On the restricted matching of graphs in surfaces

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Abstract

A connected graph $G$ with at least $2m + 2n + 2$ vertices is said to have property $E(m, n)$ if, for any two disjoint matchings $M$ and $N$ of size $m$ and $n$ respectively, $G$ has a perfect matching $F$ such that $M \subseteq F$ and $N \cap F = \emptyset$. In particular, a graph with $E(m, 0)$ is $m$-extendable. Let $\mu(\Sigma)$ be the smallest integer $k$ such that no graphs embedded on a surface $\Sigma$ are $k$-extendable. Aldred and Plummer have proved that no graphs embedded on the surfaces $\Sigma$ such as the sphere, the projective plane, the torus, and the Klein bottle are $E(\mu(\Sigma) - 1, 1)$. In this paper, we show that this result always holds for any surface. Furthermore, we obtain that if a graph $G$ embedded on a surface has sufficiently many vertices, then $G$ has no property $E(k - 1, 1)$ for each integer $k \geq 4$, which implies that $G$ is not $k$-extendable. In the case of $k = 4$, we get immediately a main result that Aldred et al. recently obtained.

Keywords: Perfect matching; Restricted matching; Extendability; Graphs in surface.

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1 Introduction

A matching of a graph $G$ is a set of independent edges of $G$ and a matching is called perfect if it covers all vertices of $G$. A connected graph $G$ with at least $2m + 2n + 2$ vertices is said to have property $E(m, n)$ (or abbreviated as $G$ is $E(m, n)$) if, for any two disjoint matchings $M$, $N \subseteq E(G)$ of size $m$ and $n$ respectively, there is a perfect matching $F$ such that $M \subseteq F$ and $N \cap F = \emptyset$. It is obvious that a graph with $E(0, 0)$ has a perfect matching. Since properties $E(m, 0)$ and $m$-extendability are equivalent, property $E(m, n)$ is somewhat a generalization

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of \(m\)-extendability. The concept of \(m\)-extendable graphs was gradually evolved from the study of elementary bipartite graphs and matching-covered graphs (i.e. each edge belongs to a perfect matching) and introduced by M.D. Plummer \([13]\) in 1980. For extensive studies on \(m\)-extendable graphs, see two surveys \([9]\) and \([10]\). A basic property is stated as follows.

Lemma 1.1. (\([13]\)) Every \(m\)-extendable graph is \((m + 1)\)-connected.

For a vertex \(v\) of a graph \(G\), let \(N(v)\) denote the neighborhood of \(v\), i.e., the set of vertices adjacent to \(v\) in \(G\), and \(G[N(v)]\) the subgraph of \(G\) induced by \(N(v)\).

Lemma 1.2. (\([4]\)) Let \(v\) be a vertex of degree \(m + t\) in an \(m\)-extendable graph \(G\). Then \(G[N(v)]\) does not contain a matching of size \(t\).

Porteous and Aldred \([15]\) introduced the concept of property \(E(m, n)\) and focussed on when the implication \(E(m, n) \rightarrow E(p, q)\) does and does not hold. From then on, the possible implications among the properties \(E(m, n)\) for various values of \(m\) and \(n\) are studied in \([6, 14, 15]\). The following three non-trivial results will be used later.

Lemma 1.3. (\([15]\)) If a graph \(G\) is \(E(m, n)\), then it is \(E(m, 0)\).

Lemma 1.4. (\([15]\)) If a graph \(G\) is \(E(m, n)\), then it is \(E(m - 1, n)\).

Lemma 1.5. (\([15]\)) If a graph \(G\) is \(E(m, 0)\) for \(m \geq 1\), then it is \(E(m - 1, 1)\).

The converse of Lemma 1.5 does not hold. For example, the join graph \(\overline{K_2} + K_{2m}\), obtained by joining each of two vertices to each vertex of the complete graph \(K_{2m}\) with edges, has property \(E(m - 1, 1)\), but is not \(m\)-extendable.

A surface is a connected compact Hausdorff space which is locally homeomorphic to an open disc in the plane. If a surface \(\Sigma\) is obtained from the sphere by adding some number \(g \geq 0\) of handles (resp. some number \(\bar{g} > 0\) of crosscaps), then it is said to be orientable of genus \(g = g(\Sigma)\) (resp. non-orientable of genus \(\bar{g} = \bar{g}(\Sigma)\)). We shall follow the usual notation of the surface of orientable genus \(g\) (resp. non-orientable genus \(\bar{g}\)) by \(S_g\) (resp. \(N_{\bar{g}}\)).

