SPECTRAL ASYMPTOTICS FOR KREIN-FELLER-OPERATORS WITH RESPECT TO RANDOM RECURSIVE CANTOR MEASURES

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Abstract. We study the limit behavior of the Dirichlet and Neumann eigenvalue counting function of generalized second order differential operators $d dµ \frac{d}{dx}$, where $µ$ is a finite atomless Borel measure on some compact interval $[a, b]$. We firstly recall the results of the spectral asymptotics for these operators received so far. Afterwards, we give the spectral asymptotics for so called random recursive Cantor measures. Finally, we compare the results for random recursive and random homogeneous Cantor measures.

1. Introduction

It is well known that $f \in C^0([a, b], \mathbb{R})$ possesses a $L_2$ weak derivative $g \in L_2(\lambda^1, [a, b])$, where $\lambda^1$ denotes the one dimensional Lebesgue measure, if and only if

$$f(x) = f(a) + \int_a^x g(y) \, dy.$$ 

Replacing the one dimensional Lebesgue measure by some measure $\mu$ leads to a generalized $L_2$ weak derivative depending on the measure $\mu$. Therefore, we let $\mu$ be a finite non-atomic Borel measure on some interval $[a, b], -\infty < a < b < \infty$. The $\mu$-derivative of $f : [a, b] \rightarrow \mathbb{R}$ for which $f^\mu \in L_2(\mu)$ exists such that

$$f(x) = f(a) + \int_a^x f^\mu(y) \, d\mu(y) \quad \text{for all } x \in [a, b]$$

is defined as the unique equivalence class of $f^\mu$ in $L_2(\mu)$. We denote this equivalence class by $\frac{df}{d\mu}$.

The Krein-Feller-operator $\frac{d}{d\mu} \frac{d}{dx} f$ is than given as the $\mu$-derivative of the $\lambda^1_{[a, b]}$-derivative of $f$.

This operator were introduced for example in [12], [15], [16], [17], [18] investgate on properties of the generated stochastic process, called quasi or gap diffusion, and related objects.

As in e.g. [1], [9], we are interested in the spectral asymptotics for generalized second order differential operators $\frac{d}{d\mu} \frac{d}{dx}$ with Dirichlet or Neumann boundary conditions, i.e. we study the equation

$$\frac{d}{d\mu} \frac{d}{dx} f = -\lambda f \quad (1)$$

with

$$f(a) = f(b) = 0 \quad \text{or} \quad f'(a) = f'(b) = 0.$$ 

For a physical motivation, we consider a flexible string which is clamped between two points $a$ and $b$. If we deflect the string, a tension force drives the string back towards its state of equilibrium. Mathematically, the deviation of the string is described by some solution $u$ of the one dimensional wave equation

$$\rho(x) \frac{\partial^2 u(t, x)}{\partial t^2} = \frac{\partial^2 u(t, x)}{\partial x^2}, \quad x \in [a, b], \ t \in [0, \infty)$$

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with Dirichlet boundary condition \( u(t,a) = u(t,b) = 0 \) for all \( t \). Hereby, \( \rho \) is given as the density of the mass distribution of the string and \( F \) as the tangential acting tension force. To solve this equation, we make the ansatz \( u(t,x) = \psi(t) \phi(x) \) and receive

\[
\frac{\psi''(t)}{F \psi(t)} = \frac{\phi''(x)}{\phi(x) \rho(x)} = -\lambda,
\]

for some constant \( \lambda \in \mathbb{R} \). In the following, we only consider the equation

\[
\frac{\phi''(x)}{\phi(x) \rho(x)} = -\lambda.
\]

Thus, we have

\[
\phi'(t) - \phi'(a) = -\lambda \int_a^t \phi(y) \, d\mu(y),
\]

where \( \mu \) is the mass distribution of the string. In other words,

\[
\frac{d}{d\mu} \frac{d}{dx} \phi = -\lambda \phi. \tag{2}
\]

This equation no longer involves the density \( \rho \), meaning that we can reformulate the problem for singular measures \( \mu \). Such a solution \( \phi \) can be regarded as the shape of the string at some fixed time \( t \). Up to a multiplicative constant, the natural frequencies of the string are given as the square root of the eigenvalues of (2).

In Freiberg [5] analytic properties of this operator are developed. There, it is shown that \( -\frac{d}{dx} \frac{d}{dx} \) with Dirichlet or Neumann boundary conditions has a pure point spectrum and no finite accumulation points. Moreover, the eigenvalues are non-negative and have finite multiplicity.

We denote the sequence of Dirichlet eigenvalues of \( -\frac{d}{dx} \frac{d}{dx} \) by \( (\lambda_{D,n}^\mu)_{n \in \mathbb{N}} \) and the sequence of Neumann eigenvalues by \( (\lambda_{N,n}^\mu)_{n \in \mathbb{N}_0} \), where we assert the eigenvalues ascending and count them according to multiplicities. Let

\[
N_D^\mu(x) := \# \left\{ i \in \mathbb{N} : \lambda_{D,i}^\mu \leq x \right\} \quad \text{and} \quad N_N^\mu(x) := \# \left\{ i \in \mathbb{N}_0 : \lambda_{N,i}^\mu \leq x \right\}.
\]

\( N_D^\mu \) and \( N_N^\mu \) are called the Dirichlet and Neumann eigenvalue counting function of \( -\frac{d}{dx} \frac{d}{dx} \), respectively. The problem of determining \( \gamma > 0 \) such that

\[
N_D^\mu_{D/N}(x) \asymp x^\gamma, \quad x \to \infty, \tag{3}
\]

is an extension of the analogous problem for the one dimensional Laplacian. The following theorem is a well-known result of Weyl [21].

**Theorem 1.1:** Let \( \Omega \subseteq \mathbb{R}^n \) be a domain with smooth boundary \( \partial \Omega \). Consider the eigenvalue problem

\[
\begin{cases}
-\Delta_{n,\Omega} u = \lambda u \quad &\text{on} \quad \Omega, \\
u_{|\partial \Omega} = 0,
\end{cases}
\]

where \( \Delta_{n,\Omega} \) denotes the Laplace operator on \( \Omega \). Then, for the Dirichlet eigenvalue counting function \( N_D^{(n,\Omega)} \) of \( \Delta_{n,\Omega} \) it holds that

\[
N_D^{(n,\Omega)}(x) = (2\pi)^{-n} c_n \ vol_n(\Omega) x^{n/2} + o \left( x^{n/2} \right), \quad x \to \infty, \tag{4}
\]

hereby \( c_n \) denotes the volume of the \( n \)-dimensional unit ball.
Choosing \( \mu = \lambda_{|[a,b]} \) leads to
\[
N_D^{\mu}(x) = N^{(1,(a,b))}_D(x) \approx x^{1/2}, \quad x \to \infty,
\]
which gives the leading order term in the Weyl asymptotics as in Theorem 1.1. (4) motivates the definition of the spectral dimension
\[
d_s(\Omega) = \frac{1}{2} \lim_{\lambda \to \infty} \frac{\log N_D^{(n,\Omega)}(\lambda)}{\log \lambda}.
\]
Which leads to
\[
d_s(\Omega) = n
\]
in Theorem 1.1. Many authors before studied the expression (5) for generalized Laplacians on p.c.f. fractals, e.g. [8], [10], [14]. In this paper, we investigate on this expression for the Krein-Feller-operator on so called random recursive Cantor sets. Therefore, we call the limit
\[
\gamma := \gamma(\mu) := \lim_{\lambda \to \infty} \frac{\log N_D^{\mu}(\lambda)}{\log \lambda}
\]
the spectral exponent of the corresponding Krein-Feller-operator.

The spectral asymptotics for Krein-Feller-operators with respect to self similar measures was developed by Fujita [9], more general by Freiberg [7] and with respect to random (and deterministic) homogeneous Cantor measures by Arzt [1].

We give an example of a random recursive Cantor set and a corresponding random recursive Cantor measure. In Section 4.1 we define the general class. The fractal is constructed as follows: we subdivide the unit interval with probability \( p \) into three intervals with equal lengths, where we remove the open middle third interval and with probability \( 1 - p \) into five intervals with equal lengths, where we remove the open second and fourth interval. In the next step, we subdivide the remaining intervals independent from each other likewise and continue the procedure. The fractal under consideration is the limiting set, called random \( \frac{1}{3} \cdot \frac{1}{5} \)-recursive Cantor set.

