Fitting of particle trajectory in magnetic dipole equatorial field and prolate epicycloid

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Abstract

In this paper, the motion of charged particles in the equatorial plane of the magnetic dipole field is analyzed systematically with the help of calculation software. We first propose the use of a physical model of the prolate epicycloid to explain and repeat the true trajectories of charged particles in a magnetic dipole field when the particle trajectory is bounded. The fitted numerical results show that the results obtained with the prolate epicycloid model are almost identical to the real trajectories of charged particles in the magnetic dipole field, which means that we can understand and describe the charged particles in the magnetic dipole from a new and more visual perspective. Not only that, when $B \sim 1/\rho^n (n>0)$, some of those case can be explained by prolate epicycloid; when $B \sim 1/\rho^n (n<0)$, the trajectories can be explained roughly by prolate hypocycloid and curtate hypocycloid. Moreover, we also study the trajectories of charged particles in magnetic quadrupole field and find that there is something similar to prolate epicycloid.

Keywords: magnetic dipole, particle motion, prolate epicycloid

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1. Introduction

The study of the trajectories of charged particles in magnetic dipole field is an agelong electromagnetism problem. There are many applications of this particle trajectories, such as auroral theory and plasma physics. Although the motion of particles in a magnetic field has been studied in some literatures\cite{1, 2}, a large part of these conclusions are described by elliptic integrals, which are quite complicated. Therefore, whether a simpler and more intuitive physical model can be used to describe the motion of particles in a magnetic field has become a question of valuable exploration.

As we all know, Prolate epicycloid\cite{5} is a fascinating curve and formed by the superposition of two rotating motions of a point on the plane, as shown in Fig.1. There is an example in the field of machinery, during the process of pinion is moving with a pure rolling along the large gear, the motion of a point outside the pinion rotating with the same angular speed as the pinion is prolate epicycloid. Meanwhile, a number of other physical phenomena can be explained by prolate epicycloid in nature. In this paper, we proposed the use of physical model of prolate epicycloid to explain the trajectories of charged particles in magnetic dipole field when particle trajectories are bounded, since they have homogeneous trajectories. The result show that, it seems worth doing as this fitting result is meaningful. What’s more, we tried to fit the trajectories of other types of axisymmetric magnetic field\cite{6} by prolate epicycloid, prolate hypocycloid and curtate hypocycloid respectively\cite{5, 7}. Last, the motion trajectory of charged particles in equatorial field of the magnetic quadrupole is studied based on the research of magnetic dipole.

2. The model of prolate epicycloid, prolate hypocycloid and curtate hypocycloid

2.1. Prolate epicycloid

There is a fixed circle with radius $a$, a moving circle with radius $b$ and a point moving in a circle of radius $R$ relative to the moving circle. We set that a
point does circular motion and maintains the same angular velocity with respect to the moving circle, while the moving circle rolling without slipping relative to the fixed circle.

When the point outside the fixed circle rolling without slipping, the track of prolate epicycloid will be formed as shown in Fig.1. The parametric equations of prolate epicycloid is

\[
\begin{align*}
    x &= (a + b) \cos \theta - R \cos \left( \frac{a + b}{b} \theta \right) \\
    y &= (a + b) \sin \theta - R \sin \left( \frac{a + b}{b} \theta \right)
\end{align*}
\]

(1)

here, \( \theta \) is revolution angle.

Figure 1: Prolate epicycloid when each parameter \( a = 16 \text{m}, b = 2 \text{m}, R = 8 \text{m} \). The parameters \( a, b, R \) are indicated.

2.2. Prolate hypocycloid and curtate hypocycloid

When the point inside the fixed circle rolling without slipping, prolate hypocycloid will be formed in Fig.2 and curtate hypocycloid will be formed when \( R < |b| \) as shown in Fig.3.

