An Improved Multi-access Coded Caching with Uncoded Placement

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Abstract

In this work, we consider a slight variant of well known coded caching problem, referred as multi-access coded caching problem, where each user has access to $z$ neighboring caches in a cyclic wrap-around way. We present a placement and delivery scheme for this problem, under the restriction of uncoded placement. Our work is a generalization of one of the cases considered in “Multi-access coded caching : gains beyond cache-redundancy” by B. Serbetci, E. Parrinello and P. Elia. To be precise, when our scheme is specialized to $z = \frac{K-1}{K\gamma}$, for any $K\gamma$, where $K$ is the number of users and $\gamma$ is the normalized cache size, we show that our result coincides with their result. We show that for the cases considered in this work, our scheme outperforms the scheme proposed in “Rate-memory trade-off for multi-access coded caching with uncoded placement” by K. S. Reddy and N. Karamchandani, except for some special cases considered in that paper. We also show that for $z = K - 1$, our scheme achieves the optimal transmission rate.

I. INTRODUCTION

The proliferation of users and their demand for high data rate content lead to a drastic increase in the high traffic volume during peak periods. In the seminal paper \[1\], Maddah-Ali and Neison proposed a coded caching scheme to relieve the traffic burden during peak hours by utilizing the ample cache memories available at the user ends. The proposed scheme tackles the under utilization of resources during off-peak hours in the placement phase by filling the cache
memories during the off-peak period so as to avail multicasting opportunities, when the demands are revealed by the users, during the delivery phase. They consider a setup consisting of a central server having access to a database comprising of a set of $N$ files of equal size and a set of $K$ users where each user has a normalized cache size $\gamma$. Each user reveals their demand, which is assumed to be a single file among the files held by the server, during the peak hours. Then, the server transmits coded symbols to all the users over an error-free link such that each user can retrieve the demanded file from the local cache content and the transmitted symbols. The overall objective is to design the placement as well as the delivery phases with minimum transmission rate such that the user demands are satisfied. A lot of research has been done during the past few years in this area \cite{2}, \cite{3} where it is assumed that each user has their own dedicated cache.

However in a lot of scenarios such as in cellular networks users can have access to multiple caches when their coverage areas overlap. Considering this possibility, recently a couple of studies have been done where each user has access to some $z$ neighboring caches in a cyclic wrap around fashion referred to as multi-access coded caching problem. Each cache is also connected to $z$ users. In \cite{4}, the authors have proposed a scheme for which the transmission rate is order optimal with respect to the information-theoretic lower bound. In \cite{5}, the authors have proposed a scheme for uncoded cache placement which has a lower transmission rate than the one proposed in \cite{4}, by mapping of the coded caching problem to the index coding problem. A lower bound on the optimal transmission rate for multi-access coded caching with $z \geq \frac{K}{2}$ over all uncoded placement policies was also provided in \cite{5}. Additionally, the exact transmission rate-memory trade-off was established for a few special cases, i.e., when $z = K - 1$, $z = K - 2$, $z = K - 3$ with $K$ even and $z = K - \frac{K}{g} + 1$ for some positive integer $g$.

For the multi-access coded caching problem, in \cite{6} the authors have proposed a novel coded caching scheme that achieve a coding gain that exceeds $K\gamma$. Two special cases are considered in \cite{6}, one is when $K\gamma = 2$ and the other is when $z = \frac{K-1}{K\gamma}$, for any $K\gamma$. For both the cases, the proposed scheme can serve, on average, more than $K\gamma$ users at a time. For the second special case, i.e., when $z = \frac{K-1}{K\gamma}$, it was proved that the achieved gain is optimal under uncoded cache placement.

**Notations:** The finite field with $q$ elements is denoted by $\mathbb{F}_q$. $[n]$ represents the set $\{1, 2, \ldots, n\}$
while \([a, b]\) represents the set \(\{a, a+1, \ldots, b\}\). The bit wise exclusive OR (XOR) operation is denoted by \(\oplus\). \([x]\) denotes the largest integer smaller or equal than \(x\) and \([x]\) denotes the smallest integer greater than or equal than \(x\). \(a \mid b\) implies \(a\) divides \(b\) and \(a \nmid b\) implies \(a\) does not divide \(b\), for some integers \(a\) and \(b\).

A. Background and Preliminaries

In this section, we formally define the multi-access coded caching problem considered in this work. The system model [4], [5], as shown as in Fig. 1, consists of a network comprising of a central server storing \(N\) files, \(W^0, W^1, W^2, \ldots, W^{N-1}\), each of size 1 unit, \(K\) users, \(U_0, U_1, \ldots, U_{K-1}\), and \(K\) caches such that

- each user is connected to \(z\) neighboring caches in a cyclic wrap-around fashion, and has access to the data stored in those caches,
- each cache has a memory size of \(M = N\gamma\) files, where \(\gamma \in \left\{\frac{k}{K}, k = 1, 2, \ldots, K\right\}\) is the normalized cache size,
- each user demands one among the \(N\) files, and
- all the \(K\) users are connected via an error-free shared link to the server.

Each cache is connected to exactly \(z\) users and each user has access to exactly \(z\) caches. Each user \(U_\alpha, \alpha \in \{0, 1, \ldots, K-1\}\) has access to all the caches in \(C_\alpha\), where, \(C_\alpha = \{\alpha, \alpha+1, \alpha+2, \ldots, \alpha+z-1\}\).

The system works in two phases- a placement phase, and a delivery phase. In the placement phase the caches are filled with the content of the files from the servers’ database prior to the users’ requests. In the delivery phase, each user \(U_\alpha\) demands a file from the database. The index of the file demanded by the user \(U_\alpha\) is denoted by \(d(\alpha)\). We denote \(d = (d(0), d(1), \ldots, d(K-1))\) as the demand vector. We concentrate on the worst case scenario where the demand of each user is distinct. When the demands are revealed by each user, the server transmits coded symbols depending upon the demand vector and cache content at each user. With the help of the server transmissions and the accessible cache content, each user \(U_\alpha\) decode the desired file \(W^{d(\alpha)}\). The multi-access coded caching problem is to develop placement and delivery schemes so as to minimize the transmission rate.
Fig. 1: Multi-access Coded Caching Network [4], [5] consisting of a central server, \( K \) users, and \( K \) caches where each user is connected to \( z \) neighboring caches.

**B. Previous Results**

The multi-access coded caching problem was introduced in [4] where the authors have proposed a coloring based achievable scheme. A new transmission rate \( R_{ic} \) was derived for multi-access coded caching problem with any \( z > 1 \) in [5] which is better than that in [4].

\[
R_{ic}(\gamma) = \begin{cases} 
K(1-z\gamma)^2, & \forall \gamma \in \{0, \frac{1}{K}, \frac{2}{K}, \ldots, \left\lfloor \frac{K}{z} \right\rfloor \frac{1}{K} \} \\
0, & \text{for } \gamma = \left\lceil \frac{K}{z} \right\rceil \frac{1}{K}
\end{cases}
\]

The lower convex envelope of the above mentioned points is also achievable through memory sharing. A lower bound on the optimal transmission rate-memory trade-off \( R_{lb}(\gamma) \) for any multi-access coded caching problem under the restriction of unencoded placement with \( z \geq \frac{K}{2} \) was derived in [5],

\[
R_{lb}(\gamma) = \begin{cases} 
K - \left( \frac{K - \frac{1}{2} + z + 1}{2K} \right) K\gamma, & \text{if } 0 \leq \gamma \leq \frac{1}{K} \\
\frac{(K-z)(K-2z+1)}{2K} (2 - K\gamma), & \text{if } \frac{1}{K} \leq \gamma \leq \frac{2}{K} \\
0, & \text{if } \gamma \geq \frac{2}{K}
\end{cases}
\]

The authors have also considered some special cases for \( z \geq \frac{K}{2} \), namely when \( z = K-1 \), \( z = K-2 \), \( z = K-3 \) when \( K \) is even, and \( z = K - \frac{K}{g} + 1 \) for some positive integer \( g \), for which an achievable scheme was provided separately, which achieves the optimal transmission rate.

In [6], the authors have considered two special cases, one is when \( K\gamma = 2 \), and the other is when \( z = \frac{K-1}{K\gamma} \) for any \( K\gamma \). For \( K\gamma = 2 \), a new scheme was proposed under certain conditions, which is better than the previous results. For the other special case considered in [6], i.e., when \( z = \frac{K-1}{K\gamma} \) for any \( K\gamma \), it was proved that the achieved transmission rate \( \frac{1}{K} \) is optimal under
uncoded cache placement.

C. Our Contributions

The contributions of this paper is summarized as follows.

- We provide an achievable scheme for multi-access coded caching problem with each cache having a normalized capacity of \( \gamma \), where \( \gamma \in \left\{ \frac{k}{K} : k = 1, 2, \cdots, K \right\} \), under the restriction of uncoded placement.

- Our work is a generalization of one of the cases considered in [6]. To be precise, when our scheme is specialized to \( z = \frac{K-1}{K\gamma} \), for any \( K\gamma \), we show that our result coincides with that in Theorem 2 in [6].

- When \( K\gamma = 2 \), the proposed scheme in [6] can be applied only if certain conditions are satisfied while our scheme does not put any restrictions.

- For the following special cases, i.e., for \( z \geq \frac{K}{2} \), when \( z = K - 2 \), \( z = K - 3 \) when \( K \) is even, and \( z = K - \frac{K}{g} + 1 \) for some positive integer \( g \), the scheme proposed in [5], performs better than ours. The authors have provided separate optimal schemes for those special cases. For all other cases considered in our work, we show that our scheme outperforms the scheme proposed in [5]. We also show that for \( z = K - 1 \), our scheme achieves the optimal transmission rate as in [5].

The paper is organized as follows. Section II provides the main result of this paper, i.e., Theorem II presents the transmission rate which is achievable. We compare our result with the previous works in the same section. Our proposed scheme is described in Section III which achieves the transmission rate presented in Theorem II. Section IV concludes our paper. Proof of correctness of the delivery scheme is given in Appendix A.

II. Main Results

We discuss our main results in this section. We characterize our result for any

\( \gamma \in \left\{ \frac{k}{K} : k = 1, 2, \cdots, K \right\} \) in Theorem II

**Theorem 1.** Consider a multi-access coded caching scenario with \( N \) files, and \( K \) users, each having access to \( z \geq 2 \) neighboring caches in a cyclic wrap-around way, with each cache having
a normalized capacity of \( \gamma \), where \( \gamma \in \{ \frac{k}{K} : k = 1, 2, \ldots, K \} \), the following transmission rate \( R_{\text{new}}(\gamma) \) is achievable:

- If \( (K - k)z \) is even, then \( R_{\text{new}}(\gamma) = 2 \sum_{r=K-kz}^{K-1} \frac{1}{1 + \left| \frac{kz}{r} \right|} \).
- If \( (K - k)z > 1 \) is odd, then \( R_{\text{new}}(\gamma) = \left( \frac{1}{1 + \left| \frac{kz}{K-kz+1} \right|} \right) + \sum_{r=K-kz+3}^{K-1} \frac{2}{1 + \left| \frac{kz}{r} \right|} \).
- If \( (K - k)z \leq 0 \), then \( R_{\text{new}}(\gamma) = 0 \).

For any \( z \), the lower convex envelope of the above mentioned points is also achievable through memory sharing.

The placement scheme and the delivery algorithm achieving the rate claimed in Theorem 1 is given in Section III with several illustrating examples.

