Local Convergence for a Regula Falsi-Type Method under Weak Convergence

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Abstract

We present a local convergence analysis for a regula falsi-type method for solving nonlinear equations. In the earlier studies such as hypotheses on the second derivative have been used to show convergence of this method. In this paper we show convergence under hypotheses only on the first derivative. Moreover, we provide a radius of convergence and computable error bounds on the distances involved. Numerical examples are also given in this study.

Keywords: Regula falsi method; Radius of convergence; Local convergence

Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x$ of equation

$$F(x) = 0,$$  (1.1)

where $F : D \subseteq S \to S$ is a nonlinear function, $D$ is a convex subset of $S$ and $S$ is $\mathbb{R}$ or $\mathbb{C}$. Newton-like methods are used for finding solution of (1.1), these methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to find estimates of the radii of convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence of the iterative procedure.

Moreover, there are some disadvantages, since one end-point is kept after step, efficient way of generating a sequence approximating $x$ cannot apply to solve equation (1.1). In the present study, we use hypotheses only on the first derivative in our local convergence analysis. Moreover, we provide a radius of convergence and computable error bounds on the distances $|x_n - x^*|$ not given in [13,14].

In order to include more general methods, we shall study instead of (1.2), method defined for each $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - (1+\gamma)\mathcal{A}_n^{-1}F(x_n)$$  (1.3)

where $x_0$ is an initial point and

$$\mathcal{A}_n = \frac{F(x_n + F(x_n)) - F(x_n)}{F(x_n)} + \frac{F(x_n + F(x_n)) - F(x_n)}{F(x_n)} + 4\mathcal{p}^2F^3(x_n)$$

where $\in S - [-1]$ and $\in S$ are given parameters. Notice that if $\alpha = 1$, we obtain Chen's method (1.2) and if $\gamma = 0$, we obtain Steffensen's method [15]. Notice in particular that the method in (27) is a special case of method (1.2). Other choices of $_{\gamma}$ are possible [16-21]. The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of method (1.3). The numerical examples are given in the concluding Section 3.

Local Convergence

We present the local convergence analysis of method (1.3) in this section. Let $U(v, \rho)$, $\mathcal{U}(v, \rho)$ the open and closed balls in $S$, respectively, with center $v \in S$ and of radius $\rho > 0$.

It is convenient for the local convergence analysis of method (1.3) that follows to define some functions and parameters. Let $L_0 > 0$, $L > 0$, $M > 0$, $\alpha \geq 0$, $\gamma \in \mathbb{R} - [-1]$ and $\mathcal{p} \in \mathbb{R}$ be given parameters. Define

$$f(x) = \begin{cases} 2x\ln x + x^4 - 3x, & x > 0 \\ 0, & x = 0 \end{cases}$$

Then, we have that

$$f'(x) = 2x\ln x + 4x^3 - 3x^2 + x$$

And

$$f''(x) = 2\ln x + 12x^2 - 6x + 3$$

Hence, function $f''$ is unbounded on $D$. Therefore, the results in [9-12] cannot apply to solve equation (1.1). In the present study, we use hypotheses only on the first derivative in our local convergence analysis.
functions on the interval \([0, +\infty)\) by
\[
g_\alpha(t) = L_\alpha(1 + \frac{M_\alpha}{2})t + |\alpha| M_\alpha \sqrt{1 + 4\rho t^2}
\]
and
\[
h_\beta(t) = g_\alpha(t) - 1,
\]
And
\[
g(t) = \frac{LM_\alpha}{2}t + |\alpha| M_\alpha \sqrt{1 + 4\rho t^2}
\]
Suppose that
\[
|\alpha| < \frac{1}{M}
\]
(2.1)
Then, we have by (2.1) that \(h_\beta(0) = |\alpha| M - 1 < 0\). We also get that \(h_\beta(t) \to +\infty \) as \(t \to +\infty\). Hence, function \(h_\beta\) has zero in the interval \((0, +\infty)\) by the Intermediate value theorem. Denote by \(r_0\) the smallest such zero. We have that
\[
r_0 < \frac{1}{L_\alpha}.
\]
Indeed, if \(r_0 > \frac{1}{L_\alpha}\) then
\[
1 - g_\alpha(g_\alpha) < L_\alpha(0) - \frac{M_\alpha}{2} + |\alpha| M_\alpha \sqrt{1 + 4\rho r_0^2} > 1
\]
which is a contradiction. Moreover, define functions on the interval \([0, r_0)\) by
\[
g_\alpha(0) = \frac{L_\alpha}{2(1 - L_\alpha)} + |\gamma| + g_\alpha(L_\alpha - 1 - g_\alpha(0))
\]
And
\[
h_\beta(t) = (1 - g_\alpha(t))L_\alpha + 2(1 - L_\alpha t) |\gamma| + g_\alpha(t) - 2(1 - L_\alpha t)(1 - g_\alpha(t))
\]
Suppose that
\[
|\gamma| < 1 - 2 |\alpha| M
\]
(2.2)
And
\[
|\gamma| < \frac{1}{2M}
\]
(2.3)
Then, we have by (2.2), (2.3) that \(g_\alpha(0) = g_\alpha(0) = |\alpha| M\) and
\[
h_\beta(0) = 2(|\gamma| + g_\alpha(0) - (1 - g_\alpha(0))) = 2(|\gamma| + M) < 0
\]
Moreover, we have that \(h_\beta(t_0) = 2(|\gamma| (1 - L_\alpha t_0) + g_\alpha(t_0)) > 0\) since \(g_\alpha(t_0) > 0\). Hence function \(h_\beta\) has zeros in the interval \((0, r_0)\). Denote by \(r_1\) the smallest such zero. Define parameter
\[
r = \frac{2}{L_\alpha + L} < \frac{1}{L_\alpha}
\]
(2.4)
Then, we have that
\[
h_\beta(t) = (1 - g_\alpha(t))(L_\alpha - 2(1 - L_\alpha r) + 2(1 - L_\alpha t) + g_\alpha(t))
\]
\[
= (|\gamma| (1 - L_\alpha r) + g_\alpha(t)) > 0
\]
Since \(L_\alpha r - 2(1 - L_\alpha r) = 0\) by (2.4) \(L_\alpha r < 1\) and \(g_\alpha(r) > 0\). Hence, we obtain that
\[
r_1 < r
\]
(2.5)
Then, we conclude that for each \(t \in [0, r_1)\)
\[
0 \leq g_\alpha(t) < 1
\]
(2.6)
\[ x_1 = x_0 + F(y) - \theta(x_0 - x_1) \]

