Convergence of the self-avoiding walk on random quadrangulations to $\text{SLE}_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity

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Abstract

We prove that a uniform infinite quadrangulation of the half-plane decorated by a self-avoiding walk (SAW) converges in the scaling limit to the metric gluing of two independent Brownian half-planes identified along their positive boundary rays. Combined with other work of the authors, this implies the convergence of the SAW on a random quadrangulation to $\text{SLE}_{8/3}$ on a certain $\sqrt{8/3}$-Liouville quantum gravity surface. The topology of convergence is the local Gromov-Hausdorff-Prokhorov-uniform topology, the natural generalization of the local Gromov-Hausdorff topology to curve-decorated metric measure spaces. We also prove analogous scaling limit results for uniform infinite quadrangulations of the whole plane decorated by either a one-sided or two-sided SAW. Our proof uses only the peeling procedure for random quadrangulations and some basic properties of the Brownian half-plane, so can be read without any knowledge of SLE or LQG.

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The SAW in this context was first studied (non-rigorously) by Duplantier and Werner [LSW04b] that in this case the SAW converges upon appropriate rescaling to the Schramm-Loewner evolution (SLE) [Sch00] with parameter $\kappa = 8/3$. This conjecture was derived by making the ansatz that the scaling limit of the SAW should be conformally invariant and satisfy a certain Markov property. The value $\kappa = 8/3$ arises because SLE$_{8/3}$ satisfies the so-called restriction property [LSW03], which is the continuum analog of the fact that a SAW conditioned to stay in a subgraph is the same as a SAW on that subgraph. This conjecture has been supported by extensive numerical simulations due to Tom Kennedy [Ken02]. Prior to the present work, no scaling limit result for the SAW in two dimensions has been proved, however.

We will study and prove scaling limit results for the SAW in two dimensions on certain types of random planar maps. The SAW in this context was first studied (non-rigorously) by Duplantier and Kostov [DK88, DK90] as a test case for the KPZ formula [KPZ88], which relates exponents for critical models on random surfaces with the corresponding exponents on planar lattices. We will establish the existence of the scaling limit of the SAW on a random planar quadrangulation, viewed as a curve-decorated metric measure space equipped with the SAW, the graph distance, and the counting measure on edges. Although the proof of this scaling limit result uses only the theory of random planar maps, the results of [GM16a] allow us to identify the limiting object with SLE$_{8/3}$ on a $\sqrt{8/3}$-Liouville quantum gravity wedge, a certain random metric measure space with the topology of the upper half-plane. We will discuss this identification further in Section 1.1.4, but first let us say more about SAW on random planar maps.
Recall that a planar map is a graph together with an embedding in the plane so that no two edges cross. Two such maps are said to be equivalent if there exists an orientation preserving homeomorphism which takes one to the other. A map is said to be a quadrangulation if every face has exactly four adjacent edges.

The theories of statistical mechanics models like the SAW on random planar maps and on deterministic lattices are equally important: both are well-motivated physically and have been studied extensively in the math and physics literature. There are many questions (such as scaling limits of various curves toward SLE) which can be asked in both the random planar map and deterministic lattice settings (in the former setting, one has to specify a topology). It is not in general clear which setting is easier to analyze. For example, a type of convergence known as peanosphere convergence [DMS14,She16b,KMSW15,GKM16,LSW17] for many different models has so far only been established in the random planar map setting; whereas the scaling limit of loop-erased random walk toward SLE$_2$ on a deterministic lattice has been established in [LSW04a], but so far this convergence has not been shown for any planar map in either the metric sense or as a curve embedded into the plane via some canonical embedding of the planar map.

The convergence of the SAW toward SLE$_{8/3}$ is particularly interesting since in both the random planar map and deterministic lattice settings, the SAW is easy to define and important both mathematically and physically; the convergence toward SLE$_{8/3}$ is supported by heuristic evidence; and, prior to this work, the convergence was not proven rigorously in either setting.

1.1.2 Gluing together random quadrangulations

We will now describe a simple construction of a finite quadrangulation decorated with a SAW and then describe the corresponding infinite volume versions of this construction.

Suppose we sample two independent uniformly random quadrangulations of the disk with simple boundary with $n$ quadrilaterals and perimeter $2l$ and then glue them together along a boundary segment of length $2s < 2l$ by identifying the corresponding edges (Figure 1, left). The conditional law of the gluing interface given the overall glued map will then be that of a SAW of length $2s$ conditioned on its left and right complementary components both containing $n$ quadrilaterals. One can also glue the entire boundaries of the two disks to obtain a map with the topology of the sphere decorated by a path whose conditional law given the map is that of a self-avoiding loop on length $2l$ conditioned on the two complementary components both containing $n$ quadrilaterals. See, for example, the discussion in [Bet15, Section 8.2] (which builds on [BG09,BBG12]) for additional explanation.

The uniform infinite half-planar quadrangulation with simple boundary (UIHPQ$_S$) is the infinite-volume local limit of uniform quadrangulations of the disk with simple boundary rooted at a boundary edge as the total number of interior faces (or interior vertices), and then the number of boundary edges, is sent to $\infty$ [CM15,CC15].

It is shown by Caraceni and Curien [CC16, Section 1.4] that the infinite volume limit of the aforementioned random SAW-decorated quadrangulations can be constructed by starting off with two independent UIHPQ$_S$’s and then gluing them together along their boundary (Figure 1, right). In this case, the gluing interface is an infinite volume limit of a SAW. There are several natural constructions leading to SAW decorated quadrangulations that one can build with these types of gluing operations:

- **Chordal SAW on a half-planar quadrangulation from 0 to $\infty$**: Glue two independent UIHPQ$_S$’s along their positive boundaries, yielding a random quadrangulation of the upper half-plane with a distinguished path from the boundary to $\infty$.

- **Two-sided SAW on a whole-plane quadrangulation from $\infty$ to 0 and back to $\infty$**: Glue two independent UIHPQ$_S$’s along their entire boundaries, yielding a random quadrangulation of the plane together with a two-sided path from $\infty$ through the root vertex and then back to $\infty$.

- **Whole-plane SAW from 0 to $\infty$ on a whole-plane quadrangulation**: Glue together the two complementary rays of the boundary of a single UIHPQ$_S$, yielding a quadrangulation of the plane together with a distinguished path from the root vertex to $\infty$. 


Figure 1: **Left:** Two independent uniformly random finite quadrangulations with boundary glued together along a boundary arc to get a uniformly random SAW-decorated quadrangulation with boundary. **Right:** The infinite-volume and boundary length limit of the left panel: two independent UIHPQ$^+$'s glued together along their positive boundary rays to obtain an infinite-volume uniform SAW-decorated quadrangulation with boundary. We prove that the scaling limit of the picture on the right exists and is equal to the metric space quotient of a pair of independent Brownian half-planes glued together along their positive boundaries. By the results of [GM16a], this limiting space can equivalently be described as a weight-4 Liouville quantum gravity wedge decorated by an independent chordal SLE$_{8/3}$ curve.

### 1.1.3 Gluing together Brownian half-planes

Building on the scaling limit result for finite uniform quadrangulations with boundary in [BM17], it was proved in [GM16b] that the UIHPQ$^+$ converges in the scaling limit to the so-called Brownian half-plane (see also [BLR17] for more general scaling limit results for half-plane quadrangulations with general boundary). This is a random metric space with boundary which has the topology of the upper-half-plane whose definition is reviewed in Section 2.3. This metric space comes with some additional structure: an area measure and a boundary length measure. One can perform each of the aforementioned gluing operations with the Brownian half-plane in place of the UIHPQ$^+$ by identifying Brownian half-planes together along their boundaries and taking a metric space quotient (see Figure 3).

The main results of the present work, stated precisely in Section 1.2, are that in each of the above three itemized cases the construction built from the UIHPQ$^+$ converges in the scaling limit to the corresponding construction built from the Brownian half-plane (see Remark 1.4 for a discussion of the case where we glue together finite quadrangulations with simple boundary). Combining this with the main results of [GM16a], we conclude that the SAW on random quadrangulations converges to SLE$_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity (LQG). We will explain this latter point in more detail just below.

The topology in which the scaling limits in this paper take place is the one induced by the *local Gromov-Hausdorff-Prokhorov-uniform (GHPU) metric* on curve-decorated metric measure spaces, which is introduced in [GM16b] and reviewed in Section 2.2 below. The local GHPU metric is the natural analog of the local Gromov-Hausdorff metric when we study metric spaces with a distinguished measure and curve. Roughly speaking, two compact curve-decorated metric measure spaces are said to be close in the GHPU metric if they can be isometrically embedded into a common metric space in such a way that the spaces are close in the Hausdorff distance, the measures are close in the Prokhorov distance, and the curves are close in the uniform distance. Two non-compact curve decorated metric measure spaces are close in the local GHPU topology if
their metric balls of radius \( r \) are close in the GHPU topology for a large value of \( r \). See Section 2.2 below for a precise definition of the local GHPU metric. (See also \([GM17a]\) for an analogous GHPU convergence result for percolation on random quadrangulations with simple boundary to SLE\(_6\) on \( \sqrt{8/3}\text{-LQG} \).)

Since we already know that the scaling limit of the UIHPQs is the Brownian half-plane, to prove our main results we need to show that the operation of passing to the scaling limit of two independent UIHPQs’ to get two independent Brownian half-planes commutes with the operation of gluing the surfaces together along their boundaries. It is natural to expect this to be the case, but proving this commutation of scaling limits and gluing operations is highly non-trivial. Indeed, it is a priori possible that paths which cross the gluing interface more than a constant-order number of times are typically much shorter than paths which cross only a constant-order number of times. If this is the case, then a subsequential scaling limit of the discrete glued maps in the GHPU topology might not coincide with the metric gluing of the scaling limits of the UIHPQs’s on either side of the SAW. Here we emphasize that distances in the continuum metric gluing are given by the infimum of the lengths of paths which cross the gluing interface at most finitely many times; see the definition of the quotient metric in Section 2.1.4.

For similar reasons, it is not a priori clear that gluing together Brownian half-planes along their boundaries produces a metric space decorated by a simple curve. The results of \([GM16a]\) imply that this is indeed the case (and identifies the law of the curve-decorated metric space with a certain type of SLE\(_{8/3}\)-decorated \( \sqrt{8/3}\text{-LQG} \) surface). As a by-product of the arguments in the present paper, we obtain another proof that the gluing interface is simple, and show that it is in fact locally reverse Hölder continuous of any exponent \( p > 3/2 \), i.e., for each fixed \( L > 0 \) there exists \( c > 0 \) such that the interface \( \eta_{zip}(t_1), \eta_{zip}(t_2) \geq c|t_2 - t_1|^p \) for each \( t_1, t_2 \in [0, L] \) (see Lemma 7.3 below). See the introduction of \([GM16a]\) for some additional discussion of the issues which can arise when performing metric gluings.

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**Figure 2:** A chart of the different components which serve as input into the proof that self-avoiding walk on random quadrangulations converges to SLE\(_{8/3}\) on \( \sqrt{8/3}\text{-LQG} \). The present article corresponds to the blue box and implies that the discrete graph gluing of random quadrangulations of the upper half-plane converge to the metric gluing of Brownian half-planes. Combined with \([GM16a]\) (i.e., the article indicated in the box immediately to the right of the blue box), this implies that the self-avoiding walk on random quadrangulations converges to an SLE\(_{8/3}\)-type path on \( \sqrt{8/3}\text{-LQG} \).
1.1.4 Identification with SLE$_{8/3}$-decorated $\sqrt{8/3}$-LQG

In order to explain how the main results of this article allow us to identify the scaling limit of the SAW with SLE$_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity (LQG), we first need to briefly discuss the basics of LQG surfaces (see Section 2.4 and the references therein for more detail). Such a surface is formally described by the metric $e^{\sqrt{8/3}h}dx \otimes dy$ where $dz \otimes dy$ is the Euclidean metric tensor and $h$ is an instance of some form of the Gaussian free field (GFF) [She07, SS13] on a domain $D \subset \mathbb{C}$. This metric tensor does not make rigorous sense since $h$ is a distribution, not a function. However, it is shown in [DS11] that one can make rigorous sense of the volume form $\mu_h = e^{\sqrt{8/3}h} dz$, where $dz$ denotes Lebesgue measure, via a regularization procedure.

It was further shown in [MS15a, MS15b, MS15c, MS16a, MS16b], building on [MS16d], that every $\sqrt{8/3}$-LQG surface can be endowed with a canonical metric space structure. Certain special $\sqrt{8/3}$-LQG surfaces are equivalent to Brownian surfaces, like the aforementioned Brownian half-plane and the Brownian map [Le 13, Mie13], in the sense that they differ by a measure-preserving isometry. In particular, the Brownian half-plane is equivalent to the so-called weight-2 quantum wedge, a $\sqrt{8/3}$-LQG surface described by a certain variant of the GFF on the upper half-plane $\mathbb{H}$.

By the main result of [MS16b], the metric space structure of a $\sqrt{8/3}$-LQG surface a.s. determines the surface. This implies in particular that the Brownian map has a canonical embedding into $\mathbb{H}$ (modulo conformal automorphisms) obtained by identifying it with a weight-2 quantum wedge parameterized by $\mathbb{H}$. Furthermore, there is an infinite family of random metric measure spaces which locally look like Brownian surfaces, obtained by considering different variants of the GFF on different domains. We provide in Section 2.4 below a more detailed exposition of LQG and its relationship to Brownian surfaces.

It is shown in [She16a, DMS14] that one can conformally weld two $\sqrt{8/3}$-LQG surfaces according to the $\sqrt{8/3}$-LQG length measure along their boundaries to get a new LQG surface, and the interface between such surfaces after welding is an SLE$_{6/3}$-type curve [She16a, DMS14]. It was proved in [GM16a] that the $\sqrt{8/3}$-LQG metric on the welded surface coincides with the metric quotient of the two smaller surfaces; such a statement is not at all obvious from the construction of the $\sqrt{8/3}$-LQG metric in [MS15b, MS16a, MS16b].

The preceding paragraph and the equivalence of the Brownian half-plane and the weight-2 quantum wedge together imply that the curve-decorated metric measure spaces obtained by gluing together Brownian half-planes which arise as the scaling limits of random SAW-decorated quadrangulations are equivalent to certain explicit $\sqrt{8/3}$-LQG surfaces decorated by SLE$_{8/3}$-type curves.

We emphasize, however, that the present work does not use any LQG machinery (see Figure 2 for the dependencies). The LQG machinery in [MS15b, MS16a, MS16b, GM16a] is what allows us to deduce the correspondence with SLE$_{8/3}$ on $\sqrt{8/3}$-LQG from the results proved here.

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1.2 Main results

In this subsection we state our main results, which say that self-avoiding walk on random quadrangulations converges to SLE$_{8/3}$ on $\sqrt{8/3}$-LQG in the half-plane (chordal), two-sided whole-plane, and one-sided whole-plane cases. The theorem statements in the three cases are similar, so the reader may wish to read just one of the statements (probably the chordal case) and skim the others. See Figure 3 for an illustration of the limiting objects in each of the three statements.

Since our convergence results are with respect to the Gromov-Hausdorff-Prokhorov-uniform (GHPU) metric, we need to work with continuous curves. To do this, we view graphs as connected metric spaces by identifying each edge with an isometric copy of the unit interval, and extend the definitions of curves from discrete intervals to continuum intervals by linear interpolation; c.f. Remark 2.2 below. A review of the definitions of the objects involved in the theorem statements (in particular, the GHPU metric, the Brownian half-plane, and the particular $\sqrt{8/3}$-LQG surfaces obtained by gluing together Brownian half-planes) can be found in Section 2.
1.2.1 Chordal case

Let \((Q_-, e_-)\) and \((Q_+, e_+)\) be independent UIHPQs’s. Let \(Q_{zip}^n\) be the infinite quadrangulation with boundary obtained by identifying each edge on the positive infinite ray of \(\partial Q_-\) (i.e., each edge to the right of \(e_-\)) with the corresponding edge of \(\partial Q_+\). Let \(\lambda_{zip} : \mathbb{N}_0 \to \mathcal{E}(Q_{zip})\) be the path in \(Q_{zip}\) corresponding to the identified boundary rays of \(Q_\pm\). Then \((Q_{zip}, \lambda_{zip})\) is the infinite-volume limit of uniform SAW-decorated quadrangulations with boundary based at the starting point of the SAW [CC16].

For \(n \in \mathbb{N}\), let \(d_{zip}^n\) be the graph metric on \(Q_{zip}\), re-scaled by \((9/8)^{1/4}n^{-1/4}\). Let \(\mu_{zip}^n\) be the measure on \(Q_{zip}^n\) which assigns to each vertex a mass equal to \((4n)^{-1}\) times its degree. Extend the path \(\lambda_{zip}\) to \([0, \infty)\) by linear interpolation (in the manner discussed above) and let \(\eta_{zip}^n(t) := \lambda_{zip}\left(\frac{3\sqrt{2}}{4}n^{1/2}t\right)\) for \(t \geq 0\).

Let \((X_{-}, d_{-}, x_{-})\) and \((X_{+}, d_{+}, x_{+})\) be a pair of independent Brownian half-planes (weight-2 quantum wedges) with marked boundary points. Let \((X_{zip}, d_{zip})\) be the metric space quotient of the disjoint union of \((X_{-}, d_{-})\) and \((X_{+}, d_{+})\) under the equivalence relation which identifies their positive boundary rays (i.e., the rays to the right of \(x_{\pm}\)) according to boundary length. Let \(\mu_{zip}\) be the measure on \(X_{zip}\) inherited from the area measures on \(X_{\pm}\). Let \(\eta_{zip} : [0, \infty) \to X_{zip}\) be the path corresponding to the identified boundary rays, each parameterized by boundary length.

By [GM16a, Corollary 1.2], \((X_{zip}, d_{zip}, \mu_{zip}, \eta_{zip})\) is equivalent as a curve-decorated metric measure space to a certain \(\sqrt{8/3}\)-LQG surface called a weight-4 quantum wedge decorated by an independent chordal SLE\(_{8/3}\) curve from 0 to \(\infty\). That is, there is a GFF-type distribution \(h_{zip}\) on \(\mathbb{H}\), which is a.s. determined by \((X_{zip}, d_{zip}, \mu_{zip})\), and a map \(X_{zip} \to \mathbb{H}\) which a.s. takes \(d_{zip}\) and \(\mu_{zip}\), respectively, to the \(\sqrt{8/3}\)-LQG metric and \(\sqrt{8/3}\)-LQG area measure, respectively, induced by \(h_{zip}\) and takes \(\eta_{zip}\) to a chordal SLE\(_{8/3}\) curve from 0 to \(\infty\) in \(\mathbb{H}\) sampled independently from \(h_{zip}\), then parameterized by \(\sqrt{8/3}\)-LQG length with respect to \(h_{zip}\). See Section 2.4 below for more details.

**Theorem 1.1.** In the setting described just above,

\[
(Q_{zip}, d_{zip}^n, \mu_{zip}^n, \eta_{zip}^n) \rightarrow (X_{zip}, d_{zip}, \mu_{zip}, \eta_{zip})
\]  

in law in the local Gromov-Hausdorff-Prokhorov-uniform topology. In other words, the scaling limit of uniform random SAW-decorated half-planar maps in the local GHP topology is a weight-4 quantum wedge decorated by an independent chordal SLE\(_{8/3}\) parameterized by \(\sqrt{8/3}\)-LQG length.

It follows from [GM16b, Theorem 1.12] that the independent UIHPQs’s \(Q_{\pm}\), equipped with their graph metric, area measure, and boundary path, (with the aforementioned scaling) converge in law to a pair of independent Brownian half-planes. Theorem 1.1 says that the metric gluing operation for the UIHPQs’s (or Brownian half-planes) commutes with the operation of taking the limit as \(n \to \infty\). A similar statement holds in the settings of Theorems 1.2 and 1.3 below.

1.2.2 Two-sided whole-plane case

Next we state a variant of Theorem 1.1 for the case when we identify two UIHPQs’s along their entire boundary (not just their positive boundary rays).

Let \((Q_+, e_+)\) and \((X_{+}, d_{+}, x_{+})\), respectively, be UIHPQs’s and Brownian half-planes as above. Let \(Q_{full}\) be the quadrangulation without boundary obtained by identifying every edge on \(\partial Q_-\) to the corresponding edge on \(\partial Q_+\) (equivalently, the map obtained by identifying the left and right boundary rays of \(Q_{zip}\)). Let \(\lambda_{full} : \mathbb{Z} \to \mathcal{E}(Q_{full})\) be the two-sided path corresponding to the identified boundary paths of \(Q_{\pm}\). Then \((Q_{full}, \lambda_{full})\) is the local limit of uniformly random SAW-decorated quadrangulations of the sphere based at a typical point of the SAW [CC16].

For \(n \in \mathbb{N}\), let \(d_{full}^n\) be the graph metric on \(Q_{full}\), re-scaled by \((9/8)^{1/4}n^{-1/4}\). Let \(\mu_{full}^n\) be the measure on \(Q_{full}^n\) which assigns to each vertex a mass equal to \((4n)^{-1}\) times its degree. Let \(\eta_{full}^n(t) := \lambda_{full}\left(\frac{3\sqrt{2}}{4}n^{1/2}t\right)\) for \(t \in \mathbb{R}\), where here we have extended \(\lambda_{full}\) in linear interpolation in the manner discussed above.

\(^1\)The reason for the subscript zip is that \((Q_{zip}, \lambda_{zip})\) is the discrete analog of the so-called quantum zipper [She16a] obtained by gluing together two LQG surfaces along an SLE\(_{8/3}\) curve.
Figure 3: **Left:** The limiting space $X_{\text{zip}}$ in Theorem 1.1, which is a weight-4 quantum wedge decorated by an independent chordal SLE$_{8/3}$ and is obtained by gluing two independent Brownian half-planes $X_{\pm}$ along their positive boundary rays according to boundary length. **Middle:** The limiting space $X_{\text{full}}$ in Theorem 1.2, which is a weight-4 quantum cone decorated by a two-sided SLE$_{8/3}$-type curve and is obtained by gluing two independent Brownian half-planes $X_{\pm}$ along their full boundaries according to boundary length. (This SLE$_{8/3}$-type path can be described as a pair of GFF flow lines [MS16c, MS13].) **Right:** The limiting space $X_{\text{cone}}$ in Theorem 1.3, which is a weight-2 quantum cone decorated by a whole-plane SLE$_{8/3}$ curve and is obtained by gluing together the left and right boundary rays of a single Brownian half-plane $X_{\infty}$ according to boundary length.

Let $(X_{\text{full}}, d_{\text{full}})$ be the metric space quotient of the disjoint union of $(X_{-}, d_{-})$ and $(X_{+}, d_{+})$ under the equivalence relation which identifies their entire boundaries according to boundary length in such a way that the marked points $x_{-}$ and $x_{+}$ are identified. Let $\mu_{\text{full}}$ be the measure on $X_{\text{full}}$ inherited from the area measures on $X_{\pm}$. Let $\eta_{\text{full}} : \mathbb{R} \to X_{\text{full}}$ be the path corresponding to the identified boundary rays, each parameterized by boundary length.

By [GM16a, Corollary 1.5], $(X_{\text{full}}, d_{\text{full}}, \mu_{\text{full}}, \eta_{\text{full}})$ is equivalent as a curve-decorated metric measure space to a $\sqrt{8/3}$-LQG surface called a *weight-4 quantum cone* decorated by a two-sided SLE$_{8/3}$-type curve in $\mathbb{C}$ passing through the origin. More precisely, there is a GFF-type distribution $h_{\text{full}}$ on $\mathbb{C}$ which is a.s. determined by $(X_{\text{full}}, d_{\text{full}}, \mu_{\text{full}})$ and a map $X_{\text{full}} \to \mathbb{C}$ which a.s. takes $d_{\text{full}}$ and $\mu_{\text{full}}$, respectively, to the $\sqrt{8/3}$-LQG metric and $\sqrt{8/3}$-LQG area measure, respectively, induced by $h_{\text{full}}$ and which takes $\eta_{\text{full}}$ to a two-sided SLE$_{8/3}$-type curve sampled independently from $h_{\text{full}}$ then parameterized according to $\sqrt{8/3}$-LQG length with respect to $h_{\text{full}}$. The law of this SLE$_{8/3}$-type curve can be sampled from as follows: first sample a whole-plane SLE$_{8/3}$ curve $\eta_1$ from $\infty$ to 0; then, conditional on $\eta_1$, sample a chordal SLE$_{8/3}$ curve $\eta_2$ from 0 to $\infty$ in $\mathbb{C} \setminus \eta_1$. Then concatenate these two curves. (These two curves can also be described as a pair of GFF flow lines [MS16c, MS13].)

**Theorem 1.2.** In the setting described just above,

$$ (Q_{\text{full}}, d_{\text{full}}^n, \mu_{\text{full}}^n, \eta_{\text{full}}^n) \to (X_{\text{full}}, d_{\text{full}}, \mu_{\text{full}}, \eta_{\text{full}}) $$

in law in the local Gromov-Hausdorff-Prokhorov-uniform topology. In other words, the scaling limit of uniform random full-planar maps decorated by a two-sided SAW in the local GHPU topology is a weight-4 quantum cone decorated by an independent two-sided SLE$_{8/3}$-type curve as described above parameterized by $\sqrt{8/3}$-LQG length.
1.2.3 One-sided whole-plane case

We next state a variant of Theorem 1.1 for the case when we glue a single UIHPQs to itself along the two sides of its boundary.

Let \((Q_s, e_s)\) be a UIHPQs. Let \(Q_{cone}\) be the quadrangulation without boundary obtained by identifying every edge on the positive ray of \(\partial Q_s\) (i.e., the ray to the right of \(e_s\)) to the corresponding edge on the negative ray of \(\partial Q_s\). Let \(\lambda_{cone} : N_0 \to E(Q_{cone})\) be the one-sided path corresponding to the identified boundary rays of \(Q_s\). Then \((Q_{cone}, \eta_{cone})\) is the local limit of uniformly random SAW-decorated quadrangulations of the sphere based at the starting point of the SAW [CC16].

For \(n \in \mathbb{N}\), let \(d^n_{cone}\) be the graph metric on \(Q_{cone}\), re-scaled by \((9/8)^{1/4}n^{-1/4}\). Let \(\mu^n_{cone}\) be the measure on \(Q^n_{cone}\) which assigns to each vertex a mass equal to \((4n)^{-1}\) times its degree. Let \(\eta^n_{cone}(t) := \lambda_{cone}\left(\frac{2^{3/2}}{9/8}n^{1/2}t\right)\) for \(t \in \mathbb{R}\), with \(\lambda_{cone}\) viewed as a continuous curve via linear interpolation, as discussed at the beginning of this subsection.

Let \((X_{cone}, d_{cone})\) be a Brownian half-plane with marked boundary point. Let \((X_{cone}, d_{cone})\) be the metric space quotient of \((X_{cone}, d_{cone})\) under the equivalence relation which identifies the positive and negative rays (i.e., the rays to the left and right of \(x_{cone}\)) of \(\partial X_{cone}\) according to boundary length. Let \(\mu_{cone}\) be the measure on \(X_{cone}\) inherited from the area measure on \(X_{cone}\). Let \(\eta_{cone} : [0, \infty) \to X_{cone}\) be the path corresponding to the identified boundary rays, each parameterized by boundary length.

By [GM16a, Corollary 1.4], the metric measure space \((X_{cone}, d_{cone}, \mu_{cone})\) is equivalent to a curve-decorated metric measure space to a \(\sqrt{8/3}\)-LQG surface called a \textit{weight-2 quantum cone} decorated by an independent whole-plane SLE\(_{8/3}\) curve. That is, there is a GFF-type distribution \(h_{cone}\) on \(\mathbb{C}\) which is a deterministic functional of \((X_{cone}, d_{cone}, \mu_{cone})\) and a map \(X_{cone} \to \mathbb{C}\) which a.s. takes \(d_{cone}\) and \(\mu_{cone}\), respectively, to the \(\sqrt{8/3}\)-LQG metric and \(\sqrt{8/3}\)-LQG area measure, respectively, induced by \(h_{cone}\) and which takes \(\eta_{cone}\) to a whole-plane SLE\(_{8/3}\) curve from 0 to \(\infty\) sampled independently from \(h_{cone}\) then parameterized according to \(\sqrt{8/3}\)-LQG length with respect to \(h_{cone}\).

**Theorem 1.3.** In the setting described just above,

\[
(Q^n_{cone}, d^n_{cone}, \mu^n_{cone}, \eta^n_{cone}) \to (X_{cone}, d_{cone}, \mu_{cone}, \eta_{cone})
\]

in law in the local Gromov-Hausdorff-Prokhorov-uniform topology. In other words, the scaling limit of uniform random full-planar maps decorated by a one-sided SAW in the local GHPU topology is a weight-2 quantum cone decorated by an independent whole-plane SLE\(_{8/3}\) parameterized by \(\sqrt{8/3}\)-LQG length.

**Remark 1.4.** In [GM17b], we obtain analogs of the results of this paper for the \textit{finite} uniform SAW-decorated planar quadrangulations obtained by gluing together finite quadrangulations with simple boundary along their boundaries. The main inputs in the proof in this case are the results of the present paper and a scaling limit result for free Boltzmann quadrangulations with simple boundary toward the Brownian disk, which is also proven in [GM17b].

1.3 Outline

In this subsection we give a moderately detailed overview of the main ideas of our proof and the content of the remainder of this article. We will only give a detailed proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are essentially identical. We will remark briefly on the proofs of these latter two theorems in Remark 7.12.

Before we describe our proof, we make some general comments.

- Our proof does not use anything from the theory of SLE or Liouville quantum gravity. In fact, the only non-trivial outside inputs are the definition of the GHPU topology, the scaling limit of the UIHPQs [GM16b], and some basic estimates for the peeling procedure of the UIHPQs (see Section 3).
- By [GM16b], we know that the two UIHPQs’s \((Q_s, e_s)\) converge in law in the local GHPU topology to the two Brownian half-planes \((X_+, d_+)\). Due to the universal property of the quotient metric (recall Section 2.1.4), we expect that the metric on any subsequential scaling limit of our glued maps \((Q_{zip}, d^n_{zip}, \mu^n_{zip}, \eta^n_{zip})\) is in some sense no larger than the metric on \(d_{zip}\) on \(X_{zip}\). It could \textit{a priori} be
strictly smaller if paths in $Q_{ zip}$ which cross the SAW $\eta^{zip}_{\text{SAW}}$ more than a constant order number of times are shorter than paths which cross only a constant order number of times. Hence most of our estimates are devoted to proving lower bounds for distances in $Q_{ zip}$ (equivalently upper bounds for the size of metric balls) and upper bounds for how often $Q_{ zip}$-geodesics cross the SAW.

- Similarly to Brownian surfaces, the random planar maps considered in this paper satisfy a scaling rule. Heuristically, a graph distance ball of radius $r \in \mathbb{N}$ typically has boundary length $\approx r^2$ and contains at most $\approx r^2$ edges of the SAW or the boundary of the map; and contains $\approx r^4$ total edges.

Before beginning the proofs of our main theorems, in Section 2 we will establish some standard notational conventions and review some background on several objects which are relevant to this paper, including the Gromov-Hausdorff-Prokhorov-uniform metric, the Brownian half-plane, and the theory of Liouville quantum gravity surfaces. The sections on the Brownian half-plane and on LQG are not used in our proofs and are provided only to make the statements and interpretations of our main results more self-contained.

The main tool in this paper is the peeling procedure for the UIHPQs, which is a means of exploring a UIHPQs one quadrilateral at a time in such a way that the law of the unbounded connected component of the unexplored region is always that of a UIHPQs. In Section 3, we will review the peeling procedure and some of the estimates for peeling which have been proven elsewhere in the literature. We will also use peeling to prove some basic estimates for the UIHPQs which will be needed later.

In Section 4, we will introduce the glued peeling process, a peeling process for the glued map $Q_{ zip} = Q_\text{SAW} \cup Q_\text{SAW}$ appearing in Theorem 1.1 which approximates the sequence of $Q_{ zip}$-graph metric neighborhoods $B_r(\mathbb{A}; Q_{ zip})$ for $r \in \mathbb{N}$ together with the points they disconnect from $\infty$. We will also prove some basic estimates for the glued peeling process using the results of Section 3.

Roughly speaking, if one is given a bounded connected initial edge set $\mathbb{A} \subset \partial Q_\text{SAW} \cup \partial Q_\text{SAW}$, the glued peeling process started from $\mathbb{A}$ is the family of quadrangulations $\{Q^{(j)}\}_{j \in \mathbb{N}_0}$ obtained as follows. We start by peeling some quadrilateral of $Q_\text{SAW}$ or $Q_\text{SAW}$ which shares a vertex with $\mathbb{A}$, and define $\hat{Q}$ to be the quadrangulation consisting of the union of this quadrilateral and all of the vertices and edges it disconnects from $\infty$ in either $Q_\text{SAW}$ or $Q_\text{SAW}$. We continue this procedure until the first time $J_0 \in \mathbb{N}_0$ that every quadrilateral which shares a vertex with $\mathbb{A}$ belongs to $Q^{(j)}$. We then continue in the same manner, except we peel quadrilaterals incident to $\partial Q_{ J_0}$ instead of quadrilaterals incident to $\mathbb{A}$. There is a natural sequence of stopping times $\{J_j\}_{j \in \mathbb{N}_0}$ associated with the glued peeling process, with the property that $J_0 = 0$ and $J_r$ is the smallest $r \in \mathbb{N}$ such that $Q^{(J_r)}$ contains every quadrilateral of $Q_{ zip}$ incident to $\partial Q^{(J_r-1)}$. One easily checks (Lemma 4.3) that the
$Q_{\text{zip}}$-graph metric ball satisfies

$$B_r(A;Q_{\text{zip}}) \subset \hat{Q}_r^J, \quad \forall r \in \mathbb{N}_0,$$

(1.4)

although the inclusion is typically strict. Hence we can use the precise estimates for peeling described in Section 3 to obtain upper bounds for the size of graph metric balls in $Q_{\text{zip}}$.

The glued peeling process is similar in spirit to the peeling by layers algorithm studied in [CL14a]; c.f. Remark 4.1. This peeling process is also introduced and studied independently in [CC16], where it is shown that the number of SAW edges contained in radius-$r$ glued peeling cluster is typically at most $O_r(r^2)$. Our estimates for the glued peeling process, described just below, are sharper than those of [CC16].

Sections 5–7 form the core of the paper. See Figure 4 for a map of how the main statements of these sections fit together. In what follows, we will provide a more detailed outline of how these statements are proved.

In Section 5, we will prove our key estimate for the glued peeling process (Proposition 5.1), which says that for $r \in \mathbb{N}$ and $p \in [1,3/2]$, the $p$th moment of the number of edges of $\partial Q_- \cup \partial Q_+$ which belong to $\hat{Q}_r^J$ (which is at least the number of SAW edges belonging to $\hat{Q}_r^J$, but could be more since not every edge of $\partial Q_-$ is identified with an edge of $\partial Q_+$) and the $p$th moment of the length of $\partial Q_r^J$ are both at most a constant times $r^{2p}$. This estimate is proven by using results from Sections 3 and 4 and the inductive manner in which the glued peeling clusters are constructed to set up various recursive relations between quantities related to the glued peeling process, then solving the recursions to obtain estimates.

Section 5.4 contains several consequences of Proposition 5.1 which imply qualitative statements about subsequential limits of the curve-decorated metric measure spaces $(Q_{\text{zip}}^n, \eta_{\text{zip}}^n, \mu_{\text{zip}}^n, \nu_{\text{zip}}^n)$ of Theorem 1.1 in the GHPU topology. In particular, we obtain an estimate (Lemma 5.10) which implies that the curve in any subsequential limit is simple; and an upper bound for the diameter of a $Q_{\text{zip}}$-metric ball with respect to the metrics on the two UIHPQs’ $Q_{\pm}$ (Lemma 5.12).

In Section 6, we will prove two estimates which will be used to identify the law of a subsequential limit of our SAW-decorated quadrangulations in the GHPU topology. Proposition 6.1 tells us that two given points of the re-scaled SAW $\eta_{\text{zip}}^n$ can typically be joined by a path which crosses $\eta_{\text{zip}}^n$ at most an $n$-independent number of times; and whose length is at most a universal constant $C$ times the $d_{\text{zip}}$-distance between the two points. Recall that the quotient metric $d_{\text{zip}}$ on $X_{\text{zip}}$ is defined in terms of paths which cross the gluing interface $\eta_{\text{zip}}$ at most a finite number of times (c.f. Section 2.1.4). Hence this result will imply that if $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{\eta})$ is a subsequential limit of $\{(Q_{\text{zip}}^n, \eta_{\text{zip}}^n, \mu_{\text{zip}}^n, \nu_{\text{zip}}^n)\}_{n \in \mathbb{N}}$ in the GHPU topology, then the metric $\tilde{d}$ is in some sense bounded below by $C^{-1}d_{\text{zip}}$. Proposition 6.2 tells us that a $d_{\text{zip}}$-geodesic between two given points of $\eta_{\text{zip}}^n$ typically spends at least a positive fraction of times time away from $\eta_{\text{zip}}^n$.

The above two propositions are proven by showing that for most edges along the SAW, there exists a “good” radius $R$ for the glued peeling process started from that edge for which a certain event occurs, and then studying the behavior of a $Q_{\text{zip}}$-geodesic when it passes through $\partial Q_r^J$. In the case of Proposition 6.1, this event corresponds to the condition that the diameters of $\partial Q_r^J \cap Q_-$ and $\partial Q_r^J \cap Q_+$ are each most $C R$, for a constant $C > 1$. In the case of Proposition 6.2, this event corresponds to the condition that a $Q_{\text{zip}}$-geodesic from $\partial Q_r^J$ to our given edge has to spend at least $\beta R$ units of time away from the SAW, for a constant $\beta \in (0, 1)$. The existence of such an $R$ is deduced from the estimates of Section 5 together with a multi-scale argument.

As explained in the earlier parts of Section 7, the results of Section 5 together with the scaling limit result for the UIHPQs [GM16b, Theorem 1.12] already imply the convergence of $(Q_{\text{zip}}^n, \eta_{\text{zip}}^n, \mu_{\text{zip}}^n, \nu_{\text{zip}}^n)$ along subsequences to a non-degenerate limiting curve-decorated metric measure space. In fact, using these results plus some abstract arguments for curve-decorated metric measure spaces from [GM16a], one can show that if $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{\eta})$ is a subsequential limit, then there exists a bijective 1-Lipschitz map $f_{\text{zip}} : X_{\text{zip}} \to \tilde{X}$ satisfying $(f_{\text{zip}})_* \mu_{\text{zip}} = \tilde{\mu}$ and $f_{\text{zip}} \circ \eta_{\text{zip}} = \tilde{\eta}$ which preserves the length of any path in $X_{\text{zip}}$ which does not hit $\eta_{\text{zip}}$ (Proposition 7.6).

In Section 7.4, we will show that the map $f_{\text{zip}}$ is an isometry as follows. The results of Section 6 discussed above imply that there are universal constants $C \geq 1$ and $\beta \in (0, 1)$ such that the following is true. The map $f_{\text{zip}}$ is a.s. Lipschitz with Lipschitz constant $C$; and almost every pair of points on $\tilde{\eta}$ can be joined by a $\tilde{d}$-geodesic which spends at least a $\beta$-fraction of its time outside of $\tilde{\eta}$.

Suppose $\gamma$ is such a geodesic. The map $f_{\text{zip}}^{-1}$ is an isometry away from $\tilde{\eta}$, so the $\tilde{d}$-length of each excursion of $\gamma$ away from $\tilde{\eta}$ is the same as the $d_{\text{zip}}$-length of the image of this excursion under $f_{\text{zip}}^{-1}$. On the other hand,
the $\tilde{d}$-length of any segment of $\gamma$ is at most $C$ times the $d_{\text{zip}}$-length of its image under $f_{\text{zip}}^{-1}$. By decomposing $\gamma$ into finitely many time intervals of total length at least $\beta/2$ during which it does not cross $\eta$ and finitely many complementary time intervals during which it may cross $\eta$, we see that the $\tilde{d}$-length of $\gamma$ is at most $(1 - \beta/2)C + \beta/2$ times the $d_{\text{zip}}$-length of $f_{\text{zip}}^{-1}(\gamma)$. Therefore, $C \leq (1 - \beta/2)C + \beta/2$, so $C = 1$. Thus any subsequential limit of the SAW-decorated quadrangulations agrees with $(X_{\text{zip}}, d_{\text{zip}}, P_{\text{zip}}, \eta_{\text{zip}})$ as elements of $\mathcal{M}_{\text{GHPU}}$.

For the convenience of the reader, we have included in Section A an index of the commonly used symbols in the paper.

2 Preliminaries

In this section we will introduce some notation and review several objects from other places in the literature which are relevant to the results of this paper. In Section 2.1, we will fix some (essentially standard) notation which we will use throughout the remainder of this article. In Section 2.2, we will review the definition of the Gromov-Hausdorff-Prokhorov-uniform metric from [GM16b] and some of its basic properties. This is the metric with respect to which the convergence in our main theorems takes place. In Section 2.3, we recall the definition of the Brownian half-plane, which can be used to construct the limiting objects in our main theorems. In Section 2.4, we review the theory of Liouville quantum gravity and explain why the limiting objects in our main theorems are equivalent to $\sqrt{8/3}$-LQG surfaces decorated by independent SLE$_{8/3}$-type curves.

We emphasize that most of the content of this paper can be understood independently of this section. In order to understand the proofs in Sections 3–6, the reader only needs to be familiar with Sections 2.1 and 2.2.

2.1 Notational conventions

2.1.1 Basic notation

We write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b \in \mathbb{R}$, we define the discrete intervals $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$ and $(a, b)_{\mathbb{Z}} := (a, b) \cap \mathbb{Z}$.

If $a$ and $b$ are two quantities, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant $C$ (independent of the parameters of interest) such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$.

2.1.2 Graphs and maps

For a planar map $G$, we write $V(G)$, $E(G)$, and $F(G)$, respectively, for the set of vertices, edges, and faces, respectively, of $G$.

By a path in $G$, we mean a function $\lambda : I \to E(G)$ for some (possibly infinite) discrete interval $I \subset \mathbb{Z}$, with the property that the edges $\{\lambda(i)\}_{i \in I}$ can be oriented in such a way that the terminal endpoint of $\lambda(i)$ coincides with the initial endpoint of $\lambda(i + 1)$ for each $i \in I$ other than the right endpoint of $I$. We define the length of $\lambda$, denoted $|\lambda|$, to be the integer $\#I$.

For sets $A_1$, $A_2$ consisting of vertices and/or edges of $G$, we write $\text{dist}(A_1, A_2; G)$ for the graph distance from $A_1$ to $A_2$ in $G$, i.e. the minimum of the lengths of paths in $G$ whose initial edge either has an endpoint which is a vertex in $A_1$ or shares an endpoint with an edge in $A_1$; and whose final edge satisfies the same condition with $A_2$ in place of $A_1$. If $A_1$ and/or $A_2$ is a singleton, we do not include the set brackets. Note that the graph distance from an edge $e$ to a set $A$ is the minimum distance between the endpoints of $e$ and the set $A$.

