Many-body level statistics of single-particle quantum chaos

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We consider a many-fermion system populating levels of a random matrix ensemble and study the corresponding many-particle level statistics. This is equivalent to many-body level statistics of the $q = 2$ complex Sachdev-Ye-Kitaev model. The spectral form factor, $K(t)$, is derived analytically using algebraic methods of random matrix theory and matched with an exact numerical simulation. Despite obvious integrability of the theory, the structure of the spectral form factor is found to be surprisingly rich, representing various degrees of spectrum rigidity on different energy scales. $K(t)$ exhibits an initial drop, followed by an exponential ramp, asymptotically approaching a plateau at large times $t_* \sim 2N$, where $N \gg 1$ is the number of single-particle levels. The width of the ramp reflects the residual repulsion of the distant many-body levels stemming from single-particle Wigner-Dyson level repulsion. We also develop a σ-model approach to calculate the spectral form factor and demonstrate that the sharp exponential ramp is due to soft diffusion-like modes. In generic interacting models, we expect dephasing processes to arise, introduce a “mass” to these modes, which would suppress the exponential ramp, and result in a linear ramp in the spectral form factor – a hallmark signature of many-body quantum chaos.

There has been growing recent interest in the foundational questions of statistical mechanics, from the eigenstate thermalization hypothesis [1–5] to many-body localization [6–12]. These questions are intimately related to the notion of quantum chaos [13]. While intuitive, quantum chaos is not easy to define. The usual approach is to associate quantum-chaotic, ergodic systems with energy spectra exhibiting Wigner-Dyson (WD) level statistics [14–18]. On the other hand, integrable (including localized) non-ergodic systems are expected to exhibit Poisson level statistics - lack of any correlation between the levels [19]. A lot is known about single-particle quantum chaos, which has been explored for a variety of systems from chaotic billiards [15, 20–23] to disordered metals [24–31], where the WD level statistics has indeed been seen using numerical and analytical theoretical techniques, as well as in experiment [32–36].

Many-body quantum chaos [37–39] is a more difficult concept to both define and study. The structure of many-body energy levels is very fine, with nearest neighbor level spacing inverse proportional to the Hilbert space size, which is exponential in system size, $N$, which itself is usually an astronomically large number in most many-body systems of interest. Furthermore, there are confusingly two types of “quantum chaos” that may be present in a many-body system. Consider for example a weakly disordered system in three dimensions. A single quantum particle moving in this random potential will exhibit WD level statistics - a classic result of Altshuler and Shklovskii [27], which is the hallmark of single-particle quantum chaos. If we embed a non-interacting $N$-particle (e.g., $N$-electron) system in this random medium, it will never thermalize due to lack of interactions. This Fermi gas is an integrable system, which is not associated with many-body quantum chaos. On the other hand, a generic interacting many-body Fermi system is expected to thermalize - i.e., exhibit an ergodic, many-body quantum-chaotic behavior (which presumably should exist with or without an initial disordered potential). Of great interest is the open problem of a transition or perhaps energy-dependent crossover from single-particle to many-body quantum chaos in the distribution of energy levels of such a system.

Stanford et al. [40, 41] recently considered many-body level statistics of a family of the SYK models [42–46] using a combination of field theory technique and numerical simulations. However, explicit analytical results are lacking even in the non-interacting case (or equivalently, SYK-2). In this work, we calculate the many-body level statistics of a single-particle quantum chaotic model. We show that the corresponding spectral form factor is not pure Poisson and retains rich structure descending from single-particle chaos of the underlying model.

Model— As a simple model exhibiting quantum chaos, we choose a Gaussian Unitary ensemble (GUE) [16, 47] of $N \times N$ Hermitian single-particle Hamiltonians, $\hat{h}$, following the distribution function:

$$P(\hat{h}) = 2^{N(N-1)/2} \left( \frac{N}{2\pi} \right)^{N^2/2} \exp \left[ -\frac{N}{2} \text{Tr} \left( \hat{h}^2 \right) \right],$$

with the local level statistics of $\hat{h}$ falling into the unitary WD class [16, 48]. Populating these single-particle energy levels with fermions $(\hat{f}_i, \hat{f}_i^\dagger)$ with a chemical potential $\mu$ then defines the many-body Hamiltonian,

$$\hat{H} = \sum_{i,j} \hat{f}_i^\dagger (\hat{h}_{ij} - \mu \delta_{ij}) \hat{f}_j.$$

This is an integrable model, and the particle number at each single-particle level is a constant of the motion.
In general, for a statistical ensemble of Hamiltonians, we can define a representative 2-point spectral form factor (SFF) [48, 49],

\[ K(t) \equiv \langle |Z(it)|^2 \rangle = \left\{ \sum_{n,m} e^{i(E_m - E_n)t} \right\}, \quad (3) \]

where \( Z(it) \equiv \text{Tr}(e^{-i\hat{H}t}) = \sum_n e^{-iE_nt}, \) with \( E_n \) being the eigenvalues of \( \hat{H} \), and the angular bracket represents ensemble averaging. It immediately follows that \( 0 \leq |K(t)| \leq L^2 \) with \( K(0) = L^2 \), and \( K(\infty) = L \) if degeneracies are statistically insignificant in the ensemble, where \( L \) is the Hilbert space size i.e. number of energy levels of the system.

The SFF is essentially a Fourier transform of the joint two-level distribution function (see also Eq. (17)). For an ensemble with Poisson statistics (independently distributed energy levels), \( K(t) \) decays from \( L^2 \) at \( t = 0 \), gradually approaching \( L \) at a time scale much smaller than the inverse mean level separation. However, Hamiltonians obeying WD statistics are characterized by level repulsion at a scale \( \Delta \) corresponding to the typical level spacing. This results in an SFF that “slopes” down below \( L \) up to around a dip time \( t \sim t_d \), where it reaches a minimum, then grows in an approximately linear “ramp”, abruptly reaching a \( K(t) = L \) “plateau” for \( t \geq t_* \). We will call \( t_d \) the plateau time. The ramp and plateau have their origins in the Fourier transform of the level repulsion component of the distribution, which implies that \( t_* \sim 1/\Delta \). To the extent that the level repulsion is given by the WD universality classes, the ramp and plateau are also universal features of quantum chaotic systems (see e.g. Refs. [16, 48, 50, 51]).

Results—For the system given by Eq. (2), which is our primary concern in this paper, we have \( N \) single-particle levels, and \( L = 2^N \). We find three approximate expressions that closely describe the SFF in different regions, in the large \( N \) limit, i.e.

\[ K(t) \approx \begin{cases} K_1(t), & 0 < t \ll O(1), \\ K_2(t), & O(1) \ll t < \eta_l < O(N/\log_2 N), \\ K_3(t), & \sqrt{2N} < t < \infty, \end{cases} \quad (4) \]

where

\[ K_1(t) = L^2 \cos^{2N} \left( \frac{\mu t}{2} \right) \exp \left[ N \left( \frac{J_1(2t)}{t} - 1 \right) \cos(\mu t) \right], \quad (5a) \]

\[ K_2(t) = \left( \frac{N}{8} e^{\gamma_E} \right)^{t/4} \exp \left[ N \frac{J_1(2t)}{t} \cos(\mu t) \right], \quad (5b) \]

\[ K_3(t) = L \exp \left[ -\frac{(4N^2 - t^2)^{3/2}}{12\pi N t} \right], \quad (5c) \]

with \( \gamma_E = 0.577... \) being the Euler-Mascheroni constant. \( K_1(t) \) describes the initial downward slope region; \( K_2(t) \) is related to the transition from an oscillatory region up to \( t \sim O((N/\ln N)^{5/3}) \) to an exponential beginning of the ramp; and \( K_3(t) \) gives the late-time ramp approaching the plateau. These approximations are illustrated in Fig. (1), where they are compared with a numerical calculation based on Eqs. (12) and (13) (“exact”), compared with the three approximate expressions \( K_{1,2,3}(t) \) given by Eqs. (5a), (5b) and (5c), respectively.

**Details of calculation**—From the definition of the SFF (Eq. (3)), it is straightforward to show that for an ensemble of Hamiltonians described by Eqs. (1) and (2), \( K(t) \) is given by

\[ K(t) = \left\{ \sum_{\alpha,\beta=1}^N e^{i(n^\alpha_k - n^\beta_k)\xi_k t} \right\} \]

\[ = 2^N \int d\varepsilon_1...d\varepsilon_N P(\varepsilon_1,...,\varepsilon_N) \prod_{k=1}^N \left( 1 + \cos(\xi_k t) \right). \quad (6) \]

Here \( n^\alpha_k \) indicates the occupation number of the \( k \)-th level for the many body state \( \alpha \), the angular bracket represents the ensemble average, and \( P(\varepsilon_1,...,\varepsilon_N) \) denotes the joint probability density of single-particle levels \( \{\varepsilon_i\} \). We have introduced the notation \( \xi_k = \varepsilon_k - \mu \).

The above equation can be rewritten as

\[ K(t) = 2^N \left[ 1 + \sum_{n=1}^N \frac{1}{n!} \bar{r}_n(t) \right], \quad (7) \]

\[ \bar{r}_n(t) = \int d\varepsilon_1...d\varepsilon_n r_n(\varepsilon_1,...,\varepsilon_n) \prod_{k=1}^n \cos(\xi_k t). \]

Here \( r_n(\varepsilon_1,...,\varepsilon_n) \) is the \( n \)-point single-particle level cor-
relation function [16] defined as
\[ R_n(\varepsilon_1, \ldots, \varepsilon_n) = \frac{N!}{(N-n)!} \int d\varepsilon_{n+1} \ldots d\varepsilon_N P(\varepsilon_1, \ldots, \varepsilon_N). \]  
(8)

Introducing the corresponding cluster function (irreducible correlation function) of the single-particle levels [16], we find, in the large N limit, the SFF is
\[ K(t) = 2^N \exp \left\{ N \frac{J_1(2t)}{t} \cos(\mu t) + A_0(t) \right\} \]
\[ + 2 \sum_{p=1}^{N} A_p(t)(-1)^p \frac{\sin(pi tp/2)}{pi tp/2} \cos(pt) \right\}. \]  
(9)

Here \( J_1(z) \) denotes the Bessel function of the first kind, and \( A_p(t) \) is given by
\[ A_p(t) \equiv -N \sum_{n=2}^{N} \frac{1}{2^n} \sum_{\sum_{i=1}^{n} \zeta_i = p} \left[ 1 - \frac{t}{2N} s(\{\zeta_i\}) \right] \]
\[ \times \Theta \left[ 1 - \frac{t}{2N} s(\{\zeta_i\}) \right], \]  
(10)

for any integer \( p \geq 0 \). \[ \sum_{\sum_{i=1}^{n} \zeta_i = p} \] represents the summation over all configurations of \( \{\zeta_i = +1\} \) obeying the constraint \( \sum_{i=1}^{n} \zeta_i = p \). \[ s(\{\zeta_i\}) \] is the difference between the maximum and the minimum elements of the sequence \( \{0, \sum_{j=1}^{n} \zeta_j\} \).

We first focus on the regime in which almost all unit step functions in the summation in Eq. (10) take the value of 1. For sufficiently small \( t = O(1) \), we can drop the \( t \)-dependent term and approximate \( A_p(t) \) by a constant, obtaining a rapid decay that corresponds to the 'slope' region of the SFF described by \( K_1(t) \) (Eq. (5a)). By contrast, for \( t \gg O(1) \), \( A_{p\geq1}(t) \) is of the order of or smaller compared with \( A_0(t) \) which assumes the form
\[ A_0(t) \approx \frac{t}{4} \left( \ln \frac{N}{8} + \gamma_E \right) - N \ln 2. \]  
(11)

As a result, the last term in the exponent of \( K(t) \) in Eq. (9), a summation of the highly oscillating functions, can be ignored, leading to \( K_2(t) \) (Eq. (5b)). For \( t = O((N/\log N)^{1/2}) \), the oscillatory Bessel function term in the exponent dominates, and the SFF continues to decay. On the other hand, for \( t \gg O((N/\ln N)^{1/2}) \), \( A_0(t) \) dominates, and the SFF shows a ramp with the asymptotic behavior of \( (Ne^{\pi E}/8)^{1/4} \). While it is not easy to determine the upper limit of validity of \( K_2(t) \) (Eq. (5b)), we can estimate an upper bound for the range of validity by noting that we must constrain \( t = O(N/\log_2 N) \) in this expression to avoid violating the condition \( K(t) \leq L^2 \).

