Slow-light solitons

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A new type of soliton with controllable speed is constructed generalizing the theory of slow-light propagation to an integrable regime of nonlinear dynamics. The scheme would allow the quantum-information transfer between optical solitons and atomic media.

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Solitons are stable wave packets occurring in many areas of the physical world, from tsunamis, flocks of migrating arctic geese to light pulses in optical fibres. Their stability stems from the non-linear balance between dispersing and focusing processes. As a feature of non-linear dynamics, the speed of a soliton may depend on its amplitude. For example, the height of a solitary water wave determines its velocity. However, once a soliton is launched there is usually no further control over its speed. Here we show how to generate optical solitons in atomic media that can be slowed down or accelerated at will. Ultimately, this method will allow the storage, retrieval and possibly the manipulation of the quantum information of solitons in media. This idea extends the simplest scheme for slow-light propagation to a genuinely non-linear regime.

In the pioneering experimental demonstrations of slow light, atomic media store the shape of light pulses in the spin states of atoms in what is known as dark states. Usually, such experiments operate in a regime where the atomic spins deviate only slightly from a default direction, a regime of linear spin waves. Here we consider strong spin modulations where, as we show, the non-linear dynamics of light and atoms create polarization solitons, see Fig. 1 for an illustration. Unlike other solitons known so far, their speed can be controlled after they have been launched, in precisely the same way as for slow light. In quantum-information applications, such slow-light solitons could complement the use of quantum solitons in fibres with the advantage of storing quantum information in media and complement methods for quantum memory with the advantages of non-linear dynamics, in particular the intrinsic stability of solitons. Strong spin polarizations can be imaged by illuminating the sample from the side with resonant light of uniform polarization. Moreover, they could be manipulated with light or magnetic fields, possibly a step from quantum-information storage to quantum-information processing in atomic vapours.

Consider a cell filled with atomic vapour or a Bose-Einstein condensate illuminated with light propagating in one direction. Only three atomic levels shall interact with the light, an excited state and two degenerate ground states, for example two hyperfine levels.

As a consequence of the conservation of angular momentum, the two circular polarizations of light couple to either one of the ground states. This type of interaction between three-level atoms and light is called A configuration. In the presence of light of constant but arbitrary polarization the atoms relax by spontaneous emission to spin states pointing in the opposite direction of the optical polarization, so-called dark states, a process known as optical pumping. The light spin-polarizes the atoms. Once the atoms are in dark states, they follow the polarization of light by themselves, without spontaneous emission, as long as their excited states are sparsely populated. Suppose that the polarization of the light has varied in time and has imprinted a spatial modulation in the atomic spin distribution. Any subsequent light incident with uniform polarization will attempt to...
re-polarize the atoms, but, in turn, is re-polarized itself. The tail-end of the spin profile is reset to the default polarization and the front-end moves. The light thus carries the spin modulation along the medium, possibly modulating it further, with a speed that depends on the intensity. When the light is switched off, the spin profile stands still, as long as it is not eroded by diffusion, depolarizing atomic collisions or the spin precession in stray magnetic fields. Illuminating the atoms will continue to move the spin wave until it reaches the boundary of the medium where the stored spin profile emerges as a polarization profile of light. The pioneering demonstrations [10, 11] of stopping light [13] have used this method [14], but with an important qualification [14]: One of the polarization modes of light dominates the other. In this regime, the spin-perturbation waves are small, with linear dynamics, and they move without changing shape [14]. In the case of strong polarization modulations, the balance between the two competing processes involved, light polarizing the atoms and the atoms polarizing the light, may generate stable polarization structures, solitons [2, 3, 4].

To develop a quantitative description of slow-light solitons, we assume that the light propagates in positive $z$ direction with amplitudes that vary slowly in comparison with the carrier frequency $\omega$. We describe the left and right circular polarization amplitudes in terms of the Rabi frequencies $\Omega^\pm(t, z)$ and $\Omega^-(t, z)$, respectively, i.e. by the transversal components of the electric field strength divided by $\hbar$ and multiplied by the dipole moment $\kappa$ of the atomic transitions the light interacts with. We distinguish atoms with different detunings $\Delta$ of the transition frequencies from resonance with the light, caused, for example, by the Doppler effect of their thermal motion. We describe the quantum state of each atom in terms of the density matrices $\rho(t, z, \Delta)$ considering only the ground states $|\pm\rangle$ and the excited state $|e\rangle$ that interact with the light. In the slowly varying envelope approximation [20] the light amplitudes obey the approximative Maxwell equations

$$\frac{\partial \Omega^\pm}{\partial t} + \frac{\partial \Omega^\pm}{\partial z} = ig \int_{-\infty}^{+\infty} \langle e | \rho | \pm \rangle \nu d\Delta, \quad g = \frac{\kappa^2 \omega}{4\hbar\varepsilon_0} n. \quad (1)$$

