GENERAL TAIL BOUNDS FOR RANDOM TENSORS SUMMATION: MAJORIZATION APPROACH

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Abstract. In recent years, tensors have been applied to different applications in science and engineering fields. In order to establish theory about tail bounds of the tensors summation behavior, this work extends previous work by considering the tensors summation tail behavior of the top $k$-largest singular values of a function of the tensors summation, instead of the largest/smallest singular value of the tensors summation directly (identity function) explored in [5]. Majorization and antisymmetric tensor product tools are main techniques utilized to establish inequalities for unitarily norms of multivariate tensors. The Laplace transform method is integrated with these inequalities for unitarily norms of multivariate tensors to give us tail bounds estimation for Ky Fan $k$-norm for a function of the tensors summation. By restricting different random tensor conditions, we obtain generalized tensor Chernoff and Bernstein inequalities.

Key words. Random Tensors, Chernoff Inequality, Bernstein Inequality, Unitarily Invariant Norm, Log-Majorization

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1. Introduction.

1.1. From Random Matrices to Random Tensors. A random matrix is a matrix-valued random variable—that is, a matrix in which some or all entries are random variables. Random matrices have played an important role in numerical linear algebra [12], quantum mechanics [4], neural networks [39], communication theory [37], robust control [16], etc. Many important properties of scientific and engineering systems can be modelled by matrix formulations. In order to consider a high-dimensional system, it is often more convenient to consider tensors, or multivariate data, instead of matrices (two-dimensional data).

In recent years, tensors have been applied to different applications in science and engineering [30]. In data processing fields, tensor theory applications include unsupervised separation of unknown mixtures of data signals [41, 25], signals filtering [26], network signal processing [33, 32, 11] and image processing [20, 18]. In wireless communication applications, tensors are applied to model high-dimensional communication channels, e.g., MIMO (multi-input multi-output) code-division [8, 44], radar communications [29, 34]. In numerical multilinear algebra computations, tensors can be applied to solve multilinear system of equations [40], high-dimensional data fitting/regression [9], tensor complementary problem [42], tensor eigenvalue problem [7], etc. In machine learning, tensors are also used to characterize data with coupling effects, for example, tensor decomposition methods have been reported recently to establish the latent-variable models, such as topic models in [2], and the method of moments for undertaking the Latent Dirichlet Allocation (LDA) in [35]. Nevertheless, all these applications assume that systems modelled by tensors are fixed and such assumption is not true and practical in problems involving tensor formulations. In recent years, there are more works beginning to develop theory about random tensors, see [13, 19, 38], and references therein. In this work, we will apply majorization techniques to establish inequalities for unitarily norms of multivariate tensors and

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these inequalities will be used to derive more general tensor Chernoff and Bernstein inequalities.

1.2. Technical Results. Majorization is an effective tool for proving norm and trace inequalities of linear operators, see [24] for more advanced topics therein. An important topic investigated by majorization technique is to answer the following question:

**Question 1**: Given a continuous function \( f \), two matrices \( C \) and \( D \), what is the relationship between the majorization order among singular values of \( f(C) \) and \( f(D) \), and the matrix norm relation of \( f(C) \) and \( f(D) \)?

In Prop. 4.4.13 in [14], they showed that the singular values of \( C \) are weakly majorized by the singular values of \( D \) if and only if \( \|C\|_\rho \leq \|D\|_\rho \) for every unitarily invariant norm \( \|\cdot\|_\rho \), e.g., Schatten \( p \)-norms and trace norm. In this previous work, they only find this relationship in Question 1 when the function \( f \) is an identity map. In another recent work [15], they apply log-majorization to find relationship in Question 1 by considering more general continuous function \( f \) and utilize this new relationship to prove multivariate generalizations of the Araki–Lieb–Thirring inequality and the Golden–Thompson inequality for unitarily invariant norm of matrices.

In this work, we generalize this approach to answer Question 1 in tensors settings. Our first main result is to prove multivariate tensor norm inequalities given by Theorem 1.1. Besides extending from matrices to tensors, our work tries to simplify some steps in the approach adopted by [15], for example, Lemma 3.5 is simplified from Lemma 15 in [15].

**Theorem 1.1.** Let \( C_i \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) be positive definite Hermitian tensors for \( 1 \leq i \leq n \) with Hermitian rank \( r \), \( \|\cdot\|_\rho \) be a unitarily invariant norm with corresponding gauge function \( \rho \). For any continuous function \( f : (0, \infty) \rightarrow [0, \infty) \) such that \( x \rightarrow \log f(e^x) \) is convex on \( \mathbb{R} \), we have

\[
(1.1) \quad \left\| f \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_\rho \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} C_i^{1+t} \right) \right\|_\rho \beta_0(t) \, dt,
\]

where \( t = \sqrt{-1} \) and \( \beta_0(t) = \frac{\pi}{2(\cosh(\pi t)+1)} \).

For any continuous function \( g(0, \infty) \rightarrow [0, \infty) \) such that \( x \rightarrow g(e^x) \) is convex on \( \mathbb{R} \), we have

\[
(1.2) \quad \left\| g \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_\rho \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_i^{1+t} \right) \right\|_\rho \beta_0(t) \, dt.
\]

There are two main technical tools required by this work to build those tensor probability bounds. The first is *Laplace transform method*, which provides a systematic way to give tail bounds for the sum of scalar random variables. In [1], the authors apply Laplace transform method to bound the largest eigenvalue with the matrix setting, i.e., the tail probability for the maximum eigenvalue of the sum of Hermitian matrices is controlled by a matrix version of the moment-generating function. They prove following:

\[
(1.3) \quad \Pr \left( \lambda_{\max} \left( \sum_{i} X_i \right) \geq \theta \right) \leq \inf_{t>0} \left\{ e^{-t\theta} \mathbb{E} \exp \left( \sum_{i} t X_i \right) \right\}.
\]

\(^1\)The exact unitarily invariant norm definition will be provided in Section 2.2.
In this work, we extend the Laplace transform method to tensors and utilize Theorem 1.1 to obtain the following Ky Fan $k$-norm bounds for the tail behavior of a function of tensors summation.

**Theorem 1.2.** Consider a sequence $\{X_j \in \mathbb{C}^{I_1 \times \cdots \times I_N} \}$ of independent, random, Hermitian tensors. Let $g$ be a polynomial function with degree $n$ and nonnegative coefficients $a_0, a_1, \ldots, a_n$ raised by power $s \geq 1$, i.e., $g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s$. Suppose following condition is satisfied:

$$g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( tg \left( \sum_{j=1}^{m} X_j \right) \right)$$  \hspace{1cm} \text{almost surely}, \hspace{1cm} (1.4)$$

where $t > 0$. Then, we have

$$\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t > 0} \left( k a_0^s + \sum_{l=1}^{n} a_l^s \sum_{j=1}^{m} \mathbb{E} \left[ \exp \left( p_j \lambda s X_j \right) \right] \right).$$  \hspace{1cm} (1.5)$$

where $\sum_{j=1}^{m} p_j = 1$ and $p_j > 0$.

If we restrict following conditions: $X_i \geq 0$ and $\lambda_{\max}(X_i) \leq R$ almost surely; to random tensors $X_i$, we can further bound the term of $\mathbb{E} \left[ \exp \left( p_j \lambda s X_j \right) \right]$ to get the following general tensor Chernoff bound provided by Theorem 1.3.

**Theorem 1.3 (Generalized Tensor Chernoff Bound).** Consider a sequence $\{X_j \in \mathbb{C}^{I_1 \times \cdots \times I_N} \}$ of independent, random, Hermitian tensors. Let $g$ be a polynomial function with degree $n$ and nonnegative coefficients $a_0, a_1, \ldots, a_n$ raised by power $s \geq 1$, i.e., $g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s$ with $s \geq 1$. Suppose following condition is satisfied:

$$g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( tg \left( \sum_{j=1}^{m} X_j \right) \right)$$  \hspace{1cm} \text{almost surely}, \hspace{1cm} (1.6)$$

where $t > 0$. Moreover, we require

$$X_i \geq 0 \hspace{1cm} \text{and} \hspace{1cm} \lambda_{\max}(X_i) \leq R \hspace{1cm} \text{almost surely.} \hspace{1cm} (1.7)$$

Then we have following inequality:

$$\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t > 0} e^{-\theta t}.$$  \hspace{1cm} (1.8)$$

$$\left\{ k a_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} k a_l^s \left[ 1 + \left( e^{m \lambda s R t} - 1 \right) \frac{1}{\sigma_1(X_j)} + C \left( e^{m \lambda s R t} - 1 \right) \Xi(X_j) \right] \right\},$$

where $C$ is a constant and $\Xi(X_j)$ is determined from the expectation of entries from the tensor $X_j$ defined by Eq. (4.16).
On the other hand, if we consider following conditions to random tensors $X_j$:

$$ E[X] = 0 \quad \text{and} \quad X_j^p \leq \frac{plA_j^2}{2} \quad \text{almost surely for } p = 2, 3, 4, \ldots $$

we will get the following generalized tensor Bernstein bound provided by Theorem 1.4.

**Theorem 1.4 (Generalized Tensor Bernstein Bound).** Consider a sequence $\{X_j \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}\}$ of independent, random, Hermitian tensors. Let $g$ be a polynomial function with degree $n$ and nonnegative coefficients $a_0, a_1, \ldots, a_n$ raised by power $s \geq 1$, i.e., $g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s$ with $s \geq 1$. Suppose following condition is satisfied:

$$ g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( t g \left( \sum_{j=1}^{m} X_j \right) \right) \quad \text{almost surely}, $$

where $t > 0$, and we also have

$$ E[X] = 0 \quad \text{and} \quad X_j^p \leq \frac{plA_j^2}{2} \quad \text{almost surely for } p = 2, 3, 4, \ldots $$

Then we have following inequality:

$$ \Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\| \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t>0} e^{-\theta t} k. $$

$$ \left\{ a_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} a_l^{ls} \left[ \frac{1}{m} + \frac{m(lst)^2\sigma_l(A_j^2)}{2(1 - mlst)} + lstCY(X_j) \right] \right\}, $$

where $C$ is a constant and $Y(X_j)$ is determined from the expectation of entries from the tensor $X_j$ defined by Eq. (4.32).

The bounds provided by Theorems 1.2, 1.3 and 1.4 are more general than those random tensor bounds given by [5] since we can consider the concentration behavior of the top $k$-largest singular values, instead of the largest singular value, and a function of tensors summation, instead of the identity map of tensors summation. We have another work based on similar majorization techniques to consider Chernoff expander bounds for random tensors without independent random tensors assumptions [6].

1.3. Paper Organization. The paper is organized as follows. Preliminaries of tensors and basic majorization notations are given in Section 2. In Section 3, we will develop theorems about majorization and log majorization with integral average which will be applied to derive bounds for unitarily norms of multivariate tensors. Compared to work [5], more generalized tensor Chernoff and Bernstein inequalities are discussed in Section 4. Finally, the conclusions are given in Section 6.

2. Fundamentals of Tensors and Majorization.

2.1. Tensors Preliminaries. Throughout this work, scalars are represented by lower-case letters (e.g., $d$, $e$, $f$, ...), vectors by boldfaced lower-case letters (e.g., $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$, ...), matrices by boldfaced capitalized letters (e.g., $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, ...), and tensors by calligraphic letters (e.g., $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, ...), respectively. Tensors are multiarrays of...
values which are higher-dimensional generalizations from vectors and matrices. Given a positive integer \( N \), let \( [N] \) denote \( \{1, 2, \ldots, N\} \). An order-\( N \) tensor (or \( N \)-th order tensor) denoted by \( \mathbf{X} \equiv (x_{i_1, i_2, \ldots, i_N}) \), where \( 1 \leq i_j = 1, 2, \ldots, I_j \) for \( j \in [N] \), is a multidimensional array containing \( \prod_{n=1}^{N} I_n \) entries. Let \( \mathbb{C}^{I_1 \times \cdots \times I_N} \) and \( \mathbb{R}^{I_1 \times \cdots \times I_N} \) be the sets of the order-\( N \) \( I_1 \times \cdots \times I_N \) tensors over the complex field \( \mathbb{C} \) and the real field \( \mathbb{R} \), respectively. For example, \( \mathbf{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) is an order-\( N \) multiarray, where the first, second, ..., and \( N \)-th dimensions have \( I_1, I_2, \ldots, \) and \( I_N \) entries, respectively. Thus, each entry of \( \mathbf{X} \) can be represented by \( x_{i_1, \ldots, i_N} \). For example, when \( N = 3 \), \( \mathbf{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3} \) is a third-order tensor containing entries \( x_{i_1, i_2, i_3} \) s.

Without loss of generality, one can partition the dimensions of a tensor into two groups, say \( M \) and \( N \) dimensions, separately. Thus, for two order-\( (M+N) \) tensors: \( \mathbf{X} \equiv (x_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathbf{Y} \equiv (y_{j_1, \ldots, j_M, k_1, \ldots, k_L}) \in \mathbb{C}^{J_1 \times \cdots \times J_M \times K_1 \times \cdots \times K_L} \), according to \cite{Ref22}, the tensor addition \( \mathbf{X} + \mathbf{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) is given by

\[
(\mathbf{X} + \mathbf{Y})_{i_1, \ldots, i_M, j_1 \times \cdots \times j_N} \equiv x_{i_1, \ldots, i_M, j_1 \times \cdots \times j_N} + y_{i_1, \ldots, i_M, j_1 \times \cdots \times j_N}.
\]

(2.1)

On the other hand, for tensors \( \mathbf{X} \equiv (x_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathbf{Y} \equiv (y_{j_1, \ldots, j_M, k_1, \ldots, k_L}) \in \mathbb{C}^{J_1 \times \cdots \times J_M \times K_1 \times \cdots \times K_L} \), according to \cite{Ref22}, the Einstein product (or simply referred to as tensor product in this work) \( \mathbf{X} \ast \mathbf{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L} \) is given by

\[
(\mathbf{X} \ast \mathbf{Y})_{i_1, \ldots, i_M, j_1 \times \cdots \times j_N, k_1, \ldots, k_L} \equiv \sum_{j_1, \ldots, j_N} x_{i_1, \ldots, i_M, j_1, \ldots, j_N} y_{j_1, \ldots, j_N, k_1, \ldots, k_L}.
\]

(2.2)

Note that we will often abbreviate a tensor product \( \mathbf{X} \ast \mathbf{Y} \) to “\( \mathbf{X} \mathbf{Y} \)” for notational simplicity in the rest of the paper. This tensor product will be reduced to the standard matrix multiplication as \( L = M = N = 1 \). Other simplified situations can also be extended as tensor–vector product (\( M > 1 \), \( N = 1 \), and \( L = 0 \)) and tensor–matrix product (\( M > 1 \) and \( N = L = 1 \)). In analogy to matrix analysis, we define some basic tensors and elementary tensor operations as follows.

