Since the pioneering work of Dyson [1], who provided an argument as to why perturbation series in quantum electrodynamics (QED) are divergent, the fundamental problem of how to reconstruct physical observables from power-series expansions with zero convergence radius has remained an active area of research [2–3]. This problem has been encountered in virtually all areas of quantum physics, such as statistical [4–6], string [7] and quantum field theories [8], as well as many-body problems of condensed matter physics [9].

The simplest examples can be found in single-particle quantum mechanics [2–4]. For instance, the perturbation expansion for the Stark Hamiltonian has zero radius of convergence [12, 13]. When a hydrogen atom is placed in a homogeneous electric field, the electronic ground state energy is shifted and broadened, thus acquiring an imaginary part which is considered to be a paradigm of nonperturbative behavior. Here we demonstrate how the low order coefficients of a divergent perturbation series can be used to obtain excellent approximations to both real and imaginary parts of the perturbed ground state eigenenergy. The key is to use analytic continuation functions with a built-in analytic structure within the complex plane of the coupling constant, which is tailor-made by means of Bender-Wu dispersion relations. In the examples discussed the analytic continuation functions are Gauss hypergeometric functions, which have been used for rather large values of the coupling constant.

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These considerations lead us to the following questions: Do we really need a large number of coefficients to predict quantities like $\Gamma(F)$? Or can it be done with just a few coefficients? In this Letter, we demonstrate that low-order approximations can, paradoxically, reproduce nonperturbative quantities like $\Gamma(F)$ with excellent accuracy even for rather large values of the perturbation strength. This is achieved by using the so-called Bender-Wu dispersion relations [10] to guess the branch cut structure of $E(F)$ for large values of $F$ to design rapidly converging strong-coupling expansions from divergent weak-coupling expansions. While promising, these approaches have remained largely unexplored.

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indeed known from Bender-Wu dispersion relations [10]: $E(F)$ possesses branch points at $F = 0$ and $F \to \pm \infty$.

Let us then start by trying to calculate $E(F)$ using a divergent series such as Eq. (1). Traditionally, the first choice is to calculate Padé approximants [18]. These are parameterized rational approximations, $E(F) \approx E_{L/M}(F)$, of the form

$$E_{L/M}(F) = \frac{\sum_{n=0}^{L} p_n f^n}{1 + \sum_{n=1}^{M} q_n f^n},$$

(2)

where the parameters $p_n$ and $q_n$ are determined by equating each order up to $L+M = N$ in the Taylor and asymptotic series of $E_{L/M}(F)$ and $E(F)$, respectively, around $F = 0$, so that $E(F) = E_{L/M}(F) + O(f^{L+M+1})$. Padé approximants and other similarly simple sequence transformations are valuable tools for analytic continuation (AC), and can work well in many cases of practical interest [17]. They provide a family of rational functions that are easily built order by order: first-order perturbation theory gives $E_{1/0}$ and $E_{0/1}$; second-order perturbation theory yields $E_{2/0}$, $E_{0/2}$ and $E_{1/1}$; etc. By studying the resulting Padé table, one can in many cases extract good approximations to the expectation value of interest. However, by approximating $E(F)$ with a rational function of $F$, one is imposing an asymptotic behavior for large values of $F$ which is in general not physical. Approximating $\Delta(F)$ can be difficult because the denominator in $E_{L/M}$ can vanish for specific values of the interaction strength. More importantly, $E_{L/M}(F)$ is a real number for real $F$, and therefore $\Gamma_{L/M}(F) = 0$. This means that the standard Padé approximants cannot work for our problem as they fail to give $\Gamma(F) \neq 0$ [12, 14]. Nevertheless, the idea behind Padé approximants is very general and it can be used to propose new approximations. For example, one can choose a parameterized analytic function $E(F) = \mathcal{E}\{h_i\}; F$ to approximate $E(F)$, fixing the parameters $\{h_i\}$ so that the Taylor series for $\mathcal{E}(F)$ is equal to the asymptotic series of $E(F)$ up to the desired order. An example is provided in Ref. [19], which considers continued-exponential approximations of the form $\exp(h_1 \exp(h_2 x \exp(h_3 x \ldots)))$, where the parameters $h_i$ are fixed from the perturbation expansion of $E(F)$, as is the case with Padé approximants. Since here we are concerned with the determination of $\Gamma(F)$, we initially aimed for a function $\mathcal{E}(F)$ with the following desirable properties: (i) it is a complex function of real $F$, with the ability to mimic the branch cut structure discussed above; (ii) it can be built from low-order perturbation theory, as Padé approximants and continued-functions are built; (iii) it is amenable to generalization by being a member of a more general family of “higher order” functions; and (iv) it is sufficiently general and flexible in order to include many possible functions as particular cases.

