QUANTITATIVE GREEN’S FUNCTION ESTIMATES FOR LATTICE QUASI-PERIODIC SCHRÖDINGER OPERATORS

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Abstract. In this paper, we establish quantitative Green’s function estimates for some higher dimensional lattice quasi-periodic (QP) Schrödinger operators. The resonances in the estimates can be described via a pair of symmetric zeros of certain functions and the estimates apply to the sub-exponential type non-resonant conditions. As the application of quantitative Green’s function estimates, we prove both the arithmetic version of Anderson localization and finite volume version of $\frac{1}{2}$-Hölder continuity of the integrated density of states (IDS) for such QP Schrödinger operators. This gives an affirmative answer to Bourgain’s problem in [Bou00].

1. Introduction

Consider the QP Schrödinger operators

\[ H = \Delta + \lambda V(\theta + n\omega)\delta_{n,n'} \text{ on } \mathbb{Z}^d, \]  

where $\Delta$ is the discrete Laplacian, $V : T^d = (\mathbb{R}/\mathbb{Z})^d \to \mathbb{R}$ is the potential and $n\omega = (n_1\omega_1, \ldots, n_d\omega_d)$. Typically, we call $\theta \in T^d$ the phase, $\omega \in [0,1]^d$ the frequency and $\lambda \in \mathbb{R}$ the coupling. Particularly, if $V = 2\cos 2\pi \theta$ and $d = 1$, then the operators (1.1) become the famous almost Mathieu operators (AMO).

Over the past decades, the study of spectral and dynamical properties of lattice QP Schrödinger operators has been one of the central themes in mathematical physics. Of particular importance is the phenomenon of Anderson localization (i.e., pure point spectrum with exponentially decaying eigenfunctions). Determining the nature of the spectrum and the eigenfunctions properties of (1.1) can be viewed as a small divisor problem, which depends sensitively on features of $\lambda, V, \omega, \theta$ and $d$. Then substantial progress has been made following Green’s function estimates based on a KAM type multi-scale analysis (MSA) of Fröhlich-Spencer [FS83].

More precisely, Sinai [Sin87] first proved the Anderson localization for a class of 1D QP Schrödinger operators with a $C^2$ cosine-like potential assuming the Diophantine frequency \(^1\). The proof focuses on eigenfunctions parametrization and the resonances are overcome via a KAM iteration scheme. Independently, Fröhlich-Spencer-Wittwer [FSW90] extended the celebrated method of Fröhlich-Spencer [FS83] originated from random Schrödinger operators case to the QP one, and

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\(^1\)We say $\omega \in \mathbb{R}$ satisfies the Diophantine condition if there are $\tau > 1$ and $\gamma > 0$ so that

\[ \|k\omega\| = \inf_{l \in \mathbb{Z}} |l-k\omega| \geq \frac{\gamma}{|k|^\tau} \text{ for } \forall k \in \mathbb{Z} \setminus \{0\}. \]
obtained similar Anderson localization result with [Sin87]. The proof however uses estimates of finite volume Green’s functions based on the MSA and the eigenvalue variations. Both [Sin87] and [FSW90] were inspired essentially by arguments of [FS83]. Eliasson [Eli97] applied a reducibility method based on KAM iterations to general Gevrey QP potentials and established the pure point spectrum for corresponding Schrödinger operators. All these 1D results are perturbative in the sense that the required perturbation strength depends heavily on the Diophantine frequency (i.e., localization holds for $|\lambda| \geq \lambda_0(V, \omega) > 0$). The great breakthrough was then made by Jitomirskaya [Jit94, Jit99], in which the non-perturbative methods for control of Green’s functions (cf. [Jit02]) were developed first for AMO. The non-perturbative methods can avoid the usage of multi-scale scheme and the eigenvalue variations. This will allow effective (even optimal in many cases) and independent of $\omega$ estimate on $\lambda_0$. In addition, such methods can provide arithmetic version of Anderson localization which means the removed sets on both $\omega$ and $\theta$ when obtaining localization have an explicit arithmetic description (cf. [Jit99, JL18] for details). In contrast, the current perturbation methods seem only providing certain measure or complexity bounds on these sets. Later, Bourgain-Jitomirskaya [BJ02] extended some results of [Jit99] to the exponential long-range hopping case (thus the absence of Lyapunov exponent) and obtained both nonperturbative and arithmetic Anderson localization. Significantly, Bourgain-Goldstein [BG00] generalized the non-perturbative Green’s function estimates of Jitomirskaya [Jit99] by introducing the new ingredients of semi-algebraic sets theory and subharmonic function estimates, and established the non-perturbative Anderson localization for general analytic QP potentials. The localization results of [BG00] hold for arbitrary $\theta \in \mathbb{T}$ and a.e. Diophantine frequencies (the permitted set of frequencies depends on $\theta$), and there seems no arithmetic version of Anderson localization results in this case. We would like to mention that the Anderson localization can also be obtained via reducibility arguments based on Aubry duality [JK16, AYZ17].

If one increases the lattice dimensions of QP operators, the Anderson localization proof becomes significantly difficult. In this setting, Chulaevsky-Dinaburg [CD93] and Dinaburg [Din97] first extended results of Sinai [Sin87] to the exponential long-range operator with a $C^2$ cosine type potential on $\mathbb{Z}^d$ for arbitrary $d \geq 1$. However, in this case, the localization holds assuming further restrictions on the frequencies (i.e., localization only holds for frequencies in a set of positive measure, but without explicit arithmetic description). Subsequently, the remarkable work of Bourgain-Goldstein-Schlag [BGS02] established the Anderson localization for the general analytic QP Schrödinger operators with $(n, \theta, \omega) \in \mathbb{Z}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ via Green’s function estimates. In [BGS02] they first proved the large deviation theorem (LDT) for the finite volume Green’s functions by combining MSA, matrix-valued Cartan’s estimates and semi-algebraic sets theory. Then by using further semi-algebraic arguments together with LDT, they proved the Anderson localization for all $\theta \in \mathbb{T}^2$ and $\omega$ in a set of positive measure (depending on $\theta$). While the restrictions of the frequencies when achieving LDT are purely arithmetic and do not depend on the choice of potentials, in order to obtain the Anderson localization it needs to remove an additional frequencies set of positive measure. The proof of [BGS02] is essentially

\[\text{i.e., Anderson localization assuming the positivity of the Lyapunov exponent. In the present context by nonperturbative Anderson localization we mean localization if } |\lambda| \geq \lambda_0 = \lambda_0(V) > 0 \text{ with } \lambda_0 \text{ being independent of } \omega.\]
two-dimensional and a generation of it to higher dimensions is significantly difficult. In 2007, Bourgain [Bou07] successfully extended the results of [BGS02] to arbitrary dimensions, and one of his key ideas is allowing the restrictions of frequencies to depend on the potential by means of delicate semi-algebraic sets analysis when proving LDT for Green’s functions. In other words, for the proof of LDT in [Bou07] there has already been additional restrictions on the frequencies, which depends on the potential \( V \) and is thus not arithmetic. The results of [Bou07] have been largely generalized by Jitomirskaya-Liu-Shi [JLS20] to the case of both arbitrarily dimensional multi-frequencies and exponential long-range hopping. Very recently, Ge-You [GY20] applied a reducibility argument to higher dimensional long-range QP operators with the cosine potential, and proved the first arithmetic Anderson localization assuming the Diophantine frequency.

Definitely, the LDT type Green’s function estimates methods are powerful to deal with higher dimensional QP Schrödinger operators with general analytic potentials. However, such methods do not provide the detailed information on Green’s functions and eigenfunctions that may be extracted by purely perturbative method based on Weierstrass preparation type theorem. As an evidence, in the celebrated work [Bou00], Bourgain developed the method of [Bou97] further to first obtain the finite volume version of \( (\frac{1}{2} - \delta) \)-Hölder continuity of the IDS for AMO. The proof shows that the Green’s functions can be controlled via certain quadratic polynomials, and the resonances are completely determined by zeros of these polynomials. Using this method then yields a surprising quantitative result on the Hölder exponent of the IDS, since the celebrated method of Goldstein-Schlag [GS01] which is non-perturbative and works for more general potentials does not seem to provide explicit information on the Hölder exponent. In 2009, by using KAM reducibility method of Eliasson [Eli92], Amor [Amo09] obtained the first \( \frac{1}{2} \)-Hölder continuity result of the IDS for 1D and multi-frequency QP Schrödinger operators with small analytic potentials and Diophantine frequencies. Later, the one-frequency result of Amor was largely generalized by Avila-Jitomirskaya [AJ10] to the non-perturbative case via the quantitative almost reducibility and localization method. In the regime of the positive Lyapunov exponent, Goldstein-Schlag [GS08] successfully proved the \( (\frac{1}{2m} - \delta) \)-Hölder continuity of the IDS for 1D and one-frequency QP Schrödinger operators with potentials given by analytic perturbations of certain trigonometric polynomials of degree \( m \geq 1 \). This celebrated work provides in fact the finite volume version of estimates on the IDS. We remark that the Hölder continuity of the IDS for 1D and multi-frequency QP Schrödinger operators with large potentials is hard to prove. In [GS01], by using the LDT for transfer matrix and the avalanche principle, Goldstein-Schlag showed the weak Hölder continuity (cf. (1.2)) of the IDS for 1D and multi-frequency QP Schrödinger operators assuming the positivity of the Lyapunov exponent and strong Diophantine frequencies. The weak Hölder continuity of the IDS for higher dimensional QP Schrödinger operators has been established in [Sch01, Bou07, Liu20]. Very recently, Ge-You-Zhao [GYZ22] proved the \( (\frac{1}{2m} - \delta) \)-Hölder continuity of the IDS for higher dimensional QP Schrödinger operators with small exponential long-range hopping and trigonometric polynomial (of degree \( m \)) potentials via the reducibility argument. By Aubry duality, they can obtain the \( (\frac{1}{2m} - \lambda) \)-Hölder continuity of the IDS for 1D and multi-frequency QP operators with a finite range hopping.
Of course, the references mentioned as above are far from complete and we refer the reader to [Bou05, MJ17, Dam17] for more recent results on the study of both Anderson localization and the Hölder regularity of the IDS for lattice QP Schrödinger operators.

1.1. Bourgain’s problems. The remarkable Green’s function estimates of [Bou00] should be not restricted to the proof of \((\frac{1}{2}-)\)-Hölder regularity of the IDS for AMO only. In fact, in [Bou00] (cf. Page 89), Bourgain made three comments on the possible extensions of his method:

1. In fact, one may also recover the Anderson localization results from [Sin87] and [FSW90] in the perturbative case;
2. One may hope that it may be combined with nonperturbative arguments in the spirit of [BG00, GS01] to establish \((\frac{1}{2}-)\)-Hölder regularity assuming positivity of the Lyapunov exponent only;
3. It may also allow progress in the multi-frequency case (perturbative or nonperturbative) where regularity estimates of the form \((0.28)^3\) are the best obtained so far.

An extension of (2) has been accomplished by Goldstein-Schlag [GS08]. The answer to the extension of (1) is highly nontrivial due to the following reasons:

- The Green’s function on good sets (cf. Section 3 for details) only has a sub-exponential off-diagonal decay estimate rather than an exponential one required by proving Anderson localization;
- At the \(s\)-th iteration step \((s \geq 1)\), the resonances of [Bou00] are characterized as

\[
\min\{\|\theta + k\omega - \theta_{s,1}\|, \|\theta + k\omega - \theta_{s,2}\|\} \leq \delta_s \sim \delta_0^{C_s}, \ C > 1.
\]

However, the symmetry information of \(\theta_{s,1}\) and \(\theta_{s,2}\) is missing. Actually, in [Bou00], it might be \(\theta_{s,1} + \theta_{s,2} \neq 0\) because of the construction of resonant blocks;
- If one tries to extend the method of Bourgain [Bou00] to higher lattice dimensions, there comes new difficulty: the resonant blocks at each iteration step could not be the cubes similar to the intervals appeared in the 1D case.

To extend the method of Bourgain [Bou00] to higher lattice dimensions and recover the Anderson localization, one has to address the above issues, which is our main motivation of this paper.

1.2. Main results. In this paper, we study the QP Schrödinger operators on \(\mathbb{Z}^d\)

\[
H(\theta) = \varepsilon \Delta + \cos 2\pi(\theta + n \cdot \omega)\delta_{n,n'}, \ \varepsilon > 0,
\]

where the discrete Laplacian \(\Delta\) is defined as

\[
\Delta(n, n') = \delta_{\|n-n'\|_1, 1}, \ ||n||_1 := \sum_{i=1}^d |n_i|.
\]

\[^3\text{i.e., a weak Hölder continuity estimate}

\[
|\mathcal{N}(E) - \mathcal{N}(E')| \leq e^{-\left(\log \frac{1}{|E-E'|}\right)^\zeta}, \ \zeta \in (0,1),
\]

where \(\mathcal{N}(\cdot)\) denotes the IDS.
For the diagonal part of (1.3), we have \( \theta \in T = \mathbb{R}/\mathbb{Z}, \omega \in [0, 1]^d \) and \( n \cdot \omega = \sum_{i=1}^d n_i \omega_i \).

Throughout the paper, we assume that \( \omega \in R_{\tau, \gamma} \) for some \( 0 < \tau < 1 \) and \( \gamma > 0 \) with

\[
R_{\tau, \gamma} = \left\{ \omega \in [0, 1]^d : \|n \cdot \omega\| = \inf_{l \in \mathbb{Z}} |l - n \cdot \omega| \geq \gamma e^{-\|n\| \tau} \text{ for } \forall n \in \mathbb{Z}^d \setminus \{0\} \right\},
\]

where

\[
\|n\| := \sup_{1 \leq i \leq d} |n_i|.
\]

We aim to extend the method of Bourgain [Bou00] to higher lattice dimensions and establish quantitative Green’s function estimates assuming (1.4). As the application, we prove the arithmetic version of Anderson localization and the finite volume version of \((\frac{1}{2} -)-\)Hölder continuity of the IDS for (1.3).

1.2.1. Quantitative Green’s function estimates. The first main result of this paper is a quantitative version of Green’s function estimates, which will imply both arithmetic Anderson localization and the finite volume version of \((\frac{1}{2} -)-\)Hölder continuity of IDS. The estimates on Green’s function are based on multi-scale induction arguments.

Let \( \Lambda \subset \mathbb{Z}^d \) and denote by \( R_\Lambda \) the restriction operator. Given \( E \in \mathbb{R} \), the Green’s function (if exists) is defined by

\[
T_\Lambda^{-1}(E; \theta) = (H_\Lambda(\theta) - E)^{-1}, \quad H_\Lambda(\theta) = R_\Lambda H(\theta) R_\Lambda.
\]

Recall that \( \omega \in R_{\tau, \gamma} \) and \( \tau \in (0, 1) \). We fix a constant \( c > 0 \) so that

\[
1 < e^{20} < \frac{1}{\tau}.
\]

At the \( s \)-th iteration step, let \( \delta_{s}^{-1} \) (resp. \( N_s \)) describe the resonance strength (resp. the size of resonant blocks) defined by

\[
N_{s+1} = \left[ \log \frac{\gamma}{\delta_s} \right], \quad \left| \log \frac{\gamma}{\delta_{s+1}} \right| = \left| \log \frac{\gamma}{\delta_s} \right|^{\delta_0}, \quad \delta_0 = \varepsilon^{\frac{1}{20}},
\]

where \( [x] \) denotes the integer part of \( x \in \mathbb{R} \).

If \( a \in \mathbb{R} \), let \( \|a\| = \text{dist}(a, \mathbb{Z}) = \inf_{l \in \mathbb{Z}} |l - a| \). For \( z = a + \sqrt{-1} b \in \mathbb{C} \) with \( a, b \in \mathbb{R} \), define \( \|z\| = \sqrt{|b|^2 + |a|^2} \). Denote by \( \text{dist}(\cdot, \cdot) \) the distance induced by the supremum norm on \( \mathbb{R}^d \). Then we have

**Theorem 1.1.** Let \( \omega \in R_{\tau, \gamma} \). Then there is some \( \varepsilon_0 = \varepsilon_0(d, \tau, \gamma) > 0 \) so that, for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( E \in [-2, 2] \), there exists a sequence \( \{\theta_s = \theta_s(E)\}_{s=0}^{s'} \subset \mathbb{C} \) \((s' \in \mathbb{N} \cup \{+\infty\})\) with the following properties. Fix any \( \theta \in T \). If a finite set \( \Lambda \subset \mathbb{Z}^d \) is \( s \)-good (cf. (e) of the Statement 3.1 for the definition of \( s \)-good sets, and Section 3 for the definitions of \( \{\theta_s\}_{s=0}^{s'} \), the sets \( P_s, Q_s, \Omega_{\Lambda}^s \)), then

\[
\|T_\Lambda^{-1}(E; \theta)\| \leq \delta_{s-1}^{-3} \sup_{\{k \in P_s \mid \Omega_s \subset \Lambda\}} \|\theta + k \cdot \omega - \theta_s\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_s\|^{-1} \leq \delta_{s-1}^{-3},
\]

\[
\|T_\Lambda^{-1}(E; \theta)(x, y)\| \leq e^{-\frac{1}{4} \log \varepsilon \|x - y\|} \text{ for } \|x - y\| > N_s^{-3}.
\]
In particular, for any finite set \( \Lambda \subset \mathbb{Z}^d \), there exists some \( \tilde{\Lambda} \) satisfying
\[
\Lambda \subset \tilde{\Lambda} \subset \left\{ k \in \mathbb{Z}^d : \text{dist}(k, \Lambda) \leq 50N_s^c \right\}
\]
so that, if
\[
\min_{k \in \Lambda^c} \min_{\sigma = \pm 1} (\| \theta + k \cdot \omega + \sigma \theta_s \|) \geq \delta_s,
\]
then
\[
\| T_{\tilde{\Lambda}}^{-1}(E; \theta) \| \leq \delta_{s-1}^{-3} \delta_s^{-2} \leq \delta_s^{-3},
\]
\[
| T_{\tilde{\Lambda}}^{-1}(E; \theta)(x, y) | \leq e^{-\frac{1}{4} \log e \| x - y \|} \text{ for } \| x - y \| > N_s^c,
\]
where
\[
\tilde{\Lambda}^* = \left\{ k \in \frac{1}{2} \mathbb{Z}^d : \text{dist}(k, \tilde{\Lambda}) \leq \frac{1}{2} \right\}.
\]

Let us refer to Section 3 for a complete description of our Green’s function estimates.

1.2.2. Arithmetic Anderson localization and Hölder continuity of the IDS. As the application of quantitative Green’s function estimates, we first prove the following arithmetic version of Anderson localization for \( H(\theta) \). Let \( \tau_1 > 0 \) and define
\[
\Theta_{\tau_1} = \{(\theta, \omega) \in \mathbb{T} \times \mathcal{R}_{\tau, \gamma} : \| 2 \theta + n \cdot \omega \| \leq e^{-\| n \|^{2/3}} \text{ holds for finitely many } n \in \mathbb{Z}^d \}.
\]

We have

**Theorem 1.2.** Let \( H(\theta) \) be given by (1.3) and let \( 0 < \tau_1 < \tau \). Then there exists some \( \varepsilon_0 = \varepsilon_0(d, \tau, \gamma) > 0 \) such that, if \( 0 < \varepsilon \leq \varepsilon_0 \), then for \( (\theta, \omega) \in \Theta_{\tau_1} \), \( H(\theta) \) satisfies the Anderson localization.

**Remark 1.1.** It is easy to check both \( \text{mes}(\mathbb{T} \setminus \Theta_{\tau_1, \omega}) = 0 \) and \( \text{mes}(\mathcal{R}_{\tau, \gamma} \setminus \Theta_{\tau_1, \theta}) = 0 \), where \( \Theta_{\tau_1, \omega} = \{ \theta \in \mathbb{T} : (\theta, \omega) \in \Theta_{\tau_1} \} \), \( \Theta_{\tau_1, \theta} = \{ \omega \in \mathcal{R}_{\tau, \gamma} : (\theta, \omega) \in \Theta_{\tau_1} \} \) and \( \text{mes}(\cdot) \) denotes the Lebesgue measure. Thus Anderson localization can be established either by fixing \( \omega \in \mathcal{R}_{\tau, \gamma} \) and removing \( \theta \) in the spirit of [Jit99], or by fixing \( \theta \in \mathbb{T} \) and removing \( \omega \) in the spirit of [BG00, BGS02].

The second application is a proof of the finite volume version of \( (\frac{1}{2} - \frac{d}{2}) \)-Hölder continuity of the IDS for \( H(\theta) \). For a finite set \( \Lambda \), denote by \#\( \Lambda \) the cardinality of \( \Lambda \). Let
\[
\mathcal{N}_\Lambda(E; \theta) = \frac{1}{\#\Lambda} \# \{ \lambda \in \sigma(H(\Lambda)) : \lambda \leq E \}
\]
and denote by
\[
\mathcal{N}(E) = \lim_{N \to \infty} \mathcal{N}_{\Lambda_N}(E; \theta) \tag{1.5}
\]
the IDS, where \( \Lambda_N = \{ k \in \mathbb{Z}^d : \| k \| \leq N \} \) for \( N > 0 \). It is well-known that the limit in (1.5) exists and is independent of \( \theta \) for a.e. \( \theta \).

**Theorem 1.3.** Let \( H(\theta) \) be given by (1.3) and let \( \omega \in \mathcal{R}_{\tau, \gamma} \). Then there exists some \( \varepsilon_0 = \varepsilon_0(d, \tau, \gamma) > 0 \) such that if \( 0 < \varepsilon \leq \varepsilon_0 \), then for any small \( \mu > 0 \) and \( 0 < \eta < \eta_0(d, \tau, \gamma, \mu) \), we have for sufficiently large \( N \) depending on \( \eta \),
\[
\sup_{\theta \in \mathbb{T}, E \in \mathbb{R}} (\mathcal{N}_{\Lambda_N}(E + \eta; \theta) - \mathcal{N}_{\Lambda_N}(E - \eta; \theta)) \leq \eta^{\frac{d}{2} - \mu}. \tag{1.6}
\]
In particular, the IDS is Hölder continuous of exponent \( \iota \) for any \( \iota \in (0, \frac{d}{2}) \).
Let us give some remarks on our results.

(1) The Green’s function estimates can be extended to the exponential long-range hopping case, and may not be restricted to the cosine potential. Except for the proof of arithmetic Anderson localization and the finite volume version of \((\frac{1}{2}−\)-Hölder regularity of the IDS, the quantitative Green’s function estimates should have potential applications in other problems, such as the estimates of Lebesgue measure of the spectrum, dynamical localization, the estimates of level spacings of eigenvalues and finite volume version of localization. We can even expect fine results in dealing with the Melnikov’s persistency problem (cf. \([\text{Bou97}]\)) by employing our Green’s function estimates method.

(2) As mentioned previously, Ge-You \([\text{GY20}]\) proved the first arithmetic Anderson localization result for higher dimensional QP operators with the exponential long-range hopping and the cosine potential via their reducibility method. Our result is valid for frequencies satisfying the sub-exponential non-resonant condition (cf. (1.4)) of Rüssmann type \([\text{Rüs80}]\), which slightly generalizes the Diophantine type localization result of \([\text{GY20}]\). While the Rüssmann type condition is sufficient for the use of classical KAM method, it is not clear if such condition still suffices for the validity of MSA method. Definitely, the localization result of both \([\text{GY20}]\) and the present work is perturbative.\(^4\) Finally, since our proof of arithmetic Anderson localization is based on Green’s function estimates, it could be improved to obtain the finite volume version of Anderson localization as that obtained in \([\text{GS11}]\).

