The Nature of Primordial Fluctuations from Anisotropic Inflation

Masa-aki Watanabe\textsuperscript{1)}, Sugumi Kanno\textsuperscript{2)}, and Jiro Soda\textsuperscript{1)}

\textsuperscript{1)} Department of Physics, Kyoto University, Kyoto, 606-8501, Japan and
\textsuperscript{2)} Centre for Particle Theory, Department of Mathematical Sciences, Durham University,
Science Laboratories, South Road, Durham, DH1 3LE, United Kingdom

(Dated: March 9, 2010)

We study the statistical nature of primordial fluctuations from an anisotropic inflation which is realized by a vector field coupled to an inflaton. We find a suitable gauge, which we call the canonical gauge, for anisotropic inflation by generalizing the flat slicing gauge in conventional isotropic inflation. Using the canonical gauge, we reveal the structure of the couplings between curvature perturbations, vector waves, and gravitational waves. We identify two sources of anisotropy, i.e. the anisotropy due to the anisotropic expansion of the universe and that due to the anisotropic couplings among variables. It turns out that the latter effect is dominant. Since the coupling between the curvature perturbations and vector waves is the strongest one, the statistical anisotropy in the curvature perturbations is larger than that in gravitational waves. We find the cross correlation between the curvature perturbations and gravitational waves which never occurs in conventional inflation. We also find the linear polarization of gravitational waves. Finally, we discuss cosmological implication of our results.

PACS numbers: 98.80.Cq, 98.80.Hw

I. INTRODUCTION

The primordial fluctuations from inflation is supposed to be statistically isotropic, Gaussian, and scale invariant. The nature of fluctuations is associated with the nature of de Sitter spacetime. However, since the expansion during inflation is not exactly de Sitter, the power spectrum is slightly tilted by the order of the slow roll parameter $\epsilon$, which characterizes the deviation of the expansion from the exact de Sitter expansion. The deviation from the Gaussianity is also known to be related to the slow roll parameter $\epsilon$. On the other hand, the statistical isotropy has been regarded as a robust prediction so far because the cosmic no-hair conjecture is thought to be robust $\epsilon$.

From an observational point of view, there are various indications that there exists statistical anisotropy in the cosmic microwave background radiation (CMB) $\epsilon$. Although the statistical significance of these anomalies is still under debate, the possibility of the statistical anisotropy certainly deserves further theoretical investigation $\epsilon$. Recently, breaking the statistical isotropy through the vector fields in an inflationary universe is proposed in the paper $\epsilon$ and extended in various ways $\epsilon$. However, if the vector field is relevant to inflation, it may also produce anisotropy in an inflationary universe whatever small it is, which seems to contradict the cosmic no-hair conjecture.

From the above perspective, it is interesting to ask if it is possible to have anisotropic inflationary universe $\epsilon$. If possible, it provides a simple mechanism to break the statistical isotropy by breaking the isotropy of the spacetime $\epsilon$. In the light of no-hair conjecture $\epsilon$, one may deny this possibility. In fact, many attempts to construct anisotropic inflationary models suffer from the instability $\epsilon$. However, recently, stable anisotropic inflationary models are found for the first time $\epsilon$. This can be regarded as a counter example to the cosmic no-hair conjecture. The interesting point is that the deviation from isotropy is related to the slow roll parameter, namely, the deviation from the exact deSitter expansion. Of course, that means the degree of the anisotropy is quite small. From the point of view of precision cosmology, however, it is worth exploring theoretical fine structure in an inflationary scenario.

In this paper, we study cosmological perturbations in an anisotropic inflationary scenario we have found. The expected phenomenology of the anisotropic inflation is as follows:

- There should be statistical anisotropy in curvature perturbations.
- There should be statistical anisotropy in gravitational waves.
- There should exist the cross correlation between curvature perturbations and gravitational waves.
- There should be linear polarization of gravitational waves.

The first item will be tested by the PLANCK $\epsilon$. The second one may be detected through B-mode polarization in the CMB $\epsilon$. The third one will imply T-B correlation in CMB $\epsilon$. The last one could be important for the future direct measurement of gravitational waves through the interferometer $\epsilon$. The purpose of this paper is to calculate the above quantities numerically and analytically and reveal the physics behind them. Since the spacetime is anisotropic, the formalism treating perturbations is non-standard. Although there are many works treating the
cosmological perturbations in an anisotropic universe \cite{26,31}, there have been several obstructions in extracting concrete predictions for CMB. The main obstruction was the lack of the concrete anisotropic cosmological models. Now, since we have such models, we have succeeded in obtaining concrete results by utilizing the canonical gauge which is a generalization of the flat slicing in the conventional isotropic inflationary scenario.

Recently, during our slow preparation of this paper, two papers have appeared on the archive \cite{32,33}. The first one \cite{32} studied the primordial perturbations in an anisotropic inflationary universe using a perturbative method. The second one \cite{33} investigated the same issue numerically. The conclusion is quite similar to ours. The main difference is the gauge used in analysis. Our canonical gauge allows us to reveal the nature of primordial fluctuations from anisotropic inflation in a transparent way. Interestingly, on the contrary to a naive expectation, all of these works including ours imply that even if the anisotropy of the universe is very small, a large statistical anisotropy in the spectrum of curvature perturbations could be created.

The organization of this paper is as follows: In section II, we review an anisotropic inflation which is caused by the inflaton coupled to the vector field. Here, we will see the anisotropy is determined by the slow roll parameter. In section III, we choose the canonical gauge which is a generalization of the flat slicing in the conventional inflation and classify perturbations in anisotropic universe based on the 2-dimensional rotation symmetry. Then, we obtain the quadratic action for perturbed quantities. In section IV, we reduce the action to that for physical variables from which we can read off the structure of couplings between those variables. Based on the reduced action, we calculate various statistical quantities numerically and analytically to reveal the nature of primordial fluctuations in anisotropic inflation. In section V, we discuss cosmological implication of our results. The final section is devoted to the conclusion. In the Appendix A, we provide a detailed derivation of the action for 2-dimensional scalar sector perturbations.

II. REVIEW OF ANISOTROPIC INFLATION

In this section, we review background solutions proposed in \cite{20}, and see how the anisotropic inflation is realized.

We consider the vector field $A_\mu$ whose kinetic term is coupled to the inflaton field $\phi$. We note that this kind of model is quite natural in the context of the supergravity \cite{34}. The action is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) - \frac{1}{4} f(\phi)^2 F_{\mu\nu} F^{\mu\nu} \right],$$

(1)

where $\kappa^2$ is the reduced gravitational constant, $g$ is the determinant of the metric, $R$ is the Ricci scalar, $V(\phi)$ is the inflaton potential, $F_{\mu\nu}$ is the field strength of the vector field defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $f(\phi)$ is a coupling function of the vector field. We assume that the background spacetime is given by the Bianchi type I metric

$$ds^2 = -dt^2 + \sum_{i=1}^3 a_i(t)^2 dx^2.$$  

As for the vector field, it can be reduced into "electric" ($F_{0i}$) and "magnetic" ($F_{ij}$) components, and here we consider only the "electric" component for simplicity. It is not hard to prove that the direction of the "electric" field does not change in time by solving its evolution equation. Without losing the generality, one can take $x$-axis in the direction of the "electric" field. Using the gauge invariance, we can express the vector field as $A_\mu dx^\mu = v(t) dx$. Thus, there exists the rotational symmetry in the $y$-$z$ plane. Given this configuration, it is convenient to parameterize the metric as follows:

$$ds^2 = -N(t)^2 dt^2 + e^{2\alpha(t)} \left[ e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (e^{2\sqrt{\sigma}(t)} dy^2 + e^{-2\sqrt{\sigma}(t)} dz^2) \right],$$

(2)

where $e^\alpha$, $\sigma$ and $\sigma_-$ are an isotropic scale factor and spatial shears, respectively. We also define the averaged Hubble parameter as $H \equiv \dot{\alpha}$. Here, the lapse function $N$ is introduced to obtain the Hamiltonian constraint. With the above ansatz, the action becomes

$$S = \int d^4x \frac{1}{N} e^{3\alpha} \left[ \frac{3}{\kappa^2} (-\dot{\alpha}^2 + \dot{\sigma}^2 + \dot{\sigma}_-^2) + \frac{1}{2} \dot{\sigma}^2 - N^2 V(\phi) + \frac{1}{2} f(\phi)^2 \dot{\phi}^2 e^{-2\alpha(t) + 4\sigma(t)} \right],$$

(3)

where an overdot denotes the derivative with respect to the physical time $t$. First, its variation with respect to $\sigma_-$ yields

$$\ddot{\sigma}_- = -3\dot{\sigma}_- \dot{\alpha} + \sigma_3 = -3\dot{\alpha} \dot{\sigma}_- .$$

(4)

This gives $\dot{\sigma}_- \propto e^{-3\alpha}$, hence, the anisotropy in the $y$-$z$ plane rapidly decays as the universe expands. Hereafter, for simplicity, we assume $\sigma_- = 0$ and set the metric to be

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left[ e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right].$$

(5)
Next, the equation of motion for $v$ is easily solved as
\[ \dot{v} = f(\phi)^{-2} e^{-\alpha - 4\sigma} p_A, \]
where $p_A$ is a constant. Taking the variation of the action with respect to $V, \alpha, \sigma$ and $\phi$ and substituting the solution $\Phi$, we obtain the following basic equations:
\[ \dot{\alpha^2} = \dot{\alpha}^2 + \frac{\kappa^2}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{p_A^2}{2} f(\phi)^{-2} e^{-4\alpha - 4\sigma} \right], \]
\[ \dot{\alpha} = -3\dot{\alpha^2} + \kappa^2 V(\phi) + \frac{\kappa^2 p_A^2}{6} f(\phi)^{-2} e^{-4\alpha - 4\sigma}, \]
\[ \dot{\sigma} = -3\dot{\sigma^2} + \frac{\kappa^2 p_A^2}{3} f(\phi)^{-2} e^{-4\alpha - 4\sigma}, \]
\[ \ddot{\phi} = -3\dot{\phi} - V_\phi + \frac{p_A^2}{3} f(\phi)^{-3} f_\phi e^{-4\alpha - 4\sigma}, \]
where the subscript in $V_\phi$ denotes derivative with respect to $\phi$. Let us check whether inflation occurs in this model. Using Eqs. (7) and (8), the equation for acceleration of the cosmic expansion is given by
\[ \frac{(e^{2\alpha})'}{e^\alpha} = \dot{\alpha} + \dot{\alpha}^2 = -2\dot{\alpha^2} - \frac{\kappa^2}{3} \dot{\phi}^2 + \frac{\kappa^2}{3} \left[ V - \frac{p_A^2}{2} f^{-2} e^{-4\alpha - 4\sigma} \right]. \]
We see that the potential energy of the inflaton needs to be dominant and the energy density of the vector field $\rho_v \equiv p_A^2 f(\phi)^{-2} e^{-4\alpha - 4\sigma}/2$ and the shear $\Sigma \equiv \dot{\sigma}$ should be subdominant for inflation to occur. Next we want to see if the anisotropy is produced during inflation. Here we look at the ratio of the shear to the expansion rate $\Sigma/H$ to characterize the anisotropy of the inflationary universe. Then, Eq. (11) reads
\[ \dot{\Sigma} = -3H\Sigma + \frac{2\kappa^2}{3} \rho_v. \]
If the anisotropy converges to a value, i.e. $\dot{\Sigma}$ becomes negligible, the terminal value should be given by
\[ \frac{\Sigma}{H} = \frac{2}{3} \mathcal{R}, \]
where we used the slow roll equation $H^2 = \kappa^2 V(\phi)/3$ derived from Eq. (7) and defined the energy ratio $\mathcal{R} \equiv \rho_v/V(\phi)$.

