WHITEHEAD GROUPS AND THE BASS CONJECTURE

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Abstract. This paper will be concerned with proving that certain Whitehead groups of torsion-free elementary amenable groups are torsion groups and related results, and then applying these results to the Bass conjecture. In particular we shall establish the strong Bass conjecture for an arbitrary elementary amenable group.

1. Introduction

Let $k$ be a field, let $\Gamma$ be a group, and let $\text{Units}^*(k\Gamma)$ denote the subgroup of $\text{Units}(k\Gamma)$ consisting of those units in $k\Gamma$ of the form $a\gamma$ with $a \in k \setminus 0$ and $\gamma \in \Gamma$. Then we shall use the notation $\text{Wh}^k(\Gamma)$ for the quotient of $K_1(k\Gamma)$ by the image of $\text{Units}^*(k\Gamma)$ under the natural homomorphism $\text{Units}^*(k\Gamma) \to K_1(k\Gamma)$. All rings will have a unity element $1$, and a ring of prime characteristic will mean a ring such that $p1 = 0$ for some prime $p$. Let $\mathcal{T}$ indicate the class of torsion abelian groups. Recall that the class of elementary amenable groups is the smallest class of groups which contains $\mathbb{Z}$ and all finite groups, and is closed under taking extensions and directed unions. Our first result is

Theorem 1.1. Let $k$ be a field of prime characteristic and let $\Gamma$ be a torsion-free elementary amenable group. Then $\text{Wh}^k(\Gamma) \in \mathcal{T}$.

Let $\hat{K}_0(R)$ denote the reduced projective class group of the ring $R$, that is $K_0(R)/\langle [R] \rangle$. Then in the situation of Theorem 1.1 it now follows from Bass’s contracted functor argument [5, Chapter XII, §7] that $\hat{K}_0(k\Gamma) \in \mathcal{T}$ and $K_i(k\Gamma) \in \mathcal{T}$ for all $i < 0$, where $k$ and $\Gamma$ are as in Theorem 1.1. However we shall prove the following more general result.

Theorem 1.2. Let $k$ be a field of prime characteristic and let $\Gamma$ be an elementary amenable group. Then $K_0(k\Gamma) \otimes \mathbb{Q}$ is generated by the images of $K_0(kG)$ as $G$ runs over the finite subgroups of $\Gamma$.

It is likely that Theorems 1.1 and 1.2 are also true for fields $k$ of characteristic 0. However our proofs depend on Lemma 2.2 which shows that for rings of prime characteristic, certain nil groups are torsion groups. This is certainly false for arbitrary rings, so our proof does not cover the case when $k$ has characteristic 0.

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Given a prime power $q$, we shall let $k_q$ indicate the field with $q$ elements. In the case $k = k_2$, we shall give a second proof of Theorem 1.2 in Section 7. This alternative proof will yield other results; for example we obtain the following variation of [20, Theorem 1.1].

**Theorem 1.3.** Let $G$ be a torsion-free virtually solvable subgroup of $GL_n(\mathbb{C})$. Then $\text{Wh}^{k_2}(G) = 0$.

We shall also prove (a 2-group is a group in which every element has order a power of 2)

**Proposition 1.4.** Let $G$ be an abelian 2-group. Then $\tilde{K}_0(k_2[G \wr \mathbb{Z}]) = 0$.

For the special case $G = C_2$ in Proposition 1.4, the group is often called the lamplighter group. Many interesting properties of this group are established in [23].

We shall apply Theorem 1.2 to obtain results on Bass’s strong conjecture [6, 4.5]. For convenience, we will restate the conjecture here (see Section 2 for the explanation of some standard notation used below).

**Conjecture 1.5** (The strong Bass conjecture). Let $k$ be an integral domain, let $\Gamma$ be a group, let $g \in \Gamma$, and let $P$ be a finitely generated projective $k\Gamma$-module. Suppose $o(g)$ is not invertible in $k$. Then $r_P(g) = 0$.

Some cases for which Conjecture 1.5 is known to be true are

1. $k = \mathbb{C}$ and $G$ a linear group [6, Proposition 6.2],
2. $k = \mathbb{Z}$ and $o(g) < \infty$ [27, Lemma 4.1],
3. $k = \mathbb{Q}$ and $G$ has cohomological dimension over $\mathbb{Q}$ at most two [10, Theorem 3.3].

We shall prove

**Theorem 1.6.** The strong Bass conjecture is true for elementary amenable groups. More precisely, let $\Gamma$ be an elementary amenable group, let $k$ be an integral domain, and let $P$ be a finitely generated projective $k\Gamma$-module. If $g \in \Gamma$ and $o(g)$ is not invertible in $k$, then $r_P(g) = 0$.

While revising this paper, we have learned that Berrick, Chatterji and Mislin [7] have proved the strong Bass conjecture for amenable groups in the case $k = \mathbb{C}$. In fact they prove a rather stronger version for the Banach space $\ell^1(G)$; on the other hand their results do not apply to the case when $k$ has nonzero characteristic. Their techniques are rather different from ours, and depend on recent work of Vincent Lafforgue [26].

One of many interesting results which Gerald Cliff proved in his important paper [8] was [8, Theorem 1], that if $k$ is a field of nonzero characteristic $p$ and $\Gamma$ is a polycyclic-by-finite group with the property that all finite subgroups have $p$-power order, then $k\Gamma$ has no nontrivial idempotents. We will use Theorem 1.2 and the techniques of Section 5 to extend Cliff’s result to elementary amenable groups.

**Theorem 1.7.** Let $p$ be a prime, let $k$ be an integral domain of characteristic $p$, and let $\Gamma$ be an elementary amenable group. Suppose every finite subgroup of $\Gamma$ has $p$-power order. Then $k\Gamma$ has no nontrivial idempotents.

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2. Preliminary results

Notation. All modules will be right modules and mappings will be written on the left. For each positive integer \( n \), we shall use the notation \( C_n \) for the cyclic group of order \( n \), \( \text{Mat}_n(R) \) for the \( n \times n \) matrices over the ring \( R \), \( \text{Mat}(R) \) for \( \bigcup_{n=1}^{\infty} \text{Mat}_n(R) \), and \( \mathcal{A} \) for the class of finitely generated virtually abelian groups. Also \( G \wr A \) will denote the restricted wreath product of the groups \( G \) and \( A \); thus \( G \wr \mathbb{Z} \) will have base group \( \bigoplus_{t \in \mathbb{Z}} G \) and quotient group \( \mathbb{Z} \). We shall let \( o(g) \) denote the order of the element \( g \in G \). In the case \( o(g) = \infty \), we shall adopt the convention that \( o(g) \) is not invertible in any ring. A \( T \)-exact sequence will mean a sequence which is exact modulo torsion abelian groups; in other words every element of the homology group at each stage of the sequence has finite order. Similarly a \( T \)-epimorphism means a homomorphism which is onto modulo torsion. Suppose \( \alpha \) is an automorphism of the ring \( R \). Then \( R_n[t] \) will denote the twisted polynomial ring over \( R \), and \( R_n[t, t^{-1}] \) will denote the twisted Laurent polynomial ring over \( R \). Following [14] §2, we define \( C(R, \alpha) \) to be the category whose objects are pairs \((P, \phi)\) where \( P \) is a finitely generated projective \( R \)-module and \( \phi \) is an \( \alpha \)-linear nilpotent endomorphism of \( P \), and whose morphisms \( g : (P_1, \phi_1) \to (P_2, \phi_2) \) are \( R \)-linear homomorphisms \( g : P_1 \to P_2 \) with \( \phi_1 = \phi_2 g \). Then we shall let \( C(R, \alpha) = K_0(C(R, \alpha)) / \langle ([R, 0]) \rangle \), and \( \tilde{C}(R, \alpha) \) denote the subgroup of \( C(R, \alpha) \) generated by elements of the form \([([R^n, \phi]) \). We remark that \( \tilde{C}(R, \alpha) \) is isomorphic to the subgroup of \( K_1(R_n[t]) \) generated by elements which are represented by matrices in \( \text{Mat}(R_n[t]) \) of the form \( I + Nt \), where \( I \) is the identity matrix and \( N \in \text{Mat}(R) \) such that \( Nt \) is nilpotent; this can be seen from [14] proof of Theorem 13 and Proposition 20]. If \( \beta \) is an automorphism of the group \( \Gamma \), then we shall let \( \beta \) also indicate the automorphism of \( k\Gamma \) induced by \( \beta \).

We need a standard description (cf. [25] §3) of elementary amenable groups in order to carry out induction arguments. If \( \mathcal{X} \) and \( \mathcal{Y} \) are classes of groups, then \( G \in \mathcal{X} \mathcal{Y} \) will mean that \( G \) has a normal subgroup \( H \) such that \( H \in \mathcal{X} \) and \( G/H \in \mathcal{Y} \), and \( G \in L\mathcal{X} \) will mean that every finitely generated subgroup of \( G \) is contained in a \( \mathcal{X} \)-group (if \( \mathcal{X} \) is closed under taking subgroups, this is equivalent to saying that every finitely generated subgroup of \( G \) is an \( \mathcal{X} \)-group). For each ordinal \( \alpha \), we define \( \mathcal{X}_\alpha \) inductively as follows.

\[
\begin{align*}
\mathcal{X}_0 &= \text{the class of finite groups}; \\
\mathcal{X}_\alpha &= (L\mathcal{X}_{\alpha-1}) \mathcal{A} \text{ if } \alpha \text{ is a successor ordinal}; \\
\mathcal{X}_\alpha &= \bigcup_{\beta<\alpha} \mathcal{X}_\beta \text{ if } \alpha \text{ is a limit ordinal}.
\end{align*}
\]

Then the proof of [25] Lemma 3.1] (see also [28] Lemma 4.9] yields the following.

Lemma 2.1.  
(i) The class of elementary amenable groups is \( \bigcup_{\alpha \geq 0} \mathcal{X}_\alpha \);  
(ii) Each \( \mathcal{X}_\alpha \) is closed under taking subgroups;  
(iii) If \( H \triangleleft G \) with \( G/H \) finite and \( H \in \mathcal{X}_\alpha \) or \( L\mathcal{X}_\alpha \), then \( G \in \mathcal{X}_\alpha \) or \( L\mathcal{X}_\alpha \) respectively.

Lemma 2.2. Let \( p \) be a prime, let \( R \) be a ring such that \( p1 = 0 \), and let \( \alpha \) be an automorphism of \( R \). Then \( \tilde{C}(R, \alpha) \) is an abelian \( p \)-group.

Proof. Let \( t \) be an indeterminant and let \( N \in \text{Mat}(R) \) such that \( (Nt)^n = 0 \) for some positive integer \( n \). Then \( I + Nt \) represents an element of \( K_1(R_n[t]) \) and we need to prove that this element has finite order. Now for \( M \in \text{Mat}(R_n[t]) \), the
binomial formula shows that \((I + M)^p = I + Mp\) because \(pM = 0\) and \(p\) divides \(\binom{p}{i}\) when \(0 < i < p\). This equation by repetition yields
\[
(I + M)^{p^n} = I + M^{p^n}.
\]
Substituting into this equation \(M = Nt\) and observing that \((Nt)^{p^n} = 0\) because \(p^n \geq n\), we obtain \((I + Nt)^{p^n} = I\). Hence \(\tilde{C}(R, \alpha)\) is generated by elements of \(p\)-power order and the result follows.

\[\square\]

**Lemma 2.3.** Let \(R\) be a ring of prime characteristic and let \(\alpha\) be an automorphism of \(R\). Then there is a natural \(\mathcal{T}\)-exact sequence
\[
K_1(R) \longrightarrow K_1(R_\alpha[t, t^{-1}]) \longrightarrow K_0(R) \xrightarrow{1-\alpha} K_0(R).
\]

**Proof.** This follows immediately from [14, Theorem 19c] and Lemma 2.2. \(\square\)

**Remark 2.4.** The constant group homomorphism \(\mathbb{Z}^n \rightarrow 1\) induces a ring homomorphism
\[
R[\mathbb{Z}^n] \rightarrow R[1] = R
\]
which splits the inclusion ring homomorphism \(R \rightarrow R[\mathbb{Z}^n]\). Hence \(K_i(R[\mathbb{Z}^n])\) is naturally a direct sum of \(K_i(R)\) and the complementary summand \(\hat{K}_i(R[\mathbb{Z}^n])\) which is the kernel of the homomorphism induced by the above splitting. The Bass-Heller-Swan formula, which is the fundamental theorem of algebraic \(K\)-theory, asserts that
\[
\hat{K}_1(R[\mathbb{Z}]) \cong K_0(R) \oplus C(R, \text{id}) \oplus C(R, \text{id}).
\]
In his contracted functor theory Bass used this formula to define the lower algebraic \(K\)-groups \(\hat{K}_n(R)\) for \(n > 0\), so that they are direct summands of \(\hat{K}_0(R[\mathbb{Z}^n])\); cf. [5, Chapter XII, §7].

**Corollary 2.5.** Let \(k\) be a field of prime characteristic and let \(\Gamma = \pi \times \mathbb{Z}\) be a group such that \(\text{Wh}^k(\pi \times \mathbb{Z}) \in \mathcal{T}\) for all \(n \geq 0\). Then \(\text{Wh}^k(\Gamma \times \mathbb{Z}) \in \mathcal{T}\) for all \(n \geq 0\).

