Studies of Transverse Momentum Dependent Parton Distributions and Bessel Weighting

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ABSTRACT: In this paper we present a new technique for analysis of transverse momentum dependent parton distribution functions, based on the Bessel weighting formalism. The procedure is applied to studies of the double longitudinal spin asymmetry in semi-inclusive deep inelastic scattering using a new dedicated Monte Carlo generator which includes quark intrinsic transverse momentum within the generalized parton model. Using a fully differential cross section for the process, the effect of four momentum conservation is analyzed using various input models for transverse momentum distributions and fragmentation functions. We observe a few percent systematic offset of the Bessel-weighted asymmetry obtained from Monte Carlo extraction compared to input model calculations, which is due to the limitations imposed by the energy and momentum conservation at the given energy/$Q^2$. We find that the Bessel weighting technique provides a powerful and reliable tool to study the Fourier transform of TMDs with controlled systematics due to experimental acceptances and resolutions with different TMD model inputs.

KEYWORDS: SIDIS, parton intrinsic transverse momentum, azimuthal moments

PREPRINT: JLAB-THY-14-1945
1 Introduction

The study of the spin structure of protons and neutrons is one of the central issues in hadron physics, with many dedicated experiments, recent (HERMES at DESY, CLAS and Hall-A at JLAB), running (COMPASS at CERN, STAR and PHENIX at RHIC), approved (JLab 12 GeV upgrade [1], COMPASS-II [2]) or planned (Electron Ion Collider [3–5]). The Transverse Momentum Dependent (TMD) parton distribution functions and fragmentation functions play a crucial role in gathering and interpreting information of a true “3-dimensional” imaging of the nucleon. These Transverse Momentum Dependent distribution and fragmentation functions (collectively here called “TMDs”) can be accessed in several types of processes, one of the most important is single particle hadron production in Semi-Inclusive Deep Inelastic Scattering (SIDIS) of leptons on nucleons. A significant amount of data on spin-azimuthal distributions of hadrons in SIDIS, providing access to TMDs has been accumulated in recent years by several collaborations, including HERMES, COMPASS and Halls A,B and C at JLab [6–15]. At least an order of magnitude more data is expected in coming years of running of JLab 12 [1].

A rigorous basis for studies of TMDs in SIDIS is provided by TMD factorization in QCD, which has been established in Refs. [16–23] for leading twist single hadron production.
with transverse momentum of the produced hadron being much smaller than the hard scattering scale, and the order of $\Lambda_{QCD}$, that is $\Lambda_{QCD}^2 < P_{h\perp}^2 \ll Q^2$. In this kinematic domain the SIDIS cross section can be expressed in terms of structure functions encoding the strong-interaction dynamics of the hadronic sub-process $\gamma^* + p \rightarrow h + X$ [24–27], which are given by convolutions of a hard scattering cross section and TMDs. However the extraction of TMDs as a function of the light-cone fraction $x$ and transverse momentum $k_{\perp}$ from single and double spin azimuthal asymmetries is hindered by the fact that observables are complicated convolutions in momentum space making the flavor decomposition of the underlying TMDs a model dependent procedure.

Based on TMD factorization theorems, experimentally measured cross sections are expressed as convolutions of TMDs where $k_{\perp}$ dependence is integrated over and related to measured value of $P_{h\perp}$. A reliable method to directly access the $k_{\perp}$ dependence of TMDs is very desirable. However, various assumptions involved in modern extractions of TMDs from available data rely on conjectures of the transverse momentum dependence of distribution and fragmentation functions [28–38] making estimates of systematic errors due to those assumptions extremely challenging.

In a paper by Boer, Gamberg, Musch, and Prokudin [39], a new technique has been proposed called Bessel weighting, which relies on a model-independent deconvolution of structure functions in terms of Fourier transforms of TMDs from observed azimuthal moments in SIDIS with polarized and unpolarized targets. In this paper, we apply the Bessel weighting procedure to present an extraction of Fourier transforms of TMDs from a Monte Carlo event generator. As an application of this procedure we consider the ratio of helicity, $g_{1L}$ and unpolarized, $f_1$ TMDs from the double longitudinally polarization asymmetry.

This paper is organized as follows: We begin our discussion in Section 2 with a brief review of the formalism of the SIDIS cross section and its representation in both momentum and Fourier conjugate $b_T$ space [39], while also presenting a description of the experimental procedure to study the TMDs based on the Bessel weighting. In Section 3 we introduce a fully differential Monte Carlo generator which has been developed to test the procedure for extraction of TMDs from SIDIS. As a test of the quality of our constructed Monte Carlo, in Section 3.2 we present a study of the Cahn effect [40, 41] contribution to the average $\langle \cos \phi \rangle$ moment in SIDIS. In Section 4 we present the extraction of the double spin asymmetry $A_{LL}(b_T)$, defined as the ratio of the difference and the sum of electro-production cross sections for anti-parallel and parallel configurations of lepton and nucleon spins using the Bessel weighting procedure. The effects of different model inputs and experimental resolutions and acceptances on extracted TMDs are investigated. Finally in Section 5 we draw some conclusions of the present analysis and outline steps for future work.
2 Extraction of TMDs using Bessel Weighting

The SIDIS cross section can be expressed in a model independent way in terms of a set of 18 structure functions [24, 25, 27, 42–44],

\[
\frac{d\sigma}{dx
dy
d\psi
d\phi_h
d|P_{h\perp}|^2} = \frac{\alpha^2 \gamma^2}{xyQ^2 2 (1 - \varepsilon)} \left( 1 + \frac{\gamma^2}{2x} \right) \left\{ F_{UU,T} + \varepsilon F_{UU,L} \\
+ \sqrt{2 \varepsilon (1 + \varepsilon)} \cos \phi_h F_{UU}^{\cos \phi_h} + \varepsilon \cos(2\phi_h) F_{UU}^{\cos 2\phi_h} \\
+ \lambda_\varepsilon \sqrt{2 \varepsilon (1 + \varepsilon)} \sin \phi_h F_{LU}^{\sin \phi_h} \\
+ S_\parallel \left[ \sqrt{2 \varepsilon (1 + \varepsilon)} \sin \phi_h F_{UL}^{\sin \phi_h} + \varepsilon \sin(2\phi_h) F_{UL}^{\sin 2\phi_h} \right] \\
+ S_\parallel \lambda_\varepsilon \left[ \sqrt{1 - \varepsilon^2} F_{LL} + \sqrt{2 \varepsilon (1 - \varepsilon)} \cos \phi_h F_{LL}^{\cos \phi_h} \right] \\
+ |S_{\perp}| \left[ \sin(\phi_h - \phi_S) \left( F_{UT,U,T}^{\sin(\phi_h - \phi_S)} + \varepsilon F_{UT,U,L}^{\sin(\phi_h - \phi_S)} \right) \right. \\
+ \varepsilon \sin(\phi_h + \phi_S) F_{UT}^{\sin(\phi_h + \phi_S)} + \varepsilon \sin(3\phi_h - \phi_S) F_{UT}^{\sin(3\phi_h - \phi_S)} \\
+ \sqrt{2 \varepsilon (1 + \varepsilon)} \sin \phi_S F_{UT}^{\sin \phi_S} + \sqrt{2 \varepsilon (1 + \varepsilon)} \sin(2\phi_h - \phi_S) F_{UT}^{\sin(2\phi_h - \phi_S)} \right] \\
+ |S_{\perp}| \lambda_\varepsilon \left[ \sqrt{1 - \varepsilon^2} \cos(\phi_h - \phi_S) F_{LT}^{\cos(\phi_h - \phi_S)} + \sqrt{2 \varepsilon (1 - \varepsilon)} \cos \phi_S F_{LT}^{\cos \phi_S} \right. \\
+ \left. \sqrt{2 \varepsilon (1 - \varepsilon)} \cos(2\phi_h - \phi_S) F_{LT}^{\cos(2\phi_h - \phi_S)} \right] \right\},
\]

(2.1)

where the first two subscripts of the structure functions \(F_{XY}\) indicate the polarization of the beam and target, and in certain cases, a third sub-script in \(F_{XY,Z}\) indicates the polarization of the virtual photon. The structure functions depend on the the scaling variables \(x, z, y\), the four momentum \(Q^2 = -q^2\), where \(q = l - l'\) is the momentum of the virtual photon, and \(l\) and \(l'\) are the 4-momenta of the incoming and outgoing leptons, respectively. \(P_{h\perp}\) is the transverse momentum component of the produced hadron with respect to the virtual photon direction.