Here $n(z)$ denotes the atom-number density profile of the medium and $\nu(\Delta)$ the relative distribution of the detuned atoms. $\varepsilon_0$ is the electric permeability of the vacuum. At this stage we neglect any atomic relaxation, but we determine later the condition when this is justified. Without relaxation the atomic density matrices evolve according to the Liouville equation in the interaction picture

$$\frac{\partial \rho}{\partial t} = -i[H, \rho],$$

$$H = -\Delta |e\rangle\langle e| - \sum_\pm \left( \frac{\Omega^\pm}{2} |\pm\rangle\langle \pm| + \frac{\Omega^\mp}{2} |\pm\rangle\langle \parallel| \right). \quad (2)$$

Despite the apparent complexity and non-linearity of the Maxwell-Liouville equations (1,2), they belong to the class of integrable systems [15]. The key element of an integrable system is a Lax pair [2, 3, 4] of matrices $U$ and $V$ that generates the equations of motion as the compatibility condition $\partial U/\partial t - \partial V/\partial z + [U, V] = 0$ for a variable complex spectral parameter, which allows the use of the Inverse Scattering Method [2, 3, 4]. For the three-level Maxwell-Liouville system (1,2) the detuning $\Delta$ serves as the spectral parameter, $\tau$ means the retarded time $t-z/c$, and $\zeta$ denotes the spatial coordinate $z/c$. One verifies that the Lax pair is [21]

$$U = -iH, \quad V = -\frac{iq}{2} \int_{-\infty}^{+\infty} \frac{\rho(\Delta')\nu(\Delta') d\Delta'}{\Delta - \Delta'}.$$ \quad (3)

Therefore, the theory of slow light is integrable [2, 3, 4], which comes in useful for finding analytic solutions, and, more importantly, which allows the existence of stable non-linear waves, solitons.

Various methods for finding soliton solutions start from a Lax pair of type (3). We use a modification of the Zakharov-Shabat dressing method [22] for a variable background field $\Omega(\tau)$ of constant polarization with the atoms being in the corresponding dark states. We obtain the single-soliton solution for the field $\Omega^\pm$ and the atoms $\rho = \psi_\alpha \psi_\alpha^\dagger$ as

$$\Omega^\pm = \Omega \varphi^\pm,$$

$$\varphi_+ = \frac{\xi - i\eta \tanh Q}{\xi + i\eta}, \quad \varphi_- = \frac{\eta \exp(-i\Phi) \sech Q}{\xi + i\eta},$$

$$Q = Q_0 - \int \frac{\eta [\xi^2 + \eta^2]}{2(\xi - \Delta)^2 + \eta^2} - \int \frac{\eta \nu d\Delta d\zeta}{2(\xi - \Delta)^2 + \eta^2},$$

$$\Phi = \Phi_0 + \int \frac{\xi \Omega^2 d\tau}{2(\xi - \Delta)^2 + \eta^2} - \int \frac{\eta \nu d\Delta d\zeta}{2(\xi - \Delta)^2 + \eta^2},$$

$$\psi_+ = \frac{(\xi + i\eta)\varphi_- - \Delta}{\xi - \Delta + i\eta}, \quad \psi_- = \frac{(\xi + i\eta)\varphi_+ - \Delta}{\xi - \Delta + i\eta}. \quad (4)$$

in the limit $\xi^2 + \eta^2 \gg |\Omega|^2$ where $\xi, \eta, Q_0, \Phi_0$ are real constants and $\Omega(\tau)$ is a complex function. One verifies that the results (4) solve the reduced Maxwell equation (1) and the approximate equation (A13) of Ref. [19] that describes the atomic dynamics. In the solution (4) the incident field is left-circularly polarized. We can, however, apply the transformation

$$\begin{pmatrix} \Omega^\prime_+ \\ \Omega^\prime_- \end{pmatrix} = B \begin{pmatrix} \Omega^+_0 \\ \Omega^-_0 \end{pmatrix}, \quad \begin{pmatrix} \psi^\prime_+ \\ \psi^\prime_- \end{pmatrix} = B^\dagger \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

with constant unitary matrices $B$ to describe an arbitrary initial polarization state. The results (4) are generalizations of some known solutions [23] to slow-soliton propagation in the presence of inhomogeneous line broadening $\nu(\Delta)$, variable spatial density $n(\zeta)$ and, more importantly, time-dependent background $\Omega(\tau)$.
Slow-light solitons turn out to be pure polarization structures that, above a critical strength, perform complete polarization twists, see Fig. 1. Let us discuss their properties based on the analytic solution (1). A single soliton depends on four real parameters $\xi, \eta, Q_0, \Phi_0$ and on one arbitrary complex function $\Omega(t)$ for which we require that $|\Omega|^2 \ll |\xi|^2 + |\eta|^2$. We obtain from the solution (1) that

