Visualizing one-dimensional non-hermitian topological phases

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Abstract

We develop a graphic approach for characterizing one-dimensional non-Hermitian topological phases. The eigenstates of energy bands are mapped to a graph on the torus, where a nontrivial topology exhibits as links. The topology of band touching exceptional points is a crucial aspect of a non-Hermitian system; the existence of exceptional point results in networks. We discuss the parity-time ($PT$) symmetric two-band models. The pseudo-anti-Hermiticity protects the band topology, and the eigenstate graphs in the exact $PT$-symmetric phase locate on the torus surface under the $PT$ symmetry protection. For the Su-Schrieffer-Heeger ladder, the eigenstate graph is a Hopf link in the gapped nontrivial phase; chiral-time symmetry protects that the movable exceptional points appear in pairs in the real-energy gapless phase, and each exceptional point splits into a pair of exceptional points when the $PT$ symmetry breaks. The proposed graphic approach is applicable in one-dimensional $N$-band models. Our findings provide insight into one-dimensional non-Hermitian topology phases through visualizing the eigenstates.

1. Introduction

Topological phases of matter are a frontier field of research [1–6]. Majorana, Dirac, and Weyl fermions stimulate the investigation of semimetals [7–9], and considerable efforts have been made to unravel the mystery of non-Hermitian physics [10–15]. In recent years, investigation into topological physics has been extended to include non-Hermitian physics [16–35]. The non-Hermitian topological systems under various combined symmetries are explored [36–41]. Parity-time ($PT$) symmetric interface states have been realized in passive optical systems [42–44], whereas edge state lasing is demonstrated in active optical systems [45–48]. The non-Hermitian topological invariants and band theory are established [49–51]. The breakdown of conventional bulk-boundary correspondence is explained by the non-Hermitian Aharonov–Bohm effect and the conventional bulk-boundary correspondence recovers under the inversion or combined-inversion symmetry [52]. Non-Hermitian skin effect [53–60], topological classifications [60–66], theories for high-order non-Hermitian topological systems [67–71], massive Dirac models [72], and non-Hermitian superfluidity [73] are developed.

In non-Hermitian topological systems, closing gaps in energy bands is usually associated with exceptional points (EPs) instead of diabolic points (DPs) [74]. EPs are a unique concept in non-Hermitian systems and feature an exotic topology related to Riemann surfaces and the non-Hermitian phase transition [75]. The band touching EP pairs split from single DPs are connected by open Fermi arcs [76]; or alternatively, non-Hermitian semimetals exhibit nodal phases with symmetry-protected EP rings and surfaces [77–84]; the corresponding energy bands in non-Hermitian gapless phases are all complex. Complex-energy bands associated EPs have been studied systematically [64] and are markedly different from the gapless phase in a Hermitian system. In one respect, EPs as topological defects usually possess a half-integer charge/winding number [50, 51, 82]. In another respect, a non-Hermitian gapless phase with a complex-energy bands has Riemann sheet band structure, an additional vorticity/winding number presents solely for non-Hermitian systems [50, 51, 60]. However, the aforementioned two topological invariants are insufficient for distinguishing all types of gapless phases arising from DPs and EPs [85].
In this paper, we propose a graphic approach for visualizing the topology of the one-dimensional (1D) phases of matter. The rich topological phases of either gapped (separable [51]) or gapless, either Hermitian or non-Hermitian, and either trivial or nontrivial phases are all distinguishable from the geometrical topologies of eigenstate graphs. The essence of eigenstate graphs differs dramatically from that of the knotted or linked nodal lines in semimetals that representing a zero-energy (equal-energy) surface [86–88]. In the proposed graphic approach, eigenstates are mapped to closed loops on the solid torus. In $PT$-symmetric two-band models, the nontrivial topology is reflected as torus links, where the $PT$ symmetry protects the graphic eigenstate loops on the torus surface. The graphic eigenstates of gapless bands arising from EPs form a network whose topological nature is characterized by its nodes, branches, and independent loops. Networks with different links and fixed or movable nodes correspond to different gapless phases.

2. Graphic eigenstate

In the theory of topological insulators [1, 2], the energy band is insufficient for determining the full topological characters of a phase of matter. However, bulk topological features are encoded in the eigenstates, which are crucial for topological characterization even in the non-Hermitian topological phase.