**Proof.** Note that [14, Theorem 21d] remains true with \(k\) in place of \(\mathbb{Z}\) and \(\text{Wh}^k\) in place of \(\text{Wh}\). So applying this result, Lemma 2.2 and the Bass-Heller-Swan formula, we obtain the exact sequence
\[
0 = \text{Wh}^k(\pi \times \mathbb{Z}^n) \otimes \mathbb{Q} \longrightarrow \text{Wh}^k(\Gamma \times \mathbb{Z}^n) \otimes \mathbb{Q} \longrightarrow \hat{K}_0(k[\pi \times \mathbb{Z}^n]) \otimes \mathbb{Q} \subseteq \text{Wh}^k(\pi \times \mathbb{Z}^{n+1}) \otimes \mathbb{Q} = 0.
\]
This proves \(\text{Wh}^k(\Gamma \times \mathbb{Z}) \in \mathcal{T}\) for all \(n \geq 0\), as required. \(\square\)

**Corollary 2.6.** Let \(R\) be a ring of prime characteristic and let \(\alpha\) be an automorphism of \(R\). Suppose the natural map \(K_0(R) \rightarrow K_0(R[s, s^{-1}])\) is a \(\mathcal{T}\)-epimorphism. Then the natural map \(K_0(R) \rightarrow K_0(R_\alpha[t, t^{-1}])\) is also a \(\mathcal{T}\)-epimorphism.
Proof. Consider the following commutative diagram

\[
\begin{array}{ccc}
K_0(R) & \longrightarrow & K_0(R_n[t, t^{-1}]) \\
\uparrow & & \uparrow \\
K_1(R[s, s^{-1}]) & \longrightarrow & K_1(R_n[s, s^{-1}, t, t^{-1}]) & \longrightarrow & K_0(R[s, s^{-1}]) & \xrightarrow{1-\alpha} & K_0(R[s, s^{-1}]) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K_1(R) & \longrightarrow & K_1(R_n[t, t^{-1}]) & \longrightarrow & K_0(R) & \xrightarrow{1-\alpha} & K_0(R)
\end{array}
\]

The squares in this diagram all commute and the 3 pairs of vertical arrows going in the opposite directions are splittings. The two horizontal sequences are \(T\)-exact by Lemma 2.3. Also in each of the first two columns, the composite of the two up vertical arrows is \(0\). A simple diagram chase using the fact that the marked up vertical arrow is a \(T\)-epi yields the result. \(\square\)

Corollary 2.7. Let \(R\) be a ring of prime characteristic, let \(\alpha\) be an automorphism of \(R\), and suppose that the natural map \(K_0(R) \rightarrow K_0(R[\mathbb{Z}^n])\) is a \(T\)-epimorphism for all \(n\). Then \(K_i(R) \in T\) and \(K_i(R_n[t, t^{-1}][\mathbb{Z}^n]) \in T\) for all \(i < 0\), and \(K_0(R) \rightarrow K_0(R_n[t, t^{-1}][\mathbb{Z}^n])\) is a \(T\)-epimorphism, for all \(n\).

Proof. We use Bass’s contracted functor theory to deduce this; in particular, we use Remark 2.4. By our assumption \(K_0(R[\mathbb{Z}^n]) \in T\) and hence \(K_i(R) \in T\) for all \(i < 0\) because \(K_i(R) \subseteq K_0(R[\mathbb{Z}^n])\) where \(n = -i\). Since

\[K_0(R[\mathbb{Z}^n]) \rightarrow K_0(R[\mathbb{Z}^{n+1}]) = K_0(R[\mathbb{Z}^n][s, s^{-1}])\]

is clearly a \(T\)-epimorphism for all \(n \geq 0\), \(f_1: K_0(R[\mathbb{Z}^n]) \rightarrow K_0(R_n[t, t^{-1}][\mathbb{Z}^n])\) is also a \(T\)-epimorphism by Corollary 2.6. Consequently so is \(f_2: K_0(R) \rightarrow K_0(R_n[t, t^{-1}][\mathbb{Z}^n])\) since it is the composite of the two \(T\)-epimorphisms \(K_0(R) \rightarrow K_0(R[\mathbb{Z}^n])\) and \(f_1\).

Setting \(n = m + (-i)\), where \(m \geq 0\) and \(0 > i\) are given, we see that

\[f_3: K_0(R_n[t, t^{-1}][\mathbb{Z}^m]) \rightarrow K_0((R_n[t, t^{-1}][\mathbb{Z}^m])[\mathbb{Z}^n]),\]

is also a \(T\)-epimorphism since \(f_2 = f_3 \circ f_4\) where \(f_4: K_0(R) \rightarrow K_0(R_n[t, t^{-1}][\mathbb{Z}^m])\). Therefore \(K_0(S[\mathbb{Z}^n]) \in T\) where \(S = R_n[t, t^{-1}][\mathbb{Z}^m]\). But \(K_i(S)\) is a direct summand of \(K_0(S[\mathbb{Z}^n])\), consequently \(K_i(R_n[t, t^{-1}][\mathbb{Z}^m]) \in T\). \(\square\)

Corollary 2.8. Let \(k\) be a field of prime characteristic, let \(\Gamma = \pi \rtimes \mathbb{Z}\) be a group such that for all \(n \geq 0\), \(K_0(k[\pi \times \mathbb{Z}^n]) \otimes \mathbb{Q}\) is generated by the images of \(K_0(kG)\) as \(G\) varies over the finite subgroups of \(\pi\). Then for all \(i < 0\) and \(n \geq 0\), it follows that \(K_i(k\pi) \in T\) and \(K_i(k[\Gamma \times \mathbb{Z}^n]) \in T\), and \(K_0(k[\Gamma \times \mathbb{Z}^n]) \otimes \mathbb{Q}\) is generated by the images of \(K_0(kG)\) as \(G\) varies over the finite subgroups of \(\Gamma\).

Proof. Let \(\alpha\) denote the automorphism of \(\pi\) determined by the conjugation action of \(\mathbb{Z}\) on \(\pi\), and let \(R = k\pi\). Note that

\[R[\mathbb{Z}^n] = k[\pi \times \mathbb{Z}^n]\]

and \(R_n[t, t^{-1}][\mathbb{Z}^n] = k[\Gamma \times \mathbb{Z}^n]\).

Thus the natural map \(K_0(R) \rightarrow K_0(R[\mathbb{Z}^n])\) is \(T\)-surjective, and we can now obtain the result from Corollary 2.7. \(\square\)
Lemma 2.9. Let \( n \geq 0 \), let \( G \) be a finite group, and let \( k \) be a field. Then \( K_i(\pi G) = 0 \) for all \( i < 0 \) and the inclusion \( G \hookrightarrow G \times \mathbb{Z}^n \) induces an isomorphism \( K_0(\pi G) \to K_0(k[G \times \mathbb{Z}^n]) \).

Proof. Let \( J \) denote the Jacobson radical of \( kG \). Since \( kG \) is an Artinian ring, \( J \) is nilpotent \([24, \text{Corollary 1, p. 39}]\) and \( kG/J \) is a semisimple Artinian ring \([24, \S \text{III.3}]\), and in particular \( kG/J \) is a regular ring. Then we have a commutative diagram

\[
\begin{array}{ccc}
K_0(kG) & \longrightarrow & K_0(k[G \times \mathbb{Z}^n]) \\
\approx & & \approx \\
K_0(kG/J) & \longrightarrow & K_0(k[G \times \mathbb{Z}^n]/J[\mathbb{Z}^n]).
\end{array}
\]

The two vertical arrows are isomorphisms by \([40, \text{Lemma II.2.2}]\) and the bottom horizontal line is an isomorphism by the Bass-Heller-Swan formula, hence the top arrow is also an isomorphism as required and the second statement is proven.

A consequence of the second statement is that \( \hat{K}_0(S[\mathbb{Z}^{-i}]) = 0 \) where \( S = kG \) and \( i < 0 \). Hence \( K_i(kG) = 0 \) since it is a direct summand of \( \hat{K}_0(kG[\mathbb{Z}^{-i}]) \). \( \square \)

Lemma 2.10. Let \( L \in \mathcal{A} \). Then \( L \) is isomorphic to a discrete cocompact subgroup of a virtually connected Lie group.

Proof. There clearly exists a non-negative integer \( n \) such that \( L \) is an extension of a finite group \( F \) by the free abelian group \( \mathbb{Z}^n \), where \( \mathbb{Z}^n \) denotes the integral lattice points in the additive group \( \mathbb{R}^n \). This extension determines an action of \( F \) on \( \mathbb{Z}^n \) and hence on \( \mathbb{R}^n \), and a cohomology class \( \theta \in H^2(F, \mathbb{Z}^n) \). Let \( \theta' \in H^2(F, \mathbb{R}^n) \) be the image of \( \theta \) and let \( G \) be the extension of \( F \) by \( \mathbb{R}^n \) determined by \( \theta' \). Then \( G \) is a virtually connected Lie group containing \( L \) as a discrete cocompact subgroup. (Note that \( \theta' = 0 \) and hence \( G = \mathbb{R}^n \rtimes F \).) \( \square \)

The subclass of \( \mathcal{A} \) consisting of the virtually cyclic groups is of particular importance to us. There is fortunately the following quite useful structure theorem for this subclass due to Scott and Wall \([36]\); cf. \([19, \text{Lemma 2.5}]\) for another proof.

Proposition 2.11. A virtually cyclic group \( \Gamma \) contains a finite normal subgroup \( F \) such that \( \Gamma/F \) is either trivial, infinite cyclic, or infinite dihedral.

3. WHITEHEAD GROUPS OF ELEMENTARY AMENABLE GROUPS

Theorem 3.1. Let \( k \) be a field of prime characteristic, and let \( \pi \triangleleft \Gamma \) be groups such that \( \Gamma/\pi \) is a crystallographic group. Suppose \( \text{Wh}^k(\pi \times \mathbb{Z}^n) \in \mathcal{T} \) for all non-negative integers \( n \) whenever \( \pi/\pi \) is a finite subgroup of \( \Gamma/\pi \). Then \( \text{Wh}^k(\Gamma) \in \mathcal{T} \).

Proof. Our proof of Theorem 3.1 follows the pattern established in \([16, 33]\). In these papers it was shown that \( \text{Wh}^R(\Gamma) = 0 \) for \( \Gamma \) a torsion-free virtually poly-\( \mathbb{Z} \) group and \( R \) any subring of \( \mathbb{Q} \). The case \( R = \mathbb{Z} \) was done in \([16]\) and the general case in \([33]\). Quinn had to develop important new geometric algebra concepts to do the general case, and these concepts are crucial in our proof of Theorem 3.1.

Notation 3.2. Let \( L \) be a crystallographic group. Then \( L \) is isomorphic to a discrete cocompact subgroup of the group of all rigid motions

\[ \text{Iso}(\mathbb{Z}^n) \cong \text{Orthog}(n) \rtimes \mathbb{R}^n \]
of some Euclidean space $\mathbb{E}^n$. The number $n$ is the rank of a torsion-free abelian subgroup of finite index in $L$, and is called the dimension of $L$ or $\dim L$. Also the image of $L$ in Orthog($n$) is a finite group $G$, called the holonomy group of $L$, and its isomorphism class is determined by $L$. The order of $G$ is called the holonomy number of $L$ and is denoted by $\#(L)$. If $S$ is a subgroup of finite index in $L$, then $S$ is also crystallographic. Furthermore if $S = L$ and the holonomy group $G_1$ of $S$ is isomorphic to a subgroup of $G$, hence

- If $G$ is cyclic, then so is $G_1$.
- $\#(S) \leq \#(L)$.

Let $L = \Gamma/\pi$ where $\Gamma$ and $\pi$ come from the statement of Theorem\,\ref{thm:holonomy} let $\phi: \Gamma \to L$ denote the natural epimorphism, and let $G$ denote the holonomy group of $L$. Frobenius induction relative to $G$ reduces the proof of Theorem\,\ref{thm:holonomy} to the case $G$ is cyclic. This follows from \cite[Corollary 2.12]{Swan} (see \cite{Fulton} for more details). It is also recommended that the reader now glance at the summary of Swan’s “Frobenius induction theory” given later in this paper in the paragraph following (3.24); in particular, see the important fact \ref{thm:primary} mentioned there. We proceed to prove Theorem\,\ref{thm:holonomy} by simultaneous induction on $\dim(L)$ and $\#(L)$, where we always assume that $G$ is cyclic. Our explicit inductive assumption is that whenever $L_0$ is a crystallographic group such that either

$$\dim(L_0) < \dim(L) \quad \text{or} \quad \dim(L_0) = \dim(L) \quad \text{and} \quad \#(L_0) < \#(L),$$

then the theorem is true for $L_0$. So primary induction on $\dim(L)$ and secondary induction on $\#(L)$.

To start the $n$th secondary induction, we need to show that Theorem\,\ref{thm:holonomy} is true when $L = \mathbb{Z}^n$ (i.e. when $\#(L) = 1$). Clearly we may assume that $n > 0$. In this case $\Gamma/\pi \cong \mathbb{Z}^n$, so there exists a normal subgroup $\Gamma_0$ of $\Gamma$ containing $\pi$ such that $\Gamma/\Gamma_0 \cong \mathbb{Z}$ and $\Gamma_0/\pi \cong \mathbb{Z}^{n-1}$. Then $\Gamma \cong \Gamma_0 \times \mathbb{Z}$, and by induction $\text{Wh}^k(\Gamma_0 \times \mathbb{Z}^m) \in \tau$ for all $m \geq 0$. We may now apply Corollary\,\ref{cor:holonomy} with $\pi = \Gamma_0$.

