The Case for 2-D Turbulence in Antarctic Data

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Abstract

In this paper we examine the data that was collected at Haley Station in Antarctica on June 22, 1987. Using a test devised by Dewan [1] we interpret the flow as one which represents two-dimensional turbulence. We also construct a model to interpret the spectrum of this data which is almost independent of the wave number for a range of frequencies.


1 Introduction

Two dimensional turbulence has been the subject of intense theoretical research \[1, 2\] and simulation experiments \[3\]. The reason for this interest stems from the fundamental differences between 3-d isotropic and 2-d turbulence. To begin with, vortex stretching is absent in 2-d as a direct consequence of Navier-Stokes equations. Furthermore in 3-d the energy cascade is from the large eddies to small one but this process reverses itself in 2-d and leads to the formation of large scale coherent eddies. Another difference between two and three dimensional turbulence exists in the inertial range of the spectrum. Kraichnan showed \[4\] that in 2-d in addition to Kolmogorov inertial range there is (due to enstrophy conservation in zero viscosity) another scaling law in the form

\[ E(k) = c \eta^{2/3} k^{-3} \]

where \( \eta \) is enstrophy dissipation rate.

While many simulations \[5, 6\] confirm these theoretical predictions the actual observation and detection of 2-d turbulence as a natural phenomena remains (as far as we know) an open questions.

One of the objectives of this paper is to weigh in the pros and cons for 2-d turbulence in the Antarctic data that was obtained by the British observation post as Haley Station in Antarctica on June 22, 1987 (for further description of this data see \[7, 8\]). The importance of these measurements stem from the fact that the flow field \( u = (u, v, w) \) and the temperatures were measured simultaneously at three different heights viz. 5m, 16m and 32m. These simultaneous readings enable us to apply a test devised by E. Dewan \[9\] for the detection of 2-d turbulence. According to this test 2-d turbulence is characterized by small values for the coherence \[20\] between the time series which represent the various meteorological variables at different heights.

From another point of view the Antarctic data represent a stably stratified medium. (According to mission records the temperature gradient with height can reach up to \( 1K/m \)). Under these circumstances Bolgiano \[10, 11\] and others \[4\] speculated about the existence of “buoyancy range turbulence” (BRT) which should lead to a flattening of the spectra in parts of the inertial range. In this paper we shall estimate the power spectrum for the data using
the usual Fourier transform and by the method of maximum entropy (briefly the reason for this duplicacy is due to the existence of “discontinuities” in the data). Both of these estimates show a spectral range in which the spectrum is almost flat and thus support the theoretical arguments that were advanced for the existence of BRT.

The plan of the paper is as follows: In section 2 we describe the method used to filter out the mean flow and waves from the data and the tests that were applied to verify that the residuals actually represent turbulence. In section 3 we apply the coherence test for 2-d turbulence and discuss its consequences. In section 4 we present a model for the power spectrum of the data and its implications. We end up in section 5 with some conclusions.

2 Data Detrending

The statistical approach to turbulence splits the flow variables $\tilde{u}, \tilde{T}$ (where $\tilde{T}$ is the temperature) into a sum

$$\tilde{u} = u + u' + u, \quad \tilde{T} = T + T' + t$$

where $u, T$ represent the mean (large scale) flow, $u', T'$ represent waves and $u, t$ “turbulent residuals” [12]

To effect such a decomposition in our data we used the Karahuman-Loeve (K-L) decomposition algorithm (or PCA) which was used by many researchers (for a review see [13]). Here we shall give only a brief overview of this algorithm within our context.

Let be given a time series $X$ (of length $N$) of some geophysical variable. We first determine a time delay $\Delta$ for which the points in the series are decorrelated. Using $\Delta$ we create $n$ copies of the original series

$$X(k), \quad X(d + \Delta), \ldots, X(k + (n - 1)\Delta).$$

(To create these one uses either periodicity or choose to consider shorter time-series). Then one computes the auto-covariance matrix $R = (R_{ij})$

$$R_{ij} = \sum_{k=1}^{N} X(k + i\Delta)X(k + j\Delta). \quad (2.1)$$
Let $\lambda_0 > \lambda_1, \ldots, > \lambda_{n-1}$ be the eigenvalues of $R$ with their corresponding eigenvectors

