SPHERICAL AVERAGED ENDPOINT STRICHARTZ ESTIMATES FOR THE TWO-DIMENSIONAL SCHRÖDINGER EQUATIONS WITH INVERSE SQUARE POTENTIAL

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1. Introduction

Strichartz estimates are crucial in handling local and global well-posedness problems for nonlinear dispersive equations (See [1] [4] [21]). For the Schrodinger equation below

\[
\begin{aligned}
&i\partial_t u - \Delta u = 0 \\
&u(x,0) = u_0(x),
\end{aligned}
\]

one considers estimates in mixed spacetime Lebesque norms of the type

\[
\|u(x,t)\|_{L^q_t L^r_x} = \left( \int \|u(\cdot, t)\|_{L^q_x}^q \, dt \right)^{1/q}.
\]

Let us define the set of admissible exponents.

**Definition 1.1.** If \( n \) is given, we say that the exponent pair \((q, r)\) is admissible if \( q, r \geq 2 \), \((q, r, n) \neq (2, \infty, 2)\) and they satisfy the relation

\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.
\]

Under this assumption, the following estimates are known.

**Theorem 1.2.** If \((q, r, n)\) is admissible, we have the estimates

\[
\|u(x, t)\|_{L^q_t L^r_x} \leq \|u_0\|_{L^2_x}.
\]

From the scaling argument, or in other words dimensional analysis, we can see that the relation (3) is necessary for inequality (4) to hold.

There is a long line of investigation on this problem. The original work was done by Strichartz (see [20] [18] [19]). A more general result was done by Ginibre and Velo (See [7]). For dimension \( n \geq 3 \), the endpoint cases \((q, r, n) = (2, \frac{2n}{n-2}, n)\) was proved by Keel and Tao [12].

The double endpoint \((q, r, n) = (2, \infty, 2)\) is proved not to be true by Montgomery-Smith (see [14]), even when we replace \( L_\infty \) norm with \( BMO \) norm. However, it can be recovered in some special setting, for example Stefanov (See [16]) and Tao (See [22]). In particular, Tao replaces \( L_\infty^\infty \) by a

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norm that takes $L^2$ average over the angular variables then $L^\infty$ norm over the radial variable.

In the present work, I want to consider the end point estimates for the Schrödinger equation with inverse square potential,

\begin{equation}
\begin{cases}
    i\partial_t u - \Delta u + \frac{a^2}{|x|^2} u = 0 \\
    u(x,0) = u_0(x),
\end{cases}
\end{equation}

where $x \in \mathbb{R}^n$, and initial data, $u_0 \in L^2$. For $n \geq 2$ the same Strichartz estimates as in Theorem 1 are proved by Planchon, Stalker, and Tahvidar-Zadeh (see [3]). They did not cover the end point cases for $n = 2$.

We use the same norm as Tao in [22]. We define the $L_\theta$ norm as follows.

**Definition 1.3.**

\begin{equation}
    \|f\|_{L_\theta}^2 := \frac{1}{2\pi} \int_0^{2\pi} |f(r \cos \theta, r \sin \theta)|^2 d\theta.
\end{equation}

The main result in this paper is the following theorem.

**Theorem 1.4.** For $x \in \mathbb{R}^2$, $a \geq 0$, suppose $u(x,t)$ satisfies the following homogeneous initial value problem,

\begin{equation}
\begin{cases}
    i\partial_t u - \Delta u + \frac{a^2}{|x|^2} u = 0 \\
    u(x,0) = u_0(x),
\end{cases}
\end{equation}

then the following apriori estimate holds

\begin{equation}
    \|u\|_{L^2_t(L^\infty_r L^\infty_\theta)} \leq C\|u_0\|_{L^2(\mathbb{R}^2)}.
\end{equation}

Let us consider the equation in polar coordinates. Write $v(r, \theta,t) = u(x,t)$ and $f(r, \theta) = u_0(x)$. We have that $v(r, \theta,t)$ satisfies the equation below,

\begin{equation}
\begin{cases}
    i\partial_t v - \partial_r^2 v - \frac{1}{r^2} \partial_r v - \frac{1}{r^2} \partial_\theta^2 v + \frac{a^2}{r^2} v = 0 \\
    v(r, \theta,0) = f(r, \theta)
\end{cases}
\end{equation}

We write the initial data as superposition of spherical harmonic functions, as follows

\[ f(r, \theta) = \sum_{k \in \mathbb{Z}} f_k(r) e^{ik\theta}. \]

Using separation of variables, we can write $v$ as a superposition,

\[ v(r, \theta,t) = \sum_{-\infty}^{\infty} e^{ik\theta} v_k(r,t), \]

where the radial functions, $v_k$, satisfy the equations below

\begin{equation}
\begin{cases}
    i\partial_t v_k - \partial_r^2 v_k - \frac{1}{r^2} \partial_r v_k + \frac{a^2+k^2}{r^2} v_k = 0, \quad k \in \mathbb{Z} \\
    v_k(r,0) = f_k(r)
\end{cases}
\end{equation}
Remark 1.5. Combining Tao’s result in [22] with the equation (10) above, we can conclude that Theorem 1.4 is true for special cases \( a \in \mathbb{N} \) and \( u \) is radially symmetric. However, the analysis in [22] does not apply to general cases.

For fixed \( r \), we take \( L_\theta \) norm and from the orthogonality of spherical harmonics, we have

\[
||v(r, \theta, t)||_{L_\theta}^2 = \sum_{k \in \mathbb{Z}} |v_k(r, t)|^2.
\]

We will prove the following lemma.

Lemma 1.6. Suppose \( v_k \) satisfies (10), for every \( k \in \mathbb{Z} \) the following apriori estimate holds.

\[
\int |v_k(r, t)|_{L_\infty}^2 dt \leq C \int_0^\infty |f_k(r)|^2 r dr,
\]

where \( C \) is a constant independent of \( k \).

