Twisted WZW Branes from Twisted REA’s

Jacek Pawełczyk\textsuperscript{a}, Rafał R. Suszek\textsuperscript{a,b,†}, Harold Steinacker\textsuperscript{c}

\textsuperscript{a} Institute of Theoretical Physics, Warsaw University, ul. Hoża 69, PL-00-681 Warsaw, Poland
\textsuperscript{b} King’s College London, Strand, London WC2R 2LS, UK
\textsuperscript{c} Sektion Physik der Ludwig–Maximilians–Universität München, Theresienstr. 37, D-80333 München, Germany

E-mail: Jacek.Pawelczyk@fuw.edu.pl, Rafal-Roman.Suszek@fuw.edu.pl, Harold.Steinacker@physik.uni-muenchen.de

Abstract: Quantum geometry of twisted Wess–Zumino–Witten branes is formulated in the framework of twisted Reflection Equation Algebras. A semiclassical relation to their untwisted counterparts, used previously to describe untwisted WZW branes, is shown to yield a quantisation rule for brane positions consistent with earlier analyses. The coideal property of the twisted Reflection Equation Algebras is employed to reproduce twisted brane distributions within the group manifold.

Keywords: (twisted) D-branes, WZW models, quantum groups, (twisted) reflection equation algebras.

\textsuperscript{*}Work supported by Polish State Committee for Scientific Research (KBN) under contract 2 P03B 001 25 (2003-2005)

\textsuperscript{†}Marie Curie Fellow
1. Introduction.

Branes on group manifolds and quotients thereof have long been at the focus of research efforts aimed at understanding the deformation of classical geometry and gauge dynamics effected by string propagation in background fluxes\(^1\). While the branes naturally lead to the concept of a curved non-commutative space (\([2, 3]\)), they are still amenable to direct investigation using diverse methods such as the Lagrangian formalism of the associated WZW models (\([3]\)) confining the branes to (twisted) conjugacy classes, effective field theory formulated in terms of the Dirac–Born–Infeld functional (\([3]\)) proving their stability, matrix models (\([2, 3]\)) providing a semi-classical picture of the geometry and gauge dynamics, renormalisation group techniques (\([2, 3, 6]\)) capturing brane condensation phenomena, K-theory (\([7, 8, 9]\)) classifying their charges and Boundary Conformal Field Theory (BCFT) offering access to their microscopic structure via the boundary state construction.

In the latter approach, (twisted) branes are identified with states in the Hilbert space of the bulk (or closed string) theory implementing (twisted) gluing conditions for chiral currents of the bulk CFT (\(a\) is an index of the adjoint representation of the horizontal Lie algebra \(g \equiv \text{Lie}G\) of the Kac–Moody algebra \(\hat{g}_\kappa\), \(n\) enumerates Laurent modes and \(B\) is a boundary state label):

\[
(J^a_n + \omega(\bar{J}^a_{-n})) |B \gg \omega = 0, \tag{1.1}
\]

where \(\omega\) is an outer automorphism of the current algebra \(\hat{g}_\kappa\) (see, e.g., \([10, 11]\)). Thus branes break the full chiral symmetry algebra \(\hat{g}_L^\kappa \times \hat{g}_R^\kappa\) of the bulk WZW to the subalgebra spanned by annihilators of \(B \gg \omega\), isomorphic to \(\hat{g}_\omega^\kappa\) (the \(\omega\)-invariant subalgebra of \(\hat{g}_\kappa\)). The above gluing condition defines a (twisted) character (or Ishibashi) state \(|B \gg \omega_c\). Upon imposing additional physical consistency conditions, passing under the general name of sewing constraints, we arrive in the case at hand at Cardy states \(|B \gg C\) solving the trivial gluing condition (\(\omega \equiv \text{id}\)), and their twisted counterparts \(|B \gg \omega_c C\) (\(\omega_c\) is an automorphism of the Dynkin diagram of \(g\)), respectively. We shall invoke certain elements of this approach in the sequel.

Non-commutative geometry entered the stage thus set in \([2]\) where a matrix model of “fuzzy” physics of untwisted branes was explicitly derived from open string perturbation theory in the large volume (or, equivalently, large level \(\kappa\)) limit and subsequently used to formulate an algebraic description of brane condensation phenomena relating distinct brane configurations associated with “fuzzy” conjugacy classes. The twisted case was then examined at great length in \([3]\), along similar lines. The semi-classical approach of \([2]\) was later extended in \([12]\) where an Ansatz for brane geometry and gauge dynamics at arbitrary level was advanced, based on the fundamental concept of quantum group symmetry, as suggested by the underlying (B)CFT, and the well-known correspondence between untwisted affine Lie algebras \(\hat{g}_\kappa\) and Drinfel’d–Jimbo quantum algebras \(U_q(g)\) (see, e.g., \([13]\)). The latter proposal was shown to successfully encode essential (untwisted) brane data such as tensions, localisations, the algebra of functions, internal gauge excitations and

\(^1\)For a review, see: \([1]\).
interbrane open string modes. It was also generalised in \[14\] to a class of orbifold backgrounds, known as simple current orbifolds $SU(N)/\mathbb{Z}_N$, whereby the basic structure of the associated matrix model, a so-called Reflection Equation Algebra$^2$ (REA) $\text{REA}_q(A_N)$, was examined extensively. The study revealed an attractive geometric picture behind the compact algebraic framework of the REA’s, which was next exploited in an explicit construction of some new quantum geometries corresponding to (fractional) orbifold branes.

In the present paper, we discuss an algebraic framework relevant to analysis at arbitrary level of twisted branes on the $SU(2n+1)$ group manifolds. Accordingly, we specialise our exposition to the case $\hat{g}_c = A^{(1)}_{2n}$ (in which $\omega_c$ is the standard $\mathbb{Z}_2$-reflection). The exposition is centred on the CFT-inspired notion of twisted quantum group symmetry, as represented by so-called twisted Reflection Equation Algebras (tREA) $\text{tREA}_q(A_{2n})$, endowed with the structure of coideal subalgebras$^3$ of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ - a quantum-algebraic counterpart of the classical subalgebra structure: $\mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{su}_{2n+1}$ \[13, 19\]. Using this structure, in conjunction with some facts from the representation theory of the twisted quantum groups $\mathcal{U}_q'(\mathfrak{so}_{2n+1}) \cong \text{tREA}_q(A_{2n})$ \[21, 22, 23, 24, 25, 26, 27\], we provide evidence of an intricate relationship between twisted boundary states \[11, 28, 29\], as defined above, and the representation theory of $\mathcal{U}_q'(\mathfrak{so}_{2n+1})$ (at $q$ a root of unity, as dictated by the CFT) embedded in that of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$. We also rederive the quantisation rule for twisted brane positions within the WZW group manifold of $SU(3)$ (originally obtained from homological analysis in \[30\]), whereby we establish - in direct analogy with the untwisted case - a simple geometric meaning of the Casimir operators of $\text{tREA}_q(A_{2n})$. Attractive as it appears, the construction presented in the paper clearly requires further corroboration and detailed study from a variety of angles: the representation theory of the relevant algebras, $\text{tREA}_q(A_{2n})$, the harmonic analysis on the associated geometries, and the gauge dynamics of twisted WZW branes that the algebras are claimed to describe. These issues are currently investigated and the outcome shall be reported in a future publication.

