A COMPARISON THEOREM FOR FINSLER SUBMANIFOLDS AND ITS APPLICATIONS

WEI ZHAO

Abstract. In this paper, we consider the conormal bundle over a submanifold in a Finsler manifold and establish a volume comparison theorem. As an application, we derive a lower estimate for length of closed geodesics in a Finsler manifold. In the reversible case, a lower bound of injective radius is also obtained.

1. Introduction

In Riemannian geometry it is an important subject to investigate the geometry of submanifolds. And there are many important local and global results, which in turn lead to a better understanding on Riemann manifolds [Che, Fr, HK, L, PT, W]. For example, the Heintze-Karcher comparison theorem [HK] plays a very important role in the global differential geometry of Riemann manifolds, which says that the upper bound for the volume of a closed Riemannian manifold can be estimated in terms of the the volume and the mean curvature of an arbitrary closed submanifold, the diameter and a lower bound for the section curvature. One of it applications is the lower estimate on the length of sample closed geodesics in a closed Riemannian manifold [Ch, HK]. More precisely, if $(M, g)$ is a closed Riemannian $m$-dimensional manifold with the section curvature $K \geq \delta$, $\text{diam}(M) \leq d$ and $\text{Vol}(M) \geq V$, then for any sample closed geodesic $\gamma$ in $M$, its length satisfies

$$L_g(\gamma) \geq \frac{(m-1)V}{c_m - 2s_\delta^{m-1} \left( \min \left\{ d, \frac{\pi}{\sqrt{\delta}} \right\} \right)},$$

where $c_m := \text{Vol}(S^m)$, $\pi/\sqrt{\delta} := +\infty$ if $\delta \leq 0$, and $s_\delta(t)$ is the unique solution to $y'' + \delta y = 0$ with $y(0) = 0$ and $y'(0) = 1$. Combining this with Kingenberg’s theorem [K], one can obtain the injectivity radius estimate of Cheeger [Ch] without using Toponogov’s comparison theorem. See [Bu, GR, Mo, MJ, Sc], etc., for more details on the Heintze-Karcher comparison theorem.

Finsler geometry, a natural generalization of Riemannian geometry, was initiated by Finsler [F] from considerations of regular problems in the calculus of variations. Recently, the geometry of Finsler submanifolds has been developed tremendously, especially in the aspects of Finsler minimal submanifolds and Minkowski submanifolds [HS, Sh2, Sh3, ST]. However, the geometry of Finsler submanifolds is much different from the one of Riemannian submanifolds. For instance, there exit
totally geodesic submanifolds which are not minimal for the Busemann-Hausdorff measure but are minimal for the Holmes-Thompson measure [AB].

Now we consider the normal bundle of a Finsler submanifold. Let \((M, F)\) be a forward complete Finsler \(m\)-manifold and let \(N\) be a connected \(k\)-dimensional submanifold of \(M, 0 \leq k \leq m\). According to [Ru, Sh5], the "normal bundle" \(\mathcal{V}N\) of \(N\) in \(M\) is defined as \(\mathcal{V}N := \cup_{x \in N} \mathcal{V}_x N\), where \(\mathcal{V}_x N := \{0\} \cup \{n \in T_x M : n \neq 0, g_n(n, X) = 0, \forall X \in T_x N\}\). It is a generally recognized principle that the normal bundle \(\mathcal{V}N\) of \(N\) in \(M\) carries much geometric information. However, in general case, \(\mathcal{V}N\) is not a vector bundle but a cone bundle [Ru, Sh5]. Apparently, it is rather hard to handle due to nonlinearity of \(\mathcal{V}N\). Recall the Legendre transformation \(\mathcal{L} : TM \rightarrow T^*M\) is a homeomorphism [BCS, Sh2]. It should be remarked that \(\mathcal{L}\) is a diffeomorphism (or isomorphism) if and only if \(F\) is Riemannian. The conormal bundle \(\mathcal{V}^*N\) of \(N\) in \(M\) is defined as the homeomorphic image of \(\mathcal{V}N\) under \(\mathcal{L}^{-1}\).

Clearly, \(\mathcal{V}^*N\) coincides with the original definition in the Riemannian case. The study of the (co-)normal bundle of a Finsler submanifold is still at its infant stage. Several people have made some fundamental contributions to this subject from various points of view [Hu, Da, Ma, Ru, Sh2, Sh3], etc.

The purpose of this paper is to investigate the conormal bundle of a Finsler submanifold and establish the Heintze-Karcher comparison theorem in Finsler geometry for both the Busemann-Hausdorff measure and the Holmes-Thompson measure.

Given \(\xi \in \mathcal{V}^*N\backslash \{0\}\), let \(H_\xi\) denote the co-mean curvature along \(\xi\), which is given explicitly in Sec.3. First, we have the following theorem.

**Theorem 1.1.** Let \((M, F)\) be a closed Finsler \(m\)-manifold with uniform constant \(\Lambda_F\) and diameter \(d\) and let \(N\) be a connected \(k\)-dimensional submanifold of \(M, 0 \leq k \leq m\). Suppose the flag curvature \(K \geq \delta\).

1. If \(k = 0\), i.e., \(N = \{x\}\), then
   \[
   \mu(M) \leq \int_{S_x M} e^{-\tau(\gamma_s(t))} d\nu_x(y) \int_0^d g_\delta^{m-1}(t) dt,
   \]
   where \(\mu\) is any volume form on \(M\) and \(d\nu_x\) is the Riemannian volume on \(S_x M\) induced by \(g_x\).

2. If \(k \geq 1\), then
   \[
   \mu(M) \leq c_{m-k-1} \cdot \Lambda_F^{(3m+k)/2} \cdot \bar{\mu}(N) \cdot \int_0^{\min\{d, \zeta(\xi_0)\}} \left( g_\delta^k - \frac{H_0}{k} g_\delta \right)^k (t) \cdot g_\delta^{m-k-1}(t) dt.
   \]
   where \(\mu\) and \(\bar{\mu}\) denote the Busemann-Hausdorff volume or the Holmes-Thompson volume on \((M, F)\) and \((N, F|_N)\), respectively and \(\xi_0 := \min_{\xi \in \mathcal{V}^*N\backslash \{0\}} H_\xi\) and \(\zeta(\xi_0)\) is the first positive zero of \(\left( g_\delta^k - \frac{H_0}{k} g_\delta \right) (t)\).

Let \(\pi : TM \rightarrow M\) be the tangent bundle over \(M\). Unlike Riemannian case, all the connections in Finsler geometry are defined on \(\pi^*TM\). Given a local coordinate system \((x^i, y^i)\) of \(TM\), let \(\nabla\) (resp. \(\Gamma^k_{ij}\)) denote the Chern connection (resp. connection coefficients), i.e.,

\[
\Gamma^k_{ij}(x, y) \frac{\partial}{\partial x^k} := \nabla_{\left(\frac{\partial}{\partial x^j}\right)} \left(\frac{\partial}{\partial x^i}\right) := \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \forall (x, y) \in TM\backslash \{0\}.
\]

In general, \(\Gamma^k_{ij}(x, y) \neq \Gamma^k_{ij}(x, z)\) if \(y \neq z\). Shen in [Sh2] introduce the \(T\)-curvature to measure the difference of Chern connection coefficients between two points in
More precisely, the T-curvature $T : TM \setminus 0 \times TM \setminus 0 \to \mathbb{R}$ is defined by
\[ T_y(v) := g_y(\nabla_v V, y) - g_y(\nabla'_v V, y), \quad v \in T_x M, \]
where $Y$ is a geodesic field such that $Y|_x = y$ and $V$ is any extension of $v$. $T = 0$ if and only if $(M, F)$ is Berwald space. In this case, $(M, F)$ is modeled on a single Minkowski space and $\Gamma_{ij}^k$ coincide with some Riemannian metric's Christoffel symbols (cf. [BCS, Sh2, Sz]).

We now consider the special case when $N = \gamma$ is a geodesic. It should be remarked that neither the Busemann-Hausdorff volume or the Holmes-Thompson volume of $\gamma$ is equal to the length of $\gamma$ unless $F|_{\gamma}$ is reversible. But we still have the following estimate.

**Corollary 1.2.** Let $(M, F)$ be a closed Finsler manifold with uniform constant $\Lambda_F$ and diameter $d$. If $K \geq \delta$ and $T \leq l$, then for any simple closed geodesic $\gamma$ in $M$,

\[ L_F(\gamma) \geq \frac{\mu(M)}{c_{m-2}(3m+1)/2 \left[ s_0^{m-1} \left( \min\left\{ d, \frac{\pi}{2\sqrt{\delta}} \right\} \right) \right] + l \int_0^d s_0^{m-1}(t)dt}, \]

where $\mu(M)$ is either the Busemann-Hausdorff volume or the Holmes-Thompson volume of $M$ and $L_F(\gamma)$ is the length of $\gamma$.

According to [Sh2, Lemma 12.2.5], Klingenberg lemma [Kl] can be extended to the case of a reversible Finsler metric. This together with the corollary above furnishes

**Corollary 1.3.** Let $(M, F)$ be a closed reversible Finsler manifold with uniform constant $\Lambda$, diameter $\leq d$, $\mu(M) \geq V$, $|K| \leq \delta$ and $T \leq l$, where $\mu(M)$ is either the Busemann-Hausdorff volume or the Holmes-Thompson volume of $M$. Then

\[ i_M \geq \min \left\{ \frac{\pi}{\sqrt{\delta}}, \frac{V}{2c_{m-2}(3m+1)/2 \left[ s_0^{m-1} \left( \min\left\{ d, \frac{\pi}{2\sqrt{\delta}} \right\} \right) \right] + l \int_0^d s_0^{m-1}(t)dt} \right\}. \]

Randers metrics are natural and important Finsler metrics which are defined as the sum of a Riemannian metric and a 1-form.

**Theorem 1.4.** Let $(M, F)$ be a compact Randers manifold with $K \geq \delta$ and let $\gamma$ be a closed geodesic in $M$. Set $b := \sup_{x \in M} \|\beta\|_\alpha$ and $b_1 := \sup_{x \in M} \|\nabla\beta\|_\alpha$. Then

\[ L_F(\gamma) \geq \frac{(1 - b)^{m+2}}{c_{m-2} (1 + b)^2 s_0^{m-1}} \left[ s_0^{m-1} \left( \min\left\{ d, \frac{\pi}{2\sqrt{\delta}} \right\} \right) \right] + b_1 \left( 2b^3 + 5b^2 - 2b + 7 \right) \int_0^d s_0^{m-1}(t)dt, \]

where

\[ s_0^{m-1} \left( \min\left\{ d, \frac{\pi}{2\sqrt{\delta}} \right\} \right) = \frac{b_1 (2b^3 + 5b^2 - 2b + 7)}{2(1 - b)^3} \int_0^d s_0^{m-1}(t)dt, \]

and $\text{Vol}_\alpha$ is the Riemannian volume of $M$ induced by $\alpha$.

