Abstract
We develop the general Theory of Cayley Hamilton algebras and we compare this with the theory of pseudocharacters. We finally characterize prime \( T \)-ideals for Cayley Hamilton algebras and discuss some of their geometry.

To the memory of T. A. Springer

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Foreword

A basic fact for an \( n \times n \) matrix \( a \) (with entries in a commutative ring) is the construction of its characteristic polynomial \( \chi_a(t) := \det(tI - a) \) and the Cayley Hamilton theorem \( \chi_a(a) = 0 \).

The notion of Cayley Hamilton algebra (CH algebras for short), see Definition 2.13, was introduced in 1987 by Procesi [18] as an axiomatic treatment of the Cayley Hamilton theorem. This was done in order to clarify the Theory of \( n \)-dimensional representations, cf. Definition 1.2, of an associative and in general noncommutative algebra \( R \) (from now on just called algebra).

The theory was developed only in characteristic 0, for two reasons, the first being that at that time it was not clear to the author if the characteristic free results of Donkin [8] and Zubkov [30] were sufficient to found the theory in general. The second reason was mostly because it looked not likely that the main theorem 2.18 could possibly hold in general.

The first concern can now be considered to have a positive solution due to the contributions of several people and we may take the book [7] as reference. As for the second, that is the main theorem in positive characteristic, the issue remains unsettled. The present author feels that it should not be true in general but has no counterexamples.

Independently, studying deformations of representations of Galois groups, the theory of pseudocharacters or pseudorepresentations was developed by several authors see [29], [28]. We shall discuss the relationships between the two approaches in Theorem 2.17.

A partial theory in general characteristics replacing the trace with the determinant, a norm, appears already in Procesi [15] and [19].

For the general definition, see Chenevier [4]. The original definition over \( \mathbb{Q} \) is through the axiomatization of a trace, see §2.1, and closer to the Theory of pseudocharacters Definition 2.15. The definition via trace is also closer to the language un Universal algebra, while the one using norms is more categorical in nature. In this paper we restrict to characteristic 0 and traces, a discussion of the general case will appear elsewhere.

1. Invariants and representations

1.1. \( n \)-dimensional representations

Let us recall some basic facts which are treated in detail in the forthcoming book with Aljadeff, Giambruno and Regev [1].

For a given \( n \in \mathbb{N} \) and a ring \( A \) by \( M_n(A) \) we denote the ring of \( n \times n \) matrices with coefficients in \( A \), by a symbol \( (a_{i,j}) \) we denote a matrix with entries \( a_{i,j} \in A \), \( i,j = 1, \ldots, n \).

In particular we will usually assume \( A \) commutative so that the construction \( A \rightarrow M_n(A) \) is a functor from the category \( \mathcal{C} \) of commutative rings to that \( \mathcal{R} \) of associative rings. To a map \( f : A \rightarrow B \) is associated a map \( M_n(f) : M_n(A) \rightarrow M_n(B) \) in the obvious way \( M_n(f)((a_{i,j})) := (f(a_{i,j})) \).
**Definition 1.2.** By an $n$–dimensional representation of a ring $R$ we mean a homomorphism $f : R \to M_n(A)$ with $A$ commutative.

The set valued functor $A \mapsto \text{hom}_R(R, M_n(A))$ is representable. That is:

**Proposition 1.3.** There is a commutative ring $T_n(R)$ and a natural isomorphism $j_A : \text{hom}_R(R, M_n(A)) \cong \text{hom}_C(T_n(R), A)$ given by the commutative diagram $f = M_n(\bar{f}) \circ j_R$.

$$
\begin{array}{ccc}
R & \xrightarrow{j_R} & M_n(T_n(R)) \\
\downarrow f & & \downarrow M_n(\bar{f}) \\
M_n(f) & \rightarrow & M_n(A)
\end{array}
$$

(1)

The map $j_R : R \to M_n(T_n(R))$ is called the universal $n$–dimensional representation of $R$ or the universal map into $n \times n$ matrices.

Of course it is possible that $R$ has no $n$–dimensional representations, in which case $T_n(R) = \{0\}$.

**Remark 1.4.** The same discussion can be performed when $R$ is in the category $\mathcal{R}_F$ of algebras over a commutative ring $F$.

Then the functor $A \mapsto \text{hom}_{\mathcal{R}_F}(R, M_n(A))$ is on commutative $F$ algebras and $T_n(R)$ is an $F$ algebra.

From now on $F$ will be a fixed commutative ring.

The construction of $j_R$ is in two steps, first one easily sees that when $R = F(\langle x_i \rangle_{i \in I})$ is a free algebra then:

**Proposition 1.5.** $T_n(R) = F[\xi^{(i)}_{h, k}]$ is the polynomial algebra over $F$ in the variables $\xi^{(i)}_{h, k}$, $i \in I$, $h, k = 1, \ldots, n$ and $j_R(x_i) = \xi_i := (\xi^{(i)}_{h, k})$ is the generic matrix with entries $\xi^{(i)}_{h, k}$.

**Definition 1.6.** The subalgebra $F_n(I) := F(\xi_i)$ of $M_n(F[\xi^{(i)}_{h, k}])$, $i \in I$, $h, k = 1, \ldots, n$ generated by the matrices $\xi_i$ is called the algebra of generic matrices.

If $I$ has $\ell$ elements we also denote $F_n(I) = F_n(\ell)$. A classical Theorem of Amitsur states that:

**Theorem 1.7.** If $F$ is a domain then $F_n(I)$ is a domain. If $\ell \geq 2$ then $F_n(\ell)$ has a division ring of quotients $D_n(\ell)$ which is of dimension $n^2$ over its center $Z_n(\ell)$.
These algebras have been extensively studied. One defines first the commutative algebra $T_n(\ell) \subset \mathbb{Z}_n(\ell)$ generated, for all $a \in F_n(\ell)$, by the coefficients $\sigma_i(a)$ of the characteristic polynomial

$$\det(t - a) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(a)t^{n-i}, \forall a \in F_n(\ell).$$

Next define $S_n(\ell) = F_n(\ell)T_n(\ell) \subset D_n(\ell)$, (called the trace algebra). One can understand $S_n(\ell)$ and $T_n(\ell)$ by invariant theory, see Theorem 1.8.

The invariant theory involved is presented in [16] when $F$ is a field of characteristic 0, and may be considered as the first fundamental theorem of matrix invariants. For a characteristic free treatment the Theorem is due to Donkin [8]. In general assume that $F$ is an infinite field:

**Theorem 1.8.** The algebra $T_n(\ell)$ is the algebra of polynomial invariants under the simultaneous action of $GL(n,F)$ by conjugation on the space $M_n(F)^\ell$ of $\ell$–tuples of $n \times n$ matrices.

The algebra $S_n(\ell)$ is the algebra of $GL(n,F)$–equivariant polynomial maps from the space $M_n(F)^\ell$ of $\ell$–tuples of $n \times n$ matrices to $M_m(F)$.

As usual together with a first fundamental theorem one may ask for a second fundamental theorem which was proved independently by Procesi [16] and Razmyslov [22] when $F$ has characteristic 0 and by Zubkov [30] in general, see the book [7].

The best way to explain this Theorem in characteristic 0, and which is the basis of the present work, is to set it into the language of universal algebra by introducing the category of algebras with trace and trace identities, see §2.1.

A general algebra $R$ can be presented as a quotient $R = F\langle x_i \rangle/I$ of a free algebra. Then $j_{F(x_i)}(I)$ generates in $M_n(F[\xi_{h,k}^{(i)}])$, $i \in I$, $h, k = 1, \ldots, n$ an ideal which is, as any ideal in a matrix algebra, of the form $M_n(J)$, with $J$ an ideal of $F[\xi_{h,k}^{(i)}]$. Then the universal map for the algebra $R$ is given by $j_R : R \to M_n(F[\xi_{h,k}^{(i)}]/J)$. By the universal property this is independent of the presentation of $R$.

Again one may add to $R$ the algebra $T_n(R)$ generated by the coefficients of the characteristic polynomial $\sigma_i(a)$, $\forall a \in j_R(R)$.

1.9. Symmetry

The functor $\text{hom}_R(R, M_n(A))$ has a group of symmetries: the projective linear group $PGL(n)$.

It is best to define this as a representable group valued functor on the category $\mathcal{C}$ of commutative rings. The functor associates to a commutative ring $A$ the group $\mathfrak{S}_n(A) := Aut_A(M_n(A))$ of $A$–linear automorphisms of the
matrix algebra $M_n(A)$. To a morphism $f : A \to B$ one has an associated morphism $f_* : \mathfrak{S}_n(A) \to \mathfrak{S}_n(B)$ and the commutative diagram:

$$
\begin{array}{ccc}
M_n(A) & \xrightarrow{g} & M_n(A) \\
\downarrow{M_n(f)} & & \downarrow{M_n(f)} \\
M_n(B) & \xrightarrow{f(g)} & M_n(B)
\end{array}
$$

One has a natural homomorphism of the general linear group $GL(n, A)$ to $\mathfrak{S}_n(A)$ which associates to an invertible matrix $X$ the inner automorphism $a \mapsto XaX^{-1}$.

The functor general linear group $GL(n, A)$ is represented by the Hopf algebra $\mathbb{Z}[x_{i, j}][d^{-1}]$, $i, j = 1, \ldots, n$ with $d = \det(X)$, $X := (x_{i, j})$ with the usual structure given compactly by comultiplication $\delta$, antipode $S$ and counit $\epsilon$:

$$\delta(X) = X \otimes X, \quad S(X) := X^{-1}, \quad \epsilon : X \to 1_n.$$

The functor $\mathfrak{S}_n(A)$ is represented by the sub Hopf algebra, of $GL(n, A)$, $P_n \subset \mathbb{Z}[x_{i, j}][d^{-1}]$ formed of elements homogeneous of degree 0. It has a basis, over $\mathbb{Z}$, of elements $ad^{-h}$ where $a$ is a doubly standard tableaux with no rows of length $n$ and of degree $h \cdot n$. For a proof see [1] Theorem 3.4.21. Finally we have an action of $\mathfrak{S}_n(A)$ on $\text{hom}_R(R, M_n(A))$ by composing a map $f$ with an automorphism $g$. One has a commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{j_R} & M_n(T_n(R)) \\
\downarrow{f} & & \downarrow{M_n(gf)} \\
M_n(A) & \xrightarrow{g} & M_n(A)
\end{array}
$$

Assume now that $R$ is an $F$ algebra so also $T_n(R)$ is an $F$ algebra and $M_n(T_n(R)) = M_n(F) \otimes_F T_n(R)$.

If $g$ is an automorphism of $M_n(F)$ set $\hat{g} := g \circ j_R$. Given $g_1, g_2$ two automorphisms of $M_n(F)$ we have when $A = T_n(R)$ and $f = j_R$ in Formula (3):

$$1 \otimes g_1 \circ g_2 \circ j_R = (g_1 \circ g_2) \otimes 1 \circ j_R = g_1 \otimes 1 \circ 1 \otimes \hat{g}_2 \circ j_R = 1 \otimes \hat{g}_2 \circ g_1 \otimes 1 \circ j_R = 1 \otimes \hat{g}_2 \circ 1 \otimes \hat{g}_1 \circ j_R$$

implies $\hat{g}_1 \circ \hat{g}_2 = \hat{g}_2 \circ \hat{g}_1$. Which implies that the map $g \mapsto \hat{g}$ is an antihomomorphism from $\text{Aut}_F(M_n(F))$ to $\text{Aut}(T_n(R))$. Finally

**Proposition 1.10.** The map $g \mapsto g \otimes \hat{g}^{-1}$ is a homomorphism from the group $\text{Aut}_FM_n(F)$ to the group of all automorphisms of $M_n(T_n(R))$. The image of $R$ under $j_R$ is formed of invariant elements.

Theorem 2.18 states that when $R$ is an $n$ Cayley–Hamilton algebra the map $j_R$ is an isomorphism to this ring of invariants. We call this statement the strong embedding theorem.
2. The trace

In this section we assume, unless otherwise specified, that the algebras are over \( \mathbb{Q} \).

2.1. Axioms for a trace

**Definition 2.2.** An associative algebra with trace, over a field \( F \) is an associative \( F \)-algebra \( R \) with a 1-ary operation

\[ t : R \to R \]

which is assumed to satisfy the following axioms:

1. \( t \) is \( F \)-linear.
2. \( t(a)b = bt(a), \forall a, b \in R. \)
3. \( t(ab) = t(ba), \forall a, b \in R. \)
4. \( t(t(a)b) = t(a)t(b), \forall a, b \in R. \)

This operation is called a formal trace. We denote \( t(R) := \{ t(a), a \in R \} \) the image of \( t \). From the axioms it follows that \( t(R) \) is a commutative \( F \) algebra which we call the trace algebra of \( R \).

**Remark 2.3.** We have the following implications:

Axiom 1) implies that \( t(R) \) is an \( F \)-submodule.
Axiom 2) implies that \( t(R) \) is in the center of \( R \).
Axiom 3) implies that \( t \) is 0 on the space of commutators \([R, R]\) .
Axiom 4) implies that \( t(R) \) is an \( F \)-subalgebra and that \( t \) is \( t(R) \)-linear.

These axioms in general are quite weak and strange traces may appear. For instance if \( R \) is a commutative \( A \) algebra any \( A \) linear map of \( R \) to \( A \) satisfies these axioms.

As examples of strange traces, to use later in order to see why Cayley–Hamilton algebras are special, the reader may use \( R = F[x]/(x^2) \) and either \( t(1) = 0, t(x) = 1 \) or \( t(1) = 1, t(x) = 1. \) On the other hand for each \( n \in \mathbb{N} \) the trace \( t(1) = n, t(x) = 0 \) is a natural trace arising from matrix theory.

The axioms are in the form of universal algebra so that in the category of algebras with trace free algebras exist, see §2.6.1.

By a homomorphism of algebras with trace \( R_1, R_2 \) we mean a homomorphism \( f : R_1 \to R_2 \) of algebras which commutes with the trace, that is \( f(t(a)) = t(f(a)) \). Clearly a homomorphism of algebras with trace \( R_1, R_2 \) induces a homomorphism of their trace algebras \( T_1, T_2 \).

Thus algebras with trace form a category which we will denote \( T_F \) or simply \( T \) when \( F \) is chosen.

In fact we may distinguish between algebras with 1 or no assumption. In general we make no special assumptions on \( t(1) \).
Definition 2.4. Given an algebra $R$ with trace, a set $S \subset R$ generates $R$ if $R$ is the smallest subalgebra closed under trace containing $S$.

