AN ALEXANDROV-FENCHEL-TYPE INEQUALITY IN HYPERBOLIC SPACE WITH AN APPLICATION TO A PENROSE INEQUALITY

LEVI LOPES DE LIMA AND FREDERICO GIRÃO

ABSTRACT. We prove a sharp Alexandrov-Fenchel-type inequality for star-shaped, strictly mean convex hypersurfaces in hyperbolic $n$-space, $n \geq 3$. The argument uses two new monotone quantities along the inverse mean curvature flow. As an application we establish, in any dimension, an optimal Penrose inequality for asymptotically hyperbolic graphs carrying a minimal horizon, with the equality occurring if and only if the graph is an anti-de Sitter-Schwarzschild solution. This sharpens previous results by Dahl-Gicquaud-Sakovich and settles, for this class of initial data sets, the conjectured Penrose inequality for time-symmetric space-times with negative cosmological constant. We also explain how our methods can be easily adapted to derive an optimal Penrose inequality for asymptotically locally hyperbolic graphs in any dimension $n \geq 3$. When the horizon has the topology of a compact surface of genus at least one, this provides an affirmative answer, for this class of initial data sets, to a question posed by Gibbons, Chruściel and Simon on the validity of a Penrose-type inequality for exotic black holes.

CONTENTS

1. Introduction 1
2. Mass-type invariants and Penrose inequalities for asymptotically locally hyperbolic manifolds 5
3. Geometric flows for hypersurfaces 14
4. The proofs of Theorems 1.1 and 1.2 15
Appendix A. The asymptotic behavior of $\mathcal{L}$ 17
Appendix B. The asymptotically locally hyperbolic case 19
References 20

1. INTRODUCTION

If $\Sigma \subset \mathbb{R}^n$ is a convex hypersurface then the Alexandrov-Fenchel inequalities say that

\[
\int_{\Sigma} \sigma_k(\kappa) d\Sigma \geq C_{n,k} \left( \int_{\Sigma} \sigma_{k-1}(\kappa) d\Sigma \right)^{\frac{n-1}{n-1-k}},
\]

where $\sigma_k(\kappa)$, $1 \leq k \leq n-1$, is the $k^{\text{th}}$ elementary symmetric function of the principal curvature vector $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of $\Sigma$ and $C_{n,k} > 0$ is a universal constant. Moreover, the equality holds in (1) if and only if $\Sigma$ is a round sphere. Classically, (1) follows from the general theory of mixed volumes, so that convexity is used in an essential way; see (32). Recently, however, Guan and Li (21) used a suitable

The first author was partially supported by a CNPq/Brazil research grant.
normalization of a certain inverse curvature flow to extend the validity of (1), with
the corresponding rigidity statement, for any Σ which is star-shaped and k-convex
(which means that σ_i(κ) ≥ 0 for i = 1, ⋯, k).

An interesting question is to establish versions of these inequalities for appro-
piate classes of hypersurfaces in more general ambient manifolds, preferably with
a corresponding rigidity statement for the case of equality. Here we focus on the
case k = 1 of (1), namely,

(2) \[ c_n \int_\Sigma H d\Sigma \geq \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{2}{n-2}}, \]

where A is the area, H = σ_1(κ) is the mean curvature,

\[ c_n = \frac{1}{2(n-1)\omega_{n-1}}, \]

and \( \omega_{n-1} \) is the area of the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). We take a first step toward solving
this problem by establishing a natural analogue of (2) for star-shaped, strictly
mean convex hypersurfaces in hyperbolic n-space, n ≥ 3; see Theorem 1.1. The
proof is partly inspired by [21] and uses two new monotone quantities for the in-
verse mean curvature flow in hyperbolic space. The precise asymptotics for this
flow, which is a key ingredient in our analysis, has been recently established by
Gerhard [18] [19], see [14] for previous work on this subject. Also, a Heintze-
Karcher-type inequality due to Brendle [4] plays a key role in our proof. We also
make use of a special case of a sharp geometric inequality by Brendle-Hung-Wang
[5]. We note that Gallego and Solanes [15] proved related isoperimetric inequali-
ties using integral-geometric methods, but their results do not seem to be sharp.

The inequality (2) has recently become relevant in the context of the Penrose
inequality for asymptotically flat graphs carrying a minimal horizon [27] [11] and
for asymptotically hyperbolic graphs carrying a constant mean curvature horizon
[13]. As an application of Theorem 1.1 we establish an optimal Penrose inequality
for asymptotically hyperbolic graphs carrying a minimal horizon, including the
rigidity statement according to which the equality holds only if \( (M, g) \) is the graph
realization of an anti-de Sitter-Schwarzschild solution; see Theorem 1.2. This Pen-
rose inequality improves recent results by Dahl-Gicquaud-Sakovich [10] and set-
tles, for this class of initial data sets, the conjectured Penrose inequality for time-
symmetric space-times with negative cosmological constant [6] [29]. We remark
that the proof of the rigidity statement follows from the arguments in a recent pa-
ter by Huang and Wu [24], as adapted to the asymptotically hyperbolic case in
[10, Section 5]; see also [12].

To explain our results, let us consider the hyperbolic n-space \( \mathbb{H}^n \) with coordi-
nates \( (r, \theta) \in \mathbb{R}^+ \times S^{n-1} \) and endowed with the metric

(3) \[ g_1 = dr^2 + \sinh^2 r \, h, \]

where r is the geodesic distance to a chosen origin corresponding to \( r = 0 \) and h is
the round metric on \( S^{n-1} \). We say that a closed, embedded hypersurface Σ ⊂ \( \mathbb{H}^n \)
is star-shaped if it can be written as a radial graph over a geodesic sphere centered
at the origin. Also, it is strictly mean convex if its mean curvature H is positive
everywhere. We also consider \( \rho_1 : \mathbb{H}^n \to \mathbb{R} \),

(4) \[ \rho_1 (r) = \cosh r. \]
With this notation at hand we can state the hyperbolic Alexandrov-Fenchel-type inequality.

**Theorem 1.1.** If $\Sigma \subset \mathbb{H}^n$ is a star-shaped and strictly mean convex hypersurface then

$$c_n \int_{\Sigma} \rho_1 H d\Sigma \geq \frac{1}{2} \left( \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-1}{n}} + \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \right),$$

where $A$ is the area of $\Sigma$. Moreover, the equality holds if and only if $\Sigma$ is a geodesic sphere centered at the origin.

We now explain the relevance of this result for a certain Penrose inequality. Recall that a Riemannian manifold $(M^n, g)$ is said to be **asymptotically hyperbolic (AH)** if there exists a compact subset $K \subset M$ and a diffeomorphism $\Psi : M - K \to \mathbb{H}^n - K_0$, where $K_0 \subset \mathbb{H}^n$ is compact, such that

$$\|\Psi_* g - g_1\|_{g_1} = O(e^{-\sigma r}), \quad \|\nabla_{g_1} \Psi_* g\|_{g_1} = O(e^{-\sigma r}),$$

as $r \to +\infty$, for some $\sigma > n/2$. We also assume that the difference between scalar curvatures, namely,

$$\mathcal{R}_g = R_g + n(n - 1),$$

is such that $\rho_1 \mathcal{R}_g$ is integrable. For any chart at infinity as in (6) it is possible to associate a mass-like invariant $m_\Psi$ which lies in $\mathbb{L}^{n+1}$, the Lorentzian space endowed with the metric

$$(z, w) = z_0 w_0 - z_1 w_1 - \cdots - z_n w_n;$$

see Section 2 for more details on this construction. It turns out that the causal character of $m_\Psi$ is invariant under coordinate changes at infinity. Moreover, the numerical invariant $m_{(M, g)}$ defined by

$$m_{(M, g)}^2 = \langle m_\Psi, m_\Psi \rangle$$

does not depend on the chart $\Psi$ and is termed the **mass of $(M, g)$**. It is natural to choose $m_{(M, g)} > 0$ if $m_\Psi$ is time-like and future directed.

The Positive Mass Conjecture in this context asserts that if $\mathcal{R}_g \geq 0$ then $m_\Psi$ is time-like and future-directed or vanishes, the latter occurring only if $(M, g)$ is isometric to $(\mathbb{H}^n, g_1)$. Equivalently, $m_{(M, g)} \geq 0$ with equality holding only for hyperbolic space. This has been proved for the spin case by Chruściel and Herzlich [7], generalizing a previous contribution by Wang [34]; see also [2] for a similar result in low dimensions with the spin condition removed. Moreover, if $M$ carries a (possibly disconnected) compact, outermost minimal boundary $\Gamma$ (a *horizon*) of area $A$, then the corresponding Penrose Conjecture says that

$$m_{(M, g)} \geq \frac{1}{2} \left( \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-1}{n}} + \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \right),$$

with equality holding only if $(M, g)$ is the (exterior) anti-de Sitter-Schwarzschild solution. We refer to Section 2 and the surveys [6] and [29] for background on this conjecture.

