Twists for duplex regions

Pedro H. Milet  Nicolau C. Saldanha

November 3, 2014

Abstract

This note relies heavily on arXiv:1404.6509 and arXiv:1410.7693. Both articles discuss domino tilings of three-dimensional regions, and both are concerned with flips, the local move performed by removing two parallel dominoes and placing them back in the only other possible position. In the second article, an integer Tw(t) is defined for any tiling t of a large class of regions R: it turns out that Tw(t) is invariant by flips. In the first article, a more complicated polynomial invariant P_t(q) is introduced for tilings of two-story regions. It turns out that Tw(t) = P_t′(1) whenever t is a tiling of a duplex region, a special kind of two-story region for which both invariants are defined. This identity is proved in arXiv:1410.7693 in an indirect and nonconstructive manner. In the present note, we provide an alternative, more direct proof.

1 Introduction

We assume that the reader is familiar to the notations of [2, 1]. In particular, a multiplex region with axis \(\vec{k}\) (or \(\vec{k}\)-multiplex) is a region of the form \(D + [0, N]\vec{k}\), where \(D \subset \mathbb{R}^2 \times \{0\}\) is simply connected and has connected interior; if \(N = 2\), the region is a \((\vec{k})\)-duplex region. The twist Tw(t) is an integer defined below for tilings of a multiplex, and the polynomial \(P_t(q)\) is defined for tilings of a duplex region via the formula (1) below.

Recall that given \(\vec{u} \in \Phi = \{\pm \vec{i}, \pm \vec{j}, \pm \vec{k}\}\), we define the \(\vec{u}\)-pretwist of a tiling \(t\) of a region \(R\) as \(T^{\vec{u}}(t) = \sum_{d_0,d_1 \in t} \tau^{\vec{u}}(d_0,d_1)\), where \(\tau^{\vec{u}}\) denotes the effect along \(\vec{u}\), or

\[
\tau^{\vec{u}}(d_0,d_1) = \begin{cases} 
\frac{1}{4} \det(\vec{v}(d_1), \vec{v}(d_0), \vec{u}), & d_1 \cap S^{\vec{u}}(d_0) \neq \emptyset \\
0, & \text{otherwise.}
\end{cases}
\]

Here \(S^{\vec{u}}(d_0)\) denotes the open \(\vec{u}\)-shade of \(d_0\), as defined in Section 3 of [1], and \(\vec{v}(d) \in \Phi\) denotes the center of the black cube contained in \(d\) minus the center of the white one.
Proposition 1. If $\mathcal{R}$ is a duplex region, then, for any tiling $t$ of $\mathcal{R}$,

$$P'_t(1) = T^i(t) = T^j(t) = T^k(t).$$

The equality $T^i(t) = T^j(t) = T^k(t)$ above is a special case of Proposition 3.3 in [1], and this value is, by definition, the twist $\text{Tw}(t)$. In this note, we give an independent proof of this equality in the particular case of duplex regions. In the aforementioned article, we present a different, shorter proof of the equality $P'_t(1) = \text{Tw}(t)$ using the connectivity of the space of domino tilings of a duplex region by flips and trits. The proof here presented is longer, but more direct.

The authors gratefully acknowledge the support from FAPERJ, CNPq and CAPES.

2 Socks and winding numbers

Let $\mathcal{R}$ be a duplex region. Consider the undirected plane graph $G$ whose vertex set is

$$\left\{ ([x], [y]) : (x^*, y^*, z^*) \text{ is the center of a cube in } R \right\},$$

and where two vertices are joined by an edge if their Euclidean distance is 1. A system of cycles, or sock, in $G$ is a directed subgraph of $G$ consisting solely of oriented simple cycles. A jewel of a sock is a vertex of $G$ that is not contained in the sock. A vertex $v = (x, y) \in \mathbb{Z}^2$ is called white (resp. black) if $x + y$ is even (resp. odd). We set $\text{color}(v) = 1$ if $v$ is black, and $-1$ if it is white.

Each tiling $t$ of $\mathcal{R}$ has a unique corresponding sock in $G$, where trivial cycles in the associated drawing of $t$ are represented as pairs of adjacent jewels: this is illustrated in Figure 1. Each sock may refer to a set of tilings, all in the same flip connected component (therefore, with the same $P'_t(1)$ and the same $\text{Tw}(t)$).

