A quaternionic braid representation
(after Goldschmidt and Jones)

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Abstract. We show that the braid group representations associated with the \((3, 6)\)-quotients of the Hecke algebras factor over a finite group. This was known to experts going back to the 1980s, but a proof has never appeared in print. Our proof uses an unpublished quaternionic representation of the braid group due to Goldschmidt and Jones. Possible topological and categorical generalizations are discussed.

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1. Introduction

Jones analyzed the images of the braid group representations obtained from Temperley–Lieb algebras in [11] where, in particular, he determined when the braid group images are finite or not. Braid group representations with finite image were also recognized in [12] and [8]. Some 15 years later the problem of determining the closure of the image of braid group representations associated with Hecke algebras played a critical role in analyzing the computational power of the topological model for quantum computation [6]. Following these developments the author and collaborators analyzed braid group representations associated with BMW-algebras [15] and twisted doubles of finite groups [5]. Partially motivated by empirical evidence the author conjectured that the braid group representations associated with an object \(X\) in a braided fusion category \(\mathcal{C}\) has finite image if, and only if, the Frobenius–Perron dimension of \(\mathcal{C}\) is integral (see e.g. Conjecture 6.6 of [22]). In [18], [25] various instances of this conjecture were verified. This current work verifies this conjecture for the braided fusion category \(\mathcal{C}(\mathfrak{sl}_3, 6)\) obtained from the representation category of the quantum group \(U_q\mathfrak{sl}_3\) at \(q = e^{\pi i / 6}\) (see [23] for details and notation). More

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generally, Jimbo’s [10] quantum Schur–Weyl duality establishes a relationship between the modular categories $\mathcal{C}(\mathfrak{sl}_k, \ell)$ obtained from the quantum group $U_q \mathfrak{sl}_k$ at $q = e^{\pi i / \ell}$ and certain semisimple quotients $\mathcal{H}_n(k, \ell)$ of specialized Hecke algebras $\mathcal{H}_n(q)$ (defined below). That is, if we denote by $X \in \mathcal{C}(\mathfrak{sl}_k, \ell)$ the simple object analogous to the vector representation of $\mathfrak{sl}_k$ then there is an isomorphism $\mathcal{H}_n(k, \ell) \cong \text{End}(X \otimes \cdots \otimes X \otimes I_X^{n-i-1})$ induced by $g_i \mapsto I_X^{i-1} \otimes e_{X,X} \otimes I_X^{n-i-1}$. In particular, the braid group representations associated with the modular category $\mathcal{C}(\mathfrak{sl}_3, 6)$ are the same as those obtained from $\mathcal{H}_n(3, 6)$. It is known that braid group representations obtained from $\mathcal{H}_n(3, 6)$ have finite image (mentioned in [6], [13], [18]), but a proof has never appeared in print. This fact was discovered by Goldschmidt and Jones during the writing of [8] and independently by Larsen during the writing of [6]. We benefitted from the notes of Goldschmidt and Jones containing the description of the quaternionic braid representation below. Our techniques follow closely those of [11], [12], [14]. The rest of the paper is organized into three sections. In Section 2 we recall some notation and facts about Hecke algebras and their quotients. The main results are in Section 3, and in Section 4 we indicate how the category $\mathcal{C}(\mathfrak{sl}_3, 6)$ is exceptional from topological and categorical points of view.

2. Hecke algebras

We extract the necessary definitions and results from [27] that we will need in the sequel.

**Definition 2.1.** The Hecke algebra $\mathcal{H}_n(q)$ for $q \in \mathbb{C}$ is the $\mathbb{C}$-algebra with generators $g_1, \ldots, g_{n-1}$ satisfying relations

(H1)$'$ $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $1 \leq i \leq n - 2$,

(H2)$'$ $g_i g_j = g_j g_i$ for $|i - j| > 1$, and

(H3)$'$ $(g_i + 1)(g_i - q) = 0$.

