A CLASS OF PSEUDOREAL RIEMANN SURFACES WITH
DIAGONAL AUTOMORPHISM GROUP

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Abstract. A Riemann surface $S$ having field of moduli $\mathbb{R}$, but not a field of
definition, is called pseudoreal. This means that $S$ has anticonformal automor-
phisms, but none of them is an involution. We call a Riemann surface $S$ plane
if it can be described by a smooth plane model of some degree $d \geq 4$ in $\mathbb{P}^2_{\mathbb{C}}$.

We characterize pseudoreal-plane Riemann surfaces $S$, whose conformal
automorphism group $\text{Aut}^+ (S)$ is $\text{PGL}_3 (\mathbb{C})$-conjugate to a finite non-trivial
group $G$ that leaves invariant infinitely many points of $\mathbb{P}^2_{\mathbb{C}}$. In particular, we
show that such pseudoreal-plane Riemann surfaces exist only if $\text{Aut}^+(S)$ is
cyclic of even order $n$ dividing the degree $d$. Explicit examples are given, for
any degree $d = 2pm$ with $m > 1$ odd, $p$ is prime and $n = d/p$.

1. Introduction

Any Riemann surface $S$ of genus $g$ can be understood by a certain irreducible
projective curve $C$ of genus $g$ over $\mathbb{C}$. A complex subfield $K$ is a field of definition
for $S$ if $C$ can be defined over $K$. The field of moduli of $S$ is the intersection of all
fields of definition for $S$.

There exists in the literature an alternative definition for the field of moduli, rel-
tyve to a given field extension $L/K$, which is commonly used; Given an irreducible
projective curve $C$ defined over a field $L$, the field of moduli of $C$ relative to $L/K$,
denoted by $M_{L/K}(C)$, is the fixed subfield of $L$ by the subgroup

$$U_{L/K}(C) := \{ \sigma \in \text{Gal}(L/K) | C \text{ isomorphic over } L \text{ to } \sigma C \},$$

where $\text{Gal}(L/K)$ denotes the group of all $K$-automorphisms of the field $L$. We are
interested in the particular case when $K = \mathbb{R}$ and $L = \mathbb{C}$.

A necessary and sufficient condition for the field of moduli of $S$ to be a field of defi-
nition was provided by Weil [26]:

**Theorem 1.1** (Weil’s criterion of descent). Let $C$ be an irreducible projective algebraic curve defined over a field $L$, and let $L/K$ be a finite Galois extension. There is an irreducible projective algebraic curve $C'$ over $K$ and an $L$-isomorphism $g : C' \times_K L \to C$ if and only if, for any $\sigma \in \text{Gal}(L/K)$, there exists an $L$-
isomorphism $f_{\sigma} : \sigma C \to C$ such that $f_{\sigma} \circ \sigma f_{\tau} = f_{\sigma \tau}$ for all $\sigma, \tau \in \text{Gal}(L/K)$. Moreover, $f_{\sigma} \circ \sigma g = g$ for all $\sigma \in \text{Gal}(L/K)$.

**Remark 1.2.** A. Weil in [26] observed that the above result is valid as long the
overfield is finitely generated over the lower field. In particular, it holds for $L = \mathbb{C}$,
$K = \mathbb{Q}$ and $C$ of genus $g \geq 2$ (cf. [6, Remark 2.2]).

A Riemann surfaces $S$ with trivial automorphism group needs to be defined over
its field of moduli, since the above condition becomes trivially true. It is also known
that the field of moduli is a field of definition for $S$ when the genus $g$ is at most 1.
However, if $g > 1$ and $\text{Aut}^+(S)$ is non-trivial, then Weil’s conditions are difficult
to be checked and so there is no guarantee that the field of moduli is a field of definition. This was first pointed out by Earle [9] and Shimura [23]. More concretely, in page 177 of [23], the first examples not definable over their field of moduli are introduced, which are hyperelliptic curves over $\mathbb{C}$ with two automorphisms. There are also examples of non-hyperelliptic curves not definable over their field of moduli in [2, 12, 15, 20]. For all these (explicit) examples, the field of moduli is always a subfield of $\mathbb{R}$, and they can be defined over an imaginary extension of degree two of the field of moduli. The present author and et. al. constructed in [6] the first (explicit) examples of Riemann surfaces, which are definable over the reals but cannot be defined over the field of moduli.

1.1. Case of hyperelliptic curves. By the work of Mestre [21], Huggins [16, 15], Lercier-Ritzenthaler [17] and Lercier-Ritzenthaler-Sijsling [19], one gets the answer in the case of hyperelliptic curves $\mathcal{C}$ over a perfect field $L$ of characteristic $p \neq 2$. We issue the next table from [19]:

| $H = \text{Aut}_{+}(\mathcal{C})/(t)$ | Conditions | The field of moduli= A field of definition |
|--------------------------------------|------------|---------------------------------------------|
| Not tamely cyclic                    |            | Yes                                         |
| Tamely cyclic with $\# H > 1$        | $g$ odd, $\# H$ odd | Yes                                         |
|                                      | $g$ even or $\# H$ even | No                                          |
| Tamely cyclic with $\# H = 1$        | $g$ odd   | Yes                                         |
|                                      | $g$ even  | No                                          |

By tamely cyclic, we mean that the group is cyclic of order not divisible by the characteristic $p$.

1.2. Case of smooth plane curves. A linear transformation $A = (a_{i,j})$ of the projective plane $\mathbb{P}^2_L$ over a field $L$ is often written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z],$$

where $\{X, Y, Z\}$ are the homogenous coordinates of $\mathbb{P}^2_L$.

**Definition 1.3.** A smooth projective curve $\mathcal{C}$ of genus $g \geq 3$ over an algebraically closed field $\overline{\mathbb{L}}$ is called a smooth plane curve of genus $g$ over $\overline{\mathbb{L}}$ if $\mathcal{C}$ is $\overline{\mathbb{L}}$-isomorphic to a non-singular plane model $F_{\mathcal{C}}(X, Y, Z) = 0$ in $\mathbb{P}^2_{\overline{\mathbb{L}}}$, where $F_{\mathcal{C}}(X, Y, Z)$ is a homogenous polynomial of degree $d$ with coefficients in $\overline{\mathbb{L}}$, and $g = (d - 1)(d - 2)/2$.

The problem for smooth plane curves was addresses by B. Huggins in [15, Chps. 6 and 7]. In particular, we have:

**Theorem 1.4.** (B. Huggins, [15, Theorem 6.4.8]) Let $\overline{T}$ be a fixed algebraic closure of a perfect field $L$ of characteristic $p \neq 2$. Then, if $\mathcal{C}$ is a smooth plane curve of genus $g \geq 3$ defined over $\overline{T}$, then the field of moduli $\mathcal{M}_{\overline{T}/L}(\mathcal{C})$ for $\mathcal{C}$, relative to the Galois extension $\overline{T}/L$, is a field of definition if the automorphism group $\text{Aut}_{+}(\mathcal{C})$ is not $\text{PGL}_3(\overline{T})$-conjugate to a diagonal subgroup of $\text{PGL}_3(\overline{T})$, or to one of the Hessian groups $\text{Hess}$, with $\ast \in \{18, 36\}$, or a semidirect product $B \rtimes A$ for some finite diagonal subgroup $A$ of $\text{PGL}_3(\overline{L})$ and a non-trivial $p$-group $B$ consisting entirely of elements of the shape $[X : \alpha X + Y : \beta X + \gamma Y + Z]$.

