High-Q states in acoustic apple-shaped resonators

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Abstract. Apples play a significant role in our culture in various points of human history: starting from Adam and Eve, going on with Judgement of Paris, it also touches such great minds as Sir Isaac Newton and Alan Turing. Beyond that apples are still extremely relevant today due to Steve Jobs. In this work we study high quality (high-$Q$) resonant states of apple-shaped resonators. We have found that quasi bound states in continuum (quasi-BICs) are possible in the linear acoustic domain. We show that quasi-BICs are of Friedrich-Wintgen type, i.e. accompanied with avoided crossings while elongating or shrinking the apple-shaped resonator. Finally, we build a concise theory based on the group theory approach utilizing Wigner’s theorem. We illustrate that only the resonator symmetry plays major role, but not particular resonator’s shape.

1. Introduction
Studying acoustic resonators is inevitable for designing acoustic metamaterials [1, 2, 3, 4], optomechanical systems [5, 6, 7, 8], and creating the materials with specified properties [9, 10]. High-$Q$ resonances based on observation of bound states in the continuum (BICs) are very important for engineering these properties and are extensively studied in optical [11, 12, 13, 14] and acoustical [15, 16, 17, 18, 19] systems. In our work, we focus on a Friedrich-Wintgen type BICs [20, 21] based on avoided crossings of two eigenstates decaying into the same radiation channel [22], which were recently discovered in optical resonators [23]. In solid-state physics, quantum chemistry and optics modes are classified by irreducible representations of a system’s symmetry group [24, 25, 26, 27]. The classification can be applied to any structure and is defined solely by its symmetry. In this work, we obtained the high-$Q$ resonances of Friedrich-Wintgen type in acoustic resonators of arbitrary shape, and discussed their nature in terms of the symmetry of the eigenmodes. We illustrate this with an example of an azimuthally symmetric apple-shaped resonator, to emphasize, that only the symmetry plays major role, and not particular resonator’s shape.

2. Theoretical approach
The actual object of this research are eigenmodes of an acoustic resonator of an arbitrary shape. There is no general solution for the eigenmode problem, only specific cases can be studied fully analytically [28] such as sphere or infinite cylinder by spherical or cylindrical harmonics correspondingly. However, knowing the resonator’s symmetry and applying Wigner’s theorem [29, 24], one can find a set of constrains, certain rules, and properties of the modes. For the sake of con-
Figure 1. Geometry of the problem. Resonator boundary is represented by equations where \( x \) and \( y \) are measured in centimeters. Initial height \( h_0 = 3 \) cm, initial radius \( R_0 = (\sqrt{3.125} - 0.25) \) cm. In this work, we scale only the height of the object by a factor \( h/h_0 \). We use density \( \rho_0 = 1.225 \) kg/m\(^3\) and speed of sound \( c_0 = 335 \) m/s for the host and \( \rho = 4.2 \cdot \rho_0, c = c_0/3 \) for the resonator. These parameters are arbitrarily chosen while the system is scalable.

For convenience we explicitly state the theorem.

**Wigner’s theorem:**

\[
\mathcal{H}(\mathbf{r})\psi(\mathbf{r}) = \epsilon\psi(\mathbf{r}) \tag{1}
\]

Suppose that a system (Hamiltonian) is invariant under the transformations of a symmetry group, then the eigenfunctions are transformed under irreducible representations of the group.

Namely, there are as many types of eigenmodes excited as there are irreducible representations of a symmetry group, defined by the resonator’s shape. A set of modes’ behavior under all symmetry transformations defines in accordance to which irreducible representation it transforms. With all of the above stated, each mode can be classified and named after its irreducible representation, which fully describes the symmetry of this mode. For the linear acoustic domain, Eq. (1) transforms to 

\[
-c(\mathbf{r})^2 \nabla^2 p(\mathbf{r}) = \omega^2 p(\mathbf{r}),
\]

where \( p \) is the acoustic pressure, \( \nabla \) is the nabla operator, \( \omega \) is the angular frequency, and \( c(\mathbf{r}) = 1/\sqrt{\beta(\mathbf{r})\rho(\mathbf{r})} \) is the speed of sound. The medium is described by the compressibility \( \beta \) and mass density \( \rho \) [28].

The key consequence of these considerations is that a coupling can appear only for the modes transformed in accordance with the same irreducible representation [30, 31], moreover, only modes of the same symmetry radiate in the same radiation channel. Every resonator has infinite number of modes of each type. Tuning the resonator’s parameters can lead to the appearance of crossings (if there is no mode coupling or weak coupling [32]), or, on the contrary, the crossings can be avoided in case of strong coupling of the modes.

### 3. Results and methods

To illustrate this approach, we considered scattering by a single apple-shaped resonator with \( D_{\infty h} \) symmetry [33, 34]. Incident plane wave is along the z-axis. This symmetry group contains an infinite number of elements and irreducible representations, but only two types of modes are excited by such normally incident plane wave: \( A_{1g} \) and \( A_{1u} \) modes. Both of these modes are invariant under rotations around the z-axis, as well as the incident plane wave, but they have opposite parity under reflection in the \( z = 0 \) plane.

We performed the numerical modelling with COMSOL Multiphysics™ simulation software. The following parameters were taken: density \( \rho_0 = 1.225 \) kg/m\(^3\) and speed of sound \( c_0 = 335 \) m/s for the host and \( \rho = 4.2 \cdot \rho_0, c = c_0/3 \) for the resonator. We do not consider shear waves or
Figure 2. (a) The dependence of eigenmodes’ frequency of the apple-shaped acoustic resonator on $h_0/h$ ratio. Two types of modes shown, $A_{1g}$ (green) and $A_{1u}$ (purple), are named depending on their irreducible representations. Line width is proportional to the Q-factor of the mode. (b) Map of the calculated scattering cross section $\sigma$ versus frequency and $h_0/h$ ratio. High-Q (red) and low-Q (blue) modes are shown with circles. (c,d,e) Dependence of Q-factor on $h_0/h$ ratio for three different pairs of modes is calculated.

viscosity, which lower the Q-factors, but theoretical considerations remain the same. Geometry of the problem is shown in Figure 1. We have computed scattering using the built-in 2D Pressure Acoustics physics interface as the problem possesses axial symmetry. Spherical wave radiation boundary condition was used in both models.

In Figure 2 eigenmodes and scattering maps of apple-shaped resonator are shown. It shows that between two modes transformed under the same representation an anticrossing can be observed, whereas between two modes transformed under different representations a crossing will be observed. In Figure 3 one of the eigenmodes of $A_{1u}$ irreducible representation is shown in more detail. As one can see, at the point II (10.4 kHz frequency, $h_0/h = 1.0225$) the Q-factor increase of one of the modes takes place. Such behavior is typical for high-Q resonances of Friedrich-Wintgen type.
Figure 3. (a) Coupling of a pair of eigenmodes versus frequency and $h_0/h$ ratio. Line width depends on the Q-factor. (b) Calculated dependence of Q-factor on $h_0/h$ ratio for high Q (red) and low Q (blue) modes of the pair. (c) Field patterns in the resonator.

4. Conclusion
We have shown that scattering of an acoustic wave on an open resonator obeys the same mechanisms as light scattering on optical cavities. We have identified that for certain geometries high-Q resonant states can be observed due to the interaction of the modes with identical symmetry. The observation of these types of states is provided by the symmetry of the resonator and mode symmetry, but not the particular shape of this structure.

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