BRST OPERATOR FOR QUANTUM
LIE ALGEBRAS AND DIFFERENTIAL CALCULUS
ON QUANTUM GROUPS

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ABSTRACT

For a Hopf algebra $A$, we define the structures of differential complexes
on two dual exterior Hopf algebras: 1) an exterior extension of $A$ and 2) an
exterior extension of the dual algebra $A^*$. The Heisenberg double of these
two exterior Hopf algebras defines the differential algebra for the Cartan dif-
fferential calculus on $A$. The first differential complex is an analog of the de
Rham complex. In the situation when $A^*$ is a universal enveloping of a Lie
(superalgebra the second complex coincides with the standard complex. The
differential is realized as an (anti)commutator with a BRST- operator $Q$. A re-
current relation which defines uniquely the operator $Q$ is given. The BRST and
anti-BRST operators are constructed explicitly and the Hodge decomposition
theorem is formulated for the case of the quantum Lie algebra $U_q(gl(N))$.

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1. Introduction

The theory of the bicovariant differential calculus on a Hopf algebra $A$ has been developed by S.Woronowicz [1] (for reviews see [2], [3], [4]) on the basis of the axiomatics for the noncommutative differential geometry proposed by A.Connes [5]. Then it was shown in [6], [7] that the Woronowicz theory could be applied to the description of differential calculi on quantum analogs of some Lie groups and can be adopted to the $R$-matrix formalism of [8]. The special example of $\text{Fun}(GL_q(N))$ has been considered in detail in [6], [9], [10], [11] (see also [4] and references therein).

In [6], [11], [4] it was established that the $GL_q(N)$- differential algebra is a Poincaré-Birkhoff-Witt (PBW) type algebra and therefore indeed describes the quantum deformation of the classical differential algebras over $GL(N)$. The $SL_q(N)$- differential algebra of PBW type has been constructed in [12] (see also [13]).

However, further analysis [14] has shown that the bicovariant differential algebras [6], [13] of the Woronowicz calculus on the $\text{Fun}(SO_q(N))$ and $\text{Fun}(Sp_q(2n))$ (actually their bicovariant subalgebras of differential 1-forms) have incorrect dimensions and are not of PBW type. Thus, these noncommutative algebras could not be interpreted as the quantum deformations of the corresponding classical differential algebras.

In this paper we relate the Woronowicz theory to a deformation of the BRST theory (for review of the BRST theory see [16]; the applications of the BRST theory to the Lie algebra cohomology theory can be found in [17], [18]). This relation has already been exploited in [19], [20] in the context of the discussion of the so-called “quantum group gauge theories” (about the $q$-group gauge theories and related noncommutative geometries see e.g. [21] and references therein). The main idea is that the Woronowicz exterior differential map $d$ acting on the exterior extension of the Hopf algebra $A$ should be generated by a nilpotent operator which is nothing but the BRST charge related to the deformed algebra of the vector fields over $A$. The construction of the explicit formula for this BRST operator is the main result of the present paper.

In the first Section we explain the notion of the quantum Lie algebra. In the second Section we collect all results about the Cartan differential calculus on Hopf algebras (this is an extension of the Woronowicz differential calculus) and show how the quantum Lie algebras appear naturally in the context of these calculi. The BRST operator for the quantum Lie algebras is constructed in Section 3. In Section 4 we consider the special case of the quantum Lie algebra $U_q(gl(N))$ in detail. The BRST and anti-BRST operators are constructed explicitly and the Hodge decomposition theorem is formulated for this case.
2. Quantum Lie Algebras

A quantum Lie algebra \([22, 1, 14, 4]\) is defined by two tensors \(C_{ij}^k\) and \(\sigma_{ij}^{mk}\) (indices belong to some set \(N\), say, \(N = \{1, \ldots, N\}\)). By definition, the matrix \(\sigma_{ij}^{mk}\) has an eigenvalue 1; one demands that \((P(1))_{ij}^{mk}C_{mk} = 0\), where \(P(1)\) is a projector on the eigenspace of \(\sigma\) corresponding to the eigenvalue 1.

By definition, a quantum Lie algebra \(\Gamma\) is generated by elements \(\chi_i, i = 1, \ldots, N\), subjected to relations

\[
\chi_i \chi_j - \sigma_{ij}^{mk} \chi_m \chi_k = C_{ij}^k \chi_k. \tag{1}
\]

Here the structure constants \(C_{ij}^k\) obey

\[
C_{ij}^k C_{lj}^m = C_{ij}^m C_{lk}^m + C_{ij}^l C_{lk}^m \quad \Leftrightarrow \quad C_{ij}^{<1|} C_{lj}^{<4|} + C_{ij}^{<3|} C_{lj}^{<4|} - C_{ij}^{<4|} C_{lj}^{<4|} = 0, \tag{2}
\]

\[
C_{ni}^k \sigma_{kq}^{pm} = \sigma_{iq}^{pm} \sigma_{ns}^{rk} C_{nk}^m \quad \Leftrightarrow \quad C_{ni}^{<1|} \sigma_{13}^{<1|} + C_{ni}^{<3|} \sigma_{13}^{<3|} = 0, \tag{3}
\]

\[
(C_{ni}^{<1|} C_{pj}^{<4|} + \delta_{q}^{p} C_{kq}^{n} C_{nj}^{k}) \sigma_{nk}^{rs} \sigma_{ns}^{rk} \sigma_{ns}^{rk} C_{nk}^m \quad \Leftrightarrow \quad (C_{ni}^{<1|} \sigma_{13}^{<1|} + C_{ni}^{<3|} \sigma_{13}^{<3|}) \sigma_{13} = 0. \tag{4}
\]

The matrix \(\sigma_{ij}^{mk}\) satisfies the Yang-Baxter equation

\[
\sigma_{12}^{jk} \sigma_{13}^{n2k3} \sigma_{1j}^{n1k2} \sigma_{ij}^{n1k2} \sigma_{12} \sigma_{13} \sigma_{12} \sigma_{13} \sigma_{12} \sigma_{13} = 0. \tag{5}
\]

In the right hand side of \([3]-[6]\) we use the FRT matrix notations \([8]\); indices \(\{1, 2, 3, \ldots\}\) are the numbers of vector spaces, e.g., \(f_1 := f_{11}^1\) is a matrix which acts in the first vector space. Additionally, we use incoming and outgoing indices, e.g., \(\Omega^{<1|} := \Omega^1\) and \(\gamma_{1>} := \gamma_{1j}\) denote a covector with one incoming index and a vector with one incoming index respectively. Thus, in this notation, the matrix \(f_1\) can be written as \(f_1 = f_{11}^{<1|}\).

Note that relations \([3]-[6]\) can be justified if we consider a monomial \(\chi_{1>} \chi_{2>} \chi_{3>}\) of degree three and reorder it in two different ways using the defining relations \([1]\). A demand that the results coincide implies

\[
0 = \left( \sigma_{12} \sigma_{23} \sigma_{12} - \sigma_{23} \sigma_{12} \sigma_{23} \right) \chi_{1>} \chi_{2>} \chi_{3>} + \left( \sigma_{23} \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \right) \chi_{1>} \chi_{2>} \chi_{3>} = \left( \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \right) \chi_{1>} \chi_{2>} \chi_{3>}, \tag{6}
\]

\[
\left( \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \right) \chi_{1>} \chi_{2>} \chi_{3>} = \left( \sigma_{23} \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \sigma_{12} \sigma_{23} \right) \chi_{1>} \chi_{2>} \chi_{3>},
\]

where \(\sigma_{ij}^{mk}\) is a covector with one outcoming and \(\sigma_{ij}^{mk}\) denote a covector with one outcoming.
This is indeed an identity: the cubic term vanishes because of (5), the quadratic terms vanish in view of (4) and (3) while the last term disappears due to the Jacobi identity (2).

