MULTIPlicity-FREE REPRESENTATIONS OF CERTAIN NILPOTENT LIE GROUPS OVER SIEGEL DOMAINS OF THE SECOND KIND

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Abstract. We investigate the multiplicity-freeness property for the holomorphic multiplier representations of affine transformation groups of a Siegel domain of the second kind. We deal with the generalized Heisenberg group and its subgroups. Necessary and sufficient conditions for a specific representation to be multiplicity-free are provided. We study the multiplicity-freeness property in relation to the geometrical notions of coisotropic action and visible action, and also the commutativity of the algebra of invariant differential operators.

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1. Introduction

In this article, we study multiplicity-free representations of certain nilpotent affine transformation groups of Siegel domains of the second kind. This class of domains includes (up to holomorphic equivalence) non-tube type bounded homogeneous domains, and as special cases, non-tube type bounded symmetric domains, and we regard these domains as Kähler manifolds by the Bergman metric.

Let $N$ and $M$ be positive integers. For a regular cone $\Omega \neq \emptyset \subset \mathbb{R}^N$ and an $\Omega$-positive Hermitian map $Q : \mathbb{C}^M \times \mathbb{C}^M \to \mathbb{C}^N$, put

$$\mathcal{D}(\Omega, Q) := \{(z, u) \in \mathbb{C}^N \times \mathbb{C}^M \mid \text{Im } z - Q(u, u) \in \Omega\},$$

which is called a Siegel domain of the second kind. For $x_0 \in \mathbb{R}^N$ and $u_0 \in \mathbb{C}^M$, let us denote by $n(x_0, u_0)$ the affine transformation

$$\mathbb{C}^N \times \mathbb{C}^M \ni (z, u) \mapsto (z + x_0 + 2iQ(u, u_0) + iQ(u_0, u_0), u + u_0) \in \mathbb{C}^N \times \mathbb{C}^M.$$
Let $G := \{ n(x_0, u_0) \mid x_0 \in \mathbb{R}, u_0 \in \mathbb{C}^M \}$. The maps $n(x_0, u_0)$ preserve $\mathcal{D}(\Omega, Q)$, and hence $G$ acts on $\mathcal{D}(\Omega, Q)$. We prove the following theorem.

**Theorem 1.1** (see Theorems 3.2, 4.7). For any $G$-equivariant holomorphic line bundle $L$ over $\mathcal{D}(\Omega, Q)$, the natural representation of $G$ on the space $\Gamma^{hol}(\mathcal{D}(\Omega, Q), L)$ of holomorphic sections of $L$ is multiplicity-free.

Let $\pi_0$ be the representation of $G$ on the space $\mathcal{O}(\mathcal{D}(\Omega, Q))$ of holomorphic functions on $\mathcal{D}(\Omega, Q)$ defined by

$$
\pi_0(g)f(z, u) := f(g^{-1}(z, u)) \quad (g \in G, f \in \mathcal{O}(\mathcal{D}(\Omega, Q)), (z, u) \in \mathcal{D}(\Omega, Q)).
$$

The representation $\pi_0$ is a simple example of the representation of $G$ in Theorem 1.1 and at least in this case, we can regard the multiplicity-freeness in a notably strong sense that any unitary representation of $G$ realized in $\mathcal{O}(\mathcal{D}(\Omega, Q))$ is multiplicity-free (see Definitions 4.1, 5.1 and also Remark 4.3). We shall make some comments on related work, which motivated our study of these representations of $G$. Gindikin [7] gives an integral expression for the Bergman kernel of $\mathcal{D}(\Omega, Q)$ (see Theorem 5.4), from which we can see a multiplicity-free direct integral decomposition of the unitary subrepresentation of $\pi_0$ on the space of holomorphic $L^2$-functions on $\mathcal{D}(\Omega, Q)$. When $\mathcal{D}(\Omega, Q)$ is homogeneous, Ishi [8, Section 4] gives the direct integral decompositions of unitary subrepresentations of $\pi_0$, which are the restrictions of representations of a maximal connected real split solvable Lie subgroup of the holomorphic automorphism group of $\mathcal{D}(\Omega, Q)$.

To show Theorem 1.1 we first classify all $G$-equivariant holomorphic line bundles over $\mathcal{D}(\Omega, Q)$. Next, using a description of the multiplicities for the restriction of a unitary representation by Corwin and Greenleaf [4], we show that any infinite dimensional irreducible unitary representation of $G$ realized in $\Gamma^{hol}(\mathcal{D}(\Omega, Q), L)$ is a coherent state representation in the sense of Lisiecki [15]. Accordingly, such a representation has a vector, on which a certain complex Lie algebra acts by scalars. We then complete the proof of Theorem 1.1 by showing such a vector in $\Gamma^{hol}(\mathcal{D}(\Omega, Q), L)$ is determined uniquely up to a constant.

Another motivation of our study of representations of $G$ is geometrical aspects of multiplicity-free representations. For instance, coisotropic action [10], spherical variety [18], and visible action [11, 13] are distinct notions that provide machinery for generating multiplicity-free representations. Not limited to the case of $G$ only, we study certain subgroups of $G$. Let $W \subset \mathbb{C}^M$ be a real subspace, and put

$$
G^W := \{ n(x, u) \mid x \in \mathbb{R}^N, u \in W \}.
$$
We first address the problem of the existence of the strongly visible action (Definition 2.1) of $G^W$.

**Theorem 1.2** (see Theorem 2.4). The action of $G^W$ on $\mathcal{D}(\Omega, Q)$ is strongly visible with respect to an involutive anti-holomorphic diffeomorphism if and only if $W$ contains a real form $W_0$ of $\mathbb{C}^M$ such that

$$(1.1) \quad \text{Im} Q(W_0, W_0) = 0.$$  

We note that, by the multiplicity-free theorem [13, Corollary 2.3], the natural representation $\pi_0|_{G^W}$ is multiplicity-free, assuming the visibility condition, which is a key step in the proof of Theorem 1.2.

Concerning the strongly visible action of a non-reductive Lie group, Kobayashi pointed out in [12] that for a Hermitian symmetric space $H/K$ of noncompact type, the action of a maximal unipotent subgroup of $H$ is strongly visible. Note that our group $G^W$ is unipotent, though it is not, in general, a maximal unipotent subgroup of $H$ in this case. For the strongly visible action of an affine automorphism group containing $G$ as a subgroup, see [1]. In contrast, we deal with smaller groups, and Theorem 1.2 reveals a nontrivial constraint concerning the strongly visible action. Note that recently, several authors establish equivalences between the strong visibility and various multiplicity-free conditions. In [16, Theorem 1.2], Tanaka proved that for a connected complex reductive group $H$ and a spherical $H$-variety $D$, the action of a compact real form of $H$ is strongly visible. For a non-reductive group, Baklouti and Sasaki [2] studied the quasi-regular representations of the Heisenberg group, and related the multiplicity-freeness of such a representation to the strongly visibility of a group action on a complex Heisenberg homogeneous space.

The condition (1.1) can be regarded as a geometrical constraint on $\mathcal{D}(\Omega, Q)$ (see [7, Proposition 1.3]), and we have the following corollary.

**Corollary 1.3.** The action of $G$ on $\mathcal{D}(\Omega, Q)$ is strongly visible with respect to an involutive anti-holomorphic diffeomorphism if and only if $\mathcal{D}(\Omega, Q)$ is holomorphically equivalent to a tube domain.

Theorem 1.1 is a part of a study of the more general group $G^W$, which will be dealt with in another paper. In this article, we focus on the special case that $\dim W = M$. For a complex manifold $D$ and a Lie group $H$ acting holomorphically on $D$, let $\mathcal{D}_H(D)$ be the algebra of $H$-invariant differential operators on $D$ with holomorphic coefficients. We prove the following theorem. While the implications $\text{(v)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(v)}$ of the theorem are included in the previous arguments about Theorem 1.2 and the proof of Theorem 2.4, the remaining
parts $((i) \iff (ii) \iff (iii) \iff (iv))$ constitute one of the main contributions of this article.

