Research Article

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A note on Kazdan–Warner equation on networks

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Abstract: We investigate the Kazdan–Warner equation on a network. In this case, the differential equation is defined on each edge, while appropriate transition conditions of Kirchhoff type are prescribed at the vertices. We show that the whole Kazdan–Warner theory, both for the noncritical and the critical case, extends to the present setting.

Keywords: Kazdan–Warner equation, network, Kirchhoff condition

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1 Introduction

The Kazdan–Warner equation

\[ \Delta u = c - he^u, \]  

(1.1)

where \( c \) is a constant and \( h \) a given function, was introduced in [15] in connection with the problem of prescribing the Gaussian curvature of a compact manifold \( M \). The resolvability of (1.1) depends on the sign of \( c \). Let \( \bar{h} \) denote the average of \( h \) on \( M \). In [15], it is shown that,

(i) if \( c = 0 \) and \( h \neq 0 \), then (1.1) is solvable if and only if \( h \) changes sign and \( \bar{h} < 0 \);
(ii) if \( c > 0 \), then (1.1) is solvable if and only if the set \( \{ h > 0 \} \) is not empty;
(iii) if \( c < 0 \), if (1.1) is solvable, then \( \bar{h} < 0 \). For \( \bar{h} < 0 \), there exists a constant \( -\infty \leq c(h) < 0 \) such that (1.1) is solvable for any \( c \in (c(h), 0) \) and not solvable for any \( c < c(h) \). Moreover, \( c(h) = -\infty \) if and only if \( h \leq 0 \) in \( M \).

If \( c < 0 \), \( c = c(h) \) is not included in the previous cases and deserves particular attention. It has been shown in [8] that, if \( c(h) > -\infty \), then (1.1) can be also solved for \( c = c(h) \).

The previous theory has been recently extended in [13, 14] to the case of a combinatorial graph. Here the Laplacian is replaced by a finite difference operator, the so-called graph Laplacian, and most of the effort is to reproduce in a finite-dimensional setting some crucial properties as the maximum principle and the Moser–Trudinger inequality.

An intermediate situation between a compact manifold and a combinatorial graph is given by a network (or metric graph) \( \Gamma \), which is given by a finite collection of vertices connected by continuous non-self-intersecting edges (see Section 2 for the precise definition). On a network, differential equation (1.1) is defined on each edge, while appropriate transition conditions of Kirchhoff type are prescribed at the vertices. In this

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paper, we obtain the same conclusions of the manifold and the graph cases, showing that the Kazdan–Warner theory remains unchanged for different classes of manifolds, also non-regular such as in the case of networks. To prove these results, we shall adapt the method by Kazdan and Warner [15, Theorem 5.3] (see also [14, Theorem 2]) and, for the critical case, some techniques of [8, 13] with some specific arguments for networks.

The study of the theory of differential equations on networks is motivated by two features: the networks provide simplified mathematical models for physics, chemistry and engineering where one can investigate phenomena in a “quasi-one-dimensional” environment. Moreover, being encompassed in the larger family of non-regular manifold, they are a first step toward the approach to more general manifolds. The recent study of differential equations on networks steamed mainly in the framework of hyperbolic systems and spectral theory (we refer the reader to the monographs [3, 5, 18] and references therein). A general theory for linear and semilinear differential equations has been developed mainly employing the variational structure of the problem, and in this framework, the natural transition conditions at the vertices are the Kirchhoff conditions (for instance, see [17–19]; see also [10, 11] for a stochastic interpretation). The theory has been also extended to some nonlinear problem such as traffic flows [12], optimal control problems through the corresponding Hamilton–Jacobi equations [4, 16] and mean field games [1, 7, 16], nonlinear Schrödinger equations [2, 9]. For nonlinear problems, different transition conditions at the vertices can arise.

The paper is organized as follows. In Section 2, we introduce some notation and preliminary results. In Sections 3, 4 and 5, we study respectively the cases \( c = 0, c > 0 \) and \( c < 0 \). In Section 5, we also discuss the critical case \( c = c(h) \).

### 2 Notation, definitions and preliminary results

A network \( \Gamma = (V, E) \) is a finite collection of points \( V := \{v_i\}_{i \in I} \) in \( \mathbb{R}^n \) connected by continuous edges \( E := \{e_j\}_{j \in J} \), where each edge does not intersect itself and any two edges can only have intersection at a vertex. For \( i \in I \), we set

\[
\text{Inc}_i := \{j \in J : e_j \text{ is incident to } v_i\}.
\]

A coordinate \( \pi_j : [0, l_j] \to \mathbb{R}^n \), with \( l_j > 0 \), is chosen to parametrize \( e_j \), i.e. \( e_j := \pi_j((0, l_j)) \). We assume that \( \Gamma \) is compact and connected, and we denote by \(|\Gamma|\) the sum of the lengths of the edges \( e_j, j \in J \).

For a function \( u : \Gamma \to \mathbb{R} \), we denote by \( u_j : [0, l_j] \to \mathbb{R} \) the restriction of \( u \) to \( e_j \), i.e. \( u(x) = u_j(y) \) for \( x \in e_j, y = \pi_j^{-1}(x) \in (0, l_j) \). Given \( v_i \in V \), we denote by \( \partial_j u(v_i) \) the oriented derivative at \( v_i \) along the arc \( e_j \) defined by

\[
\partial_j u(v_i) = \lim_{x \in e_j, x \to v_i} \frac{u_j(\pi_j^{-1}(x)) - u_j(\pi_j^{-1}(v_i))}{|\pi_j^{-1}(x) - \pi_j^{-1}(v_i)|}
\]

if the limit exists, where \( \pi_j \) is the parametrization of arc \( e_j \). For a function \( \phi : \Gamma \to \mathbb{R} \) and \( A \subset \Gamma \), we set

\[
\int_A \phi(x) \, dx := \sum_{(0, l_j) \cap \pi_j^{-1}(A)} \int_{(0, l_j) \cap \pi_j^{-1}(A)} \phi(r) \, dr.
\]

A function \( u \) is said to be continuous on \( \Gamma \) if it is continuous with respect to the subspace topology of \( \Gamma \), i.e. \( u_j \in C((0, l_j)) \) for any \( j \in J \) and \( u_k(\pi_k^{-1}(v_i)) = u_j(\pi_j^{-1}(v_i)) \) for any \( i, j, k \in \text{Inc}_i \).

