Propagators in Curved Space

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March 28, 2022

Abstract

We demonstrate how to obtain explicitly the propagators for quantum fields residing in curved space-time using the heat kernel for which a new construction procedure exists. Propagators are determined for the case of Rindler, Friedman-Robertson-Walker, Schwarzschild and general conformally flat metrics, both for scalar, Dirac and Yang-Mills fields. The calculations are based on an improved formula for the heat kernel in a general curved space. All the calculations are done in $d = 4$ dimensions for concreteness, but are easily generalizable to arbitrary $d$. The new method advocated here does not assume that the fields are massive, nor is it based on an asymptotic expansion as such. Whenever possible, the results are compared to that of other authors.

1 Introduction

The calculation of propagators of various fields in a curved background is of utmost importance in theoretical physics, since it essentially provides the only way of calculating scattering processes in, say, the early universe or around a black hole. As there is at present no fully satisfactory theory of quantum gravity, the only way of studying quantum effects in the presence of strong gravitational fields is through quantum field theory in curved spacetime, what one might call semi-classical gravity. The quantities of interest there
are first and foremost the propagators, the effective actions and the energy-momentum tensor. Therefore a lot of work have been done in the past two or three decades in this field, see e.g. [1, 2, 3, 4]. In this paper we will concentrate on the calculation of the propagators of quantum fields of spin zero, one-half and one. To the best of our knowledge this is the first time propagators of non-zero spin fields is determined in a general background. We will leave the study of gravitons on arbitrary backgrounds for future research – the main aim of this paper is quantum field theory in curved spacetime. In previous papers [5, 6] we have developed a method for the determination of generating functionals of quantum fields residing in curved spacetime. This was done using the heat kernel method and fairly reliable methods for determining this quantity were devised [5]. One could decide to determine propagators for the quantum fields from the generating functional but it turns out that one can construct the propagators directly from the heat kernel and this latter approach is the one pursued in this paper. It should be emphasized that the resulting propagator is not just the free or bare one, as the calculations put forward here are indeed non-perturbative, at least in the coupling to the background fields (usually just the gravitational field, but occasional comments are made on how to include other kinds of backgrounds, e.g. external Yang-Mills or Higgs fields).

In section 2 we relate the heat kernel of a differential operator $A$, associated with the quantum field in question, to the corresponding propagator. Then we show how to explicitly determine propagators using previous results on constructing the heat kernel. This is just the Schwinger-DeWitt proper time formalism, but we use an improved expression for the heat kernel. Section 3 is devoted to a discussion of the corresponding vacuum state and a discussion of the Hadamard condition. We exemplify the approach in section 4 where we also compare our results to that of other authors and finally provide a brief conclusion and outlook in section 5.
2 Determining the Propagator From the Heat Kernel

Call the operator of interest, which generally varies in space, \( A = A(x) \), and the corresponding eigenfunctions \( \psi_\lambda \) so that

\[
A\psi_\lambda = \lambda \psi_\lambda
\]  

(1)

The heat kernel \( G_A(x, x', \sigma) \) is the function satisfying the heat kernel equation

\[
AG_A(x, x'; \sigma) = -\frac{\partial}{\partial \sigma} G_A(x, x'; \sigma)
\]

subject to the boundary condition \( G_A(x, x'; 0) = \delta(x, x') \).

Note that this equation is satisfied by (the spectral representation of the heat kernel)

\[
G_A(x, x', \sigma) \equiv \sum_\lambda \psi_\lambda(x)\psi_\lambda^*(x')e^{-\lambda \sigma}
\]  

(2)

The Green’s function on the other hand is a solution to the equation

\[
AG(x, x') = \delta(x, x')
\]  

(3)

which is satisfied by (the spectral representation of the Green’s function)

\[
G(x, x') \equiv \sum_\lambda \psi_\lambda(x)\psi_\lambda^*(x')\lambda^{-1}
\]  

(4)

Put together with equation (2) this yields the relationship between the Green’s function and the heat kernel:

\[
G(x, x') = -\int_0^\infty d\sigma G_A(x, x'; \sigma)
\]  

(5)

(provided that \( \lambda \neq 0 \)). This equation alone shows that the usual asymptotic expansion, due to Schwinger and DeWitt, which is only valid for \( \sigma \to 0 \), cannot be relied upon to provide good propagators. Furthermore, for massless fields, the Schwinger-DeWitt expansion would lead to a manifestly divergent \( \sigma \)-integral. In the following section we will write down an improved expansion for the heat kernel valid for \( \sigma \) large as well, and which moreover leads to closed expressions for the expansion coefficients. This improved expansion is convergent even for \( m = 0 \), and the method of calculation is readily generalized to higher spin too.
2.1 Heat Kernel and Propagators for a Scalar Field

The operator involved in the case of scalar particles is the (curved space) d’Alembertian (minimally coupled scalar fields) or the d’Alembertian plus some function (non-minimally coupled scalar fields). So we express the d’Alembertian in terms of the local vierbeins, \( e^a_\mu \equiv \frac{\partial x^a}{\partial x^\mu} \) where Greek indices refer to general coordinates while Latin indices refer to the local inertial frame (the metric is \( g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \) where \( \eta_{ab} \) is the Minkowski metric, \( g \) is the determinant of the metric and \( e \) the vierbein determinant, \( e = \sqrt{|g|} \):

\[
\Box \equiv \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu) \\
= \frac{1}{e} \partial_\mu (e\eta^{ab} e^a_\mu e^b_\nu \partial_\nu) \\
= \frac{1}{e} e^m_\mu \partial_m (e\eta^{ab} e^a_\mu e^b_\nu \partial_\nu) \\
= \frac{1}{e} e^m_\mu \partial_m (e\eta^{ab} e^a_\mu e^b_\nu \partial_\nu) + \eta^{ab} e^m_\mu e^\nu_\mu e^\nu_n \partial_m \partial_n \\
= \Box_0 + \frac{1}{e} e^m_\mu (\partial_m (ee^\mu_n)) \partial_n
\]

where \( \Box_0 = \eta^{ab} \partial_a \partial_b \) is the d’Alembertian of a comoving observer the heat kernel of which is known to be \( \Box_0 = (4\pi\sigma)^{-2} e^{-\frac{\Delta(x,x')}{4\sigma}} \)

\[
G_0(x,x';\sigma) = (4\pi\sigma)^{-2} e^{-\frac{\Delta(x,x')}{4\sigma}}
\]

where \( \Delta(x,x') \equiv (\int_{x'}^x ds)^2 \), is the geodesic distance squared (i.e. half the so-called Synge world function) which is just \((x-x')^2\) in Cartesian coordinates. We will refer to \( \Box_0 \) as the “flat” d’Alembertian, and consequently to \( G_0 \) as the “flat” heat kernel. Proceed to remove the first order term of the Lagrangian by the substitution

\[
G_\Box = \tilde{G}(x,x';\sigma) e^{-\frac{1}{2} \int e^\nu_\mu (\partial_m (ee^\mu_n)) dx_n
\]

\[1\]The heat kernel \( \Box_0 \) is the straightforward covariant generalization of the result in Minkowski space-time \( G = (4\pi\sigma)^{-2} \exp(-\frac{(x-x')^2}{4\sigma}) \). In \( d \) dimensions one simply has to replace \((4\pi\sigma)^{-2}\) by \((4\pi\sigma)^{-d/2}\). The vierbeins are introduced to make this transition from flat to curved space more transparent.
The integral in this expression is always easy to calculate as the term differentiated is just the reciprocal of the term with which it’s multiplied by so that one gets a logarithm of a product of vierbein components which in turn makes the exponential the reciprocal of the square root of this product. It turns out, however, that we actually do not need the explicit form of this integral as it cancels out in the final expression.

When making the substitution (8) in equation (2) one gets a zero’th-order term which must be added to the one that was there in the first place (i.e. to $\xi R$). One thus gets the following heat kernel equation ($\xi = 0$ in the case of minimal coupling):

$$\square_0 + \frac{1}{4} \left( \frac{1}{e^\mu_m (\partial_m (e^n_e)))^2} \right) - \frac{1}{2} \partial^n \left( \frac{1}{e^\mu_m (\partial_m (e^n_e))) + \xi R \right) \tilde{G}(x, x'; \sigma) = -\partial_\sigma \tilde{G}(x, x'; \sigma)$$

(9)

written compactly as

$$(\square_0 + f_0(x)) \tilde{G}(x, x'; \sigma) = -\partial_\sigma \tilde{G}(x, x'; \sigma)$$

(10)

with

$$f_0 \equiv \xi R + \frac{1}{4} \left( e^{-\mu_m (\partial_m (e^n_e)))^2} \right) - \frac{1}{2} \partial^n \left( e^{-\mu_m (\partial_m (e^n_e))) \right)$$

(11)

The method for solving this equation devised in papers [5] makes use of the following trick:

$$\tilde{G}(x, x'; \sigma) = G_0(x, x'; \sigma) e^{-T(x,x';\sigma)}$$

(12)

Taylor expanding $T$

$$T(x, x'; \sigma) = \sum_{n=0}^{\infty} \tau_n(x, x') \sigma^n$$

(13)

one gets the following recursion relation for the coefficients

$$n \tau_n = -\square_0 \tau_{n-1} + \sum_{n'=0}^{n-1} \partial \tau_{n'} \cdot \partial \tau_{n-1-n'} + \frac{1}{2} \partial \Delta \cdot \partial \tau_n$$

(14)

One should note that this expansion is not the usual asymptotic one due to Schwinger and DeWitt (see e.g. [1]), but actually holds for all values of $\sigma$. Furthermore, as the recursion relation will show, it is actually rather easy to evaluate the coefficients, especially along the diagonal $x = x'$. The expansion shown here works for massless fields as well, contrary to the Schwinger-DeWitt expansion.
with \( \tau_0 = +\frac{1}{2} \int e^{-1} \partial_\mu (ee_\mu^n) dx^n \), as follows from the boundary condition 
\( \lim_{\sigma \to 0} G(x, x'; \sigma) = \delta(x(x')) \). Along the diagonal \( x = x' \), this recursion relation simplifies and the result is listed in [5]. In the general case \( x \neq x' \), however, the coefficients develop a dependency upon (the values of the curvature and its derivatives along) the geodesic from \( x \) to \( x' \) (assuming the existence of such a curve – otherwise \( \Delta \) has to be put equal to \( \infty \), leading to zero amplitude for the propagation). Explicitly,

\[
\tau_n(x, x') = e^{-2n \sqrt{\Delta(x,x')}} \int_0^1 g_{n-1}(\tau) e^{2n\tau} d\tau + c_n e^{-2n \sqrt{\Delta(x,x')}}
\]  
(15)

where the integral is along the abovementioned geodesic, and the functions \( g_{n-1} \) are given by

\[ g_n(x, x') = -\Box_0 \tau_n(x, x') + \sum_{n'=1}^{n} \partial \tau_{n'} \cdot \partial \tau_{n-n'} \]

with the first few functions being

\[
\begin{align*}
    g_0 &= \tilde{f}_0 \equiv f_0 + \partial^n (e^{-1} \partial_n e_\mu^n) + e^{-2} \eta^{mn} \partial_\mu (ee_\nu_m) \partial_\nu (ee_\nu^n) \\
    g_1 &= -\Box_0 \tau_1(x, x') + 2 \partial \tau_0 \cdot \partial \tau_1 \\
    g_2 &= -\Box_0 \tau_2(x, x') + (\partial \tau_1)^2 + 2 \partial \tau_0 \cdot \partial \tau_2
\end{align*}
\]

and so on, with \( f_0 \) given by [11], \( g_0 \) is not just simply \( f_0 \), as one might otherwise expect, a fact due to the boundary condition \( G(x, x'; \sigma) \to \delta(x(x')) \) for \( \sigma \to 0 \). The variable \( \tau \) is the proper-time of a comoving observer, scaled such that at \( \tau = 0 \) the observer is at position \( x \), whereas at the later time \( \tau = 1 \) (s)he is at \( x' \).

