Einstein submanifolds with parallel mean curvature

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Abstract. We provide a classification of Einstein submanifolds in space forms with flat normal bundle and parallel mean curvature. This extends a previous result due to Dajczer and Tojeiro (Tohoku Math J (2) 45:43–49, 1993) for isometric immersions of Riemannian manifolds with constant sectional curvature.

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1. Introduction. The study of isometric immersions of Riemannian manifolds with constant sectional curvature into space forms is a basic topic in submanifold theory that goes back to Cartan [1,2]. Several interesting results towards the classification of these immersions have been obtained ever since (see [4,5,9]). In particular, Dajczer and Tojeiro [3] provided a classification of all such isometric immersions with flat normal bundle and parallel mean curvature vector field.

A natural generalization of the concept of Riemannian manifolds with constant sectional curvature is the notion of manifolds with constant Ricci curvature, namely Einstein manifolds. Fialkow [7] and Thomas [15] initiated the study of isometric immersions of Einstein manifolds into space forms. Indeed, after the early work of Fialkow and Thomas, Ryan [14] gave a local classification of Einstein hypersurfaces in any space form. In arbitrary codimension, Di Scala [6] proved that Einstein real Kähler submanifolds of a Euclidean space are totally geodesic provided that they are minimal. The same conclusion still holds for minimal Einstein submanifolds with flat normal bundle in the Euclidean space (see [11]).
In the present paper, we classify isometric immersions of Einstein manifolds into a complete and simply connected Riemannian manifold $Q^c_N$ of constant sectional curvature $c$ of arbitrary codimension, with flat normal bundle and parallel mean curvature vector field. Our result, which extends the aforementioned result of Dajczer and Tojeiro [3], is stated as follows:

**Theorem 1.** Let $f: M^n \to Q^c_N, n \geq 3$, be an isometric immersion of a connected Einstein manifold with Ricci curvature $\lambda$, flat normal bundle, and parallel mean curvature vector field. Then one of the following holds:

(i) The immersion $f$ is totally umbilical.

(ii) $\lambda = 0 = c$ and

$$f(M^n) \subset S^1(r_1) \times \cdots \times S^1(r_k) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+k}.$$  

(iii) $\lambda = 0 < c$ and

$$f(M^n) \subset S^1(r_1) \times \cdots \times S^1(r_n) \subset S^3_{c}^{2n-1} \subset \mathbb{R}^{2n},$$

where $r_1^2 + \cdots + r_n^2 = 1/c$.

(iv) $\lambda = 0 > c$ and

$$f(M^n) \subset \mathbb{H}^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n) \subset \mathbb{H}^3_{c}^{2n-1} \subset \mathbb{L}^{2n},$$

where $-r_1^2 + r_2^2 + \cdots + r_n^2 = 1/c$.

(v) $\lambda = c(n-k) > 0$ and

$$f(M^n) \subset S^{m_1}(\rho_1) \times \cdots \times S^{m_k}(\rho_k) \subset S^n_{c}^{n+k-1} \subset \mathbb{R}^{n+k},$$

where $\rho_i = \sqrt{(m_i-1)/\lambda}$, $m_i \geq 2$ for all $1 \leq i \leq k$.

(vi) $f = j \circ g$, where $g$ is as in (ii), (iii), (iv), or (v) and $j$ is a totally umbilical inclusion.

2. Preliminaries. Let $f: M^n \to Q^c_N$ be an isometric immersion of an $n$-dimensional Riemannian manifold with Levi-Civita connection $\nabla$. The second fundamental form $\alpha$ of $f$ is a symmetric section of the vector bundle $\text{Hom}(TM \times TM, N_f M)$, where $N_f M$ is the normal bundle of $f$. We say that $f$ is totally umbilical if

$$\alpha(X,Y) = \langle X,Y \rangle H,$$

where $H$ is the mean curvature vector field.