We introduce some functional spaces for functions defined on the network. The space \( L^p(\Gamma) \), \( p \geq 1 \), consists of the functions that are measurable and \( p \)-integrable on each edge \( e_j, j \in J \). We set

\[
\|f\|_{L^p} := \left( \sum_{j \in J} \|f_j\|_{L^p(e_j)}^p \right)^{\frac{1}{p}}.
\]

The space \( L^\infty(\Gamma) \) consists of the functions that are measurable and bounded on each edge \( e_j, j \in J \). We set

\[
\|f\|_{L^\infty} := \sup_{j \in J} \|f_j\|_{L^\infty(e_j)}.
\]
The Sobolev space $W^{k,p}(\Gamma)$, $k \in \mathbb{N}$ and $p \geq 1$, consists of all continuous functions on $\Gamma$ that belong to $W^{k,p}(e_j)$ for each $j \in J$. We set

$$\|f\|_{W^{k,p}} := \left( \sum_{l=0}^{k} \|\partial^l f\|_{L^p}^p \right)^{\frac{1}{p}}.$$ 

As usual, we set $H^k(\Gamma) := W^{k,2}(\Gamma)$, $k \in \mathbb{N}$. The space $C^k(\Gamma)$ for $k \in \mathbb{N}$ consists of all continuous functions on $\Gamma$ that belong to $C^k(e_j)$ for $j \in J$. The space $C^k(\Gamma)$ is a Banach space with the norm

$$\|f\|_{C^k} = \max_{\beta \leq k} \|\partial^\beta f\|_{L^\infty}.$$ 

The following lemma gives a Poincaré inequality for compact networks.

**Lemma 2.1.** For every function $f \in H^1(\Gamma)$ with $\int_\Gamma f(x) \, dx = 0$, there hold

(i) $|f(x)| \leq \sqrt{|\Gamma|} \|\partial f\|_{L^2},$

(ii) $\int_\Gamma f^2(x) \, dx \leq |\Gamma|^2 \int_\Gamma |\partial f(x)|^2 \, dx.$

**Proof.** By definition of $H^1$, the function $f$ is continuous on $\Gamma$; hence there exists a point $x_0 \in \Gamma$ such that $f(x_0) = 0$. Since $\Gamma$ is connected, for any point $x \in \Gamma$, there exists a path $\gamma: (0, r) \to \Gamma$ on the network such that $\gamma(0) = x_0$, $\gamma(r) = x$, $|\gamma'(s)| = 1$ and $r \leq |\Gamma|$. Hence we have

$$|f(x)| = |f(x_0) + \int_0^r (f \circ \gamma)'(s) \, ds| \leq \int_0^r |\partial f(\gamma(s))| \, ds \leq \sqrt{r} \|\partial f\|_{L^2(\gamma)} \leq \sqrt{|\Gamma|} \|\partial f\|_{L^2}.$$

We deduce that

$$\int_\Gamma f^2(x) \, dx \leq |\Gamma|^2 \|\partial f\|_{L^2}^2.$$

We also give an analogue of the Trudinger–Moser inequality for compact networks.

**Lemma 2.2.** For any $\beta, \delta \in \mathbb{R}$ with $\delta > 0$, there exists a constant $C$ (depending only on $\beta$, $\delta$ and the network) such that, for all functions $f \in H^1(\Gamma)$ with $\int_\Gamma |\partial f|^2 \leq \delta$ and $\int_\Gamma f = 0$, there holds

$$\int_\Gamma e^{\beta f(x)} \, dx \leq C.$$

**Proof.** We adapt the argument of [14, Lemma 7]. The case $\beta \leq 0$ is obvious because $\Gamma$ has a bounded total length. Fix $\beta > 0$, and consider a function $f$ as in the statement. By Lemma (2.1) (i) and the assumption $\|\partial f\|_{L^2}^2 \leq \delta$, we have

$$\int_\Gamma e^{\beta f(x)} \, dx \leq \int_\Gamma e^{\beta |\partial f|^2} \, dx \leq e^{\beta |\Gamma| \delta} |\Gamma|.$$

We consider the Kazdan–Warner equation on the network $\Gamma$,

$$\begin{cases}
\delta^2 u = c - he^u, & x \in e_j, 
\quad j \in J, \\
u_j(\nu_i) = u_k(\nu_i), & j, k \in \text{Inc}_i, \nu_i \in V, \\
\sum_{j \in \text{Inc}_i} \partial_j u(\nu_i) = 0, & \nu_i \in V,
\end{cases} \tag{2.1}
$$

where $c$ is a given constant and $h$ is a continuous function on $\Gamma$. Note that the Kazdan–Warner equation is defined on each edge, while, at the vertices, we impose the continuity of $u$ and the Kirchhoff condition, a classical condition for differential equations defined on networks (see [18, 20]).

**Definition 2.1.** We introduce the notion of solution to problem (2.1).

(a) A strong solution to problem (2.1) is a function $u \in C^2(\Gamma)$ which satisfies (2.1) in a pointwise manner.

(b) A weak solution to problem (2.1) is a function $u \in H^1(\Gamma)$ such that

$$\int_\Gamma \partial u \partial \phi \, dx = -c \int_\Gamma \phi \, dx + \int_\Gamma he^u \phi \, dx \quad \text{for all } \phi \in H^1(\Gamma). \tag{2.2}$$
Remark 2.1. One can easily check that, if \( u \in C^2(\Gamma) \) is a weak solution of (2.1), then it is also a strong solution. Moreover, any weak solution of (2.1) is also a strong solution. Actually, a weak solution \( u \) fulfills \( \partial^2 u = c - h e^u \) in distributional sense inside each edge \( e_j \). The right-hand side of this equality is continuous; hence, by standard theory, \( u \in C^2(e_j) \) for every \( j \in J \). Being a weak solution, \( u \) also belongs to \( H^1(\Gamma) \); hence, \( u \) belongs to \( C^2(\Gamma) \) and is a classical solution of the differential equation in (2.1) inside each edge. By these last properties, integrating by parts (2.2), we obtain that \( u \) also fulfills the Kirchhoff condition in (2.1). In conclusion, \( u \) is a strong solution to (2.1).

In the next three sections, we discuss the solvability of (2.1) in the cases \( c = 0 \), \( c > 0 \) and \( c < 0 \).

3 The Kazdan–Warner equation with case \( c = 0 \)

Theorem 3.1. Assume \( c = 0 \) and \( h \neq 0 \). Then problem (2.1) has a solution \( u \) if and only if \( h \) changes sign and \( \int_\Gamma h < 0 \).

Proof. Assume that \( u \) is a solution to problem (2.1) with \( c = 0 \). We note that the hypothesis \( h \neq 0 \) prevents \( u \) to be constant. Letting \( \phi \equiv 1 \) in (2.2), we get \( \int_\Gamma h e^u \, dx = 0 \), which implies that \( h \) must change sign. Integrating \( e^{-u} \partial^2 u = -h \) by parts on \( \Gamma \), we get

\[
\int_{\Gamma} (\partial u)^2 e^{-u} \, dx + \sum_{i \in J} \sum_{j \in \text{Inc}_i} e^{-u(v_i)} \partial_i u(v_i) = -\int_{\Gamma} h \, dx.
\]

Taking advantage of the Kirchhoff condition and of the continuity of \( u \) at each vertex, we obtain

\[
\int_{\Gamma} (\partial u)^2 e^{-u} \, dx = -\int_{\Gamma} h \, dx.
\]

Since \( u \) cannot be constant, we deduce \( \int_{\Gamma} h \, dx < 0 \).

