IMPARTIAL ACHIEVEMENT GAMES ON CONVEX GEOMETRIES

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Abstract. We study a game where two players take turns selecting points of a convex geometry until the convex closure of the jointly selected points contains all the points of a given winning set. The winner of the game is the last player able to move. We develop a structure theory for these games and use it to determine the nim number for several classes of convex geometries, including one-dimensional affine geometries, vertex geometries of trees, and games with a winning set consisting of extreme points.

1. Introduction

A convex geometry is an abstract generalization of the notion of convexity on a finite set of points in Euclidean space. We study an achievement game where two players take turns selecting previously unselected points of a convex geometry until the convex closure of the jointly selected points contains all the points of a given winning set. The winner of the game is the last player able move. That is, the winner is the first player to make the convex hull of the jointly selected points a superset of the winning set.

This game is a version of a group generating game introduced by Anderson and Harary [3] and further developed in [4, 5, 6, 7, 12]. Our game is played on a different kind of mathematical object. It is also a generalization since we introduce a winning set that can be different from the base set of the mathematical object. The key tool for studying these generating games is structure equivalence introduced in [12]. Structure equivalence is an equivalence relation on the game positions that is compatible with the option structure of the positions. Taking the quotient of the game digraph by structure equivalence provides significant simplifications. We develop a structure theory for our generalization. This allows us to determine the nim number of our games for several classes of convex geometries, including one-dimensional affine geometries, vertex geometries of trees, and games with a winning set consisting of extreme points.

The structure of the paper proceeds as follows. In Section 2 we recall some basic terminology of impartial games and convex geometries. In Section 3 we describe the convex closure achievement game in detail and provide some general results. In Section 4 we introduce structure equivalence and structure diagrams. In Section 5 we determine the nim numbers of games in which the goal set is a subset of the extreme point set of the ground set. In Section 6 we determine the nim numbers of games played on vertex geometries of trees. This allows us to easily determine the nim numbers of games played on affine geometries in \( \mathbb{R} \) in Section 7 since these affine geometries are isomorphic to vertex geometries of paths. We conclude with some further questions in Section 8.
2. Preliminaries

If $f : X \to Y$ and $A \subseteq X$, then we often use the standard $f(A) := \{f(a) \mid a \in A\}$ notation for the image of $A$. The cardinality of a set $A$ is denoted by $|A|$, and we write $\text{pty}(A) := |A| \mod 2$ for the parity of a set.

2.1. Impartial games. We recall the basic terminology of impartial combinatorial games. Our general references for the subject are [2, 16].

An impartial game is a finite set $P$ of positions accompanied with a starting position and a collection $\text{Opt}(P) \subseteq P$ of options for each position $P$ such that every game finishes in finitely many steps. We can think of it as a finite acyclic digraph with the positions as vertices, where there is an arrow from a position to every option of that position. Game play is moving a token from one vertex to another along the arrows. The game ends when the token reaches a sink of the graph. The last player to move is the winner of the game.

The minimum excludant $\text{mex}(A)$ of a set $A$ of non-negative integers is the smallest non-negative integer that is not in $A$. The nim number $\text{nim}(P)$ of a position $P$ of a game is defined recursively as the minimum excludant of the nim numbers of the options of $P$. That is,

$$\text{nim}(P) := \text{mex}(\text{nim}(\text{Opt}(P))).$$

The nim number of the game is the nim number of the starting position.

A position is terminal if it has no options. A terminal position $P$ has nim number $\text{nim}(P) = \text{mex}(\text{nim}(\emptyset)) = \text{mex}(\emptyset) = 0$. A position $P$ is losing for the player about to move ($P$-position) if $\text{nim}(P) = 0$ and winning ($N$-position) otherwise. The winning strategy is to always move to an option with nim number 0 if available. This places the opponent into a losing position. The nim number is a central object of interest for impartial games. In addition to determining the outcome of the game, it also makes it easy to compute the nim number of sums of games.

2.2. Convex geometries. We recall some facts about convex geometries from [9, 11, 14].

**Definition 2.1.** A convex geometry is a pair $(S, \mathcal{K})$, where $S$ is a finite set and $\mathcal{K}$ is a family of subsets of $S$ satisfying the following properties:

1. $S \in \mathcal{K}$;
2. $K, L \in \mathcal{K}$ implies $K \cap L \in \mathcal{K}$;
3. $S \neq K \in \mathcal{K}$ implies $K \cup \{a\} \in \mathcal{K}$ for some $a \in S \setminus K$.

The sets in $\mathcal{K}$ are called convex.

The third condition is called accessibility. Following [9, 10], we do not require that the empty set and singleton sets are convex. Note that the family of complements of the convex sets in a convex geometry forms an anti-matroid.

A convex geometry determines a convex closure operator $\tau : 2^S \to 2^S$ defined by

$$\tau(A) := \bigcap\{K \in \mathcal{K} \mid A \subseteq K\}.$$

A point $a$ of a subset $A$ of a convex geometry is called an extreme point of $A$ if $a \notin \tau(A \setminus \{a\})$. The set of extreme points of $A$ is denoted by $\text{Ex}(A)$. 
Proposition 2.2. The convex closure operator of a convex geometry satisfies the following properties:

1. \( A \subseteq \tau(A) \);
2. \( A \subseteq B \) implies \( \tau(A) \subseteq \tau(B) \);
3. \( \tau(\tau(A)) = \tau(A) \);
4. \( a, b \not\in \tau(A) \) and \( a \neq b \in \tau(A \cup \{a\}) \) implies \( a \not\in (A \cup \{b\}) \);
5. \( A \) is convex if and only if \( \tau(\tau(A)) = \tau(A) \);
6. \( \tau(A) = \tau(\text{Ex}(A)) \).

The fourth property is called the anti-exchange property. So \( \tau \) is a closure operator satisfying the anti-exchange property. Note that the convex closure of the empty set might not be the empty set since we do not require the empty set to be convex.

Example 2.3. Let \( S \) be a finite subset of \( \mathbb{R}^n \). The convex subsets of \( S \) are the intersections of \( S \) with convex subsets of \( \mathbb{R}^n \). The collection \( K \) of convex subsets of \( S \) forms a convex geometry on \( S \) called the affine convex geometry. The convex closure operator is \( \tau(A) = S \cap \text{Conv}(A) \), where \( \text{Conv}(A) \) is the convex hull of \( A \).

Example 2.4. Figure 2.1 shows an example of a point set \( S = \{a, b, \ldots, f\} \) in \( \mathbb{R}^2 \). The set of extreme points of \( S \) in the affine convex geometry is \( \text{Ex}(S) = \{a, b, c, d\} \). The set of extreme points of \( A = \{a, b, c, e, f, g\} \) is \( \text{Ex}(A) = \{a, b, c\} \), so \( A = \tau(\{a, b, c\}) \). We also have \( A = \tau(\{a, b, c, e\}) = \tau(\{a, b, c, f\}) \).