Technically, $\mathcal{H}_n(q)$ is the Hecke algebra of type $A$, but we will not be considering other types so we suppress this distinction. One immediately observes that $\mathcal{H}_n(q)$ is the quotient of the braid group algebra $\mathbb{C}[B_n]$ by the relation (H1)$'$. $\mathcal{H}_n(q)$ may also be described in terms of the generators $e_i = \frac{(q-g_i)}{(1+q)}$, which satisfy

(H1) $e_i^2 = e_i$,

(H2) $e_i e_j = e_j e_i$ for $|i - j| > 1$, and

(H3) $e_i e_{i+1} e_i - q/(1+q)^2 e_i = e_{i+1} e_i e_{i+1} - q/(1+q)^2 e_{i+1}$ for $1 \leq i \leq n - 2$.

For any $\eta \in \mathbb{C}$, Ocneanu [7] showed that one may uniquely define a linear functional $\text{tr}$ on $\mathcal{H}_\infty(q) = \bigcup_{n=1}^{\infty} \mathcal{H}_n(q)$ satisfying

(1) $\text{tr}(1) = 1$, \[174\]
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(2) \( \text{tr}(ab) = \text{tr}(ba) \), and

(3) \( \text{tr}(xe_n) = \eta \text{tr}(x) \) for any \( x \in \mathcal{H}_n(q) \).

Any linear function on \( \mathcal{H}_\infty \) satisfying these conditions is called a Markov trace and is determined by the value \( \eta = \text{tr}(e_1) \). Now suppose that \( q = e^{2\pi i/\ell} \) and \( \eta = \frac{(1-q^{1-k})}{(1+q)(1-q^k)} \) for some integers \( k < \ell \). Then, for each \( n \), the (semisimple) quotient of \( \mathcal{H}_n(q) \) by the annihilator of the restriction of the trace \( \mathcal{H}_n(q) / \text{Ann} \) is called the \((k, \ell)\)-quotient. We will denote this quotient by \( \mathcal{H}_n(k, \ell) \) for convenience.

Wenzl [27] has shown that \( \mathcal{H}_n(k, \ell) \) is semisimple and described the irreducible representations \( \rho_{\lambda}^{(k, \ell)} \) where \( \lambda \) is a \((k, \ell)\)-admissible Young diagrams of size \( n \). Here a Young diagram \( \lambda \) is \((k, \ell)\)-admissible if \( \lambda \) has at most \( k \) rows and \( \lambda_1 - \lambda_k \leq \ell - k \) where \( \lambda_i \) denotes the number of boxes in the \( i \)th row of \( \lambda \). The (faithful) Jones–Wenzl representation is the sum \( \rho^{(k, \ell)} = \bigoplus_{\lambda} \rho_{\lambda}^{(k, \ell)} \). Wenzl [27] has shown that \( \rho^{(k, \ell)} \) is a \( C^* \)-representation, i.e. the representation space is a Hilbert space (with respect to a Hermitian form induced by the trace \( \text{tr} \)) and \( \rho_{\lambda}^{(k, \ell)}(e_i) \) is a self-adjoint operator. One important consequence is that each \( \rho_{\lambda}^{(k, \ell)} \) induces an irreducible unitary representation of the braid group \( B_n \) via composition with \( \sigma_i \mapsto g_i \), which is also called the Jones–Wenzl representation of \( B_n \).

