The mechanism for the generalized dual problem of network programming solving

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Abstract. The generalized dual problem (GDP) is formulated as a search task of the upper bound minimum (lower bound maximum) for the optimum obtained by solving discrete linear and nonlinear problems by the network programming method. The need for GDP solution arises when, in order to meet the applicability conditions of the method, the right side of the constraint is divided into several unknown summands. The GDP consists in finding of such a partition that ensures achievement of corresponding minimum (maximum). There is no evidence for an existence of the GDP solution for today. However, the conditions for the GDP optimum solution have been formulated. The article proposes an iterative mechanism for search solution of the generalized dual problem. This mechanism is based on the software implementation of the procedure for solving the optimization problem by means of network programming. The application of the mechanism is illustrated on the developing competences task of IT services users. It was shown that the GDP solution exists and is not the only one.

1. Introduction
Let us consider the search mechanism of the GDP optimum by the example of the task of developing competences of IT services users [1, 2]. Let us assume that \{\{p_{ji}\}|i=1,n_{j}|j=1,m}\} be a set of competency developing programs implemented by a consulting company involved in the development and implementation of IT services. Here j – the business process number, i – the number of the training program, \(p_{ji}\) – the \(ji\)-th training program for the \(j\)-th business process, \(n_{j}\) – the number of training programs for the \(j\)-th process, \(m\) – the number of business processes. Let us denote through \(c_{ji}=c(p_{ji})\) the cost of training of one user within the program \(p_{ji}\), through \(q_{ji}^k=q^k(p_{ji})\) - competence "increment" of the \(k\)-th user (estimated in points) as a result of training within the program \(p_{ji}\). We will indicate through \(k_{ji}\) the number of users of the \(j\)-th business process, through \(c^*\) – the maximum limit of funds that the management of a company can spend on the user training [3, 4].
2. The problem formalization of the network programming

We will introduce a discrete variable $x_{ji}^k$, which is equal to 1, if the $k$-th user of the $j$-th process is to be trained in accordance with the program $p_{ji}$, and equal to 0 otherwise. Then the problem of training costs optimal planning is formulated as follows:

$$q = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{k_{ji}} q_{ji}^k \rightarrow \max$$

(1)

$$c = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{k_{ji}} c_{ji}^k x_{ji}^k \leq c^*$$

(2)

$$\sum_{j=1}^{m} \sum_{i=1}^{n_j} x_{ji}^k \geq k_j^*$$,  \hspace{1em} j=1..m.

(3)

The solution of a problem (1) – (3) is a set of user training programs $\{\{x_{ji}^k|i=1,n_j||k=1,k_{ji}||j=1,m\} \}$ that maximizes the summary “increment” of users competences $q$ with a given restraint for the maximum funds limits $c^*$ allocated for training and that meets the specified restrictions on the minimum required number $k_j^*$ of users to be trained (note that cardinal number of the problem solution is equal to $\prod_{j=1}^{m} n_j^h$ [5, 6, 7]). Solution pattern of the problem (1) – (3) by means of network programming method [8, 9] includes successive solution of following evaluating problems:

- The solution of $m$ problems (for each $j=1..m$):

$$q_j = \sum_{i=1}^{n_j} \sum_{k=1}^{k_{ji}} q_{ji}^k \rightarrow \max,$$

(4)

$$c_j = \sum_{i=1}^{n_j} \sum_{k=1}^{k_{ji}} c_{ji}^k x_{ji}^k \leq c_j^*,$$

(5)

$$\sum_{i=1}^{n_j} \sum_{k=1}^{k_{ji}} x_{ji}^k \geq k_j^*.$$

(6)

- Problem solution

$$q = \sum_{j=1}^{m} q_j(x_j) \rightarrow \max$$

(7)

$$c = \sum_{j=1}^{m} c_j(x_j) \leq c^*,$$

(8)

by (m-1)-th successive integration of solution $m$ of estimation problems (4) – (6) (integration of solutions for $j=1$ and $j=2$, integration of a solution obtained with a problem solution for $j=3$ and so on).

Optimal value of the criterion obtained when solving problems (4) – (8) is the lower bound for the optimum value of the initial problem (1) – (3). The difficulty of finding the maximum of the lower bound consist in finding appropriate $c_j^*, j=1..m, \sum_{j=1}^{m} c_j^* \leq c^*$. Determining values $c_j^*$ in nonlinear problems is often one by the Lagrange multiplier method [10], which make it possible to obtain sufficiently
"good" values. Various heuristic techniques producing acceptable estimates from below are used into linear problems. It is interesting to form a mechanism for determining the set of values \( c^*_j, j=1,m \), which maximize estimates from below for the optimum obtained by implementing procedure (4) – (8), in other words solve the generalized dual problem GDP. GDP was first formulated by Burkova I V [11]. She owns also a theorem on the necessary and sufficient conditions for the GDP optimality for the case where the objective functions of estimation problems are linear. However, to date, it has not been proved that the optimal GDP solution exists in the class of linear objective functions of estimation problems.

