MOTIVIC INTEGRATION ON SPECIAL RIGID VARIETIES AND THE MOTIVIC INTEGRAL IDENTITY CONJECTURE

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Abstract. We prove in this paper the original version of Kontsevich and Soibelman’s motivic integral identity conjecture for formal functions by developing a novel framework for equivariant motivic integration on special rigid varieties. This theory is built upon our recent research on equivariant motivic integration within the realm of special formal schemes. The central element of our approach lies in demonstrating that two formal models of a given smooth rigid variety can be dominated by a third formal model. Notably, a similar assertion for quasi-compact rigid varieties was obtained by Bosch, Lütkebohmert, and Raynaud in 1993. Consequently, we establish a concept of motivic volume for a special smooth rigid variety, ensuring independence from the selection of its models. We demonstrate that this motivic volume can be extended to a homomorphism from a certain Grothendieck ring of special smooth rigid varieties to the classical Grothendieck ring of varieties. Moreover, our developed motivic volume exhibits a Fubini-type property, which recovers Nicaise and Payne’s motivic Fubini theorem for the tropicalization map.

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1. Introduction

We construct in this paper an equivariant motivic integration on rigid varieties after quickly revisiting the theory of equivariant motivic integration on on special formal schemes. As an application, we give a proof of Kontsevich and Soibelman’s motivic integral identity conjecture.

1.1. Introduced by Kontsevich in 1995, motivic integration has since become an important subject in algebraic geometry by virtue of its connection to many areas of mathematics, including mathematical physics, birational geometry, non-Archimedean geometry, tropical geometry, singularity theory, Hodge theory, model theory (see for instance, [17], [18], [2], [40], [32], [25], [37], [31], [10], [27]). Kontsevich’s method involves arc spaces and the Grothendieck ring of varieties, which brings about the birth of geometric motivic integration. Nowadays, this kind of integration becomes one of the common central objects of algebraic geometry, singularity theory, mathematical physics.

Another point of view on motivic integration known as arithmetic motivic integration was introduced by Kontsevich in 1995. This kind of integration specializes to both of arithmetic and geometric points of view, we can also refer to more recent works such as [24], [13], [14].

1.2. Let $R$ be a complete discrete valuation ring with fraction field $K$ and perfect residue field $k$. Let $\varpi \in R$ be a uniformizing parameter, which will be fixed throughout this article. We denote by $R^{sh}$ and $K^{sh}$ the strict henselizations of $R$ and $K$ respectively. A special formal $R$-scheme is a separated Noetherian adic formal scheme locally defined as $\text{Spf} A$ where $A$ is a Noetherian $R$-algebra with the largest ideal of definition $J$ such that $A/J$ is a finitely generated $k$-algebra. We associate to each special formal $R$-scheme its reduction and generic fiber. Let $\mathcal{X}$ be a special formal $R$-scheme with the largest ideal of definition $J$. The reduction $\mathcal{X}_0$ of $\mathcal{X}$ is the closed subvariety over $k$ defined by $J$. The generic fiber $\mathcal{X}_n$ of $\mathcal{X}$ was constructed by Berthelot ([31, 0.2.6]), it is a rigid variety over $K$ (see Definition 2.2). Let $\omega$ be a gauge form on the generic fiber $\mathcal{X}_n$ of $\mathcal{X}$. Let $G$ be a finite algebraic group over $k$ which acts on $\mathcal{X}$. Based on the works in [40], [32], [34], we define in [31] a notion of motivic $G$-integral of $\omega$ on $\mathcal{X}$, which generalizes the integral in [34] to equivariant setting. This integral is denoted by $\int_{\mathcal{X}} |\omega|$ and takes its value in the ring $\mathcal{M}_{\mathcal{X}_0}^G$, a localization of the Grothendieck ring of varieties. It admits a change of variable formula (Proposition 2.1). Moreover, it is additive with respect to open covers and with respect to the completions along strata of locally closed stratifications of the reduction $\mathcal{X}_0$ of $\mathcal{X}$ (Proposition 2.5).

Let $\hat{\mu}$ be the profinite group scheme of roots of unity, i.e. the projective limit of the group schemes $\mu_n = \text{Spec}(k[\xi]/(\xi^n - 1))$. For $n \in \mathbb{N}^*$ and any formal $R$-scheme $\mathcal{X}$, we put $R(n) = R[\tau]/(\tau^n - \varpi)$, $K(n) = K[\tau]/(\tau^n - \varpi)$ and $\mathcal{X}(n) := \mathcal{X} \times_R R(n)$. Then for any gauge form $\omega$ on $\mathcal{X}_n$ we consider the $\mu_n$-integral of $\omega$ on $\mathcal{X}(n)$, where the action $\mu_n$ on $\mathcal{X}(n)$ is induced from the natural action of $\mu_n$ on $R(n)$. We define the Poincaré series of $\mathcal{X}, \omega$ as

\[ P(\mathcal{X}, \omega; T) := \sum_{n \geq 1} \left( \int_{\mathcal{X}(n)} |\omega(n)| \right) T^n \in \mathcal{M}_{\mathcal{X}_0}^G[T]. \]
and prove that this series is rational if the characteristic of $k$ is zero. Let $d$ be the dimension of $X$ over $R$. We define the motivic volume of $X$ as

$$\text{MV}(X) := -\lim_{T \to \infty} P(X, \omega; T),$$

which is independent of the choice of $\omega$.

Assume that the characteristic of $k$ is zero. Let $f \in k\{x\}[y]$, with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_{m'})$ such that $f(x, 0)$ is not a non-zero constant. Let $\mathfrak{X}(f)$ denote the formal completion of $\text{Spf}(k\{x\}[y])$ along $(f)$ with the $R$-structural morphism induced by $\varpi \mapsto f$.

The motivic volume of $\mathfrak{X}(f)$ is denoted by $S_f$ and called the the motivic nearby cycle of $f$. If $f$ is a polynomial, this recovers the notion of motivic nearby cycles defined by Denef-Loeser ([17], [18]).

1.3. A rigid $K$-variety $X$ is called special if it admits a formal model which is a special formal $R$-scheme. That is, there exists a special formal $R$-scheme $X$ such that $X_\eta = X$. An action of $G$ on $X$ is an equivalent class of the pair $(X, \theta)$ consisting of a model $X$ endowed with an action $\theta$ of $G$. Two $G$-pairs $(X, \theta)$ and $(X', \theta')$ of $X$ are equivalent if there exist a formal $R$-scheme $X''$ endowed with a good $G$-action (see Section 2.3) and two $G$-equivariant morphisms $X'' \to X'$ and $X'' \to X$ such that the induced morphisms morphism $X''_\eta \to X'_\eta$ and $X''_\eta 
\to X_\eta$ are open embedding satisfying $X''(K^{sh}) = X(K^{sh})$. If $\omega$ is a gauge form on $X$, then the quantity

$$\int_{X_0} \int_X |\omega| \in \mathcal{M}_k^G$$

is independent of the choice of a representative $(X, \theta)$ and called the $G$-integral of $\omega$ on $X$ and denoted by $\int_X |\omega|$. Considering the natural action of $\mu_n$ on $X(n) := X \otimes K(n)$ we obtain Poincaré series of $X, \omega$ defined as

$$P(X, \omega; T) := \sum_{n \geq 1} \left( \int_{X(n)} |\omega(n)| \right) T^n \in \mathcal{M}_k[[T]].$$

We prove that if $X$ is bounded (see Definition 3.12), then the series $P(X, \omega; T)$ is rational.

We would like to define the motivic volume of a smooth special rigid $K$-variety $X$. If $X$ is quasi-compact, this definition was given in [37, 8.3]: the image of $\text{MV}(X)$ under the forgetful morphism $\mathcal{M}_{X_0} \to \mathcal{M}_k$ only depends on $X_\eta$. In order to define the motivic volume for smooth special rigid $K$-varieties, we need to prove that “any two formal models of $X$ can be dominated by a third”, which was asked by J. Nicaise in [34, Page 338]. The same claim for quasi-compact varieties was proved in [2], but the method thereby can not be extended to the special rigid varieties. For a proof of the statement, we need to apply the de Jong’s descent theory for closed rigid subvarieties in [15].

**Theorem 1** (Theorem 3.7). Let $X$ be a smooth special rigid $K$-variety. If $\mathfrak{X}$ and $\mathfrak{X}'$ are two formal models of $X$, then there exist another model $\mathfrak{X}''$ of $X$ together with two morphisms $h : \mathfrak{X}'' \to \mathfrak{X}$ and $h' : \mathfrak{X}'' \to \mathfrak{X}'$ such that the induced morphisms $h_\eta$ and $h'_\eta$ are isomorphisms.

Applying Theorem 1 one can show that the quantity

$$\text{MV}(X) := \int_{X_0} \text{MV}(\mathfrak{X})$$

depends only on $X$, and called the motivic volume of $X$. 


Let $\text{SSRig}_K$ denote the category of special smooth rigid $K$-varieties. A special rational subdomain of $X$ is defined locally as

$$X\left(\frac{f}{g}\right) := \{x \in X \mid |f_i(x)| \leq |g(x)|, \forall i\},$$

where $X = (\text{Spf}A)_n$ and $g, f_1, \ldots, f_n \in A$ generating the unit ideal in $A \otimes_R K$. Define $K(\text{SSRig}_K)$ the abelian group generated by the isomorphism classes $[X]$ of SSRig$_K$ modulo the relation

$$[X] = [Y] + [X \setminus Y]$$

where $Y \subseteq X$ is a special rational subdomain of $X$. The group $K(\text{SSRig}_K)$ admits a ring structure whose multiplication is induced by fiber product.

**Theorem 2** (Theorem 3.17). There exists a unique ring homomorphism

$$\text{MV}: K(\text{SSRig}_K) \rightarrow \mathbb{M}^\mu_k$$

satisfying

$$\text{MV}([X]) = \text{MV}(X)$$

for all objects $X$ of $\text{SSRig}_K$.

The motivic volume $\text{MV}$ admits the following version of Fubini theorem.

**Theorem 3** (Theorem 3.22). Let $X$ be a smooth special rigid $K$-variety with a model $\mathcal{X}$. Let $g = \{g_1, \ldots, g_r\}$ be a system of elements of $\Gamma(\mathcal{X}, \mathcal{O}_X)$. For each $\gamma \in \mathbb{Q}_{\geq 0}$ we define the variety

$$X_\gamma := \{x \in X \mid |g_i(x)| = |\varpi|\gamma_i\},$$

Then the function $\varphi_g: \mathbb{Q}_{\geq 0} \rightarrow \mathbb{M}^\mu_k$ defined as

$$\varphi_g(\gamma) = \text{MV}(X_\gamma)$$

is constructible, and moreover,

$$\text{MV}(X) = \int_{\mathbb{Q}_{\geq 0}} \varphi_g d\chi_c = \int_{\mathbb{Q}_{\geq 0}} \varphi_g d\chi'.$$

Here $\chi_c$ and $\chi'$ denote the compactly supported and bounded Euler characteristics respectively.

1.4. We explore a vital use of $\mu$-equivariant motivic integration on rigid $K$-varieties in relation to the integral identity conjecture. It is widely recognized that this conjecture serves as a foundational element in Kontsevich-Soibelman's theory of motivic Donaldson-Thomas invariants concerning noncommutative Calabi-Yau threefolds. Specifically, it directly implies the existence of these invariants, as outlined in [27]. Let us first state the conjecture, see [27, Conjecture 4.4]. Here, for a tuple of variables $x = (x_1, \ldots, x_d)$ and a new univariate $t$ we write $tx$ for $(tx_1, \ldots, tx_d)$.

**Conjecture 1** (Kontsevich-Soibelman). Let $f \in k[x, y, z]$ be a formal power series such that $f(0, 0, 0) = 0$ and $f(tx, y, z) = f(x, ty, z)$ in $k[x, y, z, t]$, where $x, y, z$ are tuples of variables $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d)$ and $z = (z_1, \ldots, z_d)$. Then $f$ is considered as an element in $k\{x\}[y, z]$ and then the identity

$$\int_{k^d} S_{\hat{f}} = \mathbb{L}^d S_{\hat{f}, 0}$$

holds in $\mathbb{M}^\mu_k$, where $\hat{f}(z) = f(0, 0, z) \in k[z]$. 

As far as we know there has been no proofs of the conjecture for the case of formal series even the special case when \( k \) is assumed to be algebraically closed. All known results are assumed \( f \) to be a polynomial. More precisely, the conjecture for polynomials was first proved by Lê [28] for the case where \( f \) is either a function of Steenbrink type or the composition of a pair of regular functions with a polynomial in two variables. In [29, Theorem 1.2], in view of the formalism of Hrushovski-Kazhdan [25] and Hrushovski-Loeser [26], Lê showed that the conjecture for polynomials holds in \( M_{\text{loc}}^\mu \), a “big” localization of \( M_k^\mu \), as soon as the base field \( k \) is algebraically closed. Nicaise and Payne [35] proved the conjecture for polynomials by proving the motivic Fubini theorem for the tropicalization map, on the foundation of [25] and tropical geometry, with assumption that \( k \) contains all roots of unity. In [30], by developing an equivariant motivic integration on varieties, we give a proof of the conjecture for polynomials.

In this article, we give a proof of Conjecture 0 (see Theorem 1.1). The work is based on our theory of equivariant motivic integration on special rigid varieties developed in Section 4 (see Section 4 for detailed arguments).

2. Equivariant motivic integration on special formal schemes

In this section we recall the theory of equivariant motivic integration for special formal schemes developed in [31]. All definitions and results are borrowed from [31] and [34] except Proposition 2.17.

2.1. Equivariant Grothendieck rings of varieties. Let \( k \) be a perfect field. Let \( S \) be a \( k \)-variety endowed with a good action of a finite algebraic group \( G \). We denote by \( \text{Var}_G^S \) the category of \( S \)-varieties \( X \) endowed with a good action of \( G \) such that the morphism \( X \to S \) is \( G \)-equivariant. By definition, \( K(\text{Var}_G^S) \) is the quotient of the free abelian group generated by the \( G \)-equivariant isomorphism classes \([X]\) in \( \text{Var}_G^S \) modulo the relations

\[
[X \to S] = [Y \to S] + [X \backslash Y \to S],
\]

for \( Y \) being \( G \)-invariant and Zariski closed in \( X \), and

\[
[X \times_k \mathbb{A}^n_k \to S, \sigma] = [X \times_k \mathbb{A}^n_k \to S, \sigma']
\]

if \( \sigma \) and \( \sigma' \) lift the same good \( G \)-action on \( X \). Together with fiber product over \( S \), \( K(\text{Var}_G^S) \) is a commutative ring with unity \( \text{id}_S \). We define the localization \( M_G^S \) of the ring \( K(\text{Var}_G^S) \) by inverting \( \mathbb{L} \) where \( \mathbb{L} \) is the class of \( \mathbb{A}^1_k \times_k S \to S \) endowed with the trivial action of \( G \).

Let \( \hat{G} \) be a group scheme over \( k \) of the form \( \hat{G} = \lim_{\to} G_i \), where \( I \) is a partially ordered set and \( \{G_i, G_j \to G_i \mid i \leq j \in I\} \) is a projective system of algebraic groups over \( k \). We define \( K_0^G(\text{Var}_S) = \lim_{\to} K_0^G(\text{Var}_S) \) and \( M_G^S = K_0^G(\text{Var}_S)[ \mathbb{L}^{-1}] \), which implies the identity \( M_G^S = \lim_{\to} M_G^{G_i} \). In particular, we may consider \( \hat{G} \) to be the profinite group scheme of roots of unity \( \hat{\mu} \), the projective limit of the group schemes \( \mu_n = \text{Spec}(k[\xi]/(\xi^n - 1)) \) and transition morphisms \( \mu_m \to \mu_n \) sending \( \lambda \) to \( \lambda^m \).

Let \( f: S \to S' \) be a morphism of algebraic \( k \)-variety. We denote by \( f^*: M_G^{S'} \to M_G^S \) the ring homomorphism induced from the fiber product (the pullback morphism), and by \( f_1: M_G^S \to M_G^{S'} \) the \( M_k^G \)-linear homomorphism defined by the composition with \( f \) (the push-forward morphism). When \( S' \) is \( \text{Spec}k \), one usually writes \( \int_S f \) instead of \( f_1 \).

2.2. Rational series. Let \( M \) be a commutative ring with unity which contains \( \mathbb{L} \) and \( \mathbb{L}^{-1} \). Let \( M[T] \) be the set of formal power series in \( T \) with coefficients in \( M \), which is a ring and also a \( M \)-module with respect to usual operations for series. Denote by \( M[T]_{sr} \) the submodule
of $\mathcal{M}[T]$ generated by 1 and by finite products of terms $\frac{L^aT^b}{1-L^aT^b}$ for $(a, b) \in \mathbb{Z} \times \mathbb{N}_{>0}$. An element of $\mathcal{M}[T]_{sr}$ is called a rational series. By [17], there exists a unique $\mathcal{M}$-linear morphism

$$\lim_{T \to \infty} : \mathcal{M}[T]_{sr} \to \mathcal{M}$$

such that for any $(a, b)$ in $\mathbb{Z} \times \mathbb{N}_{>0}$, one has

$$\lim_{T \to \infty} \frac{L^aT^b}{1-L^aT^b} = -1.$$

Let $I$ be a finite set. Let $\Delta$ be a rational polyhedral convex cones in $\mathbb{R}_{\geq 0}^I$. This means that, $\Delta$ is a convex subset of $\mathbb{R}_{\geq 0}^I$ defined by a finite number of integral linear inequalities of type $a \geq 0$ or $b > 0$ and stable by multiplication by $\mathbb{R}_{>0}$. We denote by $\bar{\Delta}$ the closure of $\Delta$ in $\mathbb{R}_{\geq 0}^I$. Let $\ell$ and $v$ be integral linear forms on $\mathbb{Z}^I$ which are positive on $\Delta \setminus \{0\}$. It follows from [23, 2.9] that

$$\lim_{T \to \infty} \sum_{k \in \Delta \cap \mathbb{N}_{>0}^I} T^\ell(k) L^{-v(k)} = \chi(\Delta)$$

in $\mathcal{M}[T]$.

We will use the notion of the Hadamard product of two formal power series. By definition, the Hadamard product of two formal power series $p(T) = \sum_{n \geq 1} p_n T^n$ and $q(T) = \sum_{n \geq 1} q_n T^n$ in $\mathcal{M}[T]$ is the series

$$p(T) \ast q(T) := \sum_{n \geq 1} p_n \cdot q_n T^n \in \mathcal{M}[T].$$

**Lemma 2.1 (33).** If $p(T)$ and $q(T)$ are rational series in $\mathcal{M}[T]$, so is $p(T) \ast q(T)$, and in this case,

$$\lim_{T \to \infty} p(T) \ast q(T) = - \lim_{T \to \infty} p(T) \cdot \lim_{T \to \infty} q(T).$$

**2.3. Special formal schemes with actions.** Let $R$ be a complete discrete valuation ring with fraction field $K$ and residue field $k$. Let $\varpi \in R$ be a uniformizer, which will be fixed throughout this article. We denote by $R^{sh}$, $K^{sh}$ the strict henselization of $R$ and $K$ respectively. A topological $R$-algebra $A$ is called special if $A$ is a Noetherian adic ring and the $R$-algebra $A/J$ is finitely generated for some ideal of definition $J$ of $A$. Let $R\{x_1, \ldots, x_m\}[y_1, \ldots, y_{m'}]$ be the subring of elements of the form

$$\sum_{\alpha} c_\alpha x^\alpha, \quad c_\alpha \in R[y_1, \ldots, y_{m'}]$$

such that $|c_\alpha| \to 0$ as $|\alpha| \to \infty$, where the norm in $R[y_1, \ldots, y_{m'}]$ is induced from the order. By [3], a topological $R$-algebra $A$ is special if and only if $A$ is topologically $R$-isomorphic to a quotient the $R$-algebra $R\{x_1, \ldots, x_m\}[y_1, \ldots, y_{m'}]$ for some $m, m' \in \mathbb{N}^*$. 