Progress in establishing (9) has been restricted so far to the case of graphs, as we now pass to explain. Recall that the metric

$$\bar{g}_1 = \rho_1^2 d\tau^2 + g_1, \quad \tau \in \mathbb{R},$$
realizes $\mathbb{H}^n \times \mathbb{R}$ as the hyperbolic $(n + 1)$-space $\mathbb{H}^{n+1}$. Using this model we then say that a complete immersed hypersurface $M \subset \mathbb{H}^{n+1}$ is asymptotically hyperbolic (AH) if there exists a compact subset $K \subset M$ such that $M - K$ can be written as a vertical graph associated to a smooth function $u : \mathbb{H}^n - K_0 \rightarrow \mathbb{R}$, where $K_0 \subset \mathbb{H}^n$ is compact, so that (6) holds for the chart $\Psi$ given by $\Psi(x, u(x)) = x, x \in M - K_0$; see Definition 2.2 below. As explained in [13], if additionally $M$ carries a minimal horizon $\Gamma$ then we may assume that $m_\Psi$ is time-like and future oriented so that after composing $\Psi$ with an isometry we have

$$m_{(M, g)} = m_\Psi(\rho_1),$$

where here we use that $\rho_1 = z_0|_{\partial \Gamma}$ if we view $\mathbb{H}^n \subset \mathbb{L}^{n+1}$ as the standard hyperboloid. Charts with this property are called balanced. Now let $M$ be balanced in the sense that nonparametric coordinates at infinity are balanced as above. Moreover, assume that $\Gamma$ lies on a totally geodesic hypersurface $P \subset \mathbb{H}^{n+1}$ defined by $\tau = \tau_0$, $\tau_0 \in \mathbb{R}$, and that $M$ meets $P$ orthogonally along $\Gamma$, so that $\Gamma$ is minimal (hence, a horizon indeed). Under these conditions and starting from (11) it is shown in [13] that

$$m_{(M, g)} = c_n \int_M \Theta \mathcal{R}_g dM + c_n \int_\Gamma \rho_1 H d\Gamma,$$

where $\Theta = (N, \partial / \partial t)$, with $N$ being the unit normal to $M$ pointing upward at infinity and $H$ being the mean curvature of $\Gamma \subset P$ with respect to its inward pointing unit normal; compare to the more general formula in (24).

We remark that if $M$ is a graph then (12) has been previously proved in [10]. If this is the case, so that $\Theta > 0$, and if we assume further that $\mathcal{R}_g \geq 0$ then we obtain from (12) that

$$m_{(M, g)} \geq c_n \int_\Gamma \rho H d\Gamma.$$

In [10] this estimate is used to obtain several sub-optimal versions of (9). For instance, assuming that $\Gamma \subset P = \mathbb{H}^n$ is $h$-convex (in the sense that all principal curvatures are at least 1) and encloses the origin of $P$, the authors show that

$$m_{(M, g)} \geq \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \frac{n-2}{h} + \sinh r_{in} \frac{A}{\omega_{n-1}} \right),$$

where $r_{in}$ is the radius of the largest geodesic ball centered at the origin and contained in the region enclosed by $\Gamma$. Notice that this only yields the conjectured inequality (9) if $\Gamma$ is a geodesic sphere centered at the origin. In view of Theorem 1.4 however, if we take $\Sigma = \Gamma$ we immediately obtain the first statement in the following theorem; a more general result, covering the locally hyperbolic case, can be found in Theorem 2.4 below. Recall that a hypersurface is said to be mean convex if its mean curvature is non-negative everywhere.

**Theorem 1.2.** Let $(M, g) \subset \mathbb{H}^{n+1}$ be a balanced AH graph carrying a minimal horizon $\Gamma$ as above. If we assume further that $\Gamma \subset P = \mathbb{H}^n$ is star-shaped (with respect to the origin) and mean convex then (9) holds if $R_0 \geq -n(n - 1)$. Moreover, the equality occurs if and only if $(M, g)$ is the graph realization of an (exterior) anti-de Sitter-Schwarzschild solution (which is obtained by taking $\epsilon = 1$ in (23) below).
As remarked above, the rigidity statement requires a separate argument and is based on results in a recent preprint by Huang and Wu [24].

This paper is organized as follows. In Section 2 we recall the definitions and main properties of mass-type invariants for asymptotically locally hyperbolic manifolds. The proofs of Theorems 1.1 and 1.2 which use the material on the inverse mean curvature flow discussed in Section 3 and in Appendix A are presented in Section 4. We observe that the Penrose inequality (9) admits a natural generalization to the case in which the geometry at infinity is asymptotically locally hyperbolic. In Appendix B we indicate how the arguments leading to the proofs of Theorems 1.1 and 1.2 can be adapted to prove a sharp Penrose-type inequality for asymptotically locally hyperbolic graphs, which is described in Theorem 2.4 below.

Acknowledgements. The work leading to this paper started when the authors were visiting IMPA on the occasion of its 2012 Summer Program and they would like to thank Prof. F. Marques for providing a wonderful scientific environment. Also, the authors would like to thank Y. Ge, P. K. Hung, A. Neves, G. Wang, M. T. Wang and Y. Wei for discussions and C. Xiong for pointing out an error in a previous version of this work. Finally, we thank the anonymous referees for suggestions that contributed to substantially improve the presentation.

2. Mass-type invariants and Penrose inequalities for asymptotically locally hyperbolic manifolds

In this section we review the main properties of mass-type invariants for asymptotically hyperbolic manifolds; see [7], [8], [23] and [30] for more details. The class of invariants presented here, which appear in natural generalizations of the classical Penrose inequality, is just a special case of a much more general construction due to Chruściel, Herzlich and Nagy [7] [8] [23]. Their definition applies in particular to the class of asymptotically locally hyperbolic manifolds we consider here.

We start by describing the corresponding locally hyperbolic (LH) reference metrics. Fix \( \epsilon = 0, \pm 1 \) and let \((N^{n-1}, h)\) be a closed space form of sectional curvature \( \epsilon \). In the product manifold \( P_\epsilon = I_\epsilon \times N \), where \( I_{-1} = (1, +\infty) \) and \( I_0 = I_1 = (0, +\infty) \), define the metric

\[
g_\epsilon = \frac{d\tilde{r}^2}{\rho_\epsilon(\tilde{r})^2} + \tilde{r}^2 h, \quad \tilde{r} \in I_\epsilon,
\]

where

\[
\rho_\epsilon(\tilde{r}) = \sqrt{\tilde{r}^2 + \epsilon}.
\]

It is easy to check that \((P_\epsilon, g_\epsilon)\) is locally hyperbolic in the sense that its sectional curvature is constant and equal to \(-1\). We also note that the manifold \( Q_\epsilon = \mathbb{R} \times P_\epsilon \), endowed with the metric

\[
\mathcal{F}_\epsilon = \rho_\epsilon^2 d\tau^2 + g_\epsilon,
\]

is locally hyperbolic as well. Moreover, if \( \tau_0 \in \mathbb{R} \) then the horizontal slice \( P_\epsilon^{\tau_0} \subset Q_\epsilon \) given by \( \tau = \tau_0 \) is totally geodesic. This follows from the fact that the vertical vector field \( \partial_\tau \) is Killing. In particular, each \( P_\epsilon^{\tau_0} \) can be naturally identified to \( P_\epsilon \). Notice that if we take \( \epsilon = 1 \) and \((N, h)\) to be a round sphere then we recover hyperbolic space as in the Introduction; compare (14)-(15) with (3)-(4) after setting \( \tilde{r} = \sinh r \).
Let \((M^n, g)\) be a complete \(n\)-dimensional Riemannian manifold, possibly carrying a compact inner boundary \(\Gamma\). For simplicity we assume that \(M\) has a unique end, say \(E\). We say that \((M, g)\) is asymptotically locally hyperbolic (ALH) if there exists a chart \(\Psi\) taking \(E\) to the end of \(P_\epsilon\) corresponding to \(\tilde{r} = +\infty\) so that, as \(\tilde{r} \to +\infty\),

\[
\|\Psi_*g - g_\epsilon\|_{g_\epsilon} + \|d\Psi_*g\|_{g_\epsilon} = O(\tilde{r}^{-\sigma}),
\]

for some \(\sigma > n/2\). We also assume that \(\mathfrak{R}_g = R_g + n(n-1)\) is such that \(\rho_\epsilon \mathfrak{R}_g\) is integrable. If \(\epsilon = 1\) and \(N = S^{n-1}\) we say that \((M, g)\) is asymptotically hyperbolic (AH).

