NUMERICAL RADIUS INEQUALITIES OF OPERATOR MATRICES FROM A NEW NORM ON $B(H)$

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Abstract. This paper is a continuation of a recent work on a new norm, christened the $(\alpha, \beta)$-norm, on the space of bounded linear operators on a Hilbert space. We obtain some upper bounds for the said norm of $n \times n$ operator matrices. As an application of the present study, we estimate bounds for the numerical radius and the usual operator norm of $n \times n$ operator matrices, which generalize the existing ones.

1. Introduction

The purpose of the present article is to study the bounds for the newly introduced 2010 $(\alpha, \beta)$-norm of $n \times n$ operator matrices, from which we obtain bounds for the numerical radius of $n \times n$ operator matrices. Let us first introduce the following notations and terminologies to be used throughout the article.

Let $H_i, H_j$ be two complex Hilbert spaces with usual inner product $\langle \cdot, \cdot \rangle$ and let $B(H_i, H_j)$ denote the space of all bounded linear operators from $H_i$ to $H_j$. If $H_i = H_j = H$ then we write $B(H_i, H_j) = B(H)$. For $T \in B(H)$, we write $Re(T)$ and $Im(T)$ for the real part of $T$ and the imaginary part of $T$, respectively, i.e., $Re(T) = \frac{T + T^*}{2}$ and $Im(T) = \frac{T - T^*}{2i}$. Let $T^*$ denote the adjoint of $T$ and let $|T|$ be the positive operator $(T^*T)^{1/2}$. Let $\sigma(T)$ denote the spectrum of $T$. The spectral radius of $T$, denoted by $r(T)$, is defined by $r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}$. The numerical range of $T$, denoted by $W(T)$, is defined as $W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}$. The usual operator norm and the numerical radius of $T$, denoted by $\|T\|$ and $w(T)$, respectively, are defined as $\|T\| = \sup \{ \|Tx\| : x \in H, \|x\| = 1 \}$ and $w(T) = \sup \{ |c| : c \in W(T) \}$. Let $M_T$ denote the usual operator norm attainment set of $T$, i.e., $M_T = \{ x \in H : \|Tx\| = \|T\|, \|x\| = 1 \}$. It is well-known that the numerical radius defines a norm on $B(H)$ and is equivalent to the usual operator norm, satisfying that for $T \in B(H)$,

$$\frac{1}{2} \|T\| \leq w(T) \leq \|T\|.$$

The study of the numerical range of an operator and the associated numerical radius inequalities are an important area of research in operator theory and it has attracted many mathematicians [1, 2, 3, 4, 5, 8] over the years. With an aim to develop better upper and lower bounds for the numerical radius, a new norm named

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as the \((\alpha, \beta)\)-norm, was introduced on \(B(\mathcal{H})\) in [10]. For \(T \in B(\mathcal{H})\), the \((\alpha, \beta)\)-norm of \(T\), denoted by \(\|T\|_{\alpha,\beta}\), is defined as:
\[
\|T\|_{\alpha,\beta} = \sup \left\{ \sqrt{\alpha \|Tx\|^2 + \beta \|Tx\|^2} : x \in \mathcal{H}, \|x\| = 1 \right\},
\]
where \(\alpha, \beta\) are real positive constants with \((\alpha, \beta) \neq (0, 0)\). We note that if \(\alpha = 1, \beta = 0\) then \(\|T\|_{\alpha,\beta} = \|T\|\), and if \(\alpha = 0, \beta = 1\) then \(\|T\|_{\alpha,\beta} = \|T\|\). Also, if we consider \(\alpha = \beta = 1\), then we have the modified Davis-Wielandt radius of \(T\), that is, \(\|T\|_{1,1} = dw^*(T)\), (see [6]). In this article, we consider \(\alpha + \beta = 1\), i.e., \(\beta = 1 - \alpha\) and explore the \(\alpha\)-norm of \(n \times n\) operator matrices, where the \(\alpha\)-norm of \(T\) is defined as:
\[
\|T\|_{\alpha} = \sup \left\{ \sqrt{\alpha \|Tx\|^2 + (1 - \alpha)\|Tx\|^2} : x \in \mathcal{H}, \|x\| = 1 \right\}.
\]

We compute the exact value of the \(\alpha\)-norm of \(2 \times 2\) operator matrices in \(B(\mathcal{H} \oplus \mathcal{H})\) of the form \(
\begin{pmatrix}
0 & X \\
0 & 0
\end{pmatrix}
\), where \(X \in B(\mathcal{H})\). We obtain some upper bounds for the \(\alpha\)-norm of \(n \times n\) operator matrices, which generalize the existing numerical radius inequalities and the usual operator norm inequalities of \(n \times n\) operator matrices. As an application our results, we estimate new upper bounds for the numerical radius and the usual operator norm of \(n \times n\) operator matrices.