The main theorem follows from the Lemma 1.6 above because of the following observation,

\[
||v(r, \theta, t)||_{L_t^2 L_r^\infty L_\theta}^2 = \int \left( \sup_{r>0} \left\{ \left( \sum_{k \in \mathbb{Z}} |v_k(r, t)|^2 \right)^{\frac{1}{2}} \right\} \right)^2 dt \\
\leq \sum_{k \in \mathbb{Z}} \int \sup_{r>0} |v_k(r, t)|^2 dt \\
\leq \int_0^\infty \sum_{k \in \mathbb{Z}} |f_k(r)|^2 r dr = \int_0^\infty \int_0^{2\pi} |f(r)|^2 rdrd\theta.
\]

The rest of the paper is devoted to the proof of Lemma 1.6.

2. Hankel Transform

The main tools will be the Fourier and Hankel transforms. We want to introduce certain well-known properties of Hankel transform which are necessary for the proof. We consider the kth mode in spherical harmonic. Let \( \nu(k)^2 = a^2 + k^2 \). We define the following elliptic operator

\[
A_\nu := -\partial_r^2 - \frac{1}{r} \partial_r + \frac{\nu^2}{r^2}.
\]

For fixed \( k \), we skip the \( k \) in the notation for convenience. Equation (10) becomes

\[
\begin{cases}
i \partial_t v + A_\nu v = 0, \\
v(r, 0) = f(r)
\end{cases}
\]
Next, we define the Hankel transform as follows.

\[ \phi^\#(\xi) := \int_0^\infty J_\nu(r\xi)\phi(r)rdr, \]

where \( J_\nu \) is the Bessel function of real order \( \nu > -\frac{1}{2} \) defined via,

\[ J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2)\pi^{1/2}} \int_{-1}^1 e^{irt}(1 - t^2)^{\nu - 1/2}dt. \]

The following properties of the Hankel transform are well known, (See [3])

Proposition 2.1.

(i) \( (\phi^\#)^\# = \phi \)

(ii) \( (A_\nu\phi)^\#(\xi) = |\xi|^2\phi^\#(\xi) \)

(iii) \( \int_0^\infty |\phi^\#(\xi)|^2\xi d\xi = \int_0^\infty |\phi(r)|^2rdr. \)

If we apply Hankel transform on equation (13), we obtain

\[ \begin{cases} i\partial_t v^\#(\xi,t) + |\xi|^2 v^\#(\xi,t) = 0, \\ v^\#(\xi,0) = f^\#(\xi) \end{cases} \]

Solving the ODE and inverting the Hankel transform, we have the formula

\[ v(r,t) = \int_0^\infty J_\nu(sr)e^{is^2t}f^#(s)sds. \]

The change of variables, \( y = s^2 \), implies

\[ v(r,t) = \frac{1}{2} \int_0^\infty J_\nu(r\sqrt{y})e^{iyt}f^#(\sqrt{y})dy. \]

Let us define the function \( h \) as follows

\[ h(y) := \begin{cases} f^#(\sqrt{y}) & y > 0 \\ 0 & y \leq 0 \end{cases} \]

Then the expression in (18) becomes

\[ v(r,t) = \frac{1}{2} \int_\mathbb{R} J_\nu(|r||y|^{1/2})h(y)e^{iyt}dy. \]

From the Proposition [2,1] we have

\[ \int_{-\infty}^\infty |h(y)|^2 dy = \frac{1}{2} \int_0^\infty |f^#(s)|^2sds = \frac{1}{2} \int_0^\infty |f(\eta)|^2\eta d\eta < \infty. \]

So, \( h \) is an \( L^2 \) function. We will work with \( h(y) \) belonging to Schwartz class. These are \( C^\infty \) functions that tend to zero faster than any polynomial at infinity, i.e.

\[ S(\mathbb{R}) = \{ f(x) \in C^\infty(x) | \sup_{x \in \mathbb{R}} \frac{d^\alpha f(x)}{d^\alpha x} < C^{\alpha\beta}x^{-\beta} \ \forall \alpha, \beta \in \mathbb{N} \}. \]
The general case of $h \in L^2$ follows by a density argument. We use smooth cut off fountions to partition the Bessel function $J_{\nu}$ as follwos,

\begin{equation}
J_{\nu}(\eta) = m_{\nu}^0(\eta) + m_{\nu}^1(\eta) + \sum_{j \gg \log \nu} m_{\nu}^j(\eta),
\end{equation}

where $m_{\nu}^0$, $m_{\nu}^1$ and $m_{\nu}^j$ are supported on $\eta < \frac{\nu}{\sqrt{2}}$, $\eta \sim \nu$ and $\eta \sim 2^j$ for $j \gg \log \nu$ respectively. Let $J_{\nu}^k = \sum_0^k m_{\nu}^j$. Equation (19) holds in the sense that we can write

\begin{equation}
v(r, t) = \lim_{k \to \infty} \frac{1}{2} \int_{\mathbb{R}} J_{\nu}^k(|y||r|^{1/2}) h(y) e^{i\nu t} dy.
\end{equation}

Substituting $h$ by the inverse Fourier formula

\begin{equation}
h(y) = \int_{\mathbb{R}} e^{i(\eta - t)y} \hat{h}(\eta - t) d\eta,
\end{equation}

and changing the order of integration, we have

\begin{equation}
v(r, t) = \lim_{k \to \infty} \int_{\mathbb{R}} \left( \frac{1}{2} \int_{\mathbb{R}} J_{\nu}^k(\sqrt{|y||r|}) e^{i\nu y} dy \right) \hat{h}_k(\eta - t) d\eta.
\end{equation}

Let us define the kernel below

\begin{equation}
K_{\nu,r}^j(\eta) = \frac{1}{2} \int_{\mathbb{R}} m_{\nu}^j(\sqrt{|y||r|}) e^{i\nu y} dy.
\end{equation}

For convenience, rename $g(y) = \hat{h}(-y)$ and define an operator

\begin{equation}
T_{\nu,r}^j[g](t) = (K_{\nu,r}^j * g)(t).
\end{equation}

Since it is a convolution, it becomes a multiplication in Fourier space. Thus, this operator has another equivalent expression

\begin{equation}
T_{\nu,r}^j[g](t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_{\nu}^j(r|\xi|^{1/2}) \hat{g}(\xi) e^{i\xi t} d\xi.
\end{equation}

Notice that both the kernel $K_{\nu,r}^j$ and the operator $T_{\nu,r}^j$ are functions of $\nu$. We can rewrite equation (23) in the following form,

\begin{equation}
v(r, t) = \lim_{k \to \infty} \sum_{j \leq k} T_{\nu,r}^j[g(\eta)](t).
\end{equation}

The main theorem in this paper will follow from the lemma below.

