Quantum Group Theory in $\tau^{(2)}$-model, Duality of $\tau^{(2)}$-model and XXZ-model with Cyclic $U_q(sl_2)$-representation for $q^n = 1$, and Chiral Potts Model

Shi-shyr Roan

Institute of Mathematics
Academia Sinica
Taipei, Taiwan
(email: maroan@gate.sinica.edu.tw)

We identify the quantum group $U_w(sl_2)$ in the $L$-operator of $\tau^{(2)}$-model for a generic $w$ as a subalgebra of $U_q(sl_2)$ with $w = q^{-2}$. In the roots of unity case, $q = q, w = \omega$ with $q^n = \omega^N = 1$, the eigenvalues and eigenvectors of XXZ-model with the $U_q(sl_2)$-cyclic representation are determined by the $\tau^{(2)}$-model with the induced $U_\omega(sl_2)$-cyclic representation, which is decomposed as a finite sum of $\tau^{(2)}$-models in non-superintegrable inhomogeneous $N$-state chiral Potts model. Through the theory of chiral Potts model, the $Q$-operator of XXZ-model can be identified with the related chiral Potts transfer matrices, with special features appeared in the $n = 2N$, e.g. $N$ even, case. We also establish the duality of $\tau^{(2)}$-models related to cyclic representations of $U_q(sl_2)$, analogous to the $\tau^{(2)}$-duality in chiral Potts model; and identify the model dual to the XXZ model with $U_q(sl_2)$-cyclic representation.

Abstract

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1 Introduction

The $\tau(2)$-model is the six-vertex model first appeared in the $N$-state chiral Potts model (CPM) in [11] as an adjacent model for the study of chiral Potts transfer matrix in the frame work of the Baxter’s $TQ$-relation [5]. Together with a set of $\tau(j)$-models and functional relations among the various transfer matrices [10], one can determine all the eigenvalues of both matrices, hence solve the eigenvalue problem of CPM [1, 6, 7, 9, 20, 25]. One also expects the eigenvector problem of CPM should equally rely on the $\tau(2)$-eigenvectors as well, as already revealed in the relation of state-correspondence in the duality of chiral Potts model [27]. Furthermore, the recent progress made on the eigenvector problem in superintegrable $N$-state CPM for odd $N$ in [28] again showed the vital role of $\tau(2)$-model about the degeneracy symmetries of $\tau(2)$-eigenspaces through its equivalent spin-$\frac{N-1}{2}$ XXZ-chain. It is known that the theory of quantum group $U_q(sl_2)$ for a generic $q$ stemmed from the study of XXZ-model as an equivalent formulation of the Yang-Baxter (YB) relation (2.3) of $L$-operator [14, 17]. In the case when $q$ is an $N$th root of unity for odd $N$, the XXZ chain with cyclic representation was identified with the chiral Potts $\tau(2)$-model in [26, 27]. The observation and arguments there can be further extended to a general setting for a quantum group $U_q(sl_2)$ with a generic $q$. In this work, we find a quantum subalgebra $U_w(sl_2)$ of $U_q(sl_2)$ for a generic $w (= q^{-2})$ (see (2.5) in the paper), with an associated $L$-operator satisfying the YB relation (2.6) of $\tau(2)$-model. The understanding of local state vectors of a statistical $\tau(2)$-model is thus reduced to the representation theory of $U_w(sl_2)$. The theory of $N$-state CPM is in essence the study of $Q$-operator associated to XXZ-chains with the cyclic representation in the root of unity case: $q = q$, $w = \omega$ a primitive $n$th and $N$th root of unity for $n \geq 3$, $N \geq 2$ respectively with the relation

$$q^{-2} = \omega, \quad q^n = \omega^N = 1, \quad n = N \text{ odd, or } n = 2N. \quad (1.1)$$

The aim of this paper is to find an explicit relationship between XXZ-chains of $U_q(sl_2)$-cyclic representation and non-superintegrable $N$-state chiral Potts model, with especial attention on even $N$ case; and explore the duality theory connected to XXZ-models with cyclic representation. Since every cyclic representation of $U_q(sl_2)$ induces a representation of $U_\omega(sl_2)$, the XXZ-model with an $U_q(sl_2)$-cyclic representation gives rise to a $\tau(2)$-model, denoted by $\tau^{(2)}(t)$ in (3.41), with the induced $U_\omega(sl_2)$-cyclic representation. Correspondingly, the ABCD-algebra, i.e. the monodromy algebra, of the XXZ-model is generated by ABCD-algebra of $\tau^{(2)}$-model and $K^\sharp$. Hence one can study the XXZ-model with a cyclic representation through its induced $\tau^{(2)}$-model. By the representation theory of $U_\omega(sl_2)$, we find that a $\tau^{(2)}$-model with the chain-size $L$ is decomposed as $(n/N)^L$ sub-models, each of which is isomorphic to some chiral-Potts $\tau^{(2)}$-model with inhomogeneous vertical rapidities. The $K^\sharp$-operator of XXZ-model gives rise to a pairing of these sub-models with identical eigenvalues and eigenvectors, also with similar $Q$-operators. By this, the eigenvalues and eigenvectors of XXZ-model are obtained , and the $Q$-operator can be identified with the chiral Potts transfer matrix. In the duality theory of CPM in [27], $\tau^{(2)}$-duality is the equality
of two $\tau^{(2)}$-models by identifying the "ordered- and disordered-” state vectors through the spin-
and-face-variable-expression of a $\tau^{(2)}$-transfer matrix [8]. We would like to examine how far the
$\tau^{(2)}$-duality in CPM can be extended to XXZ-models with cyclic representation. First, we notice
that the $\tau^{(2)}$-duality of CPM for the sub-models in the decomposition of $\tau^{(2)}(t)$ can not be carried
over to XXZ model, partly due to the fact that the duality works on transfer matrices only, not
as ABCD-algebra- or $U_\omega(sl_2)$-representations. However, we are still able to find a spin-and-face-
expression of the transfer matrix of XXZ model with cyclic representation, and identify the model
dual to it. It turns out the dual model of XXZ-chains with cyclic representation is another
type of $\tau^{(2)}$-model, $\tau^{(2)}(t)$ in (3.44), dual to the $\tau^{(2)}(t)$ inherited from XXZ-model. This shows
the dualities involved with XXZ-models all depend on the duality of $\tau^{(2)}$-models, which we expect
also serves the fundamental role in the duality of XXZ-model with a representation of $U_q(sl_2)$ other
than cyclic representations.

This paper is organized as follows. In section 2, we provide a $\hat{U}_w(sl_2)$-quantum group formula-
tion of $\tau^{(2)}$-model for a generic $w$, as a parallel theory to the quantum group $U_q(sl_2)$ in XXZ-chains
for a generic $q$. We then derive some basic properties about the monodromy-(ABCD)-
 algebras of the general $\tau^{(2)}$-model. Afterwards, we shall concern only with the root of unity case (1.1) for
the rest of this paper. Section 3 is devoted to the study of $\tau^{(2)}$-model and XXZ-model with cyclic
representation (of $\hat{U}_\omega(sl_2)$ and $U_q(sl_2)$ respectively). In subsection 3.1, we first recall the cyclic
$\mathbf{C}^n$-representation of $U_q(sl_2)$, and describe its related cyclic representations of $\hat{U}_\omega(sl_2)$; then present
a detailed structure about the representation theory of $L$-operators in $\tau^{(2)}$-model and XXZ-model
with cyclic representation. In subsection 3.2, we study the structure of XXZ-model and $\tau^{(2)}$-model
induced from cyclic $\mathbf{C}^n$-representation of $U_q(sl_2)$. Each irreducible component of $\tau^{(2)}$-model with
the induced cyclic $\mathbf{C}^n$-representation is isomorphic to some $\tau^{(2)}$-model associated to $N$-state CPM
with (possible) inhomogeneous vertical rapidities. By this, we are able to find the connection about
eigenvalues and eigenvectors between XXZ-model and $\tau^{(2)}$-model in CPM. Section 4 is devoted to
the discussion of duality of $\tau^{(2)}$-models and XXZ models with cyclic representation. First in sub-
section 4.1, we recall the $\tau^{(2)}$-duality in CPM; and then in subsection 4.2, we study the duality of
$\tau^{(2)}$-model with cyclic $\mathbf{C}^n$-representation, and find the duality between $\tau^{(2)}$- and $\tau^{(2)}$-models. In
subsection 4.3, we examine the relationship between the $\tau^{(2)}$-dualities found in the previous two
subsections. The $\tau^{(2)}$-$\tau^{(2)}$-duality agrees with the $\tau^{(2)}$-duality in CPM only in the $n = N$ odd
case, but differs in $n = 2N$ case. In subsection 4.4, we identify the $\tau^{(2)}$-model as the dual model of
XXZ model with cyclic representation using a spin-and-face-expression of the XXZ-transfer matrix.
Section 5 is the discussion of the relationship between CPM and XXZ model with $U_q(sl_2)$-cyclic
representation. First, we recall some basic notions and the duality in CPM in subsection 5.1. Then
in subsection 5.2, we identify the chiral Potts transfer matrix as the $Q$-operator of XXZ model with
cyclic representation.

**Notation:** In this paper, we use the following standard notations. For a positive integer
$N$ greater than one, $\mathbf{C}^N$ denotes the vector space of $N$-cyclic vectors with the canonical base
$|\sigma\rangle$, $\sigma \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. For a $N$th primitive root of unity $\omega$, e.g. $\omega = e^{2\pi i\frac{1}{N}}$, the Weyl operators
$X, Z$ (with respective to $\omega$) with the relations, $X^N = Z^N = 1$ and $XZ = \omega^{-1}ZX$, are defined by
$$X|\sigma\rangle = |\sigma + 1\rangle, \quad Z|\sigma\rangle = \omega^\sigma|\sigma\rangle \quad (\sigma \in \mathbb{Z}_N).$$

The Fourier basis $\{|k\rangle\}$ of $\{|\sigma\rangle\}$ is defined by
$${\hat{k}} = \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{N-1} \omega^{-k\sigma}|\sigma\rangle, \quad |\sigma\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{\sigma k}|k\rangle, \quad \sigma \in \mathbb{Z}_N,$$
with the corresponding Weyl operators, $X|k\rangle = |k + 1\rangle$, $Z|k\rangle = \omega^k|k\rangle$ satisfying $XZ = \omega^{-1}ZX$.

Then the following equality holds:
$$(X, Z) = (\hat{Z}, \hat{X}^{-1}).$$

The Fourier bases of $\otimes^L \mathbb{C}^N$ are denoted by
$$|\sigma_1, \ldots, \sigma_L\rangle := |\sigma_1\rangle \otimes \cdots \otimes |\sigma_L\rangle, \quad \text{or} \quad |k_1, \ldots, k_L\rangle := |k_1\rangle \otimes \cdots \otimes |k_L\rangle, \quad (\sigma_j, k_j \in \mathbb{Z}_N).$$

### 2 Quantum Group Theory of $\tau^{(2)}$-model

For a generic $q \in \mathbb{C}\setminus\{0, \pm 1\}$, we denote $w = q^{-2}$. It is well-known that the quantum group $U_q(sl_2)$ is the algebra generated by $K^\pm, e^\pm$ with the relations $K^\pm K^\pm = 1$ and
$$K^\frac{1}{2} e^\pm K^{-\frac{1}{2}} = q^{\pm 1} e^\pm, \quad [e^+, e^-] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Define the $L$-operator with non-zero complex parameter $\rho, \nu \in \mathbb{C}$:
$$L(s) \quad (= L(s; \rho, \nu)) = \left(\begin{array}{cccc} \rho^{-1} \nu^\frac{1}{2} s K^\frac{1}{2} - \nu^{-\frac{1}{2}} s^{-1} K^{-\frac{1}{2}} & (q - q^{-1}) e^- \\
(q - q^{-1}) e^+ & \nu^\frac{1}{2} s K^\frac{1}{2} - \rho \nu^{-\frac{1}{2}} s^{-1} K^{-\frac{1}{2}} \end{array}\right).$$

The quantum group $U_q(sl_2)$ is characterized by the YB relation,
$$R(s/s')(L(s) \otimes_{aux} 1)(1 \otimes_{aux} L(s')) = (1 \otimes_{aux} L(s'))(L(s) \otimes_{aux} 1)R(s/s'),$$
with the symmetric $R$-matrix $[14, 17]$:
$$R(s) = \left(\begin{array}{cccc} s^{-1}q - sq^{-1} & 0 & 0 & 0 \\
0 & s^{-1} - s & q - q^{-1} & 0 \\
0 & q - q^{-1} & s^{-1} - s & 0 \\
0 & 0 & 0 & s^{-1}q - sq^{-1} \end{array}\right).$$

By setting $t = s^2$, the modified $L$-operator, $-\nu^\frac{1}{2} K^\frac{1}{2} L(s)$ with the gauge transform dia$[1, -s^\nu\frac{1}{2} q]$, is expressed by
$$L(t) \quad (= L(t; \rho, \nu)) = \left(\begin{array}{cc} 1 - t \nu \rho^{-1} K^{-1} & (1 - w) E^- \\
-t \nu (1 - w) E^+ & -t \nu + \rho K^{-1} \end{array}\right),$$

with
where \( E^\pm = -q^2 K^{-1} e^\pm, E^- = K^{-1} e^- \). The quantum subalgebra of \( U_q(sl_2) \) generated by \( K^\pm, E^\pm \), with the generator-relation:

\[
KE^\pm K^{-1} = w^{\pm 1} E^\pm, \quad wE^\pm E^- E^\pm = \frac{K^2 - 1}{1 - w}, \quad (2.5)
\]

will be denoted by \( U_w(sl_2) \). Then the \( L \)-matrix in (2.4) satisfy the YB equation,

\[
R(t/t')(L(t) \otimes 1)(1 \otimes L(t')) = (1 \otimes L(t'))(L(t) \otimes 1)R(t/t') \quad (2.6)
\]

with the asymmetry \( R \)-matrix

\[
R(t) = \begin{pmatrix}
t w - 1 & 0 & 0 & 0 \\
0 & t - 1 & w - 1 & 0 \\
0 & t(w - 1) & (t - 1)w & 0 \\
0 & 0 & 0 & t w - 1
\end{pmatrix}.
\]

Indeed, the generator-relation (2.5) of \( U_w(sl_2) \) is characterized by the YB relation (2.6) of the \( L \)-operator (2.4). Hence a representation of \( U_q(sl_2) \) or \( U_w(sl_2) \) is equivalent to a representation of the \( L \)-operator (2.2) or (2.4) satisfying the respective YB relation. Two representations of \( L \)-operator (2.2) or (2.4) are equivalent if and only if the induced equivalent representations of \( U_q(sl_2) \) or \( U_w(sl_2) \) are equivalent with the same parameter \( \rho, \nu \). In this paper, we shall also make the identification of the quantum-algebra representation and \( L \)-operator representation when no confusion could arise.

