NONLINEAR V ARIANTS OF A THEOREM OF KWAPIEŃ

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Abstract. A famous result of S. Kwapieni asserts that a linear operator from a Banach space to a Hilbert space is absolutely \(1\)-summing whenever its adjoint is absolutely \(q\)-summing for some \(1 \leq q < \infty\); this result was recently extended to Lipschitz operators by Chen and Zheng. In the present paper we show that Kwapieni’s and Chen–Zheng theorems hold in a very relaxed nonlinear environment, under weaker hypotheses. Even when restricted to the original linear case, our result generalizes Kwapieni’s theorem because it holds when the adjoint is just almost summing. In addition, a variant for \(L_p\)-spaces, with \(p \geq 2\), instead of Hilbert spaces is provided.

1. Introduction

The theory of absolutely summing linear operators was originated from Grothendieck’s Resumé [13] and since the publication of the papers of Lindenstrauss, Pełczyński and Pietsch [15, 22], it has performed a fundamental role in Banach Space Theory and its applications. In the last decades the path from the linear to the nonlinear theory of absolutely summing operators was encouraged by seminal works of Pietsch [23], Farmer and Johnson [10], among others. Nowadays, nonlinear variants of absolutely summing operators have been explored in different settings with applications in different fields of pure and applied mathematics (see, for instance, [2, 21, 25] and the references therein). Let \(X, Y\) be Banach spaces over \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\) and, from now on, \(B_X\) denotes the closed unit ball of \(X\), and \(X^*\) represents its topological dual. If \(1 \leq p < \infty\), a linear operator \(T : X \to Y\) is absolutely \(p\)-summing if there exists a constant \(C\) such that

\[
\left( \sum_{i=1}^{n} \|T(x_i)\|^p \right)^{1/p} \leq C \| \langle x_i \rangle_{i=1}^{n} \|_{p,w},
\]

for every \(x_1, ..., x_n \in X\) and all positive integers \(n\), where

\[
\| \langle x_i \rangle_{i=1}^{n} \|_{p,w} := \sup_{\varphi \in B_{X^*}} \left( \sum_{i=1}^{n} |\varphi(x_i)|^p \right)^{1/p}.
\]

The class of absolutely \(p\)-summing linear operators from \(X\) to \(Y\) will be represented, as it is usual, by \(\Pi_p(X, Y)\) and the infimum of all \(C\) satisfying (11) defines a norm on \(\Pi_p(X, Y)\), denoted by \(\pi_p(T)\).

The related notion of almost summing operators will be important for our purposes. According to [3], a linear operator \(T : X \to Y\) is almost \(p\)-summing if there is a constant \(C\) such that

\[
\left( \int_0^1 \left\| \sum_{i=1}^{n} t r_i(t) T(x_i) \right\|^2 dt \right)^{\frac{1}{2}} \leq C \| \langle x_i \rangle_{i=1}^{n} \|_{p,w},
\]

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for every positive integer \( n \) and \( x_1, \ldots, x_n \in X \). Above, \( r_i \) are the Rademacher functions, defined as

\[
  r_i : [0, 1] \rightarrow \mathbb{R} \\
  t \mapsto \text{sign} (\sin 2^i \pi t).
\]

The infimum of the constants \( C \) satisfying (2) is denoted by \( \pi_{al.s.p}(T) \). The class of all almost \( p \)-summing operators is denoted by \( \Pi_{al.s.p} \). When \( p = 2 \), these operators are simply called almost summing and we write \( \Pi_{al.s} \) instead of \( \Pi_{al.s.2} \) (see [9, Chapter 12]). By [9, Proposition 12.5],

\[
  \bigcup_{1 \leq q < \infty} \Pi_q(X, Y) \subseteq \Pi_{al.s}(X, Y).
\]

For details, we refer to the classical book by Diestel, Jarchow and Tonge [9].

