ON THE REGULARITY 
OF SCALAR TYPE SPECTRAL $C_0$-SEMIGROUPS

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Abstract. We show that, for the $C_0$-semigroups of scalar type spectral operators, a well-known necessary condition for the generation of eventually norm-continuous $C_0$-semigroups, formulated exclusively in terms of the location of the spectrum of the semigroup’s generator in the complex plane, is also sufficient and, in fact, characterizes the generators of immediately norm-continuous such semigroups.

Combining characterizations of the immediate differentiability and the Gevrey ultradifferentiability of scalar type spectral $C_0$-semigroups with the generation theorem, found earlier by the author, we arrive at respective characterizations of the generation of such semigroups.

We further establish characterizations of the generation of eventually differentiable and immediately compact scalar type spectral $C_0$-semigroups also in terms of the generator’s spectrum and show that, for such semigroups, eventual compactness implies immediate.

All the obtained results are instantly transferred to the $C_0$-semigroups of normal operators.

1. Introduction

While a Hille–Yosida type characterization of the generation of eventually norm continuous $C_0$-semigroups (see Preliminaries) appears to be still unestablished (see, e.g., [3,11–14], for $C_0$-semigroups on Hilbert spaces, see [10,34] and [12]), well-known is the following necessary condition formulated exclusively in terms of the location of the spectrum of the semigroup’s generator in the complex plane.

Theorem 1.1 (Necessary Condition for the Generation of Eventually Norm-Continuous $C_0$-Semigroups [13, Theorem II.5.3]). If $A$ is a generator of an eventually norm continuous $C_0$-semigroup on a complex Banach space, then, for any $b \in \mathbb{R}$, the set

$$\{ \lambda \in \sigma(A) \mid \text{Re} \lambda \geq b \}$$

($\sigma(\cdot)$ is the spectrum of an operator) is bounded.

To a significant extent, the importance of eventually norm-continuous $C_0$-semigroups is attributed the fact, that such a semigroup $\{T(t)\}_{t \geq 0}$ with generator $A$ is subject to a weak spectral mapping theorem

(WSMT) \[ \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \setminus \{0\}, \ t \geq 0, \]

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∀ is the \textit{closure} of a set) [13, Examples V.2.2], in fact, a stronger \textit{spectral mapping theorem}
\begin{equation}
\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \ t \geq 0,
\end{equation}
\textcolor{blue}{[13, Theorem V.2.8, Proposition V.2.3]}, and with that the following \textit{spectral bound equal growth bound condition}
\begin{equation}
\text{(SBeGB) } s(A) = \omega_0,
\end{equation}
where
\[ s(A) := \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \} \ (s(A) := -\infty \text{ if } \sigma(A) = \emptyset) \]
is the \textit{spectral bound} of the generator and
\begin{equation}
\omega_0 := \inf \{ \omega \in \mathbb{R} \mid \exists M = M(\omega) \geq 1 : \|T(t)\| \leq Me^{\omega t}, \ t \geq 0 \}
\end{equation}
is the \textit{growth bound} of the semigroup [13, Proposition V.2.3, Corollary V.2.10]. The asymptotic behavior of a \( C_0 \)-semigroup satisfying the \textit{spectral bound equal growth bound condition} \textcolor{blue}{(SBeGB)} is governed by the spectral bound of its generator and, in particular, is subject to a \textit{generalized Lyapunov stability theorem} [13, Theorem V.3.6].

Eventually norm-continuous are \( C_0 \)-semigroups with certain regularity properties such as \textit{eventual differentiability}, in particular \textit{analyticity} or \textit{uniform continuity}, or \textit{eventual compactness} [29, 33] (see also [13]).

As is shown in [27], a \textit{scalar type spectral} \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) with generator \( A \) is subject to the following \textit{weak spectral mapping theorem}
\begin{equation}
\sigma(T(t)) = e^{t\sigma(A)}, \ t \geq 0,
\end{equation}
\textcolor{blue}{[27, Theorem 4.1]} (cf. [13, Corollary V.2.12]) and with that \textit{spectral bound equal growth bound condition} \textcolor{blue}{(SBeGB)} without any regularity requirement.

We show that, for the \( C_0 \)-semigroups of \textit{scalar type spectral operators}, the above necessary condition is also sufficient and, in fact, characterizes the generators of \textit{immediately norm continuous} such semigroups.

Combining the characterizations of the \textit{immediate differentiability} and \textit{Gevrey ultradifferentiability} of scalar type spectral \( C_0 \)-semigroups found in [17, 18, 23] (see also [21]) with the \textit{generation theorem} [17, Proposition 3.1], we arrive at respective characterizations of the generation of such semigroups.

We further establish characterizations of the generation of \textit{eventually differentiable} and \textit{immediately compact} scalar type spectral \( C_0 \)-semigroups also in terms of the generator’s spectrum and show that, for such semigroups, eventual compactness implies immediate.

All the obtained results are immediately transferred to the \( C_0 \)-semigroups of \textit{normal operators}.

This work continues the series of papers [17, 18, 21, 23] (see also [19, 22]) on the generation of differentiable and Carleman, in particular Gevrey, ultradifferentiable scalar type spectral \( C_0 \)-semigroups.

2. Preliminaries

Here, we concisely outline preliminaries necessary for our subsequent discourse.
2.1. $C_0$-Semigroups.

**Definition 2.1** (Norm-Continuity).

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space $(X, \| \cdot \|)$ is said to be

- *eventually norm-continuous* if there exists a $t_0 > 0$ such that the operator function
  \[ [t_0, \infty) \ni t \mapsto T(t) \in L(X) \]
  $(L(X)$ is the space of bounded linear operators on $X$) is continuous relative to the operator norm;
- *immediately norm-continuous* if the operator function
  \[ (0, \infty) \ni t \mapsto T(t) \in L(X) \]
  is continuous relative to the operator norm;
- *uniformly continuous* if the operator function
  \[ [0, \infty) \ni t \mapsto T(t) \in L(X) \]
  is continuous relative to the operator norm.

**Definition 2.2** (Differentiability).

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space $(X, \| \cdot \|)$ is said to be

- *eventually differentiable* if there exists a $t_0 > 0$ such that the semigroup is strongly differentiable on $[t_0, \infty)$, i.e., the orbit maps
  \[ [0, \infty) \ni t \mapsto T(t)f \in X \]
  are differentiable on $[t_0, \infty)$ for all $f \in X$;
- *immediately differentiable* if the semigroup is strongly differentiable on $(0, \infty)$, i.e., the orbit maps
  \[ (0, \infty) \ni t \mapsto T(t)f \in X \]
  are differentiable on $(0, \infty)$ for all $f \in X$.

