The Large-$N$ Limits of Brownian Motions on $\mathbb{GL}_N$

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May 7, 2014

Abstract

We introduce a two-parameter family of diffusion processes $(B_{r,s}^N(t))_{t \geq 0}$, $r, s > 0$, on the general linear group $\mathbb{GL}_N$ that are Brownian motions with respect to certain natural metrics on the group. At the same time, we introduce a two-parameter family of free Itô processes $(b_{r,s}(t))_{t \geq 0}$ in a faithful, tracial $W^*$-probability space, and we prove that the full process $(B_{r,s}^N(t))_{t \geq 0}$ converges to $(b_{r,s}(t))_{t \geq 0}$ in noncommutative distribution as $N \to \infty$ for each $r, s > 0$. The processes $(b_{r,s}(t))_{t \geq 0}$ interpolate between the free unitary Brownian motion when $(r, s) = (1, 0)$, and the free multiplicative Brownian motion when $r = s = \frac{1}{2}$; we thus resolve the open problem of convergence of the Brownian motion on $\mathbb{GL}_N$ posed by Biane in [2].

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*Supported by NSF CAREER Award DMS-1254807
1 Introduction

Let $\mathbb{M}_N$ denote the space of $N \times N$ complex matrices, and let $\mathbb{GL}_N$ denote the Lie group of invertible matrices in $\mathbb{M}_N$; its Lie algebra is the full matrix algebra $\mathfrak{gl}_N = \mathbb{M}_N$. The Lie algebra $\mathfrak{gl}_N$ possesses no $\text{Ad}(\mathbb{GL}_N)$-invariant inner product. By contrast, the Lie group $U_N = \{ U \in \mathbb{M}_N : UU^* = I_N \}$ of unitary matrices in $\mathbb{M}_N$ is compact, and the Hilbert-Schmidt inner product $\langle \xi, \eta \rangle = -\text{Tr}(\xi \eta^*)$ is $\text{Ad}(U_N)$-invariant on the Lie algebra $u_N = \{ \xi \in \mathbb{M}_N : \xi^* = -\xi \}$. (If we restrict to $\mathfrak{su}_N$, this is the the unique $\text{Ad}(\mathbb{SU}_N)$-invariant inner product, up to scale.) In fact, the Hilbert-Schmidt complex inner product $\langle \xi, \eta \rangle = \text{Tr}(\xi \eta^*)$ on $\mathfrak{gl}_N$ is also invariant under conjugation by elements of $U_N$.

The group $\mathbb{GL}_N$ is the complexification of $U_N$, which is to say that the Lie algebras satisfy $\mathfrak{gl}_N = u_N \oplus iu_N$. Both of the complementary real subspaces $u_N$ (skew-Hermitian matrices) and $iu_N$ (Hermitian matrices) are invariant under conjugation by elements of $U_N$. It follows immediately that the following real inner products are all $\text{Ad}(U_N)$-invariant.

**Definition 1.1.** Let $r, s > 0$. Define the real inner product $\langle \cdot, \cdot \rangle_{r,s}^N$ on $\mathfrak{gl}_N$ by

$$
\langle \xi_1 + i\eta_1, \xi_2 + i\eta_2 \rangle_{r,s}^N = -\frac{1}{r} N \text{Tr}(\xi_1 \xi_2) - \frac{1}{s} N \text{Tr}(\eta_1 \eta_2),\quad \xi_1, \xi_2, \eta_1, \eta_2 \in u_N.
$$

That is: $\langle \cdot, \cdot \rangle_{r,s}^N$ makes $u_N$ and $iu_N$ orthogonal, and its restrictions to these two orthocomplementary subspaces are positive scalar multiples of the Hilbert-Schmidt inner product.

**Remark 1.2.** The inner product $\langle \cdot, \cdot \rangle_{r,s}^N$ may alternatively be written in the form

$$
\langle A, B \rangle = \frac{1}{2} \left( \frac{1}{s} + \frac{1}{r} \right) N \text{Re} \text{Tr}(AB^*) + \frac{1}{2} \left( \frac{1}{s} - \frac{1}{r} \right) N \text{Re} \text{Tr}(AB).
$$

We scale with $N \text{Tr}$ in order to produce a meaningful limit as $N \to \infty$.

Any real inner product on $\mathfrak{gl}_N$ gives rise to a left-invariant Riemannian metric on $\mathbb{GL}_N$, and hence to a left-invariant Laplace-Beltrami operator, and associated diffusion process: the Brownian motion.

**Definition 1.3.** Let $r, s > 0$. Let $\Delta_{r,s}^N$ denote the Laplace-Beltrami operator on $\mathbb{GL}_N$ associated to the left-invariant Riemannian metric induced by the inner product $\langle \cdot, \cdot \rangle_{r,s}^N$. The Markov diffusion $B_{r,s}^N(t)$ on $\mathbb{GL}_N$, started at $B_{r,s}^N(0) = I_N$, with generator $\frac{1}{2} \Delta_{r,s}^N$, is called a $U_N$-invariant Brownian motion. Fix a probability space $(\Omega, \mathcal{F}, P)$ from which the random matrices $B_{r,s}^N(t)$ are sampled, and denote by $E = \int_{\Omega} \cdot \, dP$.

**Remark 1.4.** Since the inner product is $\text{Ad}_{U_N}$-invariant, its Laplace operator is also unitarily invariant. That is: for $f \in C^\infty(\mathbb{GL}_N)$ and $U \in U_N$, let $(\text{Ad}_U f)(A) = f(\text{Ad}_U A)$; then $\Delta_{r,s}(\text{Ad}_U^* f) = \Delta_{r,s} f$ for all $U \in U_N$. It follows that the law of the Brownian motion $B_{r,s}^N(t)$ (the heat kernel) is also invariant under the Ad*-action of $U_N$; hence, it is appropriate to call it $U_N$-invariant Brownian motion.

For convenience, we now fix a (large) probability space $(\Omega, \mathcal{F}, P)$ on which all of the random matrices $\{B_{r,s}^N(t) : r, s > 0, t \geq 0, N \in \mathbb{N} \}$ live. As usual, for random variables $F$ on $(\Omega, \mathcal{F})$, we denote $\int_{\Omega} F \, dP = E(F)$. We will characterize the large-$N$ limit of $B_{r,s}^N(t)$ as a noncommutative stochastic process. To do so, we introduce the following free stochastic processes (for a discussion of free stochastic calculus, see Section 2.2).
**Definition 1.5.** Fix $r, s \geq 0$. Let $(\mathcal{A}, t)$ be a $W^*$-probability space that contains two freely independent free semicircular Brownian motions $x(t), y(t)$. Let

$$w_{r,s}(t) = i\sqrt{r} x(t) + \sqrt{s} y(t).$$

(1.2)

The **free multiplicative Brownian motion** of parameters $r, s$, denoted $b_{r,s}(t)$, is the unique solution to the following free stochastic differential equation (FSDE):

$$db_{r,s}(t) = b_{r,s}(t) dw_{r,s}(t) - \frac{1}{2} (r-s) b_{r,s}(t) \, dt, \quad b_{r,s}(0) = 1.$$  

(1.3)

Let $\text{tr}$ denote the normalized trace, $\text{tr} = \frac{1}{N} \text{Tr}$ on $M_N$. The main theorem of this paper is as follows; it is proved in Section 6.

**Theorem 1.6.** For $r, s > 0$, the Brownian motion $(B^N_{r,s}(t))_{t \geq 0}$ on $\mathcal{U}_N$ converges, as a noncommutative stochastic process, to $(b_{r,s}(t))_{t \geq 0}$ as $N \to \infty$. That is to say: if $n \in \mathbb{N}$, $t_1, \ldots, t_n \geq 0$, and $\varepsilon_1, \ldots, \varepsilon_n \in \{1, *\}$, then

$$\lim_{N \to \infty} \mathbb{E} \text{tr} (B^N_{r,s}(t_1)^{\varepsilon_1} \cdots B^N_{r,s}(t_n)^{\varepsilon_n}) = \tau (b_{r,s}(t_1)^{\varepsilon_1} \cdots b_{r,s}(t_n)^{\varepsilon_n}).$$

This theorem resolves a conjecture left open by Biane in [2]. Indeed, let $G^N(t) = B^N_{1/2,1/2}(t)$, and let $g(t) = b_{1/2,1/2}(t)$; then (1.3) becomes

$$dg(t) = g(t) \, dw(t), \quad g(0) = 1,$$

(1.4)

where $w(t) = (y(t) + ix(t))/\sqrt{2}$ is a free circular Brownian motion. The process $g(t)$ is referred to as **free multiplicative Brownian motion** in [2] [3], where it was conjectures that $(G^N(t))_{t \geq 0}$ converges to $(g(t))_{t \geq 0}$ as a noncommutative stochastic process. Recent progress on this conjecture was made by Guillaume Cébron in [5] Theorem 4.6, where he showed that, for each fixed $t \geq 0$, the random matrix $G^N(t)$ converges in noncommutative distribution to $g(t)$. At the same time, the present author in [9] independently proved that, for each fixed $t \geq 0$, the empirical noncommutative distribution of the random matrix $B^N_{r,s}(t)$ converges almost surely to a linear functional $\varphi_{r,s}(t): \mathcal{C}(X, X^*) \to \mathbb{C}$, which is the noncommutative distribution of an operator in a tracial noncommutative probability space; it was left open whether the trace is faithful. Theorem 1.6 resolves this question as well. Our present techniques are quite different from those in [5] [9].

**Remark 1.7.** We may also consider the “special case” $(r, s) = (1, 0)$. Let $u(t) = b_{1,0}(t)$; then (1.3) becomes

$$du(t) = iu(t) \, dx(t) - \frac{1}{2} u(t) \, dt, \quad u(0) = 1$$

(1.5)

which is the FSDE for the (left) **free unitary Brownian motion**, introduced in [2]. The main theorem [2] Theorem 1] of that paper was the convergence of the Brownian motion $(U^N(t))_{t \geq 0}$ on $\mathcal{U}_N$ (with respect to to the inner product $-N \text{Tr} (\xi \eta)$) to $(u(t))_{t \geq 0}$. Some of the ideas we present here are motivated by this example.

In order to prove Theorem 1.6 we need to describe more concretely the noncommutative distribution of $b_{r,s}(t)$; to that end, we introduce the following indispensable constants.

**Theorem / Definition 1.8** ([1] [2]). For each $t \in \mathbb{R}$, there exists a unique probability measure $\nu_t$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with the following properties. For $t > 0$, $\nu_t$ is supported in the unit circle $\mathbb{U}$; for $t < 0$, $\nu_t$ is supported in $\mathbb{R}_+ = (0, \infty)$; and $\nu_0 = \delta_1$. In all cases, $\nu_t$ is determined by its moments: $\nu_0(t) \equiv 1$ and, for $n \in \mathbb{Z} \setminus \{0\}$,

$$\nu_n(t) \equiv \int_{\mathbb{C}^*} u^n \nu_t(du) = e^{-\frac{|n|t}{2}} \sum_{k=0}^{[|n|-1]} \frac{(-t)^k}{k!} |n|^{k-1} \binom{|n|}{k+1}.\quad (1.6)$$
Theorem 1.9. Let \( r, s, t \geq 0 \) and \( n \in \mathbb{N} \). Then
\[
\tau \left[ b_{r,s}(t)^n \right] = \tau \left[ b_{r,s}(t)^{sn} \right] = \nu_n((r - s)t), \tag{1.7}
\]
\[
\tau \left[ (b_{r,s}(t)b_{r,s}(t)^*)^n \right] = \nu_n(-4st), \tag{1.8}
\]
\[
\tau \left[ b_{r,s}(t)^2b_{r,s}(t)^*2 \right] = e^{4st} + 4st(1 + st)e^{(3s-r)t}. \tag{1.9}
\]

Theorem 1.9 is proved in Section 4.

Remark 1.10. Equations (1.7) and (1.8) were proved in the author’s paper [9, Theorems 1.3 & 1.5]. They are included here to show how they can be derived more directly from the limit process \( b_{r,s}(t) \). Equation (1.9) will be needed in the proof of Theorem 1.11 below.

In Section 5 we demonstrate that the process \( (b_{r,s}(t))_{t \geq 0} \) inherits all of the invariant properties from \( B_{r,s}^N(t) \) that qualify it as a Brownian motion.

Theorem 1.11. For \( r, s > 0 \) and \( N \in \mathbb{N}^* \), the \( \mathbb{G}L_N \) Brownian motion \( (B_{r,s}^N(t))_{t \geq 0} \) has independent, stationary multiplicative increments. If \( N \geq 2 \), then, with probability 1, \( B_{r,s}^N(t) \) is not a normal matrix for any \( t > 0 \).

For \( r, s \geq 0 \), the free multiplicative Brownian motion \( (b_{r,s}(t))_{t \geq 0} \) is invertible for all \( t \geq 0 \), and has freely independent, stationary multiplicative increments. If \( s = 0 \), then \( u(t) \) is unitary, and \( u(t) \equiv b_{r,0}(t/r) \) is a free unitary Brownian motion for any \( r > 0 \). If \( s > 0 \), then \( b_{r,s}(t) \) is not a normal operator for any \( t > 0 \).

Remark 1.12. We defined \( B_{r,s}^N(t) \) only for \( r, s > 0 \) (indeed, the inner product \( \langle \cdot, \cdot \rangle_N \) blows up as \( r \to 0 \) or \( s \to 0 \)). In the case \( s = 0 \), it is possible to make sense of \( B_{r,0}^N(t) \) as the solution to the matrix SDE (2.10) below. In this case, the process is degenerate on \( \mathbb{G}L_N \); in fact, \( B_{r,0}^N(t) \in \mathbb{U}_N \), and \( U^N(t) = B_{r,0}^N(t/r) \) is Brownian motion on \( \mathbb{U}_N \), as in the large-\( N \) limit.

The proof of Theorem 1.11 has two main parts: first, we show that \( B_{r,s}^N(t) \) converges to \( b_{r,s}^N(t) \) in noncommutative distribution for each fixed \( t \geq 0 \). We then use Theorem 1.11 since the increments of \( (b_{r,s}(t))_{t \geq 0} \) are freely independent, to prove convergence of the process it suffices to prove that the increments of \( (B_{r,s}^N(t))_{t \geq 0} \) are asymptotically free. The key to proving this property is the following multivariate extension of the technology in [7, Sections 3 & 4].

Theorem 1.13. Let \( n \in \mathbb{N}, t_1, \ldots, t_n \geq 0 \), and let \( B_{r,s}^{1,N}(t_1), \ldots, B_{r,s}^{n,N}(t_n) \) be independent copies of the Brownian motion \( B_{r,s}(\cdot) \) at these times. These operators possess a limit joint distribution, and, for any noncommutative polynomials \( f, g \in \mathbb{C}\langle X_1, \ldots, X_n, X_1^*, \ldots, X_n^* \rangle \), there is a constant \( C = C(r, s, t_1, \ldots, t_n, f, g) \) such that
\[
\text{Cov} \left[ \text{tr}(f(B_{r,s}^{1,N}(t_1), \ldots, B_{r,s}^{n,N}(t_n))), \text{tr}(g(B_{r,s}^{1,N}(t_1), \ldots, B_{r,s}^{n,N}(t_n))) \right] \leq \frac{C}{N^2}. \tag{1.10}
\]

Theorem 1.13 is proved in Section 3.

2 Background

In this section, we briefly outline the technology needed to prove the results in this paper: matrix stochastic calculus (particularly for invertible random matrices), the corresponding stochastic calculus in the free probability setting, and the notion of asymptotic freeness that ties the two together.

2.1 Stochastic Calculus on \( \mathbb{G}L_N \)

Let \( G \) be a Lie group, with Lie algebra \( \mathfrak{g} \). For \( \xi \in \mathfrak{g} \), the associated left-invariant vector field on \( G \) is denoted \( \partial_\xi \):
\[
(\partial_\xi f)(g) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(t\xi)), \quad f \in \mathbb{C}^\infty(G). \tag{2.1}
\]
Let $\langle \cdot , \cdot \rangle$ be a real inner product on $\mathfrak{g}$, and let $\beta$ be an orthonormal basis for $(\mathfrak{g}, \langle \cdot , \cdot \rangle)$. Then the Laplace-Beltrami operator on $G$ for the Riemannian metric induced by $\langle \cdot , \cdot \rangle$ is

$$\Delta_G = \sum_{\xi \in \beta} \partial^2_{\xi},$$

which does not depend on the particular orthonormal basis used.

