1) \\
T^2 & \text{if } (k \pm l) \neq 0.
\end{cases}
\]

Here, $\tau$ denotes an element of $SU(3)$ such that $Ad_\tau$ is an element in the Weyl group $W$.

**Proof.** For the first case, let $g \in N(U_{1,-1})$; then we also have $g \in N(T^2)$, as otherwise
\[
U_{1,-1} \subset gT^2g^{-1} \neq T^2
\]
which is impossible as $u$ is a regular element. Hence $N(U_{1,-1}) \subset N(T^2)$. It is well known that
\[
W := \frac{N(T^2)}{T^2} \cong \mathfrak{S}_3
\]
is the group of permutations on 3 elements; it acts on the Cartan subalgebra $t$ by permuting the three roots $\alpha, \beta$ and $\alpha + \beta$. The only elements fixing the subspace $t \cdot \alpha$ corresponding to $u$ are reflections about the hyperplane $u^\perp$, sending $u$ to $-u$ and swapping $\beta$ and $\alpha + \beta$, which can be represented by the the action $Ad_\tau$ with $\tau$ an appropriate element in $SU(3)$.

In the second case, an element $g \in N(U_{1,1})$ that preserves $u_{1,1}$ must also preserve the centralizer $C(U_{1,1}) = S(U(2) \times U(1)) \cong U(2)$, so that $N(U_{1,1}) \subset N(U(2)) = U(2)$; the reverse inclusion is obvious.

In the final case, we just have to note that roots and their orthogonal complements are the only eigenspaces for the elements of $W$. Hence the other regular elements in $t$ are normalized only by $T^2 \cong N(T^2)^0$. \blacksquare
As a consequence, we obtain the required isomorphisms for the coset spaces parametrizing $SU(3)$-equivariant automorphisms of principal orbits. Firstly,

$$\frac{N(U_{1,-1})}{U_{1,-1}} \cong U(1) \cup \tau U(1);$$

more explicitly, this group is generated by the matrices

$$\begin{pmatrix}
    e^{it} & 0 & 0 \\
    0 & e^{it} & 0 \\
    0 & 0 & e^{-2it}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
    0 & e^{it} & 0 \\
    -e^{it} & 0 & 0 \\
    0 & 0 & e^{-2it}
\end{pmatrix}. \quad (6)$$

For the second case,

$$\frac{N(U_{1,1})}{U_{1,1}} \cong SU(2),$$

and finally

$$\frac{N(U_{k,l})}{U_{k,l}} \cong U(1), \quad (k \pm l) \neq 0.$$

**Remark.** We can already estimate the number of $SU(3)$-equivariant diffeomorphism classes in some cases. For, if the principal stabilizer is conjugate to $U_{1,-1}$ then, thanks to Lemma 2.1 and Uchida’s condition 1, there are at most two such classes (the number of connected components of $N(U_{1,-1})/U_{1,-1}$). In the case of singular and all other regular elements there is just one $SU(3)$-diffeomorphism class.

The next information that we need is knowledge of which tubular neighborhoods can be built around a given singular orbit. To this aim, we have to determine which representations of a singular stabilizer are sphere-transitive, and associate to it the integers $k,l$ characterizing the corresponding principal stabilizer.

### 2.2 $U(2)$ representations

Let us concentrate now on the subgroup $S(U(2) \times U(1)) \cong U(2)$ of $SU(3)$, classifying its sphere-transitive real 4-dimensional representations.

First we introduce some notation. Let $\Sigma^n$ denote the complex irreducible representation of $SU(2)$ on $\mathbb{C}^2$ of dimension $n + 1$, and $A^m$ the $U(1)$-representation of weight $m$ with $m \in \mathbb{Z}$.

**Proposition 2.2.** The real 4-dimensional sphere-transitive representations $V$ of $U(2)$ are given by

$$V_{\mathbb{C}} \cong \mathbb{C}^4 \cong \Sigma^1 \otimes (A^m \oplus A^{-m}), \quad m = 2r + 1, \; r \in \mathbb{Z}.$$

If $\{u, v\}$ is a basis for $t \subset \mathfrak{su}(2) \oplus u_{1,1}$ (see (5)), then the Lie algebra of the stabilizer of a point $x \in S^3 \subset V \cong \mathbb{R}^4$ has the form $(3u + mv)^\perp.$

7
Proof. A consequence of the Peter-Weyl theorem is that a representation of $SU(2) \times U(1)$ necessarily has the form

$$V_C \cong \sum_{n,m} \Sigma^n \otimes A^m.$$  

It is straightforward to see that the only possible case in which one can obtain a sphere-transitive 4-dimensional representation is given by $\Sigma^1 \otimes (A^m + A^{-m})$. 

Now, the sums $1 + m$ and $1 - m$ must be even in order to obtain an $SU(2) \times U(1)$ representation, as $SU(2) \times U(1)$ covers $U(2)$ in a two-to-one manner. Hence $m$ must be odd.

Let us restrict the representation to the maximal torus $T^2$ contained in $U(2)$, whose Lie algebra is $t = \text{span}\{\mathbf{u}, \mathbf{v}\}$; then we obtain

$$V_C \cong (A^1 + A^{-1}) \otimes (A^m + A^{-m}) \cong A^{m+1} + A^{-m-1} + A^{-m+1} + A^{m-1}.$$  

The necessary real structure is effectively the tensor product $j \otimes j$ of the respective quaternionic structures on $\Sigma^1$ and $A^m + A^{-m}$. The latter act as the antilinear extensions of the maps $j(x, y) = (y, -x)$ and $j(e, f) = (f, -e)$ for $x, y$ a basis of $\Sigma$ and $e, f$ a basis of $A^m + A^{-m}$; the fixed point set is given by

$$V = \text{span}\{x \otimes e + y \otimes f, x \otimes f - y \otimes e, v(x \otimes e - y \otimes f), v(x \otimes f + y \otimes e)\} = \text{span}\{w_1, w_2, w_3, w_4\}.$$  

Let us consider now the corresponding Lie algebra representation. We choose the point $w_1 \in S^3$: then the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{u}_{1,1}$ acts on $w_1$ spanning the 3-dimensional tangent space of $S^3$. More explicitly, if

$$\mathfrak{su}(2) = \text{span}\{v_1, v_2, v_3\} = \text{span}\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\},$$  

then we obtain

$$v_1(w_1) = w_3, \ v_2(w_1) = w_4, \ v_3(w_1) = w_2;$$  

moreover the generator $v \in \mathfrak{u}_{1,1}$ acts by

$$v(w_1) = mw_3.$$  

Returning to the inclusion $\mathfrak{su}(2) \oplus \mathfrak{u}_{1,1} \subset \mathfrak{su}(3)$ and identifying $v_1 = \mathbf{u}$, we can restrict to the Cartan subalgebra $t$; then the subspace spanned by $w_1, w_3$ in $V$ is an irreducible $t$-submodule, and the corresponding weight can be represented via the Killing metric by the vector

$$
\mathbf{h} = \frac{1}{3} \mathbf{u} + \frac{m}{6} \mathbf{v};
$$  

its kernel, which kills the vector $w_1$, is given by the hyperplane $\mathbf{h}^\perp$: this is the Lie algebra of the stabilizer $U(1)$, hence the conclusion. $\blacksquare$
Let us analyse in more detail some examples for small \( m \): for \( m = 1 \) we get the hyperplane \( u_{0,1} \) as the stabilizer’s Lie algebra; for \( m = 3 \) we obtain a singular stabilizer \( u_{-2,1} \). For \( m \geq 5 \) we get other generic regular stabilizers, all belonging to the Weyl chambers delimited by \( u_{-2,1} \) and \( u_{1,-2} \); the limit stabilizing subalgebra for \( m \to \infty \) corresponds to \( u_{1,-1} \). The same results are obtained for \( m \leq 0 \), as the representation \( A^m \) and \( A^{-m} \) are isomorphic as real representations.

