Beyond convergence rates: exact recovery with the Tikhonov regularization with sparsity constraints

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Abstract

The Tikhonov regularization of linear ill-posed problems with an \(\ell^1\) penalty is considered. We recall results for linear convergence rates and results on exact recovery of the support. Moreover, we derive conditions for exact support recovery which are especially applicable in the case of ill-posed problems, where other conditions, e.g., based on the so-called coherence or the restricted isometry property are usually not applicable. The obtained results also show that the regularized solutions do not only converge in the \(\ell^1\)-norm but also in the vector space \(\ell^0\) (when considered as the strict inductive limit of the spaces \(\mathbb{R}^n\) as \(n\) tends to infinity). Additionally, the relations between different conditions for exact support recovery and linear convergence rates are investigated. With an imaging example from digital holography the applicability of the obtained results is illustrated, i.e. that one may check \textit{a priori} if the experimental setup guarantees exact recovery with the Tikhonov regularization with sparsity constraints.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this paper, we consider linear inverse problems with a bounded linear operator \(A : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) between two separable Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\),
\[
Af = g.
\] (1)

We are given a noisy observation \(g^\epsilon = g + \eta \in \mathcal{H}_2\) with noise level \(\|g - g^\epsilon\| \leq \epsilon\) and try to reconstruct the solution \(f\) of \(Af = g\) from knowledge of \(g^\epsilon\). We are especially interested

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in the case in which (1) is ill-posed in the sense of Nashed, i.e. when the range of \( A \) is not closed. In particular, this implies that the (generalized) solution of (1) is unstable, or in other words, that the generalized inverse \( A^\dagger \) is unbounded. In this context, regularization has to be employed to stably solve the problem [13].

We assume that the operator equation \( Af = g \) has a solution \( f^\circ \) that can be expressed sparsely in an orthonormal basis \( \Psi := \{ \psi_i \}_{i \in \mathbb{Z}} \) of \( H_1 \), i.e. \( f^\circ \) decomposes into a finite number of basis elements,

\[
 f^\circ = \sum_{i \in \mathbb{Z}} u_i^\circ \psi_i \quad \text{with} \quad u_i^\circ \in l^2(\mathbb{Z}, \mathbb{R}), \quad \text{and} \quad \left| \{ i \in \mathbb{Z} | u_i^\circ \neq 0 \} \right| < \infty.
\]

The knowledge that \( f^\circ \) can be expressed sparsely can be utilized for the reconstruction by using an \( l^1 \)-penalized Tikhonov regularization [7], i.e. an approximate solution is given as a minimizer of the functional

\[
 12 \| Af - g^\varepsilon \|_{l^2}^2 + \alpha \sum_{i \in \mathbb{Z}} |\langle f, \psi_i \rangle|, \tag{2}
\]

with regularization parameter \( \alpha > 0 \). In contrast to the classical Tikhonov functional with a quadratic penalty [13], the \( l^1 \)-penalized functional promotes sparsity since small coefficients are penalized more.

For the sake of notational simplification, we use \( l^2 = l^2(\mathbb{Z}, \mathbb{R}) \) and introduce the synthesis operator \( D : l^2 \rightarrow H_1 \), which for \( u \in l^2 \) is defined by \( Du = \sum u_i \psi_i \). With that and the definition \( K := A \circ D : l^2 \rightarrow H_2 \) we can rewrite the inverse problem (1) as \( Ku = g \).

Adopting the usual convention in convex analysis, we use the following somewhat sloppy notation:

\[
 \| \cdot \|_{l^1} : l^2 \rightarrow [0, \infty], \quad \| u \|_{l^1} = \begin{cases} \| u \|_{l^1}, & \text{if} \quad u \in l^1, \\ \infty, & \text{if} \quad u \in l^2 \setminus l^1, \end{cases}
\]

and we rewrite the \( l^1 \)-penalized Tikhonov regularization (2) as

\[
 T_\alpha(u) := \frac{1}{2} \| Ku - g^\varepsilon \|_{l^2}^2 + \alpha \| u \|_{l^1}. \tag{3}
\]

In the following, we frequently use the standard basis of \( l^2 \), which is denoted by \( \{ e_j \}_{j \in \mathbb{Z}} \). The Tikhonov functional (3) has also been used in the context of sparse recovery under the name basis pursuit denoising [5].

Daubechies et al [7] showed that the minimization of (3) is indeed a regularization and derived error estimates in a particular wavelet setting. Error estimates and convergence rates under different source conditions (SC) have been derived by Lorenz [22] and Grasmair et al [16]. In this paper, we aim at conditions that ensure that the minimizers \( u^{\alpha,\varepsilon} \in \arg \min T_\alpha(u) \) have the same support as \( u^\circ \). In the context of sparse recovery, this phenomenon is called exact recovery. One of the main applications in the field of sparse recovery is compressive sampling, a new sampling technique which allows us to sample sparse signals at low rates [3].

Our approach builds heavily on techniques and results from the field of sparse recovery from [12, 14, 15, 19, 28] some of which we transfer to the field of inverse and ill-posed problems. Very roughly spoken, the conditions for exact recovery can be divided into two classes: sharp conditions which are not practical since they rely on unknown quantities (this category includes for example Tropp’s exact recovery condition (ERC) [28, theorem 8] and the null space property (NSP) [18]). More loose conditions which are far from being necessary but seem more practical (this category includes for example conditions using incoherence [11], [28, corollary 9] and the restricted isometry property (RIP) [4]). Moreover, the latter conditions are usually not applicable for inverse and ill-posed problems (somehow due to arbitrarily small singular values of the operator). Hence, we focus on ‘intermediate’ conditions [9, 12] and...
show how they can be applied to general inverse and ill-posed problems. In particular, we contribute the following point. While it is good to know that exact recovery is possible for some regularization parameter, what one really needs is a computable recipe to choose a parameter which only uses available information, and hence, we especially treat this question on the choice of a regularization parameter which guarantees exact recovery.

The paper is organized as follows. In section 2, we summarize some properties of $\ell^1$-penalized Tikhonov minimizers and recall a stability result from [16]. In section 3, we review previous results on exact recovery in the context of sparse recovery and we illustrate our contribution. Section 4 contains the main theoretical results of the paper. In particular, known results on ERCs are transferred to ill-posed problems and we give a parameter choice rule which ensures exact recovery in the presence of noise (under appropriate assumptions). One novelty here is that we focus on conditions and a choice rule which can be verified a priori and hence are of practical relevance (and not only of theoretical relevance). In section 5, we investigate the relation between the ERC from [28], the SC and the NSP [18]. In section 6, we demonstrate the practicability of the deduced recovery condition with an example from imaging, namely, an example from digital holography. In section 7, we give a conclusion on ERCs for the Tikhonov regularization with sparsity constraints.

2. The $\ell^1$-penalized Tikhonov functional

Before we start with error estimates, we recall some basic properties of the $\ell^1$-penalized Tikhonov functional $T_\alpha$. First, we repeat a trivial characterization of the minimizer.

