Iterative Methods for Approximation of Fixed Points Via Like Contraction Mappings

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Abstract. The aim of this paper, is to study different iteration algorithms types two steps called, modified SP, Ishikawa, Picard-S iteration and M-iteration, which is faster than of others by using like contraction mappings. On the other hand, the M-iteration is better than of modified SP, Ishikawa and Picard-S iterations. Also, we support our analytic proof with a numerical example.

1. Introduction
The theory of fixed points requires a significant amount of literature since it offers valuable knowledge. Resources for solving many problems which have applications in various fields, such as Technology, economics, game theory and chemistry, etc. Nonetheless, once the presence of a fixed point in any mapping is known, so it is not an easy task to find the value of that fixed point, which is why we use iterative processes to compute them. Many iterative processes have been generated by time and it is difficult to cover all of them. The well-known Banach contraction theorem uses the Picard iteration method for fixed point approximation and some well-known Some of the iterative processes are Mann and modified SP, Ishikawa, Picard-S iteration and M-iteration. The Banach_contraction theory states that fixed point of contraction mappings can be approximated via Picard algorithm iterative. Let \( C \) be a closed convex, nonempty subset of a Banach space \( E \) and \( T: C \rightarrow C \) be a mapping, then \( T \) is called contraction mapping if there exist \( c \in [0,1) \) such that \( \|T a - T b\| \leq c\|a - b\| \) for all \( a, b \in C \). If \( c = 1 \) then we say that the mapping \( T \) is nonexpansive. We know that a point \( v \in C \) is a fixed point of \( T \) if \( T v = v \). It is well known that the algorithm scheme of Picard does not converges to a fixed point of nonresponsive mappings, so Mann[1] propose d a new algorithm iterative in 1953 to approach the fixed points of non-expansive mappings.

The iteration \((m_n)\) is defined in this algorithm as follows:
\[
m_0 \in C, \quad \forall n+1 = (1 - a_n)m_n + a_nTm_n
\]
where \((m_n)\) is a real sequence lies in \((0, 1)\), satisfying appropriate conditions There are other algorithm that many researchers are studied see[2-14].
Now, we recall some algorithm schemes (as modified SP, Ishikawa, Picard-S iteration and M-iteration). These algorithm schemes are given below:

1. The m-iteration schemes is defined as follows:
   \[ d_{n+1} = \mathcal{S}c_n \]
   \[ c_n = \mathcal{T}((1 - a_n)d_n + a_n\mathcal{T}d_n) \]
   where \( \mathcal{T} \) and \( \mathcal{S} \) are like contraction

2. The Picard-S iteration schemes is defined as follows:
   \[ e_{n+1} = \mathcal{T}((1 - a_n)Se_n + a_n\mathcal{T}z_n) \]
   \[ z_n = (1 - b_n)e_n + b_n\mathcal{T}e_n \]

3. The modified SP- iteration schemes [17,18] is defined as follows:
   \[ m_{n+1} = \mathcal{S}((1 - a_n)v_n + a_n\mathcal{T}v_n) \]
   \[ v_n = (1 - b_n)m_n + b_n\mathcal{T}m_n \]

4. The Ishikawa iteration schemes is defined as follows:
   \[ \kappa_{n+1} = (1 - a_n)\kappa_n + a_n\mathcal{T}v_n \]
   \[ v_n = (1 - b_n)\kappa_n + b_n\mathcal{S}\kappa_n \]
   \[ a_0 \in \mathcal{C}, n \in \mathbb{N} \]

1.1 Definition:[20]
A mapping \( \mathcal{T}: \mathbb{X} \to \mathbb{X} \) is called like contraction if there exist \( L \in [0,1) \) and strictly increasing and continuous function \( \emptyset: [0,1) \to [0,1) \) such that \( \emptyset(0) = 0 \), and
\[ \| \mathcal{T}x - \mathcal{T}y \| \leq L \| x - y \| + \emptyset(\| x - \mathcal{T}x \|) \]
Clearly, the class of contractive-like mappings is wider than the class of quasi-contractive mappings.