We define deletions by generalizing the definition in [11] and the notion of minor in [13].

Definition 2.5. Let \( (S, K) \) be a convex geometry and \( D \) be a subset of \( S \). The deletion of \( K \) by \( D \) is the collection

\[ K \setminus D := \{ K \subseteq S \setminus D \mid K \cup D \in K \} \]

Proposition 2.6. If \( (S, K) \) is a convex geometry and \( D \subseteq S \), then \( (S \setminus D, K \setminus D) \) is a convex geometry.

Proof. We see that \( S \setminus D \in K \setminus D \) since \( (S \setminus D) \cap D = S \in K \). If \( K, L \in K \setminus D \) then \( K \cap L \in K \setminus D \) since \( (K \cap L) \cap D = (K \cap D) \cap (L \cap D) \in K \). Now assume \( S \setminus D \not= K \in K \setminus D \). Since \( S \not= K \cup D \in K \), accessibility implies that \( (K \cup D) \cup \{a\} \in K \) for some \( a \in S \setminus (K \cup D) \). So \( a \in (S \setminus D) \setminus K \) and \( K \cup \{a\} \in K \setminus D \). \( \square \)
Figure 2.2. The convex geometries up to isomorphism on a two-element set $S = \{0, 1\}$ represented as deletions of affine convex geometries on a point set $T$ in $\mathbb{R}$. The deleted points are shown as empty circles.

Figure 2.3. Convex geometries up to isomorphism on a three-element set $S = \{a, b, c\}$ represented as deletions of affine convex geometries on a point set $T$ in $\mathbb{R}^2$. The deleted points are shown as empty circles.

Note that we do not require $D$ to be convex. This is not necessary since the empty set does not need to be convex.

**Example 2.7.** The convex geometries on a two-element set $S = \{0, 1\}$ up to isomorphism are shown in Figure 2.2. Each of these convex geometries can be represented as deletions of affine convex geometries on a subset $T$ of $\mathbb{R}$. The deleted points are shown as empty circles. For example, the third convex geometry in the table is the deletion of the affine convex geometry on $T = \{-1, 0, 1\}$ with a deleted point in $D = \{-1\}$. The last row of the table shows the extreme points of $T$.

**Remark 2.8.** The main result of [13] is that every convex geometry that contains the empty set can be represented as a deletion of an affine convex geometry by a convex set. We conjecture that a version of this representation result holds even for convex geometries in which the empty set is not closed. In this conjectured representation, the deleted set does not need to be convex.

**Example 2.9.** There are four convex geometries up to isomorphism with three points in which the empty set is not convex. They can be represented as deletions of affine convex geometries in $\mathbb{R}^2$, as shown in Figure 2.3.

3. **Convex closure achievement game**

We now provide a detailed description of the impartial convex closure achievement game $\text{GEN}(S, W)$ played on a convex geometry $(S, \mathcal{K})$ with a nonempty winning subset $W$ of $S$. In this game, two players take turns selecting previously unselected elements of $S$. In turn $k$ the next player picks $p_k$ from $S \setminus \{p_1, \ldots, p_{k-1}\}$. The game ends when the convex closure $\tau(P)$ of the jointly selected points $P = \{p_1, \ldots, p_k\}$ contains $W$. The last player to make a move wins $\text{GEN}(S, W)$. We call $P$ the current position of the game. The set of options for a nonterminal position $P$ is $\text{Opt}(P) = \{P \cup \{s\} \mid s \in S \setminus P\}$. 
An important special case is when $W = S$. For this game we use the notation $\text{GEN}(S)$ instead of the more precise $\text{GEN}(S, S)$.

**Example 3.1.** Consider $\text{GEN}(S)$ on the affine convex geometry with $S := \{-1, 0, 1\} \subseteq \mathbb{R}$ shown in Figure 3.1. If the first player selects $-1$, then the second player can select $1$ and win the game because $S = \tau\{-1, 1\}$. So this first move is a mistake for the first player.

If the first player selects $0$, then without loss of generality we can assume the second player selects $1$. Then the first player can win by selecting $-1$. So with best play the first player can win this game.

Figure 3.2 shows the full game digraph of $\text{GEN}(S)$. The numbers below the sets of chosen points indicate the nim numbers of the positions.

**Proposition 3.2.** The nim numbers of $\text{GEN}(S, \text{Ex}(W))$, $\text{GEN}(S, W)$, and $\text{GEN}(S, \tau(W))$ are the same.

**Proof.** Let $P \subseteq S$. Since $\text{Ex}(W) \subseteq W \subseteq \tau(W)$, $\tau(W) \subseteq \tau(P)$ implies $\text{Ex}(W) \subseteq \tau(P)$. On the other hand, $\tau(W) \subseteq \tau(P)$ implies $\tau(W) = \tau(\text{Ex}(W)) \subseteq \tau(\tau(P)) = \tau(P)$. Hence the positions in all three games are exactly the same. \hfill \Box

The following is an immediate consequence.

**Corollary 3.3.** The nim numbers of $\text{GEN}(S)$ and $\text{GEN}(S, \text{Ex}(S))$ are the same.

4. **Structure theory**

4.1. **Structure equivalence.** Structure equivalence is an equivalence relation on the set of game positions. This relation is compatible with the option structure and hence the nim numbers of the positions. This allows us to use a smaller quotient of the game digraph to compute nim numbers.
**Definition 4.1.** Consider the achievement game $\text{GEN}(S, W)$ and let $M \subseteq S$. If $W \subseteq \tau(M)$ then $M$ is called a **generating set**. Otherwise, $M$ is called a **non-generating set**. If $M$ is non-generating and $N$ is generating for all $M \subset N \subseteq S$, then $M$ is called **maximally non-generating**. We denote the set of maximally non-generating subsets by $\mathcal{M}$.

**Example 4.2.** Consider the affine convex geometry on $S = \{0, 1, 2, 3\} \subseteq \mathbb{R}$ and let $W = \{1, 2\}$. Then $\mathcal{M} = \{\{0\}, \{2, 3\}\}$.

**Proposition 4.3.** A maximally non-generating set $M$ is convex.

*Proof.* The closure $\tau(M)$ of $M$ is a non-generating superset of $M$ since $W \not\subseteq \tau(M) = \tau(\tau(M))$ and $M \subseteq \tau(M)$. Hence $M = \tau(M)$ since $M$ is maximally non-generating. $\Box$

**Definition 4.4.** We let

\[ \mathcal{I} := \{ \bigcap \mathcal{N} \mid \mathcal{N} \subseteq \mathcal{M} \} \]

be the set of intersection subsets. The smallest intersection subset is the **Frattini subset** $\Phi$, which is the intersection of all maximally non-generating subsets.