For $r > 0$, we define the graph metric ball $B_r(A_1; G)$ to be the subgraph of $G$ consisting of all vertices of $G$ whose graph distance from $A_1$ is at most $r$ and all edges of $G$ whose endpoints both lie at graph distance at most $r$ from $A_1$. If $A_1 = \{x\}$ is a single vertex or edge, we write $B_r(\{x\}; G) = B_r(x; G)$.
2.1.3 Quadrangulations with boundary

A quadrangulation with (connected) boundary is a (finite or infinite) planar map $Q$ with a distinguished face $f_\infty$, called the **exterior face**, such that every face of $Q$ other than $f_\infty$ has degree 4. The **boundary** of $Q$, denoted by $\partial Q$, is the smallest subgraph of $Q$ which contains every edge of $Q$ incident to $f_\infty$. The **perimeter** $\text{Perim}(Q)$ of $Q$ is defined to be the degree of the exterior face.

We say that $\partial Q$ is **simple** if the exterior face has no vertices of multiplicity strictly larger than 1. In this paper we will only consider quadrangulations with simple boundary.

A **boundary path** of $Q$ is a path $\lambda$ from $[1, \text{Perim}(Q)]_\mathbb{Z}$ (if $\partial Q$ is finite) or $\mathbb{Z}$ (if $\partial Q$ is infinite) to $\mathcal{E}(\partial Q)$ which traces the edges of $\partial Q$ (counted with multiplicity) in cyclic order. Choosing a boundary path is equivalent to choosing an oriented root edge on the boundary. This root edge is $\lambda(\text{Perim}(Q))$, oriented toward $\lambda(1)$ in the finite case; or $\lambda(0)$, oriented toward $\lambda(1)$, in the infinite case.

The uniform infinite planar quadrangulation with simple boundary (UIHPQ$_{\mathbb{S}}$) is the infinite boundary-rooted quadrangulation $(Q_{\mathbb{S}}, \varepsilon_{\mathbb{S}})$ with simple boundary which is the limit in law with respect to the Benjamini-Schramm topology [BS01] of a uniformly random quadrangulation with simple boundary (rooted at a uniformly random boundary edge) with $n$ interior vertices and $2l$ boundary edges if we first send $n \to \infty$ and then $l \to \infty$ [CM15, CC15]. It can also be constructed from the uniform infinite planar quadrangulation with general boundary (UIHPQ) by “pruning” quadrangulations which can be disconnected from $\infty$ by removing a single vertex; see [CM15, CC15, GM16b].

2.1.4 Metric spaces

Here we introduce some notation for metric spaces and recall some basic constructions. Throughout, let $(X, d_X)$ be a metric space.

For $A \subset X$ we write $\text{diam}(A; d_X)$ for the supremum of the $d_X$-distance between points in $A$.

For $r > 0$, we write $B_r(A; d_X)$ for the set of $x \in X$ with $d_X(x, A) \leq r$. We emphasize that $B_r(A; d_X)$ is closed (this will be convenient when we work with the local GHPU topology). If $A = \{y\}$ is a singleton, we write $B_r(\{y\}; d_X) = B_r(y; d_X)$.

Let $\sim$ be an equivalence relation on $X$, and let $\overline{X} = X/\sim$ be the corresponding topological quotient space. For equivalence classes $[x], [y] \in \overline{X}$, let $Q([x], [y])$ be the set of finite sequences $(x_1, y_1, \ldots, x_n, y_n)$ of elements of $X$ such that $x_1 \in [x]$, $y_n \in [y]$, and $y_i \sim x_{i+1}$ for each $i \in [1, n-1]$. Let

$$
\overline{d}_X([x], [y]) := \inf_{Q([x], [y])} \sum_{i=1}^n d_X(x_i, y_i). \tag{2.1}
$$

Then $\overline{d}_X$ is a pseudometric on $\overline{X}$ (i.e., it is symmetric and satisfies the triangle inequality), which we call the **quotient pseudometric**.

The quotient pseudometric possesses the following universal property. Suppose $f : (X, d_X) \to (Y, d_Y)$ is a 1-Lipschitz map such that $f(x) = f(y)$ whenever $x, y \in X$ with $x \sim y$. Then $f$ factors through the metric quotient to give a map $\overline{f} : \overline{X} \to Y$ such that $\overline{f} \circ p = f$, where $p : X \to \overline{X}$ is the quotient map.

For a curve $\gamma : [a, b] \to X$, the $d_X$-**length** of $\gamma$ is defined by

$$
\text{len}(\gamma; d_X) := \sup_P \sum_{i=1}^{\#P} d_X(\gamma(t_i), \gamma(t_{i-1}))
$$

where the supremum is over all partitions $P : a = t_0 < \cdots < t_{\#P} = b$ of $[a, b]$. Note that the $d_X$-length of a curve may be infinite.

For $Y \subset X$, the **internal metric** $d_Y$ of $d_X$ on $Y$ is defined by

$$
d_Y(x, y) := \inf_{\gamma \in Y} \text{len}(\gamma; d_X), \quad \forall x, y \in Y \tag{2.2}
$$
where the infimum is over all curves in \( Y \) from \( x \) to \( y \). The function \( d_Y \) satisfies all of the properties of a metric on \( Y \) except that it may take infinite values.

We say that \((X, d_X)\) is a length space if for each \( x, y \in X \) and each \( \epsilon > 0 \), there exists a curve of \( d_X \)-length at most \( d_X(x, y) + \epsilon \) from \( x \) to \( y \).

### 2.2 The Gromov-Hausdorff-Prokhorov-uniform metric

In this subsection we will review the definition of the local Gromov-Hausdorff-Prokhorov-uniform (GHPU) metric from \([GM16b]\), which is the metric with respect to which our scaling limit results hold.

We start by defining the metric in the compact case. For a metric space \((X, d)\), we let \( C_0(\mathbb{R}, X) \) be the space of continuous curves \( \eta : \mathbb{R} \to X \) which are “constant at \( \infty \),” i.e. \( \eta \) extends continuously to the extended real line \([-\infty, \infty]\). Each curve \( \eta : [a, b] \to X \) can be viewed as an element of \( C_0(\mathbb{R}, X) \) by defining \( \eta(t) = \eta(a) \) for \( t < a \) and \( \eta(t) = \eta(b) \) for \( t > b \).

- Let \( d^H_\mu \) be the \( d \)-Hausdorff metric on compact subsets of \( X \).
- Let \( d^P_\mu \) be the \( d \)-Prokhorov metric on finite measures on \( X \).
- Let \( d^U_\mu \) be the \( d \)-uniform metric on \( C_0(\mathbb{R}, X) \).

Let \( \mathcal{M}^{\text{GHPU}} \) be the set of 4-tuples \( \mathbf{x} = (X, d, \mu, \eta) \) where \((X, d)\) is a compact metric space, \( d \) is a metric on \( X \), \( \mu \) is a finite Borel measure on \( X \), and \( \eta \in C_0(\mathbb{R}, X) \).

Given elements \( \mathbf{x}_1 = (X_1, d_1, \mu_1, \eta_1) \) and \( \mathbf{x}_2 = (X_2, d_2, \mu_2, \eta_2) \) of \( \mathcal{M}^{\text{GHPU}} \), a compact metric space \((W, D)\), and isometric embeddings \( t_1 : X_1 \to W \) and \( t_2 : X_2 \to W \), we define their GHPU distortion by

\[
\text{Dis}^{\text{GHPU}}_{\mathbf{x}_1, \mathbf{x}_2}(W, D, t_1, t_2) := d^H_\mu(t_1(\eta_1), t_2(\eta_2)) + d^P_\mu((t_1)_* \mu_1, (t_2)_* \mu_2) + d^U_\mu(t_1 \circ \eta_1, t_2 \circ \eta_2).
\] (2.3)

We define the Gromov-Hausdorff-Prokhorov-Uniform (GHPU) distance by

\[
d^{\text{GHPU}}(\mathbf{x}_1, \mathbf{x}_2) = \inf_{(W, D), t_1, t_2} \text{Dis}^{\text{GHPU}}_{\mathbf{x}_1, \mathbf{x}_2}(W, D, t_1, t_2),
\] (2.4)

where the infimum is over all compact metric spaces \((W, D)\) and isometric embeddings \( t_1 : X_1 \to W \) and \( t_2 : X_2 \to W \). It is shown in \([GM16b]\) that this defines a complete separable metric on \( \mathcal{M}^{\text{GHPU}} \) provided we identify two elements of \( \mathcal{M}^{\text{GHPU}} \) which differ by a measure- and curve-preserving isometry.

We now define the local version of the GHPU metric. Following \([GM16b]\), we let \( \mathcal{M}^{\text{GHPU}}_{\infty} \) be the set of 4-tuples \( \mathbf{x} = (X, d, \mu, \eta) \) where \((X, d)\) is a locally compact length space, \( \mu \) is a measure on \( X \) which assigns finite mass to each finite-radius metric ball in \( X \), and \( \eta : \mathbb{R} \to X \) is a curve in \( X \). Note that \( \mathcal{M}^{\text{GHPU}}_{\infty} \) is not contained in \( \mathcal{M}^{\text{GHPU}} \) since elements of the former are not required to be length spaces.

Let \( \mathcal{M}^{\text{GHPU}}_{\infty} \) be the set of equivalence classes of elements of \( \mathcal{M}^{\text{GHPU}}_{\infty} \) under the equivalence relation whereby \((X_1, d_1, \mu_1, \eta_1) \sim (X_2, d_2, \mu_2, \eta_2)\) if and only if there exists an isometry \( f : X_1 \to X_2 \) such that \( f_* \mu_1 = \mu_2 \) and \( f \circ \eta_1 = \eta_2 \).

**Definition 2.1.** Let \( \mathbf{x} = (X, d, \mu, \eta) \) be an element of \( \mathcal{M}^{\text{GHPU}}_{\infty} \). For \( r > 0 \), let

\[
\mathcal{Z}^r := (-r) \vee \sup \{ t < 0 : d(\eta(0), \eta(t)) = r \} \quad \text{and} \quad \mathcal{T}^r := r \wedge \inf \{ t > 0 : d(\eta(0), \eta(t)) = r \}.
\] (2.5)

The \( r \)-truncation of \( \eta \) is the curve \( \mathfrak{B}, \eta \in C_0(\mathbb{R}, X) \) defined by

\[
\mathfrak{B}, \eta(t) = \begin{cases} 
\eta(\mathcal{Z}^r), & t \leq \mathcal{Z}^r \\
\ eta(t), & t \in (\mathcal{Z}^r, \mathcal{T}^r) \\
\eta(\mathcal{T}^r), & t \geq \mathcal{T}^r.
\end{cases}
\]

The \( r \)-truncation of \( \mathbf{x} \) is the curve-decorated metric measure space

\[
\mathfrak{B}, \mathbf{x} = (B_r(\eta(0); d), d|_{B_r(\eta(0); d)}, |\mu|_{B_r(\eta(0); d)}, \mathfrak{B}, \eta).
\]
The local GHPU metric on \( \mathbb{M}^\text{GHPU} \) is defined by
\[
d^\text{GHPU}_\infty(\mathcal{X}_1, \mathcal{X}_2) = \int_0^\infty e^{-r}(1 \wedge d^\text{GHPU}(\mathcal{B}_r, \mathcal{X}_1, \mathcal{B}_r, \mathcal{X}_2))dr
\]
where \( d^\text{GHPU} \) is as in (2.4). It is shown in [GM16b] that \( d^\text{GHPU}_\infty \) defines a complete separable metric on \( \mathbb{M}^\text{GHPU} \) provided we identify spaces which which differ by a measure- and curve-preserving isometry.

**Remark 2.2** (Graphs as elements of \( \mathbb{M}^\text{GHPU} \)). In this paper we will often be interested in a graph \( G \) equipped with its graph distance \( d_G \). In order to study continuous curves in \( G \), we need to linearly interpolate \( G \). We do this by identifying each edge of \( G \) with a copy of the unit interval \([0,1]\). We extend the graph metric on \( G \) by requiring that this identification is an isometry.

If \( \lambda \) is a path in \( G \), mapping some discrete interval \([a,b]\) to \( E(G) \), we extend \( \lambda \) from \([a,b] \) to \([a-1,b] \) by linear interpolation, so that for \( i \in [a,b] \), \( \lambda \) traces each edge \( \lambda(i) \) at unit speed during the time interval \([[i-1,i]] \).

If we are given a measure \( \mu \) on vertices of \( G \) and we view \( G \) as a connected metric space and \( \lambda \) as a continuous curve as above, then \( (G, d_G, \mu, \lambda) \) is an element of \( \mathbb{M}^\text{GHPU} \).

In the remainder of this subsection we explain how local GHPU convergence is equivalent to a closely related type of convergence which is often easier to work with, in which all of the curve-decorated metric measure spaces are subsets of a larger space. For this purpose we need to introduce the following definition, which we take from [GM16b].

**Definition 2.3** (Local HPU convergence). Let \((W, D)\) be a metric space. Let \( \mathcal{X}^n = (X^n, d^n, \mu^n, \eta^n) \) for \( n \in \mathbb{N} \) and \( \mathcal{X} = (X, d, \mu, \eta) \) be elements of \( \mathbb{M}^\text{GHPU} \) such that \( X \) and each \( X^n \) is a subset of \( W \) satisfying \( D|_{X^n} = d \) and \( D|_{X^n} = d^n \). We say that \( \mathcal{X}^n \to \mathcal{X} \) in the D-local Hausdorff-Prokhorov-uniform (HPU) sense if the following are true.

- For each \( r > 0 \), we have \( B_r(\eta^n(0); d^n) \to B_r(\eta(0); d) \) in the D-Hausdorff metric, i.e. \( \mathcal{X}^n \to \mathcal{X} \) in the D-local Hausdorff metric.
- For each \( r > 0 \) such that \( \mu(\partial B_r(\eta(0); d)) = 0 \), we have \( \mu^n|_{B_r(\eta^n(0); d^n)} \to \mu|_{B_r(\eta(0); d)} \) in the D-Prokhorov metric.
- For each \( a, b \in \mathbb{R} \) with \( a < b \), we have \( \eta^n|_{[a,b]} \to \eta|_{[a,b]} \) in the D-uniform metric.

The following result, which is [GM16b, Proposition 1.9], will play a key role in Section 7.

**Proposition 2.4.** Let \( \mathcal{X}^n = (X^n, d^n, \mu^n, \eta^n) \) for \( n \in \mathbb{N} \) and \( \mathcal{X} = (X, d, \mu, \eta) \) be elements of \( \mathbb{M}^\text{GHPU} \). Then \( \mathcal{X}^n \to \mathcal{X} \) in the local GHPU topology if and only if there exists a boundedly compact metric space \( (Z, D) \) (i.e., one for which closed bounded sets are compact) and isometric embeddings \( \mathcal{X}^n \to Z \) for \( n \in \mathbb{N} \) and \( \mathcal{X} \to Z \) such that the following is true. If we identify \( X^n \) and \( X \) with their embeddings into \( Z \), then \( \mathcal{X}^n \to \mathcal{X} \) in the D-local HPU sense.

### 2.3 The Brownian half-plane

A Brownian surface is a random metric measure space which locally looks like the Brownian map (see [Mic09, Mic14, Le 14] and the references therein for more on the Brownian map). Brownian surfaces arise as the scaling limits of uniformly random planar maps. Several specific Brownian surfaces have been constructed via continuum analogs of the Schaeffer bijection [Sch97], including the Brownian map itself, which is the scaling limit of uniform quadrangulations of the sphere [Mic13, Le 13]: the Brownian disk, which is the scaling limit of uniform quadrangulations with boundary [BM17]; the Brownian plane, which is the scaling limit of uniform infinite quadrangulations without boundary [CL14b]; and the Brownian half-plane, which is the scaling limit of uniform infinite half-planar quadrangulations [CC15, GM16b, BMR16]. See also [BMR16] for some additional Brownian surfaces which arise as scaling limits of certain quadrangulations with boundary.

The limiting objects in our main theorems are described by gluing together Brownian half-planes along their boundaries, so in this section we give a brief review of the definition of this object. We will not use most of the objects involved in this construction later in the paper, except for the definition of the area...
measure, boundary length measure, and boundary path. We review it only for the sake of making this work more self-contained. We use the construction from [GM16b]. A different construction, which we expect to be equivalent, is given in [CC15, Section 5.3] but the construction we give here is the one which was been proven to be the scaling limit of the UIHPQ and UIHPQs in [GM16b].

Let $W_\infty : \mathbb{R} \to [0, \infty)$ be the process such that $\{W_\infty(t)\}_{t \geq 0}$ is a standard linear Brownian motion and $\{W_\infty(-t)\}_{t \geq 0}$ is an independent Brownian motion conditioned to stay positive (i.e., a 3-dimensional Bessel process). For $r \in \mathbb{R}$, let

$$T_\infty(r) := \inf\{t \in \mathbb{R} : W_\infty(t) = -r\},$$

so that $r \mapsto T_\infty(r)$ is non-decreasing and for each $r \in \mathbb{R}$,

$$\{W_\infty(T_\infty(r) + t) + r\}_{t \in \mathbb{R}} \overset{d}{=} \{W_\infty(t)\}_{t \in \mathbb{R}}.$$  

Also let $T_\infty^{-1} : \mathbb{R} \to \mathbb{R}$ be the right-continuous inverse of $T$. For $s, t \in \mathbb{R}$, let

$$d_{W_\infty}(s, t) := W_\infty(s) + W_\infty(t) - 2 \inf_{u \in [s \wedge t, s \vee t]} W_\infty(u). \quad (2.7)$$

Then $d_{W_\infty}$ defines a pseudometric on $\mathbb{R}$ and the quotient metric space $\mathbb{R}/\{d_{W_\infty} = 0\}$ is a forest of continuum random trees, indexed by the excursions of $W_\infty$ away from its running infimum.

Conditioned on $W_\infty$, let $Z^0_\infty$ be the centered Gaussian process with

$$\text{Cov}(Z^0_\infty(s), Z^0_\infty(t)) = \inf_{u \in [s \wedge t, s \vee t]} \left( W_\infty(u) - \inf_{v \leq u} W_\infty(v) \right), \quad s, t \in \mathbb{R}. \quad (2.8)$$

By the Kolmogorov continuity criterion, $Z^0_\infty$ a.s. admits a continuous modification which satisfies $Z^0_\infty(s) = Z^0_\infty(t)$ whenever $d_{W_\infty}(s, t) = 0$.

Let $b_\infty : \mathbb{R} \to \mathbb{R}$ be $\sqrt{3}$ times a two-sided standard linear Brownian motion. For $t \in \mathbb{R}$, define

$$Z_\infty(t) := Z^0_\infty(t) + b_\infty(T_\infty^{-1}(t)).$$

For $s, t \in \mathbb{R}$, let

$$d_{Z_\infty}(s, t) = Z_\infty(s) + Z_\infty(t) - 2 \inf_{u \in [s \wedge t, s \vee t]} Z_\infty(u). \quad (2.9)$$

Also define the pseudometric

$$d^0_{Z_\infty}(s, t) = \inf \sum_{i=1}^{k} d_{Z_\infty}(s_i, t_i) \quad (2.10)$$

where the infimum is over all $k \in \mathbb{N}$ and all $(2k + 2)$-tuples $(t_0, s_1, t_1, \ldots, s_k, t_k, s_{k+1}) \in \mathbb{R}^{2k+2}$ with $t_0 = s$, $s_{k+1} = t$, and $d_{W_\infty}(t_{i-1}, s_i) = 0$ for each $i \in [1, k + 1]$. The Brownian half-plane is the quotient space $X_\infty = \mathbb{R}/\{d^0_{Z_\infty} = 0\}$ equipped with the quotient metric $d_{Z_\infty}$.

We write $p_\infty : \mathbb{R} \to X_\infty$ for the quotient map. The Brownian half-plane comes with a natural marked boundary point, namely $p(0)$. The area measure of $X_\infty$ is the pushforward of Lebesgue measure on $\mathbb{R}$ under $p_\infty$, and is denoted by $\mu_\infty$. The boundary of $X_\infty$ is the set $\partial X_\infty = p_\infty(\{T_\infty(r) : r \in \mathbb{R}\})$. The boundary measure of $X_\infty$ is the pushforward of Lebesgue measure on $\mathbb{R}$ under the map $r \mapsto p_\infty(T_\infty(r))$.

2.4 Liouville quantum gravity

In this subsection we review the definition of Liouville quantum gravity (LQG) surfaces and explain their equivalence with Brownian surfaces in the case when $\gamma = \sqrt{8/3}$. We do not use LQG in our proofs, but LQG is important for motivating and interpreting our main results. In particular, we will explain in this subsection why the limiting objects in our main theorem are equivalent to SLE-decorated LQG surfaces.
For $\gamma \in (0, 2)$, a Liouville quantum gravity surface with $k \in \mathbb{N}_0$ marked points is an equivalence class of $(k+2)$-tuples $(D, h, x_1, \ldots, x_k)$, where $D \subset \mathbb{C}$ is a domain; $h$ is a distribution on $D$, typically some variant of the Gaussian free field (GFF) [DS11, She07, SS13, She16a, MS16c, MS13]; and $x_1, \ldots, x_k \in D \cup \partial D$ are $k$ marked points. Two such $(k+2)$-tuples $(D, h, x_1, \ldots, x_k)$ and $(\tilde{D}, \tilde{h}, \tilde{x}_1, \ldots, \tilde{x}_k)$ are considered equivalent if there is a conformal map $f : D \rightarrow \tilde{D}$ such that

$$f(\tilde{x}_j) = x_j, \quad \forall j \in [1, k] \quad \text{and} \quad \tilde{h} = h \circ f + Q \log |f'| \quad \text{where} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \quad (2.12)$$

Several specific types of $\gamma$-LQG surfaces (which correspond to particular choices of the GFF-like distribution $h$) are studied in [DMS14], including quantum spheres, quantum disks, $\alpha$-quantum cones for $\alpha < Q$, and $\alpha$-quantum wedges for $\alpha < Q + \gamma/2$.

In this paper we will be particularly interested in $\alpha$-quantum wedges and $\alpha$-quantum cones for $\alpha < Q$, so we provide some additional detail on these surfaces. See [DMS14, Section 4.2] for a precise definition. Roughly speaking, an $\alpha$-quantum wedge for $\alpha < Q$ is the quantum surface $(\mathbb{H}, h, 0, \infty)$ obtained by starting with the distribution $h - \alpha \log |\cdot|$, where $h$ is a free-boundary GFF on $\mathbb{H}$, then zooming in near the origin and re-scaling to get a surface which describes the local behavior of this field when the additive constant is fixed appropriately. An $\alpha$-quantum cone is the quantum surface $(\mathbb{C}, h, 0, \infty)$ which is defined in a similar manner but starting with a whole-plane GFF plus an $\alpha$-log singularity rather than a free-boundary GFF plus an $\alpha$-log singularity.

Instead of the log-singularity parameter $\alpha$, one can also parameterize the spaces of quantum wedges and quantum cones by the weight parameter $w$, defined by

$$w = \gamma \left( \frac{\gamma}{2} + Q - \alpha \right), \quad \text{for wedges} \quad \text{and} \quad w = 2\gamma(Q - \alpha), \quad \text{for cones} \quad (2.13)$$

with $Q$ as in (2.12). The reason for using the parameter $w$ is that it is invariant under the cutting and gluing operations, which we will describe below.

It is shown in [DS11] that a Liouville quantum gravity surface admits a natural area measure $\mu_h$, which can be interpreted as "$e^{\gamma h(z)} \, dz$", where $dz$ is Lebesgue measure on $D$, and a length measure $\nu_h$ defined on certain curves in $D$, including $\partial D$ and SLE$_{\kappa}$-type curves for $\kappa = \gamma^2$. It was recently proven by Miller and Sheffield that in the special case when $\gamma = \sqrt{8/3}$, a $\sqrt{8/3}$-LQG surface admits a natural metric $\hat{\delta}_h$ [MS15b, MS16a, MS16b], building on [MS16d]. All three of these objects are invariant under coordinate changes of the form (2.12).

Several particular types of $\sqrt{8/3}$-LQG surfaces equipped with this metric structure are isometric to Brownian surfaces:

- The Brownian map is isometric to the quantum sphere;
- The Brownian disk is isometric to the quantum disk;
- The Brownian plane is isometric to the weight-4/3 quantum cone;
- The Brownian half-plane is isometric to the weight-2 quantum wedge.

It is shown in [MS16b] that the metric measure space structure a.s. determines the embedding of the quantum surface into (a subset of) $\mathbb{C}$. Hence a Brownian surface possesses a canonical embedding into the complex plane.

One can take the above isometry to push forward the $\sqrt{8/3}$-LQG area measure to the natural volume measure on the corresponding Brownian surface and (in the case of the disk or half-plane) one can take it to push forward the $\sqrt{8/3}$-LQG boundary length measure to the natural boundary length measure on the Brownian disk or half-plane. In particular, if we let $(\mathbb{H}, h, 0, \infty)$ be a $\sqrt{8/3}$-quantum wedge, equipped with its area measure $\mu_h$, boundary length measure $\nu_h$, and metric $\delta_h$, and we let $\eta_h : \mathbb{R} \rightarrow \mathbb{R}$ be the curve satisfying $\eta_h(0) = 0$ and $\nu_h(\eta_h([a, b])) = b - a$ for each $a < h$, then the curve-decorated metric measure spaces $(\mathbb{H}, \delta_h, \mu_h, \eta_h)$ and $(X_\infty, d_\infty, \mu_{\infty}, \eta_{\infty})$, the latter defined as in (2.11), are equivalent as elements of $\mathcal{M}_{\infty}$. It is shown in [She16a, DMS14] that one can conformally weld a weight-$w_{-}$ quantum wedge and a weight-$w_{+}$ quantum wedge together according to quantum length along their positive boundary rays (corresponding to $[0, \infty)$ in our parameterization of the quantum wedge) to obtain a weight-$\langle w_{-} + w_{+} \rangle$ quantum wedge.
decorated by an independent chiral SLE_{g/3}(w_− - 2; w_+ - 2) curve parameterized by quantum length with respect to the wedge. Similarly, one can conformally weld two such quantum wedges together according to quantum length along their entire boundary to obtain a weight-(w_− + w_+) quantum cone decorated by a two-sided chiral SLE_{g/3}-type curve parameterized by quantum length with respect to the wedge. One can also conformally weld the positive and negative boundary rays of single quantum wedge of weight w to each other according to quantum length to get a quantum cone of the same weight decorated by an independent whole-plane SLE_{g/3}(w - 2) curve.

It is proven in [GM16a] that in the case when \( \gamma = \sqrt{8/3} \), when one performs these gluing operations the \( \sqrt{8/3}\)-LQG metric on the glued surface is the metric space quotient of the metrics on the wedges being glued. Due to the equivalence between the weight-2 quantum wedge and the Brownian half-plane, we find the following (recall Figure 3):

- Gluing two independent Brownian half-planes together along their positive boundaries and embedding the resulting metric measure space into \( \mathbb{H} \) produces a weight-4 quantum wedge decorated by an independent chiral SLE_{8/3} curve.
- Gluing two independent Brownian half-planes together along their entire boundaries and embedding the resulting metric measure space into \( \mathbb{C} \) produces a weight-4 quantum cone decorated by an independent two-sided SLE_{8/3}-type curve which can be sampled as follows. First sample a whole-plane SLE_{8/3}(2) curve \( \eta_1 \) from 0 to \( \infty \); then, conditional on \( \eta_1 \), sample a chiral SLE_{8/3} curve \( \eta_2 \) from 0 to \( \infty \) in \( \mathbb{C} \setminus \eta_1 \). Then concatenate these two curves and parameterize the two-sided curve thus obtained by \( \sqrt{8/3}\)-LQG length. (This SLE_{8/3}-type path can be described as a pair of GFF flow lines [MS16c,MS13].)
- Gluing the two boundary rays of a single Brownian half-plane together along their entire boundaries and embedding the resulting metric measure space into \( \mathbb{C} \) produces a weight-2 quantum cone decorated by an independent whole-plane SLE_{8/3} curve.

Thus the limiting objects in Theorems 1.1, 1.2, and 1.3 are \( \sqrt{8/3}\)-LQG surfaces decorated by independent SLE_{8/3} curves.

## 3 Peeling of the UIHPQ with simple boundary

In this section, we will study the peeling procedure for the UIHPQs (also known as the spatial Markov property), which will be one of the key tools in the proofs of our main theorems. The idea of peeling was first used heuristically in the physics literature to study two-dimensional quantum gravity [ADJ97]. The first rigorous use of peeling was in [Ang03], in the context of the uniform infinite planar triangulation. The peeling procedure was later adapted to the case of the uniform infinite planar quadrangulation [BC13]. In this paper, we will only be interested peeling on the UIHPQs, which is also studied, e.g., in [AC15,AR15,Ric15].

In Section 3.1, we will review the definition of peeling on the UIHPQs, introduce notation for the objects involved, and review some formulas for peeling probabilities from elsewhere in the literature. Then, in Sections 3.2 and 3.3 we will use peeling to prove some particular estimates for the UIHPQs. The reader may wish to temporarily skip these last two subsections and refer back to them when the corresponding estimates are used.

### 3.1 Peeling of quadrangulations with boundary

#### 3.1.1 General definitions

Let \( Q \) be an infinite quadrangulation with simple boundary. For an edge \( e \in \mathcal{E}(\partial Q) \), let \( f(Q,e) \) be the quadrilateral of \( Q \) containing \( e \) on its boundary. The quadrilateral \( f(Q,e) \) has either two, three, or four vertices in \( \partial Q \), so divides \( Q \) into at most three connected components, whose union includes all of the vertices of \( Q \) and all of the edges of \( Q \) except for \( e \) (if \( f(Q,e) \) has an edge other than \( e \) in \( \partial Q \), this single edge counts as a connected component). Exactly one such component is infinite. These components have a natural cyclic ordering inherited from the cyclic ordering of their intersections with \( \partial Q \). We define the peeling indicator

\[
\Psi(Q,e) \in (\mathbb{N}_0 \cup \{\infty\}) \cup (\mathbb{N}_0 \cup \{\infty\})^2 \cup (\mathbb{N}_0 \cup \{\infty\})^3,
\]

(3.1)
to be the vector whose elements are the number of edges of each of these components shared by ∂Q, listed in counterclockwise cyclic order started from e. Note that if \( i \in \{1, 2, 3\} \) and the \( i \)th component of \( \mathcal{B}(Q, e) \) is \( k \), then the total boundary length of the \( i \)th connected component of \( Q \setminus f(Q, e) \) in counterclockwise cyclic order is \( k + 1 \) (resp. \( k + 2; \infty \)) if \( k \) is odd (resp. even; \( \infty \)).

We refer to \( \mathcal{B}(Q, e) \) as the peeling indicator. The procedure of extracting \( f(Q, e) \) and \( \mathcal{B}(Q, e) \) from \( (Q, e) \) will be referred to as peeling \( Q \) at \( e \). See Figure 5 for an illustration of some of the possible cases that can arise when peeling \( Q \) and \( e \).

Figure 5: An infinite quadrangulation \( Q \) with simple boundary together with three different cases for the peeled quadrilateral \( f(Q, e) \) (shown in light blue). In the left panel \( \mathcal{B}(Q, e) = \infty \). In the middle panel, \( \mathcal{B}(Q, e) = (\infty, 3) \). In the right panel, \( \mathcal{B}(Q, e) = (3, \infty, 1) \).

We now introduce notation for some additional objects associated with peeling.

- Let \( \text{Peel}(Q, e) \) be the infinite connected component of \( Q \setminus f(Q, e) \).
- Let \( \mathcal{F}(Q, e) \) be the union of the components of \( Q \setminus f(Q, e) \) other than \( \text{Peel}(Q, e) \).
- Let \( \text{Co}(Q, e) \) be the number of covered edges of \( \partial Q \), i.e., the number of edges of \( \partial Q \) which do not belong to \( \text{Peel}(Q, e) \) (equivalently, one plus the number of such edges which belong to \( \mathcal{F}(Q, e) \)).
- Let \( \text{Ex}(Q, e) \) be the number of exposed edges of \( f(Q, e) \), i.e., the number of edges of \( \text{Peel}(Q, e) \) which do not belong to \( \partial Q \) (equivalently, those which are incident to \( f(Q, e) \)).

### 3.1.2 Peeling the UIHPQ with simple boundary

In this subsection we will give explicit descriptions of the laws of the objects defined in Section 3.1.1 when we peel the uniform infinite half-plane quadrangulation with simple boundary (UIHPQs). These laws will be described in terms of the free Boltzmann partition function which is defined by

\[
3(2l) := \frac{s!(3l - 4)!}{(l - 2)!2l!}, \quad 3(2l + 1) := 0, \quad \forall l \in \mathbb{N}_0. \tag{3.2}
\]

The reason for the name is that \( 3 \) is the partition function for the so-called free Boltzmann quadrangulation with simple boundary of perimeter \( 2l \) [BG09], but we will not need this model here.

Suppose now that \( (Q_S, e_S) \) is an instance of the UIHPQs. As explained in [AC15, Section 2.3.1], the distribution of the peeling indicator of Section 3.1.1 when we peel at the root edge is described as follows.

\[
\begin{align*}
\mathbb{P}[(\mathcal{B}(Q_S, e_S) = \infty)] &= \frac{3}{5}, \\
\mathbb{P}[(\mathcal{B}(Q_S, e_S) = (k, \infty))] &= \frac{1}{12} \cdot 54^{(1-k)/2}3(k + 1), \quad \forall k \in \mathbb{N} \text{ odd} \\
\mathbb{P}[(\mathcal{B}(Q_S, e_S) = (k, \infty))] &= \frac{1}{12} \cdot 54^{-k/2}3(k + 2), \quad \forall k \in \mathbb{N}_0 \text{ even} \\
\mathbb{P}[(\mathcal{B}(Q_S, e_S) = (k_1, k_2, \infty))] &= \frac{54^{-(k_1+k_2)/2}3(k_1 + 1)3(k_2 + 1)}{54^{-(k_1+k_2)/2}3(k_1 + 1)3(k_2 + 1)}, \quad \forall k_1, k_2 \in \mathbb{N} \text{ odd.} \tag{3.3}
\end{align*}
\]

We get the same formulas if we replace \((k, \infty)\) with \((\infty, k)\) or \((k_1, k_2, \infty)\) with either \((\infty, k_1, k_2)\) or \((k_1, \infty, k_2)\) (which corresponds to changing which side of \( e_S \) the bounded complementary connected components of \( f(Q_S, e_S) \) lie on). The probabilities (3.3) are computed in [AC15, Section 2.3.1].

If we condition on \( \mathcal{B}(Q_S, e_S) \), then the connected components of \( Q \setminus f(Q_S, e_S) \) are conditionally independent. The conditional law of the unbounded connected component \( \text{Peel}(Q_S, e_S) \), rooted at one of the boundary edges it shares with \( f(Q_S, e_S) \) (chosen by some deterministic convention in the case when there is more than one such edge) is again that of a UIHPQs. This fact is referred to as the Markov property of peeling.


3.1.3 Peeling processes

Due to the Markov property of peeling, one can iteratively peel a UIHPQ\(S\) to obtain a sequence of quadrangulations which each has the law of a UIHPQ\(S\). To make this notion precise, let \((Q_S, e_S)\) be a UIHPQ\(S\). A peeling process on \(Q_S\) is a sequence of quadrangulation-edge pairs \(\{(Q^i, e^i)\}_{i=1}^{\infty}\) with \(I \in \mathbb{N}\) a possibly infinite random time, called the terminal time, such that the following is true.

1. \(Q^0 = Q_S\) and for each \(i \in [1, I]_\mathbb{Z}\), we have \(e^i \in \mathcal{E}(\mathcal{Q}^{i-1})\) and \(Q^i = \text{Peel}(Q^{i-1}, e^i)\).

2. Each edge \(e^i\) is chosen in a manner which is measurable with respect to the \(\sigma\)-algebra \(\mathcal{G}^{i-1}\) generated by the peeling indicator variables \(\mathbb{P}(Q^{j-1}, e^j)\) for \(j \in [1, i-1]_\mathbb{Z}\) and the planar map \(Q^{i-1}\). Furthermore, \(\{I \leq i\} \in \mathcal{G}^i\) for each \(i \in \mathbb{N}_0\).

It follows from the Markov property of peeling that for each \(i \in \mathbb{N}\), the conditional law of \((Q^{i-1}, e^i)\) given the \(\sigma\)-algebra \(\mathcal{G}^{i-1}\) of condition 2 on the event \(\{I \geq i - 1\}\) is that of a UIHPQ\(S\).

We will have occasion to consider several different peeling processes in this paper.

3.1.4 Estimates for peeling probabilities

In this subsection we will write down some estimates for the probabilities appearing in Section 3.1.2. Throughout, we let \((Q_S, e_S)\) be a UIHPQ\(S\).

Stirling’s formula implies that for each even \(k \in \mathbb{N}\), the free Boltzmann partition function (3.2) satisfies

\[
\mathcal{Z}(k) \asymp 54^{k/2}k^{-5/2}, \quad \text{for } k \text{ even}
\]

with universal implicit constant. From this we infer the following approximate versions of the probabilities (3.3).

\[
\begin{align*}
\mathbb{P}(\mathcal{P}(Q_S, e_S) = (k, \infty)] &\asymp k^{-5/2}, \quad \forall k \in \mathbb{N} \text{ odd} \\
\mathbb{P}(\mathcal{P}(Q_S, e_S) = (k, \infty]) &\asymp k^{-5/2}, \quad \forall k \in \mathbb{N} \text{ even} \\
\mathbb{P}(\mathcal{P}(Q_S, e_S) = (k_1, k_2, \infty]) &\asymp k_1^{-5/2}k_2^{-5/2}, \quad \forall k_1, k_2 \in \mathbb{N} \text{ odd).
\end{align*}
\]

We get the same approximate formulas if we replace \((k, \infty)\) with \((\infty, k)\) or \((k_1, k_2, \infty)\) with either \((\infty, k_1, k_2)\) or \((k_1, \infty, k_2)\).

Let \((Q_S, e_S)\) be a UIHPQ\(S\) and recall the definitions of the number of exposed edges \(\text{Ex}(Q_S, e_S)\) and the number of covered edges \(\text{Co}(Q_S, e_S)\) from Section 3.1.1. As explained in [AC15, Section 2.3.2], one has the following facts about the joint law of these random variables. We have the equality of means

\[
\mathbb{E}[\text{Co}(Q_S, e_S)] = \mathbb{E}[\text{Ex}(Q_S, e_S)],
\]

i.e. the expected net change in the boundary length of \(Q_S\) under the peeling operation is 0. We always have \(\text{Ex}(Q_S, e_S) \in \{1, 2, 3\}\), but \(\text{Co}(Q_S, e_S)\) can be arbitrarily large. In fact, there is a constant \(c_* > 0\) such that for \(k \in \mathbb{N}\),

\[
\mathbb{P}(\text{Co}(Q_S, e_S) = k] = (c_* + o_k(1))k^{-5/2}.
\]

3.2 Peeling all quadrilaterals incident to a vertex

Suppose we want to use peeling to approximate the graph-distance ball centered at a vertex \(v \in V(\partial Q)\) for a given quadrangulation \(Q\) with boundary \(\partial Q\). If \(v\) has high degree, it is a priori possible, e.g., that we need to peel a large number of edges in order to cover the graph metric ball \(B_1(v; Q)\). Similar issues arise when trying to use peeling to approximate metric balls with bigger radius. The purpose of this subsection is to show that versions of the estimates of Section 3.1.4 are still valid if instead of peeling a single edge incident to \(v\) we continue peeling edges incident to \(v\) until we disconnect \(v\) from the target edge. In particular, we will prove the following lemma.

**Lemma 3.1.** Let \((Q_S, e_S)\) be an instance of the UIHPQ\(S\) and let \(v \in V(\partial Q_S)\) be one of the endpoints of \(e_S\). Let \(Q_v\) be the set of quadrilaterals \(q \in F(Q_S)\) which are incident to \(v\) and let \(E^v_{\infty}\) (resp. \(E^v_0\)) be the set of
edges of $\partial Q_v$ lying to the left (resp. right) of $v$ which are disconnected from $\infty$ in $Q_v$ by some quadrilateral $q \in Q_v$. Then for $k \in \mathbb{N}$,

$$\mathbb{P} \left[ \# E^{L}_v = k \right] \asymp k^{-5/2} \quad (3.8)$$

with universal implicit constant. The same is true with “$R$” in place of “$L$.”

The proof of Lemma 3.1 will be a straightforward application of the following peeling process, which is illustrated in Figure 6.

**Definition 3.2** (One-vertex peeling process). Let $(Q_{\mathcal{S}}, \mathcal{S})$ be an instance of the UIHPQs and let $v \in \mathcal{V}(\partial Q_{\mathcal{S}})$. The left one-vertex peeling process of $Q_{\mathcal{S}}$ at $v$ is the sequence of infinite planar maps $(Q_{\mathcal{S}}^{i})_{i \in [0, I_{\mathcal{S}}]}$ and edges $(e_{\mathcal{S}}^{i})_{i \in [0, I_{\mathcal{S}}]}$ defined as follows.

Let $Q^{0}_{\mathcal{S}} = Q_{\mathcal{S}}$. Inductively, if $i \in \mathbb{N}$ and an infinite quadrangulation $Q^{i-1}_{\mathcal{S}}$ with simple boundary has been defined, we define $Q^{i}_{\mathcal{S}}$ as follows. If $v \notin \mathcal{V}(\partial Q^{i-1}_{\mathcal{S}})$, we set $Q^{i}_{\mathcal{S}} = Q^{i-1}_{\mathcal{S}}$. Otherwise, let $e^{i}_{\mathcal{S}}$ be the edge of $\partial Q^{i-1}_{\mathcal{S}}$ immediately to the left of $v$ and we set $Q^{i}_{\mathcal{S}} := \text{Peel}(Q^{i-1}_{\mathcal{S}}, e^{i}_{\mathcal{S}})$. We define the terminal time $I_{\mathcal{S}}$ to be the smallest $i \in \mathbb{N}$ for which $v \notin E(\partial Q^{i}_{\mathcal{S}})$. The edge $e^{I_{\mathcal{S}}}_{\mathcal{S}}$ is chosen in an arbitrary manner.

We define the right one-vertex peeling process in the same manner as above but with “left” in place of “right,” and denote the objects involved by replacing the superscript “$L$” with a superscript “$R$.”

![Figure 6: An illustration of the left one-vertex peeling process at $v$. The blue quadrilaterals incident to $v$ are enumerated by the order in which they are peeled.](image)

We note that the one-vertex peeling process is also studied in [Ric15, CC16]. We record for future reference the following elementary fact about the above peeling process.

**Lemma 3.3.** Let $(Q_{\mathcal{S}}, \mathcal{S})$ be an instance of the UIHPQs and let $v$ be one of the endpoints of $\mathcal{S}$. If $I_{\mathcal{S}}$ is the terminal time of the left one-vertex peeling process of $Q_{\mathcal{S}}$ at $v$ as in Definition 3.2, then $I_{\mathcal{S}}$ has a geometric distribution with some parameter $b \in (0, 1)$.

**Proof.** The time $I_{\mathcal{S}}$ is the smallest $i \in \mathbb{N}$ for which the peeled quadrilateral $f(Q^{i-1}_{\mathcal{S}}, e^{i}_{\mathcal{S}})$ is incident to an edge of $\partial Q^{i-1}_{\mathcal{S}}$ to the right of $e^{i}_{\mathcal{S}}$. Hence the statement of the lemma follows from (3.3) and the Markov property of peeling.

