Bright solitons in ultracold atoms

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Abstract We review old and recent experimental and theoretical results on bright solitons in Bose-Einstein condensates made of alkali-metal atoms and under external optical confinement. First we deduce the three-dimensional Gross-Pitaevskii equation (3D GPE) from the Dirac-Frenkel action of interacting identical bosons within a time-dependent Hartree approximation. Then we discuss the dimensional reduction of the GPE from 3D to 1D, deriving the 1D GPE and also the 1D nonpolynomial Schrödinger equation (1D NPSE). Finally, we analyze the bright soliton solutions of both 1D GPE and 1D NPSE and compare these theoretical predictions with the available experimental data.

Keywords Bright solitons · Ultracold atoms · Gross-Pitaevskii equation · Nonpolynomial Schrödinger equation

1 Introduction

In 1995 three experimental groups achieved Bose-Einstein condensation (BEC), i.e. the macroscopic occupation of a single-particle quantum state, cooling very dilute gases of $^{87}$Rb [1], $^7$Li [2], and $^{23}$Na [3] atoms. For these systems the BEC critical temperature is about $T_c \approx 100$ nanoKelvin and the gas made of alkali-metal atoms is in a meta-stable state which can survive for minutes. Another ground-breaking result with ultracold atoms was achieved some years later: a stationary optical lattice which traps ultracold atoms was obtained with counter-propagating laser beams inside an optical cavity [4]. The resulting potential confines neutral atoms in the minima of the lattice due to the electric...
dipole of atoms [5]. Nowadays the study of neutral atoms trapped with light is
a very hot topic of research because, changing the intensity and shape of the
optical lattice, it is possible to confine atoms in very different configurations.
One can have many atoms per site but also one atom per site [6].

The main theoretical tool for the study a pure BEC in ultracold and
dilute alkali-metal atoms is the Gross-Pitaevskii equation [7], that is a nonlinear
Schrödinger equation with cubic nonlinearity. In 1972 Shabat and Zakharov [8]
found that the 1D cubic nonlinear Schrödinger equation admits solitonic (i.e.
shape invariant) analytical solutions. If the 1D nonlinear strength is repulsive
(self-defocusing nonlinearity) one finds the localized dark solitons, while if the
nonlinear strength is attractive (self-focusing nonlinearity) one finds bright
solitons. Quite remarkably, both dark and bright solitons have been observed
experimentally with atomic BECs (see [9] for a comprehensive review). In this
paper we concentrate on bright solitons, discussing theoretical and experimental
results of this exciting field of research.

2 Gross-Pitaevskii equation

Static and dynamical properties of a pure BEC made of dilute and ultracold
atoms are very well described by the Gross-Pitaevskii equation [7]

\[ i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) + \left( N - 1 \right) \frac{4\pi \hbar^2 a_s}{m} |\psi(\mathbf{r}, t)|^2 \right] \psi(\mathbf{r}, t) , \]

where \( U(\mathbf{r}) \) is the external trapping potential, \( m \) is the mass of each atom,
and \( a_s \) is the s-wave scattering length of the inter-atomic potential. In this
equation \( \psi(\mathbf{r}, t) \) is the wavefunction of the BEC normalized to one, i.e.

\[ \int |\psi(\mathbf{r}, t)|^2 \, d^3 \mathbf{r} = 1 , \]

and such that \( \rho(\mathbf{r}) = N|\psi(\mathbf{r}, t)|^2 \) is the local number density of the \( N \)
condensed atoms.