Note that $\mathcal{N}$ can be the empty set and so $S = \bigcap \emptyset \in \mathcal{I}$.

**Example 4.5.** Consider the convex geometry $\mathcal{K} = \{\{0\}, \{0, 1\}\}$ on $S = \{0, 1\}$. If $W = \emptyset$ then $\mathcal{M} = \emptyset$, $\mathcal{I} = \{S\}$, $\Phi = S$, and $\text{nim}(\text{GEN}(S, W)) = 0$. Note that in this game there is only one game position $\emptyset$, and this position is both a starting and a terminal position. If $W = \{1\}$ then $\mathcal{M} = \{\{0\}\}$, $\mathcal{I} = \{\{0\}, S\}$, and $\Phi = \{0\}$.

**Example 4.6.** Consider the convex geometry $\mathcal{K} = \{\emptyset, \{0\}\}$ on $S = \{0\}$. If $W = \emptyset$ then $\mathcal{M} = \{\emptyset\}$, $\mathcal{I} = \{\emptyset, S\}$, and $\Phi = \emptyset$.

**Definition 4.7.** For a position $P$ of $\text{GEN}(S, W)$ let

\[ \mathcal{M}_P := \{ M \in \mathcal{M} \mid P \subseteq M \}, \quad [P] := \bigcap \mathcal{M}_P. \]

Two game positions $P$ and $Q$ are **structure equivalent** if $[P] = [Q]$. The **structure class** $X_I$ of $I \in \mathcal{I}$ is the equivalence class of $I$ under this equivalence relation.

Note that $[I] = I$ for all $I \in \mathcal{I}$ and that $I \mapsto X_I$ is a bijection from $\mathcal{I}$ to the set of structure classes.

**Example 4.8.** Let $W = S = \{-1, 0, 1\}$ as in Example 3.1. Then

\[ \mathcal{M} = \{-1, 0\}, \{0, 1\}\],

\[ \mathcal{I} = \{\{0\}, \{-1, 0\}, \{0, 1\}, S\}, \]

and $\Phi = \{0\}$. We also have

\[ X_{\{0\}} = \emptyset, \{0\}\], \quad X_{\{-1, 0\}} = \{-1\}, \{-1, 0\}\]

\[ X_{\{0, 1\}} = \{1\}, \{0, 1\}\], \quad X_S = \{-1, 1\}, \{-1, 0, 1\}\]

indicated with the gray boxes in Figure 3.2. So $[\{-1\}] = \{-1, 0\}$ and $[\emptyset] = \{0\}$. Notice the lack of arrow from $\{-1, 1\}$ to $\{-1, 0, 1\}$.
Proposition 4.9. If \([P] = [Q]\) then \(\mathcal{M}_P = \mathcal{M}_Q\).

Proof. For a contradiction, assume \(\mathcal{M}_P \neq \mathcal{M}_Q\). Then without loss of generality, there exists an \(M \in \mathcal{M}_P \setminus \mathcal{M}_Q\). That is, \(M \in \mathcal{M}\) such that \(P \subseteq M \) but \(Q \not\subseteq M\). Then there exists \(q \in Q \setminus M\). This is impossible since \(q \in Q \subseteq [Q] = [P] = \bigcap \mathcal{M}_P \subseteq M\). \(\square\)

Proposition 4.10. Let \(P, Q \in X_I \neq X_J\). If \(\text{Opt}(P) \cap X_J \neq \emptyset\) then \(\text{Opt}(Q) \cap X_J \neq \emptyset\).

Proof. Assume \(\text{Opt}(P) \cap X_J \neq \emptyset\). Then there is an \(r \in S \setminus P\) such that \(P \cup \{r\} \in X_J\). We show that \(Q \cup \{r\} \in X_J\). We have

\[
\mathcal{M}_{P \cup \{r\}} = \{M \in \mathcal{M} \mid P \cup \{r\} \subseteq M\}
= \{M \in \mathcal{M} \mid P \subseteq M\text{ and }r \in M\}
= \{M \in \mathcal{M}_P \mid r \in M\}
\]

and similarly \(\mathcal{M}_{Q \cup \{r\}} = \{M \in \mathcal{M}_Q \mid r \in M\}\). Hence

\[
[P \cup \{r\}] = \bigcap \mathcal{M}_{Q \cup \{r\}}
= \bigcap \{M \in \mathcal{M}_Q \mid r \in M\}
= \bigcap \{M \in \mathcal{M}_P \mid r \in M\}
= \bigcap \mathcal{M}_{P \cup \{r\}}
= [P \cup \{r\}] = J
\]

since \(\mathcal{M}_P = \mathcal{M}_Q\). \(\square\)

Definition 4.11. We say \(X_J\) is an option of \(X_I\) if \(X_J\) contains an option of \(I\). The set of options of \(X_I\) is denoted by \(\text{Opt}(X_I)\).

The following Lemma was proved in [12].

Lemma 4.12. If \(A\) and \(B\) are sets containing non-negative integers such that \(\text{mex}(A) \in B\), then \(\text{mex}(A \cup \{\text{mex}(B)\}) = \text{mex}(A)\).

Proposition 4.13. If \(P, Q \in X_I\) and \(\text{pty}(P) = \text{pty}(Q)\), then \(\text{nim}(P) = \text{nim}(Q)\).

Proof. Let

\[
Z := \{(P, Q) \mid [P] = [Q]\text{ and }\text{pty}(P) = \text{pty}(Q)\}.
\]

We say \((P, Q) \succeq (M, N)\) when \(P \subseteq M\) and \(Q \subseteq N\). Then \((Z, \succeq)\) is a partially ordered set with minimum element \((S, S)\). We proceed by structural induction on \(Z\). Let \((P, Q) \in Z\). The statement clearly holds if \(P = Q\). In particular, it holds if \((P, Q) = (S, S)\). Otherwise we let \(I := [P] = [Q]\) and consider several cases.

First, assume \(P \neq I \neq Q\). Then both \(P\) and \(Q\) have options in \(X_I\). In fact, \(P \cup \{s\} \in \text{Opt}(P) \cap X_I\) for each \(s \in I \setminus P\). If \(M\) and \(N\) are options of \(P\) and \(Q\) in \(X_I\) respectively, then \(\text{pty}(M) = \text{pty}(N)\). Hence \(\text{nim}(M) = \text{nim}(N)\) by induction since \((P, Q) \succeq (M, N)\) in \(Z\). If \(P\) has an option \(M\) in some \(X_J \neq X_I\), then \(Q\) also has an option \(N\) in \(X_J\) by Proposition 4.10. Since \([M] = J = [N]\) and \(\text{pty}(M) = \text{pty}(N)\), we have \((P, Q) \succeq (M, N)\).
Hence \( \text{nim}(M) = \text{nim}(N) \) by induction. This proves that \( \text{nim}(\text{Opt}(P)) = \text{nim}(\text{Opt}(Q)) \). Thus \( \text{nim}(P) = \text{nim}(Q) \).