2. Main results

We begin this section with the following proposition, the proof of which follows from the weakly unitarily invariant property of the \(\alpha\)-norm, i.e., for \(T \in B(\mathcal{H})\), \(\|U^*TU\|_{\alpha} = \|T\|_{\alpha}\) for every unitary operator \(U \in B(\mathcal{H})\) (see [10, Prop. 2.6]).

**Proposition 2.1.** Let \(A, B \in B(\mathcal{H})\). Then the following results hold:

(a) \[\left\| \begin{pmatrix}
0 & A \\
e^{i\theta} B & 0
\end{pmatrix}\right\|_{\alpha} = \left\| \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}\right\|_{\alpha}, \text{ for every } \theta \in \mathbb{R}.\]

(b) \[\left\| \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}\right\|_{\alpha} = \left\| \begin{pmatrix}
0 & B \\
A & 0
\end{pmatrix}\right\|_{\alpha}.\]

(c) \[\left\| \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}\right\|_{\alpha} = \left\| \begin{pmatrix}
B & 0 \\
0 & A
\end{pmatrix}\right\|_{\alpha}.\]

(d) \[\left\| \begin{pmatrix}
A & B \\
B & A
\end{pmatrix}\right\|_{\alpha} = \left\| \begin{pmatrix}
A - B & 0 \\
0 & A + B
\end{pmatrix}\right\|_{\alpha}.\]

Next, we estimate upper and lower bounds for the \(\alpha\)-norm of \(2 \times 2\) operator matrices in \(B(\mathcal{H} \oplus \mathcal{H})\) of the form \(
\begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix}
\), where \(X, Y \in B(\mathcal{H})\). Let us first note the following inequality for \(X \in B(\mathcal{H})\),
\[
\alpha \|Xx\|^2 + (1 - \alpha)\|Xx\|^2 \leq \|X\|^2 \|x\|^2 \text{ for all } x \in \mathcal{H} \text{ with } \|x\| \leq 1.
\]

**Theorem 2.2.** Let \(X, Y \in B(\mathcal{H})\). Then the following inequalities hold:

(i) \[\max \{\|X\|_{\alpha}, \|Y\|_{\alpha}\} \leq \left\| \begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix}\right\|_{\alpha}
\leq \max \left\{ \sqrt{\|X\|^2 + \alpha w^2(X)}, \sqrt{\|Y\|^2 + \alpha w^2(Y)} \right\}
\leq \sqrt{2} \max \{\|X\|_{\alpha}, \|Y\|_{\alpha}\}.\]
(ii) \[ \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_\alpha \leq \sqrt{\max \{\|X\|_\alpha^2,\|Y\|_\alpha^2\} + \alpha w(X)w(Y)}. \]

(iii) \[ \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_\alpha \leq \|X\|_\alpha + \|Y\|_\alpha. \]

**Proof.** (i). Let \( T = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \). Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \) and let \( \tilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} \). Clearly, \( \|\tilde{x}\| = 1 \). Therefore, we have,

\[ \sqrt{\alpha\|Xx, x\|^2 + (1 - \alpha)\|Xx\|^2} = \sqrt{\alpha\|T\tilde{x}, \tilde{x}\|^2 + (1 - \alpha)\|T\tilde{x}\|^2} \leq \|T\|_\alpha. \]

Taking supremum over all unit vectors in \( \mathcal{H} \), we get,

\[ \|X\|_\alpha \leq \|T\|_\alpha. \]

Similarly, it can be proved that

\[ \|Y\|_\alpha \leq \|T\|_\alpha. \]