In the sequel, let us use \( P(m) \) to denote the bundle on \( P = \mathbb{CP}^2 \) obtained as the twisted product by the representation \( V = [\Sigma^2 \otimes (A^m + A^{-m})] \). Clearly \( P(m) \cong P(-m) \), so we can restrict to \( m \in \mathbb{N} \).

### 2.3 \( T^2 \) representations

Let us discuss now the case of \( T^2 \) as a singular stabilizer: we need to determine its sphere transitive 2-dimensional representations in order to classify the possible tubular neighborhoods around a singular orbit of type \( F \). Let us choose for the standard Cartan subalgebra \( t \) the basis formed by

\[
\begin{pmatrix}
  i & 0 & 0 \\
  0 & -i & 0 \\
  0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & i & 0 \\
  0 & 0 & -i
\end{pmatrix}.
\]

Comparing this basis with that in (5), we note that the relation \( v = u + 2u' \) holds, and that \( u, u' \) correspond to the two roots \( \alpha, \beta \); the parallelogram \( P \) determined by \( 2\pi u \) and \( 2\pi u' \) is a fundamental domain for the maximal torus \( T^2 \), which can therefore be described as

\[
T^2 \cong \{ \exp su \times \exp tu' : s, t \in \mathbb{R} \}.
\]

The 2-dimensional sphere-transitive real \( T^2 \)-representations \( V \) are given by

\[
V \cong A^p \otimes A^q
\]

for \( p, q \in \mathbb{Z} \), with \( (p, q) \neq (0, 0) \) and \( A^p \otimes A^q \cong A^{-p} \otimes A^{-q} \). Each of them is determined by a weight \( z \) contained in \( t \) such that

\[
\langle z, u \rangle = p, \quad \langle z, u' \rangle = q.
\]

A basis for the integer lattice of such weights is given by

\[
z_1 := \frac{1}{3} v, \quad z_2 := \frac{1}{3} (2u + u')
\]

so that a generic weight has the form \( z = pz_1 + qz_2 \) for \( p, q \in \mathbb{Z} \), and the stabilizer for the corresponding representation is given by \( z^\perp \).

**Observation.** The weights described in Proposition 2.2 are of this type: in fact

\[
\frac{1}{2} u + \frac{m}{6} v = \frac{1}{2} u + \frac{m}{6} (u + 2u') = \frac{m-1}{2} z_1 + z_2
\]
for the choice \( p = (m - 1)/2 \) and \( q = 1 \) (recall that \( l \) is odd): this is just the result of the reduction from \( U(2) \) its maximal torus \( T^2 \). In fact the representation ring \( R[U(2)] \) is isomorphic to the polynomial ring \( \mathbb{Z}[\lambda_1, \lambda_2, \lambda_2^{-1}] \), whereas \( R[T^2] \) is isomorphic to \( \mathbb{Z}[\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}] \); it is well known that the inclusion \( T^2 \subset U(2) \) induces an injective map

\[
R[U(2)] \rightarrow R[T^2]
\]

(see [12]). The image of this inclusion coincides with the subring

\[
R[T^2]^W(U(2))
\]

of \( T^2 \) representations which are invariant under the Weyl group \( W(U(2)) \); the latter is isomorphic to \( \mathbb{Z}_2 \subset W(SU(3)) \), corresponding to a reflection around one of the singular hyperplanes.

In this case, generic stabilizers might not be connected:

**Lemma 2.3.** For a \( T^2 \)-representation of type \( A^p \otimes A^q \) the generic stabilizer is

\[
U(1) \times \mathbb{Z}_h
\]

where \( h = \gcd(p, q) \).

**Proof.** We can describe the representation by \( (x, y) \mapsto e^{2\pi i(px+qy)} \), where \( (x, y) \) are coordinates with respect to (7), after a suitable normalization, to be considered modulo \( \mathbb{Z}^2 \). The stabilizer is the solution of the equation

\[
px + qy = h, \quad h \in \mathbb{Z}; \tag{8}
\]

so we have a 1-dimensional solution for each \( h \). On the other hand we can choose any \( h_1 \) and \( h_2 \) in \( \mathbb{Z} \) so that \( (x + h_1, y + h_2) \) is the same solution as \( (x, y) \) on \( T^2 \), but for a different \( h \). Therefore equation (8) becomes

\[
px + qy = h - ph_1 - qh_2. \tag{9}
\]

Let us suppose that \( h = \gcd(p, q) > 0 \): then equation (9) is equivalent to \( px + qy = 0 \), as the gcd is precisely the smallest positive integer which can be obtained in the form \( ph_1 +qh_2 \). Hence the solution \( (x, y) \) for \( h \) is also the solution for 0; moreover this implies that if \( 0 < h' < h \), then the solution \( (x, y) \) for \( h' \) is not a solution for 0. This shows that the solutions are repeated modulo \( h \), so that there are precisely \( h \) distinct ones, each one isomorphic to a circle \( U(1) \): altogether they form an abelian subgroup, isomorphic to \( U(1) \times \mathbb{Z}_h \subset T^2 \). ■

We introduce some more notation at this point: we shall denote by \( \mathbb{F}(p, q) \) a tubular neighborhood of a flag manifold obtained by a slice representation \( A^p \otimes A^q \) as explained above. We observe that \( \mathbb{F}(p, q) \cong \mathbb{F}(-p, -q) \).
Table 1: Connected singular stabilizers and corresponding slice representations

|        | $SU(2)$ | $U(2)$ | $SO(3)$ | $T^2$ |
|--------|---------|--------|---------|-------|
| $\dim V$ | 3       | 4      | 3       | 2     |
| $V$    | $[\Sigma^2]$ | $[\Sigma \otimes (A^1 \oplus A^{-1})]$ | $[\Sigma^2]$ | $A^p \otimes A^q$ |

Regarding the singular stabilizers $SO(3)$ and $SU(2)$, we have a unique sphere-transitive 3-dimensional representation, namely the standard irreducible space $\mathbb{R}^3 \cong [\Sigma^2]$. The complete list of possible slice representations for each connected singular stabilizer is given in Table 1.

### 3 The classification

We are now in a position to present the main results of the paper, classifying the possible ways of gluing together tubular neighborhoods obtained from the singular orbits discussed in Section 1 and from the normal representations described in Section 2.

#### 3.1 Connected singular stabilizers

We focus first on the case that both the singular stabilizers $K_1, K_2$ are connected. Connected subgroups of $SU(3)$ are in one-to-one correspondence with Lie subalgebras of $su(3)$. Note that the two Lie subalgebras $so(3)$ and $su(2) \oplus \mathbb{R}$ are maximal subalgebras. It is also well known that $so(3)$ and $su(2)$ are the only 3-dimensional subalgebras of $su(3)$, up to conjugation.

Passing to subalgebras of $so(3)$ and $su(2) \oplus \mathbb{R}$, observe that $su(2) \cong so(3)$ does not contain any subalgebra of dimension greater than 1; therefore we obtain only other two subalgebras, both contained in $su(2) \oplus \mathbb{R}$: namely $su(2)$ and the Cartan subalgebra $t$.

Let us also list here the normalizers of each corresponding connected subgroup. Let $Z_3$ denote the center of $SU(3)$, and (again) $W \cong S_3$ its Weyl group. Then

\[
\begin{align*}
N(SU(2)) &= N(U(2)) = U(2) \\
N(SO(3)) &= SO(3) \times Z_3 \\
N(T^2) &= \bigcup_{\tau \in W} \tau T^2
\end{align*}
\]

**Remark.** We shall not treat immediately the case of a singular stabilizer $T^2$ with slice representation $A^p \otimes A^q$ and $\gcd(p,q) \neq 1$. In fact this can imply
that the second singular stabilizer is not connected even if $T^2$ is (because the principal stabilizer turns out to be not connected, see Lemma 2.3), and this situation fits better in Subsection 3.2 (see Proposition 3.5).