Now, assume \( P \neq I = Q \). In this case \( Q \) does not have any options in \( X_I \). We still have \( \text{nim}(\text{Opt}(Q)) \subseteq \text{nim}(\text{Opt}(P)) \) by Proposition 4.11. Let \( M \) be an option of \( P \) in \( X_I \). Since \( \text{pty}(M) \neq \text{pty}(I) \), \( M \) is different from \( I \). So \( M \) has an option \( R \in X_I \). Then \( \text{pty}(R) = \text{pty}(I) = \text{pty}(Q) \) and \( [R] = I = [Q] \), so \( (P, Q) \succ (R, Q) \). Hence \( \text{nim}(R) = \text{nim}(Q) \) by induction. This implies

\[
\text{mex}(\text{nim}(\text{Opt}(Q))) = \text{nim}(Q) = \text{nim}(R) \in \text{nim}(\text{Opt}(M)),
\]

where \( A := \text{nim}(\text{Opt}(Q)) \) and \( B := \text{nim}(\text{Opt}(M)) \). Thus,

\[
\text{mex}(\text{nim}(\text{Opt}(Q))) \cup \{\text{nim}(M)\} = \text{mex}(\text{nim}(\text{Opt}(Q)))
\]

by Lemma 4.12. If \( N \in \text{Opt}(P) \) then either \( N \not\in X_I \) or \( N \in X_I \). In the first case \( \text{nim}(N) \in \text{nim}(\text{Opt}(Q)) \). In the second case \( \text{nim}(M) = \text{nim}(N) \) by induction since \( (P, Q) \succ (M, N) \). Hence

\[
\text{nim}(P) = \text{mex}(\text{nim}(\text{Opt}(P)))
\]

\[
= \text{mex}(\text{nim}(\text{Opt}(Q)) \cup \{\text{nim}(M)\})
\]

\[
= \text{mex}(\text{nim}(\text{Opt}(Q))) = \text{nim}(Q)
\]

since \( \text{nim}(\text{Opt}(P)) = \text{nim}(\text{Opt}(Q)) \cup \{\text{nim}(M)\} \).

\[\Box\]

**Definition 4.14.** The *type* of the structure class \( X_I \) is

\[
\text{type}(X_I) := (\text{pty}(I), \text{nim}_0(X_I), \text{nim}_1(X_I)).
\]

If \( I = S \) then \( \text{nim}_0(X_S) := 0 \) and \( \text{nim}_1(X_S) := 0 \). If \( I \neq S \) then

\[
\text{nim}_{\text{pty}(I)}(X_I) := \text{mex}(\text{nim}_{1-\text{pty}(I)}(\text{Opt}(X_I))),
\]

\[
\text{nim}_{1-\text{pty}(I)}(X_I) := \text{mex}(\text{nim}_{\text{pty}(I)}(\text{Opt}(X_I)) \cup \{\text{nim}_{\text{pty}(I)}(X_I)\})
\]

is defined recursively. We call the recursive computation of types using the options of structure classes *type calculus*.

Note that \( X_S \) is the only structure class without options. The type of a structure class \( X_I \) encodes the parity of \( I \) and the nim numbers of the positions in \( X_I \).

**Proposition 4.15.** If \( P \in X_I \) then \( \text{nim}(P) = \text{nim}_{\text{pty}(P)}(X_I) \).

**Proof.** The statement is clearly true for \( I = S \). We use structural induction on the structure classes together with Propositions 4.10 and 4.13. Any option \( Q \) of position \( I \) is in \( X_I \) for some \( X_J \in \text{Opt}(X_I) \). On the other hand, if \( X_J \in \text{Opt}(X_I) \) then \( X_J \) contains an option \( Q \) of \( I \). The parity \( \text{pty}(Q) = 1 - \text{pty}(I) \) of \( Q \) is the opposite of the parity of \( I \). Hence \( \text{nim}(I) = \text{mex}(\text{nim}(\text{Opt}(I))) = \text{nim}_{\text{pty}(I)}(X_I) \) by induction. If \( P \in X_I \) and \( \text{pty}(P) = \text{pty}(I) \), then \( \text{nim}(P) = \text{nim}(I) = \text{nim}_{\text{pty}(P)}(X_I) \).

Now assume \( P \) is a position in \( X_I \) such that \( \text{pty}(P) = 1 - \text{pty}(I) \). The options of \( P \) have parity \( \text{pty}(I) \). Some of the options of \( P \) are positions in \( X_J \) for some \( X_J \in \text{Opt}(X_I) \). On the other hand, if \( X_J \in \text{Opt}(X_I) \) then \( X_J \) contains an option \( Q \) of \( P \). Some options of \( P \) are in \( X_I \). Hence \( \text{nim}(P) = \text{nim}_{\text{pty}(P)}(X_I) \) by induction. \[\Box\]
Note that $X_I = \{I\}$ is a possibility. In this case $X_I$ contains no position with parity 1 – pty($I$). Also note that the nim number of the game is the nim number of the starting position $\emptyset$. So $\text{nim}(\text{GEN}(S,W)) = \text{nim}(\emptyset) = \text{mex}_0(X_\emptyset)$ is the second component of type($X_\emptyset$).

4.2. Structure diagrams. The structure digraph of $\text{GEN}(S,W)$ has vertex set $\{X_I \mid I \in \mathcal{I}\}$ and arrow set $\{(X_I, X_J) \mid X_J \in \text{Opt}(X_I)\}$. We visualize the structure digraph with a structure diagram that also shows the type of each structure class. Within a structure diagram, a vertex $X_I$ is represented by a triangle pointing up or down depending on the parity of $I$. The triangle points down when pty($I$) = 1 and points up when pty($I$) = 0.

The numbers within each triangle represent the nim numbers of the positions within the structure class. The first number is the common nim number of all even positions in $X_I$, while the second number is the common nim number of all odd positions in $X_I$.

We call an automorphism $\alpha$ of the structure digraph size preserving if $|I| = |J|$ for all $\alpha(X_I) = X_J$. It is often useful to work with the quotient of the structure digraph with respect to orbit equivalence determined by the size preserving automorphisms of the structure digraph. We call the corresponding structure diagram the orbit quotient structure diagram. Since orbit equivalent structure classes have the same type, the nim number of the game can be determined using the orbit quotient structure diagram.

