MINIMAL REPRESENTATIONS OF SIMPLE REAL LIE GROUPS OF HERMITIAN TYPE
THE FOCK AND THE SCHRODINGER MODELS

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Abstract In the recent paper [A12], a unified geometric realization to the minimal representations of simple real Lie groups of non Hermitian type is given, based on the geometric setting introduced in [A11] and the analysis of the Brylinski-Kostant model, given in [AF12]. We give in this paper a unified geometric realization to the minimal representations of simple real Lie groups of Hermitian type.

Key words: Minimal representation, Lie algebra, Jordan algebra, Conformal group,

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Introduction
1. Construction process of complex simple Lie algebras and real forms of Hermitian type.
2. \( L_\mathbb{R} \)-invariant Hilbert subspaces of \( O(\Xi) \) and \( O(\Xi^\sigma) \).
3. Representations of the Lie algebra.
4. Unitary representations of the corresponding real Lie group.
5. Schrodinger model and Bargmann transform.
6. The \( \mathfrak{s}(2, \mathbb{R}) \)-case.

References
Introduction. —

In the paper [A11] the following principles were established:

1) Given a complex simple Jordan algebra $V$ of rank $r$ and dimension $n$, and a homogeneous polynomial $Q$ of degree $2r$ on $V$ ($Q(v) = \Delta(v)^2$ where $\Delta$ is the determinant on $V$) there is a covering $K$ of order one or two, of the conformal group $\text{Conf}(V,Q)$ and a cocycle $\mu: K \times V \to \mathbb{C}$ such that the corresponding cocycle representation of $K$ on the polynomial functions on $V$ leaves the space $p$, spanned by $Q$ and its translates $z \mapsto Q(z-a)$ with $a \in V$, invariant, producing an irreducible representation $\kappa$ of $K$ on $p$ (see Proposition 1.1 and Corollary 1.2 in [A11]). In particular, the group $L$, which is the preimage of the structure group of $V$ by the covering, acts on $p$ by the restriction of $\kappa$.

2) There is $\tilde{H} \in \mathfrak{g}(l)$ the center of $l = \text{Lie}(L)$ such that $d\kappa(\tilde{H})$ defines a grading of $p$ given by

\[ p = p_{-r} \oplus p_{-r+1} \oplus \cdots \oplus p_0 \oplus \cdots \oplus p_{r-1} \oplus p_r \]

where $p_i = \{p \in p \mid d\kappa(\tilde{H})p = ip\}$ is the set of homogeneous polynomials of degree $i + r$ in $p$. In particular, $p_{-r} = \mathbb{C}, p_r = \mathbb{C} \cdot Q, p_{-r+1} \simeq p_{r-1} \simeq V$ are simple $l$-modules and more precisely, when $r \neq 1$, $\mathcal{V} := p_{-r+1}$ is the dual of $V$ and $\mathcal{V}^\sigma := p_{r-1} = \kappa(\sigma)p_{-r+1}$, where $j$ is the conformal inversion on $V$ and $\sigma$ its preimage in $\tilde{K}$ by the covering map $s: \tilde{K} \to \text{Conf}(V,Q)$. In the special case $r = 1$, we denote by $\mathcal{V} = p_{-r}$ and $\mathcal{V}^\sigma = p_r$.

3) $\text{Lie}(L) \oplus \mathcal{W}$, where $\mathcal{W} = \mathcal{V} \cup \mathcal{V}^\sigma$, carries the structure of a complex simple Lie algebra $\mathfrak{g}$ (see Theorem 8.1 in [A11]). One constructs a non compact real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$, of Hermitian type, starting with a Euclidean real form $V_\mathbb{R}$ of $V$. The construction also yields a compact real form $L_\mathbb{R}$ of the group $L$. It turns out that we obtain in this way (see Table 3 in [A11]) all the simple real Lie algebras of Hermitian type which possess a strongly minimal real nilpotent orbit, i.e. such that $\mathcal{O}_{\min} \cap \mathfrak{g}_\mathbb{R} \neq \emptyset$, where $\mathcal{O}_{\min}$ is the minimal nilpotent adjoint orbit of $\mathfrak{g}$. Recall that from a point of view of representation theory, the condition that $\mathcal{O}_{\min} \cap \mathfrak{g}_\mathbb{R} \neq \emptyset$ is natural, since it is a necessary condition for a simple real Lie group with Lie algebra $\mathfrak{g}_\mathbb{R}$ to admit an irreducible unitary representation with associated complex nilpotent orbit $\mathcal{O}_{\min}$ (see Theorem 8.4 in [V91]).

Recall that when $V$ is simple, the map $v \mapsto \text{trace}(v)$ is (up to a scalar) the only linear form on $V$ which is invariant under the group $\text{Aut}(V)$ of automorphisms of $V$. Let $\tau \in p_{-r+1}$ be this linear form and denote by $\tau^\sigma = \kappa(\sigma)\tau \in p_{r-1}$, then $\tau(v) = tr(v)$ and $\tau^\sigma(v) = \Delta(v)^2 tr(-v^{-1})$.

In the Lie algebra $\mathfrak{g} = \text{Lie}(L) \oplus \mathcal{W}$, with $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\sigma$, one has

\[ [\tilde{H}, \tau] = (1-r)\tau \quad [\tilde{H}, \tau^\sigma] = -(1-r)\tau^\sigma \quad [\tau, \tau^\sigma] = \frac{1}{2} \tilde{H}. \]
The two $L$-orbits $\Xi = L.\tau \subset V$ and $\Xi^\sigma = L.\tau^\sigma \subset V^\sigma$ are conical varieties, related by $\Xi^\sigma = \kappa(\sigma)\Xi$, correspond respectively to the orbits $O_{\min} \cap V$ and $O_{\min} \cap V^\sigma$ in such a way that $O_{\min} \cap V = \Xi \cup \Xi^\sigma$. If $V$ is a matrix algebra (which means $V \neq \mathbb{C}^n$ and then $g_{\mathbb{R}} \neq \mathfrak{so}(n, 2)$), the orbits $\Xi$ and $\Xi^\sigma$ have a coordinate system $X = \{P(a) \mid a = \sum_{i=1}^r a_i c_i \in \mathcal{R}\}$, where $\{c_1, \ldots, c_r\} \subset V_{\mathbb{R}}$ is a complete system of orthogonal primitive idempotents, $P$ is the quadratic representation of $V$ and $\mathcal{R} = \{a \in \mathbb{C}^r, \text{Re}(a_1) \geq \ldots \geq \text{Re}(a_r) > 0, \text{Im}(a_1) \geq \ldots \geq \text{Im}(a_r) > 0\}$.

The map $\tilde{\pi}(\sigma) : f^\sigma \in \mathcal{O}(\Xi^\sigma) \mapsto \tilde{\pi}(\sigma)f^\sigma \in \mathcal{O}(\Xi)$, where $\tilde{\pi}(\sigma)f^\sigma(\xi) = f^\sigma(\kappa(\sigma)\xi)$, is an isomorphism. In the coordinate system, it is given by:

$$\tilde{\pi}(\sigma) : \mathcal{O}(X) \rightarrow \mathcal{O}(X), \phi(P(a)) \mapsto \phi(P(a^{-1})).$$

The $L$-actions on $\Xi$ and on $\Xi^\sigma$ yield representations $\pi_{\alpha}$ and $\pi_{\sigma}^\alpha$ of $L$ on the spaces $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ of holomorphic functions and then representations $\tilde{\pi}_{\alpha}$ and $\tilde{\pi}^\alpha_{\sigma}$ of $L$ on the space $\mathcal{O}(X)$. The isomorphism $\tilde{\pi}(\sigma)$ intertwines these representations.

Along the paper, The rank $r$ will be assumed $> 1$. The case $r = 1$ will be considered separately, at the last section.

The subspaces $\mathcal{O}_{\frac{m}{(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{\frac{m}{(r-1)}}(\Xi^\sigma)$) of fixed degree of homogeneity $-\frac{m}{(r-1)}$ in the fiber direction are irreducible $L$-modules. Homogeneity in the fiber direction allows to identify the space $\mathcal{O}_{\frac{m}{(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{\frac{m}{(r-1)}}(\Xi^\sigma)$) with the subspace $\mathcal{O}_m(X) \subset \mathcal{O}_{2m}(\mathbb{C}^r)$ of holomorphic functions on $X$, homogeneous of degree $m$. Also, the subspaces $\mathcal{O}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$) of fixed degree of homogeneity $-\frac{m}{(r-1)}-\frac{1}{2(r-1)}$ in the fiber direction are irreducible $L$-modules. Homogeneity in the fiber direction allows to identify $\mathcal{O}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$) with the subspace $\mathcal{O}_{m+\frac{1}{2}}(X) \subset \mathcal{O}_{2m+1}(\mathbb{C}^r)$ of holomorphic functions on $X$, homogeneous of degree $m + \frac{1}{2}$.

We consider the subspace $\tilde{\mathcal{O}}(X)$ of $\mathcal{O}(X)$ of the functions $\phi$ such that $\phi \circ P$ extends to a holomorphic function on $\mathbb{C}^r$. And then consider the corresponding subspaces $\tilde{\mathcal{O}}(\Xi)$ and $\tilde{\mathcal{O}}(\Xi^\sigma)$ of $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ and denote by

$$\tilde{\mathcal{O}}_{\frac{m}{(r-1)}}(\Xi) = \mathcal{O}_{\frac{m}{(r-1)}}(\Xi) \cap \tilde{\mathcal{O}}(\Xi),$$

$$\tilde{\mathcal{O}}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) = \mathcal{O}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) \cap \tilde{\mathcal{O}}(\Xi),$$

and similarly,

$$\tilde{\mathcal{O}}_{\frac{m}{(r-1)}}(\Xi^\sigma) = \mathcal{O}_{\frac{m}{(r-1)}}(\Xi^\sigma) \cap \tilde{\mathcal{O}}(\Xi^\sigma),$$

$$\tilde{\mathcal{O}}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) = \mathcal{O}_{\frac{-m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) \cap \tilde{\mathcal{O}}(\Xi^\sigma).$$
It turns out that for $m < 0$
\[ \tilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi) = \tilde{\mathcal{O}}_{-\frac{m}{r-1} - \frac{1}{2(r-1)}}(\Xi) = \{0\} \]
and
\[ \tilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi^\sigma) = \tilde{\mathcal{O}}_{-\frac{m}{r-1} - \frac{1}{2(r-1)}}(\Xi^\sigma) = \{0\}. \]

There exist $L_\mathbb{R}$-invariant inner products on the spaces
\[ \tilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi) \simeq \tilde{\mathcal{O}}_{m}(X), \quad \tilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi^\sigma) \simeq \tilde{\mathcal{O}}_{m}(X) \quad (m \geq 0), \]
which have reproducing kernels $H^{2m}$ and $H^{\alpha(m)}$, and, there exist $L_\mathbb{R}$-invariant inner products on the spaces
\[ \tilde{\mathcal{O}}_{-\frac{m}{r-1} - \frac{1}{2(r-1)}}(\Xi) \simeq \tilde{\mathcal{O}}_{m+\frac{1}{2}}(X), \quad \tilde{\mathcal{O}}_{-\frac{m}{r-1} - \frac{1}{2(r-1)}}(\Xi^\sigma) \simeq \tilde{\mathcal{O}}_{m+\frac{1}{2}}(X) \quad (m \geq 0), \]
which have reproducing kernels $H^{2m+1}$ and $H^{\alpha(m+\frac{1}{2})}$, where
\[ H(z) = \tau(\frac{1}{r} e + z\bar{z}) \text{ and } H^\sigma(z) = \tau^\sigma(\frac{1}{r} e + z\bar{z}). \]

Adding the kernels $H^{2m}$ and $H^{2m+1}$ (respectively $H^{\alpha(m)}$ and $H^{\alpha(m+\frac{1}{2})}$) with arbitrarily chosen non-negative weights yields a multiplicity free unitary $L_\mathbb{R}$-representation on a Hilbert subspace of $\mathcal{O}(\Xi)$ (respectively $\mathcal{O}(\Xi^\sigma)$).

For the representation $\rho$, the polynomials $p \in \mathcal{V}$ act by multiplication and the polynomials $p \in \mathcal{V}^\sigma$ act by differentiation on the spaces of finite sums
\[ \mathcal{O}_{\text{fin}}(\Xi) = \mathcal{O}_{\text{odd}}(\Xi) \oplus \mathcal{O}_{\text{even}}(\Xi) \]
where
\[ \mathcal{O}_{\text{even}}(\Xi) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{m}(X) \]
and
\[ \mathcal{O}_{\text{odd}}(\Xi) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{-\frac{m}{r-1} - \frac{1}{2(r-1)}}(\Xi) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{m+\frac{1}{2}}(X). \]

Similarly, for the representation $\rho^\sigma := \pi(\sigma)\rho\bar{\pi}(\sigma)$, the polynomials $p \in \mathcal{V}^\sigma$ act by multiplication and the polynomials $p \in \mathcal{V}$ act by differentiation on the spaces of finite sums
\[ \mathcal{O}_{\text{fin}}(\Xi^\sigma) = \mathcal{O}_{\text{odd}}(\Xi^\sigma) \oplus \mathcal{O}_{\text{even}}(\Xi^\sigma) \]
where
\[ \mathcal{O}_{\text{even}}(\Xi^\sigma) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi^\sigma) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{m}(X) \]
and
\[ \mathcal{O}_{\text{odd}}(\Xi^\sigma) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{-\frac{m}{r-1} - \frac{1}{2(r-1)}}(\Xi^\sigma) = \sum_{m \in \mathbb{N}} \tilde{\mathcal{O}}_{m+\frac{1}{2}}(X). \]
For $E = \tau$ and $F = \tau^\sigma$ in $\mathcal{W}$, the operators $\rho(E), \rho(F), \rho^\sigma(E), \rho^\sigma(F)$ are $L$-equivariant. Since

\[ [\hat{H}, E] = (1 - r)E \quad [\hat{H}, F] = (r - 1)F \quad [E, F] = \frac{1}{2} \hat{H} \]

then, for $H = \frac{r}{r - 1} \hat{H}$, one has

\[ [H, E] = -2E \quad [H, F] = 2F \quad [E, F] = \frac{r - 1}{4} H. \]

The number $\alpha$ is chosen such that $[\rho(E), \rho(F)] = \frac{r - 1}{4} d\pi_\alpha(H)$. Then $\rho : \mathcal{W} \to \text{End}(\mathcal{O}_{\text{fin}}(\Xi)), p \mapsto \rho(p)$ and $\rho^\sigma : \mathcal{W} \to \text{End}(\mathcal{O}_{\text{fin}}(\Xi^\sigma)), p \mapsto \rho^\sigma(p)$ complement $d\pi_\alpha$ and $d\pi_\alpha^\sigma$ to representations $d\pi_\alpha + \rho$ and $d\pi_\alpha^\sigma + \rho^\sigma$ of $\mathfrak{g} = \text{Lie}(L) + \mathcal{W}$ on $\mathcal{O}_{\text{fin}}(\Xi)$ and $\mathcal{O}_{\text{fin}}(\Xi^\sigma)$ respectively.

Furthermore, it is possible to determine a set of weights such that the restrictions of $d\pi_\alpha + \rho$ and $d\pi_\alpha^\sigma + \rho^\sigma$ to $\mathfrak{g}_R$ are infinitesimally unitary. Moreover, Nelson’s criterion can be used to show that these representations integrate to the simply connected Lie group $G_\mathbb{R}$ with Lie algebra $\mathfrak{g}_\mathbb{R}$.

The two representations so obtained are realized on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^\sigma$ of holomorphic functions on $\Xi$ and $\Xi^\sigma$. The reproducing kernel of $\mathcal{H}$ is given by $\exp(\hat{H}(z, z'))$. The space $\mathcal{H}$ is a weighted Bergman space. The norm is given by an integral on $\mathbb{C}^r$ with a weight given by $p(z) = \exp(-\hat{H}(z)) = \exp(-\text{tr}(zz))$ (up to a constant), in such a way that $\mathcal{H}$ is the classical Fock-space $\mathcal{F}(\mathbb{C}^r)$.

The paper [O76] gives a Fock model and the reproducing kernel for the minimal representation in the $\mathfrak{so}(n, 2)$-case, realized in a Hilbert space of holomorphic functions on the minimal adjoint orbit, with a norm given by an integral along the orbit. This case is (unfortunately) not covered by our paper since it does not correspond to a matrix Jordan algebra. The coordinate system of the orbit $\Xi$ is ’bigger’ than the set $X \simeq \mathbb{C}^2$ in this case, since one doesn’t have the nice property of bi-invariance of the trace under the automorphism group of the Jordan algebra.

The recent work by J. Hilgert, T. Kobayashi, J. Moller and B. Orsted, (see [HKMO12]), consists in constructing with a different method, Schrödinger and Fock models for minimal representations of Hermitian real Lie groups of tube type and the authors obtained a nice formula for the intertwinning operator, the Bargmann transform, between the two models.

This method works for almost all simple Hermitian real Lie groups which admit minimal representations (unless the $\mathfrak{so}(n, 2)$-case). Since it is based on a similar theory than that considered in the case of simple real Lie groups of non Hermitian type, we can consider that the papers [A12] and the present one give a rather unified theory for the Fock model of the minimal representations of simple real Lie groups.
1. Construction process of complex simple Lie algebras and of simple real Lie algebras of Hermitian type.

Let $V$ be a simple complex Jordan algebra with rank $r$ and dimension $n$ and $Q$ the homogeneous polynomial of degree $2r$ on $V$ given by $Q(v) = \Delta(v)^2$ where $\Delta$ is the Jordan algebra determinant. Let

$$\text{Str}(V,Q) = \{ g \in \text{GL}(V) \mid \exists \gamma(g) \in \mathbb{C}, Q(g \cdot z) = \gamma(g)Q(z) \}.$$  

The conformal group $\text{Conf}(V,Q)$ is the group of rational transformations $g$ of $V$ generated by: the translations $z \mapsto z + a$ ($a \in V$), the dilations $z \mapsto \ell \cdot z$ ($\ell \in \text{Str}(V)$), and the conformal inversion $j : z \mapsto -z^{-1}$ (see [M78]).

