How to detect fluctuating stripes in the high-temperature superconductors

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We discuss fluctuating order in a quantum disordered phase proximate to a quantum critical point, with particular emphasis on fluctuating stripe order. Optimal strategies for extracting information concerning such local order from experiments are derived with emphasis on neutron scattering and scanning tunneling microscopy. These ideas are tested by application to two model systems - the exactly solvable one dimensional electron gas with an impurity, and a weakly-interacting 2D electron gas. We extensively review experiments on the cuprate high-temperature superconductors which can be analyzed using these strategies. We adduce evidence that stripe correlations are widespread in the cuprates. Finally, we compare and contrast the advantages of two limiting perspectives on the high-temperature superconductor: weak coupling, in which correlation effects are treated as a perturbation on an underlying metallic (although renormalized) Fermi liquid state, and strong coupling, in which the magnetism is associated with well defined localized spins, and stripes are viewed as a form of micro-phase separation. We present quantitative indicators that the latter view better accounts for the observed stripe phenomena in the cuprates.

I. INTRODUCTION

Ordered states of matter are characterized by broken symmetry. Depending on various real-world details, this may be relatively easier or harder to detect experimentally, but once detected it is unambiguous. The notion that order parameter fluctuations are important in the
disordered phase proximate to an ordered state is a rather obvious extension of the idea of broken symmetry; however, this notion is more difficult to define precisely. To develop this important concept, we will discuss strategies for detecting quantum fluctuating order in the particular context of high-temperature superconductor. We will focus on the detection of stripe order in the putative stripe liquid phase of these systems. More generally, we are interested in electronic liquid crystalline states and their associated fluctuations (Kivelson et al., 1998), but the results are also easily generalized to other forms of order.

Since the whole notion of fluctuating order is based on proximity of an ordered state, it is essential first to establish the existence of the “nearby” ordered state by directly detecting the relevant broken symmetry. The ordered phase may be induced by making small changes to the chemical composition of the material, applying pressure or magnetic fields, etc. Unless an actual ordered state can be reached, it is dangerous to speculate about the effects of related fluctuating order.

Typically, the best way to detect both the broken-symmetry state and the relevant fluctuations is by measuring the appropriate dynamical structure factor, \( S(q, \omega) \). Indeed, X-ray and neutron scattering studies have provided the best evidence of ordered and fluctuating stripe phases, as we shall discuss. However, in many interesting materials, appropriate crystals are not available and so such experiments are not possible. Here, probes of local order, such as nuclear magnetic resonance (NMR), nuclear quadrupole resonance (NQR), muon spin rotation (\( \mu \)SR), and scanning tunneling microscopy (STM) techniques, may be the best available. All of these are quasi-static (i.e., nearly zero frequency) probes. In a pure quantum system, in its disordered phase, the order-parameter fluctuations are not static, but rather fluctuate with a characteristic frequency that grows with the distance \( \ell \) to the quantum critical point. Thus, unless something is done to “pin” these fluctuations, they are invisible to local probes. Such pinning is induced by boundaries, vortices, crystal-field effects, weak quenched disorder, etc.

In this article, we first discuss results obtained for solvable model systems, which we analyze in various ways to illustrate the optimal strategies for extracting information about local order. In particular, much of this discussion addresses the character of the local order in a quantum disordered phase close to a quantum critical point; we believe that the intuitions gleaned from this study are more generally valid, but without the nearness to the critical point as a small parameter, it is difficult to treat the problem in a controlled fashion. We also review in some depth, although by no means exhaustively, the experimental evidence of various forms of stripe order in the cuprate superconductors. There are several related topics that we do not cover in this article; instead, we direct the interested reader to recent reviews. Specifically, the mechanism of stripe formation is given short shrift, and the possible relevance of local stripe order to the mechanism of high temperature superconductivity and to the various remarkable normal state properties observed in the cuprates is only touched on briefly. More broadly, the context for this review is the role of “competing orders” in high temperature superconductors.

In Section II we discuss basic scaling considerations that govern the relevant length, frequency, and energy scales of fluctuating order in the neighborhood of a quantum critical point, and the consequences of the pinning induced by disorder and other perturbations. We show that it is the low frequency part of the dynamical structure factor, rather than its integral over all frequencies, that contains the clearest information concerning local order. We also define a new response function which determines the local density-of-states modulations induced by a weak-impurity potential. Finally, we sketch various possible versions of the phase diagram of a system with competing stripe and superconducting phases.

In Section III (and Appendix A), we consider a theoretically well understood system, the one dimensional electron gas (1DEG) in the presence of an impurity. We compute the local density of states (LDOS), as would be measured in STM, and the dynamical structure factor, and show by this explicit example how the general considerations articulated in the present paper are manifest in this solvable model. (The 1DEG can also be viewed as a quantum critical state associated with charge density wave (CDW) order.) In addition to its pedagogical value, this section contains explicit results that should be useful for analyzing STM experiments on quasi 1D systems with dilute impurities, such as the chain-layers

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1 For brief reviews of the current evidence on various types of stripe order in the cuprates, see (Carlson et al., 2003; Castro Neto and Morais Smith, 2003; Emery et al., 1999; Orenstein and Millis, 2000; Sachdev, 2002; Zaanen, 2000)

2 Fluctuations associated with off-diagonal (i.e., superconducting) order are best studied using different strategies; see, for example, (Carlson et al., 2000; Iguchi et al., 2001; Janko et al., 1999; Usaj et al., 2002; Wang et al., 2001; Xu et al., 2000)

3 For reviews of neutron and X-ray evidence of stripes, see (Emery et al., 1999; Tranquada, 1998a,b).

4 Recent reviews of the mechanisms of stripe formation in the cuprates and, more generally in doped antiferromagnets, often with rather different perspectives on the issue, can be found in Refs. (Carlson et al., 2003; Emery et al., 1999; Hasselmann et al., 2002; Ichioka and Machida, 1999; Vojta and Sachdev, 1999; White et al., 2002; White and Scalapino, 2000; Zaanen, 1998; Zaanen and Littlewood, 1994).

5 Various perspectives concerning the relevance of local stripe order to other properties of the high-temperature superconductor are reviewed in (Carlson et al., 2003; Castro Neto, 2001; Lorenzo et al., 2001a,b; Zaanen et al., 2001).

6 Several recent discussions of the role of competing orders in determining the phase diagram of the high temperature superconductors are contained in (Chakravarty et al., 2001b; Emery et al., 1999; Kivelson et al., 2001; Orenstein and Millis, 2000; Sachdev, 2000; Varma, 1997; Zaanen, 2000).
of YBCO (Derro et al., 2002), or STM experiments on carbon nanotubes (Hornbaker et al., 2002; Odom et al., 2002).

In Section IV we calculate the LDOS in the context of a simple model-system with quasiparticles and incipient order - the weakly-interacting electron gas in two dimensions.

In Section V we discuss applications of these ideas to experiments in the cuprates. In Section V.A we discuss diffraction studies, and in Section V.B STM studies. This latter section contains new insights concerning the optimal way to analyze STM data to extract information about fluctuating order; applying these ideas to the recent experimental results of Hoffman et al. (2002a,b), and Howald et al. (2003, 2002), reveals the existence of a nearby stripe ordered phase in optimally doped BSCCO. In Section V.C contains a discussion of experiments to detect local nematic order, and a discussion of recent STM evidence of such order on the same BSCCO surfaces. In Section V.D, we discuss indirect evidence of stripe order that comes from the “1/8 anomaly.” In Section V.E we briefly discuss other probes of stripe order, including NMR, NQR and μSR.

In Section VI, we address an important issue of perspective: In the weak coupling limit, the electronic properties of a solid are essentially determined by the quasiparticle band structure, while collective modes and various forms of order reflect slight rearrangements of the states near the Fermi surface. Often these effects can be plausibly treated in the context of a Hartree-Fock or RPA treatment of the residual interactions. Conversely, in the strong coupling limit, the physics is more simply understood in terms of interacting collective modes, such as spin waves, superconducting phase fluctuations, and the “phonons” of a charge-ordered state. Here, we discuss the interrelation between these opposite perspectives. In particular we present evidence that the stripe order observed in the cuprates and related compounds is best thought of as arising from the strong-coupling antiferromagnetism of the undoped Mott insulating parent compounds than from the more conventionally metallic physics of the strongly overdoped materials.

Our most important conclusions are embodied in five numbered “Lessons,” which are stated in Sections II, III and IV, and summarized in Section VII.

II. GENERAL CONSIDERATIONS

Since the discovery (Bednorz and Müller, 1986) of high temperature superconductivity in the cuprate superconductors, there has been intense interest in the question of how a Mott insulating antiferromagnet is converted, upon doping, into a high temperature superconductor. Generally, in electronic systems, there is a competition between the kinetic energy (Fermi pressure) which favors a uniform Fermi liquid phase with sharply defined, itinerant quasiparticles, and the Coulomb repulsion between electrons, which favors various forms of insulating magnetic and/or charge ordered states. Thus, it should not be at all surprising to find various forms of charge ordered states appearing in doped antiferromagnets. In particular, “stripes” states have consistently turned up in theoretical studies of doped antiferromagnets, including early Hartree-Fock solutions of the Hubbard model, studies of Coulomb frustrated phase separation, slave-boson mean-field theories of the t-J model, and various Monte-Carlo and DMRG studies of t-J and Hubbard models. The discovery of stripe order in La2−xSrxCuO4+δ and soon after in La1.6−xNd0.4Sr2CuO4 (Tranquada et al., 1995b) added considerable credibility to the suggestion that stripe states form an important bridge between the Mott insulator, and the more metallic state at heavy doping.

