A recipe for an unpredictable random number generator

Mónica A. García-Ñustes*, Leonardo Trujillo† and Jorge A. González‡

Centro de Física, Instituto Venezolano de Investigaciones Científicas (IVIC),
A.P. 21827, Caracas 1020-A, Venezuela

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In this work we present a model for computation of random processes in digital computers which solves the problem of periodic sequences and hidden errors produced by correlations. We show that systems with non-invertible non-linearities can produce unpredictable sequences of independent random numbers. We illustrate our result with some numerical calculations related with random walks simulations.

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Many challenging problems in computational physics are associated with reliable realizations of randomness (e.g Monte Carlo simulations). In a typical 32-bit format a maximum of $2^{32}$ floating points numbers can be represented. Therefore, a recursive function $X_{n+1} = f(X_n, X_{n-1}, \ldots, X_{n-r+1})$ acting on these numbers generates a sequence $X_0, X_1, X_2, \ldots, X_{N-1}$ which must repeat itself. It is known that for any recursive function, a digital computer can only generate periodic sequences of numbers $[1, 2, 3, 4, 5, 6, 7]$. These generators are not unpredictable.

Definition of truly unpredictable: The next values are not determined by the previous values. A process $X_n = P(\theta T Z^n)$ is said to be unpredictable if for any string of values $X_0, X_1, X_2, \ldots, X_m$ of length $m + 1$, generated using some $\theta = \theta_1$, there are other values of $\theta$ for which function $X_n = P(\theta T Z^n)$ generates exactly the same string of numbers $X_0, X_1, X_2, \ldots, X_m$, but the next value $X_{m+1}$ is different, where $m$ is any integer. Note that this kind of process cannot be expressed as a map of type $X_{n+1} = f(X_n, X_{n-1}, \ldots, X_{n-r+1})$.

All known generators (in some specific physical calculations) give rise to incorrect results.

* E-mail: mogarcia@ivic.ve
† E-mail: leo@ivic.ve
‡ E-mail: jorge@ivic.ve
because they deviate from randomness. It is trivial that any periodic process is not unpredictable. Suppose $m_T$ is the period of the generated sequence. Given any string of $m_T$ values: $X_s, X_{s+1}, ..., X_{m_T-1}$; the next value $X_{m_T}$ is always known, because the process is periodic. On the other hand, for any generator of type $X_{n+1} = f(X_n, X_{n-1}, ..., X_{n-r+1})$, given any string of $r$ values: $X_s, X_{s+1}, ..., X_{s+r-1}$; the next value $X_{s+r}$ is always determined by the previous $r$ values. Thus it is not unpredictable. So the subsequences must be correlated.

An example of this can be found in Ref. [5], where the authors have shown that using common pseudo random number generators, the produced random walks present symmetries, meaning that the generated numbers are not independent. On the other hand, the logarithmic plot of the mean distance $<d>$ versus the number of steps $N$ is not a straight line (as expected theoretically, $<d> \sim N^{1/2}$) after $N > 10^5$ (in fact, it is a rapidly decaying function). Here $d$ is defined as the end to end mean-square distance from the origin of the random walk as a function of the number of steps. Other papers on the influence of the pseudorandom number generator on random walk simulations are the following [11, 12, 13].

In the following, we will show that using non-invertible nonlinear functions, we can create an unpredictable random number generator which does not contain visible correlations while simulating a random walk with the length $10^9$.

Let us investigate the following function [8, 9, 10]:

$$X_n = P(\theta T Z^n), \quad (1)$$

where $P(t)$ is a periodic function, $\theta$ is a real number, $T$ is the period of the function $P(t)$, and $Z$ is a noninteger real number.