**Remarks 2.1.**

- If a $C_0$-semigroup is strongly differentiable for $t \geq t_0$, it is norm-continuous for $t \geq t_0$ [29, Lemma 2.1] (see also [15, Theorem 10.3.5]).
- An immediately differentiable $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, often referred to as a $C^\infty$-semigroup, is immediately norm-continuous, and furthermore, infinite differentiable on $(0, \infty)$ relative to the operator norm with
  \[ T^{(n)}(t) = A^nT(t), \quad t > 0, \]
  [29, Lemma 2.1].

**Definition 2.3** (Compactness).

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space $(X, \| \cdot \|)$ is said to be

- *eventually compact* if there exists a $t_0 > 0$ such that the operator $T(t_0)$ is compact, and hence, $T(t)$ is compact for all $t \geq t_0$ (see, e.g., [26]);
• immediately compact if the operator $T(t)$ is compact for all $t > 0$.

Remarks 2.2.

• If a $C_0$-semigroup is compact for $t \geq t_0$, it is norm-continuous for $t \geq t_0$ [29, Theorem 3.1].

Thus, an immediately compact $C_0$-semigroup is immediately norm-continuous.

• Observe that a bounded operator $A$ in a complex infinite-dimensional Banach space cannot generate a compact $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, which would be uniformly continuous, and hence, the identity operator $I$ would be compact as the uniform limit of $T(t)$ as $t \to 0^+$. The latter would contradict the infinite-dimensionality of $X$ (see, e.g., [26]).

For a linear operator $A$ generating a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ in a complex Banach space $(X, \| \cdot \|)$ the abstract Cauchy problem

\[
\begin{align*}
\text{(ACP)} \quad & \begin{cases}
y'(t) = Ay(t), \ t \geq 0, \\
y(0) = f \in X,
\end{cases}
\end{align*}
\]

is well-posed, in particular uniquely solvable, in which case the weak solutions (also called mild solutions) of the associated abstract evolution equation

\[
\begin{align*}
\text{(AEE)} \quad & y'(t) = Ay(t), \ t \geq 0,
\end{align*}
\]

are the orbit maps

\[
y(t) = T(t)f, \ t \geq 0,
\]

with $f \in X$ [1, 12, 13], whereas the classical ones are the orbit maps with $f \in D(A)$.

2.2. Scalar Type Spectral Operators.

A scalar type spectral operator is a densely defined closed linear operator $A$ in a complex Banach space $(X, \| \cdot \|)$ with strongly $\sigma$-additive spectral measure (the resolution of the identity) $E_A(\cdot)$, assigning to Borel sets of the complex plane $\mathbb{C}$ projection operators on $X$ and having the operator’s spectrum $\sigma(A)$ as its support, and the associated Borel operational calculus, assigning to each Borel measurable function $F : \sigma(A) \to \mathbb{C}$ ($\mathbb{C} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane) with $E_A(\{\lambda \in \mathbb{C} | F(\lambda) = \infty\}) = 0$ (here and whenever appropriate, 0 designates the zero operator on $X$) a scalar type spectral operator in $X$

\[
F(A) := \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)
\]

with

\[
A = \int_{\sigma(A)} \lambda \, dE_A(\lambda)
\]

[5, 6, 9].

In a complex finite-dimensional space, the scalar type spectral operators are those linear operators on the space, which furnish an eigenbasis for it (see, e.g., [6, 9])
and, in a complex Hilbert space, the scalar type spectral operators are those that are similar to the normal operators \([32]\).

Due to its strong \(\sigma\)-additivity, the spectral measure is uniformly bounded, i.e.,
\[
\exists M \geq 1 \quad \forall \delta \in \mathcal{B}(\mathbb{C}) : \|E_A(\delta)\| \leq M
\]
(\(\mathcal{B}(\mathbb{C})\) is the Borel \(\sigma\)-algebra on \(\mathbb{C}\)) (see, e.g., \([7]\)).

**Remark 2.3.** Here and henceforth, we use the same notation \(\|\cdot\|\) for the operator norm and adhere to this rather common economy of symbols also for the functional norm in the dual space \(X^*\).

By \([9\), Theorem XVIII.2.11 (c)], for a Borel measurable function \(F : \sigma(A) \to \mathbb{C}\), the operator \(F(A)\) is bounded iff
\[
E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)| < \infty,
\]
in which case, we have the estimates
\[
E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)| \leq \|F(A)\| \leq 4M E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)|,
\]
where \(M \geq 1\) is from (2.2).

In particular, this implies that a scalar type spectral operator is bounded iff its spectrum \(\sigma(A)\) is a bounded set.

For arbitrary Borel measurable function \(F : \mathbb{C} \to \mathbb{C}\), \(f \in D(F(A))\), \(g^* \in X^*\), and Borel set \(\delta \subseteq \mathbb{C}\),
\[
\int_{\delta} |F(\lambda)| \, dv(f, g^*, \lambda) \leq 4M \|E_A(\delta)F(A)f\|\|g^*\|,
\]
where \(v(f, g^*, \cdot)\) is the total variation measure of the complex-valued Borel measure \(\langle E_A(\cdot)f, g^* \rangle\), for which
\[
v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \leq 4M \|f\|\|g^*\|,
\]
where \(M \geq 1\) in (2.4) and (2.5) is from (2.2) (see, e.g., \([18, 19]\)).

The following statement allows to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures.

**Proposition 2.1** (**[16, Proposition 3.1]**).

*Let \(A\) be a scalar type spectral operator in a complex Banach space \(X\) with spectral measure \(E_A(\cdot)\) and \(F : \sigma(A) \to \mathbb{C}\) be a Borel measurable function. Then \(f \in D(F(A))\) iff*

\[
\begin{align*}
(i) \quad & \text{for each } g^* \in X^*, \int_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) < \infty \quad \text{and} \\
(ii) \quad & \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid |F(\lambda)| > n\}} |F(\lambda)| \, dv(f, g^*, \lambda) \to 0, \quad n \to \infty,
\end{align*}
\]
where $X^*$ is the dual space and $v(f, g^*, \cdot)$ is the total variation measure of $\langle E_A(\cdot)f, g^* \rangle$.

