Fulde–Ferrell State in Spin–Orbit-Coupled Superconductor: Application to Dresselhaus SOC

F. Yang · M. W. Wu

Abstract We show that in the presence of magnetic field, two superconducting phases with the center-of-mass momentum of Cooper pair parallel to the magnetic field are induced in Dresselhaus spin–orbit-coupled superconductor. Specifically, at small magnetic field, the center-of-mass momentum is induced due to the energy-spectrum distortion and no unpairing region with vanishing singlet correlation appears. We refer to this superconducting state as the drift-BCS state. By further increasing the magnetic field, the superconducting state falls into the Fulde–Ferrell state with the emergence of the unpairing regions. The observed abrupt enhancement of the center-of-mass momenta and suppression on the order parameters during the transition indicate the occurrence of a first-order phase transition. Enhanced Pauli limit and hence enlarged magnetic-field regime of the Fulde–Ferrell state, due to the spin-flip terms of the spin–orbit coupling, are revealed. We also address the triplet correlations induced by the spin–orbit coupling, and show that the Cooper-pair spin polarizations, generated by the magnetic field and center-of-mass momentum with the triplet correlations exhibit totally different magnetic-field dependences between the drift-BCS and Fulde–Ferrell states.

Keywords FFLO · Spin–orbit coupling · Superconductivity

1 Introduction

Ever since the Bardeen, Cooper and Schrieffer (BCS) mechanism of superconductivity was proposed [1], it is well established that the Cooper pair in conventional supercon-
 conductors such as Al, Pb and Nb is formed by two electrons with opposite momenta and spins near the Fermi surface. Together with the conventional $s$-wave attractive potential, a spatially uniform singlet-order parameter is realized. After that, the possibility of the unconventional superconducting state has attracted much attention.

Particularly, an exotic superconducting state characterized by Cooper pairs with a finite center-of-mass (CM) momentum is expected at large magnetic field. This was first predicted by Fulde and Ferrell (FF) [2] and a little later by Larkin and Ovchinnikov (LO) [3,4] independently in 1960s. Specifically, the presence of the magnetic field leads to the mismatched Fermi surfaces of spin-up and spin-down electrons. Consequently, near the corresponding Fermi surfaces, there exist unpairing regions in which the electron cannot find the pairing partner with opposite momentum and spin to form into a Cooper pair. When the magnetic field exceeds a critical strength, by inducing a finite CM momentum, the pairing region between the spin-up and spin-down electrons is maximized, leading to the free-energy minimized. In this case, with the rotational symmetry of the system, FF proposed an order parameter $\Delta(r) = \Delta_0 e^{iQ \cdot r}$ with the inhomogeneously broadened phase but spatially uniform amplitude [2] whereas LO referred to another order parameter $\Delta(r) = \Delta_0 \cos(Q \cdot r)$ which shows the uniform phase but spatially nonuniform amplitude [3,4]. These two types of the order parameters, now both referred to as the FFLO state [5,6], have attracted tremendous theoretical and experimental efforts for decades to verify their existence. However, up till now, the decisive experimental evidence is still in progress.

The experimental difficulty arises from several different aspects. Specifically, from the FFLO theory, the FFLO state occurs in a very narrow magnetic-field regime [2,5,6], leading to the stringent experimental requirement. The unavoidable disorder may also destroy the induced CM momentum of the Cooper pair and hence the FFLO state [7–10]. Moreover, in most superconducting materials, the destruction of superconductivity comes from the orbital effect of the magnetic field [5,6,11,12]. Hence, to realize the FFLO state, the superconductivity must survive beyond the transition point from BCS to FFLO states against the orbital pair breaking effect. Therefore, weak orbital pair breaking effect of the magnetic field is required. For this reason, a great deal of efforts has been devoted to layered superconductors such as Fe-based superconductors [13–20] as well as the two-dimensional organic superconductors [21–33] with magnetic field parallel to the superconducting layers. Additionally, superconducting heavy fermion materials [10,19,34–45] have also attracted much attention due to the drastically enhanced effective electron mass which suppresses the orbital pairing breaking effect.

Moreover, most of these materials above are unconventional superconductors with anisotropic-order parameters or Fermi surfaces, which are favorable for the FFLO state since these anisotropies stabilize the FFLO state through optimization of the CM momentum [10,46]. Hence, the growing superconducting materials with anisotropic-order parameters or Fermi surfaces have stimulated theoretical interest in the FFLO state in anisotropic systems [46–52]. Accordingly, it is natural to consider the possibility of realizing FFLO state in spin–orbit-coupled superconductors, since the interplay between the spin–orbit coupling (SOC) [53–55] and magnetic field leads to marked and anisotropic mismatch of the Fermi surfaces between spin-up and spin-down elec-
trons. During the last several years, there indeed have been several theoretical studies [56–71] in superconducting systems with SOC including noncentrosymmetric heavy fermion material [57,58,60], spin–orbit-coupled ultracold atomic gases [61–71] and two-dimensional surface superconductors [56,59] predicting the FFLO state emergence, and experiments in these systems [43,45,72–75] also indirectly indicate the existence of the FFLO state. Specifically, with both SOC and magnetic field, early studies of the FFLO state were based on a Ginzburg–Landau analysis by only keeping the intraband pairing in the helical space [57,59]. Meanwhile, the FFLO state with SOC was also discussed by using the linearized Gor'kov equations with only intraband pairing included [56,58,60]. These studies are restricted to large SOC and low magnetic field. Until recently, studies of the FFLO state with the entire regime of SOC and magnetic field are reported [61–71]. From the theoretical studies above, it is revealed that with SOC, the FF state [62,67] and LO-like one [56–60] emerge at small and large SOCs, respectively. Particularly, the induced CM momentum in the FF state is reported to be parallel [62] (perpendicular [56,59,60,68]) to the magnetic field for the Dresselhaus (Rashba) SOC, due to the broken rotational symmetry by the magnetic field and SOC. With the emergence of these states, markedly enhanced Pauli limits by the SOC are revealed in their works, in accord with the experiments in superconducting with SOC [72–75]. In addition, for the small SOC, according to the quasiparticle energy spectra, the induced FF state can be divided into gapped and gapless ones, which occur at small and large magnetic fields, respectively [62–65,68,69].

However, the theoretical works above [59–70] are based on a quasi-classical approach through numerically calculating the thermodynamic-potential extrema with respect to the CM momentum and order parameter, since it is difficult to directly obtain and self-consistently solve the gap equation by analytically diagonalizing Hamiltonian in the presence of the SOC and CM momentum. Therefore, the pairing structure and microscopic properties including the singlet pairing function and correlation are beyond the scope of the quasi-classical approach. Particularly, further study on the unpairing regions with vanishing singlet correlation, which are the hallmark of the FFLO state, is necessary. Moreover, with the numerical difficulty from multi-variable minimum problem, specific behaviors of the superconducting state around a phase transition are unclear in previous works. Furthermore, with the SOC, it is known that the triplet Cooper pairing formed by electrons with the same spin is developed [76–85]. In this case, one may expect the interplay between the triplet pairing and CM momentum in the FF state, which is also absent in the previous literature.

