Improved Algorithms for Solving Polynomial Systems over GF(2) by Multiple Parity-Counting

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Abstract. We consider the problem of finding a solution to a multivariate polynomial equation system of degree $d$ in $n$ variables over $\mathbb{F}_2$. For $d = 2$, the best-known algorithm for the problem is by Bardet et al. [J. Complexity, 2013] and was shown to run in time $O(2^{0.792n})$ under assumptions that were experimentally found to hold for random equation systems. The best-known worst-case algorithm for the problem is due to Björklund et al. [ICALP’19]. It runs in time $O(2^{0.804n})$ for $d = 2$ and $O(2^{(1-1/(2.7d))n})$ for $d > 2$.

In this paper, we devise a worst-case algorithm that improves the one by Björklund et al. It runs in time $O(2^{0.6943n})$ for $d = 2$ and $O(2^{(1-1/(2d))n})$ for $d > 2$. Our algorithm thus outperforms all known worst-case algorithms, as well as ones analyzed for random equation systems. We also devise a second algorithm that outputs all solutions to a polynomial system and has similar complexity to the first (provided that the number of solutions is not too large).

A central idea in the work of Björklund et al. was to reduce the problem of finding a solution to a polynomial system over $\mathbb{F}_2$ to the problem of counting the parity of all solutions. A parity-counting instance was then reduced to many smaller parity-counting instances. Our main observation is that these smaller instances are related and can be solved more efficiently by a new algorithm to a problem which we call multiple parity-counting.

Keywords: Multivariate equation systems, polynomial method.

1 Introduction

We study the problem of solving a system of multivariate polynomial equations over the field $\mathbb{F}_2$, which is a fundamental problem in computer science. The input to this problem consists of a system of $m$ polynomials $E = \{P_1, \ldots, P_m\}$ such that for each $j = 1, \ldots, m$, $P_j \in \mathbb{F}_2[x_1, \ldots, x_n]$ is given by its algebraic normal form (ANF) as a sum of monomials. The goal is to find a satisfying assignment to the system, namely $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \{0, 1\}^n$ such that $P_j(\bar{x}) = 0$ for every $j \in \{1, \ldots, m\}$, or to determine that such an assignment does not exist. An additional important parameter of a polynomial system is its degree $d$ which bounds the algebraic degree of its polynomials. Typically, $d$ is assumed to be a small constant.

If all polynomials in $E$ are linear (i.e., $d = 1$), then the system can be efficiently solved (e.g., by Gaussian elimination). However, the problem is known to be NP-hard already for quadratic systems (namely, $d = 2$). Moreover, assuming the exponential time hypothesis [9], there exists no subexponential time (worst-case) algorithm for this problem. Yet, devising the most efficient exponential time algorithm for solving polynomial systems over $\mathbb{F}_2$ (and quadratic systems over $\mathbb{F}_2$ in particular) is an interesting and active research problem. It is particularly relevant to the domain of cryptography, as the security of various cryptosystems (known as multivariate cryptosystems) is directly based on the conjectured hardness of solving these systems. Examples of such ciphers include HFE by Patarin [15], UOV by Kipnis, Patarin and Goubin [12], and more recent cryptosystems such as MQDSS [5] which is a second round candidate signature scheme in NIST’s post-quantum standardization project [14].
1.1 Previous Work

Several restricted classes of non-linear equation systems over $\mathbb{F}_2$ are known to have polynomial time algorithms. Examples include extremely under-determined ($n > m(m + 1)/2$) or over-determined ($m > n(n + 1)/2$) quadratic systems (in the latter case an efficient algorithm – based on Gaussian elimination – exists for most instances). Various works investigate algorithms that interpolate between one of these extreme cases and the case of $m = n$ (e.g., [18] focuses on under-determined systems).

Algebraic techniques. A common approach to the general problem of solving polynomial systems over finite fields, is to use algebraic techniques in order to find a convenient representation of the ideal generated by the polynomials in the form of a reduced Gröbner basis. Well-known algorithms for computing Gröbner bases include Buchbergers algorithm [4], F4 [7] and F5 [8], but the asymptotic complexity of these algorithms is not well-understood. At a high level, Gröbner basis algorithms and their variants represent the polynomials in the input system, along with many of their multiples, using a matrix and then attempt to reduce it via elimination techniques. Another algorithm that employs related methods is the XL algorithm [6] which was developed for cryptanalytic purposes and has led to several variants. In particular, in [21] Yang and Chen developed an XL variant whose complexity for solving quadratic equations over $\mathbb{F}_2$ with $m = n$ was shown to be $O(2^{0.875n})$ under some algebraic assumptions.

In [1], Bardet et al. devised an algorithm for solving quadratic polynomial systems over $\mathbb{F}_2$ based on a hybrid approach that combines exhaustive search over a subset of the variables with elimination techniques. Under some algebraic assumptions on the input system which were experimentally found to hold for random systems, the authors bounded the complexity of the deterministic variant of their algorithm by $O(2^{0.841n})$, while the expected complexity of the Las Vegas variant was bounded by $O(2^{0.792n})$. More recently, Joux and Vitse developed a new algorithm based on a different hybrid approach and showed that it outperforms in practice previous algorithms for a wide range of parameters [10]. However, analyzing the asymptotic complexity of this algorithm seems difficult and is a very interesting open problem.

The polynomial method. In [13] Lokshtanov et al. presented the first worst-case algorithms for solving polynomial equations over finite fields with complexity of $O(2^{(1 - \Omega(1))n})$. In particular, their randomized algorithm for solving equations over $\mathbb{F}_2$ has runtime of $O(2^{0.8765n})$ for quadratic systems and $O(2^{(1 - \frac{1}{(5d)})n})$ in general. In more recent work, Björklund et al. [3] revisited the algorithms of Lokshtanov et al. for polynomial systems over $\mathbb{F}_2$ and improved their complexity to $O(2^{0.804n})$ for $d = 2$ and $O(2^{(1 - \frac{1}{(2.7d)})n})$ in general.

These new algorithms are based on the so-called polynomial method in circuit complexity [2] that has been recently applied in algorithm design [20]. At a high level, the new algorithms represent the system $E$ by the single polynomial over $\mathbb{F}_2$,

$$F(x) = (1 + P_1(x))(1 + P_2(x)) \ldots (1 + P_m(x))$$

that evaluates to 1 (only) on solutions to $E$. However, $F(x)$ has a huge number of terms and hence is replaced with a probabilistic polynomial which agrees with it on most assignments, but has a smaller number of terms. The probabilistic polynomial is then evaluated efficiently on many carefully chosen assignments (using techniques such as FFT), leading to an exponential advantage over exhaustive search.
1.2 Our Results

In this paper we present a randomized worst-case algorithm for solving quadratic systems of degree $d$ over $\mathbb{F}_2$ with better asymptotic complexity than all previously published algorithms.

**Theorem 1.1.** There is a randomized algorithm that given a system $E$ of polynomial equations over $\mathbb{F}_2$ with degree at most $d$ in $n$ variables, finds a solution to $E$ or decides that a solution does not exist. The runtime of the algorithm is bounded by $O(2^{0.6943n})$ for $d = 2$ and by $O(2^{1-1/(2d)n})$ for $d > 2$.

We note that the stated complexity bound for $d > 2$ is somewhat loose. A more precise (but rather unwieldy) complexity bound for our algorithm is given in Theorem 3.1. For example, using the precise formula, we bound the complexities of our algorithm for $d = 3$ and $d = 4$ by $O(2^{0.8114n})$ and $O(2^{0.8633n})$, respectively.

In addition, we consider the problem of outputting all the solutions to a given system.

