A nonlinear Ramsey interferometer operating beyond the Heisenberg limit

S. Choi and B. Sundaram
Department of Physics, University of Massachusetts, Boston, MA 02125, USA

We show that a dynamically evolving two-mode Bose-Einstein condensate (TBEC) with an adiabatic, time-varying Raman coupling maps exactly onto a nonlinear Ramsey interferometer that includes a nonlinear medium. Assuming a realistic quantum state for the TBEC, namely the SU(2) coherent spin state, we find that the measurement uncertainty of the “path-difference” phase shift scales as the standard quantum limit \((1/\sqrt{N})\) where \(N\) is the number of atoms, while that for the interatomic scattering strength scales as \(1/N^{7/5}\), overcoming the Heisenberg limit of \(1/N\).

PACS numbers: 03.75.Dg,03.75.Mn,03.75.Gg

High-precision interferometry is one of the most important tools of metrology enabling one to infer various properties of the system under consideration through the measurement of the phase shift. Ramsey interferometry provides a way to extract information about the changes in the system Hamiltonian \(H\) at time \(t\) via the phase shift \(\phi = \int_0^t H(t')dt'/\hbar\). In quantum systems, it is possible to achieve measurement uncertainties approaching the Heisenberg limit \(\Delta \phi \sim 1/N\) where \(N\) is the number of particles conjugate to the phase variable \(\phi\) provided one uses carefully chosen entangled input states such as Schrödinger’s Cat state.

It has long been accepted that the Heisenberg limit is the ultimate limit to measurement; however recently it was shown using parameter estimation theory that measurement uncertainty of the order \(1/N^k\), where \(k\) is the number of parameter-sensitive terms, is possible. In particular, Bose-Einstein condensates (BECs) with two-body collisions may be able to achieve up to \(\Delta \phi \sim 1/N^2\) accuracy in measurement of atom-atom interactions through a modulation the scattering length using Feshbach resonance or density variation due to gravitational gradients. In a related recent work, the quantum limit to the measurement of atomic scattering length was studied by considering the Heisenberg exclusion principle applied to a squeezing Hamiltonian, and finding the optimal spin squeezed state generated using a separate time dependent Hamiltonian.

In this paper, we show that the temporal evolution of a TBEC such as Na atoms in the \(|F = 1, M_F = \pm 1\rangle\) hyperfine states trapped in an optical dipole trap with Raman coupling maps directly onto a nonlinear Ramsey interferometer. A nonlinear interferometer, as opposed to a normal (linear) interferometer, includes nonlinear medium in one or both arms. Such TBEC systems have already been realized experimentally, and as we show, can achieve measurement accuracy better than the Heisenberg limit. It is noted that nonlinear interferometers have been studied previously, although never in the context of BEC.

A quantum interferometer can be described in terms of the angular momentum operators as a transformation operator:

\[
\hat{I} = \hat{B}_-\hat{P}(\phi)\hat{B}_+ = e^{-i\phi\hat{J}_y}.
\] (1)

The 50/50 beam splitter and the phase shifter are given by \(\hat{B}_\pm = \exp(\pm i\pi \hat{J}_x/2)\) and \(\hat{P}(\phi) = \exp(i\phi \hat{J}_y)\) where \(\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \hat{J}_y = \frac{1}{2}(\hat{J}_+ - \hat{J}_-), \hat{J}_z = \hat{a}_1(\hat{a}_2^\dagger \hat{a}_2^\dagger)\) and \(\hat{J}_y = \frac{1}{2}((\hat{a}_1^\dagger \hat{a}_1^\dagger - \hat{a}_2^\dagger \hat{a}_2^\dagger))\) with \(\hat{a}_1\) and \(\hat{a}_2\) being the two annihilation operators for the two input modes into interferometer. For the TBEC with a Raman coupling like that considered here, the two annihilation operators \(\hat{a}_1\) and \(\hat{a}_2\) correspond to the atoms in the two hyperfine states. The time evolution operator \(\hat{U}(t)\) for this system defined by \(|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle\) is:

\[
\hat{U}(t) = \hat{R}^\dagger e^{-i\hat{R}'t}\hat{R},
\] (2)

where \(\hat{R} = e^{-i\pi(\hat{J}_- - \hat{J}_+)/4}\) and \(\hat{R}' = 2\Omega \hat{J}_- - \hat{J}_y^2\). \(\Omega\) is the tunneling coupling and \(q\) is the strength of the scattering interaction between the bosons. As shown earlier, the detuning of the laser from the transition between the two species is set to be zero to make the Hamiltonian diagonal in the \(\hat{J}_z\) representation. This also prevents the generation of an additional geometric phase on top of the dynamical phase.

