3-Rainbow index and forbidden subgraphs

Wenjing Li, Xueliang Li, Jingshu Zhang
Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
liwenjing610@mail.nankai.edu.cn; lxl@nankai.edu.cn; jszhang@mail.nankai.edu.cn

Abstract

A tree in an edge-colored connected graph $G$ is called a rainbow tree if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an $S$-tree if it connects $S$ in $G$. A $k$-rainbow coloring of $G$ is an edge-coloring of $G$ having the property that for every set $S$ of $k$ vertices of $G$, there exists a rainbow $S$-tree in $G$. The minimum number of colors that are needed in a $k$-rainbow coloring of $G$ is the $k$-rainbow index of $G$, denoted by $rx_k(G)$. The Steiner distance $d(S)$ of a set $S$ of vertices of $G$ is the minimum size of an $S$-tree $T$. The $k$-Steiner diameter $sdiam_k(G)$ of $G$ is defined as the maximum Steiner distance of $S$ among all sets $S$ with $k$ vertices of $G$. In this paper, we focus on the 3-rainbow index of graphs and find all finite families $F$ of connected graphs, for which there is a constant $C_F$ such that, for every connected $F$-free graph $G$, $rx_3(G) \leq sdiam_3(G) + C_F$.

Keywords: rainbow tree, $k$-rainbow index, 3-rainbow index, forbidden subgraphs.

AMS Subject Classification 2010: 05C15, 05C35, 05C38, 05C40.

1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1] for those not defined here.

Let $G$ be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in $G$ is called a rainbow path if no two edges of the path are colored with the same color. The graph $G$ is called rainbow connected if for any two distinct vertices of $G$, there is a rainbow
path connecting them. For a connected graph $G$, the *rainbow connection number* of $G$, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make $G$ rainbow connected. These concepts were first introduced by Chartrand et al. in [4] and have been well-studied since then. For further details, we refer the reader to a survey paper [8] and a book [9].

In [5], Chartrand et al. generalized the concept of rainbow path to rainbow tree. A tree in an edge-colored graph $G$ is called a *rainbow tree* if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an $S$-tree if it connects $S$ in $G$. Let $G$ be a connected graph of order $n$. For a fixed integer $k$ with $2 \leq k \leq n$, a *k-rainbow coloring* of $G$ is an edge-coloring of $G$ having the property that for every $k$-subset $S$ of $G$, there exists a rainbow $S$-tree in $G$, and in this case, the graph $G$ is called *k-rainbow connected*. The minimum number of colors that are needed in a $k$-rainbow coloring of $G$ is the *$k$-rainbow index* of $G$, denoted by $rx_k(G)$. Clearly, $rx_2(G)$ is just the rainbow connection number $rc(G)$ of $G$. In the sequel, we assume that $k \geq 3$. It is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$. Recently, some results on the $k$-rainbow index have been published, especially on the 3-rainbow index. We refer to [3, 6] for more details.

The *Steiner distance* $d(S)$ of a set $S$ of vertices in $G$ is the minimum size of a tree in $G$ containing $S$. Such a tree is called a Steiner $S$-tree or simply a Steiner tree. The *$k$-Steiner diameter* $sdiam_k(G)$ of $G$ is defined as the maximum Steiner distance of $S$ among all $k$-subsets $S$ of $G$. Then the following observation is immediate.

**Observation 1.** [5] For every connected graph $G$ of order $n \geq 3$ and each integer $k$ with $3 \leq k \leq n$,

$$k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1.$$  

The authors of [5] showed that the $k$-rainbow index of trees can achieve the upper bound.

**Proposition 1.** [5] Let $T$ be a tree of order $n \geq 3$. For each integer $k$ with $3 \leq k \leq n$,

$$rx_k(T) = n - 1.$$  

From above, we notice that for a fixed integer $k$ with $k \geq 3$, the difference $rx_k(G) - sdiam_k(G)$ can be arbitrarily large. In fact, if $G$ is a star $K_{1,n}$, then we have $rx_k(G) - sdiam_k(G) = n - k$.

