Masking Quantum Information on Hyperdisks

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Masking information is a protocol that encodes quantum information into a bipartite entangled state while the information is completely unknown to local system. This work explicitly studies the structure of the set of maskable states and its relation to hyperdisks. We prove that maskable qubit states locate on a single hyperdisk, though it is not true for a high dimension case. Our results may shed light on several research fields of quantum information theory, such as the structure of entangled states and the local discrimination of bipartite states.

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I. INTRODUCTION

There are a variety of no-go theorems that characterize the intrinsic gap between classical and quantum information, such as the no-cloning theorem [1], the no-deleting theorem [2] and the no-go theorem on creating the superposition of unknown states [3]. A branch of no-go theorems are related to entanglement such as the no-hiding theorem [4].

Recently, Ref.[5] proposed a masking quantum information protocol, which encodes quantum information into a bipartite entangled system, while the information is completely unknown to local systems. They derived some related concepts. Then we study the classification of high-dimensional quantum states, Ref.[6] proposed a masker using generalized control-NOT gate. Based on that masker, they study on the structure of maskable states as the sharable secret sharing scheme [12] [13] [14], so it is significant to introduce some related concepts. Then we study the classification of masking protocol, depending on the dimension of input space $n$, the Schmidt number of target states $d$ and the degeneracy of marginal states. General ways are provided to derive the structure of maskable states in different cases. Based on these methods, we show that the maskable states may live on two or more different hyperdisks if $n \geq 3$. Finally, we prove that the maskable qubit states live on a hyperdisk when the dimension of the target state is large. A characterization of maskable states for $n = 2, d \geq 2$ and $n = 3, d = 3$ is given in the last section. This result may provide another way of thinking for several fields such as local discrimination in bipartite scenario and quantum secret sharing.

II. HYPERDISK AND RELATED CONCEPTS

Let $\mathcal{H}$ be an $n$-dimensional Hilbert space and $\mathcal{B} := \{ | \phi_j \rangle \}_{j=0}^{m-1}$ be an orthonormal basis for some $m$-dimensional subspace of $\mathcal{H}$, we define a real vector for a pure state $| \psi \rangle \in \mathcal{H}$:

$$r_{\mathcal{B}}(| \psi \rangle) := (| \langle \phi_0 | \psi \rangle |, ..., | \langle \phi_{m-1} | \psi \rangle |)^T. \quad (1)$$

Notice that $r_{\mathcal{B}}(| \Psi \rangle)$ is normalized iff $| \psi \rangle \in \text{span} \{ \mathcal{B} \}$.

**Definition 1** (hyperdisk). Let $\mathcal{S}$ be a set of pure states which live in an $n$-dimensional Hilbert space $\mathcal{H}$ and $\mathcal{V} := \text{span} \{ \mathcal{S} \}$ is a subspace of $\mathcal{H}$. Then $\mathcal{S}$ forms a hyperdisk if there is a complete orthonormal basis $\mathcal{B}$ of $\mathcal{V}$ which satisfies

$$r_{\mathcal{B}}(| \psi \rangle) = r, \quad \forall | \psi \rangle \in \mathcal{S}, \quad (2)$$

$$r_{\mathcal{B}}(| \xi \rangle) \neq r, \quad \forall | \xi \rangle \in \mathcal{H} \setminus \mathcal{S}. \quad (3)$$

Here $\mathcal{B}$ is called the hyperdisk basis and $m := \dim(\mathcal{V})$ is called the dimension of hyperdisk. In the simplest case that $m = 1$, $\mathcal{S}$ consists of only one pure state. For $m = 2$, $\mathcal{S}$ can be expressed as:

$$\{ | \psi(\theta) \rangle = a | \phi_0 \rangle + be^{i\theta} | \phi_1 \rangle | \theta \in \mathbb{R} \}, \quad (4)$$

where $| \phi_0 \rangle, | \phi_1 \rangle$ are orthogonal. In Bloch representation, a 2-dimensional hyperdisk can be visualized as an intersection between the sphere and a plane which is orthogonal to crossing line of two antipodal points $| \phi_0 \rangle$ and $| \phi_1 \rangle$.
Furthermore, in general case, any pure state $|\psi\rangle$ in a given $m$-dimensional hyperdisk $\mathcal{S}$ can be expressed as:

$$|\psi(\theta)\rangle = \sum_{j=0}^{m-1} r_j \exp{(i\theta_j)} |\phi_j\rangle,$$

where $\theta_j \in [0, 2\pi)$ and $|\phi_j\rangle$ are orthonormal states.

As an example, $\mathcal{S} = \{|\Psi(\theta)\rangle := |00\rangle + e^{i\theta_0} |11\rangle + e^{i\theta_1} |22\rangle\}$ is a Schmidt hyperdisk, while $\mathcal{S}' = \{|\Psi(\theta')\rangle := |00\rangle + e^{i\theta}(11) + |22\rangle\}$ is not because the basis of $\mathcal{S}'$ is $\{|00\rangle, |11\rangle + |22\rangle\}$ and $|11\rangle + |22\rangle$ is not a Schmidt basis. In this example, $\mathcal{S}'$ is a valid hyperdisk and $\mathcal{S}' \subseteq \mathcal{S}$. It leads to another concept called sub-hyperdisk.

**Definition 3 (sub-hyperdisk).** A subset $\mathcal{S}' \subseteq \mathcal{S}$ is a sub-hyperdisk of hyperdisk $\mathcal{S}$ if $\mathcal{S}'$ is also a hyperdisk.

Any state $|\psi'(\theta)\rangle$ in an $m'$-dimensional sub-hyperdisk $\mathcal{S}'$ of $\mathcal{S}$ takes the following form:

$$|\psi'(\theta)\rangle = \sum_{k=0}^{m'-1} e^{i\theta_k} |\phi'_k\rangle,$$

where $\theta_k \in \mathbb{R}^{m'}$.

Definition 4 (regular subset of hyperdisk). A subset $\mathcal{C} \subseteq \mathcal{S}$ is a regular subset of hyperdisk $\mathcal{S}$ if

$$\mathcal{V}_C \cap \mathcal{S} = C,$$  \hspace{1cm} (8)

where $\mathcal{V}_C := \text{span}\{\mathcal{C}\}$.

