Training convolutional neural networks (CNNs) with a strict Lipschitz constraint under the $l_2$ norm is useful for provable adversarial robustness, interpretable gradients and stable training. While 1-Lipschitz CNNs can be designed by enforcing a 1-Lipschitz constraint on each layer, training such networks requires each layer to have an orthogonal Jacobian matrix (for all inputs) to prevent gradients from vanishing during backpropagation. A layer with this property is said to be Gradient Norm Preserving (GNP). To construct expressive GNP activation functions, we first prove that the Jacobian of any GNP piecewise linear function is only allowed to change via Householder (HH) transformations for the function to be continuous. Building on this result, we introduce a class of nonlinear GNP activations with learnable Householder transformations called Householder activations. A householder activation parameterized by the vector $v$ outputs $(I - 2vv^T)z$ for its input $z$ if $v^Tz \leq 0$; otherwise it outputs $z$. Existing GNP activations such as MaxMin can be viewed as special cases of HH activations for certain settings of these transformations. Thus, networks with HH activations have higher expressive power than those with MaxMin activations. Although networks with HH activations have nontrivial provable robustness against adversarial attacks, we further boost their robustness by (i) introducing a certificate regularization and (ii) relaxing orthogonalization of the last layer of the network. Our experiments on CIFAR-10 and CIFAR-100 show that our regularized networks with HH activations lead to significant improvements in both the standard and provable robust accuracy over the prior works (gain of 3.65% and 4.46% on CIFAR-100 respectively).

1 Introduction

Given a neural network $f: \mathbb{R}^d \to \mathbb{R}^k$, the Lipschitz constant $\text{Lip}(f)$ enforces an upper bound on how much the output is allowed to change in proportion to a change in the input. Previous work has demonstrated that a small Lipschitz constant is useful for improved adversarial robustness [Szegedy et al., 2014, Cissé et al., 2017], generalization bounds [Bartlett et al., 2017, Long and Sedghi, 2020], interpretable gradients [Tsipras et al., 2018] and Wasserstein distance estimation [Villani, 2008]. $\text{Lip}(f)$ also upper bounds the increase in the norm of gradient during backpropagation and can thus prevent gradient explosion during training, enabling us to train very deep networks [Xiao et al., 2018]. While heuristic methods to enforce Lipschitz constraints [Miyato et al., 2018, Gulrajani et al., 2017] have achieved much practical success, they do not provably enforce a bound on $\text{Lip}(f)$ globally and it remains challenging to achieve similar results when $\text{Lip}(f)$ is provably bounded.

Using the property: $\text{Lip}(g \circ h) \leq \text{Lip}(g) \text{Lip}(h)$, the Lipschitz constant of the neural network can be bounded by the product of the Lipschitz constant of all layers. While this allows us to construct

1Unless specified, we assume the Lipschitz constant under the $l_2$ norm in this work.
Figure 1: Illustration of the Householder activation, $\sigma_\theta$. In each colored region, $\sigma_\theta$ is linear. The Jacobian is $I$ when $(z_1, z_2)$ lies in the pink region (Case 1) and $I - 2v v^T$ in the other region (Case 2) where $v = [\sin(\theta/2) \ - \cos(\theta/2)]$. Both of these matrices are orthogonal implying $\sigma_\theta$ is GNP.

1-Lipschitz neural networks by constraining each layer to be 1-Lipschitz, Anil et al. [2018] identified a key difficulty with this approach. Because a 1-Lipschitz layer can only reduce the norm of gradient during backpropagation, backprop through each layer reduces the gradient norm, resulting in small gradient values for layers closer to the input, making training slow and difficult. To address this problem, they introduce Gradient Norm Preserving (GNP) architectures where each layer preserves the gradient norm during backpropagation. This involves constraining the Jacobian of each linear layer to be an orthogonal matrix and using a GNP activation function called GroupSort.

GroupSort activation function Anil et al. [2018] first separates the vector of preactivations $z \in \mathbb{R}^m$ into groups of pre-specified sizes, sorts each group in the descending order and then concatenates these sorted groups. Since these operations can only permute the elements of $z$, the Jacobian $\nabla_z \text{GroupSort}$ is always a permutation matrix (thus orthogonal), making GroupSort 1-Lipschitz and GNP. When the group size is 2, the resulting activation function is called MaxMin. MaxMin activation has been widely used in designing provably 1-Lipschitz Convolutional Neural Networks (CNNs) achieving impressive results for deterministic provable adversarial robustness on CIFAR-10.

For 1-Lipschitz CNNs, the robustness certificate for a sample $x$ from class $l$ is computed as $\mathcal{M}_f(x)/\sqrt{2}$ where $\mathcal{M}_f(x) = f_l(x) - \max_{i \neq l} f_i(x)$. Naturally, larger values of $f_l(x)$ and smaller values of $\max_{j \neq l} f_j(x)$ will lead to larger certificates. However, a limitation of MaxMin is that given an input $(z_1, z_2) \in \mathbb{R}^2$, the output is either $(z_1, z_2)$ or $(z_2, z_1)$ and the absolute values of output elements lie between $\min(|z_1|, |z_2|)$ and $\max(|z_2|, |z_1|)$. This bounds the activation values and hence the class logits, thereby limiting the final robustness certificates. One could scale the outputs but then the resulting function is no longer 1-Lipschitz and the certificate $\mathcal{M}_f(x)/\sqrt{2}$ is not valid.

To address this limitation, we introduce a new class of activation functions based on the Householder (HH) transformations. Recall that given $z \in \mathbb{R}^m$, the HH transformation is a linear function reflecting $z$ about the hyperplane $v^T z = 0 (||v||_2 = 1)$, given by $(I - 2vv^T)z$ where $I - 2vv^T$ is orthogonal because $||v||_2 = 1$. In this paper, we introduce a nonlinear Householder activation defined below:

$$
\sigma_v(z) = \begin{cases} 
z, & \text{if } v^T z > 0, \\
(I - 2vv^T)z, & \text{if } v^T z \leq 0.
\end{cases}
$$

First, note that since $z = (I - 2vv^T)z$ along $v^T z = 0$, $\sigma_v$ is continuous. Moreover, the Jacobian $\nabla_v \sigma_v$ is either $I$ or $I - 2vv^T$ (both orthogonal) implying $\sigma_v$ is GNP. Since these properties hold for all $v : ||v||_2 = 1$, $v$ can be learned during the training, enhancing the expressive power of the network. In fact, we prove that any GNP piecewise linear function that changes from $Q_1 z$ to $Q_2 z$ ($Q_1, Q_2$ are square orthogonal matrices) along $v^T z = 0$ must satisfy $Q_2 = Q_1 (I - 2vv^T)$ to be continuous (Theorem I). That is, HH transformations are necessary for any such function to be continuous.

However, a limitation of using $\sigma_v$ directly is that it has only 2 linear regions and is thus limited in its expressive power. To increase the expressive power, we separate the preactivation vector $z \in \mathbb{R}^m$ into groups of pre-specified sizes, sorts each group in the descending order and then concatenates these sorted groups. Since these operations can only permute the elements of $z$, the Jacobian $\nabla_z \text{GroupSort}$ is always a permutation matrix (thus orthogonal), making GroupSort 1-Lipschitz and GNP. When the group size is 2, the resulting activation function is called MaxMin. MaxMin activation has been widely used in designing provably 1-Lipschitz Convolutional Neural Networks (CNNs) achieving impressive results for deterministic provable adversarial robustness on CIFAR-10.

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into $m/2$ groups of size 2 each and apply $\sigma_v$ on each group where $v \in \mathbb{R}^2$ is allowed to be different for each group. Since each group has 2 linear regions, we get $2^{m/2}$ linear regions (same number as MaxMin). When $v \in \mathbb{R}^2$, without loss of generality, we can set $v = [\sin(\theta/2) - \cos(\theta/2)]^T$ so that $\theta$ becomes the learnable parameter. The resulting function denoted by $\sigma_\theta$ (demonstrated in Figure 1) can be obtained by substituting $v$ and $I - 2vv^T$ in equation (1) as below:

$$v = [\sin(\theta/2) - \cos(\theta/2)]^T \implies I - 2vv^T = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$ (2)

For any input $(z_1, z_2)$, the absolute values of the output elements of $\sigma_\theta(z_1, z_2)$ lie between 0 and $r = \sqrt{z_1^2 + z_2^2}$ (in contrast to $\min(|z_1|, |z_2|)$, $\max(|z_1|, |z_2|)$ for MaxMin). Since the activation is applied at each layer, when the number of layers is large, this can significantly increase the range of activation values and thus the final robustness certificates. Moreover, $\sigma_\theta$ is equivalent to MaxMin for $\theta = \pi/2$ implying that it is at least as expressive as MaxMin.

Another limitation of existing 1-Lipschitz CNNs [Li et al., 2019b, Trockman and Kolter, 2021, Singla and Feizi, 2021] is that their robustness guarantees do not scale properly with the $l_2$ radius $\rho$. For example, the provable robust accuracy of [Singla and Feizi, 2021] drops $\sim 30\%$ at $\rho = 108/255$ compared to $36/255$ on CIFAR-10 (Table 1). Moreover, these methods also perform poorly when the number of classes is large compared to 10 in the usually studied datasets: MNIST, Fashion MNIST, CIFAR-10 [LeCun and Cortes, 2010, Xiao et al., 2017, Krizhevsky, 2009]. For example, on CIFAR-100 (100 classes), the best reported standard accuracy using 1-Lipschitz CNNs is $43.71\%$ (vs $76.68\%$ for CIFAR-10) [Singla and Feizi, 2021].

To address these limitations, we first introduce a regularizer (Section 5.1) that when used with our $\sigma_\theta$ activation on CIFAR-10 dataset, (i) minimally reduces the standard accuracy (max drop of $-0.56\%$) and (ii) significantly enhances the provable robust accuracy for large $\rho = 108/255$ (min gain of $+4.96\%$) across different architectures (Table 1). Next, we introduce a procedure to certify robustness without orthogonalizing the last layer of the network, thereby increasing its expressive power (Section 5.2). On CIFAR-10, this significantly improves both the standard ($> 3\%$) and provable robust accuracy ($> 4\%$ at $\rho = 36/255$) across multiple 1-Lipschitz CNN architectures (Table 2).