**Theorem 1.4.** For a real form $W \subset \mathbb{C}^M$, the following are equivalent.

(i) The representation $\pi_0|_{GW}$ is multiplicity-free;
(ii) $\text{Im} \ Q(W, W) = 0$;
(iii) Any $G_W$-orbit in $\mathcal{D}(\Omega, Q)$ is a coisotropic submanifold;
(iv) $\mathbb{D}_{GW}(\mathcal{D}(\Omega, Q))$ is commutative;
(v) The action of $G^w$ on $\mathcal{D}(\Omega, Q)$ is strongly visible.

We shall make some remarks. As regards (iii), Huckleberry and Wurzbacher [10, Theorem 7] proved a multiplicity-free theorem for the representation of a compact Lie group on the space of holomorphic $L^2$ sections of an equivariant holomorphic line bundle over a Kähler manifold. Concerning (iv), Faraut and Thomas [5, Theorem 4] gave, for a complex manifold $D$ and a Lie group $H$ acting holomorphically on $D$, some sufficient conditions for $\mathbb{D}_H(D)$ to be commutative.

To prove (i) $\Rightarrow$ (ii), we use the aforementioned result by Corwin and Greenleaf. For the proof of (ii) $\Rightarrow$ (iii), we first show that (ii) implies the visibility of the action of $G^w$. Next we prove (iii), following the arguments for visible actions on Kähler manifolds in [11, §4.3].

The integral expression of the Bergman kernel is also an important ingredient of the proof of (ii) $\iff$ (iii).

This article is organized as follows. In Sect. [2], we fix some notations about Siegel domains and affine transformation groups, and review the notion of visible action. Then we give a proof of Theorem [1.2]. In Sect. [3], we classify isomorphism classes of $G$-equivariant holomorphic line bundles over $\mathcal{D}(\Omega, Q)$. In Sect. [4] we prove Theorem [1.1] based on the classification obtained in Sect. [3]. In Sect. [5], we prove $((i) \iff (ii) \iff (iii) \iff (iv))$ of Theorems [1.4].

### 2. Visible action on Siegel domain

The aim of this section is to provide a proof of Theorem [1.2]. We deduce constraints on $Q$ from certain invariant functions, which also separate the group orbits in $\mathcal{D}(\Omega, Q)$. We will do this by linearizing the problem. Throughout this article, for a Lie group, we denote its Lie algebra by the corresponding Fraktur small letter. Also for a vector space or a Lie algebra $V$ over $\mathbb{R}$, we denote by $V_{\mathbb{C}}$ its complexification $V \otimes_{\mathbb{R}} \mathbb{C}$.

2.1. **Generalized Heisenberg group.** Let $\Omega \subset \mathbb{R}^N$ be a regular cone, that is, a nonempty open convex cone which contains no entire straight line. Let $Q : \mathbb{C}^M \times \mathbb{C}^M \to \mathbb{C}^N$ be an $\Omega$-positive Hermitian map, i.e.
a sesquilinear map satisfying $Q(u, u) \in \overline{\Omega} \setminus \{0\}$ ($u \neq 0$), where $\overline{\Omega}$ is the closure of $\Omega$. The set $G$ is closed under composition, and has the natural structure of a Lie group, which we call the generalized Heisenberg group. We identify the Lie algebra $g$ of $G$ with $\mathbb{R}^N \oplus \mathbb{C}^M$ so that $\exp(x, u) = n(x, u)$ ($x \in \mathbb{R}^N, u \in \mathbb{C}^M$) holds. Then we have the equality
\begin{equation}
[u, v] = 4 \text{Im} Q(u, v) \quad (u, v \in \mathbb{C}^M).
\end{equation}

Put $D(\Omega) := \{z \in \mathbb{C}^N \mid \text{Im} z \in \Omega\}$.

2.2. Strongly visible action and constraints on $Q$. We first recall the notion of visibility. Suppose that a Lie group $H$ acts on a connected complex manifold $D$ by holomorphic automorphisms.

**Definition 2.1** ([11, §3]).

1. We call the action **previsible** if there exists a totally real submanifold $S$ in $D$ and a (non-empty) $H$-invariant open subset $D'$ of $D$ such that $S$ meets every $H$-orbit in $D'$.

2. A previsible action is called **visible** if $J_x(T_xS) \subset T_x(H \cdot x)$ for all $x \in S$, where $J_x \in \text{End}(T_xD)$ denotes the complex structure.

3. A previsible action is called **strongly visible** if there exists an anti-holomorphic diffeomorphism $\sigma$ of $D'$ such that
   
   - $(a) \sigma|_S = \text{id}$,
   - $(b) \sigma$ preserves each $H$-orbit of $D'$.

In this article, our focus is on the case that $D = D(\Omega, Q)$. The standard complex structure on $\mathbb{C}^M$ will be denoted by $j$.

**Remark 2.2.** The condition ([3]) is sufficient to ensure the multiplicity-freeness property for a particular representation, as discussed in [13, Corollary 2.4], and [5, p. 389] when $\sigma$ is involutive.

**Proposition 2.3.** Suppose that the action of $G$ on $D(\Omega, Q)$ is strongly visible. Then there exists an antilinear map $A : \mathbb{C}^M \to \mathbb{C}^M$ such that
\begin{equation}
Q(u, v) = Q(Au, Av) \quad (u, v \in \mathbb{C}^M).
\end{equation}

**Proof.** Let $D' \neq \emptyset \subset D(\Omega, Q)$ be a $G$-invariant open set and $\sigma : D' \to D'$ an anti-holomorphic diffeomorphism preserving all $G$-orbits in $D'$. For $(z, u) \in D'$, we write $\sigma(z, u) = (\sigma_1(z, u), \sigma_2(z, u))$ with $\sigma_1(z, u) \in \mathbb{C}^N, \sigma_2(z, u) \in \mathbb{C}^M$. Put $\overline{D'} := \{(z, u) \in \mathbb{C}^N \times \mathbb{C}^M \mid (\overline{z}, \overline{u}) \in D'\}$. Then for $(z, u, \zeta, \overline{v}) \in D' \times \overline{D'}$, we have
\begin{equation}
\frac{1}{2i}(z - \zeta) - Q(u, v) = \frac{1}{2i}(\sigma_1(\zeta, v) - \overline{\sigma_1(z, u)}) - Q(\sigma_2(\zeta, v), \sigma_2(z, u)).
\end{equation}
Expand $\sigma_1(z, u), \sigma_1(\zeta, v), \sigma_2(z, u),$ and $\sigma_2(\zeta, v)$ into Taylor series around any point $(z_0, u_0, \zeta_0, v_0) \in D' \times D'$ with respect to all the variables $z, u, \zeta,$ and $v$ and compare the coefficients of monomials of the form

$$p(z - z_0, u - u_0)q(\zeta - \zeta_0, v - v_0) \quad (p \text{ and } q \text{ are monomials of degree } 1)$$
onumber on both sides of (2.3). Then we see that there exists an anti-linear map $C_N \oplus C_M \ni (z, u) \mapsto d\sigma_2(z, u) \in C_M$ such that

$$Q(u, v) = Q(d\sigma_2(\zeta, v), d\sigma_2(z, u)).$$

Letting $u = v = 0$ and $z = \zeta,$ we see that $d\sigma_2(z, u) = d\sigma_2(0, u)$ by the $\Omega$-positivity of $Q.$

We shall present the proof of Theorem 1.2, but in a slightly different form with a superfluous condition. Let $W \subset C_M$ be a real subspace. For $\xi \in (\mathbb{R}^N)^*,$ let $\omega_\xi(\cdot, \cdot) : C_M \times C_M \to \mathbb{R}$ be the skew-symmetric bilinear form defined by

$$\omega_\xi(u, v) := \langle \xi, [u, v] \rangle \quad (u, v \in C_M).$$

Put

$$W_{\perp, \omega_\xi} := \{ u \in C_M \mid \omega_\xi(u, w) = 0 \text{ for all } w \in W \}.$$ 

Let

$$\Omega^* := \{ \xi \in (\mathbb{R}^N)^* \mid \langle \xi, y \rangle > 0 \text{ for all } y \in \overline{\Omega} \backslash \{0\} \},$$

which is also a regular cone. The precise statement of a different version of Theorem 1.2 is given as follows.