**Definition 2.2.** A special formal scheme is a separated Noetherian adic formal scheme $\mathcal{X}$ which is a finite union of open affine formal schemes of the form $\text{Spf} A$ with $A$ a Noetherian special $R$-algebra. If $\mathcal{X}$ is a special formal $R$-scheme, any formal completion of $\mathcal{X}$ is also a special formal $R$-scheme. For each special formal scheme $\mathcal{X}$, its reduction $\mathcal{X}_0$ is defined locally as $(\text{Spf} A)_0 := \text{Spec} A/J$.

In this article, a morphism between special formal $R$-schemes $f : \mathcal{Y} \to \mathcal{X}$ is an adic morphism of special formal $R$-schemes. It induces a morphism $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$ of $k$-varieties at the reduction.
level. The category of special formal $R$-schemes admits fiber products and the assignment
\[ \mathcal{X} \mapsto \mathcal{X}_0 \]
from the category of special formal $R$-schemes to the category of $k$-varieties is functorial. Furthermore, the natural closed immersion $\mathcal{X}_0 \to \mathcal{X}$ is a homeomorphism.

There is a functor of generic fibres, which associates to a special formal $R$-scheme a rigid $K$-variety. As explained in [5, 0.2.6], one first considers the affine case $\mathcal{X} = \text{Spf} A$, where $A$ is a special $R$-algebra. Denote by $J$ the largest ideal of definition of $A$ and consider for each $n \in \mathbb{N}^*$ the subalgebra $A / (\varpi^{-1} J^n)$ of $A \otimes_R K$ generated by $A$ and $\varpi^{-1} J^n$. Let $B_n := B_n(A)$ be the $J$-adic completion of $A / (\varpi^{-1} J^n)$. Then we have the affinoid $K$-algebra $C_n := B_n \otimes_R K$. The inclusion $J^{n+1} \subseteq J^n$ gives rise to a morphism of affinoid $K$-algebras $C_{n+1} \to C_n$, which in turn induces an open embedding of affinoid $K$-spaces $\text{Sp}(C_n) \to \text{Sp}(C_{n+1})$. The generic fiber $\mathcal{X}_n$ of $\mathcal{X}$ is defined to be
\[ \mathcal{X}_n = \bigcup_{n \in \mathbb{N}^*} \text{Sp}(C_n). \]

Since this construction is functorial, we obtain the generic fiber of a special formal $R$-scheme $\mathcal{X}$ by a gluing process. We call $\mathcal{X}$ a formal model of $\mathcal{X}_n$. The assignment $\mathcal{X} \mapsto \mathcal{X}_n$ is a functor from the category of special formal $R$-schemes to the category of separated rigid $K$-varieties. This functor commutes with fiber products.

Let us look at morphisms of special formal schemes of the form $\text{Spf}(R') \to \mathcal{X}$, where $R \subset R'$ is a finite extension of discrete valuation rings. Another such morphism $\text{Spf}(R'') \to \mathcal{X}$ is said to be equivalent to $\text{Spf}(R') \to \mathcal{X}$ if there exists a commutative diagram
\[
\begin{array}{ccc}
\text{Spf}(R'') & \longrightarrow & \text{Spf}(R') \\
\downarrow & & \downarrow \\
\text{Spf}(R') & \longrightarrow & \mathcal{X}
\end{array}
\]
where $R \subset R''$ is also a finite extension of discrete valuation rings. Then points of $\mathcal{X}_n$ correspond bijectively with equivalence classes of such morphisms $\text{Spf}(R') \to \mathcal{X}$.

There is a specialization map $\mathcal{X}_n \to \mathcal{X}$ defined as follows. For the affine case, the map $\text{sp}: \text{Spf}(A)_n \to \text{Spf} A$ is defined as follows. Let $x$ be in $(\text{Spf} A)_n$, and let $I \subseteq A \otimes_R K$ be the maximal ideal in $A \otimes_R K$ corresponding to $x$. Put $I' = I \cap A \subseteq A$. Then, by construction, $\text{sp}(x)$ is the unique maximal ideal of $A$ containing $\varpi$ and $I'$. If the point $x$ corresponds to the equivalence class of $\varphi: \text{Spf}(R') \to \mathcal{X}$ as above then $\text{sp}(x) = \varphi$.

If $Z$ is a locally closed subscheme of $(\text{Spf} A)_n$, $\text{sp}^{-1}(Z)$ is an open rigid $K$-subvariety of $(\text{Spf} A)_n$, which is canonically isomorphic to the generic fiber of the formal completion of $\text{Spf} A$ along $Z$ (cf. [15, Section 7.1]). In general, the construction of the specialization map $\text{sp}: \mathcal{X}_n \to \mathcal{X}$ can be generalized to any special formal $R$-scheme $\mathcal{X}$ using a gluing process (see [15]).

Let $G$ be a finite algebraic group over $k$. In this article we fix an action of $G$ of $\text{Spf} R$, i.e. an adic morphism of formal schemes $G \times_k \text{Spf} R \to \text{Spf} R$ satisfying certain conditions of an action. Let $\mathcal{X}$ be a formal $R$-scheme, with structural morphism $\mathcal{X} \to \text{Spf} R$ viewed as a morphism of formal $k$-scheme. A $G$-action on $\mathcal{X}$ is a $G$-action on the formal $k$-scheme $\mathcal{X}$ (with the $k$-scheme structure induced from $k \hookrightarrow R$) such that $\mathcal{X}$ is a $G$-equivariant $k$-morphism. A $G$-action on $\mathcal{X}$ is called good if any orbit of it is contained in an affine open formal subscheme.
2.4. Order of top forms. Let $\mathfrak{X}$ be a special formal scheme of pure relative dimension $d$. Let $R'$ be an extension of $R$ of ramification index $e$ with a fixed uniformizing parameter $\varpi'$, and denote by $K'$ its quotient field. For any $R'$-point $\psi$ of $\mathfrak{X}$, the order of a differential form $\tilde{\omega}$ in $\Omega^d_{\mathfrak{X}/R}(\mathfrak{X})$ at $\psi$ is defined as follows. Since $(\gamma^*\Omega^d_{\mathfrak{X}/R}(\mathfrak{X}))/\text{tors} \cong \Omega^d_{\mathfrak{Y}/R}(\mathfrak{Y})$ for any $R'$-scheme $\mathfrak{Y}$ which factors uniquely through the sheafification map $\Omega^d_{\mathfrak{X}/R}(\mathfrak{X})$, we have either $\gamma^*\tilde{\omega} = 0$ or $\gamma^*\tilde{\omega} = \alpha \varpi'^n$ for some nonzero $\alpha \in O_{R'}$ and $n \in \mathbb{N}$. Then we define

$$\text{ord}_{\tilde{\omega}}(\psi) = \begin{cases} \infty & \text{if } \gamma^*\tilde{\omega} = 0 \\ n & \text{if } \gamma^*\tilde{\omega} = \alpha \varpi'^n. \end{cases}$$

(2.3)

Consider the canonical injective morphism (cf. [15, Sect. 7] and [34, Sect. 2.1])

$$\Phi: \Omega^d_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K \to \Omega^d_{\mathfrak{X}/K}(\mathfrak{X}_\eta)$$

which factors uniquely through the sheafification map $\Omega^d_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K \to (\Omega^d_{\mathfrak{Y}/R}(\mathfrak{Y})/\text{tors})$.

A form $\omega$ on $\mathfrak{X}_\eta$ which lies in $\text{Im}(\Phi)$ is called $\mathfrak{X}$-bounded. Note that, if $\mathfrak{X}$ is stft, i.e. $(\varpi)$ is its largest ideal of definition, then $\Phi$ is an isomorphism as shown in [8, Proposition 1.5]. A gauge form $\omega$ on $\mathfrak{X}_\eta$ is a global section of the differential sheaf $\Omega^d_{\mathfrak{X}/K}$ such that it generates the sheaf at every point of $\mathfrak{X}_\eta$.

For any $\mathfrak{X}$-bounded gauge form $\omega$ on $\mathfrak{X}_\eta$, there exist $\tilde{\omega} \in \Omega^d_{\mathfrak{X}/R}(\mathfrak{X})$ and $n \in \mathbb{N}$ such that $\omega = \varpi^{-n}\tilde{\omega}$. Then the difference

$$\text{ord}_{\omega}(\psi) := \text{ord}_{\tilde{\omega}}(\psi) - e \cdot n$$

(2.4)

is independent of the choice of $\tilde{\omega}$ as seen in [34, Definition 5.4]).

2.5. Motivic $G$-integral on stft schemes. Let $\mathfrak{X}$ be stft, i.e. a special formal $R$-scheme whose largest ideal of definition is $(\varpi)$. Let $R_n := R/\varpi^{n+1}$ and let $X_n$ be the $k$-variety defined as $(\mathfrak{X}, O_\mathfrak{X} \otimes_R R_n)$. For any $k$-algebra $A$ we denote $L(A) = A$ if $R$ has equal characteristics and $L(A) = W(A)$ otherwise, where $W(A)$ is the ring of Witt vectors over $A$. In [21], Greenberg shows that the functor defined locally by

$$\text{Spec} A \mapsto \text{Hom}_{R_n}(\text{Spec} (R_n \otimes_{L(k)} L(A)), X_n)$$

from the category of $k$-schemes to the category of sets is presented by a $k$-scheme $\text{Gr}_n(X_n)$ of finite type such that, for any $k$-algebra $A$,

$$\text{Gr}_n(X_n)(A) = X_n(R_n \otimes_{L(k)} L(A)).$$

The varieties $\text{Gr}_n(X_n)$ together with natural truncation maps form a projective system whose limit is denoted by $\text{Gr}(\mathfrak{X})$ and called the Greenberg scheme of $\mathfrak{X}$. Note that, by [34] Proposition 3.7, any action of $G$ on $\mathfrak{X}$ induces actions of $G$ on $\text{Gr}_n(X_n)$ and $\text{Gr}(\mathfrak{X})$ such that the natural morphism $\pi_n: \text{Gr}(\mathfrak{X}) \to \text{Gr}_n(X_n)$ is $G$-equivariant. A subset of $\text{Gr}(\mathfrak{X})$ is called a $G$-invariant cylinder if it is equal to $\pi_n^{-1}(C)$ for some $G$-invariant constructible subset $C$ of $\text{Gr}_n(X_n)$. Let $C_G^G$ be the set of $G$-invariant cylinders of $\text{Gr}(\mathfrak{X})$. It follows from [34] Proposition 3.9 that there exists a unique additive mapping

$$\mu_G^\mathfrak{X}: C_G^G \to \mathcal{M}_G^\mathfrak{X}_0$$

such that for any $G$-invariant cylinder $A$ of level $n$ of $\text{Gr}(\mathfrak{X})$,

$$\mu_G^\mathfrak{X}(A) = [\pi_n(A) \to X_0] \mathbb{L}^{-(n+1)d}.$$
On the other hand, for any field extension $F$ of $k$ there is a bijection, see [32 Sect. 4.1],

$$\text{Gr}(\mathcal{X})(F) \cong \mathcal{X}(R \otimes F).$$

Therefore the function $\text{ord}_\omega$ in [2.4] defines a $\mathbb{Z}$-value function

$$\text{ord}_\omega : \text{Gr}(\mathcal{X}) \setminus \text{Gr}(\mathcal{X}_{\text{sing}}) \to \mathbb{Z} \cup \{\infty\},$$

which give rise to $G$-integrable function $\mathbb{L}^{\text{ord}_\omega} \mathcal{X}(\omega)$ in the following sense: for any $\mathcal{A}$ in $C^G_{\mathcal{X}}$ and any simple function $\alpha : \mathcal{A} \to \mathbb{Z} \cup \{\infty\}$, we say that $\mathbb{L}^{\alpha}$ is $G$-integrable, if $\alpha$ takes only finitely many values in $\mathbb{Z}$ and if all the fibers of $\alpha$ are in $C^G_{\mathcal{X}}$. We define the motivic $G$-integral of $\omega$ on $\mathcal{X}$ to be

$$\int_{\mathcal{X}} |\omega| := \sum_{n \in \mathbb{Z}} \mu_{\mathcal{X}}^G(\text{ord}_\omega^{-1}(n))\mathbb{L}^{-n} \in \mathcal{M}^G_{\mathcal{X},0},$$

2.6. Motivic $G$-integral on special formal schemes.

**Definition 2.3.** Let $G$ be a smooth algebraic group over $k$. Let $\mathcal{X}$ be a flat generically smooth special formal $R$-scheme endowed with a good $G$-action. Let $\mathcal{Z}$ be a $G$-invariant closed formal subscheme of $\mathcal{X}_0$ defined by an ideal sheaf $\mathcal{I}$. Let $\pi : \mathcal{Y} \to \mathcal{X}$ be the equivariant admissible blowup with center $\mathcal{Z}$ and let $\mathcal{U}$ be the open formal subscheme of $\mathcal{Y}$ where $\mathcal{I}\mathcal{O}_{\mathcal{Y}}$ is generated by $\varpi$, the restriction $\pi : \mathcal{U} \to \mathcal{X}$ is called the $G$-dilatation of $\mathcal{X}$ with center $\mathcal{Z}$. Let $\pi : \mathcal{U} \to \mathcal{X}$ be the $G$-dilatation of $\mathcal{X}$, i.e. the $G$-dilatation with center $\mathcal{X}_0$. Notice that, in this case, if $\mathcal{X}$ is covered by affine formal subschemes $\text{Spf} A$, then $\mathcal{U}$ can be constructed by glueing affine open formal subschemes $\text{Spf} B_1(A)$ with the notation as in Section 2.3. For any gauge form $\omega$ on $\mathcal{X}_0$, we define

$$\int_{\mathcal{X}} |\omega| := \pi_0! \int_{\mathcal{U}} |\pi^*_\omega| \text{ in } \mathcal{M}^G_{\mathcal{X},0},$$

and call it the motivic $G$-integral of $\omega$ on $\mathcal{X}$. If $\mathcal{X}$ is a generically smooth special formal $R$-scheme endowed with a good adic $G$-action, we denote by $\mathcal{X}_{\text{flat}}$ its maximal flat closed subscheme (obtained by killing $\varpi$-torsion), and define the motivic $G$-integral of a gauge form $\omega$ on $\mathcal{X}$ to be

$$\int_{\mathcal{X}} |\omega| := \int_{\mathcal{X}_{\text{flat}}} |\omega| \text{ in } \mathcal{M}^G_{\mathcal{X},0}.$$

We list below several properties of the motivic $G$-integrals.

**Proposition 2.4** (Special $G$-equivariant change of variables formula). Let $G$ be a smooth finite group scheme over $k$. Let $\mathcal{X}$ and $\mathcal{Y}$ be generically smooth special formal $R$-schemes endowed with good adic actions of $G$, and let $h : \mathcal{Y} \to \mathcal{X}$ be an adic $G$-equivariant morphism of formal $R$-schemes such that the induced morphism $\mathcal{Y}_0 \to \mathcal{X}_0$ is an open embedding and $\mathcal{Y}_0(K^{\text{sh}}) = \mathcal{X}_0(K^{\text{sh}})$. If $\omega$ is a gauge form on $\mathcal{X}_0$, then

$$\int_{\mathcal{X}} |\omega| = h_0! \int_{\mathcal{Y}} |h^*_\omega| \text{ in } \mathcal{M}^G_{\mathcal{X},0}.$$

**Proposition 2.5** (Additivity of motivic integrals). Let $\mathcal{X}$ be a generically smooth special formal $R$-scheme endowed with a good adic action of $G$ and let $\omega$ be a gauge form on $\mathcal{X}_0$.

(i) If $\{U_i, i \in I\}$ is a finite stratification of $\mathcal{X}_0$ into $G$-invariant locally closed subsets, and $\mathcal{U}_i$ is the formal completion of $\mathcal{X}$ along $U_i$, then

$$\int_{\mathcal{X}} |\omega| = \sum_{i \in I} \int_{\mathcal{U}_i} |\omega| \text{ in } \mathcal{M}^G_{\mathcal{X},0}.$$
(ii) If \( \{ \mathcal{U}_i, i \in I \} \) is a finite covering of \( G \)-invariant open subsets of \( \mathfrak{X} \), then
\[
\int_{\mathfrak{X}} |\omega| = \sum_{I' \subseteq I} (-1)^{|I'| - 1} \int_{\mathcal{U}_{I'}} |\omega| \quad \text{in} \; M^G_{\mathfrak{X}_0},
\]
where \( \mathcal{U}_{I'} = \bigcap_{i \in I'} \mathcal{U}_i \).
Here the pushforward morphisms \( M^G_{\mathcal{U}_{I'}} \to M^G_{\mathfrak{X}_0} \) are applied to the RHS in both statements.

