Scalar Solitons in Non(anti)commutative Superspace

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Abstract

We study solitonic solutions of a deformed Wess-Zumino model in 2 dimensions, corresponding to a deformation of the usual $\mathcal{N} = 1, D = 2$ superspace to the one with non-anticommuting odd supercoordinates. The deformation turns out to add a kinetic term for the auxiliary field besides the known $F^3$ term coming from the deformation of the cubic superpotential. Both these modifications are proportional to the effective deformation parameter $\lambda \equiv \det C$, where $C$ denotes the non-anticommutativity matrix. We find a modified “orbit” equation which on the EOM relates the auxiliary and the scalar components of the scalar superfield as a first order correction to the usual relation in terms of the small parameter $\lambda$. Subsequently, we obtain the modified form of the first order BPS equation for the scalar field and find its solution to first order in $\lambda$. Issues such as modification of the BPS mass formula and a non-linear realization of the $\mathcal{N} = 1$ supersymmetry are discussed.

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1 Introduction

There have been several occasions in string theory where the study of string dynamics near certain corners of the moduli space of the theory has led to some new insights in field theory. One of the most known examples is the emergence of spacetime noncommutative field theories from string theory in the presence of an NS-NS background field [1] (see e.g. [2] for a review). Another example has been provided more recently by the work of Seiberg [3] (and also [4]), inspired by earlier works of Ooguri and Vafa [5, 6] in the context of the gauge theory/ matrix model correspondence (see e.g. [7]-[9]). Both these works (as well as some others such as [10]) have derived a new type of non(anti)commutative field theories from string theory in the presence of a constant graviphoton background field. These can be formulated as field theories on a superspace whose grassmann odd supercoordinates obey a Clifford like algebra, rather than the usual (anti)commutation relations. This type of noncommutativity had also been considered earlier mainly from an algebraic point of view [11]-[13]. This is an alternative deformation in contrast to the ordinary noncommutativity among the bosonic coordinates [14]-[19].

The work by Seiberg has initiated a new trend of research on various aspects of field theories formulated on such deformed superspaces, which usually (in a 4-dimensional context) are known as $\mathcal{N} = 1/2$ theories [3, 4] (for a partial list see [20]-[29]). In particular, the deformations of some $\mathcal{N} = 1$ theories (with $\mathcal{N} = \frac{1}{2}$ SUSY), such as the WZ model and SYM theories with or without matter have been investigated in [3, 20, 21]. The deformation of $\mathcal{N} = 2$ SYM theories (with $\mathcal{N} = \frac{1}{2} + \frac{3}{2}$ SUSY) has also been considered in [4, 22]. Some perturbative aspects of this class of theories, such as their renormalizability, has been recently studied in a number of papers [23]-[28]. Most of these papers deal with formulation of deformed theories in a euclidean superspace. The Minkowskian case has been considered in ref.[29]. A combination of the noncommutativity among both the bosonic and fermionic coordinates of superspace has also been studied in [11, 24, 25].

While by now a rather rich collection of information on perturbative aspects of the deformed theories on non(anti)commutative superspaces has been obtained, yet less has been known about their non-perturbative aspects such as their solitonic solutions of the form of lumps, monopoles, instantons, etc. (see however the comments in [3]). It is interesting, for example, to know to what extent such solutions parallel their counterparts in the ordinary noncommutative field theories [30] (for a review of the latter see, e.g.,
As a first step in this direction, in this paper we explicitly work out the example of 1/2 supersymmetric BPS solutions in a deformed Wess-Zumino model in 2 dimensions, whose commutative counterpart has been studied long ago by Witten and Olive [32]. In 2 dimensional deformed WZ model, as in its 4-dimensional counterpart, a modification proportional to $F^3$ (with $F$ as the auxiliary component of the scalar superfield) arises which breaks all the $\mathcal{N} = 1$ supercharges explicitly. But here, unlike in 4-dimensional case, also a kinetic part for $F$ appears which makes it a dynamical field in contrast to the undeformed theory and it also breaks the whole supersymmetry explicitly.

The reason why we adhere to 2 spacetime dimensions is that the deformation of superspace by non-anticommuting odd coordinates in a theory in any dimensions does not violate the conditions of Derrick theorem [33], as it does not bring higher order derivatives into the spacetime formulation of the theory. As a result, scalar solitons (or lumps) can indeed do exist only in 2 spacetime dimensions just as the case in the undeformed theory. This is in contrast to the ordinary noncommutative solitons where the violation of Derrick theorem due to higher order derivatives allows for the existence of scalar solitons in higher spacetime dimensions [30].

As we mentioned above, in the 2-dimensional case, the deformation breaks both the two supersymmetries of the theory as a result of the fact that 2-dimensional SUSY generators do not anti-commute with the grassmann odd derivatives appearing in the definition of the deforming star product algebra. In the same spirit of [3, 4], we may like to call the deformed model as an $\mathcal{N} = 0$ or non-supersymmetric theory. However, we shall find that, on a certain surface relating the auxiliary and scalar fields (called as the “orbit”), the model still preserves the $\mathcal{N} = 1$ SUSY somehow non-linearly. This should be compared to a non-linear realization of the broken $\mathcal{N} = 2$ SUSY in the case of the 4-dimensional deformed $\mathcal{N} = 2$ SYM theory found in [4].

The organization of this paper is as follows: In sec.2 we review some elementary facts about the deformed algebra of functions of two grassmann odd variables. We then introduce in sec.3 the deformed generators of the $\mathcal{N} = 1$ SUSY and the corresponding superderivatives on the basis of which we construct the deformed WZ model. In sec.4 we obtain the modified BPS equation governing continuous deformation of the ordinary BPS solutions in

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[31])

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2We will work throughout this paper with a euclideanized version of the WZ model in [32] in order to adapt the notation we had used in the previous works [16]-[19].
the undeformed WZ model and solve them to first order in the deformation parameter. In sec.5 we obtain an expressions for a classical effective super-potential arising in the deformed WZ model, using which we compute the deformation of the usual BPS mass formula. Finally, in sec.6 we show that the $\mathcal{N} = 1$ SUSY has a non-linear realization in terms of an effective non-linear $\sigma$-model describing the dynamics on a relevant orbit in the deformed theory.