Combining the above two inequalities, we get the first inequality in (i). Let us now prove the second inequality in (i). Let \( z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} \) with \( \|z\| = 1 \), i.e., \( \|x\|^2 + \|y\|^2 = 1 \). Then we have,

\[ \alpha\|Tz, z\|^2 + (1 - \alpha)\|Tz\|^2 = \alpha \|Xx, x\|^2 + (1 - \alpha)\|Xx\|^2 + \|Yy\|^2 \]

\[ \leq \alpha \|Xx, x\|^2 + \alpha\|Xx\|^2 + (1 - \alpha)\|Yy\|^2 + \alpha\|Yy\|^2 \]

\[ \leq \|X\|_\alpha^2 \|x\|^2 + \|Y\|_\alpha^2 \|y\|^2 \]

\[ + \alpha\|w(X)\|\|x\|^2 + \alpha\|w(Y)\|\|y\|^2, \text{ since, } \|x\| \leq 1, \|y\| \leq 1 \]

\[ = \left\{ \|X\|_\alpha^2 + \alpha w^2(X) \right\} \|x\|^2 + \left\{ \|Y\|_\alpha^2 + \alpha w^2(Y) \right\} \|y\|^2 \]

\[ \leq \max \left\{ \|X\|_\alpha^2, \|Y\|_\alpha^2 \right\} + \alpha \|w(X)\|\|y\|^2 \]

Therefore, taking supremum over all unit vectors in \( \mathcal{H} \oplus \mathcal{H} \), we get the second inequality in (i). The remaining inequality in (i) follows from the inequalities \( \alpha w^2(X) \leq \|X\|_\alpha^2 \) and \( \alpha w^2(Y) \leq \|Y\|_\alpha^2 \). This completes the proof of (i).

(ii). From \( \alpha\|Tz, z\|^2 + (1 - \alpha)\|Tz\|^2 \leq \alpha \left( \|Xx, x\| + \|Yy, y\| \right)^2 + (1 - \alpha) \left( \|Xx\|^2 + \|Yy\|^2 \right) \), we get

\[ \alpha\|Tz, z\|^2 + (1 - \alpha)\|Tz\|^2 \leq \alpha \|Xx, x\|^2 + (1 - \alpha)\|Xx\|^2 + \alpha\|Yy, y\|^2 + (1 - \alpha)\|Yy\|^2 \]

\[ + 2\alpha \|Xx, x\| \|Yy, y\| \]

\[ \leq \alpha \|Xx, x\|^2 + \alpha\|Yy, y\|^2 + 2\alpha \|Xx\|\|y\| \]

\[ \leq \|X\|_\alpha^2 \|x\|^2 + \|Y\|_\alpha^2 \|y\|^2 \]

\[ + 2\alpha \|Xx\|w(Y)\|y\|, \text{ since, } \|x\| \leq 1, \|y\| \leq 1 \]

\[ \leq \max \left\{ \|X\|_\alpha^2, \|Y\|_\alpha^2 \right\} + \alpha \|w(X)\|\|y\|. \]

Taking supremum over all unit vectors in \( \mathcal{H} \oplus \mathcal{H} \), we get the inequality in (ii).
Lemma 2.4. The inequality in (iii) follows from the triangle inequality of the $\alpha$-norm, and by using the inequality in (ii).

In the following theorem, we obtain the exact value of the $\alpha$-norm of $2 \times 2$ operator matrices in $B(H \oplus H)$ of the form \[
\begin{pmatrix}
0 & X \\
0 & 0
\end{pmatrix},
\]
where $X \in B(H)$.

**Theorem 2.3.** Let $X \in B(H)$. Then

$$
\left\| \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \right\|_{\alpha} = \begin{cases} 
\frac{1}{\sqrt{\alpha}} \|X\| & \text{if } \alpha > \frac{1}{2}, \\
\sqrt{1 - \alpha} \|X\| & \text{if } \alpha \leq \frac{1}{2}.
\end{cases}
$$

**Proof.** Let $T = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in H \oplus H$ with $\|z\| = 1$, i.e., $\|x\|^2 + \|y\|^2 = 1$. Then $\langle Tz, z \rangle = \langle Xy, x \rangle$ and $\|Tz\| = \|Xy\|$. Now we have,

$$
\|T\|_{\alpha}^2 = \sup_{\|z\|=1} (\alpha \|Tz, z\|^2 + (1 - \alpha) \|Tz\|^2) = \sup_{\|x\|^2 + \|y\|^2 = 1} (\alpha \|Xy, x\|^2 + (1 - \alpha) \|Xy\|^2)
\leq \sup_{\|x\|^2 + \|y\|^2 = 1} (\alpha \|X\|^2 \|y\|^2 \|x\|^2 + (1 - \alpha) \|X\|^2 \|y\|^2)
= \sup_{\theta \in [0, \frac{\pi}{2}]} \|X\|^2 \sin^2 \theta (\alpha \cos^2 \theta + (1 - \alpha)).
$$

