Your Rugby Mates Don’t Need to Know your Colleagues: Triadic Closure with Edge Colors

Laurent Bulteau\textsuperscript{1}, Niels Grüttemeier\textsuperscript{2}, Christian Komusiewicz\textsuperscript{2}, and Manuel Sorge\textsuperscript{*3}

\textsuperscript{1}CNRS, Université Paris-Est Marne-la-Vallée, Paris, France, laurent.bulteau@u-pem.fr
\textsuperscript{2}Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Marburg, Germany, \{niegru,komusiewicz\}@informatik.uni-marburg.de
\textsuperscript{3}Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warsaw, Poland, manuel.sorge@mimuw.edu.pl

November 26, 2018

Abstract

Given an undirected graph $G = (V, E)$ the NP-hard \textsc{Strong Triadic Closure} (STC) problem asks for a labeling of the edges as \textit{weak} and \textit{strong} such that at most $k$ edges are weak and for each induced $P_3$ in $G$ at least one edge is weak. In this work, we study the following generalizations of STC with $c$ different strong edge colors. In \textsc{Multi-STC} an induced $P_3$ may receive two strong labels as long as they are different. In \textsc{Edge-List Multi-STC} and \textsc{Vertex-List Multi-STC} we may additionally restrict the set of permitted colors for each edge of $G$. We show that, under the ETH, \textsc{Edge-List Multi-STC} and \textsc{Vertex-List Multi-STC} cannot be solved in time $2^{o(|V|^2)}$, and that \textsc{Multi-STC} is NP-hard for every fixed $c$. We then proceed with a parameterized complexity analysis in which we extend previous fixed-parameter tractability results and kernelizations for STC [Golovach et al., SWAT ’18, Grüttemeier and Komusiewicz, WG ’18] to the three variants with multiple edge colors or outline the limits of such an extension.

1 Introduction

Social networks represent relationships between humans such as acquaintance and friendship in online social networks. One task in social network analysis is to determine the strength \cite{12,29,30,34} and type \cite{4,32,35} of the relationship signified by each edge of the network. One approach to infer strong ties goes back to the notion of \textit{strong triadic closure} due to Granovetter \cite{11,12} which postulates that, if an agent has strong relations to two other agents, then these two should have at least a weak relation. Following this assertion, Sintos and Tsaparas \cite{30} proposed to find strong ties in social networks by labeling the edges as weak or strong such that the strong triadic closure property is fulfilled and the number of strong edges is maximized.

Sintos and Tsaparas \cite{30} also formulated an extension where agents may have $c$ different types of strong relationships. In this model, the strong triadic closure property only applies to edges of the same strong type. This is motivated by the observation that agents may very well have close connections to agents that do not know each other if the relations themselves arise in segregated contexts. For

\textsuperscript{*}Supported by the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 631163.11, the Israel Science Foundation (grant no. 551145/14), and by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement number 714704.
example, it is quite likely that one’s rugby teammates do not know all of one’s close colleagues. The edge labelings with up to \( c \) strong colors that model this variant of strong triadic closure and the corresponding problem are defined as follows.

**Definition 1.** A \( c \)-labeling \( L = (S^1_L, \ldots, S^c_L, W_L) \) of an undirected graph \( G = (V, E) \) is a partition of the edge set \( E \) into \( c + 1 \) color classes. The edges in \( S^i_L, i \in [c] \), are strong and the edges in \( W_L \) are weak; \( L \) is an STC-labeling if there exists no pair of edges \( \{u,v\} \in S^1_L \) and \( \{v,w\} \in S^2_L \) such that \( \{u,w\} \notin E \).

**Multi Strong Triadic Closure (Multi-STC)**

**Input:** An undirected graph \( G = (V, E) \) and integers \( c \in \mathbb{N} \) and \( k \in \mathbb{N} \).

**Question:** Is there a \( c \)-colored STC-labeling \( L \) with \(|W_L| \leq k\)?

We refer to the special case \( c = 1 \) as **Strong Triadic Closure (STC)**, STC, and thus Multi-STC, is NP-hard [30]. We study the complexity of Multi-STC and two generalizations of Multi-STC which are defined as follows.

The first generalization deals with the case when one restricts the set of possible relations for some agents. Assume, for example, that strong edges correspond to family relations or professional relations. If one knows the profession of some agents, then this knowledge can be modeled by introducing different strong colors for each profession and constraining the sought edge labeling in such a way that each agent may receive only a strong edge corresponding to a familial relation or to his profession. In other words, for each agent we are given a list \( \Lambda \) of allowed strong colors that may be assigned to incident relationships. Formally, we arrive at the following extension of STC-labelings.

**Definition 2.** Let \( G = (V, E) \) be a graph, \( \Lambda : V \to 2^{\{1,2,\ldots,c\}} \) a mapping for some \( c \in \mathbb{N} \), and \( L = (S^1_L, \ldots, S^c_L, W_L) \) a \( c \)-colored STC-labeling. We say that an edge \( \{v,w\} \in E \) satisfies the \( \Lambda \)-list property under \( L \) if \( \{v,w\} \in W_L \) or \( \{v,w\} \in S^i_L \) for some \( \alpha \in \Lambda(v) \cap \Lambda(w) \). We call a \( c \)-colored STC-labeling \( \Lambda \)-satisfying if every edge \( e \in E \) satisfies the \( \Lambda \)-list property under \( L \).

**Vertex-List Multi Strong Triadic Closure (VL-Multi-STC)**

**Input:** An undirected graph \( G = (V, E) \), integers \( c \in \mathbb{N} \) and \( k \in \mathbb{N} \), and vertex lists \( \Lambda : V \to 2^{\{1,2,\ldots,c\}} \).

**Question:** Is there a \( \Lambda \)-satisfying STC-labeling \( L \) with \(|W_L| \leq k\)?

**Multi-STC** is the special case where \( \Lambda(v) = \{1,\ldots,c\} \) for all \( v \in V \). One might also specify a set of possible strong colors for each edge. This can be useful if for a pair of agents certain relations are not possible or implausible. For example, if two rugby players live far apart, it is unlikely that they play rugby together. This more general constraint is formalized as follows.

**Definition 3.** Let \( G = (V, E) \) be a graph, \( \Psi : E \to 2^{\{1,2,\ldots,c\}} \) a mapping for some value \( c \in \mathbb{N} \) and \( L = (S^1_L, \ldots, S^c_L, W_L) \) a \( c \)-colored STC-labeling. We say that an edge \( e \in E \) satisfies the \( \Psi \)-list property under \( L \) if \( e \in W_L \) or \( e \in S^i_L \) for some \( \alpha \in \Psi(e) \). We call a \( c \)-colored STC-labeling \( \Psi \)-satisfying if every edge \( e \in E \) satisfies the \( \Psi \)-list property under \( L \).

This leads to the most general problem of this work.

**Edge-List Multi Strong Triadic Closure (EL-Multi-STC)**

**Input:** An undirected graph \( G = (V, E) \), integers \( c \in \mathbb{N} \) and \( k \in \mathbb{N} \) and edge lists \( \Psi : E \to 2^{\{1,2,\ldots,c\}} \).

**Question:** Is there a \( \Psi \)-satisfying STC-labeling \( L \) with \(|W_L| \leq k\)?

From a more abstract point of view, in STC we are to cover all induced \( P_3 \)'s, the paths on three vertices, in a graph by selecting at most \( k \) edges, a natural graph-theoretic task. Moreover, as we discuss later, all STC-problems studied here have close ties to finding proper vertex colorings in a
related graph, the Gallai graph \[8, 26, 31\] of the input graph \(G\). Hence we are motivated to study these problems from a pure combinatorial and computational complexity point of view in addition to the known applications of Multi-STC in social network analysis \[30, 18\] and plausible applications of the two generalizations proposed above. In a nutshell, we obtain strong hardness results for VL-Multi-STC and EL-Multi-STC, showing that they cannot be solved in \(2^{o(n^2)}\) time on \(n\)-vertex graphs when assuming the Exponential Time Hypothesis (ETH) \[19\]. On the positive side, we show that previous fixed-parameter tractability and kernelization results for STC \[30, 10, 13\] can be extended even to the most general problem EL-Multi-STC when \(c\) is an additional parameter.

**Related Work.** So far, algorithmic work has focused on STC \[10, 13, 22, 30, 23\]. For example, STC is NP-hard even on graphs with maximum degree four \[22\]. Motivated by this NP-hardness, the parameterized complexity of STC was studied. The two main parameters under consideration so far are the number \(k\) of weak edges and the number \(\ell := |E| - k\) of strong edges in an STC-labeling with a minimal number of weak edges. The fixed-parameter tractability for \(k\) follows from a reduction to vertex cover \[30\]. Moreover, STC admits a \(4k\)-vertex kernel \[13\]. For \(\ell\), STC is fixed-parameter tractable but does not admit a polynomial problem kernel \[10, 13\]. Golovach et al. \[10\] considered a further type of generalization of STC where the aim is to color at most \(k\) edges weak such that each induced subgraph isomorphic to a fixed graph \(F\) has at least one weak edge. Another variant of STC asks for a labeling in which some prespecified communities are connected via strong edges \[18, 1\].

**Our Results.** To motivate our main results, we show that for all \(c \geq 1\) Multi-STC, VL-Multi-STC, and EL-Multi-STC are NP-hard. In particular, for all \(c \geq 3\), we obtain NP-hardness even if \(k = 0\). For VL-Multi-STC and EL-Multi-STC, we then show that even an algorithm that is single-exponential in the number \(n\) of vertices of the input graph is unlikely. More precisely, we show that, assuming the ETH, there is no \(2^{o(n)}\)-time algorithm for VL-Multi-STC and EL-Multi-STC even if \(k = 0\) and \(c \in O(n)\). This result is achieved by a compression of 3-CNF formulas \(\phi\) where each variable occurs in a constant number of clauses into graphs with \(O(\sqrt{|\phi|})\) vertices.

We then proceed to a parameterized complexity analysis for the three problems; see Table 1 for an overview. Since all variants are NP-hard even if \(k = 0\), we consider a structural parameter related to \(k\). This parameter, denoted by \(k_1\), is the minimum number of weak edges needed in an STC-labeling for \(c = 1\). Thus, if \(k_1\) is known, then we may immediately accept all instances with \(k \geq k_1\); in this sense one may assume \(k \leq k_1\) for Multi-STC. For VL-Multi-STC and EL-Multi-STC this is not necessarily true due to some border cases of the definition.

The parameter \(k_1\) is relevant for two reasons: First, it allows us to determine to which extent the FPT algorithms for STC carry over to Multi-STC, VL-Multi-STC, and EL-Multi-STC. Second, \(k_1\) has a structural interpretation: it is the vertex cover number of the Gallai graph of the input graph \(G\). We believe that this parameterization might be useful for other problems. The specific results are as follows. We show that Multi-STC is fixed-parameter tractable when parameterized by \(k_1\) by extending the \(4k_1\)-vertex kernelization \[13\] from STC to Multi-STC. When using \(c\) as an additional parameter this yields a \(2^{c+1}k_1\)-vertex kernel for VL-Multi-STC and EL-Multi-STC. We

| Parameter | Multi-STC | VL-Multi-STC | EL-Multi-STC |
|-----------|-----------|--------------|--------------|
| \(k\)     | FPT if \(c \leq 2\), NP-hard for \(k = 0\) for all \(c \leq 3\) | \(O((c+1)^{k_1} \cdot (cm + nm))\) time | \(W[1]\)-hard |
| \(k_1\)   | \(4k_1\)-vertex kernel | \(4k_1\)-vertex kernel | no polynomial kernel |
| \((c, k_1)\) | \(4k_1\)-vertex kernel | \(2^{c+1}k_1\)-vertex kernel | |

3
show that VL-Multi-STC and EL-Multi-STC are more difficult than Multi-STC: by showing that parameterization by $k_1$ alone leads to W[1]-hardness and that both are unlikely to admit a kernel that is polynomial in $c + k_1$. We complement these results by a providing an $O((c + 1)^{k_1} \cdot (|E| + |V| \cdot |E|))$-time algorithm for the most general EL-Multi-STC.

This work is organized as follows. In Section 2, we present our notation and specify the relation of STC and its variants to vertex coloring problems in Gallai graphs. This will provide some first running-time upper bounds and explains why $k_1$ is a natural structural parameter. In Section 3, we provide the NP-hardness results and the ETH-based lower bound for EL-Multi-STC and VL-Multi-STC. In Section 4 we provide fixed-parameter tractability and intractability results and in Section 4 we provide the kernelization algorithms.

2 Preliminaries

Notation. We consider undirected graphs $G = (V, E)$ where $n := |V|$ denotes the number of vertices and $m := |E|$ denotes the number of edges in $G$. For a vertex $v \in V$ we denote by $N_G(v) := \{u \in V \mid \{u, v\} \in E\}$ the open neighborhood of $v$ and by $N_G[v] := N(v) \cup \{v\}$ the closed neighborhood of $v$. For any two vertex sets $V_1, V_2 \subseteq V$, we let $E_G(V_1, V_2) := \{(v_1, v_2) \in E \mid v_1 \in V_1, v_2 \in V_2\}$ denote the set of edges between $V_1$ and $V_2$. For any vertex set $V' \subseteq V$, we let and $E_G(V') := E_G(V', V')$, the set of edges between the vertices of $V'$. We may omit the subscript $G$ if the graph is clear from the context. The subgraph induced by a vertex set $S$ is denoted by $G[S] := (S, E_G(S))$. A proper vertex coloring with $c$ strong colors for some $c \in \mathbb{N}$ is a mapping $a : V \to \{1, \ldots, c\}$ such that there is no edge $\{u, v\} \in E$ with $a(u) = a(v)$. Throughout this work we call a $c$-colored STC-labeling $L$ optimal (for a graph $G$ and lists $\Psi$) if $L$ is $\Psi$-satisfying and the number of weak edges $|W_L|$ is minimal. For the relevant definitions of parameterized complexity such as parameterized reduction and problem kernelization refer to the standard monographs [3, 6, 7, 27]. The Exponential Time Hypothesis (ETH) implies that 3-CNF-SAT cannot be solved in $2^{o(|\phi|)}$ time where $\phi$ denotes the input formula [19].

Gallai Graphs, c-Colorable Subgraphs, and their Relation to STC. Multi-STC can be formulated in terms of so-called Gallai graphs [8, 26, 31].

Definition 4. Given a graph $G = (V, E)$, the Gallai graph $\tilde{G} = (\tilde{V}, \tilde{E})$ of $G$ is defined by $\tilde{V} := E$ and $\tilde{E} := \{(e_1, e_2) \mid e_1$ and $e_2$ form an induced $P_3$ in $G\}$.

The Gallai graph of an $n$-vertex and $m$-edge graph has $O(m)$ vertices and $O(mn)$ edges. Gallai graphs do have restricted structure but for every graph $H$, there is a Gallai graph which contains $H$ as subgraph [26]. For $c = 1$, in other words, for STC, the relation to Gallai graphs is as follows: A graph $G = (V, E)$ has an STC-labeling with at most $k$ weak edges if and only if its Gallai graph has a vertex cover of size at most $k$ [30]. This gives an $O(1.28^k + nm)$-time algorithm by using the current fastest algorithm for Vertex Cover [2]. More generally, a graph $G = (V, E)$ has a $c$-colored STC-labeling with at most $k$ weak edges if and only if the Gallai graph of $G$ has a $c$-colorable subgraph on $m - k$ vertices [30].

In the following, we extend the relation to EL-Multi-STC by considering list-colorings of the Gallai graph. The special cases VL-Multi-STC, Multi-STC, and STC nicely embed into the construction. First, let us define the problem that we need to solve in the Gallai graph formally. Given a graph $G = (V, E)$, we call a mapping $\chi : V \to \{0, 1, \ldots, c\}$ a subgraph-c-coloring if there is no edge $\{u, v\} \in E$ with $\chi(u) = \chi(v) \neq 0$. Vertices $v$ with $\chi(v) = 0$ correspond to deleted vertices. The List-Colorable Subgraph problem is now as follows.


List-Colorable Subgraph

**Input:** An undirected graph $G = (V, E)$ and integers $c \in \mathbb{N}$, $k \in \mathbb{N}$ and lists $\Gamma : V \rightarrow 2^{\{1, \ldots, c\}}$.

**Question:** Is there a subgraph-$c$-coloring $\chi : V \rightarrow \{0, 1, \ldots, c\}$ with $|\{v \in V \mid \chi(v) = 0\}| \leq k$ and $\chi(w) \in \Gamma(w) \cup \{0\}$ for every $w \in V$?