Let \(\mu(\Sigma)\) be the smallest integer \(k\) such that no graphs embedded on the surface \(\Sigma\) are \(k\)-extendable. Dean \([4]\) presented an elegant formula that

\[
\mu(\Sigma) = 2 + \lfloor \sqrt{4 - 2\chi(\Sigma)} \rfloor, \tag{1}
\]

where \(\chi(\Sigma)\) is the Euler characteristic of a surface \(\Sigma\), i.e. \(\chi(\Sigma) = 2 - 2g\) if \(\Sigma\) is an orientable surface of genus \(g\) and \(\chi(\Sigma) = 2 - \bar{g}\) if \(\Sigma\) is a non-orientable surface of genus \(\bar{g}\). For the surfaces \(\Sigma\) with small genus such as the sphere, the projective plane, the torus and the Klein bottle, the following results show that no graphs embedded on \(\Sigma\) are \(E(\mu(\Sigma) - 1, 1)\).
Lemma 1.6. (i) (2) No planar graph is $E(2, 1)$;
(ii) (3) No projective planar graph is $E(2, 1)$;
(iii) (3) If $G$ is toroidal, then $G$ is not $E(3, 1)$;
(iv) (3) If $G$ is embedded on the Klein bottle, then $G$ is not $E(3, 1)$.

In this paper we obtain the following general result, which will be proved in next section.

Theorem 1.7. For any surface $\Sigma$, no graphs embedded on $\Sigma$ are $E(\mu(\Sigma) - 1, 1)$.

Furthermore, we obtain that if a graph $G$ embedded on a surface has enough many vertices, then $G$ has no property $E(k - 1, 1)$ for each integer $k \geq 4$. Precisely, we have the following result; its proof will be given in Section 3.

Theorem 1.8. Let $G$ be a graph with genus $g$ (resp. non-orientable genus $\bar{g}$). Then if $|V(G)| \geq \left\lfloor \frac{8g - 8}{k - 3} \right\rfloor + 1$ (resp. $|V(G)| \geq \left\lfloor \frac{4\bar{g} - 8}{k - 3} \right\rfloor + 1$), $G$ is not $E(k - 1, 1)$ for each integer $k \geq 4$.

Combining Theorem 1.8 with Lemma 1.5 we have an immediate consequence as follows.

Corollary 1.9. (17) Let $G$ be any connected graph of genus $g$ (resp. non-orientable genus $\bar{g}$). Then if $|V(G)| \geq \left\lfloor \frac{8g - 8}{k - 3} \right\rfloor + 1$ (resp. $|V(G)| \geq \left\lfloor \frac{4\bar{g} - 8}{k - 3} \right\rfloor + 1$) for any integer $k \geq 4$, $G$ is not $k$-extendable.

In particular, if we put $k = 4$ in the corollary, we can obtain the following result which is also a main theorem that Aldred et al. recently obtained.

Corollary 1.10. (1) Let $G$ be any connected graph of genus $g$ (resp. non-orientable genus $\bar{g}$). Then if $|V(G)| \geq 8g - 7$ (resp. $4\bar{g} - 7$), $G$ is not 4-extendable.

2 Proof of Theorem 1.7

For a graph $G$, the genus $\gamma(G)$ (resp. non-orientable genus $\bar{\gamma}(G)$) of it is the minimum genus (resp. non-orientable genus) of all orientable (resp. non-orientable) surfaces in which $G$ can be embedded. An embedding $\tilde{G}$ of a graph $G$ on an orientable surface $S_k$ (resp. a non-orientable surface $N_k$) is said to be minimal if $\gamma(G) = k$ (resp. $\bar{\gamma}(G) = k$) and 2-cell if each component of $\Sigma - \tilde{G}$ is homeomorphic to an open disc.

Lemma 2.1. (10) Every minimal orientable embedding of a graph $G$ is a 2-cell embedding.