First we consider the case $\alpha > \frac{1}{2}$. Then

$$
\sup_{\theta \in [0, \frac{\pi}{2}]} \|X\|^2 \sin^2 \theta (\alpha \cos^2 \theta + (1 - \alpha)) = \frac{1}{4\alpha} \|X\|^2.
$$

Therefore, $\|T\|_{\alpha}^2 \leq \frac{1}{4\alpha} \|X\|^2$. We claim that there exists a sequence $\{z_n\}$ in $H \oplus H$ with $\|z_n\| = 1$ such that

$$
\lim_{n \to \infty} (\alpha \|Tz_n, z_n\|^2 + (1 - \alpha) \|Tz_n\|^2) = \frac{1}{4\alpha} \|X\|^2.
$$

Clearly, there exists a sequence $\{y_n\}$ in $H$ with $\|y_n\| = 1$ such that $\lim_{n \to \infty} \|XYn\| = \|X\|$. Let $z_n = \frac{1}{\sqrt{\|Xy_n\|^2 + k^2}} \begin{pmatrix} Xy_n \\ ky_n \end{pmatrix}$, where $k = \sqrt{\frac{1}{2\alpha - 1} \|X\|}$. Then

$$
\lim_{n \to \infty} \alpha \|Tz_n, z_n\|^2 + (1 - \alpha) \|Tz_n\|^2 = \frac{1}{4\alpha} \|X\|^2.
$$

Therefore, $\|T\|_{\alpha} = \frac{1}{2\sqrt{\alpha}} \|X\|$ if $\alpha > \frac{1}{2}$.

Next we consider the case $\alpha \leq \frac{1}{2}$. Then

$$
\sup_{\theta \in [0, \frac{\pi}{2}]} \|X\|^2 \sin^2 \theta (\alpha \cos^2 \theta + (1 - \alpha)) = (1 - \alpha) \|X\|^2
$$

Therefore, $\|T\|_{\alpha}^2 \leq (1 - \alpha) \|X\|^2$. Proceeding as before, we can show that there exists a sequence $\{z_n\}$, $\|z_n\| = 1$ such that $\lim_{n \to \infty} (\alpha \|Tz_n, z_n\|^2 + (1 - \alpha) \|Tz_n\|^2) = (1 - \alpha) \|X\|^2$. Therefore, $\|T\|_{\alpha} = \sqrt{(1 - \alpha) \|X\|}$ if $\alpha \leq \frac{1}{2}$. \(\square\)

Our next goal is to obtain upper bounds for the $\alpha$-norm of $n \times n$ operator matrices in $B(\oplus_{i=1}^n H_i)$. We require the following lemmas for our purpose.

**Lemma 2.4.** ([7, p. 44]) Let $T = (t_{ij}) \in M_n(\mathbb{C})$ with $t_{ij} \geq 0$ for all $i, j$. Then

$$
w(T) = r(Re(T)) = \|Re(T)\|.
$$
Lemma 2.5. ([9]) Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let $x \in \mathcal{H}$. Then

$$|\langle Tx, x \rangle| \leq \|T\| \langle x, x \rangle.$$

Lemma 2.6. ([9]) Let $T \in \mathcal{B}(\mathcal{H})$ with $T \geq 0$ and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$\langle Tx, x \rangle^p \leq \langle T^p x, x \rangle, \quad \forall p \geq 1.$$

Lemma 2.7. ([10, Th. 2.1]) Let $T \in \mathcal{B}(\mathcal{H})$. Then the following inequalities hold:

$$w(T) \leq \|T\| \leq \sqrt{4 - 3\alpha} \|T\|,$$

where

$$\max \left\{ \frac{1}{2}, \sqrt{(1 - \alpha)} \right\} \|T\| \leq \|T\| \leq \|T\|,$$

Now we are in a position to prove the following inequality.

Theorem 2.8. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be Hilbert spaces. Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$. Then

$$\|T\| \alpha \leq \sqrt{\|\alpha R\|^2 + (1 - \alpha)|S|^2},$$

where $R = (r_{ij})_{n \times n}$, $r_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j \\ \frac{1}{2}(\|T_{ij}\| + \|T_{ji}\|) & \text{if } i \neq j \end{cases}$

and $S = (s_{ij})_{n \times n}$, $s_{ij} = \|T_{ij}\|$.

