Strongly disordered spin ladders

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Abstract

The effect of quenched disorder on the low-energy properties of various antiferromagnetic spin ladder models is studied by a numerical strong disorder renormalization group method and by density matrix renormalization. For strong enough disorder the originally gapped phases with finite topological or dimer order become gapless. In these quantum Griffiths phases the scaling of the energy, as well as the singularities in the dynamical quantities are characterized by a finite dynamical exponent, $z$, which varies with the strength of disorder. At the phase boundaries, separating topologically distinct Griffiths phases the singular behavior of the disordered ladders is generally controlled by an infinite randomness fixed point.

I. INTRODUCTION

Low dimensional quantum spin systems, chains and ladders are fascinating objects, which are the subject of intensive experimental and theoretical research. The main source of this activity is due to the observation that quantum fluctuations could result in qualitatively different low-energy behavior in these interacting many-body systems. It was first Haldane\textsuperscript{1} who conjectured that antiferromagnetic (AF) spin chains with integer spin have a gap in the energy spectrum (Haldane phase), whereas the spectrum of chains with half-integer spins is gapless. By now a large amount of experimental and theoretical evidence have been collected in favor of the Haldane conjecture. It has been realized by Affleck, Kennedy, Lieb and Tasaki\textsuperscript{2} (AKLT) that the ground state structure of the Haldane phase for $S=1$ is closely related to that of the valence-bond solid model, where the ground state is built up from nearest-neighbor valence bonds. The hidden, topological order in the chain is measured by the non-local string order parameter

$$O^\alpha = - \lim_{|i-j| \to \infty} \left \langle S_i^\alpha \exp \left ( i \pi \sum_{l=i+1}^{j-1} S_l^x \right ) S_j^\alpha \right \rangle, \quad (1.1)$$

where $S_i^\alpha$ is a spin-1 operator at site $i$, $\alpha = x, y, z$ and $\langle \ldots \rangle$ denotes the ground state expectation value.

Another source of activity in the field of low-dimensional quantum spin systems is due to the discovery of spin ladder materials\textsuperscript{3}. It has been realized that spin ladders with even number of legs have a gapped spectrum, whereas the spectrum of odd-leg ladders is gapless. For two-leg ladders, which are analogous objects to $S=1$ spin chains, the ground state structure can be related to nearest-neighbor valence bonds and a topological hidden order parameter, similar to that in Eq.(1.1) can be defined\textsuperscript{4}.

More recently, ladder models with competing interactions, such as with staggered dimerization\textsuperscript{5} and with rung and diagonal couplings\textsuperscript{6}, have been introduced and studied. In these models, depending on the relative strength of the couplings, there are several gapped phases with different topological order, which are separated by first-or second-order phase transition lines.

Disorder turns out to play a crucial role in some experiments on low dimensional magnets. For instance, the NMP-TCNQ compound\textsuperscript{1} can be well described by $S=1/2$ spin chains with random AF couplings. More recently, non magnetic substitutions in low dimensional oxides such as Cu$_2$O$_2$(CH$_3$)$_2$COO$^-$ (being a spin-Peierls compound), PbNi$_2$O$_4$ (being a Haldane gap compound) or Y$_2$BaNiO$_{5+\delta}$ (being a Haldane gap compound) have been the subject of intense investigations. The essential feature of these compounds is the appearance of antiferromagnetism at low temperature which can be well described by the effective low energy models introduced in Refs. \textsuperscript{19,20}. Sr(Cu$_{1-x}$Zn$_x$)$_2$O$_3$ is a realization of the two-leg ladder, and can be doped by Zn, a non magnetic ion\textsuperscript{20}. The specific heat and spin susceptibility experiments indicate that the doped system is gapless even with low doping concentrations. We note that the experimentally found phase diagram of this compound, as well as other quantities, such as staggered susceptibility have been obtained by quantum Monte Carlo simulations\textsuperscript{20}.

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Theoretically, spin chains in the presence of strong disorder can be conveniently studied by a real-space renormalization group (RG) method introduced by Ma, Dasgupta and H. D. Gupta (MDH). In this method strong bonds in the system are successively eliminated and other bonds are replaced by weaker ones through a second order perturbation calculation. As realized later by Fisher for the random spin-1/2 chain and for the related model of random transverse-field Ising spin chains, the probability distribution of the couplings under renormalization becomes broader and broader without limit and therefore the system scales into an infinite randomness fixed point (IRFP), where the MDH renormalization becomes asymptotically exact. Fisher has also succeeded to solve the fixed-point RG equations in analytical form and to show, that for any type of (non-extremely singular) initial disorder the system scales into the same IRFP. Later numerical and analytical work has confirmed Fisher’s results.