2 The deformed Grassmann Algebra

In this section we review some basic facts regarding the deformed algebra of functions of two grassmann odd variables $\theta^\alpha \equiv (\theta^+, \theta^-)$ defining a $SO(2)$ spinor, corresponding to the algebra of superfields on a deformed $\mathcal{N} = 1$, $D = 2$ euclidean superspace (we will use notations of [18, 19]). The starting point for defining the deformed algebra is the non(anti)commutation relation between the odd variables, written as the Clifford algebra

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta}, \quad (1)$$

in which $C^{\alpha\beta} = C^{\beta\alpha}$ is a constant bispinor and the hats are used to interpret $\theta$’s here as operators. Defining the algebra of functions of odd coordinates via the odd star product as [3]

$$f(\theta) \ast g(\theta) \equiv f(\theta) \exp \left( -\frac{1}{2} C^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \right) g(\theta), \quad (2)$$

one then finds a representation of the Clifford algebra (1) on the space of ordinary (anticommuting) grassmann odd variables. Here the left and right differentiations on a function $f(\theta)$ of even or odd grassmann party are simply related as

$$f(\theta) \frac{\partial}{\partial \theta^\alpha} = (-1)^{\text{deg} f} \frac{\partial}{\partial \theta^\beta} f(\theta) \quad (3)$$

where $\text{deg} f = 0, 1(\text{mod} 2)$ for $f$ even or odd, respectively. We have to stress that distinguishing between the left and right differentiations in eq.(2) is important in order to have explicit isomorphism between the algebra (2) and the Clifford algebra and in proving the associativity of the star product. In particular, it can be easily verified that

$$\theta^\alpha \ast \theta^\beta = \theta^\alpha \theta^\beta + \frac{1}{2} C^{\alpha\beta}, \quad (4)$$

$$(\theta^\alpha \ast \theta^\beta) \ast \theta^\gamma = \theta^\alpha \ast (\theta^\beta \ast \theta^\gamma) = \theta^\alpha \theta^\beta \theta^\gamma + \frac{1}{2} (C^{\alpha\beta} \theta^\gamma + C^{\beta\gamma} \theta^\alpha - C^{\gamma\alpha} \theta^\beta).$$
(Note that in our 2d case the term $\theta^\alpha \theta^\beta \theta^\gamma$ in the second relation identically vanishes.) Using the ordinary rules of the grassmann calculus one can then see that the above defining equation for the star product terminates at a finite order of $\partial/\partial \theta$’s. In particular, the exponential expansion in this equation takes the simple form

$$f(\theta) \ast g(\theta) = fg + (-1)^{\deg f} \frac{1}{2} C^{\alpha\beta} \frac{\partial f}{\partial \theta^\alpha} \frac{\partial g}{\partial \theta^\beta} - \frac{1}{4} \det C \frac{\partial^2 f}{\partial \theta^2} \frac{\partial^2 g}{\partial \theta^2}, \quad (5)$$

where now a grassmannian derivative is understood to act from the left (i.e. $\frac{\partial}{\partial \theta^\alpha} \equiv \overrightarrow{\partial}_{\theta^\alpha}$) and we have introduced the notation $\frac{\partial^2}{\partial \theta^2} \equiv \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \equiv \frac{1}{2} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta}$ (with $\theta_\alpha \equiv \epsilon_{\alpha\beta} \theta^\beta$).

Also, similar to the case of the ordinary (bosonic) star product [1, 2], one can show that the “inner product” of two functions of $\theta$ is independent of $C^{\alpha\beta}$, namely

$$\int d^2 \theta \ f(\theta) \ast g(\theta) = \int d^2 \theta \ f(\theta)g(\theta), \quad (6)$$

which corresponds to the fact that the difference

$$f(\theta) \ast g(\theta) - f(\theta)g(\theta) = \frac{\partial}{\partial \theta^\alpha} (\cdots)^\alpha \quad (7)$$

is a total grassmannian derivative not surviving the grassmannian integrations. Often we denote this equivalence relation symbolically as $f \ast g \cong fg$.

We can find explicit expressions for powers of $f(\theta)$ under the star product using the general expression given in eq.(5). For this, we first recall the basic expression

$$f_\ast^2 = f \ast f = f^2 - \frac{1}{4} \det C \left( \frac{\partial^2 f}{\partial \theta^2} \right)^2, \quad (8)$$

from which, for example, the third power of $f$ follows as

$$f_\ast^3 = f \ast (f_\ast^2) = f^2 - \frac{1}{4} \det C \left[ 3f \left( \frac{\partial^2 f}{\partial \theta^2} \right)^2 + 2 \frac{\partial f}{\partial \theta^+} \frac{\partial f}{\partial \theta^-} \frac{\partial^2 f}{\partial \theta^2} \right], \quad (9)$$

and so forth for $f_\ast^n$. We note that all such expressions can totally be written in terms of the second derivatives of $f$ and $f^2$ using the identity

$$\frac{\partial^2 f^2}{\partial \theta^2} = 2 \frac{\partial f}{\partial \theta^+} \frac{\partial f}{\partial \theta^-} + 2 f \frac{\partial^2 f}{\partial \theta^2}. \quad (10)$$
In this 2-dimensional case, further simplifications occur in computing higher powers of $f$ (or generally any function of $f$) due to the fact that $\frac{\partial^2}{\partial \theta^2}$ of every function of $\theta$ is independent of $\theta$. A generic feature of such expressions for $f^n$ is that they depend on $C^{\alpha\beta}$ only through its determinant $\det C = \frac{1}{2} C^{\alpha\beta} C_{\alpha\beta}$, which is a Lorentz invariant quantity. Also all such expressions contain a total derivative part which can be neglected upon Grassmannian integrations. Often we can separate the total derivative part of $f^n$ using the lemma $\int d^2 \theta f^* g = \int d^2 \theta f g$ and the above expression for $f^*_2$. For example, using

\[ \int d^2 \theta f^*_2 = \int d^2 \theta f^2, \]
\[ \int d^2 \theta f^*_3 = \int d^2 \theta f f^*_2 = \int d^2 \theta f \left[ f^2 - \frac{1}{4} \det C \left( \frac{\partial^2 f}{\partial \theta^2} \right)^2 \right], \]
\[ \int d^2 \theta f^*_4 = \int d^2 \theta (f^*_2)^2 = \int d^2 \theta \left[ f^2 - \frac{1}{4} \det C \left( \frac{\partial^2 f}{\partial \theta^2} \right)^2 \right]^2, \] (11)

we conclude the identities

\[ f^*_2 \cong f^2, \]
\[ f^*_3 \cong f^3 - \frac{1}{4} \det C f \left( \frac{\partial^2 f}{\partial \theta^2} \right)^2, \]
\[ f^*_4 \cong f^4 - \frac{1}{2} \det C f^2 \left( \frac{\partial^2 f}{\partial \theta^2} \right)^2 + \frac{1}{16} (\det C)^2 \left( \frac{\partial^2 f}{\partial \theta^2} \right)^4. \] (12)

As these expressions suggest, ignoring total derivative terms, the deformed version of $f^n$ (as well as for any analytic function of $f$) is always written as a polynomial (generally a power series) in powers of $\det C$. It is indeed this beauty of the star product in reducing all the $C$-dependence to the Lorentz invariant combination $\det C$ which causes a great deal of simplicity in formulation of a deformed Wess-Zumino model in 2-dimensions in later sections. This is also consistent with the fact that the deformation defined by $C^{\alpha\beta}$ does not break Lorentz invariance of the low energy field theory \[3, 5\].
3 The deformed Supersymmetry in Noncommutative $\mathcal{N} = 1$, $D = 2$ Superspace

Consider first the ordinary SUSY transformations acting on a superfield $S(x, \theta)$ via the generators\(^3\)

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\gamma_\alpha^\mu \theta^\beta \partial_\mu,$$  \hspace{1cm} (13)

which in components are written as

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\theta^\pm \partial_\pm.$$ \hspace{1cm} (14)