The Gross-Pitaevskii equation (GPE) can be deduced from the many-body
quantum Hamiltonian of \( N \) identical spinless particles

\[ \hat{H} = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \nabla_i^2 + U(\mathbf{r}_i) \right) + \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} V(\mathbf{r}_i - \mathbf{r}_j) , \]

where \( V(\mathbf{r} - \mathbf{r}') \) is the inter-atomic potential. The time-dependent Schrödinger
equation of this many-body system is given by

\[ i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = \hat{H} \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) , \]
where \( \Psi(\mathbf{r}_1, ..., \mathbf{r}_N, t) \) is the time-dependent many-body wavefunction. This time-dependent many-body Schrödinger equation is the Euler-Lagrange equation of the following many-body Dirac-Frenkel [10] action functional

\[
S = \int dt \, d^3r_1 \cdots d^3r_N \, \Psi^* (\mathbf{r}_1, ..., \mathbf{r}_N, t) \left( i\hbar \frac{\partial}{\partial t} - \hat{H} \right) \Psi (\mathbf{r}_1, ..., \mathbf{r}_N, t). \tag{5}
\]

In the case of a pure Bose-Einstein condensate one assumes all bosons in the same time-dependent single-particle orbital (i.e. a time-dependent version of the Hartree approximation [11])

\[
\Psi(\mathbf{r}_1, ..., \mathbf{r}_N, t) = \prod_{i=1}^{N} \psi (\mathbf{r}_i, t). \tag{6}
\]

Inserting this ansatz into the many-body action functional one gets

\[
S = N \int dt \, d^3r \, \psi^* (\mathbf{r}, t) \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U (\mathbf{r}) \right. \\
- \left. \frac{N-1}{2} \int d^3r' \, |\psi (\mathbf{r}', t)|^2 V (\mathbf{r} - \mathbf{r}') \right) \psi (\mathbf{r}, t). \tag{7}
\]

The Euler-Lagrange equation of the previous action functional reads

\[
\frac{i\hbar}{\partial t} \psi (\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U (\mathbf{r}) + (N-1) \int d^3r' \, |\psi (\mathbf{r}', t)|^2 V (\mathbf{r} - \mathbf{r}') \right] \psi (\mathbf{r}, t). \tag{8}
\]

This is the time-dependent Hartree equation for \( N \) identical bosons in the same single-particle state \( \psi (\mathbf{r}, t) \).

In the case of dilute gases one usually assumes (Fermi pseudopotential [12]) that

\[
V (\mathbf{r}) \simeq g \, \delta^{(3)} (\mathbf{r}) \tag{9}
\]

with \( \delta^{(3)} (\mathbf{r}) \) the Dirac delta function and, by construction,

\[
g = \int V (\mathbf{r}) \, d^3r. \tag{10}
\]

From 3D scattering theory, the s-wave scattering length \( a_s \) of the inter-atomic potential can be written (Born approximation [13]) as

\[
a_s = \frac{m}{4\pi \hbar^2} \int V (\mathbf{r}) \, d^3r. \tag{11}
\]

In this way, from Eq. (8) one obtains the time-dependent 3D GPE, Eq. (1).
3 Dimensional reduction: from 3D to 1D

From the Hartree equation (8) we have obtained the time-dependent 3D GPE. Clearly, this is the Euler-Lagrange equation of the GP action functional

\[ S = N \int dt \int d^3r \, \psi^* \left( i \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(r) - \frac{N-1}{2} g|\psi(r,t)|^2 \right) \psi(r,t). \]  

(12)

Let us now consider a very strong harmonic confinement of frequency \( \omega_\perp \) along \( x \) and \( y \) and a generic confinement \( U(z) \) along \( z \), namely

\[ U(r) = \frac{1}{2} \frac{m \omega_\perp^2}{\hbar^2} (x^2 + y^2) + U(z). \]  

(13)

On the basis of the chosen external confinement, we adopt the ansatz

\[ \psi(r,t) = f(z,t) \frac{1}{\pi^{1/2} a_\perp} \exp \left( \frac{x^2 + y^2}{2a_\perp^2} \right), \]  

(14)

where \( f(z,t) \) is the axial wave function and \( a_\perp = \sqrt{\frac{\hbar}{m \omega_\perp}} \) is the characteristic length of the transverse harmonic confinement. By inserting Eq. (14) into the GP action (12) and integrating along \( x \) and \( y \), the resulting effective action functional depends only on the field \( f(z,t) \).

