Local Grammar-Based Coding Revisited

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Abstract

We revisit the problem of minimal local grammar-based coding. In this setting, the local grammar encoder encodes grammars symbol by symbol, whereas the minimal grammar transform minimizes the grammar length in a preset class of grammars as given by the length of local grammar encoding. It has been known that such minimal codes are strongly universal for a strictly positive entropy rate, whereas the number of rules in the minimal grammar constitutes an upper bound for the mutual information of the source. Whereas the fully minimal code is likely intractable, the constrained minimal block code can be efficiently computed. In this article, we present a new, simpler, and more general proof of strong universality of the minimal block code, regardless of the entropy rate. The proof is based on a simple Zipfian bound for ranked probabilities. By the way, we also show empirically that the number of rules in the minimal block code cannot clearly discriminate between long-memory and memoryless sources, such as a text in English and a random permutation of its characters. This contradicts our previous expectations.

Key words: universal codes, grammar-based codes, block codes, mutual information, Zipf’s law

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1 Introduction

The grammar-based codes, a certain natural and successful approach to universal coding, are a two-step method [1, 2]. First, the input string is represented as a dictionary grammar, i.e., a context-free grammar that generates the input string as its sole production. Second, the dictionary grammar is encoded as the output binary string using a sort of arithmetic coding. The mapping from the input string to the dictionary grammar is called the grammar transform, whereas the mapping from the dictionary grammar to the output binary string is called the grammar encoder.

Whereas the essential idea of grammar-based coding comes from computational linguistics and artificial intelligence [3, 4, 5, 6], the basic theory of grammar-based coding bears to the work by Kieffer and Yang [1]. In particular, Kieffer and Yang demonstrated that a large class of codes with irreducible grammar transforms and a particular grammar encoder is strongly universal. Another seminal work by Charikar et al. [7] showed the NP-hardness of computing the fully minimal grammar transform, whose length was defined as the sum of the lengths of the grammar rules.

Our contribution to this field [8] was to consider a local grammar encoder, which encodes dictionary grammars symbol by symbol—in a way that seems naive and suboptimal. We also considered minimal grammar-based codes with respect to the local grammar encoder, i.e., grammar transforms whose length after the encoding is minimal, given some additional constraints on the class of considered grammars. Obviously, the local grammar encoder is far from being optimal, whereas fully minimal grammars are likely intractable. Why should we consider them then?

The main motivation for considering such minimal codes is a simple upper bound for their pointwise mutual information in terms of the number of rules in the fully minimal grammar [8]. It follows hence that if the fully minimal code is strongly universal and the Shannon mutual information or the algorithmic mutual information for a given process is high then the fully minimal grammar must have numerous rules.

Consequently, we may naively suspect that the fully minimal code may be used for quantifying hierarchical structure in the stream of symbols generated by an information source [8], see also [3, 4, 5, 6]. In particular, such a code might be used to explain power-law word frequency distributions from linguistics [9, 10, 11], such as the celebrated Zipf law, in terms of a hypothetical power-law growth of mutual information for natural language, called the Hilberg hypothesis [12, 13]—see [8, 14, 15] for more theory and [16, 17, 18, 19, 20, 21, 22, 23] for more recent experimental data.

In any case, to have such results, we need to demonstrate that the fully
minimal code is strongly universal. In fact, a proof of strong universality of the fully minimal code was presented in work [8]. It rested on the proof of universality of the minimal block code discussed by Neuhoff and Shields [24]. It should be noted that the Neuhoff and Shields code can be interpreted as a grammar-based code but it does not apply a local grammar encoder. Moreover, it should be noted that the proof in [8] was not fully general since it applied only to processes with a strictly positive entropy rate.

The aim of the present article, being a supplement to paper [8], is to fill this missing gap and to present a newer, more illuminating, and more general proof of strong universality of the minimal block code which applies a local grammar encoder and constitutes a restricted version of the fully minimal code. Our novel proof does not depend on the value of the entropy rate and is based on an extremely simple Zipfian bound for ranked probabilities, namely, \(\pi_n \leq n^{-1}\), where \(\pi_n\) is the \(n\)-th largest probability.

To pour a bucket of cold water, we will also show by means of a numerical experiment that despite our earlier expectations expressed in [8], the number of rules in the minimal block code cannot be used easily to discriminate between memoryless and long-memory sources. We will highlight this issue on an example of the collection of plays by William Shakespeare and a random permutation of its characters. For both cases, it is disputable whether the number of rules grows at a qualitatively different rate.

The above phenomenon should be contrasted with a stark difference between natural language and memoryless sources for some other word-like segmentation procedures. Namely, if we estimate the Markov order of empirical data in a consistent way [25] and we count the number of distinct substrings of the length equal to the Markov order estimate then natural language exhibits many more such substrings than memoryless sources. This empirical difference can be also linked to Hilberg’s hypothesis that claims a power-law growth of mutual information for language, see [26, 14, 15].

The further organization of this article is as follows: In Section 2, we discuss the fully minimal code, which is likely intractable. Section 3 introduces the minimal block code, a restricted version of the fully minimal code that can be computed in a reasonable time. Section 4 furnishes the exact proof of strong universality of the minimal block code, which is based on a simple Zipfian bound for ranked probabilities. In Section 5, we comment on applications of this result to estimating mutual information and quantifying hierarchical structures in empirical data. By means of a simple experiment, we show that the minimal block code cannot easily distinguish between memoryless and long-memory sources.
2 Fully minimal code

Let the input alphabet be \( X = \{1, 2, ..., m\} \). The definition of a dictionary grammar, called an admissible grammar in [1], can be stated succinctly as follows.