The difference of the parametric equations between prolate epicycloid and prolate hypocycloid is that \( b \) becomes \(-b\), as shown in equation(2) while
the parametric equations for curtate hypocycloid and prolate hypocycloid are the same.

\[
\begin{align*}
    x &= (a - b) \cos \theta - R \cos \left( \frac{a-b}{b} \theta \right) \\
y &= (a - b) \sin \theta - R \sin \left( \frac{a-b}{b} \theta \right)
\end{align*}
\]

(2)

Figure 2: Prolate hypocycloid when each parameter \(a=16\text{m}, b = 2\text{m}, R =8\text{m}\). The parameters \(a, b, R\) are indicated.

Figure 3: Curtate hypocycloid when each parameter \(a=11\text{m}, b =6\text{m}, R =3\text{m}\). The parameters \(a, b, R\) are indicated.
3. Fitting of prolate epicycloid model and particle motion in magnetic
dipole equatorial field

3.1. Differential equation of motion

We premise that the magnetic dipole is located at the origin and the magnetic
dipole equatorial plane is the xoy plane which the magnetic field direction is
perpendicular to.

According to Newton’s second law and the Lorentz equations, one obtains

\[ ma = e \begin{bmatrix} i & j & k \end{bmatrix} \begin{bmatrix} v_x & v_y & 0 \\ 0 & 0 & B_z \end{bmatrix} \]  
(3)

Next, the differential equations form of the trajectory equations of charged
particles are

\[
\begin{aligned}
    m \frac{d^2 x}{dt^2} &= e \frac{dy}{dt} B_z \\
    m \frac{d^2 y}{dt^2} &= -e \frac{dx}{dt} B_z 
\end{aligned}
\]  
(4)

The magnetic field intensity at a point in the equatorial field of magnetic
dipole can be expressed as

\[ B_z = \frac{\mu_0 P_m}{4\pi(x^2 + y^2)^{5/2}} \]  
(5)

To simplify the notation it is convenient to introduce the constant

\[ A = \frac{e\mu_0 P_m}{4\pi m} \]

So equation(4) can be described as

\[
\begin{aligned}
    \frac{d^2 x}{dt^2} &= \frac{A}{(x^2 + y^2)^{5/2}} \frac{dy}{dt} \\
    \frac{d^2 y}{dt^2} &= -\frac{A}{(x^2 + y^2)^{5/2}} \frac{dx}{dt} 
\end{aligned}
\]  
(6)

In this paper, we suppose that the magnitude of \( P_m \) is \( \frac{2\pi}{\mu_0} \times 10^{-13} \text{A}\cdot\text{m} \) and
direction is vertically upward, \( e = -1.602 \times 10^{-19} \text{C} \), \( m = 9.1 \times 10^{-31} \text{kg} \).
3.2. Fitting of prolate epicycloid model and particle motion

By contrast with Fig. 1 and the trajectory of charged particles when the particles trajectories are bounded in a magnetic dipole equatorial field, the motion trajectory and prolate epicycloid are considered to be the same trajectory.

3.2.1. An express for $\omega$

If the initial conditions are given, it can always set up a cartesian coordinate system that the particles are released on the positive $x$ axis, and the motion of charged particles is determined.

We first define the following abbreviations for this section:

- $\theta = \text{angle of revolution}$;
- $\frac{a}{b}\theta = \text{angle of rotation}$;
- $\omega = \text{angular velocity of revolution}$;
- $\overline{\omega} = \text{average angular velocity of revolution}$;
- $\frac{\theta}{b}\omega = \text{angular velocity of rotation}$;
- $v_j = \omega l = \text{revolution speed}$;
- $v_i = \frac{\theta}{b}\omega R = \text{rotation speed}$;
- $\alpha = \text{angle between } v_i \text{ and } v_j$;
- $l = \text{distance of charged particle from origin}$.