By trace ideal or simply ideal in an algebra $R$ with trace we mean an ideal closed under trace.

Remark 2.5. An algebra $R$ can be finitely generated as trace algebra but not as algebra, for instance the free trace algebra, even in one variable.

Clearly if $I \subset R$ is a trace ideal of an algebra with trace then $R/I$ is an algebra with trace and $t(R/I) = t(R)/[I \cap t(R)]$. The usual homomorphism Theorems hold in this case.

There is a twist in this definition, clearly if $I, J$ are two trace ideals also $I + J, I \cap J$ are trace ideals. As for the product we need to be careful since $IJ$ need not be closed under trace. For instance if $I$ is the ideal of positive elements in the free algebra for each variable $x$ we have $x^2 \in I^2$ but $tr(x^2) \notin I^2$.

So we should define

$$I \cdot J := IJ + R \cdot tr(IJ), \quad I^n := I \cdot I^{n-1}.$$ 

In particular we say that $I$ is nilpotent if for some $n$ we have $I^n = 0$.

Proposition 2.6. 1. Let $R$ be an algebra with trace and $T$ its trace algebra. If $U$ is a commutative $T$ algebra then $U \otimes_T R$ is an algebra with trace $t(u \otimes r) = u \otimes t(r)$ and $U \otimes 1$ is its trace algebra.

2. Given two algebras with trace $R_1, R_2$ with trace algebras $T_1, T_2$ their direct sum is $R_1 \oplus R_2$ with trace $t(r_1, r_2) := (t(r_1), t(r_2))$ and trace algebra $T_1 \oplus T_2$.

3. Finally if $R$ is an algebra with trace and $T$ is its trace algebra and $T$ has also a trace with trace algebra $T_1$ the composition of the two traces is a trace on $R$ with trace algebra $T_1$.

In particular we can apply this when $U = T_S$ is the localization of $T$ at a multiplicative set $S$.

2.6.1. The free trace algebra

The free trace algebra over a set $X$ of variables will be denoted by $\mathcal{F}_T\langle X \rangle$. By definition it is a trace algebra $\mathcal{F}_T\langle X \rangle$ containing $X$ and such that for every trace algebra $U$ the set of homomorphisms from $\mathcal{F}_T\langle X \rangle$ to $U$ are in bijection with the maps $X \to U$. It can be described as follows.

Start from the usual free algebra $F\langle X \rangle$, then consider the classes of cyclic equivalence of monomials $M$, which we formally denote $tr(M)$. The algebra $\mathcal{F}_T\langle X \rangle = F\langle x_i \rangle_{i \in I}[tr(M)]$ is the polynomial ring in the infinitely many commuting variables $tr(M)$ over the free algebra $F\langle X \rangle$. Its trace algebra is the polynomial ring $F[tr(M)]$ in the infinitely many commuting variables $tr(M)$. The map $tr : M \mapsto tr(M)$ is the formal trace.

As for the usual theory of polynomial identities we have:
**Definition 2.7.** Given a trace algebra $U$, an element $f \in F_T \langle X \rangle$ is a **trace identity** for $U$ if it vanishes under all evaluations $X \rightarrow U$.

It is a **pure trace identity** if $f \in F[tr(M)]$.

Then it is easy to see that the set of trace identities of a trace algebra $U$ is a (trace) ideal of $F_T \langle X \rangle$ which is closed under the endomorphisms (variable substitutions) of $F_T \langle X \rangle$.

Such an ideal is called a $T$–ideal. Conversely any $T$–ideal $I$ of $F_T \langle X \rangle$ is the ideal of trace identities of an algebra, namely $F_T \langle X \rangle/I$. An algebra $F_T \langle X \rangle/I$ with $I$ a $T$–ideal, is called a relatively free algebra on $X$ since it is a free algebra in the variety of algebras satisfying the identities of $I$.

Finally we define:

**Definition 2.8.** Two trace algebras are **trace PI–equivalent** of just PI–equivalent if they satisfy the same trace identities.

In particular a trace algebra is PI–equivalent to the free trace algebra modulo the $T$–ideal of its identities.

### 2.9. The Cayley–Hamilton identities

We start with the case of just one variable $X = \{x\}$ which is of special importance. In this case the trace algebra $T_X$ is the polynomial algebra in the infinitely many variables $tr(x^i), i = 0, \ldots, \infty$ while the free algebra $F_T \langle X \rangle = T_X[x]$.

It is convenient to identify the trace algebra with the **ring of symmetric functions on infinitely many variables** $\lambda_j$, with coefficients in $\mathbb{Q}[tr(1)]$.

We do this by identifying $tr(x^i)$ with the power sum $\psi_j := \sum_j \lambda_j^i$. This is compatible with matrix theory, when $x$ is an $n \times n$ matrix over $\mathbb{C}$ and $\lambda_1, \ldots, \lambda_n$ its eigenvalues we have $tr(x^i) = \sum_{j=1}^n \lambda_j^i$.

We have, in the ring $\mathbb{Q}[tr(x), \ldots, tr(x^i), \ldots]$ the formal elementary symmetric functions $\sigma_i(x)$ which correspond in this identification, to the elementary symmetric functions $e_i$.

The elementary symmetric functions are related to the power sums by the recursive formula:

$$(-1)^m \psi_{m+1} + \sum_{i=1}^m (-1)^{i-1} \psi_i e_{m+1-i} = (m + 1)e_{m+1} \quad (4)$$

We have the generating formula, for symmetric functions in $n$ variables $\lambda_j$, obtained using the Taylor expansion for $\log(1 + y)$.

$$\sum_{i=0}^n (-1)^i e_i u^i = \prod_{r=1}^n (1 - \lambda_r u) = \exp(- \sum_{j=1}^\infty \frac{\psi_j}{j} u^j).$$

We then **define** the elements $\sigma_i(x)$ in the free algebra by

$$\sum_{i=0}^\infty (-1)^i \sigma_i(x) u^i := \exp(- \sum_{j=1}^\infty \frac{tr(x^j)}{j} u^i). \quad (5)$$
We may define, for each \( n \), in the free algebra with trace in a single variable \( x \) the formal Cayley Hamilton polynomial

\[
CH_n(x) := x^n + \sum_{i=1}^{n} (-1)^i \sigma_i(x)x^{n-i}.
\] (6)

Example

\[
CH_1(x) = x - \text{tr}\,(x); \quad CH_2(x) = x^2 - \text{tr}\,(x)x + \frac{1}{2}(\text{tr}\,(x)^2 - \text{tr}\,(x^2)).
\]

\[
CH_3(x) = x^3 - \text{tr}\,(x)x^2 + \frac{1}{2}(\text{tr}\,(x)^2 - \text{tr}\,(x^2))x - \frac{1}{3}\text{tr}\,(x^3) - \frac{1}{6}\text{tr}\,(x)^3 + \frac{1}{2}\text{tr}\,(x^2)\text{tr}\,(x).
\]

2.9.1. The multilinear form

It is convenient to use also the multilinear form of the Cayley–Hamilton identity and of the symmetric functions \( \sigma_i(x) \), which can be obtained by the process of full polarization. For this, given a permutation \( \sigma \in S_m \), we decompose \( \sigma = (i_1i_2 \ldots i_h) \ldots (j_1j_2 \ldots j_\ell)(s_1s_2 \ldots s_t) \) in cycles then we set:

\[
T_{\sigma}(x_1, x_2, \ldots, x_m) = \text{tr}\,(x_{i_1}x_{i_2} \ldots x_{i_h}) \ldots \text{tr}\,(x_{j_1}x_{j_2} \ldots x_{j_\ell})\text{tr}\,(x_{s_1}x_{s_2} \ldots x_{s_t}).
\] (7)

From the basic elements \( T_{\sigma} \) of Formula (7) take \( m = k + 1 \). We may assume that the last cycle ends with \( s_t = k + 1 \) so the last factor is of the form \( \text{tr}\,((x_{s_1}x_{s_2} \ldots x_{s_{t-1}})x_{k+1}) \), hence we have that

\[
T_{\sigma}(x_1, x_2, \ldots, x_{k+1}) = \text{tr}\,(\psi_{\sigma}(x_1, x_2, \ldots, x_k)x_{k+1})
\] (9)

where \( \psi_{\sigma}(x_1, x_2, \ldots, x_k) \) is the element of \( F_{T}(X) \) given by the formula

\[
\psi_{\sigma}(x_1, x_2, \ldots, x_k)
\] (10)

Then we have the multilinear form, (see also Lew [13]):

**Proposition 2.10.** 1. For each \( k \leq n \) the polarized form of \( \sigma_k(x) \) is the expression

\[
T_{\sigma}(x_1, x_2, \ldots, x_k) = \sum_{\sigma \in S_{k}} \epsilon_{\sigma}T_{\sigma}(x_1, x_2, \ldots, x_k).
\] (11)

2. The polarized form of \( CH_n(x) \) is

\[
CH(x_1, \ldots, x_n) = (-1)^n \sum_{\sigma \in S_{n+1}} \epsilon_{\sigma}\psi_{\sigma}(x_1, x_2, \ldots, x_n).
\] (12)

Here \( \epsilon_{\sigma} \) denotes the sign of \( \sigma \).

3. \( \text{tr}\,(CH(x_1, \ldots, x_n)x_{n+1}) = (-1)^nT_{n+1}(x_1, \ldots, x_{n+1}). \) (13)
Example $n = 2$ (polarize $CH_2(x)$)

\[ CH_2(x) = x^2 - tr(x)x + det(x) = x^2 - tr(x)x + \frac{1}{2}(tr(x)^2 - tr(x^2)). \]  
\[ x_1x_2 + x_2x_1 - tr(x_1)x_2 - tr(x_2)x_1 - tr(x_1x_2) + tr(x_1)tr(x_2) = CH_2(x_1 + x_2) - CH_2(x_1) - CH_2(x_2). \]

Also from the decomposition into cosets $S_{n+1} = S_n \bigcup_{i=1}^{n} S_n(i, n + 1)$, of the symmetric group, one has the recursive formula

\[ T_{n+1}(x_1, \ldots, x_{n+1}) = T_n(x_1, \ldots, x_n)tr(x_{n+1}) - \sum_{i=1}^{n} T_n(x_1, \ldots, x_i x_{n+1}, \ldots, x_n). \]  
(15)

2.10.1. The first and second fundamental Theorem for matrix invariants

The first and second fundamental Theorems for matrix invariants may be viewed as the starting point of the Theory of Cayley–Hamilton algebras.

Theorem 2.11. The algebra $F_{T,n}(X)$ of equivariant polynomial maps from $X$–tuples of $n \times n$ matrices, $M_n(F)^X$ to $n \times n$ matrices $M_n(F)$, is the free algebra with trace modulo the $T$–ideal generated by the $n^{th}$ Cayley Hamilton polynomial and $tr(1) = n$.

\[ F_{T,n}(X) := F_T(X)/\langle CH_n(x), \ tr(1) = n \rangle. \]  
(16)

Remark 2.12. It can be shown that, if we do not set $tr(1) = n$ we have:

\[ \bigoplus_{i=1}^{n} F_{T,i}(X) = F_T(X)/\langle CH_n(x) \rangle. \]

To be concrete if $X$ has $\ell$ elements, let $A_{\ell,n}$ denote the polynomial functions on the space $M_n(F)^\ell$ (that is $A_{\ell,n} = F[\xi_{i,j,h}^{(i)}]$ is the algebra of polynomials over $F$ in $\ell n^2$ variables $\xi_{i,j,h}^{(i)}$, $i = 1, \ldots \ell$, $j,h = 1, \ldots, n$).

On this space, and hence on $A_{\ell,n}$, acts the group $PGL(n,F)$ by conjugation.

The space of polynomial maps from $M_n(F)^\ell$ to $M_n(F)$ is

\[ M_n(A_{\ell,n}) = M_n(F) \otimes A_{\ell,n}. \]

On this space acts diagonally $PGL(n,F)$ and the invariants

\[ F_{T,n}(x_1, \ldots, x_{\ell}) = M_n(A_{\ell,n})^{PGL(n,F)} = (M_n(F) \otimes A_{\ell,n})^{PGL(n,F)} \]

give the relatively free algebra in $\ell$ variables in the variety of trace algebras satisfying $CH_n(x)$. 

For the proof see [6] or [1] Proposition 12.1.12.
For the trace algebra we have \( T_n(\ell) = A^{PGL(n,F)}_{\ell,n} \). Of course we may let \( \ell \) be also infinity (of any type) and have

\[
F_{T,n}(X) = M_n(A_{X,n})^{PGL(n,F)} = (M_n(F) \otimes A_{X,n})^{PGL(n,F)}
\]

where \( A_{X,n} \) is the polynomial ring on \( M_n(F)^X \).

For a formulation and proof of these Theorems in all characteristics or even \( \mathbb{Z} \)-algebras, the Theorem of Zubkov, the reader may consult [7].

2.12.1. Cayley–Hamilton algebras

**Definition 2.13.** An algebra with trace satisfying the \( n \)-Cayley–Hamilton identity (12) and \( tr(1) = n \) will be called an \( n \)-Cayley–Hamilton algebra or \( n \)-CH algebra.

In other words an \( n \)-Cayley–Hamilton algebra is a quotient, as trace algebra, of one relatively free algebra \( F_{T,n}(X) \).

For \( n = 1 \) a 1–CH algebra is just a commutative algebra in which the trace is the identity map.

**Remark 2.14.** If \( R \) is an \( n \)-CH algebra then \( A \otimes_T R \) (2.6) is an \( n \)-CH algebra.

If \( S \subset R \) is a subalgebra with trace algebra \( U \subset T \) and \( R \) is an \( n \)-CH algebra then \( S \) is also an \( n \)-CH algebra.

Following Taylor [28], we define:

**Definition 2.15.** A pseudocharacter (or pseudorepresentation) of a group \( G \), of degree \( n \) with coefficients in a commutative ring \( A \), is a map \( t : G \rightarrow A \) satisfying the following three properties:

1. \( t(1) = n \).
2. \( t(ab) = t(ba) \), \( \forall a, b \in G \).
3. \( T_{n+1}(g_1, \ldots, g_{n+1}) = 0 \), \( \forall g_i \in G \) (Formula (11)).

Frobenius [9], discovered already that this is a property of an \( n \)-dimensional character.