**Lemma 2.2.** For $g \in L^2$, $a \geq 0$, $C_1$, $C_2$, $C_3$ independent of $\nu^2(k) = a^2 + k^2$ $k \in \mathbb{N}$, the following estimates hold.

\begin{equation}
\int_{\mathbb{R}} \sup_{r > 0} |T_{\nu,r}^0[g](t)|^2 dt \leq C_1 \int_{\mathbb{R}} |g(y)|^2 dy,
\end{equation}

\begin{equation}
\int_{\mathbb{R}} \sup_{r > 0} |T_{\nu,r}^1[g](t)|^2 dt \leq C_2 \int_{\mathbb{R}} |g(y)|^2 dy,
\end{equation}

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\begin{equation}
\int_{\mathbb{R}} \sup_{r > 0} |T^j_{\nu,r}(g)(t)|^2 \, dt \leq C_3 2^{-\frac{j}{2}} \int_{\mathbb{R}} |g(y)|^2 \, dy \text{ for } j \gg \log \nu.
\end{equation}

Notice \( \|v\|^2_{L^2(L^\infty L^0)} \) can be bounded by the sum of the left hand side terms in Lemma 2.2 and the right hand side terms are summable. Thus, Lemma 1.6 follows.

We will refer to these three cases as low frequency, middle frequency, and high frequency respectively. We will prove inequalities (28), (29), and (30) in the following sections.

### 3. Estimates for Low Frequency

Our strategy is to estimate the kernel defined in (24) and apply Hardy-Littlewood maximal inequality in this case. By changing variable \( z := r^2 y \) in (24), we can write

\begin{equation}
K^0_{\nu,r}(\eta) = \frac{1}{r^2} K^0_{\nu,1}(\eta),
\end{equation}

and therefore we have

\begin{equation}
\|K^0_{\nu,r}(\eta)\|_{L_1} = \|K^0_{\nu,1}(\eta)\|_{L_1}.
\end{equation}

We will prove the following estimate.

**Lemma 3.1.** The kernel \( K^0_{\nu,1}(\eta) \) is bounded as follows,

\begin{equation}
|K^0_{\nu,1}(\eta)| \leq \Phi^0_0(\eta),
\end{equation}

where \( \Phi^0_0 \) is an even nonnegative decaying \( L^1 \) function defined as follows.

\begin{equation}
\Phi^0_0 = \begin{cases} 
  c(1 + |\eta|)^{-1+\nu/2} & \text{when } 0 < \nu \leq 2 \\
  C(\nu)(1 + |\eta|)^{-2} & \text{when } 2 < \nu,
\end{cases}
\end{equation}

where \( C(\nu) \) is uniformly bounded.

We can see \( \|\Phi^0_0\|_{L^1} \) is finite for every \( \nu \) from a direct calculation. Since \( C(\nu) \) is uniformly bounded, \( \|\Phi^0_0\|_{L^1} \) is uniformly bounded when \( \nu > 2 \). For \( 0 < \nu \leq 2 \), \( \|\Phi^0_0\|_{L^1} = 4/\nu \). However, since \( \nu(k)^2 = a^2 + k^2 \) are discrete, we can find a universal \( L_1 \) bound for given \( a \neq 0 \). Since \( \Phi^0_0 \) is an even nonnegative decaying function, we can use the property of approximate identity and obtain

\begin{equation}
\sup_{r > 0} T^0_{\nu,r}[g](t) \leq \|\Phi^0_0\|_{L^1} M[g](t),
\end{equation}

where \( M[g](t) \) is Hardy-Littlewood maximal function of \( g \) at \( t \), defined as follows.

\begin{equation}
M(g)(t) = \sup_{r > 0} \frac{1}{|I(t,r)|} \int_{I(t,r)} |g| \, dx,
\end{equation}
where $I(t, r) = (t-r, t+r)$. Finally, we apply the Hardy-Littlewood maximal inequality

\begin{equation}
\|M(F)\|_{L^p} \leq C(p)\|F\|_{L^p} \quad 1 < p < \infty
\end{equation}

to finish the proof.

We will prove Lemma 3.1 case by case as presented in (34).

\textbf{Proof.} We need to prove $K_{\nu,1}^0(\eta)$ is bounded and decays with the power advertised in (34). We first prove the decay of the tail.

Because $m_0^\nu$ is even, we have

\begin{equation}
K_{\nu,1}^0(\eta) = \frac{1}{2} \int_{\mathbb{R}} m_\nu^0(\sqrt{\eta}) e^{i\eta y} dy = \int_0^\infty m_\nu^0(\sqrt{y}) \cos(\eta y) dy
\end{equation}

Integrate by parts to obtain

\begin{equation}
K_{\nu,1}^0(\eta) = -\frac{1}{2\eta} \int_0^\infty m_\nu'^0(y^{1/2}) y^{-1/2} \sin(\eta y) dy
\end{equation}

Differentiating the expression of the Bessel function in (15), we can find the following recursive relation for Bessel functions.

\begin{equation}
J'_\nu(r) = \nu r^{-1} J_\nu(r) - J_{\nu+1}(r)
\end{equation}

From the definition of Bessel function (15), we can see

\begin{equation}
J_\nu(r) \sim \frac{1}{\Gamma(\nu+1)} \left( \frac{r}{2} \right)^\nu \text{ if } r < \sqrt{\nu + 1}.
\end{equation}

Moreover, for all $r$ the following upper bound is true

\begin{equation}
J_\nu(r) \leq \frac{c}{\Gamma(\nu+1)} \left( \frac{r}{2} \right)^\nu.
\end{equation}

Combining (40) and (41), the integrant in (39) behaves like $\sim \nu y^{\nu-1}$, when $y \ll 1 + \nu$.

