What is a Shear Wave

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Abstract. This study shows that the traditional definition of shear wave breaks the shear stress reciprocity. From the displacement equation of motion, the shear wave should be redefined as a wave propagating at shear wave velocity, in which the displacement caused by shear deformation is neither rotational nor divergent. A displacement equation of motion is also obtained from the traditional definition of shear wave. It is concluded that the displacement corresponding to the traditionally defined shear wave is a superposition of rotational displacement field and irrotational displacement field without divergence, which are related to the local rigid body rotation and shear deformation, respectively.
Introduction

In order to study the forces and movements of materials occupying a certain space, the macroscopic medium is considered to satisfy the continuum hypothesis, in which the structures of real fluid and solid are considered to be perfectly continuous and are paid no attention to their molecular structure [1, 2]. Then, continuum mechanics is established to describe the forces and movements of continuum materials based on the basic hypothesis. The continuum mechanics considers that the dynamics of continua still satisfy Newton's law, and the element translation only needs to be considered when describing its conservation of moment of momentum [3, 4, 5]. This means that the possible rotation of nonzero-volume elements constituting continuum is ignored, and the element is treated as particle that only has translation. Based on the above hypothesis, the Cauchy stress tensor, also known as the true stress tensor, is considered to fully describe the stress state at a point under the current configuration and to be a second-order symmetric stress tensor for non-polar materials [6-9].

Continuum mechanics is widely used to describe macroscopic mechanical properties of macroscopic media, such as elastomers [7]. For a linear elastomer, the strain tensor is considered to be a second-order symmetric tensor due to shear stress reciprocity, which can be expressed by the symmetrical component of the displacement gradient [1, 10]. Bringing the stress-strain relationship and geometric equation of linear elastomer into the equation of momentum conservation in the differential form, the wave equation (or the displacement equation of motion) can be obtained [6, 7, 9]. The equation of momentum conservation in the differential form is obtained by the Gaussian flux theorem, with which a surface integral is transformed into a volume integral [7]. This means that the continuum
mechanics considers the stress field to be a non-curled field, which corresponds to the shear stress reciprocity. The wave equation should be an irrotational field. However, the displacement equation of motion described by the traditionally defined shear wave is a rotational field, which means that the equation of conservation of momentum can be a rotational field. This conflicts with the assumption that the stress field is an irrotational field in continuum mechanics. Hence, the theory of elasticity based on continuum mechanics shouldn’t have obtained the traditionally defined shear wave. This indicates that the definition of shear wave has broken the assumption that the rotation of element can be neglected and the stress field is an irrotational field in continuum mechanics. Hence, the shear wave should be redefined. This point seems to haven’t been realized so far.

Although the wave equation unintentionally replaces the shear deformation with the rigid body rotation by considering the displacement field can be treated as the superposition of divergence field and rotational field, the displacement described by the wave equation is still inconsistent with the definition of the shear wave. In the following, I will use isotropic elastomer as an example to define shear waves under the assumption of classical continuum mechanics and show why the traditionally defined shear wave breaks the shear stress reciprocity.

**Derivation of the displacement equation of motion**

Only considering the element translation, for elastomer with small deformation, the equation of motion in differential form is expressed as [7]

\[
\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \sigma = f, \quad (1)
\]
here, \(\dot{\sigma}/\dot{t}\) is the time derivative, \(\nabla\) is the vector operator del, \(\sigma\) is a symmetric second-order stress tensor, \(f\) is body force which is an irrotational field, \(\rho\) is the mass density, \(u\) is the displacement, which can be expressed by the relative displacement between two points in the neighborhood. Equation (1) is obtained by Gauss’s theorem. Therefore, an element is a material volume that the dynamic state of continuum in its subregion can be considered to be constant [11].

Since continuum mechanics doesn’t consider the rotation of elements, the shear stress is reciprocal. Therefore, the strain tensor in the theory of elasticity, which describes the deformation of elastomer under force, must satisfy shear strain reciprocity. In the theory of elasticity, the symmetric part of the displacement gradient is used to express the strain, and the anti-symmetric part of the displacement gradient is treated as the local rigid body rotation without contributing stress [7, 8]. Hence, the stress-strain relationship of elastomer is expressed as follows:

\[
\sigma = Ce, \quad (2)
\]

here, \(e\) is a second-order symmetric strain tensor, and \(C\) is a fourth-order elastic tensor. For isotropic elastomers, the stress-strain relationship in component form is expressed as [7, 9]

\[
\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}, \quad (3)
\]

where, \(\lambda\) and \(\mu\) are Lamé constants, \(\delta\) is the Kronecker delta symbol. The strain tensor is expressed with displacement as:

\[
e = \frac{1}{2}(\nabla u + \nabla u^T), \quad (4)
\]
Here, $u^T$ is the transposition of displacement. The relative displacement between two points in the neighborhood $u$ can be expressed as [12]:

$$u = e \cdot \delta r + \Omega \cdot \delta r,$$

(5)

$$\Omega = \frac{1}{2} \left( \nabla u - \nabla u^T \right),$$

(6)

Where, $\delta r$ is displacement vector of two points in the neighborhood, $\Omega$ is rotation tensor, which is an anti-symmetric tensor. It can be seen from Equation (5) that the displacement field $u^S$ should be an irrotational field if the displacement is only caused by deformation:

$$\nabla \times u^S = 0,$$

(7)

$$u^S = e \cdot \delta r.$$  

(8)

If it is further assumed that the displacement field is caused by pure shear deformation, the displacement field should be an irrotational field with zero divergence:

$$\nabla \cdot u^S = 0.$$  

(9)

This is different from the traditional understanding of displacement field, which is either divergence field or curl field [7, 9, 12, 13].