The deformed SUSY is defined as the transformations generated by acting $Q$’s on $S(x, \theta)$ via the star product operation, namely by

$$Q_\alpha * S = \left( \frac{\partial}{\partial \theta^\alpha} + i\gamma_\alpha^\mu \theta^\beta \partial_\mu \right) * S$$
$$= \frac{\partial S}{\partial \theta^\alpha} + i\gamma_\alpha^\mu \partial_\mu \left( \theta^\beta S + \frac{1}{2} C^\beta\gamma \frac{\partial S}{\partial \theta^\gamma} \right)$$
$$\equiv Q_\alpha S.$$ \hspace{1cm} (15)

This amounts to defining the deformed generators $Q_\alpha$ as

$$Q_\alpha \equiv Q_\alpha + i\frac{1}{2} \gamma_\alpha^\mu C^\beta\gamma \frac{\partial}{\partial \theta^\gamma} \partial_\mu,$$ \hspace{1cm} (16)

or in components

$$Q_+ \equiv Q_+ + i\frac{1}{2} \left( C^{++} \frac{\partial}{\partial \theta^+} + C^{+-} \frac{\partial}{\partial \theta^-} \right) \partial_+,$$
$$Q_- \equiv Q_- + i\frac{1}{2} \left( C^{-+} \frac{\partial}{\partial \theta^+} + C^{--} \frac{\partial}{\partial \theta^-} \right) \partial_-.$$ \hspace{1cm} (17)

The algebra obeyed by the deformed generators simply follows from these expressions to be

$$Q_+^2 = i\partial_+ - \frac{1}{2} C^{++} \partial_+^2,$$
$$Q_-^2 = i\partial_- - \frac{1}{2} C^{--} \partial_-^2,$$
$$\{Q_+, Q_-\} = -C^{+-} \partial_+ \partial_-.$$ \hspace{1cm} (18)

\(^3\)In the notation of [18], [19] the matrices $\gamma^\mu$ in terms of the 2-dimensional Dirac matrices are written as $\gamma^\mu = \rho^1 \rho^\mu$. 

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This can be thought as a closed algebra if one considers $\partial_+^2, \partial_-^2, \partial_+\partial_-$ on its r.h.s. as the “new” operators, which commute with $Q_+, Q_-$. The deformed supertranslation algebra (18) is in contrast to another one proposed in [17, 18] (see also [19]) via introducing a gauge connection on superspace.

In a similar fashion we can define the deformed supercovariant derivatives based on the corresponding undeformed derivatives, namely

$$D_\alpha \equiv D_\alpha - i\gamma_{\alpha\beta}^\mu C^\beta \frac{\partial}{\partial \theta}\gamma_\mu,$$

$$D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} - i\gamma_{\alpha\beta}^\mu \theta^\beta \partial_\mu.$$  

(19)

One can verify that (unlike in the undeformed case as well as in the deformed 4d case [3]) these have nonvanishing anticommutators with the deformed SUSY generators,

$$\{D_+, Q_+\} = C^{++} \partial_+^2, \quad \{D_+, Q_-\} = C^{+-} \partial_+ \partial_-,$$

$$\{D_-, Q_+\} = C^{-+} \partial_-^2, \quad \{D_-, Q_-\} = C^{-+} \partial_+ \partial_-.$$  

(20)

So, after all, any superspace lagrangian constructed on the basis of the deformed superderivatives will not be invariant under the deformed SUSY transformations. Using these derivatives, however, the most natural definition for the deformed version of the kinetic term of a real scalar superfield $S$ is written as follows

$$2\mathcal{L}_K[S] = D_+ S \ast D_- S \cong D_+ S D_- S.$$  

(21)

Unlike in ordinary noncommutative spaces, here the deformed kinetic term still contains terms depending on $C$, even after separating its total derivative part. To extract the $C$-dependence of this expression, we use the definition (19) of $D_\alpha$ together with the following identities

$$\frac{\partial f}{\partial \theta^+} \frac{\partial g}{\partial \theta^-} = \frac{\partial}{\partial \theta^\pm} \left( f \frac{\partial g}{\partial \theta^\pm} \right) \cong 0,$$

$$\frac{\partial f}{\partial \theta^+} \frac{\partial g}{\partial \theta^-} \cong -\frac{\partial f}{\partial \theta^-} \frac{\partial g}{\partial \theta^+}.$$  

(22)

from which we find

$$2\mathcal{L}_K[S] \cong D_+ S D_- S + \frac{i}{2} C^{++} D_- S \frac{\partial S}{\partial \theta^+} - \frac{i}{2} C^{-+} D_+ S \frac{\partial S}{\partial \theta^-}$$

$$+ \frac{1}{2} C^{+-} \left( \theta^+ \frac{\partial S}{\partial \theta^+} \theta_- - \theta^- \frac{\partial S}{\partial \theta^-} \theta_+ \right)$$

$$- \frac{1}{4} \det C \frac{\partial S}{\partial \theta^+} \frac{\partial S}{\partial \theta^-}.$$  

(23)
Surprisingly, we find that all the $F$-term contributions (i.e. $\theta^2$ components) to this lagrangian which are linear in $C^{\alpha\beta}$ are in the form of total derivatives and so we are left with the ordinary lagrangian plus a term proportional to $\det C$. More explicitly, by putting

$$S = \phi(x) + \bar{\psi}(x) + \frac{1}{2} \bar{\theta} F(x)$$

in the above expression, we find for its several terms

$$(D_+ S D_- S)_{\theta^2} = -\partial_+ \phi \partial_- \phi + i \psi_+ \partial_- \psi_+ + i \psi_- \partial_+ \psi_- + F^2$$

$$(D_- S \partial_+ \frac{\partial S}{\partial \theta^+})_{\theta^2} = i \partial_+ \psi_+ \partial_+ \psi_+ - F \partial_+ F$$

$$= \frac{i}{2} [\partial_-(\psi_+ \partial_+ \psi_+) - \partial_+(\psi_+ \partial_+ \psi_+)] - \frac{1}{2} \partial_+ F^2,$$

$$(D_+ S \partial_- \frac{\partial S}{\partial \theta^-})_{\theta^2} = i \partial_+ \psi_- \partial_- \psi_- - F \partial_- F$$

$$= \frac{i}{2} [\partial_+(\psi_- \partial_- \psi_-) - \partial_-(\psi_- \partial_- \psi_-)] - \frac{1}{2} \partial_- F^2,$$

$$\left( \partial^+ \partial_+ \frac{\partial S}{\partial \theta^+} - \partial^+ \partial_- \frac{\partial S}{\partial \theta^-} \right)_{\theta^2} = \partial_+ F \partial_- F.$$  \hfill (25)

Ignoring the surface terms, the final expression for the ordinary ($x$-space) $L_K$ thus becomes

$$L_K = \frac{1}{2} \left( \partial_+ \phi \partial_- \phi + \frac{1}{4} \det C \partial_+ F \partial_- F - i \psi_+ \partial_- \psi_+ - i \psi_- \partial_+ \psi_- - F^2 \right).$$  \hfill (26)

We observe that, unlike in the commutative theory, the auxiliary field $F$ becomes a dynamical field which cannot be solved in terms of the other variables algebraically. This is also in contrast to the role of the auxiliary field in SUSY field theories on the ordinary noncommutative spaces where there also this field appears through its derivatives, though to infinite order of its derivatives [15]. We note that the kinetic term of $F$ will have the correct sign (the same as for $\phi$), provided we choose $\det C > 0$. 8
3.1 The Model