3. A quaternionic representation

Consider the \((3, 6)\)-quotient \( \mathcal{H}_n(3, 6) \). The \((3, 6)\)-admissible Young diagrams have at most 3 rows and \( \lambda_1 - \lambda_3 \leq 3 \). For \( n \geq 3 \) there are either 3 or 4 Young diagrams of size \( n \) that are \((3, 6)\)-admissible, and \( \eta = \frac{(1-q^{1-3})}{(1+q)(1-q^3)} = 1/2 \) in this case. Denote by \( \varphi_n \) the unitary Jones–Wenzl representation of \( B_n \) induced by \( \rho^{(3,6)} \). Our main goal is to prove the following:

**Theorem 3.1.** The image \( \varphi_n(B_n) \) is a finite group.

We will prove this theorem by embedding \( \mathcal{H}_n(3, 6) \) into a finite dimensional algebra (Lemma 3.2) and then showing that the group generated by the images of \( g_1, \ldots, g_{n-1} \) is finite (Lemma 3.3). Denote by \([ , ]\) the multiplicative group commutator and let \( q = e^{2\pi i/6} \). Consider the \( \mathbb{C} \)-algebra \( Q_n \) with generators \( u_1, v_1, \ldots, u_{n-1}, v_{n-1} \) subject to the relations

(G1) \( u_i^2 = v_i^2 = -1 \),

(G2) \( [u_i, v_j] = -1 \) if \( |i - j| \leq 1 \),

(G3) \( [u_i, v_j] = 1 \) if \( |i - j| \geq 2 \), and

(G4) \( [u_i, u_j] = [v_i, v_j] = 1 \).
Notice that the group \( \{ \pm 1, \pm u_i, \pm v_i, \pm u_i v_i \} \) is isomorphic to the group of quaternions. We see from these relations that \( \dim(Q_n) = 2^{2n-2} \) since each word in the \( u_i, v_i \) has a unique normal form

\[
\pm u_1^{\epsilon_1} \cdots u_{n-1}^{\epsilon_{n-1}} v_1^{\nu_1} \cdots v_{n-1}^{\nu_{n-1}}
\]

(1)

with \( v_i, \epsilon_i \in \{0, 1\} \). Observe that a basis for \( Q_n \) is given by taking all \( + \) signs in (1). We define a \( \mathbb{C} \)-valued trace \( \text{Tr} \) on \( Q_n \) by setting \( \text{Tr}(1) = 1 \) and \( \text{Tr}(w) = 0 \) for any non-identity word in the \( u_i, v_i \). One deduces that \( \text{Tr} \) is faithful from the uniqueness of the normal form (1). Define

\[
s_i = -\frac{1}{2q}(1 + u_i + v_i + u_i v_i),
\]

for \( 1 \leq i \leq n - 1 \).

**Lemma 3.2.** The subalgebra \( \mathcal{A}_n \subset Q_n \) generated by \( s_1, \ldots, s_{n-1} \) is isomorphic to \( \mathcal{H}_n(3, 6) \).

**Proof.** It is a straightforward computation to see that the \( s_i \) satisfy

(B1) \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \),

(B2) \( s_is_i = s_is_j \) if \( |i - j| \geq 2 \), and

(E1) \( (s_i - q)(s_i + 1) = 0 \).

Indeed, relation (B2) is immediate from relations (G3) and (G4). It is enough to check (B1) and (E1) for \( i = 1 \). For this we compute

\[
\begin{align*}
\text{B1:} & \quad s_1^{-1} = -\frac{q}{2}(1 - u_1 - v_1 - u_1 v_1), \\
\text{B2:} & \quad s_1^{-1}u_1s_1 = u_1v_1, \quad \ s_1^{-1}v_1s_1 = u_1, \\
\text{E1:} & \quad s_1^{-1}u_2s_1 = u_2v_1, \quad \ s_1^{-1}v_2s_1 = -u_1v_1v_2,
\end{align*}
\]

