In memory of J. C. Huang, late professor of Physics at the University of Missouri-Columbia.

On the Asymptotic Character of Electromagnetic Waves in a Friedmann-Robertson-Walker Universe

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Asymptotic properties of electromagnetic waves are studied within the context of Friedmann-Robertson-Walker (FRW) cosmology. Electromagnetic fields are considered as small perturbations on the background spacetime and Maxwell’s equations are solved for all three cases of flat, closed and open FRW universes. The asymptotic character of these solutions are investigated and their relevance to the problem of cosmological tails of electromagnetic waves is discussed.

KEY WORDS: Classical General Relativity; Cosmology.

1. INTRODUCTION

It is well understood that while propagating, similar to any traveling wave, electromagnetic radiation interacts with the curvature of the spacetime [1]. Such interactions affect the propagation of these waves and cause them to scatter. The scattered waves, or tails, manifest themselves as partial backscattering when the source of the curvature is a localized object. However, when propagating throughout the universe, where the curvature exists everywhere, the scattering of electromagnetic waves due to the interaction with the background curvature occurs throughout the space and at all times. The purpose of this paper is to understand how the background curvature of spacetime affects the propagation and properties of electromagnetic waves, particularly at large distances.

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Studies of this nature have been done by Faraoni and Sonego\cite{2,3} on the analysis of the propagation of scalar fields in FRW spacetime. In those papers, the authors have presented analytical solutions to the Klein-Gordon equation with non-minimal coupling and by studying the asymptotic character of these solutions, they have shown that standard calculations of the reflection and transmission coefficients are not applicable to this case. Scialom and Philippe\cite{4} have also studied the asymptotic behavior of scalar fields near the singular points of Einstein-Klein-Gordon equation in a flat FRW universe. They determined the singular points of Einstein-Klein-Gordon equation and by analytically analyzing the asymptotic behavior of complex scalar fields, showed that these fields, similar to real scalar fields, support the existence of inflationary stage in a flat FRW universe. For vector fields, Noonan\cite{5,6,7} studied the propagation of electromagnetic fields in a curved spacetime and showed that, while propagating in a conformally flat universe, components of the Faraday tensor stay tail-free. In this paper, propagation of electromagnetic waves in the FRW spacetime is studied and the explicit asymptotic character of these field in all three cases of flat, closed, and open universes will be discussed.

To investigate the asymptotic character of electromagnetic fields one has to calculate the electric and magnetic components of these fields as measured by a standard observer, and study their properties at large distances. Here electromagnetic fields are considered as small perturbations on the background curvature, and solutions of Maxwell’s equations are presented for all three cases of flat, closed, and open FRW universes. Asymptotic properties of electromagnetic fields are then investigated by studying these solutions at large distances.

Solutions to Maxwell’s equations in a FRW universe have been presented by different authors\cite{8-11}. Deng and Mannheim\cite{12} have presented solutions to Maxwell’s equations by solving these equations for spherical components of the electric and magnetic fields, separately. More recently, Mankin et al\cite{13,14,15} have presented exact and approximate solutions to electromagnetic wave equations in a curved spacetime, based on the method proposed by Hadamard\cite{16} and by using a higher-order Green’s function for the wave equation. I adopt a very helpful method due to Skrotskii\cite{17} who realized that electromagnetic field equations in a curved spacetime can be written in a non-covariant form formally equivalent to Maxwell’s equations in a macroscopic medium in flat spacetime. The electric and magnetic properties of this medium are tied to the background curvature of spacetime\cite{17,18}. This method has been used by Mashhoon to solve Maxwell’s equations in a closed expanding FRW universe\cite{19}.

The plan of this paper is as follows. Maxwell’s equations are discussed in section 2. In section 3, solutions to Maxwell’s equations are presented for all three cases of flat, closed and open FRW universes. Section 4 has to do with
the study of the asymptotic character of electromagnetic fields, and section 5 concludes this study by reviewing the results and discussing their applications.

The metric convention \( g_{\alpha\beta} = (+, -, -, -) \) with \( ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \) is used throughout this paper. The Greek indices will indicate sums over 0, 1, 2 and 3 while the Latin indices will sum over 1, 2 and 3. The units used in this paper have been chosen such that \( \hbar = c = 1 \) where \( c \) is the speed of light.

2. MAXWELL’S EQUATIONS IN FRW SPACETIME

In a curved spacetime with metric \( g_{\alpha\beta} \), the source-free Maxwell’s equations are given by

\[
\left[ (-g)^{1/2} F^{\alpha\beta} \right]_{,\beta} = 0 ,
\]

and

\[
F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0 ,
\]

where \( F_{\alpha\beta} \) is the electromagnetic field tensor, \( g = \det(g_{\alpha\beta}) \), and \( (,) \) denotes an ordinary differentiation. In accordance with Skrotskii’s formalism [17], a background inertial frame is introduced with Cartesian coordinates in which the electric and magnetic fields are defined using the decompositions \( F_{\alpha\beta} \rightarrow (\vec{E}, \vec{B}) \), and \( \sqrt{-g} F^{\alpha\beta} \rightarrow (-\vec{D}, \vec{H}) \). That is,

\[
E_a = F_{a0} , \quad D_a = (-g)^{1/2} F^{0a} ,
\]

and

\[
B_a = \frac{1}{2} \epsilon_{abc} F_{bc} , \quad H_a = \frac{1}{2} \epsilon_{abc} (-g)^{1/2} F^{bc} ,
\]

where \( \epsilon_{abc} \) is the three-dimensional Levi-Civita symbol. Using these quantities, equations (1) and (2) can be written as [18]

\[
D_{a,a} = B_{a,a} = 0 ,
\]

and

\[
-D_{a,0} + \epsilon_{abc} H_{c,b} = B_{a,0} + \epsilon_{abc} E_{c,b} = 0 .
\]

