A Period Map for Global Derived Stacks

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Abstract

We develop the theory of Griffiths period map, which relates the classification of smooth projective varieties to the associated Hodge structures, in the framework of Derived Algebraic Geometry. We complete the description of the local period map as a morphism of derived deformation functors, following the path marked by Fiorenza, Manetti and Martinengo. In the end we show how to lift the local period map to a morphism of derived stacks, in order to construct a global version of that.

Contents

1 The Period Map as a Holomorphic Function 4
   1.1 Griffiths Period Map . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
   1.2 The Differential of the Period Mapping . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

2 The Period Map as a Morphism of Deformation Functors 7
   2.1 Deformations of $k$-Schemes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
   2.2 Mapping Cones and Deformations of Filtered Complexes . . . . . . . . . . . . . . . . 9
   2.3 Cartan Homotopies and Period Maps . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
   2.4 Flag Functors and the Fiorenza-Manetti Period Map . . . . . . . . . . . . . . . . . . . 13

3 The Period Map as a Morphism of $\infty$-Groupoids 15
   3.1 Quick Review of Derived Deformation Theory . . . . . . . . . . . . . . . . . . . . . . . 15
   3.2 The Algebraic Fiorenza-Manetti-Martinengo Period Map . . . . . . . . . . . . . . . . 20
   3.3 Affine DG$_{\geq 0}$-Categories and the Dold-Kan Correspondence . . . . . . . . . . . . . 21
   3.4 Derived Deformations of Filtered Algebraic De Rham Complexes . . . . . . . . . . . . 30
   3.5 Derived Deformations of $k$-Schemes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 32
   3.6 The Geometric Fiorenza-Manetti-Martinengo Period Map . . . . . . . . . . . . . . . . 37

4 The Period Map as a Morphism of Derived Stacks 41
Introduction

Let \( X \) be a smooth complex projective variety of dimension \( d \) and consider a family of deformations \( X \to S \) of \( X \) over some affine base \( S \); in 1968 Griffiths observed that any such Kuranishi family induces canonically a variation of Hodge structures on \( X \). More formally let

\[
0 = F^{k+1}H^k(X) \hookrightarrow F^kH^k(X) \hookrightarrow \cdots \hookrightarrow F^1H^k(X) \hookrightarrow F^0H^k(X) = H^k(X)
\]

be the Hodge filtration on cohomology and set \( b^{p,k} := \dim F^pH^k(X) \); define

\[
\text{Grass}(H^*(X)) := \prod_k \text{Grass}(b^{p,k}, H^k(X))
\]

which is a complex projective variety as so are the Grassmannians involved. Griffiths constructed the morphism

\[
\mathcal{P}^p : S \longrightarrow \text{Grass}(H^*(X))
\]

\[
t \longmapsto \prod_k F^pH^k(X_t)
\]

(0.1)

where \( X_t \) is the fibre of the family \( X \to S \) over \( t \in S \); map (0.1) is said to be the \( p \)th local period map associated to \( X \to S \). In [13] Griffiths proved that such a map is well-defined and holomorphic; he also computed its differential and showed that it is the same as the contraction map on the space \( H^1(X, \mathcal{T}_X) \) of first-order deformations of \( X \). Moreover it is possible to use map (0.1) to derive some constraints on the obstructions of \( X \).

The existence and holomorphicity of the local period map says that for any given Kuranishi family of a projective manifold \( X \) there is a canonical way to construct a variation of its Hodge structures; moreover such a correspondence seems to be compatible with the general deformation theory of the variety \( X \); prompted by this observation, in 2006 Fiorenza and Manetti described Griffiths period map in terms of deformation functors. Let

\[
\text{Def}_X : \text{Art}_C \longrightarrow \text{Set}
\]

\[
A \mapsto \{\text{deformations of } X \text{ over } A\}
\]

be the functor of Artin rings parametrizing the deformations of the variety \( X \) and recall that such a deformation functor is isomorphic to the deformation functor associated to the Kodaira-Spencer dgla \( KS_X \); in a similar way for all non-negative \( p \) define the functor of Artin rings

\[
\text{Grass}_{F^pH^*(X), H^*(X)} : \text{Art}_C \longrightarrow \text{Set}
\]

\[
A \mapsto \{A\text{-deformations of } F^pH^*(X) \text{ inside } H^*(X)\}
\]

which describes the deformations of the complex \( F^pH^*(X) \) as a subcomplex of \( H^*(X) \); this functor precisely encodes variations of Hodge structures on \( X \). In [7], [8] and [9] Fiorenza and Manetti proved the following facts:

- \( \text{Grass}_{F^pH^*(X), H^*(X)} \) is a deformation functor (in the sense of Schlessinger) and

\[
\text{Grass}_{F^pH^*(X), H^*(X)} \simeq \text{Def}_X
\]
where $C_\chi$ is the $L_\infty$-algebra defined as the cone of the inclusion of dgla’s 

$$
\chi : \text{End}^{F_p} (H^* (X)) \hookrightarrow \text{End}^* (H^* (X))
$$

with 

$$
\text{End}^{F_p} (H^* (X)) := \{ f \in \text{End}^* (H^* (X)) \mid f (F_p H^* (X)) \subseteq F_p H^* (X) \};
$$

- the map 

$$
\text{FM}^p : K S_X \longrightarrow C_\chi \\
\xi \longmapsto (l_\xi, i_\xi)
$$

where $i$ is the contraction of differential forms with vector fields and $l$ stands for the holomorphic Lie derivative, is a $L_\infty$-morphism, thus it induces a morphism of deformation functors

$$
\text{FM}^p : \text{Def}_{KS_X} \longrightarrow \text{Def}_{C_\chi};
$$

- the natural transformation

$$
\mathcal{P}^p : \text{Def}_X \longrightarrow \text{Grass}_{F_p H^* (X), H^* (X)} \quad (0.2)
$$

$$
\forall A \in \mathfrak{Art}_C \\
\text{Def}_X (A) \ni \left( \mathcal{O}_A \xrightarrow{\xi} \mathcal{O}_X \right) \longmapsto F_p H^* (X, \mathcal{O}_A) \in \text{Grass}_{F_p H^* (X), H^* (X)} (A)
$$

is a morphism of deformation functors extending Griffiths period map and the two morphisms $\text{FM}^p$ and $\mathcal{P}^p$ are canonically isomorphic.

Fiorenza and Manetti’s work shows that the $p$th local period map is actually a morphism of deformation theories, thus it commutes with all deformation-theoretic constructions: in particular all results of Griffiths about the differential of map (0.1) are easily recovered as purely formal corollaries of the preceding statements. Moreover Fiorenza and Manetti’s construction works for any proper smooth scheme of dimension $d$ over a field of characteristic 0.

As we are able to interpret Griffiths period map as a natural transformation of deformation functors, the next step would be to look at it in the context of Derived Deformation Theory: more formally one could ask whether there exist derived enhancements of the functors $\text{Def}_X$ and $\text{Grass}_{F_p H^* (X), H^* (X)}$ for which it is possible to find some natural derived extension of morphism (0.2). In 2012 Fiorenza and Martinengo approached this problem, tackling it from an entirely algebraic viewpoint. As a matter of fact they observed that the contraction of differential forms with vector fields $i$ (seen in the most general way, i.e. as a morphism of complexes of sheaves over $X$) and the Lie derivative $l$ give rise to a morphism of differential graded Lie algebras

$$
\text{FMM} : \mathbb{R} \Gamma (X, \mathcal{T}_X) \xrightarrow{(l,e)} \text{holim} \left( \text{End}^0 (\mathbb{R} \Gamma (X, \Omega^*_X)) \xrightarrow{\text{incl.}_0} \text{End}^* (\mathbb{R} \Gamma (X, \Omega^*_X)) \right) \quad (0.3)
$$

where $\text{End}^0 (\mathbb{R} \Gamma (X, \Omega^*_X))$ is the dgla made of non-negatively graded maps of the complex $\mathbb{R} \Gamma (X, \Omega^*_X)$ in itself, which can be interpreted as the dgla of all filtration-preserving maps. Notice

---

1. Here, by a slight abuse of notation, the symbol $\text{FM}^p$ is denoting both the $L_\infty$-map and the induced morphism of deformation functors.

2. This is why, by a slight abuse of notation, we are using the same symbol for both.
also that the codomain of map (0.3) is nothing but the homotopy fibre over 0 of the inclusion of $\End_{\geq 0}(\R \Gamma (X, \Omega_X^\bullet))$ into $\End(\R \Gamma (X, \Omega_X^\bullet))$. In [10] Fiorenza and Martinengo showed that map (0.3) induces a morphism of derived deformation functors
\[
\text{FMM} : \R \Def_{\R \Gamma (X, \mathcal{T}_X)} \rightarrow \R \Def_{\ho \lim\left( \End_{\geq 0}(\R \Gamma (X, \Omega_X^\bullet)) \xrightarrow{\text{incl}} \End^*(\R \Gamma (X, \Omega_X^\bullet)) \right)}
\]
whose 0-truncation FM is very close to $\text{FM}^p$ (actually $\text{FM}$ is even more interesting than $\text{FM}^p$ as it does not depend on the degree of the filtration, so it can be interpreted as a universal version of Griffiths period map).

The goal of this paper is to lift Fiorenza, Manetti and Martinengo’s work to a global level, i.e. to find a morphism of derived geometric stacks whose restriction to formal neighbourhoods is (isomorphic to) map (0.4). The crucial step in order to do this consists of giving a more geometric description of such a map, thus we will define a morphism of derived deformation functors
\[
\mathcal{P} : \R \Def_X \rightarrow \ho \Flag (\R \Gamma (X, \Omega_X^\bullet), F^* \R \Gamma (X, \Omega_X^\bullet))
\]
and prove that map (0.5) is naturally isomorphic to map (0.4); the functor $\R \Def_X$ parametrizes derived deformations of the scheme $X$ (i.e. homotopy flat families of derived schemes deforming the underived scheme $X$), while $\ho \Flag (\R \Gamma (X, \Omega_X^\bullet), F^* \R \Gamma (X, \Omega_X^\bullet))$ encodes derived deformations of the filtered complex $(\R \Gamma (X, \Omega_X^\bullet), F^*)$. Although it is intuitively quite clear what such functors should be, giving a careful definition of them reveals to be non-trivial at all and has actually lead us to develop the notions of affine differential graded category and affine simplicial category, which are probably interesting objects themselves to study. $\R \Def_X$ and $\ho \Flag (\R \Gamma (X, \Omega_X^\bullet), F^* \R \Gamma (X, \Omega_X^\bullet))$ are (essentially by construction) formal neighbourhoods of interesting derived stacks: the former is the formal neighbourhood at $X$ of the derived (non-geometric) stack of derived schemes of dimension $d$, which has been recently studied by Pridham in [34], and the latter is related to the derived geometric stack of filtered perfect complexes with trivial Ext groups in higher negative degrees, which is the main object of [6]. Finally, having such tools at our disposal, we will define a morphism between these stacks inducing map (0.5) on formal neighbourhoods: such a morphism can be thought of as a global universal version of the period map.

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1 The Period Map as a Holomorphic Function

Let $X$ be a compact connected complex Kähler manifold of dimension $d$ and consider a family of deformations $\varphi : X \rightarrow S$, i.e. a proper holomorphic submersion of complex manifolds (where the base $S$ is contractible) admitting a distinguished fibre $\varphi^{-1}(0) =: X_0 \simeq X$. Recall
that a famous result due to Ehresmann says that any such family is $C^\infty$-trivial, i.e. there exists a diffeomorphism

$$T : X \xrightarrow{\sim} X_0 \times S \simeq X \times S$$

(1.1)

over $S$ (see [43] Theorem 9.3). For all $t \in S$ let $X_t := \varphi^{-1}(t)$: Ehresmann’s trivialization (1.1) clearly induces a diffeomorphism $X_t \simeq X$ for all $t$, thus we can think of the morphism $\varphi$ as a collection of complex structures over the $C^\infty$-manifold underlying the complex variety $X$. This situation is the prototypical example of all deformation problems and was originally studied by Kodaira and Spencer.

A very natural question to ask is how the standard Hodge structures over $X$ vary with respect to the family $\varphi$; more formally, consider the cohomology algebra of $X$

$$H^*(X, \mathbb{C}) := \bigoplus_{0 \leq k \leq d} H^k(X, \mathbb{C})$$

and recall that each cohomology group $H^k(X, \mathbb{C})$ is endowed with a Hodge structure of weight $k$ defined by the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \quad H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$$

or, equivalently, by the Hodge filtration

$$0 = F^{k+1}H^k(X, \mathbb{C}) \hookrightarrow F^kH^k(X, \mathbb{C}) \hookrightarrow \cdots \hookrightarrow F^1H^k(X, \mathbb{C}) \hookrightarrow F^0H^k(X, \mathbb{C}) = H^k(X, \mathbb{C})$$

where

$$F^mH^k(X, \mathbb{C}) := \bigoplus_{p \geq m} H^{p,q}(X).$$

The question we want to address is whether the family $\varphi$ induces any interesting structure on the cohomology of the fibres.

### 1.1 Griffiths Period Map

Observe that Ehresmann’s trivialization (1.1) provides us with a diagram of isomorphisms of vector spaces

$$
\begin{array}{ccc}
H^k(X, \mathbb{C}) & \xrightarrow{\sim} & H^k(X, \mathbb{C}) \\
\parallel & & \parallel \\
H^k(X_0, \mathbb{C}) & \xrightarrow{\sim} & H^k(X_t, \mathbb{C})
\end{array}
$$

which commutes for all $k \geq 0$ and $\forall t \in S$; actually much more is true, as for all $k \geq 0$ the sheaf $\mathbb{R}^k\varphi_*\mathbb{C}$ — where $\varphi_* : \mathcal{H}(X) \to \mathcal{H}(S)$ is the push-forward functor — is seen to be a local system over $S$ isomorphic to the constant sheaf $H^k(X, \mathbb{C})$ (see [43] Section 9.2 for a more detailed explanation), thus the diagram above does not depend on the choice of the trivialization. Denote

$$h^k := \dim H^k(X, \mathbb{C}) \quad h^{p,q} := \dim H^{p,q}(X) \quad b^{p,k} := \dim F^pH^k(X, \mathbb{C}).$$

A standard argument based on the $E_1$-degeneration of the Hodge-to-De Rham spectral sequence of $X$ shows that there exists a neighbourhood of $0 \in S$ such that

$$\dim H^{p,q}(X_t) = \dim H^{p,q}(X) =: h^{p,q}$$
thus the Hodge numbers of $X$ are invariant under (infinitesimal) deformation; moreover this immediately implies that the Hodge-to-De Rham spectral sequence of such fibres degenerates at its first page\(^3\) as well (see \cite{Griffiths} Proposition 9.20).

**Definition 1.1.** (Griffiths) In the above notations define the \((p,k)\)th local period map to be
\[
\p^{p,k} : S \longrightarrow \text{Grass} \left( \mathcal{H}^{p,k}(X, \mathbb{C}) \right)
\]
\[
t \mapsto F^p H^k(X_t, \mathbb{C}).
\] (1.2)

Since $H^k(X_t, \mathbb{C})$ is canonically isomorphic to $H^k(X, \mathbb{C})$ and the Hodge numbers of $X$ are invariant under deformation, map (1.2) is well-defined; the following is a famous result of Griffiths.

**Theorem 1.2.** (Griffiths) The \((p,k)\)th local period map (1.2) is holomorphic \(\forall p \leq k\).

**Proof.** See \cite{Griffiths} Theorem 10.9. \(\square\)

1.2 The Differential of the Period Mapping

Griffiths deeply studied the differential of map (1.2), as well: in order to state his result let us review what the contraction of differential forms with vector fields and the (holomorphic) Lie derivative are. Recall that the tangent sheaf $\mathcal{T}_X$ is endowed with a natural structure of sheaf of Lie algebras (which can be considered as dgla’s concentrated in degree 0) induced by the canonical isomorphism $\mathcal{T}_X \simeq \text{Der}(\mathcal{O}_X, \mathcal{O}_X)$, while $\text{End}^*(\Omega_X^*)$ comes with a structure of sheaf of differential graded Lie algebras through the standard differential on Hom complexes and the standard Lie bracket. Now the contraction morphism is defined to be the “shifted” map of sheaves of differential graded Lie algebras
\[
i : \mathcal{T}_X \rightarrow \text{End}^* (\Omega_X^*)[-1]
\]
\[
\xi \mapsto i\xi \text{ such that } i\xi (\omega) := \xi \lrcorner \omega \text{ (on local sections)}
\] (1.3)
while the differential of map (1.3) (as an element of the complex $\text{Hom}^*(\mathcal{T}_X, \text{End}^* (\Omega_X^*)[-1])$) is by definition the Lie derivative
\[
l : \mathcal{T}_X \rightarrow \text{End}^* (\Omega_X^*)
\]
\[
\xi \mapsto \iota \xi \text{ such that } l\xi (\omega) := d(\xi \lrcorner \omega) + \xi \lrcorner (d\omega) \text{ (on local sections)}
\] (1.4)
which is a genuine morphism of sheaves of dgla’s.\(^4\)

**Theorem 1.3.** (Griffiths) The differential of map (1.2) is
\[
d\p^{p,k} = i : H^1(X, \mathcal{T}_X) \longrightarrow \text{Hom} \left( \mathcal{F}^p H^k(X, \mathbb{C}), \frac{H^k(X, \mathbb{C})}{\mathcal{F}^p H^k(X, \mathbb{C})} \right).\] (1.5)
Moreover map (1.5) actually takes values in $\text{Hom} \left( \mathcal{F}^p H^k(X, \mathbb{C}), \frac{\mathcal{F}^{p-1} H^k(X, \mathbb{C})}{\mathcal{F}^p H^k(X, \mathbb{C})} \right)$.\(^5\)

\(^3\)Up to shrinking the base $S$, the fibres of $\varphi$ are Kähler manifolds themselves (see \cite{Griffiths} Theorem 9.23).
\(^4\)By a slight abuse of notation we will tend to denote by $i$ and $l$ the morphisms that maps (1.3) and (1.4) induce on global sections and derived global sections, as well.
\(^5\)This last property is generally known as Griffiths transversality.
Proof. The theorem has been stated in relatively modern terms, but a complete proof of it is given in [43] Proposition 10.12, Lemma 10.19 and Theorem 10.21.

The \((p,k)\)th local period map \((1.2)\) depends by definition on two parameters, a cohomology one – that is \(k\) – and a filtration one – that is \(p\); we would like to encode all cohomological information about the variations of Hodge structures induced by the family \(\phi\) in a single morphism.