**Proof of Lemma 3.1.** We consider the left one-vertex peeling process of $Q_{\mathcal{S}}$ at $v$ as in Definition 3.2 and use the notation of that definition. The final (time-$I_{\mathcal{S}}$) one-vertex peeling cluster disconnects $v$ from $\infty$ in $Q_{\mathcal{S}}$, so must disconnect each edge in $E^{R}_v$ from $\infty$. Since the time-$(I_{\mathcal{S}} - 1)$ cluster does not disconnect any edge in $E^{R}_v$ from $\infty$, it follows that each edge in $E^{R}_v$ is disconnected from $\infty$ in $Q^{I_{\mathcal{S}}-1}_{\mathcal{S}}$ by the last peeled quadrilateral $f(Q^{I_{\mathcal{S}}-1}_{\mathcal{S}}, e^{I_{\mathcal{S}}}_{\mathcal{S}})$. Hence $\# E^{R}_v$ is the same as the number of quadrilaterals of $\partial Q^{I_{\mathcal{S}}-1}_{\mathcal{S}}$ lying to the right of $e^{I_{\mathcal{S}}}_{\mathcal{S}}$ which are disconnected from $\infty$ by this peeled quadrilateral.

If we condition on $\{I_{\mathcal{S}} = i\}$ for some $i \in \mathbb{N}$, then the conditional law of the quadrilateral $f(Q^{i-1}_{\mathcal{S}}, e^{i}_{\mathcal{S}})$ is the same as its conditional law given that it covers up at least one edge of $\partial Q^{i-1}_{\mathcal{S}}$ to the right of $e^{i}_{\mathcal{S}}$. By (3.3) the probability of this event is a universal constant, so the estimate (3.8) (with “$R$” in place of “$L$”) follows by taking an appropriate sum of the probabilities in (3.5). The analogous estimate for $E^{L}_v$ follows by symmetry. 

□
3.3 Peeling all quadrilaterals incident to a boundary arc

Let \((Q_S, \varepsilon_S)\) be an instance of the UIHPQ\(_S\). Let \(A^L\) and \(A^R\) be the infinite rays of \(\partial Q_S\) lying to the left and right of \(\varepsilon_S\), respectively, defined in such a way that \(\varepsilon_S \in \mathcal{E}(A^L) \setminus \mathcal{E}(A^R)\), the left endpoint of \(\varepsilon_S\) belongs to \(A^L\), and the right endpoint of \(\varepsilon_S\) belongs to \(A^R\).

The goal of this subsection is to estimate the number of edges of \(A^R\) which are disconnected from \(\infty\) by quadrilaterals incident to \(A^L\) if we disregard the “big” jumps made by the peeling process. In particular, we will prove the following lemma.

**Lemma 3.4.** Let \(\{(\hat{Q}_i^{i-1}, \hat{e}_i)\}_{i \in [1, \hat{I}]_{\mathbb{Z}}}\) be a peeling process of \(Q_S\) such that each edge \(\hat{e}_i\) is incident to some vertex in \(A^L\). For \(i \in \mathbb{N}_0\), let \(\hat{O}_i := \#(\mathcal{E}(A^R) \setminus \mathcal{E}(\partial \hat{Q}_i))\) be the number of edges of \(A^R\) which have been disconnected from \(\infty\) by step \(i\). For \(n \in \mathbb{N}\), let

\[
\hat{X}(n) := \sum_{i=1}^{\hat{I}} (\hat{O}_i - \hat{O}_{i-1}) \wedge n.
\]

For each \(n \in \mathbb{N}\) and each \(p \geq 1\),

\[
\mathbb{E}\left[\hat{X}(n)^p\right] \leq n^{p-1/2}
\]

with implicit constant depending only on \(p\).

We will first prove Lemma 3.4 for a particular peeling process which is in some sense maximal, which we now define.

**Definition 3.5 (Linear peeling process).** The linear peeling process of \(Q_S\) started from \(\varepsilon_S\) is the sequence of infinite planar maps \(\{Q^L_i\}_{i \in \mathbb{N}_0}\) and edges \(\{e^L_i\}_{i \in \mathbb{N}}\) defined as follows. Let \(Q^L_0 := Q_S\). Inductively, if \(i \in \mathbb{N}\) and \(Q^L_{i-1}\) has been defined, let \(v^L_i\) be the rightmost vertex of \(\partial Q^L_{i-1}\) which also belongs to \(A^L\). Let \(e^L_i\) be the edge of \(\partial Q^L_{i-1}\) lying immediately to the right of \(v^L_i\) and let \(Q^L_i := \text{Peel}(Q^L_{i-1}, e^L_i)\).

![Figure 7: Illustration of the left linear peeling process run for 6 units of time. Quadrilaterals are numbered in the order in which they are peeled.](image)

See Figure 7 for an illustration of the above definition. We now devote our attention to proving Lemma 3.4 in the special case of the left linear peeling process.

**Lemma 3.6.** Suppose we are in the setting of Definition 3.5. For \(i \in \mathbb{N}\), let

\[
O^L_i := \#(\mathcal{E}(A^R) \setminus \mathcal{E}(\partial Q^L_i))
\]

and for \(n \in \mathbb{N}\), let

\[
X^L(n) := \sum_{i=1}^{\infty} (O^L_i - O^L_{i-1}) \wedge n.
\]

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For each \( n \in \mathbb{N} \) and each \( p \geq 1 \),
\[
\mathbb{E}[X_L(n)^p] \leq n^{p-1/2}
\]  
(3.9)

with implicit constant depending only on \( p \).

For the proof of Lemma 3.6, we will use the following notation. For \( i \in \mathbb{N}_0 \), let
\[
G^i_L := \sigma \left( \mathcal{P} \left( Q^{i-1}_L, e^i_L \right) : j \in [1, i] \right)
\]  
(3.10)

be the \( \sigma \)-algebra generated by the first \( i \) peeling steps of the left linear peeling process.

Let \( I_0 = 0 \) and for \( m \in \mathbb{N} \), let \( I_m \) be the \( m \)th smallest \( i \in \mathbb{N} \) for which \( Q^i_L - Q^{i-1}_L \neq 0 \), or \( m = \infty \) if there are fewer than \( m \) such times \( i \). Observe that each \( I_m \) is a stopping time for the filtration (3.10). Let \( M \) be the smallest \( m \in \mathbb{N} \) for which \( I_m = \infty \).

Let \( \{ v_y \}_{y \in \mathbb{N}_0} \) be the vertices of \( A^L \), ordered from right to left. For \( y \in \mathbb{N}_0 \), let \( E_y \) be the set of edges of \( A^R \) which are disconnected from \( \infty \) by some quadrilateral of \( Q_S \) which is incident to \( v_y \), so that
\[
\mathcal{E}(Q^i_L \cap A^R) \subset \bigcup_{y=0}^{\infty} E_y, \quad \forall i \in \mathbb{N}.
\]  
(3.11)

The key observation in the proof of Lemma 3.6 is the following Markov property. For each \( i \geq 2 \), the vertex \( v^i_L \) is incident to the peeled quadrilateral \( f(Q^{i-2}_L, e^{i-1}_L) \) at the previous step. Hence for each \( m \in \mathbb{N} \) for which \( I_m < \infty \), the \( \partial Q^{I_m+1}_L \)-graph distance from \( e^{I_m+1}_L \) to \( \mathcal{E}(A^R) \cap \mathcal{E}(\partial Q^{I_m+1}_L) \) is either 0 or 1. By the Markov property of the peeling process, we find that the random variables \( O^{I_m}_L - O^{I_{m-1}}_L \) are almost i.i.d., except that for some values of \( m \) we peel started at distance 0 from \( \mathcal{E}(A^R) \cap \mathcal{E}(\partial Q^{I_{m-1}}_L) \) and for some values of \( m \) we peel started at distance 1 from \( \mathcal{E}(A^R) \cap \mathcal{E}(\partial Q^{I_{m-1}}_L) \).

Lemma 3.7. In the setting described just above, there is a universal constant \( b > 0 \) such that for each \( m \in \mathbb{N} \),
\[
\mathbb{P} \left[ M = m + 1 \mid G^m_L \right] \mathbb{1}_{\{ M > m \}} \geq b \mathbb{1}_{\{ M > m \}}.
\]  
(3.12)

Proof. By the Markov property noted just above the statement of the lemma, it suffices to show that
\[
\mathbb{P}[M = 1] > 0.
\]  
(3.13)

By Lemma 3.1, with \( E_y \) defined just above (3.11), for each \( y \in \mathbb{N} \) and each \( k \in \mathbb{N} \) we have
\[
\mathbb{P}[\#E_y \geq k] \leq (k + y)^{-3/2}
\]  
(3.14)

with universal implicit constant. Taking \( k = 1 \) and summing over all \( y \geq y_0 \), we see that there exists some \( y_0 \in \mathbb{N} \) such that
\[
\mathbb{P} \left[ f(Q_S, \lambda_S(y)) \right] \text{ does not share a vertex with } A^R, \quad \forall y \geq y_0 \geq \frac{1}{2}.
\]  
(3.15)

Furthermore, by (3.3) and the Markov property of peeling there exists \( k_0 \in \mathbb{N} \) such that with positive probability \( \partial Q^{k_0}_L \) contains no edges of \( A^R \) and the edge \( e^{k_0}_L \) lies at \( \partial Q^{k_0}_L \)-graph distance at least \( y_0 \) from \( A^R \). By (3.15) and another application of the Markov property of peeling, we obtain (3.13).

Proof of Lemma 3.6. Fix \( p \geq 1 \). We first prove a \( p \)th moment bound for \( O^{I_1} \cap n \). If \( O^{I_1} \geq k \) for some \( k \in \mathbb{N} \), then by (3.11) there exists \( y \in \mathbb{N} \) such that \( \#E_y \geq k \). By (3.14),
\[
\mathbb{P} \left[ O^{I_1}_L \geq k \right] \leq \sum_{y=0}^{\infty} (k + y)^{-3/2} \leq k^{-1/2}.
\]  

Therefore, for \( n \in \mathbb{N} \),
\[
\mathbb{E} \left[ (O^{I_1}_L \cap n)^p \right] \leq \sum_{k=1}^{n} k^{p-1} \mathbb{P} [O^{I_1}_L \geq k] \leq \sum_{k=1}^{n} k^{p-3/2} \leq n^{p-1/2}.
\]
By the Markov property described just above Lemma 3.7 (and a trivial modification to the above argument to treat the case when we start at distance 1, rather than 0, from $A^R$) we also have

$$
\mathbb{E}\left[\left((O_{L}^{m} - O_{L}^{m-1}) \cap n\right)^{p} \mid G_{L}^{m} \right] \leq n^{p-1/2}, \quad \forall m \in \mathbb{N}.
$$

(3.16)

By Lemma 3.7, for each $m \in \mathbb{N}$ the conditional law of $M$ given $G_{L}^{m}$ is stochastically dominated by $m$ plus a geometric random variable with parameter $b$ (where $b$ is as in the statement of Lemma 3.7). In particular,

$$
\mathbb{E}\left[M^{p-1} \mid G_{L}^{m}\right] \leq \sum_{t=1}^{\infty} (m + t)^{p-1}(1 - b)^{t} \leq m^{p-1}.
$$

Since $O_{L}^{m} \in G_{L}^{m}$ and $\{M \geq m\} = \{M \leq m - 1\}^{c} \in G_{L}^{m+1}$, we infer from this and (3.16) that

$$
\mathbb{E}[X_{L}(n)^{p}] = \mathbb{E}\left[\left(\sum_{m=1}^{M} (O_{L}^{m} - O_{L}^{m-1}) \cap n\right)^{p}\right]
$$

\[
\leq \sum_{m=1}^{\infty} \mathbb{E}\left[\left((O_{L}^{m} - O_{L}^{m-1}) \cap n\right)^{p} M^{p-1} \mathbb{1}_{M \geq m}\right]
\]

\[
= \sum_{m=1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[\left((O_{L}^{m} - O_{L}^{m-1}) \cap n\right)^{p} \mid G_{L}^{m}\right] \mathbb{E}[M^{p-1} \mid G_{L}^{m}] \mathbb{1}_{M \geq m}\right]
\]

\[
\leq n^{p-1/2} \sum_{m=1}^{\infty} m^{p-1}(1 - b)^{m} \leq n^{p-1/2}
\]

which is (3.9). \hfill \Box

We will now extend Lemma 3.6 to get our desired estimate for general peeling processes.

**Proof of Lemma 3.4.** Let $i \in [1, 2]$. There exists $j \in \mathbb{N}$ such that the peeled quadrilateral $\hat{f}(\hat{Q}^{i-1}, \hat{e}^{i})$ is equal to the jth peeled quadrilateral $\hat{f}(\hat{Q}^{i-1}, \hat{e}^{j})$ in the linear peeling process. Every quadrilateral of $Q_{S}$ which is incident to $A^{L}$ and which is disconnected from $\infty$ by $\hat{f}(\hat{Q}^{i-1}, \hat{e}^{j})$ is peeled by the linear peeling process before time $j$. Therefore, in the notation of Lemma 3.6,

$$
O_{L}^{j} - O_{L}^{j-1} \leq \hat{O}^{i} - \hat{O}^{i-1}.
$$

Since every quadrilateral peeled by our given peeling process is also peeled by the left linear peeling process,

$$
\sum_{i=1}^{\hat{i}} (\hat{O}^{i} - \hat{O}^{i-1}) \leq \sum_{j=1}^{\infty} (O_{L}^{j} - O_{L}^{j-1}).
$$

The preceding paragraph shows that every term in the sum on the left is greater than or equal to a unique corresponding term in the sum on the right. Hence the inequality continues to hold if we truncate each of the terms in each of the sums at level $n$. That is, $X(n) \leq X_{L}(n)$ so the statement of the lemma follows from Lemma 3.6. \hfill \Box

## 4 Peeling the glued map

In this section we will introduce a two-sided peeling process for a pair of UIHPQs’s glued together along their boundaries, which we call the **glued peeling process** and which will be an important tool in the proofs of our main theorems. The main reason for our interest in this peeling process is that it satisfies a simple Markov property (Lemma 4.2) and provides an upper bound for metric balls in the glued map (Lemma 4.3). We will also prove in Section 4.2 some basic estimates for how many edges of the boundary of our original pair of UIHPQs’s are swallowed by this peeling process. These bounds will later be used to deduce moment estimates in Section 5.
4.1 Glued peeling process

Let \((Q_-, e_-)\) and \((Q_+, e_+)\) be two independent samples of the UIHPQs. Let \(\lambda_- : \mathbb{Z} \to \mathcal{E}(\partial Q_-)\) (resp. \(\lambda_+ : \mathbb{Z} \to \mathcal{E}(\partial Q_+)\)) be the boundary path for \(Q_-\) (resp. \(Q_+\)) started from \(e_-\) (resp. \(e_+)\) and traveling to the right.

Fix gluing times \(\underline{\lambda}_-, \underline{\lambda}_+ \in \mathbb{N}\) with \(\underline{\lambda}_- \leq \underline{\lambda}_+ \wedge \underline{\lambda}_+\) and let \(Q_{zip}\) be the planar map obtained from \(Q_-\) and \(Q_+\) by identifying \(\lambda_-(x)\) with \(\lambda_+(x)\) for each \(x \in [0, \underline{\lambda}_-] \cap \mathbb{N}\) and \(\lambda_-(\underline{\lambda}_- + y)\) with \(\lambda_+(\underline{\lambda}_+ + y)\) for each \(y \in \mathbb{N}\). Taking \(\underline{\lambda}_- = \underline{\lambda}_+ = \underline{\lambda}_+\) corresponds to gluing \(Q_+\) together along their positive boundaries, which is the setting of Theorem 1.1 and the main case we are interested in. Other choices of \(\underline{\lambda}_-\) and \(\underline{\lambda}_+\) result in a “hole” in \(Q_{zip}\) with left/right boundary lengths \(\underline{\lambda}_- - \underline{\lambda}_+\) and \(\underline{\lambda}_+ - \underline{\lambda}_-\). We need to consider the case when there is such a hole due to the Markov property of our peeling process (Lemma 4.2 below).

We slightly abuse notation by identifying \(Q_-\) and \(Q_+\) with the corresponding subsets of \(Q_{zip}\), so we write \(\lambda_- (\underline{\lambda}_-) = \lambda_+ (\underline{\lambda}_+)\), etc.

Choose a finite, non-empty, connected initial edge set \(A \subset \partial Q_- \cup \partial Q_+\) (which is where we will start our peeling process). In the case when either \(x_+\) or \(x_-\) is not equal to \(\underline{\lambda}_\), we require that

\[
\lambda_- (\underline{\lambda}_-) [A, x_-] \cup \lambda_+ (\underline{\lambda}_+) [A, x_+] \subset A
\]

so that \(A\) contains every edge along the boundary of the hole in \(Q_{zip}\).

We will define a joint peeling process for \(Q_-\) and \(Q_+\) (i.e., edges of both \(Q_-\) and \(Q_+\) will be peeled), called the glazed peeling process started from \(A\), whose clusters at certain special times \(J_\ell\) contain the radius-\(r\) graph metric ball centered at \(A\) in \(Q_{zip}\). The glued peeling process will be described by a sequence of finite planar maps \(\{\hat{Q}^j\}_{j \in \mathbb{N}_0}\) contained in \(Q_{zip}\), a sequence of infinite quadrangulations with boundary \(\{Q^\pm_{\lambda}\}_{\lambda \in \mathbb{N}_0}\) contained in \(Q_{zip}\) which intersect \(\hat{Q}^j\) only along their boundaries with the property that \(Q_{zip} = \hat{Q}^0 \cup Q^\pm_{\lambda}\) for each \(\lambda \in \mathbb{N}_0\), and an increasing sequence of non-negative integer stopping times \(\{J_\ell\}_{\ell \in \mathbb{N}_0}\). We define \(\partial \hat{Q}^j = \hat{Q}^j \cap (\partial Q^\pm_{\lambda}\) for each \(\lambda \in \mathbb{N}_0\)). Note that in the case when the map \(Q_{zip}\) has a hole, the outer boundary of this hole need not be part of \(\partial \hat{Q}^j\).

To start the definition, let \(\hat{Q}^0\) be the planar map with at most two faces which is the smallest subgraph of \(\partial Q_- \cup \partial Q_+\) containing \(A\). Let \(Q^\pm_{\lambda} = Q^\pm_{\lambda\lambda}\). Also let \(J_0 = 0\).

Inductively, suppose \(j \in \mathbb{N}\), \(\hat{Q}^j, \hat{Q}^j_-,\) and \(\hat{Q}^j_+\) have been defined for \(i \leq j - 1\), and \(J_\ell\) for \(r \in \mathbb{N}_0\) has been defined on the event \(\{J_\ell \leq j - 1\}\). Let \(r_{j-1}\) be the largest \(r \in \mathbb{N}_0\) for which \(J_\ell \leq j - 1\), and suppose that \(\partial \hat{Q}^{r_{j-1}}\) shares a vertex with \(\partial Q^\pm_{\lambda\lambda}\).

Let \(\check{e}\) be an edge in \(\mathcal{E}(\partial \hat{Q}^{r_{j-1}})\) (which we recall is contained in \(\mathcal{E}(\partial Q^{r_{j-1}} \cup \partial Q^\pm_{\lambda\lambda})\)) which has at least one endpoint in \(V(\hat{Q}^{r_{j-1}})\), chosen in a manner which depends only on \(\partial Q^{r_{j-1}}\) and \(V(\hat{Q}^{r_{j-1}})\) (the precise

Figure 8: Illustration of the glued peeling process run for several peeling steps in the case when \(\underline{\lambda}_- = \underline{\lambda}_+ = \underline{\lambda}_+\) (so there is no hole) and \(A\) is a single edge \(e\). Quadrilaterals are numbered by the order in which they are peeled. Quadrilaterals peeled during the first (resp. second, third) layer are colored blue (resp. orange, purple). Disconnected regions are colored light green. Here \(J_1 = 3, J_2 = 8\), and \(J_3 = 13\). The map \(\hat{Q}^{13}\) is the union of the 13 colored quadrilaterals and the light green regions which they disconnect from \(\infty\) in either \(Q_-\) or \(Q_+\).
manner in which the edge is chosen is not important for our purposes). Such an edge exists by our inductive hypothesis. If \( \hat{e}^j \in \partial Q^{J_\xi}_{\xi} \) we set \( \xi^j = - \) and otherwise (in which case \( \hat{e}^j \in \partial Q^{J_\xi}_{\xi} \)) we set \( \xi^j = + \).

Recalling the notation of Section 3.1.1, we peel \( Q^{J_\xi}_{\xi} \) at \( \hat{e}^j \) to obtain the quadrilateral \( f(Q^{J_\xi}_{\xi}, \hat{e}^j) \) and the planar map \( \mathfrak{g}(Q^{J_\xi}_{\xi}, \hat{e}^j) \) which it disconnects from \( \infty \) in \( Q^{J_\xi}_{\xi} \). We let

\[
\begin{align*}
\hat{Q}^j &:= \hat{Q}^{J_\xi}_{\xi} \cup f(Q^{J_\xi}_{\xi}, \hat{e}^j) \cup \mathfrak{g}(Q^{J_\xi}_{\xi}, \hat{e}^j), \\
Q^j_{\xi} &:= \text{Peel}(Q^{J_\xi}_{\xi}, \hat{e}^j), \quad \text{and} \quad Q^j_{\xi \xi} := Q^{J_\xi}_{\xi - \xi}.
\end{align*}
\]

By induction \( Q^j_{\xi} \) are infinite quadrangulations with boundary, \( \hat{Q}^j \) is a finite quadrangulation with boundary (possibly with a single hole corresponding to the hole in \( Q_{\text{zip}} \)) and \( Q_{\text{zip}} = Q^j_{\xi} \cup Q^j_{+} \cup \hat{Q}^j \). If \( \partial \hat{Q}^j \) shares a vertex with \( \partial Q^{J_\xi_{j+1}}_{\xi - j} \), we declare that \( J_{r_{j+1}} > j \), and otherwise we declare that \( J_{r_{j+1}} = j \). These definitions imply that \( \partial \hat{Q}^{J_{r-1}} \) shares a vertex with \( \partial Q^j \), which completes the induction.

Define the filtration

\[
F^j := \sigma \left( \hat{Q}^j, \Psi(Q^{J_{\xi}}_{\xi}, \hat{e}^j) : i \in [1, j], \forall j \in \mathbb{N}_0, \right),
\]

where here \( \Psi(\cdot, \cdot) \) is the peeling indicator variable from Section 3.1.1 Note that \( \hat{Q}^j \) and \( \hat{e}^{j+1} \) are \( F^j \)-measurable for \( j \in \mathbb{N}_0 \), and \( J_r \) for \( r \in \mathbb{N}_0 \) is a stopping time for \( \{F^j\}_{j \in \mathbb{N}_0} \).

**Remark 4.1.** The glued peeling process described above is similar to the so-called peeling by layers algorithm for infinite planar quadrangulations or triangulations without boundary which is studied in [CL14a]. However, unlike the clusters produced by the peeling by layers algorithm, our glued peeling clusters do not closely approximate filled metric balls (instead they are just larger than metric balls) since we peel edges which are disconnected from \( \infty \) on one side of the gluing interface but not the other. Furthermore, the glued peeling process is equivalent to the peeling process introduced and studied independently of the present work in [CC16, Section 2], but the estimates proven for this process in the present paper are stronger than those in [CC16].

The glued peeling process satisfies a Markov property, described as follows.

**Lemma 4.2.** With the above definitions, the following is true for each \( F^j \)-stopping time \( \iota \). The quadrangulations \( Q^j_{\xi -} \) and \( Q^j_{\xi +} \) are conditionally independent given \( F^\iota \), and the conditional law of each is that of a UIHPQ\( \mathcal{S}_\iota \). Furthermore, if \( \iota = J_r \) for some \( r \in \mathbb{N}_0 \), then there exists \( F^{J_r} \)-measurable \( \mathfrak{X}^{J_r}, \mathfrak{X}^{J_r}, \mathfrak{X}^{J_r} \in \mathbb{N}_0 \) with \( \mathfrak{X}^{J_r} \leq \mathfrak{X}^{J_r} \leq \mathfrak{X}^{J_r} + 1 \) such that \( Q^j_{\xi +} \) are glued together in the manner described at the beginning of this subsection with this choice of gluing times and \( \{\hat{Q}^{J_r}\}_{j \in \mathbb{N}_0} \) is the set of clusters of the glued peeling process of \( Q^j_{\xi -} \cup Q^j_{\xi +} \) started from \( \Lambda = \mathcal{E}(\partial \hat{Q}^{J_r}) \).

The second statement of Lemma 4.2 is the main reason why we allow general choices of \( \mathfrak{X}, \mathfrak{X}^- \), and \( \mathfrak{X}^+ \) in the above construction—cutting out the cluster \( \hat{Q}^{J_r} \) produces a hole in \( Q_{\text{zip}} \).

**Proof of Lemma 4.2.** This is immediate from the above inductive construction of the glued peeling process and the Markov property of peeling (recall Section 3.1.2).

The following lemma is the main reason for our interest in the planar maps \( \hat{Q}^j \).

**Lemma 4.3.** For each \( r \in \mathbb{N}_0 \),

\[
B_r(\Lambda; Q_{\text{zip}}) \subset \hat{Q}^{J_r}.
\]

**Proof.** It suffices to show inclusion of the vertex sets of the graphs in (4.3), since an edge in either of these graphs is the same as an edge of \( Q_{\text{zip}} \) whose endpoints are both in the vertex set of the graph. We proceed by induction on \( r \). The base case \( r = 0 \) (in which case \( J_r = 0 \)) is true by definition. Now suppose \( r \in \mathbb{N} \) and (4.3) holds with \( r - 1 \) in place of \( r \). If we are given a vertex \( v \) of \( B_r(\Lambda; Q_{\text{zip}}) \setminus \mathcal{V}(\hat{Q}^{J_{r-1}}) \), then there is a \( w \in B_{r-1}(\Lambda; Q_{\text{zip}}) \) with \( \text{dist}(w, \Lambda; Q_{\text{zip}}) = r - 1 \). By the inductive hypothesis, \( w \) belongs to \( \mathcal{V}(\partial \hat{Q}^{J_{r-1}}) \). By definition of \( J_r \), we have \( w \notin \mathcal{V}(\partial \hat{Q}^{J_r}) \) so we must have \( v \in \mathcal{V}(\hat{Q}^{J_r}) \).
It is not clear a priori that $\hat{Q}^{J_r}$ is typically contained in a graph metric ball of radius comparable to $r$. Indeed, at each step of the glued peeling process we are allowed to peel at an edge which is only disconnected from $\infty$ on one side of the gluing interface, so $\hat{Q}^{J_r}$ can potentially be much larger than $B_{3r}(A;Q_{\text{zip}})$. However, we have the following lemma which gives an upper bound for the size of $\hat{Q}^{J_r}$ in terms of the size of its intersection with $\partial Q_- \cup \partial Q_+$. 

**Lemma 4.4.** For each $r \in \mathbb{N}_0$, 

$$
\partial \hat{Q}^{J_r} \cap Q_- \subset B_{2r} \left( \hat{Q}^{J_r} \cap \partial Q_-; \hat{Q}^{J_r} \cap Q_- \right), \tag{4.4}
$$

and similarly with “+” in place of “−.” 

**Proof.** As in the proof of Lemma 4.4, it suffices to show an inclusion of vertex sets. We proceed by induction on $r$, noting that the base case $r=0$ is trivial. Suppose $r \in \mathbb{N}$ and (4.4) holds with $r-1$ in place of $r$. Let $v \in \mathcal{V}(\partial \hat{Q}^{J_r} \cap Q_-)$. If $v \in \mathcal{V}(\partial Q_-)$, then $v \in \mathcal{V} \left( B_{0} \left( \hat{Q}^{J_r} \cap \partial Q_-; \hat{Q}^{J_r} \cap Q_- \right) \right)$, so we can assume that $v \notin \mathcal{V}(\partial Q_-)$.

Every vertex of $\hat{Q}^{J_r} \cap (Q_- \cup \partial Q_-)$ which does not belong to one of the peeled quadrilaterals $j(Q_{-1}^j,\bar{e}^j)$ for $j \in [J_{r-1} + 1, J_r] \mathbb{Z}$ with $\bar{e}^j = -$ is disconnected from $\infty$ in $Q_-$ by some such quadrilateral, so does not belong to $\partial \hat{Q}^{J_r}$. Therefore $v$ must be a vertex of one of these peeled quadrilaterals. By definition of $J_r$, this quadrilateral has a vertex in $\partial \hat{Q}^{J_{r-1}}$. Hence $v$ lies at $Q_-$-graph distance at most 2 from $\partial \hat{Q}^{J_{r-1}}$, so by the inductive hypothesis $v \in \mathcal{V} \left( B_{2r} \left( \hat{Q}^{J_r} \cap \partial Q_-; \hat{Q}^{J_r} \cap Q_- \right) \right)$. 

\( \square \)

### 4.2 Bounds for the size of jumps

Suppose we are in the setting of Section 4.1. In light of Lemma 4.4, in order to prove an upper bound for the size of $\hat{Q}^{J_r}$ we need estimates for the number of edges of $\partial Q_- \cup \partial Q_+$ (which includes the gluing interface) contained in the glued peeling clusters $\hat{Q}^j$. To this end, for $j \in \mathbb{N}_0$, let 

$$
\hat{Y}^j := \# \mathcal{E} \left( \hat{Q}^j \cap (\partial Q_- \cup \partial Q_+) \right) \tag{4.5}
$$

so that $\hat{Y}^0 = \# \mathcal{A}$. For $n \in \mathbb{N}$, also define 

$$
\hat{Y}^j_n := \sum_{i=1}^{j} \left( \hat{Y}^i - \hat{Y}^{i-1} \right) \land n \tag{4.6}
$$

so that $\hat{Y}^j_n$ is the sum of the upward jumps made by $\hat{Y}$ before time $j$ truncated at level $n$.

The goal of this subsection is to prove an upper bound for $\hat{Y}^j_n$ (which implies an upper bound for the total length of the small jumps made by $\hat{Y}$ before time $J_r$) and an upper bound for the number of big jumps made by $\hat{Y}$ before time $J_r$. These bounds will be used in Section 5 to prove various moment bounds for the glued peeling procedure. We start with our upper bound for the total length of the small jumps.

**Lemma 4.5.** In the notation of (4.6), for each $r, n \in \mathbb{N}$ and each $p \geq 1$, we have 

$$
\mathbb{E} \left[ \left( \hat{Y}^{J_r}_n \right)^p \right] \preceq (r^2 \lor n)^p \tag{4.7}
$$

with implicit constant depending only on $p$.

**Proof.** By Hölder’s inequality it suffices to prove (4.7) for $p \in \mathbb{N}$. For $r \in \mathbb{N}_0$, let $A_{l,\pm}^r$ (resp. $A_{r,\pm}^r$) be the arc of $\partial Q_{l,\pm} \cap \partial Q_{r,\pm}$ lying to the left (resp. right) of $\partial \hat{Q}^{J_r}$. Then $Q_{l,\pm}^{J_{r+1}}$ is obtained from $Q_{l,\pm}^r$ by removing some of the quadrilaterals of $Q_{l,\pm}^{J_r}$ which are incident to vertices of $\partial Q_{l,\pm}^{J_r} \setminus A_{l,\pm}^r$ together with the vertices and edged which they disconnect from $\infty$ in $Q_{l,\pm}^r$. It therefore follows from Lemma 3.4 and Lemma 4.2 (together with left/right symmetry) that for each $p \in \mathbb{N}$, 

$$
\mathbb{E} \left[ \left( \hat{Y}^{J_{r+1}}_n - \hat{Y}^{J_r}_n \right)^p \mid \mathcal{F}^{J_{r-1}} \right] \preceq n^{p-1/2} \tag{4.8}
$$
with implicit constants depending only on \( p \).

Now let \( r \in \mathbb{N} \) and for \( s \in [1, r]_{\mathbb{Z}} \), let \( X_s := \tilde{Y}^r_n - \tilde{Y}^{r-1}_n \). Then for \( p \in \mathbb{N} \) we have

\[
\mathbb{E} \left[ (\tilde{Y}^r_n)^p \right] = \mathbb{E} \left[ \left( \sum_{s=1}^{r} X_s \right)^p \right] \leq \sum_{(s_1, \ldots, s_p) \in [1, r]_{\mathbb{Z}}^p} \mathbb{E} \left[ X_{s_1} \cdots X_{s_p} \right].
\]  \hspace{1cm} (4.9)

For \( q \in [1, p]_{\mathbb{Z}} \), let \( S_q \) be the set of \( p \)-tuples \((s_1, \ldots, s_p) \in [1, r]_{\mathbb{Z}}^p \) with exactly \( q \) distinct indices. By (4.8), for \((s_1, \ldots, s_p) \in S_q \) the corresponding term in the sum on the right side of (4.9) is bounded above by \( np^{-q/2} \).

We have \( \#S_q \leq r^q \) (implicit constant depending on \( p \)) since we need to choose \( q \) of the \( r \) possible indices. Therefore, (4.9) is bounded above by a \( p \)-dependent constant times

\[
\sum_{q=1}^{p} np^{-q/2} r^q \leq (r^2 \vee n)^p.
\]

Next we turn our attention to bounding the number of \( j \in [1, J_r]_{\mathbb{Z}} \) for which \( \tilde{Y}^j - \tilde{Y}^{j-1} \) is unusually large. In particular, we will prove the following lemma.

**Lemma 4.6.** For \( r > 0 \) and \( n \in \mathbb{N} \), let \( K_r(n) \) be the number of \( j \in [1, J_r]_{\mathbb{Z}} \) for which \( \tilde{Y}^j - \tilde{Y}^{j-1} \geq n \). There is a universal constant \( a > 0 \) such that for each \( k \in \mathbb{N} \),

\[
\mathbb{P} [K_r(n) > k] \leq (an^{-1/2} r)^k.
\]

For the proof of Lemma 4.6 we will need the following notation.

**Definition 4.7.** For \( j \in \mathbb{N}_0 \), let \( \rho(j) \) be the smallest \( r \in \mathbb{N} \) for which \( J_r \geq j + 1 \). For a vertex \( v \in \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{J_{\rho(j)-1}} \cap \partial \mathcal{Q}^j) \), let \( \ell_{v, \pm} \) be graph distance from \( v \) to \( \partial \mathcal{Q}^j \setminus \partial \mathcal{Q}^j \) in \( \partial \mathcal{Q}^j \).

We note that for \( r \in \mathbb{N}_0 \), we have \( \rho(J_r) = r + 1 \).

**Lemma 4.8.** Let \( \ell \) be a stopping time for the filtration \( \{F^j\}_{j \in \mathbb{N}_0} \) of (4.2). Then for \( n \in \mathbb{N} \),

\[
\mathbb{P} \left[ \exists j \in [\ell + 1, J_{\rho(j)}]_{\mathbb{Z}} \text{ with } \tilde{Y}^j - \tilde{Y}^{j-1} \geq n \mid F^\ell \right] \leq n^{-1/2}
\]

with universal implicit constant.

**Proof.** Define \( \ell_{v, \pm} \) for \( v \in \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{J_{\rho(j)-1}} \cap \partial \mathcal{Q}^j) \) as in Definition 4.7. By the construction in Section 4.1, the time \( J_{\rho(j)} \) is the smallest \( j \geq \ell + 1 \) for which no element of \( \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{J_{\rho(j)-1}}) \) belongs to \( \partial \mathcal{Q}^j \).

Furthermore, every edge \( e^i \) for \( i \in [\ell + 1, J_{\rho(j)}]_{\mathbb{Z}} \) is incident to some vertex in \( \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{J_{\rho(j)-1}}) \). Hence if there is a \( j \in [\ell + 1, J_{\rho(j)}]_{\mathbb{Z}} \) for which \( \tilde{Y}^j - \tilde{Y}^{j-1} \geq n \), then either there is a \( v \in \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{\ell_{v, \pm}} \cap \partial \mathcal{Q}^j) \) and a quadrilateral of \( Q^j \) incident to \( v \) which disconnects at least \( \ell_{v, \pm} + n \) edges of \( \partial \mathcal{Q}^j \) from \( \infty \in \partial \mathcal{Q}^j \); or the same holds with “-” in place of “+.”

For each \( k \in \mathbb{N} \), there are at most 2 vertices \( v \in \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{J_{\rho(j)-1}} \cap \partial \mathcal{Q}^j) \) with \( \ell_{v, \pm} = k \); and the same holds with “+” in place of “-.” Consequently, we can apply Lemmas 3.1 and 4.2 to get

\[
\mathbb{P} \left[ \exists j \in [\ell + 1, J_{\rho(j)}]_{\mathbb{Z}} \text{ with } \tilde{Y}^j - \tilde{Y}^{j-1} \geq n \mid F^\ell \right] \leq \sum_{\ell \in \{-+, \} \in \mathcal{V} (\partial \mathcal{Q}^j \cap \partial \mathcal{Q}^{J_{\rho(j)-1}} \cap \partial \mathcal{Q}^j)} (\ell_{v, \pm} + n)^{-3/2} \leq \sum_{k=1}^{\infty} (n + k)^{-3/2} \leq n^{-1/2}
\]

with universal implicit constant. \( \square \)
Proof of Lemma 4.6. Let \( T_0 = 0 \) and for \( k \in \mathbb{N} \) let \( T_k \) be the \( k \)th smallest \( j \in \mathbb{N} \) for which \( \hat{Y}^j - \hat{Y}^{j-1} \geq n \).

In the notation of Definition 4.7, we have \( J_{\rho(j)+r} \geq J_r \) for each \( j, r \in \mathbb{N} \). In particular, if \( T_k \leq J_r \) then there exists \( j \in [T_{k-1}+1, J_{\rho(T_{k-1})+r}] \mathbb{N} \) with \( \hat{Y}^j - \hat{Y}^{j-1} \geq n \). By applying Lemma 4.8 for each of the stopping times \( T_{k-1}, J_{\rho(T_{k-1})}, J_{\rho(T_{k-1})+1}, \ldots, J_{\rho(T_{k-1})+r-1} \) and taking a union bound, we find that

\[
\mathbb{P}[T_k \leq J_r \mid \mathcal{F}^{T_{k-1}}] \leq n^{-1/2} r.
\]

Iterating this estimate \( k \) times yields the statement of the lemma. \qed

5 Moment bounds for the glued peeling process

Suppose we are in the setting of Section 4.1 for some choice of gluing times \( \bar{\omega}, \omega_- \), and \( \omega_+ \) and initial edge set \( \mathcal{A} \) satisfying the conditions of that section. Define the clusters \( \{\hat{Q}^j\}_{j \in \mathbb{N}_0} \), the stopping times \( \{J_r\}_{r \in \mathbb{N}_0} \), and the complementary UIHPQs \( \{Q^\pm_j\}_{j \in \mathbb{N}_0} \) for the glued peeling process of \( Q_{\text{zip}} \) started from \( \mathcal{A} \).

The main goal of this section is to prove the following upper bound for the boundary length of the clusters

\[
\mathbb{E}\left[ \frac{\# \mathcal{E}\left( \hat{Q}^j \cap (\partial Q_- \cup \partial Q_+) \right)}{r^{2/p}} \right] \leq \left( r + \left(\# \mathcal{A}\right)^{1/2} \right)^{2p} \tag{5.1}
\]

and

\[
\mathbb{E}\left[ \left( \max_{j \in [1, J_r]} \# \mathcal{E}\left( \partial \hat{Q}^j \right) \right)^p \right] \leq \left( r + \left(\# \mathcal{A}\right)^{1/2} \right)^{2p} \tag{5.2}
\]

with implicit constant depending only on \( p \).

Proposition 5.1 is our most important estimate for the glued peeling clusters. The reason why we get moments up to order 3/2 is related to the 5/2 exponent appearing in (3.7).

We will deduce several consequences of Proposition 5.1 in Section 5.4 below. Since \( \hat{Q}^j \) dominates from above a \( Q_{\text{zip}} \)-graph metric ball of radius \( r \), an upper bound on \( \hat{Q}^j \) leads to lower bounds on \( Q_{\text{zip}} \)-graph distances. This will lead to the fact that the gluing interface does not form loops at large scales, which will be quantified by a reverse Hölder continuity estimate for the gluing interface with respect to the \( Q_{\text{zip}} \)-graph metric (Lemma 5.10). We will also use (5.1) to obtain an upper bound for the size of a \( Q_{\text{zip}} \)-metric ball in terms of \( Q_{\pm} \)-metric balls (Lemma 5.12). Finally, we will obtain a lower bound for the length of a path in \( Q_{\text{zip}} \), which stays near \( \partial Q_- \cup \partial Q_+ \) (Lemma 5.13). We will use these consequences as well as Proposition 5.1 itself several times in the later sections.

The proof of Proposition 5.1 is carried out in Sections 5.1–5.3 below. One may skip the details of the proof on a first reading of the article, since this section is connected to the rest of the paper only through the statement of Proposition 5.1 and its consequences deduced in Section 5.4.

The proof of Proposition 5.1 is based on an analysis of certain discrete processes associated with the boundary lengths of the clusters \( \hat{Q}^j \), which are illustrated in Figure 9. For \( j \in \mathbb{N}_0 \), let \( \hat{Y}^j \) be the number of edges in \( \partial Q_+ \cup \partial Q_- \) which are contained in the glued peeling cluster \( Q^j \) at time \( j \), as in (4.5). Then \( \hat{Y}^0 = \# \mathcal{A} \) and (5.1) of Proposition 5.1 is equivalent to a \( p \)th moment bound for \( \hat{Y}^j \).

Let

\[
X^j_\pm := \# \left( \mathcal{E}(\partial \hat{Q}^j \cap \partial Q^\pm) \setminus \mathcal{E}(\partial Q^\pm) \right) \tag{5.3}
\]

be the number of edges of \( \partial \hat{Q}^j \) which belong to the interior of \( Q^\pm \). Also let

\[
Y^j_\pm := \# \left( \mathcal{E}(\partial Q^\pm) \setminus \mathcal{E}(\partial Q^j) \right) + \# \mathcal{A} \tag{5.4}
\]

be the number of edges of \( \partial Q^\pm \) which are disconnected from \( \infty \) in \( Q^j_\pm \) by \( \partial \hat{Q}^j \), plus the number of edges in the initial edge set. Define

\[
X^j := X^j_- + X^j_+, \quad Y^j := Y^j_- + Y^j_+, \quad \text{and} \quad Z^j := X^j - Y^j. \tag{5.5}
\]
Note that $X^0 = 0$, $Y^0 = 2\#A$, $Z^0 = -2\#A$, and
\[ \hat{Y}^j \leq Y^j \leq 2\hat{Y}^j. \] (5.6)

Furthermore, in the notation of Section 3.1 (recall also the signs $\xi^j$ from Section 4.1), we have
\[ Z^j - Z^{j-1} = (X^{\xi^j}_j - Y^{\xi^j}_j) - (X^{\xi^j-1}_j - Y^{\xi^j-1}_j) = \text{Ex}\left(Q^{\xi^j-1}_j, \hat{e}_j\right) - \text{Co}\left(Q^{\xi^j-1}_j, \xi^j\right) \] (5.7)

Hence Lemma 4.2 implies that the increments $\{Z^j - Z^{j-1}\}_{j \in \mathbb{N}}$ are i.i.d. and adapted to the filtration (4.2).

Furthermore, since the number of covered and exposed edges for a peeling step have the same expectation, we find that $Z$ is an $\mathcal{F}^j$-martingale.

The proof of Proposition 5.1 consists of three main steps, which we outline below.