If $G \subset \mathbb{M}_N$ is a linear Lie group, then the Brownian motion on $G$ (the diffusion process with generator $\frac{1}{2} \Delta_G$) may be constructed as the solution to a matrix stochastic differential equation (mSDE). Fix an orthonormal basis $\beta$ for $\mathfrak{g}$, and let $W(t)$ denote the following Wiener process in $\mathfrak{g}$:

$$W(t) = \sum_{\xi \in \beta} W_\xi(t) \xi,$$

where $\{W_\xi : \xi \in \beta\}$ are i.i.d. standard $\mathbb{R}$-valued Brownian motions. Then the Brownian motion $B(t)$ is determined by the Stratonovich mSDE

$$dB(t) = W(t) \circ dW(t), \quad W(0) = I_N. \tag{2.3}$$

While convenient for proving geometric invariance, the Stratonovich form is less welladapted to computation. We can convert (2.3) to Itô form. The result, due to McKean [11, p. 116] is

$$dB(t) = B(t) dW(t) + \frac{1}{2} B(t) \left( \sum_{\xi \in \beta} \xi^2 \right) dt, \quad B(0) = I_N. \tag{2.4}$$

See, also, [8].

Let us specialize to the case of interest, with $G = GL_N$ and $\mathfrak{g}l_N$ equipped with an $Ad_U$-invariant inner product $\langle \cdot , \cdot \rangle^r_s$. To clarify: let $\langle \cdot , \cdot \rangle_{u_N}$ denote the following real inner product on $u_N$:

$$\langle \xi, \eta \rangle_{u_N} = -N \text{Tr}(\xi \eta). \tag{2.5}$$

Then the inner product $\langle \cdot , \cdot \rangle^r_s$ on $\mathfrak{g}l_N = u_N \oplus iu_N$ is given by

$$\langle \xi_1 + i \eta_1, \xi_2 + i \eta_2 \rangle^r_s = \frac{1}{r} \langle \xi_1, \xi_2 \rangle_{u_N} + \frac{1}{s} \langle \eta_1, \eta_2 \rangle_{u_N}. \tag{2.6}$$

It is straightforward to check that, if $\beta_N$ is an orthonormal basis for $u_N$ with respect to $\langle \cdot , \cdot \rangle_{u_N}$, then

$$\beta^r_s = \left\{ \sqrt{r} \xi : \xi \in \beta_N \right\} \cup \left\{ \sqrt{s} i \xi : \xi \in \beta_N \right\} \tag{2.7}$$

is an orthonormal basis for $\mathfrak{g}l_N$ with respect to $\langle \cdot , \cdot \rangle^r_s$. Equation (2.2) and a straightforward application of the chain rule in (2.1) then shows that the Laplace-Beltrami operator is

$$\Delta^r_s = \sum_{\xi \in \beta} \left( r \partial^2_{\xi} + s \partial^2_{i\xi} \right). \tag{2.8}$$

**Remark 2.1.** In [7, 9], we used the elliptic operator

$$A^N_{s,t} = \left( s - \frac{1}{2} \right) \sum_{\xi \in \beta_N} \partial^2_{\xi} + \frac{1}{2} \sum_{\xi \in \beta_N} \partial^2_{i\xi} = \Delta^N_{s-t/2,t/2}.$$

The linear change of parameters was convenient for our discussion of the two-parameter Segal–Bargmann transform, and so all of the theorems in [9] are stated using this language as well.
In [7, Proposition 3.1], the following “magic formula” was proved. If $\beta_N$ is an orthonormal basis of $u_N$, then

$$\sum_{\xi \in \beta_N} \xi^2 = -I_N. \quad (2.9)$$

Combining this with (2.7) gives

$$\sum_{\xi \in \beta^N_{ps}} \xi^2 = -(r-s)I_N,$$

and so, by (2.4), the $U_N$-invariant Brownian motion $B^N_{r,s}(t)$ is determined by the mSDE

$$dB^N_{r,s}(t) = B^N_{r,s}(t) dW^N_{r,s}(t) - \frac{1}{2}(r-s)B^N_{r,s}(t) dt, \quad (2.10)$$

where $W^N_{r,s}(t) = \sum_{\xi \in \beta^N_{ps}} W_{\xi}(t) \xi$. It will be convenient to express this Itô process in a slightly different form. Let us choose the following orthonormal basis $\beta_N$ for $u_N$:

$$\beta_N = \left\{ \frac{1}{\sqrt{N}} E_{jj}, \frac{1}{\sqrt{2N}} (E_{jk} - E_{kj}), \frac{1}{\sqrt{2N}} i(E_{jk} + E_{kj}) : 1 \leq j < k \leq N \right\}, \quad (2.11)$$

where $E_{jk}$ is the matrix unit with a 1 in the $(j,k)$-entry and 0 elsewhere. Then it is straightforward to check that

$$W^N_{r,s}(t) = \sqrt{t} \sum_{\xi \in \beta_N} B_{\xi}(t) \xi + i\sqrt{s} \sum_{\xi \in \beta_N} B_{\xi}(t) \xi = \sqrt{t} iX^N(t) + \sqrt{s} Y^N(t),$$

where $X^N(t)$ and $Y^N(t)$ are independent GUE$_N$ Brownian motions. That is: all entries of $X^N(t)$ are independent from all entries of $Y^N(t)$; the matrices $X^N(t), Y^N(t)$ are Hermitian; and all entries $[X^N(t)]_{jk}$ and $[Y^N(t)]_{jk}$ with $1 \leq j \leq k \leq N$ are i.i.d. $\mathbb{R}$-valued Brownian motions of variance $t/N$. This is a convenient representation, due to the following easily-verified stochastic calculus rules that apply to matrix stochastic integrals with respect to (linear combinations of) $X^N(t)$ and $Y^N(t)$.

**Lemma 2.2.** Let $\Theta(t), \Theta_1(t), \Theta_2(t)$ be $\mathbb{M}_N$-valued stochastic processes that are adapted to the filtration $\mathcal{F}_t$ of $X^N(t)$ and $Y^N(t)$ (in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Then the following hold:

$$\mathbb{E}(\Theta_1(t) dX^N(t) \Theta_2(t)) = \mathbb{E}(\Theta_1(t) dY^N(t) \Theta_2(t)) = 0 \quad (2.12)$$

$$dX^N(t) \Theta(t) dX^N(t) = dY^N(t) \Theta(t) dY^N(t) = \text{tr}(\Theta(t))I_N dt \quad (2.13)$$

$$dX^N(t) \Theta(t) dY^N(t) = dY^N(t) \Theta(t) dX^N(t) = 0 \quad (2.14)$$

$$\Theta_1(t) dX^N(t) \Theta_2(t) dt = \Theta_1(t) dY^N(t) \Theta_2(t) dt = 0. \quad (2.15)$$

Moreover, let $\Theta_1(t)$ and $\Theta_2(t)$ be $\mathbb{M}_N$-valued Itô processes: solutions to mSDEs of the form

$$d\Theta(t) = f_1(\Theta(t)) dX^N(t) + f_2(\Theta(t)) dY^N(t) + g(\Theta(t)) dt,$$

for smooth functions $f_1, f_2, g_1, g_2, h : \mathbb{M}_N \to \mathbb{M}_N$. Then the following Itô product rule holds:

$$d(\Theta_1(t) \Theta_2(t)) = d\Theta_1(t) \cdot \Theta_2(t) + \Theta_1(t) \cdot d\Theta_2(t) + d\Theta_1(t) \cdot d\Theta_2(t). \quad (2.16)$$

**Remark 2.3.** As usual, we abuse notation and write stochastic integral equations in differential form. For example, the last equality in (2.12) is shorthand for

$$\mathbb{E} \left( \int_0^t \Theta_1(s) dY^N(s) \Theta_2(s) \right) = 0,$$

where the matrix stochastic integral is defined exactly as the scalar stochastic integral, using matrix multiplication in the place of scalar multiplication. Lemma 2.2 is straightforward to verify from the standard Itô calculus for vector-valued processes.
2.2 Free Stochastic Calculus

For an introduction to noncommutative probability theory, and free probability in particular, we refer the reader to [14]. We assume familiarity with noncommutative probability spaces and $W^*$-probability spaces. The reader is directed to [10] Sections 1.1–1.3 for a quick introduction to free additive (semicircular) Brownian motion. Also, we give a brief discussion of free independence at the beginning of Section 2.3 below.

Let $(\mathcal{A}, \tau)$ be a faithful, tracial $W^*$-probability space. To fix notation, for $a \in \mathcal{A}$ denote its noncommutative distribution as $\varphi_a$. I.e. letting $C(X, X^*)$ denote the noncommutative polynomials in two variables, $\varphi_a : C(X, X^*) \to \mathbb{C}$ is the linear functional

$$\varphi_a(f) = \tau(f(a, a^*)), \quad f \in C(X, X^*).$$

A free semicircular Brownian motion $x(t)$ is a self-adjoint stochastic process $(x(t))_{t \geq 0}$ in $\mathcal{A}$ such that $x(0) = 0$, $\text{Var}(x(1)) = 1$, and the additive increments of $x$ are stationary and freely independent: for $0 \leq t_1 < t_2 < \infty$, $\varphi_{x(t_2) - x(t_1)} = \varphi_{x(t_2) - x(t_1)}$, and $x(t_2) - x(t_1)$ is freely independent from the $W^*$-subalgebra $\mathcal{A} \supset \mathcal{A}_1 \equiv W^* \{x(t) : 0 \leq t \leq t_1\}$. Since $x(t)$ is a bounded self-adjoint operator, its distribution is given by a compactly-supported probability measure on $\mathbb{R}$; the freeness of increments and stationarity then implies that $\varphi_{x(t_2) - x(t_1)}$ is the semicircle law: setting $t = t_2 - t_1$,

$$\tau[(x(t_2) - x(t_1))^n] = \int_{-2\sqrt{t}}^{2\sqrt{t}} s^n \frac{1}{2\pi t} \sqrt{4t - s^2} \, ds, \quad n \in \mathbb{N}.$$ 

In [17], it was proven that, if $X^N(t)$ is a GUE$_N$ Brownian motion, then the process $(X^N(t))_{t \geq 0}$ converges to a free semicircular Brownian motion: for any $n$ and any $t_1, t_2, \ldots, t_n \geq 0$, and any noncommutative polynomial $f \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$,

$$\lim_{N \to \infty} \mathbb{E}\text{tr}(f(X^N(t_1), \ldots, X^N(t_n))) = \tau(f(x(t_1), \ldots, x(t_n))).$$

Appealing to Lemma 2.2 this paves the way to free stochastic differential equations.

Let $x(t)$ and $y(t)$ be two freely independent free semicircular Brownian motions in a $W^*$-probability space $(\mathcal{A}, \tau)$, and let $\mathcal{A}_t = W^* \{x(s), y(s) : 0 \leq s \leq t\}$. Let $\theta(t), \theta_1(t), \theta_2(t)$ be processes that are adapted to the filtration $\mathcal{A}_t$. The free Itô integral

$$\int_0^t \theta_1(s) \, dx(s) \theta_2(s)$$

is defined in precisely the same manner as Itô integrals of real-valued processes with respect to real Brownian motion: as $L^2(\mathcal{A}_t, \tau)$-limits of sums $\sum \theta_1(t_j)(x(t_j) - x(t_{j-1}))\theta_2(t_j)$ over partitions $\{0 = t_0 \leq \cdots \leq t_n = t\}$ as the partition width $\sup \{|t_j - t_{j-1}| \}$ tends to 0. Standard Picard iteration techniques show that, if $f_1, f_2, g_1, g_2, h$ are polynomials then the integral equation

$$b(t) = 1 + \int_0^t f_1(b(s)) \, dx(s) \, f_2(b(s)) + \int_0^t g_1(b(s)) \, dy(s) \, g_2(b(s)) + \int_0^t h(b(s)) \, ds,$$  \hspace{1cm} (2.17)

has a unique adapted solution $b(t) \in \mathcal{A}_t$ satisfying $b(0) = 1$. As usual, we use differential notation to express (2.17) in the form

$$db(t) = f_1(b(t)) \, dx(t) \, f_2(b(t)) + g_1(b(t)) \, dy(t) \, g_2(b(t)) + h(b(t)) \, dt,$$ \hspace{1cm} (2.18)

We refer to (2.18) as a free stochastic differential equation (fSDE). Solutions of such equations are called free Itô processes. The matrix stochastic calculus of Lemma 2.2 has a precise analogue for free Itô processes.
**Lemma 2.4.** Let \((\mathcal{A}, \tau)\) be a \(W^*\)-probability space containing two freely independent free semicircular Brownian motions \(x(t)\) and \(y(t)\), adapted to the filtration \(\{\mathcal{A}_t\}_{t \geq 0}\). Let \(\theta(t), \theta_1(t), \theta_2(t)\) be processes adapted to \(\mathcal{A}_t\). Then the following hold:

\[
\tau(\theta_1(t) dx(t) \theta_2(t)) = \tau(\theta_1(t) dy(t) \theta_2(t)) = 0 
\]

\(\tau(\theta(t) dx(t)) = dx(t) \theta(t) dy(t) = \tau(\theta(t)) dt\)

\[dx(t) \theta(t) dx(t) = dy(t) \theta(t) dy(t) = \tau(\theta(t)) dt\]

\[
\theta_1(t) dx(t) \theta_2(t) dt = \theta_1(t) dy(t) \theta_2(t) dt = 0.
\]

Moreover, if \(\theta_1(t)\) and \(\theta_2(t)\) are free Itô processes, then the following Itô product rule holds:

\[d(\theta_1(t)\theta_2(t)) = d\theta_1(t) \cdot \theta_2(t) + \theta_1(t) \cdot d\theta_2(t) + d\theta_1(t) \cdot d\theta_2(t).\] (2.23)

For a proof of Lemma 2.4 see [4].

### 2.3 Asymptotic Freeness

**Definition 2.5.** Let \((\mathcal{A}, \tau)\) be a noncommutative probability space. Unital \(*\)-subalgebras \(\mathcal{A}_1, \ldots, \mathcal{A}_m \subset \mathcal{A}\) are called free with respect to \(\tau\) if, given any \(n \in \mathbb{N}\) and \(k_1, \ldots, k_n \in \{1, \ldots, m\}\) such that \(k_{j-1} \neq k_j\) for \(1 < j \leq n\), and any elements \(a_j \in \mathcal{A}_{k_j}\) with \(\tau(a_j) = 0\) for \(1 \leq j \leq n\), it follows that \(\tau(a_1 \cdots a_n) = 0\).

Random variables \(a_1, \ldots, a_m\) are said to be freely independent of the unital \(*\)-algebras \(\mathcal{A}_j = \langle a_j, a_j^*\rangle \subset \mathcal{A}\) if they generate are free.

Free independence is a \(*\)-moment factorization property. By centering \(a_i - \tau(a_i)1_{\mathcal{A}} \in \mathcal{A}_i\), the freeness rule allows (inductively) any moment \(\tau(a_{k_1}^\varepsilon \cdots a_{k_n}^\varepsilon)\) to be decomposed as a polynomial in moments \(\tau(a_j^\varepsilon)\) in the variables separately. For example, if \(a, b\) are freely independent then \(\tau(a^\varepsilon b^\delta) = \tau(a^\varepsilon)\tau(b^\delta)\), while

\[
\tau(a^\varepsilon_1 b^\delta_1 a^\varepsilon_2 b^\delta_2) = \tau(a^\varepsilon_1)\tau(a^\varepsilon_2)\tau(b^\delta_1)\tau(b^\delta_2) + \tau(a^\varepsilon_1 a^\varepsilon_2)\tau(b^\delta_1)\tau(b^\delta_2) - \tau(a^\varepsilon_1)\tau(a^\varepsilon_2)\tau(b^\delta_1)\tau(b^\delta_2),
\]

for any \(\varepsilon, \varepsilon_1, \varepsilon_2, \delta, \delta_1, \delta_2 \in \{1, *\}\). In general, if \(a_1, \ldots, a_n\) are freely independent, then their noncommutative joint distribution \(\varphi_{a_1, \ldots, a_n}\) (a linear functional on \(\mathbb{C}\langle X_1, \ldots, X_n, X_1^*, \ldots, X_n^*\rangle\)) is determined by the individual distributions \(\varphi_{a_1}, \ldots, \varphi_{a_n}\) on \(\mathbb{C}\langle X, X^*\rangle\).

