Elliptic waveforms for inspiralling compact binaries

Balázs Mikócsi
KFKI Research Institute for Particle and Nuclear Physics, Budapest 114, P.O.Box 49, H-1525 Hungary
E-mail: mikoczi@rmki.kfki.hu

Abstract. The inspiral of supermassive black hole binary systems with high orbital eccentricity are the most promising sources for the gravitational wave observatories. The importance of elliptic gravitational waveforms in various physical scenarios has been emphasized by several authors (Wahlquist 1987, Moreno-Garrido, Buitrago and Mediavilla 1994, Martel and Poisson 1999). Taking into account the eccentricity of the orbit in the total waveform improves the parameter estimation for these sources, as it is shown by the construction and analyzation of the Fisher information matrix. In our work we use the Fourier-Bessel analysis of the Kepler motion and the stationary phase approximation of time-depend waveforms.

1. Introduction
Compact binaries of black holes and neutron stars are important sources of gravitational radiation. The coalescence of black hole binaries, for example, will be detectable by the Laser Interferometer Space Antenna (LISA) (Bender et al 1998). The sensitivity domain for this detector is $10^{-4} - 10^{-1}$ Hz which corresponds to the inspiral of two $(10^4 - 10^7) M_\odot$ black holes. For circular orbits the problem of parameter estimation for LISA has been discussed in (Cutler 1998, Vecchio 2004, Berti et al 2005, Lang and Hughes 2006, Trias and Sintes 2008).

In their work Peters and Mathews have shown that the orbit of the binary are circularizing due to gravitational radiation (Peters and Mathews 1963, Peters 1964). Several authors suppose circular orbits in the gravitational wave computations. However, this circularization does not always take place. In the presence of interstellar gas the binary system can remain eccentric until the final coalescence (Armitage and Narayan 2005). N-body simulations are also confirm the relevance of the eccentricity (Preto et al 2009). Coalescing supermassive black holes in galactic nuclei with eccentricity emit detectable wave signal significantly different from those of circular orbits (O’Leary et al 2008).

The elliptic waveform is simply derived from Einstein’s quadrupole formula (Thorne 1980). The first explicit elliptic waveform appears in (Wahlquist 1987, Martel and Poisson 1999) with true anomaly. Some authors have derived the waveform in time domain (Moreno-Garrido et al 1994 and 1995, Pierro et al 2001). The parameter estimation for eccentric binaries is discussed by (Seto 2001, Barack and Cutler 2004, Yagi and Tanaka 2009).

In this paper we demonstrate that the eccentricity of orbit is important in some physically relevant situations. We use elliptic waveforms for parameter estimation of $(10^6 - 10^6) M_\odot$ massive black hole binaries and point out the differences between circular and elliptic orbits. We introduce geometric units $G = c = 1$ and the $I, II$ notations denote two independent detectors of LISA.
2. Waveform of the eccentric binary

Consider a system of two point particles in an Keplerian elliptic orbit. The parametrization of the radial motion is

\[ r = a(1 - e^2)/(1 + e \cos \phi) \]  

(1)

with the semimajor axis \( a \), the eccentricity \( e \) and the true anomaly \( \phi \) of the Keplerian orbit. According to the quadrupole formula of the gravitational radiation the two polarization states in the transverse-traceless (TT) gauge are written as (Wahlquist 1987):

\[ h_\times(\phi) = -\frac{\mu m \cos \Theta}{a(1-e^2)D_L} \left[ (5e \sin \phi + 4 \sin 2\phi + e \sin 3\phi) \cos 2\gamma \right. \]

\[ \left. - \left( 5e \cos \phi + 4 \cos 2\phi + e \cos 3\phi + 2e^2 \right) \sin 2\phi \right], \]

(2)

\[ h_+(\phi) = -\frac{\mu m}{a(1-e^2)D_L} \left[ \left( \frac{5e}{2} \cos \phi + 2 \cos 2\phi + \frac{e}{2} \cos 3\phi + e^2 \right) \cos 2\gamma \right. \]

\[ \left. + \left( \frac{5e}{2} \sin \phi + 2 \sin 2\phi + \frac{e}{2} \sin 3\phi \right) \sin 2\phi \right] \sin 2\phi + \left( e \cos \phi + e^2 \right) \frac{\sin^2 \Theta}{1 + \cos^2 \Theta} \right], \]

(3)

where \( D_L \) is the luminosity distance, \( \Theta \) is the direction of the angular momentum (inclination), \( \gamma \) is the angle of the periastron position, \( m = m_1 + m_2 \) is the total mass and \( \mu = m_1 m_2 / m \) is the reduced mass of the system. The time dependence can be computed with the use of the Fourier-Bessel decomposition. From Eqs. (2) and (3) the polarization modes can be derived (Eqs. (17) and (18) of (Moreno-Garriodo et al 1995))

\[ h_\times(t) = -h \cos \Theta \sum_n \left[ \frac{S_n - C_n}{2} \sin(nl + 2\gamma) + \frac{S_n + C_n}{2} \sin(nl - 2\gamma) \right], \]