(3) Apparently, using the Aubry duality together with Amor’s result \([\text{Amo09}]\) has already led to the \(\frac{1}{2}\)-Hölder continuity of the IDS for higher dimensional QP operators with small exponential long-range hopping and the cosine potential assuming Diophantine frequencies. So our result of \((\frac{1}{2}−\)-Hölder continuity is weaker than that of \([\text{Amo09}]\) in the Diophantine frequencies case. However, we want to emphasize that the method of Amor seems only valid for estimating the limit \(\mathcal{N}(E)\) and provides no precise information on the finite volume quantity \(\mathcal{N}_{\Lambda}(E;\theta)\). In this context, our result (cf. (1.6)) is also new as it gives uniform upper bound on the number of eigenvalues inside a small interval. In addition, our result also improves the upper bound on the number of eigenvalues of Schlag (cf. Proposition 2.2 of \([\text{Sch01}]\)) in the special case that the potential is given by the cosine function.

1.3. Notations and structure of the paper.

- Given \(A \in \mathbb{C}\) and \(B \in \mathbb{C}\), we write \(A \lesssim B\) (resp. \(A \gtrsim B\)) if there is some \(C = C(d, \tau, \gamma) > 0\) depending only on \(d, \tau, \gamma\) so that \(|A| \leq C|B|\) (resp. \(|A| \geq C|B|\)). We also denote \(A \sim B \iff \frac{1}{C} < \frac{|A|}{|B|} < C\), and for some \(D > 0\), \(A \overset{D}{\sim} B \iff \frac{1}{CD} < \frac{|A|}{|B|} < CD\).
- The determinant of a matrix \(M\) is denoted by \(\det M\).

\(^4\)In fact, Bourgain \([\text{Bou02}]\) has proven that the non-perturbative localization can not be expected in dimensions \(d \geq 2\). More precisely, consider \(H^{(2)} = \lambda \Delta + 2 \cos 2\pi(\theta + n \cdot \omega)\delta_{n,n'}\) on \(\mathbb{Z}^2\). Using Aubry duality together with result of Bourgain \([\text{Bou02}]\) yields for any \(\lambda \neq 0\), there exists a set \(\Omega \subset \mathbb{T}^2\) of positive measure with the following property, namely, for \(\omega \in \Omega\), there exists a set \(\Theta \subset T\) of positive measure, s.t., for \(\theta \in \Theta\), \(H^{(2)}\) does not satisfy Anderson localization.
• For $n \in \mathbb{R}^d$, let $\|n\|_1 := \sum_{i=1}^d |n_i|$ and $\|n\| := \sup_{1 \leq i \leq d} |n_i|$. Denote by $\text{dist}(\cdot, \cdot)$ the distance induced by $\|\cdot\|$ on $\mathbb{R}^d$, and define
  $$\text{diam} \Lambda = \sup_{k, k' \in \Lambda} \|k - k'\|.$$ 
Given $n \in \mathbb{Z}^d$, $\Lambda_1 \subset \frac{1}{2}\mathbb{Z}^d$ and $L > 0$, denote $\Lambda_L(n) = \{k \in \mathbb{Z}^d : \|k - n\| \leq L\}$ and $\Lambda_L(\Lambda_1) = \{k \in \mathbb{Z}^d : \text{dist}(k, \Lambda_1) \leq L\}$. In particular, write $\Lambda_L = \Lambda_L(0)$.

• Assume $\Lambda' \subset \Lambda \subset \mathbb{Z}^d$. Define the relatively boundaries as $\partial^+_\Lambda \Lambda' = \{k \in \Lambda : \text{dist}(k, \Lambda') = 1\}$, $\partial^-_\Lambda \Lambda' = \{k \in \Lambda : \text{dist}(k, \Lambda \setminus \Lambda') = 1\}$ and $\partial \Lambda \Lambda' = \{(k, k') : \|k - k'\| = 1, k \in \partial^+_\Lambda \Lambda', k' \in \partial^-_\Lambda \Lambda'\}$.

• Let $\Lambda \subset \mathbb{Z}^d$ and let $T : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be a linear operator. Define $T_\Lambda = R_\Lambda T R_\Lambda$, where $R_\Lambda$ is the restriction operator. Denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\ell^2(\mathbb{Z}^d)$. Set $T_\Lambda(x, y) = \langle \delta_x, T_\Lambda \delta_y \rangle$ for $x, y \in \Lambda$. By $\|T_\Lambda\|$ we mean the standard operator norm of $T_\Lambda$. The spectrum of the operator $T$ is denoted by $\sigma(T)$. Finally, $I$ typically denotes the identity operator.

The paper is organized as follows. The key ideas of the proof are introduced in §2. The proofs of Theorems 1.1, 1.2 and 1.3 are presented in §3, §4 and §5, respectively. Some useful estimates can be found in the appendix.

2. Key ideas of the proof

The main scheme of our proof is definitely adapted from Bourgain [Bou00]. The key ingredient of the proof in [Bou00] is that the resonances in dealing with Green’s function estimates can be completely determined by the roots of some quadratic polynomials. The polynomials were produced in a Fröhlich-Spencer type MSA induction procedure. However, in the estimates of Green’s functions restricted on the resonant blocks, Bourgain applied directly the Cramer’s rule and provided estimates on certain determinants. It turns out these determinants can be well controlled via estimates of previous induction steps, the Schur complement argument and Weierstrass preparation theorem. It is the preparation type technique that yields the desired quadratic polynomials. We emphasize that this new method of Bourgain is fully free from eigenvalues variations or eigenfunctions parametrization.

However, in order to extend the method to achieve arithmetic version of Anderson localization in higher dimensions, some new ideas are required:

• The off-diagonal decay of the Green’s function obtained by Bourgain [Bou00] is sub-exponential rather than exponential, which is not sufficient for a proof of Anderson localization. We resolve this issue by modifying the definitions of the resonant blocks $\Omega_k \subset \tilde{\Omega}_k \subset \mathbb{Z}^d$, and allowing
  $$\text{diam} \; \Omega_k \sim (\text{diam} \; \tilde{\Omega}_k)^\rho, \; 0 < \rho < 1.$$ 
This sublinear bound is crucial for a proof of exponential off-diagonal decay. In the argument of Bourgain, it requires actually that $\rho = 1$. Another issue we want to highlight is that Bourgain just provided outputs of iterating resolvent identity in many places of the paper [Bou00], but did not present the details. This motivates us to write down the whole iteration arguments that is also important to the exponential decay estimate.
• To prove Anderson localization, one has to eliminate the energy $E \in \mathbb{R}$ appeared in the Green’s function estimates by removing $\theta$ or $\omega$ further. Moreover, if one wants to prove an arithmetic version of Anderson localization, a geometric description of resonances (i.e., the symmetry of zeros of certain functions appearing as the perturbations of quadratic polynomials in the present context) is essential. Precisely, at the $s$-th iteration step, using the Weierstrass preparation theorem Bourgain [Bou00] had shown the existence of zeros $\theta_{s,1}(E)$ and $\theta_{s,2}(E)$, but provided no symmetry information. Indeed, the symmetry property of $\theta_{s,1}(E)$ and $\theta_{s,2}(E)$ relies highly on that of resonant blocks $\tilde{\Omega}_k^s$. However, in the construction of $\tilde{\Omega}_k^s$ in [Bou00], the symmetry property is missing. In this paper, we prove in fact $\theta_{s,1}(E) + \theta_{s,2}(E) = 0$.

The main idea is that we reconstruct $\tilde{\Omega}_k^s$ so that it is symmetrical about $k$ and allow the center $k \in \mathbb{Z}^d$.

• In the construction of resonant blocks [Bou00], the property that $\tilde{\Omega}_k^{s'} \cap \tilde{\Omega}_k^s \neq \emptyset \Rightarrow \tilde{\Omega}_k^{s'} \subset \tilde{\Omega}_k^s$ for $s' < s$ (2.1) plays a center role. In the $1D$ case, $\tilde{\Omega}_k^s$ can be defined as an interval so that (2.1) holds true. This interval structure of $\tilde{\Omega}_k^s$ is important to get desired estimates using resolvent identity. However, to generalize this argument to higher dimensions, one needs to give up the “interval” structure of $\tilde{\Omega}_k^s$ in order to fulfill the property (2.1). As a result, the geometric description of $\tilde{\Omega}_k^s$ becomes significantly complicated, and the estimates relying on resolvent identity remain unclear. We address this issue by proving that $\tilde{\Omega}_k^s$ can be constructed satisfying (2.1) and staying in some enlarged cubes, such as

$$\Lambda_{N^2} \subset \tilde{\Omega}_k^s - k \subset \Lambda_{N^2 + 50N^2 - 1} .$$

• We want to mention that in the estimates of zeros for some perturbations of quadratic polynomials, we use the standard Róche theorem rather than the Weierstrass preparation theorem as in [Bou00]. This technical modification avoids controlling the first order derivatives of determinants and simplifies significantly the proof.

The proofs of both Theorem 1.2 and Theorem 1.3 follow from the estimates in Theorem 1.1.

3. Quantitative Green’s function estimates

The spectrum $\sigma(H(\theta)) \subset [-2, 2]$ since $\|H(\theta)\| \leq 1 + 2d \varepsilon < 2$ if $0 < \varepsilon < \frac{1}{2d}$. In this section, we fix

$$\theta \in T, \ E \in [-2, 2].$$

Write

$$E = \cos 2\pi \theta_0$$

with $\theta_0 \in \mathbb{C}$. Consider

$$T(E; \theta) = H(\theta) - E = D_n \delta_{n,n'} + \varepsilon \Delta ,$$

(3.1)
where

\[ D_n = \cos 2\pi(\theta + n \cdot \omega) - E. \]  

(3.2)

For simplicity, we may omit the dependence of \( T(E; \theta) \) on \( E, \theta \) below.

We will use a multi-scale analysis induction to provide estimates on Green’s functions. Of particular importance is the analysis of resonances, which will be described by zeros of certain functions appearing as perturbations of some quadratic polynomials. Roughly speaking, at the \( s \)-th iteration step, the set \( Q_s \subseteq \frac{1}{2} \mathbb{Z}^d \) of singular sites will be completely described by a pair of symmetric zeros of certain functions, i.e.,

\[ Q_s = \bigcup_{\sigma = \pm 1} \{ k \in P_s : \| \theta + k \cdot \omega + \sigma \theta_s \| < \delta_s \}. \]

While the Green's functions restricted on \( Q_s \) cannot be generally well controlled, the algebraic structure of \( Q_s \) combined with the non-resonant condition of \( \omega \) may lead to fine separation property of singular sites. As a result, one can cover \( Q_s \) with a new generation of resonant blocks \( \tilde{\Omega}_k^{s+1} (k \in P_{s+1}) \). It turns out that one can control \( \| T_{\tilde{\Omega}_k^{s+1}} \| \) via zeros \( \pm \theta_{s+1} \) of some new functions which are also perturbations of quadratic polynomials in the sense that

\[ \det T_{\tilde{\Omega}_k^{s+1}} \sim \delta_s^{-2} \| \theta + k \cdot \omega - \theta_{s+1} \| \cdot \| \theta + k \cdot \omega + \theta_{s+1} \|. \]

The key point is that some \( T_{\tilde{\Omega}_k^{s+1}}^{-1} \) while \( \tilde{\Omega}_k^{s+1} \) intersecting \( Q_s \) become controllable \(^5\) in the \((s + 1)\)-th step. Moreover, the completely uncontrollable singular sites form the \((s + 1)\)-th singular sites, i.e.,

\[ Q_{s+1} = \bigcup_{\sigma = \pm 1} \{ k \in P_{s+1} : \| \theta + k \cdot \omega + \sigma \theta_{s+1} \| < \delta_{s+1} \}. \]

Now we turn to the statement of our main result on the multi-scale type Green’s function estimates. Define the induction parameters as follows.

\[ N_{s+1} = \left\lfloor \log \frac{\gamma}{\delta_s} \left( \frac{1}{2^s} \right) \right\rfloor, \quad | \log \frac{\gamma}{\delta_{s+1}} | = | \log \frac{\gamma}{\delta_s} |^5. \]

Thus

\[ N_s^c - 1 \leq N_{s+1} \leq (N_s + 1)^c. \]

We first introduce the following statement.

**Statement 3.1** \((P_s (s \geq 1))\). \( \quad \)

Let

\[ Q_{s-1}^+ \subseteq \{ k \in P_{s-1} : \| \theta + k \cdot \omega + \theta_{s-1} \| < \delta_{s-1} \}, \quad Q_{s-1}^- \subseteq Q_{s-1}^+ \cup Q_{s-1}^-; \]

\[ \tilde{Q}_{s-1}^+ \subseteq \{ k \in P_{s-1} : \| \theta + k \cdot \omega + \theta_{s-1} \| < \delta_{s-1} \}, \quad \tilde{Q}_{s-1}^- \subseteq \tilde{Q}_{s-1}^+ \cup \tilde{Q}_{s-1}^-; \]

We distinguish the following two cases:

(C1)\(_{s-1}. \)

\[ \text{dist}(\tilde{Q}_{s-1}^-, Q_{s-1}^+) > 100 N_s^c, \]

(3.5)

(C2)\(_{s-1}. \)

\[ \text{dist}(\tilde{Q}_{s-1}^-, Q_{s-1}^+) \leq 100 N_s^c. \]

(3.6)

\(^5\)Even more general sets, e.g., the \((s + 1)\)-good sets remain true.
Let
\[ Z^d \ni l_{s-1} = \begin{cases} 0 & \text{if (3.5) holds true,} \\ i_{s-1} - j_{s-1} & \text{if (3.6) holds true,} \end{cases} \]
where \( i_{s-1} \in Q_{s-1}^+ \), \( j_{s-1} \in \overline{Q}_{s-1}^- \) such that \( \|i_{s-1} - j_{s-1}\| \leq 100N^c_s \) in \((C2)_{s-1}\). Set \( \Omega^i_k = \{k \} \) \( (k \in \mathbb{Z}^d) \). Let \( \Lambda \subset \mathbb{Z}^d \) be a finite set. We say \( \Lambda \) is \((s-1)\)-good iff
\begin{equation}
\begin{cases}
\{k' \in Q_s, \overline{\Omega}_k' \subset \Lambda, \overline{\Omega}_k' \subset \Omega_k' \Rightarrow \overline{\Omega}_k' \subset \Lambda \text{ for } s' < s - 1, \\
\{k \in P_{s-1} : \Omega_k^{s-1} \subset \Lambda \} \cap Q_{s-1} = \emptyset.
\end{cases}
\end{equation}

Then
\((a)\). There are \( P_s \subset Q_{s-1} \) so that the following holds true. We have in the case \((C1)_{s-1}\) that
\begin{equation}
P_s = Q_{s-1} \subset \left\{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \min_{\sigma = \pm 1} \|\theta + k \cdot \omega + \sigma \theta_{s-1}\| < \delta_{s-1} \right\}.
\end{equation}

For the case \((C2)_{s-1}\), we have
\begin{equation}
P_s \subset \left\{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \|\theta + k \cdot \omega\| < 3\delta_{s-1} \right\},
\end{equation}
or \( P_s \subset \left\{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \|\theta + k \cdot \omega + \frac{1}{2}\| < 3\delta_{s-1} \right\}.
\end{equation}

For every \( k \in P_s \), we can find resonant blocks \( \Omega_k^s, \overline{\Omega}_k^s \subset \mathbb{Z}^d \) with the following properties. If (3.5) holds true, then
\[ \Lambda_{N_s} (k) \subset \Omega_k^s \subset \Lambda_{N_s + 50N^c_{s-1}} (k), \]
\[ \Lambda_{N^c_s} (k) \subset \overline{\Omega}_k^s \subset \Lambda_{N^c_s + 50N^c_{s-1}} (k), \]
and if (3.6) holds true, then
\[ \Lambda_{100N^c_s} (k) \subset \Omega_k^s \subset \Lambda_{100N^c_s + 50N^c_{s-1}} (k), \]
\[ \Lambda_{N^c_s} (k) \subset \overline{\Omega}_k^s \subset \Lambda_{N^c_s + 50N^c_{s-1}} (k). \]

These resonant blocks are constructed satisfying the following two properties.
\((a1)\).
\begin{equation}
\begin{cases}
\Omega_k^s \cap \overline{\Omega}_k^{s'} \neq \emptyset \ (s' < s) \Rightarrow \overline{\Omega}_k^{s'} \subset \Omega_k^s, \\
\overline{\Omega}_k^s \cap \overline{\Omega}_k^{s'} \neq \emptyset \ (s' < s) \Rightarrow \overline{\Omega}_k^{s'} \subset \overline{\Omega}_k^s, \\
dist(\overline{\Omega}_k^s, \overline{\Omega}_{k'}^s) > 10 \text{ diam } \overline{\Omega}_k^s \text{ for } k \neq k' \in P_s.
\end{cases}
\end{equation}

\((a2)\). The translation of \( \overline{\Omega}_k^s \),
\[ \overline{\Omega}_k^s - k \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i \]
is independent of \( k \in P_s \) and symmetrical about the origin.
(b). $Q_{s-1}$ is covered by $\Omega_k^s$ ($k \in P_s$) in the sense that, for every $k' \in Q_{s-1}$, there exists $k \in P_s$ such that
\[ \Omega_{k'}^s \subset \Omega_k^s. \] (3.11)

(c). For each $k \in P_s$, $\Omega_k^s$ contains a subset $A_k^s \subset \Omega_k^s$ with $\#A_k^s \leq 2^s$ such that $\Omega_k^s \setminus A_k^s$ is $(s-1)$-good. Moreover, $A_k^s - k$ is independent of $k$ and is symmetrical about the origin.

(d). There is $\theta_s = \theta_s(E) \in \mathbb{C}$ with the following properties. Replacing $\theta + n \cdot \omega$ by $z + (n - k) \cdot \omega$, and restricting $z$ in
\[ \{ z \in \mathbb{C} : \min_{\sigma = \pm 1} \| z + \sigma \theta_s \| < \frac{1}{\delta_s} \}, \] (3.12)
then $T_{\Omega_k^s}$ becomes
\[ M_s(z) = T(z)_{\Omega_k^s - k} = (\cos 2\pi(z + n \cdot \omega)\delta_{n,n'} - E + \epsilon \Delta)_{\Omega_k^s - k}. \]
Then $M_s(z)_{(\Omega_k^s - k) \setminus (A_k^s - k)}$ is invertible and we can define the Schur complement
\[ S_s(z) = M_s(z)_{A_k^s - k} - R_{A_k^s - k} M_s(z) R_{(\Omega_k^s - k) \setminus (A_k^s - k)}^{-1} \]
\[ \times R_{(\Omega_k^s - k) \setminus (A_k^s - k)} M_s(z)_{A_k^s - k}. \]
Moreover, if $z$ belongs to the set defined by (3.12), then we have
\[ \max_x \sum_y |S_s(z)(x,y)| < 4 + \sum_{i=0}^{s-1} \delta_i < 10, \] (3.13)
and
\[ \det S_s(z) \delta_s^{\sim 3} \| z - \theta_s \| \cdot \| z + \theta_s \|. \] (3.14)

(e). We say a finite set $\Lambda \subset \mathbb{Z}^d$ is $s$-good iff
\[ \left\{ \begin{array}{l} k' \in Q_{s'}, \Omega_{k'}^{s'} \subset \Lambda, \Omega_{k'}^{s'+1} \subset \Omega_{k'}^{s'} \Rightarrow \Omega_{k'}^{s'+1} \subset \Lambda \text{ for } s' < s, \\
\{ k \in P_s : \Omega_k^s \subset \Lambda \} \cap Q_s = \emptyset. \end{array} \right. \] (3.15)
Assuming $\Lambda$ is $s$-good, then
\[ \| T_{\Lambda}^{-1} \| \leq \delta_s^{-3} \sup_{k \in P_s : \Omega_k^s \subset \Lambda} \| \theta + k \cdot \omega - \theta_s \|^{-1} \cdot \| \theta + k \cdot \omega + \theta_s \|^{-1} \] (3.16)
\[ < \delta_s^{-3}, \]
\[ | T_{\Lambda}^{-1}(x,y) | < e^{-\gamma_s \| x-y \|^3} \text{ for } \| x - y \| > N_s^{-3}, \] (3.17)
where
\[ \gamma_0 = \frac{1}{2} \| \log \epsilon \|, \quad \gamma_s = \gamma_{s-1}(1 - N_s^{-3-1})^3. \]
Thus $\gamma_s \gamma_0 \geq \frac{1}{2} \gamma_0 = \frac{1}{2} \| \log \epsilon \|$. (3.18)

(f). We have
\[ \left\{ k \in \mathbb{Z}^d \mid \frac{1}{2} \sum_{i=0}^{s-1} l_i : \min_{\sigma = \pm 1} \| \theta + k \cdot \omega + \sigma \theta_s \| < 10 \delta_s^{\sim 3} \right\} \subset P_s. \]

The main theorem of this section is

**Theorem 3.2.** Let $\omega \in \mathcal{R}_{\tau, \gamma}$. Then there is some $\epsilon_0(d, \tau, \gamma) > 0$ so that for $0 < \epsilon \leq \epsilon_0$, the statement $P_s$ holds for all $s \geq 1$. 
The following three subsections are devoted to prove Theorem 3.2.

3.1. **The initial step.** Recalling (3.1)–(3.2) and \( \cos 2\pi \theta_0 = E \), we have
\[
|D_n| = 2|\sin \pi(\theta + n \cdot \omega + \theta_0)\sin \pi(\theta + n \cdot \omega - \theta_0)| \\
\geq 2\|\theta + n \cdot \omega + \theta_0\| \cdot \|\theta + n \cdot \omega - \theta_0\|.
\]
Denote \( \delta_0 = \varepsilon^{1/10} \) and
\[
P_0 = \mathbb{Z}^d, \quad Q_0 = \{k \in P_0 : \min(\|\theta + k \cdot \omega + \theta_0\|, \|\theta + k \cdot \omega - \theta_0\|) < \delta_0\}.
\]
We say a finite set \( \Lambda \subset \mathbb{Z}^d \) is **0-good** iff
\[
\Lambda \cap Q_0 = \emptyset.
\]
**Lemma 3.3.** If the finite set \( \Lambda \subset \mathbb{Z}^d \) is 0-good, then
\[
\|T^{-1}_\Lambda\| < 2\|D^{-1}_\Lambda\| < \delta_0^{-2},
\]
(3.19)
\[
|T^{-1}_\Lambda(x, y)| < e^{-\gamma_0\|x-y\|_1} \text{ for } \|x - y\| > 0.
\]
(3.20)
where \( \gamma_0 = 5|\log \delta_0| = \frac{1}{2}|\log \varepsilon| \).

**Proof of Lemma 3.3.** Assuming \( \Lambda \) is 0-good, we have
\[
\|D^{-1}_\Lambda\| < \frac{1}{2}\delta_0^{-2}, \quad \|\varepsilon D^{-1}_\Lambda \Delta \Lambda\| < d\varepsilon \delta_0^{-2} < \frac{1}{2}\delta_0^{-2} < \frac{1}{2}
\]
Thus
\[
T^{-1}_\Lambda = (I + \varepsilon D^{-1}_\Lambda \Delta \Lambda)^{-1} D^{-1}_\Lambda
\]
and \((I + \varepsilon D^{-1}_\Lambda \Delta \Lambda)^{-1}\) may be expanded in the Neumann series
\[
(I + \varepsilon D^{-1}_\Lambda \Delta \Lambda)^{-1} = \sum_{i=0}^{+\infty} (-\varepsilon D^{-1}_\Lambda \Delta \Lambda)^i.
\]
Hence
\[
\|T^{-1}_\Lambda\| < 2\|D^{-1}_\Lambda\| < \delta_0^{-2},
\]
which implies (3.19).

