About Imperfect Mushroom Billiards

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Imperfections of Bunimovich mushroom Billiards are analyzed. Any experiment will be affected by such imperfections, and it will be necessary to estimate their influence. In particular some of the corners will be rounded and small deviations of the angle of the underside of the mushroom head will be considered. The analysis displayed some unexpected non-generic features. The latter leads to a transition from a perfect mushroom behavior to either an ordinary KAM scenario or an abrupt transition to complete chaos, depending on the sign of the perturbation. The former produces a fractal area of islands and chaos, in fact a KAM scenario, not associated to the large island of stability of the mushroom billiard.

I. INTRODUCTION

Billiards have become the most physical or maybe the least unphysical of examples favoured by mathematicians and mathematical physicists to study chaotic dynamics and dynamics at the transition form order to chaos. Other such examples include spaces of constant negative curvature [8] and maps [9, 11].

Bunimovich’s mushroom billiards [2] have lately received much attention [3, 4, 5, 6, 7]. They represent mixed systems which do not show the generic fractal boundary between chaotic and integrable regions displayed e.g. by twist maps. The transition between the two regions is sharp, though some sticky parabolic manifolds associated to the boundary have been found [4]. The system is also of interest because it displays surprising patterns of return times to the foot of the mushroom [7], and due to its accessibility to experiment [6]. Yet experiment implies lack of exactitude, and this has caused us to study properties of imperfect mushrooom billiards. Surprisingly we found, that the result is not only of interest

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for error control, but shows features that are interesting by themselves.

In the present paper we shall first recapitulate some properties of mushroom billiards. It was known since their introduction that they are highly non generic systems, and as such we are interested in how far from their normal behavior they drift under small modifications. We will show the significance of the departure from their perfect versions under two classes of simple modifications, which can be classified as moving angles of straight walls or rounding of corners. The second one can be regarded as a smoothing on the billiard frontier. We shall proceed to discuss the points in this order and illustrate the modifications in corresponding figures. We shall discuss the results in general terms as well as their implications for experiment.

II. BUNIMOVICH MUSHROOMS

A mushroom billiard is a classical planar billiard with remarkable dynamics. It is a Hamiltonian system in which the phase space trivially decomposes into an integrable and a chaotic part whose frontier is not a fractal. This billiard in it’s simplest version is defined as follows: Consider a boundary defined by semi-circle and the diameter that closes this semicircle. Consider furthermore an opening in this diameter to which the open side of some rectangle or triangle is attached, to form three or two more sides closing the boundary of the billiard. If this rectangle or triangle does not touch the circular part of the boundary, we shall call the resulting billiard a perfect mushroom (see fig. 2). The area limited by the circle and the two radial walls will be called the hat of the mushroom and the rest the foot. It is not important that the hat is truly a semi-circle, but it must be a sector of a circle, meaning that its straight walls must point exactly towards the center of the circle. The exact shapes of the foot is irrelevant, as long as it is starshaped, limited by straight lines and not penetrating into the hat. Other options use eliptic hats, but we shall not consider these in the present paper. The foot assures the effective free expansion of bundles of rays, the mechanism with which we achieve chaos, i.e. hyperbolicity on a subset of positive measure of phase space. For a exposition of this mechanism called defocusing, see [10]. Following these rules we can create weird billiards that will still have the most important properties of mushroom billiards, see for example the ones labeled e and f in the figure 2. For simplicity we will consider only triangular or rectangular feet and axially symmetric mushrooms, as
illustrated a and b in fig. 2. Triangular feet allow us not to worry about bouncing ball orbits (parabolic manifolds) localized entirely inside the foot.

We first introduce some notation as illustrated in fig. 1. The radius of the hat will be called $R$. $r$ will be used for the distance from the center of the circle to the farthestmost inner corner of the hat. For the Poincaré-Birkhoff plots, we will take the arc length by the following convention. The Zenith of the hat will be the zero point, and it will increase to the right along the border and decrease in the opposite direction. As the border is topologically a circle, we will use the identification of the maximum and minimum arc length at the point labeled $l_{\text{max}}$ (we are using only symmetric mushrooms, for simplicity). Further notation will be given as needed.

The dynamic of the billiard is, as usual, given by free motion inside the billiard and specular reflection at the walls. Velocity is normalized to the absolute value 1. As can be seen by image construction, the orbits originating in the hat with angular momentum larger than $r/R$ never leave the hat and are completely integrable. Their caustic will define a half circle whose radius is the absolute value of the angular momentum, a conserved quantity. Therefore $r$ is also the radius of the smallest possible caustic. Almost all other trajectories, except subsets of measure zero, enter the foot of the mushroom infinitely many times. There they experience defocusing, and are therefore chaotic. According to Bunimovich’s theorem [1], they define a subset of the phase space whose dynamic is hyperbolic, ergodic and Bernoulli, therefore chaotic in the usual sense. A Poincaré-Birkhoff section illustrates this (fig. 3). A convenient parametrization of the Poincaré-Birkhoff section is the arc length of the border and the tangential (with respect to this same border) component of the momentum of each trajectory. The map that associates to each point in this section the following bouncing point of the same trajectory is called the Birkhoff map. An example for a perfect mushroom is shown in figure 3. Due to the lack of differentiability at the corners of the billiard, some curious, but not bothersome, discontinuities appear in the integrable orbits. It can be seen how the conservation of the absolute value of angular momentum produces a foliation in which the bounces corresponding to the semicircle present a constant tangential momentum. In the following sections we will focus on this region of the section. We will call the integrable part $\mathcal{I}$ and the chaotic region $\mathcal{P}$, both in the flow and in the Birkhoff map.