One easily finds that the Euler-Lagrange equation of the axial wavefunction \( f(z,t) \) reads

\[ i \hbar \frac{\partial}{\partial t} f(z,t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + U(z) + \gamma |f(z,t)|^2 \right] f(z,t), \]  

(15)

where

\[ \gamma = \frac{(N-1)g}{2 \pi a_\perp^2} \]  

(16)

is the effective one-dimensional interaction strength and the additive constant \( \hbar \omega_\perp \) has been omitted because it does not affect the dynamics.

3.1 Bright solitons of 1D GPE

In the absence of axial confinement, i.e. \( U(z) = 0 \), the 1D GPE becomes

\[ i \hbar \frac{\partial}{\partial t} f(z,t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \gamma |f(z,t)|^2 \right] f(z,t). \]  

(17)

This is a 1D nonlinear Schrödinger equation with cubic nonlinearity. In 1972 Shabat and Zakharov [8] found that this equation admits solitonic solutions, such that

\[ f(z,t) = \phi(z-vt) \, e^{i(m\nu z - m^2 t/2 - \mu t)/\hbar}, \]  

(18)
where \( v \) is the arbitrary velocity of propagation of the solution, which has a shape-invariant axial density profile:
\[
\rho(z,t) = N|f(z,t)|^2 = N|\phi(z - vt)|^2 .
\]
Setting \( \zeta = z - vt \), the 1D stationary GP equation
\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{d\zeta^2} + \gamma|\phi(\zeta)|^2 \right] \phi(\zeta) = \mu \phi(\zeta) ,
\]
with \( \gamma < 0 \) (self-focusing), admits the bright-soliton solution
\[
\phi(\zeta) = \sqrt{m|\gamma|} \frac{2}{\hbar \sqrt{4\pi}} \text{Sech} \left[ \frac{m|\gamma|}{4\hbar^2} \zeta \right] ,
\]
with \( \text{Sech}[x] = \frac{2}{e^x + e^{-x}} \) and
\[
\mu = -\frac{m \gamma^2}{16 \hbar^2} .
\]
In Fig. 1 we plot the \( |\phi(\zeta)|^2 \) of the bright soliton for three values of the nonlinear strength \( \gamma \).

Shabat and Zakharov [8] used the inverse scattering method [14] to find the explicit expression of \( \phi(\zeta) \). Here we use a much simpler (but less general) method. Let us assume that \( \phi(\zeta) \) is real. Then the 1D stationary Gross-Pitaevskii equation can be rewritten as
\[
\phi''(\zeta) = -\frac{\partial W(\phi)}{\partial \phi} ,
\]
where
\[
W(\phi) = \frac{1}{2} \frac{m|\gamma|}{\hbar^2} \phi^4 + \frac{m\mu}{\hbar^2} \phi^2 .
\]
Thus, $\phi(\zeta)$ can be seen as the “coordinate” for a fictitious particle at “time” $\zeta$. The constant of motion of the problem reads

$$K = \frac{1}{2} \phi'(\zeta)^2 + W(\phi),$$

from which one finds

$$\frac{d\phi}{d\zeta} = \sqrt{2(K - W(\phi))}.$$  \hspace{1cm} (26)

Imposing that $\phi(\zeta) \to 0$ as $|\zeta| \to \infty$ one gets $K = 0$ and consequently

$$\frac{d\phi}{\sqrt{-2W(\phi)}} = d\zeta, \hspace{1cm} (27)$$

or explicitly

$$\frac{d\phi}{\sqrt{-\frac{m|\gamma|^2}{\hbar^2}\phi^4 + \frac{2|m|\mu}{\hbar^2}\phi^2}} = d\zeta, \hspace{1cm} (28)$$

with $\mu < 0$. Inserting the integrals one obtains

$$\int_{\phi(0)}^{\phi(\zeta)} \frac{d\phi}{\sqrt{-\frac{m|\gamma|^2}{\hbar^2}\phi^4 + \frac{2|m|\mu}{\hbar^2}\phi^2}} = \zeta. \hspace{1cm} (29)$$