**Definition 1 (dictionary grammar)** A dictionary grammar is a function

\[
G : \{m + 1, m + 2, ..., m + V_G\} \to \{1, 2, ..., m + V_G\}^*
\]

such that for every \( G(r) = (r_1, r_2, ..., r_p) \) we have \( r_i < r \). Strings \( G(r) \) for \( r < m + V_G \) are called secondary rules, whereas string \( G(m + V_G) \) is called the primary rule.

The production of a string by a dictionary grammar can be also made precise in a simple way in the next definition.

**Definition 2 (grammar expansion)** For a dictionary grammar (1), we iteratively define its expansion function

\[
G' : \{1, 2, ..., m + V_G\} \to \{1, 2, ..., m\}^*
\]

as

\[
G'(r) := r \text{ for } r \leq m \text{ and concatenation } G'(r) := G'(r_1)G'(r_2)...G'(r_p) \text{ for } G(r) = (r_1, r_2, ..., r_p).
\]

We say that a dictionary grammar \( G \) produces a string \( u \in \{1, 2, ..., m\}^* \) if \( G'(m + V_G) = u \).

In paper [8], we introduced a local grammar encoder, which encodes dictionary grammars symbol by symbol, with two additional symbols for commas, namely, 0 and \(-1\). We state its definition as follows.

**Definition 3 (local grammar encoder)** Consider a code for extended natural numbers \( \psi : \{-1, 0, 1, 2, ...\} \to \{0, 1\}^* \). The local grammar encoder \( \psi^* \) for a dictionary grammar \( G \) returns string

\[
\psi^*(G) := \psi^*(G(m + 1))\psi^*(G(m + 2))...\psi^*(G(m + V_G))\psi(-1),
\]

where \( \psi^*(r_1, r_2, ..., r_p) := \psi(r_1)\psi(r_2)...\psi(r_p)\psi(0) \).

Consequently, in [8], we investigated a minimal grammar transform that minimizes the grammar length defined via the local grammar encoder.

**Definition 4 (fully \( \psi \)-minimal code)** We define the fully \( \psi \)-minimal grammar transform \( \Gamma_\psi(u) \) as the dictionary grammar \( G \) that produces string \( u \in \{1, 2, ..., m\}^* \) and minimizes length \( |\psi^*(G)| \). Subsequently, the fully \( \psi \)-minimal code \( B_\psi : \{1, 2, ..., m\}^* \to \{0, 1\}^* \) is defined as

\[
B_\psi(u) = \psi^*(\Gamma_\psi(u)).
\]
Finding the fully $\psi$-minimal grammar for a given string may be intractable because it requires searching globally through a prohibitively large space of dictionary grammars.

The exact theory of $\psi$-minimal grammars may depend on the choice of code $\psi$. There are two natural choices, for which we coin some names here. Let log be the binary logarithm.

**Definition 5 (trivial code)** A code $\psi : \{-1, 0, 1, 2, \ldots\} \to \{0, 1\}^*$ is called trivial if it satisfies $|\psi(n)| = 1$ for $n \geq 1$ and $|\psi(n)| = 0$ for $n \leq 0$.

**Definition 6 (proper code)** A code $\psi : \{-1, 0, 1, 2, \ldots\} \to \{0, 1\}^*$ is called $m$-proper if

1. $\psi$ is prefix-free;
2. $|\psi(n)| = c_1$ for $-1 \leq n \leq m$ and some $c_1 < \infty$;
3. $|\psi(n)| \leq |\psi(n + 1)|$ for $n \geq -1$;
4. $|\psi(n)| \leq \log(n + 2) \log\log n + c_2$ for $n \geq 2$ and some $c_2 < \infty$.

Proper codes exist by the Kraft inequality.

**Theorem 1** There exists an $m$-proper code $\psi$ such that

$$
|\psi(n)| = \begin{cases} 
\left\lceil \log(m + 2) + 2 \log \log m + \log \left( \frac{1}{\log^* m} + 1 \right) \right\rceil, & -1 \leq n \leq m, \\
\left\lceil \log(n + 2) + 2 \log \log n + \log \left( \frac{1}{\log^* m} + 1 \right) \right\rceil, & n > m.
\end{cases}
$$

(5)

**Proof:** Code $\psi$ exists if the Kraft inequality is satisfied, namely, if we have

$$
\sum_{n=-1}^{\infty} 2^{-|\psi(n)|} \leq 1.
$$

(6)

Observe that $\sum_{n=2}^{\infty} \frac{1}{n \log^* n} \leq 1$ and let $c_2 := \log \left( \frac{1}{\log^* m} + 1 \right)$. We can evaluate

$$
\sum_{n=-1}^{\infty} 2^{-|\psi(n)|} \leq \frac{(m + 2)2^{-c_2}}{(m + 2) \log^2 m} + \sum_{n=m+1}^{\infty} \frac{2^{-c_2}}{(n + 2) \log^2 n}
\leq 2^{-c_2} \left( \frac{1}{\log^* m} + 1 \right) = 1.
$$

(7)

Hence the code exists. \qed

4
The \( \psi \)-minimal code with a proper code \( \psi \) is uniquely decodable by the prefix-free property of code \( \psi \). Succinctly, \( \psi \)-minimal grammars and codes with a trivial or proper code \( \psi \) will be called trivial or proper, respectively.

Computing the trivial fully minimal grammar for a given string is NP-hard, as it was demonstrated by Charikar et al. [7]. In contrast, in [8], we showed that the proper fully minimal code is strongly universal for stationary ergodic processes with a strictly positive entropy rate.

3 Minimal block code

To overcome the problem of tractability of fully minimal codes, we may restrict the class of grammars over which we perform the minimization and hope to maintain the universality of the code. A sufficiently rich class is the class of block grammars.