It turns out that each model space \(P_\epsilon\) is static in the sense that the space

\[
\mathcal{N}_{g_\epsilon} = \{ f \in C^\infty(P_\epsilon); (\Delta_{g_\epsilon} f)g_\epsilon - \nabla^2_{g_\epsilon} f + f \text{Ric}_{g_\epsilon} = 0 \}
\]

is non-trivial, as we can easily check that \(\rho_\epsilon \in \mathcal{N}_{g_\epsilon}\). If \(\Psi\) is a chart at infinity as above, we define the corresponding mass functional \(m_\Psi : \mathcal{N}_{g_\epsilon} \to \mathbb{R}\) by

\[
m_\Psi(\varphi) = \lim_{r \to +\infty} c_n \int_{N_r} \left( \varphi (\text{div}_{g_\epsilon} e - dtr_{g_\epsilon} e) - i\nu_{g_\epsilon}e + (tr_{g_\epsilon} e) d\varphi \right) (\nu_r) dN_r,
\]

where \(e = \Psi_*g - g_\epsilon\), \(\nu_r\) is the unit normal to \(N_r = \{r\} \times N\) and

\[
c_n = \frac{1}{2(n-1)\sigma_{n-1}}, \quad \sigma_{n-1} = \text{area}(N, h).
\]

If \(\Phi\) is another chart at infinity one verifies that

\[
m_\Phi(\varphi) = m_\Psi(\varphi \circ \mathcal{I}^{-1}),
\]

where \(\mathcal{I} \in \text{Isom}(P_\epsilon)\) satisfies

\[
\|\Phi \circ \Psi^{-1} - \mathcal{I}\|_{g_\epsilon} = O(\tilde{r}^{-\sigma}).
\]

Thus, in order to get a numerical invariant out of this scheme we need a detailed description of the structure of the action of \(\text{Isom}(P_\epsilon)\) on \(\mathcal{N}_{g_\epsilon}\), appearing on the right-hand side of (20).

We first consider the case \(\epsilon = 1\) and \(N_1 = S^{n-1}\), so that \(P_1 = \mathbb{H}^n\). Here, \(\mathcal{N}_{g_1}\) is generated by \(\{z_i|z_i^2 = 1\}_{i=0}^n\), where we view \(\mathbb{H}^n \subset \mathbb{L}^{n+1}\), the Lorentz space endowed with the metric (7). Since the action of \(\text{Isom}(\mathbb{H}^n)\) on \(\mathcal{N}_{g_1} = \mathbb{L}^{n+1}\) preserves (7), with \(\rho_1 = z_0\) being time-like and future oriented, it follows that the real number \(m_{(M, g)}\) defined up to a sign by (8) does not depend on the chart \(\Psi\) and is termed the mass of \((M, g)\). We note that the causal character of \(m_\Psi\) is also invariant under coordinate changes at infinity, so it is natural to choose \(m_{(M, g)} > 0\) if \(m_\Psi\) is time-like and future directed. As already observed in the Introduction, if \(M\) carries a minimal horizon \(\Gamma\) then we may assume, after composition with an isometry, that (11) holds.

In contrast to the hyperbolic case described in the previous paragraph, it is known that for \(\epsilon \leq 0\) the space \(\mathcal{N}_{g_\epsilon}\) is one-dimensional, generated by \(\rho_\epsilon\). Thus, in all the cases considered here a mass-type numerical invariant \(m_{(M, g)}\) is obtained by evaluating the right-hand side of (19) on \(\varphi = \rho_\epsilon\).

An ALH manifold \((M, g)\) as above can be thought of as the initial data set of a time-symmetric solution of the Einstein field equations with negative cosmological constant. The invariant \(m_{(M, g)}\) is then interpreted as the total mass of the solution. Physical reasoning predicts that \(m_{(M, g)}\) should have the appropriate sign under the relevant dominant energy condition, namely, \(\mathfrak{R}_g \geq 0\) (equivalently,
denoted $H$ in the sense that static assertion that the Lorentzian metric $m$ which shows that the metric $g$ is strictly positive for $\tilde{r}$ is well defined on the product $P_{\epsilon,m} = I_{\epsilon,m} \times N$, where $I_{\epsilon,m} = \{ \tilde{r}; \tilde{r} > \tilde{r}_{\epsilon,m} \}$. Moreover, it extends smoothly to the slice $\tilde{r} = \tilde{r}_{\epsilon,m}$, the so-called horizon, denoted $H_{\epsilon,m}$. This terminology can be justified as follows. The metric $g_{\epsilon,m}$ is static in the sense that $\rho_{\epsilon,m} \in \mathcal{N}_{g_{\epsilon,m}}$. It is well-known that this is equivalent to the assertion that the Lorentzian metric

\[ g_{\epsilon,m} = -\rho_{\epsilon,m}^2 d\tau^2 + g_{\epsilon,m}, \]

defined on $Q_{\epsilon,m} = \mathbb{R} \times P_{\epsilon,m}$, is a solution to the vacuum field equations with negative cosmological constant:

\[ \text{Ric}(\tilde{g}_{\epsilon,m}) = -n \tilde{g}_{\epsilon,m}. \]

Moreover, the null hypersurface $\tilde{r} = \tilde{r}_{\epsilon,m}$ defines the event horizon surrounding the central singularity $\tilde{r} = 0$. This justifies the horizon terminology and explains why $g_{\epsilon,m}$ is termed a black hole metric.

A computation shows that if $(\phi_1, \cdots, \phi_{n-1})$ are coordinates in $N_{\tilde{r}}$ then the sectional curvatures of $g_{\epsilon,m}$ are

\[ K(\partial_{\phi_i}, \partial_{\phi_j}) = -1 - (n-2) \frac{m}{\tilde{r}^{n-2}}, \]

and

\[ K(\partial_{\phi_i}, \partial_{\phi_j}) = -1 + 2m \frac{m}{\tilde{r}^{n-2}}. \]

This not only shows that $g_{\epsilon,m}$ satisfies the appropriate dominant energy condition, since its scalar curvature is $R_{g_{\epsilon,m}} = -n(n-1)$, but also suggests that $g_{\epsilon,m}$ is ALH in the sense described above. In fact, a straightforward computation gives

\[ \|g_{\epsilon,m} - g_{\epsilon}\|_{g_{\epsilon}} + \|dg_{\epsilon,m}\|_{g_{\epsilon}} = O(\tilde{r}^{-n}), \]

as expected. Using (19) with $\phi = \rho_{\epsilon}$ we finally conclude that $m(P_{\epsilon,m}; g_{\epsilon,m}) = m$, which shows that $m$ should be interpreted as the total mass of $g_{\epsilon,m}$. One immediately finds that the area $|H_{\epsilon,m}|$ of the horizon $H_{\epsilon,m}$ of $(P_{\epsilon,m}; g_{\epsilon,m})$ relates to its mass $m$ by means of

\[ m = \frac{1}{2} \left( \frac{|H_{\epsilon,m}|}{\phi_{n-1}} \right)^{\frac{n-1}{n-2}} + \epsilon \left( \frac{|H_{\epsilon,m}|}{\phi_{n-1}} \right)^{\frac{n-2}{n-1}}. \]
Thus, in analogy with the standard Penrose inequality [9], it is natural to conjecture that if \((M, g)\) is an \(n\)-dimensional ALH manifold (with respect to the reference metric \(g_r\)) carrying an outermost minimal horizon \(\Gamma\) and satisfying \(R_g \geq -n(n-1)\) everywhere then there holds

\[
\mathfrak{m}(M, g) \geq \frac{1}{2} \left( \frac{|\Gamma|}{\partial_{n-1}} \right)^{\frac{m}{2}} + \epsilon \left( \frac{|\Gamma|}{\partial_{n-1}} \right)^{\frac{m-2}{2}},
\]

with the equality occurring if and only if \(g\) is isometric to the corresponding black hole metric.

