Does the Weyl ordering prescription lead to the correct energy levels for the quantum particle on the D-dimensional sphere?

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Abstract
The energy eigenvalues of the quantum particle constrained in a surface of the sphere of D dimensions embedded in a $R^{D+1}$ space are obtained by using two different procedures: in the first, we derive the Hamiltonian operator by squaring the expression of the momentum, written in cartesian components, which satisfies the Dirac brackets between the canonical operators of this second class system. We use the Weyl ordering prescription to construct the Hermitian operators. When $D = 2$ we verify that there is no constant parameter in the expression of the eigenvalues energy, a result that is in agreement with the fact that an extra term would change the level spacings in the hydrogen atom; in the second procedure it is adopted the non-abelian BFFT formalism to convert the second class constraints into first class ones. The non-abelian first class Hamiltonian operator is symmetrized by also using the Weyl ordering rule. We observe that their energy eigenvalues differ from a constant parameter when we compare with the second class system. Thus, a conversion of the D-dimensional sphere second class system for a first class one does not reproduce the same values.

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PACS number: 11.10.Ef
Keywords: Weyl ordering, constrained systems.

1 Introduction

When one obtains the representation of the physical operators given by the algebraic expressions of the Dirac commutators of a specific theory we must pay attention if these canonical operators are Hermitian ones. If it is not true, this fact indicates that there are the so-called operator ordering problems. It is well known that many theories, for example, we can cite the nonlinear sigma model, the Skyrme model, D-dimensional sphere quantum mechanics, which will be the subject of this paper, present ordering problems. The ordering problems, in many cases, appear as a consequence of the constraints dynamic acting in the systems. Therefore, the presence of the constraints can lead to a non-trivial relations for the Dirac commutators which lead to an operator level problems for the representations of the physical operators of a peculiar theory.

Some articles are devoted to study the problem of the quantization of the free particle constrained in a surface of D-dimensional sphere. We must mention the works of Abdalla and Banerjee who used a Lagrangian reduced method to obtain the quantum Hamiltonian of the multidimensional rotor; Kleinert and Shabanov who employed the angular momentum algebra operators to calculate the energy spectrum; Foerster, Girotti and Kuhn who used the metrical formalism to described the quantum particle in a curved space.

The purpose of this paper is to calculate with more rigour the energy

\footnote{These commutators define the quantum structure of a particular theory.}
spectrum of a quantum particle lying in a surface of D-dimensional sphere. We use the second and first class Dirac methods of quantization\cite{6}. In both cases, the Weyl prescription is employed to construct the physical Hermitian operators.

This paper is organized as follows. In Section 2, we derive the energy levels of the quantum Hamiltonian constructed from the Hermitian momentum operator, written in a rectangular coordinates, which satisfies the Dirac brackets. For D=2 we verify that this energy levels are in accordance with experimental values\cite{4}. This result can suggest the power of the Dirac method quantization of the constrained systems. In Section 3, we convert the second-class constraints into the first class ones. For this, we use the non-abelian BFFT formalism\cite{7,8}. With the first class Hamiltonian we calculate the energy eigenvalues of the quantum particle which differ by a constant parameter from the obtained by Dirac brackets. In Section 4, we give the conclusions. In the Appendices we perform a brief review about the non-abelian BFFT procedure, where we calculate the Lagrangian of this new first class system.

2 Dirac brackets for the free particle on the D-dimensional sphere

The quantization of the second class constrained systems is usually performed by using the method proposed by Dirac, Bergman and co-workers\cite{6}. The constraints are classified as primary and secondary ones. Secondary constraints are obtained from the condition that primary constraints are conserved in time. One must repeat the condition requiring time derivative of secondary constraints vanish until all independent constrains are obtained. If the whole second class constraints are established, the so called Dirac bracket
for the canonical variables $A$ and $B$ is given by

$$\{A, B\}^* = \{A, B\} - \{A, \phi_\alpha\}C^{-1}_{\alpha\beta}\{\phi_\beta, B\},$$

(1)

where $\phi_\alpha$ and $\phi_\beta$ are the second class constraints and the matrix elements $C_{\alpha\beta}$ is defined by

$$C_{\alpha\beta} = \{\phi_\alpha, \phi_\beta\}.$$

(2)

The quantum mechanics commutators are given by the replacement

$$\{, \}_D \to -i [ , ].$$

Now, let us consider the dynamic of a particle in the $D$-sphere manifolds. The primary constraint is

$$\phi_1 = x_i x_i - R^2 = 0,$$

(3)

where $R$ is the radius of the sphere. With the expression of the classical Hamiltonian given by

$$H_c = \frac{1}{2} p_i p_i,$$

(4)

we obtain the secondary constraint

$$\phi_2 = x_i p_i = 0,$$

(5)

which expresses the fact that a motion on the surface of sphere has no radial component. From the expressions of primary and secondary constraints, (3) and (5), we obtain the algebra of the canonical operators which defines the quantum mechanics of a particle in a surface of $D$-dimensional sphere [3], [12]
\[ [x_i, x_j] = 0, \quad (6) \]
\[ [x_j, p_k] = i \left( \delta_{jk} - \frac{x_j x_k}{R^2} \right), \quad (7) \]
\[ [p_j, p_k] = \frac{i}{R^2} (p_j x_k - p_k x_j). \quad (8) \]