Lemma 2.2. (7) Every graph $G$ has a minimal non-orientable embedding which is 2-cell.
Let $v$ be any vertex of a graph $G$ embedded on an orientable surface of genus $g$ (resp. a non-orientable surface of genus $\bar{g}$). Define the Euler contribution of the vertex $v$ to be

$$\phi(v) = 1 - \frac{\deg(v)}{2} + \sum_{i=1}^{\deg(v)} \frac{1}{f_i},$$

where the sum runs over the face angles at vertex $v$, $f_i$ denotes the size of the $i$th face at $v$ and $\deg(v)$ denotes the degree of $v$.

**Lemma 2.3.** ([5]) Let $G$ be a connected graph 2-cellularly embedded on some surface $\Sigma$ of orientable genus $g$ (resp. non-orientable genus $\bar{g}$). Then $\sum_v \phi(v) = \chi(\Sigma)$.

For a vertex $v$, it is called a control point if $\phi(v) \geq \frac{\chi(\Sigma)}{|V(G)|}$. If $G$ is 2-cellularly embedded on the surface $\Sigma$, then $G$ must have at least one control point by Lemma 2.3.

Let $\delta(G)$ denote the minimum degree of the vertices in $G$. The following lemma is a simple observation, which gives a lower bound of $\delta(G)$ of a graph $G$ with $E(m, 1)$.

**Lemma 2.4.** If a graph $G$ is $E(m, 1)$ for $m \geq 1$, then $\delta(G) \geq m + 2$.

**Proof.** By Lemma 1.3 $G$ is $E(m, 0)$. Moreover, $\delta(G) \geq m + 1$ by Lemma 1.1. Suppose to the contrary that there exists a vertex $v$ with degree $m + 1$. Then $G[N(v)]$ cannot contain a matching of size 1 by Lemma 1.2 that is, $N(v)$ is an independent set of $G$. Let $N(v) = \{v_1, v_2, ..., v_{m+1}\}$, $V = \{v_1, v_2, ..., v_m\}$ and $R = V(G) \setminus N[v]$, where $N[v] = N(v) \cup \{v\}$. Let $G[V, R]$ be the induced bipartite graph of $G$ with bipartition $V$ and $R$. Then every vertex in $V$ is adjacent to at least $m$ vertices in $R$. Hence it can easily be seen that $G[V, R]$ has a matching $M$ of size $m$ saturating $V$. Let $N = \{vv_{m+1}\}$. Obviously, there is no perfect matching $F$ of $G$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$. This contradicts that $G$ is $E(m, 1)$. \hfill $\square$

**Proof of Theorem 1.7.** Since $\mu(\Sigma)$ increases as $g$ (resp. $\bar{g}$) does and a graph embedded on a surface with small genus must be embedded on some surface with larger genus, it suffices to prove that any graph minimally embedded on the surface $\Sigma$ is not $E(\mu(\Sigma) - 1, 1)$ by Lemma 1.4 In the following, we may assume that $G$ is minimally and 2-cell embedded on the surface $\Sigma$ by Lemmas 2.1 and 2.2.

By Lemma 1.6 the theorem holds for the surfaces $S_0$, $S_1$, $N_1$ and $N_2$. Hereafter, we will restrict our considerations on the other surfaces $\Sigma$. Consequently, $\chi(\Sigma) \leq -1$ and $\mu(\Sigma) \geq 4$.

Suppose to the contrary that $G$ is $E(\mu(\Sigma) - 1, 1)$. Then $|V(G)| \geq 2(\mu(\Sigma) + 1)$, and $\delta(G) \geq \mu(\Sigma) + 1 \geq 5$ by Lemma 2.4 Since $G$ is a 2-cell embedding on the surface $\Sigma$, it has a control point $v$. Let $y := \deg(v)$ and let $x$ be the number of the triangular faces at $v$. 

\hfill 4
Claim 1. G is not $E(y - \lceil \frac{x}{2} \rceil, 1)$.

If $x = y$ and $y$ is odd, then there is a matching of size $\lfloor \frac{x}{2} \rfloor$ in $G[N(v)]$. Hence $G$ is not $E(\lceil \frac{x}{2} \rceil, 1)$, that is, $G$ is not $E(y - \lceil \frac{x}{2} \rceil, 1)$. Otherwise, there is a matching of size $\lfloor \frac{x}{2} \rfloor$ in $G[N(v)]$. Then $G$ is not $(y - \lfloor \frac{x}{2} \rfloor)$-extendable by Lemma 1.2. Hence $G$ is not $E(y - \lceil \frac{x}{2} \rceil, 1)$ by Lemma 1.3. So the claim always holds.