The scaling variables have the standard definitions, \(x = Q^2 / 2(P \cdot q)\), \(y = (P \cdot q) / (P \cdot l)\), and \(z = (P \cdot P_h) / (P \cdot q)\). Further, in Eq. (2.1) \(\alpha\) is the fine structure constant; the angle \(\psi\) is the azimuthal angle of \(l'\) around the lepton beam axis with respect to an arbitrary fixed direction [44], and \(\phi_h\) is the azimuthal angle between the scattering plane formed by the initial and final momenta of the electron and the production plane formed by the transverse momentum of the observed hadron and the virtual photon, whereas \(\phi_S\) is the azimuthal angle of the transverse spin in the scattering plane [45]. Finally, \(\varepsilon\) is the ratio of longitudinal and transverse photon fluxes [27].
At tree-level, in a parton model factorization framework \cite{25, 27, 43}, the various structure functions in the cross section are written as convolutions of the TMDs which relate transverse momenta of the active partons and produced hadron. For our purposes, the unpolarized and double longitudinal polarized structure functions are

\[ F_{UU,T} = x \sum_a e_a^2 \int d^2 p_\perp d^2 k_\perp \delta^{(2)}(z k_\perp + p_\perp - P_{h\perp}) f^1_a(x, k_\perp^2) D^1_a(z, p_\perp^2), \] (2.2)

\[ F_{LL} = x \sum_a e_a^2 \int d^2 p_\perp d^2 k_\perp \delta^{(2)}(z k_\perp + p_\perp - P_{h\perp}) g^1_{aL}(x, k_\perp^2) D^1_a(z, p_\perp^2), \] (2.3)

where \( k_\perp \) is the intrinsic transverse momentum of the struck quark, and \( p_\perp \) is the transverse momentum of the final state hadron relative to the fragmenting quark \( k' \) (see Fig. 1). \( f^1_a(x, k_\perp^2) \), \( g^1_{aL} \) and \( D^a(z, p_\perp^2) \) represent TMD PDFs and fragmentation functions respectively of flavor \( a \), \( e_a \) is the fractional charge of the struck quark or anti-quark and the summation runs over quarks and anti-quark flavors \( a \).

Measurements of transverse momentum \( P_{h\perp} \) of final state hadrons in SIDIS with polarized leptons and nucleons provide access to transverse momentum dependence of leading twist TMDs. Recent measurements of multiplicities and double spin asymmetries as a function of the final transverse momentum of pions in SIDIS at COMPASS \cite{46}, HERMES \cite{47}, and JLab \cite{13–15} suggest that transverse momentum distributions depend on the polarization of quarks and possibly also on their flavor \cite{38} (see also discussion in Ref. \cite{48}). Calculations of transverse momentum dependence of TMDs in different models \cite{49–52} and on the lattice \cite{53, 54} also indicate that the dependence of transverse momentum distributions on the quark polarization and flavor maybe significant. Larger intrinsic transverse momenta of sea-quarks compared to valence quarks have been discussed in an effective model of the low energy dynamics resulting from chiral symmetry breaking in QCD \cite{55}.

As stated above, the various assumptions on transverse momentum dependence of distributions on spin and flavor of quarks however make phenomenological fits very challenging. To minimize these model assumptions, Kotzinian and Mulders \cite{56} suggested using so called \( P_{h\perp} \)-weighted asymmetries, where the unknown \( k_\perp \)-dependencies of TMDs are integrated out, thus providing access to moments of TMDs. However, the \( P_{h\perp} \)-weighted asymmetries introduce a significant challenge to both theory and experiment. For example, the weighting with \( P_{h\perp} \) emphasizes the kinematical region with higher \( P_{h\perp} \), where the statistics are poor and systematics from detector acceptances are difficult to control and at the same time theoretical description in terms of TMDs breaks down.

The method of Bessel weighting \cite{39} addresses these experimental and theoretical issues. First, Bessel weighted asymmetries are given in terms of simple products of Fourier transformed TMDs without imposing any model assumptions of the their transverse momentum dependence. Secondly, Bessel weighting regularizes the ultraviolet divergences resulting from unbound momentum integration that arises from conventional weighting. Further, in this paper we will demonstrate that they provide a new experimental tool to study the TMD content to the SIDIS cross section that minimize the transverse momentum model dependencies inherent in conventional extractions of TMDs. Also they suppress the kinematical regions where cross sections are small and statistics are poor \cite{39}.
We begin the review by re-writing the SIDIS cross section as a Bessel weighted integral in $b_T$ space \[39\]:

\[
\alpha^2 \frac{y^2}{x y Q^2 (1 - \varepsilon)} \left( 1 + \frac{\gamma^2}{2x} \right) \int \frac{d|b_T|}{|b_T|} F_{UU,T} \left\{ J_0(|b_T||P_{h\bot}|) F_{UU,T} + \varepsilon J_0(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
+ \sqrt{2 \varepsilon (1 + \varepsilon)} \cos \phi_h J_1(|b_T||P_{h\bot}|) F_{UU,T} \left. + \varepsilon \cos(2 \phi_h) J_2(|b_T||P_{h\bot}|) F_{UU,T} \cos(2 \phi_h) \right. \\
+ \lambda e \sqrt{2 \varepsilon (1 - \varepsilon)} \sin \phi_h J_1(|b_T||P_{h\bot}|) F_{UU,L} \\
\left. + S_\| \sqrt{2 \varepsilon (1 + \varepsilon)} \sin \phi_h J_1(|b_T||P_{h\bot}|) F_{UU,L} + \varepsilon \sin(2 \phi_h) J_2(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + |S_\| \lambda e \left[ \sqrt{1 - \varepsilon^2} \cos(\phi_h - \phi_s) J_1(|b_T||P_{h\bot}|) F_{UU,T} + \varepsilon J_2(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + \varepsilon \sin(3 \phi_h - \phi_s) J_3(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + \sqrt{2 \varepsilon (1 - \varepsilon)} \sin \phi_s J_1(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + \sqrt{2 \varepsilon (1 - \varepsilon)} \sin(2 \phi_h - \phi_s) J_2(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + |S_\| \left[ \cos(\phi_h - \phi_s) J_1(|b_T||P_{h\bot}|) F_{UU,T} + \varepsilon \cos(2 \phi_h - \phi_s) J_2(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + \varepsilon \sin(3 \phi_h - \phi_s) J_3(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + \sqrt{2 \varepsilon (1 - \varepsilon)} \cos \phi_s J_1(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. + \sqrt{2 \varepsilon (1 - \varepsilon)} \cos(2 \phi_h - \phi_s) J_2(|b_T||P_{h\bot}|) F_{UU,L} \right. \\
\left. \right\} \right) \] (2.4)

where the structure functions $F_{XX,Z}$ are now given as simple products of Fourier Transforms of TMDs.

Here we consider the unpolarized and double longitudinal structure functions,

\[
F_{UU,T} = x \sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2), \quad F_{LL} = x \sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2),
\] (2.5)

where the Fourier transform of the TMDs are defined as

\[
\tilde{f}(x, \Theta^2) = \int d^2 k_{\bot} e^{i \Theta \cdot k_{\bot}} f(x, \Theta^2) = 2\pi \int dk_{\bot} k_{\bot} J_0(|b_T||k_{\bot}|) f(x, \Theta^2),
\] (2.6)
\[ \tilde{D}(z, b_T^2) = \int d^2 p_\perp e^{i b_T \cdot p_\perp} D(z, p_\perp^2) \]
\[ = 2\pi \int dp_\perp p_\perp J_0(|b_T||p_\perp|) D(x, p_\perp^2). \] (2.7)

### 2.1 Bessel Weighting of Experimental Observables

It is now straightforward to express Bessel weighting of experimental observables. They are quantities which can be presented as simple products of Fourier transforms of distribution and fragmentation functions, allowing the application of standard flavor decomposition procedures. Noting that one can project out the unpolarized and double longitudinally polarized structure functions \( F_{LL} \) and \( F_{UU,T} \), by integrating Eq. (2.4) with the zeroth order Bessel function \( J_0(|b_T||P_{h\perp}|) \) over the transverse momentum of the produced hadron \( P_{h\perp} \), we arrive at an expression for the longitudinally polarized cross section \( \tilde{\sigma}^\pm(b_T) \) in \( b_T \)-space

\[ \tilde{\sigma}^\pm(b_T) = 2\pi \int \frac{d\sigma^\pm}{d\Phi} J_0(|b_T||P_{h\perp}|) P_{h\perp} dP_{h\perp}, \] (2.8)

where \( d\Phi \equiv dx dy dz dP_{h\perp} \) represents shorthand notation for the phase space differential and \( |b_T| \equiv b_T \), and \( |P_{h\perp}| \equiv P_{h\perp} \).

Now we form the double longitudinal spin asymmetry

\[ A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)} = \frac{\tilde{\sigma}_{LL}(b_T)}{\tilde{\sigma}_{UU}(b_T)} = \sqrt{1 - \varepsilon^2} \frac{\sum_a e_a^2 \tilde{f}_{L}^a(x, z^2 b_T^2) \tilde{D}_T^a(z, b_T^2)}{\sum_a e_a^2 \tilde{f}_{L}^a(x, z^2 b_T^2) D_T^a(z, b_T^2)}, \] (2.9)

Note that in our definition \( b_T \) is Fourier conjugate variable to \( P_{h\perp} \) [39].

