Equivariant Filter (EqF)

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Abstract—The kinematics of many systems encountered in robotics, mechatronics, and avionics are naturally posed on homogeneous spaces; i.e., their state lives in a smooth manifold equipped with a transitive Lie group symmetry. This article proposes a novel filter, the equivariant filter (EqF), by posing the observer state on the symmetry group, linearizing global error dynamics derived from the equivariance of the system, and applying EKF design principles. We show that equivariance of the system output can be exploited to reduce linearization error and improve filter performance. Simulation experiments of an example application show that the EqF significantly outperforms the EKF and that the reduced linearization error leads to a clear improvement in performance.

Index Terms—Algebra, control theory, kinematics, measurement, observers, robots, sensors.

I. INTRODUCTION

T

HE importance of Lie group symmetries in analyzing nonlinear control systems has been recognized since the 1970s [1]–[3]. Jurdjevic and Sussmann generalized the ideas of Brockett [1] around the theory of systems on matrix Lie groups to abstract Lie groups [2]. Cheng et al. [4] described necessary and sufficient conditions for observability of such systems. A comprehensive discussion of these early results can be found in Chapter 6 of Jurdjevic’s book [5].

In one of the earliest works applying Lie group symmetry for observer design, Salcudean [6] proposed a nonlinear observer for attitude estimation of a satellite using the quaternion representation of rotation. Thienel and Sanner [7] added an analysis of observability and bias estimation. Aghannan [8] proposed a general observer design methodology for Lagrangian systems by exploiting invariance properties. Driven by the emerging aerial robotics community and the need for robust and simple attitude observers, Mahony et al. [9] developed a nonlinear observer for rotation matrices posed directly on the matrix Lie group SO(3) with almost globally asymptotically stable error dynamics. In parallel, Bonnabel [10] proposed the left-invariant extended Kalman filter (EKF) and applied this to attitude estimation. Both of these observers are fundamentally derived from the symmetry properties of the underlying system and had significant impact in the robotics community. This motivated more general studies of systems on Lie groups and homogeneous spaces (smooth manifolds with transitive Lie group actions) [11]–[13]. In [14], Bonnabel et al. proposed the invariant extended Kalman filter (IEKF): A sophisticated observer design for systems on Lie groups with invariance properties. Given a system on a homogeneous space, a property of the system is termed equivariant if it changes in a compatible way under transformation of the state by the symmetry Lie group. Mahony et al. [15] considered observer design for equivariant kinematic systems on homogeneous spaces with equivariant output functions using Lyapunov design principles. This general design was extended in [16] to consider biased input measurements and general gain mappings.

Work by Barrau and Bonnabel [17] extended the IEKF from invariant systems to a broader class of “group affine” systems, again focusing on Lie groups only, and characterized the filter’s convergence properties. Lie group variational integrators, first proposed in [18], were applied to discretize an observer for rigid body attitude estimation by Itaiz and Sanyal in [19]. In [20], Johansen and Fossen proposed the exogenous Kalman filter, which Stovner et al. [21] showed to be globally exponentially stable when applied to attitude estimation by taking advantage of the invariant Lie group dynamics.

In the same spirit, Hamel and Samson [25] constructed a general Riccati observer for a class of systems of time-varying nonlinear systems and showed the local exponential stability of the origin of observer error as long as the linearized system about the true state is uniformly observable. Barrau and Bonnabel [26] provided a novel Lie group that models the classical simultaneous localization and mapping (SLAM) problem in robotics.

Parallel work [27] found the same structure and showed how it also models invariance in the SLAM problem, leading to a homogeneous state space structure. This led to recent work exploiting equivariance in visual (inertial) odometry [28]–[30] where there is no direct Lie group structure for the state space, and filter design methods for which a Lie group structure is necessary cannot be applied. The present article draws from recent work on observer design specifically targeting systems on homogeneous spaces [31]–[33].

In this article, we propose the equivariant filter (EqF): A novel filter design for equivariant kinematic systems posed on homogeneous spaces. The filter is derived by exploiting the...
Lie group symmetry of the kinematic system to derive global error coordinates. The EqF observer dynamics are defined on the symmetry Lie group; however, the correction is computed using a Riccati equation associated with linearized error kinematics about a fixed origin on the homogeneous state space. The proposed architecture fully exploits the symmetry properties of the system without requiring that the system model is posed explicitly on the Lie group and applies to any system with just the basic equivariance property. In contrast, the existing state-of-the-art observer/filter design methodologies for systems with symmetry depend on assuming properties of the system: Invariance for the constructive designs [15] or the more general group affine structure that is only defined for systems with Lie group state space for the I EkF designs [17]. Interestingly, the proposed EqF specializes to the IEKF [17] when the system considered is posed directly on a Lie group and displays the specific group affine structure required for the IEKF derivation. Barrau and Bonnabel [17] showed that the IEKF admits an exact linearization of the deterministic part of the state equation of the error dynamics, a property that the EqF shares on compatible systems, leading to significant performance gains versus an EKF derived without regard for the symmetry. In addition, the EqF is designed to accommodate symmetries that are compatible with the configuration output. When such a symmetry is used, we propose a novel approximation of the output equation that eliminates second-order error in the output linearization. Implementing the equivariant filter design methodology with this approximation leads to improved performance and we term the resulting observer the EqF*. The EqF methodology extends existing filter design methodologies for invariant systems by both applying to a broad class of systems on homogeneous spaces (rather than only systems with Lie group state space) and exploiting equivariance of the configuration output to further enhance filter performance compared to state-of-the-art.

We demonstrate the potential of the EqF and EqF* using a simple example of single-bearing estimation; i.e., the problem of determining the bearing of a fixed direction (with state space the sphere $S^2$) in an inertial frame with respect to a rotating frame with known angular velocity. There are no global coordinates on the sphere and application of the EKF requires consideration of local or embedded coordinates. Moreover, the state space is not a Lie group, precluding the direct application of filter design methods that require a Lie group state space. We simulate a classical EKF, the EqF, and the EqF* on this system. The simulation results demonstrate the known advantage [17] of exploiting symmetry versus (even a careful) EKF design in local coordinates and goes on to demonstrate a significant performance advantage for the EqF* over the standard EqF. The interested reader can also find a more tutorial exposition of equivariant observer design with a discussion of the equivariant filter in the preprint [33].

In Section II, the design of an EKF for the example of single-bearing estimation is detailed to motivate the developments in the sequel. In Section III, we define key notation and provide preliminary results. General systems on homogeneous spaces are defined and discussed in Section IV. In Section V, the notion of a lifted system is used to develop the dynamics of a global state error by exploiting the Lie symmetry. We also discuss the linearization of the error dynamics and show that the existence of an equivariant output leads to a better (lower error) linearization of the output map. In Section VI, the EqF equations are presented and we provide some insight into tuning the filter in practice.

The example problem of single-bearing estimation is revisited in Section VII. We show simulation results to demonstrate the performance of the EqF and EqF* compared to an EKF. Finally, Section VIII concludes this article. In Appendix A, we provide a step-by-step design methodology for implementing an EqF, and in Appendix B we show how the EqF specializes to the IEKF for a specific subclass of systems on Lie groups. We also provide open source code$^1$ for the implementation of the EqF for general systems based on numerical differentiation.

II. MOTIVATING EXAMPLE: SINGLE BEARING ESTIMATION

In this section we present the problem of single bearing estimation. This example has been chosen to be as simple as possible algebraically while presenting a problem where the equivariant filter approach is of interest. Consider a robot equipped with a gyroscope that measures its angular velocity $\Omega \in \mathbb{R}^3$ and a magnetometer that measures the magnetic field in the robot’s body-fixed frame $\eta \in S^2$. The noise free dynamics of $\eta$ are

$$\dot{\eta} = f_\Omega(\eta) := -\Omega^\times \eta$$

where $\Omega^\times \in \mathbb{R}^{3 \times 3}$ is the matrix

$$\Omega^\times := \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

The noise-free measurement model considered is a three-axis magnetometer

$$y = h(\eta) = c_m \eta \in \mathbb{R}^3$$

where $c_m$ is the (known) magnetic field strength. The system is nonlinear due to the manifold structure of the sphere $S^2$ comprising both its state and its measurement.

