Homological dimension formulas for trivial extension algebras

Dedicated to I. Reiten’s 75 birthday

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Abstract

Let $A = \Lambda \oplus C$ be a trivial extension algebra. The aim of this paper is to establish formulas for the projective dimension and the injective dimension for a certain class of $A$-modules which is expressed by using the derived functors $- \otimes^L \Lambda C$ and $\mathbb{R}\text{Hom}_{\Lambda}(C, -)$. Consequently, we obtain formulas for the global dimension of $A$, which gives a modern expression of the classical formula for the global dimension by Palmer-Roos and Lőfwall that is written in complicated classical derived functors.

The main application of the formulas is to give a necessary and sufficient condition for $A$ to be an Iwanaga-Gorenstein algebra.

We also give a description of the kernel $\text{Ker } \varpi$ of the canonical functor $\varpi : \text{D}^b(\text{mod } \Lambda) \to \text{Sing}^Z \Lambda$ in the case $\text{pd } C < \infty$.

Contents

1 Introduction
2 Homological algebra of finitely graded algebras
3 Homological dimensions of (unbounded) complexes (after Avramov-Foxby)
4 Projective dimension formula
5 Injective dimension formula
6 Upper triangular matrix algebras

1 Introduction

Throughout the paper $k$ denotes a commutative ring. An algebra $\Lambda$ is always $k$-algebra and a $\Lambda$-$\Lambda$-bimodule $C$ is always assumed to be $k$-central. Recall that the trivial extension algebra $A = \Lambda \oplus C$ is a direct sum $\Lambda \oplus C$ equipped with the multiplication

$$(r, c)(s, d) := (rs, rd + cs) \quad (r, s \in \Lambda, \ c, d \in C).$$

Since a trivial extension algebra is one of fundamental construction, it has been extensively studied from every aspect and homological dimension is no exception.
For instance, in [8], Chase raised a problem of determining the global dimension of an upper triangular matrix algebras $A = \begin{pmatrix} \Lambda_0 & C \\ 0 & \Lambda_1 \end{pmatrix}$ which is an example of a trivial extension algebra, in terms of $\Lambda_0, \Lambda_1$ and $C$.

The global dimension of general trivial extension algebra $A = \Lambda \oplus C$ had been studied by Fossum-Griffith-Reiten [12], Reiten [33], Palmer-Roos [31]. Finally, Lofwall [23] gave a general formula for the global dimension of $A$ in terms of $\Lambda_0, \Lambda_1$ and $C$. Thus, in particular, Chase’s problem was solved. (For the historical background we refer the readers to [12, Section 4], [31, Introduction].)

However, the methods “multiple Tor” for the Palmer-Roos-Lofwall formula was so complicated that the formula has never got attention which it ought to deserve. For instance, in the studies of homological dimensions of upper triangular algebras (e.g., [2, 9, 11, 34]) there have been no attempt to generalize the formula in such a way as to be applicable for each problem.

In this paper we establish formulas for homological dimensions of a class of $A$-modules by using homological dimension of objects of the derived category $D(\text{Mod} \Lambda)$ introduced by Avramov-Foxby [5]. As a corollary, we obtain formulas for the global dimension of $A$, which gives a modern expression of the Palmer-Roos-Lofwall formula.

We note that the formulas involve the iterated derived tensor product $C^a$ of $C$, where for $a \in \mathbb{N}$, we set

\[
C^a := \begin{cases} 
C \otimes^L_{\Lambda} C \otimes^L_{\Lambda} \cdots \otimes^L_{\Lambda} C & (a \text{-factors}) \\
\Lambda & a = 0
\end{cases}
\]

In the rest of Introduction, we explain the results of this paper by only focusing on injective dimensions. A key technique is the use of the grading with which a trivial extension algebra $A = \Lambda \oplus C$ is canonically equipped. Namely, deg $\Lambda = 0$, deg $C = 1$. Let $M$ be a graded $A$-module concentrated in degree $i = 0, 1$, i.e., $M_i = 0$ for $i \neq 0, 1$. Then, we will observe in Proposition 2.9 that its (ungraded) injective dimension and graded injective dimension coincide.

\[
\text{id}_A M = \text{gr.id}_A M.
\]

By this fact, we can pass to the study of the graded injective dimension of $M$. It is an analysis of a graded injective resolution of $M$ that naturally leads to the iterated derived tensor product $C^a$. To state our injective dimension formula, we use the derived coaction morphism $\Theta^0_M$ of $M$, that is, the morphism induced from the graded $A$-module structure on $M = M_0 \oplus M_1$.

\[
\Theta^0_M : M_0 \to \mathbb{R}\text{Hom}_\Lambda(C, M_1).
\]

For simplicity we set $\Theta^0_M := \mathbb{R}\text{Hom}_\Lambda(C^a, \Theta^0_M)$.

\[
\Theta^a_M : \mathbb{R}\text{Hom}_\Lambda(C^a, M_0) \to \mathbb{R}\text{Hom}_\Lambda(C^{a+1}, M_1).
\]

We can now formulate our injective dimension formula.

**Theorem 1.1** (Theorem 5.3). Let $M$ be a graded $A$-module such that $M_i = 0$ for $i \neq 0, 1$. Then,

\[
\text{id}_A M = \text{gr.id}_A M = \text{sup}\{\text{id}_A M_1, \text{id}_A (\text{cn } \Theta^a_M) + a + 1 | a \geq 0\}
\]

where $\text{cn } \Theta^a_M$ denote the cone of the morphism $\Theta^a_M$ in the derived category $D(\text{Mod} \Lambda)$.

\[1\text{ However, we remark that there is a subtlety about derived tensor product of bimodules. For this see Remark 4.9 and Remark 5.2.} \]
We note that this formula and a projective version given in Theorem 4.10 is established for not only a graded \( A \)-module but also an object of the derived category \( D(\text{Mod}^\mathbb{Z} A) \) satisfying the same condition.

In the case where \( M \) is concentrated in degree 0, the formula is simplified as in Corollary 5.4. We remark that such a graded \( A \)-module is nothing but a \( \Lambda \)-module regarded as an \( A \)-module via the augmentation map \( \text{aug} : A \to \Lambda, \text{aug}(r,c) := r \). The global dimension of \( A \) can be measured by such modules.

**Corollary 1.2** (Corollary 5.5).

\[
\text{gldim } A = \sup_{A} \{ \text{id}_A \text{Hom}_A(C^a, M) + a \mid M \in \text{Mod } A, \ a \geq 0 \}.
\]

We remark that the projective dimension version of the above formula which is given in Corollary 4.12 is essentially the same with the result of Palmer-Roos and Löfwall given in the aforementioned papers. We also remark that it seems that the above global dimension formula can be proved by their methods.

However, our main application of Theorem 1.1 which is a criterion of finiteness of self-injective dimension of \( A \), seems to be hard to be achieved by their methods. The graded component of the graded \( A \)-module \( A \) are \( A_0 = \Lambda \), \( A_1 = C \) and \( A_i = 0 \) for \( i \neq 0,1 \) and the derived coaction morphism \( \Theta^0_\lambda \) is the morphism

\[
\lambda_r : \Lambda \to \text{RHom}_A(C,C)
\]

induced from the left multiplication map \( \tilde{\lambda}_r : C \to C, (\tilde{\lambda}_r(r))(c) := sc (r \in \Lambda) \) (where the suffix \( r \) of \( \lambda_r \) indicate that this relates to the right self-injective dimension). As a consequence of Theorem 1.1 we obtain the following criterion.

**Theorem 1.3** (Theorem 5.10). The following conditions are equivalent:

1. \( \text{id } A_A < \infty \).

2. the following conditions are satisfied:

   - **Right ASID 1.** \( \text{id } C < \infty \).
   - **Right ASID 2.** \( \text{id cn}(\text{RHom}_A(C^a, \lambda_r)) < \infty \) for \( a \geq 0 \).
   - **Right ASID 3.** The morphism \( \text{RHom}_A(C^a, \lambda_r) \) is an isomorphism for \( a \gg 0 \).

Here “ASID” is an abbreviation of “attaching self-injective dimension”.

We would like to mention that the results of this paper came out of the study of finitely graded IG-algebras. Recall that a graded algebra is called Iwanaga-Gorenstein (IG) if it is graded Noetherian on both sides and has finite graded self-injective dimension on both sides. Representation theory of (graded and ungraded) IG-algebra was initiated by Auslander-Reiten [3], Happel [17] and Buchweitz [7], has been studied by many researchers and is recently getting interest from other areas [1, 6, 13, 20, 21, 22].

As is explained in Section 2.4 every finitely graded algebra is graded Morita equivalent to a trivial extension algebra. Hence, representation theoretic study of finitely graded algebras can be reduced to that of trivial extension algebras. By Theorem 1.3 a trivial extension algebra \( A = \Lambda \oplus C \) is IG if and only if \( C \) satisfies the right ASID conditions and the left version of them, the left ASID conditions. As is proved in Proposition 5.16 if \( \Lambda \) is IG, then the ASID conditions are simplified. In
the subsequent paper [25], we prove that in the case where \( \Lambda \) is IG, the right and left ASID conditions has a categorical interpretation. Using this interpretation we establish a relationship between the derived category \( D^b(\text{mod } \Lambda) \) of \( \Lambda \) and the stable category \( \underline{\mathbf{CM}}^Z \) of graded Cohen-Macaulay modules over \( \Lambda \) and provide several applications.

In [26], we introduce a new class of finitely graded IG-algebra called homologically well-graded (hwg) IG-algebra, and show that it posses nice characterizations from several view points. In particular, we characterize the condition that \( \Lambda = \Lambda \oplus C \) is hwg in terms of the right and left asid numbers \( \alpha_r, \alpha_l \) which is introduced in Definition 5.11.

The paper is organized as follows. In Section 2, we discuss homological algebra of finitely graded algebras. In particular, we study relationships between graded homological dimensions and ungraded homological dimensions.

In Section 3, we recall the projective dimension and the injective dimension for unbounded complexes introduced by Avramov-Foxby [5].

In Section 4, we establish the formula for the projective dimensions. The key tool is the decomposition of complexes of graded projective \( \Lambda \)-modules according to the degree of generators introduced by Orlov [30]. Analyzing the decomposition, we relate a projective resolution of \( M \) as graded \( \Lambda \)-modules with that of \( M \) as a \( \Lambda \)-module via the iterated derived tensor products \( C^\alpha \).

Among other things, in Corollary 4.18, we give a description of the kernel \( \text{Ker } \varpi \) of the canonical functor \( \varpi : D^b(\text{mod } \Lambda) \to \text{Sing}^Z \Lambda \) where \( \text{Sing}^Z \Lambda \) is the graded singular derived category of \( \Lambda \). This result also plays an important role in [25].

In Section 5, we establish the formula for the injective dimension. The key tool is the decomposition of complexes of graded injective \( \Lambda \)-modules. Analyzing the decomposition, we relate an injective resolution of \( M \) as graded \( \Lambda \)-modules with that of \( M \) as a \( \Lambda \)-module via the derived Hom functors \( R\text{Hom}_\Lambda(C^\alpha, -) \). We give a criterion that \( \Lambda = \Lambda \oplus C \) is IG in terms of \( \Lambda \) and \( C \). We see that if \( \Lambda \) is IG, then the condition is simplified.

In Section 6, we discuss an upper triangular matrix algebra \( \Lambda = \begin{pmatrix} \Lambda_0 & C \\ 0 & \Lambda_1 \end{pmatrix} \), which is an example of a trivial extension algebra. Although, Chase’s problem was already solved as a corollary of the main result of [31] and [23], we give our own answer. Since every \( \Lambda \)-module \( M \) has a canonical grading such that \( M_i = 0 \) for \( i \neq 0, 1 \), we obtain formulas of the projective dimension and the injective dimension of \( M \) in terms of \( \Lambda_0, \Lambda_1 \) and \( C \). Moreover, as immediate corollaries, we deduce other known results concerning on upper triangular algebras.

### 1.1 Notation and convention

Let \( \Lambda \) be an algebra. Unless otherwise stated, the word “\( \Lambda \)-modules” means right \( \Lambda \)-modules. We denote by \( \text{Mod } \Lambda \) the category of \( \Lambda \)-modules. We denote by \( \text{Proj } \Lambda \) (resp. \( \text{Inj } \Lambda \)) the full subcategory of projective (resp. injective) \( \Lambda \)-modules. We denote by \( \text{proj } \Lambda \) the full subcategory of finitely generated projective modules.

We denote the opposite algebra by \( \Lambda^{\text{op}} \). We identify left \( \Lambda \)-modules with (right) \( \Lambda^{\text{op}} \)-modules. A \( \Lambda \)-\( \Lambda \)-bimodule \( D \) is always assumed to be \( k \)-central, i.e., \( ad = da \) for \( d \in D, a \in k \). For a \( \Lambda \)-\( \Lambda \)-bimodule \( D \), we denote by \( D_\Lambda \) and \( _\Lambda D \) the underlying right and left \( \Lambda \)-modules respectively. So for example, \( \text{id}^{\Lambda^{\text{op}}} \Lambda \) denotes the injective dimension of \( D \) regarded a left \( \Lambda \)-module.

In the paper, the degree of graded modules \( M \) is usually denoted by the characters \( i, j, \ldots \). The degree, which is called the cohomological degree, of complexes \( X \) is usually denoted by the characters \( m, n, \ldots \).
2 Homological algebra of finitely graded algebras

In this paper, a graded algebra is always a non-negatively graded algebra \( A = \bigoplus_{i \geq 0} A_i \). A graded algebra \( A = \bigoplus_{i \geq 0} A_i \) is called \textit{finitely graded} if \( A_i = 0 \) for \( i \gg 0 \). In this Section 2, we collect basic facts about homological algebra of finitely graded algebras. Another aim is to introduce constructions \( p, t, p, \ i, s, i \) which play a central role in this paper.

