Symmetries in the Physics of Strongly Correlated Electronic Systems

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Strongly correlated electron systems require the development of new theoretical schemes in order to describe their unusual and unexpected properties. The usual perturbation schemes are inadequate and new concepts must be introduced. In our scheme of calculations, the Composite Operator Method, is possible to recover, through a self-consistent calculation, a series of fundamental symmetries by choosing a suitable Hilbert space.

Key words: Strongly Correlated Electron Systems, Hubbard Model, Symmetries, Composite Operator Method.

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The discovery of new materials with a large variety of unusual and unexpected properties [1] has opened a new era in the physics of Condensed Matter; new theoretical schemes must be developed [2]. The most important characteristic of these new systems is a strong correlation among the electrons that makes inapplicable classical schemes based on the band picture. It is necessary to pass from a single-electron physics to a many-electron physics, where the dominant part will be the correlations among the electrons. Usual perturbation schemes are inadequate and new concepts must be introduced.

Let us consider a certain Hamiltonian

\[ H = H [\varphi_1(x), \ldots, \varphi_n(x)] \]  

the set \( \{ \varphi_i \} \) denotes band-electron fields. Due to strong correlation effects the properties of the original electrons \( \{ \varphi_i \} \) are drastically changed; new excitation modes will appear and determine most of the observed properties of the system. It is natural to identify a new set of elementary excitations \( \{ \psi_i \} \) as basis for perturbative schemes. These excitations, constructed from the original electron set (in this sense we call them composite fields), are created by the interactions among the electrons;

*It is our great pleasure to dedicate this article to Professor Ihor Stasyuk on the occasion of his 60th birthday.
therefore, their properties will be determined by the dynamics, the symmetries of the
model, the boundary conditions and must be computed in a self consistent way \[3\].
This aspect introduces a new richness into the theory and will allow us to realize the
dynamics in the proper Hilbert space where the physical symmetries are conserved.
On the other hand, we know from experiments that highly correlated systems ex-
hibit an incredible variety of behaviors. It would be very hard to describe such a
complexity using the original fields, unless the exact solution of the model is avail-
able. The presence of new excitations, composite fields, introduces into the theory
the possibility to accommodate the multifariousness of experimental properties.
A theory built on the basis of new excitation modes is, by construction, a self-
consistent theory, and a procedure must be fixed. In particular, we must answer to
the following list of questions:

1. the identification of the fundamental set;
2. the statistics and the properties of the new fields;
3. the symmetry and the dynamics in terms of the new fields;
4. the representation where the new fields are realized.

We will now try to formulate a scheme of calculations in which an answer to the
previous questions can be found. Then, by considering a particular model, we will
present a practical realization of the theoretical scheme.

The new fields $\{\psi_i\}$ are generated by the interactions among the bare fields; then,
it is naturally to choose the new set as the one which naturally appears through
the equations of motion. The evolution of the original fields is described by the
Heisenberg equation

$$i \frac{\partial}{\partial t} \varphi_i(x) = [\varphi_i(x), H] = J_i [\varphi(x)]$$

(2)

Such an equation generates new fields $\{J_i[\varphi(x)]\}$, constructed as combinations of
the bare fields. By starting from these fields and by considering the new Heisenberg
equations

$$i \frac{\partial}{\partial t} \psi_i(x) = [\psi_i(x), H]$$

(3)

we generate an infinite hierarchy of composite fields. It is naturally impossible
to solve the infinite system of equations and some truncation procedure must be
adopted. Let us consider a $n$-component field

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

(4)

and let us choose the first $n - 1$ fields such as

$$i \frac{\partial}{\partial t} \psi_i(x) = [\psi_i(x), H] = \sum_{j=1}^{i+1} \gamma_{ij} (-i \nabla) \psi_j(x) \quad 1 \leq j \leq n - 1$$

(5)
The \( n^{th} \)-field \( \psi_n(x) \) is determined by the field equation of \( \psi_{n-1}(x) \). The matrix \( \gamma(-i\nabla) \) is completely determined by the dynamics. Then, we linearize the Heisenberg equation by writing

\[
i \frac{\partial}{\partial t} \psi(x) = \varepsilon(-i\nabla) \psi(x)
\]  

where the eigenvalue or energy matrix \( \varepsilon \) is self-consistently calculated by means of the equation

\[
\varepsilon(-i\nabla) \left\langle \left\{ \psi(x,t), \psi^\dagger(y,t) \right\} \right\rangle = \left\langle \left\{ [\psi(x,t), H], \psi^\dagger(y,t) \right\} \right\rangle
\]  