**Proposition 2.6** (Motivic G-integral of smooth formal R-schemes). Let \( \mathfrak{X} \) be a smooth special formal \( R \)-scheme of pure relative dimension \( d \), which is endowed with a good adic \( G \)-action. Suppose that \( \omega \) is an \( \mathfrak{X} \)-bounded gauge form on \( \mathfrak{X}_0 \). Denote by \( C(\mathfrak{X}_0) \) the set of all connected components of \( \mathfrak{X}_0 \). Then for each \( C \in C(\mathfrak{X}_0) \) the value \( \text{ord}_{\omega}(\psi) \) is independent of the choice of \( \psi \) such that \( \psi(0) \in C \) and denoted by \( \text{ord}_C(\omega) \). Assume that every \( C \in C(\mathfrak{X}_0) \) is \( G \)-invariant, then the identity
\[
\int_{\mathfrak{X}} |\omega| = \sum_{C \in C(\mathfrak{X}_0)} |C \hookrightarrow \mathfrak{X}_0| L^{-\text{ord}_C(\omega)}
\]
holds in \( M^G_{\mathfrak{X}_0} \). In particular, if \( \mathfrak{X}_0 \) is connected, then
\[
\int_{\mathfrak{X}} |\omega| = L^{-d+\text{ord}_{\mathfrak{X}_0}(\omega)} \quad \text{in} \; M^G_{\mathfrak{X}_0}.
\]

**Example 2.7.** Let \( R = k[[x]] \), \( K = k((x)) \), \( R(n) = k[[x^{1/n}]] \) and \( K(n) = k((x^{1/n})) \) for \( n \in \mathbb{N}^* \). Consider the formal schemes \( \mathfrak{X} = \text{Spf}(R[x_1, \ldots, x_d]) \) and \( \mathfrak{Y} = \text{Spf}(R[x_1, \ldots, x_d]) \) with \( \mathfrak{X}_0 = \text{Spec} k \) and \( \mathfrak{Y}_0 = \mathbb{A}^n_k \). Observe that \( \text{Spf}(R(n)\{x\}) \) and \( \text{Spf}(K(n)\{x\}) \) are endowed with the good \( \mu_n \)-action induced by \( \xi \mapsto \xi x^{1/n} \). Clearly, \( dx := dx_1 \wedge \ldots \wedge dx_d \) is a gauge form on \( \text{Spf}(K\{x\}) \), its pullback \( dx(n) \) via the natural morphism \( \text{Spf}(K(n)\{x\}) \to \text{Spf}(K\{x\}) \) is still \( dx \). Since the canonical isomorphism \( (2.3) \) is given by \( dx \otimes f \mapsto f dx \), the functions \( \text{ord}_{\omega, \mathfrak{X}}(dx(n)) \) and \( \text{ord}_{\omega^{1/n}, \mathfrak{X}(n)}(dx(n)) \) are zero constant. Thus
\[
\int_{\mathfrak{X}} |dx| = L^{-d} \in M_k^{\mu_1}, \quad \int_{\mathfrak{X}(n)} |dx(n)| = L^{-d} \in M_k^{\mu_n}.
\]
Similarly, one can show that
\[
\int_{\mathfrak{Y}} |dx| = L^{-d} \in M_k^{\mu_1}, \quad \int_{\mathfrak{Y}(n)} |dx(n)| = L^{-d} \in M_k^{\mu_n}.
\]
Now write \( R\{\frac{x}{x^{1/p}}\} \) for \( R\{x, y\}/(x - x^{1/p})y \) for new variables \( y = (y_1, \ldots, y_d) \) and a positive integer \( p \). Consider the formal schemes \( \mathfrak{U} = \text{Spf}(R\{\frac{x}{x^{1/p}}\}) \) and \( \mathfrak{U}(n) = \text{Spf}(R(n)\{\frac{x}{x^{1/p}}\}) \). Consider the gauge form \( dx \) on \( \mathfrak{U}_0 = \text{Spf}(K\{\frac{x}{x^{1/p}}\}) \). Then its pullback \( dx(n) \) via the natural morphism \( \text{Spf}(K(n)\{\frac{x}{x^{1/p}}\}) \to \text{Spf}(K\{\frac{x}{x^{1/p}}\}) \) is nothing else than \( dx \). Since \( x^{dp}d\frac{x}{x^{1/p}} \otimes 1 \mapsto dx \) via \( (2.3) \), the function \( \text{ord}_{\omega, \mathfrak{U}}(dx) \) is the constant function \( dp \) while \( \text{ord}_{\omega^{1/n}, \mathfrak{U}(n)}(dx(n)) \) is the constant function \( ndp \), hence
\[
\int_{\mathfrak{U}} |dx| = L^{-d(p+1)} \in M_k^{\mu_1}, \quad \int_{\mathfrak{U}(n)} |dx(n)| = L^{-d(np+1)} \in M_k^{\mu_n}.
\]

### 2.7. Monodromic volume Poincaré series and motivic volumes

A special formal \( R \)-scheme \( \mathfrak{X} \) is called regular if \( \mathcal{O}_{\mathfrak{X}, x} \) is regular for every \( x \in \mathfrak{X} \). By [23, Definition 2.33], a closed formal subscheme \( \mathcal{E} \) of a purely relatively \( d \)-dimensional special formal \( R \)-scheme \( \mathfrak{X} \) is called a strict normal crossings divisor if, for every \( x \) in \( \mathfrak{X} \), there exists a regular system of local parameters \( (x_0, \ldots, x_d) \) in \( \mathcal{O}_{\mathfrak{X}, x} \) such that the ideal defining \( \mathcal{E} \) at \( x \) is locally generated by \( \prod_{i=0}^d x_i^{N_i} \) for some \( N_i \in \mathbb{N}, 0 \leq i \leq d \), and such that the irreducible components of \( \mathcal{E} \) are
regular (see [34, Section 2.4] for definition of irreducibility). If $\mathcal{E}_i$ is an irreducible component of $\mathcal{E}$ which is defined locally by the ideal $x_i$, it is a fact that $N_i$ is constant when $x$ varies on $\mathcal{E}_i$. Then we have $\mathcal{E} = \sum_{i \in S} N_i \mathcal{E}_i$, where $\mathcal{E}_i$‘s are irreducible components of $\mathcal{E}$. The divisor $\mathcal{E}$ is called a tame strict normal crossings divisor if $N_i$ is prime to the characteristic exponent of $k$ for every $i$. Any special formal $R$-scheme $X$ is said to have tame strict normal crossings divisor if $X$ is regular and $\mathcal{E}_s$ is a tame strict normal crossings divisor.

Fix $i \in S$ and let $x$ be a point of $E_i$. We define the localization of $\mathcal{O}_{X,x}$ at the generic point corresponding to $\mathcal{E}_i$. Then by [34, Lemma 7.3], the $\mathcal{O}_{X,\mathcal{E}_i,x}$-module $\Omega_{X,\mathcal{E}_i,x}$ is free of rank one. For any

$$\omega \in \Omega^n_{X/R}(X)/(\pi - \text{torsion})$$

we define the order of $\omega$ along $\mathcal{E}_i$ at $x$ as the length of the $\mathcal{O}_{X,\mathcal{E}_i,x}$-module $\Omega_{X,\mathcal{E}_i,x}/(\mathcal{O}_{X,\mathcal{E}_i,x} \cdot \omega)$, and we denote it by $\text{ord}_{\mathcal{E}_i,x}\omega$.

If $\omega$ is a $X$-bounded $m$-form on $X_\eta$, there exists an integer $a \geq 0$ and an affine open formal subscheme $\mathfrak{U}$ of $X$ containing $x$, such that $\pi^n \omega$ belongs to

$$\Omega^n_{X/R}(\mathfrak{U})/(\pi - \text{torsion}) \subset \Omega^n_{X/R}(X) \otimes_R K$$

We define the order of $\omega$ along $\mathcal{E}_i$ at $x$ as

$$\text{ord}_{\mathcal{E}_i,x}\omega := \text{ord}_{\mathcal{E}_i,x}(\pi^n \omega) - aN_i$$

which is dependent of $a$ as well as $x \in E_i$ and is denoted by $\text{ord}_{\mathcal{E}_i}\omega$.

We now study the $\mu_n$-equivariant setting of volume Poincaré series and motivic volume of special formal $R$-schemes. Note that their older version (without action) was performed early in [34, Sect. 7].

For $n \in \mathbb{N}^*$, we put $R(n) = R[\tau]/(\tau^n - \varpi)$ and $K(n) = K[\tau]/(\tau^n - \varpi)$. For any formal $R$-scheme $X$, we define its ramifications as follows: $X(n) = X \times_R R(n), \mathfrak{X}_n(n) = \mathfrak{X}_n \times_K K(n)$. If $\omega$ is a gauge form on $X_\eta$, let $\omega(n)$ be its pullback via the natural morphism $\mathfrak{X}_n(n) \to \mathfrak{X}_n$, which is a gauge form on $\mathfrak{X}_n(n)$. Let $X$ be a formal $R$-scheme and $n$ in $\mathbb{N}^*$. Then there is a natural good adic $\mu_n$-action on both $\text{Spf}R(n)$ and $X(n)$ which is induced from the ring homomorphism $R(n) \to k[\xi]/(\xi^n - 1) \otimes_k R(n)$ given by $\tau \mapsto \xi \otimes \tau$. Moreover, the structural morphism of the formal $\text{Spf}R(n)$-scheme $\mathfrak{X}(n)$ is $\mu_n$-equivariant.

Remark that if $X$ is a generically smooth special formal $R$-scheme and $n \in \mathbb{N}^*$, then $X(n)$ is a generically smooth special formal $(n)$-scheme.

**Definition 2.8.** Let $X$ be a generically smooth special formal $R$-scheme, and let $\omega$ be a gauge form on $X_\eta$. The formal power series

$$P(X, \omega; T) := \sum_{n \geq 1} \left( \int_{X(n)} |\omega(n)| \right) T^n \in \mathcal{M}_{X_0}^0[[T]]$$

**Definition 2.9.** Let $X$ be a generically smooth flat special formal $R$-scheme of pure relative dimension $d$. Assume that $X$ admits a resolution of singularities $h: \mathfrak{Y} \to X$. A resolution of singularities of $X$ is a proper morphism of flat special formal $R$-schemes $h: \mathfrak{Y} \to X$, such that $h_\eta$ is an isomorphism and $\mathfrak{Y}$ is regular with $\mathfrak{Y}_s$ being a strict normal crossings divisor. The resolution of singularities $h$ is said to be tame if $\mathfrak{Y}_s$ is a tame strict normal crossings divisor. Note that if the base field $k$ has characteristic zero, then any generically smooth flat special formal $R$-scheme $X$ admits a resolution of singularities (cf. [34]). Let $\mathcal{E}_i, i \in S$, be the irreducible components of $(\mathfrak{Y}_s)_{\text{red}}$. Let $N_i$ be the multiplicity of $\mathcal{E}_i$ in $\mathfrak{Y}_s$. Put

$$E_i := (\mathcal{E}_i)_0$$
Theorem 2.10. Let \( Y \) be a generically smooth flat special formal \( R \)-scheme of pure relative dimension \( d \). Suppose that we are given a tame resolution of singularities \( \mathfrak{r}: \mathfrak{Y} \to \mathfrak{X} \) with \( \mathfrak{Y}_S = \sum_{i \in S} N_i \mathfrak{C}_i \) and an \( \mathfrak{X} \)-bounded gauge form \( \omega \) on \( \mathfrak{X}_\eta \) with order \( \alpha_i := \text{ord}_{\mathfrak{C}_i}(h_i^* \omega) \) for every \( i \in S \). If \( n \in \mathbb{N}^* \) is prime to the characteristic exponent of \( k \), then the identity

\[
\int_{\mathfrak{X}(N)} |\omega(n)| = L^{-d} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [\mathfrak{E}_I^0] \left( \sum_{k_i \geq 1, i \in I} L^{-\sum_{i \in I} k_i \alpha_i} \right)
\]

holds in \( \mathcal{M}^{\mu_{N_I}}_{\mathfrak{X}_0} \).

Corollary 2.11. Suppose that the base field \( k \) has characteristic zero. Let \( \mathfrak{X} \) be a generically smooth flat special formal \( R \)-scheme of relative dimension \( d \). Let \( h: \mathfrak{Y} \to \mathfrak{X} \) be a resolution of singularities with \( \mathfrak{Y}_S = \sum_{i \in S} N_i \mathfrak{C}_i \). Suppose that \( \omega \) is an \( \mathfrak{X} \)-bounded gauge form on \( \mathfrak{X}_\eta \) with \( \alpha_i := \text{ord}_{\mathfrak{C}_i}(h_i^* \omega) \) for every \( i \in S \). Then

\[
P(\mathfrak{X}, \omega; T) = L^{-d} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [\mathfrak{E}_I^0] \prod_{i \in I} \frac{L^{-\alpha_i} T^{N_i}}{1 - L^{-\alpha_i} T^{N_i}}.
\]

We deduce from Corollary 2.11 that the limit

\[
\lim_{T \to \infty} P(\mathfrak{X}, \omega; T) = -L^{-d} \sum_{\emptyset \neq I \subseteq S} (1 - L)^{|I| - 1} [\mathfrak{E}_I^0] \in \mathcal{M}^{\tilde{\mu}}_{\mathfrak{X}_0}
\]

is independent of the choice of the \( \mathfrak{X} \)-bounded gauge form \( \omega \). However, it depends on the choice of the uniformizer (see [34, Rem. 7.40]).
Definition 2.12. Suppose that $k$ has characteristic zero. Let $\mathcal{X}$ be a generically smooth special formal $R$-scheme of pure dimension $d$. Assume that $\mathcal{X}_n$ admits an $\mathcal{X}$-bounded gauge form. Then we define
\[ \text{MV}(\mathcal{X}) := -\mathbb{L}^d \lim_{T \to \infty} P(\mathcal{X}, \omega; T). \]
In general, take a resolution of singularities $h : \mathfrak{Y} \to \mathcal{X}$ and take a finite open cover $\{\mathcal{U}_i\}_{i \in I}$ of $\mathfrak{Y}$ such that $\mathcal{U}_i$ has pure relative dimension and $(\mathcal{U}_i)_n$ admits a $\mathcal{U}_i$-bounded gauge form for each $i$. Then, the value
\[ \text{MV}(\mathcal{X}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|+1} \text{MV}(\cap_{i \in J} \mathcal{U}_i) \in \mathbb{M}_k^d \]
only depends on $\mathcal{X}$ and is called the motivic volume of $\mathcal{X}$ (cf. [34, 7.38, 7.39]).

Proposition 2.13 (Additivity of MV). Suppose that $k$ has characteristic zero. Let $\mathcal{X}$ be a generically smooth special formal $R$-scheme. The following hold.
(i) If $\{U_i, i \in Q\}$ is a finite stratification of $\mathcal{X}_0$ into locally closed subsets, and $\mathcal{U}_i$ is the formal completion of $\mathcal{X}$ along $U_i$, then
\[ \text{MV}(\mathcal{X}) = \sum_{i \in Q} \text{MV}(\mathcal{U}_i). \]
(ii) If $\{\mathcal{U}_i, i \in Q\}$ is a finite open covering of $\mathcal{X}$, then by putting $\mathcal{U}_I = \bigcap_{i \in I} \mathcal{U}_i$, we have
\[ \text{MV}(\mathcal{X}) = \sum_{\emptyset \neq I \subseteq Q} (-1)^{|I|-1} \text{MV}(\mathcal{U}_I). \]

Example 2.14. Let $R = k[[\omega]]$, $K = k((\omega))$, $R(n) = k[[\omega^{1/n}]]$ and $K(n) = k((\omega^{1/n}))$ for $n \in \mathbb{N}^*$. Consider the formal schemes $\mathcal{X} = \text{Spf}(R[x_1, \ldots, x_d])$, $\mathfrak{Y} = \text{Spf}(R[x_1, \ldots, x_d])$ and $\mathcal{U} = \text{Spf}(R[\frac{1}{\omega^n}])$ with $R[\frac{1}{\omega^n}] := R[x, y]/(x - \omega^n y)$ for new variables $y = (y_1, \ldots, y_d)$. Using the computations in Example 2.7 we obtain
\[ \text{MV}(\mathcal{X}) = 1 \in \mathbb{M}_k^d, \quad \text{MV}(\mathfrak{Y}) = \text{MV}(\mathcal{U}) = 1 \in \mathbb{M}_k^d. \]

2.8. Motivic zeta functions and motivic nearby cycles of formal power series. Consider the mixed formal power series $R$-algebra $R\{x\}[y]$, with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m')$. Let $d = m + m'$. Let $f$ be of $k[x][y]$ such that $f(x, 0)$ is not a non-zero constant, and let $\mathcal{X}(f)$ be the formal completion of $\text{Spf}(k[x][y])$ along $(f)$. Then $\mathcal{X}(f)$ is a generically smooth special formal $R$-scheme of pure relative dimension $d - 1$, with structural morphism defined by $\omega \mapsto f$. Moreover, it follows from [31] Lemma 4.29 that
\[ (2.5) \quad \mathcal{X}(f) \cong \text{Spf} \left( R\{x\}[[y]]/(f - \omega) \right) \]
By [11] Sect. 4], we can see that $\mathcal{X}(f)$ is a formal scheme of pseudo-finite type over $k$, the sheaf of continuous differential form $\Omega^i_{\mathcal{X}(f)/k}$ is coherent for any $i$, and that there exists a morphism of coherent $\mathcal{O}_{\mathcal{X}(f)}$-modules $d\omega \wedge (\cdot) : \Omega^{d-1}_{\mathcal{X}(f)/R} \to \Omega^d_{\mathcal{X}(f)/k}$ defined by taking the exterior product with the differential $df$. Taking the “rig” functor ([15] Sect. 7]) we get a morphism of coherent $\mathcal{O}_{\mathcal{X}(f)/n}$-modules
\[ d\omega \wedge (\cdot) : \Omega^{d-1}_{\mathcal{X}(f)/n} \to (\Omega^d_{\mathcal{X}(f)/k})_{\text{rig}} \]
which is an isomorphism due to [34] Prop. 7.19. In this section we fix the gauge form $\omega$ on $\mathcal{X}(f)$ defined as $\omega = dx_1 \wedge \cdots \wedge dx_m \wedge dy_1 \wedge \cdots \wedge dy_m$ and denote by $\omega/df$ the inverse image of $\omega$ under $d\omega \wedge (\cdot)$ and call it the standard Gelfand-Leray form.
Let \( \mathfrak{h} : \mathcal{Y} \to \mathcal{X}(f) \) be a tame resolution of singularities of \( \mathcal{X}(f) \). Assume that the data of \( \mathcal{Y} \) are given as in the setting before Theorem 2.10 and that \( K_{\mathcal{Y}/\mathcal{X}(f)} = \sum_{i \in S}(\nu_i - 1)\xi_i \). Using the same argument in the proof of [34, Lem. 7.30] we get \( \text{ord}_{E}(\mathfrak{h}(\omega/df)) = \nu_i - N_i \) for all \( i \in S \). Note that these numbers do not depend on \( \omega \). Similarly as in the proof of Theorem 2.10 we have the following result.

**Proposition 2.15.** With the previous notation and hypotheses, if \( n \in \mathbb{N}^* \) is prime to the characteristic exponent of \( k \) and not \( \mathcal{Y}_s \)-linear, then the identity

\[
\int_{\mathcal{X}(f)(n)} |(\omega/df)(n)| = L^{n+1-d} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I|-1} [\tilde{E}_I^\circ] \left( \sum_{k_i \geq 1, i \in I} L^{\sum_{i \in I} k_i (N_i - \nu_i)} \right),
\]

holds in \( M_{\mathcal{X}(f)_0}^\mu \). If, in addition, \( k \) has characteristic zero, then

\[
P(\mathcal{X}(f), \omega/df; T) = L^{-d-1} \frac{LT}{1 - LT} \times \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I|-1} [\tilde{E}_I^\circ] \prod_{i \in I} \frac{L - \nu_i T^{N_i}}{1 - L^{-\nu_i} T^{N_i}},
\]

where \( * \) is the Hadamard product of formal power series in \( M_{\mathcal{X}(f)_0}^\mu \{T\} \). Moreover,

\[
\text{MV}(\mathcal{X}(f); \tilde{K}^x) = \sum_{\emptyset \neq I \subseteq S} (1 - L)^{|I|-1} [\tilde{E}_I^\circ] \in M_{\mathcal{X}(f)_0}^\mu.
\]

**Definition 2.16.** Let \( k \) be a field of characteristic zero. Let \( f \) be in \( k\{x\}[y] \) such that \( f(x, 0) \) is non-constant. Let \( x \) be a closed point in \( \mathcal{X}(f)_0 \). The **motivic zeta function** \( Z_f(T) \) of \( f \) and the **local motivic zeta function** \( Z_{f,x}(T) \) of \( f \) at \( x \) are defined as follows

\[
Z_f(T) := L^{-d-1} P(\mathcal{X}(f), \omega/df; T), \quad Z_{f,x}(T) := L^{-d-1} P(\mathcal{X}(f)/x, \omega/df; T).
\]

The **motivic nearby cycles** \( S_f \) of \( f \) and the **motivic Milnor fiber** \( S_{f,x} \) of \( f \) at \( x \) are defined as

\[
S_f := \text{MV}(\mathcal{X}(f)) \in M_{\mathcal{X}(f)_0}^\mu, \quad S_{f,x} := \text{MV}(\mathcal{X}(f)/x) \in M_k^\mu.
\]

**Proposition 2.17.** Let \( f, g \in k\{x\}[y] \) be two series such that \( f = u g \) for some unit \( u \in k\{x\}[y] \) which admits at least an \( n \)-th root for all \( n \geq 1 \). Then

\[
S_f = S_g
\]

in \( M_{X_0}^\mu \), where \( X_0 = \text{Spec } k[x]/(f(x, 0)) \).