A. Comparison of our scheme with the scheme in [5]

Theorem 1 is illustrated in Fig. 2 for \( K = 25 \) and it is compared with the scheme proposed in [5]. In Fig. 2, transmission rate vs \( \gamma \) plot is obtained for each \( z \in 2, 3, \ldots, 25 \) as \( \gamma \) varies from \( \frac{1}{25} \) to 1. It can be observed from the plot that our scheme performs better than that in [5] except for \( z = K - 2 \). For \( z = K - 2 = 23 \) (\( \frac{z}{K} = \frac{23}{25} \)), the scheme in [5] outperforms ours, since the case of \( z = K - 2 \) was considered separately in [5] and an optimal scheme for that particular case was provided in [5]. For each \( z \), the gap between the two curves is large for smaller \( \gamma \). As \( \gamma \) increases, the gap reduces and both the curves coincide eventually.

Now, we examine the case when \( k = 1 \) and \( k = 2 \). A plot for \( K = 11 \) and \( \gamma = \frac{1}{11} \) is shown in Fig. 3a. Another plot for \( K = 15 \) and \( \gamma = \frac{2}{15} \) is displayed in Fig. 3b. In both the plots, it can be seen that our scheme is better than the scheme in [5] except for one point in Fig. 3a. The gap between the transmission rate of our scheme and that in [5] is more for smaller \( z \). As \( z \) increases, the gap reduces and gradually both the curve coincides. The only point where the scheme in [5] is better that ours is when \( z = K - 2 = 9 \) in Fig. 3a. Like it was discussed before, this is since the case of \( z = K - 2 \) was considered separately in [5] and an optimal scheme for that particular case was provided in [5].

In general, for all the points mentioned in Theorem 1 our scheme outperforms the scheme proposed in [5] except for some special cases discussed in [5] which achieves the optimal
Fig. 2: $\gamma$ vs $z$ vs transmission rate plot when $K = 25$, as in Theorem 1. The transmission rate and $z$ are normalized by dividing those by $K$.

transmission rate. This is shown in Theorem 2. And also, when $z = K - 1$, our scheme achieves the optimal transmission rate which coincides with the result in [5]. This particular case is discussed in Corollary 1.
**Theorem 2.** For any $\gamma \in \{\frac{k}{K} : k = 1, 2, \ldots, K\}$, we have $R_{new}(\gamma) \leq R_{ic}(\gamma)$.

**Proof.** For any $\gamma$, we have $R_{ic}(\gamma) = K \left(1 - \frac{\gamma}{K}\right)^2 = \frac{(K-kz)^2}{K}$.

*Case (i):* When $K-kz = 1$, we have $R_{ic}(\gamma) = \frac{(K-kz)^2}{K} = \frac{1}{K} = R_{new}(\gamma)$.

*Case (ii):* When $K-kz$ is even, for any $i \in \{1, 2, \ldots, \frac{K-kz}{2}\}$, we have
\[
\frac{kz}{K-kz} \leq \frac{kz}{K-kz+i} \leq \left[\frac{kz}{K-kz+i}\right] \Rightarrow 1 + \frac{kz}{K-kz} \leq 1 + \left[\frac{kz}{K-kz+i}\right].
\]

For any $r \in \left[\frac{K-kz}{2} + 1, K-kz\right]$,
\[
\frac{K}{K-kz} \leq 1 + \left[\frac{kz}{r}\right] \Rightarrow \frac{K-kz}{K} \geq 1 + \left[\frac{kz}{r}\right] \Rightarrow \sum_{r=\frac{K-kz}{2}+1}^{K-kz} \frac{K-kz}{K} \geq \sum_{r=\frac{K-kz}{2}+1}^{K-kz} 1 \Rightarrow R_{ic}(\gamma) \geq R_{new}(\gamma).
\]

Hence, if $K-kz$ is even, then $R_{new}(\gamma) \leq R_{ic}(\gamma)$. *Case (iii):* When $K-kz > 1$ is odd, for any $i \in \{1, 2, \ldots, \frac{K-kz+1}{2}\}$, we have
\[
\frac{kz}{K-kz} \leq \frac{kz}{K-kz-i} \leq \left[\frac{kz}{K-kz-i}\right] \Rightarrow 1 + \frac{kz}{K-kz} \leq 1 + \left[\frac{kz}{K-kz-i}\right] \Rightarrow \frac{K}{K-kz} \leq 1 + \left[\frac{kz}{K-kz-i}\right].
\]

For any $r \in \left[\frac{K-kz+1}{2}, K-kz\right]$, we have
\[
\frac{K}{K-kz} \leq 1 + \left[\frac{kz}{r}\right] \Rightarrow \frac{K-kz}{K} \geq 1 + \frac{kz}{r}.
\]

Taking sum over all $r \in \left\{\frac{K-kz+1}{2} + 1, \frac{K-kz+1}{2} + 2, \ldots, K-kz\right\}$, we get
\[
\sum_{r=\frac{K-kz+1}{2}+1}^{K-kz} \frac{K-kz}{K} \geq \sum_{r=\frac{K-kz+1}{2}+1}^{K-kz} 1 \Rightarrow \frac{K-kz-1}{2} \times \frac{K-kz}{K} \geq \sum_{r=\frac{K-kz+1}{2}+1}^{K-kz} 1 \Rightarrow (K-kz-1) \times (K-kz) \geq 2 \sum_{r=\frac{K-kz+1}{2}+1}^{K-kz} \frac{1}{1 + \frac{kz}{r}}.
\]

Now considering the inequality (1) when $r = \frac{K-kz+1}{2}$, we get
\[
\frac{K-kz}{K} \geq \frac{1}{1 + \frac{kz}{r}}.
\]
| Parameters                         | Scheme proposed in [5]                                      | Our Scheme |
|-----------------------------------|-----------------------------------------------------------|------------|
| Sub-packetization Level           | \(\binom{K-kz+k-1}{k-1} \frac{K}{k}\)                     | \(K(K-1)\) |
| Transmission Rate                 | \(K(1-z\gamma)^2\)                                       | \(\frac{1}{K}\) if \((K-kz) = 1\) \[3\] \[4\] \(2 \sum_{r=\frac{K-kz}{2}+1}^{\frac{K-kz+1}{2}} \frac{1}{1+\left|\frac{r}{K}\right|}\) (if \((K-kz) > 1\) is odd) |

TABLE I: Comparison of our scheme with that in [5] when \(\gamma \in \{\frac{k}{K} : k = 1, 2, \ldots, K\}\).

Summing up the inequalities (3) and (4), we get,

\[
\frac{K-kz}{K} + \frac{(K-kz-1) \times (K-kz)}{K} \geq \frac{1}{1+\left|\frac{K-kz+1}{2}\right|} + 2 \sum_{r=\frac{K-kz}{2}+1}^{\frac{K-kz+1}{2}} \frac{1}{1+\left|\frac{r}{K}\right|}.
\]

Hence, \(\frac{(K-kz)^2}{K} \geq \frac{1}{1+\left|\frac{K-kz+1}{2}\right|} + 2 \sum_{r=\frac{K-kz}{2}+1}^{\frac{K-kz+1}{2}} \frac{1}{1+\left|\frac{r}{K}\right|} \Rightarrow R_{ic} (\gamma) \geq R_{new} (\gamma). \)

**Corollary 1.** For \(z = K-1\), we have \(R^* (\gamma) = R_{new} (\gamma)\), where \(R^* (\gamma)\) is the optimal rate of dedicated-cache coded caching setting where each cache has an augmented size equal to \(z\gamma\).

**Proof.** Note that if \(kz > K-1\), then \(R_{new} (\gamma) = 0\), since each user can access all the sub-files of all the files. Now, if \(k \geq 2\), then \(kz = 2(K-1) > K-1\) and hence \(R_{new} (\gamma) = 0\). When \(k = 1\) we have \(K-kz=1\) and hence \(R_{new} (\gamma) = \frac{1}{K} = R^* (\gamma)\).

**B. Comparison of our scheme with the scheme in [6]**

In both Fig. 3a and 3b, one specific point \((K-1, \frac{1}{K})\) is marked which corresponds to one of the cases discussed in [6], where \(z = \frac{K-1}{k}\). Considering this case, in Fig. 3a, since \(k = 1\), we have \(z = K-1\), for which a scheme was proposed in [6] which achieves the optimal transmission rate \(\frac{1}{K}\). The transmission rate obtained from our scheme coincides with that. Similarly, in Fig. 3b also, since \(k = 2\), we have \(z = \frac{K-1}{2}\), for which an optimal transmission rate achieving scheme was proposed in [6]. In general, when our scheme is specialized to \(z = \frac{K-1}{k}\), for any \(k\), our result coincides with that of Theorem 2 in [6]. This is since \(R_{new} (\gamma) = \frac{1}{K}\), as
\( K - kz = K - (K - 1) = 1 \). Hence, \( R^*(\gamma) = R_{new}(\gamma) = \frac{1}{K} \), where \( R^*(\gamma) \) is the optimal rate of coded caching setting with dedicated caches where each cache has an augmented size equal to \( z\gamma \). For \( k = 2 \), the condition under which the expression (3) in \([6]\) holds is not satisfied for \( K = 15 \). Hence, we cannot compare our result with that in \([6]\) for \( k = 2 \) and \( K = 15 \).

### III. PLACEMENT AND DELIVERY SCHEME

In this section we present our placement and delivery scheme to prove Theorem [1].

#### A. Placement Scheme

In the placement phase, we split each file \( W^n, n = \{0, 1, \ldots, N - 1\} \), into \( K \) disjoint sub-files \( W^n_\alpha, \alpha \in \{0, 1, \ldots, K - 1\} \). Let \( M_\alpha \) represent the content stored in the cache \( \alpha, \alpha \in \{0, 1, \ldots, K - 1\} \). Each cache \( \alpha \) is filled as follows: \( M_\alpha = \{W^n_{ko+j}, j \in \{0, 1, \ldots, k - 1\}, \forall n \in \{0, 1, \ldots, N - 1\}\} \). Each cache stores \( k \) sub-file from all the files, where each sub-file is of size \( \frac{1}{K} \). Hence, \( M = \frac{kN}{K} = N\gamma \), thus meeting our memory constraint.

#### B. Delivery Scheme

Each user’s demand, of one file among the \( N \) files from the central server, is revealed after the placement phase. Once the demand vector \( d \) is known, Algorithm [1] provides the transmissions done by the server when \( K - kz \) is even and Algorithm [2] provides transmissions done by the server when \( K - kz \) is odd.

If \( K - kz \) is even, we fix the value of \( t \) to be \( \frac{K-kz+2}{2} \). Then, for each \( r \) chosen in step 2, we calculate the corresponding value of \( p \) in step 3. When \( p \) is even, we split each sub-file into \( \frac{p}{2} \) parts and for each \( j \in [0, K - 1] \), we transmit one coded symbol \( T^{r-t}_j \). Otherwise, we split each sub-file into \( p \) parts and for each \( j \in [0, K - 1] \), we transmit two coded symbols \( T^{r-t}_{j,1} \) and \( T^{r-t}_{j,2} \). The idea behind Algorithm [1] is that, for each \( r \) chosen in step 2, each user can decode two of the sub-files of the demanded file. Precisely, the transmissions done by the server is in such a way that for a chosen \( r \), each user \( U_\alpha, \alpha \in \{0, 1, \ldots, K - 1\} \), can decode sub-files \( W^{d(\alpha)}_{ko+r} \) and \( W^{d(\alpha)}_{k(\alpha+z)+r-1} \). For that purpose, the sub-files are further split into several parts and the split parts are combined in an efficient way so as to form coded symbols.
Similarly, if $K - kz > 1$ and is odd, Algorithm 2 is designed in such a way that, for each $r \in \left[ t + 1, t + \frac{K-kz-1}{2} \right]$, where $t = \frac{K-kz+1}{2}$, chosen in step 2, the transmissions done by the server help each user $U_{\alpha}, \alpha \in \{0, 1, \ldots, K - 1\}$, to decode exactly two sub-files $W_{\alpha}^{d(\alpha)}$ and $W_{\alpha}^{d(\alpha)}$. Here also, the sub-files are further split into several parts and encoded appropriately.

