BROWNIAN MOTION IN A BALL IN THE PRESENCE OF SPHERICAL OBSTACLES

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Abstract. We study the problem of when a Brownian motion in the unit ball has a positive probability of avoiding a countable collection of spherical obstacles. We give a necessary and sufficient integral condition for such a collection to be avoidable.

1. Introduction

The setting in this paper is the unit ball, \( \mathbb{B} = \{ x \in \mathbb{R}^d : |x| < 1 \} \), in Euclidean space \( \mathbb{R}^d \) where \( d \geq 3 \). We study the problem of when Brownian motion in the ball has a positive probability of avoiding a countable collection of spherical obstacles and thereby reaching the outer boundary of \( \mathbb{B} \).

We denote by \( \Lambda \) a sequence of points in \( \mathbb{B} \). To each point \( \lambda \) in this sequence we associate a spherical obstacle, \( B(\lambda, r_\lambda) \), where

\[
B(\lambda, r_\lambda) = \{ x : |\lambda - x| \leq r_\lambda \},
\]

and denote by \( \partial B(\lambda, r_\lambda) \) the boundary of this obstacle. We let \( \mathcal{B} \) denote the countable collection of closed spherical obstacles,

\[
\mathcal{B} = \bigcup_{\lambda \in \Lambda} B(\lambda, r_\lambda).
\]

We assume that the spherical obstacles are pairwise disjoint, lie inside the ball \( \mathbb{B} \) and that the origin lies outside \( \mathcal{B} \). We call a collection of spherical obstacles avoidable if there is positive probability that Brownian motion, starting from the origin, hits the boundary of \( \mathbb{B} \) before hitting any of the spherical obstacles in \( \mathcal{B} \). This is equivalent to positive harmonic measure at 0 of the boundary of the unit ball with respect to the domain \( \Omega = \mathbb{B} \setminus \mathcal{B} \), consisting of the unit ball less the obstacles, that is \( \omega(0, \partial \mathbb{B}; \Omega) > 0 \).

In the setting of the unit disk, Ortega-Cerdà and Seip [7] gave an integral condition for a collection of disks to be avoidable. In [4], Carroll and Ortega-Cerdà gave an integral criterion for a configuration of balls in \( \mathbb{R}^d, d \geq 3 \), to be avoidable. Thus, it seems natural to ask if Ortega-Cerdà and Seip’s result for the disk in the plane can be extended to the ball in space. A solution to this problem is the main result of this paper.

Next, we put some restrictions on the spacing of the spherical obstacles. A collection of obstacles, \( \mathcal{B} \), is regularly spaced if it is separated, in that there exists \( \epsilon > 0 \) such that given any \( \lambda, \lambda' \in \Lambda \) with \( |\lambda| \geq |\lambda'| \), then \( |\lambda - \lambda'| > \epsilon(1 - |\lambda|) \); uniformly dense, in that there exists \( R \) with \( 0 < R < 1 \) such that for \( x \in \mathbb{B} \), the
ball $B(x, R(1 - |x|))$ contains at least one $\lambda \in \Lambda$; and finally the radius $r_\lambda = \phi(|\lambda|)$ where $\phi : [0, 1) \to [0, 1)$ is a decreasing function.

Answering a question of Akeroyd in [2], Ortega-Cerdà and Seip [7] proved the following theorem.

**Theorem A.** A collection of regularly spaced disks in the unit disk is avoidable if and only if

$$\int_0^1 \frac{dt}{(1 - t) \log((1 - t)/\phi(t))} < \infty.$$  

This theorem in [7] is expressed in terms of pseudo-hyperbolic disks. We extend Theorem A to the setting of the unit ball in $\mathbb{R}^d, d \geq 3$.

