GENERALIZED EXPONENTIAL BEHAVIOR AND TOPOLOGICAL EQUIVALENCE

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Abstract. We discuss the topological equivalence between evolution families with a generalized exponential dichotomy. These can occur for example when all Lyapunov exponents are infinite or all Lyapunov exponents are zero. In particular, we show that any evolution family admitting a generalized exponential dichotomy is topologically equivalent to a certain normal form, in the which the exponential behavior in the stable and unstable directions are multiples of the identity. Moreover, we show that the topological equivalence between two evolution families admitting generalized exponential dichotomies, possibly with different growth rates, can be completely characterized in terms of a new notion of equivalence between these rates.

1. Introduction. In the theories of differential equations and dynamical systems, the emphasis is often on the description of the qualitative behavior of the system. Sometimes this is unavoidable, due to the lack of an explicit description of the trajectories, although also often it is simply the best approach to understand some properties of the system. For example, the Grobman–Hartman theorem says that in a sufficiently small neighborhood of a hyperbolic fixed point there is a topological conjugacy between the trajectories of the system and those of its linearization. Hence, at least in a neighborhood of the hyperbolic fixed point, from the qualitative point of view it is sufficient to know the associated linear system. Certainly, there are other types of conjugacies, such as differentiable conjugacies, and they play an important role in the theory although in our setting they would be too rigid. For example, two linear flows in a finite-dimensional space are differentially conjugate if and only if the matrices of the systems are conjugate, that is, if they have the same Jordan canonical form.

In this work our main aim is to discuss the topological equivalence between (linear) evolution families with a generalized exponential dichotomy, in the sense that

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the contraction and expansion can be of the form $e^{\rho(t)}$ for an arbitrary increasing function $\rho$, instead of simply the usual exponential behavior $e^{ct}$. This type of generalized exponential behavior was considered earlier for example in [1, 4, 5]. It can occur naturally for instance when all Lyapunov exponents are infinite or all Lyapunov exponents are zero (the latter may correspond, for example, to a polynomial behavior).

Before being more detailed, we first recall briefly how evolution families appear naturally. Consider a nonautonomous linear equation

$$x' = A(t)x$$
on a Banach space $E$, where $t \mapsto A(t)$ is a continuous map with values on the set of bounded linear operators on $E$. Then each solution $x = x(t)$ can be written in the form $x(t) = U(t,s)x(s)$, for some linear operators $U(t,s)$ such that $U(t,t) = \text{Id}$ and

$$U(t,s)U(s,r) = U(t,r)$$

for all $t,s,r$. Any such family of linear operators is a particular case of an evolution family (see Section 2 for the definition). In the particular case considered here the map $(t,s) \mapsto U(t,s)$ is differentiable but we shall not require this property.

In order to describe our main results we recall briefly the notions of an exponential dichotomy and of a topological conjugacy. The classical notion of an exponential dichotomy corresponds to the existence of complementary projections $P(t)$ and $Q(t)$ for each $t$ and constants $K, \lambda > 0$ such that

$$\|U(t,s)P(s)\| \leq Ke^{-\lambda(t-s)}, \quad \|U(s,t)Q(t)\| \leq Ke^{-\lambda(t-s)}$$

for $t \geq s$. In the notion of a generalized exponential dichotomy these inequalities are replaced by

$$\|U(t,s)P(s)\| \leq Ke^{-\lambda(\rho(t)-\rho(s))}, \quad \|U(s,t)Q(t)\| \leq Ke^{-\lambda(\rho(t)-\rho(s))}$$

for $t \geq s$, thus allowing arbitrary growth rates. We also recall that two evolution families $U(t,s)$ and $V(t,s)$ are said to be topologically equivalent if there exist homeomorphisms $h_t$ (sometimes with a prescribed growth at infinity; see Section 3) such that

$$h_t \circ U(t,s) = V(t,s) \circ h_s$$

for all $t,s$. The maps $h_t$ act as a dictionary between the two dynamics and any properties depending only on the topology, such as for example the existence of dense orbits, is transferred by these maps from one dynamics to the other.

Our main results are the following:

1. Any evolution family admitting a generalized exponential dichotomy is topologically equivalent to a certain normal form, in the which the behaviors in the stable and unstable directions are multiples of the identity and so they are essentially 1-dimensional.

2. The topological equivalence between two evolution families admitting generalized exponential dichotomies, possibly with different growth rates, can be completely characterized by a new notion of equivalence between growth rates.
Our first main result says that an evolution family \( U(t,s) \) admitting a generalized exponential dichotomy can be brought by a topological conjugacy to the normal form
\[
x' = -x, \quad y' = y,
\] (1)
where \( x \) and \( y \) belong, respectively, to spaces whose dimensions are those of the stable and unstable subspaces. The proof consists of constructing explicitly the maps \( h_t \) in the notion of topological equivalence. For a nonautonomous linear differential equation on an arbitrary Banach space, it was shown in [8] that if the equation admits an exponential dichotomy (with \( \rho(t) = t \)), then the corresponding evolution family is topologically conjugate to the evolution family associated to (1) (the case of finite-dimensional spaces was considered earlier in [6]). Theorem 3.1 further generalizes this result to generalized exponential dichotomies, with a proof based on an explicit construction of conjugacies inspired in [8].

A consequence of Theorem 3.1 is that any evolution family \( U(t,s) \) with generalized bounded growth (see Section 2 for the definition) that admits a generalized exponential dichotomy is topologically equivalent to any sufficiently small linear perturbation
\[
V(t,s) = U(t,s) + \int_s^t U(t,\xi)B(\xi)V(\xi, s)\,d\xi,
\]
where \( t \mapsto B(t) \) is a continuous map with values on the set of bounded linear operators on \( E \). Indeed, it can be shown that if \( \sup_{\xi \in \mathbb{R}} \|B(\xi)\| \) is sufficiently small, then the evolution family \( V(t,s) \) also admits a generalized exponential dichotomy (with the same growth rate) and with topologically equivalent stable and unstable subspaces (see [2]). Therefore, as a consequence of Theorem 3.1, the evolution families \( U(t,s) \) and \( V(t,s) \) are topologically equivalent. In the particular case of the growth rate \( \rho(t) = t \), this result was obtained earlier in [8].

Our second main result addresses the problem of how one can characterize the notion of topological equivalence between evolution families admitting a generalized exponential dichotomy in terms of a notion of equivalence between growth rates. Two growth rates \( \rho_1 \) and \( \rho_2 \) are said to be equivalent if there exist constants \( \alpha_i, \beta_i > 0 \) for \( i = 1, 2 \) such that
\[
\rho_1(t) - \rho_1(s) \geq \alpha_1[\rho_2(t) - \rho_2(s)] - \beta_1
\]
and
\[
\rho_2(t) - \rho_2(s) \geq \alpha_2[\rho_1(t) - \rho_1(s)] - \beta_2
\]
for \( t \geq s \). It turns out that this notion characterizes completely (and is in fact equivalent to) the notion of topological equivalence between evolution families admitting generalized exponential dichotomies, possibly with different growth rates \( \rho_1 \) and \( \rho_2 \). This is the content of Theorem 4.4. As an outcome one can classify completely all 1-dimensional evolution families \( U(t,s) = e^{\rho(t) - \rho(s)} \) using the notion of equivalence (see Proposition 4.6). Finally, in Section 5 we detail briefly the relations between Bohl exponents and the notion of equivalence between growth rates.

2. Generalized exponential behavior. Let \( E \) be a Banach space and let \( \mathcal{L}(E) \) be the set of all bounded linear operators acting on \( E \). A family
\[
\mathcal{U} = \{ U(t,s) : t, s \in \mathbb{R}, \ t \geq s \}
\]
of linear operators in \( \mathcal{L}(E) \) is called an evolution family if:
1. \( U(t,t) = \text{Id} \) and
\[
U(t,s)U(s,r) = U(t,r)
\]
for \( t \geq s \geq r \);
2. for each \( s \in \mathbb{R} \) and \( x \in E \), the map \( t \mapsto U(t,s)x \) is continuous.

An evolution family \( \mathcal{U} \) is said to be reversible if \( U(t,s) \) is invertible for all \( t \geq s \).

In this case we write \( U(s,t) = U(t,s)^{-1} \) for \( t \geq s \) and identity (2) holds for all \( t,s,r \in \mathbb{R} \).

For simplicity of the exposition we shall always consider reversible evolution families.