**Remark 1.** Note that to obtain the identity (3) it is not necessary to require precisely the relations (3) and (4), only their combination enters in (4). To obtain relations (3) and (4) as consistency conditions, one has to consider the following situation. For each integer $M$, let $\chi^{(M)}_i$ be a copy of the generators $\chi_i$. Assume that the relations between different copies are given by

$$\chi^{(K)}_1 \chi^{(M)}_2 = \sigma_{12} \chi^{(M)}_1 \chi^{(K)}_2 + C^{<3|}_{[12]} \chi^{(K)}_3 \quad \text{for} \quad K < M .$$

(7)

Then, ordering in two different ways an expression $\chi^{(L)}_{(1)} \chi^{(M)}_{(2)} \chi^{(K)}_{(3)}$ with $L < M < K$, one obtains (3) in terms cubic in $\chi$, and, in lower order terms, an identity

$$\begin{align*}
     &\left[ \left( \sigma_{23} C^{<1|}_{(12)} + C^{<3|}_{(23)} \right) \sigma_{13} - \sigma_{12} \left( \sigma_{23} C^{<1|}_{(12)} + C^{<3|}_{(23)} \right) \right] \chi^{(M)}_{(1)} \chi^{(L)}_{(2)} \\
    &- \left[ C^{<1|}_{(12)} \sigma_{13} - \sigma_{23} \sigma_{12} C^{<3|}_{(23)} \right] \chi^{(K)}_{(1)} \chi^{(L)}_{(3)} \\
    &- \left( \left( 1 - \sigma_{23} \right) C^{<1|}_{(12)} - C^{<3|}_{(23)} \right) C^{<4|}_{(13)} \chi^{(L)}_{(4)} = 0 .
\end{align*}$$

(8)

This identity is completely equivalent to the set of the relations (3), (4) and (5).

**Remark 2.** The Jacobi identity (2) implies the existence of the adjoint representation, in which the generator $\chi_i$ is represented by a matrix $(\text{ad}(\chi_i))_k^j = C^{ij}_k$. Also, any quantum Lie algebra possesses a trivial one dimensional representation, in which each generator $\chi_i$ acts as zero.

**Remark 3.** The quantum Lie algebras defined by equations (1) - (3) generalize the usual Lie (super-)algebras. In the non-deformed case, when $\sigma^{m} = (-1)^{(m)(k)} \delta_j^m \delta_k^i$ is a super-permutation matrix (here $\sigma^2 = 1$ and (3) is fulfilled; $(m) = 0 \mod(2)$ is the parity of a generator $\chi_m$), equations (1) and (2) coincide with the defining relations and the Jacobi identities for Lie (super-)algebras. Equation (3) is then equivalent to the $Z_2$-homogeneity condition $C^{ij}_{jk} = 0$ for $(i) \neq (j) + (k)$. Equation (4) follows from (3).

**Remark 4.** It was noted in (1) that the relations (2) - (3) can be encoded as the Yang-Baxter equation for the matrix $S^{AB}_{CD}$ where capital latin indices $A$, $B$, ... belong to the set $0 \cup N$ (for the matrix $\sigma^{ij}_{kl}$ the indices belong to the set $N$). The matrix $S$ is defined by

$$S^{ij}_{kl} = \sigma^{ij}_{kl} , \quad S^{0j}_{kl} = C^{ij}_{kl} , \quad S^{0A}_{0B} = \delta^A_B$$

(9)

and the other components of $S$ are zeros.
3. Cartan Differential Calculus on Hopf Algebras

In this Section we explain that the quantum Lie algebras (1) appear naturally (as quantum analogs of vector fields) in the context of the bicovariant differential calculus on the Hopf algebras.

3.1 Exterior Hopf algebras

Let $A(\Delta, \epsilon, S)$ be a Hopf algebra and $A^*$ is a Hopf dual to $A$. The comultiplication and left-right $A$-coactions on $A$ are

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \equiv \Delta_{L,R}(a), \quad a \in A. \quad (10)$$

Here the Sweedler notation is used. One can define [1] the bicovariant bimodule $\Gamma^{(1)}$ over $A$ as a linear space with left-invariant basic elements $\{\omega^i\}$ such that left and right $A$-coactions on $\Gamma^{(1)}$ have the form

$$\Delta_L(\omega^i) = 1 \otimes \omega^i, \quad \Delta_R(\omega^i) = \omega^j \otimes r_{ij} \quad (11)$$

where $r_{ij} \in A$. Since any left-invariant element can be written as a linear combination of the basis elements $\{\omega^i\}$, one has

$$S(a_{(1)}) \omega^i a_{(2)} = f_{ij}^k(a) \omega^j \quad (12)$$

where $f_{ij}^k \in A^*$. Covariance of (12) under right $A$-coaction (10), (11) requires the main condition on the elements $f, r$:

$$(f_j^i \triangleright a) r_{kj}^l = r_{ij}^k (a \triangleleft f_l^j), \quad \forall a \in A. \quad (13)$$

As it was shown in [23] (for more details see [3]), one can construct the exterior Hopf algebra $\Gamma^{\wedge} = \{A \oplus \Gamma^{(1)} \oplus \Gamma^{(2)} + \ldots\}$ via the Woronowicz’s definition of the covariant exterior product

$$\omega^i \wedge \omega^j = \omega^i \otimes \omega^j - \omega^k \otimes \omega^l \sigma_{kl}^{ij} \Rightarrow \omega^{<1|} \wedge \omega^{<2|} = \omega^{<1|} \otimes \omega^{<2|} (1 - \sigma)_{12}. \quad (14)$$

where the matrix $\sigma_{kl}^{ij} = f^i_l r^l_j$ (an analogue of the permutation matrix) satisfy the Yang-Baxter equation (13) if we put $a = r^a_{mn}$ and then take the pairing with $f^m_p$ (here we need to use the explicit forms for the comultiplications $\Delta(f^i_j)$ and $\Delta(r^j_i)$, see below).

A generalization of the definition of the wedge product (14) to the case of the product of $n$ 1-forms is straightforward

$$\omega^{<1|} \wedge \omega^{<2|} \wedge \ldots \wedge \omega^{<n|} = \omega^{<1|} \otimes \omega^{<2|} \otimes \ldots \otimes \omega^{<n|} A_{1 \rightarrow n}. \quad (15)$$
Here the matrix operator $A_{1\to n}$ is an analog of the antisymmetrizer of $n$ spaces. This operator is defined inductively (see e.g. [24])

$$A_{1\to n} = \left( 1 - \sum_{k=1}^{n-1} (-1)^{n-k-1} \sigma_{k \to n} \right) A_{1\to n-1} = \left( 1 - \sum_{k=2}^{n} (-1)^{k} \sigma_{k\to n} \right) A_{2\to n} ,$$  \hspace{1cm} (16)$$

where $\sigma_{n-k} = \sigma_{n-1n} \ldots \sigma_{k+1\ldots k+2} \sigma_{kk+1} \ldots \sigma_{n-1n}$ (n > k). If the sequence of operators $A_{1\to n}$ vanishes at the step $n = h+1$ ($A_{1\to n} = 0 \forall n > h$) then the number $h$ is called the height of the matrix $\sigma_{12}$.