**Theorem 1.1.** The collection of regularly spaced closed spherical obstacles $B$ in $\mathbb{B}$ is avoidable if and only if

$$\int_0^1 \phi(t)^{d-2} \left(\frac{1}{(1 - t)^{d-1}} dt < \infty. \right.$$  

We present two proofs of this result. The first proof exploits a connection between avoidability and minimal thinness, a potential theoretic measure of the size of a set near a boundary point of a region. We learnt of this from both the paper of Lundh [6] and from Professor S.J. Gardiner. We also make use of a Wiener-type criterion for minimal thinness due to Aikawa [1].

The second proof is more direct and transparent. It is an adaptation of Ortega-Cerdà and Seip’s proof of Theorem A in [7], the key difference being that in higher dimensions we do not have the luxury of conformal mapping.

### 2. Avoidable Obstacles and Minimal Thinness

Following the notation of Lundh [6], we let $\text{SH}(\mathbb{B})$ denote the class of non-negative superharmonic functions on the unit ball and let $P_\tau$ denote the Poisson kernel at $\tau \in \partial \mathbb{B}$. For a positive superharmonic function $h$ on $\mathbb{B}$ the reduced function $\text{of} \ h$ with respect to a subset $E$ of $\mathbb{B}$ is

$$R_E^F(h)(w) = \inf \{ u(w) : u \in \text{SH}(\mathbb{B}), u(x) \geq h(x), x \in E \}$$

and the regularized reduced function $\hat{R}_E^F(h)(w) = \lim \inf_{x \to w} R_E^F(h)(x)$.

**Definition 2.1.** A set $E$ is minimally thin at $\tau \in \partial \mathbb{B}$ if there is an $x_0$ in the unit ball such that $\hat{R}_E^F(x_0) < P_\tau(x_0)$.

A nice account of reduced functions and minimal thinness may be found in [3] Page 38 ff or [4] Chapter 9.

#### 2.1. Avoidability and minimal thinness.

Lundh proves the following result in [6]. We include a brief proof for the convenience of the reader.

**Proposition 2.2.** Let $A$ be a closed subset of $\mathbb{B}$ such that $\mathbb{B}\setminus A$ contains the origin and is connected. Let $\mathcal{M} = \{ \tau \in \partial \mathbb{B} : A \text{ is minimally thin at } \tau \}$. Then the following are equivalent:

- $A$ is avoidable,
- $|\mathcal{M}| > 0$,

where $|.|$ denotes surface area on the unit ball.
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Proof. Noting that
\[ 1 = \int_{\partial B} P_\tau(x) \frac{d\tau}{|\partial B|}, \]
and taking \( h \equiv 1 \) in [4, Corollary 9.1.4], we see that
\[ \hat{R}_A^1(x) = \int_{\partial B} \hat{R}_A^1 P_\tau(x) \frac{d\tau}{|\partial B|}. \]

Also, it follows from [3, Page 653, 14.3sm] that the regularized reduced function of \( 1 \) with respect to \( A \) evaluated at \( x \) is the harmonic measure at \( x \) of \( \partial A \) in the domain \( B \setminus A \). Thus,
\[ \omega(0, \partial A, B \setminus A) = \hat{R}_A^1(0) = \frac{1}{|\partial B|} \int_{\partial B} \hat{R}_A^1 P_\tau(0) d\tau. \]

Since \( \hat{R}_A^1(0) \leq P_\tau(0) = 1 \), it follows that \( \omega(0, \partial A, B \setminus A) < 1 \) if and only if the set \( M_0 = \{ \tau \in \partial B, \hat{R}_A^1 P_\tau(0) < 1 \} \) has positive measure. In the connected domain \( B \setminus A \), the set \( M_0 \) is the same as the set \( M \). Thus, \( A \) being avoidable, that is \( \omega(0, \partial B; B \setminus A) > 0 \), is equivalent to \( M \) having positive measure. □