Generalization of the MDH approach for AF chains with larger values of the spin is not straightforward, since for not too strong initial disorder the generated new couplings could exceed the value of the already decimated ones. To handle this problem for the $S = 1$ chain Hyman and Yang and independently Monthus, Golinelli and Jolicoeur have introduced an effective model with spin-1 and spin-1/2 degrees of freedom and with random AF and ferromagnetic (F) couplings. From an analysis of the RG equations they arrived to the conclusion that the IRFP of the model will be attractive if the original distribution parametrized by the power-law form

$$
P_{\text{pow}}(J) = \frac{1}{D} J^{-1+1/D}, \quad (1.2)
$$

is strongly random, i.e. if $0 < D^{-1} < D_1^{-1}$. For weaker initial disorder the system is still gapless, which is called the gapless Haldane phase.

Theoretical work about disordered spin ladders is mainly concentrated on the weak disorder limit. Results in this direction are obtained in the weak interchain coupling limit via the bosonization approach and by the random mass Dirac fermion method. In particular a remarkable stability of the phases of the pure system against disorder with XY symmetry has been observed.

In the experimental situation, however, as described before the effect of disorder is usually strong and we are going to consider this limiting case in this paper. Our aim is to provide a general theoretical background for strongly disordered spin ladders by studying in detail several models (conventional ladder, dimerized ladder, zig-zag ladder, and the full ladder with rung and diagonal couplings), which could have experimental relevance. Since often a small change in the couplings or in the strength of disorder could cause large differences in the low-energy singular properties of the models, therefore we have studied the phase diagrams in the space of several parameters. As a method of calculation we used a numerical implementation of the MDH approach, which could treat the combined effect of disorder, frustration, correlations and quantum fluctuations, while some problems are also studied by density matrix renormalization (DMRG). In particular we have investigated the stability of the different topologically ordered phases and studied the region of attraction of the IRFP.

The structure of the paper is the following. In Sec. 2 we define different spin ladder models and present their phase diagram for non-random couplings. A short overview about the MDH RG method and its application to random spin chains are given in Sec. 3. Our results about random spin ladders are presented in Sec. 4 and discussed in Sec. 5.

II. THE MODELS AND THEIR PHASE DIAGRAM FOR NON-RANDOM COUPLINGS

We start with two spin-1/2 Heisenberg chains, labeled by $\tau = 1, 2$ and described by the Hamiltonians

$$
H_\tau = \sum_{l=1}^L J_{l,\tau} S_{l,\tau} S_{l+1,\tau}, \quad (2.1)
$$

where $S_{l,\tau}$ is a spin-1/2 operator at site $l$ and on chain $\tau$ and $J_{l,\tau} > 0$. For non-random spin chains dimerization can be introduced as

$$
J_{l,\tau} = J \left[ 1 + \gamma (-1)^{l+n(\tau)} \right], \quad 0 \leq \gamma < 1, \quad (2.2)
$$

with $n(\tau) = 0, 1$, whereas for random dimerized couplings the even and odd bonds are taken from different distribution. The pure chain without dimerization ($\gamma = 0$) has a gapless spectrum, and spin-spin correlations decay as a power for large distance, which is called as quasi-long-range-order (QLRO). Introducing dimerization for $\gamma > 0$ a gap opens in the spectrum, which is accompanied by non-vanishing dimer order, $O_{\text{dim}}^\alpha \neq 0$. This is measured as the difference between the string order parameters in Eq. (2.1) calculated with spin-1/2 moments at even (e) and odd (o) sites:

$$
O_{\text{dim}}^\alpha = O_{\text{e}}^\alpha - O_{\text{o}}^\alpha. \quad (2.3)
$$

In the following we generally consider non-dimerized chains, otherwise it is explicitly mentioned.