The connection between the previous two definitions is the following. One considers the group algebra \( A[G] \) and then extends the map \( t \) to a trace with trace algebra \( A \). Next one considers the Kernel of the trace, that is

\[
K_t := \{ a \in A[G] \mid t(ab) = 0, \ \forall b \in A[G] \}.
\]

It is then an easy fact to see that, if \( t \) is a pseudocharacter of \( G \) of degree \( n \), then \( A[G]/K_t \) is a \( n \)-Cayley Hamilton algebra. In particular if \( A \supset \mathbb{Q} \) one can apply Theorem 2.18. In general we have:

**Proposition 2.16.** If \( R \) is an algebra with trace and the trace satisfies

\[
T_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} \epsilon_{\sigma} T_\sigma(x_1, \ldots, x_{n+1}) = 0
\]

then \( CH_n(x_1, \ldots, x_n) \in K_t \) and \( R/K_t \) is an \( n \)-Cayley Hamilton algebra.
Observe that, from Formula (15) it follows that, if \( R \) satisfies the multilinear identity \( T_{n+1} \), then it satisfies \( T_j, \forall j \geq n + 1. \)

From the next Theorem 2.18 then follows:

**Theorem 2.17.** Given a pseudocharacter \( t \) of a group \( G \), of degree \( n \) with coefficients in a commutative ring \( A \supset \mathbb{Q} \) there is a commutative \( A \)-algebra \( B \) and a representation \( \rho : G \to GL_n(B) \) so that:

\[
t(g) = \text{tr}(\rho(g)), \quad \forall g \in G, \quad \ker(\rho) = K_t, \quad \rho : A[G] \to M_n(B) \quad (17)
\]

with \( K_t \) the kernel of the associated trace form.

In particular in [28] the Theorem that if \( A \) is an algebraically closed field of characteristic 0, then the pseudocharacter is associated to a unique semisimple representation of \( G \) follows from our Theorem 3.17 2).

2.17.1. The main Theorem of Cayley–Hamilton algebras

Let now \( R \) be any \( n \)-CH algebra over a field \( F \supset \mathbb{Q} \).

Choosing a set of generators \( X \) for \( R \) we may present \( R \) as a quotient of a free algebra \( F_{T,n}(X) = (M_n(F) \otimes A_{X,n})^{PGL(n,F)} \) modulo a trace ideal \( I \).

By a suitable analogue of a theorem of invariant theory one has that

\[
I(M_n(F) \otimes A_{X,n}) \cap F_{T,n}(X) = I.
\]

Now \( I(M_n(F) \otimes A_{X,n}) \) is an ideal of \( M_n(F) \otimes A_{X,n} = M_n(A_{X,n}) \) so there is an ideal \( J \subset A_{X,n} \), which is \( PGL(n,F) \) stable with

\[
I(M_n(F) \otimes A_{X,n}) = M_n(J)
\]

from which:

**Theorem 2.18.** We have a commutative diagram in which the first horizontal arrows are isomorphisms the second injective and the vertical maps surjective:

\[
\begin{array}{cccc}
F_{T,n}(X) & \overset{i}{\cong} & M_n[A_{X,n}]^{PGL(n,F)} & \longrightarrow & M_n[A_{X,n}] \\
\downarrow & & \downarrow & & \downarrow \\
R & \overset{i_R}{\cong} & M_n[A_{X,n}/J]^{PGL(n,F)} & \longrightarrow & M_n[A_{X,n}/J]
\end{array}
\]

(18)

Notice that the previous commutative diagram exists in general, and \( i \) is always an isomorphism. The fact that in characteristic 0, \( i_R \) is an isomorphism depends upon the fact that \( GL(n) \), in characteristic 0, is linearly reductive, and then the proof, see [18] or [1] Theorem 14.2.1, of this Theorem is based on the so called Reynolds’s identities.

The map \( R \overset{i_R}{\cong} M_n[A_{X,n}/J]^{PGL(n,F)} \longrightarrow M_n[A_{X,n}/J] \) is called the universal map into matrices since one has a natural isomorphism between
hom_T(R, M_n(B)) \simeq \text{hom}_C(A_{X,n}/J,B) \quad \text{(with } C \text{ the category of commutative algebras).}

The algebra \( A_{X,n}/J \), with its \( PGL(n,F) \) action, is the algebra \( T_n(R) \) of Proposition 1.3.

**Definition 2.19.** We denote by \( T_n \) the category of commutative algebras equipped with a rational \( PGL(n,F) \) action (and where the morphisms are \( PGL(n,F) \) equivariant).

The functor \( R \mapsto T_n(R) \) is from the category of \( n \)–CH algebras to the category \( T_n \).

This functor has a right adjoint \( C \mapsto R_n(C) := M_n[C]^{PGL(n,F)}; \)

\[
\text{hom}_{T_n}(T_n(R), C) \simeq \text{hom}_T(R, R_n(C)), \quad \varphi \mapsto 1 \otimes \varphi,
\]

\[
R = (M_n(F) \otimes T_n(R))^{PGL(n,F)} \xrightarrow{1 \otimes \varphi} (M_n(F) \otimes C)^{PGL(n,F)} = R_n(C).
\]

The adjoint is also given by the universal map \( \pi : T_n(R_n(C)) \to C \) corresponding to the identity map of \( R_n(C) \). Notice that from the previous Formula it follows that \( \pi \) restricted to the invariants gives isomorphism

\[
\pi : T_n(R_n(C))^{PGL(n,F)} \simeq C^{PGL(n,F)}.
\]

If \( C = T_n(S) \) for some CH algebra \( S \) Formula (2.17.1) becomes:

\[
\text{hom}_{T_n}(T_n(R), T_n(S)) \simeq \text{hom}_T(R, R_n(T_n(S))) = \text{hom}_T(R, R).
\]

In other words, always for \( \mathbb{Q} \) algebras:

**Proposition 2.20.** The functor \( R \mapsto T_n(R) \) is an equivalence between the category of \( n \)–CH algebras and a full subcategory of the category \( T_n \).

It is easy to give examples of algebras in \( T_n \) which are not of the form \( T_n(R) \). For instance take a nilpotent conjugacy class \( O \) of \( n \times n \) matrices. If the order of nilpotency of elements of \( O \) is \( k \leq n \) then one sees that \( R_n(C) = F[x]/(x^k), \ tr(x^i) = 0, \ i = 1, \ldots, k − 1. \) This is independent of the conjugacy class but depends only on \( k \).

In general the nature of \( T_n(R) \) may be quite difficult to study. For instance consider the scheme of pairs of commuting matrices that is the ring \( A_n := F[X,Y]/(XY − YX = 0) \) generated by the entries of two generic \( n \times n \) matrices \( X,Y \) modulo the ideal generated by the entries of \( XY − YX \). One easily sees that it is of the form \( T_n(R), \ R = R_n(F[x,y]/([x,y])) \). It is not known if this ring is a domain.

An important consequence of this Theorem is:

**Corollary 2.21.** An \( n \)–CH algebra \( R \) satisfies all polynomial identities of \( M_n(\mathbb{Q}) \).
Let us draw a first consequence.

**Lemma 2.22.** Let $a$ be an $n \times n$ matrix with entries in a commutative ring $A$.

1) If $a$ is nilpotent also $\text{tr}(a)$ is nilpotent.
2) If $a = a^2$ is idempotent then $\text{tr}(a)$ satisfies the monic polynomial with integer coefficients $\prod_{i=0}^{n}(x-i)$.

**Proof.**
1) Since in a commutative ring $A$ the set of nilpotent elements is the intersection of the prime ideals we are reduced to the case in which $A$ is a domain. Hence we can embed it into a field. For matrices over a field the trace of a nilpotent matrix is 0.

2) We may assume that $A$ has a 1. From the localization principle it is enough to prove this for $A$ local. In this case $A^n = aA^n \oplus (1-a)A^n$. The two projective modules of this decomposition are both free and the rank of $aA^n$ is some integer $0 \leq i \leq n$ so $\text{tr}(a) = i$ and the claim follows.

**Lemma 2.23.** Let $R \subset S$ be an inclusion of trace algebras over some field $F$. Assume that $R \cdot T(S) = S$. Then $R$ and $S$ satisfy the same trace identities.

**Proof.** Since we are in characteristic 0 by polarization it is enough to prove that every multilinear trace identity of $R$ holds in $S$. This follows from the fact that, by definition the trace of $S$ is $T(S)$ linear.

**Proposition 2.24.** Let $R$ be an $n$–CH algebra over $F$, then $R$ is trace–PI equivalent to a finite dimensional $F$ algebra.

**Proof.** First $R$ is PI-equivalent to its associated relatively free algebra $F_R(X)$ which is a quotient of the free Cayley–Hamilton algebra $F_{T,n}(X)$ associated to matrices. By Capelli’s theory this decomposes under the linear group (acting on the space spanned by $X$) in irreducible representations of height $\leq n^2$ so the same is true for $F_R(X)$ which is thus PI equivalent to the algebra on the first $n^2$ variables. We may thus assume that $R$ is finitely generated over $F$.

By Theorem 2.18 we embed $R \subset M_n(A)$, with $A$ a commutative algebra, finitely generated over $F$, and the embedding is compatible with the trace. Then $A$ can be embedded in a commutative algebra $B$ which contains a field $G \supset F$ and it is finite dimensional over $G$, see Cohen [5]. Then $S := RB \subset M_n(B)$ is a trace algebra finite dimensional over $G$ and satisfies the same trace identities with coefficients in $F$ as $R$ by Lemma 2.23.

By enlarging $G$ if necessary we may assume that $S = S/J = \oplus_i M_n(G)$ and $J = \oplus_j Ga_j$ for some finite set of elements $a_j$ and $J^h = 0$ for some $h$. Then the algebra $\tilde{R}$ generated by $\oplus_i M_n(F)$ and the elements $a_i$ is finite dimensional. We claim that the trace algebra of $\tilde{R}$ is also finite dimensional. Since trace is linear it is enough to show that $\text{tr}(a)$ is algebraic over $F$ for $a$ in a basis of $\tilde{R}$ over $F$. Now the traces of the nilpotent elements, in particular of the elements in its radical are all nilpotent so we only need to
show that the traces of the elements $e_{i,i}$ are algebraic over $F$. Now if $e = e^2$ is an idempotent $tr(e)$ satisfies the polynomial of Lemma 2.22. Finally the algebra $\mathfrak{g}$ generated by $\mathfrak{r}$ and its traces is finite dimensional and since $\mathfrak{g}G = S$ it is PI-equivalent to $S$ and hence to $R$.

2.24.1. Azumaya algebras

Azumaya algebras play a special role in this Theory. If $R$ is an Azumaya algebra of rank $n^2$ over its center $Z$ then one can prove that $T_n(R)$ is faithfully flat over $Z$ and $R \otimes Z T_n(R) = M_n(T_n(R))$. We think of $R$ as a non-split form of matrices, see [1], §10.4.1. We claim that

Proposition 2.25. If an Azumaya algebras $R$ of rank $n^2$ over its center $Z \supset \mathbb{Q}$ has a $Z$-linear trace $t$ with respect to which it is an $n$-CH algebra then $t = tr$ the usual reduced trace.

Proof. By the main Theorem 2.18 we have a trace preserving embedding of $R$ in the universal algebra $M_n(T_n(R))$ under which $t$ is the trace and since the reduced trace is independent of the splitting the claim follows. □

In fact it also follows that, under the same hypotheses with respect to $t$, if we have that $R$ is an $k$–CH algebra then $k = i \cdot n$ for some $i$ and $t = i \cdot tr$ with $tr$ the usual reduced trace.

2.26. The variety of semisimple representations

There is a geometric interpretation of Theorem 1.8. Consider the space of $\ell$–tuples of matrices $M_n(F)^\ell$ where we shall now assume that $F$ is algebraically closed of any characteristic.

We think of the space $M_n(F)^\ell$ as the space of $n$–dimensional representations of the free algebra $F\langle x_1, \ldots, x_\ell \rangle$ in $\ell$ generators, where a representation $\rho : F\langle x_1, \ldots, x_\ell \rangle \to M_n(F)$ corresponds to the $\ell$–tuple $(a_i := \rho(x_i))$.

The linear group $GL(n, F)$ acts by simultaneous conjugation, in fact this action is trivial by the scalar matrices and hence should be thought of as an action of the projective linear group

$$G := GL(n, F)/F^* = PGL(n, F).$$

Remark 2.27. 1. Clearly two $\ell$–tuples are in the same orbit if and only if the two representations are isomorphic.

In other words the study of isomorphism classes of representations is the same as the study of $G$–orbits.

2. Notice that in the language of orbits, if $\rho : F\langle x_1, \ldots, x_\ell \rangle \to M_n(F)$ is a representation, i.e. $\rho \in M_n(F)^\ell$, its stabilizer in $GL(n, F)$ is the set of elements of $GL(n, F)$ for which $gpg^{-1} = \rho$. That is the elements of $GL(n, F)$ which are in the commutant or centralizer of the image of the representation.
From the definitions it also follows that the $n$–dimensional representation $\rho : F\langle x_1, \ldots, x_\ell \rangle \to M_n(F)$ can be thought of as a representation of the relatively free CH–algebra $F_{T,n}(x_1, \ldots, x_\ell) \to M_n(F)$ compatible with the trace. Denote for simplicity $S_n(\ell) := F_{T,n}(x_1, \ldots, x_\ell)$ and with $A_{\ell,n}$ the algebra of polynomial functions on the space $M_n(F)^\ell$ as in Theorem 2.18.

By Geometric invariant theory the variety associated to the trace algebra $T_n(\ell) = A_{\ell,n}^{PGL(n,F)}$ of the algebra $S_n(\ell)$ parametrizes closed orbits, hence we should understand which representations correspond to closed orbits. This has been shown by M. Artin, [3], see [1] Chapter 14.

First let us recall that given any module $V$, over an $F$–algebra $R$, so that $V$ is of finite dimension $n$ over the field $F$, we can construct a Jordan–Hölder series $V = V_1 \supset V_2 \supset \cdots \supset V_m \supset V_{m+1} = 0$ of submodules $V_i$ so that $V_i/V_{i+1}$ is always irreducible. The semisimple representation $\bigoplus_i V_i/V_{i+1}$ does not depend, up to isomorphism, on the chain chosen, this is the Jordan–Hölder Theorem. It will be called the semisimple module associated to the given module. When we fix a basis of $\bigoplus_i V_i/V_{i+1}$ we have associated to this module a semisimple representation, whose orbit is independent of the basis.

**Theorem 2.28.** A semisimple representation associated to any given representation is in the closure of its orbit.

A representation is in a closed orbit if and only if it is semisimple.