2.2. Minimal thinness and a Wiener-type criterion. It is a standard result, see for example Aikawa [1] or Lundh [6], that a set is minimally thin at a point if and only if it satisfies a Wiener-type criterion. Let \( \{Q_k\} \) be a Whitney decomposition of the unit ball \( B \) in \( \mathbb{R}^d \) (\( d \geq 3 \)) and let \( q_k \) be the Euclidean distance from the centre, \( c_k \), of the Whitney cube \( Q_k \) to the boundary of \( B \). Let \( A \) be a subset of \( B \). Let \( \tau \) be a boundary point of \( B \) and \( \rho_k(\tau) \) be the distance from \( c_k \) to the boundary point \( \tau \). Let \( \text{cap} \) denote Newtonian capacity. Then \( A \) is minimally thin at the point \( \tau \) if and only if
\[ \sum_k \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(A \cap Q_k) < \infty. \]

In the next section, we consider this Wiener-type criterion in the particular setting of the unit ball less a collection of regularly spaced spherical obstacles.

2.3. Wiener-type criterion and integral condition. For a constant \( K > 1 \), we let \( S_j = \{ x : |x| = 1 - K^{-j} \} \) be the sphere of radius \( \rho_j = 1 - K^{-j} \) and \( B_j \) be the interior of this sphere. We denote by \( A_j \) the annulus bounded by \( S_j \) and \( S_{j-1} \), and write \( \phi_j \) for \( \phi(\rho_j) \).

Proposition 2.3. Let \( B \) be a regularly spaced collection of spherical obstacles in \( B \).

(i) If the set \( B \) satisfies the Wiener-type criterion \((2.1)\) at some point in \( \partial B \) then the integral condition \((1.1)\) holds,

(ii) The integral condition \((1.1)\) implies that \( B \) satisfies the Wiener-type criterion \((2.1)\) at all points \( \tau \in \partial B \).

Proof. We first assume that the integral condition holds and we’ll show that \((2.1)\) follows. We note that the integral condition \((1.1)\) is equivalent to
\[ \sum_{j=1}^{\infty} (\phi_j K^d)^{d-2} < \infty, \]
where $K > 1$. By the separation condition on the sequence $\Lambda$, there is an $N$ such that any cube $Q_k$ can contain no more than $N$ points in $\Lambda$. Splitting the sum in (2.1) into a sum over annuli we obtain

\[ \sum_k \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(B \cap Q_k) = \sum_{j=1}^{\infty} \sum_{k: c_k \in A_j} \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(B \cap Q_k) \]

(2.3)

\[ \leq \sum_{j=1}^{\infty} N(K^{-j})^2 \phi_j^{d-2} \sum_{k: c_k \in A_j} \frac{1}{\rho_k(\tau)^d}. \]

(2.4)

since the capacity of a ball with radius $\phi_j$ is equal to $\phi_j^{d-2}$. We now concentrate on the latter sum in (2.4). We split up the $j$th annulus $A_j$ into rings centered at the projection of $\tau$ onto the sphere $S_j$, and with radius equal to $nK^{-j}$ where we recall that $K^{-j}$ is the distance from $\tau$ to $S_j$. There are at most

\[ c_d(nK^{-j})^{d-2} \]

Whitney cubes in each ring where $c_d$ is a constant depending on the dimension, $d$. For the $n$th ring,

\[ \rho_k(\tau) \geq nK^{-j} \]

and $N_j$ rings intersect the annulus $A_j$. Thus,

\[ \sum_{k: c_k \in A_j} \frac{1}{\rho_k(\tau)^d} \leq \sum_{n=1}^{N_j} \frac{c_d n^{d-2}}{(nK^{-j})^d} \]

\[ \leq (K^{-j})^d c_d \sum_{n=1}^{N_j} \frac{1}{n^2}. \]

Thus, we see that the Wiener-type series (2.3) is convergent.