Eq. (8) means that for any linear combination $|\eta\rangle$ of states in $\mathcal{C}$, the condition $|\eta\rangle \in \mathcal{S}$ leads to $|\eta\rangle \in C$. It is obvious that every sub-hyperdisk is a regular subset. We label the $\text{dim}(C)$ as $\text{dim}(\mathcal{V}_C)$. Notice that $\mathcal{V}_C = \text{span}\{\mathcal{S}\}$ iff $\text{dim}(C) = \text{dim}(\mathcal{S})$, which leads to $C = \mathcal{S}$.

A general subset $\mathcal{G}$ of $\mathcal{S}$ can be expressed as:

$$\mathcal{G} = \bigcup_{p \in \mathcal{P}} \mathcal{S}_p,$$  \hspace{1cm} (9)

where $\{\mathcal{S}_p|p \in \mathcal{P}\}$ is the set of all hyperdisks contained in $\mathcal{G}$. Notice that this expression is valid because even a single state forms a hyperdisk. However it does not limit the number of hyperdisks in $\mathcal{G}$. Here we define the optimal hyperdisk cover $\mathcal{A}$ as the minimal subset of the index set $\mathcal{P}$ that fully covers $\mathcal{G}$. The size of $\mathcal{A}$ is called the optimal cover number. The following lemma shows the structure of $\mathcal{A}$ for 2-dimensional regular subset.

**Lemma 1.** The optimal cover number for 2-dimensional regular subset is at most 2.
Proof. In a 2-dimensional regular subset $\mathcal{C}$ of an $m$-dimensional hyperdisk $\mathcal{S}$, there are at least two states, 
\[
    |\psi_0\rangle = \sum_{j=0}^{m-1} r_j |\phi_j\rangle, |\psi_1\rangle = \sum_{j=0}^{m-1} r_j e^{i\theta_j} |\phi_j\rangle.
\]
(10)

Notice that $\{|\psi_0\rangle, |\psi_1\rangle\}$ can form a regular subset if there is no other states in $\mathcal{C}$. In this case, the regular subset $\mathcal{C}$ is consist of two 1-dimensional hyperdisks, so the optimal cover number is two. Now suppose $\mathcal{C}$ contains a third state $|\psi\rangle$. From Def.4, it can be expressed as 
\[
    |\psi\rangle = a |\psi_0\rangle + b e^{i\varphi} |\psi_1\rangle,
\]
(11)
where $a, b$ are non-zero real numbers. Furthermore, $|\psi\rangle \in \mathcal{S}$ implies $|a + be^{i(\varphi+\theta_j)}| = 1$ for all $j$. Then there is $\eta \in [0, \pi)$ such that $\cos(\eta) \equiv \cos(\varphi + \theta_j)$, it follows that $\theta_j = 2k\pi \pm \eta - \varphi$ where $k \in \mathbb{Z}$. Thus $|\psi_1\rangle$ can be reformulated as 
\[
    |\psi_1\rangle = \sum_{j\theta_j+\varphi=-\eta} r_j |\phi_j\rangle + e^{i2\eta} \sum_{j\theta_j+\varphi=-\eta} r_j |\phi_j\rangle.
\]
which together with $|\psi_0\rangle = |\phi_0^a\rangle + |\phi_1^a\rangle$ leads to $\mathcal{V}_C = \text{span}(|\phi_0^a\rangle, |\phi_1^a\rangle)$. Hence the condition $\mathcal{V}_C = \mathcal{V}_C \cap \mathcal{S}$ indicates that $\mathcal{C}$ is just a 2-dimensional hyperdisk in $\mathcal{S}$: 
\[
    |\psi(\theta)\rangle = |\phi_0^a\rangle + e^{i\theta} |\phi_1^a\rangle.
\]
(12)
It means the optimal cover number of this case is one. \qed

III. MASKING INFORMATION PROTOCOL

A masking information protocol involves three participants: a referee R and two players A and B. Each of them holds a space denoted as $\mathcal{H}_R, \mathcal{H}_A$ or $\mathcal{H}_B$. For every round of the protocol, the referee randomly chooses a pure state $|\psi\rangle$ in the set of maskable states $\mathcal{R}$ in $\mathcal{H}_R$, and puts the state $|\psi\rangle$ into a masking machine.

Definition 5 (masking machine). The linear isometry 
\[
    V_{\text{mask}} : \mathcal{H}_R \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B
\]
(13)
is a masking machine for a set of maskable states $\mathcal{R}$ in $\mathcal{H}_R$, if for any $|\psi\rangle$ in $\mathcal{R}$, the marginal states of $|\Psi\rangle = V_{\text{mask}} |\psi\rangle$ are constant, i.e. $\text{Tr}_B(|\Psi\rangle \langle \Psi|) \equiv \rho_A$ and $\text{Tr}_A(|\Psi\rangle \langle \Psi|) \equiv \rho_B$ are independent of $|\psi\rangle$.

In this way, the referee distributes $|\psi\rangle \in \mathcal{R}$ to the players without losing any quantum information. Notice that no communication is allowed between the two players, so they cannot gain any information about which state the referee has chosen.

The set of bipartite pure states $|\Psi\rangle$ shared between Alice and Bob are called the target states, denoted by $\mathcal{T}$, with $\mathcal{V}_T = \text{span}\{\mathcal{T}\}$. The marginal states $\rho_A$ and $\rho_B$ are mixed states if $\mathcal{R}$ contains more than one state, so $|\Psi\rangle$ must be entangled. In other words, this protocol encodes quantum information into non-local correlation.

The dimension of $\text{span}(\mathcal{R})$ is denoted by $n$. States that do not live in $\text{span}(\mathcal{R})$ cannot be masked, so we set $\mathcal{H}_R = \text{span}(\mathcal{R})$ without loss of generality. The isometry $V_{\text{mask}}$ restricts that $\mathcal{H}_R \simeq \mathcal{V}_T$, so $\dim(\mathcal{V}_T) = n$.

