Faster Treasure Hunt and Better Strongly Universal Exploration Sequences

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Abstract

In this paper, we investigate the explicit deterministic treasure hunt problem in a $n$-vertex network. This problem was firstly introduced by Ta-Shma and Zwick in [10] [SODA’07]. Note also it is a variant of the well known rendezvous problem in which one of the robot (the treasure) is always stationary. In this paper, we propose an $O(n^{c(1+\frac{1}{\lambda})})$-time algorithm for the treasure hunt problem, which significantly improves the currently best known result of running time $O(n^{2c})$ in [10], where $c$ is a constant induced from the construction of an universal exploration sequence in [9, 10], and $\lambda \gg 1$ is an arbitrary large, but fixed, integer constant. The treasure hunt problem also motivates the study of strongly universal exploration sequences. In this paper, we also propose a much better explicit construction for strongly universal exploration sequences compared to the one in [10].

Keywords: Design and analysis of algorithms, distributed computing, graph searching and robotics, rendezvous, strongly universal exploration sequences.

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1. Introduction

In the rendezvous problem ([3, 7, 10]), two robots are placed in an unknown environment modeled by a finite, connected, undirected graph \( G = (V, E) \). We assume that \(|V| = n\). The size of the network, i.e., the number of vertices in the graph is not known to the robots. The edges incident on a vertex \( u \in V \) are numbered \( 0, 1, 2, \ldots, \deg(u) - 1 \), in a predetermined manner, where \( \deg(u) \) is the degree of \( u \). In general, the numbering is not assumed to be consistent, i.e., an edge \((u, v) \in E\) may be the \( i \)-th edge of \( u \) but the \( j \)-th edge of \( v \), where \( i \neq j \).

When a robot is in a vertex \( u \in V \) it is told the degree \( \deg(u) \) of \( u \). However, all vertices of the same degree are not distinguishable. The robots are not allowed to put any information such as tokens or markers at the vertices that they visit. At any time step a robot is only allowed to either traverse an edge, or stay in place. When the robot is at a vertex \( u \) it may ask to traverse the \( i \)-th edge \((u, v) \in E\) of \( u \), where \( 0 \leq i \leq \deg(u) - 1 \). The robot observes itself at vertex \( v \), the another endpoint of this edge. As described before, the \( i \)-th edge of \( u \) is the \( j \)-th edge of \( v \), for some \( 0 \leq j \leq \deg(v) - 1 \). In general \( j \neq i \).

There are two different variants of the model used in the field. In the first one, the robot is told the index \( j \) of the edge it used to enter \( v \). This allows the robot to return to \( u \) at the next step, if it wants to do so. This variant of the problem is called the rendezvous problem with backtracking. In the second variant of the model, the robot observes itself at vertex \( v \) without knowing which edge it used to get there. We call this variant of the problem the general rendezvous problem.

Same as most of work [3, 7, 10] described, the main strategy is to give the two robots deterministic sequences of instructions which will guarantee that two robots would eventually meet each other, no matter in which graph they are located, and no matter when they are activated. It is, however, expected that such a meeting would happen as soon as possible. A robot is unaware of the whereabouts of another robot, even it is very close to another one in the graph. The two robots meet only when they are both active and are at same vertex at same time. In particular, the two robots may traverse the same edge but in different directions, and still miss each other.

For the deterministic solutions, it has to be assumed that two robots have different labels, e.g. \( L_1 \neq L_2 \). Without such an assumption there is no deterministic way of breaking symmetry and no deterministic strategy is possible. An example has been shown in [10] if the two robots are completely identical. Assume \( G \) is a ring on \( n \) vertices and that the edges are labeled so that out of every vertex, edge 0 goes clockwise, while edge 1 goes anti-clockwise. If the two robots start the
same time at different vertices and follow the same instructions, they would never meet! Same as the previous work [10], We also assume that the moves of the two robots are synchronous after both of them are activated. The crucial feature of this problem is that the two robots may be activated at different times which decides arbitrarily by the adversary. A meeting can happen only when both robots are active. The time complexity of any solution is bounded by the number of steps used to complete such a task, which counts from the activation of the second robot.

The treasure hunt problem is a variant of the rendezvous problem in which the robots are assigned the labels 0 and 1 and robot 0, the treasure, cannot move, which firstly introduced in [10]. As in the rendezvous problem, the treasure and the seeking robot are not necessarily activated at the same time.