For \( t = O(N) \), a considerable number of the unit step functions in Eq. (10) vanish and the above approximation is no longer valid. We instead start from the following expression, obtained using the well known technique of expressing \( P(\varepsilon_1, \ldots, \varepsilon_N) \) as a determinant of the Hermite polynomials \( H_n(x) \) [16],
\[ K(t) = 2^N \det [\delta_{jk} + M_{jk}(t)]_{j,k=1,\ldots,N}, \]  
(12)

where
\[ M_{jk}(t) = W_{jk} \left( \frac{t^2}{N} \right) \left( \frac{1}{j} \right)^{j-k} \cos(\mu t), \quad j - k \text{ is even}, \]
\[ W_{j\geq k}(\tau) = \left( \frac{\tau - \frac{1}{2}}{(j-1)!} \right) e^{-\frac{\tau}{2}} L_{j-\frac{1}{2}}(\tau), \]  
(13)

with \( W_{jk} = W_{kj} \), and \( L_n^\alpha(x) \) are the Laguerre polynomials.

To simplify them further, we require an asymptotic expression for \( L_n^\alpha(x) \) for large \( n \) with large or small \( \alpha \). For this purpose, we use a modification of the standard result given in Ref. [53],
\[ e^{-\frac{x}{2}} x^{\nu} L_n^\alpha(x) \approx \sqrt{\frac{2}{\pi}} \left( \frac{n + \alpha}{n!} \right) \sin(\varphi_n^\alpha(x)) \Theta(\nu - x) \]
\[ \frac{(x(\nu - x))^{\nu}}{(\nu - x)^{\nu}}, \]  
(14)

where \( \nu = 4n + 2\alpha + 2 \) and the precise form of \( \varphi_n^\alpha(x) \) is irrelevant for our purposes, except to know that it varies over \( x \) as well as \( n, \alpha \). One principal consequence of this approximation is that each \( M_{jk}(t) \) acquires a cutoff in \( t \) above which it vanishes:
\[ M_{jk}(t) \propto \Theta(2N(j+k) - t^2). \]  
(15)

Note that the maximum value of \( j + k \) is \( 2N \), so the \( \Theta \)-functions ensure that all the \( M_{jk}(t) \) vanish for \( t > 2N \), showing that the plateau occurs at \( t_* = 2N \). We use the matrix relation \( \det A = e^{-\text{Tr} \ln A} \) in Eq. (12), and expand the exponent in powers \( \mathfrak{M}^n \) of \( \mathfrak{M} \). Due to the rapidly oscillating sin factor in Eq. (15), we can approximate \( \text{Tr}[\mathfrak{M}^n(t)] \approx 0 \) for odd \( n \). The \( n = 2 \) term gives
\[ K(t) \approx 2^N \exp \left\{ \frac{1}{2} \sum_{j,k=1}^{N} \mathfrak{M}_{jk}^2(t) \right\}. \]  
(16)

We evaluate the sum by approximating the oscillatory contribution \( \sin^2(\varphi) \) by its mean value 1/2, which when used with Eq. (14), directly gives \( K_3(t) \) as in Eq. (5c), in the regime \( t > \sqrt{2N} \). The 'higher order' terms \( n = 4, 6 \ldots \) are guaranteed to be positive as \( \mathfrak{M} \), being symmetric, has real eigenvalues, so strictly speaking \( K_3(t) \) represents an approximate upper bound for the ramp in this regime, approaching exactness at the plateau.

Discussion.—To study the level statistics with the help of \( K(t) \), we can take the Fourier transform of Eq. (3) to get the joint density function of two many-body levels with separation \( S \), summed over the entire spectrum,
\[ \mathfrak{R}_2(S) = \int \frac{dt}{2\pi} K(|t|) e^{-iSt} - L \delta(S), \]  
(17)
where in the second term, we have subtracted off the contribution from when the two levels are identical. We will now set $\mu = 0$ as it simplifies our arguments without altering their essential content (as the ramp and plateau are $\mu$-independent). For $S \gg 1$, only the small-$t$ behavior of $K(t)$ is relevant. We therefore use the expression $K_1(t)$ from Eq. (5a) with $\mu = 0$. Expanding to leading order in $t$, we have $K_1(t) \approx L^2 \exp \left(-\frac{N}{2} t^2\right)$, which gives

$$\tilde{R}_2(S \gg 1) = \frac{L^2}{\sqrt{2\pi N}} \exp \left(-\frac{S^2}{2N}\right),$$  

(18)

showing that the many-body energy spectrum has a width $w \sim \sqrt{N}$.

At the scale $\Delta \sim N^{-1}$ of single-particle level spacings, we must account for the contribution from $K_3$ (by setting $S \approx 0$ in Eq. (18)) as well as the ramp and plateau. We note that for $t \sim 2N$, we can expand the exponent in Eq. (5c), obtaining, to leading order,

$$K_3(t < 2N) \approx L \exp \left[-\frac{(2N - t)^\frac{3}{2}}{3\pi N^{\frac{1}{2}}} \right].$$  

(19)

$K_3(t)$ is therefore comparable to $L$ only in a relatively small region of size $\sim N^{1/3} \ll 2N$. In particular, we can approximate $K_3(t)$ by a box function in $0 < t < 2N$, with magnitude $L$ and width chosen to enclose the same area up to the $t$-axis as in Eq. (19). This allows us to treat the ramp essentially as a discontinuous jump from $K(t) = 0$ to $K(t) = L$ at the plateau time $t_\ast = 2N$ i.e.

$$K_{\text{ramp}}(t) \approx L(1 - \Theta(2N\alpha - t)),$$  

(20)

where $\alpha = 1 - \frac{2}{3} \Gamma(\frac{3}{2}) \frac{3\pi}{N} + \mathcal{O}(N^{-3/5})$, as determined by integrating Eq. (19). This gives,

$$\tilde{R}_2(S \sim N^{-1}) \approx \frac{L^2}{\sqrt{2\pi N}} - \frac{2N\alpha L}{\pi} \sin(2N\alpha S) \frac{2N\alpha S}{2N\alpha S}. $$  

(21)

The local effect represented by the second term, $\Delta \tilde{R}_2(S) = \tilde{R}_2(S) - L^2/\sqrt{2\pi N}$, is plotted in Fig. (2), and compared with a numerical computation based on Eqs. (12) and (17). Eq. (21) can be contrasted with the more familiar two-level correlation function for the GUE (e.g. [16, 48, 50, 51]),

$$R_2(S \sim N^{-1}) \propto 1 - \frac{\sin^2(NS)}{N^2S^2}. $$  

(22)

The second term in both Eq. (21) and Eq. (22) contains the level repulsion effect, with any two levels least likely to have $S \ll N^{-1}$ at this scale. Unlike the single-particle GUE, where $R_2(0) = 0$, the level repulsion for the many body case, Eq. (21), isn’t total (i.e. $R_2(0) \neq 0$), and in fact, negligible compared to the actual two-level density near $S = 0$. This is essentially because the ultimate origin of this level repulsion is still the single particle level spectrum. Another interesting feature of Eq. (21) is that the second term can take both positive and negative values. This means that in addition to the level repulsion effect, there are less dominant centers of level attraction i.e. values of $S$ where levels tend to be found relative to each other with higher probability than in Poisson statistics. As $\alpha \to 1$ when $N \to \infty$, these essentially occur for values of $S$ immediately preceding the local maxima of Eq. (22).

In conclusion, we point out that an alternative to the brute-force algebraic (Hamiltonian) method to calculating the SFF, used here, is a Lagrangian formalism (a sigma model [47] or a similar path-integral approach to SYK [54, 55]). As demonstrated in Supplement Sec. III on the example of the sigma model, a non-perturbative resummation is required to recover the SFF. The zero mode fluctuation around the saddle point does detect the presence of an exponential ramp of Eq. (5b) [56], but other soft modes and massive modes are equally important and contribute to the coefficient in the exponent. One can trace the presence of the sharp ramp in the non-interacting theory to infrared divergences due to soft modes (the diffusons $(\Delta q^2 - i\omega)^{-1}$ in the theory of ordered modes, which reduce to $\omega^{-1}$ in zero-dimensional theories, such as RMT studied here and equivalently SYK-2). In generic interacting theories, we expect the appearance of a dephasing-type cut-off in the soft modes that would result in suppression of the ramp to a slower linear-in-$t$ growth (i.e., the many-body level statistics Eq. (21) is expected to change to the form described by Eq. (22)). This corresponds to many-body quantum chaos. As the Wigner-Dyson distribution would then be truly over the finely-spaced many body levels, the plateau time would be much closer to $t_\ast \sim L$. The details of the many-body SFF calculation for an interacting theory will be presented elsewhere.

