Approximations of certain classes of functions of several variables by greedy approximants in the integral metrics

We find the exact order estimates of the approximations of the classes $F_{q,r}^\psi$ of functions of several variables by greedy approximants in the integral metric. We also obtain the exact order estimates of the best n-term orthogonal trigonometric approximations of the classes $F_{q,r}^\psi$ in the integral metric.

1. Introduction. Let $d$ be a fixed natural number, let $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{Z}_+^d$ be the sets of all ordered collections $k := (k_1, \ldots, k_d)$ of $d$ real, integer and integer nonnegative numbers correspondingly. Let also $\mathbb{T}^d := [0, 2\pi]^d$ denote $d$-dimensional torus.

Further, let $L_p(\mathbb{T}^d)$, $1 \leq p < \infty$, be the space of all Lebesgue-measurable on $\mathbb{R}^d$ $2\pi$-periodic in each variable functions $f$ with finite norm

$$||f||_{L_p(\mathbb{T}^d)} := \begin{cases} \left( (2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{x \in \mathbb{T}^d} |f(x)|, & p = \infty. \end{cases}$$

Set $(k, x) := k_1 x_1 + k_2 x_2 + \ldots + k_d x_d$ and for any $f \in L_1^d$, we denote the Fourier coefficients of $f$ by

$$\hat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-i(k,x)} \, dx, \quad k \in \mathbb{Z}^d.$$ 

We denote by $l_p^N$, $N = 1, 2, \ldots$, $0 < p \leq \infty$, the space $\mathbb{R}^N$ equipped with $l_p$-(quasi-)norm that is defined for $x = \{x_k\}_{k=1}^N \in \mathbb{R}^N$ by

$$|x|_p := ||x||_{l_p} = \begin{cases} \left( \sum_{k=1}^N |x_k|^p \right)^{1/p}, & 0 < p < \infty, \\ \sup_{1 \leq i \leq N} |x_i|, & p = \infty. \end{cases}$$

Let also $\psi = \psi(t)$, $t \geq 1$, be a positive decreasing function, $\psi(0) := \psi(1)$ and $0 < q, r \leq \infty$.

In the paper, we investigate asymptotical behavior of some important approximation characteristics (in the sense of order estimates) of the classes of functions of several variables $F_{q,r}^\psi$, defined by the following equality:

$$F_{q,r}^\psi := \left\{ f \in L_1(\mathbb{T}^d) : \||\hat{f}(k)|/\psi(||k||_r)\|_{l_p(\mathbb{Z}^d)} \leq 1 \right\}.$$

If $\psi(t) = t^{-s}$, $s \in \mathbb{N}$ and $q = 1$, then $F_{q,\infty}^{\psi} := F_{q,\infty}^s$ is a set of functions whose $\alpha$th partial derivatives have absolutely convergent Fourier series. When $q = 2$, $F_{q,\infty}^s$ is equivalent (modulo constants) to the unit ball of the Sobolev class $W_2^s$.

Approximation characteristics of the classes $F_{q,r}^\psi$ for different $r \in (0, \infty]$ and for the various functions $\psi$ were investigated in the papers [1]–[8]. In particular, in [1], the authors found the exact
order estimates of the quantities of the best $n$-term trigonometric approximations of the classes $\mathcal{F}_{q,\infty}^s$, $s > 0$, in the spaces $L_p(\mathbb{T}^d)$. In [2], V.N. Temlyakov obtained the exact order estimates of approximations of these classes by $n$-term greedy approximants in the spaces $L_p(\mathbb{T}^d)$. If $\psi(t) = R^{-t}$, $R > 1$, then the exact order estimates of the quantities of the best $n$-term orthogonal trigonometric approximations in the spaces $L_p(\mathbb{T}^d)$, $2 \leq p < \infty$, of the classes $\mathcal{F}^\psi_{q,1}$ were found by V.S. Romanyuk [3].

In the case, where $\psi(t)$ is a positive function that decreases to zero no faster than some power function, the quantities of the best $n$-term trigonometric approximations and the quantities of approximations by $n$-term Greedy approximants of the classes $\mathcal{F}^\psi_{q,\infty}$ were studied in [4] and [5].

It should be noted that in [7; 8, Ch. XI] A.I. Stepanets got the exact values the best $n$-term trigonometric approximations of these classes by $\mathcal{F}^\psi_{q,r}$ in the known spaces $S^p$. These results are significantly used by us in the proof and presented in Section 4.

2. Approximation characteristics. In this section, we give the definition of the approximation quantities for the functions of the classes $\mathcal{F}^\psi_{q,r}$, which are considered in this paper. First, for further convenience, we formulate the definition of the spaces $S^p(\mathbb{T}^d)$.

The space $S^p(\mathbb{T}^d)$, $0 < p < \infty$, (see, for example, [8, Ch. XI]) is the space of all functions $f \in L_1(\mathbb{T}^d)$ such that

$$\|f\|_{S^p(\mathbb{T}^d)} := \left\| \left\{ \hat{f}(k) \right\}_{k \in \mathbb{Z}^d} \right\|_{L_p(\mathbb{T}^d)} = \left( \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^p \right)^{1/p} < \infty. \tag{2.1}$$

The functions $f \in L_1(\mathbb{T}^d)$ and $g \in L_1(\mathbb{T}^d)$ are equivalent in the space $S^p(\mathbb{T}^d)$, when $\|f - g\|_{S^p(\mathbb{T}^d)} = 0$.

Further, for $f \in L_1(\mathbb{T}^d)$, let $\{k(l)\}_{k=1}^\infty = \{k(l, f)\}_{k=1}^\infty$ denote the rearrangement of numbers $k \in \mathbb{Z}^d$ such that

$$|\hat{f}(k(1))| \geq |\hat{f}(k(2))| \geq \ldots. \tag{2.2}$$

In general case, this rearrangement is not unique. In such a case, we take any rearrangement satisfying (2.2).

In the paper, the main approximation quantities for the functions $f \in \mathcal{F}^\psi_{q,r}$ are the following quantities:

$$\|f - G_n(f)\|_X := \|f(\cdot) - \sum_{l=1}^n \hat{f}(k(l)) e^{i\langle k(l), \cdot \rangle}\|_X, \tag{2.3}$$

$$e_n^{(1)}(f)_X := \inf_{\gamma_n} \|f(\cdot) - \sum_{k \in \gamma_n} \hat{f}(k) e^{i\langle k, \cdot \rangle}\|_X, \tag{2.4}$$

and

$$e_n(f)_X := \inf_{\gamma_n, c_k} \|f(\cdot) - \sum_{k \in \gamma_n} c_k e^{i\langle k, \cdot \rangle}\|_X, \tag{2.5}$$

where $X$ is one of the spaces $L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, or $S^p(\mathbb{T}^d)$, $0 < p < \infty$, $\gamma_n$ is a collection of $n$ different vectors from the set $\mathbb{Z}^d$, $c_k$ are any complex numbers. Here, it is assumed that the embedding $\mathcal{F}^\psi_{q,r} \subset X$ is true.
The quantities \(\overline{2.5}\) and \(\overline{2.4}\) are respectively called by the best \(n\)-term trigonometric and the best \(n\)-term orthogonal trigonometric approximations of the function \(f\) in the space \(X\). The quantity \(\overline{2.3}\) is called by the approximation of the function \(f\) by greedy approximants in the space \(X\).