However, Finsler geometry is much more complicated than Riemannian geometry.
Example 1 (BCS). Let $M := \mathbb{S}^2 \times \mathbb{S}$. Let $\alpha$ be the canonical Riemannian product metric on $M$, that is, $\alpha = \sqrt{dr \otimes dr + \sin^2(r) d\theta \otimes d\theta + dt \otimes dt}$, where $(r, \theta)$ (resp. $t$) is the usual spherical coordinates on $\mathbb{S}^2$ (resp. $\mathbb{S}$). Choose a 1-form $\beta_\epsilon := \epsilon dt$, where $\epsilon \in [0,1)$. Note that $\beta_\epsilon$ is globally defined on $M$, even though the coordinate $t$ is not. Take $F_\epsilon = \alpha + \beta_\epsilon$. A direct calculation shows that $(M, F_\epsilon)$ satisfy

$$\mu_{HT}(M) = 8\pi^2, \quad K \geq 0, \quad \text{diam}(M) \leq 6\pi,$$

for all $\epsilon \in [0,1)$. Since $F_\epsilon$ is a Berwald metric, $\gamma(s) = (0,0,-s)$ is a closed geodesic of $F_\epsilon$. Clearly, $L_{F_\epsilon}(\gamma) = \pi(1-\epsilon) \to 0$ as $\epsilon \to 1$.

2. Preliminaries

In this section, we recall some definitions and properties concerned with Finsler manifolds. See [BCS, Sh5] for more details.

Let $(M, F)$ be a (connected) Finsler manifold with Finsler metric $F : TM \to [0, \infty)$. Define $S_x M := \{y \in T_x M : F(x, y) = 1\}$ and $SM := \cup_{x \in M} S_x M$. Let $(x, y) = (x^i, y^j)$ be local coordinates on $T M$, and let $\pi : TM \to M$ and $\pi_1 : SM \to M$ be the natural projections. Denote by $c_{n-1}$ the volume of the Euclidean unit $(n-1)$-sphere. Define

$$\ell^i := \frac{y^j}{F}, \quad g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad A_{ijk}(x, y) := \frac{F}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k},$$

$$\gamma^i_{jk} := \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \quad N^i_j := (\gamma^i_{jk} \ell^k - A^k_{ij}) \ell^s. \ell^s : F.$$

The Chern connection $\nabla$ is defined on the pulled-back bundle $\pi^* TM$ and its forms are characterized by the following structure equations:

1. Torsion freeness: $dx^i \wedge \omega^i_j = 0$;
2. Almost $g$-compatibility: $dg_{ij} - g_{ij} \omega^k_i - g_{ik} \omega^k_j = 2 \Delta_{jk} \omega^s_i dx^s$.

From above, it’s easy to obtain $\omega^i_j = \Gamma^i_{jk} dx^k$, and $\Gamma^i_{jk} = \Gamma^i_{kj}$.

The curvature form of the Chern connection is defined as

$$\Omega^i_j := d\omega^i_j - \omega^i_j \wedge \omega^k_j := \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + P^i_{jkl} dx^k \wedge \frac{dy^j + N^j_i dx^s}{F}.$$

Given a non-zero vector $V \in T_x M$, the flag curvature $K(y, V)$ on $(x, y) \in TM \setminus 0$ is defined as

$$K(y, V) := \frac{V^i V^j V^k R_{jkl} y^i y^j y^k}{g_y(y, y)g_y(V, V) - |g_y(y, V)|^2},$$

where $R_{jkl} := g_{is} R^s_{jkl}$. And the Ricci curvature of $y$ is defined by

$$\text{Ric}(y) := \sum_i K(y, e_i),$$

where $e_1, \ldots, e_n$ is a $g_y$-orthonormal base on $(x, y) \in TM \setminus 0$.

Given $y \in T_x M \setminus 0$, extend $y$ to a geodesic field $Y$ is a neighborhood of $x$ (i.e., $\nabla^Y V = 0$), and define $T$-curvature $T$ as

$$T_y(v) := g_y(\nabla^Y V, y) - g_y(\nabla^Y V, y), \quad v \in T_x M,$$

where $V$ is a vector field with $V_x = v$. In any local coordinates $(x^i, y^j)$,

$$T_y(v) = y^j g_{jk}(y) \{ \Gamma^k_{jm}(v) - \Gamma^k_{jm}(y) \} v^m.$$
We say \( T \leq l \) if

\[
T_y(v) \leq l \left[ \sqrt{g_y(v,v)} - g_y \left( v, \frac{y}{F(y)} \right) \right]^2 F(y),
\]

where \( y, v \in TM \setminus 0 \). Similarly, we define the bound \( T \geq l \).

Given \( y \neq 0 \), we always use \( \gamma_y(t) \) to denote a constant speed geodesic with \( \dot{\gamma}_y(0) = y \). Given any the volume form \( d\mu \) on \( M \). In a local coordinate system \((x^i)\), express \( d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n \). For \( y \in T_yM \setminus 0 \), define the distorsion of \((M, F, d\mu)\) as

\[
\tau(y) := \log \frac{\sqrt{\text{det}(g_{ij}(x,y))}}{\sigma(x)}.
\]

And we define the S-curvature \( S \) as

\[
S(y) := \frac{d}{dt} \tau(\dot{\gamma}_y(t)) \bigg|_{t=0},
\]

Two volume forms used frequently are the Busemann-Hausdorff volume form \( d\mu_{BH} \) and the Holmes-Thompson volume form \( d\mu_{HT} \), respectively. Given a local coordinate system \((x^i)\), \( d\mu_{BH} = \sigma_{BH}(x)dx^1 \wedge \cdots \wedge dx^n \) and \( d\mu_{HT} = \sigma_{HT}(x)dx^1 \wedge \cdots \wedge dx^n \), where

\[
\sigma_{BH}(x) = \frac{\text{Vol}^n(\mathbb{E}^n)}{\text{Vol}^n \{ y \in T_xM : F(x,y) < 1 \}},
\]

\[
\sigma_{HT}(x) = \frac{1}{e_{n-1}} \int_{S^n_{,M}} \det g_{ij}(x,y) \left( \sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge dy^i \wedge \cdots \wedge dy^n \right).
\]

The reversibility \( \lambda_F \) of \((M, F)\) is defined by \((Ra)\)

\[
\lambda_F := \sup_{(x,y) \in TM \setminus 0} \frac{F(x, -y)}{F(x, y)}.
\]

Clearly \( \lambda_F \geq 1 \) and \( \lambda_F = 1 \) if and only if \( F \) is reversible. The uniformity constant of \((M, F)\) is defined by \((E)\)

\[
\Lambda_F := \sup_{X, Y, Z \in SM} \frac{g_X(Y, Y)}{g_Z(Y, Y)}.
\]

Clearly, \( \lambda_F \leq \sqrt{\Lambda_F} \) and \( \Lambda_F = 1 \) if and only if \( F \) is Riemannian.

By \((Sh2)\), the Legendre transformation \( \mathcal{L} : TM \to T^*M \) is defined by

\[
\mathcal{L}(Y) = \begin{cases} 0, & Y = 0, \\ g_Y(Y, \cdot), & Y \neq 0. \end{cases}
\]

For any \( x \in M \), the Legendre transformation is a smooth diffeomorphism from \( T_xM \setminus \{0\} \) onto \( T^*_xM \setminus \{0\} \).

3. Conormal bundle

Throughout this paper, we assume that \((M, F)\) is a forward complete Finsler \( m \)-manifold and \( i : N \to M^m \) is a connected \( k \)-dimensional submanifold of \( M \), \( 0 \leq k \leq m \). The rules that govern our index gymnastics are as follows: \( i, j \) run from 1 to \( m \). \( \alpha, \beta \) run from 1 to \( k \). \( A, B \) run from \( k + 1 \) to \( m \). \( g, h \) run from \( k \) to \( m - 1 \). \( a, b \) run from 1 to \( k - 1 \).

According to \((Ru, Sh2)\), the "normal bundle" \( \mathcal{V}N \) of \( N \) is defined as

\[
\mathcal{V}N := \{ n \in TM : n = 0 \text{ or } g_n(n, TN) = 0 \}.
\]
It is remarkable that if $k \geq 2$, then $\mathcal{V}N$ is not a vector bundle unless $F$ is Riemannian. Even in the case that $k = 1$, $\mathcal{V}N$ maybe not a vector bundle unless $F$ is reversible.

Consider the following subbundle of $T^*M$

$$\mathcal{V}^*N := \{ \omega \in T^*M : i^*\omega = 0 \}.$$  

It is easy to see that $\mathcal{V}^*N = \mathcal{L}(\mathcal{V}N)$, where $\mathcal{L} : TM \to T^*M$ is the Legendre transformation. Note that $\mathcal{L}$ is a homeomorphism from $TM$ to $T^*M$ and a diffeomorphism from $TM\setminus 0$ to $T^*M\setminus 0$. Hence, $\mathcal{V}^*N$ is called the conormal bundle over $N$ in $M$.

**Example 2.** Let $F(y) = \alpha(y) + \beta(y)$ be a Randers norm on a vector space $V$, where $\alpha$ is an Euclidean norm and $\beta$ is a 1-form. Let $N = \{ vt + w : t \in \mathbb{R} \}$ be a straight line in $V$, where $v \neq 0$ and $w$ are constant vectors in $V$. Clearly, $\mathcal{V}^*N = \{ \xi : \langle \xi, v \rangle = 0 \}$ is a bundle. However, a direct calculation shows that $\mathcal{V}N = \{ n : \alpha(n)\beta(v) = -\langle v, n \rangle \}$, where $\langle \cdot, \cdot \rangle$ is the inner product induced by $\alpha$. Hence, $\mathcal{V}N$ is a vector bundle if and only if $\beta = 0$, i.e., $F = \alpha$.

Let $\pi : \mathcal{V}^*N \to N$ denote the bundle projection. Given $\eta \in \mathcal{V}^*N$, set $x = \pi(\eta)$. There exist local coordinate systems $(U_N, u^\alpha)$ and $(U_M, x^i)$ of $x$ and $i(x)$, respectively, such that $U_N \subset U_M$, $u^\alpha = x^\alpha$ and $x^\alpha|_{U_N} = 0$. Hence, for each $\xi \in \pi^{-1}(U_N)$, $\xi = \xi_A dx^A$ and therefore, $\pi^{-1}(U_N) \cong U_N \times \mathbb{R}^{n-k}$. We call $(u^\alpha, \xi_A)$ the (local) canonical coordinates on $\mathcal{V}^*N$.

For simplicity, set $\mathcal{V}^*_*N := \pi^{-1}(x)$. And we always identify $N$ with the zero section of $\mathcal{V}^*N$.