The rank of marginal states (or the Schmidt number of target states) is denoted by $d$. Then the marginal states can be written as 
\[
    \rho_A = \sum_{j=0}^{d-1} \lambda_j |\phi_j^A\rangle \langle \phi_j^A|, \rho_B = \sum_{j=0}^{d-1} \lambda_j |\phi_j^B\rangle \langle \phi_j^B|.
\]
(14)

Notice that marginal states may have some degrees of degeneracy, so the eigenstates may not be fixed. By the purification process [15], it is necessary for the target states to be expressed as 
\[
    |\Psi(\theta)\rangle = \sum_{j=0}^{d-1} \sqrt{\lambda_j} e^{i\theta_j} |\phi_j^A\phi_j^B\rangle,
\]
(15)
where $\theta_j \in [0, 2\pi)$. However, it is not a sufficient condition as these states may not live in $\mathcal{V}_T$, so we call the states in the form of Eq.(15) the legal states. The set of legal states is denoted by $\mathcal{L}$ and $\mathcal{V}_L := \text{span}(\mathcal{L})$. Without loss of generality, we set $\dim(\mathcal{H}_{AB}) = d^2$.

Here we mention that the definition of masking machine in Ref.5 is a unitary transformation $U_{\text{mask}} : \mathcal{H}_{RS} \rightarrow \mathcal{H}_{AB}$:
\[
    |\Psi\rangle = U_{\text{mask}} |\psi\rangle |s\rangle
\]
(16)
where $|s\rangle$ is a fixed state of the auxiliary system $\mathcal{H}_S$. This definition is a special case of ours where the isometry $V_{\text{mask}}$ is chosen as 
\[
    V_{\text{mask}} = U_{\text{mask}} \mathbb{I}_R \otimes |s\rangle.
\]
(17)
The advantage of our definition is that we require less parameters to describe a masking machine. We will see in the following that the classification of masking protocols is now mainly guided by the parameters $n$ and $d$.

By definition, the tuple $(V_{\text{mask}}, \rho_A, \rho_B)$ fully characterize a masking information protocol. The set of legal target states $\mathcal{L}$ is determined by $(\rho_A, \rho_B)$. The linearity of isometry $V_{\text{mask}}$ implies that the linear combination of target states belongs to $\mathcal{T}$. The set of target states can now be expressed as 
\[
    \mathcal{T} = \mathcal{V}_T \cap \mathcal{L}.
\]
(18)
In order to characterize the structure of maskable states, we focus on the structure of target states, which is isomorphic to $\mathcal{R}$.

The degeneracy of the marginal states determines the structure of the set of legal states $\mathcal{L}$. In following we divide the discussion into three parts according to degeneracy of marginal states: the non-degenerate case, the completely degenerate case and the general case.
A. Non-degenerate Case

In this case, $\rho_A$ and $\rho_B$ have fixed eigenstates. Hence the set of legal target states $\mathcal{L}_{\text{nd}}$ is a Schmidt hyperdisk $S^{AB}$ as the following form:

$$\Psi(\theta) = \sum_{j=0}^{d-1} e^{i\theta_j} \sqrt{x_j} |\phi^A_j \phi^B_j\rangle,$$

(19)

where $\theta_j \in [0, 2\pi)$ and $\dim(\mathcal{V}_{\text{nd}}) = d$. The set of target states for non-degenerate case is denoted by $\mathcal{T}_{\text{nd}}$ and $\mathcal{V}_{\text{nd}} = \text{span}\{\mathcal{T}_{\text{nd}}\}$. Because $\mathcal{T}_{\text{nd}} \subseteq \mathcal{L}_{\text{nd}}$, the target states must live on the $d$-dimensional hyperdisk $S^{AB}$.

From Eq. (8) and Eq. (18), $\mathcal{T}_{\text{nd}}$ can be expressed as a regular subset of $S^{AB}$:

$$\mathcal{T}_{\text{nd}} = \mathcal{V}_{\text{nd}} \cap S^{AB}.$$

(20)

Thus $\dim(\mathcal{V}_{\text{nd}})$ is bounded as

$$\dim(\mathcal{V}_{\text{nd}}) = n \leq d = \dim(\mathcal{V}_{\text{nd}}).$$

(21)

The equality holds iff $\mathcal{T}_{\text{nd}} = S^{AB}$, which means that the maskable states in $\mathcal{H}_R$ form a hyperdisk.

However, when $n < d$, there are situations where $\mathcal{T}_{\text{nd}}$ is not a sub-hyperdisk of $S^{AB}$, and hence the set of maskable states $\mathcal{R}_{\text{nd}}$ may not live on any hyperdisk in $\mathcal{H}_R$. For example, we consider the following non-degenerate masking protocol with $n = 3, d = 4$. Here, $\mathcal{T}_{\text{nd}}$ consists of the following states:

$$\Psi_0(\alpha) = |00\rangle + \sqrt{2} |11\rangle + e^{i\alpha} \left(\sqrt{3} |22\rangle + 2 |33\rangle\right),$$

$$\Psi_1(\beta) = |00\rangle + \sqrt{3} |22\rangle + e^{i\beta} \left(\sqrt{2} |11\rangle + 2 |33\rangle\right).$$

(22)

It follows that $\mathcal{V}_T = \text{span}\{|00\rangle + \sqrt{2} |11\rangle, \sqrt{3} |22\rangle + 2 |33\rangle, |\Phi_+\rangle\}$ is a 3-dimensional subspace of $\mathcal{H}_{AB}$, where $|\Phi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|22\rangle + 2 |33\rangle + |\Phi_+\rangle).$ Here we define the masking machine $V_{\text{mask}}$ as $|1\rangle \rightarrow |00\rangle + \sqrt{2} |11\rangle,$ $|1\rangle \rightarrow \sqrt{3} |22\rangle + 2 |33\rangle$ and $|2\rangle \rightarrow |\Phi_+\rangle$. Even though $\mathcal{T}_{\text{nd}}$ is a regular subset that belongs to a 4-dimensional hyperdisk, the set of maskable states $\mathcal{R}_{\text{nd}}$ does not live on any hyperdisk.

$$|\psi_0(\alpha)\rangle = \sqrt{3} |0\rangle + e^{i\alpha} \sqrt{7} |1\rangle,$$

$$|\psi_1(\beta)\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{3}{\sqrt{7}} |1\rangle - \frac{\sqrt{50}}{21} |2\rangle + e^{i\beta} \left(\frac{2}{\sqrt{3}} |0\rangle + \frac{4}{\sqrt{7}} |1\rangle + \frac{\sqrt{50}}{21} |2\rangle\right).$$

(23)

To see that, assume $\mathcal{R}_{\text{nd}}$ is a subset of some hyperdisk in $\mathcal{H}_R$, then the hyperdisk basis must contains either $|0\rangle$ or $|1\rangle$ because of $|\psi_0(\alpha)\rangle$. However, for $|\psi_1(\beta)\rangle$, neither $|0\rangle$ nor $|1\rangle$ is in hyperdisk basis, this is a contradiction to the assumption. Therefore, one can mask states which does not live on any single hyperdisk in $\mathcal{H}_R$, even the non-degenerate masking protocol is employed.

