Complete physical simulation of the entangling-probe attack on the BB84 protocol

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We have used deterministic single-photon two qubit (SPTQ) quantum logic to implement the most powerful individual-photon attack against the Bennett-Brassard 1984 (BB84) quantum key distribution protocol. Our measurement results, including physical source and gate errors, are in good agreement with theoretical predictions for the Rényi information obtained by Eve as a function of the errors she imparts to Alice and Bob’s sifted key bits. The current experiment is a physical simulation of a true attack, because Eve has access to Bob’s physical receiver module. This experiment illustrates the utility of an efficient deterministic quantum logic for performing realistic physical simulations of quantum information processing functions.

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In 1984 Bennett and Brassard [1] proposed a protocol (BB84) for quantum key distribution (QKD) in which the sender (Alice) transmits single-photon pulses to the receiver (Bob) in such a way that security is vouchsafed by physical laws. Since then, BB84 has been the subject of many security analyses [2], particularly for configurations that involve nonideal operating conditions, such as the use of weak laser pulses in lieu of single photons. A more fundamental question is how much information the eavesdropper (Eve) can gain under ideal BB84 operating conditions. Papers by Fuchs and Peres [3], Slutsky et al. [4], and Brandt [5] show that the most powerful individual-photon attack can be accomplished with a controlled-NOT (CNOT) gate. As illustrated in Fig. 1, Eve supplies the target qubit to the CNOT gate, which entangles it with the BB84 qubit that Alice is sending to Bob. Eve then makes her measurement of the target qubit to obtain information on the shared key bit at the expense of imposing detectable errors between Alice and Bob [3, 6].

We have recently shown [6] that this Fuchs-Peres-Brandt (FPB) entangling probe can be implemented using single-photon two-qubit (SPTQ) quantum logic in a proof-of-principle experiment. In SPTQ logic a single photon carries two independent qubits: the polarization and the momentum (or spatial orientation) states of the photon. Compared to standard two-photon quantum gates, SPTQ gates are deterministic and can be efficiently implemented using only linear optical elements. We have previously demonstrated CNOT \( \text{\\[\text{\textbackslash gate\]}\] \text{\textbackslash gate\]} \) and SWAP \( \text{\\[\text{\textbackslash gate\]}\] \text{\textbackslash gate\]} \) gates in this SPTQ quantum logic platform. SPTQ logic affords a simple yet powerful way to investigate few-qubit quantum information processing tasks.

In this work we use SPTQ logic to implement the FPB probe as a complete physical simulation of the most powerful attack on BB84, including physical errors. This is to our knowledge the first experiment on attacking BB84, and the results are in good agreement with theoretical predictions. It is only a physical simulation because the two qubits of a single photon carrier must be measured jointly, so that Eve needs access to Bob’s receiver, but not his measurement. The SPTQ probe could become a true attack if quantum nondemolition measurements were available to Eve [6]. Nevertheless, the physical simulation allows investigation of the fundamental security limit of BB84 against eavesdropping in the presence of realistic physical errors, and it affords the opportunity to study the effectiveness of privacy amplification when BB84 is attacked.

\begin{align*}
|0\rangle_C &= \cos(\pi/8)|H\rangle + \sin(\pi/8)|V\rangle , \\
|1\rangle_C &= -\sin(\pi/8)|H\rangle + \cos(\pi/8)|V\rangle . 
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\end{align*}
Having selected the error probability, $P_E$, that she is willing to create, Eve prepares her probe qubit (CNOT’s target) in the initial state

$$|T_{in}\rangle = \{(C + S)|0\rangle_T + (C - S)|1\rangle_T\}/\sqrt{2}$$

$$= \cos \theta_{in}|0\rangle_T + \sin \theta_{in}|1\rangle_T$$

where $C = \sqrt{1 - 2P_E}, \ S = \sqrt{2P_E}$, and $\{0\rangle_T, |1\rangle_T$ is the target qubit’s computational basis. After the CNOT operation—with inputs from Alice’s photon and Eve’s probe—the two qubits become entangled. For each of Alice’s four possible inputs, $|H\rangle, |V\rangle, |D\rangle$, and $|A\rangle$, the output of the CNOT gate is