Afterwards, we construct probability measures \( \mu_n, n \in \mathbb{N} \) such that \( \mu_n \) is a weighted Lebesgue measure those support is given by the \( n \)-th approximation step of the random \( \frac{1}{3} \cdot \frac{1}{5} \)-recursive Cantor set. To this end, let \( m^{(j)} = \left( m_1^{(j)}, \ldots, m_{N_j}^{(j)} \right), j = 1, 2, N_1 = 2, N_2 = 3 \) be vectors of weights, i.e. \( \sum_{i=1}^{N_j} m_i^{(j)} = 1, m_i \in (0,1), i = 1, \ldots, N_j, j = 1, 2, \mu_1 \) weights the left remaining interval by \( m_1^{(1)} \) and the right by \( m_2^{(1)} \), if we subdivided the unit interval into three parts, else it weights the left interval by \( m_1^{(2)} \), the middle interval by \( m_2^{(2)} \) and the right by \( m_3^{(2)} \). \( \mu_2 \) weights an interval by the weight of the predecessor interval multiplied by the weight according to the procedure for \( n = 1 \). Recursively, we continue this construction.
A random recursive Cantor measure $\mu(\frac{1}{3}, \frac{1}{5})$ corresponding to the $\frac{1}{3}$-$\frac{1}{5}$-recursive Cantor set is given as the weak limit of the sequence $(\mu_n)_{n \in \mathbb{N}}$.

It turns out that under some regularity conditions for the solution $\gamma > 0$ of

$$E \left( \sum_{i=1}^{N_{U_0}} \left( r_i^{(U_0)} m_i^{(U_0)} \right)^{\gamma} \right) = 1,$$

there exists a constant $C > 0$ and a random variable $W > 0$ a.s., $E W = 1$ such that

$$N_{D/N}(x)^{x^{-\gamma}} \rightarrow C W \text{ a.s.} \quad (6)$$

or there exists a deterministic periodic function $G$ such that

$$N_{D/N}^\mu(x) = (G(\log(x)) + o(1)) x^{\gamma} W \text{ a.s.,} \quad (7)$$

where $\mu$ is a random recursive Cantor measure. Hereby $U_0$ is the unique ancestor of the underlying random tree, $N_{U_0}$ is the corresponding number of self similarities, $r_i^{(U_0)}$ are the corresponding scale factors and $m_i^{(U_0)}$ are the entries of the corresponding vector of weights.

Since the eigenvalue counting functions are branching processes, they fulfill a random version of the renewal equation of [4]. The constant $C$ in (6) is given as the limit of $E \left( N_{D/N}^\mu(x)^{x^{-\gamma}} \right)$.

The random variable $W$ is the limit of the fundamental martingale of the underlying random population. The strict positivity of $W$ follows by an $x \log x$ argument, standard in branching theory.

It is an open question whether there exists a non-trivial example in (7) or not.

For the random $\frac{1}{3}$-$\frac{1}{5}$-recursive Cantor set we thus receive that either (6) or (7) is satisfied, where $\gamma > 0$ is the unique solution of

$$p \left( \left( \frac{m_1^{(1)}}{3} \right)^{\gamma} + \left( \frac{m_2^{(1)}}{3} \right)^{\gamma} \right) + (1-p) \left( \left( \frac{m_1^{(2)}}{5} \right)^{\gamma} + \left( \frac{m_2^{(2)}}{5} \right)^{\gamma} + \left( \frac{m_3^{(2)}}{5} \right)^{\gamma} \right) = 1.$$
2. Preliminaries

2.1. Definition of the Krein-Feller-Operator. Let \( \mu \) be a finite non-atomic Borel measure on \([a, b], -\infty < a < b < \infty\) and

\[
D^\mu_1 := \left\{ f : [a, b] \to \mathbb{R} : \exists f^\mu \in L_2(\mu) : \right. \\
\left. f(x) = f(a) + \int_a^x f^\mu(y) \, d\mu(y), \quad x \in [a, b] \right\}.
\]

The \( \mu \)-derivative of \( f \) is defined as the equivalence class of \( f^\mu \) in \( L_2(\mu) \). It is known (see [5, Corollary 6.4]) that this equivalence class is unique. Thus, the operator

\[
\frac{d}{d\mu} : D^\mu_1 \to L_2(\mu), \\
\frac{df}{d\mu} \mapsto [f^\mu]_{\sim}\mu
\]

is well-defined. Let

\[
D := D^\mu_2 := \left\{ f \in C^1((a, b)) \cap C^0([a, b]) : \exists (f')^\mu \in L_2(\mu) : \right. \\
\left. f'(x) = f'(0) + \int_a^x (f')^\mu(y) \, d\mu(y), \quad x \in [a, b] \right\}.
\]

The Krein-Feller-operator w.r.t. \( \mu \) is given as

\[
\frac{d}{d\mu} \frac{d}{dx} : D \to L_2(\mu), \\
\frac{df}{d\mu} \frac{d}{dx} \mapsto [(f')^\mu]_{\sim}\mu.
\]

2.2. Spectral Asymptotics for Self-Similar and Random Homogeneous Cantor Measures. As mentioned in the introduction, the spectral asymptotics for Krein-Feller-operators were discovered by [9] and [1] for special types of measures. In this section we summarize some main results. Firstly, we consider self-similar measures, treated in [9]. Therefore, let \( \mathcal{S} = \{S_1, \ldots, S_N\}, N \geq 2 \) be an iterated function system given by

\[S_i(x) = r_i x + c_i, \quad x \in [a, b],\]

whereby \( r_i \in (0, 1), c_i \in \mathbb{R} \) are constants such that the open set condition is satisfied, \( S_i[a, b] \subset [a, b] \) for all \( i \) and let \( m = (m_1, \ldots, m_N) \) be a vector of weights. As shown in [11], there exists a unique non-empty compact set \( C = C(\mathcal{S}) \subset [a, b] \) such that \( \bigcup_{i=1}^N S_i(C) = C \) and a unique Borel probability measure \( \mu = \mu(\mathcal{S}, m) \) such that \( \mu = \sum_{i=1}^N m_i \mu \circ S_i^{-1}. \) Moreover it holds supp \( \mu = C \).

We call \( C \) self-similar w.r.t. \( \mathcal{S} \) and \( \mu \) self-similar w.r.t. \( \mathcal{S} \) and \( m \). The Hausdorff dimension of \( C \) is given by the unique solution \( d \in [0, 1] \) of \( \sum_{i=1}^N r_i^d = 1 \) and it holds \( \mathcal{H}^d(C) \in (0, \infty). \) Moreover, if \( m_i = r_i^d \) for all \( i \), we have \( \mu = \mathcal{H}^d(C)^{-1} \mathcal{H}^d_{\mathcal{S}}. \) In this setting, the spectral exponent of the corresponding Krein-Feller-operator is the unique solution \( \gamma > 0 \) of \( \sum_{i=1}^N (m_i r_i)^\gamma = 1. \) For references see [9, Theorem 3.6] and [7, Theorem 4.1].

In the following, we want to relax the self similarity of the set \( C \) and the measure \( \mu. \) To this end, we take an index set \( J \) and define to each \( j \in J \) an IFS \( \mathcal{S}^{(j)} = \{S_1^{(j)}, \ldots, S_N^{(j)}\}. \) Then, we choose randomly \( j_0 \in J \) (according to some probability distribution on \( J \)) and take the image of \([a, b]\) under \( S^{(j_0)}). \) Next, we choose randomly \( j_1 \in J \) (according to the same probability distribution) and take the image of \( S_1^{(j_0)}[a, b], \ldots, S_N^{(j_0)}[a, b] \) under \( S^{(j_1)}. \) The limit of this construction is the fractal under consideration. More precise, let \( J \) be a non-empty countable set. To each \( j \in J \) let \( \mathcal{S}^{(j)} = \{S_1^{(j)}, \ldots, S_N^{(j)}\}, N_j \in \mathbb{N} \) be such that

\[S_i^{(j)}(x) = r_i^{(j)} x + c_i^{(j)}, \quad x \in [a, b], \quad i = 1, \ldots, N_j,\]
where the constants \( r^{(j)}_i \in (0, 1), c^{(j)}_i \in \mathbb{R} \) are chosen such that

\[
a = S_1^{(j)}(a) < S_2^{(j)}(b) \leq \cdots < S_{N_j}^{(j)}(b) = b. \tag{8}
\]

Further, we call \( \xi = (\xi_1, \xi_2, \ldots), \xi_i \in J \) an environment sequence and define

\[
W_n := \{1, \ldots, N_{\xi_1}\} \times \{1, \ldots, N_{\xi_2}\} \times \cdots \times \{1, \ldots, N_{\xi_n}\}, \quad n \in \mathbb{N}.
\]

The homogeneous Cantor set to a given environment sequence \( \xi \) is

\[
K^{(\xi)} := \bigcap_{n=1}^{\infty} \bigcup_{w \in W_n} \left(S^{(\xi_1)}_{w_1} \circ S^{(\xi_2)}_{w_2} \circ \cdots \circ S^{(\xi_n)}_{w_n}\right)([a, b]).
\]

Next, we define a measure \( \mu^{(\xi)} \) on \([a, b]\) to a given environment sequence \( \xi \), which generalizes the invariant measures, presented before. To this end, let \( m^{(j)} = (m^{(j)}_1, \ldots, m^{(j)}_{N_j}), j \in J \) be a vector of weights. \( \mu^{(\xi)} \) is defined as the week limit of the sequence of Borel probability measures \( \left( \mu^{(\xi)}_n \right)_{n \in \mathbb{N}} \):

\[
\mu^{(\xi)}_n := \sum_{w \in W_n} m^{(\xi_1)}_{w_1} \cdots m^{(\xi_n)}_{w_n} \mu_0 \circ \left(S^{(\xi_1)}_{w_1} \circ \cdots \circ S^{(\xi_n)}_{w_n}\right)^{-1}, \quad \mu_0 := \frac{1}{b-a} \lambda_{[a,b]}.
\]

\( \mu^{(\xi)} \) is called homogeneous Cantor measure, corresponding to \( K^{(\xi)} \). If \(|J| = 1\), then the definition of invariant sets and measures coincide with \( K^{(\xi)} \) and \( \mu^{(\xi)} \).