Equations (5) and (6) are formally equivalent to the electromagnetic field equations in a medium in flat spacetime with a dielectric constant \( \varepsilon_{ab} \) and a permutivity \( \mu_{ab} \) given by

\[
\varepsilon_{ab} = \mu_{ab} = -(-g)^{1/2} \frac{g_{ab}}{g_{00}} .
\]
In this auxiliary medium and with the assumed Cartesian coordinate system, \((\vec{E}, \vec{D})\) represent the electric fields and \((\vec{B}, \vec{H})\) are their corresponding magnetic fields. These vector fields are related via constitutive relations \([19]\)

\[
D_a = \varepsilon_{ab}E_b - (\vec{G} \times \vec{H})_a, \tag{8}
\]

\[
B_a = \mu_{ab}H_b + (\vec{G} \times \vec{E})_a, \tag{9}
\]

where \(G_a = -g_{aa}/g^{00}\).

The above-mentioned formalism is applicable to any curved spacetime. In this study the curved spacetime of interest is FRW with a metric given by

\[
(ds)^2 = (dt)^2 - S^2(t)R_0^2 \left\{ \frac{(d\mathcal{R})^2}{(1 - k\mathcal{R}^2)} + \mathcal{R}^2(d\theta)^2 + \mathcal{R}^2 \sin^2 \theta (d\phi)^2 \right\}. \tag{10}
\]

In equation (10), \(R_0\) is the radius of the model universe at some epoch \(t_0\). The expansion parameter of this model universe is given by \(S(t)\) such that \(S(t_0) = 1\). The product \(S(t)R_0\) in equation (10) is the cosmic scale factor, and \(k = -1, 0, +1\) corresponding to open, flat, and closed universes, respectively. For \(k = +1\), \(\mathcal{R}\) ranges from 0 to 1.

Introducing \(x_1, x_2\) and \(x_3\) given by \(r \sin \theta \cos \phi, r \sin \theta \sin \phi\) and \(r \cos \theta\), respectively, the line element (10) can be written as

\[
(ds)^2 = C^2(\eta) \left\{ (d\eta)^2 - [f_k(r)]^2 (\delta_{\hat{w}\hat{w}'}dx^\hat{w}dx^\hat{w}') \right\}. \tag{11}
\]

In this equation, \(\eta\), the conformal time, is given by \(dt = S(t)d\eta\), and \(C(\eta) = S(t)\). The function \(f_k(r)\) in equation (11) is equal to

\[
f_k(r) = \left[ 1 + k \left( \frac{r}{2R_0} \right)^2 \right]^{-1}, \tag{12}
\]

where \(r\) is defined via

\[
\mathcal{R} = \frac{r}{R_0} \left[ 1 + k \left( \frac{r}{2R_0} \right)^2 \right]^{-1}. \tag{13}
\]

In terms of the new coordinates \((\eta, x_1, x_2, x_3)\), the dielectric constant \(\varepsilon_{ab}\) and permittivity \(\mu_{ab}\) are given by

\[
\varepsilon_{ab} = \mu_{ab} = f_k(r)\delta_{ab}. \tag{14}
\]
Also in this new coordinate system, $G_a = 0$. Substituting these quantities in equations (5) and (6), one can show that Maxwell’s equations can be written as \[12\]
\[-i\vec{\nabla} \times \vec{I} = f_k(r) \frac{\partial \vec{I}}{\partial \eta}, \quad (15)\]
and
\[\vec{\nabla} \cdot \left[ f_k(r) \vec{I} \right] = 0, \quad (16)\]
where $\vec{I} = \vec{E} + i \vec{H}$. Equations (15) and (16) represent the electromagnetic field equations for the line element (11). It is evident from these equations that for the metric (11), electromagnetic fields are independent of $S(t)$. This is due to the explicit conformal invariance of the formalism above, and implies that in a universe with a background curvature given by equation (11), the physical parameters of electromagnetic fields vary adiabatically with the changing radius of the universe. Among the solutions of equations (15) and (16), only those that represent $\vec{E}$ and $\vec{H}$ at a conformal time $\eta$, corresponding to $t = \tilde{\tau}$, are solutions to the Maxwell’s equations in a FRW universe at time $\tilde{\tau}$.