**Proposition 2.1 (Optimality condition).** Define the set-valued sign function $\text{Sign} : \ell^2 \to \{-1, [1, +1], +1\}$, for $u \in \ell^2$, by

$$(\text{Sign}(u))_k := \begin{cases} 
{-1}, & u_k < 0, \\
{-1, +1}, & u_k = 0, \\
{+1}, & u_k > 0.
\end{cases}$$

(4)

Let $u^{\alpha, \epsilon} \in \ell^2$. Then, the following statements are equivalent:

(i) $u^{\alpha, \epsilon} \in \arg\min_{u \in \ell^2} T_\alpha(u)$.

(ii) $-K^*(Ku^{\alpha, \epsilon} - g^\epsilon) \in \alpha \text{Sign}(u^{\alpha, \epsilon})$.

(6)

**Proof.** Since the the set-valued sign function is actually the subgradient of the $\ell^1$-norm, the proof consists of noting that (ii) is just the optimality condition $0 \in \partial T_\alpha(u^{\alpha, \epsilon})$. Due to convexity of $T_\alpha$, (ii) is also sufficient. □

Another well-known characterization of a minimizer $u^{\alpha, \epsilon}$ is that it is a fixed point of $u^{\alpha, \epsilon} = S_\alpha(u^{\alpha, \epsilon} + K^*(g^\epsilon - Ku^{\alpha, \epsilon}))$, where $S_\alpha$ denotes the soft-thresholding operator, cf e.g. [7]. From this characterization or from (6) we can deduce the following. Since the range of $K^*$ is contained in $\ell^2$, any minimizer $u^{\alpha, \epsilon}$ of the $\ell^1$-penalized Tikhonov functional $T_\alpha$ is finitely supported for every $\alpha > 0$. (Note that this observation relies on the fact that $K$ is bounded on $\ell^2$.

If we model $K : \ell^1 \to H_2$ boundedly, as appropriate for normalized dictionaries, we cannot conclude that $u^{\alpha, \epsilon}$ is finitely supported, since the adjoint operator maps $K^* : H_2 \to \ell^\infty$.)

Uniqueness of the minimizer of (3) could be guaranteed by ensuring strict convexity. This holds, e.g., if $K$ is injective. A weaker property of the operator $K$, which also guarantees uniqueness (although the functional is not strictly convex), is the finite basis injectivity (FBI) [1] property defined below.
Definition 2.2. Let $K : \ell^p \rightarrow \mathcal{H}_2$ be an operator mapping into a Hilbert space $\mathcal{H}_2$. Then $K$ has the FBI property if for all finite subsets $J \subset \mathbb{Z}$ the operator restricted to $\text{span}(e_i; i \in J)$ is injective, i.e., for all $u, v \in \ell^2$ with $Ku = Kv$ and $u_k = v_k = 0$, for all $k \notin J$, it follows that $u = v$.

In inverse problems with sparsity constraints, the FBI property is used for a couple of issues concerning $\ell^p$-penalized Tikhonov functionals, for example for deduction of stability results [16, 17, 22], for derivation of efficient minimization schemes [20, 21] and for proving convergence of minimization algorithms [1, 6, 24]. A demonstrative example for an operator which possesses the FBI property but is not fully injective is the following. Denote with $\{e_i\}_{i=0}^{\infty}$ the usual real Fourier basis of $L^2([0, 1])$ and with $\{\psi_j\}_{j=1}^{\infty}$ the Haar wavelet basis of $L^2([0, 1])$ and define the operator $K : \ell^2(\mathbb{Z}) \rightarrow L^2([0, 1])$ by $Ku = \sum_{j=0}^{\infty} u_j e_j + \sum_{j=-\infty}^{-1} u_j \psi_j$. Then $K$ is clearly not injective, since any Haar wavelet can be expressed in the Fourier basis and vice versa. However, $K$ obeys the FBI property since neither any Haar wavelet is a finite linear combination of elements of the Fourier bases nor the other way round.

The FBI property is related to the so-called RIP [4] of a matrix, which is a quite common assumption in the theory of compressive sampling [2, 3]. The RIP is defined as follows. Let $A$ be an $m \times n$ matrix and let $s < n$ be an integer. The restricted isometry constant of order $s$ is defined as the smallest number $0 < c_s < 1$, such that the following condition holds for all $v \in \mathbb{R}^n$ with at most $s$ non-zero entries:

$$ (1 - c_s) \|v\|_2^2 \leq \|Av\|_2^2 \leq (1 + c_s) \|v\|_2^2. $$

Essentially, this property denotes that the matrix is approximately an isometry when restricted to small subspaces. The FBI property, however, is defined for operators acting on the sequence space and only says that the restriction to finite-dimensional subspaces is still injective and makes no assumption of the involved constants.

With $\ell^0$ we denote the vector space of all real-valued sequence with only finitely many non-zero entries. In contrast to the $\ell^p$ spaces with $p > 0$ there is no obvious (quasi-)norm available which turns $\ell^0$ into a (quasi-)Banach space. We will come back to the issue of defining a suitable topology on $\ell^0$ later. In general, the minimum-$\|\cdot\|_{\ell^0}$ solution $u^0$ of $Ku = g$ neither needs to be in $\ell^0$ nor needs to be unique. If we assume that there is a finitely supported solution $u^0 \in \ell^0$ of $Ku = g$, then the set of all solutions of $Ku = g$ is given by $u^0 + \ker K$. If $K$ possesses the FBI property, then the solution $u^0$ is the unique solution in $\ell^0$; hence $\ker K \subset \ell^0 \setminus \ell^0$. However, in general $u^0 \in \ell^0$ is not a minimum-$\|\cdot\|_{\ell^0}$ solution. In the following, we assume that $K$ possesses the FBI property and denote the unique solution of $Ku = g$ in $\ell^0$ with $u^0$.

Stability and convergence rate results for $\ell^1$-penalized Tikhonov functionals have been deduced in [7, 16, 17, 22]. The following error estimate from [16] ensures the linear convergence to the minimum-$\|\cdot\|_{\ell^0}$ solution if a certain SC is satisfied. We state it here in full detail and give explicit constants.

Theorem 2.3 (Error estimate [16, theorem 15]). Let $K$ possess the FBI property, $u^0 \in \ell^0$ with $\text{supp } u^0 = \{1\}$ be a minimum-$\|\cdot\|_{\ell^0}$ solution of $Ku = g$, and $\|g - g^0\|_{\mathcal{H}_2} \leq \varepsilon$. Let the following SC be fulfilled:

there exists $w \in \mathcal{H}_2$ such that $K^* w = \xi \in \text{Sign}(u^0)$.  \(\text{(7)}\)

Moreover, let

$$ \theta = \sup(|\xi_k|) |\xi_k| < 1 $$
and $c > 0$ such that for all $u \in \ell^2$ with $\text{supp}(u) \subset I$ it holds that
$$\|Ku\| \geq c\|u\|.$$ Then for the minimizers $u^{\alpha, \varepsilon}$ of $T_\alpha$ it holds that
$$\|u^{\alpha, \varepsilon} - u^\diamond\|_{\ell^1} \leq \frac{\|K\| + 1}{1 - \theta} \varepsilon^2 + \frac{1}{c} \frac{\|K\| + 1}{1 - \theta} (\alpha + \varepsilon).$$ (8)

In particular, with $\alpha \asymp \varepsilon$ it holds that
$$\|u^{\alpha, \varepsilon} - u^\diamond\|_{\ell^1} = O(\varepsilon).$$ (9)

Remark 2.4.
(a) Since $\xi \in \ell^2$ by definition of $K$, it is clear that $\theta < 1$.
(b) Since $K$ possesses the FBI property, the existence of $c > 0$ is ensured.
(c) To achieve the linear convergence rate (9), the SC (7) is even necessary, cf [17].