![Velocity analysis diagram of charged particles and the angles of a triangle formed by dashed line are $\alpha, \frac{\theta}{b}\alpha - \alpha, \pi - \frac{\theta}{b}\alpha$.]
From the conservation of kinetic energy we have

\[
(\omega l + \frac{a}{b} \omega R \cos \alpha)^2 + (\frac{a}{b} \omega R \sin \alpha)^2 = v^2
\]  

(7)

Fig. 4 shows the three internal angles of the triangle formed by the charged particle, the center of the moving circle and the center of the fixed circle. According to the law of cosines, the following equations are obtained

\[
(a + b)^2 + R^2 - l^2 = -2(a + b)R \cos \frac{a}{b} \theta
\]

\[
l^2 + R^2 - (a + b)^2 = 2lR \cos \alpha
\]

The result of equation (7) can be expressed as

\[
\omega^2[(a + b)^2 + (R + \frac{aR}{b})^2 + 2(a + b)(1 + \frac{a}{b})R \cos \frac{a}{b} \theta] = v^2
\]  

(8)

here, integral is used to solve this complicated equation

\[
\int_0^\theta [(a + b)^2 + (R + \frac{aR}{b})^2 + 2(a + b)(1 + \frac{a}{b})R \cos \frac{a}{b} \theta] d\theta = \int_0^t \frac{v^2}{\omega} dt + C
\]  

(9)

When \( \theta \) takes the moment of discretization like \( \theta = \frac{2\pi b}{a} n \ (n = 0, 1, 2, 3, \cdots) \), the corresponding time \( t \) and \( \theta \) are in linear relation, \( \omega \) is constant as well. In addition, the \( \theta = 0 \) is true in the case of \( t = 0 \). So the result of equation (9) is

\[
[(a + b)^2 + (R + \frac{aR}{b})^2] \theta = \frac{v^2 t}{\omega}
\]  

(10)

The average angular velocity of revolution is given by

\[
\bar{\omega} = \frac{v}{\sqrt{(a + b)^2 + (R + \frac{aR}{b})^2}}
\]

3.2.2. An express for the parameters \( a, b, R \)

During the movement of charged particles, there are the maximum distance of charged particle from the origin \( r_{max} \), the minimum distance \( r_{min} \) and the period \( T \) over the course of a loop. If take \( v < (v_e/4) \) as a premise

\[
r_{max} = \frac{2v_0}{1 + \sqrt{1 - 4(v/v_e)}}
\]
\[ r_{\text{min}} = \frac{2r_0}{1 + \sqrt{1 + 4(v/v_c)}} \]

here, \( v_c = \frac{A}{r_0} \), the distance of charged particle from the origin \( \rho = r_0 \) when charged particles are forced toward the magnetic dipole.

In the description of E. H. Avrett, when \( V = v_c/v \) is sufficiently large, the result can be obtained by the method of series expansion

\[ T = \frac{2\pi r_0}{v_c} \left( 1 + \frac{15}{2V^2} + \frac{315}{4V^4} + \cdots \right) \quad (11) \]

The parameters \( a, b, R \) satisfies the following relationship

\[ R = \frac{r_{\text{max}} - r_{\text{min}}}{2} \]

\[ a + b = \frac{r_{\text{max}} + r_{\text{min}}}{2} \]

\[ \frac{a}{b} = \frac{2\pi T}{\omega} \]

3.2.3. Fitted parametric equations of charged particles

If the direction of the initial release velocity of the charged particles is perpendicular to \( x \) axis. The initial release position must be \( r_{\text{max}} \) or \( r_{\text{min}} \), and the parametric equations of charged particles are

\[
\begin{align*}
x &= (a + b) \cos [\text{sgn}(B_z)\omega t] - \text{sgn}(A \times v)R \cos [\text{sgn}(B_z)\frac{a+b}{b}\omega t] \\
y &= (a + b) \sin [\text{sgn}(B_z)\omega t] - \text{sgn}(A \times v)R \sin [\text{sgn}(B_z)\frac{a+b}{b}\omega t]
\end{align*}
\quad (12)
\]

