A freely falling frame at the interface of gravitational and quantum realms

D V Ahluwalia-Khalilova

Ashram for the Studies of the Glass Bead Game (ASGBG), Ap. Postal C-600, Zacatecas, Zac 98060, Mexico
and
Center for Mathematical, Physical, and Biological Structure of Universe (CIU), Department of Mathematics, University of Zacatecas (UAZ), Zacatecas, Zac 98060, Mexico
E-mail: d.v.ahluwalia-khalilova@heritage.reduaz.mx

Received 4 August 2004, in final form 4 February 2005
Published 17 March 2005
Online at stacks.iop.org/CQG/22/1433

Abstract
I briefly argue for logical necessity to incorporate, besides $c$, $\hbar$, two fundamental length scales in the symmetries associated with the interface of gravitational and quantum realms. Next, in order to clear the proverbial bush, I discuss the CPT and indistinguishability issue related to recent nonlinear deformations of special relativity and suggest why algebraically well-defined extensions of special relativity do not require nonlinear deformations. That done, I suggest why the stable Snyder–Yang–Mendes Lie algebra should be considered as a serious candidate for the symmetries underlying freely falling frames at the interface of gravitational and quantum realms; thus echoing, and complementing, arguments recently put forward by Chryssomalakos and Okon. In the process I obtain concrete form of uncertainty relations which involve above-indicated length scales and a new dimensionless constant. I draw attention to the fact that because superconducting quantum interference devices can carry roughly $10^{23}$ Cooper pairs in a single quantum state, Planck-mass quantum systems already exist in the laboratory. These may be used for possible exploration of the interface of the gravitational and quantum realms.

PACS numbers: 03.30.+p, 04.50.+h, 04.60.−m

1. Introduction
When one sets out to think about quantum gravity, her/his first question perhaps ought to be: what meanings do ‘quantum’ and ‘gravity’ carry in any such theory? The fundamental significance of this question arises from the facts that [1–3]: (a) the Heisenberg’s fundamental commutators $[x, p_x] = i\hbar, \ldots$ lie at the heart of wave–particle duality and affect the entire
quantum mechanical framework of fields and particles; and that (b) Poincaré spacetime symmetries—as an algebraic representation of constancy of speed of light \(c\) for all inertial observers—not only define the notion of particles, but also suggest the equality of inertial and gravitational masses \([4]\). Furthermore \([5, 6]\), Doplicher et al and the author have independently argued that interplay of the uncertainty relations, \(\delta x \delta p_x \geq \hbar/2\) (following from the fundamental commutator) and Einsteinian gravity, renders spacetime measurements non-commutative. These observations already hint that meanings of quantum and gravity may undergo conceptual and mathematical modifications at the interface of gravitational and quantum realms (IGQR).

A precise and concrete answer to the opening question comes from the stability analysis of the associated algebraic structures \([7]\)—where it may be noted that from a physicist’s point of view a Lie algebra is considered stable if infinitesimal perturbations in its structure constants lead to isomorphic algebras (see, e.g. \([8]\)). The analysis presented in \([7]\), and now confirmed and extended in \([9]\), says in essence, that neither the Heisenberg algebra, nor the Poincaré algebra, preserves its stability at the IGQR\(^1\). The stabilized Heisenberg–Poincaré algebra asks for two additional length scales (to be discussed below) in the same manner that the stability analysis of \([7]\) shows \(1/c^2\) and \(\hbar\), without giving their numerical values, as parameters required by the stabilization of the Galilean and classical kinematics. The stabilization, as reviewed in \([7, 9]\), leads respectively to special relativity (with \(1/c^2\) as the deformation parameter), and quantum mechanics (with \(\hbar\) as the deformation parameter). For earlier references which obtain similar results, the reader’s attention is drawn to \([13, 14]\). In addition, Faddeev has made the observation that general relativity may be viewed to arise from special relativity with Newtonian gravitational constant \(G\) serving as the deformation parameter \([15]\). The paradigm of Lie-algebraic deformations to obtain stable theories claims not only historical success—in retrospect, as having the power to have predicted relativistic and quantum revolutions—but it is also the theory which identifies the fundamental constants underlying their Lie algebraic structure.

There are two other possible answers to the questions asked. The Lie-algebra deformations, leading to a modification of the Heisenberg and Poincaré algebra, considered in the above-cited works of Mendes \([7]\) and that of Chryssomalakos and Okon \([9]\) are in the classical sense of Nijenhuis and Richardson \([16]\). These deformations, being minimal in the sense that they still preserve the Lie algebraic structure, perhaps capture the essence of modifications to the notions of quantum and gravity in IGQR. It is possible that still higher-order corrections/modifications occur in the context of q-deformations or quantum groups \([17]\), an observation already made by Mendes in the concluding paragraph of \([7]\). These shall not be pursued in this paper.

Another possible answer to the question asked is offered by the phenomenological modifications of dispersion relations in such a way that two, or more, additional deformation parameters are introduced. These theories, considered under a generic misnomer ‘doubly/triply special relativity’, despite significant amount of effort and publications, continue to suffer from a lack of well-defined mathematical framework which assures self-consistency. Additionally, they have failed to provide a satisfactory spacetime, or phase space, structure. For instance, Kowalski–Glikman and Smolin have compiled a list of four

\(^1\) To avoid confusion it is to be noted that the stability of the Poincaré algebra away from the gravitational realm refers to the kinematical group of the tangent space to the spacetime manifold and not to the group of motions in the manifold itself. A ‘harmless’, i.e. devoid of physical implications, instability also exists for the Heisenberg algebra. It is an artefact of singling out \(x\), as compared, say, to \(\exp(ix)\), as a physical observable. While we do not deal here with quantum deformations, it is worth noting, as has been pointed out to us by one of the referees, that a quantum deformation of a group is a stabilization in the domain of Hopf algebras. From this point of view also neither the Poincaré nor the Heisenberg Lie algebra is stable. The reader is referred to \([10–12]\) for further details.
astrophysical and cosmological anomalies which seem to carry a single quantum–gravity origin [18]. In the same paper they suggest a new nonlinear deformation of the Poincaré algebra and put forward a ‘triply special relativity’. Chryssomalakos and Okon [19] were immediately able to show that the proposed algebra can be brought to a Lie—i.e. a linear—form by an appropriate identification of its generators, and that this linear form was the same as that arrived at by Mendes [7] and Yang [20]. The nonlinearity in the Kowalski–Glikman and Smolin proposal arises due to the implicit insistence that central charge(s) remain undeformed. Since there is no physical or mathematical justification to make this assumption, and since the motivation to introduce additional invariant scales remains unaffected by this assumption, there seems to be no reason to abandon linearity (i.e. the Lie algebraic framework). Therefore, on the positive side, physical motivations provided by the literature on ‘doubly/triply special relativity (DSR/TSR)’ are indicative of a fundamental change required for notions of quantum and gravity in IGQR. On the discouraging side, the DSR/TSR’s theoretical framework remains far from a well-defined mathematical scheme and it carries dubious/incomplete interpretational elements.