They also determined the precise values for the $k$-rainbow index of the cycle $C_n$ and the 3-rainbow index of the complete graph $K_n$.

**Theorem 1.** [5] For integers $k$ and $n$ with $3 \leq k \leq n$,

$$rx_k(C_n) = \begin{cases} 
  n - 2 & \text{if } k = 3 \text{ and } n \geq 4 \\
  n - 1 & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. 
\end{cases}$$  

**Theorem 2.** [5]  

$$rx_3(K_n) = \begin{cases} 
  2 & \text{if } 3 \leq n \leq 5 \\
  3 & \text{if } n \geq 6. 
\end{cases}$$
Let $\mathcal{F}$ be a family of connected graphs. We say that a graph $G$ is $\mathcal{F}$-free if $G$ does not contain any induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that $G$ is $X$-free, for $\mathcal{F} = \{X, Y\}$ we say that $G$ is $(X,Y)$-free, and for $\mathcal{F} = \{X, Y, Z\}$ we say that $G$ is $(X,Y,Z)$-free. The members of $\mathcal{F}$ will be referred as forbidden induced subgraphs in this context. If $\mathcal{F} = \{X_1, X_2, \ldots, X_k\}$, we also refer to the graphs $X_1, X_2, \ldots, X_k$ as a forbidden $k$-tuple, and for $|\mathcal{F}| = 2$ and 3 we also say forbidden pair and forbidden triple, respectively.

In [7], Holub et al. considered the question: For which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph $G$ satisfies $rc(G) \leq \text{diam}(G) + C_{\mathcal{F}}$, where $C_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and they gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in the following two results (where $N$ denotes the net, a graph obtained by attaching a pendant edge to each vertex of a triangle).

**Theorem 3.** [7] Let $X$ be a connected graph. Then there is a constant $C_X$ such that every connected $X$-free graph $G$ satisfies $rc(G) \leq \text{diam}(G) + C_X$, if and only if $X = P_3$.

**Theorem 4.** [7] Let $X, Y$ be connected graphs such that $X, Y \neq P_3$. Then there is a constant $C_{XY}$ such that every connected $(X,Y)$-free graph $G$ satisfies $rc(G) \leq \text{diam}(G) + C_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ ($r \geq 4$) and $Y = P_4$, or $X = K_{1,3}$ and $Y$ is an induced subgraph of $N$.

Let $k \geq 3$ be a positive integer. From Observation 1, we know that the $k$-rainbow index is lower bounded by the $k$-Steiner diameter. So we wonder an analogous question concerning the $k$-rainbow index of graphs. In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $C_{\mathcal{F}}$ such that $rx_k(G) \leq \text{sdiam}_k(G) + C_{\mathcal{F}}$ if a connected graph $G$ is $\mathcal{F}$-free ?

In general, it is very difficult to give answers to the above question, even if one considers the case $k = 4$. So, in this paper we pay our attention only on the case $k = 3$. In Sections 3, 4 and 5, we give complete answers for the 3-rainbow index when $|\mathcal{F}| = 1, 2$ and 3, respectively. Finally, we give a complete characterization for an arbitrary finite family $\mathcal{F}$.

## 2 Preliminaries

In this section, we introduce some further terminology and notation that will be used in the sequel. Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers.

Let $G$ be a graph. We use $V(G)$, $E(G)$, and $|G|$ to denote the vertex set, edge set, and the order of $G$, respectively. For $A \subseteq V(G)$, $|A|$ denotes the number of vertices in $A$, and $G[A]$ denotes the subgraph of $G$ induced by the vertex set $A$. For two disjoint subsets $X$ and $Y$ of $V(G)$, we use $E[X,Y]$ to denote the set of edges of $G$ between $X$ and $Y$. For graphs $X$ and $G$, we write $X \subseteq G$ if $X$ is a subgraph of $G$, $X \in G$ if $X$
is an induced subgraph of $G$, and $X \cong G$ if $X$ is isomorphic to $G$. In an edge-colored graph $G$, we use $c(uv)$ to denote the color assigned to an edge $uv \in E(G)$.