The overall action of the time evolution operator \(\hat{U}(t)\), Eq. (2), can clearly be mapped onto a nonlinear Ramsey interferometer with the transformation operator

\[
\hat{I} = \hat{B}_-\hat{P}(\phi_1)\hat{S}(\phi_2)\hat{B}_+ = e^{-i\phi_1 \hat{J}_y - i\phi_2 \hat{J}_y^2}/2.
\] (3)

The 50/50 beam splitter and the phase shifter are given by \(\hat{B}_\pm = \exp(-i\pi \hat{J}_y/2)\) and \(\hat{B}_+ = \hat{R}\) are the two 50/50 beam splitters, while \(\hat{P}(\phi_1) = e^{-i\phi_1 \hat{J}_y}\) and \(\hat{S}(\phi_2) = e^{i\phi_2 \hat{J}_y^2}/2\) represent, respectively, the “path-difference” phase shifter and the nonlinear medium. The phase shifts \(\phi_1 = 2\Omega t\) and \(\phi_2 = qt\) are given by the Hamiltonian dynamics. The presence of the intrinsic temporal evolution has to be taken into account in the measurement of \(\phi_1\) and \(\phi_2\) i.e. any measurement will be shifted at the rate of \(2\Omega\) and \(q\) per unit time. Since we are interested in measuring the changes in the phases \(\phi_i, i = 1, 2\), we shall redefine \(\phi_i\) to be \(\phi'_i = 2\Omega t + \phi_1\) and \(\phi'_2 = qt + \phi_2\) and concern ourselves with the measurement of \(\phi_i\).

For our input state we shall consider an SU(2) atomic coherent state or a coherent spin state (CSS), \(|\theta, \phi\rangle\) which is a reasonable quantum state representing a TBEC. It is noted that \(\hat{R}(t)|\theta, \phi\rangle = \sum_{m=-J}^J \hat{R}_m(\theta + \lambda, \phi - \beta)\).
where \( \Delta t \ket{j,m} \) where \( \mathcal{R}^j_m(\theta,\phi) \) is defined

\[
\mathcal{R}^j_m(\theta,\phi) = \left( \frac{2j}{j + m} \right)^{1/2} \cos^{j+m} \left( \frac{\theta}{2} \right) \sin^{j-m} \left( \frac{\theta}{2} \right) \times e^{i(j-m)\phi}.
\]

(4)

Since the azimuthal angle \( \phi \) simply to shifts the origin, we shall only consider CSS with \( \phi = 0 \) in this paper. Exotic input states such as the NOON or the Yurke state[7] will be considered elsewhere as they are currently not yet practical in the context of TBEC.

The simplest possible scenario is to measure the “path-difference” phase shift \( \phi_1 \) while applying a magnetic field to tune \( \phi_2 = 0 \) via the Feshbach resonance i.e. no nonlinear perturbations to the Hamiltonian. This is the standard Ramsey interferometry which has been studied extensively. The fact that a TBEC is used instead of the thermal atoms simply provides clean signals owing to the inherent long range coherence of a condensate. We will first define the positive valued operator measure \( \hat{E}(\phi) \) such that the probability density of the corresponding measurement result is: \( P(\phi) = \text{Tr}[\rho \hat{E}(\phi)] \) where \( \rho \) is the density matrix for the system. As in Ref. [11], we define the normalized phase state

\[
\ket{j,\Phi} = (2j + 1)^{-1/2} \sum_{m_x = -j}^{j} e^{im_x \Phi} \ket{j,m_x}.
\]

so that \( \hat{E}(\phi) \partial \Phi = (2j + 1) \ketbra{j,\Phi} \) and for an arbitrary input state \( \ket{\psi} \), the probability density of the measurement result is

\[
P(\Phi) = \frac{2j + 1}{2\pi} |\bra{\psi} \hat{E}^\dagger \ket{j,\Phi}|^2.
\]

(6)

With a CSS input, the phase measurement gives a probability density distribution

\[
P(\Phi) = \frac{1}{2\pi} \sum_{m_x = -j}^{j} e^{i(\Phi - 2\Delta \Omega)m_x} \times e^{i\Delta \Omega^2 / 2} R^j_m(\theta,\phi) d^j_{m_x,m_x}(\pi/2) \biggr| \biggr|^2,
\]

where \( d^j_{m_x,m_x}(\pi/2) = \langle j,m_x|j,m_x \rangle \) is the Wigner d-matrix:

\[
d^j_{m_x,m_x}(\pi/2) = \langle j,m_x|e^{-i\pi J^z/j^2}|j,m_x \rangle
\]

\[
= \frac{1}{2^{m_x}} \left[ \frac{(j - m_x)(j + m_x)!}{(j - m_x)(j + m_x)!} \right]^{1/2}
\]

\[
\times P_{j-m_x,m_x+m_x}(x = 0)
\]

