Null vector fields in spaces with affine connections and metrics. Doppler’s effect, Hublle’s effect, and aberration’s effect.

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Abstract

The notion of null (isotropic) vector field is considered in spaces with affine connections and metrics \((\mathbb{L}_n, g)\)-spaces as models of space or space-time. On its basis the propagation of signals in space-time is considered. The Doppler effect is generalized for these type of spaces. The notions of standard Doppler effect and transversal Doppler effect are introduced. On their grounds, the Hubble effect and the aberration effect appear as Doppler effects with explicit forms of the centrifugal (centripetal) and Coriolis velocity vector fields in spaces with affine connections and metrics. The upper limit of the value of the general observed shift parameter \(z\), generated by both the effects, based on the Doppler effects, is found to be \(z = \sqrt{2}\). Doppler effects, Hublle’s effect, and aberration’s effect could be used in mechanics of continuous media and in other classical field theories in the same way as the standard Doppler effect is used in classical and relativistic mechanics.

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1 Introduction

The notion of null (isotropic) vector field is related to the light propagation described in relativistic electrodynamics on the basis of special and general relativity theories \(\mathbb{II} \div \mathbb{III}\). On the other side, the notion of null (isotropic) vector field could be considered in spaces with (definite) or indefinite metric as a geometric object (contravariant vector field) with specific properties making it useful in the description of the propagation of signals in space or in space-time as well as in geometrical optics based on different mathematical models. Usually, it is assumed that a signal is propagating with limited velocity through a continuous media or in vacuum. The velocity of propagation of signals could
be a constant quantity or a non-constant quantity depending on the properties of
the space or the space-time, where the signals are transmitted and propagated.
Recently, it has been shown that a classical field theory could be considered as
a theory of continuous media with its kinematic and dynamic characteristic [4] ÷ [7]. On this basis the propagation of signals in different models of space or of
space-time is worth being investigated.

In the present paper the notion of contravariant null (isotropic) vector field is
introduced and considered in spaces with affine connections and metrics [(n, g)-
spaces]. In Section 2 the properties of null vector fields are considered on the
basis of (n − 1) + 1 representation of non-null (non-isotropic) vector fields or-
thogonal to each other. In Section 3 the notions of distance and space velocity
are discussed and their relations to null vector fields are investigated. In Section
4 the kinematic effects [longitudinal and transversal Doppler’s effects, Hubble’s
effect, and aberration’s effect] related to the kinematic characteristics of the
relative velocity and their connections with null vector fields are considered. It
is shown that the Hubble effect and the aberration effect appear as corollaries
of the standard (longitudinal) and transversal Doppler effects. On the other
side, the Hubble effect and the aberration effect are closely related to the cen-
trifugal (centripetal) and the Coriolis velocities. The results discussed in the
paper could be important from the point of view of the possible applications of
kinematic characteristics in continuous media mechanics as well as in classical
(non-quantum) field theories in spaces with affine connections and metrics.

The main results in the paper are given in details (even in full details) for
these readers who are not familiar with the considered problems. The definitions
and abbreviations are identical to those used in [6] and [7]. The reader is kindly
asked to refer to them for more details and explanations of the statements and
results only cited in this paper.

2 Null (isotropic) vector fields. Definition and
properties

2.1 Definition of a null (isotropic) vector field

Let us now consider a space with affine connections and metrics [(n, g)-space]
[8], [9] as a model of a space or of a space-time. In this space the length \( l_v \) of a
contravariant vector field \( v \in T(M) \) is defined by the use of the covariant metric
tensor field (covariant metric) \( g \in \otimes_{2s}(M) \) as

\[
g(v, v) = \pm l_v^2 , \quad l_v^2 \geq 0 .
\]  

Remark. The sign before \( l_v^2 \) depends on the signature \( Sgn \) of the covariant
metric \( g \).

The contravariant vector fields can be divided into two classes with respect
to their lengths:

- null or isotropic vector fields with length \( l_v = 0 \),  
- non-null or non-isotropic vector fields with length \( l_v \neq 0 \).
In the case of a positive definite covariant metric \( g \) \((Sgn g = \pm n, \dim M = n)\)
the null (isotropic) vector field is identically equal to zero, i.e. if \( l_v = 0 \) then
\[ v = v^i \cdot e_i \equiv 0 \in T(M), \quad v^i \equiv 0. \]

In the case of an indefinite covariant metric \( g \) \((Sgn g < n \text{ or } Sgn g > -n, \dim M = n)\)
the null (isotropic) vector field with equal to zero length \( l_v = 0 \) can
have different from zero components in an arbitrary given basis, i.e. it is not
identically equal to zero at the points, where it has been defined, i.e. if \( l_v = 0 \)
then \( v \neq 0 \in T(M), \quad v = v^i \cdot e_i \in T(M) \) and \( v^i \neq 0 \).
In a \((\mathcal{L}_n, g)\)-space the components \( g_{ij} \) of a covariant metric tensor \( g \) could be written in a local co-ordinate
system at a given point of the space as \( g_{ij} = (-1, -1, -1, ..., +1, +1, +1, ...) \)
with \( k + l = n \).

The signature \( Sgn \) of \( g \) is defined as
\[ Sgn g = -k + l = 2 \cdot l - n = n - 2 \cdot k \quad n, k, l \in \mathbb{N}, \]
where \( k = n - l, \ l = n - k \).

In the relativistic physics for \( \dim M = 4 \), the number \( l \) and \( k \) are chosen
as \( l = 1, \ k = 3 \) or \( l = 3, \ k = 1 \) so that \( Sgn g = -2 \sim (-1, -1, -1, +1) \) or
\( Sgn g = +2 \sim (+1, +1, +1, -1) \). In general, a \((\mathcal{L}_n, g)\)-space could be consider
as a model of space-time with \( Sgn g < 0 \) and \( (k > l, \ l = 1) \) or with \( Sgn g > 0 \)
and \((l > k, \ k = 1) \).

The non-null (non-isotropic) contravariant vector fields are divided into two
classes.

1. For \( Sgn g < 0 \)
   (a) \( g(v, v) = +l_v^2 > 0 \) := time like vector field \( v \in T(M) \),
   (b) \( g(v, v) = -l_v^2 < 0 \) := space like vector field \( v \in T(M) \).

2. For \( Sgn g > 0 \)
   (a) \( g(v, v) = -l_v^2 < 0 \) := time like vector field \( v \in T(M) \),
   (b) \( g(v, v) = +l_v^2 > 0 \) := space like vector field \( v \in T(M) \).

Therefore, if we do not fix a priori the signature of the space-time models
we can distinguish a \textit{time like vector field} \( u \) with
\[
g(u, u) = \begin{cases} +l_u^2 & \text{for } Sgn g < 0 \\ -l_u^2 & \text{for } Sgn g > 0 \end{cases}
\]
or \( g(u, u) = \pm l_u^2 \), and a \textit{space like vector field} \( \xi \perp \) with
\[
g(\xi \perp, \xi \perp) = \begin{cases} -l^2_{\xi \perp} & \text{for } Sgn g < 0 \\ +l^2_{\xi \perp} & \text{for } Sgn g > 0 \end{cases}
\]
or \( g(\xi \perp, \xi \perp) = \mp l^2_{\xi \perp} \). This means that in symbols \( \pm l_u^2 \) or \( \mp l_u^2 \) \((\varnothing \in T(M))\)
the sign above is related to \( Sgn g < 0 \) and the sign below is related to \( Sgn g > 0 \).

\textbf{Remark.} Since \( l_\varnothing = \sqrt{2} \), the sings in this case will be denoted as not
related to the signature of the metric \( g \).

A non-null (non-isotropic) contravariant vector field \( v \) could be represented
by its length \( l_v \) and its corresponding unit vector \( n_v = \frac{v}{l_v} \) as \( v = \pm l_u \cdot n_v \) in
contrast to a null (isotropic) vector field \( \tilde{k} \) with \( l_{\tilde{k}} = 0 \) (the sings here are not
related to the signature of the metric \( g \))

\[
v = \pm l_v \cdot n_v \quad g(v, v) = l_v^2 \cdot g(n_v, n_v) = \pm l_v^2 \quad g(n_v, n_v) = \pm 1 ,
\]

3
\[ v = \pm l_v \cdot n_v \quad , \quad g(v, v) = l_v^2 \cdot g(n_v, n_v) = \mp l_v^2 \quad , \quad g(n_v, n_v) = \mp 1 . \]

**Remark.** In the experimental physics, the measurements are related to the lengths and to the directions of a non-null (non-isotropic) vector field with respect to a frame of reference. Since different types of co-ordinates could be used in a frame of reference, the components of a vector field related to these co-ordinates cannot be considered as invariant characteristics of the vector field and on this grounds the components cannot be important characteristics for the vector fields.

After these preliminary remarks, we can introduce the notion of a null (isotropic) vector field

**Definition 1.** A contravariant vector field \( \tilde{k} \) with length zero is called null (isotropic) vector field, i.e. \( \tilde{k} := \text{null (isotropic) vector field} \) if

\[ g(\tilde{k}, \tilde{k}) = \pm l_{\tilde{k}}^2 = 0 \quad , \quad l_{\tilde{k}} = |g(\tilde{k}, \tilde{k})|^{1/2} = 0 . \quad (3) \]

### 2.2 Properties of a null (isotropic) vector field

The properties of a null (isotropic) contravariant vector field could be considered in a \( (n - 1) + 1 \) invariant decomposition of a space-time by the use of two non-isotropic contravariant vector fields \( u \) and \( \xi_{\perp} \), orthogonal to each other \[9\], i.e. \( g(u, \xi_{\perp}) = 0 \). The contravariant vectors \( u \) and \( \xi_{\perp} \) are essential elements of the structure of a frame of reference \[10\] in a space-time.

#### 2.2.1 Invariant representation of a null vector field by the use of a non-null contravariant vector field

(a) **Invariant projections of a null vector field along and orthogonal to a non-null (non-isotropic) contravariant vector field**

Every contravariant vector field \( \tilde{k} \in T(M) \) could be represented in the form

\[ \tilde{k} = \frac{1}{e} \cdot g(u, \tilde{k}) \cdot u + g[h_u(\tilde{k})] = k_{\parallel} + k_{\perp} , \quad (4) \]

where

\[ e = g(u, u) = \pm l_u^2 , \]
\[ g = g^{ij} \cdot \partial_i \cdot \partial_j , \quad \partial_i \cdot \partial_j = \frac{1}{2} \cdot (\partial_i \otimes \partial_j + \partial_j \otimes \partial_i) , \]
\[ g = g_{ij} \cdot dx^i \cdot dx^j , \quad dx^i \cdot dx^j = \frac{1}{2} \cdot (dx^i \otimes dx^j + dx^j \otimes dx^i) , \]
\[ h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u) , \quad h^u = \frac{1}{e} \cdot u \otimes u , \]
\[ g(u) = g_{\tau u} \cdot u^j \cdot dx^i , \]
\[ g[h_u(\tilde{k})] = g^{ij} \cdot h_{\tau u} \cdot \tilde{k}^i \cdot \partial_i := k_{\perp} , \quad k_{\parallel} := \frac{1}{e} \cdot g(u, \tilde{k}) \cdot u . \quad (5) \]

\[ g(k_{\parallel}, k_{\perp}) = 0 \quad , \quad g(u, k_{\perp}) = 0 . \quad (6) \]
Let us now take a closer look at the first term $k_\parallel$ of the representation of $\vec{k}$.

\[
k_\parallel = \frac{1}{\epsilon} \cdot g(u, \vec{k}) \cdot u = \pm \frac{1}{l_u^2} \cdot g(u, \vec{k}) \cdot u = \pm \frac{1}{l_u} \cdot g(u, k_\parallel) \cdot \frac{u}{l_u} . \tag{7}
\]

If we introduce the abbreviations

\[
n_\parallel = \frac{u}{l_u} , \quad \omega = g(u, \vec{k}) = g(u, k_\parallel) , \tag{8}
\]

where

\[
g(n_\parallel, n_\parallel) = \frac{1}{l_u^2} \cdot g(u, u) = \frac{1}{l_u^2} \cdot (\pm l_u^2) = \pm 1 , \tag{9}
\]

\[
\omega = g(u, \vec{k}) = g(u, k_\parallel + k_\perp) = g(u, k_\parallel) = l_u \cdot g(n_\parallel, k_\parallel) = l_u \cdot g(k_\parallel, n_\parallel) , \tag{10}
\]

\[
k_\parallel := \pm l_{k_\parallel} \cdot n_\parallel , \quad g(k_\parallel, k_\parallel) = l_{k_\parallel}^2 \cdot g(n_\parallel, n_\parallel) = l_{k_\parallel}^2 \cdot (\pm 1) = \pm l_{k_\parallel}^2 , \tag{11}
\]

\[
g(k_\parallel, n_\parallel) = \pm l_{k_\parallel} \cdot g(n_\parallel, n_\parallel) = \pm l_{k_\parallel} \cdot (\pm 1) = l_{k_\parallel} = \frac{\omega}{l_u} , \tag{12}
\]

then $k_\parallel$ could be expressed as (the signs are not related to the signature of the metric $g$)

\[
k_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel = \pm l_{k_\parallel} \cdot n_\parallel . \tag{13}
\]

The vector $n_\parallel$ is a unit vector $[g(n_\parallel, n_\parallel) = \pm 1]$ collinear to $u$ and, therefore, tangential to a curve with parameter $\tau$ if $u = \frac{d\xi}{d\tau}$.

The scalar invariant $\omega = g(u, \vec{k})$ is usually interpreted as the frequency of the radiation related to the null vector field $\vec{k}$ and propagating with velocity $u$ with absolute value $l_u$ with respect to the trajectory $x^\prime(\tau)$. In general relativity $l_u := c$ and it is assumed that the radiation is of electromagnetic nature propagating with the velocity of light $c$ in vacuum. We will come back to this interpretation in the next considerations.

