Warped Solitonic Deformations and Propagation of Black Holes in 5D Vacuum Gravity

Sergiu I. Vacaru *
Physics Department, CSU Fresno, Fresno, CA 93740-8031, USA,
and
Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física,
Instituto Superior Tecnico, Av. Rovisco Pais 1, Lisboa, 1049-001,
Portugal

D. Singleton †
Physics Department, CSU Fresno, Fresno, CA 93740-8031, USA
(November 13, 2018)

Abstract

In this paper we use the anholonomic frames method to construct exact solutions for vacuum 5D gravity with metrics having off-diagonal components. The solutions are in general anisotropic and possess interesting features such as an anisotropic warp factor with respect to the extra dimension, or a gravitational scaling/running of some of the physical parameters associated with the solutions. A certain class of solutions are found to describe Schwarzschild black holes which “solitonically” propagate in spacetime. The solitonic character of these black hole solutions arises from the embedding of a 3D soliton configuration (e.g. the soliton solutions to the Kadomtsev–Petviashvily or sine–Gordon equations) into certain ansatz functions of the 5D metric. These solitonic solutions may either violate or preserve local Lorentz invariance. In addition there is a connection between these solutions and noncommutative field theory.

Pacs 04.50.+h

I. INTRODUCTION

Recent work in string theory and brane physics has suggested new paradigms for solving things like the hierarchy problem using novel treatments of the extra dimensions that occur in these theories [1]. One approach is to view the extra dimension(s) as compactified, but

*e-mail: sergiu_vacaru@yahoo.com, vacaru@fisica.ist.utl.pt
†e-mail: dougs@csufresno.edu
at a size much larger than the Planck length. This is the “large extra dimension” scenario\[2\]. Another approach is to postulate that most particles and fields are suppressed from moving into the extra dimension(s) by an exponential “warp” factor with respect to the extra dimension(s). This is the Randall-Sundrum (RS) model\[3\].

The RS model in particular has gained considerable popularity in high energy physics, cosmology and black hole physics\[4-8\]. Recently, it was shown\[9,10\] that by considering off–diagonal metrics, which are diagonalized using anholonomic frames, that the RS and Kaluza–Klein theories become locally anisotropic and exhibit a variation or “running” of some of the constants associated with these metrics. This variation of the effective 4D constants occurred with respect to either the extra dimension coordinate or one of the angular coordinates. These anisotropic solutions also exhibited RS style warp factors, but without the need for any specific brane energy-momentum tensor – the warp factor arose from anisotropic, vacuum solutions in 5D gravity.

In refs.\[11\] the anholonomic frames method was used to construct 4D and 5D vacuum solutions with one or more of the ansatz functions which parameterized the off–diagonal metric being taken as a solitonic solution to some nonlinear equation. The two cases considered were the solitonic solutions of the 3D Kadomtsev–Petviashvili and sine–Gordon equations. (2D solitonic solutions in 4D general relativity were originally introduced in ref.\[12\]. A recent discussion can be found in ref.\[13\]. Generalizations to 3D are given in refs.\[14\]). These 3D solitons, embedded in the 5D spacetime, could be seen as solitonic, traveling pulses which might have a larger cross section for being detected as compared to standard gravitational waves.

In the present work we examine how black hole solutions are modified when combined with the anisotropic/warped/solitonic configurations of refs.\[11\]. We begin by parameterizing a 5D spacetime with many off–diagonal metric components. Into this off–diagonal metric we then embed a 4D Schwarzschild solution. Adding to this various anisotropic/warped/solitonic configurations it is found that the black hole solution can be modified in a number of interesting ways. First, the horizon can be anisotropically deformed. Along with the deformation of the horizon we find that some of the “constants” of the original Schwarzschild solution become dependent on one or more of the coordinates (e.g. the mass of the solution becomes dependent on the 5th coordinate. Second, the 3D solitonic configurations can move or propagate the black hole either in the normal uncompactified spatial dimensions, or in the extra, compactified spatial dimension.

In this paper use the term “locally anisotropic” spacetime or “anisotropic’ spacetime” for a 5D pseudo-Riemannian spacetime with an anholonomic frame structure with mixed holonomic and anholonomic variables. The anisotropy of the gravitational interactions arises as a result of the off–diagonal metrics, or, equivalently, by their diagonalized versions given with respect to anholonomic frames.

The paper has the following structure: in section II we present the theoretical framework of the anholonomic frames method. We then introduce the 5D metric ansatz form and write down the corresponding vacuum Einstein equations. We show that it is possible to embed the Kadomtsev-Petviashvili and sine-Gordon solitons into this system through one of the ansatz functions. This creates a 3D gravitational soliton. In section III a 4D Schwarzschild solution is embedded in the 5D spacetime and deformed in various ways. These deformations lead to some of the “constant” parameters of the original 4D Schwarzschild solution scaling
with respect to some of the coordinates. In section IV we focus on three classes of solutions with small deformations of the horizon. The deformation of the horizons come from the 3D soliton solutions of section II. In section V we examine solitonic black hole solutions which move in the bulk 5D spacetime. These moving black hole solutions can be given with either spherical or deformed horizons. In section VI we discuss the physical features of the various solutions and give our conclusions.

II. OFF–DIAGONAL METRICS AND 3D SOLITONS

The Schwarzschild solution in isotropic spherical coordinates is given by

$$ds^2 = \left( \frac{\hat{\rho} - 1}{\hat{\rho} + 1} \right)^2 dt^2 - \rho_g^2 \left( \frac{\hat{\rho} + 1}{\hat{\rho}} \right)^4 \left( d\hat{\rho}^2 + \hat{\rho}^2 d\theta^2 + \hat{\rho}^2 \sin^2 \theta d\varphi^2 \right),$$

(1)

The re–scaled isotropic radial coordinate is $\hat{\rho} = \rho / \rho_g$, with $\rho_g = r_g / 4$; $\rho$ is connected with the usual radial coordinate $r$ by $r = \rho (1 + r_g / 4 \rho)^2$; $r_g = 2G_{[4]}m_0 / c^2$ is the 4D Schwarzschild radius of a point particle of mass $m_0$; $G_{[4]} = 1 / M_{[4]}^2$ is the 4D Newton constant expressed via the Planck mass $M_{[4]}$ ($M_{[4]}$ may be an effective 4D mass scale which arises from a more fundamental scale of the full, higher dimensional spacetime). In the rest of the paper we set $c = 1$. The system of coordinates is $(t, \rho, \theta, \varphi)$, where $t$ is the time like coordinate and $(\rho, \theta, \varphi)$ are 3D spherical coordinates. The metric (1) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass $m_0$. It has a singularity for $r = 0$ and a spherical horizon at $r = r_g$, or at $\hat{\rho} = 1$ in the re–scaled isotropic coordinates. This solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames. This spherically symmetric solution can be deformed in various interesting ways using the anholonomic frames method. Writing down a general off–diagonal metric ansatz we first embed the above form of the Schwarzschild solutions into the metric. Then this off–diagonal metric with the embedded Schwarzschild solution is diagonalized with respect to the anholonomic frames. The resulting field equations are of a form that is relatively simple, and allows for analytical solutions which represent anisotropic deformations of the original Schwarzschild solution.

A. Off–diagonal metric ansatz

We split the 5D coordinates $u^a = (x^i, y^a)$ into coordinates $x^i$, with indices $i, j, k, ... = 1, 2, 3$, and coordinates $y^a$, with indices $a, b, c, ... = 4, 5$. Explicitly the coordinates are of the form the coordinates

$$x^i = (x^1 = \chi, \quad x^2 = \lambda = \ln \hat{\rho}, \quad x^3 = \theta) \quad \text{and} \quad y^a = (y^4 = v, \quad y^5 = p),$$

The metric interval is written as

$$ds^2 = \Omega^2(x^i, v) \hat{g}_{\alpha\beta}(x^i, v) \, du^\alpha du^\beta,$$

(2)

were the coefficients $\hat{g}_{\alpha\beta}$ are parametrized by the ansatz
The coefficients of equations (7) - (10) are given by
\[
\begin{bmatrix}
g_1 + (w_1^2 + \zeta_1^2)h_4 + n_1^2h_5 & (w_1 w_2 + \zeta_1 \zeta_2)h_4 + n_1 n_2h_5 & (w_1 w_3 + \zeta_1 \zeta_3)h_4 + n_1 n_3h_5 & (w_1 + \zeta_1)h_4 + n_1h_5 \\
(w_1 w_2 + \zeta_1 \zeta_2)h_4 + n_1 n_2h_5 & g_2 + (w_2^2 + \zeta_2^2)h_4 + n_2^2h_5 & (w_2 w_3 + \zeta_2 \zeta_3)h_4 + n_2 n_3h_5 & (w_2 + \zeta_2)h_4 + n_2h_5 \\
(w_1 w_3 + \zeta_1 \zeta_3)h_4 + n_1 n_3h_5 & (w_2 w_3 + \zeta_2 \zeta_3)h_4 + n_2 n_3h_5 & g_3 + (w_3^2 + \zeta_3^2)h_4 + n_3^2h_5 & (w_3 + \zeta_3)h_4 + n_3h_5 \\
(w_1 + \zeta_1)h_4 & (w_2 + \zeta_2)h_4 & (w_3 + \zeta_3)h_4 & 0 \\
\end{bmatrix},
\]
(3)

The metric coefficients are smooth functions of the form:
\[
g_1 = \pm 1, \quad g_{2,3} = g_{2,3}(x^2, x^3), \quad h_{4,5} = h_{4,5}(x^1, v),
\]
\[
w_i = w_i(x^i, v), \quad n_i = n_i(x^i, v), \quad \zeta_i = \zeta_i(x^i, v), \quad \Omega = \Omega(x^i, v).
\]