Let $p$ be the space of polynomials on $V$ generated by the translated $Q(z - a)$ of $Q$, with $a \in V$. Let $\kappa$ be the cocycle representation of $\text{Conf}(V,Q)$ or of a covering $K$ of order two of it on $p$, defined in [A11] and [AF12] as follows:

**Case 1**

In case there exists a character $\chi$ of $\text{Str}(V,Q)$ such that $\chi^2 = \gamma$, then let $K = \text{Conf}(V,Q)$. Define the cocycle

$$\mu(g,z) = \chi(Dg(z)^{-1}) \quad (g \in K, \ z \in V),$$

and the representation $\kappa$ of $K$ on $p$,

$$(\kappa(g)p)(z) = \mu(g^{-1},z)p(g^{-1} \cdot z).$$

The function $\kappa(g)p$ belongs actually to $p$ (see [FG96], Proposition 6.2). The cocycle $\mu(g,z)$ is a polynomial in $z$ of degree $\leq \deg Q$ and

$$(\kappa(\tau_a)p)(z) = p(z - a) \quad (a \in V),$$

$$(\kappa(\ell)p)(z) = \chi(\ell)p(\ell^{-1} \cdot z) \quad (\ell \in L),$$

$$(\kappa(j)p)(z) = Q(z)p(-z^{-1}).$$

**Case 2** Otherwise the group $K$ is defined as the set of pairs $(g,\mu)$ with $g \in \text{Conf}(V,Q)$, and $\mu$ is a rational function on $V$ such that

$$\mu(z)^2 = \gamma(Dg(z))^{-1}.$$ 

We consider on $K$ the product $(g_1,\mu_1)(g_2,\mu_2) = (g_1g_2,\mu_3)$ with $\mu_3(z) = \mu_1(g_2 \cdot z)\mu_2(z)$. For $\tilde{g} = (g,\mu) \in K$, define $\mu(\tilde{g},z) = \mu(z)$. Then $\mu(\tilde{g},z)$ is a cocycle: $\mu(\tilde{g}_1\tilde{g}_2,z) = \mu(\tilde{g}_1,\tilde{g}_2 \cdot z)\mu(\tilde{g}_2,z)$, where $\tilde{g} \cdot z = g \cdot z$ by definition.
Recall that the representation $\kappa$ of $K$ on $\mathfrak{p}$ is irreducible and 
\[(\kappa(g)p)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).\]

Observe that since the degree of $Q$ is even it follows that the inverse in $K$ of $\sigma = (j, Q(z))$ is $\sigma$.

Let $L$ be $\text{Str}(V, Q)$ in the first case or the preimage of $\text{Str}(V, Q)$ by the covering map $s : K \to \text{Conf}(V, Q)$, in the second case. It is established in [A11] that there is $\tilde{H} \in \mathfrak{z}(\mathfrak{l})$, with $\mathfrak{l} = \text{Lie}(L)$ which defines a grading of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{p}_{-r} \oplus \mathfrak{p}_{-r+1} \oplus \ldots \oplus \mathfrak{p}_0 \oplus \ldots \oplus \mathfrak{p}_{r-1} \oplus \mathfrak{p}_r,$$

where

$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid d\kappa(\tilde{H})p = jp\}$$

is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j + r$. Furthermore $\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}$, and

$$\mathfrak{p}_{-r} = \mathbb{C}, \quad \mathfrak{p}_r = \mathbb{C} Q, \quad \mathfrak{p}_{r-1} \simeq V, \quad \mathfrak{p}_{-r+1} \simeq V.$$

Observe that $\mathfrak{p}_{-r+1}$ is the space of linear forms on $V$, $\mathfrak{p}_{r-1} = \{\kappa(\sigma)p \mid p \in \mathfrak{p}_{-r+1}\}$ and for a linear form $\tau$ on $V$,

$$\kappa(\sigma)\tau(z) = \Delta(z)^2\tau(-z^{-1}).$$

Assume $r \neq 1$ and denote by $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\sigma$ where

$$\mathcal{V} = \mathfrak{p}_{-r+1}, \quad \mathcal{V}^\sigma = \mathfrak{p}_{r-1}.$$

Let’s consider the linear form $\tau : z \mapsto \text{tr}(z) = \text{trace}(z)$, which is the unique (up to a scalar) $K$-invariant linear form on $V$, with $K = \text{Aut}(V)$ the automorphism group of $V$. Then $\tau^\sigma(z) = Q(z)\tau(-z^{-1})$. Observe that in the case $r = 1$, $\tau$ equals $-\tau^\sigma$.

Denote by $E = \tau$ and $F = \tau^\sigma$ and let $X_E \in \mathfrak{k}_1$ and $X_F \in \mathfrak{k}_{-1}$ such that $E = d\kappa(X_E)1$ and $F = d\kappa(X_F)Q$.

Then $\mathfrak{g} = \mathfrak{l} \oplus \mathcal{W}$ carries a unique simple Lie algebra structure such that

\begin{align*}
(i) \quad [X, X'] &= [X, X'] \mathfrak{l} \quad (X, X' \in \mathfrak{l}), \\
(ii) \quad [X, p] &= d\kappa(X)p \quad (X \in \mathfrak{l}, p \in \mathcal{W}), \\
(iii) \quad [E, F] &= [X_E, X_F].
\end{align*}

In fact, this follows from Theorem 8.1 in [A11]: for every $p \in \mathcal{V}$, there is a unique $X_p \in \mathfrak{k}_{-1}$ such that $p = d\kappa(X_p)Q$ and for every $p^\sigma \in \mathcal{V}^\sigma$, there is a unique $X_{p^\sigma} \in \mathfrak{k}_1$ such that $p^\sigma = d\kappa(X_{p^\sigma})1$ (see Lemma 1.1 in [A11]) and one defines the bracket $[p, p^\sigma] = [X_p, X_{p^\sigma}]$. 

7
Furthermore, since $E = dk(X_E)1$ and $F = dk(X_F)Q$, then there is $a \in V$ such that for every $t \in \mathbb{C}$, $\exp(tX_F)$ is the translation $\tau_{ta} : V \to V, v \mapsto v + ta$ or a preimage $\tilde{\tau}_{ta}$ of such translation by the covering map. Furthermore, since $E = \kappa(\sigma)F$, then $\exp(tX_E)$ equals $j \circ \tau_{ta} \circ j$ or $\sigma \tau_{ta} \sigma$. Since for $z \in V$, 

$$F(z) = \frac{d}{dt}_{|t=0} Q(z + ta) = DQ(z)(a) = 2Q(z)tr(z^{-1}a)$$

then $a = -\frac{1}{2}e$, where $e$ is the unit element in $V$, and, conformally to the notations in [FK94] page 209 for the elements of the Kantor-Koecher-Tits algebra $\mathfrak{t}$,

$$X_F = \left(-\frac{1}{2}e, 0, 0\right) = (u_2, T_2, v_2)$$

and

$$X_E = (0, 0, -\frac{1}{2}e) = (u_1, T_1, v_1)$$

and it follows that

$$[X_E, X_F] = (0, 0)$$

with $T \in \mathfrak{t}_0 = I$ given by

$$T = -2(u_2 \circ v_1) = -\frac{1}{2}(e \circ e) = -\frac{1}{2}id_V = \frac{1}{2}\tilde{H}.$$ 

Now, since $[\tilde{H}, E] = (1-r)E$ , $[\tilde{H}, F] = (r-1)F$ , $[E, F] = \frac{1}{2}\tilde{H}$, then, for

$$H = \frac{2}{1-r}\tilde{H},$$

$$[H, E] = 2E \ , \ [H, F] = -2F \ , \ [E, F] = \frac{1-r}{4}H.$$ 

In the special case $r = 1$, we denote by $V = \mathfrak{p}_1$ and $V^\varphi = \mathfrak{p}_{-1}$ and by $F(z) = Q(z) = z^2$ and $E(z) = 1$.

Then, for $H = -2\tilde{H}$, one has $[H, E] = 2E$ and $[H, F] = -2F$. We consider the Lie algebra structure on $\mathfrak{g} = \mathfrak{l} \oplus V \oplus V^\varphi$ such that $[E, F] = H$. The Lie algebra $\mathfrak{g}$ is then isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. This special case will be the subject of section 6.

Recall now the real form $\mathfrak{g}_R$ of $\mathfrak{g}$ which will be considered in the sequel. It has been introduced in [A11]. We fix a Euclidean real form $V_R$ of the complex Jordan algebra $V$, denote by $z \mapsto \bar{z}$ the conjugation of $V$ with respect to $V_R$, and then consider the involution $g \mapsto \bar{g}$ of $\text{Conf}(V,Q)$ given by: $\bar{g} \cdot z = \bar{g} \cdot \bar{z}$. For $(g, \mu) \in K$ define $(g, \mu) = (\bar{g}, \bar{\mu})$, where $\bar{\mu}(z) = \mu(\bar{z})$. The involution $\alpha$ defined by $\alpha(g) = \sigma \bar{g} \sigma^{-1}$ is a Cartan involution of $K$ and $K_R := \{g \in K \mid \alpha(g) = g\}$ is a compact real form of $K$ and it follows that $L_R := L \cap K_R$ is a compact real form of $L$. Observe that, since for $g \in \text{Str}(V)$, $j \circ g \circ j = g^l$, the adjoint of $g$ with respect to the symmetric form $(w \mid w') = \tau(w\bar{w}')$, then $L_R = \{l \in L \mid s(l)s(l)' = \text{id}_V\}$. 

8
Let \( u \) be the compact real form of \( g \) such that \( l \cap u = l_{\mathbb{R}} \), the Lie algebra of \( L_{\mathbb{R}} \). Denote by \( \mathcal{W}_{\mathbb{R}} = u \cap (iu) \). Then, the real Lie algebra defined by

\[
g_{\mathbb{R}} = l_{\mathbb{R}} + \mathcal{W}_{\mathbb{R}}
\]

is a real form of \( g \) and the decomposition (*) is its Cartan decomposition.

Since the complexification of the Cartan decomposition of \( g_{\mathbb{R}} \) is \( g = l + \mathcal{W} \) and since \( \mathcal{W} = \mathcal{V} + \mathcal{V}^\sigma = d\kappa(U(l))E + d\kappa(U(l))F \) is a sum of two simple \( l \)-modules, it follows that the simple real Lie algebra \( g_{\mathbb{R}} \) is of Hermitian type. One can show that

\[
\mathcal{W}_{\mathbb{R}} = \{ p \in \mathcal{W} \mid \beta(p) = p \}
\]

where we defined for a polynomial \( p \in \mathcal{W} \), \( \bar{p} = p(\bar{z}) \), and considered the antilinear involution \( \beta \) of \( \mathcal{W} \) given by \( \beta(p) = \kappa(\sigma)p \).

Since the decomposition \( g = l + \mathcal{W} \) is the complexification of the Cartan decomposition \( g_{\mathbb{R}} = l_{\mathbb{R}} + \mathcal{W}_{\mathbb{R}} \) of the real form \( g_{\mathbb{R}} \), then it follows from the Kostant-Sekiguchi correspondence that if \( G^a \) is the adjoint group of \( g \), \( G^a_{\mathbb{R}} \) is the connected Lie subgroup of \( G^a \) with Lie algebra \( g_{\mathbb{R}} \), \( L^a \) is the connected Lie subgroup of \( G^a \) with Lie algebra \( l \) and \( L^a_{\mathbb{R}} \) the connected (maximal compact) subgroup of \( G^a_{\mathbb{R}} \) with Lie algebra \( l_{\mathbb{R}} \), then there is a bijection between the set of \( G^a_{\mathbb{R}} \)-nilpotent adjoint orbits in \( g_{\mathbb{R}} \) and the set of nilpotent \( L^a_{\mathbb{R}} \)-orbits in \( \mathcal{W} \). (Observe that, up to covering, the group \( L^a \) corresponds to \( L \) and the group \( L^a_{\mathbb{R}} \) corresponds to \( L_{\mathbb{R}} \)).

Furthermore, since \( g \) is Hermitian, one knows that every nilpotent \( L^a_{\mathbb{R}} \)-orbit in \( \mathcal{W} = \mathcal{V} + \mathcal{V}^\sigma \) is a union of two connected components, which are two nilpotent \( L^a \)-orbits, one in \( \mathcal{V} \) and the other in \( \mathcal{V}^\sigma \).

Recall that in the present case, the Kostant-Sekiguchi bijection consists in \( O \cap g_{\mathbb{R}} \mapsto O \cap \mathcal{W} \), for every adjoint nilpotent orbit \( O \) in \( g \). In the particular case of the minimal nilpotent orbit \( O_{\text{min}} \) of \( g \), one knows that \( O_{\text{min}} \cap \mathcal{W} = \tilde{\Xi} \cup \tilde{\Xi}^\sigma \), where \( \tilde{\Xi} \) (resp. \( \tilde{\Xi}^\sigma \)) is the \( L^a \)-orbit in \( \mathcal{V} \) (resp. \( \mathcal{V}^\sigma \)) of a lowest (resp. highest) weight vector of the adjoint representation of \( g \).
Observe that $E + F$ and $i(E - F)$ belong to $\mathcal{W}_R$ and one can show that
\[
\mathcal{W}_R = d\kappa(U(l_R))(E + F) + d\kappa(U(l_R))(i(E - F)),
\]
then
\[
g_R = l_R + d\kappa(U(l_R))(E + F) + d\kappa(U(l_R))(i(E - F)).
\]
Since $iH$ belongs to $l_R$ and
\[
[E + F, i(E - F)] = -2i[E, F] = \frac{r-1}{2}iH,
\]
\[
[iH, E + F] = 2i(E - F), \quad [iH, i(E - F)] = -2(E + F),
\]
then the real Lie subalgebra of $g_R$ generated by $iH, E + F, i(E - F)$ is isomorphic to $\mathfrak{su}(1, 1)$.

An other real form of the Lie algebra $g$ can be obtained naturally by considering the real analogous of $g_R$, given by
\[
\tilde{g}_R = \tilde{l}_R + \tilde{V}_R + \tilde{V}_\sigma_R
\]
where
\[
\tilde{l}_R = \text{Lie}(\text{Str}(V_R)), \tilde{V}_R = \tilde{p}_{-r+1}, \tilde{V}_\sigma_R = \tilde{p}_{r-1},
\]
\[
\tilde{p}_j = \{ p \in \tilde{p}_R | d\kappa(\tilde{H})p = jp \}
\]
and where $\tilde{p}_R$ is the real subspace of $p$ generated by the polynomials $Q(x - a)$ on $V_R$ with $a \in V_R$ and
\[
\tilde{p}_R = \tilde{p}_{-r} \oplus \tilde{p}_{-r+1} \oplus \ldots \oplus \tilde{p}_0 \oplus \ldots \oplus \tilde{p}_{r-1} \oplus \tilde{p}_r
\]
is its eigen-space decomposition under $\text{ad}(\tilde{H}) = d\kappa(\tilde{H})$, where we notice that the element $\tilde{H}$ of $\mathfrak{z}(l)$ belongs to $\mathfrak{z}(l_R)$ too.

Moreover, observe that the complex Lie algebra $g$ is given by
\[
g = l + d\kappa(U(l))(E + F)
\]
and that its real analogous Lie algebra $\tilde{g}_R$ is given by
\[
\tilde{g}_R = \tilde{l}_R + d\kappa(U(l_R))(E + F)
\]
Since $H$ belongs to $\tilde{l}_R$ and since $E(z) = \tau(z), F(z) = (\kappa(\sigma)\tau)(z)$ for $z \in V$ (for $g$) and $E(x) = \tau(x), F(x) = (\kappa(\sigma)\tau)(x)$ for $x \in V_R$ (for $\tilde{g}_R$),
\[
[E, F] = \frac{1-r}{4}H, \quad [H, E] = 2E, \quad [H, F] = -2F,
\]
then the real Lie subalgebra of $\tilde{g}_R$ generated by $H, E, F$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. 

10
Observe that the two real forms \( g_R \) and \( \tilde{g}_R \) of the Lie algebra \( g \) are not isomorphic, in general. They are isomorphic iff the maximal compact subalgebra \( I_R \) of \( g_R \) satisfies \( I_R = \tilde{I}_R \cap I_R + i(\tilde{I}_R \cap I_R) \), or equivalently, \( \tilde{I}_R = I_R \cap I_R + I_R \cap (iI_R) \). In that case, a canonical vector space isomorphism \( c : \tilde{g}_R \to g_R \) is given by:

\[
\begin{align*}
c(X) &= X \text{ for } X \in \tilde{I}_R \cap I_R, \\
c(Y) &= iY \text{ for } Y \in i((\tilde{I}_R) \cap I_R), \\
c(p) &= i(p - \kappa(\sigma)p) \text{ for } p \in \tilde{V}_R, \\
c(p^\sigma) &= \kappa(\sigma)p^\sigma + p^\sigma \text{ for } p^\sigma \in \tilde{V}_R^\sigma.
\end{align*}
\]

In particular,

\[
c(H) = iH, \quad c(E) = i(E - F), \quad c(F) = E + F.
\]

Now, recall that the Lie groups \( \text{SL}(2, \mathbb{R}) \) and \( \text{SU}(1, 1) \) are conjugate within \( \text{SL}(2, \mathbb{C}) \), that the required isomorphism is given by

\[
\text{SL}(2, \mathbb{R}) \to \text{SU}(1, 1), \quad g \mapsto g^{-1}gg_0
\]

where

\[
g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}
\]

and that this leads to an isomorphism between the Lie algebras \( \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{su}(1, 1) \) given by

\[
\text{Ad}(g_0^{-1}) : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{su}(1, 1).
\]

In particular, one has

\[
\begin{align*}
\text{Ad}(g_0^{-1})(H) &= -i(E - F), \\
\text{Ad}(g_0^{-1})(E) &= \frac{1}{2}(iH) - \frac{1}{2}(E + F), \\
\text{Ad}(g_0^{-1})(F) &= -\frac{1}{2}(iH) - \frac{1}{2}(E + F).
\end{align*}
\]

Notice that one can use the same formulas for each of the triples \((p, \kappa(\sigma)p, p, \kappa(\sigma)p)\) and \((p^\sigma, \kappa(\sigma)p^\sigma, p^\sigma, \kappa(\sigma)p^\sigma)\) for \( p \in \mathcal{V}_R \) and \( p^\sigma \in \mathcal{V}_R^\sigma \).