We will examine various cases of the parameter $\nu$.

\begin{itemize}
  \item Case 1: $0 < \nu \leq 2$
\end{itemize}

We break the integral into two parts, from 0 to $|\eta|^{-\alpha}$ and the rest and integrate by parts the latter, i.e. we write $K_{\nu,1}^0(\eta) = I_1 + B_2 + I_2$, where

\begin{align*}
I_1 &= -\frac{1}{2\eta} \int_0^{\eta^{-\alpha}} m_\nu'(y^{1/2}) y^{-1/2} \sin(\eta y) dy \\
B_2 &= -\frac{1}{2\eta^2} \cos(\eta |\eta|^{-\alpha}) m_\nu'^0(|\eta|^{-\alpha/2}) |\eta|^{\alpha/2} \\
I_2 &= -\frac{1}{4\eta^2} \int_{|\eta|^{-\alpha}}^\infty m_\nu''(y^{1/2}) y^{-1} \cos(\eta y) - m_\nu'(y^{1/2}) y^{-3/2} \cos(\eta y) dy,
\end{align*}

where $\alpha$ is a parameter to be determined later.
Estimate $I_1$ using the equation (41), we have $|I_1| \sim |\eta|^{-1-\frac{\alpha}{2}}$. Taking the absolute value, we have $|B_2| \sim \nu|\eta|^{-\frac{\alpha}{2}+\alpha-2}$. For $I_2$, we use the fact that Bessel function is the solution of the following differential equation

\[ J''_\nu(r) + \frac{1}{r}J'_\nu(r) + \left(1 - \nu^2 r^2\right)J_\nu(r) = 0. \]

Combining with the identity (40), we have

\[ J''_\nu(r) = \frac{1}{r}J'_{\nu+1}(r) - \left(1 + \frac{\nu}{r^2} - \frac{\nu^2}{r^2}\right)J_\nu(r). \]

Using (42), we can estimate the integrand in $I_2$ by $c\nu (\nu - 2)^{-2}$. Thus, we have $|I_2| \leq c\nu (\nu - 2)^{-2}$. To balance the contribution from $I_1$, $B_2$, and $I_2$, we choose $\alpha = 1$. Thus, we have $K_{0,1}(\eta) < c\nu (1 + \frac{\nu}{2})$.

**Case 2: $\nu < 2$**

We do not split the integral in this case. We can integrate by parts twice without introducing boundary terms and obtain

\[ K_0^1(\eta) = -\frac{1}{4\eta^2} \int_0^\infty \left( m_{0,\nu}^{1''}(y^{1/2})y^{-1}\cos(\eta y) - m_{0,\nu}^{1'}(y^{1/2})y^{-3/2}\cos(\eta y) \right) dy. \]

Since $m_{0,\nu}$ is supported within $[0, \nu/\sqrt{2})$, the integral is bounded by $\eta^{-2}$ multiplied by a constant namely $C(\nu) = c(\nu - 2)(\Gamma(\nu + 1))^{-1}2^{3\nu}$, where $P$ is a polynomial with finite degree. Using the Stirling’s formula

\[ \Gamma(z) = \sqrt{2\pi} \left( \frac{z}{e} \right)^z \left(1 + O\left(\frac{1}{z}\right)\right), \]

and observing that $e < 2^{3/2}$, we can see that $C(\nu)$ has a bound independent of $\nu$.

Now, we took care of the tail. The remaining task is to prove that $K_{0,1}(\eta)$ is bounded. We take absolute value of the integrant in (39)

\[ |K_{0,1}^1(\eta)| \leq \int |m_{0,\nu}(\sqrt{|y|})| dy. \]

Since $m_{0,\nu}$ is a bounded function with a compact support, we proved $K_{1,1}(\eta)$ is bounded for fixed $\nu$. Furthermore if we apply (42), we have

\[ |K_{0,1}^1(\eta)| \leq c \frac{\nu^3 \nu^\nu}{\Gamma(\nu + 1)2^{\frac{3\nu}{2}}}. \]

Using the Stirling’s formula (45) again, we can show that there is a bound independent of $\nu$. \qed

### 4. Estimates for Middle Frequency

The goal is to prove the inequality (29), namely

\[ \|T_{\nu,\tau}(g)(t)\|_{L^2_x L^\infty_t} \leq C\|g\|_{L^2}. \]
First, we want to estimate $L^\infty_r$ norm for fixed $t$. Recall the equation (26), we have

$$T^1_{\nu, r}(g)(t) = \frac{1}{\sqrt{2\pi}} \int m^1_{\nu}(r|\xi|^{\frac{1}{2}}) \hat{g}(\xi) e^{i\xi t} d\xi$$

Since composing Fourier transform with inverse Fourier transform will form identity map, we have

$$T^1_{\nu, r_0}(g)(t) = \frac{1}{\sqrt{2\pi}} \int \int m^1_{\nu}(r|\xi|^{\frac{1}{2}}) \hat{g}(\xi) e^{i\xi t} d\xi e^{ir\rho} dr e^{-i\rho r_0} d\rho.$$

Using smooth dyadic decomposition, we write $\hat{g}(\xi) = \sum \hat{g}_n(\xi)$ where $\hat{g}_n$ is supported on $(-2^{n+1}, -2^{n-1}) \cup (2^{n-1}, 2^{n+1})$. We will prove the following lemma.

