Direct comparison of quantum and simulated annealing on a fully-connected Ising ferromagnet

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We compare the performance of quantum annealing (QA, through Schrödinger dynamics) and simulated annealing (SA, through a classical master equation) on the $p$-spin infinite range ferromagnetic Ising model, by slowly driving the system across its equilibrium, quantum or classical, phase transition. When the phase transition is second-order ($p=2$, the familiar two-spin Ising interaction) SA shows a remarkable exponential speed-up over QA. For a first-order phase transition ($p \geq 3$, i.e., with multi-spin Ising interactions), in contrast, the classical annealing dynamics appears to remain stuck in the disordered phase, while we have clear evidence that QA shows a residual energy which decreases towards 0 when the total annealing time $\tau$ increases, albeit in a rather slow (logarithmic) fashion. This is one of the rare examples where a limited quantum speedup, a speedup by QA over SA, has been shown to exist by direct solutions of the Schrödinger and master equations in combination with a non-equilibrium Landau-Zener analysis. We also analyse the imaginary-time QA dynamics of the model, finding a $1/\tau^2$ behaviour for all finite values of $p$, as predicted by the adiabatic theorem of quantum mechanics. The Grover-search limit $p(\text{odd}) = \infty$ is also discussed.

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I. INTRODUCTION

Many of the complex problems of interest, notably all combinatorial optimization problems, can be generally cast as the search for the global minimum energy state of a suitable classical Ising Hamiltonian, depending on $N$ binary (spin) variables. Quantum annealing (QA) — intimately related to Adiabatic Quantum Computation (AQC) — was originally thought as an alternative route to classical annealing, employing quantum fluctuations to effectively escape, through quantum tunnelling, from unfavourable local minima in a complex energy landscape.

QA has lately become a very active field of research, due to the availability of quantum annealing programmable machines based on superconducting flux quantum bits$^{10,11}$. One of the open issues in the field is finding classes of problems where clear evidence is seen for a limited quantum speedup in the sense defined in Ref$^{12}$ i.e., a QA based computation would show a better scaling with the number of variables $N$ than a corresponding classical heuristic, for instance Simulated Annealing (SA)$^3$. The original positive results on the random Ising model$^{2,7,13}$ followed by equally encouraging results on the Travelling Salesman Problem$^{14}$ were soon followed by a rather disappointing outcome on a random Satisfiability problem$^{15}$. By now, the literature on this issue has grown enormously. A partial list is Refs. $^{12,16-18}$ and the references therein cited.

It is fair to say that the picture is far from complete. One of the notable exceptions where a quantum speedup has been clearly proven is Grover’s search problem$^{47,48}$ — searching for a given item in a unsorted database of $\mathcal{N} = 2^N$ items — where a quadratic speedup, from the classical algorithm scaling as $\sim 2^N$ to a quantum algorithm scaling as $2^{N/2}$ is known$^{12,13}$.

The Grover problem can be regarded as a particular limit$^{29}$ of a fully-connected Ising model with uniform ferromagnetic couplings, whose classical Hamiltonian reads:

$$H_C = -\frac{J N}{2} \left( \frac{1}{N} \sum_{j=1}^{N} \hat{\sigma}_j^z \right)^p ,$$

where $p \geq 2$ is an integer parameter, and we have expressed the classical spin variables in terms of $\hat{\sigma}^z$ Pauli matrices. In the limit in which one sets $p(\text{odd}) = \infty$ the model effectively describes the Grover problem: a single isolated minimal energy configuration, the fully magnetized state $|\uparrow,\uparrow,\ldots,\uparrow\rangle$ has energy $-J N/2$, and is well separated from all other configurations having zero energy.

Even for the problem defined by the Hamiltonian in Eq. (1), a definite answer for generic $p$ is still missing for dynamics, although statistical-mechanical analyses lead to a good amount of understanding of equilibrium properties$^{29,30,31}$. It is interesting to ask whether and when a quantum speedup can be proven in this particular class of simple problems. Studies on the QA of the fully-connected Ising ferromagnet have appeared in Refs. $^{49,51}$. The common setting is to supplement the classical “potential energy” $H_C$ with a transverse field term $-\Gamma(t) \sum_j \hat{\sigma}_j^x$, obtaining a time-dependent quantum Hamiltonian of the form

$$\tilde{H}_Q(t) = -\frac{J N}{2} \left( \frac{1}{N} \sum_{j=1}^{N} \hat{\sigma}_j^z \right)^p - \Gamma(t) \sum_{i=1}^{N} \hat{\sigma}_i^x ,$$

where $\Gamma(t)$ is a transverse field.
Annealing of the $p$-spin fully connected Ising model

QA (Schrödinger) dynamics: $\hat{H}_Q(t) = -\frac{JN}{2} \left( \sum_{i=1}^{N} \sigma_i^x \right)^{p} - \Gamma(t) \sum_{i=1}^{N} \sigma_i^z \quad \Gamma(t) = \Gamma_1 (1 - \frac{t}{\tau}) + \Gamma_{1/2}$

SA (Master eq.) dynamics: $H_C = -\frac{JN}{2} \left( \sum_{i=1}^{N} \sigma_i^x \right)^{p} \quad T(t) = T_1 (1 - \frac{t}{\tau}) + T_{1/2}$

| $\epsilon_N^{\text{res}}(\tau)$ | $\epsilon_N^{\text{res}}(\tau)$ |
|------------------------------|------------------------------|
| $p = 2$ (intermediate $\tau$) | $p \geq 3$ (intermediate $\tau$) |
| $\epsilon_N^{\text{res}}(\tau)$ | $\epsilon_N^{\text{res}}(\tau)$ |
| asymptotic | asymptotic |

| QA-RT | $\frac{C_{N}}{N} e^{-\tau/\tau_N} \quad (\tau_N \sim N^{2/3})$ | $\frac{C_{p}}{N} e^{-\tau/\tau_{N,p}} \quad (\tau_{N,p} \sim N^{2/3})$ |
| $\Gamma_{N}^{2} \frac{1}{\tau^{2}}$ | $\Gamma_{N}^{2} \frac{1}{p^{3} \tau^{2}}$ |
| $\sim \frac{1}{\tau^{3/2}}$ | $\sim \frac{1}{\log(\gamma/\tau)}$ |

| QA-IT | $\frac{C}{\tau}$ | $\frac{C_{N}}{N} e^{-\tau/\tau_{N,p}} \quad (\tau_{N,p} \sim N^{2/3})$ |
| $\Gamma_{N}^{2} \frac{1}{8 \tau^{2}}$ | $\Gamma_{N}^{2} \frac{1}{p^{3} \tau^{2}}$ |
| $\sim e^{-\tau/\tau_{N,p}} \quad (\tau_{N,p} \sim N^{2/3})$ |

| SA ($T_{t} > 0$) | $\frac{1}{2} e^{-\tau/\tau_{t}} \quad (N = \text{indep.} \tau_{t})$ | $\frac{C}{\tau}$ |
| $\sim \frac{1}{2} e^{-\tau/\tau_{t}} \quad (N = \text{indep.} \tau_{t})$ | $\sim \frac{1}{2} e^{-\tau/\tau_{t}} \quad (N = \text{indep.} \tau_{t})$ |

| SA ($T_{t} = 0$) | $\frac{1}{2}$ |
| $\sim e^{-\tau/\tau_{t}} \quad (N = \text{indep.} \tau_{t})$ | $\sim e^{-\tau/\tau_{t}} \quad (N = \text{indep.} \tau_{t})$ |