1.1. Previous work

Dessmark et al. [3] presented a deterministic solution of the rendezvous problem which guarantees a meeting of the two robots after a number of steps which is polynomial in \( n \), the size of the graph, \( l \), the length of the shorter of the two labels, and \( \tau \), the difference between their activation times. More specifically, the bound on the number of steps that they obtain is \( \tilde{O}(n^{5} \sqrt{\tau l} + n^{10} l) \). In the same paper, Dessmark et al. [3] also ask whether it is possible to obtain a polynomial bound that is independent of \( \tau \). Kowalski and Malinowski [7] have recently presented a deterministic solution to the rendezvous problem that guarantees a meeting after at most \( \tilde{O}(n^{15} + l^3) \) steps, which is independent of \( \tau \), and also firstly answer the open problem of [3] when backtracking is allowed. Very recently in [10], Ta-Shma, and Zwick propose a deterministic solution that guarantees a rendezvous within \( \tilde{O}(n^{5} l) \) time units after the activation of the second robot, and also uses backtracking. This is the currently best known solution. All the solutions mentioned above rely on the existence of a universal traversal sequences, introduced by Aleliunas et al. [1], and are therefore non-explicit. The first explicit solution for both rendezvous problem and treasure hunt problem can be found in [10]. This work allows backtracking, by using the explicit construction of a strongly universal exploration sequence SUES. The time complexity of the solutions for both problems is \( O(n^{2c}) \), where \( c \) is a huge constant. Other variants of the rendezvous problem could be found in [5].

Note if randomization is allowed, then both the rendezvous problem and the treasure hunt problem have the trivial solutions using a polynomial number of steps in term of the size of the graph with high probability, e.g. a random walk by Coppersmith et al. [2].
1.2. Our results

We mainly study here the explicit deterministic treasure hunt problem with backtracking in a \( n \)-vertex network. It is a variant of the well known rendezvous problem in which one of the robot (the treasure) is always stationary. We propose an \( O(n^{c(1+\frac{1}{\lambda})}) \)-time algorithm for treasure hunt problem, which significantly improves currently best known result with running time \( O(n^{2c}) \) in \([10]\), where \( c \) is a constant induced from the construction of an universal exploration sequence in \([9,10]\), \( \lambda \gg 1 \) is a constant integer. The treasure hunt problem also motivates the study of strongly universal exploration sequences. In this paper, we also propose a much better explicit construction for strongly universal exploration sequences (SUESs) compared to the one in \([10]\). The improved explicit SUESs could be also used to improve the explicit solution of the rendezvous problem in \([10]\).

2. The treasure hunt problem

The treasure hunt problem is a variant of the well known rendezvous problem in which one of the robots, the treasure, is always stationary. A seeking robot and the treasure are placed in an unknown location in an unknown environment, modeled again by a finite, connected, undirected graph. Same as in the rendezvous problem, the treasure and the seeking robot searching for it are not necessarily activated at the same time. The most difficult case of the problem is when the seeking robot is activated before the treasure. To clarify our presentation, we give a formal definition for the treasure hunt problem, which is a modified version from the rendezvous problem by Ta-Shma and Zwick \([10]\).

Formally, a deterministic solution for the general treasure hunt problem (without backtracking) is a deterministic algorithm that computes a function \( f : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \), where for \( d \geq 1 \) and \( t \geq 0 \) we have \( 0 \leq f(d, t) \leq d - 1 \). This function defines the walk carried out by the seeking robot as follows: at the \( t \)-th time unit since activation, when at a vertex of degree \( d \), use edge number \( f(d, t) \) to walk in the next step.

A deterministic solution for the treasure hunt problem with backtracking is a deterministic algorithm that computes a function \( f : \mathbb{Z}^+ \times \mathbb{Z}^+ \times (\mathbb{Z}^+)^* \rightarrow \mathbb{Z}^+ \), where for every \( d \geq 1 \), \( t \geq 0 \), and \( T \in (\mathbb{Z}^+)^* \) we have \( 0 \leq f(d, t, T) \leq d - 1 \). This function defines the walk carried out by the seeking robot as follows: at the \( t \)-th time unit since activation, when at a vertex of degree \( d \), if the sequence of edge numbers assigned to the edges that were used to enter the vertices at the previous time units is \( T \in (\mathbb{Z}^+)^* \) the robot will exit the current node using the
edge number } f(d, t, T), in the next step. In our solutions that use this model, the function } f depends on } T only through its last element, which is the same as [10].

Throughout most of this paper we shall assume that the graph } G in which the robots (seeking robot and the treasure robot) are placed is a } d-regular graph, for some } d \geq 3. Note it is easy to extend the solutions given for the } d-regular graphs to general graphs using the ideas from [3].

3. Universal and strongly universal exploration sequences

For clarity of presentation, we use the same definitions as in [10]. Let } G = (V, E) be a } d-regular graph. A sequence } \tau_1 \tau_2 \cdots \tau_k \in \{0, 1, 2, \cdots, d-1\}^k \text{ and a starting edge } e_0 = (v_{-1}, v_0) \in E \text{ define a walk } v_{-1}, v_0, \cdots, v_k \text{ as follows: For } 1 \leq i \leq k, \text{ if } (v_{i-1}, v_i) \text{ is the } s\text{-th edge of } v_i, \text{ let } e_i = (v_i, v_{i+1}) \text{ be the } (s+\tau_i)\text{-th edge of } v_i, \text{ where we assume here that the edges of } v_i \text{ are numbered } 0, 1, \cdots, d-1, \text{ and that } s + \tau_i \text{ is computed by modulo } d.

