Uncertainty Relations in the Presence of Quantum Memory for Mutually Unbiased Measurements

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In [1], uncertainty relations in the presence of quantum memory was formulated for mutually unbiased bases using conditional collision entropy. In this paper, we generalize their results to the mutually unbiased measurements. Our primary result is an equality between the amount of uncertainty for a set of measurements and the amount of entanglement of the measured state, both of which are quantified by the conditional collision entropy. Implications of this equality relation are discussed. We further show that similar equality relation can be obtained for generalized symmetric informationally complete measurements. We also derive an interesting equality for arbitrary orthogonal basis of the space of Hermitian, traceless operators.

I. INTRODUCTION

Uncertainty relations form a central part of quantum mechanics. They impose fundamental limitations on our ability to simultaneously predict the outcomes of non-commuting observables. Different approaches have been proposed to quantify these relations. The original formulation is given by Heisenberg [2] in terms of standard deviations for momentum and position operators. His result is then generalized to arbitrary observables [3]. Later it is recognized that one can express uncertainty relations in terms of entropies [4–6]. In this approach, entropy functions like the Shannon and Rényi entropies of an observable are used to quantify uncertainty (Ref. 7 is a nice survey on this topic).

Mutually unbiased bases (MUB) have many applications in quantum information theory: quantum error correction codes [8], quantum cryptography [9], and entanglement detection [10] (see review [11] and references therein). There has been of great effort and research interest in constructing the complete set of MUB. However, the existence problem of complete set of mutually unbiased bases for arbitrary dimension is still open. In [12], the authors proposed the concept of mutually unbiased measurements (MUM). These measurements contain the complete set of MUBs as a special case while the measurement operators need not be rank one projectors. They proved that a complete set of mutually unbiased measurements can be built explicitly for arbitrary finite dimension.

Uncertainty relations in the presence of quantum memory was formulated for MUBs using conditional collision entropy in [1], in which the authors gave an exact relation between the amount of uncertainty as measured by the guessing probability and the amount of entanglement as measured by the recoverable entanglement fidelity. As MUMs are natural generalizations of MUBs, one may naturally conjecture that similar uncertainty relations hold for MUMs. In this paper, we show that this is indeed the case: we generalize their results to the set of MUMs. The main result is an equality between the amount of uncertainty for a set of measurements and the amount of entanglement of the measured state, both of which are quantified by the conditional collision entropy.

The rest of this paper is organized as follows. In Sec. II, we establish the notation and briefly review the concepts of MUMs, conical 2-design, and conditional collision entropy. In Sec. III, we present our central result — an equality between the amount of uncertainty for a complete set of MUMs and the amount of entanglement of the measured state, both of which are quantified by the conditional collision entropy. We discuss several implications of this equality relation. We further show that an equality relation can be obtained for generalized symmetric informationally complete measurements in Sec. IV. We conclude in Sec. V. Some proofs are given in the Appendix.

II. PRELIMINARIES

A quantum system $A$ is associated to a Hilbert space $\mathcal{H}_A$ with some fixed orthonormal basis $\{ |s\rangle \}$. Throughout this article, we assume $\mathcal{H}_A$ is $d$-dimensional. If the underlying system is clear from context, we simply write the space as $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators, by $\mathcal{P}(\mathcal{H})$ the set of positive semidefinite operators, and by $\mathcal{D}(\mathcal{H})$ the set of density operators on $\mathcal{H}$. We use $\mathbb{1}_A$ to represent the identity operator and $\pi_A$ to represent the maximally mixed operator of system $A$. Systems with the same letter are assumed to be isomorphic: $A' \cong A$. We denote by $|\Psi_{AA'}\rangle$ the normalized maximally entangled state on system $AA'$, which has the form $|\Psi_{AA'}\rangle = \sum_s |ss\rangle$. For simplicity, we let $[d] = \{1, \cdots , d\}$. 

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A. Mutually unbiased measurements

Two orthonormal bases $B^{(1)} = \{ |\psi_x^{(1)}\rangle \}_{x \in [d]}$ and $B^{(2)} = \{ |\psi_x^{(2)}\rangle \}_{x \in [d]}$ of $\mathcal{H}$ are said to be mutually unbiased if

$$|\langle \psi_x^{(1)} | \psi_y^{(2)} \rangle| = \frac{1}{\sqrt{d}} \quad \forall x, y \in [d].$$

Intuitively, if $\mathcal{H}$ is prepared in an eigenstate of $B^{(1)}$ and measured in $B^{(2)}$, the measurement outcome is completely random. A set of orthonormal bases $\{B^{(\theta)}\}_{\theta \in \Theta}$ forms a set of MUBs if these bases are pairwise unbiased.

In a $d$-dimensional Hilbert space there are at most $d + 1$ pairwise unbiased bases [11]. This set is called a complete set of MUBs. It is open whether complete set of MUBs exists for arbitrary $d$.