The quadratic line element (2) with metric coefficients (3) can be diagonalized,
\[
\delta s^2 = \Omega^2(x^i, v)[g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta}v)^2 + h_5(\delta p)^2],
\]
with respect to the anholonomic co–frame \((dx^i, \hat{\delta}v, \delta p)\), where
\[
\hat{\delta}v = dv + (w_i + \zeta_i)dx^i + \zeta_5\delta p \quad \text{and} \quad \delta p = dp + n_i dx^i
\]
which is dual to the frame \((\delta_i, \partial_4, \partial_5)\), where
\[
\delta_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i \partial_5, \quad \partial_5 = \partial_5 - \zeta_5 \partial_4.
\]

The simplest way to compute the nontrivial coefficients of the Ricci tensor for the (14) is to do this with respect to anholonomic bases (13) and (3), which reduces the 5D vacuum Einstein equations to the following system:
\[
R_2^2 = R_3^3 = -\frac{1}{2g_2 g_3} \left[ g_3^{\bullet \bullet} - \frac{g_2^2 g_3^\bullet}{2g_2} + g_2^\prime - \frac{g_2^\prime g_3^\prime}{2g_2} + \frac{(g_2^\prime)^2}{2g_2} \right] = 0, \quad (7)
\]
\[
R_4^4 = R_5^5 = -\frac{\beta}{2h_4 h_5} = 0, \quad (8)
\]
\[
R_{4i} = -w_i \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \quad (9)
\]
\[
R_{5i} = -\frac{h_5}{2h_4} \left[ n_i^{\bullet \bullet} + \gamma n_i^\bullet \right] = 0, \quad (10)
\]

with the conditions that
\[
\Omega^{n_i/q_i} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}), \quad (11)
\]
and \(\zeta_i\) satisfies the equations
\[
\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0, \quad (12)
\]
The coefficients of equations (7) - (10) are given by
\[
\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \quad \beta = h_5^{\bullet \bullet} - h_5^\bullet [\ln \sqrt{|h_4 h_5|}]^\bullet, \quad \gamma = \frac{3h_5^\prime}{2h_5} - \frac{h_4^\prime}{h_4}. \quad (13)
\]

The various partial derivatives are denoted as \(\dot{a} = \partial a/\partial x^1, a^\bullet = \partial a/\partial x^2, a^\prime = \partial a/\partial x^3, a^* = \partial a/\partial v\). The details of the computations leading to this system are given in refs. 3,10.
This system of equations (7)–(10), (11) and (12) can be solved by choosing one of the ansatz functions (e.g. \( h_4(x^i, v) \) or \( h_5(x^i, v) \)) to take some arbitrary, but physically interesting form. Then the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way one can generate many solutions, but the requirement that the initial, arbitrary choice of the ansatz functions be “physically interesting” means that one wants to make this original choice so that the final solution generated in this way yield a well behaved solution. To satisfy this requirement we start from well known solutions of Einstein’s equations and then use the above procedure to deform this solutions in a number of ways.

The Schwarzschild solution is given in terms of the parameterization in (3) by

\[
\begin{align*}
g_1 &= \pm 1, & g_2 = g_3 &= -1, & h_4 &= h_{4[0]}(x^i), & h_5 &= h_{5[0]}(x^i), \\
w_i &= 0, & n_i &= 0, & \zeta_i &= 0, & \Omega &= \Omega_{[0]}(x^i),
\end{align*}
\]

with

\[
\begin{align*}
h_{4[0]}(x^i) &= \frac{b(\lambda)}{a(\lambda)}, & h_{5[0]}(x^i) &= -\sin^2 \theta, & \Omega^2_{[0]}(x^i) &= a(\lambda)
\end{align*}
\]

or alternatively, for another class of solutions,

\[
\begin{align*}
h_{4[0]}(x^i) &= -\sin^2 \theta, & h_{5[0]}(x^i) &= \frac{b(\lambda)}{a(\lambda)},
\end{align*}
\]

were

\[
\begin{align*}
a(\lambda) &= \rho_g^2 \left(\frac{e^\lambda + 1}{e^{2\lambda}}\right)^4 \\
b(\lambda) &= \left(\frac{e^\lambda - 1}{(e^\lambda + 1)^2}\right)^2.
\end{align*}
\]

Putting this together gives

\[
ds^2 = \pm d\chi^2 - a(\lambda) \left( d\lambda^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right) + b(\lambda) \, dt^2
\]

which represents a trivial embedding of the 4D Schwarzschild metric (1) into the 5D space-time. We now want to anisotropically deform the coefficients of (17) in the following way

\[
\begin{align*}
h_{4[0]}(x^i) &\rightarrow h_4(x^i, v) = \eta_4(x^i, v) h_{4[0]}(x^i), & h_{5[0]}(x^i) &\rightarrow h_5(x^i, v) = \eta_5(x^i, v) h_{5[0]}(x^i), \\
\Omega^2_{[0]}(x^i) &\rightarrow \Omega^2(x^i, v) = \Omega^2_{[0]}(x^i) \Omega^2_{[1]}(x^i, v).
\end{align*}
\]

The factors \( \eta_4 \) and \( \Omega^2_{[1]} \) can be interpreted as re-scaling or renormalizing the original ansatz functions. These gravitational “polarization” factors – \( \eta_{4,5} \) and \( \Omega^2_{[1]} \) – generate non-trivial values for \( w_i(x^i, v) \), \( n_i(x^i, v) \) and \( \zeta_i(x^i, v) \), via the vacuum equations (7)–(12). We shall also consider more general nonlinear polarizations which can not be expresses as \( h \sim \eta h_{[0]} \). In the next section we show how the coefficients \( a(\lambda) \) and \( b(\lambda) \) of the Schwarzschild metric can be polarized by choosing the original, arbitrary ansatz function to be some 3D soliton configuration.
B. 3D soliton deformation of metric coefficients

Vacuum gravitational 2D solitons in 4D Einstein vacuum gravity were originally investigated by Belinski and Zakharov [12]. In ref. [11] 3D solitonic configurations in 4D gravity were constructed on anisotropic Taub-NUT backgrounds. Here we show that 3D solitonic configurations can be embedded into 5D vacuum gravity.

The simplest way to construct a solitonic deformation of the off–diagonal metric in equation (3) is to take one of the “polarization” factors $\eta_4$, $\eta_5$ or the ansatz function $n_i$ as a solitonic solution of some particular non-linear equation. The rest of the ansatz functions can then be found by carrying out the integrations of equations (7)–(12).

As an example of this procedure we take $\eta_5(r, \theta, \chi)$ to be a soliton solution of the Kadomtsev–Petviashvili (KdP) equation or (2+1) sine-Gordon (SG) equation (Refs. [14] contain the original results, basic references and methods for handling such non-linear equations with solitonic solutions). In the KdP case $\eta_5(v, \theta, \chi)$ satisfies the following equation

$$\eta_5^{**} + \epsilon (\eta_5 - 6\eta_5\eta_5' + \eta_5'''') = 0, \quad \epsilon = \pm 1,$$

(18)

while in the most general SG case $\eta_5(v, \chi)$ satisfies

$$\pm \eta_5^{**} \mp \eta_5 = \sin(\eta_5).$$

(19)

We can also consider less general versions of the SG equation where $\eta_5$ depends on only one (e.g. $v$ and $x_1$) variable. We will use the notation $\eta_5 = \eta_5^{KP}$ or $\eta_5 = \eta_5^{SG}$ ($h_5 = h_5^{KP}$ or $h_5 = h_5^{SG}$) depending on if ($\eta_5$) ($h_5$) satisfies equation (18), or (19) respectively.

Having chosen a solitonic form for $h_5 = h_5^{KP,SG}$, $h_4$ can be found from

$$h_4 = h_4^{KP,SG} = h_4^{[0]} \left[ \left| \frac{h_5^{KP,SG}(v, \chi)}{h_5^{KP,SG}(x_i, v)} \right| \right]^2$$

(20)

where $h_4^{[0]}$ is a constant. By direct substitution it can be shown that equation (20) solves equation (3) with $\beta$ given by (13) when $h_5^* \neq 0$. If $h_5^* = 0$, then $h_4$ is an arbitrary function $h_4(x_i, v)$. In either case we will denote the ansatz function determined in this way as $h_4^{KP,SG}$ although it does not necessarily share the solitonic character of $h_5$. Substituting the values $h_4^{KP,SG}$ and $h_5^{KP,SG}$ into $\gamma$ from equation (13) gives, after two $v$ integrations of equation (11), the ansatz functions $n_i = n_i^{KP,SG}(v, \theta, \chi)$. Here, for simplicity, we set $g_{2,3} = -1$ so that the holonomic 2D background is trivial. In ref. [11] it was shown how to generate solutions using 2D solitonic configurations for $g_2$ or $g_3$.

In addition to imposing a solitonic form on $h_5$ it is possible to carry out a similar procedure starting with $h_4$ or $n_i$ (i.e. choose $n_4$ or $n_i$ to satisfy equation (18) or equation (19) and then use equations (7)–(12), (13) and (20) and to determine the other ansatz functions).