Definition 1. (Universal Exploration Sequences (UESs) [6, 10]) A sequence } \tau_1 \tau_2 \cdots \tau_l \in \{0, 1, \cdots, d-1\}^l \text{ is a universal exploration sequence (UES) for } d\text{-regular graphs of size at most } n \text{ if for every connected } d\text{-regular graph } G = (V, E) \text{ on at most } n \text{ vertices, any numbering of its edges, and any starting edge } (v_{-1}, v_0) \in E, \text{ the walk obtained visits all the vertices of the graph.}

Reingold [9] obtains an explicit construction of polynomial-size UES:

Theorem 1. ([9]) There exists a constant } c \geq 1 \text{ such that for every } d \geq 3 \text{ and } n \geq 1, \text{ a UES of length } O(n^c) \text{ for } d\text{-regular graphs of size at most } n \text{ can be constructed, deterministically, in polynomial time.}

Definition 2. (Strongly Universal Exploration Sequences (SUESs) [10]) A possibly infinite sequence } \tau = \tau_1 \tau_2 \cdots, \text{ where } \tau_i \in \{0, 1, \cdots, d-1\}, \text{ is a strongly universal exploration sequence (SUES) for } d\text{-regular graphs with cover time } p(\cdot), \text{ if for any } n \geq 1, \text{ any contiguous subsequence of } \tau \text{ of length } p(n) \text{ is a UES for } d\text{-regular graphs of size } n.

Let } O(n^c) \text{ be the length of a UES (see [6, 10]), the main Theorem of this section shows that strongly universal exploration sequences (SUESs) do exist and they can be constructed deterministically in polynomial time with cover time } p(n) = O(n^{c(1+\frac{1}{d})}), \text{ which significantly improves the currently best known result in [10] with } p(n) = O(n^{2c}), \text{ where } c \text{ is a constant induced from the construction}
of an universal exploration sequence in [9, 10], and $\lambda \gg 1$ is an arbitrary large, but fixed, integer constant.

In this section, we firstly give a weak solution with $p(n) = O(n^{2c})$, then we show our main result described above.

3.1. Explicit SUESs with $p(n) = O(n^{2c})$

In this section, we propose a new explicit strongly universal exploration sequence with cover time $p(n) = O(n^{2c})$ for the $d$-regular graphs of size at most $n$, where $c$ is the same constant as was used in [9, 10]. It is a weak version (a special case) of our main result in Section 3.2, but which gives more intuition on our new approaches in Section 3.2.

3.1.1. Properties of exploration sequences

A very useful property of exploration sequences [6, 10] is that walks defined by an exploration sequence can be reversed. For $\tau = \tau_1 \tau_2 \cdots \tau_k \in \{0, 1, \ldots, d-1\}^k$, we let $\tau^{-1} = \tau_k^{-1} \tau_{k-1}^{-1} \cdots \tau_1^{-1}$, where $\tau_i^{-1} = d - \tau_i \mod d$. It is not difficult to see that a walk defined by an exploration sequence $\tau$ can be backtracked by executing the sequence $0\tau^{-1}0$. Note that if $e_0, e_1, \ldots, e_k$ is the sequence of edges defined by $\tau$, starting with $e_0$, then executing $0\tau^{-1}0$, starting with $e_k$ defines the sequence $e_k, \bar{e}_k, \bar{e}_{k-1}, \ldots, \bar{e}_0, e_0$, where $\bar{e}$ is the reverse of edge $e$. Also, if $\tau$ is a universal exploration sequence for graphs with size at most $n$, then so is $0\tau^{-1}0$ starting with the last edge defined by $\tau$.

3.1.2. Construction of SUESs

Let $U_n$ be a sequence of length $n$ which is a universal exploration sequence for $d$-regular graphs of size at most $bn^{\frac{1}{c}}$, for some constants $b > 1$, and $c > 1$, which can be constructed, deterministically, in polynomial time of $n$ due to Theorem 1. We are interested in sequences $U_n$ only if $n$ is a power of 2. For the sake of technique, we can construct $U_n = U_1 U_1 U_2 U_4 \cdots U_n$. Therefore, $U_k$ is a prefix of $U_n$, for every $k = 2^i$ and $n = 2^j$, where $i < j$.

A strongly universal exploration sequence $S_n$ is a sequence defined in a recursive manner. Our approach is based on the similar idea in [10], but different interleaving components between the symbols which originate from $U_n$. We begin with $S_1 = U_1$. Assume that $U_n = u_1 u_2 \cdots u_n$ and that $n \geq 2$.

Define,

$$S_n = u_1 S_{r_1} 0 S_{r_1}^{-1} 0 u_2 S_{r_2} 0 S_{r_2}^{-1} 0 u_3 \cdots u_i S_{r_i} 0 S_{r_i}^{-1} 0 u_{i+1} \cdots u_{n-1} S_{r_{n-1}} 0 S_{r_{n-1}}^{-1} 0 u_n,$$
for every $1 \leq i < n$, we set $r_i = \langle i \rangle$, where $\langle i \rangle = \max\{2^j | 2^j \leq i, j \in \mathbb{Z}^+\}$. Similar construction strategy for $U_n$ (e.g., $U_n = U_1 U_2 U_3 \cdots U_{\frac{n}{2}}$) is also adopted to construct $S_n$. Note that the sequence $S_k$ is a prefix of $S_n$ for every $k = 2^p$ and $n = 2^j$, where $p < j$. Moreover, we also assign $r_i = r_{n-i}$, for every $1 \leq i < n$.