We choose as our model the deformed WZ action of a single scalar superfield which in general consists of the above deformed kinetic lagrangian of $S$ plus a deformed scalar superpotential $W_*(S)$. Though, for our purpose in this paper, there is no essential restriction on the form of the undeformed superpotential $W(S)$, it is most convenient to work with the cubic superpotential

$$W(S) = \frac{1}{2}mS^2 + \frac{1}{3}gS^3. \quad (27)$$

By the rules of the previous section, the deformed superpotential then takes the form

$$W_*(S) = \frac{1}{2}mS^2 + \frac{1}{3}gS^3 + \frac{1}{12}g \det C S \left( \frac{\partial^2 S}{\partial \theta^2} \right)^2. \quad (28)$$

This contributes to the $F$-term as

$$[W_*(S)]_{g^2} = FW'(\phi) - \psi_+ \psi_- W''(\phi) - \frac{1}{12}g \det C F^3. \quad (29)$$

As such, the overall lagrangian of this system,

$$\mathcal{L} = - \int d^2\theta \left( \frac{1}{2}D_+ S + D_- S + W_*(S) \right) , \quad (30)$$

takes the explicit form

$$\mathcal{L} = \frac{1}{2} \partial_+ \phi \partial_- \phi + \frac{1}{8} \det C \partial_+ F \partial_- F - \frac{1}{2} \bar{\psi} \psi + \frac{1}{12}g \det C F^3 - \frac{1}{2} F^2 - FW'(\phi) + \frac{1}{2} \bar{\psi} \psi W''(\phi). \quad (31)$$

In later sections, we will consider the $\psi = 0$ configurations of this theory which provide trivial solutions to the EOM of fermions, and will focus on analyzing its solitons.
4 Solitons

The main question we are concerned in this paper is to find the finite energy solutions of the above model which are continuously connected to the BPS solutions of the corresponding undeformed model. The bosonic part of this model, obtained by putting $\psi = 0$ in eq.(31), reads

$$\mathcal{L}[\phi, F] = \frac{1}{2} \partial_+ \phi \partial_- \phi + \frac{\lambda}{8} \partial_+ F \partial_- F + V(\phi, F),$$  \hspace{1cm} (32)

with a potential term of the form

$$V(\phi, F) = \frac{\lambda}{12} g F^3 - \frac{1}{2} F^2 - FW'(\phi),$$  \hspace{1cm} (33)

where $W'(\phi) = m \phi + g \phi^2$ and for convenience we have denoted the deformation parameter by $\lambda \equiv \det C$. This is an example of coupled scalar field theories in 2 dimensions which finding their exact solitonic solutions is not a generally well posed problem (see, however, the methods in [33]). It becomes, however easy to determine such a solution, if an “orbit” equation in the $(\phi, F)$ plane, connecting two specific “boundary points”, has been specified via some other information. The boundary points must themselves be solutions to the EOM and hence they satisfy

$$\frac{\partial V}{\partial \phi} = -FW''(\phi) = 0,$$

$$\frac{\partial V}{\partial F} = \frac{\lambda}{4} g F^2 - F - W'(\phi) = 0.$$  \hspace{1cm} (34)

A solution of these equations not depending on $\lambda$ is given by $F = W'(\phi) = 0$ (i.e. $\phi = 0$ or $-m/g$), which specifies the classical supersymmetric vacua of the undeformed theory. We choose to work with these boundary values of $(\phi, F)$ for the rest of this section.

Though we are not claiming to obtain all the solutions in this paper, in the following we will find via perturbative method a class of solutions which are continuous deformations of the standard BPS solutions. For this, let us consider the time independent solutions of this system which are governed by the following 1-dimensional lagrangian,

$$\mathcal{L}[\phi, F] = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{\lambda}{8} \left( \frac{dF}{dx} \right)^2 + V(\phi, F),$$  \hspace{1cm} (35)
describing motion of an analogue particle in 2 dimensions (with $x$ as the time variable).\footnote{Here we adopt the convention $x^\pm = \frac{i}{2}(x \pm iy)$, with $x$ the spatial coordinate and $y$ the Euclidean time.} The EOM of $\phi, F$ from this Lagrangian follow to be

$$
\frac{d^2 \phi}{dx^2} = -FW''(\phi),
$$
(36)

$$
\frac{\lambda}{4} \frac{d^2 F}{dx^2} = \frac{\lambda}{4} gF^2 - F - W'(\phi).
$$
(37)

Useful information regarding this system of equations can be obtained from a conserved quantity corresponding to the “energy” of the analogue particle described by eq.(35). This has the form

$$
\mathcal{E} = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{\lambda}{8} \left( \frac{dF}{dx} \right)^2 - V(\phi, F).
$$
(38)

The finite energy solutions of the original theory are specified as those solutions of the system (35) for which the energy of the analogue particle vanishes, i.e. $\mathcal{E} = 0$ \cite{33}. Hence solutions of interest should satisfy the “conservation equation”:

$$
\frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{\lambda}{8} \left( \frac{dF}{dx} \right)^2 = V(\phi, F).
$$
(39)

On the other hand, we know that for the undeformed theory with $\lambda = 0$, the finite energy BPS solutions are given by the following equations \cite{32}

$$
F = -W'(\phi),
$$
$$
\frac{d\phi}{dx} = \pm W'(\phi).
$$
(40)

As we mentioned, here we are interested in those solutions of (35) that are continuously connected to the solutions of eqs.(40), when $\lambda$ slightly differs from zero. Such solutions will have series expansions in powers of $\lambda$ of the form

$$
\phi(x) = \phi_0(x) + \lambda \phi_1(x) + \lambda^2 \phi_2(x) + \cdots,
$$
$$
F(x) = F_0(x) + \lambda F_1(x) + \lambda^2 F_2(x) + \cdots,
$$
(41)

with $\phi_0(x), F_0(x)$ denoting a solution of the unperturbed eqs.(40). To find a perturbative scheme for determining such solutions, we first write eq.(37).
\[ F = -W'(\phi) - \frac{\lambda}{4} \left( \frac{dF}{dx^2} - gF^2 \right). \]  

(42)

Here we limit our discussion only to first order corrections, though the procedure can indeed be extended to all orders in $\lambda$. The above equation suggests an iterative scheme by which we can first determine perturbatively the “orbit” equation, \( F = F(\phi) \), and subsequently find the expression for $d\phi/dx$ in terms of $\phi$ at each order of $\lambda$. We start iteration by putting \( F = -W'(\phi) + \mathcal{O}(\lambda) \) on the r.h.s. of eq. (42), from which follows

\[ F = -W' - \frac{\lambda}{4} \left[ -dW''(\phi) - gW'^2(\phi) + \mathcal{O}(\lambda) \right] + \mathcal{O}(\lambda^2). \] 

(43)

Now, using the EOM of $\phi$, eq.(38), and that \( W^{(3)} = 2g \), the above equation becomes

\[ F = -W' - \frac{\lambda}{4} \left[ FW'^2 - 2g \left( \frac{d\phi}{dx} \right)^2 - gW'^2 \right] + \mathcal{O}(\lambda^2) \]

\[ = -W' + \frac{\lambda}{4} \left[ W'W'^2 - 2g \left( \frac{d\phi}{dx} \right)^2 + gW'^2 \right] + \mathcal{O}(\lambda^2), \] 