**Definition 2.1.** A tensor whose entries are all zero is called a zero tensor, denoted by \( \mathbf{0} \).

**Definition 2.2.** An identity tensor \( \mathbf{I} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N} \) is defined by

\[
(\mathbf{I})_{i_1 \times \cdots \times i_N \times j_1 \times \cdots \times j_N} \equiv \prod_{k=1}^{N} \delta_{i_k, j_k},
\]

where \( \delta_{i_k, j_k} \equiv 1 \) if \( i_k = j_k \); otherwise \( \delta_{i_k, j_k} \equiv 0 \).

In order to define Hermitian tensor, the conjugate transpose operation (or Hermitian adjoint) of a tensor is specified as follows.

**Definition 2.3.** Given a tensor \( \mathbf{X} \equiv (x_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), its conjugate transpose, denoted by \( \mathbf{X}^H \), is defined by

\[
(\mathbf{X}^H)_{j_1, \ldots, j_N, i_1 \times \cdots \times i_M} \equiv x_{i_1, \ldots, i_M, j_1, \ldots, j_N}^*.
\]

(2.4)

where the star * symbol indicates the complex conjugate of the complex number \( x_{i_1, \ldots, i_M, j_1, \ldots, j_N} \).

If a tensor \( \mathbf{X} \) satisfies \( \mathbf{X}^H = \mathbf{X} \), then \( \mathbf{X} \) is a Hermitian tensor.
We will use symbol $\iota$ to represent $\sqrt{-1}$.
Following definition is about unitary tensors.

**Definition 2.4.** Given a tensor $\mathcal{U} \triangleq (u_{i_1,\ldots,i_M,i_1,\ldots,i_M}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M},$
if
$$\mathcal{U}^H \star_M \mathcal{U} = \mathcal{U} \star_M \mathcal{U}^H = \mathcal{I} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M},$$
then $\mathcal{U}$ is a unitary tensor.

In this work, the symbol $\mathcal{U}$ is reserved for a unitary tensor.

**Definition 2.5.** Given a square tensor $\mathcal{X} \triangleq (x_{i_1,\ldots,i_M,j_1,\ldots,j_M}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M},$
if there exists $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ such that
$$\mathcal{X} \star_M \mathcal{X} = \mathcal{X} \star_M \mathcal{X} = \mathcal{I},$$
then $\mathcal{X}$ is the inverse of $\mathcal{X}$. We usually write $\mathcal{X} \triangleq \mathcal{X}^{-1}$ thereby.

We also list other crucial tensor operations here. The **trace** of a square tensor is equivalent to the summation of all diagonal entries such that
$$\text{Tr}(\mathcal{X}) \triangleq \sum_{1 \leq i_j \leq I_1, j \in [M]} \mathcal{X}_{i_1,\ldots,i_M,i_1,\ldots,i_M}.$$
The **inner product** of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is given by
$$\langle \mathcal{X}, \mathcal{Y} \rangle \triangleq \text{Tr} (\mathcal{X}^H \star_M \mathcal{Y}).$$
According to Eq. (2.8), the **Frobenius norm** of a tensor $\mathcal{X}$ is defined by
$$\|\mathcal{X}\| \triangleq \sqrt{\langle \mathcal{X}, \mathcal{X} \rangle}.$$ 
From Theorem 5.2 in [28], every Hermitian tensor $\mathcal{H} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ has following decomposition
$$\mathcal{H} = \sum_{i=1}^{r} \lambda_i \mathcal{U}_i \otimes \mathcal{U}_i^H,$$
with $\langle \mathcal{U}_i, \mathcal{U}_i \rangle = 1$ and $\langle \mathcal{U}_i, \mathcal{U}_j \rangle = 0$ for $i \neq j$,
where $\lambda_i \in \mathbb{R}$ and $\otimes$ denotes for Kronecker product. The values $\lambda_i$ are named as **Hermitian eigenvalues**, and the minimum integer of $r$ to decompose a Hermitian tensor as in Eq. (2.10) is called **Hermitian tensor rank**. A **positive Hermitian tensor** is a Hermitian tensor with all Hermitian eigenvalues are positive. A **nonnegative Hermitian tensor** is a Hermitian tensor with all Hermitian eigenvalues are nonnegative. The **Hermitian determinant**, denoted as $\det_H(\mathcal{A})$, is defined as the product of $\lambda_i$ of the tensor $\mathcal{A}$.

**Axioms For Hermitian Determinant** The Hermitian determinant is a mapping from Hermitian tensor to a real number satisfying following:

1. $\det_H(\mathcal{I}) = 1$.
2. $\det_H(\mathcal{A} \star \mathcal{B}) = \det_H(\mathcal{A}) \det_H(\mathcal{B})$.

Then, we have $|\det(U)| = 1$, $\det(A) = \det(H)$ (the product of all eigenvalues)
2.2. Unitarily Invariant Tensor Norms. Let us represent the Hermitian eigenvalues of a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{l_1 \times \cdots \times l_N}$ in decreasing order by the vector $\vec{\lambda} = (\lambda_1(\mathcal{H}), \ldots, \lambda_r(\mathcal{H}))$, where $r$ is the Hermitian rank of the tensor $\mathcal{H}$. We use $\mathbb{R}_{\geq 0}$ to represent a set of nonnegative (positive) real numbers. Let $\|\cdot\|_\rho$ be a unitarily invariant tensor norm, i.e., $\|\mathcal{H} \ast \mathcal{U}\|_\rho = \|\mathcal{U} \ast \mathcal{H}\|_\rho = \|\mathcal{H}\|_\rho'$, where $\mathcal{U}$ is any unitary tensor. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the corresponding gauge function that satisfies Hölder’s inequality so that

$$\|\mathcal{H}\|_\rho = \|\Lambda(\mathcal{H})\|_\rho,$$

where $|\mathcal{H}| \overset{def}{=} \sqrt{\mathcal{H}^H \ast \mathcal{U} \mathcal{H}}$. The bijective correspondence between symmetric gauge functions on $\mathbb{R}_{\geq 0}$ and unitarily invariant norms is due to von Neumann [10].

Several popular norms can be treated as special cases of unitarily invariant tensor norm. The first one is Ky Fan like $k$-norm for tensors. For $k \in \{1, 2, \ldots, r\}$, the Ky Fan $k$-norm [10] for tensors $\mathcal{H} \in \mathbb{C}^{l_1 \times \cdots \times l_N}$, denoted as $\|\mathcal{H}\|_{(k)}$, is defined as:

$$\|\mathcal{H}\|_{(k)} = \sum_{i=1}^{k} \lambda_i(|\mathcal{H}|),$$

If $k = 1$, the Ky Fan $k$-norm for tensors is the tensor operator norm, denoted as $\|\mathcal{H}\|$. The second one is Schatten $p$-norm for tensors, denoted as $\|\mathcal{H}\|_p$, is defined as:

$$\|\mathcal{H}\|_p \overset{def}{=} (\text{Tr}|\mathcal{H}|^p)^{\frac{1}{p}},$$

where $p \geq 1$. If $p = 1$, it is the trace norm. The third one is $k$-trace norm, denoted as $\text{Tr}_k[\mathcal{H}]$, defined by [17]. It is

$$\text{Tr}_k[\mathcal{H}] \overset{def}{=} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq r} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k},$$

where $1 \leq k \leq r$. If $k = 1$, $\text{Tr}_k[\mathcal{H}]$ is reduced as trace norm.

Following inequality is the extension of Hölder inequality to gauge function $\rho$ which will be used by later to prove majorization relations.

LEMMA 2.6. For $n$ nonnegative real vectors with the dimension $r$, i.e., $b_i = (b_{i1}, \ldots, b_{ir}) \in \mathbb{R}_{\geq 0}^r$, and $\alpha > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$\rho \left( \prod_{i=1}^{n} b_{i1}^{\alpha_i}, \prod_{i=1}^{n} b_{i2}^{\alpha_i}, \ldots, \prod_{i=1}^{n} b_{ir}^{\alpha_i} \right) \leq \prod_{i=1}^{n} \rho(b_i)^{\alpha_i}$$

Proof: This proof is based on mathematical induction. The base case for $n = 2$ has been shown by Theorem IV.1.6 from [3].

We assume that Eq. (2.15) is true for $n = m$, where $m > 2$. Let $\odot$ be the component-wise product (Hadamard product) between two vectors. Then, we have

$$\rho \left( \prod_{i=1}^{m+1} b_{i1}^{\alpha_i}, \prod_{i=1}^{m+1} b_{i2}^{\alpha_i}, \ldots, \prod_{i=1}^{m+1} b_{ir}^{\alpha_i} \right) = \rho \left( \odot_{i=1}^{m+1} b_{i}^{\alpha_i} \right),$$
where \( \odot_{i=1}^{m+1} b_{i}^{\alpha_i} \) is defined as \( \left( \prod_{i=1}^{m+1} b_{i}^{\alpha_i}, \prod_{i=1}^{m+1} b_{i}^{\alpha_i}, \cdots, \prod_{i=1}^{m+1} b_{i}^{\alpha_i} \right) \) with \( b_{i}^{\alpha_i} \equiv (b_{i_1}^{\alpha_i}, \cdots, b_{i_r}^{\alpha_i}) \).

Under such notations, Eq. (2.16) can be bounded as

\[
\rho \left( \odot_{i=1}^{m+1} b_{i}^{\alpha_i} \right) = \rho \left( \left( \prod_{j=1}^{m} \frac{\alpha_j}{\sum_{j=1}^{m} \alpha_j} b_{i}^{\alpha_i} \right) \odot b_{m+1}^{\alpha_{m+1}} \right) \leq \left[ \sum_{j=1}^{m} \alpha_j \left( \prod_{i=1}^{m} b_{i}^{\alpha_i} \right) \right] \cdot \rho(b_{m+1})^{\alpha_{m+1}} \leq \prod_{i=1}^{m+1} \rho(b_{i})^{\alpha_{i}}.
\]

By mathematical induction, this lemma is proved.

\[\square\]

2.3. Antisymmetric Tensor Product. Let \( \mathcal{H} \) be a Hilbert space of dimension \( r \), \( \mathcal{L}(\mathcal{H}) \) be the set of tensors (linear operators) on \( \mathcal{H} \). Two tensors \( A, B \in \mathcal{L}(\mathcal{H}) \) is said \( A \geq B \) if \( A - B \) is a nonnegative Hermitian tensor. For any \( k \in \{1, 2, \cdots, r\} \), let \( \mathcal{H}^{\otimes k} \) be the \( k \)-th tensor power of the space \( \mathcal{H} \) and let \( \mathcal{H}^{\wedge k} \) be the antisymmetric subspace of \( \mathcal{H}^{\otimes k} \). We define function \( \wedge^k : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}^{\wedge k}) \) as mapping any tensor \( A \) to the restriction of \( A^{\wedge k} \in \mathcal{L}(\mathcal{H}^{\otimes k}) \) to the antisymmetric subspace \( \mathcal{H}^{\wedge k} \) of \( \mathcal{H}^{\otimes k} \). Following lemma summarizes several useful properties of such antisymmetric tensor products.

**Lemma 2.7.** Let \( A, B, C \in \mathcal{O}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) be tensors in \( \mathcal{L}(\mathcal{H}) \), and \( D \in \mathcal{O}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) be Hermitian tensors from \( \mathcal{H} \) with Hermitian rank \( r \). For any \( k \in \{1, 2, \cdots, r\} \), we have

1. \( (A^{\otimes k})^H = (A^H)^{\otimes k} \).
2. \( (A^{\otimes k}) \ast_N (B^{\otimes k}) = (A \ast_N B)^{\otimes k} \).
3. If \( \lim_{i \rightarrow \infty} \|A_i - A\| \rightarrow 0 \), then \( \lim_{i \rightarrow \infty} \|A_i^{\wedge k} - A^{\wedge k}\| \rightarrow 0 \).
4. \( |C| \geq O \) (zero tensor), then \( C^{\wedge k} \geq O \) and \( (C^p)^{\wedge k} = (C^{\wedge k})^p \) for all \( p \in \mathbb{R} > 0 \).
5. \( |A|^{\wedge k} \geq |A^{\wedge k}| \).
6. If \( D \geq O \) and \( D \) is invertible, \( (D^z)^{\wedge k} = (D^{\wedge k})^z \) for all \( z \in \mathbb{D} \).
7. \( \|A^{\wedge k}\| = \prod_{i=1}^{k} \lambda_i(|A|) \).

**Proof:** Facts [1] and [2] are the restrictions of the associated relations \( (A^H)^{\otimes k} = (A^{\otimes k})^H \) and \( (A \ast_N B)^{\otimes k} = (A^{\otimes k}) \ast_N (B^{\otimes k}) \) to \( \mathcal{H}^{\wedge k} \). The fact [3] is true since, if \( \lim_{i \rightarrow \infty} \|A_i - A\| \rightarrow 0 \), we have \( \lim_{i \rightarrow \infty} \|A_i^{\otimes k} - A^{\otimes k}\| \rightarrow 0 \) and the assoicated restrictions of \( A^{\otimes k} \) to the antisymmetric subspace \( \mathcal{H}^{\wedge k} \).

For the fact [4], if \( C \geq O \), then we have \( C^{\wedge k} = ((C^{1/2})^{\wedge k})^H \ast_N ((C^{1/2})^{\wedge k}) \geq O \) from facts [1] and [2]. If \( p \) is rational, we have \( (C^p)^{\wedge k} = (C^{\wedge k})^p \) from the fact [2], and the equality \( (C^p)^{\wedge k} = (C^{\wedge k})^p \) is also true for any \( p > 0 \) if we apply the fact [3] to approximate any irrational numbers by rational numbers.

