AVERAGING OF SEMIGROUPS ASSOCIATED TO DIFFUSION PROCESSES ON A SIMPLEX

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Abstract. We study the averaging of a diffusion process living in a simplex $K$ of $\mathbb{R}^n$, $n \geq 1$. We assume that its infinitesimal generator can be decomposed as a sum of two generators corresponding to two distinct timescales and that the one corresponding to the fastest timescale is pure noise with a diffusion coefficient vanishing exactly on the vertices of $K$. We show that this diffusion process averages to a pure jump Markov process living on the vertices of $K$ for the Meyer–Zheng topology. The role of the geometric assumptions done on $K$ is also discussed.

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1. Introduction

Let $E$ be a set and consider a family of $E$-valued Markovian processes \{\(X_\gamma\), $\gamma > 0$\} depending on some parameter $\gamma > 0$ whose generators can be written in the form

\[
\mathcal{L}_\gamma = \mathcal{L}^{(0)} + \gamma \mathcal{L}^{(1)}. \tag{1.1}
\]

A general question in the probabilistic and analysis literature is to understand the asymptotic behavior of $X_\gamma$ when $\gamma$ goes to infinity \(^1\) (see books and reviews \([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]\)). More precisely, given $x \in E$ and $t > 0$, we want to investigate the large $\gamma$ limit of the semigroup acting on a generic test function $f : E \to \mathbb{R}$

\[
e^{t \mathcal{L}_\gamma} f(x) = \mathbb{E}_x (f(X_\gamma^t)).
\]

A general ‘meta-theorem’ is that:

\[
e^{t \mathcal{L}_\gamma} \xrightarrow{\gamma \to +\infty} \mathcal{P} e^{t \mathcal{L}_\infty} \mathcal{P} \tag{1.2}
\]

where $\mathcal{P}$ is the spectral projector onto the kernel of the dominant generator $\mathcal{L}^{(1)}$, parallel to the image of $\mathcal{L}^{(1)}$, and $\mathcal{L}_\infty = \mathcal{P} \mathcal{L}^{(0)} \mathcal{P}$. When this quantity is well defined, one has:

\[
\mathcal{P} = \lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{s \mathcal{L}^{(1)}}.
\]

Of course such a theorem would require several assumptions to hold and may be invalid in several situations, in particular when the limit itself does not make sense. Observe also that a priori the operator $\mathcal{L}_\infty$ is not necessarily related to the generator of a Markovian process.

Let us give a formal proof of (1.2). Let denote $f_t^{\gamma} = e^{t \mathcal{L}_\gamma} f$. Dynkin’s formula gives

\[
\frac{\partial}{\partial t} f_t^{\gamma} = (\mathcal{L}^{(0)} + \gamma \mathcal{L}^{(1)}) f_t^{\gamma}. \tag{1.3}
\]

Looking for a solution of this problem of the form

\[
f_t^{\gamma} = f_t^{(0)} + \frac{1}{\gamma} f_t^{(1)} + O\left(\frac{1}{\gamma^2}\right), \tag{1.4}
\]

\(^1\)Alternatively, if $\varepsilon = \gamma^{-1}$, the aim is to understand in the long time scale $t\varepsilon^{-1}$ the behavior of the process generated by $\mathcal{L}^{(1)} + \varepsilon \mathcal{L}^{(0)}$, i.e. of the process which is a small perturbation of the process generated by $\mathcal{L}^{(1)}$.\]
injecting (1.4) in (1.3) and equating coefficients in powers of \( \frac{1}{\gamma} \) we get that for any \( t > 0 \):

\[
\mathcal{L}^{(1)} f_t^{(0)} = 0, \tag{1.5}
\]

\[
\mathcal{L}^{(1)} f_t^{(1)} = \frac{\partial}{\partial t} f_t^{(0)} - \mathcal{L}^{(0)} f_t^{(0)} \tag{1.6}
\]

Equation (1.5) gives us that \( f_t^{(0)} \) is in the kernel of \( \mathcal{L}^{(1)} \), so eventually

\[
f_t^{(0)} = \mathcal{P} f_t^{(0)} \tag{1.7}
\]

by the definition of \( \mathcal{P} \). We have also \( \mathcal{P} \mathcal{L}^{(1)} = 0 \) by the definition of \( \mathcal{P} \), so applying \( \mathcal{P} \) on the left on (1.6), we get, using (1.7) and \( \mathcal{P}^2 = \mathcal{P} \), that:

\[
0 = \mathcal{P} \mathcal{L}^{(1)} f_t^{(1)} = \mathcal{P} \frac{\partial}{\partial t} f_t^{(0)} - \mathcal{P} \mathcal{L}^{(0)} f_t^{(0)} = \frac{\partial}{\partial t} \mathcal{P} f_t^{(0)} - (\mathcal{P} \mathcal{L}^{(0)} \mathcal{P}) \mathcal{P} f_t^{(0)}.
\]

We thus have, using \( f_0^{(0)} = f \), that:

\[
f_t^{(0)} = \mathcal{P} f_t^{(0)} = e^{t \mathcal{L}^{(0)}} \mathcal{P} f = \mathcal{P} e^{t \mathcal{L}^{(0)}} \mathcal{P} f.
\]

However, the proof is only formal, and it is only in some specific situations that a rigorous proof of the ‘meta-theorem’ can be given. In the literature, this kind of theorem is sometimes called an ‘averaging principle’ and has been developed not only in the context of diffusion processes [3, 4, 6, 8, 9, 10, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44] but also, and historically first, for dynamical systems [18, 45, 46, 47, 48].

For instance, the averaging problem has been widely studied for solutions \( X^\gamma = (y^\gamma, z^\gamma) \in \mathbb{T}^{d-\ell} \times \mathbb{T}^\ell \) of a stochastic differential equation (SDE) taking the form:

\[
\begin{cases}
\displaystyle \frac{dy_i^\gamma}{dt} = \gamma b^y (z_i^\gamma, y_i^\gamma) \ dt + \sqrt{\gamma} \sigma^y (z_i^\gamma, y_i^\gamma) \ dW_t, \\
\displaystyle \frac{dz_i^\gamma}{dt} = b^z (z_i^\gamma, y_i^\gamma) \ dt + \sigma^z (z_i^\gamma, y_i^\gamma) \ dB_t
\end{cases} \tag{1.8}
\]

where all the functions \( b^y, b^z, \sigma^y, \sigma^z \) involved are smooth and \( (B_t)_{t \geq 0} \) and \( (W_t)_{t \geq 0} \) are independent Brownian motions. Here, the generator of the coupled process \( X^\gamma \) is in the form (1.1) with the subdominant generator given by:

\[
\mathcal{L}^{(0)} = \langle b^z, \nabla_z \rangle + \frac{1}{2} (\sigma^z (\sigma^z)^T \nabla_z, \nabla_z)
\]

and the dominant generator driven by:

\[
\mathcal{L}^{(1)} = \langle b^y, \nabla_y \rangle + \frac{1}{2} (\sigma^y (\sigma^y)^T \nabla_y, \nabla_y).
\]
Observe that in (1.8) there is a clear separation of time scales so that \( z^\gamma \) (resp. \( y^\gamma \)) can be identified to a slow (resp. fast) component of the coupled process \( X^\gamma \). Notice also that the first equation in (1.8), when we fix \( z^\gamma_t = \xi \), define an SDE on \( \mathbb{T}^{d-\ell} \) of invariant distribution \( \rho^\infty(dy, \xi) \).

When \( \gamma \) is big enough, for any \( t > 0 \), and for a small amount of time \( dt \), in a first approximation, \( z^\gamma_s \) is almost equal to \( z^\gamma_t = \xi \) for \( s \in [t, t+dt] \). Hence, during this time interval, \( (y^\gamma_s)_{s \in [t, t+dt]} \) is roughly equal in law to the scaled process \( (\tilde{y}^\gamma_s)_{s \in [t, t+dt]} \) where \( \tilde{y} \) is the process on \( \mathbb{T}^{d-\ell} \) with generator \( L^{(1)} \) in which the variable \( z \) has been frozen to the value \( \xi \) during this amount of time \( dt \). Therefore, for \( \gamma \) large, by the ergodic theorem, the density of \( y^\gamma_s \) is almost constant equal to \( \rho^\infty(y, \xi)dy \) during this time interval. It follows that in a second approximation \((z^\gamma_t)_{t \geq 0}\) is equal in law to \((z_t)_{t \geq 0}\), the solution of

\[
\frac{dz_t}{dt} = F(z_t)dt + A(z_t)dB_t,
\]

with

\[
\begin{cases}
F(\xi) &= \int_{\mathbb{T}^{d-\ell}} b^z(y, \xi) \rho^\infty(dy, \xi), \\
A(\xi)A(\xi)^T &= \int_{\mathbb{T}^{d-\ell}} \sigma^z(y, \xi)(\sigma^z(y, \xi))^T \rho^\infty(dy, \xi).
\end{cases}
\]

The generator of this autonomous process living in \( \mathbb{T}^\ell \) is:

\[ \mathcal{L} = \langle F, \nabla_z \rangle + \frac{1}{2} \langle AA^T \nabla_z, \nabla_z \rangle, \]

and therefore the law of \( z^\gamma_t \) is close to the probability measure \( \mu^\gamma_t = \delta_z e^{t\mathcal{L}} \) for large \( \gamma \). This argument can be made rigorous and it is proved historically in [29, 30, 31, 32], under some assumptions, that \((z^\gamma_t)_{0 \leq t \leq T}\) converges weakly to \((z_t)_{0 \leq t \leq T}\) in \( C([0, T], \mathbb{T}^\ell) \) when \( \gamma \) tends to infinity for all \( T > 0 \).

We now explain how to connect this result with the ‘meta-theorem’ stated at the beginning of the paper. The result just above implies that for any smooth function \( f : (y, z) \in \mathbb{T}^d = \mathbb{T}^{d-\ell} \times \mathbb{T}^\ell \to f(y, z) \in \mathbb{R} \) we have that

\[ \lim_{\gamma \to \infty} \mathbb{E}_{(y, z)}(f(y^\gamma_t, z^\gamma_t)) = \int_{\mathbb{T}^\ell} \left[ \int_{\mathbb{T}^{d-\ell}} f(y', z') \rho^\infty(dy', z') \right] d\mu^\gamma_t(z'). \]

By the definition of \( \mathcal{P} \) we have that:

\[ (\mathcal{P}f)(y, z) = \int_{\mathbb{T}^{d-\ell}} f(y', z) \rho^\infty(dy', z), \]
that is independent of \( y \). Hence, denoting \( \bar{f}(z) = (Pf)(y, z) \), we get
\[
\lim_{\gamma \to \infty} e^{t\mathcal{L}_\gamma} f(y, z) = \lim_{\gamma \to \infty} \mathbb{E}_{(y,z)}(f(y', z')) = \int_{\mathbb{T}} \bar{f}(z') \, d\mu^*_\gamma(z').
\]
Recalling that \( \mu^*_\gamma = \delta_{z} e^{t\mathcal{L}_\gamma} \), we obtain by formal integration by parts:
\[
\lim_{\gamma \to \infty} e^{t\mathcal{L}_\gamma} f(y, z) = e^{t\mathcal{L}} \bar{f}(z)
\]
On the other hand a trivial computation shows that:
\[
\mathcal{L}_\infty f = \mathcal{P} \mathcal{L}^{(0)} \mathcal{P} f = \mathcal{P} \mathcal{L}^{(0)} \bar{f} = \mathcal{L} \bar{f}.
\]
This proves the ‘meta-theorem’ (1.2) for this particular system.

A fundamental remark is that in this seminal example, the kernel of the dominant process \( \mathcal{L}^{(1)} \) is composed of the functions constant in the second (slow) variable, i.e. depending only of the first (fast) variable \( y \). The kernel of \( \mathcal{L}^{(1)} \) is hence infinite-dimensional. A natural question motivating this paper is then: what happens for the ‘meta-theorem’ for diffusion processes if the kernel of \( \mathcal{L}^{(1)} \) is finite dimensional generated by a basis \( \{f_1, \ldots, f_p\} \)? Our conjecture is that in general, the limiting semi-group \( e^{t\mathcal{L}_\infty} \) appearing in the ‘meta-theorem’, will be associated to a pure jump continuous time Markov process living in the space of the basis \( \{f_1, \ldots, f_p\} \). As far as we know this question has not been addressed in the literature. However, very recently, in [11], the authors show that a certain class of diffusions, related to quantum continuous measurements [49, 50, 51, 52], and having a generator in the form (1.1) with a finite-dimensional kernel for \( \mathcal{L}^{(1)} \), converge in the large \( \gamma \) limit to a pure jump continuous time Markov process on a finite space. To prove their theorem the authors develop a finite-dimensional homogenization theorem [11, Theorem 3.1] for bounded operators and, due to the specific form of the diffusion processes considered (linear drift and quadratic mobility), are able to prove the convergence by developing a tricky perturbative argument. This homogenization theorem is very similar, at least in its simplified version, to the ‘meta-theorem’ for matrices. A more natural approach, but technically much more involved, would have been in [11] to use directly their homogenization theorem for the infinitesimal generators of the diffusion processes, i.e. for unbounded linear operators (as it is done without mathematical rigor in [53]). Unfortunately the proof of [11, Theorem 3.1] seem to be difficult.

\(^2\)Of course the formulation of the question is a bit messy since we do not precise the functional spaces considered.

\(^3\)In fact in [11] a more general situation is considered with a dominant, an intermediate and a subdominant generators.
to extend for unbounded operators.

The aim of this work is to provide a first step in this direction by adapting the proof of [1, Theorem 3.1] for some (unbounded) generators of diffusion operators. In order that the kernel of $\mathcal{L}^{(1)}$ is finite-dimensional, specific properties of the corresponding diffusion process have to be imposed. We will consider diffusions living in a simplex $K \subset \mathbb{R}^n$. Moreover, we will only investigate the case where the dominant generator $\mathcal{L}^{(1)}$ does not contain any drift term, i.e. is pure noise, and such that the kernel of $\mathcal{L}^{(1)}$ is finite-dimensional. To satisfy the later condition we will assume that the volatility in $\mathcal{L}^{(1)}$ is non-negative and vanishes exactly on a finite subset $K_0$ of $K$. Extending the assumptions done here would be interesting but up to now we have not been able to do it. In this article we show that under a geometric assumption on $K_0$, the convergence of the diffusion processes on the simplex $K$ to a pure jump continuous-time Markov process on $K_0$ holds. We also give a counterexample of the theorem if this geometrical hypothesis is not verified, and thus prove the optimality of our conditions.