3. SOLUTIONS OF MAXWELL’S EQUATIONS

The isotropy and homogeneity of FRW spacetime implies that it is possible to find solutions to equations (15) and (16) for a definite angular momentum $J$, and its component along the $x_3$-axis, $M$. Denoting such solutions by $\vec{I}_{JM\sigma_k}(\vec{r}, \eta)$, one can write
\[\vec{I}_{JM\sigma_k}(\vec{r}, \eta) = \sum_{\lambda=e,m,0} \vec{T}_{JM\sigma_k}(r) \vec{Y}_{JM}^{\lambda}(\theta, \phi) \, e^{-i \sigma_k \eta}, \quad (17)\]
where $\vec{I}_{JM\sigma_k}(\vec{r}, \eta)$ have been expanded in terms of the vector spherical harmonics $\vec{Y}_{JM}^{\lambda}(\theta, \phi)$ [19,20]. In equation (17), $\sigma_k$ is a positive constant and $\lambda = e, m, 0$ representing different types of vector spherical harmonics [20]. For a definite parity and angular momentum, the electric and magnetic components of $\vec{I}_{JM\sigma_k}(\vec{r}, \eta)$ can be written as [21]
\[\vec{E}_{JM\sigma_k}^{m}(\vec{r}) = \vec{H}_{JM\sigma_k}^{e}(\vec{r}) = \vec{T}_{JM\sigma_k}(r) \vec{Y}_{JM}^{m}(\theta, \phi), \quad (18)\]
and
\[\vec{E}_{JM\sigma_k}^{e}(\vec{r}) = -\vec{H}_{JM\sigma_k}^{m}(\vec{r}) = i \left[ \vec{T}_{JM\sigma_k}(r) \vec{Y}_{JM}^{e}(\theta, \phi) + \vec{T}_{JM\sigma_k}(r) \vec{Y}_{JM}^{0}(\theta, \phi) \right], \quad (19)\]
where from equations (15) and (16), \( I_{\sigma k}^\lambda (r) \) are given by

\[
I_{\sigma k}^\lambda (r) = -\frac{1}{\sigma_k f_k(r)} \left( \frac{d}{dr} + \frac{1}{r} \right) T_{\sigma k}^m (r),
\]

(20)

\[
T_{\sigma k}^0 (r) = -\frac{1}{\sigma_k r f_k(r)} \left[ J(J + 1) \right]^{1/2} I_{\sigma k}^m (r),
\]

(21)

\[
\left[ \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) - \frac{1}{f_k(r)} \frac{d f_k(r)}{dr} \left( \frac{1}{r} + \frac{d}{dr} \right) \right. \\
\left. - \frac{J(J + 1)}{r^2} + \sigma_k^2 f_k^2(r) \right] I_{\sigma k}^m (r) = 0.
\]

(22)

The most general solution to equations (15) and (16) is a superposition of the fields (18) and (19). That is,

\[
\vec{E}(\vec{r}, \eta) = \sum_{J,M,\sigma_k} \sum_{\lambda'=e,m} \left[ a_{jM\sigma_k}^{\lambda'} (\eta) \vec{E}_{jM\sigma_k}^{\lambda'} (\vec{r}) + a_{jM\sigma_k}^{\lambda'*} (\eta) \vec{E}_{jM\sigma_k}^{\lambda'*} (\vec{r}) \right],
\]

(23)

and

\[
\vec{H}(\vec{r}, \eta) = \sum_{J,M,\sigma_k} \sum_{\lambda'=e,m} \left[ a_{jM\sigma_k}^{\lambda'} (\eta) \vec{H}_{jM\sigma_k}^{\lambda'} (\vec{r}) + a_{jM\sigma_k}^{\lambda'*} (\eta) \vec{H}_{jM\sigma_k}^{\lambda'*} (\vec{r}) \right],
\]

(24)

where * indicates complex conjugation and from equation (15), \( a_{jM\sigma_k}^{\lambda'} (\eta) \) satisfies

\[
\frac{d}{d\eta} a_{jM\sigma_k}^{\lambda'} (\eta) = -i\sigma_k a_{jM\sigma_k}^{\lambda'} (\eta).
\]

(25)

In the following, solutions of the differential equations (20) to (22) are presented for all three cases of flat, closed and open FRW universes.
3-a. Flat Universe

In a flat universe where $k = 0$, equation (22) is written as

$$
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \left( \frac{d}{dr} \right) - \frac{J(J + 1)}{r^2} + \sigma_0^2 \right] T^m_{\sigma_0}(r) = 0. \quad (26)
$$

The general solution of equation (26) can be written in terms of Hankel functions $h^{(1,2)}$. That is,

$$
T^m_{\sigma_0}(r) = A h^{(1)}_J(r\sigma_0) + B h^{(2)}_J(r\sigma_0), \quad (27)
$$

where $A$ and $B$ are constant quantities. In equation (27),

$$
h^{(1)}_J(r\sigma_0) = j_J(r\sigma_0) + in_J(r\sigma_0), \quad (28)
$$

and

$$
h^{(2)}_J(r\sigma_0) = j_J(r\sigma_0) - in_J(r\sigma_0), \quad (29)
$$

where $j_J$ and $n_J$ are spherical Bessel and Neumann functions, respectively, and $J = 0, 1, 2, 3, \ldots$. Because $n_J(r\sigma_0)$ is singular at $r = 0$, solutions of equation (26) are proportional only to spherical Bessel functions. Using

$$
\int_0^{\infty} j_\ell(r') j_{\ell'}(r') dr' = \frac{\pi}{2(2\ell + 1)} \delta_{\ell\ell'}, \quad (30)
$$

these solutions can be written as

$$
T^m_{\sigma_0}(r) = \left[ \frac{2\sigma_0(2J + 1)}{\pi} \right]^{1/2} j_J(r\sigma_0), \quad (31)
$$