**Theorem 1.2.** There is a randomized algorithm that given a system $E$ of polynomial equations over $\mathbb{F}_2$ with degree at most $d$ in $n$ variables, outputs all $K$ solutions to $E$ or decides that a solution does not exist. For an arbitrarily small $\epsilon > 0$, the runtime of the algorithm is bounded by $O\left(\max\left(2^{0.6943n}, K \cdot 2^n\right)\right)$ for $d = 2$, and by $O\left(\max\left(2^{1-1/(2d)n}, K \cdot 2^n\right)\right)$ for $d > 2$.

When $K \geq 2^{0.6943n}$ for $d = 2$ and $K \geq 2^{1-1/(2d)n}$ for $d > 2$, then our algorithm is close to the best possible (assuming the goal is to explicitly output solutions, rather than some compact representation of them).

The algorithm may be useful in various scenarios. For example, suppose the system $E$ contains several polynomials of degree $d = 2$ and additional ones with higher degrees. Then, one may attempt to find a solution to $E$ by enumerating all solutions to the system $E'$ which contains only the quadratic polynomials, and testing them on the remaining polynomials. Denoting the number of solutions to $E'$ by $K'$, the approach is generally preferable compared to analyzing the entire system at once if $K' = O(2^{0.6943n})$ (and depends on additional parameters otherwise). Moreover, by random sampling, one can obtain a sufficiently good estimate of $K'$ in order to determine the complexity in advance.

**Techniques.** We continue the line of work on polynomial method-based algorithms for solving equation systems, initiated by Lokshtanov et al. [13] and Björklund et al. [3]. In particular, we revisit the algorithm of Björklund et al. which reduced the problem of finding a solution to a polynomial system over $\mathbb{F}_2$ to a parity-counting problem of computing the overall parity of solutions $\sum_{\bar{x} \in \{0,1\}^n} F(\bar{x})$ for $F(x) = (1 + P_1(x)) \ldots (1 + P_n(x))$. By exploiting probabilistic polynomials that approximate $F(x)$, Björklund et al. reduced a parity-counting instance to many smaller instances of the same problem, where each smaller instance was obtained by fixing a variable subset to a particular value.

Our main observation is that all of these smaller parity-counting instances are related and solving them independently is wasteful. In order to exploit this observation, we define a new problem of multiple parity-counting which solves many small parity-counting instances at once. Making use of probabilistic polynomials, we reduce an instance of multiple parity-counting to only a few instances of the same problem. While the number of recursive calls is significantly reduced, it is not immediately obvious that the reduced instances are actually easier to solve compared to the original one. This is established by careful analysis and choice of parameters. Interestingly, the
multiple parity-counting problem can be used in a natural way as a sub-procedure in an algorithm that enumerates over all solutions to the system (Theorem 1.2).

We describe some preliminaries (including the Björklund et al. algorithm [3]) next, while the proofs of theorems 1.1 and 1.2 are given in sections 3 and 4, respectively.

2 Preliminaries

Given a finite set \( S \), denote by \( |S| \) its size. Given a vector \( x \in \{0, 1\}^n \), let \( \text{HW}(x) \) denote its Hamming weight. We denote by \( W^n_w \) the set \( \{ x \in \{0, 1\}^n \mid \text{HW}(x) \leq w \} \). Note that \( |W^n_w| = \sum_{i=0}^w \binom{n}{i} \) and we simply notation by denoting \( \binom{n}{\bar{i}} = \sum_{i=0}^w \binom{n}{i} \). We denote by \( W^n_w[i] \) the \( i \)’th element of the set \( W^n_w \) (according to some ordering of its elements). Finally, we denote by \([n]\) the set \( \{1, \ldots, n\}\).

2.1 Boolean Algebra

Any function \( F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \) can be described as a multilinear polynomial, whose algebraic normal form (ANF) is unique and given by \( F(x_1, \ldots, x_n) = \sum_{u \in \{0, 1\}^n} \alpha_u(F) M_u(x) \), where \( \alpha_u(F) \in \{0, 1\} \) is the coefficient of the monomial \( M_u(x) = \prod_{i=1}^n x_i^{u_i} \) (the sum and multiplication are over \( \mathbb{F}_2 \)). We differentiate between formal (vectors of) variables and assignments to these variables using overline notation. For example, an assignment to the formal variable vector \( x \) in \( F(x) \) is denoted by \( \bar{x} \) and the value of \( F \) on this assignment is \( F(\bar{x}) \).

The algebraic degree of the function \( F \) is defined as \( \max\{\text{HW}(u) \mid \alpha_u(F) \neq 0\} \). Therefore, a function \( F \) with a degree bounded by \( d \leq n \) can be described using \( \binom{n}{\bar{d}} \) coefficients.

Interpolation. Any ANF coefficient \( \alpha_u(F) \) can be interpolated by summing (over \( \mathbb{F}_2 \)) over \( 2^{\text{HW}(u)} \) evaluations of \( F \): for \( u \in \{0, 1\}^n \), define \( I_u = \{i \in [n] \mid u_i = 1\} \) and let \( S_u = \{x \in \{0, 1\}^n \mid I_x \subseteq I_u\} \). Then,

\[
\alpha_u(F) = \sum_{x \in S_u} F(\bar{x}).
\]

Indeed, among all monomials only \( M_u(x) \) attains a value of 1 an odd number of times in the expression

\[
\sum_{x \in S_u} F(\bar{x}) = \sum_{x \in S_u} \sum_{v \in \{0, 1\}^n} \alpha_v(F) M_v(\bar{x}).
\]

Therefore, a function \( F \) of degree bounded by \( d \leq n \) can be fully interpolated from its evaluations on the set \( W^d_n \).

Fact 1 (Symbolic interpolation) Let \( F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \). For some \( 1 \leq n_1 \leq n \), partition its \( n \) variables into two sets \( y_1, \ldots, y_{n-n_1}, z_1, \ldots, z_{n_1} \). Given the ANF of \( F \), factor out all the monomials that are multiplied with \( z_1 \ldots z_{n_1} \), and write it as \( F(y, z) = (z_1 \ldots z_{n_1}) F_1(y) + F_2(y, z) \), where each monomial in \( F_2(y, z) \) misses at least one variable from \( \{z_1, \ldots, z_{n_1}\} \). Then,

\[
F_1(y) = \sum_{\bar{z} \in \{0, 1\}^{n_1}} F(y, \bar{z}).
\]

The fact follows from (1) by considering the polynomial \( F_1(y) \) as the symbolic coefficient of the monomial \( z_1 \ldots z_{n_1} \). Note that if \( F(y, z) \) is of degree \( d \) then \( F_1(y) \) is of degree at most \( d - n_1 \).
The Möbius transform. Given the truth table of an arbitrary function \( F \) (as a bit vector of \( 2^n \) entries), the ANF of \( F \) can be represented as a bit vector of \( 2^n \) entries, corresponding to its \( 2^n \) coefficients. It follows from (1) that the ANF representation can be computed from the truth table of \( F \) via a linear transformation over \( \mathbb{F}_2 \). This linear transformation is known as (a specific type of) the Möbius transform and it can be computed in \( O(n \cdot 2^n) \) bit operations. The Möbius transform over \( \mathbb{F}_2^n \) coincides with its inverse which corresponds to evaluating the ANF representation of \( F \) (i.e., computing its truth table).

When dealing with a function \( F \) of low degree \( d \), the Möbius transform allows to convert the evaluations of \( F \) on the set \( W_n^d \) to its ANF representation (and vice-versa) and it can be computed in time \( O(n \cdot 2^n \cdot \binom{n}{d}) \). Even more generally, given the ANF representation of a function \( F \) of degree \( d \) such that its variables are partitioned into 2 sets, \( y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n_1} \), we can evaluate this polynomial on the set \( W_{d_1}^{n-n_1} \times \{0,1\}^{n_1} \) (for \( d_1 \geq d \)) in time

\[
O(n \cdot |W_{d_1}^{n-n_1} \times \{0,1\}^{n_1}|) = O(n \cdot 2^{n_1} \binom{n-n_1}{d_1}).
\]

This evaluation is performed by composing the Möbius transforms on \( W_{d_1}^{n-n_1} \) and \( \{0,1\}^{n_1} \). For more details on this transform, refer to [11].