Since the system is nonlinear, a nonlinear filter design is required and the EKF is the industry standard choice. To provide context for the main contribution of the article we provide a sketch of the derivation of a classical EKF on the sphere for direction estimation. In Section VII, we will provide simulation results that compare the classical EKF for this problem to the proposed EqF that is the main contribution of the article.

A. EKF for Direction Estimation

The EKF requires a system to be written in Euclidean coordinates and one must either choose local coordinates for the sphere or embed the sphere in $\mathbb{R}^3$ and introduce a state constraint in the filter design [34]. The sphere $S^2$ has many possible choices of local coordinates, some of which cover the space almost globally. However, all choices introduce significant nonlinearities in the system dynamics, leading to loss of filter performance. To avoid this and provide a state-of-the-art EKF implementation, we instead choose to implement the EKF by embedding the system dynamics (1) and measurement (2) from the sphere into $\mathbb{R}^3$ and incorporating a nonlinear state constraint as in [34]. That is, the extended noise free system in $\mathbb{R}^3$ is

$$\dot{\eta} = -\Omega^\times \eta$$

$$h(\eta) = c_m \frac{\eta}{||\eta||}$$

$$g(\eta) = ||\eta||^2 = 1$$

$^1$https://github.com/STR-ANU/auto_eqf
where \( g \) is the nonlinear state constraint, and can be treated as an additional measurement with a small associated uncertainty due to linearization error [34]. This approach is particularly attractive for direction estimation on the sphere since the embedded dynamics (1) on \( \mathbb{R}^3 \) are linear time-varying. Let \( \tilde{\eta} \in \mathbb{R}^3 \) be the EKF state estimate and define the state error and innovation \( \hat{\eta} := \eta - \tilde{\eta} \in \mathbb{R}^3 \) and \( \hat{y} := h(\eta) - h(\tilde{\eta}) \). The output linearization is given by

\[
\begin{aligned}
\hat{y} &= \frac{c_m}{|\tilde{\eta}|} \left( I_3 - \frac{\tilde{\eta} \tilde{\eta}^\top}{|\tilde{\eta}|^2} \right) \hat{\eta} + O(|\hat{\eta}|^2) \\
\text{noting that } \hat{\eta} &\in \mathbb{R}^3. \text{ The constraint innovation } \tilde{z} = g(\eta) - \tilde{g}(\hat{\eta}) \text{ has linearization}
\end{aligned}
\]  

\[
\tilde{z} = -2\hat{\eta}^\top \hat{\eta} + O(|\hat{\eta}|^2).
\]

The linearized constraint is assigned a nonzero virtual covariance to reflect the linearization error, and this choice is a design parameter without stochastic justification.

The proposed EKF described above delivers the best performance for a classical EKF design for the direction estimation problem that the authors are aware of. However, it is clear that applying a classical EKF to this problem is not straightforward. The practitioner must choose between local and embedded coordinates. The former are not globally well-defined and introduce linearization errors. The latter require the introduction and linearization of state constraints. The resulting algorithms are not intrinsic in any sense, depending on a whole sequence of design decisions, and although the performance of the filters is accepted, a question always remains about whether a different choice of coordinates, or an embedding with a different state constraint could have improved the results.

### III. Preliminaries

For a comprehensive introduction to smooth manifolds and Lie groups, the authors recommend [35].

For a smooth manifold \( M \), let \( T_c M \) denote the tangent space of \( M \) at \( c \), let \( TM \) denote the tangent bundle, and let \( \mathfrak{X}(M) \) denote the infinite dimensional linear vector space of vector fields over \( M \). Given a vector field \( f \in \mathfrak{X}(M) \), \( f(c) \in T_c M \) denotes the value of \( f \) at \( c \in M \).

Given a differentiable function between smooth manifolds \( h : M \to N \), the linear map

\[
D_h c h(\xi) : T_c M \to T_{h(c)} N
\]

\( v \mapsto D_h c h(\xi)[v] \)

denotes the differential of \( h \) with respect to the argument \( \xi \) evaluated at \( c \). The shorthand \( D_h \) is used when the argument and base point are implied. In general, the composition of two maps \( h_1 \circ h_2 \) is written \( h_1 \circ h_2 \), with \( h_1 \circ h_2 (\xi) := h_1(h_2(\xi)) \). For linear maps \( H_1, H_2 \), the composition may also be written \( H_1 \circ H_2 \) to emphasize the linearity and the link to matrix multiplication. We apply this notation frequently in the context of the chain rule

\[
\begin{aligned}
D_\xi h(h_1 \circ h_2)(\xi)[v] &= D_{h_2} h_1(\eta) \cdot D_\xi h_2(\xi)[v] \\
D(h_1 \circ h_2)[v] &= D h_1 D h_2 [v].
\end{aligned}
\]

A general Lie group is denoted \( G \) and has Lie algebra \( \mathfrak{g} \). The identity element is written \( \text{id} \in G \). For any \( X \in G \), the left and right translations by \( X \) are denoted \( L_X \) and \( R_X \), respectively, and are defined by

\[ L_X(Y) := XY, \quad R_X(Y) := YX \]

where \( XY \) denotes the group product between \( X \) and \( Y \). The adjoint map \( \text{Ad} : G \times \mathfrak{g} \to \mathfrak{g} \) is defined by

\[ \text{Ad}_X[U] := DL_X \cdot DR_{X^{-1}}[U] \]

for every \( X \in G \) and \( U \in \mathfrak{g} \).

A right action of a Lie group \( G \) on a manifold \( M \) is a smooth map \( \phi : G \times M \to M \) that satisfies

\[
\phi(Y, \phi(X, \xi)) = \phi(XY, \xi)
\]

\[
\phi(\text{id}, \xi) = \xi
\]

for any \( X, Y \in G \) and any \( \xi \in M \). We will choose all symmetries in this article to be right actions, noting that any left action can be transformed to a right action by considering the inverse parameterization of the group [32]. Right-handed symmetries are naturally associated with the body-fixed sensor suites typical for most mobile robot applications. For a fixed \( X \in G \), the partial map \( \phi_X : M \to M \) is defined by \( \phi_X(\xi) := \phi(X, \xi) \).

Likewise, for a fixed \( \xi \in M \), the partial map \( \phi_\xi : G \to M \) is defined by \( \phi_\xi(X) := \phi(X, \xi) \).

An action \( \phi \) is called transitive if, for any \( \xi, \xi' \in M \), there exists \( X \in G \) such that \( \phi(X, \xi) = \xi' \).

A homogeneous space \( M \) is a manifold with a smooth and transitive symmetry action \( \phi : G \times M \to M \), where \( G \) is a Lie group. Let \( m \) denote the dimension of \( M \). For any element \( \xi \in M \) one may always choose an \( m \)-dimensional subspace \( m \subset \mathfrak{g} \) such that \( E_{\phi_\xi}(\phi(\xi), \xi) \) is a linear isomorphism \( m \to T_\xi M \). Let \( \cdot : \mathbb{R}^m \to m \subset \mathfrak{g} \) be a linear isomorphism that identifies \( m \) with \( \mathbb{R}^m \), and let its inverse be \( \cdot : m \to \mathbb{R}^m \). Then, at least in a local neighborhood \( \mathcal{U}_\xi \) of \( \xi \), the map \( \cdot : \mathcal{U}_\xi \subset M \to \mathbb{R}^m \), defined by the unique element \( \theta(\xi) \in \mathbb{R}^m \) such that \( \phi(\exp(\theta(\xi) \cdot), \xi) = \xi \), is well defined and smooth. The map \( \cdot : \mathcal{U}_\xi \to \mathbb{R}^m \) is called a normal coordinate chart of the homogeneous space (about \( \xi \) [36]. Note that normal coordinates are not unique since the choice of subspace \( m \) and its identification with \( \mathbb{R}^m \) are arbitrary.

**Proposition III.1**: Any right action \( \phi : G \times M \to M \) induces a right action on the vector fields over \( M \), denoted \( \Phi : G \times \mathfrak{X}(M) \to \mathfrak{X}(M) \), and defined by

\[
\Phi(X, f) := D \phi_X \cdot f \circ \phi_X^{-1}
\]

for any \( f \in \mathfrak{X}(M) \) and \( X \in G \). For a fixed \( X \in G \), \( \Phi_X \) is a linear map on \( \mathfrak{X}(M) \).