Independently, as already noted above, it appears that any attempt which incorporates the gravitational effects in quantum measurement of spacetime events leads to (a) a non-commutative spacetime and (b) to the associated modification of the fundamental commutators [6, 22–24]. The latter, in the framework considered in [22, 23], leads to modification of the de Broglie wave–particle duality in such a manner that it saturates the matter wavelengths to Planck length, \( \ell_P \equiv \sqrt{\hbar G/c^3} \). That is, irrespective of the relative velocity of two inertial frames, \( \ell_P \) does not Lorentz contract. On one hand this is a physical counterpart of Mendes’ stability argument and, on the other, it immediately calls for modification of special relativity to incorporate not only the invariant \( c \) but also \( \ell_P \). When one adds to this Mendes’ stability argument for the combined Heisenberg–Poincaré algebra, one is forced to include yet another length scale. That length scale may be tentatively identified with a large-scale cosmological property governed by 
\[
\ell_C = \sqrt{\frac{3c^2}{8\pi G \rho_{\text{vac}}}} \defeq \sqrt{\frac{1}{\Lambda}},
\]
with \( \Lambda \) being the cosmological constant.

The important thing for this paper is not that the length scales take the values \( \ell_P \) and \( \ell_C \) but that there exist two length scales: one in the extreme short-distance range, and the other carrying astrophysical, or cosmological, scale. In what follows, this flexibility in the identification shall be taken as implicit.

Freely falling frames being most appropriate realms to establish the relativistic and quantum algebras, the primary aim of this paper becomes to present a concrete modification to the notion of freely falling frames at the interface of gravitational and quantum realms, and to address the related issues. In the process we shall give algebraically precise meaning to the notions of ‘quantum’ and ‘gravity’ and point out some of the most immediate implications.

In section 2, as required by the above discussion, I first attempt to clear the proverbial bush relevant to addressing the issue of freely falling frames at the interface of gravitational and quantum realms. This effort also allows us to present a systematic methodology to obtain wave equations. It applies not only to nonlinear proposals, but also to those Lie algebraic frameworks where the Lorentz sector remains intact. Then, in section 3, I return to the subject of what Lie algebraic structure the freely falling frame at the interface of gravitational and quantum realms may carry. This then provides a systematic step towards construction of a relativity for the IGQR where spacetime acquires intrinsically quantum and gravitational character. That is, even in a freely falling frame there remain intrinsically quantum and gravitational signatures. The spacetime in IGQR requires not only \( c \), but also \( \hbar, \ell_P \) and \( \ell_C \) (and possibly a new dimensionless constant \( \beta \) signifying a radical departure from some of the
quantum relativistic notions). Section 4 is devoted to concluding remarks. With the exception of section 2, where we set $\hbar$ and $c$ to be unity, we shall make them explicit in the remainder of this paper.

2. CPT and the indistinguishability issue for nonlinear deformations of special relativity

In order to take the next logical step I find it necessary to first update and close an argument which I put forward a few years ago in [25]. Here, therefore, I summarize my views on CPT and the indistinguishability issue as encountered in doubly [26–28], and now triply [18], special relativity. Before I address the matters of content, it may be useful to make a few informal remarks and attend to a matter of nomenclature. I take this liberty because, in my opinion, these remarks set the essential tone of this paper and because any future evolution of this subject should be based on a more clear and well-defined premise. On the indicated issue I do not give a categorical, or unambiguous, answer (nor is one possible in a model-independent manner) but, instead, write this paper in a manner which parallels the development of the ideas in the field and how, prematurely, one may be tempted to claim results [18, 29] which on closer examination raise troubling questions2 [19, 30–36].

2.1. Nomenclature

The special of ‘special relativity’ refers to the circumstance that one restricts to a special class of inertial observers which move with a relative uniform velocity. The general of ‘general relativity’ lifts this restriction. The ‘special’ of special relativity has nothing to do with one versus two, or three, invariant scales. It rather refers to the special class of inertial observers; a circumstance that remains unchanged in special relativity with two invariant scales. Taken to its [i]logical conclusion it would mean that a theory of general relativity with two invariant scales would be called ‘doubly general relativity’. A detailed look at the algebraic structure of ‘doubly special relativity’ [27, 28] suggests that these are nonlinear deformations of special relativity; or, in the language of [32], a change of basis, associated with a nonlinear change of generators, in the enveloping algebra. The deformations are characterized with two invariant scales. The technically appropriate, though by no means unique, nomenclature is thus: nonlinear deformations of special relativity with 2 invariant scales, i.e., NSR–2. Many of the remarks I make here, though written in the context of NSR–2, remain valid for NSR–3 as well. Taking note of this observation, the authors of [19] have expressed their opinion in the following words, ‘we think the above term is conceptually inappropriate enough to warrant its abolishment . . . ’. I agree. The term they are referring to is ‘doubly/triply special relativity’.

Yet, one may be tempted to preserve DSR, with D now meaning ‘deformed’ rather than ‘double’. But then it does not distinguish between nonlinear and linear (i.e., Lie) deformations. Nor does such an abbreviation extend naturally to triply special relativity.

2.2. A mix of deformations in algebra and transformation parameters

In NSR–2s, the underlying algebra for the rotations and boosts remains intact as the standard Lie algebra of the Lorentz group. The nonlinear deformation of the algebra is contained in the remaining sector. As long as the motivation is to obtain a special relativity with an extended number of invariants—from 1 to 2 (or even 3 as in [18])—and as long as the Lie

2 In the connection of [32], it is worthwhile noting that the nonlinearity of some commutators in quantum deformed algebra may be eliminated by means of an appropriate change of basis but then one gets again the same Hopf algebra and not a Lie algebra. For relevant details and construction of wave equations, see [37, 38].
algebra framework provides a well-defined framework [19], the physical and mathematical justification for invoking nonlinear deformations seems to be too unwarranted a break from the standard framework\(^3\). Furthermore, even though in NSR–2s the Lorentz algebra remains intact, the boost parameter suffers a modification. This mix of deformations in the underlying algebra and the associated transformation parameters complicates the theory significantly enough that no fully satisfactory version of the theory exists beyond the momentum space (and even there the many-particle sector is not devoid of the unresolved problems). That is, one is presented with an extension of special relativity without providing a full replacement of spacetime transformations, and without providing an operational meaning of the various symbols used. Thus, e.g., the phase space remains either ill defined or undefined, so also is the case with the parameter which attends to the inertial properties of the particles.

Under these circumstances the physical distinguishability issue becomes ill-defined and one can only clarify issues which do not invoke phase space. One such issue is the question of modification of the Dirac equation, and study of wave equations associated with different representation spaces. The task of this section is to show that under these circumstances one can claim all sorts of effects which depend on the deformation parameter \(\ell_P\), the Planck length. But many of these corrections carry no operational meaning. The temptation to claim \(O(\ell_P)\) corrections [39], and even to suggest a CPT violation at that order [40], should be resisted and additional conceptual and mathematical questions asked.

These claims are now established. In writing these claims I, by necessity, and to make this work as self-contained as possible, reproduce some of the results of [25, 41, 42]. Some of the mathematical aspects are similar to those presented by Agostini, Amelino-Camelia and Arzano in [39] but my interpretation differs dramatically. In what follows, I also incorporate an important phase factor, not appreciated in [39]. The neglect of that phase amounts to projecting out antiparticles from NSR–2s as I have already noted in [42].