Remark 3.3. Let $T$ be the abelian normal subgroup of the crystallographic group $L$ consisting of all pure translations, so $T = L \cap \mathbb{R}^n$ (see the notation above). Then $L$ is an extension of $G$ by $T$. This extension determines by conjugation a representation of $G$ on $T$, called the holonomy representation of $L$. It is well known that $L$ maps epimorphically onto $\mathbb{Z}$ if and only if $T^G \neq 0$, where $T^G$ denotes the subgroup fixed by $G$; cf. \cite[Lemma 1.4]{Lyndon} and \cite{Hochschild}. This fact can be proven by applying the Lyndon-Hochschild-Serre spectral sequence to the group extension $1 \to T \to L \to G \to 1$. Its $E_2^{pq}$ term is $H^p(G; H^q(T, \mathbb{Z}))$ and it converges to $H^{p+q}(L, \mathbb{Z})$. Using that $E_2^{2,0} = H^2(G, \mathbb{Z})$ is finite, one sees easily that $E_2^{0,1} = H^1(G, \mathbb{Z}) = \text{Hom}(L, \mathbb{Z})$ vanishes if and only if $H^1(L, \mathbb{Z}) = \text{Hom}(L, \mathbb{Z})$ vanishes. But $(\text{Hom}(T, \mathbb{Z}))^G$ vanishes if and only if $T^G$ vanishes.

We now consider the general inductive step, which is divided into two cases according to whether $T^G \neq 0$ or $T^G = 0$.

Case 1. $T^G \neq 0$.

By Remark\,\ref{rem:holonomy} we may write $L = L \times \mathbb{Z}$ for some $L \leq L$. Let $\Gamma_0 = \phi^{-1}(\mathcal{L})$, so $\Gamma_0/\pi \cong \mathcal{L}$. Since an $\mathcal{A}$-group is crystallographic if and only if its unique maximal finite normal subgroup is 1, we see that $\mathcal{L}$ is a crystallographic group with $\dim \mathcal{L} =$
dim \( L - 1 \). Therefore Wh\(^k\)(\( \Gamma_0 \times \mathbb{Z}^m \)) \( \in \mathcal{T} \) for all \( m \geq 0 \) because of our inductive assumption. We may now apply Corollary 2.5 to complete the inductive step in Case 1.

Case 2. \( T^G = 0 \).

The geometric algebra developed by Quinn [31, 32, 33, 34] is used crucially here, replacing the \( h \)-cobordisms used in [16]. A relatively simple example which concretely illustrates the terminology used in the remainder of the proof of Theorem 3.1 is worked out in detail in Section 7; cf. Case 2 of the proof of Corollary 7.3 where \( L \) is the infinite dihedral group \( C_2 * C_2 \). It is recommended that the reader keep this example in mind while perusing the rest of the proof of Theorem 3.1. He would also see [32, Appendix] for details about stratified systems of fibrations.

Topology now enters into our proof. Let \( M \) be a connected smooth manifold with \( \pi_1(M) = \Gamma \) and denote its universal cover by \( \tilde{M} \). Also identify \( \Gamma \) with the group of all deck transformations of \( \tilde{M} \rightarrow M \). Projection onto the first factor of \( E := \mathbb{R}^n \times \Gamma \tilde{M} \) induces a map

\[
q: E \rightarrow \mathbb{R}^n/L,
\]

where \( \Gamma \) acts on \( \mathbb{R}^n \) via \( \Gamma \rightarrow L \subseteq \text{Iso}(\mathbb{R}^n) \). This map is a stratified system of fibrations on \( \mathbb{R}^n/L \) in the sense of [32, Definition 8.2], whose strata are determined in the standard way by the holonomy group action of \( G \) on the \( n \)-torus \( \mathbb{R}^n/T \). Let \( s \) be any prime congruent to 1 mod \( \#(L) \); recall that there is an infinitude of such primes by Dirichlet’s theorem. For each such \( s \), there is an endomorphism \( \psi \) of \( L \) and a \( \psi \)-equivariant diffeomorphism \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) (relative to \( L \subseteq \text{Iso}(\mathbb{R}^n) \)), so \( fh = \psi(h)f \) for all \( h \in L \) such that

- \( |df(x)| = s|x| \) for each vector \( x \) tangent to \( \mathbb{R}^n \);
- \( \psi(T) \subseteq T \);
- \( \psi \) induces \( \text{id}_G \) on \( G = L/T \) and
- \( \psi|_T \) is multiplication by \( s \).

This is a restating of [16, Theorem 2.2] which is itself an immediate extension of the classical Epstein-Shub result [11].

**Notation 3.4.** We denote by \( L_s \) and \( T_s \) the finite quotient groups \( L/sT \) and \( T/sT \) respectively. Note that the exact sequence

\[
1 \rightarrow T_s \rightarrow L_s \rightarrow G \rightarrow 1
\]

splits, and does so uniquely up to conjugacy since \((s, |G|) = 1\). One splitting, which we shall denote by \( G_s \), is given by the image of \( \psi(L) \) in \( L_s \) under the projection \( L \rightarrow L_s \). Let \( \eta: \Gamma \rightarrow L_s \) denote the composite epimorphism

\[
\Gamma \xrightarrow{\phi} L \rightarrow L_s,
\]

and let \( \Gamma_s \subseteq \Gamma \) indicate the inverse image of \( \psi(L) \) with respect to the epimorphism \( \phi: \Gamma \rightarrow L \), so \( \Gamma_s/\pi = \psi(L) \). Observe that \( \Gamma_s = \eta^{-1}(G_s) \). Furthermore, let \( E_s \) denote

\[
\mathbb{R}^n \times_{\Gamma_s} \tilde{M},
\]

let \( \hat{f}: \mathbb{R}^n/L \rightarrow \mathbb{R}^n/\psi(L) \) denote the map induced by \( f \), and let \( q_s: E_s \rightarrow \mathbb{R}^n/L \) be the composite of the map \( E_s \rightarrow \mathbb{R}^n/\psi(L) \) induced by projection onto the first factor of \( \mathbb{R}^n \times \tilde{M} \) with the homeomorphism \( \hat{f}^{-1} \).
The map $q_s$ is also a stratified system of fibrations with the same strata as $q$. The following is an important observation.

**Remark 3.5.** Let $\alpha$ be a smooth curve in $E$ and let $\tilde{\alpha}$ be a lift of $\alpha$ to the covering space $E_s$. Then $|q_s \circ \tilde{\alpha}| \leq |q \circ \alpha|/s$, where $| \cdot |$ denotes arc length (measured via $\mathbb{R}^n$).

**Proof.** Let $\hat{\alpha}$ be a lift of $\tilde{\alpha}$ to $\mathbb{R}^n \times \tilde{M}$ which is the universal cover of both $E_s$ and $E$. Then $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)$ where $\hat{\alpha}_1$ is a smooth curve in $\mathbb{R}^n$ and $\hat{\alpha}_2$ is a smooth curve in $\tilde{M}$. Then $|q_s \circ \tilde{\alpha}|$ is by definition the arc length of $f^{-1} \circ \alpha_1$ while $|q \circ \alpha|$ is the arc length of $\alpha_1$. Now the inequality (in fact equality) asserted in Remark 3.5 follows from the fact (noted above) that $df$ stretches tangent vectors by a factor of $s$. □

Another important observation is the following.

**Remark 3.6.** Let $S$ be a cyclic subgroup of $L_s$ such that $S$ projects onto $G$ under the second map in the short exact sequence

$$1 \longrightarrow T_s \longrightarrow L_s \longrightarrow G \longrightarrow 1.$$  

Then $S \cap T_s = 1$; i.e., $S$ splits this sequence and is consequently conjugate to $G_s$.

This observation is a consequence of our assumption that $T^G = 0$ together with [15, Lemmas 1.2 and 1.4].

**Remark 3.7.** So far most of the proof of Theorem 3.1 can be repeated verbatim for the proof of Theorem 4.1 in the next section. However at this point the two proofs diverge somewhat.

We now recall some basic facts about Quinn assembly; cf. [32, Appendix] for more details. These facts will also be used in Sections 4 and 7. Let $S$ be a homotopy invariant (covariant) functor from the category of topological spaces to $\Omega$-spectra. Important examples of such functors are: $X \mapsto K(R\pi_1X)$, $X \mapsto Wh(\pi_1X)$, $X \mapsto Wh^k(\pi_1X)$, where $R$ is a ring with 1. Here $Wh(\pi)$ is the cofiber of the standard map of spectra

$$\mathbb{H}(B\pi; K(\mathbb{Z})) \longrightarrow K(\mathbb{Z}\pi)$$
defined by Loday [29] and likewise $Wh^k(\pi)$ is the cofiber of

$$\mathbb{H}(B\pi; K(k)) \longrightarrow K(k\pi)$$
defined also in [29]. Let $\mathcal{M}$ denote the category of continuous surjective maps; i.e. an object in $\mathcal{M}$ is a continuous surjective map $p: E \rightarrow B$ between topological spaces $E$ and $B$, while a morphism from $p_1: E_1 \rightarrow B_1$ to $p_2: E_2 \rightarrow B_2$ is a pair of continuous maps $f: E_1 \rightarrow E_2$, $g: B_1 \rightarrow B_2$ making the following diagram a commutative square of maps:

$$
\begin{array}{c}
E_1 \xrightarrow{f} E_2 \\
p_1 \downarrow \quad \quad \downarrow p_2 \\
B_1 \xrightarrow{g} B_2.
\end{array}
$$

Quinn [32, Appendix] constructed a functor from $\mathcal{M}$ to the category of $\Omega$-spectra which associates to the map $p$ the spectrum $\mathbb{H}(B; S(p))$ in such a way that $\mathbb{H}(B; S(p)) =$
the natural homomorphism \( \mathbb{W}_k \). In fact, the sheeted cover \( E \) that \( x \) has finite order to prove Theorem 3.1. For this purpose, recall that Quinn showed the image of \( H_\infty \) goes to zero as \( s \to \infty \).

\[
\lim_{s \to \infty} \epsilon_s = 0.
\]

In fact \( h_s \) represents an element \( \tilde{x}_s \in \mathbb{W}_h^k(\mathbb{R}^n/L, q_s, \epsilon_s) \) and \( \tilde{x}_s \) maps to \( x_s \) under the natural homomorphism \( \mathbb{W}_h^k(\mathbb{R}^n/L, q_s, \epsilon_s) \to \mathbb{W}_h^k(\Gamma_s) \). Since the strata in \( \mathbb{R}^n/L \) of the stratified systems \( q_s \) are independent of \( s \), Quinn’s stability theorem [32] \( \S 4 \) together with equation \( (3.1) \) and \[33] \( \S 3 \) yield that \( x_s \) is contained in the image of \( H_1(\mathbb{R}^n/L; \mathbb{W}_h^k(q_s)) \) under the Quinn assembly map provided \( s \) is sufficiently large. Now there is an Atiyah-Hirzebruch-Quinn spectral sequence \( E^2_{ij} \) converging to \( H_{i+j}(\mathbb{R}^n/L; \mathbb{W}_h^k(q_s)) \) such that

\[
E^2_{ij} = H_i(\mathbb{R}^n/L; \mathbb{W}_h^k(q_s))
\]

[32] Theorem 8.7. Here

\[
\mathbb{W}_h^k(\cdot) = \mathbb{W}_h^k(\cdot), \quad \mathbb{W}_h^k(\cdot) = \tilde{K}(\cdot), \quad \mathbb{W}_h^k(\cdot) = K_1(\cdot) \quad \text{if } j < 0.
\]

The stalk of the coefficient sheaf \( \mathbb{W}_h^k(q_s) \) over \( y \in \mathbb{R}^n/L \) is \( \mathbb{W}_h^k(\pi_1(q_s^{-1}(y))) \). But \( \pi \) is a subgroup of finite index in \( \pi_1(q_s^{-1}(y)) \). Using the given hypotheses with \( \tilde{\pi} = \pi_1(q_s^{-1}(y)) \) and Bass’s contracted functor theory [51] Chapter XII, \( \S 7 \), we see that

\[
\mathbb{W}_h^k(\pi_1(q_s^{-1}(y))) \in \mathcal{T} \quad \text{if } j \leq 1.
\]

Since \( \mathbb{R}^n/L \) is a finite polyhedron, we conclude that \( H_1(\mathbb{R}^n/L; \mathbb{W}_h^k(q_s)) \in \mathcal{T} \). So this discussion yields that

\[
(3.2) \quad x_s \text{ has finite order}
\]

provided \( s \) is sufficiently large. We now fix, for the remainder of the proof, a sufficiently large prime \( s \) such that

- \( s \equiv 1 \mod |G| \) and
- \( x_s \) has finite order.

We proceed to apply Frobenius induction to \( \mathbb{W}_h^k(\Gamma) \) relative to the factor group \( L_s \). Let \( \eta: \Gamma \to L_s \) denote the composite epimorphism

\[
\Gamma \xrightarrow{\phi} L \twoheadrightarrow L_s.
\]
Now $\text{Wh}^k(\eta^{-1}(S))$ is a Frobenius module over Swan’s Frobenius functor $G_0(S)$ as $S$ varies over the subgroups of $L_s$. Hence to show that $x$ has finite order, it suffices to show, for each cyclic subgroup $S$ of $L_s$, that the associated transfer of $x$ has finite order. This criterion is a consequence of\cite[Corollary 2.12]{37}. Furthermore, we need only check this condition for one group in each conjugacy class of cyclic subgroups of $L_s$.