**Proof**: For fixed \( X \in G \), \( \phi_X : M \to M \) is a diffeomorphism and the map \( \Phi(X, \cdot) \) is the push forward operator [35]. To see that \( \Phi \) is a group action, let \( X, Y \in G \) and \( f \in \mathfrak{X}(M) \). Then

\[
\Phi(Y, \Phi(X, f)) = D \phi_Y D \phi_X f \circ \phi_X^{-1} \circ \phi_Y^{-1} = D \phi_{XY} f \circ \phi_{Y^{-1} X^{-1}} = \Phi(YX, f).
\]

To prove linearity let \( c_1, c_2 \in \mathbb{R} \) and let \( f_1, f_2 \in \mathfrak{X}(M) \), then

\[
\Phi(X, c_1 f_1 + c_2 f_2) = D \phi_X (c_1 f_1 + c_2 f_2) (\phi_X^{-1}(\xi)) = c_1 D \phi_X (f_1) (\phi_X^{-1}(\xi)) + c_2 D \phi_X (f_2) (\phi_X^{-1}(\xi)) = c_1 \Phi(X, f_1)(\xi) + c_2 \Phi(X, f_2)(\xi)
\]
where the second-last line follows from the linearity of the differential $D\phi_X$.

\[ \square \]

### IV. Problem Description

#### A. Systems on Homogeneous Spaces

Let $\mathcal{M}$ be a smooth $m$-dimensional manifold termed the state space. An affine (kinematic) system on $\mathcal{M}$ may be written

$$\dot{x} = f_0(x) + \sum_i f_i(x)u_i$$

for some vector fields $f_0, f_1, \ldots, f_i \in \mathcal{X}(\mathcal{M})$ and scalar input signals $u_1, \ldots, u_i \in \mathbb{R}$. Such a system is represented by an affine system function [33]

$$f : \mathbb{L} \mapsto \mathcal{X}(\mathcal{M})$$

$$u \mapsto f_u \in \mathcal{X}(\mathcal{M})$$

(7)

where $f_u(x) := f_0(x) + \sum_i f_i(x)u_i$ and $\mathbb{L}$ is a real vector space termed the input space with $u = (u_1, \ldots, u_i) \in \mathbb{L}$ the components of $u$. We refer to $f_0$ as the drift term of the system and $f_i$ as the input vector fields. Trajectories $x(t) \in \mathcal{M}$ on a time interval $[0, \infty)$ of the system considered are solutions of the ordinary differential equation

$$\dot{x} = f_u(t)(x), \quad x(0) \in \mathcal{M}$$

(8)

with initial condition $x(0)$ and measured input signal $u(t) \in \mathbb{L}$. We will assume $u(t)$ is sufficiently smooth to ensure unique well-defined solutions for all time. The configuration output [33] for a kinematic system is a function

$$h : \mathcal{M} \mapsto \mathcal{N} \subset \mathbb{R}^n$$

(9)

where $\mathcal{N}$ is a smooth manifold termed the output space, embedded in $\mathbb{R}^n$.

Let $\mathbf{G}$ be a Lie group with Lie algebra $\mathfrak{g}$, and suppose that $\mathcal{M}$ is a homogeneous space of $\mathbf{G}$; i.e., there exists a smooth, transitive, right group action of $\mathbf{G}$ on $\mathcal{M}$

$$\phi : \mathbf{G} \times \mathcal{M} \mapsto \mathcal{M}.$$  

(10)

A lift [33] for the system function (8) is a map $\Lambda : \mathcal{M} \times \mathbb{L} \mapsto \mathfrak{g}$ satisfying

$$D_{\mathcal{M}}\phi_X(X)[\Lambda(x, u)] = f_u(x)$$

(11)

for every $x \in \mathcal{M}$ and $u \in \mathbb{L}$. Any kinematic system (8) defined on a homogeneous space admits a lift $\Lambda : \mathcal{M} \times \mathbb{L} \mapsto \mathfrak{g}$ satisfying (11) [15], [33]. In the particular case where $D_{\mathcal{M}}\phi_X(X)$ is invertible ($\phi$ is a free group action [35]), the lift is unique [33].

#### B. Equivariant Systems

Equivariance of a system is a powerful structural property that can be formulated for any system on a homogeneous space. There are many established examples of equivariant systems [9], [11], [16], [17], [19], [21], [28], [37] where exploiting the equivariant structure has led to high performance observers and filters.

A kinematic system (8) is termed equivariant if there exists a smooth right group action $\phi : \mathbf{G} \times \mathbb{L} \mapsto \mathbb{L}$, such that

$$D\phi_X f_u(x) = f_{\phi_X(u)}(\phi_x(x))$$

for all $u \in \mathbb{L}, x \in \mathcal{M}$, and $X \in \mathbf{G}$. In this case, recalling (6), one has

$$f_{\phi_X(u)}(\xi) = f_{\phi_X(u)}(\phi_X(\phi^{-1}_X(\xi)))$$

$$= D\phi_X f_u(\phi_X^{-1}(\xi))$$

$$= \phi_X f_u(\xi).$$

That is, $f_{\phi_X(u)} = \phi_X f_u$ as vector fields on $\mathcal{M}$, or equivalently, the diagram

$$\begin{array}{c}
\mathbb{L} \\
\psi_X \downarrow \quad \downarrow \psi_X \\
\mathcal{X}(\mathcal{M}) \quad \mathcal{X}(\mathcal{M}) \\
\phi \downarrow \quad \downarrow \phi_X \\
\mathcal{M} \times \mathbb{L} \quad \mathcal{M} \times \mathbb{L}
\end{array}$$

commutes for every $X \in \mathbf{G}$.

Suppose the system (8) is equivariant with state symmetry $\phi : \mathbf{G} \times \mathcal{M} \mapsto \mathcal{M}$ and input symmetry $\psi : \mathbf{G} \times \mathbb{L} \mapsto \mathbb{L}$. A lift $\Lambda$ for the system is equivariant if

$$\Lambda(\phi(X, \xi), \psi(X, u)) = \Lambda \cdot \Lambda(\xi, u)$$

(13)

for all $X \in \mathbf{G}, \xi \in \mathcal{M}$, and $u \in \mathbb{L}$, or equivalently, the diagram

$$\begin{array}{c}
\mathbb{L} \\
\Lambda \uparrow \quad \uparrow \Lambda \\
\mathcal{M} \times \mathcal{L} \quad \mathcal{M} \times \mathcal{L} \\
\phi_X \times \psi_X \downarrow \quad \downarrow \phi_X \times \psi_X \\
\mathcal{M} \times \mathbb{L} \quad \mathcal{M} \times \mathbb{L}
\end{array}$$

commutes for every $X \in \mathbf{G}$.

Remark IV.1: On a matrix Lie group where the group action is right translation, the lift is just $\Lambda(X, u) = X^{-1}X^{-1}f_u(X) \in \mathfrak{g}$. However, for a general system on a homogeneous space there may be many choices of equivariant lift. In such a case, an equivariant lift can be found by expanding the conditions (11) and (13) for the particular system, and searching for a solution for $\Lambda$ [32].

The system (7) has equivariant output if there exists an action $\rho : \mathbf{G} \times \mathcal{N} \mapsto \mathcal{N}$ satisfying

$$\rho(X, h(\xi)) = h(\phi(X, \xi))$$

(14)

for all $X \in \mathbf{G}$ and $\xi$, or equivalently, the diagram

$$\begin{array}{c}
\mathcal{M} \\
\phi_X \downarrow \quad \downarrow \phi_X \\
\mathcal{M} \quad \mathcal{M} \\
\mathcal{N} \quad \mathcal{N} \\
\rho_X \downarrow \quad \downarrow \rho_X \\
\mathcal{N} \quad \mathcal{N}
\end{array}$$

commutes for every $X \in \mathbf{G}$.