For any \(\mathfrak{N} \subset X\), we set
\[
e_n(\mathfrak{N})_X := \sup_{f \in \mathfrak{N}} e_n^+(f)_X,
\]
and
\[
e_n(\mathfrak{N})_X := \sup_{f \in \mathfrak{N}} e_n(f)_X.
\]
In general case, the quantities \(\overline{2.3}\) depend of the choice of the rearrangement satisfying \(\overline{2.2}\). So, for the unique definition, we set
\[
G_n(\mathfrak{N})_X := \sup_{f \in \mathfrak{N}} \inf_{(k(l,f))_{l=1}^\infty} \|f(\cdot) - \sum_{l=1}^n \hat{f}(k(l,f)) e^{\imath(k(l,f)\cdot)}\|_X.
\]
In \(\overline{2.6}\), for any function \(f \in \mathfrak{N}\), we consider the infimum on all rearrangements, satisfying \(\overline{2.2}\), but it should be noted that results, formulated in this paper, are also true for any other rearrangements, satisfying \(\overline{2.2}\).

Research of the quantities of the form \(\overline{2.3} \rightleftharpoons \overline{2.5}\) goes back to the paper of S.B. Stechkin \[9\]. Order estimates of these quantities on different classes of functions of one and several variables were obtained by many authors. In particular, in \[10\] and \[11\], one can be found the bibliography of papers in which the similar results were obtained.

Note that for any \(f \in L_p(\mathbb{T}^d)\),
\[
e_n(f)_{L_p(\mathbb{T}^d)} \leq e_n^+(f)_{L_p(\mathbb{T}^d)} \leq \|f - G_n(f)\|_{L_p(\mathbb{T}^d)},
\]
and by virtue of \(\overline{2.1}\), for any \(f \in S^p(\mathbb{T}^d)\),
\[
e_n(f)_{S^p(\mathbb{T}^d)} = e_n^+(f)_{S^p(\mathbb{T}^d)} = \|f - G_n(f)\|_{S^p(\mathbb{T}^d)}.
\]

3. **Main result.** The main purpose of this work is to find the dependence of the choice of the parameters \(r, \psi\) and \(q\) on the rate of convergence to zero, as \(n \to \infty\), of the approximative characteristics of the classes \(\mathcal{F}^\psi_{q,r}\).

3.1. As mentioned above, in the case, where \(\psi(t)\) is a power function, i.e., \(\psi(t) = t^{-s}, s > 0\), for all \(1 \leq p \leq \infty\), the exact order estimates of the quantities \(e_n(\mathcal{F}^\psi_{q,\infty})_{L_p(\mathbb{T}^d)}\) and \(G_n(\mathcal{F}^\psi_{q,\infty})_{L_p(\mathbb{T}^d)}\) were obtained in \[1\] and \[2\], correspondingly. In particular, from Theorem 6.1 of \[1\], it follows that for all \(s > 0\), when \(0 < q \leq 1\), and for all \(s > d(1 - \frac{1}{q})\), when \(1 < q < \infty\), the following relation is true\[2\]:
\[
e_n(\mathcal{F}^\psi_{q,\infty})_{L_p(\mathbb{T}^d)} \asymp n^{-\frac{d}{q} - \frac{1}{q} + \frac{1}{p}}, \quad 1 \leq p \leq \infty;
\]
\[2\]Here and in what follows, for positive sequences \(a(n)\) and \(b(n)\), the expression \(\text{’}a(n) \asymp b(n)\text{’}\) means that there are constants \(0 < K_1 < K_2\) such that for any \(n \in \mathbb{N}\), \(a(n) \leq K_2 b(n)\) (in this case, we write \(\text{’}a(n) \ll b(n)\text{’}\)) and \(a(n) \geq K_1 b(n)\) (in this case, we write \(\text{’}a(n) \gg b(n)\text{’}\)).
and from Theorem 3.1 of [2], it follows that for all \( s > 0 \), when \( 0 < q \leq 1 \), and for all \( s > d(1 - \frac{1}{q}) \), when \( 1 < q < \infty \),

\[
G_n(F_{q,\infty}^s)_{L_p(T^d)} \asymp \begin{cases} 
  n^{-\frac{s}{q} - \frac{1}{2} + \frac{1}{q}}, & 1 \leq p < 2, \\
  n^{-\frac{s}{q} - \frac{1}{2} + \frac{1}{q}}, & 2 \leq p < \infty.
\end{cases} \tag{3.2}
\]

From the following Theorem 3.1, in particular, it follows that the estimates of form (3.2) of the quantities \( G_n(F_{q,\infty}^s)_{L_p(T^d)} \) are satisfied for a wider set of the functions \( \psi \). Actually, this result was obtained in [5], where the case \( r = \infty \) was considered. For completeness and in view of the relative inaccessibility of [5] for English-speaking readers, in this paper, we give it with a proof.

To formulate this statement, we use the following notation: let \( B \) denote the set of all positive descending functions such that

\[
\lim_{t \to \infty} \psi(t) = 0, \tag{3.3}
\]

and for a certain number \( c > 1 \) and for all \( t \geq 1 \), the following relation is true:

\[
1 < \frac{\psi(t)}{\psi(ct)} \leq K. \tag{3.4}
\]

Here and in what follows \( K, K_1, K_2, \ldots \) are positive constants which are independent of the variable \( t \).

**Theorem 3.1.** Assume that \( 1 \leq r \leq \infty \), \( 1 \leq p < \infty \), \( 0 < q < \infty \), the function \( \psi \) belongs to the set \( B \) and moreover for \( 0 < p/(p-1) < q \) and for all \( t \), larger than a certain number \( t_0 \), \( \psi \) is convex downwards and satisfies the condition

\[
\frac{1}{\alpha(\psi, t)} \geq K_\psi > \begin{cases} 
  d(\frac{1}{2} - \frac{1}{q}), & 1 < p \leq 2, \\
  d(1 - \frac{1}{p} - \frac{1}{q}), & 2 \leq p < \infty,
\end{cases}
\]

where

\[
\alpha(\psi, t) := \frac{\psi(t)}{t|\psi'(t)|}, \quad \psi'(t) := \psi'(t+). \tag{3.5}
\]

Then

\[
G_n(F_{q,r}^\psi)_{L_p(T^d)} \asymp e_n^\psi(F_{q,r}^\psi)_{L_p(T^d)} \asymp \begin{cases} 
  \frac{\psi^{1/d}}{n^{\frac{1}{\frac{1}{p} - \frac{1}{q}}}}, & 1 \leq p \leq 2, \\
  \frac{\psi^{1/d}}{n^{\frac{1}{\frac{1}{p} - \frac{1}{q}}}}, & 2 \leq p < \infty.
\end{cases} \tag{3.6}
\]

Note that the conditions on the function \( \psi \) in Theorem 3.1 guarantee the embedding \( F_{q,r}^\psi \subset L_p(T^d) \).