In this work, we systematically investigate the properties of the FF state in the spin–orbit-coupled superconductors with a magnetic field. Specifically, by analytically obtaining the anomalous Green function (pairing), we derive the singlet correlation and hence the gap equation. Then, by self-consistently solving the gap equation, the superconducting state and its corresponding microscopic properties including the CM momentum of Cooper pairs, order parameter, quasiparticle energy spectra and singlet correlation can be determined by numerically calculating the energy minimum with respect to a single parameter, i.e., the CM momentum. Pairing structures and behaviors of the superconducting state around the phase transition are also revealed. Moreover, with the SOC, we discuss the induced triplet correlations [78,81,84,85]. In this case, the Cooper-pair spin polarizations [76,77,86–88], which are predicted in the presence
of the CM momentum and triplet correlation [87], are also investigated. In this work, we focus on superconductors with Dresselhaus SOC.

The calculation shows that at small field, the CM momentum of the Cooper pair parallel to magnetic field is induced. We show that the emerged superconducting state with the induced CM momentum here corresponds to the gapped FF one mentioned above [62–65]. However, we find that no unpairing region with vanishing singlet correlation is developed in this case, which is absent in the previous studies of the FF state with SOC [61–68]. Consequently, the emerged superconducting state here is very different from the conventional FF one without SOC, where the CM momentum is induced simultaneously with the emergence of the unpairing regions [2]. By looking into the pairing structure, it is shown that the induced CM momentum with SOC at small magnetic field is due to the energy-spectrum distortion, and hence has different origin from the case in the conventional FF state [2]. Therefore, it is more appropriate to refer to such a superconducting state, in which the CM momentum is induced but no unpairing region is developed, as the drift-BCS state. By further increasing the magnetic field, abrupt enhancement of the CM momentum and suppression on the order parameters are revealed, meaning the occurrence of the first-order phase transition. Particularly, after the transition, we find that unpairing regions with vanishing singlet correlation are induced, indicating the emergence of the FF state, resembling the conventional FF one [2]. We show that the emerged FF state here corresponds to the gapless one mentioned above [62–65]. Enhanced Pauli limit by SOC is also observed in our work and we further reveal that this enhancement is due to the spin-flip terms of the SOC, which suppress the unpairing regions. Finally, we discuss the induced triplet correlation [78,81,84,85] and Cooper-pair spin polarization. Particularly, it is found that the Cooper-pair spin polarizations [76,77,86,87] exhibit totally different magnetic-field dependences between the drift-BCS and FF states. This provides an experimental scheme to distinguish these phases through the reported magnetoelectric Andreev effect [86–88], in addition to the phase transition.

This paper is organized as follows. In Sec. 2, we introduce our model and present the calculation of the energy for the superconducting state. The specific numerical results and analytic analysis are presented in Sec. 3. We summarize in Sec. 4.

2 Model

In this section, we first present the Hamiltonian of the spin–orbit-coupled s-wave superconductor in the presence of the magnetic field. Then we give the gap equation and lay out the calculation of the energy for the superconducting state.

2.1 Hamiltonian and Gap Equation

With the magnetic field and CM momentum of Cooper pair, by defining the Nambu spinors \( \hat{\Phi}_k = \begin{bmatrix} \phi_{k+q} \phi_{k+q}^\dagger \phi_{k-q} \phi_{k-q}^\dagger \end{bmatrix}^T \), we present the Hamiltonian \( H_S \) of the spin–orbit-coupled s-wave superconductor as [2,85,87]:
\[
\hat{H}_S = \frac{1}{2} \int d\mathbf{k} \hat{\Phi}_k^\dagger \hat{H}_s(\mathbf{k}) \rho_3 \hat{\Phi}_k, 
\]

with

\[
\hat{H}_s(\mathbf{k}) = \begin{pmatrix}
\xi_{k^+} + \Omega_{k^+} \cdot \sigma & \Delta q i \sigma_2 \\
\Delta q^* i \sigma_2 & \xi_{k^-} + \Omega_{k^-} \cdot \sigma
\end{pmatrix}.
\]

Here, \( \rho_3 = \sigma_0 \otimes \tau_3; \sigma_i \) and \( \tau_i \) stand for the Pauli matrices in spin and particle-hole spaces, respectively; \( \mathbf{k}^\pm = \pm \mathbf{k} + \mathbf{q} \) with \( \mathbf{q} \) standing for the CM momentum; \( \xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - E_F \) and \( \varepsilon_{\mathbf{k}} = k^2/(2m^*) \) with \( m^* \) being the effective mass of electrons in superconductor and \( E_F \) denoting the Fermi energy; \( \Omega_{\mathbf{k}} = \mathbf{h}_k + \mathbf{h}_B \) with \( \mathbf{h}_k \) and \( \mathbf{h}_B \) representing the SOC and Zeeman energy, respectively; \( \Delta_{\mathbf{q}} = V \sum_k \langle \phi_{k+q}^\dagger \phi_{-k+q} \rangle \) stands for the order parameter of the superconducting state in the momentum space; \( \langle \cdot \rangle \) stands for the ensemble average; \( V \) is the conventional \( s \)-wave attractive potential in superconductors. It is noted that the order parameter in this case can be transformed into the FF form \[2\] \( \Delta(\tau) = \Delta_0 e^{iQ \cdot r} (Q = 2\mathbf{q}) \) in real space.

In Nambu \( \otimes \) spin space, the equilibrium Green function \[89–91\] in the momentum space is given by

\[
G_q(\mathbf{k}, \tau) = -\rho_3 \langle T_\tau \hat{\Phi}_k(\tau) \hat{\Phi}_k^\dagger(0) \rangle,
\]

where \( T_\tau \) represents the chronological-ordering operator; \( \tau \) is the imaginary time. By expressing

\[
G_q(\mathbf{k}, \tau) = \begin{pmatrix}
g_q(\mathbf{k}, \tau) & f_q(\mathbf{k}, \tau) \\
f_q^\dagger(-\mathbf{k}, \tau) & g_q^\dagger(-\mathbf{k}, \tau)
\end{pmatrix},
\]

one can obtain the normal Green function \( g_q(\mathbf{k}, \tau) \) and anomalous Green function \( f_q(\mathbf{k}, \tau) \) \[78,89–91\].

