Quasiparticle localization in superconductors with spin-orbit scattering

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We develop a theory of quasiparticle localization in superconductors in situations without spin rotation invariance. We discuss the existence, and properties of superconducting phases with localized/delocalized quasiparticle excitations in such systems in various dimensionalities. Implications for a variety of experimental systems, and to the properties of random Ising models in two dimensions, are briefly discussed.

I. INTRODUCTION

A powerful probe of the properties of a superconductor is obtained by studying the low temperature dynamics of the quasiparticles. In this context, we proposed \cite{1} that all ground state phases of disordered superconductors can be characterized, at zero temperature, by their quasiparticle transport properties. The two general possibilities are that the quasiparticle excitations may be delocalized, analogous to a metal, or be localized analogous to an insulator. Previous papers \cite{2,3} have developed a theory of localization of quasiparticles in superconductors in situations with spin rotation invariance. In this paper, we consider the case where the spin is not conserved. This may happen, for instance, in a singlet superconductor in the presence of spin-orbit scattering. Another example is provided by a triplet superconductor where the quasiparticles can exchange spin with the condensate, and hence, do not have conserved spin. Indeed, a number of superconducting systems, such as for instance the heavy fermion superconductors, are both strongly disordered and have strong spin-orbit scattering (and perhaps even triplet pairing). Thus, in order to understand the possibility of quasiparticle localization in such systems, it is necessary to develop a theory that includes spin-orbit scattering in an essential way. Besides, by analogy with what happens in normal metals, spin-orbit scattering is expected to have profound effects on localization phenomena.

Because the quasiparticle charge density is also not conserved in a superconductor, the only conserved quantity carried by the quasiparticles (at low energies) is the energy density itself. In the presence of impurity scattering, the quasiparticle charge and spin densities in such a superconductor do not diffuse, as they are not conserved. Energy diffusion is possible, though. The corresponding transport quantity is the quasiparticle thermal conductivity.

From a theoretical point of view, quasiparticles in a superconductor with non-conserved spin are more appropriately thought of as real (Majorana) fermions. Thus the problem we consider here is one of localization of Majorana fermions. While localization issues of complex (conventional) fermions have been explored in considerable detail, surprisingly, there has been very little theoretical work on corresponding issues for Majorana fermions. As we argue, the superconductor with non-conserved spin provides a natural experimental realization of such a system. We examine the possible phases (as characterized by quasiparticle transport) and the associated phase transitions. It is of interest to distinguish between situations with and without time reversal symmetry \( T \), and we will consider each separately. (In the notation of Ref. \cite{4}, these correspond to Class DIII and Class D respectively).

While the superconductor is our primary motivation, we note also that Majorana fermions arise in other contexts as well. A well-known example is the two dimensional Ising model. There have been several studies \cite{5} of the properties of the two dimensional Ising model in the presence of randomness in the bond strengths, though there are still several poorly understood issues. The implications of this work for that problem will be considered briefly towards the end of the paper.

We first show the existence, in two dimensions, of stable “metallic” and “insulating” phases inside the superconductor with delocalized and localized quasiparticle excitations respectively. The stability of the “metallic” phase in two dimensions has already been alluded to in Ref. \cite{1}. We emphasize that both phases are superconducting - they are distinguished by the nature of thermal transport due to the quasiparticles. These two phases are separated by a phase transition which is a novel “metal-insulator” transition inside a superconductor. In Ref. \cite{1}, we discussed possible experimental realizations of such phase transitions. The universal critical properties of this transition depend, of course, on whether time reversal is a good symmetry or not. In \( T \)-non-invariant systems, the insulating phases may be further characterized in terms of their values of the Hall thermal conductance. The dimensionless ratio \( \frac{3h\kappa_{xy}}{\pi k_B T} \) is quantized in units of \( 1/2 \). Phases with different values of this quantized Hall thermal conductance are topologically distinct, and are separated by phase transitions. In Figure 1, we show a schematic phase diagram. Note that, in the case with no \( T \), apart from the “metal-insulator” transition, there are also transitions between the “metal” and the “quantum Hall” phase, and from the insulator with \( \kappa_{xy}/T = 0 \) to
the “quantum Hall” phase. Further, there is a multicritical point, where all three phases come together. In three dimensional systems, “metallic” and “insulating” phases are again possible with the transition between the two being in a new universality class for localization. In the rest of the paper, we will substantiate these claims, and analyse the properties of each phase in further detail.

**SUPERCONDUCTING THERMAL INSULATOR**

**SUPERCONDUCTING THERMAL QUANTUM HALL FLUID**

**SUPERCONDUCTING THERMAL METAL**

FIG. 1. Schematic zero temperature phase diagram for the two dimensional superconductor in the absence of both spin rotation and time reversal invariances.

**II. MODELS AND GENERAL FORMALISM**

Consider a general lattice Hamiltonian for the quasiparticles in a superconductor with strong spin-orbit scattering:

\[ \mathcal{H} = \sum_{ij} t_{ij} \hat{c}_{i\alpha}^\dagger \hat{c}_{j\beta} + \Delta_{ij} \hat{c}_{i\alpha}^\dagger \hat{c}_{j\beta}^\dagger + h.c. \]  

(1)

Here \( i, j \) refer to the sites of a lattice, and \( \alpha, \beta \) are spin indices. We assume that \( t \) and \( \Delta \) are both short-ranged in space.