We can now state the main result of this section:

**Theorem 3.1.** Tables 2 and 3 list respectively all the $SU(3)$-diffeomorphism classes of 8-dimensional compact cohomogeneity-one $SU(3)$-manifolds with orbit space $[0, 1]$ such that:

— both stabilizers belong to the set \{SU(2), U(2), SO(3)\},
— one singular stabilizer is isomorphic to $T^2$ and the normal representation is $A^p \otimes A^q$ with $\gcd(p, q) = 1$.

\[
\begin{array}{|c|c|c|c|}
\hline
M_2 \setminus M_1 & S & L & \mathbb{P}(l) \\
\hline
S & 1 & 1 & \delta_l^1 \\
L & 1 & 1 & \delta_l^l \\
\mathbb{P}(m) & \delta_m^1 & \delta_m^1 & \delta_l^m + \delta_l^1 \delta_m^1 \\
\hline
\end{array}
\]

Table 2: Numbers of $SU(3)$-diffeomorphism classes of 8-manifolds: singular stabilizers $SU(2), U(2)$ and $SO(3)$ ($m, l$ odd)

\[
\begin{array}{|c|c|c|c|c|}
\hline
M_2 \setminus M_1 & \mathbb{P}(l, m) & \mathbb{P}(l) & L & S \\
\hline
\mathbb{P}(p, q) & \delta_p^p \delta_q^m & \delta_l^p \delta_q^{l-1}/2 \delta_q^1 & \delta_p^p \delta_q^1 & \delta_l^p \delta_q^1 \\
\hline
\end{array}
\]

Table 3: Numbers of $SU(3)$-diffeomorphism classes of 8-manifolds: one singular stabilizer of type $T^2$ ($(p, q) \neq (0, 0)$ and $\gcd(p, q) = 1$)

**Proof.** Let us consider these connected singular stabilizers: correspondingly we have a slice representation $V$ of dimension 3, 4, 3, 2 (see Table 1); the representations involved must again be of cohomogeneity one, or in other words the singular stabilizer $K_i$ must act transitively on the unit sphere $S^{n-1} \subset V$.

Let us analyse the possibilities case by case.

The cases of $SU(2)$ and $SO(3)$ are rather simple, as the only 3-dimensional representation of cohomogeneity one is the standard 3-dimensional irreducible representation $\mathbb{R}^3 \cong [\Sigma^2]$, as already observed at the end of Section 2 in this case the principal stabilizer turns out to be one corresponding to $U_{1,-1}$.

12
Therefore, thanks to Lemma 2.1, the normalizer is $T^2 \cup \tau T^2$ in both cases; on the other hand, it can be shown that for both the singular orbits $\mathbb{S}$ and $\mathbb{L}$, the component $\tau T^2$ of the normalizer $N(U(1))$ intersects $SU(2) \subset N(SU(2))$ and $SO(3) \subset N(SO(3))$ respectively (for instance in a point $x$ obtained by putting $t = \pi/2$ in an appropriate conjugate of the second element in (3)). Hence any $SU(3)$ equivariant automorphism of the principal orbit is diffeotopic to one which can be extended to an automorphism of the whole tubular neighborhood (see Uchida’s criteria in Section 2), so that we have a unique $SU(3)$-equivariant diffeomorphism class of $M$ containing one of $\mathbb{S}$ or $\mathbb{L}$ and another singular orbit $M_2$.

Let us discuss the case of $\mathbb{P}(m)$: Proposition 2.2 says that we have a singular stabilizer for $m = 3$ and a root stabilizer for $m = 1$, while for all other values of $m$ the stabilizer is generic regular; in all cases, except for $m = 1$, we have that $N(U(1))$ is connected, hence we have 1 possible way of gluing each of these tubular neighborhoods to others; for $m = 1$ we have that the component $\tau T^2$ does not intersect $S(U(2) \times U(1))$, so we have 2 distinct classes in this case. The fact that the two classes obtained form the two gluing maps $\phi^e$ and $\phi^r$ cannot be isomorphic follows by inspecting diagram (4). In fact all the vertical maps must be of the form $\phi^e$ in order to be defined on the whole tubular neighborhoods, and this implies that the central part of the diagram can not be commutative if we put, for instance, $\psi = \phi^r$. In general we can combine two tubular neighborhoods if and only if the principal stabilizers are conjugate; therefore $\mathbb{P}(n)$ and $\mathbb{P}(m)$ can be glued together if and only if $n = m$, and the gluing map is unique for $m \neq 1$, and there are two distinct for $m = 1$.

Let us pass now to tubular neighborhoods of type $\mathbb{F}(p, q)$ assuming that $\gcd(p, q) = 1$: there is precisely one gluing map for $(p, q) \neq (0, 1), (p, q) \neq (1, 0)$ or $(p, q) \neq (1, -1)$, as in fact in this case the normalizers $N(U(1))$ are all connected. By contrast, for the remaining representations the principal stabilizer is of type $U_{1,-1}$, hence we have at first sight two possible gluing maps. These can be used to join this tubular neighborhood to others with the same type of stabilizer; nevertheless $\tau T^2 \subset N(T^2)$, so as usual $\phi^e$ can be extended to the whole tubular neighborhood, and it is equivalent to the identity gluing map.

The list of all possible combinations is given in the Tables. ■

We end this section by examining the case in which $M/SU(3)$ is $S^1$ and the principal orbits are simply connected. Let us point out that the homogeneous manifold $\mathbb{A}_{k,l}$ is simply connected, as shown by the long exact homotopy sequence for a fibration:

$$
\cdots \pi_1(SU(3)) \longrightarrow \pi_1(\mathbb{A}_{k,l}) \longrightarrow \pi_0(U(1)) \longrightarrow \cdots.
$$

(10)
In this case there are no singular orbits and the manifold $M$ is a bundle

$$
\begin{align*}
G/H & \hookrightarrow M \\
\downarrow & \\
S^1 & 
\end{align*}
$$

where $H = U_{k,l}$ is the principal (and unique) stabilizer; the structure group for this bundle is contained in $N(H)/H$ (see [11], Th. 8.2, Ch. IV). Hence we have

**Theorem 3.2.** Let $M$ be a cohomogeneity-one $SU(3)$-manifold with $M/G \cong S^1$ and such that the principal orbit is simply connected. Then the principal orbit has the form $\mathbb{A}_{k,l}$.

Either $M \cong \mathbb{A}_{k,l} \times S^1$, which is possible for any $k,l$, or $\mathbb{A}_{k,l} = \mathbb{A}_{1,-1}$ and $M$ is a nontrivial bundle over $S^1$.

**Proof.** We can divide the proof in three cases, corresponding to the stabilizers described in Lemma [22]. First we note that the bundle structure is given by the $N(H)/H$-valued transition functions $g_1$ and $g_2$ defined on the two points $p_1$ and $p_2$, which constitute the “equator” of the base manifold $S^1$.

In the first case, $N(U_{1,-1})/U_{1,-1}$ has 2 connected components, therefore there are two possible nonequivalent choices for the maps $g_i$, giving rise to the trivial bundle and another nontrivial, respectively.

In the remaining two cases, $N(U_{k,l})/U_{k,l}$ is connected: we have a unique (trivial) bundle for $U_{1,1}$, and there are infinite nonconjugate generic $U_{k,l}$'s, giving rise to nonisomorphic generic fibres $\mathbb{A}_{k,l}$.

