A RELAXED EVALUATION SUBGROUP

TOSHIHIRO YAMAGUCHI

Abstract. Let \( f : X \to Y \) be a pointed map between connected CW-complexes. As a generalization of the evaluation subgroup \( G_n(Y, X; f) \), we will define the relaxed evaluation subgroup \( \mathcal{G}_n(Y, X; f) \) in the homotopy group \( \pi_n(Y) \) of \( Y \), which is identified with \( \mathrm{Im} \pi_n(\tilde{ev}) \) for the evaluation map \( \tilde{ev} : \text{map}(X, Y; f) \times X \to Y \) given by \( \tilde{ev}(h, x) = h(x) \). Especially we see by using Sullivan model in rational homotopy theory for the rationalized map \( f_\mathbb{Q} \) that \( \mathcal{G}_n(Y_\mathbb{Q}, X_\mathbb{Q}; f_\mathbb{Q}) = \pi_n(Y_\mathbb{Q}) \otimes \mathbb{Q} \) if the map \( f \) induces an injection of rational homotopy groups. Also we compare it with more relaxed subgroups by several rationalized examples.

1. Introduction

Let \( f : X \to Y \) and \( g : B \to Y \) be pointed maps of connected CW complexes. Recall the definition of pairing with axes \( f \) and \( g \) \([O]\), which is given by the existence of a map \( F_{g,f} : B \times X \to Y \) in the homotopy commutative diagram:

\[
\begin{array}{ccc}
B \times X & \xrightarrow{i_B} & B \\
& i_X \searrow & \downarrow f \\
& B & \rightarrow Y,
\end{array}
\]

where \( i_B(b) = (b, \ast) \) and \( i_X(x) = (\ast, x) \). In particular, when \( B = Y = X \) and \( f = g = \text{id} \), \( X \) is an H-space of the multiplication \( F_{\text{id}, \text{id}} \). In this paper, we consider whether or not there exist a section \( s : B \to B \times X \) and a map \( F : B \times X \to Y \) such that the diagram

\[
\begin{array}{ccc}
B \times X & \xrightarrow{i_B} & B \\
& i_X \searrow & \downarrow F \\
& B & \rightarrow Y,
\end{array}
\]

homotopically commutes. Here a section means a map \( s : B \to B \times X \) which satisfies \( p_B \circ s \simeq \text{id}_B \) for the projection \( p_B : B \times X \to B \) with \( p_B(b, x) = b \).

The \( n \)th Gottlieb group \( G_n(X) \) of a space \( X \) is the subgroup of the \( n \)th homotopy group \( \pi_n(X) \) of \( X \) consisting of homotopy classes of maps \( a : S^n \to X \) such that the wedge \( (a|\text{id}_X) : S^n \vee X \to X \) extends to a map \( F_a : S^n \times X \to X \) \([G]\). The \( n \)th evaluation subgroup \( G_n(Y, X; f) \) of a map \( f : X \to Y \) is the subgroup of \( \pi_n(Y) \) represented by maps \( a : S^n \to Y \) such that \( (a|f) : S^n \vee X \to Y \) extends to a map...
$F_{a,f} : S^n \times X \to Y$ inducing the homotopy commutative diagram

\[
\begin{array}{c}
\xymatrix{ S^n \times X \ar[r]^{i_X} \ar[d]_{i_{S^n}} & X \ar[d]^f \ar[dl]_{F_{a,f}} & \nonumber \\
S^n \ar[r]_a & Y & \nonumber }
\end{array}
\tag{3}
\]

which is the case of $B = S^n$ in (1). The map $F_{a,f}$ in (3) is the adjoint of a map $F_{a,f}'$ in the homotopy commutative diagram

\[
\begin{array}{c}
\xymatrix{ \text{map}_f(X,Y) \ar[d]^{ev} & \nonumber \\
S^n \ar[r]^a & Y & \nonumber }
\end{array}
\tag{3}'
\]

where $F_{a,f}'(b)(x) := F_{a,f}(b, x)$ for $b \in S^n$ and $x \in X$. Here $\text{map}_f(X,Y)$ is the connected component of $f$ in the mapping space $\text{map}(X,Y)$ from $X$ to $Y$ with the compact open topology and $ev$ is the evaluation map given by $ev(h) = h(\ast)$. Then

\[
G_n(Y, X; f) = \text{Im} (\pi_n(ev) : \pi_n(\text{map}_f(X,Y)) \to \pi_n(Y))
\]

in $\pi_n(Y)$ and therefore it is called an ‘evaluation’ subgroup of a map. Notice that (3) is a special case of the homotopy commutative diagram

\[
\begin{array}{c}
\xymatrix{ S^n \times X \ar[r]^{i_X} \ar[d]_s & X \ar[d]^f \ar[dl]_F & \nonumber \\
S^n \ar[r]_a & Y & \nonumber }
\end{array}
\tag{4}
\]

in which $\eta : X \xrightarrow{i_X} S^n \times X \xrightarrow{p_X} S^n$ is a trivial fibration with a section $s : S^n \to S^n \times X$. Here we can put $s(b) = (b, \tau(b))$ for a map $\tau : S^n \to X$.

**Definition A.** For a map $f : X \to Y$, the $n$th relaxed evaluation subgroup of $f$ is given as

\[
G_n(Y, X; f) := \{ a \in \pi_n(Y) \mid \text{there are a section } s : S^n \to S^n \times X \text{ and a map } F : S^n \times X \to Y \text{ such that } F \circ s \simeq a, f \simeq F \circ i_X \}.
\]

The map $F$ in (4) is the adjoint of a map $\tilde{F}'$ in the homotopy commutative diagram

\[
\begin{array}{c}
\xymatrix{ \text{map}_f(X,Y) \times X \ar[d]^{ev} & \nonumber \\
S^n \ar[r]^a & Y & \nonumber }
\end{array}
\tag{4}'
\]

where $\tilde{F}'(b) := (F'(b), \tau(b))$ with $F'(b)(x) = F(b, x)$ and $s(b) = (b, \tau(b))$ for $b \in S^n$ and $x \in X$. Here $ev : \text{map}(X,Y; f) \times X \to Y$ is the evaluation map given by $ev(h, x) = h(x)$. Thus
Claim. \( \mathcal{G}_n(Y, X; f) = \text{Im}(\pi_n(\tilde{ev}) : \pi_n(\text{map}_f(X, Y) \times X) \to \pi_n(Y)) \).

