TROPICAL FANO SCHEMES

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Abstract. We define a tropical version $F_d(trop X)$ of the Fano Scheme $F_d(X)$ of a projective variety $X \subseteq \mathbb{P}^n$ and prove that $F_d(trop X)$ is the support of a polyhedral complex contained in $trop G(d,n)$. In general $trop F_d(X) \subseteq F_d(trop X)$ but we construct linear spaces $L$ such that $trop F_1(X) \subsetneq F_1(trop X)$ and show that for a toric variety $trop F_d(X) = F_d(trop X)$.

1. Introduction

The classical Fano scheme of a projective variety $X \subseteq \mathbb{P}^n$ is the fine moduli space parametrising linear spaces contained in $X$. It is denoted by $F_d(X)$, with $d$ the dimension of the linear spaces, and is a subscheme of the Grassmannian $G(d,n)$ of $d$–dimensional subspaces of $\mathbb{P}^n$. Fano schemes have been intensively studied because of their geometric properties. Gino Fano [8] first introduced these schemes and mostly considered the case of hypersurfaces. Then in the 70s these schemes have been used to prove results on the irrationality of cubic threefolds [5,20]. Recently there has been new interests for Fano schemes not only in algebraic geometry [4,14–16] but also in machine learning [17] and geometric complexity theory [19].

In this paper we study a tropical version of the Fano scheme. We investigate the structure of this tropical object and relations with the classical $F_d(X)$.

The first way of obtaining a tropical version of $F_d(X)$ is to consider its tropicalization inside $trop G(d,n)$. The points of $trop F_d(X)$ are in correspondence with the tropicalization of the classical linear spaces contained in $X$. However it is not true in general that a tropicalized linear space that lies in $trop X$ is the tropicalization of a classical linear space in $X$. A famous example of this is in [23] where Vigeland proves that there are smooth surfaces in $\mathbb{P}^3$ of degree 3 whose tropicalization contains infinitely many lines. Since there are only 27 lines in the classical surfaces we deduce that these infinite tropical lines do not come from their tropicalization.

This leads us to define the second tropical version of $F_d(X)$ to be the set of tropicalized linear spaces of dimension $d$ contained in trop $X$. We call this the tropical Fano scheme and we denote it by $F_d(trop X)$. We take the first steps in studying the structure and the properties of this object that can also be used to investigate the classical Fano scheme.

Theorem 1. Let $X$ be a projective variety in $\mathbb{P}^n$. Then the tropical Fano scheme $F_d(trop X)$ is a polyhedral complex whose support is contained in trop $G(d,n)$. Moreover if $X$ is a fan then $F_d(trop X)$ is a fan.

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The two tropical versions of the Fano scheme come from two different constructions. The first is strictly linked to the algebraic variety and to its classical Fano scheme while the other only depends on the tropical variety \( \text{trop} X \). However we immediately observe that
\[
(1.1) \quad \text{trop} F_d(X) \subseteq F_d(\text{trop} X)
\]
and since Theorem 1 allows us to define a dimension for \( F_d(\text{trop} X) \) we obtain a bound for the dimension of \( F_d(X) \). A natural question arises:

**Question 2.** For which varieties \( X \) do we have \( \text{trop} F_d(X) = F_d(\text{trop} X) \)?

We start by looking at the simplest algebraic varieties: linear subspaces of \( \mathbb{P}^n \). We then analyse the case of toric varieties embedded in \( \mathbb{P}^n \) via monomial maps. These are two examples where the tropicalization can be easily described. For a linear space \( L \) the tropicalization is computed from the matroid associated to \( L \). On the other hand a monomial map can be tropicalized to a linear map from \( \mathbb{R}^r \) to \( \mathbb{R}^n \) and its image is the tropicalization of the toric variety associated to the monomial map (\cite[Corollary 3.2.13]{18}).

**Theorem 3.**

1. Let \( n \geq 5 \). If \( L \) is a generic 2-dimensional plane in \( \mathbb{P}^n \) then 
   \[ \text{trop} F_1(L) \subsetneq F_1(\text{trop} L). \]
2. If \( X \) is a toric variety in \( \mathbb{P}^n \) then \( F_d(\text{trop} X) = \text{trop} F_d(X) \).

The paper is structured as follows. In Section 2 we define the tropical Fano scheme and we give a rigorous statement of Theorem 1 (Theorem 2.3 and Corollary 2.4). We study the case of linear spaces in Section 3. We prove the first part of Theorem 3 in Theorem 3.1 and then use it to prove the strict containment in (1.1) for a generic hypersurface. In Section 4 we analyse the case of toric varieties and we prove the second part of Theorem 3 (Theorem 4.2). Finally in Section 5 we study the structure of \( F_d(\text{trop} X) \).

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### 2. Definitions of \( F_d(\text{trop} X) \)

In this section we set notation and define the tropical Fano Scheme \( F_d(\text{trop} X) \) of the tropicalization of a projective variety \( X \subseteq \mathbb{P}^n \).

Let \( k \) be a field with a surjective valuation \( v : k^* \rightarrow \mathbb{R} \) (cf. Remark 5.2) and let \( T^m \) be the torus \( (k^*)^{m+1}/k^* \) contained in \( \mathbb{P}^m \). The tropical projective space \( \text{trop} \mathbb{P}^m \) is \( (\overline{\mathbb{R}^{m+1}} \setminus \{(\infty, \ldots, \infty)\})/\mathbb{R}1 \) where \( \overline{\mathbb{R}} \) denotes \( \mathbb{R} \cup \{\infty\} \) and \( \mathbb{R}1 \) is the linear space spanned by the vector \((1, \ldots, 1)\). Let \( O \) be a \( T^m \)-orbit of \( \mathbb{P}^m \). This is the locus of
.points in \( \mathbb{P}^m \) where \( x_i = 0 \) for every \( i \) in the subset \( I \) of all coordinates and \( x_i \neq 0 \) for \( i \notin I \). Its tropicalization \( \mathcal{O} := \text{trop } \mathcal{O} \) is the locus of points \((x_0, \ldots, x_n)\) in \( \text{trop } \mathbb{P}^m \) where \( x_i = \infty \) if and only if \( i \in I \). We refer to \( \mathcal{O} \) as an orbit of \( \text{trop } \mathbb{P}^m \).

For any projective variety \( Y \subseteq \mathbb{P}^m \) the tropicalization \( \text{trop } Y \) is given by the union of \( \text{trop } Y \cap \mathcal{O} := \text{trop } (Y \cap \mathcal{O}) \) where \( \mathcal{O} \) is the unique orbit of \( \mathbb{P}^m \) such that \( \text{trop } \mathcal{O} = \mathcal{O} \) (see Section 6 in [18]). If \( Y \) is irreducible and \( \mathcal{O} \) is such that \( \dim \overline{Y \cap \mathcal{O}} = \dim Y \) then \( Y \subseteq \mathcal{O} \) and \( \text{trop } Y = \overline{Y \cap \mathcal{O}} \) in \( \text{trop } \mathbb{P}^m \) [18, Theorem 6.2.18].

