ON FAMILIES OF TRIANGULAR HOPF ALGEBRAS

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1. Introduction

It was recently discovered ([AEG], [EG1]) that the twisting procedure of finite supergroups can be used to construct new families of triangular Hopf algebras. Furthermore, it turned out that these families exhaust all triangular Hopf algebras with the Chevalley property (in particular, all previously known triangular Hopf algebras). The goal of this paper is to continue the study of these families, and in particular to give some applications outside of the theory of triangular Hopf algebras.

More specifically, we consider the family of finite-dimensional complex triangular Hopf algebras $A(G, V, u, B)$, where $G$ is a finite group, $V$ a finite-dimensional representation of $G$, $u$ a central element of order 2 in $G$ acting by $-1$ on $V$, and $B$ an element in $S^2V$. This is the simplest of the families from [AEG], [EG1], corresponding to the twists which are entirely contained in the “supernaive” part of a finite supergroup.

We start by finding the condition under which two members of such a family are isomorphic as Hopf algebras (without regard for the triangular structure). This allows us to calculate the moduli space of isomorphism classes, which often turns out to be of positive dimension. In particular, we see that this construction easily produces continuous families of pairwise non-isomorphic Hopf algebras, and hence Kaplansky’s 10th conjecture [K] that there are finitely many isomorphism classes of Hopf algebras in each dimension (which was disproved in [AS], [BDG], [G]) fails even for triangular Hopf algebras. In fact, the lowest dimension in which we get continuous triangular families is 32, which is the lowest dimension in which continuous families of Hopf algebras are known to exist [Gr]. More precisely, in dimension 32 we get three non-equivalent 1−parameter families, which are exactly the duals to the three families constructed in [Gr] (and shown to be the only continuous families of 32−dimensional pointed Hopf algebras [Gr]). In particular, we see that the families in [Gr] are cotriangular.

We also consider the question when the Hopf algebra $A(G, V, u, B)$ is twist equivalent to the Hopf algebra $A(G, V, u, B')$ by twisting of the multiplication. We show that if $(S^2V)^G = 0$ then for each $B$, there are only finitely many Hopf algebra isomorphism classes of $A(G, V, u, B')$ with this property. This implies that whenever the set of isomorphism classes is infinite (e.g. in the 32−dimensional examples of [Gr]), we obtain continuous families of Hopf algebras which are not equivalent by a twist of multiplication. This disproves a weakened form of Kaplansky 10th conjecture suggested by Masuoka [M], which states that such a family cannot exist.

Next, we study the algebra structure of $A(G, V, u, B)^*$, and show that it is isomorphic as an algebra to a direct sum of Clifford algebras. The only invariants of Clifford algebras being the dimension and the rank of the quadratic form, we see
that in each family $A(G, V, u, B)$ there are finitely many coalgebra types. Since all
the members of this family also have the same algebra type, one might wonder if
there are finitely many algebra (or, equivalently, coalgebra) types of Hopf algebras
of a given dimension. We do not conjecture this, but it would be interesting to find
a counterexample.

Finally, we apply the results of this paper to the theory of tensor categories.
Namely, we start by recalling Schauenburg’s theorem that if $H_1, H_2$ are finite-
dimensional Hopf algebras such that $\text{Rep}(H_1)$ is equivalent to $\text{Rep}(H_2)$ as a tensor
category then $H_1, H_2$ are twist equivalent (we give a simple proof of this result,
which also works in the more general case of quasi-bialgebras). This theorem im-
plies that if $(S^2V)^G = 0$ then in the family of tensor categories $\text{Rep}(A(G, V, u, B)^*)$, each
member is equivalent to at most finitely many other members. On the other
hand, there are only finitely many types of abelian categories among them (since
there are finitely many algebra types of the underlying Hopf algebras). Thus we get
an example of a finite abelian category (i.e. one equivalent to the representation cat-
egory of a finite-dimensional algebra) which admits infinitely many non-equivalent
rigid tensor structures with a fixed Grothendieck ring. As far as we know, such
examples were not previously available, and moreover it was shown by Ocneanu
(unpublished) that they cannot exist for semisimple categories.