The extension of the comultiplication ([10]) to the whole exterior algebra $\Gamma^\wedge$ is defined by [23]

$$\Delta(\omega^i) \equiv \Delta_L(\omega^i) + \Delta_R(\omega^i) = 1 \otimes \omega^i + \omega^j \otimes r^i_j ,$$  \hspace{1cm} (17)$$

$$\Delta(\omega a) = \Delta(\omega) \Delta(a) , \hspace{0.5cm} \Delta(a \omega) = \Delta(a) \Delta(\omega) ,$$  \hspace{1cm} (18)$$

where $\otimes$ is a graded tensor product and the grading is: $\text{deg}(\omega) = n$ for $\omega \in \Gamma^{(n)}$. Associativity condition for $(\omega a b)$ with respect to [14] and coassociativity condition for [17] yield the form of comultiplications for $f^i_j$ and $r^i_j$

$$\Delta(f^i_j) = f^i_k \otimes f^k_j , \hspace{0.5cm} \Delta(r^i_j) = r^k_j \otimes r^i_k .$$  \hspace{1cm} (19)$$

The other structure mappings for $f$, $r$ and $\omega$'s are obtained from [17], [19]

$$\epsilon(r^i_j) = \delta^i_j = \epsilon(f^i_j) , \hspace{0.5cm} S(f^i_j) f^k_j = \delta^i_j = S(r^i_j) r^k_j ,$$  \hspace{1cm} (17)$$

$$\epsilon(\omega^i) = 0 , \hspace{0.5cm} S(\omega^i) = -\omega^j S(r^i_j) .$$  \hspace{1cm} (20)$$

To summarize this subsection we stress that the knowledge of two sets of elements $\{r^i_j\} \in A$ and $\{f^i_j\} \in A^*$ which satisfy [13] is enough to construct a bicovariant bimodule over $A$ and then extend $A$ to the exterior Hopf algebra $\Gamma^\wedge$.

**Remark.** The space $\Gamma^{(k)}$ is a subspace in $\omega^\otimes k$ spanned by tensors $a$ of the form

$$a = \omega^{i_1} \otimes \ldots \otimes \omega^{i_k} A_{i_1\ldots i_k} \sigma_{j_1\ldots j_k}$$  \hspace{1cm} (21)$$

The formula $A_{1\ldots k} f_1 \ldots f_k = f_1 \ldots f_k A_{1\ldots k}$ implies that multiplication of the elements of $\Gamma^{(k)}$ and $A$ is compatible with [12]. Given two forms $a \in \Gamma^{(k)}$ (as in [21]) and $b \in \Gamma^{(l)}$ (with coefficients $b_{j_1\ldots j_l}$) define their wedge product $a \wedge b \in \Gamma^{(k+l)}$ to be

$$a \wedge b := \omega^{i_1} \otimes \ldots \otimes \omega^{i_k+l} A_{i_1\ldots i_k+l} c_{j_1\ldots j_k+l}$$  \hspace{1cm} (22)$$

where

$$c_{i_1\ldots i_k j_1\ldots j_l} = a_{i_1\ldots i_k} b_{j_1\ldots j_l} .$$  \hspace{1cm} (23)$$
The definition \((22), (23)\) of the wedge product of the differential forms is self-consistent. Indeed, one can change \(a\) by adding \(\delta a\), such that \(A_{1...k} \delta a = 0\). Then, \(A_{1...k+l} \delta a \wedge b = 0\), since \(A_{1...k+l}\) (in view of \((16)\)) is proportional to \(A_{1...k}, A_{1...k+l} = Z \cdot A_{1...k}\) for some \(Z\). It is straightforward to see that this tensor product is associative.

### 3.2 Dual exterior Hopf algebra

By analogy with the construction of the previous subsection one can define the bicovariant bimodule \(\Gamma^{(1)^*}\) and exterior Hopf algebra over the dual algebra \(A^*\) \((\Delta(h) = h_{(1)} \otimes h_{(2)}, h \in A^*)\) by introducing two sets of elements \(\{r_i\} \in A\) and \(\{f_{ij}\} \in A^*\) such that (cf. with \((12), (17), (20)\))

\[
\gamma_i h = (r_i^j \triangleright h) \gamma_j = h_{(1)} \gamma_j, \quad \Gamma^{(1)^*} = 1 \otimes \gamma_i + \gamma_j \otimes f_{ij},
\]

\[
\epsilon(\gamma_i) = 0, \quad S(\gamma_i) = -\gamma_j S(f_{ij}) \Rightarrow S^{-1}(\gamma_i) = -S^{-1}(f_{ij}) \gamma_j,
\]

where \(< h, a > = h(a)\) is a dual pairing for Hopf algebras \(A, A^*\) and the elements \(\{\gamma_i\}\) form the left-invariant basis for the bimodule \(\Gamma^{(1)^*}\). As above, we have from \(\Delta(\gamma_i h) = \Delta(\gamma_i) \Delta(h)\) the relation (cf. with \((13)\))

\[
(r_i^j \triangleright h) f_{ij} = f_{ij}^k (h \triangleright r_k^j), \quad \forall h \in A^*.
\]

and the wedge product in the exterior Hopf algebra \(\Gamma^{\wedge *} = A^* \oplus \Gamma^{(1)^*} \oplus \Gamma^{(2)^*} \oplus \ldots\) \((\deg(\gamma) = -n\) for \(\gamma \in \Gamma^{(n)^*}\)) is defined by

\[
\gamma_i \wedge \gamma_j = \gamma_i \otimes \gamma_j - f_{ij}^{kl} \gamma_k \otimes \gamma_l, \quad f_{ij}^{kl} := r_i^j f_{kj}^l.
\]

Let us consider the special case when \(\dim(\Gamma^{(1)}) = \dim(\Gamma^{(1)^*})\) and formulate the conditions when the exterior Hopf algebras \(\Gamma^{\wedge}\) and \(\Gamma^{\wedge^*}\) are Hopf dual to each other. One can extend the pairing for the algebras \(A\) and \(A^*\) to the non-degenerate pairing of the algebras \(\Gamma^{\wedge}\) and \(\Gamma^{\wedge^*}\):

\[
< \gamma_i, \omega^j > = \delta_i^j, \quad < \Gamma^{(n)^*}, \Gamma^{(m)} > \sim \delta^{nm}.
\]

This pairing is compatible with the grading. The relations \((17), (23)\) and \((29)\) give

\[
< h \gamma_i, a \omega^j > = < h, a > \delta_i^j, \quad h \in A^*, \quad a \in A.
\]

Now it is clear that the equations \((29)\) and \((30)\) relate the sets of elements \(\{r, f\}\) with \(\{\tau, \overline{f}\}\). Indeed, from one side we have \(< \gamma_i, \omega^k a >= < \Delta(\gamma_i), \omega^k \otimes a > =
\( f_k^i(a) \), \( \forall a \) but on the other side we deduce \( \langle \gamma_i, \omega^k a \rangle = \langle f_k^i \circ a \rangle \omega^j \rangle = f_k^i(a) \) and therefore \( f_k^i = f_k^i \). Considering the pairing \( \langle \gamma_i h, \omega^k \rangle \), we obtain \( r_k^i = r_k^j \) and, thus, \( \sigma_{ij}^k = \sigma_{ij}^l \). It leads to the definition of wedge product of \( n \) elements \( \gamma_i \) (see (28)):

\[
\gamma_1 > \wedge \gamma_2 > \ldots \wedge \gamma_n > = A_{1 \to n} \gamma_1 > \otimes \gamma_2 > \ldots \otimes \gamma_n > ,
\]

where operators \( A_{1 \to n} \) are the same as in (10).

