An improved upper bound on the maximum degree of terminal-pairable complete graphs

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Abstract

A graph $G$ is terminal-pairable with respect to a demand multigraph $D$ on the same vertex set as $G$, if there exists edge-disjoint paths joining the end vertices of every demand edge of $D$. In this short note, we improve the upper bound on the largest $\Delta(n)$ with the property that the complete graph on $n$ vertices is terminal-pairable with respect to any demand multigraph of maximum degree at most $\Delta(n)$. This disproves a conjecture originally stated by Csaba, Faudree, Gyárfás, Lehel and Schelp.

1 Introduction

The concept of terminal-pairability emerged as a practical networking problem and was introduced by Csaba, Faudree, Gyárfás, Lehel, and Shelp [2]. It was further studied by Faudree, Gyárfás, and Lehel [3, 4, 5] and by Kubicka, Kubicki and Lehel [7]. Terminal-pairable networks can be defined as follows: given a simple undirected graph $G = (V(G), E(G))$ and an undirected multigraph $D = (V(D), E(D))$ on the same vertex set $(V(D) = V(G))$ we say that $G$ can realize the edges $e_1, \ldots , e_{|E(D)|}$ of $D$ if there exist edge disjoint paths $P_1, \ldots , P_{|E(D)|}$ in $G$ such that $P_i$ joins that endpoints of $e_i$, $i = 1, 2, \ldots , |E(D)|$. We call $D$ and its edges the demand graph and the demand edges of $G$, respectively. Given $G$ and a family $\mathcal{F}$ of (demand)graphs defined on $V(G)$ we call $G$ terminal-pairable with respect to $\mathcal{F}$ if every demand graph in $\mathcal{F}$ can be realized in $G$.

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Given a simple graph $G$, one central question in the topic of terminal-pairability concerns the maximum value of $\Delta$ for which any demand graph $D$ with maximum degree $\Delta(D) \leq \Delta$ can be realized in $G$. Csaba, Faudree, Gyárfás, Lehel, and Shelp [2] studied the above extremal value for complete graphs. Let $K^q_n$ denote the set of demand multigraphs with maximum degree at most $q$ on a complete graph on $n$ vertices. One can easily verify that if $K_n$ is terminal-pairable with respect to $K^q_n$ then $q$ cannot exceed $n/2$. Indeed, consider the demand graph $D$ obtained by replacing every edge in a one-factor by $q$ parallel edges. In order to create edge-disjoint paths routing the endpoints of the demand edges, one needs to use at least two edges in $K_n$, for most of the demand edges. Thus a rather short calculation implies the indicated upper bound on $\Delta(D)$.

The same authors proved in [2] that $K^q_n$ can be realized in $K_n$ if $q \leq \frac{n}{4+2\sqrt{3}}$, and conjectured that if $n \equiv 2 \pmod{4}$, then the upper bound of $n/2$ is attainable, that is, $K_n$ is terminal-pairable with respect to $K^{n/2}_n$. This conjecture is also stated in [3]. In this note, we disprove this conjecture by showing that $q/n$ is asymptotically bounded away from $1/2$.

**Theorem 1.1.** If $K_n$ is terminal-pairable with respect to $K^q_n$, then $q \leq \frac{13}{27}n + O(1)$.

We do not know if the newly established upper bound is asymptotically sharp. To this date the best known lower bound on $q$ is $\frac{n}{3} - O(1)$ (see [6]).

## 2 Proof of Theorem 1.1

We may assume $n$ is divisible by 3. Let $q$ be an even integer. We shall construct a demand graph on $n$ vertices by partitioning the vertex set of $K_n$ into triples, each one forming a triangle where every edge has multiplicity $\frac{q}{2}$. Assume that there exists an edge-disjoint path system $P$ in $K_n$ that satisfies this demand graph. Note that $e(D) = \frac{3nq}{2}$ and at most $n$ demand edges can be realized using exactly one edge of $K_n$ or using 2 edges within its triple, thus at least $\frac{3nq}{2} - n$ demand edges correspond to path of length 2 or more in $P$. In particular, if $t$ denotes the number of paths of length 2 in $P$, then the following condition holds due to simple edge counting:

$$2t + 3\left(\frac{nq}{2} - n - t\right) \leq \frac{n(n-1)}{2},$$

that is, $t \geq \frac{n}{2}(3q - n - 5)$. Hence, if $q$ is sufficiently large (in terms of $n$), then lots of demand edges must be realized through paths of length 2 ("cherries").
For a triangle $T_i$, let $\alpha_i$ denote the number of demand edges not in $T_i$ that are resolved in a cherry through any vertex of $T_i$. Also, let $\beta_i$ be the number of demand edges of $T_i$ that are resolved via a cherry with its middle vertex lying outside of $T_i$. Observe that by simple double-counting

$$\sum_{i=1}^{n} \alpha_i + \beta_i = 2t \geq n(3q - n - 5)$$

and therefore there must exist a triangle $T_i$ with $\alpha_i + \beta_i \geq 3(3q - n - 5)$.

Note that between two distinct triangles at most 4 edges can be solved via paths of length 2 (every cherry requires two edges between the triangles in $K_n$ and we only have 9 of them). This implies that between $T_i$ and any other triangle at most $4 \cdot (\frac{n}{3} - 1)$ demand edges can be solved via cherries. Hence, $4 \cdot (\frac{n}{3} - 1) \geq 3(3q-n-5)$ which implies $q \leq \frac{13}{27}n+1$ as desired.

3 Additional Remarks

Terminal-pairability of non-complete graphs has been recently studied by Colucci et al. [1]. The authors investigated the extremal value of $q$ for which the set $K_{n,n}^q$ consisting of every demand multigraph on the complete balanced bipartite graph $K_{n,n}$ with maximum degree at most $q$, can be realized. Note that in this variant of the problem the demand graph does not have to be bipartite. It was shown in [1] that $K_{n,n}^q$ can be realized in $K_{n,n}$ as long as $q \leq (1 - o(1))\frac{n}{4}$. On the other hand, $q$ has to be smaller than $(1 + o(1))\frac{n}{3}$ since any one-factor in which every edge has multiplicity $n/3$ is not realizable in $K_{n,n}$. However, we believe that the one-factor construction does not yield a sharp bound:

**Conjecture 3.1.** There exists $\epsilon > 0$ such that if $K_{n,n}^q$ can be realized in $K_{n,n}$, then $q < (\frac{1}{3} - \epsilon)n$, for every $n$ sufficiently large.

References

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