1.2 Definition: [21]
Let \( < a_n >, < b_n > \) are sequences lies in \( \mathbb{R} \) such that \( < a_n > \) converge to \( a \), and \( < b_n > \) converge to \( b \) and
\[ L = \lim_{n \to \infty} \frac{|a_n - a|}{|b_n - b|} \]
1. If \( L = 0 \) \( \implies \) The sequence \( < a_n > \) is converge to \( a \) faster then \( < b_n > \) converge to \( b \)
2. If \( 0 < L < \infty \) \( \to \) \( < a_n > \) and \( < b_n > \) have the same rate of convergence
2. Main Results

During this section, we will study the convergence better of different types of schemes by using like contraction Mappings. Let \((N, \| . \|)\) be a Normed Space, \( \emptyset \neq C \) be a convex closed- subset of \( N \).

2.1 Theorem: Let \( T, S \) be a like contraction mappings on \( C \), suppose that the iterations M-
iteration scheme and Picard-S iteration scheme converges to \( y \) and \( y \in F(T) \cap F(S) \cap F(P) \)
, if \( 0< \lambda < a_n, b_n, c_n < 1. \forall n \in \mathbb{N} \). Then the M-iteration scheme converges faster than Picard-
S iteration scheme .

Proof: Consider the M-iteration scheme, we have
\[ \|d_{n+1} - y\| = \|Sd_n - y\| \]
\[ = L \|d_n - y\| + \phi(\|y - Sd_n\|) \]
\[ = L \|(1 - a_n)d_n + a_nTd_n\) - y\| \]
\[ \leq L \left( \|d_n - y\| + a_n \|Td_n - y\| \right) \]
\[ \leq L^2 (\|d_n - y\| + a_n \|d_n - y\| + \phi(\|y - Td_n\|)) \]
\[ = L^2 (\|d_n - y\| + a_n L \|d_n - y\|) \]
\[ = L^2 (1 - a_n)(1 - L) \|d_n - y\| \]
\[ \leq L^2 (1 - \lambda (1 - L)) \|d_n - y\| \]
\[ : \]
\[ \leq L^{2n} (1 - \lambda (1 - L))^n \|d_0 - y\| \]

Let \( A_n = L^{2n} (1 - \lambda (1 - L))^n \|d_0 - y\| \)

From Picard-S iteration scheme , we obtain
\[ \|e_{n+1} - y\| = \|T((1 - a_n)Se_n + a_nTe_n - y\| \]
\[ = L \|(1 - a_n)Se_n + a_nTe_n - y\| + \phi(\|y - Td_n\|) \]
\[ \leq L \|(1 - a_n)Se_n - y\| + \phi(\|y - Td_n\|) \]
\[ \leq L \left( (1 - a_n)(L \|e_n - y\| + \phi(\|y - Sd_n\|)) + a_n (L \|z_n - y\| + \phi(\|y - Td_n\|)) \right) \]
\[ = L((1 - a_n)L \|e_n - y\| + a_n L \|z_n - y\|) \]
\[ L((1 - a_n)L\|e_n - \gamma\| + a_nL(1 - b_n)e_n + b_nT e_n - \gamma\|) \]
\[ = L^2((1 - a_n)\|e_n - \gamma\| + a_n(1 - b_n)(e_n - \gamma) + b_n\|T e_n - \gamma\|) \]
\[ \leq L^2((1 - a_n)\|e_n - \gamma\| + a_n((1 - b_n)\|e_n - \gamma\| + b_n\|T e_n - \gamma\|)) \]
\[ \leq L^2((1 - a_n)\|e_n - \gamma\| + a_n(1 - b_n)\|e_n - \gamma\| + b_n(L\|e_n - \gamma\| + \mathcal{O}(\|\gamma - Ty\|))) \]
\[ = L^2((1 - a_n) + a_n(1 - b_n) + a_n b_n L)\|e_n - \gamma\| \]
\[ = L^2(1 - a_n b_n (1 - L))\|e_n - \gamma\| \]
\[ \leq L^2(1 - \lambda^2 (1 - L))\|e_n - \gamma\| \]
\[ \vdots \]
\[ \leq L^{2n}(1 - \lambda^2 (1 - L))^n\|e_0 - \gamma\| \]

Let \( I_n = L^{2n}(1 - \lambda^2 (1 - L))^n\|e_0 - \gamma\| \)

Here after simple compute we have

\[
\frac{A_n}{I_n} = \frac{L^{2n}(1 - \lambda(1 - L))^n\|d_0 - \gamma\|}{L^{2n}(1 - \lambda^2 (1 - L))^n\|e_0 - \gamma\|} \rightarrow 0 \quad \text{as} \ n \rightarrow \infty
\]

2.2 Theorem: Let \( T, S \) be a like contraction mappings on \( C \), suppose that the iterations M-iteration scheme and modified-SP iteration scheme converge to \( \gamma \) and \( r \in F(T) \cap F(S) \cap F(P) \), if \( 0 < \lambda < a_n, b_n, c_n < 1, \forall n \in \mathbb{N} \). Then the M-iteration scheme converges faster than modified-SP iteration scheme.