For a chain of size \( L \), we assign the \( L \)-operator (2.2) or (2.4) at the \( \ell \)th site, and form the XXZ-monodromy matrix

\[
\bigotimes_{\ell=1}^L L_\ell(s) = \begin{pmatrix}
A(s) & B(s) \\
C(s) & D(s)
\end{pmatrix}, \quad L_\ell(s) = L(s; \rho_\ell, \nu) \text{ at site } \ell; \quad (2.7)
\]

and \( \tau^{(2)} \)-monodromy matrix

\[
\bigotimes_{\ell=1}^L L_\ell(t) = \begin{pmatrix}
A(t) & B(t) \\
C(t) & D(t)
\end{pmatrix}, \quad L_\ell(t) = L(t; \rho_\ell, \nu) \text{ at site } \ell. \quad (2.8)
\]

By the relation between \( L \)-operators (2.2) and (2.4), the monodromy entries of (2.7) and (2.8) are related by

\[
\frac{A(t)}{D(t)} = (-s)L^{1/2}K^{1/2}A(s), \quad \frac{B(t)}{C(t)} = (-s)L^{1/2}K^{-1/2}q^{-1}K^{-1/2}B(s), \quad (2.9)
\]

where \( K^{1/2} := \bigotimes K_\ell^{1/2} \). Equivalently to say, the XXZ-(ABCD)-algebra (2.7) is generated by the \( \tau^{(2)} \)-(ABCD)-algebra (2.8) and \( K^{1/2} \), with the relation (2.9) between their algebra-generators. For an integer \( r \in \mathbb{Z} \), the YB relation of the monodromy matrices yields the commutating relation of the \( r \)-twisted traces:

\[
T(s) = A(s) + q^{-2r}D(s), \quad \tau^{(2)}(t) = A(tw) + w^r D(wt) \quad (2.10)
\]
for \( s \in C \) and \( t \in C \), as a one-parameter family in \( \bigotimes U_q(\mathfrak{sl}_2) \) or \( \bigotimes U_W(\mathfrak{sl}_2) \) respectively. Then we have the relation, \([T(s), K^{\frac{1}{2}}] = [\tau^{(2)}(t), K] = 0\), which indeed follow from the commutative relations of \( K^{\frac{1}{2}}, K \) and monodromy entries in (2.7) or (2.8):

\[
K^{\frac{1}{2}}A^{(s)}_{D(s)}K^{-\frac{1}{2}} = A^{(s)}_{D(s)}, \quad K^{\frac{1}{2}}C^{(s)}_{B(s)}K^{-\frac{1}{2}} = q^{C^{(s)}}_{B(s)}; \quad K^{\frac{1}{2}}A^{(t)}_{D(t)}K^{-1} = A^{(t)}_{D(t)}, \quad K^{\frac{1}{2}}C^{(t)}_{B(t)}K^{-1} = w^{-1}C^{(t)}. \]

By (2.9), the \( r \)-traces of \( \tau^{(2)} \) and \( T \) in (2.10) are related by

\[
\tau^{(2)}(t) = (-q^{-1}s)L^r_xK^{\frac{1}{2}}T(q^{-1}s), \quad t = s^2. \tag{2.11}
\]

For given \( \rho, \nu \), a \( U_q(\mathfrak{sl}_2) \)-representation on \( C^d \) gives rise to a commuting family of transfer matrices \( T(s) \) or \( \tau^{(2)}(t) \) on \( \bigotimes L \) \( C^d \) with the \( L \)-operator induced from (2.2) or (2.4) respectively. In particular, when \( \rho = 1, \nu = q^{d-2} \), the spin-\( \frac{d-1}{2} \) (highest weight) representation of \( U_q(\mathfrak{sl}_2) \) on \( C^d = \bigoplus_{k=0}^{d-1} C^k \);

\[
K^{\frac{1}{2}}(e^k) = q^{\frac{d-1}{2}k}e^k, \quad e^+(e^k) = [k]_q e^{k-1}, \quad e^-(e^k) = [d-1-k]_q e^{k+1},
\]

\[
K(e^k) = w^{-\frac{d-1}{2}+k}e^k, \quad E^+(e^k) = -w^{-\frac{d-1}{2}}[k]_q e^{k-1}, \quad E^-(e^k) = w^{-\frac{d-1}{2}}[d-1-k]_q e^{k+1}, \tag{2.12}
\]

where \([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, [n] := \frac{w^n}{1-w} \), for the XXZ chain (2.7) gives rise to the well-known homogeneous XXZ model of spin-\( \frac{d-1}{2} \) (see, e.g. [16, 21, 22] and references therein). Note that in (2.12), \( e^+(e^0) = e^- (e^{-1}) = E^+(e^0) = E^- (e^{-1}) = 0 \).

Remark. In (2.7) and (2.8), the \( L \)-operator, \( L(s) \) and \( L(t) \), are assumed with the same value of the parameter \( \nu \) in (2.2) and (2.4). However, one may also consider monodromy matrix defined by \( L(s; \rho, \nu) \) and \( L(t; \rho, \nu) \) with \( \rho \)'s distinct, and form the transfer matrix \( T(s; \{ \rho, \nu \}), \tau^{(2)}(t; \{ \rho, \nu \}) \) as in (2.10). Since the \( L \)-operators in (2.2) and (2.4) satisfy the relations,

\[
L(s; \rho, \nu) = L(\xi^1 s; \rho, \nu), \quad L(t; \rho, \nu) = L(\xi t; \rho, \nu), \quad \nu_t = \xi^2 \nu,
\]

the transfer matrices \( T(s; \{ \rho, \nu \}), \tau^{(2)}(t; \{ \rho, \nu \}) \) are reduced to those in (2.10) with the same \( \nu \) by

\[
T(s; \{ \rho, \nu \}) = T(\xi^1 s, \ldots, \xi^1 s), \quad T(s) = T(s, \ldots, s); \quad \tau^{(2)}(t; \{ \rho, \nu \}) = \tau^{(2)}(\xi^1 t, \ldots, \xi^1 t), \quad \tau^{(2)}(t) = \tau^{(2)}(t, \ldots, t). \tag{2.13}
\]

### 3 XXZ-model and \( \tau^{(2)} \)-model with Cyclic Representation

In the root of unity case (1.1), we let \( \left[ \sigma \right]'s \) or \( \left[ k \right]'s \) \((\sigma, k \in Z_N) \) of \( C^N \) be the Fourier basis in (1.2), and \((X, Z), \left( \hat{X}, \hat{Z} \right) \) the Weyl operators (with respective to \( \omega \), in (1.3)). Similarly, for the cyclic \( n \)-space \( C^n \), the Fourier bases (with respective to \( q \)) will be denoted by

\[
\left[ \hat{k} \right] = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} q^{-k \sigma} \left| \sigma \right>, \quad \left| \sigma \right> = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} q^{\sigma k} \left[ \hat{k} \right], \quad k, \sigma \in Z_n, \tag{3.1}
\]

with the Weyl operators \((X', Z'), \left( \hat{X}', \hat{Z}' \right) \) defined by

\[
X'(\sigma) = \left| \sigma + 1 \right>, \quad Z'(\sigma) = q^k \left| \sigma \right>; \quad \left( \hat{X}', \hat{Z}' \right) = \left( \left[ \hat{k} \right], q^k \left[ \hat{k} \right] \right). \]
Then the relations $X'Z' = q^{-1}Z'X'$, $\hat{X}'\hat{Z}' = q^{-1}\hat{Z}'\hat{X}'$, and the identity $(X', Z') = (\hat{Z}', \hat{X}'^{-1})$ hold.

First, we describe some special cyclic $C^N$-subspaces of $C^n$ for later use. Denote 

$$c(n) = 2^{-\frac{1-(-1)^n}{4}} = \begin{cases} 1 & \text{if } n = N \text{ odd} \\ 2^{-1/2} & \text{if } n = 2N. \end{cases}$$

**Lemma 3.1** Consider the cyclic $N$-vectors in $C^n$, 

$$|\sigma\rangle_+ := c(n)|-2\sigma\rangle, \quad |\sigma\rangle_- := c(n)|-2\sigma + 1\rangle \in C^n \quad (\sigma \in Z_N),$$

and the linear transformations 

$$\varphi_\pm : C^n \rightarrow C^N, \quad \varphi_+|\hat{k}\rangle = |\hat{k}\rangle, \quad \varphi_-|\hat{k}\rangle = q^{-k}|\hat{k}\rangle \quad (k \in Z_n). \quad (3.2)$$

Define the cyclic $N$-subspaces $C^\pm$ of $C^n$,

$$C^+ = \sum_{\sigma=1}^{N-1} C|\sigma\rangle_+, \quad C^- = \sum_{\sigma=1}^{N-1} C|\sigma\rangle_- \quad (3.3)$$

where $|\hat{k}\rangle_\pm$'s are basis of $C^\pm$, related to $|\sigma\rangle_\pm$'s by the $N$-Fourier relation:

$$|\hat{k}\rangle_\pm = \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{N-1} \omega^{-k\sigma} |\sigma\rangle_\pm, \quad |\sigma\rangle_\pm = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{k\sigma} |\hat{k}\rangle_\pm, \quad (\sigma \in Z_N).$$

Then

(i) $\varphi_- = \varphi_+ \hat{Z}'$, and $\hat{Z}' \varphi_+ = \varphi_- \hat{Z}'$.

(ii) The $C^\pm$-bases in (3.3) interchange under $C^n$-Weyl operators via

$$X' : |\sigma\rangle_+ \mapsto |\sigma\rangle_-, \quad Z' : |\sigma\rangle_- \mapsto \sigma^k|\sigma\rangle_+, \quad \hat{X}' : |\hat{k}\rangle_+ \mapsto \hat{k}_+^{\pm}, \quad \hat{Z}' : |\hat{k}\rangle_- \mapsto \hat{k}_-^{\pm}, \quad (3.4)$$

by which $C^n$-Weyl operators of $C^\pm$ are induced by

$$(X'^{-2}, Z')$, \quad $X'^{-2}, \hat{Z}'^2$ for $C^+$; \quad $X'^{-2}, q^{-1}Z'$, \quad $(q\hat{X}', \hat{Z}'^{-2})$ for $C^-$. \quad (3.5)$$

Furthermore, the $C^n$-operator $X'$ identifies the $N$-spaces $C^\pm$ by

$$X' = (\hat{Z}') : C^+ \simeq C^- \quad \text{and} \quad |\sigma\rangle_+ \mapsto |\sigma\rangle_-, \quad |\hat{k}\rangle_+ \mapsto |\hat{k}\rangle_-, \quad (3.6)$$

and $C^\pm$ are isomorphic to $C^N$ under $\varphi_\pm$ in (3.2) respectively:

$$\varphi_\pm : C^\pm \simeq C^N, \quad |\sigma\rangle_\pm \mapsto |\sigma\rangle, \quad |\hat{k}\rangle_\pm \mapsto |\hat{k}\rangle. \quad (3.7)$$

The automorphism (3.6) descends to the following $C^N$-automorphisms via $\varphi_\pm$ in (3.7):

$$C^N \varphi_+ = C^+ \varphi_+ \simeq C^N, \quad |\hat{k}\rangle \mapsto |\hat{k}\rangle; \quad C^N \varphi_- = C^- \varphi_- \simeq C^N, \quad |\hat{k}\rangle \mapsto \omega^{-k}|\hat{k}\rangle. \quad (3.8)$$

(iii) When $n = N$ odd, $C^n = C^\pm$ with $(|\sigma\rangle_-, |\hat{k}\rangle_-) = (|\sigma + \frac{N-1}{2}\rangle_+ + q^k|\hat{k}\rangle_+)$, and the projections (3.7) define isomorphisms between $C^n$ and $C^N$, preserving the Fourier basis. When $n = 2N$, $C^n = C^+ \oplus C^-$, and the kernel of (3.2) $\text{Ker}(\varphi_+) = C^\pm$. Hence $|\sigma\rangle_\pm, |\hat{k}\rangle_\pm$'s form a basis of $C^n$.

In both cases, the Weyl operators of $C^n$ and $C^N$ are related by

$$\varphi_\pm X'^{-2} = X'\varphi_\pm, q^{-\frac{11}{2}} \varphi_\pm Z' = Z'\varphi_\pm, \quad (\Leftrightarrow \quad \varphi_\pm X' = X\varphi_\pm, \varphi_\pm Z' = Z\varphi_\pm). \quad (3.9)$$
We shall also consider another kind of cyclic \(N\)-subspaces of \(C^n\).

**Lemma 3.2** Consider the cyclic \(N\)-vectors in \(C^n\),
\[
|k\rangle^{\pm} := C(n)|-2k\rangle, \quad |k\rangle^{\pm} := C(n)|-2k + 1\rangle \in C^n \quad (k \in Z_N),
\]
and the linear transformations
\[
\varphi_\pm^k : C^n \rightarrow C^N, \quad \varphi_\pm^k(|\sigma\rangle) = |\sigma\rangle, \quad \varphi_\pm^k(|\sigma\rangle) = q^{|\sigma\rangle} (\sigma \in Z_n).
\]
Define the cyclic \(N\)-subspaces \(C^{\uparrow \pm}\) of \(C^n\),
\[
C^{\uparrow +} = \sum_{k=1}^{N-1} C|k\rangle^{\uparrow +} = \sum_{k=1}^{N-1} C|k\rangle^{\uparrow +}, \quad |\sigma\rangle^{\uparrow +} := \frac{1}{2}(|\sigma\rangle + |\sigma + N\rangle),
\]
\[
C^{\uparrow -} = \sum_{k=1}^{N-1} C|k\rangle^{\uparrow -} = \sum_{k=1}^{N-1} C|k\rangle^{\uparrow -}, \quad |\sigma\rangle^{\uparrow -} := \frac{1}{2}(q^{-|\sigma\rangle} + q^{-N|\sigma\rangle + N\rangle}),
\]
with the relation of basis:
\[
|\hat{k}\rangle^{\uparrow +} = \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{N-1} \omega^{-k\sigma}|\sigma\rangle^{\uparrow +}, \quad |\sigma\rangle^{\uparrow +} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{k\sigma}|\hat{k}\rangle^{\uparrow +}, \quad (\sigma \in Z_N).
\]

Then

(i) \(\varphi_\pm^k = \varphi_\pm^k Z^{\uparrow -1}\), and \(Z\varphi^k_\pm = \varphi^k_+ Z^{\uparrow -1}\).

(ii) The \(C^n\)-Weyl operators interchange the \(C^{\uparrow \pm}\)-basis in (3.11) by
\[
X^{\prime}: \frac{|\sigma\rangle^{\uparrow +}}{|\sigma\rangle^{\uparrow +}} \mapsto \frac{|\sigma + 1\rangle^{\uparrow +}}{|\sigma + 1\rangle^{\uparrow +}}, \quad Z^{\prime}: \frac{|\sigma\rangle^{\uparrow +}}{|\sigma\rangle^{\uparrow +}} \mapsto \frac{\omega^{-\sigma\sigma}}{|\sigma\rangle^{\uparrow +}}, \quad \hat{X}: \frac{|\hat{k}\rangle^{\uparrow +}}{|\hat{k}\rangle^{\uparrow +}} \mapsto \frac{\omega^{|\hat{k}\rangle^{\uparrow +}}}{{|\hat{k}\rangle^{\uparrow +}}},
\]
by which the \(C^n\) operators are induced by
\[
(X^{\prime}, Z^{\prime-2}), \quad (\hat{X}^{\prime-2}, \hat{Z}^{\prime}) \quad \text{for} \quad C^{\uparrow +}; \quad (q^{-1}X^{\prime}, Z^{\prime-2}), \quad (\hat{X}^{\prime-2}, q^{-1}\hat{Z}^{\prime}) \quad \text{for} \quad C^{\uparrow -}.
\]

Furthermore, the \(C^n\)-operator \(Z^{\uparrow -1}\) identifies the \(N\)-spaces \(C^{\uparrow \pm}\):
\[
Z^{\uparrow -1}(= \hat{X}^{\prime}) : C^{\uparrow +} \simeq C^{\uparrow -}, \quad |\sigma\rangle^{\uparrow +} \mapsto |\sigma\rangle^{\uparrow -}, \quad |\hat{k}\rangle^{\uparrow +} \mapsto |\hat{k}\rangle^{\uparrow -},
\]
and \(C^{\uparrow \pm}\) are isomorphic to \(C^n\) under \(\varphi_\pm^k\) in (3.10) respectively:
\[
\varphi_\pm^k : C^{\uparrow +} \simeq C^n, \quad |\sigma\rangle^{\uparrow +} \mapsto |\sigma\rangle, \quad |\hat{k}\rangle^{\uparrow +} \mapsto |\hat{k}\rangle.
\]

The automorphism (3.14) descends to the following \(C^n\)-automorphisms via \(\varphi_\pm^k\) in (3.15):
\[
C^{\uparrow +} \varphi_\pm^k \simeq C^n \varphi_\pm^k \simeq C^{\uparrow -}, \quad |\sigma\rangle \mapsto |\sigma\rangle; \quad C^{\uparrow +} \varphi_\pm^k \simeq C^n \varphi_\pm^k \simeq C^{\uparrow -}, \quad |\sigma\rangle \mapsto q^{|\sigma\rangle} |\sigma\rangle.
\]

(iii) When \(n = N\) odd, \(C^n = C^{\uparrow \pm}\) with \((|\sigma\rangle^{\uparrow +}, |\hat{k}\rangle^{\uparrow +}) = (q^{-\sigma\sigma}|\sigma\rangle^{\uparrow +}, |k + \frac{N}{2}\rangle^{\uparrow +})\), and (3.15) defines the isomorphism between \(C^n\) and \(C^n\), preserving the Fourier basis. When \(n = 2N\),
\[
C^n = C^{\uparrow +} \oplus C^{\uparrow -}, \quad \text{with} \quad C^{\uparrow \pm} = \text{Ker}(\varphi_\pm^k), \quad |\sigma\rangle^{\uparrow \pm}, |\hat{k}\rangle^{\uparrow \pm}\)‘s form a basis of \(C^n\). In both cases, the Weyl operators of \(C^n\) and \(C^n\) are related by
\[
\varphi_\pm^k X^{\prime} = q^{|\hat{k}\rangle^{\uparrow \pm}} X \varphi_\pm^k, \quad \varphi_\pm^k Z^{\prime-2} = Z \varphi_\pm^k, \quad (\iff \varphi_\pm^k \hat{X}^{\prime-2} = \hat{X} \varphi_\pm^k, q^{-\frac{1}{2}} \varphi_\pm^k \hat{Z}^{\prime} = \hat{Z} \varphi_\pm^k).
\]
3.1 The $L$-operators of XXZ-model and $\tau^{(2)}$-model with cyclic representation