Inside the integral along the geodesic the differential operators \( \partial, \Box_0 \) simplify to become

\[
\begin{align*}
    \partial_p &= e_\mu^p \partial_\mu = e_\mu^p \dot{x}_\mu(\tau) \frac{\partial}{\partial \tau} = \dot{x}_p(\tau) \frac{\partial}{\partial \tau} \\
    \Box_0 &= \eta^{pq} \partial_p \partial_q = \eta^{pq} \dot{x}_p \ddot{x}_q + \dot{x}^2 \frac{\partial^2}{\partial \tau^2}
\end{align*}
\]  
(16) (17)

where \( x_\mu(\tau) \) describes the geodesic. These differentiations are consequently rather straightforward to carry out.
The coefficients $c_n$ are independent of the curve, they only depend upon the value at the end points, in fact

$$\lim_{x' \to x} \tau_n(x, x') = c_n(x) \quad \text{and} \quad \lim_{x \to x'} \tau_n(x, x') = c_n(x') \quad (18)$$

and they are thus equal to the coefficients found in [5]. They satisfy

$$(n + 1)c_{n+1} = -\Delta_0 c_n + \sum_{n'=0}^n \partial c_{n'} \cdot \partial c_{n-n'} \quad (19)$$

and thus the relevant ones are

$$
\begin{align*}
c_1 &= \tilde{f}_0 \\
c_2 &= -\frac{1}{2} \Box_0 f_0 \\
c_3 &= \frac{1}{6} \Box_0^2 f_0 - \frac{1}{3} (\partial f_0)^2 \\
&\approx -\frac{1}{3} (\partial f_0)^2
\end{align*}
$$

where we have decided to only include first and second derivatives of $f_0$ (essentially the curvature) in the last approximation. These expressions are for massless fields, masses are accommodated by adding $m^2$ to $f_0$, and therefore to $\tau_1, c_1$. Written out in full, the first non-trivial coefficient function, $g_1$, is then

$$
g_1 = -\left[ \frac{1}{2} \Delta^{-3/2} (\partial \Delta)^2 + \Delta^{-1/2} (\partial \Delta)^2 - \Delta^{-1/2} \Box_0 \Delta \right] \tau_1 + \\
\Delta^{-1/2} \Box^p \Delta \left[ \int_0^1 \partial \mu (g_0 e^{2\tau}) d\tau + \frac{1}{2} \partial \mu c_1 \right] e^{-2\sqrt{\Delta}} - \\
e^{-2\sqrt{\Delta}} \left[ \int_0^1 \Box_0 (g_0 e^{2\tau}) d\tau + \frac{1}{2} \Box_0 c_1 \right] + \\
\eta^{mn} e^{-1} \partial \mu (e e^{\mu}) \left[ -\Delta^{-1/2} (\partial_m \Delta) \tau_1 + e^{-2\sqrt{\Delta}} \left\{ \int_0^1 \partial_m (\tilde{f}_0 e^{2\tau}) d\tau + \partial_m c_1 \right\} \right]
$$

where all derivatives outside the integrals are with respect to $x$ and not $x'$, the derivatives inside the integrals, however, are to be understood as in (16-17).

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3Strictly speaking the coefficients are $\frac{1}{2} (c_n(x) + c_n(x'))$, but for simplicity we have decided not to write them in this way, thus a symmetrization in $x, x'$ is always understood. In the particular examples this is carried out explicitly.
The next coefficient function, \( g_2 \), becomes

\[
g_2 = -\left[ \Delta^{-3/2}(\partial \Delta)^2 + 4\Delta^{-1/2}(\partial \Delta)^2 - 2\Delta^{-1/2}\Box_0 \Delta \right] \tau_2 + 4\Delta^{-1/2}(\partial^p \Delta) \left[ \int_0^1 \partial_p (g_1 e^{4r}) d\tau + \frac{1}{2} \partial_p c_2 \right] e^{-4\sqrt{\Delta}} - e^{-4\sqrt{\Delta}} \left[ \int_0^1 \Box_0 (g_1 e^{4r}) d\tau + \frac{1}{2} \Box_0 c_2 \right] + \\
\left\{ \Delta^{-1}(\partial \Delta)^2 \tau_1^2 + e^{-4\sqrt{\Delta}} \left[ \int_0^1 \partial_p (g_0 e^{2r}) d\tau + \frac{1}{2} \partial_p c_1 \right]^2 - 2\Delta^{-1/2}\tau_1 e^{-2\sqrt{\Delta}}(\partial^p \Delta) \left[ \int_0^1 \partial_p (g_0 e^{2r}) d\tau + \frac{1}{2} \partial_p c_1 \right] \right\} + \eta^{mn} \partial_m (e e^p_n) \left[ -2\Delta^{-1/2}(\partial_m \Delta) \tau_2 + e^{-4\sqrt{\Delta}} \left\{ \int_0^1 \partial_m (g_1 e^{4r}) d\tau + \partial_m c_2 \right\} \right]
\]

As is apparent, the expressions quickly get somewhat involved. Note, however, that most of the needed operations are trivial – most of the terms are simple derivatives of \( \Delta \) and \( \tilde{f}_0 \). The only non-trivial part being the integral along the geodesic. By the same token, we will usually only list \( \tilde{f}_0 \) and \( \tau_1 \) in the examples to follow.

One should notice that in principle one could find the coefficients to any chosen order, we will however stick to the approximate solution with \( \tau_n \approx 0 \), \( n \geq 4 \), corresponding to \( \partial^3 R, \partial^4 R, ... \) being negligible. In fact we can make do with the even more tractable form (which is adequate\(^4\))

\[
\tilde{G}(x, x'; \sigma) \approx G_0(x, x'; \sigma) e^{-\tau_1 \sigma} (1 + \tau_2(x, x') \sigma^2 + \tau_3(x, x') \sigma^3)
= (4\pi \sigma)^{-2} e^{-\frac{\Delta(x, x')}{4\sigma}} \tau_1(x, x') \sigma (1 + \tau_2(x, x') \sigma^2 + \tau_3(x, x') \sigma^3)(20)
\]

where expression \([7]\) for the “flat” space heat kernel has been inserted.

Other modifications of the Schwinger-DeWitt asymptotic expansion exists. In fact Parker and coworkers, \([7]\), have suggested that a factor \( \exp(-\xi R \sigma) \)

\(^4\)As general relativity is only renormalizable upto one loop order, it would not be meaningful to go to higher order in this approximation; the higher order terms are essentially third and higher order derivatives and powers of the curvature, corresponding to a higher loop order. This approximation corresponds to a regime in which \( R \gg (\partial R)^2, \Box_0 R \), i.e. strong but slowly varying curvature. The other extreme \( (R \ll \partial R, \text{i.e. strongly varying curvatures, e.g. in the space-time foam}) can be treated similarly; the coefficients \( c_n \) will then be \( c_n \approx (-1)^{n-1} \frac{1}{n!} \tau_0^{n-1} f_0 \), and similarly for \( g_n, \tau_n \).
was present if one performed a partial summation of the usual expansion. The basic new thing in our approach is the actual possibility of a systematic calculation of all the coefficients in the expansion – in fact we’ve got a general, closed expression, (13). The expansion of Parker et al. differs from ours in that whereas we expand $T$ in $\exp(T)$, they expand $\exp(T)$, thus getting simply modifications of the Schwinger-DeWitt coefficients. The main point of the approach put forward in this paper is the actual ability to systematically find the expansion coefficients both for $x \neq x'$ (the general case) and for $x = x'$ (which turns out to be very simple). It is the use of the exponential which makes this simplification possible.

As pointed out by Ford and Toms, [8], there is a genuine need for non-local terms in the heat kernel (and hence a need to go beyond the Schwinger-DeWitt expansion) and as it turns out, the coefficients obtained by the method put forward here are in fact non-local. Their geometric meaning is even rather transparent as they involve the integration along a geodesic of various curvature-related quantities.

The heat kernel can also be written as a path-integral, [9], and the usual Van Vleck-Morette determinant occurs when one considers geodesics, [10]. The $\sigma$-integrals involved in determining the Green’s function from equation (5) are now of standard type leading to the following result [11]

$$G(x, x') \approx -(4\pi)^{-2} \left(2 \frac{\Delta(x, x')}{4\tau_1(x, x')}\right)^{-\frac{1}{2}} K_{-1}(\sqrt{\Delta(x, x')\tau_1(x, x')} +$$

$$\tau_2(x, x') \left(\frac{\Delta(x, x')}{4\tau_1(x, x')}\right)^{\frac{1}{2}} K_1(\sqrt{\Delta(x, x')\tau_1(x, x')}) -$$

$$\tau_3(x, x') \frac{\Delta(x, x')}{4\tau_1(x, x')} K_2(\sqrt{\Delta(x, x')\tau_1(x, x')})$$

(21)

where $K_n$ is a modified Bessel (or MacDonald) function which has the following series expansion (needed in the case of gauge bosons treated below and for the proof of the Hadamard condition being satisfied):

$$K_n(x) = (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left(\ln \frac{1}{2} x - \frac{1}{2} \psi(n+k+1) - \frac{1}{2} \psi(k+1)\right) \left(\frac{x}{2}\right)^{2k+n}$$

$$+ \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n}$$

(22)
with $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the Euler psi (or digamma) function. Using the recursion relation
\[ zK_{n-1}(z) - zK_{n+1}(z) = -2nK_n(z) \]
together with $K_{-n} = K_n$, one can reexpress (21) in terms of $K_0$ and $K_1 = -\frac{d}{dz}K_0$, only as
\[ G(x, x') = \left[ -\alpha_0(x, x') \frac{d}{dz} + \beta_0(x, x') \right] K_0(z) = \alpha_0(x, x') K_1(z) + \beta_0(x, x') K_0(z) \]
with $z = \sqrt{\Delta(x, x')\tau_1(x, x')}$. To the chosen order we have
\[ \alpha_0(x, x') = -2(4\pi)^{-2} \left( \sqrt{\frac{4\tau_1}{\Delta}} + \tau_2 \sqrt{\frac{\Delta}{4\tau_1}} + \tau_3 \sqrt{\frac{\Delta}{\tau_1^2}} \right) \]
\[ \beta_0(x, x') = -\frac{\tau_3 \Delta}{2\tau_1} (4\pi)^{-2} \]

Note that the form (23) of the Green’s function is exact, it is only the explicit expressions (24-25) for $\alpha_0, \beta_0$ which are approximate. Thus the problem of finding the Green’s function of a scalar field in some geometry, is reduced to evaluating the two functions $f_0, \tilde{f}_0$ (and from these $\tau_n$), containing the information about the metric structure. The calculation of these is, as will also be apparent from the examples to follow, rather straightforward.

We will see that also the propagators for fields with non-zero spin can be written in this form, involving only the two modified Bessel functions $K_0, K_1$. The coefficients will contain all the spin-specific data, and will be denoted by $\alpha_s, \beta_s$ for spin $s$. Only for $s = 0$ are these scalar functions, for bosons of spin $s$ they will in general be $2s$-tensors, whereas for spinors they will take values in the Clifford algebra (for Rarita-Schwinger fields, $s = 3/2$, they would carry two Lorentz indices besides their two Clifford algebra (i.e. spinor) ones, and so on).