We will focus on two cases - with and without time reversal symmetry (\( \mathcal{T} \)). If present, time reversal is imposed through an antiunitary “time-reversal” operator \( \mathcal{T} \) which transforms the \( c \) operators as:

\[ \mathcal{T} \begin{bmatrix} \hat{c}_{i\uparrow}^\dagger \\ \hat{c}_{i\downarrow} \end{bmatrix} = \begin{bmatrix} \hat{c}_{i\downarrow}^\dagger \\ -\hat{c}_{i\uparrow} \end{bmatrix} = i\sigma_y c. \]  

(2)

Note that \( \mathcal{T}\sigma^\ast \mathcal{T}^{-1} = -\sigma \) (with \( \sigma \) a vector of Pauli matrices) so that the electron spin is odd under time reversal. With time reversal invariance present we require that

\[ \mathcal{T}\mathcal{H}\mathcal{T}^{-1} = \mathcal{H}. \]  

(3)

Note that \( \mathcal{H} \) has no special symmetries (other than possibly \( \mathcal{T} \)). In particular, neither the charge nor spin is conserved. It is convenient then to work with Majorana fermions \( \eta_{1\alpha}, \eta_{2\alpha} \) defined through

\[ c_{i\alpha} = \frac{1}{\sqrt{2}} (\eta_{1i\alpha} + i\eta_{2i\alpha}). \]  

(4)

The Hamiltonian when expressed in terms of the \( \eta \) Fermions takes the form

\[ \mathcal{H} = \eta^T \mathcal{H} \eta, \]  

(5)

with \( \eta = \eta_{1a}, (a, b = 1, 2) \) and \( \mathcal{H} = H_{ij}^{ab,\alpha,\beta} \) is a matrix in \((ij), (\alpha\beta), ab\) space. By definition, \( \mathcal{H} \) is hermitian and moreover \( \mathcal{H}^T = -\mathcal{H} \), so that \( \mathcal{H} \) is pure imaginary. Thus the problem of quasiparticle localization in a superconductor with spin-orbit scattering is, in essence, one of localization of Majorana fermions. In particular, in the case where even \( \mathcal{T} \) is not a good symmetry, the Hamiltonian Eqn. 3 is the most general one describing non-interacting Majorana fermions in a disordered system.

Time reversal symmetry is easily imposed on the Majorana Hamiltonian Eqn. 3. Under the action of the antiunitary operator \( \mathcal{T} \), it is readily seen that

\[ \mathcal{T}\eta_{1a} = i(\sigma_y)_{1a}(\tau_z)_{ab} \eta_{2b}. \]  

(6)

Here \( a, b = 1, 2 \) and \( \tau \) is a Pauli matrix in \( ab \) space. The time reversal invariance condition Eqn. 3 combined with Eqn. 3 implies

\[ \sigma_y \tau_z H^* \sigma_y \tau_z = H. \]  

(7)

We note in passing that for a superconductor of spin polarized Fermions (appropriate, say, in the \( A-1 \) phase of superfluid 3 - \( He \)) in the absence of time reversal invariance, the general Hamiltonian can still be written in the form \( \mathcal{H} = \eta^T \mathcal{H} \eta \), with \( \mathcal{H} \) an antisymmetric and pure imaginary matrix in position \( (i, j) \) and “particle-hole” \( (ab) \) space. But for spinless quasiparticles the condition for time reversal invariance is different, leading to different symmetries. Specifically, with time reversal invariance the lattice Hamiltonian depends on a real symmetric hopping matrix, \( t_{ij} \), and a real antisymmetric gap matrix, \( \Delta_{ij} \). In this case, when re-expressed in terms of Majorana fermions, the Hamiltonian becomes \( \mathcal{H} = \eta^T H \eta \) with \( H = t \tau_y + i \Delta \tau_x \). This Hamiltonian matrix can equivalently be expressed as \( H = iA \tau^+ + c.c. \), with \( \tau^+ = \tau_z + i\tau_y \) and \( A = t + \Delta \) an arbitrary real matrix. This “off-diagonal” form is very different than the two cases with spinful electrons, and in fact belongs to a different symmetry class - a class studied by Gade and Wegner. Henceforth, we focus exclusively on the spinful case.