**Example 4.16.** The structure digraph and structure diagrams of $\text{GEN}(S)$ with the affine convex geometry on $S = \{-1,0,1\}$ considered in Examples 3.1 and 4.8 appear in Figure 4.1. We demonstrate type calculus by verifying that $\text{nim}(\text{GEN}(S)) = \text{nim}(\emptyset) = \text{mex}_0(X_{\emptyset}) = 1$.

We have
\[
\begin{align*}
\text{nim}_1(X_{\emptyset}) &= \text{mex}(\{\text{nim}_0(X_{-1}), \text{nim}_0(X_{0})\}) = \text{mex}(\{1\}) = 0, \\
\text{nim}_0(X_{\emptyset}) &= \text{mex}(\{\text{nim}_1(X_{-1}), \text{nim}_1(X_{0})\} \cup \{\text{nim}_1(X_{\emptyset})\}) \\
&= \text{mex}(\{2\} \cup \{0\}) = 1.
\end{align*}
\]

**Remark 4.17.** If we make a move in the achievement game on an affine convex geometry, then the rest of the game is essentially played on a convex geometry that was created from the original convex geometry by the deletion of the selected point. Since the selected points do not necessarily form a convex set, the deletion of these points may result in a convex geometry in which the empty set is not convex.

**Example 4.18.** Figure 4.2 shows the structure diagrams of the achievement game on an affine convex geometry and the effect of deletion on the structure diagrams. Note that
deletion of a point produces a structure diagram that is a sub-diagram of the original diagram. This sub-diagram starts at an option of $X_\Phi$ and it has reversed parities and nim numbers.

5. Winning subsets containing only extreme points

In this section we characterize the nim number of $\text{GEN}(S, W)$ where $W \subseteq \text{Ex}(S)$.

**Proposition 5.1.** If $W \subseteq \text{Ex}(S)$ then the set $\mathcal{M}$ of maximally non-generating subsets of $\text{GEN}(S, W)$ is $\{M_v \mid v \in W\}$, where $M_v := S \setminus \{v\}$.

**Proof.** We show that $M_v$ for $v \in W$ is a maximally non-generating set. Since $v$ is an extreme point of $S$, $v \not\in \tau(M_v)$. So $M_v$ is a non-generating set. If $M_v \subseteq N \subseteq S$ then $N = S$ is a generating set. So $M_v$ is maximally non-generating.

We show that every maximally non-generating set is $M_v$ for some $v \in W$. Let $M$ be a maximally non-generating set. Since $M$ is non-generating, there must be a $v$ in $W \setminus M$. We clearly have $M \subseteq M_v$, which means $M_v$ a non-generating superset of $M$. Thus $M = M_v$ since $M$ is maximally non-generating. \(\square\)

**Corollary 5.2.** If $W \subseteq \text{Ex}(S)$ then the set of intersection subsets of $\text{GEN}(S, W)$ is $\mathcal{I} = \{S \setminus V \mid V \subseteq W\}$.

**Proposition 5.3.** If $W = \{w\} \subseteq \text{Ex}(S)$, then

$$\text{nim}(\text{GEN}(S, W)) = \begin{cases} 1, & \text{pty}(S) = 1 \\ 2, & \text{pty}(S) = 0. \end{cases}$$
Proof. We have $\mathcal{M} = \{M\}$ and $\mathcal{I} = \{M, S\}$, where $M = S \setminus \{w\}$. If $|S|$ is odd, then $\text{nim}(\text{GEN}(S, W)) = 1$ since $M$ is even and the structure diagram is the one shown in Figure 5.1(c). If $|S|$ is even, then $\text{nim}(\text{GEN}(S)) = 2$ since $M$ is odd and the structure diagram is the one shown in Figure 5.1(a).

Definition 5.4. Let $W \subseteq \text{Ex}(S)$. For $I \in \mathcal{I}$ we define $\delta(I) := |S \setminus I|$ to be the number of goal points missing from $I$.

Proposition 5.5. If $W \subseteq \text{Ex}(S)$ with $|W| \geq 2$, then

$$\text{nim}(\text{GEN}(S, W)) = \begin{cases} 0, & \text{pty}(S) = 0 \\ 1, & \text{pty}(S) = 1 \end{cases}$$

Proof. First consider the case when $|S|$ is even. We are going to use type calculus and structural induction on the elements of $\mathcal{I}$ to show that

$$\text{type}(X_I) = \begin{cases} (0, 0, 0), & \delta(I) = 0 \\ (1, 2, 1), & \delta(I) = 1 \\ (0, 0, 1), & \delta(I) > 1 \text{ and } \delta(I) \equiv_2 0 \\ (1, 0, 1), & \delta(I) > 1 \text{ and } \delta(I) \equiv_2 1 \end{cases}$$

for all $I \in \mathcal{I}$, as shown in Figure 5.1(b). Then we will have

$$\text{nim}(\text{GEN}(S, W)) = \text{nim}(\emptyset) = \text{nim}_0(X_\Phi) = 0$$

because $\delta(\Phi) = |W| \geq 2$. If $\delta(I) = 0$ then $X_I = X_S$ and $\text{type}(X_S) = (0, 0, 0)$. If $\delta(I) = 1$ then $I = S \setminus \{w\}$ for some $w \in W$, so $\text{pty}(I) = 1 - \text{pty}(S) = 1$. Then $\text{nim}(I) = \text{nim}(\text{Opt}(I)) = \text{nim}(\text{nim}(S)) = \text{nim}(\emptyset) = 0$. Furthermore, for any $P \in X_I$ satisfying $\text{pty}(P) = 0$, we have $\text{nim}(P) = \text{nim}(\text{Opt}(P)) = \text{nim}(\emptyset) = 0$. Hence $\text{type}(X_I) = (1, 2, 1)$. 

Figure 5.1. Orbit quotient structure diagrams for $\text{GEN}(S, W)$, where $W \subseteq \text{Ex}(S)$. 

| $|S|$ is even | $|W|$ = 1 | $|S|$ is odd | $|W|$ = 4 |
|-------------|---------|-------------|---------|
| (a)         | (b)     | (c)         | (d)     |
Suppose $\delta(I) \geq 2$. We have two cases to consider. If $\pty(I) = 0$ then $\delta(I) = |S \setminus I| \equiv 2 \mod 2$, and so $\delta(J) = |S \setminus J| \equiv 1 \mod 2$ with $\type(J) \in \{(1, 2, 1), (1, 0, 1)\}$ for all $J \in \Opt(I)$ by induction. Hence

$$\type(X_I) = (0, \nim_0(X_I), \nim_1(X_I)) = (0, \mex(\nim_1(\Opt(I))), \mex(\nim_0(\Opt(I))) \cup \{\nim_0(X_I)\}) = (0, \mex(\{1\}), \mex(\{0\})) = (0, 0, 1),$$

where $A = \{0, 2\}$ or $A = \{0\}$.