2. The set of isomorphism classes of $A(G, V, u, B)$

Throughout the paper, the ground field is the field of complex numbers $\mathbb{C}$.

Let $G$ be a finite group, $V$ a finite-dimensional representation of $G$ over $\mathbb{C}$, $u \in G$
a central element satisfying $u^2 = 1$ and $u|_V = -1$, and $B \in S^2V$ a symmetric
2–tensor. Recall from [EG1] that a triangular Hopf algebra $A = A(G, V, u, B)$
with the Chevalley property can be associated to $(G, V, u, B)$ in the following way.

Let $\mathbb{C}[G \ltimes V]$ be the group algebra of the supergroup $G \ltimes V$ (i.e., the Hopf
superalgebra $\mathbb{C}[G] \ltimes \Lambda V$), and set $J := e^B \in \mathbb{C}[G \ltimes V]^{S^2}$. Then it was shown
in [AEG] that $J$ is a twist for $\mathbb{C}[G \ltimes V]$, so we can form the Hopf superalgebra
$\mathbb{C}[G \ltimes V]^J$, and then modify it using $u$ to get the Hopf algebra $A = A(G, V, u, B)$
(see [AEG, Section 3]).

Proposition 2.1. Let $A_1 = A(G_1, V_1, u_1, B_1)$ and $A_2 = A(G_2, V_2, u_2, B_2)$, with
$V_1, V_2 \neq 0$. Then the Hopf algebras $A_1, A_2$ are isomorphic if and only if there exists
a group isomorphism $\phi : G_1 \to G_2$ such that $\phi(u_1) = u_2$, and a $G_1$–isomorphism
$\eta : V_1 \to \phi^*(V_2)$ such that $(\eta \otimes \eta)(B_1) = B_2$ modulo $(S^2V_2)^G$.

Proof. Let us first prove the “if” direction. We need to show that $A(G, V, u, B) \cong
A(G, V, u, B + B')$ if $B' \in (S^2V)^G$. To this end we note that for any $g \in G$, $\alpha \in \Lambda V$, and $J = e^B$ the coproduct in $\mathbb{C}[G \ltimes V]^J$ is given by

$$\Delta^J(g) = J^{-1}\Delta(g)\Delta(\alpha)J = \Delta(g)\Delta(\alpha)e^{B-B^\sigma},$$

where $B^\sigma(x, y) := B(gx, gy)$ (here we used the fact that $B$ is even, hence central
in $\Lambda V \otimes \Lambda V$). Therefore if we replace $B$ by $B + B'$, where $B' \in (S^2V)^G$, the
isomorphism type of the Hopf algebra $A(G, V, u, B)$ does not change.

Conversely, suppose $\xi : A_1 \to A_2$ is an isomorphism of Hopf algebras. Taking
quotients by the radicals, we get an isomorphism of Hopf algebras $\mathbb{C}[G_1] \to \mathbb{C}[G_2]$, hence an isomorphism of groups $\phi : G_1 \to G_2$. Since the only non-trivial 1 : $g$
skew-primitive elements in $A_1, A_2$, arise for $g = u_1, u_2$, respectively, we must have
$\phi(u_1) = u_2$. Therefore, $\xi$ induces an isomorphism between the Hopf superalgebras
Homogeneous coefficients in the trivial bimodule $C[V_1, V_2]^J_1$ and $C[V_2 \times V_2]^J_2$, hence between the spaces $V_1, V_2$ of their primitive elements. Let us call this isomorphism $\eta$. Clearly, $\eta$ is a $G_1$-isomorphism $V_1 \to \phi^*(V_2)$. Finally, $\tilde{\epsilon}$ induces an isomorphism $C[V_1, V_2] \to C[V_2 \times V_2]^{(\eta \otimes \eta)(J_1)^{-1}}$. But this implies that $C[V_2 \times V_2]^{J_2(\eta \otimes \eta)(J_1)^{-1}}$ is cocommutative, so $J_2(\eta \otimes \eta)(J_1)^{-1} = e_B$, where $B$ belongs to $(S^2V^2)_G$.

Let us denote by $\text{Aut}(G, V)$ the group of all pairs $(\phi, \eta)$ where $\phi : G \to G$ is an automorphism and $\eta : V \to \phi^*(V)$ is a $G$-isomorphism.