If $r = t = \frac{K-kz+1}{2}$ or $K - kz = 1$, the transmissions done by the server help each server $U_{\alpha}$ to decode the sub-file $W_{\alpha}^{d(\alpha)}$ by splitting the sub-files into $\left\lfloor \frac{kz}{r} \right\rfloor + 1$ parts and encoding the parts properly. Now we will look into the transmission rate involved in this scheme. If $K - kz$ is odd, the sub-files are further split into several parts and encoded appropriately.

### Algorithm 1: Delivery scheme for multi-access coded caching problem if $\gamma \in \{ \frac{kz}{r}, 1, 2, \ldots, K \}$ and $(K - kz)$ is even.

1. Let $t = \frac{K-kz+2}{2}$.
2. For $r = t, t + 1, t + 2, \ldots, t + \frac{K-kz-2}{2}$ do
   1. Let $p = \left\lceil \frac{kz}{r} \right\rceil + 1$.
      1. If $p$ is even then
         1. Split each sub-file $W_{\alpha}^n, \alpha \in [0, K - 1], n \in [0, N - 1]$ into $\frac{p}{2}$ parts:
            \begin{align*}
            \{ W_{\alpha,i}^n, \forall i \in [0, \frac{p}{2} - 1] \}.
            \end{align*}
         2. For each $j \in \{0, 1, \ldots, K - 1\}$ do
            1. Transmit one coded symbol $T_j^{r-t}$:
               \begin{align*}
               T_j^{r-t} = \bigoplus_{i=0}^{\frac{p}{2}-1} W_{ir+j,i}^{d\left(\frac{i+1}{r}\right)} \bigoplus_{i=\frac{p}{2}}^{p-1} W_{ir+j,i-\frac{p}{2}}^{d\left(\frac{i-1}{r}\right)}
               \end{align*}
            2. End
      2. Else
         1. Split each sub-file $W_{\alpha}^n, \alpha \in [0, K - 1], n \in [0, N - 1]$ into $p$ parts:
            \begin{align*}
            \{ W_{\alpha,i}^n, \forall i \in [0, p - 1] \}.
            \end{align*}
         2. For each $j \in \{0, 1, \ldots, K - 1\}$ do
            1. Transmit coded symbols $T_{j,1}^{r-t}$ and $T_{j,2}^{r-t}$:
               \begin{align*}
               T_{j,1}^{r-t} = \bigoplus_{i=0}^{\frac{p-1}{2}} W_{ir+j,i}^{d\left(\frac{i+1}{r}\right)} \bigoplus_{i=\frac{p}{2}}^{p-1} W_{ir+j,i-\frac{p-1}{2}}^{d\left(\frac{i-1}{r}\right)}
               \end{align*}
               \begin{align*}
               T_{j,2}^{r-t} = \bigoplus_{i=0}^{\frac{p-1}{2}} W_{ir+j,i-\frac{p}{2}}^{d\left(\frac{i+1}{r}\right)} \bigoplus_{i=\frac{p}{2}}^{p-1} W_{ir+j,i}^{d\left(\frac{i-1}{r}\right)}
               \end{align*}
            2. End
   2. End
3. End

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Algorithm 2: Delivery scheme for multi-access coded caching if $\gamma \in \left\{ \frac{k}{K} : k = 1, 2, \cdots, K \right\}$ and $(K-kz)$ is odd.

1. Let $t = \frac{K-kz+1}{2}$.
2. for $r = t, t+1, t+2, \ldots, t+\frac{K-kz-1}{2}$ do
   3. Let $p = \left\lceil \frac{kz}{r} \right\rceil + 1$.
   4. if $r = t$ or $K-kz = 1$ then
      5. Split each sub-file $W^n_\alpha, \alpha \in [0, K-1], n \in [0, N-1]$ into $p$ parts:
         $\{W^n_{\alpha,i}, \forall i \in [0, p-1]\}$.
      6. for each $j \in \{0, 1, \ldots, K-1\}$ do
         7. Transmit one coded symbol $T^0_j$:
            $T^0_j = \bigoplus_{i=0}^{p-1} W^d_{ir+j,i, k}^{(i+1)r+j+k}$
         end
   7. else
      8. if $p$ is even then
         9. Split each sub-file $W^n_\alpha, \alpha \in [0, K-1], n \in [0, N-1]$ into $\frac{p}{2}$ parts:
            $\{W^n_{\alpha,i}, \forall i \in [0, \frac{p}{2}-1]\}$.
         10. for each $j \in \{0, 1, \ldots, K-1\}$ do
              11. Transmit one coded symbol $T^{r-t}_j$:
                  $T^{r-t}_j = \bigoplus_{i=0}^{\frac{p}{2}-1} W^d_{ir+j,i, \frac{k}{k}}^{(i+1)r+j+k}$
              end
      12. else
         13. Split each sub-file $W^n_\alpha, \alpha \in [0, K-1], n \in [0, N-1]$ into $\frac{p}{2}$ parts:
            $\{W^n_{\alpha,i}, \forall i \in [0, \frac{p}{2}-1]\}$.
         14. for each $j \in \{0, 1, \ldots, K-1\}$ do
              15. Transmit the two coded symbols:
                  $T^{r-t}_{j,1} = \bigoplus_{i=0}^{\frac{p-1}{2}} W^d_{ir+j,i, \frac{k}{k}}^{(i+1)r+j+k}$
                  $T^{r-t}_{j,2} = \bigoplus_{i=0}^{\frac{p-1}{2}+1} W^d_{ir+j,i, \frac{k}{k}}^{(i+1)r+j+k}$
              end
      16. else
         17. Split each sub-file $W^n_\alpha, \alpha \in [0, K-1], n \in [0, N-1]$ into $\frac{p}{2}$ parts:
            $\{W^n_{\alpha,i}, \forall i \in [0, \frac{p}{2}-1]\}$.
         18. for each $j \in \{0, 1, \ldots, K-1\}$ do
              19. Transmit the two coded symbols:
                  $T^{r-t}_{j,1} = \bigoplus_{i=0}^{\frac{p-1}{2}} W^d_{ir+j,i, \frac{k}{k}}^{(i+1)r+j+k}$
                  $T^{r-t}_{j,2} = \bigoplus_{i=0}^{\frac{p-1}{2}+1} W^d_{ir+j,i, \frac{k}{k}}^{(i+1)r+j+k}$
              end
      20. end
   8. end
end
Our scheme
Scheme in [5]

Fig. 4: $\gamma$ vs per user transmission rate plot when $K = 25$ and $z = 3$ for all the points mentioned in Theorem 1.

even, from Algorithm 1, for each $r \in \{t, t + 1, \ldots, t + \frac{K - kz - 2}{2}\}$, $t = \frac{K - kz + 1}{2}$, the amount of transmission done by the server is $2\frac{K}{p} \times \left(\frac{1}{K}\right) = \frac{2}{1 + \left\lfloor \frac{K}{K - kz + 1} \right\rfloor}$ files, accounting for a total transmission rate of $R_{\text{new}}(\gamma) = 2 \sum_{r=t}^{K - kz - 1} \frac{1}{1 + \left\lfloor \frac{K}{K - kz + 1} \right\rfloor}$. Similarly considering the case when $K - kz > 1$ and is odd, from Algorithm 2, for each $r \in \{t + 1, \ldots, t + \frac{K - kz - 1}{2}\}, t = \frac{K - kz + 1}{2}$, the amount of transmission done by the server is $2\frac{K}{p} \times \left(\frac{1}{K}\right) = \frac{2}{1 + \left\lfloor \frac{K}{K - kz} \right\rfloor}$ files. If $r = t$, the amount of transmission done by the server is $\frac{K}{p} \times \left(\frac{1}{K}\right) = \frac{1}{1 + \left\lfloor \frac{K}{K - kz} \right\rfloor}$ files. Hence the overall transmission rate is $R_{\text{new}}(\gamma) = 2 \left(\frac{1}{2\left(\left\lfloor \frac{K}{K - kz + 1} \right\rfloor + 1\right)} + \sum_{r=K - kz + 2}^{K - kz - 1} \frac{1}{1 + \left\lfloor \frac{K}{K - kz} \right\rfloor}\right)$. If $K - kz = 1$, the amount of transmission done by the server is $\frac{K}{p} \times \left(\frac{1}{K}\right) = \frac{1}{1 + \left\lfloor \frac{1}{K - kz} \right\rfloor} = \frac{1}{1 + \left\lfloor \frac{1}{1} \right\rfloor} = \frac{1}{K}$. Hence the transmission rate is $R_{\text{new}}(\gamma) = \frac{1}{K}$.

The detailed proof of the delivery scheme, i.e., the proof of correctness of Algorithm 1 and Algorithm 2 is given in Appendix A.

C. On the lower convex envelope of the achievable rates

In the placement phase, the sub-files of each file are placed in such a way that we first create a list of size $1 \times kK$ by repeating the sequence $\{0, 1, \ldots, K - 1\}$, $k$ times, i.e., $\{0, 1, \ldots, K - 1, 0, 1, \ldots, K - 1, 0, 1, 2, \ldots, K - 1, \ldots\}$. Then we fill the caches by taking $k$ items sequentially from the list, where each item on the list corresponds to the index of the sub-file. So, the first cache is filled with the first $k$ items, the second cache with the next $k$ items and so on. This
way of placing the sub-files is to make sure that any two neighboring caches get disjoint yet consecutive sub-files. It can be noted that even the first and last caches store disjoint sub-files.

In Fig. 4, per user transmission rate vs $\gamma$ plot is obtained for $K = 25$, $z = 3$ for all the points mentioned in Theorem 1 along with the lower convex envelop of all the points mentioned in Theorem 1 omitting $\gamma = \frac{2}{25}$, $\frac{3}{25}$ since the line joining the points $\gamma = \frac{1}{25}$ and $\gamma = \frac{4}{25}$ falls below those two points. In general we conjecture that points corresponding to $kz < (\frac{K-1}{2})$ except for $kz = z$ will have this characteristic. This may be mainly due to taking the ceiling operation of certain values in the rate expression. However, irrespective of this nature of some points all the points mentioned in Theorem 1 fall below the curve obtained for the scheme in [5].

D. Sub-packetization Level

For any $\gamma \in \left\{ \frac{k}{K} : k = 1, 2, \cdots, K \right\}$, depending upon the value of $K - kz$, Algorithm 1 or Algorithm 2 is used to derive the transmissions done by the server. In both Algorithm 1 and Algorithm 2, the worst case sub-packetization level is when $p$ is odd. In that case each sub-file is divided further into $p$ parts, where $p = \lceil \frac{2kz}{K-kz+1} \rceil + 1$. The value of $p$ is maximized when $r = t$, i.e., when $p = \left\lceil \frac{2kz}{K-kz+1} \right\rceil + 1$, where $i = 1$, if $k-kz$ is odd and $i = 2$, if $k-kz$ is even. To maximize $p$, the maximum value that $kz$ can take needs to be chosen, which is when $kz = K-1$. Hence the maximum value that $p$ can take is $K-1$, i.e., each sub-file is further divided into at most $K-1$ parts in the worst cast scenario. Thus, the worst cast sub-packetization level in our scheme is $K(K-1)$ while the sub-packetization level in the scheme proposed in [5] is $\binom{K-kz+k-1}{k-1} \frac{K}{k}$. This is shown in Table 1 also.