**Theorem 2.4.** The action of $G^W$ on $D(\Omega, Q)$ is strongly visible with respect to an involutive anti-holomorphic diffeomorphism if and only if $W$ contains a real form $W_0$ of $C_M$ such that

$$W_{\perp, \omega_\xi} \subset W_0 \quad (\xi \in \Omega^*) \quad \text{and} \quad \text{Im } Q(W_0, W_0) = 0.$$

**Proof.** For the “if” part, the map

$$\sigma : C_N \times C_M \ni (x + iy, u) \mapsto (-x + iy, -\overline{u}^W) \in C_N \times C_M \quad (x, y \in \mathbb{R}^N, u \in C_M)$$

defines an anti-holomorphic diffeomorphism of $D(\Omega, Q),$ where $\overline{u}^W$ denotes the complex conjugate of $u$ with respect to the real form $W_0.$ We have

$$n(2x + 4Q(w', w), 2w)(-x + iy, -w + jw') = (x + iy, w + jw') \quad (w, w' \in W_0),$$

which shows that $\sigma$ preserves each $G^W_0$-orbit in $D(\Omega, Q).$ Also, we can confirm other conditions in Definition 2.1 (3).

For the “only if” part, we have

$$W_{\perp, \omega_\xi} \subset W \quad (\xi \in \Omega^*),$$

**Remark.**
which follows from the multiplicity-free theorem (see for instance [3, Theorem 3], [13, Corollary 2.3]) together with Propositions 1.8 and 5.3 below. Also, we have $W + jW = \mathbb{C}^M$, since the action of $G^W$ is visible (see [11, Theorem 4]). Let $\xi \in \Omega^*$ and $P := W \cap jW$. Then we have

$$C^M = W \oplus jW^\perp_\xi, \quad W = W^\perp_\xi \oplus P.$$  

For $u \in \mathbb{C}^M$, let us denote by $u^{\xi}$ the projection of $u$ on $W^\perp_\xi \oplus jW^\perp_\xi$. In the proof of Proposition 2.3 we now get another relation

$$\sigma_2(z, u)^{\xi} - \overline{\sigma_2(\zeta, v)^{\xi}} = v^\xi - u^\xi \quad ((z, u, \zeta, v) \in D' \times D'),$$

where the bar denotes the complex conjugation

$$W^\perp_\xi \oplus jW^\perp_\xi \ni w + jw' \mapsto w - jw' \in W^\perp_\xi \oplus jW^\perp_\xi,$$

and hence $P$ is $A$-stable and we have

$$(Aw)^{\xi} = -\overline{w^{\xi}} \quad (w \in \mathbb{C}^M).$$

We may assume that $A$ is also involutive, and if we put

$$P_0 := \{u \in P \mid Au = -u\},$$

then

$$W_0 := \{u \in \mathbb{C}^M \mid Au = -u\} = W^\perp_\xi \oplus P_0$$

satisfies the desired properties. Indeed, it follows from (2.2) that

$$Q(w, w') = Q(Aw', Aw) = Q(w', w) \quad (w, w' \in W_0).$$

\[\square\]

**Remark 2.5.** For a certain class of domains, the multiplicity-freeness property of $\pi_0|_{G^W}$ is represented by

$$\text{Im} \; Q(W^\perp_\xi, W^\perp_\xi) = 0 \quad (\xi \in \Omega^*),$$

which will be dealt with in another paper. Note that (2.5) is always satisfied for $W = \mathbb{C}^M$ and Theorem 1.1 is consistent with this fact. On the other hand, the condition in Theorem 2.4 coincides with (2.5) when $\dim W = M$.

3. **Classification of equivariant holomorphic line bundles**

Our aim in this section is to classify the isomorphism classes of $G$-equivariant holomorphic line bundles over $\mathcal{D}(\Omega, Q)$ (see Theorem 3.2 below). Our proof is based on the Oka-Grauert principle, which shows that any holomorphic line bundles over $\mathcal{D}(\Omega, Q)$ is holomorphically trivial, and the classifying problem descends to the one for holomorphic multipliers.
Definition 3.1. A smooth function \( m : G \times D(\Omega, Q) \to \mathbb{C}^\times \) is called a multiplier if the following cocycle condition is satisfied:
\[
m(gg', (z, u)) = m(g, g'(z, u))m(g', (z, u)) \quad (g, g' \in G, (z, u) \in D(\Omega, Q)).
\]
Moreover, a multiplier \( m \) is called a holomorphic multiplier if \( m(g, (z, u)) \) is holomorphic in \((z, u) \in D(\Omega, Q)\).

For a holomorphic multiplier \( m : G \times D(\Omega, Q) \to \mathbb{C}^\times \), we define a \( G \)-equivariant holomorphic line bundle over \( D(\Omega, Q) \). Let us consider the action of \( G \) on the trivial bundle \( D(\Omega, Q) \times \mathbb{C} \) given by
\[
D(\Omega, Q) \times \mathbb{C} \ni ((z, u), \zeta) \mapsto (g(z, u), m(g, (z, u))\zeta) \in D(\Omega, Q) \times \mathbb{C} \quad (g \in G).
\]
By Chen [3], the domain \( D(\Omega, Q) \) is a Stein manifold. By the Oka-Grauert principle, every \( G \)-equivariant holomorphic line bundle over \( D(\Omega, Q) \) is isomorphic to the one defined as in (3.1). Put \( h(u, v) := \sum_{n=1}^{M} u_n v_n (u, v \in \mathbb{C}^M) \). We have the following theorem.

Theorem 3.2. The \( G \)-equivariant bundles defined as in (3.1) with the one-dimensional representations
\[
m_c : G \ni n(x_0, u_0) \mapsto e^{h(c, u_0)} \in \mathbb{C}^\times \quad (c \in \mathbb{C}^M)
\]
form a complete set of representatives of the isomorphism classes of \( G \)-equivariant holomorphic line bundles over \( D(\Omega, Q) \).

Fix \( p \in \Omega \). To prove Theorem 3.2 we first see that \( m_c (c \in \mathbb{C}^M) \) define mutually nonisomorphic \( G \)-equivariant holomorphic line bundles. Suppose that the representations \( m_c, m_{c'} (c, c' \in \mathbb{C}^M) \) define isomorphic \( G \)-equivariant holomorphic line bundles. According to the next lemma, we have
\[
m_{c'}(g, (z, u)) = f_{c,c'}(g(z, u))f_{c,c'}(z, u)^{-1}m_c(g, (z, u)) \quad (g \in G)
\]
for some holomorphic function \( f_{c,c'} : D(\Omega, \mathbb{C}) \to \mathbb{C}^\times \). The lemma below will be used also in the proof of Theorem 3.2.

Lemma 3.3 ([9, Lemma 1]). The trivial line bundles over \( D(\Omega, Q) \) together with \( G \)-actions (3.1) for two holomorphic multipliers \( m \) and \( m' \) are isomorphic as \( G \)-equivariant holomorphic line bundles if and only if there exists a holomorphic function \( f : D(\Omega, Q) \to \mathbb{C}^\times \) such that
\[
m'(g, (z, u)) = f(g(z, u))f(z, u)^{-1}m(g, (z, u)) \quad (g \in G).
\]

For a \( \mathbb{C}^N \times \mathbb{C}^M \)-valued function \( X(z, u) \) on \( D(\Omega, Q) \), we denote by \( D_X \) the differential operator defined by
\[
D_X f(z, u) := \left. \frac{d}{dt} \right|_{t=0} f((z, u) + tX(z, u)).
\]
Since \( f_{c,c'} \) is holomorphic, for any \( u_0 \in \mathbb{C}^M \), we have
\[
(D_{(0,u_0)} + iD_{(0,iu_0)}f_{c,c'})(ip,0) = 0.
\]
For a Lie group \( H \), let \( dR \) denote the representation of \( \mathfrak{h}_\mathbb{C} \) on \( C^\infty(H) \) defined by
\[
dR\left(a_1 + ia_2\right)f_0(g) := \left. \frac{d}{dt} \right|_{t=0} f_0(ge^{ta_1}) + i \left. \frac{d}{dt} \right|_{t=0} f_0(ge^{ta_2}) \quad \left( f_0 \in C^\infty(H), a_1, a_2 \in \mathfrak{h} \right).
\]
Then it follows from (3.2) that
\[
2h(c', u_0) = dR(u_0 + iju_0)m_{c'}(n(0,0))
\]
\[
= (D_{(0,u_0)} + D_{(0,iju_0)}f_{c,c'}(ip,0)f_{c,c'}(ip,0)^{-1} + dR(u_0 + iju_0)m_{c}(n(0,0))
\]
\[
= dR(u_0 + iju_0)m_{c}(n(0,0)) = 2h(c, u_0),
\]
which implies that \( c = c' \).