**Proof.** Let us denote by \( X \) the formal completion of the \( k \)-scheme \( \text{Spf } k\{x\}[y] \) along the ideal \( (f) = (g) \). Let \( \mathcal{X}(f) \) and \( \mathcal{X}(g) \) be the formal \( R \)-scheme associated to \( f \) and \( g \) respectively. Note that, \( \mathcal{X} = \mathcal{X}(f) = \mathcal{X}(g) \) as formal \( k \)-schemes. Let \( \mathcal{Y}(f) \to \mathcal{X}(f) \) be a resolution of singularities of \( \mathcal{X}(f) \). Then \( \mathcal{Y}(g) := \mathcal{Y}(f) \to \mathcal{X} \xrightarrow{\mathfrak{h}} \text{Spf } R \) is also a resolution of formal \( R \)-schemes \( \mathcal{X}(g) \) respectively. It follows from (2.5) that

\[
\mathcal{X}(f)_s \cong \text{Spf } R\{x\}[y]/(f) \cong \text{Spf } R\{x\}[y]/(g) \cong \mathcal{X}(g)_s.
\]

Therefore

\[
\mathcal{Y}(f)_s \cong \mathcal{Y}(g)_s.
\]

We now use the notation as in Definition 2.8 with \( \tilde{E}_I^\circ \) for \( \mathcal{Y}(f) \) and \( \tilde{F}_I^\circ \) for \( \mathcal{Y}(g) \). Then for each subset \( I \subset S \), there is an isomorphism \( \tilde{E}_I^\circ \cong \tilde{F}_I^\circ \) defined by

\[
(z, y) \mapsto (z, \xi, y),
\]
where \( \xi \) is induced from an \( m_\ell \)-th root of \( u \). Hence \( \text{MV}(\mathfrak{X}(f)) = \text{MV}(\mathfrak{X}(g)) \) according to Corollary 2.15.

3. Rigid analytic geometry and Motivic volume

In this section we assume that the field \( k \) has characteristic zero.

3.1. Motivic integrals on rigid analytic varieties. The motivic integral of a differential form of maximal degree \( \omega \) on a quasi-compact smooth rigid variety \( X \) was already defined by Loeser-Sebag [32], and inspired by it, Nicaise-Sebag [38] extended the notion to bounded \( \omega \) form of maximal degree 

**Definition 3.1** (Special rigid varieties). A rigid \( K \)-variety \( X \) is called (affine) special if it admits a formal model which is an (affine) special formal \( R \)-scheme. If \( X \) is affine, there exists a special \( R \)-algebra \( A \) such that \( X = (\text{Spf} A)_\eta \); we then call \( A \) a coordinate ring of \( X \).

Let \( G \) be a finite group scheme over \( k \). Let \( (\mathfrak{X}, \theta) \) be a pair consisting of a formal model \( \mathfrak{X} \) of \( X \) and a good \( G \)-action \( \theta \) on \( \mathfrak{X} \), we call \( (\mathfrak{X}, \theta) \) a \( G \)-pair of \( X \). Two \( G \)-pairs \( (\mathfrak{X}, \theta) \) and \( (\mathfrak{X}', \theta') \) of \( X \) are equivalent if there exist formal \( R \)-scheme \( \mathfrak{X}' \) endowed with a good \( G \)-action and two \( G \)-equivariant morphisms \( \mathfrak{X}' \to \mathfrak{X} \) and \( \mathfrak{X}' \to \mathfrak{X} \) such that the induced morphisms \( \mathfrak{X}' \to \mathfrak{X}' \) and \( \mathfrak{X}' \to \mathfrak{X}' \) are open embedding satisfying \( \mathfrak{X}'(K_{sh}) = X(K_{sh}) \). A good action of \( G \) (or good \( G \)-action) on a special rigid \( K \)-variety \( X \) is given by an equivalence class of \( G \)-pairs \( (\mathfrak{X}, \theta) \) of \( X \). A morphism of special \( K \)-varieties \( Y \to X \) is called \( G \)-equivariant if there exist representatives \( (\mathfrak{X}, \theta), (\mathfrak{Y}, \tau) \) and a \( G \)-equivariant morphism \( \mathfrak{Y} \to \mathfrak{X} \) such that the induced morphism \( \mathfrak{Y} \to \mathfrak{X} \) is the morphism \( Y \to X \).

**Lemma-Definition 3.2.** Let \( X \) be a smooth special rigid \( K \)-variety endowed with a good \( G \)-action. Let \( (\mathfrak{X}, \theta) \) and \( (\mathfrak{X}', \theta') \) be two equivalent \( G \)-pairs of \( X \). If \( \omega \) is a gauge form on \( X \), then

\[
\int_{\mathfrak{X}} |\omega| = \int_{\mathfrak{X}_0} \int_{\mathfrak{X}_0'} |\omega| \in M^G_k.
\]

We call this quantity the (motivic) \( G \)-integral of \( \omega \) on \( X \) and denoted by \( \int_X |\omega| \).

**Proof.** Since \((\mathfrak{X}, \theta)\) and \((\mathfrak{X}', \theta')\) are equivalent, there exist a pair \((\mathfrak{X}'', \theta'')\) and two \( G \)-equivariant morphisms \( h: \mathfrak{X}'' \to \mathfrak{X} \) and \( h': \mathfrak{X}' \to \mathfrak{X}'' \) such that the induced morphisms \( \mathfrak{X}''_0 \to \mathfrak{X}_0 \) and \( \mathfrak{X}'_0 \to \mathfrak{X}'_0 \) are open embedding satisfying \( \mathfrak{X}''(K_{sh}) = X(K_{sh}) \). By the special \( G \)-equivariant change of variables formula (cf. Proposition 2.2) we have

\[
\int_{\mathfrak{X}} |\omega| = h_0 |\mathfrak{X}_0''| \in M^G_{\mathfrak{X}_0}
\]

and

\[
\int_{\mathfrak{X}_0'} |\omega| = h'_0 |\mathfrak{X}_0''| \in M^G_{\mathfrak{X}_0'}.
\]

Hence, it follows from the fact \( \int_{\mathfrak{X}_0} h_0 = \int_{\mathfrak{X}_0''} = \int_{\mathfrak{X}_0'} h'_0 \) that

\[
\int_{\mathfrak{X}_0} \int_{\mathfrak{X}_0'} |\omega| = \int_{\mathfrak{X}_0'} \int_{\mathfrak{X}_0} |\omega| \in M^G_k.
\]

\( \Box \)
Lemma 3.3. Let $U \to X$ be a $G$-equivariant open immersion of smooth special rigid $K$-varieties such that $U(K^{sh}) = X(K^{sh})$. Then, for any gauge forms $\omega$ on $X$ and any $m \in \mathbb{N}$, the identity

$$\int_U |\omega| = \int_X |\omega|$$

holds in $\mathcal{M}_k^G$.

Proof. Let $\mathfrak{U}$ be a formal model of $U$ and by $\mathfrak{Z}$ its dilatation. Let $Z$ be the generic fiber of $\mathfrak{Z}$ which is quasi-compact. Then by definition,

$$\int_{Z(n)} |\omega(n)| = \int_{\mathfrak{Z}_n(n)} |\omega(n)| = \int_{X(n)} |\omega(n)|.$$

Moreover, for any formal model $\mathfrak{X}$ of $X$, using the isomorphism

$$\lim_{\text{models } \mathfrak{Z} \text{ of } Z} \text{Mor}(\mathfrak{Z}, \mathfrak{X}) \to \text{Mor}(\mathfrak{Z}, X)$$

in [15, (7.1.7.1)], we obtain another model $\mathfrak{Z}'$ of $Z$ and a morphism $h: \mathfrak{Z}' \to \mathfrak{X}$ such that $h_{\eta}$ is the inclusion $Z \to X$. Hence

$$\int_{Z(n)} |\omega(n)| = \int_{\mathfrak{Z}_n'(n)} |\omega(n)| = \int_{X(n)} |\omega(n)|,$$

which completes the lemma. \qed

Remark 3.4. Assume that the action of $G$ on $X$ is trivial. Then the motivic integral of $X$ defined in [38, Def. 5.10] (see also [34, Prop. 4.8], [32, Thm.-Def. 4.1.2]) is nothing but the image of the $G$-integral of $X$ under the forgetful morphism

$$\mathcal{M}_k^G \to \mathcal{M}_k.$$

Lemma 3.5. Let $X$ and $X'$ be two smooth special rigid $K$-varieties endowed with good $G$-actions. If $\omega$ and $\omega'$ are gauge forms on $X$ and $X'$ respectively, then the identity

$$\int_{X \times X'} |\omega \otimes \omega'| = \int_X |\omega| \cdot \int_{X'} |\omega'|$$

holds in $\mathcal{M}_k^G$. Here, the $G$-action on $X \times X'$ is the diagonal action.

Proof. Let $(\mathfrak{X}, \theta)$ and $(\mathfrak{X}', \theta')$ be $G$-pairs of $X$ and $X'$ respectively. Let $\pi: \mathfrak{U} \to \mathfrak{X}$ (resp. $\pi': \mathfrak{U}' \to \mathfrak{X}'$) be the $G$-dilataion of $\mathfrak{X}$ (resp. $\mathfrak{X}'$). Then $\pi \times \pi': \mathfrak{U} \times \mathfrak{U}' \to \mathfrak{X} \times \mathfrak{X}'$ is the $G$-dilataion of $\mathfrak{X} \times \mathfrak{X}'$. Let $U$ and $U'$ be the generic fibers of $\mathfrak{U}$ and $\mathfrak{U}'$, respectively. By Lemma-Definition 3.2, we have identities

$$\int_X |\omega| = \int_U |\omega|, \quad \int_{X'} |\omega'| = \int_{U'} |\omega'|$$

and

$$\int_{X \times X'} |\omega \otimes \omega'| = \int_{U \times U'} |\omega \otimes \omega'|$$

in the ring $\mathcal{M}_k^G$. Note that $U$ and $U'$ are quasi-compact rigid $K$-varieties, applying the proof of [32, Prop. 4.1.5], we obtain the identity

$$\int_{U \times U'} |\omega \otimes \omega'| = \int_U |\omega| \cdot \int_{U'} |\omega'|$$

and hence the lemma. \qed
In order to define the notion of motivic volume for general special rigid $K$-varieties we need to prove that any two models of a special rigid $K$-variety $X$ can be dominated by another model. In the case where $X$ is quasi-compact, i.e., $X$ admits an admissible covering of affinoid varieties, this is proved to be true in [7]. For a proof of the claim in general, we need to apply the descent theory developed by de Jong in [15, §7.5].

**Theorem 3.6** (de Jong’s descent theory). Let $X$ be a special $K$-rigid variety with a model $\mathcal{X}$. Consider a stratification

$$\mathcal{X}_0 = \sqcup_{i \in I} V_i$$

of $\mathcal{X}_0$ into finitely many locally closed subvarieties $V_i$ of $\mathcal{X}_0$. Let $\mathcal{V}_i$ be the formal completion of $\mathcal{X}$ along $V_i$. Let $(Z, \mathfrak{Z}_i)$ be a tuple of objects satisfying the following properties:

- $Z \subset X$ is a closed analytic subvariety of $X$,
- $\mathfrak{Z}_i$ is a closed formal subscheme of $\mathcal{V}_i$,
- the following identities hold

$$(\mathfrak{Z}_i)_\eta = \text{sp}^{-1}(V_i) \cap Z, \forall i,$$

where $\text{sp}: X \to \mathcal{X}$ denotes the specialization map.

Then there is a closed formal subscheme $\mathfrak{Z}$ of $\mathcal{X}$ such that $\mathfrak{Z}_i = \mathfrak{Z} \cap \mathcal{V}_i$ for all $i \in I$ and

$$\mathfrak{Z}_\eta = Z.$$

**Proof.** The theorem is obtained by applying [15, Prop.7.5.2] finitely many times. $\square$

**Theorem 3.7.** Let $X, Y, Z$ be smooth special rigid $K$-varieties and let $\mathfrak{M}$ be a model of $Y$.

(i) If $\phi: Z \to Y$ is a closed immersion, then there exists a model $\mathfrak{Z}$ of $Z$ and a proper morphism $\varphi: \mathfrak{Z} \to \mathfrak{M}$ such that $\varphi_\eta = \phi$.

(ii) If $\mathcal{X}$ and $\mathcal{X}'$ are two formal models of $X$, then there exist two morphisms $h: \mathcal{X}'' \to \mathcal{X}$ and $h': \mathcal{X}'' \to \mathcal{X}'$ of models of $X$ such that the induced morphisms $h_\eta$ and $h'_\eta$ are isomorphisms.

**Proof.** (i). We first prove the statement (i) for the special case when $\mathfrak{M} = \text{Spf} A$ with the largest ideal of definition $J$ of $A$ being generated by a regular system of elements in $A$. We will use the constructions in [15, §7.5] and [16].

Let $A[J^n/\varpi]$ be the subalgebra of $A \otimes_R K$ generated by $A$ and elements of form $i/\varpi$ where $i \in J^n$. Let $B_n$ be the $J$-adic completion of $A[J^n/\varpi]$ and $C_n = B_n \otimes_R K$. Note that $A[J^n/\varpi]$ is regular since $A$ is regular with a system of regular elements $x_1, \ldots, x_n, y_1, \ldots, y_m$ and $A[J^n/\varpi]/J = A/J$ is regular. Hence by [22, IV1, Lemme (17.3.8.1)] $B_n$ is regular and so normal.

Let $V_n = \text{Sp} C_n$. Let $\alpha_n$ and $\alpha'_n$ be morphisms defined as

$$\alpha_n: B_n \to C_n \xrightarrow{\alpha'_n} \Gamma(Z \cap V_n, \mathcal{O}_X)$$

where $\alpha'_n$ is induced from the inclusion $Z \cap V_n \to V_n$. We denote by $I_n = \ker \alpha_n$ the kernel of $\alpha_n$ in $B_n$ and by $I'_n$ the kernel of $\alpha'_n$ in $C_n$. Since $C_n$ is an affinoid $K$-algebra, the Maximum Modulus Principle holds for the norm $| \cdot |_{\text{sup}}$ on $C_n$ ([6, 6.2.1/4]). That is, for every $f \in C_n$, there exists $x \in \text{Sp} C_n$ such that $|f(x)| = |f|_{\text{sup}}$. This implies that for every $f \in C_n$, there exists $c \in K$ such that $|cf|_{\text{sup}} \leq 1$. Therefore $I'_n$ can be generated by power-bounded elements (an elements $f \in C_n$ is power-bounded if $|f|_{\text{sup}} \leq 1$, [6, 6.2.3/1]). Applying the isomorphism
in [15, Theorem 7.4.1] for the normal ring $B_n$ we can show that $I'_n = I'_m C_n$. Moreover, by the functoriality of $B_n$ and $C_n$ we have, for each $m > n$, the following commutative diagram

\[
\begin{array}{ccc}
B_m & \xrightarrow{\alpha'_m} & C_m \\
\downarrow & & \downarrow \\
B_n & \xrightarrow{\alpha'_n} & C_n
\end{array}
\Gamma(Z \cap V_m, \mathcal{O}_Z) \xrightarrow{\Gamma(Z \cap V_n, \mathcal{O}_Z)}
\]

which gives $I'_n = I'_m C_n$ by applying [6, 7.2/6] for the affinoid subdomain $SpC_n$ of $SpC_m$. Since $I'_n = I_n C_n$, we have $I_n = I_m B_n$. It follows from [16] (see also [15, §7.1.13]) that there exists an integer $c > 0$ and, for each $n > c$, a surjective morphism

\[\beta_n : B_n \to A/J^{n-c}\]

satisfying the following compatibility condition: for each $m > n > c$, the diagram

\[
\begin{array}{ccc}
B_m & \xrightarrow{\beta_m} & A/J^{m-c} \\
\downarrow & & \downarrow \\
B_n & \xrightarrow{\beta_n} & A/J^{n-c}
\end{array}
\]

commutes. Since $I_n = I_m B_n$, the morphisms

\[\beta_m(I_m) \to \beta_n(I_n)\]

are surjective. Define the limit ideal

\[a := \varprojlim \beta_n(I_n) = \varprojlim A/J^{n-c} = A.\]

We will show that $3 = SpfA/a$, i.e. showing that $Z = (SpfA/a)_\eta$. By definition, $Z \cap V_n = (Spf B_n/I_n)_\eta$ and $(SpfA/a)_\eta \cap V_n = (SpfA/a \otimes A B_n(A))_\eta$ since the functor $\eta$ commutes with tensor product. Therefore it suffices to show that

\[SpfA/a \otimes_A B_n(A) \cong B_n/I_n.\]

Let $\varphi_n : A \to B_n$ be the natural morphism. We will show the following identity (via $\varphi_n$)

\[(3.1)\quad I_n = a B_n.\]

Consider the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi_n} & B_n \\
\downarrow & & \downarrow & & \downarrow \\
B_m & \xrightarrow{\beta_m} & A/J^{m-c} \\
\downarrow & & \downarrow & & \downarrow \\
B_n & \xrightarrow{\beta_n} & A/J^{n-c}
\end{array}
\gamma_m
\]