[1, Theorem 3.3.10] makes a statement about the spectral exponent of the Krein-Feller-operator with respect to \( \mu^{(\xi)} \), where \( \xi \) is a deterministic environment sequence. Here, we only consider the random case. Therefore, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \( \xi = (\xi_1, \xi_2, \ldots) \) a sequence of i.i.d. \( J \)-valued random variables with \( p_j := \mathbb{P}(\xi = j) \). We denote the Dirichlet and Neumann eigenvalue counting function of the Krein-Feller-operator w.r.t. \( \mu^{(\xi)} \) by \( N^{(\xi)}_D \) and \( N^{(\xi)}_N \), respectively. Further, if \(|J| = \infty\), we need the following five technical assumptions:

\[
\sup_{j \in J} N_j < \infty, \tag{A1}
\]

\[
\inf_{j \in J} \min_{i=1, \ldots, N_j} r^{(j)}_i m^{(j)}_i > 0, \tag{A2}
\]

\[
\sup_{j \in J} \max_{i=1, \ldots, N_j} r^{(j)}_i m^{(j)}_i < 1, \tag{A3}
\]

\[
\prod_{j \in J} \sum_{i=1, \Sigma_{j=1}^{N_j} (r^{(j)}_i m^{(j)}_i) < 1}^{N_j} r^{(j)}_i m^{(j)}_i > 0, \tag{A4}
\]

\[
\prod_{j \in J} \sum_{i=1, \Sigma_{j=1}^{N_j} (r^{(j)}_i m^{(j)}_i) > 1}^{N_j} r^{(j)}_i m^{(j)}_i < \infty. \tag{A5}
\]

Under these assumptions, we obtain:

**Theorem 2.1 ([1], Corollary 3.5.1):** Let \( \gamma_h > 0 \) be the unique solution of

\[
\prod_{j \in J} \left( \sum_{i=1}^{N_j} \left( r^{(j)}_i m^{(j)}_i \right)^{\gamma_h} \right)^{p_j} = 1.
\]

Then, there exist \( C_1, C_2 > 0, x_0 > 0 \) and \( c_1(w), c_2(w) > 0 \) such that

\[
C_1 x^{\gamma_h} e^{-c_1(w) \sqrt{\log x \log \log x}} \leq N^{(\xi(w))}_D(x) \leq N^{(\xi(w))}_N(x) \leq C_2 x^{\gamma_h} e^{-c_2(w) \sqrt{\log x \log \log x}}
\]

for all \( x > x_0 \) almost surely.
3. C-M-J Branching Processes

By the construction of random recursive Cantor sets, there is a natural relation to random labelled trees. We will be able to write the eigenvalue counting function as a sum over each node of the tree, counted by some random characteristic which leads to C-M-J branching processes. This method was also used in [10]. Nerman [20] used renewal theory, based on [4], for some convergence results for C-M-J branching processes. These results can then be used to determine the asymptotic behaviour of the eigenvalue counting functions.

A C-M-J branching process is a stochastic process which counts individuals of a population according to some (maybe random) function \( \phi \). We assume that the considered population has a unique ancestor, denoted by \( \theta \). We say \( i = (i_1, ..., i_n) \) belongs to the \( n \)-th generation of the population, if the individual \( i \) is the \( i_n \)-th child of the \( i_{n-1} \)-th child of the ... of the \( i_1 \)-th child of the ancestor \( \theta \). Since a mother can give birth to a child, we say \( i \) is the mother of \( i, \) if \( \tilde{i} = (i_1, ..., i_{n-1}) \).

The generation of \( i \) is given by \( \{ i \} \). Each individual has a reproduction rate, described by a random point process \( \xi_i \) on \([0, \infty)\), i.e. an individual reproduces at time \( t \) according to \( \xi_i(t) \), for \( t \in [0, \infty) \), whereby \( \xi_i(t) \) denotes the \( \xi_i \) measure of \([0, t]\). The birth time of \( i \) is denoted by \( \sigma_i \) and is given as

\[
\begin{align*}
\sigma_\theta &= 0, \\
\sigma_i &= \sigma_{\tilde{i}} + \inf \{ u \geq 0 : \xi_i(u) \geq i_n \}.
\end{align*}
\]

Every individual has a life time \( L \). Therefore, it lives in the interval \([\sigma_i, L + \sigma_i]\) and dies at time \( L + \sigma_i \). We define the tuple \( (\xi, L, \phi) \) on some probability space \((\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})\). We call \((\xi_x, L_x, \phi_x) \) a general branching process. Let

\[
\mathcal{G}_n := \{(i_1, ..., i_n) : i_j \in \mathbb{N}, \ j = 1, ..., n\},
\]

\[
\mathcal{G} := \{0\} \cup \left( \bigcup_{n=1}^{\infty} \mathcal{G}_n \right).
\]

The probability space on which we define the C-M-J branching processes is the product space

\[
(\Omega, \mathcal{B}, \mathbb{P}) = \prod_{i \in \mathcal{G}} (\Omega_i, \mathcal{B}_i, \mathbb{P}_i),
\]

where \((\Omega_i, \mathcal{B}_i, \mathbb{P}_i)\) are copies of \((\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{\mathbb{P}})\) and contain independent copies \((\xi_i, L_i, \phi_i)\) of \((\xi, L, \phi)\). Thereby, we assume that \( \phi : \Omega \times \mathbb{R} \rightarrow [0, \infty) \) is a product measurable, separable càdlàg function on \( \mathbb{R} \). The C-M-J branching process to a given general branching process \((\xi_x, L_x, \phi_x) \) is defined by

\[
Z^\phi_t := \sum_{i \in \Sigma} \phi_i(t - \sigma_i),
\]

where \( \Sigma \) is the trace of the underlying Galton-Watson process and \( \phi_i(t) = 0 \) for \( t < 0 \). The interpretation of the process \( Z^\phi \) depends on the random characteristic \( \phi \). For \( \phi \equiv 1, \) \( Z^\phi \) describes the total number of individuals born up to and including time \( t \). In this case, we set \( T_t := Z^\phi_t \).

Further, we define \( \nu(t) := \nu([0, t]) := \mathbb{E}(\xi(t)) \) and we require that the following two properties hold:

1. There exists an \( \alpha > 0 \) such that

\[
\int_0^\infty e^{-\alpha t} \, d\nu(t) = 1.
\]

This parameter \( \alpha \) is called Malthusian parameter of the process.

2. For the Malthusian parameter \( \alpha \) holds

\[
\int_0^\infty u e^{-\alpha u} \, d\nu(u) < \infty.
\]
The following representation of $Z^\phi$ is useful for our consideration (see [13]):

$$Z^\phi_t = \phi_0(t) + \sum_{i=1}^{\xi_1(t)} Z^\phi_{t-\sigma_i}, \quad t \in [0, \infty),$$

(10)

where $\left(\{1\} Z^\phi_t\right)_t$, $i = 1, ..., \xi_0(\infty)$ are i.i.d., distributed like $\left(\{2\} Z^\phi_t\right)_t$. Also, $\left(\{i\} Z^\phi_t\right)_t$ is independent of $\xi_0$. If there will be no confusion, we will suppress the $i$ in $\phi_i$, $L_i$, etc. Further, we write $\xi(\infty)$, if we mean $\xi([0, \infty))$ and analogously for the other measures. The type of branching processes we consider is called supercritical, i.e. $\nu(\infty) > 1$. In this case the extinction probability is strictly less than 1 (see e.g. [13, Theorem 2.3.1]). In our consideration each individual will have at least two offsprings and therefore the extinction probability is 0. By $\xi_\alpha$ we denote the Laplace-Stieltjes transformation with respect to $\alpha$ of $\xi$ and by $\nu_\alpha$ its expectation, i.e.

$$\xi_\alpha(t) = \int_0^t e^{-\alpha s} d\xi(s),$$

$$\nu_\alpha(t) = E(\xi_\alpha(t)).$$

In the following we order the individuals according to their birth times, that is, if $i$ is the $n$-th individual of the population and $\sigma_i < \sigma_{(i,i)}$

for some $i \in N$ and there exists no individual $j$ such that

$$\sigma_i < \sigma_j < \sigma_{(i,i)},$$

then $(i, i)$ is the $(n+1)$-th individual. If we have several births at the same time, we sort them according to an arbitrary rule. We write $i_n$ for the $n$-th individual of the population.