Now we introduce the interchain interaction

$$
H_R = \sum_{l=1}^L J_R^l S_{l,1} S_{l,2}, \quad (2.4)
$$

which describes the usual rung coupling between the ladders (see Fig. 3a). The conventional ladder model is described by the Hamiltonian: $H = H_1 + H_2 + H_R$. In the pure model by switching on the AF rung couplings, $J_R^l = J_R > 0$, a Haldane-type gap opens above the ground state and the system has a non-vanishing even string topological order, which is measured by $H_R$.
\[ O_{\text{even}}^{\alpha} = - \lim_{|i-j| \to \infty} \left( (S_{i+1,1}^{\alpha} + S_{i+1,2}^{\alpha}) \exp \left( i\pi \sum_{l=i+1}^{j-1} (S_{l+1,1}^{\alpha} + S_{l+1,2}^{\alpha}) \right) (S_{j+1,1}^{\alpha} + S_{j+1,2}^{\alpha}) \right). \] (2.5)

For strong AF rung couplings every spin-pair on the same rung form a singlet, therefore this phase is called the rung singlet (RUS) phase.

FIG. 2. Schematic phase diagram of the two-leg AF ladder with staggered dimerization (see Fig. 1b for the definition of the couplings.) At the phase boundary between the rung singlet and dimer phases the gap vanishes.

Next, we extend our model by diagonal interchain couplings, given by the Hamiltonian term:

\[ H_Z = \sum_{l=1}^{L} J_l^Z S_{l,2} S_{l+1,1}. \] (2.6)

The complete Hamiltonian, \( H = H_1 + H_2 + H_R + H_Z \), describes a zig-zag ladder (see Fig. 1c) or can be considered as a spin chain with nearest neighbor (\( J^R_l, J^Z_l \)) and next-nearest neighbor (\( J_l \)) couplings (Fig. 1d). The pure model with \( J^R_l = J^Z_l = J_1 \) and \( J_1 = J_2 \) has two phases: a gapless phase for \( J_2/J_1 < 0.24 \) is separated from a gapped phase by a quantum phase transition point.

Finally, we extend our model by two types of diagonal couplings, which are represented by the Hamiltonian:

\[ H_D = \sum_{l=1}^{L} J_l^D (S_{l,1} S_{l+1,2} + S_{l,2} S_{l+1,1}). \] (2.7)

It is known that the pure AF diagonal ladder described by the Hamiltonian, \( H = H_1 + H_2 + H_D \) with \( J_l^D = J_D > 0 \) has a gapped spectrum 6. Its ground state is of the AKLT-type and has a non-vanishing odd string order, defined in analogy to Eq. (2.5),

\[ O^{\alpha}_{\text{odd}} = - \lim_{|i-j| \to \infty} \left( (S_{i,1}^{\alpha} + S_{i,2}^{\alpha}) \exp \left( i\pi \sum_{l=i+1}^{j-1} (S_{l,1}^{\alpha} + S_{l,2}^{\alpha}) \right) (S_{j,1}^{\alpha} + S_{j,2}^{\alpha}) \right). \] (2.8)
In the full ladder there are both rung and diagonal couplings (see Fig. 1.e) and it is described by the Hamiltonian
\[ H = H_1 + H_2 + H_R + H_D. \]
For non-random AF couplings there is a competition between rung and diagonal couplings, so that the ground state phase diagram of the system consists of two topologically distinct gapped phases (see Fig. 3). The phase transition between the two phases is of first order.

In general, the phase diagrams of the pure ladder models, in particular in Figs. 2 and 3, are modified due to the presence of quenched disorder.

**III. THE MDH RENORMALIZATION: RESULTS FOR SPIN CHAINS**

In the MDH renormalization group method for random spin-1/2 chains the random AF bonds are arranged in descending order according to their strength, and the strongest bond, say \( J_{23}, \) connecting sites 2 and 3 sets the energy scale in the problem, \( \Omega = J_{23}. \) We denote by 1 the nearest neighbor site to 2 with a connecting bond \( J_{12}, \) and similarly denote by 4 the nearest neighbor site to 3 with a connecting bond \( J_{43}. \) If \( J_{23} \) is much larger than the connecting bonds the spin pair \((2, 3)\) acts as an effective singlet. It follows that the strongly correlated singlet pair can be frozen out. Due to the virtual triplet excitations, an effective coupling \( \tilde{J}_{14} \) is generated between the sites 1 and 4. These two sites become nearest neighbors once the singlet has been eliminated. In a second-order perturbation calculation one obtains
\[ \tilde{J}_{14} = \kappa \frac{J_{12} J_{13}}{\Omega}, \quad \kappa(S = 1/2) = 1/2. \] (3.1)