where, $T^m_{\sigma_0}(r\sigma_0)$ has been normalized such that

$$
\int_0^{\infty} T^m_{\sigma_0}(r\sigma_0) T^m_{\sigma_0'}(r\sigma_0) dr = \delta_{JJ'}. \quad (32)
$$
Substituting for $I_{\lambda \sigma_0}^\lambda (r)$ from equation (31) in equations (20) and (21), one can show that

$$I_{\lambda \sigma_0}^e (r) = \left[ \frac{2\sigma_0}{\pi(2J+1)} \right]^{1/2} \left[ Jj_{j+1}(r\sigma_0) - (J+1)j_{j-1}(r\sigma_0) \right],$$  \hspace{1cm} (33)

and

$$I_{\lambda \sigma_0}^0 (r) = -\frac{1}{r} \left[ \frac{2J(J+1)(2J+1)}{\sigma_0 \pi} \right]^{1/2} j_j(r\sigma_0).$$  \hspace{1cm} (34)

Equations (31), (33) and (34) represent the $r$-dependence of the electric and magnetic fields given by equations (18) and (19). Figure 1 shows these functions for different values of $J$ and $\sigma_0$. From equation (34) one can see that
\( \mathcal{I}^0_{\sigma_0}(r) \) vanishes when \( J = 0 \). That means, \( \vec{E}_0^{e,\sigma_0}(\vec{r}) \) and \( \vec{H}_0^{m,\sigma_0}(\vec{r}) \) will be pure e-type for all values of \( \sigma_0 \). One can also see from figure 1 that for any state of \( \lambda \), when \( J \) is constant, as expected, increasing \( \sigma_0 \) results in increasing the frequency of the solutions of equation (26). This figure also shows that among different states of \( \lambda \), \( \mathcal{I}^0_{\sigma_0}(r) \) approaches zero more rapidly implying that \( \mathcal{I}^m_{\sigma_0}(r) \) and \( \mathcal{I}^e_{\sigma_0}(r) \) will have dominating effects on the behavior of \( \vec{E} \) and \( \vec{H} \) fields at large distances.

3-b. Closed Universe

In a closed FRW universe, \( k = 1 \), and therefore \( f_k(r) = f_1(r) = 1/\{1 + [r/(2R_0)]^2\} \). Equation (22), in this case, is given by

\[
\frac{d^2}{d\chi^2} \Psi^m_{\omega_1}(\chi) + \left[ \omega_1^2 - \frac{J(J+1)}{\sin^2 \chi} \right] \Psi^m_{\omega_1}(\chi) = 0 ,
\]

where \( \omega_1 = R_0 \sigma_1, \chi = 2\tan^{-1}[r/(2R_0)] \), and \( \Psi^m_{\omega_1}(\chi) = R_0 r \mathcal{I}^m_{\omega_1}(r) \). In terms of the variable \( \chi \), equation (35) resembles a Schrödinger equation with a potential function given by \( J(J+1)/\sin^2 \chi \), and eigenvalues equal to \( \omega_1^2 \). Figure 2 shows the graph of this potential function for \( J = 1, 2, 3, 4 \). It is evident from this figure that for all values of \( J \), the solutions of equation (35) must be finite for all values of \( \chi \in [0, \pi] \). Using this boundary condition, the solution of equation (35) can be written as
\[ \Psi_{m}^{J,\omega_{1}}(\chi) = 2^{J+1}J! \frac{(\omega_{1} - J - 1)!}{(\omega_{1} + J)!} \left( \sin \chi \right)^{J+1} C_{\omega_{1} - J - 1}^{J+1}(\cos \chi), \] (36)

where \( J = 1, 2, 3, \ldots \) and \( \omega_{1} = 2, 3, 4, \ldots \) representing frequencies of different modes of electromagnetic fields for the line element (11) \[8,9,19\]. In equation (36), \( C_{\nu}^{\Upsilon}(\Theta) \) are Gegenbauer polynomials \[22\] which are orthogonal over the interval \((-1, 1)\) with the weight function \( W(\Theta) = (1 - \Theta^2)^{\nu - 1/2}, \nu > -1/2\).

Orthogonality of \( C_{\omega_{1} - J - 1}^{J+1}(\cos \chi) \) requires \( \Psi_{m}^{J,\omega_{1}}(\chi) \) to be normalized such that

\[ \int_{0}^{\pi} \Psi_{m}^{J,\omega_{1}}(\chi) \Psi_{m}^{J,\omega_{1}}(\chi) d\chi = 2\pi \omega_{1} \delta_{\omega_{1} \omega_{1}'}. \] (37)