where we also used that
\[ \theta(x_0 - x_1) \in U(x_0, t \theta) \]

since \( x_0 + F(x_0) \in U(x_0, t \theta) \) where also used that \( x' + \theta(x_0 - x') \in U(x_0, t \theta) \) since
\[ |x' + \theta(x_0 - x') - x' + \theta| \leq |x_0 - x'| \leq |x_0 - x'| < \epsilon \]

We can write
\[ A_0 - F(x_0) = \frac{F(x_0 + F(x_0) - F(x_0))}{F(x_0)} \]

and (2.20) that
\[ A_0 - F(x_0) \leq \frac{1}{1 - g_1(x_0 - x')} < 1 \]

which shows (2.15) for \( n = 0 \) and that \( x_1 \) is well defined. Using method (2.11), the definition of function \( g \), (2.13) and (2.19), we get
\[ |F(x') - (A_0 - F(x'))| \leq \frac{1}{1 - g_1(x_0 - x')} < 1 \]

As in (2.20), we can get using (2.6)
\[ |F(x') - (A_0 - F(x'))| \leq \frac{1}{1 - g_1(x_0 - x')} < 1 \]

which shows (2.16) for \( n = 0 \). By simply replacing \( x_0, x_1 \) by \( x_0, x_{n+1} \), in the preceding estimates we arrive at (2.15) and (2.16). Using the estimate
\[ |x_{n+1} - x'| < |x_n - x'| < \epsilon \]

we deduce that \( x_1 \in U(x_0, t \theta) \) and lim \( k \to \infty x_k = x' \). To show the uniqueness part, let \( Q = \int_0^1 F'(x') \, d \theta \) for some \( y' \in U(x_0, t \theta) \) with \( F(y') = 0 \). Using (2.6) we get that
\[ |F(x') - (A_0 - F(x'))| \leq \frac{1}{1 - g_1(x_0 - x')} < 1 \]

It follows from (2.23) that \( Q \) is invertible. Finally, from the identity
\[ 0 = F(x') - F(y') = Q(x' - y') \]

we deduce that \( x' = y' \).

Remark 2.2.1. In view of (2.10) and the estimate
\[ \|F'(x') - F(x')\| \leq \|F'(x') - F(x') + F(x')\| \]

condition (2.13) can be dropped and \( M \) can be replaced by
\[ M(t) = 1 + L_0 t \]

2. The results obtained here can be used for operators \( F \) satisfying autonomous differential equations [22] of the form
\[ F(x) = P(F(x)) \]

where \( P \) is a continuous operator. Then, since \( F(x) = P(F(x)) = 0 \) we can apply the results without actually knowing \( x' \). For example, let \( F \) be an \( e^\lambda \).

3. It is worth noticing that method (1.2) or method (1.3) are not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [23-26]. Moreover, we can compute the
The parameters are given in Table 1.

| R  | 0.3249 |
|----|--------|
| R1 | 0.1603 |
| ξ  | 0.9999 |

Table 1: The parameters.

| R  | 0.6667 |
|----|--------|
| R1 | 0.3859 |
| ξ  | 1      |
| ξ  | 1      |

Table 2: The parameters of Define Function.

Computational order of convergence (COC) defined by

\[ \xi = \log \left( \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|} \right) / \log \left( \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|} \right) \]

or the approximate computational order of convergence

\[ \xi = \log \left( \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|} \right) / \log \left( \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|} \right) \]

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Frechet derivative of operator \( F \) [27].

**Numerical Examples**

We present two numerical examples in this section.

**Example 3.1** Let \( D = [-1, 1] \). Define function \( f \) of \( D \) by

\[ f(x) = e^x - 1. \]

Using (3.1) and \( x^* = 0 \), we get that \( L_0 = 1, L_1 = M_n = M_0 = 1, c, a = 0.1226, \gamma = 0.1667, p = 1 \). The parameters are given in Table 1.

**Example 3.2** Let \( D = [-\infty, +\infty] \). Define function \( f \) of \( D \) by

\[ f(x) = \sin(x). \]

Then we have for \( x^* = 0 \) that \( L_0 = L_n = M_0 = 1, \quad a = 0.3333, \gamma = 0.1667, p = 1 \). The parameters are given in Table 2.

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