**Definition 1.4.** In the above notations define the \(p\)th local period map to be

\[
P^p : S \longrightarrow \text{Grass}(H^*(X, \mathbb{C}))
\]

\[
t \longrightarrow \prod_k F^p H^k(X_t, \mathbb{C}).
\]

(1.6)

Notice that map \((1.6)\) is holomorphic and that its differential is still a contraction morphism, i.e.

\[
dP^p = i : H^1(X, \mathcal{T}_X) \longrightarrow \bigoplus_k \text{Hom}\left(F^p H^k(X, \mathbb{C}), \frac{H^k(X, \mathbb{C})}{F^p H^k(X, \mathbb{C})}\right).
\]

(1.7)

2 The Period Map as a Morphism of Deformation Functors

The work of Griffiths which has been described in Section 1 relates deformations of a complex smooth projective variety (or more generally complex Kähler manifold) to variations of its Hodge structures. Unfortunately the local period map \((1.6)\) is not really a morphism of deformation theories, as it depends on a given deformation of a complex variety \(X\), nonetheless its differential \((1.7)\) is very “deformation-theoretic” in nature, as it connects the space \(H^1(X, \mathcal{T}_X)\), i.e. the tangent space to the deformation functor parametrizing all deformations of \(X\), to another cohomological invariant which depends only on \(X\) rather than the special Kuranishi family over \(X\) determining map \((1.6)\). Observations like these led Fiorenza and Manetti to believe that Griffiths period map could be described as a morphism of deformation functors (in the sense of Schlessinger) whose tangent map coincided with the differential \((1.7)\).

2.1 Deformations of \(k\)-Schemes

Let \(k\) be any (non-necessarily algebraically closed) field of characteristic 0 and consider a smooth proper scheme \(X\) of dimension \(d\): these assumptions over \(X\) just algebraically resemble the analytic framework in which Griffiths studied map \((1.6)\), while the fact that the theory we are about to summarize works for any field of characteristic 0 is a consequence of Deligne’s view on Hodge Theory (for more details see [2], [3] and [4]). Notice also that by [2] Theorem 5.5 the Hodge-to-De Rham spectral sequence of the scheme \(X\) degenerates at its first page: such a property will be used several times in this paper.

Recall that the functor of deformations of \(X\) is the functor of Artin rings

\[
\begin{array}{c}
\text{Def}_X : \text{Art}_k \\
A \rightarrow \frac{\{\text{deformations of } X \text{ over } A\}}{\text{isomorphism}}
\end{array}
\]

(2.1)
where a deformation of $X$ over $A$ is a Cartesian diagram in $\mathbf{Sch}_k$

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & \text{Spec}(A)
\end{array}
$$

with $i$ a closed immersion and $p$ flat and proper; equivalently a deformation of $X$ over $A$ can be viewed as a morphism of sheaves of $A$-algebras $\mathcal{O}_A \to \mathcal{O}_X$ such that $\mathcal{O}_A$ is flat over $A$ and $\mathcal{O}_A \otimes_A k \simeq \mathcal{O}_X$. Of course, two $A$-deformations $X_1 \to \text{Spec}(A)$ and $X_2 \to \text{Spec}(A)$ of $X$ are said to be isomorphic if there is an isomorphism $X_1 \xrightarrow{\sim} X_2$ of schemes over $A$ inducing the identity on $X$: it is well-known that functor (2.1) is a deformation functor in the sense of Schlessinger (see [29] or [40] for a definition).

Now let $(l, (\cdots)_n)_{n>0}$ be a $L_\infty$-algebra over $k$ (see [29] for a definition) and recall that the deformation functor associated to $(l, (\cdots)_n)_{n>0}$ is defined to be

$$
\text{Def}_l : \text{Art}_k \xrightarrow{\text{Art}_k} \text{Set} \\
A \mapsto \text{MC}_l(A) \text{ under homotopy equivalence}
$$

where

$$
\text{MC}_l(A) := \left\{ x \in l^0[1] \otimes m_A \text{ s.t. } \sum_{n \geq 1} \frac{(x \otimes n)_n}{n!} = 0 \right\}
$$

is the set of solutions of the Maurer-Cartan equation and two elements $x_0, x_1 \in \text{MC}_l(A)$ are said to be homotopy equivalent if there exists a “path” $x(t, dt) \in \text{MC}[(t, dt)](A)$ such that $x(0) = x_0$ and $x(1) = x_1$; again, it is not hard to verify that $\text{Def}_l$ is a deformation functor in the sense of Schlessinger. Notice that, if the higher products $(\cdots)_n = 0$ for all $n > 3$, i.e. if the $L_\infty$-algebra is actually a differential graded Lie algebra (see [5] or [29] for a definition), we recover the more classical notion of deformation functor associated to a dgl.

A fundamental fact in Deformation Theory – essentially due to Kodaira, Kuranishi and Spencer and developed in many ways by several other authors – states that the functor of deformations $\text{Def}_X$ associated to a scheme $X$ which satisfies the above conditions is isomorphic to the deformation functor associated to the Kodaira-Spencer dgla of $X$, which is defined to be the differential graded Lie algebra $(KS_X, [-, -], D)$ where

$$
KS_X := \mathbb{R}\Gamma(X, \mathcal{F}_X) \simeq \Gamma \left( X, \mathcal{A}_X^{0,*} (\mathcal{F}_X) \right)
$$

$$
\left[ f d\bar{z}_i \frac{\partial}{\partial z_i}, g d\bar{z}_j \frac{\partial}{\partial z_j} \right] := d\bar{z}_i \wedge d\bar{z}_j \left( f \frac{\partial g}{\partial z_j} \frac{\partial}{\partial z_i} - g \frac{\partial f}{\partial z_i} \frac{\partial}{\partial z_j} \right).
$$

$$
D \left( \omega \frac{\partial}{\partial z_i} \right) := -\bar{\partial}(\omega) \frac{\partial}{\partial z_i}
$$

**Warning 2.1.** In this paper the Kodaira-Spencer algebra $KS_X$ will always correspond to the specific resolution given by $\Gamma \left( X, \mathcal{A}_X^{0,*} (\mathcal{F}_X) \right)$, not to any resolution computing $\mathbb{R}\Gamma(X, \mathcal{F}_X)$: as a matter of fact notice that a generic choice of functor for $\mathbb{R}\Gamma(X, \mathcal{F}_X)$ provides us with a cochain complex with no Lie algebra structure.

---

6From now on by deformation we will always mean infinitesimal deformation, i.e. a deformation over an Artinian base.
Now consider the natural transformation

\[ \mathcal{O} : \text{Def}_{KS_X} \to \text{Def}_X \]

\[ \forall A \in \Art_k \quad \text{Def}_{KS_X}(A) \ni \xi \mapsto (\mathcal{O}_\xi \to \mathcal{O}_X) \in \text{Def}_X(A). \tag{2.2} \]

where for all open \( U \subseteq X \)

\[ \mathcal{O}_\xi(U) := \left\{ f \in \mathcal{A}_X^{0,0}(U) \otimes A \text{ s.t. } \partial f = \xi \partial f \right\} \]

and the map \( \mathcal{O}_\xi \to \mathcal{O}_X \) is induced by the projection \( \mathcal{A}_X^{0,0} \to \mathcal{A}_X^{0,0} \).

**Theorem 2.2.** (Kodaira-Spencer, Kuranishi, [...] ) In the above notations, map (2.2) is an isomorphism of deformation functors.

**Proof.** There is a variety of different proofs of this result in the literature: we refer to [18] Theorem II.7.3 for a very detailed algebraic one; see also [19] Theorem 3.4. \(\square\)

### 2.2 Mapping Cones and Deformations of Filtered Complexes

The functor \( \text{Def}_X \) is the most natural candidate for the domain of a purely “deformation-theoretic” version of Griffiths period map; now we wish to understand what the codomain of such a morphism should be, i.e. we seek a deformation functor which parametrizes variations of Hodge structures over \( X \).

Let \((V, d)\) be a differential graded \( k \)-vector space and \((W, d)\) a subcomplex of its; for any \( A \in \Art_k \), consider the groups\(^7\)

\[ \text{Aut}^V(A) := \left\{ f \in \text{Hom}_A^0(V \otimes A, V \otimes A) \text{ s.t. } f \equiv \text{Id}_{(V, d)} \mod m_A \right\} \]

\[ \text{Aut}^{(V, d)}(A) := \left\{ f \in \text{Aut}^V(A) \text{ s.t. } fd = df \right\} \]

\[ \text{Aut}^{W, V}(A) := \left\{ f \in \text{Aut}^V(A) \text{ s.t. } f(W \otimes A) = W \otimes A \right\} \]

\[ \tilde{\text{Aut}}^{(V, d)}(A) := \left\{ f \in \text{Aut}^{V, d}(A) \text{ s.t. } H^*(f) \text{ is the identity on } H^*(V \otimes A, d) \right\} \]

and define the functor of deformations of \((W, d)\) inside \((V, d)\) to be the functor of Artin rings

\[ \text{Grass}_{W, V} : \Art_k \to \text{Set} \]

\[ A \mapsto \left\{ f \in \text{Aut}^V(A) \text{ s.t. } df(W \otimes A) \subseteq f(W \otimes A) \right\} \times_{\text{Aut}^{(V, d)}(A)} \text{Aut}^{W, V}(A). \tag{2.3} \]

**Remark 2.3.** Formula (2.3) is the original definition of the functor of deformations of the subcomplex \((W, d)\) as we find it in [7]; although it is quite elegant, it may not seem very intuitive, as there is no explicit reference to what a deformation of \((W, d)\) over a local Artinian \( k \)-algebra \( A \) should be. Anyway a more careful look at it immediately shows that a deformation of \((W, d)\) over \( A \) inside \((V, d)\) is a complex of free \( A \)-modules \((V \otimes A, d_A)\) such that its residue modulo \( m_A \) equals \((V, d)\) and \( d_A(W \otimes A) \subseteq W \otimes A \) (this is exactly what the “numerator” in formula

\(^7\)In this section, by a slight abuse of notation, the symbol \( d \) may indifferently denote the differential of the complex \( V \), the differential of the twisted complex \( V \otimes A \) and the differential of the endomorphism complex \( \text{End}((V, d)) \).
parametrizes); on the other hand two such deformations \((V \otimes A, d_A)\) and \((V \otimes A, d'_A)\) are isomorphic if there exists an isomorphism of cochain complexes \(\varphi\) between them such that \(\varphi(W \otimes A, d_A) = \varphi(W \otimes A, d'_A)\) and \(H^i(\varphi) = \operatorname{Id}_{H^i(V \otimes A, d)}\) for all \(i\) (this is exactly what the “denominator” in formula (2.3) parametrizes).

Now consider the graded vector spaces
\[
\begin{align*}
\operatorname{End}^* ((V, d)) & := \operatorname{Hom}^* ((V, d), (V, d)) \\
\operatorname{End}^W ((V, d)) & := \{ f \in \operatorname{End}^* ((V, d)) \text{ s.t. } f(W) \subseteq W \}.
\end{align*}
\]

They are endowed with natural structures of differential graded Lie algebras and there is an obvious inclusion
\[
\chi_{W, V} : \operatorname{End}^W ((V, d)) \hookrightarrow \operatorname{End}^* ((V, d))
\]
which is a morphism of dgla’s; recall also that the mapping cone \((C_{\chi_{W, V}}, \delta)\) of the morphism \(\chi_{W, V}\) is defined to be its homotopy cokernel, i.e. the complex
\[
\operatorname{holim}_{\leftarrow} \left( \operatorname{End}^W ((V, d)) \xrightarrow{\chi_{W, V}} \operatorname{End}^* ((V, d)) \right).
\]

More concretely, the mapping cone is given by the formulae
\[
\begin{align*}
C^i_{\chi_{W, V}} & := \operatorname{End}^W ((V, d))^i \oplus \operatorname{End}^{i-1} ((V, d)) \\
\delta ((f, g)) & := (df, \chi(f) - dg).
\end{align*}
\]

**Proposition 2.4.** (Fiorenza-Manetti) In the above notations, there is a canonical \(L_\infty\)-structure on the mapping cone \(C_{\chi_{W, V}}\).

**Proof.** See [8] Section 4 and Section 5. 

**Remark 2.5.** Fiorenza and Manetti gave two different proofs of Proposition 2.4: the first one is a very elegant but non-constructive proof based on the Homotopy Transfer Theorem (see [22] and [23], while [42] provides a gentler introduction), while the second proof relies on a careful explicit description of all the higher products defining the \(L_\infty\)-structure of \(C_{\chi_{W, V}}\); anyway, we are not reporting such formulae since they are not really needed for the sake of this paper.

Consider the natural transformation
\[
\Psi_{\chi_{W, V}} : \operatorname{Def}_{C_{\chi_{W, V}}} \longrightarrow \operatorname{Grass}_{W, V}
\]
\[
\forall A \in \mathfrak{Art}_k \quad \operatorname{Def}_{C_{\chi_{W, V}}} (A) \ni \eta \quad \mapsto (\eta (W \otimes A), d) \in \operatorname{Grass}_{W, V} (A).
\]

**Theorem 2.6.** (Fiorenza-Manetti) In the above notations, map (2.5) is an isomorphism of deformation functors; in particular \(\operatorname{Grass}_{W, V}\) is a deformation functor in the sense of Schlessinger.

**Proof.** See [7] Proposition 9.2. 

\[10\]
2.3 Cartan Homotopies and Period Maps

The work of Fiorenza and Manetti, especially Theorem 2.6 suggests that a good candidate for the codomain of a purely deformation-theoretic version of Griffiths $p^\text{th}$ local period map should be the functor $\text{Grass}_{F^p H^\ast(X,k), H^\ast(X,k)}$, where

$$H^\ast(X,k) := \mathbb{H}^\ast \left(X, \Omega^\ast_{X/k} \right)$$

is the algebraic De Rham cohomology of the scheme $X$ and $F^\ast$ is the Hodge filtration over it. Now we are almost ready to describe the actual morphism that Fiorenza and Manetti constructed to translate Griffiths period map in terms of deformation functors.

Definition 2.7. (Fiorenza-Manetti) Let $(g,d,[-,-])$ and $(l,d,[-,-])$ be two differential graded Lie algebras over $k$; a linear map $i \in \text{Hom}^{-1}(g,l)$ is said to be a Cartan homotopy if

$$\forall a,b \in g \quad i([a,b]) = [i(a) , di(b)] \quad \text{and} \quad [i(a) , i(b)] = 0^8$$

Remark 2.8. The following facts directly follow from Definition 2.7

1. The differential of a Cartan homotopy is a morphism of differential graded Lie algebras (i.e. it preserves grading and differentials);
2. The notion of Cartan homotopy is stable under composition with a dgla map and under tensorization with a differential graded commutative algebra;
3. The notion of Cartan homotopy generalizes to maps of sheaves of dgla’s;
4. Let $i : g \to l[-1]$ be a Cartan homotopy between and $l : g \to l$ its differential: $e^i$ is an homotopy between $l$ and the zero dgla morphism 0.

Example 2.9. The contraction map associated to the scheme $X$ is a Cartan homotopy of sheaves of dgla’s (see Section 1.2 for a definition in the context of complex manifolds), while derived global sections of such a map provides us with an honest Cartan homotopy of dgla’s: the latter will be a key ingredient of this paper (see Section 3.2).

The reason why we are interested in Cartan homotopies is that they behave very well with respect to mapping cones.

Proposition 2.10. (Fiorenza-Manetti) In the notations of Definition 2.7 let $l := di$; then the linear map

$$\tilde{i} : g \longrightarrow C_l \quad a \longmapsto (a, i(a))$$

is a $L_\infty$-morphism; in particular it induces a morphism between the associated deformation functors.

Proof. See [7] Proposition 7.4. \hfill \Box

8 Again, we are denoting by the same symbol the differential and the bracket of the dgla’s $g$, $l$ and $\text{Hom}^\ast(g,l)$. 

11
Now, in the notations of Proposition 2.4, set \( V := R\Gamma(X, \Omega^*_X) \), \( W := F^pR\Gamma(X, \Omega^*_X/k) \) and \( \chi^p := \chi_{V,W} \) for all \( p > 0 \); also denote by \( i \) the contraction map associated to \( X \) and by \( l \) its differential, that is the “holomorphic” Lie derivative.

**Warning 2.11.** As we did in Warning 2.1 in the case of the Kodaira-Spencer dgla, again we will always fix a specific choice of functor for \( R\Gamma(X, \Omega^*_X/k) \), i.e. the one given by the Dolbeault resolution: in other words throughout the paper we will have
\[
R\Gamma(X, \Omega^*_X/k) \simeq \Gamma(X, A^*_X, \Omega^*_X/\mathfrak{A}^*_X). \tag{2.5}
\]

**Theorem 2.12.** (Fiorenza-Manetti) The linear map
\[
fm^p : KS_X \longrightarrow C_{\chi^p}
\]
\[
\xi \mapsto (l_{\xi}, i_{\xi})
\]
is a \( L_\infty \)-morphism; in particular it induces a morphism of deformation functors
\[
fm^p : \text{Def}_{KS_X} \longrightarrow \text{Def}_{C_{\chi^p}} \tag{2.6}
\]

**Proof.** See [7] Theorem 12.1. \( \square \)

**Remark 2.13.** Recall that, as a consequence of the \( E_1 \)-degeneration of the Hodge-to De Rham spectral sequence of \( X \), the canonical inclusion of complexes \( F^pR\Gamma(X, \Omega^*_X/k) \hookrightarrow R\Gamma(X, \Omega^*_X/k) \) descends to cohomology, i.e. the induced linear map \( H^*(F^pR\Gamma(X, \Omega^*_X/k)) \rightarrow H^*(X, k) \) is injective. This is equivalent to say that for all \( p \) there is a quasi-isomorphism of complexes between \( F^pR\Gamma(X, \Omega^*_X/k) \) and \( F^pH^*(X, k) \) which in turn induces a quasi-isomorphism of dgla’s between \( \text{End}^{F^pR\Gamma(X, \Omega^*_X/k)}(\Gamma(X, \Omega^*_X/k)) \) and \( \text{End}^{F^pH^*(X, k)}(H^*(X, k)) \).

Now denote \( \hat{\chi}^p := \chi_{H^*(X,k), F^pH^*(X,k)} \): Remark 2.13 entails in particular the existence of a homotopy equivalence of \( L_\infty \)-algebras
\[
h : C_{\chi^p} \longrightarrow C_{\hat{\chi}^p}
\]
which induces, by the Basic Theorem of Deformation Theory (see [29]), an isomorphism
\[
h : \text{Def}_{C_{\chi^p}} \longrightarrow \text{Def}_{C_{\hat{\chi}^p}} \tag{2.7}
\]

between the corresponding deformation functors. In the same fashion, the natural transformation
\[
H^* : \text{Grass}_{F^pR\Gamma(X, \Omega^*_X/k), R\Gamma(X, \Omega^*_X/k)} \longrightarrow \text{Grass}_{F^pH^*(X, k), H^*(X, k)}. \tag{2.8}
\]
induced by the algebraic De Rham cohomology functor is an isomorphism: for a proof see [7] Theorem 10.6.

---

9 Here, by a slight abuse of notation, the symbol \( fm^p \) is denoting both the \( L_\infty \)-map and the induced morphism of deformation functors.
10 Notice that the case \( p = 0 \) is trivial.
11 Again, the symbol \( h \) is denoting both the \( L_\infty \)-map and the induced morphism of deformation functors.
Definition 2.14. For all $p > 0$ define the algebraic $p^{th}$ Fiorenza-Manetti local period map to be the morphism

$$\text{FM}^p : \text{Def}_{KS} \xrightarrow{\text{FM}^p} \text{Def}_{\hat{C}X^p}$$

given by the composition of maps (2.7) and (2.6).