### V. Equivariant Error System

The filter design proposed in the sequel can be applied to any equivariant system where an equivariant lift can be found. In [32], the authors showed that any system on a homogeneous space can be extended to an equivariant system, and for any equivariant system an equivariant lift can always be constructed although the resulting construction may be infinite dimensional in the input space. Thus, in principle, the proposed EqF design applies to all systems on homogeneous spaces, although the authors note that the primary systems of interest are those which are equivariant directly, or for which the extension terminates in a finite dimensional input space.
A. Observer Architecture

Consider an equivariant kinematic system (8) with state symmetry $\phi : G \times M \rightarrow M$ and input symmetry $\psi : G \times \mathbb{L} \rightarrow \mathbb{L}$. Let $\Lambda : M \times L \rightarrow g$ be an equivariant lift for this system. Given a fixed but arbitrary $\xi \in M$ and a known input signal $u(t) \in \mathbb{L}$, the lifted system [32] is defined by the ordinary differential equation (ODE)

$$\dot{X} = DL_X \Lambda(\phi(\hat{X}, \hat{\xi}), u), \quad \phi(X(0), \hat{\xi}) = \xi(0) \quad (15)$$

where $X(t) \in G$. The trajectory of the lifted system projects down to the original system trajectory [32] by

$$\phi(X(t), \hat{\xi}) \equiv \xi(t). \quad (16)$$

Define the observer state to be an element of the group $\hat{X} \in G$, and use the lifted system as the internal model for the observer dynamics

$$\dot{\hat{X}} = DL_{\hat{X}} \Lambda(\phi(\hat{X}, \hat{\xi}), u) + DR_{\hat{X}} \Delta, \quad \hat{X}(0) = id \quad (17)$$

where the correction term $\Delta$ remains to be chosen [32], [33].

The state estimate of the observer is given by the projection

$$\hat{\xi} = \phi(\hat{X}(t), \hat{\xi}).$$

Thus, the state of the observer is posed on the symmetry group rather than the state space of the system kinematics. The correction term $\Delta$ will be chosen by applying a Riccati observer to global error dynamics linearized about the fixed origin $\hat{\xi}$.

B. Global Error Dynamics

Let $\xi \in M$ be the true state of the system. Choose an arbitrary fixed origin $\hat{\xi} \in M$ and let $\hat{X} \in G$ be a state observer with dynamics given by (17). Define the global state error

$$e := \phi(\hat{X}^{-1}, \xi). \quad (18)$$

Note that $\phi(\hat{X}, \hat{\xi}) = \xi$ if and only if $e = \hat{\xi}$. Therefore, the goal of the filter design will be to drive $e \rightarrow 0$. Define the origin velocity

$$\hat{u} := \psi(\hat{X}^{-1}, u). \quad (19)$$

Note that the origin velocity $\hat{u}(t)$ can always be constructed since both $\hat{X}(t)$ and $u(t)$ are available to the observer. The action $\psi(\hat{X}^{-1}, u)$ is the equivariant system generalization of the well known adjoint action $A\xi U$ that transforms between left and right invariant algebra elements for invariant vector fields on a Lie group. In the error dynamics (20) derived below, the action $\psi_{\hat{X}^{-1}}$ transforms the measured system input into the correct representation for the error dynamics around the chosen origin $\hat{\xi}$.

Lemma VI.1: Let the global state error $e$ be defined as in (18) and the origin velocity $\hat{u}$ be defined as in (19). The dynamics of $e$ are given by

$$\dot{e} = D\phi_e \left( \Lambda(e, \hat{u}) - \Lambda(\hat{\xi}, \hat{u}) - \Delta \right) \quad (20)$$

and depend only on $e$, $\hat{u}$ and the correction $\Delta$.

Proof: Let $X(t)$ be a solution to the lifted system (15) satisfying $\phi(X(0), \hat{\xi}) = \xi(0)$. Then, $\xi \equiv \phi(X, \hat{\xi})$, and

$$e = \phi(X^{-1}, \xi) = \phi(X \hat{X}^{-1}, \hat{\xi}) = \phi(E, \hat{\xi}) \quad (21)$$

where $E := X \hat{X}^{-1}$. Computing the dynamics of $E$, one has

$$\dot{E} = DR_{\hat{X}^{-1}} DL_X \Lambda(\phi(\hat{X}, \hat{\xi}), u)$$

$$= DL_X \Lambda(\phi(\hat{X}, \hat{\xi}), u) + DL \Lambda(\phi\hat{X}, \hat{\xi}, u) + DR_{\hat{X}^{-1}} \Delta$$

$$= DL \left( \Lambda(\phi(E, \hat{\xi}), u) - \Lambda(\phi(\hat{X}, \hat{\xi}), u) \right) - DL \Delta$$

$$= DL \left( \Lambda(\phi(E, \hat{\xi}), \psi(\hat{X}^{-1}, u)) - \Lambda(\hat{\xi}, \psi(\hat{X}^{-1}, u)) \right)$$

$$- DL \Delta \quad (22)$$

where the last line follows from the equivariance of the lift. Recalling (19), the dynamics of $e$ follow from (21) and (22)

$$\dot{e} = D\phi_e \dot{\hat{X}}$$

$$= D\phi_e D\phi(\hat{X}, \hat{\xi}) DL \dot{\hat{X}}$$

$$= D\phi_e \left( \Lambda(\phi(E, \hat{\xi}), \psi(\hat{X}^{-1}, u)) - \Lambda(\hat{\xi}, \psi(\hat{X}^{-1}, u)) \right)$$

$$= D\phi_e \left( \Lambda(e, \hat{u}) - \Lambda(\hat{\xi}, \hat{u}) - \Delta \right)$$

as required.

C. Linearization

1) Error Dynamics: Let $e \in M$ denote the global state error (18), and let $\hat{u} \in L$ denote the origin velocity (19). Fix a local coordinate chart $\theta : U_{\hat{\xi}} \rightarrow \mathbb{R}^m$ where $U_{\hat{\xi}} \subset M$ is a neighborhood of the fixed origin $\hat{\xi}$ and $\theta(\hat{\xi}) = 0$. Let $e$ be the local coordinates of the state error

$$e = \theta(e). \quad (23)$$

The EqF correction is designed by linearizing the preobserver error dynamics of $e$ at zero; i.e., the dynamics (20) with the correction term $\Delta \equiv 0$ set to zero.

Lemma V.2: Let $\theta$ be local coordinates on $M$ in an open neighborhood $U_{\hat{\xi}} \subset M$ around $\hat{\xi}$. Assume $e(t) \in U_{\hat{\xi}}$ for all time. The linearized preobserver $(\Delta \equiv 0)$ dynamics of $e(t)$ about $\varepsilon = 0$ are

$$\dot{e} = \hat{A}_e \varepsilon + O(|\varepsilon|^2) \quad (24)$$

$$\hat{A}_e := D\varepsilon |\hat{\xi} \theta(\varepsilon) \cdot D\varepsilon |\hat{\xi} \theta(\varepsilon) \cdot D\varepsilon |\hat{\xi} \Lambda(e, \hat{u}) \cdot D\varepsilon |\theta^{-1}(\varepsilon). \quad (25)$$

Proof: The nonlinear preobserver dynamics of the global state error (20) in local coordinates $e = \theta(e)$ are

$$\dot{e} = D\theta \cdot D\phi_{\theta^{-1}(e)} (\Lambda(\theta^{-1}(e), \hat{u}) - \Lambda(\hat{\xi}, \hat{u})). \quad (26)$$

Clearly, $\Lambda(\theta^{-1}(0), \hat{u}) = \Lambda(\hat{\xi}, \hat{u})$ since $\theta^{-1}(0) = \hat{\xi}$. Hence, linearizing $\hat{e}$ about $\varepsilon = 0$ yields

$$D\theta \cdot D\phi_{\theta^{-1}(e)} (\Lambda(\theta^{-1}(e), \hat{u}) - \Lambda(\hat{\xi}, \hat{u}))$$

$$= D\theta \cdot D\phi_{\theta^{-1}(0)} (\Lambda(\theta^{-1}(0), \hat{u}) - \Lambda(\hat{\xi}, \hat{u})) \quad (27)$$
\[ + D_{\varepsilon}e_0 \left( D\theta \cdot D\phi_{\theta^{-1}(\varepsilon)}(\Lambda(\theta^{-1}(\varepsilon), \dot{u}) - \Lambda(\tilde{\xi}, \dot{u}) \right) e \]
\[ + O(|\varepsilon|^2) \]
\[ = D_{\varepsilon}e_0 \left( D\theta \cdot D\phi_{\theta^{-1}(\varepsilon)}(\varepsilon) \right) \cdot \left( \Lambda(\theta^{-1}(0), \dot{u}) - \Lambda(\tilde{\xi}, \dot{u}) \right) \]
\[ + D\theta \cdot D\phi_{\theta^{-1}(0)} \cdot D_{\varepsilon}e_0 \left( \Lambda(\theta^{-1}(\varepsilon), \dot{u}) - \Lambda(\tilde{\xi}, \dot{u}) \right) e \]
\[ + O(|\varepsilon|^2) \]
\[ = D\theta \cdot D\phi_{\theta}(\Lambda(e, \dot{u}) - D\theta^{-1}[e] + O(|\varepsilon|^2) \]
\[ = \hat{A}_{\varepsilon}e + O(|\varepsilon|^2). \]  

(27)

(28)

(29)

Here (27) is the first-order Taylor expansion in local coordinates about \( \varepsilon = 0 \). Equation (28) follows since \( \Lambda(\theta^{-1}(0), \dot{u}) - \Lambda(\tilde{\xi}, \dot{u}) = 0 \) and by expanding the first-order term using the product rule. Equation (29) is the result of applying \( \Lambda(\theta^{-1}(0), \dot{u}) - \Lambda(\tilde{\xi}, \dot{u}) = 0 \) to eliminate the first term, while noting that \( D_{\varepsilon}e_0\Lambda(\tilde{\xi}, \dot{u}) \equiv 0 \) and applying the chain rule to simplify the second term in (28). The final line follows from (25).