In this paper, we will estimate \( \text{Im} \pi_n(\tilde{ev}) \) in several cases according to Definition A. We note that it is a naturally generalized object of an ordinary evaluation subgroup. Indeed, for a subcomplex \( X' \) of \( X \), we can put
\[
\mathcal{G}_n(Y, X; f)(X') := \{ a \in \pi_n(Y) \mid \text{there are a section } s : S^n \to S^n \times X' \text{ and a map } F : S^n \times X \to Y \text{ such that } F \circ s \simeq a, \ f \simeq F \circ i_X \}. 
\]

Then we have
\[
\mathcal{G}_n(Y, X; f) \supset \mathcal{G}_n(Y, X; f)(X') \supset \mathcal{G}_n(Y, X; f)(*) = G_n(Y, X; f)
\]
and \( \mathcal{G}_n(Y, X; f)(X') = \text{Im}(\pi_n(\tilde{ev}) : \pi_n(\text{map}_f(X, Y) \times X') \to \pi_n(Y)) \). In the following, we see several properties of a relaxed evaluation subgroup.

**Lemma 1.1.** For a subspace \( X \) of a space \( Y \) and a map \( g : B \to Y \) with \( \text{Im} g \subset X \), there are a section \( s : B \to B \times X \) and a map \( F : B \times X \to X \) such that the diagram

\[
\begin{array}{ccc}
B \times X & \xrightarrow{i_X} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & Y
\end{array}
\]

commutes.

Indeed, put \( s(b) = (b, g(b)) \) and \( F((b, x)) = x \) for \( (b, x) \in B \times X \).

In particular, in the case of \( B = S^n \) and \( X = Y^m \), the \( m \)-skelton of \( Y \), we have by cellular approximation theorem

**Proposition 1.2.** For \( n \leq m \), \( \mathcal{G}_n(Y, Y^m; i_{Y^m}) = \pi_n(Y) \) for \( i_{Y^m} : Y^m \subset Y \). In particular, \( \mathcal{G}_n(Y, Y; id_Y) = \pi_n(Y) \).

Of course \( \mathcal{G}_n(Y, *; *) = G_n(Y, *; *) = \pi_n(Y) \) for the constant map \( * : * \to Y \) and we know \( G_n(Y) = G_n(Y, Y; id_Y) \subset \mathcal{G}_n(Y, X; f) \) for any map \( f : X \to Y \). In contrast, \( \mathcal{G}_n(Y, X; f) \subset \mathcal{G}_n(Y, Y; id_Y) \) from Proposition 1.2. Note that \( \mathcal{G}_n(Y, X; f) \) may be zero for a map \( f : X \to Y \) even if \( \pi_n(Y) \neq 0 \) (see Example 2.0 and examples in Section 3).

Also as an evaluation subgroup satisfies it, a relaxed evaluation subgroup satisfies

\[
\pi_n(g)(\mathcal{G}_n(Y, X; f)) \subset \mathcal{G}_n(Y', X; g \circ f)
\]

for a pointed map \( g : Y \to Y' \). Thus there is a map \( \pi_n(f) : \pi_n(X) = \mathcal{G}_n(X, X; id_X) \to \mathcal{G}_n(Y, X; f) \). Therefore the relaxed evaluation subgroups \( \mathcal{G}_n(X, Z; j) \) and \( \mathcal{G}_n(Y, X; f) \) are embedded into parts of the homotopy exact sequence of a fibration \( \xi : Z \xrightarrow{f} X \xrightarrow{j} Y \), respectively.

**Corollary 1.3.** For a fibration \( \xi : Z \xrightarrow{j} X \xrightarrow{f} Y \) and any \( n \), there are the sequences

\[
\pi_{n+1}(Y) \xrightarrow{\partial_{n+1}} \pi_n(Z) \xrightarrow{\pi_n(j)} \mathcal{G}_n(X, Z; j) \xrightarrow{\pi_n(f)} \pi_n(Y)
\]

and

\[
\pi_n(Z) \xrightarrow{\pi_n(j)} \pi_n(X) \xrightarrow{\pi_n(f)} \mathcal{G}_n(Y, X; f) \xrightarrow{\partial_n \circ} \pi_{n-1}(Z)
\]

which are both exact. Moreover, for the connecting map \( \tilde{\partial} : \Omega Y \to Z \) of \( \xi \),

\[
\pi_{n+1}(X) \xrightarrow{\pi_{n+1}(f)} \pi_{n+1}(Y) \xrightarrow{\partial_{n+1} \circ} \mathcal{G}_n(Z, \Omega Y; \tilde{\partial}) \xrightarrow{\pi_{n+1}(j)} \pi_n(X)
\]
is exact.

Note that, for a pointed map \( g : X' \to X \), there is an inclusion \( G_n(Y, X; f) \subset G_n(Y, X'; f \circ g) \). But it does not hold for relaxed evaluation subgroups in general.

**Lemma 1.4.** For a map \( g : B \to Y \) and a map \( f : X \to Y \) such that \( f_* : [B, X] \to [B, Y] \) is surjective, there are a section \( s : B \to B \times X \) and a map \( F : B \times X \to X \) such that the diagram

\[
\begin{array}{ccc}
B \times X & \xrightarrow{i_X} & X \\
\downarrow{s} & & \downarrow{f} \\
B & \xrightarrow{g} & Y \\
\end{array}
\]

homotopically commutes.

Indeed, there is a lift \( \tilde{g} : B \to X \) such that \( f \circ \tilde{g} \simeq g \) from the assumption. Then put \( s(b) = (b, \tilde{g}(b)) \) and \( F((b, x)) = f(x) \) for \( (b, x) \in B \times X \).