**Remark 2.1.** For \(\epsilon \leq 0\) and in the physical dimension \(n = 3\), (22) first appears in [9] as a conjectured inequality whose veracity would follow in case the use of the so-called Geroch’s monotonicity of the Hawking mass under the inverse mean curvature flow, as envisaged by Gibbons [20], could be justified. Contrary to this rather optimistic initial expectation, Neves [31] has shown that, at least in the AH case, the convergence properties of the flow at infinity are insufficient to implement Geroch’s scheme. Similar remarks should also apply in the general ALH context, even though Lee and Neves [28] have recently established that Geroch’s strategy works in the so-called ‘non-positive mass range’; see Remark 2.7. Despite these negative results, Theorem 2.4 below confirms that our methods apply to handle the special case of graphs in any dimension \(n \geq 3\).

To motivate our setting we observe that each \((P_{\epsilon, m}, g_{\epsilon, m})\) can be isometrically immersed as a radially symmetric vertical graph inside \((Q_{\epsilon, m}, \bar{g}_s)\): the defining function \(u_{\epsilon, m} : \mathbb{R} \rightarrow \mathbb{R}\) satisfies \(u_{\epsilon, m}(\bar{r}_{\epsilon, m}) = 0\) and

\[
\rho_{\epsilon}(\bar{r})^2 \left( \frac{du_{\epsilon, m}}{d\bar{r}} \right)^2 = \frac{1}{\rho_{\epsilon}(\bar{r})^2} - \frac{1}{\rho_{\epsilon}(\bar{r})^2}.
\]

It is clear from this that the horizon \(H_{\epsilon, m}\) lies on the totally geodesic horizontal slice \(P^0_{\epsilon}\), with the intersection \(M \cap P^0_{\epsilon}\) being orthogonal along \(H_{\epsilon, m}\). This motivates us to consider a more general class of hypersurfaces in \((Q_{\epsilon, m}, \bar{g}_s)\).

**Definition 2.2.** We say that a complete hypersurface \((M, g) \subset (Q_{\epsilon, m}, \bar{g}_s)\), possibly carrying a compact inner boundary \(\Gamma\), is asymptotically locally hyperbolic (ALH) if there exists a compact set \(K \subset M\) so that \(M - K\) can be written as a graph over the end \(E_0\) of the horizontal slice \(P^0\) \(\subset Q_{\epsilon}\), with the graph being associated to a smooth function \(u\) such that the asymptotic condition (17) holds for the nonparametric chart \(\Psi_s(x, u(x)) = x, x \in E_0\). Moreover, we assume that \(\rho_{\epsilon} \mathcal{R}_g\) is integrable. As usual, if \(\epsilon = 1\) and \(N = \mathbb{S}^{n-1}\) then we say that \((M, g)\) is asymptotically hyperbolic (AH).

Under these conditions, the mass \(\mathfrak{m}(M, g)\) can be computed by taking \(\Psi = \Psi_s\) in (19); as in the Introduction we assume that \(\Psi_s\) is balanced if \(\epsilon = 1\). More precisely, if the inner boundary \(\Gamma\) lies on some totally geodesic, horizontal slice \(P^0_{\epsilon}\), which we of course identify with \(P_\tau\) and moreover that the intersection \(M \cap P^0_{\epsilon}\) is orthogonal along \(\Gamma\), so that \(\Gamma \subset M\) is minimal and hence a horizon indeed, then the computations in (13) actually give the following integral formula for the mass:

\[
\mathfrak{m}(M, g) = c_n \int_M \Theta \left( R_g + n(n-1) \right) dM + c_n \int_{\Gamma} \rho_{\epsilon, m} H d\Gamma,
\]

where \(\Theta = (\partial/\partial t, N)\), \(N\) is the unit normal to \(M\), which we choose so as to point upward at infinity, and \(H\) is the mean curvature of \(\Gamma \subset P_\tau\) with respect to its inward normal.
pointing unit normal, which means that the normal points in the direction opposite to the end of $P_\epsilon$ given by $\tilde{r} = +\infty$. In particular, if $R_\gamma \geq -n(n-1)$ and $M$ is a graph $(\Theta > 0)$ then

\[
m_{(M, g)}(\Sigma, p) \geq c_n \int_{\Gamma} \rho_e H d\Gamma.
\]

We are now in a position to state the following Alexandrov-Fenchel-type inequality, which extends Theorem 1.1 to the case $\epsilon \leq 0$.

**Theorem 2.3.** Let $\Sigma \subset P_\epsilon = P$ be a compact embedded hypersurface which is star-shaped in the sense that it can be written as a radial graph over a slice $N_\epsilon = \{ \tilde{r} \} \times N$ and strictly mean convex in the sense that its mean curvature satisfies $H > 0$. Then there holds

\[
m_{(M, g)}(\Sigma, p) \geq \frac{1}{2} \left( \frac{\| \Sigma \|_n}{\partial_{n-1}} \right)^{\frac{n-1}{n}} + \epsilon \left( \frac{\| \Sigma \|}{\partial_{n-1}} \right)^{\frac{n-2}{n-1}},
\]

with the equality occurring if and only if $\Sigma$ is a slice.

By making $\Sigma = \Gamma$ and combining (25) and (26) we immediately obtain the following sharp Penrose-type inequality, which extends Theorem 1.2 to the case $\epsilon \leq 0$.

**Theorem 2.4.** If $M \subset Q_{0, \epsilon}$ is an ALH graph as above, so that its horizon $\Gamma \subset P_\epsilon^{\ast}$ is star-shaped and mean convex in the sense that $H \geq 0$, then

\[
m_{(M, g)}(\Gamma, p) \geq \frac{1}{2} \left( \frac{\| \Gamma \|_n}{\partial_{n-1}} \right)^{\frac{n-1}{n}} + \epsilon \left( \frac{\| \Gamma \|}{\partial_{n-1}} \right)^{\frac{n-2}{n-1}},
\]

with the equality holding if and only if $(M, g)$ is (congruent to) the graph realization (23) of the corresponding black hole solution.

**Remark 2.5.** Under the conditions of Theorem 2.4, the mass $m_{(M, g)}$ is always positive due to (25). In particular, if $\epsilon = -1$ the lower bounds (26) and (27) only are effective if we further assume that $|\Gamma| = \partial_{n-1}$.

As already observed, the following corollary provides a positive answer to a question posed by Gibbons [20] and Chruściel-Simon [9] for the class of initial data sets we consider.

**Corollary 2.6.** If the horizon $\Gamma$ is a surface of genus $\gamma \geq 1$ then

\[
m_{(M, g)} \geq \left( \frac{4 \pi}{\vartheta_2} \right)^{3/2} \sqrt{\frac{|\Gamma|}{16 \pi}} \left( 1 - \gamma \frac{|\Gamma|}{4 \pi} \right),
\]

where for $\gamma = 1$ we assume the normalization $\vartheta_2 = 4 \pi$. Moreover, the equality holds if and only if $(M, g)$ is (congruent to) the graph realization of the corresponding black hole solution.

**Proof.** If $\gamma \geq 2$ this follows by taking $n = 3$ and $\epsilon = -1$ in the theorem and observing that Gauss-Bonnet gives $\vartheta_2 = 4 \pi (\gamma - 1)$. If $\gamma = 1$ we take $\epsilon = 0$ and use the normalization.

The proofs of Theorems 2.3 and 2.4 are straightforward adaptations of the proofs of Theorems 1.1 and 1.2. The necessary modifications are briefly described in Appendix [B]. We note that further results in this direction have been obtained in [16].
Remark 2.7. As discussed in [9], when $\epsilon = -1$ the Kottler metrics (21) also describe static black hole solutions when the parameter $m$ becomes negative up to a certain critical value, namely,

$$m_{\text{crit}} = -\frac{(n-2)\sqrt{2}}{n\pi}.$$ 

In this regard we mention that Lee and Neves [28] used the Huisken-Ilmanen’s formulation of the inverse mean curvature flow to establish a Penrose-type inequality for conformally compact ALH 3-manifolds in this mass range. More precisely, they prove (28) with the mass replaced by the supremum of the mass aspect function, which is assumed to be non-positive along the boundary at infinity. In particular, their manifolds always have non-positive mass while our graphs necessarily satisfy $m(M,g) > 0$; see Remark 2.5. Thus, their result and Corollary 2.6 are in a sense complementary to each other.