2.1 Notation and convention for graded algebras and graded modules

Let \( A = \bigoplus_{i \geq 0} A_i \) be a graded algebra. We denote by \( \text{Mod}^\mathbb{Z} A \) the category of graded (right) \( A \)-modules \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) and \( A \)-module homomorphisms \( f: M \to N \), which, by definition, preserves degree of \( M \) and \( N \), i.e., \( f(M_i) \subset N_i \) for \( i \in \mathbb{Z} \). We denote by \( \text{Proj}^\mathbb{Z} A \) (resp. \( \text{Inj}^\mathbb{Z} A \)) the full subcategory of graded projective (resp. graded injective) modules. We denote by \( \text{proj}^\mathbb{Z} A \) the full subcategory of finitely generated graded projective modules.

For a graded \( A \)-module \( M \) and an integer \( j \in \mathbb{Z} \), we define the shift \( M(j) \in \text{Mod}^\mathbb{Z} A \) by \( (M(j))_i = M_{i+j} \). We define the truncation \( M_{\geq j} \) by \( (M_{\geq j})_i = M_i \) (\( i \geq j \)), \( (M_{\geq j})_i = 0 \) (\( i < j \)). We set \( M_{< j} := M/M_{\geq j} \) so that we have an exact sequence \( 0 \to M_{\geq j} \to M \to M_{< j} \to 0 \).

For \( M, N \in \text{Mod}^\mathbb{Z} A \), \( n \in \mathbb{N} \) and \( i \in \mathbb{Z} \), we set \( \text{EXT}^n_A(M, N)_i := \text{Ext}^n_{\text{Mod}^\mathbb{Z} A}(M, N(i)) \) and

\[
\text{EXT}^n_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{EXT}^n_A(M, N)_i = \bigoplus_{i \in \mathbb{Z}} \text{Ext}^n_{\text{Mod}^\mathbb{Z} A}(M, N(i)).
\]

We note the obvious equation \( \text{HOM}_A(M, N)_0 = \text{Hom}_{\text{Mod}^\mathbb{Z} A}(M, N) \). We denote by \( \text{Hom}_A(M, N) \) the Hom-space as ungraded \( A \)-modules. We note that there exists the canonical map \( \text{HOM}_A(M, N) \to \text{Hom}_A(M, N) \) and it become an isomorphism if \( M \) is finitely generated.

For further details of graded algebras and graded modules we refer the readers to [29].

2.2 Graded projective dimension and ungraded projective dimension of graded \( A \)-modules

In the rest of this section, \( A = \bigoplus_{i=0}^\ell A_i \) is a finitely graded algebra. We note that \( \ell \) is a natural number such that \( A_i = 0 \) for \( i > \ell + 1 \) and that it is not necessary to assume \( A_\ell \neq 0 \). For notational simplicity we set \( \Lambda := A_0 \).

The following lemma can be easily checked and is left to the readers.

\textbf{Lemma 2.1.} (1) Let \( P \) be a graded projective \( A \)-module. Then, for all \( i \in \mathbb{Z} \), the module \( (P \otimes_A \Lambda)_i \) is a projective \( \Lambda \)-module.

(2) Let \( Q \) be a projective \( \Lambda \)-module. Then, the graded module \( Q \otimes_A A \) is a graded projective \( A \)-module.

Since the grading of \( A \) is finite, the following Nakayama type Lemma follows.
Lemma 2.2. Let $M$ be a graded $A$-module. Then $M = 0$ if and only if $M \otimes_A \Lambda = 0$.

For an integer $i \in \mathbb{Z}$, we denote by $p_i : \text{Proj}^Z A \to \text{Proj} \Lambda$ the functor $p_iP := (P \otimes_A \Lambda)_i$. We define a graded $A$-module $t_iP$ to be $t_iP := (p_iP) \otimes_A A(-i)$. We may consider $p_iP$ as the space of generators of $P$ having degree $i$.

**Lemma 2.3 (cf. [24, Proposition 2.6]).** Let $P$ be an object of $\text{Proj}^Z A$. Then, we have an isomorphism of graded $A$-modules

\[(2-1) \quad P \cong \bigoplus_{i \in \mathbb{Z}} t_iP.\]

**Proof.** For simplicity, we set \( \tilde{P} := \bigoplus_{i \in \mathbb{Z}} t_iP \). We remark that $\tilde{P} = (P \otimes_A \Lambda) \otimes_A A$ and it is a graded projective $A$-module by Lemma 2.1. There are canonical surjective graded $A$-module homomorphisms $p : P \to P \otimes_A \Lambda$ and $\tilde{p} : \tilde{P} \to P \otimes_A \Lambda$. Since $P$ is a graded projective $A$-module, there exists a graded $A$-module homomorphism $f : P \to \tilde{P}$ such that $f \otimes \Lambda$ is an isomorphism. Since $\tilde{P}$ is a flat as an ungraded $A$-module, we have $(\text{Cok} f) \otimes_A \Lambda = 0$, $(\text{Ker} f) \otimes_A \Lambda = 0$. Therefore $\text{Cok} f = 0$, $\text{Ker} f = 0$ by Lemma 2.2.

Later we will use the following corollary.

**Corollary 2.4.** We have the following isomorphism for $P \in \text{Proj}^Z A$ and $M \in \text{Mod}^Z A$

\[\text{Hom}_{\text{Mod}^Z A}(P, M) \cong \prod_{i \in \mathbb{Z}} \text{Hom}_A(p_iP, M_i).\]

**Proof.** We define a map $\Phi : \text{Hom}_{\text{Mod}^Z A}(P, M) \to \prod_{i \in \mathbb{Z}} \text{Hom}_A(p_iP, M_i)$ in the following way. Let $f : P \to M$ be a graded $A$-module homomorphism. For $i \in \mathbb{Z}$, we define the $i$-th component $\Phi(f)_i : p_iP \to M_i$ of $\Phi(f)$ to be the composite map

\[\Phi(f)_i : p_iP \xrightarrow{\text{in}_i} P \xrightarrow{f} M \xrightarrow{p_i} M_i\]

where $\text{in}_i$ is a canonical inclusion and $p_i$ is a canonical projection.

We define a map $\Psi : \prod_{i \in \mathbb{Z}} \text{Hom}_A(p_iP, M_i) \to \text{Hom}_{\text{Mod}^Z A}(P, M)$ in the following way. Let $g = (g_i)_{i \in \mathbb{Z}}$ be a collection of $\Lambda$-module homomorphisms $g_i : p_iP \to M_i$. Observe that $g_i$ extends to a graded $A$-module homomorphism $\hat{g}_i : t_iP = p_iP \otimes_A A \to M$. We define $\Psi(g) : P \to M$ to be the sum

\[\Psi(g) := \sum_{i \in \mathbb{Z}} \hat{g}_i : P \cong \bigoplus_{i \in \mathbb{Z}} t_iP \longrightarrow M.\]

We can check that $\Phi$ and $\Psi$ are the inverse map to each other.

For a graded module $M$, its graded projective dimension and its (ungraded) projective dimension coincide. We remark that this statement is true for any $\mathbb{Z}$-graded algebra $A$ ([29, I.3.3.12]). For convenience of the readers, we give a proof in the case where $A$ is finitely graded.

**Proposition 2.5.** For a graded $A$-module $M$, we have $\text{pd} M = \text{gr.pd} M$.

**Proof.** The inequality $\text{pd} M \leq \text{gr.pd} M$ follows from the fact that if we forget the grading from a graded projective $A$-module, then it become a projective $A$-module.

We prove $\text{pd} M \geq \text{gr.pd} M$. We may assume $n := \text{pd} M < \infty$. Take a projective resolution $\cdots \to P^{-1} \xrightarrow{\partial^{-1}} P^0 \to M$ in $\text{Mod}^Z A$. It is enough to show that $K := \text{Ker} \partial^{-}(n-1)$ is a graded
projective $A$-module. Since $K$ is a projective $A$-module, the graded $\Lambda$-module $L := K \otimes_{\Lambda} \Lambda$ become a projective $\Lambda$-module if we forget the grading. Hence $L$ is projective as a graded $\Lambda$-module. The canonical surjection $cs : K \to L$ has a section $sc : L \to K$ in $\text{Mod}^{\mathbb{Z}} \Lambda$, which extends to a graded $A$-module homomorphism $\hat{sc} : L \otimes_{\Lambda} A \to K$. In the same way of the proof of Lemma 2.3, we can check that $\hat{sc}$ is an isomorphism.

2.3 Graded injective dimension and ungraded injective dimension of graded $A$-modules

The following Lemma can be easily checked and is left to the readers.

Lemma 2.6. (1) Let $I$ be a graded injective $A$-module. Then, for all $i \in \mathbb{Z}$, the $\Lambda$-module $\text{HOM}_{\Lambda}(\Lambda, I)_i$ is an injective $\Lambda$-module. Moreover, we have $\text{HOM}_{\Lambda}(\Lambda, I) = \text{Hom}_{\Lambda}(\Lambda, I)$ as ungraded $\Lambda$-modules.

(2) Let $J$ be an injective $\Lambda$-module. We regard $J$ as a graded $\Lambda$-module concentrated in degree 0. Then the graded $A$-module $\text{HOM}_{\Lambda}(A, J)$ is a graded injective $A$-module. Moreover we have $\text{HOM}_{\Lambda}(A, J) = \text{Hom}_{\Lambda}(A, J)$ as ungraded $A$-modules and it is an ungraded injective $A$-module.

For an integer $i \in \mathbb{Z}$, we denote by $i_i : \text{Inj}^{\mathbb{Z}} A \to \text{Inj} A$ the functor $i_i := \text{HOM}_{\Lambda}(\Lambda, I)_i$. Then, we obtain a graded injective $A$-module $s_i I := \text{HOM}_{\Lambda}(A, i_i I)(-i)$.

Lemma 2.7. Let $I$ be an object of $\text{Inj}^{\mathbb{Z}} A$. Then, we have the following isomorphism of graded $A$-modules

$$I \cong \bigoplus_{i \in \mathbb{Z}} s_i I.$$ 

Proof. First observe that the direct sum $\tilde{I} := \bigoplus_{i \in \mathbb{Z}} s_i I$ is finite in each degree. More precisely, $\tilde{I}_j = \bigoplus_{i \in \mathbb{Z}} (s_i I)_j = \bigoplus_{i+j = j} (s_i I)_j$. It follows that $\tilde{I}$ is the direct product of $\{s_i I\}_{i \in \mathbb{Z}}$ in the category $\text{Mod}^{\mathbb{Z}} A$. Therefore, $\tilde{I}$ is an injective object of $\text{Mod}^{\mathbb{Z}} A$.

It can be checked that both $I$ and $\bigoplus_{i \in \mathbb{Z}} s_i I$ contain $\text{HOM}_{\Lambda}(\Lambda, I) = \bigoplus_{i \in \mathbb{Z}} i_i I$ as an essential submodule. Hence by uniqueness of injective hull, we conclude the desired isomorphism.

The following Lemma is an injective version of Lemma 2.4. The proof is left to the readers.

Corollary 2.8. We have the following isomorphism for $I \in \text{Inj}^{\mathbb{Z}} A$ and $M \in \text{Mod}^{\mathbb{Z}} A$.

$$\text{Hom}_{\text{Mod}^{\mathbb{Z}} A}(M, I) \cong \prod_{i \in \mathbb{Z}} \text{Hom}_{\Lambda}(M_i, i_i I).$$

For a finitely graded module $M$, its graded injective dimension and (ungraded) injective dimension coincide.

Proposition 2.9. For a finitely graded $A$-module $M$, we have $\text{id} M = \text{gr.id} M$.

Proof. We claim that an injective hull $I$ of $M$ in $\text{Mod}^{\mathbb{Z}} A$ is finitely graded and is an injective hull in $\text{Mod} A$. Indeed, it can be checked that $I$ is finitely graded and $I = \bigoplus_{i \in \mathbb{Z}} s_i I$ is finite sum. Therefore it is injective as an ungraded $A$-module. It can be checked that a graded essential submodule $N \subset L$ of a graded module $L$ is an essential (ungraded) submodule (and vice versa) [23, I.3.3.13]. In particular, $M$ is an essential $A$-submodule of $I$. Thus, we conclude that $I$ is an injective hull of $M$ in $\text{Mod} A$. Using the claim, we can easily check $\text{id} M \leq \text{gr.id} M$.

Assume that $n := \text{id} M < \infty$. Take a minimal injective resolution $0 \to M \to I^0 \xrightarrow{\partial^0} I^1 \to \cdots$ in $\text{Mod}^{\mathbb{Z}} A$. Then $K = \text{Cok} \partial^{n-2}$ is finitely graded and injective as an ungraded $A$-module. It follows from the claim that $K$ is a graded injective $A$-module. Hence we conclude that $\text{id} M \geq \text{gr.id} M$. 


We point out the following immediate consequence.

**Corollary 2.10.** We have the following equation

\[ \text{id}_A = \text{gr.id}_A. \]

The graded global dimension and the (ungraded) global dimension coincide. Moreover this can be measured by the category of finitely graded modules.