Example 1. Let $N = 5$, $K = 5$, $k = 1$ and $z = 2$. The server stores 5 files: $\{W_0^0, W_1^1, W_2^2, W_3^3, W_4^4\}$. Each file $W^n$, $n \in \{0, 1, 2, 3, 4\}$, is divided into 5 sub-files: $\{W_0^n, W_1^n, W_2^n, W_3^n, W_4^n\}$. Each cache $\alpha \in \{0, 1, 2, 3, 4\}$, is filled with one sub-file $W^n_\alpha$ of each file $W^n$, $n \in \{0, 1, 2, 3, 4\}$, i.e.,

$$
M_0 = \{W_0^0, W_0^1, W_0^2, W_0^3, W_0^4\} \quad M_1 = \{W_1^0, W_1^1, W_1^2, W_1^3, W_1^4\}
$$

$$
M_2 = \{W_2^0, W_2^1, W_2^2, W_2^3, W_2^4\} \quad M_3 = \{W_3^0, W_3^1, W_3^2, W_3^3, W_3^4\}
$$

$$
M_4 = \{W_4^0, W_4^1, W_4^2, W_4^3, W_4^4\} \quad M_5 = \{W_5^0, W_5^1, W_5^2, W_5^3, W_5^4\}
$$

Each user $U_\alpha, \alpha \in \{0, 1, 2, 3, 4\}$, has access to all the caches in $C_\alpha = \{\alpha, \alpha + 1\}$. Let the demand vector be $d = (0, 1, 2, 3, 4)$. Since $K - kz = 3$ is odd, we use Algorithm 2 to find the
transmission done by the server. First we consider \( r = t = 2 \). Each sub-file \( W^n_\alpha, \forall n, \alpha \in [0, 4] \), is split into \( p = 1 + \lceil \frac{\kappa r}{r} \rceil = 2 \) parts: \( \{ W^n_{\alpha_0}, W^n_{\alpha_1} \} \). The following coded symbols are transmitted:

\[
T^0_0 = W^2_{0,0} \oplus W^4_{2,1}, \quad T^0_1 = W^3_{1,0} \oplus W^3_{3,1}, \quad T^0_2 = W^4_{2,0} \oplus W^1_{4,1}, \quad T^0_3 = W^0_{3,0} \oplus W^2_{0,1}, \quad T^0_4 = W^1_{4,0} \oplus W^3_{1,1}.
\]

Now we consider \( r = t + 1 = 3 \). Here, \( p = 1 + \lceil \frac{\kappa r}{r} \rceil = 2 \) is even. Hence, each sub-file \( W^n_\alpha, \forall n, \alpha \in [0, 4] \), is split into \( \frac{\kappa r}{2} = 1 \) part: \( \{ W^n_{\alpha_0} \} \) which is the sub-file itself. The following coded symbols are transmitted:

\[
T^1_0 = W^3_{0,0} \oplus W^3_{3,0}, \quad T^1_1 = W^4_{1,0} \oplus W^4_{4,0}, \quad T^1_2 = W^0_{2,0} \oplus W^1_{0,0}, \quad T^1_3 = W^1_{3,0} \oplus W^2_{1,0}, \quad T^1_4 = W^2_{4,0} \oplus W^3_{2,0}.
\]

Now, each user \( U_\alpha \) needs to recover the demanded file \( W^{d(\alpha)} \). Let us first consider the user \( U_0 \). The user \( U_0 \) retrieves \( W^0_{3,0} \) from \( T^0_3 \) since \( W^2_{0,1} \) is available at its cache while it retrieves \( W^0_{3,1} \) from \( T^0_1 \). The sub-file \( W^0_{2} \) - \( W^0_{2,0} \) is recovered from \( T^1_2 \) whereas the sub-file \( W^0_{4} \) - \( W^0_{4,0} \) is recovered from \( T^1_4 \). The user \( U_0 \) has decoded the file \( W^0 \) since it has retrieved all the sub-files corresponding to the file \( W^0 \). Similarly all other users can decode their demanded file as shown as in Table II. In this particular example, the transmission rate using our scheme is \( R_{\text{new}} \left( \frac{1}{3} \right) = 1.5 \) while the rate achieved in [5] is \( R_c \left( \frac{1}{3} \right) = 1.8 \).

| User \( U_\alpha \) | Sub-file \( W^{d(\alpha)} \) decoded by \( U_\alpha \) | Part of sub-file \( W^{d(\alpha)} \) decoded by \( U_\alpha \) | The coded symbol from which \( W^{d(\alpha)} \) is decoded |
|---------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| \( U_0 \)           | \( W^0_0 \)                           | \( W^0_{3,0} \)                     | \( T^0_0 \)                          |
|                     | \( W^0_1 \)                           | \( W^0_{3,1} \)                     | \( T^0_1 \)                          |
|                     | \( W^0_2 \)                           | \( W^0_{0,0} \)                     | \( T^0_2 \)                          |
|                     | \( W^0_3 \)                           | \( W^0_{4,0} \)                     | \( T^0_3 \)                          |
| \( U_1 \)           | \( W^1_0 \)                           | \( W^1_{3,0} \)                     | \( T^1_0 \)                          |
|                     | \( W^1_1 \)                           | \( W^1_{4,0} \)                     | \( T^1_1 \)                          |
|                     | \( W^1_2 \)                           | \( W^1_{0,0} \)                     | \( T^1_2 \)                          |
| \( U_2 \)           | \( W^2_0 \)                           | \( W^2_{3,0} \)                     | \( T^2_0 \)                          |
|                     | \( W^2_1 \)                           | \( W^2_{4,0} \)                     | \( T^2_1 \)                          |
|                     | \( W^2_2 \)                           | \( W^2_{0,0} \)                     | \( T^2_2 \)                          |
|                     | \( W^2_3 \)                           | \( W^2_{1,0} \)                     | \( T^2_3 \)                          |

TABLE II: Table that illustrates the decoding done by each user in Example 1.
Example 2. Let \( N = 8, K = 8, k = 1 \) and \( z = 4 \). The server stores 8 files: 
\[ \{W^0, W^1, W^2, W^3, W^4, W^5, W^6, W^7\} \]. Each file \( W^n, n \in \{0, 1, \ldots, 7\} \), is divided into \( K \) sub-files: \( \{W^0_n, W^1_n, W^2_n, W^3_n, W^4_n, W^5_n, W^6_n, W^7_n\} \). Each cache \( \alpha \in \{0, 1, \ldots, 7\} \), is filled with one sub-file \( W^n_\alpha \) of each file \( W^n, n \in \{0, 1, \ldots, 7\} \). Each user \( U_\alpha, \alpha \in \{0, 1, \ldots, 7\} \), has access to all the caches in \( C_\alpha = \{\alpha, \alpha + 1, \alpha + 2, \alpha + 3\} \). Let the demand vector be \( d = (0, 1, 2, 3, 4, 5, 6, 7) \). Since \( K - kz = 4 \) is even, we use Algorithm [7] to find the transmissions by the server. First we consider \( r = t = 3 \). We have \( p = 1 + \lceil \frac{kz}{r} \rceil = 3 \). Since \( p \) is odd, each sub-file \( W^n_\alpha, \forall n, \alpha \in [0, 7] \), is split into \( p = 3 \) parts, \( \{W^n_{\alpha,0}, W^n_{\alpha,1}, W^n_{\alpha,2}\} \). The following coded symbols are transmitted corresponding to \( r = 3 \):

- \( T^0_{1,0} = W^3_{0,0} \oplus W^5_{0,1} \oplus W^0_{0,2} \)
- \( T^0_{2,0} = W^3_{1,0} \oplus W^6_{1,2} \oplus W^0_{1,3} \)
- \( T^0_{4,1} = W^7_{1,0} \oplus W^2_{1,2} \oplus W^1_{1,3} \)
- \( T^0_{4,2} = W^7_{1,1} \oplus W^2_{0,2} \oplus W^1_{0,3} \)
- \( T^0_{5,1} = W^0_{1,0} \oplus W^2_{1,2} \oplus W^3_{1,3} \)
- \( T^0_{5,2} = W^0_{2,0} \oplus W^2_{0,2} \oplus W^3_{0,3} \)
- \( T^0_{6,1} = W^1_{2,0} \oplus W^0_{0,2} \oplus W^4_{1,3} \)
- \( T^0_{6,2} = W^1_{2,1} \oplus W^0_{0,2} \oplus W^4_{0,3} \)
- \( T^0_{7,1} = W^2_{3,0} \oplus W^0_{0,2} \oplus W^5_{1,3} \)
- \( T^0_{7,2} = W^2_{3,1} \oplus W^0_{0,2} \oplus W^5_{0,3} \)

Now we consider \( r = t + 1 = 4 \). Here, \( p = 1 + \lceil \frac{kz}{r} \rceil = 2 \) is even. Hence, each sub-file \( W^n_\alpha, \forall n, \alpha \in [0, 7] \), is split into \( \frac{p}{2} = 1 \) part: \( \{W^n_{\alpha,0}\} \) which is the sub-file itself. The following coded symbols are transmitted corresponding to \( r = 4 \):

- \( T^1_0 = W^2_{0,0} \oplus W^5_{0,2} \)
- \( T^1_1 = W^5_{1,0} \oplus W^6_{1,0} \)
- \( T^1_2 = W^6_{2,0} \oplus W^7_{0,0} \)
- \( T^1_3 = W^7_{3,0} \oplus W^0_{0,0} \)
- \( T^1_4 = W^0_{4,0} \oplus W^1_{0,0} \)
- \( T^1_5 = W^5_{5,0} \oplus W^2_{1,0} \)
- \( T^1_6 = W^2_{6,0} \oplus W^3_{2,0} \)
- \( T^1_7 = W^3_{7,0} \oplus W^4_{3,0} \)

Now, each user \( U_\alpha \) needs to recover the demanded file \( W^{d(\alpha)} \) from the above transmissions. Let us first consider the user \( U_0 \). The user \( U_0 \) retrieves \( W^0_{0,0} \) from \( T^0_{0,1} \) while it retrieves \( W^0_{1,0} \) and \( W^0_{5,2} \) from \( T^0_{0,1} \) and \( T^0_{2,2} \) respectively. Similarly, the user \( U_0 \) retrieves \( W^0_{6,0} \) and \( W^0_{6,1} \) from \( T^0_{3,1} \) and \( T^0_{0,2} \) respectively. The sub-file \( W^0_{4,0} \) is recovered from \( T^1_4 \) whereas the sub-file \( W^0_{7,0} \) is recovered from \( T^1_3 \). The user \( U_0 \) has decoded the file \( W^0 \) since it has retrieved all the sub-files corresponding to the file \( W^0 \). Similarly all other users can decode their demanded file. In this example, the transmission rate using our scheme is \( R_{\text{new}} \left( \frac{1}{8} \right) = \frac{5}{3} \) while \( R_{\text{ic}} \left( \frac{1}{8} \right) = 2 \) in [5].