Then, in the Matsubara representation \[89–91\] \( G_k(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} G_k(\tau) \), from the Gor’kov equation \[92–95\]:

\[
[i\omega_n \rho_3 - H_s(\mathbf{k})]G_q(\mathbf{k}, i\omega_n) = 1,
\]

one has

\[
(i\omega_n - \xi_{k^+} - \Omega_{k^+} \cdot \sigma) f_q(\mathbf{k}, i\omega_n) - \Delta q i \sigma_2 g_q^\dagger(\mathbf{k}, i\omega_n) = 0,
\]

\[
-\Delta q^* i \sigma_2 f_q(\mathbf{k}, i\omega_n) + (-i\omega_n - \xi_{k^-} - \Omega_{k^-} \cdot \sigma) g_q^\dagger(\mathbf{k}, i\omega_n) = 1.
\]

Here, \( \beta = 1/(k_B T) \) with \( k_B \) being the Boltzmann constant and \( T \) representing the temperature; \( \omega_n = (2n + 1)\pi k_B T \) are the Matsubara frequencies with \( n \) being integer.

Following the previous work \[87\] through multiplying Eq. (7) by \( i\sigma_2 \) from the left side, one immediately has

\[
\Delta q_i f_q(\mathbf{k}, i\omega_n) + (-i\omega_n - \xi_{k^-} + \Omega_{k^-} \cdot \sigma^*) i\sigma_2 g_q^\dagger(\mathbf{k}, i\omega_n) = i\sigma_2.
\]
Then, by using Eq. (6) to replace \( i\sigma_2 \tilde{g}_q^+(k, i\omega_n) \) in Eq. (8), the anomalous Green function can be obtained:

\[
f_q(k, i\omega_n) = [f_q^s(k, i\omega_n) + f_q^t(k, i\omega_n) \cdot \sigma] i\sigma_2,
\]

(9)

with the singlet \( f_q^s(k, i\omega_n) \) and triplet \( f_q^t(k, i\omega_n) = \left( \frac{f_{\downarrow\downarrow} - f_{\uparrow\uparrow}}{2}, \frac{f_{\downarrow\uparrow} + f_{\uparrow\downarrow}}{2}, f_{\downarrow\uparrow + \uparrow\downarrow} \right) \) pairings \([96, 97]\) written as:

\[
f_q^s(k, i\omega_n) = -\frac{\Delta_q}{w_q(k, i\omega_n)} \left[ (i\omega_n - \xi_k^+) (i\omega_n + \xi_k^-) + \Omega_k^+ \cdot \Omega_k^- - |\Delta_q|^2 \right],
\]

(10)

\[
f_q^t(k, i\omega_n) = -\frac{\Delta_q}{w_q(k, i\omega_n)} \left[ (i\omega_n - \xi_k^+) \Omega_k^- + (i\omega_n + \xi_k^-) \Omega_k^+ - i\Omega_k^+ \times \Omega_k^- \right],
\]

(11)

\[
w_q(k, i\omega_n) = \prod_{\mu=\pm} (i\omega_n - E^e_{\mu k})(i\omega_n - E^h_{\mu k}).
\]

(12)

\( E^e(h) \) \((\mu = \pm)\) stand for the quasiparticle electron (hole) energy spectra in superconductors, which can be obtained from the solutions of equation \( |f_q^s(k, \omega)|^2 - |f_q^t(k, i\omega)|^2 = 0 \) with respect to \( \omega \).

With the anomalous Green function, the singlet \( \rho_q^s \) and triplet \( \rho_q^t = (\rho_{1s=-1}^t = 1, \rho_{1s=-1}^t = -1, \rho_{1s=0}^t = 0) \) \([96–98] \) correlations are defined as

\[
\rho_q^s(k) = \frac{1}{\beta} \sum_{i\omega_n} f_q^s(k, i\omega_n),
\]

(13)

\[
\rho_q^t(k) = \frac{1}{\beta} \sum_{i\omega_n} f_q^t(k, i\omega_n).
\]

(14)

Then one immediately has the gap equation \([92–95] \):

\[
\Delta_q = V \sum_k \rho_q^s(k).
\]

(15)

Here, the summation is taken for the values of \( k \) satisfying \( |E_{\uparrow k} - E_F| < \omega_D \) and \( |E_{\downarrow k} - E_F| < \omega_D \) where \( E_{\uparrow(\downarrow)k} \) is the energy of spin-up (-down) electron with the momentum \( k \); \( \omega_D \) stands for the Debye frequency. It is noted that due to the conventional \( s \)-wave attractive potential, only singlet-order parameter exists. \(^1\) Then,
by self-consistently solving Eq. (15), the order parameter $\Delta_q$ at fixed CM momentum of Cooper pair $q$ is obtained.

2.2 Ground-State Energy

Following the previous work by FF [2], by replacing the $s$-wave attractive potential $V$ with an effective one $\lambda$ in Eq. (15), a potential-dependent order parameter $\Delta_q(\lambda)$ can be immediately obtained by self-consistently solving Eq. (15).

Then, by neglecting the Fock energy of the normal state, based on Feynman–Hellmann theorem

$$\partial_{\lambda} E = \langle \partial_{\lambda} H_\lambda(\lambda) \rangle = -\frac{|\Delta_q(\lambda)|^2}{\lambda^2},$$

(16)

the expectation value of the energy difference between the superconducting state $E_q^S$ and normal one $E_q^N$ is given by

$$\delta E_q = E_q^S - E_q^N = -\int_{V_0}^{V} d\lambda \frac{|\Delta_q(\lambda)|^2}{\lambda^2},$$

(17)

where $V_0$ is the effective attractive potential at the transition point between the superconducting state and the normal one [$\Delta_q(V_0) = 0$]. Then, by calculating the minimum of $\delta E_q$ with respect to a single parameter, i.e., the CM momentum $q$, the properties of the superconducting state including the CM momentum, order parameter, quasiparticle energy spectra, singlet and triplet correlations can all be determined. Particularly, it is noted that through self-consistently solving the gap equation in our work, the numerical difficulty from the multi-variable minimum problem mentioned in the introduction is avoided, leading to more accurate results.

Footnote 1 continued

presence of the CM momentum and SOC, all four types of Cooper pairing can be induced [85]. Nevertheless, the existence of the Cooper pairing does not guarantee the existence of the corresponding order parameter, and proper electron–electron potential is needed. For the conventional $s$-wave attractive potential we used in this work, only $s$-wave (singlet even-frequency type)-order parameter exists, since other types of the order parameters are destroyed after the summation in the gap equation $\Delta = V \sum_{i\omega_n, k} f_k(k, i\omega_n)$ due to the corresponding odd character in the frequency/momentum space. The survived $s$-wave-order parameter also agrees with the conventional BCS superconducting behaviors in the reported experiments [100, 105, 106]. Particularly, we point out that the singlet and triplet correlation discussed in our work both belong to the even-frequency type, since the odd-frequency correlation is zero after the summation in the frequency space in Eqs. (13) and (14).