3. Geometric flows for hypersurfaces

As mentioned above, the proof of Theorem 1.1 uses the inverse mean curvature flow recently studied by Gerhardt [18] [19]; see also [14]. As a preparation for the argument, let us start by considering a closed, isometrically immersed hypersurface $\Sigma \subset \mathbb{H}^n$ with unit normal $\xi$. As usual, we denote by $g_1$ both the standard metric on $\mathbb{H}^n$ and its restriction to $\Sigma$. Also, $b$ is the second fundamental form of $\Sigma$. For simplicity we set $D = \nabla_{g_1}$. Thus, if $X$ and $Y$ are vector fields tangent to $\Sigma$,

$$b(X, Y) = g_1(aX, Y) = \langle aX, Y \rangle,$$

where

$$aX = -D_{\Sigma} \xi,$$

is the shape operator. As before, we denote by $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ the principal curvature vector of $\Sigma$, so that

$$H = \sigma_1(\kappa) = \text{tr}_{g_1} b$$

is the mean curvature. It follows from the Cauchy-Schwarz inequality that

$$(n-1)|a|^2 \geq H^2,$$

with the equality occurring at a given point if and only if $\Sigma$ is umbilical there. We also consider the extrinsic scalar curvature of the immersion, namely,

$$K = \sigma_2(\kappa) = \sum_{i<j} \kappa_i \kappa_j = \frac{1}{2} \left( H^2 - |a|^2 \right).$$

Notice that these invariants are related by the Newton-MacLaurin inequality:

$$2K \leq \frac{n-2}{n-1} H^2,$$

with the equality holding at a given point only if $\Sigma$ is umbilical there [22]. We also recall the support function

$$p = \langle D\rho, \xi \rangle,$$

where we set $\rho = \rho_1$ for simplicity. This relates to $\rho$ and $H$ by means of the following Minkowski identity:

$$\Delta \rho = (n-1) \rho + Hp,$$
where $\Delta = \text{div} \circ \nabla$ is the Laplacian of $g_1|_\Sigma$. This is a consequence of the fact that the vector field $D\rho$ is conformal, that is,

$$D_X D\rho = \rho X,$$

for any vector field $X$ on $\mathbb{H}^n$. Another useful consequence of (35) is the formula

$$\text{div} (G \nabla \rho) = (n-2)\rho H + 2pK,$$

where

$$G = HI - a$$

is the Newton tensor of $a$; see [1] for further details.

We now consider an one-parameter family $X(t, \cdot) : \Sigma_{n-1} \rightarrow \mathbb{H}^n$, $t \in [0, \epsilon)$, of closed, isometrically immersed hypersurfaces evolving according to

$$\frac{\partial X}{\partial t} = F \xi,$$

where $\xi$ is the unit normal to $\Sigma_t = X(t, \cdot)$ and $F$ is a general speed function. To save notation we also denote the evolving hypersurface simply by $\Sigma$ whenever no confusion arises. The following evolution equations are well-known [35].

**Proposition 3.1.** Under the flow (38) we have:

1. The unit normal evolves as

$$\frac{\partial \xi}{\partial t} = -\nabla F;$$

2. The area element $d\Sigma$ evolves as

$$\frac{\partial}{\partial t} d\Sigma = -FH d\Sigma.$$

In particular, if $A$ is the area of $\Sigma$ then

$$\frac{dA}{dt} = -\int_\Sigma FH d\Sigma;$$

3. The mean curvature evolves as

$$\frac{\partial H}{\partial t} = \Delta F + (|\rho|^2 - (n-1))F.$$

If $\Sigma$ is star-shaped and mean convex then our conventions imply that $\xi$ is the inward pointing unit normal vector. Thus, in the model (3), $\Sigma$ can be graphically represented by means of a map of the type

$$\theta \in \mathbb{S}^{n-1} \mapsto (u(\theta), \theta) \in \mathbb{H}^n,$$

for some smooth function $u$. In particular, if $\theta = (\theta_1, \ldots, \theta_{n-1})$ is a local coordinate system on $\mathbb{S}^{n-1}$ and $E_i = \partial/\partial \theta_i$ then the tangent space to the graph is spanned by

$$Z_i = u_i \frac{\partial}{\partial r} + E_i, \quad u_i = E_i(u), \quad i = 1, \ldots, n-1,$$

so we can take

$$\xi = \frac{1}{W} \left( \frac{u^i}{\rho(u)^2} E_i - \frac{\partial}{\partial r} \right), \quad W = \sqrt{1 + \|\nabla v\|^2_{\rho}},$$

where

$$v = \varphi(u), \quad \varphi = 1/\dot{\rho},$$

and

$$D_X D\rho = \rho X,$$

for any vector field $X$ on $\mathbb{H}^n$. Another useful consequence of (35) is the formula

$$\text{div} (G \nabla \rho) = (n-2)\rho H + 2pK,$$

where

$$G = HI - a$$

is the Newton tensor of $a$; see [1] for further details.
with
\[ \dot{\rho}(u) = \sinh u; \]
see [18] or [14]. Also,
\[ p = -\frac{\sinh u}{W}. \]
(47)
Notice that \( p \leq 0 \).

From now on we assume that \( \Sigma = \Sigma_t = X(t, \cdot) \) is a one-parameter family of star-shaped, strictly mean convex hypersurfaces evolving according to (38). This assumption will be justified later on for the flows we shall consider; see Proposition 3.7 and Remark 3.6.

**Proposition 3.2.** Under the above conditions, the function \( \rho \) evolves along the flow (38) according to
\[ \frac{\partial \rho}{\partial t} = pF. \]
(48)

**Proof.** As noted above, we can graphically represent \( \Sigma \) by (43), where \( u \) is time dependent, so that (38) implies
\[ \frac{\partial u}{\partial t} = -\frac{F}{W}. \]
Since \( u = r \) along \( \Sigma \) we have
\[ \frac{\partial \rho}{\partial t} = \sinh u \frac{\partial u}{\partial t} = -\frac{\sinh u F}{W}, \]
and the result follows from (47). \( \square \)

The following proposition computes the variation of the curvature integral
\[ I(\Sigma) = \int_{\Sigma} \rho H d\Sigma \]
on the left-hand side of (5).

**Proposition 3.3.** Along the flow (38) we have
\[ \frac{dI}{dt} = 2 \int_{\Sigma} \rho F d\Sigma - 2 \int_{\Sigma} \rho K F d\Sigma. \]
(50)

**Proof.** Using Propositions 3.1 and 3.2,
\[ \frac{dI}{dt} = \int_{\Sigma} \frac{\partial \rho}{\partial t} H d\Sigma + \int_{\Sigma} \rho \frac{\partial H}{\partial t} d\Sigma + \int_{\Sigma} \rho H \frac{\partial \rho}{\partial t} d\Sigma - \int_{\Sigma} \rho H^2 F d\Sigma \]
\[ = \int_{\Sigma} \rho H F d\Sigma + \int_{\Sigma} \rho \left( \Delta F + (|a|^2 - (n-1)) F \right) d\Sigma - \int_{\Sigma} \rho H^2 F d\Sigma \]
\[ = \int_{M} \rho H F d\Sigma + \int_{\Sigma} \rho \Delta \rho d\Sigma + \int_{\Sigma} \rho \left( |a|^2 - (n-1) \right) F d\Sigma - \int_{\Sigma} \rho H^2 f d\Sigma, \]
and the result follows, after some cancelations, from (31) and (34). \( \square \)
Proposition 3.4. Along the flow (38), the support function evolves according to

\[ \frac{\partial p}{\partial t} = F \rho - \langle \nabla \rho, \nabla F \rangle. \]

As a consequence,

\[ \frac{d}{dt} \int_\Sigma p d\Sigma = n \int_\Sigma F \rho d\Sigma. \]

Proof. Using (33), (35) and (39) we compute:

\[ \frac{\partial p}{\partial t} = \frac{\partial}{\partial t} (D\rho, \xi) = F(D_\xi D\rho, \xi) + (D\rho, D_{\partial_t} \xi) = F \rho - \langle \nabla \rho, \nabla F \rangle, \]

which proves (51). Now, using this and (40),

\[ \frac{d}{dt} \int_\Sigma p d\Sigma = \int_\Sigma \frac{\partial p}{\partial t} d\Sigma + \int_\Sigma p \frac{\partial}{\partial t} d\Sigma \]
\[ = \int_\Sigma F \rho d\Sigma - \int_\Sigma \langle \nabla \rho, \nabla F \rangle d\Sigma - \int_\Sigma p F H d\Sigma \]
\[ = \int_\Sigma F \rho d\Sigma + \int_\Sigma F \Delta \rho d\Sigma - \int_\Sigma p F H d\Sigma, \]

so that (52) follows from (34). \( \square \)

The following proposition, proved in [4], plays a central role in our argument.

Proposition 3.5. If \( \Sigma \subset H^n \) is star-shaped and strictly mean convex then

\[ (n - 1) \int_\Sigma \frac{\rho}{H} d\Sigma \geq - \int_\Sigma p d\Sigma. \]

Moreover, the equality holds if and only if \( \Sigma \) is totally umbilical.