**Proposition 2.11** (cf. [29, I.7.8]). The following equations hold:

\[ \text{gldim} A = \text{grgldim} A = \sup \{ \text{gr.pd} M \mid M \text{ a finitely graded } A\text{-module.} \} = \sup \{ \text{gr.id} M \mid M \text{ a finitely graded } A\text{-module.} \} = \sup \{ \text{pd} M \mid M \text{ an } A\text{-module such that } MA_{\geq 1} = 0. \} = \sup \{ \text{id} M \mid M \text{ an } A\text{-module such that } MA_{\geq 1} = 0. \} \]

**Proof.** For simplicity, we denote by \( v_p \) the \( p \)-th value of the above equation. For example, \( v_1 := \text{gldim} A, v_2 := \text{grgldim} A \). The equation \( v_1 \geq v_2 \) follows from Proposition 2.5. It is clear that \( v_2 \geq v_3 \). Since an \( A\)-module \( M \) such that \( MA_{\geq 1} = 0 \) can be regarded as a graded \( A\)-module concentrated at 0-th degree, we can see that \( v_3 \geq v_5 \) by Proposition 2.5. We prove the inequality \( v_5 \geq v_1 \). Since an \( A\)-module \( M \) has a filtration \( M_i := MA_{\geq i} \) for \( i = 1, \ldots, \ell \), whose graded quotients \( N_i = M_i/M_{i+1} \) satisfy \( N_i A_{\geq 1} = 0 \), we conclude that \( \text{pd} M \leq v_5 \). In the same way, we can prove \( v_2 \geq v_4 \geq v_6 \geq v_1 \). \( \Box \)

### 2.4 Quasi-Veronese algebras and Beilinson algebras for finitely graded algebras

This Section 2.4 does not relate to the main subject of this paper in a strict sense. We explain an importance of trivial extension algebras by showing that every finitely graded algebra \( A \) is graded Morita equivalent to the trivial extension algebra \( \nabla A \oplus \Delta A \) equipped with the standard grading.

Let \( A \) be a finitely graded algebra. We fix a natural number \( \ell \) such that \( A_i = 0 \) for \( i \geq \ell + 1 \). (It is not necessary to assume that \( A_\ell \neq 0 \).) In this situation, we define the **Beilinson algebra** \( \nabla A \) of \( A \) (which rigorously should be called the Beilinson algebra of the pair \((A, \ell)\)) and its bimodule \( \Delta A \) to be

\[
\nabla A := \begin{pmatrix}
A_0 & A_1 & \cdots & A_{\ell-1} \\
0 & A_0 & \cdots & A_{\ell-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_0
\end{pmatrix}, \quad \Delta A := \begin{pmatrix}
A_\ell & 0 & \cdots & 0 \\
A_{\ell-1} & A_\ell & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_\ell
\end{pmatrix}
\]

where the algebra structure and the bimodule structure are given by the matrix multiplications. Then, the trivial extension algebra \( \nabla A \oplus \Delta A \) with the grading \( \deg \nabla A = 0, \deg \Delta A = 1 \) is nothing but the \( \ell \)-th quasi-Veronese algebra \( A[\ell] \) of \( A \) introduced by Mori [28, Definition 3.10].

\[ A[\ell] = \nabla A \oplus \Delta A. \]

**Remark 2.12.** In [28], the multiplication of the quasi-Veronese algebra is defined by so called the opposite of the matrix multiplication, which is different from our definition. However, this difference occurs from the notational difference.
By [28, Lemma 3.12] $A$ and $A^{[\ell]}$ are graded Morita equivalent to each other. More precisely, the functor $qv$ below gives a $k$-linear equivalence.

$$
qv : \text{Mod}^Z A \rightarrow \text{Mod}^Z A^{[\ell]},
$$

$$
vq(M) := \bigoplus_{i \in \mathbb{Z}} vq(M)_i, \quad vq(M)_i = M_{i\ell} \oplus M_{i\ell+1} \oplus \cdots \oplus M_{(i+1)\ell-1}
$$

We note that $qv$ does not commute with the degree shifts (1) of each categories. However, under this equivalence the degree shift ($\ell$) by $\ell$ of $\text{Mod}^Z A$ corresponds to the degree shift (1) by 1 of $\text{Mod}^Z A^{[\ell]}$.

We give a quick explanation. We set $V := A \oplus A(-1) \oplus \cdots \oplus A(-(\ell - 1))$. Then the set $\{V(i\ell) \mid i \in \mathbb{Z}\}$ is a set of finitely generated projective generators of the abelian category $\text{Mod}^Z A$. Therefore, in the same way of Morita theory, we can construct a $k$-linear equivalence $qv : \text{Mod}^Z A \rightarrow \text{Mod}^Z E$ by using $V$ and ($\ell$), where $E := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Mod}^Z A}(V, V(i\ell))$ is the graded algebra whose multiplication is given by the twisted composition. Now by $\text{Hom}_{\text{Mod}^Z A}(A(-j), A(-i)) \cong A_{j-i}$, we can check that $E = A^{[\ell]}$. For details, we refer [28].

We give a summary of the above consideration and collect consequences.

**Lemma 2.13.** Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded algebra. Then, $A$ is graded Morita equivalent to the trivial extension algebra $A^{[\ell]} = \nabla A \oplus \Delta A$.

Moreover the following assertions hold.

(1) $A$ is of finite global dimension if and only if so is $A^{[\ell]}$.

(2) $A$ is Iwanaga-Gorenstein if and only if so is $A^{[\ell]}$.

The definition of Iwanaga-Gorenstein algebras will be recalled in Section 5.3

3 Homological dimensions of (unbounded) complexes (after Avramov-Foxby)

In this Section 3, we recall the definition of homological dimensions of (unbounded) complexes introduced by Avramov-Foxby [5].

If the reader is only interested in suitably bounded derived categories, it is sufficient to use complexes of projective (resp. injective) modules bounded above (resp. below) in place of DG-projective (resp. DG-injective) complexes whose definition is recalled in Definition 3.1. The statements concerning on DG-projective (resp. DG-injective) complexes can be proved easily for complexes of projective (resp. injective) modules bounded above (resp. below).

For an additive category $\mathcal{A}$, we denote by $C(\mathcal{A})$ and $K(\mathcal{A})$ the category of cochain complexes and cochain morphisms and its homotopy category respectively. For complexes $X, Y \in C(\mathcal{A})$, we denote by $\text{Hom}(\mathcal{A})(X, Y)$ the Hom-complex. For an abelian category $\mathcal{A}$, we denote by $D(\mathcal{A})$ the derived category of $\mathcal{A}$.

The shift functor of a triangulated category is denoted by [1]. The cone of a morphism $f$ in a triangulated category is denoted by $\text{cn} f$.

3.1 Unbounded derived categories

We recall a projective resolution and an injective resolution of unbounded complexes. For the details we refer [18], [19], [32].
Definition 3.1. Let \( \Lambda \) be an algebra.

1. An object \( M \in \mathcal{C}(\text{Mod } \Lambda) \) is called \textit{homotopically projective} if the complex \( \text{Hom}_\Lambda^\bullet(M, N) \) is acyclic for an acyclic complex \( N \in \mathcal{C}(\text{Mod } \Lambda) \).

2. An object \( M \in \mathcal{C}(\text{Mod } \Lambda) \) is called \textit{homotopically injective} if the complex \( \text{Hom}_\Lambda^\bullet(N, M) \) is acyclic for an acyclic complex \( N \in \mathcal{C}(\text{Mod } \Lambda) \).

3. An object \( P \in \mathcal{C}(\text{Mod } \Lambda) \) is called \textit{DG-projective} if it belongs to \( \mathcal{C}(\text{Proj } \Lambda) \) and is homotopically projective.

4. An object \( I \in \mathcal{C}(\text{Mod } \Lambda) \) is called \textit{DG-injective} if it belongs to \( \mathcal{C}(\text{Inj } \Lambda) \) and is homotopically injective.

Let \( M \) be an object of \( \mathcal{C}(\text{Mod } \Lambda) \). A quasi-isomorphism \( f : P \to M \) with \( P \) DG-projective is called a \textit{projective resolution} of \( M \). We often say that \( P \) is a DG-projective resolution of \( M \) by suppressing a quasi-isomorphism \( f : P \to M \).

A quasi-isomorphism \( f : M \to I \) with \( P \) DG-injective is called an \textit{injective resolution} of \( M \). We often say that \( I \) is a DG-injective resolution of \( M \) by suppressing a quasi-isomorphism \( f : M \to I \).

It is known that every object of \( M \in \mathcal{C}(\text{Mod } \Lambda) \) has a projective resolution and an injective resolution \([32, 1.4, 1.5]\). By \([32, 8.1]\), DG-projective complexes and DG-injective complexes coincide with cofibrant DG-modules and fibrant DG-modules in \([19]\). Existence of projective resolutions and injective resolutions are noted in \([19, \text{Proposition 3.1}]\).

Let \( M \in \mathcal{C}(\text{Mod } \Lambda) \) and \( D \) a \( \Lambda \)-\( \Lambda \)-bimodule. We recall that the derived tensor product \( M \otimes^\mathbb{L}_\Lambda D \) and the derived Hom-space \( \mathbb{R}\text{Hom}_\Lambda(D, M) \) are defined by using a projective resolution \( P \xrightarrow{\sim} M \) and an injective resolution \( M \xleftarrow{\sim} I \) respectively. Namely, \( M \otimes^\mathbb{L}_\Lambda D := P \otimes_\Lambda D \) and \( \mathbb{R}\text{Hom}_\Lambda(D, M) := \text{Hom}_\Lambda^\bullet(D, I) \).

We denote by \( K_{\text{proj}}(\text{Proj } \Lambda) \) (resp. \( K_{\text{inj}}(\text{Inj } \Lambda) \)) the full subcategory of \( \mathcal{K}(\text{Mod } \Lambda) \) consisting of the homotopy classes of DG-projective (resp. DG-injective) complexes.

Then the composite functors below are equivalences of triangulated categories

\[
K_{\text{proj}}(\text{Proj } \Lambda) \hookrightarrow K(\text{Mod } \Lambda) \xrightarrow{\text{pr}} D(\text{Mod } \Lambda)
\]
\[
K_{\text{inj}}(\text{Inj } \Lambda) \hookrightarrow K(\text{Mod } \Lambda) \xrightarrow{\text{pr}} D(\text{Mod } \Lambda)
\]
where \( \text{pr} \) is a canonical projection. Moreover, the functor \( \text{pr} \) induces isomorphisms below

\[
\text{Hom}_{K(\text{Mod } \Lambda)}(P, M) \cong \text{Hom}_{D(\text{Mod } \Lambda)}(P, M)
\]
\[
\text{Hom}_{K(\text{Mod } \Lambda)}(M, I) \cong \text{Hom}_{D(\text{Mod } \Lambda)}(M, I)
\]
for \( P \in K_{\text{proj}}(\text{Proj } \Lambda) \), \( I \in K_{\text{inj}}(\text{Inj } \Lambda) \) and \( M \in K(\text{Mod } \Lambda) \), where we suppress \( \text{pr} \) in the right hand sides.

3.2 Projective dimension and injective dimension of unbounded complexes

We recall the definition of projective dimension and injective dimension for unbounded complexes introduced by Avramov-Foxby \([5]\).
Definition 3.2 ([5, Definition 2.1.P and 2.1.I and Theorem 2.4.P and 2.4.I]). (1) An object $M$ of $D({\text{Mod}} \Lambda)$ is said to have \textit{projective dimension} at most $n$ and is denoted as below if it has a projective resolution $P$ such that $P^m = 0$ for $m < -n$.

$$\text{pd}_\Lambda M \leq n.$$ 

We write $\text{pd}_\Lambda M = n$ if $\text{pd}_\Lambda M \leq n$ holds but $\text{pd}_\Lambda M \leq n - 1$ does not.

(2) An object $M$ of $D({\text{Mod}} \Lambda)$ is said to have \textit{injective dimension} at most $n$ and is denoted as below if it has an injective resolution $I$ such that $I^m = 0$ for $m > n$.

$$\text{id}_\Lambda M \leq n.$$ 

We write $\text{id}_\Lambda M = n$ if $\text{id}_\Lambda M \leq n$ holds but $\text{id}_\Lambda M \leq n - 1$ does not.

From now on, if there is no danger of confusion, we usually write the projective dimension and the injective dimension as $\text{pd}$ and $\text{id}$.

If $\text{pd}_\Lambda M \leq n$ (resp. $\text{id}_\Lambda M < n$) for $n \in \mathbb{Z}$, then we write $\text{pd}_\Lambda M = -\infty$ (resp. $\text{id}_\Lambda M = -\infty$). However, $\text{pd}_\Lambda M = -\infty \iff \text{id}_\Lambda M = -\infty \iff M = 0$ in $D({\text{Mod}} \Lambda)$ ([5, 2.3.P and I]).

Theorem 3.3 (Avramov-Foxby [5, Theorem 2.4.P and 2.4.I]). Let $M \in D({\text{Mod}} \Lambda)$ be an object and $n \in \mathbb{Z}$. Then the following assertions hold.

(1) The following conditions are equivalent.

(a) $\text{pd}_\Lambda M \leq n$.

(b) $H^m(M) = 0$ for $m < -n$ and there exists a projective resolution $P$ of $M$ such that $\text{Cok} \, d_F^{n-1}$ is a projective $\Lambda$-module where $d_F^{n-1}$ is the $(n-1)$-th differential of $P$.

(c) $H^m(M) = 0$ for $m < -n$ and for any projective resolution $P$ of $M$, $\text{Cok} \, \partial_P^{n-1}$ is a projective $\Lambda$-module.

(2) The following conditions are equivalent.

(a) $\text{id}_\Lambda M \leq n$.

(b) $H^m(M) = 0$ for $m > n$ and there exists an injective resolution $I$ of $M$ such that $\text{Ker} \, d_I^n$ is an injective $\Lambda$-module where $d_I^n$ is the $n$-th differential of $I$.