Let \(L_\infty^\infty(\Omega, \mathcal{F}, \mathbb{P}) = \bigcap_{p \geq 1} L_p(\Omega, \mathcal{F}, \mathbb{P})\), and let \(M_N \otimes L_\infty^\infty\) denote the algebra of \(N \times N\) matrices with entries in \(L_\infty^\infty(\Omega, \mathcal{F}, \mathbb{P})\). There are no non-trivial instances of free independence in the noncommutative probability space \((M_N \otimes L_\infty^\infty, \mathbb{E}, \mathbb{P})\); i.e. if \(A, B \in M_N \otimes L_\infty^\infty\) are freely independent, then at least one of them is a.s. a constant multiple of the identity matrix \(I_N\). However, asymptotic freeness abounds.

**Definition 2.6.** Let \(n \in \mathbb{N}\). For each \(n \in \mathbb{N}\), let \(A_1^n, \ldots, A_n^n\) be random matrices in \(M_N \otimes L_\infty^\infty\). Say that \((A_1^n, \ldots, A_n^n)\) are asymptotically free if there is a noncommutative probability space \((\mathcal{A}, \tau)\) containing freely independent random variables \(a_1, \ldots, a_n\) such that \((A_1^n, \ldots, A_n^n)\) converges in noncommutative distribution to \((a_1, \ldots, a_n)\).

The general mantra for producing asymptotically free random matrices is as follows.

If \(A_1^n, \ldots, A_n^n\) are random matrices whose distribution is invariant under unitary conjugation, and possess a joint limit distribution, then they are asymptotically free.

The first result in this direction was proved in [17], where the matrices \(A_j^n\) were taken to have the form \(A_j^n = U_j^n D_j^n U_j^n^{-1}\) where \(U_1^n, \ldots, U_n^n\) are independent Haar-distributed unitaries, and \(D_j^n\) are deterministic diagonal matrices with uniform bounds on their trace moments. This was later improved to include all deterministic matrices (with uniform bounds on their operator norms) in [18]; see also, [6, 20] for related results. We will use the following form of the mantra, which is a weak form of [13, Theorem 1].
Theorem 2.7. Let $A_1^N, \ldots, A_n^N$ be independent random matrices in $M_n \otimes L^\infty$, with the following properties.

1. The joint law of $A_1^N, \ldots, A_n^N$ is invariant under conjugation by unitary matrices in $U_n$.

2. There is a joint limit distribution: for any noncommutative polynomial $f \in \mathbb{C}(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*)$, there exists $\lim_{N \to \infty} \mathbb{E} \mathrm{tr}(f(A_1^N, \ldots, A_n^N, (A_1^N)^*, \ldots, (A_n^N)^*))$ exists.

3. The fluctuations are $O(1/N^2)$: for any noncommutative polynomials $f, g$ as in (2), there is a constant $C = C(f, g)$ so that

$$\mathrm{Cov} \left[ \mathrm{tr}(f(A_1^N, \ldots, A_n^N, (A_1^N)^*, \ldots, (A_n^N)^*)), \mathrm{tr}(g(A_1^N, \ldots, A_n^N, (A_1^N)^*, \ldots, (A_n^N)^*)) \right] \leq \frac{C}{N^2}.$$ 

Then $A_1^N, \ldots, A_n^N$ are asymptotically free.

Remark 2.8. [13, Theorem 1] has a much stronger assumption than (3): it also assumes that the classical cumulants $k_r$ in normalized traces of noncommutative polynomials are $o(1/N^r)$ for all $r > 2$, thus producing a so-called second-order limit distribution. However, this stronger assumption is used only to produce a stronger conclusion: that the matrices are asymptotically free of second-order. Following the proof, it is relatively easy to see that Theorem 2.7 is proved along the way, at least in the case $n = 2$. To go from 2 to general finite $n$ can be achieved by induction together with the associativity of freeness; cf. [19, Proposition 2.5.5(iii)]. See also, [12] where this is proved more explicitly in the harder case of real random matrices (where $U_n$-invariance is replaced with $\hat{O}_N$-invariance).

3 Heat Kernels on $\mathbb{GL}_n^N$

Here we generalize the technology we developed in [7] Sections 3.4 & 4.1 to independent products of heat kernel measures on $\mathbb{GL}_n^N$.

3.1 Laplacians on $\mathbb{GL}_n^N$

Let $n, N \in \mathbb{N}$. Then $\mathbb{GL}_n^N = \mathbb{GL}_n \times \cdots \times \mathbb{GL}_n$ is a Lie group of real dimension $2nN^2$. Its Lie algebra is $\mathfrak{gl}_n^N = \mathfrak{gl}_N \oplus \cdots \oplus \mathfrak{gl}_N$. For $\xi \in \mathfrak{gl}_N$, and $1 \leq j \leq n$, let $\xi_j$ denote the vector $(0, \ldots, 0, \xi, 0, \ldots, 0) \in \mathfrak{gl}_N^j$ (with $\xi$ in the $j$th component). The Lie product on $\mathfrak{gl}_n^N$ is then determined by $[\xi_j, \eta_k] = \delta_{jk} (\xi_j \eta_k - \eta_k \xi_k)$ for $1 \leq j, k \leq n$. In particular, if $j \neq k$ and $\xi, \eta \in \mathfrak{gl}_N$, then the left-invariant derivations $\partial_{\xi_j}$ and $\partial_{\eta_k}$ on $C^\infty(\mathbb{GL}_n^N)$ commute. To be clear, note that, for $f \in C^\infty(\mathbb{GL}_n^N),$

$$\left. \left( \partial_{\xi_j} f \right)(A_1, \ldots, A_n) = \frac{d}{dt} \bigg|_{t=0} f(A_1, \ldots, A_{j-1}, A_j e^{t\xi}, A_{j+1}, \ldots, A_n). \right) \quad (3.1)$$

Let $\beta_{r,s}^N$ denote an orthonormal basis for $\mathfrak{gl}_N$ (with respect to $\langle \cdot, \cdot \rangle^N_{r,s}$, as in [2,7]). For $1 \leq j \leq n$, define

$$\Delta_{r,s}^{j,N} = \sum_{\xi \in \beta_{r,s}^N} \partial_{\xi_j}^2.$$ 

(3.2)

Note that $\Delta_{r,s}^{j,N}$ and $\Delta_{r,s}^{k,N}$ commute for all $j, k$. Now, fix $t_1, \ldots, t_n > 0$. Then the operator

$$t_1 \Delta_{r,s}^{1,N} + \cdots + t_n \Delta_{r,s}^{n,N}$$

is elliptic, and essentially self-adjoint on $C^\infty_c(\mathbb{GL}_n^N)$. We may therefore use the spectral theorem to define the bounded operator

$$e^t (t_1 \Delta_{r,s}^{1,N} + \cdots + t_n \Delta_{r,s}^{n,N}) = e^{\frac{1}{2} t_1} \Delta_{r,s}^{1,N} \cdots e^{\frac{1}{2} t_n} \Delta_{r,s}^{n,N}.$$ 


Define the heat kernel measure \( \mu_{r,s; t_1,...,t_n}^{n,N} \) on \( \mathbb{GL}_N \) by

\[
\int_{\mathbb{GL}_N^n} f \, d\mu_{r,s; t_1,...,t_n}^{n,N} = \left( e^{\frac{1}{2}t_1 \Delta_{r,s}^N + \cdots + t_n \Delta_{r,s}^N} f \right)(I_N^n), \quad f \in C_c(\mathbb{GL}_N^n),
\]

where \( I_N^n = (I_N, \ldots, I_N) \in \mathbb{GL}_N^n \). In particular, let \( K_1, \ldots, K_n \subset \mathbb{GL}_N \) be compact sets; by approximating \( \mathbb{1}_{K_1 \times \cdots \times K_n} \) with a continuous function, we see that

\[
\mu_{r,s; t_1,...,t_n}^{1,N}(K_1 \times \cdots \times K_n) = \left( e^{\frac{1}{2}t_1 \Delta_{r,s}^N} \mathbb{1}_{K_1} \right)(I_N) \cdots \left( e^{\frac{1}{2}t_n \Delta_{r,s}^N} \mathbb{1}_{K_n} \right)(I_N) = \mu_{r,s; t_1}^{1,N}(K_1) \cdots \mu_{r,s; t_n}^{1,N}(K_n).
\]

Since \( \mu_{r,s; t}^{1,N} \) is the heat kernel measure on \( \mathbb{GL}_N \) corresponding to \( \Delta_{r,s}^N \), it is the distribution of the Brownian motion \( \mathbb{B}_t \), and so we have shown the following.

**Lemma 3.1.** Let \( (B_{r,s}^{1,N}(t))_{t \geq 0}, \ldots, (B_{r,s}^{n,N}(t))_{t \geq 0} \) be \( n \) independent \((r,s)\)-Brownian motions on \( \mathbb{GL}_N \). Then the joint law of the random vector \( (B_{r,s}^{1,N}(t_1), \ldots, B_{r,s}^{n,N}(t_n)) \) is \( \mu_{r,s; t_1,...,t_n}^{n,N} \).

### 3.2 Multivariate Trace Polynomials

Let \( J \) be an index set (for our purposes in this section, we will usually take \( J = \{1, \ldots, n\} \) for some \( n \in \mathbb{N} \)). Let \( \mathcal{E}_J \) denote the set of all nonempty words in \( J \times \{1, *, \} \), \( \mathcal{E}_J = \bigcup_{n \in \mathbb{N}} (J \times \{1, *, \})^n \). Let \( \mathbf{v}_J = \{v_{\varepsilon} : \varepsilon \in \mathcal{E}_J\} \) be commuting variables, and let

\[
\mathcal{P}(J) = \mathbb{C}[\mathbf{v}_J]
\]

be the algebra of (commutative) polynomials in the variables \( \mathbf{v}_J \). That is: as a \( \mathbb{C} \)-vector space, \( \mathcal{P}(J) \) has as its standard basis \( 1 \) together with the monomials

\[
v_{\varepsilon(1)} \cdots v_{\varepsilon(k)}, \quad k \in \mathbb{N}, \quad \varepsilon(1), \ldots, \varepsilon(k) \in \mathcal{E}_J,
\]

and the (commutative) product on \( \mathcal{P}(J) \) is the standard polynomial product.

We may identify monomials in \( \mathbb{C}(X_j, X_j^* : j \in J) \) with the variables \( v_{\varepsilon} \), via

\[
\Upsilon(X_{j_1}^{\varepsilon_{j_1}} \cdots X_{j_k}^{\varepsilon_{j_k}}) = v_{(j_1, \varepsilon_{j_1}) \cdots (j_k, \varepsilon_{j_k})}.
\]

Extending linearly, \( \Upsilon : \mathbb{C}(X_j, X_j^* : j \in J) \rightarrow \mathcal{P}(J) \) is a linear inclusion, identifying \( \mathbb{C}(X_j, X_j^* : j \in J) \) with the linear polynomials in \( \mathcal{P}(J) \). The algebra \( \mathcal{P}(J) \) is the “universal enveloping algebra” of \( \mathbb{C}(X_j, X_j^* : j \in J) \), in the following sense: any linear functional \( \varphi \) on \( \mathbb{C}(X_j, X_j^* : j \in J) \) extends (via \( \Upsilon \)) uniquely to an algebra homomorphism \( \tilde{\varphi} : \mathcal{P}(J) \rightarrow \mathbb{C} \). Conversely, any algebra homomorphism \( \mathcal{P}(J) \rightarrow \mathbb{C} \) is determined by its restriction to \( \mathbb{C}(X_j, X_j^* : j \in J) \), which intertwines a unique linear functional on \( \mathbb{C}(X_j, X_j^* : j \in J) \). Hence, the noncommutative distribution \( \varphi_{\{a_j : j \in J\}} \) of \( J \) random variables can be equivalently represented as an algebra homomorphism \( \mathcal{P}(J) \rightarrow \mathbb{C} \).

**Definition 3.2.** For a monomial \( \varepsilon^{(i)} \), the **trace degree** is defined to be

\[
\text{deg}(v_{\varepsilon^{(1)}} \cdots v_{\varepsilon^{(k)}}) = |\varepsilon^{(1)}| + \cdots + |\varepsilon^{(k)}|,
\]

where \( |\varepsilon| = n \) if \( \varepsilon \in (J \times \{1, *, \})^n \). More generally, if \( P \in \mathcal{P}(J) \), then \( \text{deg}(P) \) is the maximal trace degree of the monomial terms in \( P \). Define \( \text{deg}(0) = 0 \). Note that \( \text{deg}(PQ) = \text{deg}(P) + \text{deg}(Q) \), and \( \text{deg}(P + Q) \leq \max\{\text{deg}(P), \text{deg}(Q)\} \) for \( P, Q \in \mathcal{P}(J) \). For \( d \in \mathbb{N} \), denote by \( \mathcal{P}(J)_d \) the subspace

\[
\mathcal{P}_d(J) = \{ P \in \mathcal{P}(J) : \text{deg}(P) \leq d \}.
\]

Note that \( \mathcal{P}_d(J) \) is finite dimensional (if \( J \) is finite), and \( \mathcal{P}(J) = \bigcup_{d \geq 1} \mathcal{P}_d(J) \).
We now introduce a kind of functional calculus for $\mathcal{P}(J)$.

**Definition 3.3.** Let $(\mathcal{A}, \tau)$ be a noncommutative probability space. Let $J$ be an index set, and let $\{a_j : j \in J\}$ be specified elements in $\mathcal{A}$. For $n \in \mathbb{N}$, and $(J \times \{1, \ast\})^n \ni \varepsilon = ((j_1, \varepsilon_1), \ldots, (j_n, \varepsilon_n))$, define

$$a^\varepsilon \equiv a_{j_1}^{\varepsilon_1} \cdots a_{j_n}^{\varepsilon_n}.$$  

We define for each $P \in \mathcal{P}(J)$ a complex number $P_\tau(a_j : j \in J)$ as follows: for $\varepsilon \in \varepsilon_J$, $[v_\varepsilon]_P(a_j : j \in J) = \tau(a^\varepsilon)$; and, in general, the map $P \mapsto P_\tau(a_j : j \in J)$ is an algebra homomorphism from $\mathcal{P}(J)$ to $\mathbb{C}$.

In other words: $P_\tau$ is the unique algebra homomorphism extending (via $\Upsilon$) the linear functional $\varphi(a_j : j \in J)$ on $\mathbb{C}(X_j, X_j^* : j \in J)$ (i.e. the noncommutative distribution of $\{a_j : j \in J\}$).

**Example 3.4.** Let $J = \{1, 2\}$, and consider $\mathcal{P}(J) \ni P = v_{(1, \ast), (2,1), (1,1)} - 2v_{(2,1)}$, which has trace degree 3; then

$$P_\tau(a_1, a_2) = \tau(a_1^2 a_2 a_1) - 2(\tau(a_2))^2.$$

We generally refer to the functions $\{P_\tau : P \in \mathcal{P}(J)\}$ as (multivariate) **trace polynomials**.

**Notation 3.5.** For $N \in \mathbb{N}$, in the noncommutative probability space $(\mathbb{M}_N, \text{tr})$, we denote the evaluation map $P \mapsto P_{\text{tr}}$ of Definition 3.3 as $P \mapsto P_N$. Thus, if $A_1, \ldots, A_n \in \mathbb{M}_N \otimes L^\infty$, and $P$ is as in Example 3.4 then

$$P_N(A_1, \ldots, A_n) = \text{tr}(A_1^2 A_2 A_1) - 2(\text{tr}(A_2))^2,$$

which is a random variable, to be clear.

### 3.3 Intertwining Formula

The following “magic formulas” appeared as [7, Proposition 1]; note that (2.19) is a special case of (3.5).

**Proposition 3.6.** Let $\beta_N$ be an orthonormal basis for $u_N$ with respect to the inner product (2.5). Then for any $A \in \mathbb{M}_N$

\begin{align*}
\sum_{\xi \in \beta_N} \xi A \xi &= -\text{tr}(A)I_N, \quad (3.5) \\
\sum_{\xi \in \beta_N} \text{tr}(A \xi) \xi &= -\frac{1}{N^2} A. \quad (3.6)
\end{align*}

For the remainder of this section, we usually suppress the indices $r, s$ for notational convenience; so, for example, $\Delta^{\beta_N} \equiv \Delta_{\beta_N}^{\beta_N}$ for $1 \leq j \leq n$. Let $J = \{1, \ldots, n\}$ throughout.

**Theorem 3.7.** Let $j \in J$. There are collections $\{Q^j_\varepsilon : \varepsilon \in \varepsilon_J\}$ and $\{R^j_{\varepsilon, \delta} : \varepsilon, \delta \in \varepsilon_J\}$ in $\mathcal{P}(J)$ with the following properties.