2. Classical geometry of twisted WZW branes.

At the classical level, stable branes of the WZW model in the Lie group target $G$ are described by (twisted) conjugacy classes of the form:

$$\mathcal{C}^\omega(t) = \left\{ h t \omega(h^{-1}) \mid h \in G \right\}, \quad (2.1)$$

with $t$ in the "symmetric" subgroup $T^\omega$ of the maximal torus $T \subset G$, i.e. $t \in T$ with $\omega(t) = t$, whence - in particular - the conjugacy classes are invariant under $\omega$. When $G = SU(2n+1)$ and

---

$^2$Cp \[14, 16, 17\], see also: the Appendix.

$^3$Structures of this kind have long been known to arise naturally in the related context of $(1 + 1)$-dimensional massive integrable models on a half-line, with involutively twisted gluing condition for chiral symmetry currents at the boundary, cp \[20\].
\( \omega = \omega_c \) (the case of interest) the above is equivalent to \((T \text{ denotes transposition})

\[
C^{\omega_c}(t) = \{ hth^T \mid h \in SU(2n + 1) \}.
\]

(2.2)

Let \( K_t = \{ h \in G : hth^T = t \} \) be the stabiliser subgroup (in the twisted adjoint representation) of \( t \in T^\omega \). For \( t = I \), the stabiliser \( K_t \) coincides with the group \( SO(2n + 1) \). In the algebraic setup to be developed we shall encounter a quantum deformation of this group (see Sec.3). Clearly, \( C^{\omega}(t) \) can be viewed as a homogeneous space:

\[
C^{\omega}(t) \cong G/K_t.
\]

(2.3)

The twisted conjugacy classes are invariant under the twisted adjoint action of the vector subgroup \( G \cong G_V \hookrightarrow G_L \times G_R \) of the group of symmetries of the target manifold,

\[
G C^{\omega}(t) \omega(G^{-1}) = C^{\omega}(t).
\]

(2.4)

This is a classical counterpart of the symmetry breaking pattern: \( \widehat{\mathfrak{g}}_\alpha^L \times \widehat{\mathfrak{g}}_\alpha^R \rightarrow \widehat{\mathfrak{g}}^{V,\omega}_\alpha \cong \widehat{\mathfrak{g}}^{\omega}_\alpha \) mentioned under \((1.1)\).

The remaining part of the original bulk symmetry, \( G_L \times G_R \), translates - just as in the untwisted case - into invariance of the ensuing physical model under rigid one-sided translations of twisted conjugacy classes within \( G \),

\[
G_L C^{\omega}(t) G_R = G_L (G^{-1}_L) \omega(G_L) G_R = C^{\omega}(t) G.
\]

(2.5)

This reflects the residual freedom in the definition of the boundary state consisting in the choice of the inner automorphism twisting the gluing condition \((10)\).

3. Twisted Reflection Equations.

In this section, we shall discuss (quantum) algebras relevant to the description of twisted branes. The arguments we invoke are of the kind presented in \([12]\), i.e. they are based on the pattern of symmetry breaking induced by twisted branes (cp the discussion of the previous section).

Thus we propose to consider a twisted Reflection Equation (tRE):

\[
t\text{RE}^- : \quad R_{12} K_{-1} R^t_{12} K^t_{-2} = K_{-2} R^t_{12} K^t_{-1} R_{12}.
\]

(3.1)

Equations of this kind (parametrised by additional physical quantities) have long been known to describe couplings of bulk modes to the boundary in massive \((1 + 1)\)-dimensional theories on a half-line, with involutively twisted gluing condition for chiral symmetry currents at the boundary (see \([24]\) and the references within). Furthermore, the respective algebraic structures ensuing from

\[4\]In this picture, the map: \( G/K_t \rightarrow C^{\omega}(t) \), \( hK_t \mapsto h\omega(h^{-1}) \) is manifestly well-defined and bijective. Note that the left hand side is a one-sided (right) coset of \( G \).
and its dynamical counterpart from [20] share many essential features (coideal property, an intimate relation to the so-called symmetric pairs).

The twisted left-right (co)symmetries ([12]) of the tRE: $K^- \mapsto t^T K^- s$, realised in terms of $(t, s) \in G_L \otimes_R G_R \equiv SU_q(2n + 1) \otimes_R SU_q(2n + 1)$ (we have $q = e^{\pi i/(\kappa + 2n + 1)}$, as indicated by the underlying CFT), provide a quantum version of the classical left-right isometry of the group manifold, which should be a symmetry of the problem (to be broken by branes). There is another tRE with the same symmetry properties,

$$t^\text{RE}^+ : R_{21} K^+ L_1 R_{21} K^+ = K^+ L_1 R_{21} K^+ .$$

The transformation rule of $K^\pm$ reads $K^\pm \mapsto (St^T) K^\pm (Ss)^T$ ($S$ is the antipode of the Hopf algebra $SU_q(2n + 1)$). As we shall discuss in App A.2 (following [19]), the two tRE’s define the same quantum algebra $U'_{q} (\mathfrak{su}_{2n+1})$ ([22]), a quantum deformation of $\mathfrak{su}_{2n+1}$. tRE$^\pm$ differ in the manner the algebra $U'_{q} (\mathfrak{su}_{2n+1})$ is embedded in them. In view of the prominent rô le played by $SO(2n + 1)$ in the description of twisted $A_{2n}$ branes (see Sec.2.), the appearance of the latter algebra should be regarded as an encouraging fact.