Example 3. Consider an example for \( k = 2 \), where \( N = 9, K = 9 \) and \( z = 2 \). The server stores 9 files: \( \{W^0, W^1, W^2, W^3, W^4, W^5, W^6, W^7, W^8\} \). Each file \( W^n, n \in \{0, 1, \ldots, 8\} \), is divided
into 9 sub-files: \( \{W^0, W^1, W^2, W^3, W^4, W^5, W^6, W^7, W^8\} \). Each cache \( \alpha \in \{0, 1, \ldots, 8\} \), is filled with two sub-file \( \{W^0_{2\alpha}, W^n_{2\alpha+1}\} \) of each file \( W^n, n \in \{0, 1, \ldots, 8\} \). Each user \( U_\alpha, \alpha \in \{0, 1, \ldots, 8\} \), has access to all the caches in \( C_\alpha = \{\alpha, \alpha + 1\} \). Let the demand vector be \( d = (0, 2, 4, 6, 8, 1, 3, 5, 7) \). Since \( K - kz = 5 \) is odd, we use Algorithm 2 to find the transmissions done by the server. First we consider \( r = t = 3 \). We split each sub-file \( W^0_\alpha, \forall n, \alpha \in \{0, 8\} \), into \( p = 1 + \lceil \frac{kz}{2} \rceil = 3 \) parts, namely, \( \{W^0_0, W^0_{1\alpha}, W^0_{2\alpha}\} \). The following coded symbols are transmitted corresponding to \( r = 3 \):

\[
T^0_0 = W^3_0 \oplus W^6_3 \oplus W^0_6, \quad T^0_1 = W^4_1 \oplus W^7_4 \oplus W^1_7, \quad T^0_2 = W^5_2 \oplus W^8_5 \oplus W^2_8,
\]

\[
T^3_0 = W^6_3 \oplus W^9_6, \quad T^3_1 = W^7_4 \oplus W^{1\alpha}_7, \quad T^3_2 = W^8_5 \oplus W^{2\alpha}_8,
\]

\[
T^6_0 = W^9_6 \oplus W^{3\alpha}_9, \quad T^6_1 = W^{1\alpha}_1 \oplus W^{4\alpha}_4, \quad T^6_2 = W^{2\alpha}_2 \oplus W^{5\alpha}_5.
\]

Now we consider \( r = t + 1 = 4 \). We have \( p = 1 + \lceil \frac{kz}{2} \rceil = 2 \). Since \( p \) is even, each sub-file \( W^0_\alpha, \forall n, \alpha \in \{0, 8\} \), is split into \( \lceil \frac{kz}{2} \rceil = 1 \) part, \( \{W^0_\alpha\} \) which is basically the sub-file itself. The following coded symbols are transmitted corresponding to \( r = 4 \):

\[
T^1_0 = W^4_0 \oplus W^7_4, \quad T^1_1 = W^5_1 \oplus W^8_5, \quad T^1_2 = W^6_2 \oplus W^9_6, \quad T^1_3 = W^7_3 \oplus W^{1\alpha}_7, \quad T^1_4 = W^8_4 \oplus W^{2\alpha}_8,
\]

\[
T^3_0 = W^6_3 \oplus W^{3\alpha}_6, \quad T^3_1 = W^7_4 \oplus W^{4\alpha}_7, \quad T^3_2 = W^8_5 \oplus W^{5\alpha}_8, \quad T^3_3 = W^9_6 \oplus W^{2\alpha}_9.
\]

Next we consider \( r = t + 2 = 5 \). Here too, \( p = 1 + \lceil \frac{kz}{2} \rceil = 2 \) is even. Hence, each sub-file \( W^0_\alpha, \forall n, \alpha \in \{0, 8\} \), is split into \( \lceil \frac{kz}{2} \rceil = 1 \) part, \( \{W^0_\alpha\} \) which is basically the sub-file itself. The following coded symbols are transmitted corresponding to \( r = 5 \):

\[
T^2_0 = W^5_0 \oplus W^8_5, \quad T^2_1 = W^6_1 \oplus W^9_6, \quad T^2_2 = W^7_2 \oplus W^{1\alpha}_7, \quad T^2_3 = W^8_3 \oplus W^{2\alpha}_8, \quad T^2_4 = W^9_4 \oplus W^{3\alpha}_9,
\]

\[
T^5_0 = W^6_5 \oplus W^{4\alpha}_6, \quad T^5_1 = W^7_6 \oplus W^{5\alpha}_7, \quad T^5_2 = W^8_7 \oplus W^{6\alpha}_8, \quad T^5_3 = W^9_8 \oplus W^{7\alpha}_9.
\]

Now, each user \( U_\alpha \) needs to recover the demanded file \( W^{d(\alpha)} \) from the above transmissions. The user \( U_0 \) retrieves \( W^0_{6,0} \) from \( T^0_6 \) while it retrieves \( W^0_{6,1} \) and \( W^0_{6,2} \) from \( T^0_3 \) and \( T^0_0 \) respectively. Similarly, the user \( U_0 \) retrieves \( W^0_{5,0}, W^0_{7,0}, W^0_{4,0} \) and \( W^0_{8,0} \) from \( T^5_5, T^1_1, T^2_2 \) and \( T^3_3 \) respectively. The user \( U_0 \) has decoded the file \( W^0 \) since it has retrieved all the sub-files corresponding to the file \( W^0 \). Similarly all other users can decode their demanded file. In this example, the server transmission rate using our scheme is \( R_{new} \left( \frac{2}{3} \right) = \frac{2}{3} \approx 2.33 \) while \( R_{ic} \left( \frac{2}{3} \right) = \frac{25}{9} \approx 2.77 \) for the scheme in [5].
IV. Conclusion

In this work, we have presented a placement and delivery scheme for multi-access coded caching problem, under the restriction of uncoded placement, with each cache having a normalized capacity of \( \gamma \), where \( \gamma \in \{ \frac{k}{K} : k = 1, 2, \ldots, K \} \). We have shown that our work is a generalization of one of the cases considered in [6]. We have also proved that our scheme outperforms that in [5] for the cases under consideration. Here, we assume that each user has access to same number of caches and each cache is of same capacity which need not be true in practical scenarios. Hence, it is a good direction to work on when the cache sizes are heterogeneous and each user has access to random number of users.

REFERENCES

[1] M. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” *IEEE Transactions on Information Theory*, vol. 60, no. 5, pp. 2856–2867, 2014.

[2] K. Wan, D. Tuninetti, and P. Piantanida, “On the optimality of uncoded cache placement,” in *Information Theory Workshop (ITW), 2016 IEEE*, pp. 161–165.

[3] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “The exact rate-memory tradeoff for caching with uncoded prefetching,” *IEEE Transactions on Information Theory*, 2017.

[4] J. Hachem, N. Karamchandani, and S. N. Diggavi, “Coded caching for multi-level popularity and access,” *IEEE Transactions on Information Theory*, vol. 63, no. 5, pp. 3108–3141, 2017.

[5] K. S. Reddy and N. Karamchandani, “Rate-memory trade-off for multi-access coded caching with uncoded placement,” *IEEE Transactions on Communications*, Vol.68, No.6, pp. 3261-3274, 2020.

[6] B. Serbetci, E. Parrinello and P. Elia, “Multi-access coded caching: gains beyond cache-redundancy,” *2019 IEEE Information Theory Workshop (ITW), Visby, Sweden*, 2019, pp. 1-5.

APPENDIX A

Proof of Correctness of the Delivery Scheme

We provide the proof of correctness of the delivery scheme presented in Algorithm [1] and Algorithm [2].

A. Proof of Correctness of Algorithm [7]

Let \( t = \frac{K-kz}{2} + 1 \). For each user \( U_\alpha, \alpha \in [0, K-1] \), the accessible cache content is \( \{ W^{n}_{k\alpha}, W^{n}_{k\alpha+1}, \ldots, W^{n}_{k(\alpha+z)-1} \} \), \( \forall n \in [0, N-1] \). Hence, we need to prove that the user \( U_\alpha \)
can decode all other sub-files \( \{W_{r^{d(\alpha)}_{k(z+a)+i}} \mid \forall i \in [0, K - kz - 1]\} \) in order to decode the file \( W^{d(\alpha)} \). Equivalently, the user \( U_\alpha \) needs to decode all the sub-files as follows:

\[
\left\{ W_{r^{d(\alpha)}_{k(z+a)+i}} \mid \forall i \in [0, K - kz - 1] \right\} = \left\{ W_{r^{d(\alpha)}_{k(z+a)+i}} \mid \forall i \in \left[ 0, \frac{K - kz}{2} - 1 \right] \right\} \cup \left\{ W_{r^{d(\alpha)}_{k(z+a)+i}} \mid \forall i \in \left[ \frac{K - kz}{2}, K - kz - 1 \right] \right\}
\]  

(5)

Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two sets defined as \( \mathcal{P} = \left\{ W_{r^{d(\alpha)}_{k(z+a)+i}} \mid \forall i \in \left[ 0, \frac{K - kz}{2} - 1 \right] \right\} \) and \( \mathcal{Q} = \left\{ W_{r^{d(\alpha)}_{k(z+a)+i}} \mid \forall i \in \left[ \frac{K - kz}{2}, K - kz - 1 \right] \right\} \). Hence, the user \( U_\alpha \) needs to decode all the sub-files in the set \( \mathcal{P} \cup \mathcal{Q} \). By changing the variable in the set \( \mathcal{P} \), from \( i \) to \( r = K - kz - i \), we can rewrite the set \( \mathcal{P} \) as

\[
\mathcal{P} = \left\{ W_{r^{d(\alpha)}_{k(z+a)+K - kz - r}} \mid \forall r \in \left[ \frac{K - kz}{2} + 1, K - kz \right] \right\} = \left\{ W_{r^{d(\alpha)}_{kz-r}} \mid \forall r \in [t, 2t - 2] \right\}.
\]  

(6)

Similarly, we can rewrite the set \( \mathcal{Q} \) as follows, by changing the variable in the set \( \mathcal{Q} \), from \( i \) to \( r = i + 1 \).

\[
\mathcal{Q} = \left\{ W_{r^{d(\alpha)}_{k(z+a)+r-1}} \mid \forall r \in \left[ \frac{K - kz}{2} + 1, K - kz \right] \right\} = \left\{ W_{r^{d(\alpha)}_{k(z+a)+r-1}} \mid \forall r \in [t, 2t - 2] \right\}.
\]  

(7)

Hence, the user \( U_\alpha \) needs to decode all the sub-files in \( \left\{ W_{r^{d(\alpha)}_{kz-r}}, W_{r^{d(\alpha)}_{k(z+a)+r-1}} \mid \forall r \in [t, 2t - 2] \right\} \).

**Lemma 1.** Each user \( U_\alpha, \forall \alpha \in \{0, 1, \ldots, K - 1\} \), can decode the sub-files \( \left\{ W_{r^{d(\alpha)}_{kz-r}} \mid \forall r \in [t, 2t - 2] \right\} \), where \( t = \frac{K - kz + 1}{2} \), using the transmissions done as in the Step 7 or 12 in Algorithm [1] depending on whether \( 1 + \left\lceil \frac{klz}{r} \right\rceil \) is even or odd.

**Proof.** We need to prove that each user \( U_\alpha \) can decode the sub-file \( W_{r^{d(\alpha)}_{kz-r}} \), for each \( r \in \{\frac{K - kz + 2}{2}, \ldots, K - kz\} \). We prove this in two parts: first is when \( p = 1 + \left\lceil \frac{klz}{r} \right\rceil \) is even and the second is when that is odd.

**Part 1:** when \( p = 1 + \left\lceil \frac{klz}{r} \right\rceil \) is even.

We split all the sub-files into \( \frac{p}{2} \) parts as in Step 5 in Algorithm [1]. So the user \( U_\alpha \) needs to retrieve all the \( \frac{p}{2} \) parts, \( \left\{ W_{r^{d(\alpha)}_{kz-r,l}} \mid \forall l \in [0, \frac{p}{2} - 1] \right\} \), of the sub-file \( W_{r^{d(\alpha)}_{kz-r}} \) in order to decode that sub-file. The user \( U_\alpha \) retrieves \( W_{r^{d(\alpha)}_{kz-r,l}} \), for each \( l \in [0, \frac{p}{2} - 1] \), from \( T^{r-t}_{kz-r,l} \),

\[
T^{r-t}_{kz-r,l} = \bigoplus_{i \in \left[ 0, \frac{p}{2} - 1 \right]} W_{r^{d(\alpha)}_{kz-r+i}} + \bigoplus_{i \in \left[ \frac{p}{2}, p-1 \right]} W_{r^{d(\alpha)}_{kz-r+i-p+1}}
\]  

(8)
We can rewrite Eq. (8) depending on the value of \( l \) as follows.