Setting $\phi'(0) = 0$, from the definition of $K$ and using $K = 0$ one finds $W(\phi(0)) = 0$ and therefore

$$\phi(0) = \sqrt{\frac{2|m|}{|\gamma|}}. \hspace{1cm} (30)$$

After integration of Eq. (29) one gets

$$\frac{1}{\sqrt{|m|\mu}} \text{ArcSech} \left[ \sqrt{\frac{|\gamma|}{2|m|}} \phi(\zeta) \right] = \zeta \hspace{1cm} (31)$$

from which

$$\phi(\zeta) = \sqrt{\frac{2|m|}{|\gamma|}} \text{Sech} \left[ \sqrt{\frac{|m|\mu}{\hbar^2}} \zeta \right]. \hspace{1cm} (32)$$

Finally, imposing the normalization condition

$$\int dz \; \phi(\zeta)^2 = 1, \hspace{1cm} (33)$$

one obtains

$$\mu = -\frac{m \gamma^2}{16 \hbar^2}. \hspace{1cm} (34)$$
4 Improved dimensional reduction: the 1D NPSE

The bright soliton analytical solution has been obtained from the 1D GPE, which is derived from the 3D GPE assuming a transverse Gaussian with a constant transverse width $a_\perp$. A more general assumption $[15,16,17]$ is based on a space-time dependent transverse width

$$\psi(r,t) = f(z,t) \exp\left(\frac{x^2 + y^2}{2a_\perp^2 \eta(z,t)^2}\right),$$

(35)

where $f(z,t)$ is the axial wave function and $\eta(z,t)$ is the adimensional transverse width in units of $a_\perp$. From this ansatz one gets the 1D nonpolynomial Schrödinger equation (1D NPSE)

$$i\hbar \frac{\partial}{\partial t} f = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + U(z) + \frac{\gamma |f|^2}{\eta^2} + \frac{\hbar \omega_\perp}{2} \left(\frac{1}{\eta^2} + \eta^2\right)\right] f,$$

(36)

$$\eta = (1 + \gamma |f|^2)^{1/4}.$$  

(37)

In the weak-coupling regime $\gamma |f|^2 \ll 1$ one finds $\eta \simeq 1$ and the 1D NPSE becomes the familiar 1D GPE.

4.1 Bright solitons of 1D NPSE

With $U(z) = 0$ and assuming $\gamma < 0$ the NPSE admits analytical bright soliton solutions. Setting

$$f(z,t) = \phi(z - vt)e^{(mv^2/2 - \mu)/\hbar},$$

(38)

one finds the bright-soliton solution written in implicit form

$$\zeta = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 - \mu}} \arctg\left[\sqrt{\frac{\sqrt{1 - |\gamma| \phi^2} - \mu}{1 - \mu}}\right],$$

(39)

$$-\frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 + \mu}} \arctgh\left[\sqrt{\frac{\sqrt{1 - |\gamma| \phi^2} - \mu}{1 + \mu}}\right],$$

(40)

where $\zeta = z - vt$ and $|\gamma| = 2|a_s|(N - 1)/a_\perp$.