In order to realize the above situation, $\rho_v$ must be almost constant. Assuming that the vector field is subdominant in the evolution equation of the inflaton field Eq. (11) and the conventional single field slow-roll inflation is realized, one can show the coupling function $f(\phi)$ should be proportional to $e^{-2\alpha}$ to keep $\rho_v$ almost constant. In the slow roll phase, $e$-folding number $\alpha$ is related to the inflaton field $\phi$ as $d\alpha = -\kappa^2 V(\phi)d\phi/V_\phi$ as usual. Then, the functional form of $f(\phi)$ is determined as
\[ f(\phi) = e^{-2\alpha} = e^{2c^2} \int_{\phi_0}^{\phi} \frac{V_\phi d\phi}{e^\phi}. \]
For the polynomial potential $V \propto \phi^n$, for example, we have $f = e^{2\frac{c^2}{2}}$. In this scenario, the anisotropy is restricted from the condition that the vector field is negligible in Eq. (11), that is, $|p_A^2 f(\phi)^{-3} f_\phi e^{-4\alpha - 4\sigma}| \ll |V_\phi|$. Substituting Eq. (14) into this, we obtain $\mathcal{R} \ll \epsilon_V/2$, where $\epsilon_V$ is the slow-roll parameter defined by $\epsilon_V \equiv 1/2\kappa^2 (V_\phi/V)^2$. Thus, the anisotropy is constrained by $\Sigma/H \ll \mathcal{O}(\epsilon_V)$.

The above case is, in a sense, a critical one, and next we want to consider beyond it. For simplicity, we parameterize $f(\phi)$ by
\[ f(\phi) = e^{2c^2} \frac{V_\phi d\phi}{e^\phi}, \]
where $c$ is a constant parameter. Now, we look at what happens when $c > 1$. The basic equations become
\[ \frac{(e^{2\alpha})'}{e^{\alpha}} = \dot{\alpha} + \dot{\alpha}^2 = -2\dot{\alpha^2} - \frac{\kappa^2}{3} \dot{\phi}^2 + \frac{\kappa^2}{3} V(\phi) \left[ 1 - \mathcal{R} \right], \]
\[ \ddot{\phi} = -3\dot{\phi} - V_\phi \left[ 1 - \frac{2c^2}{\epsilon_V} \mathcal{R} \right]. \]
In this case, if the vector field is initially small \( R \ll \epsilon_V/2c \), then the conventional single field slow-roll inflation is realized. During this stage \( f \propto e^{-2c\alpha} \) and the vector field grows as \( \rho_v \propto e^{4(c-1)\alpha} \). Therefore, the vector field eventually becomes relevant to the inflaton dynamics Eq. (17). Nevertheless, the accelerating expansion of the universe will continue. The point is that \( R \) cannot exceed \( \epsilon_V/2c \). In fact, if \( R \) exceeds \( \epsilon_V/2c \), the inflaton field \( \phi \) does not roll down, which makes \( \rho_v = \frac{p_A^2}{2} e^{-e^{-4\alpha-4\sigma}/2} \) decrease. Hence, \( \rho_v \ll V(\phi) \) always holds. In this sense, there exists an attractor where the inflation continues even when the vector field affects the inflaton evolution. The inflaton dynamics is determined by solving the slow-roll equation:

\[
-3\dot{\phi} - V_\phi + \frac{p_A^2}{2} e^{-4\alpha-4\sigma} = 0. \tag{18}
\]

Assuming \( \dot{\phi}^2 = \kappa^2 V(\phi)/3 \), this yields

\[
\frac{d\dot{\phi}}{d\alpha} = \frac{\dot{\phi}^2}{\kappa^2 V} - \frac{2c}{\kappa^2} e^{-4\alpha-4\sigma-4\kappa^2 f V_\phi} \frac{V_\phi}{V_\phi^2} \frac{d\phi}{d\alpha} = -\frac{V_\phi}{\kappa^2 V} + e^{-4\alpha-4\sigma-4\kappa^2 f V_\phi} \frac{1}{c} \frac{1 + \text{const} e^{-4(c-1)\alpha+4\sigma}}{\kappa^2} . \tag{19}
\]

This can be integrated by neglecting the evolutions of \( V, V_\phi, \sigma \) as

\[
e^{4\alpha+4\sigma+4\kappa^2} f V_\phi \frac{d\phi}{d\alpha} = 2c e^{-2\alpha} \frac{p_A^2}{2} \kappa^2 V \left[ 1 + \text{const} e^{-4(c-1)\alpha+4\sigma} \right]^{-1} . \tag{20}
\]

Substituting this into the slow-roll equation Eq. (13), we obtain

\[
\frac{d\phi}{d\alpha} = -\frac{V_\phi}{\kappa^2 V} + e^{-4\alpha-4\sigma-4\kappa^2 f V_\phi} \frac{1}{c} \frac{1 + \text{const} e^{-4(c-1)\alpha+4\sigma}}{\kappa^2} . \tag{21}
\]

This clearly shows a transition from the conventional single field slow-roll inflationary phase, where \( d\phi/d\alpha = -V_\phi/\kappa^2 V \) holds, to what we refer to as the second inflationary phase, where the vector field is relevant to the inflaton dynamics and the inflaton gets \( 1/c \) times slower as \( d\phi/d\alpha = -V_\phi/c\kappa^2 V \). In the second inflationary phase, the energy density of the vector field becomes

\[
\rho_v = \frac{p_A^2}{2} e^{-4\alpha-4\sigma-4\kappa^2 f V_\phi} \frac{V_\phi}{V_\phi^2} = \frac{c}{2} \frac{1}{c^2} \epsilon_V V(\phi) , \tag{22}
\]

which yields the anisotropy of \( \Sigma/H = 2R/3 = (c-1)\epsilon_V/3c^2 \). From Eqs. (7) and (8), the slow-roll parameter defined in terms of the scale factor becomes

\[
\epsilon_H \equiv \frac{\dot{\epsilon}}{\dot{\alpha}^2} = -\frac{1}{\dot{\alpha}^2} \left( -\frac{1}{2} \dot{\phi}^2 + \frac{V_\phi}{3} \kappa^2 \rho_v \right) = \frac{1}{c} \frac{1}{c^2} \epsilon_V , \tag{23}
\]

where we neglected the anisotropy and used relations \( \dot{\phi}/\dot{\alpha} = d\phi/d\alpha = -V_\phi/c\kappa^2 V \) and \( \dot{\alpha}^2 = \kappa^2 V(\phi)/3 \). Thus we have a remarkable result

\[
\frac{\Sigma}{H} = \frac{1}{3} \frac{c-1}{c} \epsilon_H . \tag{24}
\]

In the next section, we will make a perturbative analysis during this second inflationary phase.

### III. CANONICAL GAUGE IN ANISOTROPIC INFLATION

Since the background is expanding anisotropically, we cannot use the standard cosmological perturbation theory. In this section, we classify perturbations under the 2-dimensional rotational symmetry and obtain the quadratic actions for 2-dimensional scalar and vector sectors. In order to grasp the meaning of variables, we start with the isotropic case and make a gauge transformation from the flat slicing gauge to the appropriate gauge for 2-dimensional classification. Then, the resultant gauge can be promoted to the anisotropic spacetime. The gauge we have chosen makes the analysis and the interpretation easier. Once the gauge is fixed, the quadratic action can be calculated.

#### A. Gauge Fixing and Classification of perturbations

First, we will start with the spatially homogeneous and isotropic universe. For simplicity, we consider flat space.

\[
ds^2 = a^2(\eta) \left[ -d\eta^2 + \delta_{ij} dx^i dx^j \right] , \tag{25}
\]
where we took a conformal time $\eta$. In that case, we can use 3-dimensional rotational symmetry to classify the perturbed metric. When we want to have diagonal quadratic action, we take the following gauge

$$ds^2 = a^2 \left[ -(1 + 2A)d\eta^2 + 2(B,i + V_i)d\eta dx^i + (\delta_{ij} + h_{ij})dx^idx^j \right]$$

(26)

where we imposed $V_i,i = 0$ and $h_{ij,j} = h_{ii} = 0$. If we ignore vector and tensor perturbations $V_i, h_{ij}$, the above gauge is called the flat slicing gauge. Now, let us move on to the Fourier space. Since there exists 3-dimensional rotation symmetry, we can take a wave number vector to be $\mathbf{k} = (k, 0, 0)$. Then, the perturbed metric has the following components:

$$\delta g_{\mu\nu} = \begin{pmatrix} -2a^2A & a^2B_{x} & a^2V_2 & a^2V_3 \\ * & 0 & 0 & 0 \\ * & * & a^2h_+ & -a^2h_\times \\ * & * & * & -a^2h_+ \end{pmatrix} \quad * \text{ is symmetric part.}$$

(27)

Here, we utilized the special choice $\mathbf{k} = (k, 0, 0)$ to solve the constraints $V_i,i = 0$ and $h_{ij,j} = h_{ii} = 0$. With the same reason, only $B_{x,x}$ remains. We defined $h_{yz} = -h_{x}, h_{yy} = -h_{xz} = h_{+}$. Now, we will pretend that we have only 2-dimensional rotation symmetry in $y$-$z$ plane. In that case, at best, we can take $\mathbf{k} = (k_x, k_y, 0)$. Hence, we make a rotation in the $x$-$y$-plane so that the wave number becomes $\mathbf{k} = (k_x, k_y, 0)$.

$$\begin{pmatrix} k_x \\ k_y \\ 0 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} k_x & -k_y & 0 \\ k_y & k_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k \\ 0 \end{pmatrix},$$

(28)

where we have a relation $k^2 = k_x^2 + k_y^2$. Under this rotation, the perturbed metric becomes

$$\delta g_{\mu\nu} = \left( -2a^2A \frac{k_x}{k}a^2B_{x} - \frac{k_y}{k}a^2V_2 \frac{k_y}{k}a^2B_{x} + \frac{k_x}{k}a^2V_2 \frac{k_x}{k}a^2B_{x} + \frac{k_x}{k}a^2V_2 a^2V_3 \right) \quad * \text{ symmetric part.}$$

(29)

To simplify the perturbations, we can make use of gauge transformation

$$\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} + \xi_{\mu,\nu} + \xi_{,\mu},$$

(30)

where the semicolon denotes the covariant derivative with respect to the background metric. Taking the parameter

$$\xi^0 = 0, \quad \xi^x = \frac{k_x}{2ik^2}h_+, \quad \xi^y = \frac{k_y}{2ik^2}h_+, \quad \xi^z = \frac{k_x}{ik_yk}h_\times,$$

we obtain

$$\delta g_{\mu\nu} = \left( -2a^2A \frac{k_x}{k}a^2B_{x} - \frac{k_y}{k}a^2V_2 + \frac{k_y}{k}a^2h_+ \right) \quad * \text{ symmetric part.}$$

(31)

It should be noted that we did not change slicing but performed only the spatial coordinate transformation. Therefore, we are still working in the flat slicing where the 3-dimensional scalar curvature vanishes.