EL-Multi-STC and List-Colorable Subgraph have the following relationship.

**Proposition 1.** An instance $(G, c, k, \Psi)$ of EL-Multi-STC is a Yes-instance if and only if $(\tilde{G}, c, k, \Psi)$ is a Yes-instance of List-Colorable Subgraph, where $\tilde{G}$ is the Gallai graph of $G$.

**Proof.** For any $c$-colored labeling $L$ for $G$ we may define a coloring $\chi_L$ of the vertices of $\tilde{G}$ by setting $\chi_L(e) := i$ for each edge in $S_L^1$, $1 \leq i \leq c$, and $\chi_L(e) = 0$ for each edge in $W_L$. By definition, the $c$-colored labeling $\chi$ is $\Psi$-satisfying if and only if $\chi_L$ satisfies the list constraints in the List-Colorable Subgraph instance, that is, $\chi_L(v) \in \Psi(v) \cup \{0\}$ for each vertex $v$. Moreover, the number of weak edges in $L$ is precisely the number of vertices in $\tilde{G}$ that receive color 0. By symmetric arguments, each subgraph-$c$-coloring $\chi$ that respects $\psi$ and has $k$ vertices $v$ such that $\chi(v) = 0$ defines a $c$-colored labeling $L_\chi$ of $\tilde{G}$ that is $\psi$-satisfying and has $k$ weak edges.

Now consider a $c$-colored $\Psi$-satisfying edge STC-labeling $L$ with at most $k$ edges in $G$. By the above, $\chi_L$ assigns at most $k$ vertices the color 0 and respects the list-constraints. Thus, it remains to show that for all adjacent vertices $u$ and $v$ in $\tilde{G}$ $\chi_L(u) \neq \chi_L(v)$ or $\chi_L(u) = 0$ or $\chi_L(v) = 0$. Assume that $\chi_L(u) \neq 0$ and $\chi_L(v) \neq 0$. Then, the edges $u$ and $v$ are colored with some strong colors $S_L^1$ and $S_L^2$. Since $u$ and $v$ are adjacent in $\tilde{G}$, $u$ and $v$ form a $P_3$ in $G$ since $L$ is a $c$-colored STC-labeling, we have $i \neq j$. Thus, $\chi(u) \neq \chi(v)$.

Conversely, consider a solution $\chi$ for the List-Colorable Subgraph instance $(\tilde{G}, c, k, \Psi)$. By the above, we directly obtain a $c$-colored labeling $L_\chi$ that is $\Psi$-satisfying and has at most $k$ weak edges. Moreover, this labeling is an STC-labeling: Consider a pair of adjacent edges $u$ and $v$ that form a $P_3$ in $G$. If $\chi(u) = 0$ or $\chi(v) = 0$, then one of the two edges is weak in $L_\chi$. Otherwise, we have $\chi(u) \neq \chi(v)$ because $u$ and $v$ are adjacent in $\tilde{G}$. Thus, $L_\chi$ assigns $u$ and $v$ to different strong colors. Hence, $L_\chi$ is an STC-labeling.

The correspondence from Proposition 1 means that we can solve EL-Multi-STC by solving List-Colorable Subgraph on the Gallai graph of the input graph. To this end we give a running time bound for List-Colorable Subgraph. The algorithm for obtaining this running time is a straightforward dynamic program over subsets. Since we are not aware of any concrete result in the literature implying this running time bound, we provide a proof for the sake of completeness.

**Proposition 2.** List-Colorable Subgraph can be solved in $O(3^m \cdot c^2(n + m))$ time. EL-Multi-STC can be solved in $O(3^m \cdot c^2mn)$ time.

**Proof.** We define a dynamic programming table $D$ with entries of the type $D[S, i]$ where $S \subseteq V$ and $i \in \{1, \ldots, c\}$. The aim is to fill $D$ such that for all entries we have $D[S, i] = ‘true’$ if there is a subgraph-$c$-coloring $\chi$ for $G[S]$ such that $\chi(v) \in \{1, \ldots, i\} \cap \Gamma(v)$ for all $v \in S$ and $D[S, i] = ‘false’$ otherwise. Then, the instance is a Yes-instance if and only if $D[S, c] = ‘true’$ for some $S$ such that $|S| \geq n - k$.

The table is initialized for $i = 1$ and each $S \subseteq V$ by setting

$$D[S, 1] := \begin{cases} ‘true’ & \text{if } S \text{ is an independent set } \land \forall v \in S : 1 \in \Gamma(v), \\ ‘false’ & \text{otherwise.} \end{cases}$$

For $i > 1$, the table entries are computed by the recurrence

$$D[S, i] := \begin{cases} ‘true’ & \text{if } \exists S' \subseteq S \text{ such that } S' \text{ is an independent set } \\
\land \forall v \in S' : i \in \Gamma(v) \\ \land D[S \setminus S', i - 1] = ‘true’, \\ ‘false’ & \text{otherwise.} \end{cases}$$
The correctness proof is straightforward and thus omitted. The running time is dominated by the time needed to fill table entries for \( i > 1 \) and can be seen as follows. For each \( i \in \{2, \ldots, c\} \) we consider all partitions of \( V \) into \( S', S \setminus S', \) and \( V \setminus S'. \) These are \( 3^n \) many. For each of them, we check in \( \mathcal{O}(c \cdot (m + n)) \) time whether \( S' \) is an independent set and whether \( i \in \Gamma(v) \) for all \( v \in S'. \)

The running time for EL-Multi-STC follows from Proposition 1 and the fact that the Gaillai graph of a graph \( G \) with \( n \) vertices and \( m \) edges has \( \mathcal{O}(m) \) vertices and \( \mathcal{O}(mn) \) edges.

3 Classical and Fine-Grained Complexity

We first observe that Multi-STC is NP-hard for all \( c. \) For \( c = 2 \) it was claimed that Multi-STC is NP-hard since in the Gallai graph this is exactly the NP-hard Odd Cycle Transversal problem \([30]\). It is not known, however, whether Odd Cycle Transversal is NP-hard on Gallai graphs. Hence, we provide a proof of NP-hardness for \( c = 2 \) and further hardness results for all \( c \geq 3. \)

**Theorem 1.** Multi-STC is NP-hard a) for \( c = 2 \) even on graphs with maximum degree four, and b) for every \( c \geq 3, \) even if \( k = 0. \)

We show Theorem 1 by proving Lemmas 1, 2, and 3 below. Note that the NP-hardness of Multi-STC for fixed \( c = 2 \) which is shown in Lemma 1 also follows from the fact that STC is NP-hard \([30]\) and the reduction in Lemma 3. But since we add large cliques in the construction of Lemma 3 this does not give us the restriction to instances with maximum degree four.

**Lemma 1.** Multi-STC is NP-hard for \( c = 2 \) even on graphs with maximum degree four.

**Proof.** We reduce from the NP-hard NAE-3SAT problem which is defined as follows. We are given a Boolean formula \( \phi \) in conjunctive normal form with clauses of size three. In the following we denote the variables of \( \phi \) by \( x_1, \ldots, x_n \) and its clauses by \( C_1, \ldots, C_m. \) We want to decide whether there is an assignment of truth values to the variables in \( \phi \) such that in each clause, there is at least one true and at least one false literal. We say such an assignment is *satisfying*. NAE-3SAT is well-known to be NP-hard \([9]\). In the following, \( K_{i,j} \) denotes the complete bipartite graph in which one part contains \( i \) vertices and the other part contains \( j \) vertices.

**Gadgets.** The reduction uses the following two gadget graphs. Each variable will be represented by the variable gadget graph as shown for \( m = 4 \) on the right in Figure 1. That is, it is a cycle on \( m \) vertices, where on each edge of the cycle we have added a path of length two to form a triangle. We call the middle vertices of these paths \( c_1, \ldots, c_m \) (see Figure 1). Below we use these vertices to connect the variable gadget, say for variable \( x, \) to the clause gadgets for those clauses that contain \( x. \)

The variable gadget has precisely the following two STC-labelings that do not color any edges weak: (1) coloring the triangles incident with \( c_i \) for odd \( i \) with color 1 and the triangles incident with \( c_i \) for even \( i \) with color 2 and (2) the coloring resulting from switching the strong colors in (1). To see this,
observe that each triangle is colored with exactly one strong color: Otherwise, there are two edges \( e, f \) in a triangle \( T \) that share an endpoint \( v \) on the inner \( m \)-vertex cycle and that have different strong colors. Consider the triangle \( U \) incident with \( v \) that is different from \( T \). The edges incident with \( v \) in \( U \) form an induced \( P_3 \) with both \( e \) and \( f \) and thus cannot have a strong color, a contradiction. Thus, indeed, each triangle has exactly one strong color. Furthermore, neighboring triangles share induced \( P_3 \)s and hence have different strong colors. Thus, there are indeed only two possible STC-labelings without weak edges for the variable gadget. These two labelings shall correspond to the truth value of the variable.

A clause will be represented by a clause gadget graph as shown on the left in Figure 1. That is, it consists of a \( K_{2,3} \) whose vertices are bipartitioned into \( \{a_1, a_2\} \) and \( \{b_1, b_2, b_3\} \), together with three edges adjoined to \( b_1, b_2, \) and \( b_3 \), respectively. We call the adjoined edges connector edges of the gadget. Below we will define the other endpoints of the connector edges in the variable gadgets.

Three crucial properties of the clause gadget are as follows.

First, (P1), each STC-labeling of the gadget uses at least three weak edges. To see this, observe that \( N[a_1] \) and \( N[a_2] \) are isomorphic to \( K_{1,3} \) and do not share any edges. Hence, each STC-labeling colors, for both \( a_1 \) and \( a_2 \) each, one of the incident edges weak. Furthermore, at least one of \( b_1, b_2, \) and \( b_3 \) is adjacent with two strong-colored edges and, since their closed neighborhoods are also isomorphic to \( K_{1,3} \), the corresponding connector edge is weak.

Second, (P2), for each permutation of \( \{b_1, b_2, b_3\} \), there is an STC-labeling of the gadget that colors the connector edge on the first vertex with 1, the one on the second with 2, and the one on the third weak and that uses at most three weak edges. By symmetry, it is enough to show this for the identity permutation. It is not hard to verify that the following labeling of the remaining edges indeed results in an STC-labeling satisfying the requirements: \( \{a_1, b_1\} \mapsto 2, \{a_1, b_2\} \mapsto \text{weak}, \{a_1, b_3\} \mapsto 1, \{a_2, b_1\} \mapsto \text{weak}, \{a_2, b_2\} \mapsto 1, \{a_2, b_3\} \mapsto 2. \)

Third, (P3), each STC-labeling of the gadget that colors at most three edges in this gadget weak, colors at least one connector edge with color 1 and another connector edge with color 2. Suppose for the sake of a contradiction that there is an STC-labeling \( L \) that does not have this property. Thus, there are two connector edges with the same color \( p \). Say without loss of generality that these two connector edges are incident with \( b_1 \) and \( b_2 \), respectively; the other cases are analogous. Note that \( p \) cannot be weak: Otherwise, since \( a_1 \) and \( a_2 \) each do not have two adjacent neighbors, their incident edges carry at least one weak color, a contradiction to the fact that \( L \) colors at most three edges in the gadget weak. Hence, the connector edge incident with \( b_3 \) is colored weak by \( L \) because not both strong colors occur among the connector edges by precondition on \( L \). Let \( q \in \{1, 2\} \setminus \{p\} \). Both the weak color and \( q \) occur among \( \{a_1, b_1\} \) and \( \{a_1, b_2\} \) and the same holds for \( \{a_2, b_1\} \) and \( \{a_2, b_2\} \) since all of these edges have a color different from \( p \) and the two pairs cannot have the same strong color. Thus, the color of both \( \{a_1, b_3\} \) and \( \{a_2, b_3\} \) is not weak since that would exceed the budget. As both \( a_1 \) and \( a_2 \) already have an incident edge colored \( q \), the color of both \( \{a_1, b_3\} \) and \( \{a_2, b_3\} \) is in fact \( p \), a contradiction. Thus, (P3) holds.

Construction. Given formula \( \phi \) we proceed as follows to construct a graph \( G \) in an instance \((G, 3m)\) of Multi-STC with budget \( 3m \), where \( m \) is the number of clauses in \( \phi \). For each variable and each clause, introduce a variable or clause gadget as described above. Then, for each clause \( C \) and each literal \( \ell \) in \( C \), pick a connector edge \( e \) in the clause gadget of \( C \) that has not been used before. Let \( x \) be the variable in \( \ell \). If \( \ell = \neg x \), then let \( d \) be the middle vertex \( c_i \) in the clause gadget for \( x \) such that \( i \) is the smallest odd index that has not been used for this purpose in another clause before. Define \( e \setminus \{b_1, b_2, b_3\} = \{d\} \). Otherwise, if \( \ell = x \), then let \( d \) be the middle vertex \( c_i \) with the smallest even index \( i \) that has not been used before and define \( e \setminus \{b_1, b_2, b_3\} = \{d\} \). This completes the construction.

Correctness. To see that a satisfying assignment for \( \phi \) implies an STC-labeling for \( G \) with at most \( 3m \) weak edges, construct an edge labeling \( L \) as follows. For each variable \( x \), if \( x \) is set to ‘true’, put \( L \), restricted to the variable gadget for \( x \), to be the STC-labeling (1) of that gadget which uses no weak color, that is, use the labeling that colors the edges incident with \( c_i \) for odd \( i \) with color 1. Otherwise, if \( x \) is set to ‘false’, then use STC-labeling (2). For each clause \( C \), pick a literal which
evaluates to ‘true’ and one which evaluates to false. Denote the corresponding variables by $x$ and $y$ and let $e_x$ and $e_y$ be the corresponding connector edges. By property (P2) of clause gadgets, there is an STC-labeling $L_C$ for the clause gadget of $C$ such that $e_x$ has color 1 and $e_y$ has color 2 and the remaining connector edge is weak. Moreover, $L_C$ has exactly three weak edges. Put $L$ restricted to the clause gadget of $C$ to be $L_C$. Clearly, the so-defined labeling $L$ uses at most $3m$ weak edges, as required.

Since $L$ was constructed from STC-labelings for the clause and variable gadgets, to see that $L$ is an STC-labeling for $G$, it is enough to consider the $P_3$s induced by a connector edge and one edge of a variable gadget. We only consider $e_x$, $e_y$ is analogous. Suppose that $x$ occurs positively in $C$. Then the satisfying assignment assigns ‘true’ to $x$. Moreover, we have defined $L$ restricted to the variable gadget for $x$ to use color 1 for the triangle incident with $c_i$ for odd $i$; thus, $L$ uses color 2 for $c_i$ with even $i$. By construction, $e_x$ is incident with some $c_j$ for even $j$. Thus, all $P_3$s involving $e_x$ and an edge of the variable gadget for $x$ use two different strong colors. The case in which $x$ occurs negatively in $C$ is analogous. We thus infer that $L$ is indeed an STC-labeling for $G$ using at most $3m$ weak edges, as required.

To see that an STC-labeling $L$ for $G$ with at most $3m$ weak edges implies a satisfying assignment for $\phi$, construct a truth assignment as follows. By property (P1) of clause gadgets, each clause gadget uses at least 3 weak edges. Thus, each variable gadget does not use any weak edges. By the property of variable gadgets, $L$ restricted to a variable gadget for can be only one of the two possibilities mentioned above. Set $x$ to ‘true’ if the variable gadget for $x$ uses color 1 for $c_i$ with odd $i$ and set $x$ to ‘false’ otherwise.

To see that the so-defined truth assignment is a satisfying assignment, consider a clause $C$. By property (P3) of clause gadgets, it contains a connector edge colored 1 and a connector edge colored 2. Let $e_x$ and $e_y$ be the corresponding connector edges and let $x$ and $y$ be the corresponding variables. Suppose that $x$ and $y$ occur positively in $C$, the other cases are analogous. In that case, $e_x$ is attached to some $c_i$ with even $i$ in $x$’s variable gadget. Hence, the triangle incident with the corresponding vertex is colored with 2. Thus, $L$ uses the STC-labeling (1) and thus, we have set $x$ to ‘true’. Similarly, $L$ uses STC-labeling (2) for the variable gadget for $y$ and thus, we have set $y$ to ‘false’. Thus, indeed, the truth assignment defined above is satisfying.

Lemma 2. \textbf{Multi-STC} is NP-hard for $c = 3$, even if both $k = 0$ and the input graph does not contain a triangle.