If $\pty(I) = 1$ then $\delta(I) \equiv 1 \mod 2$, and so $\delta(J) = |S \setminus J| \equiv 0 \mod 2$ with $\type(J) = (0, 0, 1)$ for all $J \in \Opt(I)$ by induction. Hence

$$\type(X_I) = (1, \nim_0(X_I), \nim_1(X_I)) = (1, \mex(\nim_1(\Opt(I))) \cup \{\nim_1(X_I)\}), \mex(\nim_0(\Opt(I)))) = (1, \mex(\{1\}), \mex(\{0\})) = (1, 0, 1).$$

Now consider the case when $|S|$ is odd. One can show that

$$\type(X_I) = \begin{cases} (1, 0, 0), & \delta(I) = 0 \\ (0, 1, 2), & \delta(I) = 1 \\ (1, 1, 0), & \delta(I) > 1 \text{ and } \delta(I) \equiv 2 \mod 2 \\ (0, 1, 0), & \delta(I) > 1 \text{ and } \delta(I) \equiv 0 \mod 2 \end{cases}$$

for all $I \in \mathcal{I}$, as show in Figure 5.1(d). The argument is essentially the same as the one we used in the previous case but with reversed parities. Hence

$$\nim(\text{GEN}(S,W)) = \nim(\emptyset) = \nim_0(X_{\emptyset}) = 1$$

because $\delta(\Phi) = |W| \geq 2$. \hfill \Box

The essence of the previous proof is captured in Figures 5.1(b) and (d). The parity of $|S|$ determines the direction of the triangles, while $|W|$ determines the height of the diagram.

6. Convex geometries from vertices of trees

Consider a tree graph $T$ with vertex set $S$. The vertex sets of connected subgraphs of $T$ form a convex geometry on $S$. We call this the vertex geometry of $T$. In this section we study the achievement game $\text{GEN}(S,W)$ on vertex geometries of trees.

We use the standard $N(v)$ notation for the set of vertices adjacent to a vertex $v$.

Example 6.1. The vertex geometry of the path graph $P_3$ with vertex set $S = \{1, 2, 3\}$ is $\mathcal{K} = 2^S \setminus \{\{1, 3\}\}$.

Example 6.2. Let $R := \{0, i, j, k\}$ and $D := \{-i - j, -i - k, -j - k\}$ be subsets of $\mathbb{R}^3$. The vertex geometry of the complete bipartite graph $K_{1,3}$ can be represented as a deletion of the affine convex geometry on $S = R \cup D$ by $D$. The graph $K_{1,3}$ and the point set $S$ are shown in Figure 6.1. The set of convex sets is $\mathcal{K} = 2^R \setminus \{\{i, j\}, \{j, k\}, \{i, k\}, \{i, j, k\}\}$.

Removing a vertex $w$ of a tree $T$ creates a forest that we denote by $T \setminus w$. 
Proposition 6.3. Let \((S,K)\) be the vertex geometry of a tree \(T\) and \(W \subseteq S\). Then \(w \in \text{Ex}(W)\) if and only if the elements of \(W \setminus \{w\}\) are vertices in a single connected component of \(T \setminus w\).

Proof. For each \(v \in N(w)\) let \(V_v\) be the vertex set of the connected component of \(T \setminus w\) containing \(v\). The map \(v \mapsto V_v\) is a bijection from \(N(w)\) to the collection of vertex sets of the connected components of \(T \setminus w\).

First assume \(w_1\) and \(w_2\) are elements of \(W \setminus \{w\}\) that are vertices in different connected components of \(T \setminus w\). Then \(w\) is a vertex of every connected subgraph of \(T\) that contains both \(w_1\) and \(w_2\). Hence \(w \in \tau(w_1,w_2) \subseteq \tau(W \setminus \{w\})\), so \(w \not\in \text{Ex}(W)\).

Now assume \(W \setminus \{w\}\) is contained in the vertex set of a single connected component \(V_v\) of \(T \setminus w\). Since \(V_v\) is convex and contains \(W \setminus \{w\}\), \(\tau(W \setminus \{w\}) \subseteq V_v\). So \(w \not\in \tau(W \setminus \{w\})\) since \(w \not\in V_v\). Thus \(w \in \text{Ex}(W)\). \(\square\)

6.1. Winning subsets with more than one vertex. First we consider the case when \(W\) has at least two vertices. If \(w \in \text{Ex}(W)\) then \(T \setminus w\) has exactly one component that contains some elements of \(W\). We denote the vertex set of this component by \(M_w\). It is clear from the construction that \(w \mapsto M_w : \text{Ex}(W) \to \mathcal{M}\) is a bijection. We use the notation \(V_w := S \setminus M_w\). Note that \(\{V_w \mid w \in \text{Ex}(W)\}\) is a collection of pairwise disjoint sets.

There is a bijective correspondence between the subsets of \(\text{Ex}(W)\) and the intersection subsets given by \(A \mapsto M_A : 2^{\text{Ex}(W)} \to \mathcal{T}\), where

\[
M_A := \bigcap \{M_w \mid w \in A\} = S \setminus \bigcup \{V_w \mid w \in A\}.
\]

Note that \(S = M_\emptyset\) and \(\Phi = M_{\text{Ex}(W)}\). We define the deficiency of a structure class \(M_A\) by \(\delta(M_A) := |A|\). We also define the signature of a structure class \(X_{M_A}\) to be \(\sigma(X_{M_A}) := (e,o)\), where \(e\) is the size of \(\{a \in A \mid \text{pty}(V_a) = 0\}\) and \(o\) is the size of \(\{a \in A \mid \text{pty}(V_a) = 1\}\). The signature satisfies \(\delta(M_A) = e + o\). The signature of the game is \(\sigma(X_\Phi)\).

Example 6.4. Figure 6.2 shows a tree graph with vertex set \(S = \{0,\ldots,11\}\), the lattice of intersection subsets, and the structure diagram of \(\text{GEN}(S,W)\) with winning set \(W = \{1,3,6,8\}\). The set of extreme points of \(W\) is \(\text{Ex}(W) = \{3,6,8\}\). The set of maximal non-generating sets is \(\mathcal{M} = \{M_3, M_6, M_8\}\) with \(V_3 = \{3,4,5\}\), \(V_6 = \{6,7\}\), and \(V_8 = \{8,9,10,11\}\), so that the Frattini subset is \(\Phi = M_{\{3,6,8\}} = \{0,1,2\}\). Any directed path in the structure diagram from \(X_{M_A}\) to \(X_S = X_{M_\emptyset}\) corresponds to the elements of \(A\). The signature of \(X_{M_A}\)
Figure 6.2. Tree graph, lattice of intersection subsets, and structure diagram of $\text{GEN}(S;W)$ with $W = \{1, 3, 6, 8\}$. Dotted arrows correspond to a signature change of $(0, -1)$, while solid arrows correspond to a signature change of $(-1, 0)$. The label $a$ on an arrow indicate the change from $M_A$ to $M_A\{a\}$.