The next table gives the classification of the simple real Lie algebras \( g_R \) obtained in this way. They are exactly the simple real Lie algebras of Hermitian type which satisfy the condition \( O_{\text{min}} \cap g \neq \emptyset \). It also gives the real Lie algebras \( \tilde{g}_R \).

In case of an exceptional Lie algebra \( g \), the real form \( g_R \) has been identified by computing the Cartan signature. The integer \( n \) is \( \geq 3 \).
| $V$ | $Q$ | $\ell$ | $g$ | $\mathfrak{g}_\mathbb{R}$ | $\mathfrak{l}_\mathbb{R}$ | $\mathfrak{g}_\mathbb{R}$ |
|-----|-----|-------|-----|----------------------|-----------------|-----------------|
| $\mathbb{C}$ | $z^2$ | $\mathbb{C}$ | $\mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{sl}(2, \mathbb{R})$ | $i\mathbb{R}$ | $\mathfrak{su}(1, 1)$ |
| $\mathbb{C}^n$ | $\Delta(z)^2$ | $\mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{sl}(n + 2, \mathbb{C})$ | $\mathfrak{sl}(n + 2, \mathbb{R}) \mathfrak{so}(n) \oplus i\mathbb{R}$ | $\mathfrak{so}(n, 2)$ |
| $\text{Sym}(r, \mathbb{C})$ | $\text{det}(z)^2$ | $\mathfrak{sl}(r, \mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{sp}(r, \mathbb{C})$ | $\mathfrak{sp}(r, \mathbb{R})$ | $\mathfrak{su}(r) \oplus i\mathbb{R}$ | $\mathfrak{sp}(r, \mathbb{R})$ |
| $M(r, \mathbb{C})$ | $\text{det}(z)^2$ | $\mathfrak{sl}(r, \mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{sp}(2r, \mathbb{C})$ | $\mathfrak{sp}(2r, \mathbb{R})$ | $\mathfrak{su}(r) \oplus i\mathbb{R}$ | $\mathfrak{sp}(r, \mathbb{R})$ |
| $\text{Skew}(2r, \mathbb{C})$ | $\text{det}(z)$ | $\mathfrak{sl}(2r, \mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{so}(4r, \mathbb{C})$ | $\mathfrak{so}(4r)$ | $\mathfrak{so}(2r) \oplus i\mathbb{R}$ | $\mathfrak{so}^*(4r)$ |
| $\text{Herm}(3, \mathbb{O})$ | $\text{det}(z)^2$ | $\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{e}_7(\mathbb{C})$ | $\mathfrak{e}_7(\mathbb{R})$ | $\mathfrak{e}_6(\mathbb{R}) \oplus i\mathbb{R}$ | $\mathfrak{e}_7(-25)$ |
From now on assume \( r \neq 1 \). Let \( \Xi \) and \( \Xi^\sigma \) be the \( L \)-orbits of \( \tau \) and \( \tau^\sigma \):

\[
\Xi = \{ \kappa(l)\tau \mid l \in L \} \subset V, \quad \Xi^\sigma = \{ \kappa(l)\tau^\sigma \mid l \in L \} \subset V^\sigma.
\]

They are conical varieties and related by \( \Xi^\sigma = \kappa(\sigma)\Xi \). Since \( \tau \) and \( \tau^\sigma \) are respectively a lowest weight vector and a highest weight vector for the adjoint representation of the Hermitian Lie algebra \( \mathfrak{g} = I + \mathcal{W} = \mathcal{V} + I + \mathcal{V}^\sigma \), and since, up to coverings, \( \Xi = \tilde{\Xi} \) and \( \Xi^\sigma = \tilde{\Xi}^\sigma \), then it follows from the Kostant-Sekiguchi correspondence that, up to coverings, \( \mathcal{O}_{\min} \cap \mathcal{W} = \Xi \cap \Xi^\sigma \).

Recall that the structure group \( \text{Str}(V_{\mathbb{R}}) \) of the Euclidean real form \( V_{\mathbb{R}} \) of \( V \) can be written

\[
\text{Str}(V_{\mathbb{R}}) = \text{Aut}(V_{\mathbb{R}})A_{\mathbb{R}}\text{Aut}(V_{\mathbb{R}}),
\]

where \( \text{Aut}(V_{\mathbb{R}}) \) is the automorphism group of the Jordan algebra \( V_{\mathbb{R}} \), \( A_{\mathbb{R}} = \{ P(a) \mid a \in \mathcal{R}_+ \} \) with \( \mathcal{R}_+ = \{ a = \sum_{i=1}^r a_i c_i \mid a_1 \geq \ldots \geq a_r > 0 \} \) and where \( P \) is the quadratic representation and \( \{ c_1, \ldots, c_r \} \) is a complete system of orthogonal primitive idempotents in \( V_{\mathbb{R}} \) (see [FK94], p.112). It follows by complexification that the structure group of \( V \) can be written

\[
\text{Str}(V) = \text{Aut}(V)A\text{Aut}(V)
\]

where \( \text{Aut}(V) \) is the automorphism group of \( V \) and \( A = \{ P(a) \mid a \in \tilde{\mathcal{R}} \} \), \( \tilde{\mathcal{R}} = \{ a = \sum_{i=1}^r a_i c_i \mid a_i \in \mathbb{C}, \text{Re}(a_1) \geq \ldots \geq \text{Re}(a_r) > 0, \text{Im}(a_1) \geq \ldots \geq \text{Im}(a_r) > 0 \} \).

Recall that \( jP(a)j^{-1} = P(a)' \) the adjoint of \( P(a) \) (with respect to the inner product \( \langle x \mid y \rangle = \text{tr}(xy) \)) and that for \( a \in \tilde{\mathcal{R}} \), one has \( P(a)' = P(\bar{a}) \) and \( P(a)^{-1} = P(\bar{a}^{-1}) \).

From now on assume that \( V \) is a matrix algebra. Then, using properties of the trace of matrices, one can show that for every \( k_1, k_2 \in \text{Aut}(V) \) and \( P(a) \in A \), one has \( \tau(k_1 P(a)k_2 z) = \tau(P(a)z) \) for \( z \in V \).

Using the relation \( \Delta(g \cdot v) = \text{Det}(g)\tilde{\tau} \Delta(v) \) for \( g \in \text{Str}(V) \), we deduce that a polynomial \( \xi \in \Xi \) and a polynomial \( \xi^\sigma = \kappa(\sigma)\xi \in \Xi^\sigma \) can be written

\[
\xi(z) = \kappa(l_\xi)\tau(z) = \text{Det}(P(a))^{-\tilde{\tau}} \tau(P(a)z) = \Delta(a)^{-2} \tau(P(a)z)
\]

and

\[
\xi^\sigma(z) = \kappa(l_{\xi^\sigma})\tau^\sigma(z) = \text{Det}(P(a))^{\tilde{\tau}^\sigma} \tau^\sigma(P(a^{-1})z) = \Delta(a)^2 \tau^\sigma(P(a^{-1})z).
\]

Hence we get for each of the two orbits \( \Xi \) and \( \Xi^\sigma \) a coordinate system

\[
X = A = \{ P(a) \mid a = \sum_{i=1}^r a_i c_i \in \tilde{\mathcal{R}} \}.
\]
In this coordinate system, the cocycle action of $L$ is given by the following: for $l \in L$, let $s(l) = k_1 P(a(l)) k_2$ where $s : L \to \text{Str}(V, Q)$ is the restriction of the covering map $s : K \to \text{Conf}(V, Q)$, $a(l) \in \tilde{R}, k_1, k_2 \in \text{Aut}(V)$, then

$$\kappa(l) : P(a) \mapsto P(a(l))^{-1} P(a) = P(P(a(l))^{-\frac{1}{2}} \cdot a)$$

Observe that $\kappa(\sigma)$ acts on these coordinates as follows:

$$\kappa(\sigma) : \Xi \to \Xi^\sigma, \xi(z) = \Delta(a)^{-2} \tau(P(a)z) \mapsto \xi^\sigma(z) = \Delta(a)^{2} \sigma\rho(P(a)^{-1}z).$$

Let $\alpha$ be an arbitrary real number. The group $L$ acts on the spaces $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ of holomorphic functions on $\Xi$ and $\Xi^\sigma$ respectively by:

$$(\pi_\alpha(l)f)(\xi) = \Delta(a(l))^{2\alpha} f(\kappa(l)^{(r-1)}\xi)$$

and

$$(\pi^\sigma_\alpha(l)f)(\xi^\sigma) = \Delta(a(l))^{2\alpha} f(\kappa(l)^{(r-1)}\xi^\sigma).$$

If $\xi(z) = \Delta(a)^{-2} \tau(P(a)z)$, $f \in \mathcal{O}(\Xi)$, we write $f(\xi) = \phi(P(a))$, if $\xi^\sigma(z) = \Delta(a)^{2} \sigma\rho(P(a)^{-1}z)$, $f^\sigma \in \mathcal{O}(\Xi^\sigma)$, we write $f^\sigma(\xi^\sigma) = \phi^\sigma(P(a))$.

In the coordinates $P(a)$ with $a \in \tilde{R}$, $\pi_\alpha$ and $\pi^\sigma_\alpha$ are given by

$$\tilde{\pi}_\alpha(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha} \phi(P(P(a(l))^{-\frac{r-1}{2}} \cdot a))$$

and

$$\tilde{\pi}^\sigma(\sigma) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha} \phi(P(P(a)^{-\frac{r-1}{2}} \cdot a)).$$

Consider now the isomorphism

$$\pi(\sigma) : \mathcal{O}(\Xi) \to \mathcal{O}(\Xi^\sigma), f \mapsto \pi(\sigma)f$$

where

$$\pi(\sigma)f(\xi^\sigma) = f(\kappa(\sigma)\xi^\sigma).$$

It intertwines the representations $\pi_\alpha$ and $\pi^\sigma_\alpha$ of $L$, i.e., for all $l \in L$

$$\pi(\sigma)\pi_\alpha(l) = \pi^\sigma_\alpha(l)\pi(\sigma).$$

In the coordinates $P(a)$, the isomorphism $\pi(\sigma)$ is given by:

$$\tilde{\pi}(\sigma) : \mathcal{O}(X) \to \mathcal{O}(X), \phi(P(a)) \mapsto \phi(P(a^{-1})).$$
**Example.** If $V = \text{Sym}(r, \mathbb{C})$ is the simple Jordan algebra of rank $r$ of complex symmetric matrices, and $V_{\mathbb{R}} = \text{Sym}(r, \mathbb{R})$ its Euclidean real form, then

$$g = \text{sp}(r, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A \in M(r, \mathbb{C}), B, C \in \text{Sym}(r, \mathbb{C}) \right\}$$

$$\tilde{g}_{\mathbb{R}} = \text{sp}(r, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A \in M(r, \mathbb{R}), B, C \in \text{Sym}(r, \mathbb{R}) \right\}.$$ 

One knows that the minimal orbits of $G = \text{Sp}(r, \mathbb{C})$ and $\tilde{G}_{\mathbb{R}} = \text{Sp}(r, \mathbb{R})$ in respectively $\text{sp}(r, \mathbb{C})$ and $\text{sp}(r, \mathbb{R})$ are given by

$$O = \{ X_u \mid u \in \mathbb{C}^{2r} \setminus \{0\} \}$$

and

$$O_{\mathbb{R}} = \{ X_u \mid u \in \mathbb{R}^{2r} \setminus \{0\} \}$$

where $X_u = uu^tJ$ with $J = \left( \begin{array}{cc} 0 & I_r \\ -J_r & 0 \end{array} \right)$ with $gX_ug^{-1} = X_gu$.

Furthermore, $O \cap \tilde{g}_{\mathbb{R}} = O_{\mathbb{R}} \cup (-O_{\mathbb{R}})$ is a union of two $\tilde{G}_{\mathbb{R}}$-orbits in $g_{\mathbb{R}}$.

The Cartan decomposition of $g_{\mathbb{R}}$ is given by $g_{\mathbb{R}} = l_{\mathbb{R}} + W_{\mathbb{R}}$ with

$$l_{\mathbb{R}} = \{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A \in \text{Asym}(r, \mathbb{R}), B \in \text{Sym}(r, \mathbb{R}) \} \cong \{ A + iB \mid A \in \text{Asym}(r, \mathbb{R}), B \in \text{Sym}(r, \mathbb{R}) \},$$

$$W_{\mathbb{R}} = \{ \begin{pmatrix} C & B \\ B & -C \end{pmatrix} \mid B, C \in \text{Sym}(r, \mathbb{R}) \} \cong \{ \begin{pmatrix} 0 & B + iC \\ B - iC & 0 \end{pmatrix} \mid B, C \in \text{Sym}(r, \mathbb{R}) \}.$$ 

Its complexification is $g = l + W$ where

$$l = l_{\mathbb{R}}^C = \text{su}(r)^C \cong \text{sl}(r, \mathbb{C}) = \text{str}(V)$$

and $W \cong \text{Sym}(r, \mathbb{C}) + \text{Sym}(r, \mathbb{C}) = V + V$.

(one can observe that the real algebras $\tilde{g}_{\mathbb{R}}$ and $g_{\mathbb{R}}$ are in fact isomorphic and that the isomorphism is given by a natural extension of $\text{Ad}(g_{0}^{-1})$).

Furthermore, $O \cap W = Y + \cup Y^-$, where $Y^+$ and $Y^-$ are the two orbits (corresponding to $\Xi$ and $\Xi^\sigma$), parameterized by $\mathbb{C}^r$ and given by

$$Y^+ = \{ X_u \mid u = \begin{pmatrix} x \\ ix \end{pmatrix}, x \in \mathbb{C}^r \setminus \{0\} \} = \{ \begin{pmatrix} xx^t \\ xtx \end{pmatrix}, \begin{pmatrix} xx^t \\ -ixx^t \end{pmatrix} \mid x \in \mathbb{C}^r \setminus \{0\} \}$$

and

$$Y^- = \{ X_u \mid u = \begin{pmatrix} x \\ -ix \end{pmatrix}, x \in \mathbb{C}^r \setminus \{0\} \} = \{ \begin{pmatrix} xx^t \\ xtx \end{pmatrix}, \begin{pmatrix} xx^t \\ -ixx^t \end{pmatrix} \mid x \in \mathbb{C}^r \setminus \{0\} \}.$$
2. $L_R$-invariant Hilbert subspaces of $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$. — In this section, we determine the irreducible $L_R$-invariant Hilbert subspaces of the spaces of holomorphic functions $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$. For $m \in \mathbb{Z}$, let 
\[ \mathcal{O}_{-\frac{m}{(r-1)}}(\Xi), \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi), \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma), \] 
and \[ \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) \]
be the the spaces of holomorphic functions $f$ on $\Xi$ and $f^\sigma$ on $\Xi^\sigma$ respectively such that for every $\lambda \in \mathbb{C}^*$,
\[
f(\lambda \cdot \xi) = \lambda^{-\frac{m}{(r-1)}} f(\xi), \]
\[
f(\lambda \cdot \xi) = \lambda^{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}} f(\xi) \]
and
\[
f^\sigma(\lambda \cdot \xi^\sigma) = \lambda^{-\frac{m}{(r-1)}} f^\sigma(\xi^\sigma), \]
\[
f^\sigma(\lambda \cdot \xi^\sigma) = \lambda^{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}} f^\sigma(\xi^\sigma). \]
These spaces are invariant under the representations $\pi_\lambda$ and respectively $\pi_\lambda^\sigma$. Observe that for $\xi \in \Xi$ given by $\xi(z) = \Delta(a)^{-2} \tau(P(a)z)$, and $\lambda \in \mathbb{C}^*$,
\[
\lambda \cdot \xi(z) = \Delta(\nu \cdot a)^{-2} \tau(P(\nu \cdot a)z) \text{ with } \lambda = \nu^{-2(r-1)}. 
\]
It follows that for $f$ in $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$ or in $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$, and also for $f^\sigma$ in $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$ or in $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$, their corresponding functions $\phi$ and $\phi^\sigma$ on $X$ satisfy the homogeneity properties
\[
\phi(\mu \cdot P(a)) = \mu^m \phi(P(a)) \text{ or } \phi^\sigma(\mu \cdot P(a)) = \mu^{m+\frac{1}{2}} \phi^\sigma(P(a)) 
\]
and
\[
\phi^\sigma(\mu P(a)) = \mu^m \phi^\sigma(P(a)) \text{ or } \phi^\sigma(\mu \cdot P(a)) = \mu^{m+\frac{1}{2}} \phi^\sigma(P(a)), 
\]
in such a way that the correspondences 
\[
f(\xi) \mapsto \phi(P(a)) \text{ and } f^\sigma(\xi^\sigma) \mapsto \phi^\sigma(P(a)) 
\]
map the spaces $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$ to the space
\[
\mathcal{O}_m(X) = \{ \phi \in \mathcal{O}(X) \mid \phi(\mu P(a)) = \mu^m \phi(P(a)) \} \subset \mathcal{O}_{2m}(\mathbb{C}^r) 
\]
and the spaces $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$ to the space
\[
\mathcal{O}_{m+\frac{1}{2}}(X) = \{ \phi \in \mathcal{O}(X) \mid \phi(\mu P(a)) = \mu^{m+\frac{1}{2}} \phi(P(a)) \} \subset \mathcal{O}_{2m+1}(\mathbb{C}^r). 
\]
Denote by $\pi_{\alpha,m}$, $\pi_{\alpha,m+\frac{1}{2}}$ and $\pi^\sigma_{\alpha,m}$, $\pi^\sigma_{\alpha,m+\frac{1}{2}}$ the restrictions of the representations $\pi_\lambda$ and $\pi_\lambda^\sigma$ to the spaces $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$, and denote by $\tilde{\pi}_{\alpha,m}$, $\tilde{\pi}_{\alpha,m+\frac{1}{2}}$ and $\tilde{\pi}^\sigma_{\alpha,m}$, $\tilde{\pi}^\sigma_{\alpha,m+\frac{1}{2}}$ the corresponding representations on the spaces $\mathcal{O}_m(X)$ and $\mathcal{O}_{m+\frac{1}{2}}(X)$. 