Proof. A \textit{proper edge coloring} with $c$ strong colors is a mapping $a: E \rightarrow \{1, \ldots, c\}$ such that there is no vertex $v$ and pair of edges $e, f$ incident with $v$ with $a(e) = a(f)$. Observe that in triangle-free graphs each STC-labeling without weak edges is a proper edge coloring and vice versa. Holyer [16] showed that it is NP-hard to determine whether a cubic, triangle-free graph allows for a proper edge coloring with three colors.

Lemma 3. \textbf{Multi-STC} is NP-hard for every $c \geq 3$, even if $k = 0$.

Proof. We show the following. Given a graph $G$ and integers $c$ and $k$, we can construct in polynomial time a new graph $H$ such that $G$ has an STC-labeling with $c$ strong colors and at most $k$ weak edges if and only if $H$ has an STC-labeling with $c + 1$ strong colors and at most $k$ weak edges. Starting with $H = G$, we make the following modifications to construct $H$. Let $V$ be the vertex set of $G$ and $n = |V|$. Add to $H$ a clique $C$ with $(k + 1)n$ new vertices. Partition the vertex set of $C$ into $n$ parts $C_v$, $v \in V$, of $k + 1$ vertices each. For each vertex $v \in V$ make $v$ adjacent to all vertices in $C_v$. Add $c$ cliques $U_1, \ldots, U_c$, each with $k + 1$ vertices and for each $i \in \{1, \ldots, c\}$ make all vertices of $U_i$ adjacent to all vertices of $C$. Finally, for each $i \in \{1, \ldots, c\}$ add $c$ vertices $u^i_1, \ldots, u^i_c$ and make each of them adjacent to $U_i$. This concludes the construction of $H$.

Given an STC-labeling $L$ for $G$ with $c$ strong colors $1, \ldots, c$ and at most $k$ weak edges, extend $L$ to an STC-labeling for $H$ with $c + 1$ strong colors and at most $k$ weak edges as follows. Label all edges
between $V$ and $C$ with the new strong color $c+1$ which does not occur in $L$. Label all edges in $C \cup U_1$ with strong color 1. Label all edges between $C$ and $U_i$ with strong color $i$ for each $i \in \{2, \ldots, c\}$. Finally, for each pair of integers $i, j \in \{1, \ldots, c\}$, if $i \neq j$ label the edges between $U_i$ and $u_j^i$ with strong color $j$ and, if $i = j$, with strong color $c+1$.

This results in an STC-labeling: For the sake of a contradiction, assume that there is a $P_3$ with the two edges $\{u, v\}, \{v, w\}$ such that $\{u, v\}$ and $\{v, w\}$ receive strong color $i$ and $\{u, w\}$ is not an edge in $H$. As all edges between $V$ and $C$ have strong color $c+1$, both $\{u, v\}$ and $\{u, w\}$ are not in $G$. Furthermore $i \neq c+1$ since, in this case, either $v \in V$ and $\{u, w\}$ is an edge in $H$ or $v = u_j^i$ and $\{u, w\}$ is an edge in $H$. Furthermore, since $C \cup U_1$ is a clique, $i \neq 1$. Finally, $i \notin \{2, \ldots, c\}$, since, otherwise, either $v \in U_i$ and $\{u, w\}$ is an edge in $H$, or $v = u_j^i$ and $\{u, w\}$ is an edge in $H$.

Now let $L$ be an STC-labeling for $H$ with $c+1$ strong colors and at most $k$ weak edges. We claim that $L$ restricted to the edges in $G$ is an STC-labeling with $c$ strong colors and at most $k$ weak edges. It suffices to show that there is a strong color $i \in \{1, \ldots, c+1\}$ such that for each vertex $v \in V$ at least one edge between $v$ and $C_v$ receives color $i$. To this end, first observe that, for each $i \in \{1, \ldots, c\}$, the edges between $U_i$ and $C$ have one and the same color. To see this, observe that there is a vertex $v \in U_i$ such that all its incident edges are strong. Since $v$’s neighbors $u_j^i$ are pairwise nonadjacent, the edges $\{v, u_j^i\}$ receive $c$ distinct strong colors. Since all neighbors of $v$ in $C$ are nonadjacent to the $u_j^i$, all edges between $v$ and $C$ receive the same strong color. Since this observation holds for all $U_i$, each vertex in $C$ is incident with edges between $C$ and the $U_i$ labeled with the same set $X$ of $c$ strong colors. Since $L$ is an STC-labeling, all edges between $V$ and $C$ have strong colors not in $X$. Since each vertex $v \in V$ is neighbor with at least $k+1$ vertices in $C$, at least one of the edges between $v$ and $C$ receives a strong color not from $X$, as required.

From the above construction it follows that, if MULTI-STC is NP-hard for $c$ strong colors (even if $k = 0$), then it is NP-hard for $c+1$ strong colors (even if $k = 0$), as required.

We now provide a stronger hardness result for VL-MULTI-STC and EL-MULTI-STC: we show that they are unlikely to admit a single-exponential-time algorithm with respect to the number $n$ of vertices.

Theorem 2. If the ETH is true, then VL-MULTI-STC cannot be solved in $2^{\sqrt{n}}$ time even if restricted to instances with $k = 0$.

Proof. We give a reduction from 3-SAT to VL-MULTI-STC so that the resulting graph has $O(\sqrt{|\phi|})$ vertices, where $\phi$ is the input formula and $|\phi|$ is the number of variables plus the number of clauses. By the Sparsification Lemma [19], an $2^{\sqrt{\sqrt{|\phi|}}}$-time algorithm for 3-SAT defeats the ETH and, hence, an $2^{\sqrt{|\phi|}}$-time algorithm for VL-MULTI-STC defeats the ETH as well.

Below, we use $n$ for the number of variables in $\phi$. We can furthermore assume that, in the formula $\phi$, each variable occurs in at most four clauses, since arbitrary 3-CNF formulas can be transformed in polynomial time to an equivalent formula fulfilling this restriction while only increasing the formula length by a constant factor [33]. Observe that in such instances the number of clauses in $\phi$ is at most $\frac{4}{3}n$.

Let $\phi$ be a 3-CNF formula with a set $X = \{x_1, \ldots, x_n\}$ of $n$ variables and a set $C := \{C_1, \ldots, C_m\}$ of $m \leq \frac{4}{3}n$ clauses. Let $C_j$ be a clause and $x_i$ a variable occurring in $C_j$. We define the occurrence...
number \( \Omega(C_j, x_i) \) as the number of clauses in \( \{C_1, C_2, \ldots, C_j\} \) that contain \( x_i \). If \( \Omega(C_j, x_i) = r \), we say that the \( r \)th occurrence of variable \( x_i \) is the occurrence in clause \( C_j \). Since each variable occurs in at most four clauses, we have \( \Omega(C_j, x_i) \in \{1, 2, 3, 4\} \).

We describe in three steps how to construct an equivalent instance \((G = (V, E), c = 9n+4, k = 0, \Lambda)\) for \( V\text{-}\textsc{Multi-STC} \) such that \( |V| \in \mathcal{O}(\sqrt{n}) \). First, we describe how to construct a variable gadget. Second, we describe how to construct a clause gadget. In a third step, we describe how these two gadgets are connected. Before we present the formal construction, we give some intuition.

The strong colors \( 1, \ldots, 8n \) represent the true and false assignments of the occurrences of the variables. Throughout this proof we refer to these strong colors as \( T_i^r, F_i^r \) with \( i \in \{1, \ldots, n\} \) and \( r \in \{1, 2, 3, 4\} \). The idea is that a strong color \( T_i^r \) represents a ‘true’-assignment and \( F_i^r \) represents a ‘false’-assignment of the \( r \)th occurrence of variable \( x_i \in X \). The strong colors \( 8n+1, \ldots, 9n+4 \) are auxiliary strong colors which we need for the correctness of our construction. Throughout this proof we refer to these strong colors as \( R_1, \ldots, R_n \) and \( Z_1, Z_2, Z_3, Z_4 \). In the variable gadget, there are four distinct edges \( e_1, e_2, e_3, e_4 \) for each variable \( x_i \) representing the (at most) four occurrences of the variable \( x_i \). Every such edge \( e_r \) can only be labeled with the strong colors \( T_i^r \) and \( F_i^r \). The coloring of these edges represents a truth assignment to the variable \( x_i \). In the clause gadget, there are \( m \) distinct edges such that the coloring of these edges represents a choice of literals that satisfies \( \phi \). The edges between the two gadgets make the values of the literals from the clause part consistent with the assignment of the variable part. The construction consists of five layers of vertices. In the variable gadget we have an upper- a middle- and a down layer \((U^X, M^X \text{ and } D^X)\). In the clause gadget we have an upper and a down layer \((U^C \text{ and } D^C)\). Figure 2 shows a sketch of the construction.

The Variable Gadget. The vertices of the variable gadget consist of an upper layer, a middle layer and a lower part. The vertices in the middle layer and the lower layer form a variable-representation gadget, where each edge between the two parts represents one occurrence of a variable. The vertices in the upper layer form a variable-soundness gadget, which we need to ensure that for each variable either all occurrences are assigned ‘true’ or all occurrences are assigned ‘false’. We start by describing the variable-representation gadget. Let

\[
M^X := \{ \gamma_t^r \mid t \in \{1, \ldots, \lceil \sqrt{n} \rceil \}, r \in \{1, 2, 3, 4\} \} \quad \text{be the set of middle vertices},
\]

\[
D^X := \{ \delta_t \mid t \in \{1, \ldots, \lceil \sqrt{n} \rceil + 9 \} \} \quad \text{be the set of lower vertices}.
\]

We add edges such that \( D^X \) becomes a clique in \( G \). To specify the correspondence between the variables in \( X \) and the edges in the variable-representation gadget, we define below two mappings \( \text{mid}^X : X \to \{1, \ldots, \lceil \sqrt{n} \rceil \} \) and \( \text{down}^X : X \to \{1, \ldots, \lceil \sqrt{n} \rceil + 9 \} \). Then, for each \( x_i \in X \) we add four edges \( \{ \gamma_{\text{mid}^X(x_i)}^r, \delta_{\text{down}^X(x_i)} \} \) for \( r \in \{1, 2, 3, 4\} \). We now carefully define the two mappings \( \text{mid}^X \), \( \text{down}^X \) and the vertex lists \( \Lambda(v) \) for every \( v \in M^X \cup D^X \) of the variable-representation gadget. The chosen truth assignment for each variable will be transmitted to a clause by edges between the variable and clause gadgets. To ensure that each such transmitter edge is used for exactly one occurrence of one variable, we first define the variable-conflict graph \( H^X_\phi := (X, \text{Conf}^X) \) by \( \text{Conf}^X := \{ \{x_i, x_j\} \mid x_i \text{ and } x_j \text{ occur in the same clause } C \in C \} \), which we use to define \( \text{mid}^X \) and \( \text{down}^X \). Since every variable of \( \phi \) occurs in at most four clauses, the maximum degree of \( H^X_\phi \) is at most 8. Hence, there is a proper vertex 9-coloring \( \chi : X \to \{1, 2, \ldots, 9\} \) for \( H^X_\phi \) which we compute in polynomial time by a folklore greedy algorithm. We end up with 9 color classes \( \chi^{-1}(1), \ldots, \chi^{-1}(9) \). Then, we partition each color class \( \chi^{-1}(i) \) into \( \frac{9}{\lceil \sqrt{n} \rceil} \) groups arbitrarily such that each group has size at most \( \lceil \sqrt{n} \rceil \). Let \( s \) be the overall number of such groups and let \( S := \{S_1, S_2, \ldots, S_s\} \) be the family of all such groups of vertices in \( H^X_\phi \) (each corresponding to pair of a color \( i \in \{1, \ldots, 9\} \) and a group in \( \chi^{-1}(i) \)). The following claim is directly implied by the definition of \( S \) (for part (b) observe that at most \( \lceil \sqrt{n} \rceil \) new groups are introduced during the partitioning of the color classes).

**Claim 1.** For the family \( S := \{S_1, S_2, \ldots, S_s\} \) of groups of vertices in \( H^X_\phi \), it holds that
(a) we conclude mid and therefore elements of different groups of \( S \)

Proof

\begin{align*}
\text{Claim 3.} & \\
\text{If } x_i, \ldots, x_j \text{ are adjacent in } H_\phi^X \text{, then } \text{down}^X(x_i) \neq \text{down}^X(x_j). \\
\text{Proof.} & \\
\text{By definition, } x_i \text{ and } x_j \text{ are adjacent in } H_\phi^X. \text{ Hence, } x_i \text{ and } x_j \text{ are in different color classes and therefore elements of different groups of } \mathcal{S}. & \\
\text{Claim 2.} & \\
\text{If } x_i, x_j \in X \text{ occur in the same clause } C \in \mathcal{C}, \text{ then } \text{down}^X(x_i) \neq \text{down}^X(x_j). & \\
\text{Proof.} & \\
\text{By definition, } x_i \text{ and } x_j \text{ are adjacent in } H_\phi^X. \text{ Hence, } x_i \text{ and } x_j \text{ are in different color classes and therefore elements of different groups of } \mathcal{S}. & \\
\text{Claim 3.} & \\
\text{Let } x_i, x_j \in X \text{ and } r \in \{1, 2, 3, 4\}. \text{ If } x_i \neq x_j, \text{ then} & \\
\{\gamma_{\text{mid}^X(x_i)}, \delta_{\text{down}^X(x_i)}\} & \neq \{\gamma_{\text{mid}^X(x_j)}, \delta_{\text{down}^X(x_j)}\}. \quad \square
\end{align*}
Proof. Without loss of generality, $i < j$. Obviously, the claim holds if $\down^X(x_i) \neq \down^X(x_j)$. Let $\down^X(x_i) = \down^X(x_j)$. Then, there is at least one more occurrence of $\down^X(x_i)$ in the partial sequence $\text{Seq}_i^X$ compared to $\text{Seq}_j^X$. Therefore, $\mid^X(x_i) \neq \mid^X(x_j)$. ◯

Thus we assigned a unique edge in $E(M^X, D^X)$ to each occurrence of a variable in $X$. Furthermore, the assigned edges of variables that occur in the same clause do not share an endpoint in $D^X$ (Claim 2).

We complete the description of the variable-representation gadget by defining the vertex list $\Lambda(v)$ for every $v \in M^X \cup D^X$. We set

$$\Lambda(\gamma^r_i) := \bigcup_{x_i \in X} \{T^r_i, F^r_i, R_j\} \quad \text{for every } \gamma^r_i \in M^X,$$

and

$$\Lambda(\delta_i) := \bigcup_{x_i \in X} \{T^1_i, T^2_i, T^3_i, T^4_i, F^1_i, F^2_i, F^3_i, F^4_i, Z_2\} \quad \text{for every } \delta_i \in D^X.$$

Claim 4. Let $x_i \in X$ and $r \in \{1, 2, 3, 4\}$. Then, $\Lambda(\gamma^r_{\mid^X(x_i)}) \cap \Lambda(\delta_{\down^X(x_i)}) = \{T^r_i, F^r_i\}$.

Proof. Let $\Lambda(i, r) := \Lambda(\gamma^r_{\mid^X(x_i)}) \cap \Lambda(\delta_{\down^X(x_i)})$. Obviously, $T^r_i, F^r_i \in \Lambda(i, r)$. It remains to show that there is no other strong color $Y \in \Lambda(i, r)$.

**Case 1:** $Y = Z_2$. Then, $Z_2 \not\in \Lambda(\gamma^r_i)$ and it follows $Y \not\in \Lambda(i, r)$.

**Case 2:** $Y = R_j$ with $j \in \{1, \ldots, n\}$. Then, $R_j \not\in \Lambda(\delta_i)$ and it follows $Y \not\in \Lambda(i, r)$.

**Case 3:** $Y = T^r_j$ or $Y = F^r_j$ with $r' \neq r$ and $j \in \{1, \ldots, n\}$. Then, $Y \not\in \Lambda(\gamma^r_{\mid^X(x_i)})$ and it follows $Y \not\in \Lambda(i, r)$.

**Case 4:** $Y = T^r_i$ or $Y = F^r_j$ with $i \neq j$. Assuming $T^r_i \in \Lambda(i, r)$ it follows from the definition of $\Lambda$ that there is some variable $x_i \neq x_i$ such that $\down^X(x_i) = \down^X(x_i)$ and $\mid^X(x_i) = \mid^X(x_i)$, which contradicts Claim 3. Hence, $Y \not\in \Lambda(i, r)$. ◯

Note that for each variable $x_i$ there are four edges $\{\gamma^r_{\mid^X(x_i)}', \delta_{\down^X(x_i)}' \mid r \in \{1, 2, 3, 4\}\}$ that can only be colored with the strong colors $T^r_i$ and $F^r_j$ representing the truth assignments of the four occurrences of variable $x_i$. We need to ensure that there is no variable $x_i$, where, for example, the first occurrence is set to ‘true’ ($T^1_i$) and the second occurrence is set to ‘false’ ($F^2_i$) in a $\Lambda$-satisfying STC-labeling with no weak edges. To this end, we describe how to construct the variable-soundness gadget.