2 Equation (17) should be written as $\delta E_q = -\int_{V_0}^{V} d\lambda \frac{|\Delta_q(\lambda)|^2}{\lambda^2}$ with $\lambda$ being the effective electron–electron attractive potential. Nevertheless, at a fixed CM momentum, one always has $\Delta_q(\lambda) = 0$ when the effective electron–electron potential $\lambda$ is smaller than a critical strength. This indicates that at very small effective electron–electron attractive potential, the superconductivity is destroyed and the system is in the normal state. In this situation, one has $\delta E_q = -\int_{V_0}^{V} d\lambda \frac{|\Delta_q(\lambda)|^2}{\lambda^2}$ where $\Delta_q(V_0) = 0$ and then Eq. (17) is obtained.
Table 1  Parameters used in our calculation [105]

| Parameter     | Value       |
|---------------|-------------|
| $m^*/m_0$     | 3.049$^a$   |
| $\Delta_0$ (meV) | 1.2732$^b$ |
| $\gamma$ (meV Å) | 23.28$^a$  |
| $V$ (meV)     | 0.218       |
| $E_F$ (meV)   | 70$^a$      |
| $\omega_D$ (meV) | 38.088$^b$ |
| $T$ (K)       | 0           |
| $k_F$ (Å$^{-1}$) | 0.246      |

Note that $m_0$ stands for the free electron mass. $\Delta_0$ is the order parameter at $T = 0$ K without magnetic field. $V$ is obtained by fitting $\Delta_0$. $k_F$ is the largest momentum with the Fermi energy in the absence of the magnetic field. The specific form of $\gamma_k$ is still absent in the literature. To capture the main physics, we neglect the momentum dependence of $\gamma_k$ by using the sphere average, considering the fact that the main physics occurs at the Fermi surface. This simplified SOC in Li$_2$Pd$_3$B has also been used before in the previous work $^a$Refs. [100]. $^b$ Ref. [101]

3 Numerical Results

In this section, by numerically solving Eq. (17), we discuss the properties of the superconducting state in spin–orbit-coupled superconductors. In this work, we focus on superconductors with Dresselhaus SOC. This can be realized in Li$_2$Pd$_3$B [99–106] where strong Dresselhaus SOC [105,106] $h_B = \gamma_k \mathbf{k}$ and conventional BCS superconductivity at zero magnetic field [101,103] are realized experimentally. All the material parameters used in our calculation are listed in Table 1. The magnetic field is chosen along the $z$ direction.

3.1 Orientation of CM Momentum and Pairing Structure

We first focus on the CM-momentum dependences of the energy of the superconducting state at different magnetic fields. We find that the energy difference $\delta E_\mathbf{q}$ between the superconducting state and normal one always shows isotropy with respect to the latitude of the CM momentum around the magnetic field (not shown). This is due to the spatial-rotational symmetry around the magnetic field of the system. Then, $\delta E_\mathbf{q}$ as function of the longitude $\theta$ (with respect to the magnetic field) and magnitude $q$ of the CM momentum are plotted in Fig. 1 at different magnetic fields. As seen from the figure, in the presence of the magnetic field, the minimum of $\delta E_\mathbf{q}$ is always reached at finite $q$ with $\theta = 0$ ($z$ direction). This indicates that a CM momentum parallel to magnetic field is induced in the superconducting state in the presence of the magnetic field and SOC, similar to the previous work with the same Dresselhaus SOC [62]. Small and large CM momenta are observed before [Fig. 1a, b] and after [Fig. 1c, d] $h_B = \Delta_0$ here, respectively.

Particularly, in comparison with the conventional FF state where the ground-state energy is degenerate to CM-momentum orientation and hence the FF state ($\Delta = \Delta_0 e^{i\mathbf{q}\cdot\mathbf{r}}$) with one CM momentum and LO one [$\Delta = \Delta_0(e^{i\mathbf{q}\cdot\mathbf{r}} + e^{-i\mathbf{q}\cdot\mathbf{r}})$] with two opposite CM momenta are equivalent spontaneous-symmetry breaking, with the SOC, the CM-momentum orientation is determined according to the energy minimum, leading to the FF state more favored than the LO state, in consistence with the
The energy minimum reaches at $q = 0.02$ and $\theta = 0$ in (a), $q = 0.06$ and $\theta = 0$ in (b), $q = 0.32$ and $\theta = 0$ in (c), and $q = 0.43$ and $\theta = 0$ in (d), respectively (Color figure online).

previous work through self-consistently solving Bogoliubov de Gennes equation in the real space [67]. Moreover, from the energy minimum, the uniquely determined CM momentum here is inherently robust against the impurity scattering (elastic scattering).

To illustrate the pairing structure, we further plot the energy spectra of spin-up and spin-down electrons along the $k_z$ direction (magnetic-field direction) with and without magnetic field in Fig. 2. As seen from the figure, without the magnetic field, the SOC leads to the opposite shifts of the energy spectra between spin-up (red solid curve) and spin-down (blue dotted curve) electrons along the $k_z$ direction. Then, the magnetic field causes the opposite energy shifts of the energy spectra between spin-up (red dashed curve) and spin-down (blue chain curve) electrons. In this situation, by assuming the Debye frequency $\omega_D = 0$, the Cooper pairing only occurs at the Fermi surface, as shown in Fig. 2. Then, there exist two possible types of Cooper pairings: type I, formed by spin-up electron 1 (with $-\mathbf{k}_I + \mathbf{q}$) and spin-down one 4 (with $\mathbf{k}_I + \mathbf{q}$) in Fig. 2 in favor of the CM momentum $\mathbf{q} = q_c \mathbf{z}$ ($q_c = \sqrt{|m^*\gamma|^2 + 2m^*(E_F + h_B)} - \sqrt{|m^*\gamma|^2 + 2m^*(E_F - h_B)}$); type II, formed by spin-up electron 3 (with $\mathbf{k}_I + \mathbf{q}$) and spin-down one 2 (with $-\mathbf{k}_I + \mathbf{q}$) in Fig. 2, in favor of the CM momentum $\mathbf{q} = -q_c \mathbf{z}$. Nevertheless, from Fig. 1, the CM momentum $\mathbf{q}$ is along the $\mathbf{z}$ direction, as mentioned above. This indicates the type I pairing makes the leading contribution in the determination of the CM momentum.