B. Completely Degenerate Case

In this case, $\rho_A = \rho_B = I/d$, and the set of legal states $\mathcal{L}_{\text{cd}}$ is the set of all maximally entangled states in $\mathcal{H}_{AB}$. A maximally entangled state can be expressed as

$$\psi(U) = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} U |jj\rangle = U \otimes I |\Phi_1\rangle,$$

(24)

where $|\Phi_1\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle$ and $U$ is a unitary matrix with elements $(U)_{jj} = \langle ij | \psi(U) \rangle$. $(|jj\rangle)$ is defined as the computational basis for $\mathcal{H}_A$ (or $\mathcal{H}_B$). The completely degenerate set of target states is denoted by $\mathcal{T}_{\text{cd}}$ and $\mathcal{V}_{\text{cd}} = \text{span}\{\mathcal{T}_{\text{cd}}\}$. From Eq. (18), we can define $\mathcal{T}_{\text{cd}}$ as

$$\mathcal{T}_{\text{cd}} = \mathcal{V}_{\text{cd}} \cap \mathcal{L}_{\text{cd}}$$

(25)

where $\mathcal{L}_{\text{cd}}$ is the set of legal states, i.e. the set of maximally entangled states.

Unlike the non-degenerate case, $\mathcal{L}_{\text{cd}}$ is no longer a hyperdisk. Hence $\mathcal{T}_{\text{cd}}$ may not belong to a hyperdisk anymore. The following example shows that $\mathcal{T}_{\text{cd}}$ can consist of infinite number of hyperdisks. Here we set $n = 3, d = 2$ and $\mathcal{T}_{\text{cd}} = \bigcup_{\xi, \eta} S_{\xi\eta}$, where states in $S_{\xi\eta}$ are written as:

$$|\psi_{\xi\eta}(\theta)\rangle = e^{i\theta} \left(\phi_{\xi\eta}^+ \phi_{\xi\eta}^+ + \phi_{\xi\eta}^\dagger \phi_{\xi\eta}^\dagger\right),$$

(26)

where $\left\{|\phi_{\xi\eta}^+\rangle, |\phi_{\xi\eta}^\dagger\rangle\right\}$ is one of orthonormal basis for 2-dimensional space:

$$|\phi_{\xi\eta}^+\rangle = \cos \frac{\xi}{2} |0\rangle + \sin \frac{\xi}{2} e^{i\eta} |1\rangle,$$

$$|\phi_{\xi\eta}^\dagger\rangle = \cos \frac{\xi}{2} |0\rangle - \sin \frac{\xi}{2} e^{i\eta} |1\rangle,$$

(27)

where $\xi$ and $\eta$ are continuously chosen in $[0, 2\pi)$. It follows that $\mathcal{V}_{\text{cd}} = \text{span}\{|00\rangle, |11\rangle, |01\rangle + |10\rangle\}$. Here we define masking machine $V_{\text{mask}}$ as $|0\rangle \rightarrow |00\rangle, |1\rangle \rightarrow |11\rangle$ and $|2\rangle \rightarrow |01\rangle + |10\rangle$. Then the hyperdisks $V_{\text{mask}} S_{\xi\eta}$ in $\mathcal{R}_{\text{cd}}$ are expressed as

$$|\psi_{\xi\eta}(\theta)\rangle = \cos^2 \frac{\xi}{2} |0\rangle + \sin^2 \frac{\xi}{2} e^{i\eta} |1\rangle + \frac{1}{\sqrt{2}} \sin \xi e^{i\eta} |2\rangle + e^{i\theta} \left(\sin^2 \frac{\xi}{2} |0\rangle + \cos^2 \frac{\xi}{2} e^{i\eta} |1\rangle - \frac{1}{\sqrt{2}} \sin \xi e^{i\eta} |2\rangle\right).$$

(28)

It is an example that the set of maskable states contains unlimited amount of hyperdisks. This example indicates that degeneracy of masking protocol can enhance its power.

A property of completely degenerate masking protocol is that $\mathcal{V}_{\text{cd}} = \text{span}\{\mathcal{L}_{\text{cd}}\} = \mathcal{H}_{AB}$. To prove that we use the generalized Bell states\[10\]:

$$|\Psi_{jk}\rangle := X_j Z^k \otimes I |\Phi_1\rangle,$$

(29)

where $Z$ and $X$ are generalized Pauli operators and defined as $Z |k\rangle = \exp(2k\pi i/d) |k\rangle, \ X |k\rangle = $$
\(|k+1)\bmod d\). It is easy to check that \(\{|\Psi_{ij}\}\}_{i,j=0}^{d-1}\) is a complete orthogonal basis of \(\mathcal{H}_{AB}\). Hence \(\dim(\mathcal{V}_{Tcd})\) is bounded as:

\[
\dim(\mathcal{V}_{Tcd}) = n \leq d^2 = \dim(\mathcal{V}_{Lcd}).
\]

Comparing it with Eq.\((2)\), we find that the degenerate masking protocol is more powerful than the nondegenerate counterpart.

**C. General Case**

In the general case, the marginal state \(\rho_A\) (or \(\rho_B\)) is partially degenerate. The \(j\)th degenerate eigenvalue for \(\rho_A\) and \(\rho_B\) are denoted by \(\lambda_j\). The degeneracy of \(j\)th eigenspace is \(g(j)\), and the computational basis in that subspace is \(\{|j,k\}\}_{k=0}^{g(j)-1}\). The total number of eigenspace is \(t\). Then the legal states can be expressed as:

\[
|\Psi(U)\rangle = \sum_{j=0}^{t-1} \sqrt{\lambda_j} \sum_{k=0}^{g(j)-1} U_j \otimes I |j,k\rangle |j,k\rangle
= U \otimes I |\Psi_1\rangle.
\]

where \(U\) is in the block diagonal form

\[
U = \bigoplus_{j=0}^{t-1} U_j.
\]

The \(U_j\) is \(g(j)\)-dimensional unitary matrix acts on \(j\)th degenerate subspace and \(|\Psi_1\rangle = \sum_{j=0}^{t-1} \sqrt{\lambda_j} \sum_{k=0}^{g(j)-1} |j,k\rangle |j,k\rangle\). The \(\dim(\mathcal{V}_T)\) now is limited by \(g(j)\):

\[
\dim(\mathcal{V}_T) = n \leq \sum_{j=0}^{t-1} g^2(j) = \dim(\mathcal{V}_L),
\]

where \(\sum_{j=0}^{t-1} g(j) = d\).