2.3. NSR-2 of Amelino-Camelia, and Magueijo and Smolin

The simplest of NSR–2s result from keeping the algebra of boost and rotation generators intact while modifying the boost parameter in a nonlinear manner (the deformation of algebra itself lies in the remaining sector). Specifically, in the NSR–2 of Amelino-Camelia, the boost parameter, \(\varphi\), changes from the special relativistic form

\[
\cosh \varphi = \frac{E}{m}, \quad \sinh \varphi = \frac{p}{m}, \quad \hat{\varphi} = \frac{p}{p},
\]

(1)
to [27, 43, 44]

\[
\cosh \xi = \frac{1}{\mu} \left( \frac{e^{\ell_P E} - \cosh (\ell_P m)}{\ell_P \cosh (\ell_P m/2)} \right),
\]

(2)

\[
\sinh \xi = \frac{1}{\mu} \left( \frac{pe^{\ell_P E}}{\cosh (\ell_P m/2)} \right), \quad \hat{\xi} = \frac{p}{p}.
\]

(3)

While for the NSR–2 of Magueijo and Smolin, the change takes the form [28, 44]

\[
\cosh \xi = \frac{1}{\mu} \left( \frac{E}{1 - \ell_P E} \right),
\]

(4)

\[
\sinh \xi = \frac{1}{\mu} \left( \frac{p}{1 - \ell_P E} \right), \quad \hat{\xi} = \frac{p}{p}.
\]

(5)

\(^3\) I concede that to some extent this is a matter of taste, provided one has a well-defined mathematical and interpretational scheme.
Here, $\mu$ is a Casimir invariant of NSR–2 (see equation (24) below) and is given by
\[
\mu = \begin{cases} 
\frac{2}{\ell_P} \sinh \left( \frac{\ell_P m}{2} \right) & \text{for [27]'s NSR–2} \\
\frac{m}{1 - \ell_P m} & \text{for [28]'s NSR–2.}
\end{cases}
\] (6)

The notation is that of [44]; with the minor exceptions $\lambda, \mu, \nu, m_0$ there are $\ell_P, \mu, m$ here.

Now, it is an assumption of NSR–2 theories that the nonlinear action of $\xi$ is restricted to the momentum space only. No fully satisfactory spacetime description in the context of the NSR–2 theories has yet emerged, and we are not sure if such an operationally meaningful description indeed exists. Therefore, to the extent possible, our arguments shall be confined to the momentum space.

### 2.4. Master equation for spin-1/2: Dirac case

Since the relevant underlying spacetime symmetry generators remain unchanged, much of the formal apparatus of the finite-dimensional representation spaces associated with the Lorentz group remains intact. In particular, there still exist $(1/2, 0)$ and $(0, 1/2)$ spinors. But now they transform from the rest frame to an inertial frame in which the particle has momentum, $p$, as
\[
\phi_{(1/2,0)}(p) = \exp \left( \frac{\sigma}{2} \cdot \xi \right) \phi_{(1/2,0)}(0),
\] (7)
\[
\phi_{(0,1/2)}(p) = \exp \left( - \frac{\sigma}{2} \cdot \xi \right) \phi_{(0,1/2)}(0).
\] (8)

Since the null momentum vector 0 is still isotropic, one may assume that (see p 44 of [45] and [46–48])
\[
\phi_{(0,1/2)}(0) = \zeta \phi_{(1/2,0)}(0),
\] (9)
where $\zeta$ is an undetermined phase factor. In general, the phase $\zeta$ encodes C, P, and T properties.

The interplay of equations (7)–(9) yields a master equation for the $(1/2, 0) \oplus (0, 1/2)$ spinors,
\[
\psi(p) = \begin{pmatrix} \phi_{(1/2,0)}(p) \\ \phi_{(0,1/2)}(p) \end{pmatrix},
\] (10)
to be
\[
\begin{pmatrix} -\zeta I_2 & \exp (\sigma \cdot \xi) \\ \exp(-\sigma \cdot \xi) & -\zeta^{-1} I_2 \end{pmatrix} \psi(p) = 0,
\] (11)
where $I_n$ stands for the $n \times n$ identity matrix (and $0_n$ represents the corresponding null matrix). As a check, taking $\xi$ to be $\varphi$, and after some simple algebraic manipulations, the master equation (11) reduces to
\[
\begin{pmatrix} -m \zeta^2 & E_{12} + \sigma \cdot p \\ E_{12} - \sigma \cdot p & -m \zeta^{-2} \end{pmatrix} \psi(p) = 0.
\] (12)

With the given identification of the boost parameter we are in the realm of special relativity. There, the operation of parity is well understood. Demanding parity covariance for equation (12), we obtain $\zeta = \pm 1$. Identifying
\[
\begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & -\sigma \\ \sigma & 0_2 \end{pmatrix},
\] (13)
A freely falling frame at the interface of gravitational and quantum realms 1439

with the Weyl-representation $\gamma^0$, and $\gamma^i$, respectively, equation (12) reduces to the Dirac equation of special relativity,

$$(\gamma^\mu p_\mu \mp m)\psi(\mathbf{p}) = 0. \quad (14)$$

The linearity of the Dirac equation in $p_\mu = (E, -\mathbf{p})$ is now clearly seen to be associated with two observations:

$O_1$ that, $\sigma^2 = 1$; and

$O_2$ that in special relativity, the hyperbolic functions—see equation (1)—associated with the boost parameter are linear in $p_\mu$.

In NSR–2, observation $O_1$ still holds. But, as equations (2)–(5) show, $O_2$ is strongly violated.

The extension of the presented formalism to the eigenspinors of the charge conjugation operator is more subtle [49]. The extension to $(1/2, 1/2)$ representation space to describe vector particles is less demanding and can be immediately obtained using the techniques of [50].

2.5. Master equation for higher spins

The above-outlined procedure applies to all, bosonic as well as fermionic, $(j, 0) \oplus (0, j)$ representation spaces. It is not confined to $j = 1/2$. A straightforward generalization of the $j = 1/2$ analysis immediately yields the master equation for an arbitrary spin,

$$\left(\begin{array}{cc} -\zeta^{12j+1} & \exp(2\mathbf{J} \cdot \xi) \\ \exp(-2\mathbf{J} \cdot \xi) & -\zeta^{-12j+1} \end{array}\right) \psi(\mathbf{p}) = 0, \quad (15)$$

where

$$\psi(\mathbf{p}) = \begin{pmatrix} \phi_{(j,0)}(\mathbf{p}) \\ \phi_{(0,j)}(\mathbf{p}) \end{pmatrix}. \quad (16)$$

Equation (15) contains the central result of the previous section as a special case. For studying the special relativistic limit, it is convenient to bifurcate the $(j, 0) \oplus (0, j)$ space into two sectors by splitting the $2(2j + 1)$ phases, $\zeta$, into two sets: $(2j + 1)$ phases, $\zeta^+$, and the other $(2j + 1)$ phases, $\zeta^-$. Then in the particle’s rest frame the $\psi(\mathbf{p})$ may be written as

$$\psi_h(0) = \begin{cases} u_h(0) & \text{when } \zeta = \zeta^+ \\ v_h(0) & \text{when } \zeta = \zeta^- \end{cases}. \quad (17)$$