By Eq. (2) and $\phi(v) \geq \frac{\chi(\Sigma)}{|V(G)|}$, we have

$$\frac{y}{2} \leq 1 + \sum_{i=1}^{y} \frac{1}{f_i} - \frac{\chi(\Sigma)}{|V(G)|} \leq 1 + \frac{x}{3} + \frac{y - x}{4} - \frac{\chi(\Sigma)}{2(\mu(\Sigma) + 1)},$$

which implies that

$$y \leq \frac{x}{3} + 4 - \frac{2\chi(\Sigma)}{\mu(\Sigma) + 1}.$$ 

Let

$$c := 4 - \frac{2\chi(\Sigma)}{\mu(\Sigma) + 1}. \quad (3)$$

Then $c > 4$ and $y - \lceil \frac{x}{2} \rceil \leq y - \frac{x}{2} \leq y - \frac{x}{3} \leq c$.

Claim 2. G is not $E([c] - 1, 1)$.

If $y - \lceil \frac{x}{2} \rceil \leq y - \frac{x}{2} \leq c - 1$, then $G$ is not $E([c] - 1, 1)$ by Lemma 1.2 and Claim 1.

In what follows we suppose that $y - \frac{x}{2} > c - 1$. Combining this with $y - \frac{x}{3} \leq c$, we have that $x \leq 5$, and all possible cases of pairs of non-negative integers $(x, y)$ are as follows:

$(0, [c]), (1, [c]), (1, [c] + 1), (2, [c] + 1), (3, [c] + 1), (4, [c] + 2), \text{ and } (5, [c] + 2)$.

Suppose to the contrary that $G$ is $E([c] - 1, 1)$. Then $G$ is $([c] - 1)$-extendable by Lemma 1.3 and $\delta(G) \geq [c] + 1$ by Lemma 2.1. Hence the first two cases $(0, [c])$ and $(1, [c])$ are impossible. If $y = [c] + 1$, since $\deg(v) = y = ([c] - 1) + 2$, $G[N(v)]$ cannot contain a matching of size 2 by Lemma 1.2.

For convenience, let $v_1, v_2, ..., v_y$ be the vertices adjacent to $v$ arranged clockwise at $v$ in $G$. Similar to the notation in the proof of Lemma 2.1 let $R = V(G) \setminus N[v]$ and $G[V,R]$ denote the induced bipartite graph of $G$ with bipartition $V$ and $R$, where $V \subseteq V(G) \setminus R$.

For $(x, y) = (1, [c] + 1), G[N(v)]$ cannot contain a matching of size 2. Assume that the triangular face at $v$ is $vv_1v_2$. Hence each $v_i, 3 \leq i \leq [c] + 1$, can only be adjacent to $v_1$ and $v_2$ in $N(v)$, and has at least $[c] - 2$ adjacent vertices in $R$. Let $V := \{v_3, v_4, ..., v_{[c]}\}$. Then there is a matching $M'$ of size $[c] - 2$ in $G[V, R]$. Let $M := M' \cup \{v_1v_2\}$ and $N := \{vv_{[c] + 1}\}$. Obviously, there is no perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction.

For $(x, y) = (2, [c] + 1), G[N(v)]$ cannot contain a matching of size 2, and the two triangular faces at $v$ must be adjacent. Hence we can assume that they are $vv_1v_2$ and $vv_2v_3$. Each $v_i, 4 \leq i \leq [c] + 1$, can only be adjacent to $v_2$ in $N(v)$, and has at least $[c] - 1$ adjacent
vertices in $R$. Let $V := \{v_4, v_5, \ldots, v_{|c|+1}\}$. Then we can find a matching $M'$ of size $|c| - 2$ in $G[V, R]$. Let $M = M' \cup \{v_1v_2\}$ and $N = \{vv_3\}$. Consequently, it is impossible to find a perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction.

For $(x, y) = (3, \lfloor c \rfloor + 1)$, $G[N(v)]$ contains a matching of size $\lceil \frac{3}{2} \rceil - 2$. This would be impossible.

If $y = \lfloor c \rfloor + 2$, since $\deg(v) = (\lfloor c \rfloor - 1) + 3$, $G[N(v)]$ cannot contain a matching of size 3 by Lemma 1.2. Hence $(x, y) = (5, \lfloor c \rfloor + 2)$ would also be impossible since $G[N(v)]$ contains a matching of size $\lceil \frac{5}{2} \rceil = 3$.