Definition 2.15. For all $p > 0$ define the geometric $p^{th}$ Fiorenza-Manetti local period map to be the morphism

$$\mathcal{P}^p : \text{Def}_X \xrightarrow{\mathcal{P}^p} \text{Grass}_{F^pH^*(X),H^*(X)} \xrightarrow{\mathcal{P}^p} F^pH^*(X,\mathcal{O}_A)$$

for all $A \in \text{Art}_k$.

Now we are finally ready to lift Griffiths period map to a morphism of deformation functors.

Theorem 2.16. (Fiorenza-Manetti) There is a natural isomorphism between maps $\text{FM}^p$ and $\mathcal{P}^p$, meaning that the diagram

$$\begin{array}{ccc}
\text{Def}_{KS} & \xrightarrow{\text{FM}^p} & \text{Def}_{\hat{C}X^p} \\
\downarrow{\mathcal{P}^p} & & \downarrow{\mathcal{P}^p} \\
\text{Def}_X & \xrightarrow{\mathcal{P}^p} & \text{Grass}_{F^pH^*(X),H^*(X)}
\end{array}$$

commutes. Moreover the tangent morphism to the functor $\mathcal{P}^p$ is the same as the differential (1.7) of Griffiths period map.

Proof. See [7] Theorem 12.3 and Corollary 12.5.

2.4 Flag Functors and the Fiorenza-Manetti Period Map

Both $\text{FM}^p$ and $\mathcal{P}^p$ depend on a filtration parameter: we would like to get rid of it, in order to define universal versions of the algebraic and geometric Fiorenza-Manetti period map.

Observe that the target functor of any universal version of the geometric Fiorenza-Manetti period map should not be simply the product of the deformation functors $\text{Grass}_{F^pH^*(X,k),H^*(X,k)}$, because the only deformations of the sequence $(F^pH^*(X,k))_p$ of subcomplexes of $H^*(X,k)$ which may belong to its image are those preserving the property that $F^\bullet$ is a filtration.

For this reason, let $(V,F^\bullet)$ be a filtered complex and define the flag functor associated to $(V,F^\bullet)$ to be

$$\text{Flag}^\bullet_{V} : \text{Art}_k \xrightarrow{\text{Flag}^\bullet_{V}} \text{Set}$$

$$A \mapsto \left\{(U,\mathcal{G}^p)_p \text{ s.t. } (U,\mathcal{G}^p) \in \text{Grass}_{F^p,V,V}(A), \mathcal{G}^pU \hookrightarrow \mathcal{G}^{p-1}U \right\}.$$ 

In particular, consider the functors $\text{Flag}^\bullet_{RF}(X,\Omega^*_X/k)$ and $\text{Flag}^\bullet_{RF}(X,\Omega^*_X/k)$: the same arguments used to deal with map (2.8) imply that the morphism

$$H^* : \text{Flag}^\bullet_{RF}(X,\Omega^*_X/k) \xrightarrow{H^*} \text{Flag}^\bullet_{RF}(X,\Omega^*_X/k)$$

is a natural isomorphism.
is well-defined and an isomorphism.

Let \( \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \) be the space of non-negatively graded endomorphisms of the complex \( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \) and consider the abelian dgla whose support complex is

\[
\text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \quad \text{and consider the abelian dgla whose support complex is}
\]

\[
\text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \quad \text{and consider the abelian dgla whose support complex is}
\]

Moreover denote by \( \text{End}^{\geq 0} \left( \Omega^*_X/k \right) \) the sheaf of non-negatively graded endomorphisms of the algebraic De Rham complex \( \Omega^*_X/k \).

**Remark 2.17.** The formality argument of Remark (2.13) is uniform in \( p \): in particular this means that there is a filtered quasi-isomorphism between \( \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right), F^* \right) \) and \( (H^*, (X, k), F^*) \), which in turn provides us with a weak equivalence between the dgla’s \( \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \) and \( H^* \left( \text{End}^{\geq 0} \left( \Omega^*_X/k \right) \right) \); for a more detailed explanation see [10] Section 5 and Section 6, in whose language these considerations are rephrased by asserting that

\[
\left( \text{End} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right), \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \right)
\]

is a formal pair of dgla’s.

**Proposition 2.18.** There is an isomorphism of functors

\[
\text{Flag}^{F^*}_{H^* \left( X, k \right)} \simeq \text{Def}^{\text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)}_{\text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)} \quad \text{[−1]}
\]

In particular \( \text{Flag}^{F^*}_{H^* \left( X, k \right)} \) is a deformation functor.

**Proof.** See [10] Section 5 and Section 6. \( \square \)

**Remark 2.19.** Let \( (V, F^*) \) be a filtered complex, \( \text{End}^{\geq 0} (V) \) the dgla of filtration-preserving endomorphisms of \( (V, F^*) \) and \( \chi : \text{End}^{\geq 0} (V) \rightarrow \text{End}^* (V) \) its natural inclusion in the endomorphism dgla of the complex \( V \). In general, by the constructions in Section 2.2 and especially by Theorem 2.6 we can see that the flag functor \( \text{Flag}^{F^*}_{V} \) is a deformation functor governed by the homotopy cokernel of \( \chi \); we will be more precise about this in Section 3.2.

Now we are ready to define some universal version of the Fiorenza-Manetti morphism.

**Definition 2.20.** Define the universal geometric Fiorenza-Manetti period map to be the natural transformation

\[
\mathcal{P} : \quad \text{Def}_X \quad \longrightarrow \text{Flag}^{F^*}_{H^* \left( X, k \right)}
\]

\[
\forall A \in \text{Art}_k \quad \left( \mathcal{O}_A \xrightarrow{\xi} \mathcal{O}_X \right) \quad \longrightarrow \quad \mathcal{P}_p \left( \left( \mathcal{O}_A \xrightarrow{\xi} \mathcal{O}_X \right) \right).
\]

Notice that Definition 2.15 ensures that \( \mathcal{P} \) is a well-defined morphism of functors.

Map \( \mathcal{P} \) is a good universal version of the geometric Fiorenza-Manetti period map; we would like
to complete the picture with a natural universal version of the algebraic Fiorenza-Manetti map, that is we would like to construct a morphism of differential graded Lie algebras

\[
FM : KS_X \rightarrow \operatorname{End}^* \left( \operatorname{Def} X \right)
\]

\[
\rightarrow \operatorname{End}^{\geq 0} \left( \operatorname{Def} X \right) [-1]
\]

such that the diagram

\[
\begin{array}{ccc}
\text{Def}_{KS_X} & \xrightarrow{FM} & \text{Def} \\
\downarrow i & & \downarrow 1 \\
\text{Def}_X & \xrightarrow{j} & \text{Flag}_{\mathcal{H}^*(X,k)}
\end{array}
\]

commutes: we will construct it in Section 3.2.

3 The Period Map as a Morphism of $\infty$-Groupoids

Theorem 2.16 attests two very interesting facts: the first one is that Definition 2.14 and Definition 2.15 are naturally equivalent (and this enables us to simply talk about the Fiorenza-Manetti local period map, dropping any further adjective) and the second one is that map (2.9) really extends the period mapping (1.6) to a morphism of deformation theories, as the tangent maps are the same. In this perspective, the period map is seen to play a remarkable unifying role in Deformation Theory and Hodge Theory: as a matter of fact a number of highly non-trivial classical results such as Kodaira Principle and Bogomolov-Tian-Todorov Theorem are recovered as corollaries of Theorem 2.16 (see [7], [9], [10] and [20] for more details).

Anyway the contemporary viewpoint on Deformation Theory claims that Schlessinger’s deformation functors are not the most suitable tools in order to study general local moduli problems, as they are often unable to capture most of the hidden geometry of such problems. As a matter of fact Schlessinger’s functors do not generally take into account automorphisms and higher autoequivalences of the objects they classify and in most cases they do not give a proper description of obstructions, either. Moreover the correspondence between differential graded Lie algebras and deformation functors in the context of classical Deformation Theory is not fully satisfying\footnote{Notice that an instance of such a drawback has already appeared in Section 2.2, since the mapping cone (2.4) is endowed with a non-trivial $L_\infty$-structure.}, but the most important drawback of Schlessinger’s functors for the sake of this paper is that in general they are not formal neighbourhoods of any global moduli space; this is precisely the case of the functor $\text{Def}_X$ defined in Section 2.1: there does not exist any (classical) moduli space of proper smooth schemes of dimension $d > 1$, thus for a general choice of the scheme $X$ the functor $\text{Def}_X$ cannot be describing infinitesimally any algebraic space.

3.1 Quick Review of Derived Deformation Theory

The critical aspects we have briefly listed above mark some of the reasons which have been leading to the development of Derived Deformation Theory: the rough idea behind this subject is
that Deformation Theory is not really a “categorical” subject, but rather an “\((\infty, 1)\)-categorical” one, meaning that its constructions and invariants should be homotopical (or derived) in nature. In particular the basic tools of Derived Deformation Theory should be homotopy analogues of Schlessinger’s functors, i.e. functors defined over (some subcategory of) \(\mathcal{dgArt}_k\) – not just \(\mathcal{Art}_k\) – satisfying homotopical versions of Schlessinger’s axioms and preserving the \(\infty\)-structure – actually coming from a model structure – with which the category \(\mathcal{dgArt}_k\) is endowed. Foundational work on Derived Deformation Theory includes \([11, 16, 21, 25, 27, 28, 31, 39]\), while a gentle introduction to the subject can be found in \([5]\): here we quickly review some of the main concepts just to fix notations.

There are several different ways to enhance a classical deformation functor to a derived one, giving rise to various consistent derived deformation theories; in \([31]\) Pridham proved that all these variants are homotopy equivalent\(^{13}\) thus in this paper by derived deformation functor we will always mean a \(\text{Hinich derived deformation functor}\)^{14}\# i.e. a functor

\[ \mathbf{F} : \mathcal{dgArt}_k^{\leq 0} \longrightarrow \mathcal{SSet} \]

satisfying weaker versions of Schlessinger’s axioms for classical deformation problems: for a precise definition see \([5]\) or the original paper \([16]\), but essentially \(\mathbf{F}\) is required to be \(\text{homotopic} – i.e.\) to map quasi-isomorphisms in \(\mathcal{dgArt}_k^{\leq 0}\) to weak equivalences in \(\mathcal{SSet}\) – and \(\text{homotopy-homogeneous} – i.e.\) such that for all surjections \(A \rightarrow B\) and all maps \(C \rightarrow B\) in \(\mathcal{dgArt}_k^{\leq 0}\) the natural map

\[ \mathbf{F} (A \times_B C) \longrightarrow \mathbf{F} (A) \times_{\mathbf{F}(B)} \mathbf{F} (C) \]

is a weak equivalence. In case \(\mathbf{F}\) is only \(\text{homotopy-surjecting} – i.e.\) for all tiny acyclic extension \(A \rightarrow B\) in \(\mathcal{dgArt}_k^{\leq 0}\) the induced map \(\pi_0 (\mathbf{F} (A)) \rightarrow \pi_0 (\mathbf{F} (B))\) is surjective – we will say that it is a derived \(\text{pre-deformation functor}.

All the geometry of Hinich functors is captured by certain cohomological invariants which generalize tangent spaces and obstruction theories for classical deformation functors: let us briefly recall how to construct them. Given a derived deformation functor \(\mathbf{F} : \mathcal{dgArt}_k^{\leq 0} \longrightarrow \mathcal{SSet}\), consider as in \([31]\) Section 1.6 the functor

\[ \tan \mathbf{F} : \mathcal{dgVect}_k^{\leq 0} \longrightarrow \mathcal{S Vect}_k \]

\[ V \longrightarrow \mathbf{F} (k \oplus V) \]

and recall that the \(j\)-th \(\text{generalized tangent space}\) of \(\mathbf{F}\) is said to be the group

\[ H^j (\mathbf{F}) := \pi_i (\tan \mathbf{F} (k [-n])) \quad \text{where} \quad n - i = j \]

and the definition is well-given because of \([31]\) Corollary 1.46. Generalized tangent spaces extend the underived notions of tangent and obstruction spaces in the sense that if \(\mathbf{F}\) is a derived deformation functor, the group \(H^j (\mathbf{F})\) parametrizes infinitesimal \(j\)-automorphisms associated

---

\(^{13}\) All approaches to Derived Deformation Theory are described by a well-defined \((\infty, 1)\)-category: Pridham proved that all such \((\infty, 1)\)-categories are equivalent; for more details see \([31]\).

\(^{14}\) In the literature people also refer to such functors as \(\text{formal moduli problems}\) or \(\text{formal stacks}\).

\(^{15}\) The symbol \(\times^h\) denotes the homotopy fibre product in \(\mathcal{SSet}\).
to it; in particular $H^0(F)$ encodes first-order derived deformations and $H^1(F)$ encodes second-order derived deformations, i.e. all obstructions (see [31] Section 1.6).

Now denote by $\Omega^*_{\text{DR}}(\Delta^*)$ the simplicial differential graded commutative algebra of polynomial differential forms, given in simplicial level $n$ by

$$\Omega^*_{\text{DR}}(\Delta^n) := k[x_0, x_1, \ldots, x_n, dx_0, dx_1, \ldots, dx_n]$$

where $x_0, x_1, \ldots, x_n$ live in cochain degree 0 and $dx_0, dx_1, \ldots, dx_n$ in cochain degree 1; more generally, given a simplicial set $S$, the symbol $\Omega^p_{\text{DR}}(S)$ will stand for the simplicial differential graded commutative algebra of polynomial differential forms on $S$, which is defined in dg level $p$ by

$$\Omega^p_{\text{DR}}(S) := \text{Hom}_{s\text{Set}}(S, \Omega^p_{\text{DR}}(\Delta^*))$$

Theorem 3.1. (Hinich, Pridham) The functor

$$\mathbb{R}\text{Def}_g : \mathfrak{dgArt}_{\leq 0}^k \longrightarrow s\text{Set}$$

$$A \longmapsto \mathbb{R}\text{MC}_{g \otimes \Omega^*_{\text{DR}}(\Delta^*)}(A)$$

where $\mathbb{R}\text{MC}_{g \otimes \Omega^*_{\text{DR}}(\Delta^*)}(A)$ is the simplicial set determined in level $n$ by the set

$$\mathbb{R}\text{MC}_{g \otimes \Omega^*_{\text{DR}}(\Delta^*)}(A) := \left\{ x \in \left( g \otimes \Omega^*_{\text{DR}}(\Delta^n) \otimes \mathfrak{m}_A \right)^1 \text{ s.t. } d(x) + \frac{1}{2}[x, x] = 0 \right\}.$$

The Hitchin nerve is seldom handy enough to make concrete computations. For this reason, recall that the Hitchin nerve of a dgla $g$ is defined to be the derived deformation functor

$$\mathbb{R}\text{Def}_g : \mathfrak{dgLie}_k \longrightarrow \text{Def}^\text{Hin}_k$$

$$g \longmapsto \mathbb{R}\text{Def}_g$$

is an equivalence of $(\infty, 1)$-categories, thus it induces an equivalence on the homotopy categories $\text{Ho}(\mathfrak{dgLie}_k) \simeq \text{Ho}(\text{Def}^\text{Hin}_k)$.

Proof. See [31] Corollary 4.56. □

Despite its great theoretical properties, the Hitchin nerve is seldom handy enough to make concrete computations. For this reason, recall that the (derived) Deligne groupoid associated to a differential graded Lie algebra $g$ is defined to be the formal groupoid

$$\text{Del}_g : \mathfrak{dgArt}_{\leq 0}^k \longrightarrow \text{Grpd}$$

$$A \longmapsto \left[ \text{MC}_g(A)/\text{G}_g(A) \right]$$

where

$$\text{MC}_g : \mathfrak{dgArt}_{\leq 0}^k \longrightarrow \text{Set}$$

$$A \longmapsto \left\{ x \in \left( g \otimes \mathfrak{m}_A \right)^1 \text{ s.t. } d(x) + \frac{1}{2}[x, x] = 0 \right\}$$

(3.1)

$$\text{G}_g : \mathfrak{dgArt}_{\leq 0}^k \longrightarrow \text{Grp}$$

$$A \longmapsto \exp \left( \left( g \otimes \mathfrak{m}_A \right)^0 \right)$$

(3.2)
and let
\[ \text{BDel}_g : \mathfrak{dgArt}_k^{<0} \rightarrow \mathcal{S} \text{Set} \]
denote its nerve.

**Remark 3.2.** Notice that Formula (3.1) and Formula (3.2) are just straightforward generalizations of the notions of Maurer-Cartan and gauge functor in underived Deformation Theory; these objects are used to define extended deformation functors in the sense of Manetti (see [28] or [29]). In [31] Pridham also proved that there is an equivalence of \((\infty, 1)\)-categories between \(\mathcal{D} \text{ef}_k^{\text{Man}}\) and \(\mathcal{D} \text{ef}_k^{\text{Hin}}\).

**Warning 3.3.** The nerve of the Deligne groupoid associated to a differential graded Lie algebra is a derived pre-deformation functor but not a derived deformation functor: as a matter of fact it is not homotopic in general. Moreover, although it might be a bit confusing, we will tend to refer to both \(\text{Del}_g\) and \(\text{BDel}_g\) as the Deligne groupoid associated to the differential graded Lie algebra \(g\).

Fix \(g \in \mathfrak{dgLie}_k\): we can define the functor
\[ \text{BDel}_g : \mathfrak{dgArt}_k^{<0} \rightarrow \mathcal{S} \text{Set} \]
\[ A \mapsto \text{diag} \left( \text{BDel}_g(A) \right. \]
\[ \left. \quad \text{BDel}_g \otimes \Omega(-\delta) \quad \text{BDel}_g \otimes \Omega(-\delta^2) \quad \cdots \right) \]
which is sometimes called the simplicial Deligne groupoid of \(g\).

**Theorem 3.4.** (Pridham) Let \(g\) be a differential graded Lie algebra concentrated in non-negative degrees; we have that

- the functor \(\text{BDel}_g\) is a derived deformation functor;
- the functor \(\text{BDel}_g\) is the universal derived deformation functor under \(\text{BDel}_g\);
- the functors \(\text{BDel}_g\) and \(\text{RDef}_g\) are weakly equivalent.

**Proof.** See [35] Section 3 for the proof of the first two claims, while the last statement is proven in [17] Section 3.