### 3.3 Differential \( d \) and the algebra of vector fields

Suppose that there exists a differential map \( d : \Gamma^{(n)} \to \Gamma^{(n+1)} \) (\( \Gamma^{(0)} = A \)) which squares to 0 and satisfies the Leibniz rule (32):

\[
d_2(\omega) = 0 \, , \quad d(\omega_1 \omega_2) = d(\omega_1) \omega_2 + (-1)^{\text{deg}(\omega_1)} \omega_1 d(\omega_2) .
\]

As it was shown in (1) the left and right coactions: \( \Delta_L(d(a)) = a_{(1)} \otimes d(a_{(2)}) \), \( \Delta_R(\omega) = d(a_{(1)}) \otimes a_{(2)} \) are compatible with the definition of the bicovariant bimodule \( \Gamma^{(1)} \). From (17) we obtain the ”Leibniz rule” for the comultiplication

\[
\Delta(d(a)) = (d \otimes id + id \otimes d) \Delta(a) \Rightarrow \Delta d = (d \otimes id + id \otimes d) \Delta ,
\]

which can be extended to the whole exterior algebra \( \Gamma^\wedge \) (taking into account that \( \otimes \) is the graded tensor product). Since the 1-form \( \omega = S(a_{(1)}) da_{(2)} \in \Gamma^{(1)} \) (\( \forall a \in A \)) is left-invariant, it has to be expanded over the left-invariant basis \( \{\omega^i\} \):

\[
S(a_{(1)}) da_{(2)} = \sum_i \chi_i(a) \omega^i \Rightarrow d(a) = (\chi_i \circ a) \omega^i ,
\]

where \( \chi_i(a) \) are some coefficients and \( \chi_i \in A^* \). Applying the Leibniz rules (32) and (33) to eq. (34) gives

\[
\Delta(\chi_i) = \chi_j \otimes f_j^i + 1 \otimes \chi_i ,
\]

\[
a \triangleright \chi_i = (\chi_j \circ a) r_j^i .
\]

Equations (34) and (35) lead to the definition of the antipode and the counit for the elements \( \chi_i \):

\[
\chi_i(I) = \epsilon(\chi_i) = 0 \, , \quad S(\chi_i) = -\chi_j S(f_j^i) \Rightarrow \chi_i = -S(\chi_j) f_j^i .
\]

According to [1], \( \chi_i \) are interpreted as vector fields over the Hopf algebra \( A \).

One can obtain the commutation rules for elements \( \chi_i \) with arbitrary \( h \in A^* \):

\[
\chi_i h = (r_k^i \triangleright h) \chi_k ,
\]
from the requirement that the pairing of \(A\) and \(A^*\) is non-degenerate. Indeed, we have from (36)

\[
<\chi_i, a> = <h, (\chi_k \triangleright a) \tau_k^i > = <(\tau_k^i \triangleright h) \chi_k, a> 
\]

and, since \(a \in A\) is an arbitrary element, we deduce (38). Equations (38) for \(h = \chi_j\) give the defining relations (1) for the elements \(\chi_i\):

\[
\chi_i \chi_j = (\tau_j^i \triangleright \chi_j) \chi_k = \sigma_{ij}^{mk} \chi_m \chi_k + C_{ij}^k \chi_k, \quad C_{ij}^k = <\chi_j, \tau_i^k > .
\]

(39)

The Jacobi identities (2) for the structure constants \(C_{ij}^k\) can be obtained by pairing of eq. (39) with \(\tau_n^q\). Another application of (36) is that by taking \(a = \tau_j^i\) one deduces the relation

\[
C_{ni}^k r_m^i = r_n^k r_i^j C_{mj}^k \iff C_{[12]}^k r_3 = r_1 r_2 C_{[12]}^3 ,
\]

(40)

which gives the conditions (3) by pairing with \(f_p^q\). Eqs. (40) are the invariance condition for the structure constants \(C_{ni}^k\) with respect to the rotations by matrices \(r_n^q\).

Eqs. (27), where \(\tau_j^i = r_j^i\) and \(\tau_j^i = f_j^i\), give for \(h = f_m^n\) and \(h = \chi_n\):

\[
\sigma_{12} f_1 f_2 = f_1 f_2 \sigma_{12} ,
\]

(41)

\[
(\sigma_{im}^p \chi_p + C_{im}^j) f_j^k = f_i^k f_m^p C_{jp}^k + f_i^k \chi_m ,
\]

(42)

and pairing (11) and (12) with \(r_q^r\) reproduces (3) and (4).

Now we introduce the set of elements \(a^i \in A\) such that \(<\chi_i, a^j> = \delta_i^j\). By definition we have \(S(a_{(1)}) d(a_{(2)}) = \omega^j\) and \(\epsilon(a^i) = 0\).

The Maurer-Cartan equation reflects the fact that \(d^2(a) = 0\) and this equation can be deduced as follows

\[
d \omega^k = d(S(a_{(1)}^k) d(a_{(2)}^k) = -S(a_{(1)}^k) d(a_{(2)}^k) S(a_{(3)}^k) d(a_{(4)}^k) = \]

\[
= -\chi_i(a_{(1)}^k) \chi_j(a_{(2)}^k) \omega^i \wedge \omega^j = -t_{ij}^k \omega^i \wedge \omega^j = -C_{ij}^k \omega^i \otimes \omega^j ,
\]

(43)

where \(t_{mn}^i := <\chi_m \chi_n, a^i>\) and we use relations:

\[
C_{jk}^i = (1 - \sigma)^{mn}_{jk} t_{mn}^i ,
\]

(44)

which can be obtained by pairing (39) with \(a^k\).

**Remark 1.** The action of \(\Delta_R\) on the first relation of (34) gives

\[
\chi_i(a_{(2)}) \omega^i \otimes S(a_{(1)}) a_{(3)} = \chi_i(a) \omega^i \otimes r_j^i ,
\]

(45)
and as a result the elements $r^j_i$ are expressed in terms of the generators $a^i$:

$$\chi_i(a) r^j_i = S(a_{(1)}) \chi_j(a_{(2)}) a_{(3)} \Rightarrow r^j_i = S(a_{(1)}) \chi_j(a_{(2)}) a_{(3)} ,$$

On the other hand $\chi_i(a \triangleright a) = f^j_i(a) \forall a \in A$ and, thus, the elements $r^j_i$ and $f^j_i$ (which completely define the bicovariant bimodule over $A$) are fixed by elements $a^i \in A$ and $\chi_i \in A^*$. 