(7)

The symbol \( \langle \cdots \rangle \) denotes the thermal average. Derivative operators as \( \lambda(-i\nabla) \) are defined through the relation

\[
\lambda(-i\nabla) f(x) = \int d^d y \lambda(x, y) f(y)
\]  

(8)

The rank of the energy matrix is equal to \( n \), the number of components of the vector \( \psi(x) \). When there is translational invariance we can invert Eq. (7) and it is easy to see that

\[
\varepsilon_{ij} = \left\langle \left\{ \psi_i(x), H \right\}, \psi_j^\dagger(y) \right\rangle \left\langle \left\{ \psi_i(x), \psi_j^\dagger(y) \right\} \right\rangle^{-1} = \gamma_{ij}
\]  

\begin{align}
1 \leq i \leq n-1, & \quad 1 \leq j \leq n \tag{9}
\end{align}

This approximation corresponds to the \( n \)-pole expansion of the Green’s function where finite life-time contributions are neglected. It has been proved \([4]\) that in this approximation the choice (5) for the composite operators leads to the conservation of the spectral moments. In particular, the first \( 2(n-i+1) \) spectral moments for the field \( \psi_i[1 \leq i \leq n-1] \) are conserved. This is an important property when we recall that the spectral moments are related to the spectral density function of the single-particle propagators. Also, as shown in Ref. \([4]\), the choice (5) leads to an equivalence between the \( n \)-pole approximation and the spectral density approach \([5]\), although very different results are obtained when different procedures for the self-consistency are used \([6]\).

In general the composite fields will not satisfy canonical anticommutation relations and their algebra must be calculated starting from the canonical algebra of the electron fields. Owing to this fact, the Wick theorem and the standard perturbation schemes cannot be applied. Examples of the new algebra will be presented in the second part of this article.

The properties of the new fields are fixed by a series of parameters which must be self-consistently calculated. These parameters are expressed as expectation values of composite fields. When the composite fields belong to the set they can be expressed in terms of the single-particle Green’s function and calculated by a series of coupled self-consistent equations.

However, it may happen that some of the parameters are expressed as expectation values of higher-order composite fields that do not belong to the basic set. In this case, owing to the approximation considered, the parameters are not strictly bound.
by the dynamics and there is a freedom in the procedure to fix them. At this level the powerfulness of the scheme manifests itself: one can use this freedom to choose the right representation. In the construction of a physical theory we must distinguish two levels. On one side we have the microscopic level where we are concerned with particles. The basic ingredients are the Heisenberg fields which together with the canonical commutation relations describe the dynamics. The physical laws (the equations of motion, the conservation laws, the symmetry principles) are expressed as relations among the operators. On the other side we have the macroscopic world where we are concerned with average values of operators. At the level of observation the physical laws manifest themselves as relations among matrix elements, and a suitable choice of the Hilbert space must be made. When some approximation is introduced the states are not the exact eigenstates of the Hamiltonian; the expectation values are also not the exact ones. As a consequence, the relations among the operators are generally not conserved when the expectation values are calculated. A striking example of this is the violation of the Pauli principle. A convenient way to take care of it is to operate in the representation of second quantization where the Pauli principle manifests through the algebra. It is known that in most of the approximation schemes this symmetry is violated when matrix elements are considered. Other examples of symmetries will be considered later.

The point of view adopted in this approach is that we can use the freedom in the procedure to fix the self-consistent parameters in such a way to recover the symmetries violated by the approximation. In general, a model exhibits many different symmetries and there will be a relation between the number of composite fields and the number of symmetries that can be recovered. On the physical ground one must choose which symmetries are the most important to be satisfied. In a physics dominated by strong electron correlations the Pauli principle plays a crucial role, and it is extremely relevant that the related symmetries be treated in a correct way. Therefore, in our scheme the attention is firstly put to the Pauli principle; once this is accommodated, the attention is devoted to other symmetries.