The above theorem is remarkable since it gives an error estimate for regularization with a sparsity constraints with comparably weak conditions of the operator; especially nothing is assumed about the incoherence of $K$ in either way. However, the constants in the error estimate (8) both depend on the unknown quantities $I$, $\|w\|$ and $\theta$, and are possibly large (especially $c$ can be small and $\theta$ can be close to 1).

3. Known results from sparse recovery

In [28], Tropp deduces a condition which ensures exact recovery. To formulate the statement, we need the following notations. For a subset $J \subset \mathbb{Z}$, we denote with $P_J : \ell^2 \to \ell^2$ the projection onto span\{$e_i | i \in J$\},
$$P_J u := \sum_{j \in J} u_j e_j,$$
i.e. the coefficients $j \notin J$ are set to 0 and hence supp $(P_J u) \subset J$. With that definition $K P_J : \ell^2 \to \mathcal{H}_I$ modifies the operator $K$ such that $K P_J u$ depends only on the entries $u_i$ for $i \in J$ and hence is something similar to the restriction of $K$ to span\{$e_i | i \in J$\}.

Moreover, for a linear operator $B$ we denote the pseudoinverse operator by $B^\dagger$. With these definitions we are able to formulate Tropp’s condition for exact recovery.

Theorem 3.1. Let $K$ be bounded and assume that $K P_I$ is injective; let $g^\varepsilon_J$ be the orthogonal projection of $g^\varepsilon$ to the range of $K P_I$ and denote with $u^\varepsilon_J \in \ell^2$ the unique element with supp $(u^\varepsilon_J) \subset I$ such that $g^\varepsilon_J = K u^\varepsilon_J$.

If the ERC
$$\sup_{i \in \ell^2} \|(K P_I)^J K e_i\|_{\ell^1} < 1$$ (10)
holds, then the parameter choice rule
$$\frac{\sup_{e_i \in \ell^2} |\langle g^\varepsilon - g^\varepsilon_J, K e_i \rangle|}{1 - \sup_{e_i \in \ell^2} \|(K P_I)^J K e_i\|_{\ell^1}} < \alpha < \frac{\min_{u \in \ell^2} |u^\varepsilon_J(i)|}{\|(P_I K^+ K P_I)^{-1}\|_{\ell^1, \ell^1}}$$ (11)
ensures that supp $(u^{\alpha, \varepsilon}) = \text{supp}(u^\diamond)$.

This theorem can be extracted from [28] under slightly different assumptions (basically it is theorem 8 there; however, this relies on several other results in [28]).

The applicability of this result is limited due to several terms. The expressions $\|(P_I K^+ K P_I)^{-1}\|_{\ell^1, \ell^1}$ and $\|(K P_I)^J K e_i\|_{\ell^1}$ need knowledge of $I$ which is unknown. Moreover,
the quantities $g^*_I$ (the projection of $g^*$ onto the range of $KP_I$) and $u^*_I$ (the unique element with $\text{supp}(u^*_I) \subset I$ such that $g^*_I = Ku^*_I$) are unknown and not computable without knowledge of $I$.

In the following section, we deduce a priori parameter rules that are easier to use than Tropp’s parameter choice rule (11). The idea is to get rid of the expressions $g^*_I$ and $u^*_I$, which cannot be estimated a priori. Furthermore, we will apply the techniques from [12, 19] to deal with the terms $\| (P_I K^* K P_I)^{-1} \|_{\ell^1,\ell^1}$ and $\| (K P_I)^{\dagger} K e_i \|_{\ell^1}$.

Finally, we remark that there are conditions for exact recovery (also in the presence of noise) which use the so-called coherence of the dictionary in [14, 15] and [28, corollary 9]. The conditions are much easier to check (since they only rely on inner products $\langle K e_i, K e_j \rangle$) but are also much harder to fulfill in practice.

4. Beyond convergence rates: exact recovery for ill-posed operator equations

In this section, we give an a priori parameter rule which ensures that the unknown support of the sparse solution $u^* \in \ell^0$ is recovered exactly, i.e. $\text{supp}(u^{\alpha,\epsilon}) = \text{supp}(u^*)$. We assume that $K$ possesses the FBI property, and hence $u^*$ is the unique solution of $Ku = g$ in $\ell^0$. With $I$ we denote the support of $u^*$, i.e.

$I := \text{supp}(u^*) := \{ i \in \mathbb{Z} | u^*_i \neq 0 \}$.

**Theorem 4.1** (Lower bound on $\alpha$). Let $u^* \in \ell^0$, $\text{supp}(u^*) = I$ and $g^* = Ku^* + \eta$ the noisy data. Assume that $K$ is bounded and possesses the FBI property. If the condition

$$\sup_{i \in I^c} \| (K P_I)^{\dagger} K e_i \|_{\ell^1} < 1$$

holds, then the parameter rule

$$\alpha > \frac{1 + \sup_{i \in I^c} \| (K P_I)^{\dagger} K e_i \|_{\ell^1}}{1 - \sup_{i \in I} \| (K P_I)^{\dagger} K e_i \|_{\ell^1}} \sup_{i \in \mathbb{Z}} |\langle \eta, K e_i \rangle|$$

ensures that the support of $u^{\alpha,\epsilon}$ is contained in $I$.

**Proof.** In [28, theorem 8], it is shown that condition (12) together with the parameter rule

$$\alpha > \frac{\sup_{i \in I^c} |\langle g^* - g^*_I, K e_i \rangle|}{1 - \sup_{i \in I} \| (K P_I)^{\dagger} K e_i \|_{\ell^1}}$$

ensures that the support of $u^{\alpha,\epsilon} = \arg \min T_\alpha(u)$ is contained in $I$. Recall that $g^*_I$ is the orthogonal projection of $g^*$ to the range of $K P_I$, i.e. $g^*_I = K P_I (K P_I)^{\dagger} g^*$. Since $\text{Id} - K P_I (K P_I)^{\dagger}$ equals the orthogonal projection on $\text{rg}(K P_I)^{\perp}$ and $g = Ku^* \in \text{rg}(K P_I)$, we obtain

$$g^* - g^*_I = (\text{Id} - K P_I (K P_I)^{\dagger}) g^* = (\text{Id} - K P_I (K P_I)^{\dagger})(\eta).$$