\textbf{Proof of Theorem 3.2:} We have
\[
m(n(x_0, u_0), (z, u))
\]
\[
= m(n(x_0 + 2 \text{Im } Q(u_0, u), u_0 + u), (z - iQ(u, u), 0))
\]
\[
\quad \cdot m(n(0, u), (z - iQ(u, u), 0))^{-1}
\]
\[
(3.4)
\]
\[
= m(0, u_0 + u), (z - iQ(u, u) + x_0 + 2 \text{Im } Q(u_0, u), 0)
\]
\[
\quad \cdot m(n(x_0 + 2 \text{Im } Q(u_0, u), 0), (z - iQ(u, u), 0))
\]
\[
\quad \cdot m(n(0, u), (z - iQ(u, u), 0))^{-1}
\]
\[
= m'(n(x_0, u_0)(z, u))m'(z, u)^{-1},
\]
where we put
\[
m'(z, u) := m(0, u), (z - iQ(u, u), 0))m(n(x, 0), iy - iQ(u, u)) \quad (z = x + iy, x, y \in \mathbb{R}^N).
\]

Let \( f_m(z, u) := \log m'(z, u) \) be defined on a neighborhood of a point \((x' + iy', u') \in \mathcal{D}(\Omega, Q)\). In what follows, if necessary, we may take a smaller neighborhood as the domain of definition of \( f_m \). We have
\[
f_m(n(x_0, u_0)(z, u)) - f_m(z, u) = \log m(n(x_0, u_0), (z, u)),
\]
and hence
\[
(3.6)
\]
\[
\frac{\partial}{\partial z_1} f_m(n(x_0, u_0)(z, u)) = \frac{\partial}{\partial z_1} f_m(z, u).
\]
There exists \( G_1 \in C^\infty(\Omega) \) such that \( \frac{\partial}{\partial t} f_m(z, u) = G_1(y - Q(u, u)) \), and if we put
\[
F_1(y) := -2i \int_{(y' - Q(u', u'))(y, y_2, y_3, \cdots, y_N)} G_1(t, y_2, y_3, \cdots, y_N) dt,
\]
then it follows that there exists a locally defined smooth function $K_1$ holomorphic in $z_1$ such that

$$f_m(z,u) = F_1(y - Q(u,u)) + K_1(z,u).$$

Then we get inductively that for $k = 1, 2, \cdots, N$, there exist $F_k \in C^\infty(\Omega)$ and a locally defined smooth function $K_k$ holomorphic in $z_1, z_2, \cdots, z_k$ such that

$$f_m(z,u) = F_k(y - Q(u,u)) + K_k(z,u).$$

Put $F := F_N$ and $K := K_N$.

Now (3.5) reads

$$K(n(x_0, u_0)(z,u)) - K(z,u) = \log m(n(x_0, u_0), (z,u))$$

and since $K$ is holomorphic in $z$, we have

$$\frac{\partial}{\partial u_\alpha} K(n(x_0, u_0)(z,u)) = \frac{\partial}{\partial u_\alpha} K(z,u) \quad (\alpha = 1, 2, \cdots, M).$$

Thus for $\alpha = 1, 2, \cdots, M$, there exists $H_\alpha \in C^\infty(\Omega)$ such that $\frac{\partial}{\partial u_\alpha} K(z,u) = H_\alpha(y - Q(u,u))$, and hence $\frac{\partial}{\partial u_\alpha} K(z,u) = c_\alpha \in \mathbb{C}$, as $\frac{\partial}{\partial u_\alpha} K(z,u)$ is holomorphic in $z$. It follows that $K(z,u) = h(c,u) + L(z,u)$ with $L$ holomorphic, $c \in \mathbb{C}^M$. Hence

$$f_m(z,u) = F(y - Q(u,u)) + h(c,u) + L(z,u).$$

Since $L$ is holomorphic and $\mathcal{D}(\Omega, Q)$ has a trivial homology, the locally defined functions $L$ and $F$ extend to whole $\mathcal{D}(\Omega, Q)$ in such a way that

$$e^{F(y - Q(u,u)) + L(z,u) + h(c,u)} = m'(z,u).$$

Consequently, by (3.4), we have

$$m(n(x_0, u_0), (z,u)) = e^{h(c,u_0) + L(n(x_0, u_0)(z,u)) - L(z,u)}.$$

By Lemma 3.3 our assertion follows.

4. **Representations of the generalized Heisenberg group**

In this section we show Theorem 1.1 and classify all irreducible unitary representations of $G$ realized in $\mathcal{O}(\mathcal{D}(\Omega, Q))$. According to the classification in the previous section, we can focus ourselves on representations having quite simple descriptions.

4.1. **Restriction to the center** $Z(G)$. For a complex manifold $D$, we denote by $\mathcal{O}(D)$ the space of holomorphic functions on $D$. Let $c \in \mathbb{C}^M$ and $\pi_c$ a linear representation of $G$ on $\mathcal{O}(\mathcal{D}(\Omega, Q))$ given by

$$\pi_c(g)f(z,u) := e^{h(c,u_0)}f(g^{-1}(z,u)) \quad (f \in \mathcal{O}(\mathcal{D}(\Omega, Q)), g = n(x_0, u_0) \in G).$$
Definition 4.1 (cf. [5]). For a unitary representation $(\pi, \mathcal{H})$ of $G^W$, we say $\pi$ is realized in $\mathcal{O}(\mathcal{D}(\Omega, Q))$, or $\mathcal{H}$ is a $G^W$-invariant Hilbert subspace of $\mathcal{O}(\mathcal{D}(\Omega, Q))$, given an injective continuous $G^W$-intertwining operator between $\pi$ and $\pi_c$ with respect to the compact-open topology of $\mathcal{O}(\mathcal{D}(\Omega, Q))$. In this case, we use the terminology ‘irreducible $G^W$-invariant Hilbert subspace’ when $\pi$ is irreducible.

We consider the following condition for $\pi_c|_{G^W}$.

Definition 4.2 (cf. [5], [17]). We say $\pi_c|_{G^W}$ is multiplicity-free if any two irreducible $G^W$-invariant Hilbert subspaces of $\mathcal{O}(\mathcal{D}(\Omega, Q))$ either coincide as linear spaces, and have proportional inner products, or they yield inequivalent representations of $G^W$.

Remark 4.3. According to [5, Theorem 2], when $c = 0$, the notion of multiplicity-freeness in Definition 4.2 is equivalent to the following:

Any unitary representation of $G^W$ realized in $\mathcal{O}(\mathcal{D}(\Omega, Q))$ is multiplicity-free.

For the definition of multiplicity-free unitary representation, see Definition 5.1 below.

For $\xi \in \mathfrak{g}^*$, we denote by $\pi_\xi$ an irreducible unitary representation of $G$ corresponding to the coadjoint orbit through $\xi$ by the Kirillov-Bernat correspondence. Suppose that an irreducible unitary representation $(\pi, \mathcal{H})$ of $G$ is realized in $\mathcal{O}(\mathcal{D}(\Omega, Q))$ with $\Phi_0 : \mathcal{H} \hookrightarrow \mathcal{O}(\mathcal{D}(\Omega, Q))$ and is equivalent to $\pi^\nu$ with $\nu \in \mathfrak{g}^*$. Let $K^\mathcal{H}$ be the reproducing kernel defined by $\Phi_0$, and put $K^\mathcal{H}_{(ip,0)}(z,u) := K^\mathcal{H}(z,u,ip,0)$. For $a = a_1 + ia_2 \in \mathfrak{g}_C$ with $a_1, a_2 \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, we write $\langle \xi, a \rangle = \langle \xi, a_1 \rangle + i \langle \xi, a_2 \rangle$. Let us denote by $\text{Ad}^*$ the coadjoint representation of $G$.

