A nonperturbative coupled-cluster method for quantum field theories

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Abstract

The nonperturbative Hamiltonian eigenvalue problem for bound states of a quantum field theory is formulated in terms of Dirac’s light-front coordinates and then approximated by the exponential-operator technique of the many-body coupled-cluster method. This approximation eliminates any need for the usual approximation of Fock-space truncation. Instead, the exponentiated operator is truncated, and the terms retained are determined by a set of nonlinear integral equations. These equations are solved simultaneously with an effective eigenvalue problem in the valence sector, where the number of constituents is small. Matrix elements can be calculated, with extensions of techniques from many-body coupled-cluster theory, to obtain form factors and other observables.

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I. INTRODUCTION

The nonperturbative solution of quantum field theories is a highly nontrivial task and yet is crucial for the full understanding of a strongly coupled theory such as quantum chromodynamics. Lattice methods have been quite successful in attacking this problem but suffer from the limitation to a Euclidean formulation and the lack of direct access to wave functions. Light-front Hamiltonian methods \[1, 2\] are not as well developed, but they do remain in Minkowski space and hold the promise of obtaining wave functions, as coefficients in Fock-state expansions of the Hamiltonian eigenstates.

The Hamiltonian eigenvalue problem is typically solved by first truncating Fock space. This yields a finite set of coupled equations for a finite collection of wave functions. However, the truncation induces various complications, not the least of which are uncanceled divergences and Fock-sector dependencies \[3\]. The light-front coupled-cluster (LFCC) method \[4\] avoids these complications by not truncating Fock space and instead approximating the relationship between wave functions to obtain a finite set of equations for a still infinite collection of wave functions. There is a price to be paid, of course; the resulting equations are nonlinear and the calculation of matrix elements requires particular care. The latter is greatly aided by the close similarity of the mathematics (though not the physics) of the many-body coupled-cluster method \[5\] for nonrelativistic nuclear physics and quantum chemistry \[6\].

II. OVERVIEW OF THE METHOD

The LFCC method \[4\] is designed to approximate the light-front Hamiltonian eigenvalue problem \[P^+ P^- |\psi\rangle = (M^2 + P^2_{\perp}) |\psi\rangle\] by writing the eigenstate as \[|\psi\rangle = \sqrt{Z} e^T |\phi\rangle\]. Here \[|\phi\rangle\] is a valence state with a fixed, small number of constituents; \[T\] is an operator that increases particle number and conserves all relevant quantum numbers; and \[\sqrt{Z}\] is a normalization factor. The approximation is that \[T\] is truncated to one or a few operators but the Fock space used as a basis for the eigenstate is not truncated. As a consequence of the approximation, wave functions associated with higher Fock states are restricted to what \[T\] can generate from the lower-state wave functions.

To implement this approach, we define an effective Hamiltonian \[P^+ = e^{-T} P^- e^T\] and a projection \[P_v\] onto the valence sector, with \[1 - P_v\] restricted to projection onto just enough sectors to fully constrain the form of \[T\]. The resulting LFCC equations are

\[P_v P^- |\phi\rangle = \frac{M^2 + P^2_{\perp}}{P^+} |\phi\rangle, \quad (1 - P_v) P^- |\phi\rangle = 0. \tag{2.1}\]

The second equation is essentially an auxiliary equation for \[T\], which is needed to define the effective Hamiltonian \[P^-\] used in the first, a valence eigenvalue equation.

Matrix elements are computed from the right and left eigenstates of \[P^-\]. For example, consider the expectation value for an operator \[\hat{O}\]:

\[\langle \hat{O} \rangle = \frac{\langle \phi | e^{T^\dagger} \hat{O} e^T |\phi\rangle}{\langle \phi | e^{T^\dagger} e^T |\phi\rangle}. \tag{2.2}\]

Direct computation requires an infinite sum over Fock space, which is intractable. Instead, we can borrow some mathematics from the many-body coupled-cluster method \[5\]. We
define $\mathcal{O} = e^{-T} \hat{O} e^{T}$ and $\langle \tilde{\psi} \rangle = \langle \phi | \frac{e^{T} e^{T \dagger}}{\langle \phi | e^{T} e^{T \dagger} | \phi \rangle} | \phi \rangle$. The expectation value can then be expressed as $\langle \hat{O} \rangle = \langle \tilde{\psi} | \mathcal{O} | \phi \rangle$, and the ket $| \tilde{\psi} \rangle$ is normalized as

$$
\langle \tilde{\psi}' | \phi \rangle = \langle \phi' | \frac{e^{T} e^{T \dagger}}{\langle \phi | e^{T} e^{T \dagger} | \phi \rangle} | \phi \rangle = \delta (P' - P).
$$