As a consequence of Theorem 3.4, we have that all geometric and homotopy-theoretic information concerning the Hinich nerve of a differential graded Lie algebra \(g\) are completely determined by its associated Deligne groupoid, which is a much more down-to-earth object as it is essentially a formal groupoid. Unfortunately, as \(\text{BDel}_g\) does not map quasi-isomorphisms to weak equivalences, the description of higher tangent spaces we gave above in this section is no longer valid; nonetheless Pridham found a coherent way to define good cohomology theories for derived pre-deformation functors. As a matter of fact fix a derived pre-deformation functor \(F : \mathfrak{dgArt}_k^{<0} \rightarrow \mathcal{S} \text{Set}\) and define as in [33] Section 3.3 the \(j\)-th generalized tangent space of \(F\) to be
\[ H^j(F) := \begin{cases} \pi_{-j}(F(\kappa^{[i]})) & \text{if } j \leq 0 \\ \pi_{0}(\text{tan}(F(\kappa^{[i]})))/\pi_{0}(\text{tan}(F(\text{cone}(\kappa^{[i]})))) & \text{otherwise} \end{cases} \]
which is seen to be consistent with the definition given above in this section in case \(F\) is also homotopic (see [33] Lemma 3.15).
Now fix $\mathfrak{g}$ to be a differential graded Lie algebra over $k$ concentrated in non-negative degrees and apply the above definitions to its Deligne groupoid. We have that

$$H^{-1}(BDel_{\mathfrak{g}}) = \pi_1 \left( BDel_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right) = \pi_1 \left( \left[ \text{RMC}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right] / \text{RG}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right) =$$

$$\pi_1 \left( \left[ \text{MC}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right] / \text{Gg} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right) \simeq \text{Stab}_{\text{Gg} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right)}(0)$$

but

$$\text{MC}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) = \left\{ x \otimes \varepsilon \in \mathfrak{g}^1 \otimes \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \text{ s.t. } d(x) = 0 \right\} = Z^1(\mathfrak{g}) \varepsilon$$

$$\text{Gg} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) = \exp \left( \mathfrak{g}^0 \otimes (\varepsilon) \right) \simeq \text{Id} + \mathfrak{g}^0 \varepsilon$$

and notice that the gauge action just reduces to

$$\begin{align*}
\text{Id} + \mathfrak{g}^0 \varepsilon \times Z^1(\mathfrak{g}) \varepsilon & \xrightarrow{\ast} Z^1(\mathfrak{g}) \varepsilon \\
(\text{Id} + a \varepsilon, x \varepsilon) & \mapsto (x + d(a)) \varepsilon
\end{align*} \quad (3.3)$$

therefore we get

$$\text{Stab}_{\text{Gg} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right)}(0) = \{ (\text{Id} + a \varepsilon) \in \text{Id} + \mathfrak{g}^0 \varepsilon \text{ s.t. } (\text{Id} + a \varepsilon) \ast 0 = 0 \} \simeq$$

$$\{ a \in \mathfrak{g}^0 \text{ s.t. } d(a) = 0 \} = Z^0(\mathfrak{g}) \simeq H^0(\mathfrak{g})$$

where the last identification follows from the fact that $\mathfrak{g}$ lives in non-negative degrees.

Similarly we see that

$$H^0(BDel_{\mathfrak{g}}) = \pi_0 \left( BDel_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right) =$$

$$\pi_0 \left( \left[ \text{RMC}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right] / \text{RG}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \right) = \text{MC}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) / \text{Gg} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right) \simeq Z^1(\mathfrak{g}) \varepsilon / \text{Id} + \mathfrak{g}^0 \varepsilon$$

thus the quotient of $\text{MC}_{\mathfrak{g}} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right)$ under the gauge action $\left(3.3\right)$ is given by $H^1(\mathfrak{g})$.

At last observe that $\text{cone}(k \oplus k[j - 1])$ is a path object for $k \oplus k[j]$, so the same kind of computation gives us that for all $j \geq 0$

$$H^j(BDel_{\mathfrak{g}}) = \pi_0(\tan(BDel_{\mathfrak{g}}(k[i]))) / \pi_0(\tan(BDel_{\mathfrak{g}}(\text{cone}(k[i])))) =$$

$$Z^0(\mathfrak{g} \otimes (k \oplus k[j])) / Z^0(\mathfrak{g} \otimes (k \oplus \text{cone}(k \oplus k[j - 1]))) = Z^{j+1}(\mathfrak{g}) / \mathfrak{g}^j$$

and again $\mathfrak{g}^j$ acts on $Z^{j+1}(\mathfrak{g})$ by differentials, so the quotient is $H^{j+1}(\mathfrak{g})$.

**Remark 3.5.** Let $\mathfrak{g}$ be any differential graded Lie algebra; by combining Theorem 3.4 and the above observations we have that

$$H^i(\mathbb{R}Def_{\mathfrak{g}}) \simeq H^i(BDel_{\mathfrak{g}}) = Z^{i+1} / \mathfrak{g}^i \simeq H^{i+1}(\mathfrak{g}) \quad \forall i \geq 0.$$

$$H^{-1}(\mathbb{R}Def_{\mathfrak{g}}) \simeq H^{-1}(BDel_{\mathfrak{g}}) \simeq \text{Stab}_{\text{Gg} \left( \frac{\mathfrak{k}[\varepsilon]}{(\varepsilon^2)} \right)}(0) \simeq H^0(\mathfrak{g})$$
3.2 The Algebraic Fiorenza-Manetti-Martinengo Period Map

The above considerations give many motivations to try to lift the period map from a morphism of classical deformation functors to the context of Derived Deformation Theory; Fiorenza and Martinengo started to address such a question, tackling it from an entirely algebraic viewpoint.

Let $X$ be still a proper smooth scheme of dimension $d$ over a field $k$ of characteristic $0$ and, again, consider the Cartan homotopy defined by the contraction of differential forms with vector fields

$$i : \mathcal{F}_X \rightarrow \text{End}^* \left( \Omega^*_X/k \right) [-1]$$

and the Lie derivative

$$l : \mathcal{F}_X \rightarrow \text{End}^* \left( \Omega^*_X/k \right)$$

which corresponds to the differential of $i$ in the Hom complex. Now the linear map of dgla’s

$$\tilde{i} : KS_X \simeq \mathbb{R} \Gamma \left( X, \mathcal{F}_X \right) \rightarrow \text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) [-1]$$

defined as the composition of $\mathbb{R} \Gamma \left( X, i \right)$ with the map

$$\mathbb{R} \Gamma \left( X, \text{End}^* \left( \Omega^*_X/k \right) \right) \rightarrow \text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)$$

induced by the action of derived global sections of the endomorphism sheaf of $\Omega^*_X/k$ on derived global sections of $X$ is still a Cartan homotopy: denote by $\tilde{l}$ the morphism of dgla’s induced by it, which is essentially the derived globalization of the Lie derivative. Recall that $\text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)$ is the differential graded Lie subalgebra of $\text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)$ consisting of non-negatively graded endomorphisms of the (derived global sections of the) algebraic De Rham complex, which can be thought of as filtration-preserving endomorphisms, and notice that the image of $\tilde{l}$ is contained in $\text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)$; we have the diagram of dgla’s

$$KS_X \simeq \mathbb{R} \Gamma \left( X, \mathcal{F}_X \right) \xrightarrow{\tilde{i}} \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \xrightarrow{\text{incl}} \text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)$$

where $0$ stands for the zero map. Since $\tilde{i}$ is a Cartan homotopy, by Remark 2.8 $e^{\tilde{i}}$ gives an homotopy between $\tilde{l}$ and the zero map, thus there is an induced morphism of dgla’s to the homotopy fibre

$$KS_X \xrightarrow{(\tilde{l}, e^{\tilde{i}})} \text{holim} \left( \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \xrightarrow{\text{incl}} \text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \right)$$

as observed in [10] Section 6; moreover, in the same paper, Fiorenza and Martinengo showed that

$$\text{holim} \left( \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \xrightarrow{\text{incl}} \text{End} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) \right) \simeq \text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) / \text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right) [-1]$$

and notice that by Remark 2.17 we have that

$$\frac{\text{End}^* \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)}{\text{End}^{\geq 0} \left( \mathbb{R} \Gamma \left( X, \Omega^*_X/k \right) \right)} [-1] \simeq \frac{H^* \left( \text{End}^* \left( \Omega^*_X/k \right) \right)}{H^* \left( \text{End}^{\geq 0} \left( \Omega^*_X/k \right) \right)} [-1].$$
**Definition 3.6.** Define the *universal algebraic Fiorenza-Manetti local period map* to be the morphism of deformation functors

\[
\text{FM} : \text{Def}_{K_S X} \longrightarrow \text{Def}\left(\text{End}^*\left(\mathcal{O}_{X/k}\right)\right)_{[-1]} \text{induced by map (3.4).}
\]

**Definition 3.7.** Define the *(universal) algebraic Fiorenza-Manetti-Martinengo local period map* to be the morphism of derived deformation functors

\[
\text{FMM} : \mathcal{R}\text{Def}_{K_S X} \longrightarrow \mathcal{R}\text{Def}\left(\text{End}^*\left(\mathcal{O}_{X/k}\right)\right)_{[-1]} \text{induced by map (3.4).}
\]

In Section 2.4 we described a universal version of the geometric period map (see Definition 2.20), but we did not construct its Lie-theoretic counterpart: Fiorenza and Martinengo showed that this is precisely given by map (3.4).

**Theorem 3.8.** (Fiorenza-Martinengo) The diagram

\[
\begin{array}{ccc}
\mathcal{R}\text{Def}_{K_S X} & \xrightarrow{\text{FMM}} & \mathcal{R}\text{Def}\left(\text{End}^*\left(\mathcal{O}_{X/k}\right)\right)_{[-1]} \\
\pi^0\pi_{\geq 0} & \downarrow & \pi^0\pi_{\geq 0} \\
\text{Def}_{K_S X} & \xrightarrow{\text{FM}} & \text{Def}\left(\text{End}^*\left(\mathcal{O}_{X/k}\right)\right)_{[-1]} \\
\downarrow & & \downarrow \\
\text{Def}_X & \xrightarrow{\phi} & \text{Flag}_{H^*}(X,k)
\end{array}
\]

is well-defined and commutes.

**Proof.** See [10] Section 6. □

### 3.3 Affine DG\(_{\geq 0}\)-Categories and the Dold-Kan Correspondence

Theorem 3.8 says that the morphism FMM is the correct derived enhancement of the universal Fiorenza-Manetti local period map; however the geometric interpretation of such a result is somehow indirect, thus it would be worth to find an equivalent morphism of derived deformation functors having more evident geometric meaning. Of course the key step in order to do this consists of finding the right domain and codomain for such a morphism, i.e. defining two derived deformation functors \(\mathcal{R}\text{Def}_X\) and \(\text{hoFlag}_{\mathcal{R}\Gamma}\left(\mathcal{O}_{X/k}\right)\) such that

- \(\mathcal{R}\text{Def}_X\) is weakly equivalent to \(\mathcal{R}\text{Def}_{K_S X}\) and similarly \(\text{hoFlag}_{\mathcal{R}\Gamma}\left(\mathcal{O}_{X/k}\right)\) is weakly equivalent to \(\mathcal{R}\text{Def}\left(\text{End}^*\left(\mathcal{O}_{X/k}\right)\right)_{[-1]}\).
• $R \text{Def}_X$ and $\text{hoFlag}_R^* (X, \Omega^*_X/k)$ are derived enhancements of $\text{Def}_X$ and $\text{Flag}_R^* (X, k)$, respectively.

In order to construct such functors we need some homotopy-theoretic background.

**Warning 3.9.** In this section we will deal with non-negatively graded differential graded chain structures rather than non-positively graded cochain ones, though the pictures they provide are largely equivalent; the reason for this lies in the fact that – at least in the framework of this paper – the codomain of a derived deformation functor is the simplicial model category of simplicial sets, which is more directly related to chain structures than cochain ones.

First of all, recall that the **normalization** of a simplicial $k$-vector space $(V_\bullet, \partial, \sigma_j)$ is defined to be the non-negatively graded chain complex of $k$-vector spaces $(NV, \delta)$ where

$$(NV)_n := \bigcap_i \ker (\partial_i : V_n \to V_{n-1})$$

and $\delta_n := (-1)^n \partial_n$. Moreover, given a map $f : V_\bullet \to W_\bullet$ of simplicial $k$-vector spaces, we can define the chain map

$$N(f) : NV_\bullet \longrightarrow NW_\bullet$$

identified by the relation $N(f)_n := f_n|_{NV_n}$; notice that this construction gives us a well-defined morphism of chain complexes. In the end, there is a normalization functor

$$N : s\text{Vect}_k \longrightarrow \mathcal{C}_{\geq 0}(\text{Vect}_k).$$

at our disposal.

On the other hand, let $V$ be a chain complex of $k$-vector spaces and recall that its **denormalization** is defined to be the simplicial vector space $((KV)_\bullet, \partial, \sigma_j)$ given in level $n$ by the vector space

$$(KV)_n := \prod_{\eta \in \text{Hom}_\Delta ([p], [n])} V_p [\eta],$$

where $\eta$ is surjective and $V_p [\eta] \simeq V_p$.

**Remark 3.10.** Notice that

$$(KV)_n \simeq V_0 \oplus V_1^{\oplus n} \oplus V_2^{\oplus (\binom{n}{2})} \oplus \cdots \oplus V_k^{\oplus (\binom{n}{k})} \oplus \cdots \oplus V_n^{\oplus (\binom{n}{n})}.$$

In order to complete the definition of the denormalization of $V$ we need to define face and degeneracy maps: we will describe a combinatorial procedure to determine all of them. For all morphisms $\alpha : [m] \to [n]$ in $\Delta$, we want to define a linear map $K(\alpha) : (KV)_n \to (KV)_m$; this will be done by describing all restrictions $K(\alpha, \eta) : V_p [\eta] \to (KV)_m$, for any surjective non-decreasing map $\eta \in \text{Hom}_\Delta ([p], [n])$.

For all such $\eta$, take the composite $\eta \circ \alpha$ and consider its epi-monic factorization\[16] $\epsilon \circ \eta'$, as in the diagram

$$\begin{array}{ccc}
[m] & \xrightarrow{\alpha} & [n] \\
\downarrow \eta' & & \downarrow \eta \\
[q] & \xrightarrow{\epsilon} & [p].
\end{array}$$

\[16\] The existence of such a decomposition is one of the key properties of the category $\Delta$. 

22
• if \( p = q \) (in which case \( \epsilon \) is just the identity map), then set \( K(\alpha, \eta) \) to be the natural identification of \( V_p[\eta] \) with the summand \( V_p[\eta'] \) in \( (KV)_m \);

• if \( p = q + 1 \) and \( \epsilon \) is the unique injective non-decreasing map from \( [p] \) to \( [p+1] \) whose image misses \( p \), then set \( K(\alpha, \eta) \) to be the differential \( d_p: V_p \to V_{p-1} \);

• in all other cases set \( K(\alpha, \eta) \) to be the zero map.

The above constructions determine all the simplicial vector space \( ((KV)_m, \partial_i, \sigma_j) \). As done for normalization, for any chain map \( f: V \to W \) we can define a morphism of simplicial \( k \)-vector spaces

\[
K(f): KV \to KW
\]

by setting

\[
V_0 \times V_1^{\oplus n} \times V_2^{\oplus \binom{n}{2}} \times \cdots \times V_n^{\oplus \binom{n}{n}} \xrightarrow{K(f)_n} W_0 \times W_1^{\oplus n} \times W_2^{\oplus \binom{n}{2}} \times \cdots \times W_n^{\oplus \binom{n}{n}}
\]

\[
\left(v_0, (v_1^i)_i, (v_2^j)_j, \ldots, v_n\right) \mapsto \left(f_0(v_0), (f_1(v_1^i))_i, (f_2(v_2^j))_j, \ldots, f_n(v_n)\right).
\]

Again, there is a denormalization functor

\[
K: Ch_{\geq 0}(\mathcal{V}ect_k) \to s\mathcal{V}ect_k.
\]

at our disposal.

**Theorem 3.11.** (Dold, Kan) The functors \( N \) and \( K \) form an equivalence of categories between \( s\mathcal{V}ect_k \) and \( Ch_{\geq 0}(\mathcal{V}ect_k) \).

**Proof.** See [13] Corollary 2.3 or [44] Theorem 8.4.1. \( \square \)

The Dold-Kan correspondence described in Theorem 3.11 is known to induce a number of very interesting \( \infty \)-equivalences; for instance the Eilenberg-Zilber shuffle product and the Alexander-Whitney map, which we will discuss in more details later in this Section, allow us to extend normalization and denormalization to a pair of functors

\[
N: s\mathcal{A}lg_k \rightleftarrows dg_{\geq 0}\mathcal{A}lg_k: K
\]

which is seen to be a Quillen equivalence. Moreover recall that

• a \( dg_{\geq 0} \)-category over \( k \) is a category enriched in \( Ch_{\geq 0}(\mathcal{V}ect_k) \);

• a \( k \)-simplicial category is a category enriched in \( s\mathcal{V}ect_k \);

• a simplicial category is a category enriched in \( s\mathcal{S}et \);

• a simplicial groupoid is a simplicial object in \( \mathcal{G}rp \): equivalently a simplicial groupoid is a simplicial category in which all 1-morphisms are invertible.

All the above structures form well-understood model categories; furthermore it is well-known in the homotopy-theoretic folklore that Theorem 3.11 induces a Quillen equivalence

\[
N: s\mathcal{C}at_k \rightleftarrows dg_{\geq 0}\mathcal{C}at_k: K
\]

\[17\] There is some abuse of notation in this statement.
Tabuada also constructed an explicit Quillen equivalence between $\mathcal{dg}_{\geq 0}\mathbf{Cat}_k$ and $\mathbf{sCat}$ (see [36], where an explicit proof of formula (3.6) can be found, as well).

We will use slightly more general versions of the Dold-Kan correspondence provided by Theorem 3.11 and its corollaries, so we need to develop a few technical tools.

Define $\mathbf{Aff}_k$ to be the category whose objects are $k$-vector spaces and whose morphisms are affine maps between $k$-vector spaces, i.e.

$$\text{Hom}_{\mathbf{Aff}_k}(V, W) := \{ v \mapsto f(v) + b \text{ s.t. } f \text{ linear}, b \in W \} \simeq \text{Hom}_{\mathbf{Vect}_k}(V, W) \times W.$$ 

$\mathbf{Aff}_k$ can be thought of as the category of affine spaces over $k$ and affine maps. Given $V, W \in \mathbf{Aff}_k$, define their tensor product to be

$$V \hat{\otimes} W := V \oplus W \oplus (V \otimes W) \quad (3.7)$$

where the tensor product $V \otimes W$ in the right-hand side of formula (3.7) is just the tensor product as vector spaces; in a similar way, given two affine maps

$$\phi \in \text{Hom}_{\mathbf{Aff}_k}(V, W) \quad \text{where} \quad \phi(v) := f(v) + b$$

$$\psi \in \text{Hom}_{\mathbf{Aff}_k}(U, Z) \quad \text{where} \quad \psi(u) := g(u) + d$$

the tensor product map is given by

$$\phi \otimes \psi : V \oplus W \oplus (V \otimes W) \longrightarrow U \oplus Z \oplus (U \otimes Z) \quad (u, v, x \otimes y) \quad \mapsto (u + b, v + d, f(x) \otimes g(y)). \quad (3.8)$$

Formula (3.7) and formula (3.8) determine a monoidal structure on $\mathbf{Aff}_k$: we will be more precise about this a little bit further in this Section, when dealing with $\mathcal{dg}_{\geq 0}$-affine spaces.