**Corollary 2.2.** The set of Hopf algebra isomorphism classes of $A(G, V, u, B)$, for fixed $G, V, u$ is $\frac{S^2V/(S^2V)^G}{\text{Aut}(G, V)}$.

**Remark 2.3.** It follows from Corollary 2.2 that a non-trivial continuous family of Hopf algebras $A(G, V, u, B)$, with fixed $G, V, u$, exists if and only if the quotient space $S^2V/(S^2V)^G$ modulo the action of the group $\text{Aut}(G, V)$ is infinite; for example when $\dim(S^2V/(S^2V)^G) > \dim(\text{Aut}(G, V))$.

3. **Twist equivalence classes of $A(G, V, u, B)$**

Recall that for any $B, B'$, the Hopf algebras $A(G, V, u, B)$ and $A(G, V, u, B')$ are twist equivalent as triangular Hopf algebras (see [AEG]). To the contrary, for the dual Hopf algebras we have the following theorem.

**Theorem 3.1.** Let $G, V, u, B$ be as before, and suppose $(S^2V)^G = 0$. Then there exist only finitely many $B' \in S^2V/\text{Aut}(G, V)$ such that $A(G, V, u, B)^*$ is twist equivalent to $A(G, V, u, B')^*$.

The rest of the section is devoted to the proof of Theorem 3.1.

Let $F$ be a functor and let $R^iF$ denote its $i$-th derived functor ($R^0F := F$). Recall that by a result of Grothendieck, if $Q$ is an exact functor then $R^i(QF) = QR^iF$. Recall also that $R^i\text{Hom}(\ast, N) = \text{Ext}^i(\ast, N)$.

In the following lemma we calculate the Hochschild cohomology of $C[G \ltimes V]$ with coefficients in the trivial bimodule $C$ (i.e. $abc := \varepsilon(a)e(c)b$, $a, c \in C, b \in C[G \ltimes V]$).

**Lemma 3.2.** $H^i(C[G \ltimes V], C) = (S^iV^*)^G$.

**Proof.** Let $F : \text{Rep}(C[G \ltimes V]) \to \text{Rep}(C[G])$ be the functor defined by $F(X) = \text{Hom}_{AV}(X, C)$, and let $Q : \text{Rep}(C[G]) \to \text{Vect}$ be the functor defined by $F(Y) = Y^G$. Then $Q$ is exact, and $QF(X) = \text{Hom}_{C[G]^G}(X, C)$. Therefore, on the one hand,

$$R^i(QF)(C) = \text{Ext}^i_{C[G]^G}(C, C) = H^i(C[G] \ltimes AV, C),$$

and on the other hand,

$$QR^iF(C) = \text{Ext}^i_{AV}(C, C)^G = (H^i(AV, C))^G = (S^iV^*)^G,$$

where the last equality follows from the Koszul duality.

Recall that a linear form $\Phi : H \otimes H \to C$ is called a Hopf $2-cocycle$ for $H$ if it has an inverse $\Phi^{-1}$ under the convolution product $*$ in $\text{Hom}_{C}(H \otimes H, C)$, and satisfies:

$$\sum \Phi(a_1b_1, c)\Phi(a_2, b_2) = \sum \Phi(a, b_1c_1)\Phi(b_2, c_2)$$

for all $a, b, c \in H$. 


Given a Hopf 2-cocycle $\Phi$ for $H$, one can construct a new Hopf algebra

$$(H_\Phi, m_\Phi, 1, \Delta, \varepsilon, S_\Phi)$$

as follows. As a coalgebra, $H_\Phi = H$. The new multiplication is given by

$$m_\Phi(a \otimes b) = \sum \Phi^{-1}(a_1, b_1)a_2b_2\Phi(a_3, b_3)$$

for all $a, b \in H$. The new antipode is given by

$$S_\Phi(a) = \sum \Phi^{-1}(a_1, S(a_2))S(a_3)\Phi(S(a_4), a_5)$$

for all $a \in H$.