Putting \( r = \infty \) and \( \psi(t) = t^{-s}, \ s > 0 \), from Theorem 3.1 we obtain the following corollary:

**Corollary 3.1.** Assume that \( 1 \leq p < \infty \), \( 0 < q < \infty \) and \( s \) is a positive number, which for \( 0 < p/(p-1) < q \), satisfies the condition

\[
s > \begin{cases} 
  d(\frac{1}{2} - \frac{1}{q}), & 1 < p \leq 2, \\
  d(1 - \frac{1}{p} - \frac{1}{q}), & 2 \leq p < \infty.
\end{cases}
\]

Then

\[
G_n(F_{q,\infty}^\psi)_{L_p(T^d)} \asymp e_n^\psi(F_{q,\infty}^s)_{L_p(T^d)} \asymp \begin{cases} 
  n^{-\frac{s}{q} - \frac{1}{2} + \frac{1}{q}}, & 1 \leq p < 2, \\
  n^{-\frac{s}{q} - \frac{1}{2} + \frac{1}{q}}, & 2 \leq p < \infty.
\end{cases} \tag{3.7}
\]
For $1 \leq p < \infty$, this statement complements the result mentioned above of [2] in the following sense:

- from Corollary 3.1, in particular, it follows that in the case, where $1 < q \leq \frac{p}{p-1}$, relation $[3.2]$ also holds for all $s > 0$,

- if $0 < p/(p-1) < q$ and $1 < p \leq 2$, then relation $[3.2]$ also holds for all $s$ such that $d(\frac{1}{2} - \frac{1}{q}) < s \leq d(1 - \frac{1}{q})$,

- if $0 < p/(p-1) < q$ and $2 < p < \infty$, then relation $[3.2]$ also holds for all $s$ such that $d(1 - \frac{1}{p} - \frac{1}{q}) < s \leq d(1 - \frac{1}{q})$.

Note also that if $0 < q \leq p/(p-1)$, then the conditions of Theorem 3.1 are satisfied, for example, for the function $\psi(t) = t^{-s}\ln^\varepsilon(t+e)$, where $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, as well as for the function $\psi(t) = \ln^\varepsilon(t+e)$, $\varepsilon < 0$. If $1 < p/(p-1) < q$ and $1 < p \leq 2$, then the conditions of Theorem 3.1 are satisfied for the function $\psi(t) = t^{-s}\ln^\varepsilon(t+e)$, where $\varepsilon \in \mathbb{R}$ and $s > d(\frac{1}{2} - \frac{1}{q})$. If $1 < p/(p-1) < q$ and $2 < p < \infty$, then the conditions of Theorem 3.1 are satisfied for the function $\psi(t) = t^{-s}\ln^\varepsilon(t+e)$, where $\varepsilon \in \mathbb{R}$ and $s > d(1 - \frac{1}{p} - \frac{1}{q})$.

The proof of Theorem 3.1 will be given in Section 5.

4. Order estimates for some functionals and their applications.

4.1. Let $\Psi = \Psi(k)$, $k = 1, 2, \ldots$, be a nonincreasing positive sequence such that

$$
\lim_{k \to +\infty} \Psi(k) = 0. \quad (4.1)
$$

The following Lemma 4.1 is essentially used for proving upper estimates in Theorem 3.1. This lemma gives exact order estimates for the following functionals $H_n(\Psi, s)$, which in the case, where $s \in (0, 1]$, are defined by the equality

$$
H_n(\Psi, s) := \sup_{l > n} (l - n)^{- \frac{1}{s'}} \left( \sum_{k=1}^{l} \Psi^{-s}(k) \right)^{- \frac{1}{s'}}, \quad (4.2)
$$

and for $s \in (1, \infty)$, they are defined by the equality

$$
H_n(\Psi, s) := \left( (l^* - n)^{\frac{s'}{s}} \left( \sum_{j=1}^{l^*} \Psi^{-s}(j) \right)^{- \frac{s'}{s}} + \sum_{j=l^*+1}^{\infty} \Psi^{s'}(j) \right)^{\frac{1}{s'}}, \quad (4.3)
$$

where $1/s + 1/s' = 1$,

$$
\sum_{j=1}^{\infty} \Psi^{s'}(j) < \infty, \quad (4.4)
$$

and the number $l^*$ is given by relation

$$
\Psi^{-s}(l^*) \leq \frac{1}{l^* - n} \sum_{j=1}^{l^*} \Psi^{-s}(j) < \Psi^{-s}(l^* + 1). \quad (4.5)
$$
Note that, in the terms of similar functionals, are formulated solutions of many problems of approximation theory (see, eg, [7; 8, Ch. XI; 12–18]). Therefore, the problem of finding such estimates is interesting in itself.

Let \( \nu = \{\nu_i\}_{i=0}^{\infty} \) be an increasing sequence of natural numbers, \( \nu_0 := 1, V_m = \sum_{k=0}^{m} \nu_k \), let \( I_m := I_m(\nu) = \left[ V_{m-1}, V_m \right] \), \( m = 1, 2, \ldots, \) be the set of intervals. Let also \( d \in \mathbb{N}, M_0, c_1 \) and \( c_2 \) be fixed positive numbers.

Let \( S_d(M_0) = S_d(M_0, c_1, c_2) \) denote the set of all positive nonincreasing sequences \( \Psi = \Psi(k), k = 1, 2, \ldots, \) satisfying condition (4.1), which are represented as

\[
\Psi(t) = \psi(m), \quad t \in I_m(\nu), \quad m = 1, 2, \ldots, \tag{4.6}
\]

where \( \psi \) is the decreasing sequence of different values of the sequence \( \Psi \) and the set of intervals \( I_m(\nu) \) is such that for all \( m \), greater than a certain number \( k_0 \),

\[
M_0(m - c_1)^d < V_m \leq M_0(m + c_2)^d. \tag{4.7}
\]

Without loss of generality, we assume that the sequences \( \psi \) are restrictions of certain positive continuous functions \( \psi(t) \) of continuous argument \( t \geq 1 \) on the set of natural numbers \( \mathbb{N} \).

**Lemma 4.1.** Let \( s \in (0, \infty), d \in \mathbb{N}, \) the sequence \( \Psi \) belongs to the set \( S_d(M_0) \) and the sequence of its different values is a restriction of a certain function \( \psi(t) \in B \) on the set \( \mathbb{N} \). Furthermore, in the case \( s > 1 \), we also assume that for all \( t \), greater than a certain number \( t_0 \), the function \( \psi \) is convex downwards and satisfies the condition

\[
\alpha(\psi, t) \leq K_\psi < s'/d, \quad \frac{1}{s} + \frac{1}{s'} = 1, \tag{4.8}
\]

where \( \alpha(\psi, t) \) is defined in (3.5). Then

\[
H_n(\Psi; s) \asymp \frac{\psi\left(n^{\frac{1}{s'}}\right)}{n^{\frac{1}{s}-1}}. \tag{4.9}
\]

Let us note that for any \( \Psi \in S_d(M_0) \), condition (4.8) guarantees convergence of the series in (4.4), when \( s > 1 \). Indeed, in this case, for all \( \tau \geq t_0 \),

\[
\left| \frac{\psi'(\tau)}{\psi(\tau)} \right| \geq \frac{1}{K_\psi} \geq \frac{d}{s'\tau}.
\]

After integrating each part of this relation in the range from \( t_0 \) to \( t \), we obtain \( \psi(t) \ll t^{-1/K_\psi} \ll t^{-d/s'}, t > t_0 \). Therefore, in view of (4.7), we conclude that

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \nu_k \psi'(k) \ll \sum_{k=1}^{\infty} k^{d-1} \psi'(k) \ll \int_{1}^{t} t^{d-1} \cdot t^{-s'/K_\psi} dt < \infty.
\]
Also note that in the case, where the sequences \( \Psi = \Psi(k) \) are restrictions of certain positive convex functions \( \psi(t) \) of continuous argument \( t \geq 1 \) on the set \( N \), order estimates for the functionals \( H_n(\Psi, r) \) were obtained in [17; 18] and order estimates for their integral analogues were obtained in [16].