Hence, using standard identities for the pseudoinverse and Hölder’s inequality, we obtain that for $i \in I^c$ it holds that

$$|\langle g^* - g^*_I, K e_i \rangle| = |\langle (\text{Id} - K P_I (K P_I)^{\dagger}) \eta, K e_i \rangle|$$

$$= |\langle \eta, (\text{Id} - K P_I (K P_I)^{\dagger}) K e_i \rangle|$$

$$= |\langle K^* \eta, (\text{Id} - K P_I)^{\dagger} K e_i \rangle|$$

$$\leq \| K^* \eta \|_{\ell^\infty \ell^1} \| (\text{Id} - K P_I)^{\dagger} K e_i \|_{\ell^1}.$$
Since $i \in I^*$ and $\text{supp} \left( (K P_I) K e_i \right) \subset I$ we obtain
$$\sup_{i \in I^*} \left| e_i - (K P_I) K e_i \right| = 1 + \sup_{i \in I^*} \left\| (K P_I) K e_i \right\|_{\ell^1}.$$  
Hence, using $\|K \eta\|_{\ell^\infty} = \sup_{\mathbb{Z}} |(K \eta, e_i)|$ we end in the estimate
$$\sup_{i \in I^*} \left| \left| g^i - g^i_{\ast} \right|, K e_i \right| \leq \sup_{i \in \mathbb{Z}} \left| (\eta, K e_i) \right| \left( 1 + \sup_{i \in I^*} \left\| (K P_I) K e_i \right\|_{\ell^1} \right).$$  
\hfill (14)

Thus, condition (12) together with the parameter choice rule (13) ensures that the support is contained in $I$. A direct proof without using [28] can be found in [27].

Remark 4.2. Instead of using the estimate (14), one can alternatively use another more common upper bound for $\sup_{i \in I} \left\| \left\| g^i - g^i_{\ast} \right|, K e_i \right\|$. For that note that $\text{Id} - K P_I (K P_I)$ is the orthogonal projection on $rg(K P_I)^{\perp}$. Then, since the norm of orthogonal projections is bounded by 1, we can estimate for $i \in I^*$ with the Cauchy–Schwarz inequality as follows:
$$\left| (\eta, \text{Id} - K P_I (K P_I) K e_i) \right| \leq \|\eta\|_{\ell^2} \|\text{Id} - K P_I (K P_I) K e_i\|_{\ell^2} \leq \epsilon \|K\|.$$  
\hfill (15)

In general, one cannot say which estimate gives a sharper bound, inequality (14) or inequality (15). However, in practice the noise $\eta$ often is a realization of some random variable e.g. with symmetric distribution, and hence, $\sup_{i \in \mathbb{Z}} \left| (\eta, K e_i) \right| \ll \epsilon \|K\|$ seems plausible. In this case, the estimate with Hölder’s inequality (14) gives a sharper estimate and we use (14) for the example from digital holography in section 6.

Theorem 4.1 gives a lower bound on the regularization parameter $\alpha$ to ensure $\text{sup} \left( u^{\alpha, \xi} \right) \subset \text{sup} \left( u^\ast \right)$. To guarantee $\text{sup} \left( u^{\alpha, \xi} \right) = \text{sup} \left( u^\ast \right)$ we need an additional upper bound for $\alpha$.

The following theorem leads to that purpose.

**Theorem 4.3 (Error estimate).** Let the assumptions of theorem 4.1 hold and choose $\alpha$ according to (13). Then the following error estimate is valid:
$$\|u^\ast - u^{\alpha, \xi}\|_{\ell^\infty} \leq \left( \alpha + \sup_{i \in \mathbb{Z}} \left| (\eta, K e_i) \right| \right) \left\| (P_I K^* K P_I)^{-1} \right\|_{\ell^1, \ell^1}.$$  
\hfill (16)

**Proof.** From the assumptions of theorem 4.1 we have $\text{sup} \left( u^{\alpha, \xi} \right) \subset \text{sup} \left( u^\ast \right)$. From the optimality condition (6), we know that for $u^{\alpha, \xi}$ there is a $w \in \ell^\infty$ with $\|w\|_{\ell^\infty} \leq 1$ such that
$$-K^* (K u^{\alpha, \xi} - g^\ast) = \alpha w.$$  
Hence, it holds that
$$-P_I K^* K P_I (u^{\alpha, \xi} - u^\ast) = -P_I K^* (K u^{\alpha, \xi} - g) = -P_I K^* (K u^{\alpha, \xi} - g^\ast) - P_I K^* \eta = \alpha P_I w - P_I K^* \eta.$$  
Since $\|w\|_{\ell^\infty} \leq 1$, with Hölder’s inequality we can estimate for all $j \in I$
$$\|u^{\alpha, \xi} - u^\ast\|_{\ell^1} = \|u^{\alpha, \xi} - u^\ast\|_{\ell^1} = \left| \left| (P_I K^* K P_I)^{-1} (\alpha P_I w - P_I K^* \eta), e_j \right| \right|$$
$$\leq \alpha \|P_I w, (P_I K^* K P_I)^{-1} e_j\| + \|P_I K^* \eta, (P_I K^* K P_I)^{-1} e_j\|$$
$$\leq (\alpha \|w\|_{\ell^\infty} + \|P_I K^* \eta\|_{\ell^\infty}) \left\| (P_I K^* K P_I)^{-1} \right\|_{\ell^1, \ell^1} \leq (\alpha + \sup_{i \in \mathbb{Z}} \left| (\eta, K e_i) \right|) \left\| (P_I K^* K P_I)^{-1} \right\|_{\ell^1, \ell^1}.$$  
\hfill \square

**Remark 4.4.** Due to the error estimate (16) we achieve a linear convergence rate measured in the $\ell^\infty$-norm. In finite dimensions the $\ell^p$-norms are equivalent; hence, we also obtain an estimate for the $\ell^1$ error:
$$\|u^\ast - u^{\alpha, \xi}\|_{\ell^1} \leq (\alpha + \epsilon \|K\|) \left| I \right| \left\| (P_I K^* K P_I)^{-1} \right\|_{\ell^1, \ell^1}.$$  

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Compared to the estimate (8) from theorem 2.3, quantities $\theta$ and $\|w\|$ are not present anymore. The role of $1/c$ is now played by $\| (P_i K^* K P_i)^{-1} \|_{\ell^1,\ell^1}$. However, if upper bounds on $I$ or on its size (together with structural information on $K$) are available, our estimate can give a priori checkable error estimates.

The following theorem gives a sufficient condition for the existence of a regularization parameter $\alpha$ which provides exact recovery. Due to theorem 4.3, equation (16), the regularization parameter should be chosen as small as possible.