As an illustration of the scheme we shall now consider the Hubbard model. In a standard notation this model is described by the following Hamiltonian

$$H = \sum_{ij} (t_{ij} - \mu \delta_{ij}) c^\dagger(i) c(i) + U \sum_i n^\uparrow(i)n^\downarrow(i)$$

(10)

c(i), c^\dagger(i) are annihilation and creation operators for electrons at site i in the spinor notation

$$c(i) = \begin{pmatrix} c^\uparrow(i) \\ c^\downarrow(i) \end{pmatrix} \quad c^\dagger(i) = \begin{pmatrix} c^\dagger\uparrow(i) \\ c^\dagger\downarrow(i) \end{pmatrix}$$

(11)

$$n_\sigma = c^\dagger_\sigma(i)c_\sigma(i)$$ is the number operator of electrons with spin \(\sigma = (\uparrow, \downarrow)\) at the \(i^{th}\) site. \(\mu\) is the chemical potential and is introduced in order to control the band filling \(n\). For a two-dimensional squared lattice and by restricting the analysis to first nearest neighbors, the hopping matrix \(t_{ij}\) has the form

$$t_{ij} = -4t \alpha_{ij} = -4t \frac{1}{N} \sum_k e^{i\mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)} \alpha(\mathbf{k})$$

(12)
\[ \alpha(k) = \frac{1}{2} [\cos(k_x a) + \cos(k_y a)] \] (13)

\( a \) being the lattice constant. In addition to the band term, the model contains an interaction term which approximates the interaction among the electrons. In the simplest form of the Hubbard model, the interaction is between electrons of opposite spin on the same lattice site; the strength of the interaction is described by the parameter \( U \).

The electron field \( c(i) \) satisfies the Heisenberg equation

\[ i \frac{\partial}{\partial t} c(i) = -\mu c(i) - 4t c^\alpha(i) + U \eta(i) \] (14)

where

\[ c^\alpha(i) = \sum_j \alpha_{ij} c(j) \] (15)

is the electron field on the nearest neighbor sites. We see that the dynamics has generated the composite field

\[ \eta(i) = c(i) n(i) \] (16)

The Heisenberg equation for this field will generate a new higher order composite field. The process does not stop and an infinite number of composite fields will be generated. By following the procedure mentioned above, we close the hierarchy by considering \( n \) fields and we construct the vector composite field as described by Eqs. (4) and (5). For the specific case, we consider the three-component field

\[ \psi(i) = \begin{pmatrix} \psi_1(i) \\ \psi_2(i) \\ \psi_3(i) \end{pmatrix} \] (17)

where

\[ \psi_1(i) = \xi(i) = c(i) [1 - n(i)] \]
\[ \psi_2(i) = \eta(i) = c(i) n(i) \]
\[ \psi_3(i) = \pi(i) = \frac{1}{2} \sigma^\mu n_\mu(i) c^\alpha(i) + c(i) c^\dagger \sigma^\mu(n) c(i) \] (18)

\( n_\mu(i) = c^\dagger(i) \sigma_\mu c(i) \) is the charge (\( \mu = 0 \)) and spin (\( \mu = 1, 2, 3 \)) density operator for \( c \)-electrons. We are using the following notation \( \sigma_\mu = (1, \sigma), \sigma^\mu = (-1, \sigma), \sigma \) being the Pauli matrices. The composite fields \( \{17\} \) do not satisfy a canonical algebra. For example, for the first two fields

\[ \{\xi(i), \xi^\dagger(j)\} = \delta_{ij} \left[ 1 - \frac{1}{2} \sigma^\mu n_\mu(i) \right] \]
\[ \{\eta(i), \eta^\dagger(j)\} = -\delta_{ij} \frac{1}{2} \sigma^\mu(n_\mu(i) \]
\[ \{\xi(i), \eta^\dagger(j)\} = \{\xi(i), \xi(j)\} = \{\eta(i), \eta(j)\} = 0 \] (19)
This field satisfies the Heisenberg equation

\[ i \frac{\partial}{\partial t} \psi(i) = J(i) = \begin{pmatrix} -\mu \xi(i) - 4t c^\alpha(i) - 4t \pi(i) \\ (-\mu + U) \eta(i) + 4t \pi(i) \\ -\mu \pi(i) + 4t \kappa(i) - 4t \theta(i) + U \rho(i) \end{pmatrix} \] (20)

where

\[
\begin{align*}
\kappa(i) &= \frac{1}{2} \sigma^\mu c(i) c^\dagger(i) \sigma_\mu c^\alpha(i) - \frac{1}{2} \sigma^\mu c^\alpha(i) c^\dagger(i) \sigma_\mu c(i) \\
\theta(i) &= \frac{1}{2} \sigma^\mu n_\mu(i) c^\alpha(i) c^\dagger(i) c(i) - c(i) c^\dagger(\alpha(i)) c(i) + c(i) c^\dagger(\alpha(i)) c^\alpha(i) \\
\rho(i) &= \frac{1}{2} \sigma^\mu n_\mu(i) \eta^\alpha(i) + c(i) \xi^\dagger(\alpha(i)) c(i)
\end{align*}
\] (21)