**Definition 3.1.** Given a point $x \in N$ and $\xi \in \mathcal{V}^*_*N\setminus 0$, the co-second fundamental form of $N$ along $\xi$ in $M$ is defined as

$$h_\xi(X, Y) := \langle \xi, \nabla_X Y \rangle = g_n(n, \nabla_X Y), \forall X, Y \in T_xN,$$

where $n := \mathcal{L}^{-1}(\xi)$ and $Y$ is any extension of $Y$ to a tangent vector field on $N$.

By a direct calculation, one can check that $h$ is well-defined and $h_\xi : T_xM \otimes T_xM \to \mathbb{R}$ is a symmetric bilinear form. Let $\Lambda^\xi$ denote the normal curvature defined in [Sh3]. Then $h_\xi(X, X) = -\Lambda^\xi(X) - g_n(X)$.

**Definition 3.2.** Given any $\xi \in \mathcal{V}^*_*N\setminus 0$, co-Weingarten map $\mathcal{A}^\xi : T_xN \to T_xN$ is defined as

$$\mathcal{A}^\xi(X) := -(\nabla_X \bar{n})^\top,$$

where $\bar{n} = \mathcal{L}^{-1}(\tilde{\xi})$, $\tilde{\xi}$ is an extension of $\xi$ to a co-normal vector field on $N$, and the superscript $^\top$ denotes projection to $T_xN$ by $g_n$.

**Proposition 3.3.** $\mathcal{A}^\xi$ is well-defined and

$$g_n(Y, \mathcal{A}^\xi(X)) = h_\xi(X, Y), \forall X, Y \in T_xM,$$

where $n = \mathcal{L}^{-1}(\xi)$.

**Proof.** Choose a local canonical coordinates system $(u^\alpha, \xi_A)$ around $\xi$. Let $\bar{n}$ be defined above. Thus,

$$(2.1) \quad \bar{n} = g_n^{\alpha j} \xi_A \frac{\partial}{\partial x^j} =: Z^j \frac{\partial}{\partial x^j}, \quad \nabla_X \bar{n} = \left[ X^\alpha \frac{\partial Z^j}{\partial x^\alpha} + \Gamma^j_{\alpha \beta}(n) Z^i X^\alpha \right] \frac{\partial}{\partial x^j},$$
and
\[ \frac{\partial Z^i}{\partial x^\alpha} = \xi_A \frac{\partial g^{*A}J}{\partial x^\alpha}(\xi) + g^{*A} \frac{\partial \tilde{\xi}_A}{\partial x^\alpha}. \]

Hence,
\[ g_n \left( X^\alpha \frac{\partial Z^i}{\partial x^\alpha} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j} \right) = g^*_{\beta j}(\xi) \left( \frac{\partial g^{*A}J}{\partial x^\alpha}(\xi) \right) \xi_A X^\alpha. \]

It follows from (2.1) and (2.2) that \( \mathfrak{F}^\xi \) is well-defined.

Given \( X, Y \in T_x N \). Extend \( Y \) to a tangent vector field \( \bar{Y} \) on \( N \). Then
\[ g_n(\mathfrak{F}^\xi(X), Y) = g_n(-\nabla^X \bar{\xi}, Y) = g_n(n, \nabla^X \bar{Y}) = h_\xi(X, Y). \]

\[ \Box \]

Given \( \xi \in \nu^*_x N \setminus \{0\} \). Note that \( i^*g_n \) is a Euclidean metric on \( T_x^0 N \), where \( n = L^{-1}(\xi) \). We define the co-mean curvature of \( N \) along \( \xi \) by
\[ H_\xi := \text{tr}_{i^*g_n} h_\xi. \]

It is easy to see that \( H_\xi = \sum \lambda_\alpha \), where \( \lambda_\alpha \) is the eigenvalues of \( \mathfrak{F}^\xi \). In the Riemannian case, \( h_\xi \) is the second fundamental form \( h_n \) and \( \mathfrak{F}^\xi \) is the Weingarten map \( \mathfrak{D}^n \).

Given \( \xi \in \nu^*_x N \) with \( F^*(\xi) = 1 \), let \( n := L^{-1}(\xi) \), \( T = \gamma_n(t) \), \( n^\perp = \{ X \in T_x M : g_n(n, X) = 0 \} \) and \( T^\perp N = \{ X \in T_x M : g_n(X, T_x N) = 0 \} \). The collection \( \mathfrak{T} \) of transverse Jacobi fields along the geodesic \( \gamma_n(t) \), \( t \in [0, a] \), is defined by
\[ \mathfrak{T} := \{ J : J \text{ is a Jacobi field, } g_T(T, J) = 0, J(0) \in T_x N, (\nabla^T T)J(0) + \mathfrak{F}^\xi(J(0)) \in T_x^\perp N \}. \]

It is easy to see that \( \mathfrak{T} \) is a vector space and \( (\nabla^T T)J(0) \in n^\perp \). A similar argument to the one given in [C] p. 141 shows that \( \dim(\mathfrak{T}) = m - 1 \).

**Example 3.** Let \( N, M \) and \( \gamma_n \) be as above. If the flag curvature \( K_F(T; \cdot) = k \) and \( \mathfrak{F}^\xi = \lambda \cdot \text{id} \), then the transverse Jacobi field \( J \) has the form
\[ J(t) = [s_k' - \lambda s_k](t)E(t) + s_k(t)F(t), \]
where \( E(t) \) and \( F(t) \) are two parallel vector fields along \( \gamma_n \) such that \( E(0) \in T_x N \) and \( F(0) \in T_x^\perp N \cap n^\perp \).

Let \( \mathfrak{X} \) denote the collection of all vector fields \( X \) along \( \gamma_n \) such that \( g_T(T, X) = 0 \) and \( X(0) \in T_x N \) and let \( \mathfrak{X}_0 \) consist of those elements of \( \mathfrak{X} \) that vanish at \( t = a \). On \( \mathfrak{X} \), the index is defined by
\[ I(X, Y) := -h_\xi(X(0), Y(0)) + \int_0^a g_T(\nabla^T_T X, \nabla^T_T Y) + R_T(T, X, T, Y) dt. \]

**Definition 3.4.** Let \( N, M \) and \( \gamma_n \) be as above. A point \( \gamma_n(t) \) is said to be focal to \( N \) along \( \gamma \) if there exists a nontrivial transverse Jacobi field \( J \) such that \( J(t) = 0 \).

Then we have the following lemma.

**Lemma 3.5.** Given any \( X \in \mathfrak{X}_0 \). If \( \gamma_n(t) \) has not focal points along \( \gamma_n \) on \( (0, a) \) to \( N \), then \( I(X, X) \geq 0 \) with equality if and only if \( X = 0 \).

**Proof.** From assumption, there exists \( n - 1 \) transverse Jacobi fields \( J_i \) such that \( \{ T, J_i \} \) is a frame field along \( \gamma_n \). We can suppose that \( X(t) = f^i(t)J_i(t) \) where \( f^i(a) = 0 \). Set \( A := (f^i)' \cdot J_i \) and \( B := f^i \cdot \nabla^T_T J_i \). Then \( g_T(\nabla^T_T X, \nabla^T_T X) = \).
that (also see [BCS, p. 180])

$$g_T(A, A) + g_T(B, B) + 2g_T(A, B).$$

Using the Jacobi equation, one can easily check

that (also see [BCS, p. 180])

$$g_T(B, B) + R_T(T, X, T, X)$$

(2.3) $$= \frac{d}{dt} \left[ f^i f^j g_T(\nabla^T_T J_i, J_j) - f^i (f^j)' g_T(\nabla^T_T J_i, J_j) - f^i (f^j)' g_T(\nabla^T_T J_i, J_j) \right].$$

The Lagrange identity ([BCS, p. 135]) yields

$$g_T(\nabla^T_T J_i, J_j) - g_T(J_i, \nabla^T_T J_j) = g_T(\nabla^T_T J_i, J_j)(0) - g_T(J_i, \nabla^T_T J_j)(0).$$

Recall that $g_T(\nabla^T_T J_i, J_j)(0) + \mathfrak{A}(J_j(0)) \in T^*_p N$ and $T(0) = n$. Hence,

(2.4) $$g_T(J_i(0), (\nabla^T_T J_j)(0)) = g_T(J_i(0), -\mathfrak{A}(J_j(0))) = -h_n(J_i(0), J_j(0)),$$

which implies that $g_T(\nabla^T_T J_i, J_j) = g_T(J_i, \nabla^T_T J_j)$. Thus, from (2.3), (2.4) and $f^i(a) = 0$, we have

$$I(X, X) = \int_0^a g_T(A, A) dt \geq 0,$$

with equality $A = 0$, i.e., $X = 0$. \hfill \Box

Using the lemma above, it is not hard to show

**Theorem 3.6.** Suppose that $\gamma_n(t)$ has not focal points along $\gamma_n$ on $(0, a]$ to $N$. Given $X \in X$, let $J$ denote the unique transverse Jacobi field along $\gamma_n$ such that $J(a) = X(a)$. Then $I(X, X) \geq I(J, J)$ with equality if and only if $X = J$.

The proof of the following theorem is almost the same as the one of [BCS, Proposition 7.4.1]. Hence, we omit it here.

**Theorem 3.7.** Suppose that some point $\gamma_n(t_0)$, $0 < t_0 < a$ is focal to $N$ along $\gamma_n$. Then there is $U \in X$ such that $I(U, U) < 0$.

**Remark 1.** Given an arbitrary point $p \in M \setminus N$, there exists a unit speed minimizing geodesic $\gamma_n$ from $N$ to $p$. A simple first variation argument yields $\xi := L(n) \in \mathcal{V}^* N$. If $N$ has a focal point $\gamma_n(t_0)$ along $\gamma_n$, then $d(N, p) \leq t_0$ follows from Lemma 3.5, Theorem 3.7 and the second variation of arc length formula.

4. Conormal exponential map

We define the conormal exponential map $\text{Exp}^\xi : \mathcal{V}^* N \to M$ by

$$\text{Exp}^\xi(\xi) := \exp_{\pi(t)}(\mathcal{L}^{-1}(\xi)).$$

Let $S^* M := \{ \omega \in T^* M : F^*(\omega) = 0 \}$ and $\mathcal{V}^* SN := S^* M \cap \mathcal{V}^* N$. Now we have the following

**Theorem 4.1.** For each $\eta \in \mathcal{V}^* SN$, there exists a small $\epsilon(\eta) > 0$ and an open neighborhood $W$ of $\eta$ in $\mathcal{V}^* SN$ such that $\text{Exp}^\xi_{t \xi}$ is nonsingular for all $\xi \in W$ and $t \in (0, \epsilon(\eta))$. In particular, $\text{Exp}^\xi$ is $C^1$ on $N \subset \mathcal{V}^* N$ if and only if $F$ is Riemannian.