(44)

where in the last step we have used of $F = -W'(\phi) + \mathcal{O}(\lambda)$ once again. The second term in the brackets equals $2gW'^2$ to order $\mathcal{O}(\lambda)$, as by assumption the BPS equation in our case deviates to $\frac{d\phi}{dx} = \pm W'(\phi) + \mathcal{O}(\lambda)$.\(^5\) Thus the desired equation of orbit $F = F(\phi)$ to first order in $\lambda$ becomes

\[ F = -W'(\phi) + \frac{\lambda}{4} \left[ W'^2(\phi) + 3gW'(\phi) \right] + \cdots. \] 

(45)

(Hereafter by ellipsis we mean the terms which are higher than first order in $\lambda$.) We note that to this order of $\lambda$, $F(\phi)$ is a polynomial function of $\phi$ which is divisible by the polynomial $W'(\phi)$. This is indeed a generic property of $F(\phi)$ repeated to all orders in $\lambda$. Since the roots of $W'(\phi)$ specify the SUSY vacua of the ordinary theory, after all the above expression shows that the

\(^5\)We can show this simply by putting the zeroth order equation $F = -W'(\phi) + \mathcal{O}(\lambda)$ into the EOM of $\phi$ giving rise to $\frac{d^2\phi}{dx^2} = W'(\phi)W''(\phi) + \mathcal{O}(\lambda)$ and integrating it to $(\frac{d\phi}{dx})^2 = W'^2(\phi) + \mathcal{O}(\lambda)$ using the boundary condition that $\phi(x) \to \text{const.}$ at $x \to \pm \infty$. 

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\( F = W'(\phi) = 0 \) solutions (like in the ordinary theory) still describe those vacua of the deformed theory which are invariant under the ordinary SUSY.

We now determine an expression for \( d\phi/dx \) in terms of \( \phi \) to order \( O(\lambda) \), from which the first order correction to \( \phi_0(x) \) is obtained. One way to do this is to apply the equation of orbit (eq.(45)) directly into the EOM of \( \phi \) (eq.(36)). This gives

\[
\frac{d^2\phi}{dx^2} = W'(\phi)W''(\phi) - \frac{\lambda}{4} W'(\phi)W'''(\phi) \left( W''(\phi) + 3gW'(\phi) \right) + \cdots \tag{46}
\]

Upon multiplication by 2\( d\phi/dx \) and subsequent integration this leads to

\[
\left( \frac{d\phi}{dx} \right)^2 = W'^2(\phi) - \frac{\lambda}{2} \int d\phi \left( W'(\phi)W'''(\phi) + 3gW'^2(\phi)W''(\phi) \right) + \cdots, \tag{47}
\]

where the second term on the r.h.s. is an indefinite integral. Although, the integration here is straightforward due to the polynomial nature of \( W(\phi) \), it is helpful to express it in a more explicit form. For this, we use by part integration to write

\[
\int d\phi \left( W'W'^3 + 3gW'^2W'' \right) = \int d\phi \frac{1}{2}(W'^2)'W'^2 + gW'^3 = \frac{1}{2} \left( W'^2W'^2 - \int d\phi W'^2(2W''W^{(3)}) \right) + gW'^3 = \frac{1}{2} W'^2W'^2 + \frac{g}{3} W'^3, \tag{48}
\]

where we have used the fact that \( W^{(3)} = 2g \) is constant and have adjusted the integration constant such that the overall result vanishes at the critical points of \( W(\phi) \).

We can formulate the above procedure for determining \( d\phi/dx \) to arbitrary order of \( \lambda \) in a more systematic way by forming the effective lagrangian of the single variable \( \phi(x) \), given that the desired solution obeys the known orbit equation \( F = F(\phi) \). This is most easily achieved by replacing \( F \) in the lagrangian (35), upon which it becomes

\[
\mathcal{L}_{\text{eff}}[\phi] = \frac{1}{2} \left[ 1 + \frac{\lambda}{4} F'^2(\phi) \right] \left( \frac{d\phi}{dx} \right)^2 + V(\phi, F(\phi)) = \frac{1}{2} G(\phi) \left( \frac{d\phi}{dx} \right)^2 + U(\phi). \tag{49}
\]
Here we defined the effective “metric” $G(\phi)$ and the effective scalar potential $U(\phi)$ by the quantities in the first line. Using the known expression for $F(\phi)$ it is easy to work out explicit expressions for $G(\phi), U(\phi)$ to each order of $\lambda$. For example, to first order we find the expressions
\[
G(\phi) \equiv 1 + \frac{\lambda}{4} F''^2(\phi)
= 1 + \frac{\lambda}{4} W''^2(\phi) + \cdots,
\]
\[
U(\phi) \equiv \frac{\lambda}{12} g F^3(\phi) - \frac{1}{2} F^2(\phi) - F(\phi) W'(\phi)
= \frac{1}{2} W^2(\phi) - \frac{\lambda}{12} g W^3(\phi) + \cdots.
\]

Evidently, for $\lambda \geq 0$ (as we assumed earlier), we have a positive definite metric $G(\phi) > 0$. Moreover, we see that (at least to this order of $\lambda$) the modification to the standard effective potential $U(\phi) = \frac{1}{2} W^2(\phi)$ is proportional to $W'^2(\phi)$, and so both $U(\phi)$ and its derivative $U'(\phi)$ vanish at all the critical points of the superpotential $W(\phi)$, so that $U(\phi)$ is always positive in a small vicinity of such points. However, we note that (unlike in the ordinary case), $U(\phi)$ is not a positive definite function of $\phi$ and it so may possess additional zeros given by the roots of $1 - \frac{\lambda}{6} g W'(\phi) + \cdots$. These would correspond to a class of solutions of the deformed theory which are not continuously connected to a solution of the undeformed theory via a power series expansion in $\lambda$. The critical point of $U(\phi)$ (i.e. the common roots of $U(\phi), U'(\phi)$) specify the boundary values of $\phi(x)$ at $x \to \pm \infty$ in order for the soliton to have a finite energy.

Here we are interested in those solutions of the deformed theory that have the same boundary conditions as those of the undeformed theory. Therefore we assume $\phi(x) \to \phi_{\pm \infty}$ at $x \to \pm \infty$, so that
\[
W'(\phi_{\infty}) = W'(\phi_{-\infty}) = 0.
\]
The soliton equation as found from the $E = 0$ condition now becomes
\[
E = \frac{1}{2} G(\phi) \left( \frac{d\phi}{dx} \right)^2 - U(\phi) = 0,
\]
which upon using the $O(\lambda)$ expressions of $G(\phi)$ and $U(\phi)$ leads to
\[
\pm \frac{d\phi}{dx} = \sqrt{\frac{2U(\phi)}{G(\phi)}} = W'(\phi) - \frac{\lambda}{4} \left( g W'^2(\phi) + \frac{1}{2} W'(\phi) W''(\phi) \right) + \cdots.
\]
This is just the same result we arrived in eqs. (47), (48) directly using the EOM.