We consider a 1D $PT$-symmetric non-Hermitian two-band model, the Bloch Hamiltonian reads $h_\epsilon = B \cdot \sigma$, where $B = (B_x, B_y, B_z)$ is an effective magnetic field, and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector Pauli matrix; $B_x$ and $B_y$ are real, but $B_z$ is imaginary [18, 42–45]. However, the band topology does not alter since the topological properties of $h_\epsilon$ are not affected by $B_z$, protected by pseudo-anti-Hermiticity [17, 26]. The topological phase transition and the (non)existence of edge states are unchanged even though the energy bands become gapless due to non-Hermitian phase transitions [89]. Notably, $h_\epsilon$ has $PT$ symmetry $(PT)^{-1} h_\epsilon (PT) = h_\epsilon$, where $PT = \sigma_y K$ and $K$ is the complex conjugation. A systematically geometric classification of non-Hermitian two-band system is detailed in [66]. The energy bands are tight under the influence of $B_z$ and the gapless phase arising from EPs forms upon the $PT$-symmetric phase transition. The eigenstates for the eigenvalues $\varepsilon_k^\pm = \pm B = \pm \sqrt{B_x^2 + B_y^2 + B_z^2}$ are as follows:

$$|\psi_k^\pm\rangle = \left( \begin{array}{c} \cos [\theta_\pm(k)/2] \\ \sin [\theta_\pm(k)/2] e^{-i\varphi_k}\end{array} \right),$$

with $\theta_\pm(k) = 2 \arctan |\lambda_\pm|$, and $\varphi_k = i \ln (\lambda_+ / |\lambda_+|)$, where $\lambda_\pm = (B_x \pm iB_y) / (B_z \pm \pm B)$. $\theta_\pm(k)$ and $\varphi_k$ are real periodic functions and provide all information of eigenstate $|\psi_k^\pm\rangle$ regarding the topological properties of $h_\epsilon$. $|\psi_k^\pm\rangle$ is mapped to a closed loop on the solid torus spanned by $k$, which constituted the graphic eigenstates. In the $k$-torus, $\tau_0$ is the distance from the center of the tube to the center of the torus, and $r_0$ is the radius of the tube. The two loops are plotted on the torus with $R = \tau_0$ and $r = r_0 (\sin \theta_\pm + \cos \theta_\pm)$. For the real-energy bands, which are protected by the $PT$ symmetry, we always have $\varphi_k = \pi / 2$; Two loops always locate on the torus surface ($r = r_0$). However, for the complex-energy bands, which are not protected by the $PT$ symmetry, we have $\varphi_k = i \ln ((B_x - iB_y) / \sqrt{B_x^2 + B_y^2})$ and $\theta_\pm(k) = 2 \arctan [\sqrt{(|B_x| + |B|) / (|B_x| + |B|/2)}]$; two loops are shifted outside ($r > r_0$) or inside ($r < r_0$).

In the Hermitian systems, we always have $\varphi_k = \varphi_k + \pi$; two eigenstates at a specific value of $k$ are two opposite points on the torus surface because of the orthogonality of the eigenstates. Without the DP ($B = 0$), two bands are two separate loops. By contrast, both $\varphi_k$ and $\varphi_k$ are discontinuous at the DP, exhibiting a $\pi$ jump in $\varphi_k$ and the two curves forming a knot of single loop for the gapless phases (see Appendix A for more detail) or forming network for the gapless phases (figure 1(d)). In the presence of non-Hermiticity, two real eigenstates at a specific value of $k$ are two points with arbitrary relative positions on the torus surface. Figures 1(a) and (g) are in the gapped phases; the unlinked (Hopf link) nature of the two-loop graph in figure 1(a) (figure 1(g)) shows the trivial (nontrivial) topology. The eigenstates meet on the torus surface when they coalesce at the EPs, where $\varphi_k = \varphi_k$; figures 1(b) and (f) (figures 1(c)–(e)) are in the real-energy (complex-energy) gapless phases. The two graphic eigenstates constitute a network that provides a complete topological picture of the gapless phase arising from EPs. The phase $\varphi_k$ jump $\pi$ at the origin $B_x = B_y = 0$ in figure 1(d), where the topological phase transition occurs. Figure 1 details how links are created and united.

3. Topological characterization

The unlinked (figure 1(a)) or linked (figure 1(g)) graphic eigenstate loops indicate a trivial or nontrivial topology, respectively. The rotation angle of $\varphi_k$ accumulated during a period of $k$ defines a winding number.
is gauge independent and is always valid for topological characterization. 

\[ \text{Figure 1. Graphic eigenstates of the } PT \text{-symmetric non-Hermitian SSH model, } B = (v + w \cos k, w \sin k, r\gamma) \text{ with } v = 1, \gamma = 1/2. (a) } w = 1/4, (b) } w = 1/2, (c) } w = 3/4, (d) } w = 5/4, (f) } w = 3/2, (g) } w = 7/4. \text{ In the lower panel, the black ring is } (B_x, B_y); PT \text{ symmetry is broken for } (B_x, B_y) \text{ inside the singularity ring } B_x^2 + B_y^2 = \gamma^2 \text{ in green [66]. The EPs are the red crosses where two rings intersect, and the origin (red dot) is the DP for } \gamma = 0. \]

\[ \mathcal{N}_\pm = -(2\pi)^{-1} \int_0^{2\pi} \nabla_k \varphi_\pm(k) \, dk. \] (2)

Topological phase transition induces the change of the number of edge modes; the number of edge modes at either boundary is equal to the winding number and the winding number remains valid for the gapless phase. Notably, the winding numbers for the two bands are identical either boundary is equal to the winding number and the winding number remains valid for the gapless phase.