In particular, in the case of \( B = S^n \), we have

**Proposition 1.5.** If a map \( f : X \to Y \) induces a surjection \( \pi_n(f) : \pi_n(X) \to \pi_n(Y) \), then \( G_n(Y, X; f) = \pi_n(Y) \).

Suppose that \( f_Q : X_Q \to Y_Q \) is a rationalized map with \( X \) and \( Y \) simply connected CW complexes of finite type [HMR]. We consider the relaxed evaluation subgroup of a map and more relaxed subgroups from a point of view of rational homotopy. By using Sullivan’s minimal model arguments, we show

**Theorem 1.6.** If a map \( f : X \to Y \) induces an injection \( \pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q} \) on rational homotopy groups, then \( G_*(Y_Q, X_Q; f_Q) = \pi_*(Y) \otimes \mathbb{Q} \).

From Proposition 1.5 and Theorem 1.6, we have

**Corollary 1.7.** If the homotopy fiber of a map \( f : X \to Y \) has the rational homotopy type of the Eilenberg-MacLane space \( K(\mathbb{Q}, n) \) for some \( n \), then \( G_*(Y_Q, X_Q; f_Q) = \pi_*(Y) \otimes \mathbb{Q} \).

In Section 2, we prove Theorem 1.6 after preparing of the notion of derivations of model. In Section 3, we will define more relaxed subgroups of the homotopy group of \( Y \), the trncz subgroup \( T_*(Y, X; f) \) and the sectional subgroup \( S_*(Y, X; f) \), for a map \( f : X \to Y \). They are defined by relaxing the trivial fibration \( \eta \) in (4). We compare \( G_*(Y, X; f) \) with them by several rationalized examples.

**2. Sullivan models**

We use the Sullivan minimal model \( M(Y) \) of a simply connected CW complex \( Y \) of finite type. It is a free \( \mathbb{Q} \)-commutative differential graded algebra (DGA) \((\Lambda W, dy)\) with a \( \mathbb{Q} \)-graded vector space \( W = \bigoplus_{i \geq 2} W^i \) where \( \dim W^i < \infty \) and a decomposable differential. Here \( \Lambda^+ W \) is the ideal of \( \Lambda W \) generated by elements of positive degree. Denote the degree of a homogeneous element \( x \) of \( \Lambda W \) as \(|x|\). Then \( xy = (-1)^{|x||y|}yx \) and \( d(xy) = d(x)y + (-1)^{|x|}xd(y) \). Recall \( M(Y) \) determines the rational homotopy type of \( Y \). Especially there is an isomorphism
We denote the dual element of \( a \in \pi_1(Y) \otimes \mathbb{Q} \) as \( a^* \). Put \( M(Y) = (AW, d_Y) \). Then the model of a map \( f : X \to Y \) is given by a KS-extension
\[
(W^i \cong \text{Hom}(\pi_i(Y), \mathbb{Q}). \quad \text{where} \quad M(Y) = (AW, d_Y).
\]
with \( D|_{AW} = d_Y \) and the minimal model \((AV, \mathcal{D})\) of the homotopy fiber of \( f \) or
\[
(H^*(Y; \mathbb{Q}), 0) \xrightarrow{i} (AW \otimes AV, D) \xrightarrow{\delta} (AV, \mathcal{D})
\]
when \( Y \) is formal (for example, when \( Y \) is a sphere) \([\text{FHT}]\). In general, \( D \) is not decomposable and it is decomposable if and only if \( \pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q} \) is a surjection. See \([\text{FHT}]\) for a general introduction and the standard notations.

Let \( A \) be a DGA \( A = (A^*, d_A) \) with \( A^* = \bigoplus_{i \geq 0} A^i \), \( A^0 = \mathbb{Q} \), \( A^1 = 0 \) and the augmentation \( \epsilon : A \to \mathbb{Q} \). Define \( \text{Der}_A \) the vector space of self-derivations of \( A \) decreasing the degree by \( i > 0 \), where \( \theta(xy) = \theta(x)y + (-1)^{|x|}x\theta(y) \) for \( \theta \in \text{Der}_i A \). We denote \( \bigoplus_{i \geq 0} \text{Der}_i A \) by \( \text{Der}A \). The boundary operator \( \delta : \text{Der}_A \to \text{Der}_{A-1} A \) is defined by \( \delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A \). For a DGA-map \( \phi : A \to B \), define a \( \phi \)-derivation of degree \( n \) to be a linear map \( \theta : A^* \to B^{*-n} \) with \( \theta(xy) = \theta(x)\phi(y) + (-1)^{|x|}\phi(x)\theta(y) \) and \( \text{Der}(A, B; \phi) \) the vector space of \( \phi \)-derivations. The boundary operator \( \delta_\phi : \text{Der}_n(A, B; \phi) \to \text{Der}_{n-1}(A, B; \phi) \) is defined by \( \delta_\phi(\sigma) = d_B \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A \). Note that \( \delta_\phi(\theta) = 0 \).

**Theorem 2.1.** \([\text{FHT}, \text{LS}]\) \( G_n(Y_Q, X_Q; f_Q) \cong G_n(M(Y), M(X); M(f)) = G_n((AW, d_Y), (AW \otimes AV, D)) \).

Thus \( G_n(Y_Q, X_Q; f_Q) \) is completely determined by the derivations only. For a DGA-map \( \phi : (AV, d) \to (AZ, d') \), the symbol \( (v, h) \in \text{Der}(AV, AZ; \phi) \) means the \( \phi \)-derivation sending an element \( v \in V \) to \( h \in AZ \) and the other to zero. Especially \( (v, 1) = v^* \).

