An Alternative Form of the Helmholtz Criterion in the Inverse Problem of Calculus of Variations

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AN ALTERNATIVE FORM OF THE HELMHOLTZ CRITERION IN THE INVERSE PROBLEM OF CALCULUS OF VARIATIONS

THEODORE VORONOV

To Alan Weinstein on the occasion of his 60th birthday

Abstract. We give a necessary and sufficient condition for the existence of a local solution of the inverse problem of calculus of variations in terms of the identical vanishing of the variation of a functional on an extended space (with the number of independent variables increased by one), and explain its relation with the classical Helmholtz criterion using the de Rham complex on an infinite-dimensional space of fields.

1. Introduction

This paper deals with the question, under which conditions given functions can be the variational derivatives of some functional. (The precise setup is described in §2.) This is a classical question, and there is a classical answer to it given by the so-called Helmholtz (or Helmholtz–Volterra) criterion. One can see [7] for an exposition and a historical review. The Helmholtz condition is, in fact, nothing but the closedness of a 1-form of a particular appearance on an infinite-dimensional “space of fields” on which functionals under consideration are defined, though this may be hidden in some expositions. Hence the Helmholtz criterion can be viewed as a version of an infinite-dimensional Poincaré lemma for 1-forms, and proved accordingly.

In the ordinary finite-dimensional case we know that a 1-form is closed if its integrals over paths do not change under small perturbation of a path fixing the boundary. This is also true for $k$-forms and $k$-paths. Pushed to the limit, this idea allows to express the usual de Rham differential entirely in variational terms. (For supermanifolds this is, actually, the only way one can construct an adequate analog of the Cartan–de Rham complex, see [11, 8, 9], also [4], [1], [6].) Moreover, the de Rham complex is embedded in a larger complex consisting of all Lagrangians of $k$-paths on a manifold [10].

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In this note we suggest an alternative criterion of the existence of a local solution of the inverse problem of calculus of variations in terms of the identical vanishing of the variation of a functional on an extended space (Theorem 1). This criterion is equivalent to the classical Helmholtz one. One advantage of the suggested criterion is the simplicity of application, due to the fact that, roughly speaking, it requires calculating the differential of a function, while the classical Helmholtz test requires calculating the differential of a 1-form. Theorem 1 is an immediate consequence of the existence of a natural chain transformation diminishing degrees by one and peculiar for forms on field spaces (Theorem 2). In a way, this is an “integrated” Cartan homotopy formula, with no analog on finite-dimensional manifolds. (On the other hand, the expression of the usual de Rham differential via the variation of functionals also follows from Theorem 2.)

We give a very brief outline of the appropriate de Rham complex in the Appendix. Methods of supermathematics are helpful, as usual. Clearly, forms on infinite-dimensional spaces like spaces of fields are no novel for physicists. Some of the mathematicians, on the other hand, seem to refrain from using them as legitimate objects. However, as we show, using them explicitly is very convenient and is entirely rigorous. I would like to note that a formalization based on the so-called functional forms on an infinite jet space (see [7], [2]), meant to replace the supposedly non-rigorous differential forms on field spaces, has a ineradicable defect of modelling only a subspace of the “diagonal forms” (see below). This results in the impossibility to multiply such objects and in other counterintuitive features. Working with forms as such is much better!

2. Main statements

Let $E \to M$ be a smooth fiber bundle over a smooth manifold $M$. The inverse problem of calculus of variations is to determine whether functions $A_i(x, u, u', u'', \ldots)$ can be the left hand sides of the Euler-Lagrange equations for a Lagrangian $L = L(x, u, u', \ldots)$, and to find $L$ if it exists. We denote by $x = (x^a)$ local coordinates on $M$, by $u = (u^i)$ local coordinates in the fiber of $E \to M$; then $(x, u, u', u'', \ldots) = (x^a, u^i, u^i_a, u^i_{ab}, \ldots)$ are the coordinates of jets of sections of $E \to M$. In the sequel we use the notation such as $D_a$ for total derivatives. The classical Helmholtz criterion (which we will recall below) states that, locally, a necessary and sufficient condition of the equality $A_i = \delta S/\delta u^i$ for some $S = \int L d^n x$ is the self-adjointness of a certain differential operator constructed from the functions $A_i$. We can give an alternative criterion:

**Theorem 1.** Functions $A_i(x, u, u', u'', \ldots)$ are variational derivatives of some functional $\int L d^n x$, with a Lagrangian $L$ defined locally in $u^i$, if
and only if, for the functional $\int \dot{u}^i A_i(x, u, u', u'', \ldots) d^n x dt$, defined on sections of the induced bundle $E \times \mathbb{R} \to M \times \mathbb{R}$, its variation vanishes identically for all arguments. (Dot stands for the derivative in $t$.)

Proof. Necessity. Let $A_i$ be the variational derivatives of some functional $S = \int L^a x$. Then we have $L' = \dot{u}^i A_i = \dot{u}^i \delta S/\delta u^i = D_i L + D_a f^a$ for some functions $f^a(x, u, u', \ldots)$, by the definition of variational derivative. Hence $\int L' d^n x dt$ is the integral of a total divergence, and its variation identically vanishes. Sufficiency. First we shall write down our condition

$$\delta \int \dot{u}^i A_i d^n x dt = 0$$

explicitly. By expanding and integrating by parts w.r.t. $x^a, t$ one can arrive at the following formula:

$$\delta \int \dot{u}^i A_i d^n x dt = \int d^n x dt \delta u^i(x,t) \sum \left( - \frac{\partial A_i}{\partial u^j_{a_1 \ldots a_k}} D_{a_1} \ldots D_{a_k} + (-1)^k D_{a_1} \ldots D_{a_k} \circ \frac{\partial A_j}{\partial u^i_{a_1 \ldots a_k}} \right) \dot{u}^i.$$

The symbol $\circ$ stands for the composition of operators (here we first apply the multiplication by a function and then the differentiation). Since both $\delta u^i(x,t)$ and $\dot{u}^i(x,t)$ can be arbitrary functions, we obtain the condition in the form

$$\sum_{k \geq 0} \left( \frac{\partial A_i}{\partial u^j_{a_1 \ldots a_k}} D_{a_1} \ldots D_{a_k} - (-1)^k D_{a_1} \ldots D_{a_k} \circ \frac{\partial A_j}{\partial u^i_{a_1 \ldots a_k}} \right) = 0. \quad (2)$$

Basically, there is nothing to prove now, since (2) is exactly the classical Helmholtz condition (see below). For the sake of completeness we shall supply a standard type argument. In a star-shaped domain in $u^i$, one has $A_i = \delta S/\delta u^i$ for $S = \int L^a x$, where

$$L(x, u, u', u'', \ldots) = \int_0^1 dt \dot{u}^i A_i(x, tu, tu', tu'', \ldots).$$

Indeed, by taking the variation and integrating by parts in $x^a$,

$$\delta S = \int d^n x \int_0^1 dt \delta u^i \left( A_i(x, tu, \ldots) + t \sum (-1)^k D_{a_1} \ldots D_{a_k} \frac{\partial A_j}{\partial u^i_{a_1 \ldots a_k}} (x, tu, \ldots) u^j \right).$$
which by (2) equals

$$
\int d^n x \int_0^1 dt \delta u^i \left( A_i(x, tu, \ldots) + t \sum \frac{\partial A_i}{\partial u_{a_1 \ldots a_k}} (x, tu, \ldots) u^j_{a_1 \ldots a_k} \right) =
$$

$$
\int d^n x \delta u^i \int_0^1 dt \frac{d}{dt} \left( t A_i(x, tu, \ldots) \right) = \int d^n x \delta u^i A_i(x, u, \ldots)
$$

(we have suppressed the arguments in $u^i(x), \delta u^i(x)$, etc.).