The coincidence of \( \varphi_\pm(k) \) and pointing the thumb in the direction of \( \varphi_\pm(k) \) and the argument change of \( \varphi_\pm(k) \) is associated with the topological features of the non-Hermitian two-band system. The planar vector field \( F_k(k) = (\langle \sigma_x \rangle_\pm, \langle \sigma_y \rangle_\pm) \) directly relates to the phase factor \( \varphi_\pm(k) \) of the eigenstates; and the argument change of \( \varphi_\pm(k) \) characterizes the band topology [52, 82]. The winding number associated with \( F_k(k) \) is \( (2\pi)^{-1} \int_0^{2\pi} \nabla_k \arg F_k(k) \, dk, \) which is equal to \( \mathcal{N}_\pm \) because of \( \frac{\langle \sigma_y \rangle_\pm}{\langle \sigma_x \rangle_\pm} = \tan(-\varphi_\pm(k)) \). The coincidence of the variation of \( \varphi_\pm(k) \) and the argument change of \( F_k(k) \) indicates that the band topology is reflected by the kernel of eigenstate \( \varphi_\pm(k) \). The vector field for a non-Hermitian Su-Schrieffer-Heeger (SSH) chain [17, 18, 42–45] is alternatively mapped to the Bloch sphere in [26], where different topological phases winding around the z axis of the Bloch sphere differently.

Furthermore, the generalized global Zak phase for a non-Hermitian system under a proper gauge is suitable for topological characterization [26, 44, 90, 91], this phase is equivalent to \( 2\pi \mathcal{N} \). Different topological characterizations of a topological phase must be equivalent; however, their physical implications from their original definitions might be different. Alternatively, the global Zak phase can always be zero [92]. By contrast, the vector field \( F_k(k) \) is gauge independent and is always valid for topological characterization.

Beyond the topological characterization of the winding number \( \mathcal{N} \), we have an additional network characterization for the gapless phase. The graphic eigenstates of the gapless phase arising from EPs form a network, where an EP appears as the node owing to the coalescence of the eigenstates. Each isolated band touching EP from the real-energy gapless phase in 1D splits into a pair of EPs in the broken \( PT \text{-symmetric} \) phase. By contrast, EP ring and EP surface, which inherit the topological features of Hermitian system under
symmetry protection [78–81], are created in 2D or 3D. The node is a typical feature of gapless phase arising from EPs. The network independently reflects a novel topological aspect related to the number of EPs. Thus, the proposed graphic approach is especially helpful for characterizing the gapless phase that arises from EPs.

The geometric topology of any network satisfies the fundamental fact that the numbers of branches \( b \) (a branch represents any two- terminal element), nodes \( n \) (a node connects two or more branches), and independent loops \( l \) (an independent loop must contain at least one branch that does not belong to any other loop) fulfill

\[
n = b - l + 1.
\]

The formation of the network indicates that the system is at the non-Hermitian phase transition point. The nodes in the network identifies the numbers of EPs in the energy band. For instance, \( n = 1, b = 2, l = 2 \) for the network shown in figures 1(b) and (f).

4. Non-Hermitian SSH ladder

We investigate a \( \mathcal{PT} \)-symmetric non-Hermitian SSH ladder. The non-Hermitian SSH ladder is a non-Hermitian extension of the Hermitian SSH ladder studied in [93]. The Bloch Hamiltonian reads (figure B1(a) in appendix B)

\[
h_k = \begin{pmatrix} i\gamma & w e^{ik} + ve^{-ik} + t \\
w e^{-ik} + ve^{ik} + t & -i\gamma \end{pmatrix}.
\]

We set \( t > 0 \) without loss of generality and we are not interested in \( w = v = 0 \). In the gapped phase, the graphic eigenstates are two separated loops. These loops are unlinked (figure 2(a)) in the topologically trivial phase with \( \mathcal{N} = 0 \); but form the Hopf link (figure 2(b)) in the topologically nontrivial phase with \( \mathcal{N} = \pm 1 \). The system has a gapless phase arising from DPs in the Hermitian case; at \( |w + v| = t \), we observe one fixed DP (black solid line in figure 3(a)), and two movable DPs at \( w = v \) (black dashed line in figure 3(a)) in the region of \( |w + v| > t \) that characterized by the vector field kinks [93]. One fixed DP appears at \( k_c = 0 \) or \( \pi \). The energy bands constitute a single band and the graphic eigenstates form a knot (figure 2(c)). At \( w = v \), two components of the magnetic field \( \mathbf{B} \) vanish, the band gap closes and two movable DPs appear at \( \cos k_c = t / (2v) \). In addition, two loops are separated without any intersection (figure 2(d)).