Step 1: First moment bound for the total number $J_r$ of quadrilaterals revealed by the time that $r$ layers have been peeled (Section 5.1). The idea is first to bound the conditional expectation of $J_{r+1} - J_r$ given $\mathcal{F}^j$ in terms of $X^{J_r}$ and $Y^{J_r}$. This is a natural bound since the number of quadrilaterals revealed in the $(r+1)$st layer should only depend on the total boundary length of the cluster when $r$ layers have been peeled. To complete the proof, we then deduce a recursive bound for $X^{J_r} + Y^{J_r}$, which leads to a first moment bound for $X^{J_r} + Y^{J_r}$ and thereby our desired first moment bound for $J_r$.

Step 2: Establish that $\max_{j \in [1, J_r]} (Z^j - Z^0)$ has $p$th moments for $p \in [1, 3/2)$ (Section 5.2). This will follow from the first moment bound for $J_r$ from the previous step together with a standard estimate for sums of i.i.d. heavy-tailed random variables. In particular, since $Z^j$ is a sum of i.i.d. random variables which have probability of order $k^{-5/2}$ of being equal to $k$ (recall (3.7)), it follows that $|Z^j|$ (at a deterministic time $j$) is of order $j^{2/3}$. Therefore the first moment bound for $J_r$ will indeed suffice to control the $p \in [1, 3/2)$ moments of the maximum of $Z^j - Z^0$ up to time $J_r$.

Step 3: Complete the proof of Proposition 5.1 (Section 5.3). The bound (5.1) is equivalent to a moment bound for $\hat{Y}^{J_r}$. We will prove such a bound by first bounding the moments of $\hat{Y}^{T_k \wedge J_r}$, where $T_k$ is the $k$th time at which $\hat{Y}^{J_r}$ has a macroscopic jump (i.e., at least $r^2$ edges of $\partial Q_-$ or $\partial Q_+$ are disconnected from $\infty$ simultaneously). This is done in Lemma 5.9 using a recursive argument together with our upper bound for $Z^j$ and our bound for the moments of $\hat{Y}^{J_r}$ when we skip the big jumps (Lemma 4.5). We then conclude (5.1) using Lemma 4.6, which gives an upper bound for the number of big jumps. The bound (5.2) follows easily by writing $X^j \leq Z^j + 2\hat{Y}^j$ and using (5.1) and our bound for the maximum of $Z^j$.

As we mentioned earlier, in Section 5.4 we deduce a number of consequences of Proposition 5.1.

When reading the estimates in this section, it will be helpful to keep in mind that a radius $r$ metric ball in a uniformly random quadrangulation typically has outer boundary length of order $r^2$, the glued peeling
process up to radius $r$ typically reveals of order $r^3$ quadrilaterals, and the total number of quadrilaterals cut off from $\infty$ is typically of order $r^4$.

## 5.1 First moment bounds

In this subsection we will prove recursive bounds for the number $J_r$ of quadrilaterals revealed in the glued peeling cluster $\hat{Q}^r$ when $r$ layers have been peeled and the number $\hat{Y}^J_r$ of edges in $\partial Q_- \cup \partial Q_+$ which are contained in the glued peeling cluster, also when $r$ layers have been peeled. These bounds will eventually lead to the following first moment bound for $J_r$.

**Lemma 5.2.** For each $r \in \mathbb{N}$,

$$\mathbb{E}[J_r] \leq (r + (#A)^{1/2})^3$$

with universal implicit constant.

We emphasize that the exponent 3 on the right side of (5.8) is natural because the same power arises for the number of quadrilaterals revealed when one peels a radius-$r$ metric ball rooted at a vertex on the boundary of the UIHPQs or at the root vertex of the UIPQ.

To prove Lemma 5.2, we first prove a recursive bound for the conditional expectation of $J_r + 1 - J_r$ given $\mathcal{F}_r$ in terms of $X_r + Y_r$. This comes because the number of peeling steps necessary to cover a vertex on the boundary has a geometric distribution (Lemma 5.3), and in particular has finite expectation. We then prove a first moment bound for $X_r + Y_r$ using another recursive argument (Lemma 5.6). Combining these two lemmas and summing over $s \leq r$ will imply Lemma 5.2.

**Lemma 5.3.** There is a universal constant $c_1 > 0$ such that each $r \in \mathbb{N}_0$,

$$\mathbb{E}[J_r \mid \mathcal{F}_r] \leq J_r + c_1(X_r + Y_r).$$

**Proof.** For $v \in \mathcal{V}(\partial \hat{Q}^r \cap Q_r^+)$, let $\mathcal{I}_{v,\pm}^L$ (resp. $\mathcal{I}_{v,\pm}^R$) be the terminal time of the left (resp. right) one-vertex peeling process of $Q^+_{\pm}$ at $v$ (Definition 3.2). If $v \in \mathcal{V}(\partial \hat{Q}^r \cap Q_r^\pm)$, then every quadrilateral of $Q^+_{\pm}$ incident to $v$ which is peeled by the glued peeling process between times $J_r + 1$ and $J_{r+1}$ is peeled by either the left or right one-vertex peeling process of $Q^+_{\pm}$ at $v$. Furthermore, by definition every quadrilateral which is peeled by the glued peeling process started from $A$ and targeted at $\infty$ between times $J_r$ and $J_{r+1}$ is incident to some $v \in \mathcal{V}(\partial \hat{Q}^r)$. Therefore,

$$J_{r+1} - J_r \leq \sum_{\xi \in \{-, +\}} \sum_{v \in \mathcal{V}(\partial \hat{Q}^r \cap Q^+_{\xi})} (\mathcal{I}_{v,\xi}^L + \mathcal{I}_{v,\xi}^R).$$

By Lemmas 3.3 and 4.2, $\mathbb{E}[\mathcal{I}_{v,\pm}^L + \mathcal{I}_{v,\pm}^R \mid \mathcal{F}_r]$ is bounded above by a universal constant. Hence,

$$\mathbb{E}[J_{r+1} - J_r \mid \mathcal{F}_r] \leq \# \mathcal{V}(\partial \hat{Q}^r).$$

On the other hand,

$$\# \mathcal{V}(\partial \hat{Q}^r) \leq \# \mathcal{L}(\partial \hat{Q}^r) + 2 \leq X_r + \hat{Y}_r$$

where in the last inequality we recall that $\hat{Y}_r \geq \#A \geq 1$. The statement of the lemma follows since $\hat{Y}_r \geq X_r$ (recall (5.6)). \qed

**Lemma 5.4.** For $r \in \mathbb{N}_0$ we have $\mathbb{E}[Z_{r+1} \mid \mathcal{F}_r] = Z_r$.
Proof. By Lemma 5.3, we have $\mathbb{E}[J_{r+1} - J_{r} \mid \mathcal{F}^J_{r}] < \infty$ for each $r \in \mathbb{N}$. The discussion just after (5.7) tells us that $Z$ is a $\mathcal{F}^J$-martingale. By (3.7),
\[
\mathbb{E}[|Z^j - Z^{j-1}| \mid \mathcal{F}^{j-1}] = \mathbb{E}[|Z^1 - Z^0|] < \infty
\]
for each $j \in \mathbb{N}$. Therefore, the statement of the lemma follows from Lemma 5.3 and the optional stopping theorem applied to the martingale $\{Z^j\}_{j \geq 0}$ (see [Dur10, Theorem 5.75] for the precise statement we use here).

The following lemma gives us a recursive bound for $\hat{Y}^{J_r}$, which will be used to prove a moment bound for $X^{J_r} + Y^{J_r}$. The $(X^{J_r} + Y^{J_r})^{1/2}$ term which appears on the right side below comes from peeling the quadrilaterals on the boundary of $Q_{\pm}^{J_r}$. The specific power $1/2$ arises because we are taking the mean of a distribution which equals $k$ with probability of order $(\ell + k)^{-5/2}$, where $\ell$ is the distance from the peeled edge to $\partial Q_- \cup \partial Q_+$ along $\partial \hat{Q}^{J_r}$, and then summing $\ell$ over the boundary length of the glued peeling cluster (recall (3.7)).

**Lemma 5.5.** There is a universal constant $c_2 > 0$ such that for $r \in \mathbb{N}$,
\[
\mathbb{E}\left[\hat{Y}^{J_{r+1}} \mid \mathcal{F}^{J_r}\right] \leq \hat{Y}^{J_r} + c_2(X^{J_r} + \hat{Y}^{J_r})^{1/2}.
\]  
(5.10)

*Proof.* Define $\ell^{J_r}_{v, \pm}$ for $v \in \mathcal{V}(\partial \hat{Q}^{J_r} \cap \partial Q_{\pm}^{J_r})$ as in Definition 4.7. For $v \in \mathcal{V}(\partial \hat{Q}^{J_r} \cap \partial Q_{\pm}^{J_r})$ let $E^{J_r}_{v, \pm}$ be the set of edges of $\partial Q_{\pm}^{J_r} \cap \partial Q_{\pm}^{J_r}$ which are disconnected from $\infty$ in $Q_{\pm}^{J_r}$ by the union of the quadrilaterals of $Q_{\pm}^{J_r}$ incident to $v$.

Every edge of $\hat{Q}^{J_{r+1}} \cap (\partial Q_- \cup \partial Q_+)$ which does not belong to $Q_{\pm}^{J_r} \cap (\partial Q_- \cup \partial Q_+)$ belongs to $E^{J_r}_{v, -}$ or $E^{J_r}_{v, +}$ for some $v \in \mathcal{V}(\partial Q_{\pm}^{J_r})$. Therefore,
\[
\hat{Y}^{J_{r+1}} - \hat{Y}^{J_r} \leq \sum_{\xi \in \{-, +\}} \sum_{v \in \mathcal{V}(\partial Q_{\pm}^{J_r} \cap \partial Q_{\pm}^{J_r})} \#E^{J_r}_{v, \xi}.
\]  
(5.11)

We will now bound the right side of (5.11) using peeling estimates.

If $v \in \mathcal{V}(\partial \hat{Q}^{J_r} \cap \partial Q_{\pm}^{J_r})$ and $\#E^{J_r}_{v, \pm} \geq n$ for some $n \in \mathbb{N}$, then there are at least $n + \ell^{J_r}_{v, \pm}$ edges of $\partial Q_{\pm}^{J_r}$ which are disconnected from $\infty$ in $Q_{\pm}^{J_r}$ by the union of the quadrilaterals of $Q_{\pm}^{J_r}$ incident to $v$. Therefore, Lemma 3.1 implies that
\[
\mathbb{E}\left[\#E^{J_r}_{v, \pm} \mid \mathcal{F}^{J_r}\right] \leq \sum_{n=1}^{\infty} (n + \ell^{J_r}_{v, \pm})^{-3/2} \leq (\ell^{J_r}_{v, \pm})^{-1/2}.
\]
For each $m \in \mathbb{N}$, there are at most two elements of $\mathcal{V}(\partial \hat{Q}^{J_r} \cap Q_{\pm})$ with $\ell^{J_r}_{v, \pm} = m$. Hence
\[
\mathbb{E}\left[\hat{Y}^{J_{r+1}} - \hat{Y}^{J_r} \mid \mathcal{F}^{J_r}\right] \leq \sum_{\xi \in \{-, +\}} \sum_{v \in \mathcal{V}(\partial Q_{\pm}^{J_r} \cap \partial Q_{\pm}^{J_r})} \mathbb{E}\left[\#E^{J_r}_{v, \xi} \mid \mathcal{F}^{J_r}\right]
\]  
\[
\leq \sum_{\xi \in \{-, +\}} \sum_{v \in \mathcal{V}(\partial Q_{\pm}^{J_r} \cap \partial Q_{\pm}^{J_r})} (\ell^{J_r}_{v, \xi})^{-1/2} \leq \mathbb{V}(\partial \hat{Q}^{J_r})^{1/2},
\]
where the implicit constants in $\leq$ are universal. By combining this estimate with (5.9) we conclude.

From Lemmas 5.4 and 5.5 we obtain a first moment bound for $X^{J_r} + Y^{J_r}$. As we mentioned earlier, it is natural to expect that $Q^{J_r}$ is a good approximation for a filled $Q_{\text{zip}}$-metric ball of radius $r$ hence it is natural to expect that its boundary length should be of order $r^2$.

**Lemma 5.6.** For each $r \in \mathbb{N}_0$,
\[
\mathbb{E}[X^{J_r} + Y^{J_r}] \leq (r + (\#A)^{1/2})^2
\]
with universal implicit constant.
Proof. For \( j \in \mathbb{N}_0 \), let \( W^j := 4\tilde{Y}^j + Z^j \). Since \( Y^j \leq 2\tilde{Y}^j \) (recall (5.6)) and \( Z^j = X^j - Y^j \), we have \( W^j \geq X^j + Y^j \geq 0 \). By Lemmas 5.4 and 5.5, for \( r \in \mathbb{N} \),

\[
\mathbb{E}[W^{J_{r+1}} | F^{J_r}] \leq 4\tilde{Y}^{J_r} + Z^{J_r} + 4c_2(\tilde{Y}^{J_r} + X^{J_r})^{1/2} \leq W^{J_r} + c_3(W^{J_r})^{1/2}
\]

for \( c_3 > 0 \) a universal constant. Therefore

\[
\mathbb{E}[W^{J_{r+1}}] \leq \mathbb{E}[W^{J_r}] + c_3\mathbb{E}[W^{J_r}]^{1/2}
\]

where here we have used Hölder’s inequality to move the square root outside the expectation. Iterating this estimate yields

\[
\mathbb{E}[W^{J_r}] \leq c_3 \sum_{s=0}^{r-1} \mathbb{E}[W^{J_s}]^{1/2}.
\]

(5.12)

Since \( W^{J_0} = 2\#A \), we infer from (5.12) and induction that \( \mathbb{E}[W^{J_r}] < \infty \) for each \( r \in \mathbb{N} \). Therefore, (5.12) implies that

\[
\mathbb{E}[X^{J_r} + Y^{J_r}] \leq \mathbb{E}[W^{J_r}] \leq \left(r + (\#A)^{1/2}\right)^2.
\]

Finally, we deduce our expectation bound for \( J_r \).

Proof of Lemma 5.2. By Lemmas 5.3 and 5.6, for \( s \in \mathbb{N}_0 \) we have

\[
\mathbb{E}[J_{s+1} - J_s | F^{J_s}] \leq c_0\mathbb{E}[X^{J_s} + Y^{J_s}] \leq \left(s + (\#A)^{1/2}\right)^2.
\]

Summing from \( s = 0 \) to \( s = r - 1 \) yields the statement of the lemma.

5.2 Upper bound for the martingale

We next deduce from Lemma 5.2 a tail bound for \( Z^{J_r} \) which improves on the tail bound implied by Lemma 5.6.

Lemma 5.7. For each \( C > 1 \) and \( r \geq 1 \),

\[
\mathbb{P}\left[\max_{j \in [0,J_r]} (Z^j - Z^0) > Cr^2\right] \leq (\log C)^2 C^{-3/2}
\]

(5.13)

with universal implicit constant. In particular, for each \( p \in [1,3/2) \),

\[
\mathbb{E}\left[\left(\max_{j \in [0,J_r]} (Z^j - Z^0)\right)^p\right] \leq r^{2p}
\]

(5.14)

with implicit constant depending only on \( p \).

In the statement of Lemma 5.7, we recall that \( Z^0 = -2\#A \). The tail bound in (5.13) is natural because \( |Z^j| \) is of order \( j^{2/3} \) for a deterministic value of \( j \) and \( J_r \) is typically of order \( r^3 \). For the proof of the lemma, we will need the following basic tail bound for sums of i.i.d. random variables with heavy tails. (One can skip the proof of Lemma 5.8 on a first reading.)

Lemma 5.8. Let \( \alpha \in (1,2) \) and \( b > 0 \). Let \( \{X_j\}_{j \in \mathbb{N}} \) be a sequence of i.i.d. mean-zero random variables such that \( X_j \leq b \) a.s. and for \( r > 0 \), we have \( \mathbb{P}[X_j < -r] \sim r^{-\alpha} \). Let \( S_0 = 0 \) and for \( n \in \mathbb{N} \), let \( S_n := \sum_{j=1}^n X_j \). For \( C > 0 \) and \( n \in \mathbb{N} \),

\[
\mathbb{P}\left[\max_{m \in [1,n]} S_m \geq Cn^{1/\alpha}\right] \leq a_0 e^{-a_1 C}
\]

where \( a_0, a_1 > 0 \) are constants which do not depend on \( n \) or \( C \).
Proof. Let $I_0 = 0$ and for $k \in \mathbb{N}$, inductively let
\[
I_k := \min \{ j \geq I_{k-1} + 1 : S_j > S_{I_{k-1}} \}.
\]
Note that the vectors of random variables $(X_{I_{k-1}+1}, \ldots, X_{I_k})$ for $k \in \mathbb{N}$ are i.i.d. and we always have $S_{I_k} - S_{I_{k-1}} \in (0, b]$. Furthermore, $S_{I_k} = \max_{m \in [0, I_k]} S_m$. For $n \in \mathbb{N}$ and $t > 0$, let $H^n_t := n^{-1}S_{[nt]}$. By the classical scaling limit theorem for stable processes, $H^n \to H$ in law in the local Skorokhod topology, where $H$ is an $\alpha$-stable Lévy process with only downward jumps.

Let $\sigma^n := n^{-\alpha}I_n$ and for $s \geq 0$, let $\tau_s := \inf\{ t \geq 0 : H_t = s \}$. Then
\[
H^n_{\sigma^n} = \frac{1}{n} \sum_{k=1}^{\infty} (S_{I_k} - S_{I_{k-1}})
\]
so by the law of large numbers $H^n_{\sigma^n} \to \beta$ in probability, where $\beta := \mathbb{E}[S_1] \in (0, b]$. The time $\sigma^n$ is equal to the first time that $H^n$ hits $s^n$ for some random $s^n > 0$. Since the upward jumps of $H^n$ have size at most $\beta n^{-1}$, necessarily satisfies $s^n \to \beta$ in probability. Since $H$ has no upward jumps, we infer that $\sigma^n \to \tau_\beta$ in law.

By the converse to the heavy-tailed central limit theorem,
\[
\mathbb{P}[I_1 > s] \sim s^{-1/\alpha} \quad \text{as} \quad s \to \infty.
\]
If $\max_{m \in [1, n]} S_m \geq C n^{1/\alpha}$, then $I_{\lfloor (C-b)n^{1/\alpha} \rfloor} \leq n$. We therefore have (for an appropriate $n, C$-independent constant $\tilde{a} > 0$)
\[
\mathbb{P}\left[ \max_{m \in [1, n]} S_m \geq C n^{1/\alpha} \right] \leq \mathbb{P}\left[ \max_{k \in [1, \lfloor (C-b)n^{1/\alpha} \rfloor]} (I_k - I_{k-1}) \leq n \right] \leq \left( 1 - \tilde{a} n^{-1/\alpha} \right)^{-\lfloor (C-b)n^{1/\alpha} \rfloor} \wedge 1 \leq a_0 e^{-a_1 C},
\]
with $a_0, a_1 > 0$ as in the statement of the lemma.

Proof of Lemma 5.7. Recall from the discussion just after (3.7) that the increments $Z^j - Z^{j-1}$ for $j \in \mathbb{N}$ are i.i.d. with zero mean. Furthermore, $Z^j - Z^{j-1} \leq 2$ a.s. and by (3.7), for $s \in \mathbb{N}$
\[
\mathbb{P}[Z^j - Z^{j-1} < -s] \sim s^{-3/2}.
\]
By Lemma 5.8, for $n \in \mathbb{N}$ and $A > 0$ we have
\[
\mathbb{P}\left[ \max_{j \in [0, n]} (Z^j - Z^0) > An^{2/3} \right] \leq e^{-a_1 A}
\]
for $a_1 > 0$ a universal constant.

If we are given $C > 1$ and we set $N = \lfloor (\log C)^{-2C^3/2} \rfloor$, then by (5.15)
\[
\mathbb{P}\left[ \max_{j \in [0, n]} (Z^j - Z^0) > Cr^2, J_r \leq N \right] \leq \mathbb{P}\left[ \max_{j \in [1, N]} (Z^j - Z^0) > (\log C)^{4/3} N^{2/3} \right] \leq e^{-a_1 (\log C)^{4/3}} \leq C^{-3/2}.
\]
On the other hand, Lemma 5.2 and the Chebyshev inequality imply that
\[
\mathbb{P}[J_r > N] \leq (\log C)^2 C^{-3/2}.
\]
The estimate (5.13) now follows from a union bound.

The moment bound (5.14) follows from (5.13) and the formula
\[
\mathbb{E}[W^p] = \int_0^\infty pt^{p-1} \mathbb{P}[W \geq t] dt
\]
applied to the non-negative random variable $W = r^{-2} \max_{j \in [0, J_r]} (Z^j - Z^0)$. \qed
5.3 Proof of Proposition 5.1

We now turn to complete the proof of Proposition 5.1. In view of the results of Section 4.2, in which we bounded the moments of the number \( \hat{Y}^j \) of edges of \( \partial Q_\pm \cup \partial Q_+ \) cut off from \( \infty \) by the glued peeling cluster \( Q^j \) after truncating away the macroscopic jumps, the main input in the proof of (5.1) in Proposition 5.1 is the following bound for the \( p \)th moments of \( \hat{Y} \) stopped at the times when it makes a macroscopic jump. These macroscopic jumps occur whenever the glued peeling cluster cuts off a macroscopic region from \( \infty \) upon revealing a quadrilateral which is adjacent to the gluing interface.

**Lemma 5.9.** Suppose \( c > 1 \) and \( r \in \mathbb{N} \). Let \( T_0 = T_0(c^2) = 0 \) and for \( k \in \mathbb{N} \) let \( T_k = T_k(c^2) \) be the \( k \)th smallest \( j \in \mathbb{N} \) for which \( \hat{Y}^j - \hat{Y}^{j-1} \geq cr^2 \). For each \( p \in [1, 3/2) \), there exists a constant \( A_p \geq 1 \), depending only on \( p \), such that for each \( r \in \mathbb{N} \), each \( c > 1 \) and each \( k \in \mathbb{N} \),

\[
\mathbb{E}\left[ \left( \hat{Y}^{T_k \wedge J_r} \right)^p \right] \leq A_p^k c^p \left( r + \#A^{1/2} \right)^{2p}
\]

with implicit constant depending only on \( p \).

The key point of Lemma 5.9 is that \( A_p \) and the implicit constant in (5.16) do not depend on \( c \). As we will see below, choosing \( c > 1 \) sufficiently large and applying Lemma 4.6 will allow us to cancel out the exponential factor \( A_p^k \) in (5.16) using the fact that the largest \( k \) for which \( T_k \leq J_r \) has an exponential tail (Lemma 4.6).

**Proof of Lemma 5.9.** We will prove the lemma by deriving a recursive bound for \( \mathbb{E}\left[ \left( \hat{Y}^{T_k \wedge J_r} \right)^p \right] \) in terms of \( \mathbb{E}\left[ \left( \hat{Y}^{T_{k-1} \wedge J_r} \right)^p \right] \). For \( k \in \mathbb{N} \) let \( \ell^T_k \) be the \( \partial Q_{T_k}^{T^j_{k-1}} \)-graph distance from the \( T_k \)th peeled edge \( \ell^T_k \) to \( \mathcal{E}(\partial Q_{T_k}^{T^j_k}) \setminus \mathcal{E}(\hat{Q}^{T_k}) \). Note that

\[
\ell^T_k \leq X^{T_k-1} + Y^{T_k-1} \leq Z^{T_k-1} + 4\hat{Y}^{T_k-1}.
\]

If \( k \in \mathbb{N} \) and we condition on \( \sigma(T_k) \vee \mathcal{F}^{T_k-1} \), then the conditional law of the \( T_k \)th peeling step is the same as its conditional law given that the peeled quadrilateral \( f(Q_{T_k}^{T^j_k}, \ell^T_k) \) disconnects at least \( cr^2 \) edges in \( \mathcal{E}(\partial Q_{T_k}^{T^j_k}) \setminus \mathcal{E}(\hat{Q}^{T_k}) \) from \( \infty \) in \( Q_{T_k}^{T^j_k} \). This is the case provided \( f(Q_{T_k}^{T^j_k}, \ell^T_k) \) disconnects at least \( \ell^T_k + cr^2 \) edges of \( \partial Q_{T_k}^{T^j_k} \) lying either to the left or to the right of \( \ell^T_k \) (where the choice is \( \mathcal{F}^{T_k-1} \)-measurable) from \( \infty \) in \( Q_{T_k}^{T^j_k} \). By (3.7), for \( m \in \mathbb{N} \) with \( m \geq cr^2 \) we have

\[
\mathbb{P}\left[ \hat{Y}^{T_k} - \hat{Y}^{T_{k-1}} \geq m \mid \sigma(T_k) \vee \mathcal{F}^{T_k-1} \right] \leq \left( \ell^T_k + cr^2 \right)^{3/2} (m + \ell^T_k)^{-3/2}.
\]

Therefore,

\[
\mathbb{E}\left[ \left( \hat{Y}^{T_k} - \hat{Y}^{T_{k-1}} \right)^p \mid \sigma(T_k) \vee \mathcal{F}^{T_k-1} \right] \leq \left( \ell^T_k + cr^2 \right)^{3/2} \sum_{m=\lfloor cr^2 \rfloor}^{\infty} m^{p-1} (m + \ell^T_k)^{-3/2}
\leq \left( \ell^T_k + cr^2 \right)^{3/2} \sum_{m=\lfloor cr^2 \rfloor}^{\infty} (m + \ell^T_k)^{-5/2}
\leq \left( \ell^T_k + cr^2 \right)^p \leq (Z_{T_{k-1}} \vee 0)^p + \left( \hat{Y}^{T_{k-1}} \right)^p + c^p r^{2p}.
\]

(5.17)

If \( T_k > J_r \), then \( \hat{Y}^{T_k \wedge J_r} = \hat{Y}^{T_{k-1} \wedge J_r} = 0 \). Hence (5.17) implies that

\[
\mathbb{E}\left[ \left( \hat{Y}^{T_k \wedge J_r} \right)^p \mid \sigma(T_k) \vee \mathcal{F}^{T_k-1} \right] \leq \mathbb{E}\left[ \left( \hat{Y}^{T_{k-1} \wedge J_r} \right)^p + \left( \hat{Y}^{T_k \wedge J_r} - \hat{Y}^{T_{k-1} \wedge J_r} \right)^p \mid \sigma(T_k) \vee \mathcal{F}^{T_k-1} \right]
\leq \left( \hat{Y}^{T_{k-1} \wedge J_r} \right)^p + \left( Z_{T_{k-1} \wedge J_r} \vee 0 \right)^p + \left( \hat{Y}^{T_{k-1} \wedge J_r} \right)^p + c^p r^{2p} \mathbbm{1}_{(T_k \leq J_r)}.
\]

(5.18)
By Lemma 5.7,
\[ \mathbb{E} \left[ (Z^{(T_k-1)\wedge J_r} \lor 0)^p \right] \leq r^{2p}. \] (5.19)
In the notation of (4.6), \( Y^{(T_k-1)\wedge J_r} - Y^{T_k-1\wedge J_r} \leq \hat{Y}_{cr} \) so by Lemma 4.5,
\[ \mathbb{E} \left[ (\hat{Y}^{(T_k-1)\wedge J_r})^p \right] \leq \mathbb{E} \left[ (\hat{Y}^{T_k-1\wedge J_r})^p \right] \leq \mathbb{E} \left[ (\hat{Y}^{(T_k-1)\wedge J_r})^p \right] + c r^{2p}. \] (5.20)
Taking expectations of both sides of (5.18) (ignoring the indicator function) and plugging in the estimates (5.19) and (5.20) gives
\[ \mathbb{E} \left[ (\hat{Y}^{T_k\wedge J_r})^p \right] \leq \mathbb{E} \left[ (\hat{Y}^{T_k\wedge J_r})^p \right] + c r^{2p}, \] (5.21)
implicit constants depending only on \( p \). We have \( \hat{Y}^{T_0\wedge J_r} = \hat{Y}^0 = \#A \). Hence solving the recurrence (5.21) yields (5.16).

**Proof of Proposition 5.1.** Fix \( p \in [1, 3/2) \) and \( r \in \mathbb{N} \). Let \( c > 1 \) be chosen later, depending on \( p \) and for \( k \in \mathbb{N} \) let \( T_k = T_k(cr^2) \) be the \( k \)th largest \( j \in \mathbb{N} \) for which \( \hat{Y}^j - \hat{Y}^{j-1} \geq cr^2 \), as in Lemma 5.9. Also let \( K_r = K_r(cr^2) \) be the largest \( k \in \mathbb{N} \) for which \( T_k \leq J_r \) and let \( \hat{Y}_{cr}^{J_r} \) be the sum of the truncated jumps be as in (4.6). For each \( p \in [1, 3/2) \),
\[ (\hat{Y}^{J_r})^p \leq (\hat{Y}_{cr}^{J_r})^p + (\hat{Y}^{K_r})^p \]
with implicit constant depending only on \( p \). By this and Lemma 4.5,
\[ \mathbb{E} \left[ (\hat{Y}^{J_r})^p \right] \leq c r^{2p} + \mathbb{E} \left[ (\hat{Y}^{K_r})^p \right], \] (5.22)
so it remains to bound \( \mathbb{E} \left[ (\hat{Y}^{K_r})^p \right] \).

To this end, let \( q \in (1, \frac{3}{2} p^{-1}) \). By Hölder’s inequality,
\[ \mathbb{E} \left[ (\hat{Y}^{K_r})^p \right] \leq \mathbb{E} \left[ \left( \sum_{k=1}^{K_r} \hat{Y}^{T_k \wedge J_r} \right)^q \right] \leq \sum_{k=1}^{K_r} \mathbb{E} \left[ (\hat{Y}^{T_k \wedge J_r})^p \right] \]
\[ \leq \sum_{k=1}^{\infty} \mathbb{E} \left[ K_r^{qp} \mathbbm{1}_{(K_r \geq k)} \right] \left( \sum_{k=1}^{\infty} \mathbb{E} \left[ (\hat{Y}^{T_k \wedge J_r})^q \right] \right)^{\frac{1}{q}}. \] (5.23)
By Lemma 4.6, there is a universal constant \( a > 0 \) such that the law of \( K_r \) is stochastically dominated by that of a geometric random variable with parameter \( ac^{-1/2} \). Consequently, if we take \( c > a^3 \), say, then
\[ \mathbb{E} \left[ K_r^{qp} \mathbbm{1}_{(K_r \geq k)} \right] \left( \sum_{k=1}^{\infty} \mathbb{E} \left[ (\hat{Y}^{T_k \wedge J_r})^q \right] \right)^{\frac{1}{q}} \leq c^{-bk} \]
with \( b > 0 \) and the implicit constant depending only on \( p \) and \( q \). By Lemma 5.9,
\[ \mathbb{E} \left[ (\hat{Y}^{T_k \wedge J_r})^q \right] \leq A_{qp} c^p \left( r + (\#A)^{1/2} \right)^{2p} \]
with \( A_{qp} > 1 \) and the implicit constant depending only on \( p \) and \( q \). If we choose \( c \) sufficiently large that \( c^b > A_{qp}^{1/2} \), then (5.23) implies that
\[ \mathbb{E} \left[ (\hat{Y}^{K_r})^p \right] \leq \left( r + (\#A)^{1/2} \right)^{2p}, \]
where now the implicit constant is also allowed to depend on \( c \). By combining this with (5.22) we obtain (5.1).

Next we deduce the boundary length estimate (5.2) from (5.1). For \( j \in [1, J_r]_{\mathbb{Z}} \),
\[ \# \mathcal{E} (\partial Q^j) \leq X^j + \hat{Y}^j \leq \max_{j \in [1, J_r]_{\mathbb{Z}}} (Z^j - Z^0) + 3 \hat{Y}^j, \]
where here we have used that \( X^j = Z^j + Y^j \leq Z^j + 2 \hat{Y}^j \), that \( Z^0 = -2 \#A < 0 \), and that \( \hat{Y}^j \) is monotone non-decreasing. We have a \( p \)th moment bound for \( \hat{Y}^J \) by (5.1) and a \( p \)th moment bound for \( \max_{j \in [1, J_r]_{\mathbb{Z}}} (Z^j - Z^0) \) by Lemma 5.7. 
\[ \square \]
5.4 Consequences of the moment bound

In this subsection we will deduce some consequences of Proposition 5.1 which will play an important role in later sections for controlling the large scale geometry of the gluing interface and are also of independent interest.

5.4.1 Reverse Hölder continuity estimate for the curve

Here we prove a reverse Hölder continuity estimate for the boundary path \( \lambda_\pi \) of \( Q_\pi \) with respect to the graph metric on \( Q_{\text{zip}} \), which will eventually be used to show that the gluing interface for any subsequential scaling limit of the maps \( Q_{\text{zip}} \) is a simple curve. We note that \( \lambda_\pi|_{\mathcal{N}_0} \) coincides with the SAW \( \lambda_{\text{zip}} \) and that (by symmetry) the same estimate is true with \( \lambda_\pi \) in place of \( \lambda_\pi \).

**Lemma 5.10.** Fix \( L > 0 \). For \( \delta \in (0,1) \) and \( \beta \in (0,2/3) \),

\[
\mathbb{P} \left[ \text{dist}(\lambda_\pi(x), \lambda_\pi(y); Q_{\text{zip}}) \geq \delta r, \forall x, y \in [-Lr^2, Lr^2]_{\mathbb{Z}} \text{ with } |x - y| \geq \delta^\beta r^2 \right] \geq 1 - \delta^{\frac{2}{\beta - \delta}} - o_5(1) \tag{5.24}
\]

with the rate of the \( o_5(1) \) depending only on \( L \) and \( \beta \).

**Proof.** The idea of the proof is to use Proposition 5.1 and a union bound to cover \( \lambda_\pi([-Lr^2, Lr^2]_{\mathbb{Z}}) \) by graph metric balls which do not contain any points of \( \partial Q_\pi \cup \partial Q_{\text{zip}} \), which are unusually far apart. For this purpose, the fact that we get a moment of order \( \beta \) in Proposition 5.1 is essential.

For \( \delta \in (0,1) \) and \( x \in [-Lr^2, Lr^2]_{\mathbb{Z}} \cap (\delta r^2 \mathbb{Z}) \), let \( E_\delta(x) \) be the event that the \( Q_{\text{zip}} \)-graph metric neighborhood \( B_{\delta r}(\lambda_\pi([-x - \delta^2 r^2, x]_{\mathbb{Z}}); Q_{\text{zip}}) \) does not contain \( \lambda_\pi(y) \) for any \( y \in \mathbb{Z} \) with \( |x - y| \geq (\delta^\beta - \delta)r^2 \). Also let

\[
E_\delta^c := \bigcap_{x \in [-Lr^2, Lr^2]_{\mathbb{Z}} \cap (\delta r^2 \mathbb{Z})} E_\delta(x).
\]

If \( E_\delta^c \) occurs, then by Lemma 4.3 we can find \( y \in \mathbb{Z} \) such that \( |x - y| \geq \delta^\beta r^2 \) and \( \lambda_\pi(y) \) belongs to the glued peeling cluster started from \( \mathcal{A} = \lambda_\pi([-x - \delta^2 r^2, x]_{\mathbb{Z}}) \) grown up to time \( J_{\delta r} \). Since a glued peeling cluster contains every edge of \( Q_\pi \) which it disconnects from \( \infty \), it follows that this cluster contains at least \( (\delta^\beta - \delta)r^2 \) edges of \( \partial Q_\pi \). By Proposition 5.1 and the Chebyshev inequality, for each \( p \in [1,3/2] \)

\[
\mathbb{P}[E_\delta^c] \leq \delta(2 - \beta)p^2,
\]

with the implicit constant depending only on \( p \). By the union bound,

\[
\mathbb{P}[(E_\delta^c)^c] \leq \delta(2 - \beta)p^{-2}
\]

with the implicit constant depending only on \( p \) and \( L \). Sending \( p \to 3/2 \) gives \( \mathbb{P}[(E_\delta^c)^c] \leq \delta^{\frac{2}{\beta - \delta}} - o_5(1) \), which tends to 0 as \( \delta \to 0 \) provided \( \beta < 2/3 \).

On the other hand, suppose \( E_\delta \) occurs and we are given \( x \in [-Lr^2, Lr^2]_{\mathbb{Z}} \). Choose \( x' \in [-Lr^2, Lr^2]_{\mathbb{Z}} \cap (\delta r^2 \mathbb{Z}) \) for which \( x \in [-x - \delta^2 r^2, x]_{\mathbb{Z}} \). Then

\[
B_{\delta r}(\lambda_\pi(x); Q_{\text{zip}}) \supset B_{\delta r}(\lambda_\pi([-x' - \delta^2 r^2, x']_{\mathbb{Z}}); Q_{\text{zip}})
\]

does not contain \( \lambda_\pi(y) \) for any \( y \in \mathbb{Z} \) with \( |x - y| \geq \delta^\beta r^2 \). \( \square \)

5.4.2 Hölder continuity for distances along the boundary

For our next two results (and at several later points in the paper) we need the following bound for the modulus of continuity of distances along the boundary of the UIHPQ\( Q \), which follows from the scaling limit result for the UIHPQ\( Q \) in [GM16b]. We note that the natural scaling for distances is \( r^{-1} \) while the natural scaling of boundary lengths is \( r^{-2} \).

**Lemma 5.11.** Let \((Q_\pi, \mathcal{E}_S)\) be an instance of the UIHPQ\( Q \) and let \( \lambda_S : \mathbb{Z} \to \mathcal{E}(\partial Q) \) be its boundary path. For each \( \alpha \in (0,1) \) and each \( L > 0 \), there exists \( C = C(\alpha, L) > 0 \) such that the following is true. For each \( \epsilon > 0 \), there exists \( r_* = r_*(\alpha, L, \epsilon) > 0 \) such that for \( r \geq r_* \),

\[
\mathbb{P} \left[ \frac{1}{r} \text{dist}(\lambda_S(x), \lambda_S(y); Q_\pi) \leq C \frac{x - y}{r^2} \left( \log \left( \frac{r^2}{|x - y|} \right) \right)^{\frac{1}{2}} + \epsilon, \forall x, y \in [-Lr^2, Lr^2]_{\mathbb{Z}} \right] \geq 1 - \alpha. \tag{5.25}
\]
The same holds (with a larger constant $C$) if we replace graph distances in $Q_S$ with (internal) graph distances in $B_r(\lambda_S([x, y]); Q_S)$.

Proof. Since the UIHPQ$_S$ converges to the Brownian half-plane in the local GHP topolgy [GM16b, Theorem 1.12], the first statement follows from the bound [GM16a, Lemma 3.2] for distances along the boundary of the Brownian disk and local absolute continuity of the Brownian half-plane with respect to the Brownian disk [GM16b, Proposition 4.2]. The second statement follows from the first by concatenating at most $CL^{1/2}$ paths of length at most $r$ between elements of $\lambda_S([x, y]; Q_S)$ to get a path from $x$ to $y$ which stays in $B_r(\lambda_S([x, y]); Q_S)$. \hfill $\square$

### 5.4.3 Comparison of two-sided and one-sided metric balls

In this subsection we will prove an estimate for $Q_{\text{zip}}$-metric balls in term of $Q_{\pm}$-metric balls.

**Lemma 5.12.** For each $\epsilon > 0$, there exists $R = R(\epsilon) > 0$ such that for each $r \in \mathbb{N}$ and each edge $e \in \mathcal{E}(\partial Q_-) \cap \mathcal{E}(\partial Q_+)$ chosen in some manner which depends only on $\partial Q_- \cup \partial Q_+$,

$$\mathbb{P}[B_r(e; Q_{\text{zip}}) \subset B_{Rr}(e; Q_-) \cup B_{Rr}(e; Q_+)] \geq 1 - \epsilon. \quad (5.26)$$

**Proof.** Let $\hat{Q}_r^+ \subset Q_r^+$ be the radius-$r$ peeled graph cluster with initial edge set $A = \{e\}$. By Lemma 4.3, $B_r(e; Q_{\text{zip}}) \subset \hat{Q}_r^+$. Choose $x_\pm \in \mathbb{N}_0$ such that $\lambda_\pm(x_\pm) = e$. By Proposition 5.1, there exists $L = L(\epsilon) > 0$ such that with probability at least $1 - \epsilon/2$,

$$\hat{Q}_r^+ \cap \partial Q_- \subset \lambda_-([x_- - Lr^2, x_- + Lr^2]; Q_-) \quad (5.27)$$

and the same is true with “+” in place of “−.” By Lemma 5.11, there exists $\rho = \rho(\epsilon) > 0$ such that with probability at least $1 - \epsilon/2$,

$$\operatorname{diam}(\lambda_-([x_- - Lr^2, x_- + Lr^2]; Q_-)) \leq \rho r \quad (5.28)$$

and the same is true with “+” in place of “−.”

Any vertex or edge in $B_r(e; Q_{\text{zip}})$ can be connected to $e$ by a path in $B_r(e; Q_{\text{zip}})$ of length at most $r$. By considering the segment of this path run until it first hits $\partial Q_- \cup \partial Q_+$, we see that every vertex or edge in $B_r(e; Q_{\text{zip}}) \cap Q_\pm$ lies at $Q_\pm$-graph distance at most $r$ from $B_r(e; Q_{\text{zip}}) \cap \partial Q_\pm$. Hence if (5.27) and (5.28) hold, then

$$B_r(e; Q_{\text{zip}}) \subset B_r(B_r(e; Q_{\text{zip}}) \cap \partial Q_-; Q_-) \cup B_r(B_r(e; Q_{\text{zip}}) \cap \partial Q_+; Q_+) \subset B_{(\rho + 1)r}(e; Q_-) \cup B_{(\rho + 1)r}(e; Q_+).$$

This happens with probability at least $1 - \epsilon$, so the statement of the lemma is satisfied with $R = \rho + 1$. \hfill $\square$

### 5.4.4 Lower bound for distances in a small neighborhood of the SAW

The last result of this section is a lower bound for the length of a path in $Q_{\text{zip}}$ which stays in a small neighborhood of $\partial Q_- \cup \partial Q_+$ (which we recall contains the SAW $\lambda_{\text{zip}}$ in the case when $z = x_- = x_+$, so that $Q_{\text{zip}}$ has no hole). This statement will be used in the proof of Proposition 6.9 to show that a $Q_{\text{zip}}$-geodesic is unlikely to spend too much time near $\partial Q_- \cup \partial Q_+$.

**Lemma 5.13.** Fix $L > 0$. For $\rho > 0$ and $r \in \mathbb{N}$, let $d^*_\rho$ be the (internal) graph metric on $B_{\rho r}(\lambda_-(\{-Lr^2, Lr^2\})$).

For each $\alpha, \zeta \in (0, 1)$, there exists $\rho_\alpha = \rho_\alpha(\alpha, \zeta) \in (0, 1)$ such that for each $r \in \mathbb{N}$ and each $\rho \in (0, \rho_\alpha)$, it holds with probability at least $1 - \alpha$ that

$$r^{-1} d^*_\rho(\lambda_-(x), \lambda_-(y)) \geq \rho^{-1+\zeta} \left| \frac{x - y}{r^2} \right|^{3+\zeta}, \quad \forall x, y \in [-Lr^2, Lr^2]. \quad (5.29)$$

The basic idea of the proof of Lemma 5.13 is to use Proposition 5.1 to bound for each $k \in \mathbb{N}$ the number of $Q_{\text{zip}}$-metric balls of radius $\rho r$ which contain edges of $\lambda_-(\{-Lr^2, Lr^2\})$ separated by a boundary arc of $Q_- \cup \partial Q_+$ of length greater than $2^k \rho r^2$ (Lemma 5.14). This tells us that when $\rho$ is small, $d^*_\rho$-graph distances can be bounded below in terms of the lengths of boundary arcs of $Q_- \cup \partial Q_+$. We note that we need to consider dyadic scales rather than just taking a union bound and looking at the largest possible separation between edges of $\lambda_-(\{-Lr^2, Lr^2\})$ which lie at $Q_{\text{zip}}$-distance at most $\rho r$ from each other (as in the proof of Lemma 5.10) since the latter approach does not yield a lower bound for distances which tends to $\infty$ as $\rho \to 0$. 