Let \( \mathcal{G}(d, n) \) be the Grassmannian parametrising \( d \)-dimensional projective subspaces in \( \mathbb{P}^n \). We consider it embedded via the Plücker map into \( \mathbb{P}^{\binom{n+1}{d+1}} \). Its tropicalization \( \text{trop } \mathcal{G}(d, n) \subseteq \text{trop } \mathbb{P}(\binom{n+1}{d+1}) \) parametrises tropicalized linear spaces of dimension \( d \) in \( \text{trop } \mathbb{P}^n \) ([22, Theorem 3.8], [18, Theorem 4.3.17 and Remark 4.4.2], [6]). Hence it is possible to associate to each point \( p \) of trop \( \mathcal{G}(d, n) \) a unique tropicalized linear space which we denote by \( \Gamma_p \).

**Notation 2.1.** Given two tropical varieties \( \text{trop } X, \text{trop } Y \) we write \( \text{trop } X \subseteq \text{trop } Y \) for the containment of the support of \( \text{trop } X \) in the support of \( \text{trop } Y \).

**Definition 2.2.** The **tropical Fano scheme** is the set \( F_d(\text{trop } X) \subseteq \text{trop } \mathcal{G}(d, n) \) defined by

\[
F_d(\text{trop } X) := \{ p \in \text{trop } \mathcal{G}(d, n) : \Gamma_p \subseteq \text{trop } X \}.
\]

In Section [5] we prove the following results:

**Theorem 2.3.** Let \( X \) be a projective variety in \( \mathbb{P}^n \) and \( \mathcal{O} \) be an orbit of \( \text{trop } \mathbb{P}(\binom{n+1}{d+1}) \). Then \( \text{F}_d(\text{trop } X) \cap \mathcal{O} \) is a polyhedral complex whose support is contained in the intersection \( \text{trop } \mathcal{G}(d, n) \cap \mathcal{O} \).

**Corollary 2.4.** Consider a non empty intersection \( \text{F}_d(\text{trop } X) \cap \mathcal{O} \) and let \( \mathcal{O}' \) be the unique orbit of \( \text{trop } \mathbb{P}^n \) such that \( \Gamma_p \cap \mathcal{O}' = \Gamma_p \) for all \( p \in \text{F}_d(\text{trop } X) \cap \mathcal{O} \). Then \( \text{F}_d(\text{trop } X) \cap \mathcal{O} \) is a fan if \( \text{trop } X \cap \mathcal{O}' \) is a fan.

**Remark 2.5.** Note that \( \text{trop } \mathcal{F}_d(X) \) does not have the same property described in Proposition 2.4. There are varieties \( X \subseteq \mathbb{P}^n \) such that \( \text{trop } (X \cap T^n) \) is a fan but \( \text{trop } \mathcal{F}_d(X) \cap \text{trop } T^{(n+1)} \) is not. In the next section we give an explicit example of this (Example 3.5).

### 3. Linear spaces and generic hypersurfaces

In this section we show that there exist linear spaces and hypersurfaces for which the containment \( \text{trop } F_1(X) \subseteq \text{F}_1(\text{trop } X) \) is strict. In Theorem 3.1 we prove that if \( n \geq 5 \) and \( L \) is a generic plane in \( \mathbb{P}^n \) then there exists a tropical line in \( \text{trop } L \) that is not realizable in \( L \). We then compute an explicit example of a plane \( L \subseteq \mathbb{P}^5 \) with this property and we show that \( \dim \text{trop } F_1(L) < \dim \text{F}_1(\text{trop } L) \). Finally in Proposition 3.6 we prove that the containment is strict for a **general** hypersurface \( X \) whose tropicalization has the same support as a tropical hyperplane.

**Theorem 3.1.** Let \( n \geq 5 \). There exists a semi-algebraic set in \( \mathcal{G}(2, n) \) whose points are planes \( L \subseteq \mathbb{P}^n \) such that \( \text{trop } F_1(L) \not\subseteq \text{F}_1(\text{trop } L) \).
A semialgebraic subset of an algebraic variety $X$ is a subset of $X$ that can locally be defined by finitely many Boolean operators and inequalities of the form $v(f) \leq v(g)$ where $f, g$ are algebraic functions on $X$. For example every set in $X$ that is Zariski open is also a semialgebraic set.

**Proof of Theorem 3.3** Let $L$ be the standard tropical plane in trop $\mathbb{P}^n$. This is the closure in trop $\mathbb{P}^n$ of the tropicalization of the uniform matroid of rank 3 in $\{0, 1, \ldots, n\}$, which is the fan in trop $T^n \cong \mathbb{R}^{n+1}/\mathbb{R} 1$ given by the 2-dimensional cones pos$(e_i, e_j)$ for $0 \leq i < j \leq n$ where $e_0, \ldots, e_n$ is the standard basis of $\mathbb{R}^{n+1}$. Let $\Gamma^o \subseteq \text{trop}(T^n)$ be the 1-dimensional fan whose rays are pos$(e_i + e_j)$ where $0 \leq i \neq j \leq n$. The closure of $\Gamma^o$ in trop $\mathbb{P}^n$ is a tropical line $\Gamma$ and since $\Gamma^o \subseteq L \cap \text{trop} T^n$ then $\Gamma$ is contained in $L$.

Given $p \in G(2, n)$ we denote by $L_p$ the associated plane in $\mathbb{P}^n$. We show that we can find an open semi-algebraic set $U$ in $G(2, n)$ such that for every $p \in U$ we have trop $L_p = L$ and there does not exist $\ell \subseteq L_p$ such that trop $\ell = \Gamma$.

Firstly we have that trop $L_p = L$ if and only if $p \in U_1$ where

$$U_1 = \{q \in G(2, n) : v(g) = (0, \ldots, 0)\}.$$

The plane $L_p$ induces a line arrangement $A = \{\ell_0, \ldots, \ell_n\} \subseteq \mathbb{P}^n$ given by the lines $\ell_i = L_p \cap \{x_i = 0\}$, with $x_0, \ldots, x_n$ coordinates of $\mathbb{P}^n$. Let $i, j$ be two distinct indices then we denote by $w_{i,j}$ the point of intersection of $\ell_i$ and $\ell_j$. There exists a Zariski open set $U_2$ of $G(2, n)$ such that for every $p \in \mathcal{V}$ the line arrangement induced by $L_p$ satisfies the following conditions

- (I) $\ell_i \cap \ell_j \cap \ell_k = \emptyset$ for any three distinct indices $i, j, k$;
- (II) $w_{i_0,i_1}, w_{i_2,i_3}, w_{i_4,i_5}$ are not collinear unless $\{i_0, i_1\} \cap \{i_2, i_3\} \cap \{i_4, i_5\} = \emptyset$.

Let $U$ be the set $U_1 \cap U_2$. We prove that if $p \in U$ then $\Gamma$ is not realisable in $L_p$.

Suppose there exists a line $\ell \subseteq L_p$ such that trop $\ell = \Gamma$. Let $O_{ij}$ be the orbit of $\mathbb{P}^n$ where $x_i = x_j = 0$, then by Theorem 6.3.4 in [13] we have that $\ell \cap O_{k,k+1} \neq \emptyset$ for $k = 0, \ldots, n - 1$ if $n$ is odd and for $k = 0, \ldots, n - 2$ if $n$ is even. In fact we have that trop $\ell \cap \text{pos}(e_k, e_{k+1}) = \Gamma \cap \text{pos}(e_k, e_{k+1}) = \text{pos}(e_k + e_{k+1})$. Moreover $\ell \cap O_{k,k+1} \subseteq L_p \cap O_{k,k+1} = e_k \cap \ell_k + 1 = w_{k,k+1}$ hence $\ell \cap O_{k,k+1} = w_{k,k+1}$. This implies that $w_{0,1}, \ldots, w_{n-1,n}$ (resp. $w_{0,1}, \ldots, w_{n-2,n-1}$) are collinear and if $n \geq 5$ this is a contradiction since $L_p$ satisfies condition (II).