From now on \( \mathcal{I} \) denotes an operator ideal, \( \mathcal{L} \) denotes the ideal of all bounded linear operators and \( u^* \) represents the adjoint of a linear operator \( u \). We denote, as usual, by \( \mathcal{I}_{\text{dual}} \) the dual ideal of \( \mathcal{I} \) (see [9, page 186]), that is,

\[
  \mathcal{I}_{\text{dual}}(X, Y) := \{ u \in \mathcal{L}(X, Y); \ u^* \in \mathcal{I}(Y^*, X^*) \}.
\]

In general, the adjoint of an absolutely \( p \)-summing operator may fail to be absolutely \( p \)-summing and vice-versa. For instance, considering the formal identity \( i : \ell_2 \rightarrow c_0 \), it follows that \( i^* \) is absolutely \( p \)-summing for all \( 1 \leq p < \infty \), whereas \( i \) and \( i^{**} \) aren’t absolutely \( p \)-summing. Then, the following problem arises

**Problem 1.1.** For which Banach spaces \( X \) and \( Y \) and \( 1 \leq p, q < \infty \) does the following inclusion hold

\[
\Pi_q^{\text{dual}}(X, Y) \subseteq \Pi_p(X, Y)?
\]

A remarkable result due to Kwapieni [14] shows that a linear operator \( T \) from a Banach space \( X \) to a Hilbert space \( H \) is absolutely 1-summing whenever \( T^* \) is absolutely \( q \)-summing for some \( 1 \leq q < \infty \). In other words,

**Theorem 1.2.** (see [9, Theorem 2.21]) If \( X \) is a Banach space and \( H \) is a Hilbert space, then

\[
  \bigcup_{1 \leq q < \infty} \Pi_q^{\text{dual}}(X, H) \subseteq \Pi_1(X, H).
\]

Kwapień’s Theorem was recently extended to Lipschitz operators by Chen and Zheng [7, Theorem 3.1] as we shall see next. Let \( X \) be a pointed metric space with a base point denoted by \( 0 \), and let \( Y \) be a Banach space. The Lipschitz space \( Lip_0(X; Y) \) is the Banach space of all Lipschitz operators \( T : X \rightarrow Y \) such that \( T(0) = 0 \), under the Lipschitz norm defined by

\[
Lip(T) = \sup \left\{ \frac{\|T(x) - T(y)\|}{d(x, y)} : x, y \in X, \; x \neq y \right\}.
\]

When \( Y = \mathbb{K} \), we write \( X^\# = Lip_0(X; \mathbb{K}) \). A cornerstone of the nonlinear theory of absolutely summing operators is the paper of Farmer and Johnson [10], which introduces the notion of absolutely summing operators to the Lipschitz framework as follows: a Lipschitz operator \( T : X \rightarrow Y \) is **Lipschitz \( p \)-summing** if there is a constant \( C \) such that

\[
\left( \sum_{i=1}^{n} \|T(x_i) - T(q_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X^\#}} \left( \sum_{i=1}^{n} |\varphi(x_i) - \varphi(q_i)|\right)^{\frac{1}{p}},
\]

for all \( x_1, \ldots, x_n, q_1, \ldots, q_n \in X \) and all positive integers \( n \) (for related papers on nonlinear summing operators we refer to [5, 6, 11, 16] and the references therein). According to [24], the Lipschitz
adjoint $T^\#$ of $T \in \text{Lip}_0(X;Y)$ is defined as the continuous linear operator $T^\# : Y^\# \to X^\#$ given by $T^\#(g) := g \circ T$. The variant of Kwapień’s Theorem to Lipschitz operators due to Chen and Zheng reads as follows:

**Theorem 1.3.** (see [7, Theorem 3.1]) Let $X$ be a pointed metric space and $H$ be a Hilbert space. If $T \in \text{Lip}_0(X;H)$ is such that $T^\#|_{H^*}$ is absolutely $q$-summing for some $1 \leq q < \infty$, then $T$ is Lipschitz $1$-summing.

Following this vein, other questions seem natural:

**Problem 1.4.** Are there other nonlinear extensions/generalizations of the Kwapień Theorem?

**Problem 1.5.** Are there variants of the Kwapień Theorem in which the range is not necessarily a Hilbert space?

In this paper we shall answer these problems. To state our main result we shall recall the notion of $\mathcal{L}_p$-spaces. If $\lambda > 1$ and $1 \leq p \leq \infty$, a Banach space $Y$ is an $\mathcal{L}_{p,\lambda}$-space if every finite dimensional subspace $E$ of $Y$ is contained in a finite dimensional subspace $Y_0$ of $Y$ for which there is an isomorphism $v : Y_0 \to \ell^{\dim(Y_0)}$ such that $\|v\|\|v^{-1}\| \leq \lambda$. When $Y$ is an $\mathcal{L}_{p,\lambda}$-space for a certain $\lambda$, we just say that $Y$ is an $\mathcal{L}_p$-space. It is simple to observe that Hilbert spaces are $\mathcal{L}_2$-spaces. As a consequence of the main results of the present paper we conclude that the original result of Kwapień can be improved as follows:

**Theorem 1.6.** If $X$ is a Banach space and $Y$ is an $\mathcal{L}_p$-space and $2 \leq p < \infty$, then

$\Pi_{al,s,p}(X,Y) \subseteq \Pi_1(X,Y)$.