The $SU(3)$-manifolds obtained in this way are all trivial bundles, except for the first case. ■

### 3.2 Non-connected singular stabilizers

We now conclude the classification, describing the more general situation in which the singular stabilizers are not connected. This implies that the singular orbits are not simply connected, and their respective universal covers are those described in Theorem [3.1]. Some of our arguments are inspired by those used in [4].

**Proposition 3.3.** If the connected components $K^0_i$ of the two singular stabilizers belong to the set $\{SO(3), SU(2), U(2)\}$, then both are connected: $K^0_i = K_i$ for $k = 1, 2$. 

14
Proof. Suppose that $K_1^0$ is one of the three subgroups in the list: then the codimension of the singular orbit is at least 3; a general position argument shows that $M \setminus (SU(3)/K_1)$ is simply connected, as is $M$. This complement has the same homotopy type of $SU(3)/K_2$, so $\pi_1(SU(3)/K_2) = 0$ too: this implies that the stabilizer $K_2$ is connected. By the long exact homotopy sequence for a fibration
\[ \cdots \pi_1(S^r) \longrightarrow \pi_0(H) \longrightarrow \pi_0(K_2) \cdots \] the principal stabilizer $H$ must also be connected, for $r > 1$, which is the case for all the representations involved with the three stabilizers under consideration. This implies that also $K_1$ is connected, hence the result. ■

This means that we cannot obtain new simply-connected manifolds by gluing together tubular neighborhoods unless they involve $T^2$ as $K_0^0$ for at least one $i$. We discuss now this remaining case: the new manifolds we obtain in this way are given in Table 4. We point out that the principal stabilizers turn out to be non-connected in these cases. Before that, we prove a result which corresponds to Lemma 2.1 for non-connected $H$:

**Lemma 3.4.** Consider the subgroup $U_{k,l} \times Z_h$ of $T^2 \subset SU(3)$; then
\[ N(U_{k,l} \times Z_h) = N(U_{k,l}) \]
if $U_{k,l}$ is regular; if $U_{k,l}$ is singular (for instance $U_{1,1}$) then
\[ N(U_{1,1} \times Z_h) = T^2 \cup \tau T^2. \]

**Proof.** The proof in the regular case is completely analogous to that of Lemma 2.1 for the singular case we just have to observe that if $h \neq 1$ the group contains regular elements, and the whole $T^2$ must be preserved by the normalizer of $U_{1,1} \times Z_h$. The element $\tau \in W$ that normalizes $U_{1,-1}$, reflecting the root $\alpha$, is the only one which also preserves $U_{1,1} \times Z_h$, hence the conclusion. ■

**Observation.** In this situation, we have two connected components for the normalizers of $U_{1,1} \times Z_h$ and of $U_{1,-1} \times Z_h$: if these stabilizers appear in a tubular neighborhood of type $F(p,q)$, we observe that in both cases we obtain only one $SU(3)$ diffeomorphism class, because both normalizers are contained in $N(T^2)$ (for Uchida’s criteria, see Theorem 3.1).

We pass now to the main result of this section, but before of that we recall that any subgroup $K \subset G$ is always contained in $N(K^0)$, because for any $x \in K$ the adjoint action $Ad_x$ is continuous, preserves $K$ and fixes $e$.

**Proposition 3.5.** Suppose that $K_1^0 \in \{SO(3), SU(2), U(2)\}$ and that $K_2^0 = T^2$: then $K_2 = K_2^0 = T^2$. Moreover if $K_2 = T^2$ and if the slice representation at $F$ is $A^p \otimes A^q$ there are the following possibilities:

1. if $(p,q) = (0,h)$ then $K_1 \in \{SO(3), SU(2), U(2), T^2\}$;
2. if \((p, q) = (0, h)\) for some \(h \in \mathbb{Z}, h > 1\), then \(K_1 = (SU(2) \times \mathbb{Z}_{2h})/\mathbb{Z}_2\), except in the case \(h = 3\), where also \(K_1 = SO(3) \times \mathbb{Z}_3\) is possible;

3. if \((p, q) \neq (0, h)\) and \(\gcd(p, q) = 1\) then \(K_1 \in \{T^2, U(2)\}\);

4. if \((p, q) \neq (0, h)\) and \(\gcd(p, q) \neq 1\) then \(K_0^1 = T^2\).

| \(M_2 \setminus M_1\) | \(\mathbb{S}/\mathbb{Z}_h\) | \(\mathbb{L}/\mathbb{Z}_3\) | \(\mathbb{F}(l, m)\) |
|------------------------|-----------------|-----------------|-----------------|
| \(\mathbb{F}(p, q)\)  | \(\delta^p_0\delta^h_q\) | \(\delta^p_0\delta^3_q\) | \(\delta^p_0\delta^m_q\) |

Table 4: Numbers of \(SU(3)\)-diffeomorphism classes of 8-manifolds: non-connected principal stabilizers \((\gcd(p, q) \neq 1)\)

**Proof.** The first statement follows from the same general position argument as in Proposition \ref{proposition:1}. As proved in Lemma \ref{lemma:2}, the principal stabilizer is of the form \(U_{k,l} \times \mathbb{Z}_h\), so we need to determine which of the singular stabilizers contain this subgroup.

In the first case we have \(h = 1\) and \(U_{k,l} = U_{1,-1}\), which appears as a principal stabilizer associated to any of the connected stabilizers above, with the appropriate slice representation \(V\), as already shown in Theorem \ref{theorem:3}. In this case \(K_1\) is connected, because the sphere \(S^r \subset V\) is.

For the second case we argue as follows: \(K_1\) must contain a subgroup of type \(U_{1,-1} \times \mathbb{Z}_h\), but we have to exclude \(U(2)\), because it allows only connected principal stabilizers \((h = 1)\). Another possible choice is the subgroup \(SU(2) \times \mathbb{Z}_{2h}/\mathbb{Z}_2\) where \(\mathbb{Z}_2\) is the center of \(SU(2)\); topologically it is the union of \(h\) copies of \(SU(2)\), and \(\mathbb{Z}_{2h}\) should be regarded as a subgroup of the singular \(U(1)\) centralizing \(SU(2)\) (for instance \(U_{1,1}\) for the standard immersion). We observe that the singular orbit in this case is isomorphic to \(S/\mathbb{Z}_h\). Suppose instead that \(K_1^0 = SO(3)\): then \(K_1\) must be a subgroup of the normalizer \(N(SO(3)) = SO(3) \times \mathbb{Z}_3\), which are \(SO(3)\) itself or the whole \(N(SO(3))\). The latter case in this situation corresponds for instance to the weight \((p, q) = (0, 3)\) for \(T^2\). As observed after Lemma \ref{lemma:3}, in both cases the two gluing maps \(\phi^e, \psi^r\) give rise to isomorphic \(SU(3)\)-spaces.

For the third case, the connected component \(K_1^0\) must contain \(T^2\), because only then the corresponding Lie algebra does contain the correct \(u(1)\).

Finally, in the fourth case we have to exclude \(U(2)\) because, as observed in case 2, it allows only connected principal stabilizers. ■
Observation. Two of the manifolds that are new with respect to the classification given in Theorem 3.1 come from case 2. We note that in these cases the singular stabilizer \((SU(2) \times \mathbb{Z}_{2h})/\mathbb{Z}_2\) admits as a slice representation \(V\) only the standard \(\mathbb{R}^3 \cong \Sigma^2 \otimes A^0\); in fact any \(\mathbb{Z}_h\) representation can be extended to a \(U(1)\) representation \(A^m\) with \(0 \leq m \leq h - 1\), hence \(V\) is the restriction of a \(U(2)\) representation; therefore

\[ V_C \cong \sum \Sigma^i \otimes A^m \]  

(12)
as seen in Proposition 2.2; for dimensional reasons \(\Sigma^2 \otimes A^0\) is the only possible choice. Analogous considerations hold for \(SO(3) \times \mathbb{Z}_3\).