**Remark 2.** For further consideration it is useful to introduce a slightly different basis of vector fields $\tilde{\chi}_i$ by means of the formula $d(a) = \omega^j(\tilde{\chi}_i \triangleright a)$ \cite{27}. Comparing this formula with (33) gives the relations $\chi_i = f^j_i \tilde{\chi}_j \Rightarrow \chi_j = S^{-1}(f^j_i) \chi_i$, and from \cite{23}, \cite{27} we have

$$\tilde{\chi}_i = -S^{-1}(\chi_i) \Rightarrow \Delta(\tilde{\chi}_i) = \tilde{\chi}_i \otimes 1 + S^{-1}(f^j_i) \otimes \tilde{\chi}_j .$$

By applying $S^{-1}$ to \cite{28} one obtains

$$h \tilde{\chi}_i = \tilde{\chi}_j < h(1), \quad S(r^j_i) > h(2) . \tag{45}$$

### 3.4 Heisenberg double $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$ and Cartan calculus

The differential algebra of Cartan calculus should be constructed \cite{27}, \cite{1}, \cite{25} as a Heisenberg double $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$ of algebras $\Gamma^\wedge$ and $\Gamma^\wedge\ast$. The action of $d$ on the elements of the Heisenberg double $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$ has to be extended as well.

First of all we recall that the Heisenberg double $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$ is an associative algebra which is a product of two algebras $\Gamma^\wedge$, $\Gamma^\wedge\ast$ with nontrivial $Z_2$–graded cross-multiplication rule

$$\gamma \omega = (\gamma_{(1)} \triangleright \omega) \gamma_{(2)} = (-1)^{\deg(\gamma_{(1)}) \deg(\omega_{(2)})} \omega_{(1)} < \gamma_{(1)}, \quad \omega_{(2)} > \gamma_{(2)} , \tag{46}$$

where $\omega \in \Gamma^\wedge$ and $\gamma \in \Gamma^\wedge\ast$. This rule defines the commutation relations between the elements of $\Gamma^\wedge$ and the elements of $\Gamma^\wedge\ast$.

Although $\Gamma^\wedge$ and $\Gamma^\wedge\ast$ are Hopf algebras, their Heisenberg double $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$ is not a Hopf algebra. But $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$ still possesses some covariance properties. Let us define a right $A$ - coaction and a left $A^*$ - coaction on the algebra $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$, which respect the algebra structure of $\Gamma^\wedge \triangleright \Gamma^\wedge\ast$. We denote \{$e^a\}$ and \{$e_a\}$ the dual basis elements of $A^*$ and $A$ respectively. The right $A$ - coaction and left $A^*$ - coaction on $z \in \Gamma^\wedge \triangleright \Gamma^\wedge\ast$ are defined as follows:

$$\Delta_R(z) = C (z \otimes 1) C^{-1} , \quad \Delta_L(z) = C^{-1} (1 \otimes z) C , \quad C \equiv e^a \otimes e_a . \tag{47}$$

Note that $\Delta_R(z) = \Delta(z) \forall z \in A$ and $\Delta_L(z) = \Delta(z) \forall z \in A^*$. The axioms

$$(id \otimes \Delta)\Delta_R = (\Delta_R \otimes id)\Delta_R , \quad (id \otimes \Delta_L)\Delta_L = (\Delta \otimes id)\Delta_L ,$$
\[(id \otimes \Delta_R)\Delta_L(z) = C_{13}^{-1} (\Delta_L \otimes id)\Delta_R(z) C_{13},\]
can be verified directly by using the pentagon identity [26] for \(C\).

\[C_{12} C_{13} C_{23} = C_{23} C_{12}.\]

It is clear that the coactions (47) are covariant transformations (homomorphisms) of the algebra \(\Gamma^\wedge \triangleright \Gamma^\wedge^\ast\). The inverse of the canonical element \(C\) is

\[C^{-1} = S(e^\alpha) \otimes e_\alpha = e^\alpha \otimes S(e_\alpha),\]

and \(\Delta_R\) (47) is rewritten in the form

\[\Delta_R(z) = (e^\gamma(1) z S(e^\gamma(2))) \otimes e_\gamma.\]  (48)

In particular, for \(y \in A^\ast\) we have

\[\Delta_R(y) = y(1) \otimes y(2) = (e^\gamma(1) y S(e^\gamma(2))) \otimes e_\gamma.\]  (49)

By pairing the second factor of (49) with an arbitrary \(x \in A^\ast\) we deduce the relation \(y(1) < x, y(2) > = x(1) y S(x(2))\) which is equivalent to the commutation relations for \(x, y \in A^\ast\)

\[x y = y(1) < x(1), y(2) > x(2),\]  (50)

where \(\Delta(x) = x(1) \otimes x(2)\). The inverse statement (that one can obtain (49) from (50)) is also correct. The analogs of eqs. (48), (49) and (50) for coaction \(\Delta_L\) can be deduced in the same way. Comparing the commutation relations (50) with eqs. (15) one can find as an example the right coaction \(\Delta_R\) on \(\tilde{\chi}_i\)

\[\Delta_R(\tilde{\chi}_i) = \tilde{\chi}_j \otimes S(r^j_i).\]  (51)

At the end of this Section we present some cross-commutation relations (see (16)) which will be needed below

\[[\chi_i, \omega^j] = C^j_{ik} \omega^j f^k_i, \quad [\gamma_i, \omega^j] = f^j_i, \quad [\gamma_i, a] = 0 \quad \forall a \in A.\]  (52)

4. BRST Operator for Quantum Lie Algebras

In this Section we find a bi-invariant element \(Q \in \Gamma^\wedge \triangleright \Gamma^\wedge^\ast\) (a BRST operator) which generates the differential \(d\):

\[d\omega = [Q, \omega]_\pm, \quad \forall \omega \in \Gamma^\wedge\]  (53)

The operator \(Q\) has to be of the grading 1 and obeys \(Q \wedge Q = 0\).
We change the basis of differential forms \( \{ \omega^i \} \) and consider new basics elements \( \Omega^i = \omega^j S^{-1}(f_j) \) which are convenient in view of the relation
\[
[\Omega^i, a] = 0 \quad \forall a \in \mathcal{A}.
\]
The following equations for \( \Omega^i \) are also valid (see (52)):
\[
\chi_{[2>]} \Omega^{<2]} = \Omega^{<1]} (\sigma_{12} \chi_{[1]>} + C_{[12>]}) , \quad \gamma_{[2>]} \Omega^{<2]} = -\Omega^{<1]} \sigma_{12}^{-1} \gamma_{[1]>} + I_2 ,
\]
and the definition of the wedge product for \( r \) variables \( \Omega^i \) holds:
\[
\Omega^{<r]} \ldots \Omega^{<1]} = \Omega^{<r]} \otimes \ldots \otimes \Omega^{<1]} A_{1 \to r}.
\]
Now we formulate the main result of the paper (see also [28]):

**Proposition** The BRST operator \( Q \) for the quantum algebra (1), which generate the differential (53), has the following form
\[
Q = \Omega^i \chi^i + \sum_{r=1}^{h-1} Q_{(r)},
\]
where \( h \) is the height of the operator \( \sigma_{12} \) (16).