Consider the cyclic-representation of $U_q(sl_2)$, i.e. the three-parameter family of $U_q(sl_2)$-representation on the cyclic space $C^n$ with parameters $q^\phi, q^{-\phi}$ and $q^\varepsilon$, denoted by $s_{\varepsilon,\phi,\phi'}$, and with the following expression in terms of $C^n$-Weyl operators:

$$K_{\hat{z}} = q^{\frac{\phi'(1-\phi)}{2}} \hat{Z}^t, \quad e^+ = q^\phi (q^{\phi+1} \hat{Z}^t - q^{-\phi-1} \hat{Z}^n) \hat{X}^t \quad e^- = q^{-\phi} (q^{\phi+1} \hat{Z}^t - q^{-\phi-1} \hat{Z}^n) \hat{X}^t,$$  \hspace{1cm} (3.18)

or equivalently, a expression using the spin-operators $(X', Z') = (\hat{Z}', \hat{X}'^{-1})$ (see, e.g. [12, 13] or [26, 27]). By (2.2), the $U_q(sl_2)$-representation (3.18) is equivalent to the YB relation (2.3) of the $L$-operator of XXZ-model:

$$\mathcal{L}(s) = \begin{pmatrix}
   s_{\rho}^{-\frac{1}{2}} q^{-\frac{\phi'-\phi}{2}} (1 - \omega^{-\phi-1} \hat{Z}^{-2}) \hat{X}^t & q^{-\varepsilon} (q^{\phi+1} \hat{Z}^t - q^{-\phi-1} \hat{Z}^n) \hat{X}^t \\
   q^\phi (q^{\phi+1} \hat{Z}^t - q^{-\phi-1} \hat{Z}^t) \hat{X}^t & s_{\rho}^{-\frac{1}{2}} q^{-\frac{\phi'-\phi}{2}} (1 - \omega^{-\phi-1} \hat{Z}^{-2}) \hat{X}^t
\end{pmatrix}. \hspace{1cm} (3.19)$$

The above representations are all reducible except a special one in $n = 2N$ case which is equivalent to the spin-$\frac{n-1}{2}$ representation in (2.12). The $U_q(sl_2)$-representation (3.18) induces the $C^n$-representation of $U_\omega(sl_2)$, where the generators in (2.5) are expressed by

$$K = q^{\phi'-\phi} \hat{Z}^2, \quad E^+ = q^{-\frac{\phi'-\phi}{2} + \varepsilon} (1 - \omega^{-\phi-1} \hat{Z}^{-2}) \hat{X}^t \quad E^- = q^{\frac{\phi'-\phi}{2} - \varepsilon} (1 - \omega^{-\phi-1} \hat{Z}^{-2}) \hat{X}^t. \hspace{1cm} (3.20)$$

The $L$-operator (2.4) with the $C^n$-representation (3.20),

$$L(t) = \begin{pmatrix}
   1 - t \nu q^{-\phi-\phi'} \hat{Z}^{-2} & q^{\frac{\phi'-\phi}{2} - \varepsilon} (1 - \omega^{-\phi-1} \hat{Z}^{-2}) \hat{X}^t \\
   -t \nu q^{-\phi-\phi'} + \varepsilon (1 - \omega^{-\phi-1} \hat{Z}^{-2}) \hat{X}^t & -t \nu + \rho q^{-\phi-\phi'} \hat{Z}^{-2}
\end{pmatrix}, \hspace{1cm} (3.21)$$

then satisfies the YB relation (2.6). By (3.9), representations in (3.20) descend to the following three-parameter family of cyclic $C^n$-representation of $U_\omega(sl_2)$ via $\phi_+$ in (3.2):

$$K = q^{\phi'-\phi} \hat{Z}^{-1}, \quad E^+ = q^{-\frac{\phi'-\phi}{2} + \varepsilon} (1 - \omega^{-\phi-1} \hat{Z}) \hat{X}, \quad E^- = q^{\frac{\phi'-\phi}{2} - \varepsilon} (1 - \omega^{-\phi-1} \hat{Z}) \hat{X}^{-1}. \hspace{1cm} (3.22)$$

By employing the cyclic representation (3.22) of $U_\omega(sl_2)$ on the two-parameter family of $L$-operator (2.4), one obtains the following five-parameter $L$-operators of the cyclic $C^n$-space appeared in $\tau^{(2)}$-model of the $N$-state CPM [11]:

$$L(t) = \begin{pmatrix}
   1 - t \frac{\nu}{b} \hat{Z}^{-2} & (b - \omega \frac{a}{b} \hat{Z}) \hat{X}^{-1} \\
   -t (\frac{1}{b} - \frac{a}{b^2} \hat{Z}) \hat{X} & -t \frac{1}{b} + \omega \frac{a}{b^2} \hat{Z}
\end{pmatrix}, \quad (\hat{Z}, \hat{X}) = (X, Z^{-1}), \hspace{1cm} (3.23)$$

where as in [27], the parameters $a', b', a, b, c$ are related to $\varepsilon, \phi, \phi', \rho, \nu$ by\footnote{Here the parameter $(a', b', a, b, c)$ is equal to $(\nu^2 a', \nu^2 b', \nu^2 a, \nu^2 b, c)$ in formula (4.17) of [27]. The difference is due to $L(t)$ in [27] is identified with the gauge of $-s\nu^2 K \rightarrow L(s)$ by dia[1, $-s\nu^2 q^2$] instead of dia[1, $-s\nu^2 q$] in this paper.}:

$$q^{\varepsilon + \frac{\phi'}{2}} = \left( \frac{a'^2 b'^2}{a} \right)^{\frac{1}{2}}, \quad q^{\phi + 1} = \left( \frac{a}{b} \right)^{\frac{1}{2}}, \quad q^{\phi'} = \left( \frac{b}{a} \right)^{\frac{1}{2}}, \quad \rho = q^{-1} \left( \frac{a}{b} \right)^{\frac{1}{2}}, \quad \nu = \frac{1}{b^2} \iff a = \rho^{-1} q^{\phi-\phi'}, \quad b = \nu^{-1} a, \quad c = \nu^{-1} b.$$  \hspace{1cm} (3.24)
Note that representations in (3.22), when changing $q^\varepsilon$ to $\omega^{-n}q^\varepsilon$, or equivalently, $(a', b', a, b, c)$ to $(\omega^{-n}a', \omega^n b', \omega^n a, \omega^{-n}b, c)$ for $n \in \mathbb{Z}$, are equivalent. For convenience, through a factorization of $c$, we shall express the parameter of $L$-operator in (3.23) by

$$L(t; p', p) := L(t), \quad p' = (a', b', d'), \quad p = (a, b, d), \quad c = d'd. \quad (3.25)$$

We shall also write $p'_+ := p', p_+ := p$. Through $\varphi_+$ in (3.2) and relations in (3.9), the $U_\omega(sl_2)$-representation (3.20) and $L$-operator (3.21) descend to the cyclic $\mathbb{C}^N$-representation (3.22) with $\varepsilon$ replaced by $\varepsilon - 1$, hence the $L$-operator (3.23) with parameter $(p'_-, p_-)$,

$$L(t; p'_-, p_-), \quad p'_- = (a'_-, b'_-, d'_-), \quad p_- = (a_-, b_-, d_-), \quad (3.26)$$

where $(p'_-, p_-)$ is related to $(p', p)$ by

$$(a'_-, b'_-, a_-, b_-, c_-) = (q^{-1}a', qb', qa, q^{-1}b, c), \quad (d'_-, d_-) = (d', d). \quad (3.27)$$

The relation (3.27) is equivalent to the gauge-equivalence of $L$-operators:

$$L(t; p'_-, p_-) = \text{dia}[1, q^{-1}]L(t; p'_+, p_+)\text{dia}[1, q], \quad \Leftrightarrow \quad \hat{Z}L(t; p'_+, p_+)\hat{Z}^{-1} = \text{dia}[1, q^{-1}]L(t; p'_-, p_-)\text{dia}[1, q]. \quad (3.28)$$

The $L$-operators in (3.25), (3.26) are related to the $\mathbb{C}^n$-representation in (3.20) in the following result in [26]:

**Lemma 3.3** The $N$-subspaces $\mathbb{C}^\pm$ of $\mathbb{C}^n$ in (3.3) are irreducible components of the $U_\omega(sl_2)$-representation (3.20) defined by the Weyl operators in (3.5). As $U_\omega(sl_2)$-representations, $\mathbb{C}^\pm$ is equivalent to $L(p'_+, p_+; t)$ via the morphism $\varphi_\pm$ in (3.7). For $n = N$ odd, $\mathbb{C}^n \simeq \mathbb{C}^\pm$ with the equivalence between $\mathbb{C}^\pm$ given by $|k\rangle_+ = q^k|\tilde{k}\rangle_+$. For $n = 2N$, $\mathbb{C}^n = \mathbb{C}^+ \oplus \mathbb{C}^-$, which is equivalent to $L(t; p'_+, p_+) \oplus L(t; p'_-, p_-)$, as $U_\omega(sl_2)$-representations.

**Proof.** It is obvious that $\mathbb{C}^\pm$ with the Weyl operators in (3.5) are irreducible components of $U_\omega(sl_2)$-representation $\mathbb{C}^n$ in (3.20). The rest statements follows from the relations (3.7), (3.22)-(3.26), and Lemma 3.1 (iii). \(\square\)

Through the substitution (3.24), we shall also use the parameter $(p', p)$ in (3.25) to represent the $L$-operators $L(s)$, $L(t)$ in (3.19) and (3.21):

$$L(s; p', p) = \left( \frac{b^3}{a^3 ac} \right)^{\frac{1}{2}} \begin{pmatrix} -s^{-1} \tilde{Z} + s \frac{c}{b} \tilde{Z}^{-1} & \frac{a}{(b'b)^s} \left( \frac{1}{b} \tilde{Z}' - \omega \frac{ac}{b^2} \tilde{Z}^{-1} \right) \hat{X}^{-1} \\ -\frac{(b'b)^s}{q} \left( \frac{1}{b} \tilde{Z}' - \frac{ac}{b^2} \tilde{Z}^{-1} \right) \hat{X}^{-1} & s \frac{1}{b} \tilde{Z}' - s^{-1} \frac{ac}{b^2} \tilde{Z}^{-1} \end{pmatrix}, \quad (3.29)$$

$$L(t; p', p) = \begin{pmatrix} 1 - t \frac{c}{b} \tilde{Z}^{-2} & \left( \frac{1}{b} - \omega \frac{ac}{b^2} \tilde{Z}^{-2} \right) \hat{X}^{-1} \\ -t \frac{1}{b} \tilde{Z}' - \omega \frac{ac}{b^2} \tilde{Z}^{-2} \end{pmatrix}, \quad (\tilde{Z}', \hat{X}') = (X', Z'^{-1}),$$

which are related by

$$L(t; p'p) = \text{dia}[1, -s(b'b)^{\frac{s}{2}} q] \left( -s(\omega^2 \frac{ac^2}{b^3 b^2}) \frac{1}{b} \tilde{Z}^{-1} \mathcal{L}(s; p'p) \right) \text{dia}[1, -s^{-1}(b'b)^{\frac{s}{2}} q^{-1}] \Leftrightarrow \mathcal{L}(s; p'p) = \text{dia}[1, -s^{-1}(b'b)^{\frac{s}{2}} q^{-1}] \left( -s^{-1}(\omega^2 \frac{ac^2}{b^3 b^2}) \frac{1}{b} \tilde{Z}' L(t; p'p) \right) \text{dia}[1, -s(b'b)^{\frac{s}{2}} q]. \quad (3.30)$$
Corresponding to (3.28), we have
\[
\mathcal{L}(s; p', p_-) = \hat{Z}'^{-1} L(s; p'_+, p_+) \hat{Z}', \quad \Leftrightarrow \quad \mathcal{L}(s; p'_+, p_+) = \hat{Z}' L(s; p', p_-) \hat{Z}'^{-1};
\]
\[
L(t; p', p_-) = \hat{Z}'^{-1} L(t; p'_+, p_+) \hat{Z}', \quad \Leftrightarrow \quad L(t; p'_+, p_+) = \hat{Z}' L(t; p', p_-) \hat{Z}'^{-1}.
\]
(3.31)

Indeed, by (3.16), the second relation in (3.35) is equivalent to the gauge relations in (3.28).

There is another type of cyclic \(\mathbf{C}^n\)-representations of (2.4)\(_{\mathbf{w}=\omega}\) associated to the following \(U_\omega(sl_2)\) representation:
\[
K = q^\phi - \phi \hat{Z}'^{-1}, \quad E^+ = q^{-\phi + \phi'} (1 - \omega - \phi - 1 \hat{Z}') \hat{X}^{-2}, \quad E^- = q^{-\phi + \phi'} (1 - \omega + \phi + 1 \hat{Z}') \hat{X}^2,
\]
with the \(L\)-operator
\[
L^\dagger(t; p', p) = \begin{pmatrix}
1 - t \frac{c}{b \hat{b}} & (1 - \omega \frac{c}{b \hat{b}} \hat{Z}') \hat{X}^{-2} \\
1 - t \frac{1}{b \hat{b}} & -\omega \frac{c}{b \hat{b}} + t \frac{\omega}{b \hat{b}} \hat{Z}'
\end{pmatrix}, \quad (\hat{Z}', \hat{X}') = (X', Z'^{-1}).
\]
(3.33)

The representation (3.32) and \(L\)-operator (3.33) are obtained by the replacement of \((\hat{X}', \hat{Z}')^{-2}\) in (3.20), (3.29) by \((\hat{X}'^{-2}, \hat{Z}')\). By (3.17) and (3.24), the representation (3.32) and \(L\)-operator (3.33) descend to the \(\mathbf{C}^N\)-representation (3.22) with the parameter \((\varepsilon^\dagger_\pm, \phi^\dagger_\pm, \phi'^\dagger_\pm)\) and \(L(t; p^\dagger_\pm, p^\dagger_\pm)\) in (3.23), via \(\varphi^\dagger_\pm\) in (10), where the parameters are defined by
\[
(\varepsilon^\dagger_+, \phi^\dagger_+, \phi'^\dagger_+) = (\varepsilon, \phi, \phi'), \quad p^\dagger_+ = p' = (a', b', d'), \quad p^\dagger_- = p = (a, b, d);
\]
\[
(\varepsilon^\dagger_-, \phi^\dagger_-, \phi'^\dagger_-) = (\varepsilon, \phi + \frac{1}{2}, \phi' - \frac{1}{2}), \quad p^\dagger_- = (a', b', d'), \quad p^\dagger_- = (a, b, dq).
\]
(3.34)

As in (3.35), we have
\[
L^\dagger(t; p^\dagger_-, p^\dagger_-) = \hat{X}'^{-1} L^\dagger(t; p^\dagger_+, p^\dagger_+) \hat{X}', \quad \Leftrightarrow \quad L^\dagger(t; p^\dagger_+, p^\dagger_+) = \hat{X}' L^\dagger(t; p^\dagger_-, p^\dagger_-) \hat{X}'^{-1},
\]
(3.35)

which by (3.16), is equivalent to
\[
L(t; p^\dagger_+, p^\dagger_-) = \hat{X}^\dagger L(t; p^\dagger_-, p^\dagger_+) \hat{X}^{-\dagger}, \quad \hat{X} L(t; p^\dagger_+, p^\dagger_-) \hat{X}^{-1} = \hat{X}^\dagger L(t; p^\dagger_-, p^\dagger_-) \hat{X}^{-\dagger},
\]
where \(\hat{X}^\dagger := \hat{X}'\). Note that the above relation is similar to that in (3.28), but not in a form of gauge equivalence. As in Lemma 3.3, we have the following result:

**Lemma 3.4** The \(N\)-subspaces \(\mathbf{C}^\dagger_\pm\) of \(\mathbf{C}^n\) in (3.11) are irreducible components of the \(U_\omega(sl_2)\)-representation (3.32). As \(U_\omega(sl_2)\)-representations, \(\mathbf{C}^\dagger_\pm\) is equivalent to \(L(p^\dagger_+, p^\dagger_-; t)\) via the morphism \(\varphi^\dagger_\pm\) in (3.15). For \(n = N\) odd, \(\mathbf{C}^n \simeq \mathbf{C}^\dagger_+\) with the equivalence between \(\mathbf{C}^\dagger_\pm\) given by \(|\sigma\rangle^\dagger_\pm = q^{-\sigma} |\sigma\rangle^\dagger_+\). For \(n = 2N\), \(\mathbf{C}^n = \mathbf{C}^\dagger_+ \oplus \mathbf{C}^\dagger_-\) is equivalent to \(L(t; p^\dagger_+, p^\dagger_-) \oplus L(t; p^\dagger_-, p^\dagger_-)\) as \(U_\omega(sl_2)\)-representations.