Table I: Summary of our findings for the behaviour of the residual energy $\epsilon_N^{\text{res}}(\tau)$ at the end of a linear annealing over a time $\tau$ at finite $N$ and in the thermodynamic limit $\epsilon_N^{\text{res}}(\tau)$. For a definition of the $\epsilon_N^{\text{res}}(\tau)$, see Eq. (13).

whose Schrödinger unitary evolution can be studied, exploiting permutation symmetry, for rather large values of $N \sim 1000$. The efficiency of QA is remarkably related to the nature of the transition point separating the large-$\Gamma$ quantum paramagnetic phase from the small-$\Gamma$ ferromagnetic phase. It is known that for $p = 2$ the transition is of second-order, and the resulting QA evolution improves its estimate of the ground-state energy as a power-law of the annealing time $\tau$. More precisely, one can define a residual energy per-spin, see Eq. (13) and Sec. III D for details, and show that in the thermodynamics limit $N \to \infty$ one gets $\epsilon^{\text{res}}(\tau) \sim \tau^{-3/2}$. The situation changes drastically for $p \geq 3$ because the transition turns first-order, and the QA dynamics becomes very slow. Although methods have been proposed to avoid first-order transitions for this model using non-stoquastic Hamiltonians, we focus our attention to the traditional case of a stoquastic Hamiltonian with a transverse field.

The goal of the present paper is to precisely quantify how much QA is slowed down through an accurate determination of the thermodynamic-limit behaviour of the residual energy for $p \geq 3$. Our conclusion, based on a finite-size analysis supplemented by the construction of the geometric envelope of the finite-$N$ Landau-Zener data, is that the thermodynamic limit of the residual energy is logarithmic, $\epsilon^{\text{res}}(\tau) \sim 1/\log(\gamma/\tau)$ for any finite $p$. Interestingly, we show that an imaginary-time Schrödinger annealing dynamics displays, instead, a power-law behaviour $\epsilon^{\text{res}}(\tau) \sim 1/\tau^{3/2}$ for all finite values of $p$, as predicted by the adiabatic theorem of quantum mechanics.

On the classical side, no study, to our knowledge, has tackled the single-spin-flip classical dynamics of the ferromagnetic fully-connected $p$-spin model. This is the second important goal we set for our study. We solved this problem by studying the deterministic evolution of a classical Master equation where the temperature $T$ is allowed to change in time, in the spirit of Refs. [52,53] where the same approach is applied to the Ising model in one di-
mension. We find that the final asymptotic residual energy depends significantly on the final temperature $T_i$ we set for the annealing. For $p = 2$, the classical Master equation annealing ends up being exponentially fast, $e^{\text{res}(\tau)} \sim e^{-\tau/r^*}$, hence winning over QA. For $p \geq 3$, however, the result is opposite: we have evidence that a classical annealing dynamics remains stuck, for $N \to \infty$, in the wrong (paramagnetic) sector, never attaining, even for arbitrarily long annealing times $\tau$, the correct minimal energy ferromagnetic state. Table IV deals with the imaginary-time Schrödinger equation annealing ends up being exponentially fast, $e^{\text{res}(\tau)} \sim e^{-\tau/r^*}$, hence winning over QA. For $p = 2$, which turns into 1st order for $p \geq 3$.

One quickly realises that since the Hamiltonian commutes with the total spin $S^2$, the study of the QA dynamics can be reduced to the Hilbert space subsector with the largest spin $S = N/2$, which is $(N + 1)$-dimensional, see Ref. [50]. Denoting the usual angular momentum states $|S = \frac{N}{2}, M\rangle$, where $M = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2}$, with the shorthand $|m = \frac{2M}{N}\rangle$, the amplitudes $\psi(m,t) = \langle m|\psi(t)\rangle$ of the time-evolving state vector $|\psi(t)\rangle$ obey a simple Schrödinger equation of the form (here and henceforth we set $\hbar = 1$):

$$i \frac{\partial}{\partial t} \psi(m,t) = -\frac{N}{2}Jm\psi(m,t) - \frac{N}{2} \Gamma(t) \sum_{\alpha = \pm 1} K^{(\alpha)}_m \psi(m - \frac{2\alpha}{N},t). \quad (4)$$

Here the first term originates from the classical (potential) energy $H_C$, while the second (kinetic) term involves $K^{(\pm)}_m = \sqrt{1 - m^2 + 2(1 + m)/N}$, the square-root originating from the well-known angular momentum relationship $S^2|S,M\rangle = S(S + 1) - M(M + 1)|S,M\rangle$ after rescaling by $2/N$.

B. Simulated Annealing (SA)

To tackle the single-spin-flip classical dynamics of the ferromagnetic fully-connected $p$-spin model in Eq. (1), one needs to write down a classical Master equation (ME), in the spirit of the celebrated Glauber approach to the one-dimensional Ising model, for the probability $P(\sigma, t)$ that the system is in configuration $\sigma$ at time $t$. In continuous time, such a ME would have the following linear form:

$$\frac{\partial P(\sigma, t)}{\partial t} = \sum_{\sigma'} W_{\sigma,\sigma'} P(\sigma', t) - \sum_{\sigma''} W_{\sigma'',\sigma} P(\sigma, t). \quad (5)$$

Here $W_{\sigma,\sigma'}$ is the “rate” matrix describing the transition from a configuration $\sigma'$ to $\sigma$, and the second term describes that inverse process $\sigma \to \sigma'$. The configurations $\sigma'$ which are connected to $\sigma$ by $W_{\sigma,\sigma'}$ can be regarded as neighbours of $\sigma$: we will consider only single spin-flip moves, denoting with $\sigma' = \sigma^j = (\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_N)$ the configuration which differs from $\sigma$ by a flip of the variable $\sigma_j$. Even restricting in this way the “neighbours” $\sigma'$, there is still a large freedom in the choice of $W_{\sigma,\sigma'}$. Typically, one imposes that the choice is such that the ME converges, when the temperature is kept fixed.
towards the equilibrium Boltzmann probability distribution $P^{eq}(\sigma)$. An effective way to impose this approach to equilibrium is to require that the detailed balance condition (DB) is satisfied:

$$W_{\sigma,\sigma'}P^{eq}(\sigma') = W_{\sigma',\sigma}P^{eq}(\sigma).$$

(6)

Even imposing the DB condition leaves a considerable freedom in the choice of $W_{\sigma,\sigma'}$. One possible choice is the usual Metropolis rule, very common in Monte Carlo studies, which we will not consider here because it is not analytically very convenient. A second widely used form of $W_{\sigma,\sigma'}$ satisfying DB, which we will adopt in the following, is the heat bath choice:

$$W_{\sigma,\sigma'} = \frac{e^{-\beta H_C(\sigma)}}{e^{-\beta H_C(\sigma)} + e^{-\beta H_C(\sigma')}}$$

$$= \alpha_0 \frac{e^{-\frac{\Delta E_{\sigma,\sigma'}}{2} + \frac{\beta E_{\sigma,\sigma'}}{2}}}{e^{-\frac{\Delta E_{\sigma,\sigma'}}{2} + \frac{\beta E_{\sigma,\sigma'}}{2}}},$$

(7)

where $\Delta E_{\sigma,\sigma'} = H_C(\sigma) - H_C(\sigma')$ is the classical energy difference in changing the configuration from $\sigma'$ to $\sigma$, and $\alpha_0$ is an overall rate constant which we easily reabsorb in our units of time.