A straightforward computation of the Ricci tensor gives using the Gauss equation for $f$ that

$$\text{Ric}(X,Y) = c(n-1)\langle X,Y \rangle + n\langle \alpha(X,Y), H \rangle - \sum_{i=1}^{n} \langle \alpha(X,X_i), \alpha(Y,X_i) \rangle, \ X,Y \in TM,$$

where $X_1, \ldots, X_n$ is a local orthonormal frame of the tangent bundle $TM$.

The immersion $f$ has flat normal bundle if the curvature tensor of the normal connection $\nabla^\perp$ of $N_f M$ vanishes. In this case, it is a standard fact (see [12]) that at any point $x \in M^n$ there exists a set of unique pairwise distinct normal vectors $\eta_i(x) \in N_f M(x), \ 1 \leq i \leq s = s(x)$, called the principal
normals of $f$ at $x$. Moreover, there is an associated orthogonal splitting of the tangent space as
\[ T_x M = E_1(x) \oplus \cdots \oplus E_s(x), \]
where
\[ E_i(x) = \{ X \in T_x M : \alpha(X,Y) = \langle X,Y \rangle \eta_i(x) \text{ for all } Y \in T_x M \}. \]
Hence, the second fundamental form of $f$ at $x$ is given by
\[ \alpha(X,Y) = \sum_{i=1}^{s} \langle X^i, Y^i \rangle \eta_i(x), \quad X,Y \in T_x M, \]
where $X^i$ denotes the $E_i(x)$-component of $X$.

The function $x \in M^n \mapsto s(x) \in \{1,2,\ldots,n\}$ is lower semi-continuous. Hence, if $G_k$ denotes the interior of the subset $\{ x \in M^n : s(x) = k \}$, then $\bigcup_{k=1}^{n} G_k$ is open and dense in $M^n$. On each $G_k$ the maps $x \in M^n \mapsto \eta_i(x)$, $1 \leq i \leq k$, are smooth vector fields called the principal normal vector fields of $f$ and the distributions $x \in M^n \mapsto E_i(x)$, $1 \leq i \leq k$, are smooth. The Codazzi equation on $G_k$ is easily seen to yield
\[ \langle \nabla_X Y, Z \rangle (\eta_i - \eta_j) = \langle X,Y \rangle \nabla^Z \eta_i \]
and
\[ \langle \nabla_X V, Z \rangle (\eta_j - \eta_i) = \langle \nabla^V X, Z \rangle (\eta_j - \eta_i), \]
for all $X,Y \in E_i$, $Z \in E_j$, and $V \in E_l$, where $1 \leq i \neq j \neq l \neq i \leq k$.

3. Extrinsic product of immersions. In this section, we recall the notion of extrinsic product of immersions in any space form (cf. [16]) that will be used in the proof of our main result.

A map $f: M^n \to \mathbb{R}^N$ from a product manifold $M^n = \Pi_{i=1}^{k} M_i$ is called the extrinsic product of immersions $f_i: M_i \to \mathbb{R}^{m_i}$, $1 \leq i \leq k$, if there exists an orthogonal decomposition $\mathbb{R}^N = \Pi_{i=0}^{k} \mathbb{R}^{m_i}$, with $\mathbb{R}^{m_0}$ possibly trivial, such that $f$ is given by
\[ f(x) = (v,f_1(x_1),\ldots,f_k(x_k)) \]
for all $x = (x_1,\ldots,x_k) \in M^n$ and $v \in \mathbb{R}^{m_0}$.