**Definition 7 (block grammar)** A \( k \)-block grammar is a dictionary grammar \( G \) such that every secondary rule has form \( G(r) = (R_1, R_2, ..., R_k) \) where \( R_i \leq m \) and the primary rule has form

\[
G(m + V_G) = (R_1, R_2, ..., R_l, r_1, r_2, ..., r_p, \alpha, R_{-l'}, R_{-l' + 1}, ..., R_{-1})
\]

(8)

where \( R_i \leq m \), \( r_i > m \), and \( l, l' < k \). A dictionary grammar is called a block grammar if it is a \( k \)-block grammar for a certain \( k \).

Such block grammars were considered by Neuhoff and Shields [24]. They used them to construct a certain strongly universal minimal block code which does not apply the local grammar encoder defined in the previous section.

Within the realm of local grammar encoders, we may define our own minimal block code, which is somewhat different to the Neuhoff and Shields construction.

**Definition 8 (\( \psi \)-minimal block code)** We define the \( \psi \)-minimal block grammar transform \( \Gamma^\#_\psi(u) \) as the block grammar \( G \) that produces string \( u \in \{1, 2, ..., m\}^* \) and minimizes length \( |\psi^*(G)| \). Subsequently, the \( \psi \)-minimal block code \( B^\#_\psi : \{1, 2, ..., m\}^* \to \{0, 1\}^* \) is defined as

\[
B^\#_\psi(u) = \psi^*(\Gamma^\#_\psi(u)).
\]

(9)

A very similar code to the above one was also considered in [8]. The difference is that in [8], the primary rule had form

\[
G(m + V_G) = (r_1, r_2, ..., r_p, \alpha, R_{-l'}, R_{-l' + 1}, ..., R_{-1})
\]

(10)
without the initial string \( R_1, R_2, ..., R_l \). Modifying this form to form (8) is crucial to apply the universality criterion (15) introduced in the next section. This petty difference allows to circumvent problems with the distinction between ergodicity and \( k \)-ergodicity [28].

In contrast to the trivial fully minimal code, which is NP-hard, the proper minimal block code can be provably computed in a time close to linear (with some logarithmic add-ons). For this goal, we have to consider all parsings of the input string into \( k \)-blocks and to minimize the code length over \( k \). To determine the optimal code length for each of these parsings, we notice that by inequality \( |\psi(n)| \leq |\psi(n + 1)| \), the optimal secondary rules should be sorted according to the ranked empirical distribution of \( k \)-blocks. Once such sorting is performed, the resulted code is minimal within the class of block grammars since all rules have the same length after local encoding by equality \( |\psi(n)| = c_1 \) for \( n \leq m \).

Actually, when writing paper [8], we did not notice that the proper minimal block code can be so easily computed. For this reason, we did not include in [8] the numerical experiment that we report in Section 5.

### 4 Strong universality

Since the proper minimal block code is uniquely decodable and it achieves the constrained global minimum, we can demonstrate easily that this code is strongly universal. We briefly recall what it means exactly. Let us denote the entropy rate of a stationary process \((X_i)_{i \in \mathbb{Z}}\) as

\[
h := \lim_{n \to \infty} \frac{H(X^n)}{n} = \lim_{n \to \infty} \frac{\mathbb{E} K(X^n)}{n},
\]

where \( H(X) \) is the Shannon entropy of random variable \( X \) and \( K(u) \) is the Kolmogorov complexity of string \( u \). Let \( \mathbb{X} \) be a countable alphabet. Let \( B: \mathbb{X}^* \to \{0, 1\}^* \) be a uniquely decodable code. Code \( B \) is called strongly universal for alphabet \( \mathbb{X} \) if for any stationary ergodic process \((X_i)_{i \in \mathbb{Z}}\) where \( X_i: \Omega \to \mathbb{X} \), we have

\[
\lim_{n \to \infty} \frac{|B(X^n)|}{n} = h \text{ a.s.},
\]

\[
\lim_{n \to \infty} \frac{\mathbb{E} |B(X^n)|}{n} = h.
\]

Universal codes exist if and only if the alphabet is finite [29].

There is a simple universality criterion which is based the \( k \)-th order conditional empirical entropy. We call it the conditional criterion to distinguish it from another criterion which we will introduce subsequently.
Theorem 2 (conditional universality criterion) Let $B : \mathbb{X}^* \to \{0, 1\}^*$ be a uniquely decodable code. Code $B$ is strongly universal for alphabet $\mathbb{X}$ if for any $k \geq 1$, any conditional probability distribution $\pi : \mathbb{X} \times \mathbb{X}^k \to [0, 1]$, and any string $x^n_1 \in \mathbb{X}^*$, we have

$$|B(x^n_1)| \leq C(n, k) - \log \prod_{i=k+1}^{n} \pi(x_i|x_{i-k}^{i-1}),$$

(14)

where $\lim_{k \to \infty} \limsup_{n \to \infty} C(n, k)/n = 0$.

The proof is relegated for completeness to Appendix A.

The conditional universality criterion was first apply in the celebrated Ziv inequality for proving strong universality of the Lempel-Ziv code [30, 31]. Moreover, Ochoa and Navarro [32] showed that $k$-th order conditional empirical entropy bounds the lengths of grammar-based codes that apply the encoder by Kieffer and Yang [1]. Such an inequality is sufficient to assert strong universality of the respective codes.

In contrast, for our purpose, we need a somewhat different universality criterion, which is based on the $k$-block unconditional empirical entropy. Thus we call this criterion the block criterion.

Theorem 3 (block universality criterion) Let $B : \mathbb{X}^* \to \{0, 1\}^*$ be a uniquely decodable code. Code $B$ is strongly universal for alphabet $\mathbb{X}$ if for any $k \geq 1$, any probability distribution $\pi : \mathbb{X}^k \to [0, 1]$, and any string $x^n_1 \in \mathbb{X}^*$, we have

$$|B(x^n_1)| \leq C(n, k) - \frac{1}{k} \log \prod_{i=0}^{n-k} \pi(x_{i+k}^{i+k+1}),$$

(15)

where $\lim_{k \to \infty} \limsup_{n \to \infty} C(n, k)/n = 0$.