We now assume that the set $B$ satisfies (2.1) at some arbitrary point $\tau \in \partial B$ and show that this implies the integral condition (1.1). We choose $K$ sufficiently large so that for all $j$ bigger than a fixed constant there is at least one centre of a ball in each Whitney cube, $Q_k$, in the resulting Whitney decomposition of $B$. Starting with the Wiener-type series we split it into a sum over the annuli $A_j$ and then proceed to ignore all Whitney cubes in $A_j$ except one near to the point $\tau$, for which $\rho_k(\tau) \leq K^{-j}$, as follows.

\[ \sum_k \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(B \cap Q_k) = \sum_{j=1}^{\infty} \sum_{k: c_k \in A_j} \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(B \cap Q_k) \]

\[ \geq \sum_{j=0}^{\infty} K^{-2j} \phi_j^{d-2} \frac{1}{\rho_k(\tau)^d} \]

\[ \geq \sum_{j=0}^{\infty} (\phi_j K^j)^{d-2} \]

Thus, since the Wiener-type series is convergent, (2.2) follows and so the integral condition (1.1) holds. \qed
Combining Proposition 2.2, the Wiener-type criterion (2.1) and Proposition 2.3 we have a proof of Theorem 1.1. We note that the method used in this section could also be used to give an alternative proof of Ortega-Cerdà and Seip’s Theorem A.

3. DIRECT PROOF OF THEOREM 1.1

We now give an alternative proof of Theorem 1.1 by adapting the method of Ortega-Cerdà and Seip in [7]. In dimensions higher than 2 we do not have conformal mapping, but we do have the Kelvin transform. We let

\[ x^* = \frac{\rho_j 1}{|x|^2} x \]

be the inversion of the point \( x \) in the sphere of radius \( \rho_{j+1} \). We note that \( |x||x^*| = \rho_j^2 + 1 \), and let \( \phi(|\lambda|) = \phi_\lambda \). We begin with a lemma, prove the sufficiency of the integral condition in the next subsection and the necessity in the following one.

**Lemma 3.1.** Let \( K > \max\{A, \frac{1+R}{2}\} \) and \( x \) be an arbitrary point belonging to \( S_{j-1} \). There is a centre of an obstacle, \( \lambda_x \in \Lambda \), such that \( \lambda_x \) lies in the annulus \( A_j \) bounded by \( S_{j-1} \) and \( S_j \), and

\[ |x - \lambda_x| \leq \frac{K-1}{K} |x^* - \lambda_x|. \]

**Proof.** For \( x \in S_{j-1} \), let \( x' \) be the point on the extension of the radius of \( S_j \) containing \( x \), and located halfway between \( S_{j-1} \) and \( S_j \). Then \( x' \) is a distance \( K^{-(j-1)} - \frac{K}{2^{j-1}} \) from the boundary of the ball \( B \). Since \( \Lambda \) is uniformly dense, the ball \( B(x', R(1-|x'|)) \) contains some \( \lambda_x \in \Lambda \). Also, due to the choice of \( K \), the ball \( B(x', R(1-|x'|)) \) is contained in the annulus \( A_j \). Let \( x'' \) be on the same ray as \( x \) and \( x^* \) and also on \( S_j \). We first note that \( |x - \lambda_x| \leq |x - x''| \) and \( |x^* - \lambda_x| \geq |x^* - x''| \). Also, we note that \( |x| = \rho_{j-1}, |x''| = \rho_j \) and \( |x^*| = \rho_j^2 + 1/\rho_{j-1} \). Thus,

\[ |x - \lambda_x| \leq |x - x''| = (K-1)K^{-j}. \]

Also,

\[ |x^* - \lambda_x| \geq |x^* - x''| = \frac{(1-K^{-(j+1)})^2}{1-K^{-(j-1)}} - (1-K^{-j}) \geq K^{-j+1}, \]

for \( j \geq 2 \). Thus,

\[ |x - \lambda_x| \leq \frac{K-1}{K} |x^* - \lambda_x|, \]

as required.