Furthermore, the sequence $S_n^{-1}$ differs with $S_n$ only on the symbols that originate from $U_n$ and in the alignment of the 0’s:

$$S_n^{-1} = u_n^{-1} S_{r_1} S_{r_1}^{-1} u_{n-1}^{-1} \cdots u_{i+1}^{-1} S_{r_n-i}^{-1} u_{r_n-i}^{-1} \cdots u_2^{-1} S_{r_n-1}^{-1} u_{r_n-1}^{-1}.$$

Note that $r_i \leq \sqrt{2^j}$. Thus, if $r_{\frac{n}{2}} = \sqrt{\frac{1}{2^j}}$, the first half of $S_n$ is equal to $S_{\frac{n}{2}} S_{\sqrt{\frac{1}{2^j}}} 0$, and ends with a full copy of $S_{\sqrt{\frac{1}{2^j}}}$, followed by a 0. Similarly, the second half of $S_n$ starts with a full copy of $S_{\sqrt{\frac{1}{2^j}}}^{-1}$. In the following, we bound the length of $S_n$.

**Lemma 1.** For every $n = 2^j$, where $j \geq 1$, $|S_n| < 258n$.

**Proof.** Let $s_n = |S_n|$. It is not difficult to see that $s_1 = 1$, $s_2 = 6$, $s_4 = 16$, $s_8 = 46$, $s_{16} = 126$, $s_{32} = 286$, $\cdots$, $s_{256} = 3426$. The claim that $s_n \leq 258n$ for every $n \geq 512$ then follows by using simple induction. It is not difficult to see that

$$|S_{2^i}| = |U_{2^i}| + (|U_{2^i}| - 1) \cdot 2(|S_{1}| + 1) + \sum_{j=1}^{\frac{2^i}{2}} \left[ \frac{|U_{2^i}| - 1}{2^{\frac{3j}{2}}} \right] \cdot 2(|S_{2^j}| - |S_{2^j-1}|)$$

$$s_{2^i} \leq 2^i + 4 \cdot 2^i + 2^i \cdot \sum_{j=1}^{\frac{i}{2}} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}}$$

$$= 2^i + 4 \cdot 2^i + 2^i \cdot \left( \sum_{j=1}^{4} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}} \right) + \sum_{j=5}^{\frac{2^i}{2}} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}}$$

$$\leq 5 \cdot 2^i + 2i \cdot \sum_{j=1}^{4} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}} + 2^i \cdot \sum_{j=5}^{\frac{2^i}{2}} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}}$$

$$\leq 5 \cdot 2^i + 2i \cdot \sum_{j=1}^{4} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}} + 2^i \cdot \sum_{j=5}^{+\infty} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}}$$

$$\leq 5 \cdot 2^i + 2i \cdot \sum_{j=1}^{4} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}} + 2^i \cdot \sum_{j=5}^{+\infty} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{3j}{2}}} + 2^i \cdot \sum_{j=5}^{+\infty} \frac{4}{2^{\frac{3j}{2}}}$$
The sequence \( S_n \) possesses the following interesting combinatorial property:

**Lemma 2.** Let \( k \) and \( n \geq 2k^{\frac{3}{2}} \) be powers of 2. Then, every subsequence \( T \) of \( S_n \) or \( S_{n-1} \) of length \( s_{2k^{\frac{3}{2}}} + 1 = |S_{2k^{\frac{3}{2}}}^j| + 1 \leq 516k^{\frac{3}{2}} \) contains, as a contiguous subsequence, \( S_k \) or \( 0S_k^{-1} \).

**Proof.** We prove the claim by induction on \( n \). If \( n = 2k^{\frac{3}{2}} \) then the claim is vacuously satisfied as \( S_n \) contains a full \( S_k \).

Assume, therefore, that the claim holds for every \( m = 2^j \) that satisfies \( 2k^{\frac{3}{2}} \leq m < n = 2^j \). We show that it also holds for \( n \). Let \( T \) be a subsequence of \( S_n \) of length \( s_{2k^{\frac{3}{2}}} + 1 \). Essentially the same argument works if \( T \) is a subsequence of such length of \( S_{n-1} \). We use the exactly same arguments as in Lemma 6.2 [10]. For completeness of our presentation, we reproduce the analysis from [10].

We consider the following cases:

**Case 1:** \( T \) is completely contained in a subsequence \( S_m \) or \( S_{m-1} \) of \( S_n \), for some \( m < n \).

The claim then follows immediately from the induction hypothesis.