Here we would like to comment the operator ordering problem occurring in the right-hand side of Eq. (8). This fact is solved under condition that this commutator satisfied the equation\(^2\)  
\[ [p_j, p_j] = 0. \]

The solution of the momentum operator that satisfies the canonical relations (7) and (8) is
\[ p_k = \frac{1}{i} \left[ \partial_k - \frac{x_k x_i}{R^2} \partial_i \right]. \quad (9) \]

The Laplacian operator in rectangular coordinates is\(^3\)
\[ \nabla^2 = \frac{1}{R^2} \left( \sum_{K=1}^{D-1} (\prod_{j=1}^K \sin \theta_j)^{-2} (\sin \theta_K)^{K+2-D} \frac{\partial}{\partial \theta_K} (\sin \theta_K^D - \frac{\partial}{\partial \theta_K}) + (\prod_{j=1}^{D-1} \sin \theta_j)^{-2} \frac{\partial^2}{\partial \theta^2} \right). \]

\(^4\)Notice that \( \partial_i x_i = \delta_{ii} = D + 1. \)

\(^5\)The eigenvalues of the operator \( \text{Op} \) are defined by the following equation:
\[ \text{Op} |\text{polynomial} \rangle = l |\text{polynomial} \rangle. \]
We must mention the problem of ordering that appear in Eq. (9). Due to this fact, the canonical momentum $p_k$ is not a Hermitian operator as required by quantum mechanics. However, we can solve this problem by using the Weyl ordering operator prescription\[9\]. In the case of the canonical momentum, Eq. (9), this rule expresses that the new operator must be constructed by counting all possible randomly order of the $x'$s and $\partial$. Then, the symmetric momentum operator $p_k$ reads

$$[p_k]_{\text{sym}} = \frac{1}{6i}(6\partial_k - \frac{1}{R^2}x_kx_i\partial_i - \frac{1}{R^2}x_k\partial_i x_i - \frac{1}{R^2}x_i x_k \partial_i - \frac{1}{R^2}x_i \partial_i x_k - \frac{1}{R^2}\partial_i x_k x_i - \frac{1}{R^2}\partial_i x_i x_k)$$

$$= \frac{1}{i}\left(\partial_k - \frac{1}{R^2}x_k x_i \partial_i - \left(D + \frac{2}{2R^2}\right)x_k\right). \quad (11)$$

The quantum Hamiltonian is obtained performing the square of the Hermitian operator $[p_k]_{\text{sym}}$

$$H = \frac{1}{2} [p_k]_{\text{sym}} \cdot [p_k]_{\text{sym}}$$

$$= -\frac{1}{2}\partial_k \partial_k + \frac{1}{2R^2}\left(OpOp + (D - 1)Op + \frac{D^2 - 4}{4}\right), \quad (12)$$

where the operator $Op$ is defined in ref.\[10\]. Thus, applying the Hamiltonian operator on the physical states $|\text{polynomial}\rangle$ we obtain the energy levels

$$E_l = \frac{1}{2R^2}\left[l(l + D - 1) + \frac{D^2 - 4}{4}\right]. \quad (13)$$

We would like to point out that the energy formula (13) gives no additional
constant energy for a particle on a circle [4]. D=2. This result is in agreement with the experimental energy level spacings of the hydrogen atom. Thus, for D=2, the Dirac quantization procedure together with the Weyl ordering prescription predict the correct energy levels for the quantum particle lying on the sphere.

3 The non-abelian BFFT formalism for the free particle on the D-dimensional sphere

In the appendix A we perform a brief review of the BFFT formalism and its non-abelian extension. In the appendix B we obtain the Lagrangian of this new theory.

In the Section 2, we have seen the Hamiltonian for a particle of unit mass moving on the surface of a D-dimensional sphere of radius R isometrically embedded in $\mathbb{R}^{D+1}$ is given by

$$H_c = \frac{1}{2} p_i p_i, \quad (14)$$

with the primary and the secondary constraints respectively written as

$$\phi_1 = x_i x_i - R^2 = 0, \quad (15)$$

$$\phi_2 = x_i p_i = 0. \quad (16)$$

The constraints $T_1$ and $T_2$ are of the second class. The matrix elements of their Poisson brackets read

For D=3 the extra term in Equation (13) is the same obtained in the three-sphere collective coordinates Skyrmions quantization [13].
\[ \Delta_{\alpha\beta} = \{T_\alpha, T_\beta\} = -2\epsilon_{\alpha\beta}x_i x_i, \quad \alpha, \beta = 1, 2 \quad (17) \]

where \( \epsilon_{\alpha\beta} \) is the antisymmetric tensor normalized as \( \epsilon_{12} = -\epsilon^{12} = -1 \).