**Lemma 4.1.** For $g_n \in L^2(\mathbb{R})$ such that $\hat{g}_n$ supported on $(-2^{n+1}, -2^{n-1}) \cup (2^{n-1}, 2^{n+1})$, we have the estimate

$$\|T^1_r(g_n)(t)\|_{L^\infty} \leq C \|g_n\|_{L^2},$$

where $C$ is independent of $n$.

**Proof.** On the right hand side of (49), we multiply and divide by $\sqrt{b + \rho^2 b^{-1}}$, where $b > 0$ is a parameter to be chosen later. We change the order of integration, and apply Holder’s inequality to obtain

$$|T^1_{r_0}(g_n)(t)| \leq C \left( \int \frac{|e^{-i\rho r_0(t)}|^2}{(b + \rho^2 b^{-1})} d\rho \right)^{\frac{1}{2}} \left( \int \left( \int m^1_{\nu}(r|\xi|^{\frac{1}{2}}) \hat{g}_n(\xi) e^{i\xi t} e^{ir\rho} dr \right)^2 (b + \rho^2 b^{-1}) d\rho \right)^{\frac{1}{2}}.$$

Note that the first integral on the right hand side is $\pi$ for any $b > 0$. Thus, equation (51) reduces to

$$\|T^1_{r_0}(g_n)(t)\|_{L^\infty} \leq C \left( \int \left( \int m^1_{\nu}(r|\xi|^{\frac{1}{2}}) \hat{g}_n(\xi) e^{i\xi t} e^{ir\rho} dr \right)^2 (b + \rho^2 b^{-1}) d\rho \right)^{\frac{1}{2}}.$$

We name the integral on the right hand side of (52) as $l(t)$. We distribute the sum $(b + b^{-1}\rho^2)$ and write $l^2(t) = l_1(t) + l_2(t)$, where

$$l_1(t) = \int \left( \int \int b^{\frac{1}{2}} m^1_{\nu}(r|\xi|^{\frac{1}{2}}) \hat{g}_n(\xi) e^{i\xi t} e^{ir\rho} dr \right)^2 d\rho,$$

$$l_2(t) = \int \left( \int \int b^{-\frac{1}{2}} \rho m^1_{\nu}(r|\xi|^{\frac{1}{2}}) \hat{g}_n(\xi) e^{i\xi t} e^{ir\rho} dr \right)^2 dp.$$
For \( l_2(t) \), we integrate by parts with respect to \( r \) to remove \( \rho \) and obtain,

\[
l_2(t) = \int \left| \int \int b^{-\frac{1}{2}} |\xi|^{\frac{1}{2}} (m_\nu(y))' (r|\xi|^\frac{1}{2}) \hat{g}_n(\xi) e^{i\xi t} d\xi e^{ir\rho} dr \right|^2 dp.
\]

Using the Plancherel’s theorem, we have

\[
(53) \quad l_1(t) = \int \left| \int b^{\frac{1}{2}} m_\nu(r|\xi|^\frac{1}{2}) \hat{g}_n(\xi) e^{i\xi t} d\xi \right|^2 dr
\]

\[
(54) \quad l_2(t) = \int \left| \int b^{-\frac{1}{2}} |\xi|^{\frac{1}{2}} (m_\nu(y))' (r|\xi|^\frac{1}{2}) \hat{g}_n(\xi) e^{i\xi t} d\xi \right|^2 dr.
\]

We square both sides of (52) and integrate over \( t \). Then, we change the order of integration with respect to \( r, t \), and apply Plancherel’s theorem again to obtain

\[
(55) \quad \| T^1_{r_0}(g_n)(t) \|_{L^2_t L^\infty_r}^2 \leq C \int \int \left| b^{\frac{1}{2}} m_\nu(r|\xi|^\frac{1}{2}) \hat{g}_n(\xi) \right|^2 d\xi dr
\]

\[
+ C \int \int \left| b^{-\frac{1}{2}} |\xi|^{\frac{1}{2}} (m_\nu(y))' (r|\xi|^\frac{1}{2}) \hat{g}_n(\xi) \right|^2 d\xi dr.
\]

We change the variable \( y = r|\xi|^\frac{1}{2} \). We have

\[
(56) \quad \| T^1_{r_0}(g_n)(t) \|_{L^2_t L^\infty_r}^2 \leq C \int \left( \int \frac{b}{|\xi|^\frac{1}{2}} |m_\nu(y)|^2 dy + \frac{|\xi|^\frac{1}{2}}{b} |m_\nu'(y)|^2 dy \right) |\hat{g}_n(\xi)|^2 d\xi.
\]

Use Lemma(6.1) (see appendix) which implies

\[
(57) \quad \int |m_\nu(y)|^2 dy < C, \quad \int (m_\nu(y))' \|_{L^2_t}^2 dy < C.
\]

Recall that the \( \hat{g}_n \) is supported on \((-2^{n+1}, -2^{n-1}) \cup (2^{n-1}, 2^{n+1})\). By choosing \( b = 2^\frac{\nu}{2} \), we complete the proof. \( \square \)

We proved (29) for function has bounded support in Fourier domain described above. Now we are going to discuss the general case.