In summary, in this paper, we make the following contributions:

- We prove that Householder transformations are necessary for any GNP piecewise linear function to be continuous. That is, if such a function changes from $Q_1z$ to $Q_2z$ along $v^Tz = 0$, (where $\|v\|_2 = 1$), then $Q_2 = Q_1(1 - 2vv^T)$ (Theorem 1).
- We introduce a class of piecewise linear GNP activation functions with learnable Householder transformations called Householder or HH activations. For certain settings of these transformations, HH activations are equivalent to the existing GNP activations in the literature and thus have higher expressive power, resulting in their superior performance (Tables 1 and 2).
- We introduce a regularizer that when used with our Householder activation significantly advances the provable robust accuracy with a small reduction in standard accuracy. Using LipConvnet-15 network on CIFAR-10, we achieve $+5.81\%$ improvement in provable robust accuracy (at $\rho = 108/255$) with only a $-0.29\%$ drop in standard accuracy over the existing methods (Table 1).
- We introduce a certification procedure without orthogonalizing the last linear layer that significantly enhances the standard and provable robust accuracy when the number of classes is large. Using the LipConvnet-15 network on CIFAR-100, our modification achieves gain of $+4.71\%$ in provable robust accuracy (at $\rho = 36/255$) with a gain of $+4.80\%$ in standard accuracy (Table 2).

## 2 Related work

**Provably Lipschitz convolutional neural networks**: The class of fully connected neural networks (FCNs) which are Gradient Norm Preserving (GNP) and 1-Lipschitz were first introduced by [Amir et al., 2018]. They orthogonalize weight matrices and use GroupSort as the activation function to design each layer to be GNP. While there have been numerous works on enforcing Lipschitz

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3To see this, consider $(z_1, z_2) \equiv r(\cos \theta, \sin \theta)$ or $-r(\cos \theta, \sin \theta)$. Using equations (1) and (2), the Jacobian will be $I - 2vv^T$ for one of them with the output $(r, 0)$ or $(-r, 0)$ respectively.
For a vector \( \mathbf{v} \), \( \mathbf{v}_j \) denotes its \( j \)th element. For a matrix \( \mathbf{A} \), \( \mathbf{A}_{j,:} \) and \( \mathbf{A}_{:,k} \) denote the \( j \)th row and \( k \)th column respectively. Both \( \mathbf{A}_{j,:} \) and \( \mathbf{A}_{:,k} \) are assumed to be column vectors (thus \( \mathbf{A}_{j,:} \) is the transpose of \( j \)th row of \( \mathbf{A} \)). \( \mathbf{A}_{j,k} \) denotes the element in \( j \)th row and \( k \)th column of \( \mathbf{A} \). \( \mathbf{A}_{j,:k} \) denotes the matrix containing the first \( j \) rows and \( k \) columns of \( \mathbf{A} \). The same rules are directly extended to higher order tensors. \( \mathbf{I} \) denotes the identity matrix, \( \mathbb{R} \) to denote the field of real numbers. For \( \theta \in \mathbb{R} \), \( \mathbf{J}^+(\theta) \) and \( \mathbf{J}^-(\theta) \) denote the orthogonal matrices with determinants \( +1 \) and \( -1 \) defined as follows:

\[
\mathbf{J}^+(\theta) = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}, \quad \mathbf{J}^-(\theta) = \begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{bmatrix}
\]

(3)

### 4 Householder Activation Functions

We know that given \( \mathbf{z} \in \mathbb{R}^m \), the Householder (HH) transformation reflects \( \mathbf{z} \) about the hyperplane \( \mathbf{v}^T \mathbf{x} = 0 \) where \( \|\mathbf{v}\|_2 = 1 \). The linear transformation is given by the equation \( (\mathbf{I} - 2\mathbf{vv}^T)\mathbf{z} \) where \( \mathbf{I} - 2\mathbf{vv}^T \) is orthogonal because \( \|\mathbf{v}\|_2 = 1 \). Now, consider the nonlinear function \( \sigma_\mathbf{v} \) defined below:

**Definition 1. (Householder Activation of Order 1)** The activation function \( \sigma_\mathbf{v} : \mathbb{R}^m \to \mathbb{R}^m \), applied on \( \mathbf{z} \in \mathbb{R}^m \), is called the \( m \)-dimensional Householder Activation of Order 1:

\[
\sigma_\mathbf{v}(\mathbf{z}) = \begin{cases}
\mathbf{z}, & \mathbf{v}^T \mathbf{z} > 0, \\
(\mathbf{I} - 2\mathbf{vv}^T)\mathbf{z}, & \mathbf{v}^T \mathbf{z} \leq 0.
\end{cases}
\]

(4)

Since \( \sigma_\mathbf{v} \) is linear when \( \mathbf{v}^T \mathbf{z} > 0 \) or \( \mathbf{v}^T \mathbf{z} < 0 \), it is also continuous in both cases. At the hyperplane separating the two cases i.e. \( \mathbf{v}^T \mathbf{z} = 0 \) we have: \( (\mathbf{I} - 2\mathbf{vv}^T)\mathbf{z} = \mathbf{z} - 2(\mathbf{v}^T \mathbf{z})\mathbf{v} = \mathbf{z} \) (both linear functions are equal). Thus, \( \sigma_\mathbf{v} \) is continuous \( \forall \mathbf{z} \in \mathbb{R}^m \). Moreover, the Jacobian is either \( \mathbf{I} \) or \( \mathbf{I} - 2\mathbf{vv}^T \) which are both square orthogonal matrices. Thus, \( \sigma_\mathbf{v} \) is also GNP and 1-Lipschitz. Since these properties hold for all \( \mathbf{v} \) satisfying \( \|\mathbf{v}\|_2 = 1 \), \( \mathbf{v} \) can be made a learnable parameter.
While the above arguments suggest that HH transformations are sufficient to ensure such functions are continuous, we also prove that they are necessary. That is, we prove that if a GNP piecewise linear function $g: \mathbb{R}^m \to \mathbb{R}^m$ transitions between different linear functions $Q_1z$ and $Q_2z$ (in an open set $S \subset \mathbb{R}^m$) along a hyperplane $v^Tz = 0$ (where $\|v\|_2 = 1$), then $g$ is continuous in $S$ if and only if $Q_2 = Q_1(I - 2vv^T)$. This theoretical result provides a general principle for designing piecewise linear GNP activation functions. The formal result is stated in the following Theorem:

**Theorem 1.** Given an open set $S \subset \mathbb{R}^m$, orthogonal square matrices $Q_1 \neq Q_2$, and vector $v \in \mathbb{R}^m$ ($\|v\|_2 = 1$) such that $S \cap \{z: v^Tz = 0\} \neq \emptyset$, the function $g$ defined as follows:

$$
g(z) = \begin{cases} 
Q_1z, & z \in S, v^Tz > 0, \\
Q_2z, & z \in S, v^Tz \leq 0 
\end{cases} 
$$

is continuous in $S$ if and only if $Q_2 = Q_1(I - 2vv^T)$.

Proof is in Appendix Section A.1. Note that since the matrix $I - 2vv^T$ has determinant $-1$, the above theorem necessitates that $\det(Q_1) = -\det(Q_2)$ i.e the determinant of the Jacobian must change sign whenever the Jacobian of a piecewise linear GNP activation function changes.

A major limitation of using $\sigma_\alpha$ (equation (4)) directly is that it has only 2 linear regions and is thus limited in its expressive power. In contrast, MaxMin first divides the preactivation $z \in \mathbb{R}^m$ (assuming $m$ is divisible by 2) into $m/2$ groups of size 2 each. Since each group has 2 linear regions, we get $2^{m/2}$ linear regions from the $m/2$ groups. Thus, to increase the expressive power, we similarly divide $z$ into $m/2$ groups of size 2 each and apply the 2-dimensional Householder activation function of Order 1 to each group resulting in $2^{m/2}$ linear regions (same as MaxMin). Setting $m = 2$, $Q_1 = I$, $v = [\sin(\theta/2), -\cos(\theta/2)]^T$ in Theorem 1 we get the following activation:

**Corollary 1.** The function $\sigma_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$
\sigma_\theta(z_1, z_2) = \begin{bmatrix} 1 & 0 \\
0 & 1 \\
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta 
\end{bmatrix} \begin{bmatrix} z_1 \\
z_2 
\end{bmatrix}
$$

if $z_1 \sin(\theta/2) - z_2 \cos(\theta/2) > 0$

$$
\sigma_\theta(z_1, z_2) = \begin{bmatrix} 1 & 0 \\
0 & 1 \\
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta 
\end{bmatrix} \begin{bmatrix} z_1 \\
z_2 
\end{bmatrix}
$$

if $z_1 \sin(\theta/2) - z_2 \cos(\theta/2) \leq 0$

is continuous and is called 2D Householder Activation of Order 1.

Proof in Appendix A.2. The two cases are demonstrated in Figure 1a and Figure 1b respectively. Since $\sigma_\theta$ is continuous, GNP and 1-Lipschitz $\forall \theta \in \mathbb{R}$, $\theta$ is a learnable parameter. For $\theta = \pi/2$ in equation (6), $\sigma_\theta$ is equivalent to MaxMin. Thus, $\sigma_\theta$ is at least as expressive as MaxMin.

To apply $\sigma_\theta$ to the output of a convolution layer $z \in \mathbb{R}^{m \times n \times n} (m$ is the number of channels and $n \times n$ is the spatial size), we first split $z$ into 2 tensors along the channel dimension giving the tensors: $z_{m/2;::}$ and $z_{m/2;::}$. Each of these tensors is of size $m/2 \times n \times n$ giving $n^2m/2$ groups. We use the same $\theta$ for each pair of channels (irrespective of spatial location) resulting in $m/2$ learnable parameters. We initialize each $\theta = \pi/2$ so that $\sigma_\theta$ is equivalent to MaxMin at initialization.

By construction, the output of $\sigma_\theta$ always lies in the pink region (Figure 1) and applying $\sigma_\theta$ again (using the same $\theta$ as before) results in the same output: $\sigma_\theta \circ \sigma_\theta = \sigma_\theta$. However, applying $\sigma_\theta$ to the output of $\sigma_\theta$ where $\phi \neq \theta + 2\pi n, m \in \mathbb{Z}$ further divides the pink region into two regions with different linear functions implying $\sigma_\theta \circ \sigma_\theta \neq \sigma_\theta$ (demonstrated in Figure 2a). Thus, applying $\sigma_\theta$ iteratively $n$ times (using different $\theta$s) allows us to construct HH activations with any number of linear regions. This iterative construction can however be slow when $n$ is large. To address this limitation, we derive an alternative construction of HH activations with any number of linear regions where the linear function needs to be applied only once (not $n$ times!). For this activation function, the linear function (or Jacobian) is determined efficiently based on the region in which the input $(z_1, z_2)$ lies. This construction is given in the following theorem (example in Figure 2b).