“Stripes” is a term that is used to describe unidirectional density wave states, which can involve unidirectional charge modulations (“charge stripes”) or coexisting charge and spin density order (“spin stripes”). CDW states can occur in the weak coupling limit if there are sufficiently well nested portions of the Fermi surface. From the strong coupling perspective, stripes are a real-space pattern of micro-phase separation (hence the name), driven largely by a lowering of the doped-hole kinetic energy, in which the doped holes are itinerant in metallic rivers, and the antiferromagnetic correlations of the parent insulator are preserved in between. Although the characteristic dynamics of the stripes may be quite different in these two limits, as are the possible implications for other physical properties, from a broken symmetry viewpoint there is no difference between a unidirectional CDW and a stripe-ordered state.

Since the principal purpose of the present paper is to address questions concerning the existence of local

This conclusion is in agreement with the inferences drawn from the data by Howald et al. (2003, 2002), but in disagreement with those drawn by Hoffman et al. (2002b); we will discuss the origins of the differing conclusions.
A. Stripe ordered phases

In 2D, an ordered stripe phase with the stripes running in the $y$ direction gives rise to new Bragg peaks in the electronic charge scattering (and corresponding peaks in the nuclear scattering) at $\mathbf{k} = \pm \mathbf{Q}_{\text{ch}} = (2\pi/a)(\pm \delta_{\text{ch}}, 0)$ and harmonics, where $\delta_{\text{ch}} = 1/\lambda_{\text{ch}}$ and $\lambda_{\text{ch}}$ is the charge-stripe period. Where spin order coexists with charge order, new magnetic Bragg peaks occur at harmonics of $\mathbf{k} = \mathbf{Q}_s = \mathbf{Q}_{\text{AF}} \pm \frac{1}{2}\mathbf{Q}_{\text{ch}}$ where $\mathbf{Q}_{\text{AF}} = (2\pi/a)(\frac{1}{2}, \frac{1}{2})$ is the Néel ordering vector.\(^\text{15}\)

Charge stripes break rotational symmetry and translation symmetry perpendicular to the stripes in a crystal, these are to be interpreted as breaking of the crystal symmetry group, rather than the continuous symmetries of free space. The relevant order parameter, $\langle \rho_{\mathbf{Q}_{\text{ch}}} \rangle$, is the Fourier component of the electron charge density at the ordering wave vector. If the state is, in addition, a conducting or superconducting electron fluid, it is a charge stripe smectic. Spin stripes, in addition, break spin-rotational and time reversal invariance (although a particular combination of time reversal and translation is preserved). Wherever there is spin-stripe order, there is necessarily (Zachar et al., 1998) charge-stripe order, as well. The new order parameter which distinguishes the spin stripe phase, $\langle \mathbf{s}_{\mathbf{Q}} \rangle$, is the Fourier component of the spin-density.

There is more than one possible stripe-liquid phase. In particular, there is the possibility of a "stripe nematic" phase, in which thermal or quantum fluctuations have caused the stripe-ordered state to melt (i.e., translational symmetry is restored) but orientational symmetry remains broken (i.e., a snapshot of the system is more likely to see stripe segments oriented in the $y$ than $x$ direction). The nematic shares with a charge-ordered state a precise definition in terms of broken symmetry. The order parameter can be taken to be any physical quantity transforming like a traceless symmetric tensor. For instance, the traceless part of the dielectric tensor is a measure of nematic order of the charged fluid. In two dimensions, a useful nematic order parameter is also (Chaikin and Lubensky, 1995)

\[
Q_k = \frac{S(k) - S(R[k])}{S(k) + S(R[k])},
\]

where $R$ is a rotation by $\pi/2$, and

\[
S(k) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(k, \omega)
\]

is the thermodynamic (equal-time) structure factor. More exotic states can also occur, such as a "nematic spin-nematic," which breaks rotational, spin rotational, and time reversal symmetry, but not the product of $R$ and time reversal.\(^\text{16}\)

B. Fluctuating order near a quantum critical point

We consider a system in a quantum disordered phase near to a quantum critical point beyond which the ground state would be ordered.\(^\text{17}\) To be concrete, we will discuss charge-stripe order. By the quantum disordered phase, we mean one in which there is no long-range stripe order as $T \to 0$, but there may be other forms of order, for instance superconducting order.

The charge-density dynamical structure factor, $S_{\text{ch}}(k, \omega)$, for $k$ in the neighborhood of $\mathbf{Q}_{\text{ch}}$, measures the collective fluctuations which are most sensitive to the proximity of the quantum critical point. The scaling theory of quantum critical phenomena tells us that, on the quantum disordered side of the quantum critical point, there are diverging length and time scales, $\xi \sim \ell^{-\nu}$ and $\tau \sim \ell^{-\nu\nu}$ where $\nu$ is a critical exponent, $z$ is the dynamical critical exponent, and $\ell$ is the dimensionless distance to the quantum critical point.\(^\text{18}\)

\(^{15}\) Note that, in a crystal, wave vector equalities are always to be interpreted as meaning equal up to a reciprocal lattice vector.

\(^{16}\) A nematic spin nematic is most easily pictured as a state in which the spin up and spin down electrons each form a nematic state, with their principle axes at 90° to each other.

\(^{17}\) For extremely clear reviews, see (Sachdev, 2000) and (Sachdev, 1999a), as well as (Sondhi et al., 1997).

\(^{18}\) The values of the critical exponents are “universal” in the sense that they are a small discrete set of numbers depending on some general features of the critical point. Under many circumstances, $z$ is either 1 or 2, $\nu$ depends on the effective dimension, $D + z$ and, to a lesser extent, on the symmetry of the order parameter; typically, for $D \geq 2$, $\nu$ is in the range $2/3 \geq \nu \geq 1/2$. 
In the quantum disordered phase, and in the absence of quenched disorder, the fluctuation spectrum has a characteristic frequency scale, $E_G/h$. In the absence of dissipation\textsuperscript{19}, which is the case we will treat for concreteness, $E_G$ typically is a gap in the collective mode spectrum. $E_G$ is related to the correlation time by the scaling law $E_G \sim h/\tau \sim \ell^{\nu}$. For $h\omega$ slightly larger than $E_G$, $S(\mathbf{k}, \omega)$ has a pole corresponding to a sharply defined elementary excitation; this “soft-mode” is the quantity which condenses across the quantum critical point, so its quantum numbers and characteristic wave vector directly encode the nature of the nearby ordered state. At somewhat higher energy (typically, for $h\omega > 3E_G$) $S(\mathbf{k}, \omega)$ exhibits a multi-particle continuum. Deep in the continuum the system effectively exhibits scale invariance and it behaves in much the same way as if it were precisely at the quantum critical point. Both in this high frequency regime and at the quantum critical point the notion of a quasiparticle is, truly speaking, ill-defined, although for systems with small anomalous dimension $\eta$ (the typical case for $D \geq 2$) the continuum will often exhibit features (branch-points, not poles) whose dispersion resembles that of the Goldstone modes of the ordered phase. \textsuperscript{20}

A classical critical point is described by thermodynamics alone, so none of these dynamical considerations affect the critical phenomena. In particular, it follows from the classical limit of the fluctuation-dissipation theorem that the critical phenomena. In particular, it follows from the expressions in terms of the dissipative response function, $\chi'(\mathbf{k}, \omega)$, for the susceptibility (obtained from the Kramers-Krönig relation) and $S(\mathbf{k})$ (obtained from the fluctuation-dissipation theorem):

$$
\chi(\mathbf{k}, \omega = 0) = \int \frac{d\omega}{2\pi} \chi''(\mathbf{k}, \omega)/\omega \quad (2.3)
$$

$$
S(\mathbf{k}) = \int \frac{d\omega}{2\pi} \coth \left( \frac{\beta h\omega}{2} \right) \chi''(\mathbf{k}, \omega)
$$

In the limit $T \to 0$, $\coth(\beta h\omega) \to 1$; then clearly, the factor $1/\omega$ weights the important low frequency part of the integral expression for $\chi$ much more heavily than in the expression for $S$. Consequently, it follows under fairly general circumstances (Chubukov et al., 1994) from the scaling form of $\chi''$ that $S(Q_{ch}) \sim T^{-1} \chi(Q_{ch})$, or in other words that $\chi(Q_{ch}) \sim T^{-1}$ is strongly divergent in the neighborhood of the quantum critical point, while $S(Q_{ch}) \sim T^{2-\gamma} = T^{-2(2-\gamma)}$ is much more weakly divergent, or, if perchance $\gamma \geq 2 - \eta$, not divergent at all. At criticality ($\ell = 0$), similar considerations imply that $S(\mathbf{k}, \omega)$ is a less singular function at small momentum/frequency than the corresponding susceptibility.

These considerations apply even on the ordered side of the quantum critical point, where the Bragg peak makes an exceedingly small (although sharp) contribution to the total structure factor which vanishes in proportion to $T^\nu$. By contrast, the Bragg peak constitutes the entire contribution to $S(\mathbf{k}, \omega = 0)$, and is the dominant piece of the structure factor integrated over any small frequency window which includes $\omega = 0$.

**Lesson #1:** There is an important lesson to be gleaned from this general discussion concerning the best way to analyze experiments. It is the low frequency part of $S(\mathbf{k}, \omega)$ which is most directly sensitive to the stripe order. Rather than analyze $S(\mathbf{k})$, information concerning local stripe order is best obtained by studying $\chi(\mathbf{k}, \omega = 0)$ or by analyzing the partially-frequency-integrated spectral function,

$$
\tilde{S}(\mathbf{k}, \Omega) \equiv (2\Omega)^{-1} \int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi} S_{ch}(\mathbf{k}, \omega) \quad (2.4)
$$

where on the ordered side of the critical point, the smaller $\Omega$ the better, while on the quantum disordered side, $\Omega$ must be taken larger than $E_G/h$, but not more than a few times $E_G/h$. This quantity is less severely contaminated by a “background” arising from the incoherent high energy excitations. As we will discuss in Sec. V.A, this is precisely the way local stripe order is best detected in clean, high temperature superconductors.