To implement the extended non-abelian BFFT formalism, we introduce auxiliary coordinates, one for each of the second class constraint. Let us generally denote them by \( \eta^\alpha \), where \( \alpha = 1, 2 \), and consider that the Poisson algebra of these new coordinates is given by

\[ \{\eta^\alpha, \eta^\beta\} = \omega^{\alpha\beta} = 2\epsilon^{\alpha\beta}, \quad \alpha, \beta = 1, 2. \quad (18) \]

From Eq. (A26), we have

\[ 2X_{11}X_{22} = -2x_i x_i + C_{12}^1 T_1. \quad (19) \]

After some attempts, we find that a convenient choice for these coefficients is

\[ X_{11} = R, \]
\[ X_{22} = -R, \]
\[ X_{12} = 0 = X_{21}, \]
\[ C_{12}^1 = 2, \]
\[ C_{12}^2 = 0. \quad (20) \]

Using (A4), (A6), (A11), (18) and (20), the new set of constraints is found to be

\[ \tilde{T}_1 = x_i x_i - R^2 + R\eta^1, \quad (21) \]
\[ \tilde{T}_2 = x_i p_i - R\eta^2 + \eta^1 \eta^2. \quad (22) \]
The first class constraint algebra is

\[
\{ \tilde{T}_1, \tilde{T}_1 \} = 0, \quad \{ \tilde{T}_1, \tilde{T}_2 \} = 2 \tilde{T}_1, \quad \{ \tilde{T}_2, \tilde{T}_2 \} = 0.
\]  

(23)

Next, we derive the corresponding Hamiltonian in the extended phase space. The corrections for the canonical Hamiltonian are given by Eqs. (A20) and (A28). With the objective to simplify the expression of the first class Hamiltonian, we chose an algebra for the system defined by the parameters \( B_a^b \) in (X27). We have verified that possible values are

\[
B_1^1 = B_1^2 = B_2^1 = B_2^2 = 0.
\]  

(24)

Using the inverse matrices

\[
\omega_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta},
\]

(25)

\[
X^{\alpha\beta} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix},
\]

(26)

and the algebra defined by (24), it is possible to compute the involutive first class Hamiltonian

\[
\tilde{H} = \frac{1}{2} p_i p_i(1 - \frac{\eta^1}{R}) - x_i p_i \frac{\eta^2}{R}(1 - \frac{\eta^1}{R}) + \frac{1}{2} x_i x_i \frac{\eta^2 \eta^2}{R^2}(1 - \frac{\eta^1}{R}),
\]

(27)

which satisfies the first class Poisson algebra.
\{\tilde{T}_1, \tilde{H}_2\} = 0, \quad (B_1^1 = B_1^2 = 0) \quad (28)
\{\tilde{T}_2, \tilde{H}_2\} = 0. \quad (B_2^1 = B_2^2 = 0) \quad (29)

Here we would like to remark that, contrary the results obtained by the abelian BFFT method applied to the non-linear Lagrangian theories \[1, 15\], the expression of the first class Hamiltonian (27) is a finite sum. As it was emphasized in the introduction, the possibility pointed out by Banerjee, Banerjee and Ghosh to obtain non-abelian first class theories leads to a more elegant and simplified Hamiltonian structure than usual abelian BFFT case.

Here we intend to obtain the spectrum of the extended theory. We use the Dirac method of quantization for the first class constraints [8]. The basic idea consists in imposing quantum mechanically the first class constraints as operator condition on the wave-functions as a way to obtain the physical subspace, i.e.,

\[ \tilde{T}_\alpha |\psi\rangle_{phys} = 0, \quad \alpha = 1, 2. \quad (30) \]

The operators \( \tilde{T}_1 \) and \( \tilde{T}_2 \) are

\[ \tilde{T}_1 = x_i x_i - R^2 + R\eta^1, \quad (31) \]
\[ \tilde{T}_2 = x_i p_i - R\eta^2 + \eta^1 \eta^2. \quad (32) \]

Thus, the physical states that satisfy (30) are

\[ |\psi\rangle_{phys} = \frac{1}{V} \delta(x_i p_i - R\eta^2 + \eta^1 \eta^2) \delta(x_i x_i - R^2 + R\eta^1) |\text{polynomial}\rangle, \quad (33) \]
where $V$ is the normalization factor and the ket $\text{polynomial}$ is defined by $|\text{polynomial}\rangle = \frac{1}{N(l)}(x_1 + ix_2)^l$. The corresponding quantum Hamiltonian of (27) will be indicated as

$$\tilde{H} = \frac{1}{2}p_ip_i(1 - \frac{\eta^1}{R}) - x_ip_i\frac{\eta^2}{R}(1 - \frac{\eta^1}{R}) + \frac{1}{2}x_ix_i\frac{\eta^2\eta^2}{R^2}(1 - \frac{\eta^1}{R}).$$

(34)

