OPTIMAL DIVIDEND POLICY IN AN INSURANCE COMPANY WITH CONTAGIOUS ARRIVALS OF CLAIMS

YILING CHEN
School of Mathematical Sciences,
Tongji University,
Shanghai 200092, China

BAOJUN BIAN
School of Mathematical Sciences,
Tongji University,
Shanghai 200092, China

(Communicated by Jiongmin Yong)

Abstract. In this paper we consider the optimal dividend problem for an insurance company whose surplus follows a classical Cramér-Lundberg process with a feature of self-exciting. A Hawkes process is applied so that the occurrence of a jump in the claims triggers more sequent jumps. We show that the optimal value function is a unique viscosity solution of the associated Hamilton-Jacobi-Bellman equation with a given boundary condition and declare its concavity. We introduce a barrier curve strategy and verify its optimality. Finally, some numerical results are exhibited.

1. Introduction. In the classical Cramér-Lundberg model, the arrival of a claim is modelled by a normal Poisson process with a constant intensity rate. With substantially research, this model no longer stands the test in practice. Empirical studies suggest that the claims occur in clusters, that is, a sequence of claims occur in short time following a (big) claim which occurs after a relative long quiet period of time. For example, during the financial crisis of 2008, the number of claims covered by credit insurances surged in US increased rapidly in a short time when a big claim came. The feature of clustered jumps can be described by a type of stochastic process known as Hawkes process which is a self-exciting point process firstly introduced by Hawkes[17] in 1971.

Hawkes process is very appearing in point process modeling and has wide applications in various domains. Aït Sahalia and Hurd[1] solved the optimal investment problem and gave a verification result for a logarithm utility function where Hawkes process is adopted to capture contagion features in financial market. Stabile and Torrisi[25] considered risk processes with non-stationary Hawkes claims arrivals and studied the asymptotic behavior of infinite and finite horizon ruin probabilities under light-tailed conditions on the claims. Hainaut [18] analysed the impact

2010 Mathematics Subject Classification. Primary 35J87,91B30,49J20; Secondary 49L25, 91B70, 49K20.

Key words and phrases. Insurance, optimal dividend payment, self-exciting, viscosity solution, barrier curve strategy.
of contagion between financial and non-life insurance markets on the asset-liability management policy of an insurance company.

The optimal dividend payment problem in an insurance company is classical and attracts considerable attention with techniques of modern stochastic control theory, see e.g. Schmidli [26] for a detailed review. Gradually, the dividend payment problem is studied for many various model extensions and additional practical constraints. At first, the risk reserves are subject to the classic compound Poisson risk model [14, 28]. Based on the classic risk model, various model extensions appear, see, e.g., investment [23, 11], reinsurance [3, 4, 22], capital injections [21, 27], regime-switching [19, 20]. See more processes in this field, we refer to two review articles written by Albrecher [5] and Avanzi [6] respectively. Particularly, Azcue and Muler [2] gave us a quite general survey about the dividend payment problem in the Cramér-Lundberg setting with self investment, and a band strategy was proved to be optimal. Recently, Albrecher [7] considered a two-dimensional optimal dividend problem with two insurance companies who collaborate by paying each other’s deficit when possible. On the other hand, few researchers have proposed risk models with self-exciting features in dividend payment problem of an insurance company. Then we come up with the idea to consider the dividend payment problem with Hawkes claims arrivals in an insurance company and gain a two-dimensional optimization problem.

In this paper the claim arrival is modeled by a linear Hawkes process such that the claim intensity is self-exciting and mean reversion. Gao and Zhu [15] studied the functional central limit theorems for linear Hawkes process when the initial intensity is large and applied their results to ruin probability in risk theory. In contrast, we try the dividend payment problem in an insurance company with linear Hawkes process by studying the Hamilton-Jacobi-Bellman (HJB) equation associated with our optimization problem. Previously, Chen and Bian [12] had given a brief investigation to this problem. We study the existence of the value function with respect to the associated HJB equation and characterize the optimal value function as the smallest viscosity supersolution. However, our work show that it is quite significant to prove the uniqueness of a viscosity solution to the associated HJB equation and the dividend payment policy in greater detail. The HJB equation is nonlinear degenerate integro-differential equation subject to a differential constraint with respect to two variables. We find out that a barrier strategy is no longer optimal. The optimal dividend payment strategy turns out to be decided according to a barrier curve no longer a simple point. Similar to the free boundary problem, the asymptotic properties of a free boundary are always challenging.

The main results of this paper are as follows. We provide with the existence and uniqueness of the viscosity solutions to the associated HJB equation. An introduced barrier curve strategy is verified to be optimal and the dividend strategy depends on the present surplus and the intensity rate both. A concavity property is proved to ensure that the barrier curve is decreasing with respect to the intensity rate. It turns out that the company should pay a large sum of surplus as dividends immediately to avoid the situation that there is no surplus to pay as dividends when the intensity is large since bankruptcy arrives possibly in a very short time.

The remainder of the paper is organized as follows. In section 2 we introduce the linear Hawkes process and state the basic model. Then we discuss some basic properties of the value function including the monotonicity and the growth condition. In section 3 we state that the value function is a viscosity solution of the
associated HJB equation and study its uniqueness with a boundary condition. A concavity property is also stated to further proofs. In section 4 we introduce the barrier curve and give a concrete dividend payment strategy. In section 5 we give some numerical results.

2. The statement of the problem.

2.1. The Hawkes process. A Hawkes process is a compounded Poisson process with stochastic intensity. In other words, it is a counting process with self-exciting feature. We use the notation $N_t = N(0,t]$ to denote the number of points in the interval $(0,t]$. To be exact, given a complete probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, the counting process $\{N_t\}_{t \geq 0}$ satisfies

$$
P(N_{t+\Delta t} - N_t = 1|\mathcal{F}_t) = \lambda_t \Delta t + o(\Delta t),
$$

$$
P(N_{t+\Delta t} - N_t > 1|\mathcal{F}_t) = o(\Delta t),
$$

where the claim intensity process $\lambda_t$ is as follows:

$$\lambda_t = \bar{\lambda} + (\lambda_0 - \bar{\lambda})e^{-\alpha t} + \beta \int_0^t e^{\alpha(s-t)}dN_s.$$

To provide with the ease of use, we rewrite it into the differential form

$$d\lambda_t = \alpha(\bar{\lambda} - \lambda_t)dt + \beta dN_t,$$

where $\bar{\lambda}$ is the long-run average of the jump(claim) intensity corresponding to the jump; $\alpha$ is the decay rate proposed to make the jump intensity return to the long-run average after a jump occurs; $\beta \geq 0$ is a constant indicating non-negative impact of the jump occurrence on the jump intensity.

In general, a Hawkes process differs from a doubly stochastic Poisson process since its increments are not independent and Markovian. But in our case, a commonly nontrivial kernel function of an exponential form is used. In this special case, the process $(N_t, \lambda_t)$ is Markovian and the compensated process $N_t - \int_0^t \lambda_s ds$ is a local martingale. It is known that the process is stationary if $\beta < \alpha$. Figure 1 exhibits a sample path of Hawkes process with the self-exciting feature.

![Figure 1. A sample path of Hawkes process $(N_t, \lambda_t)$ and the surplus process $X_t$ without dividends](image-url)
As one jump occurs, the claim intensity is increased by the occurred jump through the mechanics of (2). Precisely, a claim increases the probability of the next claim. As we can seen in Figure 1, a sequent jump happens more likely in a short time after a big scale claim. Meanwhile, the mean-reversion property of (2) prevents the jump intensity from explosion given $0 \leq \beta < \alpha$. For future proofs, taking the expectation of (2) and noticing the fact that $N_t - \int_0^t \lambda_s \, ds$ is a local martingale, we can write
\[
\mathbb{E}[\lambda_t] = \lambda_0 e^{(\beta-\alpha)t} + \frac{\alpha \lambda}{\alpha - \beta} (1 - e^{(\beta-\alpha)t}) \to \frac{\alpha \lambda}{\alpha - \beta}
\]
as $t \to \infty$.

2.2. The value function. In this paper we consider an insurance company that receives premiums and affords the claims with a dividend payment strategy where the claim events have the self-exciting features. We use the notation $X^l_t$ to describe the surplus process of the company
\[
X^l_t = x + pt - \sum_{i=1}^{N(\lambda_t)} Y_i - L_t,
\]
where $x$ is the initial surplus, $p$ is the premium rate. $N(\lambda_t)$ is the Hawkes process introduced by (1), and the claim sizes $Y_i$ are i.i.d. random variables with distribution $F$. We assume that the distribution $F$ has a finite expectation satisfying $p > \mathbb{E}[\lambda Y_i]$ and $F(0) = 0$. A simple simulated path of $X^l_t$ without dividends is showed in Figure 1.

The cumulative dividends the company has paid until time $t$ denoted by $L_t$ is characterized as follows:
\[
L_t = \int_0^t dL^c_s + \sum_{s \leq t, X(s+) \neq X(s)} (L_{s+} - L_s),
\]
where $L^c_s$ is a continuous and nonnegative function. In particular, $L^c_t = q_t t$, it is to say that the company pay dividends at a continuous rate $q_t$. As for discrete sum part, it allows company to pay an amount of surplus as dividends instantly.