In addition, if \( \|x - y\|_1 > i \), then
\[
((\varepsilon D^{-1}_\Lambda \Delta \Lambda)^i D^{-1}_\Lambda) (x, y) = 0.
\]
Hence
\[
|T^{-1}_\Lambda(x, y)| = \sum_{i \geq \|x-y\|_1} |((\varepsilon D^{-1}_\Lambda \Delta \Lambda)^i D^{-1}_\Lambda) (x, y)| < \delta_0^{i-\|x-y\|_1} \cdot 2.
\]
In particular,
\[
|T^{-1}_\Lambda(x, y)| < e^{-\gamma_0\|x-y\|_1} \text{ for } \|x - y\| > 0
\]
with \( \gamma_0 = 5|\log \delta_0| = \frac{1}{2}|\log \varepsilon| \), which yields (3.20). \(\square\)
3.2. Verification of $P_1$. If $\Lambda \cap Q_0 \neq \emptyset$, then the Neumann series argument of previous subsection does not work. Thus we use the resolvent identity argument to estimate $T_{\Lambda}^{-1}$, where $\Lambda$ is 1-good (1-good will be specified later) but might intersect with $Q_0$ (not 0-good).

First, we construct blocks $\Omega_k^1 (k \in P_1)$ to cover the singular point $Q_0$. Second, we get the bound estimate

$$
\|T_{\Omega_k^1}^{-1}\| < \delta_0^{-2}\|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1},
$$

where $\tilde{\Omega}_k^1$ is an extension of $\Omega_k^1$, and $\theta_1$ is obtained by analyzing the root of the equation $\det T(z - k \cdot \omega)_{\tilde{\Omega}_k^1} = 0$ about $z$. Finally, we combine the estimates of $T_{\tilde{\Omega}_k^1}^{-1}$ to get that of $T_{\Lambda}^{-1}$ by resolvent identity assuming $\Lambda$ is 1-good.

Recall that

$$
1 < e^{20} < \frac{1}{\tau}.
$$

Let

$$
N_1 = \left[ \log \frac{\gamma}{\delta_0} \left( \frac{1}{\tau} \right)^{\frac{1}{\gamma}} \right].
$$

Define

$$
Q_k^+ = \left\{ k \in \mathbb{Z}^d : \|\theta + k \cdot \omega + \theta_0\| < \delta_0 \right\}, \quad Q_0 = Q_k^+ \cup Q_k^-,
$$

$$
\tilde{Q}_0^+ = \left\{ k \in \mathbb{Z}^d : \|\theta + k \cdot \omega + \theta_0\| < \delta_0 \right\}, \quad \tilde{Q}_0 = \tilde{Q}_0^+ \cup \tilde{Q}_0^-.
$$

We distinguish three steps.

**STEP1: The case (C1)$_0$ Occurs** : i.e.,

$$
\text{dist}\left(\tilde{Q}_0^-, Q_0^+\right) > 100N_1^c. \tag{3.21}
$$

**Remark 3.1.** We have in fact

$$
\text{dist}\left(\tilde{Q}_0^-, Q_0^+\right) = \text{dist}\left(\tilde{Q}_0^+, Q_0^-\right).
$$

Thus (3.21) also implies

$$
\text{dist}\left(\tilde{Q}_0^+, Q_0^-\right) > 100N_1^c.
$$

We refer to the Appendix A for a detailed proof.

Assuming (3.21), we define

$$
P_1 = Q_0 = \left\{ k \in \mathbb{Z}^d : \min(\|\theta + k \cdot \omega + \theta_0\|, \|\theta + k \cdot \omega - \theta_0\|) < \delta_0 \right\}. \tag{3.22}
$$

Associate every $k \in P_1$ an $N_1$-block $\Omega_k^1 := \Lambda_{N_1}(k)$ and an $N_1^c$-block $\tilde{\Omega}_k^1 := \Lambda_{N_1^c}(k)$. Then $\Omega_k^1 - k \subset \mathbb{Z}^d$ is independent of $k \in P_1$ and symmetrical about the origin. If $k \neq k' \in P_1$,

$$
\|k - k'\| \geq \min\left(100N_1^c, \left|\log \frac{\gamma}{2\delta_0} \right|^{\frac{1}{\gamma}}\right) \geq 100N_1^c.
$$

Thus

$$
\text{dist}\left(\Omega_k^1, \tilde{\Omega}_{k'}^1\right) > 10\text{diam} \tilde{\Omega}_k^1 \text{ for } k \neq k' \in P_1.
$$

For $k \in Q_0^-$, we consider

$$
M_1(z) := T(z)_{\tilde{\Omega}_k^1 - k} = (\cos 2\pi(z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon \Delta)_{n \in \tilde{\Omega}_k^1 - k}
$$
defined in
\[ \{ z \in \mathbb{C} : |z - \theta_0| < \delta_0^{\frac{1}{m}} \}. \] (3.23)

For \( n \in (\bar{\Omega}_k^1 - k) \setminus \{0\} \), we have for \( 0 < \delta_0 \ll 1 \),
\[
\|z + n \cdot \omega - \theta_0\| \geq \|n \cdot \omega\| - |z - \theta_0| \\
\geq \gamma e^{-N_1^n} - \delta_0^{\frac{1}{m}} \\
\geq \gamma e^{-|\log \frac{\delta_0}{\delta_0}|} - \delta_0^{\frac{1}{m}} \\
> \delta_0^{\frac{1}{m}}.
\]

For \( n \in \bar{\Omega}_k^1 - k \), we have
\[
\|z + n \cdot \omega + \theta_0\| \geq \|\theta + (n + k) \cdot \omega + \theta_0\| - |z - \theta_0| - \|\theta + k \cdot \omega - \theta_0\| \\
\geq \delta_0^{\frac{1}{m}} - \delta_0^{\frac{1}{m}} - \delta_0 \\
> \frac{1}{2} \delta_0^{\frac{1}{m}}.
\]

Since \( \delta_0 \gg \varepsilon \), we have by Neumann series argument
\[
\left\| \left( M_1(z)_{(\bar{\Omega}_k^1 - k) \setminus \{0\}} \right)^{-1} \right\| < 3 \delta_0^{\frac{1}{m}}.
\]

Now we can apply the Schur complement lemma (cf. Lemma B.1 in the appendix) to provide desired estimates. By Lemma B.1, \( M_1(z)^{-1} \) is controlled by the inverse of the Schur complement (of \((\bar{\Omega}_k^1 - k) \setminus \{0\}\))
\[
S_1(z) = M_1(z)_{\{0\}} - R_{\{0\}} M_1(z) R_{(\bar{\Omega}_k^1 - k) \setminus \{0\}} \left( M_1(z)_{(\bar{\Omega}_k^1 - k) \setminus \{0\}} \right)^{-1} R_{(\bar{\Omega}_k^1 - k) \setminus \{0\}} M_1(z) R_{\{0\}} \\
= -2 \sin \pi (z - \theta_0) \sin \pi (z + \theta_0) + r(z) \\
= g(z)(((z - \theta_0) + r_1(z)),
\]
where \( g(z) \) and \( r_1(z) \) are analytic functions in the set defined by (3.23) satisfying
\[ |g(z)| \geq 2\|z + \theta_0\| > \delta_0^{\frac{1}{m}} \text{ and } |r_1(z)| < \varepsilon \delta_0^{-1} < \varepsilon. \]

Since
\[ |r_1(z)| < |z - \theta_0| \text{ for } |z - \theta_0| = \delta_0^{\frac{1}{m}}, \]
using Róche theorem implies the equation
\[ (z - \theta_0) + r_1(z) = 0 \]
has a unique root \( \theta_1 \) in the set of (3.23) satisfying
\[ |\theta_0 - \theta_1| = |r_1(\theta_1)| < \varepsilon, \quad |(z - \theta_0) + r_1(z)| \sim |z - \theta_1|. \]

Moreover, \( \theta_1 \) is the unique root of \( \det M_1(z) = 0 \) in the set (3.23). Since \( \|z + \theta_0\| > \frac{1}{2} \delta_0^{\frac{1}{m}} \) and \( |\theta_0 - \theta_1| < \varepsilon \), we get
\[ \|z + \theta_1\| \sim \|z + \theta_0\|, \]
which shows for \( z \) being in the set of (3.23),
\[
|S_1(z)| \sim \|z + \theta_1\| \cdot \|z - \theta_2\|,
\]
(3.24)

\[
\|M_1(z)^{-1}\| < 4 \left(1 + \|(M_1(z)_{(\bar{\Omega}_k^1 - k) \setminus \{0\}})^{-1}\| \right) \left(1 + |S_1(z)|^{-1}\right) \]
\[ < \delta_0^{-2} \|z + \theta_1\|^{-1} \cdot \|z - \theta_1\|^{-1}. \]
(3.25)
where in the first inequality we use Lemma B.1. Now, for $k \in Q^+_0$, we consider $M_1(z)$ in
\[
\{ z \in \mathbb{C} : |z + \theta_0| < \frac{1}{\delta_0^{10}} \}. \tag{3.26}
\]
The similar argument shows that $\det M_1(z) = 0$ has a unique root $\theta_1'$ in the set of (3.26). We will show $\theta_1 + \theta_1' = 0$. In fact, by Lemma C.1, $\det M_1(z)$ is an even function of $z$. Then the uniqueness of the root implies $\theta_1' = -\theta_1$. Thus for $z$ being in the set of (3.26), both (3.24) and (3.25) hold true as well. Finally, since $M_1(z)$ is 1-periodic, (3.24) and (3.25) remain valid for
\[
z \in \{ z \in \mathbb{C} : \min_{\sigma = \pm 1} |z + \sigma \theta_0| < \frac{1}{\delta_0^{10}} \}. \tag{3.27}
\]
From (3.22), we have $\theta + k \cdot \omega$ belongs to the set of (3.27). Thus for $k \in P_1$, we get
\[
\|T_{\Omega_k}^{-1}\| = \|M_1(\theta + k \cdot \omega)^{-1}\|
< \delta_0^{-2} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1}. \tag{3.28}
\]

**STEP2: The case (C2)0 Occurs:** i.e.,
\[
dist(Q^-_0, Q^+_0) \leq 100N_1^c.
\]
Then there exist $i_0 \in Q^+_0$ and $j_0 \in Q^-_0$ with $\|i_0 - j_0\| \leq 100N_1^c$, such that
\[
\|\theta + i_0 \cdot \omega + \theta_0\| < \delta_0, \|\theta + j_0 \cdot \omega - \theta_0\| < \delta_0^{10}.
\]
Denote
\[
l_0 = i_0 - j_0.
\]
Then $\|l_0\| = dist(Q_0^+, Q_0^-) = dist(Q_0^+, Q_0^-)$. Define
\[
O_1 = Q_0^- \cup (Q_0^+ - l_0).
\]
For $k \in Q_0^+$, we have
\[
\|\theta + (k - l_0) \cdot \omega - \theta_0\| < \|\theta + k \cdot \omega + \theta_0\| + \|l_0 \cdot \omega + 2\theta_0\|
< \delta_0 + \delta_0 + \delta_0^{10} < 2\delta_0^{10}.
\]
Thus
\[
O_1 \subset \left\{ o \in \mathbb{Z}^d : \|\theta + o \cdot \omega - \theta_0\| < 2\delta_0^{10} \right\}.
\]
For every $o \in O_1$, define its mirror point
\[
o^* = o + l_0.
\]
Next define
\[
P_1 = \left\{ \frac{1}{2} (o + o^*) : o \in O_1 \right\} = \left\{ o + \frac{l_0}{2} : o \in O_1 \right\}. \tag{3.29}
\]
Associate every $k \in P_1$ with a $100N_1^c$-block $\Omega^1_k := \Lambda_{100N_1^c}(k)$ and a $N_1^{c_2}$-block $\tilde{\Omega}^1_k := \Lambda_{N_1^{c_2}}(k)$. Thus
\[
Q_0 \subset \bigcup_{k \in P_1} \Omega^1_k.
\]
and $\tilde{\Omega}_k - k \subset \mathbb{Z}^d + \frac{1}{2}$ is independent of $k \in P_1$ and symmetrical about the origin. Notice that
\[
\min \left( \| \frac{l_0}{2} \cdot \omega + \theta_0 \|, \| \frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2} \| \right) = \frac{1}{2} \| l_0 \cdot \omega + 2\theta_0 \|
\]
\[
\leq \frac{1}{2} (\| \theta + i_0 \cdot \omega + \theta_0 \| + \| \theta + j_0 \cdot \omega - \theta_0 \|) < \delta_0^{\frac{1}{10^4}}.
\]
Since $\delta_0 \ll 1$, only one of
\[
\| \frac{l_0}{2} \cdot \omega + \theta_0 \| < \delta_0^{\frac{1}{10^4}}, \quad \| \frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2} \| < \delta_0^{\frac{1}{10^4}}
\]
holds true. First, we consider the case
\[
\| \frac{l_0}{2} \cdot \omega + \theta_0 \| < \delta_0^{\frac{1}{10^4}}.
\]
Let $k \in P_1$. Since $k = \frac{1}{2} (o + o^*) = (o + \frac{1}{2})$ (for some $o \in O_1$), we have
\[
\| \theta + k \cdot \omega \| \leq \| \theta + o \cdot \omega - \theta_0 \| + \| \frac{l_0}{2} \cdot \omega + \theta_0 \| < 3\delta_0^{\frac{1}{10^4}}.
\]
Thus if $k \neq k' \in P_1$, we obtain
\[
\| k - k' \| \geq \log \frac{\gamma}{6\delta_0^{\frac{1}{10^4}}} \sim N_1^{\frac{1}{5}} \gg 10N_1^{\frac{2}{5}},
\]
which implies
\[
\text{dist}(\tilde{\Omega}_k, \tilde{\Omega}_{k'}) > 10 \text{diam} \tilde{\Omega}_k \text{ for } k \neq k' \in P_1.
\]
Consider
\[
M_1(z) := T(z)\tilde{\Omega}_k - k = ((\cos 2\pi (z \cdot \omega) \delta_{n,n'} - E + \varepsilon \Delta)_{n \in \tilde{\Omega}_k - k}
\]
in
\[
\left\{ z \in \mathbb{C} : \left| z \right| < \delta_0^{\frac{1}{10^4}} \right\}.
\]
For $n \neq \pm \frac{l_0}{2}$ and $n \in \tilde{\Omega}_k - k$, we have
\[
\| n \cdot \omega \pm \theta_0 \| \geq \| \left( n \mp \frac{l_0}{2} \right) \cdot \omega \| - \| \frac{l_0}{2} \cdot \omega + \theta_0 \|
\]
\[
\geq \gamma e^{-2N_1^{\frac{1}{2}}} - \delta_0^{\frac{1}{10^4}} > 2\delta_0^{\frac{1}{10^4}}.
\]
Thus for $z$ being in the set of (3.32) and $n \neq \pm \frac{l_0}{2}$, we have
\[
\| z + n \cdot \omega \pm \theta_0 \| \geq \| n \cdot \omega \pm \theta_0 \| - |z| > \delta_0^{\frac{1}{10^4}}.
\]
Hence
\[
| \cos 2\pi (z \cdot \omega) - E | \geq \delta_0^{2 \times \frac{1}{10^4}} \gg \varepsilon.
\]
Using Neumann series argument concludes
\[
\left\| \left( M_1(z) | \tilde{\Omega}_k - k \setminus \{ \pm \frac{l_0}{2} \} \right)^{-1} \right\| < \delta_0^{-3 \times \frac{1}{10^4}}.
\]
Thus by Lemma B.1, \( M_1(z)^{-1} \) is controlled by the inverse of the Schur complement of \( \tilde{\Omega}_k \setminus \{ \pm \frac{\theta_0}{2} \} \), i.e.,

\[
S_1(z) = M_1(z)_{\{ \pm \frac{\theta_0}{2} \}} - R_{\{ \pm \frac{\theta_0}{2} \}} M_1(z)_{\tilde{\Omega}_k \setminus \{ \pm \frac{\theta_0}{2} \}} (M_1(z)_{\tilde{\Omega}_k \setminus \{ \pm \frac{\theta_0}{2} \}})^{-1} R_{\{ \pm \frac{\theta_0}{2} \}} M_1(z)_{\{ \pm \frac{\theta_0}{2} \}}.
\]

Clearly,

\[
\det S_1(z) = \det \left( M_1(z)_{\{ \pm \frac{\theta_0}{2} \}} \right) + O(\varepsilon^2 \delta_0^{-\frac{1}{10^4}})
\]

\[
= 4 \sin \pi (z + \frac{l_0}{2} \cdot \omega - \theta_0) \sin \pi (z + \frac{l_0}{2} \cdot \omega + \theta_0)
\]

\[
\times \sin \pi (z - \frac{l_0}{2} \cdot \omega - \theta_0) \sin \pi (z - \frac{l_0}{2} \cdot \omega + \theta_0) + O(\varepsilon^{1.5}).
\]

If \( l_0 = 0 \), then

\[
\det S_1(z) = -2 \sin \pi (z - \theta_0) \sin \pi (z + \theta_0) + O(\varepsilon^{1.5}).
\]

In this case, the argument is easier, and we omit the discussion. In the following, we deal with \( l_0 \neq 0 \). By (3.30) and (3.32), we have

\[
\|z + \frac{l_0}{2} \cdot \omega - \theta_0\| \geq \|l_0 \cdot \omega\| - \|\frac{l_0}{2} \cdot \omega + \theta_0\| - |z|
\]

\[
> \gamma \varepsilon^5 \gamma \left(100N_1^2\right)^{-\tau} - \frac{1}{\delta_0^{10^4}} - \frac{1}{\delta_0^{10^5}}
\]

\[
> \frac{1}{\delta_0^{10^5}},
\]

\[
\|z - \frac{l_0}{2} \cdot \omega + \theta_0\| \geq \|l_0 \cdot \omega\| - \|\frac{l_0}{2} \cdot \omega + \theta_0\| - |z|
\]

\[
> \gamma \varepsilon^5 \gamma \left(100N_1^2\right)^{-\tau} - \frac{1}{\delta_0^{10^4}} - \frac{1}{\delta_0^{10^5}}
\]

\[
> \frac{1}{\delta_0^{10^5}}.
\]

Let \( z_1 \) satisfy

\[
z_1 \equiv \frac{l_0}{2} \cdot \omega + \theta_0 \pmod{\mathbb{Z}}, \quad |z_1| = \|\frac{l_0}{2} \cdot \omega + \theta_0\| < \frac{1}{\delta_0^{10^5}}.
\]

Then

\[
\det S_1(z) \sim \|z + \frac{l_0}{2} \cdot \omega - \theta_0\| \cdot \|z - \frac{l_0}{2} \cdot \omega + \theta_0\| \cdot |(z - z_1)(z + z_1) + r_1(z)|
\]

\[
\sim \frac{1}{\delta_0^{10^5}} \cdot |(z - z_1)(z + z_1) + r_1(z)|,
\]

where \( r_1(z) \) is an analytic function in the set of (3.32) with \( |r_1(z)| < \varepsilon \ll \frac{1}{\delta_0^{10^5}} \).

Applying Rouché theorem shows the equation

\[
(z - z_1)(z + z_1) + r_1(z) = 0
\]

has exact two roots \( \theta_1, \theta_1' \) in the set of (3.32), which are perturbations of \( \pm z_1 \). Notice that

\[
\left\{ |z| < \frac{1}{\delta_0^{10^5}} : \det M_1(z) = 0 \right\} = \left\{ |z| < \frac{1}{\delta_0^{10^5}} : \det S_1(z) = 0 \right\}
\]
and $\det M_1(z)$ is an even function (cf. Lemma C.1) of $z$. Thus
\[ \theta'_1 = -\theta_1. \]

Moreover, we have
\[ |\theta_1 - z_1| \leq |r_1(\theta_1)| \frac{1}{2} < \varepsilon \frac{1}{2}, \quad |(z - z_1)(z + z_1) + r_1(z)| \sim |(z - \theta_1)(z + \theta_1)|. \]

Thus for $z$ being in the set of (3.32), we have
\[ \det S_1(z) \propto ||z - \theta_1|| \cdot ||z + \theta_1||, \tag{3.34} \]

which concludes
\[ ||S_1(z)^{-1}|| \leq C\delta_0^{-1} ||z - \theta_1||^{-1} \cdot ||z + \theta_1||^{-1}. \]

Recalling (3.33), we get since Lemma B.1
\[ ||M_1(z)^{-1}|| < 4 \left(1 + ||(M_1(z)(\bar{\Omega}_1 - k)\setminus\{0\})^{-1}|| \right) \left(1 + ||S_1(z)^{-1}||\right) \]
\[ < \delta_0^{-2} ||z + \theta_1||^{-1} \cdot ||z - \theta_1||^{-1}. \tag{3.35} \]

Thus for the case (3.30), both (3.34) and (3.35) are established for $z$ belonging to
\[ \left\{ z \in \mathbb{C} : ||z|| < \delta_0^{103} \right\}, \]

since $M_1(z)$ is 1-periodic (in $z$). By (3.31), for $k \in P_1$, we also have
\[ ||T^{-1}_{\bar{\Omega}_1}|| = ||M_1(\theta + k \cdot \omega)^{-1}|| \]
\[ < \delta_0^{-2} ||\theta + k \cdot \omega - \theta_1||^{-1} \cdot ||\theta + k \cdot \omega + \theta_1||^{-1}. \tag{3.36} \]

For the case
\[ \left\{ \frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2} < \delta_0^{103} \right\}, \tag{3.37} \]

we have for $k \in P_1$,
\[ ||\theta + k \cdot \omega - \frac{1}{2}|| < 3\delta_0^{103}. \tag{3.38} \]

Consider
\[ M_1(z) := T(z)(\bar{\Omega}_1 - k) = ((\cos 2\pi (z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon \Delta)_{\epsilon < \bar{\Omega}_1} \setminus \{\pm \omega\}_{n \in \bar{\Omega}_1} \setminus k \]

in
\[ \left\{ z \in \mathbb{C} : |z - \frac{1}{2}| < \delta_0^{103} \right\}. \tag{3.39} \]

By the similar argument as above, we get
\[ \left\| \left( M_1(z)(\bar{\Omega}_1 - k)\setminus\{\pm \omega\} \right)^{-1} \right\| < \delta_0^{-3 \times 103}. \]

Thus $M_1(z)^{-1}$ is controlled by the inverse of the Schur complement of $(\bar{\Omega}_1 - k) \setminus \{\pm \omega\}$:
\[
S_1(z) = M_1(z)\{\pm \omega\} - R\{\pm \omega\} M_1(z) R(\bar{\Omega}_1 - k)\setminus\{\pm \omega\}
\times \left( M_1(z)(\bar{\Omega}_1 - k)\setminus\{\pm \omega\} \right)^{-1} R(\bar{\Omega}_1 - k)\setminus\{\pm \omega\} M_1(z) R\{\pm \omega\}.
\]
Direct computation shows
\[
\det S_1(z) = \det \left(M_1(z) \left\{ \frac{\pm \omega}{2} \right\} \right) + O(\varepsilon^2 \delta_0^{-\frac{1}{60}}) \\
= 4 \sin \pi(z + \frac{l_0}{2} \cdot \omega - \theta_0) \sin \pi(z + \frac{l_0}{2} \cdot \omega + \theta_0) \\
\times \sin \pi(z - \frac{l_0}{2} \cdot \omega - \theta_0) \sin \pi(z - \frac{l_0}{2} \cdot \omega + \theta_0) + O(\varepsilon^{1.5}).
\]

By (3.37) and (3.39), we have
\[
\|z + \frac{l_0}{2} \cdot \omega - \theta_0\| \geq \|l_0 \cdot \omega\| - \|\frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2}\| - \|z - \frac{1}{2}\| \\
> \gamma e^{-(100N^2)} - \delta_0^{\frac{1}{120}} - \delta_0^{\frac{1}{120}},
\]
\[
\|z - \frac{l_0}{2} \cdot \omega + \theta_0\| \geq \|l_0 \cdot \omega\| - \|\frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2}\| - \|z - \frac{1}{2}\| \\
> \gamma e^{-(100N^2)} - \delta_0^{\frac{1}{120}} - \delta_0^{\frac{1}{120}},
\]
Let \(z_1\) satisfy
\[
z_1 = \frac{l_0}{2} \cdot \omega + \theta_0 \pmod{\mathbb{Z}}, \|z_1 - \frac{1}{2}\| = \|\frac{l_0}{2} \cdot \omega + \theta_0 - \frac{1}{2}\| < \delta_0^{\frac{1}{120}}.
\]

Then
\[
\det S_1(z) \sim \|z + \frac{l_0}{2} \cdot \omega - \theta_0\| \cdot \|z - \frac{l_0}{2} \cdot \omega + \theta_0\| \cdot \|(z - z_1)(z - (1 - z_1)) + r_1(z)\|
\]
\[
\sim \|z - z_1\| \cdot \|(z - (1 - z_1)) + r_1(z)\|
\]
where \(r_1(z)\) is an analytic function in the set of (3.39) with \(|r_1(z)| < \varepsilon \ll \delta_0^{\frac{1}{120}}\). Using again Rouché theorem shows the equation
\[
(z - z_1) (z - (1 - z_1)) + r_1(z) = 0
\]
has exact two roots \(\theta_1, \theta_1'\) in (3.39), which are perturbations of \(z_1\) and \(1 - z_1\). Notice that
\[
\left\{ \|z - \frac{1}{2}\| < \delta_0^{\frac{1}{60}} : \det M_1(z) = 0 \right\} = \left\{ \|z - \frac{1}{2}\| < \delta_0^{\frac{1}{60}} : \det S_1(z) = 0 \right\}
\]
and \(\det M_1(z)\) is a 1-periodic even function of \(z\) (cf. Lemma C.1). Thus
\[
\theta_1' = 1 - \theta_1.
\]
Moreover,
\[
|\theta_1 - z_1| = |r_1(\theta_1)|^\frac{1}{2} \ll \varepsilon^\frac{1}{2}, \quad |(z - z_1)(z - 1 + z_1) + r_1(z)| \sim |(z - \theta_1)(z - (1 - \theta_1))|.
\]
Thus for \(z\) belonging to the set of (3.39), we have
\[
\det S_1(z) \sim \|z - \theta_1\| \cdot \|z - (1 - \theta_1)\| = \|z - \theta_1\| \cdot \|z + \theta_1\|
\]
\[
\|M_1(z)^{-1}\| < \delta_0^{-2} \|z - \theta_1\|^{-1} \cdot \|z + \theta_1\|^{-1}.
\]
Thus for the case \((3.37)\), both \((3.34)\) and \((3.35)\) hold for \(z\) being in
\[
\{ z \in \mathbb{C} : \| z - \frac{1}{z} \| < \frac{1}{3}\delta \}
\].