The contravariant vector field $k_\perp$

\[
k_\perp = g[h_u(\vec{k})]
\]

is orthogonal to $u$ (and $k_\parallel$ respectively) part of $\vec{k}$. Since

\[
g(k_\parallel, k_\parallel) = g(\pm \frac{\omega}{l_u} \cdot n_\parallel, \pm \frac{\omega}{l_u} \cdot n_\parallel) = \frac{\omega^2}{l_u^2} \cdot g(n_\parallel, n_\parallel) = \frac{\omega^2}{l_u^2} = \pm l_{k_\parallel}^2 , \tag{14}
\]

\[
l_{k_\parallel} = \frac{\omega}{l_u} > 0 , \quad l_{k_\parallel}^2 = \frac{\omega^2}{l_u^2} , \tag{15}
\]

and

\[
g(\vec{k}, \vec{k}) = 0
\]

we have for $g(k_\perp, k_\perp)$

\[
g(\vec{k}, \vec{k}) = 0 = g(k_\parallel + k_\perp, k_\parallel + k_\perp) = g(k_\parallel, k_\parallel) + g(k_\perp, k_\perp) =
\]

\[
= \pm \frac{\omega^2}{l_u^2} + g(k_\perp, k_\perp) = \pm \frac{\omega^2}{l_u^2} + l_{k_\perp}^2 = 0 , \tag{16}
\]
and, therefore,
\[ l_{k_\perp}^2 = \frac{\omega^2}{l_u^2} , \quad l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel} . \] (17)

Remark. Since \( \omega \geq 0 \) and \( l_u > 0 \), and at the same time \( l_{k_\perp} > 0 \), and \( l_{k_\parallel} > 0 \), we have
\[ l_{k_\parallel} = \frac{\omega}{l_u} = l_{k_\perp} . \]

If we introduce the unit contravariant vector \( \tilde{n}_\perp \) with \( g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 \) then the vector \( k_\perp \) could be written as
\[ k_\perp := \mp l_{k_\perp} \cdot \tilde{n}_\perp , \] (18)
where
\[ g(k_\perp, k_\perp) = l_{k_\perp}^2 \cdot g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp l_{k_\perp}^2 = \mp \frac{\omega^2}{l_u^2} , \quad l_{k_\perp} = \frac{\omega}{l_u} . \] (19)

Therefore,
\[ k_\perp = \pm \frac{\omega}{l_u} \cdot \tilde{n}_\perp , \quad k_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel , \] (20)

where
\[ g(n_\parallel, \tilde{n}_\perp) = 0 , \quad g(k_\parallel, \xi_\perp) = 0 , \quad g(u, k_\perp) = 0 , \] (22)

\[ g(n_\parallel, k_\parallel) = \pm l_{k_\parallel} \cdot g(n_\parallel, n_\parallel) = l_{k_\parallel} = \frac{\omega}{l_u} , \] (23)

\[ g(\tilde{n}_\perp, k_\perp) = \mp l_{k_\perp} \cdot g(n_\perp, n_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp} . \] (24)

Remark. The signs not related to the metric \( g \) are chosen so to be the same with the signs related to the metric \( g \).

We have now the relations
\[ \omega = g(u, \tilde{k}) = l_u \cdot g(n_\parallel, k_\parallel) = l_u \cdot g(\tilde{n}_\perp, k_\perp) . \] (25)

If \( \tilde{n}_\perp \) is interpreted as the unit vector in the direction of the propagation of a signal in the subspace orthogonal to the contravariant vector field \( u \) and \( l_u \) is interpreted as the absolute value of the velocity of the radiated signal then \( l_u \cdot \tilde{n}_\perp \) is the path along \( \tilde{n}_\perp \) propagated by the signal in a unit time interval. Then
\[ \omega = g(u_\perp, k_\perp) , \quad u_\perp := l_u \cdot \tilde{n}_\perp , \quad g(u, u_\perp) = 0 . \] (26)

Let us now consider more closely the explicit form of \( k_\perp \)
\[ k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp . \]
(a) In 3-dimensional Euclidean space (as model of space-time of the Newtonian mechanics) the wave vector $\vec{k}$ is defined as

$$\vec{k} = \frac{2\pi}{\lambda} \cdot \vec{n},$$

where $\vec{n}$ is the unit 3-vector in the direction of propagation of a signal with absolute value of its velocity $l_u = \lambda \cdot \nu$. If we express $\lambda$ by $\lambda = l_u / \nu$ and put the equivalent expression in this for $\vec{k}$ we obtain the expression

$$\vec{k} = \frac{2\pi \cdot \nu}{l_u} \cdot \vec{n} = \frac{\omega}{l_u} \cdot \vec{n},$$

which (up to a sign depending on the signature of the metric $g$) is identical with the expression for $k_\perp$ for $n = 3$ if $k_\perp = \vec{k}$, $\vec{n} = \vec{n}_\perp$, and $\omega = 2 \cdot \pi \cdot \nu$.

(b) In 4-dimensional (pseudo) Riemannian space (as a model of space-time of the Einstein theory of gravitation) $l_u$ is interpreted as the absolute value of the velocity of light in vacuum (normalized by some authors to 1), i.e. $l_u = c$, 1. Then

$$k_\perp = \frac{\omega}{c} \cdot \vec{n}_\perp = \frac{2 \cdot \pi \cdot \nu}{\lambda \cdot \nu} \cdot \vec{n}_\perp = \frac{2 \cdot \pi}{\lambda} \cdot \vec{n}_\perp$$

and we obtain the expression for the wave vector of light propagation in general relativity, where $\vec{n}_\perp$ is the unit vector along the propagation of light in the corresponding 3-dimensional subspace of an observer with world line $x^i(\tau)$ if

$$l_u = \frac{d}{d\tau} = l_u \cdot n_\parallel, \quad n_\parallel = \frac{1}{l_u} \frac{d}{d\tau}.$$

$l_u$ is the velocity of light measured by the observer.

(c) In the general case for $k_\perp$ as

$$k_\perp = \frac{\omega}{l_u} \cdot \vec{n}_\perp,$$

$\omega$ could also be interpreted as the frequency of a signal propagating with velocity with absolute value $l_u$ in a frame of reference of an observer with world line $x^i(\tau)$. The unit vector $\vec{n}_\perp$ is the unit vector in the direction of the propagation of the signal in the subspace orthogonal to the vector $u$. The velocity of the observer is usually defined by the use of the parameter $\tau$ of the world line under the assumption that $ds = l_u \cdot d\tau$, where $ds$ is the distance of the propagation of a signal for the proper time interval $d\tau$ of the observer

$$u = \frac{d}{d\tau} = \frac{1}{l_u} \frac{d}{ds} = l_u \cdot \frac{d}{ds}.$$

Remark. Usually the velocity of a particle (observer) moving in space-time is determined by its velocity vector field $u = \frac{dx}{d\tau}$, where $\tau$ is the proper time of the observer. The parameter $\tau$ is considered as a parameter of its world line $x^i(\tau)$. By the use of $u$ and its corresponding projection metrics $h_u$ and $h^u$ a contravariant (non-null, non-isotropic) vector field $\xi$ could be represented in two parts: one part is collinear to $u$ and the other part is orthogonal to $u$

$$\xi = \frac{1}{e} \cdot g(\xi, u) \cdot u + g[h_u(\xi)] = \xi_\parallel + \xi_\perp.$$

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where
\[ \xi_{\|} = \frac{1}{e} \cdot g(\xi, u) \cdot u \quad \xi_{\perp} = \mathcal{g}[h_u(\xi)] \quad g(\xi_{\|}, \xi_{\perp}) = 0. \]

1. If an observer is moving with velocity \( v = \frac{d}{d\tau} \) on its world line \( x^i(\tau) \) then its velocity, considered with respect to the observer with velocity \( u \) and world line \( x^i(\tau) \), will have two parts \( v_{\|} \) and \( v_{\perp} \), collinear and orthogonal to \( u \) respectively at the cross point \( \tau = \tau \) of both the world lines \( x^i(\tau) \) and \( x^i(\tau) \)

\[ v = \frac{1}{e} \cdot g(v, u) \cdot u + \mathcal{g}[h_u(v)] = v_{\|} + v_{\perp}. \]

The vector \( v_{\|} \) describes the motion of the observer with velocity \( v \) along the world line of the first observer with velocity \( u \). The vector \( v_{\perp} \) describes the motion of the second observer with velocity \( v \) in direction orthogonal to the world line of the first observer. The vector \( v_{\perp} \) is the velocity of the second observer in the space of the first observer in contrast to the vector \( v_{\|} \) describing the change of \( v \) in the time of the first observer.

2. If we consider the propagation of a signal characterized by its null vector field \( \tilde{k} \) the interpretation of the vector field tangent to the world line of an observer changes. The vector field \( u = l_u \cdot n_{\|} \) is interpreted as the velocity vector field of the signal, propagating in the space-time and measured by the observer at its world line \( x^i(\tau) \) with proper time \( \tau \) as a parameter of this world line. The absolute value \( l_u \) of \( u \) is the size of the velocity of the signal measured along the unit vector field \( n_{\|} \) collinear to the tangent vector of the world line of the observer.

3. In Einstein’s theory of gravitation (ETG) both interpretations of the vector field \( u \) are put together. On the one side, the vector field \( u \) is interpreted as the velocity of an observer on its world line with parameter \( \tau \) interpreted as the proper time of the observer. On the other side, the length \( l_u \) of the vector field \( u \) is normalized either to \( \pm 1 \) or to \( \pm c = \text{const.} \) The quantity \( c \) is interpreted as the light velocity in vacuum. The basic reason for this normalization is the possibility for normalization of every non-null (non-isotropic) vector field \( u \) in the form
\[ n_u = \frac{u}{l_u} = n_{\|}, \quad \text{where} \quad l_u = | g(u, u) |^{1/2} \neq 0, \]
by the use of its different from zero length \( l_u \neq 0 \), defined by means of the covariant metric tensor \( g \).

Both the interpretations of the vector field \( u \) (as velocity of an observer or as velocity of a signal) should be considered separately from each other for avoiding ambiguities. The identification of the interpretations should mean that we assume the existence of an observer moving in space-time with velocity \( u \) and emitting (or receiving) signals with the same velocity. Such assumption does not exist in the Einstein theory of gravitation. This problem is worth to be investigated and a clear difference between both interpretations should be found. It is related to the notions of distance and velocity in spaces with affine connections and metrics.
3 Distance and velocity in a \((L_n, g)\)-space

3.1 Distance in a \((L_n, g)\)-space and its relations to the notion of velocity

1. The distance in a \((L_n, g)\)-space between a point \(P \in M\) with co-ordinates \(x^i\) and a point \(P' \in M\) with co-ordinates \(x^i + dx^i\) is determined by the length of the ordinary differential \(d\), considered as a contravariant vector field \(d = dx^i \cdot \partial_i\). If we denote the distance as \(ds\) between point \(P\) and point \(P'\) then the square \(ds^2\) of \(ds\) could be defined as the square of the length of the ordinary differential

\[
ds^2 = g(d, d) = \pm l_d^2 = g_{ij} \cdot dx^i \cdot dx^j, \quad l_d^2 \geq 0 .
\]

2. Let us now consider a two parametric congruence of curves (a set of not intersecting curves) in a \((L_n, g)\)-space

\[
x^i = x^i(\tau, r(\tau, \lambda)) = x^i(\tau, \lambda),
\]

where the function \(r = r(\tau, \lambda) \in C^r(M)\), \(r \geq 2\), depends on the two parameters \(\tau\) and \(\lambda\), \(\tau, \lambda \in \mathbb{R}\). Then

\[
dr = \frac{\partial r(\tau, \lambda)}{\partial \tau} \cdot d\tau + \frac{\partial r(\tau, \lambda)}{\partial \lambda} \cdot d\lambda
\]

and

\[
dx^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} \cdot d\tau + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \cdot \frac{\partial r(\tau, \lambda)}{\partial \tau} \cdot d\tau + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \lambda} \cdot \frac{\partial r(\tau, \lambda)}{\partial \lambda} \cdot d\lambda =
\]

\[
= \left[ \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \cdot \frac{\partial r(\tau, \lambda)}{\partial \tau} \right] \cdot d\tau + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \lambda} \cdot \frac{\partial r(\tau, \lambda)}{\partial \lambda} \cdot d\lambda
\]

or

\[
dx^i = (u^i + l_v \cdot \partial_i) \cdot d\tau + \xi^i \cdot \frac{\partial r}{\partial \lambda} \cdot d\lambda,
\]

where

\[
u^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau}, \quad l_v = \frac{\partial r(\tau, \lambda)}{\partial \tau}, \quad \xi^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r},
\]

and

\[
d = dx^i \cdot \partial_i = dx^i \cdot (u + l_v \cdot \xi) + (\partial r / \partial \lambda) \cdot d\lambda \cdot \xi,
\]

\[
\frac{d\lambda}{d\tau} = 0, \quad \frac{d\tau}{d\lambda} = 0.
\]

The change of the contravariant vector field \(d\) under the change \(d\tau\) of the parameter \(\tau\) could be expressed in the form

\[
\frac{d}{d\tau} = dx^i \cdot \partial_i = u + l_v \cdot \xi = \Pi^i \cdot \partial_i = \Pi, \quad \Pi = \frac{dx^i}{d\tau}.
\]
where the relations are valid
\[ g(\mathbf{u}, u) = g(u, u) + l_v \cdot g(\mathbf{u}, u) , \]
\[ g(\mathbf{u}, \mathbf{\xi}) = g(u, \mathbf{\xi}) + l_v \cdot g(\mathbf{u}, \mathbf{\xi}) , \]
\[ g(\mathbf{u}, \mathbf{\pi}) = g(u + l_v \cdot \mathbf{\xi}, u + l_v \cdot \mathbf{\xi}) = \]
\[ = g(u, u) + 2 \cdot l_v \cdot g(u, \mathbf{\xi}) + l_v^2 \cdot g(\mathbf{u}, \mathbf{\xi}) . \]