To practically apply this theorem one has to calculate the Euler–Lagrange expression for

$$
L_0 = \delta u^i A^i =
$$

which is linear in the derivatives involving “time”: $\dot{u}^i, \dot{u}^i_a$, etc., and set the respective coefficients to zero. This will give the equations for $A_i$.

Example. For a second-order function $A = A(x, u, u', u'')$ in the scalar case the only non-trivial equation will be

$$
\frac{\partial A}{\partial u_a} = D_b \left( \frac{\partial A}{\partial u_{ba}} \right).
$$

The necessity statement in Theorem 1 is just the equality $d^2 = 0$ in the “complex of Lagrangians” introduced in [10] (see also [5]).

The sufficiency is more delicate.

Recall the classical Helmholtz criterion (see, e.g., [7]): for functions $A_i(x, u, \ldots)$ to be variational derivatives, a necessary and sufficient condition is the formal self-adjointness of the matrix differential operator $L_{ij}$ associated with $A_i$ by the formula $L_{ij} = \sum \partial A_j/\partial u^i_{a_1 \ldots a_k} D_{a_1} \ldots D_{a_k}$, i.e.,

$$
L_{ij} = L^*_{ji} .
$$

(3)

Written explicitly, (3) is exactly equation (2). (Olver [7] deduces the local sufficiency of (3) from properties of the variational complex on jet space. A direct prove is included above for completeness.)

The main observation is that after calculating the variation, the condition (1) reduces to the same equation (2), hence (1) and (3) are equivalent.

A deeper explanation can be given as follows.

The self-adjointness condition (3), (2) is nothing but the closedness of a 1-form of a special appearance on the infinite-dimensional “space of fields” $u = \{u^i(x)\}$, i.e., the space of sections of $E$. The Helmholtz criterion can be seen as a special case of the Poincaré lemma in this infinite-dimensional situation.

Indeed, for the differential of a 1-form $A = \int d^n x \delta u^i(x) A_i(x; [u])$ we have

$$
\delta A = \frac{1}{2} \int d^n x d^n y \delta u^i(x) \delta u^j(y) \left( \frac{\delta A_j(y; [u])}{\delta u^i(x)} - \frac{\delta A_i(x; [u])}{\delta u^j(y)} \right).
$$
Hence $\delta A = 0$ means
\[
\frac{\delta A_i(y; [u])}{\delta u^i(x)} - \frac{\delta A_i(x; [u])}{\delta u^j(y)} = 0.
\] (4)
(See Appendix for a description of the corresponding de Rham complex.)

Now, functions $A_i(x, u, u', \ldots)$ can be viewed as the coefficients of a "diagonal" 1-form $A = \int d^n x \, \delta u^i(x) \, A_i(x, u, u', \ldots)$, see Appendix. It is easy to check by expanding the variational derivatives in (4) that for such kind of 1-forms the condition $\delta A = 0$ gives (2). The differential operators appearing in (2) arise from derivatives of $\delta$-functions. The proof of the local existence of $S$ simply follows the usual proof of the Poincaré lemma.

The condition (1), on the other hand, is the closedness of a 0-form. We will see that the fact that (1) and (3) give the same thing follows from the commutativity of the differentials in the de Rham complexes on field spaces with a natural homomorphism $K$ defined below.

Let $\mathcal{E}(M)$ stand for the space of sections of the bundle $E \to M$, and $\mathcal{E}(M \times \mathbb{R})$ for the space of sections of the induced bundle over $M \times \mathbb{R}$. Let $\Omega(\mathcal{E}(M))$ and $\Omega(\mathcal{E}(M \times \mathbb{R}))$ denote the corresponding algebras of forms. There is a natural odd map $K: \Omega(\mathcal{E}(M)) \to \Omega(\mathcal{E}(M \times \mathbb{R}))$ that lowers the degree by one:
\[
K \omega := \int d^n x \, dt \, u^i(x, t) \frac{\delta \omega}{\delta u^i(x)}.
\]
At the r.h.s. we treat sections on $M \times \mathbb{R}$ as families of sections on $M$. $K$ is the composition of the map $I = \int dt: \Omega(\mathcal{E}(M)) \to \Omega(\mathcal{E}(M \times \mathbb{R}))$ that sends every functional to its integral over $t$ and the interior product with the canonical vector field $\hat{u}$ on $\mathcal{E}(M \times \mathbb{R})$. Notice that $K$ is monomorphic on forms of degree $> 0$.