By \((3.38)\), for \(k \in P_1\), we obtain
\[
\| T_{\Omega_k}^{-1} \| = \| M_1(\theta + k \cdot \omega)^{-1} \|
\]
\[
< \delta_0^{-2} \| \theta + k \cdot \omega - \theta_1 \|^{-1} \cdot \| \theta + k \cdot \omega + \theta_1 \|^{-1}.
\]
(3.40)

For \(k \in P_1\), we define \(A_k^1 \subset \Omega_k^1\) to be
\[
A_k^1 := \begin{cases} 
\{ k \} & \text{case (C1)}_0 \\
\{ 0 \} \cup \{ o^* \} & \text{case (C2)}_0
\end{cases}
\] (3.41)

where \(k = \frac{1}{2} (o + o^*)\) for some \(o \in O_1\) (cf. (3.29)) in the case \((C2)_0\). We have verified \((a)_1-(d)_1\) and \((f)_1\).

**STEP3: Application of resolvent identity**

Now we verify \((e)_1\) which is based on iterating resolvent identity.

Note that
\[
\left| \log \frac{\gamma}{\delta_1} \right| = \left| \log \frac{\gamma}{\delta_0} \right|^c.
\]

Recall that
\[
Q_1^+ = \{ k \in P_1 : \| \theta + k \cdot \omega \pm \theta_1 \| < \delta_1 \}, \ \ \ Q_1 = Q_1^+ \cup Q_1^-.
\]

We say a finite set \(\Lambda \subset \mathbb{Z}^d\) is **1-good** iff
\[
\left\{ \begin{array}{l}
\Lambda \cap Q_0 \cap \Omega_k^1 \neq \emptyset \Rightarrow \Omega_k \subset \Lambda, \\
\{ k \in P_1 : \Omega_k^1 \subset \Lambda \} \cap Q_1 = \emptyset.
\end{array} \right.
\] (3.42)

**Theorem 3.4.** If \(\Lambda\) is 1-good, then
\[
\| T_{\Lambda}^{-1} \| < \delta_0^{-3} \sup_{\{k \in P_1 : \Omega_k \subset \Lambda\}} \| \theta + k \cdot \omega - \theta_1 \|^{-1} \cdot \| \theta + k \cdot \omega + \theta_1 \|^{-1},
\]
(3.43)

\[
| T_{\Lambda}^{-1}(x, y) | < e^{-\gamma_1 \| x - y \|} \text{ for } \| x - y \| > N_1^\beta.
\]
(3.44)

where \(\gamma_1 = \gamma_0(1 - N_1^\beta)^3\).

**Proof of Theorem 3.4.** Denote
\[
2\Omega_k^1 := \Lambda_{diam}(\Lambda_k^1(k)).
\]

We have

**Lemma 3.5.** For \(k \in P_1 \setminus Q_1\), we have
\[
| T_{\Omega_k^1}^{-1}(x, y) | < e^{-\gamma_0 \| x - y \|} \text{ for } x \in \partial^{-}\Omega_k^1, \ y \in 2\Omega_k^1,
\]
(3.45)

where \(\gamma_0 = \gamma_0(1 - N_1^\beta - 1)\).

**Proof of Lemma 3.5.** From our construction, we have
\[
Q_0 \subset \bigcup_{k \in P_1} A_k^1 \subset \bigcup_{k \in P_1} \Omega_k^1.
\]

Thus
\[
(\Omega_k \setminus A_k) \cap Q_0 = \emptyset,
\]
which shows $\tilde{\Omega}_k^1 \setminus A_k^1$ is 0-good. As a result, one has by (3.20),

$$|T^{-1}_{\tilde{\Omega}_k^1 \setminus A_k^1}(x, w)| < e^{-\gamma_0\|x-w\|_1} \quad \text{for } x \in \partial \tilde{\Omega}_k^1, \ w \in (\tilde{\Omega}_k^1 \setminus A_k^1) \cap 2\Omega_k^1.$$ 

Since (3.40) and $k \notin Q_1$, we have

$$\left\|T^{-1}_{\tilde{\Omega}_k^1} \right\| < \delta_0^2 \delta_1^{-2} < \delta_1^{-3}.$$ 

Using resolvent identity implies

$$|T^{-1}_{\tilde{\Omega}_k^1}(x, y)| = |T^{-1}_{\tilde{\Omega}_k^1 \setminus A_k^1}(x, y)\chi_{\tilde{\Omega}_k^1 \setminus A_k^1}(y) - \sum_{(w', y) \in \partial A_k^1} T^{-1}_{\tilde{\Omega}_k^1 \setminus A_k^1}(x, w)\Gamma(w, w')T^{-1}_{\tilde{\Omega}_k^1}(w', y)|$$

$$< e^{-\gamma_0\|x-y\|_1} + 4d \sup_{w \in \partial A_k^1} e^{-\gamma_0\|x-y\|_1\|T^{-1}_{\tilde{\Omega}_k^1}\|}$$

$$< e^{-\gamma_0\|x-y\|_1} + \sup_{w \in \partial A_k^1} e^{-\gamma_0(\|x-y\|_1 - \|y-w\|_1) + C\log\delta_1}$$

$$< e^{-\gamma_0\|x-y\|_1} + e^{-\gamma_0\left(1 - C(\|x-y\|_1 + \log\delta_1)\right)}\|x-y\|_1$$

$$< e^{-\gamma_0\|x-y\|_1} + e^{-\gamma_0\left(1 - N_1^{1/2} - C\log\delta_1\right)}\|x-y\|_1$$

$$= e^{-\gamma_0\|x-y\|_1}$$

since

$$N_1^c \lesssim \text{diam } \tilde{\Omega}_k^1 \sim \|x-y\|_1, \ \|y-w\|_1 \lesssim \text{diam } \Omega_k^1 \lesssim \left(\text{diam } \tilde{\Omega}_k^1\right)^{\frac{1}{2}}$$

and

$$|\log\delta_1| \sim |\log\delta_0|^{\nu_0} \sim N_1^{10\nu} < N_1^{\nu_0}.$$ (3.46)

We are able to prove Theorem 3.4. First, we prove the estimate (3.43) by Schur’s test. Define

$$\tilde{P}_1 = \{k \in P_1 : \Lambda \cap \Omega_k^1 \cap Q_0 \neq \emptyset\}, \ \Lambda' = \Lambda \setminus \bigcup_{k \in \tilde{P}_1} \Omega_k^1.$$ 

Then $\Lambda' \cap Q_0 = \emptyset$, which shows $\Lambda'$ is 0-good, and (3.19)–(3.20) hold for $\Lambda'$. We have the following cases.

1. Let $x \notin \bigcup_{k \in \tilde{P}_1} 2\Omega_k^1$. Thus $N_1 \leq \text{dist}(x, \partial \Lambda')$. For $y \in \Lambda$, the resolvent identity reads as

$$T^{-1}_{\Lambda}(x, y) = T^{-1}_{\Lambda'}(x, y)\chi_{\Lambda'}(y) - \sum_{(w, w') \in \partial \Lambda'} T^{-1}_{\Lambda'}(x, w)\Gamma(w, w')T^{-1}_{\Lambda'}(w', y),$$

Since

$$\sum_{y \in \Lambda'} |T^{-1}_{\Lambda'}(x, y)\chi_{\Lambda'}(y)| \leq |T^{-1}_{\Lambda'}(x, x)| + \sum_{\|y-w\|_1 > 0} |T^{-1}_{\Lambda'}(x, y)\chi_{\Lambda'}(y)|$$

$$\leq \|T^{-1}_{\Lambda'}\| + \sum_{\|y-w\|_1 > 0} e^{-\gamma_0\|x-y\|_1}$$

$$\leq 2\delta_0^{-2}$$
and
\[ \sum_{w \in \partial \Lambda'} |T_{\Lambda'}^{-1}(x, w)| \leq \sum_{\|x-w\|_1 \geq N_1} e^{-\gamma_0 \|x-w\|_1} < e^{-\frac{1}{2} \gamma_0 N_1}, \]
we get
\[ \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(x, y)| \leq \sum_{y \in \Lambda'} |T_{\Lambda'}^{-1}(x, y)\chi_{\Lambda'}(y)| + \sum_{y \in \Lambda, (w, w') \in \partial \Lambda'} |T_{\Lambda'}^{-1}(x, w)\Gamma(w, w')T_{\Lambda'}^{-1}(w', y)| \]
\[ \leq 2\delta_0^{-2} + 2d \sum_{w \in \partial \Lambda'} |T_{\Lambda'}^{-1}(x, w)| \cdot \sup_{w' \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda'}^{-1}(w', y)| \]
\[ \leq 2\delta_0^{-2} + 10 \sup_{w \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(w', y)|. \]

(2). Let \( x \in 2\Omega_k^1 \) for some \( k \in \bar{P}_1 \). Thus by (3.42), we have \( \bar{\Omega}_k^1 \subset \Lambda \) and \( k \notin Q_1 \).

For \( y \in \Lambda \), using resolvent identity shows
\[ T_{\Lambda}^{-1}(x, y) = T_{\Omega_k^1}(x, y)\chi_{\Omega_k^1}(y) - \sum_{(w, w') \in \partial \Omega_k^1} T_{\Omega_k^1}^{-1}(x, w)\Gamma(w, w')T_{\Omega_k^1}^{-1}(w', y), \]

By (3.40), (3.45) and since
\[ N_1 < \text{diam } \bar{\Omega}_k^1 \leq \text{dist}(x, \partial \bar{\Omega}_k^1), \]
we get
\[ \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(x, y)| \leq \sum_{y \in \Omega_k^1} |T_{\Omega_k^1}^{-1}(x, y)\chi_{\Omega_k^1}(y)| + \sum_{y \in \Lambda, (w, w') \in \partial \Omega_k^1} |T_{\Omega_k^1}^{-1}(x, w)\Gamma(w, w')T_{\Omega_k^1}^{-1}(w', y)| \]
\[ < \#\bar{\Omega}_k^1 \cdot \|T_{\Omega_k^1}^{-1}\| + CN_1^{\epsilon} \cdot e^{-\gamma_0 N_1} \sup_{w \in \Lambda} \sum_{y \in \Lambda} |T_{\Omega_k^1}^{-1}(w', y)| \]
\[ < CN_1^{\epsilon} \delta_0^{-2} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1} + \frac{1}{10} \sup_{w \in \Lambda} \sum_{y \in \Lambda} |T_{\Omega_k^1}^{-1}(w', y)| \]
\[ < \frac{1}{2} \delta_0^{-3} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1} + \frac{1}{10} \sup_{w \in \Lambda} \sum_{y \in \Lambda} |T_{\Omega_k^1}^{-1}(w', y)|. \]

Combining estimates of the above two cases yields
\[ \|T_{\Lambda}^{-1}\| \leq \sup_{x \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(x, y)| \]
\[ < \delta_0^{-3} \sup_{\{k \in P_1 : \bar{\Omega}_k^1 \subset \Lambda\}} \|\theta + k \cdot \omega - \theta_1\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_1\|^{-1}. \quad (3.47) \]

Now we prove the off-diagonal decay estimate (3.44). For every \( w \in \Lambda \), define its block in \( \Lambda \)
\[ J_w = \begin{cases} \Lambda_{\frac{1}{2} N_1}(w) \cap \Lambda & \text{if } w \notin \bigcup_{k \in P_1} 2\Omega_k^1, \\ \bar{\Omega_k^1} & \text{if } w \in 2\Omega_k^1 \text{ for some } k \in \bar{P}_1. \end{cases} \quad (1) \]
Then \( \text{diam } J_w \leq \max \left( \text{diam } \Lambda_{\frac{1}{2} N_1}(w), \text{diam } \bar{\Omega_k^1} \right) < 3N_1^{\epsilon} \). For (1), since
\[ \text{dist}(w, \Lambda \cap Q_0) \geq \text{dist}(w, \bigcup_{k \in \bar{P}_1} \Omega_k^1) \geq N_1, \]
we have $J_w \cap Q_0 = \emptyset$. Thus $J_w$ is 0-good. Noticing that dist$(w, \partial^- J_w) \geq \frac{1}{2}N_1$, from (3.20), we have
\[
|T_{J_w}^{-1}(w, w')| < e^{-\gamma_0\|w-w'\|_1} \text{ for } w' \in \partial^- J_w.
\]
For (2), by (3.45), we have
\[
|T_{J_w}(w, w')| < e^{-\gamma_0\|w-w'\|_1} \text{ for } w' \in \partial^- J_w.
\]
Let \(\|x - y\| > N_1^3\). Using resolvent identity shows
\[
T_{J_w}^{-1}(x, y) = T_{J_w}^{-1}(x, y)\chi_{J_w}(y) - \sum_{(w, w') \in \partial \chi_{J_w}} T_{J_w}^{-1}(x, w)\Gamma(w, w')T_{J_w}^{-1}(w', y).
\]
The first term of the above identity is zero because $y \notin J_w$ (since $\|x - y\| > N_1^3 > 3N_1^2$). It follows that
\[
|T_{J_w}^{-1}(x, y)| \leq CN_1^2e^{-\min(\gamma_0(1-2N_1^2), \gamma_0(1-N_1^2))\|x-x_1\|_1}|T_{J_w}^{-1}(x_1, y)|
\]
\[
\leq CN_1^2e^{-\gamma_0(1-N_1^2)}\|x-x_1\|_1|T_{J_w}^{-1}(x_1, y)|
\]
\[
< e^{-\gamma_0(1-N_1^2)}(\|x-x_1\|_1|T_{J_w}^{-1}(x_1, y)|)
\]
\[
< e^{-\gamma_0(1-N_1^2)}|T_{J_w}^{-1}(x_1, y)|
\]
\[
= e^{-\gamma_0\|x-x_1\|_1}|T_{J_w}^{-1}(x_1, y)|
\]
for some $x_1 \in \partial^+ J_w$, where $\gamma_0 = \gamma_0(1-2N_1^2)$. Then iterate and stop for some step $L$ such that $\|x_L - y\| < 3N_1^2$. Recalling (3.46) and (3.47), we get
\[
|T_{J_w}^{-1}(x, y)| \leq e^{-\gamma_0\|x-x_1\|_1} \ldots e^{-\gamma_0\|x_{L-1} - x_L\|_1}|T_{J_w}^{-1}(x_L, y)|
\]
\[
\leq e^{-\gamma_0(\|x-y\|_1 - 3N_1^2)}\|T_{J_w}^{-1}\|
\]
\[
< e^{-\gamma_0(1-3N_1^2 - 3)}\|x-y\|_1\delta_1^{-3}
\]
\[
< e^{-\gamma_0(1-3N_1^2 - 3)(\log N_1^2)}\|x-y\|_1
\]
\[
< e^{-\gamma_0(1-N_1^2)}\|x-y\|_1
\]
\[
= e^{-\gamma_0\|x-y\|_1}.
\]

This completes the proof of Theorem 3.4. \qed

3.3. Proof of Theorem 3.2: from $P_s$ to $P_{s+1}$.

Proof of Theorem 3.2. We have finished the proof of $P_1$ in Subsection 3.2. Assume that $P_s$ holds true. In order to complete the proof of Theorem 3.2 it suffices to establish $P_{s+1}$.

In the following, we try to prove $P_{s+1}$ holds true. For this purpose, we will establish (a) $s+1$–(f) $s+1$ assuming (a) $s$–(f) $s$. We divide the proof into 3 steps. Let
\[
Q_s^+ = \{k \in P_s : \|\theta + k \cdot \omega \pm \theta_s\| < \delta_s\}, \quad Q_s = Q_s^+ \cup Q_s^-.
\] (3.48)

and
\[
\tilde{Q}_s^+ = \left\{k \in P_s : \|\theta + k \cdot \omega \pm \theta_s\| < \delta_s^{\frac{1}{\delta_0}}\right\}, \quad \tilde{Q}_s = \tilde{Q}_s^+ \cup \tilde{Q}_s^-.
\] (3.49)
STEP1: The case (C1) Occurs: i.e.,

\[
\text{dist} \left( \bar{Q}_s^-, Q_s^+ \right) > 100N_{s+1}^c.
\] (3.50)

Remark 3.2. We can prove that

\[
\text{dist} \left( \bar{Q}_s^-, Q_s^+ \right) = \text{dist} \left( \bar{Q}_s^+, Q_s^- \right).
\]

Thus (3.50) also implies that

\[
\text{dist} \left( \bar{Q}_s^+, Q_s^- \right) > 100N_{s+1}^c.
\] (3.51)

By (3.18) and the definitions of \( Q_s^\pm \) (cf. (3.48)) and \( \bar{Q}_s^\pm \) (cf. (3.49)), we obtain

\[
Q_s^\pm = \{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \| \theta + k \cdot \omega \pm \theta_s \| < \delta_s \},
\] (3.52)

\[
\bar{Q}_s^\pm = \{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \| \theta + k \cdot \omega \pm \theta_s \| < \delta^*_s \}.
\]

Then the proof is similar to that of Remark 3.1 and we omit the details.

Assuming (3.50), then we define

\[
P_{s+1} = Q_s, \quad l_s = 0.
\] (3.53)

By (3.8) and (3.9), we have

\[
P_{s+1} \subset \left\{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s} l_i : \min_{\sigma = \pm 1} (\| \theta + k \cdot \omega + \sigma \theta_s \|) < \delta_s \right\}.
\] (3.54)

Thus from (3.51), we get for \( k, k' \in P_{s+1} \) with \( k \neq k' \),

\[
\| k - k' \| > \min \left( \log \frac{\gamma}{2\delta_s}, 100N_{s+1}^c \right) \geq 100N_{s+1}^c.
\] (3.55)

In the following, we will associate every \( k \in P_{s+1} \) with blocks \( \Lambda_{s+1}^+ \) and \( \bar{\Omega}_{k}^+ \) so that

\[
\Lambda_{N_{s+1}}(k) \subset \Omega_{k}^+ \subset \Lambda_{N_{s+1}+50N_{s+1}^c}(k),
\]

\[
\Lambda_{N_{s+1}^c}(k) \subset \bar{\Omega}_{k}^+ \subset \Lambda_{N_{s+1}^c+50N_{s+1}^c}(k),
\]

and

\[
\begin{align*}
\Omega_{k}^+ \cap \bar{\Omega}_{k'}^+ \neq \emptyset \quad (s' < s + 1) & \Rightarrow \bar{\Omega}_{k}^+ \subset \Omega_{k'}^+ \subset \Omega_{k}^+ \\
\bar{\Omega}_{k}^+ \cap \bar{\Omega}_{k'}^+ \neq \emptyset \quad (s' < s + 1) & \Rightarrow \bar{\Omega}_{k'}^+ \subset \bar{\Omega}_{k}^+ \subset \bar{\Omega}_{k'}^+ \\
\text{dist}(\bar{\Omega}_{k}^+, \bar{\Omega}_{k'}^+) & > 10 \text{diam}\bar{\Omega}_{k}^+ \text{ for } k \neq k' \in P_{s+1}.
\end{align*}
\] (3.56)

In addition, the set

\[
\bar{\Omega}_{k}^+ - k \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s} l_i
\]

is independent of \( k \in P_{s+1} \) and is symmetrical about the origin.