Let us consider the case in which \(K_i \neq K_0 = T^2\), so that \(K_i \subset N(T^2)\). Here, the two stabilizers must have the same number of connected components, otherwise the two tubular neighborhoods could not be glued together, as the principal orbits would not be isomorphic. In this case \(K_1 = K_2\) and \(\pi_1(SU(3)/K_i) \neq 0\) for \(i = 1, 2\); moreover in the long exact sequence

\[ \cdots \pi_1(SU(3)/H) \longrightarrow \pi_1(SU(3)/K_i) \longrightarrow \pi_0(S^1) \cdots \]
the bundle projections induce surjections on the respective fundamental groups. The Seifert–van Kampen Theorem tells us that this is incompatible with the simply connectedness of the manifold \(M\), so we have to exclude this case.

4 Examples

In order to present some familiar examples, the notation \(\mathcal{M}(M_1, M_2)\) indicates an 8-dimensional \(SU(3)\)-manifold obtained by gluing appropriate disk bundles over singular orbits \(M_1, M_2\) with a map \(\phi\) which may or may not be the identity.

Then we have the following remarkable identifications:

- the complex Grassmannian \(\text{Gr}_2(\mathbb{C}^4)\) is \(\mathcal{M}(\mathbb{P}, \mathbb{P})\);
- the quaternionic projective plane \(\mathbb{HP}^2\) is \(\mathcal{M}(\mathbb{P}, \mathbb{S})\);
- the exceptional Wolf space \(G_2/\text{SO}(4)\) is \(\mathcal{M}(\mathbb{P}, \mathbb{L})\);
- the product \(\mathbb{CP}^2 \times \mathbb{CP}^2 = \mathbb{P} \times \mathbb{P}\) is \(\mathcal{M}(\mathbb{P}, \mathbb{F})\);
- the Lie group \(SU(3)\) is itself \(\mathcal{M}(\mathbb{L}, \mathbb{S})\).

We describe these \(SU(3)\) spaces in a bit more detail. Recall that \(\mathbb{L} = SU(3)/SO(3)\). The first three examples are obtained by standard inclusions of \(SU(3)\) in \(SU(4), Sp(3)\) and \(G_2\), and in these cases, the normal bundle over each \(\mathbb{CP}^2 = \mathbb{P}\) is \(\mathbb{P}(1)\).

The fourth (product) case is given by the diagonal action of \(SU(3)\), where the first singular orbit consists of all couples \(([z], [z])\) of identical complex lines.
in \( \mathbb{C}^3 \), and the second consists of couples \((|z|,|w|)\) with \(|w| \subset |z|\). In this case, the slice representation \( V \) is isomorphic to the isotropy representation at \( \mathbb{P} \); in fact if \((v,v)\) is a tangent vector at \((|z|,|z|)\), with \( v \) generated by an element in \( \text{su}(3)/(\text{u}(2) \oplus \mathbb{R}) \), then normal vectors must be of the form \((v,-v)\) and give rise to the same \( U(2) \) representation. It is straightforward to check that this tubular neighborhood is of type \( \mathbb{P}(3) \).

The final case is given by a modification of the Adjoint action of \( SU(3) \) on itself, discussed in more detail in Subsection 4.1.

The case in which the two tubular neighborhoods are isomorphic and the gluing map is the identity is particularly simple. We can identify the singular orbits \( M_1 = M_2 = M \) and call the unique normal representation \( V \); the resulting manifold \( \mathcal{D}(M) = \mathcal{M}(M_1, M_2) \) is then the “double” of the disk bundle associated to \( V \). This manifold is obtained by the one-point compactification \( \mathbb{R}^n \leadsto S^n \) of the \( V \) fibres over \( M \):

\[
\begin{array}{ccc}
S^n & \overset{\iota}{\hookrightarrow} & \mathcal{D}(M) \\
\downarrow & & \downarrow \\
M & & 
\end{array}
\]  

(13)

The other singular orbit becomes the section at infinity of this new bundle.

**Proposition 4.1.** The manifolds \( \mathcal{D}(S) \) and \( \mathcal{D}(L) \) do not admit any \( SU(3) \)-invariant metric of positive sectional curvature.

**Proof.** This is just a consequence of [33, Lemma 3.2], which asserts that any even dimensional cohomogeneity-one \( G \)-manifold \( M \) with an invariant metric of positive sectional curvature has \( \chi(M) > 0 \), and of the observation

\[
\chi(\mathcal{D}(S)) = \chi(\mathcal{D}(L)) = \chi(S^3)\chi(S) = 0
\]

(recall that \( \chi(L) = \chi(S) \)). ■

### 4.1 Consimilarity

We are going now to consider a group action \( c \) of \( GL(n, \mathbb{C}) \) on itself, called **consimilarity**, defined by

\[
c(A)B := ABA^{-1}.
\]

(14)

This action naturally occurs when considering \( \text{anti} \)-linear mappings between a given vector space, of relevance in quantum theory. It also occurs in various geometrical situations (see, for example, [38]), and is intimately related to **similarity**. The mapping

\[
\Gamma: A \mapsto A\overline{A}
\]

(15)

induces a mapping between consimilarity classes and similarity classes (i.e. orbits under \( (14) \) and orbits under conjugation). Although this mapping is
not in general a bijection between the respective classes, it is true that \( \Gamma^{-1}(I) \) coincides with the consimilarity orbit

\[
\{ A \overline{A}^{-1} : A \in GL(n, \mathbb{C}) \}
\]  

of the identity. This fact is not entirely obvious, but has an easy proof [26].

Consimilarity can be restricted to \( SU(n) \subset GL(n, \mathbb{C}) \), so that \( SU(n) \) acts on itself, as in this case

\[
AB\overline{A}^{-1} = ABA^t
\]
is in \( SU(n) \) if \( A, B \) are. It is straightforward to prove that the consimilarity action of \( SU(n) \) on itself is isometric with respect to the Killing metric. The resulting action is in fact a special case of a family of actions of a Lie group \( G \) on itself, constructed using an automorphism \( \sigma \) of \( G \) (see [24], [14] and [26]).

Let us return to the case \( n = 3 \).

**Lemma 4.2.** Consimilarity is a cohomogeneity-one action of \( SU(3) \) on itself with singular orbits \( L \) and \( S = S^5 \). The former is the orbit containing the identity matrix \( I \) and coincides with \( \Gamma^{-1}(I) \cap SU(3) \).

**Proof.** It can be shown that \( \Gamma^{-1}(I) \cap SU(3) \) coincides with the set

\[
S = \{ A \in SU(3) : A = A^t \}
\]
of symmetric matrices. The map \( \xi : L \to S \) defined by

\[
A \to AA^t
\]
is well defined and surjective. It is also injective as if \( AA^t = CC^t \) then

\[
C^{-1}A = C^t(A^t)^{-1} = ((C^{-1}A)^t)^{-1}
\]
so that \( C^{-1}A \in SO(3) \). This shows that the \( c \)-orbit through the identity is \( L \).

Consider the point \( I \) and its stabilizer \( SO(3) \). The isotropy and the slice representations are determined by the decomposition

\[
\text{su}(3) = T_L \oplus V = \text{so}(3) \perp \text{so}(3) = [\Sigma^4] \oplus [\Sigma^2]
\]
as \( SO(3) \) representations. The slice representation is sphere transitive (see Table 1), and this shows that the cohomogeneity of the action is 1. For instance we can choose the normal direction determined by the matrix

\[
w = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{so}(3) = \Sigma^2.
\]

The corresponding geodesic \( B(t) = \exp(tw) \) intersects orthogonally all the \( c \)-orbits (see [24]): the second singular one is reached at \( B_s = B(\pi/4) \). In fact,
an explicit calculation shows that the stabilizer at $B_s$ is $SU(2)$; therefore the corresponding orbit is indeed $S$. ■

Observe that

$$\text{Tr}(AA) = \text{Tr}(A^2) = \text{Tr}(AA) ;$$

(17)

this implies that the image $\Gamma(SU(3))$ is contained in the hypersurface

$$\mathcal{H} := \{ B \in SU(3) : \text{Tr} B \in \mathbb{R} \}$$

(18)

of $SU(3)$. We shall investigate the resulting mapping $\Gamma : SU(3) \to \mathcal{H}$ in the next section.