Figure 6.3. Type calculus computation of the possible structure class types with deficiency less than 5. A solid arrow represents no change in parity while a dotted arrow represents a parity change.

can be computed by counting the solid and the dotted arrows along any such directed path. For example $\sigma(X_{M_{\{3,8\}}}) = (1, 1)$. Any directed path from $X_\Phi$ to $X_S$ contains one dotted and two solid arrows since the signature of the game is $\sigma(X_\Phi) = (2, 1)$.

**Proposition 6.5.** Let $(S, K)$ be the vertex geometry of a tree $T$ and $W$ be a subset of $S$ with $|W| \geq 2$. If the signature of the game is $(e, o)$, then

$$\text{nim}(\text{GEN}(S; W)) = \begin{cases} 1, & (e, o) = (1, 0) \\ 2, & (e, o) \in \{(0, 1), (1, 2)\} \\ 3, & (e, o) \in \{(1, 1), (2, 1)\} \\ 0, & \text{else} \end{cases}$$
when \( \pty(S) = 0 \), and

\[
\nim(\text{GEN}(S, W)) = \begin{cases} 
0, & (e, o) \in \{(0, 0), (1, 1)\} \\
2, & (e, o) \in \{(1, 0), (2, 0)\} \\
1, & \text{else}
\end{cases}
\]

when \( \pty(S) = 1 \).

\textbf{Proof.} Let \( X_J \) be an option of \( X_I \) and \( I = M_A \). Then \( I \cup \{v\} \in X_J \) for some \( v \in J \setminus I \). So there is a unique \( a \in A \) such that \( v \in V_a \) and \( J = M_{A \setminus \{a\}} \). So every arrow of the structure diagram corresponds to a unique element \( a \) of \( \text{Ex}(W) \). This means \( \delta(J) = |A \setminus \{a\}| = |A| - 1 = \delta(I) - 1 \).

Roughly speaking this tells us that there are no arrows between structure classes that are at the same directed distance from \( X_S \).

Assume \( I = M_A \) and \( \sigma(X_I) = (e, o) \). If \( e, o \geq 1 \) then there are \( a, b \in A \) such that \( \pty(V_a) = 0 \) and \( \pty(V_b) = 1 \). Hence

\[
\sigma(\text{Opt}(X_I)) = \begin{cases} 
\{(e - 1, 0)\}, & o = 0 \\
\{(0, o - 1)\}, & e = 0 \\
\{(e - 1, o), (e, o - 1)\}, & e, o \geq 1.
\end{cases}
\]

The result now follows from type calculus and structural induction on the structure classes. The details of the type calculus computation are shown in Figure 6.3. The signature of a structure class is \((e, o)\) where \( e \) is the number of solid arrows and \( o \) is the number of dotted arrows from the structure class to the bottom structure class along a directed path. Starting at deficiency 4 the set of possible types remains fixed by induction. \( \square \)

\textbf{6.2. Winning subsets with only one vertex.} Now we consider the case when \( W = \{w\} \) is a singleton set. For each \( v \) in the set \( N(w) \) of neighbors of \( w \), there is a unique connected component of \( T \setminus w \) that contains \( v \). The vertex set \( M_v \) of this component is a maximal non-generating set. In fact, \( \mathcal{M} = \{M_v \mid v \in N(w)\} \). Note that \( \mathcal{M} \) is a collection of disjoint subsets. The \textit{signature} of the game is \((e, o)\), where \( e \) is the size of \( \{v \in N(w) \mid \pty(M_v) = 0\} \) and \( o \) is the size of \( \{v \in N(w) \mid \pty(M_v) = 1\} \).

\textbf{Example 6.6.} Figure 6.4 shows a tree graph with vertex set \( S = \{0, \ldots, 9\} \) and the structure diagram of \( \text{GEN}(S, W) \) with winning set \( W = \{0\} \). Some of the maximal non-generating sets are \( M_1 = \{1, 2, 3\} \) and \( M_5 = \{5\} \), so that the Frattini subset is \( \Phi = \emptyset \). The signature of the game is \((e, o) = (2, 3)\).
**Proposition 6.7.** Let \((S, K)\) be the vertex geometry of a tree \(T\) and \(W = \{w\} \subseteq S\). If the signature of the game is \((e, o)\), then

\[
\text{nim}(\text{GEN}(S, W)) = \begin{cases} 
1, & o = 0 \\
2, & e = 0 \text{ and } o \geq 1 \\
3, & e, o \geq 1.
\end{cases}
\]

*Proof.* Since \(\emptyset \in X_\Phi\) and \(\{w\} \in X_S, X_S \in \text{Opt}(X_\Phi)\).

Assume \(o = 0\), so that \(\text{pty}(S) = 1\). If \(e = 0\) then the nim number of the game is 1 as shown in Figure 6.5(a) since \(S = \{w\}, M = \{\emptyset\}, \text{ and } I = \{\emptyset, S\}\). If \(e = 1\) then \(N(w) = \{v\}, M = \{M_v\}, \text{ and } I = \{M_v, S\}\). Hence the structure diagram is shown in Figure 6.5(a) and the nim number of the game is 1. If \(e \geq 2\) then \(\Phi = \emptyset\) and \(I = \{\emptyset, S\} \cup \{M_v \mid v \in N(w)\}\), so the orbit quotient structure diagram is shown in Figure 6.5(b) and the nim number of the game is 1.

Assume \(e = 0\) and \(o \geq 1\). If \(o = 1\) then \(N(w) = \{v\}, M = \{M_v\}, \text{ and } I = \{M_v, S\}\), so the structure diagram is shown in Figure 6.5(c) and the nim number of the game is 2. If \(o \geq 2\) then \(\Phi = \emptyset\) and \(I = \{\emptyset, S\} \cup \{M_v \mid v \in N(w)\}\), so the orbit quotient structure diagram is shown in Figure 6.5(d) and the nim number of the game is 2.