The explicit forms of $u_h(0)$ and $v_h(0)$ (see equation (9)) are

$$u_h(0) = \begin{pmatrix} \phi_h(0) \\ \zeta_+ \phi_h(0) \end{pmatrix}, \quad v_h(0) = \begin{pmatrix} \phi_h(0) \\ \zeta_- \phi_h(0) \end{pmatrix}, \quad (18)$$

where the $\phi_h(0)$ are defined as $\mathbf{J} \cdot \mathbf{p} \phi_h(0) = h \phi_h(0)$, and $h = -j, -j + 1, \ldots, +j$. In the parity covariant special relativistic limit, we find $\zeta^+ = +1$ while $\zeta^- = -1$. As a check, for $j = 1$, identification of $\xi$ with $\varphi$, and after implementing parity covariance, equation (15) yields

$$(\gamma^{\mu\nu} p_\mu p_\nu \mp m^2)\psi(\mathbf{p}) = 0. \quad (19)$$

The $\gamma^{\mu\nu}$ are unitarily equivalent to those of [46], and thus we reproduce bosonic matter fields with $\{C, P\} = 0$. A carefully taken massless limit then shows that the resulting equation is

4 Such an exercise was undertaken in [40] but it suffers from a set of serious interpretational issues, a matter on which I shall briefly comment in this paper. For the moment, the reader is warned that what the authors of [40] call helicity is really spin projection on the $\hat{z}$ direction. This already introduces several errors as the $\mathbf{p}$ vector in [40] is a completely general special-relativistic three momentum.
consistent with the free Maxwell equations of electrodynamics. This again casts doubt on the operational distinguishability of the $O(\ell_P)$ predictions presented in [40].

Since the $j = 1/2$ and $j = 1$ representation spaces of NSR–2 reduce to the Dirac and Maxwell descriptions, it would seem apparent (and as is often argued in similar contexts [39]—wrongly, at least to an extent, as we will soon see) that the NSR–2 contains physics beyond the linear-group realizations of special relativity. To the lowest order in $\ell_P$, equation (11) yields

$$ (\gamma^\mu p_\mu + \hat{m} + \delta_1 \ell_P) \psi(p) = 0, $$

where

$$ \hat{m} = \begin{pmatrix} -\xi_{12} & 0_2 \\ 0_2 & -\xi_{-112} \end{pmatrix} m $$

and

$$ \delta_1 = \begin{cases} \gamma^0 \left( \frac{E^2 - m^2}{2} \right) + \gamma^j p_j E & \text{for [27]'s NSR–2} \\
\gamma^\mu p_\mu (E - m) & \text{for [28]'s NSR–2.} \end{cases} $$

Similarly, the master equation presented can be used to obtain NSR–2’s counterparts for Maxwell’s electrodynamic. Unlike the Coleman–Glashow framework [51], the existing NSR–2s provide all corrections, say, to the standard model of high-energy physics, in terms of one—and not 46—fundamental constant, $\ell_P$. Had NSR–2s been operationally well defined and distinct, this would have been a remarkable power of NSR–2-motivated frameworks.

2.6. Judes–Visser variables: challenging some of the NSR–2’s claims

We now show that the NSR–2 program as implemented currently is inadvertently misleading. Sometimes this is in good humour (or, so we interpret), e.g., when the author of [52] notes ‘mathematical triviality by no means implies physical equivalence, and one may argue that it is in fact an asset’. At other times, it is simply a manifest lack of care being given to the operational meaning of various symbols one uses in his/her mathematical formalism and a total disregard for the existing literature on ‘indistinguishability’, or ‘conceptual’ issues for NSR–2 [40].

The question is what are the operationally measurable quantities in NSR–2? The $E$ is no longer the 0th component, nor is $p$ the spatial component of the 4-momentum. Neither is $m$ an invariant under the NSR–2 boosts. Their physical counterparts, as we interpret them, are Judes–Visser variables [44], $\eta^\mu \equiv (\epsilon(E, p), \pi(E, p)) = (\eta^0, \eta)$, and $\mu$. The $\epsilon(E, p)$ and $\pi(E, p)$ relate to the rapidity parameter $\xi$ of NSR–2 in the same functional form as do $E$ and $p$ to $\varphi$ of special relativity:

$$ \cosh(\xi) = \frac{\epsilon(E, p)}{\mu}, \quad \sinh(\xi) = \frac{\pi(E, p)}{\mu}, $$

where

$$ \mu^2 = [\epsilon(E, p)]^2 - [\pi(E, p)]^2. $$

They provide the most economical and physically transparent formalism for representation space theory in NSR–2. For $j = 1/2$ and $j = 1$, equation (15) yields the exact NSR-2 equations for $\psi(\pi)$:

$$ (\gamma^\mu \eta_\mu \mu + \hat{\mu}) \psi(\eta) = 0, \quad \text{where} \quad \hat{\mu} = \begin{pmatrix} -\xi_{-112} & 0_2 \\ 0_2 & -\xi_{12} \end{pmatrix} \mu, $$
A freely falling frame at the interface of gravitational and quantum realms

\[(\gamma^{\mu \nu} \eta_\mu \eta_\nu + \mu^2)\psi(\eta) = 0, \quad \text{with} \quad \mu^2 = \begin{pmatrix} -\zeta^{-1} & 0 \\ 0 & -\zeta \end{pmatrix}. \quad (26)\]

From an operational point of view the \(\eta^\mu\) and \(\mu\) are the physical observables. The old operational meaning of the symbols \(E\) and \(p\) is lost in the nonlinear realization of the boost in the momentum space\(^5\). There is covariance of the form of the considered fermionic and bosonic wave equations under the transformations

\[
m \rightarrow \mu, \quad p^\mu \rightarrow \eta^\mu, \quad \varphi \rightarrow \xi. \quad (27)\]

Thus, at the level of momentum-space wave equations, the NSR–2 and special relativistic descriptions of fermions and bosons carry identical forms. For this reason, the \(O(\ell_P)\) departures given in [39] are artifacts of the used variables. The same holds true for the \(O(\ell_P)\) corrections at the level of the vector potential given in [40]. Again, they are simply artefacts of the variables used. To be more precise, the object considered by authors of [40] Lorentz transforms as \((1/2, 1/2)\), and is represented as \(A^\mu(p)\) in [40]. On the other hand the related field strength tensor, \(F^{\mu \nu}(p)\), transforms as \((1, 0) \oplus (0, 1)\) object under Lorentz transformations. Since the latter representation space in the momentum space is shown here to be indistinguishable from its special-relativistic counterpart—see, equation (29)—the associated NSR–2’s \(A^\mu(p)\) cannot be physically different from its special-relativistic counterpart. Thus, it establishes that the Acosta–Kirchbach result on \(O(\ell_P)\) departures is an artefact of ignoring Judes–Visser variables.

Yet it should be noted that to the extent the phase space of NSR–2 remains ill-defined, one cannot make any physically significant ‘indistinguishability’ claim at the level of physical amplitudes and cross sections in NSR–2. The same remains true for NSR–3.

Given that so far a satisfactory spacetime theory of NSR–2 is lacking, we implement parity covariance by demanding covariance under \(\eta \rightarrow \eta' \equiv -\eta\). This demand transforms the above wave equations to

\[
(\gamma^{\mu \nu} \eta_\mu \eta_\nu + \mu^2)\psi(\eta) = 0, \quad (28)\]

\[
(\gamma^{\mu \nu} \eta_\mu \eta_\nu + \mu^2)\psi(\eta) = 0. \quad (29)\]

The translation of equations (28) to (29) to spacetime occurs as follows. Once \(\eta^\mu\) are accepted as physical, they immediately require the spacetime variables to be that of standard special relativity; otherwise, there is no meaning to the interpretation of \(\eta^\mu\) as corresponding to the measured and conserved energy–momentum 4-vector implied by time-translational invariance. The spacetime evolution follows with the substitution \(\eta_\mu \rightarrow i\partial_\mu\), and \(\psi(\eta) \rightarrow \exp(i\eta_\mu x^\mu)\psi(x)\).