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Lemma 5.14. Fix $L > 0$, $p \in (1, 3/2)$, and $\zeta \in (0, p - 1)$. For $r \in \mathbb{N}$, $\rho \in (0, 1)$, and $k \in \mathbb{Z}$, let $A^r_\rho(k)$ be the set of $x \in [-Lr^2, Lr^2]_\mathbb{Z}$ for which
\[
\max\{|x - y| : \lambda_-(y) \in B_{100\rho r}(\lambda_-(x); Q_{zip})\} \geq 2^kr^2r^2.
\] (5.30)

Also let
\[
E^r_\rho := \left\{ \#A^r_\rho(k) \leq 2^{-(p-\tilde{\zeta})kr^2}, \forall k \in \mathbb{N} \text{ with } 2^{-(p-1-\tilde{\zeta})k} \leq 2L\rho^\zeta \right\}.
\] (5.31)

For each $\alpha \in (0, 1)$, there exists $\rho_0 \in (0, 1)$ such that
\[
\mathbb{P}[E^r_\rho] \geq 1 - \alpha, \quad \forall \rho \in (0, \rho_0].
\] (5.32)

Proof. By Lemma 4.3, Proposition 5.1 (applied with $\lfloor \rho r \rfloor$ in place of $r$), and the Chebyshev inequality, for each fixed $x \in [-Lr^2, Lr^2]_\mathbb{Z}$,
\[
\mathbb{P}[x \in A^r_\rho(k)] \leq \frac{\rho2^{2p}\rho^{2p}}{2^{pk}\rho^{2p+2p}} = 2^{-pk}
\]
with implicit constant depending only on $p$. Therefore,
\[
\mathbb{E}[\#A^r_\rho(k)] \leq 2^{-pk}r^2,
\] (5.33)
with the implicit constant depending only on $p$ and $L$. We obtain (5.32) for small enough $\rho_0$ by applying the Chebyshev inequality to $\#A^r_\rho(k)$ for each $k \in \mathbb{N}$ with $2^{-(p-1-\tilde{\zeta})k} \leq 2L\rho^\zeta$ then taking a union bound. \hfill \Box

Proof of Lemma 5.15. Fix $L > 0$, $p \in (1, 3/2)$, and $\zeta \in (0, p - 1)$. For $\rho \in (0, 1)$ and $r \in \mathbb{N}$, let
\[
G^r_\rho := \left\{ \text{dist}(\lambda_-(x), \lambda_-(y); Q-x) \leq \rho r, \forall x, y \in [-Lr^2, Lr^2]_\mathbb{Z} \text{ with } |x - y| \leq \rho^2 + \tilde{\zeta}r^2 \right\}
\] (5.34)
and let $E^r_\rho$ be the event of Lemma 5.14. By Lemmas 5.11 and 5.14, there exists $\rho_1 = \rho_1(\alpha, p, \tilde{\zeta}) \in (0, 1)$ such that
\[
\liminf_{r \to \infty} \mathbb{P}[E^r_\rho \cap G^r_\rho] \geq 1 - \alpha, \quad \forall \rho \in (0, \rho_1].
\] (5.35)

By possibly shrinking the value of $\rho_*$ in the statement of the lemma to deal with finitely many small values of $r$, it suffices to show that for an appropriate choice of $p$ and $\zeta$ depending only on $\zeta$, (5.29) holds on $E^r_\rho \cap G^r_\rho$ for small enough $\rho \in (0, \rho_1]$ (depending only on $p, \tilde{\zeta}, L$, and $\alpha$).

Henceforth assume that $E^r_\rho \cap G^r_\rho$ occurs. Let $x, y \in [-Lr^2, Lr^2]_\mathbb{Z}$, and let $\gamma_{x,y} : [0, d^r_\rho(\lambda_-(x), \lambda_-(y))] \to \mathcal{E}(Q_{zip})$ be a $d^r_\rho$-geodesic from $\lambda_-(x)$ to $\lambda_-(y)$. We will prove a lower bound for the length of $\gamma_{x,y}$. Write
\[
N := \left\lceil \frac{1}{\rho r} d^r_\rho(\lambda_-(x), \lambda_-(y)) \right\rceil.
\] (5.36)

By definition of $d^r_\rho$, for each $j \in [1, N - 1]_\mathbb{Z}$ there exists $z_j \in [-Lr^2, Lr^2]_\mathbb{Z}$ such that
\[
\text{dist}(\lambda_-(z_j), \gamma_{x,y}([\rho r j])); Q_{zip}) \leq \rho r.
\] (5.37)

Set $z_0 = x$ and $z_N = y$, so that (5.37) holds for all $j \in [0, N]_\mathbb{Z}$ but with $2\rho$ in place of $\rho$ on the right. By the triangle inequality and since $\gamma_{x,y}$ is a geodesic for the metric $d^r_\rho$ (which is dominates the graph metric on $Q_{zip}$), for $j \in [1, N]_\mathbb{Z}$ we have
\[
\text{dist}(\lambda_-(z_{j-1}), \lambda_-(z_j); Q_{zip}) \leq 4\rho r + d^r_\rho(\gamma_{x,y}([\rho r(j - 1)]), \gamma_{x,y}([\rho r j])) \leq 6\rho r.
\] (5.38)

For $k \in \mathbb{Z}$, let
\[
X^k := \left\{ j \in [1, N]_\mathbb{Z} : 2^{k(6\rho r)^2} \leq |z_j - z_{j-1}| \leq 2^{k+1}(6\rho r)^2 \right\}.
\] (5.39)
If $z \in [-Lr^2, Lr^2]_\mathbb{Z}$ with $\lambda_-(z) \in B_{\rho r}(\lambda_-(z_j)); Q_{zip}$, then by (5.38) and the triangle inequality, $\lambda_-(z)$ lies at $Q_{zip}$-graph distance at most $7\rho r$ from each of $z_j$ and $z_{j-1}$. By the definition of $X^k$, if $j \in X^k$ then either $|z - z_j|$ or $|z - z_{j-1}|$ is at least $2^k(6\rho r)^2$, so $z \in A^r_\rho(k)$ (defined as in (5.30)).
Since we have assumed that $G^*_\rho$ occurs, for each $j \in [1, N]_\mathbb{Z}$ either

$$\lambda_-(\left[ z_j, z_j + \rho^2 + \tilde{\zeta}r^2 \right]_{\mathbb{Z}}) \quad \text{or} \quad \lambda_-(\left[ z_j - \rho^2 - \tilde{\zeta}r^2, z_j \right]_{\mathbb{Z}})$$

is contained in $B_{pr}(\lambda_-(z_j); Q_{zip})$, whence

$$\# \{ B_{pr}(\lambda_-(z_j); Q_{zip}) \cap \lambda_-([-Lr^2, Lr^2]_{\mathbb{Z}}) \} \geq \rho^2 + \tilde{\zeta}r^2. \quad (5.40)$$

By the discussion just after (5.39), whenever $j \in X^k$ and $z$ is such that $\lambda_-(z)$ belongs to the set in (5.40), it holds that $z \in A^r_j(k)$. Furthermore, by the triangle inequality and since $\gamma_{z,y}$ is a geodesic, each of the sets in (5.40) for $j \in [1, N]_\mathbb{Z}$ intersects at most 6 other such sets. Therefore,

$$\# A^r_j(k) \geq \frac{1}{6} \rho^2 + \tilde{\zeta}r^2 \# X^k.$$ 

Recalling the definition (5.31) of $E^*_\rho$, we find that for each $k \in \mathbb{N}$ with $2^{-(p-1-\tilde{\zeta})k} \leq 2L\rho\tilde{\zeta}$,

$$\# X^k \lesssim 2^{-(p-\tilde{\zeta})k} \rho^{-2-\tilde{\zeta}}. \quad (5.41)$$

To lighten notation, set $s := p - 1 - \tilde{\zeta}$. Fix a small constant $c \in (0, 1)$, to be chosen later, and let $k_0 \in \mathbb{N}$ be chosen so that

$$2^{-s k_0} r^2 \leq c \rho^2 |x - y| \leq 2^{-s(k_0 - 1)} r^2. \quad (5.42)$$

Since $|x - y| \leq 2L r^2$, we have $2^{-s k_0} \leq 2L \rho \tilde{\zeta}$. By breaking up the sum based on the value of $k$ for which $j \in X^k$ and applying (5.41), we get

$$|x - y| \leq \sum_{j \in [1, N]_\mathbb{Z}} |z_j - z_j - 1| \leq 2^{k} \rho^2 r^2 \# X^k + 2^{k_0} \rho^2 r^2 N \leq \rho^{-\tilde{\zeta}} \sum_{k = k_0}^{\infty} 2^{-s k} r^2 + 2^{k_0} \rho^2 r^2 N \leq \rho^{-\tilde{\zeta}} 2^{-s k_0} r^2 + 2^{k_0} \rho^2 r^2 N \leq c|x - y| + 2^{k_0} \rho^2 r^2 N,$$

with implicit constant depending only on $p$, $\tilde{\zeta}$, and $L$. If we choose $c$ sufficiently small, depending only on $p$, $\tilde{\zeta}$, and $L$, then we can re-arrange to get $|x - y| \leq 2^{k_0} \rho^2 r^2 N$, with the implicit constant depending on $p$, $\tilde{\zeta}$, and $L$. Recalling the definitions of $k_0$ and $N$ from (5.36) and (5.42) we see that this implies that

$$\frac{|x - y|}{r^2} \leq \rho^{1 - \frac{\tilde{\zeta}}{2}} \left| \frac{x - y}{r^2} \right|^{-\frac{1}{2}} \left( r^{-1} d^*_\rho(\lambda_-(x), \lambda_-(y)) \right)$$

and hence

$$r^{-1} d^*_\rho(\lambda_-(x), \lambda_-(y)) \geq \rho^{-1 + \frac{\tilde{\zeta}}{2}} \left| \frac{x - y}{r^2} \right|^{1 + \frac{\tilde{\zeta}}{2}}. \quad (5.43)$$

Recalling that $s = p - 1 - \tilde{\zeta}$, if we choose $p$ sufficiently close to $3/2$ and $\tilde{\zeta}$ sufficiently close to 0 we can arrange that $\tilde{\zeta}/s \leq \zeta/2$ and $1 + 1/s \leq 3 + \zeta$. Then (5.43) gives

$$r^{-1} d^*_\rho(\lambda_-(x), \lambda_-(y)) \geq \rho^{-1 + \zeta/2} \left| \frac{x - y}{r^2} \right|^{3 + \zeta}.$$ 

Hence (5.29) holds on $E^*_\rho \cap G^*_\rho$ for small enough $\rho$.

\section{Properties of geodesics in the glued map}

Throughout this section we assume that $Q_{zip} = Q_- \cup Q_+$ is as in Theorem 1.1 (equivalently, as in Section 4 with $\Xi = \Xi_\mathcal{F} = \Xi_\mathcal{S}$). We will use Proposition 5.1 to prove two qualitative properties of the graph metric on $Q_{zip}$ which will be used in Section 7.4 to identify the law of a subsequential limit (in the local GHPU

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topology) of the curve-decorated metric measure spaces in Theorem 1.1 as the metric gluing of two Brownian half-planes. Propositions 6.1 and 6.2 are the only results from this section which are needed in Section 7, so the latter section can be fully understood without reading the rest of the present section.

Our first result will eventually be used to show that any such subsequential limit can be mapped to the metric gluing of two Brownian half-planes via a bi-Lipschitz function.

**Proposition 6.1.** For each ζ ∈ (0, 1), there exists C = C(ζ) ≥ 1 such that the following is true. For each α ∈ (0, 1) and each L > 0, there exists δ* = δ*(α, L, ζ) > 0 such that for each δ ∈ (0, δ*) there exists n* = n*(α, L, ζ, δ) ∈ N such that the following holds for each n ≥ n*. Let z0, z1 ∈ [−Ln1/2, Ln1/2]2. With probability at least 1 − α, there exists a path γ in Qzip from λ−(z0) to λ−(z1) which crosses λ−([−Ln1/2, Ln1/2]2) at most 2Lδ−2 times and which has length

$$|γ| \leq C \text{dist}(λ−(z0), λ−(z1)); Qzip) + δ^{1−ζ}n^{1/4}.$$  \hspace{1cm} (6.1)

The quantity 2Lδ−2 in the proposition statement comes from the fact that in the proof, we will divide [−Ln1/2, Ln1/2]2 into 2Lδ−2 intervals of length δ2n1/2 and consider a glazed peeling cluster centered at each such interval.

When we apply Proposition 6.1, we will first rescale both sides by n−1/4, take a (subsequential) limit as n → ∞, and then finally let δ → 0. We emphasize that when we take limits in this order, we do not have to send C to infinity and get an event which occurs with probability close to 1. This is necessary because it will allow us to get a uniform Lipschitz constant for a map from a subsequential scaling limit of Qzip to the metric gluing of the scaling limits of Q− and Q+.

Our next result gives a uniform lower bound for the amount of time a Qzip-geodesic spends away from ∂Q− ∪ ∂Q+.

**Proposition 6.2.** For each ζ ∈ (0, 1), there exists β = β(ζ) > 0 such that the following is true. For each α ∈ (0, 1) and each L > 0, there exists δ* = δ*(α, L, ζ) > 0 such that for each δ ∈ (0, δ*), there exists n* = n*(α, L, ζ, δ) ∈ N such that the following holds for each n ≥ n*. Let z0, z1 ∈ [−Ln1/2, Ln1/2]2. For each Qzip-geodesic γ from λ−(z0) to λ−(z1), let Tβ(δ) be the set of times t ∈ [1, |γ|][2 such that γ(t) lies at Qzip-distance at least δn1/4 from λ−([−Ln1/2, Ln1/2]2). With probability at least 1 − α, for each such geodesic γ it holds that

$$\#Tβ(δ) \geq \beta|γ| - δ^{1−ζ}n^{1/4}.$$  \hspace{1cm} (6.2)

Note that we do not prove that the fraction of time of a Qzip geodesic spends in ∂Q− ∪ ∂Q+ tends to 0 as n → ∞. Rather, Proposition 6.2 only implies that the fraction of time that a Qzip geodesic spends in ∂Q− ∪ ∂Q+ does not tend to 1. In our application of Proposition 6.2, we will take limits in the same order as in the case of Proposition 6.1. Thus, as in the case of Proposition 6.1, we do not have to send β → 0 in order to get an event which occurs with probability close to 1.

The proofs of Propositions 6.1 and 6.2 proceed via similar arguments. We will show in Section 6.1 that, roughly speaking, the following is true. If we grow the glued peeling clusters \{Qj\}j∈N started from a given arc A ⊂ E(∂Q− ∪ ∂Q+), then with high probability there exists a radius r ∈ N which is not too much bigger than (#A)1/2 such that a certain "good" event occurs. In the case of Proposition 6.1, this event corresponds to the existence of a path which crosses ∂Q− ∪ ∂Q+ at most once and whose length is at most a constant times r. In the case of Proposition 6.2, this event amounts to the requirement that a Qzip-geodesic from ∂Q− to A make an excursion away from ∂Q− ∪ ∂Q+ of time length at least a small constant times r. In Section 6.2, we will prove Propositions 6.1 and 6.2 by arguing that most of the intersection of the geodesic with the SAW can be covered by the good scales of Section 6.1.

### 6.1 Existence of a good scale

Fix a finite connected arc A ⊂ E(∂Q− ∪ ∂Q+). Define the glued peeling clusters \{Qj\}j∈N started from A, the stopping times \{Jr\}r∈N, the complementary UHPS’s \{Q−j\}j∈N, and \{Q+j\}j∈N, and the σ-algebras \{F\}j∈N as in Section 4.1. In this subsection we will prove two lemmas to the effect that there typically exists a radius r ∈ N for which a certain good condition is satisfied. Our first lemma is needed for the proof of Proposition 6.1.
Figure 10: **Left:** Illustration of the event $E_k(C)$ used in the proof of Lemma 6.3. If $E_k(C)$ occurs, then $L_k$ is not too much bigger than $r_{k-1}^2$ and any two points in $\partial \dot{Q}^{J_{k-1}} \cap Q_\pm$ can be connected by a path of length at most $Cr_{k-1}$ which stays in $\dot{Q}^{J_{k}}$ and does not cross $\partial Q_- \cup \partial Q_+$ (two such paths are shown in blue). The paths in the figure stay in the annulus $\dot{Q}^{J_k} \setminus \dot{Q}^{J_{k-1}}$; our proof shows that we can arrange for this to be the case, but it is not necessary for the proof of Proposition 6.1. A similar comment applies in the illustration on the right. **Right:** Illustration of the event $\tilde{E}_k(C)$ used in the proof of Lemma 6.4. On $\tilde{E}_k(C)$, $L_k$ is not too much bigger than $r_{k-2}^2$; any two points in $\partial \dot{Q}^{J_{k-1}} \cap Q_\pm$ can be connected by a path of length at most $Cr_{k-1}$ which stays in $\dot{Q}^{J_k}$ and does not cross $\partial Q_- \cup \partial Q_+$; and every geodesic with respect to the internal graph metric on $\dot{Q}^{J_k}$ from a point of $\partial \dot{Q}^{J_{k-1}}$ to a point of $\partial \dot{Q}^{J_{k-2}}$ (such as the one shown in orange) must exit the $C^{-1/2} r_{k-2}$-neighborhood of $\dot{Q}^{J_k} \cap (\partial Q_- \cup \partial Q_+)$ (outlined in red in the figure).

**Lemma 6.3.** For $C > 1$, let $R(C)$ be the smallest $r \geq (\# A)^{1/2}$ for which the following are true.

1. $\text{diam} \left( \partial \dot{Q}^{J_k} \cap Q_\xi; Q_\xi \right) \leq Cr$ for each $\xi \in \{\pm\}$.

2. $\# \mathcal{L} \left( \dot{Q}^{J_k} \cap (\partial Q_- \cup \partial Q_+) \right) \leq C^2 r^2$.

For each $p \in [1, 3/2)$ there exists $C = C(p) > 1$ such that for each $S > 0$,

$$\mathbb{P} \left[ R(C) > (\# A)^{1/2} S \right] \leq S^{-2p} \quad (6.3)$$

with implicit constant depending only on $p$.

The key condition in Lemma 6.3 for the proof of Proposition 6.1 is condition 1, which says that restricting attention to paths which do not cross the gluing interface increases distances along $\partial \dot{Q}^{J_k}$ by a factor of at most $C$. This will be used in the proof of Proposition 6.1 to re-route a $Q_{zip}$-geodesic in such a way that it crosses the gluing interface a constant order number of times and its length is increased by a factor of at most $C$.

It is crucial for our purposes that the probabilistic estimate in (6.3) holds for some $p > 1$. The reason is that we will eventually take a union bound over several different choices of the initial edge set $A$ in order to cover the interval of the SAW between two specified points of the SAW by clusters of the form $\dot{Q}^{J_{R(C)}}$ for varying choices of the initial edge set $A$, most of which do not contain either of the two marked points (see Lemma 6.13). The requirement that $p \in [1, 3/2)$ comes from the fact that we get moments up to order $3/2$ in Proposition 5.1.

Our second lemma is needed for the proof of Proposition 6.2.
Lemma 6.4. For $C > 1$, let $\bar{R}(C)$ be the smallest $r \geq (\#A)^{1/2}$ for which the following are true.

1. Each $Q_{zip}$-geodesic $\gamma$ from an edge of $Q_{zip}$ lying at $Q_{zip}$-graph distance at most $(\#A)^{1/2}$ from $A$ to an edge of $\partial Q^{J_r}$ hits a vertex of $Q_{zip}$ which lies at $Q_{zip}$-graph distance at least $C^{-1}r$ from $\partial Q_− \cup \partial Q_+$.
2. $\#E\left(\hat{Q}^{J_r} \cap (\partial Q_− \cup \partial Q_+)\right) \leq C^2 r^2$.
3. $\text{diam}\left(\partial \hat{Q}^{J_r}; Q_{zip}\right) \leq Cr$.

For each $p \in [1,3/2]$ there exists $C = C(p) > 1$ such that for each $S > 0$,

$$\Pr\left[\bar{R}(C) > (\#A)^{1/2}S\right] \leq S^{-2p}$$

with implicit constant depending only on $p$.

The key condition in Lemma 6.4 for the proof of Proposition 6.2 is condition 1, which says that a $Q_{zip}$-geodesic started outside of $\hat{Q}^{J_r}$ cannot get close to $A$ without first spending at least $C^{-1}r$ units of time away from $\partial Q_− \cup \partial Q_+$. In the proof of Proposition 6.2, this will be used to show that such a geodesic cannot spend most of its time tracing $\partial Q_− \cup \partial Q_+$. As in Lemma 6.3, it is crucial here that (6.4) holds for some $p > 1$.

Lemmas 6.3 and 6.4 are the only results in this subsection which are needed for the proofs of Propositions 6.1 and 6.2, so the reader does not have to read the proofs of either before reading the rest of the paper.

To prove the lemmas, we will work with scales of approximately exponential size in $k$ and prove that for a large enough choice of $C$, the conditions in the definitions of the times $R(C)$ and $\bar{R}(C)$ of Lemma 6.3 and 6.4 have probability close to 1 to be satisfied at each scale.

More precisely, we will consider the following setup. Let $r_0 = 0$ and $L_0 = \#A$. Inductively, if $k \in \mathbb{N}$ and $r_{k-1}$ and $L_{k-1}$ have been defined, let

$$r_k := 2r_{k-1} + \lfloor L_{k-1}^{1/2} \rfloor \quad \text{and} \quad L_k := \#E\left(\partial Q^{J_{r_k}}\right) + \#E\left(\hat{Q}^{J_{r_k}} \cap \left(\partial Q^{J_{r_k-1}} \cup \partial Q^{J_{r_k-1}}\right)\right).$$

That is, $L_k$ gives the boundary length of the glued peeling cluster at the $k$th stage together with the number of edges of the gluing interface and the glued peeling cluster at the previous stage that have been separated from $\infty$. The term $\lfloor L_{k-1}^{1/2} \rfloor$ in the definition of $r_k$ is then likely to be proportional to $r_{k-1}$ (Proposition 5.1), so it is likely that the diameter of the annulus $\hat{Q}^{J_{r_k}} \setminus \hat{Q}^{J_{r_k-1}}$ is comparable to $r_{k-1}$.

To prove Lemma 6.3, in Section 6.1.1 we will define “good” events $E_k(C)$ for $k \in \mathbb{N}$ and $C > 1$ which serve to control the growth of $L_k$ and the diameter of $\partial Q^{J_{r_k-1}} \cap Q_{±}$ with respect to the graph metric on $Q^{J_{r_k}} \cap Q_{±}$ such that the conditions in the definition of $R(C)$ are satisfied with $r = r_k$ provided $E_k(C)$ occurs. It will be important that $E_k(C)$ belongs to the $\sigma$-algebra $\mathcal{F}^{J_{r_k}}$ of (4.2). We will then show that the conditional probability of $E_k(C)$ given $\mathcal{F}^{J_{r_k-1}}$ is a.s. close to 1 if the constant $C$ is chosen sufficiently large. Multiplying over several scales (so that one of the $E_k(C)$ is very likely to occur) will allow us to deduce Lemma 6.3. Lemma 6.4 is proven in Section 6.1.2 using a similar argument, with the same radii $r_k$ but with the events $E_k(C)$ replaced by different events.

The main estimate we need for the radii $r_k$ is the following lemma, which says that they typically grow at most an exponential rate.

Lemma 6.5. For each $p \in [1,3/2]$, there exists a constant $A_p > 0$ depending only on $p$ such that for each $k \in \mathbb{N}$,

$$\mathbb{E}\left[r_k^{2p}\right] \leq A_p(\#A)^p.$$

Proof. We first observe that for each $k \in \mathbb{N}$, $\{Q^{J_{r_k-1}} \setminus \hat{Q}^{J_{r_k-1}}\}_{r \geq 0}$ is the set of clusters of the glued peeling process in the glued map $Q^{J_{r_k-1}} \cup Q^{J_{r_k-1}}$ started from the initial edge set $\partial Q^{J_{r_k-1}}$, which has cardinality at most $L_{k-1} \leq r_k^2$. The cluster $\hat{Q}^{J_{r_k}}$ is obtained by growing this peeling process up to radius $r_k - r_{k-1} \leq r_k$.

By Lemma 4.2 and Proposition 5.1, we can find $\bar{A}_p > 0$, depending only on $p$, such that for each $k \in \mathbb{N}$,

$$\mathbb{E}\left[L_k^p \mid \mathcal{F}^{J_{r_k-1}}\right] \leq \bar{A}_p r_k^{2p}.$$  

(6.6)
Since \( r_{k+1} = 2r_k + \lfloor L_k^{1/2} \rfloor \),
\[
\mathbb{E}\left[ r_{k+1}^{2p} \right] \leq 2^{4p-1} \mathbb{E}\left[ r_k^{2p} \right] + 2^{2p-1} \mathbb{E}[L_k^p] \leq A_p \mathbb{E}\left[ r_k^{2p} \right]
\]
for a constant \( A_p > 1 \) as in the statement of the lemma. Since \( r_1 = (\#A)^{1/2} \), iterating this estimate \( k \) times yields the statement of the lemma. \( \square \)

### 6.1.1 Proof of Lemma 6.3

In order to prove Lemma 6.3, we will consider the following events defined in terms of the quantities \( r_k \) and \( L_k \) of (6.5). See Figure 10 for an illustration. For \( k \in \mathbb{N} \) and \( C > 8 \), let \( E_k(C) \) be the event that the following are true.

1. We have \( L_k \leq \frac{1}{2} (C^2 - 8) r_k^2 \).
2. The diameter of \( \partial \hat{Q}_{J_{r_{k-1}}} \cap Q_- \) with respect to the graph metric on \( \hat{Q}_{J_{r_k}} \cap Q_- \) is at most \( Cr_{k-1} \); and the same is true with \( \text{“}+\text{”} \) in place of \( \text{“}−\text{”} \).

Note that \( E_k(C) \) belongs to the \( \sigma \)-algebra \( \mathcal{F}_{J_{r_k}} \) defined as in (4.2) (which is why we measure distances with respect to the graph metric on \( \hat{Q}_{J_{r_k}} \cap Q_{\pm} \), rather than that on all of \( Q_{\pm} \)). Let \( K(C) \) be the smallest \( k \geq 2 \) for which \( E_k(C) \) occurs.

We now proceed to complete the proof of Lemma 6.3 following the outline indicated above by showing that the good radius \( R(C) \) occurs before \( r_{K(C)-1} \) (Lemma 6.6) and then by obtaining a uniform lower bound for the conditional probability of \( E_k(C) \) given \( \mathcal{F}_{J_{r_{k-2}}} \), which allows us to stochastically dominate \( K(C) \) by a geometric random variable.

**Lemma 6.6.** For each \( C > 8 \), we have \( R(C) \leq r_{K(C)-1} \), with \( R(C) \) as in Lemma 6.3.

**Proof.** We will show that if \( k \geq 2 \) and \( E_k(C) \) occurs, then the conditions in the definition of \( R(C) \) are satisfied for \( r = r_{k-1} \). By definition, \( r_{k-1} \geq r_1 = (\#A)^{1/2} \) for \( k \geq 2 \), so we just need to check conditions 1 and 2 in the definition of \( R(C) \).

For any \( k \in \mathbb{N} \),
\[
\text{where in the second inequality we used that } x \mapsto x^{1/2} \text{ is concave, hence subadditive. Each edge in } \mathcal{E}\left( \hat{Q}_{J_{r_k}} \cap (\partial Q_- \cup \partial Q_+) \right) \text{ belongs to } \mathcal{E}\left( \hat{Q}_{J_{r_i}} \cap (\partial Q_{J_{r_{i-1}}} \cup \partial Q_{J_{r_i}}) \right) \text{ for some } i \leq k \text{ so by (6.7),}
\]
\[
\text{#} \mathcal{E}\left( \hat{Q}_{J_{r_k}} \cap (\partial Q_- \cup \partial Q_+) \right) \leq \sum_{i=0}^{k} L_i \leq r_{k+1}^2. \tag{6.8}
\]

Since the graph metric on \( Q_- \) restricted to \( \hat{Q}_{J_{r_k}} \cap Q_- \) is bounded above by the graph metric on \( \hat{Q}_{J_{r_k}} \cap Q_- \), if \( E_k(C) \) occurs then
\[
\text{diam}\left( \partial \hat{Q}_{J_{r_{k-1}}} \cap Q_- \right) \leq Cr_{k-1}.
\]

Symmetrically, the same is true with \( \text{“}+\text{”} \) in place of \( \text{“}−\text{”} \). Furthermore, by condition 1 in the definition of \( E_k(C) \) together with (6.8) applied with \( k - 1 \) in place of \( k \),
\[
\text{#} \mathcal{E}\left( \hat{Q}_{J_{r_{k-1}}} \cap (\partial Q_- \cup \partial Q_+) \right) \leq r_k^2 \leq 8r_{k-1}^2 + 2L_k \leq C^2 r_{k-1}^2.
\]

Thus the conditions in the definition of \( R(C) \) are satisfied for \( r = r_{k-1} \). The result follows by the minimality of \( R(C) \). \( \square \)

We next prove a lower bound for the probability of the events \( E_k(C) \), which in particular implies that the time \( K(C) \) is stochastically dominated by a geometric random variable with success probability which can be made arbitrarily close to 1 by making \( C \) sufficiently large.
The relation (6.9) implies that
\[ L_{k-1} \leq C_0 r_{k-1}^2 \quad \text{and} \quad L_k \leq C_0 r_k^2. \]  
(6.9)

The relation (6.9) implies that
\[ L_k \leq C_0 \left(2r_{k-1} + \left| L_{k-1}^{1/2} \right| \right)^2 \leq C_0 (8r_{k-1}^2 + 2C_0 r_{k-1}^2) = 2C_0 (4 + C_0) r_{k-1}^2, \]
i.e. condition 1 holds for each \( C \geq 2 \sqrt{C_0(4 + C_0)} + 2. \)

Since the restriction of the graph metric on \( Q_{J^{r_{k-1}}} \) to \( Q_{J^{r_{k-1}}} \) is bounded above by the graph metric on \( Q_{J^{r_{k-1}}} \), and by Lemma 4.3,
\[ B_{r_{k-1}} \left( \partial Q_{J^{r_{k-1}}} \cap Q_{J^{r_{k-1}}}; Q_{J^{r_{k-1}}} \right) \subset Q_{J^{r_{k}}} \cap Q_{J^{r_{k}}} \]

Since the conditional law of \( Q_{J^{r_{k-1}}} \) given \( F^{J^{r_{k-1}}} \) is that of a UIHPQ, Lemma 5.11 implies that we can find \( C_1 = C_1(\alpha) > 0 \) such that with conditional probability at least \( 1 - \alpha/4 \) given \( F^{J^{r_{k-1}}} \), the diameter of \( \partial Q_{J^{r_{k-1}}} \cap Q_{J^{r_{k-1}}} \) with respect to the graph metric on \( Q_{J^{r_{k}}} \) is at most \( C_1 L_{k-1}^{1/2} \). By symmetry, the same holds with “+” in place of “−”. With conditional probability at least \( 1 - \alpha \) given \( F^{J^{r_{k-2}}} \), this condition holds for both − and + and the event in (6.9) occurs. If this is the case, then
\[ \text{diam} \left( \partial Q_{J^{r_{k-1}}} \cap Q_{J^{r_{k}}} \cap Q_{J^{r_{k}}} \right) \leq C_1 L_{k-1}^{1/2} \leq C_0 C_1 r_{k-1}, \quad \forall \xi \in \{-, +\} \]
and \( L_k \leq 2C_0 (4 + C_0) r_{k-1}^2 \). Hence \( E_k(C) \) occurs for \( C = \max \left\{ C_0 C_1, 2 \sqrt{C_0(4 + C_0)} + 2 \right\} \).

**Proof of Lemma 6.3.** By Lemma 6.6, \( R(C) \leq r_{K(C)} \) so it suffices to bound \( r_{K(C)} \) for an appropriate \( C = C(p) > 8 \). Fix \( 1 < p < p' < 3/2 \) and set \( A_p \) be as in Lemma 6.5 with \( p' \) in place of \( p \). Let \( \alpha \in (0, 1) \) be a small parameter, to be chosen later depending only on \( p \) and \( p' \), and let \( C = C(\alpha) > 8 \) be as in Lemma 6.7. By Lemma 6.7, for each \( k \in \mathbb{N} \),
\[ \mathbb{P}[K(C) > k] \leq \alpha^{k/2}. \]

For \( S > 1 \), let
\[ k_S = \frac{4p \log S}{\log \alpha} + 1 \]
so that \( \mathbb{P}[K(C) > k_S] \leq S^{-2p} \).

By Lemma 6.5 and the Chebyshev inequality,
\[ \mathbb{P} \left[ r_{k_S} > (\# \mathcal{A})^{1/2} S \right] \leq A_p^{k_S} S^{-2p'} \leq S^{-2p' + o_\alpha(1)} \]
where the rate at which the \( o_\alpha(1) \) term tends 0 as \( \alpha \to 0 \) depends only on \( p \) and \( p' \). By choosing \( \alpha \) sufficiently small (and hence \( C \) sufficiently large), depending only on \( p \) and \( p' \), we can arrange that this \( o_\alpha(1) \) is smaller than \( p' - p \). Recalling that \( R(C) \leq r_{K(C)} \), we get
\[ \mathbb{P} \left[ R(C) > (\# \mathcal{A})^{1/2} S \right] \leq \mathbb{P} \left[ r_{K(C)} > (\# \mathcal{A})^{1/2} S \right] \leq \mathbb{P} \left[ r_{k_S} > (\# \mathcal{A})^{1/2} S \right] + \mathbb{P}[K(C) > k_S] \leq S^{-2p}. \]
6.1.2 Proof of Lemma 6.4

The proof of Lemma 6.4 follows a similar outline as the proof of Lemma 6.3, but we work with different events which are somewhat more complicated. See Figure 10 for an illustration of the definition of these events.

For $k \in \mathbb{N}$, define $r_k$ and $L_k$ as in (6.5). Also let $d_k$ be the (internal) graph metric on $\hat{Q}^{J_{r_k}}$. For $C > 8$, let $\tilde{E}_k(C)$ be the event that the following are true.

1. $L_{k-2} \lor L_{k-1} \lor L_k \leq \frac{1}{2}(C^2 - 8)r_{k-2}^2$ and $r_k \leq C^{1/2}r_{k-2}$.
2. The diameter of $\partial \hat{Q}^{J_{r_k-1}} \cap Q_-$ with respect to the graph metric on $\hat{Q}^{J_{r_k}} \cap Q_-$ is at most $C r_{k-2}$; and the same is true with “+” in place of “−.”
3. No $d_k$-geodesic from a vertex of $\partial \hat{Q}^{J_{r_k-1}}$ to a vertex of $\partial \hat{Q}^{J_{r_k-2}}$ is contained in the $C^{-1/2}r_{k-2}$-neighborhood (with respect to $d_k$) of $\hat{Q}^{J_{r_k}} \cap (\partial Q_- \cup \partial Q_+)$.

As in the case of the event $E_k(C)$ from Section 6.1.1, the event $\tilde{E}_k(C)$ belongs to the $\sigma$-algebra $\mathcal{F}^{J_{r_k}}$ defined as in (4.2). Let $\tilde{K}(C)$ be the smallest $k \geq 3$ for which $\tilde{E}_k(C)$ occurs.

The following lemma, which is the analog of Lemma 6.6 in this setting, is the reason for our interest in the above events.

Lemma 6.8. For each $C > 8$, we have $\tilde{R}(C) \leq r_{\tilde{K}(C)-1}$, with $\tilde{R}(C)$ as in Lemma 6.4.

Proof. Suppose $k \geq 3$ is such that $\tilde{E}_k(C)$ occurs. We have $r_{k-1} \geq (\#A)^{1/2}$ by definition. By condition 1 in the definition of $\tilde{E}_k(C)$ together with (6.8) applied with $k-1$ in place of $k$,

$$\#\mathcal{E}(\hat{Q}^{J_{r_k-1}} \cap (\partial Q_- \cup \partial Q_+)) \leq r_k^2 \leq 8r_{k-1}^2 + 2L_k \leq C^2 r_{k-1}^2.$$ 

Thus condition 2 in the definition of $\tilde{R}(C)$ is satisfied for $r = r_{k-1}$. It is clear from condition 2 in the definition of $\tilde{E}_k(C)$ that condition 3 in the definition of $\tilde{R}(C)$ is also satisfied.

Now we will check condition 1. Let $\gamma$ be a $Q_{zip}$-geodesic from some edge of $Q_{zip}$ lying at $Q_{zip}$-graph distance at most $(\#A)^{1/2}$ from $A$ to some edge of $\partial \hat{Q}^{J_{r_k}}$. Let $t_0$ (resp. $t_1$) be the largest $t \in [1, |\gamma|_\mathcal{Z}$ such that $\gamma(t)$ has an endpoint in $\partial \hat{Q}^{J_{r_k-2}}$ (resp. $\partial \hat{Q}^{J_{r_k-1}}$). Since $r_{k-2} \geq r_1 \geq (\#A)^{1/2}$, Lemma 4.3 implies that $\gamma(1) \in \mathcal{E}(\hat{Q}^{J_{r_k-2}})$ so $t_0$ and $t_1$ exist.

The curve $\gamma|_{[t_0,t_1]}$ is a $d_k$-geodesic from a vertex of $\partial Q^{J_{r_k-2}}$ to a vertex of $\partial Q^{J_{r_k-1}}$. By condition 3 in the definition of $\tilde{E}_k(C)$, there exists $t_* \in [t_0,t_1]_\mathcal{Z}$ such that

$$d_k(\gamma(t_*), \hat{Q}^{J_{r_k}} \cap (\partial Q_- \cup \partial Q_+)) \geq C^{-1/2}r_{k-2}.$$ 

By Lemma 4.3 and the definition (6.5) of $r_k$,

$$\text{dist}(\partial \hat{Q}^{J_{r_k}}, \hat{Q}^{J_{r_k-1}}; Q_{zip}) \geq r_k - r_{k-1} \geq r_k - r_{k-2}.$$ 

Hence any path started from $\gamma(t_*)$ which exits $\hat{Q}^{J_{r_k}}$ must travel distance at least $r_{k-2}$. In particular,

$$\text{dist}(\gamma(t_*), \partial Q_- \cup \partial Q_+; Q_{zip}) \geq C^{-1/2}r_{k-2}.$$ 

By condition 1 in the definition of $\tilde{E}_k(C)$,

$$C^{-1/2}r_{k-2} \geq C^{-1}r_k \geq C^{-1}r_{k-1}.$$ 

Thus condition 1 in the definition of $\tilde{R}(C)$ is satisfied for $r = r_{k-1}$. \qed

We next have an analog of Lemma 6.7 for the events $\tilde{E}_k(C)$, which will take significantly more effort to prove.
Lemma 6.9. For each \( \alpha \in (0, 1) \), there exists \( C = C(\alpha) > 8 \) such that for each \( k \geq 3 \),

\[
P\left[ \bar{E}_k(C) \mid F^{J_{k-3}} \right] \geq 1 - \alpha.
\]

Most of the proof of Lemma 6.9 will be carried out in Lemmas 6.10–6.12 below. We give an outline before proceeding with the details. It is straightforward to obtain a lower bound for the probabilities of conditions 1 and 2 in the definition of \( \bar{E}_k(C) \), using the same argument as in the proof of Lemma 6.7 (see Lemma 6.10).

The main difficulty is proving a lower bound for the probability of condition 3. To this end, we will prove the following two statements, which correspond to Lemmas 6.11 and 6.12, respectively.

1. The \( \hat{Q}^{J_{k-2}} \)-graph distance from any point of \( \partial \hat{Q}^{J_{k-2}} \) to any point of \( \partial \hat{Q}^{J_{k-1}} \) is typically at most some constant \( A > 0 \) times \( r_{k-2} \).

2. For any given \( A > 0 \), we can find a small enough \( \rho \in (0, 1) \) such that with high conditional probability given \( F^{J_{k-3}} \), every path from \( \partial \hat{Q}^{J_{k-2}} \) to \( \partial \hat{Q}^{J_{k-1}} \) which stays in the \( \rho r_{k-2} \)-neighborhood of \( \partial \hat{Q}^{J_{k-2}} \cup \partial \hat{Q}^{J_{k-1}} \) has length at least \( Ar_{k-2} \).

The first statement is proved using the upper bounds for graph distances arising from Lemma 5.11 and Lemma 4.4, and the second is proved using the lower bound for the lengths of paths which stay near \( \partial \hat{Q} \cup \partial \hat{Q}^+ \) from Lemma 5.13.

Combining the above two statements will show that with high conditional probability given \( F^{J_{k-3}} \), a path from \( \partial \hat{Q}^{J_{k-2}} \) to \( \partial \hat{Q}^{J_{k-1}} \) which stays in the \( \rho r_{k-2} \)-neighborhood of \( \partial \hat{Q}^{J_{k-2}} \cup \partial \hat{Q}^{J_{k-1}} \) cannot be a geodesic. Indeed, such a path must have a sub-path which travels from \( \partial \hat{Q}^{J_{k-2}} \) to \( \partial \hat{Q}^{J_{k-1}} \) and stays in the \( \rho r_{k-2} \)-neighborhood of \( \partial \hat{Q}^{J_{k-2}} \cup \partial \hat{Q}^{J_{k-1}} \), so the length of the original path must be larger than the \( \hat{Q}^{J_{k}} \)-graph distance between its endpoints. This gives condition 3 with \( C = \rho^2 \).

We first give the (easy) proof that the first two conditions in the definition of \( \bar{E}_k(C) \) hold with high conditional probability given \( F^{J_{k-3}} \).

Lemma 6.10. For \( k \in \mathbb{N} \) and \( C > 8 \), let \( \bar{E}_k(C) \) be the event that conditions 1 and 2 in the definition of \( \bar{E}_k(C) \) are satisfied, i.e.,

\[
L_{k-2} \vee L_{k-1} \vee L_k \leq \frac{1}{2}(C^2 - 8)r_{k-2}^2 \quad \text{and} \quad r_k \leq C^{1/2}r_{k-2}
\]

and

\[
\text{diam} \left( \partial \hat{Q}^{J_{k-1}} \cap Q^{J_{k-1}}_\xi; \hat{Q}^{J_k} \cap Q_\xi \right) \leq Cr_{k-2}, \quad \forall \xi \in \{-, +\}.
\]

For each \( \alpha \in (0, 1) \), there exists \( C = C(\alpha) > 8 \) such that a.s. \( P \left[ \bar{E}_k(C) \mid F^{J_{k-3}} \right] \geq 1 - \alpha \).