4.2. Proof of Lemma 4.1. First, consider the case \( s \in (0, 1] \). In view of (4.6), the functionals \( H_n(\Psi, s) \) can be represented as

\[
H_n(\Psi, s) = \sup_{l > n} (l - n) \left( \sum_{j=1}^{l} \frac{1}{\psi^s(j)} \right)^{-\frac{1}{s}} = \\
= \sup_{l > n} (l - n) \left( \sum_{k=1}^{k-1} \frac{\nu_k}{\psi^s(k)} + \frac{l - V_{k-1}}{\psi^s(k)} \right)^{-\frac{1}{s}} =: \tilde{H}_n(\psi, s),
\]  

(4.10)

where \( k \) denote a number such that \( l \in I_{k} \), i.e.,

\[
V_{k-1} < l \leq V_{k}.
\]  

(4.11)

By virtue of (4.7), for all \( l > n \geq k_0 \), we see that

\[
(l/M_0)^{\frac{1}{2}} - c_2 \leq k_1 < (l/M_0)^{\frac{1}{2}} + c_1 + 1.
\]  

(4.12)

From relation (4.7), it follows that

\[
\nu_k \asymp k^{d-1},
\]  

(4.13)

therefore, for any \( r > 0 \),

\[
\sum_{k=1}^{l} \frac{\nu_k}{\psi^s(k)} \asymp \sum_{k=1}^{l} \frac{k^{d-1}}{\psi^s(k)}
\]  

(4.14)

If \( \psi \in B \), then for any \( l = 2, 3, \ldots \),

\[
\frac{l^d}{\psi^s(l)} \ll \frac{\left(\frac{l}{2}\right)^d}{\psi^s(\frac{l}{2})} \ll \sum_{l/2 \leq k \leq l} \frac{k^{d-1}}{\psi^s(k)} \leq \sum_{k=1}^{l} \frac{k^{d-1}}{\psi^s(k)} \leq \frac{l^d}{\psi^s(l)}.
\]

Hence, according to (4.14), we get

\[
\sum_{k=1}^{l} \frac{\nu_k}{\psi^s(k)} \asymp \frac{l^d}{\psi^s(l)}.
\]  

(4.15)

Further, by virtue of (4.12) and the definition of the set \( B \), we see that

\[
\psi(k_1) \asymp \psi((l/M_0)^{\frac{1}{2}}) \asymp \psi(l^{\frac{1}{2}}).
\]

and in view of (4.15), we conclude that

\[
\tilde{H}_n(\psi, s) \asymp \sup_{l > n} (l - n) \left( \frac{k_1^d}{\psi^s(k_1)} \right)^{-\frac{1}{s}} \asymp \sup_{l > n} \left( \psi(l^{\frac{1}{2}}) \frac{l - n}{l^{1/s}} \right).
\]  

(4.16)
Since the function $\psi$ is non-increasing, then
\[
\tilde{H}_n(\psi, s) \ll \psi\left(n^{\frac{1}{d}}\right) \sup_{l>n} \frac{l - n}{l^{1/s}}.
\] (4.17)

For $x > 0$, $n \in \mathbb{N}$ and $s \in (0, 1)$, the function $h(x) = h(x, s) = \frac{x - n}{x^{1/s}}$ attains its maximal value at the point $x_0 = n/(1 - s)$, and
\[
h(x_0, s) = s\left(\frac{1 - s}{n}\right)^{\frac{1}{s}}.
\] (4.18)

If $s = 1$, then the function $h(x) = h(x; 1)$ is non-decreasing and tends to 1 as $x$ increases. Therefore,
\[
\sup_{x > 0} h(x; 1) = \sup_{x > 0} \frac{x - n}{x} = \lim_{x \to +\infty} \frac{x - n}{x} = 1.
\] (4.19)

Combining (4.17)–(4.19), we obtain necessary upper estimates for the functionals $H_n(\Psi, s)$:
\[
H_n(\Psi, s) = \tilde{H}_n(\psi, s) \ll \psi\left(n^{\frac{1}{d}}\right) \frac{1}{n^{\frac{1}{s} - 1}}.
\]

Taking into account (4.16) and the fact that $\psi \in B$, we also obtain the lower estimates
\[
H_n(\Psi, s) = \tilde{H}_n(\psi, s) \simeq \sup_{l>n} \left(\psi\left(l^{\frac{1}{d}}\right) \frac{l - n}{l^{1/s}}\right) \geq \psi((2n)^\frac{1}{d}) \frac{2n - n}{(2n)^{1/s}} = \psi(n^{\frac{1}{d}}) \frac{1}{n^{\frac{1}{s} - 1}}.
\]

Now, we consider the case, when $s > 1$. To simplify the notation, we set
\[
Q_n(\Psi, l) := (l - n)\left(\sum_{j=1}^{l} \Psi^{-s}(j)\right)^{-1}, \quad l \geq n, \quad l \in \mathbb{N}.
\] (4.20)

Since for any $l > n$,
\[
Q_n(\Psi, l + 1) = Q_n(\Psi, l) + \left(\Psi^s(l + 1) - Q_n(\Psi, l)\right)\Psi^{-s}(l + 1)\left(\sum_{i=1}^{l+1} \Psi^{-s}(i)\right)^{-1}
\]
and
\[
\Psi^s(l + 1) = Q_n(\Psi, l + 1) + \left(\Psi^s(l + 1) - Q_n(\Psi, l)\right)\sum_{j=1}^{l} \Psi^{-s}(j)\left(\sum_{i=1}^{l+1} \Psi^{-s}(i)\right)^{-1},
\]
then in view of monotonicity of the function $\Psi$ and the definition of the number $l^*$ (see relation (4.5)), we conclude that for all $l \geq l^*$,
\[
Q_n(\Psi, l) > Q_n(\Psi, l + 1) > \Psi^s(l + 1),
\]
and for all $l \in [n, l^*)$,
\[
Q_n(\Psi, l) \leq Q_n(\Psi, l + 1) \leq \Psi^s(l + 1).
\]