$$\phi^i = (\phi^i_0, \ldots, \phi^i_{n-1}), \; i = 0, \ldots, n - 1.$$ 

The original time series $T$ can be reconstructed then as

$$X(j) = \sum_{k=0}^{n-1} a_k(j)\phi^k_0 \tag{2.2}$$

where

$$a_k(j) = \frac{1}{n} \sum_{i=0}^{n-1} X(j + i\Delta)\phi^k_i. \tag{2.3}$$

The essence of the K-L decomposition is based on the recognition that if a large spectral gap exists after the first $m_1$ eigenvalues of $R$ then one can reconstruct the mean flow (or the large component ( of the data by using only the first $m_1$ eigenfunctions in (2.2)). A recent refinement of this procedure due to Ghil et al [13] is that the data corresponding to eigenvalues between $m_1 + 1$ and up to the point $m_2$ where they start to form a “continuum” represent waves. The location of $m_2$ can be ascertained further by applying the tests devised by Axford [14] and Dewan [9] (see below).

Thus the original data can be decomposed into mean flow, waves and residuals (i.e. data corresponding to eigenvalues $m_2 + 1, \ldots, n - 1$ which we wish to interpret at least partly as turbulent residuals).

For the data under consideration we carried out this decomposition using a delay $\Delta$ of 1024 points (approximately 51 sec.) for all the geophysical variables. In table 1 we present the values of $m_1, m_2$ that were used in this decomposition for the flow variables at different heights. (In all cases $n = 64$).

The residuals of the time series which are reconstructed as

$$X^r(j) = \sum_{k=m_2+1}^{n-1} a_k(j)\phi^k_0 \tag{2.4}$$

contain (obviously) the measurement errors in the data. However to ascertain that they should be interpreted primarily as representing turbulence we utilize the tests devised by Axford [14] and Dewan [9]. According to these tests turbulence data (at the same location)
is characterized by low coherence between $u, v, w$ and a phase close to zero or $\pi$ between $w$ and $t$. (A phase close to $\pi/2$ is characteristic of waves). Figs. 1,2,3 show samples of the coherence between the residuals of $u, v, w$ at different heights. They demonstrate that for most frequencies the coherence is less than 0.1. Fig. 4 gives a scatter plot of the phase between $w$ and $t$ at height 5m. This figure is less definitive as there are still quite a few points in the wave sector $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. However out of the 200 points in this plot 125 are in the “turbulence sector”.

These tests show that to a large extent the residuals that were obtained from the K-L decomposition represent actual turbulence.

3 Tests for 2-d turbulence

In today literature [15] a spectral slope of $-3$ in part of the inertial range is considered to be a strong indicator for 2-d turbulence. However as noted already by Lily [5] “geophysical consideration” might modify this slope. Since the spectral plots for the flow under consideration (for sample see figs. 8,9,10) do not exhibit this dependency (except for $w$ at 16m in the low frequencies) we must resort to other tests to bolster the claim that the flow described by this data corresponds to 2-d turbulence.

To this end we utilize a test devised by Dewan [9]. According to this test inviscid two dimensional turbulence is characterized by the fact that the temporal statistical coherency [20] between the time series representing the flow variables at different altitudes is zero. With viscosity taken into account some vertical separation of the order of (10m for air) is needed for the coherency to become small. (Strong coherency with values close to one indicates a strong linear relationship between the two time series [20]).

Some typical plots for the coherency in the data is presented in figs. (5,6,7). In these plots the coherency for $w$ between the different heights is plotted for different wave numbers. We observe that for most sampled frequencies the coherency is well below 0.1 and according to Dewan [9] “these values constitute evidence for 2-d turbulence and against other types of fluctuations”.

6
4 The spectrum

Two dimensional flow of incompressible and inviscid fluid conserve both the energy $E$ and the enstrophy $\Omega$. For viscous fluid these quantities decay according to

$$-\epsilon = \frac{\partial E}{\partial t} = -2\nu\Omega, \quad -\epsilon_\omega = \frac{\partial \Omega}{\partial t} = -\nu |\nabla \omega|^2$$

(4.1)

The energy spectrum is determined therefore by both parameters $\epsilon, \epsilon_\omega$ which leads to the definition of a length scale

$$L_\omega = \left( \frac{\epsilon}{\epsilon_\omega} \right)^{1/2}$$

(4.2)

From dimensional considerations one concludes then that the energy spectrum in the inertial range must have the form

$$E(k) = f(kL_\omega)\epsilon^{2/3}k^{-5/3}$$

(4.3)

where $f$ is a function of the dimensionless variable $kL_\omega$. If at one end of the inertial range only $\epsilon$ is essential (and the effect of $\epsilon_\omega$ is negligible) then $f \cong$ constant and the energy spectrum obey Kolmogorov $5/3$ power law. If on the other end of this range $\epsilon$ is not essential then $f$ must have the form

$$f \cong (kL_\omega)^{-4/3}$$

(4.4)

and consequently

$$E(k) = C\epsilon^{2/3}_\omega k^{-3}$$

(4.5)

(where $C$ is a constant).