Note that it is straightforward to verify that if $H$ is finite-dimensional, then $\Phi \in H^* \otimes H^*$ is a Hopf 2-cocycle for $H$ if and only if it is a twist for $H^*$ (here we do not impose the counit property on the twist). So, in particular, the group $(H^*)^\times$ of invertible elements in $H^*$ acts as gauge transformations on the set of Hopf 2-cocycles for $H$.

**Proposition 3.3.** Let $H$ be any Hopf algebra which is isomorphic to $\mathbb{C}[G] \rtimes \Lambda V$ as an algebra, with the usual counit. Then there exist only finitely many Hopf 2-cocycles $\Phi$ on $H$, modulo gauge transformations, such that $H_\Phi$ is isomorphic to $\mathbb{C}[G] \rtimes \Lambda V$ as an algebra, with the usual counit.

**Proof.** Let $X$ be the space of all Hopf 2-cocycles such that $H_\Phi \cong \mathbb{C}[G] \rtimes \Lambda V$ as algebras, with the usual counit. Let $Y$ be the space of all Hopf 2-cocycles on $H$. Then $Y$ is an affine algebraic variety and $X \subseteq Y$. Let $L := (H^*)^\times$ be the group of invertible elements in $H^*$; it acts on $Y$ by gauge transformations and preserves $X$. So we have a map $\tau_x : \text{Lie}(L) \to T_x Y$, for all $x \in Y$. It is easy to see from the definition of Hopf 2-cocycles that $T_x Y = Z^2(H_x, \mathbb{C})$ (Hochschild 2-cocycles) and $\tau_x(\text{Lie}(L)) = B^2(H_x, \mathbb{C})$ (Hochschild 2-coboundaries). Therefore,

$$T_x Y/\tau_x(\text{Lie}(L)) = H^2(H_x, \mathbb{C}) = H^2(\mathbb{C}[G] \rtimes \Lambda V, \mathbb{C}) = (S^2 \Lambda V^*)^G = ((S^2 \Lambda V)^G)^* = 0$$

using Lemma 3.2. Thus, $\tau_x$ is surjective, which implies that $X/L$ is finite by Proposition 7.1. □

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 3.3, there exist finitely many gauge equivalence classes of Hopf 2-cocycles $\Phi$ such that $A(G, V, u, B_\Phi)$ is isomorphic to $\mathbb{C}[G \rtimes V]$ as an algebra, with the usual counit. This implies that there exist finitely many isomorphism classes of Hopf algebras $A(G, V, u, B')$ which are twist equivalent to $A(G, V, u, B)$ by twisting of multiplication. By Corollary 2.2, this implies the theorem.

### 4. 32-Dimensional Examples

As we remarked in the introduction, examples of non-trivial continuous families of triangular Hopf algebras occur already in dimension 32. Here we describe these families, and identify them with the families studied in [Gr].

More specifically, in [Gr] Graña classified all complex 32-dimensional pointed Hopf algebras $H$, and showed that they fall into finitely many isomorphism classes, except for three 1-parameter continuous families (a family with $G(H) = \mathbb{Z}_4 \times \mathbb{Z}_2$, and two families with $G(H) = \mathbb{Z}_8$). We will prove the following result.
Theorem 4.1. The three continuous families of 32-dimensional pointed Hopf algebras of [Gr] are of the form $A(G, V, u, B)^*$ for appropriate $G, V, u, B$.

Proof. To prove the theorem, we will construct three families of 32-dimensional pointed Hopf algebras as $A(G, V, u, B)^*$, after which it will not be difficult to see that they are identical to the families of [Gr].

1. Let $G := Z_8 = < a >$, and $\chi : Z_8 \to \mathbb{C}^*$ be the character defined by $\chi(a^m) = e^{2 \pi i m/8}$. Let $V := \chi \oplus \chi^3 = \text{sp}\{e_1\} \oplus \text{sp}\{e_2\}$, where $a \cdot e_1 = \chi(a)e_1$ and $a \cdot e_2 = \chi(a^3)e_2$, and let $u := a^4$ (so $u_{|V} = -1$). Then any $B \in S^2V$ can be represented by a symmetric $2 \times 2$ matrix with respect to the basis $(e_1, e_2)$. We claim that $A(G, V, u, B_1), A(G, V, u, B_2)$ are isomorphic if and only if $B_1 = B_2$ in $S^2V/Z_2 \ltimes D$, where $Z_2$ is generated by the matrix

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

and the result follows in a similar way to case 1.