Following the approach of Ref. [56], as recently applied on the random Ising chain problem in Ref. [57], we might transform a classical master equation into an equivalent *imaginary-time* Schrödinger problem with an appropriate effective Hamiltonian $H_{\sigma,\sigma'}$ which effectively “symmetrizes” the rate matrix $W_{\sigma,\sigma'}$ using DB, from there, one would then proceed to study such equivalent imaginary-time Schrödinger problem using the same “total spin technique” employed above for the quantum case. One might do that, but we will not do it here, for a reason that we briefly explain in Appendix B.

Now, we will directly exploit the “permutation symmetry” of the classical problem to do something that is completely equivalent to working in the “maximum spin sector”. Indeed, a common feature of the many possible configurations corresponding to a magnetization $m$ being given by $\binom{N}{N_m}$, we can define the probability distribution for the magnetization $m$ as:

$$P(m, t) = \binom{N}{N_m} P(\sigma, t).$$

(8)

Rewriting the classical ME Eq. (5) in terms of $P(m, t)$, with due account of all the binomial factors, we eventually arrive at the following “permutation symmetric” version of it:

$$-\frac{\partial P(m, t)}{\partial t} = -\frac{N}{2} \sum_{\alpha = \pm} W_m^{(\alpha)} P(m - \frac{2\alpha}{N}, t) + \frac{N}{2} \sum_{\alpha = \pm} V_m^\alpha P(m, t).$$

(9)

Again, $\frac{2}{N}$ is the change in magnetisation upon flipping a single spin, but the effective kinetic and potential term coefficients are now given by:

$$W_m^{(\alpha)} = \left(1 + \alpha m + \frac{2\alpha}{N}\right) W_{m,m+\frac{2\alpha}{N}}$$

$$V_m^\alpha = \sum_{\alpha = \pm} \left(1 + \alpha m\right) W_{m-\frac{2\alpha}{N},m}. \quad \text{(10)}$$

Notice that the transition rates $W_{m,m'}$ depend on the energy difference $E(m) - E(m')$ and also on the inverse temperature $\beta = 1/(k_B T)$. To do a SA dynamics we have to “anneal down” the temperature $T(t)$ entering in the heat-bath transition rates, thus making all the relevant ingredients entering Eq. (9) *time-dependent*: $W_{m,m'}(T(t)), W_m^\alpha(T(t))$, and $V_m(T(t))$. Eq. (9), with the prescribed time-dependent coefficients ensuing from the time-dependence of $T(t)$, is the linear system of $N+1$ coupled differential equations that we will need to solve to do SA dynamics for the fully-connected spin ferromagnet, in the spirit of a Glauber-type classical master equation.

C. Schrödinger quantum annealing in imaginary time (QA-IT)

The analogy of the classical ME in Eq. (5) with an imaginary-time Schrödinger problem is well known[50]; it can be made precise by a suitable symmetrization of the transition rate matrix $W_{\sigma,\sigma'}$ to transform it into a proper Hermitian quantum kinetic energy. Such an analogy is inspiring, as it correctly suggests, for instance, that the thermal annealing dynamics proceeds by “filtering-out” the higher excited eigenstates from the time-evolving $P(\sigma, t)$. It should, however, not be confused with the actual *imaginary-time* (IT) Schrödinger dynamics obtained by a Wick’s rotation $t \rightarrow -it$ of Eq. (4), which amounts to studying:

$$-\frac{\partial}{\partial t} \psi(m, t) = -\frac{N}{2} J m^p \psi(m, t)$$

$$-\frac{N}{2} \Gamma(t) \sum_{\alpha = \pm} K^{(\alpha)}_m \psi(m - \frac{2\alpha}{N}, t). \quad \text{(11)}$$
Although physically not relevant for the actual hardware of possible QA machines, this route is interesting from the algorithmic point of view, as many quantum Monte Carlo approaches are indeed based on an IT framework. Moreover, the filtering effect towards the ground state of the IT dynamics is beneficial within an optimization context. Indeed, recent results for an Ising chain suggest that the residual energies obtained using QA in IT are definitely below those of a standard QA in real time (RT).

D. The annealing schedule and the residual energy

To fully specify the annealing, we have to stipulate the annealing schedule we will use for the relevant parameters: the transverse field $\Gamma(t)$ for QA, and the temperature $T(t)$ for SA. Although a schedule optimization is known to be highly important — for instance, it provides the quadratic quantum speedup in a QA version of the Grover’s search problem — for the purpose of a simpler setting we will adopt here a linear schedule for both QA and SA, writing:

$$\begin{align*}
\Gamma(t) &= \Gamma_i \left(1 - \frac{t}{\tau}\right) + \Gamma_f \frac{t}{\tau} \quad \text{QA} \\
T(t) &= T_i \left(1 - \frac{t}{\tau}\right) + T_f \frac{t}{\tau} \quad \text{SA}
\end{align*}$$

On equal footing with QA, where the initial state is the ground state of a suitably large $\Gamma_i > \Gamma_c$, the SA evolution will start from the equilibrium configuration at a sufficiently large $T_i \gg T_c$, where $T_c$ is the equilibrium critical temperature. Notice that we will allow ourselves a bit more freedom in the SA case, by choosing $T_i$ to be the final temperature, which we could take to be either much less than $T_c$, or simply set to $T_i = 0$. We will see the role of this choice in discussing the SA results. In the QA case the value of $\Gamma_i$ is much less relevant, as long as $\Gamma_i \ll \Gamma_c$, and we simply set $\Gamma_i = 0$.

To characterise the annealing efficiency we study, as often done, the residual energy (per spin) at the end of annealing, defined as the difference between the total energy in the final state, minus the corresponding minimal energy. In the quantum case, the definition involves the average of the final Hamiltonian $\mathcal{H}_Q(\tau) = \mathcal{H}_C$ over the time-evolved final state $|\psi(\tau)\rangle$. In the classical case the corresponding quantity to calculate is the average of the classical $\mathcal{H}_C$ over the final probability distribution $P(\sigma, \tau)$. In both cases we write:

$$\epsilon^{\text{res}}_N(\tau) = \begin{cases} \frac{1}{N} \left[ \frac{\langle \psi(\tau) | \mathcal{H}_Q(\tau) | \psi(\tau) \rangle}{\langle \psi(\tau) | \psi(\tau) \rangle} - E_0 \right] & \text{QA} \\ \frac{1}{N} \left( \sum_m E(m) P(m, \tau) - \langle E \rangle_{\text{eq}} \right) & \text{SA} \end{cases}$$

where $E_0$ is the final minimal energy value, $E_0 = -N J/2$ in the present case, while $\langle E \rangle_{\text{eq}}$ denotes the average energy over the equilibrium distribution at the final temperature $T_i$. $\epsilon^{\text{res}}_N(\tau)$ is in general a function of $\tau$ and of the number of sites $N$. Its dependence on $p$ is implicit, but, as we shall see, the value of $p$ plays a crucial role in the following.