A map $f: M^n \to S^N(r) \subset \mathbb{R}^{N+1}$ from a product manifold $M^n = \Pi_{i=1}^{k} M_i$ into the sphere
\[ S^N(r) = \{ x \in \mathbb{R}^{N+1} : \| x \| = r \} \]
is called the extrinsic product of immersions $f_i: M_i \to S^{m_i-1}(r_i) \subset \mathbb{R}^{m_i}$, $1 \leq i \leq k$, if there exists an orthogonal decomposition $\mathbb{R}^{N+1} = \Pi_{i=0}^{k} \mathbb{R}^{m_i}$, with $\mathbb{R}^{m_0}$ possibly trivial, such that $f$ is given by
\[ f(x) = (v,f_1(x_1),\ldots,f_k(x_k)) \]
for all $x = (x_1,\ldots,x_k) \in M^n$ with $v \in \mathbb{R}^{m_0}$ and
\[ \| v \|^2 + \sum_{i=1}^{k} r_i^2 = r^2. \]
We now consider extrinsic products in the hyperbolic space
\[ \mathbb{H}^N(r) = \{ x = (x_0, \ldots, x_N) \in \mathbb{L}^{N+1} : \langle x, x \rangle = -r^2, \ x_0 > 0 \}, \]
where \( \mathbb{L}^{N+1} \) denotes the Lorentz space of dimension \( N + 1 \). In this case, there are three different types of extrinsic products called hyperbolic, elliptic, and parabolic.

A map \( f : M^n \to \mathbb{H}^N(r) \subset \mathbb{L}^{N+1} \) from a product manifold \( M^n = \prod_{i=1}^k M_i \) is called the extrinsic product of hyperbolic type of immersions \( f_1, \ldots, f_k \) if there exist an orthogonal decomposition
\[ \mathbb{L}^{N+1} = \mathbb{L}^{m_1} \times \prod_{i=2}^{k+1} \mathbb{R}^{m_i}, \]
with \( \mathbb{R}^{m_{k+1}} \) possibly trivial, and immersions
\[ f_1 : M_1 \to \mathbb{H}^{m_1-1}(r_1) \subset \mathbb{L}^{m_1} \text{ and } f_i : M_i \to S^{m_i-1}(r_i) \subset \mathbb{R}^{m_i}, \ 2 \leq i \leq k, \]
such that \( f \) is given by
\[ f(x) = (f_1(x_1), \ldots, f_k(x_k), v) \]
for all \( x = (x_1, \ldots, x_k) \in M^n \) with \( v \in \mathbb{R}^{m_{k+1}} \) and
\[ -r_1^2 + \sum_{i=2}^k r_i^2 + \|v\|^2 = -r^2. \]

A map \( f : M^n \to \mathbb{H}^N(r) \subset \mathbb{L}^{N+1} \) from a product manifold \( M^n = \prod_{i=1}^k M_i \) is called the extrinsic product of elliptic type of immersions \( f_1, \ldots, f_k \) if there exist an orthogonal decomposition
\[ \mathbb{L}^{N+1} = \prod_{i=1}^k \mathbb{R}^{m_i} \times \mathbb{L}^{m_{k+1}}, \]
a vector \( v \in \mathbb{L}^{m_{k+1}} \), and immersions \( f_i : M_i \to S^{m_i-1}(r_i) \subset \mathbb{R}^{m_i}, \ 1 \leq i \leq k, \)
such that \( f \) is given by
\[ f(x) = (f_1(x_1), \ldots, f_k(x_k), v) \]
for all \( x = (x_1, \ldots, x_k) \in M^n \) with
\[ \sum_{i=1}^k r_i^2 + \langle v, v \rangle = -r^2. \]