Conversely, we prove that, for any \( h \) which changes sign and satisfies \( \int_{\Gamma} h < 0 \), there exists a solution to (2.1). We define the set

\[
B := \left\{ v \in H^1(\Gamma) \mid \int_{\Gamma} h e^v \, dx = 0, \int_{\Gamma} v \, dx = 0 \right\}.
\]

We claim that \( B \) is not empty. Since \( h \) changes sign, there exists a point \( x_0 \in \Gamma \) such that \( h(x_0) > 0 \). By the continuity of \( h \), without any loss of generality, we can assume \( x_0 \in e_j \) for some \( j \in J \); namely, there exist \( \bar{j} \in J \) and \( y_0 \in (0, l_j) \) such that \( h(y_0) > 0 \). Moreover, still by the continuity of \( h \), there exists \( \varepsilon > 0 \) such that \( (y_0 - \varepsilon, y_0 + \varepsilon) \subset (0, l_j) \) and \( h(y) > \frac{h(y_0)}{2} \) for all \( y \in (y_0 - \varepsilon, y_0 + \varepsilon) \). Consider a function \( w \in C^2(\Gamma) \) such that \( w_j(y) = 1 \) if \( y \in \left( y_0 - \frac{\varepsilon}{2}, y_0 + \frac{\varepsilon}{2} \right) \), \( w_j(y) = 0 \) if \( y \notin \left( y_0 - \varepsilon, y_0 + \varepsilon \right) \) and \( w_j \equiv 0 \) if \( j \notin J \setminus \{j\} \). For \( \ell > 0 \), the function \( \tilde{w}(\cdot) := \ell w(\cdot) \) fulfills

\[
\int_{\Gamma} h e^{w \ell} \, dx = \int_{e_j} h e^{w \ell} \, dx + \sum_{j \in J \setminus \bar{j}} \int_{e_j} h e^{w \ell} \, dx \geq \int_{y_0 - \frac{\varepsilon}{2}}^{y_0 + \frac{\varepsilon}} h(y) e^{w \ell(y)} \, dy - \int_{\Gamma} |h| \, dx \geq \frac{\ell h(y_0) \varepsilon \ell}{2} - \int_{\Gamma} |h| \, dx > 0 \quad (3.1)
\]

provided that \( \ell \) is sufficiently large. On the other hand, for \( \ell = 0 \), we have \( w_0(x) \equiv 0 \) and, by assumptions,

\[
\int_{\Gamma} h e^{w_0(x)} \, dx = \int_{\Gamma} h \, dx < 0.
\]

Therefore, there exists \( \ell_0 > 0 \) such that \( \int_{\Gamma} h e^{w_\ell_0} = 0 \). Hence the function \( \tilde{w}(\cdot) := w_{\ell_0}(\cdot) - \int_{\Gamma} w_{\ell_0} / |\Gamma| \) belongs to \( B \), and the claim is proved.

Consider the functional

\[
\mathcal{J}(v) := \frac{1}{2} \int_{\Gamma} |\partial v|^2 \, dx \quad \text{for all } v \in B.
\]
Let \( \{v_n\}_{n \in \mathbb{N}} \) be a minimizing sequence for \( \mathcal{J} \) over \( B \), i.e. \( \lim_{n \to +\infty} \mathcal{J}(v_n) = \inf_B \mathcal{J} \). By Lemma 2.1 (ii), we have that the functions \( v_n \) are uniformly bounded in \( H^1(\Gamma) \). We deduce that, possibly passing to a subsequence, there exists \( \bar{u} \in H^1(\Gamma) \) such that, as \( n \to +\infty \), \( v_n \rightharpoonup \bar{u} \) weakly in \( H^1(\Gamma) \) and \( v_n \to \bar{u} \) uniformly on \( \Gamma \). In particular, we get that \( \bar{u} \) belongs to \( B \), and it is a minimizer of \( \mathcal{J} \) on \( B \).

We claim that \( \bar{u} \) is a strong solution to problem (2.1). Actually, by standard Lagrangian multiplier theory, there exist \( \lambda, \mu \in \mathbb{R} \) such that

\[
0 = \frac{d}{dt} \left( \mathcal{J}(\bar{u} + t\phi) - \lambda \int_{\Gamma} h e^{\bar{u} + t\phi} \, dx - \mu \int_{\Gamma} (\bar{u} + t\phi) \, dx \right) \bigg|_{t=0} = \int_{\Gamma} \partial \bar{u} \partial \phi \, dx - \lambda \int_{\Gamma} h e^{\bar{u}} \, dx - \mu \int_{\Gamma} \phi \, dx
\]

for every \( \phi \in H^1(\Gamma) \). Choosing \( \phi \equiv 1 \), since \( \bar{u} \in B \), we get \( \mu = 0 \). Arguing as in Remark 2.1, inside each edge \( e_j \), there holds \( \partial^2 \bar{u} + \lambda he^{\bar{u}} = 0 \) in distributional sense. By the continuity of \( \bar{u} \), we infer that \( \bar{u} \in C^2(\ell) \) and, since \( \bar{u} \in H^1(\Gamma) \), also that \( \bar{u} \in C^2(\Gamma) \). Moreover, \( \bar{u} \) is a strong solution to

\[
\begin{cases}
\partial^2 \bar{u} = -\lambda he^{\bar{u}}, & x \in e_j, \quad j \in \mathcal{I}, \\
\partial \bar{u} \partial \nu = 0, & j \in \text{inc} \mathcal{I}, \quad j \in \mathcal{I}, \\
\bar{u} = \bar{u}_k(\nu), & j, k \in \text{inc} \mathcal{I}, \quad j \in \mathcal{I}.
\end{cases}
\]

We claim that \( \lambda > 0 \). The function \( \bar{u} \) also solves \( e^{-\bar{u}} \partial^2 \bar{u} = -\lambda h \); integrating this relation, by Kirchhoff and continuity conditions, we get

\[
\int_{\Gamma} (\partial \bar{u})^2 e^{-\bar{u}} \, dx = -\lambda \int_{\Gamma} h \, dx.
\]

Let us first prove that the left-hand side of this equality is positive. We proceed by contradiction assuming \( \int_{\Gamma} (\partial \bar{u})^2 e^{-\bar{u}} = 0 \). Hence \( \partial \bar{u} \equiv 0 \), and in particular, \( \bar{u} \) is constant. Since \( \bar{u} \in B \), we get \( e^{\bar{u}} \int_{\Gamma} h = 0 \) contradicting the assumption \( \int_{\Gamma} h < 0 \). Therefore, the left-hand side in the last equality is positive; again by virtue of \( \int_{\Gamma} h < 0 \), the constant \( \lambda \) must be positive. Finally, the function \( u(\cdot) := \bar{u}(\cdot) + \log(\lambda) \) is a strong solution to (2.1).

\[
\square
\]

### 4 The Kazdan–Warner equation with case \( c > 0 \)

**Theorem 4.1.** Assume \( c > 0 \). Then problem (2.1) has a solution \( u \) if and only if \( h \) is positive somewhere.