A possible candidate satisfying all desirable properties (i)-(iv) is the Gauss hypergeometric function $\, _2F_1(h_1, h_2, h_3; h_4 f)$. It satisfies (i) and (ii) as it is complex (and has a branch cut) for $h_4 f > 1$, and it contains at most four parameters so it can be built from the coefficients $e_1, \ldots, e_4$. It also satisfies condition (iii) because $\, _2F_1$ can be obtained by approximating the ratio between consecutive expansion coefficients, $e_n/e_{n-1}$, by a $1/1$ Padé approximant that reproduces exactly $e_n/e_{n-1}$ for $0 < n \leq 4$. If higher-order Padé approximants could be used as well, one would obtain hypergeometric functions of higher order, $pF_q$, which are actually instances of an even “higher-order” function—the so-called Meijer-G function [15]. Finally, $\, _2F_1$ satisfies condition (iv) as it is well known that many functions are particular cases of $\, _2F_1$.

The Taylor series for $\, _2F_1$ is given by:

$$\, _2F_1(h_1, h_2, h_3; h_4 f) = \sum_{n=0}^{\infty} \frac{(h_1)_n (h_2)_n}{n! (h_3)_n} h_4^n f^n,$$

(3)

where $(h_i)_n = h_i (h_i + 1) \cdots (h_i + n - 1)$ is a so-called Pochhammer symbol. To obtain the $h_i$, one equates each order in the asymptotic series for $E(F)$ with the corresponding order in the Taylor series for $\mathcal{E}(F)$ to obtain a system of four equations with four unknowns

$$e_n = \frac{(h_1)_n (h_2)_n h_4^n}{(h_3)_n n!}, \quad 1 \leq n \leq 4.$$

(4)

Once the coefficients $h_i$ are determined, a hypergeometric approximation $E(F) \approx \mathcal{E}(F)$ for, e.g., the Stark case can be constructed as:

$$E(F) = -\frac{1}{2} \, _2F_1(h_1, h_2, h_3, h_4 f).$$

(5)

We apply this scheme to three Hamiltonians from single-particle quantum mechanics, offering illustrative examples of divergent perturbation series as well as nonperturbative behavior: the Stark Hamiltonian, $\hat{H} = -\nabla^2/2 - 1/\gamma F z$, with asymptotic series expansion described [12, 13] by Eq. (1); the cubic one-dimensional anharmonic oscillator with real perturbation [20], $\hat{H} = -\left(\partial^2/\partial x^2\right)/2 + \lambda x^2/2 + F x^3$; and the cubic one-dimensional anharmonic oscillator with imaginary perturbation [21], $\hat{H} = -\left(\partial^2/\partial x^2\right)/2 + \lambda x^2/2 + i F x^3$. Here, $\lambda$ is the force constant taken as 1/4 in the numerical analysis below. Furthermore, the perturbed ground-state eigenvalue has $\Gamma(F) \neq 0$ in the first two cases, while in the third case one has a non-Hermitian, but $PT$-symmetric [21], Hamiltonian with real eigenvalues. For simplicity, all equations are written assuming the Stark Hamiltonian problem.