Proposition 4.4. We have $K^\mathcal{H}_{(ip,0)}(z,u) = e^{-i\langle \nu,z \rangle} F(u)$ for some $F \in \mathcal{O}(\mathbb{C}^M)$.

Proof. One has

\[(\text{Ad}^*(g) \xi)|_{\mathfrak{z}(g)} = \xi|_{\mathfrak{z}(g)} \quad (g \in G)\].

From (1.1) and the description of the decomposition of an irreducible unitary representation of a nilpotent Lie group [4], we can write the disintegration of $\pi^\nu|_{Z(G)}$ as

$$\pi^\nu|_{Z(G)} \simeq \int_{\widehat{Z(G)}} n(\sigma) \sigma d\mu(\sigma) \simeq n(\sigma^\nu) \sigma^\nu,$$

with $\mu$ a Borel measure on the unitary dual $\widehat{Z(G)}$, $n(\sigma) = 0, 1, 2, \cdots, \infty$, and $\sigma^\nu$ a unitary representation of $Z(G)$ corresponding to the orbit $\{\nu|_{\mathfrak{z}(g)}\} \subset \mathfrak{z}(\mathfrak{g})^*$. On the other hand, the commutativity of $Z(G)$ implies
that every irreducible $Z(G)$-invariant Hilbert subspace of $O(D(\Omega, Q))$ is one-dimensional and given by the linear span of a function $e^{-i\langle \eta, z \rangle}F(u)$ with $\eta \in (\mathbb{R}^N)^*$ and $F \in O(\mathbb{C}^M)$. Note that the $Z(G)$-module corresponds to the orbit $\{\eta\} \subset \mathfrak{g}$. Therefore we can take an orthonormal basis $\{e_k\}_{k=1}^{n(\sigma^\nu)}$ of $\mathcal{H}$ such that $e_k(z, u) = e^{-i\langle \nu, z \rangle}F_k(u)$ with $F_k \in O(\mathbb{C}^M)$. Then we have for $(w, v) \in D(\Omega, Q)$

$$K^H(z, u, w, v) = \sum_{k=1}^{n(\sigma^\nu)} e_k(z, u)\overline{e_k(w, v)} = e^{i\langle \nu, z-w \rangle} \sum_{k=1}^{n(\sigma^\nu)} F_k(u)\overline{F_k(v)},$$

which implies the assertion.

\[\Box\]

4.2. Coherent state representation. For a unitary representation $(\tau, \mathcal{W})$ of $G$, let us regard the projective space $\mathbb{P}(\mathcal{W})$ as a (possibly infinite-dimensional) Kähler manifold and consider the action of $G$ given by $\mathbb{P}(\mathcal{W}) \ni [v] \mapsto [\tau(g)v] \in \mathbb{P}(\mathcal{W})$ ($g \in G$).

Definition 4.5 ([13, Definition 4.2]). We call $\tau$ a coherent state representation (CS representation for short) of $G$ if there exists a $G$-orbit in $\mathbb{P}(\mathcal{W})$ which is a complex submanifold of $\mathbb{P}(\mathcal{W})$ and does not reduce to a point.

The reproducing kernel $K^H$ is $G$-invariant, and hence one has

$$d\pi_c(a) K^H_{(ip, 0)} = -2h(u, c)K^H_{(ip, 0)}$$

for $a = u - iju$ with $u \in \mathbb{C}^M$, where we abbreviate $\pi_c|_H$ to $\pi_c$ and extend the differential representation $d\pi_c$ to a representation of $\mathfrak{g}_\mathbb{C}$ by the complex linearity. Let $\mathfrak{h}_-$ be the complex subalgebra of $\mathfrak{g}_\mathbb{C}$ given by

$$\mathfrak{h}_- := \mathbb{C}^N \oplus \{u + iju \mid u \in \mathbb{C}^M\} \subset \mathfrak{g}_\mathbb{C}.$$ 

Put

$$\tilde{\nu}(x, u) := \langle \nu, x \rangle + 2\text{Im} h(c, u) \ (x \in \mathbb{R}^N, u \in \mathbb{C}^M).$$

By (1.2) and Proposition 4.4, we see that $K^H_{(ip, 0)}$ solves the following equation:

$$d\pi_c(a) f = i(\tilde{\nu}, a)f \ (a \in \mathfrak{h}_-, f \in \mathcal{H}),$$

where the bar denotes the complex conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. By [14, 2., Proposition], this implies that $\pi$ is a coherent state representation of $G$, and the linear form $\xi = -\tilde{\nu}$ satisfies

$$-i\langle \xi, [a, \overline{a}] \rangle = 8\langle \xi, Q(u, u) \rangle \geq 0$$
for all \( a = u + iju \) with \( u \in \mathbb{C}^M \). Here we note that the moment map 
\( \mu_\pi : \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}^* \) defined by 
\[ \langle \mu_\pi([\psi]), x \rangle = \frac{1}{i} \frac{(d\pi(x)\psi, \psi)_\mathcal{H}}{(\psi, \psi)_\mathcal{H}} \] 
(\( x \in \mathfrak{g}, \psi \in \mathcal{H}^\infty \)) 
satisfies 
\[ \langle \mu_\pi([K^\mathcal{H}_{ip,0}]), a \rangle = \langle \tilde{\nu}, a \rangle \] 
for all \( a \in \mathfrak{h}^- \). Hence we have 
\( \mu_\pi([K^\mathcal{H}_{ip,0}]) = \tilde{\nu} \). We have the following theorem.

**Theorem 4.6.** Any infinite dimensional irreducible unitary representation of \( G \) realized in \( \mathcal{O}(\mathcal{D}(\Omega, Q)) \) is a CS representation.

### 4.3. Multiplicity-freeness.

Now we show that \( f(z, u) = e^{-i\langle \nu, z \rangle}e^{h(u,c)} \) is only the solution of (4.3) up to a constant. First, for \( a = u_0 - iju_0 \) with \( u_0 \in \mathbb{C}^M \), the equation (4.3) tells us that 
\[ (D(-2Q(u,u_0),-u_0) - iD(-2Q(u,ju_0),-ju_0)) f(z, u) = -2h(u_0, c)f(z, u). \]
Since \( f \) is holomorphic in \( z \) and \( Q \) is Hermitian, it follows that 
\[ (D(0,u_0) - iD(0,ju_0)) f(z, u) = 2h(u_0, c)f(z, u), \]
which implies that there exists \( F \in \mathcal{O}(\mathcal{D}(\Omega)) \) such that 
\[ f(z, u) = F(z)e^{h(u,c)}. \]
Next, for \( a = x_0 \in \mathbb{R}^N \), the equation (4.3) can be read as 
\[ D(-x_0,0) f(z, u) = i\langle \nu, x_0 \rangle f(z, u), \]
from which we can see that 
\[ f(z + x_0, u) = e^{-i\langle \nu, x_0 \rangle}f(z, u). \]
Combining (4.5) with (4.6), we have 
\[ f(z, u) = F(z)e^{h(u,c)} = F(ip + z - ip)e^{h(u,c)} \]
\[ = F(ip)e^{-i\langle \nu, z - ip \rangle}e^{h(u,c)}, \]
by the analytic continuation. Thus \( f(z, u) = e^{-i\langle \nu, z \rangle}e^{h(u,c)} \) is only the solution of (4.3) up to a constant. We obtain the following theorem.

**Theorem 4.7.** If an irreducible unitary representation \( (\pi, \mathcal{H}) \) of \( G \) corresponding to \( \nu \in \mathfrak{g}^* \) is realized in \( \mathcal{O}(\mathcal{D}(\Omega, Q)) \), then the reproducing kernel satisfies 
\[ K^\mathcal{H}_{ip,0}(z, u) = e^{-i\langle \nu, z \rangle}e^{h(u,c)} \]
up to a constant.