The proof is relegated to Appendix A.

An interesting detail is that the right-hand side of criterion (15) contains probabilities of overlapping blocks. If there were no overlaps and no division by $k$, we might encounter problems with the distinction between ergodicity and $k$-ergodicity [28]. Fortunately, for the successful application of criterion (15) to block codes, it is sufficient to consider all $k$ distinct shifts of blocks and take the shift that yields the shortest code.

Exactly this idea is used in the following Theorem 4 for proving universality of the proper minimal block code. We may prove that the $\psi$-minimal block code is universal for any proper code $\psi$ for extended natural numbers. Our result generalizes and simplifies the earlier result from [8]. It is as follows.
Theorem 4  The m-proper minimal block code is strongly universal for the alphabet \( X = \{1, 2, \ldots, m\} \).

The proof is presented after the proof of the auxiliary Lemma 1.

Obviously the fully \( \psi \)-minimal code is shorter than the \( \psi \)-minimal block code. In consequence, strong universality of the proper fully minimal code follows by the strong universality of the proper minimal block code. By means of our new proof of Theorem 4, it will also become obviously clear why some variations on the theme of the minimal block code discussed by Neuhoff and Shields [24] are also strongly universal. The key observation is the following extremely simple inequality which implies that ranked probabilities are upper bounded by the harmonic series.

Lemma 1 (Zipfian bound)  Let \( \pi_1 \geq \pi_2 \geq \ldots \) be a sequence of probabilities of disjoint events, i.e., \( \sum_i \pi_i \leq 1 \). Then

\[
\pi_n \leq \frac{1}{n}. \tag{16}
\]

Proof: We have \( n \pi_n \leq \sum_{j=1}^n \pi_j \leq 1 \). \( \square \)

Lemma 1 reminds of Zipf’s law \( \pi_n \propto n^{-1} \) for the rank-frequency distribution of words in natural language [9]. By Lemma 1, if we use a local grammar-based code where rules \( G(n) \) are sorted by arbitrary probabilities \( \pi_n \) then the binary identifier \( \psi(n) \) of the \( n \)-th rule will be roughly shorter than the respective minus log-probability \( -\log \pi_n \) if we have \( |\psi(n)| \approx \log n \) in general. This prompts a path to proving universality of the \( \psi \)-minimal block code. We note that the calculations below and in the paper by Neuhoff and Shields [24] apply different ideas and take distinct paths, our reasoning being simpler.

Proof of Theorem 4: It suffices to show the \( \psi \)-minimal block code satisfies universality criterion (15). We will consider a sequence of \( k \)-block grammars \( G_l \) for string \( x_1^n \) indexed by index \( l \in \{0, 1, \ldots, k-1\} \) such that:

- The secondary rules, regardless of index \( l \), define all \( k \)-blocks in the order of ranking given by the distribution \( \pi \):
  \[
  G_l(m+j) = A_j \in X^k \text{ and } \pi(A_j) \geq \pi(A_{j+1}) \text{ for } 1 \leq j \leq m^k. \tag{17}
  \]

- The primary rule of each grammar \( G_l \) defines string \( x_1^l \) using the identifiers for \( k \)-blocks shifted by \( l \) positions:
  \[
  G_l(m + m^k + 1) = (R_1, R_2, \ldots, R_l, r_1^l, r_2^l, \ldots, r_{p_l}^l, R_{-l'}, R_{-l'+1}, \ldots, R_{-1}) \tag{18}
  \]

where \( 1 \leq R_i \leq m, m < r_i < m^k \), and \( 0 \leq l, l' < k \).
We observe that none of these grammars can be better than the $\psi$-minimal block grammar for $x^n$. Hence, for any $l \in \{0, 1, ..., k - 1\}$, we may bound

$$|B^\#_\psi(x^n)| \leq |\psi^*(G_l)| \leq C(k) + \sum_{i=1}^{p_l} |\psi(r^l_i)|,$$

where $C(k) := [m^k(k+1) + 2k + 2] |\psi(m)|$.

We have inequality $|\psi(n)| \leq \log n + 2 \log \log n + c_2$ by the hypothesis and inequality $\pi(A_j) \leq 1/j$ by Lemma 1. Hence, by subadditivity of function $\log(x+1)$, we may further bound

$$\sum_{i=1}^{p_l} |\psi(r^l_i)| \leq \sum_{i=1}^{p_l} [\log r^l_i + 2 \log \log r^l_i + c_2]$$

$$\leq \sum_{i=1}^{p_l} [\log(r^l_i - m) + \log(m + 1) + 2 \log \log m^k + c_2]$$

$$\leq \frac{n}{k} [\log(m + 1) + 2 \log \log m + 2 \log k + c_2] - \sum_{i=1}^{p_l} \log \pi(A_{r^l_i - m}).$$

(20)

Denote $C(n, k) := C(k) + \frac{n}{k} [\log(m + 1) + 2 \log \log m + 2 \log k + c_2]$. Then we may bound

$$|B^\#_\psi(x^n)| \leq C(n, k) - \min_{l \in \{0, 1, ..., k - 1\}} \sum_{l=0}^{k-1} \sum_{i=1}^{p_l} \log \pi(A_{r^l_i - m})$$

$$\leq C(n, k) - \frac{1}{k} \sum_{l=0}^{k-1} \sum_{i=1}^{p_l} \log \pi(A_{r^l_i - m}) = C(n, k) - \frac{1}{k} \sum_{i=0}^{n-k} \log \pi(x^{i+k}).$$

(21)

To conclude, we observe $\lim_{k \to \infty} \limsup_{n \to \infty} C(n, k)/n = 0$. □

5 Mutual information

Let us state more motivations for Theorem 4. How can the proper minimal block code be useful if the local grammar encoding is so suboptimal? The most important motivation for this code is a bound for its pointwise mutual information in terms of the number of rules in the proper minimal block grammar. In particular, if the proper minimal block code is strongly universal and the Shannon mutual information or the algorithmic mutual information...
for a given process is large then the proper minimal block grammar must have many rules. Thus, we may be tempted to think that proper minimal block grammars—or similar ones—might be used for measuring hierarchical structure in a realization of a stochastic process. Consequently, we may think that the amount of discovered hierarchical structure might easily discriminate between memoryless and long-memory sources. However, we will show that this is not true in the particular case of proper minimal block codes.