This yields that
\[
Q_n(\Psi, l^*) = \sup_{l>n} Q_n(\Psi, l).
\] (4.21)
According to (4.5), we get
\[ \Psi(l^* + 1) > \Psi(l^*). \]
Therefore, if the function \( \Psi(t) \) is represented in the form as (4.6), then
\[ l^* = V_{k_{l^*}} = \sum_{i=0}^{k_{l^*}} \nu_i \]  
(4.22)
where \( k_{l^*} \) is defined in (4.11) for \( l = l^* \). Furthermore, in this case, the functionals \( H_n(\Psi, s), r \in (1, \infty) \) can be represented as
\[ H_n(\Psi, s) = \left( (l^* - n)^s \left( \sum_{k=1}^{k_{l^*}} \frac{\nu_k}{\psi^s(k)} \right)^{-s/r} + \sum_{k=k_{l^*}+1}^{\infty} \nu_k \psi^s(k) \right)^{1/s'} := \tilde{H}_n(\psi, s), \]  
(4.23)
where \( r \in (1, \infty), 1/s + 1/s' = 1 \) and
\[ \psi^{-s}(k_{l^*}) \leq \frac{1}{l^* - n} \sum_{j=1}^{k_{l^*}} \frac{\nu_k}{\psi^s(k)} < \psi^{-s}(k_{l^*} + 1). \]  
(4.24)
By virtue of (4.21), for the function
\[ \tilde{Q}_n(\psi, l) := (l - n) \left( \sum_{k=1}^{k_{l^*}-1} \frac{\nu_k}{\psi^s(k)} + \frac{l - V_{k_{l^*}-1}}{\psi^s(k_{l^*})} \right)^{-1}, \]  
(4.25)
where \( k_l \) is defined in (4.11), the following relation is satisfied:
\[ \sup_{l > n} \tilde{Q}_n(\psi, l) = \tilde{Q}_n(\psi, l^*) = (l^* - n) \left( \sum_{k=1}^{k_{l^*}} \frac{\nu_k}{\psi^s(k)} \right)^{-1}. \]  
(4.26)
If the function \( \Psi \) satisfies the conditions of Lemma 4.1, then similarly to the case \( s \in (0, 1] \), we show that
\[ \sum_{k=1}^{l} \frac{\nu_k}{\psi^s(k)} \asymp \frac{l^d}{\psi^s(l)} \]  
(4.27)
and
\[ \tilde{Q}_n(\psi, l^*) = \sup_{l > n} \tilde{Q}_n(\psi, l) \asymp \psi^s(n^{1/d}). \]  
(4.28)
From (4.24), for any \( \psi \in B \), we have
\[ \psi(k_{l^*}) \asymp \psi(n^{1/d}). \]  
(4.29)
From relation (4.8), it follows that for any \( t > t_0 \),
\[ \frac{1}{t} \leq K_{\psi} \frac{\psi'(t)}{\psi(t)}. \]
Integrating the left-hand and right-hand sides of this inequality in the range from a certain number \( k_0 \) to \( k_{l^*}, t_0 < k_0 < k_{l^*} \), we obtain
\[ \ln \frac{k_{l^*}}{k_0} \leq K_{\psi} \ln \frac{\psi(k_0)}{\psi(k_{l^*})}. \]  
(4.30)
Putting \( k_0 = (n/M_0)^{1/2} - c_2 \), due to (4.22) and (4.12), we conclude that \( k_0 < k_{l^*} \). Therefore, by virtue of definition of the set \( B \), relations (4.30) and (4.29), we get
\[ k_{l^*} \asymp n^{1/2}. \]  
(4.31)
Since \( \psi \in B \), then from (4.13) it follows that for any \( l \in \mathbb{N} \),

\[
\sum_{k=l+1}^{\infty} \nu_k \psi^s(k) \gg \sum_{k=l+1}^{2l} k^{d-1} \psi^s(k) \gg l^d \psi^s(l). \tag{4.32}
\]

For \( t > t_0 \), the derivative of the function \( h(t) = t^{d-1} \psi'(t) \) is of the form

\[
h'(t) = s' \psi'^s(t) t^{d-2} \left( \frac{d-1}{s'} - \frac{t|\psi'(t)|}{\psi(t)} \right).
\]

Hence, taking into account (4.8), we see that the function \( h(t) \) decreases at \( t > t_0 \). Therefore, in view of (4.13), we obtain

\[
\sum_{k=l+1}^{\infty} \nu_k \psi^s'(k) < \sum_{k=l+1}^{\infty} k^{d-1} \psi^s'(k) \ll \int_{l}^{\infty} t^{d-1} \psi^s'(t) dt. \tag{4.33}
\]

By virtue of (4.6), (4.13) and (4.4),

\[
\sum_{j=1}^{\infty} \Psi^s'(j) = \sum_{k=1}^{\infty} \nu_k \psi^s'(k) \simeq \sum_{k=1}^{\infty} k^{d-1} \psi^s'(k) < \infty. \tag{4.34}
\]

Hence, due to monotonicity of the function \( h(t) \), it follows that \( \frac{k^{d-1} \psi^s'(k)}{1/k} = k^d \psi^s'(k) \to 0 \) as \( k \to \infty \).

Further, using (4.8) and the method of integration by parts, we have

\[
\int_{l}^{\infty} t^{d-1} \psi^s'(t) dt \leq K_{\psi} \int_{l}^{\infty} t^{d} \psi'^s(t) \psi'(t) dt = \frac{K_{\psi}}{s'} l^d \psi'^s(t) dt + \frac{K_{\psi} l^d}{s'} \int_{l}^{\infty} t^{d-1} \psi^s'(t) dt. \tag{4.35}
\]

Taking into account (4.8), from the relations (4.33) and (4.35) we get the estimate

\[
\sum_{k=l+1}^{\infty} \nu_k \psi^s'(k) \ll l^d \psi^s'(l),
\]

that together with (4.32) proves the relation

\[
\sum_{k=l+1}^{\infty} \nu_k \psi^s'(k) \sim l^d \psi^s'(l). \tag{4.36}
\]

Thus, combining the relations (4.23), (4.27)–(4.35) and (4.36) and (4.31) we obtain the estimate (4.9), i.e.,

\[
H_n(\Psi, s) = \tilde{H}_n(\psi, s) \simeq \left( \psi^{1/2} \left( \frac{n^2}{\psi} \right)^{1/2} + n \psi^s(n^{1/2}) \right)^{1/s'} \simeq \psi(n^{1/2}) n^\frac{1}{2} \simeq \psi(n^{1/2}) n^{1-\frac{1}{s'}}.
\]

Lemma 4.1 is proved.