**Remark 3.2.** Note that condition (I) is satisfied by all linear spaces $L_p$ with $p \in U$. In fact $\ell_i \cap \ell_j \cap \ell_k = \emptyset$ if and only if trop $\ell_i \cap \text{trop} \ell_j \cap \text{trop} \ell_k = \emptyset$. Since trop $L_p = L$ we have that trop $\ell_i = \text{trop} L_p \cap \{x_i = \infty\} = L \cap \{x_i = \infty\}$ and by definition of $L$ the intersection trop $\ell_i \cap \text{trop} \ell_j \cap \text{trop} \ell_k$ is empty for every triple of distinct indices $i, j, k$.

In the following examples we will always assume $\mathbb{K}$ to be the field of generalised Puiseux series $\mathbb{C}((\mathbb{R}))$ with the natural valuation associated to it (see [13] Example 2.17)). The explicit computations for the tropical varieties and prevarieties are done with Tropical.m2 [11], while we use Polymake [10] and the Polyhedral package in Macaulay2 [12] to get the tree associated the tropical lines in a cone of $F_1(\text{trop} L)$. 


Example 3.3. Let $L$ be the plane spanned by the rows of the following matrix
\[
\begin{pmatrix}
0 & -271 & -92 & 0 & -13 & -54 \\
0 & -18 & -7 & -1 & 0 & -4 \\
-1 & 12293 & 4173 & 0 & 588 & 2450
\end{pmatrix}.
\]

The line arrangement $\mathcal{A} = \{ \ell_i = L \cap \{ x_i = 0 \} : i = 0, \ldots, 5 \}$ satisfies conditions (I) and (II) in the proof of Theorem 3.1. The coordinates of the point $p \in \mathbb{G}(2, 5)$ associated to $L$ are non zero complex numbers hence $\mathbf{p}(p) = (0, \ldots, 0)$. This implies that trop$L = \mathcal{L}$ hence $p \in \mathcal{U}$. The Fano scheme $F_1(L)$ is defined by the ideal
\[
\{(49p_{25} - 37p_{35} - 29p_{45}, 49p_{15} + 40p_{35} - 64p_{45}, 49p_{05} - 26p_{35} - 27p_{45},
98p_{24} - 74p_{34} + 153p_{45}, 98p_{14} + 80p_{34} + 461p_{45},
98p_{04} - 52p_{34} - 13p_{45}, 98p_{23} + 58p_{34} + 153p_{45},
98p_{13} + 128p_{34} + 461p_{35}, 98p_{03} + 54p_{34} - 13p_{35},
98p_{12} + 144p_{34} + 473p_{35} + 73p_{45}, 98p_{02} + 10p_{34} - 91p_{35} - 92p_{45},
98p_{01} - 112p_{34} - 234p_{35} - 271p_{45})
\}
\]

The tropicalization $\text{trop}F_1(L)$ is a 2-dimensional fan in trop$\mathbb{P}^5$.

The tropical Fano scheme $F_1(\text{trop} L)$ is the tropical prevareity defined by the tropical incidence relations associated to trop$L$ ([13, Theorem 1]). These are given by the Plücker relations generating $\mathbb{G}(1, 5)$ and by all tropical polynomials of the form
\[
\bigoplus_{i \in T \setminus S} p_{S \cup i} p_T \setminus i
\]
where $S \subseteq \{0, 1, 2, 3\} = T$, $|S| = 1$ and $p_T \setminus i$ are the valuations of coordinates of $p$. In this case $p_T \setminus i = 0$ for all $0 \leq i \leq 3$.

Computations show that while trop$F_1(L) \cap \text{trop}T^9$ is a 2-dimensional fan, the tropical Fano scheme $F_1(\text{trop} L) \cap \text{trop}T^9$ is a fan with 15 maximal cones of dimension 3 and 30 maximal cones of dimension 2. The rays of $F_1(\text{trop} L) \cap \text{trop}T^9$ are the same as the rays of trop$F_1(L) \cap \text{trop}T^9$ and the dimension 2 maximal cones are also cones of trop$F_1(L) \cap \text{trop}T^9$. The dimension 3 cones of $F_1(\text{trop} L) \cap \text{trop}T^9$ are the ones parametrizing tropical lines whose combinatorial type (see Section 5 for a definition) is a snow-flake tree. This is the graph in Figure 1 whose leaves are labelled by numbers from 0 to 5. The 2-dimensional faces of these cones are contained in trop$F_1(L)$. The relative interior is parametrising all tropical lines not realisable in $L$. In Figure 2 we have an example of one of these tropical lines.

In the next example we show that it is possible to realise the line $\Gamma$ in the proof of Theorem 3.1 by choosing a particular $L'$ with trop$L' = \mathcal{L}$.

Example 3.4. Let $L' \subseteq \mathbb{P}^5$ be the plane spanned by the rows of the following matrix:
\[
\begin{pmatrix}
1 & 3 & 0 & 1 & 5 & 7 \\
0 & 0 & 1 & 3 & -1 & -1 \\
1 & 4 & -1 & -3 & 0 & 0
\end{pmatrix}
\]

The line arrangement $\mathcal{A}'$ associated to $L'$ satisfies condition (I) of the proof of Theorem 3.1 and we have trop$L' = \mathcal{L} = \text{trop} L$. However $\mathcal{A}'$ does not satisfy condition
Let \( p'_{i,j} \) be the point \( L' \cap O_{i,j} \). The points \( p'_{0,1}, p'_{2,3}, p'_{4,5} \) are collinear and the line \( \ell \) passing through them is defined by the following equations

\[
x_4 - x_5 = 0, 3x_2 - x_3 = 0, 3x_1 + 4x_3 + 12x_5 = 0, 3x_0 + x_3 + 3x_5 = 0.
\]

The tropical line \( \text{trop} \ell \) is the closure in \( \text{trop} \mathbb{P}^5 \) of the fan in \( \text{trop} T^5 \) whose rays are \( \text{pos}(e_0 + e_1), \text{pos}(e_2 + e_3), \text{pos}(e_4 + e_5). \) Hence this is the tropical line \( \Gamma \) of the proof of Theorem 3.1. We now compare \( \text{trop} F_1(L') \) with \( \text{trop} F_1(L) \). The ideal associated to the Fano scheme \( F_1(L') \) is

\[
\begin{align*}
(6p_{25} - 2p_{45} - p_{45}, 6p_{15} + 8p_{35} + 97p_{45}, 6p_{05} + 2p_{35} + 25p_{45} \\
6p_{24} - 2p_{34} - p_{45}, 6p_{14} + 8p_{34} + 73p_{45}, 6p_{04} + 2p_{34} + 19p_{45} \\
6p_{23} + p_{34} - p_{35}, 6p_{13} - 97p_{34} + 73p_{35}, \\
6p_{03} - 25p_{34} + 19p_{35}, 6p_{12} - 31p_{34} + 23p_{35} - 4p_{45}, \\
6p_{02} - 8p_{34} + 6p_{35} - p_{45}, 6p_{01} + p_{34} - p_{35} - 3p_{45}
\end{align*}
\]