In the following lemma, we first give the necessary and sufficient condition for a state as in Eq.\((31)\) to live on a Schmidt hyperdisk. In general, the target states \(T\) can consist several hyperdisks. This lemma is helpful in comparing the structure of \(T\) with hyperdisks.

**Lemma 2.** A set of states \(\{|\Psi(U)\rangle\}_{U \in \mathcal{U}}\) lives on some Schmidt hyperdisk if and only if there exists a unitary matrix \(U_T\) that satisfies \([UU_T, U'_T]\) = 0, \(\forall U, U' \in \mathcal{U}\), where \(U_T\) and each \(U\) takes the block diagonal form as in Eq.\((32)\).

**Proof.** First we notice that, \(|\Psi_1\rangle\) can be expressed as:

\[
|\Psi_1\rangle = \sum_{j=0}^{t-1} \sqrt{\lambda_j} \sum_{k=0}^{g(j)-1} |\phi^*_{jk}\rangle |\phi_{jk}\rangle,
\]

where \(\{|\phi_{jk}\rangle\}_k\) is an orthonormal basis in \(j\)th eigenspace. \(|\phi^*_{jk}\rangle\) denotes the conjugate state of \(|\phi_{jk}\rangle\) such that \(\langle j', k' | \phi^*_{jk}\rangle = \langle \phi_{jk} | j', k'\rangle\), where \(\{|j', k'\rangle\}\) is the computational basis.

First we assume that \(\{|\Psi(U)\rangle\}\) lives on some Schmidt hyperdisk \(S_{AB}^T\):

\[
|\Psi(U)\rangle = \sum_j \sqrt{\lambda_j} \sum_k e^{i\theta_{jk}} |\psi_{jk}\rangle |\phi^*_{jk}\rangle.
\]

Comparing this with Eq.\((31)\), we find that \(U(\theta) = \sum_j \sum_k e^{i\theta_{jk}} |\psi_{jk}\rangle |\phi^*_{jk}\rangle\). Then we choose a unitary operator \(U_T = \sum_j \sum_k e^{i\phi_{jk}} |\phi_{jk}\rangle |\psi_{jk}\rangle\), and the commutation relation \([U(\theta)U_T, U(\theta')U_T]\) = 0 holds. Here the sufficient part has been proved.

For the necessary part, we suppose there exists a unitary operator \(U_T\) such that \([UU_T, U_T'] = 0, \forall U, U' \in \mathcal{U}\). Here \(UU_T\) is diagonal on some basis \(UU_T = \sum_j \sum_k e^{i\theta_{jk}} |\psi_{jk}\rangle |\phi_{jk}\rangle\). Without loss of generality, we choose \(U_T = \sum_j \sum_k |\phi^*_{jk}\rangle |\psi_{jk}\rangle\). It follows that

\[
U = UU_T U_T^* = \sum_j \sum_k e^{i\phi_{jk}} |\psi_{jk}\rangle |\phi^*_{jk}\rangle.
\]

By substituting above formulas into \(\{|\Psi(U)\rangle\} = U \otimes I |\Psi_1\rangle\), we arrive at Eq.\((35)\), which means that \(\{|\Psi(U)\rangle\}_{U \in \mathcal{U}}\) lives on a Schmidt hyperdisk.

**IV. STRUCTURE OF \(R\) FOR QUBITS AND QUTRITS**

In this section, we derive the structure of \(R\) for qubits and qutrits.

**A.** \(n = 2, d \geq 2\)

Here \(\mathcal{H}_R\) is restricted to be a qubit space, while the dimension of \(\mathcal{V}_T\) is not limited, we prove that the following theorem.

**Theorem 1.** For \(n = 2\) and \(d \geq 2\), \(R\) must live on a hyperdisk.

**Proof.** In the general case, we choose two different states \(|\psi_0\rangle\) and \(|\psi_1\rangle\) in \(R\). The masking machine \(V_{\text{mask}}\) can be stated as:

\[
|\psi_0\rangle \rightarrow |\psi_0\rangle = U_0 \otimes I |\Psi_1\rangle
|\psi_1\rangle \rightarrow |\psi_1\rangle = U_1 \otimes I |\Psi_1\rangle
\]

Any states in \(R\) can be written as \(|\psi\rangle = a |\psi_0\rangle + b |\psi_1\rangle\). Then the unitary matrix for target states can be expressed as \(U(a, b) = aU_0 + bU_1\). Then we choose \(U_T = U_0^\dagger\) such that \([U(a, b)U_T, U(a', b')U_T]\) = 0. From Lemma 2, the target states is on the same Schmidt hyperdisk \(S_{AB}^T\). Hence \(T\) is a 2-dimensional regular subset of \(S_{AB}^T\).

According to Lemma 1, \(T\) contains either a 2-dimensional hyperdisk or two single states. From the property (P2), \(R\) forms a hyperdisk as well in the first case. From the property (P3), \(R\) must live on some hyperdisk in the second case.

\(\square\)
This theorem extends the previous result\cite{5} that the all of qubit information cannot be masked simultaneously, as it gives a full characterization for the set of maskable qubit information.

B. $n = 3, d = 3$

For $\mathcal{H}_R$ with higher dimension, there are situations where maskable states that not live on same hyperdisk. The following theorem provides the explicit structure of the set of target states $\mathcal{T}$.

**Theorem 2.** Assume $n = d = 3$ and the set of target states $\mathcal{T}$ contains at least one 2-dimensional subhyperdisk of Schmidt hyperdisk, then $\mathcal{T}$ has 3 possible types of structure:

Type-I: $\mathcal{T}$ is a 3-dimensional Schmidt hyperdisk;

Type-II: $\mathcal{T}$ consists of two 2-dimensional subhyperdisks locating on different Schmidt hyperdisks;

Type-III: $\mathcal{T}$ has a 2-dimensional sub hyperdisk and a single state locating on different Schmidt hyperdisk.