The paper is structured as follows. In Section 2 we define and state precisely our main results and we prove them in Section 3. Section 4 is devoted to the counter-example showing in some sense the optimality of our geometric assumption on $K_0$. An appendix where a uniform ergodic theorem for martingales is proved concludes the paper.

**Notations**

Let $n \geq 1$ be an integer. The set of real-valued $n \times n$-matrix is denoted by $\mathcal{M}_n(\mathbb{R})$ and the components of a matrix $A \in \mathcal{M}_n(\mathbb{R})$ are denoted by $A_{i,j}$, $i,j \in \{1,\ldots,n\}$. In particular the coordinates of $x \in \mathcal{M}_{n,1}(\mathbb{R}) \approx \mathbb{R}^n$ are denoted by $x_i$, $i \in \{1,\ldots,n\}$. If $A = (A_{i,j})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$, the trace of $A$ is denoted by $\text{tr}(A) = \sum_{k=1}^n A_{k,k}$. For a given matrix $A$ of $\mathcal{M}_n(\mathbb{R})$, the complex conjugate of $A$ is written as $A^\dagger$. We also denote $a \cdot b = \sum_{i=1}^n a_i b_i$ the standard scalar product of $a$ and $b$ in $\mathbb{R}^n$ and by $\| \cdot \|_2$ the associated norm. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $B(x,\varepsilon)$ denotes the ball of center $x$ and radius $\varepsilon$ for the norm $\| \cdot \|_2$.

The supremum norm of a real valued function $f$ defined on a set $K \subset \mathbb{R}^n$ is denoted by $\|f\|_\infty = \sup_{x \in K} |f(x)|$. The set $\mathcal{D}(\mathbb{R}_+,K)$ is the space of càdlàg functions from $\mathbb{R}_+$ to $K \subset \mathbb{R}^n$. If $f \in C^1(\mathbb{R}_+)$, we write $\nabla_x f = (\partial_i f)_{1 \leq i \leq n}$ the gradient of $f$ and if furthermore $f \in C^2(\mathbb{R}^n)$, we
write \( H_f = (\partial^2_{i,j} f)_{1 \leq i, j \leq n} \) the Hessian of \( f \). For a vector field \( b : x \to \mathbb{R}^n \to b(x) = (b_1(x), \ldots, b_n(x)) \in \mathbb{R}^n \), we write \( \langle b, \nabla_x \rangle = \sum_{i=1}^n b_i(x) \partial_i \).

For a matrix \( A \) of \( \mathcal{M}_n(\mathbb{R}) \), one writes \( \langle A \nabla_x, \nabla_x \rangle = \sum_{i,j=1}^n A_{i,j} \partial^2_{i,j} \).

For a probability distribution \( \mu \) on \( \mathbb{R}^n \) and a real-valued bounded function \( f \) on \( \mathbb{R}^n \), we write \( \langle \mu, f \rangle = \int f \, d\mu \). For a given differential operator \( L \) (in particular for the infinitesimal generator of a Markovian process), we denote by \( t \mapsto e^{tL} \) the associated semigroup. Hence, for a given probability measure \( \mu \), an infinitesimal generator \( L \) of a Markov process and a time \( t \geq 0 \), we write \( \mu e^{tL} \) the law of the process of generator \( L \) at time \( t \).

### 2. Main result

From now on \( K \) is a simplex of \( \mathbb{R}^n \).

#### 2.1. Definitions

Let \( \gamma > 0 \), we consider the diffusion processes \( (X^\gamma_t)_{t \geq 0} := (X^\gamma_t(x))_{t \geq 0} \) solution of the following SDEs on \( K \) with initial condition \( x \in K \):

\[
dX^\gamma_t = \sqrt{\gamma} \sigma(X^\gamma_t) dW_t + b(X^\gamma_t) dt + \sigma_0(X^\gamma_t) dB_t, \quad X^\gamma_0(x) = x \in K \tag{2.1}
\]

where \( \sigma, \sigma_0 : K \to \mathcal{M}_n(\mathbb{R}) \) and \( b : K \to \mathbb{R}^n \) are Lipschitz functions on \( K \) and \( (W_t)_{t \geq 0} \) and \( (B_t)_{t \geq 0} \) are two independent Wiener processes on \( \mathbb{R}^n \).

**Remark 1.** All the results proved in this article would still hold if we considered the diffusions processes solution of:

\[dX^\gamma_t = \sqrt{\gamma} \sigma(X^\gamma_t) dW_t + b\gamma(X^\gamma_t) dt + \sigma_0(x) dB_t, \quad X^\gamma_0(x) = x \in K.
\]

where:

\[
\begin{align*}
\lim_{\gamma \to +\infty} \sigma^\gamma &= \sigma_0 \\
\lim_{\gamma \to +\infty} b^\gamma &= b.
\end{align*}
\]

However, in order not to overcharge the proofs with purely technical difficulties, we will from now on only consider (2.1).

We assume (see Remark 1) that \( \sigma, \sigma_0 \) and \( b \) are such that \( X^\gamma_t(x) \in K \) for any \( t \geq 0 \) and any \( \gamma \geq 0 \) and thus this is also true for \( (X_t)_{t \geq 0} := (X_t(x))_{t \geq 0} \) the solution of the following SDE on \( K \) with initial condition \( x \):

\[
dx_t = \sigma(X_t) dW_t, \quad X_0(x) = x. \tag{2.2}
\]
We furthermore assume for the rest of the article that \( \sigma \) is null only for a finite number of points and we denote \( K_0 = \{ x \in K, \; \sigma(x) = 0 \} \).

The infinitesimal generator of the Markov process generated by (2.1) is
\[
\mathcal{L}_\gamma = \gamma \mathcal{L}^{(1)} + \mathcal{L}^{(0)}
\]
where
\[
\mathcal{L}^{(0)} = \langle b, \nabla x \rangle + \frac{1}{2} \sigma_0^\dagger \sigma_0 \langle \nabla x, \nabla x \rangle \quad \text{and} \quad \mathcal{L}^{(1)} = \frac{1}{2} \langle \sigma \sigma^\dagger \nabla x, \nabla x \rangle
\]
are respectively the generators of the subdominant and dominant processes.

**Remark 2.** Since \( K \) is compact and \( b, \sigma \) and \( \sigma_0 \) are Lipschitz, we have that these functions are bounded on \( K \).

**Remark 3.** In view of the discussion in the introduction, equation (2.1) is not completely generic, since we assumed that the dominant process is pure noise, i.e. does not have any drift term. This implies in particular that the dominant process is a martingale.

**Remark 4.** The fact that for any \( x \in K \) one has \( X_t^\gamma(x) \in K \) and \( X_t(x) \in K \) for arbitrary times implies several constraints on \( \sigma, \sigma_0 \) and \( b \) on the boundary of \( K \). More precisely the process stays in \( K \) whatever the starting point is if and only if for any point \( x \) of a side of the simplex, \( \sigma(x) \) and \( \sigma_0(x) \) are parallel to this side and the vector \( b(x) \) is null or points to the interior of \( K \). We have thus that \( \sigma \) and \( \sigma_0 \) are null on the vertices of \( K \) since they are there parallel to two different sides.

**Remark 5.** We may ask ourselves if the results proved in this paper would still hold if we only assumed that \( K \) is a compact convex set of \( \mathbb{R}^n \). However, problems arise quickly if there is a point \( x \) of the boundary of \( K \) not in \( K_0 \) where the curvature is strictly positive. Indeed, the dominant process starting from \( x \) will then escape \( K \) with a strictly positive probability whatever the value of \( \sigma(x) \). Furthermore, even when we add a non-null drift pointing to the interior everywhere on the boundary, preventing our process from escaping \( K \), we observe that \( X^\gamma \) does not always converges in law to a process living on \( K_0 \) when \( \gamma \) approaches infinity: the limit process can for instance live in the entire boundary of \( K \). To force the limit process to live on \( K_0 \) when \( K \) is not a polytope, we have to introduce a dominant drift of a particular form, which creates new conceptual and technical difficulties we will not deal with in this article.

Before studying \( (X_t^\gamma)_{t \geq 0} \) we will first consider the dominant process \( (X_t)_{t \geq 0} \). In Theorem 1 it is proved that for all \( x \), the process \( (X_t(x))_{t \geq 0} \)
converges almost surely to a random variable $X_\infty := X_\infty(x) \in K_0$ as $t$ goes to $+\infty$. We may thus consider for any $z \in K_0$ the function:

$$H_z : x \in K \mapsto \mathbb{P}(X_\infty(x) = z) \in [0,1].$$

(2.3)

that gives the probability that $(X_t(x))_{t \geq 0}$ converges to $z$.

Now that all our objects are well defined we may state a version of the ergodic theorem for $(X_t)_{t \geq 0}$. The uniformity proved in this theorem is fundamental for the derivation of the main theorem.

**Theorem 1.** (Uniform Ergodic Theorem)

Assume that for any $z \in K_0$, the function $H_z$ is continuous. Then, for any Lipschitz function $f : K \rightarrow \mathbb{R}$, we have that:

$$e^{tc(t)} f \xrightarrow{t \rightarrow +\infty} \mathcal{P} f,$$

where the convergence is uniform in $x$ and the projector $\mathcal{P}$ is defined by:

$$\mathcal{P} f(x) = \sum_{z \in K_0} H_z(x) f(z).$$

(2.4)

**Proof.** The proof of this theorem is postponed to the Appendix. □

**Remark 6** (A crucial example). The continuity hypothesis in Theorem 1 is for example fulfilled in a specific case that is the one we are interested in this article.

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**Figure 1.** A polytope of $\mathbb{R}^n$ has at least $n + 1$ vertices.

Let us assume that the set $K_0$ is composed of $n + 1$ points of $\mathbb{R}^n$ that are affinely independent, that is to say that $K_0$ is not included in an affine hyperplane of $\mathbb{R}^n$. Then the functions $H_z$ for $z \in K_0$ are affine
functions. Indeed, since we have $n + 1$ affinely independent points on a $n$-dimensional vector space, we get that for any $z \in K_0$, there exists an affine function $f_z$ such that for all $y \in K_0$:

$$f_z(y) = \begin{cases} 
0 & \text{if } y \neq z, \\
1 & \text{if } y = z.
\end{cases}$$

Since $(X_t(x))_{t \geq 0}$ is a martingale living in a bounded space, it converges almost surely to a random variable $X_\infty(x)$ taking its values in $K_0$ (see the proof of the uniform ergodic theorem in the Appendix for more details). Thus for any $x \in K$:

$$x = \mathbb{E}(X_0(x)) = \mathbb{E}(X_\infty(x)) = \sum_{y \in K_0} \mathbb{P}_x(X_\infty = y)y.$$

Let $z \in K_0$, applying $f_z$ to last equation and using that $f_z$ is affine we get:

$$f_z(x) = f_z \left( \sum_{y \in K_0} \mathbb{P}_x(X_\infty = y)y \right) = \sum_{y \in K_0} \mathbb{P}_x(X_\infty = y)f_z(y) = \mathbb{P}_x(X_\infty = z) = H_z(x).$$

Furthermore, if we suppose that $K_0$ contains $n + 1$ affinely independent points in a $n$-dimension space, we know actually that the points of $K_0$ are exactly the extremal points of the simplex $K$ since the noise is necessarily null on them (see Remark 4), and a polytope of $\mathbb{R}^n$ has at least $n + 1$ extremal points.

2.2. Statements. Now that all the objects of the problem are now well defined, we may state our first main theorem.

**Theorem 2.** We assume that the cardinal of $K_0$ is equal to $n + 1$ and the points of this set are affinely independent. Then for any Lipschitz function $f : K \to \mathbb{R}$ and any probability measure $\mu$ on $K$, we have for any $t > 0$:

$$\mathbb{E}_\mu (f (X_t^\gamma)) \xrightarrow{\gamma \to +\infty} \mathbb{E}_\pi (\bar{f} (X_t)), $$

where $\bar{f}$ is the restriction of $f$ to $K_0$ and $(X_t^\gamma)_{t \geq 0}$ is the pure jump continuous time Markov process of the finite state space $K_0$ with initial distribution $\pi$ and with generator $\mathcal{L}$ given by

$$\mathcal{L} = (b(x) \cdot \nabla_x H_z(x))_{x, z \in K_0}, \quad \pi(z) = \int_K H_z(x)d\mu(x) \text{ for } z \in K_0.$$
Remark 7. Using the definition of \( \mathcal{P} \) in (2.4), we have for any Lipschitz function \( f : K \to \mathbb{R} \) that \( \mathcal{P}f \) is the unique affine function on \( K \) equal to \( f \) on \( K_0 \). Thus, for any affine function \( g \), we have that \( \mathcal{L}_\infty g = \mathcal{P} \mathcal{L}^{(0)} \mathcal{P} g \) is affine and we can prove (see the proof of Theorem 2) that for \( z \in K_0 \):
\[
\mathcal{L}_\infty g(z) = \mathcal{L}_\overline{\mathcal{P}} g(z).
\]
Eventually for any Lipschitz function \( f \) and any \( x \in K \):
\[
\mathcal{P} e^{t \mathcal{L} \mathcal{P}} f(x) = \sum_{z \in K_0} H_z(x) e^{t \mathcal{L} \mathcal{P}} f(z) = \sum_{z \in K_0} H_z(x) e^{t \mathcal{L} \mathcal{P}} f(z) = \sum_{z \in K_0} H_z(x) e^{t \mathcal{L} f(z)},
\]
so Theorem 2 is a reformulation of the ‘meta-theorem’ (1.2).

Remark 8. The geometrical hypothesis on \( K_0 \) may seem restrictive, but it is not clear what should be a more general statement. We will give in the last section a counter example of the theorem where \( K \subset \mathbb{R}^2 \) and \( |K_0| = 4 \).