**Theorem 4.5** (ERC in the presence of noise). Let $u^\circ \in \ell^0$ with $\text{supp} (u^\circ) = I$ and $g^\circ = Ku^\circ + \eta$ the noisy data with noise-to-signal ratio

$$r_{\eta/u} := \frac{\sup_{i \in \ell^0} |\langle \eta, K e_i \rangle|}{\min_{i \in I} |u_i^\circ|}.$$  

Assume that the operator $K$ is bounded and possesses the FBI property. Then the ERC in the presence of noise ($\varepsilon$ERC)

$$\sup_{i \in I^0} \| (K P_i)^{\dagger} K e_i \|_{\ell^1} < 1 - 2r_{\eta/u} \| (P_i K^* K P_i)^{-1} \|_{\ell^1,\ell^1}$$  

ensures that there is a suitable regularization parameter $\alpha$,

$$\frac{1 + \sup_{i \in \ell^0} \| (K P_i)^{\dagger} K e_i \|_{\ell^1}}{1 - \sup_{i \in \ell^0} \| (K P_i)^{\dagger} K e_i \|_{\ell^1}} \sup_{i \in \ell^0} |\langle \eta, K e_i \rangle| < \alpha \min_{i \in I} |u_i^\circ|$$  

$$\alpha < \frac{\| (P_i K^* K P_i)^{-1} \|_{\ell^1,\ell^1}}{1 - \sup_{i \in \ell^0} \| (K P_i)^{\dagger} K e_i \|_{\ell^1}} - \sup_{i \in \ell^0} |\langle \eta, K e_i \rangle|,$$  

which provides the exact recovery of $I$, i.e. the support of the minimizer $u^{\alpha,\varepsilon}$ coincides with $\text{supp} (u^\circ) = I$.

**Proof.** The lower bound of the parameter rule (18) ensures that $\text{supp} (u^{\alpha,\varepsilon}) \subset I$. With the error estimate (16) we see that the upper bound from the parameter rule (18) guarantees for $j \in I$ that

$$|u_j^\circ| - |u_j^{\alpha,\varepsilon}| \leq |u_j^\circ - u_j^{\alpha,\varepsilon}| < \min_{i \in I} |u_i^\circ|.$$  

Hence $|u_j^{\alpha,\varepsilon}| > |u_j^\circ| - \min_{i \in I} |u_i^\circ| \geq 0$ for all $j \in I$. The $\varepsilon$ERC (17) ensures that the interval of convenient regularization parameters $\alpha$ resulting from (18) is not empty. \hfill $\Box$

**Remark 4.6.** Theorem 4.5 gives a parameter choice rule that works without the quantities $g_j^\circ$ and $u_j^\circ$. In theorem 3.1, the existence of a parameter that ensures exact recovery is guaranteed on the condition

$$\sup_{i \in I^0} \| (K P_i)^{\dagger} K e_i \|_{\ell^1} < 1 - \frac{\sup_{i \in \ell^0} \| [g^\circ - g_j^\circ, K e_i] \|}{\min_{i \in I} |u_i^{\circ}(i)|} \| (P_i K^* K P_i)^{-1} \|_{\ell^1,\ell^1},$$  

cf equation (11). The question remains which condition is sharper, (19) or (18)? The conditions look similar with different factors on the right-hand side; however, it is not obvious which condition is sharper: indeed, provided that $\sup_{i \in \ell^0} \| (K P_i)^{\dagger} K e_i \|_{\ell^1} < 1$ holds, by using equation (14) we can estimate

$$\sup_{i \in \ell^0} \| [g^\circ - g_j^\circ, K e_i] \| \min_{i \in I} |u_i^\circ(i)| < 2 \sup_{i \in \ell^0} |\langle \eta, K e_i \rangle| \min_{i \in I} |u_i^\circ(i)|.$$  

However, it is not clear which expression, $\min_{i \in I} |u_i^\circ(i)|$ or $\min_{i \in I} |u_i^\circ|$, is smaller. Recall that $g_j^\circ$ is the orthogonal projection of $g^\circ$ to the range of $K P_i$, and $u_j^\circ$ is the unique element
with $\text{supp}(u^\circ_\varepsilon) \subset I$ such that $g^\circ_\varepsilon = Ku^\circ_\varepsilon$. Now, the noise $\eta$ with $g^\varepsilon = Ku^\circ + \eta$ can cause an increase or decrease of the values $u^\circ_\varepsilon(j)$, and hence, the coefficients $u^\circ_\varepsilon(j)$, $j \in I$ can be larger or smaller.

The results of theorems 4.3 and 4.5 can be rephrased as follows. If the regularization parameter $\alpha(\varepsilon)$ is chosen according to (18) and fulfills $\alpha \asymp \varepsilon$, then $\|u^{\alpha,\varepsilon} - u^\circ\|_\varepsilon = \mathcal{O}(\varepsilon)$ and the support of $u^{\alpha,\varepsilon}$ coincides with that of $u^\circ$. Indeed, this can be interpreted as convergence in the space $\ell^0$ with respect to the following topology.

**Definition 4.7.** We equip the spaces $\mathbb{R}^n$ with the Euclidean topology and consider them ordered by inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ in the natural way. Then, an absolutely convex and absorbent subset $U$ of $\ell^0$ is called a neighborhood of $0$ if set $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for any $n$. The topology $\tau$ on $\ell^0$ which is generated by the local base of these neighborhoods is called the topology of sparse convergence.

The definition above says that the topology of sparse convergence is generated as a strict inductive limit of the spaces $\mathbb{R}^n$. The space $\ell^0$ is turned into a complete locally convex vector space with this topology. A sequence $(u^n)$ in $\ell^0$ converges to $u$ in this topology if there is a finite set $I \subset \mathbb{N}$ such that $\text{supp}(u^n) \subset I$, for all $n \in \mathbb{N}$, and the sequence $u^n$ converges componentwise.

As a strict inductive limit of Fréchet spaces, $(\ell^0, \tau)$ is also called an LF-space and is known to be not normable, see [10, 23]. The topology of sparse convergence resembles the topology on the space of test functions $\mathcal{D}(\Omega)$ in distribution theory. (This correspondence can be pushed a little bit further by observing that the dual space of $(\ell^0, \tau)$ is the space of all real-valued sequences and plays the role of the space of distributions. We will not pursue this similarity further here.)

**Corollary 4.8 (Convergence in $\ell^0$).** In the situation of theorem 4.5 assume that $\alpha$ is chosen to fulfill the parameter choice rule (18). Then $u^{\alpha,\varepsilon} \rightharpoonup u^\circ$ in $\ell^0$ in the topology of sparse convergence.