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[1] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Advances in Physics 65, 239 (2016).
[2] M. Srednicki, Phys. Rev. E. 50, 888 (1994).
[3] M. Srednicki, Journal of Physics A: Mathematical and General 32, 1163 (1999).
[4] M. Rigol, V. Dunjko, and M. Osharian, Nature 452, 854 (2008).
[5] C. Murthy and M. Srednicki, Phys. Rev. Lett. 123, 230606 (2019).
[6] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006).
[7] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B 76, 052203 (2007).
[8] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Problems of Condensed Matter Physics, 50 (2006).
[9] J. Z. Imbrie, Journal of Statistical Physics 163, 998 (2016).
[10] V. Oganesyan and D. A. Huse, Phys. Rev. B 75, 155111 (2007).
[11] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics 6, 15 (2015).
[12] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Rev. Mod. Phys. 91, 021001 (2019).
[13] F. Haake, in Quantum Coherence in Mesoscopic Systems (Springer, 1991) pp. 583–595.
[14] O. Bohigas, M.-J. Giannoni, Lecture Notes in Physics, vol. 209, Springer, Berlin, 1984.
[15] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).
[16] M. L. Mehta, Random matrices (Elsevier, 2004).
[17] E.J. Dyson, J. Math. Phys. 3 (1962) 140.
[18] E. Wigner, Group theory: and its application to the quantum mechanics of atomic spectra, Vol. 5 (Elsevier, 2012).
[19] M. V. Berry and M. Tabor, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 356, 375 (1977).
[20] S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. 42, 1189 (1979).
[21] G. Casati, F. Valz-Gris, and I. Guarnieri, Lettere al Nuovo Cimento 28, 279 (1980).
[22] M. V. Berry, Annals of Physics 131, 163 (1981).
[23] E. B. Rozenbaum, S. Ganeshan, and I. Guarnieri, Phys. Rev. B 100, 035112 (2019).
[24] L. P. Gorkov and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. 48, 1407(1965) [Sov. Phys. JETP 21, 940 (1965)].
[25] K. Efetov, Adv. Phys. 32, 53 (1983).
[26] K. B. Efetov, A. I. Larkin, and D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. 79, 1120 (1980) [Sov. Phys. JETP 52, 568 (1980)].
[27] B. L. Altshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 91, 220 (1986)[Sov. Phys. JETP 64, 127 (1986)].
[28] J. M. Verbaarschot and M. R. Zirnbauer, Journal of Physics A: Mathematical and General 18, 1093 (1985).
[29] V. E. Kravtsov and A. D. Mirlin, Pisma, Zh. Eksp. Teor. Fiz., 60, 645 (1994) [JETP Lett. 60, 656 (1994)].
[30] A. Kamenev and M. Mézard, Phys. Rev. B 60, 3944 (1999).
[31] A. Altland and A. Kamenev, Phys. Rev. Lett. 85, 5615 (2000).
[32] M. Aßmann, J. Thewes, D. Fröhlich, and M. Bayer, Nature materials 15, 741 (2016).
[33] A. Frisch, M. Mark, K. Aikawa, F. Ferlaino, J. L. Bohn, C. Makrides, A. Petrov, and S. Kotochigova, Nature 507, 475 (2014).
[34] W. Zhou, Z. Chen, B. Zhang, C. Yu, W. Lu, and S. Shen, Phys. Rev. Lett. 105, 024101 (2010).
[35] L. Vina, M. Potemski, and W. Wang, Physics-Uspekhi 41, 153 (1998).
[36] G. E. Mitchell, A. Richter, and H. A. Weidenmüller, Reviews of Modern Physics 82, 2845 (2010).
[37] P. Kos, M. Ljubotina, and T. Prosen, Phys. Rev. X 8, 021062 (2018).
[38] B. Bertini, P. Kos, and T. Prosen, Phys. Rev. Lett. 121, 264101 (2018).
[39] R. Dübertrand and S. Müller, New Journal of Physics 18, 033009 (2016).
[40] P. Saad, S. H. Shenker, and D. Stanford, arXiv preprint arXiv:1806.06840 (2018).
[41] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, J. High Energy Phys. 2017, 118 (2017).
[42] A. Altland and D. Bagrets, Nuclear Physics B 930, 45 (2018).
[43] Y.-Z. You, A. W. Ludwig, and C. Xu, Phys. Rev. B 95, 115150 (2017).
[44] A. M. García-García, Y. Jia, J. J. Verbaarschot, et al., Phys. Rev. D. 97, 106003 (2018).
[45] P. H. C. Lau, C.-T. Ma, J. Murugan, and M. Tezuka, Physics Letters B 795, 230 (2019).
[46] P. H. C. Lau, C.-T. Ma, J. Murugan, and M. Tezuka, arXiv preprint arXiv:2003.05401 (2020).
[47] A. Kamenev and M. Mézard, Journal of Physics A: Mathematical and General 32, 4373 (1999).
[48] F. Haake, Quantum Signatures of Chaos (Springer-Verlag Berlin Heidelberg, 2010).
[49] M. V. Berry, Semiclassical theory of spectral rigidity, Proc. R. Soc. London, Ser. A 400, 229 (1985).
[50] J. Cotler, N. Hunter-Jones, J. Liu, and B. Yoshida, J. High Energy Phys. 2017, 48 (2017).
[51] J. Liu, Phys. Rev. D. 98, 086026 (2018).
[52] See Supplemental Material for the detailed derivation.
[53] H. Bateman, Higher Transcendental Functions [Volumes I-III], Vol. 2 (McGraw-Hill Book Company, 1953).
[54] A. Kitaev and S. J. Suh, J. High Energy Phys. 2018, 183 (2018).
[55] J. Maldacena and D. Stanford, Phys. Rev. D 94, 106002 (2016).
[56] This result was obtained independently by Brian Swingle, Michael Winer and Shaokai Jian using a different method (private communication).
In this supplemental material, we show the detailed derivation of the spectral form factor (SFF) for an ensemble of noninteracting fermion systems described by the Hamiltonian
\[ \hat{H} = \sum_{i,j=1}^{N} \hat{f}_i^\dagger (h_{ij} - \mu \delta_{ij}) \hat{f}_j, \] (S1)
where \( h \) is a random \( N \times N \) Hermitian matrix from the Gaussian Unitary ensemble (GUE), which has the distribution function
\[ P(h) = 2^N (\frac{N^2-1}{2\pi})^{N^2/2} \exp \left( -\frac{N}{2} \text{Tr}(h^2) \right). \] (S2)

The SFF \( K(t) \) is defined as
\[ K(t) = \langle Z(it)Z(-it) \rangle = \left\langle \sum_{\alpha,\beta} \exp[-it(E_{\alpha} - E_{\beta})] \right\rangle, \] (S3)
where \( Z(it) = \text{Tr}e^{-itH} \) is the partition function at imaginary inverse temperature \( \beta = it \), and the angular bracket stands for ensemble averaging. \( \{E_{\alpha}\} \) represent the many-body energy levels which are related to the single-particle energy levels \( \{\varepsilon_i\} \) through
\[ E_{\alpha} = \sum_{n=1}^{N} n_{i}^{\alpha} (\varepsilon_i - \mu), \quad n_{i}^{\alpha} = 0, 1. \] (S4)

Inserting Eq. (S4) into the definition of the many-body SFF \( K(t) \) Eq. (S3), we find
\[ K(t) = 2^N \left\langle \prod_{i=1}^{N} \{1 + \cos[(\varepsilon_i - \mu)t]\} \right\rangle, \] (S5)
which can be expressed as
\[ K(t) = 2^N \int d\varepsilon_1 ... d\varepsilon_N P(\varepsilon_1, ..., \varepsilon_N) \prod_{i=1}^{N} \{1 + \cos[(\varepsilon_i - \mu)t]\}. \] (S6)

Here \( P(\varepsilon_1, ..., \varepsilon_N) \) is the joint probability density function of the \( N \) single-particle energy levels \( \{\varepsilon_i\} \) and is given by
\[ P(\varepsilon_1, \varepsilon_2, ..., \varepsilon_N) = C_N \exp(-\frac{N}{2} \varepsilon_i^2) \prod_{1 \leq i < j \leq N} |\varepsilon_i - \varepsilon_j|^2, \] (S7)
where \( C_N \) is a normalization constant.

The rest of the supplement is organized as follows. In Sec. I, we discuss an approach based on the level correlation and cluster functions of the GUE, which lends itself well to estimating the initial slope region and the beginning of the ramp region (broadly, small-\( t \) behavior). In Sec. II, we instead express the SFF as a determinant involving Laguerre polynomials, which allows us to derive the ramp at very late times and the plateau time (broadly, large-\( t \) behavior). In Sec. III, we consider a path integral based \( \sigma \)-model, and examine how much of the behavior of the SFF can be extracted from this method - which has the advantage that it has a more natural generalization to systems of interacting fermions.
I. CLUSTER FUNCTION APPROACH

A. Correlation and Cluster Functions

Using the fact that the joint probability density is symmetric under the permutation of its arguments, i.e., \( P(\ldots, \varepsilon_i, \ldots, \varepsilon_j, \ldots) = P(\ldots, \varepsilon_j, \ldots, \varepsilon_i, \ldots) \), we rewrite Eq. (S6) as

\[
K(t) = 2^N \left[ 1 + \sum_{n=1}^{N} \frac{1}{n!} \int \mathcal{R}_n(\varepsilon_1, \ldots, \varepsilon_n) \prod_{i=1}^{n} \cos[(\varepsilon_i - \mu) t] \, d\varepsilon_i \right].
\]  

(S8)

Here \( \mathcal{R}_n(\varepsilon_1, \ldots, \varepsilon_n) \) is the \( n \)-point single-particle energy level correlation function, and is defined as

\[
\mathcal{R}_n(\varepsilon_1, \ldots, \varepsilon_n) = \frac{N!}{(N - n)!} \int d\varepsilon_{n+1} \ldots d\varepsilon_N P(\varepsilon_1, \ldots, \varepsilon_N).
\]  

(S9)

It gives the probability density of finding a energy level around each \( \varepsilon_i \), irrespective of the remaining levels and independent of the labeling. It has been found that, for energy level described by the GUE probability distribution (Eq. (S7)), \( \mathcal{R}_n(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n) \) is given by the determinant of the kernel \( K(\varepsilon_i, \varepsilon_j) \) \cite{S1},

\[
\mathcal{R}_n(\varepsilon_1, \varepsilon_2 \ldots \varepsilon_n) = \det [K(\varepsilon_i, \varepsilon_j)]_{i,j=1,\ldots,n},
\]  

(S10)

where \( K(\varepsilon_i, \varepsilon_j) \), in the large \( N \) limit, takes the form of

\[
K(\varepsilon_i, \varepsilon_j) = \begin{cases} 
\mathcal{R}_1(\varepsilon_i) = \frac{N}{2\pi} \sqrt{4 - \varepsilon_i^2} \Theta(2 - |\varepsilon_i|), & i = j. \\
K(\varepsilon_i - \varepsilon_j) = \frac{N \sin[N(\varepsilon_i - \varepsilon_j)]}{N(\varepsilon_i - \varepsilon_j)}, & i \neq j.
\end{cases}
\]  

(S11)

Here \( \mathcal{R}_1(\varepsilon_i) \) is the average single-particle level density and exhibits Wigner’s semicircle law. We note that the 2-point many-body SFF \( K(t) \) is given by the summation of the Fourier transform of the \( n \)-point single-particle energy level correlation function which is closely related to the single-particle \( n \)-point SFF, with \( n \) running over 1, \ldots, \( N \).

It is convenient to define the \( n \)-point cluster function \( \mathcal{T}_n(\varepsilon_1, \ldots, \varepsilon_n) \) by excluding the lower order correlation \cite{S1}:

\[
\mathcal{T}_n(\varepsilon_1, \ldots, \varepsilon_n) = \sum_{\mathcal{G}} (-1)^{n-|\mathcal{G}|} (|\mathcal{G}| - 1)! \prod_{j=1}^{|\mathcal{G}|} \mathcal{R}_{|\mathcal{G}_j|}(\{\varepsilon_k, k \in \mathcal{G}_j\}).
\]  

(S12)

Here \( \mathcal{G} \) represents a partition of the indices \{1, 2, \ldots, \( n \)\} into \( |\mathcal{G}| \) subgroups \{\( \mathcal{G}_i, i = 1, \ldots, |\mathcal{G}| \)\}, with each group \( \mathcal{G}_i \) of length \( |\mathcal{G}_i| \). It obeys the constraint \( \sum_{i=1}^{|\mathcal{G}|} |\mathcal{G}_i| = n \), with \( |\mathcal{G}| \) being the number of subgroups in the partition. From Eqs. (S12) and (S10), one can deduce the form of the \( n \)-point cluster-function \( \mathcal{T}_n \) \cite{S1}:

\[
\mathcal{T}_n(\varepsilon_1, \ldots, \varepsilon_n) = \sum_{\mathcal{P}(n)} K(\varepsilon_1, \varepsilon_2) K(\varepsilon_2, \varepsilon_3) \ldots K(\varepsilon_{n-1}, \varepsilon_n) K(\varepsilon_n, \varepsilon_1),
\]  

(S13)

where the summation is over \((n - 1)!\) cyclic permutations \( \mathcal{P}(n) \) of indices \{1, 2, \ldots, \( n \)\}.

We then define

\[
\bar{\varepsilon}_n = \int d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n \mathcal{R}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \prod_{i=1}^{n} \cos[(\varepsilon_i - \mu) t],
\]  

(S14)

\[
\bar{\varepsilon}_n = \int d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n \mathcal{T}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \prod_{i=1}^{n} \cos[(\varepsilon_i - \mu) t].
\]  

The many-body SFF \( K(t) \) can now be expressed as

\[
K(t) = 2^N \left[ 1 + \sum_{n=1}^{N} \frac{1}{n!} \bar{\varepsilon}_n \right] = 2^N \exp \left( \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n!} \bar{t}_n \right),
\]  

(S15)

where in the second equality, we have approximated the upper limit \( N \) with infinity, and use the relation between the correlation function \( \mathcal{R}_n(\varepsilon_1, \ldots, \varepsilon_n) \) and the cluster function \( \mathcal{T}_n(\varepsilon_1, \ldots, \varepsilon_n) \) (Eq. (S12)).
B. Calculation of \( \check{t}_n \)

In this section, we calculate \( \check{t}_n \) defined in Eq. (S14), which can be used to obtain the SFF \( K(t) \) using Eq. (S15). Substituting Eq. (S13) into Eq. (S14) leads to

\[
\check{t}_n = \frac{1}{2^n} \sum_{\{\zeta_i = \pm 1\}} \int d\varepsilon_1 d\varepsilon_2 ... d\varepsilon_n e^{i t \sum_{\zeta_i = 1}^{n} (\varepsilon_i - \varepsilon_j) \zeta_i} \left[ \sum_{\mathcal{P}(n)} K(\varepsilon_1, \varepsilon_2)K(\varepsilon_2, \varepsilon_3)...K(\varepsilon_{n-1}, \varepsilon_n)K(\varepsilon_n, \varepsilon_1) \right] \]
\[
= \frac{(n - 1)!}{2^n} \sum_{\{\zeta_i = \pm 1, i=1,...,n\}} I_n(\{\zeta_i\}) e^{-i t \sum_{\zeta_i = 1}^{n} \zeta_i},
\]

where \( I_n(\{\zeta_i\}) \) is defined as

\[
I_n(\{\zeta_i\}) \equiv \int d\varepsilon_1 ... d\varepsilon_n K(\varepsilon_1, \varepsilon_2)K(\varepsilon_2, \varepsilon_3)...K(\varepsilon_n, \varepsilon_1) e^{i t \sum_{\zeta_i = 1}^{n} \varepsilon_i \zeta_i}.
\]

For \( n = 1 \), it is easy to see that \( I_1(\zeta_1) \) is the Fourier transform of the average single-particle level density \( R_1(\varepsilon) \):

\[
I_1(\zeta_1) = \int d\varepsilon R_1(\varepsilon) \exp (i t \varepsilon \zeta_1) = N \frac{J_1(2t \varepsilon_0)}{t},
\]

where \( J_1(x) \) is the Bessel function of the first kind and admits the asymptotic form of \( J_1(x) = \sqrt{\frac{2}{\pi x}} \cos(x - 3\pi/4) \) for \( x \to \infty \).