**Definition 3.12.** Define a (chain) differential graded affine space over $k$ in non-negative degrees ($dg_{\geq 0}$-affine space for short) to be a pair $(A_0, V)$ where

$$V : \quad V_0 \epsilon^d V_1 \epsilon^d V_2 \epsilon^d \cdots \epsilon^d V_n \epsilon^d$$

is a non-negatively graded chain complex of $k$-vector spaces and $A_0$ is an affine space over $k$ whose difference vector space is $V_0$.

If $(A_0, V)$ and $(B_0, W)$ are $dg_{\geq 0}$-affine spaces over $k$, a morphism $\phi : (A_0, V) \rightarrow (B_0, W)$ will be a chain map which is affine in degree 0 and linear in higher degrees: more formally the set of morphisms between $(A_0, V)$ and $(B_0, W)$ is defined to be

$$\text{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)}((A_0, V), (B_0, W)) := \left\{ v \mapsto f(v) + b \text{ s.t. } f \in \text{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Vect}_k)}(V, W), b \in W_0 \right\}.$$ 

In the end we have a well-defined category of $\mathcal{dg}_{\geq 0}$-affine spaces over $k$, which we will denote as $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$.

**Remark 3.13.** We have defined the objects of $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$ as pairs where the first term is an affine space and the second term is a chain complex of vector spaces just to make the affine structure explicit; an equivalent and more compact characterization of $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$ is

$$\text{Ob} \left( \mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k) \right) := \text{Ob} \left( \mathbf{Ch}_{\geq 0}(\mathbf{Vect}_k) \right)$$

$$\text{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)}((A_0, V), (B_0, W)) \simeq \text{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Vect}_k)}(V, W) \times W_0.$$ 

In particular $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$ is a $k$-linear category.

24
The category $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$ is both complete and cocomplete: limits and colimits are constructed from those in $\mathbf{Ch}_{\geq 0}(\mathbf{Vect}_k)$. For example if $(A_0, V)$ and $(B_0, W)$ are $\text{dg}_{\geq 0}$-affine spaces their product will be just $(A_0 \times B_0, V \times W)$, where $V \times W$ is the product of $V$ and $W$ in $\mathbf{Ch}_{\geq 0}(\mathbf{Vect}_k)$ and $A_0 \times B_0$ is the affine space over $k$ whose difference vector space is $V_0 \times W_0$. We can also put a tensor structure over $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$: given two $\text{dg}_{\geq 0}$-affine spaces $(A_0, V)$ and $(B_0, W)$, define their tensor product $(A_0, V) \otimes (B_0, W)$ to be the $\text{dg}_{\geq 0}$-affine space determined by the chain complex

$$V \oplus W \oplus (V \otimes W).$$

Similarly, given

$$\phi \in \text{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)}((A_0, V), (B_0, W)) \quad \text{where} \quad \phi(v) := f(v) + b$$

$$\psi \in \text{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)}((C_0, U), (D_0, Z)) \quad \text{where} \quad \psi(u) := g(u) + d$$

the tensor product map is given by

$$\phi \otimes \psi : \quad V \oplus W \oplus (V \otimes W) \rightarrow U \oplus Z \oplus (U \otimes Z)$$

$$(u, v, x \otimes y) \rightarrow (u + b, v + d, f(x) \otimes g(y)).$$

Formula (3.9) and formula (3.10) determine a monoidal structure on $\mathbf{Ch}_{\geq 0}(\mathbf{Aff}_k)$: in particular the unit is given by the object $(*, 0)$, the associator is induced by the monoidal structure on $\mathbf{Ch}_{\geq 0}(\mathbf{Vect}_k)$ and the unitors are simply given by

$$(V \oplus 0 \oplus (V \otimes 0)) \rightarrow V$$

$$(0 \oplus V \oplus (0 \otimes V)) \rightarrow V.$$

The reader can check that the above definitions verify the pentagon and the triangle identities.

Remark 3.14. Let $(A_0, V)$ and $(B_0, W)$ be $\text{dg}_{\geq 0}$-affine spaces: notice that

$$(A_0, V) \otimes (B_0, W) \simeq A_0 \tilde{\otimes} B_0$$

where formula (3.11) is a canonical identification in $\mathbf{Aff}_k$; an analogous coherence statement holds for morphisms.

**Definition 3.15.** Define a simplicial affine space over $k$ to be just a simplicial object in $\mathbf{Aff}_k$.

Let $s\mathbf{Aff}_k$ be the category of simplicial affine spaces over $k$, i.e.

$$s\mathbf{Aff}_k := \mathbf{Aff}_k^{\Delta^{op}}.$$ 

Remark 3.16. There is a natural linearisation functor

$$L : s\mathbf{Aff}_k \longrightarrow s\mathbf{Vect}_k$$

which just deletes the non-linear part in the face and degeneracy maps defining a simplicial affine space, as well as the non-linear part of morphisms between simplicial affine spaces; in the same fashion there is a forgetful functor

$$U : s\mathbf{Vect}_k \longrightarrow s\mathbf{Aff}_k$$

which just takes (maps of) simplicial vector spaces and looks at them as (maps of) simplicial affine ones.

25
Warning 3.17. The pair of functors $(U, L)$ does not provide an adjunction between $s\mathcal{A}ff_k$ and $s\mathcal{V}ect_k$.

The category $s\mathcal{A}ff_k$ has all small limits and colimits, which are just taken levelwise; moreover define the tensor product in $s\mathcal{A}ff_k$ to be constructed by simply taking the tensor product in $\mathcal{A}ff_k$ in all levels: it is straightforward to check that this equips such a category with a monoidal structure.

Now define the normalization of a $dg_{\geq 0}$-affine space over $k$ to be the functor

$$\tilde{N} : s\mathcal{A}ff_k \longrightarrow Ch_{\geq 0}(\mathcal{A}ff_k)$$

$$A_* \mapsto (A_0, N(L(A_*)))$$

and observe that such a definition is well-given as the $0$-th term of the chain complex $N(L(A_*))$ is precisely the difference vector space of $A_0$; in other words, the normalization of a simplicial vector space does not affect the object in degree $0$, as follows from formula (3.5).

Analogously, define the denormalization of a simplicial affine space over $k$ to be the functor

$$\tilde{K} : Ch_{\geq 0}(\mathcal{A}ff_k) \longrightarrow s\mathcal{A}ff_k$$

$$(A_0, V) \mapsto A_0 \xrightarrow{\phi_0} A_0 \times V_1 \xrightarrow{\phi_0} A_0 \times V_1 \times \cdots$$

where the maps involving $A_0$ and $A_0 \times V_1$ are

$$A_0 \times V_1 \ni (a, v) \mapsto a + d(v) \in A_0$$

$$A_0 \times V_1 \ni (a, v) \mapsto a \in A_0$$

$$A_0 \ni a \mapsto (a, 0) \in A_0 \times V_1$$

and all other faces and degeneracies – which do not involve the affine space $A_0$, but rather only the vector spaces $V_i$ – are defined as done for classical denormalization (see Remark 3.10 and subsequent discussion).

In a similar way, given

$$\phi \in \text{Hom}_{Ch_{\geq 0}(\mathcal{A}ff_k)}((A_0, V), (B_0, W)) \quad \text{where} \quad \phi(v) := f(v) + b$$

the morphism $\tilde{K}(\phi)$ of simplicial affine spaces is defined in level $n$ by the affine map

$$A_0 \times V_1^{\oplus n} \times V_2^{\oplus n} \times \cdots \times V_n^{\oplus n} \longrightarrow B_0 \times W_1^{\oplus n} \times W_2^{\oplus n} \times \cdots \times W_n^{\oplus n}$$

$$\left(\left(a_0, (v_1^0)_j, (v_2^0)_j, \ldots, v_n^0\right)\right) \mapsto \left(\left(f_0(a_0) + b, (f_1(v_1^0))_j, (f_2(v_2^0))_j, \ldots, f_n(v_n^0)\right)\right)$$

We are ready to describe the generalization of Theorem 3.11 we mentioned before.

**Proposition 3.18.** The functors $\tilde{N}$ and $\tilde{K}$ form an equivalence of categories between $s\mathcal{A}ff_k$ and $Ch_{\geq 0}(\mathcal{A}ff_k)$. 

Proof. The arguments used in [44] Theorem 8.4.1 to prove the classical Dold-Kan correspondence given by Theorem 3.11 carry over to this context.

As follows for instance from the discussion in [41] Section 2.3, the normalization functor $\tilde{N}$ can be made into a lax monoidal functor via the Eilenberg-Zilber shuffle map, which is the natural transformation

$$EZ : \tilde{N} (\cdot \otimes \cdot) \longrightarrow \tilde{N} (\cdot) \otimes \tilde{N} (\cdot)$$

determined for all $A$, $B \in s{\mathfrak{Aff}}_k$ by the morphisms

$$EZ_{A, B} : \tilde{N} (A \otimes B) \longrightarrow \tilde{N} (A \otimes B)$$

where the sum runs over all $(p, q)$-shuffles

$$(\mu, \nu) := (\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q)$$

and the corresponding degeneracy maps are

$$s_\mu := s_{\mu_p} \circ \ldots \circ s_{\mu_1} \quad \quad \quad \quad \quad s_\nu := s_{\nu_q} \circ \ldots \circ s_{\nu_1}.$$ 

In the same fashion, again from [41] Section 2.3, the denormalization functor $\tilde{K}$ can also be made into a lax monoidal functor by means of the Alexander-Whitney map. The latter is defined to be the natural transformation

$$AW : \tilde{N} (\cdot \otimes \cdot) \longrightarrow \tilde{N} (\cdot) \otimes \tilde{N} (\cdot)$$

given for all $A$, $B \in s{\mathfrak{Aff}}_k$ by the morphisms

$$AW^n_{A, B} : \tilde{N} (A \otimes B) \longrightarrow \bigoplus_{p+q=n} \left( \tilde{d}^p (a) \otimes \tilde{d}_0^q (b) \right)$$

where the “front face” $\tilde{d}^p$ and the “back face” $\tilde{d}_0^q$ are induced respectively by the injective monotone maps $\tilde{d}^p : [p] \rightarrow [p + q]$ and $\tilde{d}_0^q : [q] \rightarrow [p + q]$. In particular the Alexander-Whitney map makes the normalization functor $\tilde{N}$ into a comonoidal one (again, see [41] Section 2.3, whose considerations adapt to these context). Notice also that by setting $A' := \tilde{N} (A)$ and $B' := \tilde{N} (A)$ in formula (3.12) and using the equivalence provided by Proposition 3.18, we get a version of the Alexander-Whitney transformation

$$AW : \tilde{K} (\cdot \otimes \cdot) \longrightarrow \tilde{K} (\cdot \otimes \cdot)$$

which makes the denormalization $\tilde{K}$ into a lax monoidal functor. Also we have that the composite $AW \circ EZ$ is the same as the identity, while the transformation $EZ \circ AW$ is chain homotopic to the identity: in particular the Dold-Kan equivalence provided by Proposition 3.18 is lax monoidal.

Now we are ready to introduce the notions of affine $d\geq 0$-category and affine simplicial category, which will be crucial technical tools to develop a good derived version of the period map.

---

18 There is some abuse of notation in this formula
Definition 3.19. An affine differential graded category over \( k (\mathcal{C}, \mathcal{C}_\bullet) \) (affine dg\(_{\geq 0}\)-category for short) is a category \( \mathcal{C} \) enriched over \( \mathcal{C}_{\geq 0} (\mathfrak{A}ff_k) \).

We will denote by \( \mathfrak{d}g_{\geq 0} \mathcal{C}at_{\geq 0}^{3ff} \) the \( \infty \)-category of affine dg\(_{\geq 0}\)-categories.

Let \( (\mathcal{C}, \mathcal{C}_\bullet) \) be an affine dg\(_{\geq 0}\)-category and denote by \( H_0 ((\mathcal{C}, \mathcal{C}_\bullet)) \) the (honest) category defined by the relations

\[
\text{Ob} (H_0 ((\mathcal{C}, \mathcal{C}_\bullet))) := \mathcal{C}
\]

\[
\forall X, Y \in \mathcal{C} \quad \text{Hom}_{H_0 ((\mathcal{C}, \mathcal{C}_\bullet))} (X, Y) := H_0 (\mathcal{C}_\bullet (X, Y)).
\]

Definition 3.20. An affine differential graded groupoid over \( k \) (affine dg\(_{\geq 0}\)-groupoid for short) will be an affine dg\(_{\geq 0}\)-category \( (\mathcal{C}, \mathcal{C}_\bullet) \) such that the category \( H_0 ((\mathcal{C}, \mathcal{C}_\bullet)) \) is a groupoid.

We will denote by \( \mathfrak{d}g_{\geq 0} \mathfrak{G}rp_{\geq 0}^{3ff} \) the \( \infty \)-category of affine dg\(_{\geq 0}\)-groupoids.

Remark 3.21. The notion of dg\(_{\geq 0}\)-affine space allows us to define a notion of \( \infty \)-groupoid in the differential graded context: as a matter of fact a more naive notion of dg\(_{\geq 0}\)-groupoid – intended as a dg\(_{\geq 0}\)-category where all morphisms in level 0 are isomorphism – would not really make sense as every dg\(_{\geq 0}\)-category comes with a zero morphism, which is seldom an isomorphism.

Definition 3.22. An affine simplicial category over \( k (\mathcal{C}, \mathcal{C}_\bullet) \) is a category \( \mathcal{C} \) enriched over \( s\mathfrak{A}ff_k \).

We will denote by \( s\mathcal{C}at_{\geq 0}^{3ff} \) the \( \infty \)-category of affine simplicial categories.

Let \( (\mathcal{C}, \mathcal{C}_\bullet) \) be an affine simplicial category and denote by \( \pi_0 ((\mathcal{C}, \mathcal{C}_\bullet)) \) the (honest) category defined by the relations

\[
\text{Ob} (\pi_0 ((\mathcal{C}, \mathcal{C}_\bullet))) := \mathcal{C}
\]

\[
\forall X, Y \in \mathcal{C} \quad \text{Hom}_{\pi_0 ((\mathcal{C}, \mathcal{C}_\bullet))} (X, Y) := \pi_0 (\mathcal{C}_\bullet (X, Y)).
\]

Definition 3.23. An affine simplicial groupoid over \( k \) will be an affine simplicial category \( (\mathcal{C}, \mathcal{C}_\bullet) \) such that the category \( \pi_0 ((\mathcal{C}, \mathcal{C}_\bullet)) \) is a groupoid.

We will denote by \( s\mathfrak{G}rp_{\geq 0}^{3ff} \) the \( \infty \)-category of affine simplicial groupoids.

Remark 3.24. The notion of simplicial affine space allows us to define a notion of \( \infty \)-groupoid in the \( k \)-simplicial context, just like dg\(_{\geq 0}\)-affine spaces give rise to a good notion of differential graded groupoid, as observed in Remark 3.21.

Of course any simplicial affine space has an underlying simplicial set, so an affine simplicial category over \( k \) is in particular a simplicially enriched category: more formally, there is a natural forgetful functors from \( s\mathcal{C}at_{\geq 0}^{3ff} \) to \( s\mathcal{C}at \).

The slightly extended version of the Dold-Kan equivalence given by Proposition 3.18 induces a pair of functors

\[
\tilde{N} : s\mathcal{C}at_{\geq 0}^{3ff} \rightleftarrows \mathfrak{d}g_{\geq 0} \mathcal{C}at_{\geq 0}^{3ff} : \tilde{K}^{19}
\]

\[\text{19There is some abuse of notation in this formula.}\]
where

\[
\tilde{N} : \mathbf{sCat}_k \xrightarrow{sCat^{3ff}_k} \mathbf{dg}_{\geq 0} \mathbf{Cat}_k^{3ff}
\]

\[
\forall P, Q \in \mathcal{C} \quad \xi_* (P, Q) \xrightarrow{\tilde{N} (\xi_* (P, Q))}
\]

\[
\forall P, Q, R \in \mathcal{C} \quad \begin{pmatrix}
\xi_* (P, Q) \otimes \xi_* (Q, R) \\
\xi_* (P, R)
\end{pmatrix} \quad \xrightarrow{\tilde{N} (\xi_* (P, Q)) \otimes \tilde{N} (\xi_* (Q, R))}
\]

and

\[
\tilde{K} : \mathbf{dg}_{\geq 0} \mathbf{Cat}_k^{3ff} \xrightarrow{\mathbf{sCat}_k^{3ff}} \mathbf{dg}_{\geq 0} \mathbf{Cat}_k^{3ff}
\]

\[
\forall P, Q \in \mathcal{C} \quad \xi_* (P, Q) \xrightarrow{\tilde{K} (\xi_* (P, Q))}
\]

\[
\forall P, Q, R \in \mathcal{C} \quad \begin{pmatrix}
\xi_* (P, Q) \otimes \xi_* (Q, R) \\
\xi_* (P, R)
\end{pmatrix} \quad \xrightarrow{\tilde{K} (\xi_* (P, Q)) \otimes \tilde{K} (\xi_* (Q, R))}
\]

Notice also that the \(\infty\)-equivalence given by formula (3.13) restricts to an \(\infty\)-equivalence

\[
\tilde{N} : \mathbf{sGrpd}_k^{\mathbf{Aff}} \xrightarrow{\mathbf{sSet}} \mathbf{dg}_{\geq 0} \mathbf{Grpd}_k^{\mathbf{Aff}} : \tilde{K}.
\]

At last, let us recall that there is a natural functor

\[
\hat{W} : \mathbf{sCat} \rightarrow \mathbf{sSet}
\] (3.14)

given by the right adjoint to Illusie’s Dec functor; we are not describing it explicitly as its construction is slightly technical and not really needed for the sake of this paper: the definition of \(\hat{W}\) can be found in [13] Section V.7 or [34] Section 1. Moreover in [1] Cegarra and Remedios proved that \(\hat{W}\) is weakly equivalent to the diagonal of the simplicial nerve functor. Functor (3.14) is also known to induce a right Quillen equivalence

\[
\hat{W} : \mathbf{sGrpd}_k^{\mathbf{Aff}} \rightarrow \mathbf{sSet}
\]

and – as a corollary of the results in [34] Section 1 – we also have that functor (3.14) restricts to an equivalence

\[
\hat{W} : \mathbf{sGrpd}_k^{\mathbf{Aff}} \rightarrow \mathbf{sSet}.
\]

In Section 3.4 we will apply the functor \(\hat{W}\) to interesting affine simplicial groupoids in order to define rigorously the derived deformation functor \(\text{hoFlag}^{R\Gamma \Omega^*}_{\mathbb{A}^n} (X, \Omega_X^{[n]/k})\), while the functor \(\mathbb{R} \text{Def}_X\) will be constructed in Section 3.5 by using different techniques.
3.4 Derived Deformations of Filtered Algebraic De Rham Complexes

Recall from [0] that for any \( R \) in \( \mathbf{Alg}_{k} \) or even \( \mathbf{dgAlg}_{k}^{\leq 0} \) there is a model structure on \( \mathbf{dgMod}_{R} \) modelled on the projective model structure over \( \mathbf{dgMod}_{R} \).