The contravariant vector field \( \mathbf{\pi} = \mathbf{u} \cdot \partial_i \) is usually interpreted as the velocity of an observer moving in a space-time described by a \((\mathcal{M}, g)\)-space as its model. The contravariant vector \( u \) is a tangent vector field to the curve \( x^i(\tau, \lambda = \lambda_0 = \text{const.}) = x^i(\tau, \lambda = \lambda_0 = \text{const.}) \) \( u = u^i \cdot \partial_i = \frac{\partial x^i}{\partial \tau} \cdot \partial_i \),
\[ \mathbf{\pi} = \frac{1}{g(u, u)} \cdot g(\mathbf{\pi}, u) \cdot u + g(\mathbf{\pi}, \mathbf{\pi}) . \]

The contravariant vector \( \mathbf{\xi} \) is a collinear vector to the tangent vector \( \xi \) to the curve \( x^i(\tau = \tau_0 = \text{const.}, r(\tau_0, \lambda)) = x^i(\tau \sim \tau_0 = \text{const.}, \lambda) \). Since
\[ \mathbf{\xi} = \xi^i \cdot \partial_i = \frac{\partial x^i}{\partial \lambda} \cdot \partial_i \]
then
\[ \frac{\partial x^i}{\partial r} = \frac{\partial x^i(\tau, \lambda(r, \tau))}{\partial \tau} = \frac{\partial x^i(\lambda, \tau)}{\partial \lambda} = \xi^i \cdot \partial_i , \]
where
\[ r = r(\lambda, \tau) , \lambda = \lambda(\tau, \xi) , \]
\[ \mathbf{\xi} = \xi^i \cdot \partial_i = \frac{\partial \lambda}{\partial r} \cdot \xi^i \cdot \partial_i = \frac{\partial \lambda}{\partial r} \cdot \xi \cdots \xi = \frac{\partial x^i}{\partial \lambda} \cdot \partial_i . \]

3. Further, since we wish to consider the vector field \( u \) as the velocity vector field of an observer moving at the curve \( x^i(\tau, \lambda = \lambda_0 = \text{const.}) \) interpreted as his world line, the vector field \( \xi \) (and \( \mathbf{\xi} \) respectively) could be chosen to lie in the subspace orthogonal to \( u \), i.e. \( u \) and \( \mathbf{\xi} \) could obey the condition \( g(u, \mathbf{\xi}) = 0 \) and, therefore, \( g(u, \xi) = 0 \), \( \xi = \xi_\perp = g(\xi, \xi_\perp) \), and \( \mathbf{\xi} = \mathbf{\xi}_\perp . \)

4. In the next step, we could consider the vector field \( \xi \) as a unit vector field in direction of the vector field \( \xi \), i.e.
\[ \mathbf{\xi}_\perp = n_\perp = \frac{\xi_\perp}{\l_\perp} , \quad g(u, n_\perp) = 0 , \]
\[ g(\mathbf{\xi}_\perp, \mathbf{\xi}_\perp) = g(n_\perp, n_\perp) = \frac{1}{\xi_\perp} \cdot g(\mathbf{\xi}_\perp, \mathbf{\xi}_\perp) = \mp \frac{1}{\xi_\perp} \cdot l_\perp^2 = \mp 1 , \]
\[ g(\mathbf{\xi}_\perp, \mathbf{\xi}_\perp) = g(\frac{\partial \lambda}{\partial r} \cdot \xi_\perp, \frac{\partial \lambda}{\partial r} \cdot \xi_\perp) = \left( \frac{\partial \lambda}{\partial r} \right)^2 \cdot g(\xi_\perp, \xi_\perp) = \mp \left( \frac{\partial \lambda}{\partial r} \right)^2 \cdot l_\perp^2 = \mp 1 , \]
\[ \left( \frac{\partial \lambda}{\partial r} \right)^2 \cdot l_\perp^2 = 1 , \quad l_\perp^2 = \left( \frac{\partial \lambda}{\partial r} \right)^{-2} , \quad l_\perp = \pm \left( \frac{\partial \lambda}{\partial r} \right)^{-1} . \]
After all above considerations for $\xi_\perp$ and $\bar{\xi}_\perp$ we obtain the relations

\[
\begin{align*}
g(\xi, u) &= g(u, u), \\
g(\xi, \xi_\perp) &= l_v \cdot g(\xi_\perp, \xi_\perp) = l_v \cdot g(n_\perp, n_\perp) = \mp l_v, \\
g(\xi, \bar{\xi}) &= g(u, u) + l_v^2 \cdot g(n_\perp, n_\perp) = \\
&= \pm l_u^2 \mp l_u^2 = \frac{ds^2}{d\tau^2}, \\
\frac{ds^2}{d\tau^2} &= g\left(\frac{d}{d\tau}, \frac{d}{d\tau}\right) = \pm l_u^2 \pm l_u^2 = \pm l_u^2 \cdot \left(1 - \frac{l_v^2}{l_u^2}\right).
\end{align*}
\]

Moreover,

\[
\begin{align*}
dx^i &= (u^i + l_v \cdot n_\perp^i) \cdot d\tau + \frac{\partial r}{\partial \lambda} \cdot d\lambda \cdot n_\perp^i, \\
d &= d\tau \cdot (u + l_v \cdot n_\perp) + \frac{\partial r}{\partial \lambda} \cdot d\lambda \cdot n_\perp = \\
&= d\tau \cdot u + (d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda) \cdot n_\perp = \\
&= d\tau \cdot u + dr \cdot n_\perp, \\
dr &= d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda, \\
dr(\tau, \lambda) &= l_v, \\
\frac{d\lambda}{d\tau} &= 0, \\
\bar{\xi} &= u + l_v \cdot n_\perp, \\
g(\bar{\xi}, u) &= g(u, u), \\
n_\perp &= g[h_u(n_\perp), n_\perp] ,
\end{align*}
\]

\[
\begin{align*}
ds^2 &= g(d, d) = d\tau^2 \cdot g(u, u) + (d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda)^2 \cdot g(n_\perp, n_\perp) = \\
&= \pm d\tau^2 \cdot l_u^2 \mp (d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda)^2 = \pm d\tau^2 \cdot l_u^2 \mp d\tau^2 \cdot l_v^2 = \\
&= \pm d\tau^2 \cdot l_u^2 \mp dr^2 = \mp (l_u^2 \cdot dr^2 - dr^2) = \pm d\tau^2 \cdot (l_u^2 - \frac{dr^2}{d\tau^2}) = \\
&= \pm d\tau^2 \cdot (l_u^2 - l_v^2),
\end{align*}
\]

Therefore,

\[
\begin{align*}
ds^2 &= g(d, d) = \pm d\tau^2 \cdot (l_u^2 - l_v^2), \\
\frac{ds^2}{d\tau^2} &= g\left(\frac{d}{d\tau}, \frac{d}{d\tau}\right) = g(\bar{\xi}, u) = \pm l_u^2 \cdot \left(1 - \frac{l_v^2}{l_u^2}\right).
\end{align*}
\]

5. In the non-relativistic field theories the distance between two points $P \in M$ and $\bar{P} \in M$ is defined as

\[
ds^2 = \mp dr^2
\]

and $l_u = 0$. This means that the distance between two neighboring points $P$ and $\bar{P}$ is the space distance measured between them in the rest (proper) reference frame of the observer (with absolute value $l_u$ of his velocity $u$ equal to zero).
The time parameter $\tau$ is not considered as a co-ordinate in space-time, but as a parameter, independent of the frame of reference of the observer.

6. In the relativistic field theories and especially in the Einstein theory of gravitation $dr$ is considered as the space distance between two neighboring points $P$ and $\overline{P}$ and $l_u \cdot dr$ is interpreted as the distance covered by a light signal in a time interval $d\tau$, measured by an observer in his proper frame of reference (when the observer in it is at rest). The quantity $l_u$ is usually interpreted as the absolute value $c$ of a light signal in vacuum, i.e. $l_u = c$, or $l_u$ is normalized to 1, i.e. $l_u = 1$, if the proper time interval $d\tau$ is replaced with the proper distance interval $ds = c \cdot d\tau$, i.e. $\overline{\tau} = \frac{d}{c \cdot d\tau}$ is replaced with $\overline{\tau} = \frac{d}{ds}$, $ds = c \cdot d\tau$.

Therefore, there is a difference between the interpretation of the absolute value $l_u$ of the velocity of an observer in classical and relativistic physics

(a) In classical physics, from the above consideration, it follows that $l_u = 0$ (observer at rest) and $ds = dr$ is the distance as space distance. $l_u$ is the absolute value of the velocity of the observer but the velocity of the light propagation which the observer could measure in his proper frame of reference. If we wish to interpret $l_u$ as the absolute value of the velocity of the observer himself we should assume that $l_u \neq c$ or 1 (if the observer is not moving with the speed of light).

- There is the possibility to identify $l_u$ with $l_v$ as the absolute value of the velocity of the observer at a point $P$ at his world line, measured with respect to a neighboring point $\overline{P}$ with the same proper time as the point $P$. Under this assumption, the ordinary differential becomes a null (isotropic) vector field $[g(d, d) = 0, l_d = 0, l_u = l_v \neq 0]$ in the proper frame of reference of the observer.

- We could also interpret $l_u$ as the absolute value of the velocity of the observer with respect to another frame of reference or

- we can consider $l_u$ as the absolute value of the velocity of a signal coming to the observer with velocity, different from the velocity of light. On the basis of the last assumption we can describe the propagation of signals with propagation velocity different from the velocity of light (for instance, the propagation of sound signals or (may be) gravitational signals).

If $l_v = 0$ then $\overline{\tau} = u$ and $u$ could be

- the velocity vector field $u = \frac{d}{d\tau}$ of an observer ($l_u \neq 0$, $u = l_u \cdot n_\|$) in his proper frame of reference along his world line. Since in his proper frame of reference the observer is at rest, $u$ could be interpreted as the velocity of a clock measuring the length (proper time) of the world line by the use of the parameter $\tau$ or

- the velocity of a signal detected or emitted by the observer.

3.2 Measuring a distance in $(\mathcal{L}_n, g)$-spaces

A. If the notion of distance is introduced in a space-time modeled by a $(\mathcal{L}_n, g)$-space we have to decide what is the meaning of the vector field $u$ as tangent
vector to a trajectory interpreted as the world line of an observer. On the basis of the above consideration, we have four possible answers for the meaning of the vector field $u$ as

1. Velocity vector field of a propagating signal in space-time identified with the tangent vector field $u$ at the world line of an observer. The signal is detected or emitted by the observer on his world line and the absolute value $l_u$ of $u$ is identified with the absolute value of the velocity of the signal in- or outcouming to the observer.

2. Velocity vector field of an observer moving in space-time. In this case $l_u \neq 0$ and the space-time should have a definite metric, i.e. $S g n g = \pm n$, $d i m M = n$ (for instance, motion of an observer in an Euclidean space considered as a model of space-time). The observer, moving in space-time could consider processes happened in its subspace orthogonal to his velocity. The observer will move in a flow and consider the characteristics of the flow from his own frame of reference.

3. Velocity of a clock moving in space-time and determining the proper time in the frame of reference of an observer. The velocity $u$ of the clock in space-time is with fixed absolute value $l_u$, i.e. $l_u = \text{const}$. The time interval $d\tau$ measured by the clock corresponds to the length $ds$ of its world line, i.e. $d\tau^2 = \pm \text{const} \cdot ds^2$.

Under the assumption for the constant velocity of the clock we consider in it a periodical process which indicates the time interval $d\tau$ in the proper frame of reference of the clock and of the observer respectively.

4. Velocity $u$ of a $(n - 1)$-dimensional subspace moving in time with $l_u \neq 0$. If the subspace deform in some way, the deformations reflect on the kinematic characteristics of the vector field $u$ and $u$ is used as an indicator for the changing properties of the subspace, considered as the space of an observer (laboratory) where a physical system is investigated. This type of interpretation requires not only the existence of the velocity vector field $u$ with $l_u \neq 0$ but also the existence of (at least one) orthogonal to $u$ vector field $\xi\perp$, $g(u, \xi\perp) = 0$, lying in the orthogonal to $u$ subspace $T\perp u(M)$.

All indicated interpretations could be used in solving different physical problems related to motions of physical systems in space-time.

B. After introducing the notion of distance, the question arises how a space distance between two points in a space could be measured. We could distinguish three types of measurements:

1. Direct measurements by using a measuring device (e.g. a roulette, a linear (running) meter, yard-measure-stick etc.)

2. Direct measurements by sending signals from a basic point to a fixed point of space and detecting at the basic point the reflected by the fixed point signal.

3. Indirect measurement by receiving signals from a fixed point of space without sending a signal to it.

Let us now consider every type of measurements more closely.

1. Direct measurements by using a measuring device. The space distance between two points $A$ and $B$ in a space could be measured by a second observer moving from point $A$ (where the first observer is at rest) to point $B$ in space. At the same time, the second observer moves in time from point $B$ to point $B'$. The space distance measured by the observer with world line $AA'$ could be denoted as $\Delta r = AB$ and the time period passed as $\Delta \tau = AA'$. This is a direct measurement of the space distance $AB = \Delta r$ from point $A$ to point $B$ in the
space during the time \( AA' = \triangle \tau \). It is assumed that point \( A \) and point \( B \) are at rest during the measurement. Instead of measuring the space distance \( AB \) the observers measure the space distance \( A'B' \) which exists at the time \( \tau + \triangle \tau \) if the measurement has began at the time \( \tau \) from the point of the first observer with world line \( AA' \).