The band gap shrinks as the increase of \( \gamma \) and closes at critical non-Hermiticity

\[
\gamma_c = |v - w| \sqrt{1 - t^2 / (4vw)},
\]

for \( |w + v| < 4vw / t \) except when \( w = v \), where the EPs are movable in the momentum space and appear at \( \cos k_c = -t (w + v) / (4vw) \); for \( |w + v| \geq 4vw / t \) except when \( |w + v| = t \), the band gap closes at the critical non-Hermiticity

\[
\gamma_c = |t - |w + v||,
\]

In addition, the EPs are fixed in the momentum space and appear at \( k_c = 0 \) or \( \pi \). \( \gamma_c \) for the band gap closing is depicted in figure 3(a); and \( \cos k_c \) for the location of EPs are depicted in figure 3(b). In figure 3(a), the black dashed line separates two phases with winding numbers \( \mathcal{N} = 1 \) and \( \mathcal{N} = -1 \); while the black solid line separates phases with winding numbers \( \mathcal{N} = \pm 1 \) and \( \mathcal{N} = 0 \).

Chiral-time \( \mathcal{CT} \) symmetry of \( h_k \) [18, 36, 38] protects the movable EPs in the real-energy gapless phases that appear in pairs, \( \mathcal{CT}^{-1} h_k (\mathcal{CT}) = -h_{-k}, \) where \( \mathcal{CT} = \sigma_z K \). By contrast, the \( \mathcal{PT} \) symmetry ensures that each EP in the real-energy gapless phase splits into a pair of EPs in the broken \( \mathcal{PT} \)-symmetric phase [80, 81]. For an arbitrary eigenstate \( |\psi^\lambda_\mu\rangle \), the \( \mathcal{CT} \) symmetry requires another eigenstate that satisfies \( |\psi^\lambda_{-\mu}\rangle = \mathcal{CT} |\psi^\lambda_\mu\rangle \), which leads to

\[
\varphi^\lambda(k) + \varphi^\lambda(-k) = \pm \pi.
\]

In particular, when an EP appears at \( k_c \), two eigenstates \( |\psi^\lambda_\mu\rangle \) and \( |\psi^\lambda_{-\mu}\rangle \) coalesce to one denoted as \( |\psi^\mu_\mu\rangle \). The \( \mathcal{CT} \) symmetry ensures \( |\psi^\mu_\mu\rangle = \mathcal{CT} |\psi^\mu_\mu\rangle \), i.e., the existence of a pair of EPs with zero energy for \( k_c = 0 \); \( \pi \); no EP can be separately removed, but the EP position changes based on system parameters. When \( k_c = 0 \), \( \pi \), we observe one fixed EP with \( \varphi^\mu(k_c) = \pi / 2 \) or \( -\pi / 2 \).

The topological features are clearly revealed in the graphic approach: (i) For \( \gamma < \gamma_c \), two real-energy bands are separated without an EP, and two loops are located on the torus surface without intersection (figures 2(a) and (b)). In the real-energy gapless phase, the hyperbola \( |w + v| = 4vw / t \) and the line \( w + v = 0 \) divide the \( w-v \) plane into four regions with distinct topological characters (figure 3(b)). (ii) For \( \gamma = \gamma_c \) in the region \( |w + v| < 4vw / t \), two loops have one fixed node except for \( w + v = 0 \) and \( |w + v| = t \) whose location does not change based on the parameters \( w \) and \( v \) (figure 2(e)). (iii) For \( \gamma = \gamma_c \) and \( w + v = 0 \), two loops have two fixed nodes (figure 2(f)). (iv) For \( \gamma = \gamma_c \) in the region \( |w + v| \geq 4vw / t \) except when \( w = v \), two loops are
located on the torus surface with two robust nodes (figure 2(h)) that are movable but irremovable in split of the changes in system parameters $w$ and $v$; two movable nodes merge to a single fixed node at $k_c = 0$ or $\pi$ associated with the change of network topology at hyperbola $|w + v| = 4wv/\iota$ (figure 2(g)). Notably, $n = 1(2), \ b = 2(4), \ l = 2(3)$ for the network shown in figure 2(g) (figure 2(h)). (v) In the broken $\mathcal{PT}$-symmetric phase for $\gamma > \gamma_c$, and the energy bands become complex and $\varphi_k(k) = \frac{1}{2} \text{Im}[(B_y - iB_x) \sqrt{B_y^2 + B_x^2}]$ and $\theta_k(k) = \arctan\left[\frac{\sqrt{(|B_y| + |B_x|)}}{|B_y| + |B_x|}\right]$. The graphic eigenstates and energy bands of the broken $\mathcal{PT}$-symmetric phase are depicted in figures 2(i)-(l), where the loops are detached from the torus surface.