The main conclusion of this subsection is that the ansatz (3), when treated with anholonomic frames, has a degree of freedom that allows one to pick one of the ansatz functions ($\eta_4$, $\eta_5$, or $n_i$) to satisfy some 3D solitonic equation. Then in terms of this choice all the other ansatz functions can be generated up to carrying out some explicit integrations and differentiations. In this way it is possible to build exact solutions of the 5D vacuum Einstein equations with a solitonic character.
III. BLACK HOLES WITH COORDINATE DEPENDENT PARAMETERS

In this section exact 5D vacuum solutions are constructed by first embedding the 4D spherically symmetric Schwarzschild solution into the 5D metric (2). These solutions are then deformed by fixing $h_4$ or $h_5$ to be of the KdP or SG form discussed in the last subsection (in our example we take $h_5$ to be of the solitonic form). This “renormalizes” the gravitational constant, making it develop a coordinate dependence

$$\rho_g \rightarrow \tilde{\rho}_g = \omega(x^i, v) \rho_g$$  \hspace{1cm} (21)

The polarization, $\omega(x^i, v)$, is induced by the solitonic character of $h_4$ or $h_5$ (in our example $h_5$). For simplicity, we shall omit the indices KP or SG for the coefficients of metric, since this will not result in ambiguities.

A. Solutions with trivial conformal factor $\Omega = 1$

A particular case of metric (4) with trivial conformal factor is given by

$$\delta s^2 = \left[ \pm d\chi^2 - d\lambda^2 - d\theta^2 + h_4(\delta v)^2 + h_5(\delta p)^2 \right],$$  \hspace{1cm} (22)

$$\delta v = dv + w_i(x^i, v) dx^i, \quad \delta p = dp + n_i(x^i, v) dx^i,$$

$h_4 = \eta_4(x^i, v) h_{4[0]}(x^i)$ and $h_5 = \eta_5(x^i, v) h_{5[0]}(x^i)$. The functions $h_{4[0]}(x^i)$ and $h_{5[0]}(x^i)$ are taken as in (14), or, inversely, as in (15). The polarizations, $\eta_{4,5}$ are chosen so $h_5$ is a solution of either the KdP or SG soliton configuration, and that $h_4$ is then determined in terms of this choice. If $\omega^* = \partial_v \omega \neq 0$, i.e. $h_5 \neq 0$, then the solution of equations (8) which are compatible with (9) is

$$|h_4| = h_{4[0]}^2 \left[ \left( \sqrt{|h_5(x^i, v)|} \right)^* \right]^2,$$  \hspace{1cm} (23)

where $h_{4[0]} = const$. Thus for $\eta_5^* \neq 0$, the coefficients $\eta_4$ and $\eta_5$ are related by

$$|\eta_4(x^i, v) h_{4[0]}(x^i)| = h_{4[0]}^2 h_{5[0]}(x^i) \left[ \left( \sqrt{|\eta_5(x^i, v)|} \right)^* \right]^2 .$$

If $h_5^* = 0$ (i.e. $\omega^*_\phi = 0$ and $\eta_5^* = 0$) then equation (8) is solved by any $h_4$ as a function of the variables $(x^i, v)$ i.e. $h_4(x^i, v)$. For example, for $\eta_5 = \omega^{-2}$, $\omega^* \neq 0$, and $h_{4,5[0]}$ from (14) we can express the polarization $\eta_4$ explicitly via functions $a, b$ and $\omega$,

$$|\eta_4| = h_{4[0]}^2 \frac{b(\lambda)}{\sin^2 \theta a(\lambda)} \left[ \left( \omega^{-1}(x^i, v) \right)^* \right]^2,$$  \hspace{1cm} (24)

which allows us to find the functional dependencies $h_{4,5} = h_{4,5}(a, b, \omega)$ and to compute the coefficient $\gamma(a, b, \omega)$ from (13). After two integrations with respect to $v$, the general solution of (14), expressed via the polarizations $\eta_4$ and $\eta_5$, is
\[ n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{\eta_4}{(\sqrt{|\eta_5|})^3} dv, \quad \text{for } \eta_5 \neq 0; \]
\[ = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \eta_4 dv, \quad \text{for } \eta_5^* = 0; \]
\[ = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{1}{(\sqrt{|\eta_5|})^3} dv, \quad \text{for } \eta_5^* = 0, \]

where \( n_{k[1,2]}(x^i) \) are fixed by boundary conditions. We can simplify equations (23) by re-writing them in terms \( \omega \). For instance, if \( \eta_5 = \omega^{-2}, \eta_4 \) is taken from (24) and \( \omega^* \neq 0 \), then one can write \( n_k \) as

\[ n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int (\omega^*(x^i, v))^2 \omega^{-1} dv. \]

The vacuum 5D metric (22) constructed in this way is a particular case of (3) with \( \Omega = 1, \omega_i = 0, \zeta_i = 0 \), but with non-trivial values of \( h_{4,5} = h_{4,5}(a, b, \omega) \) and \( n_k = n_k(x^i, a, b, \omega) \). These non-trivial \( h_{4,5} \) and \( n_k \) give a renormalized gravitational constant of the form \( \rho_g \to \rho_g^* = \omega \rho_g \). In the trivial limit \( \omega \to 1, n_k \to 0, h_{4,5} \to h_{4,5}(0)(x^i) \) we recover not just the trivial embedding of the Schwarzschild metric into the 5D spacetime (17), but also the following metric

\[ ds^2 = \pm d\chi^2 - d\lambda^2 - d\theta^2 - \sin^2 \theta d\varphi^2 + \frac{b(\lambda)}{a(\lambda)} dt^2. \quad (26) \]

This metric is related to the metric in equation (17) by the multiplication of the 4D part of the metric by the conformal factor \( a^{-1}(\lambda) \) and a further extension to 5D. This metric does not satisfy the vacuum 5D equations. Nevertheless, it is possible to deform such a metric, by introducing self-consistently some off-diagonal metric coefficients which make the resulting metric into a solution of the vacuum Einstein equations. The treatment of such off-diagonal metrics is that a 3D solitonic wave (satisfying the equation (18), or (19)) may polarize the metric in equation (22) to generate a 5D vacuum solution.

The metric in equation (22) still has the same horizon and singularities as the 4D Schwarzschild metric of equation (17), but it has the new feature that the gravitational constant is coordinate dependent. This coordinate dependence comes from the coordinate dependence of \( \omega(x^i, v) \).

There are two types of solutions which give an anisotropic renormalization of the gravitational constant: one where the anholonomic coordinates are chosen as \( (v = \varphi, p = t) \), and \( \omega = \omega(\chi, \lambda, \varphi), \eta_5 = \omega^{-2}(\chi, \lambda, \varphi), h_{4[0]} = -\sin^2 \theta \) and \( h_{5[0]} = b/a \). These are called the \( \varphi \)-solutions. The second form is with the anholonomic coordinates taken as \( (v = \varphi, p = \lambda) \), and \( \omega = \omega(\chi, \lambda, t), \eta_5 = \omega^{-2}(\chi, \lambda, t), h_{4[0]} = b/a \) and \( h_{5[0]} = -\sin^2 \theta \). These are called the \( t \)-solutions. The form of each of these solutions when \( \omega^* \neq 0 \) is:

\( \varphi \)-solutions : \( (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = \varphi, \quad y^5 = p = t), \]
\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = -\sin^2 \theta, \quad h_{5(0)} = \frac{b(\lambda)}{a(\lambda)}, \]
\[ h_4 = \eta_4(x^i, \varphi)h_{4(0)}(x^i), \quad h_5 = \eta_5(x^i, \varphi)h_{5(0)}(x^i), \quad \omega = \omega(\chi, \lambda, \varphi), \]

\( t \)-solutions : \( (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \varphi, \quad y^4 = v = \lambda, \quad y^5 = p = t), \]
\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = -\sin^2 \theta, \quad h_{5(0)} = \frac{b(\lambda)}{a(\lambda)}, \]
\[ h_4 = \eta_4(x^i, \varphi)h_{4(0)}(x^i), \quad h_5 = \eta_5(x^i, \varphi)h_{5(0)}(x^i), \quad \omega = \omega(\chi, \lambda, \varphi), \]

8
\[ |\eta_4| = h^2_{(0)} \frac{b(\lambda)\omega^2}{a(\lambda)\omega^4 \sin^2 \theta}, \quad \eta_5 = \omega^{-2}, \]
\[ w_i = 0, \quad \zeta_i = 0, \quad n_k \left( x^i, \varphi \right) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int (\omega^*(x^i, \varphi))^2 \omega^{-1} d\varphi. \quad (27) \]

and

\[ t\text{-solutions : } (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi), \]
\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_4(0) = \frac{b(\lambda)}{a(\lambda)}, \quad h_5(0) = -\sin^2 \theta, \]
\[ h_4 = \eta_4(x^i, t)h_4(0)(x^i), \quad h_5 = \eta_5(x^i, t)h_5(0)(x^i), \quad \omega = \omega(\chi, \lambda, t), \]
\[ \eta_4 = \omega^{-2}(x^i, t), \quad \eta_5 = \left[ \eta_{5[0]} + \frac{1}{h(0)} \sqrt{\frac{b(\lambda)}{a(\lambda)}} \int dt \omega^{-1}(x^i, t) \right]^2, \]
\[ w_i = 0, \quad \zeta_i = 0, \quad n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \eta_4 |\eta_5|^{-3/2} dt. \quad (28) \]

As a specific example of the \( \varphi \)-solution we take \( \eta_5 \) to satisfy the simplified SG equation
\[ -\ddot{\eta}_5 = \sin(\eta_5) \rightarrow \partial_\chi(\eta_5) = \sin(\eta_5). \]  The explicit form for \( \eta_5 \) is then
\[ \eta_5 = 4 \tan^{-1} \left[ e^{\pm \chi} \right] \quad (29) \]

In this case \( \eta_5^* = \partial_\chi \eta_5 = 0 \) so \( h_4 \) and \( \eta_4 \) are arbitrary functions. We will take the simple case of taking \( \eta_4 = 1 \) so that \( h_4 \) retains its original form of \( -\sin^2 \theta \). Also \( n_k \) is now given by the middle form of equation (23) and is thus fixed by the integration functions \( n_{k[1]}(x_i) \) and \( n_{k[2]}(x_i) \). For simplicity we take these functions to be zero so that \( n_k = 0 \). Thus the metric of equation (22) takes the form
\[ \delta s^2 = [\pm d\chi^2 - d\lambda^2 - d\theta^2 - \sin^2 \theta(\delta v)^2 + 4 \tan^{-1} \left[ e^{\pm \chi} \right] \frac{b(\lambda)}{a(\lambda)} (\delta t)^2], \quad (28) \]

An almost identical solution can be constructed for the \( t \)-solutions, but with \( \eta_4 \) having the form of the SG solution in equation (29).