and \( \text{trop} F_1(L') \) is a 2-dimensional fan in \( \text{trop} \mathbb{P}^5 \). Let \( L \) be the plane of Example 3.3. Since \( L = \text{trop} L' \) then \( F_1(\text{trop} L) = F_1(\text{trop} L') \) and both \( \text{trop} F_1(L') \) and \( \text{trop} F_1(L) \) are contained in \( F_1(\text{trop} L) \). All rays of \( \text{trop} F_1(L) \) are also rays of \( \text{trop} F_1(L') \) but \( \text{trop} F_1(L') \) has also an extra ray \( r \) that is not contained in \( \text{trop} F_1(L) \). The combinatorial type of the tropical lines associated to points in \( r \) is the snowflake in Figure 1. Moreover \( r \) is the barycentre of the 3-dimensional cone \( C \) of \( \text{trop} F_1(\text{trop} L) \) containing \( r \) in its relative interior. If \( C = \text{pos}(r_1, r_2, r_3) \) then \( r = \text{pos}(r_1 + r_2 + r_3) \). We have that \( C \cap \text{trop} F_1(L) \) is given by the two dimensional faces of \( C \). On the other hand \( C \cap \text{trop} F_1(L') \) is the union of the three cones \( \text{pos}(r_1, r_1 + r_2 + r_3), \text{pos}(r_2, r_1 + r_2 + r_3), \text{pos}(r_3, r_1 + r_2 + r_3) \) (see Figure 3).

In Example 3.3 we exhibit a plane \( L'' \) such that \( \text{trop}(L'' \cap T^n) \) is a fan but \( \text{trop}(F_1(L'') \cap T^{(n+1)-1}) \) is not. This shows that Proposition 2.4 does not hold if we replace \( F_1(\text{trop} X) \) with \( \text{trop} F_1(X) \).
Example 3.5. Let $L''$ be the plane in $\mathbb{P}^5$ spanned by the rows of the following matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & t & 1 & 1 \\ 1 & t+1 & 1 & 2 & t & 0 \\ 5 & 8 & 6 & 9 & 7 & 10 \end{pmatrix}.$$ 

We have that trop $L'' = \text{trop} L$ with $L$ the plane in Example 3.3 and the line arrangement $\mathcal{A}'' = \{L'' \cap O_{i,j} : 0 \leq i < j \leq 5\}$ satisfies condition (I) of proof of Theorem 3.1. Moreover the points $p''_{01} = L'' \cap O_{0,1}, p''_{23} = L'' \cap O_{2,3}$ and $p''_{45} = L'' \cap O_{4,5}$ are not collinear.

The line spanned by the first two rows of $M$ tropicalizes to a tropical line whose combinatorial type is a snowflake tree whose pairs of leaves are labelled by $i$ and $i + 1$ for $i = 0, ..., 4$. The corresponding point in trop $F_1(L'')$ is $e_{01} + e_{23} + e_{45}$ in...
\[ \mathcal{O} = \text{trop}(\mathbb{G}(1, 5) \cap T^{(5)}_e) \subseteq \mathbb{R}^{(5)} / \mathbb{R}1, \]  
where the \( e_{ij} \)'s denote the standard basis vectors of \( \mathbb{R}^{(5)} \).

We want to show that \( \text{trop} F_1(L'') \) is not a fan by proving that the ray \( \text{pos}(e_{01} + e_{23} + e_{45}) \) is not contained in \( \text{trop}(F_1(L'') \cap T^{(5)}_e) \).

By contradiction suppose \( \text{pos}(e_{01} + e_{23} + e_{45}) \subseteq \text{trop}(F_1(L'') \cap T^{(5)}_e) \) then its closure in \( \text{trop} \mathbb{P}^{(5)} \) is a point \( Q \) and it is contained in \( \text{trop} F_1(L'') \).

The point \( Q \) is in the orbit \( \mathcal{O} = \{[p_{ij}] \in \text{trop} \mathbb{P}^{(5)} - 1 : p_{01} = p_{23} = p_{45} = \infty \} \) and \( Q_{ij} = 0 \) for \( ij \neq 01, 23, 45 \). The tropical line \( \Gamma_Q \) is given by the fan in \( \text{trop} \mathbb{P}^5 \) with rays \( \text{pos}(e_0 + e_1), \text{pos}(e_2 + e_3) \) and \( \text{pos}(e_4 + e_5) \). Moreover \( \Gamma_Q \) is not realizable in \( L'' \) otherwise the points \( p''_{01}, p''_{23} \) and \( p''_{45} \) would be collinear.

Another instance where the containment \( \text{trop} F_1(X) \subseteq F_1(\text{trop} X) \) is strict is the case of general hypersurfaces whose tropicalization has the same support of a tropical linear space. An hypersurface is general if its Fano scheme of lines has dimension \( 2n - d - 3 \) (see [2, Theorem 8]).

**Proposition 3.6.** If \( X \) is a general hypersurface of degree \( d > 1 \) and the tropicalization \( \text{trop} X \) has the same support as a tropical linear space then \( \text{trop}(F_1(X)) \subsetneq F_1(\text{trop} X) \).

**Proof.** If \( L \) is a \( (n - 1) \)-dimensional linear space then the dimension of \( F_1(L) \) is \( \dim \mathbb{G}(2, n) = 2n - 4 \). By hypothesis we have that \( F_1(\text{trop} X) = F_1(\text{trop} L) \) and \( \dim F_1(\text{trop} L) \geq \dim \text{trop} F_1(L) = 2n - 4 \). On the other hand the dimension of \( \text{trop} F_1(X) \) is equal to the dimension of \( F_1(X) \) which is \( 2n - d - 3 \). Suppose \( \text{trop} F_1(X) = F_1(\text{trop} X) \) then we would have \( 2n - d - 3 \geq 2n - 4 \) but this is not the case if \( d > 1 \).

\[ \square \]

4. **Toric varieties**

In this section we look at Fano schemes of toric varieties. We prove that for these varieties the tropical Fano scheme is equal to the tropicalization of the classical Fano scheme.

Consider a toric variety \( X \) associated to a set of lattice points \( \mathcal{A} = \{a_0, \ldots, a_n\} \) with \( \mathcal{A} \subseteq \mathbb{Z}^m \times \{1\} \) and denote by \( A \) the matrix whose columns are the points in \( \mathcal{A} \). The variety \( X \) has a natural embedding in \( \mathbb{P}^n \) given by a monomial map \( \phi_A : (k^*)^m \times k^* \to \mathbb{P}^n \) (see [7, Section 2.1]). We denote the closure of the image of this map by \( X_A \). The matrix \( A \) also defines a map \( \text{trop}(\phi_A) : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1} \). By [8, Theorem 3.2.13] we have that \( \text{trop}(X_A \cap T^n) \subseteq \mathbb{R}^{n+1} / \mathbb{R}1 \) is the quotient by \( \mathbb{R}1 \) of the image of \( \text{trop}(\phi_A) \) which is the classical linear space spanned by the rows of \( A \). Since the embedding of the toric variety only depends on the row span of \( A \) ([7, Proposition 1.1.9]) it is possible to recover the ideal defining \( X_A \) from \( \text{trop}(X_A \cap T^n) \).
Example 4.1. Let $X_A \subseteq \mathbb{P}^3$ be the toric variety associated to the set of lattice points $\mathcal{A} = \{(1,1,1),(0,0,1),(0,-1,1),(1,0,1)\}$. The matrix $A$ is

$$
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
$$

and the ideal defining $X_A$ is $(xz - yw)$. The tropicalization $\text{trop}(X_A \cap T^3)$ is the quotient by $\mathbb{R}1$ of $\{(x,y,z,w) : x + z = y + w\}$ and this is equal to the quotient by $\mathbb{R}1$ of the linear span of the rows of $A$.