Let $Z$ be a rational number expressed as $Z = p/q$, where $p$ and $q$ are relative prime numbers. Now let us define the following family of sequences

$$X_n^{(k,m,s)} = P \left[ T(\theta_0 + q^m k) \left( \frac{q}{p} \right)^s \left( \frac{p}{q} \right)^n \right], \quad (2)$$

where $k$, $m$ and $s$ are non-negative integers. The parameter $k$ distinguishes the different sequences. For all sequences parametrized by $k$, the strings of $m + 1$ values $X_s, X_{s+1}, X_{s+2}, ..., X_{s+m}$ are the same. This is so because

$$X_n^{(k,m,s)} = P \left[ T\theta_0 \left( \frac{q}{p} \right)^s \left( \frac{p}{q} \right)^n + Tkp^n q^{(m-n+s)} \right] = P \left[ T\theta_0 \left( \frac{q}{p} \right)^s \left( \frac{p}{q} \right)^n \right]$$
for all $s \leq n \leq m + s$. Note that the number $k^p(n-s)q^{(m-n+s)}$ is an integer for $s \leq n \leq m + s$. So we can have an infinite number of sequences that share the same string of $m + 1$ values.

Nevertheless, the next value $X_{s+1}^{(k,m,s)} = \mathcal{P} \left[ T \theta_0 \left( \frac{q}{p} \right)^s \left( \frac{p}{q} \right)^{(s+1)} + \frac{T k_{p(m+1)}}{q} \right]$ is uncertain. In general $X_{s+1}^{(k,m,s)}$ can take $q$ different values. In addition, the value $X_{s-1}^{(k,m,s)}$, $X_{s+1}^{(k,m,s)} \mathcal{P} \left[ T \theta_0 \left( \frac{q}{p} \right)^s \left( \frac{p}{q} \right)^{(s-1)} + \frac{T k_{q(m+1)}}{p} \right]$, is also undetermined from the values of the string $X_s, X_{s+1}, X_{s+2}, ..., X_{s+m}$. There can be $p$ different possible values. In the case of a generic irrational $Z$, there are infinite possibilities for the future and the past. From the observation of the string $X_s, X_{s+1}, X_{s+2}, ..., X_{s+m}$, there is no method for determining the next and the previous values of the sequence.

But this is not the only feature of these functions. It can be demonstrated that there are no statistical correlations between $X_m$ and $X_n$ if $m \neq n$, and that they are also independent in the sense that their probability densities satisfy the relationship $P(X_n, X_m) = P(X_n)P(X_m)$\[14,15]\.

Moreover, we will show that, given the function (1), any string of sequences $X_s, X_{s+1}, ..., X_{s+r}$ constitutes a set of statistically independent random variables.

Without loss of generality, we assume that $P(t)$ has zero mean and can be expressed using the following Fourier representation $P(t) = \sum_{k=-\infty}^{\infty} a_k e^{i\pi k t}$.

We can calculate the $r$-order correlation functions [14,15]:

$$E(X_{n_1} \cdots X_{n_r}) = \int_X d\theta P(T \theta Z_{n_1}) \cdots P(T \theta Z_{n_r})$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_r=-\infty}^{\infty} a_{k_1} \cdots a_{k_r} \int_0^1 d\theta \exp \{i\pi (k_1 Z_{n_1} + \cdots + k_r Z_{n_r})T \theta \}$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_r=-\infty}^{\infty} a_{k_1} \cdots a_{k_r} \delta(k_1 Z_{n_1} + \cdots + k_r Z_{n_r}, 0), \quad (3)$$

where the coefficients $k_i$ can be different integers, and $\delta(n,m) = 1$ if $n = m$ or $\delta(n,m) = 0$ if $n \neq m$.