The $2\pi$ variation of $\varphi_k(k)$ of the graphic eigenstate in a $2\pi$ period of $k_c$ indicates a nontrivial topology and coincides with the winding number $\mathcal{N} = 1$ marked in figure 3(a). The two loops form the Hopf link for the nonzero winding number. The variation in the argument of the vector field coincides with the graphic eigenstates. Figure 4 depicts the vector field $\mathbf{F}_k = \langle \cdots \rangle$ and $\omega = \frac{1}{\vartheta_c}$ of the winding number. The imaginary energies are indicated by the dotted lines.

Figure 2. Graphic eigenstates and energy bands of the non-Hermitian SSH ladder. (a)–(d) Exact $\mathcal{PT}$-symmetric phase. (e)–(h) Gapless phase arising from EPs. (i)–(l) Broken $\mathcal{PT}$-symmetric phase. $\mathcal{N} = 1$; (a) (1/4, 1/4, 2/5), (b) (2/7, 1), (c) (1/4, 3/4, 0), (d) (2, 2), (e) (1/4, 1/4, $\gamma_c$), (f) (2, 2, $\gamma_c$), (g) (2/7, 1), and (h) (3, 3/2, $\gamma_c$). $\gamma_c$ is chosen equation (6) in (e), (f), and (g); but chosen equation (5) in (b). (i) (1/4, 1/4, 11/20), (j) (2, 2, 11/10), (k) (2, 2/7, 7/5), (l) (3, 3/2, 8/5). Each band touching EP in the middle panel splits into a pair of EPs in the broken $\mathcal{PT}$-symmetric phase. The imaginary energies are indicated by the dotted lines.

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5. Summary

We propose a novel geometric picture for the 1D non-Hermitian topological phases. Graphic eigenstates, which encoded all information regarding the system topology, are mapped on the torus, and the geometric topology of 1D closed curves in 3D space characterizes the 1D topological band. Graphs for topologically nontrivial phases are visualized as links; and gapless phases that arise from EPs form networks. Their geometric topologies reflect the topological properties of the ground state phase diagrams. The $\mathcal{PT}$ symmetry protects the topological properties of the real-energy phases and ensures that the graphic eigenstates are located on the torus surface. The $\mathcal{CT}$ symmetry protects that the EPs in the gapless phase appear in pairs. These features are then elucidated through a $\mathcal{PT}$-symmetric non-Hermitian SSH ladder that is experimentally accessible in many physical systems. In the proposed graphic approach, the band topology is visualized, the topological invariant is immediately identified, and the graphic approach is useful in the investigation of dynamical topological phase transition.

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Note added.

We acknowledge L Li for introducing their work [66] to us. We emphasis that our graphic approach completely differs from the geometric characterization in [66], where L Li et al focus on the relation between the singularity ring and the vector field of the two-band system. Here our graphic approach focuses on the visualization of the eigenstates of topological systems, which fully encoded the topological features of the systems. We emphasis that our graphic approach is applicable for both real and complex bands, for both gapped and gapless bands, and for
both Hermitian and non-Hermitian topological systems [94], and not limited to be applicable in two-band topological systems.

Appendix A. Graphic eigenstates of the SSH chain

The graphic eigenstates of the standard Hermitian SSH chain (\( \gamma = 0 \)) is depicted in figure A1(a)–(c). The graphic eigenstates of the \( \mathcal{PT} \)-symmetric non-Hermitian SSH chain (\( \gamma = 3 \)) with the fully complex-energy bands in the broken \( \mathcal{PT} \)-symmetric phase are depicted in figure A1(d)–(f). The nontrivial phase exhibits the Hopf links with winding number \( N = 1 \) (figures A1(c) and (f)), the trivial phase exhibits the unlinked two loops with winding number \( N = 0 \) (figures A1(a) and (d)), and the topological phase transition point exhibits the trivial knots of single loop (figures A1(b) and (e) in appendix A).

Appendix B. Non-Hermitian SSH ladder

As a prototype of one-dimensional non-Hermitian topological system, the \( \mathcal{PT} \)-symmetric non-Hermitian Su-Schrieffer-Heeger (SSH) model is experimentally realized in the coupled waveguides [43, 44, 95], coupled resonators [42], and polariton micropillars [46]. The essential features of the \( \mathcal{PT} \)-symmetric non-Hermitian system are captured even though the system is passive; the active non-Hermitian SSH model is realized with additional pumping [45–47]. Candidates for the experimental realization of non-Hermitian topological systems include photonic crystals [4, 6], ultracold atomic gases [5], acoustic lattices, and electric circuits [68, 96, 97]. We employ a non-Hermitian SSH ladder (figure B1(a)) to elucidate our findings. Introducing additional loss \( -2i\gamma \)
in one sublattice (the pink lattice) generates a passive non-Hermitian SSH ladder [98]; an overall decay rate $-i\gamma$ offset in both sublattices yields a $PT$-symmetric non-Hermitian SSH ladder [36]; alternatively, we can consider an active $PT$-symmetric non-Hermitian SSH ladder constituted by two coupled SSH chains incorporated gain and loss [45]. The Hamiltonian in the real space reads