In analogy with the underived case discussed in Section 3.4 Derived Deformations of Filtered Algebraic De Rham Complexes, in particular

\[
\text{hoFlag}_{\text{GR}}(X, \Omega_{X/k}^{*}), F^{\bullet} \rightarrow \Omega_{X/k}^{*} \rightarrow X/k, \quad \text{for all } F^{\bullet} \in \text{hoFlag}_{\text{GR}}(X, \Omega_{X/k}^{*}),
\]

is a surjective morphism such that the maps \( \mathcal{F} \mathcal{V} \mathcal{A}_{A} \rightarrow \Omega_{X/k}^{*} \rightarrow X/k, \quad \text{for all } \mathcal{F} \mathcal{V} \mathcal{A}_{A}, \mathcal{G} \mathcal{A}_{A} \)

is a trivial fibration and \( (\mathcal{F} \mathcal{V} \mathcal{A}_{A}, \mathcal{G} \mathcal{A}_{A}) \) is cofibrant for the model structure of \( \mathbf{dgMod}_{A} \).

In broad terms, a (derived) deformation of \( (\mathcal{F} \mathcal{V} \mathcal{A}_{A}, \mathcal{G} \mathcal{A}_{A}) \) over a differential graded Artinian algebra \( A \) is given by a filtered complex of \( A \)-modules together with a fixed quasi-isomorphism trivializing it infinitesimally.

In the above notations, let

\[
\left( hF_{\Omega_{X/k}^{*}}^{\bullet}(A), hF_{\Omega_{X/k}^{*}}^{\bullet}(A) \right)
\]

be the affine \( \mathbf{dg}_{\geq 0} \)-category defined by the formulae

\[
hF_{\Omega_{X/k}^{*}}^{\bullet}(A) := \left\{ \text{(derived) } A \text{-deformations of } \left( \mathcal{R} \mathcal{F} \mathcal{G} \mathcal{L} \mathcal{O} (X, \Omega_{X/k}^{*}), F^{\bullet} \right) \right\}
\]

and for all \( ((\mathcal{F} \mathcal{V} \mathcal{A}_{A}), \phi), ((\mathcal{G} \mathcal{A}_{A}), \phi) \in hF_{\Omega_{X/k}^{*}}^{\bullet}(A) \)

\[
\begin{align*}
\hat{hF}_{\Omega_{X/k}^{*}}^{\bullet}(A) & := \{ \Psi \in \text{Hom}^{0}((\mathcal{F} \mathcal{V} \mathcal{A}_{A}), (\mathcal{G} \mathcal{A}_{A})) \text{ s.t. } \phi \circ \Psi = \varphi \} \\
\hat{hF}_{\Omega_{X/k}^{*}}^{\bullet}(A) & := \{ \Psi \in \text{Hom}^{-1}((\mathcal{F} \mathcal{V} \mathcal{A}_{A}), (\mathcal{G} \mathcal{A}_{A})) \text{ s.t. } \phi \circ \Psi = 0 \} \\
\vdots
\end{align*}
\]

with the differential induced by the standard differential on Hom complexes.

Remark 3.26. Observe that, by the 2-out-of-3 property morphisms in \( hF_{\Omega_{X/k}^{*}}^{\bullet}(A) \) are all weak equivalences; in particular \( H_{0} \left( \left( hF_{\Omega_{X/k}^{*}}^{\bullet}(A), hF_{\Omega_{X/k}^{*}}^{\bullet}(A) \right) \right) \) is a groupoid, so the affine \( \mathbf{dg}_{\geq 0} \)-category \( \left( hF_{\Omega_{X/k}^{*}}^{\bullet}(A), hF_{\Omega_{X/k}^{*}}^{\bullet}(A) \right) \) is really an affine \( \mathbf{dg}_{\geq 0} \)-groupoid.

Now define the derived deformation functor \( \text{hoFlag}_{\mathcal{R} \mathcal{F} \mathcal{G} \mathcal{L} \mathcal{O}}^{\bullet}(X, \Omega_{X/k}^{*}) \) as

\[
\text{hoFlag}_{\mathcal{R} \mathcal{F} \mathcal{G} \mathcal{L} \mathcal{O}}^{\bullet}(X, \Omega_{X/k}^{*}) : \mathbf{d} \mathbf{g} \mathbf{A} \mathbf{r} \mathbf{t}_{k}^{\leq 0} \rightarrow \text{sSet}
\]

\[
A \mapsto \tilde{W} \left( \tilde{K} \left( \left( hF_{\Omega_{X/k}^{*}}^{\bullet}(A), hF_{\Omega_{X/k}^{*}}^{\bullet}(A) \right) \right) \right).
\]
Remark 3.27. By [34] Corollary 1.11 we have that $\text{hoFlag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k})$ is a derived enhancement of $\text{Flag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k})$, i.e.

$$\pi^0 \pi_{\leq 0} \text{hoFlag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k}) \simeq \text{Flag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k}).$$

Let $\mathbb{R} \text{Def}^{\text{RF}}_{\text{RF}}(X, \Omega^*_{X/k})$ be the derived deformation functor parametrizing derived deformations of the algebraic De Rham complex and $\mathbb{R} \text{Def}^{F^*}_{F^*} (X, \Omega^*_{X/k}) \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)$ the one parametrizing derived deformations of $F^* \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)$ as a subcomplex of $\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)$; also denote

$$\mathbb{R} \text{Def}^{F^*}_{F^*} (X, \Omega^*_{X/k}) \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right) := \bigoplus_p \mathbb{R} \text{Def}^{F^*}_{F^*} (X, \Omega^*_{X/k}) \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right).$$

Remark 3.28. By [34] Corollary 1.10, [34] Corollary 1.11 and [34] Corollary 1.12 we have that $\text{hoFlag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k}) = \text{holim} \big( \mathbb{R} \text{Def}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k}) \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right) \mathbb{R} \text{Def}^{\text{RF}}_{\text{RF}}(X, \Omega^*_{X/k}) \big)$. In particular $\text{hoFlag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k})$ is a well-defined derived deformation functor.

Lemma 3.29. The derived deformation functors $\mathbb{R} \text{Def}^{\text{RF}}_{\text{RF}}(X, \Omega^*_{X/k})$ and $\mathbb{R} \text{Def}^{\text{End}^*}_{\text{End}^*} (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right))$ are weakly equivalent.

Proof. This result is well-known in the Derived Deformation Theory folklore: we just recall the morphism

$$\mathbb{R} \text{Def}^{\text{End}^*}_{\text{End}^*} (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)) \longrightarrow \mathbb{R} \text{Def}^{\text{RF}}_{\text{RF}}(X, \Omega^*_{X/k})$$

giving the actual weak equivalence.

Note that by Theorem 3.4 it suffices to determine such a map on the (derived) Deligne groupoid $\text{BDef}^{\text{End}^*}_{\text{End}^*} (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right))$ associated to the dgla $\text{End}^* (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right))$, so define

$$\nu : \text{BDef}^{\text{End}^*}_{\text{End}^*} (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)) \longrightarrow \mathbb{R} \text{Def}^{\text{RF}}_{\text{RF}}(X, \Omega^*_{X/k})$$

for all $A \in \text{dgArt}^{\leq 0}_{\text{RF}}$ MC $\text{End}^* (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)) (A) \ni \sigma \mapsto \left[ \left( \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right) \otimes A, d + \sigma \right) \right]$

$$\text{BDef}^{\text{End}^*}_{\text{End}^*} (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right)) (A) \ni \xi \mapsto \left[ \begin{array}{c} \left( \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right) \otimes A, d + \sigma_1 \right) \\ \mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right) \otimes A, d + \sigma_2 \end{array} \right]$$

Map $\nu$ is known to be a weak equivalence: a very rigorous but quite abstract proof can be found in [30] Section 4.1, while a simpler one can be found in [29]; see also [10] Section 6. 

Proposition 3.30. The functors $\text{hoFlag}^{F^*}_{\text{RF}}(X, \Omega^*_{X/k})$ and $\mathbb{R} \text{Def}^{\text{End}^*}_{\text{End}^*} (\mathbb{R} \Gamma \left( X, \Omega^*_{X/k} \right))$ are weakly equivalent.
Proof. We want to prove that the functor \( \text{hoFlag}^{F \ast}_{\text{Flag}}(X, \Omega^\ast_{X/k}) \) is weakly equivalent to the Hinich nerve of the differential graded Lie algebra

\[
\text{holim} \left( \text{End}^{\geq 0} \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \xrightarrow{\text{incl.}} \text{End}^\ast \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \right).
\]

Recall from Section 3.2 and [10] that – as the Hodge-to-De Rham spectral sequence of \( X \) degenerates at the first page – the above homotopy limit is

\[
\text{End}^\ast \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \left[ -1 \right].
\]

As proven in [11] and [15], the functor \( \mathcal{R}\text{Def} \) (homotopically) commutes with homotopy limits, thus we have that

\[
\mathcal{R}\text{Def} \text{holim} \left( \text{End}^{\geq 0} \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \xrightarrow{\text{incl.}} \text{End}^\ast \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \right)
\]

is weakly equivalent to

\[
\text{holim} \left( \mathcal{R}\text{Def}_{\text{End}^{\geq 0}} \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \xrightarrow{\text{incl.}} \mathcal{R}\text{Def}_{\text{End}^\ast} \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \right).
\]

By Lemma 3.29 we have that \( \mathcal{R}\text{Def}_{\mathcal{R}\Gamma}(X, \Omega^\ast_{X/k}) \) is weakly equivalent to \( \mathcal{R}\text{Def}_{\text{End}^\ast} \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \) and similarly the functor \( \mathcal{R}\text{Def}_{F \ast \mathcal{R}\Gamma}(X, \Omega^\ast_{X/k}, \mathcal{R}\Gamma(X, \Omega^\ast_{X/k})) \) is seen to be weakly equivalent to \( \mathcal{R}\text{Def}_{\text{End}^{\geq 0}} \left( \mathcal{R}\Gamma \left( X, \Omega^\ast_{X/k} \right) \right) \), thus the statement follows by applying Remark 3.28.

3.5 Derived Deformations of \( k \)-Schemes

Now we want to describe the functor \( \mathcal{R}\text{Def}_X \) which parametrizes derived deformations of the scheme \( X \): the idea consists of deforming the scheme \( X \) through derived schemes instead of ordinary schemes. There are a variety of equivalent definitions of derived scheme (in particular see [25] Definition 4.5.1 and [39] Chapter 2.2 for the two most standard ways to look at it); the one we are about to recall probably is not the most elegant, but it is definitely the handiest one to make actual computations. As a matter of fact, by [32] Theorem 6.42 a derived scheme \( S \) over \( k \) can be seen as a pair \( (\pi^0S, \mathcal{O}_{S, *}) \), where \( \pi^0S \) is an honest \( k \)-scheme and \( \mathcal{O}_{S, *} \) is a presheaf of differential graded commutative algebras in non-positive degrees on the site of affine opens of \( \pi^0S \) such that:

- the (cohomology) presheaf \( \mathcal{H}^0(\mathcal{O}_{S, *}) \simeq \mathcal{O}_{\pi^0S} \);
- the (cohomology) presheaves \( \mathcal{H}^0(\mathcal{O}_{S, *}) \) are quasi-coherent \( \mathcal{O}_{\pi^0S} \)-modules.

Also, recall from [32] that a morphism \( f : A \to B \) in \( \text{dgAlg}^{\leq 0}_k \) is said to be homotopy flat if

\[
H^0(f) : H^0(A) \to H^0(B)
\]

is flat and the maps

\[
H^i(A) \otimes_{H^0(A)} H^0(B) \to H^i(B)
\]

isomorphisms.
are isomorphisms for all $i$; moreover a very useful characterization says that $f$ is homotopy flat if and only if $B \otimes_A H^0 (A)$ is (weakly equivalent to) a discrete flat $H^0 (A)$-algebra: for a proof see [31] Lemma 3.13.

Now define a derived deformation of the scheme $X$ over $A \in \mathcal{dgArt}^{\leq 0}$ to be a homotopy pull-back diagram of derived schemes

$$
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow^{h} & & \downarrow^{p} \\
\text{Spec} (k) & \rightarrow & \mathbb{R}\text{Spec} (A)
\end{array}
$$

where the map $p$ is homotopy flat; equivalently such a deformation can be seen as a morphism $\mathcal{O}_{A,*} \rightarrow \mathcal{O}_X$ of presheaves of differential graded commutative algebras over $A$ such that:

1. $\mathcal{O}_{A,*}$ is homotopy flat;
2. the induced $k$-linear morphism $\mathcal{O}_{A,*} \otimes_A^L k \rightarrow \mathcal{O}_X$ is a weak equivalence;
3. the morphism $\mathcal{O}_{A,*} \rightarrow \mathcal{O}_X$ is surjective;
4. $\mathcal{O}_{A,*}$ is cofibrant.

**Remark 3.31.** In the above notations, Condition (1) and Condition (2) are proper deformation-theoretic conditions, which resemble the ones characterizing underived deformations of schemes (see Section 2.1), while Condition (3) and Condition (4) are fibrancy-cofibrancy conditions, which are needed in order to ensure that certain maps of derived deformation functors which will arise in the rest of the paper are well-defined.

Now consider the formal groupoid

$$
\text{Del}_X : \mathcal{dgArt}^{\leq 0} \rightarrow \mathcal{Grpd}
$$

defined by the formulae

$$
\text{Del}_X (A) := \{(\text{derived}) \text{ deformations of } X \text{ over } A\}
$$

and for all $\left( \mathcal{O}_{A,*} \xrightarrow{\varphi} \mathcal{O}_X \right), \left( \mathcal{O}_{A,*}' \xrightarrow{\phi} \mathcal{O}_X \right) \in \text{Del}_X (A)$

$$
\text{Hom}_{\text{Del}_X (A)} (\varphi, \phi) := \{ \Psi \in \text{Hom}_A^0 (\mathcal{O}_{A,*}, \mathcal{O}_{A,*}') \text{ s.t. } \phi \circ \Psi = \varphi, \Psi \equiv \text{Id} \mod m_A \} \quad (3.15)
$$

**Remark 3.32.** In the notations of formula (3.15), notice that the condition $\Psi \equiv \text{Id} \mod m_A$ ensures that $\text{Del}_X (A)$ is a groupoid for all $A \in \mathcal{dgArt}^{\leq 0}$; roughly speaking, the formal groupoid $\text{Del}_X$ can be thought as some sort of (derived) Deligne groupoid associated to the scheme $X$, meaning that its role is intended to formally resemble the one played by the (derived) Deligne groupoid associated do a differential graded Lie algebra, which we described in Section 3.1.

Now consider the functor

$$
\text{BDel}_X : \mathcal{dgArt}^{\leq 0} \rightarrow \mathcal{sSet}
$$

[20][22] and [31] actually deal with homotopy flatness in terms of simplicial and $\mathcal{dg}_{\geq 0}$ chain algebras; nevertheless all definitions and arguments readily adapt to cochain algebras in non-positive degrees.
given by the nerve of Del$_X$ and define
\[ R\text{Def}_X : \mathfrak{dgArt}^0_{\mathcal{K}} \longrightarrow s\text{Set} \]
to be the right derived functor of BDel$_X$.

Remark 3.33. RDef$_X$ is a derived enhancement of Def$_X$, i.e.
\[ \pi^0\pi_{\leq 0}R\text{Def}_X \simeq \text{Def}_X. \]

The definition of Del$_X$ implies immediately that this is a derived pre-deformation functor, thus – by [33] Theorem 3.16 – RDef$_X$ turns to be a derived deformation functor. Moreover, observe that RDef$_X$ is the formal neighbourhood of the stack of k-schemes D8ch$_{\mathcal{K}/k}$ which Pridham constructed in [34]: we will discuss these facts in more details this in Section 4; in particular, it follows – using either [32] Theorem 8.8 or [34] Theorem 10.8 – that
\[ H^i(\text{BDel}_X) \simeq H^i(R\text{Def}_X) \simeq \text{Ext}^{i+1}_{\mathcal{O}_X}(\mathbb{L}_{X/k}, \mathcal{O}_X) \]

Theorem 3.34. The functors RDef$_X$ and RDef$_{KS_X}$ are weakly equivalent.

Proof. We want to construct a natural transformation
\[ R\text{Def}_{KS_X} \longrightarrow R\text{Def}_X \]
providing a weak equivalence between such derived deformation functors, i.e. an isomorphism on the level of homotopy categories.

Again, by Theorem [3.4] and the definition of RDef$_X$ it is enough to define such a morphism on BDel$_{KS_X}$, thus define the map
\[ \mu : \text{BDel}_{KS_X} \longrightarrow \text{BDel}_X \]
for all \( A \in \mathfrak{dgArt}^0_{\mathcal{K}} \), MC$_{KS_X}(A) \ni x \mapsto \left[ \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x) \right) \rightarrow \tau^{\leq 0}R\mathcal{O}_k(x) \simeq \mathcal{O}_X \right] \]

\[ \mathbb{G}_{g_{KS_X}}(A) \ni \xi \mapsto \begin{cases} \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x_1) \right) \\ \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x_2) \right) \end{cases} \]

(3.16)

where \( R\mathcal{O}_A(x) := \left( \mathfrak{s}_X^{0,+} \otimes A, \partial + l_x \right) \), \( l \) being the Lie derivative (i.e. the differential of the Cartan homotopy determined by the contraction map), \( \mathbb{Q} \) is a functorial cofibrant replacement and the map \( \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x) \right) \rightarrow \tau^{\leq 0}R\mathcal{O}_k(x) \) is induced by \( A \rightarrow A/m_A \simeq k \). Notice in particular that the surjectivity of the natural map \( A \rightarrow A/m_A \) together with the surjectivity of the canonical morphism \( \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x) \right) \rightarrow H^0(\mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x) \right)) \) – in turn due to the fact that the complex \( \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x) \right) \) lives in non-positive degrees – ensures that the morphism \( \mathbb{Q} \left( \tau^{\leq 0}R\mathcal{O}_A(x) \right) \rightarrow \mathcal{O}_X \) is surjective. For the sake of notational simplicity, in the rest of the proof we will drop any explicit reference to the choice of the cofibrant replacement \( \mathbb{Q} \), i.e. we will argue as if
The symbol \( \tau \) stands for “weakly equivalent”.

In order to show that map \( 3.16 \) is well-defined, we have to check that
\[
[\tau^{<0} \mathcal{O}_A (x) \rightarrow \tau^{<0} \mathcal{O}_k (x) \simeq \mathcal{O}_X]
\]

actually determines a derived deformation of the scheme \( X \), i.e. we need to prove that \( \tau^{<0} \mathcal{O}_A (x) \) is homotopy flat over \( A \) and \( \tau^{<0} \mathcal{O}_A (x) \otimes^L_A k \) is weakly equivalent to \( \mathcal{O}_X \) as complexes of presheaves of differential graded commutative \( k \)-algebras: this essentially means to verify that \( \mathbb{R} \mathcal{O}_A (x) \otimes^L_A \mathcal{H}^0 (A) \) is flat over \( \mathcal{H}^0 (A) \).