**Theorem 2.** The following diagram commutes:
\[
\begin{array}{ccc}
\Omega^k(\mathcal{E}(M)) & \xrightarrow{K} & \Omega^{k-1}(\mathcal{E}(M \times \mathbb{R})) \\
\delta \downarrow & & \delta \\
\Omega^{k+1}(\mathcal{E}(M)) & \xrightarrow{K} & \Omega^k(\mathcal{E}(M \times \mathbb{R}))
\end{array}
\]

**Proof.** Consider the tautological family of maps $f_t: \mathcal{E}(M \times \mathbb{R}) \to \mathcal{E}(M)$ that sends every section over $M \times \mathbb{R}$ to itself considered as a family of sections. Then $\hat{u}$ can be alternatively viewed as the time-dependent velocity vector field for $f_t$, so that $i_{\hat{u}}: \Omega(\mathcal{E}(M)) \to \Omega(\mathcal{E}(M \times \mathbb{R}))$. The homotopy formula gives
\[
(\delta i_{\hat{u}} + i_{\hat{u}} \delta) \omega = \frac{d}{dt} \int_t^\bullet \omega
\]
for any form $\omega$ on $\mathcal{E}(M)$. Hence by integrating w.r.t. $t$ we get
\[
\delta K + K \delta = 0,
\] (5)
as claimed. The r.h.s. of (5) is zero as the integral of a total derivative. The commutator has a plus sign since both \( \delta \) and \( K \) are odd.

In particular, for \( k = 1 \), we get the relation between the differentials of a 1-form \( A \) and the 0-form \( KA \): since \( K \) is monomorphic, \( \delta A = 0 \) if and only if \( K(\delta A) = -\delta(KA) = 0 \). This is precisely the relation between the classical Helmholtz criterion and our Theorem 1.

**Remark 1.** Suppose \( E \) is a bundle over a point, i.e., \( M = \{ * \} \). Then \( \Omega(E(M)) = \Omega(E) \), and Theorem 2 relates forms on a finite-dimensional manifold \( E \) with forms on the space of paths. (Without harm, \( \mathbb{R} \) can be replaced above by \( I = [0, 1] \).) Iterating, we get the commutative diagram

\[
\begin{array}{ccc}
\Omega^k(E) & \xrightarrow{K^k} & \Omega^0(\{I^k \to E\}) \\
\downarrow d & & \downarrow K \circ \delta \\
\Omega^{k+1}(E) & \xrightarrow{K^{k+1}} & \Omega^0(\{I^{k+1} \to E\})
\end{array}
\]

in which we recognize the expression of the differential of \( k \)-forms on a manifold \( E \) via the variation of functionals of \( k \)-paths.

**Remark 2.** With obvious changes, e.g., \( d^nx \) replaced by the Berezin volume element, all statements remain true for supermanifolds.

**Appendix. The de Rham complex on a space of fields**

We can consider on the space of sections of \( E \to M \) functionals of the following form: \( F[u] = \int f \, d^nx_1 \ldots d^nx_k \) and their sums, where the (formal) integration is over \( M \times \ldots \times M \) and the integrand \( f \) is allowed to depend on a finite number of derivatives of \( u_i \) at the points \( x_1, \ldots, x_k \), as well as, possibly, at some other points. Particular examples are the classical “local functionals” \( S[u] = \int L(x, u(x), \ldots) \, d^nx \) and “point functionals” such as \( u_i(x) \) or \( u^a_i(x) \). One should also allow in \( f \) products of \( \delta \)-functions and their derivatives taken at distinct points of \( M \); otherwise \( f \) should be smooth. Such functionals make an algebra closed under taking variational derivatives, which act as derivations. Variational derivatives reduce the “integrality” of a functional (the number of integrations minus the number of \( \delta \)-functions involved) by one. Without loss of generality, integrands can be considered symmetric w.r.t. the arguments corresponding to the integration points. Then the variational derivative of any such functional will be given by the usual Euler–Lagrange expression w.r.t. one point with the integration remaining over the other points. (Functionals of this kind have been considered by physicists, see, e.g., [3].)