Finally, a map \( f : M^n \to \mathbb{H}^N(r) \subset \mathbb{L}^{N+1} \) from a product manifold \( M^n = \prod_{i=1}^k M_i \) is called the extrinsic product of parabolic type of immersions \( f_1, \ldots, f_k \) if there exist \( s \in \{1, \ldots, k\} \), an orthogonal decomposition
\[ \mathbb{L}^{N+1} = \mathbb{L}^{l+1} \times \prod_{i=s+1}^{k+1} \mathbb{R}^{m_i}, \]
with \( \mathbb{R}^{m_{k+1}} \) possibly trivial, and immersions
\[ [f_i : M_i \to \mathbb{R}^{m_i}, 1 \leq i \leq s, \text{ and } [f_j : M_j \to S^{m_j-1}(r_j) \subset \mathbb{R}^{m_j}, \]
\[ s + 1 \leq j \leq k \text{ if } s < k,] \]
such that \( f \) is given by
\[ f(x) = (i(f_1(x_1), \ldots, f_s(x_s)), f_{s+1}(x_{s+1}), \ldots, f_k(x_k), v) \]
for all \( x = (x_1, \ldots, x_k) \in M^n \) with \( v \in \mathbb{R}^{m_k+1} \). Here
\[
i : \Pi_{i=1}^k \mathbb{R}^{m_i} = \mathbb{R}^{l-1} \to \mathbb{H}^l(r_1) \subset \mathbb{L}^{l+1}
\]
denotes an umbilical inclusion with
\[
-r_1^2 + \sum_{i=2}^k r_i^2 + \|v\|^2 = -r_2^2.
\]
Let \( f : M^n \to \mathbb{Q}_c^N \) be an isometric immersion of a Riemannian manifold. If \( M^n = \Pi_{i=1}^k M_i \) is a product manifold, then the second fundamental form \( \alpha \) is said to be adapted to the product structure of \( M^n \) if
\[
\alpha(X_i, X_j) = 0 \quad \text{for all} \quad X_i \in TM_i, \ X_j \in TM_j \quad \text{with} \quad 1 \leq i \neq j \leq k,
\]
where the tangent bundles \( TM_i \) are identified with the corresponding tangent distributions to \( M^n \). The next result, which is due to Moore [8] for the case \( c = 0 \) and to Molzan [10,13] for the case \( c \neq 0 \), shows that products of isometric immersions are characterized by this property among isometric immersions of Riemannian products.

**Theorem 2.** Let \( f : M^n \to \mathbb{Q}_c^N \) be an isometric immersion of a Riemannian product manifold \( M^n = \Pi_{i=1}^k M_i \) with adapted second fundamental form. Then \( f \) is an extrinsic product of isometric immersions.

**4. The proof.** Let \( f : M^n \to \mathbb{Q}_c^N \) be an isometric immersion as in Theorem 1. In the following we are working on an open subset \( G_k \) with \( k \geq 2 \).

**Lemma 3.** Around every point \( p \in G_k \) there is a neighborhood \( U \) that is a Riemannian product of Riemannian manifolds \( M_1, \ldots, M_k \). Moreover, \( f|_U \) is the extrinsic product of totally umbilical isometric immersions \( f_1, \ldots, f_k \).

**Proof.** We claim that each distribution \( E_i \), \( 1 \leq i \leq k \), is parallel, that is,
\[
\nabla_X Y \in E_i \quad \text{for all} \quad X \in TG_k, \ Y \in E_i \quad \text{and} \quad 1 \leq i \leq k.
\]
First we prove that \( \nabla_X Y \in E_i \) for all \( X, Y \in E_i \) and \( 1 \leq i \leq k \). Indeed, from (4) we have that
\[
\langle \nabla_X Y, Z \rangle (\eta_i - \eta_j) = \langle X, Y \rangle \nabla^Z \left( \eta_i - \frac{n}{2} H \right)
\]
for any \( X, Y \in E_i \) and \( Z \in E_j \) with \( j \neq i \). Thus, we obtain
\[
2 \langle \nabla_X Y, Z \rangle (\eta_i - \eta_j, \eta_i - \frac{n}{2} H) = \langle X, Y \rangle Z (\|\eta_i - \frac{n}{2} H\|^2).
\]
Using (1), (3), and the hypothesis that \( M^n \) is an Einstein manifold, we have
\[
\|\eta_i - \frac{n}{2} H\|^2 = \left\| \frac{n}{2} H \right\|^2 - \lambda + c(n-1), \quad 1 \leq i \leq k.
\]
(7)
Using (7) we observe that
\[
\langle \eta_i - \eta_j, \eta_i - \frac{n}{2} H \rangle = \left\| \eta_i - \frac{n}{2} H \right\|^2 - \langle \eta_i - \frac{n}{2} H, \eta_j - \frac{n}{2} H \rangle \neq 0.
\]
Then (6) implies that \( \nabla_X Y \in E_i \) for all \( X, Y \in E_i \) and any \( 1 \leq i \leq k \).
Now, we show that $\nabla X Y \in E_i$ for all $X \in E_j$, $Y \in E_i$ with $j \neq i$. To this aim, we consider $Y \in E_i$, $X \in E_j$, $Z \in E_l \subset E_i^\perp$ and distinguish the following two cases.