- for \( l \in \left[ 1, \frac{p}{2} - 2 \right] \), let \( j = k\alpha - (l + 1)r \).
  \[
  T_{j}^{r-t} = W_{k\alpha - r, l}^{d(\alpha)} \bigg( \sum_{i \in [0, \frac{p}{2} - 1]} W_{k\alpha + (i - l - 1)r, i}^{d\left(\frac{k\alpha + (i - l - 1)r}{k}\right)} \bigg) \bigg( \sum_{i \in \left[ \frac{p}{2}, p - 1 \right]} W_{k\alpha + (i - l - 1)r, i - \frac{p}{2}}^{d\left(\frac{k\alpha + (i - l - 1)r - (p - r)}{k}\right)} \bigg) \tag{9}
  \]

- for \( l = 0 \),
  \[
  T_{k\alpha-r}^{r-t} = W_{k\alpha - r, 0}^{d(\alpha)} \bigg( \sum_{i \in [1, \frac{p}{2} - 1]} W_{k\alpha + (i - 1)r, i}^{d\left(\frac{k\alpha + (i - 1)r}{k}\right)} \bigg) \bigg( \sum_{i \in \left[ \frac{p}{2}, p - 1 \right]} W_{k\alpha + (i - 1)r, i - \frac{p}{2}}^{d\left(\frac{k\alpha + (i - 1)r - (p - r)}{k}\right)} \bigg) \tag{10}
  \]

- for \( l = \frac{p}{2} - 1 \),
  \[
  T_{k\alpha-(\frac{p}{2})}^{r-t} = W_{k\alpha - (\frac{p}{2}), \frac{p}{2} - 1}^{d(\alpha)} \bigg( \sum_{i \in [0, \frac{p}{2} - 2]} W_{k\alpha + (i - \frac{p}{2} - 2)r, i}^{d\left(\frac{k\alpha + (i - \frac{p}{2} - 2)r}{k}\right)} \bigg) \bigg( \sum_{i \in \left[ \frac{p}{2}, p - 1 \right]} W_{k\alpha + (i - \frac{p}{2} - 2)r, i - \frac{p}{2}}^{d\left(\frac{k\alpha + (i - \frac{p}{2} - 2)r - (p - r)}{k}\right)} \bigg) \tag{11}
  \]

For any \( l \in \left[ 1, \frac{p}{2} - 1 \right] \), and \( i \in [1, l] \), we get the following inequality for the range of \( k\alpha - (i+1)r \).

\[
k\alpha - \left(\frac{p}{2}\right) r \leq k\alpha - (i+1)r \leq k\alpha - 2r.
\]

Since \( r \in \left[ \frac{K-kz}{2} + 1, K - kz \right] \), the above inequality can be written as

\[
k\alpha - r - \left(\frac{p}{2} - 1\right) r \leq k\alpha - (i+1)r \leq k\alpha - (K - kz) - 2
\]

\[
\Rightarrow k\alpha - (K - kz) - \left(\frac{p}{2} - 1\right) r \leq k\alpha - (i+1)r \leq k\alpha + kz - 2
\]

\[
\Rightarrow k\alpha + kz - \left(\frac{p}{2} - 1\right) r \leq k\alpha - (i+1)r \leq k\alpha + kz - 2.
\]
Since \( p = 1 + \left\lceil \frac{kz}{r} \right\rceil \), we have \( kz > (p - 2)r \). Hence, \( \left( \frac{p}{2} - 1 \right) r < kz \), and we can write the above inequality as

\[
k\alpha + kz - (kz - 1) \leq k\alpha - (i + 1)r \leq k\alpha + kz - 2 \Rightarrow k\alpha + 1 \leq k\alpha - (i + 1)r \leq k\alpha + kz - 2.
\]

Hence, all the sub-files \( W^n_{k\alpha - (i + 1)r} \), for any \( n \in [0, N - 1] \) and \( i \in [1, l] \) are available at the user \( U_\alpha \) in the cache. Similarly for any \( i \in [1, p - l - 1] \), and \( l \in [0, \frac{p}{2} - 1] \), the range of \( k\alpha + (i - 1)r \) is \( k\alpha \leq k\alpha + (i - 1)r \leq k\alpha + (p - 2)r \). Since \( (p - 2)r < kz \), we can rewrite it as \( k\alpha \leq k\alpha + (i - 1)r \leq k\alpha + kz - 1 \). Hence, all the sub-files \( W^n_{k\alpha + (i - 1)r} \), for any \( n \in [0, N - 1] \) and \( i \in [1, p - l - 1] \), are also available at the user \( U_\alpha \) in the cache. So, the user \( U_\alpha \) can decode \( W^d_{k\alpha - r,l} \), for each \( l \in [0, \frac{p}{2} - 1] \), from \( T^d_{k\alpha - (l + 1)r} \), since all other parts of sub-files are available at its cache. The user \( U_\alpha \) has recovered all the parts \( \{W^d_{k\alpha - r,l}, \forall l \in [0, \frac{p}{2} - 1]\} \) corresponding to the sub-file \( W^d_{k\alpha - r} \).

**Part 2:** when \( p = 1 + \left\lceil \frac{kz}{r} \right\rceil \) is odd.

We split all the sub-files into \( p \) parts as in Step 10 in Algorithm [1], where each user \( U_\alpha \) needs to retrieve all the parts \( \{W^d_{k\alpha - r,l}, \forall l \in [0, p - 1]\} \) corresponding to the sub-file \( W^d_{k\alpha - r} \) to decode \( W^d_{k\alpha - r} \). The user \( U_\alpha \) gets \( W^d_{k\alpha - r,l} \), for each \( l \in [0, \frac{p - 3}{2}] \), from \( T^d_{k\alpha - (l + 1)r} \),

\[
T^d_{k\alpha - (l + 1)r,1} = \bigoplus_{i \in [0, \frac{kz}{r}]} W^d_{k\alpha + (i - l - 1)r,i} \bigoplus_{i \in [\frac{kz}{r} - p - 1]} W^d_{k\alpha + (i - l - 1)r,i - \frac{p - 1}{2}} \tag{15}
\]

We rewrite Eq. (15) depending on the value of \( l \) as follows.

- for \( l = 0 \),

\[
T^d_{k\alpha - r,1} = W^d_{k\alpha - r,0} \bigoplus_{i \in [1, \frac{kz}{r}]} W^d_{k\alpha + (i - 1)r,i} \bigoplus_{i \in [\frac{kz}{r} - p - 1]} W^d_{k\alpha + (i - 1)r,i - \frac{p - 1}{2}} \tag{16}
\]

\[
= W^d_{k\alpha - r,0} \left( \bigoplus_{i \in [1, \frac{kz}{r}]} W^d_{k\alpha + (i - 1)r,i} \right) \bigoplus_{i \in [\frac{kz}{r} - p - 1]} W^d_{k\alpha + (i - 1)r,i - \frac{p - 1}{2}} \tag{17}
\]

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For \( l \in \left[ 1, \frac{p - 3}{2} - 1 \right] \), let \( j = k\alpha - (l + 1)r \). Then we have

\[
T_{j, r}^{r-t} = \frac{d(\alpha)}{k\alpha - r} \bigg( 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg) \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg] \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg]
\]

(18)

\[
T_{j, r}^{r-t} = \frac{d(\alpha)}{k\alpha - r} \bigg( 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg) \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg] \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg]
\]

(19)

\[
T_{j, r}^{r-t} = \frac{d(\alpha)}{k\alpha - r} \bigg( 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg) \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg] \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg]
\]

(20)

\[
\text{for } l = \frac{p - 3}{2},
\]

\[
T_{j, r}^{r-t} = \frac{d(\alpha)}{k\alpha - r} \bigg( 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg) \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg] \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg]
\]

(21)

\[
T_{j, r}^{r-t} = \frac{d(\alpha)}{k\alpha - r} \bigg( 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg) \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg] \bigg[ 1 + \frac{d(k\alpha - (i + 1)r)}{k\alpha - (i + 1)r} \bigg]
\]

(22)

For any \( l \in \left[ 1, \frac{p - 3}{2} \right] \) and \( i \in \left[ 1, l \right] \), we get the following inequality, \( k\alpha - \left( \frac{p - 1}{2} \right) r \leq k\alpha - (i + 1)r \leq k\alpha - 2r \). Since \( r \in \left[ \frac{K-kz+2}{2}, K - kz \right] \), we have

\[
k\alpha - r - \left( \frac{p - 1}{2} - 1 \right) r \leq k\alpha - (i + 1)r \leq k\alpha - (K - kz + 2)
\]

\[
\Rightarrow k\alpha - (K - kz) - \left( \frac{p - 1}{2} - 1 \right) r \leq k\alpha - (i + 1)r \leq k\alpha + kz - 2
\]

\[
\Rightarrow k\alpha + kz - \left( \frac{p - 1}{2} - 1 \right) r \leq k\alpha - (i + 1)r \leq k\alpha + kz - 2.
\]

We also know that \( \left( \frac{p-3}{2} \right) r < kz \). Hence,

\[
k\alpha + kz - (kz - 1) \leq k\alpha - (i + 1)r \leq k\alpha + kz - 2 \Rightarrow k\alpha + 1 \leq k\alpha - (i + 1)r \leq k\alpha + kz - 2.
\]
Hence, all the sub-files $W_{k\alpha-(i+1)r}^n$, for any $n \in [0, N - 1]$ and $i \in [1, l]$, are available at the user $U_\alpha$ in its cache. Similarly for any $l \in [0, \frac{p-3}{2}]$ and $i \in [1, p - l - 1]$, we have the following inequality, $k\alpha \leq k\alpha + (i - 1)r \leq k\alpha + (p - 2)r$. Since $(p - 2)r < kz$, we have $k\alpha \leq k\alpha + (i - 1)r \leq k\alpha + kz - 1$. Hence, all the sub-files $W_{k\alpha+i-1}^n$, for any $n \in [0, N - 1]$ and $i \in [1, p - l - 1]$ are also available at the user $U_\alpha$ in the cache. So, the user $U_\alpha$ can decode the part $W_{k\alpha-r,l}^{d(\alpha)}$, for each $l \in [0, \frac{p-3}{2}]$ from $T_{k\alpha-(i+1)r,1}^{r-t}$, since all other sub-files are available at its cache.

Now, the user $U_\alpha$ needs to retrieve all other parts of the sub-file $\{W_{k\alpha-r,l}^{d(\alpha)} \mid l \in \left[ \frac{p-1}{2}, p - 1 \right] \}$. The user $U_\alpha$ gets $W_{k\alpha-r,l}^{d(\alpha)}$, for each $l \in \left[ \frac{p-1}{2}, p - 1 \right]$, from $T_{k\alpha-(i+1)r,1}^{r-t}$.

\[
T_{k\alpha-(i+1)r,1}^{r-t} = \bigoplus_{i \in \left[ \frac{p-1}{2}, p - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i \bigoplus_{i \in \left[ \frac{p-1}{2}, p - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i
\]  

(23)

Depending upon the value of $l$, Eq. (23) can be rewritten as below.

\[
T_{j,2}^{r-t} = \bigoplus_{i \in \left[ \frac{p-1}{2}, p - 1 \right]} W_{k\alpha-r,l}^{d(\alpha)} r, i \bigoplus_{i \in \left[ \frac{p-1}{2}, p - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i \bigoplus_{i \in \left[ \frac{p-1}{2}, p - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i
\]  

(24)

- if $l \in \left[ \frac{p+1}{2}, p - 2 \right]$, let $j = k\alpha - (l - \frac{p-1}{2} + 1) r$. Then,

\[
T_{j,2}^{r-t} = \bigoplus_{i \in \left[ \frac{p-1}{2}, p - 1 \right]} W_{k\alpha-r,l}^{d(\alpha)} r, i \bigoplus_{i \in \left[ \frac{p+1}{2}, p - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i \bigoplus_{i \in \left[ \frac{p+1}{2}, p - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i
\]  

(25)

\[
T_{j,2}^{r-t} = \bigoplus_{i \in \left[ \frac{p+1}{2}, p - 1 \right]} W_{k\alpha-r,l}^{d(\alpha)} r, i \bigoplus_{i \in \left[ 1, l - \frac{p-1}{2} \right]} W_{k\alpha-i}^{d(\alpha)} r, i \bigoplus_{i \in \left[ 1, l - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i
\]  