**3.1. **Integral Condition (1.1) IMPLIES AVOIDABILITY.** We first assume (1.1) and show that the spherical obstacles are avoidable that is, we show that \( \omega(0, \partial B; \Omega) > 0 \). We split the collection of spherical obstacles into those with centres inside and those with centres outside a ball of radius \( r < 1 \). We let \( \Lambda_r = \{ \lambda \in \Lambda : |\lambda| > r \} \) and let

\[ \mathcal{B}_r = \bigcup_{\lambda \in \Lambda_r} B(\lambda, r_\lambda) = \bigcup_{\lambda \in \Lambda_r} B_\lambda \]

denote the infinitely many spherical obstacles with centres outside \( B(0, r) \). Also, we let \( \Omega_r = B \setminus \mathcal{B}_r \) be the champagne subregion where all obstacles have centres outside a ball of radius \( r \). We may safely ignore the finitely many spherical obstacles with centres inside the ball of radius \( r \). Thus, it is sufficient to show that \( \omega(0, \partial B; \Omega_r) > 0 \).
for some $r$ with $0 < r < 1$, which is equivalent to showing that $\omega(0, \partial B_r; \Omega_r) < 1$. We choose $r$ such that
\[
\int_r^1 \frac{\phi(t)^{d-2}}{(1-t)^{d-1}} dt < \frac{\varepsilon^d(K-1)^{d-2}}{2^{d+1}d(d-2)K^{2d-1}}
\]
and let $n_r$ be the biggest integer smaller than $1 + \log(\frac{1}{1-r})/\log K$. This ensures that $r > \lfloor 1 - K^{-(n_r - 1)} \rfloor$. We proceed as follows,
\[
\omega(0, \partial B_r; \Omega_r) = \sum_{\lambda \in \Lambda_r} \omega(0, \partial B_\lambda; \Omega_r) \leq \sum_{\lambda \in \Lambda_r} \omega(0, \partial B_\lambda; \mathbb{B} \setminus B_\lambda)
\]
\[
\leq \sum_{j=n_r}^\infty \left( \sum_{\lambda \in A_j} \omega(0, \partial B_\lambda; \mathbb{B} \setminus B_\lambda) \right).
\]
We now obtain an upper bound for the number of centres in $A_j$ and an upper bound for the contribution of an obstacle with centre in $A_j$ to the above sum. Due to the separation condition, centres of balls in $A_j$ are at least $\varepsilon K^{-j}$ apart. Thus, the number of centres in $A_j$, which is less than the volume of $A_j$ divided by the volume of a ball with radius $\varepsilon K^{-j}/2$, is less than
\[
\frac{2^d d K^2}{\varepsilon^d} K^{(d-1)j}.
\]
Next, we want an upper bound for $\omega(0, \partial B_\lambda; \mathbb{B} \setminus B_\lambda)$. We construct a suitable function $h$ that is harmonic on $\mathbb{B} \setminus B_\lambda$, continuous on its closure and also satisfies $h(x) \geq 1$, $x \in \partial B_\lambda$ and $h(x) \geq 0$, $x \in \partial \mathbb{B}$. Then, using the Maximum Principle, we obtain the required upper bound. Consider the function
\[
h(x) = 2 \left[ u_\lambda(x) - u_\lambda^*(x) \right],
\]
where
\[
u_\lambda(x) = \left[ \frac{\phi_\lambda}{|x - \lambda|} \right]^{d-2}, \quad u_\lambda^*(x) = \left[ \frac{\phi_\lambda}{|x - x^* - \lambda|} \right]^{d-2} \quad \text{and} \quad x^* = \frac{1}{|x|^2} x.
\]
We note that $u_\lambda$ and $u_\lambda^*$ are harmonic. Also, $1/2$ is a lower bound for $u_\lambda(x) - u_\lambda^*(x)$ for $x \in \partial B_\lambda$ which we show as follows. For $x \in \partial B_\lambda$, we have that $|x| \geq 1 - K^{-1}$ and $|x^* - \lambda| \geq K^{-j}$, hence
\[
u_\lambda(x) - u_\lambda^*(x) = 1 - \left[ \frac{\phi_\lambda}{|x| |x^* - \lambda|} \right]^{d-2} \geq 1 - \left[ \frac{K \phi_{j-1}}{(K-1) K^{-j}} \right]^{d-2}.
\]
It follows from (2.2) that
\[
\lim_{j \to \infty} \frac{\phi_{j-1}}{K^{-j}} = 0.
\]
Thus, there exists $N$ such that for $j > N$
\[
u_\lambda(x) - u_\lambda^*(x) > \frac{1}{2}.
\]
Thus, $h(x)$ satisfies the required criteria and is an upper bound for the harmonic measure $\omega(x, \partial B_\lambda; \mathbb{B} \setminus B_\lambda)$. 
Next, we want an upper estimate for \( h(0) \). We first note that as \( x \to 0 \), \( x^* \to \infty \) and also that \( |x||x^*| = 1 \). Thus, as \( x \to 0 \), \( u^*_\lambda(x) \to \phi^d_\lambda \). Next,