$$|\Omega_+(t,z)|^2 + |\Omega_-(t,z)|^2 = |\Omega(t-z/c)|^2, \quad (6)$$

which indicates that $|\Omega|^2$ describes the total intensity and that any incident intensity modulation propagates through the medium at the speed of light in vacuum. The parameter $Q_0$ characterizes the position and $\Phi_0$ the phase of the polarization deformation, whereas $\xi$ and $\eta$ constitute the complex spectral parameter $\xi + i\eta$ whose argument $\theta$ describes the maximal polarization deviation. Polarization twists occur when $|\theta| \geq \pi/2$. Beyond the soliton the light returns to the incident polarization state, acquiring a geometric phase [24] of $2\theta$. The atoms are in pure quantum states with probability amplitudes $\psi_0(t,z,\Delta)$. In regions of nearly uniform optical polarization the atoms are essentially in dark states, but their ability to follow a changing polarization of light is limited due to the detuning $\Delta$. Outside the medium the solution (1) for $g=0$ describes the shape of the polarization pulse required to launch the soliton. In the medium, the soliton propagates with velocity

$$v = \frac{|\Omega|^2}{2(\xi^2 + \eta^2)} \left( \int_{-\infty}^{+\infty} \frac{g\nu(\Delta) d\Delta}{(\xi - \Delta)^2 + \eta^2} \right)^{-1} \sim \frac{|\Omega|^2}{2g}, \quad (7)$$

for $v/c \ll 1$ and in the second approximate expression for $\xi^2 + \eta^2$ much larger than a characteristic detuning $\Delta^2$. We obtain from the solution (1) the length scale of the polarization profile, the soliton length

$$l_s = \frac{4(\xi^2 + \eta^2)}{|\eta| |\Omega|^2} v \sim \frac{2c(\xi^2 + \eta^2)}{|\eta| g}. \quad (8)$$

Typically for solitons [2, 8, 11], the length is adjusted to the spectral parameter. Typically $23$, the speed hardly depends on any parameters, except for the external light intensity in exactly the same way as for traditional slow light [13], allowing control over the soliton after it has been launched.

A critical issue for slow-light solitons is the potential threat of absorption, in contrast to fast self-induced-transparency solitons [2, 13] that excite and de-excite the atoms on times much shorter than the relaxation-time scales of the atoms. Consider a slow-light soliton of length $l_s$ traversing the distance $l$ in the medium. The principal relaxation mechanism is spontaneous emission from the excited state $|e\rangle$. The population of $|e\rangle$ is largest for zero detuning where $|\psi_e|^2 \sim 2c/\gamma l_s^2 v$ at the peak of the soliton. We use this case to estimate the absorption. First, we express $g$ defined in the Maxwell equations (1) in terms of the Einstein $A$ coefficient [25]. We get $g = 3/(16\pi) c A n \lambda^2$, where $\lambda$ denotes the optical wavelength $\lambda = 2\pi c/\omega$. In practice [10, 11] the excited state decays into three ground-state levels, not only the $|\pm\rangle$ selected by the propagation direction of light. We estimate the fractional loss $\eta_f$ as $3A$ multiplied by the propagation time $l/v$ times the maximal excited-state population, and obtain

$$\eta_f = \frac{32\pi l \lambda}{n \lambda^3 l_s^2} \quad (9)$$

In the experiments [10, 11] $n \lambda^3$ has been in the order of 1 or larger. Losses are thus negligible for solitons much longer than the geometric mean of wavelength and distance travelled, leaving enough room for soliton propagation. This requirement resembles the condition for traditional slow light [13] where the pulses are restricted in frequency space, which amounts to a minimal length scale in ordinary space.

Having established that slow-light solitons are feasible, we identify their quantum properties. Fundamentally, the solitons are quantum excitations of the electromagnetic field interacting with the atoms of the medium, but they are very close to the classical limit [4]. Their quantum nature manifests itself in fluctuations around the classical amplitudes. A classical soliton is stable against fluctuations except when the fluctuations occur in its degrees of freedom, its parameters, in our case in any $\alpha \in \{\xi, \eta, Q_0, \Phi_0\}$, because such parameter fluctuations transform the soliton into another stable soliton. The Goldstone modes generated by infinitesimal parameter changes define the relevant quantum modes of the soliton [4]. Therefore, we represent the quantum fields $\hat{\Omega}_\pm$ and $\hat{\psi}_a$ of light and atoms as

$$\hat{\Omega}_\pm = \Omega_\pm + \sum_\alpha \frac{\partial \Omega_\pm}{\partial \alpha} (\hat{\alpha} - \alpha),$$
$$\hat{\psi}_a = \psi_a + \sum_\alpha \frac{\partial \psi_a}{\partial \alpha} (\hat{\alpha} - \alpha). \quad (10)$$

The quantum properties of the soliton modes depend on the parameterization of the classical soliton, chosen here [4] such that they are particularly simple. We deduce the commutation relations of the mode operators $\hat{\alpha}$ from the quantum field theory of light [25] in the slowly-varying envelope approximation [20].