In particular, when $Y$ is a Hilbert space

(5) $\Pi_{al,s}(X,Y) \subseteq \Pi_1(X,Y)$.

Note that from (6) it is obvious that (5) recovers the statement of Kwapień’s Theorem. The paper is organized as follows. In Section 2 we state our main result (Theorem 2.2); in Section 3 we prove Theorem 2.2 and in Section 4 we provide applications of the main result; for instance, we shall generalize the Chen–Zheng Theorem (Theorem 1.3) following the lines of what is done in Theorem 1.6.

### 2. Main Result

We start off by recalling that the sequence of Rademacher functions $(r_i)_{i=1}^\infty$ is orthonormal; thus

(6) $\int_0^1 \left| \sum_{i=1}^n r_i(t) a_i \right|^2 dt = \sum_{i=1}^n |a_i|^2$

for all $(a_i)_{i=1}^n \in \ell^n_2$ and all positive integers $n$. The well-known Kahane Inequality shows that the spaces generated by the Rademacher functions have equivalent $L_p$ norms:

**Theorem 2.1** (Kahane inequality). Let $0 < p, q < \infty$. Then there is a constant $K_{p,q} > 0$ for which

$$\left( \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^q dt \right)^{\frac{1}{q}} \leq K_{p,q} \left( \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^p dt \right)^{\frac{1}{p}}$$

holds, regardless of the choice of a Banach space $X$ and of finitely many vectors $x_1, \ldots, x_n \in X$. 
The previous theorem, combined with [3] recovers the Khinchin inequality (see [9, page 10]), when \( q = 2 \) and \( X = K \), and in this case \( K^{-1}_{p,2} \) is usually denoted by \( A_p \); it is simple to observe that \( A_2 = 1 \) and it is well-known that \( A_1 = (\sqrt{2})^{-1} \).

In the recent years a series of works ([3, 17, 18, 19, 20]) have shown that several important results of the theory of summing operators in fact need essentially no linear structure to be valid. The proof of our main result shall rely on the abstract environment presented in [18].

From now on, unless stated otherwise, \( p \in [1, \infty) \). Let \( X \) and \( Y \) be non-void sets, \( H(X; Y) \) be a non-void family of mappings from \( X \) to \( Y \) and \( K \) be a compact Hausdorff space. Let

\[
R: K \times X \times X \longrightarrow [0, \infty) \quad \text{and} \quad S: H(X; Y) \times X \times X \longrightarrow [0, \infty)
\]

be arbitrary mappings. A mapping \( f \in H(X; Y) \) is \( RS\)-abstract \( p \)-summing if there is a constant \( C \) such that

\[
\left( \sum_{i=1}^{n} S(f, x_i, q_i)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{i=1}^{n} R(\varphi, x_i, q_i)^p \right)^{\frac{1}{p}},
\]

for all \( x_1, \ldots, x_n, q_1, \ldots, q_n \in X \) and all positive integers \( n \). We define

\[
H_{RS, p}(X; Y) = \{ f \in H(X; Y) : f \text{ is } RS\text{-abstract } p\text{-summing} \}.
\]

Let \( X \) be a pointed metric space and \( Y \) be a Banach space. We shall choose a suitable Banach space \( X^d \subseteq K^X \) containing \( X^# \) such that

\[
f^a: Y^# \longrightarrow X^d, f^a(h) := h \circ f
\]

is well-defined for all \( f \in H(X; Y) \). In general, for our applications, it will be enough to consider \( X^d = X^# \). Note that, when \( f \) is a linear operator between Banach spaces we have \( f^a(y^*) = f^*(y^*) \) for all \( y^* \in Y^* \), i.e., \( f^a \) is a kind of \textit{abstract adjoint} of \( f \). We shall need the following properties of \( R \) and \( S \):

(I) \( S(f, x, q) \leq \| f(x) - f(q) \| \) for all \( f \in H(X; Y) \) and \( x, q \in X \).