Since for every function $\phi \in \mathcal{O}(X)$, the function $a \mapsto \phi(P(a))$ is holomorphic on the open set $\tilde{R} \subset \mathbb{C}^r$, we denote by $\tilde{\mathcal{O}}(X)$ the set of such functions which extend to holomorphic functions on $\mathbb{C}^r$ and let $\tilde{\mathcal{O}}(\Xi)$ and $\tilde{\mathcal{O}}(\Xi^\sigma)$ be the spaces of the corresponding functions on $\Xi$ and $\Xi^\sigma$.

Then denote by

$$\tilde{\mathcal{O}}_m(X) = \mathcal{O}_m(X) \cap \tilde{\mathcal{O}}(X),$$

and put

$$\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi) = \tilde{\mathcal{O}}(\Xi) \cap \mathcal{O}_{\frac{m}{m-1}}(\Xi),$$

and

$$\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi^\sigma) = \tilde{\mathcal{O}}(\Xi^\sigma) \cap \mathcal{O}_{\frac{m}{m-1}}(\Xi^\sigma),$$

Observe that the spaces $\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi)$, $\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi^\sigma)$ are respectively stable by the representations $\pi_{\alpha,m}$, $\pi_{\alpha,m+\frac{1}{2}}$ and similarly, that the spaces $\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi^\sigma)$, $\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi^\sigma)$ are respectively stable by the representations $\pi_{\alpha,m}^\sigma$ and $\pi_{\alpha,m+\frac{1}{2}}^\sigma$.

**Theorem 2.1.** — (i) For $m < 0$,

$$\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi) = \mathcal{O}_{\frac{m}{m-1}}(\Xi) = \{0\}$$

and

$$\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi^\sigma) = \mathcal{O}_{\frac{m}{m-1}}(\Xi^\sigma) = \{0\}.$$

(ii) The functions $\phi \circ P$ for $\phi$ in $\tilde{\mathcal{O}}_m(X) \cup \tilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ with $m \geq 0$ are polynomials on $\mathbb{C}^r$.

(iii) The spaces $\tilde{\mathcal{O}}_m(X)$ and $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ with $m \in \mathbb{N}$ are finite dimensional, and $\pi_{\alpha,m}$, $\pi_{\alpha,m+\frac{1}{2}}$ and $\pi_{\alpha,m}^\sigma$, $\pi_{\alpha,m+\frac{1}{2}}^\sigma$ are irreducible.

**Proof.** (i), (ii) In fact, let $f$ in $\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi)$ or in $\tilde{\mathcal{O}}_{-\frac{m}{m-1}}(\Xi^\sigma)$ and $\phi$ in $\tilde{\mathcal{O}}_m(X)$ or in $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ the corresponding function on $X$. Since the function $z \in \mathbb{C}^r \mapsto \phi(P(z))$ is holomorphic at 0, it admits a power series expansion given by

$$\phi(P(z)) = \sum_{(k_1,...,k_r) \in \mathbb{N}^r} a_{k_1...k_r} z_1^{k_1} ... z_r^{k_r}.$$ 

Furthermore, using the homogeneity property, one obtains

$$\chi^{2m} \sum_{(k_1,...,k_r) \in \mathbb{N}^r} a_{k_1...k_r} z_1^{k_1} ... z_r^{k_r} = \sum_{(k_1,...,k_r) \in \mathbb{N}^r} ak_1...k_r \chi_1^{k_1} ... \chi_r^{k_r} z_1^{k_1} ... z_r^{k_r},$$

or

$$\chi^{2m+1} \sum_{(k_1,...,k_r) \in \mathbb{N}^r} a_{k_1...k_r} z_1^{k_1} ... z_r^{k_r} = \sum_{(k_1,...,k_r) \in \mathbb{N}^r} ak_1...k_r \chi_1^{k_1} ... \chi_r^{k_r} z_1^{k_1} ... z_r^{k_r}.$$
which implies that for every $\lambda$, $\lambda^{2m} = \lambda^{k_1 + \ldots + k_r}$ or $\lambda^{2m+1} = \lambda^{k_1 + \ldots + k_r}$, i.e. $k_1 + \ldots + k_r$ equals $2m$ or $2m + 1$, and then $m \geq 0$ and $\phi \circ P$ is a polynomial on $\mathbb{C}^r$.

(iii) Recall (cf. [FK94]) that $s(L) = \text{Aut}(V)AN$, $A = \{P(a) \mid a \in \tilde{R}\}$, and $N$ is the subgroup generated by the elements $\tau_0(z^{(j)}) = \exp(2z^{(j)}\circ c_j)$, with $z^{(j)} = \sum_{k=j+1}^r z_{jk} \in \oplus_{k=j+1}^r V_{jk}$ and where $V = \oplus_{j,k} V_{jk}$ is the Pierce decomposition of $V$ with respect to the complete system of primitive orthogonal idempotents $\{c_1, \ldots, c_r\}$ and where $x \circ y = L(xy) + [L(x), L(y)]$, $L(x)$ being the left multiplication operator $V \to V, v \mapsto xv$. The group $N$ is then the unipotent radical of $s(L)$.

The subspaces
\begin{align*}
\{\phi \in \tilde{O}_m(X) \mid \forall n \in N, \tilde{\pi}_{\alpha,m}(n)\phi = \phi\}, \\
\{\phi \in \tilde{O}_{m+\frac{r}{2}}(X) \mid \forall n \in N, \tilde{\pi}_{\alpha,m+\frac{r}{2}}(n)\phi = \phi\},
\end{align*}
reduce to the functions $C \cdot \Delta(z)^{\frac{2m}{2}}$ in the even case and to the functions $C \cdot \Delta(z)^{\frac{2m+1}{2}}$ in the odd case, hence are one dimensional. By the theorem of the highest weight (see [G08]), it follows that the spaces $\tilde{O}_m(X)$ and $\tilde{O}_{m+\frac{r}{2}}(X)$ are finite dimensional and irreducible for these representations.

We now define $L_\mathbb{R}$-invariant inner products on the spaces $O_{-\frac{m}{(r-1)}}(\Xi), O_{-\frac{m}{(r-1)} - \frac{1}{2(r-1)}}(\Xi)$ and $O_{-\frac{m}{(r-1)}}(\Xi^\sigma), O_{-\frac{m}{(r-1)} - \frac{1}{2(r-1)}}(\Xi^\sigma)$. Let
\begin{align*}
H(z, z') = \tau(\frac{1}{r}e + zz'), & \quad H(z) = H(z, z) = \tau(\frac{1}{r}e + zz), \\
H_\sigma(z, z') = \tau^\sigma(\frac{1}{r}e + zz'), & \quad H_\sigma(z) = H_\sigma(z, z) = \tau^\sigma(\frac{1}{r}e + zz).
\end{align*}

Recall (from the end of section 1) that for $l \in L$, one can write $s(l) = k_1 P(a(l))k_2$, where $s : L \to \text{Str}(V, Q)$ is the restriction of the covering map $s : K \to \text{Conf}(V, Q)$, $a(l) \in \tilde{R}$, $k_1, k_2 \in \text{Aut}(V)$, and that the action of the group $L$ on the coordinate system $X = A = \{P(a) \mid a = \sum_{i=1}^r a_i c_i \in \tilde{R}\}$ is given by $\kappa(l) : P(a) \mapsto P(a(l))^{-1} P(a) = P(P(a(l))^{-\frac{1}{2}} \cdot a)$ and it can be extended naturally to the same action on $\{P(a) \mid a = \sum_{i=1}^r a_i c_i \mid a_i \in \mathbb{C}\}$. 18
This induces an action of $L$ on $\mathbb{C}^r$ given by $lz = P(a(l))^{-\frac{1}{a}}z$, where $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ is identified with $\sum_{i=1}^{r} z_i c_i \in V$.

**Proposition 2.2.** — For every $l \in L_\mathbb{R}$,

$$|\Delta(a(l))| = 1$$

and

$$H(lz) = H(z), \quad H_\sigma(lz) = H_\sigma(z).$$

**Proof.** Since $l$ belongs to $L_\mathbb{R}$ iff $P(a(l))P(a(l))' = \text{id}_V$, then for every $l \in L_\mathbb{R}$, $|\Delta(a(l))| = 1$ and $\tau((lz)(t\bar{z})) = \tau(P(a(l))^{-\frac{1}{a}}zP(a(l))^{-\frac{1}{a}}\bar{z}) = \tau((P(a(l))P(a(l))')^{-\frac{1}{a}}z\bar{z}) = \tau(z\bar{z})$.

Define the two norms of a function $\phi \in \tilde{O}_m(X)$ by

$$\|\phi\|_m^2 = \frac{1}{a_m} \int_{\mathbb{C}^r} |\phi(P(z))|^2 H(z)^{-(2m)} m_0(dz),$$

$$\|\phi\|_{m,\sigma}^2 = \frac{1}{a_{m,\sigma}} \int_{\mathbb{C}^r} |\phi^\sigma(P(z))|^2 H(z)^{-\alpha(m)} m_0(dz)$$

and the two norms of a function $\phi \in \tilde{O}_{m+\frac{1}{a}}(X)$ by

$$\|\phi\|_{m+\frac{1}{a}}^2 = \frac{1}{a_{m+\frac{1}{a}}} \int_{\mathbb{C}^r} |\phi^\sigma(P(z))|^2 H(z)^{-(2m+1)} m_0(dz),$$

$$\|\phi\|_{m+\frac{1}{a},\sigma}^2 = \frac{1}{a_{m+\frac{1}{a},\sigma}} \int_{\mathbb{C}^r} |\phi^\sigma(P(z))|^2 H_\sigma(z)^{-\alpha(m+\frac{1}{a})} m_0(dz)$$

where $\alpha(m)$ is a suitable integer, $m_0(dz) = H(z)^{-(r+1)} m(dz)$ is the $L$-invariant measure on $\{ \sum_{i=1}^{r} z_i c_i \mid z_i \in \mathbb{C} \} \simeq \mathbb{C}^r$, $m(dz)$ is the Lebesgue measure and where the positive constants $a_m, a_{m+\frac{1}{a}}, a_{m,\sigma}, a_{m+\frac{1}{a},\sigma}$ are given by

$$a_m = \int_{\mathbb{C}^r} H(z)^{-(2m)} m_0(dz),$$

$$a_{m+\frac{1}{a}} = \int_{\mathbb{C}^r} H(z)^{-(2m+1)} m_0(dz),$$

and

$$a_{m,\sigma} = \int_{\mathbb{C}^r} H_\sigma(z)^{-\alpha(m)} m_0(dz),$$

$$a_{m+\frac{1}{a},\sigma} = \int_{\mathbb{C}^r} H_\sigma(z)^{-\alpha(m+\frac{1}{a})} m_0(dz).$$
PROPOSITION 2.3. —

(i) These norms are \(L_\mathbb{R}\)-invariant. Hence the normed spaces 
\(\tilde{O}_m(X), \|\cdot\|_m, (\tilde{O}_{m+\frac{1}{2}}(X), \|\cdot\|_{m+\frac{1}{2}}), (\tilde{O}_m(X), \|\cdot\|_{m,\sigma}), (\tilde{O}_{m+\frac{1}{2}}(X), \|\cdot\|_{m+\frac{1}{2},\sigma})\)
are Hilbert subspaces of \(\tilde{O}(X)\).

(ii) The reproducing kernels of the spaces
\(\tilde{O}_{-\frac{m}{(r-1)}}(\Xi), \tilde{O}_{\frac{m}{(r-1)}}(\Xi), \tilde{O}_{\frac{m}{(r-1)}}(\Xi^\sigma), \tilde{O}_{\frac{m}{(r-1)}}(\Xi^\sigma)\)
are respectively given by

\[
\tilde{K}_m(P(a), P(a')) = H(a, a')^{2m},
\]

\[
\tilde{K}_{m+\frac{1}{2}}(P(a), P(a')) = H(a, a')^{2m+1},
\]

\[
\tilde{K}_{m,\sigma}(P(a), P(a')) = H_\sigma(a, a')^{\alpha(m)},
\]

\[
\tilde{K}_{m+\frac{1}{2},\sigma}(P(a), P(a')) = H_\sigma(a, a')^{\alpha(m+\frac{1}{2})}.
\]

Proof. (i) Let \(l \in L_\mathbb{R}\). Using Proposition 2.2.,

\[
\|\pi_{\alpha, m}(l)\phi\|_m^2 = \frac{1}{a_m} \int_{\mathbb{C}^r} |\pi_{\alpha, m}(l)\phi(P(z))|^2 H(z)^{-2m} m_0(dz)
\]

\[
= \frac{1}{a_m} \int_{\mathbb{C}^r} |\Delta(a(l)) 2^\alpha \phi(P(a(l)) - z \cdot z')|^2 H(z)^{-2m} m_0(dz)
\]

\[
= \frac{1}{a_m} \int_{\mathbb{C}^r} |\phi(z')|^2 H(P(a(l)) (\frac{r-1}{r} z')^{-2m} m_0(d(P(a(l))(\frac{r-1}{r} z'))
\]

\[
= \frac{1}{a_m} \int_{\mathbb{C}^r} |\phi(z')|^2 H(P(a(l)) (\frac{r-1}{r} z')^{-2m} m_0(dz') = \|\phi\|^2_m.
\]



PROPOSITION 2.4. —

\(a_m = \frac{\pi^r}{(2m + r) \ldots (2m + 1)}, \quad a_{m+\frac{1}{2}} = \frac{\pi^r}{(2m + r + 1) \ldots (2m + 2)}\).

Proof. In fact,

\[
a_m = \int_{\mathbb{C}^r} H(z)^{-2m+r+1} m(dz) = \int_{\mathbb{C}^r} (1 + |z_1|^2 + \ldots + |z_r|^2)^{-2m+r+1},
\]

\[
= \pi^r \int_{\mathbb{R}^+} (2\rho_1) \ldots (2\rho_r)(1 + \rho_1^2 + \ldots + \rho_r^2)^{-2m+r+1} d\rho_1 \ldots d\rho_r
\]

\[
= \pi^r \frac{1}{(2m + r) \ldots (2m + 1)}.
\]
\begin{align*}
a_{m+\frac{1}{2}} &= \int_{\mathbb{C}^r} H(z)^{-2m+1+2r} m(dz) = \int_{\mathbb{C}^r} (1 + |z_1|^2 + \ldots + |z_r|^2)^{-2m+1+2r}, \\
&= \pi^r \int_{\mathbb{R}_+^r} (2\rho_1) \ldots (2\rho_r)(1 + \rho_1^2 + \ldots + \rho_r^2)^{-2m+1+2r} d\rho_1 \ldots d\rho_r \\
&= \pi^r \left( \frac{1}{2m + r + 1} \right) \ldots \left( \frac{1}{2m + 2} \right).
\end{align*}

Since the spaces \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}}(\Xi) \) and \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}}(\Xi^\sigma) \) are isomorphic to \( \tilde{O}_m(X) \), and \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}} - \frac{1}{2(\Xi^\sigma)}(\Xi) \) and \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}} - \frac{1}{2(\Xi^\sigma)}(\Xi^\sigma) \) are isomorphic to \( \tilde{O}_{m+\frac{1}{2}}(X) \), then \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}}(\Xi) \), \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}}(\Xi^\sigma) \), \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}}(\Xi^\sigma) \) become invariant Hilbert subspaces of \( \tilde{O}(\Xi) \) and \( \tilde{O}(\Xi^\sigma) \) respectively, with reproducing kernels given by:

\[ K_m(\xi, \xi') = \Phi(\xi, \xi')^{2m}, \quad K_{m+\frac{1}{2}}(\xi, \xi') = \Phi(\xi, \xi')^{2m+1}, \]

\[ K_{m,\sigma}(\xi, \xi') = \Phi_\sigma(\xi, \xi')^{\alpha(m)}, \quad K_{m+\frac{1}{2},\sigma}(\xi, \xi') = \Phi_\sigma(\xi, \xi')^{\alpha(m+\frac{1}{2})}, \]

where

\[ \Phi(\xi, \xi') = H(a, a'), \quad \Phi_\sigma(\xi, \xi') = H_\sigma(a, a') \quad (\xi = P(a), \xi' = P(a')). \]

**Theorem 2.5.** — The group \( L_\mathbb{R} \) acts multiplicity free on the spaces \( \tilde{O}(\Xi) \) and \( \tilde{O}(\Xi^\sigma) \). The irreducible \( L_\mathbb{R} \)-invariant subspaces of \( \tilde{O}(\Xi) \) and of \( \tilde{O}(\Xi^\sigma) \) are \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}}(\Xi) \), \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}} - \frac{1}{2(\Xi^\sigma)}(\Xi^\sigma) \) respectively, \( \tilde{O}_{\frac{m}{r} - \frac{1}{2}} - \frac{1}{2(\Xi^\sigma)}(\Xi^\sigma) \), with \( m \in \mathbb{N} \). If \( \mathcal{H} \subset \tilde{O}(\Xi) \) and \( \mathcal{H}^\sigma \subset \tilde{O}(\Xi^\sigma) \) are \( L_\mathbb{R} \)-invariant Hilbert subspaces, their reproducing kernels are given by

\[ K(\xi, \xi') = \sum_{m \in \mathbb{N}} c_m \Phi(\xi, \xi')^{2m} + \sum_{m \in \mathbb{N}} c_{m+\frac{1}{2}} \Phi(\xi, \xi')^{2m+1} \]

\[ K_\sigma(\xi, \xi') = \sum_{m \in \mathbb{N}} c_{m,\sigma} \Phi_\sigma(\xi, \xi')^{\alpha(m)} + \sum_{m \in \mathbb{N}} c_{m+\frac{1}{2},\sigma} \Phi_\sigma(\xi, \xi')^{\alpha(m+\frac{1}{2})}, \]

where \( (c_m), (c_{m+\frac{1}{2}}) \) and \( (c_{m,\sigma}), (c_{m+\frac{1}{2},\sigma}) \) are sequences of positive numbers such that the series

\[ \sum_{m \in \mathbb{N}} c_m \Phi(\xi, \xi')^{2m} + \sum_{m \in \mathbb{N}} c_{m+\frac{1}{2}} \Phi(\xi, \xi')^{2m+1}, \]

\[ \sum_{m \in \mathbb{N}} c_{m,\sigma} \Phi_\sigma(\xi, \xi')^{\alpha(m)} + \sum_{m \in \mathbb{N}} c_{m+\frac{1}{2},\sigma} \Phi_\sigma(\xi, \xi')^{\alpha(m+\frac{1}{2})} \]

converge uniformly on compact subsets in \( \Xi \) and \( \Xi^\sigma \) respectively.
3. Representations of the Lie algebra. —