Proof. Let $x = (x_1, x_2, \ldots, x_n) \in \oplus_{i=1}^n \mathcal{H}_i$ with $\|x\| = 1$ and let $\tilde{x} = (\|x_1\|, \|x_2\|, \ldots, \|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Now,

$$|\langle Tx, x \rangle| = \left| \sum_{i,j=1}^n \langle T_{ij} x_j, x_i \rangle \right|$$

$$\leq \sum_{i,j=1}^n |\langle T_{ij} x_j, x_i \rangle|$$

$$\leq \sum_{i=1}^n |\langle T_{ii} x_i, x_i \rangle| + \sum_{i,j=1}^n |\langle T_{ij} x_j, x_i \rangle|$$

$$\leq \sum_{i=1}^n w(T_{ii}) \|x_i\|^2 + \sum_{i,j=1}^n \|T_{ij}\| \|x_j\| \|x_i\|$$

$$= \sum_{i,j=1}^n t_{ij} \|x_j\| \|x_i\|$$

$$= \langle \tilde{T} \tilde{x}, \tilde{x} \rangle$$

$$= \langle \text{Re}(\tilde{T}) \tilde{x}, \tilde{x} \rangle + i \langle \text{Im}(\tilde{T}) \tilde{x}, \tilde{x} \rangle,$$

where $\tilde{T} = (\tilde{t}_{ij})$, $\tilde{t}_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$

Clearly, $\langle \text{Im}(\tilde{T}) \tilde{x}, \tilde{x} \rangle = 0$. So by using Lemma 2.5 and Lemma 2.6, we get

$$|\langle Tx, x \rangle| \leq \langle \text{Re}(\tilde{T}) \tilde{x}, \tilde{x} \rangle \leq \langle |\text{Re}(\tilde{T})| \tilde{x}, \tilde{x} \rangle \leq \langle |\text{Re}(\tilde{T})|^2 \tilde{x}, \tilde{x} \rangle = \langle |R|^2 \tilde{x}, \tilde{x} \rangle.$$
Also,
\[ \|Tx\|^2 = |\langle Tx, Tx \rangle| = \left| \sum_{i,j,k=1}^{n} \langle T_{kj}x_j, T_{ki}x_i \rangle \right| \leq \sum_{i,j,k=1}^{n} |\langle T_{kj}x_j, T_{ki}x_i \rangle| \leq \sum_{i,j,k=1}^{n} |\langle T_{kj}^* T_{ki}x_j, x_i \rangle| \leq \sum_{i,j,k=1}^{n} \|T_{ki}\| \|T_{kj}\| \|x_j\| \|x_i\| = \langle |S|^2 \tilde{x}, \tilde{x} \rangle. \]

Therefore,\[
\alpha |\langle Tx, x \rangle|^2 + (1 - \alpha) \|Tx\|^2 \leq \alpha \langle \|R\|^2 \tilde{x}, \tilde{x} \rangle + (1 - \alpha) \langle |S|^2 \tilde{x}, \tilde{x} \rangle = \langle (\alpha \|R\|^2 + (1 - \alpha) |S|^2) \tilde{x}, \tilde{x} \rangle \leq \|\alpha \|R\|^2 + (1 - \alpha) |S|^2 \|, \]
Taking supremum over all unit vectors in \( \bigoplus_{i=1}^{n} H_i \), we get the desired inequality. \( \square \)

As a consequence of Theorem 2.8, the following numerical radius inequality and the usual operator norm inequality can be proved quite easily.

**Corollary 2.9.** Let \( H_1, H_2, \ldots, H_n \) be Hilbert spaces. Let \( T = (T_{ij}) \) be an \( n \times n \) operator matrix, where \( T_{ij} \in B(H_j, H_i) \). Then

(i) \( w(T) \leq \min_{0 \leq \alpha \leq 1} \sqrt{|\alpha \|R\|^2 + (1 - \alpha) |S|^2|} \leq w(\tilde{T}) \)

(ii) \( \|T\| \leq \min_{0 \leq \alpha \leq 1} \frac{1}{\sqrt{\frac{1}{2} \sqrt{1 - \alpha}}} \sqrt{|\alpha \|R\|^2 + (1 - \alpha) |S|^2|} \leq \|S\| \),

where \( \tilde{T} = (\tilde{T}_{ij})_{n \times n}, \tilde{T}_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j \\ \|T_{ij}\| & \text{if } i \neq j \end{cases} \)

and \( R, S \) are same as described in Theorem 2.8.