**Theorem 2.** Given: $0 \leq \theta_0 < \theta_1 < \cdots < \theta_{2n} = 2\pi + \theta_0$ such that $\sum_{i=0}^{n-1}(\theta_{2i+1} - \theta_{2i}) = \pi$ and $\alpha_i = 2\sum_{j=0}^{i-1}(\theta_{2j+1} - \theta_{2j}) - 1)$. The function $\sigma_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, GNP and 1-Lipschitz where $\Theta = [\theta_0, \theta_1, \ldots, \theta_{2n}]$ (also called 2D Householder Activation of order $n$):

$$
\sigma_\Theta(z_1, z_2) = \begin{bmatrix} \cos \alpha_i \\
-\sin \alpha_i \\
\sin \alpha_i \\
-\cos \alpha_i 
\end{bmatrix} \begin{bmatrix} z_1 \\
z_2 
\end{bmatrix}
$$

$\theta_i \leq \phi < \theta_{i+1}$

where $\phi \in [\theta_0, \theta_{2n} = 2\pi + \theta_0)$ and $\cos(\phi) = z_1/\sqrt{z_1^2 + z_2^2}, \sin(\phi) = z_2/\sqrt{z_1^2 + z_2^2}$.
We can now construct a 1-Lipschitz neural network, \( f : \mathbb{R}^d \rightarrow \mathbb{R}^k \) (\( k \) is the number of classes) by composing 1-Lipschitz convolution layers and HH activation functions. To certify robustness for some input \( x \) with prediction \( l \), we first define the margin of prediction: \( \mathcal{M}_f(x) = \max(0, f_l(x) - \max_{i \neq l} f_i(x)) \) where \( f_i(x) \) is the logit for class \( i \) and \( l \) is the correct label. Using Theorem 7 in \cite{Li2019}, we can derive the robustness certificate (in the \( l_2 \) norm) as \( \mathcal{M}_f(x)/\sqrt{2} \). Thus, the \( l_2 \) distance of \( x \) to the decision boundary is lower bounded by \( \mathcal{M}_f(x)/\sqrt{2} \):

\[
\min_{i \neq l} \min_{f_i(x^*) = f_l(x^*)} ||x^* - x||_2 \geq \frac{\mathcal{M}_f(x)}{\sqrt{2}}
\]

### 5.1 Certificate Regularization

Our goal is to maximize the certificate \( \mathcal{M}_f(x)/\sqrt{2} \) for correctly classified inputs \( x \). However, a limitation of using cross entropy loss during training is that it is not explicitly designed to maximize the margin \( \mathcal{M}_f(x) \) and thus, the robustness certificate. That is, once the cross entropy loss becomes small, the gradients will no longer try to further increase the margin even though the network may have the capacity to learn bigger margins. To address this limitation, we add a regularization term to increase the robustness certificate per input to the usual cross entropy loss:

\[
\min_{\Omega} \mathbb{E}_{(x, l) \sim D} \left[ \ell \left( f_{\Omega}(x), l \right) - \gamma \text{relu} \left( \frac{\mathcal{M}_f(x)}{\sqrt{2}} \right) \right]
\]

In the above equation, \( f_{\Omega} \) denotes the 1-Lipschitz neural network parametrized by \( \Omega \), \( f_{\Omega}(x) \) denotes the logits for the input \( x \), \( \ell \left( f_{\Omega}(x), l \right) \) is the cross entropy loss for input \( x \) with label \( l \) and \( \gamma > 0 \) is the regularization coefficient for maximizing the certificate. We have the minus sign in front of the regularization term \( \gamma \text{relu}(\mathcal{M}_f(x)/\sqrt{2}) \) because we want to maximize the certificate while minimizing the cross entropy loss. For wrongly classified inputs, \( \mathcal{M}_f(x)/\sqrt{2} < 0 \iff \text{relu}(\mathcal{M}_f(x)/\sqrt{2}) = 0 \). This ensures that the optimization tries to increase the certificates only for the correctly classified inputs. We call the above mentioned procedure Certificate Regularization (abbreviated as CR).

### 5.2 Last Layer Normalization

Existing 1-Lipschitz neural networks constrain the weight matrices of all linear layers of the network to be orthogonal to ensure they are GNP. For the last weight matrix, \( W \in \mathbb{R}^{k \times m} \) (\( k \) is the number of classes, \( m \) is the dimension of the penultimate layer, \( m > k \)), this enforces the following constraints:

\[
\forall j, \quad ||W_{j,:}||_2 = 1, \quad i \neq j, \quad W_{j,:} \perp W_{i,:}
\]

Figure 2: Constructing Higher Order Householder activations (\( J^+ \) and \( J^- \) defined in equation \( (3) \))

Proof is in Appendix Section A.3 and more details about the Theorem in Appendix Section C.
We perform experiments under the setting of provably robust image classification on CIFAR-10 and CIFAR-100 datasets using the same 1-Lipschitz CNN architectures used by [Singla and Feizi 2021] (LipConvnet-5, 10, 15, . . . , 40) due to their superior performance over the prior works. We compare with the three orthogonal convolution layers in the literature: SOC [Singla and Feizi 2021], BCOP [Li et al. 2019b] and Cayley [Trockman and Kolter 2021] using MaxMin as the activation function. We use SOC with MaxMin as the primary baseline for comparison in the maintext due to their superior performance over prior works (BCOP, Cayley). Results using BCOP and Cayley convolutions are given in Appendix Sections E and F for completeness. We use the same implementations for these convolution layers as given in their respective github repositories. We compare the provable robust accuracy using 3 different values of the $l_2$ perturbation radius $\rho = 30/255, 72/255, 108/255$. In both Tables 1 and 2, for all networks, we use SOC as the convolution layer. The symbol HH (in Tables 1 and 2) is for the 2D Householder Activation of order 1 or $\sigma_\theta$ (defined in equation (6)). We observe negligible inference time overhead due to any of our changes (Appendix Tables B, D). All experiments were performed using 1 NVIDIA GeForce RTX 2080 Ti GPU. All networks were trained for 200 epochs with initial learning rate of 0.1, dropped by a factor of 0.1 after 100 and 150 epochs. For Certificate Regularization (or CR), we set the parameter $\gamma = 0.1$.

### 6.1 Results on CIFAR-10

In Table 1 for each architecture, the row “SOC + MaxMin” uses the MaxMin activation, the row “$\ast$ HH” uses $\sigma_\theta$ activation (replacing MaxMin) and the row “$\ast$ CR” also adds Certificate Regularization with $\gamma = 0.1$ (again using $\sigma_\theta$ as the activation). Due to the small number of classes in CIFAR-10, we do not observe significant gains using Last Layer Normalization or LLN (Appendix Table 7). Thus, we do not include LLN for any of the results in Table 1. The column, “Increase (108/255)” denotes the increase in provable robust accuracy with $\rho = 108/255$ relative to "SOC + MaxMin".

For LipConvnet-25, 30, 35, 40 architectures, we observe gains in both the standard and provable robust accuracy by replacing MaxMin with the HH activation (i.e $\sigma_\theta$). The gains in provable robust accuracy ($\rho = 108/255$) are significantly higher for deeper networks: LipConvnet-35 (3.65%) and LipConvnet-40 (4.35%) with decent gains in standard accuracy (1.71 and 1.61% respectively). We show the results of using 2D Householder activation function of order 2 in Appendix Table 6. We do not observe any significant improvements compared to using the order 1, i.e. $\sigma_\theta$ activation.
Table 1: Results for provable robustness against adversarial examples on the CIFAR-10 dataset. Results with BCOP and Cayley convolutions are in Appendix Tables 4 and 5. Adding CR further boosts the provable robust accuracy while slightly reducing the standard accuracy. Comparing "+ CR" with "SOC + MaxMin", we observe small drops in standard accuracy for LipConvnet-5, 10, ..., 30 networks (max. drop of −0.56%), and gains for LipConvnet-35 (+0.52%) and LipConvnet-40 (+0.96%). For provable robust accuracy (ρ = 108/255), we observe very significant gains of > 4.96% for all networks and > 8% for the deeper LipConvnet-35, 40 networks.

6.2 Results on CIFAR-100

In Table 2 for each architecture, the row "SOC + MaxMin" uses the MaxMin activation, "+ LLN" adds Last Layer Normalization (uses MaxMin), "+ HH" replaces MaxMin with σθ (also uses LLN), "+ CR" also adds Certificate Regularization with γ = 0.1 (uses both σθ and LLN). The column, "Increase (Standard)" denotes the increase in standard accuracy relative to "SOC + MaxMin".

By adding LLN (the row "+ LLN"), we observe gains in standard (min gain of 1.10%) and provable robust accuracy (min gain of 1.71% at ρ = 36/255) across all the LipConvnet architectures (gains relative to "SOC + MaxMin"). These gains are smallest for the LipConvnet-40 network with the maximum depth. However, replacing MaxMin with the σθ activation further improves the standard (min gain of 3.65%) and provable robust accuracy (min gain of 4.46% at ρ = 36/255) across all networks (again relative to "SOC + MaxMin"). Similar to what we observed for CIFAR-10, replacing MaxMin with σθ significantly improves the performance of the deeper LipConvnet-35, 40 networks.
| Architecture | Methods       | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase (Standard) |
|--------------|--------------|------------------|------------------|-------------------|
|              |              | 36/255           | 72/255           | 108/255           |
| LipConvnet-15| SOC + MaxMin | 42.92%           | 28.81%           | 17.93%            | 10.73%            | 4.80%             |
|              | + LLN        | 47.72%           | 33.52%           | 21.89%            | 13.76%            | +4.80%            |
|              | + HH         | 47.72%           | 33.97%           | 22.45%            | 13.81%            | +4.80%            |
|              | + CR         | 47.61%           | 34.54%           | 23.16%            | 15.09%            | +4.69%            |
| LipConvnet-20| SOC + MaxMin | 43.06%           | 29.34%           | 18.66%            | 11.20%            | _                 |
|              | + LLN        | 46.86%           | 33.48%           | 22.14%            | 14.10%            | +3.80%            |
|              | + HH         | 47.71%           | 34.22%           | 22.93%            | 14.57%            | +4.65%            |
|              | + CR         | 47.84%           | 34.77%           | 23.70%            | 15.84%            | +4.78%            |
| LipConvnet-25| SOC + MaxMin | 43.37%           | 28.59%           | 18.18%            | 10.85%            | _                 |
|              | + LLN        | 46.32%           | 32.87%           | 21.53%            | 13.86%            | +2.95%            |
|              | + HH         | 47.70%           | 34.00%           | 22.67%            | 14.57%            | +4.33%            |
|              | + CR         | 46.87%           | 34.09%           | 23.41%            | 15.61%            | +3.50%            |
| LipConvnet-30| SOC + MaxMin | 42.87%           | 28.74%           | 18.47%            | 11.21%            | _                 |
|              | + LLN        | 46.18%           | 32.82%           | 21.52%            | 13.52%            | +3.31%            |
|              | + HH         | 46.80%           | 33.72%           | 22.70%            | 14.31%            | +3.93%            |
|              | + CR         | 46.92%           | 34.17%           | 23.21%            | 15.84%            | +4.05%            |
| LipConvnet-35| SOC + MaxMin | 42.42%           | 28.34%           | 18.10%            | 10.96%            | _                 |
|              | + LLN        | 45.22%           | 32.10%           | 21.28%            | 13.25%            | +2.80%            |
|              | + HH         | 46.21%           | 32.80%           | 21.55%            | 14.13%            | +3.79%            |
|              | + CR         | 46.88%           | 33.64%           | 23.34%            | 15.73%            | +4.46%            |
| LipConvnet-40| SOC + MaxMin | 41.84%           | 28.00%           | 17.40%            | 10.28%            | _                 |
|              | + LLN        | 42.94%           | 29.71%           | 19.30%            | 11.99%            | +1.10%            |
|              | + HH         | 45.84%           | 32.79%           | 21.98%            | 14.07%            | +4.00%            |
|              | + CR         | 45.03%           | 32.57%           | 22.37%            | 14.76%            | +3.19%            |

Table 2: Results for provable robustness against adversarial examples on the CIFAR-100 dataset. Results with LipConvnet-5, 10 are in Appendix Table 8. BCOP, Cayley are in Tables 10, 11.