1. For each $\varepsilon \in \varepsilon_J$, $Q^j_\varepsilon$ is a finite sum of monomials of homogeneous trace degree $|\varepsilon|$ such that

$$\Delta^{\beta_N}([v_\varepsilon]_N) = [Q^j_\varepsilon]_N.$$

2. For each $\varepsilon, \delta \in \varepsilon_J$, $R^j_{\varepsilon, \delta}$ is a finite sum of monomials of homogeneous trace degree $|\varepsilon| + |\delta|$ such that

$$r \sum_{\xi \in \beta_N} (\partial_{\xi} [v_\varepsilon]_N)(\partial_{\xi} [v_\delta]_N) + s \sum_{\xi \in \beta_N} (\partial_{\xi} [v_\varepsilon]_N)(\partial_{\xi} [v_\delta]_N)N = \frac{1}{N^2} [R^j_{\varepsilon, \delta}]_N,$$

for any orthonormal basis $\beta_N$ of $u_N$. 

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Please note that \( Q^\pm_\varepsilon \) and \( R^\pm_{\varepsilon,\delta} \) do not depend on \( N \). The \( 1/N^2 \) in (2) comes from the magic formula (3.6), as we will see in the proof.

**Proof.** Fix \( \varepsilon_{J} \ni \varepsilon = ((j_1, \varepsilon_1), \ldots, (j_m, \varepsilon_m)) \); then \([v\varepsilon]_N(A_1, \ldots, A_n) = \text{tr}(A_{j_1}^{\varepsilon_1} \cdots A_{j_m}^{\varepsilon_m})\). Applying the product rule, for any \( \xi \in \beta_N \) we have

\[
\partial^2_{\xi_j} ([v\varepsilon]_N) = \sum_{k=1}^{m} \delta_{j,k} \text{tr}(A_{j_1}^{\varepsilon_1} \cdots (A_{j_k}^{\varepsilon_k} \xi_k) \cdots A_{j_m}^{\varepsilon_m}) \\
+ 2 \sum_{1 \leq k < \ell \leq m} \delta_{j,k} \delta_{j,\ell} \text{tr}(A_{j_1}^{\varepsilon_1} \cdots (A_{j_k}^{\varepsilon_k} \xi_k) \cdots (A_{j_\ell}^{\varepsilon_\ell} \xi_\ell) \cdots A_{j_m}^{\varepsilon_m}).
\]

(3.7) Similarly, \( \partial^2_{\xi_j} \) is given by the same formula but possibly with some minus signs in some of the terms (depending on \( \varepsilon_k, \varepsilon_\ell \)). For convenience, let \( \beta^+_{\varepsilon} = \beta_N \) and \( \beta^-_{\varepsilon} = i \beta_N \). Magic formula (3.9) gives \( \sum_{\xi \in \beta^+_{\varepsilon}} \xi^2 = \mp I_N \), and so summing over \( \beta^\pm_{\varepsilon} \) we have, for each \( k \),

\[
\sum_{\xi \in \beta^\pm_{\varepsilon}} \text{tr}(A_{j_1}^{\varepsilon_1} \cdots (A_{j_k}^{\varepsilon_k} \xi_k) \cdots A_{j_m}^{\varepsilon_m}) = \pm [v\varepsilon]_N,
\]

where the \( \pm \) on the left and right do not necessarily match (we will not keep careful track of signs through this proof). Thus, (3.7) summed over \( \beta^\pm_{\varepsilon} \) gives some integer multiple \( n_\varepsilon^\pm \) of \([v\varepsilon]_N\). Summing the terms in (3.8) over \( \xi \in \beta^\pm_{\varepsilon} \), using (3.5), yields

\[
\sum_{\xi \in \beta^\pm_{\varepsilon}} \text{tr}(A_{j_1}^{\varepsilon_1} \cdots (A_{j_k}^{\varepsilon_k} \xi_k) \cdots A_{j_m}^{\varepsilon_m}) = \pm [v_{\varepsilon_{k,\ell}}]_N [v_{\varepsilon_{k,\ell}'}]_N,
\]

where \( \varepsilon_{k,\ell} \) is a substring of \( \varepsilon \) (running between index \( k \) or \( k + 1 \) and index \( \ell - 1 \) or \( \ell \), depending on \( \varepsilon_k, \varepsilon_\ell \)) and \( \varepsilon_{k,\ell}' \) is the concatenation of the two remaining substrings of \( \varepsilon \) when \( \varepsilon_{k,\ell} \) is removed. Hence, define

\[
Q^\pm_\varepsilon = n_\varepsilon^\pm(\varepsilon) v_\varepsilon + 2 \sum_{1 \leq k < \ell \leq m} \pm \delta_{j,k} \delta_{j,\ell} v_{\varepsilon_{k,\ell}} v_{\varepsilon_{k,\ell}'}.
\]

Note that \( |\varepsilon| = |\varepsilon_{k,\ell}| + |\varepsilon_{k,\ell}'| \) for each \( k, \ell \); so \( Q^\pm_\varepsilon \) are homogeneous of trace degree \( |\varepsilon| \). The above argument shows that

\[
\sum_{\xi \in \beta^\pm_{\varepsilon}} \partial^2_{\xi_j} [v\varepsilon]_N = [Q^\pm_\varepsilon]_N,
\]

and so setting \( Q^\pm_\varepsilon = r Q^\varepsilon^+ + s Q^\varepsilon^- \) completes item (1) of the theorem.

For item (2), fix \( \varepsilon_{J} \ni \delta = ((h_1, \delta_1), \ldots, (h_p, \delta_p)) \); then \([v\delta]_N(A_1, \ldots, A_n) = \text{tr}(A_{h_1}^{\delta_1} \cdots A_{h_p}^{\delta_p})\). Thus, for \( \xi \in \beta_N \),

\[
(\partial_\xi [v\varepsilon]_N)(\partial_\xi [v\delta]_N) = \sum_{k=1}^{m} \sum_{\ell=1}^{p} \delta_{j,k} \delta_{j,\ell} \text{tr}(A_{j_1}^{\varepsilon_1} \cdots (A_{j_k}^{\varepsilon_k} \xi_k) \cdots A_{j_m}^{\varepsilon_m}) \text{tr}(A_{h_1}^{\delta_1} \cdots (A_{h_\ell}^{\delta_\ell} \xi_\ell) \cdots A_{h_p}^{\delta_p}).
\]

(3.9) (To be clear: the terms \( \delta_{j,k} \delta_{j,\ell} \) are indicator functions, not related to the string \( \delta \in \varepsilon_{J} \).) Taking \( \partial_\xi \xi_j \) instead yields the same formula, possibly with some minus signs inside the sum (depending on \( \varepsilon_k \) and \( \delta_\ell \)). We can write each term in (3.9) in the form

\[
\pm \text{tr}(\xi A^{(k)}) \text{tr}(\xi A^{(\ell)})
\]

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where $ε^{(k)}$ and $δ^{(l)}$ are certain cyclic permutations of $ε$ and $δ$. Using (3.6), summing over $ξ ∈ β_{N}^{+}$, then yields

$$
\frac{1}{N^2} \sum_{k=1}^{m} \sum_{ℓ=1}^{p} ±\delta_{j,k} δ_{j,h_{ε}[v_{ε}(k)δ(ℓ)]} f_{N},
$$

where $ε^{(k)} δ^{(ℓ)}$ denotes the concatenation; in particular, $|ε^{(k)} δ^{(ℓ)}| = |ε| + |δ|$. Thus, setting

$$
R_{ε,δ}^{j,±} = \sum_{k=1}^{m} \sum_{ℓ=1}^{p} ±\delta_{j,k} δ_{j,h_{ε}[v_{ε}(k)δ(ℓ)]}
$$

(where the ± on the two sides do not necessarily match), we have shown that

$$
\sum_{ξ ∈ β_{N}^{+}} (∂_{ξ}[v_{ε}] f_{N})(∂_{ξ}[v_{δ}] f_{N}) = \frac{1}{N^2} [R_{ε,δ}^{j,±}] f_{N}.
$$

Set $R_{ε,δ}^{j} \equiv r R_{ε,δ}^{j,±} + s R_{ε,δ}^{j,−}$; then $R_{ε,δ}^{j}$ has homogeneous trace degree $|ε| + |δ|$, and so satisfies item (2), concluding the proof of the theorem.

**Theorem 3.8 (Intertwining Formula).** For $j ∈ J$, let $\{Q_{ε}^{j}: ε ∈ Ε_{j}\}$ and $\{R_{ε,δ}^{j} : ε, δ ∈ Ε_{j}\}$ be the collections in $Φ(J)$ given in Theorem 3.7. Define the following operators on $Φ(J)$:

$$
D_{r,s}^{j} = \sum_{ε ∈ Ε_{j}} Q_{ε}^{j} ∂_{υ_ε} \quad \text{and} \quad L_{r,s}^{j} = \sum_{ε,δ ∈ Ε_{j}} R_{ε,δ}^{j} ∂_{υ_ε} ∂_{υ_δ}. \quad (3.10)
$$

Then $D_{r,s}^{j}$ and $L_{r,s}^{j}$ preserve trace degree (when $(r, s) ≠ (0, 0)$), and, for all $P ∈ Φ(J)$,

$$
Δ_{r,s}^{j,N}([P] f_{N}) = \left[ \left( D_{r,s}^{j} + \frac{1}{N^2} L_{r,s}^{j} \right) P \right] f_{N}. \quad (3.11)
$$

**Proof.** The proof is almost identical to the proof of [7, Theorem 3.26]; we repeat it here. Let $V_{N} : G_{L_{N}} → M_{n}^{Ε_{j}}$ be the map

$$(V_{N}(A_{1}, \ldots, A_{n}))(j_{1}, ε_{1}), \ldots, (j_{m}, ε_{m})) = tr(A_{j_{1}}^{ε_{1}} \cdots A_{j_{m}}^{ε_{m}}).
$$

Then, by definition, $[P] f_{N} = P ∘ V_{N}$. By the chain rule, if $ξ ∈ gl_{N}$ then

$$
∂^{2} V_{N} P_{N} = \partial^{2} P ∘ V_{N} = \sum_{ε ∈ Ε_{j}} \partial_{ξ}[v_{ε}] \left[ \left( ∂ P ∂_{υ_ε} \right) (V_{N}) \cdot \partial_{ξ}[v_{ε}] f_{N} \right]
$$

$$
= \sum_{ε ∈ Ε_{j}} \left( ∂ P ∂_{υ_ε} \right) (V_{N}) \cdot \partial^{2} V_{N} f_{N} + \sum_{ε, δ ∈ Ε_{j}} \left( ∂^{2} P ∂_{υ_ε} ∂_{υ_δ} \right) (V_{N}) \cdot \partial_{ξ}[v_{ε}] f_{N} \cdot \partial_{ξ}[v_{δ}] f_{N}
$$

from which it follows that

$$
Δ_{r,s}^{j,N} P_{N} = \sum_{ε ∈ Ε_{j}} \left( ∂ P ∂_{υ_ε} \right) (V_{N}) \cdot Δ_{r,s}^{j,N} f_{N}
$$

$$
+ \sum_{ε, δ ∈ Ε_{j}} \left( ∂^{2} P ∂_{υ_ε} ∂_{υ_δ} \right) (V_{N}) \cdot \left[r \sum_{ξ ∈ Β_{N}} (∂_{ξ}[v_{ε}](∂_{ξ}[v_{δ}]) + s \sum_{ξ ∈ Β_{N}} (∂_{ξ}[v_{δ}]) \cdot (∂_{ξ}[v_{δ}]) f_{N} \right].
$$

Combining this equation with the results of Theorem 3.7 completes the proof. □
This prompts us to define the following operators.

**Definition 3.9.** Let \( t = (t_1, \ldots, t_n) \) for some \( t_1, \ldots, t_n \geq 0 \). Define

\[
D^t_{r,s} = \frac{1}{2} \sum_{j=1}^{n} t_j D^j_{r,s}, \quad L^t_{r,s} = \frac{1}{2} \sum_{j=1}^{n} t_j L^j_{r,s}.
\]

**Corollary 3.10.** For any \( t = (t_1, \ldots, t_n) \in \mathbb{R}_{+}^n \), and \( d \in \mathbb{N} \), \( D^t_{r,s} \) and \( L^t_{r,s} \) preserve the finite dimensional space \( \mathcal{P}_d(J) \), and

\[
e^{\frac{1}{2} t_1 \Delta_{r,s}^1 + \cdots + t_n \Delta_{r,s}^n} P_N = [e^{\frac{1}{2} D^t_{r,s} + \frac{1}{N^2} c^t_{r,s} P}]_N, \quad P \in \mathcal{P}_d(J).
\]

In particular, \( e^{\frac{1}{2} D^t_{r,s} + \frac{1}{N^2} c^t_{r,s}} \) and \( e^{\frac{1}{2} L^t_{r,s}} \) are well-defined operators on the space \( \mathcal{P}(J) \).

**Proof.** Since \( D^j_{r,s} \) and \( L^j_{r,s} \) preserve trace degree, the corollary follows by expanding the exponentials as power series of operators acting on the finite dimensional spaces \( \mathcal{P}_d(J) \) and \( \mathcal{P}(J) \).

**Remark 3.11.** Since \( \Delta_{r,s}^j \) commute for \( 1 \leq j \leq n \), it is natural to expect the same holds for the intertwining operators \( D^j_{r,s} \) and \( L^j_{r,s}. \) This is true, and follows easily from examining the explicit form of the coefficients of these operators given in Theorem 3.7. One must be careful about drawing such conclusions in general, however; the map \( P \mapsto P_N \) is generally not one-to-one, due to the Cayley-Hamilton Theorem. It is asymptotically one-to-one, in the sense that its restriction to \( \mathcal{P}_d(J) \) is one-to-one for all sufficiently large \( N \) (depending on \( d \)), and this can be used to prove this commutation result. Note, however, that \( [D^j_{r,s}, L^j_{r,s}] \neq 0 \) in general.

### 3.4 Concentration of Measure

We restate a general linear algebra result here, given as \([7, \text{Lemma 4.1}]\).

**Lemma 3.12.** Let \( V \) be a finite dimensional normed \( \mathbb{C} \)-space and supposed that \( D \) and \( L \) are two operators on \( V \). Then there exists a constant \( C = C(D, L, \| \cdot \|_V) < \infty \) such that

\[
\| e^{D+eL} - e^D \|_{\text{End}(V)} \leq C |e| \text{ for all } |e| \leq 1,
\]

where \( \| \cdot \|_{\text{End}(V)} \) is the operator norm on \( V \). It follows that, if \( \psi \in V^* \) is a linear functional, then

\[
|\psi(e^{D+eL}x) - \psi(e^D x)| \leq C \|\psi\|_{V^*} \|x\|_V |e|, \quad x \in V, \ |e| \leq 1,
\]

where \( \| \cdot \|_{V^*} \) is the dual norm on \( V^* \).

Coupled with Corollary 3.10, this gives the following.

**Proposition 3.13.** Let \( P \in \mathcal{P}(J) \). Let \( t = (t_1, \ldots, t_n) \in \mathbb{R}_{+}^n \). Then there is a constant \( C = C(r, s, t, P) \) so that, for all \( N \in \mathbb{N} \),

\[
\left| \int_{\text{GL}_N} P_N d\mu_{r,s; t}^{n,N} - \left( e^{\frac{1}{2} D_{r,s}^t} P \right)(1) \right| \leq \frac{C}{N^2},
\]

where, for \( Q \in \mathcal{P}(J) \), \( Q(1) \) is the complex number given by evaluating all variables of \( Q \) at 1.

**Proof.** Let \( d = \deg(P) \); then \( P \in \mathcal{P}_d(J) \). By definition 3.3,

\[
\int_{\text{GL}_N} P_N d\mu_{r,s; t}^{n,N} = \left( e^{\frac{1}{2} D_{r,s}^t} P_N \right)(J^N).
\]
Thus concluding the proof.

Note that $\psi_1(P) = P(1)$ is a linear functional on the finite dimensional space $\mathcal{P}_d(J)$; thus the result follows from (3.13) by choosing any norm $\| \cdot \|_{\mathcal{P}_d(J)}$ on $V = \mathcal{P}_d(J)$, and setting
\[
C(r, s, t, P) = C(\mathcal{D}_r^t, \mathcal{L}_r^t, \| \mathcal{P}_d(J) \| \psi_1 \| \mathcal{P}_d(J) \| P) \| \mathcal{P}_d(J),
\]
thus concluding the proof.

We now come to the main theorems of this section.