For our main result, we need to introduce a random variable $W$ which is the almost sure limit of a martingale $(R_n)_{n \in N}$. Therefore, we define a filtration $(A_n)_{n \in N}$ on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ as follows: For $j \in \mathcal{G}$ let $P_j$ be the projection of $(\Omega, \mathcal{B})$ onto $(\Omega_j, \mathcal{B}_j)$. Then, $A_n$ is defined as the smallest $\sigma$-algebra (on $\Omega$) such that

$$\{\omega \in \Omega : i_{(1)}(\omega) = j_1, ..., i_{(n)}(\omega) = j_n\} \in A_n$$

and

$$A \cap \{\omega \in \Omega : j \in \{i_{(1)}(\omega), ..., i_{(n)}(\omega)\}\} \in A_n \quad \text{for all} A \in P^{-1}_j(B), \quad \text{for all} j \in \mathcal{G}.$$

We interpret $A_n$ as the biography of the first $n$ individuals. By construction $\sigma_{i_n}$ is $A_{n-1}$ measurable. Further, we have that $\phi_1 Z^\phi_t$ and $\xi_{i_n}$ are independent of $A_n$ for all $k > n$, $t \in \mathbb{R}$.

We remark that analogous results hold for $A_{T_t}$ (for individuals born after time $t$ such that their parents are born before or at time $t$), where $T_t$ is a stopping time with respect to the constructed filtration for fixed $t$. Let $\mathcal{H}(n)$ be the set of the first $n$ individuals of the population and

$$R_0 := 1,$$

$$R_n := 1 + \sum_{i \in \mathcal{H}(n)} \sum_{i=1}^{\xi_i(\infty)} e^{-\alpha \sigma_i} - \sum_{i \in \mathcal{H}(n)} e^{-\alpha \sigma_i}, \quad n \in \mathbb{N}.$$

**Theorem 3.1:** The process $(R_n)_{n \in \mathbb{N}}$ is a non-negative martingale with respect to $(A_n)_{n \in \mathbb{N}}$. Furthermore, there exists a random variable $W$ such that

$$R_n \xrightarrow{n \to \infty} W \quad \text{a.s.}$$

If

$$\mathbb{E}(\xi_{\alpha}(\infty) \log^+ \xi_{\alpha}(\infty)) < \infty,$$

then $W > 0$ a.s., otherwise $W = 0$ a.s.
Proof. [2, Theorem 4.1]

The case where $\phi_i$ depends on the whole line of descendants is discussed in [20, Chapter 7]. There, it is shown that Theorem 3.1 also holds.

We need a strong law of large numbers for C-M-J branching processes. For reference see [3]. For this strong law, the branching process has to satisfy the following two conditions.

**Condition 3.2:** There exists a non-increasing bounded positive integrable càdlàg function $g$ on $[0, \infty)$ such that

$$\mathbb{E} \left( \sup_{t \geq 0} \frac{\xi_\alpha(\infty) - \xi_\alpha(t)}{g(t)} \right) < \infty.$$  

**Condition 3.3:** There exists a non-increasing bounded positive integrable càdlàg function $h$ on $[0, \infty)$ such that

$$\mathbb{E} \left( \sup_{t \geq 0} \frac{e^{-\alpha t} \phi(t)}{h(t)} \right) < \infty.$$  

**Theorem 3.4 (strong law of large numbers):** Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter $\alpha$, where $\phi \geq 0$ and $\phi(t) = 0$ for $t < 0$. Then,

1. If $\nu_\alpha$ is non-lattice,

$$e^{-\alpha t} Z^{\phi}(t) \to \nu_\alpha(\infty) W \quad \text{a.s.}$$

2. If $\nu_\alpha$ is lattice with span $T$, there exists a periodic function $G$ with period $T$ such that

$$Z^{\phi}(t) = (G(t) + o(1)) e^{\alpha t} W \quad \text{a.s.}$$

$G$ is given as

$$G(t) = T \cdot \frac{\sum_{j=-\infty}^{\infty} e^{-\alpha(t+jT)} \phi(t+jT)}{\int_0^\infty t e^{-\alpha t} d\nu(t)}.$$  

4. Spectral Asymptotics for General Recursive Cantor Measures

4.1. Construction of General Recursive Cantor Measures. Let $J$ be a (possibly uncountable) index set. We define to each $j \in J$ an IFS $S^{(j)}$. Therefore, let $N_j \in \mathbb{N}, N_j \geq 2$. Then $S^{(j)} = (S_{i}^{(j)}, \ldots, S_{N_j}^{(j)})$, where we define $S_{i}^{(j)} : [a, b] \to [a, b]$ by

$$S_{i}^{(j)}(x) := r_{i}^{(j)} x + c_{i}^{(j)},$$

for some $r_{i}^{(j)} \in (0, 1), c_{i}^{(j)} \in \mathbb{R}, i = 1, \ldots, N_j$ such that

$$a = S_{1}^{(j)}(a) < S_{2}^{(j)}(a) \leq S_{2}^{(j)}(b) \leq \cdots \leq S_{N_j}^{(j)}(a) < S_{N_j}^{(j)}(b) = b.$$  

Furthermore, let $m^{(j)} = (m_{1}^{(j)}, \ldots, m_{N_j}^{(j)})$ be a vector of weights and thus, as in Chapter 2.2, an element of the index set $J$ identifies a tuple $(S^{(j)}, m^{(j)})$.

As in Chapter 3, we construct a population $I$ with unique ancestor, denoted by $\emptyset$. Every individual $i \in I$ identifies an element of $J$ which we also denote by $i$. The number of children of $i$ is $N_i$. For $i, j \in \mathcal{G}, i = (i_1, \ldots, i_m), j = (j_1, \ldots, j_n)$ we define $ij := (i_1, \ldots, i_m, j_1, \ldots, j_n)$ and, if $m > n$, $j|_m := (j_1, \ldots, j_n)$. Let $I_n$ be the $n$-th generation of $I$.  


For \( i \in I_n, i = (i_1, \ldots, i_n) \), we define
\[
m_i := m_i^{(\emptyset)} \cdots m_i^{(i_1, \ldots, i_{n-1})},
\]
\[
S_i := S_i^{(\emptyset)} \circ \cdots \circ S_i^{(i_1, \ldots, i_{n-1})}
\]
and we define analogously \( S_i^{-1} \) as the composition of the preimages of the \( S_i \).

For \( n \in \mathbb{N} \) let
\[
K_n^{(i)} := \bigcup_{i \in I_n} S_i([a, b]).
\]
The limiting set \( K^{(i)} := \bigcap_{n=1}^{\infty} K_n^{(i)} \) is called recursive Cantor set.

**Proposition 4.1:** The set \( K^{(i)} \) is compact and contains at least countably infinitely many elements, namely \( S_1(i_{n-1}) \) and \( S_1(i_1, \ldots, i_n) \), \( i_1 = 1, \ldots, N_0, \ldots, i_n = 1, \ldots, N(i_1, \ldots, i_{n-1}), n \in \mathbb{N}. \)

**Proof.** Let \( i = (i_1, \ldots, i_n) \in I_n \). For \( m \in \mathbb{N} \) let \( \bar{i} \) and \( \bar{i}' \) be two individuals of the population such that \( \bar{i} = i_1 m, m := (1, \ldots, 1) \in \mathbb{R}^m, m \in \mathbb{N} \) and \( i''_1, \ldots, i''_n = i_1, \ldots, i_n, \bar{i} = N(i_1, \ldots, i_{n-1}, N(i_1, \ldots, i_{n-1}) \) for \( k = n + 1, \ldots, n + m \). By definition, we have
\[
S_{\bar{i}}(a) = S_i(a),
\]
\[
S_{\bar{i}}(b) = S_i(b).
\]
Thus, we have \( S_i(a), S_i(b) \in K_{n+m}^{(i)} \) for all \( m \in \mathbb{N} \), which proofs the statement.

By construction, we have
\[
K^{(i)} = \bigcup_{i=1}^{N_0} S_i^{(\emptyset)} \left( K^{(\theta, i)} \right), \tag{11}
\]
where \( \theta, I \) denotes the subtree of \( I \, R \), rooted at \( (i) \).

We define the recursive Cantor measures, analogously to the homogeneous Cantor measures. Let
\[
\mu_n^{(i)}(A) := \sum_{i \in I_n} m_i \mu_0 \left( S_i^{-1}(A) \right), \quad \mu_0(A) := \frac{1}{b - a} \lambda_1^{(a, b)}(A)
\]
for all \( A \in \mathcal{B}([a, b]). \) The recursive cantor measure \( \mu^{(i)} \) to given Cantor set coded by \( I \) is defined as the weak limit of \( \left( \mu_n^{(i)} \right)_{n \in \mathbb{N}} \).