A controlled strategy is a process denoted by $l = (l_t)_{t \geq 0}$. We say a dividend strategy $l$ is admissible if it is nondecreasing, càglâd (left continuous with right limits), predictable with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, verifies $L_0 = 0$, and
\[
X_t \leq x + pt - \sum_{i=1}^{N(\lambda_t)} Y_i
\]
for any $0 \leq t < \tau^l$.

We ask for $\Delta L_t = L_{t+} - L_t \leq X^l_t$ for any $t \geq 0$ which means that the company cannot pay immediately a sum of dividends exceeding the present surplus. In other word, the ruin cannot happen due to dividends payment.

We denote by $L_{x,\lambda}$ the set of all admissible dividend strategies with initial point $x, \lambda$.

The $\tau^l$, the ruin time of the company applying the dividend strategy $l$, is defined by the following equation
\[
\tau^l = \inf \{ t > 0, X^l_t < 0 \}.
\]
Note that it can only occur at the arrival of a claim. For technical reasons, we extend the definition of the admissible dividend process as $L_t = L_{t^+}$ for $t \geq t^+$.

Then we define the value function of a control strategy $l$ by

$$V_l(x, \lambda) = \mathbb{E}_{x, \lambda} \left( \int_0^{t^+} e^{-cs} dL_s \right),$$

where $c > 0$ is the constant discount factor. The integral is interpreted pathwise in a Riemann-Stieltjes sense. Then the value function is

$$V(x, \lambda) = \sup_{l \in L_{x, \lambda}} V_l(x, \lambda).$$

The value function is the expected discounted dividends paid by the company until ruin time.

We denote $\mathbb{R}_+ = [0, +\infty)$ and for technical reasons, we define $V(x, \lambda) = 0$ for $x < 0$. Moreover, we need to assume $c > r > 0$. If not, the optimal value function will go to infinity according to Remark 2.4 introduced by Azcue and Muler[2].

The Hamilton-Jacobi-Bellman(HJB) equation associated to the above stochastic optimization problem can be derived as

$$\max \{ \mathcal{L}(V)(x, \lambda), 1 - V_x(x, \lambda) \} = 0,$$

where

$$\mathcal{L}(V)(x, \lambda) = pV_x + \alpha(\bar{\lambda} - \lambda)V_{\lambda} - (c + \lambda)V + \lambda \int_0^x V(x - \zeta, \lambda + \beta) dF(\zeta).$$

2.3. Basic properties of the value function. In the following, we give some basic properties of the value function including the growth condition, the monotonicity and the continuity.

**Proposition 1.** The optimal value function defined by (7) is well defined and satisfies

$$x \leq V(x, \lambda) \leq x + \frac{p}{c}.$$  \hfill (10)

**Proof.** The first inequality holds obviously because the company can take the present surplus as dividends immediately. For any admissible strategy $l \in L_{x, \lambda}$, we have from (5) that

$$L_t \leq \omega(t) = x + pt.$$

Then we have

$$V_l(x, \lambda) \leq \mathbb{E}_{x, \lambda} \left[ \int_0^{\infty} e^{-ct} d\omega(t) \right] = x + p \int_0^{\infty} e^{-ct} dt = x + \frac{p}{c}.$$ 

So $V(x, \lambda)$ is well defined and satisfies the second inequality of the proposition. \hfill \Box

**Proposition 2.** The value function $V(x, \lambda)$ defined by (7) is increasing in $x$ and decreasing in $\lambda$. Furthermore, $V(x, \lambda)$ is continuous in $x, \lambda$.

**Proof.** First we show that $V(x + h, \lambda) - V(x, \lambda) \geq 0$ for any $h > 0$. We assume that $l^*$ is the optimal strategy for the company whose initial reserve is $x$. Then we have $V(x, \lambda) = V_l(x, \lambda)$. As for the company whose initial reserve is $x + h$, we define a new strategy $l^*$: pay $h$ as dividend immediately, and then apply the strategy $l^*$. Then

$$V(x + h, \lambda) - V(x, \lambda) \geq V_{l^*}(x + h, \lambda) - V_{l^*}(x, \lambda)$$

$$= V_{l^*}(x, \lambda) - V_{l^*}(x, \lambda) + h \geq 0.$$ \hfill (11)
Now we show that $V(x, \lambda) - V(x, \lambda + h) \geq 0$ for any $h > 0$. We assume that $l^+$ is the optimal strategy for the company whose initial jump density is $\lambda + h$ and we have $V(x, \lambda + h) = V_{l^+}(x, \lambda + h)$. We can adopt the same strategy for the company whose initial jump density is $\lambda$ since we have no restriction on the dividend payment process $L_t$. 

We denote $\tau_{h}^{+}$ and $\tau^{+}$ respectively as the ruin time of the surplus starting from different initial point $(x, \lambda + h)$ and $(x, \lambda)$ with the same dividend payment strategy $l^+$. By a simple calculation we obtain

$$
E_{x,\lambda} [X_{t}^{l^+}] = x + pt - E[\lambda_t] t E[Y_t] - L_{t}^{l^+} \\
= x + pt - \left[ \lambda e^{(\beta-\alpha)t} + \frac{\alpha \lambda}{\alpha - \beta} (1 - e^{(\beta-\alpha)t}) \right] t E[Y_t] - L_{t}^{l^+},
$$

(12)

in which $L_{t}^{l^+}$ denotes the cumulative dividends (4) when the strategy $l^+$ is applied.

From (12), we know that $E[X_t^{l}]$ is inversely proportional to $\lambda$. According to the definition of the ruin time, we know that $E[\tau^+]$ is proportional to $E[X_t^{l}]$. So, from $\lambda + h \geq \lambda$, we have $E_{x,\lambda+h} [\tau_{h}^{+}] \leq E_{x,\lambda} [\tau^+]$ since $E_{x,\lambda+h} [X_{t}^{l^+}] \leq E_{x,\lambda} [X_{t}^{l^+}]$. In other word, the company $(x, \lambda + h)$ faces more claims and the reserve of the company goes to zero at a higher probability.

$$
V(x, \lambda) - V(x, \lambda + h) \geq V_{l^+}(x, \lambda) - V_{l^+}(x, \lambda + h) \\
\geq E_{x,\lambda} \left[ \int_0^{\tau_{h}^{+}} e^{-cs} dL_{s}^{l^+} \right] - E_{x,\lambda+h} \left[ \int_0^{\tau_{h}^{+}} e^{-cs} dL_{s}^{l^+} \right] \geq 0.
$$

(13)

Since the two companies apply the same dividend payment strategy including continuous dividend rate $q_t$ and the discrete part of dividends $L_{t^+} - L_t$ for every $t$, then the value function only depends on $E[\tau^+]$. The longer it lives, the more dividend it pays. Through the above analysis, the last inequality holds from $E_{x,\lambda+h} [\tau_{h}^{+}] \leq E_{x,\lambda} [\tau^+]$.

Now we claim that $V(x, \lambda)$ is continuous in $x, \lambda$. We assume that $l$ is the best strategy for initial point $(x + h, \lambda + h)$ and we have $V(x + h, \lambda + h) = V_{l}(x + h, \lambda + h)$. We denote $\tau_{h}^{l}$ and $\tau^{l}$ respectively as the ruin time of the surplus starting from different initial point $(x + h, \lambda + h)$ and $(x, \lambda)$ with the same dividend payment strategy $l$.

$$
|V(x + h, \lambda + h) - V(x, \lambda)| \leq \left| E \left[ \int_{\tau^{l}}^{\tau_{h}^{l}} e^{-cs} dL_{s}^{l^+} \right] \right|
$$

$$
\leq \left| E \left[ \int_{\tau^{l}}^{\tau_{h}^{l}} \tilde{q}_s ds \right] \right| + \sum_{L(s^+) \neq L(s)} \sum_{\tau^l \leq s \leq \tau_{h}^{l}} (V(X_{s^+}^{l^+}, \lambda_{s^+}) - V(X_{s}^{l}, \lambda_s)) \tag{14}
$$

where $0 \leq \tilde{q}_s < \infty$. From (12), the elementary properties of (2) and (3), we can easily say that $E[\tau_{h}^{l} - \tau^{l}] \to 0$ as $h \to 0$. As we mentioned before, an immediately sum of dividend payment can not occur at the ruin time which implies that the accumulation sum part of (14) doesn’t exist. So the right hand of the last inequality of (14) goes to 0 and then we conclude the results.
3. **Viscosity solution and uniqueness.** The result of next proposition is technical and will be used to relate the composition of our controlled surplus process \((X_t, \lambda_t)\) and the corresponding infinitesimal generator (Itô Lemma).