In the sequel, we construct representations \( \rho \) and \( \rho^\sigma \) of the Lie algebra \( g \), which will be the infinitesimal versions of the two (non unitary equivalent) minimal representations. Recall that the group \( L \) acts on the spaces \( \mathcal{O}(\Xi) \) and \( \mathcal{O}(\Xi^\sigma) \) respectively by:

\[
(\pi_\alpha(l)f)(\xi) = \Delta(a(l))^{2\alpha} f\left((\kappa(l))^{-1} \xi\right)
\]

and

\[
(\pi_\sigma^\alpha(l)f^\sigma)(\xi^\sigma) = \Delta(a(l))^{2\alpha} f^\sigma\left((\kappa(l))^{-1} \xi^\sigma\right).
\]

It follows that \( L \) acts on the space \( \mathcal{O}(X) \) by:

\[
\tilde{\pi}_\alpha(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha} \phi\left(P\left(e^{\beta(l)}\phi\left(P\left(e^{\alpha(a(l))}\right)\right)\right)\right).
\]

and

\[
\tilde{\pi}_\sigma^\alpha(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha} \phi\left(P\left(e^{\beta(l)}\phi\left(P\left(e^{\alpha(a(l))}\right)\right)\right)\right).
\]

This leads by differentiation, to representations \( d\pi_\alpha \) and \( d\pi_\sigma^\alpha \) of the Lie algebra \( l \) in the spaces \( \mathcal{O}(\Xi) \) and \( \mathcal{O}(\Xi^\sigma) \) and representations \( d\tilde{\pi}_\alpha \) and \( d\tilde{\pi}_\sigma^\alpha \) of the Lie algebra \( l \) in the space \( \mathcal{O}(X) \).

We will construct two representations \( \rho \) and \( \rho^\sigma \) of \( g = l + W \) on respectively the spaces of finite sums

\[
\mathcal{O}_{\text{fin}}(\Xi) = \sum_{m \in \mathbb{N}} \mathcal{O}_{-(r-1)m}(\Xi) \oplus \sum_{m \in \mathbb{N}} \mathcal{O}_{-(r-1)m - \frac{1}{2}(r-1)}(\Xi),
\]

and

\[
\mathcal{O}_{\text{fin}}(\Xi^\sigma) = \sum_{m \in \mathbb{N}} \mathcal{O}_{-(r-1)m}(\Xi^\sigma) \oplus \sum_{m \in \mathbb{N}} \mathcal{O}_{-(r-1)m - \frac{1}{2}(r-1)}(\Xi^\sigma)
\]

such that, for all \( X \in l \), \( \rho(X) = d\tilde{\pi}_\alpha(X) \) and \( \rho^\sigma(X) = d\tilde{\pi}_\sigma^\alpha(X) \).

We define first a representation \( \rho \) of the subalgebra generated by \( E,F,H \), isomorphic to \( \mathfrak{sl}(2,\mathbb{C}) \). In particular

\[
\rho(H) = d\tilde{\pi}_\alpha(H) = \frac{d}{dt}\bigg|_{t=0} \tilde{\pi}_\alpha(\exp(tH)).
\]

For \( \lambda \in \mathbb{C} \), denote by \( l_\lambda \) the dilation \( V \to V, v \mapsto \lambda v \). Since \( l_\lambda = P(\sqrt{\lambda}e) \), i.e. \( a(l_\lambda) = \sqrt{\lambda}e \), then, for \( \lambda = \exp(-\frac{2}{1-r}t) \),

\[
\pi_\alpha(\exp(tH)\phi(P(a)) = \Delta(a(l_\lambda))^{2\alpha} \phi(P(\sqrt{\lambda}^{1-r}e)P(a)) = \exp(2\alpha \frac{r}{r-1}t)\phi(P(e^t a))
\]
and then
\[ \rho(H)\phi(P(a)) = 2\alpha \frac{r}{r-1}(\phi \circ P)(a) + \mathcal{E}(\phi \circ P)(a) \]

where \( \mathcal{E} \) is the Euler operator \((\mathcal{E}\phi)(P(a)) = \frac{d}{du}\big|_{u=1} \phi(P(u \cdot a)) \). We denote by \( \tilde{O}_{2m}(\mathbb{C}^r) \) and \( \tilde{O}_{2m+1}(\mathbb{C}^r) \) the subspaces of \( \mathcal{O}(\mathbb{C}^r) \), images of \( \tilde{O}_m(X) \) and \( \tilde{O}_{m+rac{1}{2}}(X) \) by the isomorphism \( \phi \mapsto \phi \circ P \).

Let \( \rho(E) : \mathcal{O}_{\text{fin}}(X) \to \mathcal{O}_{\text{fin}}(X) \) be the multiplication operator which maps \( \tilde{O}_m(X) \) to \( \tilde{O}_{m+1}(X) \) and \( \tilde{O}_{m+rac{1}{2}}(X) \) to \( \tilde{O}_{m+1+rac{1}{2}}(X) \) and defined by:
\[ \rho(E) : \phi(P(z)) \mapsto \frac{i}{2} c \cdot \tau(z^2)\phi(P(z)), \]

and let \( \rho(F) : \mathcal{O}_{\text{fin}}(X) \to \mathcal{O}_{\text{fin}}(X) \) be the differential operator which maps \( \tilde{O}_m(X) \) to \( \tilde{O}_{m-1}(X) \) and \( \tilde{O}_{m+rac{1}{2}}(X) \) to \( \tilde{O}_{m-1+rac{1}{2}}(X) \) and defined by:
\[ \rho(F) : \phi(P(z)) \mapsto \frac{i}{2} c \cdot \tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z). \]

**Lemma 3.1.**
\[ [\rho(H), \rho(E)] = 2\rho(E), \]
\[ [\rho(H), \rho(F)] = -2\rho(F) \]

and \([\rho(E), \rho(F)] = \rho(H) \) if and only if \( \alpha = \frac{1}{2} \) and \( c = \frac{\sqrt{r-1}}{2} \).

*Proof.* Since \( \rho(H)\phi(P(a)) = 2\alpha \frac{r}{r-1}(\phi \circ P)(a) + \mathcal{E}(\phi \circ P)(a) \) then
\[ \rho(H)\rho(E) : \phi(P(z)) \mapsto \frac{i}{2} c(2\alpha \frac{r}{r-1} \tau(z^2)(\phi \circ P)(z)) + \mathcal{E}(\tau(z^2)(\phi \circ P)(z)), \]
\[ \rho(F)\rho(H) : \phi(P(z)) \mapsto \frac{i}{2} c\tau(z^2)(\mathcal{E}(2\alpha \frac{r}{r-1}(\phi \circ P)(a)) + \mathcal{E}(\phi \circ P)(a)), \]

and using the identity
\[ \mathcal{E}(\tau(z^2)(\phi \circ P)(z)) - \tau(z^2)\mathcal{E}(\phi \circ P)(z) = 2\tau(z^2)(\phi \circ P)(z), \]

one obtains \([\rho(H), \rho(E)] : \phi(P(z)) \mapsto 2\frac{1}{2}(\tau(z^2)(\phi \circ P)(z)) \) i.e.
\[ [\rho(H), \rho(E)] = 2\rho(E). \]
Similarly,
\[
\rho(H)\rho(F) : \phi(P(z)) \mapsto \frac{i}{2}c(2\alpha \frac{r}{r-1} + \mathcal{E})(\tau(\frac{\partial^2}{\partial z^2})\phi \circ P(z)),
\]
\[
\rho(F)\rho(H) : \phi(P(z)) \mapsto \frac{i}{2}c\tau(\frac{\partial^2}{\partial z^2})(2\alpha \frac{r}{r-1} + \mathcal{E})(\phi \circ P(z)),
\]
and using the identity
\[
\mathcal{E}\tau(\frac{\partial^2}{\partial z^2})(\phi \circ P(z)) - \tau(\frac{\partial^2}{\partial z^2})\mathcal{E}(\phi \circ P)(z) = -2\tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z),
\]
one obtains \([\rho(H), \rho(F)] : \phi(P(z)) \mapsto \frac{i}{2}c(-2(\tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z)) \text{ i.e.}
\]
\[
[\rho(H), \rho(F)] = -2\rho(F).
\]
\[
\rho(E)\rho(F) : \phi(P(z)) \mapsto -\frac{1}{4}c^2\tau(z^2)\tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z),
\]
\[
\rho(F)\rho(E) : \phi(P(z)) \mapsto -\frac{1}{4}c^2\tau(\frac{\partial^2}{\partial z^2})(\tau(z^2)(\phi \circ P)(z),
\]
and using the identity
\[
\tau(z^2)\tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z) - \tau(\frac{\partial^2}{\partial z^2})\tau(z^2)(\phi \circ P)(z) = (-2r - 4\mathcal{E})(\phi \circ P)(z),
\]
one obtains
\[
[\rho(E), \rho(F)] : \phi(P(z)) \mapsto c^2(\frac{1}{2}r + \mathcal{E})(\phi \circ P)(z).
\]
It follows that \([\rho(E), \rho(F)] = \frac{1-r}{4}\rho(H) \text{ if and only if}
\]
\[
\alpha = \frac{r-1}{4} \text{ and } c = i\sqrt{\frac{r-1}{2}}.
\]

For \(p \in \mathcal{V}\), define the multiplication operator \(\rho(p) : \mathcal{O}_{\text{fin}}(X) \to \mathcal{O}_{\text{fin}}(X)\) which maps \(\hat{O}_{m}(X)\) to \(\hat{O}_{m+1}(X)\) and \(\hat{O}_{m+\frac{1}{2}}(X)\) to \(\hat{O}_{m+1+\frac{1}{2}}(X)\) given by
\[
\rho(p) : \phi(P(z)) \mapsto -\frac{\sqrt{r-1}}{4}p(z^2)\phi(P(z)).
\]

For \(p^\sigma = \kappa(\sigma)p \in \mathcal{V}^\sigma\), define the differential operator \(\rho(p^\sigma) : \mathcal{O}_{\text{fin}}(X) \to \mathcal{O}_{\text{fin}}(X)\) which maps \(\hat{O}_{m}(X)\) to \(\hat{O}_{m-1}(X)\) and \(\hat{O}_{m+\frac{1}{2}}(X)\) to \(\hat{O}_{m-1+\frac{1}{2}}(X)\) given by
\[
\rho(p^\sigma) : \phi(P(z)) \mapsto -\frac{\sqrt{r-1}}{4}p^\sigma(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z).
\]
Observe that these definitions are consistent with those of $\rho(E)$ and $\rho(F)$ and that
$$\rho(\kappa(l)p) = \pi(\alpha(l)) \rho(p) \pi(\alpha(l^{-1})$$
and
$$\rho(\kappa(l)p^\sigma) = \pi(\alpha(l)) \rho(p^\sigma) \pi(\alpha(l^{-1})).$$

Recall that, for $X \in \mathcal{I}$, $\rho(X) = d\pi(\alpha)(X)$. Hence we get maps
$$\rho : g = \mathcal{I} \oplus \mathcal{W} \to \text{End}(\mathcal{O}_{\text{fin}}(\Xi))$$
and
$$\rho^\sigma = \tilde{\pi}(\sigma) \rho \tilde{\pi}(\sigma) : g = \mathcal{I} \oplus \mathcal{W} \to \text{End}(\mathcal{O}_{\text{fin}}(\Xi^\sigma)).$$

**Theorem 3.2.** —
(i) $\rho$ and $\rho^\sigma$ are representations of the Lie algebra $g$.
(ii) The spaces $\mathcal{O}_{\text{even}}(\Xi) = \sum_{m=0}^\infty \mathcal{O}_m(X)$, $\mathcal{O}_{\text{odd}}(\Xi) = \sum_{m=0}^\infty \mathcal{O}_{m+\frac{1}{2}}(X)$ are invariant and irreducible under $\rho$ and also under $\rho^\sigma$.
(iii) $\rho$ and $\rho^\sigma$ are sums of two irreducible representations.

**Proof.**
(i) Since $\pi(\alpha)$ is a representation of $L$, for $X, X' \in \mathcal{I}$,
$$[\rho(X), \rho(X')] = \rho([X, X']).$$
Furthermore, one can show that, for $X \in \mathcal{I}, p \in \mathcal{W},$
$$[\rho(X), \rho(p)] = \rho([X, p]).$$
Since $\mathcal{V}$ and $\mathcal{V}^\sigma$ are Abelian, It remains to show that, for $p \in \mathcal{V}, p' \in \mathcal{V}^\sigma,$
$$[\rho(p), \rho(p')] = \rho([p, p']).$$
Then, consider the map $\lambda : \mathcal{V} \wedge \mathcal{V}^\sigma \to \text{End}(\mathcal{O}_{\text{fin}}(\Xi)$, defined by
$$\lambda(p \wedge p') = [\rho(p), \rho(p')] - \rho([p, p']).$$
Since $[\rho(E), \rho(F)] = \rho(H)$, then $\lambda(E \wedge F) = 0$. It follows that, for $g \in L,$
$$\lambda(\kappa(g)E \wedge \kappa(g)F) = 0.$$
Since the representations $\kappa$ of $L$ in $\mathcal{V}$ and in $\mathcal{V}^\sigma$ are irreducible, and $E$ and $F$ are respectively lowest and highest vectors in $\mathcal{V}$ and $\mathcal{V}^\sigma$, the vector $E \wedge F$ is cyclic in $\mathcal{V} \wedge \mathcal{V}^\sigma$ for the action of $L$. Therefore $\lambda \equiv 0$.

(ii) Let $\mathcal{U} \neq \{0\}$ (resp. $\mathcal{U}' \neq \{0\}$) be a $\rho(g)$-invariant subspace of $\mathcal{O}_{\text{even}}(\Xi)$ (resp. $\mathcal{O}_{\text{odd}}(\Xi)$) . Then $\mathcal{U}$ and $\mathcal{U}'$ are $\rho(\mathcal{I})$-invariant.
Since $\mathcal{O}_{\text{even}}(\Xi) = \sum_{m=0}^{\infty} \mathcal{O}_m(X)$ and as the subspaces $\mathcal{O}_m(X)$ are $\rho(l)$-irreducible, then there exists $\mathcal{I} \subset \mathbb{N}$ ($\mathcal{I} \neq \emptyset$) such that $\mathcal{U} = \sum_{m \in \mathcal{I}} \mathcal{O}_m(X)$. Similarly, since $\mathcal{O}_{\text{odd}}(\Xi) = \sum_{m=0}^{\infty} \mathcal{O}_{m+\frac{1}{2}}(X)$ and as the subspaces $\mathcal{O}_{m+\frac{1}{2}}(X)$ are $\rho(l)$-irreducible, then there exists $\mathcal{I}' \subset \mathbb{N}$ ($\mathcal{I}' \neq \emptyset$) such that $\mathcal{U}' = \sum_{m \in \mathcal{I}'} \mathcal{O}_{m+\frac{1}{2}}(X)$. Observe that since $\rho(E)$ maps $\mathcal{O}_m(X)$ to $\mathcal{O}_{m+1}(X)$ and maps $\mathcal{O}_{m+\frac{1}{2}}(X)$ to $\mathcal{O}_{m+1+\frac{1}{2}}(X)$, it follows that if $\mathcal{U}$ (resp. $\mathcal{U}'$) contains $\mathcal{O}_m(X)$ (resp. $\mathcal{O}_{m+\frac{1}{2}}(X)$), then it contains $\mathcal{O}_{m+1}(X)$ (resp. $\mathcal{O}_{m+1+\frac{1}{2}}(X)$) too.

Furthermore, since $\rho(F)$ maps the space $\mathcal{O}_m(X)$ to $\mathcal{O}_{m-1}(X)$ and the space $\mathcal{O}_{m+\frac{1}{2}}(X)$ to $\mathcal{O}_{m-1+\frac{1}{2}}(X)$, it follows that if $\mathcal{U}$ (resp. $\mathcal{U}'$) contains $\mathcal{O}_m(X)$ (resp. $\mathcal{O}_{m+\frac{1}{2}}(X)$), then it contains $\mathcal{O}_{m-1}(X)$ (resp. $\mathcal{O}_{m-1+\frac{1}{2}}(X)$) too. Therefore $m_0 = m_0' = 0$, then $\mathcal{U} = \mathcal{O}_{\text{even}}(\Xi)$ and $\mathcal{U}' = \mathcal{O}_{\text{odd}}(\Xi)$.  