2.2 Solving Polynomial Systems over \( \mathbb{F}_2 \)

In this paper we deal with the following problem: the input is a system of \( m \) polynomial equations of degree \( d \) in the \( n \) Boolean variables \( x_1, \ldots, x_n \), denoted by

\[
E = \{P_j(x)\}_{j=1}^m,
\]

where each \( P_j(x) \) is given by its ANF. A vector \( \bar{x} \in \{0,1\}^n \) is solution to \( E \) if \( P_j(\bar{x}) = 0 \) for all \( j \in \{1, \ldots, m\} \). We consider three variants of the problem of solving the equation system.

1. In the decisional variant, the output is Boolean and defined to be 1 if and only if \( E \) has a solution.
2. In the search version, the goal is to output a (single) solution if the system is solvable, and output NULL otherwise.
3. In the exhaustive variant, the output consists of all solutions to the system.

In this paper we only consider randomized algorithms for these problems. We will assume that \( m \leq \binom{n}{d} \) (and thus is polynomial in \( n \)). Note that if \( m > \binom{n}{d} \) then \( E \) must contain linearly dependent equations that can be removed by Gaussian elimination (or if the equations are inconsistent, the system is unsolvable).

As noted by Björklund et al. in [3], the search variant reduces to the decisional variant: assuming the system has a solution, we first solve the decisional problem with \( \bar{x}_1 = 0 \) and with \( \bar{x}_1 = 1 \) and fix \( x_1 \) to a value for which a solution exists. Iteratively fixing all the values gives a solution after at most \( 2n \) calls to the decision algorithm.

2.3 Probabilistic Polynomials

A central idea in the algorithm of Lokshtanov et al. [13], and then in the followup work of Björklund et al. [3] is the use of probabilistic polynomials. In particular, these works use the following construction (generally credited to Razborov [16] and Smolensky [17]). Given \( m \) polynomial equations
of degree $d$ in the $n$ Boolean variables $x_1, \ldots, x_n$, $E = \{P_j(x)\}_{j=1}^m$, consider the polynomial

$$F(x) = (1 + P_1(x))(1 + P_2(x)) \cdots (1 + P_m(x)).$$

(2)

Note that $\bar{x}$ is a solution to $E$ if and only if $F(\bar{x}) = 1$. However, the degree of $F(x)$ is $d \cdot m$ in general, and it is typically too large to manipulate. Thus, we work with a probabilistic polynomial with a lower degree, defined as follows. Let $\ell < m$ be a parameter. For $i \in \{1, \ldots, \ell\}, j \in \{1, \ldots, m\}$, pick $\alpha_{ij} \in \{0,1\}$ uniformly at random and define $\ell$ degree $d$ polynomials as

$$R_i(x) = \sum_{j=1}^m \alpha_{ij} P_j(x).$$

For any $\bar{x} \in \{0,1\}^n$, if $F(\bar{x}) = 1$, then $R_i(\bar{x}) = 0$, whereas if $F(\bar{x}) = 0$, then there exists $j$ such that $P_j(\bar{x}) = 1$ and therefore $\Pr[R_i(\bar{x}) = 1] = \frac{1}{2}$ (the probability is over $\{\alpha_{ij}\}_{j=1}^m$). Let

$$\tilde{F}(x) = (1 + R_1(x))(1 + R_2(x)) \cdots (1 + R_\ell(x)).$$

(3)

By the above property, for any $\bar{x} \in \{0,1\}^n$, if $F(\bar{x}) = 1$ (i.e., $\bar{x}$ is a solution to $E$) then $\tilde{F}(\bar{x}) = 1$, whereas if $F(\bar{x}) = 0$ (i.e., $\bar{x}$ is not a solution to $E$) then

$$\Pr[\tilde{F}(\bar{x}) = 0] = 1 - 2^{-\ell}.$$  

(4)

Moreover, the degree of $\tilde{F}(x)$ is at most $d \cdot \ell$, which may be much lower than the degree of $F(x)$.

Next, we describe how such polynomials are used in the algorithm of Björklund et al.

2.4 The Björklund et al. Algorithm for Solving Polynomial Systems [3]

The Björklund et al. algorithm is based on a reduction to the parity-counting problem, as defined below.

Parity-Counting and Reduction from Decisional Polynomial System Solving The input to the parity-counting problem is a system of $m$ polynomials of degree $d$ over the $n$ Boolean variables $x_1, \ldots, x_n$, denoted by $E = \{P_j(x)\}_{j=1}^m$, where each $P_j(x)$ is given by its ANF. As in (2), we define the polynomial $F(x) = (1 + P_1(x)) \cdots (1 + P_m(x))$. The output is the overall parity of solutions

$$\sum_{\bar{x} \in \{0,1\}^n} F(\bar{x}).$$

The algorithm of [17] uses the Valiant-Vazirani affine hashing [19] in order to reduce the decisional problem of solving a polynomial system to several calls (whose number is polynomial in $n$) to an algorithm for the parity-counting problem. We briefly sketch this reduction below.

Obviously, if $\sum_{\bar{x} \in \{0,1\}^n} F(\bar{x}) = 1$, then the system represented by $E$ has a solution, but the opposite direction does not hold in general. The main idea (borrowed from [19]) is to add several random affine equations to the system with the goal of isolating some solution $\bar{x}$ (namely, $\bar{x}$ will be the only solution to the extended system), ensuring that the value of the parity-counting problem on this instance is 1. The number of equations that we need to add in order to guarantee success with high probability depends on the base 2 logarithm of the number of solutions to $E$, denoted by $k$, which is generally unknown. Yet, we can exhaust all $n+1$ possibilities of $k = 0, 1, \ldots, n$. 

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The Björklund et al. Parity-Counting Algorithm We summarize the Björklund et al. parity-counting algorithm. For more details and complexity analysis, refer to [3].

Define the probabilistic polynomial $\tilde{F}$ as in (3). To exploit its properties, partition the $n$ variables into 2 sets $y = y_1, \ldots, y_{n-n_1}$ and $z = z_1, \ldots, z_{n_1}$, where $n_1 < n$ is a parameter. Let $G(y) = \sum_{\bar{z} \in \{0,1\}^{n_1}} \tilde{F}(y, \bar{z})$. Writing $\tilde{F}(y, z) = (z_1 \ldots z_{n_1}) \cdot \tilde{F}_1(y) + \tilde{F}_2(y, z)$, by Fact 1, $G(y) = \tilde{F}_1(y)$ and its degree is at most $d \cdot \ell - n_1$. Then, interpolate $G(y)$ (as described at the end of this section) and evaluate it on all $\bar{y} \in \{0,1\}^{n-n_1}$. For each such $\bar{y}$, by (4) and a union bound over all $\bar{z} \in \{0,1\}^{n_1}$,

$$\Pr \left[ G(\bar{y}) = \sum_{z \in \{0,1\}^{n_1}} F(\bar{y}, \bar{z}) \right] = \Pr \left[ \sum_{z \in \{0,1\}^{n_1}} \tilde{F}(\bar{y}, \bar{z}) = \sum_{z \in \{0,1\}^{n_1}} F(\bar{y}, \bar{z}) \right] \geq 1 - 2^{n_1 - \ell}.$$

Choose $\ell = n_1 + 2$, so the computed partial parity $\sum_{\bar{z} \in \{0,1\}^{n_1}} \tilde{F}(\bar{y}, \bar{z})$ is correct with probability at least $\frac{3}{4}$. For each $\bar{y} \in \{0,1\}^{n-n_1}$ the error is reduced similarly to [13]: compute $t = \Theta(n)$ independent probabilistic polynomials $\{G_k(y)\}_{k=1}^t$ to obtain $t$ approximations of each partial parity and maintain a scoreboard of “votes” for it. Then, take a majority vote for each $\bar{y} \in \{0,1\}^{n-n_1}$ across all $t$ approximations to obtain the true partial, except with exponentially small probability.