To evaluate \( I_n(\{\zeta_i\}) \) for \( n \geq 2 \), we perform the following transformation

\[
u_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1}, & i = 1, ..., n - 1, \\ \varepsilon_n, & i = n. \end{cases}
\]

Under this transformation, one has

\[
\sum_{i=1}^{n} \varepsilon_i \zeta_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} u_j \right) \zeta_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} \zeta_j \right),
\]

which leads to

\[
I_n(\{\zeta_i\}) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} du_1 ... du_{n-1} \int_{-\pi/2}^{\pi/2} du_n K(u_1) ... K(u_{n-1})K(- \sum_{i=1}^{n-1} u_i + \sum_{i=1}^{n} \zeta_i) \int_{-\pi/2}^{\pi/2} \frac{dk}{2\pi} \exp \left[ ik \left( u + \sum_{i=1}^{n-1} u_i \right) \right]. \tag{S21}
\]

Here we have employed the box approximation explained in Ref. [S2, S3]. We note that \( K(\varepsilon_i \neq \varepsilon_j) \) takes the form of Eq. (S11) only for \( \varepsilon_i, j \) close to the origin \( \varepsilon = 0 \) such that \( R_1(\varepsilon) \) can be approximated by \( R_1(0) \). As a result, we have to impose a cut-off \( |\varepsilon| \leq \pi/2 \) which is determined by the normalization condition \( \int_{-\pi/2}^{\pi/2} R_1(0) = N \). We then extend the integration region of \( u_j, j = 1, ... , n - 1 \) to the entire space due to the presence of \( K(u_j) \) which decays rapidly as \( |u_j| \) increases.

After rewriting \( K(- \sum_{i=1}^{n-1} u_i) \) as a two variable integral

\[
K(- \sum_{i=1}^{n-1} u_i) = \int_{-\infty}^{\infty} du K(u) \delta(u + \sum_{i=1}^{n-1} u_i) = \int_{-\infty}^{\infty} du \frac{dk}{2\pi} \exp \left[ ik \left( u + \sum_{i=1}^{n-1} u_i \right) \right]. \tag{S22}
\]

and inserting it into Eq. (S21), one arrives at

\[
I_n(\{\zeta_i\}) = \int_{-\pi/2}^{\pi/2} du_n e^{i u \sum_{\zeta_i = 1}^{n} \zeta_i} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} du_1 ... du_{n-1} e^{i ku + \sum_{i=1}^{n-1} u_i (t \sum_{j=1}^{i} \zeta_j + k)} K(u) \prod_{i=1}^{n-1} K(u_i)
\]
\[
= \sin \left[ \pi \left( t \sum_{j=1}^{n} \zeta_j \right) \right] \int_{-\infty}^{\infty} \frac{dk}{2\pi} \prod_{j=1}^{n-1} \tilde{K}(k) \prod_{i=1}^{n-1} \tilde{K}(k + t \sum_{j=1}^{i} \zeta_i),
\]

\[
= \pi \left( t \sum_{j=1}^{n} \zeta_j \right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \prod_{j=1}^{n-1} \tilde{K}(k) \prod_{i=1}^{n-1} \tilde{K}(k + t \sum_{j=1}^{i} \zeta_i),
\]

\[
(S23)
\]
where $\tilde{K}(k)$ denotes the Fourier transform of the kernel $K(\varepsilon)$
\[
\tilde{K}(k) \equiv \int_{-\infty}^{\infty} du K(u)e^{iku} = \Theta \left(1 - \left|\frac{k}{N}\right|\right).
\] (S24)

Substituting the explicit form of $\tilde{K}(k)$ (Eq. (S24)) into Eq. (S23) then yields the following expression for $I_n(\{\zeta_i\})$ when $n \geq 2$:
\[
I_{n \geq 2}(\{\zeta_i\}) = N \frac{\sin \left[\frac{\pi}{2} \left(t \sum_{j=1}^{n} \zeta_j\right)\right]}{\pi \left(t \sum_{j=1}^{n} \zeta_j\right)} \left[1 - \frac{t}{2N} s(\{\zeta_i\})\right] \Theta \left[1 - \frac{t}{2N} s(\{\zeta_i\})\right].
\] (S25)

Here $s(\{\zeta_i\})$ is defined as the difference between the maximum and the minimum of the series $\{0, \sum_{i=1}^{j} \zeta_i\}$, for $j$ running over $j = 1, \ldots, n - 1$:
\[
s(\{\zeta_i\}) = \max\left\{0, \sum_{i=1}^{j} \zeta_i\right\} - \min\left\{0, \sum_{i=1}^{j} \zeta_i\right\}.
\] (S26)

**C. General expression for the SFF**

From previous calculation, we find the SFF $K(t)$ can be expressed as
\[
K(t) = 2^N \exp \left[\sum_{i=1}^{N} (-1)^{n-1} \frac{1}{n} \frac{1}{2^n} \sum_{\{\zeta_i = \pm 1\}} I_n(\{\zeta_i\})e^{-i\mu \sum_{i=1}^{n} \zeta_i}\right],
\] (S27)

where $\sum_{\{\zeta_i = \pm 1\}}$ denotes the summation over all possible configurations with $\zeta_i$ taking the value of $+1$ or $-1$ for $i = 1, \ldots, n$. $I_n(\{\zeta_i\})$ is given by Eq. (S18) and Eq. (S25) for $n = 1$ and $n \geq 2$, respectively.

At time $t = 0$ and $t \to \infty$, $I_n(\{\zeta_i\})$ takes the value of $N$ and $0$, respectively, for arbitrary configuration $\{\zeta_i\}$, which leads to $K(0) = L^2$ and $K(\infty) = L$, as expected from the definition of the SFF (Eq. (S3)). At large time $t \gg N$, we find that $I_n(\{\zeta_i\}) = 0$ for $n \geq 2$ due to presence of the unit step function, which yields
\[
K(t \gg N) = 2^N \exp \left(N \frac{J_1(2t)}{t} \cos(\mu t)\right) \approx L.
\] (S28)

To further simplify the calculation of $K(t)$, we then group together $I_n(\{\zeta_i\})$ with the same value of $p = \sum_{i=1}^{n} \zeta_i$, and rewrite $K(t)$ as
\[
K(t) = 2^N \exp \left\{N \frac{J_1(2t)}{t} \cos(\mu t) + A_0(t) + 2 \sum_{p=1}^{N} A_p(t)(-1)^p \frac{\sin(\pi tp/2)}{\pi tp/2} \cos(p\mu t)\right\},
\] (S29)

where $A_p(t)$ is defined as
\[
A_p(t) \equiv -N \sum_{n=2}^{N} \frac{1}{n} \frac{1}{2^n} \sum_{\sum_{i=1}^{n} \zeta_i = p} \left[1 - \frac{t}{2N} s(\{\zeta_i\})\right] \Theta \left[1 - \frac{t}{2N} s(\{\zeta_i\})\right].
\] (S30)

$\sum_{\sum_{i=1}^{n} \zeta_i = p}$ represents the summation over all configurations of $\{\zeta_i = \pm 1\}_{i=1}^{n}$ obeying the constraint $\sum_{i=1}^{n} \zeta_i = p$. We have used the fact that $\sum_{i=1}^{n} \zeta_i = p$ can only be satisfied for even $n - p$, and also $A_p(t) = A_{-p}(t)$ due to symmetry.

Eq. (S29) is valid at all time $t$. We now turn to the regime $t \ll N$, and approximate all unit step functions in $A_p(t)$ with 1. We note that for a small number of configurations, $s(\{\zeta_i\})$ is of the order of $N$, and $\Theta \left[1 - \frac{t}{2N} s(\{\zeta_i\})\right]$ becomes zero for $t \gg 1$. However, because of the overall factor $1/n2^n$ in the summation and the fact that the number
of configurations with \( s(\{\zeta_i\}) \sim O(N) \) is small, we ignore such situations, and set all \( \Theta \left[ 1 - \frac{t}{2N} s(\{\zeta_i\}) \right] \) in Eq. (S30) to 1. \( A_p(t) \) is linear in \( t \), and takes the form of

\[
A_p(t) = -B_p + C_p t,
\]

(S31a)

\[
B_p = N \sum_{m=m_0}^{N} \frac{1}{2m + |p|} \frac{1}{2^{2m+|p|}} \frac{(2m + |p|)!}{(m + |p|)!m!},
\]

(S31b)

\[
C_p = \sum_{n=2}^{N} \frac{1}{2^{n+1}} \sum_{i_1, \ldots, i_{|p|} = p}^{n} s(\{\zeta_i\}),
\]

(S31c)

where the lower limit \( m_0 \) in the summation of Eq. (S31b) is given by 0 for integer \( |p| > 1 \), and 1 for \( |p| \leq 1 \).

It is straightforward to see that, in the large \( N \) limit, \( B_p \) is given by

\[
B_p = \begin{cases} 
N \ln 2, & p = 0, \\
\frac{N}{2} |p|, & |p| = 1, \\
\frac{N}{|p|}, & |p| > 1.
\end{cases}
\]

(S32)

By contrast, the calculation of \( C_p \) (Eq. (S31c)) is complicated and can be mapped to a random walk problem. We rewrite the summation of \( s(\{\zeta_i\}) \) as the difference of two separate summations:

\[
\sum_{\sum_{i=1}^{n} \zeta_i = p} s(\{\zeta_i\}) = \sum_{\sum_{i=1}^{n} \zeta_i = p} s_a(\{\zeta_i\}) - \sum_{\sum_{i=1}^{n} \zeta_i = p} s_b(\{\zeta_i\}),
\]

(S33)

where

\[
s_a(\{\zeta_i\}) = \max \left\{ 0, \sum_{i=1}^{j} \zeta_i \right\}_{j=1}^{n-1}, \quad s_b(\{\zeta_i\}) = \min \left\{ 0, \sum_{i=1}^{j} \zeta_i \right\}_{j=1}^{n-1}.
\]

(S34)

**D. Calculation of \( C_0 \)**

Let us first consider the case \( p = 0 \). Because of the symmetry, the two sums on the right-hand side of Eq. (S33) are opposite with respect to each other and their difference can be further reduced to

\[
\sum_{\sum_{i=1}^{n} \zeta_i = 0} s(\{\zeta_i\}) = 2 \sum_{\sum_{i=1}^{n} \zeta_i = 0} s_a(\{\zeta_i\}),
\]

(S35)

where \( n \) has to be an even integer.