![Figure 1: A tiling with two trivial cycles; the same tiling with the two trivial cycles flipped into jewels; and the sock that corresponds to both of them.](image)

Let $t$ be a tiling of $\mathcal{R}$, and let $s$ be its corresponding sock in $G$. For $p \in \mathbb{R}^2$ and a cycle $\gamma$ of $s$, let $\text{wind}(\gamma, p)$ be the winding number of $\gamma$, thought of as a
curve in $\mathbb{R}^2$, around $p$. Clearly we can write our invariant $P_t(q)$ as

$$P_t(q) = \sum_{v \in \mathbb{Z}^2} \text{color}(v) q^{\sum_{\gamma, v \in \gamma} \text{wind}(\gamma, v)}, \quad (1)$$

where the sum in the exponent of $q$ is taken over all the cycles in $s$ that do not contain $v$.

**Lemma 2.** If $\mathcal{R}$ is a $\vec{k}$-duplex region with associated graph $G$, $\vec{u} \in \{\pm \vec{i}, \pm \vec{j}\}$ and $t$ is a tiling of $\mathcal{R}$ with corresponding sock $s$, then $T^\vec{u}(t) = P_t'(1)$.

**Proof.** Two dominoes that are not parallel to $\vec{k}$ have no effect along $\vec{u}$ on one another. Therefore, we only consider pairs of dominoes where one is parallel to $\vec{k}$, that is, refers to a jewel of $s$.

If $\gamma$ is a cycle of $s$ and $v$ is a jewel, one way of computing $\text{wind}(\gamma, v)$ is to count (with signs) the intersections of $\gamma$ with the half-line $v + [0, \infty)\vec{u}$. Thus, if $d_v$ denotes the domino containing $v$ and $d \in \gamma$ means that $d$ refers to an edge of $\gamma$, then $\text{color}(v) \text{wind}(\gamma, v) = 2 \sum_{d \in \gamma} \tau^\vec{u}(d, d_v) = 2 \sum_{d \in \gamma} \tau^\vec{u}(d_v, d)$. Thus,

$$P_t'(1) = \sum_{\gamma, v} \text{color}(v) \text{wind}(\gamma, v) = \sum_{d \in \gamma} (\tau^\vec{u}(d, d_v) + \tau^\vec{u}(d_v, d)) = T^\vec{u}(t),$$

completing the proof. \qed

**3 Charges and weights**

We now consider $T^\vec{k}$. Again, let $t$ be a tiling of a duplex region with corresponding sock $s$. Let the *charge enclosed* by a cycle $\gamma$ of $s$ be

$$\text{charge}_{\text{int}}(\gamma) = \sum_{v \notin \gamma} \text{color}(v) \text{wind}(\gamma, v),$$

so that $P_t'(1) = \sum_{\gamma \text{ cycle of } s} \text{charge}_{\text{int}}(\gamma)$. Charges can be looked at from a point of view that is more interesting for our purposes. Given $v \in \mathbb{R}^2$, consider the set of four points $\mathcal{N}_v = \{v + (\frac{k}{2}, \frac{l}{2})|k, l \in \{-1, 1\}\}$, i.e., the set of points of the form $v + (\pm \frac{1}{2}, \pm \frac{1}{2})$. The *metric weight* of a vertex $v \in \mathbb{Z}^2$ with respect to a cycle $\gamma$ of $s$ is given by

$$w_{\text{metric}}(\gamma, v) = \frac{1}{4} \sum_{u \in \mathcal{N}_v} \text{wind}(\gamma, u),$$

while the *topological weight* $w_{\text{top}}(\gamma, v)$ of $v$ is the (arithmetic) average of the set $\text{wind}(\gamma, \mathcal{N}_v) = \{\text{wind}(\gamma, u)|u \in \mathcal{N}_v\}$ (see Figure 2).
Lemma 3.

\[
\text{charge}_{\text{int}}(\gamma) = \sum_{v \in \mathbb{Z}^2} \text{color}(v) w_{\text{top}}(\gamma, v).
\]

Proof. Notice that

\[
w_{\text{top}}(\gamma, v) = \begin{cases} 
\text{wind}(\gamma, v), & \text{if } v \notin \gamma, \\
\frac{1}{2}, & \text{if } v \in \gamma \text{ and } \gamma \text{ is counterclockwise oriented}, \\
-\frac{1}{2}, & \text{if } v \in \gamma \text{ and } \gamma \text{ is clockwise oriented}.
\end{cases}
\]

In particular, \(\sum_{v \in \gamma} w_{\text{top}}(\gamma, v) \text{color}(v) = \pm \frac{1}{2} \sum_{v \in \gamma} \text{color}(v) = 0\). Hence,

\[
\text{charge}_{\text{int}}(\gamma) = \sum_{v \in \mathbb{Z}^2} \text{color}(v) w_{\text{top}}(\gamma, v).
\]

If \(s\) is the corresponding sock of a tiling \(t\) and \(\gamma\) is a cycle of \(s\), then the angle \(\angle(\gamma, v)\) of a vertex \(v \in \gamma\) is the difference between the angle of the edge of \(\gamma\) leaving \(v\) and the angle of the edge of \(\gamma\) entering \(v\), counted in counterclockwise laps. In other words, a vertex \(v\) where the curve goes straight has angle 0, whereas a vertex where a left (resp. right) turn occurs has angle \(1/4\) (resp. \(-1/4\)), as shown in Figure 3.