Because we have

\[
|A|^{\wedge k} = \left( \sqrt{A^H A} \right)^{\wedge k} = \sqrt{(A^{\wedge k})^H A^{\wedge k}} = |A^{\wedge k}|,
\]

from facts [1], [2] and [4], so the fact [5] is valid.

For the fact [6], if \( z < 0 \), the fact [6] is true for all \( z \in \mathbb{R} \) by applying the fact [4] to \( D^{-1} \). Since we can apply the definition \( D^z \equiv \exp(z \ln D) \) to have

\[
C^p = D^z \Leftrightarrow C = \exp \left( \frac{z}{p} \ln D \right),
\]

\[\square\]
where \( \mathcal{C} \geq \mathcal{O} \). The general case of any \( z \in \mathcal{C} \) is also true by applying the fact \([4]\) to \( \mathcal{C} = \exp(\frac{z}{p} \ln \mathcal{D}) \).

For the fact \([7]\) proof, it is enough to prove the case that \( \mathcal{A} \geq \mathcal{O} \) due to the fact \([5]\). Then, there exists a set of orthogonal tensors \( \{\mathcal{U}_1, \cdots, \mathcal{U}_r\} \) such that \( \mathcal{A} \cdot \mathcal{N} \mathcal{U}_i = \lambda_i \mathcal{U}_i \) for \( 1 \leq i \leq r \). We then have

\[
\mathcal{A}^\wedge^k (\mathcal{U}_{i_1} \wedge \cdots \wedge \mathcal{U}_{i_k}) = \mathcal{A} \cdot \mathcal{N} \mathcal{U}_{i_1} \wedge \cdots \wedge \mathcal{A} \cdot \mathcal{N} \mathcal{U}_{i_k}
\]

(2.20)

\[
= \left( \prod_{i=1}^{k} \lambda_i(|\mathcal{A}|) \right) \mathcal{U}_{i_1} \wedge \cdots \wedge \mathcal{U}_{i_k},
\]

where \( 1 \leq i_1 < i_2 < \cdots < i_k \leq r \). Hence, \( \|\mathcal{A}^\wedge^k\| = \prod_{i=1}^{k} \lambda_i(|\mathcal{A}|) \). \( \square \)

2.4. Majorization. In this subsection, we will discuss majorization and several lemmas about majorization which will be used at later proofs.

Let \( x = [x_1, \cdots, x_r] \in \mathbb{R}^r, y = [y_1, \cdots, y_r] \in \mathbb{R}^r \) be two vectors with following orders among entries \( x_1 \geq \cdots \geq x_r \) and \( y_1 \geq \cdots \geq y_r \), weak majorization between vectors \( x, y \), represented by \( x \prec_w y \), requires following relation for vectors \( x, y \):

\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i,
\]

(2.21)

where \( k \in \{1, 2, \cdots, r\} \). Majorization between vectors \( x, y \), indicated by \( x \prec y \), requires following relation for vectors \( x, y \):

\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \text{for } 1 \leq k < r;
\]

(2.22)

\[
\sum_{i=1}^{r} x_i = \sum_{i=1}^{r} y_i, \quad \text{for } k = r.
\]

For \( x, y \in \mathbb{R}_{\geq 0}^r \) such that \( x_1 \geq \cdots \geq x_r \) and \( y_1 \geq \cdots \geq y_r \), weak log majorization between vectors \( x, y \), represented by \( x \prec_w \log y \), requires following relation for vectors \( x, y \):

\[
\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i,
\]

(2.23)

where \( k \in \{1, 2, \cdots, r\} \), and log majorization between vectors \( x, y \), represented by \( x \prec_{\log} y \), requires equality for \( k = r \) in Eq. (2.23). If \( f \) is a single variable function, \( f(x) \) represents a vector of \([f(x_1), \cdots, f(x_r)]\). From Lemma 1 in \([15]\), we have

**Lemma 2.8.** (1) For any convex function \( f : [0, \infty) \rightarrow [0, \infty) \), if we have \( x \prec y \), then \( f(x) \prec f(y) \).

(2) For any convex function and non-decreasing \( f : [0, \infty) \rightarrow [0, \infty) \), if we have \( x \prec y \), then \( f(x) \prec f(y) \).

Another lemma is from Lemma 12 in \([15]\), we have

**Lemma 2.9.** Let \( x, y \in \mathbb{R}_{\geq 0}^r \) such that \( x_1 \geq \cdots \geq x_r \) and \( y_1 \geq \cdots \geq y_r \) with \( x \prec_{\log} y \). Also let \( y_i = [y_{i1}, \cdots, y_{ir}] \in \mathbb{R}_{\geq 0}^r \) be a sequence of vectors such that
y_{i:1} \geq \cdots \geq y_{i:r} > 0 \text{ and } y_i \to y \text{ as } i \to \infty. \text{ Then, there exists } i_0 \in \mathbb{N} \text{ and } 
abla \text{ such that } \sup_{\| \cdot \|} = 1/2 \text{ for } i \geq i_0 \text{ such that } x_{i:1} \geq \cdots \geq x_{i:r} > 0, x_i \to x \text{ as } i \to \infty, \text{ and }
abla \text{ such that } \sup_{\| \cdot \|} = 1/2.

Proof: For any function } f \text{ on } \mathbb{R}_{\geq 0}, \text{ the term } f(x) \text{ is defined as } f(x) \overset{\text{def}}{=} (f(x_1), \cdots, f(x_r)) \text{ with conventions } e^{-\infty} = 0 \text{ and } \log 0 = -\infty.

3. Multivariate Tensor Norm Inequalities. In this section, we will develop several theorems about majorization in Section 3.1, and log majorization with integral average in Section 3.2. These majorization related theorems will provide tools to establish new inequalities for unitarily norms of multivariate tensors in Section 3.3.

3.1. Majorization with Integral Average. Let } \Omega \text{ be a } \sigma\text{-compact metric space and } \nu \text{ a probability measure on the Borel } \sigma\text{-field of } \Omega. \text{ Let } C, D_\tau \in \mathbb{C}^{I_1 \times \cdots \times I_N} \text{ be Hermitian tensors with Hermitian rank } r. \text{ We further assume that tensors } C, D_\tau \text{ are uniformly bounded in their norm for } \tau \in \Omega. \text{ Let } \tau \in \Omega \to D_\tau \text{ be a continuous function such that } \sup \{ \| D_\tau \| : \tau \in \Omega \} < \infty. \text{ For notational convenience, we define the following relation:}

$$
\int_\Omega \lambda_1(D_\tau) d\nu(\tau), \cdots, \int_\Omega \lambda_r(D_\tau) d\nu(\tau) \overset{\text{def}}{=} \int \tilde{\lambda}(D_\tau) d\nu^\tau(\tau).
$$

If } f \text{ is a single variable function, the notation } f(\mathcal{C}) \text{ represents a tensor function with respect to the tensor } \mathcal{C}.

**Theorem 3.1.** Let } \Omega, \nu, C, D_\tau \text{ be defined as the beginning part of Section 3.1, and } f : \mathbb{R} \to [0, \infty) \text{ be a non-decreasing convex function, we have following two equivalent statements:}

$$
\tilde{\lambda}(\mathcal{C}) <_w \int \tilde{\lambda}(D_\tau) d\nu^\tau(\tau) \iff \| f(\mathcal{C}) \|_\rho \leq \int_\Omega \| f(D_\tau) \|_\rho d\nu(\tau),
$$

where } \| \cdot \|_\rho \text{ is the unitarily invariant norm defined in Eq. (2.11).

**Proof:** We assume that the left statement of Eq. (3.2) is true and the function } f \text{ is a non-decreasing convex function. From Lemma 2.8, we have

$$
\tilde{\lambda}(f(\mathcal{C})) = f(\tilde{\lambda}(\mathcal{C})) <_w f \left( \int \tilde{\lambda}(D_\tau) d\nu^\tau(\tau) \right).
$$

From the convexity of } f, \text{ we also have

$$
f \left( \int \tilde{\lambda}(D_\tau) d\nu^\tau(\tau) \right) \leq \int \tilde{\lambda}(f(D_\tau)) d\nu^\tau(\tau) = \int \tilde{\lambda}(f(D_\tau)) d\nu^\tau(\tau).
$$

Then, we obtain } \tilde{\lambda}(f(\mathcal{C})) <_w = \int \tilde{\lambda}(f(D_\tau)) d\nu^\tau(\tau). \text{ By applying Lemma 4.4.2 in [14] to both sides of } \tilde{\lambda}(f(\mathcal{C})) <_w, \text{ we obtain

$$
\| f(\mathcal{C}) \|_\rho \leq \rho \left( \int \tilde{\lambda}(f(D_\tau)) d\nu^\tau(\tau) \right)
$$

$$
\leq \int_\Omega \rho(\tilde{\lambda}(f(D_\tau))) d\nu(\tau) = \int_\Omega \| f(D_\tau) \|_\rho d\nu(\tau).
$$
Therefore, the right statement of Eq. (3.2) is true from the left statement.

On the other hand, if the right statement of Eq. (3.2) is true, we select a function \( f \equiv \max\{c+x,0\} \), where \( c \) is a positive real constant satisfying \( C+cI \geq O \), \( D_\tau+cI \geq O \) for all \( \tau \in \Omega \), and tensors \( C+cI, D_\tau+cI \) having Hermitian rank \( r \). If the Ky Fan \( k \)-norm at the right statement of Eq. (3.2) is applied, we have

\[
\sum_{i=1}^{k} (\lambda_i(C) + c) \leq \sum_{i=1}^{k} \int_{\Omega} (\lambda_i(D_\tau) + c)d\nu(\tau).
\]

(3.6)

Hence, \( \sum_{i=1}^{k} \lambda_i(C) \leq \sum_{i=1}^{k} \int_{\Omega} \lambda_i(D_\tau)d\nu(\tau) \), this is the left statement of Eq. (3.2).

Next theorem will provide a stronger version of Theorem 3.1 by removing weak majorization conditions.

**Theorem 3.2.** Let \( \Omega, \nu, C, D_\tau \) be defined as the beginning part of Section 3.1, and \( f : \mathbb{R} \to [0, \infty) \) be a convex function, we have following two equivalent statements:

\[
\bar{\lambda}(C) \prec f(\bar{\lambda}(A)) \iff \|f(C)\|_\rho \leq \int_{\Omega} \|f(D_\tau)\|_\rho d\nu(\tau),
\]

(3.7)

where \( \|\cdot\|_\rho \) is the unitarily invariant norm defined in Eq. (2.11).

**Proof:** We assume that the left statement of Eq. (3.7) is true and the function \( f \) is a convex function. Again, from Lemma 2.8, we have

\[
\bar{\lambda}(f(A)) = f(\bar{\lambda}(A)) \prec_w f \left( \int_{\Omega} \bar{\lambda}(D_\tau)d\nu(\tau) \right) \leq \int_{\Omega} f(\bar{\lambda}(D_\tau))d\nu(\tau),
\]

(3.8)

then,

\[
\|f(A)\|_\rho \leq \rho \left( \int_{\Omega} f(\bar{\lambda}(D_\tau))d\nu(\tau) \right) \leq \int_{\Omega} \rho \left( f(\bar{\lambda}(D_\tau)) \right) d\nu(\tau) = \int_{\Omega} \|f(D_\tau)\|_\rho d\nu(\tau).
\]

(3.9)

This prove the right statement of Eq. (3.7).

Now, we assume that the right statement of Eq. (3.7) is true. From Theorem 3.1, we already have \( \bar{\lambda}(C) \prec_w \int_{\Omega} \bar{\lambda}(D_\tau)d\nu(\tau) \). It is enough to prove \( \sum_{i=1}^{r} \lambda_i(C) \geq \int_{\Omega} \sum_{i=1}^{r} \lambda_i(D_\tau)d\nu(\tau) \). We define a function \( f \equiv \max\{c-x,0\} \), where \( c \) is a positive real constant satisfying \( C \leq cI \), \( D_\tau \leq cI \) for all \( \tau \in \Omega \) and tensors \( cI-C, cI-D_\tau \) having Hermitian rank \( r \). If the trace norm is applied, i.e., the sum of the absolute value of all eigenvalues of a Hermitian tensor, then the right statement of Eq. (3.7) becomes

\[
\sum_{i=1}^{r} \lambda_i(cI-C) \leq \int_{\Omega} \sum_{i=1}^{r} \lambda_i(cI-D_\tau) d\nu(\tau).
\]

(3.10)

The desired inequality \( \sum_{i=1}^{r} \lambda_i(C) \geq \int_{\Omega} \sum_{i=1}^{r} \lambda_i(D_\tau) d\nu(\tau) \) is established. \( \square \)
3.2. Log-Majorization with Integral Average. The purpose of this section is to consider log-majorization issues for unitarily invariant norms of Hermitian tensors. In this section, let $C, D \in C^{I_1 \times \cdots \times I_N}$ be nonnegative Hermitian tensors with Hermitian rank $r$, i.e., all Hermitian eigenvalues are positive, and keep other notations with the same definitions as at the beginning of the Section 3.1. For notational convenience, we define the following relation for logarithm vector:

$$\begin{align*}
(3.11) \quad &\left[ \int_{\Omega} \log \lambda_1(D_{\tau}) d\nu(\tau), \cdots, \int_{\Omega} \log \lambda_r(D_{\tau}) d\nu(\tau) \right] \overset{def}{=} \int_{\Omega^r} \log \bar{\lambda}(D_{\tau}) d\nu^r(\tau).
\end{align*}$$

**Theorem 3.3.** Let $C, D_{\tau}$ be nonnegative Hermitian tensors, $f : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to \log f(e^x)$ is convex on $\mathbb{R}$, and $g : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to g(e^x)$ is convex on $\mathbb{R}$, then we have following three equivalent statements:

$$\begin{align*}
(3.12) \quad &\bar{\lambda}(C) \preceq_w \exp \int_{\Omega^r} \log \bar{\lambda}(D_{\tau}) d\nu^r(\tau);
(3.13) \quad &\|f(C)\|_\rho \leq \exp \int_{\Omega} \log \|f(D_{\tau})\|_\rho d\nu(\tau);
(3.14) \quad &\|g(C)\|_\rho \leq \int_{\Omega} \|g(D_{\tau})\|_\rho d\nu(\tau).
\end{align*}$$

**Proof:** The roadmap of this proof is to prove equivalent statements between Eq. (3.12) and Eq. (3.13) first, followed by equivalent statements between Eq. (3.12) and Eq. (3.14).