We are interested in understanding the nature of energy transport (in both spinful cases, with and without \( \mathcal{T} \)) by the excitations described by the Hamiltonian. For this purpose, it is actually convenient to adopt the following trick. We consider two identical copies of the system. To describe the two copies, we introduce two Majorana fields \( \eta \) and \( \zeta \) and consider
\[ \tilde{H} = \eta^T H \eta + \zeta^T H \zeta. \]  

It is now possible to combine the \( \eta \) and \( \zeta \) fields into a single complex fermion \( f \):

\[ f = \frac{1}{\sqrt{2}} (\eta + i \zeta), \]  

\[ f^\dagger = \frac{1}{\sqrt{2}} (\eta - i \zeta). \]

Then, we have

\[ \tilde{H} = f^\dagger H f. \]  

Note that the number of \( f \) particles is conserved. It is then possible to consider transport of this conserved \( f \)-number density. This may be quantified by a conductivity \( \kappa_f \) of the \( f \) particles related to \( \sigma_f \) by a Weiedemann-Franz law (as \( T \to 0 \)):

\[ \frac{\kappa_f}{T \sigma_f} = \frac{\pi^2 k_B}{3}. \]

Since the Hamiltonian \( \tilde{H} \) represents just two identical copies of the original system described by \( H \), it is clear that the thermal conductivity \( (\kappa) \) of the \( \eta \) particles is exactly half that of the \( f \)-particles:

\[ \kappa = \frac{\kappa_f}{2}. \]

Combining this with Eqn. (12) we see that calculation of \( \kappa \) is reduced to calculation of \( \sigma_f \). Similarly, all the thermodynamic properties of \( \tilde{H} \) may be obtained by halving the corresponding property calculated with \( H \).

It is convenient to define the \( f \)-Green’s function

\[ G_{ij}^{ab, \alpha \beta}(x) = \langle \langle \langle a | \frac{1}{E-H+i \epsilon} | b \rangle \beta \rangle \alpha \rangle \rangle, \]

where \( \epsilon \) is a positive infinitesimal and \( E \) is the energy (measured from the Fermi energy). The condition \( H^* = -H \) immediately implies that

\[ G^*(E) = -G(-E), \]

so that \( G(E = 0) \) is pure imaginary. This Green’s function can be conveniently expressed, as usual, as a functional integral over Grassmann variables \( f_{ia}, f_{ia}^\dagger \) with the action

\[ S = i \int f (H - i \epsilon) f. \]

The symmetry Eqn. (10) implies that this same generating functional can be used to calculate transport quantities.

### III. SYSTEMS WITH TIME REVERSAL SYMMETRY

The discussion has so far been completely general. We now specialize to the case with \( T \) symmetry.

#### A. “Metallic” Phase

We first consider the situation in which the disorder is weak. We assume that there is a finite, non-zero mean free path \( l_\epsilon \) set by the impurity strength in such a manner that the \( f \)-particle motion is diffusive on larger scales. (In terms of the original physical system, this corresponds to diffusion of energy). We are interested in describing the effects of quantum interference on this diffusive motion. We assume also that, in the absence of quantum interference effects, the density of states at the Fermi energy of this diffusive system is finite and non-zero. In that case, it is possible to follow standard techniques to derive a replica non-linear sigma model field theory to describe the physics at length scales large compared to the mean free path \( l_\epsilon \). As the procedure is sufficiently well-known, we merely state the results. The field theory is described by the action

\[ S = \int d^d x \frac{1}{2g} \text{Tr} (\nabla O)^T (\nabla O) - c \text{Tr}(O + O^T), \]

where \( O(x) \in O(2n) \) is a \( 2n \times 2n \) orthogonal matrix-valued field with \( n \) the number of replicas. When \( \epsilon = 0 \) the action has a global \( O(2n) \times O(2n) \) symmetry, \( O \to AOB \) with \( A \) and \( B \) orthogonal matrices, which is broken down to the diagonal \( O(2n) \) \( (A^T = B) \) by the \( \epsilon \) term. The coupling constant \( g \) is inversely proportional to the bare \( f \)-conductivity \( \sigma_f \) (i.e the conductivity on the scale of the mean free path). The density of quasiparticle states at the Fermi energy is given by

\[ \rho = \frac{\rho_0}{4n} \text{Tr}(O + O^T). \]

Consider a renormalization group transformation where short distance fluctuations are integrated out, and the coordinate \( x \) is rescaled as \( x \to x' = xe^{-l} \). The leading quantum interference corrections to diffusion can now be obtained from the known \( \beta \) perturbative \( \beta \) function of this field theory in the replica limit. The result, in two dimensions, is

\[ \frac{dg}{dl} = -\frac{g^2}{4\pi}. \]
length scales. For a system of size $L$, at zero temperature, we may integrate the flow equation up to a scale $l^*$ given by $l_e e^{l^*} \sim L$ to get

$$ g(L) = \frac{g_0}{1 + \frac{2\pi}{4\pi} \ln \left( \frac{L}{l_e} \right)} \quad (21) $$

where $g_0$ is the bare value of $g$. For large $L$, this therefore gives

$$ g(L) \approx \frac{4\pi}{\ln \left( \frac{L}{l_e} \right)} \quad (22) $$

Thus $g(L)$ goes to zero logarithmically with the system size. As $\sigma_f$ is inversely proportional to $g$, it follows that $\sigma_f$ diverges logarithmically with the system size. At finite temperature, in an infinite system, it is natural to expect that the quantum interference effects will be cut-off at a finite dephasing length scale $L_0 \sim T^{-\rho}$ due to interaction effects not included in the model. We therefore have