The experimental procedure to study the structure functions in \( b_T \)-space amounts to discretizing Eq. (2.8). Now Eq. (2.9) results in an expression of sums and differences of Bessel functions for a given set of experimental events. The details on this formulation are given in Appendix A. The resulting expression for the spin asymmetry is

\[ A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\sum_{j^+}^{N^+} J_0(b_T P_{h\perp} j^+) - \sum_{j^-}^{N^-} J_0(b_T P_{h\perp} j^-)}{\sum_{j^+}^{N^+} J_0(b_T P_{h\perp} j^+) + \sum_{j^-}^{N^-} J_0(b_T P_{h\perp} j^-)} \] (2.10)

where \( j^\pm \) indicates a sum on events and where \( N^\pm \) is the number of events with positive/negative products of lepton and nucleon helicities.

The cross sections \( \tilde{\sigma}^\pm(b_T) \) can be extracted for any \( b_T \) using sums over the same set of data. These cross sections contain the same information as the cross sections, \( d\sigma/d\Phi \) in Eq. (2.8) differential with respect to the outgoing hadron momentum. The momentum dependent and the \( b_T \)-dependent representations of the cross section are related by a 2-D Fourier-transform in cylinder coordinates.

In the next Section we describe a new dedicated Monte Carlo generator which includes quark intrinsic transverse momentum within the generalized parton model.
3 Fully Differential Monte Carlo for SIDIS

A Monte Carlo generator is a crucial component in testing different experimental procedures. The Monte Carlo generator we use was developed to study partonic intrinsic motion using the framework of the so-called generalized parton model described in detail in Ref. [29]. We consider the SIDIS process

$$\ell(l) + N(P) \rightarrow \ell(l') + h(P_h) + X,$$

(3.1)

where $\ell$ is the incident lepton, $N$ is the target nucleon, and $h$ represents the observed hadron, and the four-momenta are given in parenthesis. Following the Trento conventions [45], the spatial component of the virtual photon momentum $q$ is along the positive $z$ direction and the proton momentum $P$ is in the opposite direction, as depicted in Fig. 1. In the parton model, the virtual photon scatters off an on-shell quark where the initial quark momentum $k$ and scattered quark momentum $k'$ have the same intrinsic transverse momentum component $k_\perp$ with respect to the $z$ axis, and where the initial quark has the fraction $x$ of the proton momentum. The produced hadron momentum, $P_h$ has the fraction $z$ of scattered quark momentum $k'$ in the $(\tilde{x}, \tilde{y}, \tilde{z})$ frame and $p_\perp$ is the transverse momentum component with respect to the scattered quark $k'$ (see also, Appendix C).

A great deal of phenomenological effort (see for example [29, 34, 57]) has been devoted to using the generalized parton model, with intrinsic quark transverse momentum, to account for experimentally observed spin and azimuthal asymmetries as a function of the produced hadron’s transverse momentum $P_{h\perp}$ in SIDIS processes. In order to take into account non-trivial kinematic effects arising from the standard approximations [25, 27] we develop a Monte Carlo based on the fully differential SIDIS cross section [29] which is given by,

$$\frac{d\sigma}{dxdydzd^2p_\perp d^2k_\perp d\phi_{l'}} = 2 K(x, y) J(x, Q^2, k^2_\perp) \times x \sum_a e_a^2 \left[ f_{1,a}(x, k^2_\perp) D_{1,a}(z, p^2_\perp) + \lambda \sqrt{1 - \varepsilon^2} g_{1L,a}(x, k^2_\perp) D_{1,a}(z, p^2_\perp) \right],$$

(3.2)

where the summation runs over quarks flavors, and the kinematic factors $K(x, y)$ and $\varepsilon$, and the Jacobian $J(x, Q^2, k_\perp)$ are defined in Appendix C. $\lambda$ is the product of target polarization and beam helicity ($\lambda = \pm 1$), $f_{1,a}(x, k^2_\perp)$ and $g_{1L,a}(x, k^2_\perp)$ are the unpolarized and helicity TMDs , and $D_{1,a}(z, p^2_\perp)$ is the unpolarized fragmentation function, $\phi_{l'}$ is the scattered lepton azimuthal angle $^1$. We adopt the parton kinematics in [29, 58] with the additional requirements, that the kinematics of the initial and final parton momenta are kept exact [59], and the nucleon mass is not set to zero. Also the hard scattering matrix elements are calculated for on-shell scattered partons.

In the Monte Carlo generator software, we used the general-purpose, self-adapting event generator, Foam [60], for drawing random points according to an arbitrary, user-defined distribution in $n$-dimensional space.

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$^1$Integration over $\phi_{l'}$ gives $2\pi$, since everything is symmetric along beam direction, although we need to keep it for further analysis, when one reconstructs generated events in the real experimental setup.
Figure 1. Kinematics of the process. $q$ is the virtual photon, $k$ and $k'$ are the initial and struck quarks, $k_\perp$ is the quark transverse component. $P_h$ is the final hadron with a $p_\perp$ component, transverse with respect to the fragmenting quark $k'$ direction.

3.1 Kinematical Distributions

Implementing the Monte Carlo we generate kinematical distributions in $x$, $z$, $k_\perp$, and $p_\perp$ of SIDIS events for several model inputs of TMDs. These distributions are then used to check the consistency of dependence of extracted quantities under different model assumptions, including, for example Gaussian and non-Gaussian distributions in transverse momentum.

In case the dependence is assumed to be a Gaussian, $x$ and $z$ dependent widths are assumed, so that TMDs take the following form,

$$f_1(x, k_\perp^2) = f_1(x) \frac{1}{\langle k_\perp^2(x) \rangle f_1} \exp\left(-\frac{k_\perp^2}{\langle k_\perp^2(x) \rangle f_1}\right),$$  

$$g_{1L}(x, k_\perp^2) = g_{1L}(x) \frac{1}{\langle k_\perp^2(x) \rangle_{g_1}} \exp\left(-\frac{k_\perp^2}{\langle k_\perp^2(x) \rangle_{g_1}}\right),$$  

$$D_1(z, p_\perp^2) = D_1(z) \frac{1}{\langle p_\perp^2(z) \rangle} \exp\left(-\frac{p_\perp^2}{\langle p_\perp^2(z) \rangle}\right),$$

where $f(x)$ and $D(z)$ are corresponding collinear parton distribution and fragmentation functions and the widths are $x$ and $z$ dependent functions. In our studies we adopt the modified Gaussian distribution functions and fragmentation functions from Eq. (3.3)-(3.5), in which $x$ and $k_\perp$ dependencies are inspired by AdS/QCD results [61, 62], with $\langle k_\perp^2(x) \rangle = Cx(1-x)$ and $\langle p_\perp^2(z) \rangle = Dz(1-z)$, where the constants $C$ and $D$ may be different for different flavors and polarization states (see for example [38]). Similarly such non-factorized $x,k_\perp$ distribution functions are also suggested by the diquark spectator model [63] and the NJL-jet model [36, 64].

For the $x$ and $z$ dependence in Eqs. (3.3) and (3.5) we use the parametrizations, $f_1(x) = (1-x)^3 x^{-1.313}, g_{1L}(x) = f_1(x) x^{0.7}$, and $D_1(z) = 0.8 (1-z)^2$, using input values $C = 0.54$ GeV$^2$ and $D = 0.5$ GeV$^2$. We also assume that $\langle k_\perp^2 \rangle_{g_{1L}} = 0.8 \langle k_\perp^2 \rangle_{f_1}$; this assumption is consistent with lattice studies [54] and experimental measurements [14].

As an example of a non-Gaussian $k_\perp$ distribution we implement the following one...
inspired by the shape of the resulting distribution in the light-cone quark model [65, 66]

\[ f_1(x, k^2_{\perp}) = f_1(x) / \left(1 + 20.82 k^2_{\perp} + 126.7 k^4_{\perp} + 1285 k^6_{\perp}\right). \]  

(3.6)

where the coefficients for \( g_{1L}(x, k^2_{\perp}) \) are chosen in such a way that effectively \( \langle k^2_{\perp} \rangle_{g_{1L}} / \langle k^2_{\perp} \rangle_{f_1} = 0.8 \).