First, let us develop the respective bounds rigorously. Let $V^\#(u; v)$ denote the number of rules in the minimal block grammar $\Gamma^\#(u; v)$, let $L^\#(u; v)$ denote the common length of rules in grammar $\Gamma^\#(u; v)$, and let us denote the pointwise mutual information

$$J^\#(u; v) = \left| B^\#(u) \right| + \left| B^\#(v) \right| - \left| B^\#(uv) \right|. \quad (22)$$

Having made these notations, we present the first desired bound.

**Theorem 5 (cf. [8])** For an $m$-proper minimal block code, we have

$$J^\#(u; v) \leq V^\#(uv)(L^\#(uv) + 1) |\psi(m)|. \quad (23)$$

A similar inequality holds for proper fully minimal codes, where the role of length $L^\#(uv)$ is played by the maximal length of a rule, see [8]. We recall the proof, which was stated in [8] in a more complicated notation.

**Proof of Theorem 5:** Let a block grammar $G$ be $\psi$-minimal for string $uv$. We can define grammars $G_1$ for $u$ and $G_2$ for $v$ by splitting the primary rule of grammar $G$. Namely, we put $V_{G_1} := V_{G_2} := V_G$, $G_1(r) := G_2(r) := G(r)$ for $r < m + V_G$, and

$$G_1(m + V_G) := (r_1, r_2, \ldots, r_k, R_1, R_2, \ldots, R_l), \quad (24)$$

$$G_2(m + V_G) := (R_{l+1}, R_{l+2}, \ldots, R_o, r_{k+1}, r_{k+2}, \ldots, r_p), \quad (25)$$

where

$$G(m + V_G) = (r_1, r_2, \ldots, r_p), \quad (26)$$

$$G(r_k) = (R_1, R_2, \ldots, R_o). \quad (27)$$

By the definitions of a block grammar and a proper code, we have

$$|\psi^*(G(j))| = (L^\#(uv) + 1) |\psi(m)|. \quad (28)$$
for $m < j < m + V_G$. Subsequently, we may bound

$$J^\#_{\psi}(u; v) = \left| B^\#_{\psi}(u) \right| + \left| B^\#_{\psi}(v) \right| - \left| B^\#_{\psi}(uv) \right|$$

$$\leq |\psi^*(G_1)| + |\psi^*(G_2)| - |\psi^*(G)|$$

$$= \sum_{j=m+1}^{m+V_G-1} |\psi^*(G(j))| + |\psi^*(G(r_k))| - |\psi(r_k)| + |\psi(-1)|$$

$$\leq V_G(L^\#_{\psi}(uv) + 1) |\psi(m)|$$

which is the desired claim. □

Moreover, for the proper minimal block code, the common length of rules seems to grow essentially logarithmically with the input length.

**Theorem 6** Consider an $m$-proper minimal block code and a stationary ergodic process $(X_i)_{i \in \mathbb{Z}}$ over alphabet $\mathbb{X} = \{1, 2, ..., m\}$ with entropy rate $h > 0$. Suppose that

$$\lim_{n \to \infty} \frac{V^\#_{\psi}(X^n_1) \log n}{n} = 0 \text{ a.s.}$$

Then we have

$$\lim sup_{n \to \infty} \frac{L^\#_{\psi}(X^n_1)}{\log n} \leq \frac{1}{h} \text{ a.s.}$$

**Proof:** Observe that the number of rules in the minimal block grammar $\Gamma^\#_{\psi}(u)$ is less than $|u|$. For the proper code $\psi$, we may bound

$$\left| B^\#_{\psi}(u) \right| \leq \frac{|u|}{L^\#_{\psi}(u)} |\psi(m + |u|)| + \left( V^\#_{\psi}(u) + 2 \right) \left( L^\#_{\psi}(u) + 1 \right) |\psi(m)|.$$  \hspace{1cm} (32)

Solving this inequality for $L^\#_{\psi}(u)$ yields

$$L^\#_{\psi}(u) \leq \frac{B - \sqrt{B^2 - 4AC}}{2A},$$

where

$$A := \left( V^\#_{\psi}(u) + 2 \right) |\psi(m)|,$$  \hspace{1cm} (34)

$$B := \left| B^\#_{\psi}(u) \right| - \left( V^\#_{\psi}(u) + 2 \right) |\psi(m)|,$$  \hspace{1cm} (35)

$$C := |u| |\psi(m + |u|)|.$$  \hspace{1cm} (36)
Observe that $\sqrt{1 + x} = 1 + \frac{x}{2} + O(x^2)$. Thus for $AC \ll B^2$, we obtain

$$L^\#(u) \leq \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{B}{2A} \left(1 - \sqrt{1 - \frac{4AC}{B^2}}\right) \approx \frac{C}{B}. \quad (37)$$

Hence the claim follows by the strong universality of code $B^\#$. \hfill \Box

Let us consider the Shannon mutual information $I(X; Y) := H(X) + H(Y) - H(X, Y)$ and the algorithmic mutual information $J(u; v) := K(u) + K(v) - K(u, v)$. In the sequel, Theorem 5 can be complemented with the following proposition concerning the asymptotic power-law rates of various quantities, collectively called Hilberg exponents [14].