We claim that

\[(3.2)\quad (\ker \beta_m)B_n \subset J^{m-n}B_n.\]
Indeed, let \( x \in B_m \) and let
\[
x = \sum_{i \geq 0} a_i \frac{\varpi^i}{\varpi^i}
\]
with \( a_i \in J^m \) for all \( i \geq 0 \) be a representative of \( x \). Then, the morphism \( \beta_m \) can be expressed as
\[
\beta_m(x) = a_0 \in A/J^m.
\]
If \( x \in \ker \beta_m \), then \( a_0 \in J^m - cB_n \subset J^m - nB_n \) for all \( i \geq 1 \),
which gives \( \text{(3.2)} \). We first prove the inclusion \( aB_n \subset I_n \) of \( \text{(3.1)} \). Take \( x \in a \) then
\[
\beta_m(\varphi_m(x)) = \gamma_m(x) \in \beta_m(I_m)
\]
since \( a = \lim \beta_n(I_n) \). Therefore, \( \beta_m(\varphi_m(x)) = \beta_m(x_m) \) for some \( x_m \in I_m \), which yields that
\[
\varphi_m(x) - x_m \in \ker \beta_m.
\]
Then
\[
\varphi_n(x) - \varphi_n^m(x_m) \in (\ker \beta_m)B_n,
\]
which implies, by combining \( \text{(3.2)} \) and the identity \( I_n = I_mB_n \), that
\[
\varphi_n(x) \in I_n + J^m - nB_n, \quad \forall m > n.
\]
Therefore \( \varphi_n(x) \in I_n \), since \( \cap_{m>n} J^m - nB_n = 0 \). To show the other inclusion \( I_n \subset aB_n \) of \( \text{(3.1)} \) we take \( y \in I_n \). Then, since \( I_n = I_mB_n \),
\[
y = \sum_{\nu} \varphi_n^m(x_{\nu})y_{\nu} \quad \text{for some} \quad x_{\nu} \in I_m \quad \text{and} \quad y_{\nu} \in B_n.
\]
Notice that, \( a = \lim \beta_m(I_m) \) and the system \( \beta_m(I_m) \) is surjective. It follows that, there exists, for each \( \nu \), elements \( a_{\nu} \in a \) such that \( \beta_m(x_{\nu}) = \gamma_m(a_{\nu}) = \beta_m(\varphi_m(a_{\nu})) \) and therefore
\[
x_{\nu} - \varphi_m(a_{\nu}) \in \ker \beta_m.
\]
Using the inclusion \( \text{(3.2)} \), we may deduce that
\[
\varphi_n^m(x_{\nu}) \in \varphi_n(a)B_n + J^m - nB_n, \quad \forall m > n.
\]
So \( y \in \varphi_n(a)B_n + J^m - nB_n, \quad \forall m > n \), and therefore \( y \in \varphi_n(a)B_n \) according to the identity \( \cap_{m>n} J^m - nB_n = 0 \). This proves \( \text{(3.1)} \). The identity \( \text{(3.1)} \) induces a morphism \( A/a \to B_n/I_n \) making the following diagram commutative
This gives rise to a morphism \( \phi: B_n(A) \otimes_A A/\mathfrak{a} \to B_n/I_n \) and an induced commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{p} & A/\mathfrak{a} \\
\downarrow{\varphi_n} & & \downarrow{\varphi_n} \\
B_n(A) & \xrightarrow{p_n} & B_n(A) \otimes_A A/\mathfrak{a} \\
& & \downarrow{\phi} \\
& & B_n/I_n
\end{array}
\]

It is obvious that the morphism \( \phi \) is surjective. To prove its injectivity, we take \( \bar{y} \in \ker \phi \) with a preimage \( y \in I_n \) in \( B_n \). It follows from the identity (3.1) that

\[
y = \sum_{\nu} \varphi_n(a_{\nu}) \cdot y_{\nu}
\]

for some \( a_{\nu} \in \mathfrak{a} \) and \( y_{\nu} \in B_n \). Then

\[
\bar{y} = p_n(y) = \sum_{\nu} p_n(\varphi_n(a_{\nu})) \cdot p_n(y_{\nu}) = \sum_{\nu} \varphi_n(p(a_{\nu})) \cdot p_n(y_{\nu}) = 0,
\]

and hence one obtains the injectivity of \( \phi \). This proves that the morphism

\[
\phi: B_n(A) \otimes_A A/\mathfrak{a} \to B_n/I_n
\]

is an isomorphism. Hence

\[
Z \cap V_n = (\text{Spf } B_n/I_n)_\eta = (\text{Spf } A/\mathfrak{a} \otimes_A B_n(A))_{\eta} = (\text{Spf } A/\mathfrak{a})_\eta \cap V_n
\]

for all \( n \geq 1 \), and therefore \( Z = (\text{Spf } A/\mathfrak{a})_\eta \).

We now construct a closed immersion of formal schemes \( \varphi: \mathfrak{Z} \to \mathfrak{Y} \) such that \( \mathfrak{Z}_\eta = Z \) for general \( \mathfrak{Y} \). Using resolution of singularities of \( \mathfrak{Y} \), we may assume that \( \mathfrak{Y} \) is regular and that \( \mathfrak{Y}_S = \sum_{i \in S} N_i \mathfrak{E}_i \) is a strict normal crossings divisor. We define, as in Definition 2.8, for each subset \( I \subset S \), \( E^0_I := \cap_{i \in I} E_i \setminus \cup_{j \not\in I} E_j \) and denote by \( \mathfrak{Y}_I \) the formal completion of \( \mathfrak{Y} \) along \( E^0_I \). Note that \( \mathfrak{Y}_I \) can be covered by an admissible covering of open affine formal schemes \( \mathfrak{U}_i \) such that all \( \mathfrak{U}_i \) and \( \mathfrak{U}_i \cap \mathfrak{U}_j \) satisfy the assumption of the special case considered above. Moreover, since the above construction for the affine case is functorial we may therefore glue affine pieces to obtain a closed formal subscheme \( \mathfrak{Z}_I \) of \( \mathfrak{Y}_I \) such that \( (\mathfrak{Z}_I)_\eta = (\mathfrak{Y}_I)_\eta \cap Z \). Since

\[
\mathfrak{Y}_0 = \sqcup_{I \subset S} E^0_I
\]

is a decomposition of \( \mathfrak{Y}_0 \) by locally closed subvarieties, it follows from Theorem 3.6 that the tuple \( ((\mathfrak{Z}_I)_{I \subset S}, Z) \) comes from a closed formal subscheme of \( \mathfrak{Y} \). That is, there is a closed formal subscheme \( \mathfrak{Z} \) of \( \mathfrak{Y} \) such that

\[
\mathfrak{Z}_\eta = Z \quad \text{and} \quad \mathfrak{Z}_I = \mathfrak{Z} \cap \mathfrak{Y}_I, \quad \forall I \subset S.
\]

This completes (i).

(ii) Let us denote by \( \mathfrak{X} \) the fiber product \( X \times_R X' \) and by \( Y \) its generic fiber \( \mathfrak{Y}_\eta \). By [15, 7.2.4], \( X \) is separated and \( Y \cong X \times X \), and therefore the diagonal morphism \( \phi: X \to X \times X \cong Y \) is a closed immersion. It then follows from (i) that, there exist a formal model \( \mathfrak{X}' \) and a morphism \( \varphi: \mathfrak{X}' \to \mathfrak{Y} \) such that \( \varphi_\eta = \phi \). Hence we obtain the morphisms \( h = p_1 \circ \varphi \) and \( h' = p_2 \circ \varphi \) as expected, where \( p_1: \mathfrak{X} \times_R \mathfrak{X}' \to \mathfrak{X} \) and \( p_2: \mathfrak{X} \times_R \mathfrak{X}' \to \mathfrak{X}' \) are the canonical projections. \( \square \)
Recall that, for \( n \in \mathbb{N}^* \), we put already \( R(n) = R[\tau]/(\tau^n - \omega) \), \( K(n) = K[\tau]/(\tau^n - \omega) \), and for each formal \( R \)-scheme \( X \) we denoted \( X(n) = X \times_R R(n) \). Let \( \theta_X \) be the action of \( \mu_n = \text{Spec}(k[\xi]/(\xi^n - 1)) \) on \( X(n) \) induced from the natural action of \( \mu_n \) on \( R(n) \). Let \( X \) be a smooth special rigid \( K \)-variety and let \( \omega \) is a gauge form on \( X \). We denote \( X(n) = X \times_K K(n) \) and by \( \omega(n) \) the pullback of \( \omega \) via the natural morphism \( X(n) \to X \).

**Lemma-Definition 3.8.** Let \( X \) be a smooth special rigid \( K \)-variety. Then, for any two special formal models \( \mathcal{X} \) and \( \mathcal{X}' \) of \( X \), the pairs \((\mathcal{X}(n), \theta_{\mathcal{X}})\) and \((\mathcal{X}'(n), \theta_{\mathcal{X}'})\) are \( \mu_n \)-pairs of \( X(n) \), and they are equivalent. We define the \( \mu_n \)-**integral** of \( \omega(n) \) on \( X(n) \) to be the \( \mu_n \)-integral of \( \omega(n) \) on \( X(n) \) with this action, i.e.

\[
\int_{X(n)} |\omega(n)| := \int_{X_0} \int_{X(n)} |\omega(n)| \in \mathcal{M}^\mu_k.
\]

**Proof.** The statement that \((\mathcal{X}(n), \theta_{\mathcal{X}})\) and \((\mathcal{X}'(n), \theta_{\mathcal{X}'})\) are \( \mu_n \)-pairs of \( X(n) \) is trivial since \( \mathcal{X}(n)_n = \mathcal{X}_n(n) \). By Theorem 3.7 there exist two morphisms \( h: \mathcal{X}'' \to \mathcal{X} \) and \( h': \mathcal{X}''' \to \mathcal{X}' \) of models of \( X \) such that \( h_n \) and \( h'_n \) are isomorphisms. It implies that \( \mathcal{X}'''(n)_n \cong \mathcal{X}(n)_n \) and \( \mathcal{X}''(n)_n \cong \mathcal{X}'(n)_n \). By the naturality of the \( \mu_n \)-action on the \( n \)-ramification \( R(n) \), the induced morphisms \( h(n): \mathcal{X}''(n) \to \mathcal{X}(n) \) and \( h'(n): \mathcal{X}'''(n) \to \mathcal{X}'(n) \) are \( \mu_n \)-equivariant. Hence, by definition, the \( \mu_n \)-pairs \((\mathcal{X}(n), \theta_{\mathcal{X}})\) and \((\mathcal{X}'(n), \theta_{\mathcal{X}'})\) of \( X(n) \) are equivalent. \( \Box \)

**Definition 3.9.** Let \( X \) be a smooth special rigid \( K \)-variety, and let \( \omega \) be a gauge form on \( X \). The formal power series

\[
P(X, \omega; T) := \sum_{n \geq 1} \left( \int_{X(n)} |\omega(n)| \right) T^n \in \mathcal{M}^\mu_k[T]
\]

is called the volume Poincaré series of \((X, \omega)\).

**Proposition-Definition 3.10.** Let \( X \) be a smooth special rigid \( K \)-variety, let \( \mathcal{X} \) be a special formal model of \( X \). Then the quantity

\[
\int_{\mathcal{X}_0} \text{MV}(\mathcal{X}) \in \mathcal{M}^\mu_k
\]

is independent of the model \( \mathcal{X} \). We call it the **motivic volume** of \( X \) and denote by \( \text{MV}(X) \).

**Proof. Step 1.** We assume that \( X \) admits an \( \mathcal{X} \)-bounded gauge form \( \omega \). Using resolution of singularities we can assume that \( \mathcal{X} \) is regular with strict normal crossing divisor \( \mathcal{X}_s = \sum_{i \in S} N_i \mathcal{E}_i \). From Definition 2.12 we have

\[
(3.3) \quad \int_{\mathcal{X}_0} \text{MV}(\mathcal{X}) = \sum_{\mathcal{Y}_0 \in S} (1 - \mathcal{L})^{1-1} \left[ \mathcal{E}^0 \to \text{Spec} \mathcal{K} \right] \in \mathcal{M}^\mu_k.
\]

Using Theorem 2.10 and Lemma-Definition 3.8 we have

\[
\left( \int_{X(n)} |\omega(n)| \right) = \mathcal{L}^{-d} \sum_{\emptyset \neq I \subseteq S} (\mathcal{L} - 1)^{|I|-1} \left[ \mathcal{E}^0 \to \text{Spec} \mathcal{K} \right] \sum_{k_i \geq 1} \sum_{i \in I} k_i N_i = n, \sum_{i \in I} \mathcal{L}^{-\sum_i k_i \alpha_i},
\]

with \( \alpha_i = \text{ord}_{\mathcal{E}_i}(\omega) \), thus

\[
(3.4) \quad P(X, \omega; T) = \mathcal{L}^{-d} \sum_{\emptyset \neq I \subseteq S} (\mathcal{L} - 1)^{|I|-1} \left[ \mathcal{E}^0 \to \text{Spec} \mathcal{K} \right] \prod_{i \in I} \mathcal{L}^{-\alpha_i T N_i},
\]

is independent of the model \( \mathcal{X} \). We call it the **motivic volume** of \( X \) and denote by \( \text{MV}(X) \).
from which, together with (3.3),

\[
\int_{\mathcal{X}_0} \text{MV}(\mathcal{X}) = -\mathbb{L}^d \lim_{T \to \infty} P(X, \omega; T).
\]

The equality (3.5) guarantees that \( \int_{\mathcal{X}_0} \text{MV}(\mathcal{X}) \) is independent of the model \( \mathcal{X} \) of \( X \).

**Step 2.** We do not assume that \( X \) admits an \( \mathcal{X} \)-bounded gauge form. Let \( \mathcal{X}' \) be another special formal models of \( X \). Since \( \mathcal{X} \) (resp. \( \mathcal{X}' \)) admits a resolution of singularities, we can identify \( \mathcal{X} \) (resp. \( \mathcal{X}' \)) with a resolution of singularities for it. By [34, Prop.-Def. 7.38], the special formal \( R \)-scheme \( \mathcal{X} \) (resp. \( \mathcal{X}' \)) has a finite open covering \( \{ \mathcal{U}_i \}_{i \in Q} \) (resp. \( \{ \mathcal{U}'_j \}_{j \in Q'} \)) such that each \( \mathcal{U}_i \eta \) (resp. \( \mathcal{U}'_j \eta \)) admits a \( \mathcal{U}_i \)-bounded (resp. \( \mathcal{U}'_j \)-bounded) gauge form \( \omega_i \) (resp. \( \omega'_j \)). By Proposition 2.13

\[
\text{MV}(\mathcal{X}) = \sum_{i \in Q} (-1)^{|I|-1} \text{MV}(\mathcal{U}_i)
\]

and

\[
\text{MV}(\mathcal{X}') = \sum_{j \in Q'} (-1)^{|J|-1} \text{MV}(\mathcal{U}'_j).
\]

By Theorem 3.7 there exist two morphisms \( h: \mathcal{X}'' \to \mathcal{X} \) and \( h': \mathcal{X}'' \to \mathcal{X}' \) of models of \( X \) such that \( h \eta \eta \) and \( h' \eta \eta \) are isomorphisms. Put \( \mathcal{Y}_i = h^{-1}(\mathcal{U}_i) \) and \( \mathcal{Y}'_j = h'^{-1}(\mathcal{U}'_j) \) for every \( i \in Q \) and \( j \in Q' \). By Proposition 2.13 and Lemma-Definition 3.8 we get (with some \( i \in I \) and \( j \in J \))

\[
\int_{(\mathcal{Y}_i)_{\eta}(n)} |h^* \omega_i(n)| = \int_{(\mathcal{U}_i)_{\eta}(n)} |\omega_i(n)|
\]

and

\[
\int_{(\mathcal{Y}'_j)_{\eta}(n)} |h'^* \omega'_j(n)| = \int_{(\mathcal{U}'_j)_{\eta}(n)} |\omega'_j(n)|,
\]

for every \( n \in \mathbb{N}^* \). These equalities together with the computation in Step 1 give us

\[
\int_{(\mathcal{Y}_i)_{\eta}} \text{MV}(\mathcal{Y}_i) = \int_{(\mathcal{U}_i)_{\eta}} \text{MV}(\mathcal{U}_i)
\]

and

\[
\int_{(\mathcal{Y}'_j)_{\eta}} \text{MV}(\mathcal{Y}'_j) = \int_{(\mathcal{U}'_j)_{\eta}} \text{MV}(\mathcal{U}'_j).
\]

Hence, by Proposition 2.13 as well as some above equalities, we have

\[
\int_{\mathcal{X}_0} \text{MV}(\mathcal{X}) = \int_{\mathcal{X}_0} \text{MV}(\mathcal{X}'') = \int_{\mathcal{X}_0} \text{MV}(\mathcal{X}').
\]

This means that \( \int_{\mathcal{X}_0} \text{MV}(\mathcal{X}) \) is independent of the model \( \mathcal{X} \). \( \square \)

The following is a direct consequence of the proposition.

**Corollary 3.11.** Let \( X \) be a bounded smooth special rigid \( K \)-variety which admits a \( \mathcal{X} \)-bounded gauge. Let \( h: \mathcal{Y} \to \mathcal{X} \) be a resolution of singularities of \( \mathcal{X} \). With the notation as in Definition 2.9 we have

\[
\text{MV}(X) = \sum_{\emptyset \neq I \subseteq S} (1 - \mathbb{L}_I)^{|I|-1} \left[ \bar{E}^0_I \right] \in \mathcal{M}_k^\mu.
\]

**Definition 3.12** (Bounded analytic spaces). Let \( X \) be a smooth special rigid \( K \)-variety. A differential form \( \omega \) on \( X \) is called **bounded** if it is \( \mathcal{X} \)-bounded for some formal model \( \mathcal{X} \) of \( X \). The rigid \( K \)-variety \( X \) is called **bounded** if it admits a bounded gauge form.
Notice that our definition of bounded rigid varieties is not related to that of [36].

**Proposition 3.13.** Let $X$ be a bounded smooth special rigid $K$-variety and let $Y$ be a smooth closed subvariety of $X$. Then $Y$ is also bounded. In particular, all affine smooth special rigid varieties are bounded.

**Proof.** It follows from Theorem [3.7(i)] and [31] Proposition 2.6. □

**Corollary 3.14.** Let $X$ be a bounded smooth special rigid $K$-variety which admits a bounded gauge form $\omega$. Then the volume Poincaré series $P(X,\omega; T)$ is rational and the following identity holds in $\mathcal{M}_k^\hat{\omega}$:

$$\text{MV}(X) = -L_d \lim_{T \to \infty} P(X,\omega; T).$$

We believe that Corollary [3.11] is still true when we do not assume the $X$-boundedness of a gauge form on $X$ (see the below conjecture).

**Conjecture 3.15.** Let $X$ be a smooth special rigid $K$-variety, and $\omega$ a gauge form on $X$. Then the volume Poincaré series $P(X,\omega; T)$ is rational and the identity

$$\text{MV}(X) = -L_d \lim_{T \to \infty} P(X,\omega; T)$$

holds in $\mathcal{M}_k^\hat{\omega}$.