The basic questions which the discussion of Acosta and Kirchbach’s paper [40] now provokes are

\(^5\) The adherents of NSR–2 propose \(p^\mu\) as a physical variable despite the fact that it has no known geometrical object that can be identified with; see, [53] for a detailed discussion of interpretational possibilities of this view and the questions it raises. On the other hand, the Judes–Visser variables \(\eta^\mu\) have well-defined mathematical meaning for all inertial observers (without requiring a preferred observer) and correspond to conserved quantities. The NSR–2 view that ‘measured quantities \(p^\mu\) need not be conserved, considered as a possibility in [53] to lend NSR–2 an element of viability, amounts to giving a nonlinear choice of coordinates in momentum space a far fetched, and unviable, physical interpretation. That is, I do not accept the plausibility argument of [53] that ‘unscreenable’ quantum gravitational effects may forbid \(\eta^\mu\) from being directly measured. One reason for this stance is that the ‘unavoidable’ quantum-mechanical fluctuations do not make the energy–momentum 4-vector as immeasurable; in fact, quantum framework provides precise calculational tools to predict the associated uncertainties. Had this not been the case, the Dirac equation would have not carried the empirically verified (and demanded by the Lorentz covariance) linearity in time acquired via the identification \(p_\mu \rightarrow i\beta_\mu\).

\(^6\) The result is expected to be the same for other representation spaces.
which parameter, \( m \) or \( \mu \), is connected with the CPT symmetry of the 2-scale nonlinear deformations of special relativities of [26, 28]?

and, which parameter, \( m \) or \( \mu \), in the non-relativistic Ehrenfest, weak-field limit of the 2-scale nonlinear deformations of special relativities of [26, 28] couples to gravity?

The authors of [40] do not ask these, or related, questions. Barring certain pathologies of the theories described in [26, 28], the answer to the question must be independent of the representation space. Therefore, it is sufficient to establish it for equation (28), i.e., the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) representation space. The answer constitutes nothing more than a simple and obvious exercise. Yet, to lay matters to rest, we outline the details.

First, we know from the above discussion, and further details given in [42], that in the Weyl representation the particle spinors read

\[
\begin{align*}
u_{\pm} (\eta) &= \kappa (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \left( \phi_{\pm} (0) \right), \\
\phi_{\pm} (0) &= \begin{pmatrix} \phi_{\pm} (0) \\ \phi_{\pm} (0) \end{pmatrix},
\end{align*}
\]

while the antiparticle spinors are

\[
\begin{align*}
u_{\pm} (\eta) &= \kappa (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \left( -\phi_{\pm} (0) \right), \\
\phi_{\pm} (0) &= \begin{pmatrix} \phi_{\pm} (0) \\ -\phi_{\pm} (0) \end{pmatrix}.
\end{align*}
\]

The \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) boost operator which appears in the above equations is

\[
\kappa (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \stackrel{\text{def}}{=} \begin{pmatrix} \exp (\frac{\sigma_2 \cdot \xi}{2}) \\ \exp (-\frac{\sigma_2 \cdot \xi}{2}) \end{pmatrix},
\]

and \( \phi_{\pm} (0) \) are eigenspinors of the helicity operator \( \sigma_2 \cdot \hat{\xi} \). These are connected by the charge conjugation symmetry only if inertial properties of the described particles are dictated by \( \mu \) and not \( m \). Second, taking the non-relativistic Ehrenfest limit of equation (28) in the presence of weak gravity reveals that it is \( \mu \) that couples to gravity and not \( m \).

The answer for both of the questions asked above is \( \mu \). Not \( m \). The \( m \) is simply not a physically meaningful observable of the theories proposed in [26, 28] and [18]. The asymmetry noted by Acosta and Kirchbach is superficial. It occurs only if one ignores the questions asked. As such, at this stage of analysis, there is no argument which favours CPT violation. Equations, such as (11), of [40] are thus devoid of any physical content. They arise on the one hand due to misidentification of the inertial properties of particles and, on the other, due to implicit neglect of Judes–Visser transformations [44].

The next level of clarification that is needed is that the apparent similarity between equations (28) and (29) and their special relativistic counterparts can make one conclude that there is no distinction between NSR–2s, or NSR–3 (for in the latter too, the Lorentz algebra remains undeformed). Such a claim, besides the reasons already noted, would appear to be misleading, because while in ordinary special relativity the commutator \([P_\mu, P_\nu]\) vanishes, it is not the case for NSR–2s and NSR–3. However, Chryssomalakos and Okon show [19] that the algebra of NSR–3 can be brought to the Lie, i.e., linear, form by a correct identification of its generators. If one complements the Chryssomalakos–Okon result with the observation that if one tentatively defines\(^7\) special-relativistic (1, 2, or 3 invariant scales are irrelevant, but only 'special' is relevant) theory as the one which carries the kinematical group of the tangent space to the spacetime manifold one must, in the notation of Chryssomalakos and Okon [19], take the \( R \to \infty \) limit of the algebra. In that limit, the \([P_\mu, P_\nu]\) vanishes again. So, in the momentum space, equations (28) and (29) and their special relativistic counterparts are identical.

\(^7\) A matter with which I will take issue below.
Yet, the indistinguishability issue cannot be settled on this last remark alone. The reason is that the $R \to \infty$ limit still leaves the underlying spacetime noncommutative. However, as Mendes has shown [7] that the spacetime representation of the $p_\mu$ is still $i\partial_\mu$. This implies that the configuration space form of equations (28) and (29) remains the same as in the ordinary special relativity. Yet, the extension of special relativities based on Lie-algebraic deformations are profoundly different form ordinary special-relativistic theories. The reason is that the Yang–Mendes algebra brought to attention in Chryssomalakos–Okon’s paper [19] carries a different phase space [54] and that has the potential to dramatically change the predictions of the theory at the Planck scale.

We conclude, echoing again the sentiments of [19], that the search for relativities, special or general, with additional invariant scales, while well motivated, carries no justification to go towards nonlinear extensions. These NSRs leave far too many questions unanswered with dubious justification for considering them in the first place. At this early stage there is no reason to invoke nonlinear deformations. On the other hand, the Lie, i.e. linear, deformations are well defined. The stabilized Poincaré–Heisenberg algebra is nothing but the Yang–Mendes Lie algebra [7, 20]. Special and general relativities with additional invariant scales, in my opinion, must be based on this yet unexplored Lie algebraic structure. In any case, despite differences of opinions, one thing is clear that the deformation of spacetime symmetries involving $c, \hbar, \ell_P = \sqrt{\hbar G/c^3}$ (which adds to $c$ and $\hbar$ the constant $G$), and the cosmological constant $\Lambda$—or, appropriate new combinations—shall play a profound role in any theory of quantum gravity [55, 42] and that it may already be evident in certain anomalous astrophysical and cosmological observations [18]. One caution should, however, be exercised: there is a tendency in the literature, see, e.g., [56], to identify modification of the certain algebraic commutators, or deformation of dispersions relations, with NSR-n’s. Such naive identifications confuse the issue of nonlinear versus linear (i.e. Lie) deformations. Not all modifications which lead to such departures from special relativity fall under the umbrella of NSR-n’s. In fact, as we shall momentarily see, NSR-n’s probably are not viable physical theories. Yet, more viable theories based on Lie-algebraic structures carry the spacetime/energy–momentum non-commutativity, and many other intrinsic features similar, though not the same, as to those found in NSR-n’s.