**Case 2:** \( T \) is completely contained in a subsequence \( S_m0S_m^{-1} \) of \( S_n \), for some \( m < n \).
In this case, $T = T'0T''$, where $T'$ is a suffix of $S_m$ and $T''$ is a prefix of $S_{m-1}$. Either $|T'| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$ or $|T''| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$. Assume that $|T'| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$. Another case is analogous. As $T''$ is a prefix of $S_{m-1}$, and $|T''| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$, it follows that $m \geq 2k^\frac{3}{2}$.

Now, $S_k^{-1}$ is almost a prefix of $S_{m-1}$, in the sense that they differ only in symbols that originate directly from $S_m$. In particular, a prefix of $S_{m-1}$ of length $\frac{1}{2}s_{2k^\frac{3}{2}}$, half the length of $S_{2k^\frac{3}{2}}$, ends with a full copy of $S_k$, followed by 0.

**Case 3:** $T$ contains a symbol $u_l$ of $S_n$ that originates from $U_n$.

In this case, $T = T'u_lT''$. Again, we have either $|T'| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$ or $|T''| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$. Assume again that $|T''| \geq \frac{1}{2}s_{2k^\frac{3}{2}}$. Another case is analogous. Let

$$S_{n,l} = u_lS_{r_l}0S_{r_l-1}0u_{l+1} \cdots u_{n-1}S_{r_{n-1}}0S_{r_{n-1}}^{-1}0u_n$$

be the suffix of $S_n$ that starts with the symbol $u_l$ that originates from the $l$-th symbol of $U_n$. We claim that the prefix of $S_{n,l}$ of length $\frac{1}{2}s_{2k^\frac{3}{2}}$ contains a copy of $S_k$. Let $l' = \left\lceil \frac{l}{k^\frac{3}{2}} \right\rceil k^\frac{3}{2}$ be the first index after $l$ which is divisible by $k^\frac{3}{2}$. Clearly $r_{l'} \geq k$ and hence $S_k$ is a prefix of $S_{r_{l'}}$. Thus, $S' = u_lS_{r_l}0S_{r_l-1}0 \cdots u_{l'}S_{r_{l'}}$ is a prefix of $S_{n,l}$ which ends with a complete $S_k$. As for every $l \leq i < l'$ we have $r_i = r_{i mod k^\frac{3}{2}}$, we have that $S'$ is contained in the first half of $S_{2k^\frac{3}{2}}$, and hence $|S'| \leq \frac{1}{2}s_{2k^\frac{3}{2}}$ as expected.

We are now ready to prove the following Theorem.

**Theorem 2.** If for any $n \geq 1$ of the power of 2 there exists an UES of length $O(n^c)$ for a $d$-regular graph of size at most $n$, then there is an infinite SUES for this $d$-regular graph with cover time $p(n) = O(n^\frac{c}{2})$, where $c$ is the fixed constant from the construction of an universal exploration sequence in [1]. Furthermore, the SUESs can be constructed deterministically in polynomial time.

**Proof.** Let us look at the recursive definition of $S_n$ and ignore all the recursive components of $S_j$ such as $j < n$, and their inverses, which because that $0S_{j-10}$ reverses the actions of $S_j$. The left parts are $U_n = u_1, u_2, u_3, \ldots, u_n$. However, note that $U_n$ is a UES for for $d$-regular graphs of size at least $bn^\frac{1}{2}$, for some constants $b \geq 1$ and $c > 1$ due to Theorem [1]. Theorem [1] also show that such a UES can be constructed, deterministically, in polynomial time. According to Lemma [2], we know that every subsequence $T$ of the SUES we constructed of length $s_{2n^\frac{1}{2}} + 1 = O(n^\frac{c}{2})$ contains, as a contiguous subsequence, a full copy of $S_n$. Consequently, there is an infinite SUES for $d$-regular graphs with cover time
\( p(n) = O(n^{2c}) \), where \( c \) is the fixed constant from the construction of an universal exploration sequence in [9, 10]. Furthermore, the SUESs can be constructed deterministically in polynomial time.

This thus gives us an explicit solution to the treasure hunt problem. In fact, the seeking robot just need run the SUES. The adversary will decide when the treasure is put into the graph. But note that the subsequence of a SUES with length \( p(n) \) starting at the activation point forms a UES, and then the seeking robot finds the treasure by following the instruction of the sequence.

### 3.2. Explicit SUESs with \( p(n) = O(n^{c(1+\frac{1}{\lambda})}) \)

In this section, we propose our main result, a new explicit strongly universal exploration sequence with cover time \( p(n) = O(n^{c(1+\frac{1}{\lambda})}) \), which significantly improves the currently best known result in [10] with \( p(n) = O(n^{2c}) \), where \( c \) is a fixed constant induced from the construction of an universal exploration sequence in [9, 10], and \( \lambda \gg 1 \) is an arbitrary large, but fixed, integer constant.