If $l = j$, then by using the previous argument, we have

$$\langle \nabla X Y, Z \rangle = -\langle Y, \nabla X Z \rangle = 0. \quad (8)$$

If $l \neq j$, then from (5) we obtain

$$\langle \nabla X Y, Z \rangle (\eta_l - \eta_i) = \langle \nabla Y X, Z \rangle (\eta_l - \eta_j).$$

We claim that $\eta_l - \eta_i$ and $\eta_l - \eta_j$ are pointwise linearly independent. Indeed, we assume to the contrary that $\eta_l - \eta_i = \mu(\eta_l - \eta_j)$ for some $\mu \in \mathbb{R} \setminus \{0\}$. Then

$$(1 - \mu)(\eta_l - \frac{n}{2}H) = \eta_i - \frac{n}{2}H - \mu(\eta_j - \frac{n}{2}H).$$

By taking the norms and using (7), we obtain that $\eta_i = \eta_j$, which is a contradiction. Thus

$$\langle \nabla X Y, Z \rangle = 0 \text{ for all } X \in E_j, Y \in E_i, Z \in E_l, \quad (9)$$

with $l \neq i \neq j$. Therefore, from (8) and (9), we obtain that $\nabla X Y \in E_i$ for all $X \in E_j$ and $Y \in E_i$ with $j \neq i$. This completes the proof of our claim.

Now, de Rham’s theorem implies that around every point $p \in G_k$ there is a neighborhood $U$ that is the Riemannian product of the integral manifolds $M_1, \ldots, M_k$ of the distributions $E_1, \ldots, E_k$, respectively, through a point $q \in U$. Moreover, since the second fundamental form of $f$ is adapted, Theorem 2 implies that $f|_U$ is an extrinsic product of isometric immersions $f_1, \ldots, f_k$, which due to (2) have to be totally umbilical.

**Lemma 4.** Let $m_i = \dim M_i$. Then the following holds:

$$(m_i - 1)(c + \|\eta_i\|^2) = \lambda, \quad 1 \leq i \leq k. \quad (10)$$

Moreover, if $m_i \geq 2$, then the sectional curvature of $M_i$ is

$$K_{M_i} = \frac{\lambda}{m_i - 1}. \quad (11)$$

Furthermore, if $m_i \geq 2$ for all $1 \leq i \leq k$, then $\lambda > 0$.

**Proof.** It follows from Lemma 3 and the Gauss equation that the principal normals $\eta_1, \ldots, \eta_k$ of $f$ satisfy

$$\langle \eta_i, \eta_j \rangle = -c, \quad 1 \leq i \neq j \leq k. \quad (12)$$

Equation (10) follows from our assumption, (1) and (12). If $m_i \geq 2$, then (10) and the Gauss equation imply (11).

Now, suppose that $m_i \geq 2$ for all $1 \leq i \leq k$ and assume to the contrary that $\lambda \leq 0$. Then (10) implies that

$$\|\eta_i\|^2 \leq -c \text{ for all } 1 \leq i \leq k. \quad (13)$$
and thus $c < 0$. Therefore, from (12), (13), and the Cauchy-Schwarz inequality, we obtain that

$$-c = \langle \eta_i, \eta_j \rangle \leq \|\eta_i\|\|\eta_j\| \leq -c \quad \text{for all} \quad i \neq j.$$  \hfill (14)

Hence, $\eta_j = \mu_{ij}\eta_i$ for some $\mu_{ij} > 0$ and $1 \leq i \neq j \leq k$. From (12) and (13) we have that $\mu_{ij} \geq 1$ for all $1 \leq i \neq j \leq k$. Since $\eta_j = \mu_{ij}\eta_i = \mu_{ij}\mu_{ji}\eta_j$, it follows that $\mu_{ij} \leq 1$. Therefore, $\mu_{ij} = 1$ which is a contradiction. \hfill \Box