Similar to CIFAR-10, adding CR further improves the provable robust accuracy while only slightly reducing the standard accuracy. Because LLN significantly improves the standard accuracy, we compare the standard accuracy numbers between rows "+ CR" and "+ LLN" to evaluate the drop due to CR. We observe a small drop in standard accuracy ($-0.04\%$, $-0.11\%$) only for LipConvnet-5 and LipConvnet-15 networks. For the other networks, the standard accuracy actually increases.

## 7 Conclusion

In this work, we prove that the Jacobian of any Gradient Norm Preserving (GNP) piecewise linear function is only allowed to change via Householder transformations for the function to be continuous. This provides a general principle for designing piecewise linear GNP functions. Using this result, we introduce a class of GNP activation functions called Householder (or HH) activations with learnable parameters of Householder transformations. Our HH activations generalize the existing GNP activations in the literature and achieve superior performance in different settings. In addition, we introduce a certificate regularizer and a procedure to certify robustness without orthogonalizing the last linear layer of the network that gives significant gains when combined with our activations. We also derive an efficient construction of HH activations with any arbitrary number of linear regions in 2 dimensions. Our experimental results suggest that constructing expressive HH activation functions in higher dimensions, coming up with better initialization methods and regularizers can further improve the performance of 1-Lipschitz neural networks and are interesting directions of future research.
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A Proofs

A.1 Proof of Theorem I

Proof. We first prove that if \( Q_2 = (I - 2vv^T)Q_1 \), then the function \( g \) is continuous. First, observe that for \( v^Tz > 0 \), \( g(z) = Q_1z \) which is continuous. Similarly, for \( v^Tz < 0 \), \( g(z) = Q_2z \) which is again continuous. This proves that the function \( g \) is continuous when \( v^Tz > 0 \) or \( v^Tz < 0 \).

Thus, to prove continuity \( \forall z \in S \), we must prove that:

\[
Q_1z = Q_2z \quad \forall z : v^Tz = 0
\]  

(9)

Since \( Q_2 = Q_1(I - 2vv^T) \), we have:

\[
Q_2 - Q_1 = -2Q_1vv^T
\]

(10)

\[
(Q_2 - Q_1)z = -2(Q_1vv^T)z = -2Q_1v(v^Tz)
\]

The above equation directly proves (9).

Now, we prove the other direction i.e if \( g \) is continuous in \( S \) then, \( Q_2 = Q_1(I - 2vv^T) \).

Since \( g \) is continuous for all \( z : v^Tz = 0 \), we have:

\[
Q_2z = Q_1z \quad \forall z : v^Tz = 0
\]

(11)

\[
(Q_2 - Q_1)z = 0 \quad \forall z : v^Tz = 0
\]

Since \( z \in \mathbb{R}^m \), we know that the set of vectors: \( \{ z : v^Tz = 0 \} \) spans a \( m - 1 \) dimensional subspace. Thus, the null space of \( Q_2 - Q_1 \) is of size \( m - 1 \).

This in turn implies that \( Q_2 - Q_1 \) is a rank one matrix given by the following equation:

\[
Q_2 - Q_1 = uv^T
\]

(12)

where the vector \( u \) remains to be determined.

Since \( Q_1 \) and \( Q_2 \) are orthogonal matrices, we have the following set of equations:

\[
Q_2^TQ_2 = (Q_1 + uv^T)^T(Q_1 + uv^T)
\]

(13)

\[
Q_2^TQ_2 = (Q_1 + uv^T)(Q_1 + uv^T)^T
\]

(14)

We first simplify equation (11):

\[
Q_2^TQ_2 = (Q_1^T + vu^T)(Q_1 + uv^T)
\]

(15)

\[
I = I + v(Q_1^Tu)^T + (Q_1^Tu)v^T + (u^Tv)v^T
\]

(16)

\[
0 = v(Q_1^Tu)^T + (Q_1^Tu)v^T + (u^Tv)v^T
\]

(17)

\[
= (u^Tv)v^T = v(u^TvQ_1) + (Q_1^Tu)v^T
\]

(18)

Right multiplying both sides by \( v \) and using \( \|v\|^2 = 1 \), we get:

\[
-(u^Tv)v = (u^TvQ_1)v + Q_1^Tu
\]

(19)

\[
Q_1^Tu = -(u^TvQ_1)v + (Q_1^Tu)v = \lambda v
\]

(20)

\[
u = \lambda Q_1v, \quad \text{where} \quad \lambda = -(u^TvQ_1)v
\]

(21)

Substituting \( u \) using equation (21) in equation (18), we get:

\[
Q_2 - Q_1 = \lambda Q_1vv^T
\]

(22)

\[
Q_2 = Q_1(I + \lambda vv^T)
\]

(23)

Since \( Q_2^TQ_2 = I \), we have:

\[
Q_2^TQ_2 = (Q_1(I + \lambda vv^T))^TQ_1(I + \lambda vv^T)
\]

(24)

\[
Q_2^TQ_2 = (I + \lambda vv^T)Q_2^TQ_1(I + \lambda vv^T)
\]

(25)

\[
I = (I + \lambda vv^T)(I + \lambda vv^T)
\]

(26)

\[
I = I + 2\lambda vv^T + \lambda^2 vv^T
\]

\[\Rightarrow \lambda = 0 \text{ or } \lambda = -2\]
Since $\lambda = 0$ would imply $Q_1 = Q_2$ which is not allowed by the assumption of the Theorem that $Q_1 \neq Q_2$. 
$\lambda = -2$ is the only possibility allowed.
This proves the other direction i.e:

$$Q_2 = Q_1 (I - 2vv^T)$$

A.2 Proof of Corollary [1]

Proof. Substitute $Q_1, v$ as follows in Theorem [1]

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$v = \begin{bmatrix} + \sin(\theta/2) \\ - \cos(\theta/2) \end{bmatrix}$$
$$Q_2 = I - 2vv^T$$

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \sin(\theta/2) \\ - \cos(\theta/2) \end{bmatrix} \begin{bmatrix} \sin(\theta/2) & - \cos(\theta/2) \end{bmatrix}$$

This proves the other direction i.e:

$$Q_2 = Q_1 (I - 2vv^T)$$

A.3 Proof of Theorem [2]

Proof. We are given the following:

$$\sum_{i=0}^{n-1} (\theta_{2i+1} - \theta_{2i}) = \pi, \quad \alpha_i = 2 \sum_{j=0}^{i} (-1)^j$$ (14)

Note that by definition (equation (7)), the function is linear for $\theta_i < \varphi < \theta_{i+1}$ and hence continuous. Furthermore, since $\varphi \in [\theta_0, \theta_{2n}]$, we proceed to prove continuity for the following two cases:

Case 1: $\theta_i - \epsilon < \varphi < \theta_i + \epsilon$, \quad $\epsilon > 0$, $i \geq 1$

Case 2: $\theta_0 < \varphi < \theta_0 + \epsilon$, \quad $\theta_{2n} - \epsilon < \varphi < \theta_{2n}$, \quad $\epsilon > 0$

Proof for Case 1:

Using equation (17), we know that $\sigma_{\Theta}$ realizes different linear functions for $\theta_i - \epsilon < \varphi < \theta_i$ and $\theta_i < \varphi < \theta_i + \epsilon$. Thus, for $\sigma_{\Theta}$ to be continuous, we require that the two linear functions be the same at the boundary i.e $\varphi = \theta_i$.

We first write the input $(z_1, z_2)$ in terms of shifted polar coordinates i.e: $(r \cos(\varphi), r \sin(\varphi))$ where $r = \sqrt{z_1^2 + z_2^2}$ and $\cos \varphi = z_1/\sqrt{z_1^2 + z_2^2}$, $\sin \varphi = z_2/\sqrt{z_1^2 + z_2^2}$, $\varphi \in [\theta_0, \theta_0 + 2\pi]$

Thus, the function for $\theta_i - \epsilon < \varphi < \theta_i$ is given by:

$$\sigma_{\Theta} (r \cos(\varphi), r \sin(\varphi)) = \begin{bmatrix} \cos \alpha_{i-1} \\ -(-1)^{i-1} \sin \alpha_{i-1} \end{bmatrix} \begin{bmatrix} \sin \alpha_{i-1} \\ -(-1)^{i} \cos \alpha_{i-1} \end{bmatrix} \begin{bmatrix} r \cos \varphi \\ r \sin \varphi \end{bmatrix}$$ (15)