The new coupling is thus smaller than any of the original ones. The energy scale \( \Omega \) is continuously reduced upon iterating the procedure and at the same time the probability distribution of the couplings \( P(J, \Omega) \) approaches a limiting function. In a gapless random system \( \Omega \) tends to zero at the fixed point of the transformation and the low energy tail of the distribution is typically given by:
\[ P(J, \Omega) dJ \sim \frac{1}{z} \left( \frac{J}{\Omega} \right)^{-1+1/z} \frac{dJ}{\Omega}. \] (3.2)

The dynamical exponent \( z \) determines how the length scale \( L \) scales with the time scale \( \tau: \)
\[ \tau \sim \Omega^{-1} \sim L^z. \] (3.3)

In general \( z \) is not a universal quantity: its value depends on the form of the original disorder. However \( z \) stays invariant under renormalization \( \frac{d}{d\ln \Omega}. \) Therefore one can deduce its value from the renormalized distribution in Eq. (3.2). Varying the parameters of the initial distribution one can reach a situation where the width of the distribution in Eq. (3.2) grows without limits, i.e. \( z \) formally tends to infinity. In this case, according to exact results on the random AF spin-1/2 chain, one should formally replace \( z \) in Eq. (3.2) by \( -\ln \Omega, \) so that the scaling relation in Eq. (3.3) takes the form:
\[ \ln \tau_r \sim L^\psi, \quad \psi = 1/2. \] (3.4)

This type of fixed point, where the ratio of any two neighboring bonds typically tends to zero or infinity, is called an infinite randomness fixed point (IRFP). It has been conjectured that the MDH renormalization group transformation (3.1) leads to exact results regarding the singular properties of the transformation, namely the value of \( \psi \) in Eq. (3.4) is exact.
For the random AF spin-1/2 chain, according to exact results any amount of disorder is sufficient to drive the system into the IRFP. Similarly, the gapped dimer phase will turn into a gapless random dimer phase for any amount of disorder, where the dimerization parameter is defined as

\[ \delta_{\text{dim}} = [\ln J_{\text{odd}}]_{\text{av}} - [\ln J_{\text{even}}]_{\text{av}} \]  

(3.5)

in terms of the couplings \( J_{\text{odd}} \) and \( J_{\text{even}} \) at odd and even sites, respectively. The random dimer phase is a quantum version of the Griffiths-phase, which has been originally introduced for classical disordered systems. The schematic RG-flow diagram of the random dimerized chain is drawn in Fig. 4.

![Diagram](image)

**FIG. 5.** Schematic RG phase diagram of the random AF spin-1/2 chain, as a function of the disorder strength \( D \) and the energy scale \( \Omega \). For weak disorder, \( D < D_0 \), there is a Haldane ("H") gap in the spectrum. For intermediate disorder, \( D_0 < D < D_1 \), the system is in the gapless Haldane ("GH") phase with a varying dynamical exponent \( z(D) \). For strong enough disorder, \( D > D_1 \), the system is in the random singlet phase and scales into the IRFP.

The MDH method has also been used to study the singular properties of the random AF spin-1 chain. Here we first note that the spectrum of the pure system has a Haldane gap, which is stable against weak randomness. Consequently the MDH renormalization, which is by definition a strong disorder approach, becomes valid if the initial disorder is increased over a limit, say \( D_0 \). Our second remark concerns the “projecting onto the lowest level” procedure when after decimating out a strongly coupled singlet the generated new coupling is in the form of Eq. (3.1), however with a constant of \( \kappa(1/2) = 4/3 \). Consequently at the initial RG steps the energy scale could behave non-monotonically, so that an IRFP behavior is expected only for strong enough initial disorder. To cure this problem a modified RG scheme was proposed, which is based on the principle of “projecting out the highest level”. By this method an effective Hamiltonian with spin-1 and spin-1/2 degrees of freedom has been introduced, where between the spins both AF and F couplings could be present but their distribution should respect some constraints. The RG analysis of this effective model leads to two different types of strongly disordered phases, provided the disorder of the original distribution exceeds the limiting value of \( D_0 \). For an intermediate range of disorder, so that \( D_0 < D < D_1 \) the system scales into a quantum Griffiths-phase, the so-called gapless Haldane phase, where \( z = z(D) \) is a monotonously increasing function of disorder and \( 1/z(D) > 0 \). When the strength of disorder exceeds a second limit, say \( D > D_1 > D_0 \), the dynamical exponent becomes infinite and the singular behavior of the system is controlled by the IRFP (see Fig. 5).

Till now there are no numerical estimates about the limiting disorder strengths, \( D_0 \) and \( D_1 \). For the uniform distribution, which corresponds to \( D = 1 \) in Eq. (1.2), the system is in the gapless Haldane phase with \( z \approx 1 \).