2. Direct measurements by sending signals from a basic point to a fixed point of space and detecting at the basic point the reflected by the fixed point signal. The space distance between two points \( A \) and \( B \) in a space could be measured by a sending a signal with velocity with absolute value \( u \neq 0 \). Then \( AB \) of the curve \( x^i(\tau = \tau_0, r) \) through point \( B \) is the distance \( \triangle r \) at the time \( \tau(A) = \tau_0 \) and \( \tau(B) = \tau_0 \).

\( A'B' \) of the curve \( x^i(\tau = \tau_0 + \triangle \tau, r) \) is the space distance \( \triangle r' \) at the time \( \tau(A') = \tau_1 \). At this time the signal is received at point \( B' \) which is point \( B \) at the time \( \tau_1 \), i.e. \( \tau(B') = \tau_1 \). \( B'A' \) is the space distance between \( B \) and \( A \) at the time \( \tau(A') = \tau_1 \), where \( \tau(B') = \tau_1 \), \( \tau(B'') = \tau_2 \). At the time \( \tau_2 \) the point \( B(\tau_0) \) will be moved in the time to point \( B''(\tau_2) \). The signal will be propagated

(a) for the time interval \( AA' = \tau_1 - \tau_0 \) to the point \( B' \) at the time \( \tau_1 \) at the space distance \( \triangle r = l_u \cdot (\tau_1 - \tau_0) \), where \( l_u \) is the velocity of the signal measured by the observer with world line \( AA' \).

(b) for the time interval \( A'A'' \) from point \( B' \) at the time \( \tau_1 \) to the point \( A'' \) at the time \( \tau_2 \) at a space distance \( l_u \cdot (\tau_2 - \tau_1) \). The whole space distance covered by the signal in the time interval \( AA'A'' = \triangle \tau = \tau_2 - \tau_0 \) is \( l_u \cdot (\tau_2 - \tau_0) = l_u \cdot (\tau_2 - \tau_1) + l_u \cdot (\tau_1 - \tau_0) \).

If we now assume that point \( A \) and point \( B \) are at rest to each other and the space distance between them does not change in the time then

\[
l_u \cdot (\tau_2 - \tau_1) = l_u \cdot (\tau_1 - \tau_0)
\]

and

\[
l_u \cdot (\tau_2 - \tau_0) = 2 \cdot l_u \cdot (\tau_1 - \tau_0) = 2 \cdot A'B'(\tau_1) = 2 \cdot AB(\tau_0) \, .
\]

Therefore, the space distance between point \( A \) and point \( B \) (at any time, if both the points are at rest to each other) is

\[
AB = \frac{1}{2} \cdot l_u \cdot (\tau_2 - \tau_0) \, ,
\]

where \( \triangle \tau = \tau_2 - \tau_0 \) is the time interval for the propagation of a signal from point \( A \) to point \( B \) and from point \( B \) back to point \( A \).

3. Indirect measurement by receiving signals from a fixed point of space without sending a signal to it. If the space distance between point \( A \) and point \( B \) is changing in the time and at point \( B \) there is an emitter then the frequency of the emitter will change in the time related to the centrifugal (centripetal) or Coriolis' velocity between both the points \( A \) and \( B \). Therefore, a criteria for no relative motion between two (space) points (points with one and the same proper time) could be the lack of change of the frequency of the signals emitted from the second point \( B \) to the basic point \( A \). [But there could be motions of an emitter which could so change its frequency that the changes compensate each other and the observer at the basic point \( A \) could come to the conclusion that there is no motions between points \( A \) and \( B \).]

If an emitter at point \( B(\tau_0) \) emits a signal with velocity \( v \) and frequency \( \nu \) then this signal will be received (detected) at the point \( A'(\tau_1) \) after a time
interval \( AA' = \Delta \tau = \tau_1 - \tau_0 \) by an observer (detector) moving in the time interval \( \Delta \tau \) from point \( A(\tau_0) \) to point \( A'(\tau_1) \) on his world line \( x^i(\tau) \). If the emitter is moving relatively to point \( A \) with relative velocity \( v_{rel} \) the detected at the point \( A' \) frequency \( \omega \) will differ from the emitted frequency \( \overline{\omega} \). If both the points \( A \) and \( B \) are at rest to each other then \( \overline{\omega} = \omega \).

C. The question arises how can we find the space distance between two points \( A \) and \( B \) lying in such a way in the space that only signals emitted from the one point (point \( B \)) could be detected at the basic point (point \( A \)), where an observer detects the signal from point \( B \). First of all, if we knew the propagation velocity \( l_n \) of a signal and the difference \( \omega - \overline{\omega} \) between the emitted frequency \( \omega \) and the detected frequency \( \overline{\omega} \) we can try to find out the relative velocity between the emitter (at a point \( B \)) and the observer (at a point \( A \)). For doing that we will need relations between the difference \( \omega - \overline{\omega} \) and the relative velocity between both the points. Such relations could be found on the basis of the structures of the relative velocity and its decomposition in centrifugal (centripetal) relative velocity and Coriolis’ relative velocity.

4 Kinematic effects related to the relative velocity

1. Let us now consider the change of a null vector field \( \tilde{k} \) under the influence of the relative velocity of the corresponding emitter and its frequency with respect to an observer detecting the emitted radiation by the emitter.

Let \( \vec{k}_{\perp} \) be the orthogonal to \( u \) part of the null vector field \( \vec{k} \) corresponding to the null vector field \( \tilde{k} \) after the influence of the relative velocity \( v_{rel} \)

\[
\vec{k} = \vec{k} + v_{rel}k , \quad \tilde{k} = k_{\parallel} + k_{\perp} , \quad \vec{k} = k_{\parallel} + k_{\perp} , \quad v_{rel}k = v_{rel}k_{\parallel} + v_{rel}k_{\perp} , \quad (31)
\]

\[
\vec{k}_{\perp} = k_{\perp} + v_{rel}k_{\perp} , \quad (32)
\]

where \( v_{rel}k \) depends on the relative velocity \( v_{rel} \).

If \( \vec{k} = \vec{k} + v_{rel}k \) then

\[
g(\vec{k}, \vec{k}) = g(v_{rel}k, v_{rel}k) + 2 \cdot g(\vec{k}, v_{rel}k) ,
\]

\[
g(v_{rel}k, v_{rel}k) = -2 \cdot g(\vec{k}, v_{rel}k) .
\]

If \( v_{rel}k = C \cdot \vec{k} \) then \( \vec{k} = \vec{k} + C \cdot \vec{k} = (1 + C) \cdot \vec{k} \) and \( g(\vec{k}, \vec{k}) = 0 = (1 + C)^2 \cdot g(\vec{k}, \vec{k}) = 0 \) for \( \forall C \in C^r(M) \). Therefore, the assumption that \( v_{rel}k = C \cdot \vec{k} \), \( C \neq 0 \), leads to a mapping of the vector field \( \vec{k} \) as a null vector field into a new null vector field \( \vec{k} \) under the influence of the relative velocity between emitter and detector. Then

\[
v_{rel}k = C \cdot \vec{k} = C \cdot (k_{\parallel} + k_{\perp}) , \quad (33)
\]

\[
\Delta \omega = g(u, v_{rel}k) = C \cdot g(u, \vec{k}) = C \cdot g(u, k_{\parallel}) = C \cdot \omega , \quad (34)
\]

\[
g(\vec{k}, u) = g(\vec{k}, u) + g(v_{rel}k, u) , \quad (35)
\]

\[
\overline{\omega} = \omega + C \cdot \omega = (1 + C) \cdot \omega , \quad (36)
\]

\[
\Delta \omega = C \cdot \omega = \overline{\omega} - \omega , \quad (37)
\]
\[ g_{(\text{rel}k, \vec{n}_\perp)} = g(C \cdot \vec{k}, \vec{n}_\perp) = g_{(\text{rel}k_\perp, \vec{n}_\perp)} = C \cdot g(k_\perp, \vec{n}_\perp) = \]
\[ = C \cdot \frac{\omega}{l_u}, \quad \text{(38)} \]
\[ \text{rel}k_\parallel = \pm \frac{\Delta \omega}{l_u} \cdot n_\parallel = \pm C \cdot \frac{\omega}{l_u} \cdot n_\parallel \quad \text{, \quad (40)} \]
\[ \text{rel}k_\perp = \mp \frac{\Delta \omega}{l_u} \cdot \vec{n}_\perp = \mp C \cdot \frac{\omega}{l_u} \cdot \vec{n}_\perp. \quad \text{\quad (41)} \]

The problem arises how can we find the invariant function (factor) \( C \) before \( \vec{k} \) depending on the relative velocity \( \text{rel}v \) and the velocity of the signal \( l_u \). Since \( l_{\text{rel}k_\parallel} = \text{rel}k_\perp \) we can find in this way the whole structure of \( \text{rel}k = \text{rel}k_\parallel + \text{rel}k_\perp \).

In the previous sections we have considered the representation of \( \vec{k} \) as \( \vec{k} = k_\parallel + k_\perp \), where

\[ k_\parallel = \pm l_{k_\parallel} \cdot n_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel, \quad l_{k_\parallel} = \frac{\omega}{l_u}, \]
\[ k_\perp = \mp l_{k_\perp} \cdot \vec{n}_\perp = \mp \frac{\omega}{l_u} \cdot \vec{n}_\perp, \quad l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel}. \]

The unit vector \( \vec{n}_\perp \) is orthogonal to the vector \( u \), i.e. \( g(u, \vec{n}_\perp) = 0 \) because of \( g(u, k_\parallel) = \mp l_{k_\parallel} \cdot g(u, \vec{n}_\perp) = 0, l_{k_\parallel} \neq 0, l_u \neq 0 \). Therefore, \( g(\vec{n}_\perp, \vec{n}_\perp) = \mp 1 = \mp l_{\vec{n}_\perp}^2, l_{\vec{n}_\perp} \neq 0 \).

We can represent the unit vector \( \vec{n}_\perp \) (orthogonal to \( u \)) in two parts: one part collinear to the vector field \( \xi_\perp \) (orthogonal to \( u \)) and one part orthogonal to the vectors \( u \) and \( \xi_\perp \), i.e.

\[ \vec{n}_\perp = \alpha \cdot n_\perp + \beta \cdot m_\perp, \quad \text{(42)} \]
\[ g(\vec{n}_\perp, \vec{n}_\perp) = \mp 1 = \mp l_{\vec{n}_\perp}^2, \quad l_{\vec{n}_\perp} > 0, \quad l_{\vec{n}_\perp} = 1, \quad \text{(43)} \]

where

\[ n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}}, \quad g(n_\perp, u) = 0, \quad g(n_\perp, n_\perp) = \mp 1 = \mp l_{n_\perp}^2, \quad \text{(44)} \]
\[ l_{n_\perp} > 0, \quad l_{n_\perp} = 1 \]
\[ m_\perp = \frac{v_c}{l_{v_c}}, \quad g(m_\perp, u) = 0, \quad g(m_\perp, \xi_\perp) = 0. \quad \text{(46)} \]

The vector field \( v_c \) is the Coriolis velocity vector field orthogonal to \( u \) and to the centrifugal (centripetal) velocity \( v_z \) collinear to \( \xi_\perp \). Since

\[ v_c = \pm l_{v_c} \cdot m_\perp, \quad g(v_c, v_c) = \mp l_{v_c}^2, \quad \text{\quad (47)} \]

we have

\[ g(m_\perp, m_\perp) = \mp 1 = \mp l_{m_\perp}^2, \quad l_{m_\perp} > 0, \quad l_{m_\perp} = 1. \quad \text{\quad (48)} \]

The Coriolis velocity \( v_c \) is related to the change of the vector \( \xi_\perp \) in direction orthogonal to \( u \) and \( \xi_\perp \).
Since $$\vec{n}_\perp$$ is a unit vector as well as the vectors $$n_\perp$$ and $$m_\perp$$, and, further, 

$$g(n_\perp,m_\perp) = 0$$, we obtain

$$g(\vec{n}_\perp,\vec{n}_\perp) = \mp 1 = g(\alpha \cdot n_\perp + \beta \cdot m_\perp, \alpha \cdot n_\perp + \beta \cdot m_\perp) =$$

$$= \alpha^2 \cdot g(n_\perp,n_\perp) + \beta^2 \cdot g(m_\perp,m_\perp) =$$

$$= \mp \alpha^2 \mp \beta^2$$.

Therefore,

$$\alpha^2 + \beta^2 = 1 \quad (49)$$

On the other side,

$$g(\vec{n}_\perp,n_\perp) = g(\alpha \cdot n_\perp + \beta \cdot m_\perp, n_\perp) =$$

$$= \alpha \cdot g(n_\perp,n_\perp) = \mp \alpha$$,

$$g(\vec{n}_\perp,m_\perp) = g(\alpha \cdot n_\perp + \beta \cdot m_\perp, m_\perp) =$$

$$= \beta \cdot g(n_\perp,n_\perp) = \mp \beta$$.

i.e.

$$\alpha = \mp g(\vec{n}_\perp,n_\perp) = \mp l_{\vec{n}_\perp} \cdot l_{n_\perp} \cdot \cos(\vec{n}_\perp,n_\perp) = \mp \cos(\vec{n}_\perp,n_\perp) \quad (50)$$

$$\beta = \mp g(\vec{n}_\perp,m_\perp) = \mp l_{\vec{n}_\perp} \cdot l_{m_\perp} \cdot \cos(\vec{n}_\perp,m_\perp) = \mp \cos(\vec{n}_\perp,m_\perp) \quad (51)$$

Thereby, $$\alpha$$ and $$\beta$$ appear as direction cosines of $$n_\perp$$ and $$m_\perp$$ with respect to the unit vector $$\vec{n}_\perp$$. Since

$$\cos^2(\vec{n}_\perp,n_\perp) + \cos^2(\vec{n}_\perp,m_\perp) = 1$$,

it follows that

$$\cos^2(\vec{n}_\perp,m_\perp) = 1 - \cos^2(\vec{n}_\perp,n_\perp) = 1 - \sin^2(\vec{n}_\perp,m_\perp) = \sin^2(\vec{n}_\perp,n_\perp)$$,

$$\sin^2(\vec{n}_\perp,m_\perp) = \cos^2(\vec{n}_\perp,n_\perp)$$,

$$\cos(\vec{n}_\perp,m_\perp) = \pm \sin(\vec{n}_\perp,n_\perp)$$,

$$\alpha = \mp \cos(\vec{n}_\perp,n_\perp)$$,

$$\beta = \mp \sin(\vec{n}_\perp,n_\perp)$$.