Thus, in order to obtain the spectrum of the theory, we take the scalar product, $\text{phys} \langle \psi | \tilde{H} | \psi \rangle_{\text{phys}}$, that is the mean value of the extended Hamiltonian. Then

$$\text{phys} \langle \psi | \tilde{H} | \psi \rangle_{\text{phys}} = \langle \text{polynomial} | \frac{1}{V^2} \int d\eta^1 d\eta^2 \delta(x_ix_i - R^2 + R\eta^1)\delta(x_ip_i - R\eta^2 + \eta^1\eta^2) \tilde{H}\delta(x_ip_i - R\eta^2 + \eta^1\eta^2)\delta(x_ix_i - R^2 + R\eta^1) | \text{polynomial} \rangle.$$  

(35)

Notice that due to the presence of the delta functions $\delta(x_ix_i - R^2 + R\eta^1)$ and $\delta(x_ip_i - R\eta^2 + \eta^1\eta^2)$ in (35) the scalar product can be simplified. Then, integrating over $\eta^1$ and $\eta^2$ we obtain\footnote{The regularization of delta function squared like $(\delta(x_ix_i - R^2 + R\eta^1))^2$ and $(\delta(x_ip_i - R\eta^2 + \eta^1\eta^2))^2$ is performed by using the delta relation, $(2\pi)^2\delta(0) = \lim_{\nu \to 0} \int d^2x e^{ikx} = \int d^2x = V$. Then, we use the parameter $V$ as the normalization factor.}

$$\text{phys} \langle \psi | \tilde{H} | \psi \rangle_{\text{phys}} = \langle \text{polynomial} | \frac{1}{2R^2}x_ix_jp_ip_j - \frac{1}{2R^2}x_ip_ip_j | \text{polynomial} \rangle.$$  

(36)

The final Hamiltonian operator inside the kets (36) must be Hermitian. Then, this Hamiltonian has to be symmetrized. Following the prescription of Weyl
ordering (symmetrization procedure) we can write the symmetric Hamiltonian as

\begin{equation}
\tilde{H}_{\text{sym}} = \frac{1}{2R^2} [x_i x_i p_j p_j]_{\text{sym}} - \frac{1}{2R^2} [x_i p_i x_j p_j]_{\text{sym}}, \tag{37}
\end{equation}

where \([x_i x_i p_j p_j]_{\text{sym}}\) and \([x_i p_i x_j p_j]_{\text{sym}}\) are defined as

\begin{align*}
[x_i x_i p_j p_j]_{\text{sym}} &= \frac{1}{24} (4x_i x_i p_j p_j + 4x_i p_j x_i p_j + 4x_i p_j p_j x_i) + 4p_j x_i x_i p_j + 4p_j x_i x_i p_j + 4p_j p_j x_i x_i, \tag{38} \\
[x_i p_i x_j p_j]_{\text{sym}} &= \frac{1}{24} (2x_i p_i x_j x_j + 2x_i p_j x_j x_j + 2x_i p_j p_j x_j + 2p_i x_i x_j p_j + 2p_i x_i x_j p_j + 2p_i p_j x_j x_j) + 2p_i p_j x_j x_j + 2p_i p_j x_i x_i + 2p_i x_j x_i p_j). \tag{40}
\end{align*}

Then, using the symmetric Hamiltonian operator \(\tilde{H}_{\text{sym}}\), eq. (37), the mean value \(\langle 36 \rangle\) is
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\[\langle \text{polynomial} | | \tilde{H}\text{sym} | \text{polynomial} \rangle \text{phys} = \]
\[\langle \text{polynomial} | \frac{1}{2R^2} [x_i x_i p_j p_j]_{\text{sym}} - \frac{1}{2R^2} [x_i p_i x_j p_j]_{\text{sym}} | \text{polynomial} \rangle.\]

\[\langle \text{polynomial} | \frac{1}{48R^2} (4x_i x_i p_j p_j + 4x_i p_j x_i p_j + 4x_i p_j x_i x_i p_j + 4p_j x_i x_i p_j + 4p_j x_i x_i p_j x_i + 4p_j x_i x_i p_j x_i)\]
\[+ 2x_i x_i p_j p_j + 2p_i x_i x_i p_j + 2p_i x_i x_i x_i p_j + 2p_i x_i x_i p_j x_i + 2p_i x_i x_i p_j x_i + 2p_i x_i x_i p_j x_i + 2p_i x_i x_i p_j x_i\]
\[+ 2p_i x_i x_i p_j x_i + 2p_i x_i x_i p_j x_i) | \text{polynomial} \rangle.\]

The operator \(\pi^j\) describes the momentum of free particle and its representation on the collective coordinates space \(x_i\) is given by

\[\pi^j = -i \frac{\partial}{\partial x_j}.\]

Substituting the expression (48) into (44), we obtain the energy levels, read as

\[E_l = \text{phys} \langle \psi | [\tilde{H}\text{sym}] | \psi \rangle_{\text{phys}} = \frac{1}{2R^2} \left[ l(l + D - 1) + \frac{D(D + 1)}{4} \right].\]

Comparing with the second-class energy expression, formula (13), we see that this expression differs by a constant parameter.