from which (B1) and (E1) are deduced. Thus \( \varphi(g_i) = s_i \) induces an algebra homomorphism \( \varphi : \mathcal{H}_n(q) \rightarrow Q_n \) with \( \varphi(\mathcal{H}_n(q)) = \mathcal{A}_n \). Set \( f_i = \varphi(e_i) = \frac{(q-s_i)}{(1+q)} \) and let \( b \in Q_{n-1} \), that is \( b \) is in the span of the words in \( \{u_1, v_1, \ldots, u_{n-2}, v_{n-2}\} \). The constant term of \( f_{n-1}b \) is the product of the constant terms of \( b \) and \( f_{n-1} \) since \( f_{n-1} \) is in the span of \( \{1, u_{n-1}, v_{n-1}, u_{n-1}v_{n-1}\} \), so \( \text{Tr}(f_{n-1}b) = \text{Tr}(f_{n-1})\text{Tr}(b) \). For each \( a \in \mathcal{H}_n(q) \) we define \( \varphi^{-1}(\text{Tr})(a) = \text{Tr}(\varphi(a)) \), and conclude that \( \varphi^{-1}(\text{Tr}) \) is a Markov trace on \( \mathcal{H}_n(q) \). Computing, we see that \( \text{Tr}(f_{n-1}) = 1/2 \), so that by uniqueness \( \varphi^{-1}(\text{Tr}) = \text{tr} \) as functionals on \( \mathcal{H}_n(q) \). Now if \( a \in \ker(\varphi) \) we see that \( \text{tr}(ac) = \text{Tr}(\varphi(ac)) = 0 \) for any \( c \) so that \( \ker(\varphi) \subset \text{Ann}(\text{tr}) \). On the other hand, if \( a \in \text{Ann}(\text{tr}) \) we must have \( \text{Tr}(\varphi(ac)) = \text{tr}(ac) = 0 \) for all \( c \in \mathcal{H}_n(q) \). If \( \varphi(a) \neq 0 \) then, by definition of \( \text{Tr} \) and \( \varphi \), there exists an \( a^\dagger \in \mathcal{H}_n(q) \) such that \( \text{Tr}(\varphi(a)^{\dagger} \varphi(a)) \neq 0 \) since \( \text{Tr} \) is faithful. Therefore \( \text{Ann}(\text{tr}) \subset \ker(\varphi) \). In particular, we see that \( \varphi \) induces

\[
\mathcal{H}_n(3, 6) = \mathcal{H}_n(q)/\text{Ann}(\text{tr}) \cong \varphi(\mathcal{H}_n(q)) = \mathcal{A}_n \subset Q_n. \]
**Lemma 3.3.** The group $G_n$ generated by $s_1, \ldots, s_{n-1}$ is finite.

**Proof.** Consider the conjugation action of the $s_i$ on $Q_n$. We claim that the conjugation action of $s_i$ on the words in $u_i, v_i$ is by a signed permutation. Since $s_i$ commutes with words in $u_j, v_j$ with $j \notin \{i-1, i, i+1\}$, by symmetry it is enough to consider the conjugation action of $s_1$ on the four elements $\{u_1, v_1, u_2, v_2\}$, which is given in (3). Thus we see that $G_n$ modulo the kernel of this action is a (finite) signed permutation group. The kernel of this conjugation action lies in the center $Z(Q_n)$ of $Q_n$. Using the normal form above we find that the center $Z(Q_n)$ is either 1-dimensional or 4-dimensional. Indeed, since the words

$$W = \{u_1^{\epsilon_1} \ldots u_{n-1}^{\epsilon_{n-1}} v_1^{\nu_1} \ldots v_{n-1}^{\nu_{n-1}}\}$$

for $(\epsilon_1, \ldots, \epsilon_{n-1}, \nu_1, \ldots, \nu_{n-1}) \in \mathbb{Z}_2^{2n-2}$ form a basis for $Q_n$ and $tw = \pm wt$ for $w, t \in W$ we may explicitly compute a basis for the center as those words $w \in W$ that commute with $u_i$ and $v_i$ for all $i$. This yields two systems of linear equations over $\mathbb{Z}_2$:

$$\begin{cases}
\epsilon_1 + \epsilon_2 = 0, \\
\epsilon_i + \epsilon_{i+1} + \epsilon_{i+2} = 0, & 1 \leq i \leq n - 3, \\
\epsilon_{n-2} + \epsilon_{n-1} = 0,
\end{cases}$$

and

$$\begin{cases}
v_1 + v_2 = 0, \\
v_{i-1} + v_i + v_{i+1} = 0, & 1 \leq i \leq n - 3, \\
v_{n-2} + v_{n-1} = 0.
\end{cases}$$