#### 3.2.1. Treasure sequences

A treasure sequence is an infinite sequence \( Q(\lambda) = q_1, q_2, q_3, q_4, \ldots \), based on a constant integer \( \lambda \gg 1 \) such as

\[
q_i = \sum_{j=i}^{+\infty} \left( 2^{-j} + 2^{-\left(\frac{2\lambda+1}{\lambda}j + \frac{\lambda}{\lambda+1}j - 2\right)} + 2^{-\left(\frac{\lambda+1}{\lambda}j - 2\right)} \right),
\]

where \( i \geq 1 \). It is easy to see that the treasure sequences \( Q(\lambda) \) are monotonely decreasing, \( \lim_{i \to +\infty} q_i = 0 \). We call the first element or term \( q_i < 1 \) in \( Q(\lambda) \) a golden ball, where \( i \geq 1 \). Similarly, we call the fixed index \( i \) of the golden ball of \( Q(\lambda) \) as the golden point, where \( \lambda \gg 1 \) is an arbitrary large, but fixed, integer constant.

The following Lemma follows directly.

**Lemma 3.** There exists a golden ball in the treasure sequence.

The property of the treasure sequence will be used to further reduce the length of the cover time \( p(n) \) of the SUESs later.
3.2.2. Construction of SUESs

Same as in Section 3.1.2 let $U_n$ be a sequence of length $n$ which is a UES for $d$-regular graphs of size at most $bn^\frac{c}{2}$, for some constants $b \geq 1$, and $c > 1$. And for every $k = 2^i$ and $n = 2^j$, where $i < j$, $U_k$ is a prefix of $U_n$.

We now define recursively a sequence $S_n$ of strongly universal exploration sequences. We start with $S_1 = U_1$. Assume that $U_n = u_1 u_2 \cdots u_n$ and that $n \geq 2$. Define,

$$S_n = u_1 S_{r_3} 0 S_{r_2} 0 S_{r_1} 0 u_3 \cdots u_i S_{r_3} 0 S_{r_2} 0 S_{r_1} 0 u_{i+1} \cdots u_{n-1} S_{r_{n-1}} 0 S_{r_{n-1}} 0 u_n,$$

where for every $1 \leq i < n$, we set $r_i = \langle \langle i \rangle \rangle$, where $\langle \langle i \rangle \rangle = \max \{ 2^j | 2^\lambda i \geq j, j \in \mathbb{Z}^+ \}$. Note that as $n = 2^j$, for some $j \geq 1$, for every $k = 2^p$, where $p < j$, the sequence $S_k$ is constructed as a prefix of $S_n$. Moreover, we also assign $r_i = r_{n-i}$ for every $1 \leq i < n$.

Note that $r_i \leq \frac{\lambda + 1}{i}$. Thus, if $r_\frac{\lambda}{2} = \frac{\lambda + 1}{\sqrt{2}}$, the first half of $S_n$ is equal to $\lambda \frac{u}{\sqrt{2}} 0$, and ends with a full copy of $\lambda \frac{u}{\sqrt{2}} 0$, followed by a 0. Similarly, the second half of $S_n$ starts with a full copy of $\lambda \frac{u}{\sqrt{2}} 0$.

We next bound the length of $S_n$.

Let $q_{g}^{[\lambda]}$, $g$ denote the golden ball and golden point of the treasure sequence $Q(\lambda)$, respectively. Further more, if $g \geq 2$, we set $C_{\text{finite}} = \sum_{j=1}^{g-1} 2|S_{gj}| - |S_{gj-1}|$, and $C_{\text{max}} = \max \{ y | y = \frac{s_j}{2^j}, \text{for } 1 \leq j \leq g-1 \}$, otherwise $C_{\text{finite}} = C_{\text{max}} = 0$ (e.g. $g = 1$). Consequently, $q_{g}^{[\lambda]}$, $g$, $C_{\text{finite}}$, and $C_{\text{max}}$ are constants due to the definition of the treasure sequence and the fact that only a finite number of terms are involved in the calculations, where $0 < q_{g}^{[\lambda]} < 1$, and $g \geq 1$.

We are now ready to prove the following Lemma.

**Lemma 4.** For every $n = 2^j$, where $j \geq 1$, $|S_n| < \left( \frac{5 + C_{\text{finite}}}{1 - q_{g}^{[\lambda]}} + C_{\text{max}} \right) n$.

**Proof.** Let $s_n = |S_n|$. For every $n \leq 2^g - 1$, we know it is true due to the definition of the constant $C_{\text{max}}$. The claim that $s_n < \left( \frac{5 + C_{\text{finite}}}{1 - q_{g}^{[\lambda]}} + C_{\text{max}} \right) n$ for every $n \geq 2^g$ then follows by using the induction. It is not difficult to see that

$$|S_{2^i}| = |U_{2^i}| + (|U_{2^i}| - 1) \cdot 2(|S_1| + 1) + \sum_{j=1}^{\frac{|S_{2^i}| - 1}{2(\lambda + 1) + \lambda}} \left( \frac{|U_{2^i}| - 1}{2^{(\lambda + 1) + \lambda}} \right) \cdot 2(|S_{2^j}| - |S_{2^{j-1}}|)$$
\[ s_{2^i} \leq 2^i + 4 \cdot 2^i + 2^i \cdot \sum_{j=1}^{\left\lfloor \frac{i}{2} \right\rfloor - 1} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{(j+1)^2}{2}}} + 2^i \cdot \sum_{j=g}^{\left\lfloor \frac{i}{2} \right\rfloor - 1} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{(j+1)^2}{2}}} \]