(52)

(53)

Further, since $$k_\perp = \mp l_{k_\perp} \cdot \vec{n}_\perp$$ then (see above)

$$g(k_\perp,k_\perp) = l_{k_\perp}^2 \cdot g(\vec{n}_\perp,\vec{n}_\perp) = \mp l_{k_\perp}^2$$,

$$g(\vec{n}_\perp,k_\perp) = \mp l_{k_\perp} \cdot g(\vec{n}_\perp,\vec{n}_\perp) = \mp \frac{\omega}{l_u} \cdot l_{k_\perp} = \mp \frac{\omega}{l_u} \cdot n_\parallel$$,

$$l_{k_\perp} = \frac{\omega}{l_u} \cdot n_\parallel$$,

$$k_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel$$,

where $$l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\perp}$$.

(51)

$$g(\vec{n}_\perp,k_\perp) = \mp l_{k_\perp} \cdot g(\vec{n}_\perp,\vec{n}_\perp) = \omega \cdot l_{k_\perp} = \omega \cdot l_{k_\perp} = l_{k_\perp} \cdot l_{k_\perp}$$.

2. For the contravariant null vector field $$\bar{k}$$ we have analogous relations as for the contravariant null vector field $$\bar{k}$$ (just changing $$\bar{k}$$ with $$\bar{k}$$ and $$\omega$$ with $$\bar{\omega}$$)

$$\bar{k} = k_\parallel + k_\perp$$,

$$\bar{-} = g(u,\bar{k}) \quad (55)$$
\[ \mathbf{F}_\| = \pm \frac{\bar{\omega}}{l_u} \cdot n_\| , \quad l_{k_\|} = \frac{\bar{\omega}}{l_u} , \quad (56) \]
\[ \mathbf{F}_\perp = \pm \frac{\bar{\omega}}{l_u} \cdot \bar{n}_\perp , \quad l_{k_\perp} = \frac{\bar{\omega}}{l_u} = l_{k_\|} , \quad (57) \]
\[ g(\bar{n}_\perp, \mathbf{F}_\perp) = \mp \frac{\bar{\omega}}{l_u} \cdot g(\bar{n}_\perp, \bar{n}_\perp) = \frac{\bar{\omega}}{l_u} = l_{k_\|} = l_{k_\perp} . \quad (58) \]

From \( \mathbf{k} = \bar{k} + \text{rel} k, \mathbf{F}_\perp = \bar{k}_\perp + \text{rel} k_\perp, \) and
\[ \mp \frac{\bar{\omega}}{l_u} \cdot \bar{n}_\perp = \mp \frac{\omega}{l_u} \cdot \bar{n}_\perp + \text{rel} k_\perp , \quad (59) \]

it follows that
\[ g(\bar{n}_\perp, \mathbf{F}_\perp) = g(\bar{n}_\perp, k) + g(\bar{n}_\perp, \text{rel} k) , \quad (60) \]
\[ g(\bar{n}_\perp, \mathbf{F}_\perp) = g(\bar{n}_\perp, k) + g(\bar{n}_\perp, \text{rel} k_\perp) , \quad (61) \]

The vectors \( \mathbf{F}_\perp \) and \( k_\perp \) are collinear to each other. This means that the last term \( g(\text{rel} k_\perp, \bar{n}_\perp) \) in the previous expression should be proportional to \( k_\perp \), i.e. \( \text{rel} k_\perp = C \cdot k_\perp \). At the same time, it should contain in its factor \( C \) before \( k_\perp \), a dimensionless term describing the influence of the relative velocity \( \text{rel} v \) on the null vector field \( \bar{k} \). This term should take into account the fact that \( \text{rel} v \) should act in the direction of \( \bar{n}_\perp \) if its influence is on \( k_\perp = \mp \frac{\bar{\omega}}{l_u} \cdot \bar{n}_\perp \) and as a result it leads to \( \mathbf{F}_\perp = \mp \frac{\bar{\omega}}{l_u} \cdot \bar{n}_\perp \), i.e. it leads to a new vector \( \mathbf{F}_\perp \) collinear to \( k_\perp \). On this basis, the conclusion could be made that we can define \( \text{rel} k_\perp \) in the form
\[ \text{rel} k_\perp = \frac{1}{l_u} \cdot g(\text{rel} v, \bar{n}_\perp) \cdot k_\perp . \quad (62) \]

Therefore, from the expressions
\[ \frac{\bar{\omega}}{l_u} = \frac{\omega}{l_u} + g(\text{rel} k_\perp, \bar{n}_\perp) = \frac{\omega}{l_u} + \frac{1}{l_u} \cdot g(\text{rel} v, \bar{n}_\perp) \cdot g(k_\perp, \bar{n}_\perp) = \]
\[ = \frac{\omega}{l_u} + \frac{1}{l_u} \cdot g(\text{rel} v, \bar{n}_\perp) \cdot \frac{\omega}{l_u} \]

follows the relation
\[ \bar{\omega} = \omega + \frac{1}{l_u} \cdot g(\text{rel} v, \bar{n}_\perp) \cdot \omega . \quad (63) \]

3. Since \( \text{rel} v = v_z + v_c \) and \( \bar{n}_\perp = \alpha \cdot n_\perp + \beta \cdot m_\perp \), we can find the explicit form of the term with \( g(\text{rel} v, \bar{n}_\perp) \). By the use of the relations \( \mathbf{n} \)
\[ g(\text{rel} v, \bar{n}_\perp) = g(v_z + v_c, \bar{n}_\perp) = g(v_z, \bar{n}_\perp) + g(v_c, \bar{n}_\perp) , \quad (64) \]
\[ g(v_z, \bar{n}_\perp) = g(v_z, \alpha \cdot n_\perp + \beta \cdot m_\perp) = \alpha \cdot g(v_z, n_\perp) + \beta \cdot g(v_z, m_\perp) = \]
\[ = \alpha \cdot g(v_z, n_\perp) , \quad (65) \]
\[ g(v_c, \bar{n}_\perp) = g(v_c, \alpha \cdot n_\perp + \beta \cdot m_\perp) = \alpha \cdot g(v_c, n_\perp) + \beta \cdot g(v_c, m_\perp) = \]
\[ = \beta \cdot g(v_c, m_\perp) , \quad (66) \]

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\[ v_z = \pm l_{v_z} \cdot n_\perp = H \cdot l_{\xi_\perp} \cdot n_\perp , \quad n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}} , \quad (67) \]
\[ v_c = \pm l_{v_c} \cdot m_\perp = H_c \cdot l_{\xi_\perp} \cdot m_\perp , \quad m_\perp = \frac{v_c}{l_{v_c}} . \quad (68) \]

Remark. The signs \( \pm \) before \( l_{v_z} \) and \( l_{v_c} \) are not related to the signature of the metric \( g \). They are showing the direction of \( v_z \) and \( v_c \) with respect to the units vectors \( n_\perp \) and \( m_\perp \) respectively.

For \( g(\text{rel} v, \tilde{n}_\perp) \) we obtain
\[ g(\text{rel} v, \tilde{n}_\perp) = g(v_z + v_c, \alpha \cdot n_\perp + \beta \cdot m_\perp) = \]
\[ = \alpha \cdot g(v_z, n_\perp) + \beta \cdot g(v_c, m_\perp) = \]
\[ = \alpha \cdot g(\pm l_{v_z} \cdot n_\perp, n_\perp) + \beta \cdot g(\pm l_{v_c} \cdot m_\perp, m_\perp) = \]
\[ = \pm \alpha \cdot l_{v_z} \cdot g(n_\perp, n_\perp) + \beta \cdot l_{v_c} \cdot g(m_\perp, m_\perp) = \]
\[ = -\alpha \cdot l_{v_z} \cdot \beta \cdot l_{v_c} = -(\alpha \cdot l_{v_z} + \beta \cdot l_{v_c}) . \quad (70) \]

For \( \omega = \omega + \frac{1}{l_u} \cdot g(\text{rel} v, \tilde{n}_\perp) \cdot \omega \) the relation follows
\[ \omega = \omega - \frac{1}{l_u} \cdot (\alpha \cdot l_{v_z} + \beta \cdot l_{v_c}) \cdot \omega . \quad (71) \]

Since \( l_{v_z} = \pm H \cdot l_{\xi_\perp} \) and \( l_{v_c} = \pm H_c \cdot l_{\xi_\perp} \), it follows further
\[ \omega = \omega \mp \frac{1}{l_u} \cdot (\alpha \cdot H + \beta \cdot H_c) \cdot l_{\xi_\perp} \cdot \omega . \quad (72) \]

Remark. The explicit forms of \( H \) and \( H_c \) will be given below in the considerations of the Hubble and the aberration effects.

Therefore, for \( \omega \) and \( \omega \) we have the relations
(a)
\[ \omega = \omega - \frac{1}{l_u} \cdot (\alpha \cdot l_{v_z} + \beta \cdot l_{v_c}) \cdot \omega , \]
(b)
\[ \omega = \omega \mp \frac{1}{l_u} \cdot (\alpha \cdot H + \beta \cdot H_c) \cdot l_{\xi_\perp} \cdot \omega . \]

On the basis of the relations (a) and (b) different kinematic effects could be considered related to the Doppler effect, to the Hubble effect, and to the aberration effect.

4.1 Standard (longitudinal) and transversal Doppler effects

The expression for \( \omega \) could also be written in the form
\[ \omega = \omega - \frac{1}{l_u} \cdot (\alpha \cdot l_{v_z} + \beta \cdot l_{v_c}) \cdot \omega , \quad (73) \]

where \( \omega \) is the frequency of an emitter moving with centrifugal (centripetal) velocity \( v_z = \pm l_{v_z} \cdot n_\perp \) and with Coriolis' velocity \( v_c = \pm l_{v_c} \cdot m_\perp \) relative to an observer (with detector) \( \tilde{n}_\perp \). The emitted signals propagate with velocity \( u = l_u \cdot n_\parallel \) with respect to the observer, where \( u \) is the tangent vector to the
world line of the observer (detector). The detected signals are with frequency $\omega$.

(a) If $l_{vc} = 0$ and the emitter moves only away or to the observer, i.e. if $\alpha = \pm 1$, $\beta = 0$, then

$$\omega = (1 \pm \frac{l_{vz}}{l_u}) \cdot \omega .$$  \hspace{1cm} (74)

For $\omega > \omega$

$$\omega = (1 + \frac{l_{vz}}{l_u}) \cdot \omega .$$  \hspace{1cm} (75)

For $\omega < \omega$

$$\omega = (1 - \frac{l_{vz}}{l_u}) \cdot \omega .$$  \hspace{1cm} (76)

If we express the frequencies as $\omega = 2 \cdot \pi \cdot \nu$ and $\omega = 2 \cdot \pi \cdot \nu$ we obtain

$$\nu > \nu : \nu = (1 + \frac{l_{vz}}{l_u}) \cdot \nu ,$$  \hspace{1cm} (77)

$$\nu < \nu : \nu = (1 - \frac{l_{vz}}{l_u}) \cdot \nu .$$  \hspace{1cm} (78)

The last relations represents a generalization of the \textit{standard (longitudinal) Doppler effect} in $(\mathcal{T}_n, g)$-spaces.

(b) If $l_{uc} = 0$ and the emitter moves only around an observer (detector) with the Coriolis velocity $v_c = \pm l_{vc} \cdot m_\perp$, i.e. if $\alpha = 0$, $\beta = \pm 1$, then

$$\omega = (1 \pm \frac{l_{vc}}{l_u}) \cdot \omega .$$  \hspace{1cm} (79)

For $\omega > \omega$

$$\omega = (1 + \frac{l_{vc}}{l_u}) \cdot \omega .$$  \hspace{1cm} (80)

For $\omega < \omega$

$$\omega = (1 - \frac{l_{vc}}{l_u}) \cdot \omega .$$  \hspace{1cm} (81)

If we express the frequencies as $\omega = 2 \cdot \pi \cdot \nu$ and $\omega = 2 \cdot \pi \cdot \nu$ we obtain

$$\nu > \nu : \nu = (1 + \frac{l_{vc}}{l_u}) \cdot \nu ,$$  \hspace{1cm} (82)

$$\nu < \nu : \nu = (1 - \frac{l_{vc}}{l_u}) \cdot \nu .$$  \hspace{1cm} (83)

The last relations represent a generalization of the \textit{transversal Doppler effect} in $(\mathcal{T}_n, g)$-spaces. The relations have the same forms as these for the standard (longitudinal) Doppler effect but the direction of the emitted signals changes in the time in contrast to the standard Doppler effect. In the expressions for the standard Doppler effect only the centrifugal (centripetal) velocity $v_z$ is replaced with the Coriolis velocity $v_c$ for receiving the relations for the transversal Doppler effect.

(c) If $l_{vc} \neq 0$ and $l_{vc} \neq 0$ then we have an accumulation of both types of the Doppler effect

$$\omega = [1 - (\alpha \cdot \frac{l_{vz}}{l_u} + \beta \cdot \frac{l_{vc}}{l_u})] \cdot \omega .$$  \hspace{1cm} (84)
We have considered the Doppler effects without taking into account the structures of the centrifugal (centripetal) velocity and of the Coriolis velocity.

It should be stressed that the generalized Doppler effects are a result of pure kinematic considerations of the properties of a null (isotropic) vector field by means of the kinematic characteristics of the relative velocity in spaces with affine connections and metrics.

