MARCEL RIESZ ON NÖRLUND MEANS

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ABSTRACT. We note that the necessary and sufficient conditions established by Marcel Riesz for the inclusion of regular Nörlund summation methods are in fact applicable quite generally.

INTRODUCTION

One of the simplest classes of summation methods for divergent series was introduced independently by Nörlund [2] in 1920 and by Voronoi in 1901 with an annotated English translation [5] by Tamarkin in 1932. Explicitly, let \( p_n : n \geq 0 \) be a real sequence, with \( p_0 > 0 \) and with \( p_n \geq 0 \) whenever \( n > 0 \); when \( n \geq 0 \) let us write \( P_n = p_0 + \cdots + p_n \). To each sequence \( s = (s_n : n \geq 0) \) is associated the sequence \( N^p s \) of Nörlund means defined by

\[
(N^p s)_m = \frac{P_0 s_m + \cdots + p_m s_0}{P_0 + \cdots + p_m} = \frac{1}{P_m} \sum_{n=0}^{m} p_{m-n}s_n.
\]

We say that the sequence \( s \) is \((N,p)\)-convergent to \( \sigma \) precisely when the sequence \( N^p s \) converges to \( \sigma \) in the ordinary sense, writing this as

\[
s \xrightarrow{(N,p)} \sigma
\]

or as \( s \to \sigma \) \((N,p)\); viewing the formation of Nörlund means as a summation method, when \( (s_n : n \geq 0) \) happens to be the sequence of partial sums of the series \( \sum_{n \geq 0} a_n \) we may instead say that this series is \((N,p)\)-summable to \( \sigma \) and write

\[
\sum_{n=0}^{\infty} a_n = \sigma \quad (N,p).
\]

An important question regarding Nörlund summation methods (and summation methods in general) concerns inclusion. We say that \((N,q)\) includes \((N,p)\) precisely when each \((N,p)\)-convergent sequence is \((N,q)\)-convergent to the same limit; equivalently, when each \((N,p)\)-summable series is \((N,q)\)-summable to the same sum. This relationship will be symbolized by \((N,p) \implies (N,q)\). The important notion of regularity may be seen as a special case of inclusion: the Nörlund method \((N,q)\) is said to be regular precisely when each ordinarily convergent sequence is \((N,q)\)-convergent to the same limit; that is, precisely when \((N,u) \implies (N,q)\) where \( u_0 = 1 \) and where \( u_n = 0 \) whenever \( n > 0 \). Precise necessary and sufficient conditions for the regular Nörlund method \((N,q)\) to include the regular Nörlund method \((N,p)\) were determined by Marcel Riesz and communicated to Hardy in a letter, an extract from which appeared as [3]. The line of argument indicated by Riesz in his letter was amplified by Hardy in his classic treatise ‘Divergent Series’ [1], which we recommend for further information regarding summation methods in general and Nörlund methods in particular.

Our primary purpose here is to point out that the necessary and sufficient ‘Riesz’ conditions in fact apply to Nörlund methods quite generally, without regularity hypotheses.
Inclusive Riesz Conditions

A celebrated theorem of Silverman, Steinmetz and Toeplitz gives necessary and sufficient conditions for a linear summation method to be regular, and proves to be very useful. The infinite matrix \( [c_{m,n} : m, n \geq 0] \) yields a linear summation method \( C \) by associating to each sequence \( s = (s_n : n \geq 0) \) a corresponding sequence \( t = (t_m : m \geq 0) \) given by

\[
t_m := \sum_{n=0}^{\infty} c_{m,n}s_n
\]

assumed convergent; to say that this linear summation method is regular is to say that, whenever the sequence \( s \) is convergent, the sequence \( t \) is convergent and \( \lim_{m \to \infty} t_m = \lim_{n \to \infty} s_n \). The Silverman-Steinmetz-Toeplitz theorem may now be stated as follows.