Finally, we mention the work by Westerberg et al. about renormalization of spin-1/2 Heisenberg chains with mixed F and AF couplings. In this problem, due to the presence of strong F bonds under renormalization spin clusters with arbitrary large effective moment \( S_{\text{eff}} \) are generated, such that \( S_{\text{eff}} \sim \Delta^{-\omega} \), where \( \Delta \) is the largest local gap in the system and \( \Delta \to 0 \) at the fixed point. Singularities of different physical quantities are related to the scaling exponent \( \omega \).

**IV. RENORMALIZATION OF AF SPIN LADDERS**

With a ladder geometry, spins are more interconnected than in a chain, which leads to a modification of the decimation procedure described in the previous section. As shown in Fig. 6 both spins of a strongly coupled pair, say (2, 3), are generally connected to the nearest neighbor spins, denoted by 1 and 4. After decimating out the singlet pair the new, effective coupling between 1 and 4 is of the form:

\[ j_{14}^{\text{eff}} = \kappa (J_{12} - J_{13}) (J_{43} - J_{42}) / \Omega, \quad \kappa(S = 1/2) = 1/2, \]  

(4.1)

which should replace Eq. (3.1) obtained in the chain topology, i.e. with \( J_{13} = J_{42} = 0 \). With the rule in Eq. (1.3) F couplings are also generated. As a consequence, the renormalized Hamiltonian contains both AF and F bonds. When at some step of the renormalization an F bond becomes the strongest one, it will lead to the formation of an effective spin-1 cluster. In further RG steps the system renormalizes into a set of effective spin clusters having different moments and connected by both AF and F bonds. The detailed renormalization rules have already been given in Ref. [4].
Due to the ladder topology and the complicated renormalization rules the RG equations cannot be treated analytically and one resorts to numerical implementations of the renormalization procedure. We note that a variant of the MDH renormalization has been successfully applied numerically for the two-dimensional random transverse-field Ising model (RTIM) [6-8] (also for double chains of the RTIM [9]). An IRFP has been obtained both for the 2D RFIM [9] and the double chain RFIM [6].

In practice we use a finite size version of the MDH renormalization, as for the RTIM in Ref. [3]. In this method we start with a finite ladder of $L$ sites and with periodic boundary conditions and perform the decimation procedure until one spin pair with a first gap $\Delta$ remains in the system. Since $\Delta$ plays the role of the energy scale at length scale $L$, $\Delta$ and $L$ should be related by the relation (3.3) involving the dynamical exponent $z$. Performing the above decimation for different samples the probability distribution of $\Delta$ in the small $\Delta$ limit is described by the form in Eq. (3.2), where the energy scale $\Omega$ is replaced by $L^{-z}$.

The IRFP is signalled by a diverging $z$, or more precisely the $P_L(\Delta)d\Delta$ distributions have strong $L$ dependence, so that the appropriate scaling combination is

$$\ln (L^z P_L(\Delta)) \simeq f(L^{-z} \ln \Delta), \quad (4.2)$$

which can be obtained from Eq. (3.2) by formally setting $z \simeq -\ln \Delta \sim L^z$.

In the actual calculations we have considered several 100,000 realizations of random ladders of length up to $L = 512$. Then, from the distribution of the gap at the last step of the RG iteration we have calculated the dynamical exponent $z$. The random couplings were taken from the power-law distribution in Eq. (3.2), where the strength of disorder is measured by the parameter $D$. In the following we present our results for the specific ladder models discussed in Sec. 2.

A. Random conventional ladders

We start with the conventional ladders in Fig. 1a where the couplings along the chains ($J^f$, $\tau = 1, 2$) and the couplings along the rungs ($J^R$) are taken from the same random distributions. In Fig. 6 we show the probability distribution of the gaps at the last step of the RG iteration calculated with the disorder parameter $D = 1$ (see Eq. (1.2)). As seen in the figure the small energy tail of the distribution follows the functional form given by Eq. (3.2) and the dynamical exponent $z$ given by the asymptotic slope of the distributions is finite and has only a very weak size dependence.