$$H = \sum_{j=1}^{N}(wa_j^{+}b_j + va_jb_{j+1} + ta_j^{+}b_j + \text{H.c.}) + i\gamma(a_j^{+}a_j - b_j^{+}b_j),$$

where $a_j^{+}$ ($b_j^{+}$) is the creation operator of the $j$th site in the sublattice $A$ ($B$). $H$ is a two-leg ladder with 2N sites, each leg is a $PT$-symmetric SSH chain with staggered real couplings $w$ and $v$ [42–47]; two ladder legs are coupled at the strength $t$ after one leg glided by one site. $H$ is a simple generalization of the $PT$-symmetric non-Hermitian SSH model. Applying the Fourier transformation, the Hamiltonian under periodical boundary condition is rewritten as $H = \sum_{k} h_k = \sum_{k} \mathbf{B} \cdot \mathbf{\sigma}$. The core matrix is obtained in the form of

$$h_k = \begin{pmatrix}
    i\gamma & we^{ik} + ve^{-ik} + t \\
    we^{-ik} + ve^{ik} + t & -i\gamma
\end{pmatrix}.$$

The nontrivial winding of eigenstate predicts the appearance of edge states in topologically nontrivial phase. The edge states experience either an additional attenuation $-i\gamma$ or an additional amplification $i\gamma$; the amplified edge state in the $PT$-symmetric phase is appropriate for topological lasing. We depict the topological edge state in the boundary of $PT$-symmetric phase transition. The probability distribution of edge states is depicted in figure B1(b) for the situation in figure 4(a) in the main text.

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References

[1] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[2] Qi X L and Zhang S C 2011 Rev. Mod. Phys. 83 1057
[3] Chiu C K, Teo J C Y, Schnyder A P and Ryu S 2016 Rev. Mod. Phys. 88 035005
[4] Lu L, Joannopoulos J D and Soljačić M 2014 Nat. Photon. 8 821
[5] Cooper N R, Dalibard J and Spielman I B 2019 Rev. Mod. Phys. 91 015006
[6] Ozawa T et al 2019 Rev. Mod. Phys. 91 015005
[7] Chiu C K and Schnyder A P 2014 Phys. Rev. B 90 205136
[8] Neto A H C, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 Rev. Mod. Phys. 81 109
[9] Armitage N P, Mele E J and Vishwanath A 2018 Rev. Mod. Phys. 90 015001
[10] Bender C M 2007 Rep. Prog. Phys. 70 947
[11] Moiseyev N 2011 Non-Hermitian Quantum Mechanics (Cambridge, UK: Cambridge University Press) https://books.google.com/books?hl=en&ei=&id=QwAXVv5_k7QC8oi=fnfd&pg=PR7&eq=Moiseyev+N+2011+Non-Hermitian+Quantum+Mechanics+(Cambridge,+UK:+Cambridge+University+Press)&ots=kBVQdxeVP&sig=rt83ppVxrlMKmTz2Tn7IlVe8Fg0.

[12] Feng L, El-Ganainy R and Ge I 2017 Nat. Photo. 11 752

[13] Longhi S 2017 Europhys. Lett. 120 64001

[14] Christodoulides D and Yang J 2018 Parity-time Symmetry and Its Applications (Berlin: Springer) https://link.springer.com/content/pdf/10.1007/978-3-662-53978-7.pdf

[15] El-Ganainy R, Makris K G, Khaびvikham M, Musslimani Z H, Rotter S and Christodoulides D N 2018 Nat. Phys. 14 11

[16] Rudner M S and Levitov L S 2009 Phys. Rev. Lett. 102 065703

[17] Esaki K, Sato M, Hasebe K and Kohmoto M 2011 Phys. Rev. B 84 205128

[18] Schomerus H 2013 Opt. Lett. 38 1912

[19] Zhao H, Longhi S and Feng L 2015 Sci. Rep. 5 17022

[20] Xiao L et al 2017 Nat. Phys. 13 1117

[21] Takata K and Notomi M 2018 Phys. Rev. Lett. 121 213902

[22] Lieu S 2018 Phys. Rev. B 98 115335

[23] Longhi S 2019 Phys. Rev. Lett. 122 233701

[24] Longhi S 2019 Phys. Rev. B 100 125157

[25] Jiang H, Lang L, Yang C, Zhu S L and Chen S 2019 Phys. Rev. B 100 054301

[26] Zeng Q B, Yang Y B and Xu Y https://arxiv.org/abs/1901.08060 https://arxiv.org/abs/1901.08060 arXiv:1901.08060

[27] Yokomizo K and Murakami S 2019 Phys. Rev. Lett. 123 066404

[28] Ge Z Y, Zhang Y R, Liu T, Li S W, Fan H and Nori F 2019 Phys. Rev. B 100 054105