### 2.2 Heat Kernel and Propagators for Spin 1 Gauge Bosons

The heat kernel for spin 1 gauge bosons has been determined in [3] and because it is the result of a lengthy calculation we simply quote the result.
The heat kernel equation can, in the mean field approximation chosen in \[5\], be written as\[^5\]

\[
\left[ \delta_n^a \delta_m^n \delta_p^b \right] \partial_p \partial^p + \delta_n^a \delta_m^n \partial^p (\partial_n e^{\mu p} - \partial^m e^{\mu n}) e^p_b \partial_p + \langle g f_b^a (\partial_n A^{mc} - \partial^m A^c_n) \rangle \\
+ \left( \frac{1}{2} \delta_n^m g^2 f_{bce} f_d^a A^e p A^{pd} \right) G^q_m (x, x'; \sigma) = -\delta_n^a \partial^{q} \frac{G^q_n}{\partial \sigma}
\]

(26)

where \( G = G^q_m \) is a matrix valued function. Defining

\[
\mathcal{E}^m_{np} = (\partial_n e^{\mu p} - \partial^m e^{\mu n}) e^p_n
\]

(27)

one can remove the first order term in equation (26) by substituting (the quantities in the below equation are matrices)

\[
G = \tilde{G} e^{-\frac{1}{4} \mathcal{E}}
\]

(28)

where \( \partial^p E^m_{np} = \mathcal{E}^m_{np} \). The heat equation then becomes

\[
\delta_n^m \partial_p \partial^p \tilde{G}^q_m = -\left[ \frac{1}{2} \partial_p \mathcal{E}^m_{np} + \frac{1}{4} \mathcal{E}^m_{np} \mathcal{E}^k_{np} + \langle g f_b^a (\partial_n A^{mc} - \partial^m A^c_n) \rangle \right] \tilde{G}^q_m = -\partial \tilde{G}^q_m \]

(29)

an equation of the same functional form as for the non-minimally coupled scalar field. The solution is then (in matrix notation)

\[
G(x, x'; \sigma) = G_{\Box_0} (x, x'; \sigma) e^{-A\sigma + \frac{1}{2} B\sigma^2 - \frac{1}{4} C\sigma^3}
\]

(30)

with\[^6\]

\[
A^m_n = e^{-2\sqrt{\Delta}} \int_0^1 \tilde{A}_n^m (x(\tau)) e^{2\tau} d\tau + \tilde{A}_n^m e^{-2\sqrt{\Delta}}
\]

(31)

\[
B^m_n = -\frac{1}{2} e^{-4\sqrt{\Delta}} \int_0^1 \square_0 A^m_n e^{4\tau} d\tau + \tilde{B}_n^m e^{-4\sqrt{\Delta}}
\]

(32)

\[
C^m_n \approx -\frac{1}{3} e^{-6\sqrt{\Delta}} \int_0^1 (\partial_p A^m_k) (\partial^p A^k_n) e^{6\tau} d\tau + \tilde{C}_n^m e^{-6\sqrt{\Delta}}
\]

(33)

\[^5\] In \[5\] we have calculated the effective action using the heat kernel without using a mean field approximation, but the result from \[5\], which we use here, is conceptually simpler.

\[^6\] Even though the matrices \( A, B, C \) do not in general commute, there is no problem with this solution, as their commutator will inevitably be a higher order term. But the formula is not as useful as for the scalar case due to the complications of the gauge field. On the other hand, the couplings and mean field values need not be small, as the result presented here is non-perturbative.
where

\[ \tilde{A}_m = \partial_p \epsilon_{mp}^{\alpha} + \frac{3}{4} \epsilon_{k}^{\mu \nu} \epsilon_{n}^{\rho \sigma} + \langle g f_{b}^{a} (\partial_{\alpha} A_{m}^{c} - \partial_{\nu} A_{n}^{c}) \rangle + \langle \frac{1}{2} \epsilon_{n}^{m} g^{2} f_{e b c} f_{d}^{a} A_{p}^{c} A_{p d} \rangle \]  

\[ \tilde{B}_m = \Box_{0} A_{m} \]  

\[ \tilde{C}_m \approx (\partial_{p} A_{m}^{c}) (\partial_{p} A_{n}^{c}) \]  

corresponding to \( c_1 = \tilde{f}_0, c_2, c_3 \) respectively, and

\[ \Box_{0} \equiv \partial_{p} \partial_{n} = \eta^{p q} \partial_{p} \partial_{q} = g^{\mu \nu} \partial_{\mu} \partial_{\nu} + \eta^{mn} e_{n}^{\alpha} (\partial_{\mu} e_{n}^{\alpha}) \partial_{\nu} \]  

as before, the heat kernel of which is given by equation (7). Analogously to the scalar case, the quantity \( \tilde{A} \) is essentially a curvature term. The approximation involved in the evaluation of \( \mathcal{C}, \tilde{C} \) above is, as for the scalar case, the omission of a fourth order derivative in the curvature, namely \( \Box_{0}^{2} A \).

A few comments concerning the actual calculation of this quantity are in order. Now, if the matrices \( A, B, C \) were simply diagonal, one could apply the Bessel functions directly to the diagonal values, which would then become scalar boson propagators. Even when \( A \) is diagonal (which holds for abelian fields), it will not \textit{a priori} commute with \( B, C \) and these can therefore not in general be diagonalized, leading to some technical difficulties in the practical application of this formula. In practical calculations it will of course be tempting, and, for most purposes, good enough, to ignore the \( B \) and \( C \) terms of the heat kernel (which is strictly speaking only good enough for abelian or weak fields and couplings) or to resort to numerical calculation of these higher order terms. This is unsatisfactory (and, moreover, rather cumbersome), but we have not yet been able to come up with a better solution.

Referring back to equation (7) the propagator becomes (compare with the scalar case of equation (21))

\[ D_{m}(x, x') = - (4\pi)^{-2} \left( 2 \left( \frac{1}{4} \Delta \right)^{-1/2} (A^{1/2})_{k}^{m} \left( K_{-1} (\sqrt{\Delta A}) k n \right) \right) + \frac{1}{2} \mathcal{B}_{n}^{l} \left( \left( \frac{1}{4} \Delta (A^{-1/2}) l i \right) \left( K_{2} (\sqrt{\Delta A}) l n \right) \right) \]  

(38)
which we will write as (in matrix notation)

\[ D(x, x') \equiv \alpha_1(x, x') K_1(\sqrt{\Delta \tau_1}) + \beta_1(x, x') K_0(\sqrt{\Delta \tau_1}) \]  

(39)

To the chosen order

\[ \alpha_1 = 2(4\pi)^{-2} \left( \frac{1}{4} \Delta^{-1/2} A^{1/2} + \frac{1}{4} \Delta B A^{-1/2} + \frac{1}{4} \Delta C A^{-1} \right) \]  

(40)

\[ \beta_1 = \frac{1}{2} \Delta (4\pi)^{-2} C A^{-1} \]  

(41)

with the Lorentz indices suppressed.

In order to determine the Green’s function for a Yang-Mills field, all we have to do is to evaluate the matrix \( \tilde{A} \), and from that the matrices \( \tilde{B}, \tilde{C} \). Just like the functions \( f_0, \tilde{f}_0 \) for the scalar case, these matrices only depend upon the vierbein (and the mean field, which can be expressed in terms of the vierbein as in [3]), but unlike the spin zero case, their evaluation is somewhat more complicated (but still feasible).

Lastly we have to consider the case of fermions:

### 3 Heat Kernel and Propagators for Dirac Fermions

Consider the heat equation for a Dirac field in curved space-time

\[ (\nabla - m)G_{1/2}(x, x'; \sigma) = -\frac{\partial}{\partial \sigma}G_{1/2}(x, x'; \sigma) \]  

(42)

where

\[ \nabla \equiv \epsilon^\mu_a \gamma^\mu (\partial_\mu + \omega^\mu_d \sigma_{dc}) \]  

(43)

\(^\text{7}\)It would be very easy to include, say, a coupling to a Yang-Mills field \( igA_{\mu a}^m \gamma^m T_k \), e.g. using the mean-field derived in the previous section, or a Yukawa coupling to a scalar or pseudo scalar field. The latter two would just amount to the addition of a term to the mass, whereas the Yang-Mills coupling would appear in an extra contribution to the function \( F_a \) to be introduced below. The Yang-Mills field, however, would lead to a non-diagonal form of \( \nabla^2 \), as it would now contain a term like \( \sigma^{\mu \nu} F_{\mu \nu} T_k \), i.e. a spin-magnetic field coupling. The introduction of a background torsion would lead to a similar term as shown in [3].
This equation can be rewritten as
\[(i\gamma^a(\partial_a + F_a) + i\gamma_5\gamma^a\tilde{F}_a - m)G_{1/2} = -\frac{\partial}{\partial\sigma}G_{1/2}\] (44)
where
\[F_a = 8ie^\mu\omega_\mu^{bc}\delta^d_b\eta_{ac}\] (45)
\[\tilde{F}_a = -4ie^\mu\omega_\mu^{bc}\varepsilon^d_{bc a}\] (46)
Here we have used that the \(\sigma_{\mu\nu}\), the generator of \(SO(1,3)\)-transformations when acting upon Dirac spinors, is an element of the Clifford algebra, \(\sigma_{mn} = \frac{1}{4}i[\gamma_m, \gamma_n]\). The spin connection \(\omega_\mu^{ab}\) can be written in terms of derivatives of the vierbeins. The heat kernel for a Dirac spinor takes values in the Clifford algebra, and can thus be expanded on the sixteen dimensional basis \(1, \gamma_5, \gamma_a, \gamma_5\gamma_a, \sigma_{ab}\). Due to this complication we have not been able to find an expression for the heat kernel for a Dirac fermion. We can still find the propagator, however, by noting
\[S(x,x') = (\sqrt{\xi} + m)G'(x,x')\] (47)
is an inverse of the Dirac operator provided \(G'\) satisfies
\[(\sqrt{\xi} - m)(\sqrt{\xi} + m)G' = (-\nabla^2 - m^2)G' = 1\] (48)
Noting that
\[\nabla^2 = (\Box + \xi f R)1\] (49)
where \(1\) is the unit element in the Clifford algebra (i.e. in this particular case a \(4 \times 4\) unit matrix) we see that \(G'\) is the Green’s function for a non-minimally coupled scalar field. Had we considered a spin-3/2 fermion (a Rarita-Schwinger field) we would have made the same Ansatz but with \(G'\) replaced by \(D'\), a similar propagator for a spin-1 boson.
The propagator then becomes
\[S(x,x') = (\sqrt{\xi} + m)(\alpha_0K_1 + \beta_0K_0)\]
\[\equiv \alpha_{1/2}K_1 + \beta_{1/2}K_0\] (50)
where \(\alpha_{1/2}, \beta_{1/2}\) are Clifford algebra-valued. To the chosen approximation, these new coefficient functions are
\[\alpha_{1/2} = (\sqrt{\xi} + m)\alpha_0 - i(z^{-1}\alpha_0 + \beta_0)\nabla z\] (51)
\[\beta_{1/2} = (\sqrt{\xi} + m)\beta_0 - i\alpha_0\nabla z\] (52)
with
\[ z \equiv \sqrt{\Delta \tau_1} \]  
(53)
We have now written \( S(x, x') \), the propagator for a Dirac field, on the same form as that of \( G \), the scalar field propagator.

### 3.1 On the Corresponding Vacuum and the Hadamard Condition

Now, in order for a two point function to be a propagator, it must be possible for it to be written as the expectation value of the (path ordered) product of two field operators with respect to some vacua \( \langle 0_{\text{out}} |, | 0_{\text{in}} \rangle \). As is well-known, the concept of a vacuum is somewhat complicated in a general curved space-time background, and different definitions will in general lead to different results. Some particularly important vacua are the asymptotic ones (assuming the space-time manifold to be asymptotically flat), the adiabatic (assuming sufficiently small or slowly varying curvature) and the conformal one. Other vacua, relying on the analogy with thermal field theory, are the Unruh and Hartle-Hawking ones, the applicability of these is usually restricted to Rindler and Schwarzschild space-times, however. It turns out, by construction, that our definition of a Green’s function entails a natural vacuum. Recall the spectral representation of the Green’s function
\[ G(x, x') = \sum_{\lambda} \frac{1}{\lambda} \psi_\lambda^*(x') \psi_\lambda(x) \]
In a second quantized formalism, these eigenfunctions are of course to be replaced by operators, \( \psi_\lambda(x) = \hat{\psi}_\lambda(x) |0_x\rangle \). Remembering that we’re relying on comoving coordinates when determining \( G \), we see that our vacuum is that of a freely falling observer, and is thus well-defined on any manifold, so there is no need to impose global constraints on its topology. The propagator, however, will vanish whenever \( x \) and \( x' \) are not causally related. Due to this relationship with freely falling coordinates we will refer to our vacuum as the freely falling one, or, somewhat more amusingly, the “elevator vacuum.”