**Proposition 3.** Let \(Z = (X^l_t, \lambda_t)_{t \geq 0}\) be the controlled surplus process with dividends defined by (2) and (3) with initial value \((x, \lambda)\). Let \(\tau^l\) be the corresponding ruin time, then we can write for any finite stopping time \(\tau \leq \tau^l\)

\[
e^{-ct}u(X^l_{\tau}, \lambda_{\tau}) - u(x, \lambda)
\]

\[
= \int_0^\tau e^{-cs} \left( \mathcal{L}(u)(X^l_s, \lambda_s) \right) ds + \int_0^\tau e^{-cs} \left( \mathcal{L}(u)(X^l_s, \lambda_s) \right) dL^c_s
\]

\[
- \int_0^\tau e^{-cs} dL^c_s + M_\tau
\]

\[
+ \sum_{L(s^+) \neq L(s), s \leq \tau} e^{-cs} \left( \int_0^{L(s^+) \neq L(s)} (1 - u_x(X_s^l - \alpha, \lambda_s)) d\alpha \right),
\]

where \(\mathcal{L}\) is the operator defined by (9), \(M_\tau\) is a local martingale with zero expectation.

**Proof.** We consider nonnegative smooth enough functions \(u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}\) and we extend the definition of \(u\) in \((\infty, 0) \times (\infty, 0)\) as any nonnegative constant. Since the function \(e^{-ct}u(X_t, \lambda_t)\) is continuously differentiable, using the expression (4) and the change of variables formula for finite variation process, we can write

\[
e^{-ct}u(X^l_{\tau}, \lambda_{\tau}) - u(x, \lambda)
\]

\[
= \int_0^\tau e^{-cs} \left( pu_x(X^l_s, \lambda_s) + \alpha(\lambda - \lambda_s)u_\lambda(X^l_s, \lambda_s) - cu \right) ds
\]

\[
- \int_0^\tau e^{-cs} u_x(X^l_s, \lambda_s) dL^c_s + \sum_{X^l_{s^+} \neq X^l_s, s \leq \tau} e^{-cs} \left( u(X^l_{s^+}, \lambda_s) - u(X^l_s, \lambda_s) \right)
\]

\[
+ \sum_{X^l_{s^+} \neq X^l_s, s \leq \tau} e^{-cs} \left( u(X^l_{s^+}, \lambda_s) - u(X^l_s, \lambda_{s^+}) \right).
\]

But \(X^l_{s^+} \neq X^l_s\) only at the jumps of \(L_s\), then \(X^l_{s^+} - X^l_s = -(L_{s^+} - L_s)\) and

\[
- \int_0^\tau e^{-cs} u_x(X^l_s, \lambda_s) dL^c_s + \sum_{X^l_{s^+} \neq X^l_s, s \leq \tau} e^{-cs} \left( u(X^l_{s^+}, \lambda_s) - u(X^l_s, \lambda_s) \right)
\]

\[
= - \int_0^\tau e^{-cs} u_x(X^l_s, \lambda_s) dL^c_s + \sum_{L_{s^+} \neq L_s} \left( \int_0^{L_{s^+} \neq L_s} u_x(X^l_s - \alpha, \lambda_s) d\alpha \right) e^{-cs}
\]

\[
= - \int_0^\tau e^{-cs} dL^c_s + \int_0^\tau e^{-cs} (1 - u_x(X^l_s, \lambda_s)) dL^c_s
\]

\[
+ \sum_{L_{s^+} \neq L_s} \left( \int_0^{L_{s^+} \neq L_s} (1 - u_x(X^l_s - \alpha, \lambda_s) d\alpha \right) e^{-cs}.
\]
On the other hand, $X_s^I \neq X_s^I$ only at the time of a claim, so
\[
M_\tau = \sum_{s \leq \tau, X_s^I \neq X_s^I} e^{-cs} \left( u(X_s^I, \lambda_s) - u(X_s^I, \lambda_s^-) \right)
- \lambda_\tau \int_0^\tau e^{-cs} \int_0^\infty \left( u(X_s^I, \lambda_s + \beta) - u(X_s^I, \lambda_s) \right) dF(\alpha) ds
\] (18)
is a martingale with zero expectation. Then the result can be attained by (16) using (17) and (18).

Next we state the dynamic programming principle. We omit the proof since Proposition 2 and that the pair process $(N_t, \lambda_t)$ is Markovian. The proof follows the standard methods introduced by Zhu[30]. Then we give the dynamic programming principle proposition directly.

**Proposition 4.** For any $x \geq 0, \lambda \geq 0$ and any stopping time $\tau$, we can write
\[
V(x, \lambda) = \sup_{E_{x, \lambda}} \left[ \int_0^{\tau \wedge \tau'} e^{-cs} dL_s + e^{-c(\tau \wedge \tau')} V(X_{\tau \wedge \tau'}, \lambda_{\tau \wedge \tau'}) \right].
\] (19)

### 3.1. The introduce of the viscosity solution
Since the HJB equation is a nonlinear integro-differential equation, the value function is possibly not smooth enough to satisfy the HJB equation in classical sense. Therefore, we ask for a weak solution called viscosity solution introduced by Crandall and Lions [9, 10].

**Definition 5.** A continuous function $\bar{u} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is viscosity supersolution of (8) at $(x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+$ if any continuously differentiable function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(x, \lambda) = \bar{u}(x, \lambda)$ such that $\bar{u} - \varphi$ reaches the minimum at $(x, \lambda)$ satisfies
\[
\max \{ \mathcal{L}(\varphi)(x, \lambda), 1 - \varphi_x(x, \lambda) \} \leq 0.
\] (20)

A continuous function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is viscosity subsolution of (8) at $(x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+$ if any continuously differentiable function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with $\psi(x, \lambda) = u(x, \lambda)$ such that $u - \psi$ reaches the maximum at $(x, \lambda)$ satisfies
\[
\max \{ \mathcal{L}(\psi)(x, \lambda), 1 - \psi_x(x, \lambda) \} \geq 0.
\] (21)

Finally, a continuous function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is viscosity solution of (8) if it is both a viscosity supersolution and subsolution at any $(x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+$.

In addition to the Definition 4, Crandall, Ishii and Lions[10] give another equivalent formulation of the viscosity solution definition and we introduce it for future proofs.

**Definition 6.** Given any continuous function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and any $(x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+$, the set of superdifferentials of $u$ at $(x, \lambda)$ is defined as
\[
D^{1+}u(x, \lambda) = \left\{ (d, q) : \text{such that } \lim_{h \to 0} \sup_{h} u(x + h, \lambda + h) - u(x, \lambda) - hd -hq \leq 0 \right\},
\]
and the set of subdifferentials of $u$ at $(x, \lambda)$ is defined as
\[
D^{1-}u(x, \lambda) = \left\{ (d, q) : \text{such that } \lim_{h \to 0} \sup_{h} u(x + h, \lambda + h) - u(x, \lambda) - hd -hq \geq 0 \right\}.
\]
Let us call
\[
\mathcal{L}(u, d, q) = pd - \alpha(\dot{\lambda} - \lambda)q - (c + \lambda)u + \lambda \int_0^x u(x - \zeta, \lambda + \beta) dF(\zeta).
\]
A continuous function \( \bar{u} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) is a viscosity supersolution of (8) at \((x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \) if
\[
\max\{\mathcal{L}(u, d, q), 1 - d\} \leq 0
\]
for all \((d, q) \in D^{1,2}\bar{u}(x, \lambda)\). A continuous function \( u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) is a viscosity subsolution of (8) at \((x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \) if
\[
\max\{\mathcal{L}(u, d, q), 1 - d\} \geq 0
\]
for all \((d, q) \in D^{1,2}u(x, \lambda)\).

3.2. Existence and uniqueness. In this section, we prove the existence and uniqueness of the value function defined by (7) with respect to the HJB equation (8). Here we prefer to omit the proof of existence since it can be obtained by standard technical operations introduced in Chen and Bian [12].

**Proposition 7.** The value function defined by (7) is a viscosity solution of (8) at any \((x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \).

Here we point out some key steps to the proof of this proposition. The viscosity supersolution result can be attained by taking a constant dividend payment rate and using the property of supremum in equation (7) and Proposition 3. As for the proof of the viscosity subsolution, we prove it by contradiction. Firstly, we construct an auxiliary function by convolution technical approach. Then we obtain a contradiction against to the assumption conditions by Proposition 3.

However, it is not direct to show the uniqueness which is critical to state. Then we give a comparison principle for viscosity solution \( u(x, \lambda) \) with a given boundary condition and the following conditions:

(A.1): \( u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is locally Lipschitz;

(A.2): There exists a constant \( M > 0 \) such that \( u(x, \lambda) \leq M(1 + x) \).

**Proposition 8.** If \( \bar{u} \) is a viscosity subsolution and \( \bar{u} \) is a viscosity supersolution of (8) satisfying the following conditions:

(B.1): \( \bar{u} \) and \( -\bar{u} \) satisfy conditions (A.1) and (A.2);

(B.2): \( \bar{u}(0, \lambda) \leq \bar{u}(0, \lambda), \bar{u}(x, 0) \leq \bar{u}(x, 0) \).

Then we have \( u \leq \bar{u} \) for all \((x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \).