Now let \( \mathcal{F} \) be the set of all increasing continuous functions \( \rho: \mathbb{R} \to \mathbb{R} \) with \( \rho(0) = 0 \). Given \( \rho \in \mathcal{F} \), we write
\[
\Phi(t,s) = \Phi_{\rho}(t,s) = \rho(t) - \rho(s).
\]

A reversible evolution family \( \mathcal{U} \) is said to have \( \rho \)-bounded growth if there exist constants \( M, \alpha \geq 0 \) such that
\[
\|U(t,s)\| \leq Me^{\alpha |\Phi(t,s)|}
\]
for \( t, s \in \mathbb{R} \).

The following is a characterization of bounded growth.

**Proposition 2.1.** A reversible evolution family \( \mathcal{U} \) has \( \rho \)-bounded growth if and only if there exist constants \( \varepsilon, \delta > 0 \) such that
\[
|\Phi(t,s)| \leq \delta \Rightarrow \|U(t,s)\| \leq \varepsilon
\]
for each \( t, s \in \mathbb{R} \).

**Proof.** It is clear that if \( \mathcal{U} \) has \( \rho \)-bounded growth, then (3) holds.

Now let us assume that property (3) holds. Given \( t \geq s \), let
\[
n = \left\lfloor \frac{\rho(t) - \rho(s)}{\delta} \right\rfloor,
\]
where \( \lfloor \cdot \rfloor \) denotes the integer part. Then
\[
\rho(t) = \rho(s) + n\delta + \theta, \quad \text{with} \quad \theta \in [0, \delta).
\]

By (3) we obtain
\[
\|U(t,s)\| \leq \|U(\rho^{-1}(\rho(t)), \rho^{-1}(\rho(t) - \theta))\| \cdot \|U(\rho^{-1}(\rho(t) - \theta), s)\|
\]
\[
\leq \varepsilon \prod_{k=0}^{n-1} \|U(\rho^{-1}(\rho(t) - k\delta), \rho^{-1}(\rho(t) - (k+1)\delta))\|
\]
\[
\leq \varepsilon \varepsilon^n \leq \varepsilon \varepsilon \varepsilon^{(\rho(t) - \rho(s))/\delta} = \varepsilon \varepsilon \varepsilon^{c(\rho(t) - \rho(s))},
\]
where \( c = \log \varepsilon/\delta \). One can proceed in a similar manner for \( t \leq s \) and hence, \( \mathcal{U} \) has \( \rho \)-bounded growth. \( \square \)

3. **Exponential dichotomies and topological equivalence.** In this section we consider evolution families admitting a generalized exponential behavior and we show that they can always be transformed (using a topological conjugacy) into a canonical form that expands and/or contracts the same in all directions.

A reversible evolution family \( \mathcal{U} \) is said to admit a \( \rho \)-exponential dichotomy if there exist complementary projections \( P(t) + Q(t) = \text{Id} \), for \( t \in \mathbb{R} \), and constants \( K, \lambda > 0 \) such that
\[
P(t)U(t,s) = U(t,s)P(s)
\]
and
\[ \|U(t, s)P(s)\| \leq Ke^{-\lambda\Phi(t, s)}, \quad \|U(s, t)Q(t)\| \leq Ke^{-\lambda\Phi(t, s)} \]
for \( t, s \in \mathbb{R} \) with \( t \geq s \).

We also introduce the notion of topological equivalence. Two reversible evolution families \( \mathcal{U} = \{U(t, s)\} \) and \( \mathcal{V} = \{V(t, s)\} \) are said to be topologically equivalent if there exist a continuous map \( h: \mathbb{R} \times E \to E \) and an increasing onto map \( L: \mathbb{R}^+_0 \to \mathbb{R}^+_0 \) such that:

1. \( h_t = h(t, \cdot): E \to E \) is a homeomorphism for \( t \in \mathbb{R} \);
2. \( U(t, s) \circ h_s = h_t \circ V(t, s) \) for \( t, s \in \mathbb{R} \);
3. \( \|h_t(x)\| \leq L(\|x\|) \) and \( \|h_t^{-1}(x)\| \leq L(\|x\|) \) for \( t \in \mathbb{R} \) and \( x \in E \).

Our main result shows that any reversible evolution family admitting an exponential dichotomy can be brought to an essentially one-dimensional evolution family by an appropriate topological conjugacy.

**Theorem 3.1.** Given a \( C^1 \) onto function \( \rho \in \mathcal{F} \), let \( \mathcal{U} \) be a reversible evolution family with \( \rho \)-bounded growth. If \( \mathcal{U} \) admits a \( \rho \)-exponential dichotomy, then it is
topologically equivalent to the evolution family
\[ V(t, s) = e^{\Phi(s, t)}P(0) + e^{\Phi(t, s)}Q(0). \]

**Proof.** Write \( \omega = \rho' \) (note that \( \omega \) is a continuous function). For each \( t \in \mathbb{R} \) and \( x \in E \), let
\[ \|x\|_t = \int_t^\infty \|U_P(\xi, t)x\| \omega(\xi) \, d\xi + \int_{-\infty}^t \|U_Q(\xi, t)x\| \omega(\xi) \, d\xi, \]
where
\[ U_P(t, s) = U(t, s)P(s) \quad \text{and} \quad U_Q(t, s) = U(t, s)Q(s). \]
Since \( \omega > 0 \), the map \( x \mapsto \|x\|_t \) is a norm on \( E \).

**Lemma 3.2.** For \( t \in \mathbb{R} \) and \( x \in E \), we have
\[ \frac{1}{\alpha M}\|x\| \leq \|x\|_t \leq \frac{2K}{\lambda}\|x\|. \quad (4) \]

**Proof of the lemma.** Clearly,
\[ \|x\|_t = \|P(t)x\|_t + \|Q(t)x\|_t, \]
where
\[ \|P(t)x\|_t = \int_t^\infty \|U_P(\xi, t)x\| \omega(\xi) \, d\xi \]
and
\[ \|Q(t)x\|_t = \int_{-\infty}^t \|U_Q(\xi, t)x\| \omega(\xi) \, d\xi. \]
We have
\[ \|P(t)x\|_t \leq \|x\| \int_t^\infty Ke^{-\lambda\Phi(\xi, t)} \omega(\xi) \, d\xi = \frac{K}{\lambda}\|x\| \quad (5) \]
and similarly,
\[ \|Q(t)x\|_t \leq \frac{K}{\lambda}\|x\|. \quad (6) \]
This yields the second inequality in (4). Moreover, for \( \xi \geq t \) we have
\[ \|P(t)x\|_t \omega(\xi) \leq \|U(t, \xi)\| \cdot \|U_P(\xi, t)x\| \omega(\xi) \]
\[ \leq Me^{\alpha\Phi(\xi, t)}\|U_P(\xi, t)x\| \omega(\xi), \]
for some constants $M, \alpha > 0$ (since $\mathcal{U}$ has $\rho$-bounded growth). Therefore,

$$\|P(t)x\|_t \geq \frac{\|P(t)x\|}{M} \int_t^{\infty} e^{-\alpha \Phi(\xi,t)} \omega(\xi) \, d\xi = \frac{\|P(t)x\|}{\alpha M}$$

and analogously,

$$\|Q(t)x\|_t \geq \frac{\|Q(t)x\|}{\alpha M}.$$  \hspace{1cm} (8)

The first inequality in (4) follows now readily from (5)–(6) and (7)–(8).

Let us write $U(t) = U(t,0), P = P(0)$ and $Q = Q(0)$.

**Lemma 3.3.** The following properties hold:

1. $t \mapsto \|U(t)Px\|_t$ is strictly decreasing and $t \mapsto \|U(t)Qx\|_t$ is strictly increasing;
2. given $x \in (PE \cup QE) \setminus \{0\}$, there exists a unique $t \in \mathbb{R}$ such that $\|U(t)x\|_t = 1$.

**Proof of the lemma.** Since

$$\|U(t)Px\|_t = \int_t^{\infty} \|U_P(\xi,0)x\| \omega(\xi) \, d\xi,$$  \hspace{1cm} (9)

we have

$$\frac{d}{dt}\|U(t)Px\|_t = -\|U_P(t,0)x\| \omega(t) < 0.$$  

Similarly, since

$$\|U(t)Qx\|_t = \int_{-\infty}^t \|U_Q(\xi,0)x\| \omega(\xi) \, d\xi,$$

we have

$$\frac{d}{dt}\|U(t)Qx\|_t = \|U_Q(t,0)x\| \omega(t) > 0.$$  

This establishes the first property.