1.3. Pseudoreal Riemann surfaces. The surface $\mathcal{S}$ is called real if $\mathbb{R}$ is a field of definition of it; this is equivalent for $\mathcal{S}$ to admit an anticonformal automorphism of order two (as a consequence of Weil’s descent theorem [26]). Also, the field of moduli of $\mathcal{S}$ is a subfield of $\mathbb{R}$ if and only if it is isomorphic to its complex conjugate, equivalently, if it admits anticonformal automorphisms [9, 23, 24]. Those surfaces of genus $g$ with real field of moduli corresponds to the real points of the moduli
space $\mathcal{M}_g$. Riemann surfaces whose field of moduli is real but are not real are usually called pseudoreal.

Theorem 1 in [25, p. 48] assures the existence of pseudoreal Riemann surfaces $S$ for any genus $g \geq 2$. However, it does not construct algebraic models for $S$. Moreover, once the genus $g \geq 2$ is fixed and the list of all automorphism groups acting on Riemann surfaces of this genus $g$, with their signature and generating vectors, is determined, then one can classify the automorphism groups of pseudoreal Riemann surfaces of genus $g$ by using the algorithm recently exposed in [1].

1.3.1. Pseudoreal-plane Riemann surfaces. A pseudoreal Riemann surface $S$ of genus $g \geq 3$ admitting a non-singular plane model in $\mathbb{P}_2^d$ is called pseudoreal-plane. In this case, $S$ is non-hyperelliptic of genus $g = (d-1)(d-2)/2$ for some integer $d \geq 4$ (cf. [6, Lemma 4.1]).

It has been announced (without a proof) in [15, p. 136] that pseudoreal-plane Riemann surfaces with automorphism group Hess$_{18}$ do not exist. Furthermore, we have seen explicit examples of pseudoreal-plane Riemann surfaces whose conformal automorphism group is Hess$_{36}$ or diagonal in [15, Chp. 7]. Therefore, by the virtue of Theorem 1.4, one concludes:

**Corollary 1.5.** A Riemann surface $S$ is pseudoreal-plane only if $\text{Aut}_+(S)$ is $\text{PGL}_3(\mathbb{C})$-conjugate to Hess$_{36}$ or to a diagonal group.

Because of the above results, we were wondering, once the genus $g \geq 3$ is fixed, about the existence of pseudoreal-plane Riemann surfaces $S$ of genus $g$ whose $\text{Aut}_+(S)$ is diagonal. We will see that, in contrast to pseudoreal Riemann surfaces, that the answer in general is No, that is they are not always exist. We also characterize $\text{Aut}_+(S)$ for pseudoreal-plane Riemann surfaces and give explicit examples.

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2. On conformal automorphism groups of plane Riemann surfaces

**Definition 2.1.** For a non-zero monomial $cX^iY^jZ^k$ with $c \in \mathbb{C} \setminus \{0\}$, its exponent is defined to be $\max\{i, j, k\}$. For a homogenous polynomial $F(X,Y,Z)$, the core of it is defined to be the sum of all terms of $F$ with the greatest exponent. Now, let $S_0$ be a plane Riemann surfaces, a pair $(S,H)$ with $H \leq \text{Aut}_+(S)$ is said to be a descendant of $S_0$ if $S$ is defined by a homogenous polynomial whose core is a defining polynomial of $S_0$ and $H$ acts on $S_0$ under a suitable change of the coordinates system, i.e. $H$ is conjugate to a subgroup of $\text{Aut}_+(S_0)$.

By [22, §1-10] and the proof of Theorem 2.1 in [10], we conclude:

**Theorem 2.2** (Mitchell [22], Harui [10]). Let $G$ be a subgroup of conformal automorphisms of a plane Riemann surface $S$ of degree $d \geq 4$. Then, one of the following holds:

(i) $G$ fixes a line in $\mathbb{P}_2^d$ and a point off this line.

(ii) $G$ fixes a triangle $\Delta \subset \mathbb{P}_2^d$, i.e. a set of three non-concurrent lines, and neither line nor a point is leaved invariant. In this case, $(S,G)$ is a descendant of the the Fermat curve $F_d : X^d + Y^d + Z^d = 0$ or the Klein curve $K_d : X Y^{d-1} + Y Z^{d-1} + Z X^{d-1} = 0$.

(iii) $G$ is $\text{PGL}_3(\mathbb{C})$-conjugate to a finite primitive subgroup namely, the Klein group $\text{PSL}(2,7)$, the icosahedral group $A_5$, the alternating group $A_6$, the Hessian group Hess, with $* \in \{36,72,216\}$. 

We use $\zeta_n$ for a fixed primitive $n$-th root of unity inside $\mathbb{C}$.

**Definition 2.3.** By an homology of period $n \in \mathbb{Z}_{\geq 1}$, we mean a projective linear transformation of the plane $\mathbb{P}_d^2$, which acts up to $\text{PGL}_3(\mathbb{C})$-conjugation, as

$$(X : Y : Z) \mapsto (\zeta_n X : Y : Z).$$

Such a transformation fixes pointwise a line (its axis) and a point off this line (its center). Otherwise, it is called a non-homology.

**Theorem 2.4** (Mitchell [22]). Let $G$ be a finite group of $\text{PGL}_3(\mathbb{C})$. If $G$ contains an homology of period $n \geq 4$, then it fixes a point, a line or a triangle. Moreover, the Hessian group $\text{Hess}_{216}$ is the only finite subgroup of $\text{PGL}_3(\mathbb{C})$ that contains homologies of period $n = 3$, and does not leave invariant a point, a line or a triangle.

### 2.1. Preliminaries on diagonal conformal automorphism groups.

The group of all $3 \times 3$ projective linear matrices of diagonal shapes is denoted by $D(\mathbb{C})$. A finite non-trivial group $G$, representable inside $\text{PGL}_3(\mathbb{C})$, is said to be diagonal if $\varphi(G)$ is $\text{PGL}_3(\mathbb{C})$-conjugate to a subgroup of $D(\mathbb{C})$, for some injective representation $\varphi : G \hookrightarrow \text{PGL}_3(\mathbb{C})$.

Firstly, we show:

**Proposition 2.5.** Let $G$ be a finite non-trivial diagonal group, consisting entirely of homologies. Then, it is either cyclic or conjugate to

$$\varphi_0(\mathbb{Z}/2\mathbb{Z}\times \mathbb{Z}/2\mathbb{Z}) := (\text{diag}(1,-1,1), \text{diag}(1,1,-1)).$$

**Proof.** Fix an injective representation $\varphi(G) \leq D(\mathbb{C})$ and suppose that $\varphi(G) \neq \varphi_0(\mathbb{Z}/2\mathbb{Z}\times \mathbb{Z}/2\mathbb{Z})$. Hence, there must be an homology $\phi \in \varphi(G)$ of order $m > 2$, since $\varphi_0(\mathbb{Z}/2\mathbb{Z}\times \mathbb{Z}/2\mathbb{Z})$ is the unique non-cyclic diagonal subgroup whose elements have orders at most $2$. There is no loss of generality to assume that $\phi = \text{diag}(1,1,\zeta_m)$, in particular its axis is the reference line $L_3 : Z = 0$ and its center is the reference point $P_3 = (1 : 0 : 0)$. Take $\psi \in \varphi(G) \setminus \langle \phi \rangle$ (if $\varphi(G) \setminus \langle \phi \rangle = \emptyset$, then $\varphi(G)$ is cyclic and there is nothing to prove further). If $\psi$ has a different axis from $L_3$, then $\phi^s \psi \in \varphi(G)$ is a non-homology for a suitable choice of the integer $s$ because $m > 2$; for example, write $\psi$ as $\text{diag}(1,\zeta_{m'},1)$ for some integer $m' > 1$, hence $\phi^s \psi = \text{diag}(1,\zeta_{m'},\zeta_m)$. So, we may set $s = 1$ when $m \neq m'$, and $s = 2$ otherwise. In both cases, $\phi^s \psi$ is a non-homology in $\varphi(G)$, which conflicts our assumption that $\varphi(G)$ is made entirely of homologies (Definition 2.3). Therefore, all elements of $\varphi(G)$ admit the same axis and the same center, i.e. each is of the shape $\text{diag}(1,1,\zeta_n)$ for some $n \in \mathbb{N}$. Consequently, $\varphi(G)$ is contained in the cyclic group generated by $\text{diag}(1,1,\zeta_n)$, where $n_0$ is the least common multiple of the orders of the elements of $\varphi(G)$. Thus $\varphi(G)$ is itself cyclic. \hfill $\square$