\[ = 2^i + 4 \cdot 2^i + 2^i \cdot \sum_{j=1}^{g-1} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{(j+1)^2}{2}}} + 2^i \cdot \sum_{j=g}^{\infty} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{(j+1)^2}{2}}} \]

\[ < 5 \cdot 2^i + 2^i \cdot \sum_{j=1}^{g-1} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{(j+1)^2}{2}}} + 2^i \cdot \sum_{j=g}^{\infty} \frac{2(s_{2j} - s_{2j-1})}{2^{\frac{(j+1)^2}{2}}} \]

\[ \leq 5 \cdot 2^i + C_{finite} \cdot 2^i + 2^i \cdot \sum_{j=g}^{\infty} \frac{2(s_{2j-1} + 2s_{2j})}{2^{\frac{(j+1)^2}{2}}} \]

\[ < 5 \cdot 2^i + C_{finite} \cdot 2^i + 2^i \cdot \left( \frac{5 + C_{finite}^{\frac{1}{x}}} {1 - q_g^{\left\lfloor \frac{1}{x} \right\rfloor}} + C_{max} \right) \cdot \sum_{j=g}^{\infty} \frac{2(2j-1 + 2\cdot 2^{(j-1)\frac{1}{x}})}{2^{\frac{(j+1)^2}{2}}} \]

\[ = 5 \cdot 2^i + C_{finite} \cdot 2^i + 2^i \cdot \left( \frac{5 + C_{finite}^{\frac{1}{x}}} {1 - q_g^{\left\lfloor \frac{1}{x} \right\rfloor}} + C_{max} \right) \cdot q_g^{\left\lfloor \frac{1}{x} \right\rfloor} \]

\[ < 5 \cdot 2^i + C_{finite} \cdot 2^i + 2^i \cdot C_{max} \cdot (1 - q_g^{\left\lfloor \frac{1}{x} \right\rfloor}) + 2^i \cdot \left( \frac{5 + C_{finite}^{\frac{1}{x}}} {1 - q_g^{\left\lfloor \frac{1}{x} \right\rfloor}} + C_{max} \right) \cdot q_g^{\left\lfloor \frac{1}{x} \right\rfloor} \]

\[ = 2^i \cdot \left( \frac{5 + C_{finite}^{\frac{1}{x}}} {1 - q_g^{\left\lfloor \frac{1}{x} \right\rfloor}} + C_{max} \right). \]

Using the same arguments as in Lemma 2, we can prove the following Lemma.

**Lemma 5.** Let \( k \) and \( n \geq 2k^{\frac{x+1}{x}} \) be powers of 2. Then, every subsequence \( T \) of \( S_n \) or \( S_n^{-1} \) of length \( s_{2k^{\frac{x+1}{x}}} + 1 \) contains, as a contiguous subsequence, a full of \( S_k \) or \( 0S_k^{-1} \).

**Proof.** We prove the claim by induction on \( n \). If \( n = 2k^{\frac{x+1}{x}} \) then the claim is vacuously satisfied as \( S_n \) contains a full \( S_k \).

Assume, therefore, that the claim holds for every \( m = 2^j \) that satisfies \( 2k^{\frac{x+1}{x}} \leq m < n = 2^i \). We show that it also holds for \( n \). Let \( T \) be a subsequence of \( S_n \) of length \( s_{2k^{\frac{x+1}{x}}} + 1 \). Essentially the same argument works if \( T \) is a subsequence of such length of \( S_n^{-1} \).

Same as in Lemma 2, we study the following cases:

**Case 1:** \( T \) is completely contained in a subsequence \( S_m \) or \( S_m^{-1} \) of \( S_n \), for some \( m < n \).
The claim then follows immediately from the induction hypothesis.

**Case 2:** $T$ is completely contained in a subsequence $S_m\cdot S_m^{-1}$ of $S_n$, for some $m < n$.

In this case, $T = T'0T''$, where $T'$ is a suffix of $S_m$ and $T''$ is a prefix of $S_m^{-1}$. Either $|T'| \geq \frac{1}{2}s_{2k^\lambda r_1+1}$ or $|T''| \geq \frac{1}{2}s_{2k^\lambda r_1+1}$. Assume that $|T''| \geq \frac{1}{2}s_{2k^\lambda r_1+1}$. Another case is analogous. As $T''$ is a prefix of $S_m^{-1}$, and $|T''| \geq \frac{1}{2}s_{2k^\lambda r_1+1}$, it follows that $m \geq 2k^\lambda r_1$. Now, $S_k^{-1}$ is almost a prefix of $S_m^{-1}$, in the sense that they differ only in symbols that originate directly from $S_m$. In particular, a prefix of $S_m^{-1}$ of length $\frac{1}{2}s_{2k^\lambda r_1}$, half the length of $\frac{1}{2}s_{2k^\lambda r_1}$, ends with a full copy of $S_k$, followed by 0.