Ref.[7] presented a result concerning the distribution of the allowed numbers of bounces
named *magic numbers* on the circular part of the boundary between any entry of the trajectory into the hat and its next exit. For every ratio $r/R$ only some "magic" numbers are allowed and they can be grouped, according to their angular momentum, into triads, in which the larger is sum of the other two. This result is consequence of a theorem for rigid circle maps [12]. In figure 7 this distribution is shown for a perfect mushroom with aspect ratio $r/R = 1/3$.

### III. TILTED MUSHROOMS

We now consider a first type of perturbations: We shall deal with deviations in the angle of the straight lines forming the underside of the hat with the circular boundary from the ideal value of $\pi/2$. The tilting of the underside of the hat will be shown on the right hand and measured in terms of the angle $\epsilon$. The other side of the picture refers to a perturbation discussed below.

The angle to the circular border is no longer $\pi/2$. and this leads to two different scenarios:

1) If this angle is smaller than $\pi/2$ i.e. $\epsilon < 0$ we find a transition to a mixed phase space with the usual characteristics of a twist map. We studied this case numerically and it displays the generic scenario see figure 9. These billiards can be considered as closely related to D billiards, i.e. circle sections, where we have an abrupt transition from generic twist map behaviour to total chaos passing through the integrable half-circle.

2) If this angle is larger than $\pi/2$ i.e. $\epsilon > 0$ we find ergodicity and indeed a $K$-system. In this case the prolongation of the edges, that constitute the underside of the hat will intersect at a point more distant than the center of the circle. Allmost evry trajectory originating on the circle will eventually hit these walls, which are farther away than a of the circle. By image construction, this produces an effective free expansion larger than a corresponding cord of the circle, and then, according to Bunimovich theorem [11], a set of full measure will have ergodic, mixing and hyperbolic dynamics. Note that this does not exclude the possible existence of parabolic manifolds of measure zero. Eventually, allmost all trajectories will leave the hat, as they are in the ergodic component of the phase space. The proof for the Kolmogorov property follows [3]. The map will present the appearance of a blurred Birkhoff Map of the perfect mushroom in the short range (fig. 5).

If the tilting is very small (from the order of a few thousandths of a radian), then, for
considerable times, the behavior of the billiard shadows that of a perfect mushroom. The
billiard is now a generalized stadion but, nonetheless, dynamics will be unindistinguishable from
the perfect mushroom for some period of time except for a small though finite measure set of
initial conditions. In such transient states, confusing structures do appear, as can be seen in
fig. 6 near the point \((l, v_t) = (2, 0.5)\). The arguments given in [7] for the bouncing number
distributions are no longer strictly valid, and the selectivity of escape times is no longer rigid.
Two consequences are to be expected: First we will find low frequency occurrences of small
bounce numbers outside the magic numbers, as a consequence of hitting the hole rather than
the hat by a small deviation. Second we expect depeletion of the large magic numbers, as
for many bounces on the hat the angular deviation due to the tilt accumulates. This can
also be viewed as a consequence of the fact that the parabolic manifolds near which the
bounce numbers diverge for the perfect mushrooms no longer exist. We may compare the
figures 7 and 8 in which the frequency of bounces in the circle of the hat is plotted. times,
particularly at longer times.

IV. ROUNDED EDGES IN MUSHROOMS

The second deformation of the billiard that we are going to consider is to round the inner
edges of the hat. We substitute the corners with defocusing arcs of circles, such that the
curve representing the boundary as well as its first derivative remain continuous. This kind
of deformations are called smoothings of the billiard [3]. Obviously the second derivative will
not be continuous. The caustic limiting the integrable part has a larger radius, corresponding
to the point where the straight edge toward the circle bounding the hat begins. On the other
hand the conditions required for chaos in the remaining phase space of the mushroom are
now no longer fulfilled. We thus expect a region with a mixed phase space to appear and
likely to be concentrated at the edges of the integrable island which persists outside the
larger caustic. The large island must persist due to the fact that the angular momentum is
still conserved outside the innermost caustic. Numerical studies confirm this expectation, as
can be seen in the figure 11. The mechanism that creates this mixed phase space is similar
to the one found in eccentric ring billiards.
V. ROUNDED AND TILTED MUSHROOMS