\[\text{Recently, a work of Hong, Kim and Park [19] suggest a generalized momenta definition that can conciliate the energy spectra obtained by Dirac and BFT formalisms, in the context of collective coordinates quantization of the Skyrme model.}\]
4 Conclusions

In this work we perform a study about the energy spectrum of a quantum particle constrained in a surface of the D-dimensional sphere. We use two procedures to obtain the D-sphere energy eigenvalues: in the first, we calculate the Hamiltonian by squaring the Hermitian momentum operator. This symmetrical momentum is derived, firstly, by an explicit solution of the Dirac brackets between the canonical operators, and then, employing the Weyl ordering prescription, we obtain the Hermitian momentum operator. It is clear that the square of this operator (that is proportional to the Hamiltonian) also is Hermitian one. When we put D=2 (the physical system) the energy eigenvalues are according to experimental results. The second procedure is the conversion of the second class constraints of the D-sphere system into first class ones. We use an extension of the BFFT formalism. After this, we also employ the Weyl ordering rules to obtain the first class Hermitian Hamiltonian. Its eigenvalues differs by a constant parameter compared with the spectrum energy in the second class method. Thus, in the case of D-dimensional sphere quantum mechanics, Dirac Brackets quantization of the second class constraints can lead to the correct experimental results, contrary to the predict by the non-abelian BFFT formalism.

5 Acknowledgments

We would like to thank F.I.Takahara for critical reading. This work is supported in part by FAPEMIG, Brazilian Research Council.
6 Appendix A: Brief review of the BFFT formalism and its non-abelian extension

The BFFT formalism of converting the second class constraints into the first class ones is a recent procedure used to quantize systems of second class constraints. Then, the purpose of this appendix is to exhibit a review of the BFFT formalism.

Let us consider a system described by a Hamiltonian $H_0$ in a phase space $(q^i, p^i)$ with $i = 1, \ldots, N$. It is supposed that there the system possesses only the second class constraints. Denoting them by $T_a$, with $a = 1, \ldots, M < 2N$, we arrive at the following algebra

$$\{T_a, T_b\} = \Delta_{ab}, \quad (A1)$$

where $\det(\Delta_{ab}) \neq 0$.

As it was mentioned above, the general purpose of the BFFT formalism is to convert the second class constraints into the first class ones. This goal is achieved by introducing canonical variables, one for each of the second class constraint (the connection between the number of the second class constraints and the number of the new variables should be equal in order to preserve the same number of the physical degrees of freedom in the resulting extended theory). We denote these auxiliary variables by $\eta^a$ and assume that they satisfy the following algebra

$$\{\eta^a, \eta^b\} = \omega^{ab}. \quad (A2)$$

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9 Together with the Dirac first class quantization prescription.
10 Compared with the well known Dirac brackets method.
Here $\omega^{ab}$ is a constant non-degenerate matrix ($\det(\omega^{ab}) \neq 0$). The obtainment of $\omega^{ab}$ is embodied in the calculation of the resulting first class constraints which are denoted as $\tilde{T}_a$. Of course, these constraints depend on the new variables $\eta^a$, that is

$$\tilde{T}_a = \tilde{T}_a(q,p;\eta),$$  \hspace{1cm} (A3)

and are supposed to satisfy the boundary condition

$$\tilde{T}_a(q,p;0) = T_a(q,p).$$  \hspace{1cm} (A4)

In the framework of the BFFT formalism, the characteristic property of the new constraints is that they are assumed to be strongly involutive, i.e.

$$\{\tilde{T}_a, \tilde{T}_b\} = 0.$$  \hspace{1cm} (A5)

The solution of Eq. (A5) can be achieved by considering $\tilde{T}_a$ expanded as

$$\tilde{T}_a = \sum_{n=0}^{\infty} T_a^{(n)},$$  \hspace{1cm} (A6)

where $T_a^{(n)}$ is a term of order $n$ in $\eta$. The condition of compatibility with the boundary condition (A4) requires

$$T_a^{(0)} = T_a.$$  \hspace{1cm} (A7)

Substituting the Eq. (A7) into (A3) leads to a set of equations, one for each coefficient of $\eta^n$. We list some of them below
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\{T_a, T_b\} + \{T^{(1)}_a, T^{(1)}_b\}_{(\eta)} = 0 \quad (A8)

\{T_a, T^{(1)}_b\} + \{T^{(1)}_a, T_b\} + \{T^{(1)}_a, T^{(2)}_b\}_{(\eta)} + \{T^{(2)}_a, T^{(1)}_b\}_{(\eta)} = 0 \quad (A9)

\{T_a, T^{(2)}_b\} + \{T^{(1)}_a, T^{(1)}_b\}_{(q,p)} + \{T^{(2)}_a, T_b\} + \{T^{(1)}_a, T^{(3)}_b\}_{(\eta)}

+ \{T^{(2)}_a, T^{(2)}_b\}_{(\eta)} + \{T^{(3)}_a, T^{(1)}_b\}_{(\eta)} = 0 \quad (A10)