**Theorem 7 (cf. [14])** For an $m$-proper minimal block code and for any stationary process $(X_i)_{i \in \mathbb{Z}}$ over alphabet $X = \{1, 2, ..., m\}$, we have

$$\limsup_{n \to \infty} \left(\mathbb{E} [B^\#(X^n_1)] - nh\right) = \limsup_{n \to \infty} \mathbb{E} J^\#(X^n_1; X^{2n}_{n+1})$$

$$\geq \limsup_{n \to \infty} (\mathbb{E} K(X^n_1) - nh) = \limsup_{n \to \infty} \mathbb{E} J(X^n_1; X^{2n}_{n+1})$$

$$\geq \limsup_{n \to \infty} (H(X^n_1) - nh) = \mathbb{E} I(X^n_1; X^{2n}_{n+1}), \quad (38)$$

where we apply the Hilberg exponent

$$\limsup_{n \to \infty} \left(\frac{\log S(n)}{\log n}\right). \quad (39)$$

Again, we sketch the proof to give an insight.

**Proof of Theorem 7:** The claim follows by the general identity

$$\limsup_{n \to \infty} (S(n) - ns) = \limsup_{n \to \infty} (2S(n) - S(2n)), \quad (40)$$

which holds if the limit $\lim_{n \to \infty} S(n)/n = s$ exists and $S(n) \geq ns$ [14]. \hfill \Box

Theorems 5–7 suggest that the proper minimal block code and the proper fully minimal code may be used for measuring the amount of hierarchical structure in empirical data. As it was argued in [8], such mathematical results can be linked potentially to Zipf’s law for the rank-frequency distribution of words in natural language [9]. It is so since secondary rules of some heuristic approximations of the proper fully minimal grammar for a text in natural language seem to correspond to the orthographic words, as shown empirically by de Marcken [3]. Seen in this light, Theorems 5–7 make a link between
Hilberg’s hypothesis about a power-law growth of mutual information for natural language [12, 13] and Herdan-Heaps’ law about the power-law growth of the number of distinct words in a text [10, 11], which is a corollary of Zipf’s law. For more details, see [8, 14, 15].

This particular theoretical explanation of Zipf’s law may be an overkill, however, for the simple reason that the number of rules $V_v^u$ may be of a similar order for both natural language and memoryless sources. To check it, we have performed a simple numerical experiment for the proper minimal block code with the code length given by the $m$-proper code (5). The results are shown in Figure 1, where we compare three sources: the Bernoulli(0.5) process, the collection of 35 plays by William Shakespeare, and a random permutation of characters for the same text in English. We observe that the amount of discovered structure is greater for Shakespeare’s plays than for the memoryless sources. However, the number of rules seems to grow at a power law rate in any discussed case. We can see that it grows by long leaps and bounds that hinder a more precise estimation of the trend.

These empirical results seem to contradict the fundamental intuition that memoryless sources do not exhibit any kind of a hierarchical structure. However, let us stress that the reported ambiguous behavior of the proper minimal block code should be contrasted with a prominent difference between natural language and memoryless sources for some other word-like segmentation procedures. In particular, we may estimate the Markov order of empirical data using a consistent estimator, such as one introduced by Merhav, Gutman, and Ziv [25], and we may count the number of distinct substrings of the length equal to the Markov order estimate. In such case, we may observe that natural language contains many more such substrings than memoryless sources, see [26, 14, 15]. This difference can be also linked, via the prediction by partial matching (PPM) codes [33, 34, 35], to the Hilberg hypothesis that states the power-law growth of mutual information for natural language [12].

Driven by this observation, we might want to seek for other classes of minimal grammar-based codes that provably do not discover hierarchical structures in pure randomness. To supply another negative result, we recall that irreducible grammar transforms discussed by Kieffer and Yang [1], which yield another class of universal codes, discover rich hierarchical structures in data streams generated by memoryless sources, see [36]. For this reason, these universal codes cannot discriminate between memoryless and long-memory sources. Thus, deciding whether there exist grammar-based codes that do discriminate between these sources remains an open problem. Another interesting question is whether such hypothetical codes compress better than the grammar-based codes known so far.
Figure 1: The number of rules and their length in the proper minimal block grammar transform: (1) for a realization of the Bernoulli(0.5) process, (2) for the concatenation of 35 plays by William Shakespeare (downloaded from Project Gutenberg), and (3) for a random permutation of characters for the same text in English. The lines are the least square regressions with the slopes presented in the key.
A Universality criteria

In this section, for completeness, we prove the universality criteria stated in Theorems 2 and 3. Let $B$ be a uniquely decodable code and $(X_i)_{i \in \mathbb{N}}$ be a stationary ergodic process. By the strong Barron lemma [37] and the Shannon-McMillan-Breiman theorem [38, 39, 40, 41, 42], we obtain the lower bound

$$\liminf_{n \to \infty} \frac{|B(X_1^n)|}{n} \geq \lim_{n \to \infty} \frac{-\log P(X_1^n)}{n} = h \text{ a.s.} \quad (41)$$

Similarly, by the source coding inequality we obtain

$$\liminf_{n \to \infty} \mathbb{E} \frac{|B(X_1^n)|}{n} \geq \lim_{n \to \infty} \frac{\mathbb{E}[-\log P(X_1^n)]}{n} = h. \quad (42)$$

It remains to prove the upper bounds.