Assuming all true partial parities $\sum_{\bar{z} \in \{0,1\}^{n_1}} F(\bar{y}, \bar{z})$ are correctly computed, output the total parity

$$\sum_{\bar{x} \in \{0,1\}^n} F(\bar{x}) = \sum_{\bar{y} \in \{0,1\}^{n-n_1}} \sum_{\bar{z} \in \{0,1\}^{n_1}} F(\bar{y}, \bar{z}).$$

Interpolating $G(y)$. We have

$$G(\bar{y}) = \sum_{\bar{z} \in \{0,1\}^{n_1}} \tilde{F}(\bar{y}, \bar{z}) = \sum_{\bar{z} \in \{0,1\}^{n_1}} (1 + R_1(\bar{y}, \bar{z})) \ldots (1 + R_\ell(\bar{y}, \bar{z})) = \sum_{\bar{z} \in \{0,1\}^{n_1}} (1 + R_{1|\bar{y}}(\bar{z})) \ldots (1 + R_{\ell|\bar{y}}(\bar{z})),$$

where $R_{ij|\bar{y}}(\bar{z}) = R_i(\bar{y}, \bar{z})$. Therefore, each evaluation $G(\bar{y})$ reduces to a parity-counting instance for the system $\{R_{ij|\bar{y}}(\bar{z})\}_{j=1}^\ell$, which has $\ell$ equations of degree $d$ over $n_1$ variables. Since its degree is $d \cdot \ell - n_1$, $G(y)$ can be interpolated from its evaluations on $\{\bar{y} \in W_{d \cdot \ell - n_1}^{n-n_1}\}$. Overall, interpolating $G(y)$ requires $\binom{n-n_1}{d \cdot \ell - n_1}$ recursive calls to (smaller) parity-counting instances.

3 Improved Algorithm for Solving Polynomial Systems over $\mathbb{F}_2$

In this section we prove the following stronger variant of Theorem 1.1.

Theorem 3.1. There is a randomized algorithm that given a system $E$ of polynomial equations over $\mathbb{F}_2$ with degree at most $d$ in $n$ variables, finds a solution to $E$ or decides that a solution does not exist. For an arbitrarily small $\epsilon > 0$, the runtime of the algorithm is bounded by $O(2^{(\tau(d) + \epsilon)n})$, where $\tau(d)$ is the maximum of the function

$$f_d(p) = (1 - p) \cdot H \left( \frac{(d-1)p}{1-p} \right)$$

in the interval $[0, \frac{1}{2d-1}]$, and $H(q) = -q \log q - (1 - q) \log (1 - q)$ is the binary entropy function.

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1 In the previous algorithm of Lokshatanov et al. [13], a similar polynomial to $G$ was defined, but it had a higher degree which resulted in a less efficient algorithm.
Analyzing the function \( f_d(p) = (1-p) \cdot H \left( \frac{(d-1)p}{q-p} \right) \), we use the inequality \( H(q) \leq 2 \sqrt{q(q-1)} \) to deduce \( f_d(p) \leq 2 \sqrt{(d-1)p(1-dp)} \). As the maximal value of the right hand side is \( \sqrt{\frac{d-1}{d^2}} \), obtained at \( p = \frac{1}{2d} \), we get \( \tau(d) \leq \frac{\sqrt{d-1}}{\sqrt{d}} \). Using the inequality \( \sqrt{d} - \sqrt{d-1} > \frac{1}{2\sqrt{d}} \) (for any \( d > 1 \)), we obtain \( 1 - \tau(d) \geq \frac{\sqrt{d} - \sqrt{d-1}}{\sqrt{d}} > \frac{1}{2d} \). Therefore, the asymptotic complexity of the algorithm is bounded by \( O \left( 2^{(1-1/(2d))n} \right) \), as in Theorem 1.1 for all \( d > 2 \) (since the inequality \( 1 - \tau(d) > \frac{1}{2d} \) is strict for any \( d \), we eliminate the addition of \( \epsilon \) in the exponent).

In the following we describe the algorithm of Theorem 3.1, and bound its complexity by \( O^*(2^{(\tau(d)+\epsilon)n}) \) for an arbitrarily small \( \epsilon > 0 \), where the \( O^* \) notation suppresses polynomial factors in \( n \). This is the same asymptotic bound claimed in Theorem 3.1 (as \( \epsilon > 0 \) is arbitrarily small). By the reductions outlined in sections 2.2 and 2.4, the parity-counting algorithm gives rise to an algorithm for finding a solution to polynomial systems of degree \( d \) over \( \mathbb{F}_2 \) (with a multiplicative polynomial overhead). Hence, we proceed to describe a parity-counting algorithm with complexity \( O^*(2^{(\tau(d)+\epsilon)n}) \). This algorithm is based on solving a somewhat more involved problem of multiple parity-counting, defined below.

### 3.1 The Multiple Parity-Counting Problem

The input to the problem is a system of \( m \) polynomial equations of degree \( d \) in \( n \) Boolean variables, along with non-negative integers \( n_1 \leq n \) and \( w \leq n - n_1 \). The \( n \) variables are partitioned into two sets according to \( n_1 \) and denoted as \( y_1, \ldots, y_{n-n_1}, z_1, \ldots, z_{n_1} \), while the system is denoted by \( E = \{P_j(y, z)\}_{j=1}^m \), where each \( P_j(y, z) \) is given by its ANF. We define the polynomial

\[
F(y, z) = (1 + P_1(y, z))(1 + P_2(y, z)) \cdots (1 + P_m(y, z)).
\]

The output is a vector of parities \( V \in \{0, 1\}^{(n-n_1)\cdot w} \) such that

\[
V[i] = \sum_{z \in \{0, 1\}^{n_1}} F(W_{w}^{n-n_1}[i], z).
\]

We refer to an instance of this problem as MultParityCount(\( \{P_j(y, z)\}_{j=1}^m, n_1, w \)). In Algorithm 1 we solve the parity-counting problem using an algorithm for the multiple parity-counting problem.

#### Details of the Multiple Parity-Counting Algorithm

We describe our algorithm for the multiple parity-counting problem and refer to Algorithm 2 for its pseudo-code.

The algorithm begins in a similar way to the previous related algorithms \([3,13]\) by choosing a parameter \( \ell \) and defining the probabilistic polynomial \( \bar{F}(y, z) \) as in (3). Yet, we work with an additional partition of the variables.

We continue in a similar manner to the Björklund et al. algorithm by partitioning the \( n_1 \) variables \( z_1, \ldots, z_{n_1} \) into 2 sets. Let \( n_2 < n_1 \) be a parameter. Let \( u = u_1, \ldots, u_{n_1-n_2} \) and \( v = v_1, \ldots, v_{n_2} \). Define

\[
G(y, u) = \sum_{\bar{v} \in \{0, 1\}^{n_2}} \bar{F}(y, u, \bar{v}).
\]
Fix any \( \tilde{y} \in \{0,1\}^{n-n_1} \) and \( \tilde{u} \in \{0,1\}^{n_1-n_2} \). By (4) and a union bound over all \( \tilde{v} \in \{0,1\}^{n_2} \),

\[
\Pr \left[ G(\tilde{y}, \tilde{u}) = \sum_{\tilde{v} \in \{0,1\}^{n_2}} F(\tilde{y}, \tilde{u}, \tilde{v}) \right] \geq 1 - 2^{n_2 - \ell}.
\]

Writing \( \tilde{F}(y, u, v) = (v_1 \ldots v_{n_2}) \cdot \tilde{F}_1(y, u) + \tilde{F}_2(y, u, v) \), by Fact 1, \( G(y, u) = \tilde{F}_1(y, u) \) and its degree is upper bounded by \( d \cdot \ell - n_2 \). Therefore, in order to interpolate \( G(y, u) \), it is sufficient to compute its values on the set \( W_{d \cdot \ell - n_2}^{n-n_2} \). Thus, for each \((\tilde{y}, \tilde{u}) \in W_{d \cdot \ell - n_2}^{n-n_2} \), we compute \( G(\tilde{y}, \tilde{u}) = \sum_{\tilde{v} \in \{0,1\}^{n_2}} \tilde{F}(\tilde{y}, \tilde{u}, \tilde{v}) \) and use these values to interpolate \( G(y, u) \).