(II) \( K \subseteq X^d, B_{TV}(Y^*) \subseteq K \) and

\[
|g(x) - g(q)| \leq R(g, x, q) \quad \text{for all } g \in B_{TV}(Y^*) \text{ and } x, q \in X.
\]

In the next section we shall present a nonlinear Kwapień-type theorem valid for \( L_p \)-spaces instead of just Hilbert spaces. As an illustration, when the main result of this section is restricted to the linear case, we conclude that if \( X \) is a Banach space and \( Y \) is an \( L_p \)-space, with \( 2 \leq p < \infty \), then

\[
\Pi_{dual, L_p}^d(X, Y) \subseteq \Pi_1(X, Y).
\]

Our techniques are, in general, different from the ones used in the proofs of Theorems [12] and [13]. for instance, unlike what is done in the aforementioned results, we do not use the Pietsch Domination Theorem. Our main result is the following:

\begin{theorem}
Let \( X \) be a pointed metric space, \( Y \) be an \( L_p \)-space, with \( 2 \leq p < \infty \), and \( f: X \longrightarrow Y \) an arbitrary map. If \( R \) and \( S \) satisfy (I) and (II) and \( f^a|_{Y^*} \) is almost \( p \)-summing, then \( f \) is \( RS\)-abstract \( 1 \)-summing.
\end{theorem}

In Section 4, applications of our main result are provided. For instance, choosing suitable \( R, S \) we obtain extensions of the theorems of Kwapień and Chen–Zheng (see Subsections [11] and [12]).
3. Proof of the main result

Let \( f \in \Pi_{al.s.p}^d(X,Y) \). Fix \( x_1,\ldots,x_n,q_1,\ldots,q_n \in X \) and a subspace \( Y_0 \) of \( Y \) containing \( \{f(x_1),\ldots,f(x_n),f(q_1),\ldots,f(q_n)\} \) and for which there is an isomorphism \( \nu : Y_0 \to \ell_p^m \) such that \( \|\nu\|\|v^{-1}\| \leq \lambda \). Then, using the monotonicity of the \( \ell_p \) norms, the Khinchin inequality and denoting by \( (e_k)_{k=1}^m \) the canonical basis of \( \ell_p^m = (\ell_p^m)^* \), where \( p^* \) is the conjugate of \( p \), it follows that

\[
\sum_{i=1}^n S(f, x_i, q_i) \leq \sum_{i=1}^n \|f(x_i) - f(q_i)\|
\]

By the Hahn-Banach Theorem we can extend \( \nu^*(e_k) \) to \( Y^* \), and thus

\[
\sum_{i=1}^n S(f, x_i, q_i) \leq \sqrt{2} \cdot \|v^{-1}\| \cdot \sum_{i=1}^n \int_0^1 \left| \sum_{k=1}^m f^{a^*}(e_k)(x_i) \cdot r_k(t) - \sum_{k=1}^m f^{a^*}(e_k)(q_i) \cdot r_k(t) \right| dt
\]

Combining the previous inequality with (II) we obtain

\[
\sum_{i=1}^n S(f, x_i, q_i) \leq \sqrt{2} \cdot \|v^{-1}\| \cdot \left( \int_0^1 \left\| \sum_{k=1}^m r_k(t) f^{a^*}(e_k) \right\|^2 dt \right)^{1/2} \cdot \sup_{g \in K} \sum_{i=1}^n R(g, x_i, q_i)
\]

and, using the monotonicity of the \( L_p \) norms, we have

\[
\sum_{i=1}^n S(f, x_i, q_i) \leq \sqrt{2} \cdot \|v^{-1}\| \cdot \left( \int_0^1 \left\| \sum_{k=1}^m r_k(t) f^{a^*}(e_k) \right\|^2 dt \right)^{1/2} \cdot \sup_{g \in K} \sum_{i=1}^n R(g, x_i, q_i).
\]
Since $f^a|_{H^*}$ is almost $p$-summing, it follows that $f^a|_{Y^*}$ also is almost $p$-summing and, since $\|(e_k)_{k=1}^n\|_{p,w} = 1$ in $l_{p,w}^n$, we have