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**Fig. 1** $\delta E_q$ versus longitude $\theta$ (with respect to $\mathbf{h}_B$) and magnitude $q$ of the CM momentum at (a) $h_B = 0.6\Delta_0$, (b) $h_B = 0.9\Delta_0$, (c) $h_B = 1.0\Delta_0$, and (d) $h_B = 1.2\Delta_0$. $q_0 = \sqrt{2m^*(E_F + h_B)} - \sqrt{2m^*(E_F - h_B)}$. The energy minimum reaches at $q = 0.02$ and $\theta = 0$ in (a), $q = 0.06$ and $\theta = 0$ in (b), $q = 0.32$ and $\theta = 0$ in (c), and $q = 0.43$ and $\theta = 0$ in (d), respectively (Color figure online).
The leading role of type I pairing can be understood as follows. On the one hand, the relative momentum $k_1$ in type I pairing is larger than $k_{II}$ in type II one. Then, with the larger relative momentum $k$ and hence larger density of states in Eq. (15), type I pairing makes the leading contribution to the summation with respect to the momentum space. On the other hand, from the framework of the Ginzburg–Landau theory, the free-energy densities of superconducting system reads

$$F = \alpha |\psi(\mathbf{r})|^2 + \eta \frac{1}{2} |\psi(\mathbf{r})|^4 + \frac{1}{2m} [\Pi \psi(\mathbf{r})]^n [\Pi \psi(\mathbf{r})],$$

with $\Pi = -i\hbar \nabla + 2eA$ and $\psi(\mathbf{r}) = \Delta(\mathbf{r})/V$; $\alpha$ and $\eta$ being the corresponding expansion parameters; $A$ standing for the magnetic vector potential. In the presence of the SOC, one can replace $eA$ by $A_s$ with $A_s = m\gamma s/\hbar$ for the Dresselhaus SOC and $s$ representing the spin vector of electrons ($s = \langle c_\mathbf{k}\sigma \sigma' c_{\mathbf{k}'\sigma'} \rangle$). Then, one has

$$F = \frac{\alpha}{V^2} |\Delta_{\mathbf{q}}|^2 + \frac{\eta}{2V^4} |\Delta_{\mathbf{q}}|^4 + \frac{\hbar^2 q^2}{2m} |\Delta_{\mathbf{q}}|^2 + \frac{2m\gamma^2 s^2 |\Delta_{\mathbf{q}}|^2}{\hbar^2 V^2}$$

$$+ \frac{2\gamma |\Delta_{\mathbf{q}}|^2}{V^2} \mathbf{s} \cdot \mathbf{q},$$

(19)

In the presence of a $\mathbf{k}$-linear-dependent SOC $h_\mathbf{k} = A_s \cdot \mathbf{k} [A_s = \gamma \sigma (A_s = \gamma \sigma_y e_x - \gamma \sigma_x e_y)]$ for simple Dresselhaus (Rashba) SOC, the Hamiltonian reads $H_\mathbf{k} = \frac{\hbar^2 q^2}{2m} - \mu + A_s \cdot \mathbf{k} = \frac{(\mathbf{k} + m\mathbf{A}_s/\hbar)^2}{2m} - \mu - \frac{m\gamma^2}{2\hbar^2}$, similar to the one $H_\mathbf{k} = \frac{(\mathbf{k} + eA_\mu \gamma_\mu)}{2m}$ with the electromagnetic field. Hence, in the presence of the SOC, one can replace $eA$ by $A_s$ in the Ginzburg–Landau theory. This can also be understood in another way as follows. Following the non-Abelian gauge theory in quantum field theory, one can define the covariant derivative $D_\mu = \partial_\mu + ig\mathbf{A}_\mu \gamma_\mu$ associated with the local SU(2) symmetry to replace $\partial_\mu$, similar to the replacement of $\partial_\mu$ by $D_\mu = \partial_\mu + ie\mathbf{A}_\mu$ associated with the local U(1) symmetry in Abelian gauge theory. This replacement is similar to the previous works with more detailed derivation for Rashba SOC [112].
Fig. 3  a Magnetic-field dependence of $\Delta_q$. The vertical dashed (chain) line indicates the BCS state (transition point between the drift-BCS and FF states). The inset (I) in (a) shows the CM momentum as function of magnetic field. Particularly, we find that $q = 7.5 \times 10^{-4}$ at $h_B = 0.05\Delta_0$. The inset (II) in (a) exhibits $\delta E_q$ versus $q = q_z$ at different magnetic fields. The results in inset (b) are renormalized by the maximum of $|\delta E_q|$ for each magnetic field. $q_0 = \sqrt{2m^*(E_F + h_B) - \sqrt{2m^*(E_F - h_B)}}$. b, c singlet correlations in the momentum space at $h_B = 0.9\Delta_0$ ($h_B = 1.1\Delta_0$) (Color figure online)

where we have applied $\Delta(r) = \Delta_q e^{i q \cdot r}$ in Eq. (19). With the magnetic field, the spin vector $s$ is anti-parallel to $h_B$. Hence, to obtain the free-energy minimum, the induced CM momentum $q$ should be parallel to the magnetic field ($z$ direction), in accord with the type I pairing (in favor of $q = q_z$).

3.2 Phase Diagram

In this part, we discuss the phase diagram of the superconducting state. The magnetic field dependences of the order parameter $\Delta_q$ and CM momentum $q = q_z$ are plotted
in Fig. 3a and the inset (I) of Fig. 3a, respectively. In the calculation, \( \mathbf{q} \) is chosen at the minimum of \( \delta E_{\mathbf{q}} \). As seen from Fig. 3a and inset (I), with the increase in magnetic field from zero, before reaching \( h_B = \Delta_0 \), the order parameter decreases slightly [Fig. 3a]. In the same time, a CM momentum is induced and increases from zero [inset (I)], similar to the gapped FF state in the previous works [62–65] mentioned in the introduction. Nevertheless, by plotting the singlet correlation at \( h_B = 0.9\Delta_0 \) in Fig. 3b, it is seen from the figure that two separated and complete circles with finite \( \rho^s_{\mathbf{q}}(\mathbf{k}) \), corresponding to type I (large \( k \)) and II (small \( k \)) pairings, are observed due to the presence of the SOC, and no unpairing region with vanishing \( \rho^s_{\mathbf{q}}(\mathbf{k}) \) appears when \( h_B < \Delta_0 \). As mentioned in the introduction, it is more appropriate to refer to such superconducting state in which the CM momentum is induced but no unpairing region is observed, as the drift-BCS state, since the induced CM momentum at small magnetic field here arises from the energy-spectrum distortion by the magnetic field and SOC as mentioned in Sec. 3.1, resembling the intravalley pairing in graphene [107] and transition metal dichalcogenides [108], and hence has different origin from the case in the conventional FF state without SOC [2]. Particularly, since this drift-BCS state occurs at small magnetic field, it can inherently survive against the orbital pairing breaking effect.

By further increasing the magnetic field, abrupt suppression on the order parameters [shown in Fig. 3a] and enhancement of the CM momenta [shown in the inset (I) of Fig. 3a] are observed before and after \( h_B \approx \Delta_0 \), indicating the occurrence of the first-order phase transition. The abrupt changes can be understood from the inset (II) of Fig. 3a where we plot the energy differences \( \delta E_{\mathbf{q}} \) versus \( \mathbf{q} = q\mathbf{z} \) at different magnetic fields. From the inset (II), it is seen that when \( h_B < \Delta_0 \), the minimum of \( \delta E_{\mathbf{q}} \) sits near \( q \approx 0 \) (brown dashed curve), and the minimum position \( q_m \) increases with the magnetic field by comparing the brown dashed curve at \( h_B = 0.6\Delta_0 \) with the black chain one at \( h_B = 0.8\Delta_0 \). Whereas with the increase in magnetic field when \( h_B \geq \Delta_0 \), \( \delta E_{\mathbf{q} \approx 0} \) increases and then the minimum of \( \delta E_{\mathbf{q}} \) appears around \( q \approx 0.45q_0 \), leading to abrupt changes of the order parameters and CM momenta before and after \( h_B \approx \Delta_0 \). Moreover, we plot the singlet correlation when \( h_B > \Delta_0 \) in Fig. 3c. In comparison with the two complete circles of singlet correlations at \( h_B < \Delta_0 \) [Fig. 3b], in the case with \( h_B > \Delta_0 \) [Fig. 3c], the inner circle (type II pairing) is broken around \( k_z \) axis whereas the outer circle (type I pairing) survives since the CM momentum is along the favorable orientation to type I pairing (Sec. 3.1). The appeared regions with the destroyed singlet correlations in this situation, known as the unpairing ones, are the hallmark of the emergence of the FF state [2].