Let $G$ be a connect graph. For $u, v \in V(G)$, a path in $G$ from $u$ to $v$ will be referred as a $(u, v)$-path, and, whenever necessary, it will be considered with orientation from $u$ to $v$. The distance between $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest $(u, v)$-path in $G$. The eccentricity of a vertex $v$ is $ecc(v) := \max_{x \in V(G)} d_G(v, x)$. The diameter of $G$ is $diam(G) := \max_{x \in V(G)} ecc(x)$, and the radius of $G$ is $rad(G) := \min_{x \in V(G)} ecc(x)$. One can easily check that $rad(G) \leq diam(G) \leq 2rad(G)$. A vertex $x$ is central in $G$ if $ecc(x) = rad(G)$. Let $D \subseteq V(G)$ and $x \in V(G) \setminus D$. Then we call a path $P = v_0v_1\ldots v_k$ is a $v$-$D$ path if $v_0 = v$ and $V(P) \cap D = \{v_k\}$, and $d_G(v, D) := \min_{w \in D} d_G(v, w)$.

For a set $S \subseteq V(G)$ and $k \in \mathbb{N}$, we use $N^k_G(S)$ to denote the neighborhood at distance $k$ of $S$, i.e., the set of all vertices of $G$ at distance $k$ from $S$. In the special case when $k = 1$, we simply write $N_G(S)$ for $N^1_G(S)$ and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subseteq V(G)$, we set $N_G(M) = N_G(S) \cap M$ and $N_G(x) = N_G(x) \cap M$. Finally, we will also use the closed neighborhood of a vertex $x \in V(G)$ defined by $N^k_G[x] = (\cup_{i=1}^k N^i_G(x)) \cup \{x\}$.

A set $D \subseteq V(G)$ is called dominating if every vertex in $V(G) \setminus D$ has a neighbor in $D$. In addition, if $G[D]$ is connected, then we call $D$ a connected dominating set. A clique of a graph $G$ is a subset $Q \subseteq V(G)$ such that $G[Q]$ is complete. A clique is maximum if $G$ has no clique $Q'$ with $|Q'| > |Q|$. For a graph $G$, a subset $I \subseteq V(G)$ is called an independent set of $G$ if no two vertices of $I$ are adjacent in $G$. An independent set is maximum if $G$ has no independent set $I'$ with $|I'| > |I|$.

For two positive integers $a$ and $b$, the Ramsey number $R(a, b)$ is the smallest integer $n$ such that in any two-coloring of the edges of a complete graph on $n$ vertices $K_n$ by red and blue, either there is a red $K_a$ (i.e., a complete subgraph on $a$ vertices all of whose edges are colored red) or there is a blue $K_b$. Ramsey [10] showed that $R(a, b)$ is finite for any $a$ and $b$.

Finally, we will use $P_n$ to denote the path on $n$ vertices. An edge is called a pendant edge if one of its end vertices has degree one.

3 Families with one forbidden subgraph

In this section, we characterize all possible connected graphs $X$ such that every connected $X$-free graph $G$ satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, where $C_X$ is a constant.

**Theorem 5.** Let $X$ be a connected graph. Then there is a constant $C_X$ such that every connected $X$-free graph $G$ satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, if and only if $X = P_3$.

**Proof.** We have that the graph $G$ is a complete graph since $G$ is $P_3$-free. Then from Theorem 2 it follows that $rx_3(G) \leq 3 = sdiam_3(G) + 1$. 