For the reader’s convenience, we recall the salient facts from Swan’s theory that are needed in this paper. Swan associates to each finite group $F$ a ring $G_0(F) = G_0(\mathbb{Z}F)$ with identity 1. And to each subgroup $S$ of $F$ he associates an additive group homomorphism $G_0(S) \to G_0(F)$ called induction. In particular (after fixing $F$) $1_S$ denotes the image in $G_0(F)$ of $1 \in G_0(S)$ under induction. Let $C$ be a collection of cyclic subgroups of $F$ such that each conjugacy class is represented exactly once in $C$. Then there exists a set of integers $\{n(S) \mid S \in C\}$ and a nonzero integer $n$ such that

$$(3.3) \quad n = \sum_{S \in C} n(S)1_S.$$  

This is a consequence of\cite[Theorems 2.14, 2.19 and 4.1]{37}. Now let $S(\pi)$ be any of the following functors from groups $\pi$ to abelian groups:

$$K_0(\mathbb{Z}\pi), \ K_0(\mathbb{Z}k\pi), \ K_0(\mathbb{Z}p), \ K_1(\mathbb{Z}\pi), \ K_1(\mathbb{Z}p), \ \text{Wh}(\pi), \ \text{or}\ \text{Wh}^k(\pi).$$

If $\psi: \Gamma \to F$ is a group epimorphism, then $A = S(\Gamma)$ is a (unitary) module over $G_0(F)$. Define abelian groups $A(S)$, for $S \in C$, by $A(S) = S(\psi^{-1}(S))$. Furthermore let $\sigma_S: A \to A(S)$ and $S^S: A(S) \to A$ denote the transfer and induction group homomorphisms, respectively. Then Swan proves the following “Frobenius reciprocity formula”

$$(3.4) \quad (1_S)x = \sigma_S(S^S(x))$$

for all $x \in A$ and every $S \in C$. The following important fact is an immediate consequence of formulae $(3.3)$ and $(3.4)$.

$$(3.5) \quad \text{If } S^S(x) \in T \text{ for each } S \in C, \text{ then } x \in T.$$  

And the following second important fact is also an easy consequence of formulae $(3.3)$ and $(3.5)$.

Suppose for each $S \in C$ that some nonzero (integral) multiple of $\sigma_S(x)$ is the sum of elements induced from $S(H)$ as $H$ varies over the finite subgroups of $\psi^{-1}(S)$. Then some nonzero (integer) multiple of $x$ is also a sum of elements induced from $S(F)$ as $F$ varies over the finite subgroups of $\Gamma$.

If the natural projection $\sigma: L_s \to G$ sends $S$ onto $G$, then $S$ is conjugate to $G_s$ by Remark\cite[83]{37}, but the transfer of $x$ associated to $G_s$ is $x_s$, which has finite order because of\cite[42]{37}. We are therefore left to examine the situation where $\sigma(S)$ is a proper subgroup of $G$; i.e. we must look at the transfer of $x$ to $\text{Wh}^k(\eta^{-1}(S))$. It clearly suffices to show that $\text{Wh}^k(\eta^{-1}(S)) \otimes \mathbb{Q} = 0$. For this purpose, consider the short exact sequence

$$1 \longrightarrow \pi \longrightarrow \eta^{-1}(S) \longrightarrow L(S) \longrightarrow 1$$

where $L(S)$ denotes the inverse image of $S$ under the epimorphism $L \to L_s$. Observe that $L(S)$ is a crystallographic group with
Corollary 3.8. Let \( k \) be a field of prime characteristic, and let \( \pi \triangleleft \Gamma \) be groups such that \( \Gamma/\pi \) is a crystallographic group. Suppose \( \text{Wh}^k(\hat{\pi} \times \mathbb{Z}^n) \in \mathcal{T} \) for all non-negative integers \( n \) whenever \( \hat{\pi}/\pi \) is a finite subgroup of \( \Gamma/\pi \). Then \( \text{Wh}^k(\Gamma \times \mathbb{Z}^n) \in \mathcal{T} \) for all \( n \geq 0 \).

Proof. Note that every finite subgroup of \( \Gamma/\pi \) is contained in \( \Gamma/\pi \). Since \( \Gamma/\pi \) is also a crystallographic group, we can apply Theorem 3.1 with \( \Gamma = \Gamma \), and \( \pi = \pi \).

We now use Corollary 3.6 to obtain the following result.

Corollary 3.9. Let \( k \) be a field of prime characteristic, and let \( \pi \triangleleft \Gamma \) be groups such that \( \Gamma/\pi \) is an elementary amenable group. Suppose \( \text{Wh}^k(\hat{\pi} \times \mathbb{Z}^n) \in \mathcal{T} \) for all non-negative integers \( n \) whenever \( \hat{\pi}/\pi \) is a finite subgroup of \( \Gamma/\pi \). Then \( \text{Wh}^k(\Gamma \times \mathbb{Z}^n) \in \mathcal{T} \) for all \( n \geq 0 \).

Note that Theorem 4.1 immediately follows from using Corollary 3.9 in the case \( \pi = 1 \). To see this, since 1 is the only finite subgroup of \( \Gamma/\Gamma \), we need to show that \( \text{Wh}^k(\mathbb{Z}^n) \in \mathcal{T} \) for all non-negative integers \( n \). However, we have in fact \( \text{Wh}^k(\mathbb{Z}^n) = 1 \) by the Bass-Heller-Swan theorem, so Theorem 4.1 is proven.

Proof of Corollary 3.9. We shall use the description of elementary amenable groups as described in Lemma 2.1. Let \( \alpha \) be the least ordinal such that \( \Gamma/\pi \in \mathcal{X}_\alpha \). Now \( \alpha \) cannot be a limit ordinal, and the result is clearly true if \( \Gamma/\pi \in \mathcal{X}_0 \) because then \( \Gamma/\pi \) is finite. Therefore we may assume that \( \alpha \) is a successor ordinal, and by transfinite induction that the result is true for groups in \( \mathcal{X}_{\alpha-1} \). Thus \( \Gamma \) has a normal subgroup \( \pi_1 \) containing \( \pi \) such that \( \Gamma/\pi_1 \in \mathcal{A} \) and \( \pi_1/\pi \in L\mathcal{X}_{\alpha-1} \). Now any \( \mathcal{A} \)-group maps onto a crystallographic group with finite kernel; in other words there is a finite normal subgroup \( \pi_2/\pi_1 \) of \( \Gamma/\pi_1 \) such that \( \Gamma/\pi_2 \) is crystallographic. Let \( \hat{\pi}/\pi_2 \) be any finite subgroup of \( \Gamma/\pi_2 \). Then Lemma 2.1 shows that \( \hat{\pi}/\pi \in L\mathcal{X}_{\alpha-1} \). Since Whitehead groups commute with direct limits, we see that Corollary 3.9 is true with \( \hat{\pi} \) in place of \( \Gamma \), so the result follows from Corollary 3.6.

4. \( K_0 \) of Elementary Amenable Groups

Theorem 4.1. Let \( k \) be a field of prime characteristic and let \( \pi \triangleleft \Gamma \) be groups such that \( \Gamma/\pi \) is a crystallographic group. Suppose that \( K_0(k[\hat{\pi} \times \mathbb{Z}^m]) \otimes \mathbb{Q} \) is generated by the images of \( K_0(kG) \otimes \mathbb{Q} \) as \( G \) varies over the finite subgroups of \( \hat{\pi} \) where \( \hat{\pi}/\pi \) is a finite subgroup of \( \Gamma/\pi \), for all \( m \geq 0 \). Then for every \( m \geq 0 \),

(i) \( K_i(k[\Gamma \times \mathbb{Z}^m]) \otimes \mathbb{Q} = 0 \) for all \( i < 0 \).

(ii) \( K_0(k[\Gamma \times \mathbb{Z}^m]) \otimes \mathbb{Q} \) is generated by the images of \( K_0(kG) \otimes \mathbb{Q} \) as \( G \) varies over the finite subgroups of \( \Gamma \).
Proof. Note that we need only establish assertion (ii), since Corollary\textsuperscript{3.8} shows that (i) is a consequence of (ii). Our proof of (ii) will follow that of Theorem\textsuperscript{3.1} up to Remark\textsuperscript{3.7} with minor modifications, so we will keep Notation\textsuperscript{3.2}. We need to check that the nth secondary induction starts; i.e. Theorem\textsuperscript{4.1} is true in the case $L = Z^n$ (i.e. $\#(L) = 1$). This is established by applying Corollary\textsuperscript{2.8} with $\pi = \Gamma_0$ (where $\pi \prec \Gamma_0 < \Gamma$ and $\Gamma/\Gamma_0 \cong \mathbb{Z}$).

We also keep Remark\textsuperscript{3.3} and the setup of Case 1. However for the last step in Case 1, we apply Corollary\textsuperscript{2.8} instead of Corollary\textsuperscript{2.5}.

For Case 2, we keep Notation\textsuperscript{3.4} and Remarks\textsuperscript{3.5} and\textsuperscript{3.6}. We replace the part of the proof of Theorem\textsuperscript{3.1} after Remark\textsuperscript{3.7} with the following new material.

We proceed to apply Frobenius induction relative to the factor group $L_s$. Let $x$ be an arbitrary but fixed element of $K_0(k[\Gamma \times Z^m])$. For each cyclic subgroup $S$ of $L_s$, let $x(S)$ denote the transfer of $x$ to $K_0(k[\eta^{-1}(S) \times Z^n])$. As $S$ varies over the subgroups of $L_s$, $S \to K_0(k[\eta^{-1}(S) \times Z^n])$ is a Frobenius module over Swan’s Frobenius functor $S \to G_0(ZS)$\textsuperscript{37} \textsuperscript{\S}2. In view of important fact \textsuperscript{3.4}, to prove Theorem\textsuperscript{4.1} it suffices to show the following condition.

**Condition 4.2.** Some nonzero multiple of $x(S)$ is a sum of elements induced from $K_0(kG)$ as $G$ varies over the finite subgroups of $\eta^{-1}(S)$.

Of course we only need to check Condition\textsuperscript{4.2} for one group $S$ in each conjugacy class of cyclic subgroups of $L_s$. Let $\sigma: L_s \to G$ denote the natural projection. By Remark\textsuperscript{3.6} if $\sigma(S) = G$, then $S$ is conjugate to $G_s$. Hence we need only check Condition\textsuperscript{4.2} for $G_s$ and those cyclic subgroups $S$ such that $\sigma(S) \neq G$.

We start checking Condition\textsuperscript{4.2} by considering the case where $\sigma(S) \neq G$. It clearly suffices to show that $\eta^{-1}(S)$ is a group for which Theorem\textsuperscript{4.1} has already been verified because it is lower in the lexicographic order. To see this consider the short exact sequence

$$1 \to \pi \to \eta^{-1}(S) \to L(S) \to 1$$

where $L(S)$ denotes the inverse image of $S$ under the epimorphism $L \to L_s$. Observe that $L(S)$ is a crystallographic group with

- $\dim L(S) = \dim(L)$ but
- $\#(L(S)) < \#(L)$

since $\sigma(S) \neq G$. Also notice that if $\tilde{\pi}$ is any subgroup of $\eta^{-1}(S)$ which contains $\pi$ with finite index, then $\tilde{\pi}$ satisfies conditions of Theorem\textsuperscript{4.1} because $\eta^{-1}(S) \subseteq \Gamma$ and we have assumed the same property for $\Gamma$. Since $\eta^{-1}(S)$ is lower in the lexicographic order, we conclude from our inductive assumption that $x(S)$ satisfies Condition\textsuperscript{4.2}.

It remains to show that $x(G_s)$ satisfies Condition\textsuperscript{4.2}. Denote this element by $x_s$. We will show by choosing the prime $s$ to be sufficiently large that $x_s$ also indeed satisfies Condition\textsuperscript{4.2}. Clearly we may assume that $m = 0$. Also we work with the more convenient functor $\tilde{K}_0$ instead of $K_0$. We can do this because of the following exact sequence

$$0 \to K_0(k) \to K_0(kH) \to \tilde{K}_0(kH) \to 0$$

which is natural with respect to induction as $H$ varies over the subgroups of $\eta^{-1}(S)$.

Now we assert that to show $x_s$ satisfies Condition\textsuperscript{4.2} it is enough to construct $s = s(x)$ so that $x_s$ is in the image of the assembly map

$$a_s : \mathbb{H}_0(\mathbb{R}^n/L; \mathbb{W}^k_n(q_s)) \to \tilde{K}_0(k\Gamma_s)$$
The left commutative square in (4.1) induces a group homomorphism 
\[ \sigma \] and \( \text{Wh}^{k}(\cdot) = \tilde{K}_{0}(\cdot) \), 
and \( \text{Wh}^{k}(\cdot) = K_{j}(\cdot) \text{ if } j < 0 \). This assertion can be seen as follows. Since \( L \) is a crystallographic group, \( \mathbb{R}^{n} \) has a triangulation on which \( L \) acts simplicially and induces a finite simplicial complex structure on \( B = \mathbb{R}^{n}/L \). Let \( B^{i} \) denote the \( i \)-skeleton of \( B \) and \( E_{s}^{i} = q_{s}^{-1}(B^{i}) \). Consider the following diagram:

\[
\begin{array}{cccc}
E_{s}^{0} & \subseteq & E_{s} & \xrightarrow{id} E_{s} \\
q^{*} & \downarrow & q_{s} & \downarrow \\
B^{0} & \subseteq & B & \longrightarrow *.
\end{array}
\]

The left commutative square in (4.1) induces a group homomorphism \( \sigma : \mathbb{H}_{0}(B^{0}; \text{Wh}^{k}(q_{s})) \rightarrow \mathbb{H}_{0}(B; \text{Wh}^{k}(q_{s})) \) while the right square induces the assembly map \( a_{s} \) displayed above. And \( \sigma \) fits into a (homology) exact sequence of abelian groups whose next group is \( \mathbb{H}_{0}(B, B_{0}; \text{Wh}^{k}(q_{s})) \) by Proposition 8.4. This relative homology group can be analyzed by an Atiyah-Hirzebruch-Quinn spectral sequence, constructed in Proposition 8.7, which converges to \( \mathbb{H}_{0}(B, B_{0}; \text{Wh}^{k}(q_{s})) \) with \( E_{2}^{j, -j} = H_{j}(B, B_{0}; \text{Wh}_{-j}^{k}(q_{s})) \) where \( \text{Wh}_{-j}^{k}(q_{s}) \) is the stratified system of groups \( \{ \text{Wh}_{-j}^{k}(q_{s}^{-1}(y)) \mid y \in B \} \). Now note that

\[
H_{j}(B, B_{0}; \text{Wh}_{-j}^{k}(q_{s})) = \begin{cases} 
0, & \text{if } j \notin [1, n] \\
0 \mod T, & \text{if } j > 0.
\end{cases}
\]