III. RESULTS: QA-RT VERSUS SA

We present here the results of our analysis of both QA-RT and SA for the fully-connected $p$-spin Ising ferromagnet. We distinguish the $p = 2$ case, where the transition (both classical and quantum) is second-order, from $p \geq 3$ where the classical and quantum transitions are first-order. (We will mostly present results for $p = 3$, but the higher $p$ we have explored present a similar phenomenology.) We will in the end discuss separately the Grover limit $p(\text{odd}) = \infty$. In our numerical analysis we simulated the QA-RT and SA dynamics by annealing down the driving parameter with a constant rate in a total annealing time $\tau$, see Eq. (12). To characterize the behaviour of the residual energy density Eq. (13), we solved the Schrödinger equation (4), and the classical ME Eq. (9), for several values of the total annealing time $\tau$ and increasing the number of sites $N$ up to 1024 spins. A comparison between real-time and imaginary-time Schrödinger QA is deferred to Sec. IV. We set the coupling $J = 1$ in what follows.

The QA-RT dynamics of the present model was previously studied in Ref. 50 for $p = 2$, finding $\epsilon^{\text{res}}_N(\tau) \sim \tau^{-3/2}$ in the thermodynamic limit, and in Ref. 51 for general $p$, with an exhaustive analysis which however did not go all the way to predicting the final asymptotic large-$\tau$ behaviour of $\epsilon^{\text{res}}_N(\tau)$.

A. $p = 2$

We start by presenting the results obtained for $p = 2$. This case was already discussed in Ref. 50. The analysis performed in the present work will prove crucial in dealing with $p \geq 3$. Fig. 1 shows our QA results for $p = 2$ for Schrödinger annealing in real time (RT). As first reported by Caneva et al., the QA-RT data for $\epsilon^{\text{res}}_N(\tau)$ as a function of the annealing time $\tau$ show four regimes:

1a) an initial very short-$\tau$ behaviour, where the system is essentially unable to follow the drive, and remain trapped in the paramagnetic phase;

1b) a subsequent power-law scaling which “feels” the critical point of the system, because the driving is too fast for the system to realize that, at finite-$N$, the gap $\Delta_N$ is non-vanishing, $\Delta_N > 0$: here $\epsilon^{\text{res}}_N(\tau) \sim \tau^{-3/2}$. From Fig. 1c and Fig. 2c, we see that the ground state for large $\Gamma$ is driven, upon
Figure 1: (a) Log-log plot of the residual energy density vs annealing time for QA-RT for \( p = 2 \). The QA-RT raw data show the four different regimes discussed in the text. Here \( \Gamma_i = 2 \). (b) The envelope construction illustrated: notice the large coherent oscillations of the finite-size LZ data, discussed in the text. The green thick line represents the geometrical envelope of our QA-RT finite-\( N \) curves, obtained from LZ fit (black solid lines). (c) The spectrum for \( N = 64 \). (d) The minimum gap \( \Delta_N \) (dashed lines) and the more relevant dynamical gap (solid lines) vs the transverse field \( \Gamma \). The inset shows the scaling with \( N \) of the minimum gap \( \Delta_N \) at the critical point.

2) an intermediate Landau-Zener regime where the driving is slow enough that the system “sees” that \( \Delta_N > 0 \), but not so slow to be completely adiabatic. Here the evolution is dominated by single Landau-Zener (LZ) events for an effective two-level-system describing the avoided crossing, with a gap \( \Delta_N > 0 \), between the two lowest-lying instantaneous eigenvalues. Notice that for even values of \( p \) the relevant gap is between the ground state and the second excited state, because \( \hat{H}_Q \) preserves parity. The dynamical gap is therefore finite for \( \Gamma < \Gamma_c \); it shares the same critical properties of the first equilibrium energy gap, as shown in Fig. 1(d). The LZ formula predicts that the probability of being excited across such an avoided level crossing transition goes like \( P_{LZ} = e^{-\pi \Delta_N^2 / \tau} \), which suggests that the residual energy per-spin in this intermediate regime should behave as:

\[
\epsilon_{LZ}^N(\tau) = C_N e^{-\tau / \tau^*_N},
\]

where \( \tau^*_N \propto \Delta_N^{-2} \) is a characteristic time for the LZ transition.

3) a final asymptotic regime for \( \tau \gg \Delta_N^{-2} \) — hence, only observed for finite \( N \) — which satisfies the adiabatic theorem prediction \[^{7,60}\] of \( \epsilon_{\text{res}}(\tau) \propto 1 / \tau^2 \). In the present case we find, for general value of \( p \):

\[
\epsilon_{\text{res}}(\tau \gg \Delta_N^{-2}) \approx \frac{\Gamma^2}{N^p} \frac{1}{\tau^2}.
\]

Although the final asymptotic behaviour of \( \epsilon_{\text{res}} \) for \( N \to \infty \) turns out to be, in the end, given by the power-law scaling \( \tau^{-3/2} \) discussed in 1b), the most interesting,
and in some sense revealing, regime is the intermediate-
\( \text{-} \) LZ region 2). This LZ-regime will be present also for
\( p \geq 3 \) and shows a behaviour of the residual energy which
can be cast in the form:

\[
C^L_N(\tau) = \begin{cases} 
\frac{C_2}{N} e^{-\gamma_2 \tau N^{-2z}} & \text{for } p = 2 \\
\frac{C_p}{N} e^{-\gamma_p \tau e^{-2\alpha_p N}} & \text{for } p \geq 3
\end{cases}.
\]

(16)

Here \( C_p \) is, as we have verified, very close to the first
energy gap at the end of the annealing, i.e., \( C_p \approx pJ + O(1/N) \). The difference between \( p = 2 \) and \( p \geq 3 \) resides in the way the transition gap \( \Delta_N \) closes with increasing
\( N \) as a power-law \( N^{-z} \) (with \( z = 1/3 \)) for
\( p = 2 \) (2\text{nd}-order transition), or exponentially \( e^{-\alpha_p N} \) for \( p \geq 3 \) (1\text{st}-order transition). To obtain the
thermodynamic limit asymptotic behaviour of \( c^L_{\text{res}}(\tau) \) we resort to a strategy that will prove extremely useful when \( p \geq 3 \) — less so for \( p = 2 \), where the asymptotic is
already announced by the intermediate power-law regime
\( \tau^{-3/2} \) — but we choose to illustrate here.

The strategy is based on the geometrical construction of the
envelope of all the finite-\( N \) LZ-curves, and is briefly described in Appendix A where we illustrated it with the example of the transverse-field Ising chain. Briefly, if
\( f_u(x) \) is a family of functions which smoothly depends on
some parameter \( u \), its envelope \( e(x) \) is a function which is
tangent in each point to a member of the family, a condi-
tion that is enforced by solving \( \partial_x f_u(x) = 0 \) to find \( u(x) \)
and then setting \( e(x) = f_u(x) \). In our case, we identify
\( x \mapsto \tau \), \( u \mapsto N \) (which we assume to be real, rather than integer), and \( f_u(x) \mapsto C^L_N(\tau) \). To construct the envelope
\( e^\text{env}(\tau) = e^L_N(\tau) \) we need to solve for \( \tau \) the implicit equation \( \partial_N e^L_N(\tau) = 0 \). For \( p = 2 \) the solution is very simple:

\[
N(\tau) = \frac{\tau}{2z+1}
\]

\[
e^\text{env}(\tau) = \frac{C_2}{N} e^{-2z} = \left( \frac{3}{2e^{7/2}} \right) \frac{C_2}{\tau^{3/2}}.
\]

(17)

(18)

where we have used the fact that \( z = 1/3 \). As shown in
Fig. 1(b), the method works well, although here — at variance with the transverse-field Ising chain illustrated in Appendix A — the LZ-regime is “decorated” by extra coherent oscillations which make the analysis of small-\( N \) data a bit more difficult: notice that these oscillations become less and less pronounced when \( N \) increases. The construction allows us to predict quite precisely both the scaling exponent 3/2 and even the numerical coefficient in front of the power-law from the sole analysis of the
LZ-regime.