**Lemma 5.** Assume that there exists $1 \leq i \leq k$ such that $m_i = 1$. Then $U$ is flat.

**Proof.** The proof follows from Lemmas 3 and 4. \hfill \Box

**Proof of Theorem 1:** Let $f: M^n \to Q^N_c$, $n \geq 3$, be an isometric immersion of a connected Einstein manifold with Ricci curvature $\lambda$, flat normal bundle, and parallel mean curvature vector field. If $c \neq 0$, we always view $Q^N_c$ as an umbilical hypersurface of the Euclidean space $\mathbb{R}^{N+1}$ or the Lorentzian space $\mathbb{L}^{N+1}$ according to the sign of $c$. In the following we are working on an open subset $G_k$. If $k = 1$, then $f$ is a totally umbilical immersion. In the sequel we assume that $k \geq 2$ and let $p \in G_k$. Then, according to Lemma 3 we have that there exists a neighborhood $U$ of $p$ that is a Riemannian product of Riemannian manifolds $M_1, \ldots, M_k$ and $f|_U$ is an extrinsic product of totally umbilical isometric immersions $f_1, \ldots, f_k$.

We distinguish two cases.

If there exists $1 \leq i \leq k$ such that $m_i = 1$, then the result follows from Lemma 5 and [3, Theorem 1].

We now assume that $m_i \geq 2$ for all $1 \leq i \leq k$. If $c = 0$, then each $f_i$ is an umbilical isometric immersion into $\mathbb{R}^{m_i+1}$, where $\mathbb{R}^N = \mathbb{R}^m \times \prod_{i=1}^k \mathbb{R}^{m_i+1}$. Therefore, bearing in mind (11) we obtain that

$$f(U) \subset S^{m_1}(\rho_1) \times \cdots \times S^{m_k}(\rho_k) \subset \mathbb{R}^N.$$  

If $c > 0$, then each $f_i$ is an umbilical isometric immersion into $S^{m_i+1}(r_i) \subset \mathbb{R}^{m_i+2}$, where $\mathbb{R}^{N+1} = \mathbb{R}^m \times \prod_{i=1}^k \mathbb{R}^{m_i+2}$. Thus, by using (11) we obtain that

$$f(U) \subset S^{m_1}(\rho_1) \times \cdots \times S^{m_k}(\rho_k) \subset S^N_c.$$  

Finally, if $c < 0$, then $f$ is the extrinsic product of umbilical isometric immersions $f_1, \ldots, f_k$ of either hyperbolic, elliptic, or parabolic type.

If $f_1, \ldots, f_k$ is of hyperbolic type, then $f_1$ is an umbilical isometric immersion into $\mathbb{H}^{m_1+1}(r_1) \subset L^{m_1+2}$ and each $f_i$ is an umbilical isometric immersion into $S^{m_i+1}(r_i) \subset R^{m_i+2}$, $2 \leq i \leq k$, where $L^{N+1} = L^{m_1+2} \times \prod_{i=2}^k R^{m_i+2} \times \mathbb{R}^{mk+1}$. Using (11) we obtain that

$$f(U) \subset S^{m_1}(\rho_1) \times \cdots \times S^{m_k}(\rho_k) \subset H^N_c.$$  

Clearly, $f(U)$ is contained in a totally umbilical submanifold of $H^N_c$ of positive sectional curvature.
If \( f_1, \ldots, f_k \) is of elliptic type, then each \( f_i \) is an umbilical isometric immersion into \( S^{m_i+1}(r_i) \subset \mathbb{R}^{m_i+2} \), where \( \mathbb{L}^{N+1} = \Pi_{i=1}^k \mathbb{R}^{m_i+2} \times \mathbb{L}^m \). Bearing in mind (11) we obtain that
\[
f(U) \subset S^{m_1}(\rho_1) \times \cdots \times S^{m_k}(\rho_k) \subset \mathbb{H}^N.
\]
Clearly, \( f(U) \) is contained in a flat totally umbilical submanifold of \( \mathbb{H}_c^N \).