For the remaining case $(x, y) = (4, \lfloor c \rfloor + 2)$, $G[N(v)]$ cannot contain a matching of size 3. Then the four triangular faces at $v$ must have the following two cases. Case 1. The four triangular faces are $vv_i v_{i+1}$, $1 \leq i \leq 4$. Each $v_i$, $6 \leq i \leq \lfloor c \rfloor + 2$, can only be adjacent to $v_2$ or $v_4$, and has at least $\lfloor c \rfloor - 2$ adjacent vertices in $R$. Let $V := \{v_6, v_7, \ldots, v_{|c|+2}\}$. Then we can find a matching $M'$ of size $\lfloor c \rfloor - 3$ in $G[V, R]$. Let $M = M' \cup \{v_1v_2, v_4v_5\}$ and $N = \{vv_3\}$. Obviously, there is no perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction. Case 2. The four triangular faces are $vv_i v_{i+1}$ for $i = 1, 2$ and $vv_j v_{j+1}$ for $j = t, t + 1$, where $t \neq 1, 2, 3, \lfloor c \rfloor + 1, \lfloor c \rfloor + 2$. Then each $v_i$, $i \neq 1, 2, 3, t, t + 1$ and $t + 2$, can only be adjacent to $v_2$ and $v_{t+1}$, and has at least $\lfloor c \rfloor - 2$ adjacent vertices in $R$. $v_3$ can only be adjacent to $v_1, v_2$ and $v_{t+1}$ in $G[N(v)]$, and has at least $\lfloor c \rfloor - 3$ adjacent vertices in $R$. Let $V := N(v) - \{v_1, v_2, v_t, v_{t+1}, v_{t+2}\}$. Then we can find a matching $M'$ of size $\lfloor c \rfloor - 3$ in $G[V, R]$. Set $M = M' \cup \{v_1v_2, v_{t+1}\}$ and $N = \{vv_3\}$. Obviously, there is no perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction. Hence the claim holds.

**Claim 3.** $|c| \leq \mu(\Sigma)$.

In fact, the inequality was stated in [4] without proof. Here we present a simple proof.

Owing to the expressions (1) and (3) of $\mu(\Sigma)$ and $c$, it suffices to prove that

$$\left\lfloor 4 - \frac{2\chi(\Sigma)}{3 + \lceil \sqrt{4 - 2\chi(\Sigma)} \rceil} \right\rfloor \leq 2 + \lceil \sqrt{4 - 2\chi(\Sigma)} \rceil.$$ 

Then we have the following implications:

$$\left\lfloor 4 - \frac{2\chi(\Sigma)}{3 + \lceil \sqrt{4 - 2\chi(\Sigma)} \rceil} \right\rfloor \leq 2 + \lceil \sqrt{4 - 2\chi(\Sigma)} \rceil \iff 3 - \frac{2\chi(\Sigma)}{3 + \lceil \sqrt{4 - 2\chi(\Sigma)} \rceil} < 2 + \lceil \sqrt{4 - 2\chi(\Sigma)} \rceil \iff 9 + 3\lceil \sqrt{4 - 2\chi(\Sigma)} \rceil - 2\chi(\Sigma) < (\lceil \sqrt{4 - 2\chi(\Sigma)} \rceil)^2 + 5\lceil \sqrt{4 - 2\chi(\Sigma)} \rceil + 6 \iff 4 - 2\chi(\Sigma) < (\lceil \sqrt{4 - 2\chi(\Sigma)} \rceil + 1)^2.$$ 

Since the last inequality obviously holds, the claim follows.

By the last arguments, we know that $G$ is $E(\mu(\Sigma) - 1, 1)$ but not $E(\lfloor c \rfloor - 1, 1)$. Hence $\lfloor c \rfloor - 1 > \mu(\Sigma) - 1$ by Lemma 1.4, which contradicts Claim 3. □
3 Proof of Theorem 1.8

Suppose to the contrary that $G$ is $E(k - 1, 1)$. Then $G$ is $E(k - 1, 0)$ and $\delta(G) \geq k + 1$ by Lemmas 1.3 and 2.1. We can assume that $G$ is a 2-cell embedding on the surface $S_5$ (resp. $N_g$) by Lemmas 2.1 and 2.2. In the following, we mainly prove that $\phi(v) \leq -\frac{k-3}{4}$ for any vertex $v \in V(G)$. It holds, by Lemma 2.3 we have $\chi(\Sigma) = \sum_v \phi(v) \leq -\frac{k-3}{4}|V(G)|$, which implies that $|V(G)| \leq \frac{4\chi(\Sigma)}{k-3}$ for $k \geq 4$. This contradiction to the condition establishes the theorem.