**Theorem 3.14.** Let $(B_{r,s}^1(t))_{t \geq 0}, \ldots, (B_{r,s}^n(t))_{t \geq 0}$ be independent Brownian motions on $\mathbb{G}_L_N$. Then these matrix processes have a joint limit distribution: for any $m \in \mathbb{N}, j_1, \ldots, j_m \in \{1, \ldots, n\}, t_1, \ldots, t_n \geq 0$ and $\varepsilon_1, \ldots, \varepsilon_m \in \{1, \ast\}$,
\[
\lim_{N \to \infty} \mathbb{E}r(B_{j_1, \varepsilon_1}^1(t_{j_1}) \cdot \cdots \cdot B_{j_m, \varepsilon_m}^1(t_{j_m})^{\varepsilon_m}) \quad \exists.
\]

**Proof.** Let $t = (t_1, \ldots, t_n)$. The given expected trace is computed in terms of the joint law $\mu_{r,s,t}^{n,N}$ of the independent Brownian random matrices as
\[
\mathbb{E}r(B_{j_1, \varepsilon_1}^1(t_{j_1}) \cdot \cdots \cdot B_{j_m, \varepsilon_m}^1(t_{j_m})^{\varepsilon_m}) = \int_{\mathcal{G}_L_N} \text{tr}(A_{j_1}^1 \cdots A_{j_m}^m) \mu_{r,s,t}^{n,N}(dA_1 \cdots dA_n) = \int_{\mathcal{G}_L_N} [v_\varepsilon]_N \mu_{r,s,t}^{n,N}
\]
where $\varepsilon = ((j_1, \varepsilon_1), \ldots, (j_m, \varepsilon_m))$. Proposition 3.13 thus shows that the limit as $N \to \infty$ exists, and is equal to $\left(e^{\mathcal{D}_r^t, \mathcal{L}_r^t}v_\varepsilon\right)(1)$.

**Theorem 3.15.** Let $P, Q \in \mathcal{P}(J)$, and let $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. There is a constant $C_2 = C_2(r, s, t, P, Q)$ such that
\[
\left| \text{Cov}_{\mu_{r,s,t}^{n,N}}(P_N, Q_N) \right| \leq \frac{C_2}{N^2}.
\]

**Proof of Theorem 3.15** The covariance of $C$-valued random variables $F, G$ is $\text{Cov}(F, G) = \mathbb{E}(FG) - \mathbb{E}(F)\mathbb{E}(G)$. Define $\mathcal{D}_r^t = \mathcal{D}_r^t + \frac{1}{N^2} \mathcal{L}_r^t$. From Lemma 3.16, we may write $P_N Q_N = [PQ^*]_N$, and so, from (3.3) and Corollary 3.10 we have
\[
\mathbb{E}_{\mu_{r,s,t}^{n,N}}(P_N Q_N) = \left(e^{\mathcal{D}_r^t, \mathcal{L}_r^t}P\cdot Q^*\right)(1).
\]

Similarly,
\[
\mathbb{E}_{\mu_{r,s,t}^{n,N}}(P_N) \cdot \mathbb{E}_{\mu_{r,s,t}^{n,N}}(Q_N) = \left(e^{\mathcal{D}_r^t, \mathcal{L}_r^t}P\right)(1) \cdot \left(e^{\mathcal{D}_r^t, \mathcal{L}_r^t}Q^*\right)(1).
\]
Now, set
\[
\Psi_1^N \equiv (e^{-D_{r,s}^t N} P)(1), \quad \Psi_*^N \equiv (e^{-D_{r,s}^t N} Q^*)(1), \quad \Psi_{1,*}^N \equiv (e^{-D_{r,s}^t N} (PQ^*)) (1), \quad (3.18)
\]
\[
\Psi^t \equiv (e^{-D_{r,s}^t N} P)(1), \quad \Psi_* \equiv (e^{-D_{r,s}^t N} Q^*)(1), \quad \Psi_{1,*} \equiv (e^{-D_{r,s}^t N} (PQ^*)) (1). \quad (3.19)
\]

Thus, (3.16) and (3.17) show that
\[
\text{Cov}_{\mu_{r,s}^N} (P_N, Q_N) = \Psi_{1,*}^N - \Psi_{1,*}^N \Psi_*^N. \quad (3.20)
\]

We estimate this as follows. First
\[
|\Psi_{1,*}^N - \Psi_{1,*}^N \Psi_*^N| \leq |\Psi_*^N| |\Psi_{1,*}^N| + |\Psi_{1,*}^N - \Psi_{1,*}^N| + |\Psi_{1,*}^N - \Psi_{1,*}^N \Psi_*^N| \quad (3.21)
\]

Referring to (3.19), note that $D_{r,s}^t$ is a first-order differential operator; it follows that $e^{-D_{r,s}^t N}$ is an algebra homomorphism, and so the second term in (3.21) is 0. The first term is bounded by $\frac{1}{N^2} \cdot C(r, s, t, PQ^*)$ by Proposition 3.13. For the third term, we add and subtract $\Psi_{1,*}^N \Psi_*^N$ to make the additional estimate
\[
|\Psi_{1,*}^N - \Psi_{1,*}^N \Psi_*^N| \leq |\Psi_*^N| |\Psi_{1,*}^N| + (|\Psi_1^N| + |\Psi_{1,*}^N - \Psi_{1,*}^N||) |\Psi_{1,*}^N - \Psi_{1,*}^N| \leq \frac{1}{N^2} \cdot |\Psi_*^N| C (r, s, t, P) + \left( |\Psi_1^N| + \frac{1}{N^2} \cdot C(r, s, t, P) \right) \cdot \frac{1}{N^2} \cdot C(r, s, t, Q^*)
\]
\[
= \frac{1}{N^2} \cdot (|\Psi_*^N| C (r, s, t, P) + |\Psi_{1,*}^N| C (r, s, t, Q^*)) + \frac{1}{N^2} \cdot C(r, s, t, P) C(r, s, t, Q^*). \quad (3.22)
\]

Combining (3.22) with (3.20) - (3.21) and the following discussion shows that the constant
\[
C_2 (r, s, t, P, Q) = C (r, s, t, PQ^*) + C (r, s, t, P) C(r, s, t, Q^*) + |\Psi_*^N| C (r, s, t, P) + |\Psi_{1,*}^N| C (r, s, t, Q^*) \quad (3.23)
\]
verifies (3.14), proving the proposition.

This brings us to the proof of Theorem 1.13. For convenience, we restate that the desired estimate is
\[
\text{Cov} \left[ \text{tr} (f(B_{1,r,s}^{1,N} (t_1), \ldots, B_{r,s}^{n,N} (t_n)^*)) \right] \leq \frac{C_2}{N^2}, \quad (3.24)
\]
for any $f, g \in \mathbb{C}(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*)$, for some constant $C_2 = C_2 (r, s, t, f, g)$; here $B_{1,r,s}^{1,N} (\cdot), \ldots, B_{r,s}^{n,N} (\cdot)$ are independent $(r, s)$-Brownian motions on $\mathbb{G}L_N$.

**Proof of Theorem 1.13** Setting $t = (t_1, \ldots, t_n)$, the covariance in (3.24) is precisely
\[
\text{Cov}_{\mu_{r,s}^N} ([\Upsilon (f)]_N, [\Upsilon (g)]_N)
\]
and so the result follows immediately from Theorem 3.15.

**Corollary 3.17**. Let $(B_{r,s}^{1,N} (t))_{t \geq 0}, \ldots, (B_{r,s}^{n,N} (t))_{t \geq 0}$ be independent Brownian motions on $\mathbb{G}L_N$. Then for any $t_1, \ldots, t_n \geq 0$ and any $f \in \mathbb{C}(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*)$, the random variable $\text{tr} (f(B_{r,s}^{1,N} (t_1), \ldots, B_{r,s}^{n,N} (t_n)^*))$ converges to its mean almost surely.
This follows immediately from the $O(1/N^2)$ covariance estimate of Theorem 1.13 together with Chebyshev’s inequality and the Borel-Cantelli lemma.

Finally, we note that we have proven asymptotic freeness of independent $(r, s)$-Brownian motions.

**Corollary 3.18.** Let $t_1, \ldots, t_n \geq 0$ and let $B_{r,s}^{1,N}(t_1), \ldots, B_{r,s}^{n,N}(t_n)$ be independent random matrices sampled from $(r, s)$-Brownian motion. Then these random matrices are asymptotically free.

**Proof.** As pointed out in Remark 1.4, the distribution of each $B_{r,s}^{j,N}(t_j)$ is invariant under $U_N$-conjugation. Theorems 1.13 and 3.14 then confirm all of the conditions of Theorem 2.7, which demonstrates the asymptotic freeness as claimed.

## 4 Moment Calculations

This section is devoted to the proof of Theorem 1.9. We begin by reiterating the following differential characterization of the constants $\nu_n(t)$ from (1.6).

**Lemma 4.1.** Let $\{\nu_n : n \geq 0\}$ be the functions in (1.6), and let $\varrho_n(t) = e^{\frac{1}{2}(r-s)t} \nu_n(t)$. The functions $\varrho_n$ are uniquely determined by the initial conditions $\varrho_n(0) = \nu_n(0) = 1$ for all $n$, $\varrho_1(t) \equiv 1$, and the following system of coupled linear ODEs for $n \geq 2$:

$$
\varrho'_n(t) = -n \sum_{k=1}^{n-1} k \varrho_k(t) \varrho_{n-k}(t).
$$

Indeed, in [2], this connection was the key step in identifying the distribution of a free unitary Brownian motion as the limit distribution (at each fixed time $t$) of a Brownian motion $U_N^t$ on $U_N$. It is also independently proved in [7, Lemma 5.4, Eq. (5.23)].

**Lemma 4.2.** Let $b_{r,s}(t)$ be defined by (1.3); for short, let $b = b_{r,s}(t)$. Set $a = a_{r,s}(t) = e^{\frac{1}{2}(r-s)t} b$. Then

$$
da = a \, dw,
$$

where $w = w_{r,s}(t)$ of (1.2).

**Proof.** Since $t \mapsto e^{\frac{1}{2}(r-s)t}$ is a free Itô process with $de^{\frac{1}{2}(r-s)t} = \frac{1}{2}(r-s)e^{\frac{1}{2}(r-s)t} \, dt$, (2.23) shows that

$$
da = de^{\frac{1}{2}(r-s)t} \cdot b + e^{\frac{1}{2}(r-s)t} \cdot db + de^{\frac{1}{2}(r-s)t} \cdot db.
$$

The last term is 0, while the first two simplify to

$$
da = \frac{1}{2}(r-s)e^{\frac{1}{2}(r-s)t}b \, dt + e^{\frac{1}{2}(r-s)t}(b \, dw - \frac{1}{2}(r-s)b \, dt) = a \, dw,
$$

by (1.3).

We also record the following Itô formula for $dw_{r,s}(t)$ products.

**Lemma 4.3.** Let $t \geq 0$ and let $\varepsilon, \varepsilon' \in \{1, *\}$. For any adapted process $\theta = \theta(t)$,

$$
dw^{\varepsilon} \theta \, dw^{\varepsilon'} = (s \pm r) \tau(\theta) \, dt,
$$

where the sign is $-$ if $\varepsilon = \varepsilon'$ and $+$ if $\varepsilon \neq \varepsilon'$.

Lemma 4.3 is an immediate computation from (2.20) – (2.22).
4.1 The Moments of $b_{r,s}(t)$

We use (4.1) to give a recursive formula for the powers of $a_{r,s}(t)$.

**Proposition 4.4.** For $n \in \mathbb{N}^*$,

$$
\left.
\begin{array}{c}
d(a^n) = \sum_{k=1}^{n} a^k \, dw \, a^{n-k} + (s-r) \mathbb{1}_{n \geq 2} \sum_{k=1}^{n-1} ka^k \tau(a^{n-k}) \, dt. \\
\end{array}
\right\} \tag{4.3}
$$

**Proof.** When $n = 1$, (4.3) reduces to (4.1). We proceed by induction, supposing that (4.3) has been verified up to level $n$. Then, using the Itô product rule (2.23), together with (4.1) and (4.3), gives

$$
\left.
\begin{array}{c}
d(a^{n+1}) = d(a \cdot a^n) = da \cdot a^n + a \cdot d(a^n) + da \cdot d(a^n) \\
= a \, dw \, a^n + \sum_{k=1}^{n} a^{k+1} \, dw \, a^{n-k} + (s-r) \sum_{k=1}^{n-1} ka^{k+1} \tau(a^{n-k}) \, dt + \sum_{k=1}^{n} a \, dw \, a^k \, dw \, a^{n-k}. \\
\end{array}
\right\}
$$

The first two terms combine, reindexing $\ell = k + 1$, to give $\sum_{\ell=1}^{n+1} a^\ell \, dw \, a^{n+1-\ell}$. From (4.2), the last terms are

$$
(s-r) \sum_{k=1}^{n} \tau(a^k) a^{n+1-k} \, dt
$$

which, when combined with the penultimate terms, yields (4.3) at level $n + 1$. This concludes the inductive proof.

**Corollary 4.5.** The moments of $a = a_{r,s}(t)$ are $\tau(a^n) = \varrho_n((r-s)t)$; consequently, the moments of $b = b_{r,s}(t)$ are $\tau(b^n) = \nu_n((r-s)t)$, verifying (1.7).

**Proof.** Since $a(0) = b(0) = 1$, $\tau(a(0)^n) = 1 = \varrho_n(0)$. Taking the trace of (4.3) and using (2.19), we have

$$
\left.
\begin{array}{c}
d\tau(a^n) = (s-r) \mathbb{1}_{n \geq 2} \sum_{k=1}^{n-1} k \tau(a^k) \tau(a^{n-k}) \, dt. \\
\end{array}
\right\} \tag{4.4}
$$

Thus $\frac{d}{dt} \tau(a) = 0 = \varrho'_1((r-s)t)$. If $s = r$, (4.4) asserts that $\tau(a^n) = \tau(a(0)^n) = 1 = \varrho_n(0 \cdot t)$ for all $n$. On the other hand, if $s \neq r$, let $\tilde{\varrho}_n(t) = \tau(a_{r,s}(t/(r-s))^n)$; then the chain rule applied to (4.4) shows that

$$
\tilde{\varrho}'_n(t) = -\mathbb{1}_{n \geq 2} \sum_{k=1}^{n-1} k \tilde{\varrho}_k(t) \tilde{\varrho}_{n-k}(t).
$$

By Lemma 4.1 it follows that $\tilde{\varrho}_n(t) = \varrho_n(t)$ for all $n, t \geq 0$. Hence, $\tau(a_{r,s}(t)^n) = \varrho_n((r-s)t) = e^{\frac{-1}{2}((r-s)t^2)} \nu_n((r-s)t)$, as claimed. As defined in Lemma 4.2, we therefore have

$$
\tau(b^n) = \tau(e^{-\frac{1}{2}((r-s)t)}a^n) = e^{-\frac{1}{2}((r-s)t)} \varrho_n((r-s)t) = \nu_n((r-s)t),
$$

verifying (1.7), and concluding the proof.
Lemma 4.6. Let \( c_{r,s}(t) = e^{-st}b_{r,s}(t) \); for short, let \( c = c_{r,s}(t) \). Then
\[
d(cc^*) = 2\sqrt{s} \, c \, dy \, c^*,
\]
where \( y = y(t) \).

Proof. First note that \( cc^* = e^{-2st}bb^* \). As in Lemma 4.2, we have
\[
d(cc^*) = -2s \, cc^* \, dt + e^{-2st}d(bb^*).
\]
By the Itô product rule (2.23) and (1.3),
\[
d(bb^*) = db \cdot b^* + b \cdot db^* + db \cdot db^*
\]
\[
= b \, dw \, b^* - \frac{1}{2} (r - s)bb^* \, dt + b \, dw^* \, b^* - \frac{1}{2} (r - s)bb^* \, dt + b \, dw \, dw^* \, b^*
\]
\[
= (dw + dw^*)b^* - (r - s)bb^* + (r + s)bb^*
\]
where the last equality follows from Lemma 4.3. Note that \( dw + dw^* = 2\sqrt{s} \, dy \), and so this simplifies to
\[
d(bb^*) = 2\sqrt{s} \, b \, dy \, b^* + 2s \, bb^* \, dt.
\]
Combining this with (4.6) yields the result.