\(\square\)

The following lemma describes the equivalent representations in (3.18), (3.20) or (3.32) under the change of parameter \((p', p)\).
Lemma 3.5 We define the l-twists of \( p = (a, b, d) \) by
\[
p(l) := (aq^l, bq^{-l}, d), \quad p[l] := (a, b, dq^l), \quad (l \in \mathbb{Z}_n).
\] (3.36)

Let \( \mathcal{L}(s; p', p), \mathcal{L}(t; p', p) \) and \( L^l(t; p', p) \) be the L-operators in (3.29), (3.33) with the parameter \( (\varepsilon, \phi, \phi', \rho, \nu) \) defined in (3.24). For \( l, l' \in \mathbb{Z}_n \), let \( (\varepsilon^0, \phi^0, \phi'^0, \rho^0, \nu^0) \), \( (\varepsilon^0, \phi^0, \phi'^0, \rho^0, \nu^0) \) be the parameters in (3.24) corresponding to \( (p'(-l'), p(l)) \), \( (p'[l'], p[l]) \) respectively, related to \( (\varepsilon, \phi, \phi', \rho, \nu) \) by
\[
q^\varepsilon = q^\varepsilon^{-l}, \quad q^\phi = q^{\phi+\frac{l-1}{2}}, \quad q^{-\phi'} = q^{-\phi'+\frac{l+1}{2}}, \quad \rho^0 = q^l\rho, \quad \nu = q^{l-l'}\nu;
\] (3.37)

Then

(i) The \( C^n \)-representations, \( s_{\varepsilon, \phi, \phi'} \) and \( s_{\varepsilon^0, \phi^0, \phi'^0} \) of \( U_q(sl_2) \) in (3.18) (or the induced \( \hat{U}_\omega(sl_2) \)-representation in (3.20)), are equivalent if and only if \( l - l' = 2m \) (\( m \in \mathbb{Z}_n \)), where \( s_{\varepsilon, \phi, \phi'} = (\hat{X}^m \hat{Z}^{-l}) s_{\varepsilon^0, \phi^0, \phi'^0} (\hat{X}^l \hat{Z}^{l'}) \). In particular, the L-operators in (3.29) are equivalent if and only if \( l = l' \in \mathbb{Z}_n \), where \( \mathcal{L}(s; p'(-l), p(l)) = \hat{Z}^{-l-l'} \mathcal{L}(s; p', p) \hat{Z}^{l'}, \) and \( L(t; p'(-l), p(l)) = \hat{Z}^{-l-l} L(t; p', p) \hat{Z}^{l} \).

Furthermore, the \( C^n \)-representations of \( U_q(sl_2) \) in (3.22) for \( (\varepsilon, \phi, \phi') \) and \( (\varepsilon^0, \phi^0, \phi'^0) \) are equivalent if and only if \( l = 2m, l - l' = 2m \in 2\mathbb{Z}_n \), where \( (3.22)_{\varepsilon, \phi, \phi'} = (\hat{X}^m \hat{Z}^{2m}) (3.22)_{\varepsilon^0, \phi^0, \phi'^0} (\hat{Z}^{-n} \hat{X}^{-m}) \). In particular, the L-operators in (3.23) are equivalent if and only if \( l = l' \) with \( L(t; p'(-l), p(l)) = \hat{Z}^n L(t; p', p) \hat{Z}^{-n} \).

(ii) The representations \( (3.32) \) of \( \hat{U}_\omega(sl_2) \) for \( (\varepsilon^0, \phi^0, \phi'^0, \rho^0, \nu^0) \) in (3.37) are all equivalent for \( l', l \in \mathbb{Z}_n \); the same for the L-operators \( L^l(t; p'[l'], p[l]) \) in (3.33) with the relation \( L^l(t; p'[l'], p[l]) = \hat{X}^{-l-l'} L^l(t; p', p) \hat{X}^{l'+l} \). Furthermore, the equivalence of \( C^n \)-representations (3.32) for \( (\varepsilon^0, \phi^0, \phi'^0) \) and L-operators \( L(t; p'[l'], p[l]) \) in (3.23) is given by the condition: \( l' + l = 2m \in 2\mathbb{Z}_n \), where \( L(t; p'[l'], p[l]) = \hat{X}^m L(t; p', p) \hat{X}^{-m} \).

\[ \square \]

Remark. (I) In Lemma 3.5 (i), the representations \( s_{\varepsilon, \phi, \phi'} \) for \( (p'(-l'), p(l)) \) and \( (p'(-l'+N), p(l+N)) \) are equivalent with the same \( \rho^0, \nu^0 \). The parameter \( (p'_-, p_-) \) in (3.26) is equal to \( (p'(-l), p(1)) \) with \( \mathcal{L}(s), \mathcal{L}(t) \)-relation in (3.35). When \( n = N \) odd, \( s_{\varepsilon, \phi, \phi'} \) for \( l, l' \in \mathbb{Z}_n \) are all equivalent with \( m = (l - l')(n+1)/2 \in \mathbb{Z}_n \). However, in \( n = 2N \) case, the requirement of the constraint \( l - l' = 2m \) for equivalent representations in (3.18) is necessary. Furthermore, in \( n = N \) odd case, one has \( L(t; p'(-l), p(l)) = \hat{Z}^{-l} L(t; p', p) \hat{Z}^{l}, \) hence \( L(t; p'_\pm, p_{\pm}) \) are equivalent. When \( n = 2N, \) since \( 1 \notin 2\mathbb{Z}_n \), \( L(p'_\pm, p_{\pm}; t) \) are not equivalent with non-isomorphic \( \hat{U}_\omega(sl_2) \)-representation (3.22).

(II) The \( (p^\dagger_+, p^\dagger_-) \) in (4.48) is equal to \( (p^\dagger_0, p^\dagger_1) \) in Lemma 3.5 (ii). Hence \( L(t; p^\dagger_+, p^\dagger_-) = \hat{X}^N L(t; p^\dagger_+, p^\dagger_-) \hat{X}^{-N} \) for \( n = N \) odd case. When \( n = 2N, \) \( L(t; p^\dagger_+, p^\dagger_-) \) are not equivalent.

For convenience, we shall also use the following convention to identify the index \( \pm \) in Lemma 3.3 with \( \mathbb{Z}_2 \):
\[
\gamma = \pm := \pm 1 = (-1)^i \iff i = \frac{1 - \gamma}{2} = 0, 1 \in \mathbb{Z}_2.
\] (3.38)

For a positive integer \( L, \{\pm\}^L \) will also be identified with \( \mathbb{Z}_2^L \) via
\[
i (i_1, \ldots, i_L) \in \mathbb{Z}_2^L \iff ((-1)^{i_1}, \ldots, (-1)^{i_L}) \in \{\pm\}^L.
\]
in particular, \( \tilde{0} = (0, \ldots, 0) \leftrightarrow (+, \ldots, +) \), \( \tilde{1} = (1, \ldots, 1) \leftrightarrow (-, \ldots, -) \). We shall also denote \( i + 1 := \tilde{i} + \tilde{1} \in \mathbb{Z}_2^\ell \).

3.2 The \( \tau^{(2)} \)-model and XXZ-model with cyclic representation

For a chain of size \( L \), we consider the \( \tau^{(2)} \)-model (2.10) defined by \( L \)-operator (3.23) with parameters \( \{ (p'_{\ell}, p_{\ell}) \}_{\ell=1}^\ell \):

\[
\bigotimes_{\ell=1}^L L(t; p'_{\ell}, p_{\ell}) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}, \quad \tau^{(2)}(t; \{ p'_{\ell}, \{ p_{\ell} \}) = A(t) + \omega^r D(t), \quad (3.39)
\]

satisfying the boundary condition

\[
|\sigma_{L+1} \rangle = |\sigma_1 - r \rangle \quad \iff \quad |\bar{k}_{L+1} \rangle = \omega^{-r k_1} |k_1 \rangle,
\]

and the periodic-parameter condition \( (p'_{L+1}, p_{L+1}) = (p_1, p_1) \). Then \( \tau^{(2)}(t; \{ p'_{\ell}, \{ p_{\ell} \}) \) commutes with the charge operator \( \hat{Z} := \prod_{\ell} \hat{Z}_{\ell} \), which is the same as the spin-shift operator \( X := \prod_{\ell} X_{\ell} \) with the eigenvalue \( \omega^Q \) for \( Q \in \mathbb{Z}_N \). The XXZ-model and \( \tau^{(2)} \)-model in (2.10) with the monodromy matrix (2.7), (2.8) and \( L \)-operator (2.39) will be denoted by

\[
\bigotimes_{\ell=1}^L \mathcal{L}(s; p'_{\ell}, p_{\ell}) = \begin{pmatrix} A(s) & B(s) \\ C(s) & D(s) \end{pmatrix}, \quad \mathcal{T}(s) = \mathcal{T}(s; p', p) = A(s) + q^{-r'} D(s);
\]

\[
\bigotimes_{\ell=1}^L L(t; p'_{\ell}, p_{\ell}) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}, \quad \tau^{(2)}(t) = \tau^{(2)}(t; p', p) = A(\omega t) + q^{-r} D(\omega t),
\]

satisfying the boundary condition

\[
|\sigma_{L+1} \rangle = |\sigma_1 + r' \rangle \quad \iff \quad |\bar{k}_{L+1} \rangle = q^{r' k_1} |k_1 \rangle.
\]

Then \( [\mathcal{T}(s), \hat{Z}] = [\tau^{(2)}(t), \hat{Z}^{(2)}] = 0 \), where \( \hat{Z} := \prod_{\ell} \hat{Z}_{\ell} = (X' = \prod_{\ell} X_{\ell}) \). The eigenvalue of \( \hat{Z}' \) and \( \hat{Z}^{(2)} \) will be denoted by \( qQ', \omega^Q \) for \( Q' \in \mathbb{Z}_N, Q \in \mathbb{Z}_N \) respectively. By (2.11) and (3.30), the transfer matrices in (3.41) are related by

\[
\mathcal{T}(s) = (-s)^L \hat{Z}^{(2)}(\omega^{-1} t) \iff \mathcal{T}(s) = (-s)^L \hat{Z}^{(2)}(\omega^{-1} t).
\]

Similarly, we denote the \( \tau^{(2)} \)-model in (2.10) with the \( L \)-operator (3.33) and the boundary condition (3.42) by

\[
\bigotimes_{\ell=1}^L L^\dagger(t; p'_{\ell}, p_{\ell}) = \begin{pmatrix} A^\dagger(t) & B^\dagger(t) \\ C^\dagger(t) & D^\dagger(t) \end{pmatrix}, \quad \tau^{(2)}(t) = \tau^{(2)}(t; p', p) = A^\dagger(\omega t) + \omega^{-r} D^\dagger(\omega t).
\]

The \( \hat{Z}^{(2)} \)-eigenvalue of \( \tau^{(2)}(t) \) will be denoted by \( \omega^{Q'} \).

In this subsection, we study the relation between the \( \tau^{(2)} \)- and XXZ-model in (3.39), (3.41) and (3.44). For \( \tau^{(2)}(t), \tau(s) \) in (3.41), we assume the boundary condition in (3.42) is related to (3.40) by \( r' \equiv 2r \mod n \). First, we derive the relation between (3.39) and XXZ-model in (3.41).
For \( \vec{i} = (i_1, \ldots, i_L) \in \mathbb{Z}_2^L \) with \( \mathbb{Z}_2 \) identified with \( \pm \) in (3.38), we define the following sub-quantum space of \( \tau^{(2)} \)-model:

\[
\mathcal{C}_{\vec{i}} = \bigotimes_{\ell} \mathcal{C}_\ell \subseteq \bigotimes L \mathcal{C}_\ell^L, \\
\varphi_{\vec{i}}^\ell := \bigotimes \varphi(\xi_{(\ell)}^{(-1)}), \quad (C_\ell := C^{(-1)}), \\
\varphi_{\vec{i}}^\ell(\vec{k}^L) = \bigotimes_{\ell} \varphi_{\varphi_{\vec{i}}^\ell(\xi_{(\ell)}^{(-1)})}) \mapsto (k_1, \ldots, k_L),
\]

(3.45)

where \( C^\pm, \varphi^\pm \) are in (3.3) (3.2). Then \( C_{\vec{i}}^\ell \) is a sub-representation of the ABCD algebra of \( \tau^{(2)} \)-model in (3.41). By Lemma 3.3, \( \varphi_{\vec{i}} \) in (3.45) induces an equivalence between the \( \tau^{(2)} \)-monodromy matrix on \( C_{\vec{i}}^\ell \) and \( \tau^{(2)} \)-monodromy matrix (3.39) with

\[
(p'_{\ell i}, p_{\ell i}) := (p_{(\ell)}^{(-1)}), (p_{(\ell)}^{(-1)}),
\]

(3.46)

where \( (p'_{\pm}, p_{\pm}) \) are defined in (3.25) (3.26). Note that the parameter in (3.22) for the \( \mathcal{C}^N \)-representations of \( \mathfrak{U}_u(sl_2) \) associated to \( L(p'_{\pm}, p_{\pm}) \) differ only in \( \varepsilon \) and \( \varepsilon - 1 \), in particular with the same \( \nu \) in (2.8). Through \( \varphi_{\vec{i}} \) in (3.45), \( \tau^{(2)}(t) \) on \( C_{\vec{i}}^\ell \) is equivalent to \( \tau^{(2)}(t; \{ p'_{\ell i} \}, \{ p_{\ell i} \}) \) via

\[
\tau^{(2)}(t)_{C_{\vec{i}}^\ell} = \varphi_{\vec{i}}^{-1} \cdot \tau^{(2)}(t; \{ p'_{\ell i} \}, \{ p_{\ell i} \}) \cdot \varphi_{\vec{i}}^\ell,
\]

(3.47)

with the relation \( \hat{Z}^{\ell-2} = \varphi_{\vec{i}}^\ell \cdot \hat{Z} \cdot \varphi_{\vec{i}}^\ell \). Indeed by Lemma 3.3, the structure of \( \tau^{(2)} \)-model is given by

\begin{proposition}
Let \( \tau^{(2)}(t) \) be the \( \tau^{(2)} \)-model in (3.41) with the boundary condition \( r' = 2r \) in (3.42), and \( (p'_{\ell i}, p_{\ell i}) \) the parameter defined in (3.46). Then

(i) When \( n = N \) odd, \( \bigotimes L \mathcal{C}_\ell^L = \mathcal{C}_{\vec{i}} \) for \( \vec{i} \in \mathbb{Z}_2^L \), as representations of ABCD algebra in (2.8), and \( \tau^{(2)}(t) \simeq \tau^{(2)}(t; \{ p'_{\ell i} \}, \{ p_{\ell i} \}) \) via (3.47). The equivalent relations of \( C_{\vec{i}}^\ell \)'s are induced by the isomorphism \( \mathcal{C}_{\vec{i}} \simeq \mathcal{C}^\ell : (k_1, \ldots, k_L) \mapsto (\hat{k}_1, \ldots, \hat{k}_L)_{\vec{i}} \).