**Example 3.16.** Let $R = k[\varpi], K = k((\varpi))$, $R(n) = k[\varpi^{1/n}]$ and $K(n) = k((\varpi^{1/n}))$ for $n \in \mathbb{N}^*$. Consider the formal schemes $\mathcal{X} = \text{Spf}(R[[x_1, \ldots, x_d]])$, $\mathcal{Y} = \text{Spf}(R\{x_1, \ldots, x_d\})$ and $\mathcal{U} = \text{Spf}(R(\varpi))$ with $R(\varpi) := R(x,y)/(x - \varpi^p y)$ for new variables $y = (y_1, \ldots, y_d)$. Then $\mathcal{X}_\eta = D^d$ is the $d$-dimensional “open” ball of radius 1, and $\mathcal{Y}_\eta = B^d$ and $\mathcal{U}_\eta = B^d(|\varpi^p|)$ are the $d$-dimensional “closed” balls of radius 1 and $|\varpi^p|$ respectively. It follows from Example [2.14] that

$$\text{MV}(D^d) = 1, \text{MV}(B^d) = \text{MV}(B^d(|\varpi^p|)) = L_d \in \mathcal{M}_k^\hat{\omega}.$$

### 3.2. Motivic volume morphism

Let SSRig$_K$ denote the category of special smooth rigid $K$-varieties. In this subsection, we define a (motivic volume) homomorphism from a certain Grothendieck ring of SSRig$_K$ to the ring $\mathcal{M}_k^\hat{\omega}$, which is compatible with the motivic volume defined in the previous section.

We first introduce the notion of special rational subdomains. Let $X$ be an affine special rigid $K$-variety with a coordinate ring $A$. Assume that $g, f_1, \ldots, f_n \in A$ generate the unit ideal in $A \otimes_K R$. Writing $L_g$ for $(f_1/g, \ldots, f_n/g)$ we define

$$X(L_g) := \{x \in X \mid |f_i(x)| \leq |g(x)|, \forall i\}.$$

A subvariety of $X$ in this form is called a *special rational subdomain of $X$*. In general, a subvariety $Y$ of a special rigid $K$-variety $X$ is called *special rational subdomain of $X$* if there exists a finite cover by affine special varieties $(X_i)_{i \in I}$ of $X$ such that $Y \cap X_i$ is a special rational subdomain of $X_i$ for every $i \in I$. Notice that a rational subdomain $Y$ of $X$ and its complement are also objects of SSRig$_K$, since for instance, if $X$ is affine, then $Y$ and $X \setminus Y$ have coordinate rings

$$A[z_1, \ldots, z_n]/(f_1 - z_1 g, \ldots, f_n - z_n g)$$

and

$$A[z]/(zf_1 - g) \cdots (zf_n - g),$$
For any rational subdomain \( Z \) of \( \text{SSRig}_K \) modulo the relation
\[
[X] = [Y] + [X \setminus Y]
\]
where \( Y \subseteq X \) is a special rational subdomain of \( X \). The group \( \mathcal{K}(\text{SSRig}_K) \) admits a ring structure whose multiplication is induced by fiber product.

**Theorem 3.17.** There exists a unique ring homomorphism
\[
\text{MV} : \mathcal{K}(\text{SSRig}_K) \to \mathcal{M}_k^\mu
\]
such that
\[
\text{MV}([X]) = \text{MV}(X)
\]
for all objects \( X \) of \( \text{SSRig}_K \).

**Proof.** It is obvious that \( \text{MV}(1) = 1 \). We first prove the additivity of \( \text{MV} \), i.e.
\[
\text{MV}([X]) = \text{MV}([Z]) + \text{MV}([X \setminus Z])
\]
for any rational subdomain \( Z \) of \( X \). In fact, by Proposition 2.13, we can assume that \( X \) is special affine, and let \( A \) be a coordinate ring of \( X \). By induction, we can assume that \( Z = X(\mathbb{A}^g) \) for some \( f, g \in A \) generating the unit ideal in \( A \otimes_R K \). We may assume further that \( f \) and \( g \) have no common factors in \( A \). Then \( A[z(1)]/(f - zg) \) and \( A[z]/(z f - g) \) are coordinate rings of \( Z \) and \( X \setminus Z \), respectively. We consider the admissible blow up of \( \mathbb{X} := \text{Spf} A \) along the ideal generated by \( f \) and \( g \), say, \( \mathcal{Y} \rightarrow \mathbb{X} \). Let us consider the following special \( R \)-algebras
\[
A_1 := A[z_1]/(z_1 f - g) \quad \text{and} \quad A_2 := A[z_2]/(f - zg).
\]
As described in [34 Lem. 2.18], \( \{\text{Spf} A_1, \text{Spf} A_2\} \) is an open cover of \( \mathcal{Y} \), and the intersection \( \text{Spf} A_1 \cap \text{Spf} A_2 \) is isomorphic to
\[
\text{Spf} A_1[z_1]/(z_1 f - g, f - zg) \cong \text{Spf} A_1[z_2]/(1 - z_1 z_2).
\]
We first observe that \( A_2 \) is a coordinate ring of \( Z \), and the intersection \( \text{Spf} A_1 \cap \text{Spf} A_2 \) can be identified with the formal completion of \( \text{Spf} A_1 \) along the open subset of the special fiber \( (\text{Spf} A_1)_0 \) defined by \( z_1 \neq 0 \). On the other hand, the formal completion of \( \text{Spf} A_1 \) along the closed subset defined by \( z_1 = 0 \) is \( A[z_1]/(z_1 f - g) \), a coordinate ring of \( X \setminus Z \). It then follows from Proposition 2.13 that
\[
\text{MV}(\text{Spf} A_1) = \text{MV}(\text{Spf} A_1 \cap \text{Spf} A_2) + \text{MV}(\text{Spf} (A[z_1]/(z_1 f - g)))
\]
Hence, by the additivity of the motivic volume of formal \( R \)-schemes (cf. Proposition 2.13) the following identities
\[
\begin{align*}
\text{MV}(\mathcal{Y}) &= \text{MV}(\text{Spf} A_1) + \text{MV}(\text{Spf} A_2) - \text{MV}(\text{Spf} A_1 \cap \text{Spf} A_2) \\
&= \text{MV}(\text{Spf} A_2) + \text{MV}(\text{Spf} (A[z_1]/(z_1 f - g))),
\end{align*}
\]
hold in \( \mathcal{M}_k^\mu_\mathcal{Y} \). Applying the push-forward morphism
\[
\int_{\mathcal{Y}_0} : \mathcal{M}_k^\mu_\mathcal{Y_0} \to \mathcal{M}_k^\mu
\]
we obtain the identity \( \text{MV}(X) = \text{MV}(Z) + \text{MV}(X \setminus Z) \) in \( \mathcal{M}_k^\mu \).

We are going to prove the multiplicativity of \( \text{MV} \), more precisely
\[
\text{MV}(X \times Y) = \text{MV}(X) \cdot \text{MV}(Y)
\]
where \( X \) and \( Y \) are smooth special rigid \( K \)-varieties. Indeed, we first prove this identity for bounded rigid \( K \)-varieties. Assume that \( X \) and \( Y \) are bounded (cf. Definition 3.12) of
dimension $d_1$ and $d_2$ with bounded gauge form $\omega_X$ and $\omega_Y$, respectively. Then $\omega := \omega_X \otimes \omega_Y$ is a bounded gauge form on $X \times Y$. Let $P(X, \omega_X; T)$, $P(Y, \omega_Y; T)$ and $P(X \times Y, \omega; T)$ be the volume Poincaré series of the pairs $(X, \omega_X)$, $(Y, \omega_Y)$ and $(X \times Y, \omega)$, respectively. By Lemma 3.5 for all $n \geq 1$ we have

$$\int_{(X \times Y)(n)} |\omega(n)| = \int_{X(n)} |\omega_X(n)| \cdot \int_{Y(n)} |\omega_Y(n)|$$

and therefore,

$$P(X \times Y, \omega; T) = P(X, \omega_X; T) \ast P(X, \omega_X; T).$$

where $\ast$ is the Hadamard product of formal power series in $\mathbb{M}_k$. By Corollary 3.11 we have

$$\text{MV}(X \times Y) = -\mathbb{L}^d \lim_{T \to \infty} P(X \times Y, \omega; T)$$

$$= -\mathbb{L}^d \lim_{T \to \infty} (P(X, \omega_X; T) \ast P(Y, \omega_Y; T))$$

$$= \left( -\mathbb{L}^{d_1} \lim_{T \to \infty} P(X, \omega_X; T) \right) \cdot \left( -\mathbb{L}^{d_2} \lim_{T \to \infty} P(Y, \omega_Y; T) \right)$$

$$= \text{MV}(X) \cdot \text{MV}(Y),$$

where $d = d_1 + d_2$ and the third identity is due to Lemma 2.11.

Now, we are able to prove the multiplicativity of MV for general rigid $K$-varieties. Using resolution of singularities and [34, Cor. 7.27] we show that $X$ can be covered by open bounded rigid varieties $(X_i)_{i \in I}$ such that $X_{I'} := \cap_{i \in I'} X_i$ are also bounded for all $I' \subseteq I$, and that

$$\text{MV}(X) = \sum_{I' \subseteq I} (-1)^{|I'|-1} \text{MV}(X_{I'}).$$

We also take such a cover $(Y_j)_{j \in J}$ for $Y$, then obtain a cover $(X_i \times Y_j)_{(i,j) \in I \times J}$ for $X \times Y$. Applying the bounded case we get

$$\text{MV}(X_{I'} \times Y_{J'}) = \text{MV}(X_{I'}) \cdot \text{MV}(Y_{J'})$$

for every $I' \subseteq I$ and $J' \subseteq J$, and hence

$$\text{MV}(X \times Y) = \text{MV}(X) \cdot \text{MV}(Y)$$

according to the additivity of MV.

\[ \square \]

3.3. A motivic Fubini theorem. We first slightly modify the notion of constructible functions and their integrals in [33]. Recall that a subset of $\mathbb{R}^n$ is semi-algebraic if it is a finite union of sets of forms

$$\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) = 0; Q(x_1, \ldots, x_n) > 0, \}$$

where $P(x)$ and $Q(x)$ are systems of polynomials.

Let $A$ be an abelian group and let $V$ be a semi-algebraic set. A function $\varphi : V \to A$ is called constructible if there exists a partition of $V$ into finitely many semi-algebraic subsets $\rho_1, \ldots, \rho_k$ such that $\varphi$ takes a constant value $a_i \in A$ on $\rho_i$ for each $i$. A constructible function $\varphi : V \to A$ can be written as a finite sum

$$\varphi = \sum_{i=1}^{k} a_i 1_{\rho_i}$$
where \( I_{\rho_i} \) is the characteristic function of \( \rho_i \). If \( \varphi \) is a constructible function, the Euler integral of \( \varphi \) is defined as
\[
\int_V \varphi d\chi_c = \sum_{i=1}^k a_i \chi_c (\rho_i) .
\]

Let \( f : V \to W \) be a continuous semi-algebraic map and let \( \varphi : V \to A \) be a constructible function. The push forward \( f_* \varphi \) of \( \varphi \) along \( f \) is the function \( f_* \varphi : W \to A \) defined by
\[
f_* \varphi(y) = \int_{f^{-1}(y)} \varphi d\chi_c .
\]

**Theorem 3.18** (Change of variables formula). Let \( f : V \to W \) be a continuous semi-algebraic map and let \( \varphi \) be a constructible function on \( V \). Then, \( f_* \varphi \) is also constructible and we have
\[
\int_W f_* \varphi d\chi_c = \int_V \varphi d\chi_c .
\]

**Proof.** See Statement 3.A in [42]. \( \square \)

**Definition 3.19.** Let \( \Gamma \) be subset of \( \mathbb{Q}^n \). \( \Gamma \) is called a polyhedron if it is defined by \( Ax^T \geq b^T \) for some \( A \in \text{Mat}(m,n;\mathbb{Q}) \) and \( b \in \text{Mat}(1,m;\mathbb{Q}) \). Extending this concept, a constructible subset \( \Gamma \) of \( \mathbb{Q}^n \) is a finite Boolean combination of polyhedra. Let \( \Gamma_\mathbb{R} \) denote the canonical subset of \( \mathbb{R}^n \) associated with \( \Gamma \), defined by the same system of \( \mathbb{Q} \)-linear inequalities as \( \Gamma \). A function \( \varphi : \Gamma \to A \) is called constructible if there exists a partition of \( \Gamma \) into finitely many constructible subsets \( \rho \) such that \( \varphi \) takes a constant value \( a_{\rho} \in A \) on each stratum \( \rho \). Then the Euler integral of \( \varphi \) is defined as
\[
\int_\Gamma \varphi d\chi_c := \sum_{\rho} a_{\rho} \chi_c (\rho_\mathbb{R}) .
\]

Let \( V \subset \mathbb{R}^n \) and \( W \subset \mathbb{R}^m \) be constructible subsets. A map \( f : V \to W \) is called a piecewise affine linear map if there is a partition of \( V \) into finitely many constructible subsets \( V_i \) such that \( f|_{V_i} \) is a restriction of an affine linear map on \( V_i \). Let \( \varphi : V \to A \) be a constructible function. The push forward \( f_* \varphi \) of \( \varphi \) along \( f \) is the function \( f_* \varphi : W \to A \) defined by
\[
f_* \varphi(y) = \int_{f^{-1}(y)} \varphi d\chi_c .
\]

**Theorem 3.20** (Change of variables formula). Let \( V \subset \mathbb{Q}^n \) and \( W \subset \mathbb{Q}^m \) be constructible subsets. Let \( f : V \to W \) be a piecewise affine linear map and let \( \varphi : V \to A \) be a constructible function. Then, \( f_* \varphi \) is also constructible and we have
\[
\int_W f_* \varphi d\chi_c = \int_V \varphi d\chi_c .
\]

**Proof.** By the additivity of the Euler integral, we may assume that \( f \) is an affine linear map and that \( \varphi \) is constant of value \( a \) on \( V \). Then
\[
f_* \varphi(y) = \int_{f^{-1}(y)} \varphi d\chi_c = a \chi_c (f^{-1}(y)) .
\]

On the other hand, applying the change of variables formula (Theorem 3.18) for the map \( f_\mathbb{R} : V_\mathbb{R} \to W_\mathbb{R} \) and the constructible function \( a_{1V_\mathbb{R}} \) we have
\[
\int_V \varphi d\chi_c = a_{1V} (V_\mathbb{R}) = \int_{V_\mathbb{R}} a_{1V} d\chi_c = \int_{W_\mathbb{R}} f_* (a_{1V_\mathbb{R}}) d\chi_c = \int_W f_* \varphi d\chi_c .
\]
\( \square \)
Definition 3.21. The bounded Euler characteristic $\chi'(\Gamma)$ on the class of constructible subsets $\Gamma$ in $\mathbb{Q}^n$ is defined as follows. Let $\Gamma_R$ be the canonical subset of $\mathbb{R}^n$ associated with $\Gamma$, defined by the same system of $\mathbb{Q}$-linear inequalities as $\Gamma$. The compactly supported Euler characteristic of $\Gamma_R \cap [-r, r]^n$ stabilizes for sufficiently large $r \in \mathbb{R}$. We define

$$\chi'(\Gamma) := \chi_c(\Gamma \cap [-r, r]^n)$$

for $r \gg 0$. In particular, if $\Gamma_R$ is bounded in $\mathbb{R}^n$, then $\chi'(\Gamma) = \chi_c(\Gamma_R)$. The bounded Euler characteristic is additive on disjoint unions and assigns the value 1 to every non-empty constructible, then the bounded Euler integral of $\varphi$ is defined as

$$\int_\Gamma \varphi \, d\chi' := \sum_{i \in I} a_i \chi'(\rho_i).$$

In the following we denote

$$\mathbb{R}_{>0} := \{ x \in \mathbb{R} \mid x > 0 \}$$

and similarly for $\mathbb{R}_{\geq 0}$ and $\mathbb{Q}_{\geq 0}$.

Theorem 3.22 (Motivic Fubini Theorem). Let $X$ be a smooth special rigid $K$-variety with a model $\mathcal{X}$. Let $g = \{g_1, \ldots, g_r\}$ be a system of elements of $\Gamma(\mathcal{X}, \mathcal{O}_X)$. For each $\gamma \in \mathbb{Q}^n_{\geq 0}$ we define the variety

$$X_\gamma := \{ x \in X \mid |g_i(x)| = |\omega| \}. $$

Then the function $\varphi_g: \mathbb{Q}^n_{\geq 0} \to \mathcal{M}_k^\mu$ defined as

$$\varphi_g(\gamma) = \text{MV}(X_\gamma)$$

is constructible, and moreover,

$$\text{MV}(X) = \int_{\mathbb{Q}^n_{\geq 0}} \varphi_g \, d\chi_c = \int_{\mathbb{Q}^n_{\geq 0}} \varphi_g \, d\chi'.$$

Proof. By additivity of the morphism MV we may assume that $X$ admits an $\mathcal{X}$-bounded gauge form $\omega$. Let us consider a resolution of singularities $h: \mathcal{X} \to \mathcal{X}$ of the formal $R$-scheme $\mathcal{X}$. Let $E_i$, $i \in S$, be the irreducible components of $(\mathcal{Q})_{\text{red}}$. Let $E_i := (E_i)_0$, $E_i^\circ := \bigcap_{j \in I} E_i \setminus \bigcup_{j \notin I} E_j$ and let $\tilde{E}^0_i \to E^\circ_i$ be the covering with Galois group $\mu_{N_i}$ defined locally over $\mathcal{U}_0 \cap \tilde{E}^0_i$ as in Definition 2.9. Then for each $y \in E^\circ_i$ there exists an affine neighbourhood $\mathcal{U}_l$ such that the following identity holds in $\Gamma(\mathcal{U}_l, \mathcal{O}_{\mathcal{U}_l})$

$$\tilde{f} := h^* f = u \prod_{i \in I} y_i^{N_i}$$

where $f$ denotes the structural morphism of $\mathcal{X}$, $y_i$ is a local equation of $E_i$ at $y$ and $u$ is invertible in $\Gamma(\mathcal{U}_l, \mathcal{O}_{\mathcal{U}_l})$. Moreover, we can choose a resolution of singularities of such that the following identities hold in $\Gamma(\mathcal{U}_l, \mathcal{O}_{\mathcal{U}_l})$

$$\tilde{g}_i := h^* g_j = u \prod_{i \in I} y_i^{M_{il}}, \quad \forall l = 1, \ldots, r,$$

where $u_l$ are invertible in $\Gamma(\mathcal{U}_l, \mathcal{O}_{\mathcal{U}_l})$. Note that $N_i > 0$ for all $i \in S$ while $M_{il}$ could be zero for some $i$ and $l$. Then one has
Lemma 3.23. Let $\alpha_i = \text{ord}_{\mathfrak{c}_i} \omega$ for each $i \in I$. Then the following identities hold in $M^\theta_k$:

$$
\int_{X_\gamma(n)} |\omega(n)| = L^{-d} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} \left[ \sum_{k_i \geq 1, i \in I} \sum_{N_i = n} \left[ \sum_{\sum_i k_i \alpha_i = 0} \right] \right].
$$

Therefore,

$$
\text{MV}(X_\gamma) = - \sum_{I \subseteq S, I(f) \neq \emptyset} (L - 1)^{|I(f)| - 1} \chi_c(\Delta_{I,\gamma}) \left[ \tilde{E}_I^\circ \right],
$$

where $\Delta_{I,\gamma}$ is the cone in $\mathbb{R}^I_{\geq 0}$ defined by the system of equations $\sum_{i \in I} (M_i - \gamma_i N_i) k_i = 0$.