If the internal coordinate \( \chi \) is taken to be compactified as in the standard Kaluza-Klein approach so that \( \chi = 0 \) is equivalent to \( \chi = 2\pi \) then the SG form for \( \eta_5 \) or \( \eta_4 \) has the problem of not being single valued (i.e. \( \eta_5(\chi = 0) \neq \eta_5(\chi = 2\pi) \)). If one considers the extra dimension to be uncompactified then one has the interesting feature that (via equation (21) – \( \Omega_g = \rho_g (4 tan^{-1} [e^{\pm \chi}]^{-1/2}) \) the strength of the gravitational coupling decreases as one moves off the 3D brane \( (\chi = 0) \) into the extra dimension \( (\chi \neq 0) \) for the \( +\chi \) solution in equation (29); the gravitational coupling increases as one moves off the 3D brane for the \( -\chi \) solution. In order to explain why this extra, non-compactified extra dimension is not observed one needs an overall RS-type \( \Omega \) exponential suppression as one moves in the extra dimension. We show that this is possible in the next subsection.

**B. Solutions with non-trivial conformal factor** \( \Omega \neq 1 \)

It is possible to generalize the ansatz of equation (22) by taking a nontrivial conformal factor \( \Omega = \Omega_0[0] \Omega_1[1] \). In this case the metric of (4) takes the form

9
\[
\delta s^2 = \Omega^2 [\pm d\chi^2 - d\lambda^2 - d\theta^2 + h_4(\delta v)^2 + h_5(\delta p)^2],
\]
\[
\delta v = dv + w_i(x^i, v)dx^i + \zeta_i(x^i, v)dx^i, \quad \delta p = dp + n_i(x^i, v)dx^i,
\]

where \( h_4 = \eta_4(x^i, t)h_{4(0)}(x^i) \), \( h_5 = \eta_5(x^i, t)h_{5(0)}(x^i) \) and \( \Omega_0 = \sqrt{a(\lambda)}, \Omega_{[1]} = \Omega_{[1]}(x^k, v) \).

The metric in (31) satisfies the 5D vacuum Einstein equations if the conformal factor \( \Omega \) is related to the \( h_4 \) via equation (11). This induces non-trivial values of \( \zeta_i \) from (12). The factor \( \Omega_{[1]} \) is taken to be of the form

\[
\Omega_{[1]}(x^k, v) = \exp[-k|x||\Omega_{[2]}(\lambda, \theta, v)]
\]

i.e. it contains an exponential warp factor with respect to the 5th coordinate. Next we choose \( h_4 = \Omega^2 \). This choice connects the polarization \( \eta_4 \) with the conformal factor \( \Omega_{[2]} \),

\[
a(\lambda)\Omega_{[2]}^2(\lambda, \theta, v) = \eta_4(x^i, v)h_{4(0)}(x^i),
\]

The result is that \( \zeta_i \) takes the form

\[
\zeta_i = (\Omega_{[1]}^*)^{-1} \partial_i \Omega_{[1]} + (\ln |\Omega_{[1]}|)^* \partial_i \ln \sqrt{a(\lambda)}.
\]

The rest of the procedure of constructing \( \varphi \)-solutions and \( t \)-solutions with a non-trivial conformal factor is similar to that from the previous subsection. The form of these solutions with \( \omega^* \neq 0 \) is

\( \varphi \)-solutions : \( (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = \varphi, \quad y^5 = p = t) \),

\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = -\sin^2 \theta, \quad h_{5(0)} = \frac{b(\lambda)}{a(\lambda)}, \]

\[ h_4 = \eta_4(x^i, \varphi)h_{4(0)}, \quad h_5 = \eta_5(x^i, \varphi)h_{5(0)}, \quad \omega = \omega(\chi, \lambda, \varphi), \]

\[ \Omega = \sqrt{a(\lambda)} \exp[-k|x||\Omega_{[2]}(\lambda, \theta, \varphi)], \quad \Omega_{[1]} = \exp[-k|x||\Omega_{[2]}(\lambda, \theta, \varphi)] \]

\[ \Omega_{[2]}^2 = \eta_4 (x^i, \varphi) |h_{4(0)}|a^{-1}(\lambda), \]

\[ |\eta_4| = \frac{b(\lambda)\omega^*}{\omega^2a(\lambda)\sin^2 \theta}, \quad \eta_5 = \omega^{-2}, \]

\[ w_i = 0, \quad n_k (x^i, \varphi) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int (\omega^*(x^i, \varphi))^2 \omega^{-1} d\varphi, \]

\[ \zeta_i = \zeta_i(x^i, \varphi) = (\Omega_{[1]}^*)^{-1} \partial_i \Omega_{[1]} + (\ln |\Omega_{[1]}|)^* \partial_i \ln \sqrt{a(\lambda)} \]

and

\( t \)-solutions : \( (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi) \),

\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = \frac{b(\lambda)}{a(\lambda)}, \quad h_{5(0)} = -\sin^2 \theta, \]

\[ h_4 = \eta_4(x^i, t)h_{4(0)}, \quad h_5 = \eta_5(x^i, t)h_{5(0)}, \quad \omega = \omega(\chi, \lambda, t), \]

\[ \Omega = \sqrt{a(\lambda)} \exp[-k|x||\Omega_{[2]}(\lambda, \theta, t)], \quad \Omega_{[1]} = \exp[-k|x||\Omega_{[2]}(\lambda, \theta, t)] \]

\[ \Omega_{[2]}^2 = \eta_4 (x^i, t) h_{4(0)}a^{-1}(\lambda), \]
\[ \eta_4 = \omega^{-2}(x^i, t), \quad \eta_5 = \left[ \eta_{5[0]} + \frac{1}{h(0)} \sqrt{\frac{b(\lambda)}{a(\lambda)}} \int \omega^{-1}(x^i, t) dt \right]^2, \]

\[ w_i = 0, \quad \zeta_i = 0, \quad n_k(x^i, t) = n_k[1](x^i) + n_k[2](x^i) \int \eta_4 | \eta_5 |^{-3/2} dt, \]

\[ \zeta_i = \zeta_i(x^i, t) = \left( \Omega^*_1 \right)^{-1} \partial_i \Omega_1 + \left( \ln | \Omega_1 | \right)^* \partial_i \ln \sqrt{a(\lambda)}. \]

It is straightforward to incorporate the special case solution given in equation (30) of the last subsection to the form in equation (33). The only change is that the metric in equation (30) is multiplied by an overall factor of \( \exp[-2k|\chi|] \), which exponentially confines all fields and particles (except gravity) to the 3D brane. Thus one has the variation of the gravitational coupling discussed in the last subsection along with a non-compact extra dimension. Because of the simply form for \( \eta_5 \) for this special solution the \( \zeta_i \) functions are arbitrary functions rather than being given by the forms in equation (33). The \( t \)-solution of the last subsection can also be extended to the form given in equation (34). Now the solitonic form for \( \eta_4 (\chi_4 = 4 \tan^{-1}[e^{-3\chi}]) \) has an effect not only on the strength of the gravitational coupling, but also on the overall conformal factor since \( \Omega^2 \propto \eta_4 \) from equation (28). For the \( \eta_4 = 4 \tan^{-1}[e^{-\chi}] \) solution this just enhances the exponential suppression as one moves away from \( \chi = 0 \). For the \( \eta_4 = 4 \tan^{-1}[e^{+\chi}] \) solution the solitonic \( \eta_4 \) factor can initially dominate over the \( e^{-2k|x|} \) factor as one first begins to move away from \( \chi = 0 \).

The solutions of this section were generated by starting with a 4D Schwarzschild metric and trivially embedding it into a 5D metric. It was found that by fixing certain ansatz functions of the metric to take the form of either the 3D KdP or SG soliton that the gravitational constant in 4D and various other parameters of the metric became “solitonically” renormalized. In addition these 3D soliton configurations could propagate in the 5D metric in such a manner so as to give a solitonic gravitational wave.

IV. SOLITONIC DEFORMATIONS OF HORIZONS

By considering off–diagonal metrics and using the anholonomic frames method it is possible to construct 4D and 5D black holes with non–spherical horizons (In refs. [9,11,10] solutions with ellipsoidal, toroidal, disk like, and bipolar horizons are discussed). In this section we demonstrate that it is possible to deform spherical horizons by the solitonic configurations of the last section.

We begin with the case of \( t \)-solutions which have a trivial polarization, \( \eta_5 = 1 \), but with \( \eta_4 = \eta_4(x^i, t) \), so that \( \eta_4^* \neq 0 \). This will lead to solutions with a trivial renormalization of constants, but with a small deformation of the spherical horizon.