4. Unitary representations of the corresponding real Lie group. —

We consider, for two sequences $(c_m), (c_{m+\frac{1}{2}})$ and similarly for two sequences $(c_{m,\sigma}), (c_{m+\frac{1}{2},\sigma})$ of positive numbers, an inner product on $\mathcal{O}_{\text{fin}}(\Xi)$ and an inner product on $\mathcal{O}_{\text{fin}}(\Xi^\sigma)$ such that

$$\|\phi\|^2 = \sum_{m \in \mathbb{N}} \frac{1}{c_m} \|\phi_m\|^2_m + \sum_{m \in \mathbb{N}} \frac{1}{c_{m+\frac{1}{2}}} \|\phi_{m+\frac{1}{2}}\|^2_{m+\frac{1}{2}}$$

and similarly,

$$\|\phi\|_{\sigma}^2 = \sum_{m \in \mathbb{N}} \frac{1}{c_{m,\sigma}} \|\phi_m\|^2_{m,\sigma} + \sum_{m \in \mathbb{N}} \frac{1}{c_{m+\frac{1}{2},\sigma}} \|\phi_{m+\frac{1}{2}}\|^2_{m+\frac{1}{2},\sigma},$$

for

$$\phi(P(a)) = \sum_{m \in \mathbb{N}} \phi_m(P(a)) + \sum_{m \in \mathbb{N}} \phi_{m+\frac{1}{2}}(P(a)).$$

These inner products are invariant under $L_{\mathbb{R}}$. We will determine the sequences $(c_m), (c_{m+\frac{1}{2}})$ and $(c_{m,\sigma}), (c_{m+\frac{1}{2},\sigma})$ such that these inner products are invariant under the representations $\rho$ and $\rho^\sigma$ restricted to $g_{\mathbb{R}}$, respectively. We denote by $\mathcal{H}$ and $\mathcal{H}^\sigma$ the Hilbert space completion of $\mathcal{O}_{\text{fin}}(\Xi)$ and of $\mathcal{O}_{\text{fin}}(\Xi^\sigma)$ with respect to these inner products. We will assume $c_0 = c_{\frac{1}{2}} = c_{0,\sigma} = c_{\frac{1}{2},\sigma} = 1$. 

26
Theorem 4.1. —
(i) The inner product of $\mathcal{H}$ is $g_\mathcal{R}$-invariant if
\[ c_m = \frac{1}{(2m)!} \quad \text{and} \quad c_{m+\frac{1}{2}} = \frac{1}{(2m+1)!}. \]
(ii) The reproducing kernel of $\mathcal{H}$ is given by
\[ K(\xi, \xi') = \exp(H(z, z')) \quad (\xi = P(a), \xi' = P(a')). \]

Proof. (i) Recall that $\mathcal{W}_\mathcal{R} = \{ p \in \mathcal{W} \mid \beta(p) = p \}$, where $\beta$ is the conjugation of $\mathcal{W}$, introduced at the end of Section 1. Recall also that $\beta(\kappa(g)p) = \kappa(\alpha(g))\beta(p)$.

The inner product of $\mathcal{H}$ is $g_\mathcal{R}$-invariant if and only if, for every $p \in \mathcal{W}$,
\[ \rho(p)^* = -\rho(\beta(p)). \]
But this is equivalent to the single condition
\[ \rho(F)^* = -\rho(E) \]
which is equivalent to the following two conditions:

1. For every $\phi \in \tilde{\mathcal{O}}_m(X), \quad \phi' \in \tilde{\mathcal{O}}_{m+1}(X)$,
\[ \frac{1}{c_{m+1}} (\rho(E)\phi \mid \phi') = -\frac{1}{c_m} (\phi \mid \rho(F)\phi'), \]

2. For every $\phi \in \tilde{\mathcal{O}}_{m+\frac{1}{2}}(X), \quad \phi' \in \tilde{\mathcal{O}}_{m+1+\frac{1}{2}}(X)$,
\[ \frac{1}{c_{m+1+\frac{1}{2}}} (\rho(E)\phi \mid \phi') = -\frac{1}{c_{m+\frac{1}{2}}} (\phi \mid \rho(F)\phi'). \]

Recall that the norms of $\tilde{\mathcal{O}}_m(X)$ and $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ are given by:
\[ \|\phi\|_m^2 = \frac{1}{a_m} \int_{\mathbb{C}^r} |(\phi \circ P)(z)|^2 H(z)^{-(2m)}m_0(dz), \]
and
\[ \|\phi\|_{m+\frac{1}{2}}^2 = \frac{1}{a_{m+\frac{1}{2}}} \int_{\mathbb{C}^r} |(\phi \circ P)(z)|^2 H(z)^{-(2m+1)}m_0(dz) \]
where the $L$-invariant measure is given by $m_0(dz) = H(z)^{-(r+1)}m(dz)$. Then, the required invariant measure conditions become
\[ \frac{1}{c_{m+1}a_{m+1}} \int_{\mathbb{C}^r} \tau(z^2)(\phi \circ P)(z)(\phi' \circ P)(z)H(z)^{-(2m+r+3)}m(dz), \]
\[ = -\frac{1}{c_{m}a_{m}} \int_{\mathbb{C}^r} (\phi \circ P)(z)(\tau\left(\frac{\partial^2}{\partial z^2}\right)(\phi' \circ P)(z)H(z)^{-(2m+r+1)}m(dz), \]
\[ = -\frac{1}{c_{m}a_{m}} \int_{\mathbb{C}^r} (\phi \circ P)(z)(\tau\left(\frac{\partial^2}{\partial z^2}\right)(\phi' \circ P)(z)H(z)^{-(2m+r+1)}m(dz), \]
and
\[
\frac{1}{c_{m+1}a_{m+1}} \int_{C'} \tau(z^2)(\phi \circ P)(z)(\phi' \circ P)(z)H(z)^{-(2m+4+r)} m(dz),
\]
\[
= - \frac{1}{c_{m+1}a_{m+1}} \int_{C'} (\phi \circ P)(z)(\tau(\frac{\partial^2}{\partial z^2}(\phi' \circ P)(z)H(z)^{-(2m+2+r)} m(dz),
\]
where we used
\[
\tau(\frac{\partial^2}{\partial z^2})(\phi' \circ P)(z) = (\tau(\frac{\partial^2}{\partial z^2})(\phi' \circ P)(z).
\]
By integrating by parts:
\[
\int_{C'} (\phi \circ P)(z)\tau(\frac{\partial^2}{\partial z^2})(\phi' \circ P)(z)H(z)^{-(2m+r+1)} m(dz)
\]
\[
= - \int_{C'} (\phi \circ P)(z)(\phi' \circ P)(z)(\tau(\frac{\partial^2}{\partial z^2})H(z)^{-(2m+r+1)} m(dz),
\]
and
\[
\int_{C'} (\phi \circ P)(z)\tau(\frac{\partial^2}{\partial z^2})(\phi' \circ P)(z)H(z)^{-(2m+2+r)} m(dz)
\]
\[
= - \int_{C'} (\phi \circ P)(z)(\phi' \circ P)(z)(\tau(\frac{\partial^2}{\partial z^2})H(z)^{-(2m+2+r)} m(dz),
\]
and using the relations
\[
\tau(\frac{\partial^2}{\partial z^2})(H(z)^{-(2m+r+1)}) = (2m+r+1)(2m+r+2)(\tau(z^2)H(z)^{-(2m+r+3)},
\]
and
\[
\tau(\frac{\partial^2}{\partial z^2})(H(z)^{-(2m+2+r)}) = (2m+r+2)(2m+r+3)(\tau(z^2)H(z)^{-(2m+4+r)},
\]
the invariance conditions can be written
\[
\frac{1}{c_{m+1}a_{m+1}} \int_{C'} \tau(z^2)(\phi \circ P)(z)(\phi' \circ P)(z)H(z)^{-(2m+r+3)} m(dz)
\]
\[
= \frac{(2m + r + 1)(2m + r + 2)}{c_m a_m} \int_{C'} \tau(z^2)(\phi \circ P)(z)(\phi' \circ P)(z)H(z)^{-(2m+r+3)} m(dz)
\]
28
\[ \frac{1}{c_{m+1}a_{m+1}} \int_{C} \tau(z^2)(\phi \circ P)(z)(\phi' \circ P)(z)H(z)^{-(2m+4+r)}m(dz) \]
\[ = \frac{(2m + 2 + r)(2m + 3 + r)}{c_{m+1}a_{m+1}} \int_{C} \tau(z^2)(\phi \circ P)(z)(\phi' \circ P)(z)H(z)^{-(2m+4+r)}m(dz) \]

and these are equivalent to

\[ \frac{1}{c_{m+1}a_{m+1}} = (2m + r + 1)(2m + r + 2) \frac{1}{c_{m}a_{m}} \]

and

\[ \frac{1}{c_{m+1}a_{m+1}} = (2m + 2 + r)(2m + 3 + r) \frac{1}{c_{m+1}a_{m+1}}. \]

Since from Proposition 2.6,

\[ a_{m} = \pi^{r} \frac{1}{(2m + r) \ldots (2m + 1)}, \]

and

\[ a_{m + \frac{1}{2}} = \pi^{r} \frac{1}{(2m + r + 1) \ldots (2m + 2)}, \]

it follows that

\[ \frac{c_{m+1}}{(2m + r + 2) \ldots (2m + 3)} = \frac{c_{m}}{(2m + r + 1)(2m + r + 2)(2m + r) \ldots (2m + 1)} \]

and

\[ \frac{c_{m+1+\frac{1}{2}}}{(2m + r + 3) \ldots (2m + 4)} = \frac{c_{m+\frac{1}{2}}}{(2m + 2 + r)(2m + r + 3)(2m + r + 1) \ldots (2m + 2)} \]

i.e.

\[ c_{m+1} = \frac{1}{(2m + 1)(2m + 2)}c_{m} \]

and

\[ c_{m+1+\frac{1}{2}} = \frac{1}{(2m + 2)(2m + 3)}c_{m+\frac{1}{2}}. \]

It follows that

\[ c_{m} = \frac{1}{(2m)!}, \quad c_{m+\frac{1}{2}} = \frac{1}{(2m + 1)!}. \]
(ii) By Theorem 2.5, the reproducing kernel of $\mathcal{H}$ is given by

$$
K(\xi, \xi') = \sum_{m \in \mathbb{N}} c_m H(z, z')^{2m} + \sum_{m \in \mathbb{N}} c_{m+\frac{1}{2}} H(z, z')^{2m+1}
$$

$$
= \sum_{m \in \mathbb{N}} \frac{1}{(2m)!} H(z, z')^{2m} + \sum_{m \in \mathbb{N}} \frac{1}{(2m+1)!} H(z, z')^{2m+1},
$$

$$
= \exp(H(z, z')) ,
$$

$$
= \exp(1 + \text{tr}(z \bar{z}')).
$$

A similar result can be obtained for the representation $\rho^\sigma$. One needs to calculate the constants $a_{m, \sigma}$ and $a_{m+\frac{1}{2}, \sigma}$ and then determine the suitable numbers $\alpha(m)$.

In the following, we will see that the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^\sigma$ are weighted Bergman spaces. It means that the norm of $\phi \in \mathcal{H}$ and the norm of $\phi \in \mathcal{H}^\sigma$ are given by an integral of $|\phi|^2$ with respect to positive weights.

For positive functions $p$ and $p^\sigma$ on $\mathbb{C}^r$, consider the subspaces $\mathcal{H} \subset \tilde{O}(\Xi)$ and $\mathcal{H}^\sigma \subset \tilde{O}(\Xi^\sigma)$ of $\phi$ such that

$$
\|\phi\|^2 = \int_{\mathbb{C}^r} |\phi(P(z))|^2 p(z) m(dz) < \infty,
$$

$$
\|\phi\|^2_\sigma = \int_{\mathbb{C}^r} |\phi(P(z))|^2 p^\sigma(z) m(dz) < \infty.
$$

Let $F$ and $F^\sigma$ be positive functions on $[0, \infty[$, and define

$$
p(z) = F(H(z) - 1) \quad \text{and} \quad p^\sigma(z) = F^\sigma(H^\sigma(z)).
$$

Then, using Proposition 2.2., one can show that $\mathcal{H}$ and $\mathcal{H}^\sigma$ are $L^R$-invariant (under respectively the representations $d\pi_\alpha$ and $d\pi^\sigma_\alpha$).

Furthermore, for $\phi(P(z)) = \sum_{m \in \mathbb{N}} \phi_m(P(z)) + \sum_{m \in \mathbb{N}} \phi_{m+\frac{1}{2}}(P(z))$,

$$
\|\phi\|^2 = \sum_{m \in \mathbb{N}} \frac{1}{c_m} \|\phi_m\|^2_m + \sum_{m \in \mathbb{N}} \frac{1}{c_{m+\frac{1}{2}}} \|\phi_{m+\frac{1}{2}}\|^2_{m+\frac{1}{2}},
$$

and for $\phi^\sigma(P(z)) = \sum_{m \in \mathbb{N}} \phi^\sigma_m(P(z)) + \sum_{m \in \mathbb{N}} \phi^\sigma_{m+\frac{1}{2}}(P(z))$,

$$
\|\phi^\sigma\|^2 = \sum_{m \in \mathbb{N}} \frac{1}{c_{m, \sigma}} \|\phi^\sigma_m\|^2_{m, \sigma} + \sum_{m \in \mathbb{N}} \frac{1}{c_{m+\frac{1}{2}, \sigma}} \|\phi^\sigma_{m+\frac{1}{2}}\|^2_{m+\frac{1}{2}, \sigma}.
$$
In particular, for 
\[ \phi_m(z) = \ldots z_r^{2m} \] and \[ \phi_{m+\frac{1}{2}}(z) = z_r^{2m+1}, \]

\[ \|\phi_m\|^2 = \frac{1}{c_m} \|\phi_m\|_{m}^2, \quad \|\phi_{m+\frac{1}{2}}\|^2 = \frac{1}{c_{m+\frac{1}{2}}} \|\phi_{m+\frac{1}{2}}\|_{m+\frac{1}{2}}^2, \]

and

\[ \|\phi_m\|_{\sigma}^2 = \frac{1}{c_{m,\sigma}} \|\phi_m\|_{m,\sigma}^2, \quad \|\phi_{m+\frac{1}{2}}\|_{\sigma}^2 = \frac{1}{c_{m+\frac{1}{2},\sigma}} \|\phi_{m+\frac{1}{2}}\|_{m+\frac{1}{2},\sigma}^2, \]

i.e.

\[ \int_{C^r} |\phi_m(z)|^2 p(z)m(dz) = \frac{1}{c_m a_m} \int_{C^r} |\phi_m(z)|^2 H(z)^{-2m} m_0(dz) \]

\[ \int_{C^r} |\phi_{m+\frac{1}{2}}(z)|^2 p(z)m(dz) = \frac{1}{c_{m+\frac{1}{2}} a_{m+\frac{1}{2}}} \int_{C^r} \phi_{m+\frac{1}{2}}(z)|^2 H(z)^{-(2m+1)} m_0(dz) \]

and

\[ \int_{C^r} |\phi_m(z)|^2 p_\sigma(z)m(dz) = \frac{1}{c_{m,\sigma} a_{m,\sigma}} \int_{C^r} |\phi_m(z)|^2 H_\sigma(z)^{-2m} m_0(dz) \]

\[ \int_{C^r} |\phi_{m+\frac{1}{2}}(z)|^2 p_\sigma(z)m(dz) = \frac{1}{c_{m+\frac{1}{2},\sigma} a_{m+\frac{1}{2},\sigma}} \int_{C^r} \phi_{m+\frac{1}{2}}(z)|^2 H_\sigma(z)^{-(2m+1)} m_0(dz). \]

**Theorem 4.2.** — For \( \phi \in \mathcal{H}, \)

\[ \|\phi\|^2 = \int_{C^r} |\phi(P(z)|^2 e^{-tr(z\bar{z})} m(dz). \]
Proof. a) In fact, if the norm of $\mathcal{H}$ is given by:

$$\|\phi\|^2 = \int_{C_r} |\phi(P(z))|^2 p(z)m(dz) < \infty$$

with $F$ a positive function on $[0, +\infty[$ and $p(z) = F(H(z) - 1) = F(\text{tr}(z\bar{z}))$, then

$$\int_{C_r} |\phi_m(z)|^2 p(z)m(dz) = \frac{1}{c_m a_m} \int_{C_r} |\phi_m(z)|^2 H(z)^{-2m} m_0(dz)$$

and

$$\int_{C_r} |\phi_{m+\frac{1}{2}}(z)|^2 p(z)m(dz) = \frac{1}{c_m a_m} \int_{C_r} |\phi_{m+\frac{1}{2}}(z)|^2 H(z)^{-(2m+1)} m_0(dz)$$

i.e.

$$\int_{C_r} (|z_r|^2)^{2m} F(\text{tr}(z\bar{z})) m(dz) = \frac{1}{c_m a_m} \int_{C_r} (|z_r|^2)^{2m} H(z)^{-2m} m_0(dz)$$

and

$$\int_{C_r} (|z_r|^2)^{2m+1} F(\text{tr}(z\bar{z})) m(dz)$$

$$= \frac{1}{c_m a_m} \int_{C_r} (|z_r|^2)^{2m+1} H(z)^{-(2m+1)} m_0(dz).$$

Observe that if we suppose that the function $F$ satisfies the property $F(a + b) = F(a)F(b)$, then

$$\int_{C_r} (|z_r|^2)^{2m} F(\text{tr}(z\bar{z})) m(dz)$$

$$= \pi^m \int_0^\infty (\rho_r^2)^{2m} F(\rho_r^2)(2\rho_r d\rho_r) \prod_{i=1}^{r-1} \int_0^\infty F(\rho_i^2)(2\rho_i d\rho_i)$$

and

$$\int_{C_r} (|z_r|^2)^{2m+1} F(\text{tr}(z\bar{z})) m(dz)$$

$$= \pi^m \int_0^\infty (\rho_r^2)^{2m-r+2} F(\rho_r^2)(2\rho_r d\rho_r) \prod_{i=1}^{r-1} \int_0^\infty F(\rho_i^2)(2\rho_i d\rho_i).$$
Then, by considering the new variables \( u_i = \rho_i^2 \) in the above integrals, one obtains

\[
\int_{C^r} (|z_r|^2)^{2m} F(\text{tr}(z\bar{z}))m(dz)
\]

\[
= \pi^r \int_0^\infty u_r^{2m} F(u_r)du_r \prod_{i=1}^{r-1} \int_0^\infty F(u_i)du_i
\]

and

\[
\int_{C^r} (|z_r|^2)^{2m+1} F(\text{tr}(z\bar{z}))m(dz)
\]

\[
= \pi^r \int_0^\infty u_r^{2m+1} F(u_r)du_r \prod_{i=1}^{r-1} \int_0^\infty F(u_i)du_i.
\]