Theorem 2 provides the convergence of the semigroup \( e^{t \mathcal{L}_\gamma} \) to the semigroup \( e^{t \mathcal{L}} \), that is to say the pointwise convergence in law of \( (X_\gamma^t)_{t \geq 0} \) to \( (\overline{X}_t)_{t \geq 0} \). It does not say anything about the convergence of \( (X_\gamma^t)_{t \geq 0} \) to \( (\overline{X}_t)_{t \geq 0} \) at the path level. One has for every \( \gamma > 0 \) that the paths of both processes \( (X_\gamma^t)_{t \geq 0} \) and \( (\overline{X}_t)_{t \geq 0} \) belong to \( \mathcal{D}(\mathbb{R}_+, K) \). The natural topology on this space is the Skorokhod one, but we cannot in fact expect a weak convergence for this topology. Indeed as underlined in [54, Theorem 13.4], weak convergence of processes with continuous paths in the Skorokhod topology yields a limiting process with continuous paths, and, while for \( \gamma > 0 \), \( (X_\gamma^t)_{t \geq 0} \) has continuous paths almost surely, we have that \( (\overline{X}_t)_{t \geq 0} \) does have discontinuous paths almost surely if \( \gamma > 0 \) is non-null on every vertex. To overcome this difficulty, we use the idea of [1] which is to replace the Skorokhod topology by the so-called Meyer–Zheng topology.

Let us define the Meyer–Zheng topology:

**Definition.** Consider a Euclidean space \((E, \| \cdot \|)\) and denote by \( L^0 = L^0(\mathbb{R}_+, E) \) the space of \( E \)-valued Borel functions on \( \mathbb{R}_+ \). Given a sequence \( \{w^\gamma, \gamma > 0\} \) of elements of \( L^0(\mathbb{R}_+, E) \), the following assertions are equivalent and define the convergence in Meyer–Zheng topology of \( \{w^\gamma, \gamma > 0\} \) to \( w \in L^0(\mathbb{R}_+, E) \):
• For all bounded continuous functions \( f : \mathbb{R}_+ \times E \to \mathbb{R} \):
\[
\lim_{\gamma \to +\infty} \int_0^{+\infty} f(t, w_t^\gamma) e^{-t} dt = \int_0^{+\infty} f(t, w_t) e^{-t} dt.
\]

• For \( \lambda(dt) = e^{-t} dt \), we have for all \( \varepsilon > 0 \):
\[
\lim_{\gamma \to +\infty} \lambda(\{ s \in \mathbb{R}_+ \| \| w_s^\gamma - w_s \| \geq \varepsilon \}) = 0.
\]

• \( \lim_{\gamma \to +\infty} d(w^\gamma, w) = 0 \) where \( d \) is defined by:
\[
d(w, w') = \int_0^{+\infty} \{ 1 \wedge \| w_t - w'_t \| \} e^{-t} dt.
\]

The distance \( d \) metrizes the Meyer–Zheng topology on \( \mathbb{L}^0 \) and \( (\mathbb{L}^0, d) \) is a Polish space.

We may now formulate our second main theorem:

**Theorem 3.** Under the geometrical hypothesis of Theorem 2, we have that:
\[
\lim_{\gamma \to +\infty} X^\gamma = \overline{X}, \quad \text{weakly in } (\mathbb{L}^0(\mathbb{R}_+, K), d)
\]
where \( (X_t)_{t \geq 0} \) is the continuous-time pure jump Markov process on \( K_0 \) defined in Theorem 2. In other words, for all bounded continuous function \( F : (\mathbb{L}^0(\mathbb{R}, K), d) \to (\mathbb{R}, |\cdot|) \), one has:
\[
\mathbb{E}(F(X^\gamma)) \longrightarrow_{\gamma \to +\infty} \mathbb{E}(F(\overline{X})).
\]

3. **Proof of Theorem 2 and Theorem 3**

3.1. **Proof of Theorem 2.** Let us start with a quick sketch of the proof of Theorem 2.

We will first prove Lemma 1, a technical result that essentially says that for \( t \) small and \( \gamma \) big, our process of generator \( \mathcal{L}_\gamma \) is basically the same as the one of generator \( \mathcal{L}^{(1)} \). The keystone of our proof is however Corollary 1 that is proved using the uniform ergodic theorem combined with Lemma 1. Corollary 1 says that for a given \( t \) strictly positive, \( X^\gamma_t \) de facto lives in small balls centered around the \( z \in K_0 \) when \( \gamma \) is big enough. Therefore, when \( \gamma \) is big enough, it is like \( (X^\gamma_t)_{t \geq 0} \) is jumping from a ball to another.

Corollary 1 is true under the geometrical hypothesis of Theorem 2 since it is based on the ergodic theorem which is only proved true under it (thanks to Remark 2). However if this hypothesis is sufficient, it is not necessary: Corollary 1 would still hold if we only knew that the
functions \( H_z \) defined by (2.3) are continuous.

Nevertheless, if we know that there is a jump process on \( K_0 \), we a priori do not know its form. Theorem 2 says that, for a specific geometry, our process converges "weakly" to a continuous-time jump process with state space \( K_0 \) and generator \( \overline{L} \).

To prove that, we will first simplify our problem by approximating the drift \( b \) by an affine drift \( \tilde{b} \) such that \( b|_{K_0} = \tilde{b}|_{K_0} \). This approximation is justified by using Corollary 1. For the stochastic process \( \tilde{X}_t^\gamma \) associated to the affine drift, the proof of our theorem for an affine test function \( g \) but for an arbitrary initial condition \( \mu \) is straightforward since we have thus that \( L_\gamma g = \mathcal{L}^{(0)} g \) is an affine function and \( \mathcal{P} \) preserves affine functions. Hence we deduce the theorem for affine test functions even when the drift is not affine.

Then, using Corollary 1 again, we know that \( (X_t^\gamma)_{t \geq 0} \) de facto lives in a neighborhood of \( K_0 \). Thus, instead of considering a generic test function \( f \), we will consider an affine approximation \( g \) of \( f \) such that \( f|_{K_0} = g|_{K_0} \). The function \( g \) exists thanks to our geometrical hypothesis.

Eventually, since the theorem is true for \( g \) and that one cannot distinguish \( f(X_t^\gamma) \) from \( g(X_t^\gamma) \) when \( \gamma \) is big enough, it will still holds for \( f \).

**Lemma 1.** Let \( f \) be a smooth function on \( K \) and \( \mu \) be a probability measure on \( K \). There exist \( C_1 \) and \( C_2 \) two positive constants independent of \( \mu \) such that for any \( t, \gamma, h > 0 \), we have:

\[
| \langle \mu, e^{hL_\gamma} f \rangle - \langle \mu, e^{\gamma\mathcal{L}(1)} f \rangle | \leq C_1 (h + h^2) \frac{1}{2} e^{C_2 \gamma t}.
\]

**Proof.**

It is sufficient to prove that the inequality is fulfilled for \( \mu = \delta_x \) for all \( x \in K \).

Let \( x \) be an arbitrary point of \( K \) and recall that \( X_t^\gamma := X_t^\gamma(x) \) be the solution of

\[
dX_t^\gamma = b(X_t^\gamma)dt + \sigma_0(X_t^\gamma)dW_t + \sqrt{\gamma \over h} \sigma(X_t^\gamma)dB_t,
\]

with initial condition \( x \).
We now fix $\gamma$.

We couple the process $(Z_t^h)_{t\geq 0} = (Z_t^h(x))_{t\geq 0}$ solution of:

$$Z_t^h = x + \frac{h}{\gamma} \int_0^t b(Z_u^h)du + \sqrt{\frac{h}{\gamma}} \int_0^t \sigma_0(Z_u^h)dB_u + \int_0^t \sigma(Z_u^h)dB_u,$$

with the process $(Z_t)_{t\geq 0} := (Z_t(x))_{t\geq 0}$ solution of:

$$Z_t = x + \int_0^t \sigma(Z_u)dB_u,$$

through the common Brownian motion $(B_t)_t$ is the same in all these processes. The generator of $(Z_t^h)_{t\geq 0}$ is

$$\frac{h}{\gamma} (b(x), \nabla_x) + \frac{1}{2} \frac{h}{\gamma} \langle \sigma_0 \sigma_0^\dagger \nabla_x, \nabla_x \rangle + \frac{1}{2} \langle \sigma \sigma^\dagger \nabla_x, \nabla_x \rangle,$$

that is also the generator of $(X_{\frac{\gamma}{h}t}^{\frac{\gamma}{h}t})_{t\geq 0}$ by scaling invariance of the Brownian motion. Since they share the same initial distribution, we have for all $h > 0$ that:

$$\left( X_{\frac{\gamma}{h}t}^{\frac{\gamma}{h}t} \right)_{t\geq 0} \overset{\text{d}}{=} (Z_t^h)_{t\geq 0},$$

which implies for $t = \gamma$ that:

$$X_{h}^{\gamma} \overset{\text{d}}{=} Z^{h}_{\gamma}.$$
Thus using Itô isometry, the fact that \( b \) and \( \sigma_0 \) are bounded by a constant \( M \) and that \( \sigma \) is \( k \)-Lipschitz, we have that:

\[
\mathbb{E} \left( \| Z^h_\gamma - Z_\gamma \|^2 \right) \\
\leq \mathbb{E} \left( \left\| \int_0^\gamma (\sigma(Z^h_u) - \sigma(Z_u))dB_u + \frac{\epsilon}{\gamma} \int_0^\gamma \sigma_0(Z^h_u)dB_u \right\|^2 \right) \\
+ \frac{b}{\gamma} \int_0^\gamma b(Z^h_u)du \|
\leq 3\mathbb{E} \left( \left\| \int_0^\gamma (\sigma(Z^h_u) - \sigma(Z_u))dB_u \right\|^2 \right) + 3\mathbb{E} \left( \left\| \frac{b}{\gamma} \int_0^\gamma \sigma_0(Z^h_u)dB_u \right\|^2 \right) \\
+ 3\mathbb{E} \left( \left\| \frac{\sigma_0(Z^h_u)}{\gamma} \right\| \right) \\
\leq 3\mathbb{E} \left( \left\| \int_0^\gamma (\sigma(Z^h_u) - \sigma(Z_u))du \right\|^2 \right) + 3M^2\gamma^2\frac{h^2}{\gamma^2} \\
+ 3\frac{h^2}{\gamma^2} \mathbb{E} \left( \left\| \sigma_0(Z^h_u) \right\| \right) \\
\leq 3k^2\mathbb{E} \left( \left\| Z^h_u - Z_u \right\|^2 \right) + 3M^2(h + h^2).
\]

From this point, Grönwall’s inequality gives us that for all \( h > 0 \) we have:

\[
\mathbb{E} \left( \| Z^h_\gamma - Z_\gamma \|^2 \right) \leq 3M^2(h + h^2)e^{3k^2\gamma}.
\]

Thus, for \( f \) a smooth function, that is therefore \( L \)-Lipschitz on \( K \), one has:

\[
\left| e^{h\mathcal{L}^{(1)}\gamma} f(x) - e^{\gamma\mathcal{L}^{(1)}\gamma} f(x) \right|
\leq \mathbb{E} \left( \left| f(X^\gamma_h) \right| - \mathbb{E}(f(Z_\gamma)) \right) \\
\leq L \mathbb{E} \left( \left\| Z^h_\gamma - Z_\gamma \right\|_2 \right) \\
\leq L \mathbb{E} \left( \left\| Z^h_\gamma - Z_\gamma \right\|_2 \right)^{\frac{1}{2}} \quad \text{Using Cauchy-Schwarz} \\
\leq \sqrt{3LM} \left( h + h^2 \right)^{\frac{1}{2}} e^{\frac{3}{2}k^2\gamma}.
\]

Since the constants are independent of \( x \) and \( \gamma \), this proves the theorem for \( C_1 = \sqrt{3LM} \) and \( C_2 = \frac{3}{2}k^2 \).

Combining this result with the uniform ergodic theorem (Theorem 1) gives us this crucial result:
Corollary 1. For any $t > 0$, $\eta > 0$ and $\mu$ probability measure on $K$, we have that:

$$\mathbb{P}_\mu \left( X_t^\gamma \in \bigcup_{z \in K_0} B(z, \eta) \right) \xrightarrow{\gamma \to +\infty} 1.$$ 

uniformly in $\mu$.

Proof.
We consider a smooth function $f : K \to [0, 1]$ such that:

- $f = 1$ on $\bigcup_{z \in K_0} B(z, \frac{\eta}{2}) \cap K$.
- $f = 0$ outside of $\bigcup_{z \in K_0} B(z, \eta)$.

Then we have:

$$\langle \mu, e^{tL^\gamma} f \rangle \leq \mathbb{P}_\mu \left( X_t^\gamma \in \bigcup_{z \in K_0} B(z, \eta) \right),$$

and thus it is sufficient to prove that the term on the left hand side of the last display converges to 1.

For a given positive $\gamma$ and a strictly positive $t$, we consider $\beta := \beta(\gamma, t), h := h(\gamma, t)$ such that:

$$\gamma = C_1 \beta^2 e^{2C_2 \beta t} \quad \text{and} \quad h = \frac{1}{C_1 \beta e^{2C_2 \beta t}}.$$

Thus one has that $\gamma = \frac{\beta}{h}$, that $\beta \xrightarrow{\gamma \to +\infty} +\infty$ and that $h \xrightarrow{\gamma \to +\infty} 0$.