(ii) When \( n = 2N \), \( \bigotimes L \mathcal{C}_\ell^L = \bigoplus_{\vec{i} \in \mathbb{Z}_2^L} \mathcal{C}_{\vec{i}} \) as representations of ABCD algebra in (2.8), hence relations in (3.47) give rise to the isomorphism \( \tau^{(2)}(t) \simeq \bigoplus_{\vec{i} \in \mathbb{Z}_2^L} \tau^{(2)}(t; \{ p'_{\ell i} \}, \{ p_{\ell i} \}) \).\n\end{proposition}

\( \Box \)

We now use results obtained in Proposition 3.1 to study the XXZ-model in (3.41). By (3.43), the transfer matrix \( \mathcal{T}(s) \) differs from \( \tau^{(2)}(t) \) by a scale factor and a multiple of \( \hat{Z} \), which interchanges \( \mathcal{C}_{\vec{i}}^\ell \) and \( \mathcal{C}_{\vec{i}+1}^\ell \) by (3.6). Indeed, \( \hat{Z} \) identifies the \( \tau^{(2)} \)-transfer matrix on these subspaces as follows:

\begin{lemma}
\( \tau^{(2)}(t)_{|\mathcal{C}_{\vec{i}}} = \hat{Z}^{-1}\tau^{(2)}(t)_{|\mathcal{C}_{\vec{i}+1}} \). As a consequence, the \( \tau^{(2)} \)-eigenvectors \( v_{\vec{i}} \in \mathcal{C}_{\vec{i}} \) and \( v_{\vec{i}+1} \in \mathcal{C}_{\vec{i}+1} \) with the same eigenvalue are related by \( v_{\vec{i}+1} = \hat{Z}(\tau^{(2)}(t)_{|\mathcal{C}_{\vec{i}}}) \) (up to a non-zero scale).
\end{lemma}

\begin{proof}
By (3.4), \( \hat{Z} \) induces an one-to-one correspondence between \( \mathcal{C}_{\vec{i}}^\ell \) and \( \mathcal{C}_{\vec{i}+1}^\ell \), which is related to the \( \bigotimes L \mathcal{C}_\ell^N \)-automorphism \( \Pi_{\ell} \hat{Z}_{\vec{i}} \) via the projection \( \varphi_{\vec{i}} \)'s in (3.45) as follows:

\[
\begin{array}{ccc}
\mathcal{C}_{\vec{i}}^\ell & \xrightarrow{\hat{Z}} & \mathcal{C}_{\vec{i}+1}^\ell, \\
\varphi_{\vec{i}} \downarrow & & \downarrow \varphi_{\vec{i}+1} \\
\otimes L \mathcal{C}_\ell^N & \xrightarrow{\Pi_{\ell} \hat{Z}_{\vec{i}}} & \otimes L \mathcal{C}_\ell^N, \\
\end{array}
\]

(3.45)
By (3.28), one obtains the identification
\[ \tau^{(2)}(t; \{ p'_{i\ell} \}, \{ p_{i\ell} \}) = (\prod_{\ell} \tilde{Z}^{i\ell}_{\ell}) \tau^{(2)}(t; \{ p'_{i\ell+1} \}, \{ p_{i\ell+1} \})(\prod_{\ell} \tilde{Z}^{-i\ell}_{\ell}). \]

Indeed, the gauge relation in (3.28) corresponds to the action of \( \hat{\mathcal{Z}}' \) of \( \bigotimes^L \mathbb{C}^n \), hence follows the result. \( \square \)

**Proposition 3.2** Let \( \mathcal{T}(s) \) be the XXZ-model (3.41) with the boundary condition \( r' = 2r \) in (3.42), and \( (p'_{i\ell}, p_{i\ell}) \) be the parameters in (3.46). Then

(i) When \( n = N \) odd, \( \bigotimes^L \mathbb{C}^n = \mathcal{C}^{\tilde{i}} \) for \( \tilde{i} \in (\mathbb{Z}_2)^L \), as representations of \( \text{XXZ-(ABCD-)} \text{-algebra} \) in (2.7). The XXZ-transfer matrix \( \mathcal{T}(s) \) in (3.41) is related to \( \tau^{(2)}(\omega^{-1}t; \{ p'_{i\ell} \}, \{ p_{i\ell} \}) \) by

\[ \mathcal{T}(s) = \varphi_1^{-1} \cdot (-s)^L \left( \frac{\omega \alpha' \alpha^2}{\beta^2} \right)^{-\frac{L}{4}} \tilde{Z}^{N-1}_{-2} \tau^{(2)}(\omega^{-1}t; \{ p'_{i\ell} \}, \{ p_{i\ell} \}) \cdot \varphi_1; \]

(ii) When \( n = 2N \), we define \( \mathcal{C}^{\tilde{i}} = \mathcal{C}^{\tilde{i}} \oplus \mathcal{C}^{\tilde{i}+1} \) for a coset \( \tilde{i} := \{ i, i + 1 \} \in (\mathbb{Z}_2)^L / \langle \tilde{I} \rangle \). Then one has the decomposition of \( \bigotimes^L \mathbb{C}^n \) as representations of \( \text{XXZ-algebra} \) (2.7):

\[ \bigotimes^L \mathbb{C}^n = \bigoplus \{ \mathcal{C}^{\tilde{i}} \mid \tilde{i} \in (\mathbb{Z}_2)^L / \langle \tilde{I} \rangle \}. \]

The transfer matrices \( \mathcal{T}(s) \) and \( \tau^{(2)}(t) \) on each \( \mathcal{C}^{\tilde{i}} \) are related by (3.43). Indeed, \( \tau \)-eigenvectors in \( \mathcal{C}^{\tilde{i}} \) are \( v^\gamma \pm q^{-Q} v^\gamma_{i\ell+1} \) with the eigenvalue \( \pm q^Q (-s)^L \left( \frac{\omega \alpha' \alpha^2}{\beta^2} \right)^{-\frac{L}{4}} \tau^{(2)}(\omega^{-1}t) \) and \( \mathbb{Z}_n \)-charge \( Q, Q + N \) respectively, where \( v^\gamma, v^\gamma_{i\ell+1} \) are the \( \tau^{(2)} \)-eigenvectors in Lemma 3.6 with the \( \tau^{(2)} \)-eigenvector \( \tau^{(2)}(t) \) and \( \mathbb{Z}_N \)-charge \( Q \).

**Proof.** (i) follows easily from Proposition 3.1 (i) and (3.43). When \( n = 2N \), \( \hat{\mathcal{Z}}' \) interchanges the factors in \( \mathcal{C}^{\tilde{i}} \). Therefore \( \mathcal{C}^{\tilde{i}} \) is a component of \( \bigotimes^L \mathbb{C}^n \) as representations of \( \text{XXZ-algebra} \). The result about \( \tau \)-eigenvectors in \( \mathcal{C}^{\tilde{i}} \) follows from Lemma 3.6, where \( \tau^{(2)} \)-eigenvectors \( v^\gamma, v^\gamma_{i\ell+1} \) are related by \( \hat{\mathcal{Z}}'(v^\gamma_{i\ell+1}) = \sigma^{-Q} v^\gamma_{i\ell} \). \( \square \)

**Remark.** When the pair \( (i, i + 1) \) is changed to \( (i + 1, i) \) in Lemma 3.6, \( (v^\gamma_{i\ell+1}, v^\gamma_{i\ell}) \) are replaced by \( (v^\gamma_{i\ell}, v^\gamma_{i\ell+1}) \), by which the \( \tau \)-eigenvectors \( v^\gamma \pm q^{-Q} v^\gamma_{i\ell+1} \) in Proposition 3.2 is changed to \( v^\gamma_{i\ell+1} \pm q^{Q} v^\gamma_{i\ell} \).

We now study the structure of the \( \tau^{(2)} \)-model in (3.44). For \( \tilde{i} := \{ i_1, \ldots, i_L \} \in (\mathbb{Z}_2)^L \), define the parameter and the sub-quantum space of \( \tau^{(2)} \)-model:

\[ (p_{i\ell}^{\dagger}, p_{i\ell}) := (p_{i\ell}^{\dagger}_{(-1)i\ell}, p_{i\ell}^{\dagger}_{(-1)i\ell}), \quad \mathcal{C}^{\tilde{i}} = \bigotimes^L \mathbb{C}^n, \quad (C^{(-1)i\ell}) := \mathcal{C}^{(-1)i\ell}, \]

\[ \varphi_{i\ell}^{\dagger} := \bigotimes^L \varphi^{\dagger}_{(-1)i\ell}, \quad \mathcal{C}^{\tilde{i} \pm \ell} \cong \bigotimes^L \mathbb{C}^N, \quad (|\sigma_1, \ldots, \sigma_L\rangle_{(-1)i\ell}) \equiv (|\sigma_1, \ldots, |\sigma_L\rangle)^{\dagger}_{(-1)i\ell} \mapsto |\sigma_1, \ldots, \sigma_L\rangle, \]

where \( (p_{i\ell}^{\dagger}, p_{i\ell}) \) and \( \mathcal{C}^{\tilde{i} \pm \ell} \) are defined in (3.48), (3.11), (3.10) respectively. Similar to Proposition 3.1 and Lemma 3.6, the relation between the \( \tau^{(2)} \)-models in (3.39), (3.44) is given by
Proposition 3.3 Let \( \tau^{(2)}(t) \) be the \( \tau^{(2)} \)-model in (3.44) with the boundary conditions \( r' \) in (3.42) related to \( r \) in (3.40) by \( r' \equiv -r \pmod{N} \). Then \( \tau^{(2)}(t)|_{C^{\tilde{i}}} = \varphi_{\tilde{i}}^1 \cdot \tau^{(2)}(t; \{p_{i\ell}^\dagger\}, \{p_{i\ell}\}) \cdot \varphi_{\tilde{i}}^1 \), by which \( \hat{X}_{\tilde{i}}^{-2} = \varphi_{\tilde{i}}^1 \cdot \hat{X}_\tilde{i} \cdot \varphi_{\tilde{i}}^1 \), and the following results hold:

(i) When \( \mathbf{n} = N \) odd, \( \bigotimes^L C^n = C^{\tilde{i}} \) for \( \tilde{i} \in \mathbb{Z}_2^L \), as representations of ABCD algebra in (2.8), and \( \tau^{(2)}(t) \simeq \tau^{(2)}(t; \{p_{i\ell}^\dagger\}, \{p_{i\ell}\}) \), where the equivalent relations among \( C^{\tilde{i}} \)'s are induced by \( C^{i\tilde{i}} \simeq C^{\tilde{i}}; |\sigma_1, \ldots, \sigma_L\rangle_0^\dagger \rightarrow |\sigma_1, \ldots, \sigma_L\rangle_{\tilde{i}}^\dagger \).

(ii) When \( \mathbf{n} = 2N \), \( \bigotimes^L C^n = \bigoplus_{\tilde{i} \in \mathbb{Z}_2^L} C^{\tilde{i}} \) as representations of ABCD algebra in (2.8), hence \( \tau^{(2)}(t) \simeq \bigoplus_{\tilde{i} \in \mathbb{Z}_2^L} \tau^{(2)}(t; \{p_{i\ell}^\dagger\}, \{p_{i\ell}\}) \).

(iii) \( \tau^{(2)}(t)|_{C^{\tilde{i}}} = \hat{X}_{\tilde{i}}^{-1} \tau^{(2)}(t)|_{C^{\tilde{i}}} \hat{X}_{\tilde{i}} \), hence the \( \tau^{(2)} \)-eigenvectors \( v_{\tilde{i}}^1 \in C^{\tilde{i}} \) and \( v_{\tilde{i}+1}^1 \in C^{\tilde{i}+1} \) with the same eigenvalue are related by \( v_{\tilde{i}+1}^1 = \hat{X}_\tilde{i} (v_{\tilde{i}}^1) \) (up to a non-zero scale).

\[ \square \]

4  Duality of \( \tau^{(2)} \)-models and XXZ-models with cyclic representation

In this section, we discuss the duality of \( \tau^{(2)} \)-models with cyclic representation, as a generalization of the \( \tau^{(2)} \)-duality in CPM [27].

4.1  \( \tau^{(2)} \)-duality in chiral Potts model

In this subsection, we recall the \( \tau^{(2)} \)-duality in chiral Potts model in [27]. Consider the \( \tau^{(2)} \)-model \( \tau^{(2)}(t; \{p_{i\ell}^\dagger\}, \{p_{i\ell}\}) \) in (3.39) with the boundary condition (3.40). Then \( \tau^{(2)}(t; \{p_{i\ell}^\dagger\}, \{p_{i\ell}\}) \) preserves the \( Q \)-subspace \( V_{r,Q} \) of \( \bigotimes^L C^N \) with the following (Hermitian) orthonormal bases:

\[
V_{r,Q} = \bigoplus_{k_\ell} C[k_1, \ldots, k_L] \quad \left( \sum_{\ell=1}^L k_\ell \equiv Q \pmod{N}, \quad k_{L+1} \equiv \omega^{-rL + 1} k_1 \right)
\]

\[
= \bigoplus_{n_\ell} C[n_1, \ldots, n_L] \quad \left( \sum_{\ell=1}^L n_\ell \equiv r \pmod{N}, \quad n_{L+1} \equiv \omega^{-rn_1 n_1} \right),
\]

(4.1)

where \( \{Q; n_1, \ldots, n_L\} := N^{-1/2} \sum_{\sigma_1=0}^{N-1} \omega^{-Q \sigma_1} |\sigma_1, \ldots, \sigma_L\rangle \) with \( \sigma_\ell - \sigma_{\ell+1} = n_\ell \) (see, e.g. [3, 4, 15] in \( r = 0 \) case, and [27, 28]). Define the dual correspondence between \( (r, Q) \)- and \( (r^*, Q^*) \)-spaces (with respective to local \( C^N \)-basis \( \{|\sigma\rangle , \{|\tilde{n}\rangle \} \) ([27] (3.16)):

\[
\Psi : V_{r,Q} \longrightarrow V_{r^*,Q^*}, \quad |Q; n_1, \ldots, n_L\rangle \mapsto |\tilde{n}_1, \ldots, \tilde{n}_L\rangle, \quad \left( \sum_{\ell=1}^L n_\ell \equiv r \right) \quad (r^*, Q^*) = (Q, r);
\]

(4.2)

and duality of parameters \( p \in C^3 \) ([27] (3.9)²):

\[
p = (a, b, d) \longrightarrow p^* = (a^*, b^*, d^*) := (ad, bd^{-1}, d^{-1}).
\]

(4.3)

The following lemma is used in the study of \( \tau^{(2)} \)-matrix and \( \tau^{(2)} \)-duality in CPM ([8] (2.14)-(2.15), [9] (3.48), [27](3.3)-(3.9)):

\[ ^2 \text{For the discussion of the chiral Potts model, the duality of rapidities in [27](3.9) differs here by a constant } \alpha = i^{1/4} \text{ as described in Remark of Proposition 4.1 in this paper.} \]
Lemma 4.1 Let
\[
L(t; p', p) = \begin{pmatrix}
L_0^0(t; p', p) & L_0^1(t; p', p) \\
L_0^1(t; p', p) & L_1^1(t; p', p)
\end{pmatrix}, \quad p' = (a', b', d'), p = (a, b, d),
\]
be a -operator in (3.23) with \(p', p\) in (3.25). Denote
\[
L_{m}^{\sigma''}(t; p', p) = \langle \sigma | L_{m}^{\sigma'}(t; p', p) | \sigma'' \rangle, \quad L_{n}^{k^*''}(t; p, p^*) = \langle \hat{k}^* | L_{n}^{\sigma'}(t; p^*, p^*) | \hat{k}^*'' \rangle,
\]
where \(\sigma, \sigma'', k^*, k^*'' \in \mathbb{Z}_N\), \(m, m', n, n' = 0, 1\). Define
\[
E(p)_{\sigma}^{\sigma''} := \omega^{-m_{\sigma}} F_{p}^{\sigma''}(\sigma - \sigma''), \quad E(p)_{\sigma}^{\sigma''} m' := \omega^{m_{\sigma} - m_{\sigma'} - m_{\sigma''}} F_{p}^{\sigma''}(\sigma - \sigma'', m')
\]
for \(a, b, c, d \in \mathbb{Z}_N\), where
\[
\frac{m}{b} = -\omega t, \quad F_{p}(0, 0) = 1, \quad F_{p}(0, 1) = \frac{-\omega t}{b}, \quad F_{p}(1, 0) = \frac{d}{b}, \quad F_{p}(1, 1) = \frac{-\omega ad}{b},
\]
Then (i) \(L_{m}^{\sigma''}(t; p', p) = E(p)_{m}^{\sigma''} E(p)_{\sigma}^{\sigma''}\) for \(\sigma, \sigma'' \in \mathbb{Z}_N\), \(m, m' = 0, 1\).
(ii) \(U_{p, p'}^{(d, \ell)} = 0\) if \(a - d\) or \(b - c \neq 0, 1\), and
\[
U_{p, p'}^{(d, \ell)} = L_{n}^{k^*''}(\omega t; p, p^*)
\]
whenever \(a - b = k^*, d - c = k^*'' \in \mathbb{Z}_N\).