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\[
T_{j,2}^{r-t} = \bigoplus_{i \in \left[ p-l, p+\frac{p-1}{2} - l - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i \bigoplus_{i \in \left[ p-l, p+\frac{p-1}{2} - l - 1 \right]} W_{k\alpha+i-1}^{d(\alpha)} r, i
\]  

available at cache

(26)
• if \( l = \frac{p-1}{2} \), let \( j = k\alpha - r \).

\[
T^{r-t}_{j,2} = W^{d(\alpha)}_{ka-r, \frac{p-1}{2}} \bigg( \sum_{i=\frac{p+1}{2},p-2}^{p-1} W^{d(\frac{k\alpha+(i-1)\alpha}{k})}_{ka+(i-1)\alpha,\frac{p+1}{2},p-2} \bigg) + \bigg( \sum_{i=\frac{p+1}{2},p-1}^{p-1} W^{d(\frac{k\alpha+(i-1)\alpha}{k})}_{ka+(i-1)\alpha,\frac{p+1}{2},p-1} \bigg)
\]

(27)

• if \( l = p-1 \), let \( j = k\alpha - \left( \frac{p-1}{2} \right) r \).

\[
T^{r-t}_{j,2} = W^{d(\alpha)}_{ka-r,p-1} \bigg( \sum_{i=\frac{p+1}{2},p-2}^{p-1} W^{d(\frac{k\alpha+(i-1)\alpha}{k})}_{ka+(i-1)\alpha,\frac{p+1}{2},p-2} \bigg) + \bigg( \sum_{i=\frac{p+1}{2},p-1}^{p-1} W^{d(\frac{k\alpha+(i-1)\alpha}{k})}_{ka+(i-1)\alpha,\frac{p+1}{2},p-1} \bigg)
\]

(28)

For any \( l \in \left[ \frac{p+1}{2}, p-1 \right] \) and \( i \in \left[ 1, l - \frac{p-1}{2} \right] \), we have, \( k\alpha - \left( \frac{p+1}{2} \right) r \leq k\alpha - (i+1)r \leq k\alpha - 2r \).

Since \( r \in \left[ \frac{K-kz+2}{2}, K-kz \right] \), we have

\[
k\alpha - r - \left( \frac{p+1}{2} - 1 \right) r \leq k\alpha - (i+1)r \leq k\alpha - (K-kz + 2)
\]

\[
\Rightarrow k\alpha - (K-kz) - \left( \frac{p+1}{2} - 1 \right) r \leq k\alpha - (i+1)r \leq k\alpha + kz - 2
\]

\[
\Rightarrow k\alpha + kz - \left( \frac{p-1}{2} \right) r \leq k\alpha - (i+1)r \leq k\alpha + kz - 2.
\]

We also know that \( \left( \frac{p-1}{2} \right) r < kz \). Hence,

\[
k\alpha + kz - (kz - 1) \leq k\alpha - (i+1)r \leq k\alpha + kz - 2 \Rightarrow k\alpha + 1 \leq k\alpha - (i+1)r \leq k\alpha + kz - 2.
\]

Hence, all the sub-files \( W^{n}_{k\alpha-(i+1)r} \), for any \( n \in [0, N-1] \) and \( i \in \left[ 1, l - \frac{p-1}{2} \right] \), are available at the user \( U_{\alpha} \) in the cache. Similarly for any \( l \in \left[ \frac{p-1}{2}, p-1 \right] \) and \( i \in \left[ 1, p-l-1 + \frac{p-1}{2} \right] \), we have \( k\alpha \leq k\alpha + (i-1)r \leq k\alpha + (p-2)r \). Since \((p-2)r < kz\), we have \( k\alpha \leq k\alpha + (i-1)r \leq k\alpha + kz - 1 \).

Hence, all the sub-files \( W^{n}_{k\alpha+(i-1)r} \), for any \( n \in [0, N-1] \) and \( i \in \left[ 1, p-l-1 + \frac{p-1}{2} \right] \) are also available at the user in the cache. So, the user \( U_{\alpha} \) can decode \( W^{d(\alpha)}_{k\alpha-r,l} \) for each \( l \in \left[ \frac{p-1}{2}, p-1 \right] \) from \( T^{r-t}_{k\alpha-(l-\frac{p-1}{2}-1)r,2} \), since all other parts of sub-files are available at its cache.
In short, the user $U_\alpha$ can decode the sub-file $W^d_{k\alpha-r}$, since it has retrieved all the parts of sub-files corresponding to $W^d_{k\alpha-r}$.

Lemma 2. Each user $U_\alpha$, $\alpha \in \{0, 1, \ldots, \alpha - 1\}$, can decode sub-files $\{W^d_{k(\alpha+z)+r-1}, \forall r \in [t, 2t-2]\}$, where $t = \frac{K-kz}{2} + 1$, using the transmissions done as in Step 7 or 12 in Algorithm 1 depending on whether $1 + \lceil \frac{kz}{r} \rceil$ is even or odd.

Proof. We need to prove that each user $U_\alpha$ can decode the sub-file $W^d_{k(\alpha+z)+r-1}$, for each $r \in \{\frac{K-kz+2}{2}, \ldots, K-kz\}$. First we prove this when $p = 1 + \lceil \frac{kz}{r} \rceil$ is even. Then we take up the case when $p = 1 + \lceil \frac{kz}{r} \rceil$ is even and prove this lemma.

Part 1: when $p = 1 + \lceil \frac{kz}{r} \rceil$ is even.

We split all the sub-files into $\frac{p}{2}$ parts as in Step 5 in Algorithm 1. So the user $U_\alpha$ needs to retrieve all the $\frac{p}{2}$ parts, $\{W^d_{k(\alpha+z)+r-1}, \forall l \in \left[0, \frac{p}{2} - 1\right]\}$, of the sub-file $W^d_{k(\alpha+z)+r-1}$ in order to decode that sub-file. The user $U_\alpha$ retrieves $W^d_{k(\alpha+z)+r-1}$, for each $l \in \left[0, \frac{p}{2} - 1\right]$, from $T^r_{k(\alpha+z)-1-(l+\frac{p}{2}-1)r}$.

$$T^r_{k(\alpha+z)-1-(l+\frac{p}{2}-1)r} = \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} d \left( \frac{k\alpha + (i-\frac{p}{2})r + k z - 1}{\frac{p}{2}} \right) \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2}+1)r,i}$$

We can rephrase Eq. (31) depending upon the value of $l$ as below. Let $m_\alpha = k(\alpha + z) + r - 1$.

- for $l \in \left[1, \frac{p}{2} - 2\right]$, let $j = k(\alpha + z) - 1 - (l + \frac{p}{2} - 1) \cdot r$,

$$T^r_{j_2} = W^d_{m_\alpha, l} \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2})r,i} \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2}+1)r,i}$$

$$= W^d_{m_\alpha, l} \left( \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2})r,i} \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2}+1)r,i} \right)$$

$$= W^d_{m_\alpha, l} \left( \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2})r,i} \bigoplus_{i \in \left[0, \frac{p}{2} - 1\right]} W^d_{m_\alpha + (i-\frac{p}{2}+1)r,i} \right)$$

$$= W^d_{m_\alpha, l} \left( \bigoplus_{i \in \left[1, \frac{p}{2} - l - 1\right]} W^d_{m_\alpha + (i-1)r,i} \bigoplus_{i \in \left[1, \frac{p}{2} - l\right]} W^d_{m_\alpha + (i-1)r,i} \bigoplus_{i \in \left[1, \frac{p}{2} - l\right]} W^d_{m_\alpha + (i-1)r,i} \right)$$

(32)
• for \( l = 0 \), let \( j = k(\alpha + z) - 1 - \left( \frac{p}{2} - 1 \right) r \).

\[
{\mathcal T}_{j}^{r-t} = W_{m_{\alpha},0}^{d(\alpha)} \bigoplus_{i \in [0, \frac{p}{2}-1]} \left( \bigoplus_{i \in [\frac{p}{2}, \frac{p}{2}-1]} W_{m_{\alpha}+(i-\frac{p}{2})r,i}^{d\left(\frac{ka+(1-(i-\frac{p}{2})r+k\left(\frac{p}{2}+1\right)}{k}\right)} \bigoplus_{i \in [1, \frac{p}{2}-1]} W_{m_{\alpha}+ir,i}^{d\left(\frac{ka+ir-1}{k}\right)} \right)
\]

\[
\text{available at cache}
\]

\[
\text{available at cache}
\]

\[
(35)
\]

• for \( l = \frac{p}{2} - 1 \), let \( j = k(\alpha + z) - 1 - (p - 2) r \).

\[
{\mathcal T}_{j}^{r-t} = W_{m_{\alpha},\frac{p}{2}-1}^{d(\alpha)} \bigoplus_{i \in [0, \frac{p}{2}-1]} \left( \bigoplus_{i \in [\frac{p}{2}, \frac{p}{2}-1]} W_{m_{\alpha}+(i-\frac{p}{2})r,i}^{d\left(\frac{ka+(1-(i-\frac{p}{2})r+k\left(\frac{p}{2}+1\right)}{k}\right)} \bigoplus_{i \in [1, \frac{p}{2}-1]} W_{m_{\alpha}+ir,i}^{d\left(\frac{ka+ir-1}{k}\right)} \right)
\]

\[
\text{available at cache}
\]

\[
\text{available at cache}
\]

\[
(37)
\]

The proof that all other parts are available at the caches is similar to that provided in Part 1 of Lemma 1. So, the user \( U_{\alpha} \) can decode \( W_{(k(\alpha+z)+r-1)}^{d(\alpha)} \) from \( T_{j}^{r-t} \) since all other parts of sub-files are available at its cache.

**Part 2:** when \( p = 1 + \lceil \frac{k}{p} \rceil \) is odd.

We split all the sub-files into \( p \) parts as in Step 10 in Algorithm 1 where each user \( U_{\alpha} \) needs to retrieve all the parts \( \{W_{(k(\alpha+z)+r-1)}, \forall l \in [0, p-1]\} \) corresponding to the sub-file \( W_{(k(\alpha+z)+r-1)}^{d(\alpha)} \) to decode \( W_{(k(\alpha+z)+r-1)}^{d(\alpha)} \). The user \( U_{\alpha} \) gets \( W_{(k(\alpha+z)+r-1),l}^{d(\alpha)} \), for each \( l \in \left[0, \frac{p-1}{2}\right] \), from \( T_{j,1}^{r-t} \), where \( j = (k(\alpha + z) - 1) - (l + \frac{p-1}{2} - 1) r \),

\[
T_{j,1}^{r-t} = \bigoplus_{i \in [0, \frac{p}{2}-1]} W_{(k(\alpha+z)-1)+(i-l-\frac{p-1}{2})r+\frac{r+1}{2},i}^{d\left(\frac{ka+(i-l-\frac{p-1}{2})r+\frac{r+1}{2}}{k}\right)} \bigoplus_{i \in [0, \frac{p}{2}-1]} W_{(k(\alpha+z)+r-1)+(i-l+1)r,i}^{d\left(\frac{ka+(i-l+1)r-1}{k}\right)}
\]

\[
(39)
\]

Depending upon the value of \( l \), the above equation can be rephrased as follows. Let \( m_{\alpha} = k(\alpha + z) + r - 1 \).
For $l \in \left[1, \frac{p-3}{2}\right]$, we have

$$T_{j,1}^{r-t} = W_{m_0,l}^{d(\alpha)} \left( \bigoplus_{i \in \left[1, \frac{p-1-l}{2}\right]} W_{m_0+(i-l+1)z,r,i}^{d\left(\frac{k_0+(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-1-l}{2}\right]} W_{m_0-(i-1)z,l-i}^{d\left(\frac{k_0-(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-1-l}{2}\right]} W_{m_0+(i-l+1)z,r,i}^{d\left(\frac{k_0+(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-1-l}{2}\right]} W_{m_0-(i-1)z,l-i}^{d\left(\frac{k_0-(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \right).$$