As it turns out ([18]), we need both $K^+$ and $K^-$ to construct Casimir operators for this algebra\footnote{In the case at hand, i.e. for the deformation parameter $q$ a root of unity, there are - as usual - additional central elements in the algebra, originally discovered in [24]. They shall not be considered in this paper. In particular, for $A_2$ with our subsequent choice of representation theory, they are known to carry no interesting information ([24]).}. They shall play an important part in our discussion of brane geometries (see Sec.4.2.). The Casimir operators can be cast in the form:

$$c_m := \text{tr} \left( X (DX)^{m-1} \right), \quad m \in \mathbb{Z}, 2n - 1;$$

where $X := K^- K^+$ and $D := \text{diag}(q^{-2}, q^{-2}, \ldots, 1)$, the latter being straightforwardly related to the antipode $S$ through

$$D^{-1} s D = S^2 s .$$

In the spirit of the papers [12, 14], we would like to identify branes with appropriately chosen irreducible representations of the tREA defined above (to be denoted as tREA$_{q}(A_{2n})$ henceforth). Further evidence in favour of such an assignment as well as the details of the identification shall be provided in Sect.4. For the present, though, we focus on a particular consequence of this idea: clearly, it should entail the existence of an algebraic counterpart of (2.4). And indeed, the vector part of the $G_L \otimes_R G_R$ symmetry, realised as

$$K^- \mapsto s^T K^- s \quad , \quad K^+ \mapsto (Ss) K^+ (Ss)^T$$

possesses the required properties. In addition to preserving the respective tRE’s, it also leaves the values of all $c_m$’s unchanged. This follows from the fact that under the above transformations send $X \mapsto s^T X (Ss)^T$, $XDX \mapsto s^T X (Ss)^T D s^T X (Ss)^T$ etc. and hence, upon noting
\( \mathbf{D}^{-1} (Ss)^T\mathbf{D} = \det gs = 1 \), a straightforward consequence (cp [31]) of (3.4), we readily verify

\[ c_m \mapsto \text{tr} \left( s^T X (DX)^{m-1} (Ss)^T \right). \]

That, however, leads us directly to the conclusion.

Next, we turn to the representation theory of (3.1)-(3.2). Recall that tREA\( q(A_{2n}) \) is related to a particular deformation of \( \mathfrak{so}_{2n+1} \) denoted by \( \mathcal{U}'_q(\mathfrak{so}_{2n+1}) \). The representation theory of \( \mathcal{U}'_q(\mathfrak{so}_{2n+1}) \) is known in considerable detail (see, e.g., [22, 25]). Here, we are interested only in the highest weight irreducible representations. For \( q = e^{\pi i / (\kappa + 2n + 1)} \), these are of the classical type, with the corresponding highest weights truncated to a fundamental domain in a \( (\kappa + 2n + 1) \)-dependent way outlined below. The labeling is given by standard signatures: \( \vec{m} = (m_1, m_2, \ldots, m_n) \) such that all \( m_i \)'s are integers or all are half-integers, subject to the dominance condition:

\[ m_1 \geq m_2 \geq \ldots \geq m_n \geq 0. \tag{3.6} \]

The signatures can readily be expressed in terms of the Dynkin labels of the corresponding weights:

\[ 2m_i = 2 \sum_{j=1}^{n-1} \lambda_j + \lambda_n, \quad (i < n), \quad 2m_n = \lambda_n. \tag{3.7} \]

The truncation scheme has not been worked out in all generality as of this writing. It is known (23) in the simplest case of \( \mathcal{U}'_q(\mathfrak{so}_3) \),

\[ \lambda_1 = 2m_1 \leq \kappa + 2, \tag{3.8} \]

and an inspection of the algebra \( \mathcal{U}'_q(\mathfrak{so}_5) \) and its representations (cp [23]) reveals that the candidate formula is \( \lambda_1 + \lambda_2 = m_1 + m_2 \leq \kappa + 5 \). Thus, it seems plausible that for irreducible representations of \( \mathcal{U}'_q(\mathfrak{so}_{2n+1}) \) highest weights are truncated as:

\[ \sum_{i=1}^{n} \lambda_i \leq \kappa + 2n + 1. \tag{3.9} \]

We shall return to this issue in the next section.

4. Geometry of twisted branes from the tREA.

In the present section, we unravel a number of intricate features of the tREA's introduced, indicating towards an intimate relationship between the latter and twisted branes of the WZW models of type \( A_{2n} \).

4.1 Algebraic truncation of twisted brane labels.

Let us start by recalling that the non-classical geometry of a maximally symmetric WZW brane has been successfully encoded in the representation theory of \( REA_q(\mathfrak{g}) \) ([12, 14]). A crucial rôle in

\(^6\)At the threshold, the matrix elements of the generators of \( \mathcal{U}'_q(\mathfrak{so}_5) \) develop poles. Analogous pathology occurs for \( \mathcal{U}'_q(\mathfrak{so}_3) \) and extends to Clebsch–Gordan coefficients, as well as the associated \( 6j \)-symbols.
this approach has been played by the map \( REA_q(g) \to U_q(g) \) given by \( M = L^+ M_0 S L^- \), in which \( L^\pm \) are the familiar FRT operators of \( [17] \) (see: App.A). The map provides us with a necessary tool to show that there is a one-to-one correspondence between highest weight irreducible representations of \( REA_q(g) \) and (untwisted) branes. Moreover, it gives geometrical information about branes in terms of Casimirs operators.

For the tRE, there is a similar embedding of \( tREA_q(A_{2n}) \cong U'_q(\mathfrak{so}_{2n+1}) \) in \( U_q(\mathfrak{su}_{2n+1}) \),

\[
K^- = (L^+)^T C^- L^- , \quad K^+ = S L^+ C^+ (S L^-)^T ,
\]

(4.1)

with \( C \) - a constant (\( c \)-number-valued) matrix solution of tRE. In what follows, we take \( C := \text{diag}(c_1, c_2, \ldots, c_{2n+1}) \) such that \( \lim_{q \to 1} c_i = 1 \), \( i \in \{1, 2n + 1\} \). This choice guarantees that in the classical limit, \( q \to 1 \), (4.1) defines the standard embedding: \( \mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{su}_{2n+1} \). The map (4.1) determines a branching of representations of \( U_q(\mathfrak{su}_{2n+1}) \) into those of \( U'_q(\mathfrak{so}_{2n+1}) \). Furthermore, it will help us build a geometrical picture of twisted branes in the next section. Based on these arguments, we shall devise a rule which says that twisted branes correspond to those highest weight irreducible representations of \( U'_q(\mathfrak{so}_{2n+1}) \) which show up in the branchings of highest weight irreducible representations of \( U_q(\mathfrak{su}_{2n+1}) \), defined by (4.1). We shall also provide the necessary truncation on the representation labels below.

As (4.1) are quantum analogs of the classical embedding \( \mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{su}_{2n+1} \), branchings of the classical-type representations are induced (up to truncation) by the following projection:

\[
[l_1, l_2, \ldots, l_{2n}] \mapsto [l_1 + l_2, l_2 + l_{2n-1}, \ldots, 2(l_n + l_{n+1})], \quad (n > 1)
\]

\[
[l_1, l_2] \mapsto [2(l_1 + l_2)]
\]

(4.2)

from the weight space of \( \mathfrak{su}_{2n+1} \) onto that of \( \mathfrak{so}_{2n+1} \).