### 4.2 Hubble’s effect

The Hubble’s law has been considered on the grounds of the structures of the centrifugal (centripetal) velocity \[^1\]. Its connection to the change of the frequency of an emitter could be found if we take into account the structures of the centrifugal (centripetal) velocity and the Coriolis velocity.

Let us now consider the standard (longitudinal) Doppler effect when the emitter has only centrifugal (centripetal) velocity with respect to the observer. Then \( \alpha = \pm 1 \), \( \beta = 0 \) and

\[
\bar{\omega} = (1 \pm \frac{l_{v_z}}{l_u}) \cdot \omega .
\]

Since \( \pm v_z = H \cdot l_{\xi_\perp} \), \( v_z = H \cdot l_{\xi_\perp} \cdot n_\perp \) we obtain for the frequency \( \bar{\omega} \) of the emitter

\[
\bar{\omega} = \omega + H \cdot \frac{l_{\xi_\perp}}{l_u} \cdot \omega .
\]  

The measured by the observer signals with frequency \( \omega \) and the emitted by the emitter signals with frequency \( \bar{\omega} \) are related to each other by the Hubble function \( H \) and on this basis to the Hubble law. Therefore, the radiation with frequency \( \bar{\omega} \) by the emitter could be expressed as

\[
\bar{\omega} = (1 + H \cdot \frac{l_{\xi_\perp}}{l_u}) \cdot \omega
\]

and it will be detected by the observer as radiation with frequency \( \omega \). The relative difference between both frequencies (emitted \( \bar{\omega} \) and detected \( \omega \))

\[
\frac{\bar{\omega} - \omega}{\omega}
\]

appears in the form

\[
\frac{\bar{\omega} - \omega}{\omega} = H \cdot \frac{l_{\xi_\perp}}{l_u} .
\]  

If we introduce the abbreviation

\[
z = \frac{\bar{\omega} - \omega}{\omega}
\]

we obtain the relation between the emitted frequency \( \bar{\omega} \) and the frequency \( \omega \) detected by the observer in the form

\[
\frac{\bar{\omega} - \omega}{\omega} = z , \quad \bar{\omega} = (1 + z) \cdot \omega , \quad z = H \cdot \frac{l_{\xi_\perp}}{l_u} .
\]  

The change of the frequency \( \bar{\omega} \) under the motion of the emitter with centrifugal (centripetal) velocity \( v_z \) relative to an observer is called *Hubble’s effect*. 

---

\[^1\] Hubble’s law has been considered on the grounds of the structures of the centrifugal (centripetal) velocity.
The quantity $z$ could be denoted as *observed Hubble’s shift frequency parameter*. If $z = 0$ then $H = 0$, $v_z = 0$, and there will be no difference between the emitted and the detected frequencies, i.e. $\omega = \overline{\omega}$, i.e. for $z = 0$, it follows that $\overline{\omega} = \omega$. If we take into account the explicit form of the Hubble function

$$H = \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) ,$$

(89)

then $z = 0$ leading to $H = 0$ will be the case if

$$\theta = \pm(n - 1) \cdot \sigma(n_\perp, n_\perp) .$$

(90)

If $z > 0$ the observed Hubble shift frequency parameter is called *Hubble’s red shift*. If $z < 0$ the observed Hubble’s shift frequency parameter is called *Hubble’s blue shift*. If $\omega$ and $\omega$ are known the observed Hubble’s shift frequency parameter $z$ could be found. If $\omega$ and $z$ are given then the corresponding $\omega$ could be estimated.

On the other side, from the explicit form of $z$

$$z = H \cdot \frac{l_{\xi_\perp}}{l_u} = \left[ \frac{1}{n - 1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot \frac{l_{\xi_\perp}}{l_u}$$

(91)

we could find the relation between the observed shift frequency parameter $z$ and the kinematic characteristics of the relative velocity such as the expansion and shear velocities.

*Special case* $\langle L_n, g \rangle$-spaces with shear-free relative velocity: $\sigma := 0$.  

$$z = \frac{1}{n - 1} \cdot \theta \cdot \frac{l_{\xi_\perp}}{l_u} \ , \ \ H = \frac{1}{n - 1} \cdot \theta \ .$$

(92)

*Special case* $\langle L_n, g \rangle$-spaces with expansion-free relative velocity: $\theta := 0$.

$$z = \mp \sigma(n_\perp, n_\perp) \cdot \frac{l_{\xi_\perp}}{l_u} \ , \ \ H = \mp \sigma(n_\perp, n_\perp) \ .$$

(93)

On the grounds of the observed shift frequency parameter $z$ the distance (the length $l_{\xi_\perp}$ of $\xi_\perp$) between the observer (with the world line $x^\tau(\tau)$) and velocity vector field $u = \frac{dx}{d\tau}$ and the observed object (at a distance $l_{\xi_\perp}$ from the observer) emitting radiation with null vector field $k$ could be found as

$$l_{\xi_\perp} = z \cdot \frac{l_u}{H} = \frac{\overline{\omega} - \omega}{\omega} \cdot \frac{l_u}{H} .$$

(94)

On the other side, if $z$, $H$, and $l_{\xi_\perp}$ are known the absolute value $l_u$ of the velocity vector $u$ could be found as

$$l_u = \frac{H}{z} \cdot l_{\xi_\perp} = \frac{H \cdot \omega}{\overline{\omega} - \omega} \cdot l_{\xi_\perp} .$$

(95)

*Remark*. In the Einstein theory of gravitation (ETG) the absolute value of $u$ is usually normalized to 1 or $c$, i.e. $l_u = 1$, $c$. Then the last expression could be used for experimental check up of the velocity $c$ of light in vacuum if $z$, $H$, and $l_{\xi_\perp}$ are known

$$c = \frac{H}{z} \cdot l_{\xi_\perp} = \frac{H \cdot \omega}{\overline{\omega} - \omega} \cdot l_{\xi_\perp} .$$

(96)
Since 
\[ z = \frac{1}{n-1} \cdot \theta \mp \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}} = \overline{\omega} - \frac{\omega}{\overline{\omega}} \]

it follows that
\[ l_u = \frac{\omega}{\overline{\omega} - \omega} \cdot \frac{1}{n-1} \cdot \theta \pm \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}} \quad . \quad (97) \]

Analogous expression we can find for the length \( l_{\xi_{\perp}} \) of the vector field \( \xi_{\perp} \)
\[ l_{\xi_{\perp}} = (n-1) \cdot (\frac{\overline{\omega}}{\omega} - 1) \cdot \frac{l_u}{\theta \mp (n-1) \cdot \sigma(n_{\perp}, n_{\perp})} \quad . \quad (98) \]

**Special case**: \((\overline{L}_n, g)\)-space with shear-free relative velocity: \( \sigma := 0 \).
\[ l_u = \frac{\omega}{\overline{\omega} - \omega} \cdot \frac{1}{n-1} \cdot \theta \cdot l_{\xi_{\perp}} \quad , \quad (99) \]
\[ l_{\xi_{\perp}} = (n-1) \cdot (\frac{\overline{\omega}}{\omega} - 1) \cdot \frac{l_u}{\theta} \quad . \quad (100) \]

**Special case**: \((\overline{L}_n, g)\)-space with expansion-free relative velocity: \( \theta := 0 \).
\[ l_u = \mp \frac{\omega}{\overline{\omega} - \omega} \cdot \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}} \quad , \quad (101) \]
\[ l_{\xi_{\perp}} = \mp (\frac{\overline{\omega}}{\omega} - 1) \cdot \frac{l_u}{\sigma(n_{\perp}, n_{\perp})} \quad . \quad (102) \]

By the use of the relation between the Hubble function \( H \) and the observed shift parameter \( z \) we can express the centrifugal (centripetal) velocity by means of the frequencies \( \overline{\omega} \) and \( \omega \). From
\[ v_z = H \cdot l_{\xi_{\perp}} \cdot n_{\perp} \quad , \quad H = z \cdot \frac{l_u}{l_{\xi_{\perp}}} = (\frac{\overline{\omega}}{\omega} - 1) \cdot \frac{l_u}{l_{\xi_{\perp}}} \quad , \]

it follows that
\[ v_z = z \cdot l_u \cdot n_{\perp} = (\frac{\overline{\omega}}{\omega} - 1) \cdot l_u \cdot n_{\perp} \quad . \quad (103) \]

Then
\[ g(v_z, v_z) = \mp (\frac{\overline{\omega}}{\omega} - 1)^2 \cdot l^2_u = \mp l^2_{v_z} \quad , \]
\[ l_{v_z} = \pm (\frac{\overline{\omega}}{\omega} - 1) \cdot l_u \quad , \quad \pm l_{v_z} = (\frac{\overline{\omega}}{\omega} - 1) \cdot l_u \quad , \quad (104) \]
\[ l_{v_z} > 0 \quad , \quad l_u > 0 \quad , \quad l_{\xi_{\perp}} > 0 \quad , \quad (105) \]

where (since \( l_{v_z} > 0 \))
\[ \overline{\omega} > \omega \quad : \quad l_{v_z} = (\frac{\overline{\omega}}{\omega} - 1) \cdot l_u \quad , \quad (106) \]
\[ \overline{\omega} < \omega \quad : \quad l_{v_z} = (1 - \frac{\overline{\omega}}{\omega}) \cdot l_u \quad . \quad (107) \]
4.2.1 Einstein’s theory of gravitation and the Hubble effect

In Einstein’s theory of gravitation it is assumed that \( l_u = c \) and for all other velocities \( l_{v_z} \leq c \), \( l_{v_c} \leq c \). From the above expressions, it follows that for \( \vec{\omega} = \omega \leq \frac{c}{l_u} \):

\[
\vec{\omega} = \omega \leq \frac{c}{l_u} \leq \frac{1}{\omega - 1} \leq 1 , \quad l_u > 0 ,
\]

i.e. the emitted frequency \( \vec{\omega} \) should be smaller than \( 2 \cdot \omega \)

\[
\frac{\vec{\omega}}{\omega} \leq 2 , \quad \vec{\omega} \leq 2 \cdot \omega .
\]

Then for \( \vec{\omega} > \omega \) the observed Hubble shift frequency parameter \( z \leq 1 \). For \( \vec{\omega} < \omega \):

\[
\vec{\omega} = \omega \pm \frac{l_{v_z}}{l_u} \cdot \omega \leq \frac{1}{\omega - 1} \cdot l_u \leq l_u ,
\]

\[
1 - \frac{\vec{\omega}}{\omega} \leq 1 : -\frac{\vec{\omega}}{\omega} \leq 0 : \omega \geq 0 ,
\]

i.e. \( l_{v_z} \leq l_u \) is always fulfilled because of \( \vec{\omega} \geq 0 \) and \( \omega \geq 0 \). Therefore, if the Einstein theory of gravitation is a correct theory for description of the gravitational interaction under the assumption that the absolute value \( l_u \) of the velocity of the light propagation is equal the absolute value \( c \) of the velocity of light in vacuum, then the red shift \( z \) could not be bigger that 1, i.e. \( z \leq 1 \) for \( \vec{\omega} > \omega \). Such a limit for \( z \) does not exist if \( \vec{\omega} < \omega \). If we could find experimentally that in some cases \( z > 1 \) then we should look for the reasons for this diversion from the theory of the Hubble effect based on the kinematic characteristics of the relative velocity. One of the possible reasons could be the influence of the aberration’s effect on the Hubble effect. The aberration effect is considered below.

4.3 Aberration’s effect

The aberration effect is related the transversal Doppler effect in analogous way as the Hubble effect is related to the standard (longitudinal) Doppler effect.

Let us now consider the transversal Doppler effect when the emitter has only Coriolis’ velocity with respect to the observer. Then \( \alpha = 0, \beta = \pm 1 \), and

\[
\vec{\omega} = \omega \pm \frac{l_{v_z}}{l_u} \cdot \omega , \quad v_c = \pm l_{v_e} \cdot m_\perp .
\]

(108)

The Coriolis velocity \( v_c = \pm l_{v_e} \cdot m_\perp \) could be represented in the form

\[
v_c = l_{\xi_\perp} \cdot \vec{g}[\sigma(n_\perp)] \pm \sigma(n_\perp) \cdot l_{\xi_\perp} \cdot n_\perp + l_{\xi_\perp} \cdot \vec{g}[\omega(n_\perp)] = l_{\xi_\perp} \cdot \vec{v}_c ,
\]

(109)

where

\[
\vec{v}_c = \vec{g}[\sigma(n_\perp)] \pm \sigma(n_\perp) \cdot n_\perp + \vec{g}[\omega(n_\perp)]
\]

(110)

Then

\[
v_c = l_{\xi_\perp} \cdot \vec{v}_c = \pm l_{v_e} \cdot m_\perp ,
\]

\[
\vec{v}_c = \pm l_{v_e} \cdot m_\perp ,
\]

\[
g(\vec{v}_c, \vec{v}_c) = \mp l_{v_e}^2
\]

(111)
\[ v_c = \pm l v_c \cdot m_\perp = l \xi_\perp \cdot (\pm l v_c \cdot m_\perp) = \pm l v_c \cdot l \xi_\perp \cdot m_\perp = (112) \]

\[ = H_c \cdot l \xi_\perp \cdot m_\perp , \quad (113) \]

\[ v_c = \pm l \xi_\perp \cdot l v_c \cdot m_\perp = \pm l v_c \cdot m_\perp , \quad \pm l v_c = \pm l \xi_\perp \cdot l v_c , \quad (114) \]

\[ l v_c = l \xi_\perp \cdot l v_c , \quad g(v_c, v_c) = v_c^2 = \mp l v_c = \mp l \xi_\perp \cdot l v_c . \quad (115) \]

where

\[ H_c = \pm l v_c , \quad \pm l v_c = H_c \cdot l \xi_\perp . \quad (116) \]

Then

\[ v_c = \pm l v_c \cdot l \xi_\perp \cdot m_\perp = H_c \cdot l \xi_\perp \cdot m_\perp \] (compare with \( v_z = H \cdot l \xi_\perp \cdot n_\perp ) . \]

After introducing the last (previous) expression for \( l v_c \) into the relation for \( \omega \), it follows that

\[ \omega = \omega \pm l u \cdot \omega = \omega + H_c \cdot l \xi_\perp \cdot \omega , \quad H_c = \pm l v_c . \quad (117) \]

The (invariant) function \( H_c \) is called Coriolis’ function.