Proof: For modified-SP iteration scheme, we have

\[ \|m_{n+1} - \gamma\| = \|S((1 - a_n)\sigma_n + a_n T \sigma_n) - \gamma\| \]
\[ = L\|(1 - a_n)\sigma_n + a_n T \sigma_n - \gamma\| + \mathcal{O}(\|\gamma - Sy\|) \]
\[ \leq L((1 - a_n)\|\sigma_n - \gamma\| + a_n \|T \sigma_n - \gamma\|) \]
\[ \leq L(\|(1 - a_n)\|\sigma_n - \gamma\| + a_n (L\|\sigma_n - \gamma\| + \mathcal{O}(\|\gamma - T \gamma\|))) \]
\[ = L(1 - a_n (1 - L))\|\sigma_n - \gamma\| \]
\[ = L(1 - a_n (1 - L))\|(1 - b_n)m_n + b_n T m_n - \gamma\| \]
\[ \leq L(1 - a_n (1 - L))((1 - b_n)\|m_n - \gamma\| + b_n \|T m_n - \gamma\|) \]
\[ \leq L(1 - a_n (1 - L))\((1 - b_n)\|m_n - \gamma\| + b_n (L\|m_n - \gamma\| + \mathcal{O}(\|\gamma - T \gamma\|))) \]
\[ = L(1 - a_n (1 - L))((1 - b_n)\|m_n - \gamma\| + b_n L\|m_n - \gamma\|) \]
\[
\begin{align*}
&= L(1 - a_n(1 - L))(1 - b_n(1 - L))\|m_n - \gamma\| \\
&\leq L(1 - \lambda(1 - L))(1 - \lambda(1 - L))\|m_n - \gamma\| \\
&= L(1 - \lambda(1 - L))^2\|m_n - \gamma\| \\
&\vdots \\
&\leq L^n(1 - \lambda(1 - L))^{2n}\|m_0 - \gamma\|
\end{align*}
\]

\[M_n = L^n(1 - \lambda(1 - L))^{2n}\|m_0 - \gamma\|\]

Here after simple compute we have

\[\frac{A_n}{M_n} = \frac{L^{2n}(1 - \lambda(1 - L))^n\|d_0 - \gamma\|}{L^n(1 - \lambda(1 - L))^{2n}\|m_0 - \gamma\|} \to 0 \text{ as } n \to \infty.
\]

2.3 Theorem: Let \(T, S\) be alike contraction mappings on \(C\), suppose that the iterations \(M\)-iteration scheme and ishikawa iteration scheme converge to \(\gamma\) and \(\gamma \in F(T) \cap F(S) \cap F(P)\). if \(0 < \lambda < a_n, b_n, c_n < 1, \forall n \in \mathbb{N}\). Then the \(M\)-iteration scheme converges faster than Ishikawa iteration scheme.

Proof: for Ishikawa iteration scheme, we obtain

\[\|\kappa_{n+1} - \gamma\| = \|(1 - a_n)\kappa_n + a_nT\nu_n - \gamma\|\]

\[\leq (1 - a_n)\|\kappa_n - \gamma\| + a_n\|T\nu_n - \gamma\|\]

\[\leq (1 - a_n)\|\kappa_n - \gamma\| + a_n(\|v_n - \gamma\| + \varnothing(||\gamma - T\gamma||))\]

\[= (1 - a_n)\|\kappa_n - \gamma\| + a_n\|v_n - \gamma\| + a_n\|\kappa_n - \gamma\| + b_n\|S\kappa_n - \gamma\|\]

\[\leq (1 - a_n)\|\kappa_n - \gamma\| + a_n L((1 - b_n)\|\kappa_n - \gamma\| + b_n\|S\kappa_n - \gamma\|)\]