**Proof.** Assume that \( u \) is a solution of (2.1); choosing \( \phi \equiv 1 \) as test function in (2.2), we get \( \int_{\Gamma} h e^{u} = c|\Gamma| > 0 \). Hence \( \{x \in \Gamma \mid h(x) > 0\} \neq \emptyset \).

Conversely, for any \( h \in C^0(\Gamma) \) with \( \{h > 0\} \neq \emptyset \), we prove that problem (2.1) admits at least one solution. To this end, it is expedient to introduce the set

\[
B := \left\{ v \in H^1(\Gamma) \mid \int_{\Gamma} h v \, dx = c|\Gamma| \right\}.
\]

We claim that \( B \) is not empty. For \( \ell \geq 0 \), we introduce the function \( w_\ell \) as in the proof of Theorem 3.1, while for \( \ell \leq 0 \), we set \( w_\ell \equiv \ell \). Since \( w_0 \equiv w_\ell \), the function

\[
g(\ell) := \begin{cases}
\int_{\Gamma} h e^{w_\ell} \, dx & \text{if } \ell \geq 0, \\
\int_{\Gamma} h e^{w_\ell} \, dx & \text{if } \ell < 0
\end{cases}
\]

is well defined and continuous, it fulfills \( \lim_{\ell \to +\infty} g(\ell) = +\infty \) (by virtue of estimate (3.1)) and

\[
\lim_{\ell \to -\infty} g(\ell) = \lim_{\ell \to -\infty} e^{\ell} \int_{\Gamma} h = 0.
\]

Hence there exists \( \ell \in \mathbb{R} \) such that \( g(\ell) = c|\Gamma| \), namely, \( B \neq \emptyset \).
We consider the functional

$$\mathcal{J}(u) := \frac{1}{2} \int_\Gamma |\partial u|^2 \, dx + c \int_\Gamma u \, dx$$

for all $u \in B$.

As a first step, let us prove that $\mathcal{J}$ is bounded from below in $B$. To this end, for any $u \in B$, we set $\bar{u} := \int_\Gamma u/|\Gamma|$ and $v := u - \bar{u}$. Note $\int_\Gamma v = 0$ and $\partial v = \partial u$. Since $u \in B$, it holds $\int_\Gamma h e^v \, dx = c|\Gamma| e^{-\bar{u}}$, which implies

$$\bar{u} = \log(c|\Gamma|) - \log \left( \int_\Gamma h e^v \, dx \right);$$

replacing this equality in the definition of $\mathcal{J}$, we get

$$\mathcal{J}(u) = \frac{1}{2} \|\partial u\|_{L^2}^2 + c|\Gamma| \log(c|\Gamma|) - c|\Gamma| \log \left( \int_\Gamma h e^v \, dx \right).$$

(4.1)

Let us now estimate $\int_\Gamma h e^v$; if $v$ is constant, then, by $\int_\Gamma v = 0$, it must be $v \equiv 0$ and, in particular $\int_\Gamma h e^v = \int_\Gamma h$. For $v$ non-constant, it is expedient to introduce the function $\tilde{v} := v/\|\partial v\|_{L^2}$ which verifies $\tilde{v} \in H^1(\Gamma)$, $\int_\Gamma \tilde{v} = 0$ and $\|\partial \tilde{v}\|_{L^2} = 1$. Lemma 2.1 (ii) and Lemma 2.2 guarantee that, for any $\beta \in \mathbb{R}$, there exists a constant $K_\beta$ (depending only on $\beta$) such that

$$\|\tilde{v}\|_{L^2} \leq |\Gamma|, \quad \int_\Gamma e^{\beta \tilde{v}(x)} \, dx \leq K_\beta.$$ 

For every $\epsilon$ positive, for $\beta_\epsilon := \frac{1}{|\Gamma| \epsilon}$, there holds

$$\int_\Gamma h e^\tilde{v} \, dx \leq \|h\|_{L^\infty} \int_\Gamma e^{\epsilon \|\partial \tilde{v}\|_{L^2}^2 + \frac{\epsilon^2}{4 \|\partial \tilde{v}\|_{L^2}^2}} \, dx \leq \|h\|_{L^\infty} e^{\epsilon \|\partial \tilde{v}\|_{L^2}^2} K_{\beta_\epsilon}.$$ 

Replacing this estimate in (4.1), we obtain

$$\mathcal{J}(u) \geq \frac{1}{2} \|\partial u\|_{L^2}^2 + c|\Gamma|[(\log(c|\Gamma|) - \epsilon \|\partial u\|_{L^2}^2 - \log(\|h\|_{L^\infty} K_{\beta_\epsilon})]$$

and, in particular, for $\epsilon_0 := \frac{1}{4c|\Gamma|}$,

$$\mathcal{J}(u) \geq \frac{1}{4} \|\partial u\|_{L^2}^2 + c|\Gamma|[(\log(c|\Gamma|) - \log(\|h\|_{L^\infty} K_{\beta_{\epsilon_0}})].$$ 

(4.2)

Hence the proof that $\mathcal{J}$ is bounded in $B$ from below is accomplished.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for $\mathcal{J}$ in $B$; set $\tilde{u}_n := \int_\Gamma u_n/|\Gamma|$ and $v_n := u_n - \tilde{u}_n$; hence $\partial u_n \equiv \partial v_n$, and by estimate (4.2), $\partial v_n$ is bounded in $L^2(\Gamma)$, uniformly in $n$. By Lemma 2.1 (ii), also $v_n$ is uniformly bounded in $L^2(\Gamma)$, and therefore the functions $v_n$ are uniformly bounded in $H^1(\Gamma)$. Moreover, by the definition of $\mathcal{J}$, we get that $\int_\Gamma u_n$ are uniformly bounded, and consequently also $\tilde{u}_n$ are uniformly bounded. Being $u_n = v_n + \tilde{u}_n$, also the functions $u_n$ are uniformly bounded in $H^1(\Gamma)$. Possibly passing to a subsequence, there exists $u \in H^1(\Gamma)$ such that, as $n \to +\infty$, $u_n \rightharpoonup u$ in the weak topology of $H^1(\Gamma)$, $u_n \to u$ uniformly, $u \in B$ and $\mathcal{J}(u) = \min_B \mathcal{J}$.