In our anisotropic inflation models, the available symmetry is actually small. The background metric is given by

$$ds^2_b = a(\eta)^2(-d\eta^2 + dx^2) + b(\eta)^2(dy^2 + dz^2),$$

(32)

that is, $a = e^{-2\sigma}, b = e^{\sigma}, d\eta = dt/a$. Notice that the conformal time in anisotropic inflation is the conformal time in 2-dimensional part $(\eta, x)$. Even in this anisotropic spacetime, as we have done in (31), one can take the following
where we have incorporated the anisotropy while keeping the spatial scalar curvature to be zero. As to the vector perturbations, we can take
\[
\delta A^\mu = (0, 0, 0, D) .
\]
Note that we have no residual gauge transformation and, in particular, \( D \) is a gauge invariant under abelian gauge transformations. And, as we have seen in (31), \( \Gamma \) corresponds to the cross-mode polarization of gravitational waves in the isotropic limit \( a = b \).

Using this gauge, we can calculate the quadratic action as
\[
S^\text{vector} = \int d\eta d^3 x \left[ \frac{b^4}{4a^2} \beta^2_3, x + \frac{b^2}{4} \beta^2_3, y - \frac{b^4}{2a^2} \Gamma' \beta_3, x + \frac{f^2 v' b^2}{a^2} \beta_3 D, x \right.
\]
\[
\left. - \frac{b^2}{4} r^2 + \frac{b^4}{4a^2} r'^2 - \frac{f^2 a^2}{2b^2} D, y - \frac{1}{2} f^2 D, x + \frac{f^2 D^2}{2} - \frac{f^2 v' b^2}{a^2} D' \Gamma \right] .
\]

Since the perturbed shift function \( \beta_3 \) does not have a time derivative, it is not dynamical. There are two physical degrees of freedom \( \Gamma \) and \( D \) in this 2-dimensional vector sector.

### C. 2d scalar sector

For the 2-dimensional scalar sector, we define the metric perturbations
\[
\delta g_{\mu \nu} = \left( \begin{array}{cccc} -2a^2 \Phi & a \beta_1 & a \beta_2 & 0 \\ * & 2a^2 G & 0 & 0 \\ * & * & 2b^2 G & 0 \\ * & * & * & -2b^2 G \end{array} \right) ,
\]
where we have kept the spatial scalar curvature vanishing. The scalar perturbation will be represented by \( \delta \phi \). The variable \( G \) and \( \delta \phi \) are the gauge invariant variables that correspond to the plus mode of gravitational waves and the scalar perturbations, respectively, in the isotropized limit \( a = b \). And, we set the perturbed vector to be
\[
\delta A^\mu_{\text{scalar}} = (\delta A_0, 0, J, 0) ,
\]
where we have fixed the abelian gauge by putting the longitudinal component to be zero. From these ansatz, we can calculate the quadratic action as

\[
S_{\text{scalar}} = \int d^4x d\eta \left[ \frac{b^2}{2a^2} f^2 \delta A_{0,x}^2 + \frac{f^2}{2} \delta A_{0,y}^2 + \frac{b^2}{a^2} f^2 v' (G + \Phi) \delta A_{0,x} - f^2 J' \delta A_{0,y} - 2 \frac{b^2}{a^2} f f \phi v' \delta \phi \delta A_{0,x} \\
+ \frac{1}{4} \beta_1^2, - \frac{1}{2} \frac{b'}{a} \beta_2 \Phi \beta_1 + \frac{b'}{a} \phi' \delta \phi \beta_1 + \frac{1}{4} \beta_2, + a \left( \frac{a'}{a} + \frac{b'}{b} \right) \beta_2 \Phi \right] - a \frac{a'}{a} - \frac{b'}{b} \beta_2 G_{,x} + \frac{b'}{a} v' \beta_2, - \frac{1}{2} f^2 J'^2 + \frac{f^2}{2} f^2 J_{,x}^2 + b^2 G_{,x}^2 - a^2 G_{,y}^2 - b^2 G_{,x}^2 \\
+ \frac{1}{2} b^2 \phi'^2 - \frac{a^2}{2} \delta \phi^2 - \frac{b^2}{2} \delta \phi^2 + \frac{b^2}{2a^2} (f^2 + f f \phi) v'^2 \delta \phi^2 - a^2 b^2 V \phi^2 \\
+ \frac{b^2}{2a^2} J^2 v'^2 G^2 - 2 a^2 b^2 V \Phi G - 2 b b' \Phi G - \left( \frac{b^2}{a^2} f f \phi v'^2 + a^2 b^2 V \phi \right) \delta \phi (G + \Phi) + b^2 \phi' \delta \phi' (G - \Phi) \right] \quad (39)
\]

Here, \(S_{\text{scalar}}\) consists of \(\Phi, \beta_1, \beta_2, G, \delta A_0, \delta \phi\) and \(J\). Among them, \(\Phi, \beta_1, \beta_2\) and \(\delta A_0\) are non-dynamical and can be eliminated.

**IV. THE NATURE OF PRIMORDIAL FLUCTUATIONS**

Now, we are in a position to calculate the statistical properties of primordial fluctuations from anisotropic inflation. For this purpose, we need to reduce the action to the one for physical variables. Then, we can quantize the system and specify the vacuum state. We analyze the vector sector and the scalar sector separately.

**A. Action in slow roll approximation**

First, let us eliminate non-dynamical variables from the action for the 2-dimensional vector sector. In Fourier space, the action for 2-dimensional vector sector becomes

\[
S_{\text{vector}} = \int d\eta d^3k \left[ \frac{1}{4} \left( \frac{b^4}{a^2} k^2 |k_1|^2 + b^2 k_y^2 |k_3|^2 \right) + i k_x \frac{b^4}{2a^2} \beta_3 \Gamma' - i k_x \frac{f^2 v'^2}{a^2} \beta_3 \tilde{D} \right. \\
+ \frac{b^4}{4a^2} |\Gamma'|^2 - \frac{b^2}{4} k_y^2 |\Gamma|^2 + \frac{f^2}{2} |\tilde{D}'|^2 - \frac{f^2 a^2}{2 b^2} k_y^2 |D|^2 - \frac{f^2}{2} k_x^2 |\Gamma'^2 - \frac{f^2 v'^2 b^2}{a^2} \tilde{D} \Gamma' \right], \quad (40)
\]

where we omitted the Fourier indices \(k\) for simplicity. After making the above action real manifestly and completing the square of \(\beta_3\), we obtain

\[
L_{\text{vector}} = \frac{b^4}{4a^2} k^2 \left| \beta_3 + i \frac{k_x}{k^2} \Gamma' + 2 i \frac{f^2}{b^2} k_x v' D \right|^2 + \frac{b^2 k_y^2}{4 k^2} |\Gamma|^2 - \frac{b^2}{4} k_y^2 |\Gamma|^2 \\
+ \frac{f^2}{2} |\tilde{D}'|^2 - \frac{f^2}{2} k_y^2 |D|^2 - \frac{f^2 v'^2 k_x^2}{a^2} |\tilde{D}^2| - \frac{f^2 v'^2 k_x^2}{a^2} \left( \Gamma' D^* + \Gamma D' \right) \right), \quad (41)
\]

where \(k\) is given by \(k(\eta) \equiv \sqrt{k_x^2 + a^2 k_y^2 / b^2}\) which becomes constant in the isotropic limit \(a = b\). The first squared term vanishes after substituting the equation of motion for \(\beta_3\). Now, we define canonically normalized variables as

\[
\tilde{\Gamma} \equiv \frac{b k_y}{\sqrt{2 k^2}} \Gamma, \quad \tilde{\tilde{D}} \equiv f D. \quad (42)
\]

Then, using the canonical variables and integrating by parts, we obtain the reduced action for physical variables

\[
S_{\text{vector}} = \int d\eta d^3k \left[ \frac{1}{2} |\tilde{\Gamma}'|^2 + \frac{1}{2} \left( \frac{b k}{b k} \right) |\tilde{\tilde{D}}|^2 \right] + \frac{1}{2} \frac{f v' a k_y}{b k} \left( \tilde{\Gamma}' \tilde{\tilde{D}} + \tilde{\tilde{D}} \tilde{\Gamma}' \right) \right]. \quad (43)
\]
In the isotropic limit $a = b$, $\bar{\Gamma}$ and $\bar{D}$ represent the cross-mode of gravitational waves and vector waves, respectively. The second line in the action (43) describes how both waves are interacting to each other.