### 4.3

In this section we apply Lemma 3.1 for estimates of approximative characteristics of the spaces \( S^p(\mathbb{T}^d) \). Approximative characteristics of the spaces \( S^p(\mathbb{T}^d) \) were studied by many authors (see, for example, [7; 8, Ch. XI; 21–25]). The exact values of the quantities \( e_n(\mathcal{F}^p_{q,r})_{S^p(\mathbb{T}^d)} \), as well as the
exact values of the quantities $G_n(F_{q,r}^\psi)_{Sp(T^d)}$ and $e_n^p(F_{q,r}^\psi)_{Sp(T^d)}$ (due to (2.8)), for any $0 < p, q < \infty$, were obtained by A.I. Stepanets ([7; 8, Ch. XI]). In particular, from Theorem 9.1 of [8, Ch. XI], it follows that for any $0 < q \leq p < \infty$ and for any positive function $\psi = \psi(t), t \geq 1$, satisfying condition (3.3), for all $n \in \mathbb{N}$

$$e_n^p(F_{q,r}^\psi)_{Sp(T^d)} = \sup_{l > n} (l - n)(\sum_{j=1}^{l} \bar{\psi}^{-q}(j))^{-\frac{n}{q}}, \quad (4.37)$$

where $\bar{\psi} = \bar{\psi}(j), j = 1, 2, \ldots$, is the decreasing rearrangement of the system of numbers $\psi(|k_r|), k \in \mathbb{Z}^d$. If $0 < p < q < \infty$ and the positive function $\psi = \psi(t), t \geq 0$, satisfies the condition

$$\sum_{k \in \mathbb{Z}^d} \psi^{\frac{pq}{q-p}}(|k_r|) < \infty, \quad (4.38)$$

then from Theorem 9.4 of [8, Ch. XI] it follows that

$$e_n^p(F_{q,r}^\psi)_{Sp(T^d)} = \left( (l^* - n)^{\frac{q-p}{q-p}} \left( \sum_{k=1}^{l^*} \bar{\psi}^{-q}(k) \right)^{\frac{q}{q-p}} + \sum_{k=l^*+1}^{\infty} \bar{\psi}^{\frac{pq}{q-p}}(k) \right)^{\frac{q-p}{q}}, \quad (4.39)$$

where $\bar{\psi} = \bar{\psi}(j), j = 1, 2, \ldots$, is the decreasing rearrangement of the system of numbers $\psi(|k_r|), k \in \mathbb{Z}^d$, and the number $l^*$ is defined by

$$\bar{\psi}^{-q}(l^*) \leq \frac{1}{l^* - n} \sum_{k=1}^{l^*} \bar{\psi}^{-q}(k) < \bar{\psi}^{-q}(l^* + 1).$$

Taking into account notation (4.2) and (4.3), we can write relations (4.37) and (4.39) as

$$e_n^p(F_{q,r}^\psi)_{Sp(T^d)} = H_n(\bar{\psi}, q/p), \quad 0 < p, q < \infty.$$  

Furthermore, if the number $V_m := |\hat{A}_{m,r}^d|$ of elements of the set

$$\hat{A}_{m,r}^d := \{ k \in \mathbb{Z}^d : |k_r| \leq m, \quad m \in \mathbb{N} \}. \quad (4.40)$$

for all sufficiently large $m \in \mathbb{Z}_+$ ($m$ greater than some positive number $k_0$) satisfies the following condition:

$$M_0(m - c_1)^d < V_m = |\hat{A}_{m,r}^d| \leq M_0(m + c_2)^d, \quad (4.41)$$

where $M_0, c_1$ and $c_2$ are certain positive constants, then the sequence $\bar{\psi} = \bar{\psi}(j), j = 1, 2, \ldots$, belongs to the set $S_d(M_0) = S_d(M_0, c_1, c_2)$. Thus, by virtue of Lemma 4.1, we can formulate the following statement:

**Assertion 4.1.** Assume that $0 < r \leq \infty, 1 \leq p < \infty, 0 < q < \infty$, condition (4.41) holds, the function $\psi^p(\cdot)$ belongs to the set $B$ and moreover for $0 < p < q$ and for all $t$, larger than a certain number $t_0$, $\psi^p(\cdot)$ is convex downwards and satisfies the relation

$$\frac{1}{q(\psi, t)} \geq K_\psi > d \left( \frac{1}{p} - \frac{1}{q} \right).$$

(4.42)
Then
\[ e_n(\mathcal{F}_{q,r}^\psi)_{SP(\mathbb{T}^d)} \asymp \frac{\psi(n^{\frac{r}{p}})}{n^{\frac{1}{p} - \frac{1}{r}}}. \]

It is clear that in the case, when \( r = \infty \), condition (4.41) is satisfied and \( M_0 = vol\{ k \in \mathbb{R}^d : |k|_{\infty} \leq 1 \} = 2^d \). If \( r = 1 \), then \( M_0 = vol\{ k \in \mathbb{R}^d : |k|_1 \leq 1 \} = 2^d/d! \). Therefore, in these cases, we have
\[ e_n(\mathcal{F}_{q,1}^\psi)_{SP(\mathbb{T}^d)} \asymp e_n(\mathcal{F}_{q,\infty}^\psi)_{SP(\mathbb{T}^d)} \asymp \frac{\psi(n^{\frac{1}{p}})}{n^{\frac{1}{p} - \frac{1}{r}}}. \]
(for \( p, q \) and \( \psi \), satisfying conditions of Assertion 4.1). Unfortunately, we do not know whether a similar relation for other \( r \) is valid. However, one can formulate the following corollary:

**Corollary 4.1.** Assume that \( 1 \leq p < \infty, 0 < q < \infty \), the function \( \psi^p(\cdot) \) belongs to the set \( B \) and moreover for \( 0 < p < q \) and for all \( t \), larger than a certain number \( t_0 \), \( \psi^p(\cdot) \) is convex downwards and satisfies the relation (4.42). Then for all \( 1 \leq r \leq \infty \),
\[ e_n(\mathcal{F}_{q,r}^\psi)_{SP(\mathbb{T}^d)} \asymp \frac{\psi(n^{\frac{r}{p}})}{n^{\frac{1}{p} - \frac{1}{r}}}. \]

Indeed, for any numbers \( r \in [1, \infty], 0 < q < \infty \) and for any positive decreasing function \( \psi = \psi(t), t \geq 1 \),
\[ \mathcal{F}_{q,1}^\psi \subset \mathcal{F}_{q,r}^\psi \subset \mathcal{F}_{q,\infty}^\psi. \] (4.43)
Therefore, if conditions of Corollary 4.1 are satisfied, then for all \( r \in [1, \infty] \),
\[ \frac{\psi(n^{\frac{r}{p}})}{n^{\frac{1}{p} - \frac{1}{r}}} \ll e_n(\mathcal{F}_{q,1}^\psi)_{SP(\mathbb{T}^d)} \leq e_n(\mathcal{F}_{q,r}^\psi)_{SP(\mathbb{T}^d)} \leq e_n(\mathcal{F}_{q,\infty}^\psi)_{SP(\mathbb{T}^d)} \ll \frac{\psi(n^{\frac{1}{p}})}{n^{\frac{1}{p} - \frac{1}{r}}}. \]

**5. Proof of Theorem 3.1.**

**5.1.** In this section we give the proof of Theorem 3.1, but first, we formulate one auxiliary lemma, which is interesting in itself.