By generalizing the notion of “unbiasedness”, the concept of mutually unbiased measurements (MUM) is introduced [12]. Two POVMs $P^{(1)} = \{ P_x^{(1)} \}_{x \in [d]}$ and $P^{(2)} = \{ P_x^{(2)} \}_{x \in [d]}$ are mutually unbiased if the following conditions are satisfied for all $x, x' \in [d], \theta = 1, 2$:

$$\text{Tr} \left[ P_x^{(\theta)} \right] = 1,$n
$$\text{Tr} \left[ P_x^{(1)} P_x^{(2)} \right] = \frac{1}{d},$$
$$\text{Tr} \left[ P_x^{(\theta)} P_{x'}^{(\theta)} \right] = \delta_{x,x'} \kappa + (1 - \delta_{x,x'}) \frac{1 - \kappa}{d - 1},$$

where the efficiency parameter $\kappa$ satisfies $1/d < \kappa \leq 1$. $\kappa$ determines how close the measurements operators are to rank-one projectors: $\kappa = 1$ if and only if $P^{(1)}$ and $P^{(2)}$ form two MUBs. Unlike the existence problem of a complete set of MUBs, there exists a general construction of complete set of MUMs for arbitrary finite $d$ [12].

Let $\{ F_k \}_{k \in \{d-1\}}$ be an orthogonal basis for the space of Hermitian, traceless operators acting on $\mathcal{H}$. We regard these operators as elements of a $(d + 1) \times (d - 1)$ block matrix

$$
\begin{pmatrix}
F_1 & F_2 & \cdots & F_{d-1} \\
F_d & F_{d+1} & \cdots & F_{2(d-1)} \\
\vdots & \vdots & \ddots & \vdots \\
F_{d(d-1)+1} & F_{d(d-1)+2} & \cdots & F_{d(d+1)(d-1)}
\end{pmatrix}.
$$

We relabel the block matrix by a tuple $(x, \theta) : x \in [d - 1], \theta \in [d + 1]$ based on their (column, row) location

$$
\begin{pmatrix}
F_{1,1} & F_{2,1} & \cdots & F_{d-1,1} \\
F_{1,2} & F_{2,2} & \cdots & F_{d-1,2} \\
\vdots & \vdots & \ddots & \vdots \\
F_{1,d+1} & F_{2,d+1} & \cdots & F_{d-1,d+1}
\end{pmatrix}.
$$

Based on $\{ F_x^{(\theta)} \}$, define the following $d(d + 1)$ operators

$$F_x^{(\theta)} = \begin{cases} F^{(\theta)} - \frac{d(d + \sqrt{d})}{\sqrt{d}}F_x^{(\theta)}, & x \in [d - 1], \\ (1 + \sqrt{d})F^{(\theta)}, & x = d, \end{cases}$$

where $F^{(\theta)} = \sum_{x=1}^{d-1} F_{x,x}^{(\theta)}$. Then, the operators

$$P_x^{(\theta)} = \frac{1}{d} I + t F_x^{(\theta)}, \quad x \in [d], \theta \in [d + 1]$$

form a complete set of MUMs, with the parameter $t$ chosen such that $P_x^{(\theta)} \geq 0$. The efficiency parameter $\kappa$ is then given by

$$\kappa = \frac{1}{d} + t^2 \left( 1 + \sqrt{d} \right)^2 (d - 1).$$

B. Conical 2-design

A complex projective 2-design is a set of vectors $\{ |\psi_x\rangle \}_{x \in \Sigma}$ (not necessarily normalized) lying in $\mathcal{H}_A$ such that

$$\frac{1}{|\Sigma|} \sum_{x \in \Sigma} |\psi_x\rangle \langle \psi_x| \otimes |\psi_x\rangle \langle \psi_x| = \frac{1}{d(d+1)} (I_{AA'} + F_{AA'}),$$

where $F_{AA'}$ is the swap operator defined as

$$F_{AA'} = \sum_{s,t} |s \rangle \langle t| \otimes |t \rangle \langle s|.\quad (4)$$

Complex projective designs play an important role in quantum information theory. A best known example is the complete set of MUBs. Let $\{B^{(\theta)}\}_{\theta \in [d+1]}$ be a complete set of MUBs (if exists) on $\mathcal{H}_A$. It is proved in [13] that such a set generates a complex projective 2-design:

$$\frac{1}{d+1} \sum_{x=1}^{d+1} \sum_{\theta=1}^{d} |\psi_x^{(\theta)}\rangle \langle \psi_x^{(\theta)}| \otimes |\psi_x^{(\theta)}\rangle \langle \psi_x^{(\theta)}| = I_{AA'} + F_{AA'}.\quad (5)$$

