Exact Relativistic Gravitational Field of a Stationary Counterrotating Dust Disk

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Abstract

Disks of collisionless particles are important models for certain galaxies and accretion disks in astrophysics. We present here a solution to the stationary axisymmetric Einstein equations which represents an infinitesimally thin dust disk consisting of two streams of particles circulating with constant angular velocity in opposite directions. These streams have the same density distribution but their relative density may vary continuously. In the limit of only one component of dust, we get the solution for the rigidly rotating dust disk previously given by Neugebauer and Meinel, in the limit of identical densities, the static disk of Morgan and Morgan is obtained. We discuss the Newtonian and the ultrarelativistic limit, the occurrence of ergospheres, and the regularity of the solution.

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In astrophysics thin disks of collisionless matter, so called dust, are discussed as models for certain galaxies (see e.g. [1]) or for accretion disks. A fully relativistic treatment of these models is necessary if a black hole is present since black holes are genuinely relativistic objects. But also the exact treatment of dust disks without central object would provide deep insight both in the mathematical structure of the field equations and in the physics of rapidly rotating relativistic bodies since dust disks can be viewed as a limiting case for extended matter sources. Corresponding exact solutions hold globally – in the vacuum and in the matter region – and can thus provide physically realistic testbeds for numerical codes. Since Newtonian dust disks are known to be unstable and since there are hints by numerical work (see e.g. [2]) that the same holds in the relativistic case, such solutions could be taken as exact initial data for numerical collapse calculations. Whereas the Newtonian theory of such disks is well established (see [1] and references given therein), the same holds in the relativistic case only for static disks which can be interpreted as consisting of two counter-rotating streams of matter with vanishing total angular momentum. The first disk of this type was considered by Morgan and Morgan [3]. Infinitely extended dust disks with finite mass were studied by Bičák, Lynden-Bell and Katz [1] in the static case and by Bičák and Ledvinka [4] in the stationary case. The first to construct the exact solution for a finite stationary dust disk were Neugebauer and Meinel [6] who gave the solution for the rigidly rotating dust disk which was first treated numerically by Bardeen and Wagoner [2]. They solved the corresponding boundary value problem for the Einstein equations with the help of a corotating coordinate system.

In this letter we present a class of new disk solutions to the Ernst equation where such a coordinate system cannot be used. The disks of finite radius \( \rho_0 \) consist of two counter-rotating components of dust with respective density \( \sigma^\pm(\rho) \). The angular velocity of both streams of particles is of the same constant absolute value \( \Omega \) but of different sign. The relative density \( \gamma = (\sigma^+ - \sigma^-)/(\sigma^+ + \sigma^-) \) is a constant with respect to \( \rho \) and \( \zeta \) which varies between one, the rigidly rotating dust disk [6], and zero, the static Morgan and Morgan disk [3]. Interestingly, observations [7] indicate that the galaxy NGC 4550 is built from two counter-streaming stellar components. We are able to give the explicit solution for the above configuration in dependence of the three parameters \( \rho_0, \Omega \) and \( \gamma \) which parametrize a Riemann surface of genus 2. We discuss physically interesting features like the static limit, the Newtonian regime and the ultrarelativistic limit. We study the occurrence of ergospheres and the absence of singularities in the allowed range of the physical parameters.

Newtonian dust disks

It is instructive to consider first the Newtonian case where the gravitational field is given by a scalar potential \( U \) which is a solution to the Laplace equation \( \Delta U = 0 \). The potential \( U \) has to be everywhere regular except at the disk where the balancing of the centrifugal and the gravitational force leads to boundary values for \( U \). We use dimensionless \( (\rho_0 = 1) \) cylindrical coordinates \( (\rho, \zeta, \phi) \) and put the disk in the \( \zeta = 0 \)-plane which leads at the disk to

\[
U_\rho(\rho, 0) = \Omega^2 \rho. \tag{1}
\]