**Proof.** For the sake of clarity, we use $(x, \xi)$ to denote a point $\xi \in \mathcal{V}^* N$. Given $(x_0, \eta_0) \in \mathcal{V}^* SN \subset \mathcal{V}^* N$. Let $(U_N, u^a)$ and $(U_M, x^i)$ be two local coordinate systems around $x_0$ and $i(x_0)$, respectively, such that $x^a|_{U_N} = u^a$ and $x^A|_{U_N} = 0$.

We can choose a small $\delta > 0$ such that $\text{Exp}^\xi(D) \subset U_M$, where $D = \{ (x, \eta) : t \in$
\[ [0, \delta], (x, \eta) \in V^* SU_N \}. \] Let \((x^i, y^j)\) and \((u^\alpha, \xi_A)\) be local canonical coordinates on \(TM\) and \(V^* N\), respectively. For each \((x, t \eta) \in D\), we have

\[
\text{Exp}^c_{* (x, t \eta)} \frac{\partial}{\partial u^\alpha} = \frac{\partial \text{exp}(x, L^{-1}(\xi))}{\partial u^\alpha} \bigg|_{x, \xi = t \eta} = \left[ \delta^i_\alpha + H(t, x, \eta)^i_\alpha \right] \frac{\partial}{\partial x^i},
\]

where

\[
H(t, x, \eta)^i_\alpha := \left[ \frac{\partial \text{exp}^i(x, tL^{-1}(\eta))}{\partial x^i} - \delta^i_\alpha \right] + \frac{\partial \exp^i(x, tL^{-1}(\eta))}{\partial y^k} \cdot \frac{\partial g^{A k}}{\partial u^\alpha}(x, \eta) \cdot t A.
\]

Likewise,

\[
\text{Exp}^c_{* (x, t \eta)} \frac{\partial}{\partial \xi_A} = g^{A k}(\eta) \left[ \delta^i_k + L(t, x, \eta)^i_k \right] \frac{\partial}{\partial x^i},
\]

where

\[
L(t, x, \eta)^i_k := \frac{\partial \exp^i(x, tL^{-1}(\eta))}{\partial y^k} - \delta^i_k.
\]

Clearly, \(\lim_{t \to 0^+} H(t, x, \eta)^i_\alpha = \lim_{t \to 0^+} L(t, x, \eta)^i_k = 0\), which together with (2.5) implies that \(\text{Exp}^c\) is \(C^1\) on \(N \subset V^* N\) if and only if \(F\) is Riemannian. From above, the matrix of \(\text{Exp}^c_{* (x, t \eta)}\) is

\[
S(t, x, \eta) = \left( \begin{array}{ccc} \delta^\beta_\alpha + H(t, x, \eta)^\beta_\alpha & \cdots & \delta^\beta_\alpha + H(t, x, \eta)^\beta_\alpha \\ g^{\alpha A}(\eta) + g^{\alpha k} L(t, x, \eta)^k_A & \cdots & g^{\alpha A}(\eta) + g^{\alpha k} L(t, x, \eta)^k_A \\ g^{\alpha B}(\eta) + g^{\alpha k} L(t, x, \eta)^k_B & \cdots & g^{\alpha B}(\eta) + g^{\alpha k} L(t, x, \eta)^k_B \end{array} \right).
\]

Since \(\det S(0, x_0, \eta_0) > 0\), there exists a small \(\epsilon(x_0, \eta_0) > 0\) and an open neighborhood \(\mathcal{W}\) of \((x_0, \eta_0)\) in \(V^* SN\) such that \(\text{Exp}^c_{* t \xi}\) is nonsingular for all \(\xi \in \mathcal{W}\) and \(t \in (0, \epsilon(x_0, \eta_0))\).

Let \(\pi_1 : V^* SN \to N\) be the natural projection. Thus, for each \(x \in N\), \(\pi_1^{-1}(x) := V^*_x N\) is a \((n - k - 1)\)-dimensional Minkowski unit sphere in \(T^*_x M\). Given a local coordinate system \((u^\alpha, \theta_B)\) on \(V^* SN\), where \((u^\alpha)\) are local coordinates of \(N\), and for fixed \(x = (u^\alpha), (\theta_B)\) are the local coordinates of \(V^*_x N\). Hence, we obtain a local cone coordinate system \((t, u^\alpha, \theta_B)\) on \(V^* N \backslash N\), that is, for \(\xi \in V^* N \backslash N, t = F^*(\xi)\) and \(\xi / F^*(\xi) = (u^\alpha, \theta_B)\). Define a map \(E : [0, +\infty) \times V^* SN \to M\) by \(E(t, \xi) = \text{Exp}^c(t \xi)\). It is easy to see that

\[
E_{* t \xi} \frac{\partial}{\partial t} = \left( \text{exp}_\pi(\xi) \right)_{t \xi} \cdot L^{-1}(\xi) = \xi.
\]

In particular, \(E_{* 0 \xi} = L^{-1}(\xi)\).

**Proposition 4.2.**

\[ J(t) = E_{* t \xi} \frac{\partial}{\partial u^\alpha} \]

is a transverse Jacobi field along \(\gamma_{L^{-1}(\xi)}(t)\) such that \(J(0) = \frac{\partial}{\partial u^\alpha}\) and \(\mathcal{K}(J(0)) \in T^*_x N\), where \(T := \gamma_{L^{-1}(\xi)}(0)\) and \(x := \pi(\xi)\).

**Proof.** Suppose \(\xi = (u^\beta, \theta_B)\). Set \(\gamma(s) = (u^\beta(s))\) and \(\xi(s) = (u^\beta(s), \theta_B)\), where \(u^\beta(s) = u^\beta + s \cdot \delta^\beta_\alpha, s \in (-\epsilon, \epsilon)\). Consider the variation \(\sigma(t, s) = E(t, \xi(s)) = \text{exp}_{\gamma(s)} t L^{-1}(\xi(s))\). Thus,

\[
J(t) = \frac{\partial}{\partial s} \bigg|_{s=0} \sigma(t, s) = E_{* t \xi} \frac{\partial}{\partial u^\alpha}.
\]
is a Jacobi field along $\gamma_{\mathcal{L}^{-1}(\xi)}(t)$. And $J(0) = \left. \frac{\partial}{\partial u} \right|_{s=0} \sigma(0) = \left. \frac{\partial}{\partial \theta} \right|_{s=0} \gamma(s) = \frac{\partial}{\partial \theta}$. Set $T(t, s) := \frac{\partial}{\partial \sigma}(t, s)$. Clearly, $T(t, 0) = T(t)$. And we have
\[(\nabla^T J)(0) = \nabla^T \gamma_{\mathcal{L}^{-1}(\xi)}(0) = \nabla^T \gamma_{\mathcal{L}^{-1}(\xi)}(\xi(s)),\]
which implies $(\nabla^T J)(0) - \mathcal{F}^\xi(J(0)) \in T_x^+ N$. Since $F^* (\xi(s)) = 1,$
\[g_T(T, (\nabla^T J)(0)) = g_{\mathcal{L}^{-1}(\xi)}(\xi(s)), (\nabla^T \gamma_{\mathcal{L}^{-1}(\xi)}(\xi(s))) = \frac{1}{2} J(0)(F^2(\mathcal{L}^{-1}(\xi(s)))) = 0.\]
Hence, $J$ is a transverse Jacobi field along $\gamma_{\mathcal{L}^{-1}(\xi)}(t)$. □

**Proposition 4.3.**
\[J(t) = E_{s(t, \xi)} \frac{\partial}{\partial \theta} \partial\]
is a transverse Jacobi field along $\gamma_{\mathcal{L}^{-1}(\xi)}(t)$ such that $J(0) = 0$ and $(\nabla^T J)(0) = \mathcal{L}^{-1}_{\xi}(\xi(0))$, where $T = \gamma_{\mathcal{L}^{-1}(\xi)}, \ x = \pi(\xi)$ and $\mathcal{L}^{-1}_{\xi} : T_x(T^* M) \to T_{\mathcal{L}^{-1}(\xi)}(T^* M)$ is the tangent map.

**Proof.** Suppose that $\xi = (u^\alpha, \theta_\xi)$. Set $\sigma(s) = (u^\alpha, \theta_\xi(s), s \in (-\epsilon, \epsilon)$, such that $\left. \frac{\partial}{\partial s} \right|_{s=0} \xi(s) = \frac{\partial}{\partial \theta}$. Consider the variation $\sigma(t, s) = E(t, \xi(s)) = \exp_{\pi(\xi)}(t \mathcal{L}^{-1}(\xi(s)))$.

Clearly,
\[J(t) = E_{s(t, \xi)} \frac{\partial}{\partial \theta} \partial = \left. \frac{\partial}{\partial s} \right|_{s=0} \sigma(t, s) = \left(\exp_{\pi(\xi)}(t \mathcal{L}^{-1}(\xi(s))) \mathcal{L}^{-1}_{\xi}(\xi) \frac{\partial}{\partial \theta} \partial \right).\]

Since $F(\mathcal{L}^{-1}(\xi(s))) = 1,$
\[0 = \left. \frac{d}{ds} \right|_{s=0} F^2(\mathcal{L}^{-1}(\xi(s))) = 2g_{\mathcal{L}^{-1}(\xi)}(\mathcal{L}^{-1}(\xi(s)), \mathcal{L}^{-1}_{\xi}(\xi(s))) = 0.\]

By the Guass lemma, we have
\[g_T(T, J) = g_{\mathcal{L}^{-1}(\xi)}(\mathcal{L}^{-1}(\xi), \mathcal{L}^{-1}_{\xi}(\xi) \frac{\partial}{\partial \theta} \partial) = 0.\]
□

Note that $T(t, \xi)(V^* N) = R \frac{\partial}{\partial u^\alpha} \oplus T_x(V^* SN)$, for all $t > 0$ and $\xi \in V^* SN$. As a trivial combination of Proposition 4.2 and Proposition 4.3 we get

**Lemma 4.4.** Given $t_0 > 0$ and $\xi \in V^* SN$. If there exists some $X$ such that $E_{s(t_0, \xi)} X = 0$, then $X \in T_\xi(V^* SN)$.

Let $\xi(s)$ defined as in the proof of Proposition 4.3. Then $\mathcal{L}^{-1}(\xi(s))$ is a unit normal vector for all $s$ and therefore,
\[0 = \left. \frac{d}{ds} \right|_{s=0} g_{\mathcal{L}^{-1}(\xi(s))} \left(\mathcal{L}^{-1}(\xi(s)), \frac{\partial}{\partial u^\alpha} \right) = g_{\mathcal{L}^{-1}(\xi)}(\mathcal{L}^{-1}_{\xi}(\xi(s)) \frac{\partial}{\partial \theta} \partial, \frac{\partial}{\partial u^\alpha}).\]

Hence, for each $\xi \in V^* SN$, we have
\[(T_x M, g_n) = \mathbb{R} \cdot n \oplus n^\perp = \mathbb{R} \cdot n \oplus T_x N \oplus (T^* x \cap n^\perp) = \mathbb{R} \cdot n \oplus \text{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial u^\alpha} \right\} \oplus \text{Span}_{\mathbb{R}} \left\{ \mathcal{L}^{-1}_{\xi}(\xi(s)) \frac{\partial}{\partial \theta} \partial \right\},\]
where $n := \mathcal{L}^{-1}(\xi)$. For convenience, set $e_\alpha := \frac{\partial}{\partial u^\alpha}$ and $e_\theta := \mathcal{L}^{-1}_{\xi}(\xi(s)) \frac{\partial}{\partial \theta} \partial$. 