Proposition 4.7. For \( n \in \mathbb{N}^* \),
\[
d[(cc^*)^n] = 2\sqrt{s} \sum_{k=1}^{n} (cc^*)^{k-1} c \, dy \, c^* \, (cc^*)^{n-k} + 4s \sum_{k=2}^{n-1} k \, (cc^*)^k \tau \, ((cc^*)^{n-k}) \, dt.
\]

Proof. When \( n = 1 \), (4.7) reduces to (4.6), so we proceed by induction: suppose that (4.7) has been verified up to level \( n \). Then we use the Itô product formula (2.23), together with (4.6) and (4.7), to compute
\[
d[(cc^*)^{n+1}] = d(cc^*) \cdot (cc^*)^n + cc^* \cdot d[(cc^*)^n] + d(cc^*) \cdot d[(cc^*)^n]
\]
\[
= 2\sqrt{s} \, c \, dy \, c^* \, (cc^*)^n + 2\sqrt{s} \sum_{k=1}^{n} (cc^*)^k \, c \, dy \, c^* \, (cc^*)^{n-k} + 4s \sum_{k=1}^{n-1} k \, (cc^*)^k \, \tau \, ((cc^*)^{n-k}) \, dt
\]
\[
+ 4s \sum_{k=1}^{n} c \, dy \, c^* \, (cc^*)^{k-1} \, c \, dy \, c^* \, (cc^*)^{n-k}.
\]
Reindexing \( \ell = k + 1 \), the first two terms combine to give
\[
2\sqrt{s} \sum_{\ell=1}^{n+1} (cc^*)^{\ell-1} \, c \, dy \, c^* \, (cc^*)^{n+1-\ell}.
\]
In the last term, we use (2.20) to yield
\[
dy \, c^* \, (cc^*)^{k-1} \, c \, dy = \tau \, (cc^*)^k \, \delta \, dt = \tau \, ((cc^*)^k) \, dt.
\]
Hence, reindexing \( j = n+1-k \), the final sum is
\[
4s \sum_{k=1}^{n} \tau \, ((cc^*)^k) \, (cc^*)^{n+1-k} \, dt = 4s \sum_{j=1}^{n} \tau \, ((cc^*)^j) \, (cc^*)^{n+1-j}.
\]
Also reindexing the penultimate sum with \( \ell = k+1 \), the last two sums combine to give
\[
4s \sum_{\ell=2}^{n} (\ell - 1) \, (cc^*)^\ell \, \tau \, ((cc^*)^{n+1-\ell}) \, dt + 4s \sum_{j=1}^{n} \tau \, ((cc^*)^{n+1-j}.
\]
Note that the first sum could just as well be started at \( \ell = 1 \) (since that term is 0), and these two combine to give the second term in (4.7), concluding the inductive proof.
Corollary 4.8. The moments of $cc^*$ are $\tau[(cc^*)^n] = \varrho_n(-4st)$; consequently, the moments of $bb^*$ are $\tau[(bb^*)^n] = \nu_n(-4st)$, verifying (1.8).

Proof. Since $b(0) = 1$, $\tau[(cc^*(0))^n] = 1 = \varrho_n(0)$ for all $n$. Taking the trace of (4.7), we have

$$d\tau[(cc^*)^n] = 4s1_{n\geq2} \sum_{k=1}^{n-1} k\tau[(cc^*)^k]\tau[(cc^*)^{n-k}] dt. \quad (4.8)$$

Thus $\frac{d}{dt}\tau(cc^*) = 0 = \varrho_1'(-4st)$. If $s = 0$, (4.8) asserts that $\tau[(cc^*)^n] = 1 = \varrho_n(0 \cdot t)$ for all $n$. If $s \neq 0$, let $\hat{\varrho}_n(t) = \tau[(cc^*)(-t/4s)]]$; then the chain rule applied to (4.8) shows that

$$\hat{\varrho}_n(t) = -1_{n\geq2} \sum_{k=1}^{n-1} k\hat{\varrho}_k(t)\hat{\varrho}_{n-k}(t).$$

By Lemma 4.1 it follows that $\hat{\varrho}_n(t) = \varrho_n(t)$ for all $n, t \geq 0$. Hence,

$$\tau[(cc^*)^n] = \varrho_n(-4st) = e^{\frac{t}{4s}}(-4st)\nu_n(-4st),$$

as claimed. As defined in Lemma 4.6 we therefore have

$$\tau[(bb^*)^n] = \tau[(e^{2st}cc^*)^n] = e^{-2nst}\varrho_n(-4st) = \nu_n(-4st),$$

verifying (1.8), and concluding the proof.

4.3 The Trace of $b_{r,s}(t)^2b_{r,s}(t)^*$

Finally, we calculate $\tau(b^2b^*)$. To that end, we need the following cubic moment as part of the recursive computation.

Lemma 4.9. Let $a = e^{\frac{t}{4}(r-s)}b$ as in Lemma 4.2 Then

$$\tau(a^2a^*) = (1 + 2st)e^{(s+r)t}. \quad (4.9)$$

Proof. From the Itô product rule (2.23), we have

$$d(a^2a^*) = da \cdot aa^* + a \cdot da \cdot da^* + a^2da^* + (da)^2 \cdot a^* + da \cdot a \cdot da^* + a \cdot da \cdot da^*.\quad (4.10)$$

Lemma 4.2 asserts that $da = a dw$. To compute $d\tau(a^2a^*)$, we can ignore the first three terms that have trace 0 by (2.19); the last three terms become

$$a dw a dw a^* + a dw a dw a^* + a^2 dw dw a^* = (s-r)\tau(a)aa^* dt + (s+r)\tau(a)aa^* dt + (s+r)a^2a^* dt$$

by Lemma 4.3 Taking traces, we therefore have

$$d\tau(a^2a^*) = 2s\tau(a)(aa^*) dt + (s+r)\tau(a^2a^*) dt. \quad (4.10)$$

In Corollary 4.5 we computed that $\tau(a) = \varrho_1((r-s)t) = e^{\frac{t}{2(r-s)}t}\nu_1((r-s)t)$, which, referring to (1.6), is equal to 1. Similarly, in Corollary 4.8 we calculated that $\tau(bb^*) = \nu_1(-4st) = e^{2st}$, and so $\tau(aa^*) = e^{(r-s)t}\tau(bb^*) = e^{(r-s)t}$. Hence, (4.10) reduces to the ODE

$$\frac{d}{dt}\tau(a^2a^*) = 2se^{(r+s)t} + (s+r)\tau(a^2a^*), \quad \tau(a^2a^*(0)) = 1.$$

It is simple to verify that (4.9) is the unique solution of this ODE.
Remark 4.10. As a sanity check, note that in the case \((r, s) = (1, 0)\) \((4.9)\) shows that \(\tau(b^2b^*) = e^{-\frac{3}{2}t}\tau(a^2a^*) = e^{-t/2}\). As pointed out in \((1.3)\), \(b_{1,0}(t) = u(t)\) is a free unitary Brownian motion, and so \(\tau(b^2b^*) = \tau(b)\) in this case; thus, we have consistency with \((1.6)\).

**Proposition 4.11.** Let \(a = e^{\frac{1}{2}(r-s)t}b\) as in Lemma 4.2. Then

\[
\tau(a^2a^*) = 4st(1 + st)e^{(s+r)t} + e^{2(s+r)t}
\]  
and thus \((1.9)\) holds true.

**Proof.** Expanding, once again, using the Itô product rule \((2.23)\), we have

\[
d(a^2a^*) = da \cdot aa^* - a \cdot da \cdot a^* + a^2 \cdot da \cdot a^* + a^2a^* \cdot da^*
\]  
\[+ (da)^2 \cdot a^* + da \cdot da^* \cdot a^* + da \cdot aa^* \cdot da^*
\]  
\[+ a \cdot da \cdot da^* \cdot a^* + a \cdot da \cdot a^* \cdot da^* + a^2 \cdot (da^*)^2.
\]  

The terms in \((4.12)\) all have trace 0. We simplify the terms in \((4.13)\) and \((4.14)\) using \(da = a dw\) and Lemma 4.3 as follows:

\[
(4.13) = a dw a dw a^* + a dw a dw^* a^* + a dw aa^* dw^* a^* = (s-r)\tau(a)aa^* dt + (s+r)\tau(a)aa^* dt + (s+r)\tau(aa^*)aa^* dt,
\]  
and

\[
(4.14) = a^2 dw dw^* a^* + a^2 dw a^* dw^* a^* + a^2 dw^* a^* dw^* a^* = (s+r)a^2aa^* dt + (s+r)\tau(a^*)a^2aa^* dt + (s-r)\tau(a^*)a^2a^* dt.
\]  

Taking traces, and using the fact (from Lemma 4.9) that \(\tau(a^*)\tau(a^2a^*)\) is real, this yields

\[
d\tau(a^2a^*) = 2s\tau(a)\tau(aa^*) dt + (s+r)[\tau(aa^*)]^2 dt + (s+r)\tau(aa^*) dt + 2s\tau(a^*)\tau(a^2a^*) dt.
\]  

Using \((4.9)\), together with \((1.8)\) and the fact (pointed out in the proof of Lemma 4.9) that \(\tau(a) = 1\), gives

\[
\frac{d}{dt}\tau(a^2a^*) = 4s(1+2st)e^{(s+r)t} + (s+r)e^{2(s+r)t} + (s+r)\tau(a^2a^*).
\]  

It is easy to verify that \((4.11)\) is the unique solution to this ODE with initial condition 1. Substituting \(b = e^{\frac{1}{2}(r-s)t}a\) then yields \((1.9)\).

**Remark 4.12.** Again, as a sanity check, \((1.9)\) reduces to \(\tau(b^2b^*) = 1\) when \(s = 0\); this is consistent with the fact that \(b\) is unitary in this case.

## 5 Properties of the Brownian Motions

Theorem 1.11 summarizes the main properties of both the matrix Brownian motions \(B^N_{r,s}(t)\) on \(\mathbb{M}_N\) and its limit \((b_{r,s}(t))_{t \geq 0}\). We will prove these properties separately for finite \(N\) versus the limit, although in many cases the proofs are extremely similar.
5.1 Properties of \((B_{r,s}^N(t))_{t \geq 0}\)

We begin by noting that the invertibility of \(B_{r,s}^N(t)\) follows from the mSDE (2.10).

**Proposition 5.1.** The diffusion \(B_{r,s}^N(t)\) is invertible for all \(t \geq 0\) (with probability 1); the inverse \(B_{r,s}^N(t)^{-1}\) is a right-invariant version of an \((r, s)\)-Brownian motion.

**Proof.** Fix a Brownian motion \(W_{r,s}^N(t) = \sqrt{r} t X^N(t) + \sqrt{s} Y^N(t)\) on \(\mathfrak{gl}_N\), so that \(B_{r,s}^N(t)\) is the solution of (2.10) with respect to \(W_{r,s}^N(t)\). Then define \(A_{r,s}^N(t)\) to be the solution to

\[
dA_{r,s}^N(t) = -dW_{r,s}^N(t) A_{r,s}^N(t) - \frac{1}{2} (r - s) A_{r,s}^N(t) dt. \tag{5.1}
\]

Note that \(-X^N(t)\) and \(-Y^N(t)\) are also independent GUE\(_N\) Brownian motions, so \(A_{r,s}^N(t)\) is a right-invariant version of \(B_{r,s}^N(t)\). (Indeed, the reader can readily check that, if \(\partial_\xi\) is replaced with the right-invariant derivative \(\frac{d}{dt} f(\exp(-t\xi)g)\), thus defining a right-invariant Laplacian, the associated Brownian motion satisfies (5.1).) To simplify notation, let \(W = W_{r,s}^N(t)\), \(B = B_{r,s}^N(t)\), and \(A = A_{r,s}^N(t)\). Using the Itô product rule (2.16), we have

\[
d(BA) = dB \cdot A + B \cdot dA + dB \cdot dA = B dW A - \frac{1}{2} (r - s) BA dt - B dW A - \frac{1}{2} (r - s) BA dt - B (dw)^2 A.
\]

From (2.13) – (2.15), we compute exactly as in Lemma 4.3 that \((dw)^2 = (s - r) I_N dt\). This shows that \(d(BA) = 0\). Since \(B_{r,s}^N(0) = A_{r,s}^N(0) = I_N\), it follows that \(BA = I_N\), so \(A_{r,s}^N(t) = B_{r,s}^N(t)^{-1}\), as claimed. \(\square\)

**Proposition 5.2.** The multiplicative increments of \((B_{r,s}^N(t))_{t \geq 0}\) are independent and stationary.

**Proof.** Let \(0 \leq t_1 < t_2 < \infty\), and let \(\mathcal{F}_{t_1}\) denote the \(\sigma\)-field generated by \(\{X^N(t), Y^N(t)\}_{0 \leq t \leq t_1}\). From the defining mSDE (2.10), we have

\[
B_{r,s}^N(t_2) - B_{r,s}^N(t_1) = \int_{t_1}^{t_2} B_{r,s}^N(t) dW_{r,s}^N(t) - \frac{1}{2} (r - s) \int_{t_1}^{t_2} B_{r,s}^N(t) dt,
\]

or, in other words,

\[
B_{r,s}^N(t_1)^{-1} B_{r,s}^N(t_2) = I_N + \int_{t_1}^{t_2} B_{r,s}^N(t_1)^{-1} B_{r,s}^N(t) dW_{r,s}^N(t) - \frac{1}{2} (r - s) \int_{t_1}^{t_2} B_{r,s}^N(t_1)^{-1} B_{r,s}^N(t) dt. \tag{5.2}
\]

This shows that the process \(C_N(t) = B_{r,s}^N(t_1)^{-1} B_{r,s}^N(t)\) for \(t \geq t_1\) satisfies the mSDE

\[
dC_N(t) = C_N(t) d(W_{r,s}^N(t) - W_{r,s}^N(t_1)) - \frac{1}{2} (r - s) C_N(t) dt.
\]

Note that \(W_{r,s}^N(t) - W_{r,s}^N(t_1) = \sqrt{r} t (X^N(t) - X^N(t_1)) + \sqrt{s} (Y^N(t) - Y^N(t_1))\). Since \((X^N(t) - X^N(t_1))_{t \geq t_1}\) and \((Y^N(t) - Y^N(t_1))_{t \geq t_1}\) are independent GUE\(_N\) Brownian motions, and since \(C_N(t_1) = I_N\), it follows that \((C_N(t))_{t \geq t_1}\) is a version of \((B_{r,s}^N(t))_{t \geq 0}\). This shows, in particular, that the multiplicative increments are stationary. Moreover, (5.2) shows that \(B_{r,s}^N(t_1)^{-1} B_{r,s}^N(t_2)\) is measurable with respect to the \(\sigma\)-field generated by the increments \((W_{r,s}^N(t) - W_{r,s}^N(t_1))_{t_1 \leq t \leq t_2}\), which is independent from \(\mathcal{F}_{t_1}\) (since the additive increments of \(X^N(t)\) and \(Y^N(t)\) are independent). Since all the random matrices \(B_{r,s}^N(t')\) with \(t' \leq t_1\) are \(\mathcal{F}_{t_1}\)-measurable, it follows that \((B_{r,s}^N(t))_{t \geq 0}\) has independent multiplicative increments, as claimed. \(\square\)
Proposition 5.3. For $r, s > 0$ and $N \geq 2$, with probability 1, $B^N_{r,s}(t)$ is non-normal for all $t > 0$.

Proof. Let $\mathbb{M}^\text{nor}_N$ denote the set of normal matrices. Let $\mathbb{D}_N$ denote the $2N$ (real) dimensional space of diagonal matrices in $\mathbb{M}_N$, and $\mathbb{T}_N \subset \mathbb{U}_N$ the $N$ (real) dimensional maximal torus of diagonal unitary matrices. The map $\Phi: \mathbb{D}_N \times \mathbb{U}_N \to \mathbb{M}^\text{nor}_N$ given by $\Phi(D, U) = DU^*$ is smooth, and (by the spectral theorem) surjective. Since $\Phi(D, U) = \Phi(D, TU)$ for any $T \in \mathbb{T}_N$, the map descends to a smooth surjection $\tilde{\Phi}: \mathbb{D}_N \times \mathbb{U}_N/\mathbb{T}_N \to \mathbb{M}^\text{nor}_N$. It follows that

$$\dim_{\mathbb{R}}(\mathbb{M}^\text{nor}_N) \leq \dim_{\mathbb{R}}(\mathbb{D}_N) + \dim_{\mathbb{R}}(\mathbb{U}_N/\mathbb{T}_N) = 2N + N^2 - N = N^2 + N.$$ 

Thus, as a submanifold of $\mathbb{M}_N$ (which has real dimension $2N^2$), $\text{codim}_{\mathbb{R}}(\mathbb{M}^\text{nor}_N) \geq 2N^2 - (N^2 + N) = N^2 - N$. This is $\geq 2$ for $N \geq 2$.