2. Let $G = Z_4 \times Z_2 = < a > \times < b >$. Let $\chi : Z_4 \to \mathbb{C}^*$ be the character defined by $\chi(a^m) = e^{2 \pi i m/4}$, $\chi_+ : Z_2 \to \mathbb{C}^*$ the trivial representation, and $\chi_- : Z_2 \to \mathbb{C}^*$ the non-trivial representation. Let $V := (\chi, \chi_+) \oplus (\chi, \chi_-)$, and $u := (a^2, 1)$. Then $u_{|V} = -1$. We claim that, as in the previous two cases, the set of isomorphism classes is $S^2V/Z_2 \ltimes D$. Indeed, we have $S^2V = 2\chi^2 \oplus \chi^6$, so as before we have $(S^2V)^G = 0$, and the result follows in a similar way to case 1, 2.

3. Let $G = Z_4 \times Z_2 = < a > \times < b >$. Let $\chi : Z_4 \to \mathbb{C}^*$ be the character defined by $\chi(a^m) = e^{2 \pi i m/4}$, $\chi_+ : Z_2 \to \mathbb{C}^*$ the trivial representation, and $\chi_- : Z_2 \to \mathbb{C}^*$ the non-trivial representation. Let $V := (\chi, \chi_+) \oplus (\chi, \chi_-)$, and $u := (a^2, 1)$. Then $u_{|V} = -1$. We claim that, as in the previous two cases, the set of isomorphism classes is $S^2V/Z_2 \ltimes D$. Indeed, we have $S^2V = 2(\chi^2, \chi_+) \oplus (\chi^2, \chi_-)$, so again $(S^2V)^G = 0$. Also, it is straightforward to check that in this case the only allowed non-trivial automorphism $\phi : G \to G$ is the one determined by $a \mapsto a$, $b \mapsto ba^2$, which interchanges $(\chi, \chi_+)$ with $(\chi, \chi_-)$. Hence the result follows in a similar way to cases 1, 2.

Now, comparing this with [Gr], it is easy to see that the above three families of Hopf algebras $A(G, V, u, B)^*$ exactly coincide with the families of [Gr]. In detail, the family in example 3 corresponds to the family in [Gr] with $G(H) = Z_4 \times Z_2$, and the families of examples 1 and 2 correspond to the two families in [Gr] for $G(H) = Z_8$ (Line 2 and Line 4 of Table 15, respectively). More precisely, it is not hard to check that under this correspondence, the lifting parameters $(\lambda_1, \lambda_2, \lambda_3)$ of [Gr] are related to our parameter $B$ via the formula $B = \begin{pmatrix}
\lambda_1 & \lambda_3/2 \\
\lambda_3/2 & \lambda_2
\end{pmatrix}$. The theorem is proved.

Remark 4.2. The equivalence relation between the lifting parameters $(\lambda_1, \lambda_2, \lambda_3)$ is thus defined by $(\lambda_1, \lambda_2, \lambda_3) \sim (t^2\lambda_1, s^2\lambda_2, ts\lambda_3)$, $t, s \neq 0$, and $(\lambda_1, \lambda_2, \lambda_3) \sim (\lambda_2, \lambda_1, \lambda_3)$. This is exactly the equivalence relation which occurs in [Gr] (see the web version of [Gr], math.QA/0110033).
Corollary 4.3. The 32–dimensional examples $A(G, V, u, B)^*$ fall into infinitely many twist equivalence classes.

Proof. Follows from Theorems 3.1, 4.1. □

5. THE ALGEBRA STRUCTURE OF $A(G, V, u, B)^*$

In this section we describe the algebra structure of the Hopf algebra $A(G, V, u, B)^*$. We start by recalling the definition of a Clifford algebra.

Definition 5.1. Let $V$ be a finite-dimensional vector space, and $B$ a symmetric bilinear form on $V$. The Clifford algebra $\Cl(V, B)$ is the quotient of the tensor algebra $T(V)$ by the ideal generated by all elements of the form $vw + vw - 2B(v, w)1$, for all $v, w \in V$ (e.g. $\Cl(V, 0) = \Lambda V$).