Using

\[ \frac{d}{d\Theta} C_{\nu}^{\Upsilon}(\Theta) = 2\nu C_{\nu+1}^{\Upsilon-1}(\Theta), \] (38)

and the recurrence relation

\[ 2\nu(1 - \Theta^2) C_{\nu+1}^{\Upsilon-1}(\Theta) = (\Upsilon + 2\nu - 1)C_{\nu-1}^{\Upsilon}(\Theta) - \Theta \Theta C_{\nu}^{\Upsilon-1}(\Theta), \] (39)

and also from the definition of \( \Psi_{m}^{J,\omega_{1}}(\chi) \), and equations (20) and (21), the \( r \)-dependence of the electric and magnetic fields in a closed FRW universe are given by

\[ I_{m}^{J,\omega_{1}}(\chi) = 2^{J}J! \frac{\omega_{1}!}{R_{0}^2} \left[ \frac{(\omega_{1} - J - 1)!}{(\omega_{1} + J)!} \right]^{1/2} (1 + \cos \chi) \sin^{J} \chi C_{\omega_{1} - J - 1}^{J+1}(\cos \chi), \] (40)

\[ I_{e}^{J,\omega_{1}}(\chi) = -2^{J}J! \frac{\omega_{1}!}{R_{0}^2} \left[ \frac{(\omega_{1} - J - 1)!}{(\omega_{1} + J)!} \right]^{1/2} (1 + \cos \chi) \sin^{J} \chi \]

\[ \left\{ \omega_{1} \cos \chi C_{\omega_{1} - J - 1}^{J+1}(\cos \chi) + (\omega_{1} + J) C_{\omega_{1} - J - 2}^{J+1}(\cos \chi) \right\}, \] (41)

\[ I_{0}^{J,\omega_{1}}(\chi) = -2^{J}J! \frac{J(J+1)(\omega_{1} - J - 1)!}{R_{0}^2} \left[ \frac{J(J+1)}{(\omega_{1} + J)!} \right]^{1/2} (1 + \cos \chi) \sin^{J} \chi C_{\omega_{1} - J - 1}^{J+1}(\cos \chi). \] (42)
Figure 3. From top to bottom, graphs of $R_0^2 I_{\lambda J \omega_1}^\chi (r)$ for $\lambda = m, e, 0$ and for different values of $J$ and $\omega_1$. When $\lambda = m, 0$, the solid line corresponds to $\omega_1 = J + 1$, the dashed line correspond to $\omega_1 = J + 2$ and the dotted line corresponds to $\omega_1 = J + 3$. For $\lambda = e$, the value of $\omega_1$ is equal to $J + 2$, $J + 3$ and $J + 4$ for the solid line, dashed line and the dotted line, respectively. Note the different scales on the vertical axes.

Figure 3 shows the quantity $R_0^2 I_{\lambda \omega_1}^\chi (r)$ for different values of $J$ and $\omega_1$. It is important to mention that because the index $(\omega_1 - J \pm 1)$ in $C_{\omega_1 - J \pm 1}^\chi (\cos \chi)$ can only be equal to zero or positive integers, the graphs of figure 3 contain degeneracies on $J$ and $\omega_1$. As shown in this figure, for a given value of $R_0$, $I_{\lambda \omega_1}^\chi (r)$ vanishes at $\chi = 0$ and $\pi$. This is quite expected since from figure 2, the potential function of the Schrödinger-type equation (35) approaches infinity at these two values of $\chi$. This indicates that in a model closed FRW universe, different modes of electromagnetic radiations, which correspond to different values of $\omega_1$, are extended over the entire volume of the universe. Each of these modes is a member of a complete set of eigenfunctions with amplitude that increase by increasing the frequency. An electromagnetic field is a superposition of these eigenfunctions.
Figure 4. Graph of $J(J+1)/\sinh^2 u$ against $u$ for $J = 1, 2, 3, 4$.

3-c. Open Universe

In an open FRW universe where $k = -1$, and $f_k(r) = f_{-1}(r) = 1/[1 - (r/(2R_0))^2]$, equation (22) can be written as

$$
\frac{d^2}{du^2} \Phi^m_{\omega-1}(u) + \left[ \omega^{-1}_1 - \frac{J(J+1)}{\sinh^2 u} \right] \Phi^m_{\omega-1}(u) = 0. \quad (43)
$$

In this equation, $\omega^{-1}_1 = R_0\sigma^{-1}_1$, $u = 2\tanh^{-1}[r/(2R_0)]$ and $\Phi^m_{\omega-1}(u) = R_0r I^m_{\omega-1}(r)$. Equation (43) is a Schrödinger-type differential equation with a potential function of $J(J+1)/\sinh^2 u$. Figure 4 shows this function for different values of $J$.

With appropriate change of variables, it is possible to show that equation (43) is, in fact, the differential equation of a hypergeometric function. Let

$$
\zeta^m_{\omega-1}(\xi) = (i\cosh u)^{i\omega-1}_1 \Phi^m_{\omega-1}(u), \quad (44)
$$

and

$$
\xi = \frac{1}{2} (1 - \coth u). \quad (45)
$$

Substituting for $\Phi^m_{\omega-1}(u)$ and $u$ from equations (44) and (45) in equation (43), one can show that this equation can be written as
\[
\xi(1-\xi)\frac{d^2\zeta_{\omega_{-1}}^m(\xi)}{d\xi^2} + (1-i\omega_{-1})(1-2\xi)\frac{d\zeta_{\omega_{-1}}^m(\xi)}{d\xi} + \left[J(J+1)+i\omega_{-1}(1-i\omega_{-1})\right]\zeta_{\omega_{-1}}^m(\xi) = 0. \tag{46}
\]