The effective operator $\mathcal{O}$ can be computed from its Baker–Hausdorff expansion, $\mathcal{O} = \hat{O} + [\hat{O}, T] + \frac{1}{2}[[\hat{O}, T], T] + \cdots$. The ket $| \tilde{\psi} \rangle$ is a left eigenvector of $\overline{P^+}$, as can be seen from

$$
\langle \tilde{\psi} | \overline{P^+} = \langle \phi | \frac{e^{T} P^- e^{T \dagger}}{\langle \phi | e^{T} e^{T \dagger} | \phi \rangle} = \langle \phi | \overline{P^-} = \frac{e^{T} e^{T}}{\langle \phi | e^{T} e^{T \dagger} | \phi \rangle} = \frac{M^2 + P_2}{P_{1}} \langle \tilde{\psi} |.
$$

Physical quantities can then be computed from the LFCC eigenstates.

This approach has been applied to a light-front analog of the Greenberg–Schweber model [4, 8], where the lowest-order $T$ produces the exact answer, and to QED [2], in a calculation of the dressed-electron state and its anomalous magnetic moment. In addition, a technique for including zero modes has been developed [3] and applied to $\phi^3$ and $\phi^4$ theories and to the Wick–Cutkosky model. Much of this work is less briefly summarized in various proceedings [10–12].

### III. SUMMARY

The LFCC method provides a nonperturbative Hamiltonian approach to the solution of quantum field theories that avoids Fock-space truncation and the consequent difficulties of uncanceled divergences and sector dependence. It can be applied to any regulated light-front Hamiltonian and is systematically improvable through the addition of terms to the $T$ operator, classified by the net increase in particle number and the number of annihilation operators. Zero modes can be included, to allow consideration of theories with symmetry breaking.

Work on QED continues, with consideration of additional terms to include electron-positron pairs and of the bound states of muonium and positronium. Many interesting applications remain to be considered, with the goal of understanding the method well enough to apply it to QCD. Light-front holographic QCD [13] may be a useful starting point.

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[1] P.A.M. Dirac, *Rev. Mod. Phys.* **21**, 392–399 (1949).
[2] For reviews of light-cone quantization, see M. Burkardt, *Adv. Nucl. Phys.* **23**, 1–74 (2002); S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, *Phys. Rep.* **301**, 299–486 (1998).
[3] S.S. Chabysheva and J.R. Hiller, *Ann. Phys.* **325**, 2435–2450 (2010).
[4] S.S. Chabysheva and J.R. Hiller, *Phys. Lett. B* **711**, 417–422 (2012).
[5] F. Coester, *Nucl. Phys. B* **7**, 421–424 (1958); F. Coester and H. Kümmel, *Nucl. Phys. B* **17**, 477–485 (1960).
[6] For reviews of the many-body coupled-cluster method, see R.J. Bartlett and M. Musial, *Rev. Mod. Phys. B* **79**, 291–352 (2007); T.D. Crawford, and H.F. Schaefer, *Rev. Comp. Chem. B* **14**, 33–136 (2000); R. Bishop, A.S. Kendall, L.Y. Wong, and Y. Xian, *Phys. Rev. D* **48**, 887–901 (1993); H. Kümmel, K.H. Lührmann, and J.G. Zabolitzky, *Phys. Rep.* **36**, 1–63 (1978).
[7] S.S. Chabysheva and J.R. Hiller, “An application of the light-front coupled-cluster method to the nonperturbative solution of QED,” [arXiv:1203.0250 [hep-ph]].
[8] S.J. Brodsky, J.R. Hiller, and G. McCartor, *Phys. Rev. D* **58**, 025005:1–16 (1998); O. Greenberg, and S.S. Schweber, *Nuovo Cimento B* **8**, 378–406 (1958).
[9] S.S. Chabysheva and J.R. Hiller, “Zero modes in the light-front coupled-cluster method,” [arXiv:1208.6076 [hep-ph]].
[10] J.R. Hiller and S.S. Chabysheva, *Few Body Syst. B* **52**, 315–321 (2012); S.S. Chabysheva and J.R. Hiller, *Few Body Syst. B* **52**, 323–329 (2012).
[11] J.R. Hiller, *PoS* (*QNP2012*), 113:1–6 (2012); S. Chabysheva, *PoS* (*QNP2012*), 123:1–6 (2012).
[12] J.R. Hiller, “A nonperturbative light-front coupled-cluster method,” *International Workshop on QCD Theory and Experiment: QCD@Work 2012*, Lecce, Italy, June 18-21, 2012, [arXiv:1208.6077 [hep-ph]]; S.S. Chabysheva, “An illustration of the light-front coupled-cluster method in quantum electrodynamics,” *International Workshop on QCD Theory and Experiment: QCD@Work 2012*, Lecce, Italy, June 18-21, 2012, [arXiv:1208.6080 [hep-ph]].
[13] G.F. de Teramond, and S.J. Brodsky, *Phys. Rev. Lett.* **102**, 081601:1–4 (2009).