**Eq. (3.12) $\implies$ Eq. (3.13)**

There are two cases to be discussed in this part of proof: $C, D_{\tau}$ are positive Hermitian tensors, and $C, D_{\tau}$ are nonnegative Hermitian tensors. At the beginning, we consider the case that $C, D_{\tau}$ are positive Hermitian tensors.

Since $D_{\tau}$ are positive, we can find $\varepsilon > 0$ such that $D_{\tau} \geq \varepsilon I$ for all $\tau \in \Omega$. From Eq. (3.12), the convexity of $\log f(e^x)$ and Lemma 2.8, we have

$$\begin{align*}
\bar{\lambda}(f(C)) &= f \left( \exp \left( \log \bar{\lambda}(C) \right) \right) \preceq_w f \left( \exp \int_{\Omega^r} \bar{\lambda}(D_{\tau}) d\nu^r(\tau) \right) \\
&\leq \exp \left( \int_{\Omega^r} \log f \left( \bar{\lambda}(D_{\tau}) \right) d\nu^r(\tau) \right).
\end{align*}$$

Then, from Eq. (2.11), we obtain

$$\begin{align*}
(3.16) \quad &\|f(C)\|_\rho \leq \rho \left( \exp \left( \int_{\Omega^r} \log f \left( \bar{\lambda}(D_{\tau}) \right) d\nu^r(\tau) \right) \right).
\end{align*}$$

From the function $f$ properties, we can assume that $f(x) > 0$ for any $x > 0$. Then, we have following bounded and continuous maps on $\Omega$: $\tau \to \log f(\lambda_i(D_{\tau}))$ for $i \in \{1, 2, \cdots, r\}$, and $\tau \to \log \|f(D_{\tau})\|_\rho$. Because we have $\nu(\Omega) = 1$ and $\sigma$-compactness of $\Omega$, we have $\tau_k^{(n)} \in \Omega$ and $\alpha_k^{(n)}$ for $k \in \{1, 2, \cdots, n\}$ and $n \in \mathbb{N}$ with $\sum_{k=1}^{n} \alpha_k^{(n)} = 1$ such that

$$\begin{align*}
(3.17) \int_{\Omega} \log f(\lambda_i(D_{\tau})) d\nu(\tau) &= \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^{(n)} \log f(\lambda_i(D_{\tau_k^{(n)}})), \text{ for } i \in \{1, 2, \cdots, r\};
\end{align*}$$

where $\alpha_k^{(n)}$ is the number of $\lambda_k(D_{\tau_k^{(n)}})$ for $k \in \{1, 2, \cdots, n\}$.


and

\begin{equation}
\int_{\Omega} \log \|f(D_{\tau})\|_\rho d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^{(n)} \log \|f(D_{\tau_k^{(n)}})\|_\rho .
\end{equation}

By taking the exponential at both sides of Eq. (3.17) and apply the gauge function \( \rho \), we have

\begin{equation}
\rho \left( \exp \int_{\Omega} \log f(\lambda(D_{\tau})) d\nu'(\tau) \right) = \lim_{n \to \infty} \rho \left( \prod_{k=1}^{n} f\left( \lambda(D_{\tau_k^{(n)}}) \right)^{\alpha_k^{(n)}} \right).
\end{equation}

Similarly, by taking the exponential at both sides of Eq. (3.18), we have

\begin{equation}
\exp \left( \int_{\Omega} \log \|f(D_{\tau})\|_\rho d\nu(\tau) \right) = \lim_{n \to \infty} \prod_{k=1}^{n} \|f(D_{\tau_k^{(n)}})\|_\rho^{\alpha_k^{(n)}} .
\end{equation}

From Lemma 2.6, we have

\begin{equation}
\rho \left( \prod_{k=1}^{n} f\left( \lambda(D_{\tau_k^{(n)}}) \right)^{\alpha_k^{(n)}} \right) \leq \prod_{k=1}^{n} \rho^{\alpha_k^{(n)}} \left( f\left( \lambda(D_{\tau_k^{(n)}}) \right) \right) = \prod_{k=1}^{n} \rho^{\alpha_k^{(n)}} \left( f\left( \lambda(D_{\tau_k^{(n)}}) \right) \right) = \prod_{k=1}^{n} \|f(D_{\tau_k^{(n)}})\|_\rho^{\alpha_k^{(n)}} .
\end{equation}

From Eqs. (3.19), (3.20) and (3.21), we have

\begin{equation}
\rho \left( \exp \int_{\Omega} \log f(\lambda(D_{\tau})) d\nu'(\tau) \right) \leq \exp \int_{\Omega} \log \|f(D_{\tau})\|_\rho d\nu(\tau).
\end{equation}

Then, Eq. (3.13) is proved from Eqs. (3.16) and (3.22).

Next, we consider that \( C, D_{\tau} \) are nonnegative Hermitian tensors. For any \( \delta > 0 \), we have following log-majorization relation:

\begin{equation}
\prod_{i=1}^{k} (\lambda_i(C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(D_{\tau}) + \delta) d\nu(\tau),
\end{equation}

where \( \epsilon_\delta > 0 \) and \( k \in \{1, 2, \cdots, r\} \). Then, we can apply the previous case result about positive Hermitian tensors to positive Hermitian tensors \( C + \epsilon_\delta I \) and \( D_{\tau} + \delta I \), and get

\begin{equation}
\|f(C) + \epsilon_\delta I\|_\rho \leq \exp \int_{\Omega} \log \|f(D_{\tau}) + \delta I\|_\rho d\nu(\tau)
\end{equation}

As \( \delta \to 0 \), Eq. (3.24) will give us Eq. (3.13) for nonnegative Hermitian tensors.

\textbf{Eq. (3.13)} \iff \textbf{Eq. (3.13)}

We consider positive Hermitian tensors at first phase by assuming that \( D_{\tau} \) are positive Hermitian for all \( \tau \in \Omega \). We may also assume that the tensor \( C \) is a positive
Hermitian tensor. Since if this is a nonnegative Hermitian tensor, i.e., some \( \lambda_i = 0 \), we always have following inequality valid:

\[
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \left( \int_{\Omega} \log \lambda_i(D_{\tau}) d\nu(\tau) \right)
\]

(3.25)

If we apply \( f(x) = x^p \) for \( p > 0 \) and \( \| \cdot \| \_p \) as Ky Fan \( k \)-norm in Eq. (3.13), we have

\[
\log \prod_{i=1}^{k} \lambda_i^p(C) \leq \int_{\Omega} \log \prod_{i=1}^{k} \lambda_i^p(D_{\tau}) d\nu(\tau).
\]

(3.26)

If we add \( \log \frac{1}{k} \) and multiply \( \frac{1}{p} \) at both sides of Eq. (3.26), we have

\[
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(C) \right) \leq \frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(D_{\tau}) \right) d\nu(\tau).
\]

(3.27)

From L'Hopital's Rule, if \( p \to 0 \), we have

\[
\frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(C) \to \frac{1}{k} \sum_{i=1}^{k} \log \lambda_i(C),
\]

and

\[
\frac{1}{k} \sum_{i=1}^{k} \lambda_i^p(D_{\tau}) \to \frac{1}{k} \sum_{i=1}^{k} \log \lambda_i(D_{\tau}),
\]

(3.29)

where \( \tau \in \Omega \). Applying Eqs. (3.28) and (3.29) into Eq. (3.27) and taking \( p \to 0 \), we have

\[
\sum_{i=1}^{k} \lambda_i(C) \leq \int_{\Omega} \sum_{i=1}^{k} \log \lambda_i(D_{\tau}) d\nu(\tau).
\]

(3.30)

Therefore, Eq. (3.12) is true for positive Hermitian tensors.

For nonnegative Hermitian tensors \( D_{\tau} \), since Eq. (3.13) is valid for \( D_{\tau} + \delta I \) for any \( \delta > 0 \), we can apply the previous case result about positive Hermitian tensors to \( D_{\tau} + \delta I \) and obtain

\[
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \left( \int_{\Omega} (\lambda_i(D_{\tau}) + \delta) d\nu(\tau) \right),
\]

(3.31)

where \( k \in \{1, 2, \cdots, r\} \). Eq. (3.12) is still true for nonnegative Hermitian tensors as \( \delta \to 0 \).

**Eq. (3.12) \implies Eq. (3.14)**

If \( C, D_{\tau} \) are positive Hermitian tensors, and \( D_{\tau} \geq \delta I \) for all \( \tau \in \Omega \). From Eq. (3.12), we have

\[
\bar{\lambda}(\log C) = \log \bar{\lambda}(C) \preceq_{u} \int_{\Omega} \log \bar{\lambda}(D_{\tau}) d\nu(\tau) = \int_{\Omega} \bar{\lambda}(\log D_{\tau}) d\nu'(\tau).
\]

(3.32)

If we apply Theorem 3.1 to \( \log C, \log D_{\tau} \) with function \( f(x) = g(e^x) \), where \( g \) is used in Eq. (3.14), Eq. (3.14) is implied.
If $C, D_{\tau}$ are nonnegative Hermitian tensors and any $\delta > 0$, we can find $\epsilon_\delta \in (0, \delta)$ to satisfy following:

$$
\prod_{i=1}^{k} (\lambda_i(C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int_\Omega \log (\lambda_i(D_{\tau}) + \delta) \, d\nu(\tau).
$$

Then, from positive Hermitian tensor case, we have

$$
\|g(C + \epsilon_\delta I)\|_\rho \leq \int_\Omega \|g(D_{\tau} + \delta I)\|_\rho \, d\nu(\tau).
$$

Eq. (3.34) is obtained by taking $\delta \to 0$ in Eq. (3.34).

For $\delta > 0$, and Ky Fan $k$-norm in Eq. (3.14), we have

$$
\sum_{i=1}^{k} \log (\delta + \lambda_i(C)) \leq \sum_{i=1}^{k} \int_\Omega \log (\delta + \lambda_i(D_{\tau})) \, d\nu(\tau).
$$

Then, we have following relation as $\delta \to 0$:

$$
\sum_{i=1}^{k} \log \lambda_i(C) \leq \sum_{i=1}^{k} \int_\Omega \log \lambda_i(D_{\tau}) \, d\nu(\tau).
$$

Therefore, Eq. (3.12) can be derived from Eq. (3.14).

Next theorem will extend Theorem 3.3 to non-weak version.

**Theorem 3.4.** Let $C, D_{\tau}$ be nonnegative Hermitian tensors with $\int_\Omega \|D_{\tau}^{-p}\|_\rho \, d\nu(\tau) < \infty$ for any $p > 0$, $f : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to \log f(e^x)$ is convex on $\mathbb{R}$, and $g : (0, \infty) \to [0, \infty)$ be a continuous function such that the mapping $x \to g(e^x)$ is convex on $\mathbb{R}$, then we have following three equivalent statements:

$$
\tilde{x}(C) \prec_{\log} \exp \int_{\Omega'} \log \tilde{x}(D_{\tau}) \, d\nu'(\tau);
$$

$$
\|f(C)\|_\rho \leq \exp \int_{\Omega} \log \|f(D_{\tau})\|_\rho \, d\nu(\tau);
$$

$$
\|g(C)\|_\rho \leq \int_{\Omega} \|g(D_{\tau})\|_\rho \, d\nu(\tau).
$$

**Proof:**

The proof plan is similar to the proof in Theorem 3.3. We prove the equivalence between Eq. (3.37) and Eq. (3.38) first, then prove the equivalence between Eq. (3.37) and Eq. (3.39).

Eq. (3.37) $\implies$ Eq. (3.38)

First, we assume that $C, D_{\tau}$ are positive Hermitian tensors with $D_{\tau} \geq \delta I$ for all $\tau \in \Omega$. The corresponding part of the proof in Theorem 3.3 about positive Hermitian tensors $C, D_{\tau}$ can be applied here.
For case that $\mathcal{C}, \mathcal{D}_\tau$ are nonnegative Hermitian tensors, we have
\begin{equation}
\prod_{i=1}^{k} \lambda_i(\mathcal{C}) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(\mathcal{D}_\tau) + \delta_n) \, d\nu(\tau),
\end{equation}
where $\delta_n > 0$ and $\delta_n \to 0$. Because $\int_{\Omega} \log \left( \tilde{\lambda}(\mathcal{D}_\tau) + \delta_n \right) \, d\nu(\tau) \to \int_{\Omega} \log \tilde{\lambda}(\mathcal{D}_\tau) \, d\nu(\tau)$ as $n \to \infty$, from Lemma 2.9, we can find $a^{(n)}$ with $n \geq n_0$ such that $a^{(n)}_i \geq \cdots \geq a^{(n)}_{r+1}$, $a^{(n)}_r \to \tilde{\lambda}(\mathcal{C})$ and $a^{(n)} \preceq \log \exp \int_{\Omega} \log \tilde{\lambda}(\mathcal{D}_\tau + \delta_n \mathcal{T}) \, d\nu(\tau)$.

Selecting $\mathcal{C}^{(n)}$ with $\tilde{\lambda}(\mathcal{C}^{(n)}) = a^{(n)}$ and applying positive Hermitian tensors case to $\mathcal{C}^{(n)}$ and $\mathcal{D}_\tau + \delta_n \mathcal{T}$, we obtain
\begin{equation}
\|f(\mathcal{C}^{(n)})\|_\rho \leq \exp \int_{\Omega} \log \|f(\mathcal{D}_\tau + \delta_n \mathcal{T})\|_\rho \, d\nu(\tau)
\end{equation}
where $n \geq n_0$.

There are two situations for the function $f$ near 0: $f(0^+) < \infty$ and $f(0^+) = \infty$. For the case with $f(0^+) < \infty$, we have
\begin{equation}
\|f(\mathcal{C}^{(n)})\|_\rho = \rho(f(a^{(n)})) \to \rho(f(\tilde{\lambda}(\mathcal{C}))) = \|f(\mathcal{C})\|_\rho,
\end{equation}
and
\begin{equation}
\|f(\mathcal{D}_\tau + \delta_n \mathcal{T})\|_\rho \to \|f(\mathcal{D}_\tau)\|_\rho,
\end{equation}
where $\tau \in \Omega$ and $n \to \infty$. From Fatou–Lebesgue theorem, we then have
\begin{equation}
\limsup_{n \to \infty} \int_{\Omega} \log \|f(\mathcal{D}_\tau + \delta_n \mathcal{T})\|_\rho \, d\nu(\tau) \leq \int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_\rho.
\end{equation}
By taking $n \to \infty$ in Eq. (3.41) and using Eqs. (3.42), (3.43), (3.44), we have Eq. (3.38) for case that $f(0^+) < \infty$.