$$\sum_{i=1}^n S(f, x_i, q_i) \leq \sqrt{2} \cdot \|v^{-1}\| \cdot \pi_{al,s.p}(f^a|_{Y^*}) \cdot \sup_{g \in K} \sum_{i=1}^n R(g, x_i, q_i)$$

$$\leq \sqrt{2} \cdot \|v^{-1}\| \cdot \|v^*\| \cdot \pi_{al,s.p}(f^a|_{Y^*}) \cdot \sup_{g \in K} \sum_{i=1}^n R(g, x_i, q_i)$$

$$= \sqrt{2} \cdot \|v^{-1}\| \cdot \|v\| \cdot \pi_{al,s.p}(f^a|_{Y^*}) \cdot \sup_{g \in K} \sum_{i=1}^n R(g, x_i, q_i)$$

$$\leq \sqrt{2} \cdot \lambda \cdot \pi_{al,s.p}(f^a|_{Y^*}) \cdot \sup_{g \in K} \sum_{i=1}^n R(g, x_i, q_i).$$

As a corollary, since every Hilbert space is an $L_2$-space, we have an abstract generalization of the Kwapień Theorem:

**Corollary 3.1** (Abstract Kwapień-type Theorem). Let $X$ be a pointed metric space, $H$ be a Hilbert space and $f \in \mathcal{H}(X, H)$. If $R$ and $S$ satisfy (I) and (II) and $f^a|_{H^*}$ is absolutely $q$-summing for some $1 \leq q < \infty$, then $T$ is $RS$-abstract $1$-summing.

4. Applications

In this section we show that Theorem 2.2 provides new Kwapień-type theorems for several classes of linear and nonlinear summing operators (and recovers the known results).

4.1. Absolutely summing linear operators. Let $X$ and $Y$ be Banach spaces and $T: X \rightarrow Y$ be a bounded linear operator. Considering the weak-star topology in $B_{X^*}$, and letting

$$(K, \mathcal{H}(X; Y)) = (B_{X^*}, \mathcal{L}(X; Y)),$$

$$\begin{cases}
R: K \times X \times X \rightarrow [0, \infty) \\
R(\varphi, x, q) = |\varphi(x - q)|,
\end{cases}$$

and

$$\begin{cases}
S: \mathcal{L}(X; Y) \times X \times X \rightarrow [0, \infty) \\
S(T, x, q) = \|T(x - q)\|,
\end{cases}$$

it is plain that $T$ is $RS$-abstract $p$-summing if, and only if, there is a constant $C$ such that

$$\left(\sum_{i=1}^n \|T(x_i - q_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^n |\varphi(x_i - q_i)|^p\right)^{\frac{1}{p}},$$

for all $x_1, \ldots, x_n, q_1, \ldots, q_n \in X$ and every positive integer $n$. Thus, $T$ is absolutely $p$-summing if, and only if, it is $RS$-abstract $p$-summing. Taking $X^d = X^*$, it follows that $T^a|_{Y^*} = T^*$ and the hypotheses (I) and (II) are satisfied. If $Y$ is an $L_p$-space, with $2 \leq p < \infty$, Theorem 2.2 tells us that $T$ is absolutely $1$-summing wherever $T^*$ almost $p$-summing. So, Corollary 3.1 recovers Theorem 1.6.
4.2. Lipschitz $p$-summing and $p$-dominated operators. Let $X$ be a pointed metric space with a base point denoted by 0 and let $Y$ be an $\mathcal{L}_p$-space, with $2 \leq p < \infty$. We can easily note that $T$ is Lipschitz $p$-summing if, and only if, it is $RS$-abstract $p$-summing. Considering $B_{X^\#}$ with the pointwise convergence topology,

$$ (K, \mathcal{H}(X;Y)) = (B_{X^\#}, Lip_0(X, Y)),$$

$$ R: K \times X \times X \rightarrow [0, \infty) $$

$$ R(\varphi, x, q) = |\varphi(x) - \varphi(q)|,$$

and

$$ S: Lip_0(X, Y) \times X \times X \rightarrow [0, \infty) $$

$$ S(T, x, q) = \|T(x) - T(q)\|,$$

and considering $X^d = X^\#$ we can invoke Theorem 2.2 to prove the following extension of the Chen–Zheng Theorem:

**Theorem 4.1.** Let $X$ be a pointed metric space and $Y$ be an $\mathcal{L}_p$-space, with $2 \leq p < \infty$. If $T \in Lip_0(X; Y)$ is such that $T^#_{Y^\ast}$ is almost $p$-summing, then $T$ is Lipschitz 1-summing.