The spherical horizon of non–deformed solutions is defined by the condition that the coefficient \( b(\lambda) \) from equation (15) vanish. This occurs when \( \lambda = 0 \). A small deformation of the horizon can be induced if \( e^{\lambda} = e^{i\lambda}(x^i, t) \cong 1 + \epsilon \lambda(x^i, t) \), for a small parameter \( \epsilon \ll 1 \). This renormalizes \( b \rightarrow b = b[1 + \epsilon \eta(x^i, t)] \). Setting \( \eta_5 = 1 \) and \( \eta_4 = 1 + \epsilon \eta(x^i, t) \) we require that the pair \( [\eta_4 h_{4(0)}, h_{5(0)}] \) satisfy equation (8). Since \( h_5 = h_{5(0)}(x^i) \) so that \( h_5^* = 0 \) this gives us some freedom in choosing \( \eta_4 h_{4(0)} \). We use this freedom to take \( \eta(x^i, t) \), as a solitonic solution of either the KdP equation, (18), or the SG equation (19). This choice induces a
time dependent, solitonic deformation of the horizon. It is also possible to generate solitonic
deformations of the horizon which depends on the extra spatial coordinate or an angular
coordinate. The ansatz functions \( n_i \) are determined from equation (25) with \( n_5 = 0 \)
\[
n_k \left( x^i, t \right) = n_{k[1]} \left( x^i \right) + n_{k[2]} \left( x^i \right) \left[ t + \epsilon \int \eta \left( x^i, t \right) dt \right].
\]
The form for metrics of equation (22) but with small deformations of the horizon is
\( t \)-solutions : \( (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi) \),
\[
g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = \frac{b(\lambda)}{a(\lambda)}, \quad h_{5(0)} = -\sin^2 \theta,
\]
\[
h_4 = \eta_4 (x^i, t) h_{4(0)} (x^i), \quad h_5 = h_{5(0)} (x^i), \quad \eta_4 = 1 + \epsilon \eta \left( x^i, t \right), \quad \eta_5 = \omega = 1,
\]
\( w_i = 0, \quad \zeta_i = 0, \quad n_k \left( x^i, t \right) = n_{k[1]} (x^i) + n_{k[2]} (x^i) \left[ t + \epsilon \int \eta \left( x^i, t \right) dt \right]. \) (35)
As a simple example of the above solution we can take \( \eta \) as satisfying the SG equation with
only \( t \) dependence: \( \partial_t \eta = \sin (\eta) \), which gives the solution
\[
\eta (t) = 4 \tan^{-1} \left[ \epsilon^{\pm t} \right] \quad (36)
\]
If we take the additional simplifying assumption that \( n_{k[1]} (x^i) = n_{k[2]} (x^i) = 0 \) so that
\( n_k (x^i, t) = 0 \) then the metric from equation (22)
\[
\delta s^2 = [\pm d\chi^2 - d\lambda^2 - d\theta^2 + (1 + \epsilon 4 \tan^{-1} [\epsilon^{\pm t}]) \frac{b(\lambda)}{a(\lambda)} (\delta t)^2 - \sin^2 \theta (\delta \varphi)^2], \quad (37)
\]
The \( \eta_4 (t) = 1 + \epsilon 4 \tan^{-1} [\epsilon^{-t}] \) solution starts at \( t = 0 \) with the finite value \( 1 + \epsilon \pi \) and goes
to 0 as \( t \to \infty \). Thus the horizon is initially deformed, but approaches the Schwarzschild
form as \( t \) increases. The \( \eta_4 (t) = 1 + \epsilon 4 \tan^{-1} [\epsilon^{t}] \) solution also starts at \( t = 0 \) with the value
\( 1 + \epsilon \pi \), but approaches the value \( 1 + 4 \epsilon \pi \). The deformation of the horizon increases as \( t \)
increases.
It is possible to carry out a similar procedure on metrics of the form (34) which have a
non–trivial conformal factor
\[
\Omega = \Omega_{[0]} \exp [-k |x|] \Omega_{[2]} (\lambda, \theta, v), \quad h_4 = \Omega^2
\]
The conformal factor is connected with the polarization \( \eta_4 = 1 + \epsilon \eta \left( x^i, t \right) \), by equation (32),
\[
a(\lambda) \exp [-2k |x|] \Omega_{[2]}^2 (\lambda, \theta, v) = \left[ 1 + \epsilon \eta \left( x^i, t \right) \right] h_{4(0)} (x^i).
\]
The form of this solution is:
\( t \)-solutions : \( (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi) \),
\[
g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = \frac{b(\lambda)}{a(\lambda)}, \quad h_{5(0)} = -\sin^2 \theta,
\]
\[
h_4 = \eta_4 (x^i, t) h_{4(0)}, \quad h_5 = h_{5(0)}, \quad \eta_4 = 1 + \epsilon \eta \left( x^i, t \right), \quad \eta_5 = \omega = 1,
\]
\[ \Omega = \sqrt{a(\lambda) \exp[-k|\chi|\Omega_{[2]}(\lambda, \theta, t)]}, \quad \Omega_{[1]} = \exp[-k|\chi|\Omega_{[2]}(\lambda, \theta, t)], \quad \Omega_{[2]}^2 = \eta_4(x^i, t) h_{4(0)} a^{-1}(\lambda), \]

\[ w_i = 0, \quad \zeta_i = 0, \quad n_k(x^i, t) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \left[ t + \epsilon \int \eta(x^i, t) dt \right], \]

\[ \zeta_i = \zeta_i(x^i, t) = \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]} + \left( \ln |\Omega_{[1]}| \right)^* \partial_i \ln \sqrt{a(\lambda)}. \]

Since \( h_5^* = 0 \) the function \( \eta(x^i, t) \) is arbitrary and so many different possible scenarios exist for generating small deformations of the horizon. Here we have taken \( \eta(x^i, t) \) as being a 3D soliton configuration, leading to small, solitonic deformations of the horizon. In appendix A we give the general forms for these deformed horizon solutions with \( \omega \neq 1 \) and/or \( \Omega \neq 1 \).

The solutions defined in this section and appendix A illustrate that extra dimensional, anholonomic gravitational vacuum interactions can induce two types of effects: renormalization of the physical constants and small deformations of the horizons of black holes.

**V. MOVING SOLITON–BLACK HOLE CONFIGURATIONS**

There is another class of vacuum solutions based on the 4D black hole embedded into the 5D spacetime. It is possible to combine the solitonic characteristics of the solutions discussed in section IIB with the embedded 4D black hole to generate a configuration which describes a 4D black hole propagating in the 5D bulk. These solutions can be set up so as to have the renormalization of the parameters of the metric and/or have deformed horizons.

**A. The Schwarzschild black hole propagating as a 3D Soliton**

The horizon is defined by the vanishing of the coefficient \( b(\lambda) \) from equation (14). This occurs when \( e^\lambda = 1 \). In order to create a solitonically propagating black hole we define the function \( \tau = \lambda - \tau_0(\chi, v) \), and let \( \tau_0(\chi, v) \) be a soliton solution of either the 3D KdP equation (18), or the SG equation (19). This redefines \( b(\lambda) \) as

\[ b(\lambda) \rightarrow B(x^i, v) = \frac{e^\tau - 1}{e^\lambda + 1}. \]

A class of 5D vacuum metrics of the form (22) can be constructed by parametrizing \( h_4 = \eta_4(x^i, v) h_{4(0)}(x^i) \) and \( h_5 = B(x^i, v)/a(\lambda) \), or inversely, \( h_4 = B(x^i, v)/a(\lambda) \) and \( h_5 = \eta_5(x^i, v) h_{5(0)}(x^i) \). The polarization \( \eta_4(x^i, v) \) (or \( \eta_5(x^i, v) \)) is determined from equation (23)

\[ |\eta_4(x^i, v) h_{4(0)}(x^i)| = h_{4(0)}^2 \left( \left| \frac{B(x^i, v)}{a(\lambda)} \right| \right)^* \]

or

\[ \left| \frac{B(x^i, v)}{a(\lambda)} \right| = h_{4(0)}^2 h_{5(0)}(x^i) \left( \left| \sqrt{ \eta_5(x^i, v) } \right| \right)^* \].
The last step in constructing of the form for these solitonically propagating black hole solutions is to use \( h_4 \) and \( h_5 \) in equation \((10)\) to determine \( n_k \)

\[
n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{h_4}{(\sqrt{|h_5|})^3} dv, \quad h_5^* \neq 0; \tag{39}
\]

\[
= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 dv, \quad h_5^* = 0; \tag{40}
\]

\[
= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{1}{(\sqrt{|h_5|})^3} dv, \quad h_4^* = 0,
\]

where \( n_{k[1,2]}(x^i) \) are set by boundary conditions.

The simplest version of the above class of solutions are the \( t \)-solutions, defined by a pair of ansatz functions, \([B(x^i,t), h_{5(0)}]\), with \( h_5^* = 0 \) and \( B(x^i,t) \) being a 3D solitonic configuration. Such solutions have a spherical horizon when \( h_4 = 0 \) i.e. when \( \tau = 0 \). This solution describes a propagating black hole horizon. The propagation occurs via a 3D solitonic wave form depending on the time coordinate, \( t \), and on the 5\(^{th} \) coordinate \( \chi \). The form of the ansatz functions for this solution (both with trivial and non-trivial conformal factors) is

\[
tag{t-solutions: (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi),}
\]

\[
g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad \tau = \lambda - \tau_0(\chi, t), \quad h_{5(0)} = -\sin^2 \theta, \quad h_4 = B/a(\lambda), \quad h_5 = h_{5(0)}(x^i), \quad \omega = \eta_5 = 1, \quad B(x^i,t) = \frac{e^\tau - 1}{e^\lambda + 1},
\]

\[
w_i = \zeta_i = 0, \quad n_k(x^i,t) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int B(x^i,t) dt \tag{41}
\]

and

\[
tag{t-solutions: (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi),}
\]

\[
g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad \tau = \lambda - \tau_0(\chi, t), \quad h_5 = h_{5(0)} = -\sin^2 \theta, \quad h_4 = \frac{B}{a(\lambda)}, \quad B(x^i,t) = \frac{e^\tau - 1}{e^\lambda + 1}, \quad \Omega^2_{[2]} = B a^{-1}(\lambda) \exp[2k|\chi|],
\]

\[
\Omega = \sqrt{a(\lambda) \exp[-k|\chi|] \Omega_{[2]}(\lambda, \theta, t)}, \quad \Omega_{[1]} = \exp[-k|\chi|] \Omega_{[2]}(\lambda, \theta, t),
\]

\[
w_i = \zeta_i = 0, \quad n_k(x^i,t) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int B(x^i,t) dt,
\]

\[
\zeta_i = \zeta_i(x^i,t) = (\Omega_{[1]}^*)^{-1} \partial_\lambda \Omega_{[1]} + \left[ \ln \Omega_{[1]} \right]^* \partial_i \ln \sqrt{a(\lambda)}. \tag{41}
\]

These forms are similar to the solutions in \((33)\) and \((38)\).