For the case with $f(0^+) = \infty$, we assume that $\int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_\rho \, d\nu(\tau) < \infty$ (since the inequality in Eq. (3.38) is always true for $\int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_\rho \, d\nu(\tau) = \infty$). Since $f$ is decreasing on $(0, \epsilon)$ for some $\epsilon > 0$. We claim that the following relation is valid: there are two constants $a, b > 0$ such that
\begin{equation}
a \leq \|f(\mathcal{D}_\tau + \delta_n \mathcal{T})\|_\rho \leq \|f(\mathcal{D}_\tau)\|_\rho + b,
\end{equation}
for all $\tau \in \Omega$ and $n \geq n_0$. If Eq. (3.45) is valid and $\int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_\rho \, d\nu(\tau) < \infty$, from Lebesgue’s dominated convergence theorem, we also have Eq. (3.38) for case that $f(0^+) = \infty$ by taking $n \to \infty$ in Eq. (3.41).

Below, we will prove the claim stated by Eq. (3.45). By the uniform boundedness of tensors $\mathcal{D}_\tau$, there is a constant $\kappa > 0$ such that
\begin{equation}
0 < \mathcal{D}_\tau + \delta_n \mathcal{T} \leq \kappa \mathcal{I},
\end{equation}
where $\tau \in \Omega$ and $n \geq n_0$. We may assume that $\mathcal{D}_\tau$ is positive Hermitian tensors because $\|f(\mathcal{D}_\tau)\|_\rho = \infty$, i.e., Eq. (3.45) being true automatically, when $\mathcal{D}_\tau$ is nonneg-
ative Hermitian tensors. From Eq. (2.10), we have
\[
 f(D_\tau + \delta_n \mathcal{I}) = \sum_{\nu \text{ s.t. } \lambda_\nu(D_\tau) + \delta_n < \epsilon} f(\lambda_\nu(D_\tau)) + \delta_n \mathcal{U}_\nu \otimes \mathcal{U}_\nu^H + \sum_{\nu' \text{ s.t. } \lambda_{\nu'}(D_\tau) + \delta_n \geq \epsilon} f(\lambda_{\nu'}(D_\tau)) + \delta_n \mathcal{U}_{\nu'} \otimes \mathcal{U}_{\nu'}^H \]
\[
 \leq \sum_{\nu \text{ s.t. } \lambda_\nu(D_\tau) + \delta_n < \epsilon} f(\lambda_\nu(D_\tau)) \mathcal{U}_\nu \otimes \mathcal{U}_\nu^H + \sum_{\nu' \text{ s.t. } \lambda_{\nu'}(D_\tau) + \delta_n \geq \epsilon} f(\lambda_{\nu'}(D_\tau)) + \delta_n \mathcal{U}_{\nu'} \otimes \mathcal{U}_{\nu'}^H \]
\[
 \leq f(D_\tau) + \sum_{\nu' \text{ s.t. } \lambda_{\nu'}(D_\tau) + \delta_n \geq \epsilon} f(\lambda_{\nu'}(D_\tau)) + \delta_n \mathcal{U}_{\nu'} \otimes \mathcal{U}_{\nu'}^H. \tag{3.47}
\]
Therefore, the claim in Eq. (3.45) follows by the triangle inequality for \(\|\cdot\|_p\) and \(f(\lambda_{\nu'}(D_\tau) + \delta_n) < \infty\) for \(\lambda_{\nu'}(D_\tau) + \delta_n \geq \epsilon\).

**Eq. (3.37) \iff Eq. (3.38)**

The weak majorization relation
\[
 \left(\sum_{i=1}^{k} \lambda_i(C) \right)^{\frac{1}{k}} \leq \exp\left(\int_{\Omega} \log \lambda_i(D_\tau) d\nu(\tau)\right), \tag{3.48}
\]
is valid for \(k < r\) from Eq. (3.12) \(\implies\) Eq. (3.13) in Theorem 3.3. We wish to prove that Eq. (3.48) becomes equal for \(k = r\). It is equivalent to prove that
\[
 \log \det H(C) \geq \int_{\Omega} \log \det H(D_\tau) d\nu(\tau), \tag{3.49}
\]
where \(\det H(\cdot)\) is the Hermitian determinant. We can assume that \(\int_{\Omega} \log \det H(D_\tau) d\nu(\tau) \geq -\infty\) since Eq. (3.49) is true for \(\int_{\Omega} \log \det H(D_\tau) d\nu(\tau) = -\infty\). Then, \(D_\tau\) are positive Hermitian tensors.

If we scale tensors \(C, D_\tau\) as \(aC, aD_\tau\) by some \(a > 0\), we can assume \(D_\tau \leq I\) and \(\lambda_i(D_\tau) \leq 1\) for all \(\tau \in \Omega\) and \(i \in \{1, 2, \ldots, r\}\). Then for any \(p > 0\), we have
\[
 \frac{1}{r} \left\|D_\tau^{-p}\right\|_1 = \lambda_r^{-p}(D_\tau) \leq (\det H(D_\tau))^{-p}, \tag{3.50}
\]
and
\[
 \frac{1}{p} \log \left(\frac{\|D_\tau^{-p}\|_1}{r}\right) \leq -\log \det H(D_\tau). \tag{3.51}
\]
If we use tensor trace norm, represented by \(\|\cdot\|_1\), as unitarily invariant tensor norm and \(f(x) = x^{-p}\) for any \(p > 0\) in Eq. (3.38), we obtain
\[
 \log \|C^{-p}\|_1 \leq \int_{\Omega} \log \|D_\tau^{-p}\|_1 d\nu(\tau). \tag{3.52}
\]
By adding \(\frac{1}{r}\) and multiplying \(\frac{1}{p}\) for both sides of Eq. (3.52), we have
\[
 \frac{1}{p} \log \left(\frac{\|C^{-p}\|_1}{r}\right) \leq \int_{\Omega} \frac{1}{p} \log \left(\frac{\|D_\tau^{-p}\|_1}{r}\right) d\nu(\tau). \tag{3.53}
\]
Similar to Eqs. (3.28) and (3.29), we have following two relations as \( p \to 0 \):

\[
\frac{1}{p} \log \left( \frac{\| C^{-p} \|_1}{r} \right) \to -\frac{1}{r} \log \det_H(C),
\]

and

\[
\frac{1}{p} \log \left( \frac{\| D_{\tau}^{-p} \|_1}{r} \right) \to -\frac{1}{r} \log \det_H(D_{\tau}).
\]

From Eq. (3.51) and Lebesgue’s dominated convergence theorem, we have

\[
\lim_{p \to 0} \int_{\Omega} \frac{1}{p} \log \left( \frac{\| D_{\tau}^{-p} \|_1}{r} \right) d\nu(\tau) = -\frac{1}{r} \int_{\Omega} \log \det_H(D_{\tau}) d\nu(\tau)
\]

Finally, we have Eq. (3.49) from Eqs. (3.53) and (3.56).

**Eq. (3.37) \implies Eq. (3.39)**

First, we assume that \( C, D_{\tau} \) are positive Hermitian tensors and \( D_{\tau} \geq \delta I \) for \( \tau \in \Omega \). From Eq. (3.37), we can apply Theorem 3.2 to \( \log C, \log D_{\tau} \) and \( f(x) = g(e^x) \) to obtain Eq. (3.39).

For \( C, D_{\tau} \) are nonnegative Hermitian tensors, we can choose \( a^{(n)} \) and corresponding \( C^{(n)} \) for \( n \geq n_0 \) given \( \delta_n \to 0 \) with \( \delta_n > 0 \) as the proof in Eq. (3.37) \implies Eq. (3.38). Since tensors \( C^{(n)}, D_{\tau} + \delta_n I \) are positive Hermitian tensors, we then have

\[
\left\| g(C^{(n)}) \right\|_\rho \leq \int_{\Omega} \left\| g(D_{\tau} + \delta_n I) \right\|_\rho d\nu(\tau).
\]

If \( g(0^+) < \infty \), Eq. (3.39) is obtained from Eq. (3.57) by taking \( n \to \infty \). On the other hand, if \( g(0^+) = \infty \), we can apply the argument similar to the portion about \( f(0^+) = \infty \) in the proof for Eq. (3.37) \implies Eq. (3.38) to get \( a, b > 0 \) such that

\[
a \leq \| g(D_{\tau} + \delta_n I) \|_\rho \leq \| g(D_{\tau}) \|_\rho + b,
\]

for all \( \tau \in \Omega \) and \( n \geq n_0 \). Since the case that \( \int_{\Omega} \| g(D_{\tau}) \|_\rho d\nu(\tau) = \infty \) will have Eq. (3.39), we only consider the case that \( \int_{\Omega} \| g(D_{\tau}) \|_\rho d\nu(\tau) < \infty \). Then, we have Eq. (3.39) from Eqs. (3.57), (3.58) and Lebesgue’s dominated convergence theorem.

**Eq. (3.37) \iff Eq. (3.39)**

The weak majorization relation

\[
\sum_{i=1}^{k} \log \lambda_i(C) \leq \sum_{i=1}^{k} \int_{\Omega} \log \lambda_i(D_{\tau}) d\nu(\tau)
\]

is true from the implication from Eq. (3.12) to Eq. (3.14) in Theorem 3.3. We have to show that this relation becomes identity for \( k = r \). If we apply \( \| \|_\rho = \| \|_1 \) and \( g(x) = x^{-p} \) for any \( p > 0 \) in Eq. (3.39), we have

\[
\frac{1}{p} \log \left( \frac{\| C^{-p} \|_1}{r} \right) \leq \frac{1}{p} \log \left( \int_{\Omega} \| D_{\tau}^{-p} \|_1 d\nu(\tau) \right).
\]

Then, we will get

\[
-\frac{\log \det_H(C)}{r} = \lim_{p \to 0} \frac{1}{p} \log \left( \frac{\| C^{-p} \|_1}{r} \right)
\]

\[
\leq \lim_{p \to 0} \frac{1}{p} \log \left( \int_{\Omega} \| D_{\tau}^{-p} \|_1 d\nu(\tau) \right) = -\frac{1}{r} \int_{\Omega} \log \det_H(D_{\tau}) d\nu(\tau),
\]

where
which will prove the identity for Eq. (3.59) when \( k = r \). The equality in \( =_1 \) will be proved by the following Lemma 3.5.

**Lemma 3.5.** Let \( D_\tau \) be nonnegative Hermitian tensors with \( \int_\Omega \|D_\tau^{-p}\|_p d\nu(\tau) < \infty \) for any \( p > 0 \), then we have

\[
\lim_{\rho \to 0} \left( \frac{1}{p} \log \int_\Omega \frac{\|D_\tau^{-p}\|}{r} d\nu(\tau) \right) = -\frac{1}{r} \int_\Omega \log \det H(D_\tau) d\nu(\tau) \quad (3.62)
\]

**Proof:** Because \( \int_\Omega \|D_\tau^{-p}\|_p d\nu(\tau) < \infty \), we have that \( D_\tau \) are positive Hermitian tensors for \( \tau \) almost everywhere in \( \Omega \). Then, we have

\[
\lim_{\rho \to 0} \left( \frac{1}{p} \log \int_\Omega \frac{\|D_\tau^{-p}\|}{r} d\nu(\tau) \right) = 1 \lim_{\rho \to 0} \int_\Omega \sum_{i=1}^r \log \lambda_i(D_\tau) d\nu(\tau)
\]

\[
= 2 \frac{1}{r} \int_\Omega \log \det H(D_\tau) d\nu(\tau), \quad (3.63)
\]

where \( =_1 \) is from l'Hopital's rule, and \( =_2 \) is obtained from \( \det H \) definition. \( \square \)

### 3.3. Multivariate Tensor Norm Inequalities

In this section, we will apply derived majorization inequalities for tensors to multivariate tensor norm inequalities which will be used to bound random tensor concentration inequalities in later sections. We will begin to present a Lie-Trotter product formula for tensors.