If we consider the first kind of deformation, such that it would lead to complete chaos and combine it with the second kind we get a picture, that without the previous discussion would seem very puzzeling. The large integrable island, corresponding to the conservation of the magnitude of angular momentum, disappears, as the tilt destabilizes the entire island. For the curved surfaces on the other hand the scenario is of KAM-type and will move islands slightly and possibly change some high level bifurcations, but not the qualitative aspect of this mixed phase space area. It becomes thus clear, that the scenario we saw in the previous section is not of typical KAM type for the dissolution of the mayor island. The mixed zone is truely independent of the large stable island. The latter becomes egodic for any tilting, though small angles again cause longtransients. The dispersing boundaries generate a KAM structure not correlated to the disappearance of the large island, i.e the mixed, fractal phase space region is not qualitatively changed. To see this we show the Birkhoff section of a mushroom with $\epsilon = 0.0001$ and $\rho = 1.0d0$ in the figures [10] and [12]. The fractal islands are very sensitive to tilting, not because they have a direct correlation with the disappearance of the larger islands, but because even a small angle of tilting will make a considerable perturbation at the point of the rounding. This billiard has no special properties for long times, but shows the features of a perfect mushroom billiard in transients.

VI. CONCLUSIONS AND REMARKS

We have seen that the characteristic behaviour of mushroom billiards is non-generic with respect to two possible one parameter perturbations. In the case of tilting, the “shadowing” of the integrable trajectories for fairly long times for small values of $\epsilon$, making the orbits appear as widened structures in the Birkhoff maps. The irruption of former prohibited bouncing numbers in the bouncing distribution of the hat can be understood in terms of the breaking of the symmetry of the map [7, 12].

The central point of interest is the way in which these billiards fail to be mushroom billiards. In the case of the tilted mushrooms, they become, in one direction, typical Bunimovich billiards (hyperbolic, ergodic and Bernoulli), and in the other (inverse tilting), generic KAM systems. Thus the perfect mushroom billiards is a very peculiar limiting case between
a fractal with integrable islands and global chaos, with just one clean island. The time rates for the deviations are also important, as they mark a limit for acceptable precision during experiments of the type proposed in [7]. In the case of rounded mushrooms, the presence of KAM islands could introduce noise in the experimental settings of micro wave billiards. If they were noticiable, but as we have seen, this is not the case. The departures from the magic number of bounces distribution are also a point to be noticed. As we could associate to each magic number a typical length for its path in the hat, it could be devised a setting for measuring it experimentally ([6]). Imperfections of the design could lead to the possible appearances of other bouncing numbers than the magic ones, and serve as a parameter for the cleanliness of the billiard design, yet as we deal with wave experiments, diffraction will hide any small defects.

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Figure 1: Mushroom with the notation used.

Figure 2: Different perfect mushroom billiards. We emphasize that in d,e and f the edges of the straight walls of the hat must point towards the center of the circle.
Figure 3: Birkhoff map of the flow for a perfect symmetric mushroom. The Integrable part, with the discontinuities encloses the bounces of chaotic orbits inside the hat. The blank areas on the outside will be filled with longer evolution, but access is slow, as they also contain the parabolic manifolds of the foot. Due to time reversal invariance and the axial symmetry of the billiard, the section presents fourfold symmetry. We will plot accordingly only the positive quadrant of the plot.

Figure 4: Perturbed mushroom notation conventions.
Figure 5: Birkhoff-map of the flow for a tilted mushroom. The bouncing ball orbits still make the mixing rate in their neighborhood very slow. Some integrable orbits of the untilted mushroom are mimicked for short times, but the covering will become uniform in time. The tilting parameter is $\epsilon = 0.01$ radians. The map has been iterated 500 times.
Figure 6: The fundamental region where the shadow of integrable orbits can be seen. The tilting is 0.003 radians and the map has been iterated thousand times. It is important to observe the slow mixing rate.
Figure 7: Bouncing Frequency against number of bounces for a perfect mushroom. Note the strong selectivity in allowed bouncing times. The ratio of the width of the opening against the radius of the hat is $1/3$. 
Figure 8: Bouncing Frequency against number of bounces for a tilted mushroom, $\epsilon = 0.03$. Compare with the plot [7]. Selectivity is lost at large bounce numbers.
Figure 9: An inverse tilted mushroom generates this kind of phase space, a very typical KAM structure. The tilting is $-0.2$ rad. Other tiltings generate very similar pictures.
Figure 10: Fundamental domain for the tilted and rounded mushroom, with parameters $\rho = 1.0, \epsilon = 0.00001$. Integrable islands are impossible to spot, and the tilting is too small to notice the breaking of the larger integrable effect. Mixing effects are not appreciable yet (500 iterations). Notice once more the smallness of the scale.
Figure 11: A beautiful interwoven pattern of KAM islands with centers of high period. The parameters are $\rho = 1.0, \epsilon = 0$. These islands are uncorrelated with the larger island. They owe their existence to a smoothing of the billiard [3]. Notice the scale in both axis. The islands are invisible in the full plot.
Figure 12: A close-up of figure 10 which shows the same region as figure 11 but with tilting ($\epsilon = 0.0002$).