**Definition 2.6** (The Hessian groups). By $\text{Hess}_{18}$ we mean the group of order 18 generated by $S := [X : \zeta_3 Y : \zeta_3^2 T], T := [Y : Z : X]$ and $R := [X : Z : Y]$. The group $\text{Hess}_{36}$ is the one generated by $\text{Hess}_{18}$ and $V := [X + Y + Z : X + \zeta_3 Y + \zeta_3^2 Z : X + \zeta_3^2 Y + \zeta_3 Z]$.

**Proposition 2.7.** The Hessian groups $\text{Hess}_*$, for $* \in \{18, 36\}$, are not diagonal.

**Proof.** Assume on the contrary that $\varphi(\text{Hess}_*) \leq D(\mathbb{C})$ for some $\varphi$. By Definition 2.6, $\varphi(\text{Hess}_*)$ contains a non-homology $\phi$ of order 3, and we may write it as $\text{diag}(1,\zeta_3,\zeta_3^2)$. Moreover, there should be another element $\psi$ of order 3, such that $\psi \phi = \phi \psi$. Hence $\psi$ should be of the shape $\{[Y : Z : X], [Z : X : Y]\}$ modulo $D(\mathbb{C})$. Thus $\psi \notin D(\mathbb{C})$, which conflicts the fact $\varphi(\text{Hess}_*) \leq D(\mathbb{C})$. \hfill $\square$

**Lemma 2.8** (Mitchell [22]). Let $\phi$ be a $3 \times 3$ complex projective linear transformation of order $n \in \mathbb{Z}_{\geq 1}$. 


(i) If $\phi$ is an homology, then the fixed points of $\phi$ consists entirely of its center and all points on its axis. In particular, every triangle whose set of vertices is pointwise fixed by $\phi$ contains its center as a vertex.

(ii) If $\phi$ is a non-homology, then it fixes exactly three points. In particular, there is a unique triangle whose vertices are pointwise fixed by $\phi$.

Let $\varrho(G)$ be a finite non-trivial group in $\text{PGL}_3(\mathbb{C})$. If $F_{\varrho(G)}(X,Y,Z) = 0$ is a family of plane Riemann surfaces in $\mathbb{P}^2_\mathbb{C}$, with conformal automorphism group exactly $\varrho(G)$, then isomorphisms between two curves in the same family $F_{\varrho(G)}(X,Y,Z) = 0$ (in particular with identical conformal automorphism group) are clearly given by $3 \times 3$ projective matrices in the normalizer $N_{\varrho(G)}(\mathbb{C})$ of $\varrho(G)$ in $\text{PGL}_3(\mathbb{C})$. Therefore, it is practical to compute the normalizer for each case.

**Proposition 2.9** (Normalizer). Let $\varrho(G) \leq D(\mathbb{C})$ be a finite non-trivial group.

(i) If $\varrho(G)$ contains a non-homology $\phi = \text{diag}(\zeta_n^a, \zeta_n^b, 1)$, then $N_{\varrho(G)}(\mathbb{C}) = \langle D(\mathbb{C}), H \rangle$ where $S_3$ is the symmetry group $\langle [X : Z : Y], [Z : X : Y] \rangle$ of order 6.

(ii) If $\varrho(G)$ is generated by an homology $\phi = \text{diag}(1, 1, \zeta_n)$ for some $n \in \mathbb{Z}_{\geq 2}$, then $N_{\varrho(G)}(\mathbb{C}) = \text{GL}_2, Z(\mathbb{C})$, where $\text{GL}_2, Z(\mathbb{C})$ is the group of all projective linear matrices of the shape

$$\begin{pmatrix}
\ast & \ast & 0 \\
\ast & \ast & 0 \\
0 & 0 & 1
\end{pmatrix}.$$

**Proof.** Using Lemma 2.8 and the assumption that $\varrho(G)$ is diagonal, we deduce that there is always a unique set $V$, which is fixed pointwise by $\varrho(G)$. We get $V = \{ P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0), P_3 = (0 : 0 : 1) \}$ if a non-homology is present in $\varrho(G)$, and $V$ is formed by all points of the line $L_3 : Z = 0$ and the point $P_3$, otherwise. Therefore, $V$ is also fixed by $N_{\varrho(G)}(\mathbb{C})$, and the computations are straightforward. □

3. **Pseudoreal-plane Riemann surfaces with diagonal automorphism group, fixing infinitely many points in $\mathbb{P}^2_\mathbb{C}$**

Let $\mathcal{M}_g$ be the (coarse) moduli space of smooth curves of genus $g \geq 3$ over $\mathbb{C}$.

**Definition 3.1.** By $\mathcal{M}_g^{(h)}_{n, \text{diag}}$ where $n \geq 2$ is an integer, we mean the substratum of $\mathcal{M}_g$ of plane Riemann surfaces $\mathcal{S}$ of genus $g = (d - 1)(d - 2)/2 \geq 3$, such that $\text{Aut}_+(\mathcal{S})$ is a finite diagonal subgroup of $\text{PGL}_3(\mathbb{C})$ made entirely of homologies. We also can assume that $n$ is maximal, in the sense that any automorphism of $\mathcal{S}$ has order at most $n$.

Given $\mathcal{S} \in \mathcal{M}_g^{(h)}_{n, \text{diag}}$, we know by Proposition 2.5 that $\text{Aut}_+(\mathcal{S})$ is either cyclic or $\text{PGL}_3(\mathbb{C})$-conjugate to $\langle \text{diag}(1, -1, 1), \text{diag}(1, 1, -1) \rangle$. Fix a non-singular plane model $F_\mathcal{S}(X,Y,Z) = 0$ for $\mathcal{S}$ over $\mathbb{C}$ of degree $d$, such that $\text{Aut}_+(F_\mathcal{S}) \leq D(\mathbb{C})$. Badr-Bars in [5], following the techniques of Dolgachev in [8] for genus 3 curves (see also [7, §2.1]), determined the possible cyclic groups, which can appear inside $\text{Aut}_+(F_\mathcal{S})$, moreover the associated defining equation is also given in each situation. In particular, one deduces that the stratum $\mathcal{M}_g^{(h)}_{n, \text{diag}}$ might be non-empty only if $n$ divides $d$ or $d - 1$.

For $u = 1, 2$, consider the sets

$$S(u)_n := \{ u \leq j \leq d - 1 : d - j \equiv 0 (\text{mod } n) \}.$$
Proposition 3.2. (Badr-Bars [5, Theorem 7]) The stratum \((\widetilde{M}^g_{\mathbb{P}})_{n,\text{diag}}^h\) is non-empty only if \(n\) divides \(d\) or \(d - 1\). More concretely, if \(n\mid d\), then \(F_\Sigma(X,Y,Z) = 0\) has the form
\[
C_1 : Z^d + \sum_{j \in S(1)_n} Z^{d-j}L_{j,Z} + L_{d,Z} = 0,
\]
and if \(n\mid d - 1\), we get the form
\[
C_2 : Z^{d-1}Y + \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + L_{d,Z} = 0.
\]
Here \(L_{j,Z}\) is a generic homogeneous polynomial of degree \(j\) where the variable \(Z\) does not appear.