**Lemma 4.2:** For all \( i \in I \) holds
\[
\mu^{(i)}(S_i([a, b])) = m_i.
\]

**Proof.** We write \( \mu = \mu^{(i)}, \mu_n = \mu_n^{(i)}, n \in \mathbb{N}. \) Let \( K_i := S_i([a, b]) \) for \( i \in I \). Let \( i \in I_n, j \in I_{n+m}, n, m \in \mathbb{N}. \) Because of
\[
K_i \cap K_j = \begin{cases} K_j, & \text{if } j | n = i \\ \emptyset, & \text{otherwise,} \end{cases}
\]
we get
\[
\mu_{n+m}(K_i)
= \sum_{j \in I_{n+m}} m_j \mu_0 \left( S_j^{-1}(K_i) \right)
= \sum_{j \in I_{n+m} \atop j | n = i} m_j \mu_0 \left( S_j^{-1}(K_i) \right).
\]
Because of
\[
(S_{i_1,\ldots,i_n,j_{n+1},\ldots,j_{n+m}})^{-1}(K_i)
\]
\[
= \left(S_{\emptyset}^{(i_1,\ldots,i_n,j_{n+1},\ldots,j_{n+m-1})} \circ \ldots \circ S_{\emptyset}^{(i_1,\ldots,i_n,j_{n+1},\ldots,j_{n+m-1})}\right)^{-1} \circ S_{\emptyset}^{-1}(K_i)
\]
\[
= S_{\emptyset}^i((i_1,\ldots,i_n) \circ \ldots \circ S_{\emptyset}^{(i_1,\ldots,i_n,j_{n+1},\ldots,j_{n+m-1})})^{-1} ([a,b])
\]
\[
= [a, b],
\]
we get
\[
\mu_{n+m}(K_i) = \sum_{j \in [n+m]} m_j = m_i.
\]

Analogously to (11) holds
\[
\mu^{(I)} = \sum_{i=1}^{N_0} m_i^{(0)} S_i^{(0)} \mu^{(0,I)},
\]
where \( S_i^{(0)} \mu^{(I)}(A) := \mu^{(I)} \left( S_i^{(0)} \right)^{-1}(A) \), \( A \in B([a,b]) \).

**Proof.** Let \( A \in B([a,b]) \). Then, we get
\[
\sum_{i=1}^{N_0} m_i^{(0)} \mu^{(0,I)} \left( S_i^{(0)} \right)^{-1}(A)
\]
\[
= \sum_{i=1}^{N_0} \sum_{i_1=1}^{N_{i_1}} \cdots \sum_{i_n=1}^{N_{i_n}} m_i^{(0)} m_i^{(i_1)} \cdots m_i^{(i_1,\ldots,i_n)} m_0 \left( S_{i_1,\ldots,i_n}^{-1}(A) \right)
\]
\[
= \sum_{i=1}^{N_0} \cdots \sum_{i_n=1}^{N_{i_n}} m_i^{(0)} m_i^{(i_1)} \cdots m_i^{(i_1,\ldots,i_n)} m_0 \left( S_{i_1,\ldots,i_n}^{-1}(A) \right)
\]
\[
= \mu^{(I)}(A).
\]
Taking the limit, we get the assertion. \( \square \)

With (12) we get the following lemma.

**Lemma 4.3:** Let \( i \in \{1, \ldots, N_0\} \) and \( A \in B([a,b]) \) with \( A \subseteq S_i^{(0)}([a,b]) \). Then, it holds
\[
\mu^{(I)}(A) = m_i^{(0)} (S_i^{(0)} \mu^{(0,I)})(A).
\]

### 4.2. Scaling Properties

We establish a Dirichlet-Neumann-Bracketing with which we receive the characteristic \( \phi \) for the C-M-J branching process under consideration. To this end, we need some scaling properties.

#### 4.2.1 Scaling Property of the \( L_2 \)-Norm

**Lemma 4.4:** Let \( f, g \in L_2(\mu^{(I)}) \). Then,
\[
\langle f, g \rangle_{L_2(\mu^{(I)})} = \sum_{i=1}^{N_0} m_i^{(0)} \left( f \circ S_i^{(0)}, g \circ S_i^{(0)} \right)_{L_2(\mu^{(0,I)})}.
\]
Proof. We have \( \supp \mu^{(I)} = K^{(I)} \). Together with Lemma 4.3, we get

\[
\langle f, g \rangle_{L^2(\mu^{(I)})} = \int_{[a,b]} fg \, d\mu^{(I)}
\]

\[
= \sum_{i=1}^{N_\theta} \int_{[a,b]} f \circ S_i^{(0)} g \circ S_i^{(0)} \, d \left( S_i^{(0)} \mu^{(I)} \right)
\]

\[
= \sum_{i=1}^{N_\theta} m_i^{(0)} \int_{[a,b]} f \circ S_i^{(0)} g \circ S_i^{(0)} \, d\mu^{(0,i)}
\]

\[
= \sum_{i=1}^{N_\theta} m_i^{(0)} \langle f \circ S_i^{(0)}, g \circ S_i^{(0)} \rangle_{L^2(\mu^{(0,i)})}.
\]

\[\square\]

### 4.2.2 Scaling of the Eigenvalue Counting Function - Neumann Boundary Conditions.

Let \((\mathcal{E}, \mathcal{F})\) be the Dirichlet form on \(L(\mu^{(I)})\), whose eigenvalues coincide with the Neumann eigenvalues of \(-\frac{d}{d\mu^{(I)}} \frac{d}{dx}\). Namely,

\[
\mathcal{F} = H^1(\lambda),
\]

\[
\mathcal{E}(f, g) = \int_a^b f'(x) g'(x) \, dx,
\]

see [6, Proposition 5.1]. We write \(N^{(I)}_N\) for the eigenvalue counting function of \((\mathcal{E}, \mathcal{F})\), instead of \(N_{(\mathcal{E},\mathcal{F})}\). To obtain the Neumann-Dirichlet-Bracketing, we define a new Dirichlet form \((\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)})\), introduced in [1, Chapter 3]. Let \(\tilde{\mathcal{F}}^{(I)}\) be the set of all functions \(f : [a, b] \rightarrow \mathbb{R} \) with \(f \circ S_i^{(0)} \in \mathcal{F}\) for all \(i = 1, \ldots, N_\theta\) and \(f \bigl( (S_i^{(0)}(b), S_i^{(0)}(a)) \bigr) \in H^1 \left( \lambda, \left( S_i^{(0)}(b), S_i^{(0)}(a) \right) \right) \) for all \(i = 1, \ldots, N_\theta - 1\).

With [1, Proposition 3.2.1] follows \(\mathcal{F} \subseteq \tilde{\mathcal{F}}^{(I)}\), but \(\tilde{\mathcal{F}}^{(I)} \nsubseteq \mathcal{F}\), because \(f \in \tilde{\mathcal{F}}^{(I)}\) has not to be continuous on the boundary points of \(S_i^{(0)}([a, b])\), \(i = 1, \ldots, N_\theta\). For all \(f, g \in \tilde{\mathcal{F}}^{(I)}\), we define

\[
\tilde{\mathcal{E}}^{(I)}(f, g) := \sum_{i=1}^{N_\theta} \frac{1}{r_i^{(0)}} \mathcal{E} \left( f \circ S_i^{(0)}, g \circ S_i^{(0)} \right) + \sum_{i=1}^{N_\theta - 1} \int_{S_i^{(0)}(b)} f'(t) g'(t) \, dt.
\]

Due to [1, Proposition 3.2.1] we then have for all \(f, g \in \tilde{\mathcal{F}}\), \(\tilde{\mathcal{E}}^{(I)}(f, g) = \mathcal{E}(f, g)\). Further, [1, Proposition 2.2.2] implies that the embedding \(\tilde{\mathcal{F}}^{(I)} \hookrightarrow L^2(\mu^{(I)})\) is a compact operator and thus we can refer to the eigenvalue counting function of the Dirichlet form \((\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)})\). From now on we suppress the \(I\) dependence of the Dirichlet form \((\tilde{\mathcal{E}}, \tilde{\mathcal{F}}^{(I)})\).

**Proposition 4.5:** For all \(x \geq 0\) holds

\[
N_{(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})}(x) = \sum_{i=1}^{N_\theta} N^{(I)}_N \left( r_i^{(0)} m_i^{(0)} x \right).
\]

**Proof.** Let \(f\) be an eigenfunction of \((\tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \mu^{(I)})\) with eigenvalue \(\lambda\), i.e.

\[
\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L^2(\mu^{(I)})} \quad \text{for all } g \in \tilde{\mathcal{F}}.
\]
Because \( f, g \in L_2 (\mu^{(f)}) \), we have with Lemma 4.4

\[
\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(0)}} \mathcal{E} \left( f \circ S_{\emptyset}^{(0)}, g \circ S_{\emptyset}^{(0)} \right) + \sum_{i=1}^{N_\emptyset} \int_{S_{\emptyset}^{(0)}(a)}^{S_{\emptyset}^{(0)}(b)} f'(t) g'(t) \, dt
= \lambda \sum_{i=1}^{N_\emptyset} m_i^{(0)} \left( f \circ S_{\emptyset}^{(0)}, g \circ S_{\emptyset}^{(0)} \right)_{L_2 (\mu^{(\emptyset,i)})}.
\]

(13)

Now, we show that each summand on the left side equals each summand on the right side, respectively. Therefore, let \( h \in F \) and define for each \( j \in \{1,...,N_\emptyset\} \)

\[
\tilde{h}_j(x) := \begin{cases} \h \circ S_{\emptyset, i}^{(0)-1}(x), & \text{if } x \in S_{\emptyset, i}^{(0)}([a, b]), \\ 0, & \text{otherwise}. \end{cases}
\]

Obviously, we have \( \tilde{h}_j \in F, \h \circ S_{\emptyset}^{(0)} = h \), for all \( j \in \{1,...,N_\emptyset\} \) and \( \tilde{h}_j \circ S_{\emptyset}^{(0)} = 0 \) for \( i \neq j \).