4
Let $t$ be an arbitrarily large integer, set $G^t_1 = K_{1,t}$, and let $G^t_2$ denote the graph obtained by attaching a pendant edge to each vertex of the complete graph $K_t$ (see Figure 1). We also use $K^t_i$ to denote $G^t_2$. Since $rx_3(G^t_1) = t$ but $sdiam_3(G^t_1) = 3$, $X$ is an induced subgraph of $G^t_1$. Clearly, $rx_3(G^t_2) \geq t + 2$ but $sdiam_3(G^t_2) = 5$, and $G^t_2$ is $K_{1,3}$-free. Hence, $X = K_{1,2} = P_3$. The proof is thus complete.

4 Forbidden pairs

The following statement, which is the main result of this section, characterizes all possible forbidden pairs $X, Y$ for which there is a constant $C_{XY}$ such that $rx_3(G) \leq sdiam_3(G) + C_{XY}$ if $G$ is $(X, Y)$-free. Since any $P_3$-free graph is a complete graph, we exclude the case that one of $X, Y$ is $P_3$.

**Theorem 6.** Let $X, Y \neq P_3$ be a pair of connected graphs. Then there is a constant $C_{XY}$ such that every connected $(X, Y)$-free graph $G$ satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$, if and only if (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.

The proof of Theorem 6 will be divided into two parts. We prove the necessity in Proposition 2 and then we establish the sufficiency in Theorem 7.

**Proposition 2.** Let $X, Y \neq P_3$ be a pair of connected graphs for which there is a constant $C_{XY}$ such that every connected $(X, Y)$-free graph $G$ satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$. Then, (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.

**Proof.** Let $t$ be an arbitrarily large integer, and set $G^t_3 = C_t$. We will also use the graphs $G^t_1$ and $G^t_2$ shown in Figure 1.

Consider the graph $G^t_1$. Since $sdiam_3(G^t_1) = 3$ but $rx_3(G^t_1) = t$, we have, up to symmetry, $X = K_{1,r}, r \geq 3$. Then we consider the graphs $G^t_2$ and $G^t_3$. It is easy to verify that $sdiam_3(G^t_2) = 5$ but $rx_3(G^t_2) \geq t + 2$, and $sdiam_3(G^t_3) = \lceil \frac{2}{3}t \rceil$ while $rx_3(G^t_3) \geq t - 2 \geq \frac{1}{2}(sdiam_3(G^t_3) - 1) - 2$, respectively. Clearly, $G^t_2$ and $G^t_3$ are both $K_{1,3}$-free, so neither of them contains $X$, implying that both $G^t_2$ and $G^t_3$ contain $Y$. Since the maximum common induced subgraph of them is $P_4$, we get that $Y = P_4$. This completes the proof. \qed

Next, we can prove that the converse of Proposition 2 is true.
**Theorem 7.** Let $G$ be a connected $(P_4, K_{1,r})$-free graph for some $r \geq 3$. Then $rx_3(G) \leq \text{sdiam}_3(G) + r + 3$.

**Proof.** Let $G$ be a connected $(P_4, K_{1,r})$-free graph ($r \geq 3$). Then, $\text{sdiam}_3(G) \geq 2$. For simplicity, we set $V = V(G)$. Let $S \subseteq V$ be the maximum clique of $G$.

**Claim 1:** $S$ is a dominating set.

*Proof.* Assume that there is a vertex $y$ at distance 2 from $S$. Let $yvu$ be a shortest path from $y$ to $S$, where $u \in S$. Because $S$ is the maximum clique, there is some $v \in S$ such that $vx \notin E(G)$. Thus the path $vuxy \cong P_4$, a contradiction. So $S$ is a dominating set. \hfill \blacksquare

Let $X$ be the maximum independent set of $G[V \setminus S]$ and $Y = V \setminus (S \cup X)$. Then for any vertex $y \in Y$, $y$ is adjacent to some $x \in X$. Furthermore, for any independent set $W$ of graph $G[Y]$, $|N_X(W)| \geq |W|$ since $X$ is maximum.