Before proving the lemma, we introduce the notion of $q$-separating resolution of singularities for some positive rational number $q$. Let $h: \mathfrak{y} \to \mathfrak{x}$ be a resolution of singularities of $\mathfrak{x}$. With the notation as in Section 2.3, the resolution $h$ is called $q$-separating for a given element $f \in \Gamma(\mathfrak{x}, \mathcal{O}_\mathfrak{x})$ if for all $i \neq j \in S$, the condition $\mathfrak{c}_i \cap \mathfrak{c}_j \neq \emptyset$ implies $N_i + N_j > q$. Comparison with the notion of non-linearity in [37, Definition 4.1], if $h$ is $q$-separating, then $n$ is not $\mathfrak{y}$-linear for all $n \leq q$.

Proof of the lemma. By [37, Lemma 5.2] the quantities in the right hand sides of the lemma do not change if one blows up $\mathfrak{y}$ along any intersection $E_I$. On the other hand, applying the argument in the proof of [3, Lemma 2.9], we may construct a resolution of singularities by bowing up centers of form $E_I$ which is $n$-separating for $f$. That is, if $\mathfrak{c}_i \cap \mathfrak{c}_j \neq \emptyset$ then $N_i + N_j > n$. Therefore, it suffices to prove the following identity holds in $M^\theta_k$

$$
\int_{X_\gamma(n)} |\omega(n)| = L^{-d} \sum_{N_i | n} \left[ \sum_{\sum_i k_i = 0} \tilde{E}_i^\circ \right] L^{-n \alpha_i / N_i}.
$$

Let $\mathfrak{z} := \text{Sm}(\mathfrak{y})$ the smooth locus of the normalization of $\mathfrak{y}(n)$. It follows from [34, Theorem 5.1] that

$$
\mathfrak{z} \to \mathfrak{y}(n)
$$

is a Néron smoothing (i.e. $\mathfrak{z}$ adic smooth over $R(n)$, and the induced morphism $\mathfrak{z}_n \to \mathfrak{y}(n)_n$ is an open embedding satisfying $\mathfrak{z}_n(K^{sh}) = \mathfrak{y}(n)_n(K^{sh})$) and

$$
\mathfrak{z}_0 = \bigsqcup_{N_i | n} \tilde{E}_i^\circ.
$$

Let $S_\gamma := \{ i \in S | M_i / \gamma = N_i, N_i | n \}$ and let $\mathfrak{z}_\gamma$ the formal completion of $\mathfrak{z}$ along $\bigsqcup_{i \in S_\gamma} \tilde{E}_i^\circ$. Take a point $x$ of $\mathfrak{z}_\gamma$, then by [15, 7.1.10] (see Section 2.3) $x$ corresponds to the equivalence class of $\varphi: \text{Spec} k' \to \mathfrak{z}$ for some finite extension $R'$ of $R(n)$. Then its origin $\varphi_0: \text{Spec} k \to \mathfrak{z}_0$ belongs to $\mathfrak{z}_0 = \bigsqcup_{i \in S_\gamma} \tilde{E}_i^\circ$, where $k'$ is the residue field of $R'$. Then $\varphi_0 \in \tilde{E}_i^\circ$ for some $i \in S, N_i | n$. Let $k_i := \text{ord}_{\varphi_0} y_i$ the order of $y_i$ at $\varphi$. Then

$$
\text{ord}_\varphi(g_i) = M_i k_i \text{ and } \text{ord}_\varphi(f) = N_i k_i = n.
$$

This means that

$$
|g_i(x)| = |\varphi(n)|^{M_i k_i} \text{ and } |f(x)| = |\varphi(n)|^{N_i k_i} = |\varphi(n)|^n.
$$
Therefore, it is easily verified that $\varphi_0 \in \bigsqcup_{i \in S} \tilde{E}_i^{\circ}$ if and only if $\varphi \in X_\gamma(n)$. Hence

$$(3_\gamma)_\eta = X_\gamma(n) \cap 3_\eta.$$  

Moreover, it follows from [3] Theorem 5.1 that $3_\eta(K(n)^{sh}) = X(n)(K(n)^{sh})$ and therefore

$$(3_\gamma)_\eta (K(n)^{sh}) = X_\gamma(n)(K(n)^{sh}).$$

We deduce from Lemma 3.3 that

$$\int_{X_\gamma(n)} |\omega(n)| = \int_{(3_\gamma)_\eta} |\omega(n)| = \left( \sum_{n \geq 1} \left( \sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^{\circ} \right] \left[ \sum_{n \geq 1} \left( \sum_{k_{i} \geq 1, i \in I} \mathbb{L}^{\sum_{i \in I} k_{i} \alpha_{i} T} \sum_{n \in I} k_{i} N_{i} \right) \right] \right) T^{n} \right)^{-d} \sum_{M_{ij} = \gamma_{i} N_{i}} \left[ \tilde{E}_I^{\circ} \right] \mathbb{L}^{-n \alpha_{i} / N_{i}}.$$  

This completes the first statement of the lemma. To prove the second statement we consider the Poincaré series

$$P(X_\gamma, \omega; T) = \sum_{n \geq 1} \int_{X_\gamma(n)} |\omega(n)| T^{n}$$

$$= \mathbb{L}^{-d} \sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^{\circ} \right] \left( \sum_{k_{i} \geq 1, i \in I} \mathbb{L}^{\sum_{i \in I} k_{i} \alpha_{i} T} \sum_{n \in I} k_{i} N_{i} \right) T^{n}$$

Hence, it follows from Corollary 3.11 and (2.2) that

$$\text{MV}(X_\gamma) = -\mathbb{L}^{-d} \lim_{T \to \infty} P(X_\gamma, \omega; T)$$

$$= -\sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^{\circ} \right] \lim_{T \to \infty} \sum_{k \in \Delta_{I, \gamma} \cap \mathbb{N}^{I}} \mathbb{L}^{-\sum_{i \in I} k_{i} \alpha_{i} T} \sum_{n \in I} k_{i} N_{i}$$

$$= -\sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^{\circ} \right] \chi_{E}(\Delta_{I, \gamma}).$$

We now are able to demonstrate that the function $\varphi_g: \mathbb{Q}_{\geq 0}^{\ell} \to \mathcal{M}_{k}^{\mu}$,

$$\varphi_g(\gamma) = \text{MV}(X_\gamma) = -\sum_{I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^{\circ} \right] \chi_{E}(\Delta_{I, \gamma})$$

is constructible. Let us denote by $\ell_{j}(k)$ and $\ell(k)$ the linear maps $\sum_{j} M_{ij} k_{j}$ and $\sum_{j} N_{j} k_{j}$ respectively. We also define the function $\Delta_{I}: \mathbb{R}_{\geq 0}^{\ell} \to \mathbb{R}_{\geq 0}^{\ell}$,

$$\Delta_{I}(k) := \left( \frac{\ell_{1}(k)}{\ell(k)}, \ldots, \frac{\ell_{\ell}(k)}{\ell(k)} \right).$$

Note that, the map $\Delta_{I}$ can be factorized as $\pi \circ L$ where $L: \mathbb{R}^{\ell} \to \mathbb{R}^{\ell+1}$ is the linear map defined as

$$L(k) = (\ell_{1}(k), \ldots, \ell_{\ell}(k), \ell(k)).$$
and \( \pi: \mathbb{R}^r \times \mathbb{R}_{>0} \to \mathbb{R}^r, (x_1, \ldots, x_r, z) \mapsto (\frac{x_1}{z}, \ldots, \frac{x_r}{z}) \). Applying Minkowski-Weyl theorem we can deduce that the image \( L(\mathbb{R}^r_{>0}) \) is a finite Boolean combination of polyhedral cones. Hence the image \( \Delta_I(\mathbb{R}^r_{>0}) \) is constructible. For each \( \gamma \in \mathbb{R}^r \) we consider the matrix
\[
B = (b_{i,j}(\gamma)) \in \text{Mat}(r, |I|; \mathbb{Q})
\]
with \( b_{i,j} = M_{i,j} - \gamma_i N_j \) and the sets
\[
B_n := \{ \gamma \in \mathbb{R}^r \mid \text{rank}(B) = n \}.
\]
Take an element \( j_0 \in I(f) \) and define a new matrix \( B' = (b'_{i,j}) \) as follows:
\[
b'_{i,j}(\gamma) := b_{i,j}(\gamma),
\]
and if \( j \neq j_0 \) then
\[
b'_{i,j}(\gamma) := b_{i,j}(\gamma) - \frac{N_j}{N_{j_0}} b_{i,j_0}(\gamma) = M_{i,j} - \frac{N_j}{N_{j_0}} M_{i,j_0}.
\]
Then it is easily seen that
\[
B_n = \{ \gamma \in \mathbb{R}^r \mid \text{rank}(B') = n \}
\]
and it is constructible. Let \( \{\rho_I\} \) be the constructible partition of \( \mathbb{Q}^r_{>0} \) defined as \( B_n \cap \Delta_I(\mathbb{R}^r_{>0}) \) and \( \mathbb{Q}^r_{>0} \setminus \Delta_I(\mathbb{R}^r_{>0}) \). We define \( \Lambda = \{\rho\} \) to be the intersection of all partitions \( \{\rho_I\} \) with \( I \subset S, I(f) \neq \emptyset \). We see that for every \( \gamma \in B_n \cap \Delta_I(\mathbb{R}^r_{>0}) \) one has
\[
\Delta_I^{-1}(\gamma) = \Delta_{I,\gamma} \approx \mathbb{R}_{>0}^n.
\]
Hence the function \( \varphi_g \) is constant on each set \( \rho \) of the partition \( \Lambda \). This gives the constructability of \( \varphi_g \).

We denote
\[
V := \bigsqcup_{I \subseteq S, I(f) \neq \emptyset} \mathbb{R}^r_{>0}
\]
and define a constructible function
\[
\tilde{\varphi}: V \to \mathbb{M}_{\hat{\mu}},
\]
as
\[
\tilde{\varphi} = \sum_{\emptyset \neq I \subseteq S} (|L - 1||I|^{-1} [\tilde{E}_I^r] 1_{\mathbb{R}^r_{>0}}).
\]
Then, by Corollary 3.11 and the definition of the Euler integral of \( \tilde{\varphi} \)
\[
\text{MV}(X) = \int_V \tilde{\varphi} \, d\chi_c.
\]
We also define an algebraic function
\[
\Delta: V \to \mathbb{R}^r_{\geq 0}
\]
by \( \Delta(k) = \Delta_I(k) \) if \( k := ((k_j)_{j \in I}) \in \mathbb{R}^r_{\geq 0} \). Then
\[
\Delta \star \tilde{\varphi}(\gamma) = \varphi_g(\gamma), \ \forall \gamma \in \mathbb{Q}^r_{>0}.
\]
Hence
\[
\text{MV}(X) = \int_V \tilde{\varphi} \, d\chi_c = \int_{\mathbb{R}^r_{>0}} \Delta \star \tilde{\varphi} \, d\chi_c = \int_{\mathbb{Q}^r_{>0}} \varphi_g \, d\chi_c.
\]
This completes the first equality. To prove the second equality
\[
\int_{\mathbb{Q}^r_{>0}} \varphi_g \, d\chi_c = \int_{\mathbb{Q}^r_{>0}} \varphi_g \, d\chi',
\]

Hence
\[
\Delta \star \tilde{\varphi}(\gamma) = \varphi_g(\gamma), \ \forall \gamma \in \mathbb{Q}^r_{>0}.
\]
it suffices to show that $\Delta(V)$ is bounded in $\mathbb{R}^r_{\geq 0}$. But this is obvious since
\[ \Delta(V) = \bigcup_{f \in S} \Delta_f(\mathbb{R}^r_{\geq 0}) \subset \{ \gamma \in \mathbb{R}^r_{\geq 0} \mid |\gamma| \leq M \}, \]
where
\[ M = \max \left\{ \frac{M_{ij}}{N_j} \mid i \in I, j = 1, \ldots, r \right\} . \]

**Corollary 3.24.** Let $X$ be a smooth special rigid $K$-variety with a model $X$. Let $g = \{g_1, \ldots, g_r\}$ be a system of elements of $\Gamma(X, \mathcal{O}_X)$. For each $\gamma \in \mathbb{Q}_{\geq 0}$ we define the variety
\[ X_\gamma := \left\{ x \in X \mid \max_i |g_i(x)| = |x^\gamma| \right\} . \]
Then the function $\varphi_{|g|} : \mathbb{Q}_{\geq 0} \to \mathcal{M}_k^\mu$ defined as
\[ \varphi_{|g|}(\gamma) = \text{MV}(X_\gamma) \]
is constructible, and moreover,
\[ \text{MV}(X) = \int_{\mathbb{Q}_{\geq 0}} \varphi_{|g|} d\chi_c. \]

**Proof.** We first see that the map $\max : \mathbb{Q}_r^r \to \mathbb{Q}_r$ sending $(x_1, \ldots, x_r)$ to $\max_i |x_i|$ is a piecewise affine linear map. Let $\varphi_g$ be the map defined as in Theorem 3.22. Applying Theorem 3.20 we obtain that the map $\varphi_{|g|} = \max_* (\varphi_g)$ is constructible and
\[ \text{MV}(X) = \int_{\mathbb{Q}_{\geq 0}} \varphi_{g} d\chi_c = \int_{\mathbb{Q}_{\geq 0}} \varphi_{|g|} d\chi_c. \]

The following is a direct consequence of the corollary.

**Corollary 3.25 (Motivic Vanishing Fubini theorem).** Let $X$ be a smooth special rigid $K$-variety with a model $X$. Let $g = \{g_1, \ldots, g_r\}$ be a system of elements of $\Gamma(X, \mathcal{O}_X)$. For each $\gamma \in \mathbb{Q}_{\geq 0}$ we define the variety
\[ X_\gamma := \left\{ x \in X \mid \max_i |g_i(x)| = |x^\gamma| \right\} . \]
If $\text{MV}(X_\gamma) = 0$ for all $\gamma \in \mathbb{Q}_{\geq 0}$, then $\text{MV}(X) = 0$.

### 3.4. Nicaise-Payne’s Motivic Fubini theorem for the tropicalization map.

In this section we prove the Nicaise-Payne’s Motivic Fubini theorem for the tropicalization map [35, Theorem 3.1.3]. The ground field $k$ is assumed to admit all roots of unity. Let us denote by $K'$ the field of Puiseux series $\bigcup_{n>0}k((t^{\frac{1}{n}}))$ and by $R'$ its valuation ring.

Let $X$ be a variety over an algebraic closure $\bar{K}$ of $K$. A semi-algebraic subset $S$ of $X$ is a finite Boolean combination of subsets of $X(\bar{K})$ of the form
\[ \{ x \in U(\bar{K}) \mid \text{val}(f(x)) \leq \text{val}(g(x)) \} \subset X(\bar{K}) \]
where $U$ is an affine open subvariety of $X$ and $f, g$ are regular functions on $U$. If $X$ is of the form $X_0 \times_K \bar{K}$, for some variety $X_0$ over $K$, then we say that $S$ is bfan defined over $K$ if we can write it as a finite Boolean combination of sets of the form such that $U, f$, and $g$ in (3.6).
are defined over $K$. Denoted by $K(\text{VF}_K)$ a certain Grothendieck ring of semi-algebraic sets defined over $K$. Based on the work of Hrushovski and Kazhdan [25], Nicaise and Payne have defined the motivic volume

$$\text{Vol} : K(\text{VF}_K) \to K^\mathbb{A} (\text{Var}_k)$$

satisfies the following properties.

1. Let $X$ be a smooth variety over $K$, and let $X'$ be a smooth $R'$-model of $X' := X \otimes_K K'$ such that the Galois action of $\mathbb{A}$ on $X'$ extends to a good action on $X$. Then $S = X'(R')$ is defined over $K$, and $\text{Vol}([S]) = [X_k]$ in $K^\mathbb{A} (\text{Var}_k)$.

2. Let $\Gamma$ be a constructible subset of $\mathbb{Q}^n$, for some $n \geq 0$, and set $S' = \text{trop}^{-1}(\Gamma)$. Then $S'$ is defined over $K$, and

$$\text{Vol} ([S']) = \chi' (\Gamma)(L - 1)^n$$

in $K^\mathbb{A} (\text{Var}_k)$. Here the tropicalization map trop is defined as

$$\text{trop} : \mathbb{G}^d_{m,K} \to \mathbb{Q}^d : (x_1, \ldots, x_d) \mapsto (\text{val}(x_1), \ldots, \text{val}(x_d)).$$

Let $\mathcal{X}$ be a $R$-scheme of finite type and let $\mathcal{X}$ be the completion of $\mathcal{X}$ along its special fiber $\mathcal{X}_k$. Let $\mathcal{X}_\eta$ be the generic fiber of $\mathcal{X}$. Then $\mathcal{X}$ is a formal $R$-scheme topologically of finite type and $\mathcal{X}$ is a quasi-compact rigid variety.

**Theorem 3.26.** With the above assumption, the equality

$$\text{MV}(\mathcal{X}_\eta) = \text{Vol}([\mathcal{X}(R')])$$

holds in $M^\mathbb{A}_k$.

**Proof.** Let $Y \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$ and let $\mathcal{Y}$ be the completion of $Y$ along its special fiber $\mathcal{Y}_k$. Then the induced morphism $\mathcal{Y} \to \mathcal{X}$ is a resolution of singularities of $\mathcal{X}$. Combining [35, Theorem 2.6.1.] and Corollary 3.11 we obtain

$$\text{MV}(\mathcal{X}_\eta) = \text{MV}(\mathcal{Y}_\eta) = \text{Vol}([\mathcal{Y}(R')]) = \text{Vol}([\mathcal{X}(R')])$$

in $M^\mathbb{A}_k$. □

Applying Theorem 3.22 for the system of coordinate functions $x_1, \ldots, x_d$ one obtains the following motivic Fubini theorem for the tropicalization map. Notice that in [35] the equality (3.7) is proved to hold in the ring $K^\mathbb{A}_0 (\text{Var}_k)$.