**Proof.** We prove it by contradiction. Suppose that there exist a point \( Z^* = (x^*, \lambda^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \) such that
\[
\bar{u}(Z^*) - \bar{u}(Z^*) \geq 2\delta > 0
\]
for some \( \delta > 0 \). Set \( \partial(\mathbb{R}_+ \times \mathbb{R}_+) = B_1 \cup B_2 \), where
\[
B_1 = \{ (x, 0) \mid 0 \leq x < \infty \}, \quad B_2 = \{ (0, \lambda) \mid 0 < \lambda < \infty \}.
\]

Define auxiliary functions
\[
\Phi(Z_1, Z_2) = \bar{u}(Z_1) - \bar{u}(Z_2) - \phi(Z_1, Z_2),
\]
(23)
\[
\phi(Z_1, Z_2) = k|Z_1 - Z_2|^2 + \varepsilon(|Z_1|^2 + |Z_2|^2).
\]
(24)

Due to the upper semicontinuity of \( \Phi \) and the growth condition (B.1), we have
\[
\sup\Phi(Z_1, Z_2) = \Phi(Z_1^k, Z_2^k) \leq G^k < \infty.
\]
(25)

As for these above notions, we keep \( \varepsilon \) fixed and emphasize the dependence on \( k \). From \( \Phi(Z_1^k, Z_2^k) \geq \Phi(Z^*, Z^*) \), we can write
\[
G^k \geq \bar{u}(Z^*) - \bar{u}(Z^*) - 2\varepsilon|Z^*|^2 > \delta > 0,
\]
if $\varepsilon$ small enough. Therefore,
\[ \infty > u(Z^k_t) - \bar{u}(Z^k_2) > \delta > 0. \tag{26} \]

Also, we can write from $\Phi(Z^k_t, Z^k_2) \geq \Phi(Z^*, Z^*)$ and condition (A.2) that
\[ k|Z^k_t - Z^k_2|^2 + \varepsilon(|Z^k_t|^2 + |Z^k_2|^2) \leq u(Z^k_t) - \bar{u}(Z^k_2) - u(Z^*) + \bar{u}(Z^*) + 2\varepsilon|Z^*|^2 \leq C_\varepsilon(1 + |Z^k_t| + |Z^k_2|). \tag{27} \]

Hence, we have $|Z^k_t|, |Z^k_2| \leq C_\varepsilon$ with constant $C_\varepsilon > 0$ implying $Z^k_t \to Z^*_t, Z^k_2 \to Z^*_2$ (along a subsequence) as $k$ goes to infinity. Observe that $k|Z^k_t - Z^k_2|^2 \leq C_\varepsilon$ as $k \to \infty$, so we conclude that $Z^*_t = Z^*_2 = Z^*$. Since $\Phi(Z^k_t, Z^k_2) \geq \Phi(Z^*, Z^*)$, we have
\[ k|Z^k_t - Z^k_2|^2 \leq u(Z^k_t) - \bar{u}(Z^k_2) - u(Z^*) + \bar{u}(Z^*). \]

Letting $k \to \infty$, we obtain
\[ \lim_{k \to \infty} k|Z^k_t - Z^k_2|^2 = 0. \tag{28} \]

Now we show that the maximum can not be attained at the boundary $B_1 \cup B_2$. From (25), we know that the maximum can not be attained at the infinity. If $Z^* \in B_1$, combining condition (B.2) and (26), we have
\[ 0 \geq u(x^*, 0) - \bar{u}(x^*, 0) = \lim_{k \to \infty} u(Z^k_t) - \bar{u}(Z^k_2) \geq \delta > 0, \]
which is a contradiction. So we can conclude that $Z^* \notin B_2$ by the same argument.

In order to apply the Ishii’s Lemma[10], we rewrite $\phi$ as
\[ \phi(Z_1, Z_2) = k(x_1 - x_2)^2 + k(\lambda_1 - \lambda_2)^2 + \varepsilon(x_1^2 + x_2^2 + \lambda_1^2 + \lambda_2^2). \]

Thus, there exist $Z^k_1, Z^k_2$ such that
\[ D_{Z_1}\phi = \left(\frac{2k(x^k_1 - x^k_2) + 2\varepsilon x^k_1}{2k(\lambda^k_1 - \lambda^k_2) + 2\varepsilon \lambda^k_1}\right) \in D^+ u(Z^k_1), \tag{29} \]
\[ -D_{Z_2}\phi = \left(\frac{2k(x^k_1 - x^k_2) - 2\varepsilon x^k_2}{2k(\lambda^k_1 - \lambda^k_2) - 2\varepsilon \lambda^k_2}\right) \in D^- \bar{u}(Z^k_2). \tag{30} \]

Since $u$ and $\bar{u}$ are viscosity subsolution and supersolution of (8) at $(Z^k_1, Z^k_2)$ respectively, we can write from (29), (30) and Definition 6 that
\[ 0 \leq \max \left\{ p[2k(x^k_1 - x^k_2) + 2\varepsilon x^k_1] + \alpha(\lambda - \lambda^k_1)(2k(\lambda^k_1 - \lambda^k_2) + 2\varepsilon \lambda^k_1) \right. \]
\[ - (c + \lambda^k_1)u(x^k_1, \lambda^k_1) + \lambda^k_1 \int_{x_1^k}^{x_2^k} u(x^k_1 - \zeta, \lambda^k_1 + \beta)dF(\zeta), \]
\[ 1 - 2k(x^k_1 - x^k_2) - 2\varepsilon x^k_1 \right\} \]
\[ 0 \geq \max \left\{ p[2k(x^k_1 - x^k_2) - 2\varepsilon x^k_2] + \alpha(\lambda - \lambda^k_2)(2k(\lambda^k_1 - \lambda^k_2) - 2\varepsilon \lambda^k_2) \right. \]
\[ - (c + \lambda^k_2)\bar{u}(x^k_2, \lambda^k_2) + \lambda^k_2 \int_{x_2^k}^{x_1^k} \bar{u}(x^k_2 - \zeta, \lambda^k_2 + \beta)dF(\zeta), \]
\[ 1 - 2k(x^k_1 - x^k_2) + 2\varepsilon x^k_2 \right\}. \tag{31} \]
Subtracting (32) from (31), we have
\[
0 \leq \max \left\{ 2 \varepsilon p(x^k_1 + x^k_2) - 2 \alpha k (\lambda^k_1 - \lambda^k_2)^2 + 2 \alpha \varepsilon (\lambda^k_1 + \lambda^k_2) - 2 \alpha \varepsilon [(\lambda^k_1)^2 + (\lambda^k_2)^2] - c[u(x^k_1, \lambda^k_1) - \Pi(x^k_2, \lambda^k_2)] \\
+ \lambda^k_1 \int_0^{x^k_1} u(x^k_1 - \zeta, \lambda^k_1 + \beta) - u(x^k_1, \lambda^k_1) dF(\zeta) \\
- \lambda^k_2 \int_0^{x^k_2} \Pi(x^k_2 - \zeta, \lambda^k_2 + \beta) - \Pi(x^k_2, \lambda^k_2) dF(\zeta) \right\}.
\]
(33)

From (26) and taking \( k \to \infty \) in (33), we have
\[
0 \leq \max \left\{ 4 \alpha \varepsilon \lambda^k \varepsilon - 4 \alpha \varepsilon (\lambda^k) - c[u(x^k, \lambda^k) - \Pi(x^k, \lambda^k)] \\
+ \lambda^k \int_0^{x^k} u(x^k - \zeta, \lambda^k + \beta) - u(x^k, \lambda^k) dF(\zeta) \\
- \lambda^k \int_0^{x^k} \Pi(x^k - \zeta, \lambda^k + \beta) - \Pi(x^k, \lambda^k) dF(\zeta) \right\}.
\]
(34)

From the equation (26), sending \( k \to \infty \), the inequality (27) leads us to
\[
u(Z^x) - \Pi(Z^x) \geq 2 \varepsilon |Z^x|^2.
\]
(35)

Observe that \( \Phi(Z^x, \lambda^k) \geq \Phi(Z^x, -(\zeta, \beta), Z^x, -(\zeta, \beta)) \), we can write by sending \( k \to \infty \)
\[
\lambda^k \int_0^{x^k} u(x^k - \zeta, \lambda^k + \beta) - \Pi(x^k - \zeta, \lambda^k + \beta) - u(x^k, \lambda^k) + \Pi(x^k, \lambda^k) dF(\zeta) \leq 2 \beta \varepsilon \lambda^k + 4 \beta \varepsilon (\lambda^k)^2.
\]
(36)

Using (35) in (34) and the fact that \( \beta < \alpha \), we have
\[
0 \leq \max \left\{ 2p \sqrt{\varepsilon} + 2 \alpha \lambda \sqrt{\varepsilon} + \beta^2 \sqrt{\varepsilon} - \frac{1}{2} c \delta + [p \sqrt{\varepsilon} - \frac{1}{2} c] 2 \varepsilon (\lambda^x)^2 + [\alpha \lambda \sqrt{\varepsilon} + \frac{1}{2} \beta^2 \sqrt{\varepsilon} + 2(\beta - \alpha) - \frac{1}{2} \varepsilon] 2 \varepsilon (\lambda^x)^2 \right\} < 0,
\]
(37)

if we take \( \varepsilon \) small enough, then we derive the contradiction.

3.3. Characterization results. In Section 3.2, we obtain a comparison principle result which gives us uniqueness of viscosity solutions with a given boundary condition at zero. However, the value function does not have a natural boundary condition at zero. So the uniqueness result for viscosity solution is not a verification theorem. In this section we give a verification theorem stating that if a viscosity supersolution of the HJB equation is obtained as a value function of an admissible strategy, then the supersolution should be optimal.