To evaluate this summation, for each configuration of \( \{\zeta_i\}_{i=1}^{n} \), we introduce a series \( \{x_i\} \):

\[
x_i = \begin{cases} 
0, & i = 0, \\
\sum_{j=1}^{i} \zeta_j, & i = 1, \ldots, n.
\end{cases}
\]

(S36)

where \( x_i \) represents the position at step \( i \). After each step, \( x_i \) is increased by \( \zeta_{i+1} \) which can only take the value of +1 or −1: \( x_{i+1} = x_i + \zeta_{i+1} \). Each configuration of \( \{\zeta_i\} \) satisfying the constraint \( \sum_{i=1}^{n} \zeta_i = 0 \) corresponds to a path that starts from \( x_0 = 0 \) and ends at \( x_n = 0 \) after \( n \) steps. The number of such paths is \( (n)!/(n/2)!(n/2)! \). In addition, \( s_a(\{\zeta_i\}) = \max \{x_i\}_{i=0}^{n-1} \) is the maximum position reached before the last step for path \( \{x_i\} \).

In order to calculate the summation of \( s_a(\{\zeta_i\}) \) for all the paths that start from \( x_0 = 0 \) and end at \( x_n = 0 \), we count the total number of the paths for which \( s_a(\{\zeta_i\}) \geq l \), denoted as \( N_n^{(0)}(s_a(\{\zeta_i\}) \geq l) \). If we plot \( x_i \) as a function of the step \( i \) for all the paths (see Fig. (S1)), those with \( s_a(\{\zeta_i\}) \geq l \) should cross the line \( x = l \) at least once at some intermediate steps, the smallest among which is called \( l_0 \). For each of these paths \( \{x_i\} \), we now define its reflection...
FIG. S1: Reflection path. The blue solid lines correspond to the original path defined by Eq. (S36), which begins from \((i = 0, x_0 = 0)\) and ends at \((i = n, x_n = 0)\). The red dashed lines represent the corresponding reflected path defined by Eq. (S37), which ends at \((i = n, x_n = 2l)\) instead. Starting from \(i_0\) the first step at which the original path reaches \(x_{i_0} = l\), the original path (blue solid lines) is reflected around the line \(x = l\) (gray dashed line) to obtain the reflected path (red dashed lines).

path \(\{x'_i\}\) where every step after \(i > i_0\) is reversed. The reflection path is symmetric around \(x = l\) for \(i > i_0\), while for \(i \leq i_0\), it remains the same:

\[
x'_i = \begin{cases} 
  x_i, & i \leq i_0, \\
  2l - x_i, & i > i_0.
\end{cases}
\] (S37)

As shown in Fig. (S1), the reflection path (represented by the red dashed line) then reach \(x'_n = 2l\) after \(n\) steps since the original path (represented by the blue solid line) ends at \(x_n = 0\). The total number of reflection paths that reach \(x'_n = 2l\) is the same to the number of the original paths that cross the line \(x = l\) (which starts at \(x_0 = 0\) and ends at \(x_n = 0\)), and equals the number of the paths with \(s_a(\{\zeta_i\}) \geq l\):

\[
N_n^{(0)}(s_a(\{\zeta_i\}) \geq l) = \frac{n!}{(\frac{n}{2} + l)! (\frac{n}{2} - l)!}.
\] (S38)

Here \(l\) is a positive integer with \(1 \leq l \leq n/2\).

We note that

\[
\sum_{l=1}^{n/2} \sum_{k=1}^{n/2} N_n^{(0)}(s_a(\{\zeta_i\}) = k) = \sum_{k=1}^{n/2} k N_n^{(0)}(s_a(\{\zeta_i\}) = k),
\] (S39)

where \(N_n^{(0)}(s_a(\{\zeta_i\}) = k)\) denotes the number of path with \(s_a(\{\zeta_i\}) = k\). The above equation gives

\[
\sum_{i=1}^{n/2} s_a(\{\zeta_i\}) = \sum_{i=1}^{\lfloor N/2 \rfloor} \frac{n!}{(\frac{n}{2} + l)! (\frac{n}{2} - l)!} = \frac{1}{2} \left( 2^n - \frac{n!}{(\frac{n}{2})! (\frac{n}{2})!} \right).
\] (S40)

Inserting Eq. (S40) into Eq. (S31c) while setting \(p = 0\), one obtains, in the large \(N\) limit,

\[
C_0 = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{1}{4m} \frac{1}{2m} \left[ 2^{2m} - \frac{(2m)!}{m!m!} \right] = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{1}{4m} - \frac{1}{2} \ln 2
\] (S41)

\[
= \frac{1}{4} \left[ \psi_0(N/2 + 1) + \gamma_E \right] - \frac{1}{2} \ln 2 = \frac{1}{4} \left( \ln \frac{N}{8} + \gamma_E \right),
\]

where \(\psi_0(x)\) is the digamma function and \(\gamma_E \approx 0.577\) is the Euler-Mascheroni constant.

E. Calculation of \(C_p\) for \(p > 0\)

\(C_p\) for \(p > 0\) can be evaluated in an analogous way as \(C_0\). We now consider the paths \(\{x_i\}\) (defined in Eq. (S36)) that start from \(x_0 = 0\) and end at \(x_n = p\) (therefore satisfy \(\sum_{i=1}^{n} \zeta_i = p\)), for non-negative even \(n - p\). \(s_a(\{\zeta_i\}) = \max \{x_i\}_{i=0}^{n-1}\) is again the maximum position reached before the \(n\)-th step.
Using the reflection path method introduced in the previous section, we calculate the number of such paths with \( s_a(\{\zeta_i\}) \geq l \), denoted as \( N_n^{(p)}(s_a(\{\zeta_i\}) \geq l) \). In the \( i \times x \) plane, these paths cross the line \( x = l \) at least once for \( p \leq l \leq \frac{n + p}{2} \). Their reflection paths defined by Eq. (S37) should reach \( x_n = 2l - p \) after \( n \) steps, and can be used to find the total number of path with \( s_a(\{\zeta_i\}) \geq l \) for \( p + 1 \leq l \leq \frac{n + p}{2} \):

\[
N_n^{(p)}(s_a(\{\zeta_i\}) \geq l) = \frac{n!}{(\frac{n+p}{2} + l)(\frac{n+p}{2} - l)!}.
\]

For \( l = p \), the above expression is no longer valid. Instead, setting \( l \) on the right-hand side of the above equation to \( p \) gives \( N_n^{(p)}(s_a(\{\zeta_i\}) \geq p - 1) \), which is also the total number of \( n \)-step paths with \( x_0 = 0 \) and \( x_n = p \). We have to consider the situation \( s_a(\{\zeta_i\}) = p \) separately. In this case, it is easy to see that \( x_{n-1} = p - 1 \). The number of paths that start from \( x_0 = 0 \) and end at \( x_{n-1} = p - 1 \) while satisfying \( s_a(\{\zeta_i = 1, \ldots, n\}) = p \) is equal to the number of paths that start from \( x_0 = 0 \) and end at \( x_n = p \) while satisfying \( s_a(\{\zeta_i = 1, \ldots, n\}) = p \), and is given by

\[
N_n^{(p)}(s_a(\{\zeta_i\}) = p) = N_n^{(p-1)}(s_a(\{\zeta_i\}) = p) = N_n^{(p-1)}(s_a(\{\zeta_i\}) \geq p) - N_n^{(p-1)}(s_a(\{\zeta_i\}) \geq p + 1)
\]

\[
= \frac{(n-1)!}{(\frac{n+p}{2}+l)(\frac{n+p}{2}-l)!} - \frac{(n-1)!}{(\frac{n+p}{2}+p)(\frac{n+p}{2}-p)!} = \frac{(p+2)(n-1)!}{(\frac{n+p}{2}+2)(\frac{n+p}{2}-2)!},
\]

for \( n \geq p + 4 \). Here in the last equality, we have used Eq. (S42) while replacing \( n \) with \( n - 1 \) and \( p \) with \( p - 1 \). It is straightforward to see \( N_n^{(p)}(s_a(\{\zeta_i\}) = p) \) vanishes for \( n = p \), and equals 1 for \( n = p + 2 \).

We note that, for \( n \geq p + 4 \),

\[
\sum_{l=p}^{\frac{n+p}{2}} N_n^{(p)}(s_a(\{\zeta_i\}) \geq l) = \sum_{l=p-1}^{\frac{n+p}{2}} \sum_{k=l}^{\frac{n+p}{2}} N_n^{(p)}(s_a(\{\zeta_i\}) = k) = \sum_{k=p-1}^{\frac{n+p}{2}} (k-p+2)N_n^{(p)}(s_a(\{\zeta_i\}) = k)
\]

\[
= \sum_{i=1}^{\frac{n+p}{2}} s_a(\{\zeta_i\}) - (p-2)\frac{n!}{(\frac{n+p}{2})!(\frac{n-p}{2})!},
\]

which leads to

\[
\sum_{\sum_{i=1}^{\frac{n+p}{2}} \zeta_i = p} s_a(\{\zeta_i\}) = \sum_{l=p}^{\frac{n+p}{2}} N_n^{(p)}(s_a(\{\zeta_i\}) \geq l) + N_n^{(p)}(s_a(\{\zeta_i\}) \geq p + 1) + N_n^{(p)}(s_a(\{\zeta_i\}) = p) + (p-1)\frac{n!}{(\frac{n+p}{2})!(\frac{n-p}{2})!}
\]

\[
= \sum_{l=p}^{\frac{n+p}{2}} \sum_{k=l}^{\frac{n+p}{2}} n! \frac{(n-2)!}{(\frac{n+p}{2}+l)(\frac{n+p}{2}-l)!} + (p+2)\frac{n!}{(\frac{n+p}{2}+p)(\frac{n+p}{2}-p)!} + (p-1)\frac{n!}{(\frac{n+p}{2}+2)(\frac{n+p}{2}-2)!}
\]

\[
= \sum_{l=1}^{\frac{n+p}{2}} \frac{n!}{(\frac{n+p}{2}+l)(\frac{n-p}{2})!} + p(n-1)\frac{(n-1)!}{(\frac{n+p}{2})!(\frac{n-p}{2})!},
\]

\[
\sum_{\sum_{i=1}^{\frac{n+p}{2}} \zeta_i = p} s_a(\{\zeta_i\}) \text{ takes the value of } p - 1 \text{ and } 1 + p + p^2, \text{ respectively.}
\]

Similarly, with the help of the reflection path method, we find the number of paths that begin from \( x_0 = 0 \) and end at \( x_n = p \) after \( n \) steps while satisfying \( s_b(\{\zeta_i\}) = \min \{x_i, i = 0, \ldots, n-1\} \leq -l \) (for \( 1 \leq l \leq \frac{n-p}{2} \)):

\[
N_n^{(p)}(\text{min } \{x_i\} \leq -l) = \frac{n!}{(\frac{n-p}{2} - l)(\frac{n+p}{2} + l)!},
\]

This leads to

\[
\sum_{\sum_{i=1}^{\frac{n+p}{2}} \zeta_i = p} s_b(\{\zeta_i\}) = \begin{cases} \sum_{l=1}^{\frac{n-p}{2}} \frac{n!}{(\frac{n-p}{2} - l)(\frac{n+p}{2} + l)!}, & n \geq p + 2, \\ 0, & n = p. \end{cases}
\]
Inserting the Eqs. (S45) and (S47) into Eq. (S31c), we find, for odd \( p > 0 \),
\[
C_p = \sum_{m=0}^{(N-1)/2} \frac{1}{2m+1} \frac{1}{2^{2m+1}} \left[ \sum_{l=1}^{2m} \frac{(2m+1)!}{(m+l+\frac{1}{2})!(m-l+\frac{1}{2})!} + p(2m) \frac{(2m)!}{(m+\frac{1}{2})!(m+\frac{1}{2})!} \right] \\
+ \frac{1}{p+2} \frac{1}{2^{p+3}} (2p + p^2) + \frac{1}{p+1} (p-1) \Theta(p-1),
\]
while for even \( p > 0 \)
\[
C_p = \sum_{m=\frac{p+1}{2}}^{[N/2]} \frac{1}{2m} \frac{1}{2^{2m+1}} \left[ \sum_{l=1}^{2m} \frac{(2m)!}{(m+l+\frac{1}{2})!(m-l+\frac{1}{2})!} + p(2m-1) \frac{(2m-1)!}{(m+\frac{1}{2})!(m-\frac{1}{2})!} \right] \\
+ \frac{1}{p+2} \frac{1}{2^{p+3}} (2p + p^2) + \frac{1}{p+1} (p-1).
\]
We note that the first term in Eq. (S48) is smaller than
\[
\sum_{m=0}^{(N-1)/2} \frac{1}{2m+1} \frac{1}{2^{2m+1}} 2^{2m+1} = \frac{1}{2} \sum_{m=0}^{(N-1)/2} \frac{1}{2m+1} = \frac{1}{4} \left[ \psi_0(N/2 + 1) + \gamma_E + \ln 4 \right] = \frac{1}{4} (\ln 2N + \gamma_E),
\]
while the first term in Eq. (S49) is smaller than
\[
\sum_{m=1}^{[N/2]} \frac{1}{2m} \frac{1}{2^{2m+1}} 2^{2m} = \frac{1}{4} \sum_{m=1}^{[N/2]} \frac{1}{m} = \frac{1}{4} [\psi_0(N/2 + 1) + \gamma_E] = \frac{1}{4} \left( \ln \frac{N}{2} + \gamma_E \right). 
\]

The remaining terms in both Eq. (S48) and Eq. (S49) converge in the large \( N \to \infty \) limit. As a result, one could draw the conclusion that \( C_p \) for \( p > 0 \) is of the order of or smaller compared with \( C_0 \).