\[
\frac{1}{2} h(0) = \lim_{x \to 0} |u^*_\lambda(0) - u^*_\lambda(0)| = \left( \frac{\phi^d_\lambda}{|\lambda|} \right)^{d-2} - \phi^d_\lambda = \phi^d_\lambda \left[ 1 - \frac{|\lambda|^{d-2}}{|\lambda|^{d-2}} \right]
\]

\[
\leq \left( \frac{\phi^d_{j-1}}{|\lambda|} \right)^{d-2} (d-2) \left[ K^{-(j-1)} + O(K^{-2j}) \right].
\]

Thus, for sufficiently large \( j \),

\[
h(0) \leq 4K(d-2) \left( \frac{\phi^d_{j-1}}{1 - K^{-(j-1)}} \right)^{d-2} K^{-j}.
\]

Therefore,

\[
\omega(0, \partial B_r; \Omega_r) \leq \sum_{j=n_r}^{\infty} \frac{2^d dK^2}{e^d} K^{(d-1)j} 4K(d-2) \left( \frac{\phi^d_{j-1}}{1 - K^{-(j-1)}} \right)^{d-2} K^{-j}
\]

\[
\leq \frac{2^{d+2} d(d-2)K^{2d-1}}{e^d(K-1)^{d-2}} \sum_{j=n_r}^{\infty} (\phi^d_{j-1}K^{j-1})^{d-2} < 1
\]

provided \( n_r \) is suitably selected as described at the start of the proof. Thus, \( \omega(0, \partial B_r; \Omega_r) < 1 \) and hence we see that \( \omega(0, \partial B; \Omega) > 0 \) as required.

### 3.2. Avoidability implies the integral condition \( (1.1) \).

Now we assume that \( \omega(0, \partial B; \Omega) > 0 \) and we’ll show \( (1.1) \) holds. We begin by ignoring all obstacles with centres in an annulus \( A_j \) where \( j \) is odd. We let

\[
\Omega' = \mathbb{B} \setminus \bigcup_{\lambda \in A_j, \ j \ even} B(\lambda, r_\lambda)
\]

and note that since \( \omega(0, \partial B; \Omega) > 0 \), then \( \omega(0, \partial B; \Omega') > 0 \). We choose \( K > \max \{4, \frac{1+R}{4} \} \), where \( R \) is the constant mentioned in the definition of regularly spaced. We let \( P_j \) denote the probability that Brownian motion starting at the origin hits \( S_{j+1} \) before hitting any of the obstacles with centres in \( B_j \) but not in any \( A_i \) where \( i \) is odd. We let \( Q_j \) denote the supremum of the probabilities that Brownian motion starting on \( S_{j-1} \) hits \( S_{j+1} \) before hitting any of the obstacles with centres in \( A_j \). We note that \( P_j \leq Q_j P_{j-2} \) and that therefore for \( n \) even

\[
P_n \leq P_0 \prod_{j=1, \ j \ even}^n Q_j.
\]

Since \( \omega(0, \partial B; \Omega') = \delta > 0 \), it follows that \( P_n \geq \delta \) for all \( n \) and, since \( Q_j < 1 \),

\[
\sum_{j=1, \ j \ even}^{\infty} (1 - Q_j) < \infty.
\]