$$ \sigma_f \sim \ln \left( \frac{1}{T} \right) \quad (23) $$

at the lowest temperatures. As this also determines the thermal conductivity $\kappa(T)$ of the original system, we have

$$ \kappa \sim \ln \frac{1}{T} \quad (24) $$

The considerations above establish the existence of a “metallic” phase with delocalized quasiparticle excitations in two dimensions in the model Hamiltonian Eqn. describing the superconductor in the presence of time reversal invariance, but no spin rotation invariance. In striking contrast to normal metals, quantum interference effects also lead to singular corrections to the density of states in a superconductor [3]. In our previous work [3], we demonstrated this in the spin-rotation invariant cases. We now show that the density of states is enhanced in the situation considered in this paper. In particular, we show that in two-dimensions it actually diverges in the thermodynamic limit.

To see this, consider the action Eqn. at finite $\epsilon$. The density of states is obtained from

$$ \rho = -\frac{1}{4n} \left[ \frac{\partial F}{\partial \epsilon} \right]_{\epsilon = 0} \quad (25) $$

where $F$ is the free energy density defined through

$$ \exp (-L^d F) = \int D\Omega \ exp (-S) \quad (26) $$

The flow of $\epsilon$ under the renormalization group may be obtained straightforwardly. To leading order in $g$ the result is

$$ \frac{d\epsilon}{dl} = \left( 1 + \frac{g}{8\pi} \right) \epsilon \quad (27) $$

This is readily integrated to give

$$ \epsilon(l) = \epsilon(0) \exp \left( 2l + \frac{1}{8\pi} \int_0^l g(l') dl' \right) \quad (28) $$

The free energy density scales according to

$$ F(g(0), \epsilon(0)) = e^{-2l F(g(l), \epsilon(l))} \quad (29) $$

where we have specialized to two-dimensions. For a system of size $L$, run the RG till a scale $l^*$ such that $l_e e^{l^*} \sim L$. The density of states is then

$$ \rho(L) = e^{-2l F(g(l^*), \epsilon(l^*))} \quad (30) $$

After scaling out to $l^*$ the mean free path is comparable to the system size, so that $-\frac{1}{4n} \frac{\partial F(g(l), \epsilon(l))}{\partial \epsilon(l)} \to const.$ Therefore,

$$ \rho(L) \sim \exp \left( \frac{1}{8\pi} \int_0^{l^*} dl' g(l') \right) \quad (31) $$

For large $l$, $g(l) \approx \frac{4\pi}{l_e}$. Thus, we have

$$ \rho(L) \sim \sqrt{\ln \frac{L}{l_e}} \quad (32) $$

Thus the density of states at the Fermi energy diverges. The behaviour of the density of states as a function of energy in an infinite system may also be found similarly (or simply guessed from the equation above) to be

$$ \rho(E) \sim \sqrt{\ln \frac{1}{E}} \quad (33) $$

The Einstein relation can now be used to infer that the heat diffusion constant $D$ also diverges on approaching the Fermi energy as

$$ D(E) \sim \sqrt{\ln \frac{1}{E}} \quad (34) $$

Thus the “metallic” phase has an infinite heat diffusion constant at zero temperature in two dimensions.

The divergence of the density of states has obvious consequences for the low temperature thermodynamic properties of this phase. For instance, the specific heat behaves as

$$ C(T) \sim T \sqrt{\ln \frac{1}{T}} \quad (35) $$
In three dimensional systems, the stability of the “metallic” phase can be established by simple power-counting arguments. Quantum interference effects are then irrelevant at long length scales. The thermal conductivity then goes to zero linearly with the temperature at low $T$:

$$\frac{\kappa_{3D}}{T} \rightarrow \text{const.}$$  \hfill (36)

The density of states at the Fermi energy is finite, and non-zero. However, quantum interference effects do lead to a singular $\sqrt{|E|}$ cusp in the density of states as a function of the energy (See Ref. [1] for an analogous discussion in the superconductor with conserved spin) so that

$$\rho(E) - \rho(E = 0) \sim -\sqrt{|E|}.$$  \hfill (37)

Note that the density of states increases with decreasing energy as also happens in $d = 2$.

**B. “Insulating” phase**

For strong disorder, it is possible to have a phase with localized quasiparticle excitations in any finite dimension. In the terminology of Ref. [1], this is a superconducting “thermal insulator”. In this phase, the ratio $\frac{\Gamma}{\Delta}$ goes to zero with the temperature. The density of quasiparticle states vanishes, at least as fast as $|E|$ in a finite system of size $L$. The distribution of a distribution has vanishing weight at $a = b = 0$. Then, the density of states $\rho(E)$ averaged over the disorder vanishes as $|E|$. If the distribution has vanishing vanish weight at $a = b = 0$, then $\rho(E)$ vanishes faster than linearly. Including hopping between the sites should not change this result so long as we are in the localized phase. (As with the superconductors with conserved spin, having a finite density of states requires a diverging weight at $a = b = 0$ which is presumably unphysical, and non-generic). We thus conclude that the density of quasiparticle states vanishes, at least as fast as $|E|$ in the localized phase.