Similarly, the function for $\theta_i < \varphi < \theta_i + \epsilon$ is given by:

$$\sigma_{\Theta} (r \cos(\varphi), r \sin(\varphi)) = \begin{bmatrix} \cos \alpha_i \\ -(-1)^{i} \sin \alpha_i \end{bmatrix} \begin{bmatrix} \sin \alpha_i \\ -(-1)^{i+1} \cos \alpha_i \end{bmatrix} \begin{bmatrix} r \cos \varphi \\ r \sin \varphi \end{bmatrix}$$ (16)
The difference in the function values at the boundary i.e. \( \varphi = \theta_i \), obtained by subtracting equations \((16)\) and \((15)\) is as follows:

\[
\begin{bmatrix}
\cos \alpha_i & \sin \alpha_i \\
(1)^i \sin \alpha_i & (1)^i \cos \alpha_i
\end{bmatrix}
\begin{bmatrix}
r \cos \theta_i \\
r \sin \theta_i
\end{bmatrix}
- \begin{bmatrix}
\cos \alpha_{i-1} & \sin \alpha_{i-1} \\
(1)^{i-1} \sin \alpha_{i-1} & (1)^{i-1} \cos \alpha_{i-1}
\end{bmatrix}
\begin{bmatrix}
r \cos \theta_i \\
r \sin \theta_i
\end{bmatrix}
\]

\[
= r \begin{bmatrix}
(1)^i \sin \alpha_i & (1)^i \cos \alpha_i \\
(1)^{i-1} \sin \alpha_{i-1} & (1)^{i-1} \cos \alpha_{i-1}
\end{bmatrix}
\begin{bmatrix}
\cos \theta_i \\
\sin \theta_i
\end{bmatrix}
\]

Using sum-to-product trigonometric identities, the above equals:

\[
2 \begin{bmatrix}
\sin \left(\frac{\alpha_{i-1} - \alpha_i}{2}\right) & \sin \left(\frac{\alpha_{i+1} + \alpha_i}{2}\right) \\
(1)^i \sin \left(\frac{\alpha_{i-1} - \alpha_i}{2}\right) & (1)^{i+1} \cos \left(\frac{\alpha_{i-1} - \alpha_i}{2}\right)
\end{bmatrix}
\begin{bmatrix}
\cos \theta_i \\
\sin \theta_i
\end{bmatrix}
\]

\[
= 2r \begin{bmatrix}
\sin \left(\frac{\alpha_{i-1} - \alpha_i}{2}\right) & \sin \left(\frac{\alpha_{i+1} + \alpha_i}{2}\right) \\
(1)^i \sin \left(\frac{\alpha_{i-1} - \alpha_i}{2}\right) & (1)^{i+1} \cos \left(\frac{\alpha_{i-1} - \alpha_i}{2}\right)
\end{bmatrix}
\begin{bmatrix}
\cos \theta_i \\
\sin \theta_i
\end{bmatrix}
\]

Using equation \((14)\), we directly have: \( \alpha_i = 2\theta_i - \alpha_{i-1} \). Thus, the above equation reduces to:

\[
2r \begin{bmatrix}
\sin (\theta_i - \alpha_i) & \sin (\theta_i - \alpha_i) \\
(1)^i \cos (\theta_i - \alpha_i)
\end{bmatrix}
\begin{bmatrix}
\cos \theta_i \\
\sin \theta_i
\end{bmatrix}
= 0.
\]

Hence, the linear functions given by equations \((15)\) and \((16)\) are equal at \( \varphi = \theta_i \). This proves that the function is continuous for Case 1.

**Proof for Case 2:**

Using equation \((7)\), we know that \( \sigma_\theta \) realizes different linear functions for \( \theta_0 < \varphi < \theta_0 + \epsilon \) and \( \theta_{2n} - \epsilon < \varphi < \theta_{2n} \).

Thus, for \( \sigma_\theta \) to be continuous, we require that the two linear functions be the same at the boundary i.e \( \varphi = \theta_0 \).

As before, we first write the input \((z_1, z_2)\) in terms of shifted polar coordinates i.e: \((r \cos(\varphi), r \sin(\varphi))\).

Thus, the function for \( \theta_0 < \varphi < \theta_0 + \epsilon \) is given by:

\[
\sigma_\theta \left(r \cos \varphi, r \sin \varphi \right) = \begin{bmatrix}
\cos \alpha_0 & \sin \alpha_0 \\
\sin \alpha_0 & -\cos \alpha_0
\end{bmatrix}
\begin{bmatrix}
r \cos \varphi \\
r \sin \varphi
\end{bmatrix}
\]  \(\text{(17)}\)

Similarly, the function for \( \theta_{2n} - \epsilon < \varphi < \theta_{2n} \) is given by:

\[
\sigma_\theta \left(r \cos \varphi, r \sin \varphi \right) = \begin{bmatrix}
\cos \alpha_{2n-1} & \sin \alpha_{2n-1} \\
(1)^i \sin \alpha_{2n-1} & (1)^{i+1} \cos \alpha_{2n-1}
\end{bmatrix}
\begin{bmatrix}
r \cos \varphi \\
r \sin \varphi
\end{bmatrix}
\]  \(\text{(18)}\)

Using equation \((14)\), \( \alpha_{2n-1} \) is given as follows:

\[
\alpha_{2n-1} = 2 \sum_{i=0}^{2n-1} \theta_{2n-1-i} (-1)^i = 2 \sum_{i=0}^{n-1} (\theta_{2i+1} - \theta_{2i}) = 2\pi
\]

Thus, equation \((18)\) reduces to:

\[
\sigma_\theta \left(r \cos \varphi, r \sin \varphi \right) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
r \cos \varphi \\
r \sin \varphi
\end{bmatrix}
\]  \(\text{(19)}\)

The difference in the function values at the boundary i.e \( \varphi = \theta_0 \), obtained by subtracting equations \((19)\) and \((17)\) is as follows:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
r \cos \theta_0 \\
r \sin \theta_0
\end{bmatrix}
- \begin{bmatrix}
\cos \alpha_0 & \sin \alpha_0 \\
\sin \alpha_0 & -\cos \alpha_0
\end{bmatrix}
\begin{bmatrix}
r \cos \theta_0 \\
r \sin \theta_0
\end{bmatrix}
\]

\[
= r \begin{bmatrix}
1 - \cos \alpha_0 & -\sin \alpha_0 \\
-\sin \alpha_0 & 1 + \cos \alpha_0
\end{bmatrix}
\begin{bmatrix}
\cos \theta_0 \\
\sin \theta_0
\end{bmatrix}
\]
Using the trigonometric identities: $1 - \cos(\theta) = 2\sin^2(\theta/2)$, $1 + \cos(\theta) = 2\cos^2(\theta/2)$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$, we have:

$$= r\begin{bmatrix} 2\sin^2(\frac{\alpha}{2}) & -2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2}) \\ -2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2}) & 2\cos^2(\frac{\alpha}{2}) \end{bmatrix}\begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}$$

$$= 2r\begin{bmatrix} \sin(\frac{\alpha}{2}) \\ -\cos(\frac{\alpha}{2}) \end{bmatrix}\begin{bmatrix} \sin(\frac{\alpha}{2}) \\ -\cos(\frac{\alpha}{2}) \end{bmatrix}\begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}$$

Using equation (14), we have: $\alpha_0 = 2\theta_0$. Thus, the above equation reduces to:

$$= 2r\begin{bmatrix} \sin(\theta_0) \\ -\cos(\theta_0) \end{bmatrix}\begin{bmatrix} \sin(\theta_0) \\ -\cos(\theta_0) \end{bmatrix}\begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}$$

Hence, the linear functions given by equations (17) and (19) are equal at $\varphi = \theta_0$. This proves that the function is continuous for Case 2.

\[ \square \]

### A.4 Proof of Theorem 3

**Proof.** We proceed by computing the Lipschitz constant of the function $f_i - f_l$. The gradient of the function: $f_i - f_l$, at $x$ can be computed using the chain rule:

$$\nabla_{x}(f_{i} - f_{l}) = (W_{l,.} - W_{i,.})^{T}\nabla_{x}g$$

Since $g$ is given to be 1-Lipschitz, the Lipschitz constant of $f_i - f_l$ can be computed using the above equation as follows:

$$\|\nabla_{x}(f_{i} - f_{l})\|_{2} \leq \|(W_{l,.} - W_{i,.})^{T}(\nabla_{x}g)\|_{2}$$

Thus, the distance of $x$ to the decision boundary $f_i - f_l = 0$, is lower bounded by:

$$\min_{f_{i}(x^*) = f_{l}(x^*)} \|x^* - x\|_{2} \geq \frac{f_{i}(x) - f_{l}(x)}{\|W_{l,.} - W_{i,.}\|_{2}}$$

Thus, the distance to decision boundary across all classes $i \neq l$ is lower bounded by:

$$\min_{i \neq l} \min_{f_{i}(x^*) = f_{l}(x^*)} \|x^* - x\|_{2} \geq \min_{i \neq l} \frac{f_{i}(x) - f_{l}(x)}{\|W_{l,.} - W_{i,.}\|_{2}} \quad \square$$

### B Verification that $\sigma_\theta(z_1, z_2)$ always lies on one side of the hyperplane

Consider the case: $z_1 \sin(\theta/2) - z_2 \cos(\theta/2) > 0$

In this case $\sigma_\theta(z_1, z_2) = (z_1, z_2)$ and the result follows directly.

Consider the other case: $z_1 \sin(\theta/2) - z_2 \cos(\theta/2) \leq 0$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 \cos\theta + z_2 \sin\theta \\ z_1 \sin\theta - z_2 \cos\theta \end{bmatrix}$$

$$a_1 \sin(\theta/2) - a_2 \cos(\theta/2) = (z_1 \cos\theta + z_2 \sin\theta) \sin(\theta/2) - (z_1 \sin\theta - z_2 \cos\theta) \cos(\theta/2) = z_1 \cos\theta \sin(\theta/2) - z_1 \sin\theta \cos(\theta/2) + z_2 \sin\theta \sin(\theta/2) + z_2 \cos\theta \cos(\theta/2) = -z_1 \sin(\theta/2) + z_2 \cos(\theta/2)$$

Since $z_1 \sin(\theta/2) - z_2 \cos(\theta/2) \leq 0$, we have $-z_1 \sin(\theta/2) + z_2 \cos(\theta/2) \geq 0$.

### C Higher order Householder activation functions

We know that $\text{MaxMin}(z_1, z_2) = (\max(z_1, z_2), \min(z_1, z_2))$ where $z_1, z_2 \in \mathbb{R}$. Because $\max(z_1, z_2) > \min(z_1, z_2)$, applying MaxMin again gives the same result i.e $\text{MaxMin} \circ \text{MaxMin} = \text{MaxMin}$.
MaxMin. Now consider the function $\sigma_\theta$ (discussed in the main text, given below for convenience):

$$\sigma_\theta(z_1, z_2) = \begin{cases} 
1 & \text{if } z_1 \sin(\theta/2) - z_2 \cos(\theta/2) > 0 \\
0 & \text{if } z_1 \sin(\theta/2) - z_2 \cos(\theta/2) = 0 \\
\frac{1}{2} & \text{if } z_1 \sin(\theta/2) - z_2 \cos(\theta/2) < 0 
\end{cases}$$

From Figure 2a, we observe that if $(u_1, u_2) = \sigma_\theta(z_1, z_2)$, then $(u_1, u_2)$ always lies on the right side of the hyperplane (pink colored region). In other words, $u_1 \sin(\theta/2) - u_2 \cos(\theta/2) > 0 \ \forall \ z_1, z_2$ (proof in Appendix B). This further implies that $\sigma_\theta \circ \sigma_\theta = \sigma_\theta$.