Given $\xi \in V^*SN$, let $n, e_\alpha, e_\beta$ be defined as above. Denote by $P_{t,n}$ the parallel translation along $\gamma_n$ from $T_{\gamma_n(0)}M$ to $T_{\gamma_n(t)}M$ (with respect to the Chern connection) for all $t \geq 0$. Set $T = \dot{\gamma}_n(t)$. Let $R_T := R_T(\cdot, T)T$ and
\[ R(t, n) := P_{t,n}^{-1} \circ R_T \circ P_{t,n} : n^+ \to n^+. \]
Let $A(t, n)$ be the solution of the matrix (or linear transformation) ordinary differential equation on $n^+$:
\[
\begin{cases}
A'' + R(t, y)A = 0, \\
A(0, n)e_\alpha = e_\alpha, \quad A'(0, n)e_\alpha = (\nabla_T^T J_n)(0) \\
A(0, n)e_\beta = 0, \quad A'(0, n)e_\beta = e_\beta,
\end{cases}
\]
where $A' = \frac{d}{dt}A$ and $J_n(t) = E_{s(t, \xi)}e_\alpha$. Note that $\gamma_n(t) = P_{t,n}n$. Thus, for each $X \in n^+$,
\[ g_{P_{t,n}n}(P_{t,n}n, P_{t,n}A(t, n)X) = g_{n}(n, A(t, n)X) = 0, \]
that is, $P_{t,n}A(t, n)X$ is a transverse Jacobi field along $\gamma_n$. In particular, $E_{s(t, \xi)}e_\alpha = P_{t,n}A(t, n)e_\alpha$ and $E_{s(t, \xi)}\mathcal{L}_{s\xi}e_\beta = P_{t,n}A(t, n)e_\beta$. Set $Ae_\alpha =: A_\alpha e_\beta$ and $det A := det A^2_\alpha$, where $\alpha = \alpha, \beta$. Clearly, $det A$ is independent of choice of basis for $n^+$. Then we have the following

**Proposition 4.5.** Given $\xi \in V^*SN$, let $n = \mathcal{L}^{-1}(\xi)$. The following statements are mutually equivalent:

1. $\gamma_n(t_0), 0 < t_0 < \infty$ is a focal point of $N$ along $\gamma_n$.
2. $\text{Exp}^\xi_{s(t, \xi)}$ is singular.
3. $E_{s(t, \xi)}$ is singular.
4. $det A(t_0, n) = 0$.

**Proof.** From above, we have (3) $\Rightarrow$ (4).

Define a map $\mathcal{F} : (0, +\infty) \times V^*SN \to V^*N \setminus N$ by $\mathcal{F}(t, \xi) = t\xi$. Then $\mathcal{F}$ is a diffeomorphism, since $\mathcal{F}^{-1}(\xi) = (F^*(\xi), \xi/F^*(\xi))$. Clearly, $\text{Exp}^\xi_{s(t, \xi)} = E_{s(t, \xi)}$, which implies (2) $\Leftrightarrow$ (3).

By Theorem 4.1, there exists $\epsilon(\xi) > 0$ such that $E_{s(t, \xi)}$ is nonsingular for $0 < t \leq \epsilon(\xi)$. Thus, $\{E_{s(t, \xi)}e_\alpha, E_{s(t, \xi)}\mathcal{L}_{s\xi}e_\beta\}$ form a basis for the space of the transverse Jacobi fields along $\gamma_n(t), 0 \leq t \leq \epsilon(\xi)$.

Suppose $\gamma_n(t_0)$ is a focal point. Then there exists a nontrivial transverse Jacobi field $J$ along $\gamma_n$ such that $J(t_0) = 0$. From above, $J(t) = C^\alpha E_{s(t, \xi)}e_\alpha + C^\beta E_{s(t, \xi)}\mathcal{L}_{s\xi}e_\beta$, for $0 \leq t \leq \epsilon(\xi)$ and therefore, for $t \geq 0$. Here, $C^\alpha$'s are constants not all zero. Then $J(t_0) = 0$ implies (3). And it follows from Lemma 4.4 that (1) $\Leftrightarrow$ (3). \hfill \Box

By Theorem 4.1 and Proposition 4.5, we give the following definition.

**Definition 4.6.** Given $\xi \in V^*SN$, the focal value $c_f(\xi)$ is defined by
\[ c_f(\xi) := \sup\{r > 0 : \text{no point } \gamma_{\mathcal{L}^{-1}(\xi)}(t), 0 < t < r \text{ is focal point}\} \]

**Proposition 4.7.** The function $c_f : V^*SN \to (0, +\infty]$ is lower semicontinuous. That is,
\[ \liminf_{\xi \to \xi_0} c_f(\xi) \geq c_f(\xi_0). \]

**Proof.** **Step 1:** Suppose $\text{Exp}^\xi_{s(t, \xi)}$ is nonsingular at $t\xi$, where $t > 0$ and $\xi \in V^*SN$. Then one can obtain a neighborhood $U$ of $\xi$ in $V^*SN$ and a small $\epsilon > 0$ such that

1. For each $\eta \in U$ and $s \in (t - \epsilon, t + \epsilon)$, $\text{Exp}^\xi_{s(t, \xi)}$ is also nonsingular at $sn$. 

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A COMPARISON THEOREM FOR FINSLER SUBMANIFOLDS AND ITS APPLICATIONS 11
(ii) In a local trivialization of $\mathcal{V}^*SN$, $U$ is the Cartesian product of an open ball (for the position $u^*\alpha$) in $\mathbb{R}^k$ with an open "disk" (for the position $\theta_\xi$) on the standard unit sphere $S^{n-k-1}$.

**Step 2:** Given $\xi_0 \in \mathcal{V}^*SN$ and $0 < r < c_f(\xi_0)$. Let $\varepsilon(\xi_0)$ and $\mathcal{W}$ be as in Theorem 4.1. If $r \leq \varepsilon(\xi_0)/2$, then we take $\varepsilon = \varepsilon(\xi_0)/2$ and $U = \mathcal{W}$. Now suppose $r > \varepsilon(\xi_0)/2$. For each $t \in [\varepsilon(\xi_0)/2, r]$, one has a neighborhood $U_t$ of $\xi$ and a interval $I_t = (t - \varepsilon_t, t + \varepsilon_t)$ with the properties stated in Step 1. Then one can find finitely many $\{I_{t_k}\}^k_{k=1}$ such that $\bigcup_t I_{t_k} \supseteq [\varepsilon(\xi_0)/2, r]$. Without loss of generality, we suppose that $t_1 < \cdots < t_k$ and $t_k = r$ (so $\varepsilon_{t_k} = \varepsilon_r$). Now let $U := \cap_t U_t \cap \mathcal{W}$. From above, it is easy to see that $\text{Exp}^t_{(x, \xi_t)}$ is not singular for all $t \in (0, r + \varepsilon_r)$ and $(x, \xi) \in U$, i.e., $c_f(\xi) > r + \varepsilon_r$. Clearly, $\lim_{t \to \varepsilon^{-}(\xi_0)} \varepsilon_r = 0$. Hence, $\lim inf_{\xi \to \xi_0} c_f(\xi) \geq r + \varepsilon_r$. We complete the proof by letting $r \to c_f(\xi_0)$.

Let $\xi, n, T$ and $e_\alpha, a = \alpha, g$ be as before. Now, we continue to investigate $\det A(t, n)$. For simplicity, set $J_\alpha(t) := P_{\xi n}A(t, n)e_\alpha$. A direct calculation yields

$$
\det \left[ g_T(J_\alpha(t), J_b(t)) \right] = \det \left[ g_n(A(t, n)e_\alpha, A(t, n)e_b) \right]
$$

By the Lagrange identity and L'Hospital's rule, we have

$$
\lim_{t \to 0^+} \frac{g_T(J_\alpha(t), J_b(t))}{t^2} = \lim_{t \to 0^+} \frac{g_T(\nabla_T J_\alpha(t), J_b(t))}{t} = \lim_{t \to 0^+} g_T(J_\alpha(t), \nabla_T J_b(t)) = g_n(\nabla_T J_\alpha(0), e_g, e_b).
$$

And it is easy to see that $\lim_{t \to 0^+} \frac{g_T(J_\alpha(t), J_b(t))}{t^2} = g_n(e_g, e_b)$. Hence, we have

$$
\lim_{t \to 0^+} \frac{g_T(J_\alpha(t), J_b(t))}{t^2} = \lim_{t \to 0^+} \det \left( \frac{g_T(J_\alpha(t), J_b(t))}{t^2} \right) = \det g_n(e_\alpha, e_\beta) \det g_n(e_g, e_b),
$$

which implies that

$$
\lim_{t \to 0^+} \frac{\det A(t, n)}{t^{n-k-1}} = 1.
$$

Moreover, we have the following

**Theorem 4.8.** Given $\xi \in \mathcal{V}^*SN$. Let $n = \mathcal{L}^{-1}(\xi)$ and $H_\xi = \text{tr} g_\xi(\partial_\xi)$. If the flag curvature $K(\gamma_n(t); \cdot) \geq \delta$, then $c_f(\xi) \leq \min \{\xi, \pi/\sqrt{\delta}\}$ and

$$
\det A(t, n) \leq \left( g_\delta - \frac{H_\xi}{k} g_\delta \right)^k(t) \cdot g_\delta^{n-k-1}(t), \text{ for } t \in [0, c_f(\xi)],
$$

where $\zeta$ is the first positive zero of

$$
\left( g_\delta - \frac{H_\xi}{k} g_\delta \right)^k(t)
$$

(should such a zero exist; otherwise, set $\zeta = +\infty$).