Next, we use the slow roll approximation to simplify the action. To obtain the homogeneous background metric, we integrate the following equations

\[-\frac{\dot{H}}{H^2} = \epsilon_H, \quad \frac{\Sigma}{H} = \frac{1}{3} I\epsilon_H, \quad (44)\]

by assuming $\dot{\epsilon}_H/\epsilon_H \ll a'/a$. The resultant expressions are

\[a = (-\eta)^{-1-\epsilon_H}, \quad b = (-\eta)^{-1-\epsilon_H-I\epsilon_H}. \quad (45)\]

In this approximation, the universe shows anisotropic power law inflation. We should recall, in the second inflationary phase, the variable $I$ is given by

\[I = \frac{c-1}{c}. \quad (46)\]

Note that the range $(1, \infty)$ for $c$ corresponds to $(0, 1)$ for $I$. The above approximation gives useful formula for the subsequent calculations

\[
\begin{align*}
\frac{\dot{a}^2}{a^2} &= (-\eta)^{-2} \left[ 1 + 2\epsilon_H \right], \quad \frac{\dot{a}''}{a} = (-\eta)^{-2} \left[ 2 + 3\epsilon_H \right], \quad \frac{a'}{b} = (-\eta)^{-2} \left[ 1 + 2\epsilon_H + I\epsilon_H \right], \\
\frac{\dot{b}^2}{b^2} &= (-\eta)^{-2} \left[ 1 + 2\epsilon_H + 2I\epsilon_H \right], \quad \frac{b''}{b} = (-\eta)^{-2} \left[ 2 + 3\epsilon_H + 3I\epsilon_H \right], \\
\frac{k'}{k} &= (-\eta)^{-1} \left[ -I\epsilon_H \right], \quad \frac{k''}{k} = (-\eta)^{-2} \left[ -I\epsilon_H \right].
\end{align*}
\quad (47)
\]

From the background equations Eqs. (44)-(110), it is easy to obtain

\[
\frac{f^2 v^2}{a^2} = \frac{\dot{b}^2}{b^2} + \frac{b''}{b} + \frac{\dot{a}^2}{a^2} - \frac{a''}{a} - 2\frac{a'b'}{ab}.
\quad (48)
\]

Using the formula (47), we obtain

\[
\frac{f^2 v^2}{a^2} = 3(-\eta)^{-2} I\epsilon_H. \quad (49)
\]

From Eq. (49), the background equation for the vector can be found as

\[
\left[ \frac{f^2 v^2}{a^2} \right]' = 0. \quad (50)
\]

From this equation, it is easy to deduce the relation

\[
\frac{f'}{f} = (-\eta)^{-1} \left[ -2 - 3\epsilon_H + \eta_H - 2I\epsilon_H \right], \quad (51)
\]

where $\eta_H$ is a slow-roll parameter defined by

\[
\frac{\epsilon_H}{\dot{\epsilon}_H} = 2\frac{(e^\alpha)'}{e^\alpha} \left( 2\epsilon_H - \eta_H \right) = 2(2\epsilon_H - \eta_H)(-\eta)^{-1}. \quad (52)
\]

Of course, $\eta_H$ is not related to the conformal time $\eta$. Furthermore, we obtain

\[
\frac{f''}{f} = (-\eta)^{-2} \left[ 2 + 9\epsilon_H - 3\eta_H + 6I\epsilon_H \right]. \quad (53)
\]
Substituting these results into the action, we obtain the action in the slow roll approximation

$$S_{\text{vector}} = \int d\eta d^3k \left[ \frac{1}{2} \bar{\Gamma}'^2 + \frac{1}{2} \left[ -k^2 + (-\eta)^{-2} \left\{ 2 + 3\epsilon_H + 3I\epsilon_H + 3I\epsilon_H \sin^2 \theta \right\} \right] |\bar{\Gamma}|^2 
+ \frac{1}{2} |\bar{D}'|^2 + \frac{1}{2} \left[ -k^2 + (-\eta)^{-2} \left\{ 2 + 9\epsilon_H - 3\eta_H - 6I\epsilon_H \sin^2 \theta \right\} \right] |\bar{D}|^2 
+ \frac{\sqrt{6I\epsilon_H}}{2} \left( (-\eta)^{-1} \sin \theta (\bar{\Gamma}^* \bar{D}^* + \bar{\Gamma}^* \bar{D}) \right) - \frac{\sqrt{6I\epsilon_H}}{2} \left( (-\eta)^{-1} \sin \theta (\bar{\Gamma}^* \bar{D} + \bar{\Gamma} \bar{D}^*) \right) \right],$$

(54)

where we have defined $\sin \theta \equiv k_\alpha a/k_b$. This $\theta$ represents the direction dependence. In the isotropic limit $I = 0$, the Lagrangian for $\bar{\Gamma}$ becomes the familiar one for gravitational waves in a Friedman-Lemaître universe.

In a similar way, we can derive the quadratic action for physical variables in the 2-dimensional scalar sector. The details can be found in the Appendix A. Moreover, it is straightforward to deduce the action in the slow roll approximation. The resultant action is given by

$$S_{\text{scalar}} = \int d\eta d^3k \left[ L^{GG} + L^{JJ} + L^{\phi\phi} + L^{\phi J} + L^{IJ} \right],$$

(55)

$$L^{GG} = \frac{1}{2} \left[ \bar{G}'^2 + \frac{1}{2} \left[ -k^2 + (-\eta)^{-2} \left\{ 2 + 3\epsilon_H + 3I\epsilon_H + 3I\epsilon_H \sin^2 \theta \right\} \right] |\bar{G}|^2, \right.$$  

(56)

$$L^{JJ} = \frac{1}{2} \left[ \bar{J}'^2 + \frac{1}{2} \left[ -k^2 + (-\eta)^{-2} \left\{ 2 + 9\epsilon_H - 3\eta_H - 6I\epsilon_H \sin^2 \theta \right\} \right] |\bar{J}|^2, \right.$$  

(57)

$$L^{\phi\phi} = \frac{1}{2} \left[ \delta\phi'^2 + \frac{1}{2} \left[ -k^2 + (-\eta)^{-2} \left\{ 2 + 9\epsilon_H - 3\eta_H - 6I\epsilon_H \sin^2 \theta \right\} \right] |\delta\phi|^2, \right.$$  

(58)

$$L^{\phi J} = -3I \sqrt{\frac{\epsilon_H}{1-I}} (-\eta)^{-1} \sin \theta \left( \delta\phi^* \bar{J} + \delta\phi \bar{J}^* \right) - \sqrt{\frac{6I\epsilon_H}{1-I}} (-\eta)^{-1} \sin \theta \left( \delta\phi^* \bar{J} + \delta\phi \bar{J}^* \right),$$

(59)

$$L^{IJ} = -\frac{\sqrt{6I\epsilon_H}}{2} (-\eta)^{-1} \sin \theta \left( \bar{G}^* \bar{J} + \bar{G} \bar{J}^* \right) + \frac{\sqrt{6I\epsilon_H}}{2} (-\eta)^{-1} \sin \theta \left( \bar{G}^* \bar{J} + \bar{G} \bar{J}^* \right),$$

(60)

where we defined canonical variables

$$\bar{G} \equiv \sqrt{2}bG, \quad \bar{J} \equiv \frac{f[k_\alpha]}{k} J, \quad \delta\phi \equiv b\delta\phi.$$

(62)

Here, $\bar{G}$, $\bar{J}$ and $\delta\phi$ represent the gravitational waves, the vector waves, and the scalar perturbations, respectively. The above action shows there exist the interaction among these variables. We notice the scalar part contains $I$ without suppression by a slow-roll parameter $\epsilon_H$. Therefore, to obtain the quasi-scale invariant spectrum of curvature perturbation, $I$ itself has to be small.

From the actions (55) and (54), we see there are two sources of statistical anisotropy of fluctuations. First, the statistical anisotropy of fluctuations comes from the anisotropic expansion itself. Intuitively, this can be understood from the anisotropic effective Hawking temperature $H_{\text{eff}}/2\pi$, where $H_{\text{eff}}$ denotes the effective expansion rate. Indeed, the expansion rate in the direction of the background vector is relatively small, hence the effective Hawking temperature is low. Then, this direction has less fluctuation power compared to the other directions. Thus, the effective Hawking temperature induces the anisotropy in the power spectrum of fluctuations. This effect is encoded in $\epsilon_H$. The other source of the statistical anisotropy of fluctuations comes from the couplings (55), (57) and (61) due to the background vector field. The essential structure of couplings can be understood without complicated calculations. Take a look at the following term

$$\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}f^2(\phi)F_{\mu\nu}F_{\alpha\beta}.$$

(63)

Here, we should recall the order of magnitude of background quantities

$$\frac{f^2a^2}{a^2} \sim I\epsilon_H, \quad \frac{f_\phi}{f} \sim \frac{\epsilon_H}{V_\phi} \sim \frac{1}{\sqrt{\epsilon_H}}.$$  

For example, to obtain the $J - G$ coupling, one of $F_{\mu\nu}$ have to be replaced by the background quantity $a'$. Hence, the coefficients in the $J - G$ coupling should be proportional to $f a'$ which is of the order of $\sqrt{\epsilon_H}$. This explains the
strength of the coupling in \(61\). Similarly, \(J - \delta \phi\) coupling should be proportional to \(f_\phi v'\) because we have to take the variation with respect to \(\phi\). Hence, we can estimate its magnitude to be \(\sqrt{f}\). This agrees to the interaction term \(60\). Finally, the coupling \(G - \delta \phi\) has a magnitude of the order of \(f_\phi v'^2\) which is proportional to \(I\sqrt{f}\). This shows a good agreement with the coupling \(59\). Thus, we can understand why there is a hierarchy among the couplings of the gravitational waves, the vector waves and the scalar field.

B. Numerical results

In this section, we will calculate power spectrum of various variables. To set the initial conditions, we need to quantize this system by promoting canonical variables to operators which satisfy the following canonical commutation relations

\[
\big[ Q_a(\eta, x), P_b(\eta, y) \big] = i \delta_{ab} \delta(x - y), \quad \big[ Q_a(\eta, x), Q_b(\eta, y) \big] = \big[ P_a(\eta, x), P_b(\eta, y) \big] = 0 \tag{64}
\]

where \(Q_a\) (1 \(\leq a \leq 5\) denote the variables \(\bar{\Gamma}, \bar{D}, \delta \dot{\phi}, \bar{G}, \bar{J}\) in this order, while \(P_a\) is their conjugate momentum defined by \(P_a \equiv \delta L/\delta \dot{Q}_a\). The point is that, with a given wave number, the actions \(54\) and \(55\) reduce to those of independent harmonic oscillators in the subhorizon limit \(\sim k \eta \gg 1\)

\[
S = \sum_{a=1}^{5} \frac{1}{2} \int d\eta d^3k \left[ |\dot{Q}_a|^2 - k^2 |Q_a|^2 \right]. \tag{65}
\]

We choose the Bunch-Davis vacuum state \(|0\rangle\) by imposing the following initial conditions at an initial time \(\eta_i\)

\[
Q_a(\eta_i) = \sqrt{\frac{1}{2k}} (a_{a,k} + a_{a,-k}^\dagger), \quad Q_a'(\eta_i) = -i \sqrt{\frac{1}{2k}} (a_{a,k} - a_{a,-k}^\dagger), \tag{66}
\]

where \(a_{a,k}\) is an annihilation operator whose commutation relations are given by

\[
\left[ a_{a,k}, a_{b,k'}^\dagger \right] = \delta_{ab} \delta^{(3)}(k - k'), \quad \left[ a_{a,k}, a_{b,k'} \right] = 0. \tag{67}
\]

It is easy to verify the commutation relations imposed for \(P_a\), \(Q_a\). The Bunch-Davis vacuum \(|0\rangle\) is defined by \(a_{a,k}|0\rangle = 0\).