To check the existence and finding of the GDP solution, the authors used computer-aided modelling, which is based on the software implementation of the solving procedure of problems (4) – (8).

3. The mechanism of a GDP solving by computer-based simulation methods

To find a GDP solution, we will build following iterative procedure.

- Let as assume on the first iteration:
  \[
  c^*_{ij} = c^* \frac{\sum_{i=1}^{n_i} \sum_{k=1}^{k_i} c_{jk}}{\sum_{j=1}^{m} \sum_{i=1}^{n_i} \sum_{k=1}^{k_i} c_{ij}}. 
  \]  
  (9)

According to (9), the budget is distributed in proportion to the training needs required for each user to be trained across all programs. We solve the problem (1) - (3) using the dichotomic programming method for the obtained values \( c^*_{ij}, j=1,m \). Let us assume that the set

\[
\{ \{ x^k_i | i=1,n_j \} | k=1,k_i \}, j=1,m, \theta=1,\theta_1 \}
\]  
(10)

describes the optimal problem solutions found on the first iteration (\( \theta_1 \) –the number of solutions).

- To perform the second iteration, we increase the values of each \( c^*_{ij}, j=1,m \) by a quantity of the maximum cost of the user training program of the corresponding process \( j \):
  \[
  c^*_{ij} = c^*_{ij} + \max_{i=1,n_j} c_{i}, j=1,m. 
  \]  
  (11)

The solution of problem (1) – (3) for new values \( c^*_{ij}, j=1,m \). Let us assume that the set

\[
\{ \{ x^k_i | i=1,n_j \} | k=1,k_i \}, j=1,m, \theta=1,\theta_2 \}
\]  
(12)

describes the optimal problem solutions found on the second iteration.

1. If sets (10) and (12) do not coincide, new solutions are found with equal lower bound for the global optimum. In this case, a new iteration according to paragraph 2 is necessary.

If sets (10) and (12) coincide, the search procedure of the GDP solution is complete because adding resources in the volume of maximum cost of user training programs does not increase the lower bound.

4. Example

Let us consider the case with three business processes, respectively, with three, two and two training programs for these processes, as well as with eight, seven and six users. Initial information on the training cost and cost of the competencies level increasing as a result of user training is given in table 1.
Table 1. The initial data of the demonstration example.

| P_{11} | P_{12} | P_{13} | P_{21} | P_{22} | P_{31} | P_{32} |
|--------|--------|--------|--------|--------|--------|--------|
| q_{11} | 4      | 3      | 5      | q_{21} | 3      | 4      | q_{31} |
| q_{12} | 5      | 4      | 3      | q_{22} | 4      | 3      | q_{32} |
| q_{13} | 3      | 2      | 4      | q_{23} | 2      | 4      | q_{33} |
| q_{14} | 4      | 3      | 2      | q_{24} | 3      | 5      | q_{34} |
| q_{15} | 3      | 3      | 5      | q_{25} | 2      | 4      | q_{35} |
| q_{16} | 4      | 3      | 3      | q_{26} | 3      | 5      | q_{36} |
| q_{17} | 5      | 4      | 3      | q_{27} | 4      | 5      | q_{37} |
| q_{18} | 4      | 2      | 4      | q_{28} | 4      | 5      | q_{38} |
| c_{11} | 60     | 64     | 90     | c_{21} | 54     | 90     | c_{31} |

| k_{1}  | \geq 5 | k_{2}  | \geq 3 | k_{3}  | \geq 2 |

Cardinal number of the all solutions for this example make \( \prod_{j=1}^{m} n_{j}^{k_{j}} = 3^{8} \cdot 2^{13} \).

1. Having complied with calculations according to (7), we shall obtain:

\[
\begin{align*}
\hat{c}_{11} &= \frac{1712}{3584} - 1100 \approx 528, \\
\hat{c}_{12} &= \frac{1008}{3584} - 1100 \approx 308, \\
\hat{c}_{13} &= \frac{864}{3584} - 1100 \approx 264.
\end{align*}
\]

(13)

Having solved the problem (1) – (3) for values (13), we shall obtain the only solution (\( \theta_{1}=1 \)), table 2.