In [17], it is shown that, for scalar type spectral $C_0$-semigroups, the classical Hille-Yosida-Feller-Miyadera-Phillips generation theorem (see, e.g., [13, Theorems II.3.5 and II.3.8]) acquires the following inherently qualitative form exclusively in terms of the location of the generator's spectrum in the complex plane, void of any estimates on the generator’s resolvent.

**Theorem 2.1** (Generation Theorem [17, Proposition 3.1]).

A scalar type spectral operator $A$ in a complex Banach space with spectral measure $E_A(\cdot)$ generates a $C_0$-semigroup (of scalar type spectral operators) $\{T(t)\}_{t \geq 0}$ iff there exists an $\omega \in \mathbb{R}$ such that

$$\sigma(A) \subseteq \{ \lambda \in \mathbb{C} | \text{Re} \lambda \leq \omega \}$$

in which case

$$T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} \, dE_A(\lambda), \ t \geq 0,$$

Furthermore, scalar type spectral $C_0$-semigroups are generated by scalar type spectral operators [2,28].

In [17,18,21,23], characterizations of the analyticity, immediate differentiability, and Carleman, in particular Gevrey ultradifferentiability, of scalar type spectral $C_0$-semigroups are found also purely in terms of the location of the generator’s spectrum in the complex plane. In particular, the characterizations analyticity and immediate differentiability, similarly to the above generation theorem, are obtained from their general Hille–Yosida type counterparts [29,33] (see also [4]) by dropping the corresponding estimates on the generator’s resolvent.

### 2.3. Finer Spectrum Structure.

The spectrum $\sigma(A)$ of a closed linear operator $A$ in a complex Banach space $(X, \| \cdot \|)$ is partitioned into the following pairwise disjoint subsets:

- $\sigma_p(A) := \{ \lambda \in \mathbb{C} | A - \lambda I \text{ is not one-to-one}, \ i.e., \ \lambda \text{ is an eigenvalue of } A \}$,
- $\sigma_c(A) := \{ \lambda \in \mathbb{C} | A - \lambda I \text{ is one-to-one and } R(A - \lambda I) \neq X, \text{ but } R(A - \lambda \overline{I}) = X \}$,
- $\sigma_r(A) := \{ \lambda \in \mathbb{C} | A - \lambda I \text{ is one-to-one and } R(A - \lambda I) \neq X \}$(\text{R(\cdot) is the range of an operator), called the point, continuous, and residual spectrum of } A, \text{ respectively (see, e.g., [26])}.

For a scalar type spectral operator $A$,

$$\sigma_r(A) = \emptyset$$

[24, Corollary 3.1] (see also [20]).
3. Norm Continuity

**Theorem 3.1** (Characterization of the Generation of Immediately Norm-Continuous Scalar Type Spectral $C_0$-Semigroups). A scalar type spectral operator $A$ in a complex Banach space generates an immediately norm continuous $C_0$-semigroup (of scalar type spectral operators) iff, for any $b \in \mathbb{R}$, the set
\begin{equation}
\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq b \}
\end{equation}
is bounded.

**Proof.**

"Only if" part immediately follows from the *Necessary Condition for the Generation of Eventually Norm-Continuous $C_0$-Semigroups* (Theorem 1.1).

"If" part. Suppose that a scalar type spectral operator $A$ in a complex Banach space with spectral measure $E_A(\cdot)$ satisfies condition (3.7). Then there exists an $\omega \in \mathbb{R}$ such that
\[ \sigma(A) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \leq \omega \}, \]
and hence, by the *Generation Theorem* (Theorem 2.1), $A$ generates a scalar type spectral $C_0$-semigroup
\[ T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \quad t \geq 0. \]

The premise also implies that, for each $n \in \mathbb{N}$ ($\mathbb{N} := \{1, 2, \ldots\}$ is the set of natural numbers), by the properties of the Borel operational calculus [9, Theorem XVIII.2.11 (f)],
\[ A_n := AE_A (\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq -n \}) = \int_{\sigma(A)} \lambda \chi_{\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq -n \}}(\lambda) dE_A(\lambda) \]
($\chi_\delta(\cdot)$ is the characteristic function of a set $\delta \subseteq \mathbb{C}$) is a bounded scalar type spectral operator since its spectrum $\sigma(A_n)$ is a bounded set (see Preliminaries). Indeed, by the *A.E. Weak Spectral Inclusion and Mapping Theorem* ([27, Theorem 3.1]), we have the inclusion
\[ \sigma(A_n) \subseteq \{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq -n \} \cup \{0\} \]
(see also [25]).

Thus, for each $n \in \mathbb{N}$, $A_n$ generates the uniformly continuous semigroup $\{ e^{tA_n} \}_{t \geq 0}$ of its exponentials (see Preliminaries).

By the properties of the Borel operational calculus [9, Theorem XVIII.2.11],
\begin{align*}
\int_{\sigma(A)} e^{t\lambda \chi_{\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq -n \}}(\lambda)} dE_A(\lambda) & = \int_{\sigma(A)} e^{t\lambda} \chi_{\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq -n \}}(\lambda) dE_A(\lambda) \\
& + \int_{\sigma(A)} \chi_{\{ \lambda \in \sigma(A) \mid \text{Re}\lambda < -n \}}(\lambda) dE_A(\lambda) \\
& = e^{tA} E_A (\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq -n \}) + E_A (\{ \lambda \in \sigma(A) \mid \text{Re}\lambda < -n \}), \quad t \geq 0.
\end{align*}
Hence, for each \( n \in \mathbb{N} \), by the properties of the Borel operational calculus \([9, \text{Theorem XVIII.2.11 (g)}]\), the operator function
\[
[0, \infty) \ni t \mapsto E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) e^{tA_n}
\]
\[
= E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) e^{tA} E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right)
\]
\[
= e^{tA} E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) \in L(X)
\]
is continuous relative to the operator norm on \([0, \infty)\).