Proof. This is a rather special case of the Heintze-Karcher-type inequality proved in [4], so we merely sketch the elegant argument there. The idea is to let \( \Sigma \) flow under

\[ \frac{\partial X}{\partial t} = \rho \xi, \]

so we take \( F = \rho \) in (38). Using (42), (48), (34) and (30) we see that, as long as the flow exists,

\[ \frac{\partial}{\partial t} \frac{\rho}{H} = \frac{1}{H} \frac{\partial \rho}{\partial t} - \frac{\rho}{H^2} \frac{\partial H}{\partial t} \]
\[ = \frac{p \rho}{H} - \frac{\rho}{H^2} \left( \Delta \rho + (\rho^2 - (n - 1) \rho) \right) \]
\[ = -\frac{\rho^2}{H^2} |\rho|^2 \]
\[ \leq -\frac{\rho^2}{n - 1}, \]

so that by (40),

\[ \frac{d}{dt} \int_\Sigma \frac{\rho}{H} d\Sigma \leq -\frac{n}{n - 1} \int_\Sigma \rho^2 d\Sigma. \]
Combining this with (52) we finally get
\[
\frac{d}{dt} \left( (n - 1) \int \frac{\rho}{H} d\Sigma + \int p d\Sigma \right) \leq 0,
\]
that is, the quantity within parenthesis is monotone non-increasing along the flow (54). The next step is to investigate the asymptotic behavior of solutions of (54). This might appear problematic at first sight but the key observation is that (54) is equivalent to the standard flow by \textit{inward} parallel hypersurfaces \((F = 1)\) in the conformal metric
\[
\tilde{g}_1 = \rho^{-2} g_1 = \frac{dr^2}{\cosh^2 r} + \tanh^2 r h.
\]
Thus, any solution becomes extinct in a certain finite time \(t_* > 0\) so that
\[
\lim_{t \to t_*} (n - 1) \int \frac{\rho}{H} d\Sigma + \int p d\Sigma = 0,
\]
as desired. In fact, an additional complication arises from the fact that the flow might develop singularities before the extinction time due to the appearance of cut points but, as explained in [4], a regularization procedure can be implemented to take care of this. \(\Box\)

**Remark 3.6.** It follows from the computation above that
\[
\frac{\partial}{\partial t} \frac{H}{\rho} \geq \frac{H^2}{n - 1},
\]
which implies that strict mean convexity is preserved under (54).

From now on we specialize to the flow
\[
\frac{\partial X}{\partial t} = -\frac{\xi}{H},
\]
so that \(F = -1/H\). This is the famous \textit{inverse mean curvature flow}, which has been extensively studied in a variety of contexts [17] [33] [25] [31]. Here we will make use of recent results by Gerhardt [18] [19] for evolving hypersurfaces in hyperbolic space, which we collect below.

**Proposition 3.7.** If the initial hypersurface is star-shaped and strictly mean convex then the corresponding solution is defined for all \(t > 0\) and expands the evolving hypersurfaces toward infinity while maintaining star-shapedness and strictly mean convexity. Moreover, the hypersurfaces become strictly convex exponentially fast and also more and more umbilical in the sense that
\[
\|\kappa^i_j - \delta^i_j\| \leq Ce^{-\frac{t}{n-1}}, \quad t > 0,
\]
that is, the principal curvatures are uniformly bounded and converge exponentially fast to 1. Moreover, there exists \(f : \mathbb{S}^{n-1} \to \mathbb{R}\) smooth so that, as \(t \to +\infty\), the graphing function \(u\) satisfies
\[
\lim_{t \to +\infty} \left\| u - \frac{t}{n - 1} - f(\theta) \right\|_{C^\infty(\mathbb{S}^{n-1})} = 0.
\]
In particular,
\[
\rho(u) = \cosh u = O(e^{\frac{t}{n-1}}), \quad \dot{\rho}(u) = \sinh u = O(e^{\frac{t}{n-1}}),
\]
Remark 3.8. It is claimed in [18] that the function \( f \) above is actually a constant, which means that the flow would deform the induced metric on the hypersurface to a round one after a suitable scaling. This is, however, not correct, as the concrete example in [26] shows. The correct asymptotics (57) appears in [19].

4. THE PROOFS OF THEOREMS 1.1 AND 1.2

The proof of Theorem 1.1 involves the consideration of two new monotone quantities along the solution of (55) with \( \Sigma \) as the initial hypersurface. Thus, for any closed \( \Sigma \subset H^n \) we set

\[
\mathcal{J}(\Sigma) = - \int_{\Sigma} \rho d\Sigma,
\]

\[
\mathcal{K}(\Sigma) = \omega_{n-1} A(\Sigma) \frac{n}{\omega_n},
\]

where \( A(\Sigma) = A/\omega_{n-1} \), and

\[
\mathcal{L}(\Sigma) = \frac{\mathcal{I}(\Sigma) - (n-1)\mathcal{K}(\Sigma)}{A(\Sigma) \frac{n}{\omega_n}}.
\]

To save notation, sometimes we write \( \mathcal{I}(t) = \mathcal{I}(\Sigma_t) \), etc. As we shall see below, the new monotone quantities are \( \mathcal{L} \) and \( A^{-\frac{n}{n-1}}(\mathcal{J} - \mathcal{K}) \).

**Proposition 4.1.** On a geodesic sphere we have

\[
\mathcal{L} \geq (n-1)\omega_{n-1}.
\]

Moreover, the equality holds if and only if the geodesic sphere is centered at the origin.

**Proof.** If a geodesic sphere has radius \( r \) then its area is \( A = \omega_{n-1} \sinh^{n-1} r \) and its mean curvature is \( H = (n-1) \coth r \). Furthermore, if it is centered at the origin then its support function is \( p = -\sinh r \) by (47). The equality in (63) then follows by a direct computation. On the other hand, if \( \Sigma \subset H^n \) is any geodesic sphere of radius \( r \) then

\[
K = \frac{(n-1)(n-2)}{2} \coth^2 r,
\]

so that (66) yields

\[
\int_{\Sigma} \rho H d\Sigma = -\frac{2}{n-2} \int_{\Sigma} pK d\Sigma = -(n-1) \coth^2 r \int_{\Sigma} \rho d\Sigma.
\]

Furthermore, if \( B \) is the geodesic ball bounded by \( \Sigma \), (33), (35) and the divergence theorem imply

\[
\int_{\Sigma} \rho d\Sigma = -\int_{B} \Delta_{H^n} \rho d\mathbb{H}^n = -n \int_{B} \rho d\mathbb{H}^n,
\]

so that

\[
\mathcal{I}(\Sigma) = \int_{\Sigma} \rho H d\Sigma = n(n-1) \coth^2 r \int_{\Sigma} \rho d\mathbb{H}^n.
\]

It is clear from (62) and this way of writing \( \mathcal{I}(\Sigma) \) as a volume integral involving \( \rho \) that the strict inequality in (63) holds if \( \Sigma \) is not centered at the origin. \( \square \)
Remark 4.2. Inequality (65) above just means that the inequality in Theorem 1.1 holds for any geodesic sphere, with the equality occurring if and only if it is centered at the origin.

Proposition 4.3. If the initial hypersurface $\Sigma$ in (55) is star-shaped and strictly mean convex then

$$\frac{dA}{dt} = A$$

and

$$\frac{dK}{dt} = \frac{n}{n-1}K.$$

Also,

$$\frac{dJ}{dt} \geq \frac{n}{n-1}J,$$

with the equality occurring if and only if $\Sigma$ is totally umbilical.

Proof. The relation (65) follows from (64), which is a consequence of (41) with $F = -1/H$. Also, (66) follows immediately from (52) and (53).

The above result is crucial in establishing the existence of monotone quantities for the flow (55).

Proposition 4.4. If $\Sigma$ is star-shaped and strictly mean convex then

$$\frac{d}{dt} \left( J - K \right) \geq 0,$$

along any solution of (55). Also, in any interval where $J \leq K$ there holds

$$\frac{dL}{dt} \leq 0.$$

Moreover, if the equality holds in any of these inequalities for some $t$ then $\Sigma_t$ is totally umbilical.