For constant \( \Omega \), this can be easily integrated to give \( U(\rho) = U_0 + \frac{1}{2} \Omega^2 \rho^2 \) where the constant
\( U_0 \) is related in the relativistic case to the central redshift. Notice that there are no effects due to counter-rotation in the Newtonian case since \( \Omega \) enters (1) quadratically. The solution to this problem can be constructed e.g. with Riemann-Hilbert techniques (see [8]) which lead in the case of the Laplace equation to a potential of the form

\[
U(\rho, \zeta) = -\frac{1}{4\pi i} \int_\Gamma \frac{\ln G(\tau) d\tau}{\sqrt{(\tau - \zeta)^2 + \rho^2}}
\]

which is a function on the Riemann surface of genus zero given by \( \mu_0^2(\tau) = (\tau - \zeta)^2 + \rho^2 \); here \( \ln G(\tau) \) is an analytic function and \( \Gamma \) is the covering of the imaginary axis in the upper sheet between \(-i\) and \(i\). In the Newtonian case, the situation is equatorially symmetric which leads at the disk to

\[
U(\rho, 0) = -\frac{1}{2\pi i} \int_0^{i\rho} \frac{\ln G(\tau) d\tau}{\sqrt{\tau^2 + \rho^2}}.
\]

This is an Abelian integral equation for \( \ln G \) since the left hand side is given by the integral of (1). It has the solution \( \ln G = 4\Omega^2 (\tau^2 + 1) \) where a singular ring at the rim of the disk is excluded by the condition \( G(\pm i) = 1 \). The condition implies that \( U \) is continuous at the rim, and it fixes the constant \( U_0 = -\Omega^2 \). This shows that the solution indeed only depends on the two parameters \( \rho_0 \) and \( \Omega \).

The relativistic case

The metric describing the exterior (i.e. the vacuum region) of an axisymmetric, stationary rotating body can be written in the Weyl–Lewis–Papapetrou form (see [9])

\[
ds^2 = -e^{2U}(dt + a d\phi)^2 + e^{-2U} \left(e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2\right),
\]

where \( \partial_t \) and \( \partial_\phi \) are two commuting asymptotically timelike respectively spacelike Killing vectors. In this case, the vacuum field equations are equivalent to the Ernst equation for the potential \( f \),

\[
f_{zz} + \frac{1}{2(z + z)}(f_z + f_{z}) = \frac{2}{f + f}f_z f_{\bar{z}} ,
\]

where \( f = e^{2U} + ib \) and the real function \( b \) is related to the metric functions via \( b_z = -(i/\rho)e^{4U}a_z \). Here the complex variable \( z \) stands for \( z = \rho + i\zeta \). With a solution of the Ernst equation, the metric function \( U \) follows directly from the definition of the Ernst potential whereas \( a \) and \( k \) can be obtained from \( f \) via quadratures.

The complete integrability of this equation allows in principle to give explicit solutions for boundary value problems. For two-dimensionally extended matter sources, e.g. infinitesimally thin disks, one gets global solutions since one encounters ordinary differential equations in the matter region which provide boundary data for the vacuum equations. Riemann-Hilbert problems (see e.g. [10]) generate solutions to the Ernst equation with free functions which have to be determined by the boundary data. However this is only possible uniquely in the case of Cauchy data, which would lead in general to singular solutions of the elliptic field equations. In [6] this problem was circumvented by the use of a corotating coordinate
system, a technique that is not applicable in the case of differential rotation or for several matter components.

In [10] we have shown that the matrix Riemann-Hilbert problem is gauge equivalent to a scalar problem on a Riemann surface which can be explicitly solved in the case of rational boundary data, i.e. belongs to Korotkin’s [11] finite gap solutions. The explicit form of the solution allows in principle to solve boundary value problems directly. The branch points of the Riemann surface are used as a discrete degree of freedom. Since the solutions already have the required regularity properties (see [12, 13]), there can be no problems with singularities as in the direct approach in the matrix case. The purely algebro-geometric treatment has the further advantage that it is not limited by the possibility to introduce special coordinate systems.