For arbitrary \( t_0 > 0 \) and \( n \in \mathbb{N} \), we have:
\[
\sup_{t \geq t_0} \| T(t) - E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) e^{tA_n} \|
\]
\[
= \sup_{t \geq t_0} \| e^{tA} - E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) e^{tA_n} \|
\]
\[
= \sup_{t \geq t_0} \| e^{tA} - e^{tA} E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) \|
\]
\[
= \sup_{t \geq t_0} \| e^{tA} E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) \|
\]
by \((2.3)\);
\[
= \sup_{t \geq t_0} \left\| \int_{\{ \lambda \in \sigma(A) \mid \Re \lambda < -n \}} e^{tA} dE_A(\lambda) \right\| 
\leq 4M \sup_{t \geq t_0} \sup_{\{ \lambda \in \sigma(A) \mid \Re \lambda < -n \}} \| e^{tA} \|
\]
\[
= 4M \sup_{t \geq t_0} \sup_{\{ \lambda \in \sigma(A) \mid \Re \lambda < -n \}} e^{t \Re \lambda} = 4M e^{-t_0 n} \rightarrow 0, \ n \rightarrow \infty,
\]
where the constant \( M \geq 1 \) is from \((2.2)\).

Thus, for any \( t_0 > 0 \), the sequence of operator functions
\[
\left( E_A \left( \{ \lambda \in \sigma(A) \mid \Re \lambda \geq -n \} \right) e^{tA} \right)_{n \in \mathbb{N}},
\]
continuous relative to the operator norm on \([0, \infty)\), converges uniformly to \( T(\cdot) \) in the operator norm on \([t_0, \infty)\). This implies immediate norm continuity for the semigroup \( \{ T(t) \}_{t \geq 0} \) and completes the proof. \( \square \)

We directly obtain the following

**Corollary 3.1** (Characterization of Immediately Norm-Continuous Scalar Type Spectral \( C_0 \)-Semigroups). A \( C_0 \)-semigroup (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator \( A \) is immediately norm-continuous iff, for any \( b \in \mathbb{R} \), the set
\[
\{ \lambda \in \sigma(A) \mid \Re \lambda \geq b \}
\]
is bounded.

By the **Necessary Condition for the Generation of Eventually Norm-Continuous \( C_0 \)-Semigroups** (Theorem 1.1), we also instantly arrive at

**Corollary 3.2** (Equivalence of Eventual and Immediate Norm-Continuity). If a scalar type spectral \( C_0 \)-semigroup in a complex Banach space is eventually norm-continuous, it is immediately norm-continuous.
4. Differentiability

Combining the characterizations of the immediate differentiability and Gevrey ultradifferentiability, in particular analyticity, of scalar type spectral $C_0$-semigroups found in [17, 18, 23] (see also [21]) with the Generation Theorem (Theorem 2.1) for such semigroups, we instantly obtain the subsequent characterizations of the generation of such semigroups.

**Theorem 4.1** (Characterization of the Generation of Immediately Differentiable Scalar Type Spectral $C_0$-Semigroups). A scalar type spectral operator $A$ in a complex Banach space generates an immediately differentiable $C_0$-semigroup (of scalar type spectral operators) iff there exists $\omega \in \mathbb{R}$ and, for any $b > 0$, there exists an $a \in \mathbb{R}$ such that

$$\sigma(A) \subseteq \{ \lambda \in \mathbb{C} \mid \Re \lambda \leq \min (\omega, a - b \ln |\Im \lambda|) \}.$$

See [18] for details (cf. [29]).

**Theorem 4.2** (Characterization of the Generation of Roumieu-type Gevrey Ultradifferentiable Scalar Type Spectral $C_0$-Semigroups). Let $1 \leq \beta < \infty$. A scalar type spectral operator $A$ in a complex Banach space generates a $\beta$th-order Roumieu-type Gevrey ultradifferentiable on $(0, \infty)$ $C_0$-semigroup (of scalar type spectral operators) iff there exist $b > 0$ and $a \in \mathbb{R}$ such that

$$\sigma(A) \subseteq \left\{ \lambda \in \mathbb{C} \mid \Re \lambda \leq a - b |\Im \lambda|^{1/\beta} \right\}.$$

See [18, 21] (cf. [4]).

In particular, for $\beta = 1$, we arrive at

**Corollary 4.1** (Characterization of the Generation of Analytic Scalar Type Spectral $C_0$-Semigroups). A scalar type spectral operator $A$ in a complex Banach space generates an analytic $C_0$-semigroup (of scalar type spectral operators) iff there exist $b > 0$ and $a \in \mathbb{R}$ such that

$$\sigma(A) \subseteq \{ \lambda \in \mathbb{C} \mid \Re \lambda \leq a - b |\Im \lambda| \}.$$

See [17] (cf. [33]).

A similar characterization of the generation of the $\beta$th-order Beurling-type Gevrey ultradifferentiable on $(0, \infty)$ scalar type spectral $C_0$-semigroups ($1 < \beta < \infty$) is found in [23].

For characterizations of the generation of the Carleman ultradifferentiable scalar type spectral $C_0$-semigroups, see [21, 23].

We obtain the following characterization of the generation of eventually differentiable scalar type spectral $C_0$-semigroups dropping the estimate on the resolvent of the semigroup’s generator in its general Hille–Yosida type counterpart [29, Theorem 2.5].
Theorem 4.3 (Characterization of the Generation of Eventual Differentiable Scalar Type Spectral $C_0$-Semigroups). A scalar type spectral operator $A$ in a complex Banach space generates an eventually differentiable $C_0$-semigroup \( \{T(t)\}_{t \geq 0} \) (of scalar type spectral operators) iff there exist $\omega \in \mathbb{R}$, $b > 0$, and $a \in \mathbb{R}$ such that
\[
\sigma(A) \subseteq \{ \lambda \in \mathbb{C} | \Re \lambda \leq \min(\omega, a - b \ln |\Im \lambda|) \},
\]
in which case the orbit maps \([0, \infty) \ni t \mapsto T(t)f \in X\) are differentiable on \([b^{-1}, \infty)\) for all $f \in X$.

Proof.

“Only if” part immediately follows from [29, Theorem 2.5].

“If” part. Suppose that, for a scalar type spectral operator $A$ in a complex Banach space \((X, \| \cdot \|)\) with spectral measure $E_A(\cdot)$, inclusion (4.8) holds with some $\omega \in \mathbb{R}$, $b > 0$, and $a \in \mathbb{R}$.

To prove that the $C_0$-semigroup of exponentials \(\{e^{tA}\}_{t \geq 0}\) generated by $A$ (see Preliminaries) is eventually differentiable is to show that
\[
\exists t_0 > 0 : e^{t_0A}X \subseteq D(A)
\]
[29].