5 Quotients by circle subgroups

An analogous classification of $SU(3)$ actions is possible in dimension 7, and partial results can be found in [32]. Restricting attention here to the case in which both singular orbits are $\mathbb{P} = \mathbb{C}P^2$, and both tubular neighborhoods are isomorphic to the rank 3 vector bundle $\Lambda^2 \mathbb{C}P^2$, it is not hard to show the existence of only two classes of cohomogeneity-one $SU(3)$-spaces with this data. There is a choice of gluing map between the generic orbits $F$: the identity in one case, and a map $\phi^\tau$ associated to a non-trivial element $\tau \in W$ in the other. With the latter choice, we obtain the sphere $S^7 \subset su(3)$ with the action induced by the Adjoint representation.

We now exhibit a model for the manifold obtained in the former case, denoted here by $N^7$, involving the Grassmannian $\mathcal{G}_{r3}(su(3))$ of oriented 3-dimensional subspaces of the Lie algebra $su(3)$, which is an $SU(3)$-space under the action induced by $Ad_{SU(3)}$.

**Proposition 5.1.** The manifold $N^7$ is a submanifold of $\mathcal{G}_{r3}(su(3))$ with the $SU(3)$ action induced by the Adjoint action on $su(3)$.

**Proof.** Following [36], we consider the function $f : \mathcal{G}_{r3}(su(3)) \to \mathbb{R}$ induced by the standard 3-form on $su(3)$. Thus

$$f(U) = \langle x, [y, z] \rangle,$$

where $\{x, y, z\}$ is an orthonormal basis of the 3-dimensional subspace $U \subset su(3)$. The absolute maxima and minima of $f$ are each attained on a copy of $\mathbb{C}P^2$ corresponding to the highest root embedding $su(2) \subset su(3)$ and a choice of orientation for $su(2)$. The tangent space $T_U \mathcal{G}_{r3}(su(3))$ has the form $U \otimes U^\perp$; for $U = su(2)$ it can be decomposed as

$$T_{su(2)} \mathcal{G}_{r3}(su(3)) = su(2) \otimes (\Sigma^0 + 2\Sigma^1) \cong \Sigma^2 + 2(\Sigma^1 + \Sigma^3).$$

The subspace $2\Sigma^1 = \Sigma^1 \oplus \Sigma^1$ represents the tangent space to the critical manifold $\mathbb{C}P^2$; if we choose instead the summand $\Sigma^2$, we obtain the bundle.
Λ² CP², which is therefore a subbundle of the normal bundle at both CP². In the two cases it turns out to be a stable or an unstable subbundle respectively.

The manifold \( N^7 \) is obtained from the two \( Λ² CP² \) over the two extremal \( CP² \). To see this, denote by \( \tilde{N}^7 \) the manifold obtained by considering the union of the flow lines of the vector field grad \( f \) with limit points in the two copies of \( CP² \) and tangent directions corresponding to the respective \( Σ² \). Such a flow line (without caring about the parametrization) is given by

\[
V(t) = \text{span}\{ u \cos t + v \sin t, u_2, u_3 \},
\]

with \( u, v \) as in (5) and \( \text{su}(2) = \text{span}\{ u, u_2, u_3 \} \). It is straightforward to see that the stabilizer for \( t \neq kπ \) under the \( \text{Ad}_{SU(3)} \) action is \( T² \), and for \( t = π \) the integral curve intersects the minimal critical submanifold at the same subalgebra \( \text{su}(2) \) with opposite orientation. In both cases the tangential direction of \( V(t) \) belongs to the summand \( Σ² \) at the critical points: these facts imply that the gluing map for the two tubular neighborhoods must be the identity, so \( \tilde{N}^7 \cong N^7 \).

**Remark.** We point out that \( N^7 \) is *not* homeomorphic to \( S^7 \). As the double \( D(CP²) \), it can be regarded as a $3$-sphere bundle over \( CP² \) as in (13), in contrast to \( S^7 \). Now, \( π_2(CP²) = H₂(CP², Z) = Z \), and writing the homotopy exact sequence for a fibration we obtain

\[
\cdots → π_2(S³) → π_2(N^7) → π_2(CP²) → π_1(S³) → \cdots.
\]

This implies \( π_2(N^7) = π_2(CP²) = Z \), whilst \( π_2(S^7) = 0 \).

It is shown in [32] that \( N^7 \) cannot be equipped with an invariant metric of positive curvature. Indeed, \( S^7 \) is the unique $7$-dimensional positively curved cohomogeneity-one \( G \)-manifold, if the semisimple part of \( G \) has dimension greater than $6$.

The above example is linked to the $8$-dimensional case by a moment map \( µ \) associated to the action of \( SU(3) \) on the Wolf spaces \( HP² \) and \( Gr₂(C⁴) \) (see [19]). Denoting by \( M \) either of these space, it is possible to construct from \( µ \) an equivariant map

\[
Ψ : M₀ \longrightarrow \tilde{Gr}_3(\text{su}(3)) \tag{19}
\]

defined on an open dense subset \( M₀ ⊂ M \). This construction was used in [20] in order to relate the geometry of a quaternion-Kähler manifold with the geometry of the Grassmannian \( \tilde{Gr}_3(\mathfrak{g}) \), but we cannot use the same techniques here since in the two cases considered, the differential \( Ψ_* \) is nowhere injective. Moreover the subset \( M₀ \) is strictly contained in \( M \): indeed \( M₀ = HP² \setminus CP² \) and \( M₀ = Gr₂(C⁴) \setminus CP² \) respectively, and

\[
Ψ(M₀) \subset N^7 \subset \tilde{Gr}_3(\text{su}(3)).
\]
One may ask if the map $\Psi$ could be extended equivariantly to the whole $W$ in both cases, as this happens in other significant cases (for instance $Sp(n)Sp(1)$ acting on $\mathbb{H}P^n$ or $Sp(n)$ acting on $Gr_2(C^n)$). In fact the generic fibre $\Psi^{-1}(x)$ is a circle $S^1$: the resulting $S^1$ action on $\mathbb{H}P^2$ was described in [8], and

$$\mathbb{H}P^2/S^1 \cong S^7$$

(see [6] and [7]). For the same reason we have a topological quotient

$$Gr_2(C^4)/S^1 \cong S^7.$$  

However, as observed above, $S^7$ is different from $N^7$, and it is easy to check that $\Psi$ cannot be extended equivariantly to the whole Wolf spaces $\mathbb{H}P^2$ and $Gr_2(C^4)$.

The fact that $S^7$ is a compactification of $\Lambda^2_\mathbb{C}P^2$ was used in [30] to unify the construction of various Ricci-flat metrics on complements of homogeneous spaces inside spheres. There are analogous constructions on $G_2/\text{SO}(4)$ and $SU(3)$. The descriptions at the start of Section 4 show that dense open subsets of these two manifolds can be $SU(3)$-equivariantly identified. However, the respective singular orbits $\mathbb{P}$ and $S$ are not directly related by the Hopf fibration $S \to \mathbb{P}$; indeed passing from the $\mathbb{P}$ of $G_2/\text{SO}(4)$ to the $S$ of $SU(3)$ requires a “flip” of the type considered in [22]. This is made possible by the existence of three distinct mappings $\mathbb{F} \to \mathbb{P}$, similarly exploited in the theory of harmonic maps [35].