**IV. TIME REVERSAL BROKEN SYSTEMS**

We now move on to systems without time reversal symmetry. As mentioned earlier, this corresponds to studying the general localization properties of a non-interacting disordered system of Majorana fermions. It is therefore appropriate to call the phase with extended states a “Majorana metal”, and the phase with localized states a “Majorana insulator”.

**A. Majorana metal**

To address issues such as the stability and properties of the Majorana metal, it is useful to think in terms of a replica sigma model field theory which permits a systematic study of quantum interference corrections to diffusive energy transport. This is readily done using standard techniques. The result is

$$S = \int d^2 x \frac{1}{2g} Tr (\partial Q)^2 - c Tr (\sigma y Q),$$  \hfill (38)

where $Q = V^T \sigma y V$ is a $2n \times 2n$ matrix, $V$ is an $O(2n)$ matrix and $n$ is again the replica index. When $\epsilon = 0$, the action is invariant under the group $O(2n)$, $Q \rightarrow W^T Q W$, with $W$ an orthogonal matrix. With non-zero $\epsilon$, invariance of the action requires $W = \exp(iA) \times G$ with $G$ pure imaginary and antisymmetric of the form, $W = S \sigma_y + iA$. Here $S$ is an $n$ by $n$ real symmetric matrix and $A$ is an $n$ by $n$ real antisymmetric. Since $S + iA$ is $n$ by $n$ Hermitian, the group has evidently been broken down to $U(n)$ by the $\epsilon$ term. Thus $S$ describes a non-linear sigma model on the manifold $O(2n)/U(n)$. The coupling constant $g$ is again inversely proportional to the (longitudinal) $f$-particle conductance. The density of states of the quasiparticles may be obtained from

$$\rho = \frac{1}{2n} \left< \text{Tr} (\sigma_y Q) \right>.$$  \hfill (39)

Actually, on symmetry grounds, there is a topological term allowed in the action. This follows from the observation that $\Pi_2 (O(2n)/U(n)) = Z$ is non-trivial. This will be important to understand the behaviour of the thermal Hall conductivity later.

The perturbative RG flows of the couplings $g$ and $\epsilon$ in two dimensions in the replica limit are

$$\frac{dg}{dl} = -\frac{g}{8\pi},$$  \hfill (40)

$$\frac{d\epsilon}{dl} = \left(2 + \frac{g}{8\pi}\right) \epsilon.$$  \hfill (41)

We may use these to extract the physical properties of the Majorana metal phase in exactly the same manner as in the previous section. We therefore simply state the results. First note that the (marginal) irrelevance of the coupling $g$ implies the stability of the Majorana metal in $d = 2$. As the temperature goes to zero, the (longitudinal) thermal conductivity $\kappa_{xx}$ in a finite system of size $L$ behaves as

$$\frac{\kappa_{xx}}{T} \sim \ln \left(\frac{L}{l_\epsilon}\right).$$  \hfill (42)
In an infinite system at finite $T$, where the quantum interference is cut-off by dephasing due to interaction effects, we have
\[
\frac{\kappa_{xx}}{T} \sim \ln \left( \frac{1}{T} \right).
\] (43)

Similarly, the density of states at the Fermi energy now diverges logarithmically with the system size:
\[
\rho(L) \sim \ln \frac{L}{L_c}. \tag{44}
\]

As a function of energy in an infinite system, we have
\[
\rho(E) \sim \ln \frac{1}{E}. \tag{45}
\]

This leads, for instance, to a specific heat which depends on temperature as
\[
\frac{C(T)}{T} \sim \ln \frac{1}{T}. \tag{46}
\]

Since $\kappa_{xx}/T$ and the density of states diverge in the same manner with energy, determining the behavior of the thermal diffusion coefficient would require going to second order in the perturbative RG. The diffusion coefficient could then also diverge, but not more rapidly than double logarithmically with energy.

In three dimensions, the behaviour of the superconducting “metal” phase is qualitatively the same irrespective of whether time reversal symmetry is present or not. Therefore, the discussion in the previous section of the three dimensional case applies here as well.

**B. Majorana insulator**

At strong disorder, in any dimension, it is possible to find phases where the quasiparticle excitations are localized. The (longitudinal) quasiparticle heat conductivity $\kappa_{xx}$ goes to zero rapidly with the temperature, in such a phase. We will call this the Majorana insulator, as this corresponds to a localized phase of Majorana fermions. The density of states of the Majorana insulator may be found by an argument analogous to the one in Section III B. In brief, we consider a single site Hamiltonian for two Majorana species (corresponding to a single spinless complex Fermion) which is constrained to be of the form $H = a \tau_y$ with $a$ real and random. If the distribution of $a$ has a finite, non-zero weight at $a = 0$, then there is a finite density of states at zero energy. We again expect this to hold through out the localized phase, so that the density of states at the Fermi energy can be non-zero. Note that, like in a conventional insulator in non-superconducting systems, a density of states that vanishes at the Fermi energy is also possible.