A complex projective 2-design consists of rank-one projectors. In [14] the authors introduce a generalization which shares properties with complex projective 2-design, but in which the projectors are arbitrary positive semidefinite operators. A conical 2-design is a set of positive semidefinite operators $\{A_x\}_{x \in \Sigma}$ in $\mathcal{H}_A$ satisfying

$$\sum_{x \in \Sigma} A_x \otimes A_x = k_+ I_{AA'} + k_- F_{AA'}$$

for some $k_+ \geq k_- \geq 0$. As MUMs are generalizations of MUBs, we wish similar property (that complete set of MUBs forms a complex projective 2-design) holds for MUMs. In [14], it is proved that a complete set of MUMs forms a conical 2-design

$$\frac{1}{d+1} \sum_{\theta=1}^{d+1} \sum_{x=1}^{d} P_x^{(\theta)} \otimes P_x^{(\theta)} = f(\kappa) I_{AA'} + g(\kappa) F_{AA'},\quad (6)$$

where the coefficients are given by

$$f(\kappa) = 1 + \frac{1 - \kappa}{d - 1}, \quad g(\kappa) = \frac{kd - 1}{d - 1}.\quad (7)$$

Eq. (6) can be viewed as a generalization of Eq. (5).
C. Conditional collision entropy

We use conditional collision entropy as measure of uncertainty. Let \( \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \) be a quantum state, the conditional collision entropy is defined as [13]
\[
H_2(A|B)_\rho = -\log \text{Tr}[\rho_{AB}(1_A \otimes \rho_B)^{-1/2}\rho_{AB}(1_A \otimes \rho_B)^{-1/2}].
\]
(8)
Trivializing system \( B \), we get the collision entropy of single system: \( H_2(A)_\rho = -\log \text{Tr} \rho_A^2 \).

Collision entropy admits nice operational interpretations. Let \( \rho_{XB} = \sum_n \eta_n |x_n\rangle \langle x_n| \otimes \rho_{x2} \) be a classical quantum state shared between Alice and Bob. From Bob’s view, he owns a state ensemble \( \{\rho_{x2}\} \) associated with \( \rho_{XB} \) to extract information about index \( x \). The measurement operators are given by \( M_y = \rho_B^{-1/2}(\eta_y \rho_y)\rho_B^{-1/2} \), where \( \rho_B = \text{Tr}_X \rho_{XB} = \sum_x \eta_x \rho_{x2} \). Denote by \( \mathcal{P}^{\text{pg}}(X|B)_\rho \) the probability that he can correctly guess the index \( x \) on average, then
\[
\mathcal{P}^{\text{pg}}(X|B)_\rho = \sum_x \eta_x \text{Tr}[M_x \rho_{x2}].
\]
It is proved in [10] that \( H_2(X|B)_\rho \) has the following operational interpretation:
\[
\mathcal{P}^{\text{pg}}(X|B)_\rho = 2^{-H_2(X|B)_\rho}.
\]

Now consider the fully quantum conditional collision entropy. Given state \( \rho_{AB} \), the pretty-good recoverable entanglement fidelity quantifies how well the local pretty-good recovery map \( \mathcal{R}_{B \to A'}^{\text{pg}} \) (defined in [1]) can bring \( \rho_{AB} \) to \( |\Psi_{AA'}\rangle \):
\[
\mathcal{F}^{\text{pg}}(A|B)_\rho = d_A \mathcal{F}((1_A \otimes \mathcal{R}_{B \to A'}^{\text{pg}})\rho_{AB}, |\Psi_{AA'}\rangle \langle \Psi_{AA'}|),
\]
where \( \mathcal{F}(\rho, \sigma) = (\text{Tr} \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}})^2 \) is Uhlmann’s fidelity [17]. It is proved in [1] that \( H_2(A|B)_\rho \) has the following operational interpretation:
\[
\mathcal{F}^{\text{pg}}(A|B)_\rho = 2^{-H_2(A|B)_\rho}.
\]

III. UNCERTAINTY RELATIONS FOR COMPLETE SET OF MUMS

In this section, we present uncertainty relations in the presence of memory for a complete set of mutually unbiased measurements. The main result is an equality quantifying the relation between uncertainty and entanglement, both of which are measured by conditional collision entropy.

Let \( \mathcal{P}(\theta) = \{P_x^{(\theta)}\}_{x \in [d]} \) be a MUM in \( A \) and \( \rho_{AB} \) be quantum state on \( AB \). Measuring \( \rho_{AB} \) on \( A \) by \( \mathcal{P}(\theta) \), we arrive at a classical-quantum state
\[
\omega_{X(\theta)|B} = \sum_{x=1}^d |x\rangle \langle x| \otimes \text{Tr}_A \left[ \left( P_x^{(\theta)} \otimes 1 \right) \rho_{AB} \right],
\]
where \( \mathcal{X} \) is a \( d \)-dimensional Hilbert space with \{\( |x\rangle \)\} being its standard basis. The classical register \( X \) indicates which measurement operator is performed; \( \text{Tr}_A[P_x^{(\theta)} \rho_{AB}] \) is the post-measurement state (unnormalized) left in system \( B \), conditioned on the measurement operator performed; and \( \text{Tr}[P_x^{(\theta)} \rho_{AB}] \) is probability that the measurement outcome is \( x \). We remark that the choice of \{\( |x\rangle \)\} does not affect our result as long as it forms an orthonormal basis of \( \mathcal{X} \).