In other cases, the conclusion is discussed by K. Kabin and G. Bonner in detail. \( t_{\text{min}} \) shows the time to \( \rho_{\text{min}} \) turning point and it has the following description:

\[
 t_{\text{min}} = \frac{\text{sgn}(v_{\rho_0})}{\sqrt{A^2 - v_{\perp}^2}} (R(\rho_0) - \frac{\rho_{\text{min}} + \rho_{\text{max}}}{2} (\frac{\pi}{2} - \arctan \frac{\rho_{\text{min}} + \rho_{\text{max}} - 2\rho_0}{2R(\rho_0)}))
\]

where,

\[
 R(\rho) = \frac{|v_p|\rho}{\sqrt{A^2 - v_{\perp}^2}} \quad (13)
\]
So the analytical expression of the trajectory of charged particle can be obtained as

\[
\begin{align*}
    x &= (a + b) \cos[\text{sgn}(B_z)\omega(t - t_0) + \mu] - \text{sgn}(A \times v)R \cos[\text{sgn}(B_z)\frac{a + b}{R}\omega(t - t_0) + \mu] \\
y &= (a + b) \sin[\text{sgn}(B_z)\omega(t - t_0) + \mu] - \text{sgn}(A \times v)R \sin[\text{sgn}(B_z)\frac{a + b}{R}\omega(t - t_0) + \mu]
\end{align*}
\]

(14)

where, \( t_0 = t_{\min} \) when \( \text{sgn}(A \times v) \) is positive; \( t_0 = t_{\min} + \frac{T}{2} \) when \( \text{sgn}(A \times v) \) is negative. \( \mu \) satisfies the condition that \( y(0) = 0 \).

3.2.4. Comparison and verification

C. Graef, S. Kusaka\[1\] and A. R. Juarez\[2\] take the same starting point: Using the equations of motion of charged particles in a equatorial plane of magnetic dipole in polar coordinates. The conclusion of the equations of motion is

\[
\begin{align*}
    \frac{d^2 r}{ds^2} - r \frac{d^2 \phi}{ds^2} &= -\frac{1}{r^2} \frac{d \phi}{ds} \\
    \frac{1}{r} \frac{d}{ds}(r^2 \frac{d \phi}{ds}) &= \frac{1}{r} \frac{dr}{dt} \\
    (\frac{dr}{ds})^2 + r^2 (\frac{d\phi}{ds})^2 &= 1
\end{align*}
\]

(15)

E. H. Avrett\[2\] gave an implicit description of the orbit.

\[
r = \frac{2r_0}{1 + \sqrt{1 - 4(\nu/\nu_c)\sin \alpha}}
\]

(16)

K. Kabin\[4\] came up with another way to write

\[
\rho = \frac{p_\phi + \sqrt{p_\phi^2 + 4C\nu \sin \alpha}}{2\nu \sin \alpha}
\]

(17)

Next, the fitted parametric equations will be verified. First, the initial conditions for charged particles are taken as (\( x, y, \dot{x}, \dot{y} \))=(0.15, 0, 0, -0.1). And the trajectories of charged particles are known by Mathematica in Fig.5(a). \( r_{\max} = 0.15, r_{\min} = 0.10455, \frac{\phi}{p} = 19.014 \) are known, given the trajectories of
charged particles. So the analytic expression of prolate epicycloid is

\[
\begin{align*}
  x &= 0.1272748 \cos(-0.21187x) + 0.0227252 \cos(-4.2374x) \\
  y &= 0.1272748 \sin(-0.21187x) + 0.0227252 \sin(-4.2374x)
\end{align*}
\]

which is described in Fig.5(b).

Figure 5: Comparison of real and fitted trajectories. (a). The particle trajectory in a magnetic dipole equatorial field as a movement time of 28s. (b). The trajectory of prolate epicycloid as a movement time of 28s.

We calculate the error rate of the fitted particle trajectory as

\[
\eta = \frac{t_0 - t_1}{t_0}
\]

where, \(t_0\) is the time of actual motion of charged particles, \(t_1\) is the time of fitting motion of charged particles.

By comparing Fig.5(a) and Fig.5(b), the error ratio \(\eta_1 = 0.002143\). It is clear that the fitting effect is excellent, which proves that the parametric equation obtained by our fitting is meaningful.

4. Particle motion in an axisymmetric magnetic field

The magnetic field intensity at a point in an axisymmetric magnetic field can be expressed as

\[
B_z = \frac{\mu_0 P_m}{4\pi(x^2 + y^2)^{n/2}}
\]

(18)
Especially, when $n=3$, the axisymmetric magnetic field is a magnetic dipole field.