The top equation is immediate for dimension reasons while the bottom equation is a consequence of the fact that each \( \pi_{1}(q_{s}^{-1}(y)) \) contains \( \pi \) with finite index. Consequently \( \mathbb{H}_{0}(B, B_{0}; \text{Wh}^{k}(q_{s})) \in T \) and therefore also \( \text{coker}(\sigma) \in T \). And consequently some nonzero multiple of \( x_{s} \) is in \( \text{im}(a_{s} \circ \sigma) \). Next consider the following second diagram:

\[
\begin{array}{cccc}
E_{s}^{0} & \xrightarrow{id} & E_{s}^{0} & \subseteq E_{s} \\
q^{*} & \downarrow & q_{s} & \downarrow \\
B^{0} & \longrightarrow & * & \longrightarrow *.
\end{array}
\]

The left commutative square in (4.2) induces the assembly map

\[
a_{s}^{0} : \mathbb{H}_{0}(B^{0}; \text{Wh}^{k}(q_{s})) \rightarrow \bigoplus_{v \in B^{0}} \tilde{K}_{0}(k \pi_{1}q_{s}^{-1}(v))
\]

while the right commutative square induces

\[
\tau : \bigoplus_{v \in B^{0}} \tilde{K}_{0}(k \pi_{1}q_{s}^{-1}(v)) \longrightarrow \tilde{K}_{0}(k \Gamma_{s})
\]

which is the sum of the homomorphisms induced by the group inclusions

\[
\pi_{1}(q_{s}^{-1}(v)) \subseteq \pi_{1}(E_{s}) = \Gamma_{s}
\]

for \( v \in B^{0} \). Since the concatenation of the left and right squares in (4.1) is the same square, namely

\[
\begin{array}{cccc}
E_{s}^{0} & \subseteq & E_{s} \\
q^{*} & \downarrow & q_{s} & \downarrow \\
B^{0} & \longrightarrow & *.
\end{array}
\]
as the concatenation of the corresponding squares in (4.2), we have by functoriality that 
\( a_s \circ \sigma = \tau \circ \tilde{a}_s \) and consequently that
\[
(4.3) \quad \text{some nonzero multiple of } x_s \text{ is in } \text{im}(\tau).
\]
Recall that each group \( \pi_1(q^{-1}_s(v)), v \in B^0 \), contains \( \pi \) with finite index. Hence the hypothesis of Theorem 4.1 together with fact (4.3) verifies our assertion.

Now to show that \( x_s \in \text{im}(a_s) \), it is easier to work with the Quinn assembly map at the Whitehead group level; i.e. with the map
\[
\tilde{a}_s : H_1(\mathbb{R}^n/L; \text{Wh}^k(\tilde{q}_s)) \to \text{Wh}^k(\Gamma \times \mathbb{Z}).
\]
where \( \tilde{q}_s : E_s \times S^1 \to \mathbb{R}^n/L \) is the composite of the projection \( E_s \times S^1 \to E_s \) with \( q_s : E_s \to \mathbb{R}^n/L \). In particular the following statement is true for \( \tilde{a}_s \).

**Assertion 4.3.** Given any element \( y \in \text{Wh}^k(\Gamma \times \mathbb{Z}) \), there exists a prime \( \bar{s}(y) = s \) with \( s \equiv 1 \mod |G| \), such that the transfer \( y_s \in \text{Wh}^k(\Gamma_s \times \mathbb{Z}) \) of \( y \) is in \( \text{im} \tilde{a}_s \).

Before verifying Assertion 4.3, we use it to complete the proof of Theorem 4.1. There is a natural ring homomorphism \( R[t, t^{-1}] \to \mu R \), where \( R \) is any ring with 1 and \( \mu R \) denotes its suspension, which induces the projection map in the Bass-Heller-Swan formula \( \tilde{K}_s(R[t, t^{-1}]) \to \tilde{K}_{s+1}(R) \) on the spectrum level; cf. [21, 33]. Consequently there is the following commutative diagram
\[
\begin{array}{c}
\mathbb{H}_0(\mathbb{R}^n/L; \text{Wh}^k(q_s)) \xrightarrow{\tilde{a}_s} \mathbb{K}_0(k\Gamma_s) \\
\uparrow & \uparrow \\
\mathbb{H}_0(\mathbb{R}^n/L; \text{Wh}^k(\tilde{q}_s)) \xrightarrow{\tilde{a}_s} \text{Wh}^k(\Gamma_s \times \mathbb{Z}).
\end{array}
\]
Now set \( s(x) = \bar{s}(y) \) where \( y \) is the image of \( x \) in \( \text{Wh}^k(\Gamma \times \mathbb{Z}) \) under the natural embedding
\[
\tilde{K}_0(k\Gamma) \to \text{Wh}^k(\Gamma \times \mathbb{Z}).
\]
Then by a diagram chase one sees directly that \( x_{s(x)} \) is in the image of \( a_{s(x)} \).

It remains to verify Assertion 4.3. For this purpose, recall that Quinn showed that \( y \) can be represented by a geometric isomorphism (denoted \( h \)) of geometric \( k \)-modules on the space \( E \times S^1 \) [34, §3.2]. The transfer of \( h \) (denoted \( h_s \)) to the finite sheeted cover
\[
E_s \times S^1 \to E \times S^1
\]
clearly represents the transfer of \( y \) (denoted \( y_s \)) to \( \text{Wh}^k(\Gamma_s \times \mathbb{Z}) \). But the radii \( \epsilon_s \) of these geometric isomorphisms \( h_s \) measured via \( \tilde{q}_s \) in \( \mathbb{R}^n/L \) go to zero as \( s \to \infty \); i.e.
\[
\lim_{s \to \infty} \epsilon_s = 0.
\]
In fact \( h_s \) represents an element \( \bar{y}_s \in \text{Wh}^k(\mathbb{R}^n/L, \tilde{q}_s, \epsilon_s) \) and \( \bar{y}_s \) maps to \( y_s \) under the natural homomorphism
\[
\text{Wh}^k(\mathbb{R}^n/L, \tilde{q}_s, \epsilon_s) \to \text{Wh}^k(\Gamma_s \times \mathbb{Z}).
\]
Since the strata in \( \mathbb{R}^n/L \) of the stratified system \( \tilde{q}_s \) are independent of \( s \), Quinn’s Stability Theorem [32, §4] together with equation (4.4) and [33, §3] yield that \( y_s \) is contained in the image of \( H_1(\mathbb{R}^n/L; \text{Wh}^k(\tilde{q}_s)) \), i.e. \( y_s \in \text{im}(a_s) \) when \( s \) is sufficiently large. This verifies Assertion 4.3 thus completing the proof of Theorem 4.1. \( \square \)

We now use Theorem 4.1 to obtain the following result.
Corollary 4.4. Let \( k \) be a field of prime characteristic and let \( \pi \triangleleft \Gamma \) be groups such that \( \Gamma/\pi \) is an elementary amenable group. Suppose that \( K_0(k[\hat{\pi} \times \mathbb{Z}^m]) \otimes \mathbb{Q} \) is generated by the images of \( K_0(kG) \otimes \mathbb{Q} \) as \( G \) varies over the finite subgroups of \( \hat{\pi} \) where \( \hat{\pi}/\pi \) is a finite subgroup of \( \Gamma/\pi \), for all \( m \geq 0 \). Then for every \( m \geq 0 \),

(i) \( K_i(k[\Gamma \times \mathbb{Z}^m]) \otimes \mathbb{Q} = 0 \) for all \( i < 0 \).

(ii) \( K_0(k[\Gamma \times \mathbb{Z}^m]) \otimes \mathbb{Q} \) is generated by the images of \( K_0(kG) \otimes \mathbb{Q} \) as \( G \) varies over the finite subgroups of \( \Gamma \).

Note that Theorem 1.24 immediately follows from using Corollary 4.4(ii) in the case \( \pi = 1 \). To see this, it will be sufficient to show that \( K_0(k[\hat{\pi} \times \mathbb{Z}^m]) \otimes \mathbb{Q} \) is generated by the image of \( K_0(k\hat{\pi}) \otimes \mathbb{Q} \) whenever \( \hat{\pi} \) is a finite subgroup of \( \Gamma \), for all \( m \geq 0 \). But we know this by Lemma 2.4.

Proof of Corollary 4.4. By Corollary 2.8 we need only prove Corollary 4.4(ii). We shall use the description of elementary amenable groups as described in Lemma 2.4.

Let \( \alpha \) be the least ordinal such that \( \Gamma/\pi \in \mathcal{X}_\alpha \). Now \( \alpha \) cannot be a limit ordinal, and the result is clearly true if \( \Gamma/\pi \in \mathcal{X}_0 \) because then \( \Gamma/\pi \) is finite. Therefore we may assume that \( \alpha \) is a successor ordinal, and by transfinite induction that the result is true for groups in \( \mathcal{X}_{\alpha-1} \). Thus \( \Gamma \) has a normal subgroup \( \pi_1 \) containing \( \pi \) such that \( \Gamma/\pi_1 \in A \) and \( \pi_1/\pi \in L\mathcal{X}_{\alpha-1} \). Now any \( A \)-group maps onto a crystallographic group with finite kernel; in other words there is a finite normal subgroup \( \pi_2/\pi_1 \) of \( \Gamma/\pi_1 \) such that \( \Gamma/\pi_2 \) is crystallographic. Let \( \hat{\pi}/\pi_2 \) be any finite subgroup of \( \Gamma/\pi_2 \). Then Lemma 2.4 shows that \( \hat{\pi}/\pi \in L\mathcal{X}_{\alpha-1} \). Since \( K_0 \) commutes with direct limits, Theorem 1.24(i) is true with \( \hat{\pi} = \Gamma \) by our inductive hypothesis, consequently \( K_0(k[\hat{\pi} \times \mathbb{Z}^m]) \) is generated by the images of \( K_0(kG) \otimes \mathbb{Q} \) as \( G \) varies over the finite subgroups of \( \hat{\pi} \). The result now follows from Theorem 4.1.

\[ \square \]

5. The strong Bass conjecture and idempotents

We begin this section by reviewing some results on lifting idempotents modulo a nilpotent ideal, the relationship between idempotents and projective modules, and the Hattori-Stallings trace. For more information, see [6] and [24] [III.7 and III.8]. Some of the ideas used below originate from [5] [13].

Let \( R \) be a ring and let \( d \) be a positive integer. If \( e \in \text{Mat}_d(R) \) is an idempotent matrix, then \( e \) defines by left multiplication an \( R \)-map \( R^d \to R^d \) whose image is a finitely generated projective right \( R \)-module \( P \). Conversely given a finitely generated projective \( R \)-module \( P \), then for some positive integer \( d \) we may write \( P \cong R^d \otimes Q \), and then the projection from \( R^d \) onto \( P \) associated with this direct sum yields an idempotent \( e \in \text{Mat}_d(R) \). In this situation we shall say that \( P \) and \( e \) correspond. Of course \( P \) does not determine \( e \), but if \( e' \) is another choice for \( e \), then for \( m \) sufficiently large, we may view \( e, e' \in \text{Mat}_m(R) \) with the property that \( e' = u e u^{-1} \) for some \( u \in \text{GL}_m(R) \).

Suppose now that \( N \) is a nil ideal of \( R \) (so given \( n \in N \), there exists \( s > 0 \) such that \( n^s = 0 \)). If \( \hat{e} \in \text{Mat}_d(R/N) \) is an idempotent, then by the theory of [24] [III.8], we may lift \( \hat{e} \) to an idempotent \( e \in \text{Mat}_d(R) \). Since \( N \) is contained in the Jacobson radical of \( R \), by Nakayama’s lemma a map \( P \to P' \) between finitely generated projective \( R \)-modules is an isomorphism if and only if the induced map \( P/PN \to P'/P'N \) is an isomorphism. Thus though \( \hat{e} \) does not in general determine \( e \), if \( e' \) is another choice for \( e \), then the corresponding projective \( R \)-modules \( P, P' \) are isomorphic.
Now let $k$ be a commutative ring and let $\Gamma$ be a group. If $\alpha \in \text{Mat}_d(k\Gamma)$, then we may write $\alpha = \sum_{g \in \Gamma} \alpha_g g$ where $\alpha_g \in \text{Mat}_d(k)$. Let $\text{tr}: \text{Mat}_d(k) \to k$ denote the usual trace map (i.e., the sum of the diagonal entries). Then for $g \in \Gamma$ we define $\text{Tr}_g: \text{Mat}_d(k\Gamma) \to k$ by

$$\text{Tr}_g(\alpha) = \sum_{x \sim g} \text{tr}(\alpha_x)$$

where $x \sim g$ means that $x$ is conjugate to $g$ in $\Gamma$. If $u \in \text{GL}_d(k\Gamma)$, then $\text{Tr}_d(u\alpha u^{-1}) = \text{Tr}_d(\alpha)$. Suppose now $P$ is a finitely generated projective $k\Gamma$-module and $P$ corresponds to an idempotent $e \in \text{Mat}_d(k\Gamma)$. Then we set $r_P(g) = \text{Tr}_g(e)$. This is the Hattori-Stallings trace and is well defined. Moreover if $Q \cong P$ and $h \sim g$, then $r_Q(h) = r_P(g)$.