(7)

for \( m_x - m_z > -1 \) and \( m_z + m_x > -1 \). \( P^{(\alpha,\beta)}(x) \) denote the Jacobi polynomials. Symmetries give \( d^j_{m_x,m_x} = (-1)^{m_x-m_z} d^j_{-m_x,-m_z} \). We plot the probability density in Fig. 1 at various times starting from the initial states \( \theta = \pi/4 \) (Solid line) and \( \theta = \pi/2 \) (Dashed line). The initial Dicke state \( \ket{\theta = 0} \) is not considered as it is orthogonal to the projective measurement on the phase. In order to highlight the effect of nonlinearity on the measurement of \( \phi_1 \), we plot in the left column the case of \( q = 0 \) for comparison with the corresponding results with \( q \neq 0 \) in the right column. We choose \( q = 3/N \) which corresponds to the Josephson regime[9]. The presence of nonlinearity generally degrades the performance of the interferometer as evidenced by the increase in the width of the probability distribution. It is also notable that the probability density becomes multiple peaked after a long time. This may be interpreted as the generation of a superposition state due to nonlinearity as studied by Yurke and Stoler[12]. It is clear that to use the TBEC as an effective interferometer based on projective measurement onto phase states, \( q \) needs to be minimized and the time of measurement must be kept relatively short.

The uncertainty in phase measurement can be stud-
\[
F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{d}{d\Phi} \ln P(\Phi) \right]^2 P(\Phi) d\Phi.
\]

In Fig. 2 we plot the quantity \( \log N[\Delta \Phi] \) where \( N \) is the total number of atoms and \( \Delta \Phi \) is the uncertainty in phase. We used the standard deviation \( \Delta \Phi \) calculated directly from the probability distribution \( P(\Phi) \) (Fig. 1) and the CRLB, where the CRLB effectively gives a time averaged value of the directly calculated uncertainty. It is noted that in all these figures \( \Delta \Phi \geq 1/N^{1/2} \), the standard quantum limit. It is also noted that although the uncertainty associated with the \( \theta = \pi/2 \) state is lower than that of the \( \theta = \pi/4 \) state for \( q = 0 \) it quickly loses this advantage with \( q > 0 \), indicating sensitivity to dephasing due to interatomic collisions. The \( \theta = \pi/4 \) state is therefore a more robust state for interferometry in the presence of nonlinearity.

Next, instead of projective measurement onto a phase state, we consider projective measurement of the atom number difference. The total number of atoms measured indicates the number of “input” atoms while the atom number difference, \( \langle \hat{J}_z(\phi_1', \phi_2') \rangle = \langle \psi(0) | \hat{T}^\dagger(\phi_1', \phi_2') \hat{J}_z(\phi_1', \phi_2') | \psi(0) \rangle \), allows us to infer the phase shift and is equivalent to measuring the number of atoms at each of the output port of a typical Mach-Zehnder interferometer. An analytic expression for \( \langle \hat{J}_z \rangle \) is given by [9]:

\[
\langle \hat{J}_z(\phi_1', \phi_2') \rangle = -\sum_{m=-N/2}^{N/2-1} D(\theta, m) \tan^{-1} \left( \frac{\theta - \pi/2}{2} \right) \times \cos \left( \phi_1' - \phi_2' \left( m + \frac{1}{2} \right) \right),
\]

where we have defined \( D(\theta, m) = C_{N/2+m+1}^{N/2} \), \( \cos^{2N} \left( \frac{\theta - \pi/2}{2} \right) \tan^{-2m} \left( \frac{\theta - \pi/2}{2} \right) \).

Figure 3 shows \( \langle \hat{J}_z \rangle / N \) as a function of the changes in the phase shifts \( \phi_1 \) (Left column) and \( \phi_2 \) (Right column) for the initial CSS \( |\theta = 0, \phi = 0 \rangle \) (Dashed line) and \( |\theta = \pi/4, \phi = 0 \rangle \) (Solid line) at different times \( \Omega t = \pi/4, \pi, 6\pi \) (Top, middle, bottom rows respectively).
where, as found in Ref. [3], the variance is
\[ \langle \Delta J_z \rangle^2 = \langle \hat{J}_z^2 \rangle - \langle \hat{J}_z \rangle^2 \]
with \( \langle \hat{J}_z(t) \rangle = \frac{1}{4} \sum_{m=-N/2}^{N/2-1} D(\theta, m) \tan^{-2}(\theta/2 - \pi/4)(N/2 - m) + \]
\[ D(\theta, m)(N/2 + m + 1) + \sum_{m=-N/2}^{N/2-1} [\frac{1}{4} D(\theta, m) \tan^{-2}(\theta/2 - \pi/4)(N/2 - m - 1)] \cos[2\phi_1' - 2\phi_2'(m + 1)]. \]