By contrast with the case of linear spaces we show that for toric varieties the tropical Fano scheme is the same as the tropicalization of the classical Fano scheme.

Theorem 4.2. Let $X = X_A$ be a toric variety. Then $F_d(\text{trop} X) = \text{trop} F_d(X)$.

We prove this result by showing that for each tropicalized linear space $\Gamma \subseteq \text{trop} X$ there exists a linear space $\ell \subseteq X$ that tropicalizes to it. We explicitly construct $\ell$ using Cayley structures on $\mathcal{A}$. We use results in [10, Section 3] where the authors prove that for each $s$–Cayley structure $\pi$ there exists a subvariety $Z_\pi$ of $F_s(X_A)$ and from $\pi$ it is also possible to deduce equations of the linear spaces parametrised by $Z_\pi$.

Given a set of $n + 1$ lattice points $\mathcal{A}$ in $\mathbb{Z}^n \times \{1\}$, let $L$ be the kernel of the map defined by the matrix $A$ and $e_i$ be the standard basis vectors of $\mathbb{R}^{s+1}$. If $l \in L$ we can write $l = \sum_{l_i > 0} l_i e_i - \sum_{l_i < 0} -l_i e_i$ and denote by $l^+ = \sum_{l_i > 0} l_i e_i$ and $l^- = \sum_{l_i < 0} -l_i e_i$.

We have that $l \in L$ if and only if $\sum l_i a_i = 0$. The toric variety $X_A \subseteq \mathbb{P}^n$ is generated by binomials of the form $x^l_1 - x^l_0 = \prod_{i>0} x_i^{l_i} - \prod_{i<0} x_i^{l_i}$ with $l \in L$ (13 Proposition 1.1.9).

A face $\tau$ of $\mathcal{A}$ is the intersection of a face of $\text{conv}(\mathcal{A})$ with $\mathcal{A}$. Denote by $\Delta_s$ the standard basis $\{e_0, \ldots, e_s\}$ of $\mathbb{Z}^{s+1}$.

Definition 4.3. An $s$–Cayley structure on $\pi$ is a surjective map $\pi : \tau \rightarrow \Delta_s$ such that if $l \in L$, $l_i \neq 0$ for all $i$ with $a_i \in \tau$ and $\sum_{l_i \neq 0} l_i a_i = 0$ then $\sum_{l_i \neq 0} l_i \pi(a_i) = 0$, or equivalently $\sum_{l_i > 0} l_i \pi(a_i) = \sum_{l_i < 0} -l_i \pi(a_i)$.

Example 4.4. Consider the set of lattice points $\mathcal{A}$ as in Example 4.1. A 1–Cayley structure is given by $\pi : \mathcal{A} \rightarrow \mathbb{Z}^2$ with $\pi((0,0,1)) = \pi((0,-1,1)) = e_0$ and $\pi((1,0,1)) = \pi((1,1,1)) = e_1$. An example of a surjective map $\pi : \mathcal{A} \rightarrow \Delta_1$ that is not a Cayley structure is given by $\pi : \mathcal{A} \rightarrow \mathbb{Z}$ with $\pi((1,1,1)) = \pi((0,-1,1)) = e_0$ and $\pi((0,0,1)) = \pi((1,0,1)) = e_1$. We can see that $l = (1,-1,1,-1)$ is in $L$ hence $(1,1,1) - (0,0,1) + (0,-1,1) - (1,0,1) = 0$ but if we apply $\pi$ we get $2e_1 - 2e_2 = 0$ which is a contradiction.

We now prove that given a tropicalized linear space in $\text{trop}(X \cap T^n)$ we can associate a Cayley structure on $\mathcal{A}$ to it.

Let $\Gamma$ be a $d$–dimensional tropicalized linear space in $\text{trop} T^n$ and let $M_{\Gamma}$ be the matroid associated to it. This is the matroid on $\{0,1,\ldots,n\}$ whose bases are the set $\{i_0, \ldots, i_d\}$ such that the corresponding Plücker coordinates $p_{i_0,\ldots,i_d}$ is not zero. Note that this matrix does not have loops, circuits of one element.
The recession fan of $\Gamma$ is the fan whose cones are $\text{pos}(e_{F_1}, \ldots, e_{F_{d+1}}) + \mathbb{R}1$ where $\emptyset \neq F_1 \subsetneq \ldots \subsetneq F_{d+1}$ is a maximal chain of flats of $M_T$, $e_{F_i} = \sum_{j \in F_i} e_i$ and $(e_i)_k = 1$ for $k = i$ and $(e_i)_k = 0$ otherwise.

**Proposition 4.5.** Let $X_A \subseteq \mathbb{P}^n$ be a toric variety and let $\Gamma$ be a tropicalized linear space contained in $\text{trop}(X_A \cap T^n)$. If $M_T$ has $m + 1$ non-empty minimal flats then there exists an $m$–Cayley structure on $A$.

The following is a technical lemma which will be used for the proof of Proposition 4.5.

**Lemma 4.6.** Let $\Gamma \subseteq \text{trop}T^n$ be a tropicalized linear space and $\{F_1^0, \ldots, F_m^0\}$ the set of non-empty minimal flats of $M_T$. Then

(i) there exists a unique $F_i^j$ such that $i \in F_i^j$;
(ii) $\bigcup_{j=1}^m F_i^j = \{0, \ldots, n\}$.

*Proof.* For (i) we observe that if $i \in F_i^j \cap F_i^k$ then, since there are no loops, $\{i\}$ would also be a flat but this would contradict the minimality of $F_i^j$ and $F_i^k$.

If there exists $i \in \{0, \ldots, n\}$ that is not in $\bigcup_{j=1}^m F_i^j$ then $\{i\}$ can not be a flat. This implies that it is a loop but this is a contradiction since $M_T$ has no loops. □