As a corollary of Theorem 4.7, we now show Theorem 1.1.
4.4. Classification of irreducible invariant Hilbert subspaces.
Let $\xi \in \mathfrak{g}^*$ be satisfying $|\xi|_{C^M} = 0$, which may be expressed as $\xi \in (\mathbb{R}^N)^*$, and (4.4). Put

$$N_\xi := \{u \in \mathbb{C}^M \mid \langle \xi, Q(u, u) \rangle = 0 \} \quad \text{and} \quad N_\xi^\perp := \{u \in \mathbb{C}^M \mid h(u, N_\xi) = 0 \}.$$ For a finite dimensional real vector space $V$ the pushforward measure of the Lebesgue measure. Put

$$\pi := \pi \circ \Phi \circ \Phi^{-1} \circ \pi,$$

where $\pi := d\lambda_{N_\xi^\perp}$ is normalized so that $\int_{N_\xi^\perp} e^{-2(\langle \xi, Q(u, u) \rangle)} d\lambda_{\xi}(u) = 1$.

For $s \in \mathbb{C}^M$, let $V_{s,\xi}$ be the representation of $G$ on $\mathcal{F}_\xi$ defined by

$$V_{s,\xi}(n(x_0, u_0)) F(u) := e^{-i(\langle \xi, x_0 \rangle) + 2i \text{Im} h(s, u_0)} e^{-\langle \xi, Q(u_0, u_0) \rangle} e^{2(\langle \xi, Q(u, u) \rangle)} F(u - u_0)$$

for all $u = (\mathbb{R}^N, u_0 \in \mathbb{C}^M)$.

**Proposition 4.8.** The operator $\Phi_{\xi} \in \mathcal{L}(\mathcal{F}_\xi)$ intertwines $V_{s,\xi}$ with $\pi_c$.

**Proof.** We have

$$\pi_c(\exp x_0) \Phi_{\xi} F(z, u) = \Phi_{\xi} F(z - x_0, u) = e^{h(u,c)} e^{\langle \xi, i(z - x_0) \rangle} F(u)$$

$$= e^{h(u,c)} e^{\langle \xi, iz \rangle} V_{s,\xi}(\exp x_0) F(u) = \Phi_{\xi} V_{s,\xi}(\exp x_0) F(z, u).$$

Also, we have

$$\pi_c(\exp u_0) = e^{h(u - u_0, c)} e^{h(c, u_0)} e^{\langle \xi, iz + 2Q(u, u_0) - Q(u_0, u_0) \rangle} F(u - u_0)$$

$$= e^{h(u,c)} e^{\langle \xi, iz \rangle} V_{s,\xi}(\exp u_0) F(u) = \Phi_{\xi} V_{s,\xi}(\exp u_0) F(z, u).$$

This completes the proof. \qed

Let $X_{s,\xi} \in \mathfrak{g}^*$ be given by

$$X_{s,\xi}(x, u) := -\langle \xi, x \rangle + 2 \text{Im} h(s, u).$$

For $u \in \mathbb{C}^M$, let $u_\xi \in N_\xi$, and $u_{\xi}^\perp \in N_{\xi}^\perp$ be the orthogonal projections of $u$ on $N_\xi$ and $N_\xi^\perp$, respectively.
Lemma 4.9. The image of the coadjoint orbit \( \text{Ad}^*(G)X_{\xi,s} \subset g^* \) under the Kirillov-Bernat map is the unitary equivalence class of \( V_{\xi,s} \).

Proof. First we reconstruct \( V_{\xi,s} \) via the Auslander-Kostant theory. Let
\[
p := \mathbb{C}^N \oplus (N_\xi)_C \oplus \{ u + i u \ | \ u \in N_\xi^1 \}, \quad \mathfrak{d} := p \cap g, \quad \text{and} \quad D := Z(G) \exp(N_\xi).
\]
Then \( p \) is a positive polarization of \( g \) at \( X_{\xi,s} \in g^* \) satisfying the Pukanszky condition. Let \( H(X_{\xi,s}, p, G) \) be the space of smooth functions \( \phi \) on \( G \) satisfying
\[
(4.7) \quad \phi(g \exp b) = e^{-i \langle X_{\xi,s}, b \rangle} \phi(g) \quad (g \in G, b \in \mathfrak{d}),
\]
and
\[
(4.8) \quad \int_{G/D} |\phi|^2 d\mu_{G/D} < \infty,
\]
and
\[
(4.9) \quad dR(q) \phi = -i \langle X_{\xi,s}, q \rangle \phi \quad (q \in p),
\]
where \( \mu_{G/D} \) denotes a nonzero \( G \)-invariant measure on \( G/D \). The holomorphic induced representation \( \rho = \rho(X_{\xi,s}, p, G) \) is defined by
\[
\rho(g) \phi(g_0) := \phi(g^{-1}g_0) \quad (\phi \in H(X_{\xi,s}, p, G), g, g_0 \in G).
\]
Let \( \Psi_{\xi,s} : H(X_{\xi,s}, p, G) \to \mathcal{O}(\mathbb{C}M) \) be given by
\[
\Psi_{\xi,s} \phi(u) = e^{i \langle \xi, Q(u, u) \rangle} \phi(n(0, u^\xi)) \quad (u \in \mathbb{C}M).
\]
Then \( \Psi_{\xi,s} \) gives a \( G \)-intertwining operator from \( H(X_{\xi,s}, p, G) \) onto \( F_{\xi} \).

The reproducing kernel \( K^H \), which is partially determined in Theorem \[4.7\], coincides with the reproducing kernel defined by \( \Phi_\xi \) with \( \xi|_{\mathfrak{d}(g)} = -\nu|_{\mathfrak{d}(g)} \), up to a constant. Thus if we put
\[
P := \{ \xi \in g^* \ | \ \xi \text{ satisfies } \xi|_{\mathbb{C}M} = 0 \text{ and } (4.4) \},
\]
then we have the following.

Corollary 4.10. \( \{ V_{\xi,s} \}_{\xi \in P} \) is a set of complete representatives of the equivalence classes of irreducible unitary representations of \( G \) realized in \( \mathcal{O}(\mathcal{D}(\Omega, Q)) \).

We shall complete the classification of \( V_{\xi,s} \) \( (\xi \in P, s \in \mathbb{C}M) \).

Proposition 4.11. For \( s, t \in \mathbb{C}M \), the representations \( V_{\xi,s} \) and \( V_{\xi,t} \) of \( G \) are unitarily equivalent if and only if \( s - t \in N_\xi^1 \). When this condition is satisfied, the following operator \( \psi_{s,t} \) intertwines \( (V_{\xi,s}, F_{\xi}) \)
with \((V_{\xi,t}, F_{\xi})\):

\[
\psi_{s,t} F(u) := \int_{N_{\xi}^{\perp}} F(v) e^{2i \Im h(t-s,v)} e^{2i (\xi, Q(u,v))} e^{-2i (\xi, Q(v,v))} d\lambda_{\xi}(v).
\]

**Proof.** For \(u, w \in \mathbb{C}^M\) and \(F \in F_{\xi}\), we have

\[
\psi_{n,t} V_{\xi,s}(n(0, w)) F(u) = e^{2i \Im h(s,w)} e^{- (\xi, Q(w,w))} \cdot \int_{N_{\xi}^{\perp}} F(v - w) e^{2i \Im h(t-s,v) + 2i (\xi, Q(v,w))} e^{-2i (\xi, Q(v,v))} d\lambda_{\xi}(v) = e^{2i \Im h(s-t,w)} V_{\xi,t}(n(0, w)) \psi_{s,t} F(u).
\]

This shows the “if” part of the statement. Conversely, suppose that \(V_{\xi,s} \simeq V_{\xi,t}\). By the “if” part of the statement and Lemma 1.9, we see that the unitary equivalence class of \(V_{\xi,s}\) corresponds to \(\text{Ad}^*(G) X_{\xi,s}\). From the equality \(\text{Ad}^*(G) X_{\xi,s} = \text{Ad}^*(G) X_{\xi,t}\) we see that there exists \(u \in \mathbb{C}^M\) such that

\[
\Im h((s-t)\xi, v) = (\xi, [u, v]) \quad (v \in \mathbb{C}^M).
\]

Putting \(v = j(s-t)\xi\in \mathbb{C}^M\) in the above equality, we see that \(s-t \in N_{\xi}^{\perp}\). \(\square\)

### 5. Subgroups of the Generalized Heisenberg Group

In this section we study several properties that ensure the multiplicity-freeness of \(\pi_0|_{G^W}\), especially in the case that \(\dim W = M\) as in Theorem 1.4.