Proof of Theorem 2: Let $B$ be a uniquely decodable code and $(X_i)_{i \in \mathbb{N}}$ be a stationary ergodic process. If (14) holds for any $k \geq 1$ and any conditional probability distribution $\pi : \mathbb{X} \times \mathbb{X}^k \to [0,1]$ then

$$|B(X_1^n)| \leq C(n,k) - \log \prod_{i=k+1}^n P(X_i|X_{i-1}^{i-1}) \text{ a.s.} \quad (43)$$

Consequently, the Birkhoff ergodic theorem [43, 44] yields

$$\limsup_{n \to \infty} \frac{|B(X_1^n)| - C(n,k)}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=k+1}^n \left[-\log P(X_i|X_{i-1}^{i-1}) \right]$$

$$= H(X_i|X_{i-1}^{i-1}) \text{ a.s.} \quad (44)$$

This holds for any $k \geq 1$, so

$$\limsup_{n \to \infty} \frac{|B(X_1^n)|}{n} \leq \lim_{k \to \infty} \left[ \limsup_{n \to \infty} \frac{C(n,k)}{n} + H(X_i|X_{i-1}^{i-1}) \right] = h \text{ a.s.} \quad (45)$$

Similarly,

$$\limsup_{n \to \infty} \mathbb{E} \frac{|B(X_1^n)| - C(n,k)}{n} \leq \mathbb{E} \left[-\log P(X_i|X_{i-1}^{i-1}) \right] = H(X_i|X_{i-1}^{i-1}). \quad (46)$$

This also holds for any $k \geq 1$, so

$$\limsup_{n \to \infty} \mathbb{E} \frac{|B(X_1^n)|}{n} \leq \lim_{k \to \infty} \left[ \limsup_{n \to \infty} \frac{C(n,k)}{n} + H(X_i|X_{i-1}^{i-1}) \right] = h. \quad (47)$$

Combining this with (41)–(42), we have established strong universality of code $B$. □
Proof of Theorem 3: Let $B$ be a uniquely decodable code and $(X_i)_{i \in \mathbb{N}}$ be a stationary ergodic process. If (15) holds for any $k \geq 1$ and any probability distribution $\pi : X^k \to [0, 1]$ then

$$|B(X^n)| \leq C(n, k) - \frac{1}{k} \log \prod_{i=0}^{n-k} P(X_{i+1}^{i+k}) \text{ a.s.} \quad (48)$$

Consequently, the Birkhoff ergodic theorem [43, 44] yields

$$\limsup_{n \to \infty} \frac{|B(X^n)| - C(n, k)}{n} \leq \frac{1}{k} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-k} [- \log P(X_{i+1}^{i+k})] = \frac{H(X_k)}{k} \text{ a.s.} \quad (49)$$

This holds for any $k \geq 1$, so

$$\limsup_{n \to \infty} \frac{|B(X^n)|}{n} \leq \lim_{k \to \infty} \left[ \limsup_{n \to \infty} \frac{C(n, k)}{n} + \frac{H(X_k)}{k} \right] = h \text{ a.s.} \quad (50)$$

Similarly,

$$\limsup_{n \to \infty} \frac{E |B(X^n)| - C(n, k)}{n} \leq \frac{E \left[ - \log P(X_{i+1}^{i+k}) \right]}{k} = \frac{H(X_k)}{k}. \quad (51)$$

This also holds for any $k \geq 1$, so

$$\limsup_{n \to \infty} \frac{E |B(X^n)|}{n} \leq \lim_{k \to \infty} \left[ \limsup_{n \to \infty} \frac{C(n, k)}{n} + \frac{H(X_k)}{k} \right] = h. \quad (52)$$

Combining this with (41)–(42), we have established strong universality of code $B$. \hfill \square

References

[1] J. C. Kieffer and E. Yang, “Grammar-based codes: A new class of universal lossless source codes,” IEEE Trans. Inform. Theory, vol. 46, pp. 737–754, 2000.

[2] ———, “Survey of grammar-based data structure compression,” IEEE BITS the Information Theory Magazine, pp. 1–12, 2022.

[3] C. G. de Marcken, “Unsupervised language acquisition,” Ph.D. dissertation, Massachusetts Institute of Technology, 1996.
[4] C. G. Nevill-Manning, “Inferring sequential structure,” Ph.D. dissertation, University of Waikato, 1996.

[5] C. G. Nevill-Manning and I. H. Witten, “Compression and explanation using hierarchical grammars,” *Computer J.*, vol. 40, pp. 103–116, 1997.

[6] ——, “Identifying hierarchical structure in sequences: A linear-time algorithm,” *J. Artif. Intel. Res.*, vol. 7, pp. 67–82, 1997.

[7] M. Charikar, E. Lehman, A. Lehman, D. Liu, R. Panigrahy, M. Prabhakaran, A. Sahai, and A. Shelat, “The smallest grammar problem,” *IEEE Trans. Inform. Theory*, vol. 51, pp. 2554–2576, 2005.

[8] Ł. Dębowski, “On the vocabulary of grammar-based codes and the logical consistency of texts,” *IEEE Trans. Inform. Theory*, vol. 57, pp. 4589–4599, 2011.

[9] G. K. Zipf, *The Psycho-Biology of Language: An Introduction to Dynamic Philology*. Houghton Mifflin, 1935.

[10] G. Herdan, *Quantitative Linguistics*. Butterworths, 1964.

[11] H. S. Heaps, *Information Retrieval—Computational and Theoretical Aspects*. Academic Press, 1978.

[12] W. Hilberg, “Der bekannte Grenzwert der redundanzfreien Information in Texten — eine Fehlinterpretation der Shannonschen Experimente?” *Frequenz*, vol. 44, pp. 243–248, 1990.

[13] J. P. Crutchfield and D. P. Feldman, “Regularities unseen, randomness observed: The entropy convergence hierarchy,” *Chaos*, vol. 15, pp. 25–54, 2003.

[14] Ł. Dębowski, *Information Theory Meets Power Laws: Stochastic Processes and Language Models*. Wiley & Sons, 2021.