We claim that $u$ is a solution to (2.1). By standard Lagrangian theory, there exists $\lambda \in \mathbb{R}$ such that, for every $\phi \in H^1(\Gamma)$,

$$0 = \frac{d}{dt} \left( \int_\Gamma \frac{\partial(u + t\phi)^2}{2} \, dx + c \int_\Gamma (u + t\phi) \, dx - \lambda \left( c|\Gamma| - \int_\Gamma h e^{u+t\phi} \, dx \right) \right) \bigg|_{t=0}$$

$$= \int_\Gamma \partial u \partial \phi \, dx + c \int_\Gamma \phi \, dx - \lambda \int_\Gamma h e^{u} \phi \, dx. \quad (4.3)$$

Choosing $\phi \equiv 1$, we get $c|\Gamma| = \lambda \int_\Gamma h e^{u}$; since $u \in B$, we get $\lambda = 1$. In conclusion, relation (4.3) with $\lambda = 1$ is equivalent to the definition of weak solution to (2.1).
5 The Kazdan–Warner equation with case $c < 0$

**Theorem 5.1.** Assume $c < 0$.

(i) If (2.1) has a solution, then $\int_\Gamma h < 0$.

(ii) If $\int_\Gamma h < 0$, then there exists a constant $c(h) \in [-\infty, 0)$ such that (2.1) has a solution for any $c(h) < c < 0$ and no solution for $c < c(h)$.

(iii) For $\int_\Gamma h < 0$, let $c(h)$ be defined as in (ii). Then $c(h) = -\infty$ if and only if $h \leq 0$ in $\Gamma$.

We introduce the definition of upper and lower solution to (2.1).

**Definition 5.1.** A function $u \in C^2(\Gamma)$ is said to be a lower (respectively, an upper) solution of (2.1) if

$$
\begin{cases}
\partial^2 u - c + h e^u \geq 0, & x \in e_j, j \in J, \\
\sum_{j \in \text{Inc}_i} \partial_j u(v_i) \geq 0, & v_i \in V,
\end{cases}
$$

resp.,

$$
\begin{cases}
\partial^2 u - c + h e^u \leq 0, & x \in e_j, j \in J, \\
\sum_{j \in \text{Inc}_i} \partial_j u(v_i) \leq 0, & v_i \in V.
\end{cases}
$$

In order to prove Theorem 5.1, we need some preliminary results.

**Lemma 5.1.** If there exist a lower solution $u_-$ and an upper solution $u_+$ of (2.1) such that $u_- \leq u_+$, then there exists a solution $u$ of (2.1) such that $u_- \leq u \leq u_+$.

**Proof.** Set $k_1(x) = \max\{1, -h(x)\}$ and $k(x) = k_1(x)e^{u_n(x)}$, and consider the sequence of functions $\{u_n\}_{n \in \mathbb{N}}$ defined inductively as $u_0 = u_+$ and $u_{n+1} \in C^2(\Gamma)$ the solution of

$$
\begin{cases}
\mathcal{L}u_{n+1} = f(x, u_n) - ku_n, & x \in e_j, j \in J, \\
\sum_{j \in \text{Inc}_i} \partial_j u_{n+1}(v_i) = 0, & v_i \in V,
\end{cases}
$$

(5.1)

where $\mathcal{L}u = \partial^2 u - ku$ and $f(x, u) = c - h(x)e^u$. We first observe that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is well defined. Indeed, since $k(x) \geq e^{-ku_n(x)} = : \lambda > 0$, the differential equation in (5.1) can be written as

$$
-\partial^2 u_{n+1} + H(x, u_{n+1}) + \lambda u_{n+1} = 0
$$

with $H(x, r) := (k(x) - \lambda)r + f(x, r) - k(x)u_n(x)$; hence the result in [7, Proposition 10] ensures the existence of a solution to (5.1). Moreover, by the linearity of (5.1), the maximum principle in [7, Proposition 12] guarantees the uniqueness of this solution. We claim that

$$
u_- \leq u_{n+1} \leq u_n \leq u_+ \quad \text{for any } n \in \mathbb{N}.
$$

(5.2)

Since

$$
\mathcal{L}(u_1 - u_0) = f(x, u_0) - ku_0 - \partial^2 u_0 + ku_0 \geq 0, \quad x \in e_j, j \in J,
$$

$$
\sum_{j \in \text{Inc}_i} \partial_j (u_1 - u_0)(v_i) \geq 0, \quad v_i \in V,
$$

the inequality $u_1 \leq u_0 = u_+$ on $\Gamma$ follows immediately by the maximum principle (see [7, Proposition 12]). Assuming inductively that $u_n \leq u_{n-1}$, we have

$$
\mathcal{L}(u_{n+1} - u_n) = k(x)(u_{n-1} - u_n) + h(x)(e^{u_{n-1}} - e^{u_n}) \\
\geq k_1(x)e^{u_n(x)}(u_{n-1} - u_n) - k_1(x)(e^{u_{n-1}} - e^{u_n}) \\
\geq k_1(x)(e^{u_n(x)} - e^{\xi(x)})(u_{n-1} - u_n) \quad \text{for } x \in e_j, j \in J,
$$

where $\xi(x) \in [\min(u_n(x), u_{n-1}(x))]$. By induction, we have $u_+ \geq u_{n-1}$, and recalling the condition at the vertices, we get

$$
\mathcal{L}(u_{n+1} - u_n) \geq 0, \quad x \in e_j, j \in J,
$$

$$
\sum_{j \in \text{Inc}_i} \partial_j (u_{n+1} - u_n)(v_i) \geq 0, \quad v_i \in V.
$$
We conclude again by the maximum principle that \( u_{n+1} \leq u_n \) in \( \Gamma \). We finally observe that, arguing as before, we have
\[
\begin{align*}
\mathcal{L}(u_n - u_{n+1}) &\geq k(x)(u_n - u_n) + h(x)(e^{u_n} - e^{u_n}) \geq 0, \quad x \in e_j, \; j \in J, \\
\sum_{j \in \text{Inc}_i} \partial_j (u_n - u_{n+1})(v_i) &\geq 0, \quad V_i \in V,
\end{align*}
\]
and therefore \( u_n \leq u_{n+1} \) on \( \Gamma \) for all \( n \). Hence the claim (5.2) is proved.