Assume \(e, o \geq 1\), so that \(\Phi = \emptyset\) and \(I = \{\emptyset, S\} \cup \{M_v \mid v \in N(w)\}\). So the orbit quotient structure diagram is shown in Figure 6.5(e) and the nim number of the game is 3. \(\square\)

**Corollary 6.8.** If \((S, K)\) is the vertex geometry of a tree \(T\), then \(\text{nim}(\text{GEN}(S, W)) \in \{0, 1, 2, 3\}\).

7. **Affine convex geometries in \(\mathbb{R}\)**

In this section we study the games played on affine convex geometries in \(\mathbb{R}\). It is easy to see that if \(S = \{s_1, \ldots, s_n\} \subseteq \mathbb{R}\) then the affine convex geometry on \(S\) is isomorphic to the vertex geometry of a path graph \(P_n\) with \(n\) vertices.
Proposition 7.1. Let \( S = \{s_1, \ldots, s_n\} \subseteq \mathbb{R} \) such that \( s_i < s_{i+1} \) for all \( i \) and \( W = \{s_{i_1}, \ldots, s_{i_k}\} \) such that \( i_1 < i_2 < \cdots < i_k \) and \( k \geq 2 \). Then
\[
\text{nim}(\text{GEN}(S,W)) = \begin{cases} 
0, & i_1 \equiv_2 i_k + 1 \\
1, & i_1 \equiv_2 1 \equiv_2 i_k \text{ and } n \equiv_2 1 \\
2, & i_1 \equiv_2 0 \equiv_2 i_k \text{ and } n \equiv_2 1 \\
3, & i_1 \equiv_2 i_k \text{ and } n \equiv_2 0.
\end{cases}
\]

Proof. We consider the equivalent game on the vertex geometry of the path graph \( P_n \) with vertex set \( \{1, 2, \ldots, n\} \) and \( W = \{i_1, i_2, \ldots, i_k\} \). The set of extreme points of the winning set is \( \text{Ex}(W) = \{i_{1}, i_{k}\} \). We have \( |V_{i_1}| = i_1 \) and \( |V_{i_k}| = n - i_k + 1 \). So the signature of this game is
\[
\sigma(X_\Phi) = \begin{cases} 
(1, 1), & i_1 \equiv_2 i_k + 1 \text{ and } n \equiv_2 1 \\
(0, 2), & i_1 \equiv_2 i_k + 1 \text{ and } n \equiv_2 0 \\
(0, 2), & i_1 \equiv_2 1 \equiv_2 i_k \text{ and } n \equiv_2 1 \\
(2, 0), & i_1 \equiv_2 0 \equiv_2 i_k \text{ and } n \equiv_2 1 \\
(1, 1), & i_1 \equiv_2 i_k \text{ and } n \equiv_2 0.
\end{cases}
\]
The result now follows from Proposition 6.5. \( \square \)

Proposition 7.2. Let \( S = \{s_1, \ldots, s_n\} \subseteq \mathbb{R} \) such that \( s_i < s_{i+1} \) for all \( i \) and \( W = \{s_k\} \). Then
\[
\text{nim}(\text{GEN}(S,W)) = \begin{cases} 
1, & n \equiv_2 1 \equiv_2 k \\
2, & n \equiv_2 1 \equiv_2 k + 1 \\
2, & n \equiv_2 0 \text{ and } k \in \{1, n\} \\
3, & n \equiv_2 0 \text{ and } k \in \{2, \ldots, n - 1\}.
\end{cases}
\]

Proof. We consider the equivalent game on the vertex geometry of the path graph \( P_n \) with vertex set \( V = \{1, 2, \ldots, n\} \) and \( W = \{k\} \). The neighborhood of \( k \) is \( N(k) = \{k - 1, k + 1\} \cap V \). So the signature of this game is
\[
\sigma(X_\Phi) = \begin{cases} 
(0, 0), & n = 1 = k \\
(1, 0), & n \equiv_2 1 \equiv_2 k \text{ and } k \in \{1, n\} \\
(2, 0), & n \equiv_2 1 \equiv_2 k \text{ and } k \in \{2, \ldots, n - 1\} \\
(0, 2), & n \equiv_2 1 \equiv_2 k + 1 \\
(0, 1), & n \equiv_2 0 \text{ and } k \in \{1, n\} \\
(1, 1), & n \equiv_2 0 \text{ and } k \in \{2, \ldots, n - 1\}.
\end{cases}
\]
The result now follows from Proposition 6.7. \( \square \)

Deleted affine convex geometries can be solved using the techniques of this section.

Example 7.3. Consider the deleted affine convex geometry on the set \( S = \{0, 1, \ldots, 6\} \) with deleted set \( D = \{3, 5\} \). The achievement game with winning set \( W = \{2, 6\} \) is equivalent to \( \text{GEN}(S \setminus D, W) \), so it has nim number 1. The achievement game with winning set \( W = \{1, 2\} \) is equivalent to \( \text{GEN}(\{1, 2, 4, 6\}, \{1\}) \), so it has nim number 2.
8. FURTHER DIRECTIONS

We provide some comments and propose some questions for further study.

(1) The conjecture in Remark 2.8 probably can be proved by adjusting the approach of [13].

(2) Our results may suggest that the nim number of GEN(S, W) is in the set \{0, 1, 2, 3\}. The point set in Figure 8.1 provides an example in \(\mathbb{R}^2\) where the nim number is 6. The companion web site [15] provides an example in \(\mathbb{R}^3\) where the nim number is 8. What are the possible nim numbers for convex geometries? What are the possible nim numbers for affine convex geometries? Does the answer depend on the dimension of the space?

(3) The definition of vertex geometry can be generalized to forest graphs. Our results might generalize to this setting.

(4) Consider a tree graph \(T\) with edge set \(S\). The edge sets of connected subgraphs of \(T\) form a convex geometry on \(S\). What is the nim number of GEN(S, W) played on these edge geometries?

(5) There are several ways to build a convex geometry from a partially ordered set. What can we say about GEN(S, W) played on these convex geometries?

(6) What can we say about GEN(S, W), where W is the Frattini set of GEN(S)?

(7) The original games introduced by [3] and played on a group can be generalized by allowing a winning set that is a subset of the group. This may produce interesting results for special subgroups as the winning set.

(8) It might be easier to determine the possible nim values of the avoidance game DNG(S, W) played on convex geometries. The avoidance version was also introduced by [3]. In this game it is not allowed to select an element that creates a set whose convex closure contains the winning set. In the group version of the game, Lagrange’s Theorem is a powerful restriction on the possible structure diagrams [8]. Without this restriction, the spectrum of nim numbers in the convex geometry version could be all nonnegative integers.

(9) Does the Tutte polynomial of the anti-matroid associated with a convex geometry have any information about GEN(S, W). Some connection is expected since both deletion and contraction on a convex geometry have meaningful game theoretic interpretations.
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