Figure 1 shows $\Delta(F)$ [top panels] and $\Gamma(F)$ [bottom panels] as a function of $F$ for these three Hamiltonians. In Fig. 1a, values of $\Delta(F)$ and $\Gamma(F)$ are shown for the Stark Hamiltonian. Exact results taken from Ref. [24] are compared with those calculated using the
To obtain $\Gamma(F)$ and Padé approximants is admittedly not very fair. A large number of coefficients are evaluated by using the Borel-Padé method [2]. In this method, one starts from the standard procedure [12] thus far has been to employ the hypergeometric AC of the Borel-transformed series, calculating the $h_i$ coefficients that define the hypergeometric function $\text{}_2\text{F}_1(h_1, h_2, h_3; h_4 f)$ from $e_n/n!$. The Borel-hypergeometric approximation, $E(F) \approx \mathcal{E}(F)$, is then

$$
\mathcal{E}(F) \approx -\frac{1}{2} \int_0^\infty dt e^{-\alpha t} \text{}_2\text{F}_1(h_1, h_2, h_3; ah_4 ft),
$$

and $\alpha = \sqrt{i}$ specifies the integration contour [12]. An expression somewhat similar to Eq. (7) was used in Ref. [4] as the starting point to construct convergent strong-coupling expansions, while requiring the knowledge of both $e_{n \to \infty}$ and $E(F \to \infty)$.

A comparison between the hypergeometric approximant and Padé approximants is admittedly not very fair. To obtain $\Gamma(F) \neq 0$ from Padé approximants, the standard procedure [12] thus far has been to employ the Borel-Padé method [2]. In this method, one starts from a large number of coefficients $e_n$ and evaluates the Borel-transformed coefficients $b_n = e_n/n!$, which are then employed to calculate Padé approximants $B_{L/M}(f)$ and integrals

$$
F(f) = \int_0^\infty dt B_{L/M}(ft) e^{-t}.
$$

This yields the Borel-Padé approximation, $E(F) \approx -\frac{1}{2} F(f)$. The Borel method removes $n!$ from the coefficients, sums the transformed series, and puts $n!$ back into the series by means of the Laplace transform in Eq. (6). The essence of the Borel-Padé method [2] is to perform AC on the Borel transformed coefficients and use the resulting analytic function to evaluate the integral in Eq. (6). While the Borel-Padé method allows accurate calculations of $\Gamma(F)$ from the perturbation series, it also requires very large orders of perturbation theory that are unavailable in practice and lead to accuracy issues, impeding the calculation at high values of the perturbation strength.

In the Borel-Padé method, the analytic function is a Padé approximant. In the same spirit, we use hypergeometric functions as analytic functions to construct the Borel-hypergeometric method, by performing hypergeometric AC of the Borel-transformed series, calculating the $h_i$ coefficients that define the hypergeometric function $\text{}_2\text{F}_1(h_1, h_2, h_3; h_4 f)$ from $e_n/n!$. The Borel-hypergeometric approximation, $E(F) \approx \mathcal{E}(F)$, is then