The manifold $\mathbb{G}_N$ is an open dense subset of $\mathbb{M}_N$, and the generator $\Delta_{r,s}$ is easily seen to be a non-degenerate elliptic operator on $C^\infty(\mathbb{M}_N)$. Thus, by the main theorem of [2], $\mathbb{M}^\text{nor}_N$ is a polar set for the diffusion generated by $\frac{1}{2} \Delta_{r,s}$; i.e. the hitting time of $\mathbb{M}^\text{nor}_N$ for $(B^N_{r,s}(t))_{t \geq 0}$ is $+\infty$ almost surely. This concludes the proof. \qed

Remark 5.4. If $D$ is in the open dense subset of $\mathbb{D}_N$ with all eigenvalues distinct, then the stabilizer of $D$ in $\mathbb{U}_N$ is exactly equal to $\mathbb{T}_N$; thus the map $\tilde{\Phi}$ above is generically a local diffeomorphism. It follows that $\dim_{\mathbb{R}}(\mathbb{M}^\text{nor}_N) = N^2 + N$.

Propositions 5.1–5.3 address the first half of Theorem 1.1. Let us also address Remark 1.12 here.

Proposition 5.5. For $r > 0$, $V^N(t) \equiv B^N_{r,0}(t/r)$ is Brownian motion on $\mathbb{U}_N$ with respect to the metric induced by the inner product $\langle \xi, \eta \rangle = -N\text{Tr}(\xi \eta)$ on $\mathbb{U}_N$.

Proof. Let $\beta_N$ be the basis for $\mathbb{u}_N$ defined in (2.11); then $\beta_N$ is orthonormal for the stated inner product. From (2.4) and (2.9), we see that, with $W^N(t) = \sum_{\xi \in \beta_N} B^N_{\xi}(t) \xi$, the Brownian motion $U^N(t)$ on $\mathbb{U}_N$ satisfies the mSDE

$$dU^N(t) = U^N(t) dW^N(t) - \frac{1}{2} U^N(t) dt, \quad U^N(0) = I_N.$$ 

(Note: the proof that this process takes values in $\mathbb{U}_N$ for all $t \geq 0$ follows much the same way as the proof of Proposition 5.1.) Note, as above, that $W^N(t) = iX^N(t)$ where $X^N(t)$ is a $\text{GUE}_N$ Brownian motion. Now, from (2.10), we compute that, for $r > 0$,

$$dV^N(rt) = dB^N_{r,0}(t) = \sqrt{r} i B^N_{r,0}(t) dX^N(t) - \frac{1}{2} r B^N_{r,0}(t) dt = i B^N_{r,0}(t) dX^N(rt) - \frac{1}{2} B^N_{r,0}(t) dr(t) = i V^N(rt) dX^N(rt) - \frac{1}{2} V^N(rt) dr(t),$$

using the standard space-time scaling of the Brownian motion $X^N(t)$ and the chain rule. Thus $V^N(t)$ satisfies the same mSDE, with the same initial condition, as $U^N(t)$; this proves the proposition. \qed

5.2 Properties of $(b_{r,s}(t))_{t \geq 0}$

Proposition 5.6. For all $r, s, t \geq 0$, the free multiplicative $(r, s)$-Brownian motion $b_{r,s}(t)$ is invertible; the inverse $a_{r,s}(t) = b_{r,s}(t)^{-1}$ satisfies the fSDE

$$da_{r,s}(t) = -dw_{r,s}(t) a_{r,s}(t) - \frac{1}{2} (r - s) a_{r,s}(t) dt.$$  \hfill (5.3)
Since let not normal. In this infinite-dimensional setting, we must also verify that
and so
with initial condition
We now use (1.6), (1.8), and (1.9) to expand this:
Thus, \( \tau (a_t b_t) \) satisfies the fSDE
Thus, \( a_t b_t \) satisfies the fSDE
with initial condition \( a_0 b_0 = 1 \). Notice that the fSDE \( d\theta_t = [\theta_t, dw_t] + (r-s)[\theta_t - \tau(\theta_t)] \) holds true for any constant process \( \theta_t \); thus, with initial condition \( \theta_0 = 1 \) uniquely determining the solution, we see that \( a_t b_t = 1 \) as well.

\[ d(t) b(t) = d(t) b(t) + (r-s)(a_t b_t - \tau(a_t b_t)), \]

Proposition 5.7. The multiplicative increments of \( (b_{r,s}(t))_{t \geq 0} \) are freely independent and stationary.

The proof proceeds very similarly to the proof of Proposition 5.1 using (2.20) – (2.22) instead of (2.13) – (2.15), we compute that \( d(b_{r,s}(t)) = 0 \), which shows, since \( b_{r,s}(0) = a_{r,s}(0) = 1 \), that \( b_{r,s}(t) a_{r,s}(t) = 1 \).

Proposition 5.8. \( b_{r,s}(t) \) is non-normal for all \( t > 0 \).

Proof. Let \( b_t = b_{r,s}(t) \); we compute that

\[ \tau ([b_t, b_t^*]^2) = 2\tau((b_t^*)^2) - 2\tau(b_t^2 b_t^*), \]

and so

We now use (1.6), (1.8), and (1.9) to expand this:

\[ \tau ([b_t, b_t^*]^2) - \tau(b_t^2 (b_t^*)^2) = \nu_2(-4st) - (e^{4st} + 4st(1 + st)e^{(3s-r)t}) \]
\[ = e^{4st}(1 + 4st) - (e^{4st} + 4st(1 + st)e^{(3s-r)t}) \]
\[ = 4ste^{3st}[e^{st} - (1 + st)e^{-rt}]. \]

Since \( r \geq 0, e^{-rt} \leq 1 \), and since \( s > 0, e^{st} > 1 + st \). It follows that \( \tau([b_t, b_t^*]^2) > 0 \) for \( t > 0 \), proving that \( b_t \) is not normal.

Proposition 5.9. \( u(t) = b_{r,0}(t/r) \) is a free unitary Brownian motion.

Proof. This follows immediately from Proposition 5.5 together with \( \| \theta \|_1 \). Alternatively, we can see directly that (1.3) reduces to \( du(t) = i dx(t) - \frac{1}{2} u(t) dt \) for \( u(t) = b_{1,0}(t) \), which is the defining SDE of a (left) free unitary Brownian motion, and then do a time change computation as in the proof of Proposition 5.5 for \( b_{r,0}(t/r) \).
6 Convergence of the Brownian Motions

This final section is devoted to the proof of Theorem 1.6 that the process \((B^N_{r,s}(t))_{t \geq 0}\) converges in noncommutative distribution to the process \((b_{r,s}(t))_{t \geq 0}\). We first show the convergence of the random matrices \(B^N_{r,s}(t)\) for each fixed \(t \geq 0\); the multi-time statement then follows from asymptotic freeness considerations.

6.1 Convergence for a Fixed \(t\)

We begin by noting the single-\(t\) version of Theorem 1.13, which was proved in [9, Proposition 4.13]. For any \(r, s > 0\) and \(t \geq 0\), and any noncommutative polynomials \(f, g \in \mathbb{C}(X, X^*)\), there is a constant \(C_{r,s}(t, f, g)\) such that

\[
\text{Cov} \left[ \text{tr} \left( f(B^N_{r,s}(t), B^N_{r,s}(t)^*) \right), \text{tr} \left( g(B^N_{r,s}(t), B^N_{r,s}(t)^*) \right) \right] \leq \frac{C_{r,s}(t, f, g)}{N^2},
\]

(6.1)

where \(C_{r,s}(t, f, g)\) depends continuously on \(t\).

We now proceed to prove the fixed-\(t\) case of Theorem 1.6. The idea is to compare the mSDE for \(B^N_{r,s}(t)\) to the fSDE for \(b_{r,s}(t)\), and inductively show that traces of \(*\)-moments differ by \(O(1/N^2)\), using (6.1).

**Theorem 6.1.** Let \(r, s, t \geq 0\). Let \(n \in \mathbb{N}\) and let \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, \ast\}^n\). Then there is a constant \(C_{r,s}(t, \varepsilon)\) that depends continuously on \(r, s, t\) so that

\[
|\text{E} \text{tr}(B^N_{r,s}(t)^{\varepsilon_1} \cdots B^N_{r,s}(t)^{\varepsilon_n}) - \tau(b_{r,s}(t)^{\varepsilon_1} \cdots b_{r,s}(t)^{\varepsilon_n})| \leq \frac{C_{r,s}(t, \varepsilon)}{N^2}.
\]

(6.2)

**Proof.** In the case \(n = 0\), (6.2) holds true vacuously with \(C_{r,s}(t, \varnothing) = 0\). When \(n = 1\), as computed in (1.7) we have \(\tau(b_{r,s}(t)^{\varepsilon_1}) = \nu_1((r - s)t)\), and so (6.2) follows immediately from [9, Theorem 1.3]. From here, we proceed by induction: assume that (6.2) has been verified up to, but not including, level \(n\).

Fix \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, \ast\}^n\). Let \(A^N_{r,s}(t) = e^{\frac{N}{2}(r-s)t}B^N_{r,s}(t)\), so that, following precisely the proof of Lemma 4.2 but using (2.16) instead of (2.23), we have

\[
dA^N_{r,s}(t) = A^N_{r,s}(t) dW^N_{r,s}(t).
\]

(6.3)

For convenience, denote \(A = A^N_{r,s}(t)\), and denote \(A^\varepsilon = A^{\varepsilon_1} \cdots A^{\varepsilon_n}\). Then, using the Itô product rule (2.16), we have

\[
d(A^\varepsilon) = \sum_{j=1}^{n} A^{\varepsilon_1} \cdots A^{\varepsilon_{j-1}} \cdot dA^{\varepsilon_j} \cdot A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_n} \]

(6.4)

\[+ \sum_{1 \leq j < k \leq n} A^{\varepsilon_1} \cdots A^{\varepsilon_{j-1}} \cdot dA^{\varepsilon_j} \cdot A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} \cdot dA^{\varepsilon_k} \cdot A^{\varepsilon_{k+1}} \cdots A^{\varepsilon_n}.
\]

(6.5)

From (2.14) and (6.3), the terms in (6.5) become

\[
A^{\varepsilon_1} \cdots A^{\varepsilon_{j-1}} \cdot dA^{\varepsilon_j} \cdot A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} \cdot dA^{\varepsilon_k} \cdot A^{\varepsilon_{k+1}} \cdots A^{\varepsilon_n}
\]

\[= A^{\varepsilon_1} \cdots A^{\varepsilon_{j-1}} \cdot dW^{\varepsilon_j} A^{\varepsilon_k} A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} A^{\varepsilon_k} A^{\varepsilon_{k+1}} \cdots A^{\varepsilon_n}
\]

where \(W = W^N_{r,s}(t)\), and \(1' = 1, 1'' = 1\), and \(\varphi'' = \varphi'\). As in Lemma 4.3 (2.13) - (2.15) show that, for any adapted process \(\Theta\),

\[
dW^\varepsilon \Theta dW^{\varepsilon'} = (s \pm r) \text{tr}(\Theta) \, dt
\]

(6.6)

where the sign is \(-\) if \(\varepsilon = \varepsilon'\) and \(+\) if \(\varepsilon \neq \varepsilon'\). Hence, the terms in (6.5) become

\[
(s \pm r) \text{tr}(A^{\varepsilon_{j+1}} A^{\varepsilon_{j-1}} A^{\varepsilon_{k-1}} A^{\varepsilon_k}) A^{\varepsilon_1} \cdots A^{\varepsilon_{j-1}} A^{\varepsilon_j} A^{\varepsilon_{k+1}} A^{\varepsilon_k} A^{\varepsilon_{k+1}} \cdots A^{\varepsilon_n}.
\]
Now, note that the expected value of all the terms in (6.4) is 0 by (2.12) and (6.3). Therefore, taking \( \mathbb{E} \) \( \text{tr} \) in (6.4) and (6.5), we have
\[
\frac{d}{dt} \mathbb{E} \text{tr}(A^\varepsilon) = \sum_{1 \leq j < k \leq n} (s + r) \mathbb{E} \left[ \text{tr}(A^\varepsilon A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} A^{\varepsilon_k}) \text{tr}(A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} A^{\varepsilon_k} A^{\varepsilon_{k+1}} \cdots A^{\varepsilon_n}) \right].
\]

It is possible for one of the two trace terms to be trivial, in two special cases.

- If \( j = 1 \) and \( k = n \), and if \( \varepsilon_1 = * \) and \( \varepsilon_n = 1 \), then the first trace term is equal to \( \text{tr}(A^\varepsilon) \), while the second one is just \( \text{tr}(I_N) = 1 \).

- For \( 1 \leq j < n \), if \( k = j + 1 \), and \( \varepsilon_j = 1 \) while \( \varepsilon_k = * \), then the second trace term is equal to \( \text{tr}(A^\varepsilon) \), while the first one is just \( \text{tr}(I_N) = 1 \).

In all other \((\varepsilon, j, k)\) configurations, each trace term involves a non-trivial string of length \(< n \). Note that, in both these exceptional cases, the two exponents must be different, and so the factor in front is \( s + r \). We separate out these cases as follows:
\[
\frac{d}{dt} \mathbb{E} \text{tr}(A^\varepsilon) = (s + r) \mathbb{1}_{(\varepsilon_1, \varepsilon_n) = (*, 1)} \mathbb{E} \text{tr}(A^\varepsilon) + (s + r) \sum_{j=1}^{n-1} \mathbb{1}_{(\varepsilon_j, \varepsilon_{j+1}) = (1, *)} \mathbb{E} \text{tr}(A^\varepsilon)
\]
\[
+ \sum_{1 \leq j < k \leq n} (s + r) \mathbb{E} \left[ \text{tr}(A^\varepsilon A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} A^{\varepsilon_k}) \text{tr}(A^{\varepsilon_{j+1}} \cdots A^{\varepsilon_{k-1}} A^{\varepsilon_k} A^{\varepsilon_{k+1}} \cdots A^{\varepsilon_n}) \right],
\]

where \( \sum \) indicates that the sum excludes the at-most-\( n \) terms accounted for in the special cases. Define
\[
\kappa(\varepsilon) = \mathbb{1}_{(\varepsilon_1, \varepsilon_n) = (*, 1)} + \sum_{j=1}^{n-1} \mathbb{1}_{(\varepsilon_j, \varepsilon_{j+1}) = (1, *)},
\]
and let
\[
\varepsilon_{j,k}^1 = (\varepsilon_j'', \ldots, \varepsilon_k'), \quad \varepsilon_{j,k}^2 = (\varepsilon_1, \ldots, \varepsilon_j', \varepsilon_k'', \ldots, \varepsilon_n).
\]

Thus we have shown that \( \mathbb{E} \) \( \text{tr} \)(\( A^\varepsilon \)) satisfies the ODE
\[
\frac{d}{dt} \mathbb{E} \text{tr}(A^\varepsilon) = \kappa(\varepsilon)(s + r) \mathbb{E} \text{tr}(A^\varepsilon) + \sum_{1 \leq j < k \leq n} (s + r) \mathbb{E} \left[ \text{tr}(A^{\varepsilon_{j,k}^1}) \text{tr}(A^{\varepsilon_{j,k}^2}) \right],
\]
(6.7)

where all the terms in the sum are expectations of products of traces of words in \( A \) and \( A^* \) of length strictly less than \( n \). Since \( A(0) = I_N \), the unique solution of this ODE (in terms of these functions in the sum) is
\[
\mathbb{E} \text{tr}(A_T^\varepsilon) = e^{\kappa(\varepsilon)(s+r)T} + \sum_{1 \leq j < k \leq n} (s + r) \int_0^T e^{\kappa(\varepsilon)(s+r)(T-t)} \mathbb{E} \left[ \text{tr}(A_T^{\varepsilon_{j,k}^1}) \text{tr}(A_T^{\varepsilon_{j,k}^2}) \right] dt
\]
(6.8)

where we have written \( A_t = A_{r,s}^N(t) \) to emphasize the different times of evaluation. Now returning to \( B_t = B_{r,s}^N(t) = e^{-\frac{r-s}{2}} A_t \), and noting that the total length of the two strings \( \varepsilon_{j,k}^1 \) and \( \varepsilon_{j,k}^2 \) is \( n \), the same as the length of \( \varepsilon \), this gives
\[
\mathbb{E} \text{tr}(B_T^\varepsilon) = e^{[\kappa(\varepsilon)(s+r)-\frac{r-s}{2}]T}
\]
\[
+ \sum_{1 \leq j < k \leq n} (s + r) \int_0^T e^{[\kappa(\varepsilon)(s+r)-\frac{r-s}{2}(T-t)]} e^{\frac{r-s}{2}t} \mathbb{E} \left[ \text{tr}(B_t^{\varepsilon_{j,k}^1}) \text{tr}(B_t^{\varepsilon_{j,k}^2}) \right] dt,
\]
(6.9)
Now, repeating this deviation line-by-line, we find that, setting $b_t = b_{r,s}(t)$,
\[
\tau(b_t^{\varepsilon}) = e^{[\varepsilon(s+r+1)]/2} \tau(b_t^{\varepsilon}) T + \sum_{1 \leq j < k \leq n} (s + r) \int_0^T e^{[\varepsilon(s+r+1)][T(t) - t]} e^{\varepsilon(r-s)T} \tau(b_{r,s}^{\varepsilon}) \tau(b_t^{\varepsilon}) dt. \quad (6.10)
\]