**C. How weak disorder can make life simpler**

In the absence of quenched disorder, there is no useful information available from the structure factor at fre-
In the absence of an external potential, the low-energy states are sensitive to the local density of electronic states, \( \bar{\rho} = 2 \). Although the effects of quenched randomness on the pure critical theory can be subtle (Fisher, 1992; Griffiths, 1969; McCoy and Wu, 1968), and are rarely well understood (unless disorder happens to be irrelevant), low-energy states are frequently produced.

**Lesson #2:** The upshot is that the effect of weak quenched randomness in a quantum disordered phase is to produce a low-frequency "quasi-elastic" part of the spectral function \( S(\mathbf{k}, \omega) \). In other words, disorder will eliminate the spectral gap, but will only weakly affect the partially integrated spectral function, Eq. (2.4), with the integration scale set by \( V_{\text{dis}} \). In particular, in the presence of weak disorder, the static structure factor, \( \chi_{\text{ch}}(\mathbf{k}, \omega = 0) \), should exhibit similar \( \mathbf{k} \) dependence as \( \bar{S}(\mathbf{k}, \Omega) \) of the pure system, and so can be used to reveal the nature of the nearby ordered phase.

1. Response functions

To formalize some of these notions, we consider the somewhat simpler problem of the response of the system to a weak, applied external field which couples to the order parameter. In a quantum disordered phase the existence of an order parameter can only be made apparent directly by coupling the system to a suitable symmetry-breaking field. For a charge-stripe smectic, a non-uniform potential couples to the order parameter and thus serves as a suitable symmetry breaking field. Thus, the Fourier component \( V_{\mathbf{k}} \) of a weak potential induces a non-vanishing expectation value of the order parameter which, in linear response, is

\[
\langle \rho_{\mathbf{k}} \rangle = \chi_{\text{ch}}(\mathbf{k}) V_{\mathbf{k}} + \ldots \tag{2.5}
\]

The linear response law of Eq. (2.5) is valid provided \( V_{\mathbf{k}} \) is sufficiently small. However, if the typical magnitude of \( V \) with Fourier components in the range \( |\mathbf{k} - Q_{\text{ch}}| \xi \lesssim 1 \) is not small, \( V_{\text{dis}} \gg E_G \), then the critical region is accessed and Eq. (2.5) is replaced by the law

\[
\langle \rho_{\mathbf{k}_{\text{ch}}} \rangle \sim [V_{\text{dis}}]^{1/\delta} \tag{2.6}
\]

where \( \delta = 2/(D - 2 + \eta) \) is another critical exponent.

Many local probes, including STM and NMR, are more sensitive to the local density of electronic states, \( \bar{N}(\mathbf{r}, E) \). Again, in the quantum disordered state, translation invariance implies that, in the absence of an external perturbation, \( \bar{N}(\mathbf{r}, E) = N_0(E) \) is independent of \( \mathbf{r} \). In the presence of a weak potential, it is possible to define a relation for the local density of states

\[
\bar{N}(\mathbf{k}, E) = \chi_{\text{DOS}}(\mathbf{k}, E)V_{\mathbf{k}} + \ldots \tag{2.7}
\]

where \( \bar{N}(\mathbf{k}, E) \) is the Fourier transform of \( \bar{N}(\mathbf{r}, E) \). From linear response theory it follows that

\[
\chi_{\text{DOS}}(\mathbf{k}, E) = \frac{(2\pi)^{-1}}{f(E)} \int d\mathbf{r} d\tau e^{iE\tau - i\mathbf{k} \cdot \mathbf{r}} \theta(\tau)
\]

\[
\times \left\langle \left\{ \Psi_{\sigma}^{\dagger}(\mathbf{r}, t + \tau), \Psi_{\sigma}(\mathbf{r}, \tau) \right\}, \hat{n}(0) \right\rangle \tag{2.8}
\]

where \( \theta \) is the Heaviside function, \( \Psi_{\sigma}^{\dagger} \) is the electron creation operator, and \( \hat{n} = \sum_{\sigma} \Psi_{\sigma}^{\dagger} \Psi_{\sigma} \) is the electron density operator. Note that, despite appearances, \( E \) is not a frequency variable, but is rather the energy at which the time independent density of states is measured. A simple sum-rule relates \( \chi_{\text{ch}} \) to \( \chi_{\text{DOS}} \):

\[
\chi_{\text{ch}}(\mathbf{k}) = \int dE f(E) \chi_{\text{DOS}}(\mathbf{k}, E) \tag{2.9}
\]

where \( f(E) \) is the Fermi function.

It is also interesting to consider stripe orientational (nematic) order. Since \( \chi_{\text{ch}}(\mathbf{k}) \) is a property of the uniform fluid phase, it respects all the symmetries of the underlying crystal; in particular, if rotation by \( \pi/2 \) is a symmetry of the crystal, then \( \chi_{\text{ch}}(\mathbf{k}) = \chi_{\text{ch}}(\mathcal{R}[\mathbf{k}]) \), so no information about incipient nematic order can be obtained to linear order in the applied field. However, the leading non-linear response yields a susceptibility for the nematic order parameter, defined in Eq. (2.1), as

\[
Q_{\mathbf{k}} = \int d\mathbf{p} \chi_{\text{nem}}(\mathbf{k}; \mathbf{p})|V_{\mathbf{p}} - V_{\mathbf{R}[\mathbf{p}]}||V_{-\mathbf{p}} + V_{-\mathcal{R}[\mathbf{p}]|} \tag{2.10}
\]

This non-linear susceptibility (\( \chi_{\text{nem}} \) contains a four-density correlator) diverges at the nematic to isotropic quantum critical point, and we would generally expect it to be largest for \( \mathbf{k} \approx Q_{\text{ch}} \). It is important to note that the density modulations, reflecting the proximity of stripe order, are a first order effect, while the nematic response, which differentiates \( Q_{\text{ch}} \) from \( \mathcal{R}[Q_{\text{ch}}] \), is second order, and so will tend to be weaker, even if the nematic quantum critical point is nearer at hand than that involving the stripe-ordered state.

D. Phase diagrams

When there is more than one competing ordered phase, the phase diagram can be very complicated, and moreover can be quite different depending on various microscopic details. Nonetheless, it is often useful to have a concrete realization of the phase diagram in mind, especially when thinking about issues related to order, and order parameter fluctuations. In this subsection, we sketch schematic phase diagrams which characterize some of the
ordered phases and fluctuation effects discussed above; the reader is cautioned, however, that the shape, topology, and even the number of ordered phases may vary for material-specific reasons.

Figure 1(a) represents the essential features we have in mind, in the absence of any quenched disorder. Here, there is a stripe ordered phase which occurs at low doping and low temperature, with a phase boundary which ends at a quantum critical point deep in the superconducting phase. In the region marked “fluctuating stripes,” there is significant local stripe order whose character is governed by the proximity to the quantum critical point. Notice that this fluctuation region extends into the stripe ordered phase itself; although there is true long-range stripe order in this region of the phase diagram, close to the quantum critical point the degree of striping at the local level is much greater than the small ordered component. There is no universal statement possible concerning how far the fluctuating stripe region extends beyond the ordered phase. Clearly, if the region in which there is local stripe order does not at least include the entire region of the phase diagram in which high temperature superconductivity occurs, it cannot be essential for the mechanism of superconductivity.

Under many circumstances, the stripe ordered phase will not end at a quantum critical point, but rather will terminate at a first order line (Kivelson et al., 2001). If the transition is only weakly first order, this will not affect the general considerations presented here. If it is strongly first order, fluctuation effects are much weaker, and the effects of even very weak quenched disorder are much more severe than in the case of a continuous transition. This limit warrants further study (Cardy, 1999), both for application to the cuprates and more generally; it is likely relevant for the organic superconductors.21

Figure 1(b) is a more ornate version of the phase diagram, in which all the different broken-symmetry phases discussed in the present paper are exhibited. Here, we illustrate the case in which the nematic, charge, and spin ordering occur at distinct transitions, under which circumstances the nematic ordering must precede the charge ordering, which must in turn precede the spin ordering (Zachar et al., 1998). In general, there are also distinctions between commensurate and incommensurate, diagonal and vertical, bond-centered and site centered stripe order, so the same general considerations could lead to significantly more complicated phase diagrams.

Since the superconducting order is suppressed in the vortex state, and eliminated above $H_{c2}$, studies of the magnetic-field-induced changes in the local stripe order have recently emerged as one of the best ways of deter-

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21 A first order transition of this sort has been reported in the BEDT system (Lefebvre et al., 2000) and in $(TMTSF)_2PF_6$ (Yu et al., 2002).

22 In sketching the global shape of the phase diagram, we have included the prejudice that static stripe order, especially static spin-stripe order, competes strongly with superconductivity, but that a degree of local stripe order is necessary for high temperature pairing—the latter prejudice is reflected in the vanishing of the superconducting $T_c$ as the local stripe order is suppressed upon overdoping. However, these prejudices affect only the interplay between stripe order and superconducting order in the phase diagram, and not the central features on which we focus in the present article, concerning the character of fluctuating stripe order.
maining the nature of the interplay between stripe and superconducting order. There has been a flurry of recent papers, both experimental and theoretical on this subject. Since this subject is fairly involved, and is well reviewed in the literature, we will not elaborate here on the phase diagram in a field, although we will briefly mention some of the salient findings in Sec. V.