**Example 2.2.** Consider the fibration \( S^3 \to X \xrightarrow{f} Y = S^2 \times S^2 \) whose KS-extension is given by
\[
(A(w_1, w_2, w_3, w_4), d_Y) \to (\Lambda(w_1, w_2, w_3, w_4, v), D) \to (\Lambda(v), 0),
\]
where \( |w_1| = |w_2| = 2 \), \( |w_3| = |w_4| = |v| = 3 \), \( d_Y w_1 = d_Y w_2 = 0 \), \( d_Y w_3 = w_1^2 \), \( d_Y w_4 = w_2^2 \) and \( Dv = w_1 w_2 \). Since \( D \) is decomposable, \( \pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q} \) is surjective. So we have \( G_2(Y_Q, X_Q; f_Q) = \pi_2(Y) \otimes \mathbb{Q} \) from Proposition \([\text{FHT}]\). On the other hand, from Theorem 2.1, \( G_2(Y_Q, X_Q; f_Q) = 0 \) since
\[
\delta_f((w_1, 1)) = 2(w_{i+2}, w_i) \not\in \delta_f(\text{Der}(AW, \Lambda W \otimes \Lambda^+ v))
\]
for \( i = 1, 2 \) and \( W = \mathbb{Q}(w_1, w_2, w_3, w_4) \). Note that \( f_Q \) has no section \( \{\Pi\} \).

**Proof of Theorem 2.1.** Fix an element \( a \in \pi_n(Y) \otimes \mathbb{Q} \). From the assumption, there is a DGA-projection \( p_{W,V} : (AW, d_Y) \to (\Lambda V, d_Y) \) as the model of \( f \). A model of
the (non homotopy commutative) diagram

\[
\begin{array}{c}
\begin{array}{c}
E \\
p
\end{array}
\end{array}
\xrightarrow{\ i \ } \begin{array}{c}
\begin{array}{c}
X_Q \\
f_Q
\end{array}
\end{array}
\xrightarrow{\ f_0 \ } \begin{array}{c}
\begin{array}{c}
S^n_Q \\
a
\end{array}
\end{array}
\xrightarrow{\ \sigma \ } \begin{array}{c}
\begin{array}{c}
Y_Q
\end{array}
\end{array}
\end{array}
\]

is given by

\[
\begin{array}{c}
(\Lambda x/x^2 \otimes \Lambda V, D') \xrightarrow{\ q \ } (\Lambda V, d_Y) \\
\cup
\end{array}
\xrightarrow{\ pw.V \ } \begin{array}{c}
\begin{array}{c}
(\Lambda x/x^2, 0) \xleftarrow{\ M(a) \ } (\Lambda W, d_Y)
\end{array}
\end{array}
\]

where \(|x| = n, i_{V,W} : V \subset W\) and \(p_{W,V}(\Lambda W) = \Lambda W = \Lambda V\) with \(p_{W,V} \circ i_{V,W} = id_V\). Here \(\Lambda x/x^2 = \Lambda x\) if \(n\) is odd and \(\Lambda x/x^2 = Q[x]/(x^2)\) if \(n\) is even.

We will construct a rationally trivial fibration of the form \(\eta : X_Q \to E \to S^n_Q\) together with a suitable map \(F : E \to Y_Q\), in model terms. Define a graded \(\mathbb{Q}\)-algebra map \(F' : \Lambda W \to \Lambda x/x^2 \otimes \Lambda V\) by

\[
F'(w) = \overline{w} + (-1)^{|w|} \sigma(w)x
\]

where \(\sigma \in \text{Der}_n(\Lambda W, \Lambda V; p_{W,V})\) with \((-1)^{|\sigma|} \sigma(u)x = M(a)(u)\) for \(u \in W\). Also define the differential \(D'\) by \(D'(x) = 0\) and

\[
D'_{|\Lambda V} = \overline{d_Y} - \delta_{\overline{\sigma}_y} \cdot x,
\]

where \(\overline{\sigma} \in \text{Der}_n(\Lambda V)\) is uniquely given by \(\overline{\sigma}(w) = \sigma(w)\) for \(w \in \Lambda W\) and \((\theta x)(z) := (-1)^{|\theta|} \theta(z)x\) for a derivation \(\theta\). Then \(D' \circ D' = 0\) from \(\delta_{\overline{\sigma}_y} \circ \delta_{\overline{\sigma}_y} = 0\) and \(F'\) is a DGA-map by

\[
F'd_Y(w) = \overline{d_Y}(w) + (-1)^{|w|+1} \sigma(d_Y w)x
\]

\[
= \overline{d_Y}(w) - (-1)^{|\sigma|+1} \delta_{\overline{\sigma}_y}(\overline{\sigma})(\overline{w})x + (-1)^{|\sigma|} \overline{d_Y}\sigma(w)x
\]

\[
= D'(\overline{w} + (-1)^{|\sigma|} \sigma(w)x) = D'F'(w)
\]

for \(w \in \Lambda W\). Thus we have the KS-model of \(\eta\)

\[
(\Lambda x/x^2, 0) \xrightarrow{\ i \ } (\Lambda x/x^2 \otimes \Lambda V, D') \xrightarrow{\ q \ } (\Lambda V, d_Y)
\]

and a map

\[
F' : (\Lambda W, d_Y) \to (\Lambda x/x^2 \otimes \Lambda V, D').
\]

Since \(\delta_{\overline{\sigma}_y}(\overline{\sigma}) \in \text{Der}(\Lambda V, \Lambda^+ V)\), the fibration \(\eta\) has a section \(s\) \((T)\). Moreover the definition of \(D'\) indicates the rational triviality of \(\eta\):

\[
(\Lambda x/x^2 \otimes \Lambda V, D') \cong (\Lambda x/x^2, 0) \otimes (\Lambda V, d_Y)
\]

over \((\Lambda x/x^2, 0)\) since then the homotopy class of the classifying map \(S^n\to \text{Baut}_1 X_Q\), \([\delta_{\overline{\sigma}_y}(\overline{\sigma})]\), is zero in \(\pi_n(\text{Baut}_1 X) \otimes \mathbb{Q} = H_{n-1}(\text{Der}M(X)) \mathbb{Q}\). We
can choose the model of $s$ by $M(s)(x) = x$ and $M(s)(z) = 0$ for $z \in \Lambda V$, then $M(s) \circ F' = M(a)$. Thus there is a DGA-commutative diagram

$$
\begin{array}{cccc}
(\Lambda x/x^2 \otimes \Lambda V, D') & \xrightarrow{q} & (\Lambda V, D') \\
M(s) & \downarrow & \downarrow_{pW, V} \\
(\Lambda x/x^2, 0) & \xleftarrow{M(a)} & (\Lambda W, d_Y).
\end{array}
$$

It is the rational model of (4).