**Proof.** Fix some positive number $r < c_f(\xi)$. Let $J_\alpha := P_{\xi n}Ae_\alpha$, for $a = \alpha, g$. For $s \in (0, r)$, by (6), we have

$$
\frac{(\det A')'}{\det A} = \frac{1}{2} \frac{(\det g_T(J_{\alpha}(t), J_b(t)))'}{\det g_T(J_{\alpha}(t), J_b(t))}(s).
$$
Note that \( \{ J_a(t) \} \) is a basis for the space \( \mathcal{J} \) of transverse Jacobi fields along \( \gamma_n(t) \), \( 0 \leq t \leq s \). Let \( \{ J_a(t) \} \) be another \( n - 1 \) transverse Jacobi fields such that \( \{ T(s), \tilde{J}_a(s) \} \) is a \( g_T \)-orthonormal basis. Then \( \{ \tilde{J}_a(t) \} \) is also a basis for \( \mathcal{J} \). It is easy to check that

\[
(*)_1 \quad \frac{(\det A')}{\det A}(s) = \frac{1}{2} \frac{(\det g_T(J_a, J_b))'}{\det g_T(J_a, J_b)}(s) = \frac{1}{2} \frac{(\det g_T(J_a, J_b))'}{\det g_T(J_a, J_b)}(s).
\]

A direct calculation shows

\[
(*)_2 \quad \frac{(\det g_T(\tilde{J}_a, \tilde{J}_b))'}{\det g_T(J_a, J_b)}(s) = \sum_a (g_T(\nabla^T \tilde{J}_a, \tilde{J}_a))'(s) = 2 \sum_a I_{[0, s]}(\tilde{J}_a, \tilde{J}_a),
\]

where \( I_{[0, s]} \) is the index restricted to \( \gamma_n(t) \), \( 0 \leq t \leq s \).

Consider the solution \( A_\delta(t) \) to the matrix differential equation in \( n^+ \):

\[
A_\delta' + kA_\delta = 0,
\]

with the same initial conditions as \( A(t) \). Let \( \{ f_a \} \) be a \( g_n \)-orthonormal basis for \( T_x N \) consisting of eigenvectors of the Weingarten map \( \mathcal{A}_\xi \), with respective eigenvalues \( \lambda_a \). And let \( \{ f_\delta \} \) be an orthonormal basis for \( n^+ \cap T_x^+ N \). Then we have

\[
A_\delta f_a = (s_\delta^2 - \lambda_a s_\delta)(t) \cdot f_a + C_\delta^a s_\delta(t) \cdot f_\delta,
\]

\[
A_\delta f_\delta = s_\delta(t) \cdot f_\delta,
\]

where \( C_\delta^a \)'s are constants determined by the initial conditions of \( A_k(t) \). It is easy to see that

\[
\det A_\delta(t) = s_\delta^{m-k-1}(t) \cdot \prod_{\alpha=1}^k (s_\delta^2 - \lambda_\alpha s_\delta)(t).
\]

Suppose \( r < \zeta_0 \), where \( \zeta_0 \) is the first positive zero of \( \det A_\delta(t) \). Thus, \( \{ T, P_{\gamma_n}A_\delta(t) f_a \} \)

is a basis at \( \gamma_n(t) \) for all \( t \in (0, r) \). Hence, one can find constants \( C_\delta^a \) such that \( \det C_\delta^a \neq 0 \) and \( J_\delta(t) = C_\delta^a P_{\gamma_n}A_\delta(t) f_a \). Consider the vector fields \( Y_a(t) := C_\delta^a P_{\gamma_n}A_\delta(t) f_a \). Clearly, \( \nabla^T Y_a = C_\delta^a P_{\gamma_n}A_\delta(t) f_\delta \). Theorem 3.6 then yields

\[
\sum_a I_{[0, s]}(J_a, \tilde{J}_a) \leq \sum_a I_{[0, s]}(Y_a, \tilde{Y}_a) \\
\leq \sum_a \left( -h_\xi(Y_a(0), \tilde{Y}_a(0)) + \int_0^s g_T(\nabla^T Y_a, \nabla^T Y_a) - \delta g_T(Y_a, \tilde{Y}_a) dt \right) \\
(*)_3 \quad = \sum_a g_T(\nabla^T Y_a, \tilde{Y}_a)(s).
\]

Note that \( \{ l_a := C_\delta^a \cdot f_b \} \) is also a basis for \( n^+ \). Set \( A_\delta l_a := (A_\delta)_{ba} \cdot l_b \). Since \( g_T(Y_a, Y_b)(s) = \delta_{ab}, \quad (A_\delta)_{ba}^2(s) \cdot g_n(l_a, l_c) = (A_\delta^{-1})_{ba}(s) \) and

\[
(*)_4 \quad \sum_a g_T(\nabla^T Y_a, \tilde{Y}_a)(s) = \text{tr}(A_\delta' : A_\delta^{-1})(s) = \frac{(\det A_\delta')'}{(\det A_\delta)}(s).
\]

(*1) together with (*2), (*3) and (*4) furnishes

\[
\frac{(\det A')}{\det A}(s) \leq \frac{(\det A_\delta')'}{(\det A_\delta)}(s).
\]
Since det \( \mathcal{A}(t) \sim t^{n-k-1} \sim \det \mathcal{A}_d(t) \), det \( \mathcal{A}(t) \leq \det \mathcal{A}_d(t) \) for all \( t \in [0, r] \), which implies that \( c_f(\xi) \leq \xi_0 \). The arithmetic-geometric mean inequality now yields

\[
\det \mathcal{A}(t) \leq \det \mathcal{A}_d(t) \leq \left( \sum_{\alpha} \lambda_{\alpha} \delta_{\alpha} - \frac{\sum_{\alpha} \lambda_{\alpha} \delta_{\alpha}}{k} \right)^k (t) \cdot \delta_{\delta}^{n-k-1}(t),
\]

for \( t \in [0, c_f(\xi)] \).

5. PROOF OF THEOREM 1.1

Given \( \xi \in V^*SN \setminus N \), choose a local coordinate system \((u^\alpha, \theta^\beta)\) around \( \xi \) such that \((t, u^\alpha, \theta^\beta)\) is a local cone coordinate system \((t, u^\alpha, \theta^\beta)\) on \( V^*N \setminus N \). Set \( \pi_1(\xi) = x \). It is easy to check that \( \mathcal{L}^{-1} \) is an isometry from \((T^*_x M \setminus 0, g^*_x)\) to \((T_x M \setminus 0, g_x)\). Denote by \( d\nu_x \) the Riemannian volume form on \( V^*_x SN \) induced by \( g^*_x \). Let \( n \) and \( e_\alpha \), \( a = \alpha, g \) be defined as before. Since

\[
g^*_\xi \left( \frac{\partial}{\partial \theta^\beta}, \frac{\partial}{\partial \theta^\beta} \right) = \left( \left( \mathcal{L}^{-1} \right)^* g_n \right) \left( \frac{\partial}{\partial \theta^\beta}, \frac{\partial}{\partial \theta^\beta} \right) = g_n(e_\alpha, e_\beta),
\]

we have \( d\nu_x(\xi) = \sqrt{\det g_n(e_\alpha, e_\beta)} d\Theta \), where \( d\Theta = d\theta^\alpha \). We define a \( n \)-form \( \omega \) on \((0, +\infty) \times V^*SN \) by

\[
\omega(t, \xi) = e^{-\tau(\gamma_n(t))} \det \mathcal{A}(t, n) dt \wedge \sqrt{\det g_n(e_\alpha, e_\beta)} du^1 \wedge \cdots \wedge du^k \wedge d\nu_x(\xi).
\]

A direct calculation shows that \( \omega \) is independent of the choice of chart.

Given an arbitrary point \( p \in M \setminus N \), there exists a unit speed minimizing geodesic \( \gamma_n \) from \( N \) to \( p \). By Remark 4.1, we have \( E(D) = M \), where \( D := \{(t, \xi) : \xi \in V^*SN, 0 \leq t \leq c_f(\xi)\} \).

Moreover, for each \( x_0 = E(t_0, \xi_0) \in M \) with \( 0 < t < c_f(\xi_0) \), by Proposition 4.7, there exists an open set \( Q(t_0, \xi_0) = (t_0 - \varepsilon, t_0 + \varepsilon) \times W(\xi_0) \) such that \( E|_{Q(t_0, \xi_0)} : Q(t_0, \xi_0) \to E(Q(t_0, \xi_0)) \) is a diffeomorphism. Hence, \((t \to E|_{Q(t_0, \xi_0)}, u^\alpha \circ E|_{Q(t_0, \xi_0)}^{-1}, \theta^\beta \circ E|_{Q(t_0, \xi_0)}^{-1})\) is a local coordinate system on \( E(Q(t_0, \xi_0)) \). For simplicity, we still use \((t, u^\alpha, \theta^\beta)\) or \((t, \xi)\) to denote this coordinate system. In this case,

\[
\frac{\partial}{\partial t}(t, \xi) = P_{t;\alpha n} \frac{\partial}{\partial u^\alpha}(t, \xi) = P_{t;\alpha n} \mathcal{A}e_\alpha, \quad \frac{\partial}{\partial \theta^\beta}(t, \xi) = P_{t;\alpha n} \mathcal{A}e_\beta,
\]

where \( n \) and \( e_\alpha \), \( a = \alpha, g \) are defined as before. And denote by \( \det g_{\mathcal{A} n}(E(t, \xi)) \) the determinant of \( g_{\mathcal{A} n} \).

Given a volume form \( d\mu \) on \( M \). It follows from (6) that

\[
E|_{Q(t_0, \xi_0)}^* d\mu = \sigma(t, u^\alpha, \theta^\beta) dt \wedge du^1 \wedge \cdots \wedge du^k \wedge d\Theta
\]

\[
= \sqrt{\det g_{\mathcal{A} n}(E(t, \xi))} \det g_{\mathcal{A} n}(E(t, \xi)) dt \wedge du^1 \wedge \cdots \wedge du^k \wedge d\Theta
\]

(4.2)

\[= \omega.\]

Proof of Theorem 1.1. By the argument above, we have \( \mu(M) = \mu(D^3_d) \), where \( D^3_d := \{(t, \xi) : \xi \in V^*SN, 0 < t < \min\{d, c_f(\xi)\}\} \). Let \( D_1 := \{(t, \xi) : \xi \in V^*SN, 0 < t < d < c_f(\xi)\} \), \( D_2 := \{(t, \xi) : \xi \in V^*SN, 0 < t < c_f(\xi) < d\} \) and \( D_3 := \{(d, \xi) : \xi \in V^*SN, d = c_f(\xi)\} \). Sard’s theorem then yields \( \mu(M) = \mu(E(D_1)) + \mu(E(D_2)) \).
For each $\( t, \xi \) \in D_s$, $s = 1, 2$, from above, one can find an open neighborhood $Q(t, \xi) \subset D_s$ of $\( t, \xi \)$ such that $E_{|Q(t,\xi)} : Q(t, \xi) \to E(Q(t, \xi))$ is a diffeomorphism. Hence, there is a countable covering $\{ Q(t, \xi) \}$ of $D_1 \cup D_2$. For simplicity, set $Q_i := Q(t_i, \xi_i)$ and $E_i := E|_{Q_i}$. Note that $\{ E(Q_i) \}$ is also a open covering of $E(D_1 \cup D_2)$. Let $\{ \rho_i \}$ be a partition of unity subordinate to $\{ E(Q_i) \}$. And define a sequence of nonnegative continuous functions $\rho_i : D_1 \cup D_2 \to \mathbb{R}$ by

$$\rho_i(t, \xi) := \begin{cases} \rho_i \circ E_i, & (t, \xi) \in Q_i \\ 0, & \text{else} \end{cases}$$