Proof. By (6.6) and the Chebyshev inequality, we can find \( C_1 = C_1(\alpha) > 1 \) such that it is a.s. the case that with conditional probability at least \( 1 - \alpha/2 \) given \( F^{J_{k-3}} \),

\[
L_{k-2} \leq C_1 r_{k-2}^2, \quad L_{k-1} \leq C_1 r_{k-1}^2, \quad \text{and} \quad L_k \leq C_1 r_k^2
\]

As in the proof of Lemma 6.7, if (6.12) holds then (6.10) holds for an appropriate \( C > 8 \) depending only on \( C_1 \). Again arguing as in the proof of Lemma 6.7, we deduce from Lemmas 4.3 and 5.11 that for a possibly larger choice of \( C > 8 \), the relation (6.11) also a.s. holds with conditional probability at least \( 1 - \alpha/2 \) given \( F^{J_{k-3}} \).

In the next two lemmas, we focus our attention on condition 3. The first step is an upper bound on the maximal graph distance between a vertex of \( \partial \hat{Q}^{J_{k-1}} \) and a vertex of \( \partial \hat{Q}^{J_{k-2}} \).

Lemma 6.11. For each \( \alpha \in (0, 1) \), there exists \( C = C(\alpha) > 1 \) such that for each \( k \in \mathbb{N} \), it a.s. holds with conditional probability at least \( 1 - \alpha/4 \) given \( F^{J_{k-3}} \) that

\[
\max \left\{ \text{dist} \left( v_1, v_2; \hat{Q}^{J_k} \right) : v_1 \in V \left( \partial \hat{Q}^{J_{k-1}} \right), v_2 \in V \left( \partial \hat{Q}^{J_{k-2}} \right) \right\} \leq Cr_{k-2}
\]

(6.13)
Proof. The idea of the proof is to use Lemmas 6.10 and 5.11 to bound the diameter of $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$, which contains $\partial \dot{Q}^{Jr_{k-2}} \cap \partial Q^{Jr_{k-2}}_\pm$ (see (6.16)); then use Lemma 4.4 to bound the maximal distance from a vertex of $\dot{Q}^{Jr_{k-1}}$ to $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$. We will then conclude by means of the triangle inequality.

Let $C_0 > 8$ be chosen so that the conclusion of Lemma 6.10 is satisfied with $C = C_0$ and $\alpha/3$ in place of $\alpha$. Let $A$ be the set of edges $e$ of $\partial Q^{Jr_{k-2}}_\pm \cup \partial Q^{Jr_{k-2}}_+$ which can be connected to $\partial \dot{Q}^{Jr_{k-2}}$ by an arc of $\partial Q^{Jr_{k-2}}_\pm$ or $\partial Q^{Jr_{k-2}}_+$ with length at most $C_0^2 r_{k-2}$. By (6.10), $L_k \leq \frac{1}{2} C^2_0 r_{k-2}^2$ on $\tilde{E}_k^0(C_0)$ so on $\tilde{E}_k^0(C_0)$,

$$E\left(\dot{Q}^{Jr_{k-1}} \cap \left(\partial Q^{Jr_{k-2}}_\pm \cup \partial Q^{Jr_{k-2}}_+\right)\right) \subset A.$$  

(6.14)

By Lemma 4.2, the conditional law of $Q^{Jr_{k-2}}_-$ given $\mathcal{F}^{Jr_{k-2}}$ is that of a UIHPQ$_S$, so by Lemma 5.11, we can find a constant $C_1 = C_1(\alpha) > 0$ such that the conditional probability given $\mathcal{F}^{Jr_{k-2}}$ of the following event is a.s. at least $1 - \alpha/3$:

$$\text{diam}\left(X; B_{r_{k-2}}(X; Q^{Jr_{k-2}}_-)\right) \leq C_1 r_{k-2}, \quad \forall X \subset E(A \cap \partial Q^{Jr_{k-2}}_-).$$  

(6.15)

By symmetry, the same holds with “+” in place of “−”.

Henceforth assume that $\tilde{E}_k^0(C_0)$ occurs and (6.15) is satisfied with both choices of sign, which happens with probability at least $1 - \alpha$. We will verify (6.13) for an appropriate choice of $C$.

By Lemma 4.3, $\dot{Q}^{Jr_{k-1}}$ contains the $r_{k-1} - r_{k-2} \geq r_{k-2}$-neighborhood of $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}$ with respect to the graph metric on $Q^{Jr_{k-2}}_- \cup Q^{Jr_{k-2}}_+$, which in turn contains the $r_{k-2}$-neighborhood of $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$ with respect to the graph metric on $Q^{Jr_{k-2}}_\pm$. Hence (6.14) and (6.15) together imply that

$$\text{diam}\left(\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_-; \dot{Q}^{Jr_{k}} \cap Q^{Jr_{k-2}}_-\right) \leq C_1 r_{k-2}$$  

(6.16)

and the same holds with “+” in place of “−”.

By Lemma 4.4, $\partial \dot{Q}^{Jr_{k-1}} \cap Q^{Jr_{k-2}}_\pm$ is contained in the $2(r_{k-1} - r_{k-2})$-neighborhood of $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$ with respect to the graph metric on $\dot{Q}^{Jr_{k-1}} \cap Q^{Jr_{k-2}}_\pm$ and the same holds with “+” in place of “−”. By (6.10) from the definition of $\tilde{E}_k^0(C_0)$, we have $r_{k-1} - r_{k-2} \leq C_0^{1/2} r_{k-2}$.

Consequently, each $v_1 \in \mathcal{V}(\partial \dot{Q}^{Jr_{k-1}})$ can be joined to a vertex $v'_2$ of $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$ by a path of length at most $2C_0^{1/2} r_{k-2}$ which stays in $\dot{Q}^{Jr_{k-1}} \subset \dot{Q}^{Jr_{k}}$. By (6.16), the $\dot{Q}^{Jr_{k}}$-graph distance diameter of $\dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$ is at most $C_1 r_{k-2}$. Since $\partial \dot{Q}^{Jr_{k-2}} \subset \dot{Q}^{Jr_{k-1}} \cap \partial Q^{Jr_{k-2}}_\pm$, the point $v'_2$ lies at $\dot{Q}^{Jr_{k-2}}$-graph distance at most $C_1 r_{k-2}$ from any given vertex $v_2$ of $\partial \dot{Q}^{Jr_{k-2}}$. Hence (6.13) is satisfied with $C = 2C_0^{1/2} + C_1$.

The second step in our verification of condition 3 in the definition of $\tilde{E}_k(C)$ is the following lower bound for distances when we restrict attention to paths which stay close to $\partial Q^{Jr_{k-2}}_\pm \cup \partial Q^{Jr_{k-2}}_+$, which will be deduced from Lemma 5.13.

**Lemma 6.12.** For $\rho \in (0,1)$, let $B_\rho$ be the $pr_{k-2}$-neighborhood of $\dot{Q}^{Jr_{k-1}} \cap \left(\partial Q^{Jr_{k-2}}_- \cup \partial Q^{Jr_{k-2}}_+\right)$ with respect to the internal graph metric $d_k$ on $\dot{Q}^{Jr_{k}}$. For each $A > 0$ and each $\alpha \in (0,1)$, there exists $\rho = \rho(\alpha, A) \in (0,1)$ such that for $k \in \mathbb{N}$, it is a.s. the case that the conditional probability given $\mathcal{F}^{Jr_{k-3}}$ of the following event is at least $1 - \alpha$:

$$\text{dist}\left(\partial \dot{Q}^{Jr_{k-1}} \cap B_\rho; \partial \dot{Q}^{Jr_{k-2}} \cap B_\rho; B_\rho\right) \geq Ar_{k-2}.$$  

(6.17)

**Proof.** As in the proof of Lemma 6.11, we let $C_0 > 8$ be chosen so that the conclusion of Lemma 6.10 is satisfied with $C = C_0$ and $\alpha/3$ in place of $\alpha$ and we let $A$ be the set of edges $e \in E(\partial Q^{Jr_{k-2}}_- \cup \partial Q^{Jr_{k-2}}_+)$ which can be connected to $\partial \dot{Q}^{Jr_{k-2}}$ by an arc of $\partial Q^{Jr_{k-2}}_-$ or $\partial Q^{Jr_{k-2}}_+$ with length at most $C_0^2 r_{k-2}^2$. 

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We first establish a lower bound for the number of edges along $\partial Q^{J_{r_{k-2}}}$ between $\partial Q^{J_{r_{k-2}}}$ and $\partial \hat{Q}^{J_{r_{k-2}}}$.

One expects there to be of order $r^2_{k-2}$ such edges since the graph distance from $\partial Q^{J_{r_{k-2}}}$ to $\partial \hat{Q}^{J_{r_{k-2}}}$ is of order at least $r^2_{k-2}$.

Fix a constant $C_1 > 0$ to be chosen later, in a manner depending only on $\alpha$, and let $\mathcal{A}^L$ and $\mathcal{A}^R$, respectively, be the set of edges in the length-$[C_1^{-1}r^2_{k-2}]$ arc of $\partial Q^{J_{r_{k-2}}}$ (equivalently, of $\partial Q^{J_{r_{k-2}}}$) lying immediately to the left and right, respectively, of $\hat{Q}^{J_{r_{k-2}}}$.

Lemma 5.11 applied to the UIHPQ$_S$ $Q^{J_{r_{k-2}}}$ implies that if $C_1$ is chosen sufficiently small, in a manner depending only on $\alpha$, then with conditional probability at least $1 - \alpha/3$ given $\mathcal{F}^{J_{r_{k-2}}}$, the $J^{r_{k-2}}$-diameters of the segments $\mathcal{A}^L$ and $\mathcal{A}^R$ are each at most $\frac{1}{2}r_{k-2} - 1$. If this is the case, then Lemma 4.3 implies that

$$
\mathcal{A}^L \cup \mathcal{A}^R \subset \mathcal{E}(\hat{Q}^{J_{r_{k-2}} - 1} \setminus \hat{Q}^{J_{r_{k-2}}})
$$

(6.18)

(here we note that $\frac{3}{2}r_{k-2} \leq r_{k-1} - \frac{1}{2}r_{k-2}$). Since the boundaries of $Q^{J_{r_{k-1}} - 1}$ and $Q^{J_{r_{k-1}}} - 1$ are disjoint, (6.18) implies that $L_{k-1} \geq C_1^{-1}r^2_{k-2}$ and

$$
dist(e, \partial Q^{J_{r_{k-2}}}; Q_{-}^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}) \geq C_1^{-1}r^2_{k-2}, \quad \forall e \in \mathcal{E}(\partial Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}) \setminus \mathcal{E}(Q^{J_{r_{k-2}}}).
$$

(6.19)

Note that here we are considering distances along paths in $\partial Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}$.

We now use Lemma 5.13 to prove a lower bound for graph distances in a neighborhood of $\partial Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}$ in terms of graph distances along $\partial Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}$ itself. For $\rho > 0$, let $B^\rho$ be the $\rho r_{k-2}$-neighborhood of the set $\mathcal{A}$ (defined at the beginning of the proof) with respect to the graph metric on the unexplored quadrangulation $Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}$. By Lemma 5.13 applied to the glued quadrangulation $Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}$ and with $r = [C_1^2 r_{k-2}]$ and $\zeta = 1/2$, we can find $\rho_\star = \rho_\star(\alpha) \in (0, 1/2)$ such that for $\rho \in (0, \rho_\star)$, it a.s. holds with conditional probability at least $1 - \alpha/5$ given $\mathcal{F}^{J_{r_{k-2}}}$ that

$$
dist(e_1, e_2; B^\rho) \geq C_0^{-12} \rho^{-1/2} r_{k-2} \left( \frac{1}{r_{k-2}^2} \dist(e_1, e_2; \partial Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}) \right)^{7/2}, \quad \forall e_1, e_2 \in \mathcal{A}.
$$

(6.20)

Henceforth assume that $E^\rho_\star(C_0)$ occurs, (6.19) is satisfied, and (6.20) is satisfied for some fixed $\rho \in (0, \rho_\star)$, which happens with probability at least $1 - \alpha$. We will show that (6.17) holds provided $\rho$ is chosen sufficiently small.

We claim that

$$
dist(\partial Q^{J_{r_{k-1}}} \cap B^\rho, \partial Q^{J_{r_{k-2}}}; B^\rho) \geq \left( C_0^{-12} C_1^{-7/2} \rho^{-1/2} - \rho \right) r_{k-2}.
$$

(6.21)

Note here that $\partial Q^{J_{r_{k-2}}}$ is contained in $B^\rho$ by (6.14). By (6.14), the occurrence of $E^\rho_\star(C_0)$ implies that then $\mathcal{E}(\partial Q^{J_{r_{k-1}}} \cap Q^{J_{r_{k-2}}}) \subset \mathcal{A}$ and the same holds with “$+$” in place of “$-$”. Consequently, if $v \in \mathcal{V}(\partial Q^{J_{r_{k-1}}} \cap B^\rho)$, then there is an $e \in \mathcal{E}$ with $\dist(v, e; B^\rho) \leq \rho r_{k-2}$. Since $\rho \in (0, 1/2)$ and by Lemma 4.3,

$$
\rho r_{k-2} < r_{k-1} - \frac{3}{2} r_{k-2} \leq \dist(v, \partial Q^{J_{r_{k-2}}}; Q_{zip}).
$$

Consequently, the edge $e$ cannot belong to $Q^{J_{r_{k-2}}}$.

By (6.19), the distance from $e$ to $\partial Q^{J_{r_{k-2}}}$ along $\partial Q^{J_{r_{k-2}}} \cup Q_{+}^{J_{r_{k-2}}}$ is at least $C_1^{-1} r^2_{k-2}$. By (6.20) (applied with $r = e$ and $\rho \in \mathcal{E}(\partial Q^{J_{r_{k-2}}})$) and the triangle inequality, we infer that (6.21) holds.

The relation (6.14) implies that with $B^\rho$, as in the statement of the lemma, we have $B^\rho \setminus \hat{Q}^{J_{r_{k-2}}} \subset B^\rho$. Hence (6.21) implies (6.17) upon choosing $\rho$ sufficiently small that $C_0^{-12} C_1^{-7/2} \rho^{-1/2} - \rho \geq A$.

Proof of Lemma 6.9. Given $\alpha \in (0, 1)$, choose $C_0 = C_0(\alpha) > 8$ sufficiently large that the conclusions of Lemmas 6.10 and 6.11 are satisfied with this choice of $C$ and with $\alpha/3$ in place of $\alpha$; and $\rho \in (0, 1)$ sufficiently
small that the conclusion of Lemma 6.12 is satisfied with $A = C_0$ and with $\alpha/3$ in place of $\alpha$. Then for each $k \in \mathbb{N}$, it a.s. holds with probability at least $1 - \alpha$ that conditions 1 and 2 in the definition of $E_k(C_0)$ are satisfied and both (6.13) and (6.17) hold. We now assume that this is the case for some $k \geq 3$ and deduce condition 3 for an appropriate $C \geq C_0$.

By (6.13) and (6.17), if we let $B_\rho$ be as in (6.17) then the $d_\rho$-distance from any vertex $v_1 \in \mathcal{V}(\partial \hat{Q}_{r_{k-1}})$ to any vertex $v_2 \in \mathcal{V}(\partial \hat{Q}_{r_{k-2}})$ is smaller than the distance from $\partial \hat{Q}_{r_{k-1}} \cap B_\rho$ to $\partial \hat{Q}_{r_{k-2}} \cap B_\rho$ along paths which stay in $B_\rho$. Therefore, any $d_\rho$-geodesic from $v_1$ to $v_2$ must exit $B_\rho$ before hitting $\partial \hat{Q}_{r_{k-2}}$. Since the $\rho k_{k-2}$-neighborhood of $\hat{Q}_{r_{k-2}} \cap \partial Q_j \cup \partial Q_{j+1}$ with respect to $d_\rho$ is contained in $B_\rho \cup \hat{Q}_{r_{k-2}}$, we infer that condition 3 in the definition of $\tilde{E}_k(C)$ holds for any $C \geq \rho^{-2}$. We thus obtain the statement of the lemma with $C = C_0 \lor \rho^{-2}$.

Proof of Lemma 6.4. This is deduced from Lemmas 6.5, 6.8, and 6.9 via exactly the same argument used in the proof of Lemma 6.3.

6.2 Proof of Propositions 6.1 and 6.2

In this subsection we will deduce Propositions 6.1 and 6.2 from Lemmas 6.3 and 6.4, respectively. To this end we will use the following notation. Fix $L > 0$. For $n \in \mathbb{N}$ and $\delta \in (0, 1)$, let

\[ \mathcal{I}^n(\delta) := \left\{ [x - \delta^2 n^{1/2}, x]_\mathbb{Z} : x \in [-Ln^{1/2}, Ln^{1/2}] \cap \left( \{ \delta^2 n^{1/2} \} \mathbb{Z} \right) \right\} \]  

so an element of $\mathcal{I}^n(\delta)$ is a discrete interval $I$ of length $\delta^2 n^{1/2}$ (up to rounding error). For $I \in \mathcal{I}^n(\delta)$, let $\{\hat{Q}_I\}_{J \in \mathbb{N}_0}$ be the clusters of the glued peeling process of $Q_{zip}$ started from the initial edge set $A = \lambda_-(I)$, where here we recall that $\lambda_-$ is the boundary path of $Q_-$. Also let $\{J_I\}_{I \in \mathbb{N}_0}$ be the stopping times as in Section 4.1 for these clusters and for $C > 2$, let $R_I(C)$ and $\tilde{R}_I(C)$ be the random “good” radii defined in Lemmas 6.3 and 6.4, respectively, for these clusters. To lighten notation, we abbreviate the glued peeling clusters at the good radii by

\[ Q_I(C) := \hat{Q}_I^{L, R_I(C)} \quad \text{and} \quad \tilde{Q}_I(C) := \hat{Q}_I^{\tilde{L}, \tilde{R}_I(C)} \]  

The following lemma tells us that the radii $R_I(C)$ and $\tilde{R}_I(C)$ are typically not too big, uniformly over all $I \in \mathcal{I}^n(\delta)$. It will eventually be used to show that most points of a specified boundary/SAW segment $\lambda_-(I_0, I_1|z)$ are contained in one of the good clusters $Q_I(C)$ or $\tilde{Q}_I(C)$ which does not contain either of the endpoints $\lambda_-(I_0)$ or $\lambda_-(I_1)$.

**Lemma 6.13.** For each $\zeta \in (0, 1)$, there exists $C = C(\zeta) > 1$ such that the following is true. For each $L > 0$, each $n \in \mathbb{N}$, each $\delta \in (0, 1)$, and each $z \in [-Ln^{1/2}, Ln^{1/2}]$, we have (in the notation introduced just above)

\[ \mathbb{P} \left[ \lambda_-(z) \notin \mathcal{E}(Q_I(C)), \forall I \in \mathcal{I}^n(\delta) \right] \text{ with } \text{dist}(z, I) \geq \delta^2 n^{1/2} = 1 - O_3(\delta^\zeta), \]  

where $\text{dist}(z, I)$ denotes one-dimensional Euclidean distance and the rate of the $O_3(\delta^\zeta)$ depends only on $L$ and $\zeta$. The same holds with $\tilde{Q}_I(C)$ in place of $Q_I(C)$.

**Proof.** We give the proof in the case of $Q_I(C)$; the proof for $\tilde{Q}_I(C)$ is identical. Fix $L > 1$, $\delta \in (0, 1)$, and $z \in [-Ln^{1/2}, Ln^{1/2}]$. Also fix $p \in (1, 3/2)$ and let $C = C(p) > 1$ be as in Lemma 6.3 for this choice of $p$.

By condition 2 in the definition of $R_I(C)$, if $I \in \mathcal{I}^n(\delta)$ then each edge of $Q_I(C) \cap \partial Q_-$ lies at $\partial Q_- \text{-graph distance at most } C^2 R_I(C)^2 \text{ from } I$. Hence if $R_I(C) \leq (2C)^{-1} \text{dist}(z, I)^{1/2}$ then $\lambda_-(z) \notin \mathcal{E}(Q_I(C))$. By Lemma 6.3 (applied with $\#A = \delta^2 n^{1/2}$ and $S = (2C)^{-1} \text{dist}(z, I)^{1/2}/\#A$), for $I \in \mathcal{I}^n(\delta)$,

\[ \mathbb{P}[\lambda_-(z) \in \mathcal{E}(Q_I(C))] \leq \mathbb{P}[R_I(C) > (2C)^{-1} \text{dist}(z, I)^{1/2}] \leq \frac{\delta^2 p^3 / 2}{\text{dist}(z, I)^p} \]  

with the implicit constant depending only on $p$. For each $k \in \mathbb{N}$, there are at most 2 intervals $I \in \mathcal{I}^n(\delta)$ with $\text{dist}(z, I) = \delta^2 n^{1/2}k$ (up to rounding error). Summing the estimate (6.25) over all such intervals $I$ with
dist(\(z, I\)) \(\geq \delta^2 - 3\zeta n^{1/2}\) shows that the probability in (6.24) is at most a constant (depending only on \(p, L,\) and \(\zeta\)) times

\[
\sum_{k=[\delta^{-3}\zeta]}^{[2L\delta^{-2}]} \frac{1}{kp} \leq \delta^{3(p-1)\zeta}
\]

which is at most \(\delta^\zeta\) provided we take \(p > 4/3\).

### 6.2.1 Proof of Proposition 6.1

**Step 0: setup.** See Figure 11 for an illustration and outline of the proof. Let \(\zeta \in (0, 1)\) and let \(C = C(\zeta) > 1\) be as in Lemma 6.13 for this choice of \(\zeta\). Also fix \(n \in \mathbb{N}, L > 0,\) and \(\delta \in (0, 1)\). Let \(\mathcal{I}^n(\delta)\) be as in (6.22) and for \(I \in \mathcal{I}^n(\delta)\), define the good radius \(R_I(C)\) and the cluster \(Q_I(C)\) as in (6.23) and the discussion just preceding it.

Fix \(z_0, z_1 \in [-Ln^{1/2}, Ln^{1/2}]\). Throughout most of the proof we will work on the regularity event

\[
\mathcal{E}^n := \{\lambda_-(z_1) \notin \mathcal{E}(Q_I(C)), \forall I \in \mathcal{I}^n(\delta) \text{ with } \text{dist}(z_i, I) \geq \delta^2 - 3\zeta n^{1/2}, \forall i \in \{0, 1\}\},
\]

which by Lemma 6.13 satisfies \(\mathbb{P}[\mathcal{E}^n] = 1 - O_4(\delta^\zeta)\), at a rate depending only on \(L\) and \(\zeta\).

Let \(\gamma\) be a \(Q_{zip}\)-geodesic from \(\lambda_-(z_0)\) to \(\lambda_-(z_1)\), chosen in some measurable manner. We will iteratively re-route the segment of \(\gamma\) inside each of the at most \(2L\delta^{-2}\) good clusters \(Q_I(C)\) to get a new curve segment which crosses the gluing interface at most once and whose length is at most \(2C\) times the original segment. Since there are at most \(2L\delta^{-2}\) good clusters, the final curve after all of these re-routings will have length at most \(2C|\gamma|\) (plus a small error) and cross the gluing interface at most \(2L\delta^{-2}\) times.

**Step 1: modifying \(\gamma\) near its endpoints.** We first deal with the small number of intervals \(I \in \mathcal{I}^n(\delta)\) near \(z_0\) and \(z_1\) for which \(\lambda_-(z_0)\) or \(\lambda_-(z_1)\) belongs to \(\mathcal{E}(Q_I(C))\) by replacing \(\gamma\) by a slightly different path.
γ’. Let $T_0$ (resp. $T_1$) be the largest (resp. smallest) $t \in [1, |\gamma||z]$ for which $\gamma(t)$ is incident to an edge in $\lambda_-(|z_0 - \delta^2 - 3\kappa_1 n^{1/2}, \kappa_1 + \delta^2 - 3\kappa_1 n^{1/2}|z_2])$ (resp. $\lambda_-(|z_1 - \delta^2 - 3\kappa_1 n^{1/2}, \kappa_1 + \delta^2 - 3\kappa_1 n^{1/2}|z_2])$). Let $\gamma'_0$ (resp. $\gamma'_1$) be a $Q_\perp$-graph distance geodesic from $\lambda_-(z_0)$ (resp. $\gamma(T_1)$) to $\gamma(T_1)$ (resp. $\lambda_-(z_1)$), with the sign chosen so that $\gamma(T_0)$ (resp. $\gamma(T_1)$) belongs to $Q_\perp$.

With $\tilde{E}^n$ as in (6.27), let
\[
E^n := \tilde{E}^n \cap \{ |\gamma_0| \vee |\gamma'_1| \leq \delta^{1-2\kappa} n^{1/4}\}.
\]
By Lemma 5.11 and since $P(\tilde{E}^n) = 1 - O(\delta^\kappa)$, for each $\alpha \in (0, 1)$ there exists $\delta_\alpha = \delta_\alpha(\alpha, L, \kappa) > 0$ such that for each $\delta \in (0, \delta_\alpha)$, we have $P(E^n) \geq 1 - \alpha$ for large enough $n \in \mathbb{N}$. Hence it suffices to prove the existence of a path $\tilde{\gamma}$ as in the proposition statement on the event $E^n$.

Henceforth assume that $E^n$ occurs. Let $\gamma''$ be the concatenation of $\gamma'_0, \gamma(T_0, T_1)|z_2$, and $\gamma'_1$ and let $\gamma'$ be obtained from $\gamma''$ by erasing (chronologically) any loops it makes. Then $\gamma'$ is a simple path from $\lambda_-(z_0)$ to $\lambda_-(z_1)$ whose image is a subset of $\gamma''$ and by definition of $E^n$,
\[
|\gamma'| \leq |\gamma| + 2\delta^{1-2\kappa} n^{1/4} = \text{dist}(\lambda_-(z_0), \lambda_-(z_1); Q_{\text{zip}}) + 2\delta^{1-2\kappa} n^{1/4}.
\]
(6.28)
For each edge $e$ hit by $\gamma'$, let
\[
\sigma_e := (\gamma')^{-1}(e) \in [1, |\gamma'|]|z.
\]
(6.29)
Note that this is well-defined since $\gamma'$ is simple. Also let $T'_0$ (resp. $T'_1$) be the largest (resp. smallest) $t \in [1, |\gamma'|]|z$ for which $\gamma'(t)$ is incident to an edge in $\lambda_-(|z_0 - \delta^2 - 3\kappa_1 n^{1/2}, \kappa_2 + \delta^2 - 3\kappa_1 n^{1/2}|z_2)$ (resp. $\lambda_-(|z_1 - \delta^2 - 3\kappa_1 n^{1/2}, \kappa_2 + \delta^2 - 3\kappa_1 n^{1/2}|z_2)$), and note that
\[
\gamma'([0, T'_0]|z) \subset \gamma'_0 \quad \text{and} \quad \gamma'([T'_1, |\gamma'|]|z) \subset \gamma'_1.
\]
(6.30)
so in particular $\gamma'$ does not cross $\lambda_-([-Ln^{1/2}, Ln^{1/2}]|z)$ before time $T'_0$ or after time $T'_1$.

\textbf{Step 2: inductive construction of paths.} To construct the path $\tilde{\gamma}$ in the proposition statement, we will inductively define paths $\gamma_m$ from $\lambda_-(z_0)$ to $\lambda_-(z_1)$ and times $t_m \in [1, |\gamma_m|]|z$ for $m \in \mathbb{N}_0$ with the following properties.

1. $\gamma_m|_{[t_m, |\gamma_m|]|z}$ coincides with the final segment of $\gamma'$ of the same time length.

2. The number of times that $\gamma_m|_{[1, t_m]|z}$ crosses $\lambda_-([-Ln^{1/2}, Ln^{1/2}]|z)$ is at most $m$.

3. With $\sigma_{\gamma_m(t_m)}$ as in (6.29), we have $t_m \leq 2C\sigma_{\gamma_m(t_m)}$.

We will eventually take $\tilde{\gamma} = \gamma_M$, where $M$ is the time, to be defined below, after which all of the paths $\gamma_m$ are equal. A typical step of the inductive construction is illustrated in Figure 11.

Recall the times $T'_0$ and $T'_1$ defined just above (6.30). Let $\gamma_0 = \gamma'$ and $t_0 = T'_0$. Inductively, suppose $m \in \mathbb{N}_0$ and $\gamma_m$ and $t_m$ have been defined. We will define $\gamma_{m+1}$ and $t_{m+1}$. Let
\[
\tau_m := \inf\{ t \in [t_m + 1, |\gamma_m|]|z : \gamma_m(t) \text{ is incident to } \lambda_-([-Ln^{1/2}, Ln^{1/2}]|z) \}.
\]
If $|\gamma_m| - \tau_m \leq |\gamma'| - T'_1$ (equivalently $\sigma_{\gamma_m(t_m)} \geq T'_1$) we set $\gamma_{m+1} = \gamma_m$.

Now suppose $|\gamma_m| - \tau_m > |\gamma'| - T'_1$. Let $I_m \in T^n(\delta)$ be chosen so that $\gamma_m(t_m) \in \lambda_-(I_m)$. By condition 1 in the inductive hypothesis, the definition of $T'_1$, and the definition (6.27) of $E^n$, it follows that neither $\lambda_-(z_0) = \gamma_m(1)$ nor $\lambda_-(z_1) = \gamma_m(|\gamma_m|)$ belongs to the glued peeling cluster $Q_{\lambda_m}(C)$.

Consequently, if we let $\tilde{s}_m$ (resp. $s_m$) be the first (resp. last) time $s \in [1, |\gamma_m|]|z$ such that $\gamma_m(s)$ is incident to $Q_{\lambda_m}(C)$, then $1 < \tilde{s}_m < s_m < |\gamma_m|$.

We claim that there exists a path $\beta_m$ from $\gamma_m(\tilde{s}_m)$ to $\gamma_m(s_m)$ which crosses $\lambda_-([-Ln^{1/2}, Ln^{1/2}]|z)$ at most once and which has length $|\beta_m| \leq 2C\sigma_{\gamma_m(t_m)}$. To see this, suppose without loss of generality that $\gamma_m(\tilde{s}_m) \in E(Q_-)$. By definition of $R_{\lambda_m}(C)$ (recall Lemma 6.3), there exists a path in $Q_-$ from $\gamma_m(\tilde{s}_m)$ to any other given edge $e$ of $\partial Q_{\lambda_m}(C) \cap Q_-$ with length at most $C\sigma_{\gamma_m(t_m)}$. If $\gamma_m(s_m) \in E(Q_-)$, we take $\beta_m$ to be such a path for $e = \gamma_m(s_m)$. Otherwise, we let $e$ be an edge of $\partial Q_{\lambda_m}(C) \cap Q_-$ which is incident to an edge $e'$ of $\partial Q_{\lambda_m}(C) \cap Q_+$ and concatenate a path in $Q_-$ of length at most $C\sigma_{\gamma_m(t_m)}$ from $e'$ to $\gamma_m(s_m)$. 52
Let $\gamma_{m+1}$ be the path obtained from $\gamma_m$ by replacing $\gamma_m|_{[-s_m, s_m]}$ with $\beta_m$, i.e. the concatenation of $\gamma_m|_{[1, t_m]}$, $\beta_m$, and $\gamma_m|_{[s_m, 1]}$. Also let $t_{m+1}$ be the time for $\gamma_{m+1}$ at which it finishes tracing $\beta_m$, so that $\gamma_{m+1}(t_{m+1}) = \gamma_m(s_m)$. By the inductive hypothesis $\gamma_{m+1}$ is a path from $\lambda_-(z_0)$ to $\lambda_-(z_1)$ which coincides with $\gamma$ after time $t_{m+1}$, so condition 1 in the inductive hypothesis is satisfied with $m+1$ in place of $m$.

By our choice of $\beta_m$, the path $\gamma_{m+1}|_{[1, t_{m+1}]}$ crosses $\lambda_-([-Ln^{1/2}, Ln^{1/2}])$ at most one more time than $\gamma_m|_{[1, t_m]}$, so condition 2 in the inductive hypothesis is satisfied with $m+1$ in place of $m$.

We will now check condition 3. For this, we recall the times $\sigma_e$ from (6.29). By condition 3 in the inductive hypothesis,

$$t_m \leq 2C \sigma_\gamma_m(t_m).$$

By condition 1 in the inductive hypothesis, $\gamma_m$ traces the final segment of $\gamma'$ after time $t_m$ so

$$t_m / \tilde{s}_m \leq 2C \sigma_\gamma_m(t_m / \tilde{s}_m) \quad \text{and} \quad s_m - (t_m / \tilde{s}_m) = \sigma_\gamma_m(s_m) - \sigma_\gamma_m(t_m / \tilde{s}_m) = \sigma_{\gamma_{m+1}}(t_{m+1}) - \sigma_\gamma_m(t_m / \tilde{s}_m).$$

Lemma 4.3 implies that each edge of $\partial Q_I$ lies at $Q_{zip}$-graph distance at least $R_I = 1$ from $\lambda_-(I_m)$. In particular, $s_m - (t_m / \tilde{s}_m) \geq s_m - \tau_m \geq R_I = 1$. Hence

$$t_{m+1} \leq \tilde{s}_m + |\beta_m| \leq \tilde{s}_m + 2C R_I \leq (t_{m+1} / \tilde{s}_m) + 2C(s_m - (t_m / \tilde{s}_m)).$$

By combining this with (6.31), we get

$$t_{m+1} \leq 2C \sigma_{\gamma_m}(t_{m+1} / \tilde{s}_m) + 2C \sigma_{\gamma_m}(t_{m+1} - \sigma_\gamma_m(t_m / \tilde{s}_m)) = 2C \sigma_{\gamma_m}(t_{m+1} / \tilde{s}_m),$$

which is condition 3 with $m+1$ in place of $m$. This completes the induction.

**Step 3: definition of the path $\tilde{\gamma}$.** To conclude the proof, let $M$ be the smallest $m \in \mathbb{N}$ for which $|\gamma_m| - \tau_m \leq |\gamma'| - T_1$, and note that our construction above gives $\gamma_m = \gamma_M$ for each $m \geq M$. We will now check that the conditions in the statement of the lemma are satisfied for $\tilde{\gamma} = \gamma_M$. It is clear that $\gamma_M$ is a path from $\lambda_-(z_0)$ to $\lambda_-(z_1)$. With the discrete interval $I_m$ as above, the path $\gamma'$ does not hit $\lambda_-(I_m)$ after hitting $\gamma'_m(t_m)$. Therefore, $I_m \neq I_{m-1}$ unless $m = M$. Since there are only $2\delta I^{-2}$ elements of $\mathcal{I}^n(\delta)$, we infer that $M \leq 2\delta I^{-2}$. By condition (2) for $m = M$, the path $\gamma_M|_{[1, t_M]}$ crosses $\lambda_-([-Ln^{1/2}, Ln^{1/2}])$ at most $2\delta I^{-2}$ times. By (6.30) the path $\gamma_M$ traces a segment of the one-sided geodesic $\gamma'$ after time $t_M$, so does not cross $\lambda_-([-Ln^{1/2}, Ln^{1/2}])$ after time $t_M$. Therefore $\gamma_M$ crosses $\lambda_-([-Ln^{1/2}, Ln^{1/2}])$ at most $2\delta I^{-2}$ times. By condition 3 for $m = M$ and (6.28),

$$|\gamma_M| \leq 2C |\gamma'| \leq 2C \text{dist}(\lambda_-(z_0), \lambda_-(z_1); Q_{zip}) + 4C \delta^{1-2\delta} n^{1/4}.$$

Replacing $\zeta$ with $\zeta/3$ and possibly shrinking $\delta_*$ yields (6.1) with $C = 2C(\zeta/3)$. \hfill \Box

### 6.2.2 Proof of Proposition 6.2

**Step 0: setup.** See Figure 12 for an illustration and outline of the proof. Fix $L > 0$, $z_0, z_1 \in [-Ln^{1/2}, Ln^{1/2}]/\mathbb{Z}$, and $\zeta \in (0, 1)$. For $\delta \in (0, 1)$ and $n \in \mathbb{N}$, define the set of intervals $\mathcal{I}^n(\delta)$ as in (6.22) and the clusters $\bar{Q}_I(C)$ for $I \in \mathcal{I}^n(\delta)$ and $C > 1$ as in (6.23).

The main idea of the proof is to use condition 1 of Lemma 6.4 to argue that whenever a $\mathcal{Q}_{zip}$-geodesic $\gamma$ from $\lambda_-(z_0)$ to $\lambda_-(z_1)$ gets close to one of the intervals $I \in \mathcal{I}^n(\delta)$, it must subsequently spend a positive-length interval of time away from $\partial Q_-$ and $\partial Q_+$. Some complications arise, however, since one has to bound the amount of overlap between these positive-length intervals in order to get a lower bound for how much time $\gamma$ spends away from $\partial Q_-$ and $\partial Q_+$.

We first define a regularity event which we will work on for most of the proof. For $C > 1$, let $G^\alpha = G^\alpha(C, \delta, \zeta, L, z_0, z_1)$ be the event that the following are true.

1. $z_0, z_1 \notin \bar{Q}_I(C)$ whenever $I \in \mathcal{I}^n(\delta)$ with dist$(I, \{z_0, z_1\}) \geq \delta^2 - \zeta/4 n^{1/2}$.
2. diam$(\lambda_-(I); Q_-) \leq \delta^{1-\zeta/4} n^{1/4}$ for each $I \in \mathcal{I}^n(\delta)$. 53
Assume now that \( \lambda \) is at most the total amount of time that \( \gamma \) spends in the \( \lambda \) of one of the intervals \( t \) from \( t \) to \( t \) for which \( \gamma \) is at distance at least \( \sigma \) away from the SAW (the image of this time interval under \( \gamma \) is outlined in red). Also let \( \sigma_j \) and \( \sigma_j \) be the first and last times at which \( \gamma \) enters the glued peeling cluster \( \tilde{Q}_{ij}(C) \), which are finite for most choices of \( j \) by Lemma 6.13. By definition of \( \tilde{R}_{ij}(C) \) (recall Lemma 6.4), there must be a time interval \( [t_j, \tilde{t}_j] \subset [\sigma_j, \sigma_j] \) of length at least \( C^{-1} \tilde{R}_{ij}(C) \) during which \( \gamma \) is at distance at least \( C^{-1} \delta n^{1/4} \) away from the SAW (the image of this time interval under \( \gamma \) is shown in orange). Furthermore, we have \( \sigma_j - \sigma_j \leq 2C\tilde{R}_{ij}(C) \). This leads to the conclusion \( \gamma \) spends at least a \( \frac{1}{2} C^{-2} \)-fraction of its time (minus a small error) at distance at least \( C^{-1} \delta n^{1/4} \) away from our given segment of \( \partial Q_\lambda \).

By Lemmas 5.11 and 6.13 (the latter applied with \( \zeta/100 \) in place of \( \zeta \), say), for each \( \alpha \in (0, 1) \) there exists \( n = n(\alpha, L, \zeta) \) and \( C = C(\zeta) > 1 \) such that for \( \delta \in (0, n] \), we have \( \mathbb{P}[G^n] \geq 1 - \alpha \) for large enough values of \( n \). Henceforth fix such a \( \alpha \). We will prove that for an appropriate choice of \( \beta \) and small enough \( \delta \in (0, n] \) (depending only on \( \alpha, L, \) and \( \zeta \)), the condition in the proposition statement is satisfied whenever \( G^n \) occurs.

**Step 1: the set of “bad” indices.** Assume now that \( G^n \) occurs and let \( \gamma \) be a \( Q_\text{zip} \)-geodesic from \( \lambda_\lambda \) to \( \lambda_\lambda \). For \( j \in [0, \delta^{-1} n^{-1/4}] \), let

\[
s_j := \lfloor \delta n^{1/4} j \rfloor.
\]

(6.33)

Let \( J^\alpha \) be the set of “bad” indices \( j \in [1, \delta^{-1} n^{-1/4}] \) for which there exists \( t \in [s_j-1+1, s_j] \) and \( I_j \in T^\alpha(\delta) \) for which \( z_0, z_1 \notin \tilde{Q}_{ij}(C) \) and \( \text{dist}(\gamma(t), \lambda_\lambda(I_j)); Q_\text{zip}) \leq \frac{1}{4} C^{-1} \delta n^{1/4} \).

(6.34)

If we let \( \beta \in (0, \frac{1}{4} C^{-1}] \) and let \( T^\beta_\gamma(\delta) \) be the set of times at which \( \gamma \) lies at \( Q_\text{zip} \)-distance at least \( \beta \delta n^{1/4} \) from \( \lambda_\lambda \) or \( \lambda_\lambda \), as in the proposition statement, then by condition 1 in the definition of \( G^n \), each \( t \in [1, \lfloor \gamma \rfloor] \setminus T^\beta_\gamma(\delta) \) is either contained in \( [s_{j-1} + 1, s_j] \) for some \( j \in J \) or satisfies \( \text{dist}(\gamma(t), I_\gamma) \leq \frac{1}{4} C^{-1} \delta n^{1/4} \) for one of the intervals \( I \in T^\alpha(\delta) \) with \( \text{dist}(I_\gamma, z_0, z_1) \leq \delta^{-1/4} n^{1/2} \). There are at most \( 4\delta^{-1/4} \) such intervals and by condition 2 in the definition of \( G^n \) each has \( Q_\text{zip} \)-diameter at most \( \delta^{-1/4} n^{1/4} \). Since \( \gamma \) is a geodesic, the total amount of time that \( \gamma \) spends in the \( 1/4 C^{-1} \delta n^{1/4} \)-neighborhood of the union of these \( \delta^{-1/4} \) intervals is at most

\[
\left( \frac{1}{4} C^{-1} \delta n^{1/4} + \delta^{1-\zeta/4} n^{1/4} \right) \times 4 \delta^{-1/4} n^{1/4} \leq \delta^{1-\zeta/4} n^{1/4}.
\]

Consequently,

\[
\#T^\beta_\gamma(\delta) \geq \lfloor \gamma \rfloor - \delta n^{1/4} \left( \#J + 8\delta^{-\zeta/2} \right).
\]

(6.35)

We will now use the definition of \( \tilde{R}_{ij}(C) \) from Lemma 6.4 to prove another lower bound for \( \#T^\beta_\gamma(\delta) \) which is increasing in \( \#J \). Combining this bound with (6.35) and considering the worst possible value of \( \#J \) will give (6.2).
Step 2: bad indices give rise to intervals during which \( \gamma \) is far from \( \partial Q_- \cup \partial Q_+ \). For \( j \in \mathcal{J} \), let \( \tau_j \) be the smallest \( t \in [s_j-1, s_j] \) for which (6.34) holds. By Lemma 4.3, \( \gamma(t) \in \mathcal{Q}_{\mathcal{J}, j}^{\mathcal{Q}_{\mathcal{J}}, \{\mathcal{J}, j\}} \), which is contained in \( \mathcal{Q}_{\mathcal{I}, j}(C) \) since \( \mathcal{R}_{\mathcal{I}, j}(C) \geq \#(\mathcal{L}_{\mathcal{I}, j}(C))^{1/2} \geq \delta n^{1/4} \) by definition (recall Lemma 6.4).

Also let \( \sigma_j \) (resp. \( \tau_j \)) be the first (resp. last) time \( t \in [1, \|\gamma\|_2] \) for which \( \gamma(t) \) is incident to an edge of \( \partial \mathcal{Q}_{\mathcal{I}, j}(C) \). By (6.34), \( z_0, z_1 \notin \mathcal{Q}_{\mathcal{I}, j}(C) \) so \( \sigma_j \) and \( \tau_j \) are well-defined and (by the preceding paragraph) \( \tau_j \in [\sigma_j, \tau_j] \).