The primary role of the error dynamics linearization is in the covariance propagation instantiated in the Ricatti (38) where the covariance is propagated along the preobserver trajectories [33]. Barrua and Bonnabel [17] showed that, if the manifold \( M = G \), the origin is chosen \( \xi = id \), the local coordinates are chosen \( \gamma(\varepsilon) := \log(e) \), and the system dynamics are “group affine” (cf. Appendix B), then the preobserver dynamics are exact, \( \varepsilon = \hat{A}_{\varepsilon}e \). This exact linearization of the preobserver error dynamics removes the \( O(|\varepsilon|^2) \) linearization error and significantly improves the performance of the filter by reducing linearization error in the Ricatti equation. Even without the group affine property, the equivariant structure of the error dynamics provides significant advantages over the standard EKF. The error dynamics are linearized at a single point \( \xi \) and using a single coordinate chart \( \gamma \) which can be designed intentionally to minimize the \( O(|\varepsilon|^2) \) linearization error. As we show in Lemma V.3, it is also possible to exploit the equivariant system structure to reduce linearization error in the output approximation, further improving the filter performance.

2) System Output: Consider the state error \( e \) (18), and let \( \xi \in M \) and \( \dot{X} \in G \) denote the true system state and observer state, respectively. Let \( e \in \mathbb{R}^m \) represent local coordinates for \( e \) as in (23). The output \( y = h(\xi) \) can be written
\[ h(\xi) = h(\phi(\dot{X}, e)) = h(\phi(\dot{X}(\theta^{-1}(\varepsilon)))) \]
\[ \text{(30)} \]

Note that substituting \( \varepsilon = 0 \) gives
\[ h(\phi(\dot{X}(\theta^{-1}(0)))) = h(\phi(\dot{X}(\tilde{\xi}))) = h(\hat{\xi}). \]

In common with error state Kalman filters, the output \( y = y(\varepsilon) \) is considered as a function of the error (30) while the predicted output \( \hat{y} = h(\hat{\xi}) \) is considered an independent signal. Define the output residual
\[ \hat{y} = y(\varepsilon) - \hat{y}.\]

(31)

Linearizing \( \hat{y} \) as a function of \( e \in \mathbb{R}^m \) around \( \varepsilon = 0 \) yields
\[ \hat{y} = C_{\varepsilon}e + O(|\varepsilon|^2) \]
\[ \text{(32)} \]
\[ C_{\varepsilon} := D_{\varepsilon}e_0 h(\hat{\xi}) \cdot D_{\varepsilon}e_0 \phi_{\dot{X}}(\epsilon) \cdot D_{\varepsilon}e_0 \theta^{-1}(\varepsilon). \]

(33)

The matrix \( C_{\varepsilon} \in \mathbb{R}^{n_x \times m} \), the Jacobian of (30), is termed the standard output matrix.

3) Equivariant Output Linearization: When the system has output equivariance, the linearization of the output function can be improved to obtain \( O(|\varepsilon|^3) \) error. A filter’s ability to incorporate information from measurements correctly is fundamental to its robustness and transient performance. As shown in the simulation results in Section VII, the equivariant output linearization presented below greatly improves the EqF performance both with and without the presence of noise in the measurement signals.

\[ \hat{y} = C_{\varepsilon}^* e + O(|\varepsilon|^3) \]

(34)

\[ C_{\varepsilon}^* e = \frac{1}{2} \left( D_{E}|_{\alpha} D_{E}(E, y) + D_{E}|_{\alpha} D_{E}(E, \hat{y}) \right) A_{\dot{X}}^{-1} e \]

(35)

where \( \dot{\gamma} : \mathbb{R}^m \to m \subset g \) is the identification of \( m \) with \( \mathbb{R}^m \) used in defining \( \theta \).

Proof: By construction \( \theta^{-1}(\varepsilon) = \phi(\varepsilon) = \phi(\varepsilon, \hat{\xi}) \) and \( E, y \). Recalling (30) and (31) one has

\[ \hat{y}(\varepsilon, \dot{X}) = h(\phi(\dot{X}, \phi(\varepsilon, \dot{X}))) - h(\phi(\dot{X}, \hat{\xi})). \]

Clearly, \( \hat{y}(0; \dot{X}) = 0 \). Using the equivariance of \( h \), one has

\[ h(\phi(\dot{X}, \phi(\varepsilon, \dot{X}))) \]

\[ = h(\phi(\varepsilon, \dot{X}, \hat{\xi})) \]
\[ = h(\phi(\dot{X}, \dot{X}^{-1} \exp(\varepsilon) \hat{\xi})) \]
\[ = h(\phi(\exp(\varepsilon), \hat{\xi})) \]
\[ = h(\phi(\exp(\varepsilon), \tilde{\xi}, \hat{\xi})) \]
\[ = h(\out{\exp(\varepsilon)} \tilde{\xi}, \hat{\xi}). \]

Setting \( \hat{y} = h(\hat{\xi}) \) and differentiating \( \hat{y} \) at \( \varepsilon = 0 \) in a direction \( \gamma \in \mathbb{R}^m \) yields

\[ D_{\varepsilon}e \out{\hat{y}}(x, \dot{X}) | \gamma = D_{E}|_{\alpha} D_{E}(E) A_{\dot{X}}^{-1} e. \]

Although this formula allows for an arbitrary \( \gamma \in \mathbb{R}^m \), the linearization is computed for \( \gamma = e \).

The fact that the linearization is computed in the same direction \( \varepsilon \) as the coordinates of the error can be exploited along with equivariance to obtain a second linearization point. In particular, we will compute the differential of \( y \) at \( \varepsilon = \theta(\varepsilon) \) in direction \( \varepsilon \). Note that \( \phi_{\hat{\xi}}(\exp(\varepsilon)) = \phi(\hat{X}^{-1}, \xi) \), and therefore \( \phi_{\hat{\xi}}(\exp(\varepsilon)) \hat{X} = \xi \).
\[ \frac{d}{dt} \left. \rho(\exp(t \text{Ad}_X^{-1} \varepsilon), h(\xi)) \right|_{t=0} = D_{\xi} \rho(E, y) \text{Ad}_X^{-1} \varepsilon^\wedge. \]

Although the differential is posed at the unknown error state \( \varepsilon \), it is evaluated using only the known measurement data \( y \).

Consider the Taylor expansion of the differential \( D_{x, \varepsilon} \hat{y}(x; \hat{X}) \) with respect to \( \varepsilon \) around \( \varepsilon = 0 \)

\[ D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon] = D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon] + D_{x, \varepsilon}^2 \hat{y}(x; \hat{X})[\varepsilon, \varepsilon] + O(|\varepsilon|^3) \]

and hence

\[ D_{x, \varepsilon}^2 \hat{y}(x; \hat{X})[\varepsilon, \varepsilon] = D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon] - D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon] + O(|\varepsilon|^3). \]

The result (35) follows from taking the Taylor expansion of \( \hat{y} \) with respect to \( \varepsilon \) and substituting

\[ \hat{y}(\varepsilon; \hat{X}) = \hat{y}(0; \hat{X}) + D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon] + \frac{1}{2} D_{x, \varepsilon}^2 \hat{y}(x; \hat{X})[\varepsilon, \varepsilon] + O(|\varepsilon|^3) \]

\[ = \frac{1}{2} (D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon] + D_{x, \varepsilon} \hat{y}(x; \hat{X})[\varepsilon]) + O(|\varepsilon|^3) \]

\[ = \frac{1}{2} (D_{E, \varepsilon} \rho(E, y) + D_{E, \varepsilon} \rho(E, \hat{y})) \text{Ad}_X^{-1} \varepsilon^\wedge + O(|\varepsilon|^3). \]

\[ \square \]

### VI. Equivariant Filter

Consider a kinematic system (7) with a state symmetry \( \phi : G \times M \to M \). Assume the system is equivariant with input symmetry \( \psi : G \times \mathbb{L} \to \mathbb{L} \) and has an equivariant lift \( \Lambda : M \times \mathbb{L} \to \mathbb{L} \). Let \( \xi \in \mathcal{M} \) denote the true state of the system, with trajectory determined by the measured input \( u \in \mathbb{L} \). Denote the configuration output \( y = h(\xi) \).