Let us first prove that \( \tau^{<0} \mathcal{O}_A (x) \) is weakly equivalent to \( \mathbb{R} \mathcal{O}_A (x) \). Filter the latter complex by powers of the maximal ideal \( m_A \) of \( A \), i.e. define the filtered complex \( (\mathbb{R} \mathcal{O}_A (x) , \mathcal{F} \bullet ) \) through the relation
\[
\mathcal{F}^p \mathbb{R} \mathcal{O}_A (x) := m_A^p \mathbb{R} \mathcal{O}_A (x)
\]
and take the associated graded object
\[
\text{Gr}^p (\mathcal{F} \bullet ) := \frac{m_A^p \mathbb{R} \mathcal{O}_A (x)}{m_A^{p+1} \mathbb{R} \mathcal{O}_A (x)}.
\]

Notice that Formula \( 3.17 \),
\[
\text{Gr}^p (\mathcal{F} \bullet ) \simeq \mathcal{O}_X^{0,*} \otimes \frac{m_A^p}{m_A^{p+1}}.
\]

Now consider the spectral sequence
\[
H^{p+q} \left( \mathcal{O}_X^{0,*} \otimes \frac{m_A^p}{m_A^{p+1}} \right) \simeq \bigoplus_{i+j=p+q} H^i \left( \mathcal{O}_X^{0,*} \right) \otimes H^j \left( \frac{m_A^p}{m_A^{p+1}} \right) \Rightarrow H^{p+q} (\mathbb{R} \mathcal{O}_A (x)) \quad (3.18)
\]

which converges by the Classical Convergence Theorem (see \[44\] Theorem 5.5.1); note that
\[
H^j \left( \frac{m_A^p}{m_A^{p+1}} \right) = 0 \quad \text{when } j > 0 \quad \text{and, since the “Dolbeaut” resolution } \mathcal{O}_X^{0,*} \leftrightarrow \mathcal{O}_X \text{ provides a weak equivalence between } \mathcal{O}_X^{0,*} \text{ and } \mathcal{O}_X \text{ in the category of quasi-coherent } \mathcal{O}_X \text{-modules in complexes, also } H^i \left( \mathcal{O}_X^{0,*} \right) = 0 \quad \text{when } i > 0: \text{ this means that at least one of these two terms vanishes whenever } i + j > 0, \text{ so the convergence of spectral sequence (3.18) implies that }
\]
\[
H^n (\mathbb{R} \mathcal{O}_A (x)) = 0 \quad \forall n > 0.
\]

In particular \( \mathbb{R} \mathcal{O}_A (x) \) and \( \tau^{<0} \mathbb{R} \mathcal{O}_A (x) \) are weakly equivalent.

Now we want to prove that \( \tau^{<0} \mathbb{R} \mathcal{O}_A (x) \otimes_A^L \mathcal{H}^0 (A) \) is flat over \( \mathcal{H}^0 (A) \); first notice that
\[
\tau^{<0} \mathbb{R} \mathcal{O}_A (x) \otimes^L_A \mathcal{H}^0 (A) \simeq \mathbb{R} \mathcal{O}_A (x) \otimes^L_A \mathcal{H}^0 (A) \approx \mathbb{R} \mathcal{O}_A (x) \otimes_A \mathcal{H}^0 (A) \quad \text{[21]}
\]

so it is enough to show that \( \mathbb{R} \mathcal{O}_A (x) \otimes_A \mathcal{H}^0 (A) \) is flat over \( \mathcal{H}^0 (A) \). In order to prove this let \( M \) be any \( \mathcal{H}^0 (A) \)-module, consider the complex
\[
(\mathbb{R} \mathcal{O}_A (x) \otimes_A \mathcal{H}^0 (A)) \otimes_{\mathcal{H}^0 (A)} M \simeq \mathbb{R} \mathcal{O}_A (x) \otimes_A M.
\]

\[21\] The symbol \( \approx \) stands for “weakly equivalent”.
and filter it by powers of the maximal ideal \( m_{H^0(A)} \) of \( H^0(A) \), i.e. define the filtered complex \( (\mathbb{R} \mathcal{O}_A (x) \otimes_A M, \mathcal{F}^*) \) through the relation
\[
\mathcal{F}^p (\mathbb{R} \mathcal{O}_A (x) \otimes_A M) := m_{H^0(A)}^p (\mathbb{R} \mathcal{O}_A (x) \otimes_A M).
\]
As before, the associated graded object is
\[
Gr^p (\mathcal{F}) := \frac{m_{H^0(A)}^p (\mathbb{R} \mathcal{O}_A (x) \otimes_A M)}{m_{H^0(A)}^{p+1} (\mathbb{R} \mathcal{O}_A (x) \otimes_A M)} \simeq \mathcal{A}_X^{0,*} \otimes \frac{m_{H^0(A)}^p M}{m_{H^0(A)}^{p+1} M}
\]
and there is a spectral sequence
\[
H^{p+q} \left( \mathcal{A}_X^{0,*} \otimes \frac{m_{H^0(A)}^p M}{m_{H^0(A)}^{p+1} M} \right) \implies H^{p+q} (\mathbb{R} \mathcal{O}_A (x) \otimes_A M)
\]
which still converges because of the Classical Convergence Theorem. Of course partition computation also ensures that the map
\[
\tau_{\leq 0} \mathcal{O}_A (x) \otimes_A \mathcal{O}_X \to A
\]
also a chain of canonical isomorphisms
\[
\mathcal{A}_X^{0,*} \otimes \frac{m_{H^0(A)}^p M}{m_{H^0(A)}^{p+1} M}
\]
by definition and the last one is given by the Dolbeaut Theorem. In the same fashion, there is
\[
\text{Tor}^A_n (\mathbb{R} \mathcal{O}_A (x), M) \simeq H^{-n} (\mathbb{R} \mathcal{O}_A (x) \otimes_A M) = 0 \quad \forall n \neq 0
\]
which gives us the flatness of \( \tau_{\leq 0} \mathbb{R} \mathcal{O}_A (x) \otimes_A H^0 (A) \) over \( H^0 (A) \). Notice that the same computation also ensures that the map
\[
\left[ \tau_{\leq 0} \mathbb{R} \mathcal{O}_A (x) \to \tau_{\leq 0} \mathbb{R} \mathcal{O}_k (x) \simeq \mathcal{O}_X \right]
\]
is quasi-smooth.
Now we want to prove that map \( 3.16 \) is a weak equivalence of derived deformation functors; by \( 3.11 \) Corollary 1.49 it suffices to check that such a map induces isomorphisms on generalized tangent spaces, so consider the morphisms
\[
H^i (\mu) : H^i (\text{BDef}_{KS_X}) \to H^i (\text{BDef}_X) \quad i \geq -1
\]
and notice that higher tangent maps in larger negative degrees vanish as \( KS_X \) lives only in non-negative degrees.
For all \( i \geq 0 \) we have the chain of canonical identifications
\[
H^i (\check{\mathbb{R} \text{Def}}_{KS_X}) \simeq H^i (\text{BDef}_{KS_X}) \simeq H^{i+1} (KS_X) \simeq H^{i+1} (X, \mathcal{A}_X^{0,*} (\mathcal{F}_X)) \simeq H^{i+1} (X, \mathcal{F}_X)
\]
where the first and the second isomorphism come from Remark \( 3.3 \) the third one is true just by definition and the last one is given by the Dolbeaut Theorem. In the same fashion, there is also a chain of canonical isomorphisms
\[
H^i (\text{BDef}_X) \simeq \text{Ext}^{i+1}_{\mathcal{O}_X} (\mathcal{L}_{X/k}, \mathcal{O}_X) \simeq \text{Hom}_{D(X)} (\Omega^1_{X/k} \otimes \mathcal{O}_X, \mathcal{O}_X [-i - 1]) \simeq \text{Hom}_{D(X)} (\mathcal{O}_X, \mathcal{F}_X [-i - 1]) \simeq \text{Ext}^{i+1}_{\mathcal{O}_X} (\mathcal{O}_X, \mathcal{F}_X)
\]
36
where the first isomorphism – as we discussed before – comes from the fact that \( \mathbb{R} \text{Def}_X \) is the formal neighbourhood of a derived stack of schemes, the third one is true by adjunction, while all the other ones directly follow from definitions.

Finally, for all \( i \geq 0 \) we see that the map \( H^i(\mu) \) is

\[
H^i(\mu) : H^{i+1}(X, \mathcal{T}_X) \rightarrow \text{Ext}^{i+1}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{T}_X) \\
\xi \quad \mapsto (\mathcal{O}_X \xrightarrow{\xi} \mathcal{T}_X[-i-1])
\] (3.20)

where the (cohomology class of the) degree \( i \) morphism \( \mathcal{O}_X \xrightarrow{\xi} \mathcal{T}_X[-i-1] \) is nothing but the map\(^\text{22}\) induced in \( D(X) \) by the cocycle \( \xi \); on the other hand – again by using Remark 3.5 – the map \( H^{-1}(\mu) \) turns out to be

\[
H^{-1}(\mu) : H^0(\mathcal{K}S_X) \cong \text{Stab}_{G^s_{(x,\mathcal{O}_X^0(\mathcal{F}_X))}}(0) \rightarrow H^0(X, \mathcal{T}_X) \cong \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{T}_X) \\
\text{Id} + \xi \quad \mapsto \xi.
\] (3.21)

Both map (3.21) and map (3.20) are clearly isomorphisms, so this completes the proof.

3.6 The Geometric Fiorenza-Manetti-Martinengo Period Map

Now we have all the ingredients to give a geometric interpretation of the map \( \text{FMM} \) described in Definition 3.7.

**Definition 3.35.** Define the (universal) geometric Fiorenza-Manetti-Martinengo local period map to be the morphism of derived deformation functors

\[
\mathbb{R} P : \mathbb{R} \text{Def}_X \longrightarrow \text{hoFlag}_{\mathbb{R}G}(X, \mathcal{O}_{X/k})
\]

\(\mathbb{R} P \) is the universal morphism of derived deformation functors under the morphism of derived pre-deformation functors given for all \( A \in \mathfrak{dgArt}^{\leq 0} \) by

\[
\text{BDel}_X \quad \longrightarrow \quad \text{hoFlag}_{\mathbb{R}G}(X, \mathcal{O}_{X/k})
\]

\[
[\mathcal{O}_{A, s} \xrightarrow{\varphi} \mathcal{O}_X] \approx \left[ \begin{array}{c} X' \\ \downarrow \Delta^h \\ \text{Spec}(k) \end{array} \longrightarrow \left[ \begin{array}{c} X \\ \downarrow \\ \text{Spec}(k) \end{array} \right] \right] \rightarrow \left[ \left( \left( \mathbb{R} \Gamma \left( \pi^0 X, \Omega_{X/A}^* \right), F^* \right), \tilde{\varphi} \right) \right]
\]

\[
\left[ \begin{array}{c} \mathcal{O}_{A, s}' \\ \downarrow \varphi' \\ \mathcal{O}_X \\ \downarrow \varphi'' \\ \mathcal{O}_{A, s}'' \end{array} \right] \rightarrow \left[ \left( \left( \mathbb{R} \Gamma \left( \pi^0 X, \Omega_{X''/A}^* \right), F^* \right), \tilde{\varphi}' \right) \right]
\]

where \( \tilde{\varphi} \) is the derived globalization of the natural \( A \)-linear map extending \( \varphi \) to the algebraic De Rham complex; also \( \tilde{\Psi} \) is constructed by using the same universal property.

\(^{22}\)Of course, there is some abuse of notation in this sentence.
Remark 3.36. The fibrant-cofibrant replacement properties pointed out in Remark 3.25 and Remark 3.31 ensure that the geometric Fiorenza-Manetti-Martinengo local period map described in Definition 3.35.

In the end all constructions and results we have discussed so far sum up in the following theorem.

**Theorem 3.37.** The diagram of derived deformation functors and (Schlessinger’s) deformation functors

\[
\begin{align*}
\text{RDef}_{KS_X} & \quad \xrightarrow{FMM} \quad \text{RDef}_{\text{End}^∗(\Gamma(X,\Omega^∗_{X/k}))}^{\text{End}^0(\Gamma(X,\Omega^∗_{X/k}))}\left[-1\right] \\
\text{RDef}_X & \quad \xrightarrow{\mathcal{P}} \quad \text{hoFlag}_{\text{Flag}}^{\mathcal{P}}\left(X,\Omega^∗_{X/k}\right) \\
\text{Def}_{KS_X} & \quad \xrightarrow{\mathcal{P}} \quad \text{Def}_{\text{End}^0(\mathcal{H}^∗(X,k))}^{\text{End}^{-1}(\mathcal{H}^∗(X,k))}\left[-1\right] \\
\text{Def}_X & \quad \xrightarrow{\mathcal{P}} \quad \text{Flag}_{\mathcal{H}^∗(X,k)}^{\mathcal{P}}
\end{align*}
\]

commutes up to isomorphism; in particular the morphisms \(\mathcal{P}\) and \(FMM\) are equivalent.

**Proof.** Notice that:

- the commutativity of the bottom diagram follows from Theorem 2.16 and Theorem 3.8;
- the commutativity (up to isomorphism) of the back diagram corresponds to Theorem 3.8;
- the commutativity (up to isomorphism) of the front diagram follows immediately from Remark 3.33, Remark 3.27 and the definitions of the maps \(\mathcal{P}\) and \(\mathcal{P}\);
- the commutativity (up to isomorphism) of the left hand diagram is obtained by combining Theorem 2.2, Theorem 3.34 and Remark 3.33;
- the commutativity (up to isomorphism) of the right hand diagram is obtained by combining Proposition 2.13, Proposition 3.30 and Remark 3.27.

As regards the top diagram, again by Theorem 3.4 it suffices to verify its commutativity up to isomorphism on \(\text{BDef}_{KS_X}\); moreover Remark 3.28 tells us that the derived deformation functors \(\text{RDef}_{\text{End}^∗(\Gamma(X,\Omega^∗_{X/k}))}^{\text{End}^0(\Gamma(X,\Omega^∗_{X/k}))}\left[-1\right]\) and \(\text{hoFlag}_{\text{Flag}}^{\mathcal{P}}\left(X,\Omega^∗_{X/k}\right)\) can be constructed as homotopy fibres, so it
is enough to check that the diagrams \[ (3.22) \]
\[
\begin{array}{ccc}
\text{BDef}_{K S_X} & \xrightarrow{\text{RDef}(l)} & \text{BDef}_{\text{End}^*}(\text{RG}(X, \Omega_{X/k}^*)) \\
\mu \downarrow & & \downarrow \nu \\
\text{RDef}_X \quad (X, \mathcal{O}_{A,x}) \mapsto \text{RG}(\pi^0 X, \Omega_{X/A}^*) & & \text{RDef}_X \quad \text{RG}(X, \text{RG}(X, \Omega_{X/k}^*))
\end{array}
\]

\[ \text{and} \]
\[
\begin{array}{ccc}
\text{BDef}_{K S_X} & \xrightarrow{\text{RDef}(l)} & \text{BDef}_{\text{End}^*}(\text{RG}(X, \Omega_{X/k}^*)) \\
\mu \downarrow & & \downarrow \nu \\
\text{RDef}_X \quad (X, \mathcal{O}_{A,x}) \mapsto \text{RG}(\pi^0 X, \Omega_{X/A}^*).F^*) & & \text{RDef}_X \quad \text{RG}(X, \text{RG}(X, \Omega_{X/k}^*)), \text{RG}(X, \text{RG}(X, \Omega_{X/k}^*))
\end{array}
\]

\[ \text{commute up to isomorphism. We are only going to show the commutativity of diagram (3.22), as the commutativity of diagram (3.23) is verified by a similar argument.} \]

Let us walk along its arrows: for all \( A \in \text{dgArt}_k \), an element \( x \in \text{MC}_{K S_X} (A) \) maps through \( \mu \) to \( X \to \mathbb{R}\text{Spec}(A) \) – where \( X = (X, \mathcal{O}_{A}(x)) \) – and in turn this is sent to the complex

\[ \text{RG} \left( \pi^0 X, \Omega_{X/A}^* \right) \approx \text{RG} \left( X, \Omega_{\mathcal{O}_{A}(x)/A}^* \right) \]

\[ (3.24) \]

which is an honest derived deformation over \( A \) of the algebraic De Rham complex \( \text{RG} \left( X, \Omega_{X/k}^* \right) \); on the other side, the vector \( x \) is sent to the derivation \( l_x \) and – proceeding down along map \( \nu \) – this determines the complex

\[ \left( \text{RG} \left( X, \Omega_{X/k}^* \right) \otimes A, d + l_x \right). \]

\[ (3.25) \]

We claim that complexes \[ (3.24) \] and \[ (3.25) \] are quasi-isomorphic: more precisely, we assert that the natural zig-zag

\[ \text{RG} \left( X, \Omega_{\mathcal{O}_{A}(x)/A}^* \right) \leftrightarrow (\Gamma \left( X, \mathcal{O}_{X}^* \otimes A \right), \partial + (\bar{\partial} + l_x)) \leftrightarrow \text{RG} \left( X, \Omega_{X/k}^* \right) \otimes A, d + l_x \]

\[ (3.26) \]

is a chain of quasi-isomorphisms. The right-hand morphism in diagram \[ (3.26) \] is essentially given by the resolution \[ (2.5) \]. As regards the left-hand one, this is constructed in the following way: consider the standard Dolbeaut resolution \( \mathcal{O}_{X}^* \leftrightarrow \mathcal{O}_{X} \) and twist it through the derivation \( l_x \), so get a map \( \mathcal{O}_{A}(x) \leftrightarrow \mathcal{O}_{X} \otimes A \) and hence a morphism \( \Omega_{\mathcal{O}_{A}(x)/A}^* \leftrightarrow \Omega_{X}^* \otimes A \); now just recall that \( \mathcal{O}_{X}^* \approx \Omega_{X}^* \otimes \mathcal{O}_{X} \): this provides us with a natural map \( \Omega_{\mathcal{O}_{A}(x)/A}^* \leftrightarrow \mathcal{O}_{X}^* \otimes A \), whose globalization finally gives us the left-hand map in diagram \[ (3.26) \].