Furthermore

\[
\int_{C^r} (|z_r|^2)^{2m} H(z)^{-2m}m_0(dz) =
\]

\[
\pi^r \int_0^\infty (\rho_r^2)^{2m} (2\rho_r d\rho_r) \prod_{i=1}^{r-1} \int_0^\infty (2\rho_{r-1} d\rho_{r-1}) \frac{1}{(1 + \rho_1^2 + \ldots + \rho_r^2)^{2m+r+1}}
\]

and

\[
\int_{C^r} (|z_r|^2)^{2m+1} H(z)^{-(2m+1)}m_0(dz) =
\]

\[
\pi^r \int_0^\infty (\rho_r^2)^{2m+1} (2\rho_r d\rho_r) \prod_{i=1}^{r-1} \int_0^\infty (2\rho_{r-1} d\rho_{r-1}) \frac{1}{(1 + \rho_1^2 + \ldots + \rho_r^2)^{2m+r+2}}.
\]
\[ \int_{C} (|z_r|^2)^{2m} H(z)^{-2m} m_0(dz) = \]

\[ \pi^r \int_{0}^{\infty} u_r^{2m} du_r \int_{[0, \infty[} du_1 \ldots \int_{[0, \infty[} du_{r-1} \frac{1}{(1 + u_1 + \ldots + u_r)^{2m+r+1}} \]

\[ = \pi^r \frac{1}{(2m + r)(2m + r - 1) \ldots (2m + 2)} \int_{0}^{\infty} u_r^{2m} \frac{1}{(1 + u_r)^{2m+2}} du_r \]

\[ = \pi^r \frac{(2m)!}{(2m + r)!} \]

and

\[ \int_{C} (|z_r|^2)^{2m+1} H(z)^{-2m+1} m_0(dz) = \]

\[ \pi^r \int_{0}^{\infty} u_r^{2m+1} du_r \int_{[0, \infty[} du_1 \ldots \int_{[0, \infty[} du_{r-1} \frac{1}{(1 + u_1 + \ldots + u_r)^{2m+r+2}} \]

\[ = \pi^r \frac{1}{(2m + r + 1)(2m + r) \ldots (2m + 3)} \int_{0}^{\infty} u_r^{2m+1} \frac{1}{(1 + u_r)^{2m+3}} du_r \]

\[ = \pi^r \frac{(2m + 1)!}{(2m + r + 1)!} \]

From the proof of Theorem 4.1, one has

\[ \frac{1}{a_m c_m} = (2m)! \frac{1}{\pi^r} (2m + r) \ldots (2m + 1) = \frac{1}{\pi^r} (2m + r)!, \]

and

\[ \frac{1}{a_m + \frac{1}{2} c_m + \frac{1}{2}} = (2m + 1)! \frac{1}{\pi^r} (2m + r + 1) \ldots (2m + 2) = \frac{1}{\pi^r} (2m + 1 + r)!. \]

Then

\[ \int_{C} |\phi_m(z)|^2 p(z) m(dz) = \frac{1}{\pi^r} (2m + r)! \left( \frac{(2m)!}{(2m + r)!} \right) = (2m)! \]

and

\[ \int_{C} |\phi_{m+\frac{1}{2}}(z)|^2 p(z) m(dz) = \frac{1}{\pi^r} (2m + 1 + r)! \left( \frac{(2m + 1)!}{(2m + r + 1)!} \right) = (2m + 1)! \]
It follows that the function $F$ satisfies
\[ \pi \int_0^\infty u^{2m} F(u) du = (2m)! \]
and
\[ \pi \int_0^\infty u^{2m+1} F(u) du = (2m + 1)! \]
By inversion of the Mellin transform, one obtains $F(u) = e^{-\pi u}$ and then
\[ p(z) = \exp(-\pi \text{tr}(z \bar{z})). \]

b) Let us consider the weighted Bergman space $\mathcal{H}^1$ whose norm is given by
\[ \|\phi\|_1^2 = \int_{\mathbb{C}^r} |\phi(P(z))|^2 |p(z)| m(dz). \]
By Theorem 2.5,
\[ \|\phi\|_1^2 = \sum_{m=0}^{\infty} \frac{1}{c_m^1} \|\phi_m\|_{2m}^2 + \sum_{m=0}^{\infty} \frac{1}{c_{m+\frac{1}{2}}^1} \|\phi_m\|_{2m+1}^2, \]
with
\[ \frac{1}{a_mc_m^1} = C \int_0^\infty |F(u)| u^{2m} du \]
and
\[ \frac{1}{a_{m+\frac{1}{2}}c_{m+\frac{1}{2}}^1} = C \int_0^\infty |F(u)| u^{2m+1} du \]
Obviously $c_m^1 \leq c_m$, and $c_{m+\frac{1}{2}}^1 \leq c_{m+\frac{1}{2}}$, therefore $\mathcal{H}^1 \subset \mathcal{H}$. We will show that $\mathcal{H} \subset \mathcal{H}^1$. For that we will prove that there is a constant $A$ such that $c_m \leq A \cdot c_m^1$ and $c_{m+\frac{1}{2}} \leq A \cdot c_{m+\frac{1}{2}}^1$.

Since $F(u) \geq 0$, then for $u_0 \neq 0$,
\[ \int_0^\infty |F(u)| u^{2m} du \leq \int_0^u F(u) u^{2m} du + 2 \int_0^{u_0} |F(u)| u^{2m} du \]
and
\[ \int_0^\infty |F(u)| u^{2m+1} du \leq \int_0^u F(u) u^{2m+1} du + 2 \int_0^{u_0} |F(u)| u^{2m+1} du. \]
Hence
\[ \frac{1}{c_m} \leq \frac{1}{c_m} + 2a_m u_0^{2m} \int_0^{u_0} |F(u)|du, \]

and

\[ \frac{1}{c_{m+\frac{1}{2}}} \leq \frac{1}{c_{m+\frac{1}{2}}} + 2a_{m+\frac{1}{2}} u_0^{2m+1} \int_0^{u_0} |F(u)|du. \]

It follows that the sequences \( a_m c_m u_0^{2m} \) and \( a_{m+\frac{1}{2}} c_{m+\frac{1}{2}} u_0^{2m+1} \) are bounded. Therefore there is a constant \( A \) such that

\[ \frac{1}{c_m} \leq A \frac{1}{c_m}, \quad \text{and} \quad \frac{1}{c_{m+\frac{1}{2}}} \leq A \frac{1}{c_{m+\frac{1}{2}}}, \]

and this implies that \( \mathcal{H} \subset \mathcal{H}_1 \).

A similar result can be obtained for the representation \( \rho^\sigma \). One needs to calculate the constants \( a_{m,\sigma}, a_{m+\frac{1}{2},\sigma}, c_{m,\sigma}, c_{m+\frac{1}{2},\sigma} \) and determine the function \( F_\sigma \).

Let \( G_\mathbb{R} \) be the connected and simply connected Lie group with Lie algebra \( \mathfrak{g}_\mathbb{R} \).

Using Nelson’s criterion, in a similar way than for Theorem 6.3 in [AF12], one obtains:

**Theorem 4.3.** — There is a unique unitary representation \( T \) of \( G_\mathbb{R} \) on \( \mathcal{H} \) and a unique unitary representation \( T^\sigma \) of \( G_\mathbb{R} \) on \( \mathcal{H}^\sigma \) such that \( dT = \rho \) and \( dT^\sigma = \rho^\sigma \).
5. Schrödinger model and Bargmann transform. —

We consider in this section the real form \( \tilde{g}_R \) of \( g \) introduced at the end of section 1 and given by \( \tilde{g}_R = I_R + \tilde{V}_R + \tilde{V}_R^g \), where

\[
\tilde{I}_R = \text{Lie}(\text{Str}(V_R)), \quad \tilde{V}_R = \tilde{p}_{-r+1}, \quad \tilde{V}_R^g = \tilde{p}_{r-1},
\]

and where \( \tilde{p}_R \) is the real subspace of \( p \) generated by the polynomials \( Q(x - a) \) on \( V_R \) and \( \tilde{p}_R = \tilde{p}_0 + \tilde{p}_{-1} + \ldots + \tilde{p}_0 + \ldots + p_{r-1} + \tilde{p}_r \) is its eigen-space decomposition under \( \text{ad}(H) = dk(H) \).

Let \( \tilde{\rho}_R \) and \( \tilde{\rho}_R^\sigma \) be the representations of \( \tilde{g}_R \) in the Hilbert space \( L^2(\mathbb{R}^r) \) defined in an analogous manner than the representations \( \rho \) and \( \rho^\sigma \). More precisely, since the group \( \tilde{L}_R \) (the real analogous of \( L \)) acts on the spaces \( C^\infty_c(\tilde{\Xi}_R) \) and \( C^\infty(\tilde{\Xi}_R^g) \) of \( C^\infty \) functions on the real orbits \( \Xi_R = \kappa(\tilde{L}_R)E \) and \( \Xi_R^g = \kappa(\tilde{L}_R)F \) by respectively:

\[
(\pi_{\alpha,R}(l)f)(\xi) = \Delta(a(l))^{2\alpha} f(\kappa(l)^{-1-r}\xi)
\]

and

\[
(\pi_{\alpha,R}^\sigma(l)f^\sigma)(\xi^\sigma) = \Delta(a(l))^{2\alpha} f^\sigma(\kappa(l)^{-1-r}\xi^\sigma),
\]

then \( \tilde{L}_R \) acts on the space \( C^\infty(X_R) \) of \( C^\infty \) functions on \( X_R \), the coordinate system \( X_R \) of the orbits \( \tilde{\Xi}_R \) and \( \tilde{\Xi}_R^g \) given by

\[
X_R = \{P(a) | a \in \mathcal{R}_+\} \quad \text{with} \quad \mathcal{R}_+ = \{a = \sum_{i=1}^r a_i c_i | a_1 \geq \ldots \geq a_r > 0\}
\]

by

\[
\tilde{\pi}_{\alpha,R}(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha} \phi(P(P(a))^{-\frac{1-\sigma}{2}} \cdot a))
\]

and

\[
\tilde{\pi}_{\alpha,R}^\sigma(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha} \phi(P(P(a))^{\frac{1-\sigma}{2}} \cdot a))
\]

which leads by differentiation, to representations \( d\pi_{\alpha,R} \) and \( d\pi_{\alpha,R}^\sigma \) of the Lie algebra \( \tilde{I}_R \) in the spaces \( \mathcal{C}^\infty(\tilde{\Xi}_R) \) and \( \mathcal{C}^\infty(\tilde{\Xi}_R^g) \) and representations \( d\tilde{\pi}_{\alpha,R} \) and \( d\tilde{\pi}_{\alpha,R}^\sigma \) of the Lie algebra \( \tilde{I}_R \) in the space \( \mathcal{C}(X_R) \).

The two representations \( \tilde{\rho}_R \) and \( \tilde{\rho}_R^\sigma \) of \( \tilde{g}_R = \tilde{I}_R + \tilde{V}_R \) on the spaces of finite sums

\[
\mathcal{C}_{\text{fin}}(\tilde{\Xi}_R) = \sum_{m \in \mathbb{N}} C^\infty - \frac{m}{(r-1)}(\tilde{\Xi}_R) \oplus \sum_{m \in \mathbb{N}} C^\infty - \frac{m}{(r-1)} - \frac{1}{2(r-1)}(\tilde{\Xi}_R),
\]

and respectively

\[
\mathcal{C}_{\text{fin}}(\tilde{\Xi}_R^g) = \sum_{m \in \mathbb{N}} C^\infty - \frac{m}{(r-1)}(\tilde{\Xi}_R^g) \oplus \sum_{m \in \mathbb{N}} C^\infty - \frac{m}{(r-1)} - \frac{1}{2(r-1)}(\tilde{\Xi}_R^g)
\]

are defined (similarly than the representations \( \rho \) and \( \rho^\sigma \) of \( g \)) in such a way that for all \( X \in \tilde{I}_R \), \( \tilde{\rho}_R(X) = d\tilde{\pi}_{\alpha,R}(X) \) and \( \tilde{\rho}_R^\sigma(X) = d\tilde{\pi}_{\alpha,R}^\sigma(X) \).

Then

\[
\tilde{\rho}_R(H)\phi(P(a)) = 2\alpha \frac{r}{r-1}(\phi \circ P)(a) + \mathcal{E}(\phi \circ P)(a)
\]

where \( \mathcal{E} \) is the Euler operator \( (\mathcal{E}\phi)(P(a)) = \frac{d}{du}|_{u=1} \phi(P(u \cdot a)) \).

37
Furthermore, if \( \tilde{C}^\infty_{2m}(\mathbb{R}^r) \) and \( \tilde{C}^\infty_{2m+1}(\mathbb{R}^r) \) are the subspaces of \( \mathcal{C}^\infty(\mathbb{R}^r) \), images of \( \tilde{C}^\infty_m(X) = \mathcal{C}^\infty_m(X) \cap \mathcal{C}^\infty(\mathbb{R}^r) \), \( \tilde{C}^\infty_{m+\frac{1}{2}}(X) = \mathcal{C}^\infty_{m+\frac{1}{2}}(X) \cap \mathcal{C}^\infty(\mathbb{R}^r) \) by the isomorphism \( \phi \mapsto \phi \circ P \), then the operators \( \tilde{\rho}_R(E) \) and \( \tilde{\rho}_R(F) \) are defined by:

\[
\tilde{\rho}_R(E) : \mathcal{C}_{\text{fin}}(X) \rightarrow \mathcal{C}_{\text{fin}}(X) \text{ is the multiplication operator which maps } \tilde{C}^\infty_m(X) \text{ to } \tilde{O}^\infty_{m+1}(X) \text{ and } \tilde{C}^\infty_{m+\frac{1}{2}}(X) \text{ to } \tilde{C}^\infty_{m+1+\frac{1}{2}}(X) \text{ given by:}
\]

\[
\tilde{\rho}_R(E) : \phi(P(x)) \mapsto \frac{i}{2} x \cdot \tau(x^2) \phi(P(x)),
\]

\[
\tilde{\rho}_R(F) : \mathcal{C}_{\text{fin}}(X) \rightarrow \mathcal{C}_{\text{fin}}(X) \text{ is the differential operator which maps } \tilde{C}^\infty_m(X) \text{ to } \tilde{C}^\infty_{m-1}(X) \text{ and } \tilde{C}^\infty_{m+\frac{1}{2}}(X) \text{ to } \tilde{C}^\infty_{m-1+\frac{1}{2}}(X) \text{ given by:}
\]

\[
\tilde{\rho}_R(F) : \phi(P(x)) \mapsto i \frac{c}{2} \cdot \tau(x^2) \phi \circ P(x).
\]

As in the case of \( \rho \), one has \([\tilde{\rho}_R(E), \tilde{\rho}_R(F)] = \tilde{\rho}_R([E, F]) = \frac{1-r}{4} \tilde{\rho}_R(H)\) if and only if \( \alpha = \frac{c-1}{4} \) and \( c = i \frac{\sqrt{r-1}}{2} \).

Also, for \( p \in \hat{V}_R \) and for \( p^\sigma = \kappa(\sigma)p \in \hat{V}_R^\sigma \) the operators \( \tilde{\rho}_R(p), \tilde{\rho}_R(p^\sigma) : \mathcal{C}_{\text{fin}}(X) \rightarrow \mathcal{C}_{\text{fin}}(X) \) are respectively given by:

\[
\tilde{\rho}_R(p) : \phi(P(x)) \mapsto -\frac{\sqrt{r-1}}{4} p(x^2) \phi(P(x))
\]

and

\[
\tilde{\rho}_R(p^\sigma) : \phi(P(x)) \mapsto -\frac{\sqrt{r-1}}{4} p(\frac{\partial^2}{\partial x^2}) \phi \circ P(x).
\]

Finally, one obtains two representations of the real Lie algebra \( \tilde{\mathfrak{g}}_R \):

\[
\tilde{\rho}_R : \tilde{\mathfrak{g}}_R \rightarrow \tilde{\mathfrak{h}}_R \oplus \tilde{\mathcal{W}}_R \rightarrow \text{End}(\mathcal{C}_{\text{fin}}(\tilde{\Xi}_R))
\]

and

\[
\tilde{\rho}_R^\sigma = \tilde{\pi}_R(\sigma) \tilde{\rho}_R \tilde{\pi}_R(\sigma) : \tilde{\mathfrak{g}}_R \rightarrow \tilde{\mathfrak{h}}_R \oplus \tilde{\mathcal{W}}_R \rightarrow \text{End}(\mathcal{C}_{\text{fin}}(\tilde{\Xi}_R^\sigma)).
\]

Each of \( \tilde{\rho}_R \) and \( \tilde{\rho}_R^\sigma \) is a sum of two irreducible representations realized in \( \mathcal{C}_{\text{even}}(\tilde{\Xi}_R) = \sum_{m=0}^\infty \mathcal{C}^\infty_m(X_R) \) and \( \mathcal{C}_{\text{odd}}(\tilde{\Xi}_R) = \sum_{m=0}^\infty \mathcal{C}^\infty_{m+\frac{1}{2}}(X_R) \).