Here the notations \{\cdot,\cdot\}_{(q,p)} and \{\cdot,\cdot\}_{(\eta)}, represent the parts of the Poisson bracket \{\cdot,\cdot\} corresponding to the variables \((q,p)\) and \((\eta)\), respectively. The equations above are used iteratively to obtain the corrections \(T^{(n)}(n \geq 1)\). Equation (A8) gives \(T^{(1)}\). Using this result together with the Eq. (A9), one calculates \(T^{(2)}\), and so on. Since \(T^{(1)}\) is linear in \(\eta\) one can write it as

\[ T^{(1)}_a = X_{ab}(q,p)\eta^b, \quad (A11) \]

where \(X_{ab}\) are some new quantities. Substituting this expression into (A8) and using (A1) and (A2), we obtain

\[ \Delta_{ab} + X_{ac} \omega^{cd} X_{bd} = 0. \quad (A12) \]

We notice that this equation does not define \(X_{ab}\) in a unique way, because it also contains the still unknown elements \(\omega^{ab}\). What is usually done is to choose \(\omega^{ab}\) in such a way that the new variables are unconstrained. Consequently, the consistency of the method requires an introduction of other new variables in order to transform these constraints into the first class ones. This may lead to an endless process. It is important to emphasize that \(\omega^{ab}\) can be fixed anyway. However, even if one fixes \(\omega^{ab}\), it is still not possible to obtain a unique solution for \(X_{ab}\). Let us check this point. \(\Delta_{ab}\) and \(\omega^{ab}\) are
antisymmetric quantities so expression (A12) includes $M(M-1)/2$ independent equations. On the other hand, since there is no additional symmetry involving $X_{ab}$, they should represent a set of $M^2$ independent quantities.

In the case when $X_{ab}$ does not depend on $(q,p)$, it is easily seen that the expression $T_a + \tilde{T}_a^{(1)}$ is already strongly involutive for any choice we make and we succeed in obtaining $\tilde{T}_a$. If this is not so, the usual procedure is to introduce $T_a^{(1)}$ into Eq. (A9) in order to calculate $T_a^{(2)}$ and so on. At this point one faces a problem that has been the origin of some developments of the BFFT method, including the adoption of a non-abelian constraint algebra. This occurs because we do not know a priori what is the best choice we can make to go from one step to another. Sometimes it is possible to figure out a convenient choice for $X_{ab}$ in order to obtain a first class (abelian) constraint algebra at the first stage of the process [1]. It is opportune to mention that in ref. [18], the use of a non-abelian algebra was in fact a way of avoiding to dealing with the higher orders of the iterative method.

Another point of the usual BFFT formalism is that any dynamic function $A(q,p)$ (for instance, the Hamiltonian) has also to be properly modified in order to be strongly involutive with the first class constraints $\tilde{T}_a$. Denoting the modified quantity by $\tilde{A}(q,p;\eta)$, we then have

$$\{\tilde{T}_a, \tilde{A}\} = 0.$$ \hspace{1cm} (A13)

In addition, $\tilde{A}$ has to satisfy the boundary condition

$$\tilde{A}(q,p;0) = A(q,p).$$ \hspace{1cm} (A14)

The derivation of $\tilde{A}$ is similar to what has been done in getting $\tilde{T}_a$. Therefore, we consider an expansion of the form
\[ \tilde{A} = \sum_{n=0}^{\infty} A^{(n)}, \quad (A15) \]

where \( A^{(n)} \) is also a term of order \( n \) in \( \eta \)'s. Consequently, the compatibility with Eq. (A14) requires that

\[ A^{(0)} = A. \quad (A16) \]

The combination of Eqs. (A6), (A7), (A13), (A15), and (A16) gives the equations

\[
\begin{align*}
\{ T_a, A \} + \{ T_a^{(1)}, A^{(1)} \}_{(\eta)} & = 0 \quad (A17) \\
\{ T_a, A^{(1)} \} + \{ T_a^{(1)}, A \} + \{ T_a^{(1)}, A^{(2)} \}_{(\eta)} + \{ T_a^{(2)}, A^{(1)} \}_{(\eta)} & = 0 \quad (A18) \\
\{ T_a, A^{(2)} \} + \{ T_a^{(1)}, A^{(1)} \}_{(q,p)} + \{ T_a^{(2)}, \} + \{ T_a^{(1)}, A^{(3)} \}_{(\eta)} + \{ T_a^{(2)}, A^{(2)} \}_{(\eta)} + \{ T_a^{(3)}, A^{(1)} \}_{(\eta)} & = 0 \quad (A19) \\
\vdots
\end{align*}
\]

which correspond to the coefficients of the powers 0, 1, 2, etc. . . of the variable \( \eta \). It is just a matter of algebraic work to show that the general expression for \( A^{(n)} \) reads as

\[
A^{(n+1)} = -\frac{1}{n+1} \eta^a \omega_{ab} X^{bc} G_c^{(n)}. \quad (A20)
\]

where \( \omega_{ab} \) and \( X^{ab} \) are the inverses of \( \omega^{ab} \) and \( X_{ab} \), and

\[
G_a^{(n)} = \sum_{m=0}^{n} \{ T_a^{(n-m)}, A^{(m)} \}_{(q,p)} + \sum_{m=0}^{n-2} \{ T_a^{(n-m)}, A^{(m+2)} \}_{(\eta)} + \{ T_a^{(n+1)}, A^{(1)} \}_{(\eta)}. \quad (A21)
\]
Finally, let us consider the case where the first class constraints form a non-abelian algebra, i.e.