We can solve the differential equation (53) perturbatively for \( \phi(x) = \phi_0(x) + \lambda \phi_1(x) + \cdots \) to arbitrary order in \( \lambda \). For this, let us write this equation simply as

\[
\pm \frac{d\phi}{dx} = v_0(\phi) + \lambda v_1(\phi) + \cdots, \tag{54}
\]

where

\[
v_0(\phi) \equiv W'(\phi), \quad v_1(\phi) \equiv -\frac{1}{4} W'(\phi) \left( \frac{g}{3} W'(\phi) + \frac{1}{2} W''(\phi) \right), \tag{55}
\]

and so on. The zeroth order (unperturbed) solution \( \phi_0(x) \) then satisfies the unperturbed BPS equation

\[
\pm \frac{d\phi_0}{dx} = v_0(\phi_0), \tag{56}
\]

having the standard kink/anti-kink solution \([33]\)

\[
\phi_0(x) = \frac{m}{2g} \left( \pm \tanh \frac{mx}{2} - 1 \right). \tag{57}
\]

The equation for first order correction \( \phi_1(x) \) on the other hand becomes

\[
\pm \frac{d\phi_1}{dx} - v_0'(\phi_0) \phi_1 = v_1(\phi_0). \tag{58}
\]

This is a linear ODE of first order with non-constant coefficients whose solution for \( \phi_1(x) \) is known in terms of quadratures:

\[
\phi_1 = \pm e^{\pm \int dx v'(\phi_0)} \int dx v_1(\phi_0) e^{\mp \int dx v'(\phi_0)}. \tag{59}
\]

This solution can be simplified by using \( \pm \frac{d\phi_0}{dx} = v_0(\phi_0) \) from which we obtain

\[
\pm \int dx v'_0(\phi_0) = \int dx \phi_0 \frac{v'_0(\phi_0)}{v_0(\phi_0)} = \log v_0(\phi_0), \tag{60}
\]

and therefore for \( \phi_1 \) in terms of \( \phi_0 \)

\[
\phi_1 = \pm v_0(\phi_0) \int dx \frac{v_1(\phi_0)}{v_0(\phi_0)} = v_0(\phi_0) \int dx \frac{v_1(\phi_0)}{v_0^2(\phi_0)}. \tag{61}
\]
Using the explicit expression for \(v_0(\phi), v_1(\phi)\) in terms of \(W(\phi)\) this becomes

\[
\phi_1 = -\frac{1}{4} W'(\phi_0) \int \frac{d\phi_0}{W''(\phi_0)} \left( \frac{g}{3} W'(\phi_0) + \frac{1}{2} W''(\phi_0) \right)
\]

\[
= (m\phi_0 + g\phi_0^2) \left[ -\frac{7}{12} g\phi_0 + \frac{m}{8} \log \left( \frac{m + g\phi_0}{g\phi_0} \right) + c \right].
\]  

Finally, using the known solution for \(\phi_0(x)\), this solution is translated into an explicit expression for \(\phi_1(x)\)

\[
\phi_1(x) = \pm \frac{7m^3}{96g} \left( \frac{\tanh \frac{mx}{2} - \frac{3}{2} \cosh \frac{mx}{2}}{ \cosh^2 \frac{mx}{2}} \right),
\]  

where the integration constant is chosen \(c = 0\) so that at \(x = 0\) the correction does not change the value of \(\phi(0) = -\frac{m}{2g}\), i.e. so that \(\phi_1(0) = 0\). Note that the first order correction to either of the kink/anti-kink solutions is an antisymmetric function of \(x\) and it vanishes at \(x \to \pm \infty\), as we have expected. The profiles of \(\phi_0(x), \phi_1(x)\) are plotted in figs.1,2 below.

5 The Effective Superpotential and Modification of the BPS Mass Formula

As we mentioned in the last section, solitons of interest in the deformed theory are solutions of an effective 1-dimensional system described by the
Figure 2: The profile of the $O(\lambda)$ correction to the kink soliton, $\phi_1(x)$, with $\phi_1$ measured in units of $\frac{7m^3}{96g}$.

The lagrangian (49). This lagrangian can be imagined to have originated from a non-linear $\sigma$-model on $\mathcal{N} = 1, D = 2$ superspace with the lagrangian:

$$\mathcal{L}_{\text{eff}}[\hat{S}] = \frac{1}{2} G(\hat{S}) \mathcal{D}_+ S \mathcal{D}_- S + W(\hat{S}).$$

(64)

This is a special (one-dimensional) case of the more general $\sigma$-model with multi-dimensional target space analyzed in an appendix in [16]. It is easy to see that for such a model the field theory scalar potential has the following form

$$U(\phi) = \frac{W'(\phi)}{2G(\phi)}.$$  

(65)

Conversely, we can solve for $W(\phi)$ in terms of the known functions $G(\phi)$, $U(\phi)$ as follows

$$W(\phi) = \int d\phi [2G(\phi)U(\phi)]^{1/2}$$

$$= \int d\phi \left[ \left( 1 + \frac{\lambda}{4} W'^2 + \cdots \right) \left( W'^2 - \frac{\lambda}{6} g W'^3 + \cdots \right) \right]^{1/2}$$

$$= \int d\phi W'(\phi) \left[ 1 + \frac{\lambda}{2} \left( \frac{1}{4} W'^2(\phi) - \frac{g}{6} W'(\phi) \right) + \cdots \right].$$  

(66)

$^6$Here $\hat{S}$ and $S$ have the same $\phi$ and $\psi$ components, but their auxiliary components $\hat{F}$ and $F$ are different, and indeed, unlike the latter the former is a non-dynamical field which on its EOM is algebraically solved in terms of $\phi$ as $\hat{F} = -W'(\phi)/G(\phi)$. 

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The zeroth order term is simply $W(\phi)$, while the $O(\lambda)$ term is simplified using $\int d\phi W'W'' = \frac{1}{2} W'^2 W'' - g \int d\phi W'^2$ to give the result

$$W(\phi) = W(\phi) + \lambda \left( \frac{1}{16} W'^2 - \frac{5}{24} g \int d\phi W'^2 \right) + \cdots. \quad (67)$$

Now for calculation of the mass of the deformed BPS soliton, we recall the general BPS mass formula [32, 16], which in terms of the effective $\sigma$-model (64) may be written as

$$M = \left| \int dxd^2\theta W(\bar{S}) \right| = |W(\phi_{\infty}) - W(\phi_{-\infty})|. \quad (68)$$

Using $W(\phi)$ of eq.(66) and noting that $W'(\phi_{\pm\infty}) = 0$, this formula gives

$$M = M^{(0)} + \lambda M^{(1)} + \cdots,$$

where

$$M^{(0)} = |W(\phi_{\infty}) - W(\phi_{-\infty})| = \left| \left( \frac{m^2}{2} + \frac{g}{3} \phi^2 \right)^0_{\phi=-m/g} \right| = \frac{m^3}{6g^2},$$

$$M^{(1)} = \frac{5g}{24} \int_{\phi_{-\infty}}^{\phi_{\infty}} d\phi W'^2(\phi) = \frac{5g}{24} \int_{-m/g}^{0} d\phi (m\phi + g\phi^2)^2 = \frac{1}{144} \frac{m^5}{g^2}. \quad (69)$$

In other words

$$M = \frac{m^3}{6g^2} \left( 1 + \frac{1}{24} m^2 \det C + \cdots \right). \quad (70)$$

As this expression suggests, a meaning of smallness of $C$ concerned to our computations is $|\det C| \ll 24/m^2$. Note that the usual dependence of the mass of a soliton to the coupling as $M \sim 1/g^2$ does not get modified due to the deformation of superspace. In particular, solitons still have a large/small mass at weak/strong coupling and so they can be interpreted as non-perturbative objects of field theory.