**Theorem 2.29.** The ring of invariants $T_n(\ell)$ of $\ell$–tuples of $n \times n$ matrices, is the coordinate ring of an irreducible affine variety

$$V_n(\ell) := M_n(F)^\ell //GL(n, F).$$

If $\ell \geq 2$ the variety $V_n(\ell)$ is of dimension $(\ell - 1)n^2 + 1$. Its points parametrize isomorphism classes of semisimple representations of dimension $n$ of the free algebra in $\ell$–variables or of trace compatible semisimple representations of $S_n(\ell)$.

Assume now $F$ of characteristic 0, (for the general case see [15], [19]).

The algebra $S_n(\ell)$ also has a geometric interpretation. First denote by

$$\pi : M_n(F)^\ell \to V_n(\ell) := M_n(F)^\ell //GL(n, F)$$

the quotient map associated to the inclusion $T_n(\ell) = A_{\ell,n}^{PGL(n,F)} \subset A_{\ell,n}$.

Consider a maximal ideal $m \subset T_n(\ell)$ corresponding to a point $p \in V_n(\ell)$. By Proposition 3.4 we have

$$mS_n(\ell) \cap T_n(\ell) = m$$

and the CH algebra $S_n(\ell)/mS_n(\ell)$ is finite dimensional with trace algebra $F = T_n(\ell)/m$. The points of $M_n(F)^\ell$ thought of as representations of $S_n(\ell) \to M_n(F)$ which factor through

$$\Sigma_p := S_n(\ell)/mS_n(\ell) = M_n(A_{n,\ell}/m)^{GL(n,F)}$$
are thus the points in the fiber $\pi^{-1}(p)$. In particular, the radical $J$ of $\Sigma_p$ vanishes on the closed orbit formed by semisimple representations of $\Sigma_p$. So the points of the closed orbit are representations of $\Sigma_p := \Sigma_p / J$. By Corollary 3.7 $J$ is in fact the kernel of the trace form. The algebra $\Sigma_p$ is a semisimple algebra with trace and simple as trace algebra. So, given a point $q$ in the closed orbit the corresponding representation $\rho_q : \Sigma_p \to M_n(F)$ is injective. Now a semisimple subalgebra of $M_n(F)$ is described by Proposition 3.11. We then have the finitely many strata of closed orbits associated to the lists $m_1, \ldots, m_k$ and $a_1, \ldots, a_k$ of positive integers with $\sum_j m_j a_j = n$. As we shall remark later, §3.38.1, in the quotient variety these are the smooth strata of Luna’s stratification by stabilizer type.

Of particular importance is the open set $U_{n,\ell}$ of irreducible representations that is of $\ell$–tuples of matrices which generate the algebra $M_n(F)$ ($\ell \geq 2$). On this open set the group $PGL(n, F)$ acts freely and the quotient $U_{n,\ell} / PGL(n, F)$ is smooth in $M_n(F)^\ell / GL(n, F)$, in fact except a few cases this is exactly the smooth part of $M_n(F)^\ell / GL(n, F)$. The map $\pi : U_{n,\ell} \to U_{n,\ell} / PGL(n, F)$ is a principal $PGL(n, F)$ bundle locally trivial in the étale topology, see [14].

This geometric description has a counterpart in the structure of the algebra $S_n(\ell)$ which is an Azumaya algebra of rank $n^2$ over its center exactly in the points of $U_{n,\ell} / PGL(n, F)$. This fact can be seen in a more explicit form as follows. Let $p \in U_{n,\ell}$ be a point corresponding to a surjective map $\rho : F\langle x_1, \ldots, x_\ell \rangle \to M_n(F)$. Then there are two elements $a, b \in F\langle x_1, \ldots, x_\ell \rangle$ so that $\rho(a)$ is a diagonal matrix with distinct and non-zero entries and the elements $\rho(a)^i \rho(b)^j$, $i, j = 0, \ldots, n - 1$ form a basis of $M_n(F)$. Let $\bar{a}, \bar{b}$ be the images of $a, b$ in $S_n(\ell)$ then the two invariants $D(\bar{a})$ the discriminant of $\bar{a}$ and $\Delta = \det(tr(\bar{b}^i \bar{a}^j))$ the discriminant of the basis $\bar{b}^i \bar{a}^j$ are in $T_n(\ell)$ and do not vanish on $p$ and hence on $\pi(p)$.

Let $T_n(\ell)_p := T_n(\ell)[D(\bar{a})^{-1}, \Delta^{-1}]$ and $E_p := \tilde{T}_n(\ell)_p[t] / CH_n(a)$. We claim that $E_p$ is étale over $T_n(\ell)_p$ and

$$E_p \otimes_T S_n(\ell) \simeq M_n(E_p).$$

In fact inverting $\Delta$ implies that $S_n(\ell)[\Delta^{-1}]$ is a free module over $T_n(\ell)[\Delta^{-1}]$ with basis $\bar{b}^i \bar{a}^j$, $i, j = 0, \ldots, n - 1$. Then inverting also $D(\bar{a})$ gives that the subalgebra $T_n(\ell)[D(\bar{a})\Delta^{-1}][a] \subset S_n(\ell)[D(\bar{a})\Delta^{-1}]$ is isomorphic to $E_p$. The characteristic polynomial of the element $a$ has distinct eigenvalues so that adding $a$ is a simple étale extension, cf. [21]. The left multiplication of $S_n(\ell)[D(\bar{a})\Delta^{-1}]$ on itself maps $S_n(\ell)[D(\bar{a})\Delta^{-1}]$ isomorphically to $\text{End}_{E_p}(S_n(\ell)[D(\bar{a})\Delta^{-1}])$, since $S_n(\ell)[D(\bar{a})\Delta^{-1}]$ is a free $E_p$ module with basis the elements $\bar{b}^j$, $j = 0, \ldots, n - 1$.

By the compactness of the Zariski topology one has a finite covering of $U_{n,\ell} / PGL(n, F)$ by affine open sets associated to pairs $\bar{a}_i, \bar{b}_i$ as before. Notice that the fibration $\pi : U_{n,\ell} \to U_{n,\ell} / PGL(n, F)$ is NOT locally trivial in the Zariski topology since the ring of fractions of $S_n(\ell)$ is a division algebra
and not a matrix algebra over a field as it would be if locally trivial in the Zariski topology.

3. Cayley Hamilton algebras

3.1. General theory

3.1.1. Some basic definitions

Definition 3.2. 1. A simple trace algebra is one with no proper trace ideals,
2. A prime trace algebra is one in which if $I, J$ are two trace ideals with $IJ = 0$ then either $I = 0$ or $J = 0$.
3. Finally a semiprime trace algebra is one in which if $I$ is an ideal with $I^2 = 0$ then $I = 0$.

Notice that prime implies semiprime.

Definition 3.3. Given a trace algebra $R$ the set

$$ K_R := \{ x \in R \mid t(xy) = 0, \forall y \in R \} \quad (22) $$

will be called the kernel of the trace algebra.

$R$ is called nondegenerate if $K_R = 0$.

If $I$ is a (trace ideal) in a trace algebra $R$ we set $K(I) \supset I$ to be the ideal such that $R/K(I) = K_R/I$. We call $K(I)$ the radical kernel of $I$.

Proposition 3.4. Let $R$ be an algebra with trace $tr$ and $T$ its trace ring. Assume that $tr(1)$ is invertible in $T$, then:

1. Given any ideal $I$ of $T$ we have that $IR$ is a trace ideal and $IR \cap T = I$ so $R/IR$ is an algebra with trace and trace ring $T/I$.
2. Moreover $R$ decomposes into the direct sum $T \oplus R^0$, of $T$ modules, with $R^0$ the space of trace 0 elements.
3. If $R$ is prime resp. simple as algebra with trace then $T$ is a domain resp. a field.

Proof. 1) Let $a = \sum_j t_j r_j \in T$ with $t_j \in I$, $r_j \in R$. Taking traces we have

$$ tr(a) = \sum_j t_j tr(r_j), \quad tr(a) = a \cdot tr(1) \implies a = tr(1)^{-1} \sum_j t_j tr(r_j) \in I. $$

2) The second part follows from axiom (4) as the map $x \mapsto t(1)^{-1}t(x)$ is a $T$ linear projection to $T$ with kernel $R^0$.

3) Since $tr(1)$ is invertible $T \neq \{0\}$. If $a, b \in T$ are non zero and $ab = 0$ then $aR, bR$ are two trace ideals and $aRbR = 0$ a contradiction.

If $R$ is simple and $a \in T$, $a \neq 0$ then $aR$ is a trace ideal hence $aR = R$. So there is a $b \in R$ with $ab = 1$ and then $a \cdot tr(b) = tr(1)$. Since $tr(1)$ is invertible the claim follows. \qed
Observe that an algebra $R$ can be considered as algebra with trace by setting the trace identically equal to 0.

**Proposition 3.5.** $K_R$ is the maximal trace ideal $J$ where $tr(J) = 0$.

If $t(1)$ is invertible $R/K_R$ is non degenerate.

If $R$ is an $n^{th}$–CH algebra we have $K_R^n = 0$.

**Proof.** The first part is clear, as for the second if $a \in R$ is in the kernel modulo $K_R$ we have that for all $r \in R$, $t(ar) \in K_R$ so

$$t(t(ar)) = t(ar) \cdot t(1) = 0, \quad \Rightarrow \quad t(ar) = 0$$

and the claim follows.

As for the last statement, we have that the Cayley Hamilton identity on $I$ is $x^n = 0$ so the statement follows from Razmyslov’s estimate in the so called Dubnov–Ivanov Nagata–Higman Theorem, [1] Theorem 12.2.13.

**Proposition 3.6.** Let $R$ be a $n^{th}$–CH algebra and $r \in R$ a nilpotent element, then $tr(r)$ is nilpotent.

**Proof.** By Theorem 2.18 we can embed $R$ into matrices over a commutative ring so that the trace becomes the ordinary trace. Hence the statement follows from Lemma 2.22.

**Corollary 3.7.** Let $R$ be a $n^{th}$–CH algebra with trace algebra reduced (no nonzero nilpotent elements) then if $r \in R$ is nilpotent we have $r^n = 0$, its Kernel $K_R$ is the maximal nil ideal and $K_R^n = 0$.

In particular we have

**Corollary 3.8.** 1) An $n^{th}$–CH algebra $R$ is semiprime if and only if its trace algebra is reduced and the Kernel $K_R = 0$.

2) An $n^{th}$–CH algebra $R$ is prime if and only if its trace algebra is a domain and $K_R = 0$.

**Proof.** Assume $R$ semiprime. If the trace algebra contains a non zero nilpotent element then it contains one with $a^2 = 0$ and then $Ra$ is a trace ideal with $(Ra)^2 = 0$. Also if $K_R \neq 0$ since $K_R^n = 0$ the algebra is not semiprime.

Conversely if $R$ has an ideal $I \neq 0$ with $I^2 = 0$ then for each $a \in I$ we have $tr(a)$ is nilpotent, by Proposition 3.6 hence $tr(a) = 0$ by assumption. If for all $a \in I$ we have $tr(a) = 0$ then $I \subset K_R$ hence $I = 0$.

As for the second statement let us show that the given conditions imply $R$ prime. In fact given two ideals $I, J$ with $IJ = 0$ since $t(I) \subset I, \ t(J) \subset J$ we have $t(I)t(J) = 0$. Since these are ideals and $T$ is a domain one of them must be 0. If $t(I) = 0$ then $I \subset K_R = \{0\}$ by hypothesis.

Conversely if $R$ is prime we must have $K_R = 0$ by Corollary 3.8 2). By Proposition 3.4 4) $T(R)$ is a domain.

\[^1\] $n^2$ is not the best bound, conjecturally the best is $\left(\begin{smallmatrix} n+1 \\ 2 \end{smallmatrix}\right)$ verified only for very small values of $n$. 
Finally the local finiteness property:

**Proposition 3.9.** An \( n^{\text{th}} \)-CH algebra \( R \) finitely generated over its trace ring \( T \) is a finite \( T \) module.

**Proof.** The Cayley Hamilton identity implies that each element of \( R \) is integral over \( T \) of degree \( \leq n \) then this is a standard result in PI rings consequence of Shirshov’s Lemma, see [1] Theorem 8.2.1. \( \square \)

### 3.10. Semisimple algebras

Given two lists \( \underline{m} := m_1, \ldots, m_k \) and \( \underline{a} := a_1, \ldots, a_k \) of positive integers with \( \sum_j m_j a_j = n \) consider the algebra

\[
F(\underline{m}; \underline{a}) := \bigoplus_{i=1}^k M_{m_i}(F), \quad \text{with trace } t(r_1, \ldots, r_k) = \sum_{i=1}^k tr(r_i) a_i, \quad (23)
\]

and \( tr(r_i) \) the trace as matrix.

\( F(\underline{m}; \underline{a}) \) is a subalgebra (of block diagonal matrices) of \( M_n(F) \), with the block \( M_{m_i}(F) \) repeated \( a_i \) times. Hence \( F(\underline{m}; \underline{a}) \) is an \( n \)-CH algebra, and, as trace algebra, it is simple.

Conversely

**Proposition 3.11.** If \( F \) is algebraically closed and \( S \subset M_n(F) \) is a semisimple algebra then it is one of the algebras \( F(m_1, \ldots, m_k; a_1, \ldots, a_k) \).

**Proof.** A semisimple algebra \( S \) over \( F \) is of the form \( S = \bigoplus_{i=1}^k M_{m_i}(F) \). An embedding of \( S \) in \( M_n(F) \) is a faithful \( n \)-dimensional representation of \( S \). Now the representations of \( S \) are direct sums of the irreducible representations \( F^{m_i} \) of the blocks \( M_{m_i}(F) \), and a faithful \( n \)-dimensional representation of \( S \) is thus of the form

\[
\bigoplus_i (F^{m_i})^{\otimes a_i}, \quad a_i \in \mathbb{N}, \quad a_i > 0, \quad \sum_i a_i m_i = n.
\]

For this representation the algebra \( S \) appears as block diagonal matrices, with an \( m_i \times m_i \) block repeated \( a_i \) times. The trace is then the one described in Formula (23). \( \square \)

**Theorem 3.12.** Let \( F \) be an algebraically closed field of characteristic 0 and \( S \) an \( n \)-CH algebra with trace algebra \( F \) and kernel \( K_S \).

Then \( S/K_S \) is finite dimensional, simple, and isomorphic to one of the algebras \( F(m_1, \ldots, m_k; a_1, \ldots, a_k) \).