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We now apply the Borel-hypergeometric method to approximate $\Delta(F)$ and $\Gamma(F)$ for the same three Hamiltonians studied in Fig. 1. Figure 2 demonstrates that in all three problems considered the Borel-hypergeometric method gives excellent approximations to both $\Delta(F)$ and $\Gamma(F)$, and reproduces the results given by the hypergeometric approximant. Comparing Borel-hypergeometric and hypergeometric approximants reveals that the hypergeometric approximant is Borel consistent to a very good approximation. It is well known that a convergent sum and its Borel resummation give the same result. The hypergeometric approximations discussed here clearly satisfy this highly desirable property. The careful reader will surely notice that in Fig. 2(b) the Borel-hypergeometric sum diverges for very small $F$. This is not a problem since the simple hypergeometric approximant is already well-behaved for $F \to 0$. Alternatively, one can calculate one extra order of perturbation theory and build the Borel-hypergeometric approximant from the coefficients of the once-subtracted series, $[E(F) - E(0)]/(e_1 f)$. As shown in Fig. 2(b) that procedure mitigates this minor problem, while leading to similarly accurate overall results. We emphasize that the hypergeometric and Borel-hypergeometric approaches are fourth-order approximations and thus much simpler and less expensive than the widely used Borel-Padé method [2], while being of comparable accuracy. For instance, in the case of the Stark Hamiltonian with very large $F = 0.4$, approximately 70 orders of perturbation theory were required in Ref. [12] as the input for the Borel-Padé scheme to produce $E(F = 0.4) = -0.608 - 0.200i$, which can be contrasted with our result $E(F = 0.4) = -0.609 - 0.212i$, and with numerically exact data $E(F = 0.4) = -0.613 - 0.205i$. For the Stark Hamiltonian, $F = 0.4$ a.u. $\simeq 2 \times 10^3$ MV cm$^{-1}$ corresponds to a rather large electric field.

It is easy to understand why the hypergeometric and Borel-hypergeometric method dramatically outperform the traditional Borel-Padé method [2]. To obtain $\Gamma(F) \neq 0$ one needs approximants with a branch cut in the complex $F$ plane with branch points at $F = 0$ and $F = \pm \infty$. Padé approximants typically have both poles and zeroes on the real $F$ axis, thereby lacking the correct analytic structure of $E(F)$—essential for rapid convergence. The function $\mathcal{F}_4(h_1, h_2, h_3; h_4 F)$ has a branch cut running from $h_4 F = 1$ to $h_4 F = \infty$. When calculating $\{h_i\}$ from the low-order coefficients $e_1, \ldots, e_4$, we typically obtain a large value for $h_4$, thus mimicking the correct branch cut structure in $E(F)$, as illustrated in Fig. 3.

Our findings have interesting physical implications. At the end of Ref. [11], Dyson wonders about the possibility that a series with zero radius of convergence might contain all that there is to know in a quantum system, arguing that if this were the case then an extension of QED would be needed. In the three examples considered in our study, the “extension” needed consists in supplementing the low-order information with an AC function able to mimic the branch cut structure required to obtain $\Gamma(F) \neq 0$. [8, 14].

We speculate that the approximations developed here have potential applications in nonequilibrium many-body perturbation theory [16] for condensed matter systems. For example, there is a habit of partially resumming classes of Feynman diagrams for such problems, as exemplified by the self-consistent Born approximation (SCBA) [23, 24] for electron-boson interacting systems or the self-consistent $GW$ approximation [25] for electron-electron interacting systems out of equilibrium. These
are both self-consistent in order to conserve charge current \[10\], and can therefore be viewed as “infinite order” approximations. However, they are exact only to first order because, to second order, SCBA misses the polarization bubble diagram \[24\] and the first-order vertex correction \[25\] to the Fock diagram, while the self-consistent \(GW\) approximation accounts for the former but also misses the latter. Thus, such resummation schemes are uncontrollable because an error is “summed” to all orders (starting from second order) \[11\]. An alternative could be to build the exact diagrammatic series—including vertex corrections—at low orders, and then use a suitable AC technique. The hypergeometric and Borel-hypergeometric techniques proposed here look promising in this respect.

In conclusion, by analogy with traditional Padé and Borel-Padé techniques we have developed a fourth-order hypergeometric approximant and its natural Borel extension. We demonstrate the effectiveness of this technique by summing the perturbative expansions with zero radius of convergence for three well-known examples from single-particle quantum mechanics, obtaining excellent approximations to their perturbed ground state eigenenergies, even for rather large values of the perturbation strength. The imaginary part of the perturbed eigenvalue is obtained with good precision using only fourth-order perturbation theory, thereby evading the calculation of a large number of coefficients in perturbation theory. Our results show that nonperturbative physics can be obtained from the low-order coefficients of a divergent perturbation series, as long as a carefully tailored analytic continuation technique is implemented.

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