We then generate events using the cross section from Eq. (3.2) for both Gaussian and non-Gaussian initial distributions respectively, and we show the resulting transverse momentum distributions in Figs. 2 and 3. Note that the generator we construct is implemented with on mass-shell partons and with four momentum conservation imposed. While this choice is not compulsory we adopt it as it allows us to fully reconstruct kinematics for a given event. At the same time limitations due to available phase space integration will modify the reconstructed distributions with respect to the input distributions. We analyze the effect of the available phase space in the Monte Carlo on the average \( \langle k^2_{\perp} \rangle \) for finite beam energies as a function of \( x \) by calculating the effective \( \langle k^2_{\perp} \rangle \) from the following formula,

\[ \langle k^2_{\perp}(x) \rangle = \frac{\int d^2k_{\perp} k^2_{\perp} d\sigma_{MC}}{\int d^2k_{\perp} d\sigma_{MC}} = \frac{\sum_{j=1}^{N} k^2_{\perp,j}}{N}, \]  

(3.7)

where the index \( j \) runs over the \( N \) Monte Carlo generated events. Note, \( d\sigma_{MC} \) is the cross section of the Monte Carlo simulation, that is Eq. (3.2), modified by imposing the four momenta conservation and on-shell condition for initial quark.

Indeed in Figs. 2 and 3 we find when comparing the Monte Carlo generated events with the input distributions, using Eq. (3.3) and Eq. (3.6), shown as solid black curves for a given \( x \), that the larger \( k_{\perp} \) values of the Monte Carlo events (red triangles up, 160 GeV beam energy, and blue triangles down, 6 GeV beam energy) are suppressed due to the available phase space imposed by both the finite beam energy, and four momentum conservation in the Monte Carlo. The fit of the Monte Carlo distributions for the modified Gaussian model are shown as dashed lines displayed in Fig. 2. They return the fitted values \( C = 0.527 \) and \( C = 0.444 \) for the 160 GeV and 6 GeV Monte Carlo simulations respectively. In Fig. 3 we study the effect of the non-Gaussian distribution Eq. (3.6). Integrating Eq. (3.7) over \( k_{\perp} \) gives a value of \( \langle k^2_{\perp} \rangle = 0.084 \text{GeV}^2 \), and the dashed curve represents the fit to the Monte Carlo distribution with a value of \( \langle k^2_{\perp} \rangle = 0.064 \text{GeV}^2 \) for the 6 GeV initial lepton beam energy.

In Fig. 4, the average \( \langle k^2_{\perp} \rangle \) versus \( x \) from the Monte Carlo for different incoming beam energies, for \( 0.5 < z < 0.52 \), is presented. For the modified Gaussian distribution function with the input value \( \langle k^2_{\perp}(x) \rangle = 0.54 x(1-x) \text{GeV}^2 \), the suppression of the generated \( \langle k^2_{\perp}(x) \rangle \) compared to input distributions (solid line) is greater for the lower beam energy. In Fig. 5 the constraints of four momentum conservation also affect the \( p^2_{\perp} \) distributions, which in turn also affect the observed \( P_{h\perp} \) distribution.

The systematics of the extraction of the TMDs in momentum space due to the kinematic constraints has been studied in detail using our fully differential Monte Carlo. We conclude this section with the general observation that imposing four momentum conservation in the event generator effectively modifies the initial distributions due to the
limitations of the available phase space in the generator. This deformation is more pronounced at lower energies or $Q^2$. A shift of a few percent is visible for 160 GeV cm energy, while for the lower 6 GeV cm energy the effective $\langle k_x^2 \rangle$ is lower than the input value by approximately $\sim 20\%$.

![Figure 2](image1.png)  
**Figure 2.** (Color online) The solid line is the Gaussian input distribution implemented using Eq. (3.3), with red triangles coming from the Monte Carlo at 160 GeV initial lepton energy, blue triangles coming from the Monte Carlo at 6 GeV. The dashed line represents the fit to the Monte Carlo distributions which returned values of $C = 0.527$ and $C = 0.444$ at 160 GeV and 6 GeV respectively.

![Figure 3](image2.png)  
**Figure 3.** (Color online) The solid line is the implemented non-Gaussian distribution using Eq. (3.6), with $\langle k_{x}^{2} \rangle = 0.084$ GeV$^2$, and the dashed curve represents the fit to the Monte Carlo distribution with the value of $\langle k_{x}^{2} \rangle = 0.064$ GeV$^2$ at 6 GeV initial lepton beam energy. The available phase space dictated by four momentum conservation results in a deformation of the input distribution.

3.2 The Cahn effect in the Monte Carlo Generator

As an example of an application of our constructed Monte Carlo we present a study of the Cahn effect [40, 41] contribution to the average $\langle \cos \phi \rangle$ moment in SIDIS. We generate
Monte Carlo events using the following expression for the cross section \[29\],

\[
\frac{d\sigma}{dxdydzd^2p_\perp d^2k_\perp} = K(x, y)J(x, Q^2, k_\perp^2) \sum_a f_{1,a}(x, k_\perp^2)D_{1,a}(z, p_\perp^2) \frac{s^2 + \hat{u}^2}{Q^4}
\]

where \(\hat{s} = (l + k)^2\) and \(\hat{u} = (k - l')^2\) (see Fig. 1). As stated above, in the Monte Carlo we impose four momentum conservation with target mass corrections.

In Fig. 6 we present output from the Monte Carlo using the non-factorized Gaussian distribution function and fragmentation function (Eqs. (3.3) and (3.5)). We also compare our results to the HERMES data, and Ref. [58]. It is clear that the results of our Monte Carlo are comparable to that of [58] and close to HERMES data. For the red triangles we used \(\langle k_\perp^2 \rangle = 0.54 \ x(1 - x)\) and \(\langle p_\perp^2 \rangle = 0.5 \ z(1 - z)\) GeV\(^2\). As one can see for HERMES kinematics the modified Gaussian TMDs reduces the contribution of the Cahn effect contribution to the \(\langle \cos(\phi_h) \rangle\) moment. In Ref. [58] this effect is achieved by imposing a so-called direction cut (that the quark moves in the forward direction with respect of the proton). In this Monte Carlo there are two main factors that modify the distribution; the four-momentum conservation and \(x(z)\) dependent values of \(\langle k_\perp^2 \rangle (\langle p_\perp^2 \rangle)\).

\[\text{Figure 6.} \ (\text{Color online}) \text{ The Cahn contribution in } \langle \cos(\phi_h) \rangle \text{ for } \pi^+ \text{ from modified Gaussian PDFs is presented for HERMES kinematics in comparison with [58]) and published HERMES data.}\]

In the next Section we apply the Bessel weighting formalism for the double longitudinal spin asymmetry in semi-inclusive deep inelastic scattering to data from our Monte Carlo generator.

4 Bessel Weighted Double Spin Asymmetry

In this Section, we present an extraction of the Bessel weighted double longitudinal spin asymmetry in \(b_T\) space. We also carry out a study of the accuracy of such an extraction. We use the dedicated fully differential SIDIS single hadron Monte Carlo to generate events based on the input TMDs. For simplicity we perform this comparison in a one flavor approximation.

\[\text{Figure 6.} \ (\text{Color online}) \text{ The Cahn contribution in } \langle \cos(\phi_h) \rangle \text{ for } \pi^+ \text{ from modified Gaussian PDFs is presented for HERMES kinematics in comparison with [58]) and published HERMES data.}\]

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4.1 Results from the Monte Carlo

The Monte Carlo generated events are used like experimental events to extract both the Bessel weighted asymmetry, $A_{J0}^{LL}(b_T P_{h⊥})$, and the ratio of the Fourier transform of $g_{1L}$ and $f_1$ using the Bessel weighting method described in [39]. The results are then compared to the Monte Carlo input. The Bessel moments are extracted from the Monte Carlo with 6 GeV beam energy using both the modified Gaussian type of functions (see Eqs. (3.3)-(3.5)) and power law-tail like function (see Eq. (3.6)).

The numerical results of our studies are summarized and displayed in Figs. 7 and 8 for the modified Gaussian distribution function and for the power law-tail like distribution function inputs respectively. In the left panel of Fig. 7 we show the Bessel-weighted asymmetry versus $b_T$. The blue curve labeled “BW Input”, is the asymmetry calculated analytically using the right hand side of Eq. (2.9) and the Fourier transformed input distribution functions (one can compare this with the model calculation in Ref. [67]).

We now compare various distributions generated from the Monte Carlo. We plot the generated distribution using Eq. (2.10) (full red points) labeled “BW($P_{h⊥}$) Generated”, and the black triangles labeled “BW($P_{h⊥}$) Sm + Acc”, which represents the same extraction after experimental smearing and acceptance (using the CLAS detector [68]). Next we consider the Fourier transform ratio $\tilde{g}_{1L}$ to $\tilde{f}_1$, the (green) curve with triangles up labeled “BW($k_{⊥}$)” obtained from numerically Fourier transforming the $k_{⊥}$ distributions from the

![Figure 7](image-url)
Monte Carlo generator on an event by event basis (see Eq. (2.6)),

\[ \sqrt{1 - \epsilon^2 g_{1L}(b_T)} \frac{\tilde{f}_1(b_T)}{f_1(b_T)} = \frac{\sum_{j=+}^{N^+} J_0(b_T k_{\perp j}^+)}{\sum_{j=+}^{N^+} J_0(b_T k_{\perp j}^-)} = \frac{\sum_{j=+}^{N^+} J_0(b_T k_{\perp j}^+)}{\sum_{j=-}^{N^-} J_0(b_T k_{\perp j}^-)} . \]  

(4.1)

This quantity corresponds to the right hand side of Eq. (2.9) in the one flavor approximation, where the fragmentation functions are expected to cancel out. For the Monte Carlo generated events, this cancellation is only an approximation, leading to the deviation between the red (or black) points and the green curve at large $b_T$. The reasons for the imperfect cancellation are discussed in section 4.2.