Table 2. Optimal solution of the problem (1) – (3) on the first iteration.

| q    | 70 |
|------|----|
| k    | 13 |
| c    | 1072 |
| x1   | 101110000100000100100100 |
| x2   | 10100011000010 |
| x3   | 11010001000 |

Let us define values \( \hat{c}_{j}^{*} \) for the second iteration in relation with formulas (9):

\[
\begin{align*}
\hat{c}_{21}^{*} &= 528 + 90 = 618, \\
\hat{c}_{22}^{*} &= 308 + 90 = 398, \\
\hat{c}_{23}^{*} &= 264 + 90 = 354.
\end{align*}
\]

(14)

Having solved the problem (1) – (3) for values \( \hat{c}_{j}^{*} \), (14), we shall obtain five optimum solutions. Let us assume for the third iteration:

\[
\begin{align*}
\hat{c}_{31}^{*} &= 618 + 64 = 682, \\
\hat{c}_{32}^{*} &= 398 + 54 = 452, \\
\hat{c}_{33}^{*} &= 354 + 54 = 408.
\end{align*}
\]

(15)

Having solved the problem for values \( \hat{c}_{j}^{*} \), (15), we shall obtain 6 optimum solutions, table 3.
Table 3. Optimal solution of the problem (1) – (3) on the third iteration.

|   | q 72 |   | q 72 |   | q 72 |
|---|------|---|------|---|------|
| k | 15   | 15| 14   |   |      |
| c | 1100 | 1100| 1100 |   |      |
| x1| 101110000100 | 101110000100 | 101110000100 |   |      |
|   | 001110100100 | 001110100100 | 001110100100 |   |      |
| x2| 10100010001010 | 10100010001010 | 10100010001010 |   |      |
|   | 010100010000 | 010100010001 | 110100010000 |   |      |

|   | q 72 |   | q 72 |   | q 72 |
|---|------|---|------|---|------|
| k | 14   | 15| 15   |   |      |
| c | 1100 | 1100| 1100 |   |      |
| x1| 101110000100 | 100110000100 | 100110000100 |   |      |
|   | 000110100100 | 000110100100 | 000110100100 |   |      |
| x2| 10100010000010 | 10100010001010 | 1010001000010 |   |      |
|   | 110100010001 | 110100010000 | 110100010001 |   |      |

Solution sets obtained on the second and third iteration are inconsistent.

- Let us assume for the fourth iteration:
  \[ c_{41}^* = 682 + 90 = 772, \quad c_{42}^* = 452 + 90 = 542, \quad c_{43}^* = 408 + 90 = 498. \] (16)

Having solved the problem (1) – (3) for values \( c_{4j}^*, j=1,m, \) (16), we shall obtain 6 optimal solutions, table 4.

Table 4. Optimal solution of the problem (1) – (3) on the fourth iteration.

|   | q 72 |   | q 72 |   | q 72 |
|---|------|---|------|---|------|
| k | 15   | 15| 14   |   |      |
| c | 1100 | 1100| 1100 |   |      |
| x1| 101110000100 | 101110000100 | 101110000100 |   |      |
|   | 001110100100 | 001110100100 | 001110100100 |   |      |
| x2| 1010001000010 | 10100010001010 | 1010001000010 |   |      |
|   | 010100010000 | 010100010001 | 110100010000 |   |      |

|   | q 72 |   | q 72 |   | q 72 |
|---|------|---|------|---|------|
| k | 14   | 15| 15   |   |      |
| c | 1100 | 1100| 1100 |   |      |
| x1| 101110000100 | 100110000100 | 100110000100 |   |      |
|   | 000110100100 | 000110100100 | 000110100100 |   |      |
| x2| 1010001000010 | 10100010001010 | 1010001000010 |   |      |
|   | 110100010001 | 110100010000 | 110100010001 |   |      |

Solution sets obtained on the third and fourth iteration coincide completely. They define a set of optimal solutions of the GDP. The optimal solutions of the GDP (sets of values \( c_{j}^*, j=1,m, \) delivering maximum to the lower bond of the problem solving (1) – (3)), are given in the table 5.
### Table 5. Set of solution of the generalized dual problem.

| Item | N | c_1 | c_2 | c_3 |
|------|---|-----|-----|-----|
| 1    |   | 1100| 1100| 1100|
| 2    |   | 668 | 668 | 578 |
| 3    |   | 270 | 216 | 270 |
| 4    |   | 162 | 216 | 252 |
| 5    |   |     |     |     |
| 6    |   |     |     |     |

5. Conclusion
The solution of the GDP has been achieved for four iteration of the computer simulating. It's not the only one. Many solutions are not "compact" - solutions differ significantly with the same values of q and c. You must enter an additional criterion to choose the best solution you have received.

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