**Interpolating \( G(y, u) \).** The main difference from the Björklund et al. parity-counting algorithm is in the way that the \( |W_{d \cdot \ell - n_2}^{n-n_2}| \) evaluations of \( G(y, u) \) are computed. In [3], \((y, u)\) was treated as a single vector of variables \( y' \) and each evaluation \( G(y') = \sum_{\tilde{v} \in \{0,1\}^{n_2}} \tilde{F}(y', \tilde{v}) \) was computed by a separate recursive call to the parity-counting problem.

On the other hand, we reduce the computation of all the \( |W_{d \cdot \ell - n_2}^{n-n_2}| = \binom{n-n_2}{d \cdot \ell - n_2} \) parity-counting instances (per probabilistic polynomial) to a single instance of the multiple parity-counting problem and solve it via one recursive call to

\[
\text{MultParityCount}((R_i((y, u), v))_{i=1}^\ell, n_2, d \cdot \ell - n_2).
\]

We use the vector of evaluations returned from the recursive call to interpolate \( G(y, u) \).

**Remark 3.1.** \( G(y, u) \) is interpolated by evaluating it on the set \( W_{d \cdot \ell - n_2}^{n-n_2} = W_{n_2(d-1)+2d}^{n-n_2} \) (as \( \ell = n_2+2 \)) via a multiple parity-counting instance with parameters \((n_2, n_2(d-1)+2d)\), which itself calls an instance with parameters \((n'_2, n'_2(d-1)+2d)\) for some \( n'_2 < n_2 \). Thus, the number of variables over which the polynomials are defined increases with the recursion depth, but their degree decreases (since \( d - 1 \geq 1 \)). Our choice of parameters will ensure that \( \binom{n-n_2}{d \cdot \ell - n_2} > \binom{n-n_2}{n'_2(d-1)+2d} \), so the new instance is not harder than the original one.

**Finalizing the algorithm.** After interpolating \( G(y, u) \), we evaluate it on all \( \tilde{y} \in W_{n_1}^{n-n_1} \) and \( \tilde{u} \in \{0,1\}^{n_1-n_2} \), and obtain \( \binom{n-n_1}{n_1} \cdot 2^{n_2-n_1} \) evaluations.

Recall that our goal is to return the true parities \( \sum_{\tilde{z} \in \{0,1\}^{n_1}} F(\tilde{y}, \tilde{z}) \) for each \( \tilde{y} \in W_{n_1}^{n-n_1} \). Since we choose \( \ell = n_2 + 2 \), (5) implies that for every \((\tilde{y}, \tilde{u})\) we have

\[
\Pr \left[ G(\tilde{y}, \tilde{u}) = \sum_{\tilde{z} \in \{0,1\}^{n_1}} F(\tilde{y}, \tilde{u}, \tilde{z}) \right] \geq \frac{3}{4}.
\]

Namely, we obtain the correct partial parity with probability at least \( \frac{3}{4} \).

This allows to perform error correction using scoreboards similarly to [3]. For this purpose, for a parameter \( t \), we compute probabilistic polynomials \( \{G_k(y, u)\}_{k=1}^t \) and obtain \( t \) approximations. We then perform a majority vote across all \( t \) approximations to obtain the true partial parity \( \sum_{\tilde{z} \in \{0,1\}^{n_1}} F(\tilde{y}, \tilde{u}, \tilde{z}) \) for each \((\tilde{y}, \tilde{u}) \in W_{n_1}^{n-n_1} \times \{0,1\}^{n_1-n_2}\) (except with exponentially small probability).

Assuming we obtain the true partial parities, we can compute the required output vector of parities, as for each \( \tilde{y} \in W_{n_1}^{n-n_1} \),

\[
\sum_{\tilde{z} \in \{0,1\}^{n_1}} F(\tilde{y}, \tilde{z}) = \sum_{\tilde{u} \in \{0,1\}^{n_1-n_2}} \sum_{\tilde{v} \in \{0,1\}^{n_2}} F(\tilde{y}, \tilde{u}, \tilde{v}).
\]
Parameter: $\kappa_0$

Initialization: $n_1 \leftarrow \lfloor \kappa_0 n \rfloor$

1: $V[0 \ldots 2^{n-n_1} - 1] \leftarrow \text{MultParityCount}(\{P_j(y, z)\}_{j=1}^m, n_1, n - n_1)$
2: $\text{Parity} \leftarrow 0$
3: for all $\bar{y} \in \{0, 1\}^{n-n_1}$ do
4: \quad $\text{Parity} \leftarrow \text{Parity} + V[\bar{y}]$ (sum is over $\mathbb{F}_2$)
5: return $\text{Parity}$

Algorithm 1: $\text{ParityCount}(\{P_j(x)\}_{j=1}^m)$

### 3.2 Analysis

In this section we analyze Algorithm 1, completing the proof of Theorem 3.1. The success probability analysis of this algorithm is similar to that of Björklund et al. [3]. It is given in Appendix A which shows that by selecting $t = 48n + 1$, the algorithm returns the correct parity with probability at least $1 - 2^{-n}$. We proceed with runtime analysis.

**Runtime Analysis**

Denote by $T(n_1, w)$ the runtime of $\text{MultParityCount}(\{P_j(y, z)\}_{j=1}^m, n_1, w)$ (we omit the parameters $n$ and $d$ that remain unchanged in the recursive calls). Assuming that $n_2 > 0$ and the recursive version (rather than brute force) is called,

$$T(n_1, w) = O \left(t \cdot \left(T(n_2, d \cdot \ell - n_2) + n \cdot \binom{n-n_1}{\ell} \cdot 2^{n_1-n_2} + n \cdot \binom{n-n_2}{\ell_2^{n_1} + n_2} \right) \right) = O(n \cdot T(n_2, n_2(d - 1) + 2d)) + O \left(n^2 \cdot \binom{n-n_2}{\ell_2^{n_1} + n_2} \right),$$

recalling that $\ell = n_2 + 2$, $t = 48n + 1$.

The second term corresponds to the evaluations of $G_k(y, u)$ on the set $W^{n-n_1} \times \{0, 1\}^{n_1-n_2}$ using the Möbius transform, as described in Section 2.1. The third term corresponds to the interpolation of $G_k(y, u)$ from its values on the set $W^{n-n_2}$ via the Möbius transform.

Finally, the second term dominates the runtime complexity of the remaining steps (not including the recursive call) as the complexity of updating the scoreboards (and the final majority votes) is $O(|W^{n-n_1}| \cdot 2^{n_1-n_2})$. Moreover, the ANF computation of $\{R_i(y, z)\}_{i=1}^\ell$ requires $O \left(t \cdot \ell \cdot m \cdot \binom{n}{\ell} \right) = O^*(1)$ time (assuming $m$ is polynomial in $n$).

**Runtime analysis by recursion level.** We will select the parameters such that the recursion runs for a constant number of $D + 1$ levels (where $D = D(d)$) and the last level applies brute force. Note that the $i$’th level of the recursion tree (starting from the top level, where $i = 0$) contains $O(n^i)$ nodes. We will now start indexing the recursion variables by their level of recursion $i$.