3. A stable Lie-algebraic structure for freely falling frames at the interface of gravitational and quantum realms

Mendes [7] and Chryssomalakos [8] have emphasized that stability of the underlying Lie algebras must be taken as one of the important physical criteria to consider a theory as physically viable. In the context of this newly suggested criterion, if one confines oneself only to spacetime symmetries then one notes that the algebra underlying special relativity, i.e., Poincaré, is stable modulo the remarks made earlier. However, as soon as quantum phenomena are studied one must bring in the Heisenberg algebra, which is also stable (up to a ‘harmless’ instability noted earlier), into the picture. But the Poincaré–Heisenberg algebra ceases to be stable. These observations may be further amplified by noting the following.

- In 1947 Snyder pointed out that the assumption that spacetime be a continuum is not imposed by the Lorentz invariance [57].
- Later in the same year, Yang noted that lack of translational invariance in Snyder’s framework can be rectified if spacetime is allowed to carry curvature [20]. In that same one-and-half column paper, Yang also presented the complete Lie algebra associated with the suggested modification.
• In a series of papers published in the last decade (and which remain almost unnoticed, see, e.g., [7, 58]), Mendes came to the conclusion that when Poincaré and Heisenberg algebras are considered together—as they must be in any relativistic quantum framework—they are not rigid in the mathematical sense. That is, the physical theories based on them lack certain elements of robustness, or stability. In addition, he obtained the suggested stable Lie algebra. That Lie algebra contained two additional length scales, and it was the same very algebra as was obtained by Yang in 1947. Not only this, in support of his suggestion that stability should be considered an important physical criterion he pointed out that both the ‘relativistic revolution’ and the ‘quantum revolution’ of the last century can be motivated—alas in retrospect—by the stability criterion.

• Taking note of the work by Mendes, Chryssomalakos and Okon have just noted that recently proposed triply special relativity algebra proposed by Kowalski-Glikman and Smolin [18] can be brought to a linear (i.e., Lie) form by a correct identification of its generators and that the resulting Lie algebra is precisely of Yang–Mendes form.

As, in essence, the whole story began with Snyder, one is tempted to suggest the stabilized form of the Poincaré–Heisenberg algebra, i.e. Snyder–Yang–Mendes algebra, be called by a simple acronym SYM. But this acronym has already been used widely in the literature in other contexts, such as for super Yang–Mills theories, and for that reason we shall settle for ‘Lie algebra for IGQR’.

Referring to the Introduction, one can now identify the two length scales that appear in the Lie algebra for IGQR with \( \ell_p \) and \( \ell_C \). With the indicated identifications, the Mendes-inspired work of Chryssomalakos and Okon [7, 9] suggests

\[
\begin{align*}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}), \\
[J_{\mu\nu}, P_{\lambda}] &= i(\eta_{\nu\lambda} P_{\mu} - \eta_{\mu\lambda} P_{\nu}), \\
[J_{\mu\nu}, X_{\lambda}] &= i(\eta_{\nu\lambda} X_{\mu} - \eta_{\mu\lambda} X_{\nu}), \\
[P_{\mu}, P_{\nu}] &= i\frac{\hbar^2}{\ell_C^2} J_{\mu\nu}, \\
[X_{\mu}, X_{\nu}] &= i\ell_p^2 J_{\mu\nu}, \\
[P_{\mu}, X_{\nu}] &= i\hbar\eta_{\mu\nu}\mathcal{F} + i\hbar\beta J_{\mu\nu}, \\
[P_{\mu}, \mathcal{F}] &= i\frac{\hbar}{\ell_C^2} X_{\mu} - i\beta P_{\mu}, \\
[X_{\mu}, \mathcal{F}] &= i\beta X_{\mu} - i\frac{\ell_p^2}{\hbar} P_{\mu}, \\
[J_{\mu\nu}, \mathcal{F}] &= 0,
\end{align*}
\]

as a natural candidate for a physically viable theory in the IGQR. Here \( \beta \in R \) is a new dimensionless constant. Its presence has been noted in [7, 9, 59] with differing emphasis. In this section we exhibit \( \epsilon \) and \( \hbar \) explicitly. As in the previous section, \( \eta_{\mu\nu} \) is diagonal with \( \text{diag}(1, -1, -1, -1) \), whereas \( p^\mu = (E/c, p) \) and \( p^\mu = (E/c, -p) \).

There is now a temptation to take the \( \ell_C \rightarrow \infty \) limit and identify the resulting algebra with the algebra that shall be found in freely falling frames of quantum gravity. A hint in that direction occurs in the papers of Mendes. This, in my opinion, may not be justified as

\^8 When Mendes wrote his papers it appears that while he was aware of Snyder’s paper, Yang’s important paper had escaped his attention.
then, owing to work of Yang [20], translational invariance is no longer obvious (a question which should be re-examined in any case for $\beta \neq 0$). But, more importantly, the modified zero point energy—which now need not carry the same magnitude for fermionic and bosonic fields—cannot be removed from a freely falling frame. This is as true for IGQR as for freely falling frames of special relativity endowed with any quantum field. The difference now is that the magnitude of the zero-point energy at a given angular frequency is not guaranteed to be equal (a subject which requires \textit{ab initio} calculations). It may indeed happen that the famous ‘120 orders of magnitude problem’ associated with the cosmological context in the standard Poincaré–Heisenberg framework is resolved by cancellation of the zeroth order ‘$\pm (1/2)\hbar \omega$’ contributions, and the observed cosmological constant arises due to higher-order terms. However, for such a cancellation to occur, supersymmetry would seem the most natural agent. But such a circumstance would require a non-trivial stability analysis with supersymmetry incorporated. This is not to be interpreted as a weakness of the Lie-algebraic stability paradigm but as its strength because it suggests a logical and well-defined path to be followed. Also if the above-mentioned translational invariance is lost, operational meaning for energy and momentum ceases to exist.

For these reasons, I suggest that the Lie algebra for IGQR as it is written above represents the algebra of new special relativity that underlies the IGQR. It has an intrinsic curvature. That is, the IGQR spacetime carries a curvature even in the absence of conventional sources. The source of this curvature is the vacuum energy density that defines the cosmological constant and cannot be eliminated from freely falling frames as can be justified on empirical grounds also. In this interpretation, quantum gravity is likely to arise from ‘gauging’ this algebra in precisely the same sense as in Yang–Mills gauge theories. In such a theory the notion of point particle is no longer a viable one. It is replaced by a fuzzy specification governed by spacetime noncommutativity (37). Similarly, de Broglie wave–particle duality suffers a modification due to deformation of the fundamental commutator (see below, again). The locality of the standard quantum field theory (which is based on unstable Poincaré–Heisenberg algebra) is lost; see equation (37). This latter unavoidable feature is likely to introduce an intrinsic element of CPT violation.