We would like to note that the inequalities in [1, Th. 1] and [8, Th. 1.1] follow from (i) and (ii) of Corollary 2.9, respectively.

In our next result, we obtain an upper bound for the \( \alpha \)-norm of \( n \times n \) operator matrices in terms of non-negative continuous functions on \([0, \infty)\). First we need the following lemma.

**Lemma 2.10.** ([9, Th. 5]) Let \( T \in B(H) \) and let \( f \) and \( g \) be two non-negative continuous functions on \([0, \infty)\) such that \( f(t)g(t) = t, \ \forall t \in [0, \infty) \). Then
\[ |\langle Tx, y \rangle| \leq \|f(|T|)x\| \|g(|T^*|)y\|, \ \forall x, y \in H. \]
Theorem 2.11. Let \( T = (T_{ij}) \) be an \( n \times n \) operator matrix, where \( T_{ij} \in B(H) \). Let \( f \) and \( g \) be two non-negative continuous functions on \([0, \infty)\) such that \( f(t)g(t) = t \), \( \forall t \geq 0 \). Then
\[
\|T\|_\alpha \leq \sqrt{\|\alpha|R\|^2 + (1 - \alpha)|S|^2}.
\]
where \( R = (r_{ij})_{n \times n} \), \( r_{ij} = \frac{1}{2} \left( \|f^2(T_{ij})\|^\frac{1}{2} \|g^2(T_{ij})\|^\frac{1}{2} + \|f^2(T_{ji})\|^\frac{1}{2} \|g^2(T_{ji})\|^\frac{1}{2} \right) \)
and \( S = (s_{ij})_{n \times n} \), \( s_{ij} = \|T_{ij}\| \).

Proof. Let \( x = (x_1, x_2, \ldots, x_n) \in \oplus_{i=1}^n H \) with \( \|x\| = 1 \) and let \( \hat{x} = (\|x_1\|, \|x_2\|, \ldots, \|x_n\|) \).
Clearly, \( \hat{x} \) is a unit vector in \( C^n \). Using Lemma 2.10, we get that
\[
|\langle Tx, x \rangle| = \left| \sum_{i,j=1}^n \langle T_{ij}x_j, x_i \rangle \right|
\leq \sum_{i,j=1}^n |\langle T_{ij}x_j, x_i \rangle|
\leq \sum_{i,j=1}^n \|f(|T_{ij}|)x_j\| \|g(|T_{ij}^*|)x_i\|
= \sum_{i,j=1}^n \|f^2(|T_{ij}|)x_j\|^\frac{1}{2} \|g^2(|T_{ij}^*|)x_i\|^\frac{1}{2}
\leq \sum_{i,j=1}^n \|f^2(|T_{ij}|)\|^\frac{1}{2} \|g^2(|T_{ij}^*|)\|^\frac{1}{2} \|x_i\| \|x_j\|
= \sum_{i,j=1}^n \|T_{ij}\| \|x_i\| \|x_j\|
= \langle \hat{T} \hat{x}, \hat{x} \rangle
\]
where \( \hat{T} = (\hat{t}_{ij}) \), \( \hat{t}_{ij} = \|f^2(|T_{ij}|)\|^\frac{1}{2} \|g^2(|T_{ij}^*|)\|^\frac{1}{2} \).

Proceeding similarly as in the proof of Theorem 2.8, we get
\[
|\langle Tx, x \rangle|^2 \leq \langle |R|^2 \hat{x}, \hat{x} \rangle \quad \text{and} \quad \|Tx\|^2 \leq \langle |S|^2 \hat{x}, \hat{x} \rangle.
\]

Therefore,
\[
\alpha|\langle Tx, x \rangle|^2 + (1 - \alpha)\|Tx\|^2 \leq \sum_{0 \leq \alpha \leq 1} \sqrt{\alpha|R|^2 + (1 - \alpha)|S|^2} \leq w(\hat{T}),
\]
Taking supremum over all unit vectors in \( \oplus_{i=1}^n H \), we get the desired inequality. \( \square \)

The following numerical radius inequality is an easy consequence of Theorem 2.11.