**Case 3:** $T$ contains a symbol $u_t$ of $S_n$ that originates from $U_n$.

In this case, $T = T'u_tT''$. Again, we have either $|T'| \geq \frac{1}{2}s_{2k^\lambda r_1}$ or $|T''| \geq \frac{1}{2}s_{2k^\lambda r_1}$. Assume again that $|T''| \geq \frac{1}{2}s_{2k^\lambda r_1}$. Another case is analogous. Let $S_{n,t} = u_tS_{r_1}0S_{r_1}^{-1}0u_{t+1} \cdots u_{n-1}S_{r_{n-1}}0S_{r_{n-1}}^{-1}0u_n$ be the suffix of $S_n$ that starts with the symbol $u_t$ that originates from the $l$-th symbol of $U_n$. We claim that the prefix of $S_{n,t}$ of length $\frac{1}{2}s_{2k^\lambda r_1}$ contains a copy of $S_k$. Let $l' = \left\lceil \frac{r_0 - 1}{k^\lambda} \right\rceil k^\lambda r_1$ be the first index after $l$ which is divisible by $k^\lambda r_1$. Clearly $r_{l'} \geq k$ and hence $S_k$ is a prefix of $S_{r_{l'}}$. Thus, $S' = u_tS_{r_1}0S_{r_1}^{-1}0 \cdots u_{l'}S_k$ is a prefix of $S_{n,t}$ which ends with a complete $S_k$. As for every $l \leq i < l'$ we have $r_i = r_{i \mod k^\lambda r_1}$, we have that $S'$ is contained in the first half of $\frac{1}{2}s_{2k^\lambda r_1}$, and hence $|S'| \leq \frac{1}{2}s_{2k^\lambda r_1}$ as expected.

Furthermore, by the same arguments as in Theorem 2 we have:

**Theorem 3.** If for every $n \geq 1$ of the power of 2 there is an UES of length $O(n^c)$ for $d$-regular graphs of size at most $n$, then there exists an infinite SUES for $d$-regular graphs with cover time $p(n) = O(n^{c(1+\frac{1}{d})})$, where $c$ is the constant induced from the construction of an universal exploration sequence in [10], and $\lambda \gg 1$ is an arbitrary large, but fixed, integer constant. Moreover, the SUESs can be constructed deterministically in polynomial time.

**Proof.** Let us ignore all the recursive components of $S_j$ from $S_n$ such as $j < n$, and their inverses, which because that $0S_j^{-1}0$ reverses the actions of $S_j$. The left parts are $U_n = u_1, u_2, u_3, \cdots, u_n$. Moreover, note that $U_n$ is a UES for $d$-regular graphs of size at least $bn^{\frac{1}{d}}$, for some constants $b \geq 1$ and $c > 1$ due to
Theorem 1. Theorem 1 also shows that such a UES could be constructed, deterministically, in polynomial time. According to Lemma 5, we know that every subsequence $T$ of the SUES we constructed of length $s^{\frac{\lambda+1}{\lambda}} + 1 = O(n^{\frac{\lambda+1}{\lambda}})$ contains, as a contiguous subsequence, a full copy of $S_n$. Consequently, there is an infinite SUES for $d$-regular graphs with cover time $p(n) = O(n^{\frac{\lambda+1}{\lambda}c})$, where $c$ is the fixed constant from the construction of an universal exploration sequence in [9, 10]. Furthermore, the SUESs can be constructed deterministically in polynomial time.

Finally, by employing the standard double techniques in $d$-regular graphs of size at most $n$, we get the desired result.

**Theorem 4.** If for every $n \geq 1$ there is an UES of length $O(n^c)$ for $d$-regular graphs of size at most $n$, then there exists an infinite SUES for $d$-regular graphs with cover time $p(n) = O(n^{c(1+\frac{1}{\lambda})})$, where $c$ is the fixed constant induced from the construction of an universal exploration sequence in [9, 10], and $\lambda \gg 1$ is an arbitrary large, but fixed, integer constant. Moreover, the SUESs can be constructed deterministically in polynomial time.

**Remark 1.** It is easy to extend the solutions given for the $d$-regular graphs to general graphs by using the ideas from [3, 10].

**Remark 2.** The proposed explicit SUESs could be also used to improve the running time of the explicit solution suggested for the rendezvous problem with backtracking in [10].

### 4. Conclusion and open problems

We proposed an improved explicit deterministic solution for the treasure hunt problem with backtracking. More precisely, we derived an $O(n^{c(1+\frac{1}{\lambda})})$-time algorithm for the treasure hunt, which significantly improves the currently best known result with running time $O(n^{2c})$ in [10], where $c$ is the constant induced from the construction of an universal exploration sequence in [9, 10], $\lambda \gg 1$ is an arbitrary large, but fixed, integer constant. In this work, we also proposed a much better explicit construction for strongly universal exploration sequences compared to the one in [10]. The proposed explicit SUESs could be also used to further improve the time complexity of the explicit solution addressed for the rendezvous problem with backtracking in [10].
The existence of strongly universal exploration sequences without backtracking is left as an intriguing open problem.

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