[29] Yuce C 2013 Phys. Lett. A 379 121

[30] Yuce C 2018 Phys. Rev. A 97 042118

[31] Lieu S 2018 Phys. Rev. B 97 045106

[32] Menke H and Hirschmann M M 2017 Phys. Rev. B 95 174506

[33] Carlstrom J, Stallhammar M, Budich J C and Bergholtz E J 2019 Phys. Rev. B 99 161115(R)

[34] Kawabata K, Shiozaki K and Ueda M 2018 Phys. Rev. B 98 165148

[35] Zyuzin A A and Zyuzin A Y 2018 Phys. Rev. B 99 041203(R)

[36] Ezawa M 2019 Phys. Rev. B 100 045407

[37] Ezawa M 2019 Phys. Rev. B 100 081401

[38] Ghatak A and Das T 2019 J. Phys.: Condens. Matter 31 263001

[39] Kunst F K and Dwivedi V 2019 Phys. Rev. B 99 245116

[40] Herviou L, Bardarson J H and Regnault N 2019 Phys. Rev. A 99 052118

[41] Hisbrunner M R, Philip T M and Gilbert M I 2019 Phys. Rev. B 100 081104

[42] Malzard S, Poli C and Schomerus H 2015 Phys. Rev. Lett. 115 200402

[43] Jin L 2017 Phys. Rev. A 96 032103

[44] Jin L, Wang P and Song Z 2017 Sci. Rep. 7 5903

[45] Lin S and Song Z 2017 Phys. Rev. A 96 052121

[46] Kawabata K, Ashida Y, Katsura H and Ueda M 2018 Phys. Rev. B 98 085116

[47] Ni X, Smirnova D, Podulnuy A, Leykim D, Chong Y and Khanikaev A B 2018 Phys. Rev. B 98 165129

[48] Lang L J, Wang Y, Wang H and Chong Y D 2018 Phys. Rev. B 98 094307

[49] Cancellieri E and Schomerus H 2019 Phys. Rev. A 99 033801

[50] Hou J L, Z G and Zhang C https://arxiv.org/abs/1904.05260 arXiv:1904.05260

[51] Poli C, Bellec M, Kuhl U, Mortessagne F and Schomerus H 2015 Nat. Commun. 6 6710

[52] Weimann S, Kremer M, Plotnik Y, Lumer Y, Nolte S, Makris K G, Segev M, Rechtsman M C and Szameit A 2017 Nat. Mater. 16 433

[53] Pan M, Zhao H, Miao P, Longhi S and Feng L 2018 Nat. Commun. 9 1308

[54] Parlo M, Wittke S, Hodaei H, Harari G, Bandres M A, Ren J, Rechtsman M C, Segev M, Christodoulides D N and Khajavikhan M 2018 Phys. Rev. Lett. 120 113901

[55] St-Jean P, Godbolt Y, Galopin E, Lemaître A, Orozco T, Le Gratiet L, Sagnes I, Bichlo J and Amo A 2017 Nat. Photon. 11 651

[56] Zhao H, Miao P, Teimourpour M H, Malzard S, El-Ganainy R, Schomerus H and Feng L 2018 Nat. Commun. 9 981

[57] Harari G, Bandres M A, Lumer Y, Rechtsman M C, Chong Y D, Khajavikhan M, Christodoulides D N and Segev M 2018 Science 359 82

[58] Bandres M A, Wittke S, Harari G, Parto M, Ren J, Segev M, Christodoulides D N and Segev M 2018 Science 359 eaar4003

[59] Xu Y, Wang S T and Duan L 2017 Phys. Rev. Lett. 118 045701

[60] Leykim D, Bilikh K Y, Huang C, Chong Y D and Nori F 2017 Phys. Rev. Lett. 118 040401

[61] Shen H, Zhen B and Fu L 2018 Phys. Rev. Lett. 120 146402

[62] Jin L and Song Z 2019 Phys. Rev. B 99 081103(R)

[63] Wu H C, Jin L and Song Z 2019 Phys. Rev. B 100 155117

[64] Lee T E 2016 Phys. Rev. Lett. 116 133903

[65] Yao S and Wang Z 2018 Phys. Rev. Lett. 121 086803

[66] Yao S, Song F and Wang Z 2018 Phys. Rev. Lett. 121 136802

[67] Song F, Yao S and Wang Z 2019 Phys. Rev. Lett. 123 170401

[68] Kunst F K, Edvardsson E, Budich J C and Bergholtz E J 2018 Phys. Rev. Lett. 121 026808

[69] Wang H, Ruan J and Zhang H 2019 Phys. Rev. B 99 075130

[70] Lee C H and Thomale R 2019 Phys. Rev. B 99 201103(R)

[71] Borgnia D S, Kruchkov A J and Slater R J https://arxiv.org/abs/1902.07217 arXiv:1902.07217