**Theorem 1.** The linear summation method \( C \) with matrix \( [c_{m,n} : m, n \geq 0] \) is regular precisely when each of the following conditions is satisfied:

(i) there exists \( H \geq 0 \) such that for each \( m \geq 0 \)

\[
\sum_{n=0}^{\infty} |c_{m,n}| \leq H;
\]

(ii) for each \( n \geq 0 \)

\[
\lim_{m \to \infty} c_{m,n} = 0;
\]

(iii)

\[
\lim_{m \to \infty} \sum_{n=0}^{\infty} c_{m,n} = 1.
\]

**Proof.** This appears conveniently as Theorem 2 in [1].

Now, let \((N, p)\) and \((N, q)\) be Nörlund summation methods, or Nörlunds for short. As \( p_0 \) is nonzero, the (triangular Toeplitz) system

\[
q_n = k_0 p_n + \cdots + k_n p_0 \quad (n \geq 0)
\]

is solved (recursively) by a unique sequence \( k = (k_n : n \geq 0) \) of comparison coefficients; by summation, it follows that whenever \( n \geq 0 \) also

\[
Q_n = k_0 P_n + \cdots + k_n P_0.
\]

The comparison sequence \( k \) generates a (formal) power series

\[
k(x) = \sum_{n=0}^{\infty} k_n x^n
\]

while the Nörlund sequences \( p \) and \( q \) also generate their own power series; the convolution relation \( q = k * p \) between sequences corresponds to the relation

\[
q(x) = k(x)p(x)
\]

between generating functions. We remark that if the Nörlund \((N, p)\) and \((N, q)\) are regular, their power series \( p(x) \) and \( q(x) \) converge whenever \(|x| < 1\); the nonvanishing of \( p(0) = p_0 \) then ensures that the power series \( k(x) \) converges when \(|x| \) is small.

The introduction of the sequence \( (k_n : n \geq 0) \) of comparison coefficients facilitates the following convenient expression for the Nörlund means determined by \((N, q)\) in terms of the Nörlund means determined by \((N, p)\).
Theorem 2. If \( r = (r_n : n \geq 0) \) is any sequence then

\[
(N^q r)_m = \sum_{n=0}^{\infty} c_{m,n}(N^p r)_n
\]

where if \( n > m \) then \( c_{m,n} = 0 \) while if \( n \leq m \) then \( c_{m,n} = k_{m-n}P_n/Q_m \).

Proof. Direct calculation: simply take the definition

\[
Q_m(N^q r)_m = q_0 r_m + \cdots + q_m r_0
\]

and rearrange thus

\[
k_0 p_0 r_m + \cdots + (k_0 p_m + \cdots + k_m p_0) r_0 = k_0(p_0 r_m + \cdots + p_m r_0) + \cdots + k_m(p_0 r_0)
\]

to obtain

\[
Q_m(N^q r)_m = k_0 P_m(N^p r)_m + \cdots + k_m P_0(N^p r)_0.
\]

\( \square \)

We note that this result appears in the proof of [1] Theorem 19 but is there recorded only for regular Nörlunds and established by comparing power series expansions; the argument presented here (essentially due to Nörlund) is taken from [1] Theorem 17 and comes directly from the comparison coefficients without involving regularity.

The Riesz conditions \( R_{pq} \) associated to the Nörlunds \( (N, p) \) and \( (N, q) \) may now be stated as follows:

- \( R_{pq}^1 \): there exists \( H \geq 0 \) such that for each \( m \geq 0 \)
  \[ |k_0| P_m + \cdots + |k_m| P_0 \leq HQ_m; \]

- \( R_{pq}^2 \): the sequence \( (k_m/Q_m : m \geq 0) \) converges to zero.

As mentioned in the introduction, the fact that \( R_{pq}^1 \) and \( R_{pq}^2 \) are both necessary and sufficient for the inclusion \( (N, p) \hookrightarrow (N, q) \) between regular Nörlunds appeared in [3] and was elaborated in [1] where it becomes Theorem 19. In what follows, we re-examine the line of argument taken in [3] and [1], deliberately stripping regularity hypotheses.

Henceforth, we shall write \( C_{pq} \) for the linear summation method with matrix \( [c_{m,n} : m, n \geq 0] \) expressing \( (N, q) \) in terms of \( (N, p) \) as in Theorem 2.

On the one hand, we relate inclusion \( (N, p) \hookrightarrow (N, q) \) to regularity of \( C_{pq} \).

Theorem 3. The inclusion \( (N, p) \hookrightarrow (N, q) \) holds precisely when the linear summation method \( C_{pq} \) is regular.