To conclude the paper, we identify an analogue for $SU(3)$ of the map $\Psi$ described in (19).

**Theorem 5.2.** The image $\Gamma(SU(3))$ of (15) is the hypersurface (18), and is homeomorphic to the Thom space of the vector bundle $\Lambda^2_\mathbb{C}P^2$. The restriction of $\Gamma$ to $SU(3) \setminus L$ is a principal $S^1$ bundle over $\Lambda^2_\mathbb{C}P^2$.

**Proof.** Let $SU(2) \subset SU(3)$; then

$$\mathcal{H} = \bigcup_{g \in SU(3)} Ad_g SU(2). \quad (20)$$

In fact, consider the Lie algebra $\mathfrak{su}(3)$; it is well known that any element $x \in \mathfrak{su}(3)$ belongs to the standard $t = \text{span}\{u, v\}$ (see (7)), up to conjugation. It is sufficient therefore to solve the equation

$$\text{Im} \ Tr( \exp (t u + s v)) = 0,$$

equivalent to

$$\sin(t + s) + \sin(s - t) - \sin(2s) = 0;$$

22
this has solutions \( \{ s = 0 + k\pi \} \cup \{ s = \pm t + 2k\pi \} \). These are nothing other than the three lines corresponding to \( u_{1,-1} \), \( u_{1,0} \), \( u_{0,1} \) and their translates. However, when we exponentiate, all the solutions are sent to the triplet

\[
U_{1,-1} \cup U_{1,0} \cup U_{0,1} = T^2 \cap \mathcal{H}
\]  

which are the intersections of \( T^2 \) with three conjugate copies of \( SU(2) \); the equality in (20) follows by noting that both sides are \( Ad \)-invariant. Consider again any subgroup \( SU(2) \subset SU(3) \) and let \( g \in SU(3) \); then

\[
SU(2) \cap Ad_g SU(2) = SU(2) \text{ or } \{e\}. 
\]  

In fact let us consider a point \( x \in SU(2) \cap Ad_g SU(2) \); if \( x \) is regular then it is contained in a unique maximal torus \( T^2 \); on the other hand \( x \) belongs to one of the connected components of (21), say \( U_{1,-1} \), which therefore belongs entirely to \( SU(2) \cap Ad_g SU(2) \). This implies that \( g \in N(U_{1,-1}) \), which is contained in \( N(SU(2)) \), hence we fall in the first case of (22). Suppose now that \( x \) is singular: there exists only one singular point for each copy of \( SU(2) \), namely

\[
x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

for the standard embedding of \( SU(2) \); singular elements are preserved by the adjoint action, therefore \( g \) is in the stabilizer of \( x \), which is \( U(2) = N(SU(2)) \), and we are again in in the first case of (22). If \( g \notin N(SU(2)) \) then the intersection consists of just \( e \).

This discussion proves that we can realize \( \mathcal{H} \) as the union of copies of \( SU(2) \) which share only the identity \( e \) inside \( SU(3) \). On the other hand, the singular orbit \( \mathbb{C}P^2 \) parametrizes this union; our conclusion is that \( \mathcal{H} \) is therefore isomorphic to the total space of a fibre bundle \( \mathcal{P} \) over \( \mathbb{C}P^2 \) with fibre \( SU(2) \) and with one point for each fibre identified:

\[
\mathcal{P} \leftarrow SU(2) \rightarrow \mathbb{C}P^2
\]

and \( \mathcal{H} = \mathcal{P}/\sim \), with \( e \sim e' \) if and only if \( e \) and \( e' \) are the identity of two fibres \( SU(2) \) and \( SU(2)' \) (the identity is well defined as it is fixed by the isotropy subgroup of \( \mathbb{C}P^2 \) acting on the fibres).

The Thom space of a vector bundle \( E \rightarrow M \) is obtained by a 1-point compactification of the total space \( E \). Our construction shows that \( \mathcal{H} \) is indeed the Thom space of the bundle \( \Lambda^2 \mathbb{C}P^2 \); in fact the fibre of this vector bundle is isomorphic to \( \mathfrak{su}(2) \) as a representation of the stabilizer \( U(2) \); then, consider the closed disk \( D_{\sqrt{2}\pi} \subset \mathfrak{su}(2) \); we can identify

\[
SU(3) = \exp D_{\sqrt{2}\pi}
\]
where the spheres $S^2_r$ of radius $r < \sqrt{2}\pi$ are sent $Ad_{SU(2)}$-equivariantly to 2-spheres, whilst the boundary $S^2_{\sqrt{2}\pi}$ is collapsed to a point $x$ antipodal to $e$ (see (23)). We have therefore a corresponding disk subbundle $D$, and a bundle with fibre $SU(2) \cong S^3$ obtained from the former by collapsing the boundary of each fibre to a point. The Thom space can be therefore obtained by additionally identifying all the antipodal points of the various fibres. This is precisely what happens for the hypersurface $\mathcal{H}$, but this time identifying the identities $e$ instead of the antipodal points. This is not a real difference: in fact the antipodal element $x$ belongs to the center $C(SU(2)) = \mathbb{Z}_2$, and the automorphism $SU(2) \to xSU(2)$ is $Ad_{SU(2)}$ equivariant and swaps $e$ and $x$, giving rise to isomorphic bundles with fibre $SU(2)$. The hypersurface $\mathcal{H}$ can be shown to be smooth everywhere excepted at $e$.

The image $\Gamma(SU(3))$ is contained in $\mathcal{H}$, as seen in (17): the surjectivity of $\Gamma$ can be established in the following way by equivariance: the normal geodesic $B(t)$ used in Lemma 4.2 intersects all the $c$ orbits; its image is given by $\Gamma(B(t)) = B(2t)$ which intersects all the $Ad_{SU(3)}$ orbits orthogonally, joining the two singular orbits $e$ and $\mathbb{C}P^2$. We observe that the singular orbit $L \subset SU(3)$ is collapsed to $e$.

We pass now to the last statement of the theorem: we will use an argument which is a bundle version of that discussed in Section 2 (see (3)). We can describe the tubular neighborhood $D_3 \cong SU(3) \setminus L$ around $S$ as the $[\Sigma^2] = \mathbb{R}^3$ bundle obtained by the twisted product $SU(3) \times_{SU(2)} [\Sigma^2]$; in other words the couples $(g,v) \in SU(3) \times \mathbb{R}^3$ are identified by the relation $(g,v) \sim (g',v')$ if and only if $g' = gh$, $v = h^{-1}v'$ for some $h \in SU(2)$. The space of classes $[g,v]$ is naturally a left $SU(3)$-space under the action $g'[g,v] = [g'g,v]$. Observe now that the $SU(2)$ representation $\mathbb{R}^3$ can be extended to a $U(2)$ representation of the form $[\Sigma^2] \otimes A^0$, so that the $U(1)$ centralizer of $SU(2)$ acts trivially. This implies that $D_3$ becomes also a right $U(2)$-space in the following way: an element $k \in U(2)$ acts by $k[g,v] = [gk,k^{-1}v]$. This action is well defined because $U(2) = N(SU(2))$, and it is equivariant with respect to the left $SU(3)$ action. Clearly $SU(2) \subset U(2)$ is precisely the non-effectivity kernel, so we can just consider this action a $U(2)/SU(2) = U(1)$ effective action. The quotient space $D_3/U(1)$ turns out to be a twisted product of the form $SU(3) \times_{U(2)} V$, with $V = [\Sigma^2] \otimes A^0$, which is nothing other than $\Lambda^2\mathbb{C}P^2$. The projection $\pi_U(1)$ is therefore an equivariant map

$$SU(3) \setminus L \longrightarrow \mathcal{H} \setminus \{e\}$$

as is the map $\Gamma$; the restriction to each orbit is an equivariant projection of homogeneous spaces in both cases, and an inspection of the normalizers of $SU(2)$ and $U_{1,-1}$ shows that the choice is unique, hence $\Gamma = \pi_U(1)$. ■
Observation. The proof above has identified $\Gamma$ with the quotient

$$SU(3) \backslash L \cong \mathbb{H}P^2 \setminus \mathbb{C}P^2 \rightarrow S^7 \setminus \mathbb{C}P^2 \cong \Lambda^2 \mathbb{C}P^2$$

induced by the $U(1)$ action described in [7], [30].