Since the denominator of Eq. (11) involves a function of the form
\[ \sin [\phi_1' - \phi_2' (m + \frac{1}{2})], \]
the quantity \( \langle \Delta \phi_k \rangle^2 \) is minimized for the values of \( \phi_1' = \phi_2' (m + \frac{1}{2}) = \pm \pi/2 \). This indicates that the measurement accuracy is dependent on the measurement values, where results such as \( \phi_1' = \pm \pi/2 \) and \( \phi_2' = 0 \) give optimum results. In particular, for a large number of atoms \( N \) one can approximate the coefficient \( D(\theta, m) \) by
\[ \sqrt{\frac{N}{\pi e^{-2(m-N)\sin^2 \theta}/N}}, \]
and replace the sums by integrals \( \int D(\theta, x)dx = N \) and
\[ \int N D(\theta, x)dx = \int x D(\theta, x)dx = N^2. \]
This leads to
\[ \langle \Delta J_z \rangle^2 \sim \alpha N + \beta N^2 \]
and
\[ \langle \Delta \phi_1' \rangle^2 \sim \gamma N \]
where \( \alpha, \beta, \gamma \) are constants so that:

\[ \langle \Delta \phi_k \rangle^2 \sim \frac{1}{\gamma} \left[ \alpha N^{-2k+1} + \beta N^{-2k+2} \right] \quad (11) \]

where \( k = 1, 2 \). For \( k = 1 \) one has \( \Delta \phi_1 \sim 1/N^{1/2} \) i.e. the standard quantum limit in accuracy for the measurement of \( \phi_1 \). On the other hand, it is remarkable that with \( k = 2 \) i.e. measurement of the phase shift due to the interatomic interactions, \( \Delta \phi_2 \sim 1/N^{3/2} < 1/N \), implying that, although not reaching the theoretical limit of \( 1/N^2 \), such measurement for this CSS input state has uncertainty below the Heisenberg limit. We have verified this estimate numerically; the quantity \( \log_N \Delta \phi_k \) calculated as a function of the initial angle of the CSS, \( \theta \) at the optimal values of \( \phi_1 \) and \( \phi_2 \) is plotted in Fig. 4. The solid and the dashed line represent the uncertainty in the measurement of \( \phi_1 \) and \( \phi_2 \) respectively. It is clear that the best result is obtained for \( \theta = 0 \) for the measurement of \( \phi_1 \) and \( \theta = \pi/4 \) for the measurement of \( \phi_2 \). In the bottom panel, we plot the result as a function of atom numbers for these chosen values of \( \theta \). It shows that the result is independent of number of atoms and, on average, \( \Delta \phi_1 \sim 1/N^{1/2} \) and \( \Delta \phi_2 \sim 1/N^{7/5} \) which is indeed very close to the above estimate.

In summary, we have shown that a TBEC with a Josephson-like coupling directly maps onto a nonlinear Ramsey interferometer where the phase shifts due to linear and nonlinear variations in the Hamiltonian are measured. The system is already experimentally available and the state we consider is the realistic coherent spin state rather than some exotic quantum state. It was found that projective phase measurement reaches the standard quantum limit in accuracy while, remarkably, projective number measurement of the phase shifts due to interatomic interactions were found able to overcome the Heisenberg limit, suggesting new implications for quantum metrology.

[1] S. Boixo, S. T. Flammia, C.M. Caves, and JM Geremia, Phys. Rev. Lett 98, 090401 (2007)
[2] A. M. Rey, L. Jiang, and M. D. Lukin cond-mat/0708337
[3] R. G. Unanyan, and M. Fleischhauer, Phys. Rev. Lett 90, 133601 (2003)
[4] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett 78, 586 (1997)
[5] J. Stenger, S. Inouye, D.M. Stamper-Kurn, H.-J. Miesner, A.P. Chikkatur, and W. Ketterle, Nature (London) 396, 345 (1998); H.-J. Miesner, D.M. Stamper-Kurn, J. Stenger, S. Inouye, A.P. Chikkatur, and W. Ketterle, Phys. Rev. Lett. 82, 2228 (1999)
[6] C. C. Gerry, A. Benrous, and R. A. Campos, Phys. Rev. A 66, 013804 (2002)
[7] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A 33, 4033 (1986)
[8] I. Fuentes-Guridi, J. Pachos, S. Bose, V. Vedral, and S. Choi, Phys. Rev. A 66, 022102 (2002)
[9] S. Choi, and N. P. Bigelow, Phys. Rev. A 72, 033612 (2005)
[10] F. T. Arrechi, E. Courten, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972)
[11] B. C. Sanders, and G. J. Milburn, Phys. Rev. Lett. 75, 2944 (1995)
[12] B. Yurke, and D. Stoler, Phys. Rev. A 35, 4846 (1987)