4.1. Case: $n>0$

When $n=1$, this is a special case, K. Kabin and G. Bonner obtained the motion of charged particles in this magnetic field by finding a root of a transcendental equations numerically. Here, the same initial condition is taken when $A = 5$, $(x, y, \dot{x}, \dot{y}) = (3, 0, 0, 2)$. The real trajectories and the trajectories fitted by prolate epicycloid are compared in Fig.6.

![Figure 6: Comparison of real and fitted trajectories. (a). The particle trajectory in an axisymmetric magnetic field as a movement time of 70s. (b). The trajectory of prolate hypocycloid as a movement time of 70s.](image)

The error ratio $\eta_2$ between this comparison is 0.025.

When $n=2$, initial condition is taken that $A = 0.05$, $(x, y, \dot{x}, \dot{y}) = (0.15, 0, 0, -0.1)$. The real trajectories and the trajectories fitted by prolate epicycloid are compared in Fig.7. The result of the comparison of the real and fitted trajectories shows that error ratio $\eta_3$ between this comparison is 0.005667.

After many experiments, it is found that all of those case can be explained by prolate epicycloid for any $n>0$ when the particle trajectory is bounded. And the accuracy of the fitted trajectories also increases with the increase of $n$.

4.2. Case: $n<0$

There are two mathematical curve models that can account for particle trajectories, but not for any initial condition.
4.2.1. Fitting of prolate hypocycloid and particle motion

Because turn $b$ into $-b$ in parameter equation of prolate hypocycloid relative to prolate epicycloid, the average angular velocity of prolate hypocycloid can be expressed as

$$\omega = \frac{v}{\sqrt{(a - b)^2 + (R + \frac{aR}{b})^2}}$$

Here is a typical example, we set $n = -3$, $\frac{\mu_0 B_m}{4\pi} = 10$, the initial conditions $(x, y, \dot{x}, \dot{y})$ for charged particles are $(0.15, 0, 0, -0.003)$. The same verification method was used to plot the true track and the trajectories fitted by prolate hypocycloid, as shown in Fig.8(a) and 8(b).

By comparison it is found that the two results are similar, but not completely the same, which inspires us to continue to study more optimized result.

4.2.2. Fitting of curtate hypocycloid and particle motion

We set $n = -3$, $\frac{\mu_0 B_m}{4\pi} = 10$, the initial conditions $(x, y, \dot{x}, \dot{y}) = (0.15, 0, 0, 0.2)$ for charged particles. After getting the particles trajectory, the same verification method is used to draw the track of the fitted prolate hypocycloid. Although the effect of this fit is thin for the trajectories of charged particles shown in Fig.9(a) and 9(b), it is consistent with the tendency of particles trajectories, and the trajectories of charged particles with better fitting effects will be studied later.
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5. Particle motion in a magnetic quadrupole field

Continuing the above work, we explore the motion of charged particles in a magnetic quadrupole field.

The trajectories of charged particles are shown in Fig. 10, which indicates that the track of charged particles surrounded by magnetic dipole $A$ is not the
standard track of prolate epicycloid, since magnetic dipole $B$ has an effect on charged particles. However, it indicates that the track of charged particles in the magnetic field of magnetic multipole may be related to prolate epicycloid also. Next, we will take a closer look at the motion of particles in magnetic quadrupole field, and our current research can be considered as the first step in this direction.

![Figure 10: Track of charged particles in an equatorial field of magnetic quadrupole.](image)

6. Conclusion

By establishing and fitting with the physical model of prolate epicycloid, a simple analytic expression is obtained in cartesian coordinate system. This is an excellent model for explaining the motion of charged particles in an axisymmetric magnetic field with $B \sim 1/\rho^n (n>0)$ when the particle trajectories are bounded, a particular example is the particle motion in magnetic dipole field when $n = 3$. When $B \sim 1/\rho^n (n<0)$, the trajectories can be explained roughly by prolate hypocycloid and curtate hypocycloid, and further research will continue. One notice that the particle motion in a magnetic quadrupole field is also related to prolate epicycloid. It is hoped that the results can promote the study of adiabatic invariants, the dynamics of the radiation belts and ring current particles.
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