Remark 3.3. By non-singularity, the binary form \(L_{d,Z}(X,Y)\) in Proposition 3.2 cannot have any repeated linear factors for any specializations of the parameters in \(\mathbb{C}\). Otherwise, we may assume (up to \(\mathbb{C}\)-isomorphism via a change of the variable \(X\)) that the repeated factor is the line \(X = 0\). Therefore, \(L_{d,Z}(X,Y)\) reduces to \(X^2L_{d-2,Z}(X,Y)\). However, \(d - 1 \notin S(u)_n\) for \(u = 1, 2\), since \(n > 1\). Then, \(F_\Sigma(X,Y,Z) = 0\) has the form \(Z^2G(X,Y,Z) + X^2L_{d-2,Z}(X,Y) = 0\), which is singular at the reference point \((0 : 1 : 0)\).

3.1. Curves of \((\widetilde{M}^g_{\mathbb{P}})_{n,\text{diag}}^h\) having odd signature.

Definition 3.4 (signature). Let \(\phi : C \to C/G\) be a branched Galois covering between smooth curves, and let \(y_1, ..., y_r\) be its branch points. The signature of \(\phi\) is defined as \((g_0; m_1, ..., m_r)\), where \(g_0\) is the genus of \(C/G\) and \(m_i\) is the ramification index of any point in \(\phi^{-1}(y_i)\).

R. Hidalgo [13] considered complex curves \(C\) such that the natural cover \(\pi_C : C \to C/\text{Aut}(C)\) has signature of the form \((0; m_1, m_2, m_3, m_4)\), proving that \(C\) can be defined over its field of moduli if \(m_4 \notin \{m_1, m_2, m_3\}\). Artebani-Quispe in [2] extended such a result to smooth curves of odd signature:

Definition 3.5 (odd Signature). A smooth curve \(C\) of genus \(g \geq 2\) has odd signature if the signature of the natural covering \(\pi_C : C \to C/\text{Aut}(C)\) is of the form \((0; m_1, ..., m_r)\) where some \(m_i\) appears exactly an odd number of times.

Theorem 3.6 (Artebani-Quispe, Theorem 2.5, [2]). Let \(C\) be a smooth curve of genus \(g \geq 2\) defined over an algebraically closed field \(F\). Let \(L\) be a subfield of \(F\), such that \(F/L\) is Galois. If \(C\) is an odd signature curve, then the field of moduli \(M_{F/L}(C)\) is a field of definition for \(C\).

Corollary 3.7. Let \(S\) be a plane Riemann surface in \((\widetilde{M}^g_{\mathbb{P}})_{n,\text{diag}}^h\) with \(\text{Aut}_+(S)\) cyclic of order \(n\). Then, \(S\) has odd signature if and only if \(n\) is odd and equals to either \(d\) or \(d - 1\). Moreover, in this case that \(S\) is of odd signature, \(F_\Sigma(X,Y,Z) = 0\) is defined by an equation of the form \(Z^d + L_{d,Z} = 0\) if \(n = d\), and \(Z^{d-1}Y + L_{d,Z} = 0\) if \(n = d - 1\).

Proof. By Remark 3.3, we know that the binary form \(L_{d,Z}\) factors into \(d\) distinct factors associated to \(d\) distinct roots, say \((a_i : b_i) \in \mathbb{P}^1_{\mathbb{C}}\), for \(i = 1, 2, ..., d\). Since \(\text{Aut}_+(F_\Sigma) = \langle \text{diag}(1,1,\zeta_n) \rangle\), the covering \(\pi_\Sigma : S \to S/\text{Aut}_+(S)\) is ramified exactly at the \(d\) points \(\{a_i : b_i : 0\}\) when \(n\mid d\), plus the extra point \((0 : 0 : 1)\) when \(n\mid d - 1\). This gives \(d\) branch points (resp. \(d + 1\)) with ramification index \(n\) when \(n\mid d\) (resp. \(n\mid d - 1\)). Consequently, \(F_\Sigma(X,Y,Z) = 0\) has odd signature only if \(n\mid d\) and \(d\) odd or \(n\mid d - 1\) and \(d\) even. The Riemann-Hurwitz formula reads as
\[
(d - 1)(d - 2) - 2 = n(2g_0 - 2 + d(1 - 1/n)).
\]
when \( n \mid d \) and \( d \) odd, and as 
\[(d - 1)(d - 2) - 2 = n(2g_0 - 2 + (d + 1)(1 - 1/n)),\]
when \( n \mid d - 1 \) and \( d \) even. Setting \( g_0 = 0 \) and solving for \( n \), we obtain \( n = d \) (resp. \( n = d - 1 \)). This proves the "if and only if" statement.

Lastly, for \( n = d \) (resp. \( n = d - 1 \)), the index set \( S(1)d \) (resp. \( S(2)d-1 \)) is obviously empty, and we deduce the defining equation for \( \mathcal{S} \).

\[\square\]

3.2. The stratum \((\mathcal{M}^{|h}_{g1})_{n, \text{diag}}\) with \( n \mid d - 1 \). Suppose that \((\mathcal{M}^{|h}_{g1})_{n, \text{diag}}\) is non-empty for some fixed integer \( n > 1 \) with \( n \mid d - 1 \). Let \( S \in (\mathcal{M}^{|h}_{g1})_{n, \text{diag}}\) with cyclic automorphism group. Fix a non-singular plane model \( F_S(X,Y,Z) = 0 \) of degree \( d \) over \( \mathbb{C} \) in the family \( \mathcal{C}_2 \) of Proposition 3.2, such that \( \text{Aut}_+(F_S) = \langle \text{diag}(1,1,\zeta_n) \rangle \).

Let us consider the subfamilies
\[\begin{align*}
\mathcal{C}_2^{(1)} : Z^{d-1}Y &+ \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + X^d + X^{d-2}Y^2 + \sum_{j=3}^d a_jX^{d-j}Y^j = 0, \\
\mathcal{C}_2^{(2)} : Z^{d-1}Y &+ \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + X^d + Y^d + \sum_{j=3}^{d-1} a_jX^{d-j}Y^j = 0, \\
\mathcal{C}_2^{(3)} : Z^{d-1}Y &+ \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + X^d + XY^{d-1} + \sum_{j=3}^{d-2} a_jX^{d-j}Y^j = 0, \\
\mathcal{C}_2^{(4)} : Z^{d-1}Y &+ \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + X^{d-1}Y + \sum_{j=3}^d a_jX^{d-j}Y^j = 0.
\end{align*}\]

Lemma 3.8. We have
\[\mathcal{C}_2 = \bigcup_{s=1}^4 \mathcal{C}_2^{(s)}.\]

Proof. By non-singularity, \( X^d \) or \( X^{d-1}Y \) should appear in \( L_{d,Z} \). Hence, up to rescaling the variable \( X \) and then renaming the parameters, we can split up \( \mathcal{C}_2 \), in Proposition 3.2, into two substrata defined by the equations
\[Z^{d-1}Y + \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + X^d - Y^d + \sum_{j=3}^{d-1} a_jX^{d-j}Y^j = 0\]
along with,
\[Z^{d-1}Y + \sum_{j \in S(2)_n} Z^{d-j}L_{j,Z} + X^d + X^{d-1}Y + \sum_{j=3}^{d-1} a_jX^{d-j}Y^j = 0.\]