Uncertainty relations study the unpredictability about the outcomes of many incompatible measurements. Thus in the following, we will not only measure in one fixed MUM, but with equal probability in one of \( d+1 \) MUMs. Let \( \{\mathcal{P}(\theta)\}_{\theta \in [d+1]} \) be a complete set of MUMs on system \( A \), we define the following classical-quantum state
\[
\omega_{X(\theta)B} = \frac{1}{d+1} \sum_{\theta=1}^{d+1} \sum_{x=1}^d |x\rangle \langle x| \otimes \text{Tr}_A \left[ \left( P_x^{(\theta)} \otimes 1 \right) \rho_{AB} \right] \otimes |\theta\rangle \langle \theta|_\Theta,
\]
(10)
where \( \Theta \) is an indicator specifying which measurement has been performed. The collision entropy of \( \omega_{X(\theta)B} \), with partition \( X:B, \Theta \), can be expressed as
\[
H_2(X|B\Theta) = -\log \left( \frac{1}{d+1} \sum_{\theta,x} \text{Tr}_B \left\{ \text{Tr}_A [P_x^{(\theta)} \rho_{AB}']^2 \right\} \right),
\]
(11)
where \( \rho_{AB}' = \rho_{B}^{-1/4} \rho_{AB} \rho_{B}^{-1/4} \). For the proof of Eq. (11), see Appx. A. Under this convention, the conditioned collision entropy of \( \rho_{AB} \) can be rewritten as \( H_2(A|B)_\rho = -\log \text{Tr}[\rho_{AB}^2] \). We are now ready to present our main result.

Theorem 1. Let \( \{\mathcal{P}(\theta)\}_{\theta \in [d+1]} \) be a complete set of MUMs on system \( A \). For arbitrary quantum state \( \rho_{AB} \), it holds that
\[
H_2(A|B\Theta) = \log (d+1) - \log \left( f(\kappa) + g(\kappa) 2^{-H_2(A|B)_\rho} \right),
\]
(12)
where \( \omega_{X(\theta)B} \) is defined in Eq. (10), \( f(\kappa) \) and \( g(\kappa) \) are defined in Eq. (7).

Proof. The proof is similar to the proof outlined in Appendix B of [1]. We introduce the spaces \( A' \cong A \) and \( B' \cong B \), as well as the state \( \rho_{A'B'} \cong \rho_{AB} \). Then we have
\[
(d+1)2^{-H_2(X|B\Theta)\omega} = \sum_{\theta,x} \text{Tr}_{B'} \left\{ \text{Tr}_A [P_x^{(\theta)} \rho_{AB}']^2 \right\} \]
\[
= \sum_{\theta,x} \text{Tr}_{BB'} \text{Tr}_{AA'} \left[ \left( P_x^{(\theta)} \otimes P_x^{(\theta)} \right) (\rho_{AB} \otimes \rho_{A'B'}) \mathcal{F}_{BB'} \right] \]
\[
= \text{Tr}_{BB'} \text{Tr}_{AA'} \left[ \left( \sum_{\theta,x} P_x^{(\theta)} \otimes P_x^{(\theta)} \right) (\rho_{AB} \otimes \rho_{A'B'}) \mathcal{F}_{BB'} \right] \]
\[
= \text{Tr}_{BB'} \text{Tr}_{AA'} \left[ f(\kappa) \mathcal{F}_{AA'} + g(\kappa) \mathcal{F}_{AA'} \right] \left( \rho_{AB} \otimes \rho_{A'B'} \right) \mathcal{F}_{BB'} \]
\[
= f(\kappa) \text{Tr}_{BB'} \text{Tr}_{AA'} \left[ (\rho_{AB} \otimes \rho_{A'B'}) \mathcal{F}_{BB'} \right] + g(\kappa) \text{Tr}_{BB'} \text{Tr}_{AA'} \left[ \rho_{AB} \otimes \rho_{A'B'} \right] \mathcal{F}_{BB'} \]
\[
+ g(\kappa) \text{Tr}_{BB'} \text{Tr}_{AA'} \left[ \mathcal{F}_{AA'} (\rho_{AB} \otimes \rho_{A'B'}) \mathcal{F}_{BB'} \right].
\]
(13)
(14)
In the second equality, we use the "swap trick"; for operators $M, N \in \mathcal{L}(\mathcal{H}_B)$, it holds that $\text{Tr}[MN] = \text{Tr}[(M \otimes N)\tilde{F}_{BB'}]$. In detail, we choose $M \equiv N \equiv \text{Tr}_A[P_x^\theta \rho_{AB}]$. Then

$$\text{Tr}_B[MN] = \text{Tr}_{BB'}[(M \otimes N)\tilde{F}_{BB'}] = \text{Tr}_{BB'AA'}\left[(P_x^\theta \tilde{\rho}_{AB} \otimes P_x^\theta \tilde{\rho}_{A'B'})\tilde{F}_{BB'}\right] = \text{Tr}_{BB'AA'}\left[(P_x^\theta \otimes P_x^\theta)(\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})\tilde{F}_{BB'}\right].$$