**Proof.** (29) Suppose \( r_0(t) \) realizes at least half of the supremum at every \( t \). Then, it is enough to prove the inequality

\[
(58) \quad \int |T^1_{r_0}(g)(t)|^2 dt \leq C \| g \|_{L^2}^2.
\]

We will prove (58) for an arbitrary function \( r_0(t) \). We dyadically decompose the range of \( r_0(t) \). The corresponding domains are defined as follows.

\[
(59) \quad I_k = \{ t | 2^k < r_0(t) \leq 2^{k+1} \}
\]

Since \( m^1_\nu \) is supported on \((\nu/2, 2\nu)\). We have

\[
(60) \quad \frac{\nu}{2} < r_0|\xi|^\frac{1}{2} < 2\nu.
\]
On $I_k$, by definition we have $2^k < r_0(t) \leq 2^{k+1}$. Combining (39) and (40), the integrant in the expression (49) is nonzero only when

$$2 \log_2 \nu - 2k - 4 < \log_2 |\xi| < 2 \log_2 \nu - 2k + 2.$$  

As a result, there are only 8 components in the dyadic decomposition in $\{\tilde{g}_n\}$ involved. When $t \in I_k$, we can rewrite (49)

$$T^{1}_{\nu,r_0(t)}(g)(t) = \frac{1}{\sqrt{2\pi}} \int m_{\nu}(r_0|\xi|^2) \sum_{n=n_0(k)}^{n_0+7} \tilde{g}_n(\xi)e^{i\xi t}d\xi = \sum_{n=n_0(k)}^{n_0+7} T^{1}_{\nu,r_0(t)}(g_n)(t),$$

where $n_0(k) = \lfloor 2 \log_2 \nu - 2k - 4 \rfloor$. Thus, use Cauchy-Schwartz inequality in finite sum to obtain

$$|T^{1}_{\nu,r_0(t)}(g)(t)|^2 = \left| \sum_{n=n_0(k)}^{n_0+7} T^{1}_{\nu,r_0(t)}(g_n)(t) \right|^2 \leq 8 \sum_{n=n_0(k)}^{n_0+7} \left| T^{1}_{\nu,r_0(t)}(g_n)(t) \right|^2.$$  

Combine the above with the Lemma (4.1), we have

$$\int_{I_k} |T^{1}_{\nu,r_0(t)}(g)(t)|^2 dt \leq C \sum_{n=n_0(k)}^{n_0+7} \|g_n\|^2_{L^2}.$$  

We sum over $k$.

$$\int |T^{1}_{\nu,r_0(t)}(g)(t)|^2 dt \leq C \sum_{k \in \mathbb{Z}} \sum_{n=n_0(k)}^{n_0+7} \|g_n\|^2_{L^2}.$$  

Note when we increase from $k$ to $k+1$, $n_0$ increases by 2. As a result, every $n$ only appears four times. Thus

$$\int |T^{1}_{\nu,r_0(t)}(g)(t)|^2 dt \leq 4C \sum_{n \in \mathbb{Z}} \|g_n\|^2_{L^2} \leq 4C\|g\|^2_{L^2}.$$  

This completes the proof. \qed

5. Estimates for High Frequency

The goal is to prove (30), which is equivalent to

$$\left\| \int_{\mathbb{R}} K^{j}_{r(t)}(t-\eta)g(\eta)d\eta \right\|_{L^2} \leq C2^{-\frac{4j}{3}}\|g(y)\|_{L^2},$$

for an arbitrary function $r(t)$. Using the $T^*T$ argument, we have the following lemma.

**Lemma 5.1.** The following three inequalities are equivalent.

$$\left\| \int_{\mathbb{R}} K^{j}_{\nu,r(t)}(t-\eta)g(\eta)d\eta \right\|_{L^2} \leq C2^{-\frac{4j}{3}}\|g(y)\|_{L^2}, \quad \forall g \in L^2(\mathbb{R}^1)$$
\[\left\| \int_{\mathbb{R}} K_{j}(t) F(t) dt \right\|_{L^2} \leq C 2^{-j} \| F \|_{L^2} \quad \forall F \in L^2(\mathbb{R}^1) \]

\[\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\overline{K_{j}(t')}}{K_{j}(t)} \right| (t' - \eta) d\eta F(t') dt' \right\|_{L^2} \leq C 2^{-j} \| F \|_{L^2}, \forall F \in L^2(\mathbb{R}^1). \]

**Proof.** Suppose we have (68), we want to show it implies (69). We multiply the integrant on the left hand side of (68) with arbitrary \(L^2\) function \(F(t)\), then integrate over \(t, \eta\). We apply Holder’s inequality and (68) to obtain

\[\int_{\mathbb{R}} \int_{\mathbb{R}} K_{j}(t) F(t) dt g(\eta) d\eta \leq C e^{-\frac{j}{2}} \| g \|_{L^2} \| F \|_{L^2} \]

Use the property that \(L^2\) is self-dual, i.e.

\[\| h \|_{L^2} = \sup_{f \in L^2} \frac{\int f(t) h(t) dt}{\| f \|_{L^2}}.\]

We obtain (69). Using the same argument again, we can prove (68) \(\iff\) (69).

Suppose we have (69), we will show that (70) holds. We multiply the integrant on the left hand side of (70) with arbitrary \(L^2\) function \(G(t)\) and integrate over \(\eta, t', t\). We change the order of integration and apply Holder’s inequality and (69) to obtain

\[\left| \int_{\mathbb{R}} \int_{\mathbb{R}} K_{j}(t) F(t) dt G(\eta) d\eta \right| \leq C e^{-\frac{j}{2}} \| G \|_{L^2} \| F \|_{L^2}, \]

which implies (70) by duality.

Suppose (70) holds. We multiply the integrant on the left hand side of (70) with complex conjugate of \(F(t)\), \(\bar{F}(t)\), integrate over \(\eta, t'\), and \(t\). We change the order of integration and apply Holder’s inequality and (70), we obtain (69). This completes the proof. \(\square\)

Thus, to prove (67), I have to to prove (70). Inequality (70) will follow from the following estimate.