For arbitrary $f \in X$ and $g^* \in X^*$,
\[
\int_{\sigma(A)} |\lambda| e^{b^{-1} \Re \lambda} dv(f, g^*, \lambda) = \int_{\{\lambda \in \sigma(A) | \Re \lambda < 0\}} |\lambda| e^{b^{-1} \Re \lambda} dv(f, g^*, \lambda)
\]
\[
+ \int_{\{\lambda \in \sigma(A) | \Re \lambda \geq 0\}} |\lambda| e^{b^{-1} \Re \lambda} dv(f, g^*, \lambda).
\]

Indeed,
\[
\int_{\{\lambda \in \sigma(A) | \Re \lambda \geq 0\}} |\lambda| e^{b^{-1} \Re \lambda} dv(f, g^*, \lambda)
\]
due to the boundedness of the set \(\{ \lambda \in \sigma(A) | \Re \lambda \geq 0 \}\),

which follows from inclusion (4.8), the continuity of the integrated function on \(\mathbb{C}\), and the finiteness of the measure $v(f, g^*, \cdot)$.

Further, for any $f \in X$ and $g^* \in X^*$,
\[
\int_{\{\lambda \in \sigma(A) | \Re \lambda < 0\}} |\lambda| e^{b^{-1} \Re \lambda} dv(f, g^*, \lambda) \leq \int_{\sigma(A)} (|\Re \lambda| + |\Im \lambda|) e^{b^{-1} \Re \lambda} dv(f, g^*, \lambda)
\]

since, by (4.8), for $\lambda \in \{ \lambda \in \sigma(A) | \Re \lambda < 0 \}$,
\[
\Re \lambda < 0 \text{ and } |\Im \lambda| \leq e^{b^{-1}(a - \Re \lambda)};
\]
\[
\leq \int_{\{\lambda \in \sigma(A) \mid Re \lambda < 0\}} \left[ -Re \lambda + e^{b^{-1}(a-Re \lambda)} \right] e^{b^{-1}Re \lambda} dv(f, g^*, \lambda) 
\]

since \(x \leq e^x, \ x \geq 0\);

\[
\leq \int_{\{\lambda \in \sigma(A) \mid Re \lambda < 0\}} \left[ be^{b^{-1}(-Re \lambda)} + e^{ab^{-1}}e^{b^{-1}(-Re \lambda)} \right] e^{b^{-1}Re \lambda} dv(f, g^*, \lambda)
\]

\[
= \left[ b + e^{ab^{-1}} \right] \int_{\{\lambda \in \sigma(A) \mid Re \lambda < 0\}} 1 dv(f, g^*, \lambda) \leq \left[ b + e^{ab^{-1}} \right] \int_{\sigma(A)} 1 dv(f, g^*, \lambda)
\]

\[
= \left[ b + e^{ab^{-1}} \right] v(f, g^*, \sigma(A)) \quad \text{by (2.5)};
\]

\[
\leq 4M \left[ b + e^{ab^{-1}} \right] \|f\| \|g^*\|.
\]

Also, for any \(f \in X\),

\[
(4.11) \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid |\lambda| e^{b^{-1}Re \lambda} > n\}} |\lambda| e^{b^{-1}Re \lambda} dv(f, g^*, \lambda)
\]

\[
\leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid Re \lambda < 0, \ |\lambda| e^{b^{-1}Re \lambda} > n\}} |\lambda| e^{b^{-1}Re \lambda} dv(f, g^*, \lambda)
\]

\[
+ \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid Re \lambda > 0, \ |\lambda| e^{b^{-1}Re \lambda} > n\}} |\lambda| e^{b^{-1}Re \lambda} dv(f, g^*, \lambda)
\]

\[
\to 0, \ n \to \infty.
\]

Indeed, since, due to the boundedness of the set

\[
\{\lambda \in \sigma(A) \mid Re \lambda \geq 0\}
\]

and the continuity of the integrated function on \(C\), the set

\[
\left\{ \lambda \in \sigma(A) \mid Re \lambda \geq 0, \ |\lambda| e^{b^{-1}Re \lambda} > n \right\}
\]

is empty for all sufficiently large \(n \in \mathbb{N}\), we immediately infer that

\[
\lim_{n \to \infty} \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid Re \lambda \geq 0, \ |\lambda| e^{b^{-1}Re \lambda} > n\}} |\lambda| e^{b^{-1}Re \lambda} dv(f, g^*, \lambda) = 0.
\]

Finally, as in (4.10), for an arbitrary \(f \in X\), we have:

\[
\sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid Re \lambda < 0, \ |\lambda| e^{b^{-1}Re \lambda} > n\}} |\lambda| e^{b^{-1}Re \lambda} dv(f, g^*, \lambda)
\]

\[
\leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left[ b + e^{ab^{-1}} \right] \int_{\{\lambda \in \sigma(A) \mid Re \lambda < 0, \ |\lambda| e^{b^{-1}Re \lambda} > n\}} 1 dv(f, g^*, \lambda) \quad \text{by (2.4)};
\]
\[ \leq \left[ b + e^{ab^{-1}} \right]. \]

\[ \sup_{g^* \in X^* \mid \|g^*\| = 1} 4M \| E_A(\{ \lambda \in \sigma(A) \mid \text{Re} \lambda < 0, |\lambda|e^{b^{-1}\text{Re} \lambda} > n \}) f \| \|g^*\| \]

\[ = 4M \left[ b + e^{ab^{-1}} \right] \| E_A(\{ \lambda \in \sigma(A) \mid \text{Re} \lambda < 0, |\lambda|e^{b^{-1}\text{Re} \lambda} > n \}) f \| \]

by the strong continuity of the spectral measure;

\[ \rightarrow 4M \left[ b + e^{ab^{-1}} \right] \| E_A(\emptyset) f \| = 0, \quad n \rightarrow \infty. \]

By Proposition 2.1 and the properties of the operational calculus (see [9, Theorem XVIII.2.11 (f)]), (4.10) and (4.11) jointly imply that,

\[ f \in D \left( Ae^{b^{-1}A} \right), \]

and hence,

\[ e^{b^{-1}A} f \in D(A), \]

i.e., inclusion (4.9) holds for \( t_0 = b^{-1} > 0 \), which implies that the orbit maps

\[ [0, \infty) \ni t \mapsto T(t)f \in X \]

are differentiable on \( [b^{-1}, \infty) \) for all \( f \in X \). \( \square \)

Instantly, we obtain the following

**Corollary 4.2** (Characterization of Eventually Differentiable Scalar Type Spectral \( C_0 \)-Semigroups). A \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator \( A \) is eventually differentiable iff there exist \( b > 0 \) and \( a \in \mathbb{R} \) such that

\[ \sigma(A) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \leq a - b \ln |\text{Im} \lambda| \}, \]

in which case the orbit maps

\[ [0, \infty) \ni t \mapsto T(t)f \in X \]

are differentiable on \( [b^{-1}, \infty) \) for all \( f \in X \).