In the forth equality, we use the fact that complete set of MUMs forms a conical 2-design (see Eq. (6)). Now we compute the two terms given in Eqs. (13) and (14). For the first term, we have

$$\text{Tr}_{BB'}\text{Tr}_{AA'}[(\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})\tilde{F}_{BB'}] = \text{Tr}_B[\text{Tr}_A(\tilde{\rho}_{AB})\text{Tr}_A(\tilde{\rho}_{AB})] = \text{Tr}_B[\rho_B^{1/2} \rho_B^{1/2}] = 1.$$

For the second term, we have

$$\text{Tr}_{BB'}\text{Tr}_{AA'}[\tilde{F}_{AA'}(\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})\tilde{F}_{BB'}] = \sum_{ts}\text{Tr}_{BB'}\text{Tr}_{AA'}[|t\rangle\langle s| \otimes |s\rangle\langle t| (\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'}) \tilde{F}_{BB'}]$$

$$= \sum_{ts}\text{Tr}_{BB}[|s\rangle\langle\tilde{\rho}_{AB}|t\rangle\langle t|\tilde{\rho}_{AB}|s\rangle] = \text{Tr}[\rho_{AB}] = 2^{-H_2(A\mid B)_\rho}.$$

Combining these results, we reach at

$$(d + 1)^2 H_2(A\mid B)_\omega = f(\kappa) + g(\kappa)2^{-H_2(A\mid B)_\rho}. \quad (15)$$

Rearranging the elements, we get Eq. (12). □

Following, we discuss several implications of Thm. 1. Its relation to the guessing games, its relation to the uncertainty relations expressed in bounds of sum of entropies, and its application in entanglement detection. These implications can help us gain further intuition about relation (12).

### A. Guessing games

Note that the conditional collision entropy admits an operational interpretation in terms of guessing games. Now we consider a game suited to the above MUMs situation. Bob prepares a state $\rho_{AB}$ and sends the $A$ system to Alice. She measures $A$ in one measurement randomly chosen from the complete set of MUMs, and then tells Bob which measurement has been performed (index $\theta$). Bob’s task is to guess Alice’s outcome (index $x$) using the pretty-good measurements on $B$. Thm. 1 can be understood as saying that Bob’s ability to correctly guess the outcome is quantitatively connected to the pretty-good recoverable entanglement fidelity that can be achieved by Bob. This is summarized as follows.

Lemma 2. Let $\{\mathcal{P}^{(\theta)}\}_{\theta \in [d+1]}$ be a complete set of MUMs on system $A$. For arbitrary quantum state $\rho_{AB}$, it holds that

$$\sum_{\theta=1}^{d+1} P_{\text{PG}}(X^{(\theta)}\mid B) = f(\kappa) + g(\kappa)F_{\text{PG}}(A\mid B)_\rho, \quad (16)$$

where $P_{\text{PG}}(X^{(\theta)}\mid B)_\omega$ is the pretty-good guessing probability of state $\omega_{X^{(\theta)}B}$, and $F_{\text{PG}}(A\mid B)_\rho$ is the pretty-good recoverable entanglement fidelity of state $\rho_{AB}$.

Proof. Using the operational interpretations of $H_2(X\mid B\theta)_\omega$ and $H_2(A\mid B)_\rho$, we obtain from Eq. (15) that

$$(d + 1)P_{\text{PG}}(X\mid B\theta)_\omega = f(\kappa) + g(\kappa)F_{\text{PG}}(A\mid B)_\rho.$$

Now all we need to show is the following equality

$$P_{\text{PG}}(X\mid B\theta)_\omega = \frac{1}{d + 1}\sum_{\theta=1}^{d+1} P_{\text{PG}}(X^{(\theta)}\mid B)_\omega,$$

where the LHS is evaluated on state $\omega_{X^B\theta}$, while the RHS is evaluated on the states $\omega_{X^{(\theta)}B}$. This is trivial since $\omega_{X^B\theta}$ is a uniform mixture of the states $\omega_{X^{(\theta)}B}$. □

### B. Uncertainty relations expressed in sum of entropies

Uncertainty relations are commonly expressed as lower bounds on the sum of entropies of the probability distributions induced by incompatible measurements. Using Eq. (12), we can derive an uncertainty relation of such kind in terms of sum of collision entropies.