(c) $H^m(M) = 0$ for $m > n$ and for any injective resolution $I$ of $M$, $\text{Ker} \, \partial_I^n$ is an injective $\Lambda$-module.

Later we will use the following results several times.

Theorem 3.4 ([5, 2.3.F.(5) and Theorem 4.1]). Let $M$ be a complex of $\Lambda$-modules and $D$ a complex of $\Lambda$-$\Lambda$-bimodules. The we have

$$\text{pd}_\Lambda M \otimes^L \Lambda D \leq \text{pd}_\Lambda M + \text{pd} D_\Lambda, \quad \text{id} \mathbb{R} \text{Hom}_\Lambda(D, M) \leq \text{pd} D_\Lambda + \text{id} M.$$ 

We point out the following basic properties. The proofs are left to the readers.
Lemma 3.5. (1) Let $L \rightarrow M \rightarrow N \rightarrow$ be an exact triangle in $D(\text{Mod } A)$. Then,
\[ \text{pd } M \leq \sup\{\text{pd } L, \text{pd } N\}, \quad \text{id } M \leq \sup\{\text{id } L, \text{id } N\}. \]

(2) Let $M \in D(\text{Mod } A)$ and $n \in \mathbb{Z}$. Then we have
\[ \text{pd}(M[n]) = \text{pd } M + n, \quad \text{id}(M[n]) = \text{id } N - n. \]

For a graded algebra $A$, we define the graded version of the above notions in a similar way of Definition 3.2 by using $\text{Mod}^Z A$, $\text{Proj}^Z A$ and $\text{Inj}^Z A$ in place of $\text{Mod } A$, $\text{Proj } A$ and $\text{Inj } A$. The projective dimension and the injective dimension of an object $M \in C(\text{Mod}^Z A)$ are denoted by $\text{gr.pd } M$ and $\text{gr.id } M$.

The following is a complex version of Proposition 2.5

Lemma 3.6. Let $A$ be a finitely graded algebra and $M \in D(\text{Mod}^Z A)$. Then, we have
\[ \text{gr.pd } M = \text{pd } M. \]

Proof. We have only to prove that if $P \in C(\text{Proj}^Z A)$ is DG-projective, then the underlying complex $U(P)$ of (ungraded) $A$-modules is DG-projective in $C(\text{Mod } A)$. After that, thanks to Theorem 3.3 the rest of the proof is the same with that of the Proposition 2.5.

It is enough to show that $U(P)$ is homotopically projective. For this we recall the notion of property (P) from [18]. An object $P$ of $C(\text{Proj}^Z A)$ is said to have property (P) if it has an increasing filtration $0 = P_{(0)} \subset P_{(1)} \subset \cdots$ such that each graded quotient $P_{(i)}/P_{(i-1)}$ is an object of $C(\text{Proj}^Z A)$ with 0 differential and that $\bigcup_{i \geq 0} P_{(i)} = P$. By [18, 3.1] an object $Q \in C(\text{Proj}^Z A)$ having property (P) is DG-projective in particular it is homotopically projective. We claim that every DG-projective complex $P$ is a direct summand of a complex $Q \in C(\text{Proj}^Z A)$ having property (P). Indeed, by [18, 3.1], there exists a quasi-isomorphism $f : Q \sim P$ from a complex $Q \in C(\text{Proj}^Z A)$ having property (P) such that each component $f^n : Q^n \rightarrow P^n$ is surjective. Then, since $P$ is cofibrant, $f$ has a section by [19, 3.2].

If $P \in C(\text{Proj}^Z A)$ has property (P), then so is $U(P) \in C(\text{Proj } A)$. Hence it is homotopically projective. Let $P$ be a DG-projective object of $C(\text{Proj}^Z A)$. Then by the claim, $U(P)$ is a direct summand of a homotopically projective object of $C(\text{Mod } A)$ and hence is homotopically projective.

The following is a complex version of Proposition 2.9

Lemma 3.7. Let $A$ be a finitely graded algebra and $M \in D^+(\text{Mod}^Z A)$ such that $M_i = 0$ for $|i| \gg 0$. Then, we have
\[ \text{gr.id } M = \text{id } M. \]

Proof. Under the assumption $M$ has an injective resolution $I$ such that each term $I^n$ is finitely graded. Using Theorem 3.3(b), we can prove the assertion as in the same of Proposition 2.9.

4 Projective dimension formula

In this Section 4, $A$ denotes an algebra, $C$ denotes a bimodule over it and $A = \Lambda \oplus C$ is the trivial extension algebra. We establish the projective dimension formula for an object $M \in D^b(\text{Mod}^Z A)$ such that $M_i = 0$ for $i \neq 0, 1$. The key tool is a decomposition of a complex $P \in C(\text{Proj}^Z A)$ of graded projective $A$-modules according to degree of the generators which was introduced in [30].
4.1 Decomposition of complexes of graded projective $A$-modules

4.1.1 Decomposition of graded projective $A$-modules

Recall that $p_i P = (P \otimes_A \Lambda)_i$, $t_i P := (p_i P) \otimes_A \Lambda(-i)$ for $P \in \text{Proj}^Z \Lambda$. Since now $A = \Lambda \oplus C$ is a trivial extension algebra, we have an isomorphism $t_i P = (p_i P)(-i) \oplus (p_i P) \otimes_A C(-i - 1)$ of $\Lambda$-modules. Therefore, by Lemma 2.3, we have the following isomorphisms.

\[ P \cong \bigoplus_{i \in \mathbb{Z}} t_i P \text{ (as graded } A\text{-modules)}, \]

\[ P_i \cong (t_{i-1} P)_i \oplus (t_i P)_i \cong (p_{i-1} P \otimes_A C) \oplus p_i P \text{ (as } \Lambda\text{-modules)}. \]

We discuss decomposition of a morphism $f : P \to P'$ of graded projective $A$-modules. The case where the generating spaces of $P$ and $P'$ consist of single degree can be easily deduced from Corollary 2.3.

**Lemma 4.1.** Let $P, P'$ be graded projective $A$-modules such that $P = t_i P, P' = t_j P'$ for some $i, j \in \mathbb{Z}$. Then, we have the following isomorphism

\[ \text{Hom}_{\text{Mod}^Z A}(P, P') = \text{HOM}_A(P, P')_0 \cong \begin{cases} 0 & j \neq i, i - 1, \\ \text{Hom}_A(P_i, P'_j) & j = i, \\ \text{Hom}_A(P_i, P'_{i-1} \otimes C) & j = i - 1. \end{cases} \]

Let $f : P \to P'$ be a morphism in $\text{Mod}^Z A$ with $P, P' \in \text{Proj}^Z A$. We denote by $f_{ij} : t_i P \to t_j P'$ the component of $f$ with respect to the decompositions (4-2) of $P$ and $P'$, that is, $f_{ij}$ is defined by the following composition

\[ f_{ij} : t_i P \xrightarrow{\text{in}_j} P \xrightarrow{f} P' \xrightarrow{\text{pr}_i} t_i P' \]

where $\text{in}_j$ is a canonical inclusion and $\text{pr}_i$ is a canonical projection.

Note that if $j - i \neq 0, 1$, then $f_{ij} = 0$ and that $p_i(f) = (f_{ii})_i : p_i P = (t_i P)_i \to (t_i P')_i = p_i P'$.

We set $q_i(f) := (f_{i-1,i} : p_i P \to p_{i-1} P' \otimes C)$. Then the degree $i$-part $f_i : P_i \to P'_i$ is isomorphic to

\[ \begin{pmatrix} p_{i-1}(f) \otimes C & q_i(f) \\ 0 & p_i(f) \end{pmatrix} : (p_{i-1} P \otimes_A C) \oplus p_i P \to (p_{i-1} P' \otimes C) \oplus p_i P' \]

under the isomorphisms (4-2), where we use the standard matrix multiplication

\[ \begin{pmatrix} p_{i-1}(f) \otimes C & q_i(f) \\ 0 & p_i(f) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} (p_{i-1}(f) \otimes C)(p) + q_i(f)(q) \\ p_i(f)(q) \end{pmatrix} \]

for $p \in p_{i-1} P \otimes C, q \in p_i P$.

Let $i$ be an integer. We set $t_{<i} P := \bigoplus_{j < i} t_j P$ and $t_{\geq i} P := \bigoplus_{j \geq i} t_j P$. For a graded $A$-module homomorphism $f : P \to P'$, we denote by $t_{<i} f : t_{<i} P \to t_{<i} P'$ and $t_{\geq i} f : t_{\geq i} P \to t_{\geq i} P'$ the induced morphisms. Then the canonical inclusion $t_{<i} P \to P$ and the canonical projection $P \to t_{\geq i} P$ fit into the following commutative diagram

\[ \begin{array}{ccc} 0 & \xrightarrow{t_{<i} P} & P & \xrightarrow{t_{\geq i} P} & 0 \\ \downarrow t_{<i}(f) & & \downarrow f & & \downarrow t_{\geq i}(f) \\ 0 & \xrightarrow{t_{<i} P'} & P' & \xrightarrow{t_{\geq i} P'} & 0 \end{array} \]

where the both rows are split exact.
4.1.2 Decomposition of complexes of graded projective modules

By abuse of notations, we denote the functors $p_i : C(\text{Proj}^Z A) \to C(\text{Proj} \Lambda)$ and $p_i : K(\text{Proj}^Z A) \to K(\text{Proj} \Lambda)$ induced from the functor $p_i : \text{Proj}^Z A \to \text{Proj} \Lambda$ by the same symbols.

Let $P = (\bigoplus_{n \in \mathbb{Z}} P^n, \{d^n\}_{n \in \mathbb{Z}}) \in C(\text{Proj}^Z A)$ be a complex of graded projective $A$-modules. For simplicity, we set $\partial^n_i := p_i(d^n), q^n_i := q_i(d^n)$.

Looking at graded degree $i$-part, we have the following isomorphism

$$(4-3) \quad P^n_i = (t_{i-1}P^n)_i \oplus (t_iP^n)_i \cong (p_{i-1}P^n \otimes \Lambda C) \oplus p_iP^n$$

of $\Lambda$-modules. Via this isomorphism, the differential $d^n : P^n \to P^{n+1}$ is decomposed into

$$\begin{pmatrix} \partial^n_{i-1} \otimes C & q^n_i \\ 0 & \partial^n_i \end{pmatrix} : (p_{i-1}P^n \otimes \Lambda C) \oplus p_iP^n \to (p_{i-1}P^{n+1} \otimes \Lambda C) \oplus p_iP^{n+1}. $$

Now it is clear that the canonical morphisms $p_{i-1}P \otimes \Lambda C \to P_i$, $P_i \to p_iP$ commute with the differentials. Hence we obtain an exact sequence in $C(\text{Mod} \Lambda)$

$$0 \to p_{i-1}P \otimes \Lambda C \to P_i \to p_iP \to 0$$

which splits when we forget the differentials. Moreover, the complex $P_i$ is obtained as the cone of the morphism $q_i[-1] : p_iP[-1] \to p_{i-1}P \otimes C$ which is given by the collection $\{q^n_i\}_{n \in \mathbb{Z}}$. This observation yields the following lemma.

**Lemma 4.2.** We have an exact triangle in $K(\text{Mod} \Lambda)$ and hence in $D(\text{Mod} \Lambda)$

$$p_{i-1}P \otimes \Lambda C \to P_i \to p_iP \xrightarrow{q_i} p_{i-1}P \otimes C[1].$$

In particular, $H(P)_i = H(P_1) = 0$ if and only if the morphism $q_i : p_iP \to p_{i-1}P \otimes \Lambda C[1]$ is a quasi-isomorphism.

For an integer $i$, we set $t_{\leq i} := (\bigoplus_{n \in \mathbb{Z}} t_{\leq i}P^n, \{t_{\leq i}(d^n)\}_{n \in \mathbb{Z}})$ and $t_{\geq i} := (\bigoplus_{n \in \mathbb{Z}} t_{\geq i}P^n, \{t_{\geq i}(d^n)\}_{n \in \mathbb{Z}})$. Then the canonical inclusion $t_{\leq i}P \to P$ and the canonical projection $P \to t_{\geq i}P$ form an exact sequence of $C(\text{proj}^Z A)$

$$0 \to t_{\leq i}P \to P \to t_{\geq i}P \to 0,$$

which gives an exact triangle in $K(\text{Proj}^Z A)$

$$(4-4) \quad t_{\leq i}P \to P \to t_{\geq i}P \to t_{\leq i}P[1].$$

We discuss properties of the functor $p_i : K(\text{Proj}^Z A) \to K(\text{Proj} \Lambda)$ a bit more.

**Lemma 4.3.** (1) If $P$ is DG-projective, then so is $p_iP$ for $i \in \mathbb{Z}$.

(2) Assume that $p_iP$ is DG-projective for $i \in \mathbb{Z}$ and that $p_iP = 0$ for $i \ll 0$, then so is $P$.

We use the following description of Hom-complex. We denote by $\text{Hom}^\#$ the underlying cohomologically graded $k$-modules of Hom-complexes $\text{Hom}^\bullet$.

Let $M \in C(\text{Mod}^Z A)$. The isomorphisms of Corollary 2.4 induces an isomorphism of cohomologically graded $k$-modules.

$$\Phi : \text{Hom}^\#_{\text{Mod}^Z A}(P, M) \xrightarrow{\cong} \prod_{i \in \mathbb{Z}} \text{Hom}^\#_{\Lambda}(p_iP, M_i).$$
However, this is not compatible with the canonical differentials. Via this isomorphism, the differential \( \partial_{\text{Hom}} \) of \( \text{Hom}_{\text{Mod}^z A}^\bullet(P, M) \) corresponds to the endomorphism \( \delta \) of the right hand side which is defined in the following way.