For the emitted frequency \( \bar{\omega} \) we obtain an analogous expression as in the case of the Hubble effect (only \( H \) is replaced by \( H_c \))

\[ \bar{\omega} = (1 + H_c \cdot l \xi_\perp \cdot l u ) \cdot \omega . \quad (118) \]

The detected by the observer frequency is \( \omega \) if the emitter is moving with a Coriolis velocity \( v_c \) around the observer and emitting signals with a frequency \( \bar{\omega} \). The relative difference between both the frequencies (emitted \( \bar{\omega} \) and detected \( \omega \))

\[ \frac{\bar{\omega} - \omega}{\omega} \]

appears in the form

\[ \frac{\bar{\omega} - \omega}{\omega} = H_c \cdot l \xi_\perp \cdot l u . \quad (119) \]

If we introduce the abbreviation

\[ z_c = \frac{\bar{\omega} - \omega}{\omega} \quad (120) \]

we obtain the relation between the emitted frequency \( \bar{\omega} \) and the frequency \( \omega \) detected by the observer in the form

\[ \frac{\bar{\omega} - \omega}{\omega} = z_c , \quad \bar{\omega} = (1 + z_c ) \cdot \omega , \quad z_c = H_c \cdot l \xi_\perp \cdot l u = \pm l v_c . \quad (121) \]

The change of the frequency \( \bar{\omega} \) under the motion of the emitter with Coriolis’ velocity \( v_c \) relative to an observer is called aberration’s effect.

The quantity \( z_c \) could be denoted as observed aberration’s shift frequency parameter. If \( z_c = 0 \) then \( H_c = 0, v_c = 0 \), and there will be no difference between the emitted and the detected frequencies, i.e. \( \bar{\omega} = \omega \), i.e. for \( z_c = 0 \), it follows that \( \bar{\omega} = \omega \).

If \( z_c > 0 \) the observed aberration shift frequency parameter is called aberration’s red shift. If \( z_c < 0 \) the observed aberration shift frequency parameter is called aberration’s blue shift. If \( \bar{\omega} \) and \( \omega \) are known the observed aberration shift frequency parameter \( z_c \) could be found. If \( \omega \) and \( z_c \) are given then the corresponding \( \bar{\omega} \) could be estimated.
On the other side, from the explicit form of $z_c$

$$z_c = H_c \cdot \frac{\xi_\perp}{l_u} \quad , \quad H_c = \pm l_u \quad , \quad (122)$$

we could find the relation between the observed shift frequency parameter $z_c$ and the kinematic characteristics of the relative velocity such as the shear and rotation velocities.

Since

$$v_c^2 = \mp l_c^2 = \mp l_{\xi_\perp}^2 \cdot l_{\xi_\perp} = \mp l_{\xi_\perp} \cdot H_c^2 =$$

$$= l_{\xi_\perp} \cdot \{ \tilde{\mathcal{G}}(\sigma(n), \sigma(n)) \pm [\sigma(n, n)]^2 + \tilde{\mathcal{G}}(\omega(n), \omega(n)) +$$

$$+ 2 \cdot \tilde{\mathcal{G}}(\sigma(n), \omega(n)) \} \quad , \quad (123)$$

it follows that

$$H_c^2 = \mp \{ \tilde{\mathcal{G}}(\sigma(n), \sigma(n)) \pm [\sigma(n, n)]^2 + \tilde{\mathcal{G}}(\omega(n), \omega(n)) +$$

$$+ 2 \cdot \tilde{\mathcal{G}}(\sigma(n), \omega(n)) \} \quad , \quad (124)$$

and

$$H_c = \pm \{ \tilde{\mathcal{G}}(\sigma(n), \sigma(n)) \pm [\sigma(n, n)]^2 + \tilde{\mathcal{G}}(\omega(n), \omega(n)) +$$

$$+ 2 \cdot \tilde{\mathcal{G}}(\sigma(n), \omega(n)) \}^{1/2} \quad . \quad (125)$$

**Remark.** The quantity $\omega$ in the structure of $H_c$ and $v_c$ is an antisymmetric covariant tensor of second rank $\omega \in \otimes_2(M) = \Lambda^2(M)$ interpreted as the rotation velocity tensor. It should be distinguished from the frequency $\omega$ detected by the observer. The tensor $\omega$ appears only in the structure of $H_c$ and $z_c$ in contrast to the frequency $\omega$ appearing out of these quantities.

**Special case:** ($\mathcal{L}_n, g$)-spaces with shear-free relative velocity: $\sigma := 0$.

$$z_c = \mp [\tilde{\mathcal{G}}(\omega(n), \omega(n))]^{1/2} \cdot \frac{\xi_\perp}{l_u} \quad , \quad H_c = \pm [\tilde{\mathcal{G}}(\omega(n), \omega(n))]^{1/2} \quad . \quad (126)$$

**Special case:** ($\mathcal{L}_n, g$)-spaces with rotation-free relative velocity: $\omega := 0$.

$$z_c = \pm \{ \tilde{\mathcal{G}}(\sigma(n), \sigma(n)) \pm [\sigma(n, n)]^2 \}^{1/2} \cdot \frac{\xi_\perp}{l_u} \quad , \quad (127)$$

$$H_c = \pm \{ \tilde{\mathcal{G}}(\sigma(n), \sigma(n)) \pm [\sigma(n, n)]^2 \}^{1/2} \quad . \quad (128)$$

On the grounds of the observed aberration shift frequency parameter $z_c$ the distance (the length $l_{\xi_\perp}$ of $\xi_\perp$) between the observer [with the world line $x^i(\tau)$] and velocity vector field $u = \frac{dx}{d\tau}$] and the observed object (at a distance $l_{\xi_\perp}$ from the observer) emitting radiation with null vector field $\tilde{k}$ could be found as

$$l_{\xi_\perp} = z_c \cdot \frac{l_u}{H_c} = \frac{\omega - \omega}{\omega} \cdot \frac{l_u}{H_c} \quad . \quad (130)$$

On the other side, if $z_c$, $H_c$, and $l_{\xi_\perp}$ are known the absolute value $l_u$ of the velocity vector $u$ could be found as

$$l_u = \frac{H_c}{z_c} \cdot l_{\xi_\perp} = \frac{H_c \cdot \omega}{\omega - \omega} \cdot l_{\xi_\perp} \quad . \quad (131)$$
Remark. In the Einstein theory of gravitation (ETG) the absolute value of \(u\) is usually normalized to 1 or \(c\), i.e. \(|u| = 1, c\). The last (previous) expression for \(l_u\) could be used for experimental check up of the velocity \(c\) of light in vacuum if \(z_c, H_c, \text{ and } l_{\xi\perp}\) are known

\[
c = \frac{H_c}{z_c} \cdot l_{\xi\perp} = \frac{H_c \cdot \omega}{\overline{\omega} - \omega} \cdot l_{\xi\perp} = \frac{\omega}{\overline{\omega} - \omega} \cdot \{ \pm (\overline{\sigma}(\boldsymbol{n}_{\perp}), \sigma(\boldsymbol{n}_{\perp})) \pm |\sigma(\boldsymbol{n}_{\perp}, \boldsymbol{n}_{\perp})|^2 + \overline{\sigma}(\omega(\boldsymbol{n}_{\perp}), \omega(\boldsymbol{n}_{\perp})) + 2 \cdot \overline{\sigma}(\sigma(\boldsymbol{n}_{\perp}), \omega(\boldsymbol{n}_{\perp})) \}^{1/2} \cdot l_{\xi\perp}.
\]

(132)

Since

\[
z_c = H_c \cdot \frac{l_{\xi\perp}}{l_u} = \frac{\overline{\omega} - \omega}{\omega}
\]

it follows that

\[
l_u = \frac{\omega}{\overline{\omega} - \omega} \cdot H_c \cdot l_{\xi\perp} = \frac{H_c}{z_c} \cdot l_{\xi\perp} \cdot l_{\xi\perp}.
\]

(134)

Analogous expression could be found for the length \(l_{\xi\perp}\) of the vector field \(\xi\perp\)

\[
l_{\xi\perp} = z_c \cdot \frac{l_u}{H_c} = (\frac{\overline{\omega}}{\omega} - 1) \cdot \frac{l_u}{H_c} \cdot l_{\xi\perp}.
\]

(135)

Special case: \((\mathcal{L}_n, g)\)-space with shear-free relative velocity: \(\sigma := 0\).

\[
l_u = \pm \frac{\omega}{\overline{\omega} - \omega} \cdot \{ \mp [\overline{\sigma}(\omega(\boldsymbol{n}_{\perp}), \omega(\boldsymbol{n}_{\perp})) - |\sigma(\boldsymbol{n}_{\perp}, \boldsymbol{n}_{\perp})|^2]^{1/2} \cdot l_{\xi\perp} \},
\]

(136)

\[
l_{\xi\perp} = \pm (\frac{\overline{\omega}}{\omega} - 1) \cdot \frac{l_u}{(\mp \overline{\sigma}(\omega(\boldsymbol{n}_{\perp}), \omega(\boldsymbol{n}_{\perp})) - |\sigma(\boldsymbol{n}_{\perp}, \boldsymbol{n}_{\perp})|^2)\}^{1/2} \cdot l_{\xi\perp}.
\]

(137)

Special case: \((\mathcal{L}_n, g)\)-space with rotation-free relative velocity: \(\omega := 0\).

\[
l_u = \frac{H_c}{z_c} \cdot l_{\xi\perp} = \pm \frac{\omega}{\overline{\omega} - \omega} \cdot \{ \mp [\overline{\sigma}(\sigma(\boldsymbol{n}_{\perp}), \sigma(\boldsymbol{n}_{\perp})) - (\sigma(\boldsymbol{n}_{\perp}, \boldsymbol{n}_{\perp})]^2]^{1/2} \cdot l_{\xi\perp} \},
\]

(138)

\[
l_{\xi\perp} = z_c \cdot \frac{l_u}{H_c} = \pm \frac{\overline{\omega}}{\omega} - 1 \cdot \frac{l_u}{H_c} \cdot \{ (\mp [\overline{\sigma}(\sigma(\boldsymbol{n}_{\perp}), \sigma(\boldsymbol{n}_{\perp})) - (\sigma(\boldsymbol{n}_{\perp}, \boldsymbol{n}_{\perp})]^2]^{1/2} \cdot l_{\xi\perp}.
\]

(139)

By the use of the relation between the Coriolis function \(H_c\) and the observed aberration shift parameter \(z_c\) we can express the centrifugal (centripetal) velocity \(v_c\) by means of the frequencies \(\overline{\omega}\) and \(\omega\). From

\[
v_c = H_c \cdot l_{\xi\perp} \cdot m_{\perp} \quad , \quad H_c = z_c \cdot \frac{l_u}{l_{\xi\perp}} = \frac{\overline{\omega}}{\omega} - 1 \cdot \frac{l_u}{l_{\xi\perp}} \cdot l_{\xi\perp},
\]

it follows that

\[
v_c = z_c \cdot l_u \cdot m_{\perp} = \frac{\overline{\omega}}{\omega} - 1 \cdot l_u \cdot m_{\perp}.
\]

(140)

Then

\[
g(v_z, v_z) = \mp (\frac{\overline{\omega}}{\omega} - 1)^2 \cdot l_u^2 \mp l_{v_z}^2,
\]

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\[ l_{vc} = \pm \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u \quad , \quad \pm l_{vc} = \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u \quad , \quad (141) \]

\[ l_{vc} > 0 \quad , \quad l_u > 0 \quad , \quad l_{\xi_\perp} > 0 \quad , \quad (142) \]

where (since \( l_{vc} > 0 \))

\[ \omega > \omega : l_{vc} = \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u \quad , \quad (143) \]

\[ \omega < \omega : l_{vc} = \left( 1 - \frac{\overline{\omega}}{\omega} \right) \cdot l_u \quad . \quad (144) \]

**Remark.** Analogous considerations for the Einstein theory of gravitation as in the case of the Hubble effect could be made also for the case of the aberration effect. If \( l_{vc} \leq l_u \) then from

\[ z_{vc} = \pm \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u \quad , \]

it follows that

(a) for \( z_c = + \left( \frac{\overline{\omega}}{\omega} - 1 \right) \leq \frac{\omega}{l_u} = 1 \),

(b) for \( z_c = - \left( \frac{\overline{\omega}}{\omega} - 1 \right) \geq -1 : \frac{\omega}{l_u} \leq 1 \).

If \( l_u = c \) and \( l_{vc} \leq l_u \) then for the case \( \omega > \omega \) we have the condition \( z \leq 1 \). For \( \omega < \omega \) there is no such condition because \( l_{vc} \leq l_u \) is fulfilled automatically.