According to the method given above, the equation of motion (20) is linearized as

\[ i \frac{\partial}{\partial t} \psi(i) = \sum_j \varepsilon(i,j) \psi(j) \] (22)

where the energy matrix \( \varepsilon(i,j) \) is the 3 \times 3 matrix given by

\[
\varepsilon(i,j) = \left\langle \left\{ J(i), \psi^\dagger(j) \right\} \right\rangle_{\text{E.T.}} \left\langle \left\{ \psi(i), \psi^\dagger(j) \right\} \right\rangle_{\text{E.T.}}^{-1}
\] (23)

The subscript E.T. indicates that the anticommutators are evaluated at equal time.

The physical properties can be described in terms of the thermal retarded Green’s function

\[ S(i,j) = \left\langle R \left[ \psi(i), \psi^\dagger(j) \right] \right\rangle = \frac{ia^2}{(2\pi)^3} \int_{\Omega_3} d^2k \ d\omega \ e^{i k \cdot (R_i - R_j) - i \omega (t_i - t_j)} S(k, \omega) \] (24)

where \( R \) is the usual retarded operator and the symbol \( \left\langle \cdot \cdot \cdot \right\rangle \) denotes the thermal average. By means of the linearized Heisenberg equation (22) the Fourier transform is given by

\[ S(k, \omega) = \sum_{n=1}^{3} \frac{\sigma^n(k)}{\omega - E_n(k) + i \eta} \] (25)

where the energy spectra \( E_n(k) \) are the characteristic values of the matrix \( \varepsilon(k) \), determined by the equation

\[ \sum_{m=1}^{3} a_m(k) E_n^m(k) = 0 \] (26)

The characteristic coefficients \( a_i(k) \) are defined by the following relation

\[ a_{n-k}(k) = (-)^k T_{nk} [\varepsilon(k)] \quad 0 \leq k \leq 3 \] (27)

where \( T_{nk} \) is the trace of the \( k^{th} \) order, defined as the sum of the determinants of all \( \left( \begin{array}{c} 3 \\ k \end{array} \right) \) matrices of order \( k \times k \) which can be formed by intersecting any \( k \) rows
of $\varepsilon$ with the same $k$ columns. We note that $Tr_3[\varepsilon] = Det[\varepsilon]$ and the convention $Tr_0[\varepsilon] = 1$ is used. The spectral functions are given by

$$ \sigma^n(k) = \frac{1}{b_n(k)} \sum_{m=0}^{2} E^m_n(k) \lambda^m(k) $$

(28)

where the $\lambda^n(k)$ are the $3 \times 3$ matrices:

$$ \lambda^n(k) = \sum_{m=n+1}^{3} a_m(k) \varepsilon^{m-n-1}(k) I(k) \quad 0 \leq n \leq 2 $$

(29)

and we put

$$ b_n(k) = \prod_{m=1, m \neq n}^{3} [E_n(k) - E_m(k)] $$

(30)

By standard arguments, the correlation functions can be calculated from the knowledge of the retarded Green’s function. By means of (25) we have

$$ C(i,j) = \left\langle \psi(i) \psi^\dagger(j) \right\rangle = \frac{a^2}{2(2\pi)^2} \sum_{n=1}^{3} \int d^2k e^{i k \cdot (R_i - R_j) - i E_n(k)(t_i - t_j)} \sigma^n(k) \left[ 1 + T_n(k) \right] $$

(31)

where we put

$$ T_n(k) = \tanh \left( \frac{E_n(k)}{2k_B T} \right) $$

(32)

We see that the calculation of the Green’s function requires the knowledge of the normalization matrix

$$ I(k) = \mathcal{F} \left\langle \left\{ \psi(i), \psi^\dagger(j) \right\} \right\rangle_{E.T.} $$

(33)

and of the $m$-matrix

$$ m(k) = \mathcal{F} \left\langle \left\{ J(i), \psi^\dagger(j) \right\} \right\rangle_{E.T.} $$

(34)

where $\mathcal{F}$ indicates the Fourier transform.