**Lemma 5.2.** For any \(a, b > 0\), \(t, t' \in \mathbb{R}\) we have

\[\left| \int_{\mathbb{R}} K_{j}(t) K_{b}(t' - \eta) d\eta \right| < \frac{1}{a^2} \Phi_j\left(\frac{|t - t'|}{a^2}\right)\]

where \(\Phi_j\) is even non-increasing non-negative function with

\[\| \Phi_j \|_{L^1} = \frac{1}{a^2} \Phi_j\left(\frac{y}{a^2}\right) \|_{L^1(y)} \leq C 2^{-\frac{j}{2}}\]
The estimate \( \text{[7.3]} \) does not depend on \( b \). And, \( \Phi_j \) is even non-increasing non-negative. Thus, we have

\[
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} K_{r(t)}(t - \eta) K_{r(t')}^{j}(t' - \eta) d\eta F(t') dt' \right| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{r(t)^2} \Phi_j \left( \frac{|t - t'|}{r(t)^2} \right) |F(t')| dt'
\]

\[
\leq \sup_{r > 0} \int_{\mathbb{R}} \frac{1}{r^2} \Phi_j \left( \frac{|t - t'|}{r^2} \right) |F(t')| dt' \leq \|\Phi_j\|_{L^1} M(F)(t) \leq C 2^{-\frac{\alpha}{2}} M(F)(t),
\]

where \( M(F) \) is Hardy-Littlewood maximal function of \( F \). We apply Hardy-Littlewood maximal inequality \( \text{[37]} \) to finish the proof of \( \text{[30]} \).

**Proof.** (Lemma 5.2) The kernel is in the form of inverse Fourier transform

\[
K_{\nu,r}(\eta) = \int_{\mathbb{R}} m_j^\nu(r|y|^{1/2}) e^{iy\eta} dy.
\]

By Plancherel's theorem, we have

\[
\int_{\mathbb{R}} K_{\nu,a}^{j}(t - \eta) K_{\nu,a}^{j}(t' - \eta) d\eta = \int_{\mathbb{R}} m_j^\nu(a|y|^{1/2}) m_j^\nu(b|y|^{1/2}) e^{i(t-t')y} dy.
\]

From the standard asymptotic of Bessel functions (see \( \text{[17]} \)), we have

\[
m_j^\nu(\xi) = \sum_{\pm} 2^{-j/2} e^{\pm i\xi} \psi_j^\pm(2^{-j}\xi),
\]

where \( \psi_j^\pm(\xi) \) are function supported on \( |\xi| \sim 1 \) and bounded uniformly in \( j, \nu \). We can rewrite the right hand side of (75) as a finite number of expressions of the form

\[
2^{-j} \left| \int e^{i(\pm a \pm b)|y|^{1/2} e^{i(t-t')y} \psi_j^\pm(2^{-j}a|y|^{1/2}) \psi_j^\pm(2^{-j}b|y|^{1/2}) dy \right|,
\]

where \( \pm \) sign need not agree. Since the bump functions are supported on \( (\frac{1}{2}, 2) \), this expression is identically zero except when \( \frac{1}{4} < \frac{b}{a} < 4 \). Let \( \alpha = \frac{b}{a} \)

The expression in (78) becomes,

\[
2^{-j} \left| \int e^{i(\pm a \pm b)|y|^{1/2} e^{i(t-t')y} \psi_j^\pm(2^{-j}a|y|^{1/2}) \psi_j^\pm(2^{-j}\alpha a|y|^{1/2}) dy \right|
\]

Let \( s = (t - t') \). We name the sum of this expression \( \phi_{a,\alpha}^j(s) \). By changing variable \( y = a^2 y \), we can see that

\[
\phi^j_{a,\alpha}(s) = \frac{1}{a^2} \phi^j_{1,\alpha} \left( \frac{s}{a^2} \right).
\]

By letting \( s' = \frac{s}{a^2} \), we have

\[
\|\phi^j_{a,\alpha}(s)\|_{L^1} = \|\phi^j_{1,\alpha}(s)\|_{L^1}.
\]

Now, \( \phi^j_{1,\alpha}(s) \) is finite sum of the following expressions

\[
2^{-j} \left| \int e^{i(\pm a \pm b)|y|^{1/2} e^{i(s)y} \psi_j^\pm(2^{-j}|y|^{1/2}) \psi_j^\pm(2^{-j}\alpha|y|^{1/2}) dy \right|.
\]
Changing the variable $z = 2^{-j}|y|^{1/2}$, the expression in (82) becomes

\[ 2^{j+2} \int_0^\infty e^{i(\pm 1 \pm \alpha)2^j z + s2^j z^2} \psi_j^\pm(z) \psi_j^\pm(\alpha z) z dz \]

We will prove the following lemma.

**Lemma 5.3.** $\phi_{j,\alpha}^I$ is controlled by the following function

\[ \Phi_j(s) = C \begin{cases} 
2^j s^{-1/2} & \text{when } 0 < s \leq 2^{-2j} \\
2^{j(2^j s)^{-10}} & \text{otherwise.} 
\end{cases} \]

Notice that we can estimate directly and get $\|\Phi_j\|_{L^1} \leq C 2^{j-\frac{1}{2}}$.

**Proof.** (Lemma 5.3) Take absolute value of the integrant to obtain the bound $2^j$. We will use stationary phase technique to prove the other two estimates. We call the function on the index of exponential phase. If we differentiate the phase, we get $2^j s z + (\pm 1 \pm \alpha)2^j$. Note $\alpha$ is between $1/4$ and $4$, and the product of bump functions is supported on $(1/8, 8)$. Suppose $s > 20 2^{-j}$, the derivative is never zero on the support. We get the bound $2^j (2^j s)^{-10}$ from non-stationary phase analysis. Otherwise, we note the second derivative of the phase, namely $2^{j+1} s$, is not zero. By stationary phase analysis we obtain the bound $s^{-1/2}$.