Since an eventually differentiable \( C_0 \)-semigroup is eventually norm-continuous, by the **Equivalence of Eventual and Immediate Norm-Continuity** (Corollary 3.2), we also arrive at

**Corollary 4.3** (Eventual Differentiability Implies Immediate Norm-Continuity). If a scalar type spectral \( C_0 \)-semigroup is eventually differentiable, it is immediately norm-continuous.

5. **Compactness**

Let us first prove the following characterization of compactness for the scalar type spectral operators emerging directly from the classical **Riesz-Schauder Theorem** (see, e.g., [26]).
Lemma 5.1 (Characterization of Compactness for Scalar Type Spectral Operators). A scalar type spectral operator $A \neq 0$ in a complex infinite-dimensional Banach space is compact iff the spectrum $\sigma(A)$ of $A$ consists of 0 and a nonempty countable set of nonzero eigenvalues with finite geometric multiplicities.

If the set $\sigma(A) \setminus \{0\}$ is countably infinite, for its arbitrary arrangement $\{\lambda_n\}_{n=1}^{\infty}$ (i.e., $\mathbb{N} \ni n \mapsto \lambda_n \in \sigma(A) \setminus \{0\}$ is a bijection),

$$\lambda_n \to 0, \ n \to \infty.$$

Proof.

"Only if" part follows directly from the Riesz-Schauder Theorem (see, e.g., [26]) in view of the fact that, for a scalar type spectral operator $A \neq 0$ with spectral measure $E_A(\cdot)$, due to the integral representation

$$A = \int_{\sigma(A)} \lambda dE_A(\lambda)$$

(see Preliminaries),

$$\sigma(A) \setminus \{0\} \neq \emptyset.$$

"If" part. Suppose that, a scalar type spectral operator $A \neq 0$ in a complex Banach space $(X, \| \cdot \|)$ with spectral measure $E_A(\cdot)$ is subject to the conditions of the "if" part. Then the spectrum $\sigma(A)$ is a bounded set, which implies that $A$ is bounded (see Preliminaries).

If the set $\sigma(A) \setminus \{0\} \subseteq \sigma_p(A)$

is finite, then

$$\sigma(A) \setminus \{0\} = \{\lambda_1, \ldots, \lambda_n\}$$

with some $n \in \mathbb{N}$. This implies, that

$$A = \int_{\sigma(A)} \lambda dE_A(\lambda) = \sum_{k=1}^{n} \lambda_k E_A(\{\lambda_k\}).$$

Whence, in view of the fact that each $\lambda_k$, $k = 1, \ldots, n$, is an eigenvalue with a finite geometric multiplicity, i.e., the subspace $E_A(\{\lambda_k\})X, k = 1, \ldots, n,$ is finite-dimensional, we infer that the operator $A$ is of finite rank, and hence, compact.

Now suppose that the set $\sigma(A) \setminus \{0\} \subseteq \sigma_p(A)$

is countably infinite and let $\{\lambda_n\}_{n=1}^{\infty}$ is its arbitrary arrangement. Then, by the premise,

$$\lambda_n \to 0, \ n \to \infty.$$

Whence, by the properties of the Borel operational calculus [9], we infer that, for any $n \in \mathbb{N}$,

$$\left\| A - \sum_{k=1}^{n} \lambda_k E_A(\{\lambda_k\}) \right\| = \left\| \int_{\sigma(A) \setminus \{\lambda_1, \ldots, \lambda_n\}} \lambda dE_A(\lambda) \right\|.$$
Therefore, the operator $A$ is compact as the uniform limit of the sequence
$$\left(\sum_{k=1}^{n} \lambda_k E_A(\{\lambda_k\})\right)_{n \in \mathbb{N}}$$
of finite-rank operators (see, e.g., [26]).

With all possible cases considered, the proof of the “if” part, and hence, of the entire statement, is complete. □

**Theorem 5.1** (Characterization of the Generation of Immediately Compact Scalar Type Spectral $C_0$-Semigroups). A scalar type spectral operator $A \neq 0$ in a complex infinite-dimensional Banach space generates an immediately compact $C_0$-semigroup iff $\sigma(A)$ is a countably infinite set of eigenvalues with finite geometric multiplicities, which has no finite limit points and is such that, for its arbitrary arrangement $\{\lambda_n\}_{n=1}^{\infty}$,
$$\Re \lambda_n \to -\infty, \ n \to \infty.$$
and hence,
\[ \text{Re}\lambda_n \to -\infty, \ n \to \infty. \]
Furthermore, for an arbitrary \( \mu \in \rho(A) \),
\begin{equation}
\lambda \in \sigma_p(A) \iff (\lambda - \mu)^{-1} \in \sigma_p(R(\mu, A)),
\end{equation}
in which case
\begin{equation}
\ker(A - \lambda I) = \ker(R(\mu, A) - (\lambda - \mu)^{-1} I)
\end{equation}
(see, e.g., [26]).
This, by the Characterization of Compactness for Scalar Type Spectral Operators (Lemma 5.1), implies that each eigenvalue of \( A \) has a finite geometric multiplicity.

"If" part. Suppose that, a scalar type spectral operator \( A \neq 0 \) in a complex Banach space \((X, \| \cdot \|)\) with spectral measure \( E_A(\cdot) \) is subject to the conditions of the "if" part.

Since the spectrum \( \sigma(A) \) is unbounded, the operator \( A \) is unbounded (see Preliminaries).

By the Characterization of the Generation of Immediately Norm-Continuous Scalar Type Spectral \( C_0 \)-Semigroups (Theorem 3.1), we infer that \( A \) generates an immediately norm-continuous \( C_0 \)-semigroup.