We recall (2.3), i.e. that for all $z \in K_0$ and $x \in K$, we have $H_z(x) = \mathbb{P}(X_\infty(x) = z)$, and therefore $\sum_{z \in K_0} H_z = 1$. We also recall (2.4), i.e. that for $x \in K$, we have $\mathcal{P} f(x) = \sum_{z \in K_0} f(z) H_z(x)$. Eventually, since $f$ is constant equal to 1 on $K_0$, we have that:

$$\langle \mu, \mathcal{P} f \rangle = \sum_{z \in K_0} f(z) \langle \mu, \mathcal{P} f \rangle$$

$$= \sum_{z \in K_0} \langle \mu, H_z \rangle$$

$$= \langle \mu, \sum_{z \in K_0} H_z \rangle$$

$$= \langle \mu, 1 \rangle$$

$$= 1,$$
for any \( \mu \) probability measure on \( K \). Thus one has \( \langle \mu e^{(t-h)\frac{L_\beta}{\pi}}, Pf \rangle = 1 \) for all \( \gamma \) so using Markov property:

\[
|\langle \mu, e^{tL_\gamma}f \rangle - 1| = |\langle \mu, e^{tL_\gamma}f \rangle - \langle \mu e^{(t-h)\frac{L_\beta}{\pi}}, Pf \rangle |
\]

\[
\leq |\langle \mu e^{(t-h)\frac{L_\beta}{\pi}}, e^{h\frac{L_\beta}{\pi}}f \rangle - \langle \mu e^{(t-h)\frac{L_\beta}{\pi}}, e^{\beta L^{(1)}f} \rangle |
\]

\[
+ |\langle \mu e^{(t-h)\frac{L_\beta}{\pi}}, e^{\beta L^{(1)}f} \rangle - \langle \mu e^{(t-h)\frac{L_\beta}{\pi}}, Pf \rangle |
\]

We now denote \( \mu_\gamma = \mu e^{(t-h)\frac{L_\beta}{\pi}} \) that is a probability measure. Then, using Lemma 1 we have that there exists \( C_1 \) and \( C_2 \) strictly positive constants independent of \( \gamma \) and \( \mu \) such that:

\[
|\langle \mu_\gamma, e^{h\frac{L_\beta}{\pi}}f \rangle - \langle \mu_\gamma, e^{\beta L^{(1)}f} \rangle | \leq C_1(h + h^2)^\frac{1}{2} e^{C_2\beta t}
\]

\[
\leq C_1 \sqrt{h e^{C_2\beta t}} + C_1 h e^{C_2\beta t}
\]

\[
\leq C_1 \sqrt{\frac{1}{C_1 \beta} e^{2C_2\beta t} e^{C_2\beta t} + \frac{1}{\beta} e^{-C_2\beta t}}
\]

\[
\leq \sqrt{\frac{C_1}{\beta} + \frac{1}{\beta}}
\]

\( \xrightarrow{\gamma \to +\infty} 0. \)

On the other hand, since the convergence is uniform in \( x \) in the uniform ergodic theorem, we have:

\[
|\langle \mu_\gamma, e^{\beta L^{(1)}f} \rangle - \langle \mu_\gamma, Pf \rangle | \leq \sup_{x \in K} |e^{\beta L^{(1)}f}(x) - Pf(x)| \xrightarrow{\gamma \to +\infty} 0,
\]

uniformly in \( \mu \). So eventually:

\[
\langle \mu, e^{tL_\gamma}f \rangle \xrightarrow{\gamma \to +\infty} 1,
\]

and the convergence is uniform in \( \mu \). This proves the corollary. \( \square \)

Finally, to prove our theorem on affine functions for an affine drift, we need a last technical lemma:

**Lemma 2.** We assume that \( b : x \in \mathbb{R}^n \mapsto Cx + d \in \mathbb{R}^n \) where \( C \in \mathcal{M}_n(\mathbb{R}) \) and \( d \in \mathbb{R}^n \). Then for any affine function \( g : x \in \mathbb{R}^n \mapsto v \cdot x + l \in \mathbb{R} \) with \( v \in \mathbb{R}^n \) and \( l \in \mathbb{R} \), and any \( t \geq 0 \), one has that \( e^{tL_\gamma}g \) is affine. More precisely:

\[
e^{tL_\gamma}g : x \mapsto e^{tC_1}v \cdot x + \left( v \cdot \int_0^t (e^{Cs}d)ds + l \right)
\]
Proof.

We have that for all $x \in K$:
\[
e^{t\mathcal{L}} g(x) = \mathbb{E}_x (g(X_t^\gamma)) \\
= \mathbb{E}_x (v \cdot X_t^\gamma + l) \\
= v \cdot \mathbb{E}_x (X_t^\gamma) + l.
\]
(3.1)

Applying $\mathbb{E}_x$ to the integral formulation of (2.1), we get:
\[
\mathbb{E}_x (X_t^\gamma) = x + \int_0^t \mathbb{E}_x (b(X_s^\gamma)) ds \\
= x + \int_0^t \mathbb{E}(CX_t^\gamma + d) ds \\
= x + dt + C \int_0^t \mathbb{E}_x (X_s^\gamma) ds.
\]
The associated differential equation is:
\[
\frac{\partial}{\partial t} \mathbb{E}_x (X_t^\gamma) = C \mathbb{E}_x (X_t^\gamma) + d, \quad \mathbb{E}_x (X_0^\gamma) = x.
\]
(3.2)

We can easily check that the unique solution to (3.2) is:
\[
\mathbb{E}_x (X_t^\gamma) = e^{Ct} x + \int_0^t (e^{Cs} d) ds
\]
(3.3)

Combining (3.1) and (3.3), we eventually get:
\[
e^{t\mathcal{L}} g(x) = v \cdot (e^{tC} x) + v \cdot \int_0^t (e^{Cs} d) ds + l
\]
\]

Remark 9. Let us consider an affine function $g$, equation (2.4) gives us trivially $\mathcal{P} g = g$, that is to say $\mathcal{P}$ preserves affine functions.

We may now prove our main theorem by following the path we mentioned earlier:

Proof. (Theorem 2)

We have that $b$ is Lipschitz. Since the cardinal of $K_0$ is by hypothesis $n + 1$ in a $n$-dimensional vector space, there exists a matrix $C = (c_{i,j})_{1 \leq i, j \leq n}$ and a vector $d$ such that the affine mapping:
\[
\tilde{b} : x \in \mathbb{R}^n \mapsto Cx + d \in \mathbb{R}^n,
\]
satisfies $\tilde{b}(z) = b(z)$ for all $z \in K_0$. 
We will now consider the system of equations:

\[
\begin{align*}
X_t^\gamma &= Z_0 + \int_0^t b(X_t^\gamma) dt + \int_0^t \sigma_0(X_t^\gamma) dB_t + \sqrt{\gamma} \int_0^t \sigma(X_t^\gamma) dW_t \\
Y_t^\gamma &= Z_0 + \int_0^t \tilde{b}(Y_t^\gamma) dt + \int_0^t \sigma_0(Y_t^\gamma) dB_t + \sqrt{\gamma} \int_0^t \sigma(Y_t^\gamma) dW_t
\end{align*}
\]

where the two independent Brownian motions $B_t$ and $W_t$ are the same for the two processes and $Z_0$ is a random variable of law $\mu$. By hypothesis, $X_t^\gamma$ stays in $K$ for arbitrary times, so Remark 4 gives us that $b(z)$ points to the interior of $K$ for all $z \in K_0$. But $K$ is a simplex, so each side of it is the convex envelope of its vertices, and the value of $\tilde{b}$ on a point $x$ of this side is a convex combination of the values of $b$ on its vertices: $\tilde{b}(x)$ therefore also points to the interior of $K$. Thus, Remark 4 gives us that $Y_t^\gamma$ stays in $K$ for arbitrary times.

We will denote the generators of these two processes by $L_\gamma = L^{(0)} + \gamma L^{(1)}$ and $G_\gamma = G^{(0)} + \gamma G^{(1)}$.

Let us now consider an affine function $g$. For any $\gamma > 0$ and $t > 0$ we have, using Lemma 2 that $e^{tG_\gamma}g$ is affine since $\tilde{b}$ is affine. Hence $G_\gamma e^{tG_\gamma}g = G^{(0)} e^{tG_\gamma}g$ and:

\[
\frac{\partial}{\partial t} e^{tG_\gamma}g = G_\gamma e^{tG_\gamma}g = G^{(0)} e^{tG_\gamma}g = \mathcal{P} G^{(0)} \mathcal{P} e^{tG_\gamma}g
\]

since $\mathcal{P}$ preserves linear functions (see Remark 9).

Thus, writing $G_\infty = \mathcal{P} G^{(0)} \mathcal{P}$, we have that:

\[
e^{tG_\gamma}g = e^{tG_\infty}g = \mathcal{P} e^{tG_\infty}g.
\]
We denote $g_t = e^{tG\infty}g$ for $t \geq 0$. We have for $z \in K_0$ that:

$$
\frac{\partial}{\partial t} g_t(z) = \frac{\partial}{\partial t} g_t(z) = \mathcal{P}G(0)\mathcal{P}g_t(z) = \sum_{y \in K_0} H_y(y)G(0)\mathcal{P}g_t(y) \quad \text{Using (2.4)}
$$

$$
= \tilde{b}(z) \cdot \nabla_x P g_t(z)
$$

$$
= \tilde{b}(z) \cdot \nabla_x \left( \sum_{y \in K_0} H_y(x)g_t(y) \right)(z)
$$

$$
= \sum_{y \in K_0} \left( \tilde{b}(z) \cdot \nabla_x H_y(z) \right) g_t(y)
$$

$$
= \left( \tilde{b}(x) \cdot \nabla_x H_y(x) \right)_{x,y \in K_0} (\mathcal{P}(y))_{y \in K_0}(z)
$$

$$
= \mathcal{G}g_t(z).
$$

Eventually we have that $\mathcal{G}g(z) = e^{tG_\infty}g(z)$ for all $z \in K_0$, where we remind that $\mathcal{G} = g|_{K_0}$.

Finally we have for all probability measure $\mu$:

$$
\langle \mu, e^{tG_n}g \rangle = \langle \mu \mathcal{P}, e^{tG_\infty}g \rangle = \langle \mu, e^{t\mathcal{G}g} \rangle
$$

However, if we look at the definition of $\mathcal{L}$ and $\mathcal{G}$ in Theorem 2, we notice that only the value of the drift on the $z \in K_0$ matters, therefore we have actually that $\mathcal{L} = \mathcal{G}$ so:

$$
\langle \mu, e^{t\mathcal{L}g} \rangle = \langle \mu, e^{t\mathcal{G}g} \rangle.
$$

Thus, to prove that, for any affine function $g$, we have:

$$
\langle \mu, e^{t\mathcal{L}_n}g \rangle \xrightarrow[\gamma \rightarrow +\infty]{} \langle \mu, e^{t\mathcal{G}g} \rangle,
$$

we have to show that:

$$
|\langle \mu, e^{t\mathcal{L}_n}g \rangle - \langle \mu, e^{t\mathcal{G}_n}g \rangle| = |\langle \mu, e^{t\mathcal{L}_n}g \rangle - \langle \mu, e^{t\mathcal{G}g} \rangle| \xrightarrow[\gamma \rightarrow +\infty]{} 0.
$$

First of all, let us use Dynkin’s formula for $g : x \mapsto e_i \cdot x$. We have thus $\nabla_x g = e_i$ and $H_y = 0$. Therefore, writing $X_t^i = ([X_t^i]_i)_{1 \leq i \leq n}$ and
\[ Y^\gamma_t = ([Y^\gamma_t]_j)_{1 \leq j \leq n} \text{ we have:} \]

\[
\mathbb{E}_\mu ([X^\gamma_t]_i - [Y^\gamma_t]_i) \\
= \mathbb{E}_\mu (g(X^\gamma_t)) - \mathbb{E}_\mu (g(Y^\gamma_t)) \\
= \langle \mu, g \rangle - \langle \mu, g \rangle \\
+ \mathbb{E} \left( \int_0^t (b(X^\gamma_s) \cdot \nabla_x g(X^\gamma_s) - \tilde{b}(Y^\gamma_s) \cdot \nabla_x g(Y^\gamma_s)) \, ds \right) \\
+ \frac{1}{2} \mathbb{E} \left( \int_0^t (\text{tr} (\sigma(X^\gamma_s)\sigma(X^\gamma_s)^\dagger H_g(X^\gamma_s)^\dagger)) \\
- \text{tr} (\sigma(Y^\gamma_s)\sigma(Y^\gamma_s)^\dagger H_g(Y^\gamma_s)^\dagger) \, ds \right) \\
+ \frac{1}{2} \mathbb{E} \left( \int_0^t (\text{tr} (\sigma_0(X^\gamma_s)\sigma_0(X^\gamma_s)^\dagger H_g(X^\gamma_s)^\dagger)) \\
- \text{tr} (\sigma_0(Y^\gamma_s)\sigma_0(Y^\gamma_s)^\dagger H_g(Y^\gamma_s)^\dagger) \, ds \right) \\
= \mathbb{E}_\mu \left( \int_0^t (b(X^\gamma_s) - \tilde{b}(Y^\gamma_s)) \cdot e_i \, ds \right) \\
= \mathbb{E}_\mu \left( \int_0^t (\tilde{b}(X^\gamma_s) - \tilde{b}(Y^\gamma_s)) \cdot e_i \, ds \right) + \mathbb{E}_\mu \left( \int_0^t (b(X^\gamma_s) - \tilde{b}(X^\gamma_s)) \cdot e_i \, ds \right) \\
= \mathbb{E}_\mu \left( \int_0^t \sum_{j=1}^n c_{i,j} ([X^\gamma_t]_j - [Y^\gamma_t]_j) \, ds \right) \\
+ \mathbb{E}_\mu \left( \int_0^t (b(X^\gamma_s) - \tilde{b}(X^\gamma_s)) \cdot e_i \, ds \right) \\
= \sum_{j=1}^n c_{i,j} \int_0^t \mathbb{E}_\mu ([X^\gamma_t]_j - [Y^\gamma_t]_j) \, ds + \int_0^t \mathbb{E}_\mu \left( (b(X^\gamma_s) - \tilde{b}(X^\gamma_s)) \cdot e_i \right) \, ds. \]
Thus we have:

\[ |\mathbb{E}_\mu([X_t^\gamma_i - Y_t^\gamma_i])| \]

\[ \leq \sum_{j=1}^n |c_{i,j}| \int_0^t \left| \mathbb{E}_\mu([X_s^\gamma] - [Y_s^\gamma]) \right| ds + \int_0^t \left| \mathbb{E}_\mu((b(X_s^\gamma) - \tilde{b}(X_s^\gamma)) \cdot e_i) \right| ds \]

\[ \leq \sum_{j=1}^n |c_{i,j}| \int_0^t \sup_{1 \leq k \leq n} \left| \mathbb{E}_\mu([X_s^\gamma]_k - [Y_s^\gamma]_k) \right| ds \]

\[ + \sup_{1 \leq k \leq n} \int_0^t \left| \mathbb{E}_\mu((b(X_s^\gamma) - \tilde{b}(X_s^\gamma)) \cdot e_k) \right| ds \]

\[ \leq \sum_{i,j=1}^n |c_{i,j}| \int_0^t \sup_{1 \leq k \leq n} \left| \mathbb{E}_\mu([X_s^\gamma]_k - [Y_s^\gamma]_k) \right| ds \]

\[ + \sup_{1 \leq k \leq n} \int_0^t \left| \mathbb{E}_\mu((b(X_s^\gamma) - \tilde{b}(X_s^\gamma)) \cdot e_k) \right| \]