Let $0 < \kappa_0 \leq \frac{1}{m-1}$ and $0 < \lambda < 1$ be the parameters of algorithms 1 and 2, respectively. We denote the initial value of $n_1$ in Algorithm 1 by $n_1^{(0)} = \lfloor \kappa_0 n \rfloor$. Similarly, $n_1^{(i+1)} = n_2^{(i)} = \lfloor n_1^{(i)} - \lambda n \rfloor$.

We have $w^{(0)} = n - n_1^{(0)}$, while for $i \geq 1$, $w^{(i)} = n_2^{(i-1)}(d - 1) + 2d$. Focusing on $i \geq 1$,

$$\binom{n-n_2^{(i-1)} }{w^{(i)}} \leq n^{2d} \binom{n-n_2^{(i-1)} }{n_2^{(i-1)}(d-1)+2d} \leq n^{2d+1} \binom{n-n_2^{(i-1)} }{n_2^{(i-1)}(d-1)},$$
Parameter: $\lambda$
Initialization: $n_2 \leftarrow |n_1 - \lambda n_1|, \ell \leftarrow n_2 + 2, t \leftarrow 48n + 1$
1: $V[0 \ldots |W_{w}^{n-n_1} - 1|] \leftarrow \vec{0}$ (initialize result array)
2: if $n_2 \leq 0$ then
3: $V[0 \ldots |W_{w}^{n-n_1} - 1|] \leftarrow \text{BruteForceMultParity}([P_j(y, z)]_{j=1}^{m}, n_1, w)$
4: return $V$
5: $SB[0 \ldots |W_{w}^{n-n_1} - 1| \cdot 2^n - n_2 - 1] \leftarrow \vec{0}$ (initialize scoreboards)
6: for all $k \in \{1, \ldots, t\}$ do
7: Pick $[\alpha]_{ij} \in \mathbb{F}_2^{\ell \times m}$ uniformly at random and compute $\{R_i(y, z)\}_{i=1}^{\ell} = \{\sum_{j=1}^{m} \alpha_{ij} P_j(x)\}_{i=1}^{\ell}$
8: $V_1[0 \ldots |W_{d \cdot \ell - n_2}^{n-n_2} - 1|] \leftarrow \text{MultParityCount}([R_i((y, u), v)]_{i=1}^{\ell}, n_2, d \cdot \ell - n_2)$
9: Interpolate $G_k(y, u)$: apply Möbius transform to $V_1[0 \ldots |W_{d \cdot \ell - n_2}^{n-n_2} - 1|]
10: \text{Evaluate } G_k(y, u) \text{ on } W_{w}^{n-n_1} \times \{0, 1\}^{n_2-n_2} \text{ by Möbius transform and store result in } \text{Evals}[0 \ldots |W_{w}^{n-n_1} \cdot 2^n - n_2 - 1|]
11: \text{Update scoreboards } SB[0 \ldots |W_{w}^{n-n_1} \cdot 2^n - n_2 - 1|] \text{ with Evals}
12: \text{for all } i \in \{0, \ldots, |W_{w}^{n-n_1} - 1|\} \text{ do}
13: \text{for all } \vec{u} \in \{0, 1\}^{n_2-n_2} \text{ do}
14: \text{vote } \leftarrow \text{Majority}(SB[W_{w}^{n-n_1}[i], \vec{u}])
15: \text{ } V[i] \leftarrow V[i] + \text{vote} \text{ (sum is over } \mathbb{F}_2)
16: \text{return } V$

Algorithm 2: MultParityCount([P_j(y, z)]_{j=1}^{m}, n_1, w)

where for the final inequality we assume that $2(n_2^{(i-1)}(d-1)) \leq n - n_2^{(i-1)}$. Since $d - 1 \geq 1$ and the sequence $n_2^{(i-1)}$ is decreasing as a function of $i$, it is sufficient to ensure this condition for $i = 1$, where it holds if $2\kappa_0(d - 1) \leq 1 - \kappa_0$, which is guaranteed since we choose $\kappa_0 \leq \frac{1}{2d-1}$.

Let $H(\cdot)$ be the binary entropy function. We use the fact that $\binom{u}{w} \leq 2^H(\frac{u}{w})$ and obtain

$$\binom{n-n_1^{(i)}}{\ell w^{(i)}} \leq n^{2d+1} 2^{(n-n_2^{(i-1)})H\left(\frac{n_{2}^{(i-1)}(d-1)}{n_{2}^{(i-1)}-1}\right)} = n^{2d+1} 2^{n(1-\rho_{i-1})H\left(\frac{\rho_{i-1}(d-1)}{1-\rho_{i-1}}\right)},$$

where we set $\rho_i = \frac{n_1^{(i)}}{n}$. Recalling from Theorem 3.1 that $f_d(p) = (1 - p) \cdot H\left(\frac{(d-1)p}{1-p}\right)$ we deduce

$$\binom{n-n_1^{(i)}}{\ell w^{(i)}} \leq n^{2d+1} 2^{f_d(\rho_{i-1}) n}.$$  

We also have $n_1^{(i)} - n_2^{(i)} = n_1^{(i)} - |n_1^{(i)} - \lambda n| \leq \lambda n + 1$. Plugging these into the second term of (7), we bound it by

$$O\left(n^{2d+3} \cdot 2^{(\lambda+f_d(\rho_{i-1})) n}\right).$$

Similarly, the third term of (7) is bounded by $O\left(n^{2d+3} \cdot 2^{f_d(\rho_{i}) n}\right)$. Since there are $O(n^i)$ nodes in the $i$'th level of the recursion, for $1 \leq i < D$ the total runtime for all nodes at the $i$'th level is
Plugging this into (1) and as there are \( O(n) \) parameters \( \kappa \) and as there are \( O(n) \) nodes at this level, the asymptotic total runtime at this level is

\[
O \left( n^{1+2d+3} \cdot (2^{(\lambda+f_d(\rho_{i-1}))n} + 2^{f_d(\rho_{i})n}) \right) = O^* \left( 2^{(\lambda+f_d(\rho_{i-1}))n} + 2^{f_d(\rho_{i})n} \right),
\]

(8)
as \( i < D = O(1) \). For the root node we have

\[
\binom{n-n_0}{i_w} = \binom{n-n_1}{i_w} = 2^{n-n_1} \leq 2^{1+(1-\kappa_0)n}.
\]

Plugging this into (7), its runtime is bounded by

\[
O \left( n^2 \cdot 2^{(1-\kappa_0+\lambda)n} + n^{2d+3}2^{f_d(\rho_{0})n} \right) = O^* \left( 2^{(1-\kappa_0+\lambda)n} + 2^{f_d(\rho_{0})n} \right).
\]

(9)

For \( i = D \), we solve the problem by brute force. By similar analysis, the runtime is bounded by \( O^* \left( 2^{f_d(\rho_{D-1})n+n_2^{(D)}} \right) \). Using the brute force condition \( n_2^{(D)} \leq 0 \), we obtain \( n_1^{(D)} \leq \lambda n + 1 \) and bound the runtime of each node of level \( D \) by

\[
O^* \left( 2^{(\lambda+f_d(\rho_{D-1}))n} \right),
\]

(10)
and as there are \( O(n^D) = n^{O(1)} \) nodes at this level, the asymptotic total runtime at this level is bounded similarly.

Parameter selection. The total runtime is determined by the runtime expressions at all levels, namely (8), (9) and (10). Fix a value of \( d > 1 \) and a sufficiently small \( \epsilon > 0 \). We will select the parameters \( \kappa_0, \lambda \) such that \( D = O(1) \) and the runtime is

\[
O^* \left( 2^{(\tau(d)+\epsilon)n} \right),
\]

12
where \( \tau(d) \) is the maximum of the function \( f_d(p) \) in the interval \([0, \frac{1}{2d-1}]\). Note that \( f_d(\frac{1}{2d-1}) = 1 - \frac{1}{2d-1} \), so \( \tau(d) \geq 1 - \frac{1}{2d-1} \).