We construct the EqF as follows. Let \( \hat{X} \in G \) denote the observer state. Pick an arbitrary fixed origin \( \hat{\xi} \in M \). For a general output map, set \( C_t \) to be the standard output matrix defined in (32). If the system has output equivalence then set \( C_t = C_t^* \) to be the equivariant output matrix as defined in (35).

In this case the resulting algorithm is termed the EqF\(^*\). Let \( \hat{A}_t \) denote the state matrix as defined in (25). Choose an initial value for the Riccati term \( \Sigma_0 \in S_+(m) \), where \( S_+(m) \) is the set of positive-definite symmetric \( m \times m \) matrices, and pick a state gain matrix \( M_t \in S_+(m) \) and an output gain matrix \( N_t \in S_+(n) \). Choose a right inverse \( D_{[E]} h_\phi(\hat{\xi})^\wedge \) of \( D_{[E]} h_\phi(\hat{\xi})(E) \); i.e.,

\[ D_{[E]} h_\phi(\hat{\xi})^\wedge \cdot D_{[E]} h_\phi(\hat{\xi})(E)^\wedge = \text{id}. \]

The proposed equivariant filter is given by the solution of

\[ \dot{\hat{X}} = DL_{\hat{X}} \Lambda(\phi(\hat{X}, \hat{\xi}), u) + DR_X \Delta, \quad \hat{X}(0) = \text{id} \]

\[ \Delta = D_{[E]} h_\phi(\hat{\xi})^\wedge D_{\theta} \Sigma C_t^{-1} N_t^{-1}(y - h(\phi(\hat{X}, \hat{\xi}))) \]

\[ \Sigma = A_t \Sigma + \Sigma A_t^T + M_t - \Sigma C_t^{-1} N_t^{-1} C_t \Sigma, \quad \Sigma(0) = \Sigma_0. \]

If the pair \((\hat{A}_t, C_t)\) is uniformly observable in the sense of Proposition 1 of [38], then \( \Sigma(t) \) is bounded above and below, and the Ricatti (38) is well-defined for all time [39].

Provided that the error trajectory \( \varepsilon(t) \) remains well-defined for all time \( t \geq 0 \), [40, Th. 1.1.1] provides sufficient conditions for the convergence of \( \varepsilon \to 0 \). In particular, if the error system is uniformly observable, the second derivative of the error dynamics is bounded, and the initial error is sufficiently small, then the error \( \varepsilon \) and the Lyapunov function

\[ L(t) := \varepsilon^T \Sigma^{-1} \varepsilon \]

converge exponentially to zero as \( t \to \infty \) [40].

**Remark VI.I:** Recent work by the authors [33], [37], [41] showed that the Ricatti (38) can be augmented by a curvature modification that compensates for the reset process, where the linearization point is continually translated to track the observer state, in the EKF derivation [33]. Several works have shown that, during the transient at least, an appropriate curvature modification term can improve filter performance [33], [42]. Curvature is directly connected to parallel transport on a manifold and is an additional structure that can be chosen independently from the homogeneous space structure. One possible choice is to define the normal coordinates to be flat, i.e., parallel transport on the manifold is just translation in local coordinates. The Riccati equation (38) corresponds to this choice. Such a choice has the advantage of simplicity; however, the associated affine connection will usually have nonzero torsion. The relative benefit or consequence of choosing different geometries, with different curvatures, along with symmetry or torsion of the associated connections remains an open question in equivariant systems theory.

### A. EqF Gain Tuning

The choice of gain matrices \( \Sigma_0 \), \( M_t \), and \( N_t \) can greatly influence the performance of the EqF. In the context of a Kalman–Bucy filter, \( \Sigma_0 \) reflects the uncertainty in the initial state estimate, and \( M_t \) and \( N_t \) are optimally chosen to be the intensities (covariances) of zero-mean Gaussian noise terms added to the filter dynamics and output, respectively. Similar choices can be made for the EqF. The initial value of the Riccati term \( \Sigma_0 \) can be chosen to reflect the uncertainty in the initial state estimate as expressed in the chosen local coordinates. Suppose the measured velocity \( d\mu_m = d\nu_m + d\psi_m \) and the measured output \( d\theta_m = d\psi \), where \( d\mu_m \sim \mathbb{W}(0, M_t^{\mu_m}) \) and \( d\nu_m \sim \mathbb{W}(0, N_t^{\mu_m}) \) are Wiener processes. Then, relinearizing the preobserver error dynamics (20) with \( \Delta \equiv 0 \) and output (30) yields

\[ d\varepsilon = \hat{A}_t d\varepsilon + B_t d\mu_m \]

\[ d\hat{y} = C_t d\varepsilon + d\nu_y \]

where the input matrix \( B_t \) is obtained by linearizing the preobserver error dynamics with respect to a perturbation of the measured input

\[ B_t := D_{[E]} \vartheta(\hat{\xi}) \cdot D_{[E]} h_\phi(\hat{\xi})(E) \cdot \text{Ad}_X^{-1} D_{[E]} u_n \Lambda(\hat{\xi}, u) \]

\[ = D_{[E]} \vartheta(\hat{\xi}) \cdot D_{[E]} h_\phi(\hat{\xi})(E) \cdot D_w \hat{\mu}_n \Lambda(\hat{\xi}, w) \cdot \psi \}

Based on this formulation, the EqF gain matrices can be chosen by

\[ M_t = M + B_t M_t^{\mu_m} B_t^T \]
\[ N_t = N_e + N^m_t. \]  

The matrices \( M_e \in S_+(m) \) and \( N_e \in S_+(n) \) are optimally set to zero in the case of a linear system, but can otherwise be used by the practitioner to model the error introduced to the dynamics and output by linearization.

**VII. EXAMPLE REVISITED: SINGLE BEARING ESTIMATION**

The system described in Section II of bearing estimation on the sphere provides an illustrative example of a system on a homogeneous space where the EqF design methodology may be applied. For additional examples of EqF applications, we refer the reader to [30], [31], [33].

**A. Equivariant System**

Here we describe the design preliminaries for the EqF and EqF* for the single bearing estimation problem following Algorithm 1 as described in Appendix A.

1) **State Symmetry:** Consider the Lie group of 3-D rotations

\[ \text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^\top R = I_3, \ \det(R) = 1 \}. \]

This group has a right action on the sphere \( \phi : \text{SO}(3) \times S^2 \to S^2 \) given by

\[ \phi(R, \eta) := R^\top \eta. \]  

2) **System Equivariance:** Define the map \( \psi : \text{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3 \) to be

\[ \psi(R, \Omega) := R^\top \Omega. \]  

Then, the system (1) is equivariant with respect to the state action \( \phi \) and input action \( \psi \). To see this, let \( \Omega \in \mathbb{R}^3, \eta \in S^2, \) and \( R \in \text{SO}(3) \) be arbitrary, and compute

\[ \Phi(R, f_{11})(\eta) = D\phi_R(-\Omega^\top R\eta) = -R^\top \Omega^\top R\eta = -R^\top \Omega^\top \eta = f_{11}(\eta). \]

3) **Equivariant Lift:** The Lie algebra \( \mathfrak{so}(3) \) of \( \text{SO}(3) \) can be written as the subspace of skew-symmetric matrices

\[ \mathfrak{so}(3) := \{ \omega \in \mathbb{R}^{3 \times 3} \mid \omega^\top = -\omega \}. \]

Define the candidate lift function \( \Lambda : S^2 \times \mathbb{R}^3 \to \mathfrak{so}(3) \) by

\[ \Lambda(\eta, \Omega) := \Omega^\times. \]  

To check that it is indeed a lift, evaluate the lift condition (11)

\[ D_{\eta} \Lambda_{\eta}(R) \left[ \Lambda(\eta, \Omega) \right] = D_{\eta} \Lambda_{\eta}(R) \left[ \Omega^\times \right] = (\Omega^\times)^\top \eta = -\Omega^\times \eta = f_{11}(\eta), \]

as required. Next, check the equivariance of \( \Lambda \) as in (13)

\[ \Lambda(\phi(R, \eta), \psi(R, \Omega)) = \Lambda(R^\top \eta, R^\top \Omega) = (R^\top \Omega)^\times \]

as required.