The destruction mechanism of the singlet correlation for the inner circle around \( k_z \) axis can be understood as follows in a special direction. With the induced CM momentum pointing to \( k_z \) direction (in Sec. 3.1), along \( k_z \), one has \( \mathbf{h}_k = \gamma(k_x, k_y, k_z) = \gamma(0, 0, k_z) \) where the spin-flip terms of the SOC are zero, and then the singlet correlation can be simplified into

\[
\rho^s_{\mathbf{q}}(\mathbf{k}) = \text{Tr} \left\{ \frac{\Delta_{\mathbf{q}}[n_f(E^h_{\sigma_3\mathbf{k}}) - n_f(E^e_{\sigma_3\mathbf{k}})]}{2\sqrt{(\xi_k + \sigma_3\gamma k_z)^2 + |\Delta_{\mathbf{q}}|^2}} \right\}.
\] (20)
Here, the quasiparticle electron and hole energy spectra are written as:

\begin{align}
E_{\sigma_3 k_z}^e &= \sqrt{(\xi_k + \sigma_3 \gamma k_z)^2 + |\Delta_q|^2 + \frac{k_z q}{m^*} + \mu \Omega_{qz}}, \quad (21) \\
E_{\sigma_3 k_z}^h &= -\sqrt{(\xi_k + \sigma_3 \gamma k_z)^2 + |\Delta_q|^2 + \frac{k_z q}{m^*} + \mu \Omega_{qz}}. \quad (22)
\end{align}

which are plotted in Fig. 4a. When \( k_z > 0 \), the maximum of factor \( \Delta_q / \sqrt{(\xi_k + \sigma_3 \gamma k_z)^2 + |\Delta_q|^2} \) occurs at large (small) \( k_z \) for \( \sigma_3 = -1 \) (\( \sigma_3 = 1 \)) in the summation of Eq. (20). Hence, \( \sigma_3 = -1 \) (\( \sigma_3 = 1 \)) makes the main contribution to the outer (inner) circle of the singlet correlation, e.g., type I (II) pairing, in accord with the pairing spin-down electron 4 (spin-up one 3) in Fig. 2. In this case, for \( \sigma_3 = -1 \), as shown in Fig. 4a, one always has \( E_{\sigma_3 k_z}^e > 0 \) (red solid curve) and \( E_{\sigma_3 k_z}^h < 0 \) (green dotted curve), and hence \( n_f(E_{\sigma_3 k_z}^h) - n_f(E_{\sigma_3 k_z}^e) = 1 \), leading to the finite \( \rho_{qz}^s \) for the outer circle. Whereas, for \( \sigma = 1 \), there exists the region where the quasiparticle hole energy is larger than zero (\( E_{\sigma_3 k_z}^h > 0 \)) when \( k_z > 0 \), as shown by blue dashed curve. This is due to the induced CM momentum, similar to the conventional FF state [2]. Together with \( E_{\sigma_3 k_z}^e > 0 \) (brown chain curve), one has \( n_f(E_{\sigma_3 k_z}^h) - n_f(E_{\sigma_3 k_z}^e) = 0 \) in this
region, leading to the open inner circle of the singlet correlation and hence depairing effect of Cooper pair.

By using a similar analysis, the case with \( k_z < 0 \) can also be understood. The Fermi distributions of quasiparticle electron and hole in the entire momentum space are plotted in Figs. 4b, c from full numerical results, respectively. It is found there exist two arc regions with either quasiparticle electron energy below zero or quasiparticle hole one larger than zero, which exactly correspond to the regions with vanishing singlet correlation for the inner circle shown in Fig. 3c, in consistence with the analysis above.

Furthermore, it is noted that the emerged FF state in our work corresponds to the gapless one mentioned in the introduction [62–65], since the gapless quasiparticle energy spectra \( |E^{c/h}_k| = 0 \) revealed in the gapless FF state [62–64] indicate the emergence of the unpairing regions. By the detailed study of the SOC dependence (refer to Appendix), enhanced Pauli limit and hence enlarged magnetic-field regime of the emerged FF state by the SOC is also observed in our work. We further show that this enhancement of the Pauli limit is due to the spin-flip terms of the SOC, which suppress the unpairing regions (also addressed in “Appendix”).

### 3.3 Triplet Correlation and Cooper-Pair Spin Polarization

In this section, by studying the induced \( p \)-wave triplet correlations in the pairing regions thanks to the broken space-inversion symmetry by SOC, we show that the Cooper-pair spin polarization [76,77,86,87], which is predicted to be induced by the magnetic field and CM momentum [87], exhibits totally different magnetic-field dependences in the drift-BCS and FF states due to the abrupt changes of the order parameters and CM momenta. This provides a scheme to experimentally distinguish these two phases through the reported magnetoelectric Andreev effect [86–88] in addition to the phase transition.

#### 3.3.1 Triplet Correlation

We first discuss the triplet correlations. Specifically, with the broken space-inversion symmetry by the SOC, \( p \)-wave triplet correlations are induced, plotted in Fig. 5 at different magnetic fields. From the figure, it is seen that in the drift-BCS state (\( h_B < \Delta_0 \)), two separated and complete circles with finite \( \rho_{t_s=0}^f \) [Fig. 5a], \( \rho_{t_s=1}^f \) [Fig. 5b] and \( \rho_{t_s=-1}^f \) [Fig. 5c] are observed in the momentum space, similar to the singlet case [Fig. 3b]. Moreover, for either the outer or the inner circle, it is seen that the triplet correlations show the \( p \)-wave characters: \( \rho_{t_s=0}^f \propto h_{k_z} \) and \( \rho_{t_s=\pm1}^f \propto ih_{k_y} \mp h_{k_x} \), in agreement with the previous works [78,81,84,85]. As for the FF state (\( h_B > \Delta_0 \)), as shown in Figs. 5d–f, the inner circles of the triplet correlations are open, similar to the singlet case [Fig. 3b]. Then, the triplet correlations are only observed in the pairing regions, resembling our previous work [85].