Lemma 3. Let $\{\mathcal{P}^{(\theta)}\}_{\theta \in [d+1]}$ be a complete set of MUMs on system $A$. For arbitrary quantum state $\rho_{AB}$, it holds that

$$\frac{1}{d + 1} \sum_{\theta=1}^{d+1} H_2(X^{(\theta)}\mid B)_\omega \geq \log (d + 1) - \log \left(f(\kappa) + g(\kappa)2^{-H_2(A\mid B)_\rho}\right), \quad (17)$$

where $\omega_{X^{(\theta)}B}$ is defined in Eq. (15). Coefficients $f(\kappa)$ and $g(\kappa)$ are defined in Eq. (13).

Proof. As the log function is concave, from Eq. (11) we get

$$H_2(X\mid B\theta)_\omega \leq \frac{1}{d + 1} \sum_{\theta=1}^{d+1} - \log \left(\sum_{x=1}^{d} \text{Tr}_B \left(P_x^{\theta}\tilde{\rho}_{AB}\right)^2\right) = \frac{1}{d + 1} \sum_{\theta=1}^{d+1} H_2(X^{(\theta)}\mid B)_\omega.$$

Together with Eq. (12), we prove this lemma. □
If system $B$ is trivial, Eq. (17) reduces to
\[
\frac{1}{d+1} \sum_{\theta=1}^{d+1} H_2 \left( X^{(\theta)} \right) \omega \\
\geq \log (d+1) - \log (f(\kappa) + g(\kappa) \text{Tr}[\rho_A^2]) .
\]
This inequality recovers a special case ($\alpha = 2$) of Theorem 4 in [18].

C. Entanglement detection

Entanglement is an appealing feature of quantum mechanics and has been extensively investigated in the past decades [19]. Entangled states play important roles in many quantum tasks, such as quantum teleportation [20] and dense coding [21]. Deciding whether a given quantum state is entangled is a central problem in quantum information theory and known to be computationally intractable in general [22]. Experimenters often need easy-to-implement methods to verify that their source is indeed producing entangled states [23].

Lemma 3 offers a simple strategy for detecting entanglement since it connects entanglement to uncertainty, yet it is experimentally measurable. We show that for separable states, the sum of entropies induced by complete set of MUMs has a larger lower bound, compared to that of entangled states. This bound serves as an entanglement witness, as any state that violates this bound must be necessarily entangled.

**Lemma 4.** Let \( \{ \mathcal{D}^{(\theta)} \}_{\theta \in [d+1]} \) be a complete set of MUMs on system $A$, and let \( \{ \mathcal{Q}^{(\theta)} \}_{\theta \in [d+1]} \) be an arbitrary set of $d+1$ measurements on system $B$. For arbitrary separable quantum state $\rho_{AB}$, it holds that
\[
\frac{1}{d+1} \sum_{\theta=1}^{d+1} H_2 \left( X^{(\theta)} \right) \omega \\
\geq \log (d+1) - \log (f(\kappa) + g(\kappa)) ,
\]
where \( \omega_X^{(Y)} \) is defined as
\[
\omega_X^{(Y)} = \sum_{x,y=1}^d \text{Tr} \left( [\mathcal{P}^{(\theta)} \otimes \mathcal{Q}^{(\theta)}] \rho_{AB} \right) |x\rangle \langle x| \otimes |y\rangle \langle y| .
\]

**Proof.** We use Lemma 3. It holds that
\[
\frac{1}{d+1} \sum_{\theta=1}^{d+1} H_2 \left( X^{(\theta)} \right) \omega \\
\geq \frac{1}{d+1} \sum_{\theta=1}^{d+1} H_2 \left( X^{(\theta)} \right) \omega \\
\geq \log (d+1) - \log (f(\kappa) + g(\kappa)) .
\]

The first inequality follows from the conditional collision entropy satisfies the data-processing inequality [24], the second inequality is proved in Eq. (17), while the last inequality follows from the fact that all separable states have non-negative collision entropy [1].

Lemma 4 can be used to detect entanglement. Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, Alice performs complete set of MUMs $\mathcal{D}^{(\theta)}$ on system $A$, while for each $\theta$ Bob performs a corresponding measurement $\mathcal{Q}^{(\theta)}$ on system $B$. They then evaluate the classical collision entropies $H_2(X^{(\theta)}|Y^{(\theta)})$. State $\rho_{AB}$ is entangled if
\[
\frac{1}{d+1} \sum_{\theta=1}^{d+1} H_2 \left( X^{(\theta)} \right) \omega < \log (d+1) - \log (f(\kappa) + g(\kappa)) .
\]

We remark that the choice of measurements $\mathcal{Q}^{(\theta)}$ is arbitrary. For best detection criterion, one can minimize the LHS of Eq. (18) by optimizing over all possible measurements on system $B$.

IV. UNCERTAINTY RELATIONS FOR SIM

In this section, we show that a similar equality relation in the presence of memory exists for generalized symmetric informationally complete measurements.