By [7, Proposition 10], there exists a positive constant \( C \) (independent of \( n \)) such that \( \|u_n\|_{H^1} \leq C \) and, in particular, \( \|u_n\|_{L^\infty} \leq C \) for every \( n \in \mathbb{N} \). Replacing the bounds (5.2) in the differential equation of (5.1), we obtain that \( \|\partial^2 u_n\|_{L^\infty} \) are uniformly bounded; this property and again the bounds (5.2) ensure \( \|u_n\|_{H^1} \leq C \).

The Ascoli–Arzela theorem yields that, up to passing to a subsequence, \( \{u_n\} \) converges in \( H^1(\Gamma) \) to a function \( u \in H^1(\Gamma) \) which is a weak solution to (2.1) with \( u_\leq u \leq u_+ \). Finally, by Remark 2.1, \( u \) is a classical solution to (2.1).

In the next lemma, we show that (2.1) admits a lower solution \( u_- \) for any \( c < 0 \).

**Lemma 5.2.** If \( c < 0 \), then there exists a lower solution \( u_- \) of (2.1).

**Proof.** Set \( u_- = -A \) for some constant \( A > 0 \). Then the function \( u_- \) fulfills the Kirchhoff condition in (2.1) and also
\[
\partial^2 u_- (x) - c + h(x)e^{u_-(x)} = -c + h(x)e^{-A} \geq 0, \quad x \in e_j, \; j \in J,
\]
for \( A \) sufficiently large. Hence \( u_- \) is a lower solution to (2.1).

**Proof of Theorem 5.1.** Assume that there exists a solution \( u \) of (2.1). Then, integrating (2.1) on \( \Gamma \), we get
\[
\int_{\Gamma} h(x) \; dx = \int_{\Gamma} (\partial u(x))^2 e^{-u(x)} \; dx - c \int_{\Gamma} e^{-u(x)} \; dx > 0
\]
and therefore (i).

We now assume that \( \int_{\Gamma} h(x) \; dx < 0 \). Recall that, by Lemma 5.1 and Lemma 5.2, problem (2.1) has a solution if and only if there exists an upper solution \( u_+ \). Moreover, it is easy to see that, if \( u_+ \) is an upper solution for a given \( c < 0 \), then it is also an upper solution for any \( c \) such that \( \tilde{c} \leq c < 0 \). Hence it follows that there exists a constant \( c(h) \) with \(-\infty \leq c(h) \leq 0 \) such that (2.1) admits a solution for \( c > c(h) \) and no solution for \( c < c(h) \).

We show that \( c(h) < 0 \). Let \( m \in C^2(\Gamma) \) be a solution of
\[
\begin{align*}
\partial^2 m(x) &= h(x) \; dx - h(x), \quad x \in e_j, \; j \in J, \\
\sum_{j \in \text{Inc}_i} \partial_j m(v_i) &= 0, \quad V_i \in V
\end{align*}
\]
(5.3)
(existence of a weak solution to (5.3) is proved in [7, Proposition 13], while the regularity follows by Remark 2.1), and let \( a \) be a positive constant such that
\[
\max_{x \in \Gamma} |e^{am(x)} - 1| = \frac{-\int_{\Gamma} h(x) \; dx}{2\|h(x)\|_{L^\infty}}.
\]
We define \( b = \ln(a) \), \( c = \frac{1}{2} a \int_{\Gamma} h(x) \; dx \) and \( u_+(x) = am(x) + b \). Then \( c < 0 \) and
\[
\partial^2 u_+(x) - c + h(x)e^{u_+(x)} = ah(x)(e^{am(x)} - 1) + \frac{a}{2} \int_{\Gamma} h(x) \; dx \leq a\|h(x)\|_{L^\infty}|e^{am(x)} - 1| + \frac{a}{2} \int_{\Gamma} h(x) \; dx \leq 0.
\]
Moreover, by (5.3), \( u_+ \) is continuous and verifies the Kirchhoff condition because \( m \) enjoys the same properties. Hence \( u_+ \) is an upper solution, and therefore we conclude that
\[
c(h) \leq \frac{a}{2} \int_{\Gamma} h(x) \; dx < 0.
\]
We finally prove (iii). Note that \( \int h < 0 \) ensures \( h \neq 0 \).
We first show that, if \( h < 0 \) in \( \Gamma \), then (2.1) is solvable for any \( c < 0 \), and therefore \( c(h) = -\infty \). Fixed \( c < 0 \), let \( m \) be a solution of (5.3), and choose two constants \( a, b \) such that \( a \int \Gamma h(x) \, dx < c \) and \( e^{am(x)+b} - a > 0 \) for \( x \in \Gamma \). We show that the function \( u_+(x) = am(x) + b \) is an upper solution of (2.1). Indeed, there holds
\[
\partial^2 u_+(x) - c + h(x)e^{u_+(x)} = a \int \Gamma h(x) \, dx - ah(x) - c + h(x)e^{am(x)+b} \leq h(x)(e^{am(x)+b} - a) \leq 0,
\]
while the continuity and the Kirchhoff conditions for \( u_+ \) come again from those of \( m \). Hence \( u_+ \) is an upper solution to (2.1), and therefore, for any \( c < 0 \), there exists a solution to (2.1).

Conversely, let us prove that \( c(h) = -\infty \) implies \( h < 0 \) in \( \Gamma \). To this end, as in [13, Theorem 2.3], we argue by contradiction assuming that \( |h| > 0 \) is not empty. For any \( c < 0 \), let \( u \) be a solution to (2.1) (whose existence is ensured by \( c(h) = -\infty \)), and let \( \phi_c \in C^2(\Gamma) \) be a solution to the problem
\[
\begin{cases}
  \partial^2 \phi_c + c\phi_c = h, & x \in \mathcal{E}, j \in J, \\
  \sum_{j \in \text{Inc}_i} \partial_j \phi_c(v_j) = 0, & v_j \in \mathcal{V} 
\end{cases}
\]
(whose existence is ensured by [7, Proposition 10]). We claim
\[
\phi_c(x) \geq e^{-u(x)} > 0 \quad \text{for all } x \in \Gamma.
\]
In order to prove this relation, by the maximum principle [7, Proposition 12], it suffices to prove that \( e^{-u} \) is a lower solution to (5.4). Actually, there holds
\[
\partial^2 (e^{-u}) + ce^{-u} = e^{-u}[-\partial^2 u + |\partial u|^2 + c] = e^{-u}[he^u + |\partial u|^2] \geq h;
\]
moreover, \( e^{-u} \) is continuous and satisfies the Kirchhoff condition because \( u \) does it. Hence our claim is proved.