Such \( \bar{\Omega}_{k}^+ \) and \( \bar{\Omega}_{k}^+ \) can be constructed by the following argument (We only consider \( \bar{\Omega}_{k}^+ \) since \( \bar{\Omega}_{k}^+ \) is discussed by the similar argument). Fixing \( k_0 \in Q_s^+ \), we start from

\[
J_{0,0} = \Lambda_{N_{s+1}}(k_0).
\]
Define
\[ H_r = (k_0 - P_{s+1} + P_{s-r}) \cup (k_0 + P_{s+1} - P_{s-r}) \quad (0 \leq r \leq s - 1). \]
Notice that by (3.54), we have \( k_0 - P_{s+1} \in \mathbb{Z}^d \) and, \( P_{s-r} \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-r-1} l_i \) since (3.8)–(3.9). Thus
\[ H_{s-r} \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-r-1} l_i. \]
Define inductively
\[ J_{r,0} \subset J_{r,1} \subset \cdots \subset J_{r,t_r} := J_{r+1,0}, \]
where
\[ J_{r,t+1} = J_{r,t} \cup \left( \bigcup_{h \in H_r} \Lambda_{2N^2_{s-r}}(h) \right) \]
and \( t_r \) is the largest integer satisfying the \( \subset \) relationship (the following argument shows that \( t_r < 10 \)). Thus
\[ h \in H_r, \; \Lambda_{2N^2_{s-r}}(h) \cap J_{r+1,0} = \emptyset \Rightarrow \Lambda_{2N^2_{s-r}}(h) \subset J_{r+1,0}. \]
(3.57)
For \( k \in k_0 - P_{s+1} \), we have since (3.54)
\[ \min \left( \| \tilde{k} \cdot \omega \|, \| \tilde{k} \cdot \omega + 2\theta_s \| \right) < 2\delta_s. \]
For \( k' \in P_{s-r} \), we get since (3.8) and (3.9) that
\[ \min_{\sigma = \pm 1} \left( \| \theta + k' \cdot \omega + \sigma \theta_{s-r-1} \| \right) < \delta_{s-r-1} \text{ if } (C1)_{s-r} \text{ holds true}, \]
(3.58)
\[ \| \theta + k' \cdot \omega \| < 3\delta_{s-r-1} \text{ or } \| \theta + k' \cdot \omega + \frac{1}{2} \| < 3\delta_{s-r-1} \text{ if } (C2)_{s-r} \text{ holds true}. \]
(3.59)
Thus for \( h \in k_0 - P_{s+1} + P_{s-r} \), we obtain for (3.58),
\[ \min_{\sigma = \pm 1} \left( \| \theta + h \cdot \omega + \sigma \theta_{s-r-1} \|, \| \theta + h \cdot \omega + 2\theta_s + \sigma \theta_{s-r-1} \| \right) < 2\delta_{s-r-1} \]
and for (3.59),
\[ \min \left( \| \theta + h \cdot \omega \|, \| \theta + h \cdot \omega + \frac{1}{2} \|, \| \theta + h \cdot \omega + 2\theta_s \|, \| \theta + h \cdot \omega + \frac{1}{2} + 2\theta_s \| \right) < 4\delta_{s-r-1}. \]
Notice that \( k_0 + P_{s+1} - P_{s-r} = 2k_0 - (k_0 - P_{s+1} + P_{s-r}) \) is symmetrical to \( k_0 - P_{s+1} + P_{s-r} \) about \( k_0 \). Thus, if a set \( \Lambda \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-r-1} l_i \) contains 10 distinct elements of \( H_r \), then
\[ \text{diam } \Lambda \geq \log \left| \frac{\gamma}{8\delta_{s-r-1}} \right| \gg 100N_{s-r}^2. \]
(3.60)
We claim that \( t_r < 10 \). Otherwise, there exist distinct \( h_t \in H_r \) (1 \( \leq t \leq 10 \)), such that
\[ \Lambda_{2N^2_{s-r}}(h_1) \cap J_{r,0} = \emptyset, \; \Lambda_{2N^2_{s-r}}(h_t) \cap \Lambda_{2N^2_{s-r}}(h_{t+1}) = \emptyset. \]
In particular,
\[ \text{dist}(h_t, h_{t+1}) \leq 4N_{s-r}^2. \]
Thus
\[ h_t \in \Lambda_{40N_s^c} (0) + h_1 \ (1 \leq t \leq 10). \]

This contradicts (3.60). Thus we have shown
\[ J_{r+1,0} = J_{r,t_c} \subset \Lambda_{40N_s^c} (J_{r,0}). \quad (3.61) \]

Since
\[ \sum_{r=0}^{s-1} 40N_s^c < 50N_s^c, \]

we find \( J_{s,0} \) to satisfy
\[ \Lambda_{N_{s+1}^c} (k_0) = J_{0,0} \subset J_{s,0} \subset \Lambda_{50N_s^c} (J_{0,0}) \subset \Lambda_{N_{s+1}^c + 50N_s^c} (k_0). \]

Now, for any \( k \in P_{s+1} \), define
\[ \tilde{\Omega}_k^{s+1} = J_{s,0} + (k - k_0). \quad (3.62) \]

Using \( k - k_0 \in \mathbb{Z}^d \) and \( \tilde{\Omega}_k^{s+1} \subset \mathbb{Z}^d \) yields
\[ \Lambda_{N_{s+1}^c} (k) \subset \tilde{\Omega}_k^{s+1} \subset \Lambda_{N_{s+1}^c + 50N_s^c} (k). \]

We are able to verify (3.56). In fact, since (3.55) and \( 50N_s^c \ll N_{s+1}^c \), we get
\[ \text{dist}(\tilde{\Omega}_k^{s+1}, \tilde{\Omega}_{k'}^{s+1}) > 10 \text{diam} \tilde{\Omega}_k^{s+1} \text{ for } k \neq k' \in P_{s+1}. \]

Assume that for some \( k \in P_{s+1} \) and \( k' \in P_{s'} \) (\( 1 \leq s' \leq s \))
\[ \tilde{\Omega}_k^{s+1} \cap \tilde{\Omega}_{k'}^{s'} \neq \emptyset. \]

Then
\[ \left( \tilde{\Omega}_k^{s+1} + (k_0 - k) \right) \cap \left( \tilde{\Omega}_{k'}^{s'} + (k_0 - k) \right) \neq \emptyset. \quad (3.63) \]

From
\[ \Lambda_{N_{s'}^c} (k') \subset \tilde{\Omega}_{k'}^{s'} \subset \Lambda_{N_{s'}^c + 50N_s^c} (k') \subset \Lambda_{1.5N_s^c} (k'), \]
\[ \tilde{\Omega}_k^{s+1} + (k_0 - k) = J_{s,0} \text{ and } (3.63), \text{ we obtain } \]
\[ J_{s,0} \cap \Lambda_{1.5N_s^c} (k' + k_0 - k) \neq \emptyset. \]

Recalling (3.61), we have
\[ J_{s,0} \subset \Lambda_{50N_{s'}^c} (J_{s-s'+1,0}). \]

Thus
\[ \Lambda_{50N_{s'}^c} (J_{s-s'+1,0}) \cap \Lambda_{1.5N_s^c} (k' + k_0 - k) \neq \emptyset. \]

From \( 50N_{s'}^c \ll 0.5N_{s'}^c \), it follows that
\[ J_{s-s'+1,0} \cap \Lambda_{2N_s^c} (k' + k_0 - k) \neq \emptyset. \]

Since \( k' \in P_{s'} \), we have \( k' + k_0 - k \in H_{s-s'} \), and by (3.57),
\[ \Lambda_{2N_s^c} (k' + k_0 - k) \subset J_{s-s'+1,0} \subset J_{s,0}. \]

Hence
\[ \tilde{\Omega}_{k'}^{s'} \subset \Lambda_{2N_s^c} (k') \subset J_{s,0} + (k - k_0) = \tilde{\Omega}_k^{s+1}. \]
Next, we will show $\tilde{\Omega}^{s+1}_k - k$ is independent of $k$. For this, recalling (3.62) and from $l_i \in \mathbb{Z}^d$, $\tilde{\Omega}^{s+1}_k \subset \mathbb{Z}^d$, $k \in P_{s+1} \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^s l_i$, we obtain that

$$\tilde{\Omega}^{s+1}_k - k \subset \mathbb{Z}^d - \frac{1}{2} \sum_{i=0}^s l_i = \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^s l_i$$

and

$$\tilde{\Omega}^{s+1}_k - k = J_{s,0} + (k - k_0) - k = \tilde{\Omega}^{s+1}_{k_0} - k_0$$

is independent of $k$. Finally, we prove the symmetry property of $\tilde{\Omega}^{s+1}_k$. The definition of $H_r$ implies that it is symmetrical about $k_0$, which implies all $J_{r,t}$ is symmetrical about $k_0$ as well. In particular, $\tilde{\Omega}^{s+1}_{k_0} = J_{s,0}$ is symmetrical about $k_0$, i.e., $\tilde{\Omega}^{s+1}_{k_0} - k_0$ is symmetrical about origin. In summary, we have established (a)$_{s+1}$ and (b)$_{s+1}$ in the case (C1)$_s$.

Now we turn to the proof of (c)$_{s+1}$. First, in this construction we have for every $k' \in Q_s$ ($= P_{s+1}$),

$$\tilde{\Omega}^{s'}_{k'} \subset \Omega^{s+1}_{k'}.$$

For every $k \in P_{s+1}$, define

$$A^{s+1}_k = A^s_k.$$

Then $A^{s+1}_k \subset \Omega^s_k \subset \Omega^{s+1}_k$ and $\# A^{s+1}_k = \# A^s_k \leq 2^s$. It remains to show $\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k$ is $s$-good, i.e.,

$$l' \in Q_{s'}, \tilde{\Omega}^{s'}_{l'} \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k), \tilde{\Omega}^{s'}_{l'} \subset \Omega^{s+1}_k \Rightarrow \tilde{\Omega}^{s+1}_{l'} \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)$$

for $s' < s$,

$$\left\{ l \in P_s : \tilde{\Omega}^{s'}_l \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k) \right\} \cap Q_s = \emptyset.$$

Assume that

$$l' \in Q_{s'}, \tilde{\Omega}^{s'}_{l'} \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k), \tilde{\Omega}^{s'}_{l'} \subset \Omega^{s'}_{l'}.$$

We have the following two cases. The first case is $s' \leq s - 2$, In this case, since $\emptyset \neq \tilde{\Omega}^{s'}_{l'} \subset \Omega^{s+1}_k \cap \Omega^{s+1}_{k'}$, we get by using (3.56) that $\tilde{\Omega}^{s+1}_{l'} \subset \Omega^{s+1}_k$. Assuming

$$\tilde{\Omega}^{s+1}_{l'} \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k) \neq \emptyset,$$

then $\tilde{\Omega}^{s+1}_{l'} \cap \Omega^s_k \neq \emptyset$. Thus from (3.10) (since $s' + 1 < s$), one has $\tilde{\Omega}^{s+1}_{l'} \subset \Omega^s_k$, which implies $\tilde{\Omega}^{s+1}_{l'} \subset (\tilde{\Omega}^s_k \setminus A^s_k)$. Since $\tilde{\Omega}^{s+1}_k \setminus A^s_k$ is $(s-1)$-good, we get

$$\tilde{\Omega}^{s+1}_{l'} \subset (\tilde{\Omega}^s_k \setminus A^s_k) \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k).$$

This contradicts (3.64). We then consider the case $s' = s - 1$. From $\tilde{\Omega}^{s+1}_{l'} \subset \Omega^s_{l'} \cap A^s_{l'} \neq \emptyset$, then $l = k$ and $\tilde{\Omega}^{s+1}_{l'} \subset (\tilde{\Omega}^s_k \setminus A^s_k)$. This contradicts

$$\left\{ l \in P_{s-1} : \tilde{\Omega}^{s+1}_{l'} \subset (\tilde{\Omega}^s_k \setminus A^s_k) \right\} \cap Q_{s-1} = \emptyset$$

because $(\tilde{\Omega}^s_k \setminus A^s_k)$ is $(s-1)$-good. Finally, if $l \in Q_s$ and $\tilde{\Omega}^s_{l'} \subset \tilde{\Omega}^{s+1}_k$, then $l = k$ since $k$ is the only element of $Q_s$ such that $\tilde{\Omega}^s_{k} \subset \tilde{\Omega}^{s+1}_k$ by the separation property of $Q_s$. As a result, $\tilde{\Omega}^s_{k} \not\subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k)$, which implies

$$\left\{ l \in P_s : \tilde{\Omega}^s_{l'} \subset (\tilde{\Omega}^{s+1}_k \setminus A^{s+1}_k) \right\} \cap Q_s = \emptyset.$$

Moreover, the set

$$A^{s+1}_k - k = A^s_k - k$$
is independent of $k \in P_{s+1}$ and symmetrical about the origin since the induction assumptions on $A^s_k$ of the $s$-th step. This finishes the proof of (c)$_{s+1}$ in the case (C1)$_s$.

In the following, we try to prove (d)$_{s+1}$ and (f)$_{s+1}$ in the case (C1)$_s$. For the case $k \in Q^s_k$, we consider the analytic matrix-valued function

$$M_{s+1}(z) := T(z)\tilde{\Omega}^{s+1}_{k+1-} = (\cos 2\pi (z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon\Delta)_{n \in \tilde{\Omega}^{s+1}_{k+1-} - k}$$

defined in

$$\{ z \in \mathbb{C} : |z - \theta_s| < \frac{\delta_s}{\theta_s}\}.$$ (3.65)

If $k' \in P_s$ and $\tilde{\Omega}^s_{k'} \subset (\tilde{\Omega}^{s+1}_{k+1} \setminus A^{s+1}_{k+1})$, then $0 \neq |k' - k| \leq 2N^{s+1}_s$. Thus

$$\|\theta + k' \cdot \omega - \theta_s\| \geq \|(k' - k) \cdot \omega\| - \|\theta + k \cdot \omega - \theta_s\| \geq \gamma e^{-(2N^{s+1}_s)^{-1}} - \delta_s \geq \gamma e^{-2} \frac{\log \frac{1}{\lambda_{s+1}}}{\lambda_{s+1}} - \delta_s \geq \frac{\delta_s}{\theta_s}. $$

By (3.51), we have $k' \notin \tilde{Q}^+_s$ and thus

$$\|\theta + k' \cdot \omega + \theta_s\| > \frac{\delta_s}{\theta_s}.$$ (3.66)

One may restate (3.66) as

$$\left\| \left( M_{s+1}(\theta + k \cdot \omega)_{(\tilde{\Omega}^{s+1}_{k+1} \setminus A^{s+1}_{k+1}) - k} \right)^{-1} \right\| < \frac{1}{2}\frac{\delta^{2\times 100}_s}{\theta_s}. $$

Notice that

$$|z - (\theta + k \cdot \omega)| \leq |z - \theta_s| + \|\theta + k \cdot \omega - \theta_s\| \leq \frac{\delta_s}{\theta_s} + \delta_s \ll \frac{\delta_s}{\theta_s}.$$ (3.67)

Thus by Neumann series argument, we can show

$$\left\| \left( M_{s+1}(z)_{(\tilde{\Omega}^{s+1}_{k+1} \setminus A^{s+1}_{k+1}) - k} \right)^{-1} \right\| < \frac{1}{2}\frac{\delta^{2\times 100}_s}{\theta_s}. $$ (3.68)

We may then control $M_{s+1}(z)^{-1}$ by the inverse of

$$S_{s+1}(z) = M_{s+1}^{-1}(z) A^{s+1}_{k+1} - k - R_{A^{s+1}_{k+1} - k} M_{s+1}(z) R_{(\tilde{\Omega}^{s+1}_{k+1} \setminus A^{s+1}_{k+1}) - k}.$$

Our next aim is to analyze the function $\text{det} S_{s+1}(z)$. Since $A^{s+1}_{k+1} - k = A^s_k - k \subset \tilde{\Omega}^s_k - k$ and $\text{dist} (\Omega^s_k, \partial^+ \tilde{\Omega}^s_k) > 1$, we obtain

$$R_{A^{s+1}_{k+1} - k} M_{s+1}(z) R_{(\tilde{\Omega}^{s+1}_{k+1} \setminus A^{s+1}_{k+1}) - k} = R_{A^s_k - k} M_{s+1}(z) R_{(\tilde{\Omega}^s_k \setminus A^s_k) - k}.$$
Thus for $N_c^3$, we have

$$S_{s+1}(z) = M_{s+1}(z)A_k - R_{A_k} M_{s+1}(z)R_{A_k} \left( \hat{\Omega}_k \setminus A_k \right) - k$$

$$\times \left( M_{s+1}(z)(\hat{\Omega}_k \setminus A_k)^{e+1} - k \right)^{-1} R_{A_k} M_{s+1}(z)R_{A_k} - k.$$ 

Since $\hat{\Omega}_k \setminus A_k$ is $(s - 1)$-good and by (3.16)–(3.17), we get

$$\|T^{-1}_{\hat{\Omega}_k \setminus A_k}\| < \delta_s^{-3},$$

$$\left| T^{-1}_{\hat{\Omega}_k \setminus A_k}(x, y) \right| < e^{-\gamma_s - 1}\|x - y\|^1 \text{ for } \|x - y\| > N_s^c.$$ 

Equivalently,

$$\left\| \left( M_{s+1}(\theta + k \cdot \omega)(\hat{\Omega}_k \setminus A_k)^{-k} \right)^{-1} \right\| < \delta_s^{-3},$$

$$\left| \left( M_{s+1}(\theta + k \cdot \omega)(\hat{\Omega}_k \setminus A_k)^{-k} \right)^{-1}(x, y) \right| < e^{-\gamma_s - 1}\|x - y\|^1 \text{ for } \|x - y\| > N_s^c.$$ 

In the set defined by (3.67), we claim that

$$\left| \left( M_{s+1}(z)(\hat{\Omega}_k \setminus A_k)^{-k} \right)^{-1}(x, y) \right| < \delta_s^{10} \text{ for } \|x - y\| > N_s^c.$$ 

**Proof of the Claim (i.e., (3.71)).** Denote

$$T_1 = M_{s+1}(\theta + k \cdot \omega)(\hat{\Omega}_k \setminus A_k)^{-k}, \ T_2 = M_{s+1}(z)(\hat{\Omega}_k \setminus A_k)^{-k}.$$ 

Then $D = T_1 - T_2$ is diagonal so that $\|D\| < 4\pi\delta_s^{10}$ since (3.67). Using Neumann series expansion yields

$$T_2^{-1} = (I - T_1^{-1}D)^{-1} = \sum_{i=0}^{\infty} \left( T_1^{-1}D \right)^i T_1^{-1}.$$ 

Since (3.69) and (3.70), we have

$$\left| T_1^{-1}(x, y) \right| < \delta_s^{-3}e^{-\gamma_s - 1(\|x - y\|^1 - N_s^c)}.$$ 

Thus for $\|x - y\| > N_s^c$ and $0 \leq i \leq 200$,

$$\left| (T_1^{-1}D)^i T_1^{-1}(x, y) \right| \leq \left( 4\pi\delta_s^{10} \right)^i \sum_{w_1, \ldots, w_i} |T_1(x, w_1) \cdots T_1(w_{i-1}, w_i)T_1(w_i, y)|$$

$$< \left( 4\pi\delta_s^{10} \right)^i C N_s^c d^s \delta_s^{-3(i+1)}e^{-\gamma_s - 1(\|x - y\|^1 - (i+1)N_s^c)}$$

$$< \delta_s^{i(i-1)}e^{-\gamma_s - 1(N_s^c - (i+1)N_s^c)}.$$ 

From $2 < \gamma_s - 1$, $201N_s^c - 1 < \frac{1}{2}N_s^c$ and $|\log \delta_s| \sim |\log \delta_s - 1|^c \sim N_s^{c10} \sim N_s^{c15} < N_s^c$, we get

$$e^{-\gamma_s - 1(N_s^c - (i+1)N_s^c)} < e^{-N_s^c} < \delta_s^{20}.$$ 

Hence

$$\sum_{i=0}^{200} \left| (T_1^{-1}D)^i T_1^{-1}(x, y) \right| < \frac{1}{2} \delta_s^{10}.$$
For $i > 200$, 
\[
|((T_1^{-1}D)^i T_1^{-1})(x,y)| < \left(4\pi\delta_s^{-1}\right)^i \delta_{s-1}^{-3(i+1)} < \delta_s^{-10}\delta_s^{-(r-200)}.
\]
Thus
\[
\sum_{i>200} |((T_1^{-1}D)^i T_1^{-1})(x,y)| < \frac{1}{2}\delta_s^{10}.
\] (3.74)
Combining (3.72), (3.73) and (3.74), we get
\[
|T_2^{-1}(x,y)| < \delta_s^{10} \text{ for } \|x-y\| > N^{c_1}_s.
\]
This completes the proof of (3.71). \(\square\)

Denote $X = (\Omega^+_k \setminus A^+_k) - k$ and $Y = (\Omega^{k+1}_k \setminus A^{k+1}_k) - k$. Let $x \in X$ satisfy $\text{dist}(x, A^+_k - k) \leq 1$. By resolvent identity, we have for any $y \in Y$,
\[
(M_{s+1}(z)Y)^{-1}(x,y) - \chi_X(y)(M_{s+1}(z)X)^{-1}(x,y)
\]
\[
= -\sum_{(w,w') \in \partial_X Y} (M_{s+1}(z)X)^{-1}(x,w)\Gamma(w,w')(M_{s+1}(z)Y)^{-1}(w',y). \tag{3.75}
\]
Since
\[
\text{dist}(x,w) \geq \text{dist}(A^+_k - k, \partial^{-}\Omega^+_k - k) - 2 > N_s > N^{c_1}_s,
\]
(3.68) and (3.71), the right hand side (RHS) of (3.75) is bounded by $CN_s^{-2}d_s^{-1}\delta_s^{10} < \delta_s^{9}$. It then follows that
\[
R_{A^+_k - k} M_{s+1}(z) R_X (M_{s+1}(z)Y)^{-1}
\]
\[
= R_{A^+_k - k} M_{s+1}(z) R_X (M_{s+1}(z)X)^{-1} R_X + O(\delta_s^{9}).
\]
As a result,
\[
R_{A^+_k - k} M_{s+1}(z) R_X (M_{s+1}(z)Y)^{-1} R_X M_{s+1}(z) R_{A^+_k - k}
\]
\[
= R_{A^+_k - k} M_{s+1}(z) R_X (M_{s+1}(z)X)^{-1} R_X M_{s+1}(z) R_{A^+_k - k} + O(\delta_s^{9})
\]
\[
= R_{A^+_k - k} M_{s}(z) R_X (M_{s}(z)X)^{-1} R_X M_{s}(z) R_{A^+_k - k} + O(\delta_s^{9})
\]
and
\[
S_{s+1}(z) = M_{s}(z) A^+_k - k - R_{A^+_k - k} M_{s}(z) R_X (M_{s}(z)X)^{-1} R_X M_{s}(z) R_{A^+_k - k} + O(\delta_s^{9})
\]
\[
= S_s(z) + O(\delta_s^{9}),
\]
which implies (3.13) for the $(s+1)$-th step. Recalling (3.65) and (3.12), we have since (3.14)
\[
\det S_s(z) \sim \|z - \theta_s\| \cdot \|z + \theta_s\|.
\]
By Hadamard’s inequality, we obtain
\[
\det S_{s+1}(z) = \det S_s(z) + O((2^s)^210^2\delta_s^{9})
\]
\[
= \det S_s(z) + O(\delta_s^{9}),
\]
where we use the fact that \( \#(A^k_s - k) \leq 2^s \), (3.13) and \( \log \log |\log \delta_s| \sim s \). Notice that
\[
\|z + \theta_s\| \geq \|\theta + k \cdot \omega + \theta_s\| - \|z - \theta_s\| - \|\theta + k \cdot \omega - \theta_s\|
\]
\[
> \delta_s^{\frac{1}{m}} - \delta_s^{\frac{1}{2}} - \delta_1
\]
\[
> \frac{1}{2} \delta_s^{\frac{1}{m}}.
\]
Then we have
\[
\det S_{s+1}(z) \frac{\delta}{\delta z} (z - \theta_s) + r_{s+1}(z),
\]
where \( r_{s+1}(z) \) is an analytic function defined in (3.65) with \( |r_{s+1}(z)| < \delta^7_s \). Finally, by the Róche theorem, the equation
\[
(z - \theta_s) + r_{s+1}(z) = 0
\]
has a unique root \( \theta_{s+1} \) in the set defined by (3.65) satisfying
\[
|\theta_{s+1} - \theta_s| = |r_{s+1}(\theta_{s+1})| < \delta^7_s, \quad |(z - \theta_s) + r_{s+1}(z)| \sim |z - \theta_{s+1}|.
\]
Moreover \( \theta_{s+1} \) is also the unique root of \( \det M_{s+1}(z) = 0 \) in the set defined by (3.65). From \( |z + \theta_s| > \frac{1}{2} \delta_s^{\frac{1}{m}} \) and \( |\theta_{s+1} - \theta_s| < \delta^7_s \), we have
\[
|z + \theta_s| \sim |z + \theta_{s+1}|.
\]
Thus if \( z \) belongs to the set defined by (3.65), we have
\[
\det S_{s+1}(z) \frac{\delta}{\delta z} \sim |z - \theta_{s+1}| \cdot |z + \theta_{s+1}|. \tag{3.76}
\]
Since \( |\log \delta_{s+1}| \sim |\log \delta_s|e^3 \), we get \( \delta_{s+1}^{\frac{1}{m}} < \delta_s^{\frac{1}{2}} \). Recalling (3.65), then (3.76) remains valid for \( z \) satisfying
\[
|z - \theta_{s+1}| < \delta_{s+1}^{\frac{1}{m}}.
\]
For \( k \in Q^*_s \), one considers
\[
M_{s+1}(z) := T(z)\tilde{\gamma}^{s+1}_{k+1-k} = (\cos 2\pi(z + n \cdot \omega)\delta_{n, n'} - E + \varepsilon\Delta)_{n \in \tilde{\gamma}^{s+1}_{k+1-k}}
\]
for \( z \) being in
\[
\{ z \in \mathbb{C} : |z + \theta_s| < \delta_s^{\frac{1}{m}} \}. \tag{3.77}
\]
The same argument shows that det \( M_{s+1}(z) = 0 \) has a unique root \( \theta'_{s+1} \) in the set defined by (3.77). Since det \( M_{s+1}(z) \) is an even function of \( z \), we get \( \theta'_{s+1} = -\theta_{s+1} \). Thus if \( z \) belongs to the set defined by (3.77), we also have (3.76). In conclusion, (3.76) is established for \( z \) belonging to
\[
\left\{ z \in \mathbb{C} : \min_{\sigma = \pm 1} \|z + \sigma \theta_{s+1}\| < \delta_{s+1}^{\frac{1}{m}} \right\},
\]
which proves (3.14) for the \((s+1)\)-th step. Combining \( l_s = 0 \), (3.52)–(3.53) and the following
\[
\|\theta + k \cdot \omega \pm \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{m}}, \quad |\theta_{s+1} - \theta_s| < \delta^7_s \Rightarrow \|\theta + k \cdot \omega \pm \theta_s\| < \delta_s,
\]
we get
\[
\left\{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s} l_i : \min_{\sigma = \pm 1} \|\theta + k \cdot \omega + \sigma \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{m}} \right\} \subset P_{s+1},
\]
which proves (3.18) at the \((s + 1)\)-th step. Finally, we want to estimate \(T_{\tilde{\Omega}_{k+1}}^{-1}\). For \(k \in P_{s+1}\), by (3.54), we obtain
\[
\theta + k \cdot \omega \in \{ z \in \mathbb{C} : \min_{\sigma = \pm 1} \| z + \sigma \theta_s \| < \delta_s^{\frac{1}{N}} \},
\]
which together with (3.76) implies
\[
\begin{align*}
&\left| \det(T_{\bar{A}_k}^{-1} - R_{A_k}^{-1}TR_{\tilde{\Omega}_{k+1}}^{-1}\text{A}_k^{-1}T_{\bar{A}_k}^{-1}R_{\tilde{\Omega}_{k+1}}^{-1}\text{A}_k^{-1}TR_{A_k}^{-1}) \right| \\
&= | \det S_{s+1}(\theta + k \cdot \omega) | \\
&\geq \frac{1}{C} \delta_s \| \theta + k \cdot \omega - \theta_{s+1} \| \cdot \| \theta + k \cdot \omega + \theta_{s+1} \|.
\end{align*}
\]
By Cramer’s rule and Hadamard’s inequality, one has
\[
\| (T_{A_k}^{-1} - R_{A_k}^{-1}TR_{\tilde{\Omega}_{k+1}}^{-1}\text{A}_k^{-1}T_{\bar{A}_k}^{-1}R_{\tilde{\Omega}_{k+1}}^{-1}\text{A}_k^{-1}TR_{A_k}^{-1})^{-1} \| < C2^{s}10^{s} \delta_s^{-1} \| \theta + k \cdot \omega - \theta_{s+1} \|^{-1} \cdot \| \theta + k \cdot \omega + \theta_{s+1} \|^{-1}.
\]
From Schur complement argument (cf. Lemma B.1) and (3.66), we get
\[
\| T_{\tilde{\Omega}_{k+1}}^{-1} \| < 4 \left( 1 + \| T_{\bar{A}_k}^{-1}\text{A}_k^{-1} \| \right)^2 \\
\times \left( 1 + \| (T_{A_k}^{-1} - R_{A_k}^{-1}TR_{\tilde{\Omega}_{k+1}}^{-1}\text{A}_k^{-1}T_{\bar{A}_k}^{-1}R_{\tilde{\Omega}_{k+1}}^{-1}\text{A}_k^{-1}TR_{A_k}^{-1})^{-1} \| \right) \\
< \delta_s^{-2} \| \theta + k \cdot \omega - \theta_{s+1} \|^{-1} \cdot \| \theta + k \cdot \omega + \theta_{s+1} \|^{-1}.
\] (3.78)