### F. Results

From previous calculations, we find the SFF \( K(t) \) can be expressed as Eq. (S29), where \( A_p(t) \) admits the form of \( A_p(t) = -B_p + C_p t \) in the regime of \( t \ll N \). The coefficient \( B_p \) is given by Eq. (S32), while \( C_p \) is given by the formulas in Eq. (S48) and Eq. (S49) for odd and even positive integer \( p \), respectively.

In the regime \( t \ll 1 \), \( C_p t \) is much smaller compared with \( B_p \), and as a result \( A_p(t) \) can be approximated by the constant \(-B_p\). Combining Eq. (S32) and Eq. (S29), we find
\[
K(t) = 2^N \exp \left\{ N \frac{J_1(2t)}{t} \cos(\mu t) - N \ln 2 - 2N \sum_{p=1}^{N} \frac{1}{p} \frac{\sin(\pi tp/2)}{\pi tp/2} \cos(p\mu t) - N \frac{\sin(\pi t/2)}{\pi t/2} \cos(\mu t) \right\} \\
= 2^N \exp \left\{ N \left( \frac{J_1(2t)}{t} - 1 \right) \cos(\mu t) + N \ln \left( 1 + \cos(\mu t) \right) / 2 \right\},
\]
where in the second equality, we have approximated \( \sin(\pi tp/2) / \pi tp/2 \) by 1 for \( t \ll 1 \).

For \( 1 \ll t \ll N \), \( A_p(t)(-1)^p \sin(\pi tp/2) / (\pi tp/2) \cos(p\mu t) \) is much smaller compared with \( A_0(t) \), and is highly oscillating as a function of \( p \geq 1 \). Moreover, the overall sign factor \((-1)^p\) is oscillating as well. Therefore, the summation of \( A_p(t)(-1)^p \sin(\pi tp/2) / (\pi tp/2) \cos(p\mu t) \), i.e., the last term in the exponent of Eq. (S29), is negligible, and we have
\[
K(t) = \exp \left[ N \frac{J_1(2t)}{t} \cos(\mu t) + t / 4 \left( \ln \frac{N}{8} + \gamma_E \right) \right]. 
\]

For time \((N/\ln N)^{2/5} \ll t \ll N\), the first term in the exponent of the above equation is much smaller compared with the second term there \([A_0(t)]\), and as a result, \( K(t) \) grows as \( K(t) = (Ne^{\gamma_E}/8)^{t/4} \) in this regime. By contrast, for \( 1 \ll t \ll (N/\ln N)^{2/5} \), the first term dominants, and the SFF decays as \( K(t) = \exp \left( N \frac{J_1(2t)}{t} \cos(\mu t) \right) \).
In summary, using the cluster function approach, we find the many-body SFF \( K(t) \) decays rapidly as
\[
L^2 \exp \left\{ N \left( \frac{J(t^2)}{t} - 1 \right) \cos(\mu t) + N \ln \left( \frac{1 + \cos(\mu t)}{2} \right) \right\}
\]
for early times \( t \ll 1 \), and then it continues to drop as
\[
L \exp \left[ N \frac{J(t^2)}{t} \cos(\mu t) \right]
\]
for \( 1 \ll t \ll (N/\ln N)^{2/5} \). As \( t \) becomes larger and lies within the regime \( (N/\ln N)^{2/5} \ll t \ll N \), \( K(t) \) starts to grow instead of decaying, and exhibits a ramp that scales as \( (Ne^{\gamma_E}/8)^{1/4} \). Finally, for \( t \gg N \), the SFF should reach the plateau at \( K(t) = L \). However, it is difficult to extract the behavior for \( K(t) \) around \( t = N \) due to the presence of the unit step function in the expression of \( A_p(t) \). In fact, we note that at \( t \sim 8N/\log_2(Ne^{\gamma_E}/8) \), the ramp result \( (Ne^{\gamma_E}/8)^{1/4} \) exceeds \( L^2 \) which is the upper bound for the many-body SFF. Therefore, the expression in Eq. (S53) is valid only for some \( t < \eta \) where \( \eta < O(N/\log_2 N) \). In the following, we use a different approach to study the behavior of the many-body SFF \( K(t) \) around \( t = O(N) \).

II. LAGUERRE POLYNOMIALS APPROACH

A. The SFF in determinant form

The Wigner-Dyson distribution for \( N \) GUE levels is given by Eq. (S7). Following Ref. [S1], we express this in terms of arbitrary polynomials \( C_k(x) \) of degree \( k - 1 \) (with arbitrary coefficients, which we will later compensate for by normalizing \( P(\varepsilon_1, ..., \varepsilon_N) \)). Introducing permutation operators \( I, J \) that act on \( \{1, ..., N\} \) and the symbol \( \epsilon_{\{i\}}^{(j)} = +1, -1 \) if \( I, J \) are of the same or opposite parities (respectively), we have
\[
P(\varepsilon_1, ..., \varepsilon_N) = \left( \prod_i e^{-N\varepsilon_i^2/2} \right) \left( \sum_{i,j} \epsilon_{\{i\}}^{(j)} \prod_{k=1}^N C_k(\varepsilon_{I_k})C_k(\varepsilon_{J_k}) \right)
\]
(S54)
\[
= \sum_{i,j} \epsilon_{\{i\}}^{(j)} \prod_{k=1}^N e^{-N\varepsilon_k^2/2}C_{I_k}(\varepsilon_k)C_{J_k}(\varepsilon_k),
\]
(S55)
again up to normalization. The operators \( i, j \) in the second line are the inverses of \( I, J \) from the corresponding terms in the first line.

It is convenient to choose \( \mathcal{C}_k(x) = \mathcal{H}_{k-1} \left( \sqrt{\frac{2}{N}} x \right) \), where \( \mathcal{H}_k(x) = (2^k k! \sqrt{\pi})^{-1/2} H_k(x) \) are the normalized Hermite polynomials (satisfying \( \int dx e^{-x^2} \mathcal{H}_i(x) \mathcal{H}_j(x) = \delta_{ij} \)). In that case, using Theorem 5.7.1 in Ref. [S1] to perform the integrals in the normalization condition,
\[
\int d\varepsilon_1 ... d\varepsilon_N P(\varepsilon_1, ..., \varepsilon_N) = 1,
\]
(S56)
we can show that the normalized Wigner-Dyson distribution is
\[
P(\varepsilon_1, ..., \varepsilon_N) = \frac{(N/2)^N}{N!} \sum_{i,j} \epsilon_{\{i\}}^{(j)} \prod_{k=1}^N e^{-N\varepsilon_k^2/2} \mathcal{H}_{k-1} \left( \sqrt{\frac{N}{2}} \varepsilon_k \right) \mathcal{H}_{k-1} \left( \sqrt{\frac{N}{2}} \varepsilon_k \right).
\]
(S57)

Inserting Eq. (S57) into Eq. (S6) and noting that all the integrals over the \( \varepsilon_k \) are of the same form, we obtain for the SFF (with the replacement \( \varepsilon_k \sqrt{N/2} \rightarrow x \) in each factor),
\[
K(t) = 2^N \frac{1}{N!} \sum_{i,j} \epsilon_{\{i\}}^{(j)} \prod_{k=1}^N \int dx e^{-x^2} \mathcal{H}_{k-1}(x) \mathcal{H}_{k-1}(x) \left( 1 + \cos \left( x \sqrt{\frac{2}{N}} - \mu \right) t \right).
\]
(S58)

Using the orthonormality of the \( \mathcal{H}_k(x) \), and defining,
\[
\mathcal{M}_{jk}(t) = \int dx e^{-x^2} \left\{ \mathcal{H}_{j-1}(x) \mathcal{H}_{k-1}(x) \cos \left[ x \sqrt{\frac{2}{N}} - \mu \right] t \right\},
\]
(S59)
and identifying \( (1/N!) \sum_{i,j} \epsilon_{\{i\}}^{(j)} A_{ij} = \det A \) for a matrix \( A \), we obtain,
\[
K(t) = 2^N \det \left[ \delta_{jk} + \mathcal{M}_{jk}(t) \right]_{j,k=1, ..., N}.
\]
(S60)
Expanding out \( \cos((\varepsilon - \mu)T) \) = \( \cos(\varepsilon T) \cos(\mu t) + \sin(\varepsilon T) \sin(\mu t) \), the resulting integrals can be evaluated with the help of results from standard tables (e.g. 7.388(6,7) in Ref. [S4]), and we get

\[
M_{j\geq k}(t) = W_{jk} \left( \frac{t^2}{N} \right) F_{jk}(\mu t),
\] (S61)

with \( M_{kj}(T) = M_{jk}(T) \), where

\[
W_{j\geq k}(\tau) = \sqrt{\frac{(k-1)!}{(j-1)!}} \tau^{\frac{j-k}{2}} e^{-\frac{\tau}{2}} L_{j-1}^{k-1}(\tau),
\] (S62)

with \( L_n^\alpha(x) \) denoting the Laguerre polynomials, and

\[
F_{jk}(\mu t) = \begin{cases} (-1)^{\frac{j-k}{2}} \cos(\mu t), & j - k \text{ is even}, \\ (-1)^{\frac{j-k-1}{2}} \sin(\mu t), & j - k \text{ is odd}. \end{cases}
\] (S63)

Using \( \det A = e^{\text{Tr} \ln A} \) in Eq. (S60) and expanding \( \ln A \) in a power series, we get

\[
K(t) = 2^N \exp \left\{ -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Tr} \left[ M^n(t) \right] \right\} .
\] (S64)

**B. The asymptotic behavior of Laguerre polynomials**

Now we will provide some additional background for the approximate form of Laguerre polynomials \( L_n^\alpha(x) \) used in Eq.(15) of the main text. Specifically, we are interested in the form for large \( n \), with large or small \( \alpha \). Defining \( \nu = 4n + 2\alpha + 2 \), the general behavior of the Laguerre polynomials can be split into the oscillatory region, \( x < \nu \), and the monotonic region, \( x > \nu \) [S5]. In the oscillatory region, we consider the leading term of Eq.(8) from Sec.10.15 in Ref. [S5], which amounts to

\[
e^{-\frac{\pi}{4} x^2} L_n^\alpha(x) \approx \sqrt{\frac{\nu}{\pi}} \left( \frac{\nu}{4} \right)^{\frac{\nu}{4}} \frac{\sin(\varphi_n^\alpha(x))}{(x(x-\nu))^\frac{\nu}{4}},
\] (S65)

where we have absorbed the overall sign into the oscillatory factor \( \sin(\varphi_n^\alpha(x)) \) whose specific form is unimportant for our purposes. In the monotonic region, we have (Eq.(15) form Sec.10.15 in Ref. [S5]),

\[
e^{-\frac{\pi}{4} x^2} L_n^\alpha(x) \approx \sqrt{\frac{\nu}{\pi}} \left( -1 \right)^n e^{-\frac{1}{2} \sqrt{x(x-\nu)} + \frac{\pi}{2} \cosh^{-1} \sqrt{\frac{x}{x(x-\nu)}}},
\] (S66)

which corresponds to a rapid decay to zero. Both of these expressions are valid for \( n \to \infty \), but without a corresponding \( \alpha \to \infty \) limit.