**Proof of Corollary 1.7.** Put the model of homotopy fiber $(\Lambda V, 0)$. When $Dv$ is decomposable, we have from Proposition 1.5. Also when $Dv$ is not decomposable, there is a surjection $M(Y) = (\Lambda W, d_Y) \rightarrow M(X) = (\Lambda V, d_X)$ with $\dim V = \dim W - 1$. Then we have from Theorem 1.6.

**Remark 1.** In the proof of above, the fibration $\eta: X_Q \xrightarrow{\eta} E_P S^n_Q$ is trivial, that is, there is a homotopy commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{i} & X_Q \\
\downarrow_i & & \downarrow_i \\
S^n_Q & \xrightarrow{p \circ i} & S^n_Q \times X_Q
\end{array}
$$

where $g$ is a homotopy equivalence. The model $M(g)$ is given by $id - \sigma \cdot x$, which is a quasi-isomorphism. But the map $g$ does not induce the homotopy commutative diagram with a section $s$ of $p$

$$
\begin{array}{ccc}
E & \xrightarrow{i} & X_Q \\
\downarrow{s} & & \downarrow{s} \\
S^n_Q & \xrightarrow{p \circ s} & S^n_Q \times X_Q
\end{array}
$$

in general. Therefore, even if $\pi_*(f) \otimes \mathbb{Q}$ is surjective or injective, we can not induce $G_*(Y_Q, X_Q; f_Q) = \pi_*(Y) \otimes \mathbb{Q}$ in general. For example, see Example 2.2 or Example 2.5, respectively.

In the followings, we consider some examples, whose models are given by $M(s)(x) = x$ and $M(s)(z) = 0$ for $z \in M(X)$ as in the proof of Theorem 1.6. The index of an element means the degree.

**Example 2.3.** For $n > 0$, $G_*(S^n_Q, S^n_Q; id_{S^n_Q}) = \pi_n(S^n) \otimes \mathbb{Q}$ is given by the following commutative diagrams with $|x| = n$.

$$(n: \text{odd}) \quad (\Lambda x/x^2 \otimes \Lambda w_n, 0) \xrightarrow{q} (\Lambda w_n, 0)$$

$$
\begin{array}{ccc}
(\Lambda x/x^2, 0) & \xrightarrow{M(s)} & (\Lambda w_n, 0) \\
\downarrow_{M(s)} & & \downarrow_{M(s)}
\end{array}
$$

$$(n: \text{even}) \quad (\Lambda x/x^2 \otimes \Lambda w_n, 0) \xrightarrow{q} (\Lambda w_n, 0)$$
where $M(s)(x) = x$, $M(s)(w_n) = 0$ and $F(w_n) = w_n + cx$ for $M(a)(w_n) = cx$ with $c \in \mathbb{Q}$.

\[
(n : \text{even}) \quad (\Lambda x/x^2 \otimes \Lambda(w_n, w_{2n-1}), D') \xrightarrow{\delta} (\Lambda(w_n, w_{2n-1}), d_Y)
\]

\[
M(s) \quad F \quad M(a)
\]

where $d_Y w_n = 0, d_Y w_{2n-1} = w_n^2$, $D' w_n = 0, D' w_{2n-1} = w_n^2 + 2cw_n x, F(w_n) = w_n + cx$ and $F(w_{2n-1}) = w_{2n-1}$.

**Example 2.4.** For the Hopf fibration $S^3 \to S^7 \xrightarrow{\eta} S^4$, we know $G_4(S^4_\mathbb{Q}; S^7_\mathbb{Q}; f_\mathbb{Q}) = \pi_4(S^4) \otimes \mathbb{Q}$ \cite{LS2}. Indeed, the KS-extension is given by

\[
(\Lambda(w_4, w_7), d_Y) \to (\Lambda(w_4, w_7, v_3), D) \to (\Lambda(v_3), 0)
\]

with $d_Y w_4 = 0, d_Y w_7 = w_4^2$ and $Dv_3 = w_4$. Then, from Theorem 2.1, $G_4(S^4_\mathbb{Q}; S^7_\mathbb{Q}; f_\mathbb{Q}) = G_4(\Lambda(w_4, w_7, d_Y), (\Lambda(w_4, w_7, v_3), D)) = \mathbb{Q}(w_4^4)$ since $\delta_f((w_4, 1 - 2(w_7, v_3)) = 0$. Thus we have

\[
G_4(S^4_\mathbb{Q}, S^7_\mathbb{Q}; f_\mathbb{Q}) = G_4(S^4_\mathbb{Q}, S^7_\mathbb{Q}; f_\mathbb{Q}) = \pi_4(S^4) \otimes \mathbb{Q} \cong \mathbb{Q}.
\]

Also for the product fibration $S^3 \to X = S^7 \times S^4 \xrightarrow{f \times id} S^4 \times S^4 = Y$ with the trivial fibration $\ast \to S^4 \to S^4$, we have $G_4(Y_\mathbb{Q}, X_\mathbb{Q}; (f \times id)_\mathbb{Q}) \cong \mathbb{Q}$ and $G_4(Y_\mathbb{Q}, X_\mathbb{Q}; (f \times id)_\mathbb{Q}) = \pi_4(Y) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q}$.