Now, a vector at a “point” \( \{u^i(x)\} \) is, of course, a section of \( u^*T^\text{vert}E \) (the pull-back of the vertical tangent bundle). A vector field can be
formally written as

\[ \eta = \int d^n x \eta^i(x; [u]) \frac{\delta}{\delta u^i(x)}, \]

which is a functional of \( u \) taking values in \( u^*T^\text{vert} E \).

The simplest way to define forms on this infinite-dimensional space is to consider functionals of pairs of fields \((u, \delta u)\), where \( \delta u \) is odd. (For supermanifolds, \( \delta u \) should have parity opposite to that of \( u \).) More precisely, \( \delta u \) is a section of \( u^*\Pi T^\text{vert} E \), where \( \Pi \) is the parity reversion functor. Forms make an algebra. Analogs of usual formulae hold, viz.,

\[ \delta = \int d^n x \delta u^i(x) \frac{\delta}{\delta u^i(x)} \]  

(6)

for the differential (the "exterior variation"), and

\[ i_\eta = \int d^n x \eta^i(x; [u]) \frac{\delta}{\delta \delta u^i(x)} \]  

(7)

for the interior product with a vector field \( \eta \), where at the r.h.s. of (7) stands the variational derivative w.r.t. the odd field \( \delta u^i(x) \). One can easily see that the homotopy formula and the Poincaré lemma (with the usual proof) hold as on ordinary manifolds; in particular, as a form \( \sigma \) such that \( \delta \sigma = \omega \) if \( \delta \omega = 0 \) in a star-shaped domain one can take

\[ \sigma[u, \delta u] = \int_0^1 dt \int d^n x u^i(x) \frac{\delta \omega}{\delta \delta u^i(x)} [tu, t\delta u], \]

similarly to the ordinary case.

**Remark 3.** “Functional forms” on the infinite jet space \( J^\infty(E) \), considered in [7], can be interpreted as corresponding to a special subclass of all forms on the space of sections. We call a \( k \)-form \( \omega = \omega[u, \delta u] \) on the space of sections,

\[ \omega = \int d^n x_1 \ldots d^n x_k \delta u^{i_1}(x_1) \ldots \delta u^{i_k}(x_k) \omega_{i_1 \ldots i_k}(x_1, \ldots, x_k; [u]), \]  

(8)

diagonal if all coefficients \( \omega_{i_1 \ldots i_k}(x_1, \ldots, x_k; [u]) \) are supported at the diagonal \( x_1 = \ldots = x_k = x \) and they are point functionals of \( u \) at the same point \( x \). Hence a diagonal form can be re-written (non-canonically) with only one integration as

\[ \omega = \int d^n x \delta u_{A_1}^{i_1}(x) \ldots \delta u_{A_k}^{i_k}(x) \omega_{A_1 \ldots A_k}^{i_1 \ldots i_k}(x, u(x), u'(x), \ldots) \]

where \( A_1, \ldots, A_k \) are multi-indices. Thus it can be identified with a “functional form” on jet space as defined in [7]. “Functional forms” appear in the jet-theoretic approach [7], [2] as elements of the term \( E_1 \) of the spectral sequence of the bicomplex \( \Omega^* (J^\infty(E)) \). Effective restriction by diagonal forms, as well as by similar diagonal multivector fields appearing under the guise of “functional multivectors” [7], is a
fundamental limitation of this approach. The subspace of diagonal forms is closed under the differential. However, to be able to multiply forms or consider duality with multivector fields, one has to work with arbitrary forms. Even when results concern only the diagonal forms, using arbitrary forms gives a clearer picture and busts intuition.

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