However from Figure 2b in the maintext, we observe that if we use a different angle $\phi$ where $\phi \neq \theta + 2n\pi$ for some $n \in \mathbb{Z}$, then $\sigma_\phi(u_1, u_2) \neq (u_1, u_2)$ for all $(u_1, u_2)$ in the pink colored region ($u_1 \sin(\theta/2) - u_2 \cos(\theta/2) > 0$). This motivates us to construct the function $\sigma^{(n)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as follows:

$$\sigma^{(n)} = \sigma_\theta \circ \sigma_\theta \circ \sigma_\theta \ldots \circ \sigma_\theta$$

Clearly, $\sigma^{(n)}$ has a larger number of linear regions than $\sigma_\theta$ and thus expected to have more expressive power. However, a drawback of using $\sigma^{(n)}$ is that it requires a sequential application of $\sigma_\theta$ which can be expensive if the number of iterations $n$ is large. To address this limitation, first observe that $\sigma_\theta$ realizes the same linear function for $(z_1, z_2)$ and $(cz_1, cz_2)$ when $c > 0$ i.e $\nabla_{(z_1, z_2)} \sigma_\theta = \nabla_{(cz_1, cz_2)} \sigma_\theta$. Since $\sigma_\theta$ is piecewise linear, $\sigma_\theta(cz_1, cz_2) = \sigma_\theta(z_1, z_2)$. Thus, the input of the next function in the iteration is scaled by $c$ as well and its linear function (or the Jacobian) remains unchanged. By induction, same holds for all the subsequent iterations. By chain rule, the Jacobian of composition of functions is equal to the product of Jacobian of each individual function. Since the Jacobian of each function is unchanged on scaling by $c > 0$, the Jacobian $\nabla_{(z_1, z_2)} \sigma^{(n)}$ also remains unchanged: $\nabla_{(z_1, z_2)} \sigma^{(n)} = \nabla_{(cz_1, cz_2)} \sigma^{(n)}$. This suggests that it is possible to determine the Jacobian $\nabla_{(z_1, z_2)} \sigma^{(n)}$ for the input $(z_1, z_2)$ by first converting to the polar coordinates $(\sqrt{z_1^2 + z_2^2}, \varphi)$ and then using the phase angle $\varphi$ alone (where $\cos(\varphi) = \frac{z_1}{\sqrt{z_1^2 + z_2^2}}$, $\sin(\varphi) = \frac{z_2}{\sqrt{z_1^2 + z_2^2}}$).

This motivates us to construct another GNP piecewise linear activation that only depends on the phase of the input but unlike $\sigma_\theta$, it is allowed to have more than 2 linear regions without requiring a sequential application. This construction is given in the following theorem (example in maintext).

**Theorem 2.** Given: $0 \leq \theta_0 < \theta_1 < \ldots < \theta_{2n} = 2\pi + \theta_0$ such that $\sum_{i=1}^{n-1}(\theta_{2i+1} - \theta_{2i}) = \pi$ and $\alpha_i = 2 \sum_{j=0}^{i}(-1)^{i-j}$. The function $\sigma_\Theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is continuous, GNP and 1-Lipschitz where $\Theta = [\theta_0, \theta_1, \ldots, \theta_{2n-1}]$ (also called Householder Activation of order $n$ in 2 dimensions):

$$\sigma_\Theta(z_1, z_2) = \begin{bmatrix} \cos \alpha_i & \sin \alpha_i \\ -\sin \alpha_i & \cos \alpha_i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where $\varphi \in [\theta_0, \theta_{2n}]$ and $\cos(\varphi) = \frac{z_1}{\sqrt{z_1^2 + z_2^2}}$, $\sin(\varphi) = \frac{z_2}{\sqrt{z_1^2 + z_2^2}}$.

Using the definition of $\alpha_i$, $\alpha_{2n-1}$ can be computed as follows:

$$\alpha_{2n-1} = 2 \sum_{j=0}^{2n-2} \theta_{2n-1-j} - \theta_{2n-2-j} = 2 \sum_{j=0}^{n-1} (\theta_{2j+1} - \theta_{2j}) = 2\pi$$

$$\sigma_\Theta(z_1, z_2) = \begin{bmatrix} \cos \alpha_{2n-1} & \sin \alpha_{2n-1} \\ -\sin \alpha_{2n-1} & \cos \alpha_{2n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

By continuity, $\sigma_\Theta(z_1, z_2) = (z_1, z_2)$ for $\varphi = \theta_0$. Thus if we set $\theta_0 = 0$, $\sigma_\Theta$ is fixed to be identity when $\varphi = 0$ (or $z_2 = 0$). However, a learnable $\theta_0$ offers the flexibility of choosing arbitrary intervals around the $\varphi = \theta_0$ to be the identity function (while of course allowing $\theta_0 = 0$). Since we can choose any interval of $2\pi$ for the phase angle, we choose $\varphi \in [\theta_0, \theta_{2n} = \theta_0 + 2\pi]$ instead of the usual $[0, 2\pi)$ to allow this possibility. We call $(\sqrt{z_1^2 + z_2^2}, \varphi)$, the shifted polar coordinates.
Additionally, we make the following observations about Theorem 2. First, by construction \( \sigma_\theta \) has 2n linear regions. Second, since \( \sum_{i=0}^{n-1} (\theta_{2i+1} - \theta_{2i}) = \pi \), the sum of angles subtended by linear regions with determinant −1 equals \( \pi \). This in turn implies that sum of angles subtended by linear regions with determinant +1 must also equal \( \pi \). Third, again using \( \sum_{j=0}^{n-1} (\theta_{2j+1} - \theta_{2j}) = \pi \), we know that only 2n − 1 of the 2n parameters in \([\theta_0, \theta_1, \theta_2, \ldots, \theta_{2n-1}]\) can be chosen independently implying that \( \sigma_\theta \) has 2n − 1 learnable parameters. In contrast, \( \sigma^{(n)} \) has only n learnable parameters. Fourth, when \( n = 1 \), \( \sigma_\theta \) reduces to the original \( \sigma_\rho \) activation.

Because every function of the form \( \sigma^{(n)} \) (equation 20) can have potentially 2n linear regions, while \( \sigma_\theta \) has only 2n linear regions, \( \sigma_\theta \) cannot express every function of the form \( \sigma^{(n)} \). The primary benefit of using \( \sigma_\theta \) is that it can be easily applied by first determining the angle \( \varphi \) (using shifted polar coordinates), the region \([\theta_i, \theta_{i+1}]\) to which \( \varphi \) belongs and the Jacobian for this region. This requires 1 multiplication with the Jacobian instead of \( n \) required for \( \sigma^{(n)} \).

### D Extension to higher dimensions

We introduced Householder activation function of Order 1 in \( m \) dimensions in main text Definition. However, it suffers from the limitation that it has only 2 linear regions thus limiting its expressive power. The construction given in Appendix Section allows more than 2 linear regions but is valid only for 2 dimensional inputs. This motivates us to construct Householder activations that depend on all the \( m \) components of input \( z \in \mathbb{R}^m \), \( m \geq 3 \) while allowing for more than 2 linear regions.

A straightforward way of constructing such an activation function is to apply an orthogonal matrix \( Q \in \mathbb{R}^{m \times m} \), followed by dividing the output \( Qz \) into groups of size 2 each and then applying \( \sigma_\theta \) to each group. However, since 1-Lipschitz neural networks involve multiplication with an orthogonal weight matrix followed by GNP activation anyway, this construction is trivial because it does not lead to additional gains in expressive power over using 2 dimensional \( \sigma_\theta \) activation functions.

Recall that the function \( \sigma_v \) is given by the following equation:

\[
\sigma_v(z) = \begin{cases} 
z, & \text{if } v^Tz > 0, \\
(I - 2vv^T)z, & \text{if } v^Tz \leq 0.
\end{cases}
\]

By a similar analysis as for the 2-dimensional case (Figure 2 in maintext), a repeated application of \( \sigma_v \) leads to increased number of linear regions and thus higher expressive power. This motivates us to construct the function \( \sigma^{(m,n)} : \mathbb{R}^m \to \mathbb{R}^m \) by applying the function \( \sigma_v \) (equation 21) \( n \) times iteratively with different learnable parameter \( \nu \) at each iteration:

\[
\sigma^{(m,n)} = \sigma_v \circ \sigma_v \circ \sigma_v \ldots \circ \sigma_v.
\]

Since \( \sigma_v \) realizes the same linear function for both the inputs \( z \) and \( cz \) i.e \( \nabla_{cz} \sigma_v = \nabla_{cz} \sigma_v \) when \( c > 0 \), \( \sigma^{(m,n)} \) satisfies this property as well. This suggests that it is possible to determine the Jacobian of \( \sigma^{(m,n)} \) for the given input \( z \) by projecting \( z \) onto a unit sphere: \( z/\|z\|_2 \). Moreover, we want our constructed function to have at least \( 2n \) linear regions while requiring \( k \) iterations of \( \sigma_v \) where \( k \) is independent of \( n \). This motivates the following open question:

**Open Problem.** Can non-trivial order-\( n \) (\( n > 1 \)) householder activation functions with 2n linear regions be constructed for \( m \) dimensional input (\( m > 2 \)) using \( k \) iterations of \( \sigma_v \) where \( k \) is independent of \( n \) (but may depend on \( m \))?

### E Additional results on CIFAR-10

The rows "BCOP", "Cayley" and "SOC (baseline)" all use the MaxMin activation function. "SOC + HH" replaces MaxMin with 2D Householder activation of order 1 (\( \sigma_\theta \)), "+ CR" adds Certificate Regularization (CR) with \( \gamma = 0.1 \) (while using \( \sigma_\theta \) as the activation function).

In Table 6, the row "SOC + HH(2)" uses Householder activation of order 2 (\( \sigma_\theta \) defined in Theorem 2), "+ CR" adds Certificate Regularization (CR) with \( \gamma = 0.1 \) (while using the HH activation of order 2 i.e \( \sigma_\theta \) as the activation function).