Later, the system evolves according to the actions \(54\) and \(55\) and the variables \(Q_a(\eta)\) are given by linear combination of \(a_{b,k}\) as

\[
Q_{a,k}(\eta) = \sum_{b=1}^{5} \left[ c_{ab}(\eta) a_{b,k} + c_{ba}(\eta) a_{b,-k}^\dagger \right], \tag{68}
\]

where the transfer matrix \(c_{ab}\) depends on the wave number \(k\) and the time \(\eta\). The time dependence is determined by solving the classical equations of motion with the initial conditions given by the coefficients in Eqs.\(66\). And the expectation values of operators in this vacuum state are evaluated as

\[
\langle 0 | Q_{a,k}(\eta) Q_{b,p}(\eta) | 0 \rangle = \sum_{d=1}^{5} c_{ad} c_{bd}^* \delta(k + p). \tag{69}
\]

Especially, we are interested in the power spectrum of the scalar perturbations, cross mode and plus mode of gravitational waves, the cross correlation between scalar perturbations and the plus mode of gravitational waves, and the linear polarization of gravitational waves. The power spectrum of scalar perturbations is given by

\[
\langle 0 | \delta \phi_{k}(\eta) \delta \phi_{p}(\eta) | 0 \rangle = \sum_{d=3}^{5} |c_{3d}(\eta)|^2 \delta(k + p) \equiv P_{\delta \phi}(k) \delta(k + p), \tag{70}
\]

where we took into account the fact that the 2-dimensional vector sector is decoupled from the scalar sector. The power spectrum of the cross and plus mode of gravitational waves read

\[
\langle 0 | \tilde{\Gamma}_{k}(\eta) \tilde{\Gamma}_{p}(\eta) | 0 \rangle = \sum_{d=1}^{2} |c_{1d}(\eta)|^2 \delta(k + p) \equiv P_{\Gamma}(k) \delta(k + p), \tag{71}
\]

\[
\langle 0 | \tilde{G}_{k}(\eta) \tilde{G}_{p}(\eta) | 0 \rangle = \sum_{d=3}^{5} |c_{4d}(\eta)|^2 \delta(k + p) \equiv P_{G}(k) \delta(k + p). \tag{72}
\]
Here, we used the decoupling of the 2-dimensional scalar sector and the vector sector. We can also calculate the cross correlation between the plus mode of gravitational waves and the scalar perturbations

$$\langle 0 | \delta \phi_k(\eta) \bar{G}_p(\eta) | 0 \rangle = \sum_{d=3}^5 c_{3d}(\eta) c_{4d}^*(\eta) \delta(k + p) \equiv P_{\delta \phi G}(k) \delta(k + p).$$

(73)

The linear polarization of gravitational waves can be calculated once the power spectrum of the cross and plus mode are calculated.

FIG. 1: Evolution of anisotropy of curvature perturbations. Here we depicted the anisotropy $P_{\delta \phi}(k)|_{\theta = \pi/2}/P_{\delta \phi}(k)|_{\theta = 0} - 1$ as a function of time. We set e-folding number to be zero at the time of horizon crossing of the given mode. The both axes are taken in log scale.

FIG. 2: Evolution of anisotropy in gravitational waves. Here we depicted the anisotropy $P_X(k)|_{\theta = \pi/2}/P_X(k)|_{\theta = 0} - 1$, where $X = \Gamma, G$, as a function of time. The axes are in log scale. As one can see, the difference between two modes is quite small.

Now, we numerically calculate the transfer matrix $c_{ab}$ with parameters $\epsilon_H = 10^{-2}$, $I = 10^{-5}$ and evaluate various statistical quantities. First of all, we need to calculate the statistical anisotropy in the curvature perturbations. We depict the time evolution of the anisotropy in the curvature perturbation

$$\frac{P_{\delta \phi}(k)|_{\theta = \pi/2}}{P_{\delta \phi}(k)|_{\theta = 0}} - 1$$

(74)

as a function of time in Fig. 1. Taking look at Fig. 1 we see the anisotropy grows with the squared of the e-folding number. The amplitude is larger than the expected one due to the coupling proportional to $\sqrt{I}$ which is larger than
FIG. 3: Evolution of cross correlation between curvature perturbations and the plus mode of gravitational waves. Here we depicted the value $-P_{\delta \phi G}(k)_{\theta = \pi/2}/P_{\delta \phi}(k)_{\theta = \pi/2}$ as a function of the time. The axes are in log scale.

FIG. 4: Evolution of the linear polarization of gravitational waves. Here we depicted the value $P_X(k)_{\theta = \pi/2}/P_{\Gamma}(k)_{\theta = \pi/2} - 1$ as a function of time. The axes are in log scale.

the expected one $\sqrt{\epsilon H}$. Similarly, in Fig. 2 we show the power spectrum of the cross and plus mode of gravitational waves

$$\frac{P_X(k)_{\theta = \pi/2}}{P_X(k)_{\theta = \pi/2}} - 1,$$

where $X = \Gamma, G$. Next, we are interested in the cross correlation between the curvature perturbations and the gravitational waves. In Fig. 3 we depict the cross correlation between the curvature perturbation and the plus mode of gravitational waves which is normalized by the power spectrum of scalar perturbations

$$\frac{P_{\delta \phi G}(k)_{\theta = \pi/2}}{P_{\delta \phi}(k)_{\theta = \pi/2}}$$

In Fig. 3 we see the cross correlation between the scalar perturbations and the gravitational waves grows quadratically in the e-folding number. The amplitude is proportional to the coupling between $G$ and $\delta \phi$, i.e., $I \sqrt{\epsilon H}$. The difference between the power spectrum of two polarization modes normalized by the power spectrum of the plus mode of
characterizes the linear polarization of the gravitational wave. As you can see from Fig[2] and Fig[3] the anisotropy of power spectrum grows quadratically in the e-folding number, while the difference between the two polarization modes shows the growth of higher power in e-folding number.

C. Analytical Estimation

In this section, we treat the anisotropy perturbatively and estimate its magnitude by using perturbation in the interaction picture. In the interaction picture, the expectation value for a physical quantity $\mathcal{O}(\eta)$ is given by

$$
\langle \text{in} | \mathcal{O}(\eta) | \text{in} \rangle = \left\langle 0 \left| \left[ \hat{T} \exp \left( i \int_{\eta_0}^{\eta} H_I(\eta') d\eta' \right) \right] \mathcal{O}(\eta) \left[ \hat{T} \exp \left( -i \int_{\eta_0}^{\eta} H_I(\eta') d\eta' \right) \right] \right| 0 \right\rangle,
$$

where $|\text{in}\rangle$ is an in vacuum in the interaction picture, $\hat{T}$ and $\hat{T}$ denote a time-ordered and an anti-time-ordered product and $H_I$ denotes the interaction part of Hamiltonian in this picture. This is equivalent to the following

$$
\langle \text{in} | \mathcal{O}(\eta) | \text{in} \rangle = \sum_{N=0}^{\infty} i^N \int_{\eta_0}^{\eta} d\eta_N \int_{\eta_0}^{\eta} d\eta_N-1 \cdots \int_{\eta_0}^{\eta} d\eta_1 \langle 0 | [H_I(\eta_1), [H_I(\eta_2), \cdots [H_I(\eta_N), \mathcal{O}(\eta)] \cdots]] 0 \rangle.
$$

The successive approximation is made by truncating the summation at a certain order $N$. In our analysis, we assume the noninteracting part of Hamiltonian to be that of free fields in deSitter spacetime

$$
L_0 = \sum_{n=1}^{5} \left[ \frac{1}{2} |Q_n|^2 \right],
$$

and the operators in the interaction picture are given by

$$
Q_{n,k}(\eta) = u(\eta)a_{n,k} + u(\eta)^* a_{n,-k}^*,
$$

$$
u(\eta) = \sqrt{\frac{1}{2k}} e^{-i k \eta} \left( 1 - i \frac{\eta}{k} \right),
$$

And the rest of the Lagrangian [54]-[61] is regarded as the interaction part $L_I = L^{(2)} - L_0$. To see the leading effect on the anisotropy in the scalar perturbation, which is of the order of $\mathcal{O}$, we evaluate the correction due to the interaction given by

$$
H_I^{\phi J} = \int d^3k \left[ -L^{\phi J} \right]
$$

$$
= \int d^3k \left[ -\sqrt{\frac{6I}{1-I}} (-\eta)^{-1} \sin \theta \left( \delta \phi^J \bar{J} + \delta \phi^J \bar{J} \right) + \sqrt{\frac{6I}{1-I}} (-\eta)^{-2} \sin \theta \left( \delta \phi^J \bar{J} + \delta \phi^J \bar{J} \right) \right].
$$

Note that in the analogy with the slow-roll parameter in the ordinary slow-roll inflation, the term proportional to $I \sin^2 \theta \delta \phi^J \bar{J}$ in [58] can be expected to give the anisotropy $\delta \langle \text{in} | \delta \phi_k \delta \phi_p | \text{in} \rangle / \langle 0 | \delta \phi_k \delta \phi_p | 0 \rangle \sim \sin^2 \theta \mathcal{N}(k)$ where $\mathcal{N}(k)$ is the e-folding number from the horizon exit. Thus, the leading correction comes from the interaction through the term $H_I^{\phi J}$. The leading correction is given by

$$
\delta \langle \text{in} | \delta \phi_k(\eta) \delta \phi_p(\eta) | \text{in} \rangle = i^2 \int_{\eta_0}^{\eta} d\eta_2 \int_{\eta_0}^{\eta} d\eta_1 \left\langle 0 \left| [H_I^{\phi J}(\eta_1), [H_I^{\phi J}(\eta_2), \delta \phi_k(\eta) \delta \phi_p(\eta)] \right] 0 \right\rangle.
$$

Using Eqs.[51] and commutation relations for the creation and annihilation operators, we obtain the anisotropy expressed as follows

$$
\frac{\delta \langle \text{in} | \delta \phi_k \delta \phi_p | \text{in} \rangle}{\langle 0 | \delta \phi_k \delta \phi_p | 0 \rangle} = \frac{24I}{1-I} \sin^2 \theta \int_{\eta_0}^{\eta} d\eta_2 \int_{\eta_0}^{\eta} d\eta_1 \frac{8}{|u(\eta)|^2} \text{Im} \left[ (-\eta_2)^{-1} u'(\eta_2) u^*(\eta) + (-\eta_2)^{-2} u(\eta_2) u^*(\eta) \right] \times \text{Im} \left[ u(\eta_1) u^*(\eta_2) \left\{ (-\eta_1)^{-1} u'(\eta_1) u^*(\eta) + (-\eta_1)^{-2} u(\eta_1) u^*(\eta) \right\} \right].
$$
where Im denotes the imaginary part. Substituting the function form of $u$ and introducing time variables $\chi \equiv k\eta$, $\chi_1 \equiv k\eta_1$ and $\chi_2 \equiv k\eta_2$, we have

$$
\frac{\delta\langle in | \tilde{\phi}_k \tilde{\phi}_p | in \rangle}{\langle 0 | \tilde{\phi}_k \tilde{\phi}_p | 0 \rangle} (\chi) = \frac{6I}{1 - I} \sin^2 \theta \int_{-1}^{1} d\chi_2 \int_{-1}^{1} d\chi_1 \frac{8}{1 + \frac{1}{(\chi_2)^2}} \frac{1}{\chi_1 \chi_2} \left[ \cos(-\chi_2 + \chi) - \sin(-\chi_2 + \chi) \frac{1}{\chi} \right] 
$$

$$
\times \left[ \cos(-2\chi_1 + \chi + \chi_2) \left( \frac{1 + \frac{1}{\chi_1}}{\chi_1} - \frac{1}{\chi_2} + \frac{1}{\chi_1 \chi_2} \right) 
+ \sin(-2\chi_1 + \chi + \chi_2) \left( -\frac{1}{\chi_1 \chi_2} + \frac{1}{\chi_1} - \frac{1}{\chi_2} \right) \right].
$$

(86)

The contribution to the integral from the subhorizon $-\chi_1 \gg 1$ is negligible. In the limit of superhorizon $-\chi_1 \ll 1$, we also have $-\chi_2 \ll 1, -\chi \ll 1$. Hence, the integrand in Eq. (86) approximately becomes $8/\chi_1 \chi_2$. Thus, the anisotropy can be evaluated as

$$
\frac{\delta\langle in | \tilde{\phi}_k \tilde{\phi}_p | in \rangle}{\langle 0 | \tilde{\phi}_k \tilde{\phi}_p | 0 \rangle} (\chi) = \frac{6I}{1 - I} \sin^2 \theta \int_{-1}^{1} d\chi_2 \int_{-1}^{1} d\chi_1 \frac{8}{\chi_1 \chi_2}
$$

$$
= \frac{24I}{1 - I} \sin^2 \theta \ N^2(k),
$$

(87)

where $N(k) \equiv -\ln(-k\eta)$ is the $e$-folding number from the horizon exit.