Moreover, \( \tilde{h}_j |_{(S_{\emptyset}^{(0)}(b), S_{\emptyset, i}^{(0)}(a))} = 0, j = 1, ..., N_\emptyset, i = 1, ..., N_\emptyset - 1 \). With \( g = \tilde{h}_j \), we then have in (13)

\[
\frac{1}{r_j^{(0)}} \mathcal{E} \left( f \circ S_{\emptyset}^{(0)}, h \right) = \lambda m_j^{(0)} \left( f \circ S_{\emptyset}^{(0)}, \h \right)_{L_2 (\mu^{(\emptyset,i)})}.
\]

Because this equation holds for all \( h \in F, f \circ S_{\emptyset}^{(0)} \) is an eigenfunction of the Dirichletform \((\mathcal{E}, F, \mu^{(0,f)})\) with eigenvalue \( r_j^{(0)} m_j^{(0)} \lambda \) for all \( j = 1,...,N_\emptyset \).

Now, let \( \lambda > 0 \), s.t. for \( i = 1,...,N_\emptyset, r_i^{(0)} m_i^{(0)} \lambda \) is an eigenvalue of \((\mathcal{E}, F, \mu^{(0,f)})\) with eigenfunction \( f_i \), say. This means,

\[
\mathcal{E}(f_i, g) = r_i^{(0)} m_i^{(0)} \lambda \langle f_i, g \rangle_{L_2 (\mu^{(\emptyset,i)})},
\]

for all \( g \in F \). Let

\[
f(x) := \begin{cases} f_i \circ S_{\emptyset}^{(0)-1}(x), & \text{if } x \in S_{\emptyset, i}^{(0)}([a, b]) \text{ for some } i \in \{1,...,N_\emptyset\} \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( f \in F \) and \( f \circ S_{\emptyset}^{(0)} = f_i, i = 1,...,N_\emptyset \) and therefore

\[
\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(0)}} \mathcal{E} \left( f \circ S_{\emptyset}^{(0)}, g \right) = \lambda \sum_{i=1}^{N_\emptyset} m_i^{(0)} \left( f \circ S_{\emptyset}^{(0)}, g \right)_{L_2 (\mu^{(\emptyset,i)})},
\]

for all \( g \in F \). Since for \( g \in F \) we have by definition of \( F, \tilde{g} \circ S_{\emptyset}^{(0)} \in F, i = 1,...,N_\emptyset, \) we get

\[
\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(0)}} \mathcal{E}(f \circ S_{\emptyset}^{(0)}, \tilde{g} \circ S_{\emptyset}^{(0)}) = \lambda \sum_{i=1}^{N_\emptyset} m_i^{(0)} \left( f \circ S_{\emptyset}^{(0)}, \tilde{g} \circ S_{\emptyset}^{(0)} \right)_{L_2 (\mu^{(\emptyset,i)})}.
\]

But the left side of this equation is equal to \( \mathcal{E}(f, \tilde{g}) \), because \( f'|_{(S_{\emptyset}^{(0)}(b), S_{\emptyset, i}^{(0)}(a))} = 0 \), for all \( i = 1,...,N_\emptyset - 1 \). With Lemma 4.4 we then have

\[
\mathcal{E}(f, \tilde{g}) = \lambda \langle f, \tilde{g} \rangle_{L_2 (\mu^{(\emptyset,i)})},
\]

for all \( \tilde{g} \in F \). Therefore, \( \lambda \) is an eigenvalue of \((\mathcal{E}, F, \mu^{(f)})\) with corresponding eigenfunction \( f \). Using this, we can easily conclude the claim. \( \square \)
4.2.3 Scaling of the Eigenvalue Counting Function - Dirichlet Boundary Conditions.

Let \((\mathcal{F}_0, \mathcal{E})\) be the Dirichlet form on \(L_2(\mu^{(I)})\) whose eigenvalues coincide with the Dirichlet eigenvalues of \(-\frac{d}{d\mu} \frac{d}{dx}\). Meaning, \(\mathcal{E}\) is defined as before and

\[
\mathcal{F}_0 := \{ f \in \mathcal{F} : f(a) = f(b) = 0 \}.
\]

We write \(N_D\) instead of \(N(\mathcal{F}_0, \mathcal{E})\). Again, we define a new Dirichletform \((\mathcal{E}, \tilde{\mathcal{F}}_0^{(I)})\) on \(L_2(\mu^{(I)})\) and suppress the \(I\) dependence of

\[
\tilde{\mathcal{F}}_0^{(I)} := \{ f \in \mathcal{F}_0 : f(x) = 0 \text{ for } x \in \left( S_i^{(\emptyset)}(b), S_{i+1}^{(\emptyset)}(a) \right), \ i = 1, \ldots, N_0 - 1 \}.
\]

Further, we use the notation \(\mathcal{E}\) for \(\mathcal{E}|_{\mathcal{F}_0 \times \mathcal{F}_0}\).

**Proposition 4.6:** For all \(x \geq 0\) we have

\[
N_{(\mathcal{E},\mathcal{F}_0,\mu^{(I)})}(x) = \sum_{i=1}^{N_0} N_{D}^{(\emptyset, I)} \left( r_i^{(\emptyset)} m_i^{(\emptyset)} x \right).
\]

**Proof.** Let \(f\) be an eigenfunction of \((\mathcal{E}, \mathcal{F}_0, \mu^{(I)})\) with eigenvalue \(\lambda\). Then

\[
\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L^2(\mu^{(I)})},
\]

for all \(g \in \mathcal{F}_0\). Therefore, we have with [1, Proposition 3.2.1] and Lemma 4.4,

\[
\sum_{i=1}^{N_0} \frac{1}{r_i^{(\emptyset)}} \mathcal{E} \left( f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)} \right) + \sum_{i=1}^{N_0-1} \int_{S_i^{(\emptyset)}(a)}^{S_{i+1}^{(\emptyset)}(b)} f'(t) g'(t) \, dt
\]

\[
= \lambda \sum_{i=1}^{N_0} m_i^{(\emptyset)} \langle f, g \rangle_{L^2(\mu^{(\emptyset, I)})}.
\]

For \(h \in \mathcal{F}_0\) we define

\[
\tilde{h}_j(x) := \begin{cases} h \circ S_i^{(\emptyset)^{-1}}(x), & \text{if } x \in S_i^{(\emptyset)}([a, b]) \\ 0, & \text{otherwise}. \end{cases}
\]

Because \(h \in \mathcal{F}_0\), it follows \(\tilde{h}_j \in \mathcal{F}_0\) and \(\tilde{h}_j \circ S_j^{(\emptyset)} = h\) for \(j = 1, \ldots, N_0\) and \(\tilde{h}_j \circ S_j^{(\emptyset)} = 0\), if \(i \neq j\). Hence,

\[
\frac{1}{r_i^{(\emptyset)}} \mathcal{E} \left( f \circ S_j^{(\emptyset)}, h \right) = \lambda m_i^{(\emptyset)} \langle f \circ S_j^{(\emptyset)}, \tilde{h}_j \rangle_{L^2(\mu^{(\emptyset, I)})},
\]

for all \(j = 1, \ldots, N_0\). Therefore, \(\lambda r_i^{(\emptyset)} m_i^{(\emptyset)}\) is an eigenvalue of \((\mathcal{E}, \mathcal{F}_0, \mu^{(\emptyset, I)})\) with eigenfunction \(f \circ S_i^{(\emptyset)}\), \(i = 1, \ldots, N_0\).