Proposition 9. Let \( \Pi \) be a viscosity supersolution of (8) satisfying condition (A.2), then \( V(x, \lambda) \leq \Pi(x, \lambda) \) for any \((x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \). Specifically, the value function is characterized as the smallest viscosity supersolution.

This proposition will be used in the future to verify the optimality of a candidate value function with a given strategy. As for the proof, we refer to the previous work Chen and Bian[12, Proposition 6].
3.4. Concavity preserving. There are few results for the concavity of the viscosity solutions in HJB equation. Alvarez et al. [8] established the convexity of viscosity solution of elliptic PDEs with the constraint boundary conditions. We discuss the concavity of the viscosity solutions using comparison technique [24, 29]. The concavity will be used in the Proposition 13 to show that the barrier is nonincreasing in the intensity rate.

Next, we want to show that $V$ is a concave function in $x$ for all $\lambda$. In order to obtain the needed result, we first introduce an auxiliary function and an associated equivalent viscosity solution statement. Let $v = -V$ and then we can conclude apparently that

- $V$ is the viscosity solution of (8) if and only if $v$ is the viscosity solution of
  $$\max\{-v_x - \alpha(\lambda - \lambda)v_\lambda + (\epsilon + \lambda)v - \lambda \int_0^x v(x - \alpha, \lambda + \beta)dF(\alpha), 1 + v_x \} = 0. \quad (38)$$

- $V$ is concave if and only if $v$ is convex.

Next we prove that $v$ is convex in $x$ for all $\lambda$ by contradiction.

**Proposition 10.** Let $V$ be the viscosity solution of (8) satisfying (A.1) and (A.2). Then $V(x, \lambda)$ is concave in $x$ for each $\lambda$.

**Proof.** It is enough to show that $v$ is convex in $x$ for all $\lambda$ and we prove it by contradiction with HJB equation (38). Suppose that $v$ is not convex in $x$, then we have

$$v(\theta \bar{x} + (1 - \theta)\bar{y}, \bar{\lambda}) \geq \theta v(\bar{x}, \bar{\lambda}) + (1 - \theta)v(\bar{y}, \bar{\lambda}) + 2\delta \quad (39)$$

for some $0 \leq \bar{x} < \bar{y} < \infty, 0 \leq \bar{\lambda} < \infty, \theta \in [0, 1]$ and $\delta > 0$. Let

$$\Psi(x, y, z, \lambda) = v(z, \lambda) - \theta v(x, \lambda) - (1 - \theta)v(y, \lambda) - \omega(x, y, z, \lambda),$$

for $(x, y, z, \lambda) \in \mathbb{R}^4_+$, where

$$\omega(x, y, z, \lambda) = \frac{k}{2}(\varepsilon - \theta x - (1 - \theta)y)^2 + \frac{\varepsilon}{2}(x^2 + y^2 + z^2 + \lambda^2) + \frac{\varepsilon}{\lambda},$$

for $k > 1$ and $\varepsilon \in (0, 1]$. Let

$$F_k = \sup_{\mathbb{R}^4_+} \Psi(x, y, z, \lambda).$$

From that $v$ satisfies (A.2) and the continuity of $\Psi(x, y, z, \lambda)$, we see that $F_k < \infty$ and there exists $(x_k, y_k, z_k, \lambda_k) \in \mathbb{R}^4_+$ (here we suppress the dependency on $\varepsilon$) such that $F_k = \Psi(x_k, y_k, z_k, \lambda_k)$ and

$$F_k \geq v(\theta \bar{x} + (1 - \theta)\bar{y}, \bar{\lambda}) - \theta v(\bar{x}, \bar{\lambda}) - (1 - \theta)v(\bar{y}, \bar{\lambda})$$

$$- \frac{\varepsilon}{2}(x^2 + y^2 + (\theta \bar{x} + (1 - \theta)\bar{y})^2 + \bar{\lambda}^2) - \frac{\varepsilon}{\bar{\lambda}} \geq \delta > 0 \quad (40)$$

for any $\varepsilon$ that is small enough. This implies

$$v(z_k, \lambda_k) \geq \theta v(x_k, \lambda_k) + (1 - \theta)v(y_k, \lambda_k) + \delta \quad (41)$$

for any $k > 1$ and $\varepsilon$ small enough. From (A.2), we know that the upper bound is not related to $\lambda$. We can write from $\Psi(x_k, y_k, z_k, \lambda_k) \geq \Psi(\bar{x}, \bar{y}, \theta \bar{x} + (1 - \theta)\bar{y}, \bar{\lambda})$ and
(A.2) that
\[
\frac{k}{2}(z_k - \theta x_k - (1 - \theta)y_k)^2 + \frac{\varepsilon}{2}(x_k^2 + y_k^2 + z_k^2 + \lambda_k^2) + \varepsilon \leq 3C_\varepsilon(1 + x_k + y_k + z_k)
\]
which implies the existence of finite positive constant \(C_\varepsilon\) such that \(0 \leq x_k, y_k, \lambda_k \leq C_\varepsilon\). Owing to this, we have that there exists a subsequence, still denoted by \((x_k, y_k, z_k, \lambda_k)\), which converges to some \((x, y, z, \lambda) \in \mathbb{R}_+^4\) as \(k \to \infty\) for each fixed \(\varepsilon\).

Observe that
\[
\frac{k}{2}(z_k - \theta x_k - (1 - \theta)y_k)^2 \leq (\Psi(x_k, y_k, z_k, \lambda_k))^{\varepsilon}_k,
\]
we conclude that \(z_k - \theta x_k - (1 - \theta)y_k \to 0\) as \(k \to \infty\) and \(z_k = \theta x_k + (1 - \theta)y_k\).

Using \(\Psi(x_k, y_k, z_k, \lambda_k) \geq \Psi(x, y, z, \lambda)\), we have
\[
\frac{k}{2}(z_k - \theta x_k - (1 - \theta)y_k)^2 \leq v(z_k, \lambda_k) - v(z, \lambda) + \theta v(x, \lambda) - \theta v(x_k, \lambda_k)
+ (1 - \theta)v(y, \lambda) - (1 - \theta)v(y_k, \lambda_k)
+ \frac{\varepsilon}{2}(x_k^2 + y_k^2 + z_k^2 + \lambda_k^2)
- \frac{\varepsilon}{2}(x_k^2 + y_k^2 + z_k^2 + \lambda_k^2)
+ \frac{\varepsilon}{2}(x_k - x) - \frac{\varepsilon}{2}(x_k - x_k).
\]

From the continuity of \(v\) yields
\[
k(z_k - \theta x_k - (1 - \theta)y_k)^2 \to 0, \text{ as } k \to \infty \text{ for each fixed } \varepsilon.
\]

From (42), for fixed \(\varepsilon > 0\) and \(x_k, y_k, z_k \leq C_\varepsilon\), the left side of (42) goes to infinity as \(\lambda_k \to 0\) because of the term \(\frac{\varepsilon}{\lambda_k}\). However, the right side of (42) goes to zero as \(\lambda_k \to 0\). This is a contradiction.

We know that \(\Psi(x, y, z, \lambda) = v(z, \lambda) - \theta v(x, \lambda) - (1 - \theta)v(y, \lambda) - \frac{\varepsilon}{2}(x^2 + y^2 + z^2 + \lambda^2) - \frac{\varepsilon}{\lambda}\) is decreasing in \(x, y, z\), combining \(z_k = \theta x + (1 - \theta)y\), the maximum point may be \(x = y = z = 0\). We can send \(k \to \infty\) in (41), we obtain
\[
v(0, \lambda) \geq v(0, \lambda) + (1 - \theta)v(0, \lambda) + \delta
\]
which is a contradiction.

Then we have \(0 < x_k, y_k, z_k < \infty, 0 < \lambda_k < \infty\) for any \(k \) large enough. Ishii’s Lemma[10] implies that
\[
(k(z_k - \theta x_k - (1 - \theta)y_k) - \frac{1}{\theta}z_k - \frac{1}{\theta}z(\lambda_k - \frac{1}{\lambda_k})) \in D^- v(x_k, \lambda_k),
\]
\[
(k(z_k - \theta x_k - (1 - \theta)y_k) - \frac{1}{1 - \theta}z_k - \frac{1}{1 - \theta}z(\lambda_k - \frac{1}{\lambda_k})) \in D^- v(y_k, \lambda_k),
\]
\[
(k(z_k - \theta x_k - (1 - \theta)y_k) + \varepsilon z_k, \varepsilon(\lambda_k - \frac{1}{\lambda_k})) \in D^+ v(z_k, \lambda_k).
\]
Since \( v \) is viscosity solution of (38), we can write from Definition 6 that

\[
\max \left\{ -p \left[ k(z_k - \theta x_k - (1 - \theta)y_k) - \frac{1}{\theta} \varepsilon x_k \right] - \alpha(\bar{\lambda} - \lambda_k)(\frac{1}{\theta} \varepsilon(\lambda_k - \frac{1}{\lambda_k^2})) + (c + \lambda_k)v(x_k, \lambda_k) \right. \\
- \lambda_k \int_0^{x_k} v(x_k - \zeta, \lambda_k + \beta)dF(\zeta), 1 + (k(z_k - \theta x_k - (1 - \theta)y_k) - \frac{1}{\theta} \varepsilon x_k) \right\} \geq 0,
\]