For the anisotropy of both two polarizations of gravitational waves, the similar calculations give

$$
\frac{\delta\langle in | \tilde{\Gamma}_k \tilde{\Gamma}_p | in \rangle}{\langle 0 | \tilde{\Gamma}_k \tilde{\Gamma}_p | 0 \rangle} = \frac{\delta\langle in | \tilde{G}_k \tilde{G}_p | in \rangle}{\langle 0 | \tilde{G}_k \tilde{G}_p | 0 \rangle} = 6I_H \sin^2 \theta \ N^2(k),
$$

(88)

where we used the interaction term in the action [53] for $\tilde{\Gamma}$ and that in [61] for $\tilde{G}$. It is interesting to calculate the cross correlation. The leading contribution comes from $H_I^{G\tilde{G}}$ and $H_I^{\tilde{G}G}$. The result is as follows:

$$
\frac{\langle in | \delta \tilde{\phi}_k \tilde{G}_p | in \rangle}{\langle 0 | \delta \tilde{\phi}_k \tilde{G}_p | 0 \rangle} \approx -24I \sqrt{\frac{\epsilon_H}{1 - I}} N^2(k).
$$

(89)

As we will soon see, this might give a detectable number. On the other hand, it turns out that the linear polarization of the gravitational wave must be the effect of the order higher than $I$.

V. COSMOLOGICAL IMPLICATION

Now, we are in a position to discuss cosmological implication of an anisotropic inflationary scenario. As we have listed up in section I, there are many interesting phenomenology in the anisotropic inflation. Here, we summarize our results:

- There exists statistical anisotropy in curvature perturbations of the order of $24I N^2(k)$.
- There exists statistical anisotropy in gravitational waves of the order of $6I\epsilon_H N^2(k)$.
- These exists the cross correlation between curvature perturbations and gravitational waves of the order of $-24I \sqrt{\epsilon_H} N^2(k)$.
- There is small linear polarization of gravitational waves.

Due to the interaction on superhorizon scales, there is an enhancement factor $N^2(k)$ in the above quantities. Because of this enhancement, even when the anisotropy of the spacetime is quite small, say $\Sigma/H \sim 10^{-7}$ in our example, the statistical anisotropy imprinted in primordial fluctuations cannot be negligible in precision cosmology.

Let us be more precise. The anisotropy is often parameterized by

$$
P(k) = P(k) \left[ 1 + g_* \sin^2 \theta \right].
$$

(90)

We have estimated $g_* \approx 24I N^2(k)$ which could be large. The current observational limit of the statistical anisotropy for the curvature perturbations is characterized by $g_* < 0.3$. Let us suppose $g_* = 0.2$. Then, we have the
anisotropy in the gravitational waves with $g_* \simeq 10^{-3}$. The linear polarization is of the order of $10^{-6}$. Using the definition of curvature perturbations $\zeta = \delta \varphi / \sqrt{2} \epsilon_H$, one can translate the cross correlation between scalar perturbations and gravitational waves to that between curvature perturbations and gravitational waves normalized by the power spectrum of curvature perturbations:

$$\langle \xi_k G_p | n \rangle \langle 0 | \xi_k \zeta_p | 0 \rangle \simeq -20 I N^2 (k) \epsilon_H \sim - g_* \epsilon_H \sim -2 \times 10^{-3}, \quad (91)$$

where we used $g_* \sim 0.2$ and $\epsilon_H \sim 10^{-2}$. Since the current constraints on the $TB/TE$ ratio is of the order of $10^{-2}$ [1], we need to improve the accuracy by one more order, which might be achieved by PLANCK.

In [33], it is pointed out that the sign of $g_*$ is different from the observed one. However, it might be possible to modify the model so that the sign of $g_*$ is flipped. For example, we can consider two vector fields. Then, the orthogonal direction to the plane determined by two vectors becomes a preferred direction. In this case, we can expect the sign of $g_*$ becomes opposite. We can also utilize anti-symmetric tensor fields to achieve the same aim. The details will be reported elsewhere.

VI. CONCLUSION

We have studied the statistical nature of primordial fluctuations from an anisotropic inflation which is realized by a vector field coupled to an inflaton. First, we have classified metric fluctuations according to the 2-dimensional rotational symmetry. To choose a convenient gauge in an anisotropic universe, we have started from the flat slicing gauge in an isotropic universe and made an appropriate gauge transformation to get a canonical gauge in an anisotropic universe. This gauge choice has made the subsequent analysis and the interpretation of the variables easier. Using the canonical gauge, we have revealed the structure of the couplings between curvature perturbations, vector waves, and gravitational waves. We found that there are two sources for anisotropy, i.e. the anisotropy due to the anisotropic expansion of the universe and that due to the anisotropic couplings among variables. It turned out that the latter effect is dominant. We have numerically obtained power spectra. We also presented analytical formula using in-in formalism. Since the coupling between the curvature perturbations and vector waves is the strongest one, the anisotropy in the curvature perturbations is larger than that in gravitational waves. More interestingly, we found the cross correlation between curvature perturbations and gravitational waves which is peculiar to anisotropic inflation. We also found the linear polarization of gravitational waves. Although there are several mechanism to produce circular polarization in the primordial gravitational waves [35], this is the first example which realized the linear polarization in the primordial gravitational waves.

We have only considered power spectrum for simplicity. However, as is pointed out in the paper [8], the statistical anisotropy could appear in the non-Gaussianity strongly and modify the shape of the bispectrum and trispectrum. Hence, it is interesting to study non-Gaussianity in anisotropic inflation models.

We can extend anisotropic inflation in various ways. Although we have investigated a chaotic inflation in this paper, it is easy to extend the analysis to other inflation models. It is possible to incorporate multi-vector fields. From the string theory point of view, it is intriguing to consider anti-symmetric tensor field. It is also interesting to consider other Bianchi type models [36] in the context of anisotropic inflation.

Acknowledgments

SK would like to thank the YITP members in Kyoto for warm hospitality. A part of this work was done while SK was visiting YITP supported by JSPS Grant-in-Aid for Scientific Research (A) 21244033. SK is supported by an STFC rolling grant. JS is supported by the Japan-U.K. Research Cooperative Program, Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture of Japan No.18540262, Grant-in-Aid for Scientific Research on Innovative Area No.21111006 and the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence”.

Appendix A: derivation of reduced action for 2d scalar sector

In this appendix, we present a detailed derivation of the action for the 2-dimensional scalar sector. Here, we need to eliminate non-dynamical fields $\delta A_0$, $\beta_1$, $\beta_2$ and $\Phi$. 
First, let us see the terms concerning $\delta A_0$ in Fourier space
\[
\frac{b^2}{2a^2} f^2 k^2 |\delta A_0|^2 + \delta A_0^* \left[ -ik_x \frac{b^2}{2a^2} f^2 v' (G + \Phi) + ik_y \frac{f^2}{2} J' + ik_x \frac{b^2}{a^2} f f_x v' \delta \phi \right] + \text{c.c.} .
\]
(A1)

By completing the square as
\[
\frac{b^2}{2a^2} f^2 k^2 \left| \delta A_0 - \frac{ik_x v'}{k^2} \left( G + \Phi - 2 \frac{f \phi}{f} \delta \phi \right) + \frac{ik_y a^2}{k^2} J' \right|^2 - \frac{b^2}{2a^2} f^2 \left| k_x v' \left( G + \Phi - 2 \frac{f \phi}{f} \delta \phi \right) - \frac{a^2}{b^2} k_y J' \right|^2 ,
\]
(A2)

the variable $\delta A_0$ can be eliminated because the first squared term vanishes after substituting the equation of motion for $\delta A_0$. Similarly, the terms containing $\beta_1$ are given by
\[
\frac{1}{4} k_y^2 |\beta_1|^2 + \beta_1^* \left[ - \frac{1}{4} k_x k_y \beta_2 + ik_x \frac{b^2}{2a} b \Phi - ik_x \frac{b^2 \phi'}{2a} \delta \phi \right] + \text{c.c.} .
\]
(A3)

Completing the square gives
\[
\frac{k_y^2}{4} \left| \beta_1 + \frac{k_x}{k_y} \beta_2 + 4 \frac{k_x b b'}{k_y^2} \frac{b}{a} \Phi - 2 k_x \frac{b^2 \phi'}{k_y^2} \delta \phi \right|^2 - \frac{k_y^2}{4} \left| \beta_2 - 4 \frac{1}{k_y} \frac{b b'}{a} \Phi + 2 \frac{1}{k_y} \frac{b^2 \phi'}{a} \delta \phi \right|^2 .
\]
(A4)

Again, we can ignore the first term. Taking a look at the terms related to the variable $\beta_2$, we see only the linear term in $\beta_2$ appears as
\[
\beta_2 \left[ \frac{a}{2} \left( \frac{a'}{a} + \frac{b'}{b} + 2 \frac{b^2}{a^2} k_y^2 \right) i k_y \Phi - \frac{a}{2} \left( \frac{a'}{a} - \frac{b'}{b} \right) i k_y G - \frac{a}{2} \frac{g^2 b^2}{k_y^2} \frac{2 a^2}{a^2} i k_y \delta \phi + \frac{f^2 v'}{2a} ik_x J \right] + \text{c.c.} .
\]
(A5)