**Example 2.5.** Suppose that the model of a map $f : X \to Y$ is given by the projection

\[
p_{W,V} : M(Y) = (\Lambda(w_3, w_5, w_7, w_9), d_Y) \to (\Lambda(w_3, w_7, w_9), d_Y) = M(X)
\]

where $p_{W,V}(w_5) = \overline{w}_5 = 0$, $d_Y w_5 = d_Y w_3 = d_Y w_7 = 0$, $d_Y w_7 = w_3 w_9, d_Y w_9 = w_3 w_7$ and $d_Y w_3 = d_Y w_7 = 0$ and $d_Y w_9 = w_3 w_7$. Then we have $G_3(Y_\mathbb{Q}, X_\mathbb{Q}; f_\mathbb{Q}) = G_3(Y_\mathbb{Q}, X_\mathbb{Q}; f_\mathbb{Q}) = 0$ by the direct calculations of derivations. But we have $G_3(Y_\mathbb{Q}, X_\mathbb{Q}; f_\mathbb{Q}) = \pi_4(Y) \otimes \mathbb{Q}$ from Theorem 1.6.

**Example 2.6.** When $f : S^3 \times S^3 \to S^6$ is the map collapsing $S^3 \cup S^3$, then we show

\[
G_6(S^6, S^3 \times S^3; f) = 0.
\]

By degree arguments, any fibration $\eta : S^3 \times S^3 \xrightarrow{i} E \xrightarrow{p} S^6$ is rationally trivial, especially $D = 0$ in the KS-extension

\[
(Q[w_6]/(w_6^2), 0) \to (Q[w_6]/(w_6^2) \otimes \Lambda(u_3, v_3), D) \to (\Lambda(u_3, v_3), 0),
\]

where $H^*(S^6; \mathbb{Q}) = Q[w_6]/(w_6^2)$ and $M(S^3 \times S^3) = (\Lambda(u_3, v_3), 0)$. In particular $E_\mathbb{Q} \cong (S^6 \times S^3 \times S^3)_\mathbb{Q}$.

For $a \neq 0 \in \pi_6(S^6)$, suppose that there exists a map $F : E \to S^6$ in

\[
\begin{array}{ccc}
E^i & \xrightarrow{i} & S^3 \times S^3 \\
\downarrow F & \searrow f & \\
S^6 & \xrightarrow{a} & S^6,
\end{array}
\]
Then
\[ M(F_{f,g})(w_6) = cx + u_3v_3 \]
for some non-zero \( c \in \mathbb{Q} \) associated to \( a \). Then \( w_6^2 = 0 \) in \( H^{12}(S^6; \mathbb{Q}) \) but
\[ [M(F_{f,g})(w_6)]^2 = [2cxu_3v_3] \neq 0 \]
in \( H^{12}(E; \mathbb{Q}) = H^{12}(S^6 \times S^3 \times S^3; \mathbb{Q}) \). It is a contradiction.

3. THE OTHER SUBGROUPS

Further we will relax Definition A. For pointed maps \( f : X \to Y \) and \( g : B \to Y \), consider whether or not there exists the homotopy commutative diagram
\[
\begin{array}{c}
E \\
\downarrow \eta \\
B \\
\end{array} \xleftarrow{i} \begin{array}{c} X \\
\downarrow f \\
Y \\
\end{array}
\]
in which \( \eta : X \xrightarrow{i} E \xrightarrow{p} B \) is a fibration with the section \( s \).

Recall that a fibration \( Z \xrightarrow{i} X \xrightarrow{p} Y \) is said to be tncz (totally non-cohomologous to zero) if \( H^*(X) \cong H^*(Z) \otimes H^*(Y) \) as \( H^*(Y) \)-modules by \( p^* : H^*(Y) \to H^*(X) \).

**Definition B.** For a map \( f : X \to Y \), the \( n \)th tncz subgroup of \( f \) is given as
\[
T_n(Y, X; f) := \{ a \in \pi_n(Y) \mid \text{there are a tncz fibration } \eta : X \xrightarrow{i} E \xrightarrow{p} S^n \text{ with} \}
\[
a \text{ a section } s \text{ and a map } F : E \to Y \text{ such that } F \circ s \simeq a, \ f \simeq F \circ i \}.
\]

**Definition C.** For a map \( f : X \to Y \), the \( n \)th sectional subgroup of \( f \) is given as
\[
S_n(Y, X; f) := \{ a \in \pi_n(Y) \mid \text{there are a fibration } \eta : X \xrightarrow{i} E \xrightarrow{p} S^n \text{ with} \}
\[
a \text{ a section } s \text{ and a map } F : E \to Y \text{ such that } F \circ s \simeq a, \ f \simeq F \circ i \}.
\]

**Remark 2.** Note that \( S_n(Y, X; f) \) is a group. For \( a, b \in S_n(Y, X; f) \), there is a homotopy commutative diagram
\[
\begin{array}{c}
E_{a+b} \xrightarrow{h} E_a \cup_X E_b \\
\downarrow \text{pull-back} \\
S^n \\
\end{array} \xleftarrow{i} \begin{array}{c} X \\
\downarrow f \\
Y \\
\end{array}
\]
where \( q \) is the pinching comultiplication and the dotted arrow \( G \) is given by the universality of push-out from \( F_{a,f} : E_a \to Y \) and \( F_{b,f} : E_b \to Y \). A section \( s : S^n \to E_{a+b} \) is given by \( s(x) := (x, s'(q(x))) \) for a section \( s' : S^n \vee S^n \to E_a \cup_X E_b \) with \( G \circ s' \simeq (a|b) \). Also \( i_{a+b} : X \to E_{a+b} \) is given by the universality of pull-back.
from \( i : X \to E_a \cup X E_b \) and \( * : X \to S^n \). Thus we have a homotopy commutative diagram

\[
\begin{array}{ccc}
E_{a+b} & \xrightarrow{f} & X \\
\downarrow{S} & \downarrow{Goh} & \downarrow{f} \\
S^n & \xrightarrow{a+b} & Y.
\end{array}
\]

That is \( a + b := (a|b) \circ q \in S_n(Y, X; f) \). When \( \eta_a : X \to E_a \to S^n \) and \( \eta_b : X \to E_b \to S^n \) are trivial (tncz), the fibration \( X \to E_{a+b} \to S^n \) is trivial (tncz). Thus \( G_n(Y, X; f) \) is a group too.