In order to quantitatively assess the deviation between the curves in the left panel of Fig. 7, we plot ratios of these values (see right panel). The red points represent the deviation from unity that is due to the imperfect cancellation of the fragmentation function. The black triangles represent the same after experimental smearing and acceptance are taken into account. Finally the open blue squares represent the deviation between the analytic result from the input distributions and the Monte Carlo generated events, Eq. (2.10).

The error bars in $b_T$ space for each point give the statistical standard deviation (see Appendix B for more details). If we use the same data set to integrate over $P_{h\perp}$ for all $b_T$ points, the errors in $b_T$ space are correlated and one should calculate the correlation/covariance matrix before performing any global fit to the data points. To circumvent this problem, we used different Monte Carlo samples for each $b_T$-point to avoid correlations in the error calculations.
4.2 Interpretation of the Results

One primary question addressed in this study is how robust the Bessel-weighting technique behaves under simulating real experimental conditions. Comparing the round (red) data points with the triangular (black) ones in Figs. 7 and 8, we see that switching on experimental smearing and acceptance in our simulation does not change the results significantly.

An altogether different question concerns the validity of the generalized parton model at the relatively low beam energies available in experiments today. The generalized parton model is an approximation that assumes certain components of the intrinsic parton momenta are suppressed for large beam energies and can thus be integrated out from the distributions. This becomes apparent in Eqns. (2.2) and (2.3), where a delta function is present only in the two transverse dimensions. An explicit four-momentum conservation law embedded in the formula of the cross section is thus lost. As an example of a particularly striking consequence of this, one observes that there is no explicit mechanism that prevents events at values of $p_{h\perp}$ larger than allowed by the finite beam energy. Naturally, the lower the beam energy becomes, the more serious these inaccuracies of the generalized parton model have to be taken.

Our Monte Carlo simulation allows us to take the factorized form of the generalized parton model cross section Eq. (3.2) as a basis and then to impose full four-momentum conservation for the partons, assuming the initial quarks are on-shell with non-zero mass. We also take a non-zero target mass into account. This procedure does not necessarily lead to a more accurate description of the underlying physics, because it still rests on the simplified picture of the generalized parton model and involves the approximation of an on-shell quark. Nonetheless, implementing these modifications can serve us to get an indication for the magnitude of uncertainties resulting from the aforementioned kinematic approximations in the generalized parton model. Analyzing our MC results with four momenta conservation and target mass correction, we are able to distinguish two effects in the left panels of Figs. 7 and 8:

1. **Solid (blue) curve versus triangular (green) data points:** The distributions realized in the MC simulation differ from the input distributions. In the MC, the four-momentum conservation does not allow the variables $k_\perp$ and $p_\perp$ in Eq. (3.2) to be sampled independently over the whole integration range, as it would have to be done to reproduce the unmodified generalized parton model Eqns. (2.2) and (2.3). The actual $k_\perp$ and $p_\perp$ distributions realized by the MC differ from the analytic input distributions Eqns. (3.3)-(3.6) noticeably, especially in their widths. This has already been observed in Fig. 2. The solid (blue) curve in the left panel of Figs. 7 and 8 is calculated from the input distributions according to the generalized parton model; the FFs on the right hand side of Eq. (2.9) cancel exactly in the single flavor scenario. Thus the solid curve can be compared to the triangle shaped (green) data points, which have also been calculated from a ratio of TMD PDFs, Eq. (4.1), albeit with the actual distributions realized in the MC.

2. **Triangular (green) data points vs. circular (red) data points:** inadequacy
of the generalized parton model to describe the data. In a single flavor scenario, the distribution functions $\tilde{D}_a^1$ cancel exactly on the right hand side of Eq. (2.9). Therefore, there should not be any difference between the full asymmetry $A_{LL}^{J_0(b_T p_{h\perp})}(b_T)$ of Eqs. (2.9), (2.10) and the ratio of TMD PDFs Eq. (4.1). However, we do observe a difference between the circular (red) data points and the triangular (green) data points in the left panels of Figs. 7 and 8. Again, the four-momentum conservation we have implemented is the reason for the observed difference. Since $k_{\perp}$ and $p_{\perp}$ are no longer sampled in accordance with Eqns. (2.2) and (2.3), the right hand side of Eq. (2.9) looses the prerequisites for its derivation and is violated to some degree. Therefore, we see only an incomplete cancellation of FFs for the Monte Carlo events.

To an experimentalist who is concerned about systematic errors attributed to the observables he or she extracts, the first of the two effects above is not an issue. The purpose of the generalized parton model is to provide a parametrization of the data one observes. Any effect of the underlying scattering mechanism that can be absorbed into the distributions does not contradict the validity of the model. The only concern one might have is that the distributions become beam energy/$Q^2$ dependent, an issue that should be addressed using TMD evolution equations.

On the other hand, the second effect presented above can be taken as an indication for systematic uncertainties. If, indeed, the physical reality does not generate events in accordance with the functional shape of the generalized parton model, then using the model for the extraction of distributions necessarily involves systematic errors. Again, we point out that it is unclear whether the modifications we have implemented in our MC bring us closer to the physical reality. Nonetheless, the modifications are reasonable and so we believe they can give us a hint about the order of magnitude of systematic errors from the corresponding approximations in the model. One can then estimate that for calculations such as those performed in Ref. [67], systematic errors in the comparison with experimental data for $b_T < 6 \text{GeV}^{-1}$ are of the order of a few percent. For the data with $b_T > 6 \text{GeV}^{-1}$, the effects of four-momentum conservation (difference between red and green points) becomes more pronounced, and a fit of data using the generalized parton model without manifest four-momentum conservation therefore becomes less accurate.

5 Conclusions

We have presented the first studies of Bessel-weighted asymmetries using a multi-dimensional Monte Carlo generator based on the fully differential cross section for TMD studies using the tree level parton model [29]. Two models have been used in the simulation; a modified Gaussian and a power law tail, for the distribution and fragmentation functions. The Bessel-weighted sums of double polarization observables, in particular, provide access to transverse momentum dependencies of partonic distributions $f_1$ and $g_{1L}$. Bessel-weighted asymmetries (described in [39]) have been extracted from the generated Monte Carlo events and studies of systematic uncertainties have been performed. We observe a few percent
systematic offset of the Bessel-weighted asymmetry obtained from Monte Carlo extraction compared to input model calculations, which is due to the limitations imposed by the energy and momentum conservation at the given energy/$Q^2$.

We find that the Bessel weighting technique provides a powerful and reliable tool to study the Fourier transform of TMDs with controlled systematics due to experimental acceptances and resolutions with different TMD models inputs. We plan to expand our studies with more advanced parton shower and fragmentation mechanisms, as well as to include nuclear modifications in our Monte Carlo and extraction procedure.

A Monte Carlo generator including spin-orbit correlations, quark-gluon interactions and correlations between the current and target fragmentation region, which is applicable in a wide range of kinematics, will be crucial for both experimental techniques and phenomenology of Fourier transformed TMDs. Moreover, evolution equations for the distributions are typically formulated directly in coordinate (Fourier) space [16–23]. Phenomenological studies can then performed in this space, see for example [69, 70]. Thus, the study of the scale dependence of Bessel weighted asymmetries should prove important in studies of evolution of TMDs. For the above stated reasons we propose Bessel weighted asymmetries as clean observables to study the scale dependence of TMD PDFs and FFs at existing (HERMES, COMPASS, JLab) and future facilities (Electron Ion Collider, JLab 12 GeV).

Acknowledgments

This work is supported by the U.S. Department of Energy under Contract No. DE-AC05-06OR23177 (H.A., A.P., & P.R.), No. DE-FG02-07ER41460 (L.G.), Science without borders young talent program from CAPES (contract number 150324 da CAPES), EU FP7 (HadronPhysics3, Grant Agreement number 283286) (M.A.), and the Italian Istituto Nazionale di Fisica Nucleare (M.A., H.A., E.De-S., M.M. & P.R.). We thank M. Anselmino, D. Boer, S. Brodsky, U. D’Alesio, D. Hasch, A. Kotzinian, H. Matevosyan, and S. Melis for useful and stimulating discussions.

A Bessel Weighting

In this Appendix we review the Bessel weighting framework, and the procedure to calculate the Bessel-weighted asymmetry for the longitudinally polarized beam and target, for a given set of experimental events which is expressed in Eq. (2.10).