Fig. 2 reports the axial probability density $\rho(\zeta) = |\phi(\zeta)|^2$ of the bright soliton obtained by using the 3D GPE (full line), the 1D NPSE (dotted line), and the 1D GPE (dashed line). In the weak-coupling limit ($\gamma \phi^2 \ll 1$) one finds that the NPSE bright-soliton solution reduces the the 1D GPE one. This is clearly shown in the figure. However, contrary to the 1D GPE bright soliton, the 1D NPSE bright soliton does not exist anymore, collapsing to a Dirac delta function, at

$$\gamma_c = \left(\frac{2a_s(N - 1)}{a_\perp}\right)_c = -\frac{4}{3}.$$
This analytical result is in extremely good agreement with the numerical solution of the 3D GPE [16]. Indeed, Fig. 2 shows that up to the collapse the density profile obtained with the NPSE is very close to the 3D GPE one.

5 Bright solitons in experiments with ultracold atoms

In 2002 there were two relevant experiments [19,20] about bright solitons with BECs made of $^7$Li atoms. Both experiments used the technique of Fano-Feshbach resonance [18] to tune the s-wave scattering length $a_s$ of the interatomic potential by means of an external constant magnetic field.

Khaykovich et al. [19] reported the production of bright solitons in an ultracold $^7$Li gas. The interaction was tuned with a Feshbach resonance from repulsive to attractive before release in a one-dimensional optical waveguide, which is attractive in the transverse direction but expulsive in the longitudinal direction. Propagation of the soliton without dispersion over a macroscopic distance of 1.1 millimeter was observed in the case of attractive interaction.
In their experiment, Khaykovich et al. [19] measured the root mean square size $\sigma$ of the longitudinal width versus the propagation time for three values of $a_s$: $a_s = 0$, $a_s = -0.11$ nm, and $a_s = -0.21$ nm. Fig. 3 shows the experimental data [19] and our numerical results obtained with the time-dependent 1D NPSE [21]. The agreement between experiment and theory is quite good. The numerical results are obtained by using a finite-difference Crank-Nicolson scheme with predictor-corrector [22]. Very recently, various open access and high performance codes have been developed for the solution of the time-dependent GPE [23].

Strecker et al. [20] reported the formation of a train of bright solitons of $^7$Li atoms in a quasi-one-dimensional optical trap by a sudden change in the sign of the scattering length from positive to negative. The solitons were set in motion by offsetting the optical potential, and were observed to propagate in the longitudinal harmonic potential for many oscillatory cycles without spreading.

We successfully simulate the soliton train formation of Ref. [20] by using the time-dependent 3D GPE [24]. In Fig. 4 we plot the probability density in the longitudinal direction $\rho(z)$ of the BEC made of $10^4$ $^7$Li atoms. Initially there is a stable condensate of $^7$Li atoms with a large positive scattering length.
Fig. 4 Axial density profile $\rho(z)$ of the BEC made of $10^4 \ ^7\text{Li}$ atoms obtained by solving the 3D GPE. For $t < 0$ the scattering length is $a_s = 100a_B$, while for $t \geq 0$ we set $a_s = -3a_B$ with $a_B$ the Bohr radius. Length $z$ is in units of the characteristic length $a_z = \sqrt{\hbar/(m\omega_z)}$ of the weak axial harmonic confinement of frequency $\omega_z$. Time $t$ in units of $1/\omega_z$. Density $\rho$ in units of $1/a_z$. Adapted from [24].

$a_s$ but at time $t = 0$ the scattering length $a_S$ is switched to a negative value. A number $N_S$ of bright solitons is produced and this can be interpreted in terms of the modulational instability of the time-dependent macroscopic wave function of the Bose condensate [25]. An estimate of the number $N_s$ of bright solitons which are generated is

$$N_s = \frac{\sqrt{N|a_s|L}}{\pi a_\bot},$$

(41)

where $a_s$ is the final negative scattering length, $N$ is the total number of atoms, $a_\bot$ is the characteristic length of transverse harmonic confinement and $L$ is the initial longitudinal length of the quasi-1D BEC [24]. This formula, based on the analysis of the imaginary Bogoliubov spectrum of elementary excitations (see [24] for details), is in good agreement with both experimental results and numerical simulations.