These quantities are calculated in appendix and depends on a series of parameters, that can be so listed:

1. external parameters as the temperature $T$ and the electron density $n = \left\langle c^\dagger(i) c(i) \right\rangle$;
2. model parameters as $U$ and $t$;
3. self-consistent parameters that can be calculated in terms of elements of the Green’s function, as $\mu$ and $\Delta$;
4. self-consistent parameters expressed as expectation values of composite fields out of the basis (17), as $p$, $I^0_{33}$, $I^0_{33}$, $m^0_{33}$, $m^0_{33}$.
For the latter a procedure of self-consistency must be fixed. In the Composite Operator Method (COM) we take advantage of this freedom and we fix the parameters in such a way that the Hilbert space has the right properties to conserve the relations among matrix elements imposed by symmetry laws. In a physics dominated by a high correlation among the electrons, the first attention should be put to the requirement that the approximation does not violate the symmetry required by the Pauli principle. Let us consider the correlation matrix \( \langle \xi(i) \eta^\dagger(i) \rangle \); when we take equal points the algebra leads to the following relations

\[
\langle \xi(i) \eta^\dagger(i) \rangle = 0
\]

\[
\langle \xi(i) \pi^\dagger(i) \rangle = \langle c^\alpha(i) \xi^\dagger(i) \rangle
\]

\[
\langle \eta(i) \pi^\dagger(i) \rangle = -\langle c^\alpha(i) c^\dagger(i) \rangle
\]

among matrix elements of the Green’s function. These relations constitute a set of coupled self-consistent equations which will be satisfied by an appropriate choice of the parameters.

The recovery of the Pauli principle does not exhaust all the degrees of freedom and we have place to accommodate other symmetries. An intrinsic symmetry of the Hubbard model is the pseudospin \( SU(2) \) symmetry, which is nothing also that the invariance under the particle-hole transformation. The generators of this transformation are given by the total pseudospin operators

\[
P^+ = \sum_i (-)^i c^\dagger(i) c^\dagger(i)
\]

\[
P^- = \sum_i (-)^i c(i) c^\dagger(i)
\]

\[
P_z = \frac{1}{2} \sum_i [n(i) - 1]
\]

These operators satisfy the \( SU(2) \) algebra

\[
[P^+, P^-] = 2P_z \quad [P^\pm, P_z] = \mp P^\pm
\]

and the Heisenberg equations

\[
i \frac{\partial}{\partial t} P^\pm = \pm (2\mu - U) P^\pm
\]

\[
i \frac{\partial}{\partial t} P_z = 0
\]

Let us consider the thermal retarded Green’s function

\[
P^{+-}(t - t') = \langle R \left[ P^+(t) P^-(t') \right] \rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} P^{+-}(\omega)
\]

By means of the equation of motion \([38]\) we obtain for the correlation function

\[
\frac{1}{N} \langle P^+(t) P^-(t') \rangle = \frac{(n-1)e^{-i(2\mu-U)(t-t')}}{1 - e^{-\beta(2\mu-U)}}
\]
This is an exact result which relates the pseudospin correlation function to the particle number \( n \) and it is a manifestation of the intrinsic symmetry.

Another important symmetry is given by the conservation of the current density. By defining the charge \( \rho(i) \) and current \( j(i) \) densities as

\[
\rho(i) = e c^\dagger(i) c(i) \tag{41}
\]
\[
j(i) = -i e a^2 c^\dagger(i) \left( \nabla - \nabla \right) c(i) \tag{42}
\]

it is immediate to obtain by means of the Heisenberg equation (14) the conservation law

\[
\nabla \cdot j(i) + \frac{\partial}{\partial t} \rho(i) = 0 \tag{43}
\]

The symmetry content of the algebraic equation (43) manifests at level of observation as relations among matrix elements once a choice of the physical space of states has been made. Indeed, by defining the causal charge and current functions as

\[
\chi_{ab}(i, j) = \langle T[g_a(i) g_b(i)] \rangle = \frac{i a^2}{(2\pi)^3} \int d^2k \, d\omega \, e^{i k \cdot (R_i - R_j) - i \omega (t_i - t_j)} \chi_{ab}(k, \omega) \tag{44}
\]

where

\[
g_a(i) = \begin{cases} 
\rho(i) & \text{for } a = 0 \\
n_x(i) & \text{for } a = x \\
n_y(i) & \text{for } a = y 
\end{cases} \tag{45}
\]

we can derive a series of Ward-Takahashi identities connecting current-current, charge-current and charge-charge propagators. One of those reads as follows

\[
i a \omega \chi_{00}(k, \omega) = \left[ 1 - e^{-i k_x a} \right] \chi_{x0}(k, \omega) + \left[ 1 - e^{-i k_y a} \right] \chi_{y0}(k, \omega) \tag{46}
\]

In the static approximation of the Composite Operator Method the charge, current, spin, pseudospin correlation function can be connected to convolutions of single-particle propagators. This occurrence is related to a linearized dynamics together with the choice of occupation dependent electronic excitations as basic fields [3]. Once these calculations have been performed, Eqs. (36), (40) and (46) constitute a set of five coupled self-consistent equations which can be satisfied by an appropriate choice of the five parameters \( p, I_{33}^0, I_{33}^\alpha, m_{33}^0, m_{33}^\alpha \).