\[
\{\hat{T}_a, \hat{T}_b\} = C^c_{ab} \hat{T}_c. \tag{A22}
\]

The quantities \(C^c_{ab}\) are the structure constants of the non-abelian algebra. These constraints are considered to satisfy the same previous conditions given by (A3), (A4), (A6), and (A7). But now, instead of Eqs. (A8)-(A10), we obtain

\[
C^c_{ab} T_c = \{T_a, T_b\} + \{T_a^{(1)}, T_b^{(1)}\}(\eta) \tag{A23}
\]

\[
C^c_{ab} T_c^{(1)} = \{T_a, T_b^{(1)}\} + \{T_a^{(1)}, T_b\} \\
\quad + \{T_a^{(1)}, T_b^{(2)}\}(\eta) + \{T_a^{(2)}, T_b^{(1)}\}(\eta) \tag{A24}
\]

\[
C^c_{ab} T_c^{(2)} = \{T_a, T_b^{(2)}\} + \{T_a^{(1)}, T_b^{(1)}\}(\eta) \\
\quad + \{T_a^{(2)}, T_b^{(1)}\}(\eta) + \{T_a^{(3)}, T_b^{(1)}\}(\eta) \tag{A25}
\]

The use of these equations is the same as before, i.e., they shall work iteratively. Equation (A23) gives \(T^{(1)}\). With this result and Eq. (A24) one calculates \(T^{(2)}\), and so on. To calculate the first correction, we assume it is given by the same general expression (A11). Introducing it into (A23), we now get

\[
C^c_{ab} T_c = \Delta_{ab} + X_{ac} \omega^{cd} X_{bd}. \tag{A26}
\]

Of course, the same difficulties concerning the solutions of Eq. (A12) also apply here, with the additional problem of choosing the appropriate struc-
ture constants \( C_{ab}^c \). To obtain the embedding Hamiltonian \( \tilde{H}(q,p,\eta) \) one cannot use the simplified version discussed for the abelian case (embodied into Eq. (A22)) because the algebra is not strong involutive anymore. Thus we start from the fact that the new Hamiltonian \( \tilde{H} \) and the new constraints \( \tilde{T}_a \) satisfy the relation

\[
\{\tilde{T}_a, \tilde{H}\} = B^b_a \tilde{T}_b, \tag{A27}
\]

where the coefficients \( B^b_a \) are the structure constant of the non-abelian algebra. The involutive Hamiltonian is considered to satisfy the same conditions (A14)-(A16). We then obtain that the general correction \( H^{(n)} \) is given by a relation similar to (A20), but now the quantities \( G^{(n)}_a \) are given by

\[
G^{(n)}_a = \sum_{m=0}^{n} \{T^{(n-m)}_a, H^{(m)} \}_{(q,p)} + \sum_{m=0}^{n-2} \{T^{(n-m)}_a, A^{(m+2)} \}_{(\eta)} + \{T^{(n+1)}_a, A^{(1)} \}_{(\eta)} - B^b_a T^{(n)}_c. \tag{A28}
\]

7 Appendix B: the vacuum functional of the model and the Lagrangian that corresponds to the non-abelian first class Hamiltonian

In this section, we intend to find the Lagrangian that leads to this new theory. A consistent way of doing this is by means of the path integral formalism, where the Faddeev procedure [17] has to be used. Let us identify the new variables \( \eta^a \) as a canonically conjugate pair \((\phi, \pi_\phi)\) in the Hamiltonian formalism,
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\[ \eta^1 \rightarrow 2\phi, \]
\[ \eta^2 \rightarrow \pi_\phi, \]  \( \text{(B1)} \)
satisfying (18). Then, the general expression for the vacuum functional reads

\[ Z = N \int [d\mu] \exp\{i \int dt[\dot{x}_i p_i + \phi\pi_\phi - \bar{H}]\}, \]  \( \text{(B2)} \)

with the measure \([d\mu]\) given by

\[ [d\mu] = [dx_i][dp_i][d\phi][d\pi_\phi]|det\{,\}| \]
\[ \delta(x_i x_i - R^2 + 2R\phi) \delta(x_i p_i - R\pi_\phi + 2\phi\pi_\phi) \prod_\alpha \delta(\bar{\Lambda}_\alpha), \]  \( \text{(B3)} \)