Finally, Fig. 1(c) shows the effect of weak disorder on the phase diagram illustrated in Fig. 1(b). Phase transitions involving breaking of spatial symmetries are generally rounded by quenched disorder. The resulting crossovers should have a glassy character i.e., the apparent transition temperature is frequency-dependent. Thus, the only true phase transitions in the presence of disorder are a superconducting and a spin-glass transition, although the spin-glass should have a local stripe order, the additional “pinning” effect of quenched disorder are a superconducting and a spin-glass transition. In the absence of quenched disorder the spin-freezing temperature manifests itself in these critical systems. In general, we present results for a spin-rotation-invariant Luttinger liquid, where the charge Luttinger parameter is in the range $0 < K < 1$. So long as the charge Luttinger parameter is in the zero disorder limit.

### III. ONE-DIMENSIONAL LUTTINGER LIQUID

In this section, we show how the general features described above play out in the case of the 1DEG. For simplicity, we present results for a spin-rotation-invariant Tomonaga-Luttinger liquid (TLL); in a forthcoming publication (Bindloss et al., 2003) we will provide details of the derivations, and will also discuss the Luther-Emery liquid (LEL), i.e., the spin-gap case. In this section we will describe the most salient aspects of the ways fluctuating order manifests itself in these critical systems. In Appendix A we give more details of this theory. (We use units in which $\hbar = k_B = 1$.)

Both the TLL and the LEL can be regarded as quantum critical systems with dynamic critical exponent $z = 1$. So long as the charge Luttinger parameter is in the range $0 < K < 1$ (which is typically the case for repulsive interactions), the charge susceptibility of the TLL diverges as $k \to 2k_F$ as

$$\chi(2k_F + q) \sim |q|^{-g},$$

where $k_F$ is the Fermi wave vector and $g = 1 - K_e$. This system can rightly be viewed as a quantum critical CDW state. (The CDW fluctuations are still stronger in the LEL case, where the susceptibility exponent is replaced by $K_e - 2$.) Thus, it is the perfect laboratory for testing the validity of the general scaling considerations of Section II.

The charge-density structure factor of the TLL is the Fourier transform of the more readily evaluated space time structure factor

$$S(k, \omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \ S(x, t) e^{i k x - i \omega t}$$

where the charge correlation function $S(r, t)$ is given by

$$S(r, t) = S_0(r, t) + [e^{i 2 k_F r} S_{2k_F}(r, t) + c.c.] + [e^{i 4 k_F r} S_{4k_F}(r, t) + c.c.] + \ldots$$

Explicit expressions for $S_0, S_{2k_F}, \ldots$ are given in the literature. These expressions can be Fourier transformed (although in general, this must be done numerically) to yield expressions for the dynamical structure factor. Since the LL is quantum critical, this (and other) correlation functions have a scaling form; for example, near $2k_F$ (i.e., for small $q$)

$$S(2k_F + q, \omega) = \frac{1}{v_c} \left( \frac{D}{v q} \right)^g \Phi_{2k_F} \left( \frac{\omega}{v q}, \frac{\omega}{k_F T} \right)$$

where $D$ is an ultraviolet cutoff, i.e., the band width of the TLL. The scaling function $\Phi_{2k_F}$ of Eq. (3.4) depends implicitly on the dimensionless parameters $K_e$ and $\theta(x)$, the charge and spin velocities. (Spin rotation invariance constrains the spin Luttinger exponent $K_e$ to be equal to 1.) To exhibit the general features we are interested in, while making the Fourier transform as simple as possible, we can consider the limit $T = 0$ and set $v_s = v_c = v$. Then, for $q$ small

$$S(2k_F + q, \omega) \sim \frac{1}{v} \left( \frac{D}{v q} \right)^g \theta \left( \frac{\omega^2}{v^2 q^2} - 1 \right)^{-g/2}$$

where $\theta(x)$ is the step function.

For fixed $\omega$, as a function of $q$, the dynamic structure factor exhibits a multi-particle continuum for $-\omega/v \leq q \leq \omega/v$, but it does have singular structure, which can be thought as an image of the Goldstone (phason) modes one would find were there true CDW order:

$$S(2k_F + q, \omega) \sim |\omega - v|/|q|^{-g/2} \text{ as } |q| \to \omega/v$$

The equal-time structure factor can also be readily computed by integrating the dynamical structure factor; as $q \to 0$,

$$S(2k_F + q) \sim A - A' (|q|/2)^{K_e}$$
where $A$ and $A'$ are numbers of order 1 and $\alpha$ is a short distance cutoff which we take to be $\alpha \sim v_F / D$. As promised, this singularity is much weaker than that exhibited by $\chi$, and indeed the $2k_F$ component of the structure factor remains finite (but not differentiable) as $q \to 0$. (See Fig. 2.)

![Graph of S(k) and chi(k, \omega = 0)](image)

**FIG. 2** The $T = 0$ charge structure factor, $S(k)$ (solid line), and the static susceptibility, $\chi(k, \omega = 0)$ (dashed line), of the TLL with $v_s = v_c$ and $K_c = 0.5$.

We now turn to the spatial structure in the 1DEG induced by a single impurity at the origin. For $K_c$ in this range, the $2k_F$ component of an impurity potential, whose amplitude we will denote by $\Gamma$, is a relevant perturbation (Kane and Fisher, 1992, 1994) with (boundary) scaling dimension $d = 1/2(K_c + 1)$. Thus, there is a crossover energy scale $T_K \propto \Gamma^{2/(1-K_c)}$ such that excitations with energy large compared to $T_K$ see a weak backscattering potential, and are thus only weakly perturbed by the impurity. Conversely, for energies low compared to $T_K$ the system is controlled (Kane and Fisher, 1992, 1994) by the fixed point at $\Gamma \to \infty$. However, at this fixed point, the high energy cutoff is replaced by a renormalized cutoff, $D \to T_K$.

We begin, therefore, by considering the limit $\Gamma = \infty$, i.e., a semi-infinite system with $x \geq 0$ and the boundary condition that no current can flow past $x = 0$. For finite $\Gamma$, this solution is applicable for all energies $|E| \ll T_K$. The Fourier transform of the impurity induced LDOS

$$N(k, E) = N^>(k, E) + N^< (k, E)$$

$$N^< (k, E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_0^{\infty} dx \ g^< (x, x; t) \ e^{i(Et - kx)}$$

(3.8)

can be computed using exact expressions for the appropriate single hole Green function (Eggert, 2000; Eggert et al., 1996; Mattsson et al., 1997):

$$g^< (x, x; t) = \sum_\sigma \langle \Psi^\dagger_\sigma (x, t) \Psi_\sigma (x, 0) \rangle$$

(3.9)

$$\equiv g_0 (x, t) + [e^{i2k_F x} g_{2k_F} (x, t) + e^{-i2k_F x} g_{-2k_F} (x, t)] + \ldots$$

where $g_0 (x, t)$ is the long wavelength part and $g_{2k_F} (x, t)$ is the $2k_F$ part; the single electron piece can be computed similarly from $g^>$, or more simply from the relation

![Graph of STM and ARPES spectra](image)

**FIG. 3** Thermally scaled STM spectra $N(k, E)$ in the neighborhood of $k = 2k_F$ - (a) and (c) - and ARPES spectra - (b) and (d) - near $k = k_F$. In all cases, $v_s / v_c = 4$ and $K_c = 1$; for (a) and (b), $K_c = 0.5$ ($b = 1/16$) while for (c) and (d) $K_c = 0.17$ ($b = 1/2$). In (a) and (c), the density of states oscillations are induced by an impurity scatterer at the origin with $T_K \gg E$. In the STM spectra, the real and imaginary parts are represented by the solid and dashed lines, respectively. The curves for $E/T = 10$ and $E/T = 20$ are offset by 1 and 2 respectively, in arbitrary units.
between their respective spectral densities, \( N^<(k, E) = e^{-E/T} N>(k, E) \). We will be interested in the \( 2k_F \) component which clearly contains information about CDW correlations in this semi-infinite 1DEG, and for which a general expression in space and time has been given by Eggert and others (Eggert, 2000; Eggert et al., 1996; Mattsson et al., 1997). As with the structure factor, the \( 2k_F \) part of \( N(k, E) \) can also be expressed in terms of a scaling function \( \Phi \) as (see Appendix A)

\[
N(q + 2k_F, E) = \frac{B}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} \Phi \left( \frac{2E}{\alpha v_c q} \frac{E}{T} \right) \tag{3.10}
\]

where \( b = (1 - K_c)^2/8K_c \) and \( B \) is a dimensionless constant; we have left implicit the dependence on the dimensionless parameters \( K_c \) and \( v_c/v_s \).

The scaling form of \( N(k, E) \), given in Eq. (3.10), expresses the fact that the Luttinger liquid is a quantum critical system. The spectrum of the Luttinger liquid and the existence of a charge-ordered state induced by the impurity dictate entirely the structure of the scaling function \( \Phi(x, y) \): it has a multi-soliton continuum for both right and left moving excitations, each with leading thresholds associated with one-soliton states carrying separate spin and charge quantum numbers and moving at their respective velocities, as well as a non-propagating feature associated with the pinning of the CDW order by the impurity. Thus, the STM spectrum exhibits all the striking features of the Luttinger liquid: spin-charge separation and quantum criticality, i.e., fluctuating order. The beauty of the 1DEG is that much of this can be worked out explicitly.

In Figs. 3a and 3c, we show the scaling function computed numerically for various representative values of the parameters. The plots show the real and imaginary parts of \( N(2k_F + q, E) \) at fixed \( E/T \) as a function of the scaled momentum, \( v_c q / T \), for \( |q| \ll k_F \). For comparison, we also show (Orgad et al., 2001) in Figs. 3b and 3d, the single hole spectral function \( A(k_F + q, \omega) \) that would be measured in an angle resolved photoemission experiment (ARPES) on the same system in the absence of an impurity.