According to Chen and Zheng [8], a Lipschitz operator $T: X \rightarrow Y$ between Banach spaces is **Lipschitz $p$-dominated** if there exist a Banach space $Z$ and an absolutely $p$-summing linear operator $L: X \rightarrow Z$ such that

$$ \|T(x) - T(q)\| \leq \|L(x) - L(q)\| \text{ for all } x, q \in X. $$

(7)

As a consequence of our main theorem we also conclude that if a Lipschitz operator $T: X \rightarrow Y$ satisfies (7) for a certain $L: X \rightarrow Z$, where $Z$ is an $\mathcal{L}_p$-space, with $2 \leq p < \infty$, then $T$ is Lipschitz 1-dominated whenever $L^\ast$ is almost $p$-summing.

4.3. Absolutely $p$-summing $\Sigma$-operators. According to Angulo-López and Fernández-Unzueta [1, 12], given Banach spaces $X_1, \ldots, X_n$, the set

$$ \Sigma_{X_1, \ldots, X_n} := \{ x_1 \otimes \cdots \otimes x_n \in X_1 \otimes \cdots \otimes X_n : x_i \in X_i, i = 1, \ldots, n \}$$

is the metric space of decomposable tensors endowed with the metric induced by the projective tensor norm. It is called the **metric Segre cone** of $X_1, \ldots, X_n$.

If $T : X_1 \times \cdots \times X_n \rightarrow Y$ is a multilinear mapping and $Y$ is a vector space, we denote by $\hat{T} \in L(X_1 \otimes \cdots \otimes X_n; Y)$ the unique linear mapping satisfying that for every $x_i \in X_i$, with $i \in \{1, \ldots, n\}$ and $T(x_1, \ldots, x_n) = \hat{T}(x_1 \otimes \cdots \otimes x_n)$.

**Definition 4.2 (see [1]).** If $X_1, \ldots, X_n$ are Banach spaces and $Y$ is a vector space, an operator $f : \Sigma_{X_1, \ldots, X_n} \rightarrow Y$ is a $\Sigma$-operator if there exists a multilinear operator $T \in L(X_1, \ldots, X_n, Y)$ such that $f = \hat{T}|_{\Sigma_{X_1, \ldots, X_n}}$.

We denote

$$ L(\Sigma_{X_1, \ldots, X_n}, Y) = \{ f : \Sigma_{X_1, \ldots, X_n} \rightarrow Y : f \text{ is a } \Sigma \text{-operator} \},$$

and by $L(\Sigma_{X_1, \ldots, X_n})$ we denote the space of scalar-valued continuous $\Sigma$-operators endowed with the Lipschitz norm, which happens to be a dual Banach space in which we shall consider the weak-star topology.
Definition 4.3 (see [1]). Let \( X_1, \ldots, X_m, Y \) be Banach spaces. A bounded \( \Sigma \)-operator \( f : \Sigma_{X_1,\ldots,X_m} \to Y \) is absolutely \( p \)-summing if there is a constant \( C \) so that

\[
\left( \sum_{i=1}^{n} \| f(x_i) - f(q_i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{\mathcal{L}(\Sigma_{X_1,\ldots,X_m})}} \left( \sum_{i=1}^{n} | \varphi(x_i) - \varphi(q_i) |^p \right)^{\frac{1}{p}},
\]

for every natural number \( n \) and all \( x_i, q_i \in \Sigma_{X_1,\ldots,X_m} \), with \( i = 1, \ldots, n \).