[15] ——, “A refutation of finite-state language models through Zipf’s law for factual knowledge,” *Entropy*, vol. 23, p. 1148, 2021.

[16] R. Takahira, K. Tanaka-Ishii, and Ł. Dębowski, “Entropy rate estimates for natural language—a new extrapolation of compressed large-scale corpora,” *Entropy*, vol. 18, no. 10, p. 364, 2016.

[17] J. Hestness, S. Narang, N. Ardalani, G. Diamos, H. Jun, H. Kianinejad, M. Patwary, M. Ali, Y. Yang, and Y. Zhou, “Deep learning scaling is predictable, empirically,” 2017, https://arxiv.org/abs/1712.00409.
[18] M. Hahn and R. Futrell, “Estimating predictive rate-distortion curves via neural variational inference,” *Entropy*, vol. 21, p. 640, 2019.

[19] M. Braverman, X. Chen, S. M. Kakade, K. Narasimhan, C. Zhang, and Y. Zhang, “Calibration, entropy rates, and memory in language models,” in *2020 International Conference on Machine Learning (ICML)*, 2020.

[20] J. Kaplan, S. McCandlish, T. Henighan, T. B. Brown, B. Chess, R. Child, S. Gray, A. Radford, J. Wu, and D. Amodei, “Scaling laws for neural language models,” 2020, https://arxiv.org/abs/2001.08361.

[21] T. Henighan, J. Kaplan, M. Katz, M. Chen, C. Hesse, J. Jackson, H. Jun, T. B. Brown, P. Dhariwal, S. Gray, et al., “Scaling laws for autoregressive generative modeling,” 2020, https://arxiv.org/abs/2010.14701.

[22] D. Hernandez, J. Kaplan, T. Henighan, and S. McCandlish, “Scaling laws for transfer,” 2021, https://arxiv.org/abs/2102.01293.

[23] K. Tanaka-Ishii, *Statistical Universals of Language: Mathematical Chance vs. Human Choice*. Springer, 2021.

[24] D. Neuhoff and P. C. Shields, “Simplistic universal coding,” *IEEE Trans. Inform. Theory*, vol. IT-44, pp. 778–781, 1998.

[25] N. Merhav, M. Gutman, and J. Ziv, “On the estimation of the order of a Markov chain and universal data compression,” *IEEE Trans. Inform. Theory*, vol. 35, no. 5, pp. 1014–1019, 1989.

[26] Ł. Dębowski, “Is natural language a perigraphic process? The theorem about facts and words revisited,” *Entropy*, vol. 20, no. 2, p. 85, 2018.

[27] P. Elias, “Universal codeword sets and representations of the integers,” *IEEE Trans. Inform. Theory*, vol. 21, pp. 194–203, 1975.

[28] R. M. Gray, *Probability, Random Processes, and Ergodic Properties*. Springer, 2009.

[29] L. Györfi, I. Páli, and E. C. van der Meulen, “There is no universal source code for infinite alphabet,” *IEEE Trans. Inform. Theory*, vol. 40, pp. 267–271, 1994.

[30] J. Ziv and A. Lempel, “A universal algorithm for sequential data compression,” *IEEE Trans. Inform. Theory*, vol. 23, pp. 337–343, 1977.
[31] T. M. Cover and J. A. Thomas, *Elements of Information Theory, 2nd ed.* Wiley & Sons, 2006.

[32] C. Ochoa and G. Navarro, “Repair and all irreducible grammars are upper bounded by high-order empirical entropy,” *IEEE Trans. Inform. Theory*, vol. 65, pp. 3160–3164, 2019.

[33] J. G. Cleary and I. H. Witten, “Data compression using adaptive coding and partial string matching,” *IEEE Trans. Comm.*, vol. 32, pp. 396–402, 1984.

[34] B. Y. Ryabko, “Prediction of random sequences and universal coding,” *Probl. Inform. Transm.*, vol. 24, no. 2, pp. 87–96, 1988.

[35] B. Ryabko, “Compression-based methods for nonparametric density estimation, on-line prediction, regression and classification for time series,” in *2008 IEEE Information Theory Workshop, Porto*. Institute of Electrical and Electronics Engineers, 2008, pp. 271–275.

[36] Ł. Dębowski, “Menzerath’s law for the smallest grammars,” in *Exact Methods in the Study of Language and Text*, P. Grzybek and R. Köhler, Eds. De Gruyter Mouton, 2007, pp. 77–85.

[37] A. R. Barron, “Logically smooth density estimation,” Ph.D. dissertation, Stanford University, 1985.

[38] C. Shannon, “A mathematical theory of communication,” *Bell Syst. Tech. J.*, vol. 30, pp. 379–423, 623–656, 1948.

[39] L. Breiman, “The individual ergodic theorem of information theory,” *Ann. Math. Statist.*, vol. 28, pp. 809–811, 1957.

[40] K. L. Chung, “A note on the ergodic theorem of information theory,” *Ann. Math. Statist.*, vol. 32, pp. 612–614, 1961.

[41] A. R. Barron, “The strong ergodic theorem for densities: Generalized Shannon-McMillan-Breiman theorem,” *Ann. Probab.*, vol. 13, pp. 1292–1303, 1985.

[42] P. H. Algoet and T. M. Cover, “A sandwich proof of the Shannon-McMillan-Breiman theorem,” *Ann. Probab.*, vol. 16, pp. 899–909, 1988.

[43] G. D. Birkhoff, “Proof of the ergodic theorem,” *Proc. Nat. Acad. Sci. Uni. Stat. Amer.*, vol. 17, pp. 656–660, 1932.
[44] A. M. Garsia, “A simple proof of E. Hopf’s maximal ergodic theorem,”
*J. Math. Mech.*, vol. 14, pp. 381–382, 1965.