*Proof of Proposition 4.5.* Let $\Gamma$ be a tropicalized linear space contained in $\text{trop}(X \cap T^n)$ and $F_1^0, \ldots, F_m^0$ the non-empty minimal flats of $M_T$. The ray $\text{pos}(e_{F_i})$ of $\Gamma$ is contained in $\text{trop}(X \cap T^n)$ for all $i$ hence the vectors $e_{F_1^0}, \ldots, e_{F_m^0}$ are part of a set of generators for the linear space trop$(X \cap T^n)$. Lemma 4.6 implies that they are linearly independent vectors in $\mathbb{R}^{n+1}$. The linear span in $\mathbb{R}^{n+1}/\mathbb{R}1$ of $e_{F_1^0}, \ldots, e_{F_m^0}$ is equal to the linear span of $e_{F_1}, \ldots, e_{F_m}$ and $(1, \ldots, 1)$. Hence we can assume that $e_{F_1}, \ldots, e_{F_m}$ are the first $m$ rows of $A$ and $e_{F_i^0}$ is the unique among $e_{F_1}, \ldots, e_{F_m}$ with last coordinate equal to $1$. The columns of $A$ are the points of $A$ and by Lemma 4.6 they can be partitioned in $m + 1$ sets $A_0, \ldots, A_m$. The set $A_i$, for $i = 0, \ldots, m - 1$, is given all points whose coordinates $(p_0, \ldots, p_n)$ are such that $p_i = 1$ and $p_j = 0$ for all $0 \leq j \neq i \leq m$. The set $A_m$ is given by the points whose first $m$ coordinates are zero. We have that $A_0 \cup \ldots \cup A_m = A$. In fact by Lemma 4.6 for any $i$ there exists a unique $e_{F_i^j}$ such that $(e_{F_i^j})_i = 1$. This implies for each point $(p_0, \ldots, p_n)$ in $A$ (equivalently each column of $A$) there exists a unique $0 \leq i \leq m$ such that $p_i = 1$. Since each $e_{F_i^j}$ has at least one coordinate equal to $1$ we have that $A_0 \cup \ldots \cup A_{m-1} \subseteq A$. Moreover since the first $m$ rows of $A$ are $e_{F_1}, \ldots, e_{F_m}$ we have that the last column of $A$ has first $m$ entries equal to zero. Hence $A_m \neq \emptyset$ and $A = \cup_{i=0}^m A_i$. We define $\pi : A \to \Delta_m$ to be the map that sends the points in $A_i$ to $e_{r+1} \in \mathbb{Z}^{m+1}$. This map is an $m$–Cayley structure on $A$. In fact let $1 \in L$ with $1 = l^+ - l^- = \sum_{l_i > 0} l_i e_i - \sum_{l_i < 0} l_i e_i$ and $\{i : l_i \neq 0\} = \{i : a_i \in A\}$ then we have

$$\sum_{l_i > 0, a_i \in A_0} l_i a_i + \ldots + \sum_{l_i > 0, a_i \in A_m} l_i a_i = \sum_{l_i < 0, a_i \in A_0} -l_i a_i + \ldots + \sum_{l_i < 0, a_i \in A_m} -l_i a_i.$$
We need to prove that 
\[ \sum_{l_i > 0, a_i \in A_0} l_i \pi(a_i) + \ldots + \sum_{l_i > 0, a_i \in A_m} l_i \pi(a_i) = \sum_{l_i > 0, a_i \in A_0} -l_i \pi(a_i) + \ldots + \sum_{l_i < 0, a_i \in A_m} -l_i \pi(a_i). \]

By definition of \( \pi \) we have that 
\[ \sum_{l_i > 0, a_i \in A_0} l_i \pi(a_i) + \ldots + \sum_{l_i > 0, a_i \in A_m} l_i \pi(a_i) = ( \sum_{l_i > 0, a_i \in A_0} l_i, \ldots, \sum_{l_i > 0, a_i \in A_m} l_i ) \]
and 
\[ \sum_{l_i < 0, a_i \in A_0} -l_i \pi(a_i) + \ldots + \sum_{l_i < 0, a_i \in A_m} -l_i \pi(a_i) = ( \sum_{l_i < 0, a_i \in A_0} -l_i, \ldots, \sum_{l_i < 0, a_i \in A_m} -l_i ). \]

Consider \((p_0, \ldots, p_n) = \sum_{l_i > 0, a_i \in A_0} l_i a_i + \ldots + \sum_{l_i > 0, a_i \in A_m} l_i a_i = \sum_{l_i > 0, a_i \in A_0} -l_i a_i + \ldots + \sum_{l_i < 0, a_i \in A_m} -l_i a_i. \) The first coordinate \( p_0 \) is given by the first coordinate of \( \sum_{l_i > 0, a_i \in A_0} l_i a_i \) that is \( \sum_{l_i > 0, a_i \in A_0} l_i \) or equivalently by the first coordinate of \( \sum_{l_i < 0, a_i \in A_0} -l_i a_i \) that is \( \sum_{l_i < 0, a_i \in A_0} -l_i. \)

From this we obtain \( \sum_{l_i > 0, a_i \in A_1} l_i = \sum_{l_i < 0, a_i \in A_1} -l_i. \) In the same way we have
\[ \sum_{l_i > 0, a_i \in A_1} l_i = \sum_{l_i < 0, a_i \in A_1} -l_i, \ldots, \sum_{l_i > 0, a_i \in A_m-1} l_i = \sum_{l_i < 0, a_i \in A_m-1} -l_i. \]

Since \( p_n = \sum_{l_i > 0} l_i = \sum_{l_i < 0} -l_i \) we can also deduce that 
\[ \sum_{l_i > 0, a_i \in A_0} l_i - \sum_{l_i < 0, a_i \in A_0} l_i = \sum_{l_i > 0, a_i \in A_m} l_i - \sum_{l_i < 0, a_i \in A_m} l_i \]

therefore 
\[ ( \sum_{l_i > 0, a_i \in A_0} l_i, \ldots, \sum_{l_i > 0, a_i \in A_m} l_i ) = ( \sum_{l_i < 0, a_i \in A_0} -l_i, \ldots, \sum_{l_i < 0, a_i \in A_m} -l_i ). \]

\[ \square \]

**Example 4.7.** Let \( A \) be the set given by the columns of the matrix \( A \) where 
\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 7 & 3 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \]

The toric variety \( X_A \) is defined by the ideal \((x_2x_3 - x_1^2) \subseteq \mathbb{C}[x_0, x_1, x_2, x_3, x_4]. \) The tropical line \( \Gamma_1 \) spanned by \((0, 1, 0, 0, 0) \) is contained in \( \text{trop}(X_A \cap T^3). \) In the case of tropical lines the cones \( \text{pos}(e_F^0) + \mathbb{R}1, \ldots, \text{pos}(e_F^m) + \mathbb{R}1 \) are exactly the rays of \( \Gamma. \) We can define a \( 1-\text{Cayley} \) structure associated to \( \Gamma_1 \) by sending the set 
\[ A_0 = \{(0, 1, 2, 1), (0, 0, 7, 1), (0, 0, 3, 1), (0, 0, 5, 1)\} \]
to \( e_0 \) and \( A_1 = \{(1, 0, 1, 1)\} \) to \( e_1. \)

We also notice that the tropical line \( \Gamma_2 \) whose rays are \( \text{pos}(1, 0, 0, 0), \text{pos}(0, 1, 0, 0, 0) \) and \( \text{pos}(-1, -1, 0, 0, 0) \) is contained in \( \text{trop}(X_A \cap T^3). \) The \( 2-\text{Cayley} \) structure associated to \( \Gamma_1 \) is the map sending \( A_0 = \{(0, 1, 2, 1)\} \) to \( e_0, \ A_1 = \{(1, 0, 1, 1)\} \) to \( e_1, \)
\[ A_2 = \{(0, 0, 7, 1), (0, 0, 3, 1), (0, 0, 5, 1)\} \] to \( e_2. \)
Proof of Theorem 4.2. We will prove that given a tropicalized linear space $\Gamma \subseteq \text{trop} X$ there exists a linear space $\ell'$ in $X$ such that trop$\ell' = \Gamma$.

Assume that $\Gamma$ is in trop$(X \cap O)$ with $O$ an orbit of $\mathbb{P}^n$. We can consider $Y = X \cap O$ as a subvariety of $\overline{O} \cong \mathbb{P}^s$ with $s = \dim \overline{O}$. The variety $Y$ is also a toric variety and we denote by $\mathcal{A}$ the set of lattice points associated to it.