The embedding (4.1) carries also information about the relevant truncation of the twisted brane labels. It shows that the coproduct \( \Delta \) (which defines tensoring of representations), induced from the coproduct of \( U_q(\mathfrak{su}_{2n+1}) \) via (4.1), has the property:

\[
\Delta tREA_q(A_{2n}) = U_q(\mathfrak{su}_{2n+1}) \otimes tREA_q(A_{2n}),
\]

(4.3)

which turns \( U'_q(\mathfrak{so}_{2n+1}) \) into a left coideal subalgebra of \( U_q(\mathfrak{su}_{2n+1}) \). As a consequence, generic representations of \( tREA_q(A_{2n}) \) cannot be tensored. In spite of this, tensoring can still be defined\(^7\) for pairs \( (R_V, R_{\bar{m}}) \), where \( R_V \) is the fundamental representation of signature \( \bar{m} = (1, 0, 0, \ldots, 0) \). The ensuing tensor product for classical type representations is, up to truncation, the same as the tensor product of representations of \( \mathfrak{so}_{2n+1} \).

\(^7\)What saves the situation is the \( U_q(\mathfrak{su}_{2n+1}) \)-comodule structure on \( tREA_q(A_{2n}) \), with the coaction reproduced by the comultiplication (4.3) in \( U_q(\mathfrak{su}_{2n+1}) \) with respect to the embedding (4.1). As a consequence, the left \( U_q(\mathfrak{su}_{2n+1}) \)-comodule structure induced on \( tREA_q(A_{2n}) \) can be used to introduce the restricted tensor product in a manner explained in [27].
In reconstructing the branchings in the simplest case of \( \mathfrak{so}_3 \hookrightarrow \mathfrak{su}_3 \), we encounter another intricacy: as the Dynkin labels of the \( \mathcal{U}_q(\mathfrak{so}_3) \) highest weights \( [\lambda_1] \in P_+(A_1) \) labeling direct summands in consecutive tensor products traverse the threshold: \( 2\lambda_1 \leq \kappa \), the quantum dimensions of the corresponding modules, regarded here as submodules of \( \mathcal{U}_q(\mathfrak{su}_3) \), vanish. Let us be more specific. Consider an irreducible representation \( R_m \) of \( \mathcal{U}_q(\mathfrak{so}_3) \) obtained from restricted tensoring so that \( R_m \hookrightarrow R_\Lambda \), as dictated by branching rules for the irreducible representation \( R_\Lambda \) of \( \mathcal{U}_q(\mathfrak{su}_3) \) (\( \Lambda \in P_+(A_2) \)).

Using the familiar notion of quantum trace for \( \mathcal{U}_q(\mathfrak{su}_3) \) we may then introduce the quantum dimension (\( \rho \) is the Weyl vector of \( \mathfrak{su}_3 \))

\[
\dim q R_m := \text{tr}_q | R_m \hookrightarrow R_\Lambda \mathbb{1} = \text{tr}_q | R_m \hookrightarrow R_\Lambda q^{-2H_\rho}, \quad q = e^{\frac{2\pi i}{\kappa+1}} \tag{4.4}
\]

of the submodule \( R_m \). Above, we simply take the trace of the operator \( q^{-2H_\rho} \in \mathcal{U}_q(\mathfrak{su}_3) \) over the subspace of the \( \mathcal{U}_q(\mathfrak{su}_3) \)-module \( R_\Lambda \) identified with \( R_m \). The above definition of the quantum dimension clearly makes sense in that it depends on \( m \) only. Thus (\( [x]_q := \frac{q^x - q^{-x}}{q-q^{-1}} \) and \( m(\lambda) \) is given by \((3.7)\))

\[
\dim q R_{m(\lambda)} = [\lambda_1 + 1]_q, \tag{4.5}
\]

from which it is clear that the quantum dimension vanishes for \( \lambda_1 = \kappa + 2 \) (note that \( \lambda_1 \in 2\mathbb{N} \)) and we may consistently decouple \( \mathcal{U}_q(\mathfrak{so}_3) \)-representations of highest weight bigger than \( \kappa \) by restricting to irreducible representations of a non-vanishing quantum dimension in consecutive tensor products. Recall that it is precisely the requirement of a non-vanishing quantum dimension that distinguishes physically relevant\(^8\) irreducible representations of \( \mathcal{U}_q(\mathfrak{g}) \). Note also that \((4.5)\) is actually proportional to the tension of a twisted brane (\([3, 8]\)) so that the restriction acquires a clear physical meaning.

The above argument, together with the analysis of some numerical tests for algebras of higher rank, showing analogous regularities, motivates the imposition of the following cutoff on the restricted tensor product \( R_V \otimes R_{\vec{m}} \) in the general case:

\[
2 \sum_{i=1}^{n-1} \lambda_i + \lambda_n \leq \kappa. \tag{4.6}
\]

Clearly, the effect described requires further examination. We are planning to return to this issue in a future paper.

Rather amazingly, the truncation defined above actually restricts the set of irreducible representations of \( \mathcal{U}_q(\mathfrak{so}_{2n+1}) \) appearing in the branchings to those corresponding to admissible twisted boundary labels (i.e. to the fractional symmetric affine weights of \([13]\)) under the mapping:

\[
\Psi : [\lambda_1, \lambda_2, \ldots, \lambda_n] \rightarrow \frac{1}{2} \left[ \lambda_{n-1}, \ldots, \lambda_1, \frac{\kappa - \lambda_n}{2} - \sum_{i=1}^{n-1} \lambda_i, \frac{\kappa - \lambda_n}{2} - \sum_{i=1}^{n-1} \lambda_i, \lambda_1, \ldots, \lambda_{n-1} \right], \tag{4.7}
\]

\(^8\)That is corresponding to integrable highest weight representations of Kac–Moody algebras \( \hat{\mathfrak{g}}_\kappa \).
originally proposed in \cite{28} and further discussed in \cite{3}. The latter establishes a correspondence between twisted boundary labels and highest weights of $\mathfrak{so}_{2n+1}$ for $\kappa \in 2\mathbb{N}^*$. 