The principal difference is that, when applying the free Itô product rule \([2,27]\), the trace $\tau$ factors through completely, while in the matrix Itô product rule \([2,16]\), only the trace $\text{tr}$ factors through, while the expectation $\mathbb{E}$ does not. Thus, the desired quantity (on the left-hand-side of \((6.2)\)) at time $T$ is equal to
\[
\sum_{1 \leq j < k \leq n} (s + r) \int_0^T e^{[\varepsilon(s+r+1)][T(t) - t]} e^{\varepsilon(r-s)T} \left( \mathbb{E} \left[ \text{tr}(B_{t,j,k}^{\varepsilon}) \text{tr}(B_{t}^{\varepsilon}) \right] - \tau(b_{r,s}^{\varepsilon}) \tau(b_t^{\varepsilon}) \right) dt. \quad (6.11)
\]

Again to simplify notation, fix $j$, $k$ in the sum and let $B_{t}^{\varepsilon} = B_{t,j,k}^{\varepsilon}$ and $b_t^{\varepsilon} = b_{t,j,k}^{\varepsilon}$ for $\ell = 1, 2$. Then we expand the difference as
\[
\mathbb{E}[\text{tr}(B_{t}^{\varepsilon}) \text{tr}(B_{t}^{\varepsilon})] - \tau(b_{t}^{\varepsilon}) \tau(b_{t}^{\varepsilon}) = \text{Cov}[\text{tr}(B_{t}^{\varepsilon}), \text{tr}(B_{t}^{\varepsilon})] + \mathbb{E} \text{tr}(B_{t}^{\varepsilon}) \text{tr}(B_{t}^{\varepsilon}) - \tau(b_{t}^{\varepsilon}) \tau(b_{t}^{\varepsilon}), \quad (6.12)
\]

and the last two terms may be written (by adding and subtracting $\tau(b_{t}^{\varepsilon}) \mathbb{E} \text{tr}(B_{t}^{\varepsilon})$) as
\[
\mathbb{E} \text{tr}(B_{t}^{\varepsilon}) \text{tr}(B_{t}^{\varepsilon}) - \tau(b_{t}^{\varepsilon}) \tau(b_{t}^{\varepsilon}) = \mathbb{E} \text{tr}(B_{t}^{\varepsilon}) \cdot [\mathbb{E} \text{tr}(B_{t}^{\varepsilon}) - \tau(b_{t}^{\varepsilon})] + \tau(b_{t}^{\varepsilon}) \cdot [\mathbb{E} \text{tr}(B_{t}^{\varepsilon}) - \tau(b_{t}^{\varepsilon})]. \quad (6.13)
\]

We now appeal to the inductive hypothesis. By construction, all the terms in the sum $\sum$ have both strings $\varepsilon_{j,k}$ and $\varepsilon_{k,j}$ of length strictly $< n$. As such, the inductive hypothesis yields that $|\mathbb{E} \text{tr}(B_{t}^{\varepsilon}) - \tau(b_{t}^{\varepsilon})| \leq C^{\varepsilon}(t)$ for constants $C^{\varepsilon}(t)$ that depend continuously on $t$ (and all of the hidden parameters $r, s, \varepsilon$). It follows, in particular, that the constants $\mathbb{E} \text{tr}(B_{t}^{\varepsilon})$ are uniformly bounded in $N$ and $t \in [0, T]$. Thus, the terms in \((6.13)\) are bounded by $C(t)/N^2$ for some constant $C(t)$ that is uniformly bounded in $t \in [0, T]$. By \((6.11)\), the covariance term in \((6.12)\) is also bounded by $C(t)/N^2$ for such a constant $C(t)$. Integrating $C(t) + C(t)$ times the relevant exponentials, summed over $j, k$, in \((6.11)\) now shows that the whole expression is $\leq C^\varepsilon(T)/N^2$ for some constant $C^\varepsilon(T)$ that depends continuously on $T$. This concludes the proof.

Remark 6.2. In \([9, \text{Theorem 1.6}]\), the author showed that there exists a linear functional $\varphi_{r,s}^\varepsilon : \mathbb{C}[X, X^*] \to \mathbb{C}$ so that \((6.2)\) holds with $\varphi_{r,s}^\varepsilon(X^{(1)} \cdots X^{(n)})$ in place of $\tau(b_{r,s}(t)^{e_{1}} \cdots b_{r,s}(t)^{e_{n}})$; the upshot of the present theorem is to identify this linear functional as the noncommutative distribution of $b_{r,s}(t)$. In particular, it lives in a faithful, normal, tracial $W^*$-probability space, which could not be easily proved using the techniques in \([9]\).

6.2 Asymptotic Freeness and Convergence of the Process

In this final section, we use the freeness of the increments of $b_{r,s}(t)$ and the asymptotic freeness of the increments of $B_{r,s}(t)$, together with Theorem \ref{6.1} to prove Theorem \ref{1.6}. We begin with some preliminary lemmas.

Lemma 6.3. Let $\varepsilon_1, \ldots, \varepsilon_n \in \{1, *, \}$, and let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be a noncommutative polynomial. Given any permutation $\sigma \in \Sigma_n$, there is a noncommutative polynomial $g \in \mathbb{C}[X_1, \ldots, X_n, X_1^*, \ldots, X_n^*]$ with the following property. If $b_1, \ldots, b_n$ are invertible random variables in a noncommutative probability space, and $a_1 = b_1, a_2 = b_1^{-1} b_2, \ldots, a_n = b_1^{-1} b_n$ are the corresponding multiplicative increments, then
\[
f(b_1^{\varepsilon_1}, \ldots, b_n^{\varepsilon_n}) = g(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*).
\]
Proof. For $1 \leq j \leq n$, write

$$b_j = b_1(b_1^{-1}b_2) \cdots (b_{j-1}^{-1}b_j) = a_1a_2 \cdots a_j.$$  \hfill (6.14)

Let $f_\sigma(X_1, \ldots, X_n) = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$; then

$$f(b_\sigma, \ldots, b_\sigma) = f(b_{\sigma(1)}^{-1}, \ldots, b_{\sigma(n)}^{-1}).$$

In each variable, expand the term $b_\sigma$ using (6.14) (to the $\epsilon_{\sigma^{-1}(j)}$ power); this yields the polynomial $g$. \hfill \□

The next lemma uses the language of Section 3.2 to give a more precise formulation of how free independence reduces the calculation of joint moments to separate moments.

Lemma 6.4. Given any $n \in \mathbb{N}$ and any noncommutative polynomial $g \in \mathbb{C} \langle X_1, \ldots, X_n, X_1^*, \ldots, X_n^* \rangle$, there is an $m \in \mathbb{N}$ and a collection $\{P_{i,k} \mid 1 \leq j \leq n, 1 \leq k \leq m\}$ of elements of $\mathcal{P}$ with the property that, if $(\mathcal{A}, \tau)$ is a noncommutative probability space, and $a_1, \ldots, a_n \in \mathcal{A}$ are freely independent, then

$$\tau(g(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*)) = \sum_{k=1}^{m} P_{1,k}(a_1) \cdots P_{n,k}(a_n).$$  \hfill (6.15)

Here $\mathcal{P}$ denotes the polynomial space $\mathcal{P}(J)$ with the index set $J$ a singleton. The proof of Lemma 6.4 is contained in the proof of [14, Lemma 5.13]. The idea is to center the variables and proceed inductively. The exact machinery of how $P_{i,k}$ are computed from $g$ is the business of the rich theory of free cumulants, which is the primary topic of the monograph [14].

Now, suppose $A_1^N, \ldots, A_n^N$ are $N \times N$ random matrices that are asymptotically free; cf. Definition 2.6. This means precisely that $(A_1^N, \ldots, A_n^N) \to (a_1, \ldots, a_n)$ in noncommutative distribution, for some freely independent collection $a_1, \ldots, a_n$ in a noncommutative probability space $(\mathcal{A}, \tau)$. In other words, for any noncommutative polynomial $g \in \mathbb{C} \langle X_1, \ldots, X_n, X_1^*, \ldots, X_n^* \rangle$,

$$\lim_{N \to \infty} \mathbb{E} \text{tr}(g(A_1^N, \ldots, A_n^N, (A_1^N)^*, \ldots, (A_n^N)^*)) = \tau(g(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*))$$

$$= \sum_{k=1}^{m} P_{1,k}(a_1) \cdots P_{n,k}(a_n)$$

where the second equality is Lemma 6.4. Note that $P_{\tau,k}(a)$ is a polynomial in the trace moments of $a, a^*$, and by assumption of convergence of the joint distribution, we also therefore have $(P_{\mathbb{E}\text{tr}(A_j^N)}) \to P_{\tau,k}(a_j)$ as $N \to \infty$. Hence, we can alternatively state asymptotic freeness as

$$\lim_{N \to \infty} \mathbb{E} \text{tr}(g(A_1^N, \ldots, A_n^N, (A_1^N)^*, \ldots, (A_n^N)^*)) = \lim_{N \to \infty} \sum_{k=1}^{m} P_{\mathbb{E}\text{tr}(A_1^N)} \cdots P_{\mathbb{E}\text{tr}(A_n^N)}.$$  \hfill (6.16)

We now stand ready to prove Theorem 1.6.

Proof of 1.6. For convenience, denote $B_{t,\sigma}^N(t) = B_t$ and $b_{t,\sigma}(t) = b_t$. Fix $t_1, \ldots, t_n \geq 0$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{1, *\}$. Fix a permutation $\sigma \in \Sigma_n$ such that $t_{\sigma(1)} \leq \cdots \leq t_{\sigma(n)}$ and let $t'_j = t_{\sigma(j)}$. Let

$$A_1 = B_{t_1'}, A_2 = B_{t_2'}^{-1}B_{t_2}, \ldots, A_n = B_{t_n'}^{-1}B_{t_n}$$

be the increments for the partition $t'_1 \leq \cdots \leq t'_n$. Using Lemma 6.3, we can write

$$\mathbb{E} \text{tr}(B_{t_1'} \cdots B_{t_n'}^n) = \mathbb{E} \text{tr}(g(A_1, \ldots, A_n, A_1^*, \ldots, A_n^*))$$  \hfill (6.17)
where \( g \in \mathbb{C}(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*) \) is determined by \( \sigma \) and \( \varepsilon_1, \ldots, \varepsilon_n \).

By Proposition 5.2, the increments \( A_j \) are independent; moreover, their stationarity means that \( A_j \) has the same distribution as \( B_{\Delta t'_j} \), where \( \Delta t'_1 = t'_1 \) and \( \Delta t'_j = t'_j - t'_{j-1} \) for \( 1 < j \leq n \). Thus, by Corollary 3.18 \( A_1, \ldots, A_n \) are asymptotically free. In addition, the equality of distributions means that all *-moments of \( A_j \) are equal to the same *-moments of \( B_{\Delta t'_j} \). Thus, combining (6.16) and (6.17), we have

\[
\lim_{N \to \infty} \text{Etr}(B_{t_1}^{\varepsilon_1} \cdots B_{t_n}^{\varepsilon_n}) = \lim_{N \to \infty} \sum_{k_1=1}^m P_1^{1,k}(B_{\Delta t'_1}) \cdots P_n^{\cdot,k}(B_{\Delta t'_n}).
\]

From Theorem 6.1, we therefore have

\[
\lim_{N \to \infty} \text{Etr}(B_{t_1}^{\varepsilon_1} \cdots B_{t_n}^{\varepsilon_n}) = \sum_{k_1=1}^m P_1^{1,k}(b_{\Delta t'_1}) \cdots P_n^{\cdot,k}(b_{\Delta t'_n}).
\]

Now, by Proposition 5.7, the increments \( b_{\Delta t'_j} \) are freely independent and stationary; so letting

\[
a_1 = b_{t'_1}, \quad a_2 = b_{t'_2}^{-1} b_{t'_2}, \quad \ldots, \quad a_n = b_{t'_n}^{-1} b_{t'_n}
\]

we see that \( \{b_{\Delta t'_1}, \ldots, b_{\Delta t'_n}\} \) have the same joint distribution as \( \{a_1, \ldots, a_n\} \). Thus

\[
\lim_{N \to \infty} \text{Etr}(B_{t_1}^{\varepsilon_1} \cdots B_{t_n}^{\varepsilon_n}) = \sum_{k_1=1}^m P_1^{1,k}(b_{\Delta t'_1}) \cdots P_n^{\cdot,k}(b_{\Delta t'_n}) = \sum_{k_1=1}^m P_1^{1,k}(a_1) \cdots P_n^{\cdot,k}(a_n),
\]

and by the definition (6.15) of \( P_{j,k}^{\cdot,j} \), this yields

\[
\lim_{N \to \infty} \text{Etr}(B_{t_1}^{\varepsilon_1} \cdots B_{t_n}^{\varepsilon_n}) = \tau(g(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*)).
\]

Finally, by the definition (6.17) of \( g \), we conclude that

\[
\lim_{N \to \infty} \text{Etr}(B_{t_1}^{\varepsilon_1} \cdots B_{t_n}^{\varepsilon_n}) = \tau(b_{t_1}^{\varepsilon_1} \cdots b_{t_n}^{\varepsilon_n}),
\]

concluding the proof.

Acknowledgments

The author wishes to thank Philippe Biane, Bruce Driver, Pat Fitzsimmons, and Jamie Mingo for useful conversations.

References

[1] Bercovici, H., and Voiculescu, D. Lévy-Hinčin type theorems for multiplicative and additive free convolution. Pacific J. Math. 153, 2 (1992), 217–248.

[2] Biane, P. Free Brownian motion, free stochastic calculus and random matrices. In Free probability theory (Waterloo, ON, 1995), vol. 12 of Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 1997, pp. 1–19.

[3] Biane, P. Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems. J. Funct. Anal. 144, 1 (1997), 232–286.
[4] Biane, P., and Speicher, R. Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory Related Fields* 112, 3 (1998), 373–409.

[5] Cébron, G. Free convolution operators and free Hall transform. *Preprint* (April 2013). arXiv:1304.1713.

[6] Collins, B. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. *Int. Math. Res. Not.*, 17 (2003), 953–982.

[7] Driver, B. K., Hall, B. C., and Kemp, T. The large-$N$ limit of the Segal-Bargmann transform on $\mathbb{U}_N$. *J. Funct. Anal.* (2013). To appear.

[8] Gordina, M. Quasi-invariance for the pinned Brownian motion on a Lie group. *Stochastic Process. Appl.* 104, 2 (2003), 243–257.

[9] Kemp, T. Heat kernel empirical measures on $\mathbb{U}_N$ and $\mathbb{G}_N$. *Preprint* (June 2013). arXiv:1306.2140.

[10] Kemp, T., Nourdin, I., Peccati, G., and Speicher, R. Wigner chaos and the fourth moment. *Ann. Probab.* 40, 4 (2012), 1577–1635.

[11] McKean, Jr., H. P. *Stochastic integrals*. Probability and Mathematical Statistics, No. 5. Academic Press, New York, 1969.

[12] Mingo, J., and Popa, M. Real second order freeness and Haar orthogonal matrices. *J. Math. Phys.* 54, 051701 (2013).

[13] Mingo, J. A., Śniady, P., and Speicher, R. Second order freeness and fluctuations of random matrices. II. Unitary random matrices. *Adv. Math.* 209, 1 (2007), 212–240.

[14] Nica, A., and Speicher, R. *Lectures on the combinatorics of free probability*, vol. 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[15] Ramasubramanian, S. Hitting of submanifolds by diffusions. *Probab. Theory Related Fields* 78, 1 (1988), 149–163.

[16] Robinson, D. W. *Elliptic operators and Lie groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[17] Voiculescu, D. Limit laws for random matrices and free products. *Invent. Math.* 104, 1 (1991), 201–220.

[18] Voiculescu, D. A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Internat. Math. Res. Notices*, 1 (1998), 41–63.

[19] Voiculescu, D. V., Dykema, K. J., and Nica, A. *Free random variables*, vol. 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[20] Xu, F. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.* 190, 2 (1997), 287–307.