If we take \( \lambda \) to be sufficiently small, then optimizing the parameters amounts to balancing the exponent terms \( 1 - \kappa_0 \) in (9) and the remaining terms of the form \( f_d(\rho_i) \).

We choose \( \lambda = \epsilon, \kappa_0 = 1 - \tau(d) \). Recall that we run brute force once \( n_2^{(D)} \leq 0 \). Since \( n_2^{(i)} \leq (\kappa_0 - (i + 1)\lambda)n \), then \( n_2^{(D)} \leq (\kappa_0 - (D + 1)\epsilon)n \) and therefore \( D \leq \frac{\kappa_0}{\epsilon} - 1 = O(1) \) as required.

As \( \kappa_0 n \geq n_1^{(0)} \geq n_2^{(0)} \geq \ldots \geq n_2^{(D-1)} \) and \( \kappa_0 = 1 - \tau(d) \leq \frac{1}{2d-1} \), then \( \rho_i = \frac{n_2^{(i)}}{n} \in [0, \frac{1}{2d-1}] \) for all \( i \in \{0, \ldots, D - 1\} \), and therefore (by the definition of \( \tau(d) \)), \( 2f_d(\rho_i)n \leq 2^{\tau(d)n} \). Since \( \lambda = \epsilon \), each of the expressions (8), (9) and (10) is bounded by \( O^*(2^{\tau(d) + \epsilon n}) \) as claimed.

Finally, it is possible to reduce the complexity to \( O^*(2^{\tau(d) + o(1)n}) \). For example, by choosing \( \lambda = \Theta\left(\frac{1}{\log n}\right) \), \( D \) will no longer be constant, but the total number of recursive calls is still subexponential.

### 4 Exhaustively Solving Polynomial Equations

In this section we prove the following stronger variant of Theorem 1.2.

**Theorem 4.1.** There is a randomized algorithm that given a system \( E \) of polynomial equations over \( \mathbb{F}_2 \) with degree at most \( d \) in \( n \) variables, outputs all \( K \) solutions to \( E \) or decides that a solution does not exist. For an arbitrarily small \( \epsilon > 0 \), the runtime of the algorithm is bounded by

\[
O\left(\max\left(2^{(\tau(d) + \epsilon)n}, K \cdot 2^\epsilon n\right)\right),
\]

where \( \tau(d) \) is defined in Theorem 3.1.

The algorithm’s pseudo-code is given in Algorithm 4. We proceed with a detailed description that bounds the complexity by \( O^*(\max(2^{(\tau(d) + \epsilon)n}, K \cdot 2^\epsilon n)) \) (which is sufficient for establishing the bound of Theorem 4.1).

Our approach isolates solutions similarly to the Valiant-Vazirani affine hashing [19]. However, the affine hashing generally isolates only one solution at a time, and applying it to exhaust all solutions will be inefficient unless their number is very small. Thus, we apply a variant of the affine hashing that isolates and outputs many solutions in parallel.

Recall that Algorithm 1 partitions the variables into two sets \( (y, z) \) according to a parameter \( n_1 \) and calls \( \text{MultParityCount}\left(\{P_j(y, z)\}_{j=1}^{m}, n_1, w\right) \). If the returned parity for a specific \( \bar{y} \in \{0, 1\}^{n-n_1} \) is 1 and the output is correct, then there exists a solution \( \bar{x} = (\bar{y}, \bar{z}) \) to \( E \) for some unknown \( \bar{z} \in \{0, 1\}^{n_1} \). We call the solution \( \bar{x} = (\bar{y}, \bar{z}) \) to \( E \) isolated (with respect to the variable partition \( (y, z) \)), if for any \( \bar{z}' \neq \bar{z}, (\bar{y}, \bar{z}') \) is not to a solution to \( E \).

We first describe how to output all isolated solutions with respect to \( (y, z) \) using a total of \( n_1 + 1 \) calls to \( \text{MultParityCount} \). We will assume that all returned parities by \( \text{MultParityCount} \) calls are correct (by our parameter selection, this will hold except with negligible probability).

**Outputting isolated solutions.** After running \( \text{MultParityCount} \) once, let us momentarily assume that all \( \bar{y} \in \{0, 1\}^{n-n_1} \) for which the returned parity is 1 (namely, \( \sum_{z \in \{0, 1\}^{n_1}} F(\bar{y}, z) = 1 \)) correspond to isolated solutions. The remaining \( n_1 \) bits of these solutions can be recovered one-by-one using \( n_1 \) additional calls to \( \text{MultParityCount} \), where in call \( i \), we fix variable \( z_i \) to 0 in \( \{P_j(y, z)\}_{j=1}^{m} \) (all calls are with respect to the same partition \( (y, z) \), but \( z_i \) is fixed to 0 in call \( i \)).
For $\tilde{y} \in \{0,1\}^{n-n_1}$, $i \in \{1,\ldots,n_1\}$, $b \in \{0,1\}$, let us denote

$$U(\tilde{y},i,b) = \sum_{z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n \in \{0,1\}^{n_1-1}} F(\tilde{y},z_1,\ldots,z_{i-1},b,z_{i+1},\ldots,z_n).$$

By running MultParityCount with $z_i = 0$ we derive $U(\tilde{y},i,0)$ for all $\tilde{y} \in \{0,1\}^{n-n_1}$. Since

$$U(\tilde{y},i,0) + U(\tilde{y},i,1) = \sum_{\tilde{z} \in \{0,1\}^{n_1}} F(\tilde{y},\tilde{z}),$$

then assuming $\sum_{\tilde{z} \in \{0,1\}^{n_1}} F(\tilde{y},\tilde{z}) = 1$, exactly one of the expressions $U(\tilde{y},i,0)$ and $U(\tilde{y},i,1)$ has a value of 1, and the assignment of $z_i$ in this expression is the value of $z_i$ in the isolated solution whose prefix is $\tilde{y}$.

Some of the 1 parities returned by the first MultParityCount call may not correspond to an isolated solution, but rather to an odd number of solutions which is larger than 1. In this case, the procedure for such a $\tilde{y}$ may result in a “false positive”. Thus, we need to test that each output is indeed a solution to $E$.

**Isolating solutions.** It remains to describe how to isolate solutions. For this purpose, we perform a change of variables by first selecting a uniform $n \times n$ invertible matrix $B \in \mathbb{F}_2^{n \times n}$. We replace $x_i$ in all its occurrences in $E$ by the linear expression $\sum_{j=1}^n B[i][j]v_j$ over the new variables $(v_1,\ldots,v_n)$. Note that we have $x_i = (Bv)_i$ so and $x = Bv$ as vectors of variables.

Since the change of variables is linear, the result is a system $E'$ of the same algebraic degree as $E$ over the new variables. $E'$ is equivalent to $E$ in the sense that any solution $\bar{v}$ to $E'$ corresponds to a solution $\bar{x} = B\bar{v}$ to $E$ which can be computed efficiently from $\bar{v}$ by linear algebra (and vise-versa). Thus, when we find an isolated solution $\bar{v}$ to $E'$ (with respect to some variable partition), we output $B\bar{v}$ as a solution to $E$.

For a parameter $r$, we will run the above algorithm for $r$ iterations, each time performing a new and independent change of variables and outputting the isolated solutions with respect to the new variable set. Below we select the parameters and complete the analysis.

**4.1 Analysis**

**Runtime analysis.** The change of variables requires recomputing the ANF of all polynomials. Each polynomial in $E$ has at most $n^d$ monomials, while the substitution and ANF computation for each monomial of degree $d$ requires $O(n^d)$ time. Therefore, the total runtime of this step is $O(m \cdot n^{2d}) = O^*(1)$. Additional linear algebra computations also have complexity $O^*(1)$. Thus, the runtime of each of the $r$ iterations is dominated by the calls to MultParityCount.