Our numerical tests indicate towards yet another astonishing consequence of the algebraic construction advocated above. As we impose the cutoff (4.6) on twisted weights (which can consistently be done over the entire $P^*_{\kappa}(A_{2n})$ for $\kappa \in 2\mathbb{N}^*$), the classical branching of $\mathfrak{su}_{2n+1}$-weights over the fundamental affine alcove appears to take the form:

$$R_\Lambda \longrightarrow \bigoplus_\lambda (n^{\psi(c)}_\lambda)^{\psi(\lambda)} R_{\tilde{\mu}^{(\lambda)}},$$

where $n^{\psi(c)}_\lambda$ are the so-called twisted fusion rules of the underlying BCFT (\cite{29}). While the statement is based entirely on the analysis of a number of specific cases ($n \in \{1, 2, 3\}$ and several values of the level), the emergence of non-trivial concatenations of cutoffs leading invariably to the expected result lends strong support to it. The pattern (4.8) can next be compared with the results of an explicit computation of scalar products of the untwisted boundary state $|\Lambda \gg C\rangle$ with admissible twisted boundary states $|\Psi(\lambda) \gg \omega_\kappa C\rangle$, whereby perfect agreement is observed\(^9\). The last observation, together with our earlier discussion of the significance of the coideal structure (4.3) of $\text{tREA}_q(A_{2n}) \hookrightarrow \text{REA}_q(A_{2n})$, provides significant evidence in favour of the following identification:

$$\text{the twisted brane } |\Psi(\lambda) \gg \omega_\kappa C\rangle \sim R_{\tilde{\mu}(\lambda)} \text{ of } \text{tREA}_q(A_{2n}).$$

We shall corroborate it further in the next section. Meanwhile, though, we note that it actually associates (through (4.8) and the twisted-untwisted boundary state products) the trivial representation, $R_0$, with the dimensionally reduced twisted brane as the unique one having a non-vanishing overlap with (i.e. containing) the pointlike untwisted branes localised at the $2n + 1$ points in $SU(2n + 1)$ corresponding to the elements of the centre $Z(SU(2n + 1)) \cong \mathbb{Z}_{2n+1}$. It also seems worth pointing out, after \cite{3}, that the representations admitted by (4.6) are among those representations of the algebra $\mathfrak{so}_{2n+1}$ which can be integrated to representations of the group $SO(2n + 1)$. Both these facts shall be of prime relevance to the discussion of the next section.

### 4.2 Brane localisation from Casimir eigenvalues.

We are not aware of any natural embedding $\text{tREA}_q(A_{2n}) \hookrightarrow \text{REA}_q(A_{2n})$. Recall that - following \cite{12} - we assign to the latter algebra the rôle of the quantised algebra of functions on the group manifold. Thus, the lack of such a map prevents us from giving a direct geometrical meaning to various quantities associated with $\text{tREA}$’s, e.g. to their Casimir operators. Luckily, the situation is not hopeless. We may employ (4.1) and the map $\text{REA}_q(A_{2n}) \rightarrow \mathcal{U}_q(\mathfrak{su}_{2n+1})$, (A.8), to construct a map $\text{tREA}_q(A_{2n}) \rightarrow \text{REA}_q(A_{2n})$ order by order in the parameter $1/\kappa$. Using the above expansion we shall express the quadratic Casimir operator $c_1$ of $\text{tREA}_q(A_{2n})$ in terms of the $\mathbf{M}$-variables, that is in terms of solutions to the (untwisted) RE (cp \cite{12, 14}). All approximate equalities below are up to terms of higher order in the expansion parameter. We also choose $C := I$.\(^9\)

\(^9\)We will discuss this fact and detail the relevant BCFT computation in an upcoming paper.
First, notice that $K_{ii}^\pm \approx I$ for all $i \in 1, 2n + 1$. Hence $c_1 \approx \sum_i I + \sum_{i>j} K_{ij}^- K_{ji}^+$. Upon subtracting the trivial part, we then define

$$\tilde{c}_1 := \sum_{i>j=1}^{2n+1} K_{ij}^- K_{ji}^+.$$  (4.10)

We also have $K_{ij}^- \approx \sum_{j<k<i} L_{ki}^+ L_{kj}^-$ and $K_{ji}^+ \approx \sum_{j<k<i} SL_{ik}^+ SL_{jk}^-$. Using the results from App.D of [14] we list the relevant (leading) terms of the $L^\pm$-operators:

$$L_{ij}^- \approx \lambda E_{ji}, \quad SL_{ij}^- \approx \lambda E_{ji}, \quad i < j$$

$$L_{ij}^+ \approx -\lambda E_{ji}, \quad SL_{ij}^+ \approx \lambda E_{ji}, \quad j < i$$

$$L_{ii}^\pm \approx I, \quad SL_{ii}^\pm \approx I,$$

with $E_{ij}$ defined as in [14] (their explicit form is not relevant here). The above yield

$$K_{ij}^- \approx \lambda (E_{ij} - E_{ji}) \approx -K_{ji}^+, \quad j < i$$

and - since $M_{ij} \approx \lambda E_{ji}$ for $i \neq j$ - we conclude that

$$K_{ij}^\pm \approx M_{ij} - M_{ji}.$$  (4.13)

Thus

$$\tilde{c}_1 \approx -\sum_{i>j=1}^{2n+1} (M_{ij} - M_{ji})^2 = \frac{1}{2} \text{tr}(M - M^T)^2.$$  (4.14)

At this stage, we may already evaluate the Casimir operator on a particular irreducible representation $R_{\vec{m}}$ of tREA$_q(A_{2n})$. Thus we rewrite the left hand side after [22, 23] in terms of components of the signature vector $\vec{m}$ labeling the irreducible representation chosen, whereby we obtain

$$\tilde{c}_1|_{R_{\vec{m}}} = q^{2n-1} \lambda^2 \sum_{j=1}^n \left( \left[ m_{n+1-j} + j - \frac{1}{2} \right]_q^2 - \left[ j - \frac{1}{2} \right]_q^2 \right).$$  (4.15)

On the present level of generality, we may draw one encouraging conclusion: the Casimir clearly vanishes on the trivial representation of the tREA, $R_\vec{0}$, and with our choice of truncation of admissible irreducible representations, ([14]), it is also the unique$^{10}$ representation with this property. Indeed, using $[x]^2 - [y]^2 \equiv [x - y]_q [x + y]_q$ we may recast ([1.15]) in the form:

$$\tilde{c}_1|_{R_{\vec{m}}} = q^{2n-1} \lambda^2 \sum_{j=1}^n [m_{n+1-j}]_q [m_{n+1-j} + 2j - 1]_q.$$  (4.16)

It is now clear that with the $m_i$'s constrained as in (1.4) all summands in (4.10) are positive for $\vec{m} \neq \vec{0}$ (cp (3.7) and (3.6)). However, in this distinguished case $\vec{m} = \vec{0}$, in which $K_{ij}^\pm$ reduce to

$^{10}$Note that (3.9) does not guarantee the uniqueness.
we obtain \( \text{tr}(\mathbf{M} - \mathbf{M}^T)^2 = 0 \), solved \( \mathbf{M} \) are symmetric matrices. This conforms with the known results for the dimensionally reduced brane \([8]\) to which we consequently associate the zero \( \mathcal{U}'_q(\mathfrak{so}_{2n+1}) \)-signature. Equivalently, from the (co)isometry \([3,7]\) of irreducible representations of \( \mathcal{U}'_q(\mathfrak{so}_{2n+1}) \) we conclude that the geometry defined by \( R_q \) is encoded in the twisted \( SU_q(2n+1) \)-comodule algebra: \( \mathbf{C} \mapsto s^T \mathbf{C} s \) and therefore it describes the twisted (quantum) conjugacy class of the group unit. Once again, then, we reproduce the BCFT picture in a manner consistent with our former identification \([1,9]\).