\[
(46)
\]

\[
\max \left\{ -p \left[ k(z_k - \theta x_k - (1 - \theta)y_k) - \frac{1}{1 - \theta} \varepsilon y_k \right] - \alpha(\bar{\lambda} - \lambda_k)(\frac{1}{1 - \theta} \varepsilon(\lambda_k - \frac{1}{\lambda_k^2})) + (c + \lambda_k)v(y_k, \lambda_k) \right. \\
- \lambda_k \int_0^{y_k} v(y_k - \zeta, \lambda_k + \beta)dF(\zeta), 1 + (k(z_k - \theta x_k - (1 - \theta)y_k) - \frac{1}{1 - \theta} \varepsilon y_k) \right\} \geq 0.
\]

\[
(47)
\]

\[
\max \left\{ -p [k(z_k - \theta x_k - (1 - \theta)y_k + \varepsilon z_k] - \alpha(\bar{\lambda} - \lambda_k)\varepsilon(\lambda_k - \frac{1}{\lambda_k^2}) + (c + \lambda_k)v(z_k, \lambda_k) \right. \\
- \lambda_k \int_0^{z_k} v(z_k - \zeta, \lambda_k + \beta)dF(\zeta), 1 + (k(z_k - \theta x_k - (1 - \theta)y_k + \varepsilon z_k) \right\} \leq 0.
\]

\[
(48)
\]

Here we note that \( V \) is the viscosity supersolution of (8) if and only if that \( v \) is the viscosity subsolution of (38). Next, we take (48) minus \( \theta \) times (46) minus \( (1 - \theta) \) times (47), then we receive that

\[
0 \geq \max \left\{ -p \varepsilon x_k + y_k + z_k - 3\alpha \varepsilon \lambda_k + 3\alpha \varepsilon \lambda_k^2 + 3\alpha \varepsilon \frac{\bar{\lambda} - \lambda_k}{\lambda_k^2} \right. \\
+ \varepsilon(\lambda_k - \frac{1}{\lambda_k}) + (c + \lambda_k)v(x_k, \lambda_k) \right. \\
- \lambda_k \int_0^\infty [v(z_k - \zeta, \lambda_k + \beta) - \theta v(x_k - \zeta, \lambda_k + \beta) - (1 - \theta)v(y_k - \zeta, \lambda_k + \beta)] \\
- [v(z_k, \lambda_k) - \theta v(x_k, \lambda_k) - (1 - \theta)v(y_k, \lambda_k)]dF(\zeta), \\
\varepsilon(x_k + y_k + z_k) \right\}.
\]

\[
(49)
\]

It has to be mentioned that the integral upper bound \( \infty \) is reasonable. \( \int_0^\infty v(x - \zeta)dF(\zeta) = \int_0^\infty v(x - \zeta)dF(\zeta) \) if we extend the \( v \) into \( \mathbb{R} \) and let \( v(x, \lambda) = 0 \) if \( x < 0 \).

From (40) and \( z_\varepsilon = \theta x_\varepsilon + (1 - \theta)y_\varepsilon \), by sending \( k \to \infty \), we have

\[
v(z_\varepsilon, \lambda_\varepsilon) - \theta v(x_\varepsilon, \lambda_\varepsilon) - (1 - \theta)v(y_\varepsilon, \lambda_\varepsilon) \geq \frac{\varepsilon}{2}(x_\varepsilon^2 + y_\varepsilon^2 + z_\varepsilon^2 + \lambda_\varepsilon^2).
\]

\[
(50)
\]

Observing that \( \Psi(x_k, y_k, z_k, \lambda_k) \geq \Psi(x_k - \zeta, y_k - \zeta, z_k - \zeta, \lambda_k + \beta) \), we can write by sending \( k \to \infty \) that

\[
-k_\varepsilon \int_0^\infty [v(z_\varepsilon - \zeta, \lambda_\varepsilon + \beta) - \theta v(x_\varepsilon - \zeta, \lambda_\varepsilon + \beta) - (1 - \theta)v(y_\varepsilon - \zeta, \lambda_\varepsilon + \beta)] \\
- [v(z_\varepsilon, \lambda_\varepsilon) - \theta v(x_\varepsilon, \lambda_\varepsilon) - (1 - \theta)v(y_\varepsilon, \lambda_\varepsilon)]dF(\zeta) \]

\[
\geq -\varepsilon \beta \lambda_\varepsilon^2 - \frac{\varepsilon}{2} \beta^2 \lambda_\varepsilon + \frac{\varepsilon \beta}{\lambda_\varepsilon + \beta}.
\]

\[
(51)
\]
Therefore, using (50), (51) in (49) and sending \( k \to \infty \), we have

\[
0 \geq \max \left\{ \frac{1}{2} \alpha \eta - \frac{3}{2}p\sqrt{\varepsilon} - \frac{1}{2} (3\alpha \bar{\lambda} + \frac{1}{2} \beta) \sqrt{\varepsilon} + \left( \frac{1}{4} c - \frac{1}{2} p\sqrt{\varepsilon} \right) \varepsilon (x^2 + y^2 + z^2) + 3\alpha \varepsilon \frac{\bar{\lambda} - \lambda_e}{\lambda_e^2} + \left[ \frac{1}{4} e - \frac{1}{2} (3\alpha \bar{\lambda} + \frac{1}{2} \beta) \sqrt{\varepsilon} + \alpha - \beta \right] \varepsilon \lambda_e^2, \varepsilon (x_e + y_e + z_e) \right\} > 0,
\]

if \( \varepsilon \) small enough and the assumption \( \alpha > \beta \). Here we mention that the term

\[ 3\alpha \varepsilon \frac{\bar{\lambda} - \lambda_e}{\lambda_e^2} \to 0^- \text{ when } \lambda_e \to \infty. \]

Then we derive the contradiction. □

4. Optimal dividend payment strategy.

4.1. The divided partition. According to the HJB equation (8) we can split \( \mathbb{R}_+ \times \mathbb{R}_+ \) into three subsets.

**Definition 11.** Let us define \( \mathcal{P} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \) where

- \( \mathcal{A} = \{ (x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ such that } 1 - V_x = 0 \text{ and } \mathcal{L}(V) = 0 \} \),
- \( \mathcal{B} = \{ (x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ such that } 1 - V_x = 0 \text{ and } \mathcal{L}(V) < 0 \} \),
- \( \mathcal{C} = \{ (x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ such that } 1 - V_x < 0 \text{ and } \mathcal{L}(V) = 0 \} \).

This partition indicates an associated dividend payment strategy showing as follows.

- In the case that \( (x, \lambda) \in \mathcal{A} \), then we pay an amount of surplus as dividends so as to let the \( (x, \lambda) \) go to \( (\rho(\lambda), \lambda) \) along the barrier curve until the first-claim arrival \( \tau_1 \). Specially, we take \( L_t^{x,\lambda} = pt \) up to the first-claim arrival \( \tau_1 \) when \( (x, \lambda) = (\rho(\lambda), \lambda) \). Then follow the strategy according to the initial surplus \( (X_{\tau_1}, \lambda_{\tau_1}) \).
- If \( (x, \lambda) \in \mathcal{B} \), we pay \( x - \rho(\lambda) \) as dividends immediately. Afterwards, follow the strategy according to the \( (\rho(\lambda), \lambda) \in \mathcal{A} \).
- If \( (x, \lambda) \in \mathcal{C} \), we pay no dividends up to the exit time \( \tau_C \) of \( \mathcal{C} \). Then follow the strategy corresponding to initial surplus \( (X_{\tau_C}, \lambda_{\tau_C}) \).

Denote \( \mathcal{P} \) by \( (\mathcal{A}, \mathcal{B}, \mathcal{C}) \). The family \( l(\mathcal{P}) = \{ L_t^{x,\lambda} \} \) indicates a dividend payment strategy introduced above corresponding to the partition \( \mathcal{P} \). Now we show that it is the optimal dividend payment strategy. Particularly, we show that \( V_{l(\mathcal{P})} = V \).

**Theorem 12.** Let \( l(\mathcal{P}) \) be the strategy corresponding to the partition \( \mathcal{P} \). Then we have \( V_{l(\mathcal{P})} = V \).

**Proof.** As a result of Proposition 9, it is enough to show that \( V_{l(\mathcal{P})} \) is a viscosity supersolution of (8). When \( (x, \lambda) \in \mathcal{A} \), owing to the definition of \( \mathcal{A} \), we could find test function \( \varphi \) such that \( V'(x, \lambda) = \varphi'(x, \lambda) = 1 \) and \( \mathcal{L}(\varphi) = 0 \). Then it suits the inequality (20), so we know that \( V_{l(\mathcal{P})} \) is a viscosity supersolution of (8). When \( (x, \lambda) \in \mathcal{B} \), we can also find a test function such that \( V'(x, \lambda) = \varphi'(x, \lambda) = 1 \) and \( \mathcal{L}(\varphi) < 0 \) which also suits the inequality (20). The same conclusion can be attained by taking \( V'(x, \lambda) = \varphi'(x, \lambda) < 1 \) and \( \mathcal{L}(\varphi) = 0 \) when \( (x, \lambda) \in \mathcal{C} \). Now we conclude the result. □
4.2. The barrier curve. The barrier strategy is a classical optimal strategy in dividend payment problem when only one surplus process is considered. Here we refer to Gerber [16] to show more details recently. A barrier strategy with barrier \( b \) pays out dividends whenever the surplus is above \( b \), so that the surplus level stays at \( b \), and pays no dividends below that barrier \( b \).