Now we establish the second property. It follows from (9) that $\|U(t)Px\|_t \to 0$ when $t \to \infty$. On the other hand, for $t \leq \xi \leq 0$ we have

$$\|Px\| \omega(\xi) \leq \|U_P(0,\xi)\| \cdot \|U_P(\xi,0)x\| \omega(\xi)$$

$$\leq Ke^{\lambda \rho(\xi)} \|U_P(\xi,0)x\| \omega(\xi)$$

and thus,

$$\|U(t)Px\|_t \geq \int_t^0 \|U_P(\xi,0)x\| \omega(\xi) \, d\xi$$

$$\geq \frac{\|Px\|}{K} \int_t^0 e^{-\lambda \rho(\xi)} \omega(\xi) \, d\xi = \frac{\|Px\|}{K \lambda} e^{-\lambda \rho(t)}.$$  

Therefore, $\|U(t)Px\|_t = \infty$ when $t \to -\infty$, provided that $Px \neq 0$. One can show in a similar manner that

$$\lim_{t \to -\infty} \|U(t)Qx\|_t = \infty \quad \text{and} \quad \lim_{t \to -\infty} \|U(t)Qx\|_t = 0,$$

provided that $Qx \neq 0$. The desired property follows now readily from the first property.

We continue with the proof of the theorem. Define functions

$$h^P_t : PE \to P(t)E \quad \text{and} \quad h^Q_t : QE \to Q(t)E,$$
respectively, by
\[ h^P_t(x) = \begin{cases} \frac{U(t)x}{\|U(\rho^{-1}(\rho(t) + \log\|x\|))x\|^{\rho^{-1}(\rho(t) + \log\|x\|)}} & x \neq 0, \\ 0 & x = 0 \end{cases} \] (10)
and
\[ h^Q_t(x) = \begin{cases} \frac{U(t)x}{\|U(\rho^{-1}(\rho(t) - \log\|x\|))x\|^{\rho^{-1}(\rho(t) - \log\|x\|)}} & x \neq 0, \\ 0 & x = 0 \end{cases} \] (11)
Finally, define \( h_t: E \to E \) by
\[ h_t(x) = h^P_t(Px) + h^Q_t(Qx). \]
In order to show that the maps \( h_t \) give the desired topological conjugacy, we divide the proof into several steps.

**Step 1. Invariance.** We first show that
\[ h_t(e^{\Phi(s,t)}Px + e^{\Phi(t,s)}Qx) = U(t,s)h_s(x). \] (12)
We have
\[ h^P_t(e^{-\rho(t)}Px) = \frac{e^{-\rho(t)}U(t)Px}{\|U(\rho^{-1}(s))Px\|^{\rho^{-1}(s)}} = \frac{U(t)Px}{\|U(\rho^{-1}(\log\|Px\|))Px\|^{\rho^{-1}(\log\|Px\|)}} = U(t)h^P_0(Px), \]
where
\[ s = \rho(t) + \log\|e^{-\rho(t)}Px\| = \log\|Px\|, \]
and analogously,
\[ h^Q_t(e^{\rho(t)}Qx) = U(t)h^Q_0(Qx). \]
Adding the former identities, we obtain
\[ h_t \circ V(t) = U(t) \circ h_0, \]
where \( V(t) = V(t,0) \). This readily implies that
\[ h_t \circ V(t,s) = U(t,s) \circ h_s \]
and (12) holds.

**Step 2. Injectivity of the maps \( h_t \).** Assume that \( h^P_t(Px) = h^P_t(Py) \). Then
\[ \frac{Px}{\|U(\tau_1)P\xi\|_{\tau_1}} = \frac{Py}{\|U(\tau_2)P\xi\|_{\tau_2}} = P\xi \in PE, \] (13)
where
\[ \tau_1 = \rho^{-1}(\rho(t) + \log\|Px\|) \]
and
\[ \tau_2 = \rho^{-1}(\rho(t) + \log\|Py\|). \]
Therefore,
\[ \|U(\tau_1)P\xi\|_{\tau_1} = \|U(\tau_2)P\xi\|_{\tau_2} = 1, \]
which implies that \( \tau_1 = \tau_2 \). Hence, \( \|Px\| = \|Py\| \) and it follows from (13) that
\[ c := \|U(\tau_1)Px\|_{\tau_1} = \|U(\tau_2)Py\|_{\tau_2}. \]
Therefore, \( Px = Py = cP\xi. \)

The injectivity of the maps \( h^P_t \) can be proved in a similar manner. This readily implies that the maps \( h_t \) are one-to-one.

**Step 3. Surjectivity of the maps \( h_t \).** Take \( P(t)y \in P(t)E \). If \( h^P_t(Px) = P(t)y, \) then

\[
\| U(\tau)Px \|_{\tau} = PU^{-1}(t)y,
\]

with

\[
\tau = \rho^{-1}(\rho(t) + \log\|Px\|).
\]

Therefore \( \| U(\tau)PU^{-1}(t)y \|_{\tau} = 1 \) (the existence and uniqueness of \( \tau \) is guaranteed by Lemma 3.3). By (14), we obtain

\[
e^{\Phi(\tau,t)} = \| Px \| = \| U(\tau)Px \|_{\tau} \| PU^{-1}(t)y \|,
\]

that is,

\[
\| U(\tau)Px \|_{\tau} = \frac{\| Px \|}{\| PU^{-1}(t)y \|} = \frac{e^{\Phi(\tau,t)}}{\| PU^{-1}(t)y \|}.
\]

Moreover,

\[
Px = \| U(\tau)Px \|_{\tau} PU^{-1}(t)y = e^{\Phi(\tau,t)} PU^{-1}(t)y
\]

Similarly, if \( h^Q_t(Qx) = Q(t)y, \) then

\[
Qx = e^{\Phi(\tau,\eta)} QU^{-1}(t)y \| QU^{-1}(t)y \|
\]

for a unique \( \eta \) (its existence and uniqueness is guaranteed by Lemma 3.3). This shows that \( h^P_t \) and \( h^Q_t \) are invertible, with inverses

\[
(h^P_t)^{-1}(P(t)y) = e^{\Phi(\tau,t)} PU^{-1}(t)y \| PU^{-1}(t)y \|
\]

and

\[
(h^Q_t)^{-1}(Q(t)y) = e^{\Phi(\tau,\eta)} QU^{-1}(t)y \| QU^{-1}(t)y \|.
\]

The inverse \( h^{-1}_t : E \rightarrow E \) is now given by

\[
h^{-1}_t(y) = (h^P_t)^{-1}(P(t)y) + (h^Q_t)^{-1}(Q(t)y).
\]

**Step 4. Existence of the map \( L \).** By (4) and (10), for

\[
\tau = \rho^{-1}(\rho(t) + \log\|Px\|)
\]

we have

\[
\| h^P_t(Px) \| = \| U(\tau)Px \|_{\tau} \leq \alpha M \| U(\tau)Px \|_{\tau}
\]

If \( \| Px \| \leq 1 \), then since \( e^{\Phi(\tau,t)} = \| Px \| \) we have \( \tau \leq t \) and

\[
\| h^P_t(Px) \| \leq \alpha M \| U_p(t,\tau) \| \cdot \| U(\tau)Px \|
\]

\[
\leq \alpha M e^{-\lambda \Phi(t,\tau)}
\]

\[
= \alpha MK \| Px \|^\lambda
\]

\[
\leq \alpha MK^{1+\lambda} \| x \|^\lambda
\]
If $\|Px\| > 1$, then $\tau > t$ and

$$\|h_t^P(Px)\| \leq \alpha M \|U(t, \tau)\| \leq \alpha M^2 e^{\alpha \Phi(\tau, t)}$$

$$= \alpha M^2 \|Px\|^\alpha \leq \alpha M^2 K^\alpha \|x\|^\alpha.$$ 

Therefore,

$$\|h_t^P(Px)\| \leq L_1(\|x\|),$$

where

$$L_1(\theta) = \max\{\alpha M K^{1+\lambda} \theta^\lambda, \alpha M^2 K \alpha \theta^\alpha\}. \tag{17}$$

Similarly, by (11), we obtain

$$\|h_t^Q(Qx)\| = \frac{\|U(t)Qx\|}{\|U(\eta)Qx\|} \leq M \alpha \frac{\|U(t)Qx\|}{\|U(\eta)Qx\|},$$

where

$$\eta = \rho^{-1}(\rho(t) - \log\|Qx\|),$$

or equivalently $\|Qx\| = e^{\Phi(t, \eta)}$. If $\|Qx\| \leq 1$, then $t \leq \eta$ and

$$\|h_t^Q(Qx)\| \leq \alpha M \frac{\|U(t)Qx\|}{\|U(\eta)Qx\|} \leq \alpha M \frac{\|U_Q(t, \eta)\| \cdot \|U(\eta)Qx\|}{\|U(\eta)Qx\|} \leq \alpha M K e^{-\lambda \Phi(\eta, t)} \leq \alpha M K \|Qx\|^\lambda \leq \alpha M K^{1+\lambda} \|x\|^\lambda$$

On the other hand, if $\|Qx\| > 1$, then $t > \eta$ and

$$\|h_t^Q(Qx)\| \leq \alpha M \frac{\|U(t)Qx\|}{\|U(\eta)Qx\|} \leq \alpha M \frac{\|U(t, \eta)\| \cdot \|U(\eta)Qx\|}{\|U(\eta)Qx\|} \leq \alpha M^2 e^{\alpha \Phi(t, \eta)} \leq \alpha M^2 \|Qx\|^\alpha \leq \alpha M^2 K^\alpha \|x\|^\alpha.$$ 

Therefore,

$$\|h_t^Q(Qx)\| \leq L_1(\|x\|),$$

with $L_1$ as in (17).