As a simple example of the above solutions we take \( \tau_0 \) to satisfy the SG equation \( \partial_\chi \tau_0 - \partial_t \tau_0 = \sin(\tau_0) \). This has the standard propagating kink solution

\[
\tau_0(\chi, t) = 4 \tan^{-1} [\pm \gamma(\chi - V t)] \tag{42}
\]

where \( \gamma = (1-V^2)^{-1/2} \) and \( V \) is the velocity at which the kink moves into the extra dimension \( \chi \). To obtain the simplest form of this solution we also take \( n_{k[1]}(x^i) = n_{k[2]}(x^i) = 0 \) as in our
previous examples. This example can be easily extended to the solution in equation (11), with a non-trivial conformal factor that gives an exponentially suppressing factor, \( \exp[-2k|\chi|] \). In this manner one has an effective 4D black hole which propagates from the 3D brane into the non-compact, but exponentially suppressed extra dimension, \( \chi \).

In appendix B we give the forms for two classes of solutions with \( \Omega = 1 \) and for two classes with \( \Omega \neq 1 \). These solutions contain various combinations of the previous solutions: solitonically propagating horizons; deformed horizons; renormalized constants.

The solutions constructed in this section and appendix B describe propagating 4D Schwarzschild black holes in a bulk 5D spacetime. The propagation arises from requiring that certain of the ansatz functions take a 3D soliton form. In the simplest version of these propagating solutions the parameters of the ansatz functions are constant, and the horizons are spherical. It was also shown that such propagating solutions could be formed with a renormalization of the parameters and/or deformation of the horizons.

VI. CONCLUSIONS AND DISCUSSION

In this work, we have addressed the issue of constructing various classes of exact solutions which generalize the 4D Schwarzschild black hole metric to a bulk 5D vacuum spacetime.

Throughout the paper the method of anholonomic frames was applied to a general, off–diagonal 5D metric ansatz in order to integrate the Einstein equations and determine the ansatz functions. In this way a host of different solutions were investigated which had a number of interesting features such as a “renormalization” of the parameters of the ansatz, deformation of the horizons, and self–consistent solitonic propagation of black holes in the bulk 5D spacetime. The essential features of each of these solutions would remain unchanged even if we had started with ellipsoidal or toroidal rather than spherical horizons (see ref. [9] for discussions on black holes with ellipsoidal or toroidal horizons), introduced matter sources, considered cosmological constants, or used an anisotropic Taub NUT background [11,10].

In the first part of the paper, we considered solutions with renormalized “constants” – \( \rho_g \rightarrow \bar{\rho}_g = \omega \rho_g \) where \( \rho_g = r_g/4 \) and \( r_g \) is the Schwarzschild radius. The induced coordinate dependence of \( \rho_g \) was connected with the extra dimension and/or the anholonomic gravitational interaction. Since \( \rho_g \) depends on Newton’s constant, the value of the point mass, and the speed of light one could interpret this induced coordinate dependence as a variation of Newton’s constant, an anisotropic mass, or a variation in the speed of light/gravity waves. In the first case we obtain a Randall-Sundrum like correction of Newton’s force law [9,10]. In our case this arises from the use of anholonomic frames instead of a brane configurations. In the second case, we obtain an anisotropic gravitational interaction coming from the anisotropic effective mass. In the third case, we can have metrics which violate local Lorentz symmetry since light and gravitational waves could have propagation speeds which are not constant [5].

In the second part of the paper, we analyzed configurations where 3D solitons in 5D vacuum gravity induced small “solitonic” deformations of the horizon. The simplest examples were “pure” deformations which did not renormalize the constants. It was possible to combine the horizon deformation effects with the renormalization of the constants considered in
the first part of the paper. The anisotropy of these configurations was connected with either an angular coordinate or the time-like coordinate.

The third part of the paper was devoted to solutions which described 4D black holes moving solitonicly in the 5D spacetime. The solitonic character of these solutions came from setting the form of one of the metric ansatz functions to either the 3D KdP or SG soliton configuration. It was shown that these propagating black hole solutions could be combined with the effects discussed in the first two parts of the paper, generating propagating black holes with renormalized constants and/or deformed horizons.

Mathematically this paper demonstrates the usefulness of the anholonomic frames method, developed in refs. [9,11,10], in constructing solutions to the 4D and 5D Einstein vacuum field equations with off–diagonal metrics. By introducing anholonomic transforms we were able to diagonalize the metrics and simplify the system of equations in terms of the ansatz functions. These solutions describe a generic anholonomic (anisotropic) dynamics coming from the off–diagonal metrics. They also generalize the class of exact 4D solutions with linear extensions to the bulk 5D gravity given in ref. [7].

We emphasize that the constructed solutions do not violate the conditions of the Israel and Carter theorems [17] on spherical symmetry of black hole solutions in asymptotically flat spacetimes which were formulated and proved for 4D spacetimes.

Salam, Strathee and Peracci [18] explored the association between gauge fields and the coefficients of off–diagonal metrics in extra dimensional gravity. In the present paper we have used anholonomic frames with associated nonlinear connections of 5D and 4D (pseudo) Riemannian spaces [3,11,10], to find anisotropic solutions with running constants. These solutions may point to Lorentz violations effects in Yang–Mills and electrodynamic theories induced from 5D vacuum gravity. (Refs. [19] gives a general analysis of the problem for Planck–scale physics and string theory). In addition the conditions of frame anholonomy can be associated with a specific noncommutative relation. This indicates that these solutions with generic anisotropy may mimic violations of Lorentz symmetry similar to that found in noncommutative field theories [20].

Our results should be compared with the recent work of ref. [8] where the possibility of obtaining localized black hole solutions in brane worlds was investigated. In ref. [8] this was accomplished by introducing a dependence of the 4D line element on extra dimension. It was concluded that for either an empty bulk or a bulk containing scalar or gauge fields that no conventional type of matter could support such a dependence. For a particular diagonal ansatz for the 5D line–element it was found that an exotic, shell–like, distribution of matter was required. The off–diagonal metrics and anholonomic frames considered in the present work are more general and get around the restrictions in [8]. We find that it is possible to generate warped factors and/or anisotropies in 5D vacuum gravity by taking off–diagonal metrics and using the anholonomic frames method. It is not necessary to generate these effects by some brane configurations with specific energy–momentum tensors.

As a concluding remark, we note that the black hole solutions of the type considered in this work suggest that the localizations to 4D in extra dimension gravity depends strongly not only on the nature of the bulk matter distribution, but also on the type of 5D metrics used: for diagonal metrics one needs exotic bulk matter distributions in order to localize black hole solutions; for off–diagonal metrics with associated anholonomic frames, one can construct localized black hole solutions with various effect (renormalization of constants,
deformed horizons, solitonically propagating black hole configurations) by considering an anholonomic 5D vacuum gravitational dynamics. Matter distribution, and gravitational and matter field interactions on an effective 4D spacetime can be modeled as induced by off-diagonal metrics and anholonomic frame dynamics in a vacuum extra dimension gravity.

Acknowledgments

S. V. thanks C. Grojean and I. Giannakis for useful observations and discussions on anisotropic extra dimension black hole solutions. The work is supported by a 2000–2001 California State University Legislative Award and a NATO/Portugal fellowship grant at the Technical University of Lisbon.

APPENDIX A: SOLITONIC DEFORMATIONS OF HORIZONS WITH NON-TRIVIAL POLARIZATION AND CONFORMAL FACTOR

In this appendix we collect the expressions for the solitonically deformed horizons with non-trivial polarizations and/or conformal factors.

1. $\omega \neq 1$ and $\Omega = 1$

The form of solutions (27) and (28) can be modified so as to allow configurations with both renormalizations of constants and small deformations of the horizon. The simplest way of doing this is to introduce horizon deformations into $\eta_4$ and $\eta_5$ via the modification of $\omega$ and $b$ as

$$\tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, v \right) \right] \quad \text{and} \quad \tilde{b} = b \left[ 1 + \epsilon \eta \left( x^i, v \right) \right]. \quad (A1)$$

The remaining calculations leading to the final form of the ansatz functions are similar to those in subsections IIB and IIIA. We will just write down the final form of the ansatz functions

$$\varphi\text{-solutions : } (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = \varphi, \quad y^5 = p = t),$$

$$g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = -\sin^2 \theta, \quad h_{5(0)} = \frac{b(\lambda)}{a(\lambda)},$$

$$h_4 = \eta_4(x^i, \varphi)h_{4(0)}(x^i), \quad h_5 = \eta_5(x^i, \varphi)h_{5(0)}(x^i),$$

$$|\eta_4| = h_{4(0)}^2 \frac{\tilde{b}(\tilde{\omega}^*)^2}{\tilde{\omega}^4 a(\lambda) \sin^2 \theta}, \quad \eta_5 = \tilde{\omega}^{-2}, \quad \omega = \omega(\chi, \lambda, \varphi),$$

$$\tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, \varphi \right) \right], \quad \tilde{b} = b \left[ 1 + \epsilon \eta \left( x^i, \varphi \right) \right],$$

$$w_i = 0, \quad \zeta_i = 0, \quad n_k \left( x^i, \varphi \right) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int (\tilde{\omega}^*(x^i, \varphi))^2 \omega^{-1} d\varphi.$$ and
functions in equations (33) and (34) generates metrics of the form (31) with deformed conformal factor. Taking effective polarizations as in (A1) and recalculting the ansatz and

\[ \tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, t \right) \right], \quad \tilde{b} = b \left[ 1 + \epsilon \eta \left( x^i, t \right) \right], \]

\[ w_i = 0, \quad \zeta_i = 0, \quad n_k \left( x^i, t \right) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \eta_4|\eta_5|^{-3/2} dt. \]

In the limit \( \epsilon \to 0 \), \( \eta_4 = \eta_5 = 1 \) and \( n_k = 0 \) and we obtain a 4D metric which is a conformally transformed version of the Schwarzschild metric in equation (2).