The variation with respect to $\beta_2$ gives
\[
\Phi = \frac{1}{\lambda} \left[ \left( \frac{a'}{a} - \frac{b'}{b} \right) G + \frac{k^2 b^2}{k y^2 a^2} \phi' \delta \phi - \frac{k_x}{k_y} f^2 v' \right] .
\]
(A6)

Here we have defined
\[
\lambda = \frac{a'}{a} + \frac{b'}{b} + 2 \frac{k^2 b^2}{k y^2 a^2} b' .
\]
(A7)

After substituting this result into the action, we obtain the action for physical variables:
\[
S^{(2)\text{scalar}} = \int d\eta d^3 k \left[ L^{GG} + L^{\phi \phi} + L^{JJ} + L^{G \phi} + L^{\phi J} + L^{JJ} \right] .
\]
(A8)

The term $L^{GG}$ is given by
\[
L^{GG} = \frac{b^2}{2} |G|^2 - b^2 k^2 |G|^2 - \frac{k_y^2}{k^2} f^2 v'^2 |G|^2 - \frac{b^2}{2a^2} \frac{f^2 v'^2 k^2}{\lambda^2} \left( \frac{a'}{a} - \frac{b'}{b} \right)^2 |G|^2
\]
\[- \frac{1}{\lambda} \frac{b^2 v'^2 k^2}{a^2} V \left( \frac{a'}{a} - \frac{b'}{b} \right) \left[ \frac{a'}{a} - \frac{b'}{b} \right] |G|^2 - \frac{1}{\lambda} \frac{b^2 v'^2 k^2}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \left( \frac{a'}{a} - \frac{b'}{b} \right) \left\{ \frac{1}{\lambda} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right\} \left| G \right|^2 .
\]
(A9)

Using the canonically normalized variable $\bar{G} = \sqrt{2b} G$, we finally have
\[
L^{GG} = \frac{1}{2} \left| \bar{G} \right|^2 + \frac{b^2}{2} \bar{G}^2 + \frac{k_x^2}{2k^2} f^2 v'^2 \bar{G}^2 - \frac{1}{\lambda} \left( \frac{a'}{a} - \frac{b'}{b} \right)^2 \left( \frac{a'}{a} - \frac{b'}{b} \right) \left( \frac{a'}{a} - \frac{b'}{b} \right) \left\{ \frac{1}{\lambda} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right\} \left| \bar{G} \right|^2 .
\]
(A10)
The term $L^{\phi \phi}$ reads

$$L^{\phi \phi} = \frac{b^2}{2} |\delta \phi|^2 - \frac{b^2}{2} k_2^2 |\delta \phi|^2 - \frac{a^2 b^2}{2} V_{\phi \phi} |\delta \phi|^2$$

$$- \frac{b^2}{a^2} f_{\phi \phi}^2 v^2 \frac{\phi^2}{k_2^2} |\delta \phi|^2 + \frac{b^2}{2a^2} (f f_{\phi \phi} + f_{\phi}^2) v^2 |\delta \phi|^2 - \frac{b^4 k_2^2}{a^2} \phi' \phi |\delta \phi|^2$$

$$- \frac{\phi'}{\lambda^2} \left( a^2 b^2 V + \frac{b^2 f_{\phi \phi}^2 v^2 \frac{\phi^2}{k_2^2}}{2a^2} + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \right) \left( \frac{k^2}{a^2} \right) |\delta \phi|^2$$

$$+ \frac{\phi'}{\lambda} \left( -a^2 b^2 V \phi - f f_{\phi \phi} \frac{b^2 v^2}{a^2} + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \phi' + 2 \frac{k_2^2 f_{\phi \phi} v^2}{a^2} \right) \left( \frac{k^2}{a^2} \right) \frac{b^2}{2} \frac{\phi' \phi^2}{k_2^2} + \frac{1}{2} \left\{ \frac{b^2}{\lambda} \frac{\phi' \phi^2}{k_2^2} \right\} |\delta \phi|^2$$

(A11)

Using the canonically normalized variable $\tilde{\phi} \equiv b \delta \phi$, we can write

$$L^{\phi \phi} = \frac{1}{2} |\tilde{\phi} |^2 + \frac{1}{2} \left[ - k^2 + \frac{b''}{b} - a^2 V_{\phi \phi} - \frac{f^2 v^2}{2} \frac{k^2}{k^2} + (f f_{\phi \phi} + f_{\phi}^2) \frac{v^2}{a^2} - \frac{k^2}{2a^2} \phi' \phi^2 \right.$$  

$$- \frac{2}{\lambda^2} \left( a^2 V + \frac{f^2 v^2}{2a^2} \frac{k^2}{k^2} + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \right) \left( \frac{k^2}{a^2} \right) |\delta \phi|^2$$

$$+ \frac{2}{\lambda} \left( -a^2 \phi V - \frac{f f_{\phi \phi} v^2}{2a^2} + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \phi' + 2 \frac{k_2^2 f_{\phi \phi} v^2}{a^2} \right) \left( \frac{k^2}{a^2} \right) \frac{b^2}{2} \frac{\phi' \phi^2}{k_2^2} + \frac{1}{2} \left\{ \frac{b^2}{\lambda} \frac{\phi' \phi^2}{k_2^2} \right\} |\delta \phi|^2$$

(A12)

Similarly, the term $L^{J J}$ reads

$$L^{J J} = \frac{f^2}{2} \frac{k^2}{k^2} |J |^2 - \frac{f^2}{2} \frac{k^2}{k^2} |J |^2$$

$$- \left\{ a^2 b^2 V + \frac{f^2 b^2 v^2}{2a^2} \frac{k^2}{k^2} + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \phi^2 - \frac{b^2}{2} \frac{1}{a^2} \right\} \left( \frac{b}{k} \right)^2 \left( \frac{\phi}{a} \right)^2 \right\} |J |^2$$

(A13)

Using the canonically normalized variable $\tilde{J} \equiv \frac{n^2 k}{k} J$ , we obtain

$$L^{J J} = \frac{1}{2} |\tilde{J} |^2 - \frac{1}{2} \left[ - k^2 + \left( \log \frac{k}{f} \right)^2 - \left( \log \frac{k}{f} \right)^2 \right.$$  

$$+ 2 \left\{ a^2 b^2 V + \frac{f^2}{2a^2} \frac{k^2}{k^2} f^2 v^2 + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \phi^2 - \frac{b^2}{2} \frac{1}{a^2} \right\} \left( \frac{b}{k} \right)^2 \left( \frac{\phi}{a} \right)^2 \right\} |\tilde{J} |^2$$

(A14)

The mixed term $L^{G \phi}$ is given by

$$L^{G \phi} = \left[ - \frac{b^4}{2a^2} f^2 v^2 \frac{k_2^2}{k_2^2} \phi' \phi' \frac{a^2}{a^2} \frac{b'}{b'} + 2b b^2 \phi' \phi' \frac{a^2}{a^2} \frac{b'}{b'} \lambda \left( \frac{a'}{a} - \frac{b'}{b} \right) \right.$$  

$$- \frac{\phi'}{\lambda^2} \left( a^2 b^2 V + \frac{f^2 b^2 v^2}{2a^2} \frac{k_2^2}{k_2^2} + 4 \frac{k_2^2 b^2}{k_2^2} \frac{1}{a^2} \right) \left( \frac{a'}{a} - \frac{b'}{b} \right) \left( \frac{k^2}{a^2} \right) \frac{b^2}{2} \frac{\phi' \phi^2}{k_2^2}$$

$$- \frac{a^2 b^2}{2} \frac{V}{\lambda} \left( \frac{a'}{a} - \frac{b'}{b} \right) + b b' \phi' \phi' \frac{a^2}{a^2} \frac{b^2}{b^2} \frac{f^2 v^2}{a^2} \frac{k_2^2}{k_2^2} + 2 \frac{b^2}{\lambda} \frac{\phi' \phi^2}{a^2} \frac{a^2}{a^2} \frac{b^2}{b^2} \left( \frac{a'}{a} - \frac{b'}{b} \right)^2$$

$$- \frac{2}{\lambda} f f_{\phi \phi} \frac{v^2}{a^2} \frac{b'}{b'} + 2 \frac{b^2}{\lambda} \frac{f f_{\phi \phi} v^2}{a^2} \frac{k_2^2}{k_2^2} \frac{1}{2} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right\} (G \delta \phi + G^* \delta \phi)$$

(A15)
Or, using variables \( \tilde{G}, \tilde{\phi} \), we have

\[
\sqrt{2}L^{G\phi} = \left[ -\frac{b'^2}{2a'f}v'^2\frac{k_y^2}{k_y^2} \phi' \alpha' \left( \frac{a'}{a} - \frac{b'}{b} \right) \right. \\
- \phi' \frac{\lambda}{\lambda^2} \left( 2a'^2V + \frac{f'^2v'^2}{2a''} \frac{k_y^2}{k_y^2} + 4 \frac{k_y^2}{k_y^2} \frac{b'^2}{2a''} \right) \frac{\lambda}{\lambda^2} \left( \frac{a'}{a} - \frac{b'}{b} \right) \left( \frac{a'}{a} - \frac{b'}{b} \right) \right] \left( \frac{a'}{a} - \frac{b'}{b} \right) \bar{J}G \bar{G}, \\
\left. - \frac{a'^2}{\mu^2} V \phi \left( \frac{a'}{a} - \frac{b'}{b} \right) + b' \phi' \bar{b}^2 \frac{k_y^2}{k_y^2} b'^2 + \frac{2 b'^2}{\mu^2} \left( \frac{a'}{a} - \frac{b'}{b} \right)^2 \left( G\bar{G} + G^*\phi \right) \right].
\]

(A16)

The terms containing \( \phi \) and \( J \) are given by

\[
L^{\phi J} = \frac{k_xk_y}{2k^2} f'v'^2 \left\{ \frac{\phi' b'^2}{\mu^2} \frac{k_y^2}{k_y^2} - \frac{f'}{f} \right\} \left( J\bar{\phi}' + J^*\phi \right) \\
+ \left( a'^2b^2V + \frac{b'^2}{2a'^2} f'^2v'^2 \frac{k_y^2}{k_y^2} + 4 \frac{b'^2}{2a'^2} \frac{k_y^2}{k_y^2} \right) \frac{v'}{f} \frac{f'}{f} \frac{k_x}{\mu^2} \frac{\bar{b}^2}{\bar{b}^2} \left( J\phi^* + J^*\phi \right) \\
+ \left\{ \frac{1}{2} \left( \frac{b'^2}{a'^2} f'^2v'^2 + a'^2b'^2V \phi \right) - \frac{b'^2}{a'^2} f'^2v'^2 \frac{k_y^2}{k_y^2} - 2 \frac{b'^2}{a'^2} \frac{k_y^2}{k_y^2} \phi' \right\} \frac{v'}{f} \frac{f'}{f} \frac{k_x}{\mu^2} \left( J\phi^* + J^*\phi \right) \\
+ \frac{b'^2}{a'^2} \frac{f'^2}{f} \frac{k_x}{\mu^2} \left( J\phi^* + J^*\phi \right) + \frac{b'^2}{a'^2} \frac{f'^2}{f} \frac{k_x}{\mu^2} \frac{\bar{b}^2}{\bar{b}^2} \left( J\phi^* + J^*\phi \right) .
\]