Rather astonishingly, it turns out that we may extract further information from the semiclassical result \([3,14]-[3,15]\), whereby we gain some insight into its physical meaning. To these ends we specialise the formulæ to the simplest physically relevant\(^{11}\) case: \( n = 1 \). Plugging into \([3,14]\) the explicit classical parameterisation of twisted conjugacy classes of \( G = SU(3) \),

\[
M_\theta = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

and comparing with \([3,15]\) we get the relation:

\[
-8 \sin^2 \theta = 2\lambda^2 [\lambda_1/2]_q [\lambda_1/2 + 1]_q,
\]

where - as previously - \( \lambda_1 = 2m_1 \in \mathbb{N} \).\(^{22}\) We can regard \([4,18]\) as a quantisation condition for brane positions. For \( 1 \ll \lambda_1 \ll \kappa \) it yields

\[
\theta \approx \frac{\lambda_1 \pi}{2\kappa}.
\]

Clearly, the above rule retains its validity for \( \lambda_1 = 0 \), hence we may expect it to be generally applicable in the large \( \kappa \) limit.

The significance of the classical limit \([4,19]\) of our quantum-algebraic result follows from the fact that it is amenable to direct comparison with the data on twisted brane localisation which can be found in the literature\(^{12}\). Thus we compare \([4,19]\) with the homological analysis of \([3,11]\). The latter yields a quantisation rule \( (E(x) \text{ denotes the integral part of } x)\):

\[
\theta = \frac{(2n - \kappa)\pi}{2\kappa}, \quad n \in E\left(\frac{\kappa}{2}\right), \kappa,
\]

which falls in perfect agreement with \([4,19]\) (for even \( \kappa \)) and, consequently, lends support to our proposal. Indeed, upon restricting in \([4,15]\) to the integer-spinned irreducible representations of \( \mathcal{U}'_q(\mathfrak{so}_3) \), the two quantisation formulæ become fully equivalent. The latter representations, on the other hand, are precisely the ones that appear in (truncated) branchings of the irreducible representations of \( \text{REA}_q(A_2) \) used in \([14]\) in the description of untwisted branes, as indicated by \([4,8]\).

\(^{11}\)The classical \( SU(2) \) has no non-trivial diagram automorphisms.

\(^{12}\)As for exact BCFT data of, e.g., \([8]\) it unavoidably becomes obscured by the conventions adopted in the original papers. They differ from ours in the choice of the representative of the class of automorphisms implementing the Dynkin diagram reflection on the group level.
5. Summary and conclusions.

In the present paper, we have discussed a class of quantum algebras, the twisted Reflection Equation Algebras \( t\text{REA}_q(A_{2n}) \), in reference to twisted boundary states of WZW models for the groups \( SU(2n+1) \) and the associated brane worldvolumes wrapping (classically) twisted conjugacy classes within the group manifolds of the \( SU(2n+1) \). The framework, developed as a straightforward extension of the previous constructions based on the untwisted Reflection Equation Algebras \( \text{REA}_q(A_{2n}) \) for untwisted WZW branes, is a novel proposal for a compact algebraic description of the twisted branes. Our study provides several arguments in favour of its profound relationship to the CFT of twisted boundary states: classical-type irreducible representations of \( t\text{REA}_q(A_{2n}) \) enjoy a (co)symmetry that quantises the twisted adjoint symmetry of the boundary states (the starting point of the construction) and in so doing they realise a symmetry breaking scenario analogous to the BCFT one (cp the introductory remarks under (1.1)); the eigenvalues of the Casimir operators of \( t\text{REA}_q(A_{2n}) \) returned by these irreducible representations admit a simple physical interpretation in terms of quantum localisation rules for twisted brane geometries, shown to reproduce the known result for the simplest case of \( SU(3) \) in the semiclassical approximation allowing for an explicit embedding \( t\text{REA}_q(A_{2n}) \hookrightarrow \text{REA}_q(A_{2n}) \); the representation theory of \( t\text{REA}_q(A_{2n}) \), endowed with a restricted tensor product structure remarked upon under (4.3), seems to reproduce twisted brane density distributions within the quantum manifolds of the \( SU(2n+1) \) upon imposing a truncation on admissible highest weights (labels of the irreducible representation of \( t\text{REA}_q(A_{2n}) \)); the truncation is identical with the one suggested in [28] in the BCFT context and appears to manifest itself through the vanishing of quantum dimensions of the \( t\text{REA}_q(A_{2n}) \)-modules from (directly) outside of the truncated range).

In conclusion, we believe that there are sound reasons to regard the \( t\text{REA}'s \) as natural building blocks of quantum-algebraic matrix models for twisted branes on the \( SU(2n+1) \) WZW manifolds. While encouraged by the results obtained hitherto, we are aware of numerous questions that our study leaves unanswered, the most obvious having been listed already in the Introduction. We intend to return to them in a future publication.

Acknowledgments

The authors would like to thank the organizers of the 2004 ESI Workshop on "String theory on non-compact and time-dependent backgrounds" where part of this work was done. R.R.S. gratefully acknowledges useful discussions with T. Quella. It is also a pleasure to thank the Theoretical Physics Group at King’s College London, and in particular Andreas Recknagel, for interest in the project and providing a stimulating atmosphere in the final stage of this work.
A. (Twisted) Reflection Equations.