![FIG. 6. Singlet formation and decimation in the ladder geometry.](image)

Repeating the calculation for other values of $D$ we obtain a set of $D$-dependent dynamical exponents which are represented on Fig. 7. For strong disorder we obtain $z(D) < D$, which means that disorder is reduced in the course of the renormalization. In the terminology of Motrunich et al. [4], this system is a finite randomness system, as opposed to the infinite randomness systems that will be considered in the next subsections. For weak disorder the dynamical exponent predicted by the approximative MDH renormalization is lowered below one for $D < D_0 \approx 1$. Here we argue that in this region the effect of disorder is irrelevant, so that the system is in the gapped RUS phase. Indeed, in a pure quantum system, where scaling in time and space is isotropic, the dynamical exponent is $z_{\text{pure}} = 1$. Similarly, for disorder induced gapless systems, where disorder in the time direction is strictly correlated, the dynamical exponent can not be smaller, than in the pure system, so that $z_{\text{dis}} \geq z_{\text{pure}} = 1$. Consequently, if the disorder induced dynamical exponent is $z_{\text{dis}} < 1$, then disorder could only influence the correction to scaling behavior, but the system stays gapped. In view of this remark $D_0$ can be
considered as the lower limiting value of the disorder, where the conventional finite randomness behavior ends.

So that the phase diagram of random conventional two-leg spin ladders consists of two phases: a gapped RUS (Haldane) phase and a random gapless Haldane phase. The latter is characterized by a finite dynamical exponent $z(D)$ for any strong but finite initial disorder. Consequently there is an important difference with the random AF spin-1 chain which flows into the IRFP above a finite critical value of randomness (see Fig. [3]).

The random critical point of this system is situated at $J_{R}^{R}/J_{max} = 2$, where the critical behavior is governed by an IRFP, so that the dynamical exponent, $z$, is formally infinity. For any other values of the couplings, $J_{R}^{R}/J_{max}$, the system is in the random dimer phase, where the dynamical exponent is finite and coupling dependent. To see the general tendency of finite-size convergence of the $z$ exponent around the critical point we have repeated the calculation at the $D = 1$, $\gamma = 1$ case for the random $XX$-chain and the numerical finite-size results are compared in Fig. [4] with the exact value of the dynamical exponent, as given by the solution of the equation:

$$J_{R}^{R}/J_{max} = 2 \left( \frac{D^2}{D^2 - z^{-2}} \right)^{-\frac{1}{2}}$$

known from Ref. [37,38]. As seen in Fig. [4] the dynamical exponents of random $XX$- and Heisenberg-ladders have very similar coupling dependence and one expects the same type of divergence at the critical point for all values of $\gamma$. In Fig. [5] we illustrate the scaling behavior far from the maximum the convergence of the data is faster. To decide about the possible limiting value of $z$, in particular at the maximum of the curves, we analyze the behavior for $\gamma = 1$ in Fig. [6], which is just a dimerized random chain, the properties of which are exactly known by some extent.

**B. Random ladders with staggered dimerization**

In this subsection we consider conventional ladders with staggered dimerization having a dimerization parameter, $0 \leq \gamma \leq 1$, in Eq. (2.2). The different type of couplings in the ladder are taken from the power-law distribution in Eq. (1.2), each having the same disorder of couplings in the ladder are taken from the power-law dynamical exponent, $D$, for any strong but finite initial disorder. Consequently there is an important difference with the random AF spin-1 chain which flows into the IRFP above a finite critical value of randomness (see Fig. [3]).

For the rung couplings, $0 < J_{l}^{R} < J_{max}^{R}$ for the rung couplings, $0 < J_{l}^{weak} < (1 - \gamma)J_{max}^{R}$ and $0 < J_{l}^{strong} < (1 + \gamma)J_{max}^{R}$ for the weaker and stronger chain couplings, respectively. For a fixed value of $\gamma$ and $D$ we have calculated the finite-size dependent effective dynamical exponent, $z$, as a function of the coupling ratio, $J_{R}^{R}/J_{max}^{R}$.

As shown in Fig. [3] the effective exponents have the same type of qualitative behavior for different values of the dimerization parameter, $\gamma$. In each case the curves have a maximum at some value of the couplings, where the finite-size dependence is the strongest, whereas more
of the gap at the transition point, i.e. at the maximum of the curves in Fig. 9b.

![FIG. 10. a) Probability distribution of the first gap at the transition point of the random conventional ladder with staggered dimerization, D = 1, γ = 0.5, J_{max}^R/J_{max} = 1.1. The distributions become broader and broader with L, which signals infinite randomness behavior. b) Scaling plot in terms of the scaling combination in Eq. (3.4).](image)

**FIG. 10.** a) Probability distribution of the first gap at the transition point of the random conventional ladder with staggered dimerization, \( D = 1, \gamma = 0.5, J_{max}^R/J_{max} = 1.1 \). The distributions become broader and broader with \( L \), which signals infinite randomness behavior. b) Scaling plot in terms of the scaling combination in Eq. (3.4).