None of the results in Tables 4, 5 and 6 include Last Layer Normalization (LLN).
## Table 3: Inference times for various networks on the complete test dataset of CIFAR-10 with 10000 samples. None of these networks include Last Layer Normalization (LLN).

| Architecture     | Running times (seconds) MaxMin | HH       |
|------------------|-------------------------------|----------|
| LipConvnet-5     | 3.7864                        | 3.86     |
| LipConvnet-10    | 5.3677                        | 5.6014   |
| LipConvnet-15    | 7.234                         | 7.3503   |
| LipConvnet-20    | 9.536                         | 9.3753   |
| LipConvnet-25    | 11.0512                       | 11.2692  |
| LipConvnet-30    | 12.5135                       | 13.6866  |
| LipConvnet-35    | 14.5539                       | 15.0921  |
| LipConvnet-40    | 17.1907                       | 17.1928  |

## Table 4: Results for provable robustness on the CIFAR-10 dataset using shallow networks. None of these results include Last Layer Normalization (LLN).

| Architecture     | Methods       | Standard Accuracy | Provable Robust Acc. ($\rho = $) | Increase |
|------------------|---------------|-------------------|-----------------------------------|----------|
| LipConvnet-5     | BCOP          | 74.25%            | 58.01% 40.34% 25.21% -1.88%      |
|                  | Cayley        | 72.37%            | 55.92% 38.65% 24.27% -2.82%      |
|                  | SOC (baseline)| 75.78%            | 59.18% 42.01% 27.09% (+0%)       |
|                  | SOC + HH      | 76.30%            | 60.12% 43.20% 27.39% +0.30%      |
|                  | + CR          | 75.31%            | 60.37% 45.62% 32.38% +5.29%      |
| LipConvnet-10    | BCOP          | 74.47%            | 58.48% 40.77% 26.16% -2.99%      |
|                  | Cayley        | 74.30%            | 57.99% 40.75% 25.93% -3.22%      |
|                  | SOC (baseline)| 76.45%            | 60.86% 44.15% 29.15% (+0%)       |
|                  | SOC + HH      | 76.86%            | 61.52% 44.91% 29.90% +0.75%      |
|                  | + CR          | 76.23%            | 62.57% 47.70% 34.15% +5.00%      |
| LipConvnet-15    | BCOP          | 73.86%            | 57.39% 39.33% 24.86% -4.80%      |
|                  | Cayley        | 71.92%            | 54.55% 37.67% 23.50% -6.16%      |
|                  | SOC (baseline)| 76.68%            | 61.36% 44.28% 29.66% (+0%)       |
|                  | SOC + HH      | 77.41%            | 61.92% 45.60% 31.10% +1.44%      |
|                  | + CR          | 76.39%            | 62.96% 48.47% 35.47% +5.81%      |
| LipConvnet-20    | BCOP          | 69.84%            | 52.10% 34.75% 21.09% -9.99%      |
|                  | Cayley        | 68.87%            | 51.88% 35.56% 21.72% -9.36%      |
|                  | SOC (baseline)| 76.90%            | 61.87% 45.79% 31.08% (+0%)       |
|                  | SOC + HH      | 76.99%            | 61.76% 45.59% 30.99% -0.09%      |
|                  | + CR          | 76.34%            | 62.63% 48.69% 36.04% +4.96%      |
Table 5: Results for provable robustness against adversarial examples on the CIFAR-10 dataset. None of these results include Last Layer Normalization (LLN).

| Architecture | Methods       | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase (108/255) |
|--------------|---------------|-------------------|----------------------------------|--------------------|
| LipConvnet-25| BCOP          | 68.47%            | 49.92% 31.99% 18.62%            | -9.98%            |
|              | Cayley        | 64.00%            | 45.55% 29.24% 16.99%            | -11.61%           |
|              | SOC (baseline)| 75.24%            | 60.17% 43.55% 28.60%            | (+0%)             |
|              | SOC + HH      | 76.37%            | 61.50% 44.72% 29.83%            | +1.23%            |
|              | + CR          | 75.21%            | 61.98% 47.93% 34.92%            | +6.32%            |
| LipConvnet-30| BCOP          | 64.11%            | 43.39% 25.02% 12.15%            | -15.90%           |
|              | Cayley        | 58.83%            | 38.68% 22.07% 10.68%            | -17.37%           |
|              | SOC (baseline)| 74.51%            | 59.06% 42.46% 28.05%            | (+0%)             |
|              | SOC + HH      | 75.25%            | 59.90% 43.85% 29.35%            | +1.30%            |
|              | + CR          | 74.23%            | 60.64% 46.51% 34.08%            | +6.03%            |
| LipConvnet-35| BCOP          | 63.05%            | 41.71% 23.30% 11.36%            | -15.84%           |
|              | Cayley        | 53.55%            | 32.37% 16.18% 6.33%             | -20.87%           |
|              | SOC (baseline)| 73.73%            | 58.50% 41.75% 27.20%            | (+0%)             |
|              | SOC + HH      | 75.44%            | 61.05% 45.38% 30.85%            | +3.65%            |
|              | + CR          | 74.25%            | 61.30% 47.60% 35.21%            | +8.01%            |
| LipConvnet-40| BCOP          | 60.17%            | 38.86% 21.20% 9.08%             | -15.05%           |
|              | Cayley        | 51.26%            | 27.90% 12.06% 3.81%             | -20.32%           |
|              | SOC (baseline)| 71.63%            | 54.39% 37.92% 24.13%            | (+0%)             |
|              | SOC + HH      | 73.24%            | 58.12% 42.24% 28.48%            | +4.35%            |
|              | + CR          | 72.59%            | 59.04% 44.92% 32.87%            | +8.74%            |
| Architecture | Methods          | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase (108/255) |
|--------------|------------------|-------------------|----------------------------------|-------------------|
|              |                  | (36/255)          | (72/255)                         | (108/255)         |
| LipConvnet-5 | SOC + HH$^{(2)}$ | 75.85%            | 59.66%                           | 42.68%            | 27.44% +0.35% |
|              | + CR             | 74.85%            | 60.56%                           | 44.96%            | 31.98% +4.59% |
| LipConvnet-10| SOC + HH$^{(2)}$| 76.80%            | 61.44%                           | 44.91%            | 29.70% +0.55% |
|              | + CR             | 76.30%            | 62.11%                           | 47.85%            | 34.49% +5.34% |
| LipConvnet-15| SOC + HH$^{(2)}$| 77.41%            | 62.21%                           | 45.11%            | 30.49% +0.83% |
|              | + CR             | 75.73%            | 62.21%                           | 47.92%            | 35.26% +5.60% |
| LipConvnet-20| SOC + HH$^{(2)}$| 76.69%            | 61.58%                           | 45.39%            | 30.89% -0.19% |
|              | + CR             | 75.72%            | 62.61%                           | 48.30%            | 35.29% +4.21% |
| LipConvnet-25| SOC + HH$^{(2)}$| 76.12%            | 61.24%                           | 44.81%            | 29.63% +1.03% |
|              | + CR             | 75.38%            | 61.94%                           | 47.67%            | 34.22% +5.62% |
| LipConvnet-30| SOC + HH$^{(2)}$| 75.09%            | 60.01%                           | 44.22%            | 29.39% +1.34% |
|              | + CR             | 74.88%            | 61.23%                           | 46.63%            | 34.02% +5.97% |
| LipConvnet-35| SOC + HH$^{(2)}$| 73.93%            | 58.61%                           | 42.29%            | 28.47% +1.27% |
|              | + CR             | 74.14%            | 60.72%                           | 46.67%            | 34.64% +7.44% |
| LipConvnet-40| SOC + HH$^{(2)}$| 70.90%            | 54.96%                           | 38.94%            | 24.90% +0.77% |
|              | + CR             | 72.28%            | 57.67%                           | 43.00%            | 30.66% +6.53% |

Table 6: Results for provable robustness on CIFAR-10 using HH activation of Order 2 ($\sigma_2$). Increase (108/255) is calculated with respect to SOC baseline (from Tables 4, 5). None of these results include Last Layer Normalization (LLN).

| Architecture | Methods          | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase (Standard) |
|--------------|------------------|-------------------|----------------------------------|---------------------|
|              |                  | (36/255)          | (72/255)                         | (108/255)           |
| LipConvnet-5 | SOC (no LLN)    | 75.78%            | 59.18%                           | 42.01%              | 27.09% (+0%) |
|              | SOC + LLN        | 75.78%            | 59.58%                           | 42.45%              | 27.20% +0.00% |
| LipConvnet-10| SOC (no LLN)    | 76.45%            | 60.86%                           | 44.15%              | 29.15% (+0%) |
|              | SOC + LLN        | 76.69%            | 61.08%                           | 44.04%              | 29.19% +0.24% |
| LipConvnet-15| SOC (no LLN)    | 76.68%            | 61.36%                           | 44.28%              | 29.66% (+0%) |
|              | SOC + LLN        | 76.84%            | 61.94%                           | 45.51%              | 30.28% +0.16% |
| LipConvnet-20| SOC (no LLN)    | 77.05%            | 61.87%                           | 45.79%              | 31.08% (+0%) |
|              | SOC + LLN        | 76.71%            | 61.44%                           | 44.92%              | 30.19% -0.34% |
| LipConvnet-25| SOC (no LLN)    | 75.24%            | 60.17%                           | 43.55%              | 28.60% (+0%) |
|              | SOC + LLN        | 76.54%            | 61.21%                           | 44.18%              | 29.47% +1.30% |
| LipConvnet-30| SOC (no LLN)    | 74.51%            | 59.06%                           | 42.46%              | 28.05% (+0%) |
|              | SOC + LLN        | 74.26%            | 58.97%                           | 41.82%              | 26.93% -0.25% |
| LipConvnet-35| SOC (no LLN)    | 73.73%            | 58.50%                           | 41.75%              | 27.20% (+0%) |
|              | SOC + LLN        | 74.32%            | 59.05%                           | 42.34%              | 28.14% +0.59% |
| LipConvnet-40| SOC (no LLN)    | 71.63%            | 54.39%                           | 37.92%              | 24.13% (+0%) |
|              | SOC + LLN        | 74.03%            | 58.27%                           | 41.75%              | 27.12% +2.40% |

Table 7: Results for provable robustness on the CIFAR-10 dataset with and without LLN.
F Additional Results on CIFAR-100

All results in Tables 10, 11 and 12 include Last Layer Normalization (LLN).

The rows "BCOP", "Cayley" and "SOC (baseline)" all use the MaxMin activation function. "SOC + HH" replaces MaxMin with 2D Householder activation of order 1 ($\sigma_1$). "+ CR" adds Certificate Regularization (CR) with $\gamma = 0.1$ (while using $\sigma_1$ as the activation function).