The question of when a candidate for a two point function is actually a physically valid propagator, has been studied intensively by Bernard S. Kay and coworkers, see [4] for a review. They impose what they call the Hadamard
condition, which, in the heuristic formulation of DeWitt and Brehme \cite{12} (see also Wald \cite{3}), states that for \( x' \approx x \)

\[
G(x, x') \sim \frac{\sqrt{\Delta_{VV}M(x, x')}}{8\pi^2} \left( \frac{4}{\Delta(x, x')} + v(x, x') \ln \frac{1}{2} \Delta(x, x') + w(x, x') \right)
\]

(54)

with \( \Delta_{VV}M \) the Van Vleck-Morette determinant

\[
\Delta_{VV}M(x, x') \equiv - \left( g(x) g(x') \right)^{-1/2} \operatorname{det} \left( -\frac{1}{2} \nabla_\mu \nabla_\nu \Delta(x, x') \right)
\]

and where \( v, w \) are smooth functions as \( x' \to x \). Kay and Wald, \cite{4}, show that this is a physically reasonable requirement, and give a much more rigorous formulation of it. We want to show that our candidate, \( G(x, x') \) (and hence also \( D_m^n(x, x') \) and \( S(x, x') \) for spin \( \neq 0 \)), meets this condition.

Now, writing \( G(x, x') \) in the compact form

\[
G(x, x') = -2(4\pi)^{-2} \left[ \alpha(x, x') K_1(z) + \beta(x, x') K_0(z) \right]
\]

with \( z = \sqrt{\Delta(x, x') \tau_1(x, x')} \) and with \( \alpha, \beta \) as given earlier in terms of \( \Delta, \tau_n \), and using

\[
K_0(z) = - \ln \frac{1}{2} z + \psi(1) + O(z)
\]
\[
K_1(z) = 2z^{-1} + O(1)
\]

we get

\[
G(x, x') = \tilde{\alpha}(x, x') \Delta^{-1} - \beta(x, x') \ln \frac{1}{2} \Delta + \gamma(x, x')
\]

(55)

where

\[
-2(4\pi)^{-2} \alpha(x, x') = \tilde{\alpha}(x, x') \Delta^{-1/2} + \text{finite terms}
\]

(56)

and \( \gamma(x, x') \) is well-behaved as \( x \to x' \) as is \( \tilde{\alpha}(x, x') \) and \( \beta(x, x') \). Our candidate thus has the right form, and hence meets the Hadamard condition. Notice, however, that we escape the logarithmic divergence as \( \beta = O(\Delta) \), i.e. while we do have a logarithmic term it does not lead to a divergence as \( x' \to x \). This is a most fortunate characteristic of our vacuum.

The way one usually finds a propagator, \cite{1}, is by using the positive frequency solutions to the equation of motion – the restrictions on the frequency being
there to ensure causal propagation. With the method put forward here, this requirement can be relaxed as we no longer need to know a complete set of solutions. Causality, moreover, is taken care of not by a process of time ordering, which might be difficult to define rigorously in a general background over large distances, but in terms of path ordering, or, if one wishes, in terms of the proper time of the comoving observer. We thus get a causal propagator by restricting attention to time-like (for massive fields) or null geodesics (for massless ones).

We will make a few comments on the comparison of our propagator with that of other authors, if such result exist (to the best of our knowledge), and it will turn out, that the results presented here disagree with all other calculations, due to the different notions of vacua. As our propagator does satisfy the Hadamard condition it is physically valid. Moreover, it is more general and apparently higher-order. It has for instance been argued that an adiabatic vacuum for spatially flat Friedman-Robertson-Walker geometries is only a Hadamard state if it is taken to infinite adiabatic order, [24]. Let us finally make a comment on the Schwinger-DeWitt expansion,

\[ G(x, x'; \sigma) \sim (4\pi\sigma)^{1/2}e^{-m^2\sigma - \frac{m}{\sigma}} \sum_{n=0}^{\infty} a_n(x, x')\sigma^n \]

Our expansion can be written in a similar form, by simply Taylor expanding the exponential. Denoting the coefficients by \( b_n \), we have

\[
\begin{align*}
\quad b_0 &= 1 \\
\quad b_1 &= \tau_1 \\
\quad b_2 &= \frac{1}{2}\tau_1^2 + \tau_2 \\
\quad b_3 &= \frac{1}{3!}\tau_1^3 + 2\tau_2\tau_1 + \tau_3
\end{align*}
\]

and so on, but \( b_n \neq a_n \) in general, a difference stemming from the different nature of the two expansions. At a first glance, our coefficients seem to depend only on the curvature scalar and not on the Ricci or Riemann-Christoffel tensors, but actually these tensors appear through the derivatives of the geodesic distance squared. It turns out, as we have emphasized earlier, that it is exactly this use of an exponential that makes a closed expression for the expansion coefficients possible. An expression, moreover, valid for the general case \( x \neq x' \) and reducing to a rather simple recursion relation for \( x = x' \).
This latter case is important for practical calculations of, say, the renormalized energy-momentum tensor, [1, 2], and with the usual Schwinger-DeWitt expansion or with the modification due to Parker et al., [3], only the first few can be found. On the contrary with the use of the exponential expansion suggested here, one can in principle (and even in practice) systematically find the coefficients to whatever order one wishes. We have used this possibility to calculate effective actions and energy-momentum tensors elsewhere [6], and further work is in progress on the applications of this expansion.

When studying coincidence limits, it is useful to evaluate $\Delta$ for $x, x'$ infinitesimally close. We thus consider two points $x, x'$ with $x'_\mu = x_\mu + \epsilon t_\mu$ (in practice only one component of $t_\mu$ will be non-zero). We can then approximate the geodesic by a straight line with a small correction. Letting, for a non-null geodesic, $\Sigma = t_\mu t_\mu$, as is common in the point splitting method, [1, 3], we get

$\tau_1 = \frac{1}{2}(e^2 + 1)\tilde{f}_0(x) + \epsilon \left[ (1 - e^2 - \sqrt{\Sigma})\tilde{f}_0(x) + \frac{1}{2}\epsilon t_\mu \partial_\mu \tilde{f}_0(x) \right] + O(\epsilon^2)$

$\tau_2 = \frac{1}{2}\Box_0 f_0(x)(1 - 4\epsilon \sqrt{\Sigma}) + \frac{1}{32}(e^4 - 1)\epsilon t_\mu \Box_0 \partial_\mu \Box_0 f_0(x) - \frac{1}{4}\epsilon t_\mu \partial_\mu \Box_0 f_0(x) + O(\epsilon^2)$

$\tau_3 = \frac{1}{144}(e^6 - 1)\epsilon t_\mu \Box_0 \partial_\mu \Box_0 f_0(x) - \frac{1}{6}\epsilon t_\mu \partial_\mu (\partial_\nu f_0(x))^2 - \frac{1}{3}(1 - 6\epsilon \sqrt{\Sigma})(\partial_\nu f_0(x))^2 + O(\epsilon^2)$

A quick application of the propagators found in this paper is the calculation of mean fields. As we have just proven $G(x, x')$ is divergent as $x' \to x$, but we can use $\zeta$-function regularization to obtain a finite value for $\langle \phi(x)^2 \rangle = G(x, x)$, the mean field. Going back to the expression of the propagator as an integral of the heat kernel, equation (5), then we see that the $x' \to x$ limits gives a $\Gamma(-1)$-singularity. Such divergencies are to be removed by taking principal values, [25]. We therefore get

$\langle \phi(x)^2 \rangle_{\text{reg}} = (4\pi)^{-2} \left[ - (\gamma - 1)\tau_1 + \tau_2 \tau_1^{-1} + \tau_3 \tau_1^{-2} + ... \right]$ (60)

as the final result.
4 Examples

To illustrate the technique developed above, we have chosen a few important examples. The first is Rindler space-time, possibly the simplest non-trivial example, and used for instance as an approximation to the Schwarzschild solution. The second and third examples are probably the most important ones from a purely theoretical point of view, as they are the classical solutions of Friedman-Robertson-Walker and Schwarzschild respectively. We finish off with a general conformally flat space-time (e.g. de Sitter space). For simplicity, we will just give the case of minimally coupled, massless scalar fields. Furthermore, we will just list the simplest expressions, i.e. $\tau_1, A$ and the coefficients $c_1, c_2, c_3, \tilde{A}, \tilde{B}, \tilde{C}$, the remaining terms (including all those for the fermions) follow by suitable differentiations, which, although very simple to carry out, leads to rather involved expressions.

4.1 Rindler Space-Time

The Rindler space-time is given by the line element

$$ds^2 = (gz)^2 dt^2 - dx^2 - dy^2 - dz^2$$

(61)

where $g$ is some constant, it is conformal to a wedge in Minkowski space-time and represents an accelerated observer in flat space. With this the vierbeins can be chosen to be

$$e^a_0 = \begin{pmatrix} gz \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^a_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^a_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^a_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(62)

A geodesic in this space-time takes the form

$$t(\tau) = -\frac{t' - t}{\sinh a^2} \sinh \frac{\tau}{a^2} + t$$

(63)

$$x(\tau) = (x' - x)\tau + x$$

(64)

$$y(\tau) = (y' - y)\tau + y$$

(65)

$$z(\tau) = a\sqrt{\ln \cosh \left( \frac{\tau}{a^2} \right)} + b$$

(66)
with \(a, b\) two constants given by

\[
a \sqrt{b} = z \quad a \sqrt{\ln \cosh a^{-2}} + b = z'
\] (67)

Thereby

\[
\sqrt{\Delta} = \int_0^1 \left[ g^2 a^2 (\ln \cosh \tau a^{-2} + b) \left( \frac{t' - t}{\sinh a^{-2}} \right)^2 \cosh^2 \tau a^{-2} - (x' - x)^2 - (y' - y)^2 - \frac{1}{4} \tanh \frac{\tau}{a^{-2}} (\ln \cosh \frac{\tau}{a^2} + b)^{-1/2} \right]^{1/2} d\tau
\] (68)

which we have not been able to reduce any further. In practice, though, since the integration is over a compact interval, it is fairly easy to carry out numerically.