Let us now illustrate our results for the SA dynamics.
First of all, the residual energy depends crucially on the
final annealing temperature \( T_f \), leading to very different results depending on whether \( T_f > 0 \) or \( T_f = 0 \). Results for \( T_f = 0 \) and \( T_f > 0 \) are reported in Figs. 2(a) and 2(b), respectively. Notice that the data bear no memory of the
properties of the critical point, as the dynamical spectral gap between the lowest eigenstate and the first accessible excited state opens-up again after the critical point crossing, see spectrum and spectral gaps in Fig. 2(c,d). The “filtering” effect is therefore very effective and we observe a size-independent decrease of \( e^\text{res} \) with \( \tau \), which is compatible with an exponential decay for large annealing times \( \tau \). Notice also the striking absence of any asymptotic adiabatic regime, as opposed to the \( 1/\tau^2 \) behaviour of QA. Indeed, had we stopped our SA at a final \( T_f > 0 \) we would have observed an adiabatic power-law tail of the form

\[
e^\text{res}(t = \tau) \sim \langle \phi_0|H_C|\phi_{\text{res}} \rangle c_{\text{res}}(\tau)
\]

\[
c_{\text{res}}(\tau) = \frac{2T_f}{\tau} (\langle \phi_{\text{res}}|\partial_{\beta} H|\phi_0 \rangle)_{T_f},
\]

(19)

where \( c_{\text{res}}(\tau) \) is the projection of the probability \( \mathbb{P}(m, t = \tau) \) over the first relevant lowest-lying left eigenstate of the transition matrix \( W_{\sigma, \sigma'} \) (see Appendix C for details). Differently from the usual result of the adiabatic theorem, the power-law pre-factor \( \langle \phi_{\text{res}}|\partial_{\beta} H|\phi_0 \rangle \) decreases exponentially fast as \( T_f \to 0 \), see Eqs. (C19),(C20). As it turns out, the ultimate behaviour of SA for \( T_f = 0 \) is exponential. What makes SA so different from QA is the dependence of the Hamiltonian from the annealing parameter: in QA \( \hat{H}_Q \) depends linearly on the transverse field \( \Gamma \), while in SA the dependence of the transition rates \( W_{m,m'} \) from the temperature \( T \) is highly non-linear, through the Boltzmann weights.

**B. \( p \geq 3 \)**

In this case the corresponding equilibrium model undergoes a first-order phase transition, as compared to the case \( p = 2 \) where the transition is of second order. This difference has an enormous impact on the efficiency of the annealing protocols. Let us now focus on the most interesting situation \( p \geq 3 \), where the transition is first order and the optimization process becomes hard\([19,21]\). Here we observe three distinct regimes for the QA-RT evolution, see Fig. 2(a). The first (short \( \tau \)) and the third (asymptotic adiabatic behaviour) are strict analogues of regime
1a and 3 observed for \( p = 2 \). The intermediate LZ-regime 2) is now very clear, but only visible in a limited range of \( N \sim 24 \div 64 \): for larger \( N \) it would occur in a re-
region of annealing times \( \tau \) prohibitive hard to simulate, since the critical gap \( \Delta_N \), and therefore the characteristic time \( \tau_N \sim \Delta_N^{-2} \), now scale exponentially\([19,21]\) with \( N \), see Eq. (16). The absence of coherent oscillations observed in Sec. III A is due to the sharp closing of the gaps between successive eigenvalues of the spectrum — Figs. 3(c) and (d) — which rapidly decouples the two states involved in the
LZ approximation. The analytical construction of the envelope is now a bit more involved, since the equation defining \( N(\tau) \) — the implicit solution of \( \partial_N e^L_N(\tau) = 0 \) — cannot be solved explicitly. Nevertheless, one can show
Figure 2: (a) $\epsilon_{res}^N(\tau)$ vs annealing time $\tau$ for SA at $p = 2$, with vanishing final temperature $T_f = 0$. The data show almost no size dependence and follow asymptotically an exponential relaxation. (b) $\epsilon_{res}^N(\tau)$ for $N = 32$ in log-log scale, for different values of the final temperature $T_f$. In (a) and (b) the initial temperature is $T_i = 2$. For $T_f > 0$ the asymptotic adiabatic regime $\tau^{-1}$ suggested by Eq. (19) is visible. (c) The instantaneous spectrum of the classical master equation for $N = 128$. (d) The minimum gap relevant for the classical master equation dynamics (solid line) and for equilibrium properties (dashed lines).

that:

$$e^{2\alpha_p N(\tau)} \approx \frac{2\alpha_p C_p}{\log(\gamma_p \tau) + \log(\log(\gamma_p \tau))} e^{-\frac{1}{\log(\gamma_p \tau)}},$$

where, see Eq. (16), $\alpha_p$ enters in the exponential closing of the critical gap $\Delta_N \sim e^{-\alpha_p N}$ and $\gamma_p$ is the overall rate-constant entering the LZ expression (16). ($C_p \approx pJ$ is again very close to the lowest gap $\Delta(\Gamma = 0)$ at the end of the annealing process.)

One should pause here to appreciate the power of the envelope construction. If we look at the raw data in Fig. 3(a), it would be hopeless to try to fit the slight (but evident) decline of the $N = 512$ annealing data with some inverse power of a logarithm of $\tau$ plus (unknown) corrections. But if you extract (by a simple fitting) the relevant ingredients from the LZ-regime of the annealing data for moderate $N = 24 \div 64$, see Fig. 3(b), this will unambiguously signal, if you assume that an envelope exists, the asymptotic result in Eq. (20). The envelope construction is, in a way, a powerful “telescope” for data that you would never be able to observe directly. Summarising, consistently with Refs. 49, 51, first-order transitions lead to exponentially small gaps and to a problem that is hard for QA, but we can precisely quantify this hardness by stating that, for $N \to \infty$, the residual error decreases as $\epsilon^{inv}(\tau) \sim 1/\log(\gamma_p \tau)$ as the annealing time $\tau$ grows. Such a statement applies to all finite values of $p$. The Grover limit should be considered separately.