The fact that the Heisenberg’s fundamental commutator (38) undergoes non-trivial modifications with $\mathcal{F}$ ceasing to be central, and $\beta \neq 0$, has the following immediately identifiable consequence: the position–momentum Heisenberg uncertainty relations get modified. For example,

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} |\langle \mathcal{F} \rangle|, \quad (42)$$

while $\Delta x \Delta p_y$ no longer vanishes, but instead is given by

$$\Delta x \Delta p_y \geq \frac{\beta \hbar}{2} |\langle J_z \rangle|. \quad (43)$$

That is, $\Delta x \Delta p_x$ is sensitive to $\mathcal{F}$; while sensitivity to $\beta$ is carried in $\Delta x \Delta p_y$. Furthermore, in the usual notation, one has the following representative expression for the product of uncertainties in position measurements:

$$\Delta x \Delta y \geq \frac{\ell_P^2}{2} |\langle J_z \rangle|, \quad (44)$$

with

$$\Delta p_x \Delta p_y \geq \frac{\hbar}{2\ell_P^2} |\langle J_z \rangle| \quad (45)$$
complementing equation (44) for momentum measurements (the expectation value, denoted by \(\langle \cdots \rangle\) in the above expressions, is with respect to states that arise in a (yet to be fully formulated) quantum field theory based on the Lie algebra for IGQR). In addition

- the de Broglie wave–particle duality must undergo a profound modification;
- the notion and magnitude of zero-point energy must suffer corrections;
- the equal-energy spacing of vibrational and rotational diatomic states must undergo well-defined corrections;
- the concept of particle is now defined via the Casimir invariants for the Lie algebra for IGQR;
- the quantization procedure requires an \textit{ab initio} formulation where the demand of locality is abandoned.

This circumstance immediately asks for a detailed investigation of these modifications/corrections not only to examine the possibility of experimental confirmation of the suggested Lie algebra by precision experiments but also to place limits on \(\beta\) from existing data. The last two inferences are directly related. For \(\beta = 0\), they are confirmed by Mendes [21]. An examination of momentum-space wave equations along the lines presented in section 2, with explicit modification to \('\ p_{\mu} \rightarrow i\hbar \partial_{\mu} '}\ consistent with the algebra (33)–(41), should yield \(\beta, \ell_P, \and \ell_C\) dependent corrections to standard model physics while at the same time defining any possible modification to the principle of equivalence\(^9\). This exercise should also allow us to study possible violation of CPT induced by the Lie algebra for IGQR. Apart from carrying intrinsic worth, the result on possible CPT violation carries enormous relevance for the LSND excess event anomaly [60, 61] and would be the most natural explanation if MiniBOONE confirms LSND result [62].

These are all welcome features which must be investigated in the new context of the Lie algebra for IGQR. In principle, any deviations from the standard zero-point energy can be studied by precision laboratory experiments involving \textit{Planck-mass} oscillators of superconducting quantum interference devices (SQUIDs). This can be readily seen from the fact that SQUIDs carry superconducting currents with \textit{temperature-tunable} superconducting mass

\[
m_s(T) \sim f(T)Na m_c,
\]

behaving as one quantum object. In the above equation, \(N_a \approx 6 \times 10^{23} \text{ mole}^{-1}, m_c \approx 2 \times 0.9 \times 10^{-27} \text{ gm, and } f(T)\) encodes the fraction of the available electrons that are in a superconducting Cooper state at temperature, \(T\). Sufficiently below the critical temperature, \(f(T)\) may approach unity, thus allowing \(m_s(T)\) to easily reach the Planck mass, \(m_P\). The experimental challenge would then be to invoke \(m_s(T)\) rather than \(m_c\). For the simplified case of \(\beta = 0\), purely on dimensional grounds, for \(m_s \sim m_P\) the departures from \('\pm (1/2)\hbar \omega'\) are expected to be of the order \(m_P \omega^2 \ell_P^2\).\(^10\) Therefore, by coupling two SQUIDs, set to oscillating super-currents at slightly different (angular) frequencies, \(\omega_1\) and \(\omega_2\), one may be able to observe the phase difference

\[
\frac{1}{\hbar} m_P (\omega_2^2 - \omega_1^2)\ell_P^2 t,
\]

thus giving a laboratory signature of quantum gravity.

The cosmological constant, as evaluated with modified zero-point energies within the context of the proposed Lie algebra for IGQR, is likely to remain at variance with data.

\(^9\) In this context a parenthetic observation may be made that now the notion of inertial mass undergoes an unavoidable change as \(P \nu P^\mu\) is no longer the Casimir for the (33)–(41).

\(^10\) This estimate is consistent with calculations given in [21].
4. Concluding remarks

The notion of free fall was originally established within the Newtonian and Galilean framework, and only later did it take its new form in general relativity by accepting Poincaré spacetime symmetries as symmetries of a freely falling frame. For historical reasons, foundational considerations on the subject which shaped its eventual evolution were confined to essentially the non-quantum realm. All problems that came to arise when one confronted the interface of gravitational and quantum realms were for a long time (with exception of recent years) considered as technical in nature and were expected to go away.

A hint that the notion of freely falling frame may be in need of revision can be deciphered from the following observations (which are to be taken in heuristic spirit, and should not be used against the concrete suggestions made in section 3).

(i) There exist freely falling frames where ‘gravitationally-induced force’ vanishes while gravitationally-induced quantum mechanical phases do not. Such considerations can be easily dismissed as the consequent red-shift of flavour oscillation clocks has no local observability. In challenge, one can argue that nothing in physics forces one to local observability alone; and this challenge becomes stronger if one considers astrophysical and cosmological scenarios. That is, from an operational point of view freely falling frames exist in which gravitation cannot be gauged away, though it can be made unobservable for a local observer\(^\text{11}\).

(ii) An unremovable and intrinsic zero-point energy in freely falling frames also speaks of inherent quantum and gravitational nature of such frames and that any theory in the IGQR must respect this circumstance. The ‘120 orders of magnitude’ problem associated with the induced cosmological constant in the standard Heisenberg–Poincaré based framework is perhaps an artifact of the wrong choice of algebra associated with spacetime symmetries in freely falling frames.

(iii) Every consideration on spacetime measurement that allows gravitational effects asks for non-commutative spacetime or a modification of spacetime at the Planck scale; see, e.g., [5, 6, 65–67].

(iv) The position and momentum operators commute in the Poincaré algebra, while some of them do not in the Heisenberg algebra. This a priori independence of these two operators in the Poincaré and Heisenberg algebras in fact already suggests an element of conceptual incompatibility if one envisages a unification of these notions. The suggested Lie algebra for IGQR in fact achieves this merging naturally. Heisenberg and Poincaré algebras no longer maintain their separate existence but are unified in one stable Lie algebra.

By modifying the notion of a freely falling frame as carrying a stabilized Heisenberg–Poincaré algebra, i.e., the Lie algebra for IGQR, to define spacetime symmetries one generalizes the notion of spacetime where its characterization requires not only \(c\), but also \(\hbar\), \(\ell_p\), and \(\ell_C\); and possibly \(\beta\). The small scale, as defined by the Planck length \(\ell_p\), and the large scale, as encoded in \(\ell_C\), are no longer part of separate realms but intermingle in the new notion of spacetime symmetries. The resulting spacetime has unavoidable quantum and gravitational features in which the notion of particles suffers a modification. Such primitive concepts as mass and spin no longer remain the same but undergo a well-defined change via

\(^{11}\) For a further discussion of some these matters, see, e.g., [63, 64], and references therein.
Casimir invariants associated with the Lie algebra for IGQR. By modifying the notion of invariant mass, one’s understanding of inertia and the equivalence principle will require an \textit{ab initio} investigation; while by modifying the Heisenberg algebra one’s notion of quantum undergoes a well-defined, though not yet fully explored, change.