**C. Majorana quantum Hall phase**

In the absence of time reversal symmetry, there is potentially another transport property which can be used to distinguish zero temperature phases: the thermal Hall conductivity $\kappa_{xy}$. In particular, in the Majorana insulator in two dimensions, the ratio $\kappa_{xx}/T$ approaches quantized values as the temperature goes to zero. Phases with different values of the quantized thermal Hall conductance are topologically distinct and are separated by phase transitions. In recent work with X.G. Wen [10], we studied the physics of this novel quantum Hall system in some detail.

From the point of view of the replica non-linear sigma model field theory discussed earlier in this section, the existence of insulating phases with quantized thermal Hall conductance can be attributed to the presence of a topological term in the action. This takes the form
\[
S_{top} = \int d^2x \frac{\theta}{16\pi} Tr (Q \partial_x Q \partial_y Q). \tag{47}
\]

The quantity multiplying $\theta$ is equal to $i \times \text{integer}$ for any given configuration of the $Q$ field. Thus the partition function is periodic under $\theta \rightarrow \theta + 2\pi m$ for any integer $m$. Physically, just as in the Pruisken field theory for the conventional integer quantum Hall transition, $\theta$ is proportional to the (bare) value of the $f$-particle Hall conductivity $\sigma_{xy}^f$. As shown in Ref. [10], this is in turn proportional to the ratio of the thermal Hall conductivity to the temperature.

**V. PHASE DIAGRAM AND TRANSITIONS**

We now discuss the (zero temperature) phase diagram and phase transitions for the dirty superconductor with non-conserved spin. Again, we consider the cases with and without $T$ separately.

**A. Time reversal invariant systems**

As argued in the previous section, “metallic” and “insulating” phases are possible in both $d = 2$ and $d = 3$. A schematic phase diagram as a function of the bare thermal conductivity is as shown below in Figure 2.

\[ \text{SC- THERMAL METAL} \quad \underbrace{\ldots}_{\kappa_{xy}^f} \quad \text{SC- THERMAL INSULATOR} \]

FIG. 2. Schematic zero temperature phase diagram for the three dimensional superconductor in the presence of spin orbit scattering.
The critical points separating the two phases are in new universality classes for Anderson localization. There are some interesting differences between the scaling properties of the two and three dimensional systems. In three dimensions, the ratio \( \frac{\kappa}{T} \), the thermal diffusion coefficient, \( D_{xy} \), and the density of states \( \rho_0 \) at the Fermi energy are finite constants in the “metal” and are zero in the “insulator”. It is therefore natural to expect that these will go to zero continuously (with some universal critical exponent) as the transition is approached from the “metallic” side. In two dimensions however, in the “metal”, these quantities are all infinite in the limit of zero temperature and infinite system size. (They are zero in the “insulator”). Exactly at the transition point, conventional scaling arguments imply a constant value for \( \frac{\kappa}{T} \). The behaviour of the density of states and diffusion coefficient is more unclear. It is perhaps natural to suggest that these will be a constant, though we certainly cannot rule out other possibilities.

B. Time reversal non-invariant systems

We first consider the three dimensional case. The phase diagram is similar to that above for the time reversal invariant system. However, as the density of states may be non-zero in the “insulator”, we expect that it generically is non-zero at the transition point as well.

The two dimensional case has a richer phase diagram due to the possibility of quantum Hall phases. For simplicity, we show, in Figure 1, only the two insulating phases with \( \kappa_{xy} = 0 \) and \( \frac{\kappa_{xy}}{T} = \frac{1}{2} \) (in units of \( \frac{e^2k_B^2}{4m} \)). Note that there are three distinct phase transitions, and a multicritical point in this phase diagram. First consider the Majorana metal-insulator transition. Both \( \frac{\kappa}{T} \) and \( \rho_0 \) are infinite in the Majorana metal. In the Majorana insulator, \( \frac{\kappa}{T} \) is zero while \( \rho_0 \) may be non-zero. As in the \( \mathcal{T} \) invariant case, conventional scaling arguments imply a finite value for \( \frac{\kappa}{T} \) at the transition point. For the density of states, it is natural to guess that it is finite at the transition, though again it is hard to rule out other possibilities.

Now consider the transition from the Majorana metal to the quantum Hall phase. This transition is in the same universality class as the transition from the Majorana metal to the insulator with \( \kappa_{xy} = 0 \). The quantum Hall phase is a Majorana insulator with \( \frac{\kappa_{xy}}{T} = \frac{1}{2} \). The distinction between these two insulating phases is due to the presence of (heat) current carrying edge states - these are expected to be unimportant in determining the properties of the transition to the metal. Below, we will provide a more formal argument in support of this claim.