Non-trivial solutions to (4) only exist if $3 \mid n$ since we must have $\epsilon_1 = \epsilon_2 = \epsilon_{n-2} = \epsilon_{n-1} = 1$ as well as $\epsilon_i = 0$ if $3 \nmid i$ and $\epsilon_j = 1$ if $3 \nmid j$ and similarly for (5). Thus $Z(Q_n)$ is $\mathbb{C}$ if $3 \nmid n$ and is spanned by $U, V$ and $UV$ where $U = \prod_{3 \nmid i} u_i$ and $V = \prod_{3 \mid i} v_i$ if $3 \mid n$. The determinant of the image of $s_i$ under any representation is a 6th root of unity and hence the same is true for any element $z \in Z(Q_n) \cap G_n$. Thus for $3 \mid n$ the image of any $z \in Z(Q_n) \cap G_n$ under the left regular representation is a root of unity times the identity matrix, and thus has finite order. Similarly, if $3 \nmid n$, the restriction of any $z \in Z(Q_n) \cap G_n$ to any of the four simple components of the left regular representation is a root of unity times the identity matrix and so has finite order. So the group $G_n$ itself is finite.

This completes the proof of Theorem 3.1.

**Remark 3.4.** The proof of Lemma 3.3 shows that the projective image of $G_n$ is a (non-abelian) subgroup of the full monomial group $G(2, 1, 4^{n-1})$ of signed $4^{n-1} \times 4^{n-1}$ matrices. The main goal of this paper is to verify [22], Conjecture 6.6, in this case, but with further effort one could determine the group $G_n$ more precisely. It is suggested
in [13] that $G_n$ is an extension of $PSU(n-1, \mathbb{F}_2)$ so that

$$|G_n| \approx \frac{1}{3} 2^{(n-1)(n-2)/2} \prod_{i=1}^{n-1} (2^i - (-1)^i),$$

but that such a result has not appeared in print. Modulo the center, the generators $s_i$ have order 3 so that $G_n/Z(G_n)$ is a quotient of the factor group $B_n/\langle \sigma_1^3 \rangle$ (here $\sigma_i$ are the usual generators of $B_n$). For $n \leq 5$, Coxeter [1] has shown that these quotients are finite groups and determined their structure. In particular, the projective image of $B_5/\langle \sigma_1^3 \rangle$ is $PSU(4, \mathbb{F}_2)$, so $G_5$ is an extension of this simple group. A strategy for showing $G_n$ is an extension of $PSU(n-1, \mathbb{F}_2)$ for $n > 5$ would be to find an $(n-1)$-dimensional invariant subspace of $Q_n$ so that the restricted action of the braid generators is by order 3 pseudo-reflections (projectively). A comparison of the dimensions of the simple $H_n(3, 6)$-modules with those of $PSU(n-1, \mathbb{F}_2)$ indicates that one must also restrict to those $n$ not divisible by 3.

4. Concluding remarks, questions and speculations

The category $\mathcal{C}(sl_3, 6)$ does not seem to have any obvious generalizations. We discuss some of the ways in which $\mathcal{C}(sl_3, 6)$ appears to be exceptional by posing a number of (somewhat naïve) questions which we expect to have negative answers.