From Eq. (2.4) the SIDIS cross section written in terms of the Fourier transformed TMD PDFs and FFs [39] for the leading twist unpolarized and doubly longitudinal polarized structure functions is given by

$$
d\sigma \over dxdyd\psi dzdP_{h\perp}^2 = K(x,y) \int \frac{db_T b_T}{2\pi} J_0(b_T P_{h\perp}) \left( F_{UU,T}(b_T) + S_\parallel \lambda e\sqrt{1 - \varepsilon^2} F_{LL}(b_T) \right)
$$

(A.1)

where $K(x,y)$ is given below in Eq. (C.2) and where $|P_{h\perp}| \equiv P_{h\perp}$ and $|b_T| \equiv b_T$. 

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Using the Bessel weighting procedure, which in this case amounts to weighting with $J_0$, we write the cross section $\tilde{\sigma}(B_T)$ in $B_T$ space, in terms of the structure functions $^2F_{UU,T}$ and $F_{LL}$

\[
\tilde{\sigma}(B_T) = 2\pi \int dP_{h\perp}P_{h\perp}J_0(B_TP_{h\perp}) \frac{d\sigma}{dx dy d\psi dz d\phi_h dP_{h\perp}P_{h\perp}} \\
= 2\pi \int dP_{h\perp}P_{h\perp}J_0(B_TP_{h\perp}) \int \frac{db_Tb_T}{2\pi} J_0(b_TP_{h\perp}) \left( F_{UU,T} + S_\parallel \lambda_e \sqrt{1 - \varepsilon^2} F_{LL} \right) \\
= K(x, y) \left( F_{UU,T} + S_\parallel \lambda_e \sqrt{1 - \varepsilon^2} F_{LL} \right), \tag{A.2}
\]

where the structure functions in $b_T$ space are given by the products of Fourier transformed TMDs $^2F$,

\[
F_{UU,T} = x \sum_a \epsilon_a^2 f_1^a(x, z^2b_T^2) \tilde{D}_1^a(z, b_T^2), \quad F_{LL} = x \sum_a \epsilon_a^2 \tilde{g}_{1L}^a(x, z^2b_T^2) \tilde{D}_1^a(z, b_T^2). \tag{A.3}
\]

Labeling the cross section with $\pm$ for the double longitudinal spin combinations $S_\parallel \lambda_e = \pm 1$ we have

\[
\tilde{\sigma}^\pm(b_T) = K(x, y) \left( F_{UU,T} \pm \sqrt{1 - \varepsilon^2} F_{LL} \right). \tag{A.4}
\]

The Bessel weighted double spin asymmetry is $b_T$ space is,

\[
A_{LL}^{J_0(b_TP_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)} = \frac{\tilde{\sigma}_{LL}(b_T)}{\tilde{\sigma}_{UU}(b_T)} = \sqrt{1 - \varepsilon^2 \sum_a \epsilon_a^2 \tilde{g}_{1L}^a(x, z^2b_T^2) \tilde{D}_1^a(z, b_T^2) / \sum_a \epsilon_a^2 f_1^a(x, z^2b_T^2) \tilde{D}_1^a(z, b_T^2)}. \tag{A.5}
\]

Now we derive the formula to extract Bessel-weighted asymmetries by means of an event by event weighting in $P_{h\perp}$, while binning in $x$, $y$, and $z$. First we express the unpolarized and doubly polarized helicity structure functions in $B_T$ space as

\[
F_{UU,T} = \frac{1}{K(x, y)} \int dP_{h\perp}P_{h\perp}J_0(b_TP_{h\perp}) \left( \frac{d\sigma^+}{d\Phi} + \frac{d\sigma^-}{d\Phi} \right) \]

\[
F_{LL} = \frac{1}{K(x, y) \sqrt{1 - \varepsilon^2}} \int dP_{h\perp}P_{h\perp}J_0(b_TP_{h\perp}) \left( \frac{d\sigma^+}{d\Phi} - \frac{d\sigma^-}{d\Phi} \right), \tag{A.6}
\]

using the shorthand notation for the differential phase space factor $d\Phi \equiv dx dy d\psi dz dP_{h\perp}P_{h\perp}$. Re-expressing the cross sections in terms of the number of events in the differential phase space “volume”, Eq. (A.6) is given by,

\[
F_{UU,T} = \frac{1}{K(x, y)} \int dP_{h\perp}P_{h\perp}J_0(b_TP_{h\perp}) \left( \frac{1}{N_0^+} \frac{dn^+}{d\Phi} + \frac{1}{N_0^-} \frac{dn^-}{d\Phi} \right) \tag{A.7}
\]

and

\[
F_{LL} = \frac{1}{K(x, y) \sqrt{1 - \varepsilon^2}} \int dP_{h\perp}P_{h\perp}J_0(b_TP_{h\perp}) \left( \frac{1}{N_0^+} \frac{dn^+}{d\Phi} - \frac{1}{N_0^-} \frac{dn^-}{d\Phi} \right) \tag{A.8}
\]

$^2$We have suppressed the dependence on the phase space variables $x, y, z$. 

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where \(dn^\pm\) are the number of events in a differential phase space volume, \(d\Phi\), and \(N_0^\pm\) is the standard normalization factor, that is the product of the number of beam and target particles with \(\pm\) polarization per unit target area.

Now we discretize the momentum integration in Eq. (A.7) and (A.8) for a fixed phase space cell in \(x, y, z\) such that the corresponding differential \(dx\,dy\,dz\) becomes the bin volume \(\Delta x\Delta y\Delta z\). Eqs. (A.7) and (A.8) thus become

\[
\begin{align*}
F_{UU,T} &= x \sum_a e^2_a \tilde{f}_1(x, z^2 b_T^2) \tilde{D}_1(z, b_T^2) \\
&= \frac{1}{2} \left\{ \frac{1}{N_0^+} \sum_{i \in \text{bin}[x,y,z]} \frac{J_0(b_T P_{h\perp i}) \Delta n_i^+}{K(x, y)} + \frac{1}{N_0^-} \sum_{i \in \text{bin}[x,y,z]} \frac{J_0(b_T P_{h\perp i}) \Delta n_i^-}{K(x, y)} \right\} \frac{1}{\Delta x\Delta y\Delta z},
\end{align*}
\]

(A.9)

and

\[
\begin{align*}
F_{LL} &= x \sum_a e^2_a \tilde{g}_1(x, b_T^2) \tilde{D}_1(z, b_T^2) \\
&= \frac{1}{2} \left\{ \frac{1}{N_0^+} \sum_{i \in \text{bin}[x,y,z]} \frac{J_0(b_T P_{h\perp i}) \Delta n_i^+}{K(x, y)\sqrt{1 - \varepsilon^2}} + \frac{1}{N_0^-} \sum_{i \in \text{bin}[x,y,z]} \frac{J_0(b_T P_{h\perp i}) \Delta n_i^-}{K(x, y)\sqrt{1 - \varepsilon^2}} \right\} \frac{1}{\Delta x\Delta y\Delta z}.
\end{align*}
\]

(A.10)

where we sum over the discrete momentum index \(i\), and \(\Delta n_i^\pm\) are the number of events for polarization \(\pm\) as a function of \(P_{h\perp i}\).

Substituting Eqs. (A.9) and (A.10) into Eq. (A.4), the experimental procedure to calculate the Bessel weighted asymmetry, \(A_{\text{LL}}^{J_0(b_T P_{h\perp})}(b_T)\), becomes,

\[
A_{\text{LL}}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)} \equiv \frac{\tilde{S}^+ - \tilde{S}^-}{\tilde{S}^+ + \tilde{S}^-}
\]

(A.11)

where \(j^\pm\) indicates a sum on events and where \(N^\pm\) is the number of events with positive/negative products of lepton and nucleon helicities for a given \(x, y\) and \(z\), and where \(\tilde{S}^\pm\) is the sum over events for \(\pm\) helicities.
B  Error calculations

To get an error estimate for the binned Bessel-weighted sums it is straightforward to show that the error for the Bessel-weighted sums are,

$$\Delta \tilde{\sigma}^{\pm}(b_T) \simeq \Delta S^{\pm} = \sqrt{\sum_{i=1}^{N^{\pm}} J_{0}^{2}(b_T p_{h \perp i})}. \quad (B.1)$$

and

$$\Delta A_{J 0}^{\pm}(b_T p_{h \perp}) = \sqrt{1 - A_{J 0}^{2}(b_T)} \left(\frac{(\Delta S^{+})^2 + (\Delta S^{-})^2}{2}\right) \quad (B.2)$$

where $A_{J 0}^{\pm}(b_T p_{h \perp})$ is given by Eq. (A.11).