Now we observe that by (15),

$$\|((h_t^P)^{-1}(P(t)y))\| = e^{\Phi(\tau, t)}, \tag{18}$$

where $\tau$ is determined by the identity

$$1 = \|U(\tau)PU^{-1}(t)y\| = \|U_P(\tau, t)y\|$$

$$= \int_{\tau}^{\infty} \|U_P(\xi, 0)U^{-1}(t)y\| \omega(\xi) \, d\xi$$

$$= \int_{\tau}^{\infty} \|U_P(\xi, t)y\| \omega(\xi) \, d\xi.$$
If \( t \geq \tau \), then
\[
\int_{\tau}^{t} \| U_P(\xi, t)y \| \omega(\xi) \, d\xi = \int_{\tau}^{\infty} \| U_P(\xi, t)y \| \omega(\xi) \, d\xi - \int_{t}^{\infty} \| U_P(\xi, t)y \| \omega(\xi) \, d\xi
\]
\[
= 1 - \| P(t)y \|_t \leq \int_{\tau}^{t} \| U(\xi, t) \| \cdot \| P(t)y \| \omega(\xi) \, d\xi
\]
\[
\leq M \| P(t)y \| \int_{\tau}^{t} e^{\alpha \Phi(\xi, \tau)} \omega(\xi) \, d\xi
\]
\[
= \frac{M}{\alpha} \| P(t)y \| (e^{\alpha \Phi(\xi, \tau)} - 1).
\]

Using (6) we obtain
\[
1 - \frac{K}{\lambda} \| P(t)y \| \leq 1 - \| P(t)y \|_t \leq \frac{M}{\alpha} \| P(t)y \| (e^{\alpha \Phi(\xi, \tau)} - 1),
\]
which leads to
\[
e^{\Phi(\tau, t)} \leq \left( \frac{M\lambda - K\alpha}{M\lambda} + \frac{\alpha}{MK\|y\|} \right)^{-1/\alpha}.
\]

By (18) we conclude that
\[
\|(h_P^t)^{-1}(P(t)y)\| \leq \left( \frac{M\lambda - K\alpha}{M\lambda} + \frac{\alpha}{MK\|y\|} \right)^{-1/\alpha}
\]
\[
= \|y\|^{1/\alpha} \left( \frac{MK\lambda}{K(M\lambda - K\alpha)\|y\| + \alpha\lambda} \right)^{1/\alpha}
\]
\[
\leq \|y\|^{1/\alpha} \left( \frac{MK\lambda}{\alpha\lambda} \right)^{1/\alpha} = \|y\|^{1/\alpha} \left( \frac{MK}{\alpha} \right)^{1/\alpha}
\]
\[
\text{(without loss of generality one can always assume that } M\lambda > K\alpha).\]

On the other hand, if \( t < \tau \), then
\[
1 = \| U_P(\tau, t)y \| = \int_{\tau}^{\infty} \| U_P(\xi, t)y \| \omega(\xi) \, d\xi
\]
\[
\leq K\|y\| \int_{\tau}^{\infty} e^{-\lambda \Phi(\xi, \tau)} \omega(\xi) \, d\xi
\]
\[
= \frac{K}{\lambda} \|y\| e^{-\lambda \Phi(\tau, t)}.
\]

Therefore,
\[
\|(h_P^t)^{-1}P(t)y\| = e^{\Phi(\tau, t)} \leq \|y\|^{1/\lambda} \left( \frac{K}{\lambda} \right)^{1/\lambda},
\]
\[
\text{(20)}
\]

By (19) and (20), we obtain
\[
\|(h_P^t)^{-1}P(t)y\| \leq L_2(\|y\|),
\]
where
\[
L_2(\|y\|) = \max \left\{ \|y\|^{1/\alpha} \left( \frac{MK}{\alpha} \right)^{1/\alpha}, \|y\|^{1/\alpha} \left( \frac{K}{\lambda} \right)^{1/\lambda} \right\}.
\]
\[
\text{(21)}
\]

Similarly, by (16), we have
\[
\|(h_Q^t)^{-1}(Q(t)y)\| = e^{\Phi(t, \eta)},
\]
where η is determined by the identity
\[ 1 = \int_{-\infty}^{\eta} \|U_Q(\xi,0)U^{-1}(t)y\|\omega(\xi) \, d\xi = \int_{-\infty}^{\eta} \|U_Q(\xi,t)y\|\omega(\xi) \, d\xi. \]

One can show in an analogous manner that if \( t < \eta \), then
\[ \|(h_t^Q)^{-1}(Q(t)y)\| \leq \|y\|^{1/\alpha} \left( \frac{MK}{\alpha} \right)^{1/\alpha} \]
and if \( t \geq \eta \), then
\[ \|(h_t^Q)^{-1}(Q(t)y)\| \leq \|y\|^{1/\lambda} \left( \frac{K}{\lambda} \right)^{1/\lambda}. \]

Therefore,
\[ \|(h_t^Q)^{-1}Q(t)y\| \leq L_2(\|y\|), \]
with \( L_2 \) as in \([21]\).

The above estimates yield the desired result. Indeed,
\[ L(\theta) = 2 \max\{L_1, L_2\} \]
is increasing, \( L(0) = 0 \) and \( L(\theta) \to \infty \) when \( \theta \to \infty \). Moreover,
\[ \|h_t(x)\| \leq L(\|x\|) \quad \text{and} \quad \|h_t^{-1}(x)\| \leq L(\|x\|). \]
This concludes the proof of the theorem. \( \square \)

4. Equivalence relations. In this section we consider a notion of equivalence between the growth rates in \( F \) and we show that it characterizes completely the notion of topological equivalence between evolution families (not necessarily admitting an exponential dichotomy).

4.1. Basic notions. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be almost increasing if there exists a constant \( c \geq 0 \) such that
\[ f(t) - f(s) \geq -c \]
for \( t \geq s \). One can obtain plenty examples of almost increasing functions from functions that are increasing on neighborhoods of infinity.

Example 4.1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function that is increasing on neighborhoods of \( -\infty \) and \( \infty \). Then \( f \) is almost increasing. Indeed, assume that for some \( \alpha \leq \beta \) the function \( f \) is increasing on \( (-\infty, \alpha] \) and on \( [\beta, \infty) \). Since \( f \) is continuous, we have
\[ \inf_{t \in [\alpha, \beta]} f(t) = f(a) \quad \text{and} \quad \sup_{t \in [\alpha, \beta]} f(t) = f(b) \]
for some \( a, b \in [\alpha, \beta] \). For \( t \geq s \) we have:
1. if \( t, s \in (-\infty, \alpha] \) or \( t \in [\beta, \infty) \), then \( f(t) \geq f(s) \);
2. if \( s \in (-\infty, \alpha] \) and \( t \in [\alpha, \beta] \), then
\[ f(t) - f(s) = f(t) - f(\alpha) + f(\alpha) - f(s) \geq f(t) - f(\alpha) \geq f(a) - f(b); \]
3. if \( s \in [\alpha, \beta] \) and \( t \in [\beta, \infty) \), then
\[ f(t) - f(s) = f(t) - f(\beta) + f(\beta) - f(s) \geq f(\beta) - f(s) \geq f(a) - f(b); \]
4. if \( t, s \in [\alpha, \beta] \), then \( f(t) - f(s) \geq f(a) - f(b) \);
5. if $s \in (-\infty, \alpha]$ and $t \in [\beta, \infty)$, then
\[
f(t) - f(s) = f(t) - f(\beta) + f(\beta) - f(\alpha) - f(s) \\
\geq f(\beta) - f(\alpha) \geq f(\alpha) - f(b).
\]

Now we define a binary relation $\preceq$ on the set $\mathcal{F}$ by
\[
\rho_1 \preceq \rho_2 \quad \text{if and only if} \quad \rho_2 - \alpha \rho_1 \text{ is almost increasing}
\]
for some constant $\alpha > 0$. Equivalently,
\[
\rho_1 \preceq \rho_2 \quad \text{if and only if} \quad \Phi_2(t, s) - \alpha \Phi_1(t, s) \geq -c
\]
for $t \geq s$, where $\Phi_1 = \Phi_{\rho_1}$ and $\Phi_2 = \Phi_{\rho_2}$. Moreover, $\rho \preceq \rho$ and if $\rho_1 \preceq \rho_2 \preceq \rho_3$, then $\rho_1 \preceq \rho_3$.