4.1. Link invariants. From any modular category one obtains (quantum) link invariants via Turaev’s approach [26]. The link invariant $P_{L}^\prime (q, \eta)$ associated with $\mathcal{C}(sl_k, \ell)$ is (a variant of) the HOMFLY-PT polynomial ([7], where a different choice of variables is used). For the choices $q = e^{2\pi i/6}$ and $\eta = 1/2$ corresponding to $\mathcal{C}(sl_3, 6)$ the invariant has been identified [16]:

$$P_{L}^\prime (e^{2\pi i/6}, 1/2) = \pm i (\sqrt{2})^{\text{dim} H_1 (T_L; \mathbb{Z}_2)},$$

where $T_L$ is the triple cyclic cover of the three sphere $S^3$ branched over the link $L$. There is a similar series of invariants for any odd prime $p$: $\pm i (\sqrt{p})^{\text{dim} H_1 (D_L; \mathbb{Z}_p)}$, where $D_L$ is the double cyclic cover of $S^3$ branched over $L$ (see [16] and [8]). It appears that this series of invariants can be obtained from modular categories $\mathcal{C}(so_p, 2p)$. This has been verified for $p = 3, 5$ (see [8] and [12]) and we have recently handled the $p = 7$ case (unpublished, using results in [29]).

**Question 4.1.** Are there modular categories with associated link invariant

$$\pm i (\sqrt{p})^{\text{dim} H_1 (T_L; \mathbb{Z}_p)}?$$

In [15] it is suggested that if the braid group images corresponding to some ribbon category are finite then the corresponding link invariant is *classical*, i.e. equivalent to a homotopy-type invariant. Another formulation of this idea is found in [24] in which *classical* is interpreted in terms of computational complexity.
4.2. Fusion categories and $II_1$ factors. The category $\mathcal{C}(\mathfrak{sl}_3, 6)$ is an integral fusion category, that is the simple objects have integral dimensions. The categories $\mathcal{C}(\mathfrak{sl}_k, \ell)$ are integral for $(k, \ell) = (3, 6)$ and $(k, k + 1)$ but no other examples are known (or believed to exist). $\mathcal{C}(\mathfrak{sl}_3, 6)$ has six simple (isomorphism classes of) objects: $\{X_i, X_i^*\}^3_{i=1}$ of dimension 2 (dual pairs), three simple objects $1, Z, Z^*$ of dimension 1, and one simple object $Y$ of dimension 3. The Bratteli diagram for tensor powers of the generating object $X_1$ is given in Figure 1. It is shown in [4] that $\mathcal{C}$ is an integral fusion category if, and only if, $\mathcal{C} \cong \text{Rep}(H)$ for some semisimple finite dimensional quasi-Hopf algebra $H$, so in particular $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \text{Rep}(H)$ for some quasi-triangular quasi-Hopf algebra $H$. One wonders if strict coassociativity can be achieved:

**Question 4.2.** Is there a (quasi-triangular) semisimple finite dimensional Hopf algebra $H$ with $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \text{Rep}(H)$?

Other examples of integral categories are the representation categories $\text{Rep}(D^\omega G)$ of twisted doubles of finite groups studied in [5] (here $G$ is a finite group and $\omega$ is a 3-cocycle on $G$). Any fusion category $\mathcal{C}$ with the property that its Drinfeld center $Z(\mathcal{C})$ is equivalent as a braided fusion category to $\text{Rep}(D^\omega G)$ for some $\omega, G$ is called group-theoretical (see [4], [19]). The main result of [5] implies that if $\mathcal{C}$ is any braided group-theoretical fusion category then the braid group representations obtained from $\mathcal{C}$ must have finite image. In [18] we showed that $\mathcal{C}(\mathfrak{sl}_3, 6)$ is not group-theoretical and in fact has minimal dimension (36) among non-group-theoretical integral modular categories.

**Question 4.3.** Is there a family of non-group-theoretical integral modular categories that includes $\mathcal{C}(\mathfrak{sl}_3, 6)$?