A set of $d^2$ positive-semidefinite operators \( \{ P_x \}_{x \in [d^2]} \) in $\mathcal{H}$ is called a generalized symmetric informationally complete measurement (SIM) if [25]

- It is a POVM: $P_x \geq 0$ and $\sum_{x=1}^{d^2} P_x = 1$; and,
- It is symmetric: \( \forall x \in [d^2], \text{Tr}[P_x]\sigma = \eta, \forall x \neq y, \text{Tr}[P_x P_y] = \frac{1}{d^2 - 1} \),

$\eta$ is the efficiency parameter defining the “type” of a general SIM, whose range is $1/d^2 < \eta \leq 1/d^2$. There exists a general method to construct the set of all generalized SIMs [25]. In [14], it is proved that every SIM forms a conical 2-design
\[
\sum_{x=1}^{d^2} P_x \otimes P_x = l(\eta) \mathbb{1}_{AA'} + r(\eta) \mathbb{1}_{AA'},
\]
where the coefficients are given by
\[
l(\kappa) = \frac{1}{d^2} - \eta, \quad r(\kappa) = \frac{d^2 \eta - 1}{d(d^2 - 1)} .
\]

Let $\mathcal{P} = \{ P_x \}_{x \in [d^2]}$ be a generalized SIM on system $A$, and $\rho_{AB}$ be quantum state on $AB$. Measuring $\rho_{AB}$ on $A$ by $\mathcal{P}$, we obtain the following classical-quantum state:
\[
\omega_{X^B} = \sum_{x=1}^{d^2} |x\rangle \langle x| \otimes \text{Tr}_A [(P_x \otimes \mathbb{1}_B) \rho_{AB}] ,
\]
where $\mathcal{H}_X$ is a $d^2$-dimensional Hilbert space with $\{|x\rangle\}$ being its standard basis. Classical register $X$ indicates
which measurement operator is performed; $\text{Tr}_A[P_x \rho_{AB}]$ is the post-measurement state (unnormalized) left in system $B$, conditioned on the measurement operator performed; and $\text{Tr}[P_x \rho_{AB}]$ is probability that the measurement outcome is $x$. We are now ready to present an equality relation for SIM with collision entropy.

**Theorem 5.** Let $\mathcal{P} = \{P_x\}_{x \in \chi}^\mathcal{X}$ be a SIM on system $A$. For arbitrary quantum state $\rho_{AB}$, it holds that

$$H_2(X|B)_{\omega} = -\log \left[ (l(\eta) + r(\eta) 2^{-H_2(A|B)_\rho}) \right], \quad (22)$$

where $\omega_{XB}$ is defined in Eq. (21). $l(\eta)$ and $r(\eta)$ are defined in Eq. (20).

The proof of Thm. 5 is identical to that of Thm. 1 with Eq. (19) substituted appropriately. Now we discuss the case of symmetric informationally complete measurements (Thm. 5). Let

$$H_2(X|B)_{\omega} = \log \left[ d(d+1) \right] - \log \left[ 1 + 2^{-H_2(A|B)_\rho} \right],$$

which is exactly the Corollary 2 proved in [1]. Trivializing system $B$, Eq. (22) becomes

$$H_2(X)_\omega = \log \left( \frac{d(d^2-1)}{(d^3-1) \text{Tr}[\rho_A^2] + (1-d\eta)d} \right). \quad (23)$$

This is an equality relation for SIM without quantum memory. Eq. (23) recovers and tightens a special case ($\alpha = 2$) of Proposition 3 in [27]. We remark that Eq. (22) can also be used to detect entanglement, using the fact that all separable states have non-negative collision entropy [1].

**V. CONCLUSIONS**

In summary, we derive several uncertainty relations in the presence of quantum memory for different set of measurements. Our results are generalizations and extensions of [1]. In that paper, uncertainty relations in the presence of quantum memory was formulated for MUBs using the conditional collision entropy. In this paper, we prove an equality between the amount of uncertainty for a set of measurements and the amount of entanglement of the measured state, both of which are quantified by the conditional collision entropy (Thm. 1). Our result relies on the fact that complete set of mutually unbiased measurements forms a conical 2-design. Several implications of this equality relation are discussed, among which the entanglement detection method may be of interest from the experiment’s point of view. Using similar techniques, we further prove an equality relation for generalized symmetric informationally complete measurements (Thm. 5). By investigating the relation between the construction of complete set of MUMs and the conical 2-design, we derive an interesting equality for arbitrary orthogonal basis of the space of Hermitian, traceless operators (Lemma 6). This equality may be helpful for studying conical designs. We hope our results can shed lights on the study of MUMs and inspire new relations quantifying the relation between uncertainty and entanglement.