**4) Output Equivariance:** The action \( \rho : \text{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3 \) given by

\[ \rho(R, y) = R^\top y \]

ensures the system (2) has output equivariance since

\[ \rho(R, h(\eta)) = R^\top (c_m \eta) = c_m R^\top \eta = h(\phi(R, \eta)). \]

5) **Origin and State Error:** Fix the origin element \( \hat{\eta} = e_1 \in S^2 \), and let \( \hat{R} = \hat{R}(\hat{\eta}) \) denote the observer state. The global state error is given by

\[ e = \phi(\hat{R}^{-1}, \eta) = \hat{R}\eta \]

where \( \eta \in S^2 \) is the true state of the system.

Since the system exhibits output equivariance, we choose to use normal coordinates. Explicitly, define

\[ m := \{ v^\top \in \mathbb{R}^{3 \times 3} \mid v \in \mathbb{R}^2 \} \subset \mathfrak{so}(3) \]

\[ (v_2, v_3)^\times := (0, v_2, v_3)^\times \]

where the indices \( (v_2, v_3) \in \mathbb{R}^2 \) are chosen to correspond to the associated indices for the embedding into \( \mathfrak{so}(3) \). Then, the normal coordinates for \( S^2 \) about \( e_1 \) are given by

\[ \partial(e) := -\text{atan2}(e_1, e_2) \]

\[ \partial^{-1}(e) := \phi(\exp(e^\times), e_1). \]

In this system, \( D_{E|u}\phi_{e_1}(E) \) is not invertible

\[ D_{E|u}\phi_{e_1}(E)[\omega^\times] = \frac{d}{dt} \bigg|_{t=0} \phi(\exp(t\omega^\times), e_1) = -\omega^\times e_1 = e_1^\top \omega. \]

We propose the following right inverse [required in (37)]:

\[ D_{E|u}\phi_{e_1}(E)D_{E|u}\phi_{e_1}(E)[u] = e_1^\top (0 \quad u_3 \quad -u_2)^\top = (0 \quad u_2 \quad u_3)^\top = u. \]

**6) EqF Matrices:** The EqF matrices are obtained by specializing the general matrix formulas to the specific example. The state matrix \( A_t \) (24), the input matrix \( B_t \) (42), the standard output matrix \( C_t \) (33), and the equivariant output matrix \( C_t^* = C_t^* \) (35) are given by

\[ A_t = 0_{2 \times 2}, \quad B_t = \begin{pmatrix} 0_{2 \times 1} & I_2 \end{pmatrix} \hat{R}, \quad C_t = \hat{y}^\times \hat{R}^\top \begin{pmatrix} 0_{2 \times 2} \end{pmatrix}, \quad C_t^* = \frac{1}{2} (\hat{y}^\times + \hat{y}^\times) \hat{R}^\top \begin{pmatrix} 0_{2 \times 2} \end{pmatrix}. \]
The state matrix $\hat{A}$ and output matrix $C_t$ can be compared to the EKF matrices (1) and (4), respectively, derived in Section II.

### B. EqF Implementation

We implement the EqF equations (36)–(38) as detailed in Algorithm 2 in Appendix A. The observer dynamics are given by specializing (36)

$$\dot{y} = DL_R \Lambda(\phi(\hat{R}, e_1)) + DR_R \Delta$$

$$= \hat{R} \Omega + \Delta \hat{R}$$

where the correction term $\Delta$ is computed according to (37).

### C. Simulation Results

To verify the observer design for this example, we performed a simulation of a robot rotating with an angular velocity $\Omega(t) = (0.1 \cos(2t), 0.2 \sin(t), -0.1 \cos(1.5t))$ rad/s, where $t$ is the simulation time in seconds. The initial state, the measured angular velocity, and the measured output were chosen by

$$\eta(0) = \frac{e_1 + \mu_0}{|e_1 + \mu_0|}, \quad \mu_0 \sim N(0, 2.0^2 I_3)$$

$$\Omega_m = \Omega + \mu_u, \quad \mu_u \sim N(0, 0.01^2 I_3)$$

$$y_m = y + \nu_y, \quad \nu_y \sim N(0, 0.05^2 I_3)$$

(50)

respectively. The state $\eta(t)$ was then computed by integrating

$$\frac{d}{dt} \eta(t) = f_{\Omega(t)}(\eta(t)) = -\Omega(t)^\top \eta(t).$$

The EqF gain matrices were chosen according to the procedure outlined in Section VI-A.

We also implemented an EKF as described in Section II to compare its performance to that of the EqF. The system (1, 2), the EqF equations (36)–(38), and the EKF were all implemented in python(3) and integrated for 5.0 s using Euler integration with a time step of 0.01 s. Both the EqF and EKF were given the initial estimate $\eta(0) = e_1$.

In order to verify the local exponential convergence of the proposed filters in the absence of noise, we performed a simulation with the gyroscope and magnetometer noise set to zero, i.e., $\mu_u = 0, \nu_y = 0$. Fig. 1 shows the absolute angle $\tilde{\theta}$ between the estimated direction and true direction and the Lyapunov value (39) for each filter, where

$$\tilde{\theta} := \arccos((\hat{\eta})^\top \eta).$$

(51)

The results demonstrate the performance of each filter under ideal conditions where the only error is due to the difference between the initial state $\eta(0)$ and the initial estimate $e_1$. That is, the true initial bearing was drawn from the distribution described in (50), but the measurements were taken to be $\Omega_m = \Omega$ and $y_m = y$ exactly. It is clear to see that the EqF is locally exponentially convergent, and that the EqF exhibits faster initial convergence.

We also performed 500 Monte Carlo simulations with noise added to the velocity, measurement, and initial conditions, generated according to (50). Fig. 2 shows the distribution of the absolute angle error (51) between the estimated direction and true direction for each filter, as well as the Lyapunov value (39) for each of the filters.

Fig. 3 shows the error introduced by different linearizations of the output. Each point $\eta \in S^2$ other than $-e_1$ has been mapped to $\mathbb{R}^2$ using spherical coordinates, and each heatmap shows, for a given filter, the absolute difference between the true measurement residual $\tilde{y} = y - h(e_1)$ and the linearized measurement residual $C_t \hat{\theta}(\eta)$. Note that $\hat{\theta}(\eta) := \eta - e_1$ in the case of the EKF. The EqF and EKF show comparable performance, with the EKF having slightly higher error near $e_1$ and the EqF having slightly higher error further away from $e_1$. The output linearization used by the EqF is superior to the other filters, both near $e_1$ as well as far from it, as expected from Lemma V.3.

The example system shown is of interest for several reasons. First, the system is defined on a homogeneous space of a Lie group rather than on a Lie group itself, necessitating the development of a lifted system to apply equivariant observer design methods. This also precludes the application of the popular IIEKF [17] as it is exclusively defined for group affine systems.
on Lie groups and not on the more general class of equivariant systems on homogeneous spaces. Second, the system has a symmetry compatible with the output function, enabling the use of Lemma V.3 to improve the output linearization. Fig. 3 shows that the improved output linearization results in a significant reduction in linearization error over the whole space, and Figs. 1 and 2 clearly show the positive effect of this improvement on filter performance.

Overall, the simulations demonstrate clearly that the EqF* outperforms the EqF, which in turn outperforms the EKF. The relative margin of improvement may appear small; however, it must be understood in the context of a problem chosen for simplicity and the implementation of an EKF that was designed carefully to exploit the specific structure available, specifically using embedding coordinates in order to obtain linear time varying state dynamics (1). Conversely, the EqF and EqF* were derived following the standard methodology outlined in Appendix A. For more complex problems, where there is no longer structure that can be exploited to design a clever EKF, the relative performance gap is expected to increase.