For our purposes, we may approximate the right hand side of Eq. (S66) by zero, and cover the entire region of \( x < \nu \) and \( x > \nu \) by the single approximate expression,

\[
e^{-\frac{\pi}{4} x^2} L_n^\alpha(x) \approx \sqrt{\frac{\nu}{\pi}} \left( \frac{\nu}{4} \right)^{\frac{\nu}{4}} \frac{\sin(\varphi_n^\alpha(x)) \Theta(\nu-x)}{(x(x-\nu))^\frac{\nu}{4}}.
\] (S67)

We see numerically (cf. Fig.(S2)) that Eq. (S67) is not a good approximation for larger \( \alpha \) (e.g. \( \alpha = O(n) \)), given that the \( \alpha \to \infty \) limit was not taken in the above standard results. We will need an expression that is valid for any \( \alpha \leq n \) for our application. Based on the form of Eq. (S62), we instead try the approximate expression,

\[
e^{-\frac{\pi}{4} x^2} L_n^\alpha(x) \approx \sqrt{\frac{\nu}{\pi}} \frac{(n+\alpha)!}{n!} \frac{\sin(\varphi_n^\alpha(x)) \Theta(\nu-x)}{(x(x-\nu))^\frac{\nu}{4}}.
\] (S68)

Numerically, it appears that this expression works better than Eq. (S67) in the desired range of \( n, \alpha \) (cf. Fig.(S2)), and also has the advantage of greatly simplifying the evaluation of the first term in the exponent of Eq. (S64).
FIG. S2: Comparison of approximations for the Laguerre polynomials $L_\alpha^n(x)$ for $n = 200, \alpha = 100$. "Standard envelope" refers to Eq. (S67), and "Modified envelope" to Eq. (S68), with the oscillatory factor $\sin(\varphi_n^\alpha(x))$ dropped and the overall sign set to +1 in both cases.

C. Evaluating the Trace

Here, we will evaluate the trace in Eq. (S64) for the $n = 2$ term, obtaining the late-time ramp. As noted in the main text, due to the oscillatory term in Eq. (S68), we can approximate all terms with odd $n$ to zero. Also, $M$ is a symmetric matrix, and therefore has real eigenvalues. $\text{Tr}[M^n]$ is then the sum of the $n$-th power of eigenvalues, and for even $n$ must always be positive. While it is hard to precisely evaluate the terms for $n = 4, 6, \ldots$, we can definitively say that their contribution decreases the SFF from the $n = 2$ estimate. Therefore, evaluating the $n = 2$ term alone would give us an (approximate) upper bound for the SFF. We have

$$\text{Tr}[M^2(t)] = \sum_{j,k=1}^{N} M_{jk}^2(t). \quad (S69)$$

It is convenient to separate out the sum into groups of terms with even or odd $j - k$,

$$\text{Tr}[M^2(t)] = \sum_{\text{even } j-k}^{N} M_{jk}^2(t) + \sum_{\text{odd } j-k}^{N} M_{jk}^2(t), \quad (S70)$$

As the sums range over $1, \ldots, N$, we expect the indices $j, k$ to typically be large when $N$ is large. Using Eqs. (S61), (S62), (S63) to write an explicit expression for $M_{jk} = M_{kj}$,

$$M_{j\geq k}(t) = \sqrt{\frac{(k-1)!}{(j-1)!}} \left( \frac{t^2}{N} \right)^{\frac{j-k}{2}} e^{-\frac{t^2}{2N} L_{k-1}^\alpha(x)} \begin{cases} (-1)^{\frac{j-k}{2}} \cos(\mu t), & j - k \text{ is even,} \\ (-1)^{\frac{j-k-1}{2}} \sin(\mu t), & j - k \text{ is odd,} \end{cases} \quad (S71)$$

together with the asymptotic expression for Laguerre polynomials, Eq. (S68), gives

$$M_{j \geq k}^2(t) = \frac{2\Theta(\nu_{j+k} - \tau)}{\pi \sqrt{\tau(\nu_{j+k} - \tau)}} \sin^2(\varphi_{j+k}(\tau)) \begin{cases} \cos^2(\mu t), & j - k \text{ is even,} \\ \sin^2(\mu t), & j - k \text{ is odd,} \end{cases} \quad (S72)$$

where $\tau = t^2/N$, and $\nu_{j+k} \approx 2(j + k)$ for large $j, k$.

Now, we consider the first sum in Eq. (S70), with $j - k$ restricted to be even. We transform the summation to the variables $b = (j + k)/2$ (necessarily an integer) and $c = j - k$; $b$ here must be an integer as $j - k$ being even requires
that \( j + k \) is also even. With \( \nu_{j+k} = 4b \), we can write

\[
\sum_{j,k=1}^{N} \mathcal{H}_{jk}^2 (t) = \sum_{b,c}^{\text{even}} \sum_{j,k=1}^{N} \frac{2 \Theta(4b - \tau)}{\pi \sqrt{\tau(4b - \tau)}} \sin^2 (\varphi_{jk}(\tau)) \cos^2 (\mu t) \quad (S73)
\]

where in the second line, we have assumed that \( \sin^2 (\varphi_{jk}(\tau)) \) oscillates several times over regions of nearly constant \((4b - \tau)^{-1/2}\) (for a given \( \tau \)), as the latter is typically a slowly varying function - this allows us to replace \( \sin^2 (\varphi_{jk}(\tau)) \) by its mean value over an oscillation i.e. 1/2. As the expression is now completely independent of \( c \), the sum over \( c \) just corresponds to accounting for the number of elements with the same \( b \) i.e. the number of elements of an \( N \times N \) matrix on the anti-diagonal given by \( j + k = 2b \). This number is given by \( 1 + 2 \min(b - 1, N - b) \) as a function of \( b \), which we can approximate by \( 2 \min(b, N - b) \) as \( N \gg 1 \) and \( b \) is typically \( O(N) \) over most of the sum. We can additionally replace the sum with an integral as the summand/integrand is not a rapidly varying function of \( b \), getting

\[
\sum_{j,k=1}^{N} \mathcal{H}_{jk}^2 (t) \approx \frac{2}{\pi \sqrt{\tau}} \cos^2 (\mu t) \left[ \int_{0}^{\frac{\pi}{2}} \frac{b}{\sqrt{4b - \tau}} \Theta(4b - \tau) + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{N - b}{\sqrt{4b - \tau}} \Theta(4b - \tau) \right] . \quad (S75)
\]

These are elementary integrals, and a straightforward calculation gives (as expressed in terms of \( t \) rather than \( \tau \)),

\[
\sum_{j,k=1}^{N} \mathcal{H}_{jk}^2 (t) \approx \frac{\Theta(2N - t)}{6\pi N t} \cos^2 (\mu t) \left[ (4N^2 - t^2)^{3/4} - 2(2N^2 - t^2)^{3/2} \Theta(\sqrt{2N} - t) \right] . \quad (S76)
\]

For the term with odd \( j - k \), almost exactly the same reasoning goes through, except \( b \) must now be summed over half-integers (i.e. odd \( j + k \)), which is a negligible difference when one replaces the sum with an integral in the large \( N \) limit. We therefore get

\[
\sum_{j,k=1}^{N} \mathcal{H}_{jk}^2 (t) \approx \frac{\Theta(2N - t)}{6\pi N t} \sin^2 (\mu t) \left[ (4N^2 - t^2)^{3/4} - 2(2N^2 - t^2)^{3/2} \Theta(\sqrt{2N} - t) \right] . \quad (S77)
\]

These are to be summed over to obtain the desired trace as in Eq. (S70), and with the trigonometric identity \( \cos^2 x + \sin^2 x = 1 \) removing all dependence of the expression on the chemical potential \( \mu \), we obtain

\[
\text{Tr} [\mathcal{H}^2 (t)] \approx \frac{\Theta(2N - t)}{6\pi N t} \left[ (4N^2 - t^2)^{3/4} - 2(2N^2 - t^2)^{3/2} \Theta(\sqrt{2N} - t) \right] , \quad (S78)
\]

leading directly to

\[
K(t) \approx 2^N \exp \left\{ -\frac{\Theta(2N - t)}{12\pi N t} \left[ (4N^2 - t^2)^{3/2} - 2(2N^2 - t^2)^{3/2} \Theta(\sqrt{2N} - t) \right] \right\} , \quad (S79)
\]

We expect this expression to hold for \( t > \eta_2 = O(N) \) as the approximations we have made i.e. the approximation for Laguerre polynomials in (S68), and the assumption that our range covers several oscillations of the polynomials, are valid only for \( t \sim O(N) \). If we are interested only in \( t \) near \( t_s = 2N \), we get a simple form of the approach of the ramp towards the plateau:

\[
K(t) \approx 2^N \exp \left[ -\frac{(4N^2 - t^2)^{3/2}}{12\pi N t} \Theta(2N - t) \right] \quad (S80)
\]

We expect this form to be valid at least in \( \sqrt{2N} < t < \infty \), owing to the dropped second term from the exponent of Eq. (S79).
III. σ-MODEL APPROACH

In this section, we introduce a σ-model approach to investigate the SFF $K(t)$ for the ensemble of noninteracting GUE Hamiltonian (Eqs. (S1) and (S2)). We construct a path integral formula for $K(t)$ and connect it with the cumulant expansion discussed in Sec. I. Then we derive a σ-model, from which we argue that the SFF can be recovered by performing a non-perturbative summation. This σ-model approach can be directly generalized to the interacting case, and is therefore especially useful for the investigation of the structure of the many-body energy levels for many-body chaotic systems.

The starting point is the path integral formula for the SFF:

$$K(t) = \left\langle \int \mathcal{D}(\bar{\psi}, \psi) \exp \left\{ \sum_{a,b=L,R} \int_0^t dt' \bar{\psi}_a(t') \left[ (i\partial_{t'} \pm \mu \sigma^3) \delta_{ij} - h_{ij} \sigma^3 \right]_{ab} \psi_b(t') \right\} \right\rangle.$$  \hspace{1cm} (S81)

Here the integrals over $\psi^L$ and $\psi^R$, respectively, lead to $Z(it)$ and $Z(-it)$. $\sigma$ indicates the Pauli matrix in the $L/R$ space. The fermionic field $\psi$ is subject to the boundary condition $\psi(t) = -\psi(0)$. After the Fourier transform, the above equation reduces to

$$K(t) = \left\langle \int \mathcal{D}(\bar{\psi}, \psi) \exp \left\{ i \sum_n \bar{\psi}_{i,n} \left[ (\omega_n + \mu \sigma^3) \delta_{ij} - h_{ij} \sigma^3 \right]_{ab} \psi_{j,n} \right\} \right\rangle,$$  \hspace{1cm} (S82)

where the subscript $n$ represents the Matsubara frequency $\omega_n = \frac{2\pi}{\tau}(n + \frac{1}{2})$, and we have employed the convention that repeated indices imply summation.