Let \( g = (g_i)_{i \in \mathbb{Z}} \) be a homogeneous element of the above graded module of cohomological degree \( n \). Then the \( i \)-th component \( \delta(g)_i \) of \( \delta(g) \) defined to be

\[
\delta(g)_i := \partial_{\text{Hom}}(g_i) - (-1)^n g_{i-1} \otimes C \circ q_i
\]

where \( \partial_{\text{Hom}} \) is the differential of \( \text{Hom}_A^\bullet(p_i P, M_i) \). Therefore, we have the following lemma.

**Lemma 4.4.** \( \Phi \) gives an isomorphism between the complexes

\[
\text{Hom}_{\text{Mod}^z A}^\bullet(P, M) \cong \prod_{i \in \mathbb{Z}} \text{Hom}_A^\#(p_i P, M_i, \delta)
\]

**Proof of Lemma 4.3.** (1) For \( i \in \mathbb{Z} \) and \( M \in \text{C(Mod } A) \), we denote by \( I_i(M) \in \text{C(Mod }^z A) \) the complex such that \( I_i(M)_i = M \) and \( I_i(M)_j = 0 \) for \( j \neq i \). If \( M \) is acyclic, then so is \( I_i(M) \). We have an isomorphism \( \text{Hom}_A^\bullet(P, M) \cong \text{Hom}_{\text{Mod}^z A}^\bullet(p_i P, I_i(M)) \) of Hom-complexes. Therefore if \( P \) is homotopically projective, then so is \( p_i P \).

(2) Let \( M \in \text{C(Proj } A) \) be an acyclic complex. By description of \( \delta \), via \( \Phi^{-1} \), the family \( \{ F_j := \prod_{i \geq j} \text{Hom}_A^\#(p_i P, M_i) \mid j \in \mathbb{Z} \} \) gives a decreasing filtration of the complex \( \text{Hom}_{\text{Mod}^z A}^\bullet(P, M) \), which is exhaustive by assumption. Since \( F_j / F_{j+1} = \text{Hom}_A^\#(p_j P, M_j) \) is acyclic for \( j \in \mathbb{Z} \), so is \( \text{Hom}_{\text{Mod}^z A}^\bullet(P, M) \).

Let \( P' \in \text{C(Proj } A) \) be another complex. We denote by \( q' \) what \( q \) for \( P' \). Then, via the isomorphism induced from \( \Phi \)

\[
\text{Hom}_{\text{Mod}^z A}^\#(P, P') \cong \prod_{i \in \mathbb{Z}} \text{Hom}_A^\#(p_i P, p_{i-1}P' \otimes C) \oplus \prod_{i \in \mathbb{Z}} \text{Hom}_A^\#(p_i P, p_i P')
\]

the canonical differential of the left hand side corresponds to \( \begin{pmatrix} \partial_{\text{Hom}} & F \\ 0 & \partial_{\text{Hom}} \end{pmatrix} \) where two \( \partial_{\text{Hom}} \)'s are the canonical differential of Hom-complexes and \( F \) is a morphism of degree 1 defined in the following way.

Let \( g = (g_i)_{i \in \mathbb{Z}} \) be a homogeneous element of \( \prod_{i \in \mathbb{Z}} \text{Hom}_A^n(p_i P, p_i P') \) of degree \( n \). Then the \( i \)-th component \( F(g)_i \) of \( F(g) \) is defined to be

\[
F(g)_i := q' \circ g_i - (-1)^n (g_{i-1} \otimes C) \circ q.
\]

Therefore, canonical inclusion and canonical projection yield an exact sequence in \( \text{C(Proj } k) \) which splits when we forget the differentials

\[
(4-5) \quad 0 \rightarrow \prod_{i \in \mathbb{Z}} \text{Hom}_A^\bullet(p_i P, p_{i-1}P' \otimes \Lambda C) \rightarrow \text{Hom}_{\text{Mod}^z A}^\bullet(P, P') \xrightarrow{(p_i)} \prod_{i \in \mathbb{Z}} \text{Hom}_A^\bullet(p_i P, p_i P') \rightarrow 0.
\]

Moreover, the complex \( \text{Hom}_{\text{Mod}^z A}^\bullet(P, P') \) is the cone of the morphism

\[
F : \prod_{i \in \mathbb{Z}} \text{Hom}_A^\bullet(p_i P, p_i P')[-1] \rightarrow \prod_{i \in \mathbb{Z}} \text{Hom}_A^\bullet(p_i P, p_{i-1}P' \otimes C).
\]
Lemma 4.5. (1) A complex $P \in \mathcal{C}(\text{Proj}^Z A)$ is null-homotopic if and only if so is $p_iP$ for $i \in \mathbb{Z}$.

(2) A morphism $f : P \to P'$ is homotopy equivalence if and only if so is $p_i(f)$ for $i \in \mathbb{Z}$.

Proof. Since the functor $p_i : K(\text{Proj}^Z A) \to K(\text{Proj} \Lambda)$ is triangulated, “only if” part of (1) and (2) are clear.

We prove “if” part of (1). Recall that a complex $X \in \mathcal{C}(\text{Mod} \Lambda)$ is null-homotopic if and only if $\text{Hom}_A^*(X,Y)$ is acyclic for $Y \in \mathcal{C}(\text{Mod} \Lambda)$ if and only if $\text{Hom}_A^*(X,X)$ is acyclic and the same is true in graded situation. Now, it is clear that “if” part of (1) follows from the exact sequence (4-5).

We prove “if” part of (2). By assumption $p_i(\text{cn} f) \cong \text{cn}(p_i f)$ is null-homotopic. Hence, the cone $\text{cn}(f)$ is null-homotopic by (1).

4.1.3 Construction of graded projective complexes and morphisms between them

Let $(\{Q_i\}_{i \in \mathbb{Z}}, \{q_i\}_{i \in \mathbb{Z}})$ be a pair of a collection of objects $Q_i \in \mathcal{C}(\text{Proj} \Lambda)$ and a collection of morphisms $q_i : Q_i \to Q_{i-1} \otimes \Lambda C[1]$ in $\mathcal{C}(\text{Mod} \Lambda)$.

We set

$$P^n := \bigoplus_{i \in \mathbb{Z}} Q^n_i \otimes \Lambda A(-i), \quad d^n_p := \begin{pmatrix} \partial^n_{i-1} \otimes C & q^n_i \\ 0 & \partial^n_i \end{pmatrix}$$

where $\partial^n_i$ denotes the $n$-th differential of $Q_i$. Then the pair $P = \left( \bigoplus_{n \in \mathbb{Z}} P^n, \{d^n_p\} \right)$ is a complex of graded projective $\Lambda$-modules such that $p_iP = Q_i$ and $q_i(d_P) = q_i$.

Let $P'$ be the object of $\mathcal{C}(\text{Proj}^Z A)$ associated to another pair $(\{Q'_i\}_{i \in \mathbb{Z}}, \{q'_i\}_{i \in \mathbb{Z}})$.

Lemma 4.6. Let $\{g_i : Q_i \to Q'_i\}$ be a collection of morphisms in $\mathcal{C}(\text{Proj} \Lambda)$. Then the following conditions are equivalent.

(1) There exists a morphism $f : P \to P'$ in $\mathcal{C}(\text{Proj}^Z A)$ such that $p_i(f) = g_i$ for $i \in \mathbb{Z}$

(2) The morphisms $q'_i \circ g_i$ and $(g_{i-1} \otimes C[1]) \circ q_i$ are homotopic to each other.

Proof. Taking the cohomology long exact sequence of (4-5), we obtain the exact sequence of Hom-spaces of cochain complexes

$$\text{Hom}_{\text{Mod}^Z A}(P, P') \xrightarrow{H((p_i))} \prod_{i \in \mathbb{Z}} \text{Hom}_{\Lambda}(Q_i, Q'_i) \xrightarrow{H(F)} \prod_{i \in \mathbb{Z}} \text{Hom}_{\Lambda}(Q_i, Q'_{i-1} \otimes \Lambda C[1]).$$

Therefore, the condition (1) is equivalent to the condition that there exists $h \in \prod_{i \in \mathbb{Z}} \text{Hom}_{\Lambda}^{-1}(Q_i, Q_{i-1} \otimes C[1])$ such that $F(g) = \partial_{\text{Hom}}(h)$ where $g := (g_i)_{i \in \mathbb{Z}}$. It can be checked that the latter condition is equivalent to the condition (2).

Combining Lemma 4.3 and Lemma 4.6, we obtain the following lemma and corollary.

Lemma 4.7. Let $P \in \mathcal{C}(\text{Proj}^Z A)$. Assume that a complex $Q'_i \in \mathcal{C}(\text{Proj} \Lambda)$ homotopic to $p_iP$ is given for all $i \in \mathbb{Z}$. Then, there exists $P' \in \mathcal{C}(\text{Proj}^Z A)$ homotopic to $P$ such that $p_iP' = Q'_i$ in $\mathcal{C}(\text{Proj} \Lambda)$ for $i \in \mathbb{Z}$.

Proof. Let $g_i : p_iP \to Q'_i$ be a homotopy equivalence and $h_i : Q'_i \to p_iP$ be a homotopy inverse. We define $q'_i$ to be the composite morphism below

$$q'_i : Q'_i \xrightarrow{h_i} P_i \xrightarrow{q_i} P_{i-1} \otimes \Lambda C[1] \xrightarrow{g_i \otimes C[1]} Q'_{i-1} \otimes \Lambda C[1].$$

Then the morphisms $q'_i \circ g_i$ and $(g_{i-1} \otimes C[1]) \circ q_i$ are homotopic to each other. We denote by $P'$ the complex constructed from the collection $(\{Q'_i\}_{i \in \mathbb{Z}}, \{q'_i\}_{i \in \mathbb{Z}})$ By Lemma 4.6 there exists a morphism $f : P \to P'$ in $\mathcal{C}(\text{Proj}^Z A)$ such that $p_i(f) = g_i$. By Lemma 4.3, $f$ is a homotopy equivalence.
Corollary 4.8. An object $P \in K(\text{Proj}^\mathbb{Z}A)$ belongs to $K^b(\text{proj}^\mathbb{Z}A)$ if and only if $p_iP$ belongs to $K^b(\text{proj} A)$ for all $i \in \mathbb{Z}$ and $p_iP = 0$ in $K(\text{Proj} A)$ for $|i| \gg 0$.

4.2 Projective dimension formula

Let $M$ be a graded $A$-module. Then the graded $A$-module structure induces a $\Lambda$-module homomorphism $\xi_{M,i} : M_i \otimes_A C \rightarrow M_{i+1}$ for $i \in \mathbb{Z}$, which we call the action morphism of $M$.

If a graded $A$-module $M$ satisfies $M_{<0} = 0$, then $(M \otimes_A C)_0 \cong M_0 \otimes_A C$. Moreover, applying $M \otimes_A -$ to the canonical inclusion $\text{can} : C \hookrightarrow A(1)$ and looking at the degree 0-part, we obtain the 0-th action morphism

$$\bar{\xi}_{M,0} : M_0 \otimes_A C \cong (M \otimes_A C)_0 \xrightarrow{(M \otimes \text{can})_0} (M \otimes_A A(1))_0 \cong M_1.$$ 

We remark that since these constructions are natural, it works for a complex $M \in C(\text{Mod}^\mathbb{Z}A)$. Let $M$ be an object of $D(\text{Mod}^\mathbb{Z}A)$. We denote a representative of $M$ by the same symbol $M \in C(\text{Mod}^\mathbb{Z}A)$. We define a morphism $\Xi_M : M_0 \otimes^L_A C \rightarrow M_1$ in $D(\text{Mod} A)$ to be the composite morphism

$$\Xi_M^0 : M_0 \otimes^L_A C \xrightarrow{\text{can}} M_0 \otimes_A C \xrightarrow{\bar{\xi}_{M,0}} M_1$$

where $\text{can}$ is a canonical morphism. It is easy to see that this morphism is well-defined.

For $a \geq 1$ and $L \in D(\text{Mod} A)$, we set

$$L \otimes^L_A C^a := (\cdots ((L \otimes^L_A C) \otimes^L_A C) \cdots \otimes^L_A C) \quad (a\text{-times}).$$

We also set $L \otimes^L_A C^0 := L$.

Remark 4.9. The above notation is only the abbreviation of the iterated application of the endofunctor $- \otimes^L_A C$ of $D(\text{Mod} A)$. We will not deal with the complex $C^a = C \otimes^L_A C \otimes^L_A \cdots \otimes^L_A C$ of $\Lambda$-$\Lambda$-bimodules. If $\Lambda$ is a projective module over a base ring $k$, then the standard definition for $C^a$ works well. For example, we have $(M \otimes^L_A C) \otimes^L_A C = M \otimes^L_A (C \otimes^L_A C)$. However, in general, the standard definition does not work well. To settle such a situation we need to use differential graded algebras. See [33, Remark 1.12] for similar discussion.

The following is the main theorem of this section. For simplicity we set $\Xi_M^a := \Xi_M^0 \otimes^L_A C^a$.

$$\Xi_M^a : M_0 \otimes^L_A C^{a+1} \rightarrow M_1 \otimes^L_A C^a.$$

Theorem 4.10. Let $M \in D(\text{Mod}^\mathbb{Z}A)$ such that $M_i = 0$ for $i \neq 0,1$. Then we have

$$\text{pd}_A M = \sup_{\Lambda} \{ \text{pd}_\Lambda M_0, \text{pd}_\Lambda (\text{can} \Xi_M^a) + a \mid a \geq 0 \}.$$ 

In particular, we have $\text{pd}_A M < \infty$ if and only if the following conditions are satisfied:

1. $\text{pd}_\Lambda M_0 < \infty$.