However, according to the stochastic control theorem, the optimal control must be related to the present surplus and intensity. Therefore, the barrier is turned out to be a curve rather than a single point in our case.

The partition \( \mathcal{A} \) defined in Definition 11 indicates a function as follows.

There exists a function \( \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that \( 1 - V_x(\rho(\lambda), \lambda) = 0 \) and \( \mathcal{L}(V)(\rho(\lambda), \lambda) = 0 \). We call

\[ \mathcal{A} = \{ (\rho(\lambda), \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \} \]

as the barrier curve.

Next we give a short proposition to state the monotonicity of the barrier curve.

**Proposition 13.** \( \rho(\lambda) \) is nonincreasing in \( \lambda \).

*Proof.* We prove it by contradiction. Suppose that \( \rho(\lambda) \) is not nonincreasing in \( x \), then there exist \( 0 \leq \lambda_1 \leq \lambda_2 < \infty \) such that

\[ \rho(\lambda_1) < \rho(\lambda_2). \]

Then we have the following facts:

- \( \{ x > \rho(\lambda_2), \lambda_2 \} \subset \mathcal{B} \).
- \( \{ 0 < x < \rho(\lambda_2), \lambda_2 \} \subset \mathcal{C} \).

Then we have \( (\rho(\lambda_1), \lambda_2) \in \mathcal{C} \) and

\[ V_x(\rho(\lambda_1), \lambda_2) > V_x(\rho(\lambda_2), \lambda_2) = 1 \]  \( \quad (53) \)

However, since \( V \) is concave with respect to \( x \) according to Proposition 10, we have \( V_x(\rho(\lambda_1), \lambda_2) \leq V_x(\rho(\lambda_2), \lambda_2) \), then we obtain the contradiction. \( \square \)

4.3. A barrier curve strategy. In this section we give some figures to illustrate how the optimal dividend strategy works. For simplicity, the barrier curve \( \rho \) is taken as a simple decreasing linear function showed in Figure 2(a) \( \mathcal{A} \) in dashed line.

The long-run average of the claim intensity \( \bar{\lambda} \) is taken as 0.5. Meanwhile, we split the \( \mathbb{R}_+ \times \mathbb{R}_+ \) into the following partitions:

- \( \mathcal{A} = \{ (\rho(\lambda), \lambda) | \lambda \in \mathbb{R}_+ \} \);
- \( \mathcal{B}_1 = \{ (x, \lambda) | x > \rho(\lambda), \lambda > \bar{\lambda} \} \);
- \( \mathcal{B}_2 = \{ (x, \lambda) | x > \rho(\lambda), \lambda < \bar{\lambda} \} \);
- \( \mathcal{C}_1 = \{ (x, \lambda) | x < \rho(\lambda), \lambda > \bar{\lambda} \} \);
- \( \mathcal{C}_2 = \{ (x, \lambda) | x < \rho(\lambda), \lambda < \bar{\lambda} \} \).

Then we have \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \) and \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \), which indicates four kinds of initial points \( (x_0, \lambda_0) \) exhibited in Figure 2 (a).

Now we begin at a state \( A = (x_A, \lambda_A) \) which is belong to \( \mathcal{B}_1 \). There might be three potential routes showed in Figure 2 (b) (c) (d). If the intensity rate decay mildly, the way forward may be \( A \rightarrow B \rightarrow D \). On the other cases, we refer to the route \( A \rightarrow B \rightarrow C \rightarrow D \) when the intensity rate decay properly and the route \( A \rightarrow B \rightarrow E \rightarrow D \) when rapidly. The instructions are given for each trip.

- \( A \rightarrow B \): pay \( x_A - x_B \) as dividends immediately.
- \( B \rightarrow C \): pay no dividends. \( \lambda_t \) return downwards to the long-run average \( \bar{\lambda} \) and \( X_t \) grows linearly at growth rate \( p \) until it touches the barrier curve.
OPTIMAL DIVIDEND POLICY WITH CONTAGIOUS ARRIVALS OF CLAIMS

Figure 2. Several optimal dividends payment strategy examples

$C \to D$: pay dividends $pt_{CD} - \rho(\lambda_D) + \rho(\lambda_C)$ in which $t_{CD} = \frac{1}{\alpha} \ln \frac{\lambda_D - \lambda}{\lambda_C - \lambda}$ on the purpose to let the surplus $X_t$ stay at the barrier curve. $\lambda_t$ returns downwards to the long-run average $\bar{\lambda}$.

$B \to D$: it is similar to the operations when we face to the situation $C \to D$.

$D$: when it comes to point $D$, we apply $L = pt$ such that the surplus $X_t$ and the intensity $\lambda_t$ stay unchanged until a claim comes. And we have to mention that the point $D$ is the prime time to pay the dividend. An insurance company usually stays at the state $D$ when the intensity rate is small since claim comes at low probability. The longer we stay at the point $D$, the more dividends can be taken out.

$B \to E$: pay no dividends. $\lambda_t$ returns downwards to the long-run average $\bar{\lambda}$ and $X_t$ grows linearly at growth rate $p$ until it touches the long-run average $\bar{\lambda}$.

$E \to D$: pay no dividends. $\lambda_t$ stays unchanged and the surplus $X_t$ grows linearly at growth rate $p$ until a claim comes or the arrival of $D$.

This time we begin at the partition $C_2$ and we select two typical points $F$ and $G$. Figure 2 (f) (g) propose the routes $F \to E \to D$ if the intensity rate grows rapidly and the route $G \to H \to D$ if it grows slowly.

$F \to E$: pay no dividends. The intensity $\lambda_t$ grows to the long-run average $\bar{\lambda}$ and $X_t$ grows linearly at growth rate $p$. And $E \to D$ part has been introduced before.

$G \to H$: pay no dividends. The intensity $\lambda_t$ and $X_t$ grows until they hit the barrier curve.

$H \to D$: pay dividends $pt_{HD} - \rho(\lambda_D) + \rho(\lambda_H)$ in which $t_{HD} = \frac{1}{\alpha} \ln \frac{\lambda_D - \lambda}{\lambda_H - \lambda}$ on the purpose to let the surplus $X_t$ stay at the barrier curve. $\lambda_t$ returns upwards to the long-run average $\bar{\lambda}$.

When we start from the point $I \in B_2$, we point out a possible route $I \to J \to D$ illustrated in Figure 2 (h).

$I \to J$: pay $x_I - x_J$ as dividends immediately and $J \to D$ is similar to $H \to D$.

When a claim comes, the pattern of change can be $D \to K \to L \to M$ or $D \to N \to O$ which are showed in Figure 2 (i) and (j).
\[ D \to K: \text{pay no dividends and a claim comes. The surplus jumps downwards but the intensity process jumps upwards mildly so that it do not exceed the barrier curve. On the other hand, one may jump to the point } N \text{ showed in Figure 2 (j) which jumps upwards fiercely so as to cross the barrier curve. Under this circumstance } D \to N \to O, \text{ we should pay } x_N - x_O \text{ as dividends immediately so as to let the state point hit the barrier curve. Then we meet an interesting case in which dividends paid when a claim comes. This goes against common sense. We proffer an explanation that the insurance company must pay the dividends when the claim intensity rate increases so high since the company may go bankruptcy because of the claim events for a short time to come. When we start from the point } K \in C_1, \text{ we pay no dividends and the state change } K \to L \text{ is similar to } B \to C. \]

\[ L \to M: \text{a big claim happened and then the company went bankrupt. Of course, there are many other forward routes in practical. But most of typical routes have been introduced before. In fact, possible paths are combinations of the trips we introduced. For example, a company may have a path } G \to H \to D \to N \to O(K) \to L \to M \text{ when taking the dividends strategy mentioned in Section 4.1.} \]

5. **Numerical examples.** In this section we show some numerical results of the optimal value functions and the optimal strategies. However, we have to mention that it is rather hard to construct a appropriate numerical scheme with our HJB equation. Furthermore, the absence boundary condition at zero also hampers the construction when the difference scheme method is adopted. In Azcue and Muler[2], they numerically solved \( \mathcal{L}(W) = 0 \) with a given boundary condition at zero. Then they construct a new function \( U(x) = W(x)/W(x^*) \) where \( x^* \) is the barrier point. Finally they proved that \( U(x) \) coincided with \( V \). However, this approach is no longer in force in our case. For the sake of simplification, we provide with a reasonable guess about the boundary condition and complete a basic numerical simulation. For the numerical approach to this kind of HJB equation in our problem, we refer to the method proposed by Forsyth and Labahn[13].