Suppose $G$ is a group, $\theta: G \to \Gamma$ is a homomorphism, and $Q$ is a finitely generated projective $kG$-module. Then $Q \otimes_{kG} k\Gamma$ is a finitely generated projective $k\Gamma$-module and we have the following induction formula:

$$r_{Q \otimes_{kG} k\Gamma}(g) = \sum_{x \# g} r_Q(x),$$

where $x \# g$ means that $\theta(x)$ is conjugate to $g$ in $\Gamma$, and we choose only one $x$ from each $G$-conjugacy class. This means in particular that if $g$ is not conjugate to any element of $\theta(G)$, then $r_{Q \otimes_{kG} k\Gamma}(g) = 0$.

Now let $S$ be a complete commutative local ring with maximal ideal $J$ such that $S/J \cong k$. Suppose $P$ is a finitely generated projective $k\Gamma$-module. Then $P$ corresponds to an idempotent $e \in \text{Mat}_d(k\Gamma)$ for some positive integer $d$. Now $k\Gamma \cong S\Gamma/J\Gamma$ and $J^n\Gamma/J^{n+1}\Gamma$ is a nil ideal (in fact even the square of the ideal is 0) of $S\Gamma/J^{n+1}\Gamma$ for all positive integers $n$. Set $e_1 = e$ and let

$$\sim: \text{Mat}_d(S\Gamma/J^{n+1}\Gamma) \to \text{Mat}_d(S\Gamma/J^n\Gamma)$$

denote the natural epimorphism. We shall also let $\sim$ denote the natural epimorphism $S\Gamma/J^{n+1}\Gamma \to S\Gamma/J^n\Gamma$. Thus starting with $n = 1$, we may inductively lift $e_n \in \text{Mat}_d(S\Gamma/J^n\Gamma)$ to an idempotent $e_{n+1} \in \text{Mat}_d(S\Gamma/J^{n+1}\Gamma)$ such that $\sim e_{n+1} = e_n$. In particular

$$\sim \text{Tr}_g(e_{n+1}) = \text{Tr}_g(e_n)$$

for all positive integers $n$ and for all $g \in \Gamma$. This means that for each $g$, the sequence $(\text{Tr}_g(e_n))$ yields a well defined element of $S$, which we shall denote by $\hat{\text{Tr}}_g(e)$. Though the $e_n$ are not uniquely determined by $e$, we do know that if $(e'_n)$ is another such sequence of idempotents, then

$$\text{Tr}_g(e'_n) = \text{Tr}_g(e_n)$$

for all $n$, consequently $\hat{\text{Tr}}_g(e)$ does not depend on the choice of liftings for the $e_n$. Suppose $e' = ueu^{-1}$ where $u \in \text{GL}_d(k\Gamma)$. Then we may lift the $u$ to units $u_n \in \text{GL}_d(S\Gamma/J^n\Gamma)$ to obtain a sequence of idempotents $u_n e_n u_n^{-1} \in \text{Mat}_d(S\Gamma/J^n\Gamma)$ which lift the idempotent $e'$. This shows that $\hat{\text{Tr}}_g(e) = \hat{\text{Tr}}_g(e')$. Now set $\hat{r}_P(g) = \hat{\text{Tr}}_g(e)$. Then the above discussion shows that $\hat{r}_P(g)$ is well defined and if $\sim: S\Gamma \to k\Gamma$ is the natural epimorphism, then

$$\hat{r}_P(g) = r_P(g).$$

Also if $U$ is a finitely generated projective $S\Gamma$-module such that $U/UJ \cong P$, then $r_{U/g} = \hat{r}_P(g)$. For $\Gamma$ finite, such a $U$ will always exist (this well-known fact
can be seen from the above, because the sequence of lifted idempotents $e_n$ will yield an idempotent of Mat$_d(S\Gamma)$, and we can let $U$ be the projective $S\Gamma$-module corresponding to this idempotent; of course if $\Gamma$ is infinite, this will not work because the supports of the lifted idempotents $e_n$ may become arbitrarily large). Finally if $\theta: G \to \Gamma$ is a group homomorphism and $Q$ is a finitely generated projective $kG$-module, then from (5.4) we get the corresponding induction formula for lifted traces:

$\hat{r}_{Q\otimes_k k\Gamma}(g) = \sum_{x \neq g} \hat{r}_{Q}(x)$. \hspace{1cm} (5.2)

Remark 5.1. If in addition $p$ is a prime, $k$ is a field of characteristic $p$, $S$ is a Noetherian integral domain, and $\phi$ is the Brauer character of $P$, then $\zeta(g)\hat{r}_{P}(g) = \phi(g)$, where $\zeta(g)$ is the order of the centralizer of $g$ in $\Gamma$.

Suppose $Q$ is a finitely generated projective $k\Gamma$-module. Then $Q$ corresponds to an idempotent $f \in \text{Mat}_c(k\Gamma)$ for some positive integer $c$ and $e \oplus f$ corresponds to $P \oplus Q$; here $e \oplus f$ means the element in $\text{Mat}_{d+c}(k\Gamma)$

$$\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$ \hspace{1cm} (5.3)

Thus if $(f_n)$ is a sequence of idempotents corresponding to $Q$, then $(e_n \oplus f_n)$ is a sequence of idempotents corresponding to $P \oplus Q$. Since $\text{Tr}_g(e_n \oplus f_n) = \text{Tr}_g(e_n) + \text{Tr}_g(f_n)$ and $\hat{r}_P(g)$ is well defined, we deduce that

$$\hat{r}_{P\oplus Q}(g) = \hat{r}_P(g) + \hat{r}_Q(g).$$

We can now prove (retaining the notation above)

**Lemma 5.2.** Let $p$ be a prime, let $k$ be a field of characteristic $p$, let $\Gamma$ be an elementary amenable group, let $P$ be a finitely generated projective $k\Gamma$-module, and let $g \in \Gamma$. Suppose all positive integers are nonzero divisors in $S$. If $p|o(g)$ or $o(g) = \infty$, then $\hat{r}_P(g) = 0$. In particular, $r_P(g) = 0$.

**Proof.** By Theorem 1.2 there is a positive integer $n$, a finitely generated free $kG$-module $Q$, finite subgroups $G_1, \ldots, G_m$ of $\Gamma$, and for each positive integer $i \leq m$ a finitely generated projective $kG_i$-module $Q_i$ such that

$$P^n \oplus Q \cong \bigoplus_{i=1}^m Q_i \otimes_{kG_i} k\Gamma \oplus Q.$$ \hspace{1cm} (5.4)

Therefore

$$n\hat{r}_P(g) = \sum_{i=1}^m \hat{r}_{Q_i\otimes_k k\Gamma}(g).$$

Since $n$ is a nonzero divisor in $S$, it is sufficient to show that $\hat{r}_{Q_i\otimes_k k\Gamma}(g) = 0$ for all $i$. Now $G_i$ is finite, so there exists a finitely generated projective $S\Gamma_i$-module $Q_{iJ}$ such that $Q_{iJ}/Q_i \cong Q_i$. Then applying (5.2), we have

$$\hat{r}_{Q_i\otimes_k k\Gamma}(g) = \sum_{x \neq g} r_{Q_i}(x).$$

In the case $o(g) = \infty$, this sum is empty and hence equal to zero. Also $r_{Q_i}(x) = 0$ whenever $p|\sigma(x)$ (this is really [9, Theorem 59.7(ii)] and Remark 5.1). Thus the result follows. \hfill $\square$
Suppose $k$ is a finite field of characteristic $p$ and order $q$ where $q$ is a power of $p$. Let $\zeta$ be a primitive $(q-1)$th root of unity (in the algebraic closure of $\mathbb{Q}_p$) and set $S = \mathbb{Z}_p[\zeta]$. Then $S$ is a complete local ring with residue field $k$.

Using Lemma 5.2 we can now prove Theorem 1.6.

Proof of Theorem 1.6. If $P$ corresponds to the idempotent $e$, then $e \in k'\Gamma$ for some finitely generated subring $k'$ of $k$. This means we may assume that $k$ is finitely generated as a ring. We have two cases to consider, depending on whether $k$ has characteristic $p$ or characteristic 0.

Case 1. $k$ has characteristic $p$. Suppose by way of contradiction that $g \in \Gamma$, $o(g)$ is not invertible in $k$, yet $r_P(g) \neq 0$. Then either $p|o(g)$ or $o(g) = \infty$. Since $k$ is a finitely generated integral domain, it has a maximal ideal $M$ such that the image of $r_P(g)$ in $k/M$ is nonzero [3, Ex. 24, p. 71, Chapter 5]. Also $k/M$ will be a finite field [3, Ex. 6, p. 84, Chapter 7] of characteristic $p$. Thus by Remark 5.3 there is a complete local ring $S$ with residue field $k/M$, and we can now obtain a contradiction by applying Lemma 5.2 and the proof is complete.

Case 2. $k$ has characteristic 0. Since $o(g)$ is not invertible in $k$, we may choose a prime $p$ such that $pk \neq k$ and either $p|o(g)$ or $o(g) = \infty$. Let $J$ be a maximal ideal in $k$ containing $p$, and let $S$ denote the completion of $k$ with respect to $J$. Then $S$ is a complete local ring containing $k$ with residue field of characteristic $p$. Also $k$ is a Noetherian ring [3, Corollary 7.7, p. 81], so by [3, Ex. 4, p. 114, Chapter 10] it has property that all positive integers are nonzero divisors in $S$. We now obtain a contradiction by applying Lemma 5.2 and the proof is complete. \hfill \square

Proof of Theorem 1.4. Suppose $e$ is a nontrivial idempotent in $k\Gamma$ and write $e = \sum_{g \in \Gamma} e_g g$ where $e_g \in k$ for all $g \in \Gamma$. Since $e_g = 0$ for all but finitely many $g$, by replacing $k$ with the subring generated by $\{e_g \mid e_g \neq 0\}$, we may assume that $k$ is a finitely generated integral domain. Choose $h \in G \setminus 1$ such that $e_h \neq 0$, which is possible because $e$ is nontrivial. Then $k$ has a maximal ideal $M$ such that the image of $e_h$ in $k/M$ is nonzero [3, Ex. 24, p. 71, Chapter 5]. Also $k/M$ will be a finite field [3, Ex. 6, p. 84, Chapter 7] of characteristic $p$. Therefore we may assume that $k$ is a finite field and hence by Remark 5.3 there is a complete local ring $S$ with residue field $k$. Now write $1 = e + f$ where $f$ is also an idempotent in $k\Gamma$. Let $P, Q$ be finitely generated projective $k\Gamma$-modules which correspond to $e, f$ respectively, and let $\theta : \Gamma \rightarrow 1$ denote the natural epimorphism. Then $P \otimes_{k\Gamma} k \oplus Q \otimes_{k\Gamma} k \cong k$, so without loss of generality we may assume that $P \otimes_{k\Gamma} k = 0$ and we see that $\hat{r}_{P \otimes_{k\Gamma} k}(1) = 0$. Since $\hat{r}_P(g) = 0$ for all $g \in \Gamma \setminus 1$ by Lemma 5.2, we deduce from the induction formula 5.2 that $\hat{r}_P(1) = 0$. If $H$ is a finite subgroup of $\Gamma$, then $H$ is a finite $p$-group, hence $kH$ is a local ring and so all projective $kH$-modules are free. Therefore $P^n$ is a stably free $k\Gamma$-module for some positive integer $n$ by Theorem 1.2, consequently $P^n \oplus (k\Gamma)^a \cong (k\Gamma)^b$ for some integers $a, b$. Since $\hat{r}_P(1) = 0$, this is only possible if $a = b$ and we now have $(k\Gamma)^a \cong (k\Gamma)^b \oplus P^n$.

Since $e$ is nonzero by assumption, we see that $P^n$ is nonzero (and also $a \neq 0$). Therefore $(k\Gamma)^a$ is not a directly finite $k\Gamma$-module [22, §6.B], hence by [22, Lemma 6.9] the ring $\text{Mat}_a(k\Gamma)$ is not directly finite. This means that there exist $r, s \in

Remark 5.3. Suppose $k$ is a finite field of characteristic $p$ and order $q$ where $q$ is a power of $p$. Let $\zeta$ be a primitive $(q-1)$th root of unity (in the algebraic closure of $\mathbb{Q}_p$) and set $S = \mathbb{Z}_p[\zeta]$. Then $S$ is a complete local ring with residue field $k$.\hfill \square
Mat\(a(k\Gamma)\) such that \(rs = 1\) but \(sr \neq 1\). By [2, Theorem 3.2], all matrix rings over the group algebra \(kG\) of an amenable group \(G\) are directly finite. Therefore Mat\(a(k\Gamma)\) is directly finite, a contradiction and the result is proven. \(\Box\)

6. Whitehead groups of solvable linear groups

In this section we shall prove Theorem 1.3. The proof is very similar to [20, Theorem 1.1], which showed that Wh\((G) = 0\) when \(G\) is a virtually solvable linear group, so we will not give full details. For the rest of this section, we set \(k = k_2\).

We also need the following consequence of Theorem 7.1(i).

**Lemma 6.1.** Let \(H \triangleleft G\) be groups such that \(G/H \in \mathcal{A}\), and let \(p: G \to G/H\) denote the natural epimorphism. If Wh\(^k(p^{-1}(S) \times \mathbb{Z}^n) = 0\) for all virtually cyclic subgroups \(S\) of \(G/H\) and for all non-negative integers \(n\), then Wh\(^k(G) = 0\).

**Proof.** This follows from Lemma 2.10 and Theorem 7.1(i). \(\Box\)

**Proof of Theorem 1.3.** Since Whitehead groups commute with direct limits, we may assume that \(G\) is finitely generated. Also we may assume that \(G > 1\). We have two cases to consider.