**Lemma 5.1.** Assume that \( 2 \leq p < \infty, n \in \mathbb{N}, \gamma_n = \{ k_1, k_2, \ldots, k_n \} \) is a collection of \( n \) vectors \( k_i \in \mathbb{Z}^d \) such that \( \gamma_n \subset [-cn^{\frac{1}{d}}, cn^{\frac{1}{d}}]^d \), where \( c \) is a positive number. Then
\[ \left\| \sum_{k \in \gamma_n} e^{i(k, \cdot)} \right\|_{L_p(\mathbb{T}^d)} \asymp n^{1 - \frac{1}{p}}. \] (5.1)

**Proof.** By virtue of the Hausdorff–Young theorem (see, for example, [26, p. 16]), we get the upper estimate:
\[ \left\| \sum_{k \in \gamma_n} e^{i(k, \cdot)} \right\|_{L_p(\mathbb{T}^d)} \ll \left\| \sum_{k \in \gamma_n} e^{i(k, \cdot)} \right\|_{SP(\mathbb{T}^d)} = n^{1 - \frac{1}{p}}. \] (5.2)
Let us obtain the lower estimate. Based on the known trigonometric formulas, we have
\[ \left\| \sum_{k \in \gamma_n} e^{i(k, \cdot)} \right\|_{L_p(\mathbb{T}^d)} \asymp \left( \int_{\mathbb{T}^d} \left| \sum_{k \in \gamma_n} e^{i(k, x)} \frac{e^{i(1, x)} - 1}{e^{i(1, x)} - 1} \right|^p dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{T}^d} \left| \sum_{k \in \gamma_n} e^{i(k+1, x)} - e^{i(k, x)} \frac{e^{i(1, x)} - 1}{e^{i(1, x)} - 1} \right|^p dx \right)^{\frac{1}{p}} = \]
\[
= \left( \int \left| \sum_{k \in \gamma_n} \frac{(\cos((k_1 + 1)x_1 + \ldots + (k_d + 1)x_d) + i \sin((k_1 + 1)x_1 + \ldots + (k_d + 1)x_d))}{\cos(x_1 + \ldots + x_d) + i \sin(x_1 + \ldots + x_d) - 1} \right|^p \, dx \right)^{\frac{1}{p}}
\]

Further, we set
\[
\left| \sum_{k \in \gamma_n} \frac{(\cos(k_1 x_1 + \ldots + k_d x_d) + i \sin(k_1 x_1 + \ldots + k_d x_d))}{\cos(x_1 + \ldots + x_d) + i \sin(x_1 + \ldots + x_d) - 1} \right|^p \, dx = \left( \int \left| \sum_{k \in \gamma_n} \frac{-2 \sin \frac{x_1 + \ldots + x_d}{2}}{\cos(x_1 + \ldots + x_d) + i \sin(x_1 + \ldots + x_d) - 1} \right|^p \, dx \right)^{\frac{1}{p}} + 2i \sum_{k \in \gamma_n} \frac{2 \sin \frac{x_1 + \ldots + x_d}{2} \cos(\frac{(2k_1+1)x_1 + \ldots + (2k_d+1)x_d}{2})}{\cos(x_1 + \ldots + x_d) + i \sin(x_1 + \ldots + x_d) - 1} \right|^p \, dx \right)^{\frac{1}{p}}.
\]

Hence, using the definition of the module, after simplifications we obtain
\[
\left| \left| \sum_{k \in \gamma_n} e^{i(k_\cdot \cdot)} \right| \right|_{L_p(T^d)} \gtrsim \left( \int \left( \sum_{k \in \gamma_n} \frac{\sin^2 \frac{x_1 + \ldots + x_d}{2}}{\sin^2 \frac{x_1 + \ldots + x_d}{2} + \frac{x_1 + \ldots + x_d}{2}} \cos((k_1 - j_1)x_1 + \ldots + (k_d - j_d)x_d) \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}
\]

Further, we set
\[
l = l(\gamma_n) = \max_{k,j \in \gamma_n} |k_m - j_m|.
\]

For all \( x \in [0, \pi/(2\beta)] \), \( \cos \beta x \geq 1 - \frac{2\beta}{\pi} x \) and \( \cos \alpha x \geq \cos \beta x \), where \( 0 < \alpha < \beta \). Therefore, from relation (5.3) we obtain
\[
\left| \left| \sum_{k \in \gamma_n} e^{i(k_\cdot \cdot)} \right| \right|_{L_p(T^d)} \gtrsim \left( \int \ldots \int \left( n + \sum_{k \in \gamma_n} \sum_{j \not= k} (1 - \frac{2l}{\pi}(x_1 + \ldots + x_d)) \right)^{\frac{p}{2}} \, dx \ldots \, dx \right)^{\frac{1}{p}}
\]

\[
\gtrsim \left( \int \ldots \int \left( n^2 - \frac{2l}{\pi} n(n-1)(x_1 + \ldots + x_d) \right)^{\frac{p}{2}} \, dx \ldots \, dx \right)^{\frac{1}{p}}
\]

\[
\gtrsim n \left( \int \ldots \int \left( 1 - \frac{2l}{\pi}(x_1 + \ldots + x_d) \right)^{\frac{p}{2}} \, dx \ldots \, dx \right)^{\frac{1}{p}} = n \left( \frac{n}{2l} \right)^{\frac{d}{p}} \left( \int \ldots \int \left( 1 - (x_1 + \ldots + x_d) \right)^{\frac{p}{2}} \, dx \ldots \, dx \right)^{\frac{1}{p}} \approx n l^{-\frac{d}{p}}. \tag{5.4}
\]

Since \( \gamma_n \subset [-cn^{\frac{1}{d}}, cn^{\frac{1}{d}}]^d \), then \( l = l(\gamma_n) \ll n^{\frac{1}{d}} \). Thus, indeed, the following estimate is true:
\[
\left| \left| \sum_{k \in \gamma_n} e^{i(k_\cdot \cdot)} \right| \right|_{L_p(T^d)} \gtrsim n^{1 - \frac{1}{p}}.
\]
Lemma is proved.

5.2. Now, we can prove Theorem 3.1.

**Upper estimates.** If $2 \leq p < \infty$, then using the Hausdorff–Young theorem and relation (2.3), we get

$$
\sup_{f \in F_{q,r}^\psi} ||f - G_n(f)||_{L_p(T^d)} \ll \sup_{f \in F_{q,r}^\psi} ||f - G_n(f)||_{L_p(T^d)} = e_n(F_{q,r}^\psi)_{L_p(T^d)},
$$

(5.5)

where $\frac{1}{p} + \frac{1}{p'} = 1$. In the case, where $1 \leq p \leq 2$, we have

$$
\sup_{f \in F_{q,r}^\psi} ||f - G_n(f)||_{L_p(T^d)} \ll \sup_{f \in F_{q,r}^\psi} ||f - G_n(f)||_{L_2(T^d)} = e_n(F_{q,r}^\psi)_{L_2(T^d)},
$$

(5.6)

Thus, to obtain the required upper estimates, it is sufficient to use Corollary 4.1.

**Lower estimate.** Let $T_m$, $m \in \mathbb{N}$, denote the set of all polynomials of the form as

$$
T_m(x) = \sum_{|k|_\infty \leq m} \hat{T}_m(k) e^{i(k,x)},
$$

and let $A_q(T_m)$, $0 < q < \infty$, denote the subset of all polynomials $T_m \in T_m$ such that $||T||_{S^q(T^d)} \leq 1$.