Suppose $M_\Gamma$ has $l + 1$ minimal flats. By Lemma 4.5 we have that there exists a $l$–Cayley structure $\pi$ on $\mathcal{A}$. Let $Z_\pi$ be the subvariety of $F_1(Y)$ associated to $\pi$ (see [16, Section 3, Section 4]). This is the closed torus orbit of the linear space $L$ generated by $v_0, \ldots, v_l \in \mathbb{R}^{s+1}$ where

$$ (v_j)_i = \begin{cases} 1 & \text{if } \pi(a_j) = e_1 \\ 0 & \text{else} \end{cases}. $$

Let $\Gamma'$ be the translation of $\Gamma$ to the origin. There exists a point $p$ in trop$Y$ such that $\Gamma = \Gamma' + p$. The vectors $e_{F_1^p}, \ldots, e_{F_1^{p_m}}$ generate a linear space $L$ and $\Gamma \subseteq L + p$.

We have that $\mathcal{L} = L$. In fact by definition of the $(v_j)_i$ and by construction of $\pi$ in Lemma 4.5 the matrix

$$ \begin{pmatrix} v_1 \\ \vdots \\ v_l \end{pmatrix} $$

is equal to the submatrix of $A$ given by the first $l$ rows. The equations of $L$ are codim$L$ binomials of type $x_i - x_j$ for pairs $(i, j)$ with $0 \leq i \neq j \leq m$, hence trop$L = L$. Moreover there exists $t \in T^{\dim Y}$ such that trop$(t \cdot L) = L + p$.

We show that $\Gamma$ is the tropicalization of a linear space in $t \cdot L$ hence in $Y$. Using the equations of $L$ we can choose $x_0, \ldots, x_l \in \{x_0, \ldots, x_s\}$ such that for any $q \in L$ we have $q_i = q_j$ for $j \in \{0, \ldots, l\}$. This implies that the projection $\phi = \phi_{x_0, \ldots, x_l} : \mathbb{P}^s \to \mathbb{P}^l$ induces an isomorphism between $L$ and $\mathbb{P}^l$. Let $\psi^{-1}$ be its inverse. Since $\phi$ and $\phi^{-1}$ are linear monomial maps then trop$(\phi) = \phi$ and trop$(\phi^{-1}) = \phi^{-1}$. Consider the linear space $\phi(\Gamma') \subseteq \text{trop} \mathbb{P}^l$. This linear space is realizable in $\mathbb{P}^l$, that is there exists $\ell' \in \mathbb{P}^l$ such that trop$\ell' = \phi(\Gamma')$. Now $\phi^{-1}(\ell') \subseteq L \subseteq Y$ and trop$(\phi^{-1}(\ell')) = \text{trop}(\phi^{-1})(\text{trop}(\ell')) = \Gamma'$. If we consider $\ell = t \cdot \phi(\ell')$ then trop$\ell) = \Gamma$.

Example 4.8. Consider the toric variety $X_\mathcal{A}$ of Example 4.7. We use the proof of Theorem 4.2 to compute the lines $\ell_1, \ell_2$ in $X_\mathcal{A}$ that tropicalize to $\Gamma_1$ and $\Gamma_2$ respectively. The line $\ell_1$ is the line $L$ associated to the 1–Cayley structure $\pi_1$. Its defining equations are $x_0 - x_2 = 0, x_2 - x_3 = 0, x_3 - x_4 = 0$. The tropical line $\Gamma_2$ is contained in the linear space $L$ defined by $x_2 - x_3 = 0, x_3 - x_4 = 0$. Consider the projection $\phi = \phi_{x_0, x_1, x_2} : \mathbb{P}^5 \to \mathbb{P}^2$ then $\phi(\Gamma_2)$ is the tropical line in $\mathbb{R}^3/\mathbb{R}1$ with rays pos$(1, 0, 0), \text{pos}(0, 1, 0), \text{pos}(0, 0, 1)$ and it is the tropicalization of the line $V(x_0 + x_1 + x_2)$. Applying $\phi$ we get that $\ell_2$ is defined by $(x_0 + x_1 + x_2, x_2x_3 - x_4^2, x_3 - x_4)$.

5. Proof of Theorem 2.3 and Proposition 2.4

In this section we prove Theorem 2.3 by showing that there exists a polyhedral structure on each $F_d(\text{trop} X) \cap O$. 

The key point in the proof of Theorem 4.2.9 is the identification of \( \text{trop} \mathcal{G}(d, n) \cap \mathcal{O} \) with the subfan of the secondary fan \( \Sigma \) of the matroid polytope \( P_M \) (see [18, Definition 4.2.9]). We see in the following paragraph that \( M \) is the uniform matroid associated to \( \text{trop} \mathcal{G}(d, n) \cap \mathcal{O} \). The cones of this subfan are the intersection of \( \text{trop} \mathcal{G}(d, n) \cap \mathcal{O} \) with the cones of \( \Sigma \) and the subdivisions associated to these cones are the matroid subdivisions (see [18, §4.4] for a definition).

The space \( \text{trop} \mathcal{G}(d, n) \cap \text{trop} T^{(n+d+1)-1} \) was first studied by Speyer and Sturmfels in [22] and can be identified with a subfan of the secondary fan of the uniform matroid of rank \( d + 1 \) on \( \{0, 1, \ldots, n\} \) [18, §4.4]. The same interpretation of \( \text{trop} \mathcal{G}(d, n) \cap \mathcal{O} \) can be extended to the case where \( \mathcal{O} \) is any orbit of \( \text{trop} \mathbb{P}^{(n+d+1)-1} \). This is done in the forthcoming paper of Cueto and Corey [6]. In particular they show that

\[
\mathcal{G}(d, n) \cap \overline{\mathcal{O}} \cong \mathcal{G}(1, n') \times \prod_{j \in J} T^j
\]

where \( n' < n \) and \( J \subseteq \mathbb{N} \) with \( |J| < \infty \). The isomorphism between them is a map \( \psi = \pi \times f \) where \( \pi \) is a projection and \( f \) is a monomial map. Hence it is possible to consider the tropicalization of this map to get

\[
\text{trop} \mathcal{G}(d, n) \cap \overline{\mathcal{O}} \cong \text{trop} \mathcal{G}(d, n') \times \prod_{j \in J} \text{trop} T^j.
\]

Let \( M' \) be the uniform matroid of rank \( d + 1 \) on \( \{0, 1, \ldots, n'\} \). We can identify \( \text{trop} \mathcal{G}(d, n) \cap \mathcal{O} \) with a product of a subfan of the secondary fan of \( P_{M'} \) with \( \prod_{j \in J} \text{trop} T^j = \mathbb{R}^{\sum_{j \in J} j} / \mathbb{R}1 \).

This identification induces a polyhedral structure on \( \text{trop} \mathcal{G}(d, n) \) given by the union of cones \( C_T \) where each \( T \) is a different matroid subdivision. Consider \( p \) in the relative interior \( C_p^\circ \), the corresponding tropical linear space \( \Gamma_p \). We say that the combinatorial type of \( \Gamma_p \) is \( T \). If \( p \) is contained in \( \in C_T \setminus C_p^\circ \) then the combinatorial type of \( \Gamma_p \) is \( T' \) where \( T' \) is the matroid subdivision associated to a cone \( C_{T'} \) in the boundary of \( C_T \) such that \( p \in C_{T'}^\circ \). Note that if \( C_T \) is in the boundary of \( C_T' \) then a cell \( \sigma \) in \( C_T \) is either equal to a cell in \( C_T' \) or it is obtained by subdividing a cell \( \sigma' \) of \( C_T' \). In the second case all the cells in the subdivision of \( \sigma' \) are cells of in \( C_T' \). For the case of \( \text{trop} \mathcal{G}(1, n) \) instead of \( T \) one considers the corresponding tree with \( n' \leq n \) labelled leaves. In fact in this case the polyhedral complex dual to the subdivision has the coarsest polyhedral structure.