**Proof.** Passing to \( S/K_S \) we may assume that \( K_S = 0 \). Let us first assume that \( S \) is finite dimensional, then by Corollary 3.7 we have that \( S \) is a semisimple algebra so it is of the form \( S = \bigoplus_{i=1}^k M_{m_i}(F) \). Since \( S \) is an \( n \)-CH algebra it is a quotient of one of the free algebra \( S_n(m) \) for some \( m \).
Restricted to the trace algebra $T_n(m)$ this gives a point $p$ in the quotient variety, thus $S$ as trace algebra is of the form $\Sigma_p$ and the statement follows from the discussion of §2.26.

Now let us show that $S$ is finite dimensional. For any choice of a finite set of elements $A = \{a_1, \ldots, a_k\} \subset S$ let $S_A$ be the subalgebra generated by these elements. By a standard theorem of PI theory, since $S$ is algebraic of bounded degree each $S_A$ is finite dimensional. Then if $J_A$ is the radical of $S_A$ we have, by the previous discussion, that $\dim S_A/J_A \leq n^2$. Let us choose $A$ so that $\dim S_A/J_A$ is maximal. We claim that $S = S_A$ and $J_A = 0$.

We have that $J_A$ is the kernel of the trace form of $S_A$.

First let us show that $J_A \subset K_S$ the Kernel of the trace form of $S$. If $a \in J_A$ let $r \in S$ we need to show that $tr(ra) = 0$. If $r \in S_A$ this is the previous statement, if $r \not\in S_A$ then $S_{A,r} \supseteq S_A$ and we claim that $J_{A,r} \supsetneq J_A$, in fact otherwise $\dim S_{A,r}/J_{A,r} > \dim S_A/J_A$ a contradiction.

Then by the previous argument $tr(ar) = 0$ so $a \in K_S$ but, since $S$ is simple, $K_S = 0$ and $J_A = 0$. Next if $S_A \neq S$ we have again some $S_{A,r} \supset S_A$ and now $J_{A,r} \neq 0$ a contradiction.

3.13. The trace algebra is a field

We want to study general CH algebras over a field $F$ such that the values of the trace are in $F$.

First some examples.

If $R$ is a finite dimensional simple $F$ algebra, then we have $R = M_k(D)$ with $D$ a division ring finite dimensional over its center $G$ which is also finite dimensional over $F$. Let $\dim_G D = h^2$, $\dim_F G = \ell$.

The algebra $R$ is endowed with a canonical reduced trace which is a composition of two traces

$$tr_{R/F} = tr_{R/G} \circ tr_{G/F}$$

The reduced trace $tr_{R/G}$ can be defined as follows. We take an algebraic closure $\bar{G}$ of $G$ then. if $a \in M_k(D)$:

$$\dim_G D = h^2 \implies M_k(D) \otimes_G \bar{G} = M_k(\bar{G})$$

and the trace $tr_{R/G}(a) := tr(a \otimes 1)$ as matrix.

It is an easy fact that $tr_{R/G}(a) \in G$. As for $tr_{G/F}(g)$, $g \in G$ one takes the trace of the multiplication by $g$ a $\ell \times \ell$ matrix over $F$.

If the characteristic of $F$ is 0 (or in general if $G$ is separable over $F$) we have

$$G \otimes_F \bar{G} = \bar{G}^\ell, \ g \otimes 1 = (\lambda_1, \ldots, \lambda_\ell) \implies tr(g) = \sum_i \lambda_i.$$

Proposition 3.14. If the characteristic of $F$ is 0 this is a $k \cdot h \cdot \ell$ CH algebra.

Proof. We have

$$M_k(D) \otimes_F \bar{G} = M_k(D) \otimes_G (G \otimes_F \bar{G}) = M_k(D) \otimes_G (\bar{G}^\ell) = M_k(\bar{G})^\ell \subset M_k(h \cdot \ell)(\bar{G})$$

and the reduced trace is induced by the trace of $M_k(h \cdot \ell)(\bar{G})$. □
Consider a general semisimple algebra finite dimensional over $F$

$$R = \bigoplus_{i=1}^{p} M_{k_i}(D_i), \quad \dim_{G_i} D_i = h_i^2, \quad \dim_{F_i} G_i = \ell_i$$

where $G_i$ is the center of the division algebra $D_i$. Given positive integers $a_i, i = 1, \ldots, p$ we may define the trace

$$t(r_1, \ldots, r_p) := \sum_{i=1}^{p} a_i tr(r_i), \quad r_i \in M_{k_i}(D_i), \quad tr(r_i) \in F$$

the reduced trace

(24)

**Theorem 3.15.** The algebra $R = \bigoplus_{i=1}^{p} M_{k_i}(D_i)$ with the previous trace is an $n$–CH algebra with $n = \sum a_i n_i, \quad n_i = k_i h_i \ell_i$.

Conversely any trace on $R$ which makes it into an $n$–CH algebra for some $n$ is of the previous form.

**Proof.** In one direction the statement is clear. The reduced trace $tr(r_i)$ is the ordinary trace associated to an embedding of $M_{k_i}(D_i)$ into $n_i \times n_i$ matrices over the algebraic closure $\bar{F}$. So the trace of formula (24) is the ordinary trace associated to an embedding of $R$ into $n \times n$ matrices over $\bar{F}$.

For the second, such a trace induces a trace in

$$R \otimes_F \bar{F} = \bigoplus_{i=1}^{p} M_{k_i}(D_i) \otimes_F \bar{F} = \bigoplus_{i=1}^{p} M_{k_i,h_i}(\bar{F})^\ell_i$$

for which this algebra is still $n$–CH. For such an algebra we know, Theorem 3.12, that there are some weights $a_{i,j} \in \mathbb{N}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, \ell_i$ associated to all the blocks $M_{k_i,h_i}(\bar{F})$ for which the trace is given by a Formula analogous to formula (24).

We only need to show that, for each $i$ the $\ell_i$ weights $a_{i,j}, \quad j = 1, \ldots, \ell_i$ are equal, set $a_i := a_{i,j}$. This follows from the fact that the Galois group of $\bar{F}$ over $F$ preserves the trace and permutes the $\ell_i$ summands $M_{k_i,h_i}(\bar{F})$. $\square$

**Remark 3.16.** Observe, from Formula (25) that $\dim_{F} R \leq n^2$ and further if $\dim_{F} R = n^2$ then $R$ is a central simple $F$ algebra and $R \otimes_F \bar{F} = M_n(\bar{F})$.

We can now generalize Theorem 3.12

**Theorem 3.17.**

1. If $S$ is a simple trace algebra, and $tr(1)$ is invertible, its trace algebra is a field $F$.

2. If $S$ is an $n$–CH algebra with trace algebra a field $F$ and $K_S = 0$ then $S$ is finite dimensional over $F$, simple and isomorphic to one of the algebras of Theorem 3.15.

3. If $R$ is an $n$–CH algebra with trace algebra a field $F$ and $S := R/K_R$ is of rank $n^2$ over its center $F$ then $K_R = 0$ and $S$ is simple as algebra.

**Proof.**

1) The fact that, if $S$ is a simple trace algebra, the trace algebra is a field follows from Proposition 3.4.

2) This will follow if we can show that the kernel of $S \otimes_F \bar{F}$ is 0, where $\bar{F}$ is an algebraic closure of $F$. 

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Now consider an element $\sum_{i=1}^{k} s_i \otimes f_i \in K_{S \otimes_F F}$ with the $f_i \in \bar{F}$ linearly independent over $F$.

If $a \in S$ we have

$$0 = tr(a \sum_{i=1}^{k} s_i \otimes f_i) = \sum_{i=1}^{k} tr(as_i) \otimes f_i.$$

Since the $f_i \in \bar{F}$ are linearly independent over $F$ and we have $tr(as_i) \in F$, this implies that for all $a$ and for all $i$ we have $tr(as_i) = 0$. So $s_i$ is in the kernel of $S$ which is assumed to be 0 hence $s_i = 0$ and $K_{S \otimes_F F} = 0$.

3) As in part 2) we may reduce to the case $F$ algebraically closed so that, from Remark 3.16, $R/K_R = M_n(F)$. At this point we have several options, first we reduce to $R$ finitely generated so finite dimensional over $F$ and hence by Wedderburn’s principal theorem we have $M_n(F) \subset R$. Thus $R = M_n(A)$ with $A$ an algebra with Jacobson radical $J$ and $A/J = F$. Then since $R$ satisfies the PI of $n \times n$ matrices it follows that $A$ is commutative. Finally by the main theorem one has that $A$ must be the trace algebra so by hypothesis $A = F$.

Another option is to use the fact that in the previous geometric picture the irreducible representations form the part of the spectrum of the quotient which is smooth and where the quotient is a principal fibration. \hfill \square

3.18. The Spectrum

Proposition 3.19. 1) If $R$ is a semiprime $n$–CH algebra with trace algebra $A$ and $a \in A$ is not a zero divisor in $A$ then $a$ is not a zero divisor in $R$.

2) If $R$ is a prime algebra with trace, the trace algebra $A$ is a domain and $R$ is torsion free relative to $A$.

Proof. 1) Let $J := \{r \in R \mid ar = 0\}$, then $J$ is an ideal and we claim it is nil hence by hypothesis 0. In fact taking trace $0 = t(ar) = at(r)$ implies $t(r) = 0$ for all $r \in J$, since $r$ satisfies its CH it must be $r^n = 0$.

2) If $a \in A$ and $J := \{r \in R \mid ar = 0\}$ then both $J$ and $Ra \neq 0$ are trace ideals. We have $RaJ = 0$ from this the claim. \hfill \square

Theorem 3.20. If $R$ is a prime $n$–CH algebra with trace algebra $A$ and $G$ is the field of fractions of $A$ we have $R \otimes_A G$ is a simple $n$–CH algebra with trace algebra $G$ and $R \subset R \otimes_A G$.

Proof. Since $R$ is torsion free over $A$ we have $R \subset R \otimes_A G$. Clearly the trace algebra of $R \otimes_A G$ is $G$ so, by Theorem 3.17 it is enough to show that the kernel is 0. Since $G$ is the field of fractions of $A$ each element of $R \otimes_A G$ is of the form $r \otimes g$, if this element is in the kernel then $r$ is in the Kernel of $R$ hence $r = 0$. \hfill \square

Remark 3.21. As a consequence $R \otimes_A G = \bigoplus_{i=1}^{k} S_i$ with $S_i$ simple and finite dimensional over $G$. We see that, if $R$ is prime as $n$–CH algebra then, the ideal $\{0\} = \bigcap_{i=1}^{k} P_i$ is the intersection of the finitely many minimal prime ideals $P_i = R \cap \bigoplus_{j=1, j \neq i}^{k} S_j$. Finally $R/P_i \otimes_A G = S_i$. 

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**Corollary 3.22.** A prime $n$–CH algebra $R$ is PI equivalent to one of the algebras $F(m; a)$ of Formula (23).

**Proof.** Clearly $R$ is PI equivalent, with the notations of Theorem 3.20, to its ring of fractions $R \otimes_T G$ which in turn is PI equivalent to $R \otimes_T \overline{G}$ with $\overline{G}$ the algebraic closure of $G$. Then by Theorem 3.12 $R \otimes_T \overline{G}$ is one of the algebras $\overline{G}(m; a)$ which, as $F$ algebra is PI equivalent to $F(m; a)$.

In any associative algebra $R$ one can define the spectrum of $R$ as the set of all its prime ideals, it is equipped with the Zariski topology.

For commutative algebras the spectrum is a contravariant functor with $f : A \to B$ giving the map $P \mapsto f^{-1}(P)$. But in general a subalgebra of a prime algebra need not be prime and the functoriality fails for non commutative algebras.

For algebras with trace $R$ we may define:

$$\text{Spec}_t(R) := \{P \mid P \text{ is a prime trace ideal}\}.$$ 

If $T \subset R$ is the trace algebra we have the map $j : \text{Spec}_t(R) \to \text{Spec}(T)$, $P \mapsto P \cap T$. For an $n$–CH algebra $R$ we have the remarkable fact:

**Proposition 3.23.** The map $j : \text{Spec}_t(R) \to \text{Spec}(T)$, $P \mapsto P \cap T$ is a homeomorphism, its inverse is $p \mapsto K(pR)$.

**Proof.** First, by Proposition 3.4 1. and any trace ideal $I \subset R$ we have that $I \cap T = t(I) = K(I) \cap T$. In fact if $a \in I \cap T$ we have $a = t(at(1)^{-1})$ and, if $a \in K(I)$ we have $t(a) \in I$.

We first show that, for an $n$–CH algebra $R$, the ideal $K(pR)$, is prime.

This follows from the characterization of prime algebras, Corollary 3.8 2), since $t(R/K(pR)) = T/p$ a domain and the kernel of $R/K(pR)$ is $\{0\}$.

So the composition in one direction is the identity $p = K(pR) \cap T$.

If $P$ is a prime ideal we need to show that $P = K((P \cap T)R)$. We certainly have $P \supset K((P \cap T)R)$ so it is enough to show that, if $P \supset Q$ are two prime ideals and $P \cap T = Q \cap T$ then $P = Q$. In fact in $R/Q$ we have $t(P/Q) = 0$ which implies $P/Q = 0$.

So for $n$–CH algebras the spectrum is also a contravariant functor setting

$$f : A \to B, \ P \mapsto K(f^{-1}(P)).$$

In particular let $p$ be a prime ideal of $T$ and consider the local algebra $T_p$ and

$$R_p := R \otimes_T T_p$$

we have then:

**Corollary 3.24.** $R_p$ is a local ring with maximal ideal $K(R_p p)$.

**Theorem 3.15** describes the possible residue and trace simple algebras $S_p := R_p/K(R_p p)$.  

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One also has, Remark 3.21, that there are only a finite number of prime (not trace invariant) ideals $P_i$ of $R$ with $P_i \cap T = p$. Moreover $Sp$ is the direct sum of the simple rings of fractions of $R/P_i$. This is a strong form of going up and lying over of commutative algebra in this general setting.

A special case is when $Sp$ is simple as algebra and of rank $n^2$ over its center. In this case one can apply Artin’s Theorem and deduce that

**Proposition 3.25.** If $Sp$ is simple as algebra and of rank $n^2$ over its center than $R_p$ is a rank $n^2$ Azumaya algebra over its center $T_p$.

Let us analyze general prime ideals, not necessarily closed under the trace.

**Proposition 3.26.** Let $S$ be an $n$–CH algebra with trace algebra $A$, $P$ an algebra ideal of $S$ which is prime and $p = P \cap A$. Let $F$ be the field of fractions of $A/p$.

Then $S/P \otimes_A F = S/P \otimes_{A/p} F$ is a simple algebra and $P$ is one of the minimal primes of the prime trace algebra $S/K(pS)$.