There are several things to note about these plots:

1. Because of the quantum critical scaling form, high energy and low temperature are equivalent. Note, however, that in interpreting the large \( E/T \) spectra as representative of the low-temperature behavior of the system, it is important to remember that the thermal scaling of the \( k \)-axis of the figure hides the fact that all features of the spectrum are becoming sharper as \( T \rightarrow 0 \); this is made apparent in Fig. 4.

2. It is clear that there are right-dispersing features of the scaling functions characterized by the spin and charge velocities. The interference between dispersing features of the ARPES spectrum near \( k_F \) and \(-k_F \) can be loosely thought of as giving rise to the dispersing features in the STM spectrum; indeed, as \( K_c \rightarrow 1 \) (the non-interacting limit), \( \text{Re}\{N(2k_F + q, E)\} \propto A(k_F - q/2, -E) \) at fixed \( E > 0 \). However, it is also clear from the figure that the stronger the interactions, the less direct is the resemblance between \( N(2k_F + q, E) \) and \( A(k_F - q/2, -E) \).

3. There is also a very weak feature in the spectrum, visible only at quite large \( E/T \) (compare Figs. 3 and 4), which disperses in the opposite direction (left) to the main features of the spectra with velocity \(-v_c \). Because these spectra are shown only for \( k \)’s in the neighborhood of \(+2k_F \) (or \(+k_F \) for \( A(k, E) \)), there is no symmetry between right and left moving excitations. If we showed the spectra on a larger scale, there would of course be the mirror symmetric spectra near \(-2k_F \) (or \(-k_F \)) as required by Kramer’s theorem.

4. In the STM spectrum, but not in the ARPES spec-

\textsuperscript{26} Details of the asymptotic analytic behaviors of the singularities exhibited in these plots can be found in Appendix A.
trum, there is a feature near \( q = 0 \) which does not disperse with increasing energy; this is directly related to the pinned CDW order. Note that for \( K_c = 0.5 \), this feature is weak in Fig. 3, and only becomes prominent at very large \( E/T \), as shown in Fig. 4; but this is the most important feature of the data if one is interested in evidence of pinned CDW order.

**Lesson #3:** From this explicit example we learn that dispersing features in an STM measurement that resemble the interference effects that arise from non-interacting quasiparticles do not necessarily imply the existence of well defined quasiparticles!

It is interesting to compare \( N(k, E) \) with the LDOS averaged over some energy scale,

\[
\tilde{N}(2k_F + q, E) \equiv E^{-1} \int_{-E}^{0} d\epsilon N(2k_F + q, \epsilon). \tag{3.11}
\]

Note that at \( T = 0, E\tilde{N}(k, E) \rightarrow \langle p_k \rangle \) as \( E \rightarrow \infty \). However, the expression we have used for \( N(k, E) \) was derived for an infinite strength scattering potential, and so is only valid up to energies of the order of \( T_K \ll D \). This integrated quantity is shown in Fig. 5 for representative parameters\(^{27} \) at \( T = 0 \), and in the limit as \( q \rightarrow 0 \), \( \tilde{N}(2k_F + q, T_K) \) has the asymptotic behavior

\[
\tilde{N}(2k_F + q, T_K) \propto \left( \frac{1}{T_K} \right)^{2b} \left( \frac{T_K}{v_c q} \right)^{2.5} \tag{3.12}
\]

Both the integrated induced tunneling density of states and the induced tunneling density of states at fixed voltage \( E \) have a singular behavior of the form \( q^{-2.5} \) (See Appendix A). However, the big difference is that \( N(k, E) \) has dispersing singularities in addition to the non-dispersing singularity at \( k \rightarrow 2k_F \), while the singularity at \( k \rightarrow 2k_F \) is the only singular feature of \( \tilde{N}(k, T_K) \).

Thus, \( \tilde{N}(k, T_K) \) is more easily analyzed for evidence of an almost ordered CDW state.

At energies \( E > T_K \), the response of the system to the presence of the impurity is weak (i.e., proportional to \( \Gamma \)), and can be computed in perturbation theory. For instance, in the weak-impurity limit \( T_K \rightarrow 0 \) the integrated response of the system over all energies of states is dominated by the high energy, perturbative regime where at \( T = 0 \) we find, in agreement with the sum rule of Eq. (2.9), (see also Appendix A)

\[
\tilde{N}(2k_F + q, D) = \chi_{\text{ch}}(2k_F + q) \Gamma + O(\Gamma^2) \tag{3.13}
\]

By comparing the weak \( \Gamma \) behavior of \( \tilde{N}(2k_F + q, D) \) (Eq. (3.13)) with its behavior in the opposite \( \Gamma \rightarrow \infty \) limit,

\[\delta N(x, E) \sim E^{1.5} \quad \text{is always large at low energies } |E| < v/x \text{ compared to the background DOS of the clean TLL}\]

\[N(E) \sim E^{-1} \quad \text{at high energies } |E| \gg v/x. \tag{3.15}\]

---

\(^{27}\) For technical reasons, what is actually plotted in Fig. 5 is \( \tilde{N}(k) \equiv \lim_{T_K \rightarrow \infty} \int_{-\infty}^{\infty} dE [1 - f(E)] N(k, E), \) where \( f \) is the Fermi function, but the distinction between this and \( \tilde{N}(k, T_K) \) is not important here.
This is yet another illustration of the way impurities enhance the low energy effects of fluctuating order.

IV. TWO-DIMENSIONAL FERMI LIQUID

In this section, we consider the application of these ideas to the case of weakly-interacting electrons in 2D. To obtain explicit results, we will consider electrons with a quadratic dispersion $c_k = \hbar^2 k^2 / 2m$.

Although, as seen in Eq. 2.8, $\chi_{\text{DOS}}$ is a two particle correlator, for non-interacting quasiparticles, it can be expressed as a convolution (Byers et al., 1993; Polkovnikov et al., 2002a) of two single-particle Green functions:

$$
\chi_{\text{DOS}}(k, E) = -\frac{1}{\pi} \text{Im} \left\{ \int \frac{d\mathbf{q}}{(2\pi)^d} G(k+q, E) G(q, E) \right\}
$$

(4.1)

Therefore, in weakly-interacting systems, it can be analyzed to obtain information about the single-particle spectrum. Explicitly, it is easy to see that for free electrons in 2D $\chi_{\text{DOS}}(k, E)$ is given by

$$
\chi_0(k, E) = \frac{m}{\pi \hbar^2} \frac{\theta(c_k - 4E)}{\sqrt{c_k(c_k - 4E)}}
$$

(4.2)

Here, the subscript 0 is introduced for later convenience to signify $\chi$ in the non-interacting limit, and once again $\theta(x)$ is the Heaviside (step) function. For fixed $E$ as a function of $k$, this quantity diverges along curves in $k$ space where $c_k = 4E (|k| = 2k_F)$. Note that the fact that $\chi_0$ vanishes for $c_k < 4E$ is a peculiarity of the 2D case with infinite quasiparticle lifetime. In 3D,

$$
\chi_0(k, E) = \frac{1}{8\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{1}{\sqrt{c_k}} \ln \left| \frac{2\sqrt{E + \sqrt{c_k}}}{2\sqrt{E - \sqrt{c_k}}} \right|
$$

(4.3)

which also has a singularity as $c_k \to 4E$, but is non-zero (and positive) for all $E > 0$. In contrast, as shown in Section III, in a 1D Luttinger liquid the induced density of states has a phase jump as $k$ crosses any of the propagating or non-propagating singularities. It is straightforward to see that both in 2D and 3D a finite lifetime of the quasiparticles leads to a rounding of the singularities and that in 2D it also leads to a positive induced density of states for all $E > 0$. It is also simple to verify that to non-linear order in the external potential, the induced density of states in 2D may or may not have a phase jump across the singularity depending on the details of the perturbing potential. For instance, the induced LDOS produced by a single impurity can be computed (from the impurity $t$-matrix); it exhibits sign reversal ($\pi$ phase shift) across the singularity at $c_k = 4E$ for a repulsive potential, but not for an attractive one. In contrast, we will show below, that any $k$ space structure that arises from proximity to a quantum critical point is derived from the susceptibility, $\chi(k)$, which is real and positive, so it always produces a signal whose phase is constant.

Lesson # 4: The density of states modulations induced by weak disorder in a non-interacting metal are quite different in character from those expected from the proximity to a CDW quantum critical point, both in that they disperse strongly as a function of energy, and the peak intensities lie along curves (surfaces in 3D) in $k$ space, as opposed to the structure associated with isolated points in $k$ space expected from near critical fluctuations.

The charge susceptibility itself, $\chi_0(k)$, is well known from many studies of the 2DEG. It has an extremely weak non-analyticity, whenever $|q| = 2k_F$

$$
\chi_0(q) = \frac{m}{2\pi \hbar^2} \left( 1 - \theta(q - 2k_F) \frac{\sqrt{q^2 - 4k_F^2}}{q} \right)
$$

(4.4)

The inverse Fourier transform of this meager non-analyticity is what gives rise to the famous Friedel oscillations in the neighborhood of an isolated impurity,

$$
\langle \rho(r) \rangle \sim \frac{1}{r^2} \cos(2k_F r).
$$

(4.5)

What this means is that, for all intents and purpose, the Friedel oscillations are all but invisible in the Fourier transform of any conceivable STM experiment on a simple metal in $D > 1$. Some non-trivial method of data analysis is necessary instead (Briner et al., 1998; Sprunger et al., 1997).