**STEP2: The case \((C2)\) Occurs:** i.e.,
\[
\text{dist} \left( \bar{Q}_s^-, Q_s^+ \right) \leq 100N_{s+1}^c.
\]
Then there exist \(i_s \in Q_s^+ \) and \(j_s \in \bar{Q}_s^- \) with \(\| i_s - j_s \| \leq 100N_{s+1}^c \), such that
\[
\| \theta + i_s \cdot \omega + \theta_s \| < \delta_s, \quad \| \theta + j_s \cdot \omega - \theta_s \| < \delta_s^{\frac{1}{N}}.
\]
Denote
\[
l_s = i_s - j_s.
\]
Using (3.8) and (3.9) yields
\[
Q_s^+, \bar{Q}_s^- \subset P_s \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i.
\]
Thus \(i_s \equiv j_s \) (mod \(\mathbb{Z}^d\)) and \(l_s \in \mathbb{Z}^d\). Define
\[
O_{s+1} = Q_s^- \cup (Q_s^+ - l_s).
\] (3.79)
For every \(o \in O_{s+1}\), define its mirror point
\[
o^* = o + l_s.
\]
Then we have
\[
O_{s+1} \subset \left\{ o \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \| o + \theta \cdot \omega - \theta_s \| < 2\delta_s^{\frac{1}{N}} \right\}
\]
and
\[ O_{s+1} + l_s \subset \left\{ o^* \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i : \| \theta + o^* \cdot \omega + \theta_s \| < 2\delta_s^{\frac{1}{100}} \right\}. \]

Then by (3.18), we obtain
\[ O_{s+1} \cup (O_{s+1} + l_s) \subset P_s. \tag{3.80} \]

Define
\[ P_{s+1} = \left\{ \frac{1}{2} (o + o^*) : o \in O_{s+1} \right\} = \left\{ o + \frac{l_s}{2} : o \in O_{s+1} \right\}. \tag{3.81} \]

Notice that
\[
\min \left( \| \frac{l_s}{2} \cdot \omega + \theta_s \|, \| \frac{l_s}{2} \cdot \omega + \frac{1}{2} \| \right) = \frac{1}{2} \| \frac{l_s}{2} \cdot \omega + 2\theta_s \|
\]
\[
\leq \frac{1}{2} \| \theta + s_s \cdot \omega + \theta_s \| + \| \theta + l_s \cdot \omega - \theta_s \| < \delta_s^{\frac{1}{100}}.
\]

Since \( \delta_s \ll 1 \), only one of the following occurs. First, we consider the case
\[ \| \frac{l_s}{2} \cdot \omega + \theta_s \| < \delta_s^{\frac{1}{100}}. \tag{3.82} \]

Let \( k \in P_{s+1} \). Since \( k = o + \frac{l_s}{2} \) for some \( o \in O_{s+1} \) and (3.82), we get
\[ \| \theta + k \cdot \omega \| \leq \| \theta + o \cdot \omega - \theta_s \| + \| \frac{l_s}{2} \cdot \omega + \theta_s \| < 3\delta_s^{\frac{1}{100}}, \]
which implies
\[ P_{s+1} \subset \left\{ k \in \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s} l_i : \| \theta + k \cdot \omega \| < 3\delta_s^{\frac{1}{100}} \right\}. \tag{3.83} \]

Moreover, if \( k \neq k' \in P_{s+1} \), then
\[
\| k - k' \| > \left| \log \frac{\gamma}{6\delta_s^{\frac{1}{100}}} \right| \sim N_{s+1}^{\gamma} \gg 10N_{s+1}^{\gamma^2}.
\]

Similar to the proof appeared in \textbf{STEP1} (i.e., the (C1) case), we can associate \( k \in P_{s+1} \) the blocks \( \Omega_k^{s+1} \) and \( \tilde{\Omega}_k^{s+1} \) with
\[
\Lambda_{100N_{s+1}}(k) \subset \Omega_k^{s+1} \subset \Lambda_{100N_{s+1} + 50N_{s+2}}(k),
\]
\[
\Lambda_{N_{s+1}^{\gamma}}(k) \subset \tilde{\Omega}_k^{s+1} \subset \Lambda_{N_{s+1}^{\gamma} + 50N_{s+2}}(k)
\]
satisfying
\[
\begin{cases}
\Omega_k^{s+1} \cap \tilde{\Omega}_{k'}^{s+1} \not= \emptyset \ (s' < s + 1) \Rightarrow \Omega_{k'}^{s+1} \subset \Omega_k^{s+1}, \\
\tilde{\Omega}_k^{s+1} \cap \tilde{\Omega}_{k'}^{s+1} \not= \emptyset \ (s' < s + 1) \Rightarrow \tilde{\Omega}_{k'}^{s+1} \subset \tilde{\Omega}_k^{s+1}, \\
\text{dist}(\tilde{\Omega}_k^{s+1}, \tilde{\Omega}_{k'}^{s+1}) > 10 \text{diam} \tilde{\Omega}_k^{s+1} \text{ for } k \neq k' \in P_{s+1}.
\end{cases} \tag{3.84}
\]

In addition, the set
\[
\tilde{\Omega}_k^{s+1} - k \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s} l_i
\]
is independent of $k \in P_{s+1}$ and symmetrical about the origin. Clearly, in this construction, for every $k' \in Q_s$, there exists $k = k' - \frac{l_s}{2}$ or $k' + \frac{l_s}{2} \in P_{s+1}$, such that

$$\tilde{\Omega}_{k'} \subset \Omega_k^{s+1}.$$

For every $k \in P_{s+1}$, we have $o, o^* \in P_s$ since (3.80). Define

$$A_{s+1}^k = A_o^s \cup A_{o^*}^s,$$

where $o \in O_{s+1}$ and $k = o + o^*$ (cf. (3.81)). Then

$$A_{s+1}^k \subset \Omega_o^s \cup \Omega_{o^*}^s \subset \Omega_k^{s+1},$$

$$\# A_{s+1}^k = \# A_o^s + \# A_{o^*}^s \leq 2^{s+1}.$$

Now we will verify that $(\tilde{\Omega}_k^{s+1} \setminus A_{s+1}^k)$ is s-good, i.e.,

$$\left\{ l' \in Q_s : \tilde{\Omega}_{l'} \subset (\tilde{\Omega}_k^{s+1} \setminus A_{s+1}^k), \tilde{\Omega}_{l'} \subset \Omega_l^{s'+1} \Rightarrow \tilde{\Omega}_{l'} \subset (\tilde{\Omega}_k^{s+1} \setminus A_{s+1}^k) \text{ for } s' < s \right\} \cap Q_s = \emptyset.$$

For this purpose, assume that

$$l' \in Q_s, \tilde{\Omega}_{l'} \subset (\tilde{\Omega}_k^{s+1} \setminus A_{s+1}^k), \tilde{\Omega}_{l'} \subset \Omega_l^{s'+1}.$$

If $s' \leq s - 2$, since $\emptyset \neq \tilde{\Omega}_{l'} \subset \tilde{\Omega}_l^{s'+1} \cap \tilde{\Omega}_k^{s+1}$, we have by (3.84) $\tilde{\Omega}_l^{s'+1} \subset \tilde{\Omega}_k^{s+1}$.

If $\tilde{\Omega}_l^{s'+1} \cap A_{s+1}^k \neq \emptyset$, then we have $\tilde{\Omega}_l^{s'+1} \cap A_{s}^o \neq \emptyset$ or $\tilde{\Omega}_l^{s'} \cap A_{o^*}^s \neq \emptyset$. Thus by (3.10) $(s' + 1 < s)$, we get $\tilde{\Omega}_l^{s} \subset \tilde{\Omega}_o^s$ or $\tilde{\Omega}_l^{s+1} \subset \tilde{\Omega}_{o^*}^s$, which implies $\tilde{\Omega}_l^{s} \subset (\tilde{\Omega}_o^s \setminus A_o^s)$ or $\tilde{\Omega}_l^{s+1} \subset (\tilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$. Thus we have either $\tilde{\Omega}_l^{s} \subset (\tilde{\Omega}_o^s \setminus A_o^s) \subset (\tilde{\Omega}_k^{s+1} \setminus A_{s+1}^k)$ or $\tilde{\Omega}_l^{s+1} \subset (\tilde{\Omega}_{o^*}^s \setminus A_{o^*}^s) \subset (\tilde{\Omega}_k^{s+1} \setminus A_{s+1}^k)$ since both $(\tilde{\Omega}_o^s \setminus A_o^s)$ and $(\tilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$ are (s-1)-good. This is a contradiction. If $s' = s - 1, \tilde{\Omega}_l^{s-1} \subset \tilde{\Omega}_o^s$ and $\tilde{\Omega}_{l'} \subset A_{s+1}^k \neq \emptyset$, then either $l = o$ or $l = o^*$, thus $\tilde{\Omega}_l^{s-1} \subset (\tilde{\Omega}_o^s \setminus A_o^s)$ or $\tilde{\Omega}_{l'} \subset (\tilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$. This contradicts

$$\left\{ l \in P_{s-1} : \tilde{\Omega}_l^{s-1} \subset (\tilde{\Omega}_o^s \setminus A_o^s) \right\} \cap Q_{s-1} = \emptyset,$$

since both $(\tilde{\Omega}_o^s \setminus A_o^s)$ and $(\tilde{\Omega}_{o^*}^s \setminus A_{o^*}^s)$ are (s-1)-good. Finally, if $l \in Q_s$ and $\tilde{\Omega}_l^{s-1} \subset \tilde{\Omega}_o^s$, then $l = o$ or $l = o^*$. Thus $\tilde{\Omega}_l \not\subset (\tilde{\Omega}_o^s \setminus A_{s+1}^k)$, which implies

$$\left\{ l \in P_s : \tilde{\Omega}_l \subset (\tilde{\Omega}_o^s \setminus A_{s+1}^k) \right\} \cap Q_s = \emptyset.$$

Moreover, we have

$$A_{s+1}^k - k = (A_o^s - k) \cup (A_{o^*}^s - k)$$

$$= \left( (A_o^s - o) - \frac{l_s}{2} \right) \cup \left( (A_{o^*}^s - o^*) + \frac{l_s}{2} \right)$$

is independent of $k \in P_{s+1}$ and symmetrical about the origin.

Now consider the analytic matrix-valued function

$$M_{s+1}(z) := T(z)\tilde{\Omega}_k^{s+1} - k = (\cos 2\pi (z + n \cdot \omega )\delta_{n,n'} - E + \varepsilon \Delta)_{n\in\tilde{\Omega}_k^{s+1} - k}$$

defined in

$$\{ z \in \mathbb{C} : |z| < \frac{1}{\delta_s^{s+1}} \}.$$  

(3.85)
If $k' \in P_s$ and $\Omega_k^* \subset (\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})$, then $k' \neq o, o^*$ and $\|k' - o\|, \|k' - o^*\| \leq 4N_k^{s+1}$.

Thus
\[
\|\theta + k' \cdot \omega - \theta_s\| \geq \|((k' - o) \cdot \omega) - (\theta + o \cdot \omega - \theta_s)\|
\geq \gamma e^{-4N_k^{s+1}} - 2s \frac{1}{\theta}
\geq \gamma e^{-4\log \frac{\delta_s}{\omega}} - 2s \frac{1}{\theta}
> \delta_s^{\frac{1}{104}},
\]
and
\[
\|\theta + k' \cdot \omega + \theta_s\| \geq \|((k' - o^*) \cdot \omega) - (\theta + o^* \cdot \omega + \theta_s)\|
\geq \gamma e^{-4N_k^{s+1}} - 2s \frac{1}{\theta}
\geq \gamma e^{-4\log \frac{\delta_s}{\omega}} - 2s \frac{1}{\theta}
> \delta_s^{\frac{1}{104}}.
\]

By (3.16), we have
\[
\|T_{\bar{\Omega}_k^{s+1} \setminus A_k^{s+1}}^{-1} \| < \delta_s^{-3} \sup_{\{k' \in P_s; \Omega_k^* \subset (\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})\}} \|\theta + k' \cdot \omega - \theta_s\|^{-1} \cdot \|\theta + k' \cdot \omega + \theta_s\|^{-1}
< \frac{1}{2}\delta_s^{\frac{3}{104}}.
\] (3.86)

One may restate (3.86) as
\[
\left\| \left( M_{s+1}(\theta + k \cdot \omega)_{(\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})-k} \right)^{-1} \right\| < \frac{1}{2}\delta_s^{\frac{3}{104}}.
\]

Since
\[
\| z - (\theta + k \cdot \omega) \| \leq |z| + \|\theta + k \cdot \omega\|
< \delta_s^{\frac{1}{10^4}} + 3\delta_s^{\frac{1}{10^4}} < 2\delta_s^{\frac{1}{10^4}} < \delta_s^{\frac{3}{104}},
\] (3.87)
we obtain using Neumann series argument
\[
\left\| \left( M_{s+1}(z)_{(\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})-k} \right)^{-1} \right\| < \delta_s^{\frac{3}{104}}.
\] (3.88)

We may control $M_{s+1}(z)^{-1}$ by the inverse of
\[
S_{s+1}(z) = M_{s+1}(z)_{A_k^{s+1}-k} - R_{A_k^{s+1}-k} M_{s+1}(z) R_{(\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})-k} \left( M_{s+1}(z)_{(\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})-k} \right)^{-1} R_{(\bar{\Omega}_k^{s+1} \setminus A_k^{s+1})-k} M_{s+1}(z) R_{A_k^{s+1}-k}.
\]

Our next aim is to analyze $\det S_{s+1}(z)$. Since $A_k^{s+1} - k = (A_o^s - k) \cup (A_o^{s*} - k)$, $A_o^s - k \subset \Omega_o^s - k$, $A_o^{s*} - k \subset \Omega_o^{s*} - k$ and
\[
\text{dist}(\Omega_o^s - k, \Omega_o^{s*} - k) > 10 \text{diam} \Omega_o^s,
\]
we have
\[
M_{s+1}(z)_{A_k^{s+1}-k} = M_{s+1}(z)_{A_o^s-k} \oplus M_{s+1}(z)_{A_o^{s*}-k}.
\]
From dist($\Omega^s_o, \partial^+ \tilde{\Omega}^s_o$) and dist($\Omega^s_o, \partial^+ \tilde{\Omega}^s_o$, $> 1$, we have
\[
R_{A^s_o - k} M_{s+1} (z) R_{\tilde{\Omega}^s_o \setminus A^s_o \setminus k} = R_{A^s_o - k} M_{s+1} (z) R_{\tilde{\Omega}^s_o \setminus A^s_o \setminus k},
\]
\[
R_{A^s_o - k} M_{s+1} (z) R_{\tilde{\Omega}^s_o \setminus A^s_o \setminus k} = R_{A^s_o - k} M_{s+1} (z) R_{\tilde{\Omega}^s_o \setminus A^s_o \setminus k}.
\]
Denote
\[
X = (\tilde{\Omega}^s_o \setminus A^s_o) - k, \quad X^* = (\tilde{\Omega}^s_o \setminus A^s_o) - k, \quad Y = (\tilde{\Omega}^s_o \setminus A^s_o) - k.
\]
Then direct computations yield
\[
S_{s+1} (z) = M_{s+1} (z) A^s_o \setminus k \oplus M_{s+1} (z) A^s_o \setminus k - (R_{A^s_o - k} \oplus R_{A^s_o - k}) M_{s+1} (z) R_Y M_{s+1} (z)^{-1} R_Y M_{s+1} (z) R_{A^s_o - k}
\]
\[
= \left( M_{s+1} (z) A^s_o \setminus k - R_{A^s_o - k} M_{s+1} (z) R_Y M_{s+1} (z)^{-1} R_Y M_{s+1} (z) R_{A^s_o - k} \right)
\]
\[
\oplus \left( M_{s+1} (z) A^s_o \setminus k - R_{A^s_o - k} M_{s+1} (z) R_Y M_{s+1} (z)^{-1} R_Y M_{s+1} (z) R_{A^s_o - k} \right) \cdot
\]
(3.89)

Since $\tilde{\Omega}^s_o \setminus A^s_o$ is $(s - 1)$-good, we have by (3.16)–(3.17)
\[
\left\| T_{\tilde{\Omega}^s_o \setminus A^s_o}^{-1} \right\| < \delta_{s-1}^{-3},
\]
\[
\left\| T_{\tilde{\Omega}^s_o \setminus A^s_o}^{-1} (x, y) \right\| < e^{-\gamma_{s-1} \| x - y \|_1} \text{ for } \| x - y \| > N_{s-1}^{c_4}.
\]
In other words,
\[
\left\| (M_{s+1} (\theta + k \cdot \omega) X)^{-1} \right\| < \delta_{s-1}^{-3}, \quad (3.90)
\]
\[
\left\| (M_{s+1} (\theta + k \cdot \omega) X)^{-1} (x, y) \right\| < e^{-\gamma_{s-1} \| x - y \|_1} \text{ for } \| x - y \| > N_{s-1}^{c_4}. \quad (3.91)
\]
From the approximation (3.87), we deduce by the same argument as (3.71) that
\[
\left\| (M_{s+1} (\theta + k \cdot \omega) X)^{-1} (x, y) \right\| < \delta_{s}^{10} \text{ for } \| x - y \| > N_{s-1}^{c_4}. \quad (3.92)
\]
Let $x \in X$ and dist($x, A^s_o \setminus k$) $\leq 1$. By resolvent identity, we have for any $y \in Y$,
\[
(M_{s+1} (z) Y)^{-1} (x, y) - \chi_X (y) (M_{s+1} (z) X)^{-1} (x, y)
\]
\[
= - \sum_{(w, w') \in \partial^+ \Omega^s_o} (M_{s+1} (z) X)^{-1} (x, w) \Gamma (w, w') (M_{s+1} (z) Y)^{-1} (w', y). \quad (3.93)
\]
From
\[
\text{dist}(x, w) \geq \text{dist}(A^s_o - k, \partial^+ \tilde{\Omega}^s_o - k) - 2 > N_{s} > N_{s-1}^{c_4},
\]
(3.88) and (3.92), the RHS of (3.93) is bounded by $C N_{s-1}^{c_3} \delta_{s}^{-2} \delta_{s}^{10} < \delta_{s}^{9}$. It follows that
\[
R_{A^s_o - k} M_{s+1} (z) R_X (M_{s+1} (z) X)^{-1} = R_{A^s_o - k} M_{s+1} (z) R_X (M_{s+1} (z) Y)^{-1} R_X + O(\delta_{s}^{9}).
\]
Similarly,
\[
R_{A^s_o - k} M_{s+1} (z) R_X \ast (M_{s+1} (z) Y)^{-1} = R_{A^s_o - k} M_{s+1} (z) R_X \ast (M_{s+1} (z) X) \ast (M_{s+1} (z) X)^{-1} R_X \ast + O(\delta_{s}^{9}).
\]
Recalling (3.89), we get
\[
S_{s+1} (z) = \left( M_{s+1} (z) A^s_o \setminus k - R_{A^s_o - k} M_{s+1} (z) R_X (M_{s+1} (z) X)^{-1} R_X \ast (\tilde{\Omega}^s_o \setminus A^s_o \setminus k) \ast \right)
\]
\[
\oplus \left( M_{s+1} (z) A^s_o \setminus k - R_{A^s_o - k} M_{s+1} (z) R_X \ast (M_{s+1} (z) X)^{-1} R_X \ast + O(\delta_{s}^{9}) \right)
\]
\[
= S_{s} (z - l_{s} \omega) + S_{s} (z + l_{s} \omega) + O(\delta_{s}^{9}). \quad (3.94)
\]
From (3.82) and (3.85), we have
\[
\|z - \frac{l_s}{2} \cdot \omega - \theta_s\| \leq |z| + \|\frac{l_s}{2} \cdot \omega + \theta_s\| < \delta_s^{1/10} + \delta_s^{1/10} < \delta_s^{1/10},
\]
\[
\|z + \frac{l_s}{2} \cdot \omega + \theta_s\| < |z| + \|\frac{l_s}{2} \cdot \omega + \theta_s\| < \delta_s^{1/10} + \delta_s^{1/10} < \delta_s^{1/10}.
\]
Thus both \(z - \frac{l_s}{2} \cdot \omega\) and \(z + \frac{l_s}{2} \cdot \omega\) belong to the set defined by (3.12), which together with (3.14) implies
\[
\det S_s(z - \frac{l_s}{2} \cdot \omega) \delta_s^{-1} ||(z - \frac{l_s}{2} \cdot \omega) - \theta_s\| \cdot \|(z - \frac{l_s}{2} \cdot \omega) + \theta_s\| = 0.
\] (3.95)
\[
\det S_s(z + \frac{l_s}{2} \cdot \omega) \delta_s^{-1} \|(z + \frac{l_s}{2} \cdot \omega) - \theta_s\| \cdot \|(z + \frac{l_s}{2} \cdot \omega) + \theta_s\| = 0. 
\] (3.96)

Moreover,
\[
\det S_{s+1}(z) = \det S_s(z - \frac{l_s}{2} \cdot \omega) \cdot \det S_s(z + \frac{l_s}{2} \cdot \omega) + O((2^{s+1})^2 2^{2^{s+1}} \delta_s^2)
\]
\[
= \det S_s(z - \frac{l_s}{2} \cdot \omega) \cdot \det S_s(z + \frac{l_s}{2} \cdot \omega) + O(\delta_s^2)
\] (3.97)
since \(#(A_k^{s+1} - k) \leq 2^{s+1}\), (3.13) and \(\log \log |\log \delta_s| \sim s\). Notice that
\[
\|z + \frac{l_s}{2} \cdot \omega - \theta_s\| \geq \|l_s \cdot \omega\| - \|z - \frac{l_s}{2} \cdot \omega - \theta_s\|
\]
\[
> \gamma e^{-(100N\gamma)^{r'}} - \delta_s^{1/10}
\]
\[
> \delta_s^{1/10},
\]
\[
\|z - \frac{l_s}{2} \cdot \omega + \theta_s\| \geq \|l_s \cdot \omega\| - \|z + \frac{l_s}{2} \cdot \omega + \theta_s\|
\]
\[
> \gamma e^{-(100N\gamma)^{r'}} - \delta_s^{1/10}
\]
\[
> \delta_s^{1/10}.
\] (3.99)

Let \(z_{s+1}\) satisfy
\[
z_{s+1} \equiv \frac{l_s}{2} \cdot \omega + \theta_s \pmod{\mathbb{Z}}, \quad |z_{s+1}| = \|\frac{l_s}{2} \cdot \omega + \theta_s\| < \delta_s^{1/100}.
\] (3.100)

From (3.95)–(3.99), we get
\[
\det S_{s+1}(z) \delta_s \cdot (z - z_{s+1}) \cdot (z + z_{s+1}) + r_{s+1}(z),
\]
where \(r_{s+1}(z)\) is an analytic function in the set defined by (3.85) with \(|r_{s+1}(z)| < \delta_s^7\).