Notice that $\mathcal{C}(\mathfrak{sl}_3, 6)$ has a ribbon subcategory $\mathcal{D}$ with simple objects $1, Z, Z^*$ and $Y$. The fusion rules are the same as those of $\text{Rep}(\mathfrak{sl}_4)$: $Y \otimes Y \cong 1 \oplus Z \oplus Z^* \oplus Y$. However $\mathcal{D}$ is not symmetric and $\mathcal{C}(\mathfrak{sl}_3, 6)$ has smallest dimension among modular categories containing $\mathcal{D}$ as a ribbon subcategory (what Müger would call a minimal modular extension [17]). One possible generalization of $\mathcal{C}(\mathfrak{sl}_3, 6)$ would be a minimal modular extension of a non-symmetric ribbon category $\mathcal{D}_n$ similar to $\mathcal{D}$ above. That is, $\mathcal{D}_n$ should be a non-symmetric ribbon category with $n$ 1-dimensional simple objects $\mathbf{1} = Z_0, \ldots, Z_{n-1}$ and one simple $n$-dimensional object $Y_n$ such that $Y_n \otimes Y_n \cong Y_n \oplus \bigoplus_{i=0}^{n-1} Z_i$ and the $Z_i$ have fusion rules like $Z_n$. For $\mathcal{D}_n$ to exist even at the generality of fusion categories one must have $n = p^\alpha - 1$ for some prime $p$ and integer $\alpha$ by [3], Corollary 7.4. However, V. Ostrik [20] informs us that these categories do not admit non-symmetric braidings except for $n = 2, 3$. So this does not produce a generalization. A pair of hyperfinite $II_1$ factors $A \subset B$ with index $[B : A] = 4$ can be constructed from $\mathcal{C}(\mathfrak{sl}_3, 6)$ (see [28], Section 4.5). The corresponding principal graph is the Dynkin diagram $E_6^{(1)}$ the nodes of which we
This principal graph can be obtained directly from the Bratteli diagram in Figure 1 as the nodes in the 6th and 7th levels and the edges between them. Hong [9] showed that any $II_1$ subfactor pair $M \subset N$ with principal graph $E_6^{(1)}$ can be constructed from some $II_1$ factor $P$ with an outer action of $\mathfrak{A}_4$ as $M = P \rtimes \mathbb{Z}_3 \subset P \rtimes \mathfrak{A}_4 = N$. Subfactor pairs with principal graph $E_7^{(1)}$ and $E_8^{(1)}$ can also be constructed (see e.g. [21]). We ask:

**Question 4.4.** Is there a unitary non-group-theoretical integral modular category with principal graph $E_7^{(1)}$ or $E_8^{(1)}$?

Even a braided fusion category with such a principal graph would be interesting, and have interesting braid group image. Notice that the subcategory $\mathcal{D}$ mentioned
above plays a role here as $A_4$ corresponds to the Dynkin diagram $E_6^{(1)}$ in the McKay correspondence. A modular category $\mathcal{C}$ with principal graph $E_7^{(1)}$ (resp. $E_8^{(1)}$) would contain a ribbon subcategory $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) with the same fusion rules as $\text{Rep}(S_4)$ (resp. $\text{Rep}(A_5)$). Using [2], Lemma 1.2, we find that such a category $\mathcal{C}$ must have dimension divisible by 144 (resp. 3600). The ribbon subcategory $\mathcal{F}_2$ must have symmetric braiding (D. Nikshych’s proof: $\text{Rep}(A_5)$ has no non-trivial fusion subcategories so if it has a non-symmetric braiding, the Müger center is trivial. But if the Müger center is trivial it is modular, which fails by [2], Lemma 1.2). This suggests that for $E_8^{(1)}$ the answer to Question 4.4 is “no.” There is a non-symmetric choice for $\mathcal{F}_1$ (as V. Ostrik informs us [20]), with Müger center equivalent to $\text{Rep}(S_3)$. By [17], Proposition 5.1, a minimal modular extension $\mathcal{C}$ of such an $\mathcal{F}_1$ would have dimension 144.

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