#### 5.1. Multiplicity-free unitary representation.

**Definition 5.1.** For a unitary representation \((\tau, W)\) of \(G^W\), we say \(\tau\) is *multiplicity-free* if the ring \(\text{End}_{G^W}(W)\) of continuous endomorphism commuting with \(G^W\) is commutative.

**Remark 5.2.** The notion of multiplicity-freeness in Definition 5.1 can be rephrased in terms of the multiplicity function determined by the direct integral decomposition (see [11, Proposition 1.5.1]).

**Proposition 5.3.** Let \(\xi \in P\). The unitary representation \(V_{\xi,c}\) is multiplicity-free when restricted to the subgroup \(G^W \subset G\) if and only if for any \(u \in W^{\perp \omega_{\xi}}\) there exists \(w \in W\) such that \(u + w \in N_{\xi}\).
Proof. Let proj : \( g^* \rightarrow (g^W)^* \) be the natural projection. We can write disintegration of \( V_{\xi,c}|_{GW} \) as

\[
V_{\xi,c}|_{GW} \simeq \int_{GW} n(\sigma)\sigma d\mu(\sigma)
\]

with \( \mu \) a Borel measure on \( \hat{GW} \) and \( n(\sigma) \) the number of \( G^W \)-orbits in \( \text{proj}^{-1}(\text{Ad}^*(G^W)(\eta + \eta')) \cap \text{Ad}^*(G)(-\xi) \) when \( \sigma \) corresponds to

\[
\text{Ad}^*(G^W)(\eta + \eta') \in (\mathbb{R}^N)^* \oplus W^* \quad (\eta \in (\mathbb{R}^N)^*, \eta' \in W^*)
\]

by the Kirillov-Bernat map. For \( u \in \mathbb{C}^M \), let \( \xi_u \subset (\mathbb{C}^M)^* \) be given by

\[
\langle \xi_u, u' \rangle := \langle \xi, [u, u'] \rangle \quad (u' \in \mathbb{C}^M).
\]

We have

\[
\text{Ad}^*(G)X_{\xi,c} = \{ X_{\xi,c} + \xi_u \in (\mathbb{R}^N)^* \oplus (\mathbb{C}^M)^* \mid u \in \mathbb{C}^M \}
\]

and

\[
\text{Ad}^*(G^W)(\eta + \eta') = \{ \eta + (\eta' + \eta_w|w) \in (\mathbb{R}^N)^* \oplus W^* \mid w \in W \}
\]

Suppose \( \text{proj}^{-1}(\text{Ad}^*(G^W)(\eta + \eta')) \cap \text{Ad}^*(G)X_{\xi,c} \neq \emptyset \) and take an element \( X_{\xi,c} + \xi_{u_0} \) with \( u_0 \in \mathbb{C}^M \). Then we have \(-\xi = \eta \) and

\[
\text{proj}^{-1}(\text{Ad}^*(G^W)(\eta + \eta')) \cap \text{Ad}^*(G)X_{\xi,c} = \{ X_{\xi,c} + \xi_u \mid u \in u_0 + W^\perp \omega_c \}
\]

Since the above set contains the coadjoint orbit

\[
\text{Ad}^*(G^W)(X_{\xi,c} + \xi_{u_0}) = \{ X_{\xi,c} + \xi_u \mid u \in u_0 + W \},
\]

the number of \( G^W \)-orbits in the set equals 1 if and only if for any \( u \in W^\perp \omega_c \) there exists \( w \in W \) such that \( u + w \in N_\xi \). Moreover, according to Remark [5.2], the condition holds if and only if \( \text{End}_{GW}(V_{\xi,c}) \) is commutative. \( \Box \)

5.2. Coisotropic action and invariant differential operators.

We see an integral expression for the Bergman kernel of \( \mathcal{D}(\Omega, Q) \). Let \( \lambda \) denote the Lebesgue measure or the measure defined on the dual space via the standard inner product. For \( \xi \in \Omega^* \), let

\[
I(\xi) := \int_{\Omega} e^{-2\langle \xi, y \rangle} d\lambda(y) \quad \text{and} \quad I_Q(\xi) := \int_{\mathcal{C}^M} e^{-2\langle \xi, Q(u, u) \rangle} d\lambda(u).
\]

Let \( L^2_a(\mathcal{D}(\Omega, Q)) := L^2(\mathcal{D}(\Omega, Q), \lambda) \cap \mathcal{O}(\mathcal{D}(\Omega, Q)) \).

Theorem 5.4 ([7]). The reproducing kernel \( K \) of \( L^2_a(\mathcal{D}(\Omega, Q)) \) is given by

\[
K(z, u, w, v) := \frac{1}{(2\pi)^N} \int_{\Omega^*} e^{i\langle \xi, z - w - 2iQ(u, v) \rangle} I(\xi)^{-1}I_Q(\xi)^{-1} d\lambda(\xi).
\]
\[ g_{k\ell} = \frac{\partial^2}{\partial z_k \partial \overline{z}_\ell} \log K(z, u, z, u) = -(2\pi)^{-2N} K^{-2}(z, u, z, u) \left\{ (2\pi)^N K(z, u, z, u) \right. \]
\[ \cdot \int_{\Omega^*} \langle \xi, e_k \rangle \langle \xi, e_l \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \]
\[ - \int_{\Omega^*} \langle \xi, e_k \rangle \langle \xi, e_l \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \]
\[ \cdot \int_{\Omega^*} \langle \xi, e_l \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \right\}, \]

(5.1)

\[ g_{k\alpha} = \frac{\partial^2}{\partial z_k \partial u_\alpha} \log K(z, u, z, u) = \frac{1}{(2\pi)^{2N} K(z, u, z, u)^2} \left\{ (2\pi)^N K(z, u, z, u) \right. \]
\[ \cdot \int_{\Omega^*} 2i \langle \xi, e_k \rangle \langle \xi, Q(u, f_\alpha) \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \]
\[ - \left( \int_{\Omega^*} i \langle \xi, e_k \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \right) \]
\[ \cdot \left( \int_{\Omega^*} 2 \langle \xi, Q(u, f_\alpha) \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \right) \right\}, \]

(5.2)

and

\[ g_{\alpha\overline{\beta}} = \frac{\partial^2}{\partial u_\alpha \partial u_\overline{\beta}} \log K(z, u, z, u) = 2(2\pi)^{-N} K^{-1}(z, u, z, u) \]
\[ 2 \int_{\Omega^*} \langle \xi, Q(u, f_\beta) \rangle \langle \xi, Q(f_\alpha, u) \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \]
\[ + \int_{\Omega^*} \langle \xi, Q(f_\alpha, u) \rangle \langle \xi, f_\beta \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \]
\[ - \frac{4}{(2\pi)^{2N} K(z, u, z, u)^2} \int_{\Omega^*} \langle \xi, Q(u, f_\beta) \rangle \langle \xi, Q(f_\alpha, u) \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) \]
\[ \cdot \int_{\Omega^*} \langle \xi, Q(u, f_\beta) \rangle e^{i(\xi, z - \overline{\xi} - 2iQ(u, u))} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi). \]

(5.3)
Now we show Theorem 1.4. Instead of the original condition (i), we shall consider the following condition:

(i') For any \( c \in \mathbb{C}^M \), the representation \( \pi_c|_{G^W} \) is multiplicity-free.

Note that the proof of (i) \( \Rightarrow \) (ii) can be seen from the one of (i') \( \Rightarrow \) (ii) below. For \( u \in \mathbb{C}^M \), we denote by \( \overline{\pi}^W \) the complex conjugation of \( u \) with respect to \( W \subset \mathbb{C}^M \).