The magnetic-field dependences of the maximum of the singlet and triplet correlations in the momentum space are plotted in Fig. 5g. From the figure, it is seen that in either the drift-BCS state (\( h_B < \Delta_0 \)) or the FF one (\( h_B > \Delta_0 \)), \( \rho_{t_s=0}^f \) (dashed
Fig. 5  a–f momentum dependence of $\rho_{s=0}^t$ and $\rho_{s=\pm 1}^t$ in the drift-BCS state at $h_B = 0.9\Delta_0$ and FF one at $h_B = 1.1\Delta_0$ correspondingly. g Maxima of the singlet and triplet correlations in the momentum space as function of $h_B$. The vertical chain line in (g) indicates the transition point between the drift-BCS and FF states (Color figure online)
curve with dots) and $\rho^t_{s=1}$ (dashed curve with triangles) are comparable to the singlet one (solid curve with crosses) when $h_B < 1.7\Delta_0$. Moreover, it is also noted that $\rho^t_{s=1} \neq \rho^t_{s=-1}$ when $h_B \neq 0$, indicating the generation of the Cooper-pair spin polarization by the magnetic field and CM momentum, as predicted in the previous work by Tkachov [87].

Nevertheless, even with the large $p$-wave triplet correlations (compared with the singlet one) and the generation of the Cooper-pair spin polarization, the $p$-wave spin-polarized superfluid is still absent, because of the vanishing $p$-wave triplet-order parameter:

$$\Delta'(k) = \sum_{k'} V_{k-k'} \rho^t(k'),$$

by the $s$-wave attractive potential $V_{k-k'} = V$ and $p$-wave character: $\rho^t(k') = -\rho^t(-k')$. However, it is proposed recently in ultracold atom systems [109,110] that in the presence of the triplet correlation, one can rapidly introduce $k$-dependent attractive potential $V_{k-k'}$ through the Feshbach resonance, and then non-vanishing $p$-wave superfluid is immediately obtained, at least just after the introduction of the potential. Additionally, it is also theoretically reported [111] that the fluctuations of the incipient parity-breaking order can generate an attractive pairing interaction $V_{kk'}$ in odd-parity channel, leading to finite $p$-wave triplet superfluid. With these approaches to introduce attractive potential in the triplet channel, the $p$-wave spin-polarized superfluid can be expected in the spin–orbit-coupled systems with a magnetic field, according to our results and analysis above.

### 3.3.2 Cooper-Pair Spin Polarization

Next, we show that due to the abrupt changes in order parameters and CM momenta between the drift-BCS and FF states, the induced Cooper-pair spin polarizations mentioned in Sec. 3.3.1, exhibit totally different magnetic-field dependences in these two phases.

Specifically, as mentioned in Sec. 3.3.1, with the induced triplet correlation, the Cooper-pair spin polarizations, defined as [76,77,86,87]

$$P_c(k) = |\rho^t_{s=1}(k)|^2 - |\rho^t_{s=-1}(k)|^2,$$

are induced by the magnetic field, plotted in Fig. 6 at different magnetic fields. As seen from the figure, in the drift-BCS state [Fig. 6a] and pairing regions of the FF one [Fig. 6b], it is seen that $P_c(k) \propto k_x^2$, which can be understood from Eq. (14). Specifically, from Eqs. (14) and (24), one has

$$P_c(k) \propto |f^{\uparrow\uparrow}(0, k)|^2 - |f^{\downarrow\downarrow}(0, k)|^2 = i f^t_q(0, k) \times f^{t*}_q(0, k) |z| .$$

Then, from Eq. (11), one immediately finds that

$$P_c(k) \propto |h_k \times (h_k \times \Omega_q)| |\Delta_q|^2 = (h_{kx}^2 + h_{ky}^2) \Omega_q |\Delta_q|^2,$$

in accord with the numerical results (Fig. 6).
The magnetic-field dependence of the maximum of the Cooper-pair spin polarization $P_{c}^{\text{max}}$ in the momentum space is plotted in Fig. 7. As seen from the figure, with the increase of $h_B$, in comparison with the linear increase of $P_{c}^{\text{max}}$ in the drift-BCS state ($h_B < \Delta_0$), $P_{c}^{\text{max}}$ first increases and then decreases in the FF one, leading to a peak observed around $1.4\Delta_0$. Particularly, near the phase transition point, the increase of $P_{c}^{\text{max}}$ with magnetic field in the drift-BCS state is faster than that in the FF one, as shown in Fig. 7. This character provides a scheme to experimentally detect the phase transition point through the reported magnetoelectric Andreev effect [86–88]. Additionally, the different magnetic-field dependences of $P_{c}^{\text{max}}$ between the drift-BCS and FF states [i.e., linear increase (peak structure) of $P_{c}^{\text{max}}$ during the entire magnetic-field regime of the drift-BCS (FF) state], also provide a scheme to experimentally distinguish these two phases.

From Eq. (26), the magnetic-field dependence of $P_{c}^{\text{max}}$ can be clearly understood. Specifically, in the drift-BCS states ($h_B < \Delta_0$), with $\Omega_{qz} \approx h_B$ and the marginal variation of $\Delta_q$ [Fig. 3a], $P_{c}^{\text{max}}$ increases linearly with $h_B$. As for the FF state, with the increase of $h_B$ at $\Delta_0 < h_B < 1.5\Delta_0$, although $\Delta_q$ is suppressed [Fig. 3a], $\Omega_{qz}$ ($\propto h_B + h_{qz}$) is markedly enhanced due to the increased CM momentum $q$ [inset (I) of Fig. 3a]. By the stronger enhancement from $\Omega_{qz}$ than the suppression...
Fig. 7 Maximum of Cooper-pair spin polarization in the momentum space versus magnetic field. The inset shows $P_{c}^{\text{max}}/|\Delta q|^2$ as function of $h_B$. The vertical dashed line indicates the transition point between the drift-BCS and FF states (Color figure online).

from $|\Delta q|^2$ in Eq. (26) at $\Delta_0 < h_B < 1.5\Delta_0$, $P_{c}^{\text{max}}$ increases nonlinearily with $h_B$. This can be justified by plotting $P_{c}^{\text{max}}/|\Delta q|^2 (\propto \Omega_{qz})$ versus $h_B$ in the inset of Fig. 7, from which it is seen that $P_{c}^{\text{max}}/|\Delta q|^2$ is markedly enhanced at $\Delta_0 < h_B < 1.5\Delta_0$. By further increasing $h_B$ after 1.5$\Delta_0$, $q$ becomes saturated at 0.47$q_0z$ [inset (I) of Fig. 3a], and hence the suppression of $\Delta q$ leads to the marked decrease of $P_{c}^{\text{max}}$.