We always can assume \( a_2 = 0 \) in the first component, by a change of variables of the shape \( X \mapsto X - \frac{a_3}{a_2}X \) and then renaming the parameters in order to obtain the fourth component \( \mathcal{C}_2^{(4)} \). In the same way, we may set \( a_1 = 0 \) in the second component through \( X \mapsto X - \frac{a_4}{a_3}X \) and renaming the parameters next. Moreover, if \( a_2 = 0 \), then we split it up according to whether the monomial \( Y^d \) appears or not (if it does not appear, then \( XY^{d-1} \) does, by non-singularity). Therefore, we get \( \mathcal{C}_2^{(2)} \) and \( \mathcal{C}_2^{(3)} \) respectively, up to rescaling \( Y \) and \( Z \). Finally, when \( a_2 \neq 0 \), we rescale \( Y \) and \( Z \) to have \( \mathcal{C}_2^{(1)} \).

\[\square\]

Remark 3.9. A priori, the index set \( S(2)_n \) is empty if and only if \( \text{diag}(1,1,\zeta_{d-1}) \in \text{Aut}_+(\mathcal{C}_2) \). In this case, \( n = d - 1 \) and the subfamily \( \mathcal{C}_2^{(4)} \) is not irreducible anymore, since it factors as \( Y \cdot G(X,Y,Z) = 0 \). Hence, \( \mathcal{C}_2 \) reduces to \( \bigcup_{s=1}^4 \mathcal{C}_2^{(s)} \) if \( n = d - 1 \).
Theorem 3.10. Let $S \in (\widetilde{M}_g^{P1})^h_{n,\text{diag}}$ with $n \mid d - 1$ and $\text{Aut}_+(S) \simeq \mathbb{Z}/n\mathbb{Z}$. If $\mathbb{R}$ is the field of moduli for $C$, relative to the Galois extension $\mathbb{C}/\mathbb{R}$, then it is also a field of definition.

Proof. By Proposition 3.2, we may take a non-singular plane model $F_S(X, Y, Z) = 0$ for $S$ in $\mathbb{P}_C^2$ of degree $d$ in the family $C_2$, such that $\text{Aut}_+(F_S) = \langle \text{diag}(1, 1, \zeta_n) \rangle$. Hence $N_{\text{Aut}_+(F_S)}(C)$ equals to $\text{GL}_{2,Z}(\mathbb{C})$, using Proposition 2.9. Because of the monomial term $Z^{d-1}Y$ in the defining equation for $F_S(X, Y, Z) = 0$, the action of the normalizer $\text{GL}_{2,Z}(\mathbb{C})$ is trivial except possibly an isomorphism of the shape $[\alpha X + \beta Y : \gamma Y : Z] \in \text{PGL}_3(\mathbb{C})$.

We will show that $\beta = 0$ and so $S$ is isomorphic to its complex conjugate $\sigma S$ via an isomorphism $\phi_\sigma = \text{diag}(1, \lambda, \mu)$ where $\lambda$ and $\mu$ are roots of unity. In particular, $\phi_\sigma$ satisfies the Weil’s condition of descent ($\phi_\sigma \circ \sigma \phi_\sigma = 1$), and $\mathbb{R}$ is a field of definition, see Theorem 1.1. First, by construction, the union decomposition $C_2 = \bigcup_{s=1}^d C_2^{(s)}$ in Lemma 3.8 is well-defined up to $\mathbb{C}$-isomorphisms, that is even $[\alpha X + \beta Y : \gamma Y : Z]$ does not define an isomorphism between two curves in two distinct components. Second, if $C$ belongs to any of the subfamilies $C_2^{(s)}$ for $s = 1, 2, 3$, then $\alpha^{d-1}\beta = 0$, since $X^{d-1}Y$ does not appear in the defining equation $F_S(X, Y, Z) = 0$. Thus $\beta = 0$, because $[\alpha X + \beta Y : \gamma Y : Z]$ must be invertible. We mimic the argument by switching the monomial term $X^{d-1}Y$ with $X^{d-2}Y^2$ for $C_2^{(4)}$.

3.3. The stratum $(\widetilde{M}_g^{P1})^h_{n,\text{diag}}$ with $n \mid d$ and $d$ odd.

Theorem 3.11. Let $S \in (\widetilde{M}_g^{P1})^h_{n,\text{diag}}$ with $n \mid d$ and $\text{Aut}_+(S) \simeq \mathbb{Z}/n\mathbb{Z}$, where $d \geq 5$ is odd. If $\mathbb{R}$ is the field of moduli for $S$, relative to the Galois extension $\mathbb{C}/\mathbb{R}$, then it is also a field of definition.

Proof. Theorem 3.6 and Corollary 3.7 gives the result when $S \in (\widetilde{M}_g^{P1})^h_{d,\text{diag}}$, i.e. when $n = d$. Therefore, we can suppose that $1 < n < d$. By Proposition 3.2, we may take a non-singular plane model $F_S(X, Y, Z) = 0$ in $\mathbb{P}_C^2$ in the family $C_1$, whose automorphism group $\text{Aut}_+(F_S) = \langle \text{diag}(1, 1, \zeta_n) \rangle$. More precisely, we have an equation of the form

$$C'_1 : Z^d + \sum_{1 \leq t < d/n} Z^{d-tn}L_{tn,Z} + L_{d,Z} = 0,$$

such that $L_{tn,Z} \neq 0$ for some $1 \leq t < d/n$.

We first show the next observation:

Observation. Let $C'_{1,0} : Z^d + L_{d,Z} = 0$. Then, for any isomorphism $\phi_\sigma : \sigma C'_{1,0} \to C'_{1,0},$

we can find $\eta_\sigma \in \langle \text{diag}(\zeta_d^{-1}, \zeta_d^{-1}, 1), 1 \rangle$ and an isomorphism $\sigma \phi_\sigma : \sigma C'_1 \to C'_1$, such that $\eta_\sigma \circ \phi_\sigma$ and $\phi_\sigma$ have the same action on $C'_1$.

Proof. Because $N_{\text{Aut}_+(F_S)}(C) = \text{GL}_{2,Z}(\mathbb{C})$ (Proposition 2.9), then the geometric fibers which are isomorphic are obtained through the action of $\text{GL}_{2,Z}(\mathbb{C})$. It is also clear that an element $\phi \in \text{GL}_{2,Z}(\mathbb{C})$, which acts non-trivially on the family $C'_{1,0}$, also has non-trivial action on $C'_1$. The converse is true, unless $\phi \in \langle \text{diag}(\zeta_d^{-1}, \zeta_d^{-1}, 1) \rangle$. Therefore, the number of isomorphic geometric fibers in $C'_{1,0}$ is exactly the same number of those in the family $C'_1$, coming by the action of $\text{GL}_{2,Z}(\mathbb{C}) \setminus \langle \text{diag}(\zeta_d^{-1}, \zeta_d^{-1}, 1) \rangle$. 