Proof. By (65) and (66) we get

$$\frac{d}{dt} (J - K) \geq \frac{n}{n-1} (J - K),$$

which by (64) clearly yields (67). Moreover, by (50) with $F = -1/H$,

$$\frac{dI}{dt} = 2 \int_{\Sigma} \frac{\rho K}{H} d\Sigma + 2J,$$

so that by (32),

$$\frac{dI}{dt} \leq \frac{n-2}{n-1} I + 2J.$$

From (65), after a rearrangement of terms, we get

$$\frac{d}{dt} (I - (n-1)K) \leq \frac{n-2}{n-1} (I - (n-1)K) + 2(J - K),$$

which reduces to

$$\frac{d}{dt} (I - (n-1)K) \leq \frac{n-2}{n-1} (I - (n-1)K),$$

whenever $J \leq K$. In the presence of (64), this immediately gives (68). Finally, if the equality holds in either (67) or in (68) then it holds in (53) as well. \qed
We start the proof of Theorem 1.1 by noticing that in [5, Theorem 1.1] the authors establish a sharp geometric inequality for strictly mean convex, star-shaped hypersurfaces in the anti-deSitter-Schwarzschild space. By sending the mass parameter to zero, it follows from their work that if we set

\[ M = \frac{I - (n - 1)J}{A^{n-2}}, \]

then there holds

\[ M(\Sigma) \geq (n - 1)\omega_{n-1}, \]

for any \( \Sigma \subset \mathbb{H}^n \) strictly mean convex and star-shaped, with the equality holding if and only if \( \Sigma \) is a geodesic sphere centered at the origin. Notice that this implies (5) whenever \( J(\Sigma) \geq K(\Sigma) \), so we may assume that \( J(\Sigma) < K(\Sigma) \).

We now let \( \Sigma \) flow under (55). In case \( J(\Sigma) > K(\Sigma) \) for some \( t > 0 \), let \( t_0 \) be the first value of the time parameter so that \( J(\Sigma_{t_0}) = K(\Sigma_{t_0}) \). Notice that \( t_0 \) exists because by (67) the quantity \( A^{-\frac{n}{n-2}}(J - K) \) is monotone nondecreasing along any solution of (55). Since \( J(\Sigma_t) \leq K(\Sigma_t) \) for \( t \leq t_0 \), it then follows from Proposition 4.4 that \( L(\Sigma) \geq L(\Sigma_{t_0}) = M(\Sigma_{t_0}) \geq (n - 1)\omega_{n-1} \), where we used (70) in the last step. Thus, our main inequality (5) is also established in this case, so it remains to consider the case in which \( J(\Sigma_t) < K(\Sigma_t) \) for any \( t > 0 \). However, if this is the case then it follows again by Proposition 4.4 that \( L \) is monotone nonincreasing for all \( t > 0 \). But by Proposition A.1 we have

\[ \liminf_{t \to +\infty} L(t) \geq (n - 1)\omega_{n-1}, \]

so that

\[ L(0) \geq (n - 1)\omega_{n-1}, \]

which is just a rewriting of (5). Finally, we note that whenever the equality holds then it also holds in (55), which implies that \( \Sigma \) is a geodesic sphere necessarily centered at the origin by Remark 4.2. This completes the proof of Theorem 1.1.

As remarked in the Introduction, the Penrose inequality in Theorem 1.2 follows immediately from (13) and the Alexandrov-Fenchel inequality in Theorem 1.1. On the other hand, the rigidity statement follows from the arguments in [10, Section 5]. This completes the proof of Theorem 1.2.

### Appendix A. The Asymptotic Behavior of \( L \)

In this appendix we present a proof of the following proposition, which provides the expected limiting estimate for the quantity \( L \) along solutions of the inverse mean curvature flow. This asymptotic behavior is used in the proof of Theorem 1.1.

**Proposition A.1.** If \( \Sigma_t \) is a solution of (55) with \( \Sigma_0 \) strictly mean convex and star-shaped then

\[ \liminf_{t \to +\infty} L(\Sigma_t) \geq (n - 1)\omega_{n-1}. \]

We write the evolving hypersurfaces as graphs of a function \( u = u(t, \theta), \theta \in S^{n-1} \). Recall that \( \rho(u) = \cosh u \) so that \( \dot{\rho}(u) = \sinh u \) and

\[ \rho^2 = \dot{\rho}^2 + 1. \]
Also, if $v = \varphi(u)$, $\varphi = 1/\hat{\rho}$, as in (46), then it follows from (58) and (59) that

\begin{equation}
|\nabla v|_h + |\nabla^2 v|_h = O(e^{-\frac{t}{n-1}})
\end{equation}

and

\begin{equation}
|\rho(u) - \hat{\rho}(u)| = O(e^{-\frac{t}{n-1}}).
\end{equation}

Moreover, by (45),

\begin{equation}
W^{-1} = 1 - \frac{1}{2}|\nabla v|_h^2 + O(e^{-\frac{t}{n-1}}).
\end{equation}

The induced metric is

\begin{equation}
g_{1ij} = \hat{\rho}^2 (h_{ij} + v_iv_j),
\end{equation}

so that

\begin{equation}
\sqrt{\det g_1} = \hat{\rho}^{n-1} \sqrt{\det h} \left(1 + \frac{1}{2}|\nabla v|_h^2 + O(e^{-\frac{t}{n-1}})\right),
\end{equation}

so we get the expansions

\begin{equation}
\mathcal{A}(\Sigma_t) = \int \hat{\rho}^{n-1} + O(e^{\frac{(n-3)t}{n-1}}),
\end{equation}

and

\begin{equation}
\mathcal{A}(\Sigma_t)^{\frac{n-2}{n-1}} = \left(\int \hat{\rho}^{n-1}\right)^{\frac{n-2}{n-1}} + O(e^{\frac{(n-4)t}{n-1}}),
\end{equation}

where

\begin{equation}
\int = \frac{1}{\omega_{n-1}} \int
\end{equation}

and the integration is over $\mathbb{S}^{n-1}$.

Recall that our intention is to estimate from below the function

\begin{equation}
L(\Sigma_t) = \frac{\int_{\Sigma_t} \rho H d\Sigma_t - (n-1)\omega_{n-1} (\mathcal{A}(\Sigma_t))^{\frac{n-2}{n-1}}}{\mathcal{A}(\Sigma_t)^{\frac{n-2}{n-1}}}.\end{equation}

In terms of $v$, the second fundamental form of the evolving hypersurface is

\begin{equation}
b_{ij} = \frac{\hat{\rho}}{W} (\rho (h_{ij} + v_iv_j) - (\nabla^2 v)_{ij}).
\end{equation}

Notice also that by (76) the inverse metric is

\begin{equation}
g_1^{ij} = \hat{\rho}^{-2} \left(h_{ij} - \frac{v^i v^j}{W^2}\right),
\end{equation}

where $v^i = h^{ij}v_j$, so that the shape operator is

\begin{equation}
a^i_j = g_1^{ik}b_{kj} = \frac{\rho}{W\hat{\rho}} \delta^i_j - \frac{1}{W\hat{\rho}} \hat{h}^{ik}(\nabla^2 v)_{kj},
\end{equation}

where

\begin{equation}
\hat{h}^{ij} = h^{ij} - \frac{v^i v^j}{W^2},
\end{equation}

and from this we see that

\begin{equation}
\rho H = (n-1)W^{-1}\frac{\hat{\rho}^2}{\hat{\rho}} - W^{-1}\frac{\rho}{\hat{\rho}} \Delta v + O(e^{-\frac{2t}{n-1}}).
\end{equation}
Thus, if we combine this with (77) and (75) we obtain
\[
\int_{\Sigma_t} \rho H d\Sigma_t = (n-1) \int \rho^2 \rho_n^2 - \int \rho^{n-1} \Delta v + O(e^{\frac{n-3}{n-1}}) \\
= (n-1) \int \rho^2 \rho_n^2 + (n-1) \int \rho^{n-2} (\nabla \rho, \nabla v)_H + O(e^{\frac{n-3}{n-1}}) \\
= (n-1) \int \rho^2 \rho_n^2 + (n-1) \int \rho^2 \rho_n^2 [\nabla v]_H^2 + O(e^{\frac{n-3}{n-1}}).
\]

On the other hand, by Hölder inequality and (77) we find that
\[
\omega_{n-1} A(\Sigma_t)^{\frac{n-1}{n}} \leq \int (\sqrt{\det g_t})^{\frac{n}{n-1}} \\
= \int \rho^n + \frac{n}{2(n-1)} \int \rho^n |\nabla v|_H^2 + O(e^{\frac{n-3}{n-1}}).
\]
Thus, by (72) we obtain
\[
\int_{\Sigma_t} \rho H d\Sigma_t - (n-1) \omega_{n-1} A(\Sigma_t)^{\frac{n-1}{n}} \geq (n-1) \int \rho^{n-2} + \frac{n-2}{2} \int \rho^n |\nabla v|_H^2 \\
+ O(e^{\frac{n-3}{n-1}}) \\
= (n-1) \int \rho^{n-2} \\
+ \frac{n-2}{2} \int \rho^{n-4} |\nabla \rho|_H^2 + O(e^{\frac{n-3}{n-1}}),
\]
so if we take (79) into account we see that proving (71) amounts to checking that
\[
(n-1) \int \rho^{n-2} + \frac{n-2}{2} \int \rho^{n-4} |\nabla \rho|_H^2 \geq (n-1) \omega_{n-1}^{\frac{n-2}{n-1}} \left( \int \rho^{n-1} \right)^{\frac{n-2}{n-1}}.
\]
But, as observed in [5], this is an immediate consequence of a sharp Sobolev type inequality by Beckner [3]. This completes the proof of Proposition A.1.