6. **Appendix: Estimate of Bessel Function around $\nu$**

**Lemma 6.1.** For the Bessel function $J_\nu$ of positive order $\nu$ and when $\frac{1}{2} \nu \leq r \leq 2 \nu$, we have the following estimates

\[ J_\nu(r) \leq C \nu^{-\frac{1}{3}} \left( 1 + \nu^{-\frac{1}{3}} |r - \nu| \right)^{-\frac{1}{3}} \]
\[ J'_\nu(r) \leq C \nu^{-\frac{1}{2}}. \]

**Proof.** We have an integral representation for the Bessel function of order $\nu > -\frac{1}{2}$ (see [23]).

\[ J_\nu(r) = A_\nu(r) - B_\nu(r), \]

where

\[ A_\nu(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(\nu \sin \theta - \nu \theta)} d\theta \]
\[ B_\nu(r) = \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu t - r \sinh(t)} dt. \]

We can see

\[ B_\nu(r) < \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu t} dt < C \nu^{-1}. \]
So, we only need to estimate $A_\nu$. We will accomplish this using stationary phase for two different cases $r > \nu$ and $r \leq \nu$. Let us consider the case when $r > \nu$ first. Call the phase in (88) $\phi(\theta) = r \sin \theta - \nu \theta$. We differentiate the phase, $\phi'(\theta) = (r \cos \theta - \nu)$. We find $\phi' = 0$ at $\pm \theta_0$, where $\theta_0 = \cos^{-1}(\frac{\nu}{r})$.

In order to obtain the estimate, we will break the integral into a small neighborhood around these points and the rest, that is

\begin{align}
N_\varepsilon &:= \{ \theta : |\theta + \theta_0| < \varepsilon \} \\
S_\varepsilon &:= [-\pi, \pi]/N_\varepsilon
\end{align}

Since the integrant is in (88) is bounded, we have

\begin{equation}
|\int_{N_\varepsilon} e^{-i(r \sin \theta - \nu \theta)}d\theta| < c\varepsilon.
\end{equation}

On $S_\varepsilon$, we integrate by parts,

\begin{equation}
\int_{S_\varepsilon} e^{-i(r \sin \theta - \nu \theta)}d\theta = \frac{e^{i(r \sin \theta - \nu \theta)}}{i(r \cos \theta - \nu)}\left\{\pi, \theta_0 \pm \varepsilon\right\} + \int_{S_\varepsilon} e^{i(r \sin \theta - \nu \theta)}r \sin \theta d\theta.
\end{equation}

All terms in the expression above are controlled by $c|\nu \cos(\theta_0 \pm \varepsilon) - \nu|^{-1}$.

We want to balance the contribution from $N_\varepsilon$ and $S_\varepsilon$ by choosing proper $\varepsilon$, such that

\begin{equation}
\varepsilon \sim |\nu \cos(\theta_0 \pm \varepsilon) - \nu|^{-1}.
\end{equation}

Using trigonometric identities, $\cos(\theta_0 \pm \varepsilon) = \cos(\theta_0) \cos(\varepsilon) \mp \sin(\theta_0) \sin(\varepsilon)$, and the definition of $\theta_0$, we have,

\begin{equation}
|r \cos(\theta_0 \pm \varepsilon) - \nu| = |\nu \cos \varepsilon - \sqrt{r^2 - \nu^2} \sin \varepsilon - \nu|.
\end{equation}

When $\varepsilon$ is small, (95) is approximately $\frac{\varepsilon}{2} \varepsilon^2 + \varepsilon \sqrt{r^2 - \nu^2}$. Thus, we have the two estimates

\begin{align}
|r \cos(\theta_0 \pm \varepsilon) - \nu|^{-1} &\leq 2\nu^{-1} \varepsilon^{-2}, \\
|r \cos(\theta_0 \pm \varepsilon) - \nu|^{-1} &\leq \varepsilon^{-1}(r^2 - \nu^2)^{-\frac{3}{4}}.
\end{align}

When $r - \nu$ is small, (96) is sharper. We pick $\varepsilon \sim \nu^{-\frac{3}{4}}$. When $r - \nu$ is big, (97) is sharper. We pick optimal $\varepsilon \sim (r^2 - \nu^2)^{-\frac{1}{4}}$. Since $r \leq 2\nu$, we have $|(r^2 - \nu^2)^{-\frac{1}{4}}| < (3\nu)^{-\frac{1}{4}}(r - \nu)^{-\frac{1}{4}}$. Thus, we have proven (88) for the case $r \geq \nu$.

Now, we will discuss the case when $r \leq \nu$. When $\nu - \nu^{-\frac{1}{4}} < r < \nu$, we follow the analysis above by choosing $\theta_0 = 0$, $\varepsilon = \nu^{-\frac{1}{4}}$. We have an estimate $J_\nu(r) \leq C\nu^{-\frac{1}{4}}$. When $\nu - \nu^{-\frac{1}{4}} > r$, we use non-stationary phase, we obtain $J_\nu(r) \leq C\frac{1}{|\nu - r|^{\frac{1}{4}}} \leq C\nu^{-\frac{1}{4}}|\nu - r|^{-\frac{1}{4}}$.

The remaining task is to prove (89). Considering the derivative of $J_\nu$, we can show

\begin{equation}
|B_\nu'(r)| = |\frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu t - r \sinh(t)} \sinh(t) dt| \leq c \int_0^\infty e^{-\nu t} dt \leq \frac{c}{\nu}.
\end{equation}
So, we only need to estimate
\begin{equation}
A_
u'(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r\sin\theta - \nu\theta)} i \sin \theta d\theta
\end{equation}

When \( r > \nu \), break the integral into two as we did in the previous case.

\begin{equation}
\left| \frac{1}{2\pi} \int_{N_\varepsilon} e^{i(r\sin\theta - \nu\theta)} i \sin \theta d\theta \right| < C \varepsilon
\end{equation}

Integrate by parts for the integral on \( S_\varepsilon \), and use trigonometric identity and Taylor expansion. Then, we can find it is controlled by
\begin{equation}
\frac{c_1 \sqrt{r - \nu} + c_2 \varepsilon}{\nu^{\frac{3}{2}} + c_3 \sqrt{r - \nu}}
\end{equation}

If we balance between integral on \( N_\varepsilon \) and \( S_\varepsilon \), we get optimal \( \nu^{-\frac{1}{2}} \). For the case \( r \leq \nu \), we can apply similar ideas as in the proof of (85). □

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