If \( f_1, \ldots, f_k \) is of parabolic type, then there exist \( s \leq k \) such that each \( f_i \) is an umbilical isometric immersion into \( \mathbb{R}^{m_i+1} \) for \( 1 \leq i \leq s \) and each \( f_j \) is an umbilical isometric immersion into \( S^{m_j+1}(r_j) \subset \mathbb{R}^{m_j+2} \) for \( s+1 \leq j \leq k \) if \( s < k \), where \( \mathbb{L}^{N+1} = \mathbb{L}^l \times \Pi_{i=s+1}^k \mathbb{R}^{m_i+2} \times \mathbb{R}^{m_k+1} \). Thus, from (11) we obtain that
\[
f(U) \subset i(\Pi_{i=1}^s S^{m_i}(\rho_i)) \times \Pi_{i=s+1}^k S^{m_i}(\rho_i) \subset \mathbb{H}_c^N.
\]
Again in this case \( f(U) \) is contained in a flat totally umbilical submanifold of \( \mathbb{H}_c^N \).

Finally, since \( M^n \) is connected and the above different type of submanifolds cannot be smoothly attached, we have that \( f(M^n) \) is an open subset of one of the above and this completes the proof. \( \square \)

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**References**

[1] E. Cartan, Sur les variétés de courbure constante d’un espace euclidien ou non-euclidien, Bull. Soc. Math. France 47 (1919), 125–160.
[2] E. Cartan, Sur les variétés de courbure constante d’un espace euclidien ou non-euclidien, Bull. Soc. Math. France 48 (1920), 132–208.
[3] M. Dajczer and R. Tojeiro, Submanifolds of constant sectional curvature with parallel or constant mean curvature, Tohoku Math. J. (2) 45 (1993), 43–49.
[4] M. Dajczer and R. Tojeiro, Isometric immersions and the generalized Laplace and elliptic sinh-Gordon equations, J. Reine Angew. Math. 467 (1995), 109–147.
[5] M. Dajczer and R. Tojeiro, On compositions of isometric immersions, J. Differential Geom. 36 (1992), 1–18.
[6] A. J. Di Scala, Minimal immersions of Kähler manifolds into Euclidean spaces, Bull. Lond. Math. Soc. 35 (2003), 825–827.
[7] A. Fialkow, Hypersurfaces of a space of constant curvature, Ann. of Math. (2) 39 (1938), 762–785.
[8] J. D. Moore, Isometric immersions of riemannian products, J. Differential Geom. 5 (1971), 159–168.
[9] J. D. Moore, Submanifolds of constant positive curvature I, Duke Math. J. 44 (1977), 449–484.
[10] S. Molzan, Extrinsische Produkte und symmetrische Untermannigfaltigkeiten in Standardraumen konstanter und konstanter holomorpher Krümmung, Doctoral Thesis, Köln, 1983.
[11] S. Nölker, Isometric immersions with homothetical Gauss map, Geom. Dedicata 34 (1990), 271–280.

[12] H. Reckziegel, Krümmungsflächen von isometrischen Immersionen in Räume konstanter Krümmung, Math. Ann. 223 (1976), 169–181.

[13] H. Reckziegel, Hypersurfaces with parallel Ricci tensor in spaces of constant curvature, Results Math. 27 (1995), 113–116. Festschrift dedicated to Katsumi Nomizu on his 70th birthday (Leuven, 1994; Brussels, 1994).

[14] P. J. Ryan, Homogeneity and some curvature conditions for hypersurfaces, Tohoku Math. J. (2) 21 (1969), 363–388.

[15] T. Y. Thomas, On closed spaces of constant mean curvature, Amer. J. Math. 58 (1936), 701–704.

[16] R. Tojeiro, A decomposition theorem for immersions of product manifolds, Proc. Edinb. Math. Soc. 59 (2016), 247–269.

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