$$|H\rangle|T_{in}\rangle \rightarrow |H\rangle|T_{out}\rangle \equiv |H\rangle|T_0\rangle + |V\rangle|T_E\rangle,$$

$$|V\rangle|T_{in}\rangle \rightarrow |V\rangle|T_{out}\rangle \equiv |V\rangle|T_0\rangle + |H\rangle|T_E\rangle,$$

$$|D\rangle|T_{in}\rangle \rightarrow |D\rangle|T_{out}\rangle \equiv |D\rangle|T_0\rangle - |A\rangle|T_E\rangle,$$

$$|A\rangle|T_{in}\rangle \rightarrow |A\rangle|T_{out}\rangle \equiv |A\rangle|T_1\rangle - |D\rangle|T_E\rangle,$$

where $|T_0\rangle, |T_1\rangle$, and $|T_E\rangle$ are defined in the target qubit’s computational basis (see Fig. 2(b)) as:

$$|T_0\rangle \equiv \left(\frac{C}{\sqrt{2}} + \frac{S}{2}\right)|0\rangle_T + \left(\frac{C}{\sqrt{2}} - \frac{S}{2}\right)|1\rangle_T,$$

$$|T_1\rangle \equiv \left(\frac{C}{\sqrt{2}} - \frac{S}{2}\right)|0\rangle_T + \left(\frac{C}{\sqrt{2}} + \frac{S}{2}\right)|1\rangle_T,$$

$$|T_E\rangle \equiv \frac{S}{2}(|0\rangle_T - |1\rangle_T).$$

Consider the case in which Bob measures in the same basis that Alice employed and his outcome matches what Alice sent. Then, according to Eqs. 3–6, the target qubit is projected into either $|T_0\rangle$ or $|T_1\rangle$. After Alice and Bob compare their basis selections over the classical channel, Eve can learn about their shared bit value by distinguishing between the $|T_0\rangle$ and $|T_1\rangle$ output states of her target qubit. To do so, she employs the minimum error probability receiver for distinguishing between $|T_0\rangle$ and $|T_1\rangle$ by performing a projective measurement along $|0\rangle_T$ and $|1\rangle_T$. Eve can then correlate the measurement of $|0\rangle_T$ ($|1\rangle_T$) with $|T_0\rangle$ ($|T_1\rangle$). Note that this projective measurement is not perfect unless $|T_0\rangle$ and $|T_1\rangle$ are orthogonal and hence coincide with the target’s computational basis, $|0\rangle_T$ and $|1\rangle_T$.

Of course, Eve’s information gain comes at a cost: Eve has caused an error event whenever Alice and Bob choose a common basis and Eve’s probe output state is $|T_E\rangle$. When Alice sent $|H\rangle$ and Bob measured in the $H-V$ basis, Eq. 3 then shows that Alice and Bob will have an error event if the measured output state is $|V\rangle|T_E\rangle$. The probability that this will occur is $P_{E} = \langle T_E|T_E\rangle = \frac{S^2}{2}$.

For the other three cases in Eqs. 4–6, the error event corresponds to the last term in each expression. Therefore the conditional error probabilities are identical, and hence $P_E$ is the unconditional error probability.

To quantify Eve’s information gain, we use the Rényi information that she derives about the sift events in which Bob correctly measures Alice’s qubit. Let $B = 0, 1$ and $E = 0, 1$ denote the ensembles of possible bit values that Bob and Eve receive on an error-free sift event. The Rényi information (in bits) that Eve learns about each error-free sift event is

$$I_R = -\log_2 \left( \sum_{b=0}^{1} P^2(b) \right) + \sum_{b=0}^{1} P(e) \log_2 \left( \sum_{b=0}^{1} P^2(b|e) \right),$$

where $\{P(b), P(e)\}$ are the a priori probabilities for Bob’s and Eve’s bit values, and $P(b|e)$ is the conditional probability for Bob’s bit value to be $b$ given that Eve’s is $e$. According to Bayes’ rule, $P(b|e)$ can be calculated in terms of $P(b,e)$ which is the probability of both Bob’s bit to be $b$ and Eve’s bit to be $e$:

$$P(b|e) = \frac{P(b,e)}{\sum_{b=0}^{1} P(b,e)},$$

where $P(b = 0, e) = |\langle H | \otimes \tau(e)|H_{out}\rangle|^2$ and $P(b = 1, e) = |\langle V | \otimes \tau(e)|V_{out}\rangle|^2$ in the case of the $H-V$ basis ($|H\rangle$ corresponds to $b = 0$). A similar calculation for the $D-A$ basis, with $|D\rangle$ corresponding to $b = 0$ then leads to the final result

$$I_R = \log_2 \left( 1 + \frac{4P(E)(1-2P_E)}{(1-P_E)^2} \right).$$

Ideally, Eve’s Rényi information is the same for both the $H-V$ and $D-A$ bases, but in actual experiments it may differ, owing to differing equipment errors in each basis.
beams were separated by \( \sim \) not pumped Sagnac interferometric down-conversion source. We used a bidirectionally by cascading three cnot gates: a momentum-controlled swap gate on the target-qubit basis followed by single-photon detection (D1) through a polarizing beam splitter (PBS) P1 along \( H \). Q1 was used to compensate an intrinsic phase shift \( \xi \) imposed by the swap gate on the target-qubit basis \( \{ \vert 0 \rangle_T, \vert 1 \rangle_T \} \). We have independently measured \( \xi \approx 88^\circ \). Therefore, the \( R_E \) operation prepared the momentum qubit in \( \vert T'_0 \rangle \) with a Q1-imposed phase shift \( \xi \)

\[
\vert T'_0 \rangle \equiv \cos \theta_0 \vert 0 \rangle_T + e^{i \xi} \sin \theta_0 \vert 1 \rangle_T . \tag{13}
\]

The extra phase shift of the swap gate would bring Eve’s probe qubit to be in \( \vert T'_0 \rangle \) of Eq. 2.

After the swap gate, \( R_A \) and \( R_{\pi/8} \) were combined in a single operation. The p-cnot gate had the same phase shift problem as the swap gate, so we used a HWP (H2) and a QWP (Q2) to compensate for this phase shift and to impose the required rotation. After H2 and Q2 Alice’s qubit becomes

\[
\vert \Psi_A \rangle \equiv \cos \theta_A \vert 0 \rangle_C + e^{i \chi} \sin \theta_A \vert 1 \rangle_C ; \tag{14}
\]

where \( \chi \approx 98^\circ \) is the compensating phase shift and \( \theta_A \) is \(-22.5^\circ, 22.5^\circ, 0^\circ, 112.5^\circ \) for \( \vert H \rangle, \vert D \rangle, \vert V \rangle \) or \( \vert A \rangle \), respectively, as shown in Fig. 2(a). Similarly we combined \( R_{\pi/8} \) and \( R_B \) into a single HWP (H3) in Fig. 3 and a PBS (P2) was used by Bob to analyze the polarization of Alice’s qubit.

Eve measured her qubit by a projective measurement along the \( \vert 0 \rangle_T \sim \vert 1 \rangle_T \) (spatially, \( R-L \)) basis. A HWP (H4/H5) was placed in the \( R \) or \( L \) beam path, as indicated in Fig. 3 so that the \( R \) and \( L \) beams would be distinguished by their orthogonal polarizations. This polarization tagging simplified their measurements by a PBS (P3/P4) and single-photon detectors. The four detectors uniquely identified the two qubits of photon 2. For example, D2 (D3) indicates \( R \) (\( L \)) for Eve’s qubit, and either D2 or D3 suggests \( H \) polarization after Bob’s analyzer. Therefore, in our physical simulation, the joint measurement by Bob and Eve yields Bob’s polarization information and Eve’s momentum information.

In data collection, we measured coincidences between D1 and one of the detectors for photon 2. Table I shows two data sets for Alice’s input of \( D \) and \( A \) polarizations.

| Alice \( P_E \) | Coincidence | Estimated | Expected |
|-----------------|-------------|-----------|-----------|
| \( |D\rangle \) | \( 0.1 \) | 2840 2420 9664 32592 | 0.058 0.086 0.196 0.661 | 0.050 0.050 0.167 0.733 |
| \( 0.33 \) | 7512 3946 1512 30916 | 0.152 0.192 0.031 0.625 | 0.167 0.167 0.667 |
| \( |A\rangle \) | \( 0.1 \) | 8480 34492 4088 2052 | 0.173 0.702 0.083 0.042 | 0.167 0.733 0.050 0.050 |
| \( 0.33 \) | 1096 32360 9384 6564 | 0.022 0.655 0.19 0.133 | 0 0.667 0.167 0.167 |