Corollary 2.12. Let \( T = (T_{ij}) \) be an \( n \times n \) operator matrix, where \( T_{ij} \in B(H) \). Let \( f \) and \( g \) be non-negative continuous functions on \([0, \infty)\) such that \( f(t)g(t) = t \), \( \forall t \geq 0 \). Then
\[
w(T) \leq \min_{0 \leq \alpha \leq 1} \sqrt{\alpha|R|^2 + (1 - \alpha)|S|^2} \leq w(\hat{T}),
\]
where \( R, S \) are same as described in Theorem 2.11 and \( \hat{T} = (\hat{t}_{ij})_{n \times n} \), \( \hat{t}_{ij} = \|f^2(|T_{ij}|)\|^\frac{1}{2} \|g^2(|T_{ij}^*|)\|^\frac{1}{2} \).
We would like to note that the inequality in [4, Th. 3.1] follows from Corollary 2.12.

In our next theorem, we obtain an upper bound for the $\alpha$-norm of $n \times n$ operator matrices.

**Theorem 2.13.** Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in B(\mathcal{H})$. Let $f, g$ be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$, $\forall t \geq 0$. If $p \geq 1$, then

$$||T||_\alpha^p \leq \sqrt{\alpha(R)^{2p} + (1 - \alpha)(S)^{2p}},$$

where $r_{ij} = \left\{ \begin{array}{ll} \frac{1}{2} \left\| f^2(|T_{ii}|) + g^2(|T_{ii}|) \right\| & \text{if } i = j \\ \left( \frac{1}{2} \left\| f^2(|T_{ij}|) \right\|^2 + \left\| g^2(|T_{ij}|) \right\|^2 \right)^{\frac{1}{2}} & \text{if } i \neq j, \end{array} \right.$

$R = (r_{ij})_{n \times n}$ and $S = (s_{ij})_{n \times n}$.

**Proof.** Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ with $||x|| = 1$ and let $\tilde{x} = (||x_1||, ||x_2||, \ldots, ||x_n||)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Using Lemma 2.10, we get that

$$|\langle Tx, x \rangle| = \sum_{i,j=1}^{n} \langle T_{ij} x_j, x_i \rangle$$

$$\leq \sum_{i,j=1}^{n} |\langle T_{ij} x_j, x_i \rangle|$$

$$\leq \sum_{i,j=1}^{n} \| f(|T_{ij}|) \| |x_j| \| g(|T_{ij}^*|) \| x_i \|$$

$$= \sum_{i,j=1}^{n} \left( f^2(|T_{ij}|) x_j \right)^{\frac{1}{2}} \left( g^2(|T_{ij}^*|) x_i \right)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} \frac{1}{2} \left( f^2(|T_{ii}|) x_i x_i + g^2(|T_{ii}^*|) x_i x_i \right)$$

$$+ \sum_{i,j=1, i \neq j}^{n} \left( f^2(|T_{ij}|) x_j x_i \right)^{\frac{1}{2}} \left( g^2(|T_{ij}^*|) x_i x_i \right)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} \frac{1}{2} \left( f^2(|T_{ii}|) + g^2(|T_{ii}^*|) \right) x_i x_i$$

$$+ \sum_{i,j=1, i \neq j}^{n} \left( f^2(|T_{ij}|) x_j x_i \right)^{\frac{1}{2}} \left( g^2(|T_{ij}^*|) x_i x_i \right)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} \frac{1}{2} \left( f^2(|T_{ii}|) + g^2(|T_{ii}^*|) \right) ||x_i||^2$$

$$+ \sum_{i,j=1, i \neq j}^{n} \left( f^2(|T_{ij}|) \right)^{\frac{1}{2}} \left( g^2(|T_{ij}^*|) \right)^{\frac{1}{2}} ||x_i|| ||x_j||$$

$$\leq \sqrt{\alpha(R)^{2p} + (1 - \alpha)(S)^{2p}},$$
Now proceeding similarly as in the proof of Theorem 2.14 and using Lemma 2.6, we obtain
\[
\|T x\|^{2 p} \leq (\|S\|^{2 p} x, 0) \leq (\|S\|^{2 p} x, x).
\]
By convexity of \(t^p, p \geq 1\), it follows that
\[
(\alpha \|T x\|^2 + (1 - \alpha)\|T x\|^{2 p}) \leq \left(\alpha (\|T x\|^{2 p} + (1 - \alpha)\|T x\|^{2 p})\right)
\]
\[
\leq \left(\alpha (\|R\|^{2 p} x, x) + (1 - \alpha) (\|S\|^{2 p} x, x)\right)
\]
\[
= \left(\alpha \|R\|^{2 p} + (1 - \alpha) \|S\|^{2 p}\right) x, x)
\]
\[
\leq \|\alpha \|R\|^{2 p} + (1 - \alpha) \|S\|^{2 p}\|.
\]
Therefore, taking supremum over all unit vectors in \(\oplus_{i=1}^{n} \mathcal{H}\), we get the desired inequality. \(\square\)

We simply state the following result and omit its proof, as it can be completed using similar arguments as given in the proof of Theorem 2.13.