The simplest surface leading to equatorially symmetric Ernst potentials is of genus 2 and of the form $\nu^2(K) = (K - P_0)(K - \bar{P}_0)(K^2 - E_1^2)(K^2 - \bar{E}_1^2)$ where $P_0 = -iz$ is a moving branch point related to the physical coordinates whereas the $z$–independent branch points are given by $E_1 = -(\alpha_1 + i\beta_1)$ with positive $\alpha_1$ and $\beta_1$. We introduce the standard quantities associated with a Riemann surface, with the cut system of figure 1, namely the 2 differentials of the first kind $d\omega_i$ normalized by $\oint_{a_i} d\omega_j = 2\pi i \delta_{ij}$, the Abel map $\omega_i(P) = \int_{P_0}^P d\omega_i$ which is defined uniquely up to periods, the Riemann matrix $\Pi$ with the elements $\pi_{ij} = \oint_{b_i} d\omega_j$, and the theta function $\Theta(z) = \sum_{N \in \mathbb{Z}^2} \exp \left\{ \frac{1}{2} \langle \Pi N, N \rangle + \langle z, N \rangle \right\}$. The normalized (all $a$–periods zero) differentials of the third kind with poles in $p$ and $q$ and residues $+1$ respectively $-1$ are denoted by $d\omega_{pq}$. Then one can show that

$$\tilde{f}(\rho, \zeta) = \frac{\Theta(\omega(\infty^+) + u)}{\Theta(\omega(\infty^-) + u)} \exp \left\{ \frac{1}{2\pi i} \oint_{\Gamma} \ln G(\tau) d\omega_{\infty^+\infty^-}(\tau) \right\} \right),$$

where the two-dimensional vector $u$ has the components $u_i = \frac{1}{2\pi i} \int_{\Gamma} \ln G d\omega_i$ and $\Gamma$ as in the Newtonian case, is a solution to the Ernst equation which is everywhere regular except at the disk if $\Theta(\omega(\infty^-) + u) \neq 0$. We can also give the metric function $a$ (see [11] and [13]) and
Notice that solutions of the form (6) are characterized by one real valued function $G$ and two real numbers, e.g. $\alpha$ and $\beta$ defined by $E_1 =: \alpha + i \beta$ where $\beta$ has to be positive.

Dust disks within general relativity are best described with the help of Israel’s covariant formalism [15] where the regions $\zeta > 0$ and $\zeta < 0$ are matched at the hypersurface $\zeta = 0$. The surface stress-energy tensor of the disk, for counter-rotating dust $S_{ab} = \sigma^+ v_a^+ v_b^+ + \sigma^- v_a^- v_b^-$ where $v^\pm$ is the velocity of the respective component of the the two-dimensional dust, is related to the jump of the extrinsic curvature at the disk. If we put $Z = (a + \gamma/\Omega)e^{2U}$, $\lambda = 2\Omega^2 e^{-2U_0}$ and $\delta = (1 - \gamma^2)/\Omega^2$, this leads via the Einstein equations to the boundary conditions

$$
\left( e^{2U} \right)_\zeta = \frac{Z^2 + \rho^2 + \delta e^{4U}}{2\rho Z} b_\rho, \quad b_\zeta = \frac{e^{2U}}{Z} - \frac{Z^2 + \rho^2 + \delta e^{4U}}{2\rho Z} \left( e^{2U} \right)_\rho.
$$

The second equation may be integrated to give $Z^2 - \rho^2 + \delta e^{4U} = \frac{2}{3} e^{2U}$. Notice that $\lambda$ can be viewed as a parameter that indicates how relativistic the situation is: the larger it is, the larger is the angular velocity and the central redshift.

We can now state the following theorem which is the main result of this letter.