Given $\( t, \xi \) \in D_1 \cup D_2$,

$$\sum_i \rho_i(t, \xi) = \sum_{\{ i : (t, \xi) \in Q_i \}} \rho_i(t, \xi) = \sum_{\{ i : (t, \xi) \in Q_i \}} \rho_i(E(t, \xi)) \leq \sum_i \rho_i(E(t, \xi)) = 1,$$

From above, we have

$$\mu(M) = \sum_i \int_{E_i(Q_i)} \rho_i \cdot d\mu = \sum_i \int_{Q_i} (\rho_i \circ E_i) \cdot \omega$$

$$= \sum_{\{ i : Q_i \subset D_1 \}} \int_{Q_i} \rho_i \cdot \omega + \sum_{\{ i : Q_i \subset D_2 \}} \int_{Q_i} \rho_i \cdot \omega$$

$$\leq \sum_{\{ i : Q_i \subset D_1 \}} \int_{D_1} \rho_i \cdot \omega + \sum_{\{ i : Q_i \subset D_2 \}} \int_{D_2} \rho_i \cdot \omega$$

(7)

**Case 1:** $k = 0$, i.e., $N = \{ x \}$. Note that $V_x^*SN = S_x^*M$ and $\mathcal{L} : (T_x M \setminus 0, g_x) \to (T_x^* M \setminus 0, g^*_x)$ is an isometry. For convenience, we also use $d\nu_x$ to denote the Riemannian volume form induced by $g_x$ on $S_x M$. It now follows Theorem 4.8 that

$$\mu(M) \leq \int_{S_x M} e^{-\tau(\gamma_x(t))}d\nu_x(y) \int_y^d g^{m-1}_x(t)dt.$$

**Case 2:** Let $d\bar{\mu}$ and $d\bar{\mu}$ denote either the Busemann-Hausdorff volume forms or Holmes-Thompson volume forms induced by $F$ and $F|_N$, respectively. Set $d\bar{\mu} := \sigma_N du^1 \wedge \cdots \wedge du^k$. Hence,

$$\mu(M) \leq \int_0^{\min\{ d, c_f(\xi) \}} \left( g^*_\delta - \frac{H_x}{k} g^*_\delta \right)^k(t) \cdot g^{n-k-1}_\delta(t)dt \int_{V_x^*SN} e^{-\tau(\gamma_{\xi}^{-1}(\xi)(t))} d\nu_x(\xi)$$

$$\cdot \int_N \sqrt{\det g_{\xi}^{-1}(\xi)} \left( \frac{\partial}{\partial u^\alpha} \cdot \frac{\partial}{\partial u^\beta} \right) du^1 \cdots du^k(x).$$

Choose $\xi_0 \in V_x^*SN$ such that $H_{\xi_0} = \min_{\xi \in V_x^*SN} H_{\xi}$. Denote by $\zeta(\xi)$ the first positive zero of $\left( g^*_\delta - \frac{H_x}{k} g^*_\delta \right)(t)$. Theorem 4.8 yields that $c_f(\xi) \leq \zeta(\xi) \leq \zeta(\xi_0)$ for all $\xi \in V_x^*SN$. By Proposition 7.1 and Proposition 7.3, we have

$$\mu(M) \leq c_{m-k-1} \cdot A_F^{(3m+k)/2} \cdot \bar{\mu}(N) \cdot \int_0^{\min\{ d, \zeta(\xi_0) \}} \left( g^*_\delta - \frac{H_{\xi_0}}{k} g^*_\delta \right)^k(t) \cdot g^{m-k-1}_\delta(t)dt.$$
Proof of Corollary 1.2. Without loss of generality, we can suppose that \( N \) is a unit speed geodesic. Then for each \( \xi \in \mathcal{V}^*SN \), we have
\[
-H_{\xi} = \left[ g_n(X, X) \right]^{-1} T_n(X) \leq \frac{\left[ \sqrt{g_n(X, X)} - g_n(X, n) \right]^{2} F(n)}{g_n(X, X)} = t,
\]
where \( n = L^{-1}(\xi) \) and \( X = \frac{\partial}{\partial u} \) is the tangent vector field of \( N \). Then we have
\[
\mu(M) \leq c_{m-2} \cdot \Lambda_F^{(3m+1)/2} \cdot \ell \cdot \left[ s_{\delta}^{m-1} \left( \min \left\{ \frac{d}{2\sqrt{\delta}} \right\} \right) \right] + l \cdot \int_{0}^{d} s_{\delta}^{m-1}(t)dt.
\]

\[\Box\]

6. Randers metric

Let \((M, F)\) be a compact Randers manifold and let \( \gamma(t), 0 \leq t \leq \ell \), be a unit speed closed geodesic in \( M \). Set \( b := \sup_{x \in \gamma([0, \ell])} \| \beta \|_\alpha \) and \( b_1 := \sup_{x \in \gamma([0, \ell])} \| \nabla \beta \|_\alpha \).

For each \( t \), there exists a local coordinate \((u, x^4)\) such that \( \frac{\partial}{\partial u} = \gamma(t) \) and \( x^4_{\gamma(t)} = 0 \). Given \( \xi \in \mathcal{V}(\gamma)SN \), let \( n = L^{-1}(\xi) \). It is easy to check that
\[
e^{-\tau_B H(\gamma_n(t))} \leq (1 + b)^{(m+1)/2} \quad \text{and} \quad e^{-\tau_{BH}(\gamma_{b}(t))} \leq (1 - b)^{-(m+1)/2}.
\]
From Example 1, we have
\[
g_n \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = \frac{1}{\alpha(n)} \left[ \alpha^2 \left( \frac{\partial}{\partial u} \right)^2 - \frac{\left( \frac{\partial}{\partial u}, n \right)^2}{\alpha^2(n)} \right] = \frac{F\left(-\frac{\partial}{\partial u}\right)}{1 - \beta(n)} \leq \frac{1 + b}{(1 - b)^2}.
\]
By the formula of T-curvature, we have
\[
T_n \left( \frac{\partial}{\partial u} \right) \leq \frac{b_1(2b^3 + 5b^2 - 2b + 7)}{2(1 - b)^3}
\]
Hence, we have the following

Theorem 6.1. Let \((M, F)\) be a compact Randers manifold with \( K \geq \delta \) and let \( \gamma \) be a closed geodesic in \( M \). Set \( b := \sup_{x \in M} \| \beta \|_\alpha \) and \( b_1 := \sup_{x \in M} \| \nabla \beta \|_\alpha \). Then
\[
L_F(\gamma) \geq \frac{(1 - b)^{m + 1}}{c_{m-2}(1 + b)^{\frac{m + 1}{2}}} \max \left\{ \frac{\mu_B(\beta)}{(1 + b)^{m + 1}}, (1 - b)^{m + 1} \operatorname{Vol}_\alpha(M) \right\},
\]
where
\[
\mathcal{G}(b, b_1, \delta, d, m) = \frac{s_{\delta}^{m-1} \left( \min \left\{ \frac{d}{2\sqrt{\delta}} \right\} \right)}{m - 1} + \frac{b_1(2b^3 + 5b^2 - 2b + 7)}{2(1 - b)^{m + 1}} \int_{0}^{d} s_{\delta}^{m-1}(t)dt,
\]
and \( \operatorname{Vol}_\alpha \) is the Riemannian volume of \( M \) induced by \( \alpha \).

Remark 2. In fact, we can take \( b := \sup_{x \in \gamma} \| \beta \|_\alpha \) and \( b_1 := \sup_{x \in \gamma} \| \nabla \beta \|_\alpha \).

7. Non-Riemannian example

In [CH], Cheeger showed the existence of the lower bound for the length of sample close geodesics in a closed Riemannian manifold in terms of an upper bound for the diameter and lower bounds for the volume and the curvature. However, this is not true in Finsler geometry. First, we introduce the notations in this section.
Let \( M = \mathbb{S}^n \times \mathbb{S} \) \((n \geq 2)\), and \( \alpha \) be the canonical Riemannian product metric on \( M \). Let \( \beta_\varepsilon \) be as in Example 1. Choose a smooth function \( \phi \) defined on \((-1,1)\) such that

\[
\begin{align*}
(i) \lim_{s \to -1} \phi(s) &= 0, \quad \sup_{s \in (-1,1)} \phi(s) < +\infty; \\
(ii) \phi(s) &= \phi'(s) + (t^2 - s^2)\phi''(s) > 0, \text{ for } t \in [0,1], \|s\| \leq t < 1.
\end{align*}
\]

Define \( F_\varepsilon := \alpha \phi(s) \), \( s = \beta / \alpha \). It follows from [CS] that \( F_\varepsilon \) is a Finsler metric on \( M \) for all \( \varepsilon \in [0,1] \). Note that \( \beta \) is parallel corresponding \( \alpha \). A direct calculation shows the spray of \( F_\varepsilon \) is the same as the one of \( \alpha \). Hence, it is easy to check that the flag curvature \( K_{F_\varepsilon} \geq 0 \) for all \( \varepsilon \in [0,1] \). Let \( (r, \theta^\alpha, t) \) be the local coordinate system of \( M \), where \((r, \theta^\alpha)\) (resp. \( t \)) is the spherical coordinates of \( \mathbb{S}^n \) (resp. \( \mathbb{S} \)).

Thus, \( \gamma(t) = (0,0,-t) \) is a geodesic of \( F_\varepsilon \). (i) yields that \( L_{F_\varepsilon}(\gamma) \to 0 \), where \( \varepsilon \to 1 \).