The following is a characterization of the binary relation in (22).

**Proposition 4.2.** Given $\rho_1, \rho_2 \in \mathcal{F}$, we have $\rho_1 \preceq \rho_2$ if and only if there exist constants $\varepsilon, \delta > 0$ such that
\[
\Phi_2(t, s) < \delta \Rightarrow \Phi_1(t, s) < \varepsilon
\]
for each $t \geq s$. In this case,
\[
\Phi_2(t, s) > \frac{\delta}{\varepsilon} \Phi_1(t, s) - \delta \quad \text{for} \quad t \geq s.
\]

**Proof.** If $\rho_1 \preceq \rho_2$, then there exist constants $\alpha, c \geq 0$ such that
\[
\Phi_2(t, s) \geq \alpha \Phi_1(t, s) - c
\]
for $t \geq s$. Now take $\delta > 0$ and let $\varepsilon = (\delta + c)/\alpha$. Clearly, if $\Phi_2(t, s) < \delta$, then $\Phi_1(t, s) < \varepsilon$.

In the other direction, we first assume that $\lim_{t \to \infty} \rho_2(t) = \infty$. For $\varepsilon, \delta > 0$ as in (23), we consider the sequence $t_n$ defined recursively by
\[
t_0 = 0, \quad \rho_2(t_{k+1}) - \rho_2(t_k) = \delta \quad \text{for} \quad k \geq 0.
\]
Since $\rho_2(t_n) - \rho_2(t_0) = n\delta$, we have $t_n \to \infty$ when $n \to \infty$. Now take $t \geq s$ and let
\[
n_0 = \left\lfloor \frac{\rho_2(s)}{\delta} \right\rfloor, \quad n = \left\lceil \frac{\rho_2(t)}{\delta} \right\rceil.
\]
Clearly, $n \geq n_0$, $t_{n_0} \leq s < t_{n_0+1}$ and $t_n \leq t < t_{n+1}$. We consider two cases. If $n = n_0$, then $t_{n_0} \leq s \leq t < t_{n_0+1}$. Therefore,
\[
\rho_2(t) - \rho_2(s) < \rho_2(t_{n_0+1}) - \rho_2(t_{n_0}) < \delta.
\]
By (23), we have $\rho_1(t) - \rho_1(s) < \varepsilon$ and thus,
\[
\rho_2(t) - \rho_2(s) > \frac{\delta}{\varepsilon} - \delta > \frac{\delta}{\varepsilon}(\rho_1(t) - \rho_1(s)) - \delta.
\]
This establishes (24). On the other hand, if $n \geq n_0 + 1$, then
\[
t_{n_0} \leq s \leq t_{n_0+1} \leq \cdots \leq t_n \leq t < t_{n+1}.
\]
By (23), we have
\[
\rho_1(t) - \rho_1(s) = \rho_1(t) - \rho_1(t_n) + \sum_{k=n_0+1}^{n-1} [\rho_1(t_{k+1}) - \rho_1(t_k)] + \rho_1(t_{n_0+1}) - \rho_1(s)
\]
\[
< \sum_{k=n_0}^{n} (\rho_1(t_{k+1}) - \rho_1(t_k)) < (n - n_0)\varepsilon.
\]
On the other hand,
\[ \rho_2(t) - \rho_2(s) > \rho_2(t_n) - \rho_2(t_{n_0+1}) = (n - n_0 - 1)\delta \]
\[ = (n - n_0)\delta - \delta = \delta\frac{n - n_0}{\varepsilon} - \delta \]
\[ > \frac{\delta}{\varepsilon}(\rho_1(t) - \rho_1(s)) - \delta. \]
Again this establishes (24).

Now we assume that \( \lim_{t \to \infty} \rho_2(t) = l < \infty \). When \( \rho_2 \) is constant, it follows from (23) that \( \rho_1(t) - \rho_1(s) < \varepsilon \) for \( t \geq s \). Therefore,
\[ \rho_2(t) - \rho_2(s) = 0 = \frac{\delta}{\varepsilon} - \delta > \frac{\delta}{\varepsilon}(\rho_1(t) - \rho_1(s)) - \delta. \]
Otherwise, let \( \varepsilon, \delta > 0 \) be as in (23). Without loss of generality, one can always assume that \( m = \lceil \delta \rceil \geq 1 \). In this case, the sequence \( t_n \) defined by (25) is finite, say
\[ t_0 = 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = \infty. \]
Given \( t \geq s \), for \( n, n_0 \) as in (26) we have \( t_{n_0} \leq s < t_{n_0+1} \) and \( t_n \leq t < t_{n+1} \).

Again we consider two cases. If \( n = n_0 < m \) or \( n_0 + 1 \leq n < m \), then the former arguments establish (24). On the other hand, if \( n = n_0 = m \), then \( t \geq s \geq t_m \).

Since \( m = \lceil \delta \rceil \), we have \( l < (m + 1)\delta \). Moreover, since \( \rho_2(t_m) = m\delta \), we obtain
\[ \rho_2(t) - \rho_2(s) < l - \rho_2(t_m) < (m + 1)\delta - m\delta = \delta. \]

By (23), we have \( \rho_1(t) - \rho_1(s) < \varepsilon \) and hence,
\[ \rho_2(t) - \rho_2(s) > \frac{\delta}{\varepsilon} - \delta > \frac{\delta}{\varepsilon}(\rho_1(t) - \rho_1(s)) - \delta, \]
which again establishes (24). \( \square \)

On the set \( F \) we define an equivalence relation that is related to the notion of topological equivalence. Two functions \( \rho_1, \rho_2 \in F \) are said to be equivalent and we write \( \rho_1 \sim \rho_2 \) if \( \rho_1 \preceq \rho_2 \) and \( \rho_2 \preceq \rho_1 \). For example, the functions \( \rho_1(t) = t^3 \) and \( \rho_2(t) = t^3 + t \) are equivalent. Using Proposition 4.2, we readily obtain a characterization of the notion of equivalence.

4.2. Alternative characterization. The following is an alternative characterization of the notion of equivalence between growth rates.

**Proposition 4.3.** Two onto functions \( \rho_1, \rho_2 \in F \) are equivalent if and only if there exist increasing invertible continuous functions \( \varphi, \psi : \mathbb{R} \to \mathbb{R} \) such that
\[ \Phi_1(t, s) \leq \varphi(\Phi_2(t, s)) \] 
and
\[ \Phi_2(t, s) \leq \psi(\Phi_1(t, s)) \]
for \( t, s \in \mathbb{R} \).

**Proof.** Assume that inequality (27) holds true and consider \( \varepsilon, \delta > 0 \) such that \( \varepsilon = \varphi(\delta) \). If \( \Phi_2(t, s) < \delta \), then \( \Phi_1(t, s) < \varepsilon \) for each \( t \geq s \). Hence, applying Proposition 4.2, we conclude that \( \rho_1 \preceq \rho_2 \). One can show in a similar manner that \( \rho_2 \preceq \rho_1 \).

Now assume that \( \rho_1 \) and \( \rho_2 \) are equivalent. By Proposition 4.2, there exist constants \( \alpha_i, \beta_i > 0 \), for \( i = 1, 2 \), such that
\[ \Phi_1(t, s) \geq \alpha_1 \Phi_2(t, s) - \beta_1 \] 
(29)
Proof of the lemma. We shall write equivalent. It follows from Lemma 4.5.