**Proof.** We have $P \supset K(pS)$ since $K(pS)$ is nilpotent modulo $pS$. Thus we have a surjective map $S/K(pS) \to S/P$ which induces a surjective map $S/K(pS) \otimes_A F \to S/P \otimes_A F$ with $F$ the field of fractions of $A/p$. Since $S/P \otimes_A F = S/P \otimes_{A/p} F$ is a prime algebra and $S/K(pS) \otimes_A F = \oplus_i S_i$ is a direct sum of simple algebras we must have $S/P \otimes_A F = S_i$ for some index $i$ and the claim follows. □

**Corollary 3.27.** Let $S$ be an $n$–CH algebra with trace algebra $A$, $M$ an algebra ideal of $S$ which is maximal and $m = M \cap A$. Then $m$ is a maximal ideal of $A$.

**Proof.** We have that $S/M$ is a simple algebra integral over $A/m$ hence its center, a field, is integral over $A/m$. The statement that $A/m$ is a field follows from the going up theorem. □

**Lemma 3.28.** Let $S$ be an $n$–CH algebra with trace algebra $A$, $M$ an algebra ideal of $S$ so that $S/M$ is simple of dimension $n^2$ over its center and $m = M \cap A$. Then $M = mS$.

**Proof.** This follows from the previous Proposition. Let $\bar{S} := S/mS$. $\bar{S}$ is an $n$–CH algebra with trace algebra $F = A/mA$ which is a field by the previous Lemma. The statement then follows by part 3) of Theorem 3.17. □

**Theorem 3.29.** Let $S$ be an $n$–CH algebra with trace algebra $A$ a local ring with maximal ideal $m$. Let $M$ be an algebra ideal of $S$ so that $S/M$ is simple of dimension $n^2$ over its center, then $M = mS$ and $S$ is a rank $n^2$ Azumaya algebra over $A$. 
Proof. The fact that $M = mS$ follows from the previous Lemma.

The fact that $S$ is a rank $n^2$ Azumaya algebra over $A$ then follows from Artin’s Theorem, since, as any $n$–CH algebra, $S$ satisfies all polynomial identities of $n \times n$ matrices and no quotient satisfies the polynomial identities of $n - 1 \times n - 1$ matrices.

3.30. The relatively free algebra of a simple algebra

Given a finite dimensional $n$–CH algebra $R$ over a field $F$ and with trace in $F$, its relatively free algebra $F_R(\ell)$ in $\ell$ variables can be described by the method of generic elements by fixing a basis $u_1, \ldots, u_m$ of $R$ over $F$ and introducing $\ell m$ variables $\xi_{i,j}$ and generic elements

$$\xi_i := \sum_{j=1}^{m} \xi_{i,j} u_j, \ i = 1, \ldots, \ell.$$ 

Then $F_R(\ell)$ is the subalgebra of $R \otimes_F F[\xi_{i,j}]$ generated by the generic elements $\xi_i$ and then closing it under trace. The trace algebra $T_R(\ell)$ of $F_R(\ell)$ is contained in the polynomial ring $F[\xi_{i,j}]$, it is finitely generated over $F$ and the algebra $F_R(\ell)$ is a finitely generated torsion free module over $T_R(\ell)$ (Proposition 8.2.4 and Theorem 8.2.5 of [1]).

Thus if $G_R(\ell)$ is the field of fractions of $T_R(\ell)$ we may construct the algebra

$$H_R(\ell) := F_R(\ell) \otimes_{T_R(\ell)} G_R(\ell) \subset R \otimes_F F(\xi_{i,j})$$  

with trace in $G_R(\ell)$ hence finite dimensional over $G_R(\ell)$ by Proposition 3.9.

We want to study the relatively free algebra in some $\ell$ variables for a simple algebra with trace as in Theorem 3.15.

Up to PI equivalence we can assume that the algebra $R$, with trace $t$, is one of the algebras $F(m; a)$ of Formula (23). We describe such an algebra $R$ ordering the multiplicities $a_i$ as follows. We have $s$ integers $\ell_1 < \ell_2 < \cdots < \ell_s$ and $s$ partitions $\ell_i := 1^{h_{i,1}}2^{h_{i,2}}\cdots \ell_i^{h_{i,i}}$, so that $R$ is a direct sum of the algebras

$$R = \bigoplus_{i=1}^{s} A_{\ell_i}, \ A_{\ell_i} := \bigoplus_j M_{j}(F)^{\oplus h_{j,i}}, \ t(a) = \ell_i \ tr(a), \ a \in M_{j}(F)^{\oplus h_{j,i}}.$$  

In this case a first general fact is

Lemma 3.31. Each algebra $R$ of Formula 23 can be generated by 2 elements.

Proof. For a matrix algebra $M_n(F)$ a choice of two generators is a diagonal matrix $D$ with distinct entries and the matrix $T_n$ of the cyclic permutation $(1, 2, \ldots, n)$. Just with $D$ we generate all diagonal matrices. So if $R = \bigoplus_j M_{h_j}(F)$ let $D = \bigoplus_j D_j$ with all the entries distinct and $X = \bigoplus_j C_{h_j}$. $D$ and $X$ clearly generate $R$. 

\]
Proposition 3.32. If $\ell \geq 2$ then, $(H_R(\ell) = F_R(\ell) \otimes_{T_R(\ell)} G_R(\ell))$:

$$\dim G_R(\ell) H_R(\ell) = \dim_F R,$$

(28)

$$H_R(\ell) \otimes G_R(\ell) F(\xi_{i,j}) = R \otimes_F F(\xi_{i,j}).$$

(29)

The algebra $F_R(\ell)$ is prime and its ring of fractions $H_R(\ell)$ is a simple algebra with trace algebra $G_R(\ell)$.

Proof. A semisimple trace algebra $R$ is characterized by the fact that the trace form $tr(ab)$ is non degenerate. If $m = \dim_F R$ we have that $m$ elements $u_1, \ldots, u_m \in R$ form a basis of $R$ if and only if the determinant of the $m \times m$ matrix $tr(u_i u_j)$ is different from 0.

Moreover, by the previous Lemma, $R$ can be generated by 2 elements. Therefore there are $m$ monomials $M_i$, $i = 1, \ldots, m$ in $\ell \geq 2$ generic elements such that the determinant $\Delta$ of the matrix $tr(M_i M_j)$ is non zero. If we invert $\Delta$ we then have that the $m$ monomials $M_i$, $i = 1, \ldots, m$ form a basis of $F_R(\ell)[\Delta^{-1}] = F_R(\ell) \otimes_{T_R(\ell)} T_R(\ell)[\Delta^{-1}]$ over $T_R(\ell)[\Delta^{-1}]$. Hence they also form a basis of $F_R(\ell) \otimes F(\xi_{i,j}) F(\xi_{i,j}) C R \otimes_F F(\xi_{i,j})$. These two spaces have the same dimension over $F(\xi_{i,j})$ so they coincide.

The final statement follows from this or just from the fact that the trace form is non degenerate.

For $\ell = 1$ the algebra $F_R(1)$ is commutative generated by a single generic element which is semisimple hence $F_R(1)$ is also prime and will be described later.

If $R$ is not semisimple the dimension of $F_R(\ell) \otimes T_R(\ell) G_R(\ell)$ may grow to infinity with $\ell$ as the simplest example shows.

Consider the algebra of dual numbers $F[\epsilon]$, $\epsilon^2 = 0$ with trace of the multiplication, $tr(a + b \epsilon) = 2a$. It is a 2 CH algebra.

Denote the generic elements $\xi_i = x_i + y_i \epsilon$. The trace algebra is $F[x_1, \ldots, x_\ell]$, and $G_R(m) = F(x_1, \ldots, x_\ell)$, we have

$$f(\xi_1, \ldots, \xi_\ell) = f(x_1, \ldots, x_\ell) + \left( \sum_{j=1}^\ell \frac{\partial f}{\partial x_j} y_j \right) \epsilon.$$

$$H_R(\ell) = G_R(\ell) \oplus_{j=1}^\ell G_R(\ell) y_j \epsilon, \quad \dim_{G_R(\ell)} H_R(\ell) = \ell + 1.$$

This phenomenon is strictly tied to the fact that $R$ is not semisimple.

We have now the following classification of prime $T$–ideals.

Theorem 3.33. A $T$–ideal $I$ of $S_n(\ell)$ is prime if and only if it is the ideal of trace identities of one of the algebras $F(m_1, \ldots, m_k; a_1, \ldots, a_k)$.

Proof. We have just seen that the $T$–ideal of one of these algebras is prime. Conversely if $I$ is prime it is the $T$–ideal of identities of the prime algebra $S_n(\ell)/I$. By Corollary 3.22 a prime algebra is PI equivalent to one of the algebras $F(m_1, \ldots, m_k; a_1, \ldots, a_k)$ hence $I = I(m_1, \ldots, m_k; a_1, \ldots, a_k)$. □
Let us then set:

\[ J(m_1, \ldots, m_k; a_1, \ldots, a_k) := I(m_1, \ldots, m_k; a_1, \ldots, a_k) \cap T_n(\ell). \] (30)

By Proposition 3.23 we have:

\[ I(m; a) = \{ x \in S_n(\ell) \mid tr(xS_n(\ell)) \subset J(m; a) \}. \] (31)

The notion of \(T\)-ideal can be defined also for ideals in the trace algebra \(T_n(\ell)\).

If \(J \subset T_n(\ell)\) is a \(T\)-ideal then the set \(I(J) := \{ x \in S_n(\ell) \mid tr(x \cdot S_n(\ell)) \subset J \}\) is clearly also a \(T\)-ideal. We deduce that

**Corollary 3.34.** A \(T\)-ideal \(J\) of \(T_n(\ell)\) is prime if and only if it is the ideal of pure trace identities of one of the algebras \(F(m_1, \ldots, m_k; a_1, \ldots, a_k)\).

**Proof.** Let \(I(J)\) be defined as before. It is enough to show that \(I(J)\) is also prime.

Let \(A, B \supset I(J)\) be two trace ideals so that \(AB \subset I(J)\).

We thus have \(tr(A)tr(B) \subset J\) and then, since \(J\) is prime, for instance \(tr(A) \subset J\) so that \(A \subset I(J)\). \(\square\)

For \(R = F(m; a)\) denote by \(G_R(\ell)\) the field of fractions of \(T_R(\ell)\). The notations are:

\[ F_R(\ell) = S_n(\ell)/I(m; a), \quad T_R(\ell) = T_n(\ell)/J(m; a), \]

\[ S_n(\ell)/I(m; a) \otimes_{T_n(\ell)/J(m; a)} G_R(\ell) = F_R(\ell) \otimes_{T_R(\ell)} G_R(\ell) = H_R(\ell). \] (32)

The geometric interpretation of these ideals of the ring of invariants of matrices will be developed in §3.38.1, Theorem 3.46 and Corollary 3.52.

The next Theorem is a generalization of Theorem 1.7 of Amitsur.

**Theorem 3.35.** Let \(R\) be as in Formula 27 and \(\ell \geq 2\) then

\[ H_R(\ell) \cong \bigoplus_{i=1}^{s} D_{h_{ij}} \] (33)

with \(D_{h_{ij}}\) a division algebra of dimension \(j^2\) over its center \(F_j\) which contains \(G_R(\ell)\) and has degree \([F_j : G_R(\ell)] = h_{ij}\). The trace is given by a Formula as in 24.

**Proof.** Given any pair \(h, k\) of integers there is a division ring \(L_{h,k}\) with center \(G_{h,k}\) and \(\dim_{G_{h,k}} L_{h,k} = h^2\) furthermore one can choose \(G_{h,k}\) as to contain a subfield \(F_{h,k}\) with \([G_{h,k} : F_{h,k}] = k\) all these contain \(F\). Then \(L_{h,k}\) equipped with the trace \(tr_{L_{h,k}/F_{h,k}} = tr_{L_{h,k}/G_{h,k}} \circ tr_{G_{h,k}/F_{h,k}}\) is PI equivalent to \(M_h(F)^{\otimes k}\). Then to compute the relatively free algebra we may replace \(R\) by \(S := \bigoplus_{i=1}^{s} \bigoplus_{j} L_{j,h_{ij}}\) with the appropriate trace and \(F_R(\ell) \simeq F_S(\ell)\). The same argument as before gives, using a basis of \(S\) for the generic elements

\[ F_S(\ell) \otimes_{T_R(\ell)} F(\xi_{i,j}) \simeq S \otimes_F F(\xi_{i,j}) = \bigoplus_{i=1}^{s} \bigoplus_{j} L_{j,h_{ij}} \otimes_F F(\xi_{i,j}). \]

Each \(L_{j,h_{ij}} \otimes_F F(\xi_{i,j})\) is still a division algebra and the claim follows. \(\square\)
Let us develop the simplest example for the algebra $R = F^\oplus s$. Take as trace $tr(r_1, \ldots, r_s) = \sum_{i=1}^s a_i x_i$ for $s$ integers $a_1 \leq a_2 \leq \ldots \leq a_s$ with $\sum a_i = n$. Then $R$ is an $n$–CH algebra.

The corresponding free algebra, for $\ell = 1$ is described as follows. Consider $s$ variables $x_1, \ldots, x_s$ over $\mathbb{Q}$ (or $\mathbb{C}$) and let $G := \mathbb{Q}(x_1, \ldots, x_s)$ be the field they generate. Let $X$ be a diagonal $n \times n$ matrix with entries $x_i$ a number $a_i$ of times. The relatively free algebra of $R$ in one variable is the trace algebra generated over $\mathbb{Q}$ by $X$.

First the simplest case $s = 2, a_1 = 1, a_2 = 2, n = 3, \ x_1 = u, x_2 = x_3 = v$.

The characteristic polynomial of $X$ is:

$$(t - u)(t - v)^2 = t^3 - (u + 2v)t^2 + (2uv + v^2)t - uv^2.$$  

Set $a = (u + 2v); \ b = (2uv + v^2); \ c = uv^2$. These 3 elements generate the algebra $A$ of traces.