Let $d = \deg(v) = k + m$. Then $m \geq 1$. For convenience, we assume that $v_1, v_2, ..., v_d$ are the vertices adjacent to $v$ arranged clockwise at $v$ in $G$. There are three cases to be considered.

Case 1. $m \geq 3$. Since $d = (k - 1) + m + 1$ and $G$ is $(k - 1)$-extendable, $G[N(v)]$ cannot contain a matching of size $m + 1$ by Lemma 1.2. If there are at most $2m$ triangular faces at $v$, then we have

$$\phi(v) \leq 1 - \frac{d}{2} + \frac{2m}{3} + \frac{k + m - 2m}{4} = \frac{-3k - m + 12}{12} \leq \frac{-3k - 3 + 12}{12} = \frac{3 - k}{4}.$$

Otherwise, there are exactly $2m + 1$ triangular faces at $v$ and $d = 2m + 1 = 2k + 1$. Let $M := \{v_i, v_{i+1} | 1 \leq i \leq 2m - 1, i \text{ is odd}\}$ and $N := \{v_i v_{2m+1}\}$. Then there exists no perfect matching $F$ such that $M \subseteq F$ and $N \cap F = \emptyset$. But $G$ is $E(k - 1, 1)$, a contradiction.

Case 2. $m = 2$. Since $d = (k - 1) + 3$, $G[N(v)]$ cannot contain a matching of size 3 by Lemma 1.2. Hence there are at most four triangular face at $v$. It suffices to prove that there are at most three triangular face at $v$. If so, we have that $\phi(v) \leq 1 - \frac{k+2}{2} + \frac{2}{3} + \frac{k-1}{4} = \frac{3-k}{4}$.

Suppose to the contrary that there are exactly four triangular faces at $v$. Then there are two cases to be considered.

Subcase 2.1. The four triangular faces are $vv_i v_{i+1}$, where $1 \leq i \leq 4$. Then each $v_i$, $6 \leq i \leq k + 2$, can only be adjacent to $v_2$ and $v_4$ and has at least $k + 1 - 3 = k - 2$ adjacent vertices in $R$, where $R = V(G) \setminus N[v]$. Let $V := \{v_6, v_7, ..., v_{k+2}\}$. Then we can find a matching $M'$ of size $k - 3$ in the induced bipartite graph $G[V, R]$ of $G$. Let $M := M' \cup \{v_1 v_2, v_4 v_5\}$ and $N := \{v v_3\}$. Obviously, there is no perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, which contradicts the supposition that $G$ has property $E(k - 1, 1)$.

Subcase 2.2. The four triangular faces are $vv_i v_{i+1}$, $1 \leq i \leq 2$, and $vv_j v_{j+1}$, $t \leq j \leq t + 1$, where $t \neq 1, 2, 3, k + 1, k + 2$. Then for each $v_i$, $i \neq 1, 2, 3, t, t + 1$ and $t + 2$, it can only be adjacent to $v_2$ and $v_4$, and has at least $k - 2$ adjacent vertices in $R$. For the vertex $v_3$, it can only be adjacent to $v_1$, $v_2$ and $v_{t+1}$ in $N(v)$. Then it has at least $k - 3$ adjacent vertices in $R$. Let $V ::= N(v) - \{v_1, v_2, v_t, v_{t+1}, v_{t+2}\}$. Then we can find a matching $M'$ of size $k - 3$ in the induced bipartite graph $G[V, R]$ of $G$. Let $M := M' \cup \{v_1 v_2, v_t v_{t+1}\}$ and $N ::= \{v v_{t+2}\}$.  

Consequently, it would be impossible to find a perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction.