Proof. Assume \( (N, p) \hookrightarrow (N, q) \). Let \( s = (s_n : n \geq 0) \) be any sequence. Note that \( s = N^p r \) for a unique sequence \( r = (r_n : n \geq 0) \) found by recursively solving the triangular Toeplitz system

\[
P_n s_n = p_0 r_n + \cdots + p_n r_0 \quad (n \geq 0).
\]

According to Theorem 2 if \( m \geq 0 \) then

\[
t_m := \sum_{n=0}^{\infty} c_{m,n} s_n = \sum_{n=0}^{\infty} c_{m,n}(N^p r)_n = (N^q r)_m.
\]

Now, let \( s \to \sigma \): then \( N^p r \to \sigma \) (by choice of \( r \)) hence \( r \stackrel{(N,p)}{\to} \sigma \) (by definition of \( (N, p) \)-convergence) so that \( r \stackrel{(N,q)}{\to} \sigma \) (by the \( (N, p) \hookrightarrow (N, q) \) assumption) whence \( N^q r \to \sigma \) (by definition of \( (N, q) \)-convergence); that is, \( t \to \sigma \). This proves that \( C_{pq} \) is regular.
Assume that \( C_{pq} \) is regular. Let \( r = (r_n : n \geq 0) \) be \((N,p)\)-convergent to \( \sigma \): then \( N^p r \rightarrow \sigma \) so Theorem 2 and the regularity of \( C_{pq} \) yield \( N^q r \rightarrow \sigma \); thus, \( r \) is \((N,q)\)-convergent to \( \sigma \) also. This proves \((N,p) \hookrightarrow (N,q)\). \( \Box \)

On the other hand, we relate regularity of \( C_{pq} \) to the Riesz conditions \( R_{pq}^1 \) and \( R_{pq}^2 \) are satisfied.

**Theorem 4.** The linear summation method \( C_{pq} \) is regular precisely when the Riesz conditions \( R_{pq}^1 \) and \( R_{pq}^2 \) are satisfied.

**Proof.** Assume \( C_{pq} \) to be regular and invoke Theorem 1. Part (i) furnishes \( H \geq 0 \) such that for each \( m \geq 0 \)

\[
\frac{|k_m| P_0 + \cdots + |k_0| P_m}{Q_m} = \sum_{n=0}^{m} \frac{|k_{m-n} P_n|}{Q_m} = \sum_{n=0}^{\infty} |c_{m,n}| \leq H
\]

whence \( R_{pq}^1 \) holds. Part (ii) says that \( \lim_{m \to \infty} c_{m,n} = 0 \) for each \( n \geq 0 \); in particular,

\[
0 = \lim_{m \to \infty} c_{m,0} = \lim_{m \to \infty} \frac{k_m}{Q_m} P_0
\]

whence \( R_{pq}^2 \) holds.

Assume that \( R_{pq}^1 \) and \( R_{pq}^2 \) are satisfied. Theorem 1(i) holds because

\[
\sum_{n=0}^{\infty} |c_{m,n}| = \sum_{n=0}^{m} \frac{|k_{m-n} P_n|}{Q_m} = \frac{|k_m| P_0 + \cdots + |k_0| P_m}{Q_m} \leq H
\]

on account of \( R_{pq}^1 \). Theorem 1(ii) holds because

\[
|c_{m,n}| = \left| \frac{k_{m-n} P_n}{Q_m} \right| \leq \left| \frac{k_{m-n}}{Q_{m-n}} \right| P_n \to 0 \quad \text{as} \quad m \to \infty
\]

on account of \( R_{pq}^2 \). Finally, Theorem 1(iii) holds simply because of the relation

\[ Q_m = k_0 P_m + \cdots + k_m P_0. \]

Theorem 1 now guarantees that \( C_{pq} \) is regular. \( \Box \)

In conclusion, the Riesz conditions \( R_{pq} \) are both necessary and sufficient for the inclusion \((N,p) \hookrightarrow (N,q)\) without any assumptions of regularity.

**Theorem 5.** Let \((N,p)\) and \((N,q)\) be any Nörlund methods. The inclusion \((N,p) \hookrightarrow (N,q)\) holds precisely when the Riesz conditions \( R_{pq}^1 \) and \( R_{pq}^2 \) are satisfied.

**Proof.** Simply combine Theorem 4 and Theorem 1. \( \Box \)

**REFERENCES**

[1] G.H. Hardy, _Divergent Series_, Clarendon Press, Oxford (1949).

[2] N.E. Norlund, _Sur une application des fonctions permutables_, Lunds Universitets Arsskrift (2) Volume 16 Number 3 (1920) 1-10.

[3] M. Riesz, _Sur l’équivalence de certaines méthodes de sommation_, Proceedings of the London Mathematical Society (2) 22 (1924) 412-419.

[4] P.L. Robinson, _Finite Nörlund Summation Methods_, arXiv 1712.06744 (2017).

[5] G.F. Voronoi, _Extension of the Notion of the Limit of the Sum of Terms of An Infinite Series_, Annals of Mathematics (2) Volume 33 Number 3 (1932) 422-428.

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