Here the operators \( Q_{(r)} \) are given by
\[
Q_{(r)} = \Omega^{<r+1]} \Omega^{<r]} \ldots \Omega^{<1]} X_{|1 \ldots r+1>}^{<1 \ldots r]} \gamma_{|1> \ldots r>}
\]
(\( \text{the wedge product is implied} \)); \( X_{|1 \ldots r+1>}^{<1 \ldots r]} \) are tensors which satisfy the following recurrent relations
\[
A_{1 \to r+1} X_{|1 \ldots r+1>}^{<1 \ldots r]} A_{1 \to r} = A_{1 \to r+1} \left( (-1)^r \sigma_{r+1 \to 1} - 1 \right) X_{|2 \ldots r+1>}^{<2 \ldots r]} A_{2 \to r}
\]
with the initial condition \( A_{12} X_{|12>}^{<0]} = -C_{|12>}^{<0]} \).

**Proof.** We have to verify the conditions \( [Q, a] = d a \ (\forall a \in \mathcal{A}) \) and \( [Q, \omega^i] = d \omega^i \ (\forall \omega^i \in \Gamma^{(1)}) \) and the identity \( Q^2 = 0 \). The proof of the condition \( [Q, a] = d a \ (\forall a \in \mathcal{A}) \) is straightforward since \( \Omega^i \) and \( \gamma_i \) commute with all \( a \in \mathcal{A} \). So we need only to prove \( [Q^i \chi^i, a] = d a \) which follows immediately from relations (54). The other conditions are valid only if the recurrent relations (57) are fulfilled. We will give the complete proof of this statement elsewhere.

**Remark 1.** The first term \( \Omega^i \chi^i = \omega^i \chi_i \) in (54) is a bi-invariant element (see (11), (51)). One can prove that all other terms \( Q_{(r)} \) are also bi-invariants.

**Remark 2.** For general \( \sigma_{ij}^k \) and \( C_{jk}^i \) it is rather difficult to solve equations (57) explicitly. However for the case \( \sigma^2 = 1 \) the main equations (57) become
simpler and the general solution for $Q$ can be found. Indeed the relation (57) for $r = 2$ gives

$$A_{1 \rightarrow 3} X_{|123\rangle}^{<12|} (1 - \sigma_{12}) = A_{1 \rightarrow 3} (\sigma_{23} \sigma_{12} - 1) X_{|123\rangle}^{<2|}.$$  \hspace{1cm} (58)

For $\sigma^2 = 1$ we have $A_{1 \rightarrow 3} (\sigma_{23} \sigma_{12} - 1) = 0$ and therefore $Q_{(r)} = 0$ for $r \geq 2$. Thus the BRST operator (55) has the familiar form

$$Q = \Omega_{|1\rangle}^{<1|} - \Omega_{|2\rangle}^{<2|} \otimes \Omega_{|1\rangle}^{<1|} C_{|12\rangle}^{<1|}.$$  \hspace{1cm} (59)

In general, for $\sigma^2 \neq 1$, the sum in (55) will be limited only by the height $h$ of the operator $\sigma$.

Below we present an explicit form for $Q$ for the standard quantum deformation $A^* = U_q(gl(N))$ of the universal enveloping algebra of the Lie algebra $gl(N)$ ($\sigma^2 \neq 1$ in this case).

**Remark 3.** Here we give expressions for first two coefficients $X_{|123\rangle}^{<12|}$ and $X_{|1234\rangle}^{<123|}$. Substitution of the initial condition $A_{12} X_{|12\rangle}^{<0|} = -C_{|12\rangle}^{<0|}$ into (58) gives

$$A_{123} X_{|123\rangle}^{<12|} A_{12} = [C_2 + \sigma_1 \sigma_2 C_1 \delta_3] A_{12}$$  \hspace{1cm} (59)

where we have used the concise notation

$$\sigma_n = \sigma_{n+1}, \quad C_n = C_{|n+1\rangle}^{<n|}, \quad \delta_n = \delta_{|n\rangle}^{<n-1|}.$$  \hspace{1cm} (60)

The analogous formula for the next coefficient is

$$A_{1234} X_{|1234\rangle}^{<123|} A_{123} = [(C_3 + \sigma_2 \sigma_3 C_2 \delta_4) - \sigma_1 \sigma_2 \sigma_3 (C_2 + \sigma_1 \sigma_2 C_1 \delta_3) \delta_4] A_{123}.$$  \hspace{1cm} (61)

To obtain (58) and (61) it is convenient to rewrite eqs. (2) - (3) in the form (see notation (60)):

$$C_1 \delta_3 C_1 = \sigma_2 C_1 \delta_3 C_1 + C_2 C_1, \quad C_1 \delta_3 \sigma_1 = \sigma_2 \sigma_1 C_2,$$

$$(\sigma_2 C_1 \delta_3 + C_2) \sigma_1 = \sigma_1 (\sigma_2 C_1 \delta_3 + C_2), \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2.$$

5. **BRST and anti-BRST operators for quantum linear algebra $U_q(gl(N))$.**
5.1 BRST operator for $U_q(gl(N))$

The quantum algebra $U_q(gl(N))$ is defined (as a Hopf algebra) by the relations

\[
\hat{R} L_2^+ L_1^- = L_2^+ L_1^- \hat{R}, \quad \hat{R} L_2^+ L_1^- = L_2^- L_1^+ \hat{R}, \quad (62)
\]

\[
\Delta(L^\pm) = L^\pm \otimes L^\pm, \quad \varepsilon(L^\pm) = 1, \quad S(L^\pm) = (L^\pm)^{-1}, \quad (63)
\]

where elements of the $N \times N$ matrices $(L^\pm)^i_j$ are generators of $U_q(gl(N))$; the matrices $L^+$ and $L^-$ are respectively upper and lower triangular, their diagonal elements are related by $(L^+_i)^i_i (L^-_i)^i_i = 1$ for all $i$. The matrix $\hat{R}$ is defined as $\hat{R} := \hat{R}_{12} = P_{12} R_{12}$ ($P_{12}$ is the permutation matrix) and the matrix $R_{12}$ is the standard Drinfeld-Jimbo $R$-matrix for $GL_q(N)$,

\[
R_{12} = R_{j_1,j_2}^{i_1,i_2} = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} (1 + (q - 1) \delta_{i_1}^{i_2}) + (q - q^{-1}) \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \Theta_{i_1 i_2}, \quad (64)
\]

where

\[
\Theta_{ij} = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i \leq j. \end{cases}
\]

This $R$-matrix satisfies the Hecke condition $\hat{R}^2 = \lambda \hat{R} + 1$, where $\lambda = (q - q^{-1})$ and $q$ is a parameter of deformation.

The generators of the algebra $A^*$ are defined by the formula

\[
\chi^i_j = \frac{1}{\lambda} \left[ (D^{-1})^i_k - (D^{-1})^j_i f^i_k \right]. \quad (65)
\]

Here $f^i_k = L^{-1}_k S(L^+_i)$ and the numerical matrix $D$ can be found by means of relations

\[
Tr_2 R_{12} \Psi_{23} = P_{13} = Tr_2 \Psi_{12} \hat{R}_{23}, \quad D_1 := Tr_2 \Psi_{12} \Rightarrow Tr_1 (D_1^{-1} \hat{R}^{-1}) = 1_2,
\]

where $Tr_1$ and $Tr_2$ denote the traces over first and second spaces.