Figure 3: Illustration of the angle of a vertex.
The boundary charge of a curve $\gamma$ is $\text{charge}_\partial(\gamma) = \sum_{v \in \gamma} \angle(\gamma, v) \color(v)$. Notice that

$$T^\partial(t) = \sum_{\gamma \text{ cycle of } s} \text{charge}_\partial(\gamma).$$

(2)

We now set out to prove that, for each $\gamma$, $\text{charge}_\partial(\gamma) = \text{charge}_{\text{int}}(\gamma)$, which will complete the proof of Proposition 1.

**Lemma 4.** For each cycle $\gamma$ of $s$,

$$\sum_{v \in \mathbb{Z}^2} w_{\text{metric}}(\gamma, v) \color(v) = 0.$$

**Proof.**

$$\sum_{v \in \mathbb{Z}^2} w_{\text{metric}}(\gamma, v) \color(v) = \sum_{u \in \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})} \left( \frac{1}{4} \text{wind}(\gamma, u) \sum_{v \in N_u} \color(v) \right) = 0.$$

\[\square\]

**Lemma 5.** If $v \in \mathbb{Z}^2$,

$$w_{\text{top}}(\gamma, v) - w_{\text{metric}}(\gamma, v) = \begin{cases} \angle(\gamma, v), & \text{if } v \in \gamma, \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** Figure 2 illustrates most of the elements needed in this proof. If $v \notin \gamma$, then $\text{wind}(\gamma, u) = \text{wind}(\gamma, v)$ for each $u \in \mathcal{N}_v$, so $w_{\text{metric}}(\gamma, v) = w_{\text{top}}(\gamma, v) = \text{wind}(\gamma, v)$.

If $v \in \gamma$ and the curve goes straight at $v$, then, for some $k$, two points in $\mathcal{N}_v$ have winding number $k$ and the other two have winding number $k + 1$, so $w_{\text{metric}}(\gamma, v) = \frac{1}{4}(k + k + (k + 1) + (k + 1)) = \frac{1}{2}(k + (k + 1)) = w_{\text{top}}(\gamma, v)$.

If $v \in \gamma$ and the curves turns left at $v$, then the winding numbers are $k, k, k, k + 1$, so $w_{\text{top}}(\gamma, v) - w_{\text{metric}}(\gamma, v) = 1/4 = \angle(\gamma, v)$. Analogously, $w_{\text{top}}(\gamma, v) - w_{\text{metric}}(\gamma, v) = -1/4 = \angle(\gamma, v)$ if $\gamma$ turns right at $v$. \[\square\]

**Lemma 6.** For each cycle $\gamma$ of $s$, $\text{charge}_\partial(\gamma) = \text{charge}_{\text{int}}(\gamma)$.

**Proof.** By Lemma 5, the boundary charge of $\gamma$ can also be written as

$$\text{charge}_\partial(\gamma) = \sum_{v \in \gamma} \angle(\gamma, v) \color(v)$$

$$= \sum_{v \in \mathbb{Z}^2} (w_{\text{top}}(\gamma, v) - w_{\text{metric}}(\gamma, v)) \color(v)$$

$$= \sum_{v \in \mathbb{Z}^2} w_{\text{top}}(\gamma, v) \color(v) - \sum_{v \in \mathbb{Z}^2} w_{\text{metric}}(\gamma, v) \color(v)$$

$$= \sum_{v \in \mathbb{Z}^2} w_{\text{top}}(\gamma, v) \color(v) = \text{charge}_{\text{int}}(\gamma),$$
the fourth equality holding because of Lemma 4; the last equality is Lemma 3.

Proof of Proposition 1. Lemma 6 and Equation (2) imply that $P'_t(1) = T^k(t)$. Together with Lemma 2, this proves the result.

Remark 7. Proposition 1 and Proposition 3.6 in [1] establish Remark 10.1 in [2].

References

[1] Pedro H Milet and Nicolau C Saldanha. Domino tilings of three-dimensional regions: flips, trits and twists. arXiv preprint arXiv:1410.7693, 2014.

[2] Pedro H Milet and Nicolau C Saldanha. Flip invariance for domino tilings of three-dimensional regions with two floors. arXiv preprint arXiv:1404.6509, 2014.

Departamento de Matemática, PUC-Rio
Rua Marquês de São Vicente, 225, Rio de Janeiro, RJ 22451-900, Brazil
milet@mat.puc-rio.br
saldanha@puc-rio.br