Complete metrics of holonomy $\text{Spin}(7)$, invariant under a $\text{Spin}(5)$ action, have been discovered on the positive spin bundle over $S^4$ [13]; more recently other metrics of this type have been constructed on 4-dimensional vector bundles over $\mathbb{C}P^2$ (see [21]). These bundles belong to the family we have denoted by $\mathbb{P}(l)$ (see Proposition 2.2). In a future article, we hope to use the examples of this paper to construct new special geometries in dimensions 7 and 8, by gluing together tubular neighborhoods that arise in our classification, adapting invariant structures to appropriate conditions at the boundaries.

Acknowledgements. The author wishes to thank S. Salamon for his constant support and encouragement. He is also grateful to Y. Nagatomo and F. Podestà for essential help in getting this paper underway, and to F. Lonegro, M. Pontecorvo and A. Di Scala for additional input. The paper was written whilst the author was a recipient of a grant within the research projects “Geometria Riemanniana e strutture differenziabili” (University of Rome La Sapienza) and “Geometria delle varietà differenziabili” (University of Florence).

References

[1] A.V. Alekseevsky, D.V Alekseevsky: Riemannian $G$-manifold with one-dimensional orbit space, Ann. Global Anal. Geom. 11 (1993), 197–211.
[2] A.V. Alekseevsky, D.V Alekseevsky: $G$-manifold with one-dimensional orbit space, Adv. in Sov. Mat. 8 (1992), 1–31.
[3] D.V. Alekseevsky: Compact quaternion spaces, Functional Anal. Appl., 2 (1968), 106–114.
[4] D. V. Alekseevsky, F. Podestà: Compact cohomogeneity one Riemannian manifolds of positive Euler characteristic and quaternionic Kähler manifolds, Geometry, Topology, Physics. Proceedings of the First USA-Brazil Workshop, Campinas 1996 (B. N. Apanasov et al. eds.), de Gruyter, Berlin, 1997, 1–33.
[5] S. Aloff, N. Wallach: An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures, Bull. A.M.S. 81 (1975), 93–97.
[6] M. Atiyah, J. Berndt: Projective planes, Severi varieties and spheres, Surveys in Differential Geometry VIII, Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck, International Press, Somerville, MA, 2003, 1–27.
[7] M. Atiyah, E. Witten: M-theory dynamics on a manifold of $G_2$ holonomy, Adv. Theor. Math. Phys. 6 (2002) 1–106
[8] F. Battaglia: Circle actions and Morse theory on quaternion-Kähler manifolds, J. London Math. Soc. 59 (1999), 345–358.
[9] L. Bérard Bergery: Sur de nouvelles variétés riemanniennes d’Einstein, Publications de l’Institut E. Cartan 4 (Nancy, 1982), 1–60.
[10] A. Besse: Einstein Manifolds, Springer-Verlag, 1987.
[11] G.E. Bredon: Introduction to compact transformation groups, Number 46 in Pure and Applied Mathematics, Academic Press, 1972.
[12] T. Bröcker, T. tom Dieck: Representations of Compact Lie Groups, Springer, 1985.
[13] R.L. Bryant, S.M. Salamon: On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989), 829–850.
[14] L. Conlon: The topology of certain spaces of paths on a compact symmetric space, Trans. Amer. Math. Soc. 112 (1964), 228–248.
[15] A. Dancer, A.F. Swann: Quaternionic Kähler manifolds of cohomogeneity one, Int. J. Math. 10 (1999), 541–570.
[16] A. Dancer, A.F. Swann: Hyperkähler metrics of cohomogeneity one, J. Geom. and Phys. 21 (1997), 218–230.
[17] A. Dancer, M.Y. Wang: Painlevé expansions, cohomogeneity one metrics and exceptional holonomy. Comm. Anal. Geom. 12 (2004), 887–926.
[18] A. Fino, M. Parton, S. Salamon: Families of strong KT structures in six dimensions, Comment. Math. Helv. 79 (2004), 317–340.
[19] K. Galicki, B. Lawson: Quaternionic reduction and quaternionic orbifolds, Mat. Ann. 282 (1988), 1–21.
[20] A. Gambioli: Latent quaternionic geometry, math.DG/0604219, to appear in Tokyo J. Math.
[21] S. Gukov, J. Sparks: M-Theory on Spin(7) manifolds, Nucl. Phys. B 625 (2002), 3–69.
[22] S. Gukov, J. Sparks, D. Tong: Conifold transitions and five-brane condensation in M-theory on Spin(7) manifolds, Class. Quantum Grav. 20 (2003), 665–705.
[23] R. Harvey, H.B. Lawson: Calibrated Geometries, Acta Math. 148 (1982), 47–157
[24] E. Heintze, R. Palais, C.-L. Terng, G. Thorbergsson: Hyperpolar actions on symmetric spaces, Geometry, topology and physics for Raoul Bott, (S.-T. Yau, ed.), International Press, Cambridge, (1995)
[25] N. Hitchin: Stable forms and special metrics, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), Contemp. Math. 288, 70–89.

[26] R.A. Horn, C.R. Johnson: Matrix Algebra, Cambridge Univ. Press, 1985.

[27] D. Joyce: Compact hypercomplex and quaternionic manifolds. J. Diff. Geom. 35 (1992), 743–761.

[28] P.Z. Kobak, A.F. Swann: Quaternionic geometry of a nilpotent variety, Math. Ann. 297 (1993), 747–764.

[29] A.A. Kollross: A classification of hyperpolar and cohomogeneity one actions, Trans. Am. Math. Soc. 354 (2001) 571–612.

[30] R. Miyaoka: Bryant-Salamon’s $G_2$ manifolds and the hypersurface geometry, math-ph/0605074

[31] P.S. Mostert: On a compact Lie group action on manifolds, Ann. Math. 65 (1957), 447–455.

[32] F. Podestà, L. Verdiani: Positively curved 7-dimensional manifolds, Quart. J. Math. Oxford 50 (1999), 497–504

[33] F. Podestà, L. Verdiani: Totally geodesic orbits of isometries, Ann. Global Anal. Geom., 16 (1998), 399–412.

[34] Y.S. Poon, S.M. Salamon: Eight-dimensional quaternionic-Kähler manifolds with positive scalar curvature, J. Diff. Geom. 33 (1991), 363–378.

[35] S.M. Salamon: Minimal surfaces and symmetric spaces. Differential geometry (Santiago de Compostela, 1984), 103–114, Res. Notes in Math. 131, Pitman, Boston, MA, 1985.

[36] A.F. Swann: Homogeneous twistor spaces and nilpotent orbits, Math. Ann. 313 (1999), 161–188.

[37] F. Uchida: Classification of compact tranformation groups on cohomology complex projective spaces with codimension one orbits, Japan J. Math., Vol. 3 (1977), 141–189.

[38] J.A. Wolf: Complex Homogeneous contact structures and quaternionic symmetric spaces, J. Math. Mech. 14 (1965), 1033–1047.

Dipartimento di Matematica “G. Castelnuovo”, Università “La Sapienza”, Piazzale A. Moro, 2 - 00185 Roma - Italy
E-mail address: gibioli@mat.uniroma1.it

27