Quantum field theories may start from a fundamental quantum commutator and a classical mode decomposition with an appropriate scalar product. Here we describe the field by two components $\hat{A}_\pm(\mathbf{r}, t)$ of the vector potential in the Coulomb gauge [25] and SI units with the equal-time commutator [25]

$$[\hat{A}_\pm(t,z_1), \varepsilon_0 \sigma \partial_\pm \hat{A}_\pm(t,z_2)/\partial t] = i \hbar \delta(z_1 - z_2)$$

where $\sigma$ denotes the longitudinal cross section of the medium. In the slowly-varying envelope approximation [25] the Rabi
frequencies $\Omega_{\pm}$ are, apart from a prefactor $\kappa/\hbar$, the components of the electric field strength $-\partial A_{\pm}/\partial t$ that propagate with the carrier $\exp(-i\omega t)$. Therefore we define $\hat{\Omega}_{\pm} = \kappa/(2\hbar) \exp(-i\omega t) (i\omega A_{\pm} - \partial \dot{A}_{\pm}/\partial t)$ and get the commutation relation in retarded time

$$[\hat{\Omega}_{\pm}(\tau_1, \zeta), \hat{\Omega}_{\pm}(\tau_2, \zeta)] = \frac{\kappa^2 \omega}{2c_0 \hbar \sigma c} \delta(\tau_1 - \tau_2).$$

(11)

Inspired by Lai’s and Haus’ quantum theory of self-induced-transparency solitons $\mathcal{Q}$ we define the scalar product of the fluctuation modes in the style of Poisson brackets

$$\{\alpha, \beta\} \equiv \frac{1}{8\hbar} \int_{-\infty}^{+\infty} \sum_{\pm} \left( \frac{\partial \Omega_{\pm}^* \partial \Omega_{\pm}}{\partial \alpha \partial \beta} - \frac{\partial \Omega_{\pm} \partial \Omega_{\pm}^*}{\partial \alpha \partial \beta} \right) d\tau.$$  

(12)

One verifies by straightforward but lengthy calculations that the solutions $\mathcal{Q}$ are designed such that all $\{\alpha, \beta\}$ vanish, except $\{\hat{Q}_0, \hat{\xi}\} = \{\hat{\Phi}_0, \eta\} = \{-\eta, \phi_0\} = 1$. These orthonormality conditions serve to deduce the commutation relations of the fluctuation operators from the commutator $\mathcal{Q}$. We re-scale the quantum spectral parameter as $\xi + i\eta = (\xi + i\eta_0) \kappa^2 \omega/(16\xi_0 \hbar \sigma c)$. Following the procedure by Lai and Haus $\mathcal{Q}$ we obtain the Heisenberg commutation relations

$$i = [\hat{Q}_0, \hat{\xi}] = [\hat{\Phi}_0, \eta] \quad 0 = [\hat{\xi}, \eta] = [\hat{Q}_0, \hat{\phi}_0] = [\hat{\xi}_0, \eta] = [\hat{\phi}_0, \xi].$$

(13)

The components of the quantum spectral parameter, $\hat{\xi}$ and $\hat{\eta}$, thus represent the canonical momenta of two independent Heisenberg pairs with $\hat{Q}_0$ and $\hat{\Phi}_0$ as their partners. This implies that the position $Q_0$ of the soliton is complementary to $\hat{\xi}$ and the phase $\Phi_0$ of the polarization twist is complementary to its magnitude $\eta$. The quantum properties of these two modes encode the quantum information $\mathcal{Q}$ of the soliton, regardless of whether it is moving or frozen. The overall amplitude $\Omega$ plays the role of the control beam $\mathcal{Q}$, $\mathcal{Q}$. When the soliton is stopped by switching off the incident light the quantum information is stored in the medium, since the atomic modes $\partial \nu_{\alpha}/\partial \alpha$ do not vanish when $\Omega(\tau) \rightarrow 0$. Switching on the light will set the soliton in motion and will release the stored quantum information in the polarization of the light emerging at the boundary of the medium.

**Summary.** The theory of slow light $\mathcal{Q}$ is integrable $\mathcal{Q}$. A slow-light soliton is a polarization structure propagating with a speed that is proportional to the total intensity of the incident light. Entire solitons can be stopped and retrieved in atomic media with their quantum information stored in atomic coherences.

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