We thus have a scheme of calculations in which it is possible to recover, through a self-consistent calculation, a series of fundamental symmetries by choosing a suitable Hilbert space.

Detailed calculations will be presented elsewhere.
Appendices

A. The normalization matrix

From the definition (33) and by means of the canonical algebra for the $c$-electrons it is straightforward to see that for a paramagnetic ground state the normalization matrix has the following expression

$$I(k) = \begin{pmatrix} I_{11}(k) & 0 & I_{13}(k) \\ 0 & I_{22}(k) & I_{23}(k) \\ I_{13}(k) & I_{23}(k) & I_{33}(k) \end{pmatrix}$$  \hspace{1cm} (A.1)$$

with

$$
I_{11}(k) = 1 - \frac{n}{2} \\
I_{13}(k) = \Delta + (p - I_{22}) \alpha (k) \\
I_{22}(k) = \frac{n}{2} \\
I_{23}(k) = -\Delta - p \alpha (k) \\
I_{33}(k) = I_{33}^0 + \alpha (k) I_{33}^\alpha
$$ \hspace{1cm} (A.2)

The quantities introduced in (A.2) are so defined

$$
n = \langle c^\dagger (i) c(i) \rangle = 2 \left[ 1 - \langle \xi (i) \xi^\dagger (i) \rangle - \langle \eta (i) \eta^\dagger (i) \rangle \right]$$

$$
\Delta = \langle \xi^\alpha (i) \xi^\dagger (i) \rangle - \langle \eta^\alpha (i) \eta^\dagger (i) \rangle$$

$$
p = \frac{1}{4} \left[ \langle n^\alpha (i) n^\mu (i) \rangle - \langle [\xi^\dagger (i) \eta^\dagger (i)]^\alpha \eta^\mu (i) \xi^\dagger (i) \rangle \right]
$$ \hspace{1cm} (A.3)

$n$ is the average number of electrons per site; $\Delta$ and $p$ are static correlation function between nearest neighbor sites. In particular, the parameter $p$ describes intersite charge, spin and pair correlations. In the calculation of $I_{33}(k)$ only the nearest neighbor contributions have been retained

$$\langle \{ \pi (i), \pi^\dagger (j) \} \rangle \approx \delta_{ij} I_{33}^0 + \alpha_{ij} I_{33}^\alpha$$ \hspace{1cm} (A.4)

B. The $m$-matrix

At first we note that time translational invariance requires the $m$-matrix be hermitian

$$m(k) = \begin{pmatrix} m_{11}(k) & m_{12}(k) & m_{13}(k) \\ m_{12}(k) & m_{22}(k) & m_{23}(k) \\ m_{13}(k) & m_{23}(k) & m_{33}(k) \end{pmatrix}$$ \hspace{1cm} (B.1)
From the definition (34) and by making use of the expressions (20–21) for the source current it is possible to calculate

\begin{align}
m_{11}(k) &= -[\mu + 4t \alpha(k)] I_{11} - 4t I_{13} \\
m_{12}(k) &= -4t \alpha(k) I_{22} - 4t I_{23} \\
m_{13}(k) &= -[\mu + 4t \alpha(k)] I_{13} - 4t [\alpha(k) I_{23} + I_{33}] \\
m_{22}(k) &= -(\mu - U) I_{22} + 4t I_{23} \\
m_{23}(k) &= -(\mu - U) I_{23} + 4t I_{33} \\
m_{23}(k) &= m^0_{33} + \alpha(k) m^\alpha_{33} 
\end{align}

(B.2)

In the calculation of \(m_{33}(k)\) only the nearest neighbor contributions have been retained

\[ \langle \{ J_3(i), \pi^\dagger(j) \} \rangle \simeq \delta_{ij} m^0_{33} + \alpha_{ij} m^\alpha_{33} \]  

(B.3)

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