For stratified medium Obukov [17] introduced the temperature inhomogeneity dissipation rate

$$\epsilon_T = 2\chi \int_0^\infty k^2 E_T(k)dk$$

(4.6)

where $E_T$ is the temperature spectra and $\chi$ is the heat conductivity of the medium. He further postulated that the turbulent component of $T$ is dependent on this parameter.
For the (stratified) Antarctic medium we would like to enlarge the domain of this postulate to include the velocity components of the flow. This enables us to introduce the buoyancy (length) scale $L_B = (\alpha g)^{-3/2} \epsilon_T^{-3/4}$

$$L_B = (\alpha g)^{-3/2} \epsilon_T^{-3/4}$$  \(4.7\)

where $(\alpha g)$ is the buoyancy parameter. The existence of this second length scale for stratified two dimensional flow lead us to replace (4.3) by

$$E(k) = f(kL_\omega, kL_B) \epsilon^{2/3} k^{-5/3}$$  \(4.8\)

However since stratification and enstrophy conservation are independent of each other we infer that $f$ must have the form

$$f \simeq (kL_\omega)^r (kL_B)^s. \quad (4.9)$$

It follows then that the spectral dependence on $k$ is given by

$$E(k) \sim k^{r+s-5/3}. \quad (4.10)$$

We conclude therefore that various combinations of $r, s$ are possible and this will lead to different spectral dependencies on $k$.

Thus if

$$E(k) \sim k^{-q}$$

and the dissipation $\epsilon$ is negligible we must have then

$$r + s = 5/3 - q, \quad \frac{r}{2} + \frac{5}{4}s + \frac{2}{3} = 0$$

which yields

$$r = \frac{33 - 15q}{9}, \quad s = \frac{15q - 18}{9}. $$

From the spectral plots for the data under consideration we see that (approximately)

$$E(k) \sim k^0$$

for a large segment of the inertial range which is characteristic of the “buoyancy range turbulence” as predicted by Bolgiano [10, 11].
It is interesting to note in this context that Kriachen [19] already observed that the “energy spectrum of the flow depends on the details of the nonlinear interaction embodied in the equations that govern the flow and can not be deduced solely from the symmetries, invariances and dimensionality of the equations”.

Finally we would like to observe that the data under consideration contains some discontinuities. These can change completely the asymptotic behavior of the spectrum. To demonstrate this assume that the data is described by

\[ D(x) = CH(x - x_0) + g(x) \]  

where \( g(x) \) is a smooth function whose Fourier transform (FT) decays exponentially and \( H(x) \) is the Heaviside function

\[
H(x) = \begin{cases} 
  1 & x \geq 0 \\
  0 & x < 0. 
\end{cases}
\]

Differentiating (4.11) we have

\[ D'(x) = C\delta(x - x_0) + g'(x) \]  

and the FT of (4.12) is

\[ \tilde{D}'(k) = C + \tilde{g}'(k) \]  

The FT of \( D \) is obtained then by dividing (4.13) by \( k \) which shows clearly that the asymptotic behavior of \( \tilde{D}(k) \) is proportional to \( k^{-1} \).

We conclude then that a proper filter for the removal of these discontinuities from the data is needed in order to obtain the true spectrum of the turbulent residuals. Such a filtering algorithm is given by the \( K - L \) decomposition which was described in Sec. 2.

5 Conclusion

Using the coherency test advanced by Dewan we are able to characterize the flow under consideration as one that has the characteristics of 2-d turbulence. One stumbling block for
this interpretation is the absence of $-3$ slope in part of the inertial range. To explain this we introduced a model that takes into account the stratification of this flow. This model shows that when buoyancy effects are taken into account different slopes of $E(k)$ are possible. Thus we believe that we introduced evidence for the interpretation of this spectra as one belonging to BTR.

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|         | $m_1$ | $m_2$ |
|---------|-------|-------|
| u at 5m | 2     | 42    |
| v at 5m | 2     | 26    |
| w at 5m | 2     | 30    |
| T at 5m | 4     | 26    |
| u at 16m| 2     | 42    |
| v at 16m| 2     | 40    |
| w at 16m| 3     | 37    |
| T at 16m| 2     | 41    |
| u at 32m| 4     | 48    |
| v at 32m| 1     | 40    |
| w at 32m| 4     | 51    |
| T at 32m| 2     | 42    |

Table 1