In fact, the parameter choice rule (18) is not an a priori parameter rule $\alpha = \alpha(\varepsilon)$, since it depends on the noise $\eta$ and on unknown quantities such as $I$ and $\min|u^\circ_\varepsilon|$. However, the term $\sup_{i \in I} |\langle \eta, Ke_i \rangle|$ is related to the noise level and it can be estimated by $\varepsilon \|K\|$, cf. remark 4.2. The term $\min |u^\circ_\varepsilon|$, i.e. the smallest non-zero entry in the unknown solution, may be estimated from below in several applications. Due to the expressions $\| (P_1 K^* K P_1)^{-1} \|_\varepsilon$, $\varepsilon$, and $\sup_{i \in I} \| (K P_1)^i Ke_i \|_\varepsilon$, the $\varepsilon$ERC (17) is hard to evaluate, especially since the support $I$ is unknown. Therefore, we follow [12] and give another sufficient recovery condition which is, on the one hand, weaker in the sense that it is easier to satisfy (and implies the $\varepsilon$ERC, and hence, is less powerful than the ERC) and, on the other hand, is easier to evaluate in practice since it depends only on inner products of images of $K$ restricted to $I$ and $I^2$. For the sake of an easier presentation we define according to [12, 19]

$$\text{COR}_I := \sup_{i \in I} \sum_{j \in I} |\langle Ke_i, Ke_j \rangle| \quad \text{and} \quad \text{COR}_{I^2} := \sup_{i \in I} \sum_{j \in I} |\langle Ke_i, Ke_j \rangle|.$$ 

**Theorem 4.9 (Neumann ERC in the presence of noise).** Let $u^\circ \in \ell^0$ with $\text{supp}(u^\circ) = I$ and $g^\varepsilon = Ku^\circ + \eta$ the noisy data with noise-to-signal ratio $r_{\eta/u}$. Assume that the operator norm of $K$ is bounded by 1 and that $K$ possesses the FBI property. Then, the Neumann ERC in the presence of noise (Neumann $\varepsilon$ERC)

$$\text{COR}_I + \text{COR}_{I^2} < \min_{i \in I} \|Ke_i\|^2_{l^2} - 2r_{\eta/u}$$

(20)
ensures that there is a suitable regularization parameter $\alpha$,

\[
\min_{i \in I} \|K e_i\|_{H^2}^2 - \text{COR}_I + \text{COR}_{I^c} \leq \sup_{i \in I} |\langle \eta, K e_i \rangle| < \alpha
\]

\[
\alpha < \left( \min_{i \in I} \|K e_i\|_{H^2}^2 - \text{COR}_I \right) \min_{i \in I} |u^\circ_i| - \sup_{i \in I} |\langle \eta, K e_i \rangle|,
\]

which provides the exact recovery of $I$, i.e. the support of $u^\alpha,\varepsilon$ coincides with $\text{supp}(u^\circ) = I$.

**Proof.** For the deduction of conditions (20) and (21) from conditions (17) and (18), respectively, one splits the operator $P_I K^* K P_I$ into diagonal and off-diagonal and uses $\|K\|_{H^2} \leq 1$ and a Neumann series expansion for its inverse, following the techniques from [12, 19]. □

**Remark 4.10.** By the assumptions of theorem 4.9, the operator norm of $K$ is bounded by 1, i.e. $\|K e_i\|_{H^2} \leq 1$ for all $i \in \mathbb{Z}$. Hence, to ensure the Neumann $\varepsilon$ ERC (20), one has necessarily for the noise-to-signal ratio $r_{\eta/u} < 1/2$. For a lot of examples one can normalize $K$, so that $\|K e_i\|_{H^2} = 1$ holds for all $i \in \mathbb{Z}$. We do this for the example from digital holography in section 6. In this case the Neumann $\varepsilon$ ERC (20) reads

$$\text{COR}_I + \text{COR}_{I^c} < 1 - 2r_{\eta/u}.$$  

This condition coincides with the result presented in [9] for the orthogonal matching pursuit.

**Remark 4.11.** We remark that the correlations $\text{COR}_I$ and $\text{COR}_{I^c}$ can be estimated from above, with $N := \|u^\circ\|_{\ell^0}$, by

$$\text{COR}_I \leq (N - 1) \mu \quad \text{and} \quad \text{COR}_{I^c} \leq N \mu.$$  

Consequently, the ERC in terms of the coherence parameter $\mu$ can easily be deduced from conditions (20) and (21). This will result in Fuchs’ ERC from [15].

### 5. Relations between recovery conditions and the SC

In this section, we compare the different conditions which have been used. As we have seen in theorems 2.3 and 4.3 both the SC (7) and the ERC (10) lead to a linear convergence rate under an appropriate parameter choice rule. However, the latter also leads to exact recovery. One should note that the ERC and the SC are crucially different in some sense. The ERC is a uniform condition in the sense that it uses a given support $I$ and, hence, also leads to a result which holds for all vectors with that support. The SC, on the other hand, depends on a particular sign pattern $\text{Sign}(u^\circ)$ and, hence, leads to a result which holds for all vectors with that sign pattern. We may hence strengthen the SC to a ‘uniform source condition’ (uniform SC) as follows:

for all $u^\circ$ with $\text{supp}(u^\circ) \subset I$ there exists $w \in \mathcal{H}_2$ such that $K^* w \in \text{Sign}(u^\circ)$.

As it turns out, the ERC does not only imply the uniform SC but even a ‘uniform strict source condition’.

**Proposition 5.1** (ERC $\Rightarrow$ uniform strict SC). Let $I$ be finite and let $K P_I$ be injective. Then, the ERC (10) implies the following uniform strict SC:

for all $u^\circ$ with $\text{supp}(u^\circ) \subset I$ there exists $w \in \mathcal{H}_2$:

\[
\begin{align*}
\{ P_I K^* w = P_I \text{ sign}(u^\circ) \\
\| P_{I^c} K^* w \|_{\ell^\infty} < 1.
\end{align*}
\]

(22)
Proof.  Let \( u^\diamond \) be such that \( \text{supp}(u^\diamond) \subseteq I \). Since \( K P_I \) is injective, the operator \( P_I K^* : \mathcal{H}_2 \to \ell^2(I) \) is surjective, and hence, the equation

\[
P_I K^* w = P_I \text{sign}(u^\diamond)
\]

has a solution which can be expressed as

\[
w = (P_I K^* P_I K^*)^\dagger P_I \text{sign}(u^\diamond).
\]

Now, it remains to check whether for \( j \notin I \) it holds that \( |\langle K^* w, e_j \rangle| < 1 \): with the Hölder inequality and the ERC it follows that

\[
|\langle K^* w, e_j \rangle| = |\langle (P_I K^* P_I K^*)^\dagger P_I \text{sign}(u^\diamond), e_j \rangle| \\
= |\langle P_I \text{sign}(u^\diamond), (P_I K^* P_I K^*)^\dagger P_I K e_j \rangle| \\
\leq \| P_I \text{sign}(u^\diamond) \|_{\ell^\infty} \| (P_I K^* P_I K^*)^\dagger P_I K e_j \|_{\ell^1} < 1.
\]

Finally, \( |\langle K^* w, e_j \rangle| \to 0 \) for \( j \to \infty \) and hence \( \| P_{\mathcal{L}} K^* w \|_{\ell^\infty} < 1 \). \hfill \Box

It should be noted that a similar result appears in [17, theorem 4.7]. There it is shown that a linear convergence rate for the minimizers \( u^{\alpha,\varepsilon} \) already implies that the strict SC holds, and hence, by theorem 4.3 ERC implies strict SC.