In Table 12 the row "SOC + HH(2)" uses Householder activation of order 2 ($\sigma_2$ defined in Theorem 2) "+ CR" adds Certificate Regularization (CR) with $\gamma = 0.1$ (while using the HH activation of order 2 i.e $\sigma_2$ as the activation function).

| Architecture | Methods      | Standard Accuracy | Provable Robust Acc. ($\rho = \sigma$) | Increase (Standard) |
|--------------|--------------|-------------------|----------------------------------------|---------------------|
| LipConvnet-5 | SOC + MaxMin | 42.71% 27.86% 17.45% 9.99% | _                                      |                     |
|              | + LLN        | 45.86% 31.93% 21.17% 13.21% +3.15% |                          |                     |
|              | + HH         | 46.36% 32.64% 21.19% 13.12% +3.65% |                          |                     |
|              | + CR         | 45.82% 32.99% 22.48% 14.79% +3.11% |                          |                     |
| LipConvnet-10| SOC + MaxMin | 43.72% 29.39% 18.56% 11.16% | _                                      |                     |
|              | + LLN        | 46.88% 33.32% 22.08% 13.87% +3.16% |                          |                     |
|              | + HH         | 47.96% 34.30% 22.35% 14.48% +4.24% |                          |                     |
|              | + CR         | 47.07% 34.53% 23.50% 15.66% +3.35% |                          |                     |

Table 8: Results for provable robustness against adversarial examples on the CIFAR-100 dataset.

| Architecture | MaxMin (no LLN) | MaxMin (LLN) | HH (LLN) |
|--------------|-----------------|--------------|----------|
| LipConvnet-5 | 3.7568          | 3.5002       | 3.6673   |
| LipConvnet-10| 5.3714          | 5.5482       | 5.5533   |
| LipConvnet-15| 7.3092          | 7.2595       | 7.3127   |
| LipConvnet-20| 9.005           | 9.2043       | 9.308    |
| LipConvnet-25| 10.9321         | 10.7868      | 11.726   |
| LipConvnet-30| 12.3198         | 13.2168      | 13.6275  |
| LipConvnet-35| 14.427          | 14.575       | 15.7069  |
| LipConvnet-40| 16.0911         | 16.2535      | 17.1015  |

Table 9: Inference times for various networks on the CIFAR-100 test dataset. Similar to CIFAR-10 (in Table 3), these numbers are for the whole test dataset with 10000 samples.
| Architecture | Methods      | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase (108/255) |
|--------------|--------------|-------------------|---------------------------------|-------------------|
|              |              | 36/255 | 72/255 | 108/255 |                      |
| LipConvnet-5 | BCOP         | 46.31% | 31.55% | 20.34% | 12.52% | -0.69% |
|              | Cayley       | 44.61% | 31.01% | 19.84% | 12.43% | -0.78% |
|              | SOC (baseline) | 45.86% | 31.93% | 21.17% | 13.21% | (+0%) |
|              | SOC + HH     | 46.36% | 32.64% | 21.19% | 13.12% | -0.09% |
|              | + CR         | 45.82% | 32.99% | 22.48% | 14.79% | +1.58% |
|              | BCOP         | 45.36% | 31.71% | 20.48% | 12.40% | -1.47% |
|              | Cayley       | 45.79% | 31.91% | 20.69% | 12.78% | -1.09% |
|              | SOC (baseline) | 46.88% | 33.32% | 22.08% | 13.87% | (+0%) |
|              | SOC + HH     | 47.96% | 34.30% | 22.35% | 14.48% | +0.61% |
|              | + CR         | 47.07% | 34.53% | 23.50% | 15.66% | +1.79% |
| LipConvnet-10 | BCOP        | 43.70% | 30.11% | 19.85% | 12.29% | -1.47% |
|              | Cayley       | 45.05% | 31.60% | 20.32% | 12.93% | -0.83% |
|              | SOC (baseline) | 47.72% | 33.52% | 21.89% | 13.76% | (+0%) |
|              | SOC + HH     | 47.72% | 33.97% | 22.45% | 13.81% | +0.05% |
|              | + CR         | 47.61% | 34.54% | 23.16% | 15.09% | +1.33% |
| LipConvnet-15 | BCOP        | 39.77% | 27.20% | 17.44% | 10.49% | -3.61% |
|              | Cayley       | 39.68% | 26.93% | 17.06% | 10.48% | -3.62% |
|              | SOC (baseline) | 46.86% | 33.48% | 22.14% | 14.10% | (+0%) |
|              | SOC + HH     | 47.71% | 34.22% | 22.93% | 14.57% | +0.47% |
|              | + CR         | 47.84% | 34.77% | 23.70% | 15.84% | +1.74% |

Table 10: Results for provable robustness on the CIFAR-100 dataset using shallow networks. All of these results include Last Layer Normalization (LLN).
| Architecture  | Methods         | Standard Accuracy | Provable Robust Acc. ($\rho = \frac{108}{255}$) | Increase (108/255) |
|--------------|----------------|-------------------|-----------------------------------------------|-------------------|
| LipConvnet-25| BCOP            | 34.15%            | 21.57% 13.52% 7.97% -5.89%                   |                   |
|              | Cayley          | 33.93%            | 21.93% 13.68% 8.19% -5.67%                   |                   |
|              | SOC (baseline)  | 46.32%            | 32.87% 21.53% 13.86% (+0%)                   |                   |
|              | SOC + HH + CR   | 47.70% 34.09% 23.41% 15.61% +1.75%          |                   |
| LipConvnet-30| BCOP            | 29.73%            | 18.69% 10.80% 6% -7.52%                      |                   |
|              | Cayley          | 28.67%            | 18.05% 10.43% 6.09% -7.43%                   |                   |
|              | SOC (baseline)  | 46.18%            | 32.82% 21.52% 13.52% (+0%)                   |                   |
|              | SOC + HH + CR   | 46.80% 33.72% 22.70% 14.31% +0.79%          |                   |
| LipConvnet-35| BCOP            | 25.65%            | 14.88% 8.47% 4.30% -8.95%                   |                   |
|              | Cayley          | 27.75%            | 16.37% 9.52% 5.40% -7.85%                   |                   |
|              | SOC (baseline)  | 45.22%            | 32.10% 21.28% 13.25% (+0%)                   |                   |
|              | SOC + HH + CR   | 46.81% 32.80% 21.55% 14.13% +0.88%          |                   |
| LipConvnet-40| BCOP            | 30.66%            | 18.68% 10.46% 5.92% -6.07%                   |                   |
|              | Cayley          | 25.54%            | 14.91% 8.37% 4.40% -7.59%                   |                   |
|              | SOC (baseline)  | 42.94%            | 29.71% 19.30% 11.99% (+0%)                   |                   |
|              | SOC + HH + CR   | 45.84% 32.79% 21.98% 14.07% +2.08%          |                   |

Table 11: Results for provable robustness on the CIFAR-100 dataset using deeper networks. All of these results include Last Layer Normalization (LLN).
| Architecture | Methods | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase |
|---------------|---------|------------------|-------------------------------|-----------|
|               |        | 36/255 | 72/255 | 108/255 | (108/255) |
| LipConvnet-5  | SOC + HH^{(2)} + CR | 46.61% | 32.50% | 21.34% | 13.22% | +0.01% |
| LipConvnet-10 | SOC + HH^{(2)} + CR | 47.47% | 33.32% | 21.84% | 13.75% | -0.01% |
| LipConvnet-15 | SOC + HH^{(2)} + CR | 47.19% | 33.67% | 22.36% | 13.78% | -0.09% |
| LipConvnet-20 | SOC + HH^{(2)} + CR | 47.86% | 33.97% | 22.78% | 14.59% | +0.73% |
| LipConvnet-25 | SOC + HH^{(2)} + CR | 46.23% | 32.64% | 21.95% | 14.00% | +0.48% |
| LipConvnet-30 | SOC + HH^{(2)} + CR | 46.06% | 32.35% | 21.33% | 13.65% | +0.40% |
| LipConvnet-35 | SOC + HH^{(2)} + CR | 43.81% | 30.59% | 20.08% | 12.56% | +0.57% |
| LipConvnet-40 | SOC + HH^{(2)} + CR | 43.81% | 30.59% | 20.08% | 12.56% | +0.57% |

Table 12: Results for provable robustness on CIFAR-100 using HH activation of Order 2 ($\sigma_2$). Increase (108/255) is calculated with respect to SOC baseline (from Tables 10, 11). All of these results include Last Layer Normalization (LLN).

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| Architecture | Methods | Standard Accuracy | Provable Robust Acc. ($\rho =$) | Increase |
|---------------|---------|------------------|-------------------------------|-----------|
|               |        | 36/255 | 72/255 | 108/255 | (Standard) |
| LipConvnet-5  | SOC (no LLN) | 42.71% | 27.86% | 17.45% | 9.99% | (+0%) |
|               | SOC + LLN | 45.86% | 31.93% | 21.17% | 13.21% | +3.15% |
| LipConvnet-10 | SOC (no LLN) | 43.72% | 29.39% | 18.56% | 11.16% | (+0%) |
|               | SOC + LLN | 46.88% | 33.32% | 22.08% | 13.87% | +3.16% |
| LipConvnet-15 | SOC (no LLN) | 42.92% | 28.81% | 17.93% | 10.73% | (+0%) |
|               | SOC + LLN | 47.72% | 33.52% | 21.89% | 13.76% | +4.80% |
| LipConvnet-20 | SOC (no LLN) | 43.06% | 29.34% | 18.66% | 11.20% | (+0%) |
|               | SOC + LLN | 46.86% | 33.48% | 22.14% | 14.10% | +3.80% |
| LipConvnet-25 | SOC (no LLN) | 43.37% | 28.59% | 18.18% | 10.85% | (+0%) |
|               | SOC + LLN | 46.32% | 32.87% | 21.53% | 13.86% | +2.95% |
| LipConvnet-30 | SOC (no LLN) | 42.87% | 28.74% | 18.47% | 11.21% | (+0%) |
|               | SOC + LLN | 46.18% | 32.82% | 21.52% | 13.52% | +3.31% |
| LipConvnet-35 | SOC (no LLN) | 42.42% | 28.34% | 18.10% | 10.96% | (+0%) |
|               | SOC + LLN | 45.22% | 32.10% | 21.28% | 13.25% | +2.80% |
| LipConvnet-40 | SOC (no LLN) | 41.84% | 28.00% | 17.40% | 10.28% | (+0%) |
|               | SOC + LLN | 42.94% | 29.71% | 19.30% | 11.99% | +1.10% |

Table 13: Results for provable robustness on the CIFAR-100 dataset with and without LLN.