Let us now turn to the crucial question: is SA better or worse than QA-RT for $p \geq 3$, where a first-order transition is present? The residual energy data when $T_f = 0$, shown in Fig. 4(a), display now a considerable size-dependence, at variance with the $p = 2$ case, and are
Figure 3: (a) Log-log plot of the residual energy density vs annealing time for QA-RT for $p = 3$. For first order phase transition the characteristic annealing times $\tau_N^* \propto N$, hence only for moderate $N$ the LZ-regime is clearly visible. Here the initial transverse field is set to $\Gamma_\text{i} = 2$. (b) Detail of the exponential fit on the regions in which the LZ approximation holds, used for the envelope construction. The envelope $1 / \log(\gamma \tau)$ (thick line) gives an estimate of the large-$\tau$ behaviour of the residual energy in the thermodynamic limit $N \to \infty$. (c) The instantaneous spectrum vs the transverse field $\Gamma$, close to the transition, for $N = 64$. One can clearly see the series of avoided level crossings encountered by the paramagnetic phase coming from the right. (d) The minimum gap close to the critical point.

welldescribedbyanexponentialrelaxationoftheform

$$\epsilon_N^\text{res}(\tau) \simeq \frac{1}{2} e^{-\tau/\tau_N^*}, \quad (21)$$

where the characteristic time-scale $\tau_N^*$ increases now as a power-law of $N$, for instance $\tau_N^* \sim \sqrt{N}$ for $p = 3$, see Fig. 4(b). Notice that the pre-factor in front of the exponential is very close to 1/2, with a negligible size dependence, at variance with the $C_p/N$ pre-factor appearing in the LZ-regime of QA-RT. To explain these features, one apply Kramer’s theory to describe the escape of the probability from the paramagnetic free-energy minimum at $T \gg T_c$. The equilibrium free-energy density $f(m, T)$ of the classical $p$-spin model is simply given by $f(m, T) = -(J/2)m^p - Ts(m)$ where the entropy density $s(m)$ originates from the binomial coefficients and is given by

$$s(m) = \log 2 - \frac{(1-m)}{2} \log(1-m) - \frac{(1+m)}{2} \log(1+m).$$

Fig. 4(c) shows $f(m, T)$ for $p = 3$ at different temperatures above and below $T_c$. The presence of a free energy barrier, separating the ferromagnetic and paramagnetic phases when $0 < T < T_c$, is visible also in the spectrum of the transition matrix. Indeed the two lowest eigenvalues are very close (exponentially) even for temperature well below the transition point, while the first excited state is almost identical to the high temperature equilibrium state, as shown in Fig. 4(d). This suggests that the dynamics is effectively described by a two-level approximation in which one considers the distribution as peaked in either one of the two minima of $f(m, T)$. Let us call $P_0(t)$ the probability that $P(m, t)$ is inside the paramag-
netic valley around $m = 0$ at time $t$ (formally, we should sum over all the $m$ values inside the paramagnetic valley, say below a given value $m_B$ marking the barrier point), and $P_1(t)$ the probability that the system is close to the ferromagnetic minimum at or near $m = 1$. Before the transition temperature $T_c$ is reached, $P_1(t)$ is zero (or even undefined, if the minimum is not formed), while for $T < T_c P_1(t)$ starts growing, but the transition rate to return back to the (wrong) paramagnetic minimum becomes increasingly small. All in all, we are justified in writing a master equation for $P_0(t)$ alone in the Kramer’s escape form:

$$\frac{d}{dt} P_0(t) \simeq -A e^{-\frac{N \Delta f(t)}{k_B T(t)}} P_0(t),$$

(22)

where $N \Delta f(t)$ is the free energy barrier separating the paramagnetic and the ferromagnetic minima for $T(t) < T_c$, while the rate $A$ can be taken to be a constant (to within leading exponentials). Given $f(m, T)$ we can calculate the free-energy barrier $\Delta f(t)$. When $T$ is small, more precisely for $T \to 0$, the position of the barrier maximum $m_B$ is close to $m = 0$, and, by expanding the entropy $s(m)$, one can easily show that

$$m_B \sim \left(\frac{2T}{Jp}\right)^{\frac{1}{p-2}}.$$

Hence the barrier height $\Delta f(t) = f(m_B, T(t)) - f(0, T(t))$ can be approximated, in the same regime, as:

$$\Delta f(t) = \frac{J(p-1)}{2} \left(\frac{2k_B T(t)}{Jp}\right)^{\frac{p}{p-2}}.$$

(23)

We need to solve Eq. (22) with the initial conditions $T(0) = T_c, P_0(0) = 1$ and the usual linear schedule for
the temperature annealing, see Eq. (12). The initial evolution for \( T(t) > T_c \) can be neglected, since the system is already in the stable minimum and it remains always close to the equilibrium distribution.) To do that, it is useful to use directly the temperature as an independent variable, with the substitution \( \frac{d}{dt} \to -\frac{T_c}{T} \frac{d}{dt} \), which allows us to rewrite Eq. (22) as

\[
\frac{d}{dT} \log(P_0) = \frac{A\tau}{T_c} e^{-N/(\frac{\epsilon}{p})} (\frac{Jp}{2})^{\frac{1}{\tau}} .
\]

Integrating between \( T_c \) and \( T_l = 0 \) we get:

\[
\log(P_0(T = 0)) = -\frac{A\tau}{T_c} \int_0^{T_c N^{-\frac{1}{2}}} dy \, e^{-N/(\frac{\epsilon}{p})} (\frac{Jp}{2})^{\frac{1}{\tau}} \, dy
\]

where we made the substitution \( N \, T \, \tau^{\frac{1}{2}} = y \tau^{\frac{1}{2}} \) in order to eliminate the dependence on \( N \) from the integrand. Notice that for any \( p \geq 3 \) the integrand is a function that drops exponentially fast to zero, so that we can send the upper integration limit to infinity, making a negligible error of the order \( O(e^{-N}) \):

\[
\log(P_0(\tau)) \simeq -\frac{A\tau}{T_c N^{-\frac{1}{2}}} \int_0^{\infty} dy \, e^{-\frac{1}{\frac{\epsilon}{p}}} (\frac{Jp}{2})^{\frac{1}{\tau}} .
\]

Therefore we predict that the final excitation probability behaves as

\[
P_0(\tau) \simeq e^{-\tau/(\Gamma N)} ,
\]

with a characteristic time \( \tau_N \) increasing polynomially with the system size \( N \), as

\[
\tau_N = \frac{T_c}{\bar{A}} \frac{N^{\frac{1}{2}}}{2} ,
\]

where \( \bar{A} \) contains both the factor \( A \) and the contribution from the integral in Eq. (23). This prediction is in very good agreement with our numerical data, see Fig. 1(b). The final outcome of this calculation suggests that in the thermodynamic limit \( N \to \infty \) the system is trapped in the paramagnetic minimum, i.e. \( P_0(\tau) \sim 1 \), hence the residual energy per-spin remains stuck at the value \( \sim 1/2 \):

\[
\epsilon_{N \to \infty}(\tau) \simeq \frac{1}{2} .
\]

C. \( p = \infty \): The Grover limit

Some final remarks are necessary to explain the connection with the Grover problem, i.e. \( p(\text{odd}) = \infty \). If we apply our analysis to this situation, we find that both QA-RT — without an optimized schedule — and SA are characterized by a time scale that diverges as \( 2^N \), in analogy with what is known in the literature \( \cite{12,13} \). Therefore for \( N \to \infty \) none of the two processes is able to attain the correct minimal energy configuration. While this is clear from our SA analysis — indeed, our arguments suggest that SA is stuck even for finite \( p \geq 3 \) — the conclusion it is slightly more subtle for QA-RT. Here it is important to notice that the logarithmically decreasing envelope in Eq. (20) originates essentially from the fact that the LZ form \( \epsilon_N^{\pm}(\tau) = \frac{C_p}{N} e^{-\tau/\tau_N} \) has a characteristic \( 1/N \) pre-factor, since the constant \( C_p \simeq \Delta(\Gamma = 0) \simeq Jp \) is essentially the (finite) excitation energy that the system reaches when the important LZ transition is precisely that occurring at the critical point. The fact that \( C_p \) is not extensive with \( N \) is crucial. This condition, however, is valid only for finite \( p \). For \( p(\text{odd}) = \infty \), indeed, the system has only three eigenvalues when \( \Gamma = 0 \) = 0, which is massively degenerate, \( 2^{N-1} \)-fold, and \( \pm N/2 \). Hence \( \Delta(\Gamma = 0) \propto N \) and the envelope will no longer be a logarithmically decreasing function of \( \tau \). In conclusion, even QA is “stuck” and guaranteed to fail for \( N \to \infty \) if we first send \( p \to \infty \): the possible battle against a classical algorithm is on the time-scale needed to reach the solution for \( N \). And here a QA schedule optimization \( \cite{14} \) can lead to the celebrated Grover quadratic speed-up \( \cite{15} \).