Another important condition in the context of sparse recovery is the so-called null space property (NSP). An operator \( K : \ell^2 \to \mathcal{H}_2 \) is said to have the NSP for the set \( I \subseteq \mathbb{N} \) if for any \( u \in \ker K, u \neq 0 \) it holds that

\[
\| P_I u \|_{\ell^1} < \| P_{\mathcal{L}} u \|_{\ell^1}.
\]

The importance of the NSP comes from the following theorem on the performance of \( \ell^1 \)-minimization.

**Theorem 5.2** ([18, theorems 2, 3]). Any vector \( u^\diamond \) with \( \text{supp}(u^\diamond) \subset I \) is the unique solution of

\[
\min_u \| u \|_{\ell^1} \text{ s.t. } K u = K u^\diamond
\]

if and only if \( K \) fulfills the NSP for the set \( I \).

However, the NSP is implied by the uniform strict SC.

**Proposition 5.3.**  The uniform strict SC (22) implies the NSP (23).

Proof.  For any \( u \in \ker K \) and any \( v \in \mathcal{H}_2 \) it holds that

\[
0 = \langle K u, v \rangle = \langle u, K^* v \rangle.
\]

Now, we define \( u^\diamond \) by

\[
i \in I : \text{sign}(u^\diamond_i) = -\text{sign}(u_i), \quad i \notin I : u^\diamond_i = 0.
\]

Due to (22) we can find \( w \) such that \( K^* w \in \text{Sign}(u^\diamond) \) and moreover \( \| P_{\mathcal{L}} w \|_{\ell^\infty} < 1 \). Using this \( w \) instead of \( v \), we obtain from the definition of \( u^\diamond \) and the Hölder inequality

\[
0 = \langle u, K^* w \rangle = \sum_{i \in I} u_i \text{sign}(u^\diamond_i) + \sum_{i \notin I} u_i (K^* w)_i < -\| P_I u \|_{\ell^1} + \| P_{\mathcal{L}} u \|_{\ell^1},
\]

which shows the assertion. \hfill \Box
The fact that the strict SC is an important condition in this context was already observed in [4, section II]. Combining their argumentation there with theorem 5.2 one obtains another proof of proposition 5.3.

Since there are plenty of conditions which are related to the performance of $\ell^1$-minimization, we end this section with an illustration of the implications between different conditions in the context of this paper. First, the obvious implication between the different ‘ERCs’:

\[
\varepsilon\text{ERC} \Rightarrow \text{Neumann } \varepsilon\text{ERC} \Rightarrow \text{ERC} \Rightarrow \text{Neumann ERC}
\]

And then the relation of ERC, SC and NSP:

\[
\text{ERC} + KP_I \text{ injective } \Rightarrow \text{uniform strict SC} \Rightarrow \text{NSP}
\]

\textbf{Remark 5.4.} The implication ‘ERC + KP_I injective } \Rightarrow \text{NSP’ has already been observed in [18, remark 4]. Moreover, [18, example 1] shows that the converse implication does not hold. We postpone further investigation of converse implications to future work.

6. Application of ERCs to digital holography

To apply the Neumann $\varepsilon$ERC (20), one has to know the support $I$. In this case, there would be no need to apply complex reconstruction methods. One may just solve the restricted least-squares problem. For deconvolution problems, however, with a certain prior knowledge, it is possible to evaluate the Neumann $\varepsilon$ERC (20) \textit{a priori}, especially when the support $I$ is not known exactly.

In the following, we use the Neumann $\varepsilon$ERC (20) exemplarily for an inverse convolution problem as it is used in digital holography of particles [8, 25]. The presentation relies on [9] and we reproduce it here for the sake of completeness in a compact style. In digital holography, the hologram corresponds to the diffraction patterns of the illuminated particles. The hologram is recorded digitally on a charge-coupled device, from the diffraction patterns the size and the distribution of particles are reconstructed.

We consider the case of spherical particles, which is of significant interest in applications such as fluid mechanics. We model the particles $j \in \{1, \ldots, N\}$ as opaque disks $B_r(x_j, y_j, z_j)$ with center $(x_j, y_j, z_j) \in \mathbb{R}^3$ and radius $r$. Hence, the source $f^\circ$ is given as a sum of characteristic functions

\[
f^\circ = \sum_{j=1}^{N} u_j^\circ \chi B_r(x_j, y_j, z_j) =: \sum_{j=1}^{N} u_j^\circ \chi_j.
\]

The real values $u_j^\circ$ are amplitude factors of the diffraction pattern that in practice depend on experimental parameters.

The forward operator $K : \ell^2 \to L^2(\mathbb{R}^2)$, which maps the coefficients $u_j$ to the corresponding digital hologram, is well modeled by a bidimensional convolution $*$ with
respect to \((x, y)\). In the following, \(i\) represents the imaginary unit. Let \(h_z\) constitute the Fresnel function defined by

\[
h_z(x, y) = \frac{1}{i\lambda z} \exp\left(i \frac{\pi}{\lambda z} ||R||^2\right), \quad \text{with} \quad R := (x, y).
\]

With that, the hologram of a particle at position \((x_j, y_j, z_j)\) and hence the corresponding operator response \(Ke_j\) has the following form [25]:

\[
(Ke_j)(x, y) = \frac{2}{\pi r^2} \chi_{Br}(x - x_j, y - y_j) * \Re(h_z(x - x_j, y - y_j)).
\]

The factor \(2/(\pi r^2)\) assures \(Ke_j\) to be unit-normed, cf [9].

The first step to evaluate the Neumann \(\varepsilon\)ERC (20) is to calculate the correlation \(|\langle Ke_i, Ke_j \rangle|\) with distance \(\varrho_{j,i} := (x_j - x_i, y_j - y_i)\). In the following, we assume that all particles are located in a plane parallel to the detector, i.e. \(z := z_i\) is constant for all \(i\). In [9], it has been shown that the correlation in digital holography can be estimated by the following majorizing function \(M: \mathbb{R}^+ \to \mathbb{R}\), with known constants \(b_L \approx 0.6748\), \(c_L \approx 0.7857\) and \(C(d)\) denoting the area of the intersection of two circles with radius \(r\) and distance \(d\):

\[
|\langle Ke_i, Ke_j \rangle| |\varrho_{j,i}| \leq M(|\varrho_{j,i}|) := \frac{C(|\varrho_{j,i}|)}{\pi r^2} \left(1 + \frac{1}{4} \min\left\{b_L^2, c_L^2 \left(\frac{\lambda z}{2\pi r}\right)^{\frac{3}{2}}, \|\varrho_{j,i}\|^{-\frac{3}{2}}\right\}\right) \min\left\{1, \frac{2\lambda z}{\pi} \|\varrho_{j,i}\|^{-2}\right\},
\]

which is monotonically decreasing in \(\|\varrho_{j,i}\|\), cf [9].