It is worthwhile considering the effects of interactions on this picture. For weak enough interactions, the effects can be treated in a Hartree-Fock approximation. Thus, if we absorb any interaction-induced changes in the band parameters into a renormalized band structure, the only change in the above analysis is that the external perturbation, $V_k$ in Eqs. (2.5) and (2.7) must be replaced by an effective potential, $V_k \to V_k + \hat{U} \rho_k$, where $U$ (which can be weakly $k$ dependent) is the strength of the electron-electron repulsion. This leads to the usual RPA expression for the susceptibilities of the interacting system, and to

$$
\chi_{\text{ch}}(k) = [1 - U_k \chi_0(k)]^{-1} \chi_0(k).
$$

$$
\chi_{\text{DOS}}(k, E) = [1 - U_k \chi_0(k)]^{-1} \chi_0(k, E).
$$

(4.6)

29 In practice, the distinction between an $N(k, E)$ which is peaked along curves, indicative of quasiparticle interference effects, and an $N(k, E)$ peaked at isolated “ordering” vectors, $k = Q$, may not always be straightforward to establish in experiment. Consider the case in which there is an anisotropy of strength $\alpha$ in the effective mass of the 2DEG, i.e., $c_k = \hbar^2 (k_x^2 + \alpha k_y^2) / 2m$. In the limit of large anisotropy, $\alpha \gg 1$, when the effects of the finite $k$ resolution of actual experiments are taken into account, apparent peak-like structures can emerge. If we represent the effect of finite resolution by integrating the expression in Eq. 4.2 over a range of momenta around different points along the ellipse $c_k = 4E$, the integrated expression is peaked near $k = \pm 2\sqrt{2mE} \hat{x}_x$, where it is a factor of $\sqrt{\alpha}$ larger than where it is minimal near $\sqrt{\alpha} k = \pm 2\sqrt{2mE} \hat{y}_y$. 


(It is the fact that $E$ is a probe energy, not a frequency, which is responsible for the fact that it is simply $\chi_0(k)$, rather than a frequency dependent factor, which appears in the expression for $\chi_{\text{DOS}}(k, E)$.) Not surprisingly, since $\chi_0(k)$ is finite for all $k$, for small $U$ there is little qualitative difference between $\chi_{\text{DOS}}(k)$ and $\chi(k)$. However, if we imagine, as is often done (although we are not aware of any reason it is justified) that this RPA expression applies qualitatively for larger magnitudes of $U$, then as a function of increasing magnitude of $U$ we would eventually satisfy a Stoner criterion for a CDW $U_k \chi_0(k) = 1$. Here, due to the relatively weak $k$ dependence of $\chi_0(k)$, the CDW ordering vector $Q_{\text{ch}}$ is determined as much by the $k$ dependence of $U_k$ as by effects intrinsic to the 2DEG.

Following this line of analysis, let us consider the behavior of $\chi_{\text{DOS}}(k, E)$ in the quantum disordered phase close to such a putative Stoner instability, where $\chi_{\text{ch}}(k)$ is highly peaked at $k = Q_{\text{ch}}$. For $k$ far from the ordering vector, both the $k$ and $E$ dependence of $\chi_{\text{DOS}}$ are determined largely by $\chi_0(k, E)$. However, at fixed voltage, $\chi_{\text{DOS}}(k, E)$ will also exhibit a peak at a $k \approx Q_{\text{ch}}$, with a voltage dependent intensity proportional to $\chi_0(Q_{\text{ch}}, E)$. Note, particularly, that as long as there is a finite correlation length associated with the incipient order, i.e., so long as the peak in $\chi_{\text{ch}}(k)$ has a finite width, the corresponding peak in $\chi_{\text{DOS}}(k, E)$ will not generally occur precisely at $Q_{\text{ch}}$. To illustrate this point, here and in Fig. 6 we adopt as a simple phenomenological model, $\chi_{\text{ch}}(k) = A(\xi k)^2 \exp\left(-\frac{\xi^2}{2}(|k| - Q_{\text{ch}})^2\right)$, where $\xi$ is the stripe correlation length. From Eq. (4.6), it follows that in addition to the singularity inherited from $\chi_0(k, E)$, $\chi_{\text{DOS}}(k, E)$ has a peak at a momentum $k$ which satisfies

$$k = Q_{\text{ch}} - \xi^2 \left[\frac{\epsilon_k - 2E}{\epsilon_k - 4E}\right] \nabla_k \log|\epsilon_k|. \quad (4.7)$$

In short, the peak associated with incipient order is weakly dispersing (especially at energies far from any quasiparticle resonance condition) but so long as $\xi$ is finite, the peak is never “non-dispersing.” (In contrast, the 1D example discussed above is quantum critical, so $\xi$ is infinite and the feature associated with fluctuating order is strictly non-dispersive.)

Although the calculations are somewhat more involved (and therefore must be implemented numerically), the same sort of weak coupling analysis can be carried out in the superconducting state. Oscillations induced by a mean-field with period 4, representing stripes with various internal structures in a d-wave superconductor, were carried out by Podolsky et al. (2002). As expected, the stripes induce oscillations in the LDOS with the period 4, but with energy dependences which reflect both the quasiparticle dispersion, and the specific stripe structure assumed. Impurity induced oscillations in the LDOS in a d-wave superconductor were computed by Byers et al. (1993) and Wang and Lee (2003), and the effects of proximity to a stripe-ordered state, at RPA level, were investigated by Polkovnikov et al. (2002b). In all cases, interference and stripe-related effects interact in fairly complex ways that require detailed analysis to disentangle.

Lesson #5: Depending on what regions of $k$ space are probed, the STM spectrum can either be dominated largely by band structure effects, or by the incipient CDW order. The most singular enhancement of the STM signal is expected to occur at energies such that a dispersing feature reflecting the underlying band structure passes through the ordering wave-vector.

V. REGARDING EXPERIMENTS IN THE CUPRATES

Even where broken symmetry associated with stripe order has ultimately been proven to exist, establishing this fact has often turned out to be difficult for a number of practical reasons. In addition, quenched disorder is always a relevant perturbation (in the renormalization group sense), macroscopic manifestations of broken spatial symmetries are sharply defined only in the zero disorder limit. Nevertheless, the existence of some form of order which coexists with superconductivity has implications for the phase diagram which can, in principle and sometimes in practice, be tested by macroscopic measurements. A particularly revealing set of phenomena occur when the strength of the superconducting order is modulated by the application of an external magnetic field.\(^{30}\) Moreover, as discussed in

\(^{30}\) For experimental studies, see (Ando, 2002; Ando et al., 1997; Hawthorn et al., 2003; Hoffman et al., 2002a; Katano et al., 2000; Khaykovich et al., 2002; Lake et al., 2001, 2002; Liu et al., 2003; Ono et al., 2000; Tyler et al., 1998); for theoretical analysis, see (Arovas et al., 1997; Chakravarty et al., 2001b; Demler et al.,
Sec. V.C, from measurements of macroscopic transport anisotropies, electronic nematic order (e.g., point-group symmetry breaking) has been identified beyond all reasonable doubt in quantum Hall systems (Du et al., 1999; Lilly et al., 1999a), and very compelling evidence for its existence has been reported in the last year in under-doped La$_{2-x}$Sr$_x$CuO$_4$ and YBa$_2$Cu$_3$O$_{6+y}$ (Ando, 2002; Ando et al., 2002b).

However, most searches for stripe order rely on more microscopic measurements, especially elastic neutron scattering. One aspect of this that has caused considerable confusion is that, unless an external perturbation (such as weak crystalline orthorhombicity) aligns the stripes in one direction, one generally finds equal numbers of $y$ directed and $x$ directed domains, leading to quartets of apparently equivalent Bragg peaks, rather than the expected pairs. Fortunately, where sufficiently long-ranged order exists, it is possible, by carefully analyzing the scattering data, to distinguish this situation from a situation in which the peaks arise from a more symmetric “checkerboard” pattern of translation-symmetry breaking. From elastic neutron scattering (i.e., measurements of both the magnetic and the nuclear $S(\mathbf{k},\omega = 0)$), and to a lesser extent from X-ray scattering, it has been possible to establish the existence of stripe-ordered phases in a wide range of members of the lanthanum cuprate family of high-temperature superconductor: La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ over the whole range of doped hole concentrations $^{31}$ from $0.05 < x < 0.2$, La$_{2-x}$Ba$_x$CuO$_4$ (Fujita et al., 2002), La$_{2-x}$Sr$_x$CuO$_4$ for $^{32}$ $0.02 < x < 0.13$, and La$_2$CuO$_{4+\delta}$ (Lee et al., 1999) (including optimally doped material with a doped hole concentration $\approx 0.15$ and a superconducting $T_c$ of 42 K).

Exciting preliminary evidence of charge-stripe order has been recently reported, as well, from elastic neutron scattering studies on underdoped YBa$_2$Cu$_3$O$_{6+y}$ (with $y = 0.35$ and $T_c = 39$ K, (Mook et al., 2002)) and optimally doped YBa$_2$Cu$_3$O$_{6+y}$ (with $y = 0.93$ and $T_c = 93$ K, (Mook, 2002)). And, it is worth mentioning, stripe order has also been similarly detected in a number of non-superconducting doped antiferromagnets, including $^{33}$ La$_{2-x}$Sr$_x$NiO$_{4+\delta}$ and the colossal magneto-resistance manganites. $^{34}$

Conversely, it is important to note that so far no evidence of stripe order, or incipient stripe order, has been found in any of the electron-doped cuprate superconductors. Indeed, all the low-energy magnetism that has been reported to date (Yamada et al., 2003) is peaked at the commensurate ordering wavevector, $\mathbf{Q}_{AP}$, rather than at an incommensurate wavevector. This is rather strong evidence that at least spin-stripe order is absent in these materials. This does not rule out a possible role for charge inhomogeneity; perhaps the electron-doped materials are more prone to form “bubble phases” i.e., crystalline phases with more than one doped electron per unit cell (Fogler et al., 1996; Seul and Andelman, 1995).