Very recently, the formation of matter-wave soliton trains by modulational instability was experimentally reexamined by Nguyen et al. [26]. They used a nearly nondestructive imaging technique to follow the dynamics of these trains finding that the modulation instability is driven by noise and neighboring solitons interact repulsively during the initial formation of the soliton train.
These findings are indeed in full agreement with our theoretical predictions based on the numerical simulation of 3D GPE and 1D NPSE \[24\].

6 Conclusions

In this paper we have explicitly derived the analytical solution of the 1D bright soliton from the one-dimensional Gross-Pitaevskii equation (1D GPE), which, in turn, is obtained from the 3D GPE assuming a transverse Gaussian with a constant width \(a_{\perp}\). We have then shown that a more general assumption, with a space-time dependent transverse width, gives rise to the 1D nonpolynomial Schrödinger equation (1D NPSE). 1D NPSE admits bright solitons which collapse at a critical interaction strength, in good agreement with the findings of full 3D GPE. Both 3D GPE and 1D NPSE are reliable tools to reproduce the available experimental data \[19,20,26\] of BEC bright solitons made of alkali-metal atoms. The experimental study of bright solitons in ultracold atoms is still a hot topic, as is evident considering the very recent experiments on the train of bright solitons \[26\] and on attractive two-component bosonic mixtures \[27\]. Moreover, in the last few years, it has been suggested that atomic bright solitons can be produced with attractive two-component mixtures \[28\] \[29\], with space-dependent scattering lengths \[30\], and also with artificial spin-orbit and Rabi couplings \[31,32,33\]. These remarkable theoretical predictions need experimental confirmation.