**Claim 2:** There is a vertex $v \in S$ such that $v$ is adjacent to all the vertices in $X$.

*Proof.* Suppose that the claim fails. Let $u$ be the vertex of $S$ with the largest number of neighbors in $X$. Set $X_1 = N_X(u)$, $X_2 = X \setminus X_1$. Then, $X_2 \neq \emptyset$ according to our assumption. Pick a vertex $w$ in $X_2$. Then, $uw \notin E(G)$. Let $v$ be a neighbor of $w$ in $S$. For any vertex $z$ in $X_1$, $G[w, v, u, z]$ can not be an induced $P_4$, so $vz$ must be an edge of $G$. Thus, $N_X(v) \supseteq N_X(u) \cup \{w\}$, contradicting the maximum of $u$. \hfill \blacksquare

Let $z$ be the vertex in $S$ which is adjacent to all the vertices of $X$. Set $X = \{x_1, x_2, \ldots, x_\ell\}$. Then, $0 \leq \ell \leq r - 1$ since $G$ is $K_{1,r}$-free. Now we demonstrate a 3-rainbow coloring of $G$ using at most $\ell + 6$ colors. Assign color $i$ to the edge $zx_i$, and $i + 1$ to the edge $x_iy$ where $1 \leq i \leq \ell$ and $y \in Y$. Color $E[S, Y]$ with color $\ell + 2$ and $E(G[Y])$ with color $\ell + 3$. Give a 3-rainbow coloring of $G[S]$ using colors from $\{\ell + 4, \ell + 5, \ell + 6\}$. And color the remaining edges arbitrarily (e.g., all of them with color 1). Next, we prove that this coloring is a 3-rainbow coloring of $G$.

Let $W = \{u, v, w\}$ be a 3-subset of $V$.

(i) $\{u, v, w\} \subseteq S \cup X$. There is a rainbow tree containing $W$.

(ii) $\{u, v\} \subseteq S \cup X, w \in Y$. We can find a rainbow tree containing an edge in $E[S, Y]$ that connects $W$.

(iii) $u \in S \cup X, \{v, w\} \subseteq Y$.

a) If $vw \in E(G)$, then there is a rainbow tree containing the edge $vw$ that connects $W$.

b) If $vw \notin E(G)$, then we have $|N_X(\{v, w\})| \geq |\{v, w\}| = 2$. So there are two vertices $x_i$ and $x_j(i \neq j)$ in $X$ adjacent to $v$ and $w$, respectively. As $i + 1 \neq j + 1$, so either $i + 1 \neq c(zu)$ or $j + 1 \neq c(zu)$. Without loss of generality, we assume that $i + 1 \neq c(zu)$ and $s$ is a neighbor of $w$ in $S$. Then there is a rainbow tree containing the edges $zu, uw, sw, sz$ if $u = x_i$ or the edges $zu, zz_i, x_i v, sw, sz$ if $u \neq x_i$.  

6
a neighbor of $w$.

Theorem 8. Let $F$ be a family of connected graphs with $|F| = 3$ such that $F \not\subseteq F'$ for any $F' \in \mathcal{F}_1 \cup \mathcal{F}_2$. Then there is a constant $C_F$ such that every connected $F$-free graph $G$ satisfies $rx_3(G) \leq sdiam_3(G) + C_F$, if and only if $F \in \mathcal{F}_3$.

First of all, we prove the necessity of the triples given by Theorem 8.

Proposition 3. Let $X, Y, Z \not\subseteq P_3$ be connected graphs, $\{X, Y, Z\} \not\subseteq F'$ for any $F' \in \mathcal{F}_2$, for which there is a constant $C_{XYZ}$ such that every connected $(X, Y)$-free graph $G$
satisfies \( rx_3(G) \leq sdiam_3(G) + C_{XYZ} \). Then, (up to symmetry) \( X = K_{1,r}(r \geq 3), Y \subseteq K^h_s(s \geq 3), \) and \( Z = P_t(t > 4) \).