**Lemma 3.6.** Let \( m \in \mathbb{N} \) and \( (\mathcal{L}_k)_{k=1}^m \) be a finite sequence of bounded tensors with dimensions \( \mathcal{L}_k \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \), then we have

\[
\lim_{n \to \infty} \left( \prod_{k=1}^m \exp \left( \frac{\mathcal{L}_k}{n} \right) \right)^n = \exp \left( \sum_{k=1}^m \mathcal{L}_k \right) \quad (3.64)
\]

**Proof:** We will prove the case for \( m = 2 \), and the general value of \( m \) can be obtained by mathematical induction. Let \( \mathcal{L}_1, \mathcal{L}_2 \) be bounded tensors act on some Hilbert space. Define \( \mathcal{C} \overset{\text{def}}{=} \exp((\mathcal{L}_1 + \mathcal{L}_2)/n) \), and \( \mathcal{D} \overset{\text{def}}{=} \exp(\mathcal{L}_1/n) *_M \exp(\mathcal{L}_2/n) \). Note we have following estimates for the norm of tensors \( \mathcal{C}, \mathcal{D} \):

\[
\|\mathcal{C}\|, \|\mathcal{D}\| \leq \exp \left( \frac{\|\mathcal{L}_1\| + \|\mathcal{L}_2\|}{n} \right) = \left[ \exp (\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) \right]^{1/n}. \quad (3.65)
\]

From the Cauchy-Product formula, the tensor \( \mathcal{D} \) can be expressed as:

\[
\mathcal{D} = \exp(\mathcal{L}_1/n) *_M \exp(\mathcal{L}_2/n) = \sum_{i=0}^{\infty} \frac{(\mathcal{L}_1/n)^i}{i!} *_M \sum_{j=0}^{\infty} \frac{(\mathcal{L}_2/n)^j}{j!} = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{\mathcal{L}_1}{i!} *_M \frac{\mathcal{L}_2}{(l-i)!} \right)^{l-i} \quad (3.66)
\]
then we can bound the norm of $C - D$ as

$$
\|C - D\| = \left\| \sum_{i=0}^{\infty} \frac{([L_1 + L_2]/n)^i}{i!} - \sum_{i=0}^{\infty} \frac{\sum_{l=0}^{n-i} \frac{L_1^i}{i!} \cdot \frac{L_2^{l-i}}{(l-i)!}}{i!} \right\|
$$

$$
\leq \left\| \sum_{i=2}^{\infty} k^{-i} \left( \frac{[L_1 + L_2]^i}{i!} - \sum_{i=0}^{\infty} \frac{\sum_{l=0}^{n-i} \frac{L_1^i}{i!} \cdot \frac{L_2^{l-i}}{(l-i)!}}{i!} \right) \right\|
$$

$$
\leq \frac{1}{k^n} \left( \exp(\|L_1\| + \|L_2\|) + \sum_{i=2}^{\infty} \sum_{l=0}^{n-i} \frac{\|L_1\|^i}{i!} \cdot \frac{\|L_2\|^{l-i}}{(l-i)!} \right)
$$

$$
\leq \frac{1}{n^2} \left( \exp(\|L_1\| + \|L_2\|) + \sum_{i=2}^{\infty} \sum_{l=0}^{n-i} (\|L_1\| + \|L_2\|)^l \right)
$$

$$
\leq 2 \exp(\|L_1\| + \|L_2\|).
$$

(3.67)

For the difference between the higher power of $C$ and $D$, we can bound them as

$$
\|C^n - D^n\| = \left\| \sum_{l=0}^{n-1} C^m (C - D) C^{n-l-1} \right\|
$$

$$
\leq \left( \exp(\|L_1\| + \|L_2\|) \right) \cdot n \cdot \|L_1 - L_2\|,
$$

(3.68)

where the inequality $\leq_1$ uses the following fact

$$
\|C^l\| \|D\|^{n-1} \leq \exp(\|L_1\| + \|L_2\|) \frac{n-1}{n} \leq \exp(\|L_1\| + \|L_2\|),
$$

(3.69)

based on Eq. (3.65). By combining with Eq. (3.67), we have the following bound

$$
\|C^n - D^n\| \leq \frac{2 \exp(\|L_1\| + \|L_2\|)}{n}.
$$

(3.70)

Then this lemma is proved when $n$ goes to infinity. □

Below, new multivariate norm inequalities for tensors are provided according to previous majorization theorems.

**Theorem 3.7.** Let $C_i \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be positive Hermitian tensors for $1 \leq i \leq n$ with Hermitian rank $r$, $\|\cdot\|_\rho$ be a unitarily invariant norm with corresponding gauge function $\rho$. For any continuous function $f : (0, \infty) \to [0, \infty)$ such that $x \to \log f(e^x)$ is convex on $\mathbb{R}$, we have

$$
\left\| f \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_\rho \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} C_i^{1+t} \right) \right\|_\rho \beta_0(t) dt,
$$

(3.71)

where $\beta_0(t) = \frac{\pi}{2(\cosh(\pi t) + 1)}$.

For any continuous function $g(0, \infty) \to [0, \infty)$ such that $x \to g(e^x)$ is convex on $\mathbb{R}$, we have

$$
\left\| g \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_\rho \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_i^{1+t} \right) \right\|_\rho \beta_0(t) dt.
$$

(3.72)
**Proof:** From Hirschman interpolation theorem [36] and \( \theta \in [0,1] \), we have

\[
(3.73) \quad \log |h(\theta)| \leq \int_{-\infty}^{\infty} \log |h(it)|^{1-\theta} \beta_{1-\theta}(t)dt + \int_{-\infty}^{\infty} \log |h(1+it)|^{\theta} \beta_{\theta}(t)dt,
\]

where \( h(z) \) be uniformly bounded on \( S \triangleq \{ z \in \mathbb{C} : 0 \leq \Re(z) \leq 1 \} \) and holomorphic on \( S \). The term \( d\beta_{\theta}(t) \) is defined as:

\[
(3.74) \quad \beta_{\theta}(t) \overset{\text{def}}{=} \frac{\sin(\pi \theta)}{2\theta(\cos(\pi t) + \cos(\pi \theta))}.
\]

Let \( H(z) \) be a uniformly bounded holomorphic function with values in \( \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \). Fix some \( \theta \in [0,1] \) and let \( U, V \in \mathbb{C}^{I_1 \times \cdots \times I_N \times 1} \) be normalized tensors such that \( \langle U, H(\theta \ast_N V) \rangle = \| H(\theta) \| \). If we define \( h(z) \) as \( h(z) \overset{\text{def}}{=} \langle U, H(z) \ast_N V \rangle \), we have following bound: 

\[
|h(z)| \leq \| H(z) \| \text{ for all } z \in S.
\]

From Hirschman interpolation theorem, we then have following interpolation theorem for tensor-valued function:

\[
(3.75) \quad \log \| H(\theta) \| \leq \int_{-\infty}^{\infty} \log \| H(it)\|^{1-\theta} \beta_{1-\theta}(t)dt + \int_{-\infty}^{\infty} \log \| H(1+it)\|^{\theta} \beta_{\theta}(t)dt.
\]

Let \( H(z) = \prod_{i=1}^{n} C_i^z \). Then the first term in the R.H.S. of Eq. (3.75) is zero since \( H(it) \) is a product of unitary tensors. Then we have

\[
(3.76) \quad \log \left\| \prod_{i=1}^{n} C_i^t \right\|^t \leq \int_{-\infty}^{\infty} \log \left\| \prod_{i=1}^{n} C_i^{1+it} \right\|^t \beta_{\theta}(t)dt.
\]

From Lemma 2.7, we have following relations:

\[
(3.77) \quad \left| \prod_{i=1}^{n} (\wedge^k C_i)^{t} \right|^t = \wedge^k \left| \prod_{i=1}^{n} C_i^t \right|^t,
\]

and

\[
(3.78) \quad \left| \prod_{i=1}^{n} (\wedge^k C_i)^{1+it} \right| = \wedge^k \left| \prod_{i=1}^{n} C_i^{1+it} \right|.
\]

If Eq. (3.76) is applied to \( \wedge^k C_i \) for \( 1 \leq k \leq r \), we have following log-majorization relation from Eqs. (3.77) and (3.78):

\[
(3.79) \quad \log \lambda \left( \left| \prod_{i=1}^{n} C_i^t \right|^t \right) \times \int_{-\infty}^{\infty} \log \lambda \left( \left| \prod_{i=1}^{n} C_i^{1+it} \right|^t \right) \beta_{\theta}(t)dt.
\]

Moreover, we have the equality condition in Eq. (3.79) for \( k = r \) due to following identities:

\[
(3.80) \quad \det_H \left| \prod_{i=1}^{n} C_i^t \right|^t = \det_H \left| \prod_{i=1}^{n} C_i^{1+it} \right| = \prod_{i=1}^{n} \det_H C_i.
\]
At this stage, we are ready to apply Theorem 3.4 for the log-majorization provided by Eq. (3.79) to get following facts:

\[(3.81) \quad \left\| f \left( \prod_{i=1}^{n} C_i^s \right) \right\|_{\rho} \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} C_i^{1+it} \right) \right\|_{\rho} \beta_\theta(t) dt, \]

and

\[(3.82) \quad \left\| g \left( \prod_{i=1}^{n} C_i^s \right) \right\|_{\rho} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_i^{1+it} \right) \right\|_{\rho} \beta_\theta(t) dt. \]

From Lie product formula for tensors given by Lemma 4.1, we have

\[(3.83) \quad \left\| \prod_{i=1}^{n} C_i^s \right\|_{\rho} \rightarrow \exp \left( \sum_{i=1}^{n} \log C_i \right). \]

By setting \( \theta \rightarrow 0 \) in Eqs. (3.81), (3.82) and using Lie product formula given by Eq. (3.83), we will get Eqs. (3.71) and (3.72).

4. Generalized Tensor Chernoff and Bernstein Inequalities. In this section, we first utilize Theorem 3.7 and Laplace transform method to obtain Ky Fan \( k \)-norm concentration bounds for a function of tensors summation in Section 4.1. Then, generalized tensor Chernoff and Bernstein bounds are discussed in Section 4.2 and Section 4.3, respectively.

4.1. Ky Fan \( k \)-norm Tail Bound. We begin by introducing following two lemmas about Ky Fan \( k \)-norm inequalities for the product of tensors (Lemma 4.1) and the summation of tensors (Lemma 4.2).

**Lemma 4.1.** Let \( C_i \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) with Hermitian rank \( r \) and let \( p_i \) be positive real numbers satisfying \( \sum_{i=1}^{m} \frac{1}{p_i} = 1 \). Then, we have

\[(4.1) \quad \left\| \prod_{i=1}^{m} C_i^s \right\|_{(k)} \leq \prod_{i=1}^{m} \left( \| C_i \|_{sp_i} \right)^{\frac{1}{p_i}} \leq \sum_{i=1}^{m} \frac{\| C_i \|_{sp_i}}{p_i}, \]

where \( s \geq 1 \) and \( k \in \{1, 2, \cdots, r\} \).

**Proof:** Since we have

\[(4.2) \quad \left\| \prod_{i=1}^{m} C_i^s \right\|_{(k)} = \sum_{j=1}^{k} \lambda_j \left( \prod_{i=1}^{m} C_i^s \right) = \sum_{j=1}^{k} \lambda_j^s \left( \prod_{i=1}^{m} C_i \right) = \sum_{j=1}^{k} \sigma_j^s \left( \prod_{i=1}^{m} C_i \right), \]

where we have orders for eigenvalues as \( \lambda_1 \geq \lambda_2 \geq \cdots \), and singular values as \( \sigma_1 \geq \sigma_2 \geq \cdots \).

From Lemma 2.7, we have

\[(4.3) \quad \left\| \left( \prod_{i=1}^{m} C_i \right)^{\wedge k} \right\| = \prod_{j=1}^{k} \sigma_j \left( \prod_{i=1}^{m} C_i \right). \]
From the fact that the norm is submultiplicative and majorization theory, we will have

$$\sum_{j=1}^{k} \sigma_j^s \left( \prod_{i=1}^{m} C_i \right) \leq \sum_{j=1}^{k} \left( \prod_{i=1}^{m} \sigma_j^s (C_i) \right).$$

(4.4)

Then, we can apply Hölder’s inequality to Eq. (4.4) and obtain

$$\sum_{j=1}^{k} \left( \prod_{i=1}^{m} \sigma_j^s (C_i) \right) \leq \prod_{i=1}^{m} \left( \sum_{j=1}^{k} \sigma_j^{sp_i} (C_i) \right)^{\frac{1}{p_i}} = \prod_{i=1}^{m} \left( \sum_{j=1}^{k} \lambda_j^{sp_i} (|C_i|) \right)^{\frac{1}{p_i}}$$

(4.5)

$$= \prod_{i=1}^{m} \left( \sum_{j=1}^{k} \lambda_j^r (|C_i|^{sp_i}) \right)^{\frac{1}{p_i}} = \prod_{i=1}^{m} \left( \|C_i|^{sp_i}\|_{(k)} \right)^{\frac{1}{p_i}}$$

The second inequality in Eq. (4.1) is obtained by applying Young’s inequality to numbers $\|C_i|^{sp_i}\|_{(k)}$ for $1 \leq i \leq m$. This completes the proof. □

**Lemma 4.2.** Let $C_i \in C_1^{I_1} \times \cdots \times I_N \times I_1 \times \cdots \times I_N$ with Hermitian rank $r$, then we have

$$\left\| \sum_{i=1}^{m} C_i \right\|_s^{(k)} \leq m^{s-1} \sum_{i=1}^{m} \|C_i|^{s}\|_{(k)}$$

(6.6)

where $s \geq 1$ and $k \in \{1, 2, \cdots, r\}$.

**Proof:** Since we have

$$\left\| \sum_{i=1}^{m} C_i \right\|_s^{(k)} = \sum_{j=1}^{k} \lambda_j^s \left( \left| \sum_{i=1}^{m} C_i \right|^s \right) = \sum_{j=1}^{k} \lambda_j^s \left( \prod_{i=1}^{m} C_i \right) = \sum_{j=1}^{k} \sigma_j^s \left( \sum_{i=1}^{m} C_i \right),$$

where we have orders for eigenvalues as $\lambda_1 \geq \lambda_2 \geq \cdots$, and singular values as $\sigma_1 \geq \sigma_2 \geq \cdots$.

From Theorem G.1.d. in [24] and Theorem 3.2 in [21], we have Fan singular value majorization inequalities:

$$\sum_{j=1}^{k} \sigma_j^s \left( \sum_{i=1}^{m} C_i \right) \leq \sum_{j=1}^{k} \sum_{i=1}^{m} \sigma_j^s (C_i),$$

(4.8)

where $k \in \{1, 2, \cdots, s\}$. Then, we have

$$\sum_{j=1}^{k} \sigma_j^s \left( \sum_{i=1}^{m} C_i \right) \leq \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j^s (C_i) \right)^s \leq m^{s-1} \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j^s (C_i) \right)$$

$$= m^{s-1} \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j^s (|C_i|) \right) = m^{s-1} \sum_{j=1}^{k} \sum_{i=1}^{m} \sigma_j (|C_i|^s)$$

(4.9)

$$= m^{s-1} \sum_{i=1}^{m} \|C_i|^{s}\|_{(k)}$$

Now, we are ready to present our main theorem about Ky Fan $k$-norm probability bound for a function of tensors summation.

□
Theorem 4.3. Consider a sequence \( \{X_j \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}\} \) of independent, random, Hermitian tensors. Let \( g \) be a polynomial function with degree \( n \) and nonnegative coefficients \( a_0, a_1, \ldots, a_n \) raised by power \( s \geq 1 \), i.e., \( g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s \). Suppose following condition is satisfied:

\[
(4.10) \quad g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( t g \left( \sum_{j=1}^{m} X_j \right) \right) \quad \text{almost surely},
\]

where \( t > 0 \). Then, we have

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t,p_j} e^{-\theta t} \left( k a_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} a_l^s \mathbb{E} \left\| \exp \left( \sum_{j=1}^{m} p_j l X_j \right) \right\|_{(k)} \right) .
\]

(4.11)

where \( \sum_{j=1}^{m} \frac{1}{p_j} = 1 \) and \( p_j > 0 \).