Here we mention that these types of structure are invariant under local unitary. The type-I structure has a general form as:

$$|\Psi(\theta)\rangle = \sqrt{\lambda_0}|00\rangle + e^{i\alpha_1}\sqrt{\lambda_1}|11\rangle + e^{i\alpha_2}\sqrt{\lambda_2}|22\rangle,$$  \hspace{1cm} (37)

The only possible type of $\mathcal{T}$ for $n = d$ non-degenerate maskers is Schmidt hyperdisk as we have discussed before, so type-I can be achieved by non-degenerate masker. But for degenerate case, the structure is complicated because of the symmetry. The type-II structure can generally be written as:

$$|\Psi_0(\alpha)\rangle = \sqrt{\lambda_1}|00\rangle + e^{i\alpha}|\sqrt{\lambda_1}|11\rangle + \sqrt{\lambda_2}|22\rangle,$$

$$|\Psi_1(\beta)\rangle = \sqrt{\lambda_1}\left(|\phi_{01}^\pm|00\rangle + e^{i\beta}|\phi_{01}^+|01\rangle + \sqrt{\lambda_2} |22\rangle \right),$$ \hspace{1cm} (38)

where $\{|\phi_{01}^\pm\rangle, |\phi_{01}\rangle\}$ and $\{ |\psi_{01}^+\rangle, |\psi_{01}\rangle\}$ are two orthogonal bases for subspace $\text{span} \{|00\rangle, |11\rangle\}$. These two orthogonal bases have a property that $|\langle 0|\phi_{01}^+\rangle| = |\langle 0|\psi_{01}^+\rangle|$. Notice that only the degenerate maskers can achieve type II structure. Type-III structure can be expressed as:

$$|\Psi_0(\alpha)\rangle = |00\rangle + e^{i\alpha}|11\rangle + |22\rangle,$$

$$|\Psi'\rangle = \cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}\left(e^{i\varphi_0}|10\rangle + e^{i\varphi_1}|01\rangle \right) |\psi_{12}^\pm\rangle + e^{i(\varphi_0 + \varphi_1)} \left( e^{i\eta_1}|2\rangle |\psi_{12}^-\rangle - \cos\frac{\theta}{2}|1\rangle |\psi_{12}^+\rangle \right),$$ \hspace{1cm} (39)

where $\{ |\psi_{12}^+\rangle, |\psi_{12}^-\rangle \}$ is an orthogonal basis for subspace $\text{span} \{|11\rangle, |22\rangle\}$. The type-III structure can only be achieved in the completely degenerate case. For the complete proof for Theorem 2, see appendix A and B.

It is worth mention that if we neglect the assumption that $\mathcal{T}$ contains at least one 2-dimensional Schmidt hyperdisk, then $\mathcal{T}$ can have structures other than the above three types. An example is that the set of target states contains infinite amount of states:

$$|\Phi_1\rangle, Z \otimes I |\Phi_1\rangle, X \otimes I |\Phi_1\rangle,$$ \hspace{1cm} (40)

where $X$ and $Z$ are generalized Pauli matrices. Another example is

$$|\Psi_\eta\rangle = \sqrt{\lambda_1/2} \left( e^{i\eta}|00\rangle + e^{-i\eta}|11\rangle \right) + \sqrt{\lambda_1/2i} (|01\rangle + |10\rangle) + \sqrt{\lambda_2}|22\rangle,$$ \hspace{1cm} (41)

here we find that $\mathcal{T}$ does not contain any non-trivial hyperdisk (i.e. the dimension of hyperdisk is larger than one).

V. CONCLUSION

We have studied the relation between hyperdisks and the structure of maskable states. In this work, unambiguous definition of the concept hyperdisk is introduced with another concept called regular subsets of hyperdisk. A method is given to deal with the classification of masking protocol by using above concepts.

We prove that in spite of the conjecture that maskable states must live on same hyperdisk holds for qubit information with finite dimensional entanglement, it fails for general case. The masking protocol on qutrit information with entanglement can be characterized with the number of hyperdisk. In general setting, the non-degenerate masking protocol can mask several independent hyperdisks for qutrit information, together with the set of completely degenerate maskable states may contain unlimited amount of hyperdisks.

There are still some questions originating from the conjecture: whether the upper bound of optimal hyperdisk covering number on regular subset of hyperdisk is equal to its dimension of spanning space; the structure of mixed state masking protocol remains unknown. These questions may provide some research directions for the future work.

Acknowledgments

This work was supported by NSFC under Grants No. 11774205, and the Young Scholars Program of Shandong University. Note added: in submitting our paper, we just notice a new online PRA paper [Phys. Rev. A 100, 030304], which proves a result similar to Theorem 1 in our paper.
[1] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299(5886):802–803, 1982.

[2] Arun Kumar Pati and Samuel L. Braunstein. Impossibility of deleting an unknown quantum state. *Nature*, 404(6774):164–165, 2000.

[3] Michał Oszmianiec, Andrzej Grudka, Michal Horodecki, and Antoni Wójcik. Creating a superposition of unknown quantum states. *Phys. Rev. Lett.*, 116:110403, Mar 2016.

[4] Samuel L. Braunstein and Arun K. Pati. Quantum information cannot be completely hidden in correlations: Implications for the black-hole information paradox. *Phys. Rev. Lett.*, 98:080502, Feb 2007.

[5] Kavan Modi, Arun Kumar Pati, Aditi Sen(De), and Ujjwal Sen. Masking quantum information is impossible. *Phys. Rev. Lett.*, 120:230501, Jun 2018.

[6] Mao-Sheng Li and Yan-Ling Wang. Masking quantum information in multipartite scenario. *Phys. Rev. A*, 98:062306, Dec 2018.

[7] Bo Li, Shu-han Jiang, Xiao-Bin Liang, Xiao-Bin Liang, Xianqing Li-Jost, and Karol Horodecki. Quantum entanglement splitting. *Rev. Mod. Phys.*, 81:865–942, Jun 2009.

[8] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*, 10th Anniversary Edition. 2011.

[9] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, 81:865–942, Jun 2009.