\[\leq (1 - a_n)\|\kappa_n - \gamma\| + a_n L((1 - b_n)\|\kappa_n - \gamma\| + b_n(L\|\kappa_n - \gamma\| + \varnothing(||\gamma - S\gamma||)))\]

\[= (1 - a_n)\|\kappa_n - \gamma\| + a_n L((1 - b_n)\|\kappa_n - \gamma\| + b_n L\|\kappa_n - \gamma\|)\]

\[= (1 - a_n \lambda(1 - L) - La_n b_n(1 - L))\|\kappa_n - \gamma\|\]

\[\vdots \\
\leq (1 - \lambda(1 - L) - L \lambda^2(1 - L))\|\kappa_n - \gamma\| \\
\vdots \\
\leq (1 - \lambda(1 - L) - L \lambda^2(1 - L))^n\|\kappa_0 - \gamma\|\]
\[ Y_n = (1 - \lambda (1 - L) - L \lambda^2 (1 - L))^n \| k_0 - \gamma \| \]

Here after simple compute we have

\[ A_n \sim \frac{L^{2n} (1 - \lambda (1 - L))^n}{(1 - \lambda (1 - L) - L \lambda^2 (1 - L))^n} \| d_0 - \gamma \| \to 0 \quad \text{as} \quad n \to \infty \]

2.4 Corollary: Let \( T, S, P \) be a almost contraction mappings on \( C \). Suppose that the iterations M-iteration scheme and Picard-S iteration scheme converge to \( \gamma \in F(T) \cap F(S) \cap F(P) \) where \( 0 < \lambda \leq a_n, b_n, c_n < 1 \), \( \forall n \in N \). Then the M-iteration scheme converges faster than Picard-S iteration scheme.

2.5 Corollary: Let \( T, S \) be a most contraction mappings on \( C \). Suppose that the iterations M-iteration scheme and modified-SP iteration scheme converge to \( \gamma \in F(T) \cap F(S) \cap F(f) \cap F(P) \) where \( 0 < \lambda \leq a_n, b_n, c_n < 1 \), \( \forall n \in N \). Then the M-iteration scheme converges faster than modified-SP iteration scheme.

2.6 Corollary: Let \( T, S \) be almost contraction mappings on \( C \). Suppose that the iterations M-iteration scheme and Ishikawa iteration scheme converge to \( \gamma \in F(T) \cap F(S) \) where \( 0 < \lambda \leq a_n, b_n, c_n < 1 \), \( \forall n \in N \). Then the M-iteration scheme converges faster than Ishikawa iteration scheme.

Example 2.7: Let us define a function \( T: [0,50) \to [0,50) \) by \( T(x) = \sqrt{2x + 3} \). Then clearly \( T \) is like contraction map. Let \( a_n = 0.85 \) and \( b_n = 0.65 \) for all \( n \). Set the stop parameter to \( \| x_n - 3 \| \leq 10^{-9} \), where 3 is fixed point of \( T \). The iterative values for initial value \( x_0 = 2.5 \) are given in Table 1.

| \( x_n \) | M-iteration | Picard-S | Modified-SP | Ishikawa   |
|----------|-------------|----------|-------------|------------|
| 1        | 2.5         | 2.5      | 2.5         | 2.5        |
| 2        | 2.975050148 | 2.963696219 | 2.957872619 | 2.842483572 |
| 3        | 2.998796654 | 2.997445966 | 2.996543701 | 2.950868727 |
| 4        | 2.999942056 | 2.999820714 | 2.999717041 | 2.984721468 |
| 5        | 2.999997210 | 2.999987416 | 2.999976839 | 2.995253188 |
| 6        | 2.999999865 | 2.9999999116 | 2.999998104 | 2.99852566 |
| 7        | 2.999999993 | 2.999999938 | 2.999999845 | 2.999542117 |
| 8        | 2.999999999 | 2.999999959 | 2.999999887 | 2.99985780 |
| 9        | 3           | 2.999999999 | 2.999999999 | 2.999955839 |
Conclusion

In this section, a two-step iterative process is studied which is used for fixed-point approximation for the like contraction mappings. Moreover, we show that the M-iteration converges faster than modified SP, Ishikawa and Picard-S iterations. However, we not only give the evidence analytically, but also verify its authenticity with an example.

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