Proof. Let \( t \) be an arbitrarily large integer, and let \( G'_1, G'_2, G'_3 \) be the graphs defined in the proof of Proposition 2.

Firstly, we consider the graph \( G'_1 \). Up to symmetry, we have \( X = K_{1,r}, r \geq 3 \) (for the case \( r = 2 \) is excluded by the assumptions). Secondly, we consider the graph \( G'_2 \). The graph \( G'_2 \) does not contain \( X \), since it is \( K_{1,3} \)-free. Thus, up to symmetry, we have \( G'_2 \) contains \( Y \), implying \( Y \subseteq K^h_s \) for some \( s \geq 3 \) (for the case \( s \leq 2 \) is excluded by the assumptions). Finally, we consider the graphs \( G'_3 \) and \( G'_{3+1} \). Clearly, they are \((K_{1,3}, K^h_s)-\)free, so both of them contain neither \( X \) nor \( Y \). Hence, we get that \( Z = P_{t} \) for some \( \ell > 4 \) (for the case \( \ell \leq 4 \) is excluded by the assumptions).

This completes the proof. \( \square \)

It is easy to observe that if \( X \subseteq X' \), then every \( (X', Y, Z) \)-free graph is also \((X', Y, Z)\)-free. Thus, when proving the sufficiency of Theorem \( \text{[8]} \) we will be always interested in maximal triples of forbidden subgraphs, i.e., triples \( X, Y, Z \) such that, if replacing one of \( X, Y, Z \), say \( X \), with a graph \( X' \neq X \) such that \( X' \subseteq X' \), then the statement under consideration is not true for \((X', Y, Z)\)-free graphs.

For every vertex \( c \in V(G) \) and \( i \in \mathbb{N} \), we set \( \alpha_i(G, c) = \max\{|M|, M \subseteq N^i_G(c), M \text{ is independent}\} \) and \( \alpha^0_i(G, c) = \max\{|M^0|, M^0 \subseteq N^i_G(c), M^0 \text{ is independent}\} \).

Lemma 1. \( \text{[2]} \) Let \( r, s, i \in \mathbb{N} \). Then there is a constant \( \alpha(r, s, i) \) such that, for every connected \((K_{1,r}, K^h_s)\)-free graph \( G \) and for every \( c \in V(G) \), \( \alpha_i(G, c) < \alpha(r, s, i) \).

We use the proof of Lemma \( \text{[1]} \) to get the following corollary concerning \( \alpha^0_i(G, c) \) for each integer \( i \geq 1 \).

Corollary 1. Let \( r, s, i \in \mathbb{N} \). Then there is a constant \( \alpha^0(r, s, i) \) such that, for every connected \((K_{1,r}, K^h_s)\)-free graph \( G \) and for every \( c \in V(G) \), \( \alpha^0_i(G, c) < \alpha^0(r, s, i) \).

Proof. For the sake of completeness, here we give a brief proof concentrating on the upper bound of \( \alpha^0_i(G, c) \). We prove the corollary by induction on \( i \).

For \( i = 1 \), we have \( \alpha^0(r, s, 1) = r \), for otherwise \( G \) contains a \( K_{1,r} \) as an induced subgraph.