Fundamental parameter settings are as follows:

\[ \alpha = 1 \quad \bar{\lambda} = 0.5 \quad \beta = 0.2 \quad p = 10 \quad c = 0.2 \quad F(x) = 1 - e^{-x}. \]

And we consider a zonal region \( \{x \in [0, 5], \lambda \in [0, 1.5]\} \) composing a grid with \( h = 0.1: \)

\[ x^i = ih \quad i = 0, 1, \cdots, N; \]
\[ \lambda_j = jh \quad j = 0, 1, \cdots, M; \]
\[ Q = \{(x^i, \lambda_j)| i = 0, 1, \cdots, N, j = 0, 1, \cdots, M\}, \]
\[ V_j^i = V(x^i, \lambda_j). \]

With regard to the boundary condition, combining the upper and lower bounds of the value function (10), we have made the following assumptions:

\[ V_0^i = \frac{p}{c}e^{-\lambda_j}, \quad V_N^i = x^N + \frac{p}{c}e^{-\lambda_j}, \]
\[ V_0^i = V^\epsilon(x^i), \quad V_M^i = x^i + \frac{p}{c}e^{-\lambda_M}. \]
where \( V^c \) is the solution of the following equation

\[
\max \{ pV_x - (c + \lambda)V + \lambda \int_0^x V(x - \zeta)dF(\zeta), 1 - V_x \} = 0.
\]

To be more precise, the intensity process \( \lambda \) is constant. In our case, we take \( \lambda = 0.1 \) for the sake of getting a distinct result. In other words, the value at \( \lambda = 0 \) is filled by an one dimensional problem solution when the claim intensity \( \lambda \) is taken as a constant 0.1.

We show in Figure 3 the optimal value function and in Figure 4 the barrier curve. We can see in Figure 4, the \( \rho(\lambda) \) is non-increasing in \( \lambda \) as we have proved in Proposition 13. It declines slow when \( \lambda \) is small but drops quickly when \( \lambda \) is large. When the claim intensity is small, the claim comes infrequently which indicating that the company can stay at a high level of surplus. When the claim intensity is large, the company should pay a large sum of surplus as dividends immediately. If not, the company may ruin rapidly and has no surplus to pay as dividends. Owing to the self-exciting feature, the claim comes at a higher probability when \( \lambda > \bar{\lambda} \), then the barrier curve declines rapidly. A higher claim probability indicates a lower \( \rho(\lambda) \).

![Figure 3: The value function](image1)

![Figure 4: The fitting barrier curve](image2)

![Figure 5: The value of \( V \) and \( V^c \) with \( \lambda = 0.5 \) and associated barrier points](image3)
In Figure 5 we show the value function $V$ in solid line and $V^c$ in dashed line when $\lambda = 0.5$. $V$ is less than $V^c$ even we take the same $\lambda = 0.5$. The explanation comes from the exist of self-exciting term. The company runs into claims more frequently comparing to the case that $\lambda$ is a constant. Meanwhile, we observe that the barrier point in self-exciting case, or accurately the barrier $x^*$, is less than that in constant case even the same $\lambda$ is taken. A self-exciting case suggests a higher claim probability than the constant case because every happened claim increased the probability of the next claim. Then the self-exciting case have a smaller barrier even we begin at the same claim intensity.

Barrier curve under different parameter settings are illustrated with Figure 6. As we can see in Figure 6(A), a higher decay rate $\alpha$ indicates a more dividend payment at the same claim intensity rate when it is larger than the long-run average of the claim intensity. What’s more, it is at a lower risk since the claim intensity return back to the average more quickly which suggests a larger dividends payment. Figure 6 (B) shows us that a larger long-run average of the claim intensity implies a higher barrier curve. This indicates that you should pay less as dividends since you are at a higher risk of the claims. Figure 6(C) exhibits that one should pay more surplus as dividends when the premium rate is smaller. When $p$ is larger, the company has strong profitability and should pay less dividends to have the adequate surplus
to afford further dividends. If you pay too much at once, you may not have enough surplus to cope with the claim to come. In other words, you have a less life time to pay dividends with a lower survival probability. Figure 6(D) tells us that a larger discount factor implies a lower barrier curve. When the discount rate $c$ is quite high, we should pay a large sum of surplus as dividends as soon as possible to maintain the value whatever the claim intensity is.

Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments and suggestions, which helped to improve significantly the quality of this paper.

REFERENCES

[1] Y. Aït-Sahalia and T. R. Hurd, Portfolio choice in markets with contagion, *Journal of Financial Econometrics*, 14 (2015), 1–28.

[2] P. Azcue and N. Muler, Optimal investment policy and dividend payment strategy in an insurance company, *The Annals of Applied Probability*, 20 (2010), 1253–1302.

[3] S. Asmussen, B. Højgaard and M. Taksar, Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation, *Finance and Stochastics*, 4 (2000), 299–324.

[4] P. Azcue and N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, *Mathematical Finance*, 15 (2005), 261–308.

[5] H. Albrecher and S. Thonhauser, Optimality results for dividend problems in insurance, *RACSAM-Revista de la Real Academia de Ciencias Exactas*, 103 (2009), 295–320.

[6] B. Avanzi, Strategies for dividend distribution: A review, *North American Actuarial Journal*, 13 (2009), 217–251.

[7] H. Albrecher P. Azcue and N. Muler, Optimal dividend strategies for two collaborating insurance companies, *Advances in Applied Probability*, 49 (2017), 515–548.

[8] O. Alvarez J. M. Lasry and P. L. Lions, Convex viscosity solutions and state constraints, *Journal de Mathématiques Pures et Appliquées*, 76 (1997), 265–288.

[9] M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Transactions of the American Mathematical Society*, 277 (1983), 1–42.

[10] M. G. Crandall and P. L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bulletin of the American mathematical society*, 27 (1992), 1–67.

[11] Y. Chen and B. Bian, Optimal investment and dividend policy in an insurance company: A varied bound for dividend rates, *Discrete & Continuous Dynamical Systems-Series B*, 24 (2019), 5083–5105.

[12] Y. Chen and B. Bian, Optimal dividend policies for compound poisson process with self-exciting, working paper.

[13] P. A. Forsyth and G. Labahn, Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance, *Journal of Computational Finance*, 11 (2007), 1–43.

[14] H. U. Gerber and E. S. W. Shiu, On optimal dividend strategies in the compound Poisson model, *North American Actuarial Journal*, 10 (2006), 76–93.

[15] X. Gao and L. Zhu, Large deviations and applications for Markovian Hawkes processes with a large initial intensity, *Bernoulli*, 24 (2018), 2875–2905.

[16] H. U. Gerber, X. S. Lin and H. Yang, A note on the dividends-penalty identity and the optimal dividend barrier, *ASTIN Bulletin: The Journal of the IAA*, 36 (2006), 489–503.

[17] A. G. Hawkes, Spectra of some self-exciting and mutually exciting point processes, *Biometrika*, 58 (1971), 83–90.

[18] D. Hainaut, Contagion modeling between the financial and insurance markets with time changed processes, *Insurance: Mathematics and Economics*, 74 (2017), 63–77.

[19] Z. Jiang and M. Pistorius, Optimal dividend distribution under Markov regime switching, *Finance and Stochastics*, 16 (2012), 449–476.

[20] Z. Jiang, Optimal dividend policy when cash reserves follow a jump-diffusion process under Markov-regime switching, *Journal of Applied Probability*, 52 (2015), 209–223.

[21] N. Kulenko and H. Schmidli, Optimal dividend strategies in a Cramér-Lundberg model with capital injections, *Insurance: Mathematics and Economics*, 43 (2008), 270–278.
[22] H. Meng and T. K. Siu, Optimal mixed impulse-equity insurance control problem with reinsurance, *SIAM Journal on Control and Optimization*, 49 (2011), 254–279.

[23] J. Paulsen, Optimal dividend payments and reinvestments of diffusion processes with both fixed and proportional costs, *SIAM Journal on Control and Optimization*, 47 (2008), 2201–2226.

[24] H. Pham, Optimal stopping of controlled jump diffusion processes: A viscosity solution approach, *Journal of Mathematical Systems, Estimation and Control*, 8 (1998), 1–27.

[25] G. Stabile and G. L. Torrisi, Risk processes with non-stationary Hawkes claims arrivals, *Methodology and Computing in Applied Probability*, 12 (2010), 415–429.

[26] H. Schmidli, *Stochastic Control in Insurance*, Springer, New York, 2008.

[27] H. Schmidli, On capital injections and dividends with tax in a classical risk model, *Insurance: Mathematics and Economics*, 71 (2016), 138–144.

[28] S. Thonhauser and H. Albrecher, Optimal dividend strategies for a compound Poisson process under transaction costs and power utility, *Stochastic Models*, 27 (2011), 120–140.

[29] Y. Wang, B. Bian and J. Zhang, Viscosity solutions of Integro-Differential equations and passport options in a Jump-Diffusion model, *Journal of Optimization Theory and Applications*, 161 (2014), 122–144.

[30] H. Zhu, *Dynamic Programming and Variational Inequalities in Singular Stochastic Control*, Ph. D Thesis, Brown University, 1992.

Received September 2019; revised December 2019.

E-mail address: yichen@tongji.edu.cn

E-mail address: bianbj@tongji.edu.cn