### 4.4 Accumulation of Hubble’s effect and aberration’s effect

If the relative velocity \( v_{rel} \) of the emitter is a superposition of the centrifugal (centripetal) velocity \( v_z \) and the Coriolis velocity \( v_c \) with respect to the observer, i.e. if

\[ v_{rel} = v_z + v_c \quad , \quad v_z \neq 0 \quad , \quad v_c \neq 0 \quad , \quad (145) \]

then both the Hubble and the aberration effects influence each other. From the general expression for the emitted frequency \( \omega \) as function of the detected frequency \( \omega \)

\[ \overline{\omega} = \left[ 1 - (\alpha \cdot \frac{l_{vc}}{l_u} + \beta \cdot \frac{l_{vc}}{l_u}) \right] \cdot \omega \quad , \quad (146) \]

after substituting \( \pm l_{vc} \) with \( \pm l_{vc} = H \cdot l_{\xi_\perp} \), \( l_{vc} = \pm H \cdot l_{\xi_\perp} \), and \( \pm l_{vc} = H_c \cdot l_{\xi_\perp} \), \( l_{vc} = \pm H_c \cdot l_{\xi_\perp} \), we obtain

\[ \overline{\omega} = \omega \mp (\alpha \cdot H + \beta \cdot H_c) \cdot \frac{l_{\xi_\perp}}{l_u} \cdot \omega \quad , \quad (147) \]

The change of the frequency of the emitter is caused by both the velocities \( v_z \) and \( v_c \). Instead of the Hubble function \( H \) (Hubble’s effect) or of the aberration function \( H_c \) (aberration’s effect) a combination of both functions appear in the expression for \( \overline{\omega} \).

If we, further, express \( H \) and \( H_c \) with \( z \) and \( z_c \) respectively

\[ H = z \cdot \frac{l_u}{l_{\xi_\perp}} \quad , \quad H_c = z_c \cdot \frac{l_u}{l_{\xi_\perp}} \quad , \quad 28 \]
we obtain for $\bar{\omega}$

$$\bar{\omega} = \omega \mp (\alpha \cdot z + \beta \cdot z_c) \cdot \frac{l_u}{l_{\xi_\perp}} \cdot \frac{l_{\xi_\perp}}{l_u} \cdot \omega =$$

$$\omega \mp (\alpha \cdot z + \beta \cdot z_c) \cdot \omega \ . \tag{148}$$

and

$$\frac{\bar{\omega} - \omega}{\omega} = \mp (\alpha \cdot z + \beta \cdot z_c) \ . \tag{149}$$

Both the Hubble effect and the aberration effect could compensate each other if

$$\alpha \cdot z + \beta \cdot z_c = 0 \ , \tag{150}$$

i.e. if

$$z_c = -\frac{\alpha}{\beta} \cdot z \ , \tag{151}$$

or if

$$z_c = -\frac{\alpha}{\beta} \cdot z = H_c \cdot \frac{l_{\xi_\perp}}{l_u} = -\frac{\alpha}{\beta} \cdot H \cdot \frac{l_{\xi_\perp}}{l_u} \ , \tag{152}$$

$$H_c = -\frac{\alpha}{\beta} \cdot H \ . \tag{153}$$

Under the above conditions $[z_c = -(\alpha/\beta) \cdot z$ or $H_c = -(\alpha/\beta) \cdot H$] there will be no change of the frequency $\bar{\omega}$ of the emitter. The same frequency $\bar{\omega} = \omega$ will also be detected by the observer.

Since $\alpha = \mp \cos (\bar{n}_\perp, n_\perp)$ and $\beta = \mp \sin (\bar{n}_\perp, n_\perp)$ the relations for $z_c$ and $H_c$ will take the form

$$z_c = -\frac{\alpha}{\beta} \cdot z = \frac{\cos (\bar{n}_\perp, n_\perp)}{\sin (\bar{n}_\perp, n_\perp)} \cdot H \cdot \frac{l_{\xi_\perp}}{l_u} =$$

$$= -H \cdot \frac{l_{\xi_\perp}}{l_u} \cdot \cotg (\bar{n}_\perp, n_\perp) \ , \tag{154}$$

$$H_c = -\frac{\alpha}{\beta} \cdot H = -\frac{\cos (\bar{n}_\perp, n_\perp)}{\sin (\bar{n}_\perp, n_\perp)} \cdot H =$$

$$= -H \cdot \cotg (\bar{n}_\perp, n_\perp) \ . \tag{155}$$

If $\cotg (\bar{n}_\perp, n_\perp) = 0$ then $z_c = 0$, $H_c = 0$, and $\bar{\omega} = \omega$.

Denoting

$$\frac{\bar{\omega} - \omega}{\omega} = z_{gen} \ , \tag{158}$$

we obtain

$$z_{gen} = \mp (\alpha \cdot z + \beta \cdot z_c) \ . \tag{159}$$

The quantity $z_{gen}$ is the general observed shift parameter as a result of both effects. Then

$$z_{gen} = H_{gen} \cdot \frac{l_{\xi_\perp}}{l_u} = \mp (\alpha \cdot z + \beta \cdot z_c) = \mp (\alpha \cdot H + \beta \cdot H_c) \cdot \frac{l_{\xi_\perp}}{l_u} =$$

$$= \frac{H \cdot \cos (\bar{n}_\perp, n_\perp) + H_c \cdot \sin (\bar{n}_\perp, n_\perp)}{l_u} \cdot \frac{l_{\xi_\perp}}{l_u} = \tag{160}$$

$$= z \cdot \cos (\bar{n}_\perp, n_\perp) + z_c \cdot \sin (\bar{n}_\perp, n_\perp) \ . \tag{161}$$

$$= z \cdot \cos (\bar{n}_\perp, n_\perp) + z_c \cdot \sin (\bar{n}_\perp, n_\perp) \ . \tag{162}$$

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The general observed shift parameter $z_{gen}$ could take values $z_{gen} \geq 0$. These values depend on the motion of the emitter relative to the observer. In some cases, when $z_{gen} = 0$, the observer could not find any difference between the emitted frequency $\bar{\omega}$ and the detected frequency $\omega$ despite of the relative motion between emitter and observer. In these cases, the effect of a motion with centrifugal (centripetal) velocity of the emitter will be compensated by the effect, generated by a motion with Coriolis' velocity.

From the relations

\[ z_{gen} = H_{gen} \cdot l_{\xi_\perp} \quad , \quad z_{gen} = \frac{\bar{\omega} - \omega}{\omega} \quad , \]

we can find $l_{\xi_\perp}$ if $z_{gen}$, $H_{gen}$, and $l_u$ are given. Then

\[ l_{\xi_\perp} = \frac{z_{gen}}{l_{\xi_\perp}} \cdot l_u = \left( \frac{\bar{\omega} - 1}{\omega} \right) \cdot \frac{1}{H_{gen}} \cdot l_u \quad . \] (164)

In the same way, if $z_{gen}$, $H_{gen}$, and $l_{\xi_\perp}$ are known the absolute value $l_u$ of the velocity vector $u$ could be determined as

\[ l_u = \frac{H_{gen}}{z_{gen}} \cdot l_{\xi_\perp} = H_{gen} \cdot \frac{\bar{\omega}}{\omega - \omega} \cdot l_{\xi_\perp} \quad . \] (165)

Remark. From the relation

\[ l_u = \frac{1}{z_{gen}} \cdot (\alpha \cdot l_{v_z} + \beta \cdot l_{v_c}) \quad , \quad l_{v_z} \geq 0 \quad , \quad l_{v_c} \geq 0 \quad , \]

under the condition that $l_u > 0$, it follows that

\[ - \frac{1}{z_{gen}} \cdot (\alpha \cdot l_{v_z} + \beta \cdot l_{v_c}) > 0 \quad . \] (169)

Therefore, either $z_{gen} < 0$ and $\alpha \cdot l_{v_z} + \beta \cdot l_{v_c} > 0$ or $z_{gen} > 0$ and $\alpha \cdot l_{v_z} + \beta \cdot l_{v_c} < 0$.

(a) For $z_{gen} > 0$, it follows that

\[ \alpha > 0 : 0 < l_{v_z} < -\frac{\beta}{\alpha} \cdot l_{v_c} \quad , \quad -\frac{\beta}{\alpha} > 0 \quad , \]

\[ \alpha < 0 : l_{v_z} > -\frac{\beta}{\alpha} \cdot l_{v_c} \quad , \quad l_{v_c} > 0 \quad . \] (171)
(b) For $z_{gen} < 0$, it follows that
\[
\alpha \cdot l_{vz} + \beta \cdot l_{ve} > 0 \quad , \quad \alpha \cdot l_{vz} > -\beta \cdot l_{ve} \quad ,
\]
\[
\alpha > 0 \quad , \quad l_{vz} > -\frac{\beta}{\alpha} \cdot l_{ve} \quad , \quad l_{vz} > 0 \quad ,
\]
\[
\alpha < 0 \quad , \quad 0 < l_{vz} < -\frac{\beta}{\alpha} \cdot l_{ve} \quad , \quad -\frac{\beta}{\alpha} > 0 \quad .
\] (173)

If, in addition, it is assumed that $l_{vz} \leq l_u$, $l_{ve} \leq l_u$, then
(a) for $z_{gen} > 0$, it follows that
\[
0 < z_{gen} \cdot l_u = -(\alpha \cdot l_{vz} + \beta \cdot l_{ve}) \quad , \quad 0 < z_{gen} = -(\alpha \frac{l_{vz}}{l_u} + \beta \frac{l_{ve}}{l_u}) \quad ,
\]
\[
0 < -(\alpha \frac{l_{vz}}{l_u} + \beta \frac{l_{ve}}{l_u}) < -(\alpha \cdot l_u + \beta \cdot l_u) = -(\alpha + \beta) \quad ,
\] (174)
\[
\alpha + \beta < 0 \quad , \quad \alpha + \sqrt{1 - \alpha^2} < 0 \quad , \quad 0 < \sqrt{1 - \alpha^2} < -\alpha \quad ,
\]
\[
0 < 1 - \alpha^2 < \alpha^2 \quad , \quad 2 \cdot \alpha^2 > 1 \quad , \quad \alpha^2 > \frac{1}{2} \quad , \quad |\alpha| > \frac{\sqrt{2}}{2} \quad .
\] (176)

(b) for $z_{gen} < 0$, it follows that
\[
0 < \alpha \frac{l_{vz}}{l_u} + \beta \frac{l_{ve}}{l_u} < \alpha \frac{l_u}{l_u} + \beta \frac{l_u}{l_u} = \alpha + \beta \quad ,
\] (177)
\[
\alpha + \beta > 0 \quad , \quad \alpha + \sqrt{1 - \alpha^2} > 0 \quad , \quad \sqrt{1 - \alpha^2} > -\alpha \quad ,
\]
\[
1 - \alpha^2 > \alpha^2 \quad , \quad 1 > 2 \cdot \alpha^2 \quad , \quad \alpha^2 < \frac{1}{2} \quad , \quad |\alpha| < \frac{\sqrt{2}}{2} \quad .
\] (179)

Let $0 < z_{gen} \leq k_0 = -(\alpha + \beta) = -(\alpha + \sqrt{1 - \alpha^2})$. We can find the value of $\alpha$ for which $k_0$ is a real number and $k_0 > 0$. Since
\[
k_0 = -\alpha - \sqrt{1 - \alpha^2} \quad , \quad k_0 + \alpha = -\sqrt{1 - \alpha^2} \quad ,
\]
\[
k_0^2 + 2 \cdot k_0 \cdot \alpha + \alpha^2 = 1 - \alpha^2 \quad ,
\]
\[
\alpha^2 + k_0 \cdot \alpha + \frac{1}{2} \cdot (k_0^2 - 1) = 0 \quad ,
\]
\[
\alpha_{1,2} = \frac{1}{2} \cdot (-k_0 \pm \sqrt{k_0^2 - 2 \cdot (k_0^2 - 1)}) \quad ,
\]
\[
\alpha_{1,2} = \frac{1}{2} \cdot (-k_0 \pm \sqrt{2 - k_0^2}) \quad .
\]

If $\alpha_{1,2}$ are real numbers then $k_0^2 \leq \frac{2}{2}$ and $|k_0| \leq \sqrt{2}$. Therefore, if $0 < z_{gen} \leq k_0 = -(\alpha + \beta)$ then $0 < z_{gen} \leq \sqrt{2}$ and $\alpha = -(\sqrt{2}/2)$ for $k_0 = \sqrt{2}$.

The general observed shift parameter $z_{gen}$ could not have values bigger than $\sqrt{2}$ if $z_{gen} > 0$. This means that if we could measure values of $z_{gen} > \sqrt{2}$ the values of the general observed shift parameter could not be explained only on the basis of the existing Doppler effect. Other physical reasons should be taken into account if $z_{gen} > \sqrt{2}$.
5 Conclusion

In the present paper we have considered the notion of null (isotropic) vector field in spaces with affine connections and metrics. On the basis of the notion of centrifugal (centripetal) and Coriolis’ velocities the notions of standard (longitudinal) and transversal Doppler effects are introduced and considered in spaces with affine connections and metrics. On the other side, by the use of the Hubble law, leading to the introduction of the Hubble effect and the aberration effect, some connections between the kinematic characteristics of the relative velocity and the Doppler effects, the Hubble effect, and the aberration effect are investigated. It is shown that the Hubble effect and the aberration effect are corollaries of the standard and transversal Doppler effects. The Hubble effect and the aberration effect could influence each other and a general effect as a results of both effects could be considered. The upper limit of the general observed shift parameter $z_{gen}$ if both the effects appear is estimated at $z_{gen} = \sqrt{2}$. This means that values of the general observed shift parameter bigger than $\sqrt{2}$ and found experimentally could not be explained only on the basis of the existing Doppler effects. In such cases, other physical reasons should be taken into account.

The Doppler effects, the Hubble effect, and the aberration effect are considered on the grounds of purely kinematic considerations. It should be stressed that the Hubble and the Coriolis functions $H$ and $H_c$ are introduced on a purely kinematic basis related to the notions of relative velocity and to the notions of centrifugal (centripetal) and Coriolis’ velocities. Its dynamic interpretations in a theory of gravitation depends on the structures of the theory and the relations between the field equations and both the functions. In this paper it is shown that notions the specialists use to apply in theories of gravitation and cosmological models could have a good kinematic grounds independent of any concrete classical field theory. Doppler effects, Hubble’s effect, and aberration’s effect could be used in mechanics of continuous media and in other classical field theories in the same way as the standard Doppler effect is used in classical and relativistic mechanics.

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