By Rouché theorem, the equation
\[
(z - z_{s+1})(z + z_{s+1}) + r_{s+1}(z) = 0
\]
has exactly two roots \(\theta_{s+1}, \theta'_{s+1}\) in the set defined by (3.85), which are perturbations of \(\pm z_{s+1}\), respectively. Notice that
\[
\left\{|z| < \delta_s^{1/10} : \det M_{s+1}(z) = 0\right\} = \left\{|z| < \delta_s^{1/10} : \det S_{s+1}(z) = 0\right\}
\]
and \(\det M_{s+1}(z)\) is an even function of \(z\). Thus
\[
\theta'_{s+1} = -\theta_{s+1}.
\]
Moreover, we get
\[ |\theta_{s+1} - z_{s+1}| \leq |r_{s+1}(\theta_{s+1})|^{\frac{1}{2}} < \delta_{s}^{3} \]  
(3.101)
and
\[ |(z - z_{s+1})(z + z_{s+1}) + r_{s+1}(z)| \sim |(z - \theta_{s+1})(z + \theta_{s+1})|. \]
Thus for \( z \) being in the set defined by (3.85), we have
\[ \text{det} S_{s+1}(z) \delta_{s}^{\frac{1}{2}} \sim |z - \theta_{s+1}| \cdot \|z + \theta_{s+1}\|. \]  
(3.102)
Since \( \delta_{s+1}^{\frac{1}{2}} < \frac{1}{2}\delta_{s}^{\frac{1}{2}} \), by combining (3.100) and (3.101), we get
\[ \{ z \in \mathbb{C} : \min_{\sigma = \pm 1} |z + \sigma\theta_{s+1}| < \delta_{s+1}^{\frac{1}{2}} \} \subset \{ z \in \mathbb{C} : |z| < \delta_{s+1}^{\frac{1}{2}} \}. \]
Hence (3.102) also holds true for \( z \) belonging to
\[ \{ z \in \mathbb{C} : \|z \pm \theta_{s+1}\| < \delta_{s+1}^{\frac{1}{2}} \}, \]
which proves (3.14) for the \((s + 1)\)-th step.

Notice that
\[ \|\theta + k \cdot \omega + \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{2}}, \quad |\theta_{s+1} - z_{s+1}| < \delta_{s}^{3} \Rightarrow \|\theta + k \cdot \omega + \frac{k_{s}}{2} + \theta_{s}\| < \delta_{s}. \]
Thus if
\[ k \in \mathbb{Z}^{d} + \frac{1}{2} \sum_{i=0}^{s} l_{i} \text{ and } \|\theta + k \cdot \omega + \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{2}}, \]
then
\[ k + \frac{l_{s}}{2} \in \mathbb{Z}^{d} + \frac{1}{2} \sum_{i=0}^{s-1} l_{i} \text{ and } \|\theta + (k + \frac{l_{s}}{2}) \cdot \omega + \theta_{s}\| < \delta_{s}. \]
Thus by (3.52), we have \( k + \frac{l_{s}}{2} \in Q_{s}^{+} \). Recalling also (3.79) and (3.81), we have \( k \in P_{s+1} \). Thus
\[ \left\{ k \in \mathbb{Z}^{d} + \frac{1}{2} \sum_{i=0}^{s} l_{i} : \|\theta + k \cdot \omega + \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{2}} \} \subset P_{s+1}. \]
Similarly,
\[ \left\{ k \in \mathbb{Z}^{d} + \frac{1}{2} \sum_{i=0}^{s} l_{i} : \|\theta + k \cdot \omega - \theta_{s+1}\| < 10\delta_{s+1}^{\frac{1}{2}} \} \subset P_{s+1}. \]
Hence we prove (3.18) for the \((s + 1)\)-th step.

Finally, we will estimate \( T_{\Omega_{k}^{s+1}}^{-1} \). For \( k \in P_{s+1} \), we have by (3.83)
\[ \theta + k \cdot \omega \in \left\{ z \in \mathbb{C} : \|z\| < \delta_{s}^{\frac{1}{10}} \right\}. \]
Thus from (3.102), we obtain
\[ \left| \text{det}(T_{A_{k}^{s+1}} - R_{A_{k}^{s+1}} T_{\Omega_{k}^{s+1}}^{-1} A_{k}^{s+1} T_{A_{k}^{s+1}}^{-1} R_{\Omega_{k}^{s+1}} A_{k}^{s+1} T_{A_{k}^{s+1}}^{-1}) \right| \]
\[ = \left| \text{det} S_{s+1}(\theta + k \cdot \omega) \right| \]
\[ \geq \frac{1}{C} \delta_{s} \|\theta + k \cdot \omega - \theta_{s+1}\| \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|. \]
Using Cramer’s rule and Hadamard’s inequality implies
\[
\|(T_{A_k^{s+1}} - R_{A_k^{s+1}} T \tilde{\Omega}_k^{s+1} \setminus A_k^{s+1}) T_{\tilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} \|^{-1} \leq C 2^{s+1} 10^{s+1} \delta_{s+1}^{-1} \|\theta + k \cdot \omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|^{-1}.
\]
Recalling Schur complement argument (cf. Lemma B.1) and (3.86), we get
\[
\|(T_{\Omega_k^{s+1}})^{-1}\| < 4 \left(1 + \|(T_{\Omega_k^{s+1}})^{-1}\|\right)^2
\times \left(1 + \|(T_{A_k^{s+1}} - R_{A_k^{s+1}} T \tilde{\Omega}_k^{s+1} \setminus A_k^{s+1}) T_{\tilde{\Omega}_k^{s+1} \setminus A_k^{s+1}} \|^{-1}\right)
\cdot \|\theta + k \cdot \omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|^{-1}.
\]
(3.103)

For the case
\[
\left\|l_n \cdot \omega + \theta - \frac{1}{2}\right\| < \delta_{s+1}^{-1},
\]
(3.104)
we have
\[
P_{s+1} \triangleright \left\{k \in \mathbb{Z}^d : \frac{1}{2} \sum_{i=0}^{s} l_i : \|\theta + k \cdot \omega - \frac{1}{2}\| < 3 \delta_{s+1}^{-1}\right\}.
\]
(3.105)

Thus we can consider
\[
M_{s+1}(z) := T(z) \tilde{\Omega}_{k-1} = (\cos 2\pi(z + n \cdot \omega)\delta_{n,n'} - E + \varepsilon \Delta)_{n \in \tilde{\Omega}_{k-1}}
\]
in
\[
\left\{z \in \mathbb{C} : \left|z - \frac{1}{2}\right| < \delta_{s+1}^{-1}\right\}.
\]
(3.106)
By the similar arguments as above, we obtain both \(\theta_{s+1}\) and \(1 - \theta_{s+1}\) belong to the set defined by (3.106). Moreover, all the corresponding conclusions in the case of (3.82) hold for the case (3.104). Recalling (3.78), estimate (3.103) holds for the case (3.104) as well.

**STEP3: Application of resolvent identity**

Finally, we aim to establish (e)_{s+1} by iteratively the resolvent identity.

Recall that
\[
\left|\log \frac{\gamma}{\delta_{s+1}}\right| = \left|\log \frac{\gamma}{\delta_s}\right|^{\epsilon_0}.
\]

Define
\[
Q_{s+1} = \left\{k \in P_{s+1} : \min_{\sigma = \pm 1} \|\theta + k \cdot \omega + \sigma \theta_{s+1}\| < \delta_{s+1}\right\}.
\]

Assume the finite set \(\Lambda \subset \mathbb{Z}^d\) is \((s+1)-\text{good}, i.e.,
\[
\left\{k' \in Q_{s'}, \tilde{\Omega}_{k'} \subset \Lambda, \tilde{\Omega}_{k'} \subset \tilde{\Omega}_{k'}^{s'+1} \supseteq \tilde{\Omega}_{k}^{s'+1} \subset \Lambda \text{ for } s' < s + 1, \right\}
\]
\[
\{k \in P_{s+1} : \tilde{\Omega}_k^{s+1} \subset \Lambda\} \cap Q_{s+1} = \emptyset.
\]
(3.107)

It remains to verify the implications (3.16) and (3.17) with \(s\) being replaced with \(s + 1\).

For \(k \in P_t\) (1 \(\leq t \leq s + 1\), denote by
\[
2 \Omega_k^t := \Lambda_{\text{diam} \Omega_k^t}(k)
\]
the “double"-size block of $\Omega_k$. Define moreover

$$P_t = \{k \in P_t : \exists k' \in Q_{t-1} \text{ s.t.}, \Omega_{k'}^{-1} \subset A, \Omega_k^{-1} \subset \Omega_k \} \quad (1 \leq t \leq s + 1). \quad (3.108)$$

**Lemma 3.6.** For $k \in P_{s+1} \setminus Q_{s+1}$, we have

$$|T_{\Omega_k}^{-1}(x, y)| < e^{-\gamma_s ||x-y||_1} \quad \text{for } x \in \partial^{-} \Omega_k^{s+1} \text{ and } y \in 2\Omega_k^{s+1}, \quad (3.109)$$

where $\tilde{\gamma}_s = \gamma_s (1 - \frac{2}{s+1})$.

**Proof of Lemma 3.6.** Notice first that

$$\text{dist}(\partial^{-} \Omega_k^{s+1}, 2\Omega_k^{s+1}) \gtrsim \text{diam} \Omega_k^{s+1} \gg N_{s+1} \gg N_s^3.$$  

Since $\Omega_k^{s+1} \setminus A_k^{s+1}$ is $s$-good, we have by (3.17)

$$|T_{\Omega_k}^{-1}(x, y)| < e^{-\gamma_s ||x-w||_1} \quad \text{for } x \in \partial^{-} \Omega_k^{s+1}, \ w \in (\Omega_k^{s+1} \setminus A_k^{s+1}) \cap 2\Omega_k^{s+1}.$$  

From (3.103) and $k \notin Q_{s+1}$, we obtain

$$\|T_{\Omega_k}^{-1}\| < \delta_{s+1}^2 \delta_{s+1}^{-2} < \delta_{s+1}^3.$$  

Using resolvent identity implies (since $x \in \partial^{-} \Omega_k^{s+1}$)

$$|T_{\Omega_k}^{-1}(x, y)| = \left| T_{\Omega_k}^{-1}(x, y) \chi_{\Omega_k}^{s+1}(x, y) - \sum_{(w', w) \in \partial A_k^{s+1}} T_{\Omega_k}^{-1}(x, w) \Gamma(w, w') T_{\Omega_k}^{-1}(w', y) \right|$$

$$< e^{-\gamma_s ||x-y||_1} + 2d \cdot 2^{s+1} \sup_{w \in \partial A_k^{s+1}} e^{-\gamma_s ||x-w||_1} \|T_{\Omega_k}^{-1}\|$$

$$< e^{-\gamma_s ||x-u||_1} + \sup_{w \in \partial A_k^{s+1}} e^{-\gamma_s \left( ||x-y||_1 + ||y-w||_1 ||y-w||_1 \right)}$$

$$< e^{-\gamma_s ||x-u||_1} + e^{-\gamma_s \left( ||x-y||_1 + ||y-w||_1 \right)}$$

$$< e^{-\gamma_s \left( 1 - \frac{1}{2^{s+1}} \right) ||x-y||_1}$$

$$= e^{-\gamma_s ||x-y||_1},$$

since

$$N_{s+1}^c \lesssim \text{diam} \Omega_k^{s+1} \sim ||x - y||_1, \ ||y - w||_1 \lesssim \text{diam} \Omega_k^{s+1} \lesssim \left( \text{diam} \Omega_k^{s+1} \right)^{\frac{1}{2}}$$

and

$$| \log \delta_{s+1} | \sim | \log \delta_s | e^c \sim N_{s+1}^{c_{10}} < N_{s+1}^3.$$  \quad (3.110)

This proves the lemma. \qed

Next we consider the general case and will finish the proof of $(e)_{s+1}$. Define

$$\Lambda' = \Lambda \setminus \bigcup_{k \in P_{s+1}} \Omega_k^{s+1}.$$  

We claim that $\Lambda'$ is $s$-good. In fact, for $s' \leq s - 1$, assume $\Omega_k^{s'} \subset \Lambda'$, $\Omega_k^{s'} \subset \Omega_k^{s'+1}$ and $\Omega_k^{s'+1} \cap \left( \bigcup_{k \in P_{s+1}} \Omega_k^{s+1} \right) \neq \emptyset$. Thus by (3.84), we obtain $\Omega_k^{s'+1} \subset \bigcup_{k \in P_{s+1}} \Omega_k^{s+1}$,
which contradicts \( \overline{\Omega}_{k'} \subset \Lambda' \). If there exists \( k' \) such that \( k' \in Q_s \) and \( \overline{\Omega}_{k'} \subset \Lambda' \subset \Lambda \), then by (3.107) there exists \( k \in P_{s+1} \), such that 

\[
\overline{\Omega}_{k'} \subset \Omega_{k+1}^s \subset \Lambda.
\]

Hence recalling (3.108), one has \( k \in \overline{P}_{s+1} \) and 

\[
\overline{\Omega}_{k'} \subset \bigcup_{k \in P_{s+1}} \Omega_{k+1}^s.
\]

This contradicts \( \overline{\Omega}_{k'} \subset \Lambda' \). We have proven the claim. As a result, the estimates (3.16) and (3.17) hold true with \( \Lambda \) replaced by \( \Lambda' \). We can now estimate \( T_{\Lambda'}^{-1} \). For this purpose, we have the following two cases.

1. Assume that \( x \notin \bigcup_{k \in \overline{P}_{s+1}} \Omega_{k+1}^s \). Then \( N_{s+1}^3 \ll N_{s+1} \leq \text{dist}(x, \partial_{\Lambda} \Lambda') \). For \( y \in \Lambda \), using resolvent identity shows 

\[
T_{\Lambda'}^{-1}(x,y) = T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y) - \sum_{(w,w') \in \partial_{\Lambda} \Lambda'} T_{\Lambda'}^{-1}(x,w)\Gamma(w,w')T_{\Lambda'}^{-1}(w',y).
\]

Since

\[
\sum_{y \in \Lambda'} |T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y)| \leq \sum_{|x-y| \leq N_{s+1}^3} |T_{\Lambda'}^{-1}(x,y)| + \sum_{|x-y| > N_{s+1}^3} |T_{\Lambda'}^{-1}(x,y)| \\
\leq N_{s+1}^3 \cdot |T_{\Lambda'}^{-1}| + \sum_{|x-y| > N_{s+1}^3} e^{-\gamma s\|x-y\|} \\
\leq 2N_{s+1}^3 \delta_{s-1} \delta_{s}^{-2} \\
< \frac{1}{2} \delta_{s},
\]

and 

\[
\sum_{w \in \partial_{\Lambda} \Lambda'} |T_{\Lambda'}^{-1}(x,w)| \leq \sum_{\|x-w\| \geq N_{s+1}} e^{-\gamma s\|x-w\|} < e^{-\frac{1}{2} \gamma s N_{s+1}},
\]

we get

\[
\sum_{y \in \Lambda} |T_{\Lambda}^{-1}(x,y)| \leq \sum_{y \in \Lambda'} |T_{\Lambda'}^{-1}(x,y)\chi_{\Lambda'}(y)| + \sum_{y \in \Lambda, (w,w') \in \partial_{\Lambda} \Lambda'} |T_{\Lambda'}^{-1}(x,w)\Gamma(w,w')T_{\Lambda'}^{-1}(w',y)| \\
\leq \frac{1}{2} \delta_{s}^{-3} + 2d \sum_{w \in \partial_{\Lambda} \Lambda'} |T_{\Lambda'}^{-1}(x,w)| \cdot \sup_{w' \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda'}^{-1}(w',y)| \\
\leq \frac{1}{2} \delta_{s}^{-3} + \frac{1}{10} \sup_{w' \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda'}^{-1}(w',y)|.
\]

(2) Assume that \( x \in \Omega_{k+1}^s \) for some \( k \in \overline{P}_{s+1} \). Then by (3.107), we have \( \Omega_{k+1}^s \subset \Lambda \) and \( k \notin Q_{s+1} \). For \( y \in \Lambda \), using resolvent identity shows 

\[
T_{\Lambda}^{-1}(x,y) = T_{\Omega_{k+1}^s}^{-1}(x,y)\chi_{\Omega_{k+1}^s}(y) - \sum_{(w,w') \in \partial_{\Lambda} \Omega_{k+1}^s} T_{\Omega_{k+1}^s}^{-1}(x,w)\Gamma(w,w')T_{\Lambda}^{-1}(w',y).
\]

By (3.103), (3.109) and 

\[
N_{s+1} < \text{diam} \Omega_{k+1}^s \lesssim \text{dist}(x, \partial_{\Lambda} \Omega_{k+1}^s),
\]

we have
we have
\[
\sum_{y \in \Lambda} |T_{\Lambda}^{-1}(x, y)| \leq \sum_{y \in \Lambda} |T_{\tilde{\Omega}_k}^{-1}(x, y)\chi_{\tilde{\Omega}_{k+1}^c}(y)| + \sum_{y \in \Lambda, (w, w') \in \partial \tilde{\Omega}_k} |T_{\tilde{\Omega}_k}^{-1}(x, w)\Gamma(w, w')T_{\Lambda}^{-1}(w', y)|
\]
\[
< \#\tilde{\Omega}_{k+1}^c \cdot |T_{\tilde{\Omega}_k}^{-1}| + CN_{s+1}^c e^{-\tilde{\gamma}_s N_{s+1}} \sup_{w' \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(w', y)|
\]
\[
< C N_{s+1}^c \delta_s^{-2} \|\theta + k \cdot \omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|^{-1} + \frac{1}{10} \sup_{w' \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(w', y)|
\]
\[
< \frac{1}{2} \delta_s^{-3} \|\theta + k \cdot \omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|^{-1} + \frac{1}{10} \sup_{w' \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(w', y)|.
\]
Combining the above two cases, we obtain
\[
\|T_{\Lambda}^{-1}\| \leq \sup_{x \in \Lambda} \sum_{y \in \Lambda} |T_{\Lambda}^{-1}(x, y)|
\]
\[
< \delta_s^{-3} \sup_{\{k \in P_{s+1}, \tilde{\Omega}_{k+1}^c \subseteq \Lambda\}} \|\theta + k \cdot \omega - \theta_{s+1}\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_{s+1}\|^{-1}.
\]
(3.111)

Finally, we turn to the off-diagonal decay estimates. From (3.11), (3.107) and (3.108), it follows that for \( k' \in \tilde{P}_t \cap Q_t \) \((1 \leq t \leq s)\) there exists \( k \in \tilde{P}_{t+1} \) such that
\[
\tilde{\Omega}_k' \subseteq \Omega_{k+1}^t
\]
and
\[
\tilde{P}_{s+1} \cap Q_{s+1} = \emptyset.
\]
Moreover,
\[
\bigcup_{1 \leq t \leq s+1} \tilde{\Omega}_k \subseteq \Lambda.
\]
Hence for any \( w \in \Lambda \), if
\[
w \in \bigcup_{k \in \tilde{P}_t} 2\Omega_{k+1}^t,
\]
then there exists \( 1 \leq t \leq s+1 \) such that
\[
w \in \bigcup_{k \in \tilde{P}_t \setminus Q_t} 2\Omega_k^t.
\]
For every \( w \in \Lambda \), define its block in \( \Lambda \)
\[
J_w = \begin{cases} 
\Lambda_{2N_1}(w) \cap \Lambda & \text{if } w \notin \bigcup_{k \in \tilde{P}_t} 2\Omega_{k+1}^t, \\
\tilde{\Omega}_k & \text{if } w \in 2\Omega_k^t \text{ for some } k \in \tilde{P}_t \setminus Q_t.
\end{cases}
\]
Then \( \text{diam} J_w \leq \text{diam} \tilde{\Omega}_{k+1}^c < 3N_{s+1}^c \). For (1), we have \( J_w \cap Q_0 = \emptyset \) and \( \text{dist}(w, \partial \Lambda J_w) \geq \frac{1}{2} N_1 \). Thus
\[
|T_{J_w}^{-1}(w, w')| < e^{-\gamma_0 \|w-w'\|_1} \text{ for } w' \in \partial \Lambda J_w.
\]
For (2), by (3.109), we have
\[
|T_{J_w}(w, w')| < e^{-\gamma_{s-1} \|w-w'\|_1} \text{ for } w' \in \partial \Lambda J_w.
\]
Let \( \|x - y\| > N_{s+1}^3 \). The resolvent identity reads as
\[
T_\Lambda^{-1}(x, y) = T_{J_x}^{-1}(x, y) \chi_{J_x}(y) - \sum_{(w, w') \in \partial_x J_x} T_{J_x}^{-1}(x, w) \Gamma(w, w') T_{J_x}^{-1}(w', y).
\]
The first term in the above identity is zero since \( \|x - y\| > N_{s+1}^3 \) (so that \( y \notin J_x \)). It follows that
\[
|T_\Lambda^{-1}(x, y)| \leq C N_{s+1}^3 e^{-\min\{\gamma_0(1-2N_{s+1}^{-1}), \gamma_{s+1}(1-N_{s+1}^{-1})\} \|x - x_1\|} |T_\Lambda^{-1}(x_1, y)|
\]
for some \( x_1 \in \partial_x J_x \), where \( \gamma_s = \gamma_s(1 - N_{s+1}^{-1})^2 \). Iterate the above procedure and stop it for some \( L, \|x_L - y\| < 3N_{s+1}^2 \). Recalling (3.110) and (3.111), we get
\[
|T_\Lambda^{-1}(x, y)| \leq e^{-\gamma_s \|x - x_1\|} e^{-\gamma_3 \|x_1 - x_2\|} \cdots e^{-\gamma_3 \|x_{L-1} - x_L\|} |T_\Lambda^{-1}(x_L, y)|
\]
This gives the off-diagonal decay estimates.