A. Connection with Cumulant Expansion

Before moving to the derivation of σ-model, in this section we integrate out the fermionic field first before ensemble averaging, in an attempt to see the connection between the current approach and cluster function approach described in Sec. I. After integrating out the fermionic field $\psi$ in Eq. (S82), we have

$$K(t) = \left\langle \exp \left\{ \sum_{n=1}^N \ln \left\{ -it \left[ \omega_n + \mu - \varepsilon_i \right] \right\} + \ln \left\{ it \left[ -\omega_n + \mu - \varepsilon_i \right] \right\} \right\} \right\rangle,$$  \hspace{1cm} (S83)

where $\varepsilon_i$ is the eigenvalue of the Hermitian matrix $h$. Carrying out the summation over the Matsubara frequency $\omega_n = \frac{2\pi}{\tau}(n + \frac{1}{2})$, we find the above equation reduces to Eq. (S5), which is the starting point of the cluster function approach. It can be rewritten as

$$K(t) = \left\langle \exp \left[ N \int_{-\infty}^{\infty} d\omega \ln \left[ 2 + 2 \cos \left( t\omega - \mu \right) \right] \nu(\omega) \right] \right\rangle,$$  \hspace{1cm} (S84)

where $\nu(\omega) = \frac{1}{N} \sum_{i=1}^N \delta(\omega - \varepsilon_i)$ is the density of state (DOS). We then introduce a new correlation function of the single-particle energy levels:

$$\tilde{R}_n(\omega_1, ..., \omega_n) = N^n \left\langle \nu(\omega_1) ... \nu(\omega_n) \right\rangle,$$  \hspace{1cm} (S85)

and the corresponding cluster function $\tilde{T}_n(\omega_1, ..., \omega_n)$. The relationship between $\tilde{R}_n$ and $\tilde{T}_n$ is equivalent to that between $R_n$ and $T_n$ in Eq. (S12). We emphasize that $\tilde{R}_n$ is defined slightly differently from $R_n$ and contains several extra $\delta$-functions.

A cumulant expansion of Eq. (S84) leads to

$$\ln K(t) = \int_{-\infty}^{\infty} d\omega \ln \left[ 2 + 2 \cos \left( t\omega - \mu \right) \right] \tilde{T}_1(\omega)$$

$$- \frac{1}{2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \ln \left[ 2 + 2 \cos \left( t\omega - \mu \right) \right] \ln \left[ 2 + 2 \cos \left( t\omega' - \mu \right) \right] \tilde{T}_2(\omega, \omega') + ...$$

$$+ (-1)^{n-1} \frac{1}{n!} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \prod_{k=1}^n \{d\omega_k \ln \left[ 2 + 2 \cos \left( t\omega_k - \mu \right) \right] \tilde{T}_n(\omega_1, ..., \omega_n) + ... .$$

By connecting $\tilde{T}_n$ and $T_n$, this equation can be proven to be equivalent to the previous cumulant expansion Eq. (S15) utilized in Sec. I.
B. σ-model for the SFF

The ensemble average in Eq. (S82) can be re-expressed as

\[
K(t) = \frac{1}{\mathcal{D}h e^{-\frac{N}{2} \text{Tr} h^2}} \mathcal{D}(\psi, \psi) \exp \left\{ \frac{i}{N} \sum_n \bar{\psi}_n \left[ (\omega_n + \mu \sigma^3) - h \sigma^3 \right] \psi_n \right\}
\]

\[
= \int \mathcal{D}(\psi, \psi) \exp \left\{ i \sum_n \bar{\psi}_n \left[ (\omega_n + \mu \sigma^3) - \frac{1}{2N} \bar{\psi}_{j,m} \sigma^3 \psi_{i,m} \psi_{i,n} \sigma^3 \psi_{j,n} \right] \right\}
\]

\[
= \frac{1}{\mathcal{D}Q e^{-\frac{N}{2} \text{Tr} Q^2}} \int \mathcal{D}Q e^{-\frac{N}{2} \text{Tr} Q^2} \mathcal{D}(\bar{\psi}, \psi) \exp \left\{ i \sum_{n,m} \bar{\psi}_{i,n} \left\{ (\omega_n \sigma^3 + \mu) \delta_{nm} - i Q_{nm} \right\} \psi_{i,m} \right\}. \tag{S87}
\]

In the 2nd equality, we have integrated over \( h \) first, resulting in a quartic interaction term. Then, in the 3rd equality, this quartic term is decoupled by a Hermitian matrix field \( Q \). Integrating over the fermionic field \( \psi \) in Eq. (S87), we obtain the \( \sigma \)-model:

\[
K(t) = \int \mathcal{D}Q \exp \left( -S[Q] \right) \int \mathcal{D}Q e^{-\frac{N}{2} \text{Tr} Q^2}, \quad S[Q] = \frac{N}{2} \text{Tr} Q^2 - N \text{Tr} \ln \left\{ -t \left[ i(\hat{\omega} \sigma^3 + \mu) + Q \right] \sigma^3 \right\}, \tag{S88}
\]

where \( \hat{\omega} \) represents a diagonal matrix with elements given by \( \hat{\omega}_{nm}^{ab} = \delta_{ab} \delta_{nm} \).

We note that this \( \sigma \)-model is largely similar to the one derived by Kamenev and Mézard in Ref. [S6], from which they recovered the semi-circle law \( R_1(\omega) \) as well as the two-point level correlation function \( R_2(\omega, \omega') \) by considering contribution from the saddle point and the quadratic order fluctuations around it.

C. Saddle Point and Fluctuations

In the following, we will describe the zero, soft and massive modes from the fluctuations around the saddle point and discuss briefly their contribution to the SFF. The saddle point equation can be obtained by taking the variation of the action \( S[Q] \) with respect to \( Q \):

\[
Q_{sp} = \left[ i(\hat{\omega} \sigma^3 + \mu) + Q_{sp} \right]^{-1}, \tag{S89}
\]

and its solution can be expressed as \( Q_{sp} = U^{-1} \Lambda U \). Here \( \Lambda \) is a diagonal matrix with element

\[
\Lambda_{nm}^{ab} = \begin{cases} 
\frac{i}{2} \left[ - (\zeta_n \omega_n + \mu) + \sqrt{ (\zeta_n \omega_n + \mu)^2 - 4 } \right] \delta_{nm}, & |\zeta_n \omega_n + \mu| > 2, \\
\frac{1}{2} \left[ - i (\zeta_n \omega_n + \mu) + s_n^a \sqrt{ 4 - (\zeta_n \omega_n + \mu)^2 } \right] \delta_{nm}, & |\zeta_n \omega_n + \mu| \leq 2,
\end{cases} \tag{S90}
\]

where \( \zeta_{L/R} = \pm 1 \). \( s_n^a \) can take the value of +1 or −1, resulting in various diagonal matrices \( \Lambda^{(s)} \) (see Ref. [S6]) which are all essential for the calculation of the SFF. \( U \) is a directly product of multiple \( U(2) \) rotation matrices, each of which applies in the space of \( L, \omega_n \) and \( R, -\omega_n \). In other words, \( U_{nm}^{ab} \) is nonzero only when \( n = m \) and \( a \neq b \), or \( n = m \) and \( a = b \).

We then consider fluctuations around the saddle point. We express \( Q \) as \( Q = Q_{sp} + U^{-1}(\delta Q/\sqrt{N})U \) and insert it into the action \( S[Q] \) (Eq. (S88)). Expanding in terms of \( \delta Q \) up to the quadratic order leads to

\[
K(t) = \sum_s e^{-S[\Lambda^{(s)}]} \int \mathcal{D}\delta Q \exp \left( -\frac{1}{2} \sum_{a_{lmn}} \delta Q_{nm}^{ab} \left[ 1 + G_a(\omega_n) G_b(\omega_m) \right] \delta Q_{mn}^{ba} \right) \int \mathcal{D}Q e^{-\frac{N}{2} \text{Tr} Q^2}, \tag{S91}
\]

where

\[
G_a(\omega_n) \equiv \left[ i (\zeta_n \omega_n + \mu) + (\Lambda^{(s)})_{n m} \right]^{-1}, \tag{S92}
\]

and \( \sum_s \) represents a summation over various diagonal saddle points \( \Lambda^{(s)} \) (Eq. (S90)) which are not connected by the rotation \( \Lambda^{(s')} = U^{-1} \Lambda^{(s)} U \).
We find various modes for $\delta Q_{nm}^{ab}$ depending on whether $s_n^a$ is equal or opposite to $s_m^b$. We focus on the case where both $|\zeta_\omega \omega + \mu|$ and $|\zeta_\omega \omega + \mu|$ are smaller than 2. When $s_n^a = -s_m^b$, the kernel of the quadratic action $1 + G_a(\omega_n)G_b(\omega_m)$ is massless, meaning it vanishes when $\zeta_\omega \omega = \zeta_\omega \omega$. For small but nonzero $\zeta_\omega \omega - \zeta_\omega \omega$, the kernel can be approximated by

$$1 + G_a(\omega_n)G_b(\omega_m) = \frac{-i(\zeta_\omega \omega_n - \zeta_\omega \omega_n)}{\sqrt{4 - (\zeta_\omega \omega_n + \mu)^2}} + O((\zeta_\omega \omega_n - \zeta_\omega \omega_n)^2).$$

(S93)

These correspond to the soft modes. For the extreme case when $\zeta_\omega \omega_n = \zeta_\omega \omega_m$, $1 + G_a(\omega_n)G_b(\omega_m)$ vanishes and the associated modes are the zero modes. By contrast, when $s_n^a = s_m^b$, the kernel has a mass. It remains nonzero when $\zeta_\omega \omega_n = \zeta_\omega \omega_m$, and is approximately

$$1 + G_a(\omega_n)G_b(\omega_m) = 1 + \frac{4}{[(i(\zeta_\omega \omega_n + \mu) + \sqrt{4 - (\zeta_\omega \omega_n + \mu)^2}]^2} + O((\zeta_\omega \omega_n - \zeta_\omega \omega_n)).$$

(S94)

for small $\zeta_\omega \omega_n - \zeta_\omega \omega_m$. These modes are the massive modes.

Integrating over $\delta Q$ in Eq. S91, one can see that, at the quadratic level, each mode contributes to the SFF a factor of $1/\sqrt{1 + G_a(\omega_n)G_b(\omega_m)}$ which becomes divergent for the zero modes. Higher order fluctuations are therefore required to regularize this infrared divergence. Integration over the zero mode yields a factor proportional to the volume of the saddle point manifold (see discussion around Eq. 31 in Ref. [S7] and the Appendix therein). With proper rescaling which accounts for the normalization constant, i.e., the denominator in Eq. (S88), one can see that the zero modes give rise to a factor of $N^{\text{const.} \times t}$, reproducing a behaviour similar to Eq. (S53) obtained by the cluster function approach (Sec. I). The contribution from the remaining soft modes and massive modes is comparable, and is important to determine the coefficient in the exponent of the $N^{\text{const.} \times t}$ ramp. To see that, one may need to consider fluctuations beyond the quadratic order, and a calculation similar to that in Refs. [S6] and [S7] can reproduce each term one by one in the cumulant expansion Eq. (S86) for the nonzero soft modes and massive modes. We note that a summation over $N \to \infty$ terms as in the cluster function approach (Sec. I) is also required to find the total contribution from nonzero modes.

One of the advantages of this $\sigma$-model approach is that the level correlation functions can be obtained automatically, while in the previous approach we directly employ Mehta’s result [S1]. Most importantly, this $\sigma$-model approach can be generalized to the interacting case, and is more suitable for investigating the many-body level statics of interacting models. In the presence of interactions, we expect that the soft modes acquire a mass term, which results in a drastically different behavior of the SFF. The detailed calculation will be presented in a separate study.

[S1] M. L. Mehta, Random matrices (Elsevier, 2004).
[S2] J. Cotler, N. Hunter-Jones, J. Liu, and B. Yoshida, J. High Energy Phys. 2017, 48 (2017).
[S3] J. Liu, Phys. Rev. D. 98, 086026 (2018).
[S4] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products (Academic press, 2014).
[S5] H. Bateman, Higher Transcendental Functions [Volumes I-III], vol. 2 (McGraw-Hill Book Company, 1953).
[S6] A. Kamelev and M. Mézard, Journal of Physics A: Mathematical and General 32, 4373 (1999).
[S7] A. Kamenev and M. Mézard, Phys. Rev. B 60, 3944 (1999).