Introducing \(p = J+1-i\omega_{-1}, \ q = -J-i\omega_{-1}, \) and \(s = 1-i\omega_{-1}\) equation (46) represents the differential equation of a hypergeometric function [22] with the general form of

\[
\xi(1-\xi)\frac{d^2\zeta_{\omega_{-1}}^m(\xi)}{d\xi^2} + \left[ s - (p+q+1)\xi \right]\frac{d\zeta_{\omega_{-1}}^m(\xi)}{d\xi} - pq\zeta_{\omega_{-1}}^m(\xi) = 0. \tag{47}
\]

Equation (47), as the general differential equation of a hypergeometric function, has three regular singular points at \(\xi = 0, 1, \) and \(\infty.\) In the neighborhood of each of these singularities, equation (47) has two independent solutions [22]. Since for all values of \(0 < u < \infty,\) from equation (45), \(-\infty < \xi < 0,\) among all these solutions, only the ones in the neighborhood of \(\xi = 0\) are valid solutions of equation (47). Within this range, equation (47) has only one regular singular point at \(\xi = 0,\) and its general solution is given by [22]

\[
\zeta_{\omega_{-1}}^m(\xi) = BF(p, q, s, \xi) + D\xi^{1-s}F(p-s+1, q-s+1, 2-s, \xi). \tag{48}
\]

In this equation,

\[
F(p, q, s, \xi) = 1 + \frac{pq}{s}\frac{\xi}{1!} + \frac{pq}{s}\frac{(p+1)(q+1)}{(s+1)2!} + \ldots, \tag{49}
\]

is the hypergeometric function, defined within its circle of convergence \(|\xi| < 1,\) and \(B\) and \(D\) are constant quantities. It is necessary to mention that solution (48) converges conditionally at \(\xi = -1,\) since \(\text{Re}(s-p-q) = 0 [23].\)

To determine the values of the two constant quantities \(B\) and \(D,\) it is more convenient to study the asymptotic character of \(\Phi_{\omega_{-1}}^m(u)\) at \(u \to 0\) and \(\infty.\) Using the identity

\[
F(p, q, s, \xi) = (1-\xi)^{s-p-q}F(s-p, s-q, s, \xi), \tag{50}
\]

and considering that, as shown by equation (49), \(F(p, q, s, \xi)\) is symmetric on \(p\) and \(q,\) equation (48) can be written as

\[
\zeta_{\omega_{-1}}^m(\xi) = BF(1-\xi)^{i\omega_{-1}}F(-J, J+1, 1-i\omega_{-1}, \xi)
\]

\[
+ D\xi^{i\omega_{-1}}F(-J, J+1, 1+i\omega_{-1}, \xi). \tag{51}
\]

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From equations (44) and (45), one can show that equation (51) simplifies to
\[
\Phi^m_{\omega_{-1}}(u) = B \left( \frac{-i e^u}{2} \right)^{i \omega_{-1}} F(-J, J + 1, 1 - i \omega_{-1}, \xi) \\
+ D \left( \frac{i e^{-u}}{2} \right)^{i \omega_{-1}} F(-J, J + 1, 1 + i \omega_{-1}, \xi).
\]
(52)

As shown in figure 4, the potential function of the Schrödinger-type equation (43) approaches infinity when \( u \to 0 \), and it tends to zero when \( u \to \infty \). That implies, the solutions of equation (43) are finite for all values of \( u \in (0, \infty) \). The general solution should therefore be a superposition of ingoing and outgoing radiation as in equation (52). However, the physical interpretation of the FRW universes based on the big bang model implies that all the radiation must be outgoing. That means, when \( u \to \infty \), the only possible solution for equation (43) is an outgoing wave [2, 24]. This requires no incoming radiation in equation (52). However, when \( u \to \infty \), the quantity \( \xi \) tends to zero, \( F(p, q, s, \xi) \to 1 \), and the second term of equation (52) represents a pure incoming wave. The condition above implies that, \( D = 0 \), and as a result, the solution of equation (43) is written as
\[
\Phi^m_{\omega_{-1}}(u) = B \left( \frac{-i e^u}{2} \right)^{i \omega_{-1}} F(-J, J + 1, 1 - i \omega_{-1}, \xi).
\]
(53)

Setting \( B = 1 \), equation (53) represents an outgoing radiation with frequency \( \omega_{-1} \). From the definition of \( \Phi^m_{\omega_{-1}}(u) \), and from equations (20) and (21), the \( r \)-dependent terms of electromagnetic fields are now given by
\[
I^m_{\omega_{-1}}(r) = \frac{1}{r R_0} X(r) F(-J, J + 1, 1 - i \omega_{-1}, \xi(r)),
\]
(54)
\[
I^e_{\omega_{-1}}(r) = -\frac{i}{r R_0} X(r) \left\{ F(-J, J + 1, 1 - i \omega_{-1}, \xi(r)) \\
+ \frac{i R_0^2 J(J + 1)}{2 r^2 \omega_{-1} (1 - i \omega_{-1})} [f_{-1}(r)]^{-2} F(-J + 1, J + 2, 2 - i \omega_{-1}, \xi(r)) \right\},
\]
(55)
and
\[
I^0_{\omega_{-1}}(r) = -\frac{1}{r^2 \omega_{-1}} \left[ J(J + 1) \right]^{1/2} [f_{-1}(r)]^{-1} \\
X(r) F(-J, J + 1, 1 - i \omega_{-1}, \xi(r)).
\]
(56)
Figure 5. From top to bottom, graphs of $R_0^2|I^\lambda_{\omega_{-1}}(r)|$ for $\lambda = m, e, 0$, and for different values of $J$ and $\omega_{-1}$. The solid line corresponds to $\omega_{-1} = 1$, the dashed line corresponds to $\omega_{-1} = 2$ and the dotted line corresponds to $\omega_{-1} = 3$. Note different scales on vertical axes.