For the $GL_q(N)$ $R$-matrix the explicit expression for the $D$-matrix is:

\[
(D^{-1})^j_i = q^{2(N-j)+1} \delta^j_i, \quad Tr(D^{-1}) = \frac{q^{2N} - 1}{q - q^{-1}}. \quad (66)
\]

It is convenient to write down the commutation relations for the differential algebra in terms of generators

\[
L^i_j = (L^+_i)^k_j S((L^-)^h_j) = \delta^i_j - \lambda S^{-1}(\chi^i_k) D^k_j,
\]

\[
J^i_n = -S^{-1}(f^i_k) \gamma^i_k D^i_n, \quad \omega^i_j = \Omega^i_m f^m_j.
\]

The indices now are pairs of indices; the roles of the elements $\chi_i$, $\gamma_j$ and $\Omega^k$ are played by the generators $\chi^i_j$, $\gamma^i_j$ and $\Omega^i_j$ respectively.
The commutation relations are \([10], [4], [25]\):

\[
\hat{\omega}_2 \hat{R}^{-1} \hat{\omega}_2 \hat{R} = -\hat{R}^{-1} \hat{\omega}_2 \hat{R}^{-1} \hat{\omega}_2, \quad \hat{\omega}_2 \hat{R} \hat{L}_2 \hat{R} = \hat{R} \hat{L}_2 \hat{R} \hat{\omega}_2, \quad (67)
\]

\[
\hat{\omega}_2 \hat{R} J_2 \hat{R} + \hat{R} J_2 \hat{R} \hat{\omega}_2 = -\hat{R}, \quad \hat{L}_2 \hat{R} L_2 \hat{R} = \hat{R} \hat{L}_2 \hat{R} L_2, \quad (68)
\]

\[
J_2 \hat{R} L_2 \hat{R} = \hat{R} \hat{L}_2 \hat{R} J_2, \quad J_2 \hat{R} J_2 \hat{R} = -\hat{R}^{-1} J_2 \hat{R} J_2. \quad (69)
\]

To write down the whole differential algebra over \(GL_q(N)\) we need to add the generators \(T_i^j\) of the quantum group \(Fun(GL_q(N))\) with commutation relations

\[
\hat{\omega}^2 \hat{R} = -\hat{R}^{-1} \hat{\omega}^2 \hat{R}^{-1}, \quad (70)
\]

\[
J_1^2 \hat{R} J_2 \hat{R} = \hat{R} J_1^2 \hat{R} J_2, \quad (71)
\]

The BRST operator \(Q\) for the differential algebra (67) – (71) can be constructed with the help of the formula (55) and has the following form \([28]\):

\[
Q = \text{Tr}_q \left( \omega \left( L - 1 \right) / \lambda - \omega L (\omega J) + \lambda \omega L (\omega J)^2 - \lambda^2 \omega L (\omega J)^3 + \ldots \right) \quad (72)
\]

\[
= \text{Tr}_q \left( \omega \left( \frac{L - 1}{\lambda} - \omega L (\omega J) (1 + \lambda \omega J)^{-1} \right) = -\frac{1}{\lambda} \text{Tr}_q(\omega) + \frac{1}{\lambda} \text{Tr}_q(\Theta), \quad (73)
\]

where \(\Theta = \omega L (1 + \lambda \omega J)^{-1}\) and \(\text{Tr}_q(Y) := \text{Tr}(D^{-1} Y)\) is a quantum trace. The sum in eq. (72) is finite due to the fact that monomials in \(\omega\)'s of the order \(N^2 + 1\) are equal to zero (since the exterior algebra (67) of forms on the quantum group \(GL_q(N)\) is a flat deformation of the classical algebra \([3], [4]\)).

One can check directly that the BRST operator \(Q\) given by (73) satisfies:

\[
Q^2 = 0, \quad [Q, L] = 0, \quad (74)
\]

\[
[Q, T] = T \omega \equiv dT, \quad [Q, \omega]_+ = -\omega^2 \equiv d\omega, \quad (75)
\]

\[
[Q, J]_+ = \frac{1}{\lambda}(1 - L). \quad (76)
\]

The (anti)commutator with the BRST operator \(Q\) (relations (73)) defines the exterior differential operator over \(GL_q(N)\); it provides the structure of the de Rham complex \(\Omega(GL_q(N))\) on the subalgebra with generators \(T^i_j\) and \(\omega^i_j\) (the de Rham complex \(\Omega(GL_q(N))\) has been firstly considered by Yu.I.Manin, G.Maltsiniotis and B.Tsygan \([3]\)).

The last relation (74) is an analog of the Cartan identity. To obtain relations (74) – (76) one has to use the invariance property of the quantum trace:

\[
\text{Tr}_q(X) 1_2 = \text{Tr}_q(\hat{R}^{\pm 1} X_2 \hat{R}^{\mp 1})
\]
and relations
\[ \hat{R} \Theta_2 \hat{R}^{-1} \omega_2 = -\omega_2 \hat{R}^{-1} \Theta_2 \hat{R} , \tag{77} \]
\[ \hat{R} \Theta_2 \hat{R}^{-1} \omega_2 = -\Theta_2 \hat{R}^{-1} \Theta_2 \hat{R}^{-1} , \tag{78} \]
\[ \hat{R}^{-1} \Theta_2 \hat{R} L_2 = L_2 \hat{R} \Theta_2 \hat{R}^{-1} , \quad \Theta_1 T_2 = T_2 \hat{R}^{-1} \Theta_2 \hat{R} , \tag{79} \]
\[ J_2 \hat{R} \Theta_2 \hat{R}^{-1} + \hat{R}^{-1} \Theta_2 \hat{R} J_2 = -L_2 (1 + \lambda \omega J)_2^{-1} \hat{R}^{-1} (1 + \lambda \omega J)_2 , \tag{80} \]
which follow from (67)-(71).

**Remark 1.** The operator \( Q \) given by (73) has the correct classical limit for \( q \to 1 (\lambda \to 0, L \to 1 + \lambda \hat{\chi}, \omega \to \hat{\omega}, J \to -\hat{\gamma}) \):
\[ Q \to Q_{cl} = \text{Tr}(\hat{\omega} \hat{\chi} + \hat{\omega}^2 \hat{\gamma}) = \text{Tr}(\hat{\omega} X - \hat{\omega} \hat{\gamma} \hat{\omega}) , \tag{81} \]
where
\[ X := \hat{\chi} + \hat{\omega} \hat{\gamma} + \hat{\gamma} \hat{\omega} \]
and the classical algebra is
\[ [\hat{\omega}_2, \hat{\gamma}_1]_+ = P_{12} , \quad [\hat{\omega}_2, \hat{\omega}_1]_+ = 0 = [\hat{\gamma}_2, \hat{\gamma}_1]_+ , \tag{82} \]
\[ [X_2, X_1] = P_{12}(X_2 - X_1) , \tag{83} \]
\[ [X_2, \hat{\omega}_1] = 0 = [X_2, \hat{\gamma}_1] . \tag{84} \]