**Theorem 2.14.** Let \(T = (T_{ij})\) be an \(n \times n\) operator matrix, where \(T_{ij} \in \mathcal{B}(\mathcal{H})\). Let \(f\) and \(g\) be two non-negative continuous functions on \([0, \infty)\) such that \(f(t)g(t) = t\), \(\forall t \geq 0\). Then
\[
\|T\|_{\alpha} \leq \sqrt{\|\alpha \|R\|^2 + (1 - \alpha) \|S\|^2},
\]
where \(r_{ij} = \begin{cases} \frac{1}{2} \|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\| & \text{if } i = j \\ \frac{1}{2} \|f^2(|T_{ij}|)\| \frac{1}{2} \|g^2(|T_{ji}^*|)\| + \frac{1}{2} \|f^2(|T_{ji}|)\| \frac{1}{2} \|g^2(|T_{ji}^*|)\| & \text{if } i \neq j \end{cases}\]
\(R = (r_{ij})_{n \times n}\) and \(S = (s_{ij})_{n \times n}\), \(s_{ij} = \|T_{ij}\|\).

The following numerical radius inequality follows easily from Theorem 2.14 by using Lemma 2.7.
Corollary 2.15. Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H})$. Let $f$ and $g$ be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$, $\forall \ t \geq 0$. Then

$$w(T) \leq \min_{0 \leq \alpha \leq 1} \sqrt{\|\alpha|R|^2 + (1 - \alpha)|S|^2\|},$$

where $R, S$ are same as described in Theorem 2.14.

Remark 2.16. In particular, if we consider $\alpha = 1$ in Corollary 2.15 then using Lemma 2.4, we get

$$w(T) \leq w(\tilde{T}),$$

where $\tilde{T} = (\tilde{t}_{ij})_{n \times n}$, $\tilde{t}_{ij} = \begin{cases} \frac{1}{2} \|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\| & \text{if } i = j \\ \frac{1}{2} \|f^2(|T_{ij}|)\|^{\frac{1}{2}} \|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} & \text{if } i \neq j. \end{cases}$

Note that the existing inequality in [4, Th. 3.3] follows from Corollary 2.15.

REFERENCES

1. A. Abu-Omar and F. Kittaneh, Numerical radius inequalities for $n \times n$ operator matrices, Linear Algebra Appl. 468 (2015) 18-26.
2. P. Bhunia, S. Bag and K. Paul, Numerical radius inequalities of operator matrices with applications, Linear Multilinear Algebra 69(9) (2021) 1635-1644.
3. S. Bag, P. Bhunia and K. Paul, Bounds of numerical radius of bounded linear operators using t-Aluthge transform, Math. Inequal. Appl. 23(3) (2020) 991-1004.
4. P. Bhunia and K. Paul, Some improvements of numerical radius inequalities of operators and operator matrices, Linear Multilinear Algebra, (2020). https://doi.org/10.1080/03081087.2020.1781037
5. P. Bhunia, S. Bag and K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials, Linear Algebra Appl. 573 (2019) 166-177.
6. P. Bhunia, D. Sain and K. Paul, On the Davis-Wielandt shell of an operator and the Davis-Wielandt index of a normed linear space. https://arxiv.org/abs/2006.15323
7. R.A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
8. J.C. Hou and H.K. Du, Norm inequalities of positive operator matrices, Integral Equations Operator Theory 22 (1995) 281-294.
9. F. Kittaneh, Notes on some inequalities for Hilbert Space operators, Publ. Res. Inst. Math. Sci. 24 (1988) 283-293.
10. D. Sain, P. Bhunia, A. Bhanja and K. Paul, On a new norm on $\mathcal{B}(\mathcal{H})$ and its applications to numerical radius inequalities, (2021). https://doi.org/10.1007/s43034-021-00138-5

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