In equations (54), (55), and (56),

$$X(r) = \left[ -\frac{i}{2} \left( \frac{1 + \varsigma(r)}{1 - \varsigma(r)} \right) \right]^{i\omega_{-1}},$$  \hspace{1cm} (57)

$$\varsigma(r) = \left( r/2R_0 \right), \text{ and the derivatives of the hypergeometric function } \mathcal{F}(p, q, s, \xi(r)) \text{ have been replaced by}$$

$$\frac{d}{dr} \mathcal{F}(p, q, s, \xi(r)) = \frac{pq}{s} \mathcal{F}(p + 1, q + 1, s + 1, \xi(r)) \left[ \frac{d\xi(r)}{dr} \right]. \hspace{1cm} (58)$$

Figure 5 shows the product of the absolute values of complex functions $I^\lambda_{\omega_{-1}}(r)$ and $R_0^2$ in terms of $\varsigma$, and for different values of $J$ and $\omega_{-1}$. It is necessary to mention that in an open FRW universe, $\mathcal{R} \in (0, \infty)$, which from equation (13)
Figure 6. Graphs of $R_0^2|\mathcal{I}_{\omega_{-1}}^\lambda (r)|$ for $\lambda = m$ (left column), $\lambda = e$ (right column), and for different values of $J$ and $\omega_{-1}$. The solid line corresponds to $J = 0$, the dashed line corresponds to $J = 1$ and the dotted line corresponds to $J = 5$. As mentioned in the text, the quantities $R_0^2|\mathcal{I}_{\omega_{-1}}^m (2R_0)|$ and $R_0^2|\mathcal{I}_{\omega_{-1}}^e (2R_0)|$ are independent of the value of $J$ and are equal to $0.5 \exp(\omega_{-1} \pi/2)$. For $\omega_{-1} = 1$ this value is $\sim 2.4$ (top graphs), for $\omega_{-1} = 2$, it is $\sim 11.57$ (middle graphs), and when $\omega_{-1} = 3$, it attains the value of $\sim 55.66$ (bottom graphs). Note different scales on vertical axes.
implies $0 < r < 2R_0$. As shown here, for a given value of the radius of universe, $T^\lambda_{\omega \gamma_1}(r)$ approaches zero for $\lambda = 0$, and a non-zero constant value for $\lambda = e, m$. Using identities $X^Y = \exp(Y \ln X)$, and $\pm i = \exp[\pm i(\pi/2 + 2\pi l)]$, and choosing the principal branch on the complex plane, one can show that this constant value is equal to $0.5 \exp(\omega_1 \pi/2)$ as demonstrated in figure 6.

4. Asymptotic Character of $\vec{E}(\vec{r}, \eta)$ and $\vec{H}(\vec{r}, \eta)$

To study the asymptotic behavior of electromagnetic fields, one has to evaluate the magnitude of $\vec{E}$ and $\vec{H}$ for large values of $R$. To do so, imagine a set of fundamental observers in FRW universes, at rest in space with constant spatial coordinates $x_1, x_2$ and $x_3$ as given by equation (11). At these coordinates, the measured components of the Faraday tensor are given by

$$F_{(\rho\nu)} = F_{\alpha\beta} \Lambda^\alpha_{(\rho)} \Lambda^\beta_{(\nu)}, \quad (59)$$

where

$$\Lambda^\alpha_{(0)} = \left(C^{-1}(\eta), 0, 0, 0\right), \quad (60)$$

$$\Lambda^\alpha_{(1)} = \left(0, C^{-1}(\eta)f_k^{-1}(r), 0, 0\right), \quad (61)$$

$$\Lambda^\alpha_{(2)} = \left(0, 0, C^{-1}(\eta)f_k^{-1}(r), 0\right), \quad (62)$$

$$\Lambda^\alpha_{(3)} = \left(0, 0, 0, C^{-1}(\eta)f_k^{-1}(r)\right), \quad (63)$$

are the orthonormal tetrad frames associated with these observers calculated from equation

$$g^{\alpha\beta} = \eta^{(\rho\nu)}_M \Lambda^\alpha_{(\rho)} \Lambda^\beta_{(\nu)}. \quad (64)$$

In equation (64), $\eta^{(\rho\nu)}_M$ is the metric of the Minkowski spacetime and $g^{\alpha\beta}$ represents the line element of FRW universes as given by equation (11). Using equations (3) and (4) and also equations (7) to (9), the components of Faraday tensor $F_{(\rho\nu)}$ can be written as

$$F_{(\rho\nu)} = C^{-2}(\eta) \left[f_k(r)\right]^{-1} \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H_3 & -H_2 \\ -E_2 & -H_3 & 0 & H_1 \\ -E_3 & H_2 & -H_1 & 0 \end{pmatrix}. \quad (65)$$
In equation (65), \((E_1, E_2, E_3)\) and \((H_1, H_2, H_3)\) represent the components of the electric and magnetic fields \(\vec{E}(\vec{r}, \eta)\) and \(\vec{H}(\vec{r}, \eta)\), as given by equations (23) and (24), along \(x_1\)-, \(x_2\)-, and \(x_3\)-axis. To study the asymptotic behavior of \(\vec{E}\) and \(\vec{H}\), one has to study the properties of these components for large values of \(R\). Such a study is meaningful only within the context of a flat and an open FRW universe. In a closed FRW universe, as shown in section 3-b, at any given epoch, the radial parts of the solution of equations (15) and (16) extend within the entire volume of the universe. This characteristic of \(I^{\lambda}_{\omega k}(r)\) makes the study of the asymptotic behavior of electromagnetic fields in a closed FRW universe implausible.