In this appendix, we discuss chosen properties of three RE's:

\[ tRE^- : R_{12}^{-} K_{12}^{-1} K_{2}^{-1} = K_{12}^{-1} K_{2}^{-1} R_{12}^{-}, \]  
(A.1)

\[ tRE^+ : R_{21}^{+} K_{21}^{+1} K_{1}^{+1} = K_{21}^{+1} K_{1}^{+1} R_{21}^{+}, \]  
(A.2)

\[ RE_0 : R_{12} M_{1} R_{21} M_{2} = M_{2} R_{12} M_{1} R_{21}, \]  
(A.3)

appearing in the paper. In the formulæ above, \( R \) is a bi-fundamental realisation of the standard universal \( \mathcal{R} \)-matrix of the relevant quantum group \( U_q(su_{2n+1}) \), \( R \equiv (R_V \otimes R_V) (\mathcal{R}) \), satisfying the celebrated Quantum Yang–Baxter Equation (see, e.g., [13, 31]). The operator-valued matrix \( K \) (resp. \( M \)) generates the twisted (resp. untwisted) Reflection Equation Algebras, \( tRE_A q (A_{2n})^+ \) (resp. \( RE_A q (A_{2n}) \)) whose quantum group comodule structure and relation to twisted (resp. untwisted) quantum algebra \( U'_q(su_{2n+1}) \) (resp. \( U_q(su_{2n+1}) \)) shall be discussed in the sequel.

A.1 Symmetries of the RE’s and their relation to \( U_q(su_{2n+1}) \).

The three RE’s of interest enjoy the following (twisted) left-right (co)symmetries which are crucial for their applicability in an effective description of branes in WZW models (\( S \) is the antipode of the Hopf algebra \( SU_q(2n + 1) \)):

\[ K^- \mapsto t^T K^- s, \quad K^+ \mapsto (St) K^+ (Ss)^T, \]  
(A.4)

\[ M \mapsto tMSs, \]  
(A.5)

where

\[ \begin{align*}
R_{12} s_1 s_2 &= s_2 s_1 R_{12}, & R_{12} t_1 t_2 &= t_2 t_1 R_{12}, & R_{12} t_1 s_2 &= s_2 t_1 R_{12},
\end{align*} \]  
(A.6)

are the defining relations of (two copies of) the quantum group \( SU_q(2n + 1) \) associated to the \( \mathcal{R} \)-matrix \( R \).

Solutions to the three RE’s under study can straightforwardly be realised in terms of generators of the extended quantum universal enveloping algebra \( U_q(su_{2n+1}) \) through

\[ K^- = (L^+)^T C^- L^- \quad \text{and} \quad K^+ = (S L^+)^C (S L^-)^T, \]  
(A.7)

\[ M = L^+ M_0 S L^-, \]  
(A.8)

where \( C^\pm \) and \( M_0 \) denote respective (arbitrary) constant solutions (c-number-valued matrices) and \( L^\pm \) are the familiar FRT-operators ([17]). The existence of the homomorphisms thus defined
enables us to use the well-known representation theory of the quantum algebra $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ to induce a representation theory of the (t)REA’s. In particular, the relevant (specialised) representation theory of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ has been studied at some length in [14].

A.2 The two embeddings $\mathcal{U}'_q(\mathfrak{so}_{2n+1}) \hookrightarrow \text{tREA}_q(A_{2n})^\mp$.

The twisted quantum orthogonal algebra $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$, considered originally by Gavrilik and Klimyk in [21], is defined by the following commutation relations:

\begin{equation}
[\Pi_i, \Pi_j] = 0 \quad \text{if} \quad |i - j| > 1,
\end{equation}

\begin{equation}
\Pi_i^2 \Pi_j - [2]_q \Pi_i \Pi_j \Pi_i + \Pi_j \Pi_i^2 = -\Pi_j \quad \text{if} \quad |i - j| = 1,
\end{equation}

satisfied by its generators $\Pi_i$, $i \in 1, 2n + 1$. In the classical limit, $q \rightarrow 1$, the above relations reproduce the standard defining relations of $\mathcal{U}(\mathfrak{so}_{2n+1})$. They differ, on the other hand, from the defining relations of the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{so}_{2n+1})$ (of Drinfel’d and Jimbo) associated to the universal $\mathcal{R}$-matrix for $\mathfrak{so}_{2n+1}$ (e.g.[31]).

In addition to the above generators, we define after [18] the operators $\Pi_{\mp ij}$, $1 \leq i < j \leq 2n + 1$ through:

\begin{equation}
\Pi_{\mp i+1,i} := \Pi_i,
\end{equation}

\begin{equation}
\Pi_{\mp ji} := \Pi_{\mp jk} \Pi_{\mp ki} - q_{\mp 1} \Pi_{\mp ki} \Pi_{\mp jk} \quad \text{for arbitrary} \quad i < k < j.
\end{equation}

It is then a matter of straightforward algebra to verify that the elements of the two operator-valued solutions to (A.1)-(A.2) provide a realisation of the algebra of $\Pi_{\mp ij}$’s. More precisely, we have the identification:

\begin{equation}
K_{\mp ij}^+ = \lambda q^{2n-j} \Pi_{\mp ij}^- \quad , \quad K_{\mp ij}^- = -\lambda q^{2n+1-j} \Pi_{\mp ij}^+,
\end{equation}

establishing a homomorphism $\mathcal{U}'_q(\mathfrak{so}_{2n+1}) \hookrightarrow \text{tREA}_q(A_{2n})^\mp$. This, together with the explicit mappings $\text{tREA}_q(A_{2n})^\mp \rightarrow \mathcal{U}_q(\mathfrak{su}_{2n+1})$, (A.7), embeds $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ in $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ as the so-called coideal subalgebra ([19]). Its representation theory, both of classical and non-classical type, has been discussed in great detail in a series of papers [21, 22, 25, 27], also in relation to the representation theory of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$. An important conclusion following from that analysis is that we can effectively restrict to $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$-irreducible representations of the classical type as long as we are dealing with classical-type irreducible representations of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ (branching into the former).

References

[1] V. Schomerus, "Lectures on branes in curved backgrounds", Class.Quant.Grav. 19 (2002) 5781 [arXiv:hep-th/0209241].
A. Yu. Alekseev, A. Recknagel and V. Schomerus, "Non-commutative world-volume geometries: branes on $SU(2)$ and fuzzy spheres", JHEP 9909 (1999) 023 [arXiv:hep-th/9908040].

A. Yu. Alekseev, A. Recknagel and V. Schomerus, "Brane dynamics in background fluxes and non-commutative geometry", JHEP 0005 (2000) 010 [arXiv:hep-th/0003187].

A. Yu. Alekseev, A. Recknagel and V. Schomerus, "Non-commutative gauge theory of twisted D-branes", Nucl. Phys. B646 (2002) 127 [arXiv:hep-th/0205123].

A. Yu. Alekseev and V. Schomerus, "D-branes in the WZW model", Phys. Rev. D 60 (1999) 061901 [arXiv:hep-th/9812193].

K. Gawedzki, "Conformal field theory: a case study" [arXiv:hep-th/9904145].

C. Bachas, M. R. Douglas and C. Schweigert, "Flux stabilization of D-branes", JHEP 0005 (2000) 048 [arXiv:hep-th/0003037].