Theorem 4.4. Given onto functions \( \rho_1, \rho_2 \in \mathcal{F} \), let \( \mathcal{U}_i \) be a reversible evolution family with \( \rho_i \)-bounded growth, for \( i = 1, 2 \). If \( \mathcal{U}_i \) admits a \( \rho_i \)-exponential dichotomy with projections \( P_i + Q_i = \text{Id} \), for \( i = 1, 2 \), then \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are topologically equivalent if and only if:

1. the subspaces \( P_1 E \) and \( P_2 E \) are homeomorphic;
2. the subspaces \( Q_1 E \) and \( Q_2 E \) are homeomorphic;
3. the functions \( \rho_1 \) and \( \rho_2 \) are equivalent.

Proof. We start with an auxiliary result. Given \( \rho_1, \rho_2 \in \mathcal{F} \), consider the evolution families \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) formed by the linear operators

\[
\mathcal{U}_1(t,s) = e^{\rho_1(t-s)} \text{Id}, \quad \mathcal{U}_2(t,s) = e^{\rho_2(t-s)} \text{Id}.
\]

We shall write \( \mathcal{U}_1 \sim \mathcal{U}_2 \) to mean that \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are topologically equivalent.

Lemma 4.5. We have \( \mathcal{U}_1 \sim \mathcal{U}_2 \) if and only if the functions \( \rho_1 \) and \( \rho_2 \) are equivalent.

Proof of the lemma. We first assume that the evolution families are topological equivalent. It follows from

\[
h_t(e^{\Phi_1(t,s)} x) = e^{\Phi_2(t,s)} h_s(x)
\]

that

\[
h_s(x) = e^{\Phi_2(s,t)} h_t(e^{\Phi_1(t,s)} x).
\]

Since \( \|h_t(x)\| \leq L(\|x\|) \), we obtain

\[
\|h_s(x)\| \leq e^{\Phi_2(s,t)} L(e^{\Phi_1(t,s)} \|x\|).
\]

(33)

Moreover, since \( \|h_s^{-1}(x)\| \leq L(\|x\|) \), replacing \( x \) by \( h_s^{-1}(x) \) in (33) yields that

\[
\|x\| \leq e^{\Phi_2(s,t)} L(e^{\Phi_1(t,s)} \|h_s^{-1}(x)\|) \leq e^{\Phi_2(s,t)} L(e^{\Phi_1(t,s)} L(\|x\|))
\]

4.3. Exponential dichotomies and equivalence. Now we relate in an optimal manner the notions of equivalence and of topological equivalence for evolution families that admit exponential dichotomies.

Theorem 4.4. Given onto functions \( \rho_1, \rho_2 \in \mathcal{F} \), let \( \mathcal{U}_i \) be a reversible evolution family with \( \rho_i \)-bounded growth, for \( i = 1, 2 \). If \( \mathcal{U}_i \) admits a \( \rho_i \)-exponential dichotomy with projections \( P_i + Q_i = \text{Id} \), for \( i = 1, 2 \), then \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are topologically equivalent if and only if:

1. the subspaces \( P_1 E \) and \( P_2 E \) are homeomorphic;
2. the subspaces \( Q_1 E \) and \( Q_2 E \) are homeomorphic;
3. the functions \( \rho_1 \) and \( \rho_2 \) are equivalent.

Proof. We start with an auxiliary result. Given \( \rho_1, \rho_2 \in \mathcal{F} \), consider the evolution families \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) formed by the linear operators

\[
\mathcal{U}_1(t,s) = e^{\rho_1(t-s)} \text{Id}, \quad \mathcal{U}_2(t,s) = e^{\rho_2(t-s)} \text{Id}.
\]

We shall write \( \mathcal{U}_1 \sim \mathcal{U}_2 \) to mean that \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are topologically equivalent.

Lemma 4.5. We have \( \mathcal{U}_1 \sim \mathcal{U}_2 \) if and only if the functions \( \rho_1 \) and \( \rho_2 \) are equivalent.

Proof of the lemma. We first assume that the evolution families are topological equivalent. It follows from

\[
h_t(e^{\Phi_1(t,s)} x) = e^{\Phi_2(t,s)} h_s(x)
\]

that

\[
h_s(x) = e^{\Phi_2(s,t)} h_t(e^{\Phi_1(t,s)} x).
\]

Since \( \|h_t(x)\| \leq L(\|x\|) \), we obtain

\[
\|h_s(x)\| \leq e^{\Phi_2(s,t)} L(e^{\Phi_1(t,s)} \|x\|).
\]

(33)

Moreover, since \( \|h_s^{-1}(x)\| \leq L(\|x\|) \), replacing \( x \) by \( h_s^{-1}(x) \) in (33) yields that

\[
\|x\| \leq e^{\Phi_2(s,t)} L(e^{\Phi_1(t,s)} \|h_s^{-1}(x)\|) \leq e^{\Phi_2(s,t)} L(e^{\Phi_1(t,s)} L(\|x\|))
\]
and
\[ e^{\Phi_2(t,s)}\|x\| \leq L(e^{\Phi_1(t,s)}L(\|x\|)). \]
Taking \( \|x\| = 1 \) we obtain
\[ e^{\Phi_2(t,s)} \leq L(e^{\Phi_1(t,s)}L(1)). \]
Now take \( \varepsilon, \delta > 0 \) such that \( e^\delta = L(e^\varepsilon L(1)) \). If \( \Phi_1(t,s) < \varepsilon \) for some \( t \geq s \), then \( \Phi_2(t,s) < \delta \). By Proposition 4.2, we conclude that \( \rho_2 \leq \rho_1 \). One can show in a similar manner that \( \rho_1 \leq \rho_2 \) and hence, \( \rho_1 \) and \( \rho_2 \) are equivalent.

Now we assume that \( \rho_1 \) and \( \rho_2 \) are equivalent. By Proposition 4.3, there exist functions \( \varphi, \psi : \mathbb{R} \to \mathbb{R} \) satisfying (27) and (28) for \( t, s \in \mathbb{R} \). We define \( \varepsilon : \mathbb{R} \to \mathbb{R} \) by
\[ \varepsilon(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases} \]
and \( h_t : \mathbb{R} \to \mathbb{R} \) by
\[ h_t(x) = \begin{cases} \varepsilon(x)e^{\rho_2(t) - \rho_2\rho_1^{-1}(\rho_1(t) - \log|\|x\||)}, & x \neq 0, \\ 0, & x = 0 \end{cases} \]
for each \( t \in \mathbb{R} \). One can easily verify that \( h_t \) is invertible, with inverse
\[ h_t^{-1}(x) = \begin{cases} \varepsilon(x)e^{\rho_1(t) - \rho_1\rho_2^{-1}(\rho_2(t) - \log|\|x\||)}, & x \neq 0, \\ 0, & x = 0. \end{cases} \]
The continuity of the functions \( h_t \) and \( h_t^{-1} \) at 0 follows directly from
\[ \lim_{x \to 0} h_t(x) = \lim_{x \to 0} h_t^{-1}(x) = 0. \]
Moreover,
\[ h_t(e^{\Phi_1(t,s)}x) = \varepsilon(x)e^{\rho_2(t) - \rho_2\rho_1^{-1}(\rho_1(t) - \log e^{\Phi_1(t,s)}|x|)} \]
\[ = \varepsilon(x)e^{\rho_2(t) - \rho_2\rho_1^{-1}(\rho_1(t) - \Phi_1(t,s) - \log|\|x\||)} \]
\[ = \varepsilon(x)e^{\rho_2(t) - \rho_2\rho_1^{-1}(\rho_1(s) - \log|\|x\||)} \]
\[ = \varepsilon(x)e^{\rho_2(s) - \rho_2\rho_1^{-1}(\rho_1(s) - \log|\|x\||)}e^{\Phi_2(t,s)} \]
\[ = e^{\Phi_2(t,s)}h_s(x). \]

It remains to show that there exists a map \( L \) as in the notion of topological equivalence. For \( x \neq 0 \) we have
\[ \|h_t(x)\| = e^{\rho_2(t) - \rho_2\rho_1^{-1}(\rho_1(t) - \log|\|x\||)} = e^{\rho_2(t) - \rho_2(s)} = e^{\Phi_2(t,s)}, \]
where \( s = \rho_1^{-1}(\rho_1(t) - \log|\|x\||) \). Therefore, \( \|x\| = e^{\Phi_1(t,s)} \) and using inequality (28) we obtain
\[ \|h_t(x)\| = e^{\Phi_2(t,s)} \leq e^{\psi(\Phi_2(t,s))} = e^{\psi(\log|x|)} = L_1(\|x\|), \]
where
\[ L_1(\theta) = \begin{cases} e^{\psi(\log \theta)}, & \theta > 0, \\ 0, & \theta = 0. \end{cases} \]
A similar argument can be used to show that \( \|h_t^{-1}(x)\| \leq L_2(\|x\|) \), where
\[ L_2(\theta) = \begin{cases} e^{\psi(\log \theta)}, & \theta > 0, \\ 0, & \theta = 0. \end{cases} \]
Letting

\[ L(\|x\|) = \max \{ L_1(\|x\|), L_2(\|x\|) \}, \]

we obtain

\[ \|h_t(x)\| \leq L(\|x\|) \quad \text{and} \quad \|h_t^{-1}(x)\| \leq L(\|x\|). \]

This concludes the proof of the lemma.