By condition 1 in the definition of \( \mathcal{R}_{\mathcal{I}, j}(C) \), for each \( j \in \mathcal{J} \) the path \( \gamma \) must hit a vertex of \( Q_{\text{zip}} \) which lies at \( Q_{\text{zip}} \)-distance at least \( C^{-1} \mathcal{R}_{\mathcal{I}, j}(C) \) from \( \mathcal{L}_{\mathcal{I}, j}(C) \) between times \( \tau_j \) and \( \tau_j \), so must spend at least \( \frac{1}{4} C^{-1} \mathcal{R}_{\mathcal{I}, j}(C) \) units of time at distance greater than \( \frac{1}{4} C^{-1} \mathcal{R}_{\mathcal{I}, j}(C) \). Consequently, if we let \( [\bar{t}_j, \bar{t}_j] \subset [\tau_j, \|\gamma\|_2] \) be the largest discrete interval contained in \( [\tau_j, \|\gamma\|_2] \) such that

\[
\text{dist} \left( \gamma([\bar{t}_j, \bar{t}_j]), \mathcal{L}_{\mathcal{I}, j}(C) \right) \geq \frac{1}{4} C^{-1} \delta n^{1/4},
\]

then

\[
\bar{t}_j - t_j \geq \frac{1}{2} C^{-1} \mathcal{R}_{\mathcal{I}, j}(C) \quad \text{and} \quad t_j \leq \sigma_j.
\]  

Step 3: bounding the overlap. By (6.36), each of the intervals \( [t_j, \bar{t}_j] \) is contained in \( T^2_\gamma(\delta) \) for \( \beta \in (0, \frac{1}{4} C^{-1}] \), but it is possible that these intervals overlap. In order to get a lower bound for \#\( T^2_\gamma(\delta) \), we need to prove an upper bound for the amount of overlap.

We first argue that if \( j, j' \in \mathcal{J} \) with \( [t_j, \bar{t}_j] \cap [t_{j'}, \bar{t}_{j'}] \neq \emptyset \), then \( [t_j, \bar{t}_j] = [t_{j'}, \bar{t}_{j'}] \). Indeed, by maximality of \( [t_j, \bar{t}_j] \) we must have \( t_{j'} = \bar{t}_j \). On the other hand, since \( \gamma(\tau_{j'}) \) lies at \( Q_{\text{zip}} \)-distance at most \( \frac{1}{4} C^{-1} \delta n^{1/4} \) from \( \mathcal{L}_{\mathcal{I}, j}(C) \), we have \( \tau_{j'} \notin [t_{j'}, \bar{t}_{j'}] \) and similarly with \( j \) and \( j' \) interchanged, so again by maximality \( t_{j'} = \bar{t}_j \).

We next claim that

\[
\# \{ j' \in \mathcal{J} : [t_{j'}, \bar{t}_{j'}] \cap [t_j, \bar{t}_j] \neq \emptyset \} \leq \frac{4C^2}{\delta n^{1/4}} (\bar{t}_j - t_j), \quad \forall j \in \mathcal{J}.
\]  

Indeed, suppose \( j \in \mathcal{J} \). Condition 3 in the definition of \( \mathcal{R}_{\mathcal{I}, j}(C) \) implies that the \( Q_{\text{zip}} \)-diameter of \( \partial \mathcal{Q}_{\mathcal{I}, j}(C) \) is at most \( C \mathcal{R}_{\mathcal{I}, j}(C) \), so since \( \gamma \) is a geodesic and \( \gamma(\sigma_j) \) and \( \gamma(\tau_j) \) are incident to \( \partial \mathcal{Q}_{\mathcal{I}, j}(C) \),

\[
0 \leq t_j - s_j - 1 \leq \sigma_j - t_j \leq 2 C \mathcal{R}_{\mathcal{I}, j}(C).
\]  

If \( j' \in \mathcal{J} \) for which \( [t_{j'}, \bar{t}_{j'}] \cap [t_j, \bar{t}_j] \neq \emptyset \) (equivalently, \( [t_{j'}, \bar{t}_{j'}] = [t_j, \bar{t}_j] \) by the preceding paragraph) then we have that,

\[
0 \leq t_j - s_{j'} - 1 = t_{j'} - s_{j'} - 1 \quad (\text{since } t_j = t_{j'})
\leq 2 C \mathcal{R}_{\mathcal{I}, j'}(C) \quad (\text{by (6.39) with } j' \text{ instead of } j)
\leq 4 C^2 (t_{j'} - t_{j'}) \quad (\text{by (6.37) with } j' \text{ instead of } j)
= 4 C^2 (\bar{t}_j - \bar{t}_{j'}) \quad (\text{since } t_j = t_{j'} \text{ and } \bar{t}_j = \bar{t}_{j'}).
\]

Therefore, every such \( j' \in \mathcal{J} \) satisfies \( s_{j'} - 1 \in [t_j - 4 C^2 (t_{j'} - t_j), t_j] \) so (recalling (6.33)) the number of such \( j' \) is at most the right side of (6.38).

Step 4: conclusion. Since \( [t_j, \bar{t}_j] \subset T^2_\gamma(\delta) \) for each \( j \in \mathcal{J} \) and each \( \beta \in (0, \frac{1}{4} C^{-1}] \), we infer from (6.38) that for each such \( \beta \),

\[
\#T^2_\gamma(\delta) \geq \sum_{j \in \mathcal{J}} \# \{ j' \in \mathcal{J} : [t_{j'}, \bar{t}_{j'}] \cap [t_j, \bar{t}_j] \neq \emptyset \} \geq \frac{\delta n^{1/4}}{4 C^2 \#\mathcal{J}} \#\mathcal{J}.
\]  

By (6.35) and (6.40),

\[
\#T^2_\gamma(\delta) \geq \max \left\{ \frac{\delta n^{1/4}}{4 C^2 \#\mathcal{J}, |\gamma| - \delta n^{1/4} (\#\mathcal{J} + 8 \delta^{-2/4})} \right\}.
\]
By considering the value of $\#J$ for which the right side is minimized, we get

$$\#T_\beta^\circ(\delta) \geq \frac{|\gamma| - 8\delta^{1-\zeta/2}n^{1/4}}{1 + 4C^2}.$$

We now conclude by choosing $\beta \leq \min\{\frac{1}{4}C^{-1}, (1 + 4C^2)^{-1}\}$ and shrinking $\delta_*^* \in (0, \delta_*^* \cup \delta_*^*)$ such that $8\delta^{1-\zeta} \leq \delta^{1-\zeta}$ for each $\delta \in (0, \delta_*^*)$. \hfill $\blacksquare$

## 7 Proof of main theorems

In this section we will complete the proof of Theorem 1.1. At the end, we will briefly remark on the minor adaptations necessary to prove Theorem 1.2 and Theorem 1.3 in Remark 7.12. See Figure 13 for an illustration of the objects involved in the proof.

We will begin in Section 7.1 by introducing some notation and establishing tightness of the 4-tuples $(Q_{zip}, d^n_{zip}, \mu^n_{zip}, \eta^n_{zip})$ in the local GHP topology. By the Prokhorov theorem and since we already know from [GM16b] that both of the 4-tuples $(Q_{\pm}, d^n_{\pm}, \mu^n_{\pm}, \eta^n_{\pm})$ converge to Brownian half-planes in the local GHP topology, we can find a random element $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{\eta}) \in \mathbb{M}^G_{\text{GHP}}$ coupled with the Brownian half-planes $(X_\pm, d_\pm)$ and a subsequence $N$ along which the joint law of $(Q_{zip}, d^n_{zip}, \mu^n_{zip}, \eta^n_{zip})$, $(Q_-, d^n_-, \mu^n_-, \eta^n_-)$ converges in the local GHP topology to the joint law of $(\tilde{X}, \tilde{d}, \tilde{\eta}, \tilde{\mu})$, $(X_-, d_-, \mu_-, \eta_-)$, and $(X_+, d_+, \mu_+, \eta_+)$.

In the remainder of the section we fix such a subsequential limit and aim to show that $(\tilde{X}, \tilde{d}, \tilde{\mu}, \tilde{\eta})$ is equivalent (as a curve-decorated metric measure space) to our desired limiting space $(X_{zip}, d_{zip}, \mu_{zip}, \eta_{zip})$, which we recall is obtained as the quotient of $(X_-, d_-)$ and $(X_+, d_+)$ identified along their positive boundary rays. We now give a brief overview of how this is accomplished.

We begin in Section 7.2 by using the results of Section 5.4 to establish some qualitative properties of the subsequential limiting curve $\tilde{\eta}$. Namely, we will deduce that $\tilde{\eta}$ is necessarily transient (i.e., it is a.s. the case that the amount of time it spends in any compact set is finite), simple, and satisfies $\tilde{\mu}(\tilde{\eta}) = 0$.

In Section 7.3, we show that there is a bijective, 1-Lipschitz, curve-preserving, measure-preserving map $f_{zip} : X_{zip} \to \tilde{X}$ which preserves the length of each path in $X_{zip}$ which does not hit the gluing interface $\eta_{zip}$. The reason that such a map $f_{zip}$ exists is the universal property of the quotient metric (see Section 2.1.4) which shows that $(X_{zip}, d_{zip})$ is the largest metric space structure which one can put on the topological quotient space obtained by identifying $X_\pm$ along their positive boundary rays which is compatible with the metrics $d_\pm$ on $X_\pm$. In order to apply this property, however, we need to check that the subsequential limiting space $\tilde{X}$ admits a decomposition analogous to the decomposition $X_{zip} = X_- \cup X_+$.

More precisely, we will show in Lemma 7.7 that there are 1-Lipschitz, measure-preserving maps $f_\pm : (X_\pm, d_\pm) \to (\tilde{X}, \tilde{d})$ which preserve the lengths of curves which do not touch the gluing interface and satisfy $f_-(X_-) \cup f_+(X_+) = \tilde{X}$ and $f_-(X_-) \cap f_+(X_+) = \tilde{\eta}$ (see Figure 13 for an illustration of these maps). The existence of $f_\pm$ is established using limiting arguments and elementary metric space theory.

In Section 7.4, we conclude the proof by showing that the above map $f_{zip}$ is in fact an isometry, so that $X_{zip} = \tilde{X}$ as curve-decorated metric measure spaces. This is the most interesting part of the argument, and is based on the results of Section 6. In particular, we first deduce from Proposition 6.1 that $f_{zip}^{-1}$ is a.s. $C$-Lipschitz for some deterministic constant $C \geq 1$ and from Proposition 6.2 that each $\tilde{d}$-geodesic spends at least a $\beta$-fraction of its time outside of $\tilde{\eta}$ for some $\beta \in (0, 1)$.

We then look at a $\tilde{d}$-geodesic $\gamma$ and decompose it as the union of finitely many segments during which it does not hit $\tilde{\eta}$, with total length $a|\gamma| \geq (\beta/2)|\gamma|$; and finitely many complementary segments with total length $(1 - a)|\gamma| \leq (1 - \beta/2)|\gamma|$ (see Figure 14 for an illustration). This latter collection of segments in particular contains the intersection of $\gamma$ with $\tilde{\eta}$. Since $f_{zip}$ preserves the lengths of paths which do not hit $\eta_{zip}$, the total $d_{zip}$-length of the images of the first collection of segments under $f_{zip}^{-1}$ is equal to $a|\gamma|$. Since $f_{zip}^{-1}$ is $C$-Lipschitz, the total $d_{zip}$-length of the images of the second collection of segments under $f_{zip}^{-1}$ is at most $C(1 - a)|\gamma|$. Hence $|f_{zip}^{-1}(\gamma)| \leq a|\gamma| + C(1 - a)|\gamma|$. Since $C \geq 1$, the right hand side is maximized when we make $a \geq \beta/2$ as small as possible. Consequently, $|f_{zip}^{-1}(\gamma)| \leq (\beta/2)|\gamma| + C(1 - \beta/2)|\gamma|$. Since $\gamma$ was an arbitrary geodesic, this implies that $f_{zip}^{-1}$ is a.s. $(\beta/2 + C(1 - \beta/2))$-Lipschitz. As $\beta \in (0, 1)$, we can therefore take $C = 1$. 

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7.1 Setup and tightness

For the proof of Theorem 1.1, we will use a slightly modified version of the notation used in the theorem statement where we define the n-th rescaled objects with respect to a different copy of the map. The reason for this is that we will be applying the Skorokhod representation theorem for weak convergence, so that we can couple everything together on the same probability space.

For \( n \in \mathbb{N} \), let \( Q^n_{\text{zip}} = Q^n \cup Q^n_{\pm} \) be a copy of the quadrangulation \( Q_{\text{zip}} \) from Theorem 1.1. We view \( Q^n_{\text{zip}} \) and \( Q^n_{\pm} \) as connected metric spaces by identifying each edge with an isometric copy of the unit interval in \( \mathbb{R} \), as in Remark 2.2. Let \( d^n, d^n_+ \), and \( d^n_{\text{zip}} \) be the graph metrics on \( Q^n, Q^n_{\pm} \), and \( Q^n_{\text{zip}} \), respectively, re-scaled by \( (9/8)^{1/4}n^{-1/4} \). Let \( \mu^n, \mu^n_+ \), and \( \mu^n_{\text{zip}} \) be the measures on \( Q^n, Q^n_{\pm} \), and \( Q^n_{\text{zip}} \), respectively, which assign to each vertex a mass equal to \( (4^n)^{-1} \) times its degree.

Let \( \lambda^n_{\pm} \) be the boundary paths of \( Q^n_{\pm} \), respectively, started from the root edge, viewed as paths from \( \mathbb{R} \) to \( Q^n_{\pm} \) in the manner of Remark 2.2. For \( t \in \mathbb{R} \), let \( \eta^n_{\pm}(t) := \lambda^n_{\pm} \left( \frac{2^{7/2}n^{1/2}t}{3} \right) \). Also let \( \eta^n_{\text{zip}} := \eta^n_{\pm}|_{[0, \infty)} \), which is equal to \( \eta^n_{\pm}|_{[0, \infty)} \) since \( Q^n_{\pm} \) are glued together along \( \lambda^n_{\pm} ([0, \infty)) \) to obtain \( Q^n_{\text{zip}} \).

As in Theorem 1.1, let \((X_-, d_-) \) and \((X_+, d_+) \) be a pair of independent Brownian half-planes. Let \( \eta_{\pm} \) be the area measure on \( X_{\pm} \) and let \( \eta_{\pm} : \mathbb{R} \rightarrow \partial X_{\pm} \) be the parameterization of \( \partial X_{\pm} \) according to boundary length, normalized so that \( \eta_{\pm}(0) \) is the marked point of \( \partial X_{\pm} \). Let \((X_{\text{zip}}, d_{\text{zip}}) \) be the metric space quotient of the disjoint union of \((X_-, d_-) \) and \((X_+, d_+) \) under the equivalence relation which identifies \( \eta_{-}(t) \) with \( \eta_{+}(t) \) for each \( t \geq 0 \). Also let \( \mu_{\text{zip}} \) be the measure on \( X_{\text{zip}} \) which restricts to the pushforward of \( \mu_{\pm} \) under the quotient map on the image of \( X_{\pm} \) under the quotient map. Let \( \eta_{\text{zip}} : [0, \infty) \rightarrow X_{\text{zip}} \) be the path which is the image of \( \eta_{-}([0, \infty)) \) (equivalently \( \eta_{+}([0, \infty)) \)) under the quotient map.

Define the curve-decorated metric measure spaces

\[
\Omega^n_{\text{zip}} := (Q^n_{\text{zip}}, d^n_{\text{zip}}, \mu^n_{\text{zip}}, \eta^n_{\text{zip}}) \quad \Omega^n_{\pm} := (Q^n_{\pm}, d^n_{\pm}, \mu^n_{\pm}, \eta^n_{\pm}) \\
\mathcal{X}_{\text{zip}} := (X_{\text{zip}}, d_{\text{zip}}, \mu_{\text{zip}}, \eta_{\text{zip}}) \quad \mathcal{X}_{\pm} := (X_{\pm}, d_{\pm}, \mu_{\pm}, \eta_{\pm}).
\] (7.1)

See Figure 13 for an illustration of the above objects (plus some additional objects, to be introduced later).

By [GM16b, Theorem 1.12], the joint law of \((\Omega^n_{\text{zip}}, \Omega^n_{\pm})\) converges in the local GHPU topology to the joint law of \((\mathcal{X}_{\pm}, \mathcal{X}_{\text{zip}}) \). We want to establish tightness of \( \{\mathcal{X}_{\text{zip}}\}_{n \in \mathbb{N}} \) in the local GHPU topology. For this purpose we record the following estimate which will also be used several times in later subsections.

**Lemma 7.1.** For \( n \in \mathbb{N} \) and \( R, \rho > 0 \), let \( G^n(\rho, R) \) be the event that the following are true.

1. \( \eta^n_{\text{zip}}([R^2, \infty)) \cap B(\eta^n_{\text{zip}}(0); d^n_{\text{zip}}) = \emptyset. \)
2. \( B_\rho(\eta^n_{\text{zip}}(0); d^n_{\text{zip}}) \subset B_R(\eta^n_{\text{zip}}(0); d^n_{\text{zip}}) \cup B_R(\eta^n_{+}(0); d^n_{+}). \)

For each \( \epsilon \in (0, 1) \) and each \( \rho > 0 \), there exists \( R = R(\rho, \epsilon) > \rho \) such that for each \( n \in \mathbb{N} \),

\[
\mathbb{P}[G^n(\rho, R)] \geq 1 - \epsilon.
\]

**Proof.** This is immediate from Proposition 5.1 and Lemma 5.12. \( \square \)

**Lemma 7.2.** The laws of the curve-decorated metric measure spaces \( \Omega^n_{\text{zip}} \) are tight in the local GHPU topology.

**Proof.** Fix \( \epsilon \in (0, 1) \) and \( \rho > 0 \). Let \( R = R(\rho, \epsilon) > \rho \) be chosen so that the conclusion of Lemma 7.1 is satisfied and let \( G^n(\rho, R) \) for \( n \in \mathbb{N} \) be as in that lemma. Since \((\Omega^n_{\pm}, d^n_{\pm}, \mu^n, \eta^n_{\pm})\) each converge to non-degenerate limits in the local GHPU topology, we can find \( C, N, \delta > 0 \) depending only on \( \rho \) and \( \epsilon \) such that for each \( n \in \mathbb{N} \), it holds with probability at least \( 1 - \epsilon \) that the following are true.

1. \( G^n(\rho, R) \) occurs.
2. \( B_R(\eta^n_{\pm}(0); d^n_{\pm}) \) can be covered by at most \( N \) \( d^n_{\pm} \)-metric balls of radius \( \epsilon \).
3. \( \mu^n_{\pm}(B_R(\eta^n_{\pm}(0); d^n_{\pm})) \leq C. \)
4. \( d^n_{\pm}(\eta^n_{\pm}(s), \eta^n_{\pm}(t)) \leq \delta \) whenever \( s, t \in [0, R^2] \) with \( |s - t| \leq \epsilon \).

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5. The previous three conditions also hold with “+” in place of “−.”

Since ρ and ϵ are arbitrary the desired tightness therefore follows from [GM16b, Lemma 2.12].

By Lemma 7.2, [GM16b, Theorem 1.12], and the Prokhorov theorem, for any sequence of positive integers tending to ∞, there exists a subsequence N, a curve-decorated metric measure space \( \bar{X} = (\bar{X}, \bar{d}, \bar{\mu}, \bar{\eta}) \) ∈ \( \mathcal{M}_{\text{GHPU}}^\infty \), and a coupling of \( \bar{X} \) with \( (X_-, X_+) \) such that

\[
(Q_n, d_n, \mu_n) \rightarrow (\bar{Q}_n, \bar{d}_n, \bar{\mu}_n) \rightarrow (\bar{Q}_n, \bar{d}_n, \bar{\mu}_n)
\]

in law in the local GHPU topology as \( N \ni n \rightarrow \infty \). Henceforth fix \( N \), \( \bar{X} \), and a coupling as above. We will show that in any such coupling, \( \bar{X} = X_{\text{zip}} \) a.s.

By the Skorokhod representation theorem, we can couple the sequence \( \{(Q_n, Q_n^-, Q_n^+)\}_{n \in N} \) with \( (\bar{X}, X_-, X_+) \) in such a way that the convergence in (7.2) holds a.s. For our proofs, it will be convenient to
embed the spaces $\Omega^n_{zip}$ (resp. $\Omega^n_\tau$, $\Omega^n_+$) for $n \in \mathcal{N}$ together with their limiting space $\bar{X}$ (resp. $\bar{X}_-$, $\bar{X}_+$) into a common metric space.

By [GM16b, Lemma 2.4], we can a.s. find (random) boundedly compact (i.e., closed bounded subsets are compact) metric spaces $(Z_{zip}, D_{zip})$, $(Z_-, D_-)$, and $(Z_+, D_+)$ and isometric embeddings

$$
\iota^\circ_n: (Q^n_{zip}, d^n_{zip}) \to (Z_{zip}, D_{zip}), \quad \iota^\circ_n: (\bar{X}, \bar{d}) \to (Z_{zip}, D_{zip}),
$$

$$
\iota^\pm_n: (Q^n_\pm, d^n_\pm) \to (Z_\pm, D_\pm), \quad \iota^\pm_n: (X_\pm, d_\pm) \to (Z_\pm, D_\pm)
$$
such that a.s.

$$
\iota^\circ_n(\Omega^n_{zip}) \to \iota(\bar{X}) \quad \text{and} \quad \iota^\circ_n(\Omega^n_\pm) \to \iota(\bar{X}_\pm)
$$
in the $D_{zip}$- and $D_\pm$-local Hausdorff-Prokhorov-uniform (HPU) topologies, respectively (Definition 2.3).

To lighten notation, we henceforth identify $\Omega^n_{zip}$, $\Omega^n_\tau$, $\Omega^n_+$, and $\bar{X}$ with their images under $\iota^\circ_n$ and $\iota$, respectively (so that $Q^n_{zip} = Q^n_\tau \cup Q^n_+$ and $\bar{X}$ are subsets of $Z_{zip}$, etc.). We similarly identify $X_\pm$ with their images under $\iota^\pm_n$ and (since we have already identified $\Omega^n_\mp$ with their images under $\iota^\mp_n$) we define

$$
\hat{\Omega}^\pm_n = \left( \hat{Q}^n_\pm, \hat{d}^n_\pm, \hat{\mu}_n^\pm, \hat{\eta}^\pm_n \right) = \iota^\pm_n(\Omega^n_\pm)
$$

Let

$$
f^n_\pm := \iota^\circ_n \circ (\iota^\pm_n)^{-1} : \hat{Q}^n_\pm \to Q^n_\pm \quad (7.3)
$$

be the map taking us from the embedding of $\hat{Q}^n_\pm$ into $Z_\pm$ to its embedding into $Z_{zip}$. Since $\iota_n^\pm$ is an isometry and $d^n_{zip}|Q^n_\pm$ is dominated by the internal metric $d^n_\pm$, each $f^n_\pm$ is 1-Lipschitz with respect to the metric $\hat{d}^n_\pm$ on the domain and the metric $d^n_{zip}$ on the range. Furthermore, $f^n_\pm(\hat{\Omega}^n_\pm) = \Omega^n_\pm$.

We will continue to use the objects introduced just above throughout the remainder of this section. See Figure 13 for an illustration of these objects.

### 7.2 Basic properties of the subsequential limiting curve

Throughout the remainder of this section we continue to use the notation introduced in Section 7.1. We will prove some basic facts about the curve $\eta$ associated with the subsequential limiting object $\bar{X}$ which follow relatively easily from estimates for $Q_{zip}$ proven earlier in the paper. In particular, we will show that it is simple, transient, and has zero $\mu$-mass.

We first show that the subsequential limiting curve $\eta$ is simple, which is a consequence of Lemma 5.10.

**Lemma 7.3.** Almost surely, the curve $\eta$ is simple. In fact, for each $L > 0$ and each $p > 3/2$, there a.s. exists $c > 0$ such that for each $\tau_1, \tau_2 \in [0, L]$, 

$$
\overline{d}(\eta(\tau_1), \eta(\tau_2)) \geq c|\tau_1 - \tau_2|^p. \quad (7.4)
$$

Once we have concluded the proof of Theorem 1.1, Lemma 7.3 will imply that the limiting curve $\eta_{zip}$ in that theorem is simple and that its inverse is locally Hölder continuous of any exponent strictly smaller than $2/3$. The simplicity of $\eta_{zip}$ can also be deduced from [GM16a, Corollary 1.2], but the Hölder continuity statement is not obvious from either the Brownian or the $\sqrt{8}/3$-LQG descriptions of $(X_{zip}, d_{zip}, \mu_{zip}, \eta_{zip})$.

**Proof of Lemma 7.3.** Fix $L > 0$ and $p > 3/2$ and let $\beta = 1/p \in (0, 2/3)$. Since $\eta_{zip} \to \bar{\eta}$ uniformly on compact subsets of $[0, \infty)$, we can take the scaling limit of the estimate of Lemma 5.10 to find that

$$
\mathbb{P}[E_\delta] \geq 1 - \delta^{\tilde{\xi}((2-\beta)-2+o(1))},
$$

where

$$
E_\delta := \left\{ \overline{d}(\eta(\tau_1), \eta(\tau_2)) \geq \delta, \forall \tau_1, \tau_2 \in [0, L] \text{ with } |\tau_1 - \tau_2| \geq \delta^{\beta} \right\}.
$$

By the Borel-Cantelli lemma, there a.s. exists $K \in \mathbb{N}$ such that $E_{2^{-k}}$ occurs for each $k \geq K$. If this is the case and $\tau_1, \tau_2 \in [0, L]$ are distinct times with $|\tau_1 - \tau_2| \leq 2^{-\beta K}$, choose $k \geq K$ such that $2^{-\beta(k+1)} \leq |\tau_1 - \tau_2| \leq 2^{-\beta k}$. Then since $E_{2^{-k}}$ occurs,

$$
\overline{d}(\eta(\tau_1), \eta(\tau_2)) \geq 2^{-k} \geq |\tau_1 - \tau_2|^p
$$
with implicit constant depending only on \( \beta \). On the other hand, if \( |\tau_1 - \tau_2| \geq 2^{-\beta K} \), then since \( E_{2^{-K}} \) occurs,

\[
\tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) \geq 2^{-K} \geq |\tau_1 - \tau_2|^p
\]

with implicit constant depending on \( K \). Hence (7.4) holds.

Next we check transience of \( \tilde{\eta} \).

**Lemma 7.4.** Almost surely, the curve \( \tilde{\eta} \) is transient. That is, for each \( \rho > 0 \) there exists \( R > 0 \) such that \( B_\rho(\tilde{\eta}(0); \tilde{d}) \cap \tilde{\eta}([R^2, \infty)) = \emptyset \).

**Proof.** Fix \( \rho > 0 \) and \( \epsilon \in (0,1) \). By Lemma 7.1, we can find \( R > 0 \) such that for each \( n \in \mathbb{N} \), the probability of the event \( G^n(\rho, R) \) defined in that lemma is at least \( 1 - \epsilon \). Hence except on an event of probability at most \( \epsilon \), there is a sequence \( N_0 \subseteq N \) such that \( G^n(\rho, R) \) occurs for each \( n \in N_0 \). If this is the case, then for each \( n \in N_0 \) and each \( T \geq R^2 \), we have \( B_\rho(\eta_\text{zip}(0); d^n_\text{zip}) \cap \eta_\text{zip}([R^2, T]) = \emptyset \). Since \( \Omega^n_\text{zip} \to \tilde{X} \) in the \( D_\text{zip} \)-local HPU topology, it must be the case that \( B_\rho(\tilde{\eta}(0); \tilde{d}) \cap \tilde{\eta}([R^2, T]) = \emptyset \) for each \( T \geq R^2 \). Thus \( B_\rho(\tilde{\eta}(0); \tilde{d}) \cap \tilde{\eta}([R^2, \infty)) = \emptyset \).

Finally we show that \( \tilde{\eta} \) has zero mass.

**Lemma 7.5.** Almost surely, \( \tilde{\mu}(\tilde{\eta}) = 0 \).

**Proof.** Fix \( T > 0 \) and \( \epsilon \in (0,1) \). By Lemma 5.11, we can find \( \rho = \rho(T, \epsilon) > 0 \) such that for each \( n \in \mathbb{N} \), it holds with probability at least \( 1 - \epsilon/3 \) that

\[
\eta^n_\text{zip}([0, T]) \subseteq B_\rho(\eta^n_\text{zip}(0); d^n_\text{zip}). \tag{7.5}
\]

By Lemma 7.1, we can find \( R = R(T, \epsilon) > 0 \) such that for each \( n \in \mathbb{N} \), it holds with probability at least \( 1 - \epsilon/3 \) that

\[
\eta^n_\text{zip}([R^2, \infty)) \cap B_{\rho + 1}(\eta^n_\text{zip}(0); d^n_\text{zip}) = \emptyset. \tag{7.6}
\]

If \( n \in \mathbb{N} \) is such that (7.5) and (7.6) hold, \( \delta \in (0,1) \), and \( x \in Q^n_\pm \cap B_\rho(\eta^n_\text{zip}(0); d^n_\text{zip}) \), then any path of length at most \( \delta \) from \( x \) to a point of \( Q^n_\mp \) must pass through \( \eta^n_\text{zip}([0, R^2]) \). Therefore, with probability at least \( 1 - 2\epsilon/3 \),

\[
B_\delta(\eta^n_\text{zip}([0, T]); d^n_\text{zip}) \subseteq B_\delta(\eta^n_\mp([0, R^2]); d^n_\mp) \cup B_\delta(\eta^n_\pm([0, R^2]); d^n_\pm), \quad \forall \delta \in (0,1).
\]

Since \( X^n_\pm \to X_\pm \) in law in the local GHPU topology and \( \mu_\pm(\eta_\pm) = 0 \), there exists \( \delta \in (0,1) \) such that with probability at least \( 1 - \epsilon/3 \),

\[
\mu^n_\mp(B_\delta(\eta^n_\mp([0, R^2]); d^n_\mp)) + \mu^n_\pm(B_\delta(\eta^n_\pm([0, R^2]); d^n_\pm)) \leq \epsilon.
\]

Hence with probability at least \( 1 - \epsilon \),

\[
\mu^n_\text{zip}(B_\delta(\eta^n_\text{zip}([0, T]); d^n_\text{zip})) \leq \epsilon.
\]

Since \( X^n_\text{zip} \to X_\text{zip} \) in law in the local GHPU topology as \( N \ni n \to \infty \), we see that

\[
P\left[ B_\delta(\tilde{\eta}([0, T])); \tilde{d} \right] \leq \epsilon \geq 1 - \epsilon.
\]

The lemma follows since we took \( T > 0 \) and \( \epsilon \in (0,1) \) arbitrary.

### 7.3 A 1-Lipschitz map from \( X_\text{zip} \) to \( \tilde{X} \)

A priori, we do not have any explicit relationship between the Brownian half-planes \( X_\pm \) and the subsequential limiting metric space \( \tilde{X} \). In this subsection, we will prove that there is a map from the desired limiting space \( X_\text{zip} \) (which we recall is the metric gluing of \( X_\pm \)) to \( \tilde{X} \) which satisfies several properties.
Proposition 7.6. Almost surely, there exists a bijective 1-Lipschitz map $f_{zip} : (X_{zip}, d_{zip}) \to (\tilde{X}, \tilde{d})$ such that $f_{zip} \circ \eta_{zip} = \tilde{\eta}$ and $(f_{zip})_*\mu_{zip} = \tilde{\mu}$. Furthermore, the $d_{zip}$-length of any curve $\gamma$ in $X_{zip}$ which does not intersect $\eta_{zip}$ is the same as the $d$-length of $f_{zip}(\gamma)$.

Once Proposition 7.6 is established, the only remaining step to show that $X_{zip} = \tilde{X}$ as curve-decorated metric measure spaces, and thereby finish the proof of Theorem 1.1, is to show that $f_{zip}$ also preserves the lengths of paths which cross $\eta_{zip}$. This will be accomplished in Section 7.4 using Propositions 6.1 and 6.2. We note that Section 7.4 does not use anything from the present subsection except Proposition 7.6.

The proof of Proposition 7.6 uses only the qualitative properties of the curve $\tilde{\eta}$ established in Section 7.2 and elementary metric space theory. The main step in the proof is establishing the existence of $f_{\pm} : X_\pm \to \tilde{X}$ which are subsequential limits of the maps $f_{\pm} : \tilde{Q}^n_{\pm} \to Q^n_{\pm}$ (see Figure 13) and which give us a decomposition of $\tilde{X} = f_-(X_-) \cup f_+(X_+)$ analogous to the decomposition $X_{zip} = X_- \cup X_+$.

Proposition 7.7. Almost surely, there exist 1-Lipschitz homeomorphisms $f_{\pm} : (X_{\pm}, d_{\pm}) \to (\tilde{X}, \tilde{d})$ such that the following are true.

1. We have $f_-(X_-) \cup f_+(X_+) = \tilde{X}$ and $f_-(X_-) \cap f_+(X_+) = \tilde{\eta}$.
2. $(f_{\pm})_*\mu_{\pm} = \tilde{\mu}$ and $f_{\pm} \circ \eta_{\pm}|_{[0, \infty)} = \tilde{\eta}$.
3. Let $\tilde{d}_{\pm}$ be the internal metric of $\tilde{d}$ on $f_{\pm}(X_{\pm}) \setminus \tilde{\eta}$. Then each $f_{\pm}$ is an isometry from $(X_{\pm} \setminus \eta_{\pm}([0, \infty)), d_{\pm})$ to $(f_{\pm}(X_{\pm}) \setminus \tilde{\eta}, \tilde{d}_{\pm})$.

Before we give the proof of Proposition 7.7, let us explain why it implies Proposition 7.6.

**Proof of Proposition 7.6.** Fix maps $f_{\pm} : X_{\pm} \to \tilde{X}$ satisfying the conditions of Proposition 7.7. Endow $X_- \sqcup X_+$ with the metric $d_- \sqcup d_+$ which restricts to $d_{\pm}$ on $X_{\pm}$ and satisfies $(d_- \sqcup d_+)(x_-, x_+) = \infty$ for $x_- \in X_-$ and $x_+ \in X_+$. Let $f_- \sqcup f_+$ be the map from $X_- \sqcup X_+$ to $\tilde{X}$ which restricts to $f_{\pm}$ on $X_{\pm}$. Then $f_- \sqcup f_+$ is 1-Lipschitz from $(X_- \sqcup X_+, d_- \sqcup d_+)$ to $(\tilde{X}, \tilde{d})$. By condition 2 of Proposition 7.7, $f_-(\eta_{-}(t)) = f_+(\eta_{+}(t)) = \tilde{\eta}(t)$ for each $t \geq 0$. That is, $(f_- \sqcup f_+)(x) = (f_- \sqcup f_+)(y)$ whenever $x, y \in X_- \sqcup X_+$ are identified under the equivalence relation defining $X_{zip}$. By the universal property of the quotient metric, there exists a 1-Lipschitz map $f_{zip} : (X_{zip}, d_{zip}) \to (\tilde{X}, \tilde{d})$ satisfying $f_{zip}|_{X_{\pm}} = f_{\pm}$ (where here we identify $X_{\pm}$ with their images under the quotient map $X_- \sqcup X_+ \to X_{zip}$).

By condition 1 of Proposition 7.7, $f_{zip}$ is surjective. Since each $f_{\pm}$ is injective and $\tilde{\eta}$ is a simple curve (Lemma 7.3), $f_{zip}$ is injective. By condition 2, $f_{zip} \circ \eta_{zip} = \tilde{\eta}$ and by this same condition together with Lemma 7.5, $(f_{zip})_*\mu_{zip} = \tilde{\mu}$. The length-preserving condition for $f_{zip}$ follows from condition 3 of Proposition 7.7 since each curve in $X_{zip}$ which does not intersect $\eta_{zip}$ is contained in either $X_-$ or $X_+$ and since $f_{zip}|_{X_{\pm}} = f_{\pm}$.

In the rest of this subsection we prove Proposition 7.7. The proof is elementary but somewhat technical since we need to check that several properties of the maps $f^n_{\pm}$ are preserved under taking subsequential limits. Since Proposition 7.6 is the only result from this subsection used in Section 7.4, the reader can safely skip the rest of this subsection on a first read.

We start by establishing existence of subsequential limits of the maps $f^n_{\pm}$ and proving some basic properties.

**Lemma 7.8.** Almost surely, there exist 1-Lipschitz maps $f_{\pm} : (X_{\pm}, d_{\pm}) \to (\tilde{X}, \tilde{d})$ and a random subsequence $\mathcal{N}' \subset \mathcal{N}$ such that the following are true.

1. Suppose given a subsequence $\mathcal{N}'' \subset \mathcal{N}'$, a sequence of points $x^n \in \tilde{Q}^n_{\pm}$ for $n \in \mathcal{N}''$, and $x \in X_-$ such that $D_-(x^n, x) \to 0$ as $\mathcal{N}'' \ni n \to \infty$. Then $D_{zip}(f_-^n(x^n), f_-(x)) \to 0$ as $\mathcal{N}'' \ni n \to \infty$; and the same holds with "$+$" in place of "$-$".

2. For each sequence $x^n \to x$ as in condition 1,
\[
\lim_{\mathcal{N}'' \ni n \to \infty} d^n_{\pm}(x^n, \eta^n_{\pm}) = \lim_{\mathcal{N}'' \ni n \to \infty} d^n_{zip}(f^n_{\pm}(x^n), \eta^n_{zip}) = d_{\pm}(x, \eta_-([0, \infty))) = \tilde{d}(f_-(x), \tilde{\eta}).
\]
3. For each subsequence \(N'' \subset N'\) and each sequence of points \(y^n \in Q^n_{\pm} \) for \(n \in N''\) such that \(y^n \to y \in \tilde{X}\) (with respect to \(D_{\text{zip}}\)) as \(N'' \ni n \to \infty\), we can find a compact subset \(A\) of \(Z_{\pm}\) such that \((f^n_{\pm})^{-1}(y^n) \in A\) for each \(n \in N''\).

4. We have \(f_- (X_-) \cup f_+ (X_+) = \tilde{X}\). In fact, for each \(\rho > 0\), there exists \(R > 0\) such that

\[
B_{\rho} \left( \eta(0); \tilde{d} \right) \subset f_- (B_R (\eta_- (0); d_-)) \cup f_+ (B_R (\eta_+ (0); d_+)).
\]

(7.7)

\textbf{Proof.} Fix \(\epsilon \in (0, 1)\). We will show that the objects in the statement of the lemma exist on an event of probability at least \(1 - \epsilon\).

For \(n \in N\) and \(\rho, R > 0\), define the event \(G^n(\rho, R)\) as in Lemma 7.1. By that lemma, for each \(k \in N\), there exists \(R_k > 0\) such that

\[
P \left[ G^n(k, R_k) \right] \geq 1 - 2^{-k} \epsilon.
\]

Let

\[
\tilde{G}^n := \bigcap_{k=1}^{\infty} G^n(k, R_k) \quad \text{and} \quad \tilde{G} := \bigcap_{n \in N} \bigcup_{m \geq n} \tilde{G}^n
\]

so that \(\tilde{G}\) is the event that \(\tilde{G}^n\) occurs for infinitely many \(n \in N\) and \(\tilde{G}^n\) and \(\tilde{G}\) each have probability at least \(1 - \epsilon\). We will check that the conditions in the statement of the lemma on \(\tilde{G}\).

\textbf{Step 1: existence of} \(f_{\pm}\) \text{ and proof of condition 1.} The maps \(f^n_{\pm}\) are 1-Lipschitz and for each \(\rho > 0\), we have \(B_{\rho} (\tilde{\eta}_\pm^n (0); \tilde{d}^n) \to B_{\rho} (\eta_\pm (0); d_\pm)\) in the \(D_{\pm}\)-Hausdorff metric. Furthermore, for each \(n \in N\) and each \(\rho > 0\) we have \(f^n_{\pm} (B_{\rho} (\tilde{\eta}_\pm^n (0); \tilde{d}^n)) \subset B_{\rho} (\eta_\pm^n (0); d_\pm^n)\), which converges to \(B_{\rho} (\eta (0); d)\) in the \(D_{\text{zip}}\)-Hausdorff metric. In particular, each \(f^n (B_{\rho} (\tilde{\eta}_\pm^n (0); \tilde{d}^n))\) is contained in an \(n\)-independent compact subset of \(Z_{\text{zip}}\). By [GM16b, Lemma 2.1], applied along a subsequence of \(N\) for which \(G^n\) occurs, on the event \(G\) there exists a subsequence \(N'' \subset N\) and maps \(f_{\pm}\) as in the statement of the lemma such that condition 1 is satisfied and \(G^n\) occurs for each \(n \in N''\).

\textbf{Step 2: proof of condition 2.} By symmetry it suffices to check condition 2 for \(f_-\). Suppose given a subsequence \(N''\) of \(N'\), a sequence of points \(x^n \in \tilde{Q}^n_{\pm}\) for \(n \in N''\), and an \(x \in X_-\) with \(x^n \to x\). We know by condition 1 that \(f^n(x^n) \to f_-(x)\). Since each \(f^n\) is an isometry from \((\tilde{Q}^n, \tilde{d}^n)\) to \((Q^n, d^n)\) and each path from \(Q^n\) to \(Q^n_{\pm}\) in \(Q^n_{\text{zip}}\) must pass through \(\eta^n_{\pm}\), for each \(n \in N''\) we have

\[
d^n_{\pm} (f^n(x^n), \eta^n) = \tilde{d}^n_{\pm} (x^n, \tilde{\eta}'_\pm^n).
\]

(7.8)

Since \(x^n \to x\) there exists \(k \in \mathbb{N}\) such that \(x^n \in B^k_{\tilde{d}^n_{\pm}} (\tilde{\eta}'_\pm^n (0); \tilde{R})\) for each \(n \in N''\). Since \(d^n_{\pm} \leq d^n_{\text{zip}}\), also \(f^n(x^n) \in B^k_{\tilde{d}^n_{\text{zip}}} (\eta^n_{\pm}(0); d^n_{\text{zip}})\) for each \(n \in N''\). By condition 1 in Lemma 7.1 for \(G^n(2k, R_{2k})\), for each \(n \in N''\) and each \(\tilde{R} \geq R_{2k}\), each of the quantities (7.8) is equal to

\[
\tilde{d}^n_{\text{zip}} \left( x^n, \tilde{\eta}'_\pm^n ([0, \tilde{R}^2]) \right) = d^n_{\text{zip}} \left( x^n, \eta^n_- ([0, \tilde{R}^2]) \right).
\]

(7.9)

By the transience of \(\tilde{\eta}\) (Lemma 7.3) and of \(\eta_-\) (which is immediate from the definition of the Brownian half-plane), we can a.s. find \(\tilde{R} \geq R_{2k}\) such that

\[
d_-(x, \eta_- ([0, \infty))) = d_- (x, \eta_- ([0, \tilde{R}^2])) \quad \text{and} \quad \tilde{d} (f_-(x), \tilde{\eta}) = \tilde{d} \left( f_-(x), \tilde{\eta} ([0, \tilde{R}^2]) \right).
\]

Since \(\tilde{Q}^n_{\text{zip}} \to X_-\) in the \(D_1\)-local HPU topology and \(\Omega^n_{\text{zip}} \to \tilde{X}\) in the \(D_{\text{zip}}\)-local HPU topology, we can take a limit along the subsequence \(N''\) in (7.9) to get condition 2.