We have then:

$$a^2 - 3b = (u - v)^2; \ ab - 9c = 2v(u - v)^2; \ 9c + a^3 - 4ab = u(u - v)^2,$$

$$\implies 2v = \frac{ab - 9c}{a^2 - 3b}; \ u = \frac{9c + a^3 - 4ab}{a^2 - 3b}.$$  

Moreover

$$3v^2 - 2av + b = 0; \ u^2 - 4au + a^2 - 4b = 0,$$  

and $a, b, c$ are the restrictions of the elementary symmetric functions to one of the 3 planes where two coordinates are equal, therefore they satisfy as equation the discriminant a polynomial of degree 4 and weight 6:

$$3a^2b - 162abc + 243c^2 - 12a^2b^2 + 18a^3c + 36b^3.$$  

The element $X$, besides satisfying its characteristic polynomial of degree 3, satisfies its minimal polynomial $(t - u)(t - v)$ which can be made into a polynomial with coefficients in $A$ by multiplying it by $(a^2 - 3b)^2$.

$$((a^2 - 3b)X - (9c + a^3 - 4ab)((a^2 - 3b)X - \frac{1}{2}(ab - 9c)), \quad (34)$$

The relative free algebra with trace of $R$, in 1 variable, is the algebra generated by $X, a, b, c$ modulo these two relations.

The relation (34) decomposes as a product of two factors hence the ring of fractions of $R$ is the direct sum of two fields isomorphic to $\mathbb{Q}(u, v)$ and under this isomorphism $X \mapsto (u, v)$ the trace on $\mathbb{Q}(u, v) \oplus \mathbb{Q}(u, v)$ is $(a, b) \mapsto a + 2b$.

We return to the general case of $X$ a diagonal $n \times n$ matrix with entries $p$ variables $x_i$ each a number $a_i$ of times. Let $\mathbb{Q}[X]_T := \mathbb{Q}[X, tr(x^i)], \ i = 1, \ldots, n$ the algebra with trace generated by $X$ and $T(X) = \mathbb{Q}[tr(x^i)]$ its trace
algebra. Then \( T(X) \) is generated by the coefficients of the characteristic polynomial of \( X \) i.e. setting \( m := \sum_i a_i \):

\[
h(t; x_1, \ldots, x_p) : \prod_{i=1}^p (t - x_i)^{a_i} = t^n + \sum_{j=1}^n (-1)^j \alpha_j t^{n-j}
\]  

(35)

and \( X \) is integral over \( T(X) \).

In the special case in which all \( a_i = 1 \) we have that \( T(X) \) is the algebra of symmetric functions \( \mathbb{Q}(x_1, \ldots, x_n)^{S_n} = \mathbb{Q}[e_1, \ldots, e_n] \) with \( e_i \) the elementary symmetric functions

\[
h(t; x_1, \ldots, x_n) : \prod_{i=1}^n (t - x_i) = t^n + \sum_{j=1}^n (-1)^j e_j t^{n-j}
\]

and \( T(X)[X] = \mathbb{Q}[e_1, \ldots, e_n][t]/(h(t; x_1, \ldots, x_n)) \).

In general let \( F \subset G = \mathbb{Q}(x_1, \ldots, x_p) \) be the field of fractions of \( T(X) \). We have that \( X \) is algebraic over \( F \) and \( F[X] \) is semisimple. In order to understand \( F[X] \) we can use the fact that it is a CH algebra and \( tr(X^2) = \sum_i a_j x_i^2 \in F \). In fact these elements generate \( F \) over \( \mathbb{Q} \). We need a simple Lemma:

**Lemma 3.36.** Let \( L \) be a field of characteristic 0 and \( f(t) \in L[t] \) a polynomial whose distinct roots are \( \alpha_1, \ldots, \alpha_k \) (they may appear with multiplicities).

Then the polynomial \( \bar{f}(t) := \prod_{i=1}^k (t - \alpha_i) \) has coefficients in \( L \).

**Proof.** Decompose \( f(t) = \prod_{i=1}^j g_i(t)^{b_i} \) with the \( g_i(t) \in L[t] \) irreducible and distinct. Then, since \( L \) has characteristic 0, all the \( g_i(t) \) have distinct roots and clearly \( f(t) = \prod_{i=1}^j g_i(t) \). \( \square \)

Applying this Lemma to the polynomial \( h \) of Formula (35) we have that

\[
\mathbb{Q}(x_1, \ldots, x_p)^{S_p} \subset F \subset \mathbb{Q}(x_1, \ldots, x_p),
\]

The set of the \( p \) variables \( x_i \) is partitioned into the \( s \) equivalence classes of the equivalence relation \( x_i \equiv x_j \iff a_i = a_j \). Each class with \( k_i \) elements, \( A_i := \{ x_j \mid k_1 + \ldots + k_{i-1} + 1 \leq j \leq k_1 + \ldots + k_i \} \).

In other words, up to reordering the \( x_i's \), we have strictly positive integers \( k_1, k_2, \ldots, k_s \) and also \( 0 < a_1 < a_2 < \ldots < a_s \) so that \( n = \sum k_i a_i \) and

\[
h(t; x_1, \ldots, x_p) = \prod_{i=1}^s \left( \prod_{j=k_1+\ldots+k_i+1}^{k_1+\ldots+k_i} (t - x_i)^{a_j} \right)
\]  

(36)

As for \( F \), by Galois Theory we have \( F = \mathbb{Q}(x_1, \ldots, x_p)^H \) where \( H \) is the subgroup of \( S_p \) fixing the polynomial \( h(t; x_1, \ldots, x_p) \). This is clearly the Young subgroup \( \prod_{i=1}^s S_{A_i} \) product of the symmetric groups on the disjoint sets \( A_j \). Setting \( L := \mathbb{Q}(x_1, \ldots, x_p)^{S_p} \) and \( F_i := L(x_j; j \in A_i)^{S_{A_i}} \) we thus have
Proposition 3.37.
\[ F = F_1 \otimes_L F_2 \otimes_L \cdots \otimes_L F_{s-1} \otimes_L F_s \]  
(37)

\[ F[X] = \oplus_{i=1}^{s} F[t] / \prod_{i=k_1+\ldots+k_{i-1}+1}^{k_1+\ldots+k_i} (t - x_i) = \oplus_{i=1}^{s} G_i. \]  
(38)

The element \( X \) corresponds to \((x_1,\ldots,x_s)\) with \( x_i \) the class of \( t \) in the \( i^{th} \) summand \( G_i \).

The trace is
\[ \text{tr}(r_1,\ldots,r_s) = \sum_{i=1}^{s} a_i \text{tr}_{G_i \setminus F}(r_i). \]

As for the trace algebra \( T(X) = \mathbb{Q}[\text{tr}(X^i)] \), this is also generated by the coefficients of the characteristic polynomial of \( X \), Formula (35). The matrix \( X \) is the direct sum \( X = \oplus_{i=1}^{s} X_i^{k_i} \) of the diagonal matrices \( X_i \) of size \( k_i \) having as entries the variables \( x_j, j \in A_i \). If we let \( T_i(X) = T(X_i) \) we have that \( T(X_i) \) is the polynomial ring in the \( k_i \) elementary symmetric functions in these variables and finally:

**Theorem 3.38.** The integral closure of \( T(X) \) in its field of fractions \( F \) is
\[ T(X) = T(X_1) \otimes_{\mathbb{Q}} T(X_2) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} T(X_{s-1}) \otimes_{\mathbb{Q}} T(X_s) \]  
(39)

The integral closure of \( T(X)[X] \) in the semisimple algebra \( F[X] \) is
\[ \oplus_{i=1}^{s} T(X)[t] / \prod_{i=k_1+\ldots+k_{i-1}+1}^{k_1+\ldots+k_i} (t - x_i). \]  
(40)

3.38.1. The general case \( \ell \geq 2 \)

We assume \( F \) algebraically closed. Let us first recall some general facts for which we may refer to Springer [27]. Let \( G \) be a reductive group and \( V \) a linear representation, one has that the ring of invariants \( S[V^*]^G \) is finitely generated so it is the coordinate ring of some irreducible variety denoted by \( V/\!/G \). The inclusion \( S[V^*]^G \subset S[V^*] \) corresponds to a quotient map \( \pi : V \to V/\!/G \) which is surjective and such that each fiber contains a unique closed orbit. The stabilizer of a point of a closed orbit is also reductive, one says that two stabilizers have the same type if they are conjugate. One has only finitely many types of stabilizers and by Luna’s theory, [14], the quotient variety is stratified into the smooth strata \( \Sigma_H \) formed by all points in \( V/\!/G \) corresponding to the stabilizer types \( H \). That is the closed orbit in \( \pi^{-1}(p) \) is \( G \)–isomorphic to \( G/H \) or, equivalently, there is a point \( q \in \pi^{-1}(p) \) with stabilizer \( H \). Finally the closure of a stratum \( \Sigma_H \) is the union of the strata \( \Sigma_K \) where \( H \) is conjugate to a subgroup of \( K \).
Our case is the space $M_n(F)\ell$ acted by conjugation by the linear group. By a Theorem of M. Artin in this case, the non-0 closed orbits correspond to those $\ell$-tuples of matrices which generate a semisimple algebra, and such an algebra is conjugate to one of the algebras $F(m; a) = F(m_1, \ldots, m_k; a_1, \ldots, a_k)$, $\sum m_i a_i = n$ of Formula (23). We assume from now on that $\ell \geq 2$ so, from Lemma 3.31, all of these algebras appear.

**Proposition 3.39.** The stabilizer of a $\ell$-tuple of matrices generating the algebra $F(m; a)$ is the group $G(a; m)$ of invertible elements in the centralizer algebra conjugate to $F(a; m)$ of dimension $\sum a_i^2$.

**Proof.** In fact an element $g \in GL(n, F)$ fixes the $\ell$ tuple if and only if it lies in the centralizer of the algebra $F(m; a)$ which is an algebra of type $F(a; m)$ by the double centralizer Theorem. $\square$

**Corollary 3.40.** In the case of $M_n(F)\ell$ we have that the stabilizer types are of the form $G(a; m)$.

We denote by $\Sigma^\ell(a; m) = \Sigma^\ell(a_1, \ldots, a_k; m_1, \ldots, m_k)$ the union of the closed orbits with stabilizer type $G(a; m)$. The variety $\Sigma^\ell(a; m)$ can be described as follows.

**Lemma 3.41.** A closed orbit $O$, with stabilizer type $G(a; m)$ intersects the space $F(m; a)^\ell$ in a single orbit under the group $\text{Aut}(m; a)$ of automorphisms of $F(m; a)$.

**Proof.** By definition there is a point $p := (a_1, \ldots, a_\ell)$ in the orbit with stabilizer exactly $G(a; m)$. Thus the elements $\{a_1, \ldots, a_\ell\}$ lie in $F(m; a)$ the centralizer of $G(a; m)$. Moreover these elements must generate $F(m; a)$, again by the double centralizer Theorem, since they generate a semisimple algebra with centralizer $F(a; m)$. Assume that $p, q \in F(m; a)^\ell$ are in the same orbit $q = g \cdot p$. Since $p$ and $q$ generate $F(m; a)$ we must have that $g$ fixes $F(m; a)$ and of course it is an isomorphism.

Conversely it is enough to show that:

**Lemma 3.42.** Each automorphism of $F(m; a)$ is the restriction of an inner automorphism of matrices.

**Proof.** The automorphism group $\text{Aut}(m; a) = G_0 \rtimes H$, $\dim \text{Aut}(m; a) = \sum m_i^2 - k$ (41) is the semidirect product of its connected component $G_0 = \prod_{i=1}^k PGL(m_i, F)$ of inner automorphisms and the finite group $H$ product of the symmetric groups permuting the blocks relative to indices $(m_i, a_i)$ which are equal between each other. These are also given by conjugation by permutation matrices. $\square$
Proposition 3.43.

\[ \dim \Sigma^\sharp(a; m) = n^2 + (\ell - 1)\left(\sum_{i=1}^{k} m_i^2\right) - \sum_{i=1}^{k} a_i^2 + k. \]  \hspace{1cm} (42)

Proof. For each \( p \in \Sigma^\sharp(a; m) \) denote by \( F(p) \) the semisimple subalgebra it generates, by hypothesis conjugate to \( F(m; a) \).

The set of subalgebras of \( M_n(F) \) conjugate to \( F(m; a) \) is an orbit, in the Grassmann variety of subspaces of \( M_n(F) \), under \( GL(n, F) \). The subgroup

\[ \tilde{G}(m; a) := \{ g \in GL(n, F) \mid g(F(m; a)) = F(m; a) \} \]

fits into an exact sequence

\[ 0 \longrightarrow G(a; m) \longrightarrow \tilde{G}(m; a) \longrightarrow Aut(m; a) \longrightarrow 0. \]

Thus this orbit of \( F(m; a) \) has dimension

\[ \dim GL(n, F) - \dim \tilde{G}(a; m) = n^2 - \sum_{i} a_i^2 - \sum_{i} m_i^2 + k. \]

The variety \( \Sigma^\sharp(a; m) \) fibers, in a \( GL(n, F) \) equivariant way over this orbit.

We can take as fiber the points \( p \in \Sigma^\sharp(a; m) \) with \( F(p) = F(m; a) \). Thus \( p \in \mathcal{O}(m; a, \ell) \) the open set of \( F(m; a)^\ell \) of \( \ell \)-tuples which generate \( F(m; a) \). We have \( \dim \mathcal{O}(m; a, \ell) = \ell \sum_i m_i^2 \), hence the claim.

We now pass to the Luna strata \( \Sigma(a; m) \) in the quotient variety.

Proposition 3.44.

\[ \dim \Sigma(a; m) = (\ell - 1)\left(\sum_{i=1}^{k} m_i^2\right) + k. \]  \hspace{1cm} (43)

Proof. This follows from the previous Proposition and the fact that the closed orbits of type \( (a; m) \) have dimension \( n^2 - \sum_i a_i^2 \).

Consider a maximal proper subalgebra \( \oplus_j M_{p_j}(F) \subset \oplus_i M_{m_i}(F) \). This is obtained by splitting one of the \( m_i \) say \( m_1 = p + q \) and replacing \( M_{m_1}(F) \) with \( M_{p}(F) \oplus M_{q}(F) \). The dimension of the corresponding stratum is

\[ \dim \Sigma(a; m) - (\ell - 1)(p + q)^2 + (\ell - 1)(p^2 + q^2) + 1 = \dim \Sigma(a; m) - (\ell - 1)2pq + 1 \]

We have \( (\ell - 1)2pq - 1 = 1 \) if and only if \( \ell = 2, p = q = 1 \) so we have:
Theorem 3.45. Except in the case \( \ell = 2 \) and one of the blocks is \( 2 \times 2 \) matrices the complement of the stratum \( \Sigma(a;m) \) in its closure has codimension \( \geq 2 \).

In this case the regular functions on \( \Sigma(a;m) \) give the integral closure of the functions on \( \Sigma(a;m) \).

For two \( 2 \times 2 \) matrices the complement of the open stratum is the hypersurface of equation \( \det(XY - YX) \).