We notice that \( i \) does not appear in the last expression, thus we have:

\[ \sup_{1 \leq k \leq n} \left| \mathbb{E}_\mu([X_t^\gamma]_k - [Y_t^\gamma]_k) \right| \leq \|C\|_1 \int_0^t \sup_{1 \leq k \leq n} \left| \mathbb{E}_\mu([X_s^\gamma]_k - [Y_s^\gamma]_k) \right| ds \]

\[ + \sup_{1 \leq k \leq n} \int_0^t \left| \mathbb{E}_\mu((b(X_s^\gamma) - \tilde{b}(X_s^\gamma)) \cdot e_k) \right| . \]

We notice that for \( t \in [0, T] \), we have

\[ \sup_{1 \leq k \leq n} \int_0^t \left| \mathbb{E}_\mu((b(X_s^\gamma) - \tilde{b}(X_s^\gamma)) \cdot e_k) \right| \leq C_T^\gamma \]

where

\[ C_T^\gamma = \sup_{1 \leq k \leq n} \int_0^T \left| \mathbb{E}_\mu \left( (b(X_s^\gamma) - \tilde{b}(X_s^\gamma)) \cdot e_k \right) \right| . \quad (3.4) \]

Finally, writing \( g_\gamma : t \in \mathbb{R}_+ \mapsto \sup_{1 \leq k \leq n} \left| \mathbb{E}_\mu(X_{t,k}^\gamma - Y_{t,k}) \right| \in \mathbb{R}_+ \), we have, for \( t \in [0, T] \), that:

\[ g_\gamma(t) \leq \|C\|_1 \int_0^t g_\gamma(s) ds + C_T^\gamma, \]

so Grönwall’s lemma gives that for \( t \in [0, T] \):

\[ 0 \leq g_\gamma(t) \leq C_T^\gamma e^{\|C\|_1 t}. \quad (3.5) \]

But Corollary 1 gives that for any \( s > 0 \), we have that \( X_t^\gamma \) lives around points of \( K_0 \) when \( \gamma \) is big enough, and we have defined \( \tilde{b} \) in such a
way that \( \bar{b}_{|K_0} = b_{|K_0} \). Eventually we have for all \( s > 0 \):

\[
\left| \mathbb{E}_\mu \left( (b(X^\gamma_s) - \bar{b}(X^\gamma_s)) \cdot e_k \right) \right| \xrightarrow{\gamma \to +\infty} 0.
\]

This quantity is bounded by a constant (that is integrable on \([0, T]\)), by definition of \( C_T^\gamma \) in (3.4) the dominated convergence theorem gives us eventually:

\[
C_T^\gamma \xrightarrow{\gamma \to +\infty} 0.
\]

Therefore, by (3.5) we finally have:

\[
g^\gamma(t) \xrightarrow{\gamma \to +\infty} 0.
\]

Thus, for \( g : x \in \mathbb{R}^n \mapsto v \cdot x + l \in \mathbb{R} \), where \( v = (v_i)_{1 \leq i \leq n} \in \mathbb{R}^n \) and \( e \in \mathbb{R} \), one has:

\[
\left| \langle \mu, e^{tL^\gamma}g \rangle - \langle \mu, e^{tG^\gamma}g \rangle \right| = \left| \mathbb{E}_\mu (g(X^\gamma_t)) - \mathbb{E}_\mu (g(Y^\gamma_t)) \right|
\]

\[
= \left| \mathbb{E}(X^\gamma_t - Y^\gamma_t) \cdot v \right|
\]

\[
= \left| \sum_{1 \leq i \leq n} v_i \mathbb{E}(X^\gamma_t - Y^\gamma_t)_i \right|
\]

\[
= \sup_{1 \leq i \leq n} \left| \mathbb{E}(X^\gamma_t - Y^\gamma_t)_i \right| \times \sum_{1 \leq j \leq n} \left| v_j \right|
\]

\[
= g^\gamma(t) \|v\|_1 \xrightarrow{\gamma \to +\infty} 0
\]

Furthermore, since the result of Corollary 1 is uniform in \( \mu \), this convergence result is uniform in \( \mu \) for \( g \) affine and \( t > 0 \) given.

Eventually, we have that for any affine function \( g \):

\[
\langle \mu, e^{tL^\gamma}g \rangle \xrightarrow{\gamma \to +\infty} \langle \bar{\mu}, e^{t\bar{L}^\gamma} \bar{g} \rangle
\]

\[
= \sum_{z \in K_0} \sum_{y \in K_0} \bar{\mu}(y) e^{t\bar{L}^\gamma}(y, z) \bar{g}(z).
\]

We will now use the geometrical hypothesis of the theorem to conclude the proof for a Lipschitz test function \( f \). Since we have \( n + 1 \) independent points in a \( n \)-dimensional vector space, there exists an affine function \( g \) such that \( f_{|K_0} = g_{|K_0} \).

We thus have \( \bar{f} = \bar{g} \), so in particular we have proved that:

\[
\langle \mu, e^{tL^\gamma}g \rangle \xrightarrow{\gamma \to +\infty} \langle \bar{\mu}, e^{t\bar{L}^\gamma} \bar{g} \rangle = \langle \bar{\mu}, e^{t\bar{L}^\gamma} \bar{f} \rangle.
\]
Therefore if we prove that:
\[
\left| \langle \mu, e^{t\mathcal{L}_\gamma} f \rangle - \langle \mu, e^{t\mathcal{L}_\gamma} g \rangle \right| \xrightarrow{\gamma \to +\infty} 0,
\]
we will be done with the theorem.

Let \( \varepsilon > 0 \), we consider \( \eta > 0 \) sufficiently small in order that the balls \( B(z, \eta) \) for \( z \in K_0 \) are disjoint and such that:
\[
\sup_{z \in K_0} \sup_{x \in B(z, \eta)} |f(x) - g(x)| \leq \varepsilon.
\]

On the other hand, Corollary 1 gives that for \( \gamma \) big enough
\[
\mathbb{P}_\mu \left( X_{\gamma t} \notin \bigcup_{z \in K_0} B(z, \eta) \right) \leq \varepsilon.
\]

Eventually, we have for \( \gamma \) big enough that:
\[
\left| \langle \mu, e^{t\mathcal{L}_\gamma} f \rangle - \langle \mu, e^{t\mathcal{L}_\gamma} g \rangle \right| = |\mathbb{E}_\mu((f - g)(X_{\gamma t}))|
\leq (\|f\|_\infty + \|g\|_\infty) \mathbb{P}_\mu \left( X_{\gamma t} \notin \bigcup_{z \in K_0} B(z, \eta) \right) + \sum_{z \in K_0} \mathbb{P}_\mu \left( X_{\gamma t} \in B(z, \eta) \right) \sup_{x \in B(z, \eta)} |f(x) - g(x)|
\leq (\|f\|_\infty + \|g\|_\infty) \varepsilon + \varepsilon.
\]

This proves the theorem. \( \square \)

3.2. Proof of Theorem 3. Now that our Theorem 2 is proven, let us give a sketch of the proof of Theorem 3.

We know that for all \( \gamma > 0 \) our trajectory \((X_{\gamma t})_{t \geq 0}\) are elements of \( \mathcal{D} \), so [2] gives us a powerful criterion to show that this family of processes is tight in \( L^0(\mathbb{R}, K) \). Thus, Prokhorov’s theorem gives us that the set of the laws of these processes is a relatively compact subset of the space of probability measures on \( L^0(\mathbb{R}, K) \) for the topology of weak convergence.

To prove Theorem 3, we therefore just have to show that if a subsequence of \((X_{\gamma t})_{t \geq 0}\) strongly converges to \( \beta \in L^0 \), then its limit is necessarily \((X_t)_{t \geq 0}\). It is in fact enough to prove that for any \( r \in \mathbb{N}^* \) and for almost any sequence \( 0 \leq t_1 \leq \ldots \leq t_r < +\infty \) one has that the laws of \((X_{t_1}, \ldots, X_{t_r})\) and \((\beta_{t_1}, \ldots, \beta_{t_r})\) are the same.

Let us start with the following result:
Theorem 4. [2, Theorem 4] Let $E$ be an Euclidean space. If $X$ is an $E$-valued stochastic process with natural filtration $(\mathcal{F}_t, t \geq 0)$, then for any $\tau \in \mathbb{R}_+$, its conditional variation on $[0, \tau]$ is defined as:

$$V_\tau(X) := \sup_{0 = t_0 < t_1 < \ldots < t_k = \tau} \sum_{i=0}^{k-1} \mathbb{E} \left( \| \mathbb{E}(X_{t_{i+1}} - X_{t_i}) | \mathcal{F}_{t_i} \| \right)$$

Consider an index set $I$ and a family $(X^\gamma, \gamma > 0)$ of processes living in $D(\mathbb{R}_+, E)$ which satisfy:

$$\sup_{\gamma > 0} \left| V_\tau(X^\gamma) + \mathbb{E} \left( \sup_{0 \leq t \leq \tau} X^\gamma_t \right) \right| < +\infty,$$

for all $\tau > 0$. Then the family of laws of the $X^\gamma$ is tight for the Meyer–Zheng topology and all the limiting points are supported in $D(\mathbb{R}_+, E)$.

The previous theorem is the main tool for proving:

Lemma 3. The family of processes:

$$(X^\gamma_t)_{t \geq 0}, \gamma > 0$$

is tight in the Polish space $(\mathbb{L}^0(\mathbb{R}_+, K), d)$ and all limiting points are supported on càdlàg paths.

Proof.

Since for any $\gamma > 0$ and any $t \geq 0$, one has $X^\gamma_t \in K$ almost surely where $K$ is compact, we only have to prove that the conditional variation in Meyer–Zheng’s Theorem 4 is bounded on segments. For fixed $\tau > 0$, we have in our case:

$$V_\tau(X^\gamma) := \sup_{0 = t_0 < t_1 < \ldots < t_k = \tau} \sum_{i=0}^{k-1} \mathbb{E} \left( \| \mathbb{E}(X^\gamma_{t_{i+1}} - X^\gamma_{t_i}) | \mathcal{F}_{t_i} \| \right).$$

Equivalently, thanks to [2] Eq. (4),(5) and following paragraph,

$$V_\tau(X^\gamma) = \sup_{\| \varphi \|_2 \leq C_K} \int_0^\tau \mathbb{E} \left( \langle \varphi_t, dX^\gamma_t \rangle \right)$$

where the supremum is taken over the simple predictable process taking value in the ball of center 0 and of radius $C_K := \sup_{z \in K} \| z \|_2$. It follows from (2.1), using that the mean of a Brownian motion is equal to 0, that:

$$V_\tau(X^\gamma) = \sup_{\| \varphi \| \leq C_K} \int_0^\tau \mathbb{E} \left( \langle \varphi_t, b(X^\gamma_t) \rangle \right) dt.$$

So Cauchy-Schwarz inequality gives us:

$$V_\tau(X^\gamma) \leq C_K \int_0^\tau \mathbb{E} (\| b(X^\gamma_t) \|_2) dt.$$
Since $b$ is bounded by $M$, this quantity is trivially bounded by $C_K \times M \times \tau$ for all $\gamma > 0$, and we have our result using Theorem 4. □

Now that we proved that $(X^\gamma)_{\gamma > 0}$ is tight, we only have to show that if a subsequence converges, it converges to $\overline{X}$ to conclude. So from now on we assume that $(X^\gamma_p)_{p \in \mathbb{N}}$ converges weakly to some $\beta$, where $(\gamma_p)_{p \in \mathbb{N}}$ is an unbounded sequence in $]0, +\infty[$. Theorem 4 gives us furthermore that the trajectories of $\beta$ are almost surely càdlàg.

In order to prove that the law of $\beta$ is the law of $\overline{X}$, we will in fact show that almost all their finite-dimensional distributions are the same, that is to say that for all $r \in \mathbb{N}^*$, for almost all $0 \leq t_1 \leq \ldots \leq t_r < +\infty$ and for all $f$ continuous function on $K^r$ one has:

$$\mathbb{E} (f(\beta_{t_1}, \ldots, \beta_{t_r})) = \mathbb{E} (f(\overline{X}_{t_1}, \ldots, \overline{X}_{t_r})).$$

Let us start with a linearization trick:

Lemma 4. For all $f$ continuous on $K^r$, there exists $F : (\mathbb{R}^n)^r \longrightarrow \mathbb{R}$ a $r$-linear function such that:

$$\int_{0, +\infty}^r \mathbb{E} (f(X^\gamma_{t_1}, \ldots, X^\gamma_{t_r})) - \mathbb{E} (F(X^\gamma_{t_1}, \ldots, X^\gamma_{t_r})) \lambda^{\otimes r} (dt_1, \ldots, dt_r) \longrightarrow 0 \quad p \rightarrow +\infty$$

where we remind that $\lambda(dt) = e^{-t}dt$.