### 4.1.1 Scalar Field Propagator

The quantities \(f_0, \tilde{f}_0\), needed to find the scalar field propagator, are seen to be (for a massless minimally coupled field – masses and non-minially couplings can be accomodated by adding \(\xi R + m^2\) to these functions)

\[
f_0 = -\frac{3}{4} z^{-2}
\] (69)

\[
\tilde{f}_0 = -\frac{7}{4} z^{-2}
\] (70)

From which we get the function \(\tau_1\) to be equal to (remember \(g_0 = c_1 = \tilde{f}_0\))

\[
\tau_1(x, x') = -\frac{7}{4} e^{-2 \sqrt{\Delta}} a^2 \int_0^1 \frac{e^{2\tau}}{\ln \cosh \frac{\tau}{a^2} + b} d\tau - \frac{7}{8} (z^{-2} + z'^{-2}) e^{-2 \sqrt{\Delta}}
\] (71)

As usual, the remaining coefficients \(\tau_2, \tau_3, \ldots\) are found by differentiation of this quantity in appropriate ways. The remaining \(c_n\)'s are found to be

\[
c_2 = -\frac{9}{4} z^{-4} 
\] \[
c_3 = -\frac{1}{4} z^{-6}
\] (72)

For a massless scalar field, an expression for the propagator is known already [13], namely

\[
G'(x, x') = \frac{\xi_4}{4 \pi^2 z z' \sinh \xi_4 (\xi_4^2 - g^2 (t - t')^2)}
\] (73)
with
\[ \xi_4 \equiv \text{Arccosh} \frac{z^2 - z'^2 + (x - x')^2 + (y - y')^2}{2zz'} \]

Defining coordinates
\[ x_0 = g^{-1}e^{g\xi} \sinh g\tau \quad x_3 = g^{-1}e^{g\xi} \cosh g\tau \]

the propagator can be rewritten as
\[ G'(x, x') = -\frac{1}{4\pi^2} \frac{g^2 e^{g\xi}}{\sinh \chi} \sum_{n=0}^{\infty} \frac{\chi}{g^2(\tau - \tau' + i\beta)^2 - \chi^2 - i\epsilon} \]

where \( \beta = \frac{2\pi}{g} \) is an inverse temperature (appearing due to the Davies-Fulling-Hawking-Unruh effect, [1, 2]) and
\[ \cosh \chi = \cosh g(\xi - \xi') + \frac{1}{2} g^2 a^{-g(\xi + \xi')}( (x_1 - x'_1)^2 + (x_2 - x'_2)^2) \]

The integer \( n \) is a kind of winding number. For \( x_2 = x'_2 = x_3 = x'_3 = 0 \) the geodesic distance becomes, [13]
\[ \Delta = 4g^{-2} \sinh^2 \left( \frac{1}{2} g(\tau - \tau') \right) \]

For simplicity we can set \( x = x' = 0 \), \( y = y' = 0 \) to get a function of \( t - t' \) and \( z, z' \) only (notice that neither \( G \), nor \( G' \) are functions of \( z - z' \) but depends on more complicated combinations of \( z, z' \)). The quantity \( \xi_4 \) is then simply
\[ \xi_4 = \text{Arccosh} \frac{z^2 - z'^2}{2zz'} \]

Taylor expanding around \( z = z', t = t' \) we get
\[ G'(x, x') = -\frac{1}{2\pi^3 z'^2} + \frac{2g^2(t - t')^2}{\pi^5 z'^2} + \]
\[ (z - z') \left( \frac{\pi - 2}{2\pi^4 z'^3} + \frac{2g^2(t - t')^2(6 - \pi)}{\pi^6 z'^3} \right) + \]
\[ (z - z')^2 \left( \frac{6\pi - 3\pi^2 - 8}{4\pi^5 z'^4} + \frac{3g^2(t - t')^2(16 - 6\pi + \pi^2)}{\pi^7 z'^4} \right) + O((z - z')^3, (t - t')^3) \]

Since there is no singularity, this function does not satisfy the Hadamard condition, and can thus at most be considered as an approximation to a physical quantity.
4.1.2 Gauge Boson Propagator

In order to write down the propagator of a Yang-Mills field, we need to evaluate the matrices $A, B, C$. As we do not have a tractable expression for the mean fields we cannot, however, write down these explicitly; only their curvature (or abelian) contribution can be found explicitly. For the first of these matrices we then get

$$\tilde{A}_m^n = \frac{3}{4} \begin{pmatrix} z^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-2} \end{pmatrix} + \text{mean field terms} \quad (74)$$

and similarly for the remaining matrices

$$\tilde{B}_m^n = -\frac{9}{2} \begin{pmatrix} z^{-4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-4} \end{pmatrix} + \text{mean field terms} \quad (75)$$

$$\tilde{C}_m^n = \frac{9}{4} \begin{pmatrix} z^{-6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-6} \end{pmatrix} + \text{mean field terms} \quad (76)$$

Thereby the coefficient $A$ becomes

$$\tilde{A}_m^n = \frac{3}{7} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tau_1^{(s=0)} + \text{mean field terms} \quad (77)$$

and quite similarly for $B, C$

$$\tilde{B}_m^n = \frac{3}{7} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tau_2^{(s=0)} + \text{mean field terms} \quad (78)$$

$$\tilde{C}_m^n = \frac{9}{49} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tau_3^{(s=0)} + \text{mean field terms} \quad (79)$$
Since these matrices are all diagonal (the only possibly non-diagonal term comes from \( (g f^a_b c (\partial_n A^{mc} - \partial^m A^c_n)) \), i.e. one of the two terms we cannot handle analytically), we can actually apply the Bessel functions to them without any great difficulty to simply obtain the Bessel function of the (scalar) element multiplied by the constant \( 4 \times 4 \) matrix appearing in \( A, B, C \). Since the coefficient of proportionality between the vector and scalar case is not the same for all three terms (it is \( 3/7 \) for the first two, and the square of this for the last) we do not get just the scalar propagator multiplied by a constant \( 4 \times 4 \) matrix, instead we get the coefficients \( \alpha_1, \beta_1 \) to be

\[
\alpha_1 = 2(4\pi)^{-2} \left( \sqrt{\frac{3}{7}} \left( \frac{\Delta}{4\tau_1} \right)^{-1/2} + \sqrt{\frac{3}{7}} \left( \frac{\Delta}{4\tau_1} \right)^{1/2} \tau_2 - \frac{3 \Delta}{74\tau_1\tau_3} \right) \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \text{mean field terms} \tag{80}
\]

\[
\beta_1 = (4\pi)^{-2} \frac{3 \Delta}{72\tau_1\tau_3} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \text{mean field terms} \tag{81}
\]

with \( \tau_n \) given by the \( s = 0 \) case. For an abelian field such as the Maxwell field, the mean field terms become irrelevant, and the expressions above becomes exact.

### 4.2 Heat Kernel and Propagators in the Friedman-Robertson-Walker Metric

An important case is the cosmologically interesting Friedman-Robertson-Walker space-time

\[
ds^2 = dt^2 - a^2(t)(d\chi^2 - f^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)) \tag{82}
\]

in which the vierbeins can be taken to read

\[
e_0^a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_1^a = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad e_2^a = \begin{pmatrix} 0 \\ a f \\ 0 \end{pmatrix}, \quad e_3^a = \begin{pmatrix} 0 \\ 0 \\ a f \sin \theta \end{pmatrix} \tag{83}
\]
4.2.1 Scalar Field Propagator

With the vierbeins given by equation (83) we get

\[ f_0 = \frac{1}{4}(\frac{1}{e} e^\mu (\partial_m (e e^\mu)))^2 - \frac{1}{2} \partial^n (\frac{1}{e} e^m \partial_m (e e^n)) \]

\[ = \frac{1}{4} a^{-2}(3\dot{a} - a^2 f^{-1} f' - a^2 \cot \theta)^2 - \frac{1}{2} (3 \sin \theta \partial_t (a^{-1} \dot{a}) - \dot{a} (f' \sin \theta + f \cos \theta)) \]

\[ + \frac{1}{2} a^{-2}(-3\dot{a} f^{-2} f' + a^2 \partial_\chi (f^{-2} f')) + \frac{1}{2} \sin \theta \]

(84)

\[ \tilde{f}_0 = \frac{1}{4} a^{-2}(3\dot{a} - a^2 f^{-1} f' - a^2 \cot \theta)^2 + \frac{1}{2} (3 \sin \theta \partial_t (a^{-1} \dot{a}) - \dot{a} (f' \sin \theta + f \cos \theta)) \]

\[ - \frac{1}{2} a^{-2}(-3\dot{a} f^{-2} f' + a^2 \partial_\chi (f^{-2} f')) - \frac{1}{2} \sin \theta \]

(85)

We note that \( f_0 \) and \( \tilde{f}_0 \) only differ in some relative signs. For the geodesic distance \( \Delta \) we have not been able to find an analytical expression; in general one would then have to find this numerically.

The coefficient \( \tau_1 \) becomes

\[ \tau_1 = e^{-2\sqrt{\Sigma}} \int_0^1 e^{2\tau} \left[ \frac{1}{4} a^{-2} \left( 3\dot{a} - a^2 f^{-1} f' - a^2 \cot \theta \right)^2 + \frac{1}{2} (3 \sin \theta \partial_t \ln a - \dot{a} (f' \sin \theta + f \cos \theta)) \right. \]

\[ - \frac{1}{2} a^{-2}(-3\dot{a} f^{-2} f' + a^2 \partial_\chi (f^{-2} f')) - \frac{1}{2} \sin \theta \left. \right] d\tau + \frac{1}{2} (\tilde{f}_0(x) + \tilde{f}_0(x')) e^{-2\sqrt{\Sigma}} \]

(86)

For massless conformally coupled scalar fields some results are known \[21, 1, 13\] in the static case; for \( K = -1 \) one has

\[ G'_{\text{stat},-}(x, x') = \frac{c \sinh(\chi - \chi')}{4\pi^2 c} \frac{\chi - \chi'}{((\chi - \chi')^2 + (\eta - \eta' - i\epsilon)^2)} \]

(87)

while for \( K = +1 \) (the Einstein universe) one has

\[ G'_{\text{stat},+}(x, x') = (8\pi^2 c(\cos(\eta - \eta' - i\epsilon) - \cos(\chi - \chi'))^{-1} \]

(88)

with \( \eta \) the conformal time, and \( c \) given by \( R = 6K/c \). In order to compare these results with ours, we must first notice that in the static case with \( \theta = \theta', \phi = \phi' \) the geodesics are trivial

\[ t(\tau) = a\tau + b \quad \chi(\tau) = \alpha\tau + \beta \]

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with these simplifications, the coefficient $\tau_1$ becomes

$$\tau_1 = e^{-2\sqrt{\Delta}} \int_0^1 e^{2\tau} \left[ \frac{1}{4} a_0^2 (f^{-1} f' + \cot \theta)^2 - \frac{1}{2} \partial_\chi (f^{-2} f') - \frac{1}{2} \sin \theta \right] d\tau + \tilde{f}_0 e^{-2\sqrt{\Delta}}$$

Now, since the geodesics are trivial, we can actually find the geodesic distance

$$\Delta = (t - t')^2 - a_0^2 (\chi - \chi')^2$$

where $a_0$ is some constant (the value of the scale factor).

Taylor expanding the functions $G_{\text{stat}, \pm}'$ around $\eta = \eta', \chi = \chi'$ taking $\theta = \theta', \phi = \phi'$ by symmetry, one finds

$$G_{\text{stat}, -}' = \frac{1}{4c\pi^2 (\eta - \eta')^2} - (\chi - \chi') \left( \frac{1}{4c\pi^2 (\eta - \eta')^4} + \frac{1}{24c\pi^2 (\eta - \eta')^2} \right) + O((\eta - \eta')^3, (\chi - \chi')^3)$$

$$G_{\text{stat}, +}' = \frac{-1}{4c\pi^2 (\eta - \eta')^2} - \frac{1}{48c\pi^2} \left( \frac{1}{160c^2 (\eta - \eta')^2} - \frac{1}{24c\pi^2 (\eta - \eta')^4} + \frac{1}{24c\pi^2 (\eta - \eta')^2} + \frac{11}{2880c\pi^2} + \frac{31}{120960c\pi^2} \right) +$$

$$O((\eta - \eta')^3, (\chi - \chi')^3)$$

We notice that the singularity goes like $(\eta - \eta')^{-n}$, and is thus independent of $\chi, \chi'$, we seem only to get a singularity when both $\eta - \eta'$ and $\chi - \chi'$ tend to zero. On the other hand, Bunch and Davies, [15], have proven that the propagators obtained by a mode sum has the following structure in the coincidence limit $x' \rightarrow x$ even in the non-static limit.

$$G' = \frac{-1}{16c^2 \pi^2 \Sigma} (1 + O(\epsilon^2))$$

and thus it seems to satisfy the Hadamard condition. It has been shown, however, by Pirk, [24], at least for spatially flat Friedman-Robertson-Walker space-times that adiabatic vacua are Hadamard states if and only if they are of infinite adiabatic order. Such subtleties are absent in our approach.