(40)

available at cache

available at cache

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available at cache

(41)

for $l = 0$,

$$T_{j,1}^{r-t} = W_{m_0,0}^{d(\alpha)} \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0+(i-1)z,r,i}^{d\left(\frac{k_0+(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0-(i-1)z,l-i}^{d\left(\frac{k_0-(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0+(i-1)z,r,i}^{d\left(\frac{k_0+(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0-(i-1)z,l-i}^{d\left(\frac{k_0-(i-1)z}{k}\right) + \frac{r+kz-1}{k}} \right) \right).$$

(42)

available at cache

available at cache

available at cache

available at cache

(43)

for $l = \frac{p-1}{2}$,

$$T_{j,1}^{r-t} = W_{m_0,\frac{p-1}{2}}^{d(\alpha)} \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0+(i+1-p)z,r,i}^{d\left(\frac{k_0+(i+1-p)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0-(i+1-p)z,l-i}^{d\left(\frac{k_0-(i+1-p)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0+(i+1-p)z,r,i}^{d\left(\frac{k_0+(i+1-p)z}{k}\right) + \frac{r+kz-1}{k}} \right) \left( \bigoplus_{i \in \left[1, \frac{p-3}{2}\right]} W_{m_0-(i+1-p)z,l-i}^{d\left(\frac{k_0-(i+1-p)z}{k}\right) + \frac{r+kz-1}{k}} \right) \right).$$

(44)

available at cache

available at cache

available at cache

available at cache

(45)

The proof that all other parts are available at the caches is in similar lines to that given in Part 2 in Lemma 1. So, the user $U_\alpha$ can decode some $W_{m_0}^{d(\alpha)}$, for each $l \in \left[0, \frac{p-1}{2}\right]$ from $T_{j,1}^{r-t}$, $j = (k(\alpha + z) - 1) - (l + \frac{p-1}{2} - 1) r$, since all other sub-files are available at its cache.

Now, the user $U_\alpha$ needs to retrieve all other parts $\{W_{m_0}^{d(\alpha)}\}_{k(\alpha+z)+r-1,l}$, $\forall l \in \left[\frac{p+1}{2}, p-1\right]$. The user
Depending upon the value of $l$, the above equation is rewritten as below. Let $m_{\alpha} = k(\alpha + z) + r - 1$.

- for $l \in \left[\frac{p+3}{2}, p - 2\right]$,

$$T_{j, 2}^{r-t} = W_{m_{\alpha}, l}^{d(\alpha)} \bigg( \bigoplus_{i \in [0, \frac{l-1}{2}]} W_{m_{\alpha} + (i-l)r, i + \frac{l-1}{2}}^{d[\frac{(kn + k\alpha + (i-2l) + 1)r-1}{k}]} \bigoplus_{i \in [\frac{l+1}{2}, p-1] \setminus \{l\}} W_{m_{\alpha} + (i-l)r, i}^{d[\frac{(kn + k\alpha + (i-2l) + 1)r-1}{k}]} \bigg)$$

$$= W_{m_{\alpha}, l}^{d(\alpha)} \bigg( \bigoplus_{i \in [1, p-l-1]} W_{m_{\alpha} + (i-1)r + \frac{l-1}{2}}^{d[\frac{(kn + k\alpha + (i+2l-1)r-1}{k}]} \bigg) \bigg( \bigoplus_{i \in [\frac{l+1}{2}, l-1]} W_{m_{\alpha} + (i-1)r, i}^{d[\frac{(kn + k\alpha + (i-2l) + 1)r-1}{k}]} \bigg)$$

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available at cache

- for $l = \frac{p+1}{2}$,

$$T_{j, 2}^{r-t} = W_{m_{\alpha}, \frac{p+1}{2}}^{d(\alpha)} \bigg( \bigoplus_{i \in [0, \frac{l-1}{2}]} W_{m_{\alpha} + (i-1)r + \frac{l-1}{2}}^{d[\frac{(kn + k\alpha + (i+2l-1)r-1}{k}]} \bigg) \bigg( \bigoplus_{i \in [\frac{l+1}{2}, p-1] \setminus \{l\}} W_{m_{\alpha} + (i-1)r, i}^{d[\frac{(kn + k\alpha + (i-2l) + 1)r-1}{k}]} \bigg)$$

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- for $l = p - 1$,

$$T_{j, 2}^{r-t} = W_{m_{\alpha}, p-1}^{d(\alpha)} \bigg( \bigoplus_{i \in [0, \frac{l-1}{2}]} W_{m_{\alpha} + (i-1)r + \frac{l-1}{2}}^{d[\frac{(kn + k\alpha + (i+2l-1)r-1}{k}]} \bigg) \bigg( \bigoplus_{i \in [\frac{l+1}{2}, p-1] \setminus \{l\}} W_{m_{\alpha} + (i-1)r, i}^{d[\frac{(kn + k\alpha + (i-2l) + 1)r-1}{k}]} \bigg)$$

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available at cache

So, the user $U_{\alpha}$ can decode $W_{k(\alpha + z) + r-1, l}^{d(\alpha)}$ for each $l \in \left[\frac{p+1}{2}, p - 1\right]$ from $T_{j, 2}^{r-t} ; j = k(\alpha + z) - 1 - (i - 1)r$, since all other parts of sub-files are available at its cache. The proof that all other
parts are available at the caches is similar to that given in Part 2 in Lemma 1. In short, the user \( U_\alpha \) can decode the sub-file \( W_{d(\alpha)}^{k(\alpha+z)+r-1} \), since it has retrieved all the parts of the sub-file corresponding to \( W_{d(\alpha)}^{k(\alpha+z)+r-1} \).

From Lemma 1 and 2 when \( K - kz \) is even, each user \( U_\alpha \) can decode all the sub-files, \( \{W_{d(\alpha)}^{k(\alpha+z)+i}, \forall i \in [0, K - kz - 1]\} \), corresponding to its demanded file \( W_{d(\alpha)} \), which are not available in its cache.

**B. Proof of Correctness of Algorithm 2**

Let \( t = \frac{K - kz + 1}{2} \). For each user \( U_\alpha, \alpha \in [0, K - 1] \), the accessible cache content is \( \{\{W_{d(\alpha)}^{n(k(\alpha+z)+r-1)}, \forall n \in [0, N - 1]\} \). Hence, we need to prove that the user \( U_\alpha \) can decode all other sub-files \( \{W_{d(\alpha)}^{k(\alpha+z)+i}, \forall i \in [0, K - kz - 1]\} \) in order to decode the file \( W_{d(\alpha)} \).

Equivalently, like in the proof of correctness of Algorithm 1, if \( K - kz > 1 \), then the user \( U_\alpha \) needs to decode all the sub-files \( \{W_{d(\alpha)}^{k(\alpha+z)+r-1}, \forall r \in [t + 1, 2t - 1]\} \), if \( K - kz > 1 \).

The user needs to decode \( \{W_{d(\alpha)}^{k(\alpha+z)} \}, \) if \( K - kz = 1 \). The set \( \{W_{d(\alpha)}^{k(\alpha+z)} \} \) can be rewritten as

\[
\{W_{d(\alpha)}^{k(\alpha+z)} \} = \{W_{d(\alpha)}^{k(\alpha+z)-K}, kz > r \} = \{W_{d(\alpha)}^{k(\alpha+z)-K}, kz > r \} = \{W_{d(\alpha)}^{k(\alpha+z)-K}, kz > r \}.
\]

Hence, the user needs to decode \( \{W_{d(\alpha)}^{k(\alpha+z)} \}, \) if \( K - kz = 1 \)

**Lemma 3.** Each user \( U_\alpha, \alpha \in \{0, 1, \ldots, K - 1\} \), can decode the sub-files \( \{W_{d(\alpha)}^{r(\alpha-z)+r-1}, \forall r \in [t + 1, 2t - 1]\} \), where \( t = \frac{K - kz + 1}{2} \), using the transmissions done as in Step 14 or 19 in Algorithm 2 depending on whether \( 1 + \left\lceil \frac{kz}{r} \right\rceil \) is even or odd.

**Proof.** The proof is similar to the proof of Lemma 1. Hence we omit the proof.

**Lemma 4.** Each user \( U_\alpha, \alpha \in \{0, 1, \ldots, K - 1\} \), can decode the sub-files \( \{W_{d(\alpha)}^{k(\alpha+z)+r-1}, \forall r \in [t + 1, 2t - 1]\} \), where \( t = \frac{K - kz + 1}{2} \), using the transmissions done as in Step 14 or 19 in Algorithm 2 depending on whether \( 1 + \left\lceil \frac{kz}{r} \right\rceil \) is even or odd.

**Proof.** We have omitted the proof since it is similar to the proof of Lemma 2.

**Lemma 5.** Each user \( U_\alpha, \alpha \in \{0, 1, \ldots, K - 1\} \), can decode the sub-file \( W_{d(\alpha)}^{k(\alpha+z)} \), where \( t = \frac{K - kz + 1}{2} \), using the transmissions done as in Step 8 in Algorithm 2.
Proof. We split all the sub-files into \( p \) parts as in Step 5 in Algorithm \( \mathbf{2} \). So the user \( U_\alpha \) needs to retrieve all the \( p - 1 \) parts, \( \{ W^d(\alpha)_{k\alpha - t, l}, \forall l \in [0, p - 1] \} \), of the sub-file \( W^d(\alpha)_{k\alpha - t} \) in order to decode that sub-file. The user \( U_\alpha \) retrieves each sub-file \( W^d(\alpha)_{k\alpha - t, l}, l \in [0, p - 1] \), from \( T^0_j \), where \( j = k\alpha - (l + 1)t \),

\[
T^0_j = \bigoplus_{i \in [0, p-1]} W^d(\frac{k\alpha + (i-l)t}{k})_{k\alpha + (i-l-1)t, i}
\]

Depending upon the value of \( l \), the above equation can be rewritten as follows.

- for \( l \in [1, p - 2] \),

\[
T^0_j = \bigoplus_{i \in [0, l-1]} W^d(\frac{k\alpha + (i-l)t}{k})_{k\alpha + (i-l-1)t, i} \bigoplus_{i \in [l+1, p-1]} W^d(\frac{k\alpha + (i-l)t}{k})_{k\alpha + (i-l-1)t, i}
\]

- for \( l = 0 \),

\[
T^0_j = W^d(\alpha)_{k\alpha - t, 0} \bigoplus_{i \in [1, p-1]} W^d(\frac{k\alpha + i}{k})_{k\alpha + (i-1)t, i}
\]

- for \( l = p - 1 \),

\[
T^0_j = \bigoplus_{i \in [0, p-2]} W^d(\frac{k\alpha + (i-p+1)t}{k})_{k\alpha + (i-p)t, i} \bigoplus_{i \in [1, p-1]} W^d(\frac{k\alpha - i}{k})_{k\alpha - (i+1)t, p-1-i}
\]

The proof that all other parts are available at the caches is similar to that provided in Part 1 of Lemma 1. So, the user \( U_\alpha \) can decode \( W^d(\alpha)_{k\alpha - t, l} \) for each \( l \in [0, p - 1] \), from \( T^0_{k\alpha - (l+1)t} \) since all other parts of sub-files are available at its cache. Hence it can retrieve the sub-file \( W^d(\alpha)_{k\alpha - t} \). \( \Box \)

From Lemma \( \mathbf{3} \) \( \mathbf{4} \) and \( \mathbf{5} \) when \( K - kz \) is odd, each user \( U_\alpha \) can decode all the sub-files, \( \{ W^d(\alpha)_{k\alpha + kz + i}, \forall i \in [0, K - kz - 1] \} \), corresponding to its demanded file \( W^d(\alpha) \), which are not available in its cache.