Proof. Let us first introduce the function $F$. We write $z_0, \ldots, z_n$ the elements of $K_0$ and for all $i = (i_1, \ldots, i_r) \in [0, n]^r$ we write $F_i = f(z_{i_1}, \ldots, z_{i_r})$. We notice that for all $(x_1, \ldots, x_r) \in \{z_i, i \in [0, n]\}^r$ one has:

$$f(x_1, \ldots, x_r) = \sum_{i \in [0, n]^r} F_i \prod_{k=1}^r \chi_{z_{i_k}}(x_k)$$

$$= \sum_{i \in [0, n]^r} F_i \prod_{k=1}^r f_{z_{i_k}}(x_k)$$

$$=: F(x_1, \ldots, x_r),$$

where the functions $f_{z_{i_k}}$ are the one mentioned in Remark 6. Since they are all linear, the function $F$ is clearly a continuous $r$-linear map.
Now [2, Theorem 6] applied to $f$ and $F$ says exactly that:

$$\int_{[0, +\infty]^r} |\mathbb{E}(f(X_{t_1}^{\gamma_p}, \ldots, X_{t_r}^{\gamma_p})) - \mathbb{E}(f(\beta_{t_1}, \ldots, \beta_{t_r}))| \lambda^\otimes r(dt_1, \ldots, dt_r) \xrightarrow{p \to +\infty} 0$$

(3.7)

and:

$$\int_{[0, +\infty]^r} |\mathbb{E}(F(X_{t_1}^{\gamma_p}, \ldots, X_{t_r}^{\gamma_p})) - \mathbb{E}(F(\beta_{t_1}, \ldots, \beta_{t_r}))| \lambda^\otimes r(dt_1, \ldots, dt_r) \xrightarrow{p \to +\infty} 0$$

(3.8)

We will now invoke an immediate consequence of Corollary 1: for any sequence $0 \leq t_1 \leq \ldots \leq t_r < +\infty$ and for any $\varepsilon > 0$ one has, using $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$, that:

$$\mathbb{P}_\mu \left( X_{t_1}^\gamma \in \bigcup_{z \in K_0} B(z, \eta), \ldots, X_{t_r}^\gamma \in \bigcup_{z \in K_0} B(z, \varepsilon) \right)$$

$$\geq \mathbb{P}_\mu \left( X_{t_1}^\gamma \in \bigcup_{z \in K_0} B(z, \varepsilon), \ldots, X_{t_{r-1}}^\gamma \in \bigcup_{z \in K_0} B(z, \varepsilon) \right)$$

$$+ \mathbb{P}_\mu \left( X_{t_r}^\gamma \in \bigcup_{z \in K_0} B(z, \varepsilon) \right) - 1$$

$$\geq \left( \sum_{k=1}^{r} \mathbb{P}_\mu \left( X_{t_k}^\gamma \in \bigcup_{z \in K_0} B(z, \varepsilon) \right) \right) - (r - 1) \quad \text{by recurrence}$$

$$\xrightarrow{\gamma \to +\infty} 1.$$  

Thus, since $X^\gamma_{\mathfrak{p}}$ converges weakly to $\beta$, one has that $(\beta_{t_1}, \ldots, \beta_{t_r}) \in K_0^r$ almost surely, so eventually one has almost surely:

$$f(\beta_{t_1}, \ldots, \beta_{t_r}) = F(\beta_{t_1}, \ldots, \beta_{t_r}).$$

Since this is true almost surely, the expectancy of these two quantities are equal, and combining (3.7) and (3.8) we find:

$$\int_{[0, +\infty]^r} |\mathbb{E}(f(X_{t_1}^{\gamma_p}, \ldots, X_{t_r}^{\gamma_p})) - \mathbb{E}(F(X_{t_1}^{\gamma_p}, \ldots, X_{t_r}^{\gamma_p}))| \lambda^\otimes r(dt_1, \ldots, dt_r) \xrightarrow{p \to +\infty} 0$$

(3.9)

Therefore, in the large $p$ limit, every continuous function $f$ in $r$ variables can be replaced by its $r$-linearization. \qed
The $r$-linearization of $f$ is a sum of terms of the form $\prod_{k=1}^{r} f_{z_{ik}}$, so we will first study these elementary bricks. One has for $F = \prod_{k=1}^{r} f_{z_{ik}}$ that:

$$
\mathbb{E}_\pi(F(X_{t_1}, \ldots, X_{t_r})) = \mathbb{E}_\pi\left(\prod_{i=1}^{r} f_{z_{ik}}(X_{t_k})\right)
$$

$$
= \mathbb{E}_\pi\left(\mathbb{E}_\pi\left(\prod_{i=1}^{r} f_{z_{ik}}(X_{t_k}) | X_{t_{r-1}} = z_{i_{r-1}}\right)\right)
$$

$$
= \mathbb{E}_\pi(f_{z_i}(X_{t_i}) | X_{t_{r-1}} = z_{i_{r-1}}) \times \mathbb{E}_\pi\left(\prod_{i=1}^{r-1} f_{z_{ik}}(X_{t_k})\right)
$$

$$
= \mathbb{E}_\pi\left(f_{z_1}(X_{t_1})\right) \times \prod_{k=2}^{r} \mathbb{E}_\mu\left(f_{z_k}(X_{t_k} | X_{t_{k-1}} = z_{i_{k-1}})\right)
$$

$$
= \langle \pi, \epsilon^{t_1} f_{z_1} \rangle \prod_{k=2}^{r} \langle \delta_{z_{i_{k-1}}}, e^{(t_k - t_{k-1})L} f_{z_k} \rangle \text{ by Theorem } 3
$$

$$
= \sum_{i=0}^{n} \pi(z_i) \prod_{k=1}^{r} e^{(t_k - t_{k-1})L} f_{z_k}(z_{i_{k-1}})
$$

where we implicitly assumed that $i_0 = i$.

To prove that this is the limit of $\mathbb{E} \left(\prod_{k=1}^{r} f_z(X_{t_k}^{p_k})\right)$ for almost any sequence $0 \leq t_1 \leq \ldots \leq t_r$, we will prove by induction that:

$$
0 = \lim_{p \to +\infty} \int_{0=t_0 \leq t_1 \leq \ldots \leq t_r < +\infty} \lambda^{\otimes r}(dt_1, \ldots, dt_r)
$$

$$
\left| \mathbb{E} \left(\prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{p_k})\right) - \sum_{i=0}^{n} \pi(z_i) \prod_{k=1}^{r} e^{(t_k - t_{k-1})L} f_{z_k}(z_{i_{k-1}}) \right| (3.10)
$$

For $r = 1$, this is trivial since the integrand is bounded by 2 that is integrable on $((\mathbb{R}_+)^r, \lambda^{\otimes r})$ and it converges to 0 pointwisely using Theorem 2 we conclude using the dominated convergence theorem.

We assume that (3.10) is proven for $r \in \mathbb{N}^*$ given. Let $(\mathcal{F}_t, t \geq 0)$ be the natural filtration. For all $0 \leq t_1 \leq \ldots \leq t_{r+1} < +\infty$, the tower
property of conditional expectation and then the Markov property imply:

\[ \mathbb{E} \left( \prod_{k=1}^{r+1} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \right) = \mathbb{E} \left( \prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \times \mathbb{E}(f_{z_{i_{r+1}}}(X_{t_{r+1}}^{\gamma_p})|F_{t_r}) \right) \]

\[ = \mathbb{E} \left( \prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \times \langle \delta_{X_{t_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle \right) \]

We have therefore:

\[ \left| \mathbb{E} \left( \prod_{k=1}^{r+1} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \right) - \mathbb{E} \left( \prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \times \langle \delta_{z_{i_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle \right) \right| \]

\[ = \left| \mathbb{E} \left( \prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \times \langle \delta_{X_{t_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle \right) - \mathbb{E} \left( \prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{\gamma_p}) \times \langle \delta_{z_{i_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle \right) \right| \]

\[ \leq \mathbb{E} \left( |f_{z_{i_{r}}}(X_{t_r}^{\gamma_p})| \times \left| \langle \delta_{X_{t_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle - \langle \delta_{z_{i_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle \right| \right) \]

(3.11)

where the penultimate inequality follows \(|f_{z_{i_{r}}}(X_{t_r}^{\gamma_p})| \leq 1\). We want to show that (3.11) converges to 0. In order to do that, we will consider approximations of \(\langle \delta_{X_{t_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle\) and \(\langle \delta_{z_{i_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle\) where \(\gamma_p\) only appears in the bras.

The key idea is the one used in the proof of Theorem 2 instead of considering a complex drift \(b\), we approximate our process of generator \(\mathcal{L}_\gamma\) with a one of generator \(\mathcal{G}_\gamma\) whose drift \(\tilde{b}\) is linear. Since \(f_{z_{i_{r+1}}} \) is affine and the convergence in (3.6) is uniform in \(\mu\), one has:

\[ \left| \langle \delta_{X_{t_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle - \langle \delta_{X_{t_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{G}_\gamma f_{z_{i_{r+1}}}} \rangle \right| \xrightarrow{p \to +\infty} 0, \]

and

\[ \left| \langle \delta_{z_{i_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{L}_\gamma f_{z_{i_{r+1}}}} \rangle - \langle \delta_{z_{i_r}^{\gamma_p}}, e^{(t_{r+1}-t_r)\mathcal{G}_\gamma f_{z_{i_{r+1}}}} \rangle \right| \xrightarrow{p \to +\infty} 0, \]
where the first convergence is uniform in $\omega$ in the sample space.

We can therefore replace $\mathcal{L}_\gamma$ by $\mathcal{G}_\gamma$ in (3.11), but since $\tilde{b}$ is linear, Lemma 2 gives us:

$$\langle \delta_{z_t}, e^{(t_{r+1} - t_r)\mathcal{L}_p} f_{z_{r+1}} \rangle = \langle \delta_{z_t}, e^{(t_{r+1} - t_r)\mathcal{L}(0)} f_{z_{r+1}} \rangle$$

and

$$\langle \delta_{X_t^{gp}}, e^{(t_{r+1} - t_r)\mathcal{L}_p} f_{z_{r+1}} \rangle = \langle \delta_{X_t^{gp}}, e^{(t_{r+1} - t_r)\mathcal{L}(0)} f_{z_{r+1}} \rangle$$

We eventually integrate (3.11) and use the last two equalities:

$$\limsup_{p \to +\infty} \int_{0 = t_0 \leq t_1 \leq \ldots \leq t_{r+1} < +\infty} \lambda^{\otimes r}(dt_1, \ldots, dt_{r+1})$$

$$\left| \mathbb{E} \left( \prod_{k=1}^{r+1} f_{z_{ik}}(X_{t_k}^{gp}) \right) - \mathbb{E} \left( \prod_{k=1}^{r} f_{z_{ik}}(X_{t_k}^{gp}) \times \langle \delta_{z_t}, e^{(t_{r+1} - t_r)\mathcal{L}_p} f_{z_{r+1}} \rangle \right) \right|$$

$$\leq \limsup_{p \to +\infty} \int_0^{+\infty} \mathbb{E} \left( |f_{z_t}(X_{t}^{gp})| \times \left( |\delta_{X_t^{gp}}, e^{(t_{r+1} - t_r)\mathcal{L}(0)} f_{z_{r+1}}(X_{t}^{gp})| \right) \right) \lambda(dt_{r+1})$$

$$= \int_0^{+\infty} \mathbb{E} \left( \left| f_{z_t}(\beta_{t_r}) \times \left( \langle \delta_{\beta_{t_r}}, e^{(t_{r+1} - t_r)\mathcal{L}(0)} f_{z_{r+1}} \rangle \right. \right. \right) \lambda(dt_{r+1})$$

$$= 0,$$

the last equality being a consequence of $f_{z_t}(\beta_{t_r}) = \mathbb{I}_{z_t}(\beta_{t_r})$ for all $t_r$, so that the product with the other term vanished necessarily. Now invoking the induction hypothesis with $r$:

$$0 = \limsup_{p \to +\infty} \int_{0 = t_0 \leq t_1 \leq \ldots \leq t_{r+1} < +\infty} \left| \sum_{i=0}^{n} \prod_{k=1}^{r+1} e^{(t_k - t_{k-1})\mathcal{L}_p} f_{z_k}(z_{ik-1}) \right| \lambda^{\otimes r}(dt_1, \ldots, dt_{r+1})$$

Combining the last limits, we prove the claim for $r + 1$. 
Using that $F$ is a sum of terms of the form $\prod_{k=1}^{r} f_{z_{ik}}$, equation (3.10) implies:

$$0 = \lim_{p \to +\infty} \int_{0=t_0 \leq t_1 \leq \ldots \leq t_r < +\infty} \left| \mathbb{E} \left( F(X_{t_1}^{\gamma_p}, \ldots, X_{t_r}^{\gamma_p}) \right) \right|$$

$$- \sum_{i \in [0,n]^r} F_i \sum_{i=0}^{n} p(z_i) \prod_{k=1}^{r} e^{(t_k-t_{k-1})} \mathbb{E} f_{z_k}(z_{i_{k-1}})$$

This last result combined with equations (3.8) and (3.9) implies that for $\lambda^{\otimes r}$-almost every (so for Lebesgue-almost every) sequence $0 \leq t_1 \leq \ldots \leq t_r$ and for any function $f$ continuous, we have:

$$\lim_{p \to +\infty} \mathbb{E} \left( f(X_{t_1}^{\gamma_p}, \ldots, X_{t_r}^{\gamma_p}) \right) = \mathbb{E} \left( f(\beta_{t_1}, \ldots, \beta_{t_r}) \right)$$

Thus, almost all the finite-dimensional distributions of $\beta$ and $X$ are the same. Finally, [2, Theorem 6] yields that $\beta$ has the same law as the processes $X$ where $\overline{X}$ is the Markov process on $K_0$ of generator $\overline{L}$ and of initial condition $\overline{\mu}$. This proves Theorem 3.

4. Discussion on the geometrical hypothesis

The proof of our theorem massively use affine approximations, first of the drift then of our test functions, and thus falls apart if we do not assume that $K_0$ is composed of $n + 1$ affinely independent points.

We may however ask ourselves if it is only a technical hypothesis, or if it is completely essential, that is to say Theorem [2] is false when it is not fulfilled. In this section, we actually construct a counterexample of our theorem with a compact set $K$ of $\mathbb{R}^2$ and a diffusion process on it where the cardinal of $K_0$ is equal to $4 > 2 + 1$.

Let us give a graphic representation of our counterexample:
Here, our compact $K$ is the solid triangle of vertices $A(-1, 0)$, $B(1, 0)$, and $C(0, 4)$. Let us define the dominant noise coefficient $\sigma$. For $(x, y) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1]$, we take:

$$\sigma(x, y) = \left( x^2(1 - x)(1 + x) + y^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If we furthermore take for $x \in [-1, 1]$:

$$\sigma(x, 0) = \left( x^2(1 - x)(1 + x) + y^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we have defined a smooth function on $[-1, 1] \times \{0\} \cup \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1]$ that is null in $A, B$ and $O$. While it is not very interesting to write it down explicitly, it is really not hard to find a smooth extension of $\sigma$ that will be null only on $A, B, C$ and $O$. We have thus $K_0 = \{A, B, C, O\}$. 
To understand where the problem is, it may be interesting to consider the functions $H_z$ for $z \in K_0$. The key hypothesis of the (uniform) ergodic theorem was that the functions $H_z$ were continuous on $K$. We claim this is false in our example.