IV. RESULTS: QA IN REAL-TIME VERSUS IMAGINARY-TIME

Let us now consider the case of the imaginary-time QA Schrödinger dynamics, to contrast it to the physical real-time QA. The behaviour of the two dynamics turns out to be very different, as already found in the Ising chain case \( \cite{16} \). In particular, the closing of the gap at the transition point is highly irrelevant to the QA-IT dynamics: as the gap re-opens and stays finite in the whole ferromagnetic phase with \( \Gamma < \Gamma_c \), the dynamics will filter very effectively the instantaneous ground state; the scaling of the residual energy can be easily explained through the adiabatic theorem for imaginary-time processes \( \cite{17} \), which predicts an asymptotic \( 1/\tau^2 \) behaviour. Indeed, as predicted in Ref. \( \cite{17} \) and clearly visible from the numerical results in Fig. 3, the two QA dynamics share the same asymptotic regime: the only caveat is that while QA-IT obeys it without restrictions, QA-RT does it only for \( \tau \gg \Delta_N^{-2} \), hence only for finite \( N \). For QA-IT, the predictions of the adiabatic theorem \( \cite{17} \) imply that for essentially all values of \( \tau \)

\[
\epsilon_{\text{res}}(\tau) \simeq \frac{\Gamma}{p} \frac{1}{\tau^2} ,
\]

with negligible size-corrections.

V. DISCUSSION AND CONCLUSIONS

In this work we compared classical thermal annealing (SA) and quantum annealing (QA) — both in real time and imaginary time — on a fully-connected \( p \)-spin Ising
A slow decay of the residual energy with the annealing time, where the system evolution can be effectively described by the interference of the two lowest energy levels (Landau-Zener regime). For $p \geq 3$, this analysis predicts a slow decay of the residual energy with the annealing time $\tau$, with an inverse-logarithmic behaviour $1/\log(\tau)$, which means a limited quantum speedup as defined in Ref.12. We would emphasize that the present conclusion has been drawn by direct solutions of the Schrödinger and master equations in combination with a non-equilibrium analysis based on the Landau-Zener formula, whereas most studies so far have been based on numerical simulations, a notable exception being the one-dimensional case discussed in Ref.57. The envelope method is hence particularly useful to study the asymptotic regime of systems that cross a first-order phase transition, where usually one can solve the dynamics only for small values $N$ and $\tau$.

A possible line of investigation, which we leave to future work, concerns the study of the open-system quantum dynamics of the same model, to elucidate the competition between thermal effects due to the environment, and genuine quantum tunnelling effects as discussed recently in Ref.45.

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Appendix A: The envelope construction to finite-$N$ annealing data

The strategy is based on the geometrical construction of the envelope of all the finite-$N$ LZ-curves. Briefly, if $f_u(x)$ is a family of functions which smoothly depends on some parameter $u$, its envelope $e(x)$ is a function which is tangent at each point to a member of the family, a condition that is enforced by solving $\partial_u f_u(x) = 0$ to find $u(x)$ and then setting

$$e(x) = f_{u(x)}(x).$$

(A1)

If we now identify $u \rightarrow N$ (which we assume to be real, rather than integer), $x \rightarrow \tau$, and $f_u(x) \rightarrow \epsilon_{LZ}^N(\tau)$, we can construct the geometrical envelope of these finite-$N$ Landau-Zener annealing data. We illustrate in Fig. how this procedure works in the transverse-field Ising chain, where the well-known Kibble-Zurek form of the asymptotic residual energy is $\epsilon_{\text{asymp}}(\tau) \sim 1/\sqrt{\tau}$.
verified that this term cannot be neglected in the present analysis. As it turns out, we have
which originates from the time derivative of the equilibrium state, i.e., \( \sum_{\sigma'} K_{\sigma,\sigma'} P_{\sigma}(t) = 0 \), which ensures probability conservation. Since we want to use Eq. (C1) to describe a thermal process, we impose that the thermal distribution \( P_{\text{eq}}(\sigma) = e^{-\beta H_C(\sigma)}/Z \) is the equilibrium state, i.e., \( \sum_{\sigma'} K_{\sigma,\sigma'} P_{\text{eq}}(\sigma') = 0 \). \( P_{\text{eq}}(\sigma') \) is therefore a right eigenvector of \( K_{\sigma,\sigma'} \) with null eigenvalue \( E_0 = 0 \). The left eigenstate \( \tilde{P}_0(\sigma) \) corresponding to \( E_0 = 0 \) is the row vector with all elements equal to 1.

In general, \( K_{\sigma,\sigma'} \) has a basis of right eigenvectors \( P_n(\sigma) \) and left eigenvectors \( \tilde{P}_n(\sigma) \), which satisfy the following equations

\[
\sum_{\sigma'} K_{\sigma,\sigma'} P_n(\sigma') = E_n P_n(\sigma), \\
\sum_{\sigma} \tilde{P}_n(\sigma) K_{\sigma,\sigma'} = E_n \tilde{P}_n(\sigma'), \\
\sum_{\sigma} \tilde{P}_m(\sigma) P_n(\sigma) = \delta_{m,n}.
\]

To study the spectrum of the operator \( K_{\sigma,\sigma'} \), it is useful to apply a standard symmetrization procedure to map the transition rate matrix \( K_{\sigma,\sigma'} \) into a symmetric operator \( \mathcal{H}_{\sigma,\sigma'} \). The new eigenstates are related to the right and left eigenvectors of \( K_{\sigma,\sigma'} \) by

\[
\sum_{\sigma'} \mathcal{H}_{\sigma,\sigma'} \phi_n(\sigma') = E_n \phi_n(\sigma), \\
\tilde{P}_n(\sigma) = \sqrt{P_{\text{eq}}(\sigma)} \phi_n(\sigma), \\
P_n(\sigma) = \frac{1}{\sqrt{P_{\text{eq}}(\sigma)}} \phi_n(\sigma).
\]