**Case 3.** $m = 1$. Since $d = (k - 1) + 2$, $G[N(v)]$ cannot contain a matching of size 2. Then there are at most two triangular faces at $v$. It suffices to prove that there are no triangular faces at $v$. If so, $\phi(v) = 1 - \frac{d}{2} + \sum_{i=1}^{d} \frac{1}{i} \leq 1 - \frac{d}{2} + \frac{d}{4} = \frac{3k - 1}{4}$.

If there is exactly one triangular face at $v$, suppose that it is $vv_1v_2$. Since $G[N(v)]$ cannot contain a matching of size 2, each vertex $v_i$, where $3 \leq i \leq k$, can only be adjacent to $v_1$ and $v_2$ in $N(v)$. Consequently, it is adjacent to at least $k - 2$ vertices in $R$. Let $V = \{v_3, v_4, ..., v_k\}$. Then we can find a matching $M'$ of size $k - 2$ in the induced bipartite graph $G[V, R]$ of $G$. Set $M = M' \cup \{v_1v_2\}$ and $N = \{vv_{k+1}\}$. Then there is no perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction.

Otherwise, there are exactly two triangular faces at $v$, which must be adjacent faces, say $vv_iv_{i+1}$, where $i = 1, 2$. Then each vertex $v_i$, where $4 \leq i \leq k + 1$, can only be adjacent to $v_2$ in $N(v)$, and is adjacent to at least $k - 1$ vertices in $R$. Let $V := \{v_4, v_5, ..., v_{k+1}\}$. Then we can find a matching $M'$ of size $k - 2$ in the induced bipartite graph $G[V, R]$ of $G$. Set $M := M' \cup \{v_1v_2\}$ and $N := \{vv_3\}$. Then there is no perfect matching $F$ satisfying that $M \subseteq F$ and $N \cap F = \emptyset$, a contradiction.

**References**

[1] R.E.L. Aldred, Ken-ichi Kawarabayashi, M.D. Plummer, On the matching extendability of graphs in surfaces, J. Combin. Theory Ser. B 98 (2008) 105-115.

[2] R.E.L. Aldred and M.D. Plummer, On restricted matching extension in planar graphs, in: 17th British Combin. Conference (Canterbury 1999), Discrete Math. 231 (2001) 73-79.

[3] R.E.L. Aldred and M.D. Plummer, Restricted matching in graphs of small genus, Discrete Math. 308 (2008) 5907-5921.

[4] N. Dean, The matching extendability of surfaces, J. Combin. Theory Ser. B 54 (1992) 133-141.

[5] H. Lebesgue, Quelques conséquences simples de la formule d’Euler, J. Math. 9 (1940) 27-43.

[6] A. McGregor-Macdonald, The $E(m, n)$ property, M.S. Thesis, University of Otago, 2000.

[7] T. Parsons, G. Pica, T. Pisanski and A. Ventre, Orientably simple graphs, Math. Slovaca 37 (1987) 391-394.
[8] M.D. Plummer, A theorem on matchings in the plane, in: L.D. Andersen, et al. (Eds.), Proceedings of a Conference in Memory of Gabriel Dirac, Ann. Discrete Math. Vol. 41, North-Holland, Amsterdam, 1989, pp. 347-354.

[9] M.D. Plummer, Extending matchings in graphs: a survey, Discrete Math. 127 (1994) 277-292.

[10] M.D. Plummer, Extending matchings in graphs: an update, Cong. Numer. 116 (1996) 3-32.

[11] M.D. Plummer, Matching extension and the genus of a graph, J. Combin. Theory Ser. B 44 (1986) 329-337.

[12] M.D. Plummer, Matching extension in bipartite graphs, in: Proceedings of the Seventeenth Southeastern Conference on Combinatorics, Graph Theory and Computing, in: Congr. Numer., LIV Utilitas Math., Winnipeg, 1986, pp. 245-258.

[13] M.D. Plummer, On n-extendable graphs, Discrete Math. 31 (1980) 201-210.

[14] M. Porteous, Generalizing matching extensions, M.A. Thesis, University of Otago, 1995.

[15] M. Porteous and R. Aldred, Matching extensions with prescribed and forbidden edges, Austral. J. Combin. 13 (1996) 163-174.

[16] J.W.T. Youngs, Minimal imbeddings and the genus of a graph, J. Math. Mech. 12 (1963) 303-315.

[17] Heping Zhang, Qiuli Li and Wenwen Liu, Notes on the matching extendability of graphs on surfaces, submitted.