VIII. Conclusion

The EqF proposed in this article is a nonlinear observer that exploits symmetry properties of equivariant kinematic systems posed on homogeneous spaces. The key contributions of this article are as follows.

1) Proposing a general filter design for equivariant systems with linearized dynamics about a fixed origin point rather than the time-varying state estimate.

2) Demonstrating how output equivariance leads to an approximation of the output map for the EqF that has third-order linearization error (rather than the usual quadratic linearization error), improving filter robustness and transient performance.

3) Providing a simple example of the EqF application and simulation results where the EqF clearly outperforms the traditional EKF.

It is important to note that the example in the present article is chosen to be as simple as possible. It may appear that the mathematical overhead of the EqF is not warranted. Recent works [28]–[30] provide other examples where the EqF is the only symmetry-based filter that can be applied.

The appendices provide details on how to derive and implement the EqF, and the conditions under which the EqF specializes to the IEKF.

APPENDIX A
FILTER IMPLEMENTATION

The following algorithms are used to design the EqF for a given system.

Algorithm 1: EqF Design Preliminaries.

1: Find a Lie group \( \mathbf{G} \) and state action \( \phi : \mathbf{G} \times \mathcal{M} \to \mathcal{M} \).
2: Check that the system is equivariant and compute the input symmetry \( \psi : \mathbf{G} \times \mathcal{L} \to \mathcal{L} \).
3: Construct an equivariant lift \( \Lambda : \mathcal{M} \times \mathcal{L} \to \mathfrak{g} \).
4: Check if there exists an action \( \alpha : \mathbf{G} \times \mathcal{N} \to \mathcal{N} \) such that the configuration output is equivariant.
5: Choose an origin \( \xi \in \mathcal{M} \), choose a local coordinate chart \( \vartheta \) about \( \xi \), and fix a right-inverse \( D_X [\alpha \phi_\xi (X)] \)
   of \( D_X [\alpha \phi_\xi (X)] \).
6: Initialise the observer state \( \hat{X}(t) = \text{id} \) and the Riccati term \( \Sigma(0) = \Sigma_0 \in \mathcal{S}_+(m) \).

While the filter is presented in continuous time, in practice the equations must be implemented through numerical integration. Given input and output measurements \( u \in \mathcal{L} \) and \( y = h(\xi) \in \mathcal{N} \) at a given time, the steps in Algorithm 2 are executed. For the majority of applications, Euler integration or a higher order Runge–Kutta method is appropriate. Izadi and Sanyal [19] showed the effectiveness of Lie group variational integrators [18] for discretizing an observer for invariant attitude dynamics, and a similar approach may also be applied to discretizing the EqF for certain systems.

In some cases it may be difficult to compute an explicit algebraic expression for the velocity action \( \psi \). The following Lemma provides a way to implement the EqF without the need to derive \( \psi \).

Lemma A.1: The linearized state matrix \( \hat{A}_t \) defined in (25) can be written

\[
\hat{A}_t = D_\xi |\vartheta(\epsilon)D_\xi |\phi_\xi (\cdot) \cdot D_\xi |\phi_\xi (\epsilon) \cdot D_\xi |\psi_\xi (\cdot) - 1 \cdot \epsilon.
\]

(52)
Algorithm 2: EqF Design Implementation.
1: Compute the origin velocity \( \dot{u} = \psi_{X^{-1}}(u) \) and use this to obtain the state matrix \( \hat{A}_t \) (25) (cf. Lemma A.1).
2: Compute the standard output matrix \( C_t \) (33) or (preferably) the equivariant output matrix \( C_t^\ast \) (35).
3: Choose state and output gain matrices \( M_t \in S_+ (m) \) and \( N_t \in S_+ (n) \).
4: Update the observer state \( \hat{X}(t) \) and Riccati state \( \Sigma(t) \) by numerically approximating equations (36-38).

**Proof:** Recall the equivariant lift condition (13). It follows that
\[
D_E|_{\partial \psi(X)}(\Lambda) = D_E|_{\partial \psi(X)}(\Lambda) = D_E|_{\partial \psi(X)}(\Lambda) = D_E|_{\partial \psi(X)}(\Lambda) = D_E|_{\partial \psi(X)}(\Lambda).
\]

Then, the expression (52) follows from the definition of \( \hat{A}_t \) in (25). Unlike (25), the expression (52) depends only the measured signal \( u \in \mathbb{L} \) and not on the origin velocity \( \dot{u} = \psi_{X^{-1}}(u) \in \mathbb{L} \).

While the definitions of \( \hat{A}_t \) in (25) and (52) are equivalent, they present different challenges in practical implementation of the filter equations. Using the definition (25) requires an explicit algebraic expression for the velocity action \( \psi \), which may be challenging to compute. On the other hand, (52) is independent of the algebraic expression of \( \psi \), but requires the differentials \( D_\xi|_{\psi_{X^{-1}}(\xi)} \) and \( D_\xi|_{\phi_{\chi}(\xi)} \) to be recomputed at different state elements \( \hat{\xi} \in M \) for each iteration. The expression (52) is particularly useful in situations where the equivariant velocity extension of a system is infinite [32].

**APPENDIX B**

**SPECIALIZATION TO IEKF**

The EqF specializes to an IEKF [17] for a certain subclass of equivariant systems; specifically those with group affine dynamics on a Lie group, where the origin is chosen to be the identity, and where the local coordinates are chosen to be the exponential map in the EqF implementation.

Consider a system function \( f : \mathbb{L} \to X(\mathbb{G}) \) where \( \mathbb{G} \) is the torsor of an \( m \)-dimensional matrix Lie group \( \mathbb{G} \subset \text{GL}(d) \). Right translation \( R : \mathbb{G} \times \mathbb{G} \to \mathbb{G} \), defined by \( R_X(B) = BX \) is a smooth, transitive right action of \( \mathbb{G} \). Suppose the system is equivariant with respect to \( R \) and some velocity action \( \psi \), i.e.,
\[
D_R f_u(P) = f_{\psi_X(u)}(P)X.
\]

for all \( P \in \mathbb{G}, X \in \mathbb{G} \), and \( u \in \mathbb{L} \). Define the lift \( \Lambda : \mathbb{G} \times \mathbb{L} \to \mathbb{g} \) to be
\[
\Lambda(P, u) = P^{-1} f_u(P)
\]
where \( P^{-1} \) is understood as a matrix inverse.

Let \( P \in \mathbb{G} \) denote the true state of the system. Choose the origin element, \( P = I_d \), to be the identity matrix and let \( X \in \mathbb{G} \) denote the state of the observer, with dynamics
\[
\frac{d}{dt} \hat{X} := \hat{X} \Lambda(\hat{X}, u) + \Delta \hat{X} = \hat{X} \Lambda(X, u) + \Delta \hat{X} = f_u(\hat{X}) + \Delta \hat{X}.
\]

In [17], Barrault and Bonnabel showed (see also [32, Remark 7.1]) that if the system \( f \) is “group affine,” then the dynamics of the error \( E := PX^{-1} \) depend only on \( u \) and not the origin velocity \( \dot{u} \)
\[
\hat{E} = f_u(E) - E f_u(I_d) - E \Delta.
\]

Let \( \vartheta : \mathbb{U}_{I_d} \to \mathbb{R}^m \) be the normal coordinates for \( \mathbb{G} \) about \( I_d \) so that \( \vartheta = \vartheta(E) = \log(E)^\ast \). The preobserver (\( \Delta \equiv 0 \)) dynamics of \( \varepsilon \) about \( \varepsilon = \vartheta \) are exactly
\[
\dot{\varepsilon} = \hat{A}_t \varepsilon
\]
\[
\hat{A}_t \varepsilon = (D_E|_{\vartheta(E)}(\Lambda, \vartheta)(\partial E)^\ast)^\ast.
\]

Let the output space \( \mathcal{N} \) be the Euclidean space \( \mathbb{R}^n \), and let the configuration output \( h : \mathbb{G} \to \mathbb{R}^n \) be any map; i.e., \( h \) is not necessarily equivariant. Then, the EqF equations (36)-(38) with standard output matrix specializes to the IEKF proposed in [17].

In [17], the preobserver dynamics of \( \varepsilon \) are shown to be exactly linear as a consequence of the novel “group affine” property. In this case the IEKF is locally asymptotically stable with a constant convergence radius [17].

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