Define

$$U^X := \{\alpha^{(r,r')} \mid t \in \{1, \ldots, \lceil \sqrt{n} \rceil + 9\}, (r, r') \in \{1, 2, 3, 4\}^2, r \neq r'\}$$

to be the set of upper vertices. We add edges such that the vertices in $U^X$ form a clique in $G$. To specify the correspondence between the variables and the edges in the variable-soundness gadget, we define below a mapping $\text{up}^X : X \to \{1, 2, \ldots, \lceil \sqrt{n} \rceil + 9\}$. The main idea of the variable-soundness gadget is that for each variable $x_i \in X$ and each pair $(r, r') \in \{1, 2, 3, 4\}$ there are four edges between the vertices $\gamma^r_i, \gamma^{r'}_i$ and the vertices $\alpha^{(r,r')}, \alpha^{(r',r)}$ of $U^X$ which can not all be strong in a $\Lambda$-satisfying STC-Labeling if $\{\gamma^r_{\mid^X(x_i)}, \delta_{\down^X(x_i)}\}$ has strong color $T^r_i$ and $\{\gamma^{r'}_{\mid^X(x_i)}, \delta_{\down^X(x_i)}\}$ has strong color $F^{r'}_i$. (Recall that we do not allow weak edges.) To this end, we assign a set of 12 endpoints in $U^X$ to each variable $x_i$. We need to ensure in particular that two variables $x_i, x_j$ with $\mid^X(x_i) = \mid^X(x_j)$ do not use the same endpoints in $U^X$. We define $\text{up}^X(x_i) := \down^X(x_i)$. The following claim directly follows from Claim 3.

Claim 5. Let $x_i, x_j \in X$ with $x_i \neq x_j$. If $\mid^X(x_i) = \mid^X(x_j)$, then $\text{up}^X(x_i) \neq \text{up}^X(x_j)$.

We add the following edges between the vertices of $M^X$ and $U^X$: For every variable $x_i$, every $r \in \{1, 2, 3, 4\}$, and every $r' \in \{1, 2, 3, 4\} \setminus \{r\}$ we add the edges $\{\alpha^{(r,r')}_{\text{up}^X(x_i)}, \gamma^{r'}_{\mid^X(x_i)}\}$ and $\{\alpha^{(r,r')}_{\text{up}^X(x_i)}, \gamma^r_{\mid^X(x_i)}\}$.
We complete the description of the variable-soundness gadget by defining the vertex lists \( \Lambda(v) \) for each \( v \in U^X \). We set
\[
\Lambda(\alpha_i^{(r,r')}) := \bigcup_{x_j \in X} \{ T_i^x, F_i^x, R_i, Z_i \} \quad \text{for every } \alpha_i^{(r,r')} \in U^X.
\]

**Claim 6.** Let \( x_i \in X \), let \( r, r' \in \{1, 2, 3, 4\} \), and let \( r' \in \{1, 2, 3, 4\} \setminus \{r\} \). Then
\[
\quad a) \ \Lambda(\alpha_{up^X(x_i)}^i) \cap \Lambda(\gamma_{mid^X(x_i)}^r) = \{ T_i^r, R_i \}, \text{ and } \\
\quad b) \ \Lambda(\alpha_{up^X(x_i)}^i) \cap \Lambda(\gamma_{mid^X(x_i)}^{r'}) = \{ F_i^{r'}, R_i \}.
\]

**Proof.** We first prove statement (a). Let \( \Lambda(i, r, r') := \Lambda(\alpha_{up^X(x_i)}^i) \cap \Lambda(\gamma_{mid^X(x_i)}^r) \). Clearly, \( T_i^r, R_i \in \Lambda(i, r, r') \). It remains to show that there is no other strong color \( Y \in \Lambda(i, r, r') \). Recall that
\[
\Lambda(\gamma_i^r) := \bigcup_{x_j \in X} \{ T_j^r, F_j^r, R_j \},
\]

In the following case distinction we consider every possible strong color \( Y \in \Lambda(\gamma_{mid^X(x_i)}^r) \).

**Case a.1:** \( Y = R_j \) or \( Y = T_j^r \) for some \( j \neq i \). Then, there is a variable \( x_j \neq x_i \) with \( \text{mid}^X(x_j) = \text{mid}^X(x_i) \). It follows by Claim 5 that \( \text{up}^X(x_j) \neq \text{up}^X(x_i) \) and therefore \( R_j, T_j^r \notin \Lambda(\alpha_{up^X(x_j)}^i) \). Hence, \( Y \notin \Lambda(i, r, r') \).

**Case a.2:** \( Y = F_j^r \). Then, since
\[
\{ F_j^p \mid p \in \{1, 2, 3, 4\}, t \in \{1, \ldots, n\} \} \cap \Lambda(\alpha_{up^X(x_i)}^i) \subseteq \{ F_1^{r'}, F_2^{r'}, \ldots, F_n^{r'} \} \quad \text{and } r' \neq r
\]
we conclude \( F_j^r \notin \Lambda(\alpha_{up^X(x_j)}^i) \). Hence, \( Y \notin \Lambda(i, r, r') \).

Next, we prove statement (b) which works analogously. Let \( \Lambda(i, r, r') := \Lambda(\alpha_{up^X(x_i)}^i) \cap \Lambda(\gamma_{mid^X(x_i)}^{r'}) \). Clearly, \( \{ F_i^{r'}, R_i \} \subseteq \Lambda(i, r, r') \). It remains to show that there is no other color \( Y \in \Lambda(i, r, r') \).

**Case b.1:** \( Y = R_j \) or \( Y = F_j^{r'} \) for some \( j \neq i \). Then, analogously to Case a.1 we conclude that \( Y \notin \Lambda(i, r, r') \).

**Case b.2:** \( Y = T_j^{r'} \). Then, since
\[
\{ T_j^p \mid p \in \{1, 2, 3, 4\}, t \in \{1, \ldots, n\} \} \cap \Lambda(\alpha_{up^X(x_i)}^i) \subseteq \{ T_1^{r'}, T_2^{r'}, \ldots, T_n^{r'} \} \quad \text{and } r' \neq r
\]
we conclude \( T_j^{r'} \notin \Lambda(\alpha_{up^X(x_j)}^i) \). Hence, \( Y \notin \Lambda(i, r, r') \). This completes the proof of Claim 6. \( \diamond \)

This completes the description of the variable gadget. For an illustration of the variable-representation and the variable-soundness gadget for some variable \( x_i \) see Fig. 3. We continue with the description of the clause gadget.

The Clause Gadget. The clause gadget consists of an upper part and a lower part. Let \( U^C := \{ \eta_i \mid i \in \{1, \ldots, 12\lceil \sqrt{n} \rceil + 1\} \} \) be the set of upper vertices and \( D^C := \{ \theta_i \mid i \in \{1, \ldots, \lceil \sqrt{n} \rceil \} \} \) be the set of lower vertices. We add edges such that \( U^C \) and \( D^C \) each form cliques in \( G \).

Recall that for some clause \( C_j \in C \) and a variable \( x_i \) occurring in \( C_j \) the occurrence number \( \Omega(C_j, x_i) \) is defined as the number of clauses in \( \{ C_1, C_2, \ldots, C_j \} \) that contain \( x_i \). Below we define two mappings \( \text{up}^C : C \rightarrow \{1, 2, \ldots, 12\lceil \sqrt{n} \rceil + 1\} \), \( \text{down}^C : C \rightarrow \{1, 2, \ldots, \lceil \sqrt{n} \rceil \} \), and vertex lists \( \Lambda : V \rightarrow 2^{\{1, \ldots, c\}} \).

Then, for each clause \( C_j \in C \) we add an edge \( \{ \eta_{\text{up}^C(C_j)}, \theta_{\text{down}^C(C_j)} \} \). Next, we ensure that this edge
with similar arguments as Claim C contains a variable \( x_i \) and a clause \( C_j \) such that \( \text{down}^X(x_i) = \text{down}^X(x_j) \) then \( \text{up}^C(C_j) = \text{up}^C(C_j) \).

**Proof.** By definition, \( C_{j_1} \) and \( C_{j_2} \) are adjacent in \( H^C_\phi \). Hence, \( C_{j_1} \) and \( C_{j_2} \) are elements of different color classes and therefore \( \text{up}^C(C_{j_1}) \neq \text{up}^C(C_{j_2}) \).
Next, we define $down^C$ analogously to $up^X$. To this end consider the finite sequence $Seq_1^m = (up^X(C_1), up^X(C_2), \ldots, up^X(C_n))$ and define $down^Y(C_j)$ as the number of occurrences of $up^X(C_j)$ in the finite sequence $Seq_1^i := (up^X(C_1), \ldots, up^X(C_j))$. From the fact that each color class contains at most $\lceil \sqrt{n} \rceil$ elements, we conclude $down^Y(C_j) \leq \lceil \sqrt{n} \rceil$. The following claim is similar to Claim 3.

**Claim 8.** Let $C_i, C_j \in \mathcal{C}$. If $C_i \neq C_j$, then $\{\eta_{up^X(C_i)}, \theta_{down^Y(C_j)}\} \neq \{\eta_{up^X(C_j)}, \theta_{down^Y(C_i)}\}$.

**Proof.** Without loss of generality, $i < j$. The claim obviously holds if $up^X(C_i) \neq up^X(C_j)$, so let $up^X(C_i) = up^X(C_j)$. Then, there is at least one more occurrence of $up^X(C_i)$ in the partial sequence $Seq_1^i$ than in $Seq_1^j$. Therefore $down^Y(C_i) \neq down^Y(C_j)$. $\diamondsuit$

We complete the description of the clause gadget by defining the vertex lists $\mathcal{A}(v)$ for every $v \in U^C \cup D^C$. For a given clause $C_j \in \mathcal{C}$ we define the color set $\mathfrak{X}(C_j)$ and the literal color set $\mathfrak{L}(C_j)$ of $C_j$ by

$$\mathfrak{X}(C_j) := \{T_i^{\Omega(C_j,x_i)} : x_i \text{ occurs in } C_j\},$$

$$\mathfrak{L}(C_j) := \{T_i^{\Omega(C_j,x_i)} : x_i \text{ occurs as a positive literal in } C_j\} \cup \{F_i^{\Omega(C_j,x_i)} : x_i \text{ occurs as a negative literal in } C_j\}.$$ 

Note that $\mathfrak{L}(C_j) \subseteq \mathfrak{X}(C_j)$. The vertex lists for the vertices in $U^C \cup D^C$ are defined as

$$\mathcal{A}(\eta_t) := \bigcup_{C_j \in \mathcal{C}, up^X(C_j) = t} \mathfrak{X}(C_j) \cup \{Z_3\} \quad \text{for every } \eta_t \in U^C,$$

$$\mathcal{A}(\theta_t) := \bigcup_{C_j \in \mathcal{C}, down^Y(C_j) = t} \mathfrak{L}(C_j) \cup \{Z_4\} \quad \text{for every } \theta_t \in D^C.$$ 

**Claim 9.** Let $C_j \in \mathcal{C}$. Then, $\mathcal{A}(\eta_{up^X(C_j)}) \cap \mathcal{A}(\theta_{down^Y(C_j)}) = \mathfrak{L}(C_j)$.

**Proof.** Let $\mathcal{A}(\eta) := \mathcal{A}(\eta_{up^X(C_j)}) \cap \mathcal{A}(\theta_{down^Y(C_j)})$. Since $\mathfrak{L}(C_j) \subseteq \mathfrak{X}(C_j)$ it holds that $\mathfrak{L}(C_j) \subseteq \mathcal{A}(\eta)$. It remains to show that there is no other strong color $Y \in \mathcal{A}(\eta) \setminus \mathfrak{L}(C_j)$.

**Case 1:** $Y \notin \{Z_3, Z_4\}$. Then, since $Z_3 \notin \mathcal{A}(\theta_{down^Y(C_j)})$ and $Z_4 \notin \mathcal{A}(\eta_{up^X(C_j)})$ it follows that $Y \notin \mathcal{A}(\eta)$.

**Case 2:** $Y \notin \{Z_3, Z_4\}$. Assume towards a contradiction that $Y \in \mathcal{A}(\eta)$. From $Y \in \mathcal{A}(\theta_{down^Y(C_j)})$ it follows that there is a clause $C_{j_2}$ with $down^Y(C_{j_2}) = down^Y(C_j)$ and $Y \in \mathfrak{L}(C_{j_2})$. It holds that $C_{j_1} \neq C_{j_2}$, since otherwise $Y \in \mathfrak{L}(C_j)$, which contradicts the choice of $Y$. From $Y \in \mathcal{A}(\eta_{up^X(C_j)})$ it follows that there is a clause $C_{j_1}$ with $up^X(C_{j_2}) = up^X(C_j)$ and $Y \notin \mathfrak{X}(C_{j_2})$. By the definition of $\mathfrak{X}$ and $\mathfrak{L}$ there exists a variable $x_i$ that occurs in $C_{j_1}$ and $C_{j_2}$ such that $Y = T_i^{\Omega(C_{j_1}, x_i)} = T_i^{\Omega(C_{j_2}, x_i)}$ or $Y = F_i^{\Omega(C_{j_1}, x_i)} = F_i^{\Omega(C_{j_2}, x_i)}$. We conclude $\Omega(C_{j_1}, x_i) = \Omega(C_{j_2}, x_i)$ and therefore $C_{j_2} = C_{j_1} \neq C_{j_2}$. Then, the fact that $up^X(C_{j_1}) = up^X(C_j)$ and $down^Y(C_{j_2}) = down^Y(C_j)$ contradicts Claim 8 and therefore $Y \notin \mathcal{A}(\eta)$. $\square$

**Connecting the Gadgets.** We complete the construction of $G$ by describing how the vertices of the variable gadget and the vertices of the clause gadget are connected. The idea is to define edges between the vertices in $D^X$ and $U^C$ that model the occurrences of variables in the clauses.

Let $x_{i_1}, x_{i_2}$, and $x_{i_3}$ be the variables that occur in some clause $C_j$. Then, we add the following edges: $\{(\delta_{down^X(x_{i_1})}, \eta_{up^X(C_j)}), \delta_{down^X(x_{i_2})}, \eta_{up^X(C_j)}\}$, and $\{(\delta_{down^X(x_{i_3})}, \eta_{up^X(C_j)}\}$, and $\{\delta_{down^X(x_{i_1})}, \eta_{up^X(C_j)}\}$. We do this for every clause $C_j \in \mathcal{C}$.

The idea is that an edge $\{(\delta_{down^X(x_{i_1})}, \eta_{up^X(C_j)}\}$ transmits the truth value of a variable $x_{i_1}$ to a clause $C_j$, where $x_{i_1}$ occurs as a positive or negative literal. The following claim states that the possible
strong colors for such edge are only $$T_i^{\Omega(C_j,x_i)}$$ and $$F_i^{\Omega(C_j,x_i)}$$, which correspond to the truth assignment of the $$\Omega(C_j,x_i)$$-th occurrence of $$x_i$$.

**Claim 10.** Let $$C_j \in C$$ be a clause and let $$x_i \in X$$ be some variable that occurs in $$C_j$$. Then $$\Lambda(\delta_{\text{down}}^x(x_i)) \cap \Lambda(\eta_{\text{up}}^c(C_j)) = \{T_i^{\Omega(C_j,x_i)}, F_i^{\Omega(C_j,x_i)}\}$$.

**Proof.** Let $$\Lambda(i,j) := \Lambda(\delta_{\text{down}}^x(x_i)) \cap \Lambda(\eta_{\text{up}}^c(C_j))$$. Obviously, $$\{T_i^{\Omega(C_j,x_i)}, F_i^{\Omega(C_j,x_i)}\} \subseteq \Lambda(i,j)$$. It remains to show that there is no strong color $$Y \in \Lambda(i,j) \setminus \{T_i^{\Omega(C_j,x_i)}, F_i^{\Omega(C_j,x_i)}\}$$.

**Case 1:** $$Y = Z_3$$ or $$Y = Z_2$$. Since $$Z_3 \not\in \Lambda(\delta_{\text{down}}^x(x_i))$$ and $$Z_2 \not\in \Lambda(\eta_{\text{up}}^c(C_j))$$ we have $$Y \not\in \Lambda(i,j)$$.