Let us consider $H_O$: on the one hand, one has obviously that $H_O(O) = 1$. On the other hand, let us consider the dominant diffusion process $(X_t)_{t \geq 0}$ solution of (2.2), starting from $(0, y)$ with $y > 0$. Then, for $\omega$ an element of the sample space, if $X_t(\omega)$ converges to $O$ when $t$ approaches infinity, for any $\varepsilon > 0$, one has $X_t(\omega) \in B(O, \varepsilon)$ for all $t$ big enough. But in fact, the red lines of our drawing are insurmountable obstacles: since the noise on the $y$-axis is null on it, if $X_t(\omega)$ is on one of them at a given time $t$, in order to decrease along the $y$-axis it has to go out of the red box and enter in it again. Yet we know that $X_t(\omega) \in B(O, \epsilon)$ for $t$ big enough, and thus $X_t(\omega)$ cannot get out of the box $[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]$. Therefore, we have

$$X_t(\omega) \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \{0\}$$

for $t$ big enough, that is to say we reached the bottom line in finite time. But since we have to quit the box to get to the line, there exists $T$ such that

$$X_T(\omega) \in \left( \left[ -1, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 1 \right] \right) \times \{0\}.$$

Starting from any point of this set, the probability of converging to $O$ is less or equal to $\frac{1}{2}$, so using Markov strong property (the process in homogeneous in time), we have that the probability of converging to $O$ starting from $(0, y)$ is less or equal to $\frac{1}{2}$ for any $y > 0$: our function $H_O$ cannot be continuous.

**Remark 10.** If the noise is Lipschitz, it is in fact impossible to reach the border in finite time, so we have actually $H_z(x, y) = 0$ for all $y > 0$: the state $O$ is unreachable except if one starts from the bottom line.

We easily see that Theorem 2 cannot hold in this situation, since our objects are not even well defined: to compute matrix $L$, one has to consider the quantity:

$$b(O) \cdot \nabla H_O(O),$$

that is not well defined if $b$ is non null on the $y$-axis.

Furthermore, even if our process nevertheless was converging to a stochastic process, there are situations where we know that the latter could not be a continuous-time Markov process with state space $K_0$. 

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We consider the noise $\sigma$ defined above (that is only null on $A$, $B$, $C$ and $O$), and a smooth drift $b$ that checks:

$$
\begin{align*}
&b(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in [-\frac{1}{2}, \frac{1}{2}] \times [0, 1], \\
0 & \text{outside the box } [-\frac{5}{9}, \frac{5}{9}] \times [0, \frac{10}{9}].
\end{cases}
\end{align*}
$$

It is easy to prove the existence of such a drift. We assume that the stochastic process $(X_t^\gamma)_{t \geq 0}$ starting from $O$ of generator $L^\gamma$ converges in the weak sense of Theorem 2 to the Markov process $(\tilde{X}_t)_{t \geq 0}$ starting from $O$ of state space $K_0$ and of generator $\mathcal{L}$, and we will get a contradiction.

The idea of the proof is the following: since $(\tilde{X}_s)_{s \geq 0}$ is a Markov process starting from $O$, we have $\tilde{X}_t \in O$ with a probability approaching 1 for a time $t > 0$ small enough. Let us now consider the process $(X_s^\gamma)_{s \geq 0}$ when $\gamma$ is big enough. During $[0, t]$ it first enters the box $[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]$ pushed by the drift in $O$, then, since the noise inside is very strong, it quits the red box and, from the exit point, finally approaches $A$, $B$ or $C$ with a probability greater than a strictly positive quantity. Since this one is independent of $\gamma$ and $t$ when $\gamma$ is big enough, it will contradict the fact that it should converge to 0 when $\gamma$ approaches infinity and $t$ approaches 0.

We consider a smooth function $f$ such that $f(O) = 1$. We assume furthermore that $0 \leq f \leq 1$ and that $f$ is null outside $B(O, \frac{1}{8})$. We know that there exists $t > 0$ small enough such that:

$$
\langle \delta_O, e^{te^\gamma f} \rangle \geq 1 - 2^{-10}.
$$

Hence for $\gamma$ big enough:

$$
P \left( X_t^\gamma \in B(O, \frac{1}{8}) \right) \geq \langle \delta_O, e^{te^\gamma f} \rangle \geq 1 - 2^{-9} \quad (4.1)
$$

Let us now consider the process $(X_s^\gamma)_{s \geq 0}$ starting from $O$ of generator $L^\gamma$. We write $T^\gamma$ the hitting time of the boundary of our red square:

$$
\inf \left\{ s > 0, X_s^\gamma \in \left\{ -\frac{1}{2} \right\} \times [0, 1] \cup \left\{ \frac{1}{2} \right\} \times [0, 1] \cup \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \{1\} \right\}.
$$

We may now prove that it is arbitrarily small with a high probability when $\gamma$ approaches infinity. Indeed let $g_\varepsilon$ be a smooth function such that $g_\varepsilon(A) = g_\varepsilon(B) = g_\varepsilon(C) = g_\varepsilon(O) = 1$ and we furthermore assumed
that \( 0 \leq g_{\varepsilon} \leq 1 \) and that \( g_{\varepsilon} \) is null outside of the balls of center \( z \in K_0 \) and of radius \( \varepsilon \). Then for any \( s > 0 \) and for any \( \varepsilon \) one has that

\[
\mathbb{P} \left( X_s^\gamma \in \bigcup_{z \in K_0} B(z, \varepsilon) \right) \geq \langle \delta_O, e^{s\mathcal{L}_0} g_{\varepsilon} \rangle \xrightarrow{\gamma \to +\infty} \langle \delta_O, e^{s\mathcal{L}} g_{\varepsilon} \rangle = 1. \quad (4.2)
\]

Now let us assume that there exists an increasing sequence \((\gamma_k)_k\) going to infinity such that

\[
\mathbb{P}(T_{\gamma_k} > s) \geq \alpha > 0.
\]

We notice that for \( \omega \) in the sample space such that \( T_{\gamma_k}(\omega) > s \), using that \( \sigma \cdot e_y = 0 \) into the red box, we have that:

\[
X_s^\gamma(\omega) \cdot e_y = \int_0^s b(X_u^\gamma(\omega)) du \cdot e_y = s.
\]

Since \( X_s^\gamma(\omega) \) is in the red box, we have thus:

\[
\mathbb{P} \left( X_s^\gamma_k \notin \bigcup_{z \in K_0} B(z, \frac{s}{2}) \right) \geq \mathbb{P}(T_{\gamma_k} > s) \geq \alpha
\]

for all \( k \in \mathbb{N} \), but it contradicts (4.2) when \( k \) approaches infinity.

In conclusion, for all \( s > 0 \), one has for \( \gamma \) large enough that \( \mathbb{P}(T_{\gamma} \leq s) \geq \frac{1}{2} \), and we notice that for \( \omega \) such that \( T_{\gamma}(\omega) \leq s \), one has

\[
X_{T_{\gamma}(\omega)}^\gamma(\omega) \cdot e_y = \int_0^{T_{\gamma}(\omega)} b(X_u^\gamma(\omega)) \cdot e_y du = T_{\gamma}(\omega) \leq s.
\]

We now consider the two points \( R_{-1}(\frac{-1}{2}, 0) \) and \( R_1(\frac{1}{2}, 0) \). We just proved that choosing \( \gamma \) big enough, we can make sure that \( (X_u^\gamma)_{u \geq 0} \) reaches a point on the border of the red box arbitrarily close to \( R_{-1} \) or \( R_1 \) during \([0, t]\) with a probability at least \( \frac{1}{2} \). We will thus make two approximations: we will first approach during a short time the diffusion process starting from the point \( x_\gamma = X_{T_{\gamma}}^\gamma \) with the dominant one starting from the same point (using Lemma 1), and then approach the dominant process starting from \( x_\gamma \) with the one starting from \( R_{-1} \) or \( R_1 \).

Since both cases are symmetric, from now on we assume that \( x_\gamma \in \{\frac{1}{2}\} \times [0, 1] \). We consider \((X_s(z))_{s \geq 0}\) the process of generator \( \mathcal{L}^{(1)} \) starting from \( z \in K \). We have with a probability \( \frac{1}{2} \) that \( X_s(R_1) \xrightarrow{s \to +\infty} B \), so there exists \( \beta > 0 \) such that:

\[
\mathbb{P} \left( X_{\beta}(R_1) \in B(B, \frac{1}{32}) \mid X_{\infty} = B \right) \geq \frac{3}{4}.
\]
and therefore:
\[
\mathbb{P} (X_{\beta}(R_1) \in B(B, \frac{1}{32})) \\
\geq \mathbb{P} (X_{\beta}(R_1) \in B(B, \frac{1}{32}) | X_{\infty} = B) \mathbb{P}(X_{\infty}(R_1) = B) \\
\geq \frac{3}{4} \times \frac{1}{2} \\
\geq \frac{3}{8}.
\]  
(4.3)

We now use the dependence on initial conditions: there exists $\eta \in ]0, \frac{t}{2}[$ such that for all $x \in B(R_1, \eta)$, one has:
\[
\sup_{u \in [0,\beta]} \mathbb{E}(\|X_u(R_1) - X_u(x)\|_2) \leq \frac{1}{128}.
\]
so by Markov’s inequality, for all $x \in B(R_1, \eta)$:
\[
\mathbb{P} \left( \|X_{\beta}(R_1) - X_{\beta}(x)\|_2 \geq \frac{1}{32} \right) \leq \frac{1}{8}. 
\]  
(4.4)

We now use (4.2): for $\gamma$ large enough, taking $s = \eta$, we have, with a probability at least $\frac{1}{2}$, that $T_\gamma \leq \eta$ and thus $X_{T_\gamma}^{\gamma} \in B(R_1, \eta)$ (respectively $B(R_{-1}, \eta)$).

We now write $\gamma = \frac{\beta}{n}$, and we take $\gamma$ big enough so that, using Lemma 1, we have for all $x \in B(R_1, \eta)$:
\[
\mathbb{E} \left( \|X_{T_{\gamma}}^{\frac{\beta}{n}}(x) - X_{\beta}(x)\|_2 \right) \leq \frac{1}{128}
\]
and therefore:
\[
\mathbb{P} \left( \|X_{T_{\gamma}}^{\frac{\beta}{n}}(x) - X_{\beta}(x)\|_2 \geq \frac{1}{16} \right) \leq \frac{1}{8}. 
\]  
(4.5)

Finally we have using (4.3), (4.4) and (4.3):
\[
\mathbb{P} \left( X_{h}^{\gamma}(x) \notin B(B, \frac{1}{8}) \right) \\
\leq \mathbb{P} \left( \{X_{\beta}(R_1) \notin B(B, \frac{1}{32})\} \cup \{\|X_{\beta}(R_1) - X_{\beta}(x)\|_2 \geq \frac{1}{32}\} \right) \\
\cup \{\|X_{\beta}(x) - X_{T_{\gamma}}^{\frac{\beta}{n}}(x)\|_2 \geq \frac{1}{16}\} \\
\leq \frac{5}{8} + \frac{1}{8} + \frac{1}{8} \\
\leq \frac{7}{8}.
\]
Thus we have:

\[
P \left( \{ X_{T_{\gamma} + h} \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}) \} \cap \{ T_{\gamma} \leq \frac{t}{2} \} \right) \\
\geq P \left( \{ X_{T_{\gamma} + h} \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}) \} \cap \{ T_{\gamma} \leq \eta \} \right) \\
\geq P \left( X_{T_{\gamma} + h} \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}) \mid T_{\gamma} \leq \eta \right) P \left( T_{\gamma} \leq \eta \right) \\
\geq \frac{1}{8} \times \frac{1}{2} \\
\geq \frac{1}{16}. \quad (4.6)
\]

We thus have with a probability at least \( \frac{1}{16} \) that at a time \( T_{\gamma} + h < t \) our process will be in balls of radius \( \frac{1}{8} \) of centers \( A \) or \( B \), this result being true for all \( \gamma \) big enough. We just have to prove that during \( [T_{\gamma} + h, t] \), our process stays, with a fixed strictly positive probability, far from the ball \( B(O, \frac{1}{8}) \) to find a contradiction with (4.1). The idea of the proof is to show that if one is close to \( B \), the probability that it stays close to \( B \) forever is strictly positive.

We consider \( S \) the stopping defined by:

\[
S(x) = \inf \{ s \geq 0, X_s(x) \notin B(B, \frac{1}{8}) \}.
\]

We notice that the definition \( S \) involves the dominant process and not the general one, and for a good reason: as long as our process lives outside the box \([-\frac{5}{9}, \frac{5}{9}] \times [0, \frac{10}{9}]\), the drift is null and our complex process acts exactly like the dominant one. We have for all \( x \in B(B, \frac{1}{8}) \) that \( (X_{S(x)} \wedge u(x))_{u \geq 0} \) is a martingale, so we have that:

\[
x = E(X_0(x)) = E(X_{S(x)}(x)),
\]

with \( S(x) \) being potentially infinite. Since \( (X_u(x))_{u \geq 0} \) converges (it is a bounded martingale), if \( S(x)(\omega) = +\infty \) then \( X_\infty(x)(\omega) = B \). We have therefore, rewriting the last equation, that:

\[
x = P(S(x) = +\infty)\mathbb{E}(S(x) < +\infty)\mathbb{E}(X_{S(x)}(x) \mid S(x) < +\infty). \quad (4.7)
\]

Using basic geometry we have that \( W(x) = \mathbb{E}(X_{S(x)}(x) \mid S(x) < +\infty) \), being the barycenter of points from \( \partial B(B, \frac{1}{8}) \cap K \), checks

\[
\| W(x) - B \|_2 \geq \frac{\sqrt{2}}{8}. \quad (4.8)
\]

Since \( x \in B(B, \frac{1}{8}) \), one has using (4.7) and (4.8) that:

\[
P(S(x) = +\infty) \geq \frac{\sqrt{2} - 1}{\sqrt{2}} > \frac{1}{4} \quad (4.9)
\]
Our dominant process starting from \( x \in B(B, \frac{1}{8}) \) does not quit \( B(B, \frac{1}{8}) \) with a probability at least \( \frac{1}{4} \). Yet, if it never quits \( B(B, \frac{1}{8}) \), it will never reach \( B(O, \frac{1}{8}) \), so using (4.6) and (4.9) we get for \( \gamma \) big enough:

\[
\mathbb{P}\left( X_t^\gamma \notin B(O, \frac{1}{8}) \right) \\
\geq \mathbb{P}\left( \{ X_t^\gamma \notin B(O, \frac{1}{8}) \} \cap \{ X_{T_\gamma+h}^\gamma \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}) \} \cap \{ T_\gamma \leq \frac{t}{2} \} \right) \\
\geq \mathbb{P}\left( X_t^\gamma \notin B(O, \frac{1}{8}) | X_{T_\gamma+h}^\gamma \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}), T_\gamma \leq \frac{t}{2} \right) \\
\geq \mathbb{P}\left( \{ X_{T_\gamma+h}^\gamma \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}) \} \cap \{ T_\gamma \leq \frac{t}{2} \} \right) \\
\geq \mathbb{P}\left( S(X_{T_\gamma+h}^\gamma) = +\infty | X_{T_\gamma+h}^\gamma \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}), T_\gamma \leq \frac{t}{2} \right) \\
\geq \mathbb{P}\left( \{ X_{T_\gamma+h}^\gamma \in B(A, \frac{1}{8}) \cup B(B, \frac{1}{8}) \} \cap \{ T_\gamma \leq \frac{t}{2} \} \right) \\
\geq \frac{1}{4} \times \frac{1}{16} \\
\geq 2^{-6} \tag{4.10}
\]

Finally (4.11) gives us \( \mathbb{P}\left( X_t^\gamma \notin B(O, \frac{1}{8}) \right) \leq 2^{-9} \) and (4.10) gives us \( \mathbb{P}\left( X_t^\gamma \notin B(O, \frac{1}{8}) \right) \geq 2^{-6} \): Contradiction.