Proof: Let \( t > 0 \) be a parameter to be chosen later. Then

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) = \Pr \left( \left\| \exp \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \exp (\theta t) \right) \leq_1 \exp (-\theta t) \mathbb{E} \left( \left\| \exp \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \right) \leq_2 \exp (-\theta t) \mathbb{E} \left( \left\| g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \right\|_{(k)} \right)
\]

(4.12)

where \( \leq_1 \) uses Markov’s inequality, \( \leq_2 \) requires condition provided by Eq. (4.10).

We can further bound the expectation term in Eq. (4.11) as

\[
\mathbb{E} \left( \left\| g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \right\|_{(k)} \right) \leq_3 \mathbb{E} \int_{-\infty}^{\infty} \left\| g \left( \prod_{j=1}^{m} e^{(1+i\tau) t X_j} \right) \right\|_{(k)} \beta_0(\tau) d\tau
\]

(4.13)

\[
\leq_4 (n + 1)^{s-1} \left( k a_0^s + \sum_{l=1}^{n} a_l^s \mathbb{E} \int_{-\infty}^{\infty} \left\| \prod_{j=1}^{m} e^{(1+i\tau) l X_j} \right\|_{(k)} \beta_0(\tau) d\tau \right),
\]

where \( \leq_3 \) from Eq. (3.72) in Theorem 3.7, \( \leq_4 \) is obtained from function \( g \) definition and Lemma 4.2. Again, the expectation term in Eq. (4.13) can be further bounded
by Lemma 4.1 as

\[
E \int_{-\infty}^{\infty} \left\| \prod_{j=1}^{m} e^{(1+i\pi)\tau X_j} \right\|_{(k)} I_s \beta_0(\tau) d\tau \leq E \int_{-\infty}^{\infty} \sum_{j=1}^{m} \left\| \frac{e^{tX_j} p_j}{p_j} \right\|_{(k)} \beta_0(\tau) d\tau
\]

(4.14)

Note that the final equality is obtained due to that the integrand is independent of the variable \( \tau \) and \( \int_{-\infty}^{\infty} \beta_0(\tau) d\tau = 1. \)

Finally, this theorem is established from Eqs. (4.12), (4.13), and (4.14). \( \square \)

Remarks: The condition provided by Eq. (4.12) can be achieved by normalizing tensors \( X_j \) through scaling.

4.2. Generalized Tensor Chernoff Bound. Following lemma is about Ky Fan \( k \)-norm bound for the exponential of a random tensor with the constraint of the maximum singular value.

Lemma 4.4. Given a nonnegative Hermitian random tensor \( X \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) with \( \lambda_{\text{max}}(X) \leq \mathbb{R} \), \( \frac{X + X^*}{2} \) is defined as \( \mathbb{E} \left( \frac{X + X^*}{2} \right) \), and \( \frac{X - X^*}{2} \) is defined as \( \mathbb{E} \left( \frac{X - X^*}{2} \right) \) respectively. Then, we have following bound about the expectation value of Ky Fan \( k \)-norm for the random tensor \( \exp(\theta X) \)

\[
\mathbb{E} \left\| \exp(\theta X) \right\|_{(k)} \leq k \left\{ 1 + (e^\theta - 1) \left[ \sigma_1 \left( \frac{X + X^*}{2} \right) + \sigma_1 \left( \frac{X - X^*}{2} \right) \right] \right\} \leq (e^\theta - 1) C \Xi(X),
\]

(4.15)

where \( \theta \) is a real number, \( C \) is a constant, \( x_{i,j} \) and \( y_{i,j} \) are entries of matrices obtained from unfolded random real tensors \( \frac{X + X^*}{2} - \frac{X + X^*}{2} \) and \( \frac{X - X^*}{2} - \frac{X - X^*}{2} \), respectively.

The matrices from unfolded tensors are obtained by the method presented in Section 2.2 [21]. For notation simplicity, the term \( \Xi(X) \) is defined as

\[
\Xi(X) \overset{\text{def}}{=} \left\{ \max_i \left( \sum_j \mathbb{E} x_{i,j}^2 \right)^{1/2} + \max_j \left( \sum_i \mathbb{E} x_{i,j}^2 \right)^{1/2} + \left( \sum_{i,j} \mathbb{E} x_{i,j}^4 \right)^{1/4} \right\}.
\]

(4.16)
**Proof:** Consider the function $f(x) = e^{\theta x}$. Since $f$ is convex, we have
\begin{equation}
(4.17) \quad f(x) \leq f(0) + [f(R) - f(0)]x \quad \text{for } x \in [0, R].
\end{equation}

Because all eigenvalues of $\mathcal{X}$ lie in $[0, R]$, we will get
\begin{equation}
(4.18) \quad e^{\theta \mathcal{X}} \preceq \mathcal{I} + (e^{\theta R} - 1)\mathcal{X}.
\end{equation}

From majorization inequality for eigenvalues of two positive definite tensors, we have
\begin{equation}
(4.19) \quad \mathbb{E} \|\exp(\theta \mathcal{X})\|_{(k)} = \sum_{l=1}^{k} \mathbb{E}\sigma_l(\exp(\theta \mathcal{X})) \\
\leq \sum_{l=1}^{k} \mathbb{E}\sigma_l(\mathcal{I} + (e^{\theta R} - 1)\mathcal{X}) \leq k\mathbb{E}\sigma_1(\mathcal{I} + (e^{\theta R} - 1)\mathcal{X})
\end{equation}
where $\sigma_l(\cdot)$ is the $l$-th largest singular value.

From Theorem G.1.d in [24] and Theorem 3.2 in [21], we have $\sigma_1(\mathcal{A} + \mathcal{B}) \leq \sigma_1(\mathcal{A}) + \sigma_1(\mathcal{B})$ for two complex tensors $\mathcal{A}$ and $\mathcal{B}$. Then, we can bound $\mathbb{E}\sigma_1(\mathcal{I} + (e^{\theta R} - 1)\mathcal{X})$ as
\begin{align*}
\mathbb{E}\sigma_1(\mathcal{I} + (e^{\theta R} - 1)\mathcal{X}) &\leq \sigma_1(\mathcal{I}) + (e^{\theta R} - 1)\mathbb{E}\sigma_1(\mathcal{X}) \\
&= 1 + (e^{\theta R} - 1)\mathbb{E}\sigma_1\left(\frac{\mathcal{X} + \mathcal{X}^*}{2} + \frac{i}{2} \mathcal{X} - \mathcal{X}^*\right) \\
&\leq 1 + (e^{\theta R} - 1)\mathbb{E}\sigma_1\left(\frac{\mathcal{X} + \mathcal{X}^*}{2}\right) + (e^{\theta R} - 1)\mathbb{E}\sigma_1\left(\frac{i}{2} \mathcal{X} - \mathcal{X}^*\right) \\
&\leq 1 + (e^{\theta R} - 1) \left[ \sigma_1\left(\frac{\mathcal{X} + \mathcal{X}^*}{2}\right) + \mathbb{E}\sigma_1\left(\frac{\mathcal{X} + \mathcal{X}^*}{2} - \frac{\mathcal{X} - \mathcal{X}^*}{2}\right) \right] + (e^{\theta R} - 1) \left[ \sigma_1\left(\frac{\mathcal{X} - \mathcal{X}^*}{2}\right) + \mathbb{E}\sigma_1\left(\frac{\mathcal{X} - \mathcal{X}^*}{2} - \frac{\mathcal{X} + \mathcal{X}^*}{2}\right) \right].
\end{align*}

This lemma is proved by using Theorem 2.5 in [31] to bound $\mathbb{E}\sigma_1\left(\frac{\mathcal{X} + \mathcal{X}^*}{2} - \frac{\mathcal{X} - \mathcal{X}^*}{2}\right)$ and $\mathbb{E}\sigma_1\left(\frac{\mathcal{X} - \mathcal{X}^*}{2} - \frac{\mathcal{X} + \mathcal{X}^*}{2}\right)$ since $\mathbb{E}\left(\frac{\mathcal{X} + \mathcal{X}^*}{2} - \frac{\mathcal{X} - \mathcal{X}^*}{2}\right) = \mathbb{E}\left(\frac{\mathcal{X} - \mathcal{X}^*}{2} - \frac{\mathcal{X} + \mathcal{X}^*}{2}\right) = O$. □

Following theorem is presented to provide the general tensor Chernoff bound under Ky Fan $k$-norm.

**Theorem 4.5 (Generalized Tensor Chernoff Bound).** Consider a sequence $\{\mathcal{X}_j \in \mathbb{C}^{I_1 \times \cdots \times I_N} \}$ of independent, random, Hermitian tensors. Let $g$ be a polynomial function with degree $n$ and nonnegative coefficients $a_0, a_1, \ldots, a_n$, raised by power $s \geq 1$, i.e., $g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s$ with $s \geq 1$. Suppose following condition is satisfied:
\begin{equation}
(4.21) \quad g \left( \exp \left( t \sum_{j=1}^{m} \mathcal{X}_j \right) \right) \geq \exp \left( tg \left( \sum_{j=1}^{m} \mathcal{X}_j \right) \right) \quad \text{almost surely,}
\end{equation}

where $t > 0$. Moreover, we require
\begin{equation}
(4.22) \quad \mathcal{X}_i \preceq O \quad \text{and} \quad \lambda_{\max}(\mathcal{X}_i) \leq R \quad \text{almost surely.}
\end{equation}
Then we have following inequality:

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_{(k)} \geq \theta \right) \leq (n+1)^{s-1} \inf_{t \geq 0} e^{-\theta t} .
\]

(4.23) \quad \left\{ ka_t^s + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{k a_t^s}{m} \left[ 1 + (e^{m l s R t} - 1) \frac{\sigma_1}{\sigma_1} + C \left( e^{m l s R t} - 1 \right) \Xi(X_j) \right] \right\},

where \( \overline{\sigma_1} = \left[ \sigma_1 \left( \frac{X_j + X_j}{2} \right) \right] \). Let us define following three terms \( A_1(t), A_2(t) \) and \( A_3(t) \) as

\[
A_1(t) = ka_t^s + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{k a_t^s}{m} \left[ 1 + (e^{m l s R t} - 1) \overline{\sigma_1} + C \left( e^{m l s R t} - 1 \right) \Xi(X_j) \right],
\]

\[
A_2(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} k l s R k a_t^s \left( \overline{\sigma_1} + C \Xi(X_j) \right) e^{m l s R t},
\]

(4.24) \quad A_3(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} k m (l s R)^2 a_t^s \left( \overline{\sigma_1} + C \Xi(X_j) \right) e^{m l s R t}.

If we have \( \theta^2 A_1(t) - 2 \theta A_2(t) + A_3(t) > 0 \) for \( t > 0 \), then the bound in Eq. (4.23) is a convex function with respect to \( t \) and the minimizer, denoted as \( t_{opt} \), is the solution of the following equation:

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{k a_t^s}{m} e^{m l s R t_{opt}} \left[ (m l s R + \theta)(\overline{\sigma_1} + C \Xi(X_j)) \right] = \theta \left[ ka_t^s + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{k a_t^s}{m} \left( 1 - \overline{\sigma_1} - C \Xi(X_j) \right) \right]
\]

(4.25)

**Proof:** From Theorem 4.3 and Lemma 4.4, we will have the bound given by Eq. (4.23) by taking \( p_j = m \).

The convexity condition, \( \theta^2 A_1(t) - 2 \theta A_2(t) + A_3(t) > 0 \), of this generalized Chernoff bound is obtained by setting the second derivative of Eq. (4.26) with respect to \( t \) greater than zero.

\[
e^{-\theta t} \left\{ ka_t^s + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{k a_t^s}{m} \left[ 1 + (e^{m l s R t} - 1) \overline{\sigma_1} + C \left( e^{m l s R t} - 1 \right) \Xi(X_j) \right] \right\}.
\]

(4.26)

Similarly, the optimizer \( t_{opt} \) is obtained by setting the first derivative of Eq. (4.26) with respect to \( t \) to zero.

**4.3. Generalized Bernstein Tensor Bound.** In this section, we will present a generalized tensor Bernstein bound, and we will begin with a lemma to bound exponential of a random tensor.
LEMMA 4.6. Suppose that $\mathcal{X}$ is a random Hermitian tensor that satisfies

\begin{equation}
\mathcal{X}^p \leq \frac{p!A^2}{2} \quad \text{almost surely for } p = 2, 3, 4, \ldots,
\end{equation}

where $A$ is a fixed Hermitian tensor. Then, we have

\begin{equation}
e^{t\mathcal{X}} \leq I + t\mathcal{X} + \frac{t^2 A^2}{2(1-t)} \quad \text{almost surely},
\end{equation}

where $0 < t < 1$.

**Proof:** From Tayler series of the tensor exponential expansion, we have

\begin{equation}
e^{t\mathcal{X}} = I + t\mathcal{X} + \sum_{p=2}^{\infty} \frac{t^p \mathcal{X}^p}{p!} \leq I + t\mathcal{X} + \sum_{p=2}^{\infty} \frac{t^p A^2}{2} = I + t\mathcal{X} + \frac{t^2 A^2}{2(1-t)}.
\end{equation}

Therefore, this Lemma is proved.

Following lemma is about Ky Fan $k$-norm bound for the exponential of a random tensor with subexponential constraints.