**Case 2:** $$Y = T_i^t$$ or $$Y = F_i^t$$ with $$t \neq i$$ and $$r \in \{1, 2, 3, 4\}$$. If $$Y \not\in \Lambda(\eta_{\text{up}}^c(C_j))$$, then obviously $$Y \not\in \Lambda(i,j)$$. Thus, let $$Y \in \Lambda(\eta_{\text{up}}^c(C_j))$$. Then, by the definition of the color set $$\mathcal{X}(\cdot)$$, there is a clause $$C_j'$$ containing a variable $$x_t \neq x_i$$ with $$\text{up}^c(C_j') = \text{up}^c(C_j)$$ and $$\Omega(C_j', x_i) = r \neq \Omega(C_j, x_i)$$. It follows that $$C_j' \neq C_j$$ which contradicts Claim 7. Hence, $$Y \not\in \Lambda(i,j)$$.

**Case 3:** $$Y = T_i^r$$ or $$Y = F_i^r$$ with $$r \neq \Omega(C_j, x_i)$$. Assume towards a contradiction that $$Y \in \Lambda(\delta_{\text{down}}^x(x_i))$$. Assume towards a contradiction that $$Y \in \Lambda(\delta_{\text{down}}^x(x_i))$$. Then, by the definition of the color set $$\mathcal{X}(\cdot)$$, there is a clause $$C_j$$ containing $$x_i$$ such that $$\text{up}^c(C_j') = \text{up}^c(C_j)$$ and $$\Omega(C_j', x_i) = r \not\in \Omega(C_j, x_i)$$. Otherwise, if $$C_j' \neq C_j$$, then it follows by Claim 7 together with the fact that $$\text{up}^c(C_j') = \text{up}^c(C_j)$$ that down$$^X(x_i) \neq$$ and thus $$Y \not\in \Lambda(\delta_{\text{down}}^x(x_i))$$. Therefore, $$Y \not\in \Lambda(i,j)$$. ♦

This completes the description of the construction and basic properties of the VL-Multi-STC instance $$(G, 9n + 4, 0, \Lambda)$$. Note that $$G$$ has $$\mathcal{O}(\sqrt{n})$$ vertices. It remains to show the correctness of the reduction.

**Correctness.** We now show that there is a satisfying assignment for $$\phi$$ if and only if there is a $$(9n + 4)$$-colored $$\Lambda$$-satisfying STC-labeling $$L$$ for $$G$$ with strong color classes

$$S_L^{T_r}, S_L^{F_r}, S_L^{R_r}, S_L^{Z_r} \quad \text{for all } t \in \{1, \ldots, n\} \text{ and } r \in \{1, 2, 3, 4\},$$

and $$W_L = \emptyset$$.

($$\Rightarrow$$) Let $$\Lambda : X \rightarrow \{\text{true}, \text{false}\}$$ be a satisfying assignment for $$\phi$$. We describe step-by-step to which strong color classes we add the edges of $$G$$ so that we obtain a $$\Lambda$$-satisfying STC-labeling.

First, we describe to which strong color classes we add the edges in $$E(U^X \cup M^X \cup D^X)$$ of the variable gadget. Let $$e := \{\delta_{\text{down}}^x(x_i), \gamma_{\text{mid}}^x(x_i)\}$$ be an edge of the variable-representation gadget for some $$x_i \in X$$ and $$r \in \{1, 2, 3, 4\}$$. We add $$e$$ to $$S_{L}^{T_r}$$ if $$A(x_i) = \text{true}$$ or to $$S_{L}^{F_r}$$ if $$A(x_i) = \text{false}$$. In both cases, $$e$$ satisfies the $$\Lambda$$-list property by Claim 4. Next, let $$e_1 := \{\gamma_{\text{mid}}^x(x_i), \alpha_{\text{up}}^{x}(x_i)\}$$, and $$e_2 := \{\gamma_{\text{mid}}^x(x_i), \alpha_{\text{up}}^{x}(r,r')\}$$ be two edges of the variable-soundness gadget for some $$x_i \in X$$, $$r \in \{1, 2, 3, 4\}$$ and $$r' \in \{1, 2, 3, 4\} \setminus \{r\}$$. We add $$e_1$$ to $$S_{L}^{R_r}$$ if $$A(x_i) = \text{true}$$ or to $$S_{L}^{T_r}$$ if $$A(x_i) = \text{false}$$. Further, we add $$e_2$$ to $$S_{L}^{F_{r'}}$$ if $$A(x_i) = \text{true}$$ or to $$S_{L}^{R_{r'}}$$ if $$A(x_i) = \text{false}$$. In each case, $$e_1$$ and $$e_2$$ satisfy the $$\Lambda$$-list property by Claim 6. For the remaining edges of the variable-gadget we do the following: We add all edges of $$E(U^X)$$ to $$S_{L}^{Z_1}$$ and all edges of $$E(D_X)$$ to $$S_{L}^{Z_2}$$. Obviously, this does not violate the $$\Lambda$$-list property.

Second, we describe to which strong color classes we add the edges in $$E(U^C \cup D^C)$$ of the clause gadget. Let $$C_j \in C$$ be a clause. Since $$A$$ satisfies $$\phi$$, there is some variable $$x_i$$ occurring in $$C_j$$, such that the assignment $$A(x_i) = \text{true}$$ satisfies the clause $$C_j$$. Let $$r := \Omega(C_j, x_i)$$. We add the edge $$\{\eta_{\text{up}}^c(C_j), \theta_{\text{down}}^c(C_j)\}$$ to $$S_{L}^{T_{r}}$$ if $$A(x_i) = \text{true}$$ or to $$S_{L}^{F_{r}}$$ if $$A(x_i) = \text{false}$$. In both cases, the edge satisfies the $$\Lambda$$-list property by Claim 9. For the remaining edges of the clause gadget we do the following: We add all edges of $$E(U^C)$$ to $$S_{L}^{Z_3}$$ and all edges of $$E(D^C)$$ to $$S_{L}^{Z_4}$$. Obviously, this does not violate the $$\Lambda$$-list property.

Third, we describe to which strong color classes we add the edges in $$E(D^X, U^C)$$ between the two gadgets. Let $$C_j \in C$$ be a clause and let $$x_i$$ be some variable occurring in $$C_j$$. Let $$r := \Omega(C_j, x_i)$$. Let
We add the edge \{δ_{down}^{X}(x), η_{up}^{C}(c_j)\} to \(S_L^{T'}\) if \(A(x) = true\) or to \(S_L^{T'}\) if \(A(x) = false\). This edge satisfies the \(Λ\)-list property by Claim 10.

We have now added every edge of \(G\) to exactly one strong color class of \(L\), such that \(L\) is \(Λ\)-satisfying. It remains to show that there is no induced \(P_3\) subgraph containing two edges \{\(u,v\)\} and \{\(v,w\)\} from the same strong color class. In the following case distinction we consider every possible induced \(P_3\) on vertices \(u,v,w\) where \(v\) is the central vertex.

Case 1: \(v \in U^X\). Then, \(v = α_i^{r,r'}\) for some \(t \in \{1, \ldots, \lceil \sqrt{n} \rceil + 9\}\), \(r \in \{1,2,3,4\}\) and \(r' \in \{1,2,3,4\} \setminus \{r\}\). Note that the vertices in \(U^X\) are not adjacent to vertices in \(D^X\), \(U^C\) and \(D^C\). It suffices to consider the following subcases.

Case 1.1: \(u \in U^X\). Then, \{\(u,v\)\} \(\in S_L^{2}L\). If \(w \in U^X\), then the vertices \(u,v,w\) do not form an induced \(P_3\), since \(U^X\) is a clique in \(G\). If \(w \notin U^X\), then \{\(v,w\)\} \(\notin S_L^{2}L\). Hence, there is no STC-violation.

Case 1.2: \(u,w \in M^X\). Then, there are variables \(x_i\) and \(x_j\) with \(up^{X}(x_i) = up^{X}(x_j) = t\) and \(u = γ_{mid}^r(x_i), w = γ_{mid}^q(x_j)\) for some \(p,q \in \{r,r'\}\). We need to consider the following subcases.

Case 1.2.1: \(x_i \neq x_j\). Then \(i \neq j\). By Claim 6 it holds without loss of generality that \(Λ(u) \cap Λ(v) \subset \{T_r^i,F_r^i,T_r^{r'},F_r^{r'},R_1\}\) and \(Λ(v) \cap Λ(w) \subset \{T_r^j,F_r^j,T_r^{r'},F_r^{r'},R_1\}\). Since \(L\) is \(Λ\)-satisfying, the edges \{\(u,v\)\} and \{\(v,w\)\} are elements of different strong color classes. Thus, there is no STC-violation.

Case 1.2.2: \(x_i = x_j\). Then, \(p \neq q\), since otherwise \(u = v\). Without loss of generality, we have \(u = γ_{mid}^r(x_i), w = γ_{mid}^q(x_j)\). If \(A(x_i) = true\), it follows that \{\(u,v\)\} \(\in S_L^{R}\) and \{\(v,w\)\} \(\in S_L^{R'}\). Otherwise, if \(A(x_i) = false\), it follows that \{\(u,v\)\} \(\in S_L^{T'}\) and \{\(v,w\)\} \(\in S_L^{R}\). In both cases, the vertices in \(M^X\) are not adjacent to vertices in \(D^C\), \(D^C\) and \(M^X\). It suffices to consider the following subcases.

Case 2.1: \(u,w \in U^X\) or \(u,w \in D^X\). Then, since \(U^X\) and \(D^X\) are cliques in \(G\), the vertices \(u,v,w\) do not form an induced \(P_3\) in \(G\). Hence, there is no STC-violation.

Case 2.2: \(u \in U^X\) and \(w \in D^X\). Then, there are variables \(x_i\) and \(x_j\) with \(mid^X(x_i) = mid^X(x_j) = t\) and \(u \in \{α_{up}^r(x_i), α_{up}^r(x_j)\}, w = δ_{down}(x_j)\) for some \(r' \neq r\). We need to consider the following subcases.

Case 2.2.1: \(x_i \neq x_j\). Then, \(i \neq j\). Without loss of generality it holds by Claim 6 that \(Λ(u) \cap Λ(v) \subset \{T_r^i,F_r^i,T_r^{r'},F_r^{r'},R_1\}\) for some \(r' \neq r\) and by Claim 4 that \(Λ(v) \cap Λ(w) \subset \{T_r^j,F_r^j\}\). Since \(L\) is \(Λ\)-satisfying, the edges \{\(u,v\)\} and \{\(v,w\)\} are elements of different strong color classes. Thus, there is no STC-violation.

Case 2.2.2: \(x_i = x_j\). Then, if \(A(x_i) = true\) it follows that \{\(u,v\)\} \(\in S_L^{R} \cup S_L^{T'}\) and \{\(v,w\)\} \(\in S_L^{T'}\). If \(A(x_i) = false\) it follows that \{\(u,v\)\} \(\in S_L^{R} \cup S_L^{T'}\) and \{\(v,w\)\} \(\in S_L^{R}\). In both cases the edges \{\(u,v\)\} and \{\(v,w\)\} are elements of different strong color classes. Thus, there is no STC-violation.

Case 3: \(v \in D^X\). Then \(v = δ_{t}\) for some \(t \in \{1, \ldots, \lceil \sqrt{n} \rceil + 9\}\). Note that the vertices in \(D^X\) are not adjacent to vertices in \(U^X\) and \(D^C\). It suffices to consider the following subcases.

Case 3.1: \(u \in D^X\). Then, \{\(u,v\)\} \(\in S_L^{2}L\). If \(w \in D^X\), the vertices \(u,v,w\) do not form an induced \(P_3\), since \(D^X\) is a clique in \(G\). If \(w \notin D^X\) it follows \{\(v,w\)\} \(\notin S_L^{2}L\). Hence, there is no STC-violation.

Case 3.2: \(u,w \in U^C\). Then, the vertices \(u,v,w\) do not form an induced \(P_3\), since \(U^C\) forms a clique.

Case 3.3: \(u,w \in M^X\). By Claim 4, all edges \{\(v,y\)\} \(\in E(\{v\}, M^X)\) have distinct possible strong colors in \(Λ(v) \cap Λ(y)\). Since \(L\) is \(Λ\)-satisfying, the edges \{\(u,v\)\} and \{\(v,w\)\} are elements of different strong color classes.

Case 3.4: \(u \in M^X\) and \(w \in U^C\). Then \(u = γ_{mid}^r(x_i)\) for some \(x_i \in X\) with \(down^X(x_i) = t\) and \(r \in \{1,2,3,4\}\). Moreover, \(w = η_{up}^{C}(c_j)\) for some clause \(C_j\) containing a variable \(x\) with \(down^X(x) = t\). We need to consider the following subcases.

Case 3.4.1: \(x_i \neq x\). Then, \(i \neq i'\) and by Claim 4 we have \(Λ(u) \cap Λ(v) = \{T_r^i,F_r^i\}\) and by
Claim 10 we have $\Delta(v) \cap \Delta(w) = \{T'_v, F'_v\}$ with $r' = \Omega(C_j, x_v)$. Then, since $L$ is $A$-satisfying, $\{u, v\}$ and $\{v, w\}$ are not elements of the same strong color class.

**Case 3.4.2:** $x_i = x_v'$. Then, if $A(x_i) = \text{true}$ it follows that $\{u, v\} \in S_{L}^{T'}$ and $\{v, w\} \in S_{L}^{F'}$ for some $r' \in \{1, 2, 3, 4\}$. If $A(x_i) = \text{false}$ it follows that $\{u, v\} \in S_{L}^{F'}$ and $\{v, w\} \in S_{L}^{T'}$ for some $r' \in \{1, 2, 3, 4\}$. In both cases $\{u, v\}$ and $\{v, w\}$ are elements of different strong color classes.

**Case 4:** $v \in U^c$. Then $v = \eta_t$ for some $t \in \{1, \ldots, 12\lfloor \sqrt{n} \rfloor + 1\}$. Note that the vertices in $U^c$ are not adjacent to vertices in $U^X$ and $M^X$. It suffices to consider the following subcases.

**Case 4.1:** $u \in U^c$. Then, $\{u, v\} \in S_{L}^{Z_4}$. If $w \in U^c$, the vertices $u, v, w$ do not form an induced $P_3$ since $U^X$ is a clique in $G$. If $w \notin U^c$ it follows that $\{v, w\} \notin S_{L}^{Z_4}$. Hence, there is no STC-violation.

**Case 4.2:** $u, w \in D^X$ or $u, w \in D^c$. Then, the vertices $u, v, w$ do not form an induced $P_3$, since $D^X$ and $D^c$ form cliques in $G$.

**Case 4.3:** $u \in D^X$ and $w \in D^c$. Then, there is a clause $C_j$ with $\text{up}^c(C_j) = t$ and a clause $C_j'$ containing a variable $x_i$ with $\text{up}^c(C_j') = t$ and $u = \delta^c_{\text{down}}(x_i), w = \theta^c_{\text{down}}(C_j')$. We consider the following subcases.

**Case 4.3.1:** $C_j \neq C_j'$. Then, since $\text{up}^c(C_j) = \text{up}^c(C_j')$ it follows by Claim 7 that $C_j$ and $C_j'$ do not share a variable. Hence, $x_i$ does not occur in $C_j$ and therefore $T^p_{i}(C_j, x_i), F^p_{i}(C_j, x_i) \notin \Sigma(C_j)$. Thus, by Claims 9 and 10 and the fact that $L$ is $A$-satisfying, the edges $\{u, v\}$ and $\{v, w\}$ are elements of different strong color classes.

**Case 4.3.2:** $C_j = C_j'$. Let $r := \Omega(C_j, x_i)$. If $\{v, w\} \notin S_{L}^{T'} \cup S_{L}^{F'}$, the edges $\{u, v\}$ and $\{v, w\}$ are elements of different color classes. Thus, there is no STC-violation. If $\{v, w\} \in S_{L}^{T'} \cup S_{L}^{F'}$ it follows by the construction of $L$ that $C_j$ is satisfied by the assignment $A(x_i)$. Without loss of generality assume that $x_i$ occurs as a positive literal in $C_j$. Then, $A(x_i) = \text{true}$. This implies $\{v, w\} \in S_{L}^{T'}$ and $\{u, v\} \in S_{L}^{F'}$. Hence, $\{u, v\}$ and $\{v, w\}$ are elements of different strong color classes.

**Case 5:** $v \in D^c$. Then, $v$ is not adjacent to any vertices in $U^X, M^X$ or $D^X$. Hence, we need to consider the following cases.