A very general form for the propagator of a massive, conformally coupled scalar field in the flat Friedmann-Robertson-Walker universe has been found by Charach and Parker, [16], and it contains the older results of Narimi and
Azuma, [17], or Chitre and Hartle, [18], as special cases. This general formula is

\[ G(x, x') = \frac{i}{(2\pi)^2} \int e^{ik \cdot (x-x')} H^{(2)}_{ik} (mt_>) \left[ \frac{B_k^*}{C_k^*} e^{2\pi k} H^{(2)}_{ik} (mt_>) + H^{(1)}_{ik} (mt_<) \right] d^3k \]

in terms of Hänkel functions. The coefficients \( B_k, C_k \) are restricted by \( |C_k|^2 e^{-k\pi} - |B_k|^2 e^{k\pi} = \frac{\pi}{4} \), for various choices of these one then gets the aforementioned results by Narimi and Azuma and by Banerjee and Hartle.

In the above formula \( t_> \) is the largest of \( t, t' \), while \( t_< \) is the smallest of the two. That this actually satisfies the Hadamard condition in this formulation is far from clear. Furthermore, the limit of \( m \rightarrow 0 \) seems to be ill-defined as the Hänkel functions go like \( z^{-\nu} \) where \( z \) is the argument and \( \nu \) the order,

contrary to this, our formula is valid for as well massive as massless fields.

Moreover, since the expression for \( G(x, x') \) proposed in this paper does not involve an integral over the order of Bessel functions but only integrals of geometric quantities along geodesics, it might be more convenient from a practical viewpoint.

For finite temperature a result has been found by Banerjee and Mallik, [19].

### 4.2.2 The Gauge Boson Propagator

In order to find the propagator of a spin 1 field we must first find the matrix \( \tilde{\mathbf{A}} \) as defined above. Inserting the vierbeins (83) one can find

\[
\tilde{\mathbf{A}}_n^m = \begin{pmatrix}
3a^{-2} \dot{a}^2 & 2a^{-1}f^{-1}\dot{a}f' & a^{-1}\dot{a}\cot\theta & 0 \\
2a^{-1}f^{-1}\dot{a}f' & a^{-2}\dot{a}^2 + 2f^{-2}f'^2 & f^{-1}f'\cot\theta & 0 \\
a^{-1}\dot{a}\cot\theta & f^{-1}f'\cot\theta & a^{-2}\dot{a}^2 f^{-2}f'^2 + \cot^2\theta & 0 \\
0 & 0 & 0 & a^{-2}\dot{a}^2 f^{-2}f'^2 + \cot^2\theta \\
\end{pmatrix} + \text{mean field dependent terms}
\]

The matrix \( \tilde{\mathbf{B}} \) is then \( \Box_0 = \eta^{mn} \partial_m \partial_n \) applied to this. As we do not have a very tractable expression for the mean fields we will only list its curvature contribution together with that of \( \tilde{\mathbf{C}} \) in table I.
The coefficient $A$ becomes similarly

$$A_m = e^{-2\sqrt{\Delta}} \int_0^1 \left( \begin{array}{cccc} \frac{3\dot{\theta}^2}{\dot{a}^2} & \frac{2\dot{f}f'}{a^2} & \frac{\ddot{a}}{a} \cot \theta & 0 \\ \frac{2\dot{f}f'}{a^2} & \frac{\dot{f}^2}{a^2} + \frac{2f'^2}{f^2} & \frac{f'}{f} \cot \theta & 0 \\ \frac{\ddot{a}}{a} \cot \theta & \frac{f'}{f} \cot \theta & \frac{\dot{a}^2f'^2}{a^2f^2} + \cot^2 \theta & 0 \\ 0 & 0 & 0 & \frac{\dot{a}^2f'^2}{a^2f^2} + \cot^2 \theta \end{array} \right) e^{2\tau} d\tau$$

$$+ \frac{1}{2}(\tilde{A}_m^n(x) + \tilde{A}_m^n(x')) e^{-2\sqrt{\Delta}} + \text{mean field terms} \quad (90)$$

Again, for a static Friedman-Robertson-Walker space-time this simplifies immensely to give

$$A_m^m = e^{-2\sqrt{\Delta}} \int_0^1 \left( \begin{array}{cccc} 3 & 0 & 0 & 0 \\ 0 & 2f^{-2}f'^2 & f^{-1}f' \cot \theta & 0 \\ 0 & f^{-1}f' \cot \theta & \cot^2 \theta & 0 \\ 0 & 0 & 0 & \cot^2 \theta \end{array} \right) e^{2\tau} d\tau +$$

$$\frac{1}{2}(\tilde{A}_m^n(x) + \tilde{A}_m^n(x')) e^{-2\sqrt{\Delta}}$$

with (as for the scalar case)

$$\Delta = (t - t')^2 - a^2(\chi - \chi')^2$$

Thus the Green’s function for spin one contains essentially the same integrals as for spin zero, albeit in a somewhat simpler configuration.

4.3 Heat Kernel and Propagators in Schwarzschild Space

The Schwarzschild metric is given by

$$ds^2 = h(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2d\Omega \quad (91)$$

with

$$h(r) = 1 - \frac{2M}{r} \quad (92)$$
where $M$ denotes the mass of the object generating the gravitational field. Thus the vierbeins can be taken as

\[
e_0^a = \begin{pmatrix} \sqrt{h} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_1^a = \begin{pmatrix} 0 \\ h^{-1/2} \\ 0 \\ 0 \end{pmatrix}, \quad e_2^a = \begin{pmatrix} 0 \\ 0 \\ r \\ 0 \end{pmatrix}, \quad e_3^a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ r \sin \theta \end{pmatrix}
\]

(93)

According to Chandrasekhar, [14], the time-like geodesics are given by

\[
\frac{dt}{d\tau} = \frac{E}{h(r)} \\
\left( \frac{dr}{d\tau} \right)^2 = E^2 - h(r)(1 + \frac{L^2}{r}) \\
\frac{d\theta}{d\tau} = 0 \\
\frac{d\phi}{d\tau} = \frac{L}{r^2}
\]

where $E$ is a constant (the energy) and where $L$ is the angular momentum. For the radial case $L = 0$, one finds

\[
t = E\sqrt{r_0^3 2M} \left[ \frac{1}{2}(\tau + \sin \tau) + (1 - E^2)\tau \right] + 2M \log \frac{\tan \frac{1}{2}\tau_H + \tan \frac{1}{2}\tau}{\tan \frac{1}{2}\tau_H - \tan \frac{1}{2}\tau}
\]

\[
r = r_0 \cos^2 \frac{1}{2}\tau
\]

where

\[
r_0 \equiv \frac{2M}{1 - E^2}
\]

is the original position (the geodesics describe matter falling into the black hole, or whatever is creating the gravitational field) and where

\[
\tau_H = 2\arcsin E
\]

Null geodesics, describing massless particles, are somewhat easier. The radial solution is

\[
t = \pm r_+(\tau) + c_\pm \\
r = \pm E\tau + c_\pm
\]

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with

\[ r_* = r + 2M \log \left( \frac{r}{2M} - 1 \right) \]

and where \( c_\pm \) are constants.

### 4.3.1 Scalar Field Propagator

For the scalar field we only need to evaluate the two quantities \( f_0, \tilde{f}_0 \). With the vierbeins as given above, this is an easy task and we get

\[
f_0 = r^{-3}(r - 2M) + \frac{1}{4} M^2 r^{-3}(r - 2M)^{-1} + Mr^{-3} + \frac{3}{2} r^{-2} - M r^{-4}(r - 2M)^{-1} + \frac{1}{2} M^2 r^{-4}(r - 2M)^{-2} + M(r - 2M)^{-1} \tag{94}
\]

\[
\tilde{f}_0 = r^{-3}(r - 2M) + \frac{1}{4} M^2 r^{-3}(r - 2M)^{-1} + Mr^{-3} - \frac{3}{2} r^{-2} + Mr^{-4}(r - 2M)^{-1} - \frac{1}{2} M^2 r^{-4}(r - 2M)^{-2} - M(r - 2M)^{-1} \tag{95}
\]

which can then be inserted into the definition of \( \tau_1 \)

\[
\tau_1 = e^{-2\sqrt{\Delta}} \int_0^1 \tilde{f}_0 e^{2r} d\tau + \frac{1}{2} (\tilde{f}_0(r) + \tilde{f}_0(r')) e^{-2\sqrt{\Delta}}
\]

for a radial null-geodesic, the integral over \( \tau \) can actually be carried out and we get

\[
\int_0^1 \tilde{f}_0(r(\tau)) e^{2r} d\tau = \frac{1}{2} c(E-c-2M) - \frac{1}{4} E (E-2M) - 2M^2 + \frac{1}{2} e^2 (c + E + 2M) + \frac{1}{4} c^2 E (E + 2M) + 2M^2 e^2 + 8E^{-1} M^3 e^{-2(c-2M)/E} \times (Ei(2(c + E - 2M)/E) - Ei(2(c - 2M)/E)) + M (\frac{3}{2} c^{-1} E^{-2} - \frac{1}{2} e^2 E^{-2} (2c + 3E)(c + E)^{-2} + 2E^{-3} e^{-2c/E} \times (Ei(2c/E) - Ei(2 + 2c/E)) -
\]

29
\[
\frac{3}{2} \left[ (cE)^{-1} - e^2 E^{-1} (c + E)^{-1} + 2e^{-2c/E} E^{-2} (\text{Ei}(2 + 2c/E) - \text{Ei}(2c/E)) \right] + \\
M \left[ - \frac{1}{24c^3 E^3 M^3} \left( 3c^2 E (E + 2M) + 3cE^2 M + 4M^2 (E^2 + 2c^2 + cE) \right) \\
e^2 (24E^3 M^3 (c + E)^3)^{-1} \left( 3c^2 E (E + 2E^2 + 2M) + 3E^4 + E^3 M (9 + 15c) + \\
4M^2 (4E^2 + 2c^2 + 5cE) \right) + (12M^3 E^4)^{-1} e^{-2c/E} (3E^2 + 6EM + 8M^2) \times \\
(\text{Ei}(2c/E) - \text{Ei}(2 + 2c/E)) + \\
(16EM^4)^{-1} e^{-2c/E} (\text{Ei}(2c/E) - \text{Ei}(2 + 2c/E)) + \\
\text{Ei}(2(c + E - 2M)/E) e^{-4M/E} - \text{Ei}(2(c - 2M)/E) e^{-4M/E} \right] - \\
\frac{1}{2} M^2 \times \left[ (12EM^2 c^3)^{-1} - (16E(2M - c)M^4)^{-1} + \\
e^2 (16E(2M - E - c)M^4)^{-1} - e^2 (12EM^2 (c + E)^3)^{-1} + \\
\frac{1}{8} (EM^3)^{-1} e^{-2c/E} (\text{Ei}(2 + 2c/E) - \text{Ei}(2c/E)) + \\
(48c^2 E^3 M^4)^{-1} \left( c(9E^2 + 12EM + 8M^2) + 6E^2 M + 4EM^2 \right) + \\
e^2 (48EM^3 (c + E)^2 M^4)^{-1} \left( 9E^2 (c + E + 2M) + 12cEM + 4M^2 (3E + 2c) \right) + \\
(48E^4 M^4)^{-1} e^{-2c/E} (\text{Ei}(2c/E) - \text{Ei}(2 + 2c/E)) + \\
(8E^2 M^5)^{-1} e^{-2(c - 2M)/E} (\text{Ei}(2(c + E - 2M)/E) - \text{Ei}(2(c - 2M)/E)) \right] - \\
ME^{-1} e^{-2(c - 2M)/E} (\text{Ei}(2(c + E - 2M)/E) - \text{Ei}(2(c - 2M)/E))]
\]

which looks very complicated but is actually just a combination of polynomials, exponentials and exponential integral functions.