We note that \( 1 - Q_j \) is the infimum over \( x \in S_{j-1} \) of the probability that Brownian motion starting at \( x \) hits a ball with centre in \( A_j \) before hitting \( S_{j+1} \). Thus, if we consider only a single ball near \( x \), say \( B_{\lambda_x} \) where \( \lambda_x \) is the centre of the ball near \( x \) as described in Lemma 3.1, then

\[
1 - Q_j \geq \inf_{x \in S_{j-1}} \omega(x; \partial B_{\lambda_x}; B_{j+1} \setminus B_{\lambda_x}).
\]
Thus, we need a lower bound for \( \omega(x, \partial B_{\lambda x}; B_{j+1} \setminus B_{\lambda x}) \). We want a suitable function \( h_j \) that is harmonic on \( B_{j+1} \setminus B_{\lambda x} \), continuous on its closure and also satisfies
\[
h_j(y) \leq 1, \quad y \in \partial B_{\lambda x} \quad \text{and} \quad h_j(y) \leq 0, \quad y \in S_{j+1}.
\]
Then we can again avail of the Maximum Principle to obtain the required lower bound. Consider the function
\[
h_j(y) = u_\lambda(y) - u_\lambda^*(y),
\]
where
\[
u_\lambda(y) = \left[ \frac{\phi_\lambda}{|y - \lambda x|} \right]^{d-2}, \quad u_\lambda^*(y) = \left( \frac{\rho_{j+1}}{|y|} \right)^{d-2} \left[ \frac{\phi_\lambda}{|y^* - \lambda x|} \right]^{d-2}
\]
and \( y^* = \frac{\rho_{j+1}}{|y|^2} y \). Then \( h_j(y) \) satisfies the required criteria as both \( u_\lambda \) and \( u_\lambda^* \) are harmonic, \( h_j \leq u_\lambda = 1 \) on \( \partial B_{\lambda x} \), and \( u_\lambda = u_\lambda^* \) on \( S_{j+1} \). Next, we want a lower estimate for \( h_j \) at the point \( x \in S_{j-1} \). With the help of Lemma 3.1,
\[
u_\lambda(x) - u_\lambda^*(x) = \left[ \frac{\phi_\lambda}{|x - \lambda x|} \right]^{d-2} - \left( \frac{\rho_{j+1}}{\rho_{j-1}} \right)^{d-2} \left[ \frac{\phi_\lambda}{|x^* - \lambda x|} \right]^{d-2}
\]
\[
\geq \left( \frac{\phi_j}{|x - \lambda x|} \right)^{d-2} \left[ 1 - \left( \frac{\rho_{j+1}}{D \rho_{j-1}} \right)^{d-2} \right],
\]
where \( D = K/(K - 1) > 1 \). Then for sufficiently large \( j \), namely \( j \) where
\[
\frac{\rho_{j+1}}{\rho_{j-1}} < \frac{1 + D}{2},
\]
we find that
\[
u_\lambda(x) - u_\lambda^*(x) \geq c \left( \frac{\phi_j}{|x - \lambda x|} \right)^{d-2},
\]
where \( c \) is some positive constant.

By (3.1), we find that for \( x \in S_{j-1} \),
\[
\omega(x, \partial B_{\lambda x}; B_{j+1} \setminus B_{\lambda x}) \geq h_j(x) = u_\lambda(x) - u_\lambda^*(x) \geq c(K - 1)^{2-d}(\phi_j K^j)^{d-2}.
\]
It now follows from (3.2) that
\[
\sum_{j=1, \text{ even}}^{\infty} (\phi_j K^j)^{d-2} < \infty.
\]
Similarly it may be shown that
\[
\sum_{j=1, \text{ odd}}^{\infty} (\phi_j K^j)^{d-2} < \infty,
\]
and so
\[
\sum_{j=1}^{\infty} (\phi_j K^j)^{d-2} < \infty.
\]
Hence, (1.1) holds and the proof is complete.

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