Proof of (i) \( \iff \) (ii), (ii) \( \iff \) (iii), (ii) \( \iff \) (iv) of Theorem 1.4 ((i') \( \Rightarrow \) (ii))
Let \( \xi \in \Omega^* \). Then \( \omega_\xi \) is nondegenerate and \( N_\xi = \{ 0 \} \). Owing to Faraut and Thomas [5, Theorem 2], it follows from Proposition 4.8 that \( V_{\xi,0} \) is a multiplicity-free representation of \( G^W \). By Proposition 5.3 we have \( W^{\perp,\omega_\xi} \subset W \). Since \( \dim_{\mathbb{R}} W = \frac{1}{2} \dim_{\mathbb{R}} \mathbb{C}^M \), we have also \( W \subset W^{\perp,\omega_\xi} \), i.e. \( \langle \xi, \text{Im} Q(W, W) \rangle = 0 \). Since this holds for every \( \xi \in \Omega^* \) and \( \Omega^* \) is also a regular cone, we conclude that \( \text{Im} Q(W, W) = 0 \).

(ii) \( \Rightarrow \) (i') From (2.1) we see that \( (5.5) \) and \( (5.6) \) we see that the function \( C_{z,u} \) determined up to a constant and \( \overline{\pi}^W \) from which we conclude that the function \( C_{z,u} \) is uniquely determined up to a constant and \( C_{z,u} \) as in (5.4).

(ii) \( \Rightarrow \) (iii) First we show that the action of \( G^W \) on \( D(\Omega, Q) \) is visible with the totally real submanifold \( S := (i \mathbb{R}^N \times jW) \cap D(\Omega, Q) \).
It is easy to see that every \( G^W \)-orbit meets \( S \). For \( s := (iy, jw) \in S \), we have
\[
n(x + 2Q(w, w'), w')s = (x + iy + iQ(w', w'), jw + w'),
\]
which implies that $J_sT_sS \subset T_sG^W_s$. Next let $T_sS^\perp$ denote the orthogonal complement of $T_sS$ with respect to the Bergman metric. We show that $J_sT_sS \subset T_sS^\perp$, which implies $T_sS^\perp \subset J_sT_sS$ and hence (iii), owing to Kobayashi [11, Theorem 7]. We see from (5.1) that $\text{Im} g_{k\bar{\tau}} = 0 (1 \leq k, l \leq N)$. The equality (5.3) tells us that $\text{Im} g_{a\bar{\beta}} = 0 (1 \leq \alpha, \beta \leq M)$ under the condition (ii). Also (5.2) implies that $\text{Im} g_{k\tau} = 0 (1 \leq k \leq N, 1 \leq \alpha \leq M)$. We conclude that $J_sT_sS \subset T_sS^\perp$.

(iii) $\Rightarrow$ (ii) For $u = 0$, the assumption together with (5.3) implies that

$$\int_{\Omega^*} \langle \xi, \text{Im} Q(f_{\alpha}, f_{\beta}) \rangle e^{-2(\xi,y)} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) = 0 \quad (y \in \Omega, 1 \leq \alpha, \beta \leq M).$$

Taking directional derivative in the direction $\text{Im} Q(f_{\alpha}, f_{\beta}) \in \mathbb{R}^N$, we get

$$\int_{\Omega^*} \langle \xi, \text{Im} Q(f_{\alpha}, f_{\beta}) \rangle^2 e^{-2(\xi,y)} I(\xi)^{-1} I_Q(\xi)^{-1} d\lambda(\xi) = 0 \quad (y \in \Omega, 1 \leq \alpha, \beta \leq M),$$

from which we see that $\text{Im} Q(f_{\alpha}, f_{\beta}) = 0 (1 \leq \alpha, \beta \leq M)$.

(ii) $\Leftrightarrow$ (iv) For vector-valued functions $a : D(\Omega, Q) \to \mathbb{C}^N$ and $b : D(\Omega, Q) \to \mathbb{C}^M$, we shall use the notation

$$a(z, u) \frac{\partial}{\partial z} + b(z, u) \frac{\partial}{\partial u} := \sum_{k=1}^N a_k(z, u) \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^M b_\alpha(z, u) \frac{\partial}{\partial u_\alpha}.$$

For $l = 1, 2, \ldots$, let $D_{G^W_l}(D(\Omega, Q)) \subset D_{G^W_l}(D(\Omega, Q))$ denote the space of $l$-th order invariant differential operators. We first show that

(5.7)

$$D_{G^W_l}(D(\Omega, Q)) = \left\{ \left(2iQ(b, \bar{u}^W) + A\right) \frac{\partial}{\partial z} + b \frac{\partial}{\partial u} \right| A \in \mathbb{C}^N, b \in \mathbb{C}^M \right\}.$$

Let $a(z, u) \frac{\partial}{\partial z} + b(z, u) \frac{\partial}{\partial u} \in D_{G^W_l}(D(\Omega, Q))$. Then for $w \in W$, we have

$$a(n(x, w)(z, u)) = a(z, u) + 2iQ(b(z, u), w),$$

and

(5.8)

$$b(n(x, w)(z, u)) = b(z, u).$$

Letting $x_0 = -x$ in (5.8), we see that $b$ does not depend on $z$, since it does not depend on $x$. Then we can see that $b$ does not depend on $u$ also, i.e. $b \in \mathbb{C}^M$ is a constant function. We have

$$a(z, u) = a(n(x, w)(iy - 2iQ(u, w) + iQ(w, w), jw'))$$

$$= 2iQ(b, w) + a(iy - 2iQ(u, w) + iQ(w, w), jw'),$$
which implies that $a$ does not depend on $z$, and we may write $a(z, u) = a(u) = 2iQ(b, w) + a(jw')$. By the analytic continuation, we see that $a(u) = 2iQ(b, u^W) + A$ for some $A \in \mathbb{C}^N$. Therefore we obtain (5.7).

Next, for $b, d \in \mathbb{C}^M$, we have the following bracket relation of $\mathcal{D}^\mathcal{G}_W(\mathcal{D}(\Omega, Q))$:

\[
\left[ 2iQ(b, u^W) \frac{\partial}{\partial z} + b \frac{\partial}{\partial u}, 2iQ(d, u^W) \frac{\partial}{\partial z} + d \frac{\partial}{\partial u}, \ldots \right] = \{ 2iQ(d, u^W) - 2iQ(b, u^W) \} \frac{\partial}{\partial z},
\]

which shows (iv) $\Rightarrow$ (ii). Conversely, we show that under (ii) the ring $\mathcal{D}_W^\mathcal{G}(\mathcal{D}(\Omega, Q))$ is generated by $\mathcal{D}_1^\mathcal{G}_W(\mathcal{D}(\Omega, Q))$, and hence (iv) holds. We show that each $\mathcal{D}_l^\mathcal{G}_W(\mathcal{D}(\Omega, Q))$ is generated by $\mathcal{D}_1^\mathcal{G}_W(\mathcal{D}(\Omega, Q))$ by induction on $l$. Let $D \in \mathcal{D}_l^\mathcal{G}_W(\mathcal{D}(\Omega, Q))$. Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers, and for $m = (m_1, m_2, \ldots, m_M) \in \mathbb{Z}_{\geq 0}^M$, we put $|m| := m_1 + m_2 + \cdots + m_M$. Then there exist holomorphic functions $f_m (m \in \mathbb{Z}_{\geq 0}^M, |m| = l)$ on $\mathcal{D}(\Omega, Q)$ and $l-1$-th order differential operators $D_k (k = 1, 2, \ldots, N)$, generated by $\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \ldots, \frac{\partial}{\partial z_N}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_M}$, with holomorphic coefficients such that

\[
D = \sum_{\substack{m \in \mathbb{Z}_{\geq 0}^M \cap \mathbb{Z}_0^M, \ |m| = l}} f_m(z, u) \prod_{\alpha=1}^M \left( 2iQ(e_\alpha, u^W) \frac{\partial}{\partial z} + \frac{\partial}{\partial u_\alpha} \right)^{m_\alpha} + \sum_{k=1}^N D_k \frac{\partial}{\partial z_k}.
\]

From this expression, it follows that all $f_m (m \in \mathbb{Z}_{\geq 0}^M, |m| = l)$ are constant functions, and $D_k (k = 1, 2, \ldots, N)$ are $G_W$-invariant, and hence $D$ is generated by $\mathcal{D}_1^\mathcal{G}_W(\mathcal{D}(\Omega, Q))$ by assumption.

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