**Case 5.1:** $u, w \in U^c$ or $u, w \in D^c$. Then, the vertices $u, v, w$ do not form an induced $P_3$ since $U^c$ and $D^c$ are cliques in $G$.

**Case 5.2:** $u \in D^c$ and $w \in U^c$. Then, $\{u, v\} \in S_{L}^{Z_4}$ and $\{v, w\} \notin S_{L}^{Z_4}$. Hence, there is no STC-violation.

This proves that $L$ is an $A$-satisfying STC-labeling for $G$ with no weak edges, which completes the first direction of the correctness.

($\Leftarrow$) Conversely, let $L$ be a $(9n + 4)$-colored $A$-satisfying STC-labeling for $G$. We show that $\phi$ is satisfiable. We define an assignment $A : C \rightarrow \{\text{true}, \text{false}\}$ by

$$A(x_i) := \begin{cases} \text{true} & \text{if } \{\delta^c_{\text{down}}(x_i), \gamma^c_{\text{mid}}(x_i)\} \in S_{L}^{T'}, \text{ and} \\ \text{false} & \text{if } \{\delta^c_{\text{down}}(x_i), \gamma^c_{\text{mid}}(x_i)\} \in S_{L}^{F'}. \end{cases}$$

The assignment is well-defined due to Claim 4. The following claim states that, if there is one occurrence $r \in \{1, 2, 3, 4\}$ of some variable $x_i$ that is assigned ‘true’ (or ‘false’ respectively) so is the first occurrence of $x_i$. We obtain this statement by using the variable-soundness gadget.

**Claim 11.** Let $x_i \in X$ and $r \in \{2, 3, 4\}$.

a) If $\{\delta^c_{\text{down}}(x_i), \gamma^c_{\text{mid}}(x_i)\} \in S_{L}^{T'},$ then $\{\delta^c_{\text{down}}(x_i), \gamma^c_{\text{mid}}(x_i)\} \in S_{L}^{T'}$.

b) If $\{\delta^c_{\text{down}}(x_i), \gamma^c_{\text{mid}}(x_i)\} \in S_{L}^{F'},$ then $\{\delta^c_{\text{down}}(x_i), \gamma^c_{\text{mid}}(x_i)\} \in S_{L}^{F'}$. 

18
Proof. We first show (a). Let \(\delta_{\text{down}^{X}(x_i)}; \gamma^{r}_{\text{mid}^{X}(x_i)}\) \(\in S^{T_{r}}_{L}\). Consider the vertex \(\alpha^{(r,1)}_{\text{up}^{X}(x_i)}\). By Claim 6 we have

\[
\Lambda(\alpha^{(r,1)}_{\text{up}^{X}(x_i)}) \cap \Lambda(\gamma^{r}_{\text{mid}^{X}(x_i)}) = \{T_{i}, R_{i}\},
\]

and

\[
\Lambda(\alpha^{(r,1)}_{\text{up}^{X}(x_i)}) \cap \Lambda(\gamma^{1}_{\text{mid}^{X}(x_i)}) = \{F_{i}, R_{i}\}.
\]

Note, that the vertices \(\delta_{\text{down}^{X}(x_i)}; \gamma^{r}_{\text{mid}^{X}(x_i)}; \alpha^{(r,1)}_{\text{up}^{X}(x_i)}\) form an induced \(P_{3}\) in \(G\). By the fact that \(L\) is a \(\Lambda\)-satisfying STC-labeling with no weak edges it holds that \(\{\gamma^{r}_{\text{mid}^{X}(x_i)}, \alpha^{(r,1)}_{\text{up}^{X}(x_i)}\} \in S^{R}_{L}\). Then, since the vertices \(\gamma^{r}_{\text{mid}^{X}(x_i)}, \alpha^{(r,1)}_{\text{up}^{X}(x_i)}\), and \(\gamma^{1}_{\text{mid}^{X}(x_i)}\) form an induced \(P_{3}\), the same argument implies \(\{\alpha^{(r,1)}_{\text{up}^{X}(x_i)}, \gamma^{1}_{\text{mid}^{X}(x_i)}\} \in S^{P}_{L}\). Then, since \(\Lambda(\delta_{\text{down}^{X}(x_i)}) \cap \Lambda(\gamma^{1}_{\text{mid}^{X}(x_i)}) = \{T_{i}, F_{i}\}\) by Claim 4 and the fact that \(\delta_{\text{down}^{X}(x_i)}; \gamma^{1}_{\text{mid}^{X}(x_i)}; \alpha^{(r,1)}_{\text{up}^{X}(x_i)}\) form an induced \(P_{3}\) it follows that \(\{\delta_{\text{down}^{X}(x_i)}; \gamma^{1}_{\text{mid}^{X}(x_i)}\} \in S^{T_{r}}_{L}\) as claimed.

Statement (b) can be shown with the same arguments by considering the vertex \(\alpha^{(1,r)}_{\text{up}^{X}(x_i)}\) instead of \(\alpha^{(r,1)}_{\text{up}^{X}(x_i)}\).

Next we use Claim 11 to show that every clause is satisfied by \(A\). Let \(C_{j} \in \mathcal{C}\) be a clause. Then, there is an edge \(e_{1} := \{\eta_{\text{up}^{X}(C_{j})}; \theta_{\text{down}^{X}(C_{j})}\} \in E\). By Claim 9 we have \(\Lambda(\eta_{\text{up}^{X}(C_{j})}) \cap \Lambda(\theta_{\text{down}^{X}(C_{j})}) = \mathcal{L}(C_{j})\).

Since \(L\) is \(\Lambda\)-satisfying it follows \(e_{1} \in S^{Y}_{L}\) for some \(Y \in \mathcal{L}(C_{j})\).

Consider the case \(Y = T_{r}^{i}\) for some variable \(x_{i}\) that occurs positively in \(C_{j}\) and \(r = \Omega(C_{j}, x_{i})\). We show that \(A(x_{i}) = \text{true}\). Since \(x_{i}\) occurs in \(C_{j}\) there is an edge \(e_{2} := \{\delta_{\text{down}^{X}(x_{i})}; \eta_{\text{up}^{X}(C_{j})}\} \in E\) which can only be an element of the strong classes \(S^{T_{r}}_{L}\) or \(S^{P^{r}}_{L}\) due to Claim 10. Since \(e_{1}\) and \(e_{2}\) form an induced \(P_{3}\) and \(L\) is an STC-labeling we have \(e_{2} \in S^{P^{r}}_{L}\). The edge \(e_{3} := \{\delta_{\text{down}^{X}(x_{i})}; \gamma^{r}_{\text{mid}^{X}(x_{i})}\}\) forms an induced \(P_{3}\) with \(e_{2}\) and can only be an element of the strong classes \(S^{T_{r}}_{L}\) or \(S^{P^{r}}_{L}\) by Claim 4. Hence, \(e_{3} \in S^{T_{r}}_{L}\). By Claim 11 we conclude \(\{\delta_{\text{down}^{X}(x_{i})}; \gamma^{r}_{\text{mid}^{X}(x_{i})}\} \in S^{T_{r}}_{L}\) and therefore \(A(x_{i}) = \text{true}\). Hence, \(C_{j}\) is satisfied by \(A\).

For the case \(Y = F_{r}^{i}\) we can use the same arguments to conclude \(A(x_{i}) = \text{false}\). Hence, \(A\) satisfies every clause of \(\phi\).

Note that in the instance constructed in the proof of Theorem 2, every edge has at most three possible strong colors and \(c \in \mathcal{O}(n)\). This implies the following.

Corollary 1. If ETH is true, then

a) EL-Multi-STC cannot be solved in \(2^{o(|V|^{2})}\)-time even if restricted to instances \((G, c, k, \Psi)\) where \(k = 0\) and \(\max_{e \in E} \Psi(e) = 3\).

b) VL-Multi-STC cannot be solved in \(c^{O(|V|^{2}/\log |V|)}\)-time even if restricted to instances where \(k = 0\).

4 Parameterized Complexity

The most natural parameter is the number \(k\) of weak edges. The case \(c = 1\) (STC) is fixed-parameter tractable [30]. For \(c = 2\), we also obtain an FPT algorithm: one may solve Odd Cycle Transversal in the Gallai graph \(\tilde{G}\) which is fixed-parameter tractable with respect to \(k\) [3]. This extends to EL-Multi-STC with \(c = 2\) by applying standard techniques for the Odd Cycle Transversal instance \((\tilde{G}, k)\). In contrast, for every fixed \(c \geq 3\), Multi-STC is NP-hard even if \(k = 0\). Thus, FPT algorithms for the parameters \(c, k\), or even \((c, k)\) are unlikely. We thus define the parameter \(k_{1}\)
and analyze the parameterized complexity of \((VL-/EL-)\text{MULTI-STC}\) regarding the parameters \(k_1\) and \((c,k_1)\).

**Definition 5.** Let \(G = (V,E)\) be a graph with a 1-colored STC-labeling \(L = (S_L,W_L)\) such that there is no 1-colored STC-Labeling \(L' = (S_L',W_L')\) for \(G\) with \(|W_L'| < |W_L|\). Then \(k_1 = k_1(G) := |W_L|\).

Note, that for a given graph \(G\), the value \(k_1\) equals the size of a minimal vertex cover of the Gallai graph \(\tilde{G}\) [30]. First, we provide a simple FPT algorithm for \(\text{EL-MULTI-STC}\) parameterized by \((c,k_1)\), which is the most general of the three problems. The main idea of the algorithm is to solve \text{LIST-COLORABLE SUBGRAPH}\ on the Gallai graph of the input graph which is equivalent due to Proposition 1.

**Theorem 3.** \(\text{EL-MULTI-STC}\) can be solved in \(O((c+1)^{k_1} \cdot (cm+nm))\) time.

**Proof.** Let \((G,c,k,\Psi)\) be an instance of \(\text{EL-MULTI-STC}\). The first step is to compute the Gallai graph \(\tilde{G} = (\tilde{V},\tilde{E})\) of \(G\) which has \(m\) vertices and at most \(nm\) edges. We describe an algorithm that solves \text{LIST-COLORABLE SUBGRAPH}\ on \((\tilde{G},c,k,\Psi)\) in \(O((c+1)^s \cdot (|\tilde{V}| \cdot c + |\tilde{E}|))\) time, where \(s = k_1\) denotes the size of a minimum vertex cover.

Let \(S \subseteq \tilde{V}\) be a size-\(s\) vertex cover of \(\tilde{G}\), which can be computed in \(O(1.28^s + sn)\) time [2]. Moreover, let \(I := \tilde{V} \setminus \tilde{S}\) denote the remaining independent set. We now compute if \(\tilde{G}\) has a subgraph-\(c\)-coloring \(a : \tilde{V} \rightarrow \{0,1,\ldots,c\}\) with \(|\{v \in \tilde{V} | a(v) = 0\}| \leq k\).

First, we enumerate all possible mappings \(a_S : S \rightarrow \{0,1,\ldots,c\}\). Observe that there are \((c+1)^s\) such mappings. For each \(a_S\) we check if \(a_S(v) \in \Psi(v) \cup \{0\}\) for all \(v \in S\). Furthermore, we check in \(O(|\tilde{V}| \cdot c + |\tilde{E}|)\) time if \(a_S\) is a subgraph-\(c\)-coloring for \(G[S]\). If this is not the case, then discard the current \(a_S\). Otherwise, go on to the next step.

Next, we check if it is possible to extend \(a_S\) to a mapping \(a : \tilde{V} \rightarrow \{0,1,\ldots,c\}\) that is a proper subgraph-\(c\)-coloring for \(\tilde{G}\). For each vertex \(v \in I\) we check if \(P_v := \Psi(v) \setminus \bigcup_{w \in N_G(v)} \{a_S(w)\}\) is not empty. In this case we set \(a(v) = p\) for some arbitrary \(p \in P_v\). If \(P_v = \emptyset\) we set \(a(v) = 0\). This can be done in \(O(|\tilde{V}| \cdot c + |\tilde{E}|)\) time. The resulting mapping \(a : \tilde{V} \rightarrow \{0,1,\ldots,c\}\) is obviously a subgraph-\(c\)-coloring for \(\tilde{G}\), since \(a_S\) is a subgraph-\(c\)-coloring for \(G[S]\) and every \(v \in I\) has a color \(a(v)\) distinct from all vertices in \(N(v) \subseteq S\).

It remains to check if the total amount of \(v \in \tilde{V}\) with \(a(v) = 0\) is at most \(k\). The overall running time of the algorithm is \(O((c+1)^s \cdot (nc + m))\) as claimed.

Recall that \(k_1 = s\), \(|\tilde{V}| = m\) and \(|\tilde{E}| \leq nm\). Therefore, we can solve \(\text{EL-MULTI-STC}\) in \(O((c+1)^{k_1} \cdot (cm+nm))\) time.

Next, we conclude that \(\text{MULTI-STC}\) parameterized by \(k_1\) alone is fixed-parameter tractable. To this end we observe the following relationship between \(c\) and \(k_1\).

**Lemma 4.** Let \(G = (V,E)\) be a graph. For all \(k \in \mathbb{N}\) and \(c > k_1\) it holds that \((G,c,k)\) is a Yes-instance for \(\text{MULTI-STC}\).

**Proof.** Let \(c > k_1\). Then there exists an STC-labeling \(L = (S_L,W_L)\) for \(G\) with one strong color and \(|W_L| = k_1\). Let \(e_1,e_2,\ldots,e_{k_1}\) be the weak edges of \(L\). We define a \(c\)-colored labeling \(L^+ := (S^+_L,\ldots,S^+_L, W^+_L)\) by

\[
W^+_L := \emptyset \text{ and } S^+_L := \begin{cases} 
\{e_1\} & \text{for } 1 \leq i \leq k_1, \\
S_L & \text{for } i = k_1 + 1, \\
\emptyset & \text{for } k_1 + 1 < i \leq c.
\end{cases}
\]

Since \(c > k_1\), every edge of \(G\) is labeled by \(L^+\). Because \(L\) is an STC-labeling, there is no induced \(I_3\) containing two edges from \(S^+_L = S_L\). Moreover, since \(|S^+_L| \leq 1\) for \(i \neq k_1 + 1\), the labeling \(L^+\) satisfies STC. Since \(|W^+_L| = 0\) it holds that \((G,c,k)\) is a Yes-instance for \(\text{MULTI-STC}\) for every \(k \in \mathbb{N}\).
**Theorem 4.** Multi-STC can be solved in \(O((k_1 + 1)^{k_1} \cdot (k_1 m + nm))\) time.

**Proof.** Let \((G, c, k)\) be an instance of Multi-STC. Consider the running time \(O((c + 1)^{k_1} \cdot (cm + n^3))\) of the algorithm from Theorem 3. If \(c > k_1\) then \((G, c, k)\) is a Yes-instance by Lemma 4, we only need to consider instances with \(c \leq k_1\). Hence, we can solve Multi-STC in \(O((k_1 + 1)^{k_1} \cdot (m^2 + n^3))\) time.

Lemma 4 states a relationship between \(c\) and \(k_1\), which leads to an FPT result for Multi-STC parameterized only by \(k_1\). It is natural to ask if some similar approach yields an FPT result for EL-Multi-STC parameterized by \(k_1\) alone. We now show that there is little hope by proving that VL-Multi-STC is W[1]-hard if parameterized by \(k_1\) alone. We prove the W[1]-hardness by giving a parameterized reduction from Set Cover parameterized by dual, which is defined as follows.

**Set Cover**

**Input:** A finite universe \(U \subseteq \mathbb{N}\), a family \(F \subseteq 2^U\) and an integer \(t \in \mathbb{N}\).

**Question:** Is there a subfamily \(F' \subseteq F\) with \(|F'| \leq t\) such that \(\bigcup_{F \in F'} F = U\)?

The W[1]-hardness of Set Cover parameterized by dual follows from a classical reduction from Independent Set [20]. We provide it here for the sake of completeness.

**Proposition 3.** Set Cover parameterized by \(|F| - t\) is W[1]-hard.

**Proof.** The classical problem Independent Set asks if for a given graph \(G = (V, E)\) there is a subset \(V' \subseteq V\) of size at least \(s\) such that the vertices in \(V'\) are pairwise non-adjacent in \(G\). It is known to be W[1]-hard when parameterized by \(s\) [3].

Let \((G = (V, E), s)\) be an instance of Independent Set. We construct a Set Cover-instance \((U, F, t)\) as follows. Set \(U := E, F := \{F_v \mid v \in V\}\) with \(F_v := \{\{v, u\} \mid u \in N(v)\}\) and \(t = |V| - s\). Note that \(|F| = |V|\), hence \(|F| - t = |V| - (|V| - s) = s\).