Consequently, the action of any isomorphism $\sigma: C'_{1,0} \to C'_{1,0}$ can always be extended to an action $\tilde{\sigma}: C'_1 \to C'_1$. In particular, the composition $\tilde{\sigma} \circ \sigma^{-1}$ acts trivially on $Z^d + L_{d,Z} = 0$, that is $\tilde{\sigma} \circ \sigma^{-1} = 1$ for all $1 \leq t < d/n$. That is, $\tilde{\sigma} = \sigma_{\tau}^{-1} \circ \sigma_{\tau} = \tilde{\sigma}$ for some $d$-th root of unity $\eta_\tau$. Hence, it satisfies $(\sigma L_{tn,Z})(X,Y) = L_{tn,Z}(\tilde{\sigma}(X,Y)) = \epsilon_{\tau}^{-tn}L_{tn,Z}(\tilde{\sigma}(X,Y))$, for all $1 \leq t < d/n$. That is,

\[
L_{tn,Z}(X,Y) = (\sigma L_{tn,Z})(X,Y) = \sigma(\epsilon_{\tau}^{-tn}L_{tn,Z}(\tilde{\sigma}(X,Y)))
\]

Next, by the aid of Theorem 3.6 and Corollary 3.7, we have an isomorphism $\phi_\sigma: C'_{1,0} \to C'_{1,0}$ in $\text{GL}_2(Z(C))$, satisfying the Weil’s cocycle condition of descent $(\phi_\sigma \circ \sigma_{\tau} = 1)$, see Theorem 1.1. Using the previous observation, we also have an isomorphism $\tilde{\phi}_\sigma := \eta_\tau \circ \phi_\sigma: C'_1 \to C'_1$ in $\text{GL}_2(Z(C))$, where $\eta_\tau := \text{diag}(\epsilon_{\tau}^{-1}, \epsilon_{\tau}^{-1}, 1)$ for some $d$-th root of unity $\epsilon_{\tau}$. Hence, it satisfies $L_{tn,Z}(X,Y) = L_{tn,Z}(\tilde{\phi}_\sigma(X,Y)) = \epsilon_{\tau}^{-tn}L_{tn,Z}(\tilde{\phi}_\sigma(X,Y))$.

So $\tilde{\phi}_\sigma$ satisfies the Weil’s condition of descent as well, and $\mathbb{R}$ is a field of definition. \(\square\)

3.4. The stratum $(M^h_{g})_{n, \text{diag}}$ with $n \mid d$ and $d$ even. There is no plane Riemann surfaces $\mathcal{S}$ of genus 3 with automorphism group $\text{PGL}_2(C)$-conjugate to $\langle \text{diag}(1,1,1) \rangle$, see [11] or [7] for more details. Hence, the stratum $(M^h_{g})_{4, \text{diag}}$ is empty, and we have nothing to say in this case.

Take $m, r \in \mathbb{N}$ such that $2mr > 5$ and $r$ is odd when $m$ does. Consider a binary form $G(X,Y) \in \mathbb{C}[X,Y] \setminus \mathbb{R}[X,Y]$ given by

$G(X,Y) := \prod_{i=1}^{r}(X^m - a_iY^m)(X^m + \sigma a_iY^m),$

for some $a_1, ..., a_r \in \mathbb{C}$ such that the next conditions hold: $G(X,1)$ has no repeated zeros, the map $[\alpha: \beta] \mapsto \beta: \alpha$ does not map the zero set of $G(X,1)$ into itself, for any root of unity $\zeta$ we should have $\{\zeta a_i, -\zeta/a_i\} \neq \{\zeta a_i, -\zeta/a_i\}$, and when $m = 3$, the map $[\alpha: \beta] \mapsto [-\alpha + (1 + \sqrt{3})\beta: (1 + \sqrt{3})\alpha + \beta]$ does not map the zero set of $G(X,1)$ into itself.

**Proposition 3.12.** (B. Huggins, [15, Chapter 7, §1]) Following the above notations, let $\mathcal{S}$ be a plane Riemann surface of degree $> 5$ given in $\mathbb{P}_C^2$ by an equation of the form

$F_\mathcal{S}(X,Y) := Z^{2mr} - G(X,Y) = 0$

Then, the automorphism group $\text{Aut}_+(F_\mathcal{S})$ is diagonal and equals $\langle \text{diag}(\zeta_m, 1, 1), \text{diag}(1, \zeta_m, 1), \text{diag}(1, 1, \zeta_m) \rangle$.

Moreover, the field of moduli $\mathcal{M}_{2g}(\mathcal{S}) = \mathbb{R}$, but it is not a field of definition.

**Corollary 3.13.** For any integer $d = 2(2k + 1) \geq 6$, pseudoreal-plane Riemann surfaces in $(M^h_{g})_{d, \text{diag}}$ exist. In other words, there exist pseudoreal-plane Riemann surfaces $\mathcal{S}$ of genus $g = (d-1)(d-2)/2$ with $\text{Aut}_+(\mathcal{S})$ conjugated to $\langle \text{diag}(1,1,1) \rangle$.

**Proof.** Let $m = 1$ and $r = 2k + 1$ in Proposition 3.12. \(\square\)
3.4.1. The stratum $\overline{(M^P_g)^h}_{d,\text{diag}}$ with $d$ even and $p|d$ is prime. It remains yet the study of $\overline{(M^P_g)^h}_{n,\text{diag}}$ when the degree $d \geq 4$ is even and $n$ divides $d$ properly. In this case, the field of moduli does not need to be a field of definition. The first (explicit) example appears for genus $g = 3$ curves, and we address the reader to [2, §4, Lemma 4.2, Proposition 4.3], for a smooth plane quartic curve over $C$, not definable over its field of moduli $R$, and whose conformal automorphism group is $Z/2Z$. In what follows, we generalize this construction for higher degrees:

**Example 3.14.** Take $d = 2pm$ an integer with $m \geq 3$ odd and $p$ is a prime number. Define a Riemann surface $S$ in $P^2_C$ by an equation of the form

$$F_S(X, Y, Z) := Z^d + Z^2 g(X, Y) - f(X, Y) = 0,$$

where

$$g(X, Y) := \prod_{i=1}^{\frac{m-1}{2}} (X - a_i Y)(X + \frac{1}{a_i} Y),$$

$$f(X, Y) := \prod_{i=1}^{\frac{m}{2}} (X - b_i Y)(X + 1/b_i Y),$$

with $a_i \in R \setminus \{0\}$, for $1 \leq i \leq (p-1)d/2p$, and $b_i \in C \setminus \{0\}$, for $1 \leq i \leq d/2$. Suppose also that $f(X, Y)$ and $g(X, Y)$ have no repeated zeros. We also choose the $a_i$'s in the way that $g(X, Y)$ is not $\psi$-invariant or $\psi_{a,b}$-invariant for any $\psi : (X : Y) \mapsto (Y : cX)$ and $\psi_{a,b} : (X : Y) \mapsto (X + aY : bX - Y)$ in PGL$_2(C)$.

**Theorem 3.15.** For any integer $d = 2pm$ with $m \geq 3$ odd and $p$ is prime, pseudoreal-plane Riemann surfaces $S$ in $\overline{(M^P_g)^h}_{d,\text{diag}}$ exist. That is, there exist pseudoreal-plane Riemann surfaces of genus $g = (d - 1)(d - 2)/2$ with conformal automorphism group conjugated to $(\text{diag}(1, 1, \zeta_{d/p}))$.

**Proof.** Let $S$ be a complex curve defined by an equation of the form (3.1) such that $\prod_{i=1}^{\frac{m}{2}} b_i \in R$. We first show that $S$ has no singular points in $P^2_C$. Since $F_S(X, 0, Z) = Z^d + (X^{p-1}Z)^2 - X^d = 0$ has no repeated zeros, the common zeros of $F_S(X, 0, Z)$ and $\frac{\partial F_S}{\partial X}(X, 0, Z) = Z^2 g(X, 1) - f'(X, 1)$ and $\frac{\partial^2 F_S}{\partial X^2}(X, 1, Z) = \frac{d}{p^2}Z^{2-1}(pZ^{1-1}g(X, 1))$. But $f(X, Y)$ is square free, then $(X : 1 : 0)$ gives no singularities on $F_S(X, Y, Z) = 0$. Furthermore, if we substitute $g(X, 1) = -pZ^{1-1}X$ into $F_S(X, 1, Z) = \frac{\partial F_S}{\partial X}(X, 1, Z) = 0$, we get that $S$ is singular only if $f(X, 1)g(X, 1)p = (1 - p)f'(X, 1)^p$, that is when $f(X, 1)$ has repeated zeros, a contradiction. Hence $S$ is plane.