Acknowledgments

It is my pleasure to thank the following colleagues for remarks on the penultimate draft of this paper: Giovanni Amelino-Camelia, Chryssomalis Chryssomalakos, Daniel Grumiller, Jerzy Kowalski-Glikman, Jerzy Lukierski, Matt Visser and the three CQG referees for their advice and questions. In part, this work is supported by Consejo Nacional de Ciencia y Tecnolog\'ia (CONACyT, Mexico) grant E-32067.

References

[1] Diarc P A M 1958 \textit{The Principles of Quantum Mechanics} (Oxford: Oxford University Press)
[2] Wigner E P 1962 Unitary representations of the inhomogeneous Lorentz group including reflections \textit{Group Theoretical Concepts and Methods in Elementary Particle Physics (Lectures of the Istanbul Summer School of Theoretical Physics)} ed F Gursey (New York: Gordon and Breach)
[3] Weinberg S 1995 \textit{The Quantum Theory of Fields Vol. I: Foundations} (Cambridge: Cambridge University Press)
[4] Weinberg S 1964 Photons and gravitons in S matrix theory: derivation of charge conservation and equality of gravitational and inertial mass \textit{Phys. Rev.} \textbf{135} B1049
[5] Doplicher S, Fredenhagen K and Roberts J E 1994 Space-time quantization induced by classical gravity \textit{Phys. Lett. B} 331 39 (Preprint DESY 94-065)
[6] Ahluwalia D V 1994 Quantum measurements, gravitation, and locality \textit{Phys. Lett. B} 339 301 (Preprint gr-qc/9308007)
[7] Mendes R V 1994 Deformations, stable theories and fundamental constants \textit{J. Phys. A: Math. Gen.} \textbf{27} 8091
[8] Chryssomalakos C 2001 Stability of Lie superalgebras and branes \textit{Mod. Phys. Lett. A} \textbf{16} 197 (Preprint hep-th/0102134)
[9] Chryssomalakos C and Okon E 2004 Generalized quantum relativistic kinematics: a stability point of view \textit{Int. J. Mod. Phys. D} \textbf{13} 2003 (Preprint hep-th/0410212)
[10] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 The three-dimensional Euclidean quantum group $E(3)_q$ and its $R$-Matrix \textit{J. Math. Phys.} \textbf{32} 1159
[11] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 The quantum Heisenberg group $H(1)_q$ \textit{J. Math. Phys.} \textbf{32} 1155
[12] Celeghini E, Giachetti R, Sorace E and Tarlini M 1990 Three-dimensional quantum groups from contraction of SU(2)$_q$ \textit{J. Math. Phys.} \textbf{31} 2548
[13] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1977 Quantum mechanics as a deformation of classical mechanics \textit{Lett. Math. Phys.} \textbf{1} 521
[14] Flato M 1982 Deformation view of physical theories \textit{Czech. J. Phys. B} \textbf{32} 472
[15] Faddeev L D 1988 \textit{Asia-Pacific News} \textbf{3} 21
Faddeev L D 1989 \textit{Frontiers in Physics: High Technology and Mathematics} ed H A cerdeira and S Lundqvist (Singapore: World Scientific) pp 238–46
[16] Nijenhuis A and Richardson R W Jr 1967 Deformations of Lie algebra structures \textit{J. Math. Mech.} \textbf{17} 89
[17] Majid S 2000 \textit{Foundations of Quantum Group Theory} (Cambridge: Cambridge University Press)
[18] Kowalski-Glikman J and Smolin L 2004 Triply special relativity \textit{Phys. Rev. D} \textbf{70} 065020 (Preprint hep-th/0406276)
[19] Chryssomalakos C and Okon E 2004 Linear form of 3-scale relativity algebra and the relevance of stability \textit{Int. J. Mod. Phys. D} \textbf{13} 1817 (Preprint hep-th/0407080)
[20] Yang C N 1947 On quantized space-time \textit{Phys. Rev.} \textbf{72} 874
[21] Mendes R V 2000 Geometry, stochastic calculus and quantum fields in a non-commutative space-time \textit{J. Math. Phys.} \textbf{41} 156 (Preprint math-ph/9907001)
[22] Kempf A, Mangano G and Mann R B 1995 Hilbert space representation of the minimal length uncertainty relation \textit{Phys. Rev. D} \textbf{52} 1108 (Preprint hep-th/9412167)
[50] Ahluwalia D V and Kirchbach M 2001 (1/2,1/2) representation space: an ab initio construct Mod. Phys. Lett. A 16 1377 (Preprint hep-th/0101009)
[51] Coleman S R and Glashow S L 1999 High-energy tests of Lorentz invariance Phys. Rev. D 59 116008 (Preprint hep-ph/9812418)
[52] Magueijo J 2003 New varying speed of light theories Rep. Prog. Phys. 66 2025 (Preprint astro-ph/0305457)
[53] Liberati S, Sonego S and Visser M 2005 Interpreting doubly special relativity as a modified theory of measurement Phys. Rev. D 71 045001 (Preprint gr-qc/0410113)
[54] Mendes R V 2004 Some consequences of a noncommutative space-time structure Preprint hep-th/0406013
[55] Amelino-Camelia G, Smolin L and Starodubtsev A 2004 Quantum symmetry, the cosmological constant and Planck scale phenomenology Class. Quantum Grav. 21 3095 (Preprint hep-th/0306134)
[56] Freidel L, Kowalski-Glikman J and Smolin L 2004 2+1 gravity and doubly special relativity Phys. Rev. D 69 044001 (Preprint hep-th/0307085)
[57] Snyder H S 1947 Quantized space-time Phys. Rev. 71 38
[58] Mendes R V 1996 Quantum mechanics and non-commutative spacetime Phys. Lett. A 210 232
[59] Khruschev V V and Lezno A N 2003 Relativistically invariant Lie algebras for kinematic observables in quantum space-time Grav. Cosmol. 9 159 (Preprint hep-th/0207082)
[60] Ahluwalia D V 1998 Reconciling super-Kamiokande, LSND, and Homestake neutrino oscillation data Mod. Phys. Lett. A 13 2249 (Preprint hep-ph/9807267)
[61] Murayama H and Yanagida T 2001 LSND, SN1987A, and CPT violation Phys. Lett. B 520 263 (Preprint hep-ph/0010178)
[62] Ray H L (MiniBooNE Collaboration) 2004 Current status of the MiniBooNE experiment Preprint hep-ex/0411022
[63] Ahluwalia D V and Burgard C 1998 Interplay of gravitation and linear superposition of different mass eigenstates Phys. Rev. D 57 4724 (Preprint gr-qc/9803015)
[64] Konno K and Kasai M 1998 General relativistic effects of gravity in quantum mechanics: a case of ultra-relativistic, spin 1/2 particles Prog. Theor. Phys. 100 1145
[65] Garay L J 1995 Quantum gravity and minimum length Int. J. Mod. Phys. A 10 145 (Preprint gr-qc/9403008)
[66] Sasakura N 2000 Space-time uncertainty relation and Lorentz invariance J. High Energy Phys. JHEP05(2000)015 (Preprint hep-th/0001161)
[67] Collins J, Perez A, Sudarsky D, Urrutia L and Vucetich H 2004 Lorentz invariance: an additional fine-tuning problem Phys. Rev. Lett. 93 191301 (Preprint gr-qc/0403053)