**APPENDIX B. THE ASYMPTOTICALLY LOCALLY HYPERBOLIC CASE**

In this appendix we briefly describe how the argument leading to Theorem 1.2 can be easily adapted to recover its generalization given by Theorem 2.4.

Clearly, the key step is to prove the Alexandrov-Fenchel-type inequality (26) in Theorem 2.3. We start by observing that Propositions 3.2, 3.3 and 3.4 remain true with $\rho$ replaced by $\rho_\epsilon$, since their proofs only uses that the ambient manifold is locally hyperbolic and carries the conformal field $\nabla g, \rho_\epsilon$; see [35]. Also, since this ambient manifold satisfies the structural conditions in the main result in [4], the analogue of Proposition 3.5 also holds true. Taken together, these facts imply that the analogue of Proposition 4.4 still holds true, so that the proof of (26) boils down to checking that
\[
\lim_{t \to +\infty} \mathcal{L}(\Sigma_t) \geq (n-1) \vartheta_{n-1} \epsilon,
\]
where $\Sigma_t$ is the solution to the inverse mean curvature flow having $\Sigma$ as initial hypersurface; compare with (71). We note that the left-hand side of (82) makes sense
because the analogue of Proposition 3.7 remains true, which follows straightforwardly from the methods in [18, 19]. Taking into account that (72) should be replaced by
\[ \rho^2 = \dot{\rho}^2 + \epsilon, \]
we see that (82) reduces to proving that
\[ (n - 1)\epsilon \int N \rho^{n-2} + \frac{n}{2} \int N \rho^{n-1} |\nabla \rho|_h \geq (n - 1)\epsilon \omega_{n-1} \left( \int N \rho^{n-1} \right)^\frac{n-1}{n}; \]
come to (81). Since the validity of this inequality is immediate for \( \epsilon = 0, -1 \), and the rigidity statement follows from a simple adaptation of the arguments in [10, Section 5], the proof of Theorem 2.4 is completed.

REFERENCES

[1] L. J. Aliás, J. H. S. de Lira, and J. M. Malacarne. Constant higher-order mean curvature hypersurfaces in Riemannian spaces. J. Inst. Math. Jussieu, 5(4):527–562, 2006.
[2] L. Andersson, M. Cai, and G. J. Galloway. Rigidity and positivity of mass for asymptotically hyperbolic manifolds. Ann. Henri Poincaré, 9(1):1–33, 2008.
[3] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. (2), 138(1):213–242, 1993.
[4] S. Brendle. Constant mean curvature surfaces in warped product manifolds. Publ. Math. Inst. Hautes Études Sci., 117:247–269, 2013.
[5] S. Brendle, P.-K. Hung, and M.-T. Wang. A Minkowski inequality for hypersurfaces in the Anti-deSitter-Schwarzschild manifold. ArXiv e-prints, Sept. 2012.
[6] P. T. Chruściel and H. Friedrich, editors. The Einstein equations and the large scale behavior of gravitational fields. Birkhäuser Verlag, Basel, 2004. 50 years of the Cauchy problem in general relativity.
[7] P. T. Chruściel and M. Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. Pacific J. Math., 212(2):231–264, 2003.
[8] P. T. Chruściel and G. Nagy. The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times. Adv. Theor. Math. Phys., 5(4):697–754, 2001.
[9] P. T. Chruściel and W. Simon. Towards the classification of static vacuum spacetimes with negative cosmological constant. J. Math. Phys., 42(4):1779–1817, 2001.
[10] M. Dahl, R. Gicquaud, and A. Sakovich. Penrose type inequalities for asymptotically hyperbolic graphs. Ann. Henri Poincaré, 14(5):1135–1168, 2013.
[11] L. L. de Lima and F. Girão. The ADM mass of asymptotically flat hypersurfaces. Trans. Amer. Math. Soc., electronically published on October 3, 2014, DOI: http://dx.doi.org/10.1090/S0002-9947-2014-05902-3 (to appear in print).
[12] L. L. de Lima and F. Girão. A rigidity result for the graph case of the Penrose inequality. ArXiv e-prints, May 2012.
[13] L. L. de Lima and F. Girão. Positive mass and Penrose type inequalities for asymptotically hyperbolic hypersurfaces. Gen. Relativity Gravitation, 47(3):47:23, 2015.
[14] Q. Ding. The inverse mean curvature flow in rotationally symmetric spaces. Chin. Ann. Math. Ser. B, 32(1):77–44, 2011.
[15] E. Gallego and G. Solanes. Integral geometry and geometric inequalities in hyperbolic space. Differential Geom. Appl., 22(3):315–325, 2005.
[16] Y. Ge, G. Wang, J. Wu, and C. Xia. A Penrose inequality for graphs over Kottler space. Calc. Var. Partial Differential Equations, 52(3-4):755–782, 2015.
[17] C. Gerhardt. Flow of nonconvex hypersurfaces into spheres. J. Differential Geom., 32(1):299–314, 1990.
[18] C. Gerhardt. Inverse curvature flows in hyperbolic space. J. Differential Geom., 89(3):487–527, 2011.
[19] C. Gerhardt. Inverse curvature flows in hyperbolic space. ArXiv e-prints, Jan. 2011.
[20] G. W. Gibbons. Some comments on gravitational entropy and the inverse mean curvature flow. Classical Quantum Gravity, 16(6):1677–1687, 1999.
[21] P. Guan and J. Li. The quermassintegral inequalities for k-convex starshaped domains. Adv. Math., 221(5):1725–1732, 2009.
[22] G. H. Hardy, J. E. Littlewood, and G. Pólya. Inequalities. Cambridge, at the University Press, 1952. 2d ed.
[23] M. Herzlich. Mass formulae for asymptotically hyperbolic manifolds. In AdS/CFT correspondence: Einstein metrics and their conformal boundaries, volume 8 of IRMA Lect. Math. Theor. Phys., pages 103–121. Eur. Math. Soc., Zürich, 2005.

[24] L.-H. Huang and D. Wu. The equality case of the Penrose inequality for asymptotically flat graphs. Trans. Amer. Math. Soc., 367(1):31–47, 2015.

[25] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differential Geom., 59(3):353–437, 2001.

[26] P.-K. Hung and M.-T. Wang. Inverse mean curvature flows in the hyperbolic 3-space revisited. ArXiv e-prints, June 2014.

[27] M.-K. G. Lam. The Graphs Cases of the Riemannian Positive Mass and Penrose Inequalities in All Dimensions. ArXiv e-prints, Oct. 2010.

[28] D. A. Lee and A. Neves. The Penrose inequality for asymptotically locally hyperbolic spaces with nonpositive mass. ArXiv e-prints, Oct. 2013.

[29] M. Mars. Present status of the Penrose inequality. Classical Quantum Gravity, 26(19):193001, 59, 2009.

[30] B. Michel. Geometric invariance of mass-like asymptotic invariants. J. Math. Phys., 52(5):052504, 14, 2011.

[31] A. Neves. Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds. J. Differential Geom., 84(1):191–229, 2010.

[32] R. Schneider. Convex bodies: the Brunn-Minkowski theory, volume 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1993.

[33] J. I. E. Urbas. On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z., 205(3):355–372, 1990.

[34] X. Wang. The mass of asymptotically hyperbolic manifolds. J. Differential Geom., 57(2):273–299, 2001.

[35] X.-P. Zhu. Lectures on mean curvature flows, volume 32 of AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2002.

Universidade Federal do Ceará (UFC), Departamento de Matemática, Campus do Pici, Av. Humberto Monte, s/n, Bloco 914, 60455-760, Fortaleza, CE, Brazil.

Email address: levi@mat.ufc.br

Universidade Federal do Ceará (UFC), Departamento de Matemática, Campus do Pici, Av. Humberto Monte, s/n, Bloco 914, 60455-760, Fortaleza, CE, Brazil.

Email address: fred@mat.ufc.br