C  Equations in the Monte Carlo

Eq. 3.2 in the Monte Carlo is implemented in the form,

$$\frac{d\sigma}{dxdydz} = \frac{1}{2} K(x, y) J(x, Q^2, k_{1 \perp}^2) \times x \sum_{a} e_a^2 \left[ f_a(x_{LC}, k_{1 \perp}^2)D_{1,a}(z_{LC}, p_{2 \perp}) + \lambda \sqrt{1 - \varepsilon^2} g_{1L,a}(x, k_{1 \perp}^2)D_{1,a}(z, p_{2 \perp}) \right] \quad (C.1)$$

where the summation runs over quarks flavors and,

$$K(x, y) = \frac{\alpha^2}{xyQ^2} \frac{y^2}{2(1 - \varepsilon)} \left(1 + \frac{\gamma^2}{2x} \right), \quad \varepsilon = \frac{1 - y - \frac{1}{3}\gamma^2y^2}{1 - y + \frac{1}{2}y^2 + \frac{1}{4}\gamma^2y^2}, \quad (C.2)$$

and the Jacobian $J$ is given by

$$J(x, Q^2, k_{1 \perp}^2) = \frac{x}{x_{LC}} \left(1 + \frac{x^2}{x_{LC}Q^2} \right)^{-1}. \quad (C.3)$$

In above equations $x$ is the Bjorken variable, while $x_{LC} = k^-/P^-$ is the light-cone (LC) fraction of the proton momentum carried by the quark $k$ [29]. The quark four momentum is given by,

$$k_0 = x_{LC}P' + \frac{k_{1 \perp}^2}{4x_{LC}P'}, \quad (C.4)$$

$$k_x = k_{1 \perp} \cos(\phi_k), \quad k_y = k_{1 \perp} \sin(\phi_k), \quad k_z = -x_{LC}P' + \frac{k_{1 \perp}^2}{4x_{LC}P'}, \quad (C.5)$$

where $k_0$ is the quark energy, and $k_{(x,y,z)}$ are the $x$, $y$ and $z$ components of the quark momentum in the CM frame of virtual photon and proton. $P' = 0.5(E_p + |P_{pz}|)$, where proton energy in the CM is $E_p = \sqrt{P_{pz}^2 + M^2}$. Taking into account the nucleon mass, the on-shell condition for the final quark implies

$$x_{LC} = \frac{x}{x_N} \left(1 + \sqrt{1 + \frac{4k_{1 \perp}^2}{Q^2}} \right), \quad x_N = 1 + \sqrt{1 + \frac{4M^2x_N^2}{Q^2}}, \quad (C.6)$$

- 19 -
where $k_\perp$ is the parton transverse momentum. The scattered quark momentum $k'$ is constructed using $k' = k + q$ (see Fig 1) and $p_\perp$ is the transverse momentum of the hadron with respect to the scattered quark $k'$. $\phi_k$ is the initial quark azimuthal angle Fig 1. $z_{LC} = P_{h\perp}/k^+$ is the light-cone fraction of the quark momentum carried by the resulting hadron in the $(\tilde{x}, \tilde{y}, \tilde{z})$-system [29], where $\tilde{z}$ is aligned along the scattered quark $k'$ Fig 1. The final hadron momentum is constructed using,

$$P_{h\tilde{x}} = p_\perp \cos(\phi), \quad P_{h\tilde{y}} = p_\perp \sin(\phi), \quad P_{h\tilde{z}} = z_{LC} k' - \frac{p_\perp^2 + M_h^2}{4z_{LC} k'_0}$$

were $\phi$ is the angle between quark and hadron planes. $\phi_h$ is the angle between leptonic and hadronic planes according to the Trento convention and $P_{h\perp}$ is the final hadron transverse momentum [45]. The final hadron SIDIS variables $\phi_h$, $P_{h\perp}$ and $z$ are calculated after event generation. Here we should note, that theoretical or phenomenological DFs and fragmentation functions are expected to be in the light cone coordinate system (see Eq. C.1). Motivated by the fact that $x_{LC} \simeq x$ and $z_{LC} \simeq z$ is a widely used approximation we implement $f_1(x)$ and $D_1(z)$ instead of $f_1(x_{LC})$ and $D_1(z_{LC})$ in Monte Carlo.

References

[1] J. Dudek, R. Ent, R. Essig, K. Kumar, C. Meyer, et. al., *Physics Opportunities with the 12 GeV Upgrade at Jefferson Lab*, Eur.Phys.J. *A*48 (2012) 187, [arXiv:1208.1244].

[2] COMPASS Collaboration Collaboration, F. Gautheron et. al., *COMPASS-II Proposal*, CERN-SPSC-2010-014.

[3] H. Gao, L. Gamberg, J. Chen, X. Qian, Y. Qiang, et. al., *Transverse Spin Structure of the Nucleon through Target Single Spin Asymmetry in Semi-Inclusive Deep-Inelastic (e,e'π±) Reaction at Jefferson Lab*, Eur.Phys.J.Plus *126* (2011) 2, [arXiv:1009.3803].

[4] D. Boer, M. Diehl, R. Milner, R. Venugopalan, W. Vogelsang, et. al., *Gluons and the quark sea at high energies: Distributions, polarization, tomography*, arXiv:1108.1713.

[5] A. Accardi, J. Albacete, M. Anselmino, N. Armesto, E. Aschenauer, et. al., *Electron Ion Collider: The Next QCD Frontier - Understanding the glue that binds us all*, arXiv:1212.1701.

[6] HERMES Collaboration, A. Airapetian et. al., *Single-spin asymmetries in semi-inclusive deep-inelastic scattering on a transversely polarized hydrogen target*, Phys. Rev. Lett. *94* (2005) 012002, [hep-ex/0408013].

[7] COMPASS Collaboration, V. Y. Alexakhin et. al., *First measurement of the transverse spin asymmetries of the deuteron in semi-inclusive deep inelastic scattering*, Phys. Rev. Lett. *94* (2005) 202002, [hep-ex/0503002].

[8] COMPASS Collaboration, E. S. Ageev et. al., *A new measurement of the collins and sivers asymmetries on a transversely polarised deuteron target*, Nucl. Phys. *B765* (2007) 31–70, [hep-ex/0610068].

[9] HERMES Collaboration, A. Airapetian et. al., *Effects of transversity in deep-inelastic scattering by polarized protons*, Phys. Lett. *B693* (2010) 11–16, [arXiv:1006.4221].
[10] The COMPASS Collaboration, M. G. Alekseev et. al., Measurement of the Collins and Sivers asymmetries on transversely polarised protons, Phys. Lett. B692 (2010) 240–246, [arXiv:1005.5609].

[11] M. G. Alekseev et. al., Azimuthal asymmetries of charged hadrons produced by high- energy muons scattered off longitudinally polarised deuterons, Eur. Phys. J. C70 (2010) 39–49, [arXiv:1007.1562].

[12] COMPASS Collaboration Collaboration, C. Adolph et. al., Transverse spin effects in hadron-pair production from semi-inclusive deep inelastic scattering, Phys. Lett. B713 (2012) 10–16, [arXiv:1202.6150].

[13] H. Mkrtchyan et. al., Transverse momentum dependence of semi-inclusive pion production, Phys. Lett. B665 (2008) 20–25, [hep-ph/0709.3020].

[14] CLAS Collaboration, H. Avakian et. al., Measurement of Single and Double Spin Asymmetries in Deep Inelastic Pion Electroproduction with a Longitudinally Polarized Target, Phys. Rev. Lett. 105 (2010) 262002, [hep-ex/1003.4549].

[15] The Jefferson Lab Hall A Collaboration Collaboration, X. Qian et. al., Single Spin Asymmetries in Charged Pion Production from Semi-Inclusive Deep Inelastic Scattering on a Transversely Polarized 3He Target, Phys.Rev.Lett. 107 (2011) 072003, [arXiv:1106.0363]. 6 pages, 2 figures, 2 tables, published in PRL.

[16] J. C. Collins and D. E. Soper, Back-to-back jets in qcd, Nucl. Phys. B193 (1981) 381.

[17] J. C. Collins and D. E. Soper, Parton Distribution and Decay Functions, Nucl.Phys. B194 (1982) 445.

[18] J. C. Collins, D. E. Soper, and G. F. Sterman, Transverse Momentum Distribution in Drell-Yan Pair and W and Z Boson Production, Nucl.Phys. B250 (1985) 199.

[19] X. Ji, J. Ma, and F. Yuan, Qcd factorization for semi-inclusive deep-inelastic scattering at low transverse momentum, Phys. Rev. D71 (2005) 034005, [hep-ph/0404183].

[20] J. C. Collins and A. Metz, Universality of soft and collinear factors in hard- scattering factorization, Phys. Rev. Lett. 93 (2004) 252001, [hep-ph/0408249].

[21] A. Bacchetta, D. Boer, M. Diehl, and P. J. Mulders, Matches and mismatches in the descriptions of semi- inclusive processes at low and high transverse momentum, JHEP 08 (2008) 023, [hep-ph/0803.0227].

[22] S. M. Aybat and T. C. Rogers, TMD Parton Distribution and Fragmentation Functions with QCD Evolution, Phys.Rev. D83 (2011) 114042, [arXiv:1101.5057].