Finally, consider the transition between the insulator with \( \kappa_{xy} = 0 \) and the one with \( \frac{\kappa_{xy}}{T} = \frac{1}{2} \). A theory for this transition is obtained by considering a particular regularized of the two phases. Consider a system of spinless fermions paired into a \( p_x + ip_y \) superconducting state in two dimensions. In Ref. [10], it was shown that such a superconductor has a quantized thermal Hall conductivity \( \frac{\kappa_{xy}}{T} = \frac{1}{2} \) if the chemical potential is positive (the “BCS” limit), and has \( \kappa_{xy} = 0 \) if the chemical potential is negative (“molecular” limit). Thus, as the chemical potential is varied through zero, there is a transition from the Majorana quantum hall phase to the Majorana insulator. Ref. [10] also examined the theory for this transition, and argued that, at least at weak disorder, it is correctly described by a theory of relativistic Majorana fermions with random mass. It is well-known that the random mass is irrelevant at the pure free Majorana fixed point. Thus the critical theory is known exactly in this case. If we make the important assumption that there is a unique fixed point describing the Majorana insulator-quantum Hall transition, then the arguments above identify it with the free relativistic Majorana fixed point. There is, however, some reason to question this assumption - see Section VI.

Some more insight into the phase diagram and the transitions comes from considering the properties of the replica non-linear sigma model field theory describing the quasiparticles in the two dimensional superconductor with spin orbit scattering, and no \( \mathcal{T} \). This is described by the action Eqn. 38 supplemented with the topological term Eqn. 39. The small \( g \) regime is described by the perturbative calculations of the previous section. Note that the presence of the topological term does not affect those results. Indeed, the parameter \( \theta \) plays no role in perturbation theory, and does not renormalize. Thus there is a line of fixed points in the \( g, \theta \) plane at \( g = 0 \) with \( \theta \) arbitrary. As the value of \( \theta \) is proportional to the value of \( \frac{\kappa}{T} \), this ratio varies continuously in the Majorana metal.

Now consider the behaviour of the sigma model at large \( g \). It is natural to expect that, in this limit, the physics is captured by a strong-coupling expansion, and that the resulting phase corresponds, physically, to a localized phase. (For conceptual purposes, it may be convenient to think in terms of a lattice regularized version of the field theory.) First consider \( \theta = 0 \). The corresponding localized phase has \( \kappa_{xy} = 0 \). Similarly, the insulator with \( \theta = 2\pi \) has \( \frac{\kappa}{T} = \frac{1}{2} \). Just as in other sigma model field theories with topological terms, the large \( g \) phase for \( 0 \leq \theta < \pi \) is expected to be continuously connected to the large \( g \) phase at \( \theta = 0 \), i.e it is an “insulator” with \( \kappa_{xy} = 0 \). Similarly the large \( g \) phase with \( \pi < \theta \leq 2\pi \) is expected to be continuously connected to the large \( g \) phase with \( \theta = 2\pi \), i.e an “insulator” with \( \frac{\kappa}{T} = \frac{1}{2} \). The transition between the two localized phases occurs at \( \theta = \pi \). The small \( g \) “metallic” phase is separated from these localized phases at large \( g \) by a phase boundary. The symmetry \( \theta \to 2\pi - \theta \) implies that this phase boundary be symmetric about the \( \theta = \pi \) line.
It is therefore natural to try to identify various phases of the Ising model with corresponding \((T = 0)\) phases in the two dimensional superconductor with no spin rotation or time reversal invariance. First consider the pure Ising model. Both the high and low temperature phases correspond to gapped phases in terms of the Majorana fermions. Thus these correspond to insulating phases of the Majorana fermions. Disorder in the Ising model, obtained by making some of the bonds random, would tend to fill up the gap. However, if the disorder is weak, the resulting low energy states will be strongly localized. Thus, at weak disorder, both the high and low temperature phases of the Ising model correspond to localized phases of the Majorana fermions. How then do we distinguish between the two? The distinction is topological, with the quantized value of the thermal Hall conductance differing by \(1/2\) between the two phases. Strong support for this suggestion is obtained by examining the properties of the system near the transition between the two phases. In the pure Ising model, the critical point is described by a massless relativistic Majorana theory. Moving off criticality introduces a mass for the fermions with the sign of the mass distinguishing the two phases. The result that the dimensionless \(\kappa_{xy}\) jumps by \(1/2\) at the transition can now be established by direct calculation. The critical theory is identical to the one describing the transition, at weak disorder, in the \(p_x + i p_y\) superconductor of spinless fermions between the BCS and molecular limits. We thus identify the Ising transition at weak disorder with the thermal quantum Hall transition.

Now consider introducing strong (bond) randomness into the Ising model. If all the bonds are still positive, the transition is believed to be always controlled by the pure Ising fixed point. The situation is more interesting if some of the bonds are made negative. As the concentration of negative bonds is increased, there is a multicritical point (the Nishimori point, see Figure 4) on the phase boundary, beyond which the transition is no longer controlled by the pure Ising fixed point. The properties of this multicritical point have been the focus of a number of investigations over the last many years. Despite this, there is no detailed understanding of the theory of this point, or of the properties at the phase boundary at lower temperatures. What does the the Nishimori point correspond to in the language of the Majorana fermions? There appear to be two distinct possibilities - we will discuss evidence supporting either below.