We have the sequence of inclusions of groups:

\[
G_n(f) \subset G_n(f) \subset T_n(f) \subset S_n(f) \subset \pi_n(Y)
\]

for a map \( f : X \to Y \).

We consider them under some conditions on \( X \). For the KS-extension of a fibration \( \eta : X \to E \to S^n \), we see \( D' - d_X = 0 \) if \( \pi_{\geq n}(X) \otimes \mathbb{Q} = 0 \). Thus

**Proposition 3.1.** If \( \pi_{\geq n}(X) \otimes \mathbb{Q} = 0 \), \( S_n(Y, X; f) = G_n(Y, X; f) \).

For example, \( G_6(f_Q) = G_6(f_Q) = T_6(f_Q) = S_6(f_Q) = 0 \) for the map \( f \) in Example 2.6. We know that a fibration \( X \to E \to B \) is homotopically trivial if the classifying map \( B \to \text{Baut}X \) is homotopically trivial if the constant map. Therefore we have

**Proposition 3.2.** If \( \pi_n(\text{Baut}X) \otimes \mathbb{Q} = 0 \), \( G_n(Y, X; f_Q) = S_n(Y, X; f_Q) \).

In Example 3.4 and Example 3.5 below, \( \pi_n(\text{Baut}X) \otimes \mathbb{Q} \neq H_{n-1}(\text{Der}M(X)) \neq 0 \). But it is known that any fibration over \( CP^m \), the \( m \)-dimensional complex projective space, is rationally tncz. In general, we note

**Lemma 3.3.** For \( n > 1 \), any fibration with fiber \( X \) over \( S^n \) is rationally tncz if and only if the map \( \rho : H_{n-1}(\text{Der}M(X)) \to \text{Der}n_{n-1}H^*(X; \mathbb{Q}) \) with \( \rho([\sigma])([w]) = [\sigma(w)] \) is zero. Here \( \text{Der}n_{n-1}H^*(X; \mathbb{Q}) \) means the derivations of the graded algebra \( H^*(X; \mathbb{Q}) \) decreasing the degree by \( n-1 \) (POT 9.7.2).

**Proof.** The KS-extension of a fibration \( X \to E \to S^n \) is given by the differential \( Dv = dv + \sigma(v)x \) for some \( [\sigma] \in H_{n-1}(\text{Der}M(X)) \) with the differential \( d(M) \) and \( v \in M(X) \). Then an element \([w] \) of \( H^*(X; \mathbb{Q}) \) is extend to an element \([w + w'x] \) of \( H^*(E; \mathbb{Q}) \) if and only if \( dv' = \sigma(w) \).

**Example 3.4.** For the associated fibration \( S^2 \to CP^3 \xrightarrow{f} S^4 \) of the Hopf fibration \( S^3 \to S^7 \xrightarrow{f} S^4 \), the KS-model is given by

\[
(\Lambda(w_4, w_7), dv_4) \to (\Lambda(w_4, w_7, v_2, v_3), D) \to (\Lambda(v_2, v_3), d)
\]

where \( dv_4 = 0, dv_7 = w_4^2, Dv_2 = dv_2 = 0, dv_3 = v_2^2 \) and \( Dv_3 = v_2^2 - w_4 \). Also \( M(CP^3) \cong (\Lambda(v_2, w_7), dx) \) with \( dx v_2 = 0, dx w_7 = v_2^3 \) and then \( M(f)(w_4) = v_2^3, M(f)(w_7) = w_7 \). Then we have

\[
T_4(S^4, CP^3; f) = \pi_4(S^4) \otimes \mathbb{Q} = \mathbb{Q}.
\]
Indeed, for \( a \in \pi_4(S^4) \otimes \mathbb{Q} \) with \( M(a)(w_4) = cx \) \((c \in \mathbb{Q})\) and \( M(a)(w_7) = 0 \), put

\[
D'v_2 = 0, \quad D'w_7 = v_2^4 + 2cv_2^2x
\]

and

\[
F(w_4) = v_2^2 + cx, \quad F(w_7) = w_7
\]
in

\[
\begin{array}{ccc}
(\Lambda x/x^2 \otimes \Lambda(v_2, w_7), D') & \stackrel{F}{\longrightarrow} & (\Lambda(v_2, w_7), dx) \\
\downarrow & & \downarrow M(f) \\
(\Lambda x/x^2, 0) & \stackrel{M(a)}{\longrightarrow} & (\Lambda(w_4, w_7), dy).
\end{array}
\]

Thus \( a \in \mathcal{T}_4(S^4_2, \mathbb{CP}^2_2; f_Q) \). On the other hand, \( \mathcal{G}_4(S^4_2, \mathbb{CP}^2_2; f_Q) = 0 \) since \((\Lambda x/x^2 \otimes \Lambda(v_2, w_7), D')\) can not be isomorphic to \((\Lambda x/x^2, 0)(\Lambda(v_2, w_7), dx)\) over \((\Lambda x/x^2, 0)\) for any \( D' \).

**Example 3.5.** For the map \( f: \mathbb{CP}^2 \to S^4 \) collapsing the 2-cell, \( M(f): M(S^4) = (\Lambda(w_4, w_7), dy) \to (\Lambda(v_2, v_5), dx) = M(\mathbb{CP}^2) \) is given by \( M(f)(w_4) = v_2^2 \) and \( M(f)(w_7) = v_2v_5 \). Then we have

\[
\mathcal{T}_4(S^4_2, \mathbb{CP}^2_2; f_Q) = \pi_4(S^4) \otimes \mathbb{Q} = \mathbb{Q}.
\]