Unfortunately, the geodesic distance \( \Delta \) is rather difficult to find analytically in Schwarzschild space-time, only radial geodesics are easily obtained. In many applications, though, these actually suffice. Along these geodesics the differential operators appearing in the expressions for \( \tau_2, \tau_3 \) are

\[
\partial_\rho = e^\mu_\rho \dot{x}^\mu \frac{\partial}{\partial \tau} = (h^{-1/2} \dot{t} - h^{1/2} \dot{r}) \frac{\partial}{\partial \tau} \quad (96)
\]

\[
\Box_0 = \eta^{pq} \partial_\rho \partial_q = (h^{-1} \ddot{t} - h \dot{r} \ddot{r}) \frac{\partial}{\partial \tau} + (h^{-1} \ddot{r}^2 - \dot{r}^2) \frac{\partial^2}{\partial \tau^2} \quad (97)
\]

Fairly simple expressions. But we will not write down the explicit form for \( \tau_2, \tau_3 \) which follows from the application of these operators as the results are too involved.
Candelas and Jensen, [20], have also found a propagator for the Schwarzschild background. They use the Hartle-Hawking vacuum (i.e. the Green’s functions is demanded to vanish as the spatial distance tends to infinity and to be periodic in imaginary time), with this they arrive at

\[
G(-i\tau, r, \theta, \phi, -i\tau', r', \theta', \phi') = \frac{i}{32\pi^2M^2} \left[ \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \gamma) P_l(\xi_<) Q_l(\xi_>) + \sum_{n=1}^{\infty} \frac{1}{n} \cos(n\kappa(\tau - \tau')) \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \gamma) p_n^l(\xi_<) q_n^l(\xi_> \right] \]

where \(P_l, Q_l\) are Legendre polynomials, \(p_n^l, q_n^l\) are solutions to

\[
\left[ \frac{d}{d\xi}(\xi^2 - 1) \frac{d}{d\xi} - l(l + 1) - \frac{n^2(\xi + 1)^4}{16(\xi^2 - 1)} \right] f = 0
\]

where \(\xi = \frac{r}{M} - 1\) and

\[
\xi_<= \min(\xi, \xi')
\]
\[
\xi_> = \max(\xi, \xi')
\]
\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')
\]

They prove that

\[
\lim_{x' \to x} G(x, x') = \frac{i}{8\pi^2\Delta(x, x')} + \frac{i}{12(8\pi M)^2} \frac{1 - \left(\frac{2M}{r}\right)^4}{1 - \frac{2M}{r}} + \frac{iF(r)}{(8\pi M)^2}
\]

with \(F(r)\) small. Like our results, then, Candelas and Jensen finds a Green’s function with only the \(\Delta^{-1}\)-singularity, whether they also get a \(\Delta \log \Delta\) term is, however, not so clear, since not much is known about \(q_n^l, p_n^l\) and the sums involving these and/or Legendre polynomials is difficult to actually carry out. If one calculate the finite part of our propagator as \(x' \to x\), one finds

\[
-\frac{M}{32\pi^2} \left( 48M^3 - M^2r(48 + 52M^2) + Mr^2(19 + 159M^2) - 158M^2r^3 + 66Mr^4 - r^5(10 - 4M^2) - 2Mr^6 \right) (8r^7(r - 2M)^3)^{-1} + \frac{1}{32\pi^2} \left( 48M^3 - M^2r(48 + 52M^2) + Mr^2(19 + 159M^2) - 158M^2r^3 +
\]

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\[66Mr^4 - r^5(10 - 4M^2) - 2Mr^6 \right)^2 (24r^{10}(r - 2M)^6)^{-1} -
\]
\[
\frac{1}{32\pi^2} \left(240M^4 - M^3r(384 + 216M^2) + M^2r^2(198 + 666M^2) - Mr^3(30 + 793M^2) +
461M^2r^4 - 132Mr^5 + r^6(15 - 4M^2) + 2Mr^7\right)(8r^7(r - 2M)^3)^{-1}
\]
i.e. one finds more terms which diverge as one approaches the Schwarzschild radius. A discrepancy likely to be due to (1) the different vacua (we do not have an explicit temperature) and (2) the different orders of the respective approximations.

4.3.2 Gauge Boson Propagator

The curvature contribution to the matrix \(\tilde{A}\) is in this case, with the vierbeins as given above, easy to find and we get

\[
\tilde{A}^n_m = \frac{1}{2} \begin{pmatrix}
    h^2(\partial_r h^{-1/2})^2 & 0 & 0 & 0 \\
    0 & h^2(\partial_r h^{-1/2})^2 + 2\frac{h^{1/2}h r^2}{2r^2} & \frac{h^{1/2}h r^2}{2r^2} & 0 \\
    0 & \frac{h^{1/2}h r^2}{2r^2} & h^{1/2}h r^2 & 0 \\
    0 & 0 & 0 & \frac{h^{1/2}h r^2}{2r^2}
\end{pmatrix}
\]

+ mean field terms

(98)

whereby the matrices \(\tilde{B}, \tilde{C}\) become as follows

\[
\tilde{B}^n_m = -\frac{1}{4} \begin{pmatrix}
    2M^2(27M^2 - 32Mr + 10r^2) & 0 & 0 & 0 \\
    0 & 2(243M^4 - 400M^3r + 240M^2r^2 - 62Mr^3 + 6r^4) & 0 & 0 \\
    0 & 0 & \frac{(2M - r)(10M - 4r - 10M \cos 2\theta + 3r \cos 2\theta) \cot \theta \csc^2 \theta}{\sqrt{1 - \frac{2M}{r}r^6}} & 0 \\
    0 & 0 & 0 & \frac{54M^2 - 38Mr + 6r^2}{r^6} \cot \theta \csc^2 \theta + \frac{(14M - 6r) \cot \theta}{r^4} + \frac{2r^2 - 2M^2 - 2Mr}{r^5} - \frac{(14M - 6r) \cot \theta}{r^5} + \frac{\cot \theta \csc^2 \theta}{r^4}
\end{pmatrix}
\]

(99)

+ mean field terms

where we, for reasons of space, have split-up the matrix \(\tilde{B}\) and

\[
\tilde{C}^n_m \approx \frac{1}{12} \times
\]

32
\[
\begin{pmatrix}
\frac{4M^4(2r-3M)^2}{r^7(r-2M)^3} & 0 & 0 & 0 \\
0 & \frac{4(27M^3-34Mr+14M^2-2r)^2}{(r-2M)^5} & \frac{(5M-2r)^2\cot^2\theta}{r^7} + \frac{(r-2M)\csc^4\theta}{r^7} & 0 \\
0 & \frac{(5M-2r)^2\cot^2\theta}{r^6(r-2M)^2} + \frac{(r-2M)\csc^4\theta}{r^7} & \frac{6M-2r}{r^6} + \frac{4\cot^2\theta}{r^7} + \frac{\csc^4\theta}{r^7} & 0 \\
0 & 0 & 0 & \frac{6M-2r}{r^6} - \frac{4\cot^2\theta}{r^7} + \frac{\csc^4\theta}{r^7}
\end{pmatrix}

+ \text{mean field terms}
\]

And this can then be inserted into the formula (39) for the propagator \(D_m^n(x, x')\).

### 4.4 Conformally Flat Space-Times

As our last example we will consider a conformally flat space-time with line element

\[
d s^2 = C(\eta)^2(d\eta^2 - \delta_{ij}dx^idx^j)
\]

(101)

This could for instance be de Sitter space or a general spatially flat Friedman-Robertson-Walker metric in arbitrary dimensions. For Friedman-Robertson-Walker space-time the conformal time \(\eta\) is defined to be \(\int a^{-1}dt\) and \(C(\eta)\) is just the usual scale factor \(a(t)\) expressed in this new coordinate, whereas for de Sitter space-time \(\eta = -\alpha \exp(-t/\alpha)\) where \(\alpha = \sqrt{12/R}\) with \(R\) the curvature scalar (i.e. a constant in this case). The calculations will be carried out in \(d = 4\) dimensions, but can of course easily be generalized to any \(d\).

For the metric as given above, we can choose the vierbeins to be

\[
e^0_a = \begin{pmatrix} C(\eta) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^1_a = \begin{pmatrix} 0 \\ C(\eta) \\ 0 \\ 0 \end{pmatrix}, \quad e^2_a = \begin{pmatrix} 0 \\ 0 \\ C(\eta) \\ 0 \end{pmatrix}, \quad e^3_a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ C(\eta) \end{pmatrix}
\]

(102)

#### 4.4.1 Scalar Boson Case

The d’Alembertian can be written as

\[
\Box = \Box_0 + \mathcal{E}_a \partial^a = \Box_0 + 3 \frac{\dot{C}}{C^3} \partial_\eta
\]

(103)
where $\dot{C} = \partial_\eta C$. We then have
\begin{align}
 f_0 &= \frac{21}{4} \dot{C}^2 C^{-4} - \frac{3}{2} \ddot{C} C^{-3} \\
 \tilde{f}_0 &= \frac{33}{4} \dot{C}^2 C^{-4} + \frac{3}{2} \ddot{C} C^{-3}
\end{align}

whereby we get
\begin{align}
 \tau_1 &= e^{-2\sqrt{\Delta}} \int_0^1 \left( \frac{33}{4} \dot{C}^2 C^{-4} + \frac{3}{2} \ddot{C} C^{-3} \right) e^{2\tau} d\tau \\
 &\quad + \frac{1}{4} \left( \frac{33}{2} \dot{C}(\eta)^2 C(\eta)^{-4} + \frac{33}{2} \dot{C}(\eta')^2 C(\eta')^{-4} + 3\ddot{C}(\eta) C(\eta)^{-3} + 3\ddot{C}(\eta') C(\eta')^{-3} \right) e^{-2\sqrt{\Delta}} 
\end{align}

and the coefficients $c_2, c_3$ are found to be
\begin{align}
 c_2 &= -\frac{3}{4} \left( C^{-5} C^{(4)} + C^{-6} (C + 1) \dot{C} \dot{C} \right) - 3C^{-6} \dddot{C} - \frac{1}{4} C^{-7} (51C + 174) \dot{C}^2 \dddot{C} \\
 &\quad - 21C^{-7} \dot{C}^4 - \frac{105}{2} C^{-8} \dot{C}^4 \\
 c_3 &= \frac{147}{4} C^{-10} \dddot{C}^2 \dddot{C}^2 - 147C^{-11} \dot{C}^4 \dddot{C} + 147C^{-12} \dot{C}^{-6}
\end{align}

It turns out that in the particular case of de Sitter space-time, another formula for the propagator of a scalar field is known in the literature [1, 22], namely
\begin{align}
 G'(x, x') = \frac{1}{16\pi\alpha^2} \left( \frac{1}{4} - \nu^2 \right) \sec \pi \nu F\left( \frac{3}{2} + \nu, \frac{3}{2} - \nu; 2; 1 + \frac{(\eta - \eta' - i\epsilon)^2 - (x - x')^2}{4\eta\eta'} \right)
\end{align}

with
\begin{align}
 \nu^2 \equiv \frac{9}{4} - 12(m^2 R^{-1} + \xi)
\end{align}

Clearly this cannot be the final answer as it diverges for a massless minimally coupled field (in this case $\nu = \frac{3}{2}$). Let us nonetheless compare it with our result obtained by the method described in this paper. The first thing we
\footnote{For thermal fields, a propagator was also found by Chardury and Mallik, [23], but it will not be discussed here.}
notice is that, even on the level of series expansion as in equation (22), our expression is manifestly divergent as \( x \to x' \) (due to the \( \Delta^{-1}, \log \Delta \) terms), whereas the expression ([109]) only diverges when all the terms in the series expansion of the hypergeometric function is included (it then goes to \( \Gamma(-1) \)). More explicitly, by letting \( x \to x' \) along a non-null geodesic with tangent \( t^\mu \), \( t^\mu t_\mu \equiv \Sigma = \pm 1 \), one has [22] (for \( d = 2 \), but the form is the same in any dimension, only the coefficients change)

\[
G(x, x') \approx -(4\pi)^{-1} \left[ 2\gamma + \ln \frac{1}{2} \epsilon^2 \Sigma R + \psi(\frac{1}{2} + \mu) + \psi(\frac{1}{2} - \mu) + \frac{1}{6} \epsilon^2 \Sigma R - \epsilon^2 \Sigma (\xi R + m^2) \left( 2\gamma + \ln \frac{1}{2} \epsilon^2 \Sigma R + \psi(\frac{3}{2} + \mu) + \psi(\frac{3}{2} - \mu) - 2 \right) \right]
\]

with

\[
\mu = \sqrt{\frac{1}{4} - 2\xi - m^2 \alpha^2}
\]