[72] Longhi S 2019 Phys. Rev. Research 1 023013

[73] Gong Z, Ashida Y, Kawabata K, Takasak K, Higashikawa S and Ueda M 2018 Phys. Rev. X 8 031079 https://journals.aps.org/prx/abstract/10.1103/PhysRevX.8.031079

[74] Kawabata K, Higashikawa S, Gong Z, Ashida Y and Ueda M 2019 Nat. Commun. 10 297

[75] Zhou H and Lee Y 2019 Phys. Rev. B 100 235112

[76] Kawabata K, Shiozaki K, Ueda M and Sato M 2019 Phys. Rev. X 9 041015 https://journals.aps.org/prx/abstract/10.1103/PhysRevX.9.041015
[64] Kawabata K, Bessho T and Sato M 2019 Phys. Rev. Lett. 123 066405
[65] Liu C-H, Jiang H and Chen S 2019 Phys. Rev. B 99 125103
[66] Li L, Lee C H and Gong J 2019 Phys. Rev. B 100 075403
[67] Liu T, Zhang Y R, Ai Q, Gong Z, Kawabata K, Ueda M and Nori F 2018 Phys. Rev. Lett. 122 076801
[68] Ezawa M 2019 Phys. Rev. B 99 121411(R)
   Ezawa M 2019 Phys. Rev. B 99 201411(R)
[69] Lee C H, Li L and Gong J 2019 Phys. Rev. Lett. 123 016805
[70] Luo X W and Zhang C 2019 Phys. Rev. Lett. 123 075601
[71] Edwardsson E, Kunst F K and Bergholtz E J 2019 Phys. Rev. B 99 081302(R)
[72] Rui W B, Zhao Y X and Schnyder A P 2019 Phys. Rev. B 99 241110(R)
[73] Yamamoto K, Nakagawa M, Adachi K, Takasan K, Ueda M and Kawakami N 2019 Phys. Rev. Lett. 123 126801
[74] Chen W, Özdemir S K, Zhao G, Wiersig J and Yang L 2017 Nature 548 192 (2017)
[75] Miri M A and Alù A 2019 Science 363 eaat7709
[76] Zhou H, Peng C, Yoon Y, Hsu C W, Nelson K A, Fu L, Joannopoulos J D, Soljačić M and Zhen B 2018 Science 359 1009
[77] González J and Molina R A 2017 Phys. Rev. B 96 045437
   Molina R A and González J 2018 Phys. Rev. Lett. 120 146601
[78] Budich J C, Carlström J, Kunst F K and Bergholtz E J 2019 Phys. Rev. B 99 041406(R)
[79] Yoshida T, Peters R, Kawakami N and Hatsuigui Y 2019 Phys. Rev. B 99 121101(R)
[80] Zhou H, Lee J Y, Liu S and Zhen B 2019 Optica 6 190
[81] Okugawa R and Yokoyama T 2019 Phys. Rev. B 99 041202(R)
[82] Lin S, Jin L and Song Z 2019 Phys. Rev. B 99 165148
[83] Rui W B, Hirschmann M M and Schnyder A P https://arxiv.org/abs/1907.10417 arXiv:1907.10417
[84] Cerjan A, Huang S, Chen K P, Chong Y and Rechtsman M C 2019 Nat. Photon. 13 623
[85] Jin L, Wu H C, Wei B B and Song Z https://arxiv.org/abs/1908.10512 arXiv:1908.10512
[86] Yang Z and Hu J 2019 Phys. Rev. B 99 081102(R)
[87] Carlström J, Stilhammar M, Budich J C and Bergholtz E J 2019 Phys. Rev. B 99 161115(R) (2019)
[88] Lee C H, Li G, Liu Y, Tai T, Thomale R and Zhang X https://arxiv.org/abs/1812.02011 arXiv:1812.02011
[89] Zhang K L, Wu H C, Jin L and Song Z 2019 Phys. Rev. B 100 045141
[90] Liang S D and Huang G Y 2013 Phys. Rev. A 87 012118
[91] Jiang H, Yang C and Chen S 2018 Phys. Rev. A 98 052116
[92] Nesterov A I and Aceves de la Cruz F 2008 J. Phys. A: Math. Theor. 41 485304
[93] Li C, Lin S, Zhang G and Song Z 2017 Phys. Rev. B 96 125418
[94] Yang X M, Jin L and Song Z https://arxiv.org/abs/1906.09016 arXiv:1906.09016
[95] Zeuner J M, Rechtsman M C, Plotnik Y, Lumer Y, Nolte S, Rudner M S, Segev M and Staneit A 2015 Phys. Rev. Lett. 115 040402
[96] Ezawa M https://arxiv.org/abs/1902.03716 https://arxiv.org/abs/1904.03823 arXiv:1904.03823
[97] Lee C H, Imhof S, Berger C, Bayer F, Brehm J and Molenkamp L W 2016 Commun. Phys. 1 139
[98] Guo A, Salamo G J, Duchesne D, Morandotti R, Volatier–Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 Phys. Rev. Lett. 103 093902