Now denote

\[ \mathcal{O}_{A}(x)(n) := (\mathcal{O}_{X}^* \otimes A, \bar{\partial} + l_x) \]

and observe that to show that the zig-zag \( (3.26) \) is really a chain of quasi-isomorphisms it suffices to prove that the complexes \( \mathcal{O}_{A}(x)(n) \) and \( \Omega_{\mathcal{O}_{A}(x)/A}^n \) are weakly equivalent. As already done\[^{23}\]

\[^{23}\text{here maps } \mu \text{ and } \nu \text{ are the morphisms defined in Theorem (3.34) and Lemma (3.29) respectively.} \]

39
in the proof of Theorem 3.34, filter them by powers of the maximal ideal \(m_A\), i.e. consider the filtrations

\[
\mathcal{F}^p (R\mathcal{O}_A(x) (n)) := m_\mathcal{O}_A (x) (n) \quad \Rightarrow \quad \text{Gr}^p (\mathcal{F}) \simeq \mathcal{O}_X^{m,n} \otimes m_A^{p+1} \\
\mathcal{F}^p (\Omega^n_{R\mathcal{O}_A(x)}) := m_\mathcal{O}_A \Omega^n_{R\mathcal{O}_A(x)} \quad \Rightarrow \quad \text{Gr}^p (\mathcal{F}) \simeq \Omega^n_{\mathcal{O}_X} \otimes m_A^{p+1}
\]

which kill the twisting \(l_x\). Now observe that

\[
\mathcal{O}_X^{m,n} \simeq \mathcal{O}_X^{m,n} \simeq \mathcal{O}_X^{m,n}
\]

where the first quasi-isomorphism is induced by the Dolbeaut resolution \(\mathcal{O}_X^{m,n} \leftarrow \mathcal{O}_X\) and the second one is true basically by definition of \(\mathcal{O}_X^{m,n}\), so in particular \(H^m (\mathcal{O}_X^{m,n}) = H^m (\mathcal{O}_X^{m,n})\) for all \(m\). Finally, look at the induced spectral sequences: we have

\[
H^{p+q} \left( \mathcal{O}_X^{m,n} \otimes m_A^{p+1} \right) \simeq \bigoplus_{i+j=p+q} (H^i (\mathcal{O}_X^{m,n}) \otimes H^j (m_A^{p+1})) \Rightarrow H^{p+q} (R\mathcal{O}_A (x) (n))
\]

and

\[
H^{p+q} \left( \Omega^n_{\mathcal{O}_X} \otimes m_A^{p+1} \right) \simeq \bigoplus_{i+j=p+q} (H^i (\mathcal{O}_X^{m,n}) \otimes H^j (m_A^{p+1})) \Rightarrow H^{p+q} (\Omega^n_{R\mathcal{O}_A(x)})
\]

so the complexes \(\Omega^n_{\mathcal{O}_X}\) and \(R\mathcal{O}_A (x) (n)\) are quasi-isomorphic as their cohomologies are computed by the same spectral sequence.

Now let us look at diagram (3.22) on the level of morphisms; a gauge element \(\xi\) in the Kodaira-Spencer differential graded Lie algebra associated to \(X\) maps through \(R\text{Def} (l)\) to \(l_{\xi}\), which in turn induces by \(\nu\) the morphism of complexes

\[
\text{RDef} (X \otimes e^{\xi}) : \left( \text{RDef} (X \otimes \Omega^n_{\mathcal{O}_X}) \otimes A, d + l_{x_1} \right) \to \left( \text{RDef} (X \otimes \Omega^n_{\mathcal{O}_X}) \otimes A, d + l_{x_2} \right).
\]

In a similar way, the gauge \(\xi\) determines via \(\mu\) the morphism of complexes

\[
e^{\xi} : R\mathcal{O}_A (x_1) \to R\mathcal{O}_A (x_2)
\]

which in turn induces through the bottom arrow in diagram (3.22) the morphism

\[
\text{RDef} (X_1 \otimes e^{\xi}) : \text{RDef} (X_1 \otimes \Omega^n_{\mathcal{O}_X}) \to \text{RDef} (X_2 \otimes \Omega^n_{\mathcal{O}_X})
\]

therefore we end up with a diagram

\[
\begin{array}{ccc}
\text{RDef} (X_1 \otimes e^{\xi}) & \to & \text{RDef} (X_1 \otimes \Omega^n_{\mathcal{O}_X}) \\
\downarrow & & \downarrow \\
\text{RDef} (X, e^{\xi}) & \to & \text{RDef} (X, \Omega^n_{\mathcal{O}_X})
\end{array}
\]

\[
(3.27)
\]

\[\text{There is some abuse of notation in these formulas.}\]
Notice that the right hand square of diagram \([3.27]\) commutes because the morphism \(\Gamma(X, e^l)\) is induced by \(R\Gamma(X, e^l)\) via the standard Dolbeaut resolution; as regards the left hand square, consider for all \(n\) the unglobalized diagram

\[
\begin{align*}
\Omega^n & \quad \Omega^n \oplus A, \bar{\partial} + l \downarrow \downarrow \\
\Omega^n & \quad \Omega^n \oplus A, \bar{\partial} + l \downarrow \downarrow 
\end{align*}
\]

and again filter all complexes by powers of the maximal ideal \(m_A\) in order to kill the derivations \(l_1, l_2\) and hence the gauge \(l_\xi\): we end up with a sequence of commutative diagrams

\[
\begin{align*}
\Omega^n & \quad \Omega^n \oplus \frac{m_A}{m_A^2}, \bar{\partial} \\
\Omega^n & \quad \Omega^n \oplus \frac{m_A}{m_A^2}, \bar{\partial}
\end{align*}
\]

therefore diagram \([3.28]\) has to commute and so does diagram \([3.27]\), as well. This observation completes the proof.

\[\square\]

### 4 The Period Map as a Morphism of Derived Stacks

Theorem \([3.37]\) gives the ultimate picture of the local period map as a deformation-theoretic morphism, since it explains how the Fiorenza-Manetti map lifts naturally to the context of Derived Deformation Theory. Anyway, despite being entirely canonical, the Fiorenza-Manetti-Martinengo map\(^{25}\) is still a local morphism: concretely this means that it provides a fully satisfying description of the behaviour of “derived variations of the Hodge structures” associated to some nice \(k\)-scheme \(X\) with respect to the infinitesimal derived deformations of the scheme itself, but this map is not able to give us any global information, i.e. it does not provide significant relations between the associated global (derived) moduli stacks.

Foundational work on higher stacks and Derived Algebraic Geometry includes \([25]\), \([26]\), \([32]\), \([37]\) and \([39]\): here we only recall that a crucial property of derived deformation functors is that these are formal neighbourhoods of global derived stacks; more formally let

\[
\mathcal{F}: \mathcal{DAlg}_{\leq k} \to sSet
\]

be a (possibly non-geometric) derived stack over \(k\) and \(x\) a point on it: the formal neighbourhood of \(F\) at \(x\) defined as

\[
\tilde{F}_x: \mathcal{DArt}_{\leq k} \longrightarrow sSet
\]

\[
A \longrightarrow \mathcal{F}(A) \times h_{\mathcal{F}(k)}(x)
\]

is a derived deformation functor. This result is well-known in the Derived Algebraic Geometry folklore: a proof of it is hidden somewhere in \([25]\) and \([39]\); see also \([37]\) and \([31]\).\(^{25}\)

\(^{25}\)Again, Theorem \([3.37]\) allows us to drop any further adjective.
In the following $X$ will denote again a smooth proper scheme over $k$ of dimension $d$ and $n$ will be any positive integer; let

$$\text{Stack}_{n/k}^\mathcal{D} : \mathfrak{M}_k \longrightarrow \mathfrak{s}\text{Cat}$$

$$A \mapsto \text{Stack}_{n/k}^\mathcal{D}(A) := \text{simplicial category of algebraic } n\text{-spaces over } A$$

be the simplicial (underived) moduli functor classifying (underived) 0-stacks of dimension $n$ over $k$. Pridham showed that such a functor induces a derived (non-geometric) stack $\mathcal{D}\text{Sch}_{n/k}$ parametrizing derived schemes over $k$ of dimension $n$ (see [34] Example 3.36 for a detailed construction).

**Remark 4.1.** The derived deformation functor $\mathbb{R}\text{Def}_X$ is the formal neighbourhood of the derived stack $\mathcal{D}\text{Sch}_{d/k}$ at $X$.

Now consider the (underived) simplicial moduli functor

$$\mathfrak{M}^n : \mathfrak{M}_k \longrightarrow \mathfrak{s}\text{Cat}$$

$$A \mapsto \mathfrak{M}(A) := \text{simplicial category of perfect } A\text{-modules in complexes } \mathcal{E}$$

such that $\text{Ext}^i_A(\mathcal{E}, \mathcal{E}) = 0$ for $i < -n$

which classifies perfect $k$-modules in complexes with trivial Ext groups in higher negative degrees. Such a moduli problems has been deeply studied by many authors (see for Example [24], [34] and [38]) and it is now known to induce a truncated derived geometric stack $\mathbb{R}\text{Perf}^n_k$ parametrizing perfect $k$-modules in complexes (see [6] Section 3 for a detailed construction).

**Remark 4.2.** The derived deformation functor $\mathbb{R}\text{Def}_{\mathbb{R}\Gamma(X, \Omega_X/k)}$ is the formal neighbourhood of the derived stack $\mathbb{R}\text{Perf}^{2d}_k$ at $\mathbb{R}\Gamma(X, \Omega_X/k)$.

In the same fashion, it has been shown that the simplicial moduli functor

$$\mathfrak{M}_{\text{filt}}^n : \mathfrak{M}_k \longrightarrow \mathfrak{s}\text{Cat}$$

$$A \mapsto \mathfrak{M}(A) := \text{simplicial category of filtered } A\text{-modules in complexes } (\mathcal{E}, \mathcal{P}^\bullet)$$

such that:

- $\mathcal{P}^\bullet$ has finite length
- $\mathcal{P}^p \mathcal{E}$ is perfect for all $p$
- $\text{Ext}_A^i(\text{Rees}(\mathcal{E}, \mathcal{P}^\bullet), \text{Rees}(\mathcal{E}, \mathcal{P}^\bullet))^G = 0$ for $i < -n$

induces a derived geometric truncated stack $\mathbb{R}\text{Filt}^n_k$ classifying filtered perfect $k$-modules in complexes (see [6] Section 2.3 for a detailed construction)

**Remark 4.3.** The derived deformation functor $\mathbb{R}\text{Def}_{\mathbb{R}\Gamma(X, \Omega_X/k), \mathbb{R}\Gamma(X, \Omega_X^* /k)}$ is the formal neighbourhood of $\mathbb{R}\text{Filt}^{2d}_k$ at the pair $\left(\mathbb{R}\Gamma(X, \Omega_X^* /k), \mathcal{F}^\bullet\right)$.

**Remark 4.4.** Fix $W$ to be a complex of finite-dimensional $k$-vector spaces and, as done in [6] Section 2.4, consider the derived geometric stack

$$\mathcal{D}\text{Flag}^{2d}_k(W) := \text{holim} \left( \mathbb{R}\text{Filt}^{2d}_k \xrightarrow{(\mathcal{E}, \mathcal{P}^\bullet) \mapsto \mathcal{E}} \mathbb{R}\text{Perf}^{2d}_k \xrightarrow{\text{const}_W} \right)$$

26There are many interesting geometric substacks of it: see [34] Section 3 for more details.

27The symbol Rees stands for the Rees functor: for more details see [6] Section 1.5.
where \(^\text{const}_W\) denotes the constant morphism sending any filtered complex to \(W\); Remark 4.2 and Remark 4.3 together with Remark 3.28 imply that the derived deformation functor \(\text{hoFlag}^F\) is the formal neighbourhood of the derived stack \(\mathcal{D}\text{Flag}^{2d}_k\left(\mathbb{R}\Gamma\left(X, \Omega^*_X/k\right)\right)\) at \(\left(\mathbb{R}\Gamma\left(X, \Omega^*_X/k\right), F^*\right)\), i.e. at the point in the moduli stack determined by the Hodge filtration.

The derived stacks \(\mathcal{D}\text{Sch}_{d/k}\) and \(\mathcal{D}\text{Flag}^{2d}_k\left(\mathbb{R}\Gamma\left(X, \Omega^*_X/k\right)\right)\) allow us to define a global version of the period map.

**Definition 4.5.** Define the (universal) global period map to be the morphism of derived stacks

\[
\mathbb{R}P : \mathcal{D}\text{Sch}_{d/k} \longrightarrow \mathcal{D}\text{Flag}^{2d}_k\left(\mathbb{R}\Gamma\left(X, \Omega^*_X/k\right)\right)
\]

\[
Y \mapsto \left(\mathbb{R}\Gamma\left(\pi^0Y, \Omega^*_Y/A\right), F^*\right)
\]

for all \(A \in \mathfrak{dgAlg}_{\leq 0}^\leq\).

The following result completes our study of the period map.

**Theorem 4.6.** The diagram of derived stacks and derived deformation functors

\[
\begin{array}{ccc}
\mathcal{D}\text{Sch}_{d/k} & \xrightarrow{\mathbb{R}P} & \mathcal{D}\text{Flag}^{2d}_k\left(\mathbb{R}\Gamma\left(X, \Omega^*_X/k\right)\right) \\
\downarrow \text{formal neighbourhood inclusion} & & \downarrow \text{formal neighbourhood inclusion} \\
\mathbb{R}\text{Def}_X & \xrightarrow{\mathbb{R}P} & \text{hoFlag}^F\left(X, \Omega^*_X/k\right)
\end{array}
\]

is well-defined and commutes.

**Proof.** The fact that the diagram is well-defined is precisely the content of Remark 4.1 and Remark 4.4; the commutativity is readily verified just walking along the arrows, as done in the proof of Theorem 3.37. \(\square\)

**Notations and conventions**

- If \(i \geq 0\) \(\Delta^i\) is the \(i\)-th standard simplicial simplex
- \(\text{diag}(-) = \text{diagonal of a bisimplicial set}\)
- \(k = \text{fixed field of characteristic } 0\), unless otherwise stated
- If \(A\) is a (possibly differential graded) local Artin ring, \(m_A\) will be its unique maximal (possibly differential graded) ideal
- \(R = \text{fixed (possibly differential graded) commutative unital } k\)-algebra, unless otherwise stated
- If \(R\) is a commutative unital ring then \(\underline{R}\) is the constant sheaf of stalk \(R\)
If \((V^*, d)\) is a cochain complex (in some suitable category) then \((V[n]^*, d[n])\) will be the cochain complex such that \(V[n]^j := V^{j+n}\) and \(d[n]^j = d^{j+n}\).

- \(\mathcal{G}_m\) = multiplicative group scheme over \(k\)
- \(X\) = smooth proper scheme over \(k\) of finite dimension, unless otherwise stated
- \(\mathcal{O}_X\) = structure sheaf of \(X\)
- \(\mathcal{T}_X\) = tangent sheaf of \(X\)
- \(\mathcal{A}_X^{0,*}\) = “Dolbeaut” complex of \(X\)
- \(\mathcal{A}_X^{*,*}\) = double complex of \(\bar{k}\)-valued forms on \(X\)
- \(\Omega^*_X/k\) = algebraic De Rham complex of \(X\)
- \(F^*\) = Hodge filtration on \(\Omega^*_X/k\) or cohomology, unless otherwise stated
- \(\mathcal{L}^{X/k}\) = (absolute) cotangent complex of \(X\) over \(k\)
- \(\mathfrak{Sh}(X)\) = category of sheaves of abelian groups over \(X\)
- \(\mathcal{D}(X)\) = derived category of \(X\)
- \(\Delta\) = category of finite ordinal numbers
- \(\mathcal{A}\mathfrak{lg}_k\) = category of commutative associative unital algebras over \(k\)
- \(\mathcal{A}\mathfrak{ff}_k\) = category of (linear) affine spaces over \(k\)
- \(\mathcal{A}\mathfrak{rt}_k\) = category of local Artin algebras over \(k\)
- \(\mathcal{C}^{\geq 0}(\mathfrak{vect}_k)\) = model category of chain complexes of vector spaces over \(k\) in non-negative degrees
- \(\mathcal{C}^{\geq 0}(\mathfrak{aff}_k)\) model category of (chain) \(\mathfrak{dg}\geq 0\)-affine spaces over \(k\)
- \(\mathfrak{Def}_{\mathfrak{h}}^{\mathfrak{h}}\) = \(\infty\)-category of Hinich derived deformation functors (over \(k\))
- \(\mathfrak{Def}_{\mathfrak{m}}^{\mathfrak{m}}\) = \(\infty\)-category of Manetti extended deformation functors (over \(k\))
- \(\mathfrak{dg}_{\geq 0}\mathcal{A}\mathfrak{lg}_k\) = model category of (chain) differential graded commutative algebras over \(k\) in non-negative degrees
- \(\mathfrak{dg}_{\geq 0}\mathcal{C}\mathfrak{at}_k\) = model category of (chain) differential graded categories over \(k\)
- \(\mathfrak{dg}_{\geq 0}\mathcal{C}\mathfrak{at}_k^{\mathfrak{aff}}\) = \(\infty\)-category of affine (chain) differential graded categories over \(k\)
- \(\mathfrak{dg}_{\geq 0}\mathcal{G}\mathfrak{rp}\mathfrak{d}_k^{\mathfrak{aff}}\) = \(\infty\)-category of affine (chain) differential graded groupoids over \(k\)
- \(\mathfrak{dg}_{\leq 0}\mathcal{A}\mathfrak{lg}_k^{\leq 0}\) = model category of (cochain) differential graded commutative algebras over \(k\) in non-positive degrees
- \(\mathfrak{dg}_{\leq 0}\mathcal{A}\mathfrak{rt}_k\) = model category of (cochain) differential graded local Artin algebras over \(k\)
- \(\mathfrak{dg}_{\leq 0}\mathcal{A}\mathfrak{rt}_k^{\leq 0}\) = model category of (cochain) differential graded local Artin algebras over \(k\) in non-positive degrees
- \(\mathfrak{dg}_{\leq 0}\mathcal{L}\mathfrak{ie}_k\) = model category of (cochain) differential graded Lie algebras over \(k\)
- \(\mathfrak{dg}_{\leq 0}\mathfrak{M}\mathfrak{od}_R\) = model category of \(R\)-modules in (cochain) complexes
- \(\mathfrak{dg}_{\leq 0}\mathfrak{vect}_k^{\leq 0}\) = model category of (cochain) differential graded vector spaces over \(k\) in non-positive degrees
• $\mathcal{FdoMod}_R = \text{model category of filtered } R\text{-modules in (cochain) complexes}$
• $\mathcal{Grpd} = \text{2-category of groupoids}$
• $\mathcal{Set} = \text{category of sets}$
• $\mathcal{Sch}_k = \text{category of schemes over } k$
• $\mathcal{sAff}_k = \text{model category of simplicial affine spaces over } k$
• $\mathcal{sAlg}_k = \text{model category of simplicial commutative associative unital algebras over } k$
• $\mathcal{sCat} = \text{model category of simplicial categories}$
• $\mathcal{sCat}_k = \infty\text{-category of } k\text{-simplicial categories over } k$
• $\mathcal{sCat}^{\text{aff}}_k = \infty\text{-category of affine simplicial categories over } k$
• $\mathcal{sGrpd} = \text{model category of simplicial groupoids}$
• $\mathcal{sGrpd}^{\text{aff}}_k = \infty\text{-category of affine simplicial groupoids over } k$
• $\mathcal{sSet} = \text{simplicial model category of simplicial sets}$
• $\mathcal{sVect}_k = \text{model category of simplicial vector spaces over } k$
• $\mathcal{Vect}_k = \text{category of vector spaces over } k$

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