**Remark 2.** The differential complex \( \Omega(GL_q(N)) \) gives rise to the de Rham cohomology groups \( H^p(GL_q(N)) \) of the quantum groups \( GL_q(N) \). The \( q \)-analogs of the basic generators for the de Rham cohomology ring \( H^*(GL_q(N)) \) can be chosen as
\[ \Omega^{(n)} := \text{Tr}_q(\omega^n) \quad (n = 1, 3, 5, \ldots, 2N - 1) . \tag{85} \]
These generators satisfy \[ \Omega^{(n)} := \text{Tr}_q(\omega^{n+1}) = 0, \quad [\Omega^{(n)}, \Omega^{(k)}]_+ = 0 . \tag{86} \]

### 5.2 Anti-BRST operator and quantum Laplacian

In the same way as we deduce the explicit formula (72)-(73) for the BRST operator \( Q \), one can construct the anti-BRST operator \( Q^* \) for the algebra \( U_q(gl(N)) \):
\[ Q^* = \text{Tr}_q \left( J (L^{-1} - 1)/\lambda + J L^{-1} J \omega \right) = \frac{1}{\lambda} \left( \text{Tr}_q(\Theta^*) - \text{Tr}_q(J) \right) , \tag{86} \]
where $\Theta^* = J L^{-1} (1 + \lambda J \omega)$. The operator $Q^*$ satisfies

\begin{align}
(Q^*)^2 &= 0, \quad [Q^*, L] = 0, \\
[Q^*, T] &= q^{2N} T J, \quad [Q^*, J^+] = -q^{2N} J^2, \tag{87} \\
[Q^*, \omega] &= q^{2N} \left( W \frac{1 - L^{-1}}{\lambda} \bar{W} - \lambda \omega J^2 \omega \right). \tag{88}
\end{align}

Here the factor $q^{2N}$ appeared because of the following identities (see (66))

\[Tr q_1(R) = Tr q_1(R^{-1} + \lambda 1) = q^{2N} 1_2.\]

For convenience we introduce new operators

\[W = (1 + \lambda \omega J), \quad \bar{W} = (1 + \lambda J \omega).\]  

To prove eqs. (87), (88) and (89) we used relations

\begin{align}
\Theta_1^* T_2 &= T_2 \hat{R} \Theta_2^* \hat{R}^{-1}, \quad \hat{R}^{-1} \Theta_2^* \hat{R} J_2 = -J_2 \hat{R} \Theta_2^* \hat{R}^{-1}, \\
\hat{R} \Theta_2^* \hat{R} \Theta_1^* &= -\Theta_2^* \hat{R} \Theta_2^* \hat{R}^{-1}, \quad \hat{R}^{-1} \Theta_2^* \hat{R} L_2 = L_2 \hat{R} \Theta_2^* \hat{R}^{-1}, \tag{89} \\
\hat{R} \Theta_2^* \hat{R}^{-1} \omega_2 + \omega_2 \hat{R}^{-1} \Theta_2^* \hat{R} &= -(W L^{-1} \bar{W})_2 \hat{R}. \tag{90}
\end{align}

Using eqs. (88) and (89) one can define the dual differential $d^*$ as an (anti)commutator with the anti-BRST operator $Q^*$.

Now we define the current matrix $U_j^*$:

\[U := W L^{-1} \bar{W} = -q^{-2N} [Tr_q(\Theta^*), \omega]^+,\]

which is a quantum analog of the matrix $X$ (81) and satisfies the reflection equation

\[\hat{R}^{-1} U_2 \hat{R}^{-1} U_2 = U_2 \hat{R}^{-1} U_2 \hat{R}^{-1}.\]

For this current matrix we deduce the following commutation relations

\[\hat{R} U_2 \hat{R}^{-1} \omega_2 = \omega_2 \hat{R}^{-1} U_2 \hat{R}, \quad \hat{R}^{-1} U_2 \hat{R} J_2 = J_2 \hat{R} U_2 \hat{R}^{-1}. \tag{92}\]

These relations are correct quantum analogs of the commutativity conditions (84) ($U$ is represented as $U = 1 + \lambda X$ where $X \rightarrow X$ for $q \rightarrow 1$), i.e.

\[X_{1>} \omega_{2>} = \sigma_{12} \omega_{2>} X_{1>}, \quad X_{1>} J_{2>} = \sigma_{12} J_{2>} X_{1>},
\]

where $\sigma_{12}$, $\sigma_{12}$ are braid matrices defined by (24) converted to the permutation matrices for $q = 1$. Moreover, the transformation of the left-invariant current $U$ into the right invariant current $\Xi$:

\[\Xi = T W L^{-1} \bar{W} T^{-1} = T U T^{-1},\]
leads to the exact commutativity conditions: $\{\Xi_2, \omega_1\} = 0$, $\{\Xi_2, J_1\} = 0$.

By definition the quantum Laplace operator is (here it is enough to use relations (74) - (76))

$$
\Delta := Q^* Q + Q Q^* = \lambda^{-2} Tr_q \left( L - 2 + L^{-1} W - \lambda J L^{-1} W \omega \right),
$$

(93)
The Laplacian $\Delta$ is a BRST and anti-BRST invariant operator:

$$
[Q, \Delta] = 0 = [Q^*, \Delta],
$$

and it generalizes the Casimir operator for the universal enveloping algebra $U_q(gl(N))$.

Taking into account the identity

$$
-\lambda Tr_q \left( L^{-1} W \omega \right) = Tr_q \left( (q^{2N} - 1) L^{-1} W + \lambda q^{2N} \omega J L^{-1} W \right),
$$

one obtains a remarkable expression for the quantum Laplacian via the current $U$:

$$
\Delta = \frac{1}{\lambda^2} Tr_q \left( L + q^{2N} U - 2 \right).
$$

Now the formulation of the Hodge decomposition theorem is in order. Consider the space of polynomials in the variables $\omega_{i_1} \ldots \omega_{i_k}$ and $T_{i_1} \ldots T_{i_k}$ with complex coefficients $\psi_{i_1 \ldots i_k}$,

$$
|\Psi \rangle := \Psi[T, \omega] = \sum_{k=0}^{K} \sum_{r=0}^{k} Tr_{1 \ldots k} (T_1 \ldots T_r \omega_{r+1} \ldots \omega_k \psi_{1 \ldots k})
$$

(for some $K$). Here the case $r = 0$ corresponds to the arbitrary polynomial in $\omega$'s which independent of $T$'s. The vector fields $L^i_j = \delta^i_j$ and inner derivatives $J^i_j$ act on the zero-order monomial $|0 \rangle := 1$ from the left as anihilation operators:

$$(L^i_j - \delta^i_j) |0 \rangle = 0, \quad J^i_j |0 \rangle = 0.$$  

This defines the left action of the BRST, anti-BRST and Laplace operators on the polynomials $|\Psi \rangle$. Now the decomposition theorem can be formulated:

**Theorem.** Any polynomial $\Psi[T, \omega]$ can be decomposed into a sum of BRST-exact, co-exact and harmonic polynomials:

$$
|\Psi \rangle = |\Omega \rangle + Q \cdot |\chi \rangle + Q^* \cdot |\Phi \rangle,
$$

where $\Delta |\Omega \rangle > 0$.

The proof of this theorem is straightforward and analogous to the proof of the decomposition theorem in the case of the classical Lie algebras (see e.g. [17], [18]).

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