Let us assume for the moment that there holds
\[
\lim_{c \to -\infty} c\phi_c(x) = h \quad \text{for all } x \in \Gamma;
\]
then this property would contradict \( \phi_c \geq 0 \) in \( |h| > 0 \), accomplishing the proof. Therefore, it remains to prove (5.5); to this end, we observe that the operator \( A := -\partial^2 \), coupled with Kirchhoff condition, is a maximal monotone operator (for the precise definition and main properties, we refer the reader to the monograph [6, p. 101]). Applying [6, Proposition VII.2(c)] with \( \lambda \) equal to \( \frac{1}{c} \), we obtain \( \lim_{c \to -\infty} u_c = h \), where \( u_c \) solves \((I + \frac{1}{c})(u_c) = h\). By linearity, we get \( u_c = c\phi_c \), accomplishing the proof of (5.5).

\[\Box\]

### 5.1 The critical case \( c = c(h) \)

**Proposition 5.1.** For \( \int \Gamma h < 0 \) and \( c(h) > -\infty \), problem (2.1) with \( c = c(h) \) admits a solution.

**Proof.** Note that Theorem 5.1 (iii) ensures that \( h \) changes sign (and obviously, \( h \neq 0 \)). Given a decreasing sequence \( \{c_k\}_{k \in \mathbb{N}} \) with \( c(h) < c_k < 0 \) converging to \( c(h) \) as \( k \to +\infty \), we consider
\[
\begin{cases}
  \partial^2 u = c_k - he^{u}, & x \in \mathcal{E}, j \in J, \\
  \sum_{j \in \text{Inc}_i} \partial_j u(v_j) = 0, & v_j \in \mathcal{V} 
\end{cases}
\]
(5.6)
The idea is to show that a sequence of solutions \( u_k \) of (5.6), appropriately chosen, converges for \( k \to \infty \) to a solution of (2.1) with \( c = c(h) \).

**Lemma 5.3.** For each \( k \in \mathbb{N} \), there exist a lower solution \( \phi_k \equiv -A \in \mathbb{R} \) and an upper solution \( \psi_k \) to (5.6) with \( \psi_k > \phi_k \).

**Proof.** To show the existence of a lower solution, it suffices to argue as in Lemma 5.2 choosing \( A \) sufficiently large so that
\[
-c_k + h(x)e^{-A} \geq -c_k - \|h\|_{L^\infty}e^{-A} =: \delta > 0.
\]
(5.7)
For the upper solution, we choose $\psi_k$ as a solution to (2.1) with $c$ replaced by any $\tilde{c}_k \in (c(h), c_k)$ (whose existence is established in Theorem 5.1).

Finally, it remains to prove the inequality $\psi_k > -A$ (see also [21, Claim 3] for a similar argument). Denoting by $\tilde{x}$ a minimum point of $\psi_k$ on $\Gamma$, we claim that $\psi_k(\tilde{x}) > -A$.

Assume first that $\tilde{x} \in e_j$ for some $j \in J$. The first equation in (2.1) yields

$$h(\tilde{x}) e^{\psi_k(\tilde{x})} = \tilde{c}_k - \partial^2 \psi_k(\tilde{x}) \leq \tilde{c}_k < 0$$

and, in particular, $h(\tilde{x}) < 0$. On the other hand, the function $\phi_k \equiv -A$ satisfies $h(\tilde{x}) e^{-A} > \tilde{c}_k$. The last three relations give $e^{\psi_k(\tilde{x})} - e^{-A} > 0$, which is equivalent to $\psi_k(\tilde{x}) > -A$.

Assume now $\tilde{x} = \nu_i$ for some $i \in I$ and, for later contradiction, $\psi_k(\nu_i) \leq -A$. Observe that, for any $j \in \text{Inc}_i$, the restriction of $\psi_k$ to $e_j$ attains its minimum at $\nu_i$, and consequently, $\partial_j \psi_k(\nu_i) \geq 0$. Taking into account the Kirchhoff condition in (2.1), we deduce $\partial_j \psi_k(\nu_i) = 0$ for all $j \in \text{Inc}_i$. On the other hand, by (5.7) and the continuity of $h$, there exists $\eta > 0$ such that

$$\tilde{c}_k + \|h\|_{L^\infty} e^{-A+\eta} < -\frac{\delta}{2}. \quad (5.8)$$

Moreover, by the continuity of $\psi_k$ and $\psi_k(\nu_i) \leq -A$, (5.8) ensures

$$\partial^2 \psi_k(x) = \tilde{c}_k - h(x) e^{\psi_k(x)} \leq \tilde{c}_k + \|h\|_{L^\infty} e^{-A+\eta} < -\frac{\delta}{2} < 0$$

for any $x \in e_j$ sufficiently close to $\nu_i$. In conclusion, near $\nu_i$, the function $\partial_j \psi_k$ is strictly decreasing with $\partial_j \psi_k(\nu_i) = 0$, and therefore $\psi_k$ is strictly decreasing. This fact contradicts that $\psi_k$ attains its minimum at $\nu_i$.

**Lemma 5.4.** Fix $k \in \mathbb{N}$. The minimum of the problem

$$\inf \{ J_k(u) : u \in H^1(\Gamma), -A \leq u(x) \leq \psi_k(x) \text{ for all } x \in \Gamma \}, \quad (5.9)$$

where

$$J_k(u) := \frac{1}{2} \int_{\Gamma} |\partial u|^2 \, dx + c_k \int_{\Gamma} u \, dx - \int_{\Gamma} h e^u \, dx,$$

is attained by some function $\tilde{u}$ with

$$-A < \tilde{u} < \psi_k. \quad (5.10)$$

Moreover, $\tilde{u}$ is a solution of (5.6).

**Proof.** Let $\{\nu_n\}_n$ be a minimizing sequence for $J_k$. Then there holds

$$J_k(\nu_n) + o(1) \leq J_k(-A) = c_k(-A) |\Gamma| - e^{-A} \int_{\Gamma} h \leq C$$

for some constant $C$ (independent of $k$). Moreover, we have

$$C + o(1) \geq J_k(\nu_n) = \frac{1}{2} \int_{\Gamma} |\partial \nu_n|^2 \, dx + c_k \int_{\Gamma} \nu_n \, dx - \int_{\Gamma} h e^{\nu_n} \, dx \geq \frac{1}{2} \int_{\Gamma} |\partial \nu_n|^2 \, dx + c_k \int_{\Gamma} \psi_k \, dx - \|h\|_{L^\infty} \int_{\Gamma} e^{\psi_k} \, dx,$$