Second, one easily checks that $F_S(X, Y, Z) = 0$ is isomorphic to its complex conjugate $(\overline{F_S})(X, Y, Z) = 0$ via the isomorphism $\phi_\sigma := [\overline{Y} : X : \zeta_{d/p}Z]$. Consequently, $R$ is the field of moduli for $S$, relative to $C/R$. If we assume that our claim on $\text{Aut}_+(S)$ is true, then $R$ is not a field of definition for $S$. To see this, let $\phi'_R$ be any isomorphism. Then, $\phi_\sigma \circ \phi'^{-1}_R \in \text{Aut}_+(S)$, and so $\phi'_R \circ \phi_\sigma^{-1} \in \text{Aut}_+(S)$, and so $\phi'_R \circ \phi_\sigma^{-1} = \text{diag}(1, 1, \zeta_{d/p})$ for some integer $0 \leq r < d/p$. However, any such $\phi'_R$ does not satisfy Weil’s condition of descent ($\phi'_R \circ \sigma \phi'_R = 1$) in Theorem 1.1, since $\phi'_R \circ \sigma \phi'_R = \text{diag}(1, 1, -1) \neq 1$. Thus $S$ is pseudoreal.

Now, we prove the claim on $\text{Aut}_+(S)$ by using quite similar techniques as in [3, 4, 5]. Obviously, $\psi := \text{diag}(1, 1, \zeta_{d/p}) \in \text{Aut}_+(F_S)$ is an homomorphism of order $d/p \geq 4$. Therefore, $\text{Aut}_+(F_S)$ fixes a point, a line or a triangle, by Theorem 2.4.
In particular, it is not conjugate to any of the finite primitive group mentioned in Theorem 2.2-(iii).

Now, we treat each of the following subcases:

(i) A line $L \subset \mathbb{P}^2$ and a point $P \not\in L$ are leaved invariant: By [10, Theorem 2.1], we can think about $\text{Aut}_+(F_S)$ in a short exact sequence

$$
\begin{array}{cccc}
1 & \mathbb{C}^* & \text{GL}_2(Y) & \phi \hookrightarrow \text{PGL}_2(C) & 1 \\
1 & [\psi] & \text{Aut}_+(F_S) & \hookrightarrow G & 1
\end{array}
$$

where $G$ is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order $m \leq d - 1$, a Dihedral group $D_{2m}$ of order $2m$ with $m|(d-2)$, one of the alternating groups $A_4$, $A_5$, or to the symmetry group $S_4$. Any such $G$, which is not cyclic, contains an element of order 2. Let $\psi' \in \text{Aut}_+(F_S)$ such that $g(\psi')$ has order 2. Then, $g(\psi')$ has the shape $\psi_{a,b}$ for some $a, b, c \in \mathbb{C} \setminus \{0\}$, which is absurd by our assumptions on $g(X,Y)$. Consequently, $G = g(\text{Aut}_+(F_S))$ is cyclic, generated by the image of a specific $\psi_G \in \text{GL}_2(Y)(\mathbb{C})$. This would lead to a polynomial expression of $b_i's$ in terms of the $a_i's$, hence we still have infinitely many possibilities to choose the $b_i's$ such that $f(X,Y)$ not $\langle g(\psi_G) \rangle$-invariant. In particular, $|G| = 1$ and $\text{Aut}_+(F_S)$ is $\text{PGL}_2(C)$-conjugate to $\langle \text{diag}(1,1,\zeta_{d/p}) \rangle$.

(ii) A triangle $\Delta$ is fixed by $\text{Aut}_+(F_S)$ and neither a line nor a point is leaved invariant: It follows by the proof of Theorem 2.1 in [10] that $\langle S, \text{Aut}_+(S) \rangle$ is a descendant of the Fermat curve $F_d$ or the Klein curve $K_d$ as in Theorem 2.2. Note that $d/p$ does not divide $|\text{Aut}_+(F_d)| = 3(d^2 - 3d + 3)$, e.g. [10, Propositions 3.5]. Therefore, $\langle S, \text{Aut}_+(S) \rangle$ is not a descendant of $K_d$. Hence, $\exists \phi \in \text{PGL}_2(C)$ where $H := \phi^{-1}\text{Aut}_+(F_S)\phi \leq \text{Aut}_+(F_d)$. It is also well known (e.g. [10, Proposition 3.3]) that $\text{Aut}_+(F_d)$ is a semidirect product of $S_3 = \langle T := [Y : Z : X], R := [X : Z : Y] \rangle$ acting on $(\mathbb{Z}/d\mathbb{Z})^2 = \langle [\zeta_d X : Y : Z], [X : \zeta_d Y : Z] \rangle$. Thus any element of $\phi^{-1}\text{Aut}_+(S)\phi$ has the shape $DRT^j$, for some $0 \leq i \leq 1$ and $0 \leq j \leq 2$ and $D$ is of diagonal shape in $\text{PGL}_2(C)$. It is straightforward to check that any $DRT^j$ with $j \neq 0$ has order $3 < d/p$. Thus $\phi^{-1}\phi$ has also a diagonal shape, and then we may take $\phi$ in the normalizer of $\langle \psi \rangle$, up to a change of variables in $\text{Aut}_+(F_d)$. In this case, we can think about $\text{Aut}_+(F_S)$ in the commutative diagram

$$
\begin{array}{cccc}
1 & \mathbb{Z}/d\mathbb{Z} & \text{Aut}_+(F_d) & \phi \rightarrow S_3 \\
1 & \text{Ker}(\phi|_{\mathbb{R}}) & \psi \rightarrow H & G := \text{Im}(\phi|_{\mathbb{R}}) & 1
\end{array}
$$

The variable $Z$ in the transformed defining equation via $\phi$ appears exactly as the original equation in the statement. Hence, $G$ is at most cyclic of order 2, since otherwise $H$ must have an element of the shape $[\zeta_d^s Y : \zeta_d^t Z : X]$ or $[\zeta_d^s Z : \zeta_d^t X : Y]$ for some integers $s, t$, which is not possible. For the same reason, $G$ is then generated by a certain $g([\zeta_d^s Y : \zeta_d^t X : Z])$, and as before, it only requires to exclude finitely many possibilities of $\{b_i \subset \mathbb{C}\setminus\{0\}$ such that $f(\phi(X,Y))$ is not $\langle g([\zeta_d^s Y : \zeta_d^t X : Z]) \rangle$-invariant, where $\phi$ is the restriction of $\phi$ on $\mathbb{C}[X,Y]$. 
3.5. **On the stratum for \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).** It remains the study when \( \text{Aut}_+ (S) \) is conjugate to the Klein four group \( \langle \text{diag}(1, -1, 1), \text{diag}(1, 1, -1) \rangle \).

We first generalize [4, Lemma 10]:

**Proposition 3.16.** Let \( S \) be a plane Riemann surface of odd degree \( d \geq 5 \). Then, \( \text{Aut}_+ (S) \) cannot be \( \text{PGL}_3 (\mathbb{C}) \)-conjugate to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), \( A_4 \), \( S_4 \) or \( A_5 \).