With the estimate (25), we come to a resolution bound for droplets jet reconstruction, as e.g. used in [25]. Here, monodisperse droplets (i.e. they have the same size, shape and mass) were generated and emitted on a straight line parallel to the detector plane. This configuration eases the computation of the \(\varepsilon\)Neumann ERC. We define that the particles are located at some grid points

\[
\Delta Z := \{i \in \mathbb{Z} | i/\Delta \in \mathbb{Z}\},
\]

where the parameter \(\Delta\) describes the grid refinement. Assume that the particles have the minimal distance

\[
\rho := \min_{i, j \in \text{supp}(u^{\diamond})} \|\varrho_{j,i}\| \in \Delta \mathbb{N};
\]

then the sums of correlations \(\text{COR}_f\) and \(\text{COR}_{\varepsilon}\) can be estimated from above. W.l.o.g. we fix one particle at the origin and estimate with the worst case that the other particles appear at a distance of \(j\rho\) to the origin, with \([-N/2] \leq j \leq [N/2]\). Then, for \(\rho > \Delta\) we obtain

\[
\text{COR}_f = \sum_{i \in \mathbb{Z}, j \neq i} |\langle Ke_i, Ke_j \rangle| \leq 2 \sum_{j=1}^{[N/2]} M(j\rho),
\]

\[
\text{COR}_{\varepsilon} = \sum_{i \in \mathbb{Z}, j \neq i} |\langle Ke_i, Ke_j \rangle| \leq \sum_{\Delta \leq d \leq \rho, j = -[N/2]} M((j\rho - i)).
\]

Consequently, we can formulate an estimate for the Neumann \(\varepsilon\)ERC (20).
Figure 1. Simulated holograms of spherical particles. On the left-hand side the noiseless signal is displayed. For reconstruction, the noisy signal on the right-hand side is used. The dots correspond to the true location of the particles. The existence of a suitable regularization parameter is guaranteed by condition (26) of proposition 6.1 and hence the $\ell^1$-penalized Tikhonov regularization recovered all particles exactly.

**Proposition 6.1** (Neumann $\varepsilon$ERC for Fresnel-convolved characteristic functions). An estimate from above for the Neumann $\varepsilon$ERC (20) for characteristic functions convolved with the real part of the Fresnel kernel is for $\rho > \Delta$

$$2 \sum_{j=1}^{[N/2]} \mathcal{M}(j\rho) + \sup_{1 \leq i < \rho \Delta} \sum_{j=-[N/2]}^{[N/2]} \mathcal{M}(|j\rho - i\Delta|) < 1 - 2r\eta/u.$$  

(26)

This means that there is a regularization parameter $\alpha$ which allows the exact recovery of the support with the $\ell^1$-penalized Tikhonov regularization if the above condition is fulfilled.

**Remark 6.2.** In comparison, the ERC in terms of the coherence parameter $\mu = \sup_{i \neq j} |\langle Ke_i, Ke_j \rangle|/\langle \varrho_{j,i} \rangle \leq \sup_{i \neq j} \mathcal{M}(\|\varrho_{j,i}\|) = \mathcal{M}(\Delta)$ according to Fuchs’ ERC from [15] appears as

$$(2N - 1) \mathcal{M}(\Delta) < 1 - 2r\eta/u,$$

see remark 4.11. This condition is significantly worse than (26), since asymptotically it holds that $\mathcal{M}(\|\varrho_{j,i}\|) \sim \|\varrho_{j,i}\|^{-4}$.

Condition (26) of proposition 6.1 seems not to be easy to handle due to the upper bound $\mathcal{M}$ from (25). However, in practice all parameters are known, and one can compute a bound via approximating from large $\rho$. As soon as the sum is smaller than $1 - 2r\eta/u$, it is guaranteed that the $\ell^1$-penalized Tikhonov regularization can recover exactly.

We apply the Neumann $\varepsilon$ERC (26) to simulated data of droplet jets. For the simulation we use a red laser of wavelength $\lambda = 0.6328$ $\mu$m and a distance of $z = 200$ mm from the camera. The particles have a diameter of 100 $\mu$m and for the corresponding grid we choose a refinement of 25 $\mu$m. Those parameters correspond to that of the experimental setup used in [25, 26].

After applying the digital holography model, we add Gaussian noise of different noise levels and in each case of zero mean. For the coefficients $u^2_i$, we choose a setting which implies $u^2_i \approx 10$ for all $i \in I$. Figures 1 and 2 show simulated holograms with different distances $\rho$ and different noise-to-signal ratios $r\eta/u$. For all noisy examples in the right columns of figures 1 and 2, it was manually possible to find a regularization parameter $\alpha$ so that all
Figure 2. Simulated holograms of spherical particles. In the left column the noiseless signals are displayed. For reconstruction, the noisy signals of the right column are used. The dots correspond to the true location of the particles. The existence of a suitable regularization parameter cannot be shown with condition (26) of proposition 6.1; however, it is possible to find a regularization parameter so that the \(\ell^1\)-penalized Tikhonov regularization recovered all particles exactly. The reason why condition (26) is not fulfilled is that in the top image, the particles have a too small distance to each other, and at the bottom, the image was manipulated with unrealistically large noise.

particles were recovered exactly. For minimization of the Tikhonov functional we used the iterated soft-thresholding algorithm [7].

However, only for the image in figure 1 (\(\rho \approx 721 \mu m\)) condition (26) of proposition 6.1 holds; hence the existence of a suitable regularization parameter was guaranteed. For the examples in figure 2, the existence of a regularization parameter which ensures exact recovery cannot be shown, i.e. condition (26) of proposition 6.1 is not valid. In the top image of figure 2, the particles have a too small distance to each other (\(\rho \approx 360 \mu m\)), and even for the noiseless case condition (26) is not fulfilled. The bottom image of figure 2 (\(\rho \approx 721 \mu m\)) was manipulated with unrealistically large noise, so that condition (26) is violated, too.

7. Conclusion

With papers [16] and [17], the analysis of a priori parameter rules for \(\ell^1\)-penalized Tikhonov functionals seemed completed. On the common parameter rule \(\alpha \approx \varepsilon\), linear, i.e. the best possible, convergence is guaranteed. In this paper, we have gone beyond this question by presenting a parameter rule which ensures exact recovery of the unknown support of \(u^\diamond \in \ell^0\).
Moreover, on that condition we achieve a linear convergence rate measured in the $\ell^1$-norm that comes with \textit{a priori} checkable error constants, which are easier to handle than the ones from [16]. A side product of our analysis is the proof of convergence in $\ell^0$ in the topology of sparse convergence.

Section 5 analyzes some implications between different conditions for exact recovery. However, in most cases it remains open whether the reverse implications also hold and we postpone this investigation to future work.

Granted, to apply the Neumann $\varepsilon$ERC (20) and the Neumann parameter rule (21) one has to know the support $I$. However, with a certain prior knowledge the correlations

$$\text{COR}_I := \sup_{i \in I} \sum_{j \in I, j \neq i} |\langle K e_i, K e_j \rangle| \quad \text{and} \quad \text{COR}_{I^c} := \sup_{i \in I^c} \sum_{j \in I} |\langle K e_i, K e_j \rangle|$$

can be estimated from above \textit{a priori}, especially when the support $I$ is not known exactly. That way it is possible to obtain \textit{a priori} computable conditions for exact recovery. In section 6, it has been exemplarily for characteristic functions convolved with a Fresnel function. This shows the practical relevance of the condition.

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