References

1. M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, Science 269, 198201 (1995).
2. C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions, Phys. Rev. Lett. 75, 16871690 (1995).
3. K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Bose-Einstein condensation in a gas of sodium atoms, Phys. Rev. Lett. 75, 3969397 (1995).
4. M. Greiner, O. Mandel, T. Esslinger, T.W. Hansch, and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms, Nature 415, 39-44 (2002).
5. I. Bloch, Ultracold quantum gases in optical lattices, Nature 1, 23-30 (2005).
6. O. Morsch and M Oberthaler, Dynamics of Bose-Einstein condensates in optical lattices, Rev. Mod. Phys. 78, 179 (2006).
7. E.P. Gross, Structure of a quantized vortex in boson systems, Nuovo Cim. 20, 454 (1961); L.P. Pitaevskii, Vortex lines in an imperfect bose gas, Sov. Phys. JETP. 13, 451 (1961).
8. A Shabat and V Zakharov, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP. 34, 62 (1972).
9. P.G. Kevrekidis, D.J. Frantzeskakis, and R. Carretero-Gonzalez (Eds), Emergent Nonlinear Phenomena in Bose-Einstein Condensates. Theory and experiments (Springer, Berlin, 2007).
10. P.A.M. Dirac, Note on exchange phenomena in the Thomas atom, Math. Proc. Cambridge Phil. Soc. 26, 376385 (1930); Y.I. Frenkel, Wave mechanics. Advanced General Theory (Clarendon Press, Oxford, 1934).
11. D.R. Hartree, The Wave Mechanics of an Atom with a Non-Coulomb Central Field. Part I. Theory and Methods, Math. Proc. Cambridge Phil. Soc. 24, 89 (1928).
12. E. Fermi, Motion of neutrons in hydrogenous substances, Ricerca Scientifica 7, 1352 (1926).
13. M. Born, Quantenmechanik der Stossvorgänge, Zeit. fur Physik 38, 803 (1926).
14. M. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform (SIAM, Philadelphia, 1981).
15. L. Salasnich, A. Parola, and L. Reatto, Effective wave equations for the dynamics of cigar-shaped and disk-shaped Bose condensates, Phys. Rev. A 65, 043614 (2002).
16. L. Salasnich, A. Parola, and L. Reatto, Condensate bright solitons under transverse confinement, Phys. Rev. A 66, 043603 (2002).
17. S.K. Adhikari and L. Salasnich, Effective nonlinear Schrödinger equations for cigar-shaped and disc-shaped Fermi superfluids at unitarity, New J. Phys. 11, 023011 (2009).
18. U. Fano, On the absorption spectrum of a noble gas near the limit of the discrete spectrum, Nuovo Cim. 156, 12 (1935); H. Feshbach, Unified theory of nuclear reactions, Ann. Phys. 5, 357 (1958); U. Fano, Effects of Configuration Interaction on Intensities and Phase Shifts, Phys. Rev. 124, 1866 (1961).
19. L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L.D. Carr, Y. Castin, C. Salomon, Formation of a Matter-Wave Bright Soliton, Science 296, 1290-1293 (2002).
20. K.E. Strecker, G.B. Partridge, A.G. Truscott, and R.G. Hulet, Formation and propagation of matter-wave soliton trains, Nature 417, 150 (2002).
21. L. Salasnich, Dynamics of a BEC bright soliton in an expulsive potential, Phys. Rev. A 70, 053617 (2004).
22. E. Cerboneschi, R. Mannella, E. Arimondo, and L. Salasnich, Oscillation Frequencies for a Bose Condensate in a Triaxial Magnetic Trap, Phys. Lett. A 249, 495 (1998).
23. L.E. Young-S., P. Muruganandam, S.K. Adhikari, V. Loncar, D. Vudragovic, and Antun Balaz, OpenMP GNU and Intel Fortran programs for solving the time-dependent Gross-Pitaevskii equation, Comput. Phys. Commun. 220, 503 (2017).
24. L. Salasnich, A. Parola, and L. Reatto, Modulational Instability and Complex Dynamics of Confined Matter-Wave Solitons, Phys. Rev. Lett. 91, 080405 (2003).
25. L.D. Carr, C.W. Clark, and W.P. Reinhardt, Stationary solutions of the one-dimensional nonlinear Schrödinger equation. II. Case of attractive nonlinearity, Phys. Rev. A 62, 063611 (2000).
26. J.H.V. Nguyen, D. Luo, R.G. Hulet, Formation of matter-wave soliton trains by modulational instability, Science 356, 422 (2017).
27. P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell, Bright soliton to quantum droplet transition in a mixture of Bose-Einstein condensates, e-print arXiv:1710.11079, G. Semeghini, G. Persoli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Self-bound quantum droplets in atomic mixtures, e-print arXiv:1710.10890.
28. L. Salasnich and B.A. Malomed, Vector solitons in nearly one-dimensional Bose-Einstein condensates, Phys. Rev. A 77, 053610 (2008); S.K. Adhikari, B.A. Malomed, L. Salasnich, and F. Toigo, Spontaneous symmetry breaking of Bose-Fermi mixtures in double-well potentials, Phys. Rev. A 81, 053630 (2010).
29. D.S. Petrov, Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture, Phys. Rev. Lett. 115, 155302 (2015).
30. Y.V. Kartashov, B.A. Malomed, and L. Torner, Solitons in nonlinear lattices, Rev. Mod. Phys. 83, 247 (2011).
31. V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, and D. E. Pelinovsky, Matter-wave bright solitons in spin-orbit coupled Bose-Einstein condensates, Phys. Rev. Lett. 110, 264101 (2013).
32. L. Salasnich and B.A. Malomed, Localized modes in dense repulsive and attractive Bose-Einstein condensates with spin-orbit and Rabi couplings, Phys. Rev. A 87, 063625 (2013); A. Cappellaro, T. Macrì, G.F. Bertacca, and L. Salasnich, Equation of state and self-bound droplet in Rabi-coupled Bose mixtures, Sci. Rep. 7, 13358 (2017).
33. H. Sakaguchi, B. Li, and B.A. Malomed, Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space, Phys. Rev. E 89, 032920 (2014).