**Theorem:**
The solution of the boundary value problem (7) for the Ernst equation is given by an Ernst potential of the form (6) with

$$
\alpha = -1 + \frac{\delta}{4}, \quad \beta^2 = \frac{1}{\lambda^2} + \delta - \frac{\delta^2}{4}
$$

and

$$
G(\tau) = \frac{\sqrt{(\tau^2 - \alpha)^2 + \beta^2 + \tau^2 + 1}}{\sqrt{(\tau^2 - \alpha)^2 + \beta^2 - (\tau^2 + 1)}}.
$$

We briefly sketch the proof which uses in principle only fundamental theorems of algebraic geometry which can be found in the standard literature (see e.g. [16, 17]). The main idea is to establish the relations between the real and the imaginary part of the Ernst potential enforced by the given Riemann surface independently of $G^\dagger$. This selects the class of boundary value problems that can be solved on a certain Riemann surface. We note that the steps below can all be performed uniquely what implies that the form of the solution is unique at least among genus 2 solutions.

**Proof:**

1. The Jacobi inversion theorem ensures that the equations $\omega(X) - \omega(D) = u$ where $X = X_1 + X_2$ and $D = E_1 + (-E_1)$ are divisors can always be solved for the divisor $X$. The divisor $X$ is called non-special if the solution is unique (how to overcome problems with special divisors is discussed in [14]). If we introduce this divisor in (4), we end up with

The simplest example for this concept is provided by the function $e^{iw}$ which is the analogue of theta functions on the Riemann surface of genus zero. In this case we have of course the simple relation for the real and imaginary part that $\cos^2 w + \sin^2 w = 1$ for all $w$. 

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the algebraic formulation of the hyperelliptic solutions in [18] and [19].

2. The reality of the $u_i$ leads to the condition $\omega(X) + \omega(\bar{D}) = \omega(\bar{X}) + \omega(D)$. Abel’s theorem implies that this condition is equivalent to the existence of a rational function on $\Sigma_2$ with zeros in $X + \bar{D}$ and poles in $\bar{X} + D$. We can thus express $X$ via $be^{-2U}$ and a second real quantity as the solution of a system of algebraic equations.

3. If we introduce the divisor $X$ in the expression for the metric function $a$ ([11, 10]) we can use so-called root functions (see [17]) to express $X$ via $a$ and $be^{-2U}$ alone.

4. Differentiating with respect to the physical coordinates and using the equatorial symmetry in the equatorial plane, we arrive at a set of equations which contain only $e^{2U}$, $a$, $b$ and their derivatives. Using the boundary conditions (7) together with the definition of $b$, we can show that these equations are identically satisfied.

5. The constant $\beta$ is used as an integration constant in the integration of the second equation in (7).

6. The function $G$ is determined as in the Newtonian case as the solution of an Abelian integral equation, e.g. the equation following from $\omega_1(X) - \omega_1(D) = u_1$. The regularity condition $G(\pm i) = 1$ then fixes $\alpha$.

Discussion:
The above disk solutions have several interesting limiting cases. The simplest is of course $\delta = 0$, the rigidly rotating dust disk [6], here in the form [12]. Mathematically more intriguing is the limit of the Morgan and Morgan disk. It was argued in [12] that the hyperelliptic solutions of the form (6) are all non-static. This holds of course only in the case of regular Riemann surfaces. A real Ernst potential, i.e. a solution which belongs to the static Weyl class, is only possible on a degenerated Riemann surface where the branch points coincide. This is a well-known limiting case of algebro-geometric solutions of integrable equations in which the soliton solutions are obtained. Since the solutions via Bäcklund transformations correspond to the ‘solitonic’ solutions of the Ernst equation, Korotkin [11] was able to obtain the Kerr solution in such a limit.