Case 1. \(G\) is virtually nilpotent.

Since \(G\) is also finitely generated, we see that it is a virtually poly-\(\mathbb{Z}\) group. Hence it is a discrete cocompact subgroup of a virtually connected Lie group by a result of Auslander and Johnson [3, Theorem 1]. And therefore \(G\) satisfies the FIC in dimensions \(\leq 1\) because of Theorem 7.1. Applying this to the situation where \(\phi = \text{id}: \Gamma \to \Gamma\), we conclude that Wh\(^k(G) = 0\) since

\[
\text{Wh}^k(C) = K_0(kC) = K_i(kC) = 0
\]

for all \(i < 0\) when \(C = 1\) or \(\mathbb{Z}\). Since Wh\(^k(\ .\) commutes with direct limits and we have not used, so far, the assumption that \(G\) is a subgroup of GL\(_n\)(\(C\)), the following additional remark is true.

**Remark 6.2.** If \(N\) is any torsion-free virtually nilpotent group, then Wh\(^k(N) = 0\) and also \(\tilde{K}_0(kN) = 0\).

To see the “and also” assertion, notice that \(\tilde{K}_0(kN)\) is a subgroup of Wh\(^k(N \times \mathbb{Z})\) [5, Chapter XII, §7] and that \(N \times \mathbb{Z}\) is also both torsion-free and virtually nilpotent.

Case 2. \(G\) is not virtually nilpotent.

By a theorem of Malcev [35, 15.1.4], there exists \(H \triangleleft G\) such that \(H\) is nilpotent and \(G/H \in \mathcal{A}\) (though \(H\) will not be finitely generated in general).

Let \(C/H\) be a virtually cyclic subgroup of \(G/H\), let \(n\) be a non-negative integer, and set \(D = C \times \mathbb{Z}^n\). By Lemma 6.1 it will suffice to show that Wh\(^k(D) = 0\). Set \(E = H \times \mathbb{Z}^n\). Then \(E\) is a normal nilpotent subgroup of \(D\) and \(D/E\) is virtually cyclic. Since \(D/E\) is virtually cyclic, Proposition 2.11 yields that it has a finite normal subgroup \(F/E\) such that \(D/F\) is either trivial, infinite cyclic or infinite dihedral. For any finite subgroup \(N/F\) of \(D/F\) (e.g. \(N = F\)), the group \(N\) is torsion-free virtually nilpotent and hence

\[
\text{Wh}^k(N) = \tilde{K}_0(kN) = 0
\]
by Remark 6.2. In particular if $D/F = 1$, this shows that $\text{Wh}^k(D) = 1$. In the other two cases, we now apply Waldhausen’s results and we shall adopt his terminology. We shall also require the following well known result, which for example follows immediately from [20, Lemma 3.1].

**Lemma 6.3.** Let $G$ be a finitely generated torsion-free virtually nilpotent group. Then $kG$ is regular Noetherian.

Using Lemma 6.3 and [39, Theorem 19.1(iii)], we see that $kF$ is regular coherent. If $D/F$ is infinite cyclic, then the result follows from [39, Corollary 17.2.3]. On the other hand if $D/F$ is infinite dihedral, then the result follows from [39, Corollary 17.1.3].

7. **$k_2$ Group Rings**

**Theorem 7.1.** The FIC in dimensions $\leq 1$ is true for the group $\Gamma$ relative to the functor $K(k_2)$ in the following cases.

(i) $\Gamma$ is a discrete cocompact subgroup of a virtually connected Lie group.

(ii) $\Gamma = H \wr \mathbb{Z}$ where $H$ is any finite abelian group.

The precise statement is that given any epimorphism $\phi: \pi \rightarrow \Gamma$, then the assembly map

$$H_i(B; K(k_2\Phi)) \rightarrow K_i(k_2\pi)$$

induces an isomorphism for all $i \leq 1$. Here $B = E/\Gamma$ where $E$ is a universal $\Gamma$-space with respect to the class of virtually cyclic subgroups of $\Gamma$; $E$ is the universal cover of an Eilenberg-Mac Lane space $K(\pi, 1)$ and $\Phi: E \times \pi \rightarrow B$ is induced by the projection $E \times E \rightarrow E$. Also $\pi$ acts on $E$ via $\phi$.

**Caveat.** The “FIC in dimensions $\leq 1$” defined in Theorem 7.1 is somewhat weaker than the straightforward generalization of the FIC which was formulated in [18, §1.7] and restated in [20, §7]. In the notation of [20], the FIC allows an arbitrary free and properly discontinuous $\Gamma$-space $Y$, while $Y = E/\ker \phi$ in Theorem 7.1.

Using Theorem 7.1, we are able to prove Corollary 7.3, which clearly implies Theorem 7.1 in the case $k = k_2$. It now follows that we have a second proof of Theorem 1.2 in the case $k = k_2$. Also Theorem 7.1 enables us to prove Proposition 1.4. On the other hand, we do not know how to prove Theorem 7.1 in the case $k = k_3$. See [18, Appendix] for details about universal $\Gamma$-spaces relative to a class of subgroups of $\Gamma$.

**Remark 7.2.** Formal homology properties show that Theorem 7.1 is equivalent to the statement that the assembly map

$$H_i(B; \text{Wh}_k(\Phi)) \rightarrow \text{Wh}_i(\pi)$$

induces an isomorphism for all $i \leq 1$. Here $\text{Wh}_k(\pi)$ is the cofiber of the standard map of spectra $\mathbb{H}(B\pi; K(k)) \rightarrow K(k\pi)$ defined by Loday [20] (which is the Quinn assembly map associated to the fibration id: $B\pi \rightarrow B\pi$; cf. [33]). Here $k = k_2$.

**Proof of Proposition 1.4.** Since $\tilde{K}_0$ commutes with direct limits, we may assume that $G$ is a finite abelian 2-group. In view of Theorem 7.1(ii), it will now be sufficient to show that $\tilde{K}_0(k_2H) = 0$ and $K_i(k_2H) = 0$ for every virtually cyclic subgroup $H$ of $G \wr \mathbb{Z}$ and for every $i < 0$. But the only virtually cyclic subgroups of
Case 1. So we have two cases to consider. Case 2.

Then \( \Gamma = \ldots \) This case would follow from an argument similar to that given for Case 1 using \([39]\) we have \( \ldots \) \( \text{but more explicit} \), and our notation will be consistent with that used there. Thus following argument which reduces Case 2 also to Lemma 2.2. We proceed via a Frobenius induction argument which is similar to the ones in Theorems 3.1 and 4.1, some evidence for this suspicion is given by \([19, \text{Theorem 2.6}] \). Hence we have opted instead for the esoteric groups are “dominated” by those already analyzed. Some evidence for these more esoteric groups are “dominated” by those already analyzed. (Some evidence for this suspicion is given by \([19, \text{Theorem 2.6}] \).) Hence we have opted instead for the following argument which reduces Case 2 also to Lemma 2.2. We proceed via a Frobenius induction argument which is similar to the ones in Theorems 3.1 and 4.1 (but more explicit), and our notation will be consistent with that used there. Thus we have

\[ q: E \longrightarrow \mathbb{R}/C_2 \ast C_2 = [0, 1] \]

with the action of \( C_2 \ast C_2 \) on \( \mathbb{R} \) generated by the two reflections: \( a \) in 0 and \( b \) in 1. Note that \( s \) can be any odd prime in this case, since \( \#(L) = 2 \). The endomorphism \( \psi \) of \( C_2 \ast C_2 \) is explicitly described by the formula

\[ \psi(a) = a \quad \text{and} \quad \psi(b) = (ba)^{s-1}b. \]

Likewise \( f: \mathbb{R} \rightarrow \mathbb{R} \) is given by

\[ f(x) = sx \quad \text{for all} \ x \in \mathbb{R}. \]
The quotient group \( L_s \) of \( L \) is now the finite dihedral group \( D_{2s} \) of order \( 2s \); it is gotten from \( C_2 \ast C_2 \) by adding the extra relation \((ab)^s = 1\). Note that \( \psi(L) \) is the subgroup of index \( s \) in \( C_2 \ast C_2 \) generated by \( a \) and \((ba)^s \), and is of course isomorphic to \( C_2 \ast C_2 \) via \( \psi \). Also \( \Gamma_s = \phi^{-1}(\psi(L)) \) is a subgroup of index \( s \) in \( \Gamma \).

The smooth 5-manifold \( E_s \) is the space \( \mathbb{R} \times \Gamma, M \) and the map \( q_s : E_s \to [0,1] \) is the composition of

\[
E_s \to \mathbb{R}/\psi(K) = [0,s]
\]

with the homeomorphism

\[
\hat{f}^{-1} : [0,s] \to [0,1]
\]

given by multiplication by \( 1/s \). It is a stratified system of fibrations with \( \{0,1\} \) and \( (0,1) \) the strata for \([0,1]\). We will apply Frobenius induction with respect to the finite quotient group \( L_s \) of \( \Gamma \). Since we only need to obtain results modulo the class \( T \), it suffices to consider only the maximal cyclic subgroups of \( L_s \) and we need only one representative from each conjugacy class. These can be chosen to be the cyclic subgroups \( \langle a \rangle \) and \( \langle ab \rangle \). Note that \( |\langle a \rangle| = 2 \) and \( |\langle ab \rangle| = s \). Let \( \eta : \Gamma \to L_s \) denote the composite epimorphism

\[
\Gamma \xrightarrow{\phi} L \to L_s.
\]

We will be concerned with the subgroups \( \eta^{-1}(\langle a \rangle) \) and \( \eta^{-1}(\langle ab \rangle) \). Note that \( \eta^{-1}(\langle a \rangle) = \Gamma_s \) and \( \phi \eta^{-1}(\langle ab \rangle) \cong \mathbb{Z} \). Denote \( \eta^{-1}(\langle ab \rangle) \) by \( \Gamma(s) \). Hence the already verified Case 1 yields that \( K_0(k[\Gamma(s) \times \mathbb{Z}^n]) \otimes \mathbb{Q} \) is generated by the images of \( K_0(kG) \otimes \mathbb{Q} \) as \( G \) varies over the finite subgroups of \( \Gamma(s) \). Thus by Corollary \ref{cor1} and the important fact \( \text{[3,4]} \), to complete the proof in Case 2, it suffices to associate an odd prime \( s = s(x) \) to each element \( x \in K_0(k[\Gamma \times \mathbb{Z}^n]) \) where \( n \geq 0 \), such that the transfer \( x_s \) of \( x \) to \( K_0(k[\Gamma_s \times \mathbb{Z}^n]) \) satisfies the following:

There exists a positive integer \( m \) such that \( m x_s \) is the sum of elements induced from \( K_0(kG) \) as \( G \) varies over the finite subgroups of \( \Gamma_s \).

We may assume that \( n = 0 \) because the \( \mathbb{Z}^n \) factor is a “dummy variable”, and we work with the more convenient functor \( K_0 \) instead of \( K_0 \); this is clearly equivalent (cf. Remark \ref{remark2}).

Notice that it is enough to construct \( s = s(x) \) so that \( x_s \) is in the image of the assembly map

\[
\alpha_s : \mathbb{H}_0([0,1]; \text{Wh}^k(q_s)) \to \tilde{K}_0(k\Gamma_s).
\]

To see this note that the Atiyah-Hirzebruch-Quinn spectral sequence together with our assumption of Corollary \ref{cor1} yields that any element in \( \text{im}(\alpha_s) \) has a nonzero multiple which is the sum of elements induced from \( K_0(kG) \) as \( G \) varies over the finite subgroups of \( \Gamma \). (See the second paragraph before Assertion 4.3 for more details.)

On the other hand, it was shown in the proof of Theorem \ref{thm1} that an analogous statement with \( \tilde{K}_0 \) replaced by \( \text{Wh}^k \) is true. To be explicit given any \( y \in \text{Wh}^k(\Gamma \times \mathbb{Z}) \), there is an odd prime \( \bar{s}(y) = s \) such that the transfer \( y_s \in \text{Wh}^k(\Gamma_s \times \mathbb{Z}) \) of \( y \) is in the image of the assembly map

\[
\bar{\alpha}_s : \mathbb{H}_1([0,1]; \text{Wh}^k(q_s)) \to \text{Wh}^k(\Gamma_s \times \mathbb{Z}).
\]
where \( q_s : E_s \times \mathbb{S}^{1} \to [0,1] \) is the composite of the projection \( E_s \times \mathbb{S}^{1} \to E_s \) with \( q_s : E_s \to [0,1] \). Since the projection map from \( \text{Wh} \) to \( \tilde{K}_0 \) in the Bass-Heller-Swan formula can be naturally defined on the spectrum level, there is the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}_0 \left( [0,1]; \text{Wh}^k(q_s) \right) & \xrightarrow{a_s} & \tilde{K}_0(k\Gamma_s) \\
\uparrow & & \uparrow \\
\mathbb{H}_1 \left( [0,1]; \text{Wh}^k(q_s) \right) & \xrightarrow{a_s} & \text{Wh}^k(\Gamma_s \times \mathbb{Z}).
\end{array}
\]

Now set \( s(x) = \bar{s}(y) \) where \( y \) is the image of \( x \) in \( \text{Wh}^k(\Gamma \times \mathbb{Z}) \) under the natural embedding

\[
\tilde{K}_0(k\Gamma) \to \text{Wh}^k(\Gamma \times \mathbb{Z}).
\]

Then by a diagram chase, one sees directly that \( x_{s(x)} \) is in the image of \( a_{s(x)} \). \( \square \)

**Proof of Theorem 7.1.** We start by making some general constructions of spectra arising from a unital ring homomorphism by applying the \( K \)-theory functor.