From equations (18) and (19), and for given values of \(J, M, \sigma_k\) and \(\lambda\), the \(r\)-dependence of electromagnetic fields is given by \(I^{\lambda}_{\omega k}(r)\). As a result, the measured Faraday tensor \(F_{(\rho(\nu)}\) will have an \(r\)-dependence of the form \(I^{\lambda}_{\omega k}(r)/f_k(r)\). In a flat FRW universe, \(R = r/R_0\), \(f_0(r) = 1\), and from equations (31), (33) and (34), for a constant value of \(\sigma_0\), the functions \(I^{\lambda}_{\omega k}(r)\) are proportional to spherical Bessel functions \(j_j(r)\). For large values of \(R\), the asymptotic values of these functions are given by [22,25]

\[
j_j(r') \sim \frac{1}{r'} \sin \left( r' - \frac{J\pi}{2} \right).
\]

Equation (66) indicates that, as expected, in a flat FRW universe, electromagnetic fields approach zero at large distances with amplitudes decreasing as \(R^{-1}\). A result that is also seen from figure 1. In an open FRW, as indicated by equations (54), (55) and (56), \(I^{\lambda}_{\omega k}(r)\) approach constant values as \(r \rightarrow 2R_0\). Therefore from equation (65), the measured fields approach zero since \([f_0(r)]^{-1} = \{1 - \frac{r}{(2R_0)^2}\} \rightarrow 0\). It follows that, similar to a flat universe, in an open FRW universe also, measured fields approach zero as \(R^{-1}\) as \(R \rightarrow \infty\).

5. SUMMARY

The results of a study of the asymptotic properties of electromagnetic waves in a FRW universe were presented. Electromagnetic fields were considered as small perturbations on the background curvature of FRW spacetime, and the metric of the universe was written in form of a line element that, at a conformal time corresponding to a given epoch, is electromagnetically equivalent to FRW metric. The solutions of Maxwell’s equations were obtained for all three cases of flat, closed and open FRW universes.

The isotropy and homogeneity of FRW metric allows writing the solutions of the electromagnetic field equations in terms of vector spherical harmonics. Expansions to generalized spherical harmonics have also been presented by
Mankin et al [13-15,26], and by Laas et al [27] to construct exact and approximate solutions to electromagnetic wave equations in a curved spacetime using higher-order Green’s function method.

It was shown that using appropriate transformations, the equations governing the \( r \)-dependence of the electromagnetic fields for the type-\( m \) vector spherical harmonics could be written in form of a one-dimensional Schrödinger-type equation whose eigenvalues represent the frequencies of different modes of electromagnetic radiations. In a closed FRW universe, the solutions of the above-mentioned equation resemble standing waves, extended over the entire volume of the universe, implying that the study of their asymptotic character would be implausible.

The asymptotic values of electromagnetic fields in a flat and an open FRW universe were obtained by calculating these fields in the local frame of an observer at large distances. Analysis of the results indicated that, as expected, these fields tend to zero as \( R^{-1} \) at large distances.

An application of the results of the analysis presented here would be in the study of the tails of electromagnetic waves in FRW universes. As shown by Noonan [5,6,7], the vector potential of an electromagnetic wave has a non-zero tail when propagating in a curved spacetime. However, in conformally flat universes such as FRW, because of the conformal invariance of Maxwell’s equations, the components of Faraday’s tensor have no tails [5,6]. Such a conclusion can trivially be made in a flat universe where the background spatial curvature is non-existent, and can be attributed to the fact that in a (3+1) flat universe, the four dimensional Green’s function of the spacetime has a delta-function character, and as a result, only motions along the light cone are supported. For a closed FRW universe, as mentioned in section 3-b, because the solutions of its corresponding Schrödinger-type differential equation resemble standing waves, the concept of tails is not applicable. However, such studies are quite plausible in an open FRW universe. For a specific electromagnetic wave, in this case, the tail-free nature of the components of Faraday tensor can be examined by expanding these components in terms of \( \vec{E}^\lambda_{\mu\nu\sigma}(\vec{r}) \) and \( \vec{H}^\lambda_{\mu\nu\sigma}(\vec{r}) \) as given by equations (18) and (19), and calculating their asymptotic values for large values of \( \mathcal{R} \). For given values of \( J, M \) and \( \sigma_0 \), the \( r \)-dependence of these fields are given by \( \mathcal{I}^\lambda_{\mu\nu\sigma}(r)/f_{-1}(r) \). As shown in section 4, for large values of \( \mathcal{R} \), these quantities approach zero as \( \mathcal{R}^{-1} \), indicating the tail-free nature of electromagnetic waves.

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