Let, to the contrary, \( i \) be the smallest integer for which \( \alpha^0(r, s, i) \) does not exist (i.e., \( \alpha^0_i(G, c) \) can be arbitrarily large), choose a graph \( G \) and a vertex \( c \in V(G) \) such that \( \alpha^0_i(G, c) \geq (r - 2)R(s(2r - 3), \alpha^0(r, s, i - 1)) \), and let \( M^0 = \{x_1^0, \ldots, x_k^0\} \subseteq N^i_G(c) \) be an independent set in \( G \) of size \( \alpha^0_i(G, c) \). Obviously, \( k \geq (r - 2)R(s(2r - 3), \alpha^0(r, s, i - 1)) \). Let \( Q_j \) be a shortest \((x_j^0, c)\)-path in \( G \), \( j = 1, \ldots, k \). We denote \( M^1 \subseteq N^{-1}_G(c) \) the set of all successors of the vertices from \( M^0 \) on \( Q_j \), \( j = 1, \ldots, k \), and \( x_j^1 \) the successor of \( x_j^0 \) on \( Q_j \) (note that some distinct vertices in \( M^0 \) can have a common successor in \( M^1 \)). Every vertex in \( M^1 \) has at most \( r - 2 \) neighbors in \( M^0 \) since \( G \) is \( K_{1,r} \)-free. 8
Thus, $|M^1| \geq \frac{k}{2} \geq R(s(2r-3), a^0(r, s, i-1))$. By the induction assumption and the definition of Ramsey number, $G[M^1]$ contains a complete subgraph $K_{s(2r-3)}$. Choose the notation such that $V(K_{s(2r-3)}) = \{x_1^1, \ldots, x_{s(2r-3)}^1\}$, and set $\hat{M}^0 = N_M^0(K_{s(2r-3)})$. Using a matching between $K_{s(2r-3)}$ and $\hat{M}^0$, we can find in $G$ an induced $K_h^s$ with vertices of degree 1 in $\hat{M}^0$, a contradiction. For more details about finding the $K_h^s$, we refer the reader to [2].

Armed with Corollary 1, we can get the following important theorem.

**Theorem 9.** Let $r \geq 3, s \geq 3$, and $\ell > 4$ be fixed integers. Then there is a constant $C(r, s, \ell)$ such that every connected $(K_{1, r}, K_h^s, P_\ell)$-free graph $G$ satisfies $rx_3(G) \leq sdiam_3(G) + C(r, s, \ell)$.

**Proof.** We have $diam(G) \leq \ell - 2$ since $G$ is $P_\ell$-free. Let $c$ be a central vertex of $G$, i.e., $ecc(c) = rad(G) \leq diam(G) \leq \ell - 2$. And we set $S_i = \cup_{j=1}^{i} N_G^j[c]$ for an integer $i \geq 1$.

**Claim:** $rx_3(G[S_i \cup N_G^{i+1}(c)]) \leq rx_3(G[S_i]) + \alpha^0_{i+1}(G, c) + 3$

**Proof.** Let $X = \{x_1, x_2, \ldots, x_{\alpha^0_{i+1}(G, c)}\}$ be the maximum independent set of $N_G^{i+1}(c)$ and $Y = N_G^{i+1}(c) \setminus X$. Then for any vertex $y \in Y$, $y$ is adjacent to some $x \in X$ and $s \in S$. Further more, for any independent set $W$ of graph $G[Y]$, we have $|N_X(W)| \geq |W|$ since $X$ is maximum.

Now we demonstrate a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$ using at most $k + \alpha^0_{i+1}(G, c) + 3$ colors, where $k = rx_3(G[S_i])$. We color the edges of $G[S_i]$ using colors 1, 2, $\ldots$, $k$. Color $E[S_i, Y]$ with color $k + 1$ and $E(G[Y])$ with color $k + 2$. And assign color $j + k + 2$ to the edges $E[\{x_j\}, S_i]$, and $j + k + 3$ to the edges $E[\{x_j\}, Y]$ where $1 \leq j \leq \alpha^0_{i+1}(G, c)$. With the same argument as the proof of Theorem 7, we can prove that this coloring is a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$.