where the inequality is due to the constraint $-A \leq \nu_n \leq \psi_k$. We deduce that $\|\partial \nu_n\|_{L^2}$ are uniformly bounded; on the other hand, also $\|\nu_n\|_{L^\infty}$ are uniformly bounded. Therefore, the sequence $\{\nu_n\}_n$ is uniformly bounded in $H^1(\Gamma)$. We infer that, possibly passing to a subsequence, there exists $\tilde{u} \in H^1(\Gamma)$ with $-A \leq \tilde{u} \leq \psi_k$ such that $\nu_n \to \tilde{u}$ uniformly and $\nu_n \rightharpoonup \tilde{u}$ weakly in $H^1$. By the lower semicontinuity of $J_k$, we get $J_k(\tilde{u}) \leq \liminf_n J_k(\nu_n)$; hence $\tilde{u}$ is a minimum point for (5.9). Let us assume for the moment that inequalities (5.10) hold true; then, by standard Lagrange multipliers method, we have $\partial_t J_k(\tilde{u} + t \phi)|_{t=0} = 0$ for any $\phi \in H^1(\Gamma)$, from which we
get (2.2). Arguing as in Remark 2.1, we get that \( \bar{u} \) is a strong solution to (5.6). Therefore, it remains to prove the inequalities in (5.10); since the proofs are similar, we shall only provide the one for the first relation. As a first step, we claim that it cannot exist an interval \( [x_1, x_2] \) for some \( j \) such that \( \bar{u} = -A \) on \( [x_1, x_2] \). Actually, for later contradiction, assume that there exists such an interval \( [x_1, x_2] \). Without any loss of generality, we can assume \( x_1 = 0 \). We introduce the function

\[
\bar{u}(x) := \begin{cases} -A + \alpha \min\{x, x_2 - x\} & \text{if } x \in (0, x_2), \\ \bar{u}(x) & \text{otherwise}, \end{cases}
\]

where \( \alpha > 0 \) is a constant so small that \( \bar{u} \) is admissible for (5.9) and to be suitably chosen later on. We have

\[
J_k(\bar{u}) - J_k(\bar{u}) \geq -\frac{a^2 x_2}{2} - \frac{ac_k x_2^2}{4} - \|h\|_{L^\infty} \int_0^{x_2} \left(e^\bar{u} - e^{\bar{u}}\right) dx.
\]

By the Lagrange theorem, for every \( x \in (0, \frac{x_2}{2}) \), there exists \( \xi \in (0, x) \) such that

\[
e^{\bar{u}(x)} - e^{\bar{u}(x)} = ae^{-A+\alpha \xi} x \leq ae^{-A+\alpha \frac{x_2}{2}} \frac{x_2^2}{2};
\]

by symmetry, there holds also

\[
e^{\bar{u}(x)} - e^{\bar{u}(x)} \leq ae^{-A+\alpha \frac{x_2}{2}} x \frac{x_2^2}{2} \text{ for every } x \in \left(\frac{x_2}{2}, x_2\right).
\]

Replacing these inequalities in the previous one, we get

\[
J_k(\bar{u}) - J_k(\bar{u}) \geq -\frac{a^2 x_2}{2} + a[-c_k - 2\|h\|_{L^\infty} e^{-A+\alpha \frac{x_2}{2}} \frac{x_2^2}{4}]
\]

For \( A \) sufficiently large and \( \alpha \) sufficiently small, we get a contradiction. Finally, we claim that, if \( \bar{u} < -A \) on some interval \( (x_1, x_2) \) for some \( j \), then \( \bar{u} > -A \) on \( \bar{e}_j \). Indeed, let us assume \( -A < \bar{u} < \psi_k \) on some \( (\bar{x}_1, \bar{x}_2) \) and, by later contradiction, \( \bar{u}(x_2) = -A \). Then \( \phi := \bar{u} + A \) attains a minimum at \( x_2 \). Moreover, as before, by the Lagrange multipliers method, \( \bar{u} \) solves (5.6) on \( (x_1, x_2) \). Then, for \( x \to x_2 \), by (5.7), we have

\[
\partial^2 \phi = c_k - he^{\bar{u}} = c_k - he^{-A} + o(1) < 0,
\]

which provides a contradiction.

We can now conclude the proof of Proposition 5.1. Denote by \( u_k, k \in \mathbb{N} \), a solution of (5.6) given by Lemma 5.4. Assume for the moment that the sequence \( \{u_k\} \) is bounded in \( H^1(\Gamma) \). Hence there exists \( u \in H^2(\Gamma) \) such that, as \( k \to +\infty \), up to a subsequence, \( u_k \to u \) in the weak topology of \( H^1(\Gamma) \) and \( u_k \to u \) uniformly. Passing to the limit in the weak formulation of (5.6), we get that \( u \) is a weak, and therefore also a strong, solution to (2.1) with \( c = c(h) \).

It remains to prove that \( \{u_k\} \) is bounded in \( H^1 \). To this end, fix \( 0 < \delta < \max \Gamma \), an interval \( D \) inside some edge \( \bar{e}_j \) such that \( D \subset \{|h(x)| \geq \delta\} \) and a point \( \bar{x} \in D \); by the same arguments as in [8, p. 743] (note that we can use [8, Lemma 2.1] because any solution of the equation in \( D \) is also a solution in a 2-dimensional domain), we get that the \( u_k \)'s are uniformly bounded in \( D \). Therefore, the functions \( w_k(x) := u_k(x) - u_k(\bar{x}) \) satisfy \( w_k(\bar{x}) = 0 \), and there exists \( C_1 > 0 \) such that \( |u_k(\bar{x})| \leq C_1 \) for any \( k \). Arguing as in Lemma 2.1 (i), we get

\[
\|w_k\|_{L^\infty} \leq |\Gamma|^\frac{1}{2} \|w_k\|_{L^2} = |\Gamma|^\frac{1}{2} \|u_k\|_{L^2},
\]

and we deduce

\[
\|u_k\|_{L^\infty} \leq |u_k(\bar{x})| + \|w_k\|_{L^\infty} \leq C_1 + |\Gamma|^\frac{1}{2} \|u_k\|_{L^2}.
\]

On the other hand, choosing \( \phi \equiv 1 \) as test function in the weak formulation of (5.6), we get

\[
\int_\Gamma he^{u_k} dx = c_k |\Gamma|.
\]

Since \( c_k \) are negative, relations (5.11) with \( \nu_{\alpha} = u_k \) and (5.13) entail

\[
C \geq \frac{1}{2} \int_\Gamma |\partial u_k|^2 dx + c_k \int_\Gamma u_k dx - \int_\Gamma he^{u_k} dx \geq \frac{\|\partial u_k\|^2_{L^2}}{2} + c_k \int_\Gamma |u_k| dx - c_k |\Gamma|
\]

\[
\geq \frac{\|\partial u_k\|^2_{L^2}}{2} + c_k C_1 |\Gamma| + c_k |\Gamma|^\frac{1}{2} \|u_k\|_{L^2} - c_k |\Gamma|,
\]

where the last inequality is due to (5.12). Hence \( \partial u_k \) are uniformly bounded in \( L^2 \); by (5.12), the \( u_k \)'s are uniformly bounded in \( L^\infty \) and consequently also in \( H^1 \).
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