4 Summary

In summary, we have systematically investigated the properties of the FF state with an induced CM momentum in the Dresselhaus spin–orbit-coupled superconductor in the presence of the magnetic field. Differing from the previous theoretical works [61–65] where the study is based on the numerical calculation of the free-energy minimum with respect to the CM momentum and order parameter, in our work, by analytically obtaining the anomalous Green function and hence the gap equation, the superconducting state can be determined by computing the energy minimum with respect to a single parameter, i.e., the CM momentum. Moreover, from the obtained anomalous Green function, properties of the superconducting state including quasiparticle energy spectra, singlet and triplet correlations, behaviors of the CM momentum and order parameter at the phase transition are also addressed in our work.

Specifically, it is found that with the SOC, the CM momentum parallel to the magnetic field is induced at small magnetic field, similar to the gapped FF state in the previous works [62–65]. Nevertheless, we have shown that two complete circles of the singlet correlation due to the SOC are observed in the momentum space, and no unpairing region with vanishing singlet correlation appears. This is very different from the conventional FF state without SOC, where the CM momentum is induced simultaneously with the emergence of the unpairing regions [2]. By further studying the pairing structure, it is shown that the induced CM momentum with SOC at small magnetic field is due to the energy-spectrum distortion, resembling the intravalley pairing.
in graphene [107], and transition metal dichalcogenides [108] and hence has different origin from the case in conventional FF state [2]. Therefore, it is more appropriate to refer to such superconducting state, in which the CM momentum is induced but no unpairing region is developed, as the drift-BCS state. By further increasing the magnetic field, abrupt enhancement of the CM momenta and suppression on the order parameters are observed, indicating the occurrence of the first-order phase transition. Particularly, we find that open circle of the singlet correlation, i.e., unpairing region with vanishing singlet correlation, is induced after the phase transition, showing the emergence of the FF state. It is further shown that induced unpairing regions arises from the quasiparticle electron and hole with energies below and larger than zero, respectively, indicating the emerged FF state here corresponds to the gapless one in the previous works [62–65]. Enhanced Pauli limit and hence enlarged magnetic-field regime of the emerged FF state are also observed in our work, and we demonstrate that the enhancement of the Pauli limit is due to the spin-flip terms of the SOC, which suppress the unpairing regions.

Finally, we discuss the possibilities of experimentally detecting the drift-BCS and FF states with SOC. Up till now, the emergence of the FF phase has been demonstrated mainly by indirect signatures such as thermodynamic characters including evidences like latent heat [24] and discontinuous magnetization [25] for the first-order phase transition, anomalous thermal conductivity [22], enhanced Pauli limit [27] and measurements of the temperature and angular dependencies of the upper magnetic field [14,17], and microscopic characters including anomaly in the NMR relaxation rate [29] and existence of the spin polarization from the NMR detection [26]. However, the direct observation of the inhomogeneously broadened phase (CM momentum) for the FF state is so far still absent. Very recently, it was proposed that with the SOC, due to the CM momentum, the Cooper-pair spin polarization [76,77,86,87] is induced, which can be experimentally detected through the reported magnetoelectric Andreev effect [86–88]. We discuss the triplet correlation induced by the SOC, and show that in the presence of the triplet correlation and CM momentum, the induced Cooper-pair spin polarizations exhibit totally different magnetic-field dependences between the drift-BCS and FF states. This difference between the drift-BCS and FF states, arising from the abrupt changes in order parameters and CM momenta, provides a scheme to experimentally detect and distinguish these two phases through the magnetoelectric Andreev effect, in addition to the phase transition.

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Appendix A: Soc Dependence

In this part, we address the SOC dependence of the superconducting state. The magnetic field dependences of the order parameter $\Delta_q$ and CM momentum $q = qz$ are plotted in Fig. 8a and the inset of the same figure at different SOCs, respectively. As seen from the figure, with the increase in the SOC strength, the Pauli limit is enhanced.
Fig. 8 a $\Delta q$ versus $h_B$ at different SOCs. The inset shows $q = qz$ as function of $h_B$ at different SOCs. In the calculation, $\gamma = \alpha \gamma_0$ with $\gamma_0$ being the SOC strength in Li$_2$Pd$_3$B. b $\Delta q$ versus $h_B$ at $\alpha = 1$. Triangles (Squares) in (b) the spin-flip terms of the SOC are included (removed) by setting $h_k = \gamma k$ ($h_k = \gamma k_z$) (Color figure online)

and hence the regime where the FF occurs is enlarged, in accord with the previous experiments [72–75] and prediction [61].

The enhancement of the Pauli limit is due to the spin-flip terms of the SOC (perpendicular to $h_B$). This can be seen from Fig. 8b where we plot $\Delta q$ versus $h_B$ with and without the spin-flip terms of the SOC. From the figure, it is seen that in comparison with the case at $h_k = \gamma k$ (triangles), in the situation without the spin-flip terms of the SOC ($h_k = \gamma k_z$), as shown by squares, the Pauli limit is markedly suppressed and the FF state occurs in a narrow magnetic-field regime. This is because that away from the $k_z$ axis, the spin-flip terms of the SOC couple the quasiparticle electrons $E^{e+}_{+k}$ and $E^{e-}_{-k}$ (holes $E^{h+}_{+k}$ and $E^{h-}_{-k}$) with different spin polarizations in Fig. 4a. Consequently, due to $E^{e+}_{+k} > 0$ ($E^{h-}_{-k} < 0$) shown in Fig. 4a, the regions with $E^{e-}_{-k} < 0$ ($E^{h+}_{+k} > 0$) mentioned in Sec. 3.2, i.e., the unpairing regions, are suppressed, leading to the suppressed Pauli limit.

Furthermore, it is noted that at very small SOC $\gamma = 0.1 \gamma_0$ (shown by triangles), the FF state occurs in a narrow regime $0.7 \Delta_0 < h_B < 0.8 \Delta_0$, close to the conventional FF one without SOC ($0.66 \Delta_0 < h_B < 0.8 \Delta_0$) [2]. With the increase in the SOC,
the variation of $\Delta_q$ at the phase transition between the drift-BCS and FF states is suppressed. Particularly, when we extend to a large SOC $\gamma = 5\gamma_0$ [shown by pentagons in Fig. 8a], the variation of $\Delta_q$ at the phase transition ($h_B = \Delta_0$) becomes nearly indistinguishable but still exists. This case is very similar to Fig. 4b in Ref. [62]. This suppressed variation of $\Delta_q$ at the phase transition is due to the enhancement of the CM momentum in the drift-BCS state by the SOC, leading to the close CM momenta between the drift-BCS ($h_B < \Delta_0$) and FF ($h_B > \Delta_0$) states, as shown by pentagons in the inset of Fig. 8a.

Nevertheless, with the larger SOC $\gamma > 5\gamma_0$ ($\omega_D, \Delta_0 \ll \gamma k_F$), the mean-field theory $\Delta_q = V \sum_k \langle \phi_{\uparrow k+q} \phi_{\downarrow k+q} \rangle$ in colinear space is inappropriate due to the large spin-flip terms of the SOC. In this case, pairing structure and the generation of the CM momentum should be discussed in the helix space [59,60], which is beyond the scope of our work.

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