One can show that the maximal compact subgroup \( \tilde{L}^c_R \) of \( \tilde{L}_R \) acts unitarily on the Hilbert space \( L^2(\mathbb{R}^r) \) under the representations

\[
\tilde{\pi}_{\alpha,R}(l) : \phi(x) \mapsto \Delta(a(l))^{2\alpha} \phi(P(a(l))^\frac{1-r}{2} \cdot x))
\]

and

\[
\tilde{\pi}_{\alpha,R}^\sigma(l) : \phi(x) \mapsto \Delta(a(l))^{2\alpha} \phi(P(a(l))^{\frac{1-r}{2} \cdot x}).
\]

In fact, \( l \) belongs to \( \tilde{L}^c_R \) off \( ll' = id \) (where \( l' \) is the adjoint of \( l \) with respect to \( \text{tr}(xy) \)) and it follows that for every such \( l, |\text{Det}(P(a(l)))| = 1. \)
Moreover, the Hilbert space $L^2(\mathbb{R}^r)$ is $\tilde{g}_R$-invariant under the representation $\tilde{\rho}_R$. In fact, this is equivalent to the two conditions

$$<\tilde{\rho}_R(E)\phi, \phi'> = -<\phi, \tilde{\rho}_R(E)\phi'>$$

and

$$<\tilde{\rho}_R(F)\phi, \phi'> = -<\phi, \tilde{\rho}_R(F)\phi'>$$

for every $C^\infty$ functions with compact support $\phi, \phi'$ on $\mathbb{R}^r$, where

$$<\psi, \psi'> = \int_{\mathbb{R}^r} \psi(x)\overline{\psi'(x)}m(dx)$$

is the scalar product in $L^2(\mathbb{R}^r)$ and $m(dx)$ is the Lebesgue measure. But

$$<\tilde{\rho}_R(E)\phi, \phi'> = \int_{\mathbb{R}^r} i\frac{c}{2}\tau(x^2)(\phi \circ P)(x)\overline{\phi'(x)}m(dx)$$

and by an integration by parts one has

$$<\tilde{\rho}_R(F)\phi, \phi'> = \int_{\mathbb{R}^r} i\frac{c}{2}\tau(x^2)(\phi \circ P)(x)\overline{\phi'(x)}m(dx)$$

Finally, using Nelson’s criterion, one can show that the representation $\tilde{\rho}_R$ integrates to a unitary representation $\tilde{T}_R$ of the simply connected Lie group $\tilde{G}_R$ with Lie algebra $\tilde{g}_R$.

If there is a Lie algebra isomorphism $c : \tilde{g}_R \rightarrow g_R$, then one can suppose that it restricts to $\text{Ad}(g_0^{-1}) : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{su}(1, 1)$. Then, similarly than in the $\mathfrak{sl}(2, \mathbb{R})$-case (see section 6), the Bargmann transform given by

$$B : L^2(\mathbb{R}^r) \rightarrow \mathcal{F}(\mathbb{C}^r), \phi \mapsto \psi,$$

with

$$\psi(z) = \int_{\mathbb{R}^r} e^{-\frac{i}{2}\tau(x^2)+\tau(xz)} \phi(x)dx,$$

intertwines the representations $\tilde{\rho}_R$ and $\rho_R \circ c$, more precisely the operators $\tilde{\rho}_R(E), \tilde{\rho}_R(F), \tilde{\rho}_R(H)$ with $\rho_R(\text{Ad}(g_0^{-1})E), \rho_R(\text{Ad}(g_0^{-1})F), \rho_R(\text{Ad}(g_0^{-1})H)$ respectively.

In the case $\tilde{g}_R \simeq g_R$, the representations $(\tilde{T}_R, L^2(\mathbb{R}^r))$ and $(T_R, \mathcal{F}(\mathbb{C}^r))$ are respectively the Schrödinger and the Fock model of one of the two minimal representations of the real Lie group $G_R \simeq G_R$. 

39
6. The \(\mathfrak{sl}(2,\mathbb{R})\)-case. —

In the special case \(r = 1\), since \(\tau^\sigma(z) = -\tau(z) = -z\), we denote by \(F = Q \in \mathcal{V} = \mathfrak{p}_1\) and \(E = 1 \in \mathcal{V}^\sigma = \mathfrak{p}_{-1}\). Then for \(H = -2\tilde{H}\), one has
\[
[H, E] = 2E \quad \text{and} \quad [H, F] = -2F
\]
and then consider the Lie algebra structure on \(\mathfrak{g} = \mathfrak{t} \oplus \mathcal{V} \oplus \mathcal{V}^\sigma\) such that \([E, F] = H\). In this case \(\mathfrak{g}\) is isomorphic to \(\mathfrak{sl}(2, \mathbb{C})\) and the real form \(\mathfrak{g}_\mathbb{R}\)

\text{is isomorphic to} \(\mathfrak{su}(1, 1)\). The structure group of \(V = \mathbb{C}\) is \(\text{Str}(V, Q) = \mathbb{C}^*\) acting by dilations \(l_\lambda\) and, since for \(\lambda \in \mathbb{C}^*, Q(\lambda \cdot z) = \lambda^2 Q(z)\), then \(K = \text{Conf}(V, Q)\) and \(L = \text{Str}(V, Q)\). The orbits \(\Xi\) and \(\Xi^\sigma\) are given by
\[
\Xi = \{\kappa(l_z^{-1})Q \mid z \in \mathbb{C}^*\} = \{z \cdot Q \mid z \in \mathbb{C}^*\},
\]
\[
\Xi^\sigma = \{\kappa(l_z^{-1})1 \mid z \in \mathbb{C}^*\} = \{\frac{1}{z} \cdot 1 \mid z \in \mathbb{C}^*\}.
\]

The variety \(X\) is here given by \(X = \{z \in \mathbb{C}^* \mid \text{Re}(z) > 0\}\) and then \(\tilde{O}(X) = \mathcal{O}(\mathbb{C})\), the space of holomorphic funtions on \(\mathbb{C}\). Since every \(\xi \in \Xi\) can be written \(\xi(v) = (\sqrt{z})^{-2}Q(P(\sqrt{z})v)\) and every \(\xi^\sigma \in \Xi^\sigma\) can be written \(\xi^\sigma(v) = (\sqrt{z})^2Q(P(\sqrt{z}^{-1})v)\) where \(P(u) = u^2\) is the quadratic representation of \(V = \mathbb{C}\), we deduce that the coordinates \(P(a)\) correspond here to \(P(a) = z\), i.e. \(a = \sqrt{z}\) and the isomorphism \(\tilde{\pi}(\sigma)\) maps \(\phi(z)\) to \(\phi(\frac{1}{z})\). It follows that for \(m \in \mathbb{N}\)
\[
\tilde{O}_m(\Xi) = \mathcal{O}_{2m}(\mathbb{C}) = \{\psi \in \mathcal{O}(\mathbb{C}) \mid \phi(\mu \cdot z) = \mu^{2m}\phi(z)\} = \mathbb{C} \cdot z^{2m}
\]
and
\[
\tilde{O}_{m+\frac{1}{2}}(\Xi) = \mathcal{O}_{2m+1}(\mathbb{C}) = \{\psi \in \mathcal{O}(\mathbb{C}) \mid \phi(\mu \cdot z) = \mu^{2m+1}\phi(z)\} = \mathbb{C} \cdot z^{2m+1}.
\]

Then \(\mathcal{O}_{\text{fin}}(\Xi) = \{\psi(z) = \sum_{k=0}^{n} a_k z^k \mid a_k \in \mathbb{C}\}\). The norms on the spaces \(\tilde{O}_m(\Xi)\) and \(\tilde{O}_{m+\frac{1}{2}}(\Xi)\) are respectively given by:
\[
\|\phi\|^2_m = \frac{1}{a_m} \int_{\mathbb{C}} |\phi(z)|^2 H(z)^{-2m} m_0(dz),
\]
\[
\|\phi\|^2_{m+\frac{1}{2}} = \frac{1}{a_{m+\frac{1}{2}}} \int_{\mathbb{C}} |\phi(z)|^2 H(z)^{-2m-1} m_0(dz)
\]
where \(H(z) = \tau(e + z\bar{z}) = 1 + |z|^2\), \(m_0(dz) = \frac{1}{(1+|z|^2)^2} dz\) and where the constants \(a_m\) and \(a_{m+\frac{1}{2}}\) are given by

40
\[ a_m = \int_{\mathbb{C}} (1 + |z|^2)^{-(2m+2)} \, dz = \frac{\pi}{2m+1}, \]
\[ a_{m+\frac{1}{2}} = \int_{\mathbb{C}} (1 + |z|^2)^{-(2m+3)} \, dz = \frac{\pi}{2m+2}. \]

The reproducing kernel of the space \((\tilde{O}_m(\Xi), \| \cdot \|_m)\) is given by
\[ K(z, z') = (1 + z \bar{z}')^{2m}. \]

The reproducing kernel of the space \((\tilde{O}_{m+\frac{1}{2}}(\Xi), \| \cdot \|_{m+\frac{1}{2}})\) is given by
\[ K(z, z') = (1 + z \bar{z}')^{2m+1}. \]

The representation \(\pi_\alpha\) of \(L\) on \(\tilde{O}(\Xi) = \mathcal{O}(\mathbb{C})\) is given by
\[ \pi_\alpha(l_\lambda) \phi(z) = \lambda^\alpha \phi(\lambda^{-1} \cdot z). \]

It follows in particular that for \(\lambda = e^{-t}\), we have \(\pi_\alpha(l_\lambda) \phi(z) = e^{-t \alpha} \phi(e^t \cdot z)\), then
\[ d\pi_\alpha(H) \phi(z) = -\alpha \phi(z) + \mathcal{E} \phi(z). \]

The representation \(\rho\) is given by
\[ \rho(E) \phi(z) = \frac{i}{2} z^2 \phi(z), \]
\[ \rho(F) \phi(z) = \frac{i}{2} \frac{\partial^2}{\partial z^2} \phi(z) \]
and
\[ \rho(H) \phi(z) = d\pi_\alpha(H) \phi(z). \]

Since
\[ z^2 \frac{\partial^2}{\partial z^2} \phi(z) - \frac{\partial^2}{\partial z^2} (z \phi(z)) = -2 \phi(z) - 4 \mathcal{E} \phi(z), \]
it follows that for \(\alpha = -\frac{1}{2}\), one has
\[ [\rho(E), \rho(F)] = d\pi_\alpha(H). \]

The invariance condition \(\rho(F)^* = -\rho(E)\) is equivalent to the two following conditions:
\[
\frac{1}{a_{m+1}c_{m+1}} \int_C z^2 \phi(z) \overline{\phi'(z)} (1 + |z|^2)^{-(2(m+1)+2)} dz = \\
- \frac{1}{a_{m} c_{m}} \int_C \phi(z) \frac{\partial^2}{\partial z^2} \overline{\phi'(z)} (1 + |z|^2)^{-(2m+2)} dz
\]
and
\[
\frac{1}{a_{m+1+\frac{1}{2}} c_{m+1+\frac{1}{2}}} \int_C z^2 \phi(z) \overline{\phi'(z)} (1 + |z|^2)^{-(2(m+1)+3)} dz = \\
- \frac{1}{a_{m+\frac{1}{2}} c_{m+\frac{1}{2}}} \int_C \phi(z) \frac{\partial^2}{\partial z^2} \overline{\phi'(z)} (1 + |z|^2)^{-(2m+3)} dz,
\]
which are equivalent to
\[
\frac{1}{a_{m+1}c_{m+1}} \int_C z^2 \phi(z) \overline{\phi'(z)} (1 + |z|^2)^{-(2m+4)} dz = \\
- \frac{1}{a_{m} c_{m}} \int_C \phi(z) \frac{\partial^2}{\partial z^2} \overline{\phi'(z)} (1 + |z|^2)^{-(2m+2)} dz
\]
and
\[
\frac{1}{a_{m+1+\frac{1}{2}} c_{m+1+\frac{1}{2}}} \int_C z^2 \phi(z) \overline{\phi'(z)} (1 + |z|^2)^{-(2m+5)} dz = \\
- \frac{1}{a_{m+\frac{1}{2}} c_{m+\frac{1}{2}}} \int_C \phi(z) \frac{\partial^2}{\partial z^2} \overline{\phi'(z)} (1 + |z|^2)^{-(2m+3)} dz.
\]
Since
\[
\frac{\partial^2}{\partial z^2} (1 + |z|^2)^{-(2m+2)} = (2m + 2)(2m + 3) z^2 (1 + |z|^2)^{-(2(m+1)+2)},
\]
and
\[
\frac{\partial^2}{\partial z^2} (1 + |z|^2)^{-(2m+3)} = (2m + 3)(2m + 4) z^2 (1 + |z|^2)^{-(2(m+1)+3)},
\]
it follows, after an integration by parts, that the invariance conditions can be written
\[
\frac{1}{a_{m+1}c_{m+1}} = (2m + 2)(2m + 3) \frac{1}{a_{m} c_{m}}
\]
and
\[
\frac{1}{a_{m+1+\frac{1}{2}} c_{m+1+\frac{1}{2}}} = (2m + 3)(2m + 4) \frac{1}{a_{m+\frac{1}{2}} c_{m+\frac{1}{2}}}.
\]
Finally, using the formulas \(a_m = \pi \frac{1}{2m+1}\) and \(a_{m+\frac{1}{2}} = \pi \frac{1}{2m+2}\), we get
\[(2m + 3)\frac{1}{c_{m+1}} = (2m + 2)(2m + 3)(2m + 1)\frac{1}{c_m}\]

and

\[(2m + 4)\frac{1}{c_{m+1 + \frac{1}{2}}} = (2m + 3)(2m + 4)(2m + 2)\frac{1}{c_{m + \frac{1}{2}}}\]

i.e.

\[\frac{1}{c_{m+1 + \frac{1}{2}}} = (2m + 3)(2m + 2)\frac{1}{c_{m + \frac{1}{2}}}\]

and it follows that

\[c_{m+\frac{1}{2}} = \frac{1}{(2m + 1)!}.\]

The representation \(dp_{\frac{1}{2}} + \rho\) integrates to a unitary representation of \(\widetilde{SU}(1, 1)\) in the Hilbert space representation \(\mathcal{H}\), the completion of \(O_{\text{fin}}(\Xi)\) with respect to the norm given for \(\phi = \sum_{m \in \mathbb{N}} \phi_m + \sum_{m \in \mathbb{N}} \phi_{m + \frac{1}{2}}\), by

\[\|\phi\|^2 = \sum_{m \in \mathbb{N}} \frac{1}{c_m} \|\phi_m\|_{2m}^2 + \sum_{m \in \mathbb{N}} \frac{1}{c_{m + \frac{1}{2}}} \|\phi_{m + \frac{1}{2}}\|_{2m+1}^2,\]

and the reproducing kernel of \(\mathcal{H}\) is given by:

\[\mathcal{K}(z, z') = \sum_{m \in \mathbb{N}} \frac{1}{m!} H(z, z')^m = e^{H(z,z')}\]

Furthermore the Hilbert space \(\mathcal{H}\) is a weighted Bergman space and its norm is given by

\[\|\phi\|^2 = \int \|\phi(z)\|^2 e^{-|z|^2} m(dz)\]

and it follows that \(\mathcal{H} = \mathcal{F}(\mathbb{C})\), the classical Fock-space on \(\mathbb{C}\).

Moreover, \(\tilde{g}_\mathbb{R} = \mathfrak{sl}(2, \mathbb{R})\) and the representation \(\tilde{\rho}_\mathbb{R}\) in \(L^2(\mathbb{R})\) is determined by

\[\tilde{\rho}_\mathbb{R}(E)\phi(x) = \frac{i}{2} x^2 \phi(x),\]

\[\tilde{\rho}_\mathbb{R}(F)\phi(x) = \frac{i}{2} \frac{\partial^2}{\partial x^2} \phi(x),\]

\[\tilde{\rho}_\mathbb{R}(H)\phi(x) = \frac{1}{2} \phi(x) + E\phi(x)\]

which is the Schrödinger model of the Segal-Shale-Weil representation.
Furthermore, $g_R = \mathfrak{su}(1, 1)$ and the restricted representation $\rho_R$ of $\rho$ to $\mathfrak{g}_R$ in $\mathcal{F}(\mathbb{C})$ is determined by

$$\rho_R(E + F)\psi(z) = \frac{i}{2} z^2 \psi(z) + \frac{i}{2} \frac{\partial^2}{\partial z^2} \psi(z),$$

$$\rho_R(i(E - F))\psi(z) = -\frac{1}{2} z^2 \psi(z) + \frac{1}{2} \frac{\partial^2}{\partial z^2} \psi(z),$$

$$\rho_R(iH)\psi(z) = \frac{i}{2} \psi(z) + i\mathcal{E}\psi(z)$$

which is the Fock-model of the Segal-Shale-Weil representation.

One knows that the Bargmann transform

$$\mathcal{B} : L^2(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{C}), \phi \mapsto \psi,$$

with

$$\psi(z) = e^{\frac{z^2}{4}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-z)^2} \phi(x) dx,$$

intertwines the representations $\tilde{\rho}_R$ and $\rho_R \circ \text{Ad}(g_0^{-1})$, more precisely $\tilde{\rho}_R(E), \tilde{\rho}_R(F), \tilde{\rho}_R(H)$ with $\rho_R(\text{Ad}(g_0^{-1})E), \rho_R(\text{Ad}(g_0^{-1})F), \rho_R(\text{Ad}(g_0^{-1})H)$ respectively, which are given by

$$\rho_R(\text{Ad}(g_0^{-1})E)\psi(z) = \rho_R\left(\frac{1}{2}iH - \frac{1}{2}(E + F)\right)\psi(z)$$

$$= -\frac{i}{2} \psi(z) - i\mathcal{E}\psi(z) - \frac{i}{4} z^2 \psi(z) - \frac{i}{4} \frac{\partial^2}{\partial z^2} \psi(z),$$

$$\rho_R(\text{Ad}(g_0^{-1})F)\psi(z) = \rho_R\left(-\frac{1}{2}iH - \frac{1}{2}(E + F)\right)\psi(z)$$

$$= \frac{i}{2} \psi(z) + i\mathcal{E}\psi(z) - \frac{i}{4} z^2 \psi(z) - \frac{i}{4} \frac{\partial^2}{\partial z^2} \psi(z),$$

$$\rho_R(\text{Ad}(g_0^{-1})H)\psi(z) = \rho_R(i(F - E))\psi(z)$$

$$= -\frac{1}{2} \frac{\partial^2}{\partial z^2} \psi(z) + \frac{1}{2} z^2 \psi(z).$$

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