Now, let \(r_i^{(\emptyset)} m_i^{(\emptyset)}\) be an eigenvalue of \((\mathcal{E}, \mathcal{F}_0, \mu^{(\emptyset, I)})\) for some \(\lambda > 0\) with corresponding eigenfunction \(f_i\), \(i = 1, \ldots, N_0\). Therefore, we have

\[
\mathcal{E}(f_i, g) = r_i^{(\emptyset)} m_i^{(\emptyset)} \lambda \langle f_i, g \rangle_{L^2(\mu^{(\emptyset, I)})}
\]

for all \(g \in \mathcal{F}_0\). Let

\[
f(x) := \begin{cases} f_i \circ S_i^{(\emptyset)^{-1}}(x), & \text{if } x \in S_i^{(\emptyset)}([a, b]), \text{ for some } i \in \{1, \ldots, N_0\} \\ 0, & \text{otherwise}. \end{cases}
\]
Since $f_i \in F_0$, we have $f \in F_0$ and because of $f \circ S_i^{(0)} = f_i$, $i = 1, ..., N_0$, we have
\[
\sum_{i=1}^{N_0} \frac{1}{r_i} E \left( f \circ S_i^{(0)}, g \right) = \lambda \sum_{i=1}^{N_0} m_i^{(0)} \left( f \circ S_i^{(0)}, g \right)_{L^2(\mu^{(\omega)}i)},
\]
for all $g \in F_0$. For $\tilde{g} \in F_0$, we have $\tilde{g} \circ S_i^{(0)} \in F_0$, $i = 1, ..., N_0$. Analogously to the case with Neumann boundary conditions we get with [1, Proposition 3.2.1] and Lemma 4.4,
\[
E(f, \tilde{g}) = \lambda \langle f, \tilde{g} \rangle_{L^2(\mu^{(\omega)})}.
\]
Hence, $\lambda$ is an eigenvalue of $(E, F_0, \mu^{(I)})$ with eigenfunction $f$ and, as before, we can now easily conclude the claim.

Since $(E, F, \mu^{(I)})$ is an extension of $(E, F_0, \mu^{(I)})$ and $(E, F_0, \mu^{(I)})$ is an extension of $(E, F_0, \mu^{(I)})$, we get the following corollary.

**Corollary 4.7:** For all $x \geq 0$ holds
\[
\sum_{i=1}^{N_0} N_i^{(0),i}(x) \leq N_i(x) \leq \sum_{i=1}^{N_0} N_i^{(0),i}(x).
\]

### 4.3. Spectral Asymptotics

We define a probability space $(\Omega, B, \mathbb{P})$ in which every atomic event indicates a random tree $I$. Let $(\tilde{\Omega}, \tilde{B}, \tilde{\mathbb{P}})$ be a probability space and $\tilde{U}_i$, $i \in \mathcal{G}$ be i.i.d. $I$-valued random variables. The probability space we are interested in is defined as in (9), meaning
\[
(\Omega, B, \mathbb{P}) = \prod_{i \in \mathcal{G}} (\Omega_i, B_i, \mathbb{P}_i),
\]
whereby $(\Omega_i, B_i, \mathbb{P}_i)$ are copies of $(\tilde{\Omega}, \tilde{B}, \tilde{\mathbb{P}})$. We set $U_i = \tilde{U}_i \circ P_i$, $i \in \mathcal{G}$, where $P_i$ is the projection map onto the $i$-th component. $\omega \in \Omega$ indicates a random tree $I(\omega)$. If $(i_1, ..., i_n) = i \in \mathcal{G}$ is such that $N_{i_1,i_2, ..., i_{n-1}}(\omega) < i_n$, then in the infinite tree $I(\omega)$, the $i_n$-th child of $(i_1, ..., i_{n-1})$ is never born, i.e. $i \notin I(\omega)$. If we refer to the Dirichlet/Neumann eigenvalue counting function, we write $N_{D/N}^{(\omega)}$ instead of $N_{D/N}^{(I(\omega))}$. Also, we write $\theta_i(\omega)$, if we mean the sub tree $\theta_i(\omega)$ of $I(\omega)$, rooted at $i \in I(\omega)$, is measurable.

We consider C-M-J branching processes with
\[
(\xi_i, L_i) = \left( \sum_{i_1=1}^{U_{i_1}} \delta_{-\log \left( r_{i_1}^{-1} m_i^{0} \right)}, \max_{i \in \{1, ..., U_i\}} \left( -\log \left( r_i^{-1} m_i^{0} \right) \right) \right),
\]
whereby $\delta\nu(\cdot)$ denotes the Dirac delta function $\delta(\cdot - y)$. Let $(z_t)_t$ denote the C-M-J branching process to the random characteristic
\[
\hat{\phi}(t) = \xi_t(\infty) = \xi_i(t).
\]
Then $z_t$ denotes the number of individuals born after time $t$ to mothers born before or at time $t$. We assume that Condition 3.2 and Condition 3.3 are satisfied and thus there exists a random variable $W$ such that
\[
\lim_{t \to \infty} e^{-\alpha t} z_t = W \nu^\phi_\alpha(\infty) \quad a.s., \quad \nu^\phi_\alpha(\infty) := \frac{\int_0^\infty e^{-\alpha t} E(\hat{\phi}(t)) dt}{\int_0^\infty t d\nu_\alpha(t)},
\]
or there exists a periodic function $G^\phi_\alpha$ such that
\[
z_t = W e^{\alpha t} \left( G^\phi_\alpha + o(1) \right) \quad a.s.
If we assume that \( \mathbb{E}N^2 < \infty \), we have
\[
\mathbb{E}(\xi_\alpha(\infty) \log^+ \xi_\alpha(\infty)) < \infty.
\]
Hence, by Theorem 3.1, \( W > 0 \) a.s. For the rest of this chapter we denote by \( W \) this random variable.

With Corollary 4.7 we have for each \( x \geq 0 \)
\[
\sum_{i=1}^{N(t)} N^D_N(i^{(U)} m_i^{(U)} x) \leq N^D_N(x) \leq \sum_{i=1}^{N(t)} N^D_N(i^{(U)} m_i^{(U)} x).
\]

We consider the scaling property
\[
N^D_N(i^{(U)} m_i^{(U)} x) \leq N^D_N(x).
\]

We suppress the \( \omega \) dependence and define \( X_D(t) := N_D(e^t) \).

Therefore, we have
\[
\sum_{i=1}^{\xi_\phi(\infty)} X_D(t - \sigma_i) \leq X_D(t) \ a.s.
\]

As in [10] we extend the branching processes to \( \{X^\phi(t) : -\infty < t < \infty\} \), where
\[
X^\phi(t) := \sum_{i \in I} \phi_{\theta_0}(t - \sigma_i)
\]
and \( \phi_{\theta_0} \) is defined for all \( t \in \mathbb{R} \) and \( \omega \in \Omega \). For our purposes it is enough that \( \phi_{\theta_0} \) is bounded and \( \phi_{\theta_0}(t) = 0 \) for all \( t < t_0(\omega) \), for some \( t_0(\omega) \in \mathbb{R} \). As for the C-M-J branching processes, we have
\[
X^\phi(t) = \phi_{\theta_0}(t) + \sum_{i=1}^{\xi_\phi(\infty)} (i) X^\phi(t - \sigma_i),
\]
where \( \{ (i) X^\phi(t) \}_{t}, i = 1, ..., \xi_\phi(\infty) \) are branching processes with characteristic \( \phi \) with the assumption that the population has initial ancestor \( (i) \). Moreover, \( (i) X^\phi \) are i.i.d. copies of \( X^\phi \), distributed like \( X^\phi \) and independent of \( U_\emptyset \) and \( \xi_\emptyset \). We will suppress \( (i) \), if it will not cause confusion.

We want to give a representation of \( X_D \) such that \( X_D = X^\phi \) for some bounded \( \phi \). Let
\[
\eta(t) := X_D(t) - \sum_{i=1}^{\xi_\phi(\infty)} X_D(t - \log \tau_1(i))
\]
and
\[
\tilde{\eta}(t) := \eta(t) \mathbb{1}_{t \geq 0} + \sum_{i=1}^{\xi_\phi(\infty)} (i) X_D(t - \sigma_i) \mathbb{1}_{\{0 \leq t < \sigma_i\}}.
\]

Then, we have \( X_D = X^\eta \) and \( X^\tilde{\eta}(t) = \mathbb{1}_{[0,\infty)}(t) X^\eta(t) \) and thus both processes have the same asymptotic behavior as \( t \) tends to infinity.
Lemma 4.8: Assume that

$$E N_{U_0}^2 < \infty.$$ 

Then, the Malthusian parameter of the process \( \{X_D(t) : t \in \mathbb{R}\} \) is the unique solution \( \gamma > 0 \) of

$$E \left( \sum_{i=1}^{N_{U_0}} \left( r_i(U_\theta) m_i(U_\theta) \right)^\gamma \right) = 1.$$ 

If \( \nu \) is non-lattice, then

$$\lim_{t \to \infty} X_D(t) e^{-\gamma t} = v_\gamma^0(\infty) W \quad \text{a.s.},$$

where

$$v_\gamma^0(\infty) := \frac{\int_0^\infty e^{-\gamma t} E(\tilde{\eta}(t)) \, dt}{\int_0^\infty t e^{-\gamma t} d\nu(t)}.$$ 

If \( \nu \) is lattice with period \( T \), then

$$X_D(t) = (G_\gamma(t) + o(1)) e^{\gamma t} W \quad \text{a.s.},$$

where \( G \) is a periodic function with period \( T \), given by

$$G_\gamma(t) = T \cdot \sum_{j=-\infty}^{\infty} e^{-\gamma(t+jT)} E(\tilde{\eta}(t+jT)) \int_0^\infty t e^{-\gamma t} d\nu(t).$$