Recall that a \( T \)-ideal in the free algebra with trace \( F\langle x_1, \ldots, x_\ell \rangle[t(M)] \) is an ideal stable under the semigroup \( S \) of variable substitutions or endomorphisms.

Such a semigroup induces for all \( n \)

1. a semigroup \( S_n \) of variable substitutions or endomorphisms in each of the quotient algebras \( F_{T,n}\langle x_1, \ldots, x_\ell \rangle \)
2. an opposite semigroup \( S_n^{op} \) of regular maps on the variety \( M_n(F)^\ell \) commuting with the \( PGL(n,F) \) action
3. a semigroup of regular maps in the quotient variety \( M_n(F)^\ell \) commuting with the \( PGL(n,F) \) action.

Let \( G \) be the (infinite dimensional) group of automorphisms of the free algebra in the variables \( X \).

Then, for each \( n \), the group \( G \) induces a group \( G_n \) of regular automorphisms on the variety \( M_n(F)^X \) commuting with \( PGL(n,F) \); therefore it preserves the Luna stratification.

Given a \( T \)-ideal \( M_n(A_{\ell,n})^{PGL(n,F)} \) it defines a \( PGL(n,F) \) stable subalgebra conjugate to \( F_{T,n}(x_1, \ldots, x_\ell) \).

Theorem 3.46. The ideal \( J(m_1, \ldots, m_k; a_1, \ldots, a_k) \) is the ideal of \( T_n(\ell) \) vanishing on the subvariety \( \Sigma(a;m) \).

The ideal \( I(m_1, \ldots, m_k; a_1, \ldots, a_k) \) is the ideal of \( S_n(\ell) \) vanishing on the subvariety \( \Sigma(a;m) \).

Proof. First \( J(m_1, \ldots, m_k; a_1, \ldots, a_k) \) vanishes on the set of \( \ell \)-tuples generating a subalgebra conjugate to \( F(m;a) \) so it vanishes on \( \Sigma(a;m) \). Conversely the ideal of \( T_n(\ell) \) vanishing on the subvariety \( \Sigma(a;m) \) by the previous remark is a \( T \)-ideal. It is prime and contained in \( J(m_1, \ldots, m_k; a_1, \ldots, a_k) \). Then since the prime \( T \)-ideals are in 1 to 1 correspondence with the strata it must coincide with \( J(m_1, \ldots, m_k; a_1, \ldots, a_k) \). \( \square \)

We next analyze in some detail these ideals. Consider the space \( F(m;a)^\ell \) and inside it the open set \( O(a;m) := \Sigma^\ell(a;m) \cap F(m;a)^\ell \) of generating \( \ell \)-tuples.

We have that, Lemma 3.42, two points in \( O(a;m) \) are in the same orbit under \( Aut(m;a) \) if and only if they are in the same \( PGL(n,F) \) orbit so we
have a factorization of quotient maps,

\[
\begin{array}{ccc}
F(m; a)^\ell /& & M_n(F)^\ell \\
\downarrow f & & \downarrow f \\
F(m; a)^\ell //\text{Aut}(m; a) & \longrightarrow & M_n(F)^\ell //\text{PGL}(n, F)
\end{array}
\]  

(44)

such that, by Lemma 3.41, the map

\[
\tilde{i} : \mathcal{O}(m; a)^\ell //\text{Aut}(m; a) \to \Sigma(a; m)
\]

is bijective and, since \(\Sigma(a; m)\) is smooth it must be an isomorphism.

As a consequence, setting \(A_{(m)}^\ell = \bigotimes_{i=1}^k A_{m_i, \ell}\) to be the algebra of polynomial functions on \(F(m; a)^\ell = \bigoplus M_{m_i}(F)\) we have:

**Theorem 3.47.** The map \(\pi : F(m; a)^\ell //\text{Aut}(m; a) \to \Sigma(a; m)\) is the normalization. The invariant algebra \(A_{(m)}^\ell\) is the normalization of the algebra \(T_n(\ell)/J(m; a)\).

**Proof.** By Formula (41)

\[
F(m; a)^\ell //\text{Aut}(m; a) = \left(F(m; a)^\ell //G_0\right) //H
\]

(45)

\[
F(m; a)^\ell //G_0 = \prod_{i=1}^k M_{m_i}^\ell //\text{PGL}(m_i, F).
\]

(46)

This last variety is the variety of \(k\)-tuples of semisimple representations each of dimensions \(m_i\). As \(F(m; a)^\ell //\text{Aut}(m; a)\) is normal and by the previous remark the map \(\pi\) is birational, it only needs to be verified that the map is finite. For this it is enough to see that the map

\[
F(m; a)^\ell // \prod_{i=1}^k \text{PGL}(m_i, F) \to F(m; a)^\ell //\text{Aut}(m; a) \to \Sigma(a; m)
\]

is finite. This follows from the set theoretic description of the two varieties as semisimple representations since a semisimple representation can be presented as a direct sum only in finitely many ways.

We need now a general fact, take an algebra with trace \(R\). Change the trace by multiplication by \(a \in F\) and set \(R_a\) to be the algebra \(R\) with this new trace.

One has that trace identities of \(R_a\) correspond bijectively to trace identities of \(R\) by the isomorphism of the free algebra with trace, \(\varphi_a\) mapping \(\varphi_a : t(M) \to a \cdot t(M)\). In particular if \(R\) is a \(k\) Cayley–Hamilton algebra and \(a \in \mathbb{N}\), one has that \(R_a\) is an \(n = a \cdot k\) Cayley–Hamilton algebra, but there is still an isomorphism between the two spaces of trace identities so the \(n = a \cdot k\) Cayley–Hamilton identity for \(R_a\) is the \(a\) power of the transformed Cayley–Hamilton identity for \(R\). A special case is the algebra \(F(k; a)\) which is just \(M_k(F)\) with trace \(t(r) = a \cdot tr(r)\).
Proposition 3.48. The relatively free algebra in \( \ell \) variables of \( F(k; a) \) is \( S_k(\ell)_a \), i.e. \( S_k(\ell) \) with the new trace \( a \cdot \text{tr}(r) \).

Now we introduce the algebra \( S_\ell(m; a) \) of \( \text{Aut}(m; a) \) equivariant maps from \( F(m; a) \) to \( F(m; a) \). Denote by \( A_{(m); \ell} \) the algebra of polynomial functions on \( F(m; a) \) so that:

\[
S_\ell(m; a) = (A_{(m); \ell} \otimes F(m; a))^{\text{Aut}(m; a)}.
\] (47)

By Lemma 3.31, as soon as \( \ell \geq 2 \) the center \( T_\ell(m; a) \) of \( S_\ell(m; a) \) is

\[
T_\ell(m; a) = (A_{(m); \ell} \otimes F^k)^{\text{Aut}(m; a)}
\] (48)

with \( F^k \) the center of \( F(m; a) \).

Lemma 3.49. The kernel of the restriction of the functions \( S_n(\ell) \) (resp \( T_n(\ell) \)) to \( F(m; a)^\ell \) is \( I(m; a) \) (resp \( J(m; a) \)).

The algebra \( S_n(\ell) \) (resp \( T_n(\ell) \)) maps, under the restriction of the functions to \( F(m; a)^\ell \), to the algebra \( S_\ell(m; a) \) (resp \( T_\ell(m; a) \)).

Proof. By Theorem 3.46 the ideal \( I(m_1, \ldots, m_k; a_1, \ldots, a_k) \) is the ideal of \( S_n(\ell) \) vanishing on the subvariety \( \Sigma(a; m) \).

Since \( PGL(n; F)(F(m; a)^\ell) \) is dense in \( \Sigma(a; m) \) and the elements of \( S_n(\ell) \) are \( PGL(n; F) \) equivariant the first statement follows.

As for the second recall that the elements \( p \in F(m; a)^\ell \) are the fixed points of the invertible elements of the centralizer. Thus, under a \( PGL(n; F) \) equivariant map, such a point \( p \) is sent to \( F(m; a) \). Moreover since \( \text{Aut}(m; a) \) is induced by a subgroup of \( PGL(n; F) \) the second statement also holds. \( \square \)

So next we must analyze the algebra \( S_\ell(m; a) \). The \( k \) indices decompose into \( \ell \) subsets \( I_j \) each of some cardinality \( u_j \) where the pairs \( (m_i; a_i) \) are equal to some \((m(j), a(j)) \) and \( H = \prod_{j=1}^u S_{u_j} \), of Formula (41), where \( S_{u_j} \) permutes \( u_j \) factors of some type \( M_{m(j), a(j)}(F)^\ell \).

Theorem 3.50. The algebra \( S_\ell(m; a) \) is isomorphic to

\[
S_\ell(m; a) \cong \bigoplus_{j=1}^{u_j} H_j S_{m(j); \ell} \otimes_{T_{m(j); \ell}} A_{(m); \ell}^{G_0}.
\] (49)

It contains the algebra, with \( R = F(m; a) \):

\[
S_n(\ell)/I(m; a) \otimes_{T_n(\ell)/J(m; a)} A_{(m); \ell}^{\text{Aut}(m; a)} \cong F_R(\ell) \otimes_{T_R(\ell)} A_{(m); \ell}^{\text{Aut}(m; a)}.
\] (50)

Proof. This we do in two steps, first we consider the larger algebra \( S_\ell^0(m; a) \) of \( G_0 \) (Formula (41)) equivariant maps from \( F(m; a)^\ell \) to \( F(m; a) \):

\[
S_\ell^0(m; a) = (F(m; a) \otimes F A_{(m); \ell})^{G_0} = \bigoplus_{i=1}^k (M_{m_i}(F) \otimes A_{(m); \ell})^{G_0}
\]

\[
= \bigoplus_{i=1}^k (M_{m_i}(F) \otimes A_{m_i; \ell})^{PGL(m_i) \otimes_{T(m_i; \ell)} A_{(m_i); \ell}}
\]
This is the direct sum of the algebras of $\text{PGL}(m_i, F)$ equivariant maps from $M_{m_i}(F)^\ell$ to $M_{m_i}(F)$ that is the usual trace algebras of generic matrices extended to $A^G_{\text{PGL}(m_i); \ell} = T_{m_1}(\ell) \otimes T_{m_2}(\ell) \otimes \ldots \otimes T_{m_k}(\ell)$. If the summand relative to $i$ corresponds to the pair $(m_i, a_i)$, the trace in $S_{m_i}(\ell) \otimes T_{m_i}(\ell) A^G_{\text{PGL}(m_i); \ell}$ is the ordinary trace multiplied by $a_i$, by proposition 3.48. Formula (49) follows from Formula (51). The group $H = \prod_{i=1}^k S_{u_i}$ acts on the algebra of Formula (51) by permuting the summands of each term $\bigoplus_{i=1}^k S_{m(j)}(\ell) \otimes T_{m_i}(\ell) A^G_{\text{PGL}(m_i); \ell}$ through $S_{u_i}$ hence the claim of Formula (49). The map $S_{\ell}(\ell) \to \bigoplus_{i=1}^k S_{m_i}(\ell)$ given by restricting to $F(m; a)$ induces a map to $S_{\ell}(m; a)$ with kernel $I(m; a)$ by Lemma 3.49 and $I(m; a) \cap T_{\ell}(\ell) = J(m; a)$.

As for $\left( \bigoplus_{i=1}^k S_{m(j)}(\ell) \otimes T_{m_i}(\ell) A^G_{\text{PGL}(m_i); \ell} \right)^{S}_{u_i}$ we have the following general fact. Let $R = A^{\oplus h} = A_1 \oplus A_2 \oplus \ldots \oplus A_h$ be a direct sum of algebras over $\mathbb{Q}$ and $G$ a finite group acting on $R$ permuting the summands transitively and $H$ the subgroup of $G$ fixing the first summand (and permuting the others). Thus choosing $g_i \in G$ with $g_i \cdot A_1 = A_i$ we have that $G = \bigcup_{i=1}^h g_i H$ is the coset decomposition.

**Proposition 3.51.** The projection $\pi_1 : R = A^{\oplus h} \to A$ on the first summand induces an isomorphism between $R^G$ and $A^H$.

**Proof.** Let $(a_1, a_2, \ldots, a_h) \in R^G$, if $h \in H$ we have

$$h \cdot (a_1, a_2, \ldots, a_h) = (h \cdot a_1, a_2, \ldots, a_h) = (a_1, a_2, \ldots, a_h) \implies a_1 \in A^H.$$

Next since $G$ permutes the summands transitively if $a_1 = 0$ then $a_i = g_i a_1 = 0$, Vi so $\pi_1$ is injective.

Finally $\pi_1$ is surjective since if $a \in A^H$ we have

$$\frac{1}{|H|} \sum_{g \in G} g a = \frac{1}{|H|} \sum_{i=1}^h \sum_{h \in H} g_i h a = (a, g_2 a, g_3 a, \ldots, g_h a) \in R^G.$$

Assume that the $u_j$ indices which correspond to the pair $m(j), a(j)$ are $v + 1, \ldots, v + u_j$, then write

$$T_{m_1}(\ell) \otimes T_{m_2}(\ell) \otimes \ldots \otimes T_{m_k}(\ell) = B \otimes T_{v+1}(\ell) \otimes T_{v+2}(\ell) \otimes \ldots \otimes T_{v+u_j}(\ell) \otimes C$$

and $S_{u_j}$ permutes the factors $T_{v+1}(\ell) \otimes T_{v+2}(\ell) \otimes \ldots \otimes T_{v+u_j}(\ell)$.

**Corollary 3.52.**

$$\left( \bigoplus_{i=1}^k S_{m(j)}(\ell) \otimes T_{m_i}(\ell) A^G_{\text{PGL}(m_i); \ell} \right)^{S}_{u_j} \simeq S_{m(j)}(\ell) \otimes_F B \otimes (T_{v+2}(\ell) \otimes \ldots \otimes T_{v+u_j}(\ell))^{S_{u_j-1}} \otimes C.$$

(52)
In particular, this last algebra is a domain and we have a better understanding of Theorem 3.35. We leave to the reader to verify
\[
F_R(\ell) = S_n(\ell)/I(\mathfrak{m}; \mathfrak{a}) \subset S_\ell(\mathfrak{m}; \mathfrak{a}) \subset F_R(\ell) \otimes_{T_R(\ell)} G_R(\ell) \cong \bigoplus_{i=1}^s \bigoplus_j D_{h_{i,j}}
\] (53)

The algebra of Formula (49) may be viewed as a form of integral closure of the algebra of Formula (50). Compare with Theorem 3.41.

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