6 Non-linear Realization of $\mathcal{N} = 1$ Supersymmetry

So far we have stressed that the deformed the $\mathcal{N} = 1$ theory in 2d explicitly breaks all the supersymmetries that underlie the original undeformed theory. We have noticed, however, that a class of solutions of the deformed theory can still be described as the 1/2 BPS solutions of some other (undeformed) $\mathcal{N} = 1$ supersymmetric $\sigma$-model with a metric and superpotential deviating from those of the original model by corrections in powers of the small parameter $\lambda$. The simple fact that these solutions can be described
as the one-half SUSY solutions of an $\mathcal{N} = 1$ theory raises the question that whether the deformed $\mathcal{N} = 0$ theory realizes the underlying $\mathcal{N} = 1$ SUSY somehow non-linearly.

To find the answer, let us first write the complete form of the effective $\mathcal{N} = 1$ non-linear $\sigma$-model on the orbit of $F = F(\phi)$, including its fermions, that is

$$
\mathcal{L}_{\text{eff}}[\phi, \psi] = \frac{1}{2}G(\phi)\partial_+ \phi \partial_- \phi + U(\phi) - \frac{i}{2}\bar{\psi} \partial_\phi \psi + \frac{1}{2}W''(\phi)\bar{\psi} \psi,
$$

(71)

with $G(\phi), U(\phi)$ given in terms of $W(\phi)$ as in eqs.(50). Actually this $\mathcal{L}_{\text{eff}}$ is not written in the standard form of an $\mathcal{N} = 1$ non-linear $\sigma$-model with the auxiliary fields integrated out using their EOM. To find a precise match let us consider a generic $\sigma$-model on one-dimensional target space defined by the action

$$
I[\tilde{S}] = -\int d^2 x d^2 \theta \left( \frac{1}{2} \tilde{G}(\tilde{S}) \mathcal{D}_+\tilde{S} \mathcal{D}_-\tilde{S} + \tilde{W}(\tilde{S}) \right).
$$

(72)

By carrying the $\theta$-integrations and subsequently by eliminating the auxiliary field $\tilde{F}$ via its EOM, $\tilde{F} = -\tilde{W}(\tilde{\phi})/\tilde{G}(\tilde{\phi})$, we find the ordinary space lagrangian

$$
\tilde{\mathcal{L}}[\tilde{\phi}, \tilde{\psi}] = \frac{1}{2}\tilde{G}(\tilde{\phi})\partial_+ \tilde{\phi} \partial_- \tilde{\phi} + \frac{\tilde{W}'(\tilde{\phi})}{2\tilde{G}(\tilde{\phi})} \left( \frac{\tilde{W}''(\tilde{\phi})}{\sqrt{\tilde{G}(\tilde{\phi})}} \right)' \tilde{\psi} \tilde{\psi}.
$$

(73)

We now demand that $\mathcal{L}_{\text{eff}}$ takes the same form as $\tilde{\mathcal{L}}$ via a suitable map between their field contents. A simple comparison of similar terms between the two lagrangians then suggests that this map must be of the form

$$
\tilde{\phi} = \tilde{\phi}(\phi), \quad \tilde{\psi} = \frac{1}{\sqrt{\tilde{G}(\phi)}} \psi.
$$

(74)

The unknown function $\tilde{\phi}(\phi)$ is to be determined along with the metric $\tilde{G}(\tilde{\phi})$ and superpotential $\tilde{W}(\tilde{\phi})$ of the $\sigma$-model. The equivalence between the two
lagrangians then implies the following conditions

\[
\begin{align*}
\tilde{G}(\tilde{\phi}) \left( \frac{d\tilde{\phi}}{d\phi} \right)^2 &= G(\phi), \\
\frac{\tilde{W}'^2(\tilde{\phi})}{2\tilde{G}(\tilde{\phi})} &= U(\phi), \\
\sqrt{\tilde{G}(\tilde{\phi})} \left( \frac{\tilde{W}'(\tilde{\phi})}{\sqrt{\tilde{G}(\tilde{\phi})}} \right)' &= W''(\phi).
\end{align*}
\]

(75)

Here the primes in each case denote differentiations with respect to the specified variable. These set of equations determine both the map \( \tilde{\phi} = \tilde{\phi}(\phi) \) and the unknown functions \( \tilde{G}(\tilde{\phi}) \), \( \tilde{W}(\tilde{\phi}) \) by quadratures. For this, using the last two equations in (75), we first write

\[
\sqrt{\tilde{G}(\tilde{\phi})} \left( \sqrt{2U(\phi)} \right)' \frac{d\phi}{d\tilde{\phi}} = W''(\phi).
\]

(76)

Define for convenience the (known) function of \( \phi \),

\[
H(\phi) = \frac{\left( \sqrt{2U(\phi)} \right)'}{W''(\phi)}.
\]

(77)

Then, combining the previous equation with eqs.(75), we obtain

\[
\begin{align*}
\tilde{G}(\tilde{\phi}(\phi)) &= \frac{\sqrt{G(\phi)}}{H(\phi)}, \\
\tilde{W}'(\tilde{\phi}(\phi)) &= \sqrt{\frac{2U(\phi)\sqrt{G(\phi)}}{H(\phi)}}, \\
\left( \frac{d\tilde{\phi}}{d\phi} \right)^4 &= G(\phi)H^2(\phi).
\end{align*}
\]

(78)

These in principle specify the solutions for \( \tilde{G}, \tilde{W} \) as functions of \( \tilde{\phi} \), provided we have integrated and inverted the last equation for \( \tilde{\phi}(\phi) : \)

\[
\tilde{\phi} = \int d\phi \sqrt[4]{G(\phi)H^2(\phi)}.
\]

(79)

Although it is possible to write down the explicit forms of these solutions perturbatively in powers of \( \lambda \), it turns out that for determining the
non-linearly deformed SUSY transformations of $\phi$, $\psi$ (corresponding to the ordinary transformations of $\tilde{\phi}$, $\tilde{\psi}$) we do not need to know these explicit solutions. To see how this works, let us begin with recalling the ordinary SUSY transformations of $\tilde{\phi}$, $\tilde{\psi}$, i.e. \[ (\ref{eq:80}) \]

\[
\begin{align*}
\delta_\epsilon \tilde{\phi} &= \bar{\epsilon} \tilde{\psi}, \\
\delta_\epsilon \tilde{\psi} &= - \left( \partial_\phi \tilde{\phi} + \frac{\tilde{W}'(\tilde{\phi})}{G(\tilde{\phi})} - \frac{1}{4} \frac{\tilde{G}'(\tilde{\phi})}{G(\tilde{\phi})} \tilde{\psi} \tilde{\psi} \right) \epsilon.
\end{align*}
\]