2. $\text{pd}_\Lambda (\text{can} \Xi_M^a) < \infty$ for $a \geq 0$.

3. $\Xi_M^a$ is an isomorphism for $a \gg 0$. 
As a corollary, we obtain the following formula, from which the projective version of Theorem \ref{thm:main} follows.

**Corollary 4.11.** For $M \in \mathcal{D}(\text{Mod } \Lambda)$, we have

\[
\text{pd}_A M = \sup \{ \text{pd}_A (M \otimes^L_A C^a) + a \mid a \geq 0 \}.
\]

Combining this corollary with Proposition \ref{prop:main}, we see the next corollary.

**Corollary 4.12.** We have the equation

\[
\text{gldim } A = \sup \{ \text{pd}_A (M \otimes^L_A C^a) + a \mid M \in \text{Mod } \Lambda, a \geq 0 \}.
\]

We need preparations.

**Lemma 4.13.** Let $M$ be an object of $\mathcal{D}(\text{Mod } A)$ such that $\text{M}_{<0} = 0$ and $P \in \mathcal{C}(\text{Proj } \Lambda)$ a projective resolution of $M$.

Then the following assertions hold.

1. $t_{<0} P$ is null-homotopic and hence $t_{\geq 0} P$ is homotopic to $P$ via the canonical morphism.
2. The complex $p_0 P$ is a projective resolution of $M_0$ as complexes of $\Lambda$-modules.
3. For $N \in \mathcal{D}(\text{Mod } \Lambda^{op})$, we have $(M \otimes^L_A N)_i = 0$ for $i < 0$ and $(M \otimes^L_A N)_0 = M \otimes^L_A N$.
4. We have $p_i P = (M \otimes^L_A \Lambda)_i$ in $\mathcal{D}(\text{Mod } A)$.

**Proof.** We only prove (1). Proof of the other statements are left to reader. First note that for a complex $L \in \mathcal{C}(\text{Mod } \Lambda)$ such that $L_{<0} = 0$, we have $\text{Hom}_{\mathcal{D}(\text{Mod } \Lambda)}(t_{<0} P, L) = \text{Hom}_{K(\text{Mod } \Lambda)}(t_{<0} P, L) = 0$ by Lemma \ref{lem:main}. Since $P$ is quasi-isomorphic to $M$, we have

\[
\text{Hom}_{K(\text{Mod } \Lambda)}(t_{<0} P, P) = \text{Hom}_{\mathcal{D}(\text{Mod } \Lambda)}(t_{<0} P, P) = \text{Hom}_{\mathcal{D}(\text{Mod } \Lambda)}(t_{<0} P, M) = 0.
\]

Therefore, using the exact triangle \ref{exact_triangle} we see that $t_{<0} P$ is a direct summand of $t_{\geq 0} P[-1]$ in $K(\text{Mod } \Lambda)$. Since $\text{Hom}_{K(\text{Mod } \Lambda)}(t_{<0} P, t_{\geq 0} P[-1]) = 0$, we conclude that $t_{<0} P = 0$ in $K(\text{Mod } \Lambda)$. \hfill \Box

The following lemma gives a relationship between $p_i P$ and $\Xi_M^a$.

**Lemma 4.14.** Let $M$ be an object of $\mathcal{D}(\text{Mod } \Lambda)$ such that $M_i = 0$ for $i \neq 0$, $1$ and $P \in \mathcal{C}(\text{Proj } \Lambda)$ a projective resolution of $M$. Then in $\mathcal{D}(\text{Mod } A)$ we have

\[
p_i P \cong \begin{cases} 
\text{cn}_M^{i-1}[i-1] & i \geq 1, \\
M_0 & i = 0, \\
0 & i < 0.
\end{cases}
\]

**Proof.** The case $i \leq 0$ is clear from Lemma \ref{lem:main}.

Applying $M \otimes^L_A - \otimes^L_A C^a$ to the canonical exact sequence $0 \rightarrow C(-1) \rightarrow A \rightarrow \Lambda \rightarrow 0$, we obtain an exact triangle

\[
M \otimes^L_A C^{a+1}(-1) \rightarrow M \otimes^L_A C^a \rightarrow M \otimes^L_A C^a \rightarrow .
\]

We remark that the degree 1-part of the first morphism is identified with

\[
\Xi^a_M : M_0 \otimes^L_A C^{a+1} \cong (M \otimes^L_A C^{a+1}(-1))_1 \rightarrow (M \otimes^L_A C^a)_1 \cong M_1 \otimes^L_A C^a.
\]

18
By looking degree $i \geq 1$ part, we obtain isomorphisms
\[
(M \otimes_A L^a)_{i} \cong \text{cn}[\Xi_M^a : M_0 \otimes_A L^a C^{a+1} \to M_1 \otimes_A L^a C^a],
\]
\[
(M \otimes_A L^a)_{i} \cong (M \otimes_A L^{a+1})_{i-1}[1]
\]
for $i \geq 2$.

Therefore, for $i \geq 1$ we have
\[
p_i P \cong (M \otimes_A L)_{i} \cong (M \otimes_A L C^{a+1})_{i-1}[1]
\]
\[
\cong \cdots \cong (M \otimes_A L (C^{i-1})_{i-1} = \text{cn} \Xi_{M}^{i-1}[i-1].
\]

Proof of Theorem 4.10. For simplicity we set
\[
m := \text{pd} M, \ n := \sup_{\Lambda} \{ \text{pd} M_0, \text{pd}(\text{cn} \Xi_{M}^{a}) + a \mid a \geq 0 \}.
\]

We prove $m \geq n$. We may assume that $m < \infty$. Let $P$ be a projective resolution of $M$ as a complex of graded $A$-modules such that $P^{<m} = 0$. It follows from Lemma 4.3 and Lemma 4.14, that $p_0 P$ is a projective resolution of $M_0$ and that $p_i P$ is a projective resolution of $\text{cn} \Xi_{M}^{i-1}[i-1]$ for $i \geq 1$. Hence we prove the desired inequality.

We prove $m \leq n$. We may assume that $n < \infty$. Let $Q_0$ be a projective resolution of $M_0$ in $\mathcal{C} (\text{Mod } \Lambda)$ such that $Q_0^{<n} = 0$ and $Q_i$ a projective resolution of $\text{cn} \Xi_{M}^{i-1}[i-1]$ in $\mathcal{C} (\text{Mod } \Lambda)$ such that $Q_i^{<n} = 0$ for $i \geq 1$. It follows from Lemma 4.3, Lemma 4.7 and Lemma 4.14 that there exists a projective resolution $P \in \mathcal{C} (\text{Mod } \mathbb{Z})$ of $M$ such that $p_i P = Q_i$ for $i \geq 0$ and $p_i P = 0$ for $i < 0$. Since $P^{<n} = 0$, we prove the desired inequality.

4.2.1 Finiteness of global dimension

Using Corollary 4.11 we can deduce the following result.

Corollary 4.15. Let $A = \Lambda \oplus C$ be a trivial extension algebra. Then the following conditions are equivalent.

(1) gldim $A < \infty$.

(2) gldim $\Lambda < \infty$ and $C^a = 0$ for some $a > 0$.

Proof. (1) $\Rightarrow$ (2). Since gldim $\Lambda \leq$ gldim $A$, we have gldim $\Lambda < \infty$. Applying Corollary 4.11 to $\Lambda$, we see that $C^a = 0$ for $a >$ gldim $\Lambda$.

(2) $\Rightarrow$ (1). Let $M$ be a $\Lambda$-module. For $b \in \mathbb{N}$ we have the following inequality by Theorem 3.4
\[
\text{pd}_\Lambda M \otimes L^b C^b \leq \text{pd}_\Lambda M + b \text{pd}_\Lambda C.
\]

Therefore, if $C^a = 0$, then we have
\[
\text{pd}_\Lambda M = \sup_{\Lambda} \{ \text{pd}_\Lambda M \otimes L^b C^b + b \mid 0 \leq b \leq a - 1 \} \leq a(\text{gldim } \Lambda + 1) - 1.
\]

Hence by Proposition 2.111 gldim $A \leq a(\text{gldim } \Lambda + 1) - 1$.

We can reprove the characterization of the case gldim $A \leq 1$ which was first proved by I. Reiten [33, Proposition 2.3.3] (see also [23, Corollary 5.3]). The details of proof is left to the readers.
Corollary 4.16. We have \( \text{gldim} \ A \leq 1 \) if and only if the following conditions are satisfied:

1. \( \text{gldim} \ A \leq 1 \).
2. The left \( \Lambda \)-module \( \Lambda C \) is flat.
3. \( M \otimes_{\Lambda} C \) is projective for \( M \in \text{Mod} \ \Lambda \).
4. \( C \otimes_{\Lambda} C = 0 \).

4.3 A criterion of perfection of complexes of graded \( \Lambda \)-modules

Recall that an object of \( K^b(\text{proj} \ \Lambda) \) (resp. \( K^b(\text{proj} \ \mathbb{Z} \ A) \)) is called a perfect complex (resp. graded perfect complex). Using the same method with the proof of Theorem 4.10, we can prove a criterion that an object \( M \in D(\text{Mod} \ \Lambda) \) is a graded perfect complex.

Theorem 4.17. An object \( M \in D(\text{Mod} \ \Lambda) \) belongs to \( K^b(\text{proj} \ \mathbb{Z} \ A) \) if and only if \( M \otimes_{\Lambda} C^a \) belongs to \( K^b(\text{proj} \ \Lambda) \) for \( a \geq 0 \) and \( M \otimes_{\Lambda} C^a = 0 \) for \( a \gg 0 \).

Proof. In the case where \( M_i = 0 \) for \( i \neq 0 \), Lemma 4.14 become
\[
p_i, \mathcal{P} \cong \begin{cases} M \otimes_{\Lambda} C^a[i] & i \geq 0, \\ 0 & i < 0. \end{cases}
\]

Now the proof is the same with that of Theorem 4.10 except that we need to use Corollary 4.8 instead of Lemma 4.7. \( \square \)

4.3.1 Applications

Assume that \( C_{\Lambda} \) belongs to \( K^b(\text{proj} \ \Lambda) \). Then the endofunctor \( - \otimes_{\Lambda} C^a \) of \( D(\text{Mod} \ \Lambda) \) preserves \( K^b(\text{proj} \ \Lambda) \) for \( a \geq 0 \). We denote the restriction functor by \( (\- \otimes_{\Lambda} C^a)|_{\mathcal{K}} \).

Observe that there exists the following increasing sequence of thick subcategories of \( K^b(\text{proj} \ \Lambda) \).
\[
\text{Ker}(\- \otimes_{\Lambda} C)|_{\mathcal{K}} \subset \text{Ker}(\- \otimes_{\Lambda} C^2)|_{\mathcal{K}} \subset \cdots \subset \text{Ker}(\- \otimes_{\Lambda} C^a)|_{\mathcal{K}} \subset \cdots .
\]

Applying Theorem 4.17, we obtain the following result.

Corollary 4.18. Under the above situation, we have
\[
D(\text{Mod} \ \Lambda) \cap K^b(\text{proj} \ \mathbb{Z} \ A) = K^b(\text{proj} \ \Lambda) \cap K^b(\text{proj} \ \mathbb{Z} \ A) = \bigcup_{a \geq 0} \text{Ker}(\- \otimes_{\Lambda} C^a)|_{\mathcal{K}}.
\]

This result plays an important role in [25], since in the case where \( A \) is Noetherian, it gives a description of the kernel of the canonical functor
\[
\varpi : D^b(\text{mod} \ \Lambda) \to D^b(\text{mod} \ \mathbb{Z} \ A) \to \text{Sing} \mathbb{Z} \ A = D^b(\text{mod} \ \mathbb{Z} \ A)/K^b(\text{proj} \ \mathbb{Z} \ A)
\]
where \( \text{Sing} \mathbb{Z} \ A \) is the graded singular derived category of \( A \).

As in the same way of Corollary 4.15, we can prove the following Corollary by using Theorem 4.17.

Corollary 4.19. Assume that \( A = \Lambda \oplus C \) is Noetherian. Then the following conditions are equivalent.
(1) $\mathbb{K}^b(\text{proj}^Z A) = D^b(\text{mod}^Z A)$.

(2) $\mathbb{K}^b(\text{proj} \Lambda) = D^b(\text{mod} \Lambda)$ and $C^a = 0$ for some $a > 0$.

Proof. Assume that the condition (1) holds. Then since $D^b(\text{mod} \Lambda) \subset D^b(\text{mod}^Z A)$, the first condition of (2) follows from Theorem [4.17]. Applying Theorem [4.17] to $M = \Lambda$, we obtain $C^a = 0$ for $a \gg 0$.

We prove (2) $\Rightarrow$ (1). Since $C_\Lambda$ is finitely generated, the derived tensor product $(- \otimes^L_C)_{|K}$ has its value in $D^b(\text{mod} \Lambda)$. It follows from the first condition that the functor $- \otimes^L_C$ is an endofunctor of $D^b(\text{mod} \Lambda)$. Therefore, $D^b(\text{mod} \Lambda)$ is contained in $\mathbb{K}^b(\text{proj}^Z A)$ by Theorem [4.17]. Since the graded degree shift (1) is an exact autofunctor of $\mathbb{K}^b(\text{proj}^Z A)$, $D^b(\text{mod} \Lambda)(i)$ is also contained in $\mathbb{K}^b(\text{proj}^Z A)$ for $i \in \mathbb{Z}$. Since every object of $D^b(\text{mod}^Z \Lambda)$ is constructed by extensions from objects of $\{D^b(\text{mod} \Lambda)(i) \mid i \in \mathbb{Z}\}$, it is contained in $\mathbb{K}^b(\text{proj}^Z A)$.