Proof. Let

$$f(s) := E \left( \sum_{i=1}^{N_{U_0}} \left( r_i(U_\theta) m_i(U_\theta) \right)^{s} \right).$$

By dominated convergence, we see \( f : [0, \infty) \to \mathbb{R} \) is continuous and because \( r_i^{(j)} m_i^{(j)} < 1 \) for all \( j \in J, i = 1, ..., N_j \), \( f \) is strictly decreasing. Because \( N_j \geq 2, j \in J \), we have

$$f(0) \geq 2$$

and

$$\lim_{s \to \infty} f(s) = 0.$$ 

By continuity, there exists \( \gamma > 0 \) such that \( f(\gamma) = 1 \). Furthermore, \( \gamma \) is the unique solution strictly bigger than zero and also the Malthusian Parameter of the general branching process under consideration. The first moment of \( \nu_\gamma \) is finite, since \( E N_{U_0} < \infty \). With \( g(t) = t^{-2} \wedge 1 \) Condition 3.2 is satisfied since

$$E \left( \sup_{t \geq 0} \frac{\xi_\gamma(\infty) - \xi_\gamma(t)}{g(t)} \right) \leq E \left( \sup_{t \geq 0} \int_t^\infty \frac{1}{g(s)} d\xi_\gamma(s) \right) \leq \sup_{t \geq 0} \left\{ (1 \vee t^2) e^{-\gamma t} \right\} E N_{U_0} < \infty.$$ 

By [19, Lemma 4.10] there exists a deterministic constant \( \tilde{c} > 0 \) such that

$$X_D(t) \leq \tilde{c} e^t.$$ 

(15)

Further, from the Dirichlet-Neumann-bracketing follows that

$$0 \leq \eta(t) \leq \sum_{i=1}^{N_{U_0}} \left( N_N^{(0, I)} \left( r_i^{(0)} m_i^{(0)} e^t \right) - N_D^{(0, I)} \left( r_i^{(0)} m_i^{(0)} e^t \right) \right).$$
With [6, Proposition 5.5]
\[ N_D(x) \leq N_N(x) \leq N_D(x) + 2, \]
we thus receive
\[ \eta(t) \leq 2N_{U_\emptyset}. \tag{16} \]
Taking together (15) and (16), we receive
\[ \tilde{\eta}(t) \leq cN_{U_\emptyset}, \]
for some deterministic \( c > 0 \). Therefore, Condition 3.3 follows with
\[ h(t) = e^{-\gamma t}. \] The Lemma then follows from Theorem 3.4.

\[ \square \]

**Theorem 4.9:** Assume that
\[ \mathbb{E}N_{U_\emptyset}^2 < \infty \]
and let \( \gamma > 0 \) be the unique solution of
\[ \mathbb{E}\left( \sum_{i=1}^{N_{U_\emptyset}} \left( r_i(U_\emptyset) m_i(U_\emptyset) \right) \right)^\gamma = 1. \]
Then,

1. If \( \nu \) is non-lattice, then
\[ \lim_{x \to \infty} N_{D/N}(x) x^{-\gamma} = \nu^\circ(\infty) W, \quad \text{a.s.,} \]
where
\[ \nu^\circ(\infty) := \frac{\int_{-\infty}^\infty e^{-\gamma t} \mathbb{E}(\tilde{\eta}(t)) dt}{\int_{0}^\infty t e^{-\gamma t} d\nu(t)}. \]

2. If the support of \( \nu \) lies in a discrete subgroup of \( \mathbb{R} \), then
\[ N_{D/N}(x) = (G(\log(x)) + o(1)) x^\gamma W, \quad \text{a.s.,} \]
where \( G \) is a periodic function with period \( T \), given by
\[ G(t) = T \cdot \frac{\sum_{j=-\infty}^{\infty} e^{-\gamma(t+jT)} \mathbb{E}(\tilde{\eta}(t+jT))}{\int_{0}^{\infty} t e^{-\gamma t} d\nu(t)}. \]

**Proof.** For the Dirichlet eigenvalue counting function, we simply rescale Lemma 4.8 by \( x = \log(t) \) and hence the claim follows. The assertion for the Neumann eigenvalue counting function follows from the identity
\[ N_D(x) \leq N_N(x) \leq N_D(x) + 2, \]
see [6, Proposition 5.5].

\[ \square \]
4.4. Comparison between Random Recursive and Random Homogeneous Cantor Measures. We have seen the construction of the recursive Cantor sets and the corresponding recursive Cantor measures. Then, we randomized these sets and measures and showed that under some regularity conditions the spectral exponent for the corresponding Krein-Feller-operator is almost surely given by the unique solution $\gamma_r > 0$ of

$$
E \left( \sum_{i=1}^{N_{U_0}} (r_i^{(U_0)} m_i^{(U_0)})^{\gamma_r} \right) = 1.
$$

In Theorem 2.1 we recalled the results of [1] about the spectral asymptotics for Krein-Feller-operators w.r.t. random homogeneous Cantor measures. The next proposition relates $\gamma_r$ to $\gamma_h$, where we assume that conditions (A1)-(A5) are satisfied.

**Proposition 4.10:** With the notation above and in Theorem 2.1, we have $\gamma_h \leq \gamma_r$ and equality if and only if there exists $\alpha > 0$ such that

$$
\sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^\alpha = 1, \quad \text{for all } j \in J.
$$

**Proof.** Let $x_j(\alpha) := \sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^\alpha$, $j \in J$. With Jensen’s inequality, we receive

$$
\sum_{j \in J} p_j \log (x_j(\alpha)) \leq \log \left( \sum_{j \in J} p_j x_j(\alpha) \right).
$$

Since log is strictly increasing, we have equality if and only if $x_i(\alpha) = x_j(\alpha) = 1$ for all $i, j \in J$. Now, let (17) not be satisfied. Then,

$$
0 = \sum_{j \in J} p_j \log (x_j(\gamma_h)) < \log \left( \sum_{j \in J} p_j x_j(\gamma_h) \right).
$$

As $\log \left( \sum_{j \in J} p_j x_j(\alpha) \right)$ decreases as $\alpha$ increases, the assertion follows. \qed

**Remark 4.11:** If $U_i = U_j$ for all $i, j \in I$ such that $|i| = |j|$, then the corresponding recursive Cantor measure is homogeneous. However, Theorem 4.9 makes no statement about the spectral asymptotics w.r.t. homogeneous Cantor measures, since the probability that $\mu^{(i)}$ is homogeneous is 0.

**Example 4.12:** Let $J$ be countable and $p_j := \mathbb{P}(U_0 = j) \in (0, 1)$, $j \in J$. Further, assume that $r_1^{(j)} = \ldots = r_{N_j}^{(j)}$, $m_1^{(j)} = \ldots = m_{N_j}^{(j)}$ for all $j \in J$. Therefore, $m_i^{(j)} = \frac{1}{N_j}$, $i = 1, \ldots, N_j$ for all $j \in J$. Let $r := r_{U_0}$ and $N := N_{U_0}$. If the conditions (A1)-(A5) are satisfied, then the spectral exponent for the Krein-Feller-operator w.r.t. the corresponding random homogeneous Cantor measure is given by

$$
\gamma_h := \frac{E \log N}{E \log (N/r)},
$$

see [1, Page 64]. The spectral exponent for the Krein-Feller-operator w.r.t. the corresponding random recursive Cantor measure is given by the unique solution $\gamma_r > 0$ of

$$
E (N (r/N)^{\gamma_r}) = 1.
$$

If not $(r/N)^{\alpha} = 1/N$ for some $\alpha > 0$, for almost all $\omega \in \Omega$, we thus have

$$
0 = \log \left( \sum_{j \in J} p_j N_j \left( r_1^{(j)} / N_j \right)^{\gamma_r} \right) \leq \sum_{j \in J} p_j \log \left( N_j \left( r_1^{(j)} / N_j \right)^{\gamma_r} \right) = E \log (N (r/N)^{\gamma_r}).
$$
Therefore,

\[ \gamma_h = \frac{\mathbb{E} \log N}{\mathbb{E} \log(N/r)} < \gamma_r. \]

Coming back to the \( \frac{1}{3} \)-recursive Cantor set from the introduction and let \( p = \frac{3}{5} \), \( m_1^{(1)} = m_2^{(1)} = \frac{1}{2}, m_1^{(2)} = m_2^{(2)} = \frac{1}{3} \). Then, the spectral exponent for the Krein-Feller-operator w.r.t. the corresponding random recursive Cantor measure is given as the unique solution \( \gamma_r > 0 \) of

\[ \left( \frac{1}{6} \right)^{\gamma_r} + \left( \frac{1}{15} \right)^{\gamma_r} = \frac{5}{6}. \]

Numerically, we get \( \gamma_r \approx 0.396403 \).

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