Note that the algebra $\Cl(V, B)$ has a unique structure of a superalgebra, determined by requiring $V$ to be odd. Recall also that $\Cl(V, B)$ has a filtration determined by letting $v \in V$ have degree 1, so that $gr\Cl(V, B) = \Lambda V$, and hence $\dim(\Cl(V, B)) = 2^{\dim(V)}$.

Finally, recall that if $V_1, V_2$ are finite-dimensional vector spaces, and $B_1, B_2$ are symmetric bilinear forms on $V_1, V_2$ respectively, then $\Cl(V_1 \oplus V_2, B_1 \oplus B_2) = \Cl(V_1, B_1) \otimes \Cl(V_2, B_2)$, where $\otimes$ denotes the tensor product of superalgebras.

The main result of this section is the following theorem.

Theorem 5.2. $A(G, V, u, B)^* \cong \bigoplus_{h \in G/<u>} \Cl\left( V^* \oplus \mathbb{C}, \left( \begin{array}{cc} B - B^h & 0 \\ 0 & 1 \end{array} \right) \right)$, where $h = \{g, gu\}$, and $B^h$ denotes $B^g = B^{gu}$.

In particular, Theorem 5.2 implies the following.

Corollary 5.3. The Hopf algebra $A(G, V, u, B)$ is pointed if and only if $B \in (S^2V)^G$; i.e., $A(G, V, u, B) \cong A(G, V, u, 0)$ as Hopf algebras.

Corollary 5.4. The algebras $A(G, V, u, B)^*$, with fixed $G, V, u$, fall into finitely many isomorphism classes.

The rest of the section is devoted to the proof of Theorem 5.2.

Proposition 5.5. Let $V$ be a finite-dimensional vector space, and let $B \in S^2V$. Define the supercoalgebra $A_{V, B}$ to be $\Lambda V$ as a vector space, with comultiplication $\Delta_B$ given by $\Delta_B(\varphi) = \Delta(\varphi) e^B$, $\varphi \in \Lambda V$, where $\Delta$ denotes the usual comultiplication in $\Lambda V$. Then, $A_{V, B} \cong \Cl(V^*, B)$ as superalgebras.

Remark 5.6. By a super(co)algebra we mean a (co)algebra with an action of $\mathbb{Z}_2$, and the dual algebra is taken in the usual (rather than super) sense.

Proof. It is clear that $A_{V_1 \oplus V_2, B_1 \oplus B_2} = A_{V_1, B_1} \otimes A_{V_2, B_2}$, where $\otimes$ denotes the tensor product of supercoalgebras. Since Clifford algebras have a similar property, and since any symmetric bilinear form is a direct sum of 1–dimensional forms, it is sufficient to prove the result when $\dim(V) = 1$. But then the result is easy. □

Let us now determine the algebra structure of the superalgebra $A^*$, where $A := \mathbb{C}[G \ltimes V]^J$ and $J := e^B$. First, note that $A = \bigoplus_{g \in G} A_g$, where $A_g := gAV$, so $A^* = \bigoplus_{g \in G} A_g^*$. 


**Proposition 5.7.** The following hold:
1. $A_g$ is a subcoalgebra for all $g \in G$, so $A_g^*$ is a subalgebra for all $g \in G$ and $A^* = \bigoplus_{g \in G} A_g^*$ as an algebra.
2. $A_g^* \cong \text{Cl}(V^*, B - B^g)$, as superalgebras.

**Proof.** The first part is clear from equation (1), and the second part follows from Proposition 5.3.

Assume now that we have fixed a central element $u \in G$ such that $u^2 = 1$ and $u|_V = -1$. Then we can consider the Hopf algebra $A(G, V, u, B)$. We now wish to study the algebra structure of $A(G, V, u, B)^*$.