Putting $(\tilde{\phi}, \tilde{\psi})$ in terms of $(\phi, \psi)$ in these equations we find

\[
\begin{align*}
\frac{d\tilde{\phi}}{d\phi} \delta_\epsilon \phi &= \bar{\epsilon} \psi \sqrt{G}, \\
\delta_\epsilon \left( \frac{\psi}{\sqrt{G}} \right) &= - \left( \sqrt{G(\phi)} \frac{d\phi}{d\phi} + \frac{\tilde{W}'(\phi)}{G(\phi)} - \frac{1}{4} \frac{\tilde{G}'(\phi)}{G^2(\phi)} \tilde{\psi} \tilde{\psi} \right) \epsilon.
\end{align*}
\]

Expanding the variation on the l.h.s. of the second equation above and using the first equation, together with eqs.\((\ref{eq:78})\), we arrive at the desired transformations:

\[
\begin{align*}
\delta_\epsilon \phi &= \frac{\bar{\epsilon} \psi}{\sqrt{G(\phi)}}, \\
\delta_\epsilon \psi &= - \left( \sqrt{G(\phi)} \partial_\phi \phi + \sqrt{2U(\phi)} \right) \epsilon,
\end{align*}
\]

which leads to known expressions without really requiring to solve eqs.\((\ref{eq:75})\). These must be of course accompanied by the transformation equation of the auxiliary field $F$, which using the orbit equation $F = F(\phi)$ is found to be

\[
\delta_\epsilon F = \frac{F'(\phi)}{\sqrt{G(\phi)}} \bar{\epsilon} \psi. \tag{\ref{eq:83}}
\]

These deformed transformations can explicitly be expanded in powers of $\lambda$ and can be checked that they are actually symmetries of the lagrangian \((\ref{eq:71})\) to arbitrary order in $\lambda$. Also, one can check that the equation $\delta_\epsilon \psi = 0$ (the equation $\delta_\epsilon \phi = 0$ becomes trivial for $\psi = 0$) leads to the same eq.\((\ref{eq:53})\) which we derived previously for the deformation of the BPS solutions.

**References**

[1] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909, 032 (1999) [arXiv:hep-th/9908142].
[2] M.R. Douglas and N.A. Nekrasov, “Noncommutative field theory,” Rev. Mod. Phys. 73, 977 (2001) [arXiv:hep-th/0106048].

[3] N. Seiberg, “Noncommutative superspace, N = 1/2 supersymmetry, field theory and string theory,” JHEP 0306, 010 (2003) [arXiv:hep-th/0305248].

[4] N. Berkovits and N. Seiberg, “Superstrings in graviphoton background and N = 1/2 + 3/2 supersymmetry,” JHEP 0307, 010 (2003) [arXiv:hep-th/0306226].

[5] H. Ooguri and C. Vafa, “The C-deformation of gluino and non-planar diagrams,” arXiv:hep-th/0302109.

[6] H. Ooguri and C. Vafa, “Gravity induced C-deformation,” arXiv:hep-th/0303063.

[7] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” arXiv:hep-th/0208048.

[8] R. Dijkgraaf and C. Vafa, “On geometry and matrix models,” Nucl. Phys. B 644, 21 (2002) [arXiv:hep-th/0207106].

[9] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” Nucl. Phys. B 644, 3 (2002) [arXiv:hep-th/0206255].

[10] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, “Non-commutative superspace from string theory,” arXiv:hep-th/0302078.

[11] S. Ferrara, M.A. Lledó, “Some Aspects of Deformations of Supersymmetric Field Theories,” JHEP 0005 (2000) 008, hep-th/0002084.

[12] D. Klemm, S. Penati and L. Tamassia, “Non(anti)commutative superspace,” Class. Quant. Grav. 20, 2905 (2003).

[13] J. Bagger and I.Giannakis, “Spacetime supersymmetry in a nontrivial NS-NS superstring background,” Phys. Rev. D 65, 046002 (2002) [arXiv:hep-th/0107260].

[14] C.S. Chu and F. Zamora, “Manifest supersymmetry in non-commutative geometry,” JHEP 0002, 022 (2000) [arXiv:hep-th/9912153].
[15] S. Terashima, “A note on superfields and noncommutative geometry,” Phys. Lett. B 482, 276 (2000) [arXiv:hep-th/0002119].

[16] R. Abbaspour, “Realization Of Central Charges In Theories With Generalized Noncommutative Supersymmetry,” JHEP 0305 (2003) 023.

[17] R. Abbaspour, “Generalized Noncommutative Superalgebras,” Mod. Phys. Lett. A 18 (2003) 587.

[18] R. Abbaspour, “Generalized noncommutative supersymmetry from a new gauge symmetry,” arXiv:hep-th/0206170.

[19] R. Abbaspour, “Noncommutative supersymmetry in two dimensions,” Int. J. Mod. Phys. A 18, 855 (2003) [arXiv:hep-th/0110005].

[20] S. Terashima and J.T. Yee, “Comments on noncommutative superspace,” arXiv:hep-th/0306237.

[21] S. Ferrara, M.A. Lledo and O. Macia, “Supersymmetry in noncommutative superspaces,” arXiv:hep-th/0307039.

[22] T. Araki, K. Ito and A. Ohtsuka, “Supersymmetric gauge theories on noncommutative superspace,” arXiv:hep-th/0307076.

[23] R. Britto, B. Feng and S.J. Rey, “Deformed superspace, N = 1/2 supersymmetry and (non)renormalization theorems,” arXiv:hep-th/0306215.

[24] R. Britto, B. Feng and S.J. Rey, “Non(anti)commutative superspace, UV/IR mixing and open Wilson lines,” arXiv:hep-th/0307091.

[25] O. Lunin and S.J. Rey, “Renormalizability of non(anti)commutative gauge theories with N = 1/2 supersymmetry,” arXiv:hep-th/0307275.

[26] R. Britto and B. Feng, “N = 1/2 Wess-Zumino model is renormalizable,” arXiv:hep-th/0307165.

[27] M.T. Grisaru, S. Penati and A. Romagnoni, “Two-loop renormalization for nonanticommutative N = 1/2 supersymmetric WZ model,” arXiv:hep-th/0307099.

[28] A. Romagnoni, “Renormalizability of N = 1/2 Wess-Zumino model in superspace,” arXiv:hep-th/0307209.
[29] M. Chaichian and A. Kobakhidze, “Deformed N = 1 supersymmetry,” arXiv:hep-th/0307243.

[30] R. Gopakumar, S. Minwalla and A. Strominger, “Noncommutative solitons,” JHEP 0005, 020 (2000) [arXiv:hep-th/0003160].

[31] J.A. Harvey, “Komaba lectures on noncommutative solitons and D-branes,” arXiv:hep-th/0102076.

[32] E. Witten, D. Olive, “Supersymmetry Algebras That Include Topological Charges,” Phys.Lett. B 78 (1978) 97

[33] R. Rajaraman, “Solitons and instantons: an introduction to solitons and instantons in quantum field theory,” North-Holland, Amsterdam 1989.