(i) For the problem of localization of Majorana fermions in two dimensions, the assumption that there is a unique fixed point controlling the thermal quantum Hall transition leads to it’s identification with the free relativistic Majorana theory. This assumption, which is perhaps natural in the fermion language, then points to the identification of the Nishimori point with the multicritical point separating the Majorana metal, insulator, and the quantum Hall phase. If true, then it is natu-
eral to expect that the ferromagnetic phase transition in the Ising model at low temperatures below the Nishimori point is actually the Majorana insulator-metal transition. Some evidence in support of this is provided by the numerical results of Ref. [14] which found no signs of localization of the fermions in the non-ferromagnetic phase at low temperature close to the phase boundary. On the other hand, note that the paramagnetic phase at high temperature and weak disorder corresponds to a localized phase. If the scenario outlined above is correct, then we are led to infer the existence of a finite temperature phase transition outside the ferromagnetic phase in the Ising model associated with a delocalization of the Majorana fermions. It is unclear what this transition means in the Ising language, and even whether it happens at all. A natural candidate would have been a transition to a spin glass phase at low temperature - however there is strong numerical evidence for the absence of spin glass order at finite temperature in two dimensional Ising systems. It seems possible that a delocalized (“metallic”) phase of Majorana fermions would correspond to a phase in which both the Ising spin and the dual Ising disorder parameter are simultaneously zero.

FIG. 4. Phase diagram of the two dimensional Ising model with random bond strengths. $p$ is the concentration of negative bonds, and $T$ is the temperature. The point $N$ is the Nishimori multicritical point. SG refers to the spin glass phase which is believed to exist only at zero temperature.

(ii) A different scenario is obtained by dropping the assumption that there is a unique fixed point controlling the thermal quantum Hall transition. Instead, for the Ising model, we assume that the only finite temperature phase transition is associated with the destruction of the ferromagnetic order. If the concentration of negative bonds is large enough to destroy the ferromagnetism at zero temperature, we assume that the resultant state is a spin glass. These assumptions are perhaps reasonable expectations for the Ising model. Then, the finite temperature ferromagnetic and paramagnetic phases are, in the fermion language both localized phases which are distinguished by their thermal Hall conductance. The existence of the Nishimori point then implies the existence of a multicritical point (at strong disorder) in the phase boundary between these two localized phases. Then, the thermal quantum Hall transition will be controlled by the free relativistic massless Majorana theory at weak disorder, but by a different fixed point at strong disorder. The spin glass phase in the Ising model presumably corresponds to the Majorana metal. However, it may seem a bit puzzling, in this scenario, why the Majorana metal which is a stable phase of disordered Majorana fermions in two dimensions is only realized in a set of measure zero (a line at $T = 0$) in the Ising phase diagram.

Which one of these two scenarios is actually realized and the resultant consequences both for the Ising model and the Majorana localization problem, we leave as an intriguing open question.

VII. DISCUSSION

In this paper, we have discussed the physics of localization of quasiparticles in a superconductor in situations where the quasiparticle spin is not a good quantum number. Our discussion was entirely based on models of non-interacting quasiparticles - including the effects of interactions is an important and interesting issue which we leave for future work. Even within the non-interacting theory, we have found a rather rich phase diagram, and a number of as yet unexplored phase transitions. Several experimental systems to which this work is of relevance can be imagined - here we briefly mention a few.

(i) A particularly attractive prospect for probing the physics discussed here is in a Type $II$ superconductor in a strong magnetic field in the presence of spin-orbit scattering impurities. At low fields, the system will be in the superconducting “insulator” phase. With increasing field, under conditions discussed in Ref. [1], there will be a transition (at zero temperature) to a superconducting “metal” phase. This transition can be probed, for instance, by measurements of the low temperature quasiparticle heat conductivity [1].

(ii) A number of recent experiments [15] have measured low temperature heat transport by the quasiparticles in the heavy fermion superconductors. We note that as these systems typically have strong spin-orbit scattering, the results of this paper are of potential impor-
tance. Some of the heat conductivity measurements have been motivated by the possibility of identifying the correct pairing symmetry in these superconductors. Such identification could be seriously hampered by the localization issues discussed here. In particular, if the disorder is strong enough to localize the quasiparticles, it is hard to infer whether the pure system has a gap to quasiparticle excitations or not from heat transport measurements alone.

(iii) The properties of superfluid He$^3$ in porous media have been the subject of some experimental studies. In this context, it is interesting to ask if the fermionic He$^3$ quasiparticles in the superfluid are localized or delocalized at zero temperature. This may, perhaps, again be probed by heat transport experiments. It also interesting to consider the properties of superfluid He$^3$ on a disordered two dimensional substrate. In the A phase, at weak disorder, a thermal quantum Hall effect is predicted. With increasing disorder, it is possible (though not necessary) that there is a zero temperature phase transition where the quantization of the thermal Hall conductivity is destroyed before the superfluidity is destroyed. If this happens, this would be an experimental realization of the thermal quantum Hall transition discussed in the previous section.

Finally, the general problem of localization of Majorana fermions arises in other contexts as well - a specific example being the random bond Ising model in two dimensions as discussed briefly in Section VI. Progress in the localization problem may therefore provide a route to further our understanding of the random bond Ising model.

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