When all $n_i$ are even, the following equation is satisfied

$$E(X_{s_1}^{n_1}X_{s+1}^{n_2} \cdots X_{s+r}^{n_r}) = E(X_{s_1}^{n_1})E(X_{s+1}^{n_2}) \cdots E(X_{s+r}^{n_r}). \quad (4)$$

The main problem in this equation occurs when one of the numbers $n_i$ is odd. In this case, the correlations $E(X_{s_1}^{n_1}X_{s+1}^{n_2} \cdots X_{s+r}^{n_r})$ must be zero. A nonzero correlation in Eq.(4) exists only for the sets $(n_1, n_2, \ldots, n_r)$ that satisfy the equation $k_1 Z_{n_1} + \cdots + k_r Z_{n_r} = 0$. For a typical real number $Z$, this equation is never satisfied.
If we use non-invertible nonlinear functions, type of (1), we can implement a Truly Random Number Generator (TRNG). In this case, we propose the following function

\[ X_n = [\theta Z^n] \mod 1 \] (5)

Function (5) is an example of the general case \( X_n = P[\theta T Z^n] \) studied in this paper. We have shown that the subsequences \( X_s, X_{s+1}, \ldots, X_{s+r} \) constitutes a set of statistically independent random variables. The particular case of function (5) is well–known to produce uniformly distributed numbers [8, 9, 10].

Now we will formulate a central limit theorem. Using theorems proved in previous studies [14, 15, 16, 17, 18, 19] and the results obtained from this paper, we obtain the following formula: If \( Z \) is a generic real number and \( X_n = 2(Y_n - 1/2) \), \( Y_n = [\theta Z^n] \mod 1 \), then

\[
\lim_{r \to \infty} P\left\{ \alpha < \frac{X_1 + X_2 + \cdots + X_r}{\sqrt{r}} < \beta \right\} = \frac{1}{\sqrt{\pi}} \int_{\alpha}^{\beta} e^{-\xi^2} d\xi. \] (6)

The Gaussian distribution of the sums is correct even for other functions \( X_n = P[\theta Z^n] \), where \( P(t) \) is periodic. This has been shown in numerical simulations [14].

The numbers \( X_n = [\theta Z^n] \mod 1 \) are uniformly distributed [8, 9, 10]. We can simulate different stochastic processes (with different distributions) using different functions \( X_n = P[\theta T Z^n] \). As \( \rho(X_n) = 1, \rho(X_{n+1}) = 1, \rho(X_n, X_{n+1}) = 1 \), it is trivial that they are independent.

It is interesting to check the theoretical predictions using numerical simulations of the behavior of different stochastic processes.

For instance, let us study the function

\[ U_n = \cos[2\pi \theta Z^n]. \] (7)

All the moments and higher–order correlations can be calculated exactly [14, 15].

For odd \( m \):

\[ E(U_n^m) = 0. \] (8)

If any \( n_i \) is odd, then

\[ E(U_{s+1}^{n_1} \cdots U_{s+r}^{n_r}) = 0 \] (9)

Suppose now that all \( n_i \) are even:

\[
E(U_{s+1}^{n_1} \cdots U_{s+r}^{n_r}) = 2^{-(n_0 + n_1 + \cdots + n_r)} \left( \binom{n_0}{\frac{n_0}{2}} \binom{n_1}{\frac{n_1}{2}} \cdots \binom{n_r}{\frac{n_r}{2}} \right), \] (10)
\[ E(U_{s}^{n_{0}}) = 2^{-n_{0}} \left( \binom{n_{0}}{\frac{n_{0}}{2}} \right), \] (11)

\[ E(U_{s+1}^{n_{1}}) = 2^{-n_{1}} \left( \binom{n_{1}}{\frac{n_{1}}{2}} \right), \ldots, \] (12)

\[ E(U_{s+r}^{n_{r}}) = 2^{-n_{r}} \left( \binom{n_{r}}{\frac{n_{r}}{2}} \right). \] (13)

Note that the condition for independence is satisfied

\[ E(U_{s}^{n_{0}}U_{s+1}^{n_{1}} \cdots U_{s+r}^{n_{r}}) = E(U_{s}^{n_{0}})E(U_{s+1}^{n_{1}}) \cdots E(U_{s+r}^{n_{r}}), \] (14)

for all integer \( n_{0}, n_{1}, \ldots n_{r} \).

We have performed extensive numerical simulations that confirm the values of these moments and the independent conditions.

An additional checking is the following.