(4)

\[ h_+(t) = -\frac{h}{2} \sum_n \left[ \sin^2 \Theta A_n \cos nl + (1 + \cos^2 \Theta) \right. \]

\[ \left. \times \left( -\frac{S_n - C_n}{2} \cos(nl + 2\gamma) + \frac{S_n + C_n}{2} \cos(nl - 2\gamma) \right) \right], \]

(5)

with \( h = 4\mu m/aD_L \) and

\[ S_n = -\frac{2(1-e^2)^{1/2}}{e} J'_n(ne) + \frac{2(1-e^2)^{3/2}}{e^2} n J_n(ne), \]

(6)

\[ C_n = -\frac{2-e^2}{e^2} J_n(ne) + \frac{2(1-e^2)}{e} J'_n(ne), \quad A_n = J_n(ne), \]

(7)

here \( J'_n(ne) = n[J_{n-1}(ne) - J_{n+1}(ne)]/2 \) and \( l \) is the mean anomaly defined by the Kepler equation

\[ l = \xi - e \sin \xi = 2\pi \nu (t - \tau), \]

(8)

where \( \xi \) is the eccentric anomaly and \( \nu \) is the Keplerian orbital frequency (thereafter we set \( \tau = 0 \)). In the circular limit (\( e \rightarrow 0 \)) the elliptic waveforms (2, 3) reduce to

\[ h_\times'(t) = -h_c \cos \Theta \sin \psi, \quad h_+'(t) = \frac{h_c}{2} (1 + \cos^2 \Theta) \cos \psi, \]

(9)

with the phase of the circular orbit \( \psi = 2\pi \int (f/\dot{f}) df \) and \( h_c = 4\mu (m \pi f)^{2/3} / D_L \) (using the expressions \((2\pi \nu)^3 a^3 = m \) (Kepler’s third law) and \( f = 2\nu \) (Peters and Mathews 1963)).
Figure 1. Distribution of the angular resolution $\Delta \Omega_3$ for circular (left) and eccentric (right) waveforms. The results are valid for 1-year observation of $(10^9 - 10^5) M_\odot$ black hole binaries at $z = 1$ ($D_L = 6.4\text{Gpc}$) by LISA. We have used initial position according to (Cutler 1998) and neglected the Doppler phase. The initial eccentricity is $e_0 = 0.1$ in the eccentric case (the evolution of the eccentricity due to gravitation radiation was given in (Peters 1964) to the leading order as $e(\nu) = e_0 (\nu/\nu_0)^{-10/18}$, where $\nu_0$ is the initial frequency, the frequency range is the same in both cases). The histograms show the results of two Monte Carlo simulations with $10^3$ sources randomly located in the sky.

3. Fourier transformation of the waveform

The frequency dependence of the waveform can be computed with its Fourier transformation in the stationary phase approximation (SPA) (Moreno-Garriodo et al 1994): if $g(t)$ is an arbitrary function of time, then

$$
\int_{-\infty}^{\infty} h[\nu(t)] e^{i \nu t} dt = \sqrt{\frac{2\pi}{g(t_0)}} h(\nu) e^{i [g(t_0) + \frac{\pi}{4}]} ,
$$

where $\nu = \nu (t_0)$ is the time of the sampling point at the given frequency. We ignore the periastron advance ($\dot{\gamma} = 0$ in Eqs. (4) and (5)) and the Doppler phase. In this case the polarization modes are ($\gamma = 0$):

$$
h_\times (\nu)_n = -\frac{\mathcal{M}^{5/6} (2\nu)^{-7/6}}{\pi^{2/3} D_L} k(n, e) \cos \Theta S_n e^{i \Phi_+} ,
$$

$$
h_+ (\nu)_n = -\frac{\mathcal{M}^{5/6} (2\nu)^{-7/6}}{2\pi^{2/3} D_L} k(n, e) \left[ \sin^2 \Theta A_n + (1 + \cos^2 \Theta) C_n \right] e^{i \Phi_-} ,
$$

where $\mathcal{M} = \mu^{3/5} m^{2/5}$ is the chirp mass, $\Psi_\pm = 2\pi n t(\nu) - \Phi(\nu) \pm \pi/4$ (where $t(\nu)$ and $\Phi(\nu)$ can be computed from the evolution of orbital frequency due to gravitational radiation (Yunes et al 2009)) is the “eccentric phasing function” (for circular orbits the phasing is $\Psi = 2\pi f t_c - \phi_c - \pi/4 + 3/4 (8\pi \mathcal{M} f_c)^{-5/3}$, where $t_c$ and $\phi_c$ is the coalescing time and phase) and