Indeed, for \( a \in \pi_4(S^4) \otimes \mathbb{Q} \) with \( M(a)(w_4) = cx \) \((c \in \mathbb{Q})\) and \( M(a)(w_7) = 0 \), put

\[
D'(v_2) = 0, \quad D'(v_5) = v_2^4 + 2cv_2x
\]

and

\[
F(w_4) = v_2^2 + cx, \quad F(w_7) = v_2v_5
\]
in

\[
\begin{array}{ccc}
(\Lambda x/x^2 \otimes \Lambda(v_2, v_5), D') & \stackrel{F}{\longrightarrow} & (\Lambda(v_2, v_5), dx) \\
\downarrow & & \downarrow M(f) \\
(\Lambda x/x^2, 0) & \stackrel{M(a)}{\longrightarrow} & (\Lambda(w_4, w_7), dy).
\end{array}
\]

Thus \( a \in \mathcal{T}_4(S^4_2, \mathbb{CP}^2_2; f_Q) \). On the other hand, we have \( \mathcal{G}_4(S^4_2, \mathbb{CP}^2_2; f_Q) = 0 \) since \((\Lambda x/x^2 \otimes \Lambda(v_2, v_5), D')\) can not be isomorphic to \((\Lambda x/x^2, 0)(\Lambda(v_2, v_5), dx)\) over \((\Lambda x/x^2, 0)\) for any \( D' \).

**Example 3.6.** Put \( \xi: \Omega X \xrightarrow{i} LX \xrightarrow{\xi} X \) the fibration of free loops, in which \( \Omega X \) is the loop space and \( LX = map(S^1, X) \) is the free loop space of a simply connected space \( X \). It has the section \( s: X \to LX \) with \( s(z) \) the constant loop at a point \( z \) in \( X \). Consider the case that \( X = S^2 \). Then \( S_2(LS^2, \Omega S^2; i) \ni s \) since we can choose \( F = id_{LS^2} \) as

\[
\begin{array}{ccc}
LS^2 & \xrightarrow{i} & \Omega S^2 \\
\downarrow s & & \downarrow i \\
S^2 & = & LS^2.
\end{array}
\]

Thus we have \( S_2(LS^2, \Omega S^2; i) \neq 0 \). Especially, we see \( S_2(LS^2_2, \Omega S^2_2; i_Q) \neq 0 \) since \( s \) is the torsion free generator of \( \pi_2(LS^2) \).

But \( \mathcal{T}_2(LS^2_2, \Omega S^2_2; i_Q) = 0 \). Indeed, the KS-model of \( \xi \) is given by

\[
(\Lambda(x, y), dy) \to (\Lambda(x, y, \overline{x}, \overline{y}), D) \to (\Lambda(\overline{x}, \overline{y}), 0)
\]
where \( M(S^2) = (\Lambda(x,y),d_Y) \) with \( @|x| = 2, |y| = 3, d_Y x = 0, d_Y y = x^2, |\bar{\tau}| = 1, |\bar{\eta}| = 2, D(\bar{\tau}) = 0, \) and \( D(\bar{\eta}) = 2x \bar{\tau} \) [VS]. For the KS-model of a fibration \( \eta : \Omega S^2 \rightarrow E \rightarrow S^2 \) with a section is given as
\[
(\Lambda(x,y),d_Y) \rightarrow (\Lambda(x,y,\bar{\tau},\bar{\eta}),D') \rightarrow (\Lambda(x,y,\bar{\eta}),0)
\]
where \( D'(\bar{\tau}) = 0 \) and \( D'(\bar{\eta}) = cx \bar{\tau} \) for some \( c \in \mathbb{Q} \) by the degree arguments. If \( c = 0 \), there does not exist a map \( F : (\Lambda(x,y,\bar{\tau},\bar{\eta}),D) \rightarrow (\Lambda(x,y,\bar{\tau},\bar{\eta}),D') = (\Lambda(x,y,\bar{\eta}),d_Y) \) that we want. If \( c \neq 0 \), it is not rationally tncz since
\[
H^*(\Omega S^2;\mathbb{Q}) = \Lambda(\bar{\tau},\bar{\eta})
\]
and
\[
H^*(E;\mathbb{Q}) \cong \mathbb{Q}[x,\{u_i\}_{i>0}] \otimes \Lambda(\bar{x},x\bar{x},\{xu_i\},\{xu_iu_j\}) \cup \mathbb{Q}[x].
\]

**Example 3.7.** Put \( \xi : S^1 \rightarrow K \rightarrow S^1 \) the fiber bundle with total space a Klein bottle \( K \). Then \( G_1(K,S^1,j) = G_1(K,S^1,j) = \mathcal{T}_1(K,S^1,j) = \mathbb{Z} \) and \( S_1(K,S^1,j) = \pi_1(K) \).

**References**

[FH] Y. Félix and S. Halperin, *Rational LS category and its applications*, Trans. A.M.S. 273 (1982) 1-38

[FHT] Y. Félix, S. Halperin and J.-C. Thomas, *Rational homotopy theory*, Springer G.T.M. 205 [2001].

[FOT] Y. Félix, J. Oprea and D. Tanré, *Algebraic models in geometry*, Oxford 17 [2008]

[G] D.H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. 91 (1969) 729-756

[HMR] P. Hilton, G. Mislin and J. Reitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies 15 (1975)

[LS] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G-sequences*, J. Pure Appl. Algebra 209 (1) (2007) 159-171

[LS2] G. Lupton and S. B. Smith, *The evaluation subgroup of a fibre inclusion*, Topology and its Applications 154 (2007) 1107-1118

[O] N. Oda, *Pairings and copairings in the category of topological spaces*, Publ. Res. Inst. Math. Sci. Kyoto Univ. 28 (1992) 83-97

[S] D. Sullivan, *Infiniteesimal computations in topology*, Publ. IHES 47 (1978) 269-331

[T] J.-C. Thomas, *Rational homotopy of Serre fibrations*, Ann. Inst. Fourier, Grenoble 31 (1981) 71-90

[VS] M. Vigué-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*, ibid. 11 (1976) 633-644

[WK] M. H. Woo and J. R. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. 21 (1984) 109-120

Faculty of Education, Kochi University, 2-5-1, Kochi, 780-8520, Japan

E-mail address: tyamag@kochi-u.ac.jp