2. \( \omega \neq 1 \) and \( \Omega \neq 1 \)

In a similar manner we can construct metrics with deformed horizons and a non-trivial conformal factor. Taking effective polarizations as in (A1) and recalculating the ansatz functions in equations (A3) and (A4) generates metrics of the form (4) with deformed horizons. With \( \omega^* \neq 0 \) the form for these solutions is

\[ \varphi \text{-solutions : } (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = \varphi, \quad y^5 = p = t), \]

\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = \frac{b(\lambda)}{a(\lambda)}, \quad h_{5(0)} = -\sin^2 \theta, \]

\[ h_4 = \eta_4(x^i, \varphi)h_{4(0)}, \quad h_5 = \eta_5(x^i, \varphi)h_{5(0)}, \quad \omega = \omega(\chi, \lambda, \varphi), \]

\[ \Omega = \sqrt{a(\lambda)} \exp[-k|x|][\Omega_{[2]}(\lambda, \theta, \varphi), \quad \Omega_{[1]} = \exp[-k|x|][\Omega_{[2]}(\lambda, \theta, \varphi), \]

\[ \Omega^2_{[2]} = \eta_4(x^i, \varphi) |h_{4(0)}| a^{-1}(\lambda), \quad \tilde{b} = b \left[ 1 + \epsilon \eta \left( x^i, \varphi \right) \right], \]

\[ |\eta_4| = h_{4(0)}^2 \tilde{b} \tilde{\omega}^2 a(\lambda) \sin^2 \theta, \quad \eta_5 = \tilde{\omega}^{-2}, \quad \tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, \varphi \right) \right], \]

\[ w_i = 0, \quad n_k \left( x^i, \varphi \right) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int (\tilde{\omega}^*(x^i, \varphi))^2 \omega^{-1} d\varphi, \quad (A4) \]

\[ \zeta_i = \zeta_i(x^i, \varphi) = \left( \Omega^*_{[1]} \right)^{-1} \partial_i \Omega_{[1]} + \left( \ln |\Omega_{[1]}| \right)^* \partial_i \ln \sqrt{a(\lambda)} \]

and

\[ t \text{-solutions : } (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi), \]

\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = \frac{b(\lambda)}{a(\lambda)}, \quad h_{5(0)} = -\sin^2 \theta, \]

\[ h_4 = \eta_4(x^i, t)h_{4(0)}, \quad h_5 = \eta_5(x^i, t)h_{5(0)}, \quad \omega = \omega(\chi, \lambda, t), \quad (A5) \]

\[ \Omega = \sqrt{a(\lambda)} \exp[-k|x|][\Omega_{[2]}(\lambda, \theta, t), \quad \Omega_{[1]} = \exp[-k|x|][\Omega_{[2]}(\lambda, \theta, t) \]

\[ \Omega^2_{[2]} = \eta_4(x^i, t) |h_{4(0)}| a^{-1}(\lambda), \quad \tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, t \right) \right], \]
\[
\eta_4 = \tilde{\omega}^{-2}(x^i, t), \quad \eta_5 = \left[ \eta_{5[0]} + \frac{\sqrt{b}}{h(0) \sqrt{a(\lambda)}} \int \tilde{\omega}^{-1}(x^i, t) dt \right]^2, \quad w_i = \zeta_i = 0,
\]

\[
n_k(x^i, t) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \eta_4 |\eta_5|^{-3/2} dt, \quad \tilde{b} = b \left[ 1 + \epsilon \eta \left( x^i, t \right) \right],
\]

\[
\zeta_i = \zeta_i(x^i, t) = \left( \Omega^*_{[1]} \right)^{-1} \partial_i \Omega_{[1]} + \left( \ln |\Omega_{[1]}| \right)^* \partial_i \ln \sqrt{a(\lambda)}.
\]

The solutions constructed in this subsection, (A4) and (A5), reduce to the trivial embedding of the Schwarzschild metric in the 5D spacetime (1) in the limit when \( \epsilon \to 0, \eta_4 = \eta_5 = 1 \) and \( n_k = 0 \).

**APPENDIX B: SOLITONIC PROPAGATING BLACK HOLES WITH DEFORMED HORIZONS AND/OR NON-TRIVIAL CONFORMAL FACTOR**

In this appendix we collect the expressions for the solitonically propagating black holes with deformed horizons and/or non-trivial conformal factors.

1. Solitonic propagating and deformed black holes, \( \Omega = 1 \)

To the solitically moving black holes of section V we add the effects of renormalization of constants and horizon deformation found in sections III and IV. This is accomplished by modifying the function \( B(x^i, v) \) and \( \omega \) in a manner similar to (A1)

\[
\tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, v \right) \right] \quad \text{and} \quad \tilde{B} = B \left( x^i, v \right) \left[ 1 + \epsilon \eta \left( x^i, v \right) \right]
\]

and, for \( \varphi \)-solutions, write

\[
h_4 = \eta_4 \left( x^i, \varphi \right) h_{4(0)} \quad \text{and} \quad h_5 = \frac{\tilde{B} \left( x^i, \varphi \right)}{a(\lambda) \tilde{\omega}^2},
\]

or for \( t \)-solutions

\[
h_4 = \frac{\tilde{B} \left( x^i, t \right)}{a(\lambda) \tilde{\omega}^2} \quad \text{and} \quad h_5 = \eta_5 \left( x^i, t \right) h_{5(0)}.
\]

The polarization \( \eta_4(x^i, \varphi) \), or \( \eta_5(x^i, t) \), is found by equation (23), and the coefficients \( n_{i4}(x^i, v) \) are found by integrating as in equation (39) for \( h_4^* \neq 0 \) and \( h_5^* \neq 0 \). The forms for exact solutions are

\[
\varphi \text{-solutions: } (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = \varphi, \quad y^5 = p = t), \quad g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = - \sin^2 \theta, \quad h_4 = \eta_4(x^i, \varphi) h_{4(0)}(x^i), \quad h_5 = \frac{\tilde{B}(x^i, \varphi)}{a(\lambda) \tilde{\omega}^2} \quad |\eta_4| = \frac{h_{10}^2}{a(\lambda) \sin^2 \theta} \left[ \frac{\sqrt{B}}{\tilde{\omega}} \right]^4,
\]

\[
\omega = \omega(\chi, \lambda, \varphi), \quad \tau = \lambda - \tau_0(\chi, \varphi), \quad \tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \epsilon \eta \left( x^i, \varphi \right) \right],
\]

19
\[ \tilde{B} = B \left[ 1 + \eta (x^i, \varphi) \right], \quad B \left( x^i, \varphi \right) = \frac{(e^\tau - 1)}{(e^\lambda + 1)}, \]
\[ w = \zeta = 0, \quad n_k \left( x^i, \varphi \right) = n_k[1](x^i) + n_k[2](x^i) \int \frac{h_4}{\sqrt{|h_5|^3}} \, d\varphi. \]

and

\[ t\text{-solutions :} \quad (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = \nu = t, \quad y^5 = p = \varphi), \]
\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{5(0)} = -\sin^2 \theta, \quad h_5 = \eta_5(x^i, t)h_{5(0)}(x^i), \]
\[ h_4 = \frac{\tilde{B}(x^i, t)}{a(\lambda) \tilde{\omega}^2}, \quad \eta_5 = \left[ \eta_5[0] + \frac{1}{h_5(0)\sqrt{a(\lambda)}} \int dt \sqrt{\frac{\tilde{B}(x^i, t)}{\tilde{\omega}(x^i, t)}} \right]^2, \]
\[ \omega = \omega(\chi, \lambda, t), \quad \tau = \lambda - \tau_0(\chi, t), \quad \tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \eta_5(x^i, t) \right], \]
\[ \tilde{B} = B \left[ 1 + \eta_5(x^i, t) \right], \quad B \left( x^i, t \right) = \frac{(e^\tau - 1)}{(e^\lambda + 1)}, \quad (B2) \]
\[ w_4 \zeta_i = 0, \quad n_k \left( x^i, t \right) = n_k[1](x^i) + n_k[2](x^i) \int \frac{h_4}{\sqrt{|h_5|^3}} \, dt. \]

In the limits \( \epsilon \to 0, \eta_4 = \eta_5 = 1, B \to b \) and \( n_k = 0 \) we obtain a 4D metric trivially extended into 5D, which is a conformally transformed version of the Schwarzschild metric \((26)\).