Substituting Equations (2)-(4) into Equation (1) and ignoring the body force, the equation of motion expressed by displacement can be obtained:

$$(\lambda + \mu) \nabla (\nabla \cdot u) + \mu \nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2}.$$  

(10)

This equation is also called elastic wave equation or Navier’s equation.
**Definition of shear wave**

The stress-strain relationship (Equation (2)) is often called generalized Hooke's law. Then, Equation (10) can also be regarded as the motion equation of the generalized spring proton model. This means that the displacement in Equation (10) should be caused only by deformation. Therefore, the displacement field in Equation (10) should be irrotational. When only shear deformation occurs to the elastomer, Equation (10) can be reduced to the following equation:

\[
\mu \nabla^2 \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}.
\]  

Equation (11)

The displacement field in Equation (11) is neither rotational nor divergent. From Equation (11), it is obtained that a shear wave should be defined as a wave propagating at shear wave velocity, in which the displacement field caused by shear deformation is neither rotational nor divergent. To some extent, the shear wave can also be regarded as a special longitudinal wave with shear wave velocity and vertical strain not equal to zero. From the definition of shear wave in Equation (10), the displacement perpendicular to the propagation direction can both be non-zero as long as the displacement field is neither rotational nor divergent.

The definition is different from the traditionally defined shear wave.

In the traditional understanding of displacement field, the displacement field can be expressed as the superposition of curl field and divergence field, and the following formula is always true [7, 9, 12, 13]:

\[
\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}.
\]  

Equation (12)

Substituting Equation (12) in to Equation (10), the following formula is obtained:
\begin{equation}
(\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}.
\end{equation}

Equation (12) is considered to be an alternative form of the displacement equation of motion. As the local rigid body rotation does not contribute to force, the displacement in Equation (10) is independent to the rotation of the local rigid body. Therefore, Equation (11) is not true. With Equation (11), the shear deformation, which produces a displacement field without rotation or divergence, is actually replaced by the local rigid body rotation. Hence, the wave propagating at shear wave velocity defined in Equation (10) is different from the shear wave expressed with Equation (13). As the relation between displacement field and rotation tensor is as follows [4]:

\begin{equation}
\nabla \times \mathbf{u} = \mathbf{\varepsilon} : \mathbf{\Omega}.
\end{equation}

Equation (13) can be also written as:

\begin{equation}
(\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \nabla \times \left( \frac{1}{2} \mathbf{\varepsilon} : 2\mu \mathbf{\Omega} \right) = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}.
\end{equation}

From Equation (15), it is seen that the local rigid body rotation contributes to stress. This means that Equation (13) breaks the hypothesis that the local rigid body rotation does not contribute to stress in elastic theory. Therefore, the shear wave defined by Equation (13) corresponds to the rotation of elements. Under the assumption of classical continuum mechanics, in which the element is treated as particle that only has translation, the shear wave defined by Equation (13) does not exist.

In order to explain the traditional definition of shear wave without breaking the traditional assumption of continuum mechanics, we can think that the traditional definition of shear wave omits the displacement along the propagation direction. The movement of
the particle along the vertical direction at the equilibrium position is caused by the stretching and contraction of the elastic body aroused by the deformation in the vertical propagation direction. This omission misleads the understanding of the definition of shear waves.

**The wave equation from traditionally defined shear wave**

Although the traditional definition of shear wave can be explained under the assumption of shear stress reciprocity, the interpretation is mathematically unacceptable. The shear stress reciprocity is obtained from the assumption that the elements forming continuum can be considered as particles whose rotation can be neglected and motion can be described by Newton's law. The assumption has been widely accepted by researchers [1, 2, 5-10]. However, according to the traditional definition of shear wave, the displacement field caused by shear wave is a rotational field. This means that the acceleration field and stress field are rotational fields. In other words, the traditional definition of shear wave breaks the assumption of the motion of elements in the classical continuum mechanics. With an open attitude, below I interpret the deformation of elastomer by assuming that the traditional definition of shear wave is objective.

The traditional definition of shear wave considers that the displacement field caused by shear wave is perpendicular to its propagation direction. Suppose a plane shear wave propagates along the $x_1$ coordinate and the displacement is along $x_2$, it is obtained that the strain tensor and stress tensor at one point have only one component ($e_{12} = u_{2,1}$ and $\sigma_{12} = \mu u_{2,1}$). By decomposing the stress and strain tensors for the traditionally defined shear wave, it can be easily obtained that the traditionally defined shear wave is a superposition of shear
deformation and local rigid body rotation. Therefore, the traditionally defined shear wave in elastomer cannot be explained with Equation (10) or Equation (13) due to Equation (10) only considered the shear deformation and Equation (13) only considered the local rigid body rotation. From the traditional definition of shear wave, it is obtained that the element rotation should be consider to describe the motion of continuum and the shear stress reciprocity shouldn’t be a prerequisite. The displacement equation of motion including the traditionally defined shear wave should be expressed as:

\[
(\lambda + \mu) \nabla (\nabla \cdot u) + \mu \nabla^2 u^S - \mu \nabla \times \nabla \times u + f = \rho \frac{\partial^2 u}{\partial t^2},
\]

(16)

where, \( u^S \) is the irrotational displacement, which is caused by the deformation.

**Conclusion**

In summary, this study shows that shear waves should be defined as waves propagating at shear wave velocity, in which the displacement field caused by shear deformation is neither rotational nor divergent. This definition is different with the traditionally defined shear wave. By explain the stress and strain states, it is obtained that the traditional definition of shear wave breaks the shear stress reciprocity.

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