The problem of the parametrisation of the solution via $\lambda$ and $\delta$ which is enforced by the solution process is that the dependence on the physical parameters $\Omega$ and $\gamma$ is concealed by a factor $e^{2U_0}$ which is itself a hyperelliptic function of $\delta$ and $\lambda$. In the static limit, we get however the simple relation $\delta = 2(1 + \sqrt{1 + 1/\lambda^2})$ where $\beta = 0$. This means that the branch points coincide pairwise on the real axis since $\alpha > 0$ in this case. The Ernst potential thus simplifies to

$$U(\rho, \zeta) = -\frac{1}{4\pi} \int_{\Gamma} \frac{\ln G(\tau) d\tau}{\sqrt{(\tau - \zeta)^2 + \rho^2}}$$

where $G = 1 - 4(\tau^2 + 1)/\delta$, i.e. the expected potential of the Morgan and Morgan disk in the form [20]. This disk thus provides a nice example where a stationary solution of the Einstein equations for an extended body is linked continuously to a static solution via a parameter.

The Newtonian limit of the disks is reached for $\lambda \to 0$. In this case, we have $G \approx 1$ which implies that the solution approaches Minkowski spacetime as expected. Since $\ln G \approx$
2\lambda(\tau^2 + 1) and \(U\) is given by (1), we get the above Newtonian limit. As expected, counter-rotation i.e. \(\delta\) does not play a role in this case. The other extreme limit is the ultrarelativistic limit. The corresponding limit of the Morgan and Morgan disk is of course static. It is reached for \(\delta \to 4\) or \(\lambda \to \infty\) and describes a disk where the matter streams rotate with the velocity of light (see e.g. [20]). The density of the matter diverges at the origin where the central redshift diverges, too, but the mass of the disk remains finite. For the non-static disks, the exterior of the disk can be interpreted as the extreme Kerr solution as in [2, 12]. The disk can thus be viewed as being hidden behind the horizon of the Kerr metric. The exact form of these extreme configurations will be the subject of further investigation.

In [12, 10] we have given the necessary conditions for the occurrence of ergospheres. Numerically we find that the ergospheres always form first in the disk. If more matter is counter-rotating, ergospheres are less likely to appear since they are due to gravitomagnetic effects which vanish in the static limit of two counter-rotating streams with identical density. The necessary condition for an ergosphere to reach the rim of the disk is that \(\lambda\) takes the value \(2/(1 - \delta)\) which is only possible for \(\delta < 1\). For larger values of \(\delta\), ergospheres thus cannot meet the equatorial plane outside the disk. In figure 2 we have plotted for given \(\delta\) the values \(\lambda, \rho\) of the common points of the ergosphere and the disk. For small values of \(\lambda\), there will be no ergospheres since we are close to the Newtonian regime. For increasing \(\lambda\), there will form a ring-like ergosphere at some radius \(\rho\). If \(\lambda\) is increased further, the then roughly toroidal ergosphere hits the disk at two values of \(\rho\). The larger value of \(\rho\) reaches the rim of the disk at a finite value of \(\lambda\) for \(\delta < 0\) or is strictly smaller than 1 for \(\delta \geq 1\). The inner radius of the ergosphere will hit the axis in the ultrarelativistic limit for \(\lambda = \lambda_c\).

It makes little sense to increase the parameter \(\lambda\) beyond this critical value \(\lambda_c\) where the solution still formally exists. These regions are most probably non-physical since the solutions may have negative ADM-mass (see [12]) and have singularities. The allowed parameter
range is thus $0 < \lambda < \lambda_c$ and $0 \leq \delta \leq 2(1 + \sqrt{1 + 1/\lambda^2})$. It can be shown that the regularity condition $\Theta(\omega(\infty^-) + u) \neq 0$ is always fulfilled in the allowed parameter range.

Thus we have shown that it is possible by purely algebro-geometric methods to construct a solution to the Ernst equation which describes a non-static counter-rotating dust disk. The fact that this is possible without the use of special coordinate systems gives rise to the hope that astrophysically interesting objects like dust disks with differential rotation or with a central black hole or disks with surface tensions can be constructed in selected cases. The open question is whether the resulting algebraic equations can still be handled analytically.

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