Furthermore, by the Spectral Mapping Theorem for the Resolvent ([13, Theorem V.1.13]), for an arbitrary \( \lambda \in \rho(A) \),
\[ \sigma(R(\lambda, A)) \setminus \{0\} = (\sigma(A) - \lambda)^{-1}, \]
and hence, the spectrum \( \sigma(R(\lambda, A)) \) of \( R(\lambda, A) \) consists of 0 (because \( A \) is unbounded) and a nonempty countably infinite set \( \sigma(R(\lambda, A)) \setminus \{0\} \) such that, for its arbitrary arrangement \( \{\lambda_n\}_{n=1}^\infty \),
\[ \lambda_n \to 0, \ n \to \infty. \]

By (5.12) and (5.13), we also infer that the set set \( \sigma(R(\lambda, A)) \setminus \{0\} \) consists of the eigenvalues of \( R(\lambda, A) \) with finite geometric multiplicities.

By the the Characterization of Compactness for Scalar Type Spectral Operators (Lemma 5.1), we conclude that the resolvent \( R(\lambda, A) \) is compact for all \( \lambda \in \rho(A) \), which along with the immediate norm-continuity of the \( C_0 \)-semigroup generated by \( A \), by [29, Theorem 3.3], implies that the latter is immediately compact. \( \square \)

Instantly, we obtain

Corollary 5.1 (Characterization of Immediately Compact Scalar Type Spectral \( C_0 \)-Semigroups). A \( C_0 \)-semigroup (of scalar type spectral operators) on complex infinite-dimensional Banach space generated by a scalar type spectral operator \( A \) is immediately compact iff \( \sigma(A) \) is a countably infinite set of eigenvalues with finite geometric multiplicities, which has no finite limit points and is such that, for its arbitrary arrangement \( \{\lambda_n\}_{n=1}^\infty \),
\[ \text{Re}\lambda_n \to -\infty, \ n \to \infty. \]
We also have the following analogue of the *Equivalence of Eventual and Immediate Norm-Continuity* (Corollary 3.2).

**Corollary 5.2** (Equivalence of Eventual and Immediate Compactness).
If a scalar type spectral $C_0$-semigroup in a complex infinite-dimensional Banach space is eventually compact, its is immediately compact.

*Proof.* Suppose that a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex infinite-dimensional Banach space generated by a scalar type spectral operator $A \neq 0$ with spectral measure $E_A(\cdot)$ is compact for $t \geq t_0$ with some $t_0 > 0$.

Since $T(t) = e^{tA}, t \geq t_0,$ (see Preliminaries) is a compact operator, in view of the infinite dimensionality of the underlying space, its inverse

$$T^{-1}(t) = e^{-tA} := \int_{\sigma(A)} e^{-t\lambda} dE_A(\lambda)$$

is necessarily unbounded (see, e.g., [26]), which implies that the scalar type spectral operator $A$ has an unbounded, in particular infinite, spectrum $\sigma(A)$, and hence, is unbounded (see Preliminaries).

Since, by the *Weak Spectral Mapping Theorem* ([27, Theorem 4.1]),

(5.14) $$\sigma(T(t)) = e^{t\sigma(A)}, \ t \geq 0,$$

we infer that $\sigma(A)$ has no finite limit points. Otherwise, by the continuity of the exponential function $e^{t\omega}$, the spectrum $\sigma(T(t_0))$ of the compact operator $T(t_0)$ would have a nonzero finite limit point, which is impossible by the *Characterization of Compactness for Scalar Type Spectral Operators* (Lemma 5.1).

Being an infinite set in $\mathbb{C}$ with no finite limit points, the spectrum $\sigma(A)$ is countably infinite.

Further, since $\sigma(A)$ is the support for the spectral measure $E_A(\cdot)$ (see Preliminaries) and consist of isolated points,

$$\forall \lambda \in \sigma(A) : \ E_A(\lambda) \neq 0,$$

and hence,

$$\sigma(A) = \sigma_p(A),$$

[9].

Since the operator $T(t) = e^{tA}$ is compact for each $t \geq t_0$, considering the $2\pi i$-periodicity (i is the imaginary unit) of the exponential function $e^t$, we infer that there exists a $t_1 \geq t_0$ such that the spectrum $\sigma(T(t_1))$ is an infinite set. Indeed, assuming that the set

$$\sigma(T(t_0)) = e^{t_0\sigma(A)}$$

(see (5.14)) is finite, we conclude that the infinite set $\sigma(A)$ contains an infinitely subset consisting of the points of the form

$$\lambda_0 + it_0^{-1}2\pi n, \ n \in \mathbb{Z},$$

with some $\lambda_0 \in \mathbb{C}$, mapped under $e^{t_0}$ into the same point of $\sigma(T(t_0))$. Then, for any $t > t_0$, these points are mapped under $e^t$ into infinitely many pints of

$$\sigma(T(t)) = e^{t\sigma(A)}.$$
ON THE REGULARITY OF SCALAR TYPE SPECTRAL $C_0$-SEMIGROUPS

Let $\{\lambda_n\}_{n=1}^{\infty}$ be an arbitrary arrangement of $\sigma(A)$. Then, by the Characterization of Compactness for Scalar Type Spectral Operators (Lemma 5.1),

$$e^{t_1 \text{Re} \lambda_n} = |e^{t_1 \lambda_n}| \to 0, \; n \to \infty.$$  

Therefore,

$$\text{Re} \lambda_n \to -\infty, \; n \to \infty.$$  

Furthermore, since, for arbitrary $\lambda \in \sigma_p(A)$ and $f \in \ker(A - \lambda I)$,

$$y(t) := e^{t \lambda} f, \; t \geq 0,$$

is the unique eigenvalue solution of the Cauchy problem

$$\begin{cases} y'(t) = Ay(t), \; t \geq 0, \\ y(0) = f, \end{cases}$$

(see Preliminaries), we infer that

$$T(t) f = e^{t \lambda} f, \; t \geq 0,$$

and hence,

$$f \in \ker (T(t) - e^{t \lambda} I), \; t \geq 0,$$

in particular, $f \in \ker (T(t_0) - e^{t_0 \lambda} I)$, which implies that

$$\ker(A - \lambda I) \subseteq \ker (T(t_0) - e^{t_0 \lambda} I).$$

Whence, since, by the Characterization of Compactness for Scalar Type Spectral Operators (Lemma 5.1), $\ker (T(t_0) - e^{t_0 \lambda} I)$ is finite-dimensional, we conclude that each eigenvalue of $A$ has a finite geometric multiplicity.