[23] J. Collins, Foundations of perturbative QCD, . Foundations of Perturbative QCD, Cambridge, UK: Univ.Pr.

[24] A. Kotzinian, New quark distributions and semiinclusive electroproduction on the polarized nucleons, Nucl. Phys. B441 (1995) 234–248, [hep-ph/9412283].

[25] P. J. Mulders and R. D. Tangerman, The complete tree-level result up to order 1/q for polarized deep-inelastic lepton production, Nucl. Phys. B461 (1996) 197–237, [hep-ph/9510301].

[26] J. Levelt and P. J. Mulders, Time reversal odd fragmentation functions in semiinclusive scattering of polarized leptons from unpolarized hadrons, Phys. Lett. B338 (1994) 357–362, [hep-ph/9408257].
[27] A. Bacchetta et al., Semi-inclusive deep inelastic scattering at small transverse momentum, JHEP 02 (2007) 093, [hep-ph/0611265].

[28] D. de Florian, R. Sassot, and M. Stratmann, Global analysis of fragmentation functions for pions and kaons and their uncertainties, Phys.Rev. D75 (2007) 114010, [hep-ph/0703242].

[29] M. Anselmino et al., The role of Cahn and Sivers effects in deep inelastic scattering, Phys. Rev. D71 (2005) 074006, [hep-ph/0501196].

[30] D. Amrath, A. Bacchetta, and A. Metz, Reviewing model calculations of the Collins fragmentation function, Phys. Rev. D 71 (2005) 114018, [hep-ph/0504124].

[31] A. Bacchetta, L. P. Gamberg, G. R. Goldstein, and A. Mukherjee, Collins fragmentation function for pions and kaons in a spectator model, Phys. Lett. B659 (2008) 234–243, [hep-ph/0707.3372].

[32] H. H. Matevosyan, A. W. Thomas, and W. Bentz, Calculating kaon fragmentation functions from the nambu–jona-lasinio jet model, Phys. Rev. D 83 (Apr, 2011) 074003.

[33] M. Hirai, S. Kumano, T.-H. Nagai, and K. Sudoh, Determination of fragmentation functions and their uncertainties, Phys. Rev. D75 (2007) 094009, [hep-ph/0702250].

[34] M. Anselmino, M. Boglione, U. D’Alesio, A. Kotzinian, S. Melis et al., Sivers Effect for Pion and Kaon Production in Semi-Inclusive Deep Inelastic Scattering, Eur.Phys.J. A39 (2009) 89–100, [arXiv:0805.2677].

[35] H. H. Matevosyan, W. Bentz, I. C. Cloet, and A. W. Thomas, Collins Fragmentation Function within NJL-jet Model, Phys.Rev. D86 (2012) 034025, [arXiv:1205.5813].

[36] H. H. Matevosyan, W. Bentz, I. C. Cloet, and A. W. Thomas, Transverse Momentum Dependent Fragmentation and Quark Distribution Functions from the NJL-jet Model, Phys.Rev. D85 (2012) 014021, [arXiv:1111.1740].

[37] A. Casey, I. C. Cloet, H. H. Matevosyan, and A. W. Thomas, Dihadron Fragmentation Functions from the NJL-jet model and their QCD Evolution, Phys.Rev. D86 (2012) 114018, [arXiv:1207.4267].

[38] A. Signori, A. Bacchetta, M. Radici, and G. Schnell, Investigations into the flavor dependence of partonic transverse momentum, JHEP 1311 (2013) 194, [arXiv:1309.3507].

[39] D. Boer, L. Gamberg, B. Musch, and A. Prokudin, Bessel-Weighted Asymmetries in Semi Inclusive Deep Inelastic Scattering, JHEP 1110 (2011) 021, [arXiv:1107.5294].

[40] R. N. Cahn, Azimuthal Dependence in Leptoproduction: A Simple Parton Model Calculation, Phys. Lett. B78 (1978) 269.

[41] R. N. Cahn, Critique of parton model calculations of azimuthal dependence in lepton production, Phys. Rev. D40 (1989) 3107–3110.

[42] M. Gourdin, Semiclusive reactions induced by leptons, Nucl. Phys. B49 (1972) 501–512.

[43] D. Boer and P. J. Mulders, Time-reversal odd distribution functions in lepton production, Phys. Rev. D57 (1998) 5780–5786, [hep-ph/9711485].

[44] M. Diehl and S. Sapeta, On the analysis of lepton scattering on longitudinally or transversely polarized protons, Eur. Phys. J. C41 (2005) 515–533, [hep-ph/0503023].

[45] A. Bacchetta, U. D’Alesio, M. Diehl, and C. A. Miller, Single-spin asymmetries: The trento conventions, Phys. Rev. D70 (2004) 117504, [hep-ph/0410050].
COMPASS Collaboration, C. Adolph et. al., Hadron Transverse Momentum Distributions in Muon Deep Inelastic Scattering at 160 GeV/c, Eur.Phys.J. C73 (2013) 2531, [arXiv:1305.7317].

HERMES Collaboration Collaboration, A. Airapetian et. al., Multiplicities of charged pions and kaons from semi-inclusive deep-inelastic scattering by the proton and the deuteron, Phys.Rev. D87 (2013) 074029, [arXiv:1212.5407].

M. Anselmino, M. Boglione, J. Gonzalez H., S. Melis, and A. Prokudin, Unpolarised Transverse Momentum Dependent Distribution and Fragmentation Functions from SIDIS Multiplicities, JHEP 1404 (2014) 005, [arXiv:1312.6261].

Z. Lu and B.-Q. Ma, Sivers function in light-cone quark model and azimuthal spin asymmetries in pion electroproduction, Nucl. Phys. A741 (2004) 200–214, [hep-ph/0406171].

B. Pasquini, S. Cazzaniga, and S. Boffi, Transverse momentum dependent parton distributions in a light-cone quark model, Phys. Rev. D78 (2008) 034025, [hep-ph/0806.2298].

C. Bourrely, F. Buccella, and J. Soffer, Semiinclusive DIS cross sections and spin asymmetries in the quantum statistical parton distributions approach, Phys.Rev. D83 (2011) 074008, [arXiv:1008.5322].

P. Hagler, B. U. Musch, J. W. Negele, and A. Schafer, Intrinsic quark transverse momentum in the nucleon from lattice QCD, Europhys. Lett. 88 (2009) 61001, [hep-lat/0908.1283].

B. U. Musch, P. Hagler, J. W. Negele, and A. Schafer, Exploring quark transverse momentum distributions with lattice QCD, Phys.Rev. D83 (2011) 094507, [arXiv:1011.1213].

P. Schweitzer, M. Strikman, and C. Weiss, Intrinsic transverse momentum and parton correlations from dynamical chiral symmetry breaking, JHEP 1301 (2013) 163, [arXiv:1210.1267].

J. Collins, T. Rogers, and A. Stasto, Fully unintegrated parton correlation functions and factorization in lowest-order hard scattering, Phys.Rev. D77 (2008) 085009, [arXiv:0708.2833].
[63] L. P. Gamberg, G. R. Goldstein, and M. Schlegel, *Transverse Quark Spin Effects and the Flavor Dependence of the Boer-Mulders Function*, Phys. Rev. D77 (2008) 094016, [hep-ph/0708.0324].

[64] H. H. Matevosyan, A. Kotzinian, and A. W. Thomas, *Studies of Azimuthal Modulations in Two Hadron Fragmentation of a Transversely Polarised Quark*, Phys.Lett. B731 (2014) 208–216, [arXiv:1312.4556].

[65] B. Pasquini and S. Boffi, *Electroweak structure of the nucleon, meson cloud and light-cone wavefunctions*, Phys.Rev. D76 (2007) 074011, [arXiv:0707.2897].

[66] B. Pasquini, S. Boffi, A. Efremov, and P. Schweitzer, *Transverse momentum dependent parton distributions and azimuthal asymmetries in light-cone quark models*, arXiv:0912.1761.

[67] Z. Lu and B.-Q. Ma, *Quark helicity distributions in transverse momentum space and transverse coordinate space*, Phys.Rev. D87 (2013) 034037, [arXiv:1212.6864].

[68] CLAS Collaboration Collaboration, B. Mecking et. al., *The CEBAF Large Acceptance Spectrometer (CLAS)*, Nucl.Instrum.Meth. A503 (2003) 513–553.

[69] A. V. Konychev and P. M. Nadolsky, *Universality of the Collins-Soper-Sterman nonperturbative function in gauge boson production*, Phys.Lett. B633 (2006) 710–714, [hep-ph/0506225].

[70] C. Aidala, B. Field, L. Gamberg, and T. Rogers, *Limits on TMD Evolution From Semi-Inclusive Deep Inelastic Scattering at Moderate Q*, Phys.Rev. D89 (2014) 094002, [arXiv:1401.2654].