We now turn to the core problem: Given a system which in the absence of quenched disorder or explicit symmetry breaking terms is in an isotropic fluid state, how is the existence of substantial local stripe order identified?

### A. Diffraction from stripes

Peaks in $S(\mathbf{k},\omega)$ at the characteristic stripe ordering vectors indicate a degree of local stripe order. The $k$ width of these peaks can be interpreted as an indication of the spatial extent of local stripe order, and the low frequency cutoff as an indication of the typical stripe fluctuation frequency. So long as there is no spontaneous symmetry breaking, $S(\mathbf{k},\omega)$ necessarily respects all the point-group symmetries of the crystal, and thus will necessarily always show peaks at quartets of $k$ values, never the pairs of $k$ values of a single-domain stripe ordered state.

Low frequency spin fluctuations with relatively sharp peaks at incommensurate wave vectors were detected many years ago in inelastic neutron scattering studies (Cheong et al., 1991; Thurston et al., 1989, 1992) of “optimally-doped” $(x \approx 0.15, T_c \sim 38$ K) La$_{2-x}$Sr$_x$CuO$_4$. However, not until the discovery (Tranquada et al., 1995b) of “honest” stripe-ordered phases in the closely related compound La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ was the interpretation of these peaks as being due to stripe fluctuations made unambiguously clear. For instance, as shown in Fig. 7, the magnetic structure factor at low temperature and small but finite frequency, $\hbar\omega = 3$ meV, looks very similar (in absolute magnitude and width) in both La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ with $x = 0.12$, where elastic scattering indicating statistically ordered stripes has been detected (Tranquada et al., 1995b), and in La$_{2-x}$Sr$_x$CuO$_4$ with $x = 0.15$, where no such static order is discernible (Yamada et al., 1995). As discussed in the previous section, this is precisely the expected behavior near a quantum critical point, where presumably the partial substitution of La by Nd has moved the system from slightly on the quantum disordered to slightly on the ordered side of a stripe ordering quantum critical point. $^{35}$ We can now confidently characterize

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$^{31}$ Ichikawa et al. (2000); Tranquada et al. (1997); Wakimoto et al. (2001)

$^{32}$ Kimura et al. (1999); Matsuda et al. (2000); Niedermayer et al. (1998); Suzuki et al. (1998); Wakimoto et al. (1999, 2001)

$^{33}$ Kajimoto et al. (2003); Lee and Cheong (1997); Tranquada (1998a); Yoshizawa et al. (2000)

$^{34}$ Mori et al. (1998a,b); Radaelli et al. (1999); Wang et al. (2000)

$^{35}$ This interpretation is further supported by the finding that the magnetic $S(\mathbf{k},\omega)$ in near optimally doped La$_{2-x}$Sr$_x$CuO$_4$
La$_{2-x}$Sr$_x$CuO$_4$ over an extremely broad range of doping as either being in a stripe-ordered, or a nearly ordered stripe-liquid phase.

An important test of this idea comes from studies of the changes in $S(k,\omega)$ produced by weak disorder. Specifically, in Fig. 8, we compare the low frequency (Matsuda et al., 1993) and elastic (Hirota et al., 1999) pieces of the magnetic structure factor of La$_{2-x}$Sr$_x$CuO$_4$ in the presence and absence of a small concentration (1.2%) of Zn impurities. (Zn substitutes for Cu.) As one might have expected, the Zn slightly broadens the k space structure, although not enormously (Kimura et al., 1999; Tranquada et al., 1999a). Most dramatically, the Zn “pins” the stripe fluctuations, in the sense that what appears only as finite frequency fluctuation effects in the Zn-free material, is pushed to lower frequencies and even to $\omega = 0$ by the quenched disorder.

has scaling properties consistent with its being dominated by a nearby stripe ordering quantum critical point; see (Aeppli et al., 1997).

$\kappa$ It is certain that one effect of the Zn impurities is to pin the charge stripe fluctuations. It may also be that the missing Cu spin plays an important role in slowing the spin-fluctuations in the neighborhood of the Zn; the potential importance of this form of coupling, based on the behavior of a Kondo impurity in a system close to a magnetic quantum critical point, has been stressed in (Sachdev et al., 1999).

FIG. 7 Comparison of constant-energy scans at $\hbar\omega = 3$ meV through an incommensurate magnetic peak (along path shown in inset) for (a) La$_{1.86}$Sr$_{0.15}$CuO$_4$ and (b) La$_{1.84}$Nd$_{0.2}$Sr$_{0.2}$CuO$_4$. Both scans are at $T = 40$ K $> T_c$. Measurement conditions are described in (Tranquada et al., 1999b).

FIG. 8 Comparison of magnetic scattering measurements in La$_{2-x}$Sr$_x$CuO$_4$ with and without Zn; all scans are along $Q = (1/4 + h, 1/4, 0)$, measured in reciprocal lattice units. (a) Scan at $E = 1.5$ meV and $T = 7$ K, and (b) difference between elastic scans measure at $T = 50$ K, both for La$_{1.86}$Sr$_{0.14}$Cu$_{0.985}$Zn$_{0.015}$O$_4$ ($T_c = 19$ K) (Hirota, 2000). (c) Scan at $E = 2$ meV and $T = 38$ K (Yamada et al., 1998), and (d) elastic scans at $T = 1.5$ K (circles) and $50$ K (triangles) (Kimura et al., 1999), for La$_{1.86}$Sr$_{0.15}$CuO$_4$ ($T_c = 38$ K). Measurement conditions were different for each panel; see references for details.

The issue still remains actively debated whether or not various fluctuation effects seen in neutron scattering studies of the other widely studied families of high temperature superconductors, especially YBa$_2$Cu$_3$O$_{6+y}$ and Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$, can be associated with stripe fluctuations. We will not review this debate here, but will touch on it again in Secs. V.B and V.C, below.

One question arises (Chen et al., 2002) concerning how we can distinguish a fluctuating stripe phase from a fluctuating checkerboard phase (which breaks translation symmetry but preserves the symmetry under exchange of x and y). Of course, the strongest indication that stripes, rather than checkerboards are responsible for the observed fluctuations comes from the presence of “nearby” stripe-ordered phases, and the absence of any clear evidence of actual ordered checkerboard phases in any doped antiferromagnet, to date. However, in theoretical studies (Chen et al., 2002; Fogler et al., 1996; Löw et al., 1994; Moessner and Chalker, 1996) both types of order appear to be close to each other in energy, so this argument should not be given undue weight.

Where both spin and charge peaks are observed, it turns out that there is a straightforward way to distinguish. From Landau theory, it follows (Zachar et al., 1998) that there is a preferred relation between the spin
and charge-ordering wave-vectors, \( \mathbf{Q}_s + \mathbf{Q}_s' = \mathbf{Q}_{ch} \). For stripe order, this means that the spin and charge wave-vectors are parallel to each other, and related (up to a reciprocal lattice vector) by the relation \( 2\mathbf{Q}_s = \mathbf{Q}_{ch} \). However, in the case of checkerboard order, the dominant spin-ordering wave-vector is not parallel to the charge-ordering wave vectors; if \( \mathbf{Q}_{ch} = (2\pi/a)(\pm \delta_{ch}, 0) \) and \( \mathbf{Q}_{ch}' = (2\pi/a)(0, \pm \delta_{ch}) \), the corresponding spin-ordering vectors, \( \mathbf{Q}_s = \mathbf{Q}_{AF} \pm (2\pi/a)(\frac{1}{2}\delta_{ch}, \frac{1}{2}\delta_{ch}) \) and \( \mathbf{Q}_s' = \mathbf{Q}_{AF} \pm (2\pi/a)(\frac{1}{2}\delta_{ch}, -\frac{1}{2}\delta_{ch}) \) satisfy the requisite identities. Thus, the relative orientation of the spin and charge peaks can be used to distinguish fluctuating stripe order from fluctuating checkerboard order (Tranquada et al., 1999a).

B. STM measurements and stripes

STM is a static probe and thus cannot detect any structure associated with fluctuating order unless something pins it \(^{37}\). Density or “Friedel oscillations” (Friedel, 1958) in simple metals produced by the presence of a defect are directly related to Fermi-surface-derived non-analyticities in the susceptibility, \( \chi(k) \). However, “generalized Friedel oscillations” can occur in more diverse systems in which the relevant structure in \( \chi(k) \) is not directly related to any feature of a Fermi surface. For instance, a bosonic superfluid on a lattice close to a second order transition to an insulating, bosonic crystalline phase would exhibit generalized Friedel oscillations with the characteristic wave length of the roton-minimum - these oscillations, in a very direct sense, would image the fluctuating crystalline order present in the fluid state.

There are a few important features of generalized Friedel oscillations which follow from general principles. If the liquid state is proximate to a highly anisotropic state, such as a stripe state, the values of \( \mathbf{k} = \mathbf{Q} \) at which \( \chi \) has maxima will reflect the pattern of spatial symmetry breaking of the ordered state, but \( \chi(k) \) will respect the full point-group symmetry of the crystal unless the liquid state spontaneously breaks this symmetry, e.g., is a nematic. So, the generalized Friedel oscillations around a point impurity in a stripe-liquid phase will inevitably form a checkerboard pattern, unless some form of external symmetry breaking field is applied (Polkovnikov et al., 2002b).