**Theorem 5.** VL-Multi-STC parameterized by \(k_1\) is W[1]-hard, even if \(k = 0\). VL-Multi-STC parameterized by \((c, k_1)\) does not admit a polynomial kernel unless \(NP \subseteq coNP/poly\).

**Proof.** We give a parameterized reduction from Set Cover parameterized by \(|F| - t\) which is W[1]-hard due to Proposition 3. For a given Set Cover-instance \((U, F, t)\) we describe how to construct an equivalent VL-Multi-STC-instance \((G = (V, E), c, k, A)\) with \(k_1 \leq |F| - t\) and \(k = 0\). Assume \(F = \{F_1, \ldots, F_{|F|}\}\). We define the vertex set \(V\) of graph \(G\) by \(V := U \cup Z \cup \{a\}\) with \(Z := \{z_i \mid t + 1 \leq i \leq |F|\}\). We add edges, such that \(U\) becomes a clique in \(G\). Define the edge set of \(G\) by \(E := E(U) \cup E_{ua} \cup E_{za}\) with \(E_{ua} := \{\{u, a\} \mid u \in U\}\) and \(E_{za} := \{\{z, a\} \mid z \in Z\}\). Note that \(|E_{za}| = |Z| = |F| - t\).

We let \(c := |F| + 1\) and define the lists \(A\) as

\[
A(v) := \begin{cases} 
\{i \mid v \in F_i\} \cup \{|F| + 1\} & \text{if } v \in U, \\
\{1, 2, \ldots, |F|\} & \text{if } v \notin U.
\end{cases}
\]

Our intuition for this construction should be, that the vertex \(a\) "selects" sets from \(F\) by labeling the edges in \(E_{ua}\). The edges in \(E_{za}\) ensure, that there are exactly \(t\) different strong colors left for the edges in \(E_{ua}\).

We first show that \(k_1 \leq |F| - t\). Let \(e_1, e_2 \in E\) be the edges of an induced \(P_5\) in \(G\). Since \(U \cup \{a\}\) is a clique by construction, at least one of the edges \(e_1\) or \(e_2\) has one endpoint in \(Z\), hence it belongs to \(E_{za}\). Since every \(P_3\) in \(G\) contains at least one edge from \(E_{za}\) it follows that defining \(E_{za}\) as weak edges and \(E(U) \cup E_{ua}\) as strong edges yields an STC-labeling with one strong color. This labeling has \(|E_{za}| = |F| - t\) weak edges, hence \(k_1 \leq |F| - t\).
It remains to show that \((U, \mathcal{F}, t)\) has a solution \(\mathcal{F}'\) of size \(t\) if and only if \(G\) has a \(\Lambda\)-satisfying STC-labeling \(L = (S^L_1, \ldots, S^{|\mathcal{F}|+1}_L, W_L)\) with \(W_L = \emptyset\).

Let \(\mathcal{F}' \subseteq \mathcal{F}\) be a set cover of \(U\) with \(|\mathcal{F}'| = t\). Without loss of generality let \(\mathcal{F}' = \{F_1, F_2, \ldots, F_t\}\). We define an STC labeling \(L = (S^L_1, \ldots, S^{|\mathcal{F}|+1}_L, \emptyset)\) as follows. We start by defining the classes \(S^t_i\) for \(i \in \{t + 1, \ldots, |\mathcal{F}| + 1\}\). We set
\[
S^{|\mathcal{F}|+1}_L := E(U) \text{ and } S^t_i := \{\{u, z\} \mid \text{for every } t + 1 \leq i \leq |\mathcal{F}|\}.
\]

Note, that \(S^t+1_L \cup \cdots \cup S^{|\mathcal{F}|+1}_L = E(U) \cup E_{za}\), so by defining the strong color classes \(S^t+1_L, \ldots, S^{|\mathcal{F}|+1}_L\) we have labeled all edges in \(E(U) \cup E_{za}\). We proceed to show that this definition does not violate the STC property and every edge in \(E(U) \cup E_{za}\) satisfies the \(\Lambda\)-list property. Since \(U\) is a clique by construction, there is no induced \(P_3\) containing two edges from \(S^{|\mathcal{F}|+1}_L\) violating STC in \(E(U)\). Moreover, since all sets \(S^t_L, \ldots, S^{|\mathcal{F}|}_L\) contain exactly one element, there is obviously no STC violation in \(E_{za}\). For every vertex \(u \in U\) it holds that \(|\mathcal{F}| + 1 \in \Lambda(u)\), hence the \(\Lambda\)-list property is satisfied for every \(e \in E(U)\). Since \(\{1, 2, \ldots, |\mathcal{F}|\} = \Lambda(u) = \Lambda(z_{t+1}) = \cdots = \Lambda(z_{|\mathcal{F}|})\), the edges in \(E_{za}\) also satisfy the \(\Lambda\)-list property.

We now label the edges in \(E_{ua}\) by defining the sets \(S^1_L, \ldots, S^T_L\). Recall that \(\mathcal{F}' = \{F_1, \ldots, F_t\}\) is a set cover of size \(t\). We set \(S^1_L := \{\{u, a\} \mid u \in F_1\}\) and \(S^t_L := \{\{u, a\} \mid u \in F_t \setminus (F_1 \cup \cdots \cup F_{t-1})\}\) for each \(i \in \{2, \ldots, t\}\). Obviously, each edge of \(E_{ua}\) is an element of at most one of the sets \(S^1_L, \ldots, S^T_L\). Since \(\mathcal{F}'\) is a set cover, we know that \(F_1 \cup \cdots \cup F_t = U\). It follows that every edge in \(E_{ua}\) is an element of exactly one of the sets \(S^1_L, \ldots, S^T_L\). Since \(U \cup \{a\}\) forms a clique, no edge in \(E_{ua}\) violates STC. From the definition of \(\Lambda\) we know that \(\Lambda(a) = \{1, \ldots, |\mathcal{F}|\}\) and for every \(u \in U\) it holds that \(i \in \Lambda(u)\) if \(u \in F_i\). Hence, every edge in \(E_{ua}\) satisfies the \(\Lambda\)-list property. It follows that \(L\) is a \(c\)-colored STC-labeling with \(W_L = \emptyset\) such that every edge satisfies the \(\Lambda\)-list property under \(L\), which proves the first direction of the equivalence.

Conversely, let \(L = (S^1_L, \ldots, S^{|\mathcal{F}|+1}_L, \emptyset)\) be a \(c\)-colored STC-labeling for \(G\) such that every edge of \(G\) satisfies the \(\Lambda\)-list property. We will construct a set cover \(\mathcal{F}' \subseteq \mathcal{F}\) with \(|\mathcal{F}'| \leq t\). We focus on the vertex \(a\) and its incident edges. Those are exactly the edges of \(E_{ua} \cup E_{za}\). Since there are no weak edges, we know that all those edges are elements of strong color classes \(S^1_L\). Since \(L\) is an STC-labeling and every pair of edges \(e, e' \in E_{za}\) forms a \(P_3\), it follows by \(|E_{za}| = |\mathcal{F}| - t\) that those edges are elements of \(|\mathcal{F}| - t\) distinct color classes. By the fact that there is no edge between the vertices of \(U\) and \(Z\), it also holds that there is no \(e \in E_{ua}\) that is an element of the same strong color class as some \(e' \in E_{za}\). Otherwise, \(e\) and \(e'\) form a \(P_3\) with the same strong color which contradicts the fact that \(L\) is an STC-labeling. It follows that the edges in \(E_{ua}\) are elements of at most \(t\) distinct strong color classes, since \(|\Lambda(a)| = |\mathcal{F}|\) and every edge of \(G\) satisfies the \(\Lambda\)-list property under \(L\). Without loss of generality we can assume that those strong color classes are \(S^1_L, \ldots, S^T_L\). Recall that \(\mathcal{F} = \{F_1, F_2, \ldots, F_{|\mathcal{F}|}\}\). We define
\[
\mathcal{F}' := \{F_1, F_2, \ldots, F_t\}.
\]

Obviously, \(|\mathcal{F}'| = t\). It remains to show that \(\mathcal{F}'\) is a set cover. From the fact that all edges of \(G\) satisfy the \(\Lambda\)-list property under \(L\), we conclude that for every \(u \in U\) there is a \(j \in \{1, \ldots, t\}\) such that \(j \in \Lambda(u)\). Since \(\Lambda(u) = \{i \mid u \in F_i\} \cup \{|\mathcal{F}| + 1\}\) for all \(u \in U\) by construction, it follows that every \(u \in U\) is an element of one of the sets \(F_1, F_2, \ldots, F_t\). Hence, \(U = F_1 \cup F_2 \cup \cdots \cup F_t\).

Then, \(\mathcal{F}'\) is a set cover of size \(t\), which completes the proof that \(\text{VL-MULTI-STC}\) parameterized by \(k_1\) is \(W[1]\) hard even if \(k = 0\).

A closer look at the instance \((G, c, k, \Lambda)\) for \(\text{VL-MULTI-STC}\) constructed from an instance \((U, \mathcal{F}, t)\) for \(\text{Set Cover}\) in the proof of Theorem 5 reveals that \(c = |\mathcal{F}| + 1\) and \(k_1 \leq |\mathcal{F}| - t\). It follows that \(c + k_1 \leq 2|\mathcal{F}| + 1\), so the construction is a polynomial-parameter transformation from \(\text{Set Cover}\) parameterized by \(|\mathcal{F}|\) to \(\text{VL-MULTI-STC}\) parameterized by \((c, k_1)\). By the fact that \(\text{Set Cover}\)
Corollary 2. **VL-Multi-STC** parameterized by \((c,k_1)\) does not admit a polynomial kernel unless \(NP \subseteq coNP/poly\) [5] we obtain the following.

**On Problem Kernelization.** Since EL-Multi-STC is a generalization of VL-Multi-STC, we conclude from Corollary 2 that there is no polynomial kernel for EL-Multi-STC parameterized by \((c,k_1)\) unless \(NP \subseteq coNP/poly\) and thus we give a \(2^{c+1} \cdot k_1\)-vertex kernel for EL-Multi-STC. To this end, we define a new parameter \(\tau\) as follows. Let \(I := (G,c,k,\Psi)\) be an instance of EL-Multi-STC. Then \(\tau := |\Psi(E) \setminus \{\emptyset\}|\) is defined as the number of different non-empty edge lists occurring in the instance \(I\). It clearly holds that \(\tau \leq 2^c - 1\).

For this kernelization we use critical cliques and critical clique graphs [28]. These concepts were also used to obtain linear-vertex kernels for Cluster Deletion [14] and STC [13]. The kernelization described here generalizes the linear-vertex kernel for STC.

Definition 6. A **critical clique** of a graph \(G\) is a clique \(K\) where the vertices of \(K\) all have the same neighbors in \(V \setminus K\), and \(K\) is maximal under this property. Given a graph \(G = (V,E)\), let \(K\) be the collection of its critical cliques. The **critical clique graph** \(C\) of \(G\) is the graph \((K, E_C)\) with \(\{K_i, K_j\} \in E_C \iff \exists u \in K_i, v \in K_j : \{u, v\} \in E\).

For a critical clique \(K\) we let \(N(K) := \bigcup_{K' \in N_C(K)} K'\) denote the union of its neighbor cliques in the critical clique graph and \(N^2(K) := \bigcup_{K' \in N_C(K)} K'\) denote the union of the critical cliques at distance exactly two from \(K\). The critical clique graph can be constructed in \(O(n+m)\) time [17].

Critical cliques are an important tool for EL-Multi-STC, because every edge between the vertices of some critical clique is not part of any induced \(P_3\) in \(G\). Hence, each such edge \(e\) is strong under any STC-Labeling unless \(\Psi(e) = \emptyset\). In the following, we will distinguish between two types of critical cliques. We say that \(K\) is **closed** if \(N(K)\) forms a clique in \(G\) and that \(K\) is open otherwise. We will see that the number of vertices in open critical cliques is at most \(2k_1\). The main reduction rule of this kernelization describes how to deal with large closed critical cliques. Before we give the concrete rules we provide a useful property of closed critical cliques.

**Lemma 5.** Let \(G = (V,E)\) be a graph, let \(K\) be a closed critical clique in \(G\), and let \(\Psi : E \rightarrow 2^{\{1,\ldots,c\}}\) a mapping for some \(c \in \mathbb{N}\). Moreover, let \(v \in N(K)\) and \(E' \subseteq E\{\{v\}\} K\) such that all \(e' \in E'\) have the same strong color list under \(\Psi\). Then, there is an optimal STC-labeling \(L = (S_{L1}, S_{L2}, \ldots, S_{Lq}, W_L)\) for \(G\) and \(\Psi\) such that \(E' \subseteq A\) for some \(A \subseteq \{S_{L1}, S_{L2}, \ldots, S_{Lq}, W_L\}\).

**Proof.** Pick an optimal STC-labeling \(L = (S_{L1}, S_{L2}, \ldots, S_{Lq}, W_L)\) such that \(\max_i |S_{Li}|\) is largest possible. Without loss of generality, by renaming, assume that \(S_{L1}\) is the largest strong color class. We claim that there are no two elements \(e', e'' \in E'\) with \(e' \in S_{L1}\) and \(e'' \notin S_{L1}\). Suppose such elements \(e', e''\) as before exist. We define a new labeling \(\hat{L} = (S_{L1}', S_{L2}', \ldots, S_{Lq}', W_{L})\) by \(S_{Li}' := S_{Li} \cup \{e''\}\), \(W_L := W_L \setminus \{e''\}\) and \(S_{Li}' := S_{Li} \setminus \{e''\}\) for \(i \in \{2,\ldots,q\}\) and show that \(\hat{L}\) is an optimal STC-labeling.

Let \(e'' \in E\) such that \(e''\) and \(e''\) are the edges of an induced \(P_3\) in \(G\). Since \(K \cup N(K)\) forms a clique by the definition of closed critical cliques, it follows that \(\tau \in E\{\{v\}, N^2(K)\}\). Note that \(\tau\) also forms an induced \(P_3\) with \(e'\). Observe that \(\tau \notin S_{L1}'\) since \(L\) is an STC-labeling and \(e' \in S_{L1}'\). Hence, \(\hat{L}\) does not violate STC.

From the definition of \(E'\) and the fact that \(e' \in S_{L1}'\) we know that \(1 \in \Psi(e') = \Psi(e'')\). Hence, \(\hat{L}\) is \(\Psi\)-satisfying. If \(e'' \notin W_L\), it follows that \(W_L = W_L\) and since \(W_L\) is minimal, so is \(W_L\). Otherwise, if \(e' \in W_L\), it follows that \(|W_L| < |W_L|\) which contradicts the fact that \(L\) is optimal. It follows that \(\hat{L}\) is an optimal STC-labeling with \(e', e'' \in S_{L1}'\). That is, \(|S_{L1}'| > |S_{L1}'|\), a contradiction to the choice of \(L\).
Algorithm 1 EL-Multi-STC kernel reduction

1: Input: \( G = (V, E) \) graph, \( K \subseteq V \) closed critical clique in \( G \)
2: for each \( v \in \mathcal{N}(K) \) do
3: for each \( \psi \in \{\Psi(e) \neq \emptyset \mid e \in E\{v\}, K\} \) do
4: \( i := 0 \)
5: for each \( w \in N(v) \cap K \) do
6: if \( \Psi\{v, w\} = \psi \) then
7: Mark \( w \) as important
8: \( i := i + 1 \)
9: if \( i = |E\{v\}, \mathcal{N}^2(K)\| \) then
10: break
11: Decrease the value of \( k \) by the number of edges \( e \) that are incident with a deleted vertex \( u \) and \( \Psi(e) = \emptyset \).
12: Delete all vertices \( u \in K \) which are not marked as important from \( G \)
13: Decrease the value of \( k \) by the number of edges \( e \) that are incident with a deleted vertex \( u \) and \( \Psi(e) = \emptyset \).

Rule 1. If \( G \) has a closed critical clique \( K \) with \( |K| > \tau \cdot |E(\mathcal{N}(K), \mathcal{N}^2(K))| \), then apply Algorithm 1 on \( G \) and \( K \).

Proposition 4. Rule 1 is safe and can be applied in polynomial time.

Proof. Let \( (G = (V, E), c, k, \Psi) \) be an instance for EL-Multi-STC and let \( K \) be a closed critical clique. We show that Algorithm 1 applied on \( G \) and \( K \) runs in \( O(n^3) \) time and produces an equivalent instance \( (G' = (V', E'), c, k', \Psi') \) for EL-Multi-STC.