We proceed with the proof of the theorem. Let us consider the evolution families \( \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_2' \) formed by the linear operators

\[ V_i(t, s) = e^{\Phi_i(t, s)} P_i + e^{\Phi_i(t, s)} Q_i, \quad i = 1, 2, \]

and

\[ V_i'(t, s) = e^{\Phi_i(t, s)} P_2 + e^{\Phi_i(t, s)} Q_2. \]

By Theorem 3.1, we have \( U_i \sim V_i \) for \( i = 1, 2 \).

We first assume that properties 1–3 hold. Since the subspaces \( P_1E \) and \( P_2E \) are homeomorphic, their unit spheres \( S(P_1E) \) and \( S(P_2E) \) are also homeomorphic. The same happens to the spheres \( S(Q_1E) \) and \( S(Q_2E) \). Let

\[ f: S(P_1E) \to S(P_2E) \quad \text{and} \quad g: S(Q_1E) \to S(Q_2E) \]

be homeomorphisms. We define maps \( F: P_1E \to P_2E \) and \( G: Q_1E \to Q_2E \) by

\[ F(x) = \begin{cases} \|x\| f \left( \frac{x}{\|x\|} \right), & x \in P_1E \setminus \{0\}, \\ 0, & x = 0 \end{cases} \]

and

\[ G(x) = \begin{cases} \|x\| g \left( \frac{x}{\|x\|} \right), & x \in Q_1E \setminus \{0\}, \\ 0, & x = 0. \end{cases} \]

One can easily verify that \( F \) and \( G \) are homeomorphisms, with inverses

\[ F^{-1}(x) = \begin{cases} \|x\| f^{-1} \left( \frac{x}{\|x\|} \right), & x \in P_2E \setminus \{0\}, \\ 0, & x = 0 \end{cases} \]

and

\[ G^{-1}(x) = \begin{cases} \|x\| g^{-1} \left( \frac{x}{\|x\|} \right), & x \in Q_2E \setminus \{0\}, \\ 0, & x = 0. \end{cases} \]

Then \( H = F \oplus G \) is a homeomorphism of \( E \), with inverse \( H^{-1} = F^{-1} \oplus G^{-1} \). We have

\[ H(V_i(t, s)x) = F \left( e^{\Phi_1(t, s)} P_1x + e^{\Phi_1(t, s)} Q_1x \right) \]

\[ = e^{\Phi_1(t, s)} F(P_1x) + e^{\Phi_1(t, s)} G(Q_1x) \]

\[ = \left( e^{\Phi_1(t, s)} P_2 + e^{\Phi_1(t, s)} Q_2 \right) H(x) = V_i'(t, s) H(x) \]

and hence, \( \mathcal{V}_1 \sim \mathcal{V}_2' \). Since \( \rho_1 \) and \( \rho_2 \) are equivalent, by Lemma 4.5, we have \( \mathcal{V}_2 \sim \mathcal{V}_2' \). Moreover, since \( \mathcal{V}_1 \sim \mathcal{V}_2 \), we obtain \( \mathcal{V}_1 \sim \mathcal{V}_2 \) and since \( U_i \sim V_i \) for \( i = 1, 2 \) (by Theorem 3.1), we conclude that \( U_1 \sim U_2 \).

Now we assume that \( U_i \sim U_2 \). Since \( U_i \sim V_i \) for \( i = 1, 2 \), we obtain \( \mathcal{V}_1 \sim \mathcal{V}_2 \). Hence, there exists a family of homeomorphisms \( h_t: E \to E \) such that

\[ h_t(e^{-\rho_1(t)} P_1 x + e^{\rho_1(t)} Q_1 x) = e^{-\rho_2(t)} P_2 h_0(x) + e^{\rho_2(t)} Q_2 h_0(x). \]

Replacing \( x \) by \( P_1 x \), we obtain

\[ h_t(e^{-\rho_1(t)} P_1 x) = e^{-\rho_2(t)} P_2 h_0(P_1 x) + e^{\rho_2(t)} Q_2 h_0(P_1 x). \]
Since
\[
\lim_{t \to \infty} h_t(e^{-\rho_2(t)}P_1x) = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{-\rho_2(t)}P_2h_0(P_1x) = 0,
\]
we obtain
\[
\lim_{t \to \infty} e^{\rho_2(t)}Q_2h_0(P_1x) = 0.
\]
But this happens if and only if \(Q_2h_0(P_1x) = 0\) or, equivalently,
\[
P_2h_0(P_1x) = h_0(P_1x).
\]
Therefore,
\[
h_0(P_1E) \subseteq P_2E. \tag{35}
\]
Now we rewrite identity (34) in the form
\[
e^{-\rho_1(t)}P_1h_0^{-1}(x) + e^{\rho_1(t)}Q_1h_0^{-1}(x) = h_t^{-1}(e^{-\rho_2(t)}P_2x + e^{\rho_2(t)}Q_2x).
\]
Replacing \(x\) by \(P_2x\), we obtain
\[
e^{-\rho_1(t)}P_1h_0^{-1}(P_2x) + e^{\rho_1(t)}Q_1h_0^{-1}(P_2x) = h_t^{-1}(e^{-\rho_2(t)}P_2x).
\]
Since
\[
\lim_{t \to \infty} e^{-\rho_1(t)}P_1h_0^{-1}(P_2x) = 0 \quad \text{and} \quad \lim_{t \to \infty} h_t^{-1}e^{-\rho_2(t)}(P_2x) = 0,
\]
we obtain
\[
\lim_{t \to \infty} e^{\rho_1(t)}Q_1h_0^{-1}(P_2x).
\]
This implies that \(Q_1h_0^{-1}(P_2x) = 0\) or equivalently \(P_1h_0^{-1}(P_2x) = h_0^{-1}(P_2x)\). Therefore,
\[
h_0^{-1}(P_2E) \subseteq P_1E. \tag{36}
\]
Relations (35) and (36) imply that the subspaces \(P_1E\) and \(P_2E\) are homeomorphic, via \(h_0\). One can show in an analogous manner that the subspaces \(Q_1E\) and \(Q_2E\) are homeomorphic. Proceeding as above, one can now show that \(V_1 \sim V_2\) and hence \(V_2 \sim V_2'\). Thus, there exists a family of homeomorphisms \(g_t: E \to E\) such that
\[
g_t(e^{\Phi_1(s,t)}P_2x + e^{\Phi_1(s,t)}Q_2x) = (e^{\Phi_2(s,t)}P_2 + e^{\Phi_2(s,t)}Q_2)g_s(x).
\]
As above, one can show that the subspaces \(P_2E\) and \(Q_2E\) are homeomorphic. Moreover,
\[
g_t(e^{\Phi_1(s,t)}P_2x) = e^{\Phi_2(s,t)}g_s(P_2x) \quad \text{and} \quad g_t(e^{\Phi_1(s,t)}Q_2x) = e^{\Phi_2(s,t)}g_s(Q_2x).
\]
Applying Lemma 4.5 to the subspaces \(P_2E\) and \(Q_2E\), we conclude that the functions \(\rho_1\) and \(\rho_2\) are equivalent. \(\square\)

As an outcome of our approach one can classify completely all 1-dimensional evolution families \(U(t, s) = e^{\rho(t) - \rho(s)}\). The following is an immediate consequence of Proposition 4.3 and Lemma 4.5.

**Proposition 4.6.** Given \(\rho_1, \rho_2 \in F\), the following properties are equivalent:

1. the functions \(\rho_1\) and \(\rho_2\) equivalent;
2. there exist increasing invertible continuous functions \(\varphi\) and \(\psi\) satisfying (27) and (28);
3. \(e^{\rho_1(t) - \rho_1(s)}\) and \(e^{\rho_2(t) - \rho_2(s)}\) are topologically equivalent.
5. Generalized Bohl exponents. In this section we detail the relations between Bohl exponents and the notion of equivalence between growth rates. We refer the reader to [3] for the classical notion of Bohl exponent.