\textbf{Step 3: proof of condition 3.} We check the condition for \(f_+\) (which, again, suffices by symmetry). Suppose we are given \(N'' \subset N'\) and \(y^n \in Q^n_{\pm} \) for \(n \in N''\) such that \(y^n \to y \in \tilde{X}\). Since \(Q^n_{\pm} \subset Q^n_{\text{zip}}\) and the latter
converges to $\tilde{X}$ in the $D_{\text{zip}}$-local Hausdorff metric, we can find $k \in \mathbb{N}$ such that $y^n \in B_k(\eta^n_{\text{zip}}(0); d^n_{\text{zip}}) \cap Q^n_-$ for each $n \in \mathbb{N}^*$.

By condition 2 of Lemma 7.1 for the event $G^n(k, R_k)$, we have $y^n \in B_{R_k}(\eta^n_{\text{zip}}(0); d^n_{\text{zip}})$ for each $n \in \mathbb{N}^*$, so $(f^n)^{-1}(y^n) \in B_{R_k}(\tilde{\eta}^n_{\text{zip}}(0); \tilde{d}^n)$ for each such $n$. Since $\tilde{Q}^n_+ \to X_-$ in the $D_-$-local Hausdorff metric, there is a compact subset $A$ of $Z_-$ such that $B_{R_k}(\tilde{\eta}^n_{\text{zip}}(0); \tilde{d}^n) \subset A$ for each $n \in \mathbb{N}^*$. Thus condition 3 is satisfied.

**Step 4: proof of condition 4.** Suppose given $\rho > 0$ and $y \in B_{\rho}(\tilde{y}(0); \tilde{d})$. Since $Q^n_+ \to X$ in the $D_-$-Hausdorff metric, we can find a sequence of points $y^n \in Q^n_{\text{zip}}$ for $n \in \mathbb{N}^*$ such that $y^n \to y$. Either there is a subsequence $\mathbb{N}^*$ of $\mathbb{N}^*$ such that $y^n \in Q^n_+$ or $y^n \in Q^n_{\text{zip}}$ for each $n \in \mathbb{N}^*$. Suppose we are in the former situation. Then condition 3 implies that after passing to a further subsequence, we can arrange that $(f^n)^{-1}(y^n)$ converges to some $x \in X_-$ with respect to $D_-$. By condition 1, $f_-(x) = y$, which gives the first part of condition 4. To obtain (7.7), choose $k \in \mathbb{N}$ with $k > \rho$. Then $y^n \in B_{\rho}(\eta^n_{\text{zip}}(0); d^n_{\text{zip}})$ for large enough $n \in \mathbb{N}^*$, so since $\tilde{G}^n$ occurs for each $n \in \mathbb{N}^*$, $(f^n)^{-1}(y^n) \in B_{R_k}(\tilde{\eta}^n_{\text{zip}}(0); \tilde{d}^n)$. Therefore $y \in B_{R_k}(\eta_-(0); d_-)$.

From Lemma 7.8, we can deduce the following further properties of the maps $f_{\pm}$.

**Lemma 7.9.** Let $f_\pm : (X_\pm, d_\pm) \to (\tilde{X}, \tilde{d})$ and $\mathbb{N}' \subset \mathbb{N}$ be 1-Lipschitz maps and a subsequence satisfying the conditions of Lemma 7.8. Almost surely, the following conditions are satisfied.

1. $f_- \circ \eta_-|_{[0, \infty)} = f_+ \circ \eta_+|_{[0, \infty)} = \tilde{\eta}$.
2. $f_-(X_-) \cap f_+(X_+) = \tilde{\eta}$.
3. For each $x \in X_\pm \setminus \eta_\pm((0, \infty))$ and each $0 < \rho < \frac{1}{3}d_\pm(x, \eta_\pm([0, \infty)))$, the map $f_\pm|_{B_\rho(x; d_\pm)}$ is an isometry onto $B_\rho(f_\pm(x); d)$ (with the metric $d_\pm$ on the domain and the metric $\tilde{d}$ on the range).
4. For $x$ and $\rho$ as in condition 3, we have $\mu_\pm(A) = \mu(\tilde{f}(\pm(A))$ for each Borel set $A \subset B_\rho(x; d_\pm)$.

**Proof.** *Step 1: proof of condition 1.* Each $t \geq 0$ and each $n \in \mathbb{N}'$, we have $f^n_\pm(\tilde{\eta}^n_{\text{zip}}(t)) = \eta^n_\pm(t) = \eta^n_{\text{zip}}(t)$. Furthermore, $D_\pm(\tilde{\eta}^n_\pm(t), \eta_\pm(t)) \to 0$ and $D_{\text{zip}}(\eta^n_{\text{zip}}(t), \tilde{\eta}(t)) \to 0$ as $\mathbb{N}' \ni n \to \infty$. Therefore, condition 1 of Lemma 7.8 implies that $f_\pm(\eta^n(t)) = \tilde{\eta}(t)$, i.e. condition 1 holds.

**Step 2: proof of condition 2.** By condition 1, we have $f_- (X_-) \cap f_+(X_+) \supset \tilde{\eta}$, so we just need to check the reverse inclusion. If $z \in f_- (X_-) \cap f_+(X_+)$, then there exists $x^n \in \tilde{Q}^n_+$ and $x^n_\pm \in \tilde{Q}^n_{\text{zip}}$ for $n \in \mathbb{N}'$ such that $f^n_\pm(x^n_\pm) \to z$ and $f^n_\pm(x^n_\pm) \to z$. This implies that $\tilde{d}^-_{\text{zip}}(f^n_\pm(x^n_\pm), f^n_\pm(x^n_\pm)) \to 0$, so since $\tilde{Q}^n_{\text{zip}}$ intersect only along $\eta^n_{\text{zip}}$,

$$
\tilde{d}^n_{\text{zip}}(x^n_\pm, \tilde{\eta}^n_{\text{zip}}([0, \infty))) = d^n_{\text{zip}}(f^n_\pm(x^n_\pm), \eta^n_{\text{zip}}(t)) \to 0.
$$

(7.10)

By condition 3 of Lemma 7.8, we can find $x^n_\pm \in X_\pm$ such that $x^n_\pm \to x_\pm$. By condition 1 of Lemma 7.8, $f_\pm(x_\pm) = z$. By (7.10) and condition 2 of Lemma 7.8, $z \in \tilde{\eta}$.

**Step 3: proof of condition 3.** By symmetry it suffices to do this for $f_-$. Let $x \in X_- \setminus \eta_-(0, \infty))$ and $0 < \rho < \frac{1}{3}d_-(x, \eta_-(0, \infty)))$ and choose $\epsilon \in (0, 1)$ such that $0 < \epsilon < \frac{1}{3}d_-(x, \eta_-(0, \infty)))$. Let $y_1, y_2 \in B_\rho(x; d_-)$ and choose points $x^n_1, y^n_1, y^n_2 \in \tilde{Q}^n_-$ for $n \in \mathbb{N}'$ such that $D_-(x^n_1, x) \to 0$ and $D_-(y^n_i, y_i) \to 0$ for $i \in \{1, 2\}$. By condition 1 in Lemma 7.8, $D_{\text{zip}}(f^n_-(y_1^n), f^n_-(y_2^n)) \to 0$.

Since $\tilde{Q}^n_- \to \tilde{X}_-$ in the $D_-$-local HPU topology and by condition 2 of Lemma 7.8, for large enough $n \in \mathbb{N}'$,

$$
\tilde{d}^n_-(x^n, \tilde{\eta}^n_-(0, \infty))) > 3\rho + 3\epsilon, \quad \text{and} \quad \tilde{d}^n_-(x^n, y^n_i) < \rho + \epsilon, \quad \forall i \in \{1, 2\}.
$$

If this is the case, then $y^n_1$ and $y^n_2$ are $\tilde{d}^-_{\text{zip}}$-closer to each other than to $\eta^n_-(0, \infty))$, so by the triangle inequality and since every path from $Q^n_-$ to $Q^n_-$ in $Q^n_{\text{zip}}$ must pass through $\eta^n_-(0, \infty))$,

$$
\tilde{d}^n_{\text{zip}}(y^n_1, y^n_2) = d^n_-(f^n_-(y^n_1), f^n_-(y^n_2)) = d^n_{\text{zip}}(f^n_-(y^n_1), f^n_-(y^n_2)).
$$

Taking a limit as $n \to \infty$ shows that $d_-(y_1, y_2) = \tilde{d}(f_-(y_1), f_-(y_2))$. Therefore $f_-$ is distance-preserving on $B_\rho(x; d_-)$.

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We still need to show that \( f_-(B_\rho(x; d_-)) = B_\rho(f_-(x); \tilde{d}) \). It is clear from the preceding paragraph that \( f_-(B_\rho(x; d_-)) \subset B_\rho(f_-(x); \tilde{d}) \), so we just need to prove the reverse inclusion. Since \( f_-(x^n) \to f_-(x) \) and \( \Omega_{zip}^n \to \tilde{\mathcal{X}} \) in the \( D_{zip} \)-local HPU topology,

\[
B_\rho(f_-(x^n); d_{zip}^n) \to B_\rho(f_-(x); \tilde{d})
\]  

(7.11)
in the \( D_{zip} \)-Hausdorff topology. By (7.11), for each \( z \in B_\rho(f_-(x); \tilde{d}) \), there exists a sequence of points \( z^n \in B_\rho(f_-(x^n); d_{zip}^n) \) for \( n \in \mathcal{N} \) such that \( z^n \to z \). By condition 2 of Lemma 7.8 and our choice of \( \rho \), for large enough \( n \in \mathcal{N} \), \( z^n \) is \( d_{zip}^n \)-closer to \( f_-(x^n) \) than to \( \eta_{zip}^n \), so \( z^n \in Q_- \) and

\[
d_{zip}^n(f_-(x^n), z^n) = d_{zip}^n(f_-(x^n), z^n) = \tilde{d}_{zip}^n(x^n, (f^-)^{-1}(z^n)).
\]  

(7.12)

By condition 3 of Lemma 7.8, there is a subsequence \( \mathcal{N}' \) of \( \mathcal{N} \) and a \( y \in X_- \) such that \( (f^-)^{-1}(z^n) \to y \) as \( \mathcal{N}' \ni n \to \infty \). By condition 1 of Lemma 7.8, \( f_-(y) = z \). The left side of (7.12) converges to \( d_{zip}^n(f_-(x), z) \leq \rho \) and the right side converges to \( d_-(x, y) \). Therefore \( y \in B_\rho(x; d_-) \) so since our initial choice of \( z \) was arbitrary, we obtain condition 3.

**Step 4: proof of condition 4.** Let \( x \), \( \rho \), and \( x^n \in \tilde{\mathcal{Q}}^n \) be as above and choose \( \rho' > \rho \) such that \( \rho' < \frac{1}{3}d_-(x, \eta_{zip}^n([0, \infty))) \) and

\[

\mu_{zip}(\partial B_\rho'(x; d_-)) = \tilde{\mu}(\partial B_\rho'(f_-(x); \tilde{d})) = 0.
\]

By this condition together with the local HPU convergence \( \tilde{\mathcal{Q}}^n \to \tilde{\mathcal{X}} \) and \( \Omega_{zip}^n \to \tilde{\mathcal{X}} \) as \( \mathcal{N}' \ni n \to \infty \),

\[

\tilde{\mu}_n |_{B_{\rho'}(x^n; \tilde{d}_n)} \to \mu_- |_{B_{\rho'}(x; d_-)}
\]

(7.13)
in the \( D_--\)Prokhorov metric and

\[

\mu_{zip}^n |_{B_{\rho'}(f_-(x^n); d_{zip}^n)} \to \tilde{\mu} |_{B_{\rho'}(f_-(x); \tilde{d})}
\]

(7.14)
in the \( D_{zip} \)-Prokhorov metric.

Conditional on everything else, for \( n \in \mathcal{N}' \) let \( w^n \) be sampled uniformly from \( \tilde{\mu}_n |_{B_{\rho'}(x^n; \tilde{d}_n)} \) (normalized to be a probability measure) and let \( w \) be sampled uniformly from \( \mu_- |_{B_{\rho'}(x; d_-)} \) (normalized to be a probability measure). By (7.13) \( w^n \to w \) in \( w \), so by the Skorokhod representation theorem, we can couple together \( \{w^n\}_{n \in \mathcal{N}'} \) and \( w \) in such a way that a.s. \( w^n \to w \) as \( \mathcal{N}' \ni n \to \infty \). By condition 1 of Lemma 7.8, \( f_-(w^n) \to f_-(w) \). By our choice of \( \rho' \), for each sufficiently large \( n \in \mathcal{N}' \),

\[
B_{\rho'}(f_-(x^n); d_{zip}^n) = B_{\rho'}(f_-(x^n); d_{zip}^n).
\]

For such an \( n \) the law of \( f_-(w^n) \) is that of a uniform sample from \( \mu_{zip}^n |_{B_{\rho'}(f_-(x^n); d_{zip}^n)} \). By (7.14), the law of \( f_-(w) \) is that of a uniform sample from \( \tilde{\mu} |_{B_{\rho'}(f_-(x); \tilde{d})} \). We similarly infer from (7.13) and (7.14) that

\[
\mu_- (B_{\rho'}(x; d_-)) = \tilde{\mu}(B_{\rho'}(f_-(x); \tilde{d})).
\]

Therefore,

\[
(f_-)_* (\mu_- |_{B_{\rho'}(x; d_-)}) = \tilde{\mu} |_{B_{\rho'}(f_-(x); \tilde{d})},
\]

which implies condition 4 for \( f_- \). By symmetry, the analogous relation holds for \( f_+ \). \( \square \)

Now we can establish the main desired properties of the maps \( f_{\pm} \).

**Proof of Proposition 7.7.** Let \( f_{\pm} : (X_\pm, d_\pm) \to (\tilde{X}, \tilde{d}) \) and \( \mathcal{N}' \subset \mathcal{N} \) be 1-Lipschitz maps and a subsequence satisfying the conditions of Lemma 7.8. We will check the conditions of the proposition statement for \( f_- \); the statement for \( f_+ \) follows by symmetry.

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Condition 1 follows from condition 4 of Lemma 7.8 together with condition 2 of Lemma 7.9. By condition 1 of Lemma 7.9, \( f_\tau \circ \eta - \eta \) is injective. By condition 4 in Lemma 7.9, \( (f_\tau)_* \mu_\nu = \mu_{f_\tau(x_\nu \setminus \eta)} \) and by Lemma 7.5, \( \mu(\eta) = 0 \). Therefore \( (f_\tau)_* \mu_\nu = \mu_{f_\tau(x_\nu \setminus \eta)} \), i.e. condition 2 holds.

Next we check that \( f_\tau \) is a homeomorphism onto its image. We first argue that \( f_\tau \) is injective. Indeed, condition 3 of Lemma 7.9 implies that \( f_\tau(x) \neq f_\tau(y) \) whenever \( x, y \in X_\nu \) and either \( x \) or \( y \) does not belong to \( X_\nu \setminus \eta - ([0, \infty)) \). By condition 1 of Lemma 7.9 and Lemma 7.3, \( f_\tau|_{\eta - ([0, \infty))} \) is injective, so \( f_\tau \) is injective.

The relation (7.7) of Lemma 7.8 implies that if \( \{x_j\}_{j \in \mathbb{N}} \) is a sequence of points in \( X_\nu \) which tends to \( \infty \), then for each \( \rho > 0 \), \( f_\tau(x_j) \) lies outside of \( B_\rho(\eta(0); d) \) for large enough \( j \in \mathbb{N} \). Therefore \( f_\tau \) is a homeomorphism from \( X_\nu \) to \( f_\tau(X_\nu) \) (equipped with the restriction of \( d \), not \( d_\nu \)). In particular, \( f_\tau \) restricts to a homeomorphism from \( X_\nu \setminus \eta - ([0, \infty)) \) to \( f_\tau(X_\nu \setminus \eta) \).

Finally, we check condition 3. Given \( x \in X_\nu \setminus \eta - ([0, \infty)) \), let \( 0 < \rho < \frac{1}{3} d_\nu(x, \eta - ([0, \infty))) \). By condition 3 of Lemma 7.9, \( f_\tau \) maps \( B_\rho(x; d) \) isometrically onto \( B_\rho(f_\tau(x); d) \). The image of any finite continuous path \( \gamma \) in \( f_\tau(X_\nu) \) which does not hit \( \tilde{\eta} \) can be covered by finitely many balls of the form \( B_\rho(f_\tau(x); d) \) for \( x \in X_\nu \setminus \eta - ([0, \infty)) \) and \( 0 < \rho < \frac{1}{3} d_\nu(x, \eta - ([0, \infty))) \). The \( d_\nu \)-length of \( \gamma \) is determined by its restriction to the time intervals which it spends in these balls. Consequently, this \( d_\nu \) length is the same as the \( d_\nu \)-length of \( f_\tau^{-1}(\gamma) \). Therefore, \( f_\tau \) is an isometry from \( (X_\nu \setminus \eta - ([0, \infty)), d) \) to \( (f_\tau(X_\nu \setminus \eta), \tilde{d}_\nu) \).

\[ \square \]

### 7.4 Proof of Theorem 1.1

In this subsection we will conclude the proof of Theorem 1.1 by showing that \( \tilde{X} = X_{zip} \) as elements of \( \mathcal{M}_{\text{GHP}} \). In order to prove this, it remains only to show that the map \( f_{zip} \), of Proposition 7.6 does not decrease distances. This will be accomplished using the results of Section 6. We first use Proposition 6.1 to show that \( f_{zip}^{-1} \) is a.s. Lipschitz with a deterministic Lipschitz constant.

**Lemma 7.10.** Let \( f_{zip} : X_{zip} \to \tilde{X} \) be as in Proposition 7.6. There is a universal constant \( C \geq 1 \) such that almost surely

\[ d_{zip}(x, y) \leq C \tilde{d}(f_{zip}(x), f_{zip}(y)), \quad \forall x, y \in X_{zip}. \]  

(7.15)

**Proof.** Let \( C \geq 1 \) be chosen so that the conclusion of Proposition 6.1 holds for \( \zeta = 1/2 \). Also let \( \tau_1, \tau_2 \geq 0 \). We will now take limits of the paths produced in Proposition 6.1 to show that almost surely

\[ d_{zip}(\eta_{zip}(\tau_1), \eta_{zip}(\tau_2)) \leq C \tilde{d}(\eta(\tau_1), \eta(\tau_2)). \]  

(7.16)

To this end, fix \( \alpha \in (0, 1) \). By Lemma 7.1, we can find \( R = R(\tau_1, \tau_2, \alpha) > 0 \) and \( L = L(\tau_1, \tau_2, \alpha) > 0 \) such that for each \( n \) in our original subsequence \( \mathcal{N} \), it holds with probability at least \( 1 - \alpha/2 \) that

\[ d_{zip}(\eta_{zip}(\tau_1), Q_{zip} \setminus B_R(\eta_{zip}(0); d_{zip})) \geq 2Cd_{zip}(\eta_{zip}(\tau_1), \eta_{zip}(\tau_2)), \quad \forall i \in \{1, 2\}. \]  

(7.17)

and

\[ B_R(\eta_{zip}(0); d_{zip}) \cap \eta_{zip}(\{0, L\}) = \emptyset. \]  

By Proposition 6.1 and our choice of \( C \), we can find \( \delta_0 = \delta(\tau_1, \tau_2, \alpha) > 0 \) such that for each \( \delta \in \{0, \delta_0 \} \) and each sufficiently large \( n \in \mathcal{N} \), it holds with probability at least \( 1 - \alpha/2 \) that there is a path \( \tilde{\gamma}^n \) from \( \eta_{zip}^n(\tau_1) \) to \( \eta_{zip}^n(\tau_2) \) in \( Q_{zip} \), which crosses \( \eta_{zip}^n([0, L]) \) at most \( 2L \delta^{-2} \) times and which has \( d_{zip} \)-length at most \( Cd_{zip}(\eta_{zip}^n(\tau_1), \eta_{zip}^n(\tau_2)) + \delta^{1/2} \). Let \( \tilde{E}_\delta \) be the event that (7.17) and (7.18) hold and such a path \( \tilde{\gamma}^n \) exists; and let \( \tilde{E}_\delta \) be the event that \( E_{\tau_0}^n \) occurs for infinitely many \( n \in \mathcal{N} \), so that \( \mathbb{P}[\tilde{E}_\delta] \geq 1 - \alpha \).

Now fix \( \delta \in \{0, \delta_0 \} \) and suppose that \( \tilde{E}_\delta \) occurs. Let \( N := [2L \delta^{-2}] \). For \( n \in \mathcal{N} \) for which \( E_{\tau_0}^n \) occurs, let \( \tilde{\gamma}^n \) be a path as in the definition of \( E_{\tau_0}^n \). By (7.17) and (7.18), if we choose \( \delta < C\delta_0 \) then \( \tilde{\gamma}^n \cap \eta_{zip}^n \subset \eta_{zip}^n([0, L]) \). Let \( s_0^\delta = 0, s_\nu^\delta = \text{len}(\tilde{\gamma}^n; d_{zip}) \), and for \( j \in [1, N - 1]_\mathbb{Z} \) let \( s_j^\delta \) be the \( j \)th smallest time \( s \in [0, \text{len}(\tilde{\gamma}^n; d_{zip})] \) at which \( \tilde{\gamma}^n \) crosses \( \eta_{zip}^n \); or \( s_\nu^\delta = \text{len}(\tilde{\gamma}^n; d_{zip}) \) if there are fewer than \( j \) such times.

By our choice of \( \tilde{\gamma}^n \), for each \( j \in [1, N]_\mathbb{Z} \) we can choose \( \xi^\delta \in \{-, +\} \) such that \( \tilde{\gamma}^n([s_j^\delta - 1, s_j^\delta]) \subset Q\xi^\delta \). Then
for \( n \in \mathcal{N} \) such that \( E_\delta^n \) occurs,

\[
C_{\text{zip}}(\eta_{\text{zip}}(\tau_1), \eta_{\text{zip}}(\tau_2)) + \delta^{1/2} \geq \sum_{j=1}^{N} d_{\text{zip}}(\tilde{\gamma}^n(s_{j-1}^n), \tilde{\gamma}^n(s_j^n))
\]

\[
= \sum_{j=1}^{N} \tilde{d}_{\xi_j}((f_{\xi_j}^n)^{-1}(\tilde{\gamma}^n(s_{j-1}^n)), (f_{\xi_j}^n)^{-1}(\tilde{\gamma}^n(s_j^n))), \tag{7.19}
\]

where \( f_{\xi_j}^n : \tilde{Q}_{\pm}^n \rightarrow Q_{\pm}^n \subset Q_{\text{zip}}^n \) are the maps defined in (7.3).

By compactness and since each point \( \tilde{\gamma}^n(s_j^n) \) lies in \( \eta_{\text{zip}}([0, L]) \), on the event \( E_\delta \) we can a.s. find a random subsequence \( \mathcal{N}' \subset \mathcal{N} \) such \( E_\delta^n \) occurs for each \( n \in \mathcal{N}' \) and for each \( j \in [0, N]_\mathbb{Z} \), there exists \( \xi_j \in \{-, +\} \) and \( t_j \in [0, L]_\mathbb{Z} \) such that the following is true. We have \( \xi_j^n = \tilde{\xi}_j^n = \xi_j \) for each \( n, n' \in \mathcal{N}' \) and each \( j \in [0, N]_\mathbb{Z} \) and

\[
\lim_{\mathcal{N}' \ni n \rightarrow 0} D_{\xi_j}((f_{\xi_j}^n)^{-1}(\tilde{\gamma}^n(s_j^n)), \hat{\eta}_{\xi_j}(t_j)) = \lim_{\mathcal{N}' \ni n \rightarrow 0} D_{\xi_j}((f_{\xi_j}^n)^{-1}(\tilde{\gamma}^n(s_j^n)), \hat{\eta}_{\xi_j}(t_j-1)) = 0 \tag{7.20}
\]

for each \( j \in [1, N]_\mathbb{Z} \). Note that the reason why we can take the second limit to be zero as well is that \( \eta^n(t) = \eta^n(t) = \eta_{zip}(t) \) for each \( t \geq 0 \).

We have

\[
d_{\text{zip}}(\eta_{\text{zip}}(\tau_1), \eta_{\text{zip}}(\tau_2)) \to \tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) \quad \text{as} \quad \mathcal{N} \ni n \rightarrow \infty
\]

so taking the limit of the left and right sides of (7.19) along the subsequence \( \mathcal{N}' \) and applying (7.20) shows that on \( E_\delta \),

\[
C_{\text{zip}}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) + \delta^{1/2} \geq \sum_{j=1}^{N} d_{\xi_j}((\eta_{\xi_j}(t_j-1), \eta_{\xi_j}(t_j))
\]

The right side of this inequality is at least

\[
\sum_{j=1}^{N} d_{\text{zip}}(\eta_{\text{zip}}(t_j-1), \eta_{\text{zip}}(t_j)) \geq d_{\text{zip}}(\eta_{\text{zip}}(\tau_1), \eta_{\text{zip}}(\tau_2)).
\]

Sending \( \delta \to 0 \) and then \( \alpha \to 0 \) shows that (7.16) holds a.s. for each fixed \( \tau_1, \tau_2 \geq 0 \).

The relation (7.16) holds a.s. for a dense set of times \( \tau_1, \tau_2 \geq 0 \), by continuity it holds a.s. for all such times simultaneously, i.e. the left inequality in (7.15) holds whenever \( x, y \in \eta_{\text{zip}} \). By the last statement in Proposition 7.6, the \( d_{\text{zip}} \)-length of any path in \( X_{\text{zip}} \) which does not hit \( \eta_{\text{zip}} \) except at its endpoints is the same as the \( d \)-length of its image under \( f_{\text{zip}} \). By decomposing a geodesic between given points \( x, y \in X_{\text{zip}} \) into two paths which hit \( \eta_{\text{zip}} \) only at their endpoints and a path between two points of \( \eta_{\text{zip}} \), we obtain (7.15). \( \square \)

In order to show that the Lipschitz constant in Lemma 7.10 is equal to 1, we will use a lower bound for the amount of time that a \( d \)-geodesic spends away from \( \tilde{\eta} \). This bound will be deduced from Proposition 6.2.

**Lemma 7.11.** There is a universal constant \( \beta \in (0, 1) \) such that the following is true. Fix distinct \( \tau_1, \tau_2 \geq 0 \). Almost surely, there exists a \( d \)-geodesic \( \gamma \) from \( \tilde{\eta}(\tau_1) \) to \( \tilde{\eta}(\tau_2) \) such that with \( T_\gamma = \{ t \in [0, d(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) ] : \gamma(t) \notin \tilde{\eta} \} \)

\[
|T_\gamma| \geq \beta \tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)). \tag{7.21}
\]

where here \( |\cdot| \) denotes Lebesgue measure.

**Proof.** Let \( \beta \in (0, 1) \) be chosen so that the conclusion of Proposition 6.2 holds. We will show that (7.21) holds with \( \beta/2 \) in place of \( \beta \) by applying Proposition 6.2 and using that geodesics behave well under Gromov-Hausdorff convergence.

Since Proposition 6.2 is proven only for a fixed time interval \([-Ln^{1/2}, Ln^{1/2}]_\mathbb{Z} \), we first need to ensure that we can restrict attention to such a time interval. By Lemma 7.1, we can find \( R = R(\tau_1, \tau_2, \alpha) > 0 \) and \( L = L(\tau_1, \tau_2, \alpha) > 0 \) such that for each \( n \in \mathcal{N} \), it holds with probability at least \( 1 - \alpha/2 \) that

\[
d_{\text{zip}}(\eta_{\text{zip}}(\tau_1), Q_{\text{zip}}^n \setminus B_R(\eta_{\text{zip}}(0); d_{\text{zip}})) \geq 2d_{\text{zip}}(\eta_{\text{zip}}(\tau_1), \eta_{\text{zip}}(\tau_2)), \quad \forall i \in \{1, 2\}. \tag{7.22}
\]
By Proposition 6.2 and our choice of $\beta$, we can find $\delta_\ast = \delta_\ast(t_1, t_2, \alpha) > 0$ such that for each $\delta \in (0, \delta_\ast]$ and each sufficiently large $n \in \mathcal{N}$, it holds with probability at least $1 - \alpha/2$ that every $d_{\text{zip}}$-geodesic $\gamma^n$ from $\eta^n_{\text{zip}}(t_1)$ to $\eta^n_{\text{zip}}(t_2)$ spends at least $\beta d_{\text{zip}}(\eta^n_{\text{zip}}(t_1), \eta^n_{\text{zip}}(t_2)) - \delta_\ast/2$ units of time at $d_{\text{zip}}$-distance at least $\beta \delta$ away from $\eta^n_{\text{zip}}([0, L])$. Let $F^n_{\delta}$ be the event that this is the case and (7.22) and (7.23) hold; and let $F_{\delta}$ be the event that $F^n_{\delta}$ occurs for infinitely many $n \in \mathcal{N}$, so that $\mathbb{P}[F_{\delta}] \geq 1 - \alpha$.

On $F_{\delta}$, choose for each $n \in \mathcal{N}$ such that $F^n_{\delta}$ occurs a $d_{\text{zip}}$-geodesic $\gamma^n$ from $\eta^n_{\text{zip}}(t_1)$ to $\eta^n_{\text{zip}}(t_2)$. By the Arzéla-Ascoli theorem, on $F_{\delta}$ we can a.s. find a random subsequence $\mathcal{N}' \subset \mathcal{N}$ such that $F^n_{\delta}$ occurs for each $n \in \mathcal{N}'$ and a $\tilde{d}$-geodesic $\gamma$ from $\tilde{\eta}(t_1)$ to $\tilde{\eta}(t_2)$ such that $\gamma^n \to \gamma$ in the $D_{\text{zip}}$-uniform topology. This geodesic $\gamma$ spends at least $\beta \tilde{d}(\tilde{\eta}(t_1), \tilde{\eta}(t_2)) - \delta_\ast/2$ units of time away from $\tilde{\eta}([0, L])$. By passing to the limit in (7.22) and (7.23), we see that $\gamma$ cannot hit $\tilde{\eta}([0, L])$, so spends at least $\beta \tilde{d}(\tilde{\eta}(t_1), \tilde{\eta}(t_2)) - \delta_\ast/2$ units of time away from $\tilde{\eta}$.

For each $\delta \in (0, \delta_\ast]$, a geodesic $\gamma$ as in the preceding paragraph exists with probability at least $1 - \alpha$. By Lemma 7.3, we can choose $\delta \in (0, \delta_\ast]$ such that with probability at least $1 - \alpha$, we have $\delta_\ast/2 \leq (\beta/2) \tilde{d}(\tilde{\eta}(t_1), \tilde{\eta}(t_2))$. Then with probability at least $1 - 2\alpha$, there exists a $\tilde{d}$-geodesic $\gamma$ satisfying (7.21). Since $\alpha$ is arbitrary, we conclude.

![Figure 14: Illustration of the proof of Theorem 1.1. Fix times $t_1, t_2 \geq 0$ and consider a $\tilde{d}$-geodesic $\gamma$ from $\tilde{\eta}(t_1)$ to $\tilde{\eta}(t_2)$. By Lemma 7.11 there exists $\beta \in (0, 1)$ such that $\gamma$ a.s. spends at least a $\beta$-fraction of its time away from $\tilde{\eta}$, so we can find finitely many excursion intervals $[\bar{s}_1, s_1], \ldots, [\bar{s}_N, s_N]$ during which $\gamma$ does not cross $\tilde{\eta}$ whose total length is at least $\beta/2$ times the $\tilde{d}$-length of $\gamma$ (purple). The $\tilde{d}$-length of the restriction of $\gamma$ to each such excursion interval is equal to the $d_{\text{zip}}$-length of its pre-image under $f_{\text{zip}}$ since $f_{\text{zip}}$ preserves the lengths of paths which stay entirely on one of the two sides of $\tilde{\eta}$ (Proposition 7.6). The $\tilde{d}$-length of each of the blue intermediate segments is at most $C$ times the $d_{\text{zip}}$-length of its pre-image under $f_{\text{zip}}$ by Lemma 7.10. Hence the $d_{\text{zip}}$-length of $f_{\text{zip}}^{-1}(\gamma)$ is at most $(1 - \beta/2)C + \beta/2$ times the $\tilde{d}$-length of $\gamma$. Since this is true for almost every pair of times $(t_1, t_2)$, we get $C \leq (1 - \beta/2)C + \beta/2$, so $C = 1$.]

We are now ready to prove our main theorem. See Figure 14 for an illustration of the proof.

**Proof of Theorem 1.1.** The map $f_{\text{zip}} : X_{\text{zip}} \to \tilde{X}$ constructed in Proposition 7.6 is 1-Lipschitz, surjective, and pushes forward $\mu_{\text{zip}}$ to $\tilde{\mu}$ and $\eta_{\text{zip}}$ to $\tilde{\eta}$. We will show that $f_{\text{zip}}$ does not decrease distances, so is an isometry. This will imply that $\tilde{X}$ and $X_{\text{zip}}$ are equivalent elements $\mathcal{M}_{\infty}^{\text{GHP}}$. Since $X_{\text{zip}}^{n} \to \tilde{X}$ in the local GHPU topology as $N \ni n \to \infty$ and our initial choice of subsequence (from which $N$ was extracted) was arbitrary, this will imply that $X_{\text{zip}}^{n} \to X_{\text{zip}}$ a.s. in the local GHPU topology.

By Lemma 7.10, there is a universal constant $C \geq 1$ such that a.s.

$$d_{\text{zip}}(x, y) \leq C \tilde{d}(f_{\text{zip}}(x), f_{\text{zip}}(y)), \quad \forall x, y \in X_{\text{zip}}.$$ (7.24)

Suppose that $C \geq 1$ is the smallest universal constant for which this is the case. We will show that in fact $C = 1$. 67
Let $\beta \in (0, 1)$ be chosen so that the conclusion of Lemma 7.11 is satisfied. Almost surely, for each distinct $\tau_1, \tau_2 \geq 0$ there is a $\tilde{d}$-geodesic $\gamma$ from $\tilde{\eta}(\tau_1)$ to $\tilde{\eta}(\tau_2)$ which spends at least a $\beta$-fraction of its time away from $\tilde{\eta}$. For such a geodesic $\gamma$, we can choose finitely many times

$$0 = s_0 < \tilde{s}_1 < s_1 < \cdots < \tilde{s}_N < s_N < \tilde{s}_{N+1} = \tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2))$$

such that the following hold. For each $j \in [0, N]_\mathbb{Z}$, there exists $t_j \geq 0$ such that $\gamma(\tilde{s}_j) = \tilde{\eta}(t_j)$; for each $j \in [1, N+1]_\mathbb{Z}$, there exists $t_j \geq 0$ such that $\gamma(s_j) = \tilde{\eta}(t_j)$; each segment $\gamma((\tilde{s}_j, s_j))$ for $j \in [1, N]_\mathbb{Z}$ is contained in either $f_{zip}(X_-) \setminus \tilde{\eta}$ or $f_{zip}(X_+) \setminus \tilde{\eta}$; and

$$\sum_{j=1}^N (s_j - \tilde{s}_j) \geq (\beta/2)\tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)). \quad (7.25)$$

We note that the total length of the complementary segments of $\gamma$ (during which it might cross $\tilde{\eta}$ many times) satisfies

$$\sum_{j=0}^N (\tilde{s}_{j+1} - s_j) = \tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) - \sum_{j=1}^N (s_j - \tilde{s}_j). \quad (7.26)$$

By (7.24) and since $\gamma$ is a $\tilde{d}$-geodesic,

$$d_{zip}(\eta_{zip}(t_j), \eta_{zip}(\tilde{t}_{j+1})) \leq C(\tilde{s}_{j+1} - s_j), \quad \forall j \in [0, N]_\mathbb{Z}. \quad (7.27)$$

The curve $\gamma$ is a $\tilde{d}$-geodesic and each segment $\gamma((\tilde{s}_j, s_j))$ is contained in one of $f_{zip}(X_\pm) \setminus \tilde{\eta}$. By the last assertion of Proposition 7.6, $f_{zip}|_{X_\pm \setminus \eta_{zip}}$ is an isometry with respect to the internal metric of $d_{zip}$ on $X_\pm \setminus \eta_{zip}$. Therefore,

$$d_{zip}(\eta_{zip}(\tilde{t}_j), \eta_{zip}(t_j)) = s_j - \tilde{s}_j, \quad \forall j \in [1, N]_\mathbb{Z}. \quad (7.28)$$

By (7.26), (7.27), and (7.28),

$$d_{zip}(\eta_{zip}(\tau_1), \eta_{zip}(\tau_2)) \leq C \sum_{j=0}^N (\tilde{s}_{j+1} - s_j) + \sum_{j=1}^N (s_j - \tilde{s}_j)
= C \left( \tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) - \sum_{j=1}^N (s_j - \tilde{s}_j) \right) + \sum_{j=1}^N (s_j - \tilde{s}_j).$$

Since $C \geq 1$, the above expression only gets larger when we replace the term $\sum_{j=1}^N (s_j - \tilde{s}_j)$ by its lower bound of $(\beta/2)\tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2))$ from (7.25). We therefore have that

$$d_{zip}(\eta_{zip}(\tau_1), \eta_{zip}(\tau_2)) \leq C(1 - \beta/2)\tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)) + (\beta/2)\tilde{d}(\tilde{\eta}(\tau_1), \tilde{\eta}(\tau_2)). \quad (7.29)$$

The inequality (7.29) holds a.s. for a dense set of pairs of times $\tau_1, \tau_2 \geq 0$. By the same argument used at the end of the proof of Lemma 7.10, we infer that (7.24) holds a.s. with $(1 - \beta/2)C + \beta/2$ in place of $C$. By the minimality of $C \geq 1$,

$$C \leq (1 - \beta/2)C + \beta/2,$$

therefore $C = 1$. \hfill \square

Remark 7.12 (Whole-plane cases). In this remark we explain the modifications to our proof of Theorem 1.1 which are needed to prove Theorems 1.2 and 1.3.

The proofs of both of these theorems are essentially identical to the proof of Theorem 1.1. In the case of Theorem 1.2 (resp. Theorem 1.3) one considers the glued peeling procedure as in Section 4 on a pair of UIHPQs’s glued together along their entire boundaries (resp. a single UIHPQ with its positive and negative boundary rays glued together), rather than on a pair of UIHPQs’s glued together along their positive boundary rays. These variants of the glued peeling procedure possess a Markov property analogous to that of the glued peeling procedure considered in Section 4. This enables us to apply the results of Section 3 to
estimate these glued peeling procedures, and we find that all of the statements and proofs in Sections 4, 5, and 6 hold in this setting with only cosmetic changes. We can then apply the results of these sections together with exactly the same argument given in Section 7 to prove Theorems 1.2 and 1.3. Note that we have only a single UIHPQs in the setting of Theorem 1.3, so in this case there is only one map in the analogs of the aforementioned results.

A Index of notation

Here we record some commonly used symbols in the paper, along with their meaning and the location where they are first defined.

- \( Q_-, Q_+, Q_S \): UIHPQs’s; Section 1.2 (c.f. Section 2.1.3).
- \( \lambda_-, \lambda_+, \lambda_S \): boundary path of the UIHPQ; Section 1.2.
- \( Q_{zip} \): SAW-decorated map obtained by gluing \( Q_\pm \); Section 1.2 (c.f. Section 5).
- \( \lambda_{zip} \): gluing interface (=SAW); Section 1.2.
- \( x_\pm = (X_\pm, d_\pm, \mu_\pm, \eta_\pm) \): Brownian half-planes; Section 1.2.
- \( Q_{zip}^n = (Q_{zip}^n, q_{zip}^n, \mu_{zip}^n, \eta_{zip}^n) \): re-scaled glued map; Section 1.2 (c.f. (7.1)).
- \( X_{zip} = (X_{zip}, d_{zip}, \mu_{zip}, \eta_{zip}) \): space obtained by gluing \( X_\pm \); Section 1.2 (c.f. (7.1)).
- \( \Psi(Q, e) \): peeling indicator; (3.1).
- \( f(Q, e) \): peeled quadrilateral; Section 3.1.1.
- \( \text{Peel}(Q, e) \): unbounded component of \( Q \setminus f(Q, e) \); Section 3.1.1.
- \( \mathfrak{S}(Q, e) \): union of bounded components of \( Q \setminus f(Q, e) \); Section 3.1.1.
- \( \text{Co}(Q, e) \): # of edges of \( \partial Q \) disconnected from \( \infty \) by \( f(Q, e) \); Section 3.1.1.
- \( \text{Ex}(Q, e) \): # of exposed edges of \( f(Q, e) \); Section 3.1.1.
- \( A \): subset of \( \partial Q_- \cup \partial Q_+ \) where we start glued peeling process; Section 3.1.1.
- \( \hat{Q}^j \): glued peeling cluster; Section 4.1.
- \( Q^j_\pm \): unexplored UIHPQs’s in the glued peeling procedure; Section 4.1.
- \( J_j \): number of peeled quadrilaterals before the glued peeling process reaches radius \( r \); Section 4.1.
- \( \xi_j \): sign indicating whether \( Q_- \) or \( Q_+ \) is peeled at step \( j \); Section 4.1.
- \( F^j \): filtration for glued peeling process; Section 4.1.
- \( \hat{Y}^j \): number of edges of \( \hat{Q}^j \cap (\partial Q_- \cup \partial Q_+) \); (4.5).
- \( \tilde{Y}^j ; \text{sum of jumps of } \hat{Y}^j \text{ truncated at level } n; (4.6). \)
- \( X^j_\pm \): outer boundary length of \( \hat{Q}^j \cap Q_\pm \); (5.3).
- \( Y^j_\pm \): number of edges of \( Q_\pm \) disconnected from \( \infty \) by \( \hat{Q}^j \); (5.4).
- \( X^j \) and \( Y^j \): union of \( X^j_\pm \) and \( Y^j_\pm \); (5.5).
- \( Z^j \): \( X^j - Y^j \); (5.5).
- \( R(C) \): good radius used in proof of Proposition 6.1; Lemma 6.3.
- \( \tilde{R}(C) \): good radius used in proof of Proposition 6.2; Lemma 6.4.
- \( r_k \) and \( L_k \): radii and boundary lengths for clusters used in Section 6.1; (6.5).
- \( T^n(\delta) \): set of \( \lfloor 2L\delta^{-2} \rfloor \) discrete intervals of length \( \lfloor \delta^2 n^{1/2} \rfloor \in [-Ln^{1/2}, Ln^{1/2}] \); (6.22).
- \( Q_I(C) \) and \( \hat{Q}_I(C) \): good-radius glued peeling cluster for \( I \in \mathbb{T}^n(\delta) \); (6.23).
- \( \tilde{N} \): subsequence along which we have GHPU convergence; Section 7.1.
- \( \mathfrak{X} = (\mathfrak{X}, \mu, \tilde{\mu}, \tilde{\eta}) \): subsequential limiting space; Section 7.1.
- \( (Z_\pm, D_\pm), (Z_{zip}, D_{zip}) \): big metric spaces obtained using Proposition 2.4; Section 7.1.
- \( \tilde{\mathfrak{X}}^n = (\tilde{Q}^n_+, D_\pm, \mu^n_\pm, \eta^n_\pm) \): re-scaled UIHPQs’s viewed as subspaces of \( (Z_\pm, D_\pm) \); Section 7.1.
- \( f^*_\pm \): identity map \( Z_\pm \supset \tilde{Q}^n_\pm \rightarrow Q_\pm \subset Z_{zip} \); Section 7.1.
- \( f_\pm \): subsequential limits of \( f^*_\pm \); Lemma 7.8.
- \( f_{zip} \): 1-Lipschitz map from \( X_{zip} \) to \( \mathfrak{X} \); Proposition 7.6.
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