There is another form of spatial modulation of the density of states, one with a period which disperses as a function of the probe energy, which is sometimes (incorrectly, we believe) referred to in the STM literature as Friedel oscillations. This latter effect, which was first demonstrated by Crommie et al. (1993), is produced by the elastic scattering of quasiparticles of given energy off an impurity. The resulting interference between scattered waves leads to variations of the local density of states at wave vectors \( \mathbf{Q} = \mathbf{k} - \mathbf{k}' \), where \( \mathbf{k} \) and \( \mathbf{k}' \) are the wave-vectors of states with energy \( E = \epsilon(\mathbf{k}) = \epsilon(\mathbf{k}') \), as determined by the band structure, \( \epsilon(k) \). Generalized versions of these oscillations can occur even when there are no well defined quasiparticles, so long as there are some elementary excitations of the system with a well-defined dispersion relation, as is shown in Sec. III.

Thus, in STM studies of cuprates we would expect stripe correlations to make an appearance as generalized Friedel oscillations, while quasiparticle-like interference is a distinct phenomenon that could also be present. The observation of a checkerboard pattern with a 4a period about vortex cores in Bi2Sr2CaCu2O8+δ by Hoffman et al. (2002a) is provocative evidence for pinned charge stripes. These results motivated Howald et al. (2003, 2002) to search for similar evidence of stripes in Bi2Sr2CaCu2O8+δ with no applied field. The discussion below is focused on approaches for distinguishing and enhancing modulations due to generalized Friedel oscillations associated with such incipient order. Before continuing, though, we should note that compelling evidence for a “quasiparticle” interference response has been reported by Hoffman et al. (2002b) and by McElroy et al. (2003). Those experimental results are not in direct conflict with those of Howald et al. (2003, 2002); however, there has been some controversy over interpretation, regarding whether the observed spatial modulations in the tunneling conductance are explained entirely by the interference mechanism, or whether interference modulations coexist with generalized Friedel oscillations. We will return to that controversy in a subsection below.

Modulations which reflect the spectrum of elementary excitations are distinguishable (Polkovnikov et al., 2002a) from those related to incipient order in a variety of ways: whereas incipient order produces effects peaked near isolated ordering vectors, \( \mathbf{Q} \), the single particle effects are peaked along extremal curves in \( \mathbf{k} \) space which disperse as a function of \( E \). Peaks in the Fourier transformed LDOS, \( N(k, E) \), produced by incipient order tend to be phase coherent while other features either have a random phase, or a phase that is strongly energy dependent. Indeed, the phase information may be the best way to distinguish the consequences of incipient order from interference effects. Naturally, in the presence of true long range charge order \( N(k, E) \) should exhibit sharp (resolution-limited) peaks which reflect the charge density modulation in real space. This effect has been seen in STM measurements of the quasi two-dimensional incommensurate CDW system 1T-TaS2, lightly doped with Nb, by Dai et al. (1991).

To demonstrate the effect of the phase, assume we have measured \( N_L(r, E) \), the LDOS at a particular bias voltage, \( V = E/e \) on a sample of size \( L \times L \). Its Fourier transform, \( N_L(k, E) \), has an arbitrary \( k \)-dependent but \( E \)-independent phase, \( e^{-ikr_0} \), which depends on the choice of origin of coordinates, \( r_0 \). As discussed above, \( N_L(k, E) \)

\(^{37}\) Hoffman et al., 2002a; Howald et al., 2003; Hudson et al., 1999; Lang et al., 2000; Pan et al., 2000a,b, 2001; Renner et al., 1998; Sprunger et al., 1997; Yazdani et al., 1999)
can be expected to have contributions from incipient order (with wave vector \( \mathbf{Q} \)), and dispersing quasiparticles. Integrating the signal over a finite energy window yields the quantity \( \tilde{N}(\mathbf{k}, E) \), in which the contributions from incipient order from the entire range of integration add constructively, while other features tend to interfere destructively.\(^{38}\) This mode of analysis is particularly useful if an energy window can be found in which none of the dispersing features expected on the basis of band structure considerations have wave vectors equal to the expected ordering wave vector, \( \mathbf{Q} \).

Indeed, Howald et al. (2002) have already demonstrated the first half of this point in STM studies of a very slightly overdoped sample of \( \text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta} \) \((T_c = 86)\). They identified a peak which in \( N(\mathbf{k}, E) \) at approximately \( \mathbf{k} = \pm \mathbf{Q}_{ch} \approx \pm (2\pi/a)(0.25, 0) \) and \( \mathbf{k} = \pm R[\mathbf{Q}_{ch}] = \pm (2\pi/a)(0, 0.25) \) (see Fig. 9), and showed that at these wave numbers the phase of \( N(\mathbf{k}, E) \) is energy independent for \( E \) between 0 and 40 meV, at which point the amplitude crosses through zero \( i.e., \) the sign of the signal changes. The constancy of the phase implies that the location in real-space of the density of states modulation is fixed at all energies, which strongly indicates that it reflects pinned incipient order.

However, as explained above, the existence of incipient order can be magnified if we integrate the LDOS as a function of energy. In Fig. 9(a) we show data from Howald et al. (2003, 2002) in which the values of \( dI/dV \) were obtained by a Fourier transform of a real space image at various voltages, \( V = E/e \), on a patch of surface of size \( L = 160 \text{ Å} \). The origin of coordinates was chosen that \( N_L(\mathbf{k}, E) \) is real and positive for \( \mathbf{k} - \mathbf{Q}_{ch} \). In the figure \( \mathbf{k} \) is taken to lie along the \((1, 0)\) direction and what is shown is the real part of \( N_L(\mathbf{k}, E) \) (the imaginary part if generally small and noisy.) In Fig. 9b, we show the same data averaged over energy from 0 to \( E \), \( I/V \propto \tilde{N}_L(\mathbf{k}, E) \). It is apparent that integration enhances the strength of the peak at \( \mathbf{Q}_{ch} \), and depresses the remaining signal, especially when \( E \) is smaller than the maximum superconducting gap value, \( \Delta_0 \approx 35 \text{ meV} \). Indeed, precisely this same integration technique was used previously by Hoffman et al. (2002a) to enhance the magnetic-field induced “checkerboard” pattern in vortex cores.

1. Differing interpretations of the STM spectra

As mentioned in the introduction, there is controversy in the literature concerning the interpretation of the STM spectra, although there seems to be only relatively minor disagreements concerning the data itself. Specifically,

\(^{38}\) It is important to emphasize that both the phase and amplitude information encoded in \( N_L(\mathbf{k}, E) \) are physically significant. The commonly used power spectrum, \( |N_L(\mathbf{k}, E)|^2 \), will not lead to destructive interference of propagating features when integrated over energies.
Specifically, if we assume that there are well-defined quasiparticles with energy obtained by solving the Bogoliubov-de Gennes equations for a d-wave superconductor, then for each energy $E$ there are eight values $q_i$ (two in the neighborhood of each nodal point) which simultaneously satisfy the equations $\epsilon(q_i) = 0$ (i.e., they lie on the normal state Fermi surface) and $\Delta q_i = E(q_i) = E$ (i.e., the quasiparticle creation energy is $E$). Peaks in $N(k, E)$ will then occur at the various distinct values of $K_{ij} = q_i - q_j$, of which there are 7 distinct (up to symmetry) values. For instance, the two wave vectors with smallest magnitudes are $k_{12} = 2q_x(1, 0)$ and $k_{13} = \sqrt{2}(q_x - q_y) \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ which come from the interference between the state $q_i = (q_x, q_y)$ and, respectively, the states at $q_2 = (-q_x, q_y)$ and $q_3 = (q_y, q_x)$. The positions of the peaks in $N(k, E)$ thought to correspond to $k_{12}$ and $k_{13}$ from the work Hoffman et al. (2002b) are reproduced in Fig. 10. Given the quasiparticle interpretation of the STM spectrum, then from the observed location of any two distinct peak positions $k_{ij}$ at a given energy, it is possible to reconstruct the positions of all eight locations in the Brillouin zone, $q_i$ which give rise to all the expected peaks at this energy. Thus, where 7 distinct peaks are observed (as are reported by McElroy et al. (2003) in some ranges of energy), the quasiparticle spectrum is highly over-constrained, and this provides a very stringent self-consistency check on the quasiparticle interpretation. According to McElroy et al. (2003), the peaks they observe generally pass this consistency check. The experimental case for dispersive features in $N(k, E)$ that are consistent with the quasiparticle interference mechanism is quite persuasive.

The controversy over interpretation revolves around the degree to which quasiparticle interference is sufficient to explain all of the structure in $N(k, E)$. Examining Fig. 10(a), one can see that for energies below ~20 meV, $k_{12}$ lies close to the expected stripe ordering vector, $Q_{ch}$. This is consistent with the experimental measurements and analysis by Howald et al. (2002), who had interpreted this behavior as evidence for stripe order; in contrast, (Hoffman et al., 2002b) argued that the continuity with the dispersive signal favors a single mechanism based on band structure effects.

In weighing these alternatives, it is useful to consider a second consistency check that the quasiparticle picture should satisfy. While, as shown in Eq. (4.1), $N(k, E)$ is actually a measure of a two-particle response function, to the extent that it is dominated by single-particle effects it should be expressible in terms of a convolution of single-particle Green functions, as in Eq. (4.1). Thus, it should be consistent with the behavior of the spectral function measured in ARPES. In evaluating this connection, significant challenges arise with respect to the sufficiency of the quasiparticle interpretation.

In the first place, ARPES consistently reveals strong deviations from a non-interacting line shape. Well below $T_c$, the measured energy distribution curve (EDC) at fixed $q$ on the Fermi surface consists of a dispersing "quasiparticle-like" peak with a small weight, $Z \ll 1$, and a broad, and largely featureless multiparticle continuum which contains most of the spectral weight. Moreover, the quasiparticle peak is always anomalously broad with a distinctly non-Fermi liquid temperature and energy dependence (Gweon et al., 2001; Orgad et al