The distributions in Fig. 10a become broader and broader with the size and the effective dynamical exponent is increasing with the size without limits. An appropriate scaling collapse of the gap-distributions has been obtained in Fig. 10b, where the scaling variable in Eq. (3.4), with \( \psi = 1/2 \) is used. Similar type of infinite randomness behavior is observed at other points of the critical lines, with the same exponent \( \psi = 1/2 \), which turned out to be universal. Thus we conclude that the random conventional ladder with staggered dimerization has two Griffiths-type gapless phases, the random dimer phase and the random rung singlet phase, which are separated by a random critical line, along which there is infinite randomness behavior. For different disorder parameter, \( D \), the position of the random critical line is modified, generally stronger disorder is in favor of the random rung singlet phase, see Fig. 11.

We note that the previously studied random conventional ladder is contained as a special point in this phase diagram at \( J_{max}^R/J_{max} = 1 \) and \( \gamma = 0 \). This point is in the random rung singlet phase for any value of \( D \), thus the dynamical exponent is finite in accordance with the previous results.

**C. Random zig-zag ladders**

For the zig-zag ladders the nearest neighbor couplings \( (J_l^R = J_2^l \equiv J_1^l) \) are taken from the power-law distribution in Eq. (1.2) with the coupling \( J_1^l \) within the range \( 0 < J_1^l < J_{max}^l \). Similarly, the next-nearest neighbor couplings \( (J_1^l \equiv J_2^l) \) are taken from the same type of power-law distribution and the range of couplings is now \( 0 < J_2^l < J_{max}^l \).

![FIG. 11. Phase diagram of random conventional ladders with staggered dimerization for different disorder parameters. The random dimer (RD) phase and the random rung singlet (RRS) phase are separated by a random critical line of infinite randomness behavior.](image)

**FIG. 11.** Phase diagram of random conventional ladders with staggered dimerization for different disorder parameters. The random dimer (RD) phase and the random rung singlet (RRS) phase are separated by a random critical line of infinite randomness behavior.

The calculated dynamical exponent, \( z \), as shown in Figs. 12 has its maximum at \( J_{max}^2/J_{max}^1 = 0 \) and around this point one can observe strong finite-size dependence, the range of which is wide, in particular for weak disorder (see Fig. 13). At \( J_{max}^2/J_{max}^1 = 0 \), where the zig-zag ladder reduces to a random AF chain, the system is in the IRFP, thus the extrapolated value of the dynamical exponent is formally infinity. Given the strong finite-size corrections in the numerical RG data of the dynamical exponent, it is difficult to decide whether the IRFP behavior of the zig-zag ladders is extended to a finite region of the couplings \( J_{max}^2/J_{max}^1 > 0 \) or whether this region shrinks to a single point only. The first scenario may be related to the existence of a gapless phase of the pure model for \( J^2/J^1 < 0.24 \).
To discuss this issue we have calculated the dynamical exponent by an independent method based on density matrix renormalization. In principle, the dynamical exponent is related to the distribution of the first gap, $\Delta$, in the small $\Delta$ limit, see in Eq.(3.2) with $\Delta$. The asymptotic slope of the distribution, indicated by the straight line, is the inverse of the dynamical exponent, see of the random zig-zag ladder using a power-law distribution $L=1$, $J^2_{\max}/J^1_{\max} = 0.5$), for different lengths of the ladder, $L$. The asymptotic slope of the distribution, indicated by the straight line, is the inverse of the dynamical exponent, see Eq.(4.4).

Thus the dynamical exponent $z$ can be obtained from an analysis of the small $m_s$ tail of the distribution, as illustrated in Fig. 13 where the distribution function of $\ln m_s$ is given in a log-log plot for different lengths of the ladder. As seen in this Figure the slope of the distribution is well defined for larger systems, from which one can obtain an accurate estimate for the dynamical exponent, which is finite. Repeating the calculation for other values of the coupling ratio, $J^2_{\max}/J^1_{\max}$, we have obtained a set of the dynamical exponents, which are plotted in Fig. 14. These accurate DMRG data show that the extrapolated values of the effective exponents calculated by the numerical RG-method are finite for any $J^2_{\max}/J^1_{\max} > 0$. Consequently the random zig-zag ladder has just one IRFP at $J^2_{\max}/J^1_{\max} = 0$, whereas the system in the region of $J^2_{\max}/J^1_{\max} > 0$ is in a gapless random dimer phase. In view of the numerical results in Fig. 12, where $z_{\infty}$ seems to stay over $z_{\text{pure}} = 1$, it is quite probable that the random dimer phase exists for any small value of the disorder.