$$
k(n, e) = \sqrt{\frac{5(1 - e^2)^{7/2}}{48n (1 + 34 e^2 + 45 e^4)}} .
$$

Then the frequency domain waveforms detected by the I. and II. LISA detectors are

$$
h^{I,II}_I (\nu) = \sqrt{3} \sum_n \left[ F^{I,II}_\times h_\times (\nu)_n + F^{I,II}_+ h_+ (\nu)_n \right] ,
$$

where the antenna-beam pattern for the LISA was given in (Cutler 1998). The $F^{I,II}_\times$ antenna-beam pattern are rather complicated and time dependent.
4. Parameter estimation
The definition of the Fisher information matrix is

\[
\Gamma_{ab} = 2 \sum_{I,II} \max_{\nu_{\min}} \int \frac{\partial \tilde{h}^{I,I}(\nu)}{\partial \lambda_a} \frac{\partial \tilde{h}^{I,I}(\nu)}{\partial \lambda_b} S_h(\nu) d\nu ,
\]

where \( \tilde{h} \) is the complex conjugate of \( h \), \( S_h(\nu) \) is the noise spectra of the detector (we used the LISA sensitivity data file) and \( \lambda_a \) are the parameters of the waveform. The correlation matrix \( \Sigma \), which is the inverse of the Fisher matrix, \( \Sigma = (\Gamma)^{-1} \), defines the parameter errors: \( \Delta \lambda_a = \sqrt{\Sigma_{aa}}. \)

The error of the angular resolution is

\[
\Delta \Omega_S = 2\pi \sqrt{(\Delta \mu_S \Delta \phi_S)^2 - \langle \delta \mu_S \delta \phi_S \rangle^2} ,
\]

with \( \mu_S = \cos \theta_S \), \( \langle \delta \lambda_a \delta \lambda_b \rangle = \Sigma_{ab} \) and the location of the binary source \( (\mu_S, \phi_S) \). For the estimation of the parameter errors we have considered \( (10^6 - 10^8) M_\odot \) black holes binaries with \( e_0 = 0.1 \). Our frequency domain results are in agreement with those in (Cutler 1998) (1-year observation up to the innermost stable circular orbit). We have used 9 parameters in the elliptic case: \( M, D_L, t_c, \phi_c, \mu_S, \phi_S, \mu_L, \phi_L \) and \( e_0 \).

5. Summary
In this work we have analyzed eccentric waveforms of Keplerian orbits. During the calculations the periastron advance and the Doppler phase modulation in the phasing were neglected. We have presented how the results of parameter estimation change due to the eccentricity by computing the angular resolutions for \( 10^3 \) binaries with Monte Carlo simulation (Fig. 1).

6. Acknowledgment
This work was supported by the Hungarian Scientific Research Fund (OTKA) grant No. NI68228. I would like to thank Mátéyás Vasúth and Péter Forgács for their help during my work.

7. References
Armitage P J, Natarajan P 2005 Astrophys. J. 632 921
Barack L and Cutler C 2004 Phys. Rev. D 69 082005
Bender P L et al. 1998 LISA Pre-Phase A Report 2nd ed. See http://lisa.jpl.nasa.gov
Berti E, Buonanno A, and Will C M 2005 Phys. Rev. D 71 084025
Cutler C 1998 Phys. Rev. D 57 7089
Lang R and Hughes S A 2006 Phys. Rev. D 74 122001
Martel K and Poisson E 1999 Phys. Rev. D 60 124008
Moreno-Garrido C, Mediavilla E and Buitrago J 1995 Mon. Not. R. Astron. Soc. 274 115
Moreno-Garrido C, Buitrago J, and Mediavilla E 1994 Mon. Not. R. Astron. Soc. 266 16
O’Leary R M, Kocsis B and Loeb A 2008 Mon. Not. R. Astron. Soc. 395 2127
Peters P C and Mathews S 1963 Phys. Rev. 131 435
Peters P C 1964 Phys. Rev. 136 1224
Pierro V, Pinto I M, Spallucci E D, Laserra E and Recano F 2001 Mon. Not. R. Astron. Soc. 325 358
Preto M, Berentzen I, Berczik P, Merritt D and Spurzem R 2009 J. Phys.:Conf. Ser. 154 012049
Seto N 2001 Phys. Rev. Lett. 87 251101
Thorne K S 1980 Rev. Mod. Phys. 52 299
Trias M and Sintes A M 2008 Phys. Rev. D 77 024030
Yagi K and Tanaka T 2009 Constraining alternative theories of gravity by gravitational waves from precessing eccentric compact binaries with LISA gr-qc/0906.4269
Yunes N, Arun K G, Berti E and Will C M 2009 Phys. Rev. D 80 084001
Vecchio A 2004 Phys. Rev. D 70 042001
Wahlquist H 1987 Gen. Rel. Grav. 19 1101
http://www.srl.caltech.edu/~shane/sensitivity/