Next, we analyze the success probability as a function of the parameters $n_1, r$.

**Success probability analysis.** Fix a solution $\bar{x}$ to $E$. Under a change of variables $B$, it is transformed into a solution $(\tilde{y},\tilde{z}) = \bar{v} = B^{-1}\bar{x}$ to $E'$. We will lower bound the probability that $\bar{v}$ is isolated by the change of variables, namely, for any $\bar{z} \neq \tilde{z}$, we require that $(\tilde{y},\tilde{z}')$ is not a solution to $E'$. Let $\bar{x}'$ be another solution to $E$. Then $B^{-1}\bar{x}' = (\tilde{y},\tilde{z}')$ for $\tilde{z}' \neq \tilde{z}$ if and only if $B^{-1}(\bar{x} + \bar{x}') = (\bar{0},\bar{z} + \bar{z}')$ and $\bar{z} + \bar{z}' \neq \bar{0}$, namely, $B^{-1}(\bar{x} + \bar{x}')$ is a vector in a specific $n_1$-dimensional subspace. Since $B^{-1}$ is itself a uniform invertible linear transformation, any non-zero vector is mapped to this space with probability at most $2^{n_1-n}$. Taking a union bound over all $K$ solutions to $E$,

$$\Pr[B^{-1}\bar{x} \text{ is an isolated solution to } E' \text{ with respect to } (y,z)] \geq 1 - K \cdot 2^{n_1-n}. $$

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Parameters: $n_1$

Initialization: $r \leftarrow 2n$

1: for all $k \in \{1, \ldots, r\}$ do
2: Sample uniform invertible matrix $B \in \mathbb{F}_2^{n \times n}$
3: $\{Q_j(v)\}_{j=1}^{m} \leftarrow \text{ChangeVariables}(B, \{P_j(x)\}_{j=1}^{m})$
4: $ZV[0] \{0 \ldots 2^{n-1} - 1\} \leftarrow \overrightarrow{0}$ (init mult parity array per $z_i$)
5: $ZV[0] \{0 \ldots 2^{n-1} - 1\} \leftarrow \text{MultParityCount}(\{Q_j(y, z)\}_{j=1}^{m}, n_1, n - n_1)$
6: for all $i \in \{1, \ldots, n_1\}$ do
7: $ZV[i] \{0 \ldots 2^{n-1} - 1\} \leftarrow \text{MultParityCount}(\{Q_j(y, z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_{n_1})\}_{j=1}^{m}, n_1 - 1, n - n_1)$
8: for all $\hat{y} \in \{0, 1\}^{n-n_1}$ do
9: if $ZV[0][\hat{y}] = 1$ then
10: sol $\leftarrow \hat{y}$
11: for all $i \in \{1, \ldots, n_1\}$ do
12: $p_0 \leftarrow ZV[i][\hat{y}]$
13: if $p_0 = 1$ then
14: sol $\leftarrow$ sol $\parallel$ 0 (concatenate bit to solution)
15: else
16: sol $\leftarrow$ sol $\parallel$ 1
17: if $B \cdot$ sol is a solution to $\{P_j(x)\}_{j=1}^{m}$ then
18: output $B \cdot$ sol

Algorithm 4: ExhaustSolutions($\{P_j(x)\}_{j=1}^{m}$)

Parameter selection. We choose $n_1$ so the isolation probability above is at least $\frac{1}{2}$ (except with exponentially small probability). Then, setting $r = 2n$, each solution is isolated at least once with probability at least $1 - 2^{-2n}$. Consequently, by a union bound over all $K$ solutions to $E$, all of them will be output with probability at least $1 - 2^{-n}$.

The choice of $n_1$ for which the isolation probability is $\frac{1}{2}$ depends on $K$ which is unknown. However, by random sampling (using a standard Chernoff bound), the fraction of solutions $\frac{K}{n_1}$ can be estimated up to a multiplicative factor of 2 in complexity $O^*(\frac{2^n}{K})$ and exponentially small error probably. For $K = \Omega(2^{0.6943n})$ this requires $O^*(2^{0.3057n})$ time. In particular, we calculate in time $O^*(2^{0.3057n})$ a value $\tilde{K}$ such that if $\tilde{K} \leq 2^{\tau(d)n-2}$ then $K \leq 2^{\tau(d)n-1}$, while if $\tilde{K} > 2^{\tau(d)n-2}$, then $\frac{K}{\tilde{K}} \leq 2K$ (except with exponentially small probably).

Therefore, if $\tilde{K} \leq 2^{\tau(d)n-2}$, we set $n_1 = \lceil (1 - \tau(d))n \rceil$ (as chosen in Section 3.2 for Algorithm 1) and the complexity remains $O^*(2^{\tau(d)+\epsilon}n)$. Otherwise, $\tilde{K} > 2^{\tau(d)n-2}$ and we set $n_1 = \lceil n - \log \tilde{K} - 2 \rceil$. The complexity becomes $O^*(2^{n-n_1+\epsilon n}) = O^*(\tilde{K} \cdot 2^\epsilon) = O^*(K \cdot 2^\epsilon)$ (due to the factor $O^*(2^{1-\epsilon_0+\lambda})$ in the runtime expression for the root node (9)).

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A Success Probability Analysis of Algorithm 1

The algorithm is guaranteed to be correct if the scoreboard majority votes are equal to the corresponding parities in the top level multiple parity-counting instance and in all the recursive calls. We choose $t$ such that each scoreboard majority is correct, except with exponentially small probability.

For $(\bar{y}, \bar{u}) \in W_w^{n-n_1} \times \{0,1\}^{n_1-n_2}$, denote $F'(\bar{y}, \bar{u}) = \sum_{\bar{v} \in \{0,1\}^{n_2}} F(\bar{y}, \bar{u}, \bar{v})$. Also, denote $SB[\bar{y}, \bar{u}]$ the scoreboard entry for $(\bar{y}, \bar{u})$ and by $M[\bar{y}, \bar{u}]$ the majority for this entry. Recall from (6) that

$$\Pr[G_k(\bar{y}, \bar{u}) = F'(\bar{y}, \bar{u})] \geq \frac{3}{4}$$

holds independently for each $k \in \{1, \ldots, t\}$. Since $SB[\bar{y}, \bar{u}]$ is the sum of the $t$ random variables $G_k(\bar{y}, \bar{u})$,

$$E[SB[\bar{y}, \bar{u}] | F'(\bar{y}, \bar{u}) = 1] \geq \frac{3}{4} t, \text{ and}$$

$$E[t - SB[\bar{y}, \bar{u}] | F'(\bar{y}, \bar{u}) = 0] \geq \frac{3}{4} t.$$  

A standard Chernoff bound for a random variable $X$ that is a sum of independent and identically distributed random variables states that for every $0 < \delta < 1$,

$$\Pr[X \leq (1 - \delta) E[X]] \leq \exp \left( -\frac{\delta^2 E[X]}{2} \right).$$

Since all random variables $\{G_k(\bar{y}, \bar{u})\}_{k=1}^t$ are independent and identically distributed, we apply this bound with $\delta = 1/3$ and obtain

$$\Pr[SB[\bar{y}, \bar{u}] > \frac{t}{2} | F'(\bar{y}, \bar{u}) = 1] \geq 1 - \exp \left( -\frac{t}{24} \right), \text{ and}$$

$$\Pr[t - SB[\bar{y}, \bar{u}] \geq \frac{t}{2} | F'(\bar{y}, \bar{u}) = 0] \geq 1 - \exp \left( -\frac{t}{24} \right),$$

given that $t$ is odd.

For $t = 48n + 1$, we obtain

$$\Pr[M[\bar{y}, \bar{u}] = F'(\bar{y}, \bar{u})] \geq 1 - 2^{-2n}.$$  

Taking a union bound over all scoreboard entries computed by the algorithm (whose number is smaller than $2^n$), we bound its error probability by $2^{-n}$. 

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