2. Solitonic propagating and deformed black holes, \( \Omega \neq 1 \)

We can revise \((B1)\) and \((B2)\) so as to generate metrics of the class \((31)\), but with nontrivial conformal factors. The forms for these solutions with \( \omega^* \neq 0 \) are

\[ \varphi\text{-solutions :} \quad (x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = \nu = \varphi, \quad y^5 = p = t), \]
\[ g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{4(0)} = -\sin^2 \theta, \quad h_4 = \eta_4(x^i, \varphi)h_{4(0)}(x^i), \]
\[ h_5 = \frac{\tilde{B}(x^i, \varphi)}{\tilde{\omega}^2 a(\lambda)}, \quad |\eta_4| = \frac{h_{5(0)}^2}{a(\lambda) \sin^2 \theta} \left[ \left( \frac{\sqrt{\tilde{B}}}{\tilde{\omega}} \right)^* \right]^2, \]
\[ \omega = \omega(\chi, \lambda, \varphi), \quad \tau = \lambda - \tau_0(\chi, \varphi), \quad \tilde{\omega}^{-2} = \omega^{-2} \left[ 1 + \eta_4 \left( x^i, \varphi \right) \right], \]
\[ \tilde{B} = B \left[ 1 + \eta_4 \left( x^i, \varphi \right) \right], \quad B \left( x^i, \varphi \right) = \frac{(e^\tau - 1)}{(e^\lambda + 1)}, \quad (B3) \]
\[ \Omega = \sqrt{a(\lambda)} \exp[-k|\chi||\Omega_{[2]}(\lambda, \theta, \varphi)], \quad \Omega_{[1]} = \exp[-k|\chi||\Omega_{[1]}(\lambda, \theta, \varphi)], \]
\[ \Omega_{[2]}^2 = \eta_4 \left( x^i, \varphi \right) |h_{4(0)}| a^{-1}(\lambda), \]
\[ w_4 \zeta_i = 0, \quad n_k \left( x^i, \varphi \right) = n_k[1](x^i) + n_k[2](x^i) \int \frac{h_4}{\sqrt{|h_5|^3}} \, d\varphi, \]
\[ \zeta_i = \zeta_i(x^i, \varphi) = \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]} + \left( \ln |\Omega_{[1]}|^* \right) \partial_i \ln \sqrt{a(\lambda)} \]

and
\( t \)-solutions: \((x^1 = \chi, \quad x^2 = \lambda, \quad x^3 = \theta, \quad y^4 = v = t, \quad y^5 = p = \varphi), \quad g_1 = \pm 1, \quad g_2 = g_3 = -1, \quad h_{5(0)} = -\sin^2 \theta, \quad h_5 = \eta_5(x^i, t)h_{5(0)}(x^i), \)

\[
\eta_5 = \left[ \eta_{5[0]} + \frac{1}{h_{5(0)}\sqrt{a(\lambda)}} \int dt \frac{\sqrt{B}}{\bar{\omega}(x^i, t)} \right]^2,
\]

\[
\bar{B} = B \left[ 1 + \epsilon \eta \left( x^i, t \right) \right], \quad B \left( x^i, t \right) = \frac{(e^\tau - 1)}{(e^\lambda + 1)}.
\]

\[
\Omega = \sqrt{a(\lambda)} \exp[-k|\chi|\Omega_{[2]}(\lambda, \theta, t)], \quad \Omega_{[1]} = \exp[-k|\chi|\Omega_{[2]}(\lambda, \theta, t)], \quad \Omega_{[2]} = \eta_4 \left( x^i, t \right) |h_{4(0)}|a^{-1}(\lambda), \quad \Omega_{[2]}^2 = \eta_4 \left( x^i, t \right) |h_{4(0)}|a^{-1}(\lambda), \quad (B4)
\]

\[
w_i = \zeta_i = 0, \quad n_k \left( x^i, t \right) = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{h_4}{\sqrt{|h_5|}} dt.
\]

The solutions of equations (B3) and (B4) reduce to the trivial embedding of the 4D Schwarzschild metric into 5D spacetime from equation (I) in the limit \( \epsilon \to 0, \eta_4 = \eta_5 = 1, B \to b \) and \( n_k = 0 \).
REFERENCES

[1] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724; P. Horava and E. Witten, Nucl. Phys. B460 (1996) 506; A. Lukas, B. A. Ovrut and D. Waldram, Phys. Rev. D 60 (1999) 086001.

[2] I. Antoniadis, Phys. Lett. B246 (1990) 317; N. Arkani–Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 (1998) 263; Phys. Rev. D 59 (1999) 086004; I. Antoniadis, N. Arkani–Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436 (1998) 257; N. Arkani–Hamed, S. Dimopoulos and J. March–Russel, Phys. Rev. D 63 (2001) 064020.

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370; Phys. Rev. Lett. 83 (1999) 4690; M. Gogberashvili, Hierarchy problem in the shell–Universe model, hep-ph/9812296; Europhys. Lett. 49 (2000) 396; Mod. Phys. Lett. A14 (1999) 2025.

[4] A. Chamblin, S. W. Hawking, and H. S. Reall, Phys. Rev. D 61 (2000) 065007; P. Kraus, JHEP 9912, 011 (1999); J. Garriga and M. Sasaki, Phys. Rev. D 62 (2000) 043523; R. Emparan, G. T. Horowitz, and R. C. Myers, JHEP 0001, 007 (2000); A. Chamblin, C. Csaki, J. Erlich, and T. J. Hollowood, Phys. Rev. D 62 (2000) 044012; R. Maartens, Geometry and Dynamics of the Brane–Words, gr–qc/0101059; N. Dadhich, R. Maartens, P. Papadopoulos and V. Rezania, Phys. Lett. B 487, 1 (2000); S. B. Giddings, E. Katz, and L. Randall, JHEP 0003 (2000) 023.

[5] C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B604 (2001) 312; gr–qc/0105114.

[6] C. Csaki, M. L. Graesser and G. D. Kribs, Phys. Rev. D 63 (2001) 065002.

[7] I. Giannakis, and Hai-cang Ren, Phys. Rev. D 63 (2001) 065015.

[8] P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov, Phys. Lett. B468 (1999) 31; Phys. Rev. D 61 (2000) 106004; P. Kanti and K. Tamvakis, hep-th/0110298.

[9] S. Vacaru, hep–th/0110250, 0110284; gr–qc/0001020.

[10] S. Vacaru, hep–ph/0106268; S. Vacaru and E. Gaburov, hep–th/0108063; S. Vacaru and D. Gontsa, hep–th/0109114; S.Vacaru and O. Tintareanu–Mirchea, Nucl.Phys. B626, 239 (2002); S. Vacaru, P. Stavrinos and E. Gaburov, gr–qc/0106068; S. Vacaru and D. Singleton, J. Math. Phys., 43, 2486 (2002)

[11] S. Vacaru, JHEP 0104 (2001) 009; Ann. Phys. (NY) 290, 83 (2001); S. Vacaru, D. Singleton, V. Botan and D. Dotenco, Phys. Lett. B 519 (2001) 249; S. Vacaru and F. C. Popa, Class. Quant. Grav. 18, 4921 (2001).

[12] V. A. Belinski and V. E. Zakharov, Soviet. Phys. JETP, 48 (1978) 985–994 [translated from: Zh. Eksper. Teoret. Fiz. 75 (1978) 1955–1971, in Russian]

[13] V. Belinski and E. Verdaguer, Gravitational Solitons (Cambridge University Press, 2001)

[14] B. B. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR, 192, 753 (1970); V. S. Dryuma, Pis’ma JETP, 19, 753 (1974); V. E. Zakharov and A. B. Shabat, Funk. Analiz i Ego Prilojenia [in Russian] 8, 43 (1974); B. Harrison, Phys. Rev. Lett., 41, 1197 (1978); G. Liebrandt, Phys. Rev. Lett. 41, 435 (1978); G. B. Whitham, J. Phys. A. Math., 12, L1 (1979).

[15] S. Vacaru, J. Math. Phys. 37 (1996) 508; Ann. Phys. (NY) 256, (1997) 39; Nucl. Phys. B 494 (1997) 590; JHEP 9809 (1998) 011; S. Vacaru, Interactions, Strings and Isotopies in Higher Order Anisotropic Superspaces (Hadronic Press, Palm Harbor, USA, 1997); S. Vacaru and Yu. Goncharenko, Int. J. Theor. Phys. 34 (1995) 1955; S. Vacaru, I. Chiosa, and N. Vicol, in Noncommutative Structures in Mathematics and Physics,
[16] L. Landau and E. M. Lifshits, *The Classical Theory of Fields*, vol. 2 (Nauka, Moscow, 1988) [in russian]; S. Weinberg, *Gravitation and Cosmology*, (John Wiley and Sons, 1972).

[17] W. Israel, *Phys. Rev.* 164, 1776 (1967); B. Carter, *Phys. Rev. Lett* 26, 331 (1971); D. C. Robinson, *Phys. Rev. Lett* 34, 905 (1975); M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, 1996).

[18] A. Salam and J. Strathdee, *Ann. Phys.* (NY) 141, (1982) 316; R. Percacci and S. Randjbar–Daemi, *J.Math.Phys.* 24, (1983)807.

[19] V. A. Kostelecky and S. Samuel, *Phys. Rev. D* 39 (1989) 683; *Phys. Rev. Lett* 63 (1989) 224; *Phys. Rev. D* 63 (2001) 046007; V. A. Kostelecky, M. Perry, and R. Potting, *Phys. Rev. Lett* 84 (2000) 4541; V. A. Kostelecky and M. Mewes, [hep-ph/0111026](http://arxiv.org/abs/hep-ph/0111026).

[20] I. Mocioiu, M. Pospelov and R. Roiban, *Phys. Lett.* B 489 (2000) 390; S. M. Carroll et al., *Phys. Rev. Lett* 87 (2001) 141601; Z. Guralnik et al., [hep-th/0106044](http://arxiv.org/abs/hep-th/0106044); A. Anisimov et al., [hep-th/0106386](http://arxiv.org/abs/hep-th/0106386).