We get that one cannot write the potential limit of our stochastic process of generator \( L_\gamma \), as \( \gamma \) approaches infinity, as a pure jump Markov process of state space \( K_0 \).

**Remark 11.** In this counterexample, we notice that \( O \) belongs to the segment \([A, B]\). More generally, it is easy to build a counterexample of the theorem when the \((n+2)^{th}\) element of \( K_0 \) belongs to the convex envelope of the other points: if our extra point is in the interior of \( K \), we just have to squeeze it between two red boxes like the one of our counterexample to find similar contradictions. We have however not yet found a counterexample of our theorem when there is at least \( n+2 \) points but we assumed that the the points of \( K_0 \) are exactly the extremal points of \( K \).

**Remark 12.** One may say that our counterexample is a pathological one since our point \( O \) is unreachable if one does not start from a point of the boundary, that is say unreachable from almost any starting point. We may indeed imagine that our limit process could be, from almost any starting point, a Markov chain on the points \( \{A, B, C\} \). However, and
it is quite surprising, a point that almost surely does not exist for the dominant process, may in fact exists for the complex process. Indeed, if one takes on the compact $K$ of our counterexample:

$$b(A) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b(B) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b(C) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad b(O) = 0,$$

and the same $\sigma$ as above, then we observe in the limit process an increasing accumulation of the mass on $O$, whatever the starting point is. We cannot thus make as if $O$ was not existing.

**Appendix: Proof of the Ergodic Theorem (Theorem 1)**

We denote by $\mathcal{P}(K)$ the set of probability measures on $K$. We recall that $K$ is equipped with the usual Euclidean norm $\| \cdot \|_2$. For $\mu, \nu \in \mathcal{P}(K)$, the (first) Wasserstein’s distance between $\mu$ and $\nu$ is defined as:

$$d_W(\mu, \nu) = \sup \left\{ \int_K \varphi(z)(d\mu(z) - d\nu(z)) \mid \varphi \text{ Lipschitz, Lip}(\varphi) \leq 1 \right\}.$$

We recall that $(\mathcal{P}(K), d_W)$ is Polish space and that $d_W$ metrizes the weak convergence on $\mathcal{P}(K)$.

From now on we write $H(x, \cdot) \in \mathcal{P}(K)$ the law of $X_\infty(x)$. We have then by definition of the functions $H_z$ that for any Lipschitz function $f$ on $K$:

$$\mathcal{P}f(x) = \sum_{z \in K_0} H_z(x)f(z) = \int_K f(z)H(x, dz).$$

If we assume that the functions $H_z$ are continuous, we have easily that the function $x \in K \mapsto H(x, \cdot) \in \mathcal{P}(K)$ is continuous.

**Remark 13.** In this article we only consider $K_0$ finite, but our proof of the uniform ergodic theorem would still work whatever is the form of the null set of $\sigma$ as long as the dominant process is pure noise (that is to say $(X_t)_{t \geq 0}$ is a martingale) and the function $x \mapsto H(x, \cdot)$ is continuous (where $H(x, \cdot)$ is still the law of $X_\infty(x)$).

Let us start with a technical lemma:

**Lemma 5.** If $x \in K \mapsto H(x, \cdot) \in \mathcal{P}(K)$ is continuous, then for every $\alpha, h > 0$, there exists $\beta > 0$ such that for every $z, y \in K$,

$$d_W(\delta_z, H(y, \cdot)) \leq \beta \quad \text{implies} \quad \mathbb{P}(X_\infty(y) \in B(z, h)) \geq 1 - \alpha.$$

**Proof.**

We consider a smooth function $\zeta_z$ taking values in $[0, 1]$ such that $\zeta_z(z) = 0$ and the restriction of $\zeta_z$ to the complement of $B(z, h)$ is
1. Being smooth, it is $L$-Lipschitz, therefore $\frac{1}{L} \zeta_z$ is 1-Lipschitz. Eventually since $\zeta_z(z) = 0$:
\[
P(X_\infty(y) \notin B(z, h)) \leq \mathbb{E}(\zeta(X_\infty(y)))
\leq L \left( \mathbb{E}\left(\frac{1}{L} \zeta_z(X_\infty(y))\right) - \mathbb{E}\left(\frac{1}{L} \zeta_z(z)\right)\right)
\leq \mathbb{L}W(H(y, \cdot), \delta_z).
\]
We therefore only have to choose $\beta \leq \frac{h}{L}$ to conclude. The constant $L$ is independent of $z$, since we could perform the same computation for $z' \in K$ considering $\zeta_{z'} = \zeta_z(\cdot - (z' - z))$ that is also $L$-Lipschitz. \hfill \square

We may now prove the uniform ergodic theorem:

Proof. (Theorem \[1\])
We will first prove the simple convergence of our process.

Since $\mathcal{L}^{(1)}$ is just noise, one has for all $x$ and for all $i \in [1, n]$ that
\[
X_{t,i}(x) = X_t(x) \cdot e_i
\]
is a martingale that lives in the compact space $K$ and is therefore bounded.

The martingale convergence theorem implies that for any $x$, $(X_{t,i}(x))_{t \geq 0}$ converges almost surely and in $L^2$ to $[X_\infty(x)]_i$ as $t$ goes to infinity. Let us prove that $(X_{t,i}(x))_{t \geq 0}$ converges necessarily to one of the $z \in K_0$. From now on the $x$ will be implicit.

Using Dambis, Dubins-Schwarz’s theorem, there exists an extension $\tilde{\Omega}$ of our probability space $\Omega$ and a Brownian motion on this space $\beta$ such that:
\[
X_{t,i} = \beta_{X_i} t
\]
where the quadratic variation of $X_i$ is given by:
\[
\langle X_i \rangle_t = \sum_{j=1}^n \int_0^t \sigma_{i,j} (X_s)^2 \, ds.
\]
Thus, since for almost any $\omega \in \tilde{\Omega}$ one has that $X_{t,i}(\omega)$ converges when $t$ tends to infinity, we have that $\langle X_i \rangle_t(\omega)$ converges.

But $\langle X_i \rangle_t$ is a sum of $n$ integrals from 0 to $t$ of positive functions, that is to say $n$ functions increasing with $t$ and thus for all $i, j \in [1, n]$ one
has:
\[ t \mapsto \int_0^t \sigma_{i,j}(X_s(\omega))^2 \, ds \]
converges and thus
\[ \sigma_{i,j}(X_t(\omega)) \xrightarrow{t \to +\infty} 0. \]
Eventually one has that:
\[ \sigma(X_\infty(\omega)) = 0, \]
and thus \( X_\infty(\omega) \in K_0. \)

Therefore we get that for all \( x \in K \):
\[
\mathbb{E}_x (f(X_\infty)) = \int_K f(z) H(x, dz) \\
= \sum_{z \in K_0} \mathbb{P}_x(X_\infty = z) f(z) \\
= \mathcal{P}_f(x).
\]
Finally since \( f \) is \( L \)-Lipschitz for some \( L \geq 0 \):
\[
\mathbb{E}_x (f(X_t)) - \mathcal{P}_f(x) \leq L \mathbb{E} (\|X_t(x) - X_\infty(x)\|_2) \\
\leq L \mathbb{E} (\|X_t(x) - X_\infty(x)\|_2)^{\frac{1}{2}} \\
\xrightarrow{t \to +\infty} 0,
\]
by using the theorem of convergence of martingales in \( L^2 \).

We thus proved the pointwise convergence but not the uniform one.

We notice that for any \( t > 0, x, x' \in K \), one has:
\[
\left| \mathcal{L}^{(1)} e^{t \mathcal{L}} f(x') - \mathcal{P} f(x') \right| \\
\leq L \mathbb{E} (\|X_t(x) - X_t(x')\|_2) + L \mathbb{E} (\|X_t(x) - X_\infty(x)\|_2) \\
+ L \mathbb{E} (\|X_\infty(x) - X_\infty(x')\|_2) \\
\leq L \mathbb{E} (\|X_t(x) - X_t(x')\|_2)^{\frac{1}{2}} + L \mathbb{E} (\|X_t(x) - X_\infty(x)\|_2)^{\frac{1}{2}} \\
+ L \mathbb{E} (\|X_\infty(x) - X_\infty(x')\|_2)^{\frac{1}{2}} \\
\leq L \mathbb{E} (\|X_t(x) - X_\infty(x)\|_2)^{\frac{1}{2}} + 2L \mathbb{E} (\|X_\infty(x) - X_\infty(x')\|_2)^{\frac{1}{2}}
\]
where we used in the last inequality that \( \|X_t(x) - X_t(x')\|_2^2 \) is a submartingale, and thus its expectation is increasing with \( t \).
Let \( \varepsilon > 0 \), using the last inequality, we have that if there is \( \eta_x > 0 \) such that for all \( x' \in B(x, \eta_x) \) one has:

\[
\mathbb{E} \left( \| X_\infty(x') - X_\infty(x) \|_2^2 \right) \leq \varepsilon^2,
\]

(4.11)

then for \( t > 0 \) such that

\[
\mathbb{E} \left( \| X_t(x) - X_\infty(x) \|_2^2 \right)^{\frac{1}{2}} \leq \varepsilon,
\]

we will have for all \( x' \in B(x, \eta_x) \):

\[
\left| e^{tL(1)} f(x') - \mathcal{P} f(x') \right| \leq 3L \varepsilon.
\]

Since \( K \) is a compact set, it can be covered by a finite number of balls of the form \( B(x, \eta_x) \), yet we have for all \( x \) that:

\[
\mathbb{E} \left( \| X_t(x) - X_\infty(x) \|_2^2 \right) \xrightarrow{t \to +\infty} 0,
\]

so eventually there exists \( M > 0 \) such that for all \( t > M \) and for all \( x' \in K \) we have:

\[
\left| e^{tL(1)} f(x') - \mathcal{P} f(x') \right| \leq 3L \varepsilon,
\]

and the theorem is proved.

We will now prove (4.11). Let \( \alpha, h \) be in \( ]0, 1[ \), Lemma 5 gives us that there exists \( \beta > 0 \), such that

\[
d_{W}(\delta_z, H(y, \cdot)) \leq \beta \quad \text{implies} \quad \mathbb{P}(X_\infty(y) \in B(z, h)) \geq 1 - \alpha.
\]

The function \( x \in K \mapsto H(x, \cdot) \in \mathcal{P}(K) \) is continuous and since \( K \) is compact, Heine’s theorem gives that it is uniformly continuous. Therefore there exists \( a > 0 \) such that \( |x - y| \leq a \) implies \( d_{W}(H(x, \cdot), H(y, \cdot)) \leq \beta \). Since for \( z \in K_0 \) it holds that \( H(z, \cdot) = \delta_z \), we get that for all \( z \in K_0 \):

\[
|z - y| \leq a \quad \text{implies} \quad \mathbb{P}(X_\infty(y) \in B(z, h)) \geq 1 - \alpha \quad (4.12)
\]

We have that \( (X_t(x))_t \) converges almost surely to \( X_\infty(x) \), therefore there exists \( M > 0 \) such that \( \mathbb{P}(X_M(x) \in B(X_\infty(x), \frac{a}{2})) \geq 1 - \alpha \). We now use the dependence on initial conditions: there exists \( \eta_x > 0 \) such that for all \( x' \in B(x, \eta_x) \), we have:

\[
\mathbb{P} \left( |X_M(x') - X_M(x)| \geq \frac{a}{2} \right) \leq \alpha.
\]

Therefore:

\[
\mathbb{P}(X_M(x') \in B(X_\infty(x), a)) \geq 1 - 2\alpha. \quad (4.13)
\]
By homogeneity of our process, we have:
\[
P(\{X_\infty(x') \in B(X_\infty(x), h)\})
= P(\{X_\infty(X_M(x')) \in B(X_\infty(x), h)\})
\geq P(\{X_\infty(X_M(x')) \in B(X_\infty(x), h)\} | X_\infty(x') \in B(X_\infty(x), a))
\times P(X_M(x') \in B(X_\infty(x), a))
\geq (1 - \alpha)(1 - 2\alpha) \quad \text{by (4.12) and (4.13)}
\geq 1 - 3\alpha - 2\alpha^2
\geq 1 - 5\alpha.
\]

Finally, if \( R = \sup_{z \in K} \|z\|_2^2 \), we have for all \( x' \in B(x, \eta_x) \):
\[
\mathbb{E}(\|X_\infty(x) - X_\infty(x')\|^2_2)
\leq 2R \mathbb{P}(X_\infty(x') \notin B(X_\infty(x), h)) + h^2 \mathbb{P}(X_\infty(x') \in B(X_\infty(x), h))
\leq 2R \times 5\alpha + h^2.
\]

We then choose \( h \) and \( \alpha \) small enough so that the last term is inferior to \( \varepsilon^2 \), which proves (4.11) and thus our theorem.

\[\square\]

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