Proof. (i) follows from (3.23) and the definition of \(E(p)_{m}^{\sigma''} E(p)_{\sigma}^{\sigma''}\). One also finds
\[
U_{p, p'}^{(d, \ell)} = \sum_{m = 0}^{1} \omega^{m(d - b)}(-\omega t)^{a - d - m} F_{p}(a - d, m) F_{p'}(b - c, m)
\]
whose non-zero values are determined by entries of
\[
L(\omega t; p^*, p^{*'}) = \begin{pmatrix}
1 - \omega t \frac{1}{bb' \hat{Z}} \hat{X}^{-1} & -\omega t \frac{d}{bb'} \hat{X}^{-1}
\\
-\omega t \frac{d}{bb'} \hat{X}^{-1} - \omega t \frac{dd'}{bb'} \hat{X}
\end{pmatrix}.
\]
Then (ii) follows. □

Using Lemma 4.1 and (4.4), one finds the product form of \(\tau(2) (t; \{p'_{i}\}, \{p_{i}\})\) ([9] (3.44) \(k = 0, j = 2\), [8] (2.16)):
\[
\tau(2)(t; \{p'_{i}\}, \{p_{i}\}) = \prod_{\ell = 1}^{L} U_{p_{\ell}, p'_{\ell + 1}}^{(d, \ell)}(\sigma_{\ell''}^{\sigma_{\ell''}}, \sigma_{\ell + 1}^{\sigma_{\ell + 1}}),
\]
where \(U_{p, p'}^{(d, \ell)}\) are defined in (4.4). Note that by the boundary condition (3.40), \(E(p'_{L+1})_{m}^{\sigma'} \sigma_{L+1}^{\sigma'} = \omega^{m_{\sigma'}} E(p'_{m})_{\sigma}^{\sigma'}\), which contributes the \(\omega^{m_{\sigma'}}\)-factor in (3.39).

Proposition 4.1 ([27] Proposition 3.1) Let \(p_{\ell}^{*}, p'_{\ell}^{*}\) be the dual of \(p_{\ell}, p'_{\ell}\) in (4.3), and \(\Psi\) be the dual correspondence between \(V_{r, Q}\) and \(V_{r, Q'}\) with \((r^{*}, Q^{*}) = (Q, r)\) in (4.2). Then \(\tau(2)(t; \{p'_{i}\}, \{p_{i}\})\) on \(V_{r, Q}\) is equivalent to \(\tau(2)(t; \{p^{*}_{i}\}, \{p^{*'}_{i}\})\) on \(V_{r, Q'}\) by
\[
\tau(2)(t; \{p'_{i}\}, \{p_{i}\}) = \Psi^{-1} \tau(2)(t; \{p^{*}_{i}\}, \{p^{*'}_{i}\}) \Psi.
\]
Lemma 4.2  

We now study the duality of \(\tau^{(2)}\). By (4.6) and Lemma 4.1, one finds

\[
\langle Q; n_1, \ldots, n_L | \tau^{(2)}(t; \{p'_{i_1}\}, \{p_{i_1}\}) | Q; n'_1, \ldots, n'_L \rangle = \langle \hat{n}_1, \ldots, \hat{n}_L | \tau^{(2)}(t; \{p'_{i_1}\}, \{p^*_{i_1+1}\}) | \hat{n}'_1, \ldots, \hat{n}'_L \rangle.
\]

Then the result follows. □

Remark. One may modify the duality of parameters in (4.3) by defining \(p^* = (\alpha a, \alpha b d^{-1}, d^{-1})\), where \(\alpha\) is a scale constant. Then \(U_{p,p'}(d_{ib})_{in}\) in (4.4) = \(\text{dia}[1, \alpha^{-1}]L(\omega t^*; p^*, p'^*_{i_1+1})\text{dia}[1, \alpha]\) with \(t^* = \alpha^2 t\), by which the duality (4.7) again holds if \(\tau^{(2)}(t; \{p'_{i_1}\}, \{p'^*_{i_1+1}\})\) in (4.7) is changed to \(\tau^{(2)}(t^*; \{p'_{i_1}\}, \{p'^*_{i_1+1}\})\).

4.2 Duality of \(t^{(2)}\)-models with cyclic representation

We now study the duality of \(\tau^{(2)}\)-models \(\tau^{(2)}(t; p', p), \tau^{(1)}(t; p', p)\) in (3.41), (3.44) with the boundary condition (3.42). As in (4.1), these \(\tau^{(2)}\)-models preserve the \(Q'\)-subspace \(V_{r', Q'}\) of \(\bigotimes L^1_m \mathbf{C}^n\), which is generated by the following orthonormal bases:

\[
V_{r', Q'} = \bigoplus_{k_1} \mathbf{C}[\hat{k}_1, \ldots, \hat{k}_L] \quad (\sum_{\ell=1}^L k_\ell \equiv Q' \text{ (mod } n), \hat{k}_{L+1} \equiv r'k_1 \hat{k}_1),
\]

\[
= \bigoplus_{n_L} \mathbf{C}[Q'; n_1, \ldots, n_L] \quad (\sum_{\ell=1}^L n_\ell \equiv -r' \text{ (mod } n), n_{L+1} \equiv -Q' n_1 n_1).
\]

where \(|Q'; n_1, \ldots, n_L) := n^{-1/2} \sum_{\ell=1}^{n-1} q^{-Q' \sigma_1} |\sigma_1, \ldots, \sigma_L)\) with \(\sigma_\ell - \sigma_{\ell+1} = n_\ell\). There is the dual correspondence between \((r', Q')\) and \((r^*, Q^*)\)-spaces (with respective to the local \(\mathbf{C}^n\)-basis \(|\sigma)\}, \{\hat{n})\})\) with \((r^*, Q^*) = (-Q, r)\):

\[
\Psi' : V_{r', Q'} \rightarrow V_{r^*, Q^*}, \quad |Q'; n_1, \ldots, n_L) \mapsto |\hat{n}_1, \ldots, \hat{n}_L), \quad (\sum_{\ell=1}^L n_\ell \equiv -r \text{ (mod } n)).
\]

As the \(\tau^{(2)}\)-model in Lemma 4.1, the \(L\)-operator of \(\tau^{(2)}\)- and \(\tau^{(1)}\)-model can be decomposed into a product form:

Lemma 4.2  

Let

\[
\begin{aligned}
L(t; p', p) &= \begin{pmatrix} L^0_0(t; p', p) & L^0_1(t; p', p) \\ L^1_0(t; p', p) & L^1_1(t; p', p) \end{pmatrix}, \\
L(t^*) &:= \begin{pmatrix} L^1_0(t^*; p', p) & L^1_1(t^*; p', p) \\ L^0_1(t^*; p', p) & L^0_0(t^*; p', p) \end{pmatrix},
\end{aligned}
\]

be \(L\)-operator in (3.35), (3.33) with \(p', p\) in (3.25). Denote

\[
L^{m' \sigma''}(t; p', p) = \langle |\sigma| L^{m'}_m(t; p', p) |\sigma'')\rangle, \\
L^{m \sigma''}(t; p', p) = \langle |\sigma| L^{m'}_m(t; p', p) |\sigma''\rangle,
\]

where \(\sigma, \sigma'' \in \mathbf{Z}_m\), \(m, m' = 0, 1\). Define

\[
\begin{aligned}
E(p')_{m; \sigma} := q^{-m \sigma} F_p(-\frac{\sigma + \sigma'}{2}, m), \\
E(p)^{m''}_{m; \sigma} := q^{m' \sigma''} \eta_{m+\sigma''/2} F_p(-\frac{\sigma + \sigma''}{2}, m'); \\
E(p')^\dagger_{m; \sigma} := \omega^{-m \sigma} F_p(\sigma - \sigma'', m), \\
E(p)^\dagger_{m; \sigma''} := \omega^{m' \sigma''} \eta_{m''} F_p(\sigma - \sigma'', m'); \\
U_{p, p'}(d_{ib}) := \sum_{m=0}^{1} E(p')^{d}_{m} E(p)^{c}_{m; b}. \\
U_{p, p'}^\dagger(\hat{d}_{ib}) := \sum_{m=0}^{1} E(p)^{d}_{m} E(p')^{c}_{m; b},
\end{aligned}
\]

for \(a, b, c, d \in \mathbf{Z}_n\) with \(\frac{\eta_a}{\eta_b}\) and \(F_p(\alpha, m)\) in (4.5). Then

(i) The \((m, m')\)th entry of \(L, L^\dagger\)-operator for \(m, m' = 0, 1\) are expressed by

\[
L^{m' \sigma''}_m(\omega t; p', p) = E(p')^{m'; \sigma''} E(p)^{c}_{m; \sigma} m', \\
L^{m \sigma''}_m(\omega t; p', p) = E(p')^{d; \sigma''} E(p)^{c}_{m; \sigma} m',
\]
where \( \sigma, \sigma'' \in \mathbb{Z}_n \).

(ii) \( U_{\rho, \rho'} \) and \( U_{\rho, \rho'}^\dagger \) are expressed by

\[
U_{\rho, \rho'}(d|_{\rho}) = \begin{cases} 
\langle \langle \mathcal{K}_{\rho}^L \rangle_{n'}(\omega t; p^*, p^*) \rangle | \mathcal{K}_{\rho''} \rangle & \text{if } (a - d, b - c) = (-2n, -2n'), n, n' = 0, 1; \\
0 & \text{otherwise.}
\end{cases}
\]

\[
U_{\rho, \rho'}^\dagger(d|_{\rho}) = \begin{cases} 
\langle \langle \mathcal{K}_{\rho}^L \rangle_{n'}(\omega t; p^*, p^*) \rangle | \mathcal{K}_{\rho''} \rangle & \text{if } (a - d, b - c) = (n, n'), n, n' = 0, 1; \\
0 & \text{otherwise.}
\end{cases}
\]

where \( a - b = k^*, d - c = k''^* \in \mathbb{Z}_n \), and \( p^*, p^* \) are the dual of \( p, p' \) in (4.3).

**Proof.** By the definition of \( E(p'), E(p) \) and \( E^\dagger(p'), E^\dagger(p) \) in (4.10), the spin-operator expressions of (3.29) (3.33) yield relations in (i). (ii) follows from the expression of \( U_{\rho, \rho'}, U_{\rho, \rho'}^\dagger \) in (4.10):

\[
U_{\rho, \rho'}(d|_{\rho}) = \sum_{m=0}^{q} \omega^{m(d-b)}(-\omega t)^{-a+d-m} F_p(-\omega t, d, m, m) F_p(b, c, m),
\]

\[
U_{\rho, \rho'}^\dagger(d|_{\rho'}) = \sum_{m=0}^{q} \omega^{m(d-b)}(-\omega t)^{-a+d-m} F_p(a - d, m) F_p(b - c, m).
\]

whose non-zero values are entries of the following respective \( L \)-operator:

\[
\begin{pmatrix}
1 - \omega t \frac{b}{bb} \tilde{Z}' & (\frac{d}{bb} - \omega \frac{a'd'}{bb} \tilde{Z}') \tilde{X}'^2 \\
-\omega t \frac{b}{bb} \tilde{Z}' & -\omega t \frac{b}{bb} \tilde{Z}'
\end{pmatrix} = L(t; p^*, p^*).
\]

\[\square\]

Using Lemma 4.2 (i) and (4.10), one finds the product form of \( \tau^{(2)}(t; p', p), \tau^{(2)}(t; p', p) \):

\[\tau^{(2)}(t; p', p)\{\sigma^L\}_{\{\sigma_i\}} = \prod_{t=1}^{L} U_{\rho, \rho'}(\sigma^L | \sigma_{t+1}) \times \tau^{(2)}(t; p', p)\{\sigma^L\}_{\{\sigma_i\}} = \prod_{t=1}^{L} U_{\rho, \rho'}(\sigma^L | \sigma_{t+1}).\]

By the boundary condition (3.42), one finds \( E(p') \mid \sigma_{L+1} = q^{-mr} E(p') \mid \sigma_1 \), \( E^\dagger(p') \mid \sigma_{L+1} = \omega^{-mr} E(p') \mid \sigma_1 \), which provide the factor of the second term in (3.41) or (3.44). By (4.11), one finds

\[\langle \langle Q'; n_1, \ldots, n_L \rangle \tau^{(2)}(t; p', p) \langle Q'; n'_1, \ldots, n'_L \rangle = \langle \langle \hat{n}_1, \ldots, \hat{n}_L \rangle \tau^{(2)}(t; p^*, p^*) \langle \hat{n}'_1, \ldots, \hat{n}'_L \rangle \rangle;\]

\[\langle \langle Q'; n_1, \ldots, n_L \rangle \tau^{(2)}(t; p', p) \langle Q'; n'_1, \ldots, n'_L \rangle = \langle \langle \hat{n}_1, \ldots, \hat{n}_L \rangle \tau^{(2)}(t; p^*, p^*) \langle \hat{n}'_1, \ldots, \hat{n}'_L \rangle \rangle.\]

Hence we obtain the duality between \( \tau^{(2)} \)-models in (3.41), (3.44) as \( \tau^{(2)}(t; \{p'_i\}, \{p_i\}) \) in Proposition 4.1:

**Proposition 4.2** Let \( p^*, p^* \) be the dual of \( p, p' \) in (4.3), and \( \Psi' \) be the dual correspondence between \( V_{r^*, Q^*} \) and \( V_{r^*, Q^*'} \) with \( (r^*, Q^*) = (-Q, -r) \) in (4.9). Then \( \tau^{(2)}(t; p, p), \tau^{(2)}(t; p', p) \) on \( V_{r^*, Q^*} \) is equivalent to \( \tau^{(2)}(t; p^*, p^*), \tau^{(2)}(t; p^*, p^*) \) on \( V_{r^*, Q^*'} \) respectively by

\[\tau^{(2)}(t; p, p) = \Psi^{(2)}(t; p^*, p^*) \Psi', \quad \tau^{(2)}(t; p', p) = \Psi^{(2)}(t; p^*, p^*) \Psi'.\]
4.3 Comparison of $\tau^{(2)}$-duality and $t^{(2)}$-duality

By Proposition 3.1 and 3.3, $T^{(2)}(t)$ and $T^{(2)}(t)$ can be decomposed as a sum of $\tau^{(2)}$-models via the subspaces $C^{x}_r$, $C^{x}_r$ of $\otimes \mathbb{C}^n$ for $i \in \mathbb{Z}^L$ in (3.45) (3.48) respectively. Indeed, by (3.47), $T^{(2)}(t)\mid_{C^x_i} \simeq T^{(2)}(t; \{p_i\}, \{p_i\})$ with $\{p_i\}$ in (3.46) and the boundary condition $r' \equiv 2r \mod n$. Similarly, $T^{(2)}(t)\mid_{C^x_i} \simeq T^{(2)}(t; \{p_i\}, \{p_i\})$ with $\{p_i\}$ in (3.48) and the boundary condition $r' \equiv -r \mod n$. One can lift the basis of $V_{r,Q}$ in (4.1) to $C^{x}_r$, $C^{x}_r$, denoted by

$$C^{x}_r \subseteq V_{r,Q}^{r'}$$, only if $Q = Q'$, $2r = r'$;
$$C^{x}_r \subseteq V_{r,Q}^{r'}$$, only if $i_0 = 0 : r' = -r, -2Q + \sum \ell n_\ell \equiv Q'$,

in particular, $C^{x}_r$ is not contained in any $V_{r,Q}^{r'}$. Hence in $n = 2N$ case, the $T^{(2)}$-duality in Proposition 4.2, with the $\tau^{(2)}$-decomposition of $T^{(2)}(t)$, $T^{(2)}(t)$ in Proposition 3.1, 3.3, is different from the $\tau^{(2)}$-duality in Proposition 4.1. Indeed, the difference can also be seen in the inconsistency of parameter of $\tau^{(2)}$-duality in Proposition 4.2:

$$T^{(2)}(t; p_i, p_i)\mid_{C^x_i} \simeq \tau^{(2)}(t; \{p_i\}, \{p_i\})$$, $T^{(2)}(t; p_i, p_i)\mid_{C^x_i} \simeq \tau^{(2)}(t; \{p_i\}, \{p_i\})$;

$$T^{(2)}(t; p_i, p_i)\mid_{C^x_i} \simeq \tau^{(2)}(t; \{p_i\}, \{p_i\})$$, $T^{(2)}(t; p_i, p_i)\mid_{C^x_i} \simeq \tau^{(2)}(t; \{p_i\}, \{p_i\})$.