TABLE I: Data samples, estimated probabilities, and theoretical values for \( D \) and \( A \) inputs with Bob using the same basis as Alice, and for predicted error probabilities \( P_E = 0.1, \text{and} \ 0.33 \). Column 1 shows the state Alice sent and column 2 shows the predicted error probability \( P_E \). "Coincidence" columns show coincidence counts over a 40-s interval. "Estimated" columns show the measured coincidence counts normalized by the total counts of all four detectors, and "Expected" shows the theoretical values under ideal operating conditions.
and compares with the expected values for the ideal case. From the raw data, we calculate the Rényi information $I_R$ based on Eq. (10), and Fig. 5 plots $I_R$ as a function of the error probability $P_E$ for the ideal case (solid squares, solid line), for the measured values with inputs in the $H$–$V$ basis (solid diamonds, solid line), and for the measured values with inputs in the $D$–$A$ basis (solid triangles, solid line). We note that no background counts were subtracted and the coincidence window was $\sim 1$ ns.

In the ideal case with $P_E = 0$, Eve gets no information, $I_R = 0$, and Alice and Bob have no error bits. However, due to experimental errors such as imperfect gate fidelities, we measured $\sim 5\%$ of the sifted bits had errors. For $P_E = 1/3$, Eve gains perfect information, $I_R = 1$, but in our experiment, Eve gained a maximum $I_R = 0.9$ or, on average, Eve gained $95\%$ of the correct information about Bob’s error-free sifted bits.

To understand the errors involved in the experiment, we model our experimental setup with some non-ideal parameters. We assume that the phases $\xi$ in Eq. (13) and $\chi$ in Eq. (14) could be inaccurate, and similarly for the setting of $\theta_A$ in Eq. (13) that might be caused by the wave plates. We also model the unitary p-cnot gate as

$$
\begin{pmatrix}
\cos \alpha & ie^{-i\delta} \sin \alpha & 0 & 0 \\
ie^{i\delta} \sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & -ie^{-i\delta} \sin \alpha & \cos \alpha \\
0 & 0 & \cos \alpha & -ie^{-i\delta} \sin \alpha
\end{pmatrix},
$$

(15)

where $\alpha = 0$ and $\delta = 0$ for an ideal p-cnot gate. Finally we assume that Bob’s HWP (H3) setting of $\theta_B$ was not perfect such that

$$
|H\rangle \rightarrow \cos \theta_B|0\rangle_C - \sin \theta_B|1\rangle_C, \\
|V\rangle \rightarrow \sin \theta_B|0\rangle_C + \cos \theta_B|1\rangle_C,
$$

(16)

where $\theta_B$ should equal $22.5^\circ$ ($-22.5^\circ$) in the $H$–$V$ ($D$–$A$) basis.

We fit the data by minimizing the differences between 96 measurements and the calculated numbers based on this error model. The fitting results show that $\Delta \xi \simeq 3^\circ$, $\Delta \chi \simeq -11^\circ$, $\Delta \theta_A(H, D, V, A) = \{3.2^\circ, 0.9^\circ, -0.7^\circ, -2.3^\circ\}$, $\alpha = 12.3^\circ$, $\delta = 3.6^\circ$, $\Delta \theta_B(H/V, D/A) = \{-1.8^\circ, 0^\circ\}$. As expected, the phase errors are relatively small and those associated with $\theta_A$ and $\theta_B$ are within the resolution of the rotating mounts housing the wave plates. The non-zero $\alpha$ also agrees with the measured classical visibility of $94\%$ for the p-cnot gate. We plot the fitted $I_R$ based on this model in Fig. 5 for the $H$–$V$ basis (open diamonds, dashed line) and the $D$–$A$ basis (open triangles, dashed line).

In summary, we have demonstrated experimentally the first complete physical simulation of the entangling-probe attack, showing that Eve can gain Rényi information of up to $0.9$ under realistic operating conditions, including a cnot gate that does not have an ultrahigh fidelity. Our results suggest the possible amount of information gain by Eve with current technology and the need to evaluate the required level of privacy amplification.

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