Now, in this case \( \Delta = \frac{1}{2} \Sigma \epsilon^2 \), so what we capture here is only the \( \log \Delta \) divergence and not the more common \( \Delta^{-1} \) one, consequently it does not satisfy the Hadamard condition. The result ([109]) follows from a mode summation using an adiabatic vacuum, the failure to meet the Hadamard condition is then likely to be a result of not going to infinite adiabatic order. The adiabatic vacuum has the advantage of being very similar to flat space-time, but is, alas, only possible in some space-times. The Green’s function following from the heat kernel as put forward here, on the other hand, comes with a “natural vacuum”, which apparently works in all cases. The difference between our result and ([109]) can thus come from many sources:

- Ambiguity in the definition of a vacuum ([1] uses adiabatic vacuum which does not always work).
- Ambiguity in the definition of a propagator (the Green’s function as defined in this paper is rather unique, and from this all propagators – positive frequency, advanced, retarded, Whightmann etc. – can be found).
- The approximations made (our result is highly non-pertubative and seems to include more terms than in ([109]).
This also summarizes the differences found in the other specific geometries.

4.4.2 Gauge Boson Case

The tensor
\[ \mathcal{E}_{mp}^{\eta} \equiv \eta^{mq} (\partial_{m} e_{q}^{\mu} - \partial_{q} e_{m}^{\mu}) e_{p}^{\mu} \]
is found to have the following non-vanishing components only
\[ \mathcal{E}_{0i}^{ii} = \mathcal{E}_{0i}^{0i} = - \dot{C} C^{-1} \] \hspace{1cm} (110)
with no summation over the index \( i = 1, 2, 3 \) implied. From this we get the mean-field independent part of \( \tilde{A} \) to be
\[ \tilde{A}_{m}^{i} = - \frac{9}{4} \delta_{m}^{i} \dot{C}^{2} C^{-2} + \text{mean field terms} \] \hspace{1cm} (111)
whereby
\[ A_{n}^{m} = - \frac{9}{4} \delta_{n}^{m} e^{-2\sqrt{\Delta}} \int_{0}^{1} \left( \frac{d}{d\eta} \ln C \right)^{2} e^{2\tau} d\tau \]
\[ - \frac{9}{16} \delta_{n}^{m} e^{-2\sqrt{\Delta}} \left( \dot{C}(\eta)^{2} C(\eta)^{-2} + \dot{C}(\eta')^{2} C(\eta')^{-2} \right) + \text{mean field terms} \] \hspace{1cm} (112)
For the matrix-coefficients \( \tilde{B}, \tilde{C} \) we get similarly
\[ \tilde{B}_{n}^{m} = \frac{9}{8} \delta_{n}^{m} \left( 2 C^{-4} \dot{C}^{2} + 2 C^{-4} \dot{C} \ddot{C} - 10 C^{-5} \dot{C}^{2} \ddot{C} + 6 C^{-6} \dot{C}^{4} + 2 \dot{C}^{2} \ddot{C} C^{-4} - 2 C^{-5} \dot{C}^{4} \right) + \text{mean field terms} \] \hspace{1cm} (113)
\[ \tilde{C}_{n}^{m} = \frac{81}{12} \delta_{n}^{m} \left( \dot{C} \ddot{C} C^{-2} - C^{-3} \dot{C}^{3} \right)^{2} + \text{mean field terms} \] \hspace{1cm} (114)
And once more one can get the remaining terms by simple differentiation of these.

5 Conclusion and Outlook

We have developed a way to determine the propagators of quantum fields in curved space-time explicitly. It is, for practical reasons, presently only
possible to calculate an approximation to the propagator. This approximation, while excellent for scalar and spinor fields and relatively good for spin 1 gauge bosons is, in the latter case, particularly for non-abelian gauge fields, rather complicated to work with so one might hope to change this by altering the approach by which we found this expression. In all the cases, though, the practical calculation of the Green’s function, was reduced to that of calculating a few functions (for spin zero and one half, \( f_0, \tilde{f}_0 \)) or three matrices (for vector bosons, \( \tilde{A}, \tilde{B}, \tilde{C} \)) up to interactions with other fields than the gravitational background.

It turned out, however, that our non-perturbative propagators, though satisfying the Hadamard condition and having a very simple physical meaning, differed from what other authors had found in various cases. This difference can be traced back to the uniqueness of the vacuum as presented here, and which differs, in general, from the adiabatic or conformal vacuum usually used. The ambiguity in the definition of a vacuum is related to the insufficiency of the naive particle concept in curved space-times. By always referring to a local comoving observer, we seem to have arrived at a better defined particle concept, since we are then able to relate to a flat space-time at each point along the trajectory, and of course, in flat space-time there is no problem with the particle concept, as the general success of ordinary quantum field theory shows. It should furthermore be emphasized that the method put forward in this paper is very general, as we do not to assume any particular feature of the metric. If the space-time under consideration does show some particular features (such as weak or slowly varying curvature, asymptotic or conformal flatness and so on), then of course, the calculation of the propagator by our method can be simplified, or rather adjusted, by then only including the necessary terms in the general expansion of the heat kernel. As these coefficients are actually rather easy to find this adjustment is straightforward in each particular case.

Looking at our results for, say, Friedman-Robertson-Walker and Schwarzschild geometries, we notice the appearance of terms which survive if the flat space limit is understood as \( a(t) \to \infty \) and \( M \to 0 \) respectively. These effects are most likely due to the difference in topologies; for Friedman-Robertson-Walker space-times we still have \( K = \pm 1, 0 \) even in the limit \( a \to \infty \), and in the Schwarzschild case, the limit \( M \to 0 \) corresponds to Minkowski space with the origin removed. It is not at present clear what the physical implications of such corrections are, however. When doing perturbation theory
in flat space-time, one should of course in principle take into account that we do not live in Minkowski space, and thus use, say, the \( a \to \infty \) limit of a Friedman-Robertson-Walker propagator. One would expect any such effect to be very small, but it is not \emph{a priori} certain as the results presented here are in deed non-perturbative, and even though the perturbative effect is very small, the true, non-perturbative effect need not be. Given the present day limitations on accelerator power, the only chance of seeing a quantum gravitational effect, or an effect coming from the coupling of the quantum fields to the space-time geometry, is either by studying the early universe and/or black holes or to look for some non-perturbative effect.

Naturally, the method of calculating the heat kernel as presented here in this paper is in no way restricted to quantum field theory in curved space-time. It should also be of considerable mathematical interest, as it is known that the heat kernel of the Laplace-Beltrami operator contains information about the large scale structure of the underlying manifold. Non-perturbative quantum field theory in ordinary Minkowski space is of course just a particular example, and the uses there include quark-gluon plasma.

The uses for the propagators include refinements in determining the spectrum of Hawking emission from black holes as well as the emission of gravitational waves from the space-time singularity (the Big Bang), applications that we hope to entertain in subsequent papers. Another application, which we have not yet considered, is the evaluation of the trace-anomaly of the energy-momentum tensor in general. Lastly we should note that, having obtained expressions for the propagators in this paper and for the one-loop effective actions in a previous one, we are now in a position to find the effective action to any loop order (at least formally). A paper on this is in progress.

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| $(m, n)$ | $B_m^n$ | $C_m^n$ |
|---------|---------|---------|
| (0, 0)  | $18 \left( \frac{\dot{a}}{a} \right)^4 - 30 \frac{\dddot{a}^2}{a^3} + 2 \left( \frac{\ddot{a}}{a} \right)^2 + 2 \frac{\dddot{a}^2}{a^2}$ | $\frac{9}{4} \left( \frac{\dddot{a}^2}{a^2} - \frac{\dot{a}}{a} \right)^2 + \frac{1}{4} a^{-2} \left( \frac{\ddot{a}}{a} - \frac{\dddot{a}}{a} \right)^2 \left[ \left( \frac{f''}{f} \right)^2 + f^{-2} \cot \theta ight] + \frac{1}{4} a^2 a^{-6} \left( \frac{f'''}{f} \right)^2 + a^2 a^{-6} f^{-2} \sin^{-4} \theta$ |
| (0, 1)  | $2 f'' \left( 2 \left( \frac{\dot{a}}{a} \right)^3 - 3 \frac{\dddot{a}^2}{a^3} + \frac{\dddot{a}^{(3)}}{a} \right)$ | $\frac{1}{2} \left( \frac{f''}{f} - \left( f' \right)^2 \right) \left( \frac{f'''}{f} - \left( f' \right)^3 \right) \frac{\dot{a}^3}{a^3} + \frac{\dot{a} f'}{a^2 \sin^2 \theta}$ |
| (0, 2)  | $\cot \theta \left( 2 \left( \frac{\dot{a}}{a} \right)^3 - 3 \frac{\dddot{a}^2}{a^3} + \frac{\dddot{a}^{(3)}}{a} \right)$ | $\frac{1}{4} \frac{\dddot{a}^3}{a^3} \left( \frac{\dddot{a}}{a} - \frac{\dot{a}}{a} \right) \cot \theta + \frac{3}{4} \frac{\dddot{a}^3}{a^3} \left( \frac{\dddot{a}}{a} - \frac{\dddot{a}}{a} \right) \cot \theta$ |
| (1, 1)  | $6 \left( \frac{\dot{a}}{a} \right)^4 - 10 \frac{\dddot{a}^2}{a^3} + 2 \left( \frac{\ddot{a}}{a} \right)^2 + 2 \frac{\dddot{a}^2}{a^2}$ | $\frac{1}{4} \left( \frac{\dot{a}}{a} - \left( \frac{\ddot{a}}{a} \right)^2 \right) \left( f' \right)^2 + \frac{1}{4} \left( \frac{\dddot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right)^2 \cot \theta$ |
| (1, 2)  | $a^{-2} \cot \theta \left( 2 \left( \frac{f''}{f} \right)^3 - 3 \frac{f'''}{f} + \frac{f''''}{f} \right)$ | $\frac{1}{4} \cot \theta \frac{f'''}{f} \left( \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} \right)^2 \cot \theta + \frac{3}{4} \left( \frac{f''}{f} - \left( f' \right)^2 \right) \left( f'' \right) \left( f' \right) a^{-2} \cot \theta$ |
| (2, 2)  | $9 \left( \frac{\dot{a}}{a} \right)^4 - 10 \frac{\dddot{a}^2}{a^3} + 2 \left( \frac{\ddot{a}}{a} \right)^2 + 2 \frac{\dddot{a}^2}{a^2}$ | $\frac{1}{4} \left( \frac{\dddot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right)^2 + \frac{1}{4} \frac{\dddot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right)^2 \cot \theta$ |
| (3, 3)  | as (2, 2) | \[ \frac{1}{8} a^{-2} f^{-2} \frac{\cos^2 \theta}{\sin^2 \theta} \] |

Table 1: The mean field independent contribution to the symmetric matrices $B$ and $C$ for a Friedman-Robertson-Walker space-time.