Thus, by the Characterization of the Generation of Immediately Compact Scalar Type Spectral $C_0$-Semigroups (Theorem 5.1), the $C_0$-semigroup generated by $A$ is immediately compact. \hfill $\Box$

6. The Case of Normal $C_0$-Semigroups

For the important particular case of the $C_0$-semigroups of normal operators, we instantly obtain the following counterparts of the corresponding characterizations established for $C_0$-semigroups of scalar type spectral operators.

**Corollary 6.1** (Characterization of the Generation of Immediately Norm-Continuous Normal $C_0$-Semigroups). A normal operator $A$ in a complex Hilbert space generates an immediately norm continuous $C_0$-semigroup (of normal operators) iff, for any $b \in \mathbb{R}$, the set

$$\{ \lambda \in \sigma(A) \mid \text{Re} \lambda \geq b \}$$

is bounded.

**Remark 6.1.** In view of the fact that, for a normal operator $A$ in a complex Hilbert space,

$$\|R(\lambda, A)\| = \frac{1}{\text{dist}(\lambda, \sigma(A))}, \; \lambda \in \rho(A),$$
where
\[
\text{dist}(\lambda, \sigma(A)) := \inf_{\mu \in \sigma(A)} |\mu - \lambda| > 0,
\]
(see, e.g., [8,30]), the prior corollary is consistent with the known characterizations of immediate and eventual norm continuity of \(C_0\)-semigroups in complex Hilbert spaces ([34, Theorem 1] and [10, Theorem 4], respectively).

**Corollary 6.2** (Characterization of Immediately Norm-Continuous Normal \(C_0\)-Semigroups). A \(C_0\)-semigroup (of normal operators) on a complex Hilbert space generated by a normal operator \(A\) is immediately norm-continuous iff, for any \(b \in \mathbb{R}\), the set
\[
\{ \lambda \in \sigma(A) \mid \text{Re}\lambda \geq b \}
\]
is bounded.

**Corollary 6.3** (Equivalence of Eventual and Immediate Norm-Continuity). If a normal \(C_0\)-semigroup in a complex Hilbert space is eventually norm-continuous, it is immediately norm-continuous.

**Corollary 6.4** (Characterization of the Generation of Roumieu-type Gevrey Ultradifferentiable Normal \(C_0\)-Semigroups). Let \(1 \leq \beta < \infty\). A normal operator \(A\) in a complex Hilbert space generates a \(\beta\)th-order Roumieu-type Gevrey ultradifferentiable on \((0, \infty)\) \(C_0\)-semigroup (of normal operators) iff there exist \(b > 0\) and \(a \in \mathbb{R}\) such that
\[
\sigma(A) \subseteq \left\{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \leq a - b|\text{Im}\lambda|^{1/\beta} \right\}.
\]

**Corollary 6.5** (Characterization of the Generation of Beurling-type Gevrey Ultradifferentiable Normal \(C_0\)-Semigroups). Let \(1 < \beta < \infty\). A normal operator \(A\) in a complex Hilbert space generates a \(\beta\)th-order Beurling-type Gevrey ultradifferentiable on \((0, \infty)\) \(C_0\)-semigroup (of normal operators) iff, for any \(b > 0\), there exists an \(a \in \mathbb{R}\) such that
\[
\sigma(A) \subseteq \left\{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \leq a - b|\text{Im}\lambda|^{1/\beta} \right\}.
\]

Cf. [23, Corollary 4.1].

**Corollary 6.6** (Characterization of the Generation of Eventually Differentiable Normal \(C_0\)-Semigroups). A normal operator \(A\) in a complex Hilbert space generates an eventually differentiable \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) (of normal operators) iff there exist \(\omega \in \mathbb{R}\), \(a > 0\), and \(b > 0\) such that
\[
\sigma(A) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \leq \min (\omega, a - b \ln|\text{Im}\lambda|) \},
\]
in which case the orbits
\[
[0, \infty) \ni t \mapsto T(t)f \in X
\]
are strongly differentiable on \([b^{-1}, \infty)\) for all \(f \in X\).

**Corollary 6.7** (Characterization of Eventually Differentiable Normal \(C_0\)-Semigroups). A \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) (of normal operators) on a complex Hilbert space generated by a normal operator \(A\) is eventually differentiable iff there exist \(a > 0\) and \(b > 0\) such that
\[
\sigma(A) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \leq a - b \ln|\text{Im}\lambda| \},
\]
in which case the orbits

\[ [0, \infty) \ni t \mapsto T(t)f \in X \]

are strongly differentiable on \([b^{-1}, \infty)\) for all \(f \in X\).

**Corollary 6.8** (Characterization of Compactness for Normal Operators).

A normal operator \(A \neq 0\) in a complex infinite-dimensional Hilbert space is compact iff the spectrum \(\sigma(A)\) of \(A\) consists of 0 and a nonempty countable set of nonzero eigenvalues with finite geometric multiplicities.

If the set \(\sigma(A) \setminus \{0\}\) is countably infinite, for its arbitrary arrangement \(\{\lambda_n\}_{n=1}^{\infty}\) (i.e., \(\mathbb{N} \ni n \mapsto \lambda_n \in \sigma(A) \setminus \{0\}\) is a bijection),

\[ \lambda_n \to 0, \ n \to \infty. \]

**Corollary 6.9** (Characterization of the Generation of Immediately Compact Normal \(C_0\)-Semigroups). A normal operator \(A \neq 0\) in a complex infinite-dimensional Hilbert space generates an immediately compact \(C_0\)-semigroup (of normal operators) iff \(\sigma(A)\) is a countably infinite set of eigenvalues with finite geometric multiplicities, which has no finite limit points and is such that, for its arbitrary arrangement \(\{\lambda_n\}_{n=1}^{\infty}\),

\[ \Re \lambda_n \to -\infty, \ n \to \infty. \]

**Corollary 6.10** (Characterization of Immediately Compact Normal \(C_0\)-Semigroups). A \(C_0\)-semigroup (of normal operators) on a complex infinite-dimensional Hilbert space generated by a normal operator \(A\) is immediately compact iff \(\sigma(A)\) is a countably infinite set of eigenvalues with finite geometric multiplicities, which has no finite limit points and is such that, for its arbitrary arrangement \(\{\lambda_n\}_{n=1}^{\infty}\),

\[ \Re \lambda_n \to -\infty, \ n \to \infty. \]

**Corollary 6.11** (Equivalence of Eventual and Immediate Compactness). If a normal \(C_0\)-semigroup in a complex infinite-dimensional Hilbert space is eventually compact, it is immediately compact.

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