From Theorem 5.2 of [1], it follows that for any $0 < q < \infty$, $1 \leq p < \infty$, $m = 1, 2, \ldots$ and $n = ((2m + 1)^d - 1)/2$,

$$
e_n(A_q(T_m))_{L_p(T^d)} \geq Kn^{1/2 - 1/q}.
$$

For a fixed $m \in \mathbb{N}$, consider the set

$$
\psi(dm)A_q(T_m) = \{ T \in T_m : ||T||_{S^q(T^d)} \leq \psi(dm) \}.
$$

Due to monotonicity $\psi$, for any polynomial $T \in \psi(dm)A_q(T_m)$ we have

$$
\sum_{k \in \mathbb{Z}^d} \left| \frac{\hat{T}(k)}{\psi(|k|_1)} \right|^q \leq \sum_{|k|_\infty \leq m} \left| \frac{\hat{T}(k)}{\psi(|d|_\infty)} \right|^q \leq \sum_{|k|_\infty \leq m} \left| \frac{\hat{T}(k)}{\psi(dm)} \right|^q \leq 1
$$

Therefore, $\psi(dm)A_q(T_m)$ is contained in the set $F_{q,1}^\psi$. In view of definition of the set $B$, for all $m = 1, 2, \ldots$ and $n = ((2m + 1)^d - 1)/2$, we obtain

$$
e_n(F_{q,1}^\psi)_{L_p(T^d)} \geq e_n(\psi(dm)A_q(T_m))_{L_p(T^d)} \geq K \psi(dm)n^{\frac{1}{2} - \frac{1}{q}} \geq K_1 \psi(n^{\frac{1}{4}})n^{\frac{1}{2} - \frac{1}{q}}.
$$

(5.7)

Taking into account the relations (2.7) and (4.43), monotonicity of the quantity $e_n$ and inclusion $\psi \in B$, we see that for all $1 \leq p < \infty$ and all $1 \leq r \leq \infty$,

$$
\sup_{f \in F_{q,r}^\psi} ||f - G_n(f)||_{L_p(T^d)} \gg e_n(F_{q,r}^\psi)_{L_p(T^d)} \gg e_n(F_{q,r}^\psi)_{L_p(T^d)} \gg e_n(F_{q,1}^\psi)_{L_p(T^d)} \gg \psi(n^{\frac{1}{2}}) n^{\frac{1}{2} - \frac{1}{q}}.
$$

In the case, where $2 < p < \infty$, for the quantities $e_n(F_{q,r}^\psi)_{L_p(T^d)}$ and $\sup_{f \in F_{q,r}^\psi} ||f - G_n(f)||_{L_p(T^d)}$, this estimate can be improved. For this purpose, consider the function

$$
f_1(x) = C_1(n) \sum_{|k|_1 \leq (2n/M_0)^{1/d}} e^{i(k,x)},
$$
where \( M_0 = 2^d/d! \) and
\[
C_1(n) = \left( \sum_{|k_1| \leq [(2n/M_0)^{1/d}]} \psi^{-q}(|k_1|) \right)^{-\frac{1}{q}}.
\]
It is obviously that \( f_1 \in \mathcal{F}_{q,1}^\psi \). Due to (4.41), the number \(|\Delta_{m,1}^d|\) of elements of the set
\[
\Delta_{m,1}^d := \{ k \in \mathbb{Z}^d : |k_1| = m, \ m \in \mathbb{N} \}.
\]
for all sufficiently large \( m \) satisfies the condition
\[
M_0(m - c_3)^{d-1} < |\Delta_{m,1}^d| = |\Delta_{m,1}^d| - |\Delta_{m-1,1}^d| \leq M_0(m - c_4)^{d-1}, \quad (5.8)
\]
where \( c_3 \) and \( c_4 \) are some positive numbers. Therefore,
\[
C_1^{-q}(n) = \sum_{|k_1| \leq [(2n/M_0)^{1/d}]} \psi^{-q}(|k_1|) \leq \sum_{k=1}^{[(2n/M_0)^{1/d}]} k^{d-1} \psi^q(k).
\]
Since \( \psi \in B \), then for any \( l = 2, 3, \ldots \),
\[
\frac{l^d}{\psi^q(l)} \ll \frac{(l/2)^d}{\psi^q(l/2)} \ll \sum_{l/2 \leq k \leq l} k^{d-1} \psi^q(k) \leq \sum_{k=1}^{l} k^{d-1} \psi^q(k) \ll \frac{l^d}{\psi^q(l)}.
\]
This yields
\[
C_1(n) \asymp \psi((2n/M_0)^{1/d})/[(2n/M_0)^{1/d}]^d \asymp \psi(n^\frac{d}{2})/n^\frac{d}{2}.
\]
In view of (4.41) and Lemma 5.1, for any collection \( \gamma_n \subset \mathbb{Z}^d \) and the polynomial \( \sum_{k \in \gamma_n} \hat{f}_1(k)e^{ik} \), we obtain
\[
\left\| f_1 - \sum_{k \in \gamma_n} \hat{f}_1(k)e^{ik} \right\|_{L_p(\mathbb{T}^d)} = C_1(n) \left\| \sum_{|k_1| \leq [(2n/M_0)^{1/d}]} \sum_{k \not\in \gamma_n} e^{ik} \right\|_{L_p(\mathbb{T}^d)} \gg C_1(n)n^{1-\frac{1}{p}} \asymp \psi(n^\frac{d}{2})/n^{\frac{d}{2} + \frac{1}{p} - 1},
\]
Therefore, for all \( 2 \leq p < \infty \), the following estimates are true:
\[
\sup_{f \in \mathcal{F}_{q,r}^\psi} \left\| f - G_n(f) \right\|_{L_p(\mathbb{T}^d)} \gg e_n^{\frac{1}{r}}(\mathcal{F}_{q,r}^\psi)_{L_p(\mathbb{T}^d)} \gg e_n^{\frac{1}{r}}(\mathcal{F}^\psi_{q,1})_{L_p(\mathbb{T}^d)} \gg e_n^{\frac{1}{r}}(f_1)_{L_p(\mathbb{T}^d)} \gg \psi(n^\frac{d}{2})/n^{\frac{d}{2} + \frac{1}{p} - 1}.
\]
Theorem 3.1. is proved.

**Remark 5.1.** Combining the relations (5.5) and (5.7), as well as the relations (5.9) and (5.11), taking into account Corollary 4.1 and relation (2.7), we conclude that in the case, where \( 1 \leq r \leq \infty \), \( 0 < q < \infty \) and the function \( \psi \) satisfies conditions of Theorem 3.1, for all \( 1 \leq p \leq 2 \),
\[
e_n(\mathcal{F}_{q,r}^\psi)_{L_p(\mathbb{T}^d)} \asymp \frac{\psi(n^\frac{1}{r})}{n^{\frac{d}{2} - \frac{1}{2}}},
\]
and for all \( 2 < p < \infty \),
\[
\frac{\psi(n^\frac{1}{r})}{n^{\frac{d}{2} - \frac{1}{2}}} \ll e_n(\mathcal{F}_{q,r}^\psi)_{L_p(\mathbb{T}^d)} \ll \frac{\psi(n^\frac{1}{r})}{n^{\frac{d}{2} + \frac{1}{p} - 1}}.
\]
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