Now let us focus on Eq. (C1), when it describes a process with a time-dependent temperature \( T(t) \). It is possible to expand the probability \( P(\sigma, t) \) on the right eigenvector basis of \( K_{\sigma,\sigma'} \)

\[
P(\sigma, t) = \sum_n c_n(t) P_n(\sigma) e^{\int_t^{t'} E_n(\tau')d\tau'},
\]

where the dependence of the vectors \( P_n \) and the eigenvalues \( E_n \) on time comes from \( T(t) \). Henceforth we will not write it explicitly. Eq. (C1) thus becomes

\[
-\frac{\partial}{\partial t} P(\sigma, t) = -\sum_n c_n(t) E_n P_n(\sigma) + c_n \partial_t P_n(\sigma) - E_n P_n(\sigma)
\]

\[
= \sum_n c_n(t) E_n P_n(\sigma) c_n E_n dt'.
\]
The exponential term is written so that the "trivial" diagonal evolution cancels, leaving an equation for the coefficients $c_n(t)$. After some steps identical to those exploited in the usual quantum adiabatic theorem\textsuperscript{[4]}, we obtain

$$
\dot{c}_m(t) + \sum_n c_n(t) \left( \tilde{P}_m | \partial_t P_n \right) e^{eT(E_n-E_m)(t)} = 0 , \tag{C8}
$$

where $( | )$ is the scalar product in the space of configurations $\sigma$. Now assume that the system at $t = 0$ is in the thermal equilibrium state, so that we can neglect all other contributions in the sum in Eq. (C8). This is accomplished by setting $c_n = \delta_{n,0}$. Therefore we write

$$
-\dot{c}_m(t) \simeq \left( \tilde{P}_m(t) | \partial_t P^{eq}(t) \right) e^{-\frac{T_f}{\tau} \Delta m_0 dt'}, \tag{C9}
$$

exploiting the fact that $P_0(\sigma, t) = P^{eq}(\sigma, t)$ and defining $\Delta m_0 = E_m - E_0 = E_m$. The next step is to change variable in the integral over the gap, in order to extract the adiabatic parameter $T$. If the temperature is a linear decreasing function of time, its derivative can be written as $-\frac{T_1}{\tau}$, where $\tau$ is the total time of the process and $1/\tau$ will be used as the "small" parameter for the power series expansion

$$
e^{-\frac{T_f}{\tau} \Delta m_0 dt'} = e^{-\frac{T_1}{\tau} \frac{\tau}{\tau} \Delta m_0 dt'} . \tag{C10}
$$

Eq. (C9) can be solved by an integration by part to obtain the leading order in powers of $1/\tau$

$$
c_m(\tau) \simeq -\frac{T_1}{\tau} \left[ B(T_f) - B(T_i) e^{-\frac{T_f}{\tau} \Delta m_0 dt'} \right] + O \left( \frac{1}{\tau^2} \right), \tag{C11}
$$

where the temperature coefficients $B(T)$ are defined through $B(T) = \left( \tilde{P}_m | \partial_t P^{eq}(T) \right)$, evaluated at temperature $T$. Since the exponential term in Eq. (C11) adds a vanishing contribution for large values of $\tau$, the solution can be further approximated as

$$
c_m(\tau) \simeq -\frac{T_1}{\tau} \left( \tilde{P}_m(T) | \partial_t P^{eq}(T) \right)_{T=T_1} , \tag{C12}
$$

The difficulty in applying the adiabatic expansion to a master equation lies in the evaluation of the term $\left( \tilde{P}_m(T) | \partial_t P^{eq}(T) \right)$. This is more easily accomplished by means of the symmetrized eigenstates. We start by noticing that

$$
\partial_\beta P^{eq}(\sigma) = P^{eq}(\sigma) \left( H_C(\sigma) - \langle H_C \rangle^{eq} \right) . \tag{C13}
$$

The term with the average energy is hence neglected because it is proportional to the identity matrix and therefore it gives no contribution inside the scalar product. Using the relation between the left eigenvector of $K_{\sigma,\sigma'}$ and the eigenstate of $H$, we can write

$$
\left( \tilde{P}_m | \partial_\beta P^{eq} \right) = -\sum_\sigma \frac{\phi_m(\sigma)}{\sqrt{P^{eq}(\sigma)}} P^{eq}(\sigma) H_C(\sigma)
$$

$$
- \langle \phi_m | H_C | \phi_0 \rangle . \tag{C14}
$$

On the other hand, if we substitute directly the eigenvectors $\phi_m$ instead of $P_m$ and $\tilde{P}_m$, we obtain

$$
\partial_\beta P^{eq}(\sigma) = \frac{\partial_\beta (\sqrt{P^{eq}(\sigma)} \phi_0(\sigma))}{\sqrt{P^{eq}(\sigma)}}
$$

$$
= -\frac{1}{2} \left( H_C - \langle H_C \rangle^{eq} \right) \phi_0(\sigma) + \sqrt{P^{eq}} \partial_\beta \phi_0(\sigma) .
$$

So the scalar product $\left( \tilde{P}_m | \partial_\beta P^{eq} \right)$ can be written as

$$
\left( \tilde{P}_m | \partial_\beta P^{eq} \right) = -\frac{1}{2} \left( \phi_m | H_C | \phi_0 \right) + \langle \phi_m | \partial_\beta \phi_0 \rangle . \tag{C15}
$$

Comparing Eqs. (C14) and (C15), one immediately notices that the term $-\langle \phi_m | H_C | \phi_0 \rangle$ appears in both, which implies that:

$$
-\frac{1}{2} \left( \phi_m | H_C | \phi_0 \right) = \langle \phi_m | \partial_\beta \phi_0 \rangle , \tag{C16}
$$

so that our final expression becomes

$$
\left( \tilde{P}_m | \partial_\beta P^{eq} \right) = 2 \langle \phi_m | \partial_\beta H | \phi_0 \rangle = -\frac{2 \langle \phi_m | \partial_\beta H | \phi_0 \rangle}{\Delta m_0} . \tag{C17}
$$

Therefore it finally possible to write the asymptotic solution for the coefficients $c_m(\tau)$ using the effective Hamiltonian $H$, at a generic final temperature $T_f$

$$
c_m(\tau) \simeq \frac{2T_1}{\tau} \langle \phi_m | \partial_\beta H | \phi_0 \rangle_{T=T_1} . \tag{C18}
$$

The adiabatic expansion for the solution of a classical master equation leads to a final expression analogous to what is obtained for quantum IT dynamics\textsuperscript{[7]}

Although we found numerically more convenient to solve directly the Master Equation, the symmetrized form of the transition matrix $W_{\sigma,\sigma'}$, i.e., the effective "quantum" Hamiltonian $H$, is useful to study the behavior of the adiabatic solution. After the permutation symmetry is exploited to describe the system in terms of its magnetization $m$, the effective Hamiltonian reads

$$
H_{m,m'} = \begin{cases}
\frac{N}{2} \sum_{\alpha=\pm} \frac{\sqrt{1+\frac{m_0+\frac{\Delta m}{2}}{\cosh(\frac{\Delta E_m}{2})}}}{1+\frac{\Delta m}{2}} m' = m , \\
-\frac{N}{2} \sum_{\alpha=\pm} \frac{\sqrt{1+\frac{m_0+\frac{\Delta m}{2}}{\cosh(\frac{\Delta E_m}{2})}}}{1+\frac{\Delta m}{2}} m' = m + 1 .
\end{cases}
$$

In the adiabatic expansion $H_{m,m'}$ appears through its inverse-temperature derivative, which is

$$
\partial_\beta H_{m,m'} = \begin{cases}
\frac{N}{2} \sum_{\alpha=\pm} \frac{\sqrt{1+\frac{m_0+\frac{\Delta m}{2}}{\cosh(\frac{\Delta E_m}{2})}}}{1+\frac{\Delta m}{2}} m' = m , \\
\frac{N}{2} \sum_{\alpha=\pm} \frac{\sqrt{1+\frac{m_0+\frac{\Delta m}{2}}{\cosh(\frac{\Delta E_m}{2})}}}{1+\frac{\Delta m}{2}} m' = m + 1 .
\end{cases}
$$

It is easy to see that if one takes the limit $\beta \to \infty$ ($T \to 0$) all the matrix elements $H_{m,m'}$ vanish exponentially, whence the absence of an adiabatic expansion in powers of $\frac{1}{\tau}$ when $T_i = 0$ in an annealing process.