The probability density of \( U_{n} \) is \( \rho(U) = \frac{1}{\pi \sqrt{1-U^2}} \). Define \( V_{n} = U_{n+1} \). The probability density of \( V_{n} \) is \( \rho(V) = \frac{1}{\pi \sqrt{1-V^2}} \). We have checked both theoretically and numerically that \( \rho(U, V) = \frac{1}{\pi^2 \sqrt{(1-U^2)(1-V^2)}} \), that is \( \rho(U, V) = \rho(U)\rho(V) \). This can be observed in Fig. 1 and Fig. 2. In order to avoid computation problems, we have used the following procedure. We

**FIG. 1:** Probability densities for random variables \( U_{n} = \cos[2\pi \theta Z^{n}] \) and \( V_{n} = U_{n+1} \). a)\( \rho(U) = \frac{1}{\pi \sqrt{1-U^2}} \); b)\( \rho(V) = \frac{1}{\pi \sqrt{1-V^2}} \).
change parameter $\theta$ after each set of $M$ values of $X_n$, where $M$ is the maximum number for which there are not overflow problems, in such a way that the next value of $X_{n+1}$ is obtained with the new $\theta$. Let us define

$$\theta_s = A(C_s + X_s) + 0.1$$

where $C_s$ is a sequence obtained using the digits of the Champernowne’s number $0.1234567891011\ldots$ (i.e., $0.123456$, $C_1 = 0.234567$, $C_2 = 0.345678$, $C_3 = 0.456789$, $C_4 = 0.567891$, and so on. This sequence is nonperiodic. Index $s$ is the order number of $\theta$ in such a way that $s = 0$ corresponds to the $\theta$ used for the first set of $M$ sequence values $X_1, X_2, \ldots, X_M$; $s = 1$ for the second set $X_{M+1}, X_{M+2}, \ldots, X_{2M}$, and so on. $X_0$ represents the TRNG’s seed.

Using this method we have generated a very long sequence of random numbers without computational problems.

To test function (5) as a truly random number generator, we have implemented a random walk simulation program in C++. We have made a sampling test of a random walk with $N = 10^9$ steps with 100 realizations with different initial seeds. The mean distance $\langle d \rangle$ was calculated every 1000 steps of the random walk.

The Champernowne sequence of numbers used in the generator was produced previously by a short C++ program, who created a sequence of a maximum of 40000 Champernowne’s numbers. If a larger amount of values to $C_s$ is necessary, it can be obtained using a segment code that uses the 40 thousand values already stored in $C_s$ and mixing them, e.g the algo-
A logarithm takes the first value of the series \( C_1 \), the third \( C_3 \) and so on, and adds them at the end of the series, obtaining that \( C_{s+1} = C_1, C_{s+2} = C_3, \ldots \); if more values are necessary, this procedure or cycle is repeated but now skipping two values \( C_1, C_3, C_5, \ldots \) three values \( C_1, C_4, C_7 \) and so on. In this way, we can make the \( C_s \) sequence as large as we wish.

We present a logarithmic plot of the mean distance \( < d > \) versus the number of steps \( N \) with \( N = 10^9 \) steps with \( A = 6.9109366 \) and \( Z = \pi/2 \) (See Fig. 3). We can verify that there is no deviation from the theoretical straight line, even for \( N \gg 10^5 \) steps, which is a very good test of the reliability of the Random Number Generator used in the random walk simulations.

![Logarithmic plot of the mean distance](image)

**FIG. 3:** Logarithmic plot of the mean distance \( < d > \) versus the number of steps \( N = 10^9 \) steps. a) for generator [5], b) the same simulation for a generator of type \( X_{n+1} = aX_n \mod T \).

We have presented a random number generator based on the properties of non–invertible transformations of truncated exponential functions. The obtained random process is unpredictable in the sense that the next values are not determined by the previous values. We have applied this generator to the numerical simulation of statistically independent random variables. In the simulation of a random walk with the length \( 10^9 \), the random process does not contain visible correlations.

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