From the proof of Corollary 1, it follows that $\alpha^0_1(G, c) \leq r - 1$ and $\alpha^0_0(G, c) \leq (r-2)R(s(2r-3), a^0(r, s, i-1)) - 1$ for each integer $i \geq 2$. Let $R(r, s) = \sum_{i=2}^{\infty} R(s(2r-3), a^0(r, s, i-1))$. Recall that $ecc(c) \leq \ell - 2$. Repeated application of Claim gives the following:

$$rx_3(G) \leq rx_3(G[N_G^{ecc(c)-1}(c)]) + \alpha^0_{ecc(c)}(G, c) + 3$$

$$\leq \cdots$$

$$\leq rx_3(c) + \alpha^0_1(G, c) + \cdots + \alpha^0_{ecc(c)}(G, c) + 3ecc(c)$$

$$\leq 0 + r + (r-2)R(r, s) + 2(\ell - 2)$$

$$\leq sdiam_3(G) + (r-2)(R(r, s) + 1) + 2(\ell - 1).$$

Thus, we complete our proof.

**Remark** The same as the remark in Section 4: for $i \geq 1$, every time $\alpha^0_{i+1}(G, c) \geq 4$ happens, we can save one color in the Claim of Theorem 9.
6 Forbidden $k$-tuples for any $k \in \mathbb{N}$

Let $\mathcal{F} = \{X_1, X_2, X_3, \ldots, X_k\}$ be a finite family of connected graphs with $k \geq 4$ for which there is a constant $k_F$ such that every connected $\mathcal{F}$-free graph satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$. Let $t$ be an arbitrarily large integer, and let $G_1^t, G_2^t$ and $G_3^t$ be defined in Proposition 2. For the graph $G_1^t$, up to symmetry, we suppose that $X_1 = K_r, r \geq 3$ (for the case $r = 2$ has been discussed in Section 3). Then, we consider the graphs $G_2^t$ and $G_3^t$. Notice that $G_2^t$ and $G_3^t$ are both $K_{1,3}$-free, so neither of them contains $X_1$, implying that $G_2^t$ or $G_3^t$ contains $X_i$, where $i \neq 1$. We may assume that $X_2$ is an induced subgraph of $G_2^t$. If $G_3^t$ contains $X_2$, then $X_2 = P_4$, which is just the case in Section 4. So we turn to the case that $G_3^t$ contains $X_i$ for some $i > 2$. Now consider the graphs $G_3^t, G_3^{t+1}, G_3^{t+2}, \ldots, G_3^{t+k}$, each of which contains at least one of $X_3, X_4, \ldots, X_k$ as an induced subgraph due to the analysis above. So it is forced that at least one of these $X_i (i \geq 3)$ is isomorphic to $P_l$ for some $l \geq 5$, which goes back to the case in Section 5. Thus, the conclusion comes out.

Theorem 10. Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press, London and Basingstoke, 1976.

[2] J. Brousek, P. Holub, Z. Ryjáček, P. Vrána, *Finite families of forbidden subgraphs for rainbow connection in graphs*, Discrete Math. 339(9)(2016), 2304-2312.

[3] Q. Cai, X. Li, Y. Zhao, *The 3-rainbow index and connected dominating sets*, J. Combin. Optim. 31(2016), 1142-1159.

[4] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(1)(2008), 85-98.

[5] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55(2010), 360-367.

[6] L. Chen, X. Li, K. Yang, Y. Zhao, *The 3-rainbow index of a graph*, Discuss. Math. Graph Theory 35(1)(2015), 81-94.

[7] P. Holub, Z. Ryjáček, I. Schiermeyer, P. Vrána, *Rainbow connection and forbidden subgraphs*, Discrete Math. 338(10)(2015), 1706-1713.

[8] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs: A survey*, *Graphs & Combin.* 29(2013), 1–38.
[9] X. Li, Y. Sun, *Rainbow Connections of Graphs*, New York, SpringerBriefs in Math., Springer, 2012.

[10] F.P. Ramsey, *On a problem of formal logic*, Proc. London Math. Society 30(1920), 264-286.