*Proof.* By [10, 22], a \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \subset \text{PGL}_3 (\mathbb{C}) \), giving invariant a non-singular plane model \( F_S (X,Y,Z) = 0 \) of degree \( d \geq 4 \) over \( \mathbb{C} \), should fix a point not lying on \( S \) or \( (S, \text{Aut}_+ (S)) \) must be a descendant of the Fermat curve \( F_d : X^d + Y^d + Z^d = 0 \) or the Klein curve \( K_d : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0 \). But if \( d \) is odd, then 4 does not divide \( |\text{Aut}_+ (F_d)| = 6d^2 \) and \( |\text{Aut}_+ (K_d)| = 3(d^2 - 3d + 3) \), e.g. [10, Propositions 3.3, 3.5]. In particular, \( (S, \text{Aut}_+ (S)) \) can not be a descendant of \( F_d \) or \( K_d \), and we can think about \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), in a short exact sequence of the form \( 1 \rightarrow N = 1 \rightarrow H \rightarrow H \rightarrow 1 \), where \( H \) is \( \text{PGL}_2 (\mathbb{C}) \)-conjugate to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), by Harui’s main result [10, Theorem 2.1]. Let \( H = \langle \eta_1, \eta_2 \rangle \leq \text{PGL}_2 (\mathbb{C}) \) acts on the variables \( Y,Z \), then we can assume, up to conjugation of groups in \( \text{PGL}_2 (\mathbb{C}) \), that \( \eta_1 = \text{diag}(1, -1) \) and \( \eta_2 = [aY + bZ : cY - aZ] \). Because \( \eta_1 \eta_2 \eta_1 = \eta_1 \eta_2 \), we get \( \eta_2 = \text{diag}(1, 1) \) or \( [bZ : cY] \) for some \( bc \neq 0 \). Therefore, \( S \) should have a non-singular plane model of the form \( Z^{d-1}L_{1,Z} + Z^{d-3}L_{3,Z} + \ldots + Z^2L_{d-2,Z} + L_{d,Z} = 0 \), and \( Y^{d-1}L_{1,Y} + Y^{d-3}L_{3,Y} + \ldots + Y^2L_{d-2,Y} + L_{d,Y} = 0 \) simultaneously. This reduces \( F_S (X,Y,Z) \) to \( X \cdot G(X,Y,Z) \) for some homogeneous polynomial of degree \( d - 1 \), a contradiction to non-singularity. Hence, there is no plane Riemann surface \( S \) of odd degree \( d \) with \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \leq \text{Aut}_+ (S) \). The other part of the statement is immediate, since any of these group contains a subgroup isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Now, let \( S \) be a plane Riemann surface of even degree \( d \geq 4 \) with

\[
(\text{diag}(1, -1, 1), \text{diag}(1, 1, -1), \text{diag}(1, -1, 1), \text{diag}(1, 1, -1), \text{diag}(1, -1, 1), \text{diag}(1, 1, -1))
\]

acting on a non-singular plane model \( F_S (X,Y,Z) = 0 \) over \( \mathbb{C} \). It is obvious that \( F_S (X,Y,Z) = 0 \) must be of the form

\[
X^d + Y^d + Z^d + \sum_{0 \leq s, t, u \leq (d/2) - 1 \atop s + t + u = d/2} \alpha_{s,t,u} (X^s Y^t Z^u)^2 = 0,
\]

for \( \alpha_{s,t,u} \in \mathbb{C} \).

**Example 3.17.** The family \( C_{a,b,c} \) defined by

\[
X^4 + Y^4 + Z^4 + aX^2 Y^2 + bX^2 Z^2 + cY^2 Z^2 = 0,
\]

with \( a^2, b^2, c^2 \) are pairwise distinct, and \( a^2 + b^2 + c^2 - abc, a^2, b^2, c^2 \neq 4 \), has automorphism group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

**Theorem 3.18.** (Artebani-Quispe, [2, §4]) Let \( S \) be plane Riemann surface in the family \( C_{a,b,c} \) above. Then, \( S \) is not pseudovaral.

**Remark 3.19.** Let \( G \) be the group acting on the triples \((a,b,c) \in \mathbb{C}^3\), generated by

\[
g_1(a,b,c) := (b,a,c), \quad g_2(a,b,c) := (c,b,a), \quad g_3(a,b,c) := (a,-b,-c).
\]

E. W. Howe [14, Proposition 2] observed that any isomorphism between \( C_{a,b,c} \) and \( C_{g(a,b,c)} \), for \( g \in G \), is defined over \( \mathbb{Q}(i) \).
We will show that Theorem 3.18 is true in general:

**Theorem 3.20.** Let \( S \) be a plane Riemann surface of even degree \( d \geq 4 \) whose conformal automorphism group is \( \text{PGL}_3(\mathbb{C}) \)-conjugate to \( \langle \text{diag}(1, -1, 1), \text{diag}(1, 1, -1) \rangle \). Then, \( S \) is not pseudoreal.

**Proof.** One easily checks that the normalizer of \( \langle \text{diag}(1, -1, 1), \text{diag}(1, 1, -1) \rangle \) in \( \text{PGL}_3(\mathbb{C}) \) is \( N = \langle [X : Z : Y], [Y : Z : X], D(\mathbb{C}) \rangle \). If \( M(\mathbb{C}/\mathbb{R}) = R \), then we must have an isomorphism \( \phi_\sigma: ^{\sigma}S \to S \) in \( N \). Consequently, \( \phi_\sigma \circ ^{\sigma}\phi_\sigma \in \text{Aut}_+(S) \), and hence \( \phi_\sigma \) reduces to one of the shapes

\[
\text{diag}(1, \lambda, \mu), [X : \lambda Z : \mu Y], [\lambda Y : \mu X : Z], \text{ or } [\lambda Z : Y : \mu X],
\]

where \( \lambda, \mu \in \mathbb{C} \setminus \{0\} \). Moreover, \( F_S(X, Y, Z) = 0 \) has core \( X^d + Y^d + Z^d \) as in (3.2), thus \( \lambda, \mu \) are \( d \)-th root of unity in \( \mathbb{C} \). Now, we treat each of the above cases:

If \( \phi_\sigma = \text{diag}(1, \lambda, \mu) \) with \( \lambda \) and \( \mu \) are roots of unity in \( \mathbb{C} \), then \( \phi_\sigma \circ \sigma \phi_\sigma = 1 \).

That is, \( \phi_\sigma \) satisfies Weil’s condition of decent in Theorem 1.1, and \( R \) is a field of definition for \( S \).

If \( \phi_\sigma = [X : \lambda Z : \mu Y] \), then \( F_S(X, Y, Z) = 0 \) has the form

\[
X^d + Y^d + Z^d + \sum_{0 \leq s, t, u \leq (d/2) - 1} X^{2s}(YZ)^{2t}(\alpha_{s,t,u}Y^{2u} + \beta_{s,t,u}Z^{2u}) = 0, \tag{3.3}
\]

Moreover, \( \phi_\sigma \circ \sigma \phi_\sigma = \text{diag}(1, -1, 1, \mu^{-1}) \in \langle \text{diag}(1, -1, 1), \text{diag}(1, 1, -1) \rangle \). Therefore, \( \lambda = \pm \mu \) and \( \phi_\sigma = [\epsilon X : Z : \pm Y] \) for some \( d \)-th root of unity \( \epsilon \). In particular, \( \sigma \beta_{s,t,u} = \epsilon^2 \alpha_{s,t,u} \), and equation (3.3) reduces to

\[
X^d + Y^d + Z^d + \sum_{0 \leq s, t, u \leq (d/2) - 1} X^{2s}(YZ)^{2t}(\alpha_{s,t,u}Y^{2u} + \epsilon^{-2s} \alpha_{s,t,u}Z^{2u}) = 0.
\]

In this case, we can take \( \phi_\sigma = [\epsilon X : Z : Y] \), which satisfies \( \phi_\sigma \circ \sigma \phi_\sigma = 1 \). So, \( R \) is a field of definition for \( S \) by Theorem 1.1.

The remaining situations can be treated symmetrically. \( \square \)

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