Since \( |\mathcal{N}(K)|, |\mathcal{N}(v) \cap K| \leq n \) and \( |\{\Psi(e) \neq \emptyset \mid e \in E\{v\}, K\}| \leq |K| \leq n \), the given algorithm clearly runs in \( O(n^3) \) time. It remains to show that the produced instance \( I' := (G' = (V', E'), c, k', \Psi') \) is equivalent to \( I := (G = (V, E), c, k, \Psi) \).

Let \( D_v \subseteq V \) be the set of vertices that were deleted by Algorithm 1, let \( D_E \) be the set of edges that are incident with some \( v \in D_v \) and let \( D^0_E \subseteq D_E \) be the set of edges \( e \in D_E \) with \( \Psi(e) = \emptyset \). We have

\[
G' = (V \setminus D_v, E \setminus D_E), \quad k' = k - |D^0_E|, \quad \text{and} \quad \Psi' = \Psi|_{E \setminus D_E}.
\]

We also define \( K' := K \setminus D_v \) as the modified critical clique in \( G' \).

Let \( L = (S^1_L, S^2_L, \ldots, S^c_L, W_L) \) be a \( \Psi \)-satisfying STC-labeling for \( G \) such that \( |W_L| \leq k \). We define a labeling \( \hat{L} = (S^1_L, \ldots, S^c_L, W_L) \) by \( W_L := W_L \setminus D_E \) and \( S^i_L := S^i_L \setminus D_E \) for each \( i \in \{1, \ldots, c\} \). From the fact that \( L \) is \( \Psi \)-satisfying, it follows that \( \hat{L} \) is \( \Psi' \)-satisfying. It also holds that

\[
|W_L| = |W_L \setminus D_E| = |W_L| - |W_L \cap D_E| \leq k - |D^0_E| = k',
\]

since \( D^0_E \subseteq W_L \cap D_E \). It remains to prove that \( \hat{L} \) does not violate STC. Assume there is an induced \( P_3 \) on vertices \( u, v, w \in V' \) with edges \( \{u, v\}, \{v, w\} \in S^L \) for some \( 1 \leq i \leq c \). It follows that \( \{u, w\} \in D_E \), since \( L \) is an STC-labeling. Then, by the definition of \( D_E \), at least one of the edges \( u \) or \( w \) was deleted by the algorithm. This contradicts the fact that \( u, w \in V' = V \setminus D_v \). It follows that \( \hat{L} \) is a \( \Psi' \)-satisfying STC-labeling for \( G' \) with at most \( k' \) weak edges.

Conversely, let \( \hat{L} = (S^1_L, \ldots, S^c_L, W_L) \) be a \( \Psi' \)-satisfying STC-labeling for \( G' \) such that \( |W_L| \leq k - |D^0_E| \). We define a \( \Psi \)-satisfying STC-labeling \( L \) for \( G \), with \( |W_L| \leq k \). We start with a claim about the vertices in \( \mathcal{N}(K') \). Consider a fixed vertex \( v \in \mathcal{N}(K') \) and a set \( K_v \subseteq K' \) such that all edges in \( E(\{v\}, K_v) \) have the same strong color list \( \psi \neq \emptyset \) under \( \Psi' \).

Claim 12. If \( |K_v| \geq |E(\{v\}, \mathcal{N}^2(K'))| \), then we can assume that \( E(\{v\}, K_v) \subseteq S^i_L \) for some \( 1 \leq i \leq c \).
Proof. Since $K'$ is a closed critical clique, we can assume by Lemma 5, that either all edges in $E(\{v\}, K_v)$ are weak or have the same strong color under an $\hat{L}$. It remains to consider the case, where $E(\{v\}, K_v) \subseteq W_L$.

Let $E(\{v\}, K_v) \subseteq W_L$. Note that, whenever an edge $e \in E(\{v\}, K_v)$ forms an induced $P_3$ with another edge $e'$, it follows that $e' \in E(\{v\}, N^2(K'))$. Let $i \in \psi$. We define a new labeling $P = (S^i_p, \ldots, S^\psi_p, W_P)$ for $G'$ by

$$S^i_p := S^i_L \cup E(\{v\}, K_v) \setminus E(\{v\}, N^2(K')),$$
$$W_P := W_L \setminus E(\{v\}, K_v) \cup (S^i_L \cap E(\{v\}, N^2(K'))),$$
$$S^j_p := S^j_L \text{ for all } j \neq i.$$

From $|K_v| \geq |E(\{v\}, N^2(K'))|$ we conclude

$$|W_P| = |W_L| - |E(\{v\}, K_v)| + |S^i_L \cap E(\{v\}, N^2(K'))|$$
$$\leq |W_L| - |K_v| + |E(\{v\}, N^2(K'))|$$
$$\leq |W_L|.$$

Moreover, $P$ clearly is $\Psi'$-satisfying. It remains to show that $P$ is an STC-labeling, which means that there is no induced $P_3$ containing an edge $e \in E(\{v\}, K_v) \subseteq S^i_p$ and another edge $e' \in S^i_p$. As mentioned above, the edges in $E(\{v\}, K_v)$ only form an induced $P_3$ with edges from $E(\{v\}, N^2(K'))$. By the construction of $P$, no edge from $E(\{v\}, N^2(K'))$ belongs to $S^i_p$. Hence, $P$ is an STC-labeling.

Now, we define the labeling $L$ for $G$ by extending $\hat{L}$. We set $W_L := W_L \cup D_E^0$. Since $|W_L| \leq k - |D_E^0|$, it holds that $|W_L| \leq k$. It remains to label all edges in $D_E \setminus D_E^0$. Let $u$ be some fixed vertex in $D_V$ and $v \in N(u)$ such that $\{u, v\} \notin D_E^0$.

**Case 1:** If $v \in K$, then the edge $\{u, v\}$ is an edge between two vertices of a critical clique. Since $\{u, v\} \notin D_E^0$, there is some $i \in \Psi(\{u, v\})$. Hence, $\{u, v\}$ satisfies the $\Psi$-list property if we add $\{u, v\}$ to $S^i_L$. Since $\{u, v\}$ is not part of any $P_3$ this does not violate STC.

**Case 2:** If $v \in N(K)$, then there is a set $Y \subseteq K'$ containing at least $|E(\{v\}, N^2(K'))|$ vertices distinct from $u$ such that $\Psi(\{v, y\}) = \Psi(\{v, u\})$ for every $y \in Y$. Otherwise, $u$ would have been marked as important by Algorithm 1, which contradicts the fact that $u \in D_V$. From Claim 12 we know that all edges in $E(\{v\}, Y)$ are elements of the same strong color class $S^i_L$ for some $i \in \{1, \ldots, c\}$. We set $S^i_L := S^i_L \cup \{u, v\}$. Clearly, $\{u, v\}$ satisfies the $\Psi$-list property under $L$. Moreover, this does not violate STC, since there are no edges in $S^i_L \cap E(\{v\}, N^2(K))$, since $E(\{v\}, Y) \subseteq S^i_L$ and $\hat{L}$ is an STC-labeling.

It follows that $L$ is a $\Psi$-satisfying STC-labeling for $G$ with $|W_L| \leq k$. 

We now consider instances which are reduced regarding Rule 1, which are instances where no more application of Rule 1 is possible. The following upper-bound of the size of closed critical cliques is important for the kernel result.

**Lemma 6.** Let $(G, c, k, \Psi)$ be a reduced instance for EL-MULTI-STC. For every critical closed clique $K$ in $G$ it holds that $|K| \leq \tau \cdot |E(N(K), N^2(K))|$.

**Proof.** We prove this Lemma by having a closer look at the vertices that were not deleted by Algorithm 1. Note that the algorithm is applied on every closed critical clique $K$ with $|K| > \tau \cdot |E(N(K), N^2(K))|$. Every vertex that was not marked as important in Line 7 of the algorithm is deleted from $G$. Note that there are at most $\tau$ possible images $\psi$ of $\Psi : E \rightarrow 2^{1, \ldots, c}$. By Lines 7
and 9 and 10 it holds that for every \( v \in N(v) \) the algorithm marks at most \( \tau \cdot |E(\{v\}, N^2(K))| \) vertices of \( K \). It follows that there are at most

\[
\tau \cdot \sum_{v \in N(K)} |E(\{v\}, N^2(K))| = \tau \cdot |E(N(K), N^2(K))|
\]

marked vertices in \( K \), since \( \{E(\{v\}, N^2(K)) \mid v \in N(v)\} \) forms a partition of \( E(N(K), N^2(K)) \). Hence, \(|K| \leq \tau \cdot |E(N(K), N^2(K))|\) for every closed critical clique \( K \) in \( G \).

**Theorem 6.** EL-Multi-STC admits a problem kernel with at most \( (\tau + 1) \cdot 2k_1 \) vertices.

**Proof.** Let \((G = (V, E), c, k, \Psi)\) be a reduced instance for EL-Multi-STC and let \( L = (S_L, W_L) \) be an optimal 1-colored STC-labeling for \( G \). By the definition of \( k_1 \) we have \(|W_L| = k_1\). Note that \( L \) does not have to satisfy any list properties. We show that \(|V| \leq (\tau + 1) \cdot 2k_1\).

The overall number of vertices in open critical cliques is at most \( 2k_1[13] \). Let \( K \) be some closed critical clique. We now transform the graph \( G \) into a modified graph \( G' \) in the following way. We replace every closed critical clique \( K \) with a critical clique \( K' \) such that \(|K'| = \frac{K}{\tau} \). From Lemma 6 we know that for every closed critical clique \( K \) in \( G \) it holds that \(|K| \leq \tau \cdot |E(N(K), N^2(K))|\). It follows that for every closed critical clique \( K' \) in \( G' \) it holds that \(|K'| \leq |E(N(K), N^2(K))|\). As shown previously, this implies that the overall number of vertices in closed critical cliques in \( G' \) is at most \( 2k_1[13, Proof of Theorem 1 in the long version]\). Hence, the overall number of vertices in closed critical cliques in \( G \) is at most \( \tau \cdot 2k_1 \), which gives us \(|V| \leq 2k_1 + \tau \cdot 2k_1 = (\tau + 1) \cdot 2k_1\).

Recall that for any EL-Multi-STC instance \((G, c, k, \Psi)\) we have \( \tau \leq 2^e - 1 \). Also, Multi-STC is the special case of EL-Multi-STC where every edge has the list \( \{1, 2, \ldots, c\} \), and thus \( \tau = 1 \). These two facts imply the following.

**Corollary 3.** EL-Multi-STC admits a problem kernel with at most \( 2^{e+1}k_1 \) vertices. Multi-STC admits a problem kernel with at most \( 4k_1 \) vertices.

**Acknowledgment**

This study was initiated at the 7th annual research retreat of the Algorithmics and Computational Complexity group of TU Berlin, Darlingerode, Germany, March 18th–23rd, 2018.

**References**

[1] van Bevern, R., Tsidulko, O.Y., Zschoche, P.: Facility location under matroid constraints: fixed-parameter algorithms and applications. CoRR abs/1806.11527 (2018), http://arxiv.org/abs/1806.11527

[2] Chen, J., Kanj, I.A., Xia, G.: Improved upper bounds for vertex cover. Theor. Comput. Sci. 411(40-42), 3736–3756 (2010)

[3] Cygan, M., Fomin, F.V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Parameterized Algorithms. Springer (2015)

[4] Diehl, C.P., Namata, G., Getoor, L.: Relationship identification for social network discovery. In: Proc. 22nd AAAI. pp. 546–552. AAAI Press (2007)

[5] Dom, M., Lokshtanov, D., Saurabh, S.: Kernelization lower bounds through colors and IDs. ACM Transactions on Algorithms 11(2), 13:1–13:20 (2014)
[6] Downey, R.G., Fellows, M.R.: Fundamentals of Parameterized Complexity. Texts in Computer Science, Springer (2013)

[7] Flum, J., Grohe, M.: Parameterized Complexity Theory. Texts in Theoretical Computer Science. An EATCS Series, Springer (2006)

[8] Gallai, T.: Transitiv orientierbare Graphen. Acta Math. Hung. 18(1–2), 25–66 (1967)

[9] Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman (1979)

[10] Golovach, P.A., Heggernes, P., Konstantinidis, A.L., Lima, P.T., Papadopoulos, C.: Parameterized aspects of strong subgraph closure. In: Proc. 16th SWAT). LIPIcs, vol. 101, pp. 23:1–23:13. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2018)

[11] Granovetter, M.: The strength of weak ties. Am. J. Sociol. 78, 1360–1380 (1973)

[12] Granovetter, M.: The strength of weak ties: A network theory revisited. Sociological theory pp. 201–233 (1983)

[13] Grittemeier, N., Komusiewicz, C.: On the relation of strong triadic closure and cluster deletion. In: Proc. 44th WG. LNCS, vol. 11159, pp. 239–251. Springer (2018), full version available at https://arxiv.org/abs/1803.00807

[14] Guo, J.: A more effective linear kernelization for cluster editing. Theor. Comput. Sci. 410(8-10), 718–726 (2009)

[15] Hajnal, A., Szemerédi, E.: Proof of a conjecture of P. Erdős. Combinatorial theory and its applications 2, 601–623 (1970)

[16] Holyer, I.: The NP-Completeness of Edge-Coloring. SIAM J. Comput. 10(4), 718–720 (1981)

[17] Hsu, W., Ma, T.: Substitution decomposition on chordal graphs and applications. In: Proc. 2nd ISA. LNCS, vol. 557, pp. 52–60. Springer (1991)

[18] Huang, H., Tang, J., Wu, S., Liu, L., Fu, X.: Mining triadic closure patterns in social networks. In: Proc. 23rd WWW. pp. 499–504. ACM (2014)

[19] Impagliazzo, R., Paturi, R., Zane, F.: Which problems have strongly exponential complexity? J. Comput. Syst. Sci. 63(4), 512–530 (2001)

[20] Karp, R.M.: Reducibility among combinatorial problems. In: Complexity of Computer Computations. pp. 85–103 (1972)

[21] Kierstead, H.A., Kostochka, A.V., Mydlarz, M., Szemerédi, E.: A fast algorithm for equitable coloring. Combinatorica 30(2), 217–224 (2010)

[22] Konstantinidis, A.L., Nikolopoulos, S.D., Papadopoulos, C.: Strong triadic closure in cographs and graphs of low maximum degree. Theor. Comput. Sci. 740, 76–84 (2018)

[23] Konstantinidis, A.L., Papadopoulos, C.: Maximizing the Strong Triadic Closure in Split Graphs and Proper Interval Graphs. In: Proc. 28th ISAAC. LIPIcs, vol. 92, pp. 53:1–53:12. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik (2017)

[24] Kowalik, L., Lauri, J., Socała, A.: On the fine-grained complexity of rainbow coloring. SIAM J. Discrete Math. 32, 1672–1705 (2018)
[25] Kowalik, L., Socala, A.: Tight lower bounds for list edge coloring. In: Proc. 16th SWAT. LIPIcs, vol. 101, pp. 28:1–28:12. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2018)

[26] Le, V.B.: Gallai graphs and anti-gallai graphs. Discrete Mathematics 159(1-3), 179–189 (1996)

[27] Niedermeier, R.: Invitation to Fixed-Parameter Algorithms. Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press (2006)

[28] Protti, F., Dantas da Silva, M., Szwarcfiter, J.L.: Applying modular decomposition to parameterized cluster editing problems. Theory Comput. Syst. 44(1), 91–104 (2009)

[29] Rozenshtein, P., Tatti, N., Gionis, A.: Inferring the strength of social ties: A community-driven approach. In: Proc. 23rd KDD. pp. 1017–1025. ACM (2017)

[30] Sintos, S., Tsaparas, P.: Using strong triadic closure to characterize ties in social networks. In: Proc. 20th KDD. pp. 1466–1475. ACM (2014)

[31] Sun, L.: Two classes of perfect graphs. Journal of Combinatorial Theory, Series B 53(2), 273–292 (1991)

[32] Tang, J., Lou, T., Kleinberg, J.M.: Inferring social ties across heterogeneous networks. In: Proc. 5th WSDM. pp. 743–752. ACM (2012)

[33] Tovey, C.A.: A simplified NP-complete satisfiability problem. Discr. Appl. Math. 8(1), 85–89 (1984)

[34] Xiang, R., Neville, J., Rogati, M.: Modeling relationship strength in online social networks. In: Proc. 19th WWW. pp. 981–990. ACM (2010)

[35] Yang, J., Leskovec, J.: Community-affiliation graph model for overlapping network community detection. In: Proc. 12th ICDM. pp. 1170–1175. IEEE Computer Society (2012)