**Lemma 4.7.** Given a Hermitian random tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ with $\mathbb{E}\mathcal{X} = 0$ and

\begin{equation}
\mathcal{X}^p \leq \frac{p!A^2}{2} \quad \text{almost surely for } p = 2, 3, 4, \ldots,
\end{equation}

where $A$ is a positive Hermitian tensor. Then, we have following bound about the expectation value of Ky Fan $k$-norm for the random tensor $\exp(\theta \mathcal{X})$

\begin{equation}
\mathbb{E} \|\exp(\theta \mathcal{X})\|_k \leq k \left\{ 1 + \frac{\theta^2}{2(1-\theta)} \sigma_1(A^2) + \right.
\end{equation}

\begin{equation}
\left. \theta C \left[ \max_i \left( \sum_j \mathbb{E} x_{i,j}^2 \right)^{1/2} + \max_j \left( \sum_i \mathbb{E} x_{i,j}^2 \right)^{1/2} + \left( \sum_{i,j} \mathbb{E} x_{i,j}^4 \right)^{1/4} \right] + \right.
\end{equation}

\begin{equation}
\left. \theta C \left[ \max_i \left( \sum_j \mathbb{E} y_{i,j}^2 \right)^{1/2} + \max_j \left( \sum_i \mathbb{E} y_{i,j}^2 \right)^{1/2} + \left( \sum_{i,j} \mathbb{E} y_{i,j}^4 \right)^{1/4} \right] \right\}
\end{equation}

\begin{equation}
\equiv k \left\{ 1 + \frac{\theta^2}{2(1-\theta)} \sigma_1(A^2) + \theta C \Upsilon(\mathcal{X}) \right\},
\end{equation}

where $\theta$ is a positive number in the range $(0,1)$, $C$ is a constant, $x_{i,j}$ and $y_{i,j}$ are entries of the unfolded random real tensors $\frac{\mathcal{X} + \mathcal{X}^*}{2}$ and $\frac{\mathcal{X} - \mathcal{X}^*}{2}$, respectively, from Eq. (4.16). The matrices from unfolded tensors are obtained by the method presented in Section 2.2 [21]. For notation simplicity, the term $\Upsilon$ is defined as

\begin{equation}
\Upsilon(\mathcal{X}) \equiv \left[ \max_i \left( \sum_j \mathbb{E} x_{i,j}^2 \right)^{1/2} + \max_j \left( \sum_i \mathbb{E} x_{i,j}^2 \right)^{1/2} + \left( \sum_{i,j} \mathbb{E} x_{i,j}^4 \right)^{1/4} \right] +
\end{equation}

\begin{equation}
\left. \max_i \left( \sum_j \mathbb{E} y_{i,j}^2 \right)^{1/2} + \max_j \left( \sum_i \mathbb{E} y_{i,j}^2 \right)^{1/2} + \left( \sum_{i,j} \mathbb{E} y_{i,j}^4 \right)^{1/4} \right],
\end{equation}

where random tensor $\mathcal{X}$ has zero tensor as its mean.
**Proof:** From Lemma 4.6 and majorization inequality for eigenvalues of two positive definite tensors, we have

\[
\mathbb{E} \|\exp(\theta X)\|_2 = \sum_{l=1}^{k} \mathbb{E} \sigma_l(\exp(\theta X)) \\
\leq \sum_{l=1}^{k} \mathbb{E} \sigma_l \left( I + \theta X + \frac{\theta^2 A^2}{2(1-\theta)} \right) \leq k \mathbb{E} \sigma_l \left( I + \theta X + \frac{\theta^2 A^2}{2(1-\theta)} \right)
\]

where \( \sigma_l(\cdot) \) is the \( l \)-th largest singular value.

From Theorem G.1.d in [24] and Theorem 3.2 in [21], we have \( \sigma_1(A+B) \leq \sigma_1(A) + \sigma_1(B) \) for two complex tensors \( A \) and \( B \). Then, we can bound \( \mathbb{E} \sigma_1 \left( I + \theta X + \frac{\theta^2 A^2}{2(1-\theta)} \right) \) as

\[
\mathbb{E} \sigma_1 \left( I + \theta X + \frac{\theta^2 A^2}{2(1-\theta)} \right) \leq 1 + \frac{\theta^2}{2(1-\theta)} \sigma_1(A^2) + \theta \mathbb{E} \sigma_1(X) \\
= 1 + \frac{\theta^2}{2(1-\theta)} \sigma_1(A^2) + \theta \mathbb{E} \sigma_1 \left( \frac{X - X^*}{2} \right) \\
\leq 1 + \frac{\theta^2}{2(1-\theta)} \sigma_1(A^2) + \theta \mathbb{E} \sigma_1 \left( \frac{X + X^*}{2} \right) + \theta \mathbb{E} \sigma_1 \left( \frac{X - X^*}{2} \right) \\
= 1 + \frac{\theta^2}{2(1-\theta)} \sigma_1(A^2) + \theta \mathbb{E} \sigma_1 \left( \frac{X + X^*}{2} \right) + \theta \mathbb{E} \sigma_1 \left( \frac{X - X^*}{2} \right)
\]

This lemma is proved by using Theorem 2.5 in [31] to bound \( \mathbb{E} \sigma_1 \left( \frac{X + X^*}{2} \right) \) and \( \mathbb{E} \sigma_1 \left( \frac{X - X^*}{2} \right) \) since \( \mathbb{E} \left( \frac{X + X^*}{2} \right) = \mathbb{E} \left( \frac{X - X^*}{2} \right) = \mathcal{O}. \)

**THEOREM 4.8 (Generalized Tensor Bernstein Bound).** Consider a sequence \( \{X_j \in \mathbb{C}^{l_1 \times \cdots \times l_N} \} \) of independent, random, Hermitian tensors. Let \( g \) be a polynomial function with degree \( n \) and nonnegative coefficients \( a_0, a_1, \ldots, a_n \) raised by power \( s \geq 1 \), i.e., \( g(x) = (a_0 + a_1 x + \cdots + a_n x^n)^s \) with \( s \geq 1 \). Suppose following condition is satisfied:

\[
g \left( \exp \left( t \sum_{j=1}^{m} X_j \right) \right) \geq \exp \left( t g \left( \sum_{j=1}^{m} X_j \right) \right) \quad \text{almost surely,}
\]

where \( t > 0 \), and we also have

\[
\mathbb{E} X_j = \mathcal{O} \quad \text{and} \quad X_j^p \leq \frac{p! A_j^2}{2} \quad \text{almost surely for} \quad p = 2, 3, 4, \ldots.
\]

Then we have following inequality:

\[
\Pr \left( \left\| g \left( \sum_{j=1}^{m} X_j \right) \right\|_2 \geq \theta \right) \leq (n + 1)^{s-1} \inf_{t > 0} e^{-\theta t k}.
\]

\[
\left\{ a_0^s + \sum_{l=1}^{m} \sum_{j=1}^{m} a_{l j}^s \left[ \frac{1}{m} + \frac{m(t)^2 \sigma_1(A_j^2)}{2(1 - mlst)} + lst Cy(X_j) \right] \right\}.
\]
Let us define following three terms \( B_1, B_2 \) and \( B_3 \) as

\[
B_1(t) = k\sigma_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} k\sigma_0^s \left[ \frac{1}{m} + \frac{m(lst)^2\sigma_1(A_j^2)}{2(1 - mst)} + lst\mathcal{Y}(X_j) \right],
\]

\[
B_2(t) = \sum_{l=1}^{n} \sum_{j=1}^{m} k\sigma_0^s \left[ \frac{(4lm^2s^2t - 3m^3s^3m^2t^2)\sigma_1(A_j^2)}{2(1 - mst)^2} + lst\mathcal{Y}(X_j) \right],
\]

\[
B_3(t) = \sum_{l=1}^{n} \sum_{j=1}^{m} k\sigma_0^s \left[ \frac{2lm^2s^2t}{2(1 - mst)^3} \right].
\]

(4.38) \( B_3(t) \) is obtained by taking first and second derivatives of Eq. (4.40) with respect to \( t \) through tedious algebraic manipulations.

If we have \( \theta^2B_1(t) - 2B_2(t) + B_2(t) > 0 \) for \( 0 < t < \frac{1}{mst} \) and \( Cls\mathcal{Y}(X_j) < \frac{a}{m} \) for \( 1 \leq l \leq n \) and \( 1 \leq j \leq m \), then the bound in Eq. (4.37) is a convex function with respect to \( t \) and the minimizer, denoted as \( t_{\text{opt}} \), of this bound is the solution of the following equation

\[
B_2(t_{\text{opt}}) = B_1(t_{\text{opt}})\theta.
\]

**Proof:** From Theorem 4.3 and Lemma 4.7, we will have the bound given by Eq. (4.37) by taking \( p_j = m \).

The convexity condition of this generalized Bernstein bound and its optimizer \( t_{\text{opt}} \) are obtained by taking first and second derivatives of Eq. (4.40) with respect to \( t \) through tedious algebraic manipulations.

\[
e^{-\theta t} \left\{ a_0^s + \sum_{l=1}^{n} \sum_{j=1}^{m} a_l^s \left[ \frac{1}{m} + \frac{m(lst)^2\sigma_1(A_j^2)}{2(1 - mst)} + lst\mathcal{Y}(X_j) \right] \right\}.
\]

The condition of \( \theta^2B_1(t) - 2B_2(t) + B_2(t) > 0 \) for \( 0 < t < \frac{1}{mst} \) is obtained by having the second derivative of Eq. (4.40) positive. The optimizer \( t_{\text{opt}} \) is obtained by setting the first derivative of Eq. (4.40) to zero, which is Eq. (4.39). The convexity condition of Eq. (4.40) and \( Cls\mathcal{Y}(X_j) < \frac{a}{m} \) for \( 1 \leq l \leq n \) and \( 1 \leq j \leq m \) make sure that there is a unique real solution for \( t_{\text{opt}} \) between 0 and \( \frac{1}{mst} \). \( \square \)

5. Covariance Tensor Characterization by Generalized Tensor Chernoff Inequality. In this section, we will try to apply generalized tensor Chernoff inequality derived in Section 4.2 to bound Ky Fan norm of covariance tensor. In [23], Marques et al. provide a comprehensive introduction to the spectral analysis and estimation of graph stationary processes based on graph signal processing (GSP). We extend their settings from vectors/matrices used in traditional GSP to hypergraph signal processing, where tensors are applied to characterize high-dimensional signals [43].

Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) be a directed hypergraph with nodes set \( \mathcal{V} \) and directed edges set \( \mathcal{E} \) such that if there exists a hyperedge between two sets of \( M \) nodes \( (i_1, \ldots, i_M, j_1, \ldots, j_M) \in \mathcal{E} \). We associate \( \mathcal{G} \) with the hypergraph shift operator (HGSO) \( \mathcal{S} \), defined as an square tensor with dimensions \( I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M \) whose entry \( s(i_1, \ldots, i_M, j_1, \ldots, j_M) \neq 0 \) if \( (i_1, \ldots, i_M, j_1, \ldots, j_M) \in \mathcal{E} \). We introduce a hypergraph filter \( \mathcal{H} : \mathbb{C}^{I_1 \times \cdots \times I_M} \rightarrow \mathbb{C}^{I_1 \times \cdots \times I_M} \), defined as a linear hypergraph signal operator with the form

\[
\mathcal{H} \triangleq \sum_{k=0}^{K-1} h_k S^k
\]

(5.1)
where $h_k$ are scaler coefficients. The covariance tensor of output signals $x$ after itering white input signals by hypergraph filter shown in Eq. (5.1) will be expressed as

$$C_x(h) = H^H \ast_M H = \sum_{k=0,k'=0}^{K-1} h_k h_{k'} S^k \ast_M (S^H)^k$$

(5.2)

where $\ast =_1$ is true if HGSO $S$ is a symmetric tensor, i.e., $s_{i_1, \ldots, i_M, j_1, \ldots, j_M} = s_{j_1, \ldots, j_M, i_1, \ldots, i_M}$. The coefficients $\gamma_k \overset{\text{def}}{=} \sum_{k'+k''=k} h_{k'} h_{k''}$.

It is shown by the work \cite{27} that although the correlation information of signal is given by the dense tensor, the actual relation is easier to be described by the more sparse tensor $S$. Examples about relationships between the HGSO and the covariance tensor $C_x(h)$ include

- $C_x(h) = \frac{1}{2(K-1)} \sum_{k=0}^{K-1} \gamma_k S^k$, as in graph filtering;
- $C_x(h) = S^{-1}$, as in in conditionally independent Markov random fields;
- $C_x(h) = (I - S)^{-2}$, as in symmetric structural equation models with white exogenous inputs.

In the sequel, we will bound the Ky Fan norm for the covariance tensor $C_x(h)$ when $h = [h_0, h_1]$. In random environment, suppose HGSO $S$ is obtained by sample average as

$$S = \frac{1}{m} \sum_{j=1}^{m} X_j = \sum_{j=1}^{m} X'_j,$$

(5.3)

where $X' = \frac{X}{m}$. Since the graph filter coefficients are $h = [h_0, h_1]$, from Eq. (5.2), the corresponding polynomial relation between $C_x(h)$ and $S$ is

$$C_x(h_0, h_1) = h_0^2 + 2h_0 h_1 S + h_1^2 S^2,$$

(5.4)

which is the polynomial function $g(x) = (a_0 + a_1 x + a_2 x^2)^2 = h_0^2 + 2h_0 h_1 x + h_1^2 x^2$ in Theorem 4.5. We assume that random sampled tensors $X'_j$ are identical distributed as $X'$ are satisfy Eq. (4.21) and Eq. (4.22). Then we have following bound of Ky Fan norm for the covariance $C_x[h_0, h_1]$ from Theorem 4.5:

$$\Pr \left( \|C_x([h_0, h_1])\|_{(k)} \geq \theta \right) \leq \inf_{t>0} k e^{-\theta t}$$

(5.5)

$$\left\{ h_0^2 + 2 \sum_{l=1}^{2} a_l^2 \left[ 1 + (e^{\pi Rl} - 1) \sigma_1(X') + C(e^{\pi Rl} - 1) \Xi(X') \right] \right\},$$

where $\sigma_1(X') \overset{\text{def}}{=} \left[ \sigma_1 \left( \frac{X' + X'}{2} \right) + \sigma_1 \left( \frac{X' - X'}{2} \right) \right]$ and $\Xi(X)$ is defined as (4.16).

6. Conclusions. In this work, we generalize previous work by considering the tail bound behavior of the top $k$-largest singular values of a function of random tensors summation. In previous work, we only considered the tail bound behavior of the
largest singular value of tensors summation directly (identity function). Majorization and antisymmetric tensor products are our main gadgets used to derive bounds for unitarily norms of multivariate tensors. Then, we apply Laplace transform method to these bounds of unitarily norms of multivariate tensors to obtain Ky Fan $k$-norm concentration inequalities for a function of tensors summation. Under this approach, generalized tensor Chernoff and Bernstein inequalities are special cases of Ky Fan $k$-norm concentration inequalities obtained by restricting different random tensors conditions.

Possible future works include considering a more general unitarily invariant norm instead of Ky Fan $k$-norm in our tail bounds and other functions of random tensors summation besides the power of polynomials.

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