(A17)

The above action can be rewritten by using variables \( \phi \equiv b\phi \) and \( J = \frac{f'^2}{f}J \) as

\[
bL^{\phi J} = \frac{k_y}{2k^2} f'v'^2 \left\{ \frac{\phi' b'^2}{\mu^2} \frac{k_y^2}{k_y^2} - \frac{f'}{f} \right\} \left( J\bar{\phi}' + J^*\phi \right) \\
+ \left( a'^2b^2V + \frac{b'^2}{2a'^2} f'^2v'^2 \frac{k_y^2}{k_y^2} + 4 \frac{b'^2}{2a'^2} \frac{k_y^2}{k_y^2} \right) \frac{v'}{f} \frac{f'}{f} \frac{k_x}{\mu^2} \frac{\bar{b}^2}{\bar{b}^2} \left( J\phi^* + J^*\phi \right) \\
+ \left\{ \frac{1}{2} \left( \frac{b'^2}{a'^2} f'^2v'^2 + a'^2b'^2V \phi \right) - \frac{b'^2}{a'^2} f'^2v'^2 \frac{k_y^2}{k_y^2} - 2 \frac{b'^2}{a'^2} \frac{k_y^2}{k_y^2} \phi' \right\} \frac{v'}{f} \frac{f'}{f} \frac{k_x}{\mu^2} \left( J\phi^* + J^*\phi \right) \\
+ \frac{b'b'^2}{a'^2} \frac{f'}{f} \frac{k_x}{\mu^2} \left( J\phi^* + J^*\phi \right) + \frac{b'b'^2}{a'^2} \frac{f'}{f} \frac{k_x}{\mu^2} \frac{\bar{b}^2}{\bar{b}^2} \left( J\phi^* + J^*\phi \right) .
\]

(A18)

Finally, the terms containing \( J \) and \( G \) reads

\[
L^{JG} = \left\{ \frac{k_xk_y}{2k^2} f'^2 + \left( \frac{a'}{a} - \frac{b'}{b} \right) \right\} \left( J^*G + J^*G \right) \\
+ \left( a'^2b^2V + \frac{b'^2}{2a'^2} f'^2v'^2 \frac{k_y^2}{k_y^2} \right) \frac{v'}{f} \frac{f'}{f} \frac{k_x}{\mu^2} \frac{\bar{b}^2}{\bar{b}^2} \left( JG^* + J^*G \right) + \left( J^*G + J^*G \right) \left\{ \frac{f'^2}{f} k_x \frac{a'}{a} \right\} \left( J^*G^* + J^*G^* \right) \\
+ \left( a'^2b^2V + \frac{b'^2}{2a'^2} f'^2v'^2 \frac{k_y^2}{k_y^2} \right) \frac{v'}{f} \frac{f'}{f} \frac{k_x}{\mu^2} \frac{\bar{b}^2}{\bar{b}^2} \left( JG^* + J^*G \right) + \left( J^*G^* + J^*G^* \right) .
\]

(A19)
Using variables $\bar{G} = \sqrt{2b}G$, $\bar{J} = \frac{\bar{k}_v}{k}J$, we obtain

$$\sqrt{2b}L^{\bar{G}} = \left\{ \frac{k_v f v'}{2k} + \frac{k_y f v'}{2k} \left( \frac{a'}{a} - \frac{b'}{b} \right) + b b' \left( \frac{f (f - 1)}{\lambda} \right) \right\} \left( \bar{J} \bar{G}^* + J^* \bar{G} \right)$$

$$+ \left\{ \frac{k_y}{2k^2} f^2 v' + \frac{k_y}{2k^2} \frac{f^2 v'}{\lambda} \left( \frac{a'}{a} - \frac{b'}{b} \right) + b b' \left( \frac{f^2 (f - 1)}{\lambda} \right) \right\} \left( \frac{k}{f} \right) \left( \bar{J} G^* + J^* G \right)$$

$$+ \left( \frac{a^2 b^2 v}{2a^2} + \frac{b^2}{2a^2} f^2 v^2 k^2 \right) \left( \frac{f}{k} v \right) \left( \bar{J} \bar{G}^* + J^* \bar{G} \right)$$

$$+ \left(\frac{a^2 b^2 V}{2a^2} + \frac{b^2}{2a^2} f^2 v^2 k^2 \right) \left( \frac{f}{k} v \right) \left( \bar{J} \bar{G}^* + J^* \bar{G} \right).$$

(A20)

Thus, we have obtained the action for the physical variables $G$, $J$, and $\delta \phi$.

It is useful to check if $J$, $G$, and $\delta \phi$ decouple to each other in the special case $k_y = 0$ for which there exists rotational symmetry in $y - z$ plane. Indeed, we find

$$L^{G \phi} = L^{\phi J} = L^{G J} = 0.$$  

(A21)

The decoupling occurs because the helicity conserves in this special case. The other parts of action become

$$L^{G G} = \frac{1}{2} \left| \bar{G}' \right|^2 + \frac{1}{2} \left( \bar{k}_v^2 + \frac{b''}{b} \right) \left| \bar{G} \right|^2,$$

(A22)

$$L^{\phi \phi} = \frac{1}{2} \left( \delta \phi' \right)^2 + \frac{1}{2} \left( - k_v^2 + \frac{b''}{b} - a^2 V_{\phi \phi} - \frac{v^2 (3 f_\phi^2 - f f_{\phi \phi})}{a^2} - \frac{b^2 f_{\phi}^2 \phi'^2}{b^2 \phi^2} \right) \left[ a^2 V + \frac{f^2 v^2}{2a^2} \right],$$

(A23)

$$L^{J J} = \frac{1}{2} \left| J' \right|^2 + \frac{1}{2} \left( \bar{k}_v^2 + \frac{f''}{f} - \frac{f f_{\phi}^2}{a^2} \right) \left| J \right|^2.$$  

(A24)
[10] L. H. Ford, Phys. Rev. D **40**, 967 (1989).
[11] N. Kaloper, Phys. Rev. D **44**, 2380 (1991).
[12] S. Kawai and J. Soda, Phys. Rev. D **59**, 063506 (1999) [arXiv:gr-qc/9807060].
[13] J. D. Barrow and S. Hervik, Phys. Rev. D **73**, 023007 (2006) [arXiv:gr-qc/0511127].
[14] J. D. Barrow and S. Hervik, Phys. Rev. D **81**, 023513 (2010) [arXiv:0911.3805 [gr-qc]].
[15] L. Campanelli, Phys. Rev. D **80**, 063006 (2009) [arXiv:0907.3703 [astro-ph.CO]].
[16] A. Golovnev, V. Mukhanov and V. Vanchurin, JCAP **0806**, 009 (2008) [arXiv:0802.2068 [astro-ph]].
[17] S. Kanno, M. Kimura, J. Soda and S. Yokoyama, JCAP **0808**, 034 (2008).
[18] L. Ackerman, S. M. Carroll and M. B. Wise, Phys. Rev. D **75**, 083502 (2007).
[19] B. Himmetoglu, C. R. Contaldi and M. Peloso, [arXiv:0809.2779 [astro-ph]]; B. Himmetoglu, C. R. Contaldi and M. Peloso, [arXiv:0812.1231 [astro-ph]]; B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. D **80**, 123530 (2009) [arXiv:0909.3524 [astro-ph.CO]].
[20] M. a. Watanabe, S. Kanno and J. Soda, Phys. Rev. Lett. **102**, 191302 (2009) [arXiv:0902.2833 [hep-th]].
[21] S. Kanno, J. Soda and M. a. Watanabe, JCAP **0912**, 009 (2009) [arXiv:0908.3509 [astro-ph.CO]].
[22] A. R. Pullen and M. Kamionkowski, Phys. Rev. D **76**, 103529 (2007); N. E. Groeneboom and H. K. Eriksen, Astrophys. J. **690**, 1807 (2009); C. Armendariz-Picon and L. Pekowsky, [arXiv:0807.2687 [astro-ph]].
[23] D. Baumann et al. [CMBPol Study Team Collaboration], [arXiv:0811.3919 [astro-ph]].
[24] V. Gluscevic and M. Kamionkowski, [arXiv:1002.1308 [astro-ph.CO]].
[25] N. Seto, S. Kawamura and T. Nakamura, Phys. Rev. Lett. **87**, 221103 (2001) [arXiv:astro-ph/0108011].
[26] K. Tomita and M. Den, Phys. Rev. D **34**, 3570 (1986).
[27] P. K. S. Dunsby, Phys. Rev. D **48**, 3562 (1993).
[28] H. Noh and J. C. Hwang, Phys. Rev. D **52**, 1970 (1995).
[29] T. S. Pereira, C. Pitrou and J. P. Uzan, JCAP **0709**, 006 (2007) [arXiv:0707.0736 [astro-ph]]; C. Pitrou, T. S. Pereira and J. P. Uzan, JCAP **0804**, 004 (2008) [arXiv:0801.3596 [astro-ph]].
[30] A. E. Gumrukcuoglu, C. R. Contaldi and M. Peloso, JCAP **0711**, 005 (2007) [arXiv:0707.4179 [astro-ph]].
[31] B. Himmetoglu, [arXiv:0910.0243 [astro-ph.CO]].
[32] T. R. Dulaney and M. I. Gresham, [arXiv:1001.2301 [astro-ph.CO]].
[33] A. E. Gumrukcuoglu, B. Himmetoglu and M. Peloso, [arXiv:1001.4088 [astro-ph.CO]].
[34] J. Martin and J. Yokoyama, JCAP **0801**, 025 (2008).
[35] A. Lue, L. M. Wang and M. Kamionkowski, Phys. Rev. Lett. **83**, 1506 (1999) [arXiv:astro-ph/9812088]; S. Alexander and J. Martin, Phys. Rev. D **71**, 063526 (2005) [arXiv:hep-th/0411020]; M. Satoh, S. Kanno and J. Soda, Phys. Rev. D **77**, 023526 (2008); M. Satoh and J. Soda, JCAP **0809**, 019 (2008); Y. f. Cai and Y. S. Piao, Phys. Lett. B **657**, 1 (2007) [arXiv:gr-qc/0701114]; C. R. Contaldi, J. Magueijo and L. Smolin, Phys. Rev. Lett. **101**, 141101 (2008) [arXiv:0806.3082 [astro-ph]]; T. Takahashi and J. Soda, Phys. Rev. Lett. **102**, 231301 (2009) [arXiv:0904.0554 [hep-th]].
[36] P. P. Dechant, A. N. Lasenby and M. P. Hobson, Phys. Rev. D **79**, 043524 (2009) [arXiv:0809.4335 [gr-qc]].