Let \( \mathcal{U} \) be an evolution family. The upper and lower \( \rho \)-Bohl exponents of \( \mathcal{U} \) are defined, respectively, by

\[
\overline{B}(\rho, \mathcal{U}) = \limsup_{t \to \infty} \frac{\log \|U(t, s)\|}{\Phi(t, s)}
\]

and

\[
\underline{B}(\rho, \mathcal{U}) = \liminf_{s \to -\infty} \frac{\log \|U(t, s)\|}{\Phi(t, s)}.
\]

More precisely, \( \overline{B}(\rho, \mathcal{U}) \) is the infimum of all numbers \( \gamma \in \mathbb{R} \) such that

\[
\frac{\log \|U(t, s)\|}{\Phi(t, s)} < \gamma
\]

for \( t, s \in \mathbb{R} \), with \( \min\{t - s, s\} \) sufficiently large. The number \( \overline{B}(\rho, \mathcal{U}) \) is defined in a similar manner. Clearly, \( \mathcal{U} \) has \( \rho \)-bounded growth whenever \( \overline{B}(\rho, \mathcal{U}) < \infty \) and \( \underline{B}(\rho, \mathcal{U}) > -\infty \).

The following result establishes some basic relations between the Bohl exponents of equivalent functions.

Proposition 5.1. Assume that \( \rho_1 \sim \rho_2 \) and let \( B_i = \overline{B}(\rho_i, \mathcal{U}), \) for \( i = 1, 2 \). Then:

1. \( B_1 \in (0, \infty) \) if and only if \( B_2 \in (0, \infty) \);
2. \( B_1 = 0 \) if and only if \( B_2 = 0 \);
3. \( B_1 \in (-\infty, 0) \) if and only if \( B_2 \in (-\infty, 0) \).

Proof. 1. Assume that \( B_1 \in (0, \infty) \). Since \( \rho_1 \sim \rho_2 \), there exist constants \( a, b, c, d > 0 \) such that

\[
\Phi_1(t, s) \geq a\Phi_2(t, s) - b \quad \text{and} \quad \Phi_2(t, s) \geq c\Phi_1(t, s) - d
\]

for \( t \geq s \). Hence, for each \( \varepsilon > 0 \) there exists \( M_1 > 0 \) such that

\[
\|U(t, s)\| \leq M_1e^{(\varepsilon + B_1)\Phi_2(t, s)}
\]

\[
\leq M_1e^{(\varepsilon + B_1)(\frac{a}{c}\Phi_2(t, s) + \frac{d}{c})}
\]

\[
\leq M_1e^{\frac{d}{c}(\varepsilon + B_1)}e^{\frac{a}{c}\Phi_2(t, s)},
\]

which shows that \( (\varepsilon + B_1)/c \geq B_2 \). In particular, \( B_2 < \infty \).

Now we show that in fact \( B_2 \in (0, \infty) \). Indeed, if \( B_2 < 0 \), then for each \( \varepsilon > 0 \) there exists \( M_2 > 0 \) such that

\[
\|U(t, s)\| \leq M_2e^{(\varepsilon + B_2)\Phi_2(t, s)}
\]

\[
\leq M_2e^{-(\varepsilon + B_2)(-c\Phi_1(t, s) + d)}
\]

\[
= M_2e^{-d(\varepsilon + B_2)}e^{c(\varepsilon + B_2)\Phi_1(t, s)}
\]

for \( t \geq s \). Therefore, \( c(\varepsilon + B_2) \geq B_1 \) and taking \( \varepsilon \) sufficiently small yields that \( B_1 < 0 \). On the other hand, if \( B_2 = 0 \), then for each \( \varepsilon > 0 \) there exists \( M_2 > 0 \) such that

\[
\|U(t, s)\| \leq M_2e^{\varepsilon\Phi_2(t, s)} \leq M_2e^{\varepsilon(\frac{a}{c}\Phi_2(t, s))} = M_2e^{\frac{a}{c}\Phi_2(t, s)}
\]

for \( t \geq s \), which readily implies that \( B_1 \leq \varepsilon/a \). Hence \( B_1 \leq 0 \).

2. If \( B_1 = 0 \), then for each \( \varepsilon > 0 \) there exists \( M_1 > 0 \) such that

\[
\|U(t, s)\| \leq M_1e^{\varepsilon\Phi_1(t, s)} \leq M_1e^{\varepsilon\Phi_2(t, s) + \frac{d}{c}} = M_1e^{\frac{a}{c}\Phi_2(t, s)}
\]
for $t \geq s$. Therefore, $B_2 \leq \varepsilon/c$ which implies that $B_2 \leq 0$. In order to show that

$B_2 = 0$, assume that $B_2 < 0$ and take $\varepsilon > 0$ sufficiently small so that $B_2 + \varepsilon < 0$. Then

$$
\|U(t, s)\| \leq M_2 e^{(\varepsilon+B_2)\Phi_2(t, s)}
$$

for $t \geq s$, which yields that $B_1 \leq (\varepsilon + B_2)/a < 0$, a contradiction.

3. Finally, assume that $B_1 \in (-\infty, 0)$ and take $\varepsilon > 0$ so that $B_1 + \varepsilon < 0$. Then there exists $M_1 > 0$ such that

$$
\|U(t, s)\| \leq M_1 e^{(\varepsilon+B_1)\Phi_1(t, s)}
$$

for $t \geq s$ and $B_2 \leq a(\varepsilon + B_1) < 0$. If $B_2 = -\infty$, then for each $n \in \mathbb{N}$ there exists $M_n > 0$ such that $\|U(t, s)\| \leq M_n e^{-n\Phi_2(t, s)}$ for $t \geq s$. Therefore,

$$
\|U(t, s)\| \leq M_n e^{-n\Phi_1(t, s)} e^{nd} = M'_n e^{-nc\Phi_1(t, s)},
$$

where $M'_n = M_n e^{nd}$. This implies that $B_1 = -\infty$, which contradicts to the initial hypothesis.

The following result is a partial converse of Proposition 5.1 when the Bohl exponents are limits.

**Proposition 5.2.** If $B_1, B_2 \in (0, \infty)$ or $B_1, B_2 \in (-\infty, 0)$ and the Bohl exponents are limits, then $\rho_1 \sim \rho_2$.

**Proof.** Since

$$
\frac{\Phi_1(t, s)}{\Phi_2(t, s)} = \frac{\log \|U(t, s)\|}{\log \|U'(t, s)\|},
$$

we have

$$
\lim_{t \to s, s \to \infty} \frac{\Phi_1(t, s)}{\Phi_2(t, s)} = \lim_{t \to s, s \to \infty} \frac{\log \|U(t, s)\|}{\log \|U'(t, s)\|} = \frac{B_2}{B_1} \in (0, \infty).
$$

This implies that there exists $\varepsilon > 0$ such that

$$
\frac{\Phi_1(t, s)}{\Phi_2(t, s)} \frac{B_2}{B_1} \leq \varepsilon
$$

and

$$
\frac{\Phi_2(t, s)}{\Phi_1(t, s)} \frac{B_1}{B_2} \leq \varepsilon
$$

for $t \geq s$. Therefore,

$$
\frac{1}{\varepsilon + B_1/B_2} \leq \frac{\Phi_1(t, s)}{\Phi_2(t, s)} \leq \varepsilon + \frac{B_2}{B_1},
$$

which implies that $\rho_1 \sim \rho_2$. $\Box$
We emphasize that there exists no result analogous to that in Proposition 5.2 when $B_1 = B_2 = 0$. Indeed, let

$$U(t, s) = e^{\sqrt{t} - \sqrt{s}}, \quad \Phi_1(t, s) = t - s, \quad \Phi_2(t, s) = t^2 - s^2.$$ 

Clearly,

$$B_1 = \lim_{t \to s, s \to \infty} \log \frac{\|U(t, s)\|}{\Phi_1(t, s)} = \lim_{t \to s, s \to \infty} \frac{\sqrt{t} - \sqrt{s}}{t - s} = 0$$

and

$$B_2 = \lim_{t \to s, s \to \infty} \log \frac{\|U(t, s)\|}{\Phi_2(t, s)} = \lim_{t \to s, s \to \infty} \frac{\sqrt{t} - \sqrt{s}}{t^2 - s^2} = 0.$$ 

On the other hand, one can easily verify that $\rho_1$ and $\rho_2$ are not equivalent.

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