In $n = N$ odd case, one can identify $\otimes \mathbb{C}^n$ with $C^0$ or $C^\uparrow$ in Proposition 3.1 or 3.3.

**Proposition 4.3** When $n = N$ odd, one has the identical quantum subspaces, $V_{r,Q}^{r'} = C^{x}_r$ with $Q' \equiv Q \equiv -2Q^\uparrow, r' \equiv 2r = -r^\uparrow \mod n$, and the identification of basis elements in (4.8), (4.13):

$$|\tilde{k}_1, \ldots, \tilde{k}_L\rangle_0 = |\tilde{k}_1, \ldots, \tilde{k}_L\rangle, \quad |Q; n_1, \ldots, n_L\rangle_0 = |Q'; -2n_1, \ldots, -2n_L\rangle, \quad |(-2k^\uparrow_1, \ldots, -2k^\uparrow_L)\rangle_0 = |Q'; n_1, \ldots, n_L\rangle,$$

with $\sum \ell k_\ell \equiv Q, \sum \ell k^\uparrow_\ell \equiv Q^\uparrow, \sum \ell n_\ell \equiv r, \sum \ell n^\uparrow_\ell \equiv r^\uparrow \mod n$. With the identification of $\tau^{(2)}$-models in Proposition 3.1 and 3.3,

$$(\tau^{(2)}(t; \{p\}, \{p\}), \tau^{(2)}(t; \{p\}, \{p\})) \simeq (T^{(2)}(t; p, p), T^{(2)}(t; p, p)) \simeq (T^{(2)}(t; p, p), T^{(2)}(t; p, p)),$$

the duality relations in (4.12), (4.7) are the same.
Proof. By (3.3), (3.11) and (4.14), we obtain (4.15). On the dual space in (4.12), one has \( V'_\tau, Q' = C^0_{\tau, Q} \) with \( Q^* \equiv Q^* = -2Q^{\tau^\dagger}, r^* \equiv 2r^* \equiv -r^{\tau^\dagger} \), and the identification of basis elements:

\[
\begin{align*}
|\tilde{k}_1, \ldots, \tilde{k}_L\rangle_0 &= |\tilde{k}_1, \ldots, \tilde{k}_L\rangle, & |Q^*; n_1^*, \ldots, n_L^*\rangle_0 &= |Q^*; -2n_1^*, \ldots, -2n_L^*\rangle, \\
|\tilde{k}_1^\dagger, \ldots, \tilde{k}_L^\dagger\rangle_0^\dagger &= -2k_1^\dagger, \ldots, -2k_L^\dagger, & |Q^{\tau^\dagger}; n_1^\dagger, \ldots, n_L^\dagger\rangle_0^\dagger &= |Q^*; n_1^\dagger, \ldots, n_L^\dagger\rangle,
\end{align*}
\]

with \( \sum_{\ell=1}^L k_\ell^* \equiv Q^*, \sum_{\ell=1}^L k_\ell^{\tau^\dagger} \equiv Q^{\tau^\dagger} \), \( \sum_{\ell=1}^L n_\ell^* \equiv r^*, \sum_{\ell=1}^L n_\ell^{\tau^\dagger} \equiv r^{\tau^\dagger} \). Then the following conditions are equivalent,

\[
(r^{\tau^\dagger}, Q^*) = (-Q, -r) \iff (r^{\dagger}, Q^*) = (Q, r) \iff (r^*, Q^*) = (Q^*, r^*).
\]

Then the dual correspondence \( \Psi' \) in (4.9) becomes

\[
|Q; n_1, \ldots, n_L\rangle_0 \mapsto |n_1, \ldots, n_L\rangle_0^\dagger, & |Q^*; n_1^\dagger, \ldots, n_L^\dagger\rangle_0^\dagger \mapsto |n_1^\dagger, \ldots, n_L^\dagger\rangle_0,
\]

by which, the dualities (4.12) and (4.7) are equivalent. \( \Box \)

Other than the situation in Proposition 4.3, the \( \tau(2) \)-duality (4.7) in Proposition 4.1 can not be lifted to the duality between \( \tau^{(2)}(t; p', p) \) and \( \tau^{(2)}(t; p^*, p'^*) \). Neither between \( \tau^{(2)}(t; p', p) \) and \( \tau^{(2)}(t; p^*, p'^*) \). In the latter case, the \( \tau(2) \)-model dual to (3.47) is \( \tau(2) (t; \{ p_{i \ell}^* \}, \{ p_{i \ell+1}^* \}) \), whose \( L \)-operator at \( \ell \)th site in (3.39) is defined by the parameter

\[
(p_{i \ell}^*, p_{i \ell+1}^*) = (p_{i \ell}^*; p_{i \ell}^*, p_{i \ell}^*, p_{i \ell+1}^*, p_{i \ell+1}^*).
\]

i.e. the \( l \)-twist (3.36) of \( p^* \) and \( p'^* \) with the identification \( p_{(l-1)}^* = p^*(l), p_{(l-1)}'^* = p'^*(l) \) for \( l = 0, 1 \). Hence \( \tau^{(2)}(t; \{ p_{i \ell}^* \}, \{ p_{i \ell+1}^* \}) \) is not a component of the decomposition of \( \tau(2) (t; p^*, p'^*) \) in Proposition 3.1. Since XXZ-model is related to \( \tau(2) \)-model by (3.43), the \( \tau(2) \)-duality in Proposition 4.1 can not be lifted to a duality among XXZ-models. Indeed, we shall shown in the next subsection, the dual model of XXZ-model \( \mathcal{T}(s; p', p) \) is given by \( \tau^{(2)}(t; p^*, p'^*) \), so the essence of duality lies in the duality among \( \tau(2) \)-models.

### 4.4 Dual model of XXZ-model with cyclic representation

In this subsection, we derive the dual model of the XXZ-model \( \mathcal{T}(s) \) in (3.41) with the boundary condition (3.42). As the \( L \)-operator of \( \tau(2) \)-model in Lemma 4.2, the \( L \)-operator of XXZ-model \( \mathcal{T}(s) \) can be decomposed in the following product form:

**Lemma 4.3** Let

\[
\mathcal{L}(s; p', p) = \begin{pmatrix} \mathcal{L}^+(s; p', p) & \mathcal{L}^-(s; p', p) \\ \mathcal{L}^+(s; p', p) & \mathcal{L}^-(s; p', p) \end{pmatrix}, \quad \mathcal{L}^+(t; p^*, p'^*) = \begin{pmatrix} \mathcal{L}^+(t; p^*, p'^*) & \mathcal{L}^-(t; p^*, p'^*) \\ \mathcal{L}^+(t; p^*, p'^*) & \mathcal{L}^-(t; p^*, p'^*) \end{pmatrix}
\]

be the \( L \)-operator in (3.29), (3.33) respectively, where \( p' = (a', b', d') \), \( p = (a, b, d) \) and \( p^*, p'^* \) are the dual of \( p, p' \) in (4.3), (here we use \( \gamma = \pm (= \pm 1) \) in (3.38) as the auxiliary index of the \( L \)-operator). Denote

\[
\mathcal{L}_{\gamma^\prime}^{\sigma''}(s) = \langle \sigma | \mathcal{L}_{\gamma'}^{\prime}(s; p', p) | \sigma'' \rangle, \quad \mathcal{L}_{\delta'}^{\delta'''}(t) = \langle \bar{k}^* | \mathcal{L}_{\delta'}^{\delta'}(t; p^*, p'^*) | \bar{k}'''' \rangle.
\]
where \( \sigma, \sigma'', k^*, k''\in \mathbb{Z}_n \), \( \gamma, \gamma', \delta, \delta' = \pm \). Define

\[
\mathcal{E}(p')_{\gamma}^{\sigma''} := (bb')^{-m}q^{-m\sigma}(-s)^{-(1+m)}F_p'(\sigma - \sigma'', \gamma), \quad (m = 1-\gamma), \\
\mathcal{E}(p''\sigma'')_{\gamma'} := (bb')^{m}q^{m_\infty}(-s)^{m_\infty}F_p(\sigma - \sigma'', \gamma'), \quad (m' = 1-\gamma'); \\
U_{p,p'}(d_{a|b}) := \sum_{\gamma=\pm} \mathcal{E}(p')_{\gamma}^{d_{a}} \mathcal{E}(p''\sigma')_{\gamma}^{d_{b}}, \\
\text{for } a, b, c, d \in \mathbb{Z}_n, \text{ where } \eta = -s^2, \quad F_p(\pm, +) = (\frac{b^3}{\eta^2})^{2}, \quad F_p(+, -) = (\frac{b^3}{\eta^2})^{2}, \quad F_p(-, +) = (\frac{b^3}{\eta^2})^{2}, \quad F_p(\alpha, \gamma) = 0 \text{ if } \alpha \neq \pm 1. \\
(i) \quad \mathcal{L}_q^{\sigma''}(s) = \mathcal{E}(p')_{\gamma}^{\sigma''}, \quad \mathcal{E}(p''\sigma')_{\gamma}^{\sigma''}, \quad \text{for } \sigma, \sigma'' \in \mathbb{Z}_n, \gamma, \gamma' = \pm. \\
(ii) \quad U_{p,p'}(d_{a|b}) = 0 \text{ if } a - d \text{ or } b - c \neq \pm 1, \quad \text{and} \\
U_{p,p'}(d_{a|b}) = -s^{-1}(\frac{b^3}{\alpha^3})^2 \sum_{\gamma=\pm} q^{1-\gamma}q^{a-d+1}F_p(a - d, \gamma)F_p'(b - c, \gamma),
\]

of which the non-zero values form the following \( L \)-operator

\[
-s^{-1}(\frac{b^3}{\alpha^3})^2 \sum_{\gamma=\pm} q^{1-\gamma}q^{a-d+1}F_p(a - d, \gamma)F_p'(b - c, \gamma),
\]

Then (ii) follows. \( \square \)

By Lemma 4.3 (i), \( \mathcal{T}(s; p', p) \) can be expressed in the product form:

\[
\mathcal{T}(s; p', p)_{\{\sigma''\}_{\ell+1}} = \prod_{\ell=1}^{L} U_{p,p'}(\sigma''_{\ell+1}),
\]

with the relation \( \mathcal{E}(p')_{\gamma}^{\sigma''_{\ell+1}} = q^{-\frac{(1-\gamma)\eta c}{2}}(\mathcal{E}(p')_{\gamma}^{\sigma''})_q \) by (3.42), which contributes the \( q^{-\gamma} \)-factor in (3.41). Then Lemma 4.3 (ii) yields the relation

\[
\langle \langle Q'; n_1, \ldots, n_L | T(s; p', p) | Q'; n'_1, \ldots, n'_L \rangle \rangle = (-s)\mathcal{T}(s; p', p)_{\{\sigma''\}_{\ell+1}} \mathcal{E}(p')_{\gamma}^{\sigma''_{\ell+1}}.
\]

Hence up to a scale function, \( \mathcal{T}(2)(\omega^{-1}; p^*, p^*) \) is the dual model of \( \mathcal{T}(s; p', p) \):

**Proposition 4.4** Let \( p^*, p^{**} \) be the dual of \( p, p' \) in (4.3), and \( \Psi' \) be the dual correspondence between \( V_{r', Q'} \) and \( V_{r', Q'}^{*} \) with \( \Psi' (r'; Q') = -Q, -(r) \) in (4.9). Then \( \mathcal{T}(s; p', p) \) on \( V_{r', Q'} \) is equivalent to \( (-s)^{-L}(\frac{b^3}{\alpha^3})^2 \mathcal{T}(2)(\omega^{-1}; p^*, p^*) \) on \( V_{r', Q'}^{*} \) by

\[
\mathcal{T}(s; p', p) = (-s)^{-L}(\frac{b^3}{\alpha^3})^2 \mathcal{T}(2)(\omega^{-1}; p^*, p^*) \Psi'.
\]
By Proposition 4.2, $\tau_{t}^2(t; p', p)$ is also the dual model of $\tau_{t}^2(t; p', p)$, which is related to $\mathcal{T}(s; p', p)$ by (3.43). Note that the duality in (4.6) or (4.12) is the identification of dual $\tau_{t}^2$-transfer matrices, not as the ABCD-algebra representation in (2.8). In particular, the operator $\tilde{Z}'$ in (3.43) is not preserved under the duality correspondence $\Psi'$ in (4.12). This explains the reason why $\tau_{t}^2(t; p', p)$ serves the dual model for both $\tau_{t}^2(t; p', p)$ and $\mathcal{T}(s; p', p)$.

5 Inhomogeneous Chiral Potts Model and XXZ model with cyclic $U_q(sl_2)$-representation

Let $\omega, q$ be roots of unity in (1.1). For convenience, we may assume $\omega = e^{2\pi i/n}$, and $q = -\omega^{-1/2}$ when $n = N$ odd, and $q = \pm \omega^{-1/2}$ when $n = 2N$, where $\omega := e^{\frac{2\pi i}{N}}$. By the gauge equivalence and the scaling of spectral parameter, the five parameters in the $L$-operator of $\tau_{t}^2$-model in (3.23) (3.25) can be reduced to the 3-parameter family represented by two rapidities in the same $k'$-curve of CPM model [24]:

\[(a', b', a, b, c) = (x_{p'}, y_{p'}, x_p, y_p, \mu_{p', p}), \quad p' = p' = (x_{p'}, y_{p'}, \mu_{p'}), \quad p = p = (x_p, y_p, \mu_p), \quad (5.1)\]

with $p, p' \in \mathbb{W}_{k'}$, defined by

\[\mathbb{W}_{k'}(= \mathbb{W}_{k', k}) : \quad kx^N = 1 - k'\mu^{-N}, k'y^N = 1 - k'\mu^N, \quad (k'^2 \neq 0, 1, k^2 + k'^2 = 1) \quad (5.2)\]

(see, e.g., [1, 9]). Here, $\mathbb{W}_{k'}$ is represented by $\mathbb{W}_{k', k}$ or $\mathbb{W}_{k', -k}$, which are isomorphic via the transformation $(x, y, \mu) \mapsto (\pm \omega^{\mp 1}x, \pm \omega^{\mp 1}y, \mu)$.