Appendix A: Proof of structure of partially degenerate masker when $n = 3, d = 3$

Assume there is a 2-dimensional sub-hyperdisk of Schmidt hyperdisk $S_0$ in $T$, then the remain degree of freedom is one for our choice of state $|\Psi\rangle \in T \setminus S_0$. There are two possible choices of $S_0$:

$$
|\Psi_0(\alpha)\rangle = \sqrt{\lambda_1} (|00\rangle + |11\rangle) + e^{i\alpha} \sqrt{\lambda_2} |22\rangle,
$$

$$
|\Psi_0(\alpha)\rangle = \sqrt{\lambda_1} |00\rangle + e^{i\alpha} \left( \sqrt{\lambda_1} |11\rangle + \sqrt{\lambda_2} |22\rangle \right). \tag{A1}
$$

For the first choice, $S_0$ does not live on a fixed Schmidt hyperdisk. No matter which state $|\Psi\rangle \in L \setminus S_0$ we choose, $S_0$ and $|\Psi\rangle$ always live on the same Schmidt hyperdisk, it leads $T$ to form type-I structure.

For the second choice, $S_0$ lives on a fixed Schmidt hyperdisk $S$, so we assume $|\Psi\rangle \notin S$ in following context. Here we set $|\Psi\rangle$ to live on $S_1 \subset T$, where $S_1$ is a 2-dimensioni sub-hyperdisk of another Schmidt hyperdisk $S'$:

$$
|\Psi'(\theta)\rangle = \sqrt{\lambda_1} \left( |\phi_{01}^+\rangle + e^{i\theta_1} |\phi_{01}^+\rangle \right) + e^{i\theta_2} \sqrt{\lambda_2} |22\rangle, \tag{A2}
$$

where $\{|\phi_{01}^+\rangle, |\phi_{01}^-\rangle\}$ and $\{|\psi_{01}^+\rangle, |\psi_{01}^-\rangle\}$ are an orthonormal basis for subspace span$\{|00\rangle, |11\rangle\}$ and defined as:

$$
|\phi_{01}^+\rangle := \cos \frac{\theta_0}{2} |0\rangle + \sin \frac{\theta_0}{2} e^{i\varphi_0} |1\rangle, \quad |\phi_{01}^-\rangle := \sin \frac{\theta_0}{2} e^{-i\varphi_0} |0\rangle - \cos \frac{\theta_0}{2} |1\rangle, \tag{A3}
$$

$$
|\psi_{01}^+\rangle := \cos \frac{\theta_1}{2} |0\rangle + \sin \frac{\theta_1}{2} e^{i\varphi_1} |1\rangle, \quad |\psi_{01}^-\rangle := \sin \frac{\theta_1}{2} e^{-i\varphi_1} |0\rangle - \cos \frac{\theta_1}{2} |1\rangle.
$$

There are two possible settings of $S_1$:

$$
|\Psi_1(\alpha)\rangle = \sqrt{\lambda_1} \left( |\phi_{01}^+\psi_{01}^-\rangle + e^{i\eta} |\phi_{01}^+\psi_{01}^-\rangle \right) + \sqrt{\lambda_2} e^{i\beta} |22\rangle, \quad |\Psi_1(\alpha)\rangle = \sqrt{\lambda_1} \left( |\phi_{01}^+\psi_{01}^-\rangle + e^{i\beta} |\phi_{01}^+\psi_{01}^-\rangle \right) + \sqrt{\lambda_2} e^{i\eta} |22\rangle. \tag{A4}
$$
though the first setting breaks the condition \(\dim(V_T) = 3\). In the second setting, we define

\[
\begin{align*}
|\Phi_0\rangle & := |\phi_{01}^+ \psi_{01}^-\rangle, & |\Phi_1\rangle & := \sqrt{\lambda_1} |\phi_{01}^- \psi_{01}^+\rangle + \sqrt{\lambda_2} e^{i\eta} |22\rangle, \\
|\Phi'_0\rangle & := |\Phi_0\rangle - \langle 00|\Phi_0\rangle |00\rangle, & |\Phi'_1\rangle & := |\Phi_1\rangle - \langle 00|\Phi_1\rangle |00\rangle,
\end{align*}
\]

(A5)

thus \(|\Phi'_0\rangle, |\Phi'_1\rangle\) and \(\sqrt{\lambda_1} |11\rangle + \sqrt{\lambda_2} |22\rangle\) must not be linear independent (otherwise the dimension of their spanning space will be greater than 3), it follows that \(\theta_0 = \theta_1 = \theta\) and \(\eta = 0\). Therefore a clearer form can be derived:

\[
\begin{align*}
\langle \Psi_1 (\beta) \rangle & = e^{i\beta} \sqrt{\lambda_1} \left( \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi_0} |1\rangle \right) \left( \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi_1} |1\rangle \right) \\
& + \sqrt{\lambda_1} \left( \sin \frac{\theta}{2} e^{-i\varphi_0} |0\rangle - \cos \frac{\theta}{2} |1\rangle \right) \left( \sin \frac{\theta}{2} e^{-i\varphi_1} |0\rangle - \cos \frac{\theta}{2} |1\rangle \right) + \sqrt{\lambda_2} |22\rangle
\end{align*}
\]

(A6)

Then we have an complete orthogonal basis \(\{ |00\rangle, \sqrt{\lambda_1} |11\rangle + \sqrt{\lambda_2} |22\rangle, |\Phi_\perp\rangle \}\) for \(V_T\), where \(|\Phi_\perp\rangle\) is written as

\[
|\Phi_\perp\rangle = \sin \theta \sqrt{\lambda_1} \left( e^{-i\varphi_0} |01\rangle + e^{-i\varphi_1} |10\rangle \right) + \frac{\lambda_1 \lambda_2 (1 - \cos \theta)}{\lambda_1 + \lambda_2} \left( \frac{1}{\sqrt{\lambda_1}} |11\rangle - \frac{1}{\sqrt{\lambda_2}} |22\rangle \right),
\]

(A7)

which means that \(S_1\) is unique for certain tuple \((\theta, \varphi_0, \varphi_1)\). It follows that \(T = S_0 \cup S_1\) forms a type-II structure. Moreover, this description of \(V_T\) covers all of subspaces over \(V_L\) (since \((1 - \cos \theta)/\sin \theta \in \mathbb{R}^+\) for \(\theta \in (0, \pi)\)), so the set of partially degenerate target state must be either type-I or type-II (under the constraint on Schmidt hyperdisk).