□

Remark 4.20. It seems likely that we can generalize Theorem [4.10] for an object $M \in D(\text{Mod}^Z \Lambda)$ such that $M_i = 0$ for $|i| \gg 0$ by using the derived action map.

The graded $\Lambda$-module structure on $M$ induces the derived action morphism $\rho_i : M_i \otimes^L_C \rightarrow M_{i+1}$ for $i \in \mathbb{Z}$. Note that we have $\rho_{i+1} \circ (\rho_i \otimes^L_C) = 0$. Hence we obtain a complex $\mathcal{X}_a$ of objects of $D^b(\text{Mod} \Lambda)$

$$
\mathcal{X}_a : M_{i_0} \otimes^L_C C^a \xrightarrow{\rho_{i_0} \otimes^L_C C^{a-1}} M_{i_0+1} \otimes^L_C C^{a-1} \xrightarrow{\rho_{i_0-1} \otimes^L_C C^{a-2}} \cdots \xrightarrow{\rho_{i_0-a}} M_a
$$

where we set $i_0 := \min\{i \mid M_i \neq 0\}$. Using the totalization $\text{tot}(\mathcal{X}_a)$ of the complex $\mathcal{X}_a$, we may be able to establish a projective dimension formula for $M$. However we should care about a subtlety of the totalization of objects of a triangulated category (see [13]).

We are content with Theorem [4.17] which is sufficient for our applications.

5 Injective dimension formula

In this Section 5, again, $\Lambda$ denotes an algebra, $C$ denotes a bimodule over it and $A = \Lambda \oplus C$ is the trivial extension algebra. We establish the injective dimension formula for an object $M \in D^b(\text{Mod}^Z \Lambda)$ such that $M_i = 0$ for $i \neq 0, 1$. The key tool is a decomposition of a complex $I \in C(\text{Inj}^Z \Lambda)$ of graded injective $A$-modules according to degree of the cogenerators.

Since the arguments are dual of that of Section 4, we leave the details of proofs to the readers.

In Section 5.3 we discuss a condition that $A$ is an Iwanaga-Gorenstein algebra.

5.1 Decomposition of complexes of graded injective $A$-modules

5.1.1 Decomposition of graded injective $A$-modules

Let $I$ be a graded injective $A$-module. Recall that $i_i := \text{HOM}_A(\Lambda, I)_i$, $\mathfrak{s}_i I := \text{HOM}_A(A, i_i)(-i)$. Since now $A = \Lambda \oplus C$ is a trivial extension algebra, we have an isomorphism $\mathfrak{s}_i I \cong \text{Hom}_A(C, i_i)(-i + 1) \oplus (i_i)(-i)$ as graded $\Lambda$-modules.

By Lemma [2.7] we have the following isomorphisms

$$
I \cong \bigoplus_{i \in \mathbb{Z}} \mathfrak{s}_i I \quad (\text{as graded } A\text{-modules}),
$$

$$
I_i \cong (\mathfrak{s}_i I)_i \oplus (\mathfrak{s}_{i+1} I)_i \cong i_i \oplus \text{Hom}_A(C, i_{i+1} I) \quad (\text{as } \Lambda\text{-modules}).
$$

(5-6)
As in Section 4.1.1, we may decompose the degree $i$-part $f_i$ of a morphism $f : I \to I'$ of graded injective modules under the isomorphism of (5-6).

$$\begin{pmatrix} i_i(f) \\ 0 & \text{Hom}_A(C, i_{i+1}(f)) \end{pmatrix} : i_iI \oplus \text{Hom}_A(C, i_{i+1}I) \to i_{i'} \oplus \text{Hom}_A(C, i_{i+1}I').$$

5.1.2 Decomposition of complexes of graded injective $A$-modules

As in Section 4.1.2 and Section 4.1.3 we may construct from a complex $I$ of graded injective $A$-modules a complex $i_iI$ of injective $A$-modules and prove the following lemma.

**Lemma 5.1.** Let $I$ be a complex of graded injective $A$-modules. Then, the following assertions hold.

1. We have an exact triangle in $K(\text{Mod} \Lambda)$ (and in $D(\text{Mod} \Lambda)$)

$$\text{Hom}_A(C, i_{i+1}I)[-1] \xrightarrow{j_i} i_iI \to I_i \to \text{Hom}_A(C, i_{i+1}I)$$

where $j_i$ is the morphism given by the collection $\{i_i(d_i^n)\}_{n \in \mathbb{Z}}$.

In particular, $H(I)_i = 0$ if and only if the morphism $j_i : \text{Hom}_A(C, i_{i+1}I)[-1] \to i_iI$ is a quasi-isomorphism.

2. Assume that a complex $J_i'$ in $C(\text{Inj} \Lambda)$ homotopic to $i_iI$ is given for $i \in \mathbb{Z}$. Then there exists $I' \in C(\text{Inj}^2 \Lambda)$ homotopic to $I$ such that $i_iI' = J_i'$ in $C(\text{Inj} \Lambda)$ for $i \in \mathbb{Z}$.

5.2 Injective dimension formula

Let $M$ be a graded $A$-module. Then the graded $A$-module structure induces a $\Lambda$-module homomorphism $\tilde{\theta}_{M,i} : M_i \to \text{Hom}_A(C, M_{i+1})$ for $i \in \mathbb{Z}$, which we call the coaction morphism of $M$.

If a graded $A$-module $M$ satisfies $M_{i+1} = 0$, then $(\text{HOM}_A(C, M))_1 \cong \text{Hom}_A(C, M_1)$. Moreover, applying $\text{HOM}_A(-, M)$ to the canonical inclusion $\text{can} : C \hookrightarrow A(1)$ and looking the degree 1-part, we obtain the 1-st coaction morphism

$$\tilde{\theta}_{M,0} : M_0 \cong \text{HOM}_A(A(1), M)_1 \xrightarrow{\text{HOM(\text{can}, M)_1}} \text{HOM}_A(C, M)_1 \cong \text{Hom}_A(C, M_1).$$

We remark that since these constructions are natural, it works for a complex $M \in C(\text{Mod}^2 \Lambda)$.

Let $M$ be an object of $D(\text{Mod}^2 \Lambda)$. We denote a representative of $M$ by the same symbol $M \in C(\text{Mod}^2 \Lambda)$. We define a morphism $\Theta_M^0 : M_0 \to \mathbb{R} \text{Hom}_A(C, M_1)$ in $D(\text{Mod} \Lambda)$ to be the composite morphism

$$\Theta_M^0 : M_0 \xrightarrow{\tilde{\theta}_{M,0}} \text{Hom}_A^*(C, M_1) \xrightarrow{\text{can}} \mathbb{R} \text{Hom}_A(C, M_1)$$

where $\text{can}$ is a canonical morphism. It is easy to see that this morphism is well-defined.

For $a \geq 1$ and $L \in D(\text{Mod} \Lambda)$, we set

$$\mathbb{R} \text{Hom}_A(C^a, L) := \mathbb{R} \text{Hom}_A(C, \cdots \mathbb{R} \text{Hom}_A(C, \mathbb{R} \text{Hom}_A(C, L)) \cdots) \ (a\text{-times}).$$

We also set $\mathbb{R} \text{Hom}_A(C^0, L) := L$.

**Remark 5.2.** The above notation is only the abbreviation of the iterated application of the functor $\mathbb{R} \text{Hom}_A(C, -)$. We will not deal with the complex $C \otimes_A^L C \otimes_A^L \cdots \otimes_A^L C$ of $A$-$\Lambda$-bimodules.
The following is the main theorem of this section. For simplicity we set $\Theta^a_M := \mathbb{R}\text{Hom}_\Lambda(C^a, \Theta^0_M)$.

$$\Theta^a_M : \mathbb{R}\text{Hom}_\Lambda(C^a, M_0) \to \mathbb{R}\text{Hom}_\Lambda(C^{a+1}, M_1)$$

**Theorem 5.3.** Let $M$ be a graded $A$-module such that $M_i = 0$ for $i \neq 0, 1$. Then,

$$\text{gr.id}_A M = \sup \{ \text{id}_A M_1, \text{id}_A (\text{cn}_A \Theta^a_M) + a + 1 \mid a \geq 0 \}.$$ 

In particular, we have $\text{gr.id}_A M < \infty$ if and only if the following conditions are satisfied:

1. $\text{id}_A M_1 < \infty$.
2. $\text{id}_A \text{cn}(\Theta^a_M) < \infty$ for $a \geq 0$.
3. $\Theta^a_M$ is an isomorphism for $a \gg 0$.

As a corollary, we obtain the following formula, from which Theorem 1.1 follows.

**Corollary 5.4.** Let $M \in \text{D}^+(\text{Mod } \Lambda)$. Then,

$$\text{id}_A M = \sup \{ \text{id}_A \mathbb{R}\text{Hom}_\Lambda(C^a, M) + a \mid M \in \text{Mod } \Lambda, a \geq 0 \}.$$ 

Combining this corollary with Proposition 2.11 we see the next corollary.

**Corollary 5.5.** We have the equation

$$\text{gldim } A = \sup \{ \text{id}_A (\mathbb{R}\text{Hom}_\Lambda(C^a, M)) + a \mid M \in \text{Mod } \Lambda, a \geq 0 \}.$$ 

We can prove Theorem 5.3 in the same way of Theorem 4.10 by using following Lemmas, which are dual of Lemma 4.13 and Lemma 4.14.

**Lemma 5.6.** Let $M$ be a graded $A$-module such that $M_i = 0$ for $i \neq 0, 1$ and $I$ an injective resolution of $M$ in $\text{Mod}^\mathbb{Z} A$. Then,

1. $s > 1 I$ is null-homotopic and hence $s \leq 1 I$ is homotopic to $I$ via the canonical morphism.
2. $i_1 I$ is an injective resolution of $M_1$ as complexes of $\Lambda$-modules.
3. We have $\mathbb{R}\text{Hom}_\Lambda(A, M)_i = i_1 I$ in $\text{D}($Mod $\Lambda)$.
4. For $L \in \text{D}($Mod $\Lambda)$, we have $\mathbb{R}\text{Hom}_\Lambda(L, M)_1 = \mathbb{R}\text{Hom}_\Lambda(L, M_1)$.

**Lemma 5.7.** Let $M$ be as in Theorem 5.3 and $I \in \text{C}(\text{Inj}^\mathbb{Z} A)$ an injective resolution of $M$. Then, we have the following isomorphisms in $\text{D}($Mod $\Lambda)$.

$$i_1 I \cong M_1, \quad i_{-i} I \cong \text{cn}(\Theta^i_M)[-i - 1] \quad \text{for } i \geq 0$$
Remark 5.8. We give a remark similar to Remark 4.20. It seems likely that we can generalize Theorem 5.3 for an object $M \in D(\text{Mod}^\mathbb{Z}_A)$ such that $M_i = 0$ for $|i| \gg 0$ by using the derived coaction map.

By adjunction, the derived action morphism $\rho_i : M_i \otimes^L_A C \to M_{i+1}$ induces the derived coaction morphism $\Theta'_i : M_i \to \mathbb{R}\text{Hom}_A(C, M_{i+1})$. Note that we have $\mathbb{R}\text{Hom}_A(C, \Theta'_i) \circ \Theta'_i = 0$. Hence we obtain a complex $\mathcal{Y}_a$ of objects of $D^b(\text{Mod} \Lambda)$

$$
\mathcal{Y}_a : M_{j_0-a} \xrightarrow{\Theta_{j_0-a}} \mathbb{R}\text{Hom}_A(C, M_{j_0-a+1}) \to \cdots \xrightarrow{\mathbb{R}\text{Hom}(C^{a-1}, \Theta'_{j_0-1})} \mathbb{R}\text{Hom}_A(C^a, M_{j_0})
$$

where we set $j_0 := \max\{i \mid M_i \neq 0\}$. Using the totalization $\text{tot}(\mathcal{Y}_a)$ of the complex $\mathcal{Y}_a$, we may be able to establish an injective dimension formula for $M$. However we should care about a subtlety of the totalization of a complex of objects of a triangulated category (see [14]).

We are content with Theorem 5.3 which is sufficient for our applications.

Remark 5.9. We have obtained formulas for projective dimension and injective dimension. If we only deal with a finite dimensional algebra $\Lambda$ over a field $k$ and a suitably bounded complex of finite dimensional modules over it, then we can deduce one from the other by using the natural isomorphism

$$
M \otimes^L_A \mathbb{D}(N) \to \mathbb{D}\mathbb{R}\text{Hom}_A(M, N)
$$

for $M, N \in D^-\text{(mod} \Lambda)$ where $\mathbb{D}(-) = \text{Hom}_k(-, k)$ denotes the $k$-duality.

5.3 A criterion of Iwanaga-Gorensteinness

In this Section 5.3, we discuss a condition that a trivial extension algebra $A = \Lambda \oplus C$ is an Iwanaga-Gorenstein (IG) algebra.

5.3.1 Right asid bimodules

Before recalling the definition of IG-algebra, we give a condition that $A$ has finite right self-injective dimension. For this purpose, we give a description of the morphism $\Theta^a_A$.

We define a morphism $\lambda_r : \Lambda \to \text{Hom}_A(C, C)$ by the formula $\lambda_r(x)(c) := xc$ for $x \in \Lambda$ and $c \in C$. We denote the composite morphism $\lambda_r = \text{can} \circ \lambda_r$ in $D(\text{Mod} \Lambda)$ where $\text{can}$ is the canonical morphism $\text{Hom}_A(C, C) \to \mathbb{R}\text{Hom}_A(C, C)$

$$
\lambda_r : \Lambda \xrightarrow{\tilde{\lambda}_r} \text{Hom}_A(C, C) \xrightarrow{\text{can}} \mathbb{R}\text{Hom}_A(C, C).
$$

We can check that $\lambda_r = \Theta^0_A$. Therefore, by Corollary 2.10 and Theorem 5.3 we deduce the following result.