5.1 Duality in chiral Potts model

In chiral Potts model, the $\tau_{t}^2$-model (3.39) are defined by the $L$-operators with the parameter $p'_{i_k} = p'_{i_k}, p_{i_k} = p_{i_k} \in \mathbb{W}_{k'}$ in (5.2):

\[\tau_{t}^2(t) = \tau_{t}^2(t; \{p'_{i_k}\}, \{p_{i_k}\}), \quad t := x_q y_q \quad \text{for} \quad q \in \mathbb{W}_{k'}. \quad (5.3)\]

It is known that the $Q$-operator of $\tau_{t}^2$-model (5.3) in the theory of Baxter’s $TQ$-relation [5] is the $L \otimes \mathbb{C}^N$-transfer matrix in CPM [8, 9, 23, 24][3]:

\[T(q)_{\{\sigma\}, \{\sigma'\}} = T(q; \{p'_{i_k}\}, \{p_{i_k}\})_{\{\sigma\}, \{\sigma'\}} = \prod_{k=1}^L W_{pq}(\sigma_{k} - \sigma'_{k}) W_{p_{k+1}q}(\sigma_{k+1} - \sigma'_{k}), \quad (5.4)\]

\[\tilde{T}(q)_{\{\sigma\}, \{\sigma'\}}, \quad \tilde{T}(q; \{p'_{i_k}\}, \{p_{i_k}\})_{\{\sigma\}, \{\sigma'\}} = \prod_{k=1}^L \tilde{W}_{pq}(\sigma_{k} - \sigma''_{k}) \tilde{W}_{p_{k+1}q}(\sigma'_{k} - \sigma''_{k+1}), \quad (5.4)\]

where $q \in \mathbb{W}_{k'}$ in (5.2), $\sigma, \sigma', \sigma'' \in \mathbb{Z}_N$ are the spin-basis in (1.2), and $W_{pq}, \tilde{W}_{pq}$ are the Boltzmann weights in CPM [10]:

\[
\begin{align*}
W_{pq}(\sigma) & = (\mu_{pq})^\sigma \prod_{j=1}^\sigma y_q - \omega^j x_p, \\
\tilde{W}_{pq}(\sigma) & = (\mu_{pq})^\sigma \prod_{j=1}^\sigma y_q - \omega^j x_p
\end{align*}
\]

(5.5)

Footnote:

[3] We shift the index of $p'_{i_k}$ in [27] (2.16) by one, where $p'_{i_k}$ is equal to $p'_{i_k+1}$ of (5.4) here.
with \( W_{p,q}(0) = \overline{W}_{p,q}(0) = 1 \). Here both (5.3) and (5.4) are with the boundary condition (3.40) and the periodic-vertical rapidities \((p_{L+1}', p_{L+1}) = (p_1, p_1)\). The star-triangle relation of Boltzmann weights \([2, 3, 10, 18, 19]\) yields

\[
T(q)\hat{T}(r) = \frac{(f_{p'q}f_{pr})^LT(r)\hat{T}(q)}{f_{p'q}f_{pr}}, \quad \hat{T}(q)\hat{T}(r) = \frac{(f_{pq}f_{pr})^LT(r)\hat{T}(r)T(q)}{f_{pq}f_{pr}},
\]

with the commutative relations,

\[
[T(q)\hat{T}(r), \hat{T}(q')\hat{T}(r')] = [T(q)\hat{T}(r), T(q')\hat{T}(r')] = 0
\]

for \( q, r, q', r' \in M_{k'} \), where \( f_{pq} = (\frac{\text{det}N(W_{pq}(i-j))}{\prod_{n=0}^N W_{pq}(n)})^{1/N} \). Then \( T(q)\hat{T}(q) \) can be diagonalized via two invertible \( q \)-independent matrices \( P_B, P_W \), i.e. \( P_W^{-1}T(q)P_B = T_{\text{diag}}(q) \), \( P_B^{-1}\hat{T}(q)P_W = \hat{T}_{\text{diag}}(q) \) diagonal so that \( \hat{T}_{\text{diag}}(q) = T_{\text{diag}}(q)W_{pq}(f_{pq})^TD \) for some \( q \)-independent diagonal matrix \( D \) \([9] (2.32)-(2.34), (4.46), [8] (2.10)-(2.13)\). The Baxter’s \( TQ \)-relation is in the form of \( \tau^2 \)-relation between (5.3) and (5.4) \([8] (3.15), [9] (4.31), [25] (2.31)-(2.32)\):

\[
\tau^2(q)T(Uq) = \varphi_q T(q) + \omega^q \varphi(q')XT(U^2q),
\]

\[
\tau^2(q)T(U'q) = \omega^{q'} \varphi(q')XT(q) + \varphi(q')T(U'^2q),
\]

where \( U, U' \) are automorphisms of (5.2) defined by \( U(x, y, \mu) = (\omega x, y, \mu) \), \( U'(x, y, \mu) = (x, \omega y, \mu) \), and \( \varphi_q(= \varphi(p_1'), \{ p_i \}; q), \varphi_q'(= \varphi(p_1'), \{ p_i \}; q), \varphi_q(= \varphi(p_1'), \{ p_i \}; q), \varphi_q(= \varphi(p_1'), \{ p_i \}; q) \) are functions defined by

\[
\varphi_q = \prod_{(p_{q'} - t_q)(y_{p_{q'}} - x_{q'})} \omega_{p_{q'}p_{q'}}(t_{p_{q'}} - t_q)(x_{p_{q'}} - x_q), \quad \varphi_q' = \prod_{(p_{q'} - t_q)(x_{p_{q'}} - y_q)} \omega_{p_{q'}p_{q'}}(t_{p_{q'}} - t_q)(y_{p_{q'}} - y_q),
\]

We now discuss the duality of \( \tau^2 \)-model (5.3) in CPM. In order to preserve the vertical rapidities in (5.3), we modify the parameter-duality (4.3) by a factor \( \alpha = i^{\frac{1}{N}} \) in the Remark of Proposition 4.1, so that the parameter-duality defines the dual rapidities between \( k' \)- and \( k'^{-1} \)-curves \( M_{k'} \) and \( M_{1/k'} \) \([27] (3.9)\):

\[
M_{k'} \sim M_{1/k'}, \quad p = (x_p, y_p, \mu_p) \mapsto p^* = (x_{p'}, y_{p'}, \mu_{p'}) := (i^{\frac{1}{N}} x_p \mu_p, i^{\frac{1}{N}} y_p \mu_p^{-1}, \mu_p^{-1}).
\]

Then the \( \tau^2 \)-duality (4.7) takes the following form:

\[
\tau^2(t; \{ p_{i'} \}, \{ p_i \}) = \Psi^{-1} \tau^2(t^*; \{ p_{i'}^* \}, \{ p_i^* \}) \Psi, \quad t^* = (-1)^\frac{1}{N} t.
\]

The duality (5.8) can be extended to the duality of CPM. Indeed, \( T, \hat{T} \) in (5.4) commute with the charge operator \( \hat{Z}(= X) \), hence preserve the \( Q \)-subspace \( V_{r,Q} \) in (4.1). The Boltzmann weights (5.5) for dual rapidities in (5.7) are related by \( \overline{W}_{pq}(f)(k) = W_{p,q}(k) \), \( \overline{W}_{pq}(f)(0) = \overline{W}_{p,q}(N - k) \), where \( \overline{W}_{pq}(f)(k) = \frac{\text{det}N(W_{pq}(i-j))}{\prod_{n=0}^N W_{pq}(n)} \sum_{\sigma=0}^{N-1} \omega^{k\sigma} W_{pq}(\sigma) \), \( W_{pq}(f)(k) = \frac{\text{det}N(W_{pq}(i-j))}{\prod_{n=0}^N W_{pq}(n)} \sum_{\sigma=0}^{N-1} \omega^{k\sigma} W_{pq}(\sigma) \) are the Fourier transform of Boltzmann weights in (5.5), \([27] (3.17)\). By \([27] \) Theorem 3.1, when \((r^*, q^*) = (Q, r)\), the
chiral Potts transfer matrices (5.4), \( T(q; \{ p'_\ell \}, \{ p_\ell \}), \tilde{T}(q; \{ p'_\ell \}, \{ p_\ell \}) \) on \( V, Q \) and \( T(q^*; \{ p'_\ell \}, \{ p'_{\ell+1} \}), \tilde{T}(q^*; \{ p'_\ell \}, \{ p'_{\ell+1} \}) \) on \( V^*, Q^* \), are equivalent via the dual correspondence \( \Psi \) in (4.2):

\[
T(q^*; \{ p'_\ell \}, \{ p'_{\ell+1} \}) = \left( \prod_{\ell} \frac{W^{(q'_\ell)}_{i+p'_\ell} \, W_{-i+p_{\ell+1}}(0)}{W^{(q_\ell)}_{i+p_{\ell}} \, W_{-i+p_{\ell+1}}(0)} \right) \Psi T(q; \{ p'_\ell \}, \{ p_\ell \}) \Psi^{-1},
\]

\[
\tilde{T}(q^*; \{ p'_\ell \}, \{ p'_{\ell+1} \}) = \left( \prod_{\ell} \frac{W^{(q'_\ell)}_{i+p'_\ell} \, W_{-i+p_{\ell+1}}(0)}{W^{(q_\ell)}_{i+p_{\ell}} \, W_{-i+p_{\ell+1}}(0)} \right) \Psi \tilde{T}(q; \{ p'_\ell \}, \{ p_\ell \}),
\]

(5.9)

where \( p'_\ell, p^*_{\ell}, q^* \in \mathcal{W}_{t/k'}^1 \) are the dual of \( p_\ell, p'_\ell, q \in \mathcal{W}_{k'} \) in (5.7).

### 5.2 Chiral Potts transfer matrix as the \( Q \)-operator of XXZ-model with cyclic \( U_q(sl_2) \)-representation

In this subsection, we study the \( T^{(2)} \)- and XXZ-model, (3.41) and (3.44) with parameter in (5.1):

\( p' = p', p = p \in \mathcal{W}_{k'} \). Then \( (p'_-, p_-) \) in (3.27) and \( (p^\uparrow_-, p^\uparrow_-) \) in (3.48) become

\[
p'_- = (q^{-1} x_{p'}, q y_{p'} \mu_{p'}), \quad p_- = (qx_p, q^{-1} y_p \mu_p), \quad p^\uparrow_- = p', \quad p^\uparrow_- = (x_p, y_p q \mu_p),
\]

which are elements in \( \mathcal{W}_{k'} \) in \( n = N \) odd case, denoted by \( p'_-, p_-, p^\uparrow_-, p^\uparrow_- \). In \( n = 2N \) case, \( p'_-, p_- \) are elements in \( \mathcal{W}_{k', -k} \), identified with \( p'_- = (x_{p'}, \omega^{-1} y_{p'}, \mu_{p'}), p_- = (\omega^{-1} x_p, y_p \mu_p) \in \mathcal{W}_{k'}(= \mathcal{W}_{k', k}) \); and \( p^\uparrow_- = p^\uparrow_+ \in \mathcal{W}_{-k'} \). Hence parameters in (3.27) and (3.48) are identified with

\[
\begin{align*}
 n &= N \text{ odd :} \quad p'_- = (q^{-1} x_{p'}, q y_{p'} \mu_{p'}), \quad p_- = (qx_p, q^{-1} y_p \mu_p), \quad p^\uparrow_- = p', \quad p^\uparrow_- = (x_p, y_p q \mu_p), \\
 &\quad (p'_-, p_-, p^\uparrow_- \in \mathcal{W}_{k'}); \\
 n &= 2N : \quad p'_- = (x_{p'}, \omega^{-1} y_{p'} \mu_{p'}), \quad p_- = (\omega^{-1} x_p, y_p \mu_p), \quad p^\uparrow_- = p', \quad p^\uparrow_- = (x_p, y_p q \mu_p), \\
 &\quad (p'_-, p_-, p^\uparrow_- \in \mathcal{W}_{k'}, \quad p^\uparrow_- \in \mathcal{W}_{-k'}). \quad (5.10)
\end{align*}
\]

Over the subspace \( \mathcal{C}^\uparrow \) of \( \otimes L^\uparrow \mathcal{C}^n \) in (3.45) for \( \tilde{t} \in \mathbb{Z}_L^\uparrow \), the equivalence of \( T^{(2)}(t) = T^{(2)}(t; p', p) \) and \( \tau^{(2)} \)-model in (3.47) becomes

\[
\tau^{(2)}(t) \mid_{\tilde{t}} \simeq \tau^{(2)}(t; \{ p'_\ell \}, \{ p_\ell \}), \quad p'_\ell = p'_{(-1)^\ell}, p_\ell = p_{(-1)^\ell} \in \mathcal{W}_{k'}.
\]

(5.11)

Hence we may regard \( T(q; \{ p'_\ell \}, \{ p_\ell \}), \tilde{T}(q; \{ p'_\ell \}, \{ p_\ell \}) \) in (5.4) as the \( Q \)-operator of \( \tau^{(2)}(t) \mid_{\tilde{t}} \), with \( \tau^{(2)}T \)-relation induced from (5.6). In \( n = N \) odd case, by Proposition 3.1 (i), \( \otimes L^\uparrow \mathcal{C}^n = \mathcal{C}^\uparrow \); then by Proposition 3.2 (i), the XXZ-model \( T(s, p', p) \) in (3.41) can be identified with \( \tau^{(2)}(t) \) up to some scale function, by which one may consider \( T(q; \{ p'_\ell \}, \{ p_\ell \}), \tilde{T}(q; \{ p'_\ell \}, \{ p_\ell \}) \) as the \( Q \)-operator of \( T(s, p', p) \). On the other hand, since \( p^\uparrow_-, p^\uparrow_- \in \mathcal{W}_{k'} \) by (5.10), the \( Q \)-operators of the equivalent models in Proposition 3.3 (i), \( \tau^{(2)}(t) \mid_{\tilde{t}} \simeq \tau^{(2)}(t; \{ p'_\ell \}, \{ p_\ell \}) \) with \( p^\uparrow_\ell = p^\uparrow_{(-1)^\ell}, p^\uparrow_\ell = p^\uparrow_{(-1)^\ell} \in \mathcal{W}_{k'} \), are \( T(q; \{ p'_\ell \}, \{ p^\uparrow_\ell \}), \tilde{T}(q; \{ p'_\ell \}, \{ p^\uparrow_\ell \}) \). In particular, when \( \tilde{t} = 0 \), where \( p^\uparrow_\ell = p^\uparrow_\ell = p^\uparrow_\ell = p \) for all \( \ell \), the duality in Proposition 4.3 extends to the duality of \( Q \)-operators with \( p'_\ell = p', p_\ell = p, p^\uparrow_\ell = p^\uparrow, p^\uparrow_\ell = p^\uparrow \). In \( n = 2N \) case, since \( p^\uparrow_\ell \in \mathcal{W}_{k'}, p^\uparrow_- \in \mathcal{W}_{-k'} \) in (5.10), the vertical rapidities \( p^\uparrow_\ell, p^\uparrow_\ell \) of the \( \tau^{(2)} \)-model equivalent to \( \tau^{(2)}(t) \) on the component \( \mathcal{C}^\uparrow_{\tilde{t}} \) in Proposition 3.3 (ii) are not all in the
same $k'$-curve when $\bar{t} \neq 0$, hence the chiral Potts transfer matrix fails to be the $Q$-operator of $\tau^{(2)}$-model. For $\tau^{(2)}$-model in Proposition 3.1 (ii), since the parameter $\nu$ in (3.24) corresponding to $(p'_-,p_-)$ in (3.27) and $(p'_+,p_-)$ in (5.10) differ by $\omega$ in $n = 2N$ case, $\tau^{(2)}(t;\{p'_{\ell_{i}}\},\{p_{\ell_{i}}\})$ in (5.11) and $\tau^{(2)}(t;\{p'_{\ell_{i}}\},\{p_{\ell_{i}}\})$ in (3.47) are related by the relation (2.13) with $\xi_{\ell} = \omega^{1/2}$. Therefore, the $Z_{w}$-identification of $\tau^{(2)}(t;\{p'_{\ell_{i}}\},\{p_{\ell_{i}}\})$ and $\tau^{(2)}(t;\{p'_{\ell_{i}+1}\},\{p_{\ell_{i}+1}\})$ in (5.11) via the change of local spectral parameters at site $\ell_{t}$, $t \mapsto \omega^{-(1)^{1/2}}t$. Then $T(q;\{p'_{\ell}\},\{p_{\ell}\}),\hat{T}(q;\{p'_{\ell}\},\{p_{\ell}\})$ are identified with $T(q;\{p'_{\ell}+1\},\{p_{\ell}+1\}),\hat{T}(q;\{p'_{\ell}+1\},\{p_{\ell}+1\})$ with the compatible $\tau^{(2)}$-relation (5.6) via the change of variable $q \in \mathcal{M}_{k'}$ at site $\ell$: $(x_{q},y_{q},\mu_{q}) \mapsto (x_{q},\omega^{-1}y_{q},\mu_{q})$ according to $(p'_{\ell_{i}},p_{\ell_{i}}) = (p'_{\ell},p_{\ell})$. Therefore, the two pairs of chiral Potts transfer matrices, $(T(q;\{p'_{\ell}\},\{p_{\ell}\}),\hat{T}(q;\{p'_{\ell}\},\{p_{\ell}\}))$ and $(\hat{T}(q;\{p'_{\ell}\},\{p_{\ell}\}),\hat{T}(q;\{p'_{\ell}+1\},\{p_{\ell}+1\}))$, form the $Q$-operator of $T(s;p',p)$ over the component $\mathcal{C}[L]$ of $\otimes_{i=1}^{N-1} \mathbb{C}^{2}$ in Proposition 3.2 (ii).

6 Concluding Remarks

In this paper, we first in section 2 characterize the quantum group $U_{w}(sl_{2})$ in the $L$-operator of $\tau^{(2)}$-model for a generic $w$ as a subalgebra of the quantum group $U_{q}(sl_{2})$ in XXZ model with $q^{-2} = w$. Then we study the XXZ model with cyclic representation of $U_{q}(sl_{2})$, and its related $\tau^{(2)}$-models in the root of unity case (1.1). Through the representation theory of $U_{\omega}(sl_{2})$, we obtain the structure of XXZ- and $\tau^{(2)}$-models, and their relationship with non-superintegrable $N$-state CPM in sections 3 and 5. We also study the duality of XXZ- and $\tau^{(2)}$-models with cyclic representation in section 4, and find the fundamental role of $\tau^{(2)}$-models in the duality theory. One expects a similar structure will appear again in XXZ models defined by other representations of $U_{q}(sl_{2})$, not only for the cyclic representation in this work. The representation theory should provide a more direct access to the eigenvector problem of models related to $\tau^{(2)}$-model, as indicated in [28] about the eigenvectors of superintegrable chiral Potts model. A program along this line is now under consideration, and progress is expected. We leave the discussion to future work.

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