Appendix B: Proof for structure of completely degenerate masker when \(n = 3, d = 3\)

Since completely degenerate masker contains all possible case for partially degenerate masker, we mainly discuss type-III structure in this section. In the completely degenerate case, a 2-dimensional sub-hyperdisk \(S_0\) of Schmidt hyperdisk always takes the form as below,

\[
|\Psi_0 (\alpha)\rangle = |00\rangle + e^{i\alpha} (|11\rangle + |22\rangle).
\]

(B1)

Hence we set the \(V_T\) as span\(\{ |00\rangle, |11\rangle + |22\rangle, |\Psi\rangle \}\), where \(|\Psi\rangle\) is an arbitrary maximally entangled state

\[
|\Psi\rangle := \cos \frac{\theta}{2} |00\rangle + \sin \frac{\theta}{2} \left( e^{i\varphi_0} |\phi^+_{12}\rangle |0\rangle + e^{i\varphi_1} |0\rangle |\psi^+_{12}\rangle \right) \\
+ e^{i(\varphi_0 + \varphi_1)} \left( e^{i\eta} |\phi^-_{12}\rangle |\psi^-_{12}\rangle \right)
\]

(B2)

where

\[
|\phi^+_{12}\rangle := \cos \frac{\nu_0}{2} |1\rangle + \sin \frac{\nu_0}{2} e^{i\omega_0} |2\rangle, & |\phi^-_{12}\rangle := \sin \frac{\nu_0}{2} |1\rangle - \cos \frac{\nu_0}{2} e^{i\omega_0} |2\rangle, \\
|\psi^+_{12}\rangle := \cos \frac{\nu_1}{2} |1\rangle + \sin \frac{\nu_1}{2} e^{i\omega_1} |2\rangle, & |\psi^-_{12}\rangle := \sin \frac{\nu_1}{2} |1\rangle - \cos \frac{\nu_1}{2} e^{i\omega_1} |2\rangle.
\]

(B3)

To prove that there is only one single state in \(T \setminus S_0\) (as the setting of type-III structure), first we have to define

\[
|\Phi_\perp\rangle = |\Psi\rangle - \langle 00|\Psi\rangle |00\rangle - \frac{1}{2} (\langle 11\rangle + \langle 22\rangle) |\Psi\rangle (|11\rangle + |22\rangle),
\]

(B4)

then show that \(|\Psi\rangle\) and \(|\Phi_\perp\rangle\) have one-to-one correspondence. By using the property that the structure of \(T\) is invariant under local unitary, we construct \(U_{12}\) as

\[
U_{12} = |0\rangle \langle 0| + |1\rangle \langle \phi^+_{12} |2\rangle \langle \phi^-_{12}|,
\]

(B5)

apply \(U_{12} \otimes U^*_{12}\) on \(|\Psi\rangle\) (notice that local unitary does not have effect on \(S_0\)), and get a simplified form of the maximally entangled state \(|\Psi\rangle\)

\[
|\Psi'\rangle = \cos \frac{\theta}{2} |00\rangle + \sin \frac{\theta}{2} \left( e^{i\varphi_0} |10\rangle + e^{i\varphi_1} |01\rangle |\psi^+_{12}\rangle \right) \\
+ e^{i(\varphi_0 + \varphi_1)} \left( e^{i\eta} |2\rangle |\psi^-_{12}\rangle \right)
\]

(B6)
where
\[ |\psi_{12}^+\rangle := \cos \frac{\nu}{2} |1\rangle + \sin \frac{\nu}{2} e^{i\omega} |2\rangle, \quad |\psi_{12}^-\rangle := \sin \frac{\nu}{2} |1\rangle - \cos \frac{\nu}{2} e^{i\omega} |2\rangle. \quad (B7) \]

The set of $|\Psi'\rangle$ can be divide into two parts by $\nu$:

(1) For $\nu \in (0, \pi]$, $|\Psi'\rangle$ is determined by the tuple $(\theta, \nu, \varphi_0, \varphi_1, \omega, \eta)$:
\[
|\Psi'\rangle = \cos \frac{\theta}{2} |00\rangle + \sin \frac{\theta}{2} \left( e^{i\varphi_0} |10\rangle + e^{i\varphi_1} \cos \frac{\nu}{2} |01\rangle + e^{i(\varphi_1+\omega)} \sin \frac{\nu}{2} |02\rangle \right) \\
+ e^{i(\varphi_0+\varphi_1)} \left[ e^{i\eta} \left( \sin \frac{\nu}{2} |21\rangle - e^{i\omega} \cos \frac{\nu}{2} |22\rangle \right) - \cos \frac{\theta}{2} \left( \cos \frac{\nu}{2} |11\rangle + e^{i\omega} \sin \frac{\nu}{2} |12\rangle \right) \right] , \quad (B8) \]

which leads to the unique $|\Phi'_\perp\rangle$:
\[
|\Phi'_\perp\rangle = \sin \frac{\theta}{2} \left( |10\rangle + e^{i(\varphi_1-\varphi_0)} \cos \frac{\nu}{2} |01\rangle + e^{i(\varphi_1-\varphi_0+\omega)} \sin \frac{\nu}{2} |02\rangle \right) \\
+ e^{i\varphi_1} \sin \frac{\nu}{2} \left( e^{i\eta} |21\rangle - e^{i\omega} \cos \frac{\theta}{2} |12\rangle \right) - \frac{1}{2} e^{i\varphi_1} \cos \frac{\nu}{2} \left( \cos \frac{\theta}{2} - e^{i(\eta+\omega)} \right) (|11\rangle - |22\rangle) , \quad (B9) \]

Here we show that type-III structure can be achieved in the completely degenerate case.

(2) For $\nu = 0$, $|\Psi'\rangle$ is determined by the tuple $(\theta, \varphi_0, \varphi_1, \eta)$:
\[
|\Psi'\rangle = \cos \frac{\theta}{2} |00\rangle + \sin \frac{\theta}{2} \left( e^{i\varphi_0} |10\rangle + e^{i\varphi_1} |01\rangle \right) - e^{i(\varphi_0+\varphi_1)} \left( \cos \frac{\theta}{2} |11\rangle + e^{i\eta} |22\rangle \right) . \quad (B10) \]

But $|\Psi'\rangle$ does not lead to the unique $|\Phi'_\perp\rangle$. In fact, this situation is just a special case of the type-II structure shown in the last section. Finally, we can conclude that there is only 3 types of $T$ under the constraint on Schmidt hyperdisk.