D. Random $J_1-J_2$ ladders

The full ladder, as represented in Fig. 14c has three different type of couplings: $J_1$, $J^R_1$ and $J^D_1$. Here we consider a special case of this model, when the chain ($J_1$) and rung ($J^R_1$) couplings are taken from the same power-law distribution with a disorder parameter $D$ and having a range of $0 < J_1, J^R_1 < J^2_{\max}$. On the other hand the diagonal couplings are taken from the same type of power-law distribution and are within the interval $0 < J^D_1 < J^2_{\max}$. This model, having first- and second-neighbor interactions, is called a $J_1-J_2$ ladder. We have calculated the finite-size dynamical exponents as a function of the coupling ratio $J^2_{\max}/J^1_{\max}$ for different strengths of disorder (see Fig. 14). These curves show similar qualitative behavior as those calculated for the random conventional ladders with staggered dimerization in Figs. 13 so that we can draw similar conclusions.

The extrapolated position of the maximum of the $z$ curves is identified as a quantum critical point with infinite randomness behavior. Indeed, repeating the calculation as indicated for the dimerized ladder model in Fig. 14 we obtained a scaling behavior as in Eq.(4.4) with an exponent which is compatible with $\psi = 1/2$. The random quantum critical point separates two gapless Haldane phases, having odd and even topological order, respectively.
IRFP are asymptotically exactly known from analytical calculations in random AF spin chains\cite{29,30}. Thus the general phase diagram consists of Griffiths-type phases with different topological order separated by a random critical point of the IRFP type. The zig-zag ladder is an exception, where there is just one Griffiths phase and the random critical point is located at its boundary.

Next we turn to discuss about possible cross-over effects when the strength of disorder is varied. These problems can not be directly studied by the simple strong disorder RG method, however, from arguments considering the sign of $z_{\text{dis}} - z_{\text{pure}}$ and from analogous investigations on quantum spin chains\cite{31,32,33}, we can suggest the following picture. Originally gapped phases could stay gapped for weak disorder and become gapless only if the strength of disorder exceed some finite limiting value, as seen for the random conventional ladder. However, for frustrated ladders, such as the zig-zag and the $J_1 - J_2$ ladders, any small amount of disorder seems to bring the system into a random gapless phase. At a phase boundary, such as in the staggered dimerized ladder and the $J_1 - J_2$ model, the random critical behavior is of the IRFP type, probably for any small amount of disorder.

At this point we comment on the similarity of the low-energy behavior of spin chains with $2S = \text{odd}$ ($2S = \text{even}$) spins and that of spin-1/2 ladders with $n = \text{odd}$ ($n = \text{even}$) legs. If the pure systems are gapless, i.e. $2S = n = \text{odd}$, strong enough disorder is expected to bring both systems into the IRFP. For $2S = n = \text{odd} \geq 3$, there is a limiting disorder strength, $D_c(n)$, below which the system is described by a conventional random fixed point with $z < \infty$. On the other hand for $2S = n = \text{even}$ we have only a partial analogy: for weak disorder both systems are gapped, which turn into a gapless Griffiths-type phase for stronger disorder. While the random ladder stays in this conventional random phase for any strength of the disorder the random spin chain will turn into a IRFP behavior at some finite limiting randomness. This type of infinite randomness behavior can, however, be seen for frustrated even-leg $J_1 - J_2$ ladders at the transition point. We can thus conclude that random ladders with even and odd number of legs belong to different universality classes.

Finally, we comment on random square lattice antiferromagnets, which can be obtained in the limit when the number of legs, $n$, goes to infinity. By increasing $n$ the value of the limiting strength, $D_c(n)$, is expected to increase, too, both for $n = \text{even}$ (for frustrated ladders) and $n = \text{odd}$. In the limit, $n \to \infty$, $D_c(n)$ very probably tends to infinity, so that the critical behavior of that system is described by a conventional random fixed point. Work is in progress to verify this scenario and to obtain a general physical picture about the low-energy properties of random two-dimensional antiferromagnets.

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