J. Pawelczyk, "SU(2) WZW D-branes and their noncommutative geometry from DBI action", JHEP 0008 (2000) 006 [arXiv:hep-th/0003057].

P. Bordalo, S. Ribault and C. Schweigert, "Flux stabilization in compact groups", JHEP 0110 (2001) 036 [arXiv:hep-th/0108201].

C. Bachas and M. R. Gaberdiel, "Loop operators and the Kondo problem", JHEP 0411 (2004) 065 [arXiv:hep-th/0411067].

S. Fredenhagen and V. Schomerus, "Branes on group manifolds, gluon condensates, and twisted K-theory", JHEP 0104 (2001) 007 [arXiv:hep-th/0012164].

J. M. Maldacena, G. W. Moore and N. Seiberg, "D-brane instantons and K-theory charges", JHEP 0111 (2001) 062 [arXiv:hep-th/0108100].

P. Bouwknegt, P. Dawson and A. Ridout, "D-branes on group manifolds and fusion rings", JHEP 0212 (2002) 065 [arXiv:hep-th/0210302].

V. Braun, "Twisted K-theory of Lie groups", JHEP 0403 (2004) 029 [arXiv:hep-th/0305178].

M.R. Gaberdiel and T. Gannon, "The charges of a twisted brane", JHEP 0401 (2004) 018 [arXiv:hep-th/0311242].

A. Recknagel and V. Schomerus, "D-branes in Gepner models", Nucl. Phys. B531 (1998) 185 225 [arXiv:hep-th/9712186].

A. Recknagel, V. Schomerus, "Boundary deformation theory and moduli spaces of D-branes", Nucl. Phys. B545 (1999) 233 [arXiv:hep-th/9811237].

L. Birke, J. Fuchs and C. Schweigert, "Symmetry breaking boundary conditions and WZW orbifolds", Adv. Theor. Math. Phys. 3 (1999) 671 [arXiv:hep-th/9905038].

J. Pawelczyk and H. Steinacker, "Matrix description of D-branes on 3-spheres" JHEP 0112 (2001) 018 [arXiv:hep-th/0107265].

J. Pawelczyk and H. Steinacker, "A quantum algebraic description of D-branes on group manifolds" Nucl. Phys. B638 (2002) 433–458 [arXiv:hep-th/0203110].

J. Pawelczyk and H. Steinacker, "Algebraic brane dynamics on SU(2): excitation spectra", JHEP 0312 (2003) 010 [arXiv:hep-th/0305226].

A. Pressley and V. Chari, Guide To Quantum Groups (Cambridge University Press, Cambridge, 1995).
[14] J. Pawelczyk, R. R. Suszek, "Quantum Matrix Models for Simple Current Orbifolds"
[arXiv:hep-th/0310289].

[15] E. Sklyanin, "Boundary conditions for integrable quantum systems", J. Phys. A21 (1988) 2375.

[16] P. P. Kulish, R. Sasaki and C. Schwiebert, "Constant solutions of reflection equations and quantum
groups", J.Math.Phys. 34 (1993) 286–304 [arXiv:hep-th/9205039].
P. P. Kulish and E. K. Sklyanin, "Algebraic structures related to reflection equations", J.Phys. A25
(1992) 5963–5976, [arXiv:hep-th/9209054].
P. P. Kulish and R. Sasaki, "Covariance properties of reflection equation algebras", Prog.Theor.Phys.
89 (1993) 741–762, [arXiv:hep-th/9212007].

[17] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, "Quantization of Lie groups and Lie
algebras", Leningrad Math.J. 1 (1990) 193–225, Algebra Anal. 1 (1989) 178–206.

[18] M. Noumi, T. Umeda, M. Wakayama, "Dual pairs, spherical harmonics and a Capelli identity in
quantum group theory", Compos. Math. 104 (1996) 227.

[19] M. Noumi, "Macdonald’s symmetric polynomials as zonal spherical functions on some quantum
homogeneous spaces" [arXiv:math.QA/9503224].
M. Noumi, T. Sugitani, "Quantum symmetric spaces and related q-orthogonal polynomials"
[arXiv:math.QA/9503225].

[20] G. W. Delius, N. J. MacKay, B. J. Short, "Boundary remnant of Yangian symmetry and the structure
of rational reflection matrices", Phys.Lett. B522 (2001) 335, Erratum, ibid. B524 (2002) 401
[arXiv:hep-th/0109115].

[21] A. M. Gavrilik, A. U. Klimyk, "q-Deformed Orthogonal and Pseudo-Orthogonal Algebras and Their
Representations", Lett.Math.Phys. 21 (1991) 215 [arXiv:math.QA/0203201].

[22] A. M. Gavrilik, N. Z. Iorgov, "q-Deformed algebras \( U_q(so_n) \) and their representations"
[arXiv:q-alg/9709036].
A. U. Klimyk, "Classification of irreducible representations of the q-deformed algebra \( U'_q(so_n) \)"
[arXiv:math.QA/0110038].
N. Z. Iorgov, A. U. Klimyk, "The nonstandard deformation \( U'_q(so_n) \) for q a root of unity"
[arXiv:math.QA/0007105].

[23] A. M. Gavrilik, N. Z. Iorgov, "On the Casimir Elements of q-Algebras \( U'_q(so_n) \) and Their Eigenvalues
in Representations" [arXiv:math.QA/9911201].

[24] M. Havlíček, A. U. Klimyk, S. Pošta, "Central elements of the algebras \( U'_q(so_m) \) and \( U'_q(iso_m) \)",
Czech.J.Phys. 50 (2000) 79.

[25] M. Havlíček, S. Pošta, "On the classification of irreducible finite-dimensional representations of
\( U'_q(so_3) \) algebra", J.Math.Phys. 42 (2001) 472.

[26] A. U. Klimyk, "The nonstandard q-deformation of enveloping algebra \( U(so_n) \): results and problems",
Czech.J.Phys. 51 (2001) 331.

[27] N. Z. Iorgov, "On tensor products of representations of the non-standard q-deformed algebra
\( U'_q(so_n) \)", J. Phys. A34 (2001) 3095.
[28] T. Quella, "Branching rules of semi-simple Lie algebras using affine extensions", *J.Phys.* **A35** (2002) 3743 [arXiv:math-ph/0111020].

[29] T. Quella, I. Runkel, C. Schweigert, "An algorithm for twisted fusion rules", *Adv.Theor.Math.Phys.* **6** (2002) 197 [arXiv:math.QA/0203133].

[30] S. Stanciu, "An illustrated guide to D-branes in SU(3)" [arXiv:hep-th/0111221].

[31] A. Klimyk, K. Schmüdgen, *Quantum Groups and Their Representations* (Springer-Verlag, 1998).