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**Appendix A: Correctness of Eq. (11)**

Here we prove Eq. (11) in the main text. We shall first compute $\omega_B\Theta$:

$$\omega_B\Theta = \text{Tr}_X \omega_{XB\Theta}$$

$$= \frac{1}{d+1} \sum_{\theta=1}^{d+1} \left( \sum_{x=1}^{d} \text{Tr}_A[P_x(\theta) \rho_{AB}] \right) \otimes |\theta\rangle \langle \theta| \Theta$$

$$= \rho_B \otimes \frac{1}{d+1} \sum_{\theta=1}^{d+1} |\theta\rangle \langle \theta| \Theta = \rho_B \otimes \pi_A \Theta.$$

Then

$$H_2(X|B\Theta)_{\omega} = -\log \left( \left( \frac{1}{\sqrt{d+1}} \sum_{\theta,x} |x\rangle \langle x| \otimes \rho_B^{-1/4} \text{Tr}_A[P_x(\theta) \rho_{AB}] \rho_B^{-1/4} \otimes |\theta\rangle \langle \theta| \Theta \right)^2 \right)$$

$$= -\log \left( \sum \left( \rho_B^{-1/4} \text{Tr}_A[P_x(\theta) \rho_{AB}] \rho_B^{-1/4} \right)^2 \right) = -\log \left( \frac{1}{d+1} \sum_{\theta,x} \text{Tr}_B \left[ \text{Tr}_A[\rho_{AB} P_x(\theta)]^2 \right] \right).$$

The third and fourth equality follows from that $\{|x\rangle\}$ and $\{|\theta\rangle\}$ are orthogonal, the last equality follows because
$P_x^{(θ)}$ only affects on system $A$, while $ρ_B^{-1/4}$ only affects system $B$.

**Appendix B: An equality for operator basis**

Based on the fact that a complete set of MUMs forms a canonical 2-design, we prove an interesting equality for arbitrary orthogonal basis for traceless hermitian operators acting on $H_A$. This equality has a similar form to the canonical 2-design.

**Lemma 6.** Let $\{F_k\}_{k∈[d^2-1]}$ be arbitrary orthogonal basis for the space of Hermitian, traceless operators acting on $H_A$. It holds that

$$
\sum_{k=1}^{d^2-1} F_k \otimes F_k = \mathbb{F}_{AA'} - \frac{1}{d} \mathbb{1}_{AA'}, \quad (B1)
$$

where $A' ≃ A$ and $\mathbb{F}_{AA'}$ is the swap operator defined in Eq. [3].

**Proof.** The proof relies heavily on the construction of complete set of MUMs. Using the relation between $P_x^{(θ)}$ and $F_x^{(θ)}$, we have

$$
\begin{align*}
&d+1 \sum_{θ=1}^{d} \sum_{x=1}^{d} P_x^{(θ)} \otimes P_x^{(θ)} \\
&= d+1 \sum_{θ=1}^{d} \sum_{x=1}^{d} \left( \frac{1}{d} \mathbb{1} + tF_x^{(θ)} \right) \otimes \left( \frac{1}{d} \mathbb{1} + tF_x^{(θ)} \right) \\
&= \frac{1}{d} \mathbb{1}_{AA'} + t \left( \mathbb{1}_{A} \otimes \hat{F}_A + \hat{F}_A \otimes \mathbb{1}_{A'} \right) \\
&+ t^2 \sum_{θ=1}^{d+1} \sum_{x=1}^{d} F_x^{(θ)} \otimes F_x^{(θ)},
\end{align*}
$$

where $\hat{F}$ is defined as $\hat{F} = \sum_{θ=1}^{d+1} \sum_{x=1}^{d} F_x^{(θ)}$. By definition one has $\hat{F} = 0$. Using the relation between $F_x^{(θ)}$ and $F_{x,θ}$, it can be shown that (through tedious calculation)

$$
\sum_{θ=1}^{d+1} \sum_{x=1}^{d} F_x^{(θ)} \otimes F_x^{(θ)} = (d+1)^2 \sum_{θ=1}^{d+1} \sum_{x=1}^{d} F_{x,θ} \otimes F_{x,θ}.
$$

As $F_{x,θ}$ are just rearrangements of $F_k$, we have

$$
\begin{align*}
&d+1 \sum_{θ=1}^{d} \sum_{x=1}^{d} F_x^{(θ)} \otimes F_x^{(θ)} \\
&= \left( 1 + \frac{1}{d} \right) \mathbb{1}_{AA'} + t^2 (d+\sqrt{d})^2 \sum_{k=1}^{d^2-1} F_k \otimes F_k \\
&= \left( 1 + \frac{1}{d} \right) \mathbb{1}_{AA'} + \frac{kd-1}{d-1} \sum_{k=1}^{d^2-1} F_k \otimes F_k, \quad (B2)
\end{align*}
$$

where the second equality follows from Eq. [3]. Comparing Eq. [6] and Eq. [12], we obtain the following equality for arbitrary orthogonal basis for the space of Hermitian, traceless operators acting on $H_A$:

$$
\sum_{k=1}^{d^2-1} F_k \otimes F_k = \mathbb{F}_{AA'} - \frac{1}{d} \mathbb{1}_{AA'}.
$$

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