where \(\bar{\Lambda}_\alpha\) are the gauge fixing conditions corresponding to the first class constraints \(\bar{T}_\alpha\) and the term \(|det\{,\}|\) represents the determinant of all constraints of the theory, including the gauge-fixing ones. The quantity \(N\) that appears in (B2) is an usual normalization factor. Substituting the Hamiltonian (27) into (B2), the vacuum functional reads

\[ Z = N \int [dx_i][dp_i][d\phi][d\pi_\phi]|det\{,\}| \delta(x_i x_i - R^2 + 2R\phi) \]
\[ \delta(x_i p_i - R\pi_\phi + 2\phi\pi_\phi) \prod_\alpha \delta(\bar{\Lambda}_\alpha) \exp\{i \int dt[\dot{x}_i p_i + \phi \pi_\phi \]
\[ - \frac{1}{2} p_i p_i (1 - \frac{2\phi}{R}) + x_i p_i \frac{\pi_\phi}{R} (1 - \frac{2\phi}{R}) - \frac{1}{2} x_i x_i \frac{\pi_\phi^2}{R^2} (1 - \frac{2\phi}{R})\}. \]  \( \text{(B4)} \)

Using the delta functions \(\delta(x_i x_i - R^2 + 2R\phi)\) and \(\delta(x_i p_i - R\pi_\phi + 2\phi\pi_\phi)\), and exponentiating the last one with Fourier variable \(\xi\), we obtain
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\[
Z = N \int [dx_i][dp_i][d\phi][d\pi_\phi][d\xi]\text{det}\{,\}\mid \delta(x_i x_i - R^2 + 2R\phi) \prod_{\alpha} \delta(\tilde{\Lambda}_\alpha) \\
\exp\{i \int dt [\dot{x}_i p_i - \frac{1}{2} p_i^2 (1 - \frac{2\phi}{R}) + \xi x_i p_i \\
+ \frac{1}{2} (1 - \frac{2\phi}{R})^2 \pi_\phi^2 - (\dot{\phi} - \xi R (1 - \frac{2\phi}{R})) \pi_\phi)\}\}. \quad (B5)
\]

Integrating over \(\pi_\phi\), we arrive at

\[
Z = N \int [dx_i][dp_i][d\phi][d\xi]\text{det}\{,\}\mid \delta(x_i x_i - R^2 + 2R\phi) \prod_{\alpha} \delta(\tilde{\Lambda}_\alpha) \\
\frac{1}{1 - \frac{2\phi}{R}} \exp\{i \int dt [-\frac{1}{2} (1 - \frac{2\phi}{R}) p_i^2 + (\dot{x}_i + x_i \xi) p_i \\
- \frac{1}{2} (1 - \frac{2\phi}{R})^2 \pi_\phi^2 + \frac{R\phi}{(1 - \frac{2\phi}{R})} \xi - \frac{1}{2} R^2 \xi^2]\}. \quad (B6)
\]

Performing the integration over \(p_i\), we obtain

\[
Z = N \int [dx_i][d\phi][d\xi]\text{det}\{,\}\mid \delta(x_i x_i - R^2 + 2R\phi) \prod_{\alpha} \delta(\tilde{\Lambda}_\alpha) \\
\frac{1}{1 - \frac{2\phi}{R}} \sqrt{\frac{1}{1 - \frac{2\phi}{R}}} \exp\{i \int dt \frac{1}{2} \frac{\dot{x}_i \dot{x}_i}{(1 - \frac{2\phi}{R})} - \frac{1}{2} \frac{\dot{\phi} \dot{\phi}}{(1 - \frac{2\phi}{R})^2} \\
+ (1 - \frac{2\phi}{R}) (x_i \dot{x}_i + R\phi) \xi\}. \quad (B7)
\]

Finally, the integration over \(\xi\) leads to

\[
Z = N \int [dx_i][d\phi]\text{det}\{,\}\mid \delta(x_i x_i - R^2 + 2R\phi) \delta(\dot{x}_i + R\phi) \prod_{\alpha} \delta(\tilde{\Lambda}_\alpha) \\
\sqrt{\frac{1}{1 - \frac{2\phi}{R}}} \exp\{i \int dt \frac{1}{2} \frac{\dot{x}_i \dot{x}_i}{1 - \frac{2\phi}{R}} - \frac{1}{2} \frac{\dot{\phi} \dot{\phi}}{(1 - \frac{2\phi}{R})^2}\}, \quad (B8)
\]
where the new δ function that appear into the expression (B8) was obtained after integration over ξ. We notice that it does not represent any new restriction over the coordinates of the theory and leads to a consistency condition of constraint \( T_1 \). From the vacuum functional (B8), we identify the extended Lagrangian

\[
L = \frac{1}{2} \frac{\dot{x}_i \dot{x}_i}{(1 - \frac{2\phi}{R})} - \frac{1}{2} \frac{\dot{\phi} \dot{\phi}}{(1 - \frac{2\phi}{R})^2}. \tag{B9}
\]

Putting the extended variables, in the phase space, \( \phi \) and \( \pi_\phi \) equal to zero, we obtain the original Lagrangian. This result indicates the consistency of the theory.

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