In what follows we call an open polyhedron a set of the form \( P \setminus \partial P \) where \( P \) is a polyhedron and \( \partial P \) is its boundary. For example the open square with vertices \((0,0,1), (1,0,1), (1,0,0), \) and \((0,0,0)\) in \( \mathbb{R}^3 \) is an open polyhedron.

**Proof of Theorem 2.3.** We prove that \( F_d(\text{trop} X) \cap \mathcal{O} \) can be written as the union of finitely many polyhedra, denoted by \( F_T \), and hence the common refinement of these polyhedra is the polyhedral complex structure on \( F_d(\text{trop} X) \cap \mathcal{O} \).

There are two key points in the proof. The first is that the complement of a polyhedron is the union of open polyhedra and second that the projection of an open polyhedron is an open polyhedron. Secondly it is crucial to describe the polyhedral
structure of a tropical linear space from its Plücker coordinates. In the following we will start by showing this last point.

Let $T$ be a combinatorial type of tropical linear spaces associated to the relative interior of a cone $C_T \subseteq \text{trop } \mathbb{G}(d, n) \cap \mathcal{O}$. Consider $p \in C_T$ then the tropical linear space $\Gamma_p$ is a subcomplex of the dual complex to a subdivision $T'$ of $P_M$, where $M$ is the uniform matroid associated to $\text{trop } \mathbb{G}(d, n) \cap \mathcal{O}$ and $C_{T'}$ is a face of $C_T$. This implies that $\Gamma_p = \bigcup_i C_i(p)$ and each cell $C_i(p)$ in $\text{trop } \mathbb{P}^n$ has the following form

$$\{ x \in \text{trop } \mathbb{P}^n : A(i, T)x^t \leq f(p) \text{ and } B(i, T)x^t = g(p) \}$$

where $A(i, T)$ and $B(i, T)$ are matrices with entries in $\mathbb{R}$ and $f(p), g(p)$ are vectors whose entries are linear forms in the coordinates of $p$, that depend only on $T$ and not on $p$. Note that if $p \in C_T \setminus C^o_T$ then $p \in C_{T'} \subset C_T$ hence some of the $C_i(p)$ might be the same. These are dual to the cell of $T'$ that is subdivided in $T$.

We are now ready to define $F_T$. This is the set

$$F_T = \{ p \in C_T : \Gamma_p \subseteq \text{trop } X \}$$

hence

$$F_d(\text{trop } X) \cap \mathcal{O} = \bigcup_T F_T$$

where the union is over all combinatorial types $T$ associated to the relative interior of the maximal cones of $\text{trop } \mathbb{G}(d, n) \cap \mathcal{O}$.

The tropical linear space $\Gamma_p$ is contained in $\text{trop } X$ if and only if for every $i$ we have $C_i^o(p) \subseteq \text{trop } X$, that is

$$F_T = \bigcap_i \{ p \in C_T : C_i^o(p) \subseteq \text{trop } X \}$$

where $\Gamma_p = \bigcup_i C_i^o(p)$ and $C_i^o(p)$ is the relative interior of a cell $C_i(p)$ of $\Gamma_p$. Denote by $F_{C_i}$ the set $\{ p \in C_T : C_i^o(p) \subseteq \text{trop } X \}$. We show that this set is open and is the union of open polyhedra.

Consider the set

$$\tilde{F}_{C_i} := \{ (p, x) \in C_T \times \text{trop } \mathbb{P}^n : x \in C_i^o(p) \text{ and } x \notin \text{trop } X \} \subseteq (\text{trop } \mathbb{G}(d, n) \cap \mathcal{O}) \times \text{trop } \mathbb{P}^n.$$

Firstly we observe that $x \notin \text{trop } X$ if and only if $x$ is in the complement of any cell of $\text{trop } X$ that is

$$(5.1) \quad x \notin \text{trop } X \iff x \in \bigcap_{\sigma \text{ cell of trop } X} \sigma^c.$$  

The complement of a polyhedron is a union of open polyhedra hence the term on the left of (5.1) is the union of finitely many open polyhedra.

Since

$$\tilde{F}_{C_i} = \{ (p, x) \in C_T \times \text{trop } \mathbb{P}^n : x \in C_i^o(p) \} \cap \{ (p, x) \in C_T \times \text{trop } \mathbb{P}^n : x \notin \text{trop } X \}$$

we obtain that $\tilde{F}_{C_i}$ is the union of finitely many open polyhedra. Moreover this is also the case for $\pi(\tilde{F}_{C_i})$ where $\pi$ is the projection

$$\pi : \text{trop } \mathbb{G}(d, n) \cap \mathcal{O} \times \text{trop } \mathbb{P}^n \to \text{trop } \mathbb{G}(d, n) \cap \mathcal{O}.$$
We can describe \( \pi(\tilde{F}_{C_i}) \) in the following way
\[
\pi(\tilde{F}_{C_i}) = \{ p \in C_T : \exists x \in C_i(p) \text{ such that } x \not\in \text{trop } X \}.
\]
The set \( F_{C_i} \) is the complement of \( \pi(\tilde{F}_{C_i}) \) hence it is closed and it is the union of finitely many polyhedra. This proves that \( F_T \) is the union of finitely many polyhedra and hence the same holds for \( F_d(\text{trop } X) \cap \mathcal{O} \).

Remark 5.2. It is not necessary to have a surjective valuation \( \nu \). Let \( G = \nu(\mathbb{k}) \) be the value group of \( \nu \) and assume \( G \subseteq \mathbb{R} \). Then for any variety \( X \subseteq \mathbb{P}^n \) we have that each face of \( \text{trop } X \) is a \( \nu(\mathbb{k}) \)-polyhedron, so it is defined by linear equalities and inequalities with coefficients in \( \nu(\mathbb{k}) \). In particular if \( \Gamma \) is a tropical linear space then the inequalities defining the cells have coefficients in \( \nu(\mathbb{k}) \). This implies that the set \( F_T \) is not a union of polyhedra but it is the intersection of this union with \( \nu(\mathbb{k}^*)^m \). Let \( \mathcal{O} \) be an orbit of \( \text{trop } \mathbb{P}^{n+1} \) then we can define \( F_d(\text{trop } X) \cap \mathcal{O} \) to be the Euclidean closure of \( \bigcup_T F_T \).

The structure of the tropical Fano scheme is strictly connected to the structure of the tropical variety \( \text{trop } X \).

Proof of Corollary 2.4. The polyhedral structure on \( F_d(\text{trop } X) \cap \mathcal{O} \) is the common refinement of the \( F_T \). In the case in which \( \text{trop } X \cap \mathcal{O}' \) is a fan we get that \( F_T \) is the union of finitely many cones for every \( T \). This can be seen from the construction of each \( F_T \) in the proof of Theorem 2.3. Then the common refinement of these cones for every \( T \) gives a fan structure on \( F_d(\text{trop } X) \cap \mathcal{O} \).

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