**Notation.** We let \( f : R \to S \) denote a unital ring homomorphism between rings \( R \) and \( S \) both of which contain \( 1 \), and we let \( f_* : \overline{K}(R) \to \overline{K}(S) \) denote the map of \( K \)-theory spectra induced by \( f \) with \( \overline{K}(f) \) denoting the cofiber spectrum of \( f_* \). If \( G \) is a group, then \( f^G : RG \to SG \) denotes the ring homomorphism induced by \( f \). Note that

\[
f^G_* : \overline{K}(RG) \to \overline{K}(SG)
\]

denotes the corresponding map of spectra whose cofiber is denoted by \( \overline{K}(f^G) \).

Composition with the fundamental group determines three spectra valued functors \( F_R, F_S, F_f \) from topological spaces \( X \) to spectra explicitly given by the formulae

\[
F_R(X) = \overline{K}(R\pi_1(X)), \quad F_S(X) = \overline{K}(S\pi_1(X)), \quad F_f(X) = \overline{K}(f^{\pi_1(X)})
\]

and a natural transformation

\[
f_*^{\pi_1(X)} : F_R(X) \to F_S(X).
\]

Let \( p : E \to B \) denote a simplicially stratified fibration (cf. [15, p. 254]). Then \( p \) determines three new spectra by applying Quinn’s *Homology with stratified coefficients* construction to \( p \) (cf. [32, Appendix], [18, §1.2 and §1.4]), namely

\[
\mathbb{H}(B; F_R(p)), \quad \mathbb{H}(B; F_S(p)), \quad \mathbb{H}(B; F_f(p)),
\]

and the natural transformation determines a (coefficient) map of spectra

\[
f_B : \mathbb{H}(B; F_R(p)) \to \mathbb{H}(B; F_S(p))
\]

whose cofiber spectrum is \( \mathbb{H}(B; F_f(p)) \). This fact follows from the explicit construction of \( \mathbb{H}(B; S(p)) \) given in [32, Definition 8.1] (where \( X \to S(X) \) is an arbitrary covariant functor from spaces to spectra) using that the cofiber spectra are defined by a mapping cone construction; cf. [11, p. 154]. And Quinn’s construction [32, Definition 8.1] is applicable because of [32, Proposition 8.4].

For any functor \( F \) from spaces to spectra, let

\[
a_F : \mathbb{H}(B; F(p)) \to F(E)
\]
denote the Quinn assembly map (of spectra); cf. [32, Appendix]. Since assembly respects natural transformations of functors, we have the following two step (homotopy) commutative ladder of spectra:

\[
\begin{array}{cccccc}
K(R\pi_1(E)) & \longrightarrow & K(S\pi_1(E)) & \longrightarrow & K(f\pi_1(E)) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{H}(B; F\pi_1) & \longrightarrow & \mathbb{H}(B; F\pi_2) & \longrightarrow & \mathbb{H}(B; f\pi_1) \\
\end{array}
\]

Ladder (7.1) yields two long exact sequences in homotopy and a commutative ladder connecting them.

Recall there is a fiber square of unital ring epimorphisms

\[
\begin{array}{cccccc}
Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
C^2 & \longrightarrow & \pi \\
\end{array}
\]

where \( f_1 \) and \( g_1 \) send the generator of \( C^2 \) to 1 and \(-1\) respectively. This square induces the fiber square

\[
\begin{array}{cccccc}
(ZC_2)\pi & \longrightarrow & Z\pi \\
\uparrow & & \uparrow \\
Z\pi & \longrightarrow & k_2\pi \\
\end{array}
\]

Let the simplicially stratified fibration \( p \) be the projection map

\[
p: E \times \Gamma \longrightarrow B = E/\Gamma
\]

where \( B\pi \rightarrow B\pi \) is the covering space of the Eilenberg-Mac Lane space \( B\pi \) for the group \( \pi \) corresponding to the normal subgroup \( \ker \phi \) of \( \pi \).

Now consider the commutative ladder of homology exact sequences determined by ladder (7.1) in the special case where \( f: R \rightarrow S \) is \( f_1: ZC_2 \rightarrow Z \) (and \( p: E \rightarrow B \) is as specified in (7.4)). Note that \( a_{F\pi_1} \) and \( a_{F\pi_2} \) are the assembly maps in the FIC for \( \Gamma \) relative to the functor \( K(Z) \) and the epimorphisms \( \phi: \pi \rightarrow \Gamma \) and \( \phi: C_2 \times \pi \rightarrow \Gamma \), respectively, where \( \phi \) is the composition of \( \phi \) with the canonical epimorphism \( C_2 \times \pi \rightarrow \pi \). But these assembly maps induce isomorphisms in homotopy in all dimensions \( \leq 1 \), by [18, Theorem 2.1] for Theorem 7.1(i) and by [20, Corollary 4.2] for Theorem 7.1(ii). (Here are some more details concerning the first assertion in the last sentence. Remark 7.2, replacing \( k \) by \( Z \), reduces it to the same assertion with the spectrum \( K \) replaced by the spectrum \( Wh \). But there is a natural map of spectra

\[
\psi_2: P_\pi(\cdot) \longrightarrow Wh_\pi(\cdot)
\]

where \( P(\cdot) \) is the topological pseudo-isotopy spectrum. And \( \psi_2 \) induces an isomorphism of \( \pi_i \) for all \( i \leq -1 \). In particular \( Wh(\pi_1 X) = \pi_{-1} P(X) \). See [17, p. 775, Proof of Corollary 0.7] for a detailed account of an analogous assertion.) Consequently so does \( a_{F\pi_1} \). We will soon show in Lemma 7.4 that this is also true.
for $a_{F_{1_2}}$. To do this, first note that the fiber square (7.2) determines a natural transformation

$$g^X : F_{f_1}(X) \to F_{f_2}(X)$$

making the following ladder of spectra maps commutative

$$
\begin{array}{ccc}
F_{ZC_2}(X) & \xrightarrow{f_1(X)} & F_Z(X) \\
\downarrow & & \downarrow \\
F_2(X) & \xrightarrow{g_2(X)} & F_{h_2}(X) \\
\downarrow & & \downarrow \\
F_2(X) & \xrightarrow{f_2(X)} & F_{f_2}(X)
\end{array}
$$

The map of “coefficients” induced by this natural transformation is denoted by

$$\hat{g} : \mathbb{H}(B; F_{f_1}(p)) \to \mathbb{H}(B; F_{f_2}(p))$$

and fits into the following (homotopy) commutative diagram of spectra maps:

$$
\begin{array}{ccc}
\mathbb{K}(f_1^\pi(\mathcal{E})) & \xrightarrow{g^E} & \mathbb{K}(f_2^\pi(\mathcal{E})) \\
\uparrow a_{F_{f_1}} & & \uparrow a_{F_{f_2}} \\
\mathbb{H}(B; F_{f_1}(p)) & \xrightarrow{\hat{g}} & \mathbb{H}(B; F_{f_2}(p)).
\end{array}
$$

**Lemma 7.4.** In this square both $g^E$ and $\hat{g}$ induce isomorphisms in homology in all dimensions $\leq 1$.

Before proving Lemma 7.4, note that it directly implies the fact about $a_{F_{1_2}}$ asserted above; i.e. we have the following consequence

**Corollary 7.5.** The map $a_{F_{1_2}}$ (as well as $a_{F_{f_1}}$) of spectra induces an isomorphism in homology in all dimensions $\leq 1$.

**Proof of Lemma 7.4.** We start by establishing the general fact that $g^X$ induces an isomorphism in homology in all dimensions $\leq 1$ for every path connected space $X$, and for $E$ in particular. But this is merely a reformulation of Bass’s excision property which states that the relative $K$-groups $K_i(R, I)$ for a two-sided ideal $I$ in a ring $R$ depend, for all $i \leq 0$, only on the ring structure of $I$ (not on $R$). Then note that

$$K_i(f_1^\pi) = K_{i-1}(\mathbb{Z}[C_2 \times \pi], \ker(f_1^\pi)) \quad \text{and} \quad K_i(f_2^\pi) = K_{i-1}(\mathbb{Z}[\pi], \ker(f_2^\pi))$$

where $\pi = \pi_1(X)$. But $\ker(f_2^\pi) \cong \ker(f_1^\pi)$ as rings, because $\mathcal{F}_2$ is a fiber square of epimorphisms.

To establish the statement about $\hat{g}$, we use a variant of the comparison theorem for the map induced by $\hat{g}$ between the Atiyah-Hirzebruch-Quinn spectral sequences for $\mathbb{H}(B; F_{f_1}(p))$ and $\mathbb{H}(B; F_{f_2}(p))$, respectively. By the general fact about $g^X$ demonstrated above, we get that this map is an isomorphism between $E^{ij}_2$-terms for all $j \leq 1$. From this one shows (by induction on $n$) that it induces an isomorphism between $E^{ij}_n$-terms for all $i + j \leq 1$ and an epimorphism between $E^{n,i}_{1,2-i}$-terms for all $i \geq 1$. Consequently $\hat{g}$ induces an isomorphism in homology in all dimensions $\leq 1$. □
Now consider the commutative ladder of homology exact sequences determined by ladder \( (7.1) \) in the new special case where \( f: R \rightarrow S \) is \( f_2: \mathbb{Z} \rightarrow k_2 \) (and \( p: E \rightarrow B \) is as in \( (7.1) \)). Note that \( \alpha_{F_{k_2}} \) is the assembly map mentioned in Theorem \( (7.1) \). Since \( \alpha_{F_2} \) and \( \alpha_{F_{k_2}} \) both induce isomorphisms in homology in all dimensions \( \leq 1 \) because of \([18] \) Theorem 2.1, \([20] \) Corollary 4.3 and Corollary \([16] \), we conclude from the 5-lemma that \( \alpha_{F_{k_2}} \) induces an isomorphism in homology in all dimensions \( \leq 0 \) and an epimorphism in dimension 1.

To complete the proof of Theorem \( (7.1) \) it remains to show that \( \alpha_{F_{k_2}} \) induces a monomorphism in dimension 1. To do this we make use of Milnor’s Mayer-Vietoris element \( x \) by ladder \( (7.1) \) in the new special case where \( K \) is as in \( (7.1) \). Let \( \bar{z} \) be an assembly map \( (7.5) \) described above, \( z \) pulls back to an element \( \hat{x} \in K_1(\mathbb{Z}[\mathbb{C}_2 \times \pi]) \) solving equations 1 and 2 of \( (7.5) \). Since \( K_1(f_2^\pi) (x) = 0 \), there exists an element \( \hat{x} \in K_1(\mathbb{Z}[\mathbb{C}_2 \times \pi]) \) satisfying

\[
\begin{align*}
1. & \quad K_1(g_1^\pi)(\hat{x}) = x \\
2. & \quad K_1(f_1^\pi)(\hat{x}) = 0.
\end{align*}
\]

\( (7.5) \)

**Notation.** Let \( A_{C_2}, A_2, \) and \( A_k \) denote the group homomorphisms induced by the assembly maps \( \alpha_{F_{C_2}}, \alpha_{F_2} \) and \( \alpha_{F_{k_2}} \), respectively, in 1-dimensional homology. Also let \( \bar{f}_1, \bar{f}_2, \bar{g}_1, \) and \( \bar{g}_2 \) denote the homomorphisms in 1-dimensional homology induced by the spectra maps \( (f_1)_B, (f_2)_B, (g_1)_B \) and \( (g_2)_B \), respectively.

Let \( z \in \ker(A_k) \). Once we show that \( z = 0 \), we will have completed the proof of Theorem \( (7.1) \). By the second commutative ladder of homology exact sequences described above, \( z \) pulls back to an element \( y \in \mathbb{H}_1(B; F_2(\mathbb{Z})) \); i.e. \( \bar{f}_2(y) = z \). Now let \( x = A_2(y) \) and note that \( K_1(f_2^\pi)(x) = 0 \). Hence there is an element \( \hat{x} \in K_1(\mathbb{Z}[\mathbb{C}_2 \times \pi]) \) solving equations 1 and 2 of \( (7.5) \). Since \( A_{C_2} \) is an isomorphism, there exists

\[ \hat{y} \in \mathbb{H}_1(B; F_{2C_2}(\mathbb{Z})) \]

such that \( A_{C_2}(\hat{y}) = \hat{x} \). Note that equations 1 and 2 of \( (7.5) \) yield

1. \( \bar{g}_1(\hat{y}) = y \)
2. \( \bar{f}_1(\hat{y}) = 0 \)

since \( A_2 \) is an isomorphism and the following two diagrams commute:

\[
\begin{array}{ccc}
\mathbb{H}_1(B; F_{2C_2}(\mathbb{Z})) & \xrightarrow{\bar{g}_1} & \mathbb{H}_1(B; F_2(\mathbb{Z})) \\
\mathbb{H}_1(B; F_{2C_2}(\mathbb{Z})) & \xrightarrow{\bar{f}_1} & \mathbb{H}_1(B; F_2(\mathbb{Z})) \\
A_{C_2} & \downarrow & \downarrow A_2 \\
K_1(\mathbb{Z}[\mathbb{C}_2 \times \pi]) & \xrightarrow{K_1(g_1^\pi)} & K_1(\mathbb{Z}[\pi]) \\
K_1(\mathbb{Z}[\mathbb{C}_2 \times \pi]) & \xrightarrow{K_1(f_1^\pi)} & K_1(\mathbb{Z}[\pi]).
\end{array}
\]

But \( \bar{g}_2 \circ \bar{f}_1 = \bar{f}_2 \circ \bar{g}_1 \) because of square \( (7.2) \), hence

\[ 0 = \bar{g}_2(\bar{f}_1(\hat{y})) = \bar{f}_2(\bar{g}_1(\hat{y})) = \bar{f}_2(y) = z. \]

\[ \square \]

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