Local Semicircle Law and Complete Delocalization for Wigner Random Matrices

László Erdös¹, Benjamin Schlein¹,*, Horng-Tzer Yau²,**

¹ Institute of Mathematics, University of Munich, Theresienstr. 39, D-80333 Munich, Germany.
E-mail: lerdos@math.lmu.de; schlein@math.lmu.de
² Department of Mathematics, Harvard University, Cambridge, MA 02138, USA.
E-mail: htyau@math.harvard.edu

Received: 18 March 2008 / Accepted: 19 June 2008
Published online: 24 September 2008 – © Springer-Verlag 2008

Abstract: We consider $N \times N$ Hermitian random matrices with independent identical distributed entries. The matrix is normalized so that the average spacing between consecutive eigenvalues is of order $1/N$. Under suitable assumptions on the distribution of the single matrix element, we prove that, away from the spectral edges, the density of eigenvalues concentrates around the Wigner semicircle law on energy scales $\eta \gg N^{-1} (\log N)^{8}$. Up to the logarithmic factor, this is the smallest energy scale for which the semicircle law may be valid. We also prove that for all eigenvalues away from the spectral edges, the $\ell^\infty$-norm of the corresponding eigenvectors is of order $O(N^{-1/2})$, modulo logarithmic corrections. The upper bound $O(N^{-1/2})$ implies that every eigenvector is completely delocalized, i.e., the maximum size of the components of the eigenvector is of the same order as their average size.

In the Appendix, we include a lemma by J. Bourgain which removes one of our assumptions on the distribution of the matrix elements.

1. Introduction

The Wigner semicircle law states that the empirical density of the eigenvalues of a random matrix is given by the universal semicircle distribution. This statement has been proved for many different ensembles, in particular for the case when the distributions of the entries of the matrix are independent, identically distributed (i.i.d.). To fix the scaling, we normalize the matrix so that the bulk of the spectrum lies in the energy interval $[-2, 2]$, i.e., the average spacing between consecutive eigenvalues is of order $1/N$. We now consider a window of size $\eta$ in the bulk so that the typical number of eigenvalues is of order $N \eta$. In the usual statement of the semicircle law, $\eta$ is a fixed number

* Supported by Sofja-Kovalevskaya Award of the Humboldt Foundation. On leave from Cambridge University, UK.
** Partially supported by NSF grant DMS-0602038.
independent of \( N \) and it is taken to zero only after the limit \( N \to \infty \). This can be viewed as the largest scale on which the semicircle law is valid. On the other extreme, for the smallest scale, one may take \( \eta = k/N \) and take the limit \( N \to \infty \) followed by \( k \to \infty \).

If the semicircle law is valid in this sense, we shall say that the local semicircle law holds. Below this smallest scale, the eigenvalue distribution is expected to be governed by the Dyson statistics related to sine kernels. The Dyson statistics was proved for many ensembles (see [1, 4] for a review), including Wigner matrices with Gaussian convoluted distributions [7].

In this paper, we establish the local semicircle law up to logarithmic factors in the energy scale, i.e., for \( \eta \sim N^{-1}(\log N)^{8} \). The result holds for any energy window in the bulk spectrum away from the spectral edges. In [5] we have proved the same statement for \( \eta \gg N^{-2/3} \) (modulo logarithmic corrections). Prior to our work the best result was obtained in [2] for \( \eta \gg N^{-1/2} \). See also [6] and [8] for related and earlier results. As a corollary, our result also proves that no gap between consecutive bulk eigenvalues can be bigger than \( C(\log N)^{8}/N \), to be compared with the expected average \( 1/N \) behavior given by Dyson’s law.

It is widely believed that the eigenvalue distribution of the Wigner random matrix and the random Schrödinger operator in the extended (or delocalized) state regime are the same up to normalizations. Although this conjecture is far from the reach of the current method, a natural question arises as to whether the eigenvectors of random matrices are the same up to normalizations. More precisely, if \( \mathbf{v} = (v_1, \ldots, v_N) \) is an \( \ell^2 \)-normalized eigenvector, \( \|\mathbf{v}\| = 1 \), we say that \( \mathbf{v} \) is completely delocalized if \( \|\mathbf{v}\|_{\infty} = \max |v_j| \) is bounded from above by \( CN^{-1/2} \), the average size of \( |v_j| \). In this paper, we shall prove that all eigenvectors with eigenvalues away from the spectral edges are completely delocalized (modulo logarithmic corrections) in probability. Similar results, but with \( CN^{-1/2} \) replaced by \( CN^{-1/3} \) were proved in [5]. Notice that our new result, in particular, answers (up to logarithmic factors) the question posed by T. Spencer whether \( \|\mathbf{v}\|_{4} \) is of order \( N^{-1/4} \).

Denote the \((i, j)^{th}\) entry of an \( N \times N \) matrix \( H \) by \( h_{ij} = h_{ji} \). When there is no confusion, we omit the comma between the two subscripts. We shall assume that the matrix is Hermitian, i.e., \( h_{ij} = \bar{h}_{ji} \). These matrices form a Hermitian Wigner ensemble if

\[
h_{ij} = N^{-1/2}[x_{ij} + \sqrt{-1} y_{ij}], \quad (i < j), \quad \text{and} \quad h_{ii} = N^{-1/2} x_{ii},
\]

where \( x_{ij}, y_{ij} (i < j) \) and \( x_{ii} \) are independent real random variables with mean zero. We assume that \( x_{ij}, y_{ij} (i < j) \) all have a common distribution \( \nu \) with variance \( 1/2 \) and with a strictly positive density function: \( d\nu(x) = (\text{const.})e^{-\nu(x)}dx \). The diagonal elements, \( x_{ii} \), also have a common distribution, \( d\bar{\nu}(x) = (\text{const.})e^{-\bar{\nu}(x)}dx \), that may be different from \( d\nu \). Let \( \mathbb{P} \) and \( \mathbb{E} \) denote the probability and the expectation value, respectively, w.r.t the joint distribution of all matrix elements.

We need to assume further conditions on the distributions of the matrix elements in addition to (1.1).

**C1)** The function \( g \) is twice differentiable and it satisfies

\[
\sup_{x \in \mathbb{R}} g''(x) < \infty.
\]

**C2)** There exists a \( \delta > 0 \) such that

\[
\int e^{\delta x^2} d\bar{\nu}(x) < \infty.
\]