For convergence, set

\[
\mathcal{T}(s) := \phi(s) \cdot (\phi(s) - \phi'(s))^{n-1}[\phi(s) - \phi'(s) + (t^2 - s^2)\phi''(s)].
\]

Then we have the following

**Theorem 7.1.** Let \((M, F_\varepsilon)\) be as above. Suppose one of the following conditions is true:

(i) \( \lim_{t \to 1} \phi(\varepsilon \cos t) \geq C_1 \), for almost every \( t \in [0,\pi] \);  

(ii) \( \lim_{t \to 1} \mathcal{T}(\varepsilon \cos t) \geq C_2 \), for almost every \( t \in [0,\pi] \);  

(iii) \( \phi(s) := \mathcal{T}(s) - 1 \) is odd function;

Here \( C_1 \) and \( C_2 \) are positive constants. Then \( K_{\varepsilon} \geq 0 \), \( \mu_{\varepsilon}(M) \geq V \) and \( \text{diam}_{\varepsilon}(M) \leq D \) for all \( \varepsilon \in [0,1] \), where \( \mu_{\varepsilon} \) denote the\n
there exists a geodesic \( \gamma \) of \((M, F_\varepsilon)\) such that

(i) yields that

Let \( M, \alpha \) and \( \beta_\varepsilon \) be as in Example 1. Let \( \phi \) be a smooth positive function on \([-1,1]\) such that \( \phi(s) - \phi'(s) + (t^2 - s^2)\phi''(s) > 0 \), for all \( t \in [0,1] \) and \( |s| \leq t < 1 \). Define a function \( F_\varepsilon : TM \to [0, +\infty) \) by

\[
F_\varepsilon := \alpha \phi(s), \quad s = \frac{\beta_\varepsilon}{\alpha}.
\]

By [CS], \( F_\varepsilon \) is a Berwald metric for all \( \varepsilon \in [0,1] \). Since \( M \) is compact and \( \phi \) is defined on \([-1,1]\), \( \text{diam}_{F_\varepsilon}(M) \leq \text{diam}_{\alpha}(M) \cdot \max_{s \in [-1,1]} \phi(s) \). By [Ce] [CS], it is easy to check that \( K_{F_\varepsilon} \geq 0 \) and \( \mu_{\varepsilon}(M) = C \), where \( C \) is a positive constant only dependent on \( \phi \), and \( \mu_{\varepsilon} \) denote either the Busemann-Hausdorff volume or the Holmes-Thompson volume of \((M, F_\varepsilon)\).

8. **Appendix**

**Proposition 8.1.** Let \((M, F)\) be a Finsler \( m \)-manifold with finite uniform constant \( \Lambda_{F} \). Let \( d\mu \) denote either the Busemann-Hausdorff volume form or the Holmes-Thompson volume form on \( M \). Then the distortion \( \tau \) of \( d\mu \) satisfy \( e^{-\tau(y)} \leq \Lambda_{F}^m \), for all \( y \in SM \).

**Proof.** Given \( z \in T_x M \setminus \{0\} \), let \((x^i)\) be local coordinates around \( z \) and let \( B_F(x) := \{ y = y^i \frac{\partial}{\partial x^i} : F(x, y) < 1 \} \). Set \( \det g_{ij}(x, z_1) := \max_{y \in S_x M} \det g_{ij}(x, y) \) and
\[ \det g_{ij}(x, z_1) : \min_{y \in S_x M} \det g_{ij}(x, y). \] By [Wu2] Proposition 3.1, Proposition 4.1, we have
\[ \frac{\det g_{ij}(x, z_1)}{\det g_{ij}(x, z_2)} \leq \Lambda_F^m, \quad \Lambda_F^{-m/2} \leq \frac{\text{Vol}_{g_{z_1}}(B_F(x))}{\text{Vol}(\mathbb{B}^m)} \leq \Lambda_F^{m/2}. \]

(1): Suppose \( d\mu \) is the Busemann-Hausdorff volume form. Hence,
\[ \sqrt{\det g_{ij}(x, z)} \text{Vol}(B_F(x)) \geq \sqrt{\det g_{ij}(x, z_2)} \int_{B_F(x)} \sqrt{\det g_{ij}(x, z_1)} dy^1 \wedge \cdots \wedge dy^m \geq \Lambda_F^{-m} \text{Vol}(\mathbb{B}^m), \]
where \( B_F(x) := \{ y \in T_x M : F(x, y) < 1 \} \). Then
\[ e^{-\tau(z)} = \frac{\text{Vol}(\mathbb{B}^m)}{\sqrt{\det g_{ij}(x, z)} \text{Vol}(B_F(x))} \leq \Lambda_F^m. \]

(2): Suppose \( d\mu \) is the Holmes-Thompson volume form. Thus,
\[ e^{-\tau(z)} = \frac{1}{\text{Vol}(\mathbb{B}^m)} \int_{B_F(x)} \det g_{ij}(x, y) dy^1 \wedge \cdots \wedge dy^n \leq \frac{(\det g_{ij}(x, z_1)) \text{Vol}_{g_{z_1}}(B_F(x))}{(\det g_{ij}(x, z_2)) \text{Vol}(\mathbb{B}^m)} \leq \Lambda_F^m. \]

\[ \square \]

**Proposition 8.2.** Let \((M, F)\) be a Finsler manifold with finite uniform constant \( \Lambda_F \). Then the uniform constant of \( F^* \) is still \( \Lambda_F \).

**Proof.** Let \( \Lambda_{F^*} \) denote the the uniform constant of \( F^* \). Given \( y, z \in S_x M \). For each \( X \in T_x M \setminus 0 \), we have

\[ \Lambda_{F^*}^{-1} \leq \frac{g_y(X, X)}{g_z(X, X)} \leq \Lambda_F. \]

Choose a \( g_z \)-orthonormal basis \( \{ e_i \} \) for \( T_x M \) consisting of eigenvectors of \( g_y \) with respective eigenvalues \( \lambda_i \). Then we have

\[ \Lambda_{F^*}^{-1} \leq \frac{\sum_i \lambda_i (X^i)^2}{\sum_i (X^i)^2} \leq \Lambda_F, \]

where \( X = X^i e_i \). Hence, we have \( \Lambda_{F^*}^{-1} \leq \lambda_i \leq \Lambda_F \), for all \( i \). Therefore,

\[ \Lambda_{F^*}^{-1} \leq \frac{\sum_i \lambda_i^{-1} (\dot{\xi})^2}{\sum_i (\dot{\xi})^2} \leq \Lambda_F, \]

where \( \xi = (\dot{\xi}) \neq 0 \). Let \( \omega^i \) be the dual basis of \( \{ e_i \} \). Note that
\[ g_{\xi(y)} = g_{ij}^* e_i \otimes e_j = \sum_i \lambda_i^{-1} e_i \otimes e_i, \quad g_{\xi(z)}^* = \sum_i e_i \otimes e_i. \]

Hence, we have
\[ \Lambda_{F^*}^{-1} \leq \frac{g_{\xi(y)}(\xi, \xi)}{g_{\xi(z)}(\xi, \xi)} \leq \Lambda_F, \]
for all \( \xi \in T_x M \setminus 0 \), which implies that \( \Lambda_{F^*} \leq \Lambda_F \). Likewise, one can show \( \Lambda_F \leq \Lambda_{F^*} \). \( \square \)
Proposition 8.3. Let \((M, F)\) be a Finsler \(m\)-manifold with finite uniform constant \(\Lambda_F\) and \(N\) be \(k\)-dimensional submanifold of \(M\). For each \(x \in N\), we have 
\[ \nu_x(\mathcal{V}_x^*SN) \leq c_{m-k-1} \cdot \Lambda_F^{(m-k)/2}. \]
Moreover, if \(F = \alpha + \beta\) is a Randers metric, then 
\[ \nu_x(\mathcal{V}_x^*SN) \leq c_{m-k-1} \cdot (1 - b(x))^{-\frac{m+k}{2}}, \]
where \(b(x) := \|\beta\|_{\alpha}(x)\).

Proof. For each \(x \in N\), let \((u^a, \xi_A)\) be a canonical coordinate system around \(\mathcal{V}_x^*N\). Then we have 
\[ \mathcal{V}_x^*SN = \{\xi = \xi_A dx^A : F^*(x, \xi) = 1\}. \]
Clearly,
\[ dv_x(\xi) = \sqrt{\det g^{AB}_x} \left( \sum_A (-1)^{A+1} \xi_A dx_{k+1} \wedge \cdots dx_A \wedge \cdots dx_m \right). \]
Set \(\mathcal{V}_x^*BN := \{\xi = \xi_A dx^A : F^*(x, \xi) < 1\}\). Then we have
\[ \text{Vol} \left( \mathcal{V}_x^*BN \right) = \int_{\mathcal{V}_x^*BN} \sqrt{\det g^{AB}_x} dx_{k+1} \wedge \cdots dx_m = \frac{1}{m-k} \int_{\mathcal{V}_x^*SN} dv_x. \]
A sample argument based on [Wu2, Proposition 4.1] and Proposition 6.2 shows
\[ \nu_x(\mathcal{V}_x^*SN) \leq c_{m-k-1} \cdot \Lambda_F^{(m-k)/2}. \]
If \(F = \alpha + \beta\) is a Randers metric, then \(F^* = \alpha^* + \beta^*\) is also a Randers metric. Denote by \(\Sigma\) the subspace \(\{(\xi_A, \xi_B) : \xi_A = 0, \forall A\}\) of \(T_x^*M\). And set \(F^*|_\Sigma = : \tilde{F}^* =: \tilde{\alpha}^* + \tilde{\beta}^*\). Clearly,
\[ \frac{1}{2}(\tilde{F}^*)(\xi_A, \xi_B) = g^{AB}_{\xi}, \]
where \(\xi \in \Sigma \setminus 0\). By [Sh2], we have
\[ \sup_{\xi \in \Sigma \setminus 0} \frac{\beta^*(\xi)}{\alpha^*(\xi)} \leq \sup_{\xi \in T_x^*M \setminus 0} \frac{\beta^*(\xi)}{\alpha^*(\xi)} = \|\beta^*\|_{\alpha^*} = b(x), \]
which implies that
\[ \det g^{AB}_{\xi} = (\det \alpha^{AB})(\tilde{F}^*(\xi))^{m-k+1} \leq (\det \alpha^{AB})(1 + b(x))^{m-k+1}. \]
By using the argument given in [Sh2], we obtain
\[ \int_{\mathcal{V}_x^*SN} dv_x \leq \frac{c_{m-k-1}}{(1 - b(x))^{\frac{m+k}{2}}}. \]

Proposition 8.4. Let \(F = \alpha + \beta\) be a Randers metric, were \(\alpha(y) = \sqrt{a_{ij}y^iy^j}\) and \(\beta(y) = b_ij y^j\). Let \(b_{ij}\) denote the covariant derivative corresponding with \(\alpha\). Set
\[ r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}), \quad s^i_j := a^{ik}s_{kj}, \quad s_j := b_is^i_j, \quad e_{ij} := r_{ij} + b_is_j + b_js_i. \]
The we have the following
\[ T_y(v) = -2 \left( \frac{c_{11}}{2F(y)} - s_1 \right) + 2s_{01} + \frac{1}{\alpha(y)} \left( \frac{e_{00}}{2F(y)} - s_0 \right) \left( \frac{\alpha(v) + \langle v, y \rangle}{\alpha(y)} \right) \]
\[ . \left( \frac{\alpha(v)\alpha(y) - \langle v, y \rangle}{\alpha(y)} \right), \]
where the index "0" (resp. "1") means the contraction with \(y^i\) (resp. \(v^i\)).
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Center of Mathematical Sciences, Zhejiang University, Hangzhou, China

E-mail address: zhaowei008@yahoo.cn