Theorem 5.10. The following conditions are equivalent:

1. $\text{id} A_A < \infty$.

2. the following conditions are satisfied:

- Right ASID 1. $\text{id} C^a < \infty$.

- Right ASID 2. $\text{id} \text{can}(\mathbb{R}\text{Hom}_A(C^a, \lambda_r)) < \infty$ for $a \geq 0$. 

24
Right ASID 3. The morphism $\mathbb{R}\text{Hom}_{\Lambda}(C^a, \lambda_r)$ is an isomorphism for $a \gg 0$.

**Definition 5.11.** (1) A $\Lambda$-$\Lambda$-bimodule $C$ is called a right asid (attaching self injective dimension) bimodule if $\text{id}_{A_A} < \infty$.

(2) For a right asid bimodule $C$, we define the right asid number $\alpha_r$ to be

$$\alpha_r := \min\{a \geq 0 \mid \mathbb{R}\text{Hom}_{\Lambda}(C^a, \lambda_r) \text{ is an isomorphism}\}.$$

By Lemma 5.6 and Lemma 5.7, we obtain description of the right asid numbers $\alpha_r$.

**Corollary 5.12.** Let $\Lambda$ be an algebra, $C$ a right asid bimodule over $\Lambda$ and $A = \Lambda \oplus C$ the trivial extension algebra. We regard a minimal injective resolution $I$ of $A$ as a complex. Then, we have

$$\alpha_r = \max\{a \geq -1 \mid \mathbb{R}\text{Hom}_A(\Lambda, A)_{-a} \neq 0\} + 1 = \max\{a \geq -1 \mid i_a I \neq 0\} + 1.$$

Dually, we define the notion of left asid bimodule and the left asid number $\alpha_\ell$ for them by using the left version $\lambda_\ell$ of $\lambda_r$.

$$\lambda_\ell : \Lambda \to \mathbb{R}\text{Hom}_{\Lambda^{\text{op}}}(C, C).$$

A $\Lambda$-$\Lambda$-bimodule $C$ is called asid if it is both right and left asid bimodules.

### 5.3.2 A criterion of Iwanaga-Gorensteinness

Recall that an algebra is called IG if it is Noetherian on both sides and has finite self-injective dimension on both sides. We note that by Zaks’ Theorem, right self-injective dimension and left self-injective dimension of an IG-algebra $A$ coincide.

The notion of graded IG algebra is defined for graded algebras in the same way. It is easy to see that for a finitely graded algebra $A = \bigoplus_{i=0}^\ell A_i$, graded IG-ness and IG-ness as an ungraded algebra are the same condition. Moreover in this case, we have the following equations by Corollary 2.10

$$\text{id}_{A_A} = \text{id}_{A^{\text{op}}} A = \text{gr.id}_{A_A} = \text{gr.id}_{A^{\text{op}}} A.$$

It is worth mentioning the following fact which is obtained as a combination of known results.

**Proposition 5.13.** A (not necessary finitely) graded algebra $A = \bigoplus_{i \geq 0} A_i$ is graded IG if and only if it is (ungraded) IG.

**Proof.** This is a consequence of the following two results. A graded ring $A$ is graded left Noetherian if and only if it is (ungraded) left Noetherian by [29] II 3.1. For a (graded) Noetherian ring $A$, we have $\text{id}_{A_A} \leq \text{gr.id}_{A_A} \leq \text{id}_{A_A} + 1$. The first inequality is well-known. The second is due to Van den Bergh (see [36, Theorem 12]).

A trivial extension algebra $A = \Lambda \oplus C$ is Noetherian on both sides if and only if so is $\Lambda$ and $C_\Lambda$ and $\Lambda C$ are finitely generated. Hence by Theorem 5.10 and its left version, we obtain the following criterion of IG-ness of a trivial extension algebra.

**Theorem 5.14.** Let $\Lambda$ be a Noetherian algebra and $C$ a $\Lambda$-$\Lambda$-bimodule which is finitely generated as right and as left $\Lambda$-modules respectively. The trivial extension algebra $A = \Lambda \oplus C$ is an Iwanaga-Gorenstein algebra if and only if the following conditions are satisfied:

(1) $C$ satisfies the conditions right ASID 1,2,3.
(2) $C$ satisfies the left ASID conditions below

Left ASID 1. $\text{id}_{\Lambda^{\text{op}}} C < \infty$.

Left ASID 2. $\text{id}_{\CN} \RN_{\Lambda^{\text{op}}}(C^a, \lambda_\ell) < \infty$ for $a \geq 0$.

Left ASID 3. The morphism $\RN_{\Lambda^{\text{op}}}(C^a, \lambda_\ell)$ is an isomorphism for $a \gg 0$.

In [26], we characterize an asid bimodule $C$ of $\alpha_r = 0, \alpha_\ell = 0$ as a cotilting bimodule over $\Lambda$ in the sense of Miyachi [27].

**Proposition 5.15** ([26]). Let $\Lambda$ and $C$ as in Theorem 5.14. A $\Lambda$-$\Lambda$-bimodule $C$ is an asid bimodule of $\alpha_r = 0, \alpha_\ell = 0$ if and only if it is a $\Lambda$-$\Lambda$-cotilting bimodule.

In [26], we introduce a new class of finitely graded IG-algebra called homologically well-graded (hwg) IG-algebra, and show that it possesses nice characterizations from several viewpoints. Among other things, we show that a trivial extension algebra $A = \Lambda \oplus C$ is hwg IG if and only if $C$ is an asid bimodule with $\alpha_r = 0, \alpha_\ell = 0$.

In [25], we investigate a condition that $A = \Lambda \oplus C$ is IG from a categorical viewpoint in the case where $\Lambda$ is IG. We close this section by showing that in this case the above condition become much simpler.

**Proposition 5.16.** Let $\Lambda$ and $C$ be as in Theorem 5.14. Assume moreover that $\Lambda$ is IG and $C$ satisfies the conditions right and left ASID 1. Then, $C$ also satisfies the conditions right and left ASID 2.

We use the following fundamental observation due to Iwanaga.

**Proposition 5.17** ([15]). Let $\Lambda$ be an IG-algebra and $M$ a finitely generated $\Lambda$-module. Then,

$$\text{pd}_{\Lambda} M < \infty \iff \text{id}_{\Lambda} M < \infty.$$ 

**Proof of Proposition 5.16.** We only prove that $C$ satisfies the right ASID condition 2. The left ASID condition 1 together with the left version of Proposition 5.17 implies that $\text{pd} C < \infty$. Hence by Theorem 3.14 the functor $\RN_{\Lambda}(C, -)$ preserves finiteness of injective dimension. Therefore $\RN_{\Lambda}(C^a, \Lambda), \RN_{\Lambda}(C^a, C)$ have finite injective dimension. Now the assertion follows from Lemma 3.5.

6 Upper triangular matrix algebras

An important example of a trivial extension algebra is an upper triangular matrix algebra. As applications of Theorem 4.10 and Theorem 5.3, we obtain the projective dimension formula and the injective dimension formula for a module over an upper triangular matrix algebra.

Let $A = \begin{pmatrix} \Lambda_0 & C \\ 0 & \Lambda_1 \end{pmatrix}$ be an upper triangular matrix algebra where $\Lambda_0, \Lambda_1$ are algebras and $C$ is a $\Lambda_0$-$\Lambda_1$-bimodule. It is well-known that an $A$-module $M$ can be identified with a triple $(M_0, M_1, \xi_M)$ consisting of a $\Lambda_0$-module $M_0$, a $\Lambda_1$-module $M_1$ and a $\Lambda_1$-module homomorphism $\xi_M : M_0 \otimes_{\Lambda_0} C \rightarrow$
We note that \( \xi_M \) induces a \( \Lambda_0 \)-module homomorphism \( \theta_M : M_0 \to \text{Hom}_{\Lambda_1}(C, M_1) \) via \( \otimes \)-Hom adjunction. We set

\[
\Xi_M : M_0 \otimes_{\Lambda_0} C \xrightarrow{\text{can}} M_0 \otimes_{\Lambda_0} C \xrightarrow{\xi_M} M_1,
\]

\[
\Theta_M : M_0 \xrightarrow{\theta_M} \text{Hom}_{\Lambda_1}(C, M_1) \xrightarrow{\text{can}} \text{RHom}_{\Lambda_1}(C, M_1)
\]

where \text{can} in both rows denote appropriate canonical morphisms.

**Proposition 6.1.** The following equations hold.

\[
\text{pd}_A M = \sup_{\Lambda_0} \{ \text{pd} M_0, \text{pd} \text{can} \Xi_M \}
\]

\[
\text{id}_A M = \sup_{\Lambda_1} \{ \text{id} M_1, \text{id} \text{can} \Theta_M + 1 \}
\]

We note that a related result was obtained by Asadollahi-Salarian [2, Theorem 3.1, 3.2].

**Proof.** We set \( \Lambda = \Lambda_0 \times \Lambda_1 \) and \( e_0 := (1_{\Lambda_0}, 0), e_1 := (0, 1_{\Lambda_1}) \). We equip \( C \) with a \( \Lambda \)-\( \Lambda \)-bimodule structure in the following way

\[
(a_0, a_1)c(b_0, b_1) := a_0 cb_1.
\]

Then the algebra \( A \) is the trivial extension algebra \( \Lambda \oplus C \).

Let \( M \) be an \( A \)-module. Then \( M_0 \) and \( M_1 \) above are obtained as \( M_0 := Me_0, M_1 := Me_1 \) and \( M = M_0 \oplus M_1 \) can be regarded as a graded \( A \)-module whose degree 0-part is \( M_0 \) and degree 1-part is \( M_1 \). Now the action map \( \tilde{\xi}_{M,0} \) coincides with the above \( \xi_M \) and the coaction map \( \tilde{\theta}_{M,0} \) coincides with the above \( \theta_M \).

Since \( C \otimes_{\Lambda_1} C = 0, M_1 \otimes_{\Lambda_0} C = 0, \text{RHom}_{\Lambda_1}(C, M_0) = 0 \), we deduce the desired formula from Theorem 4.10 and Theorem 5.3.

Observe that an \( A \)-module \( M \) fits into a canonical exact sequence below

\[
0 \to M_1 \to M \to M_0 \to 0.
\]

Hence, \( \text{pd}_A M \leq \sup_{\Lambda_0} \{ \text{pd} M_0, \text{pd} M_1 \} \) and \( \text{id}_A M \leq \sup_{\Lambda_1} \{ \text{id} M_0, \text{id} M_1 \} \). Therefore as a corollary, we obtain an answer to Chase’s problem of determining the global dimension of \( A \).

**Corollary 6.2.** The following equations hold.

\[
\text{gldim} A = \sup_{\Lambda_0} \{ \text{pd} M_0, \text{pd} M_0 \otimes_{\Lambda_0} C + 1, \text{pd} M_1 \mid M_0 \in \text{Mod} \Lambda_0, M_1 \in \text{Mod} \Lambda_1 \}
\]

\[
= \sup_{\Lambda_1} \{ \text{id} M_1, \text{id} \text{RHom}_{\Lambda_1}(C, M_1) + 1, \text{id} M_0 \mid M_0 \in \text{Mod} \Lambda_0, M_1 \in \text{Mod} \Lambda_1 \}
\]

We deduce the following well-known result.

**Corollary 6.3.** An upper triangular matrix algebra \( A = \begin{pmatrix} \Lambda_0 & C \\ 0 & \Lambda_1 \end{pmatrix} \) is of finite global dimension if and only if so are \( \Lambda_0, \Lambda_1 \).

The next result was essentially proved by Enochs-Cortes-Izurdiaga-Torrecillas [11, Theorem 3.1].

**Corollary 6.4.** (1) Assume that \( C_{\Lambda_1} \) has finite projective dimension. Then an \( A \)-module \( M \) has finite projective dimension if and only if so do the \( \Lambda_0 \)-module \( M_0 \) and the \( \Lambda_1 \)-module \( M_1 \).
Assume that $\Lambda_0 C$ has finite flat dimension. Then an $A$-module $M$ has finite injective dimension if and only if so do the $\Lambda_0$-module $M_0$ and the $\Lambda_1$-module $M_1$.

We can also deduce the following result which was obtained by X-W. Chen [9]. We remark that the self-injective dimension of an upper triangular algebra was studied by [34].

**Corollary 6.5.** Let $\Lambda_0, \Lambda_1$ be IG-algebras and $C$ a $\Lambda_0$-$\Lambda_1$-bimodule which is finitely generated as both left $\Lambda_0$-module and right $\Lambda_1$-module. Then, the upper triangular matrix algebra $A = \begin{pmatrix} \Lambda_0 & C \\ 0 & \Lambda_1 \end{pmatrix}$ is IG if and only if $\text{pd} \, \Lambda_0 C < \infty$ and $\text{pd} \, C \Lambda_1 < \infty$.

Using Proposition 6.1 repeatedly, we obtain the following result which will be used in the subsequent paper [25] and [26].

**Corollary 6.6.** Let $A = \bigoplus_{i=0}^\ell A_i$ be a finitely graded Noetherian algebra. Assume that the degree 0-part $A_0$ is IG and that the $A_0$-$A_0$-bimodule $A_i$ has finite projective dimension on both sides for $i = 1, \ldots, \ell$. Then the Beilinson algebra $\nabla A$ is IG and the $\nabla A$-$\nabla A$-bimodule $\Delta A$ has finite projective dimension on both sides.

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