Let us choose, in our abstract framework, \( X \) as the pointed metric space \( \Sigma_{X_1,\ldots,X_m} \) with the base point \( 0 = 0 \otimes \cdots \otimes 0 \), and

\[
(K, \mathcal{H}(X;Y)) = \left( B_{\mathcal{L}(\Sigma_{X_1,\ldots,X_m})}, \text{Lip}_0(X,Y) \right),
\]

where

\[
\begin{align*}
R : K \times X \times X &\to [0, \infty) \\
R(\varphi, x, q) &= | \varphi(x) - \varphi(q) |,
\end{align*}
\]

and

\[
\begin{align*}
S : \text{Lip}_0(X,Y) \times X \times X &\to [0, \infty) \\
S(f, x, q) &= \| f(x) - f(q) \|.
\end{align*}
\]

Letting \( Y \) be an \( \mathcal{L}_p \)-space, with \( 2 \leq p < \infty \), \( X^d = X^\# = \mathcal{L}(\Sigma_{X_1,\ldots,X_m}) \) we have that a bounded \( \Sigma \)-operator \( f \in \text{Lip}_0(X,Y) \) is absolutely \( p \)-summing if, and only if, it is \( RS \)-abstract \( p \)-summing. As \( f^a = f^\# \), Theorem 2.2 and Corollary 3.1 provide Kwapień-type theorems for this class of operators.

4.4. Lipschitz \( p \)-summing operators at one point. The next definition is motivated by the notion of absolutely summing arbitrary mappings between Banach spaces (see [17] and the references therein).

Definition 4.4. Let \( X \) be a normed vector space and \( Y \) be a Banach space. A Lipschitz operator \( T : X \to Y \) is Lipschitz \( p \)-summing at the point \( w \in X \) if there is a constant \( C \) such that

\[
\left( \sum_{i=1}^{n} \| T(w + x_i) - T(w + q_i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_X^\#} \left( \sum_{i=1}^{n} | \varphi(x_i) - \varphi(q_i) |^p \right)^{\frac{1}{p}},
\]

for every positive integer \( n \) and every \( x_1, \ldots, x_n, q_1, \ldots, q_n \in X \).

Note that Lipschitz \( p \)-summability at 0 is precisely the notion of Lipschitz \( p \)-summability (see [1]). Given \( T \in \text{Lip}(X,Y) \), we define \( T_w : X \to Y \) by

\[
T_w(x) = T(w + x) - T(w)
\]

and it is simple to check that \( T \in \text{Lip}(X,Y) \) if, and only if, \( T_w \in \text{Lip}(X,Y) \) and \( \text{Lip}(T) = \text{Lip}(T_w) \).

Note also that

\[
T \text{ is Lipschitz } p \text{-summing at } w \text{ if, and only if, } T_w \text{ is Lipschitz } p \text{-summing.}
\]

Considering

\[
w - \text{Lip}(X,Y) := \{ T_w : T \in \text{Lip}(X,Y) \},
\]

and choosing \( K = B_X^\# \), with the pointwise convergence topology, \( \mathcal{H}(X;Y) = w - \text{Lip}(X,Y) \) and \( \mathcal{R}, \mathcal{S} \) defined by

\[
\begin{align*}
\mathcal{R} : B_X^\# \times X \times X &\to [0, \infty) \\
\mathcal{R}(\varphi, x, q) &= | \varphi(x) - \varphi(q) |,
\end{align*}
\]

and

\[
\mathcal{S} : B_X^\# \times X \times X &\to [0, \infty) \\
\mathcal{S}(f, x, q) &= \| f(x) - f(q) \|.
\]

and
\[
\begin{align*}
S: w - \text{Lip}(X,Y) \times X \times X & \to [0, \infty) \\
S(T_w, x, q) = \|T_w(x) - T_w(q)\|,
\end{align*}
\]
we conclude that a $T_w \in w - \text{Lip}(X,Y)$ is RS-abstract $p$-summing if, and only if, $T_w$ is Lipschitz $p$-summing (or equivalently, $T$ is Lipschitz $p$-summing at $w$).

If $Y$ is an $\mathcal{L}_p$-space, with $2 \leq p < \infty$, $X^d = X^\#$, then the abstract adjoint $(T_w)^\#|_{Y^*}$ coincides with the Lipschitz adjoint $(T_w)^\#$. Consequently, we have the following theorem

**Theorem 4.5.** Let $X$ be a normed vector space, $Y$ be an $\mathcal{L}_p$-space, with $2 \leq p < \infty$, and $T \in \text{Lip}(X;Y)$. If $(T_w)^\#|_{Y^*}$ is almost $p$-summing, then $T$ is Lipschitz 1-summing at $w$.

Note that the Chen–Zheng theorem generalizes the result of Kwapień, since it shows that when $X$ is a normed space we can replace $\text{Lip}_0(X;Y)$ by $\text{Lip}(X;Y)$.

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