We have completed the proof of Theorem 3.2. \( \square \)

4. **Arithmetic Anderson localization**

As an application of Green’s function estimates of previous section, we prove the arithmetic version of Anderson localization below.

**Proof of Theorem 1.2.** Recall first
\[\Theta_{\gamma} = \{ (\theta, \omega) \in \mathbb{T} \times \mathcal{R}_{\tau, \gamma} : \text{the relation } \|2\theta + n \cdot \omega\| \leq e^{-\|n\|^{\tau}} \text{ holds for finitely many } n \in \mathbb{Z}^d \},\]
where \( 0 < \tau < \gamma \).

We prove for \( 0 < \varepsilon \leq \varepsilon_0, \omega \in \mathcal{R}_{\tau, \gamma} \) and \( (\theta, \omega) \in \Theta_{\gamma} \), \( H(\theta) \) has only pure point spectrum with exponentially decaying eigenfunctions. Let \( \varepsilon_0 \) be given by Theorem 3.2. Fix \( \omega \) and \( \theta \) so that \( \omega \in \mathcal{R}_{\tau, \gamma} \) and \( (\theta, \omega) \in \Theta_{\gamma} \). Let \( E \in [-2, 2] \) be a generalized eigenvalue of \( H(\theta) \) and \( u = \{u(n)\}_{n \in \mathbb{Z}^d} \neq 0 \) be the corresponding generalized eigenfunction satisfying \( |u(n)| \leq (1 + \|n\|)^d \). From Schrödinger’s theorem, it suffices to show \( u \) decays exponentially. For this purpose, note first there exists (since \( (\theta, \omega) \in \Theta_{\gamma} \)) some \( s \in \mathbb{N} \) such that
\[
\|2\theta + n \cdot \omega\| > e^{-\|n\|^{\tau}} \text{ for all } n \text{ satisfying } \|n\| \geq N_s.
\]
We claim that there exists $s_0 > 0$ such that, for $s \geq s_0$,

$$\Lambda_{2N^{c_4} s} \cap \left( \bigcup_{k \in Q_s} \tilde{\Omega}_k^s \right) \neq \emptyset. \quad (4.2)$$

For otherwise, then there exist a subsequence $s_i \to +\infty$ (as $i \to \infty$) such that

$$\Lambda_{2N^{c_4} s_i} \cap \left( \bigcup_{k \in Q_{s_i}} \tilde{\Omega}_k^{s_i} \right) = \emptyset. \quad (4.3)$$

Then we can enlarge $\Lambda_{N^{c_4} s_i}$ to $\tilde{\Lambda}_i$ satisfying

$$\Lambda_{N^{c_4} s_i} \subset \tilde{\Lambda}_i \subset \Lambda_{N^{c_4} s_i + 50N^{c_2}},$$

and

$$\tilde{\Lambda}_i \cap \tilde{\Omega}_{k'}^s \neq \emptyset \Rightarrow \tilde{\Omega}_{k'}^s \subset \tilde{\Lambda}_i \text{ for } s' \leq s \text{ and } k \in P_s.$$

From (4.3), we have

$$\tilde{\Lambda}_i \cap \left( \bigcup_{k \in Q_{s_i}} \tilde{\Omega}_k^{s_i} \right) = \emptyset,$$

which shows $\tilde{\Lambda}_i$ is $s_i$-good. As a result, for $n \in \Lambda_{N^{c_4} s_i}$, since $\text{dist}(n, \partial \tilde{\Lambda}_i) \geq \frac{1}{2} N^{c_4}_{s_i} > N^{c_3}_{s_i}$, we have

$$|u(n)| \leq \sum_{(n', n'') \in \partial \tilde{\Lambda}_i} |T_{\tilde{\Lambda}_i}^{-1}(n, n') u(n'')| \leq 2d \sum_{n' \in \partial \tilde{\Lambda}_i} |T_{\tilde{\Lambda}_i}^{-1}(n, n')| \sup_{n'' \in \partial \tilde{\Lambda}_i} |u(n'')| \leq C N^{2c_4} e^{-\frac{1}{2} s} \to 0.$$

From $N_{s_i} \to +\infty$, it follows that $u(n) = 0$ for $\forall n \in \mathbb{Z}^d$. This contradicts $u \neq 0$, and the claim is proved.

Next, define

$$U_s = \Lambda_{8N^{c_4} s} \setminus \Lambda_{4N^{c_4}}, \quad U^*_s = \Lambda_{10N^{c_4} s} \setminus \Lambda_{3N^{c_4}}.$$

We can also enlarge $U^*_s$ to $\tilde{U}^*_s$ so that

$$U^*_s \subset \tilde{U}^*_s \subset \Lambda_{50N^{c_2}}(U^*_s),$$

and

$$\tilde{U}^*_s \cap \tilde{\Omega}_{k'}^{s'} \neq \emptyset \Rightarrow \tilde{\Omega}_{k'}^{s'} \subset \tilde{U}^*_s \text{ for } s' \leq s \text{ and } k \in P_{s'}.$$

Let $n$ satisfy $\|n\| > \max(4N^{c_4} s, 4N^{c_4} s_0)$. Then there exists some $s \geq \max(s, s_0)$ such that

$$n \in U_s. \quad (4.4)$$

By (4.2), without loss of generality, we may assume

$$\Lambda_{2N^{c_4} s} \cap \tilde{\Omega}_k^s \neq \emptyset.$$
for some $k \in Q_s^+$. Then for $k \neq k' \in Q_s^+$, we have

$$\|k - k'\| > \left| \log \frac{\gamma}{2\delta_s} \right|^\frac{1}{\tau} \geq N_{s+1}^c \gg \text{diam}\tilde{\mathcal{U}}_s^*.$$ 

Thus

$$\tilde{\mathcal{U}}_s^* \cap \left( \bigcup_{l \in Q_s^+} \tilde{\Omega}_l^s \right) = \emptyset.$$ 

Now, if there exists $l \in Q_s^-$ such that

$$\tilde{\mathcal{U}}_s^* \cap \tilde{\Omega}_l^s \neq \emptyset,$$

then

$$N_s < N_{s+1}^c \leq \|l\| - \|k\| \leq \|l + k\| \leq \|l\| + \|k\| < 11N_{s+1}^c.$$ 

Recalling

$$Q_s \subset P_s \subset \mathbb{Z}^d + \frac{1}{2} \sum_{i=0}^{s-1} l_i,$$

we have $l + k \in \mathbb{Z}^d$. Hence by (4.1),

$$e^{-11N_{s+1}^c \tau} < \|2\theta + (l + k) \cdot \omega\| \leq \|\theta + l \cdot \omega - \theta_s\| + \|\theta + k \cdot \omega + \theta_s\| < 2\delta_s.$$ 

This contradicts

$$|\log \delta_s| \sim N_{s+1}^c \gg N_{s+1}^c \tau.$$ 

We thus have shown

$$\tilde{\mathcal{U}}_s^* \cap \left( \bigcup_{l \in Q_s} \tilde{\Omega}_l^s \right) = \emptyset.$$ 

This implies $\tilde{\mathcal{U}}_s^*$ is s-good.

Finally, recalling (4.4), we have

$$\text{dist}(n, \partial - \tilde{\mathcal{U}}_s^*) \geq \min \left( 10N_{s+1}^c - |n|, |n| - 3N_{s+1}^c \right) - 1 \geq \frac{1}{5} \|n\| > N_{s+1}^c.$$ 

Then

$$|u(n)| \leq \sum_{(n', n'') \in \partial \tilde{\mathcal{U}}_s^*} |T_{\tilde{\mathcal{U}}_s^*}^{-1}(n, n')u(n'')|$$

$$\leq 2d \sum_{n' \in \partial - \tilde{\mathcal{U}}_s^*} |T_{\tilde{\mathcal{U}}_s^*}^{-1}(n, n')| \cdot \sup_{n'' \in \partial + \tilde{\mathcal{U}}_s^*} |u(n'')|$$

$$\leq CN_{s+1}^c \cdot e^{-\frac{1}{5} \gamma_s \|n\|}$$

$$\leq C\|n\| 2N_{s+1}^c \cdot e^{-\frac{1}{5} \gamma_s \|n\|}$$

$$< e^{-\frac{1}{5} \gamma_s \|n\|},$$

which yields the exponential decay $u$.

We complete the proof of Theorem 1.2.
Remark 4.1. Assume that for some $E \in [-2, 2]$, the inductive process stops at a finite stage (i.e., $Q_s = \emptyset$ for some $s < \infty$). Then for $N > N_s^c$, we can enlarge $\Lambda_N$ to $\tilde{\Lambda}_N$ with

$$\Lambda_N \subset \tilde{\Lambda}_N \subset \Lambda_{N+50N_s^2},$$

and

$$\tilde{\Lambda}_N \cap \tilde{\Omega}_k^s \neq \emptyset \Rightarrow \tilde{\Omega}_k^s \subset \tilde{\Lambda}_N \text{ for } s' \leq s \text{ and } k \in P_{s'}.$$

Thus $\tilde{\Lambda}_N$ is s-good. For $n \in \Lambda_N^s$, since $\text{dist}(n, \partial \tilde{\Lambda}_N) > N_s^c$, we have

$$|u(n)| \leq \sum_{(n', n'') \in \partial \Lambda_N} |T_{\Lambda_N^{-1}}(n, n')u(n'')|$$

$$\leq 2d \sum_{n' \in \partial \tilde{\Lambda}_N} |T_{\Lambda_N^{-1}}(n, n')| \sup_{n'' \in \partial \tilde{\Lambda}_N} |u(n'')|$$

$$\leq CN^{2d} \cdot e^{-\frac{T}{2} \gamma} \to 0.$$

Hence such $E$ is not a generalized eigenvalue of $H(\theta)$.

5. $(\frac{1}{2}-)$-Hölder continuity of the IDS

In this section, we apply our estimates to obtain $(\frac{1}{2}-)$-Hölder continuity of the IDS.

Proof of Theorem 1.2. Let $T$ be given by (3.1). Fix $\mu > 0$, $\theta \in \mathbb{T}$ and $E \in [-2, 2]$. Let $\varepsilon_0$ be defined in Theorem 3.2 and assume $0 < \varepsilon \leq \varepsilon_0$. Fix

$$0 < \eta < \eta_0 = \min \left( e^{-\left(\frac{\pi}{2}\right) \gamma}, e^{-|\log \delta_0|} \right). \quad (5.1)$$

Denote by $\{\xi_r : r = 1, \ldots, R\} \subset \text{span}(\delta_n : n \in \Lambda_N)$ the $\ell^2$-orthonormal eigenvectors of $T_{\Lambda_N}$ with eigenvalues belonging to $[-\eta, \eta]$. We aim to prove that for sufficiently large $N$ (depending on $\eta$),

$$R \leq (\# \Lambda_N) \eta^2 - \mu.$$

From (5.1), we can choose $s \geq 1$ such that

$$|\log \delta_s| < |\log \eta| < |\log \delta_s|.$$

Enlarge $\Lambda_N$ to $\tilde{\Lambda}_N$ so that

$$\Lambda_N \subset \tilde{\Lambda}_N \subset \Lambda_{N+50N_s^2}$$

and

$$\tilde{\Lambda}_N \cap \tilde{\Omega}_k^s \neq \emptyset \Rightarrow \tilde{\Omega}_k^s \subset \tilde{\Lambda}_N \text{ for } s' \leq s \text{ and } k \in P_{s'}.$$

Define further

$$\mathcal{K} = \left\{ k \in P_s : \tilde{\Omega}_k^s \subset \tilde{\Lambda}_N, \min_{\sigma = \pm 1} (\|\theta + k \cdot \omega + \sigma \theta s\|) < \eta^2 - \frac{T}{2} \right\}$$

and

$$\tilde{\Lambda}_N = \tilde{\Lambda}_N \setminus \bigcup_{k \in \mathcal{K}} \Omega_k^s.$$

Thus by (3.10), we obtain

$$k' \in Q_{s'}, \tilde{\Omega}_k^s \subset \tilde{\Lambda}_N', \Omega_k^s \subset \mathcal{K} \Rightarrow \tilde{\Omega}_k^{s'} \subset \tilde{\Lambda}_N' \text{ for } s' < s.$$
Since $|\log \eta| < |\log \delta_s| = \log \delta_{s+1} < \frac{1}{2} \log \frac{N}{\delta}$, we get from the resolvent identity

$$
||T_{\Lambda_N}^{-1}|| < \delta_s^{-3} \sup_{\{k \in \mathbb{N} : \Omega_k \subset \Lambda_N\}} \|\theta + k \cdot \omega - \theta_s\|^{-1} \cdot \|\theta + k \cdot \omega + \theta_s\|^{-1}
$$

where the last inequality follows from (5.1).

By the uniform distribution of \{n \cdot \omega\}_{n \in \mathbb{Z}^d} in T, we have

$$
\#(\tilde{\Lambda}_N \setminus \tilde{\Lambda}_N') \leq \#\Omega_k \cdot \#K
\leq C N^{cd} \cdot \# \left\{ k \in \mathbb{Z}^d + \sum_{i=0}^{s-1} l_i : \|k\| \leq N + 50N_s^c \cdot \min_{\sigma = \pm 1} (\|\theta + k \cdot \omega + \sigma \theta_s\|) < \eta^{1/2 - \frac{\mu}{2}} \right\}
\leq C N^{cd} \cdot \eta^{1/2 - \frac{\mu}{2}} (N + 50N_s^c)^d
\leq C N^{cd} \cdot \eta^{1/2 - \frac{\mu}{2}} \#\Lambda_N
$$

for sufficiently large $N$.

For a vector $\xi \in \mathbb{C}^\Lambda$ with $\Lambda \subset \mathbb{Z}^d$, we define $\|\xi\|$ to be the $\ell^2$-norm. Assume $\xi \in \{\xi_r : r \leq R\}$ be an eigenvector of $T_{\Lambda_N}$. Then

$$
\|T_{\Lambda_N}\xi\| = \|R_{\Lambda_N}T\xi\| \leq \eta.
$$

Hence

$$
\eta \geq \|R_{\tilde{\Lambda}_N}T_{\Lambda_N}\xi\| = \|R_{\tilde{\Lambda}_N}TR_{\tilde{\Lambda}_N}^\perp \xi + R_{\tilde{\Lambda}_N}TR_{\Lambda_N \setminus \tilde{\Lambda}_N} \xi - R_{\tilde{\Lambda}_N \setminus \Lambda_N} T_\xi\| \tag{5.3}
$$

Applying $T_{\tilde{\Lambda}_N}^{-1}$ to (5.3) and (5.2) implies

$$
\left\| R_{\tilde{\Lambda}_N} \xi + T_{\tilde{\Lambda}_N}^{-1} \left( R_{\tilde{\Lambda}_N} TR_{\Lambda_N \setminus \tilde{\Lambda}_N} \xi - R_{\tilde{\Lambda}_N \setminus \Lambda_N} T_\xi\right) \right\| < \frac{1}{2} \tag{5.4}
$$

Denote

$$
H = \text{Range} T_{\tilde{\Lambda}_N}^{-1} \left( R_{\tilde{\Lambda}_N} TR_{\Lambda_N \setminus \tilde{\Lambda}_N} - R_{\tilde{\Lambda}_N \setminus \Lambda_N} T \right).
$$

Then

$$
\dim H \leq \text{Rank} T_{\tilde{\Lambda}_N}^{-1} \left( R_{\tilde{\Lambda}_N} TR_{\Lambda_N \setminus \tilde{\Lambda}_N} - R_{\tilde{\Lambda}_N \setminus \Lambda_N} T \right)
\leq \#(\tilde{\Lambda}_N \setminus \tilde{\Lambda}_N') + \#(\tilde{\Lambda}_N \setminus \Lambda_N)
\leq C N^{cd} \cdot \eta^{1/2 - \frac{\mu}{2}} \#\Lambda_N + C N^{cd} N^{d-1}
\leq C N^{cd} \cdot \eta^{1/2 - \frac{\mu}{2}} \#\Lambda_N.
$$

Denote by $P_H$ the orthogonal projection to $H$. Applying $I - P_H$ to (5.4), we get

$$
\|R_{\tilde{\Lambda}_N} \xi - P_H R_{\tilde{\Lambda}_N} \xi\|^2 = \|R_{\tilde{\Lambda}_N} \xi\|^2 - \|P_H R_{\tilde{\Lambda}_N} \xi\|^2 \leq \frac{1}{4}.
$$

Before concluding the proof, we need a useful lemma.

**Lemma 5.1.** Let $H$ be a Hilbert space and let $H_1, H_2$ be its subspaces. Let \{\xi_r : r = 1, \ldots, R\} be a set of orthonormal vectors. Then we have

$$
\sum_{r=1}^R \|P_{H_1} P_{H_2} \xi_r\|^2 \leq \dim H_1.
$$
Proof of Lemma 5.1. Denote by $\langle \cdot, \cdot \rangle$ the inner product on $H$. Let $\{\phi_i\}$ be the orthonormal basis of $H_1$. By Parseval’s equality and Bessel’s inequality, we have

$$\sum_{r=1}^{R} \|P_{H_1} P_{H_2} \xi_r\|^2 = \sum_{r=1}^{R} \sum_{i} |\langle \phi_i, P_{H_2} \xi_r \rangle|^2$$

$$= \sum_{i} \sum_{r=1}^{R} |\langle P_{H_2} \phi_i, \xi_r \rangle|^2$$

$$\leq \sum_{i} \|P_{H_2} \phi_i\|^2$$

$$\leq \sum_{i} \|\phi_i\|^2 \leq \dim H_1.$$  

□

Finally, it follows from Lemma 5.1 that

$$R = \sum_{r=1}^{R} \|\xi_r\|^2 = \sum_{r=1}^{R} \|R_{\Lambda_N} \xi_r\|^2 + \sum_{r=1}^{R} \|R_{\Lambda_N \setminus \tilde{\Lambda}_N} \xi_r\|^2$$

$$\leq \frac{1}{4} R + \sum_{r=1}^{R} \left( \|P_H R_{\tilde{\Lambda}_N} \xi_r\|^2 + \|R_{\Lambda_N \setminus \tilde{\Lambda}_N} \xi_r\|^2 \right)$$

$$\leq \frac{1}{4} R + \dim H + \#(\Lambda_N \setminus \tilde{\Lambda}_N)$$

$$\leq \frac{1}{4} R + C N_s^c \cdot \eta^{\frac{1}{2} - \frac{s}{2}} \#\Lambda_N.$$

Hence we get

$$R \leq C N_s^c \cdot \eta^{\frac{1}{2} - \frac{s}{2}} \#\Lambda_N \leq \eta^{\frac{1}{2} - \mu} \#\Lambda_N.$$

We finish the proof of Theorem 1.3. □

Remark 5.1. In the above proof, if the inductive process stops at a finite stage (i.e., $Q_s = \emptyset$ for some $s$) and $|\log \delta_s| \leq |\log \eta|$. Then $\tilde{\Lambda}_N$ is s-good and

$$\|T_{\Lambda_N}^{-1}\| < \delta_s^{-3} \delta_s^{-2} \leq \frac{1}{2} \eta^{-1},$$

which implies

$$R \leq \frac{4}{3} \#(\tilde{\Lambda}_N \setminus \Lambda_N) \leq C N_s^{c_d} N^{-1} \#\Lambda_N.$$

Letting $N \to \infty$, we get $\mathcal{N}(E + \eta) - \mathcal{N}(E - \eta) = 0$, which means $E \notin \sigma(H(\theta))$.

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Appendix A.

Proof of Remark 3.1. Let $i \in Q^+_0$ and $j \in \tilde{Q}^-_0$ satisfy
\[ \| \theta + i \cdot \omega + \theta_0 \| < \delta_0, \| \theta + j \cdot \omega - \theta_0 \| < \delta_0^{-1}. \]

Then (1.4) implies $1, \omega, \ldots, \omega_d$ are rational independent and \( \{ k \cdot \omega \}_{k \in \mathbb{Z}^d} \) is dense in $T$. Thus there exists $k \in \mathbb{Z}^d$ such that $\| 2\theta + k \cdot \omega \|$ is sufficiently small with
\[ \| \theta + (k - j) \cdot \omega + \theta_0 \| \leq \| 2\theta + k \cdot \omega \| + \| \theta + j \cdot \omega - \theta_0 \| < \delta_0^{-1}, \]
\[ \| \theta + (k - i) \cdot \omega - \theta_0 \| \leq \| 2\theta + k \cdot \omega \| + \| \theta + i \cdot \omega + \theta_0 \| < \delta_0. \]

We obtain then $k - j \in \tilde{Q}^+_0$ and $k - i \in Q^-_0$, which implies
\[ \text{dist} \left( \tilde{Q}^+_0 \setminus Q^-_0 \right) \leq \text{dist} \left( \tilde{Q}^-_0 \setminus Q^+_0 \right). \]

The similar argument shows
\[ \text{dist} \left( \tilde{Q}^+_0 \setminus Q^-_0 \right) \geq \text{dist} \left( \tilde{Q}^-_0 \setminus Q^+_0 \right). \]

We have shown
\[ \text{dist} \left( \tilde{Q}^+_0 \setminus Q^-_0 \right) = \text{dist} \left( \tilde{Q}^-_0 \setminus Q^+_0 \right). \]

\[ \square \]

Appendix B.

Lemma B.1 (Schur Complement Lemma). Let $A \in \mathbb{C}^{d_1 \times d_1}, D \in \mathbb{C}^{d_2 \times d_2}, B \in \mathbb{C}^{d_1 \times d_2}, D \in \mathbb{C}^{d_2 \times d_1}$ be matrices and
\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]

Assume further that $A$ is invertible and $\| B \|, \| C \| \leq 1$. Then we have
(1).
\[ \det M = \det A \cdot \det S, \]

where
\[ S = D - CA^{-1}B \]

is called the Schur complement of $A$.

(2). $M$ is invertible iff $S$ is invertible, and
\[ \| S^{-1} \| \leq \| M^{-1} \| < 4 \left( 1 + \| A^{-1} \| \right)^2 \left( 1 + \| S^{-1} \| \right). \]  \hfill (B.1)

Proof of Lemma B.1. Direct computation shows
\[ M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \]

which implies (B.1).

\[ \square \]
Appendix C.

Lemma C.1. Let \( l \in \mathbb{Z}^d \) and let \( \Lambda \subset \mathbb{Z}^d + l \) be a finite set which is symmetrical about the origin (i.e., \( n \in \Lambda \iff -n \in \Lambda \)). Then

\[
\det T(z)_\Lambda = \det ((\cos 2\pi (z + n \cdot \omega) \delta_{n,n'} - E + \varepsilon \Delta)_{n \in \Lambda}
\]

is an even function of \( z \).

Proof of Lemma C.1. Define the unitary map

\[
U_\Lambda : \ell^2(\Lambda) \longrightarrow \ell^2(\Lambda) \text{ with } (U\phi)(n) = \phi(-n).
\]

Then

\[
U_\Lambda^{-1}T(z)_\Lambda U_\Lambda = ((\cos 2\pi (z - n \cdot \omega) \delta_{n,n'} - E + \varepsilon \Delta)_{n \in \Lambda} = T(-z)_\Lambda,
\]

which implies

\[
\det T(z)_\Lambda = \det T(-z)_\Lambda.
\]

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