**Corollary 3.27** (Nicaise-Payne’s Motivic Fubini theorem for the tropicalization map). Let $Y$ be a variety over $R$. Let $d$ be a positive integer and let $S$ be a semi-algebraic subset of $\mathbb{G}^d_{m,R} \times_R Y$. Denote by

$$\pi : \mathbb{G}^d_{m,K} \times_K Y \to \mathbb{G}^d_{m,K}$$

the projection morphism. Then the function

$$(\text{trop} \circ \pi)_* 1_S : \mathbb{Q}^d \to M^\mathbb{A}_K : w \mapsto \text{Vol} (S \cap (\text{trop} \circ \pi)^{-1}(w))$$

is constructible, and

$$(3.7) \quad \text{Vol}(S) = \int_{\mathbb{Q}^d} (\text{trop} \circ \pi)_* 1_S d\chi'.$$
Proof. We prove only for the case when \( Y = \text{Spec} K \), the proof of the general case is similar. Assume that \( S \) is defined by

\[
\text{val}(f(x)) \leq \text{val}(g(x)),
\]

where \( f, g \in R[x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}] \) as in (3.40). Let \( \mathfrak{x} \) denote the formal spectrum of the ring

\[
R\{x_1, \ldots, x_d, z\}[x_1^{-1}, \ldots, x_d^{-1}]/(zf-g).
\]

Then

\[
X := \mathfrak{x}_\eta = \{ x \in E^d \mid 0 < |x_i| \leq 1 \forall i, |f| \geq |g| \}.\]

Consider the system \( g = \{x_1, \ldots, x_d\} \) in \( \Gamma(\mathfrak{x}, \mathcal{O}_\mathfrak{x}) \) and the function \( \varphi_g : \mathbb{Q}_{\geq 0}^d \to M_\mu^k \) defined as in Theorem 3.22. It follows from Theorem 3.26 that \( MV(X) = \text{Vol}(S) \), and for each \( w \in \mathbb{Q}_{\geq 0}^d \)

\[
MV(X_w) = \text{Vol}(S \cap (\text{trop} \circ \pi)^{-1}(w)),
\]

where

\[
X_w := \{ x \in X \mid |x_i(x)| = |\varpi|^w \}.
\]

Hence the corollary follows from Theorem 3.22.

\[\square\]

3.5. Generalized Poincaré series and Rationality. Consider a smooth special rigid \( K \)-variety denoted as \( X \), equipped with a model \( \mathfrak{x} \). Let \( g = \{g_1, \ldots, g_r\} \) be a system of elements of \( \Gamma(\mathfrak{x}, \mathcal{O}_\mathfrak{x}) \). For each \( \gamma \in \mathbb{Q}_{\geq 0}^r \) we define the varieties

\[
X_\gamma := \{ x \in X \mid \max_i |g_i(x)| = |\varpi|^{\gamma} \}
\]

and

\[
X_{\geq \gamma} := \{ x \in X \mid |g_i(x)| \leq |\varpi|^{\gamma} \}.
\]

Assume that \( X \) admits an \( \mathfrak{x} \)-bounded gauge form \( \omega \). Introduce \( \ell(n, m) \) as a linear form on \( \mathbb{R}^2 \) with the condition \( \ell(n, m) \geq 0 \) for \( m \leq \gamma n \). We proceed to define the following generalized Poincaré series.

\[
P(T) := P(X, \omega, \gamma, \ell; T) := \sum_{0 \leq m \leq \gamma n} \mathbb{L}^{-\ell(n, m)} \int_{X_{m/n}(n)} |\omega(n)| T^n \text{ in } M_\mu^k[T]
\]

**Theorem 3.28.** The series \( P(X, \omega, \gamma, \ell; T) \) is rational, its limit is independent of \( \ell \) and therefore equal to \( -\mathbb{L}^{-d}MV(X_{\geq \gamma}) \).

**Proof.** Let us consider a resolution of singularities \( h : \mathfrak{y} \to \mathfrak{x} \) of the formal \( R \)-scheme \( \mathfrak{x} \) as in the proof of Theorem 3.22. Let \( \mathfrak{e}_i, i \in S, \) be the irreducible components of \( (\mathfrak{y})_{\text{red}} \). Let \( E_i := (\mathfrak{e}_i)_0, E_i^\circ := \cap_{i \in I} E_i \setminus \cup_{j \notin I} E_j \) and let \( \tilde{E}_i^\circ \to E_i^\circ \) be the covering with Galois group \( \mu_{N_i} \) defined locally over \( \mathfrak{u}_0 \cap E_i^\circ \) as in Definition 2.39. Then for each \( y \in E_i^\circ \) there exists an affine neighbourhood \( \mathfrak{u}_I \) such that the following identity holds in \( \Gamma(\mathfrak{u}_I, \mathcal{O}_{\mathfrak{u}_I}) \)

\[
\tilde{f} := h^* f = u \prod_{i \in I} y_i^{N_i}
\]

\[
\bar{g}_l := h^* g_l = u_l \prod_{i \in I} y_i^{M_i^l}, \forall l = 1, \ldots, r,
\]

where \( \tilde{f} \) denotes the structural morphism of \( \mathfrak{x} \), \( y_i \) is a local equation of \( E_i \) at \( y \) and \( u, u_l \) is invertible in \( \Gamma(\mathfrak{u}_I, \mathcal{O}_{\mathfrak{u}_I}) \). The following lemma is proved by using the same argument as in the proof of Lemma 3.23.
Lemma 3.29. Let $\omega$ be a $\mathcal{X}$-bounded gauge form on $X_n$, and put $\alpha_i = \text{ord}_{e_i} \omega$ for each $i \in I$. Then the following identities hold in $\mathcal{M}_k^\mathcal{X}$:

$$
\int_{X_{\geq \gamma}(n)} |\omega(n)| = L^{-d} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [E_I^\varphi] \left( \sum_{k_i \geq 1, i \in I \atop \sum_{i \in I} k_i N_i = n} L^{-\sum_{i \in I} k_i \alpha_i} \right).
$$

Applying the lemma we have

$$
P(T) = \sum_{m \leq \gamma n} L^{-\ell(n,m)} \int_{X_{m/n}(n)} |\omega(n)| T^n
$$

$$
= L^{-d} \sum_{m \leq \gamma n} L^{-\ell(n,m)} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [E_I^\varphi] \left( \sum_{k_i \geq 1, i \in I \atop \sum_{i \in I} k_i N_i = n} L^{-\sum_{i \in I} k_i \alpha_i} \right) T^n
$$

$$
= L^{-d} \sum_{j=1}^r \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [E_I^\varphi] S_{I,j}(T),
$$

where

$$
S_{I,j}(T) = \sum_{k \in \Delta_{I,j}} \prod_{i \in I} (L^{-\alpha'_i, i} T^{N_i})^{k_i} \text{ with } \alpha'_i = \alpha_i + \ell(N_i, M_{il}),
$$

and $\Delta_{I,j}$ is defined inductively as follows

$$
\Delta_I = \{ k = (k_1, \ldots, k_I) \in \mathbb{N}_{>0}^I \mid \min_{i \in I} \sum_{i \in I} M_{il} k_i \leq \gamma \sum_{i \in I} N_i k_i \},
$$

$$
\Delta_{I,j} = \{ k \in \Delta_I \mid \sum_{i \in I} M_{il} k_i \leq \sum_{i \in I} M_{ij} k_i, \forall j \geq 2 \},
$$

$$
\Delta_{I,l} = \{ k \in \Delta_I \mid \sum_{i \in I} M_{il} k_i \leq \sum_{i \in I} M_{il} k_i, \forall j \geq l, \sum_{i \in I} M_{il} k_i < \sum_{i \in I} M_{ij} k_i, \forall j < l \}.
$$

It follows from (2.2) that $\lim_{T \to \infty} S_{I,j}(T) = \chi(\Delta_{I,j})$ and therefore

$$
\lim_{T \to \infty} P(X, \omega, \gamma, \ell; T) = L^{-d} \sum_{j=1}^r \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [E_I^\varphi] \chi(\Delta_{I,j})
$$

$$
= L^{-d} \sum_{\emptyset \neq I \subseteq S} (L - 1)^{|I| - 1} [E_I^\varphi] \chi(\Delta_I),
$$
which is clearly independent of \( \ell \). On the other hand, as computed above for \( \ell = 0 \), we have

\[
P(X, \omega, \gamma, 0; T) = \mathbb{L}^{-d} \sum_{m \leq \gamma n} \sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^0 \right] \left( \sum_{\substack{k_i \geq 1 \in I \in n \in \min_j \sum_{i \in I} k_i M_{ij} = m}} \mathbb{L}^{-\sum_{i \in I} k_i \alpha_i} \right) T^n
\]

\[
= \mathbb{L}^{-d} \sum_{n \geq 1} \sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^0 \right] \left( \sum_{\substack{k_i \geq 1 \in I \in n \in \min_j \sum_{i \in I} k_i M_{ij} \leq \gamma n}} \mathbb{L}^{-\sum_{i \in I} k_i \alpha_i} \right) T^n
\]

\[
= P(X_{\geq \gamma}, \omega; T).
\]

Hence

\[
\lim_{T \to \infty} P(X, \omega, \gamma, \ell; T) = \lim_{T \to \infty} P(X_{\geq \gamma}, \omega; T) = -\mathbb{L}^{-d} \text{MV}(X_{\geq \gamma}).
\]

4. Kontsevich-Soibelman’s Integral Identity Conjecture

In this section, using the theory of motivic integration on rigid varieties developed in the previous section, we prove the following theorem which is known as Kontsevich-Soibelman’s Integral Identity Conjecture (27).

**Theorem 4.1.** Let \( f \in \mathbb{k}[x, y, z] \) with \( x = (x_1, \ldots, x_{d_1}), y = (y_1, \ldots, y_{d_2}) \) and \( z = (z_1, \ldots, z_{d_3}) \) be a formal power series such that \( f(tx, y, z) = f(x, ty, z) \) in \( \mathbb{k}[x, y, z, t] \). Then \( f \) is a series in \( \mathbb{k}\{x\}[y, z] \) and the identity

\[
(4.1) \quad \int_{\mathbb{k}^{d_1}} S_f = \mathbb{L}^{d_1} S_{f,0}
\]

holds in \( \mathcal{M}^{\bar{\alpha}}_k \), where \( \tilde{f}(z) = f(0,0, z) \in \mathbb{k}[z] \).

**Proof.** Let \( \mathfrak{X}(f) \) and \( \mathfrak{X}(\tilde{f}) \) be the formal \( \mathcal{R} \)-schemes associated to \( f \) and \( \tilde{f} \) respectively. Then \( \mathfrak{X}(f)_0 \cong \mathbb{k}^{d_1} \) and \( \mathfrak{X}(\tilde{f})_0 = \text{Spec} \). By (2.5),

\[
\mathfrak{X}(f) \cong \text{Spf} \left( \mathcal{R} \{x\}[y, z]/(f - \varpi) \right) \quad \text{and} \quad \mathfrak{X}(\tilde{f}) \cong \text{Spf} \left( \mathcal{R}[z]/(\tilde{f} - \varpi) \right).
\]

It follows from Definition 2.10 that

\[
S_f = \text{MV}(\mathfrak{X}(f)) \in \mathcal{M}^{\bar{\alpha}}_{\mathbb{k}^{d_1}} \quad \text{and} \quad S_{f,0} = \text{MV}(\mathfrak{X}(\tilde{f})) \in \mathcal{M}^{\bar{\alpha}}_k.
\]

Hence, the conjecture is equivalent to

\[
(4.2) \quad \text{MV}(\mathfrak{X}(f)) = \mathbb{L}^{d_1} \text{MV}(\mathfrak{X}(\tilde{f})) \quad \text{in} \quad \mathcal{M}^{\bar{\alpha}}_k.
\]

Note that \( \mathfrak{X}(f) \) is a closed analytic subvariety of \( \mathcal{B}^{d_1} \times D^{d_2 + d_3} = (\text{Spec} \mathcal{R}\{x\}[y, z])_\eta \);

\[
\mathfrak{X}(f) = \left\{ u = (x, y, z) \in \mathcal{B}^{d_1} \times D^{d_2 + d_3} \mid f(u) = \varpi \right\}.
\]
Lemma 4.2. Let $X$ be the rigid $K$-variety defined by
$$X := \left\{ u = (x, y, z) \in B^{d_1} \times D^{d_2 + d_3} \mid |f(u) - \bar{\omega}| < |\omega| \right\}.$$ Then $MV(X) = MV(\mathfrak{X}(f)_\eta)$ in $M^\mu_k$.

Proof. Observe that $X$ is an affine special smooth rigid varieties with a coordinate ring
$$A = R\{x\}[y, z, s]/(f(x, y, z) - \bar{\omega} - \omega s) = R\{x\}[y, z, s]/(g(x, y, z, s) - \bar{\omega})$$
where $g(x, y, z, s) = f(x, y, z)/(1 + s)$. Since $1 + s$ admits at least one $n$-root for all $n \geq 1$, it follows from Proposition 2.17 that
$$S_f = S_g \text{ in } M^\mu_{k, d_1}$$
and hence $MV(X) = MV(\mathfrak{X}(\tilde{f})_\eta)$ in $M^\mu_k$, where $\tilde{f} = f$ in $R\{x\}[y, z, s]$. Moreover, there is an isomorphism of rigid varieties
$$\mathfrak{X}(\tilde{f})_\eta \cong \mathfrak{X}(f)_\eta \times D^1,$$
which is induced by the natural isomorphism
$$R\{x\}[y, z, s]/(\tilde{f}(x, y, z, s) - \bar{\omega}) \cong R\{x\}[y, z]/(f(x, y, z) - \omega) \otimes_R R[s].$$
It follows that
$$MV(X) = MV(\mathfrak{X}(\tilde{f})_\eta) = MV(\mathfrak{X}(f)_\eta) \cdot MV(D^1) = MV(\mathfrak{X}(f)_\eta)$$
in $M^\mu_k$. □

We now decompose $X$ into special rational subdomains $X_0, X'$
$$X_0 := \{ u \in X \mid |x(u)| \cdot |y(u)| < |\omega| \}$$
and
$$X' := \{ u \in X \mid |x(u)| \cdot |y(u)| \geq |\omega| \}.$$ Here $|x(u)| := \max \{|x_i(u)| \mid i = 1, \ldots, d_1\}$ and similarly for $|y(u)|$.

Lemma 4.3. The identity
$$MV(X_0) = L^{d_1} MV(\mathfrak{X}(\tilde{f})_\eta)$$
holds in $M^\mu_k$.

Proof. We write
$$f(x, y, z) = \sum_{|\alpha| = |\beta| > 0} a_{\alpha, \beta, \gamma} x^\alpha y^\beta z^\gamma + \tilde{f}(z).$$
Then, by denoting $u := (x, y, z)$,
$$X_0 = \left\{ u \in B^{d_1} \times D^{d_2 + d_3} \mid |f(u) - \bar{\omega}| < |\omega|, |x| \cdot |y| < |\omega| \right\}$$
$$= \left\{ u \in B^{d_1} \times D^{d_2 + d_3} \mid |\tilde{f}(z) - \bar{\omega}| < |\omega|, |x| \cdot |y| < |\omega| \right\}$$
$$\cong \left\{ (x, y) \in B^{d_1} \times D^{d_2} \mid |x| \cdot |y| < |\omega| \right\} \times \left\{ z \in D^{d_3} \mid |\tilde{f}(z) - \bar{\omega}| < |\omega| \right\}.$$ Applying Lemma 4.3 to $\tilde{f}$, we obtain that
$$MV\left( \left\{ z \in D^{d_3} \mid |\tilde{f}(z) - \bar{\omega}| < |\omega| \right\} \right) = MV(\mathfrak{X}(\tilde{f})_\eta).$$
It suffices to prove that \( \text{MV}(V) = \mathbb{L}^{d_1} \) with
\[
V := \left\{ (x, y) \in B^{d_1} \times D^{d_2} \mid |x| : |y| < |\varpi| \right\}.
\]
In fact, \( V = V_1 \sqcup V_2 \) where
\[
V_1 = \left\{ (x, y) \in V \mid |x| \leq |\varpi| \right\} \quad \text{and} \quad V_2 = \left\{ (x, y) \in V \mid |x| > |\varpi| \right\}.
\]
We see that
\[
V_1 \cong B^{d_1}(0, |\varpi|) \times D^{d_2}
\]
so \( \text{MV}(V_1) = \mathbb{L}^{d_1} \). To compute \( \text{MV}(V_2) \) we decompose \( V_2 \) into special rational subdomains
\[
V_2 = \bigcup_{i=1}^{d_1} W_i,
\]
with \( W_1 := \{ u \in V_2 \mid |x| = |x_1| \} \) and for \( 1 < i \leq d_1 \),
\[
W_i := \left\{ (x, y) \in V_2 \mid |x(u)| = |x_i| > |x_j|, \forall j < i \right\}.
\]
It is seen that the morphism \( W_1 \to W_i^1 \times \{ \xi \in B^1 \mid |\xi| > |\varpi| \} \) which sends \( (x, y) \) to \( (x_1^{-1}x, x_1y, x_1) \) is an isomorphism, where
\[
W_i^1 := \{ u \in V_2 \mid x_1 = 1 \}.
\]
Therefore
\[
\text{MV}(W_1) = \text{MV}(W_1^1) \cdot \text{MV} \left( \{ \xi \in B^1 \mid |\xi| > |\varpi| \} \right) = 0.
\]
Using the same argument we may prove that \( \text{MV}(W_i) = 0 \) for all \( 1 \leq i \leq d_1 \), and hence
\[
\text{MV}(V_2) = \sum_{i=1}^{d_1} \text{MV}(W_i) = 0.
\]
This completes the lemma. \( \Box \)

**Lemma 4.4.** *The identity*

\[
\text{MV}(X') = 0
\]

*holds in \( \mathcal{M}^k \).*

**Proof.** We decompose \( X \) into special rational subdomains \( X = X^{(1)} \sqcup \ldots \sqcup X^{(d_1)} \) where
\[
X^{(1)} := \{ u \in X \mid |x(u)| = |x_1(u)| \} \quad \text{and} \quad \text{for } 1 < i \leq d_1,
\]
\[
X^{(i)} := \{ u \in X \mid |x(u)| = |x_i(u)| > |x_j(u)|, \forall j < i \}.
\]
Let us consider the system of elements \( \{ g = (x_iy_j), i = 1, \ldots, d_1, j = 1, \ldots, d_2 \} \) in \( A = \Gamma(X, \mathcal{O}_X) \).
\[
X'^{(i)} := X^{(i)} \cap X' = \left\{ u \in X^{(i)} \mid |g(u)| \geq |\varpi| \right\}.
\]
Then, for every \( \gamma \in \mathbb{Q} \), the set
\[
X'^{(i)} := \left\{ u \in X'^{(i)} \mid |g(u)| = |\varpi|^{\gamma} \right\}
\]
is empty for all \( \gamma > 1 \). Assume that \( \gamma \leq 1 \), then there is an isomorphism of rigid varieties
\[
X'^{(i)} \cong Y'^{(i)} \times \{ \xi \in B^1 \mid |\xi| > |\varpi|^{\gamma} \},
\]
sending \( (x, y, z) \) to \( (x_1^{-1}x, x_1y, x_1), \) where
\[
Y'^{(i)} := \{ u \in X'^{(i)} \mid x_i(u) = 1 \}.
\]
Since $\text{MV}\left(\{\xi \in B^1 \mid |\xi| > |\varpi|^\gamma\}\right) = 0$ (see, Example 3.16), it follows that $\text{MV}(X'^{(i)}) = 0$ for all $i$. Applying the vanishing Fubini theorem (Corollary 3.25) we have $\text{MV}(X'^{(i)}) = 0$ for all $i$, and hence

$$\text{MV}(X') = 0.$$ 

The lemma follows. □

Combining Lemmas 4.2, 4.3 and 4.4 we obtain the following identities in $\mathcal{M}^\mu_k$

$$\text{MV}(\mathcal{X}(f)_{\eta}) = \text{MV}(\mathcal{X}(X) = \text{MV}(X_0) + \text{MV}(X') = L^{d_1}\text{MV}(\mathcal{X}(\tilde{f})_{\eta}),$$

which give (4.2) and hence the theorem. □

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