Recall that if $(C, \Delta)$ is a supercoalgebra, then one can define a coalgebra $(\overline{C}, \overline{\Delta}) = (C, \Delta)$ such that $\overline{C} = \mathbb{C}[\mathbb{Z}_2] \otimes C = C \oplus uC$, (i.e. $\mathbb{Z}_2$ is generated by $u$, with $u^2 = 1$) and $\overline{\Delta}(ux) = (u \otimes u)(\Delta(x)$ and $\overline{\Delta}(x) = \Delta_0(x) - (-1)^{p(x)}(u \otimes 1)\Delta_1(x)$, for all $x \in C$, where $p(x)$ denotes the parity of $x$, $\Delta_1(x) \in C \otimes C_1$ and $\Delta_0(x) \in C \otimes C_0$. Then, we have the following two straightforward results.

**Lemma 5.8.** $\overline{C}^* = \mathbb{C}[\mathbb{Z}_2] \ltimes C^*$, where $\mathbb{Z}_2$ acts on $C^*$ by parity.

**Proposition 5.9.** Let $\overline{\Delta}^{\mathbb{Z}}$ be the coproduct of $A(G, V, u, B)$. Then, under $\overline{\Delta}^{\mathbb{Z}}$, $A_g \oplus A_{gu}$ is a subcoalgebra, for all $g \in G$. Moreover, $(A_g \oplus A_{gu}, \overline{\Delta}^{\mathbb{Z}}) = (A_g, \overline{\Delta}^{\mathbb{Z}})$.

**Proof.** Follows from Proposition 5.7(1).

We can now prove Theorem 5.3.

**Proof of Theorem 5.3.** For any $h \in G/\langle u \rangle$, set $\overline{A}_h := A_g \oplus A_{gu}$, where $g \in G$ represents $h$. By Propositions 5.7(2) and Lemma 5.8,

$$\overline{A}_h^* = \mathbb{C}[\mathbb{Z}_2] \ltimes \text{Cl}(V^*, B - B^h).$$

But this is by definition, equal to $\text{Cl}(V^* \oplus \mathbb{C}, \left( \begin{array}{cc} B - B^h & 0 \\ 0 & 1 \end{array} \right))$, as desired.

6. **Finite Abelian Categories with Infinitely Many Tensor Structures**

In this section we show that a finite abelian category (i.e. a category of the form $\text{Rep}(A)$, $A$ a finite-dimensional algebra) may admit infinitely many non-equivalent tensor structures with a fixed Grothendieck ring.

We start by formulating a theorem of Schauenburg. Although it will not be needed, we will give a slightly generalized version of this theorem.

**Theorem 6.1.** Two finite-dimensional quasi-bialgebras $H_1, H_2$ are twist equivalent if and only if the categories $\text{Rep}(H_1), \text{Rep}(H_2)$ are tensor equivalent.

**Proof.** Using the same proof as in Section 4.2 of [EG], one can show that if there exists a tensor equivalence $\text{Rep}(H_1) \to \text{Rep}(H_2)$, which preserves ordinary dimensions of objects, then $H_1, H_2$ are twist equivalent. So, it is sufficient to show that in the finite-dimensional case, this property of a tensor equivalence is automatically satisfied.

Consider the Grothendieck ring of $\text{Rep}(H_1)$, with the distinguished basis formed by the irreducible objects $V_1, \ldots, V_n$. Let $V \in \text{Rep}(H_1)$, and consider the operator $L_V$ on the Grothendieck ring of $\text{Rep}(H_1)$, given by left multiplication by $V$ ($W \mapsto V \otimes W$). Then $L_V$ is represented by a matrix whose entries are non-negative.
integers. This matrix has an eigenvector \((\dim(V_1), \ldots, \dim(V_n))\), with positive entries, corresponding to the eigenvalue \(\dim(V)\). But then, by the Frobenius-Perron Theorem, \(\dim(V)\) is the largest real eigenvalue of \(L_V\). Therefore, \(\dim(V)\) is preserved by any tensor equivalence, as desired.

Remark 6.2. Theorem 6.1 was proved in [S] (by a different method) under the assumption that \(H_1, H_2\) are finite-dimensional Hopf algebras.

As a consequence of Schauenburg’s theorem we have the following.

Theorem 6.3. If \((S^2V)^G = 0\) and \(|S^2V/\text{Aut}(G, V)|\) is infinite then for generic \(B \in S^2V\), the abelian category \(\text{Rep}(A(G, V, u, B)^*)\) has infinitely many distinct rigid tensor structures, with the same Grothendieck ring as the tensor category \(\text{Rep}(A(G, V, u, B)^*)\).

Remark 6.4. For example, it is the case in the examples of Section 4 if \(\lambda_1, \lambda_2, \lambda_3\) is a generic complex number.

Remark 6.5. As we already mentioned in the introduction, Ocneanu showed that such examples cannot exist for finite abelian semisimple categories. On the other hand, it is well known that there exist semisimple abelian categories which are not finite (i.e. have infinitely many irreducible objects) and admit infinitely many distinct tensor structures with the same Grothendieck ring. For example, let \(\mathfrak{g}\) be a finite-dimensional complex semisimple Lie algebra. Then the representation category of \(U_q(\mathfrak{g})\), for a generic complex number \(q\), is the same as the representation category of \(U(\mathfrak{g})\), as an abelian category. However, as tensor categories, these categories are distinct (up to the symmetry \(q \rightarrow q^{-1}\)), although their Grothendieck rings are the same.

7. Appendix

The argument used in the proof of Proposition 3.3 relies on the following standard proposition from algebraic geometry.

Proposition 7.1. Let \(Y\) be an algebraic variety with an algebraic action of an algebraic group \(L\). Let \(Y_s\) be the subset of \(Y\) consisting of the points \(y\) where the natural linear map \(\text{Lie}(L) \rightarrow T_yY\) is surjective. Then \(Y_s/L\) is finite.

Let us prove Proposition 7.1, for the convenience of the reader.

Lemma 7.2. Suppose \(Z \subseteq Y\) are algebraic varieties \((Z\) is locally closed in \(Y\)), and at a point \(y \in Z\), one has \(T_yZ = T_yY\). If \(Z\) is smooth, there exists a neighborhood \(U\) of \(y\) in \(Y\) such that \(U \subseteq Z\).

Proof. It is suffices to show that the inclusion \(Z \rightarrow Y\) induces an isomorphism of local rings \(g : O_y(Y) \rightarrow O_y(Z)\). Let \(d := \dim(T_yZ) = \dim(T_yY)\). It is clear that \(\dim(O_y(Y)) \geq d = \dim(T_yY)\), so \(y\) is a smooth point of \(Y\). The rest follows from the algebraic version of the inverse function theorem.

Lemma 7.3. For any point \(y \in Y_s\), \(Ly\) is open in \(Y\).

Proof. Clearly, the lemma follows from Lemma 7.2: we can take \(Z := Ly\).

Proof of Proposition 7.1. The proposition follows from Lemma 7.3 since it implies that the orbits of \(L\) on \(Y_s\) are disjoint open subsets of \(Y\), so the number of such orbits has to be finite.
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References

[AEG] N. Andruskiewitsch, P. Etingof, S. Gelaki, Triangular Hopf algebras with the Chevalley property, Michigan J. Math, to appear, math.QA/0008232.
[AS] N. Andruskiewitsch, H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order $p^3$ J. Algebra 209 (1998), 658–691.
[BDG] M. Beattie, S. Dascalescu, L. Grunenfelder, On the number of types of finite-dimensional Hopf algebras, Invent. Math. 136 (1999), 1–7.
[EG] P. Etingof, S. Gelaki, On cotriangular Hopf algebras, Amer. J. Math. 123 (2001), 699–713.
[EG1] P. Etingof, S. Gelaki, Classification of finite-dimensional triangular Hopf algebras with the Chevalley property, Math. Res. Lett. 8 (2001), 249–255.
[G] S. Gelaki, Pointed Hopf algebras and Kaplansky’s 10th conjecture, J. Algebra 209 (1998), 635–657.
[Gr] M. Graña, Pointed Hopf algebras of dimension 32, Comm. Alg. 28 (2000), 2935–2976.
[K] I. Kaplansky, Bialgebras, Lecture Notes in Math., Department of Mathematics, University of Chicago, Chicago, Ill., (1975).
[M] A. Masuoka, Defending the negated Kaplansky conjecture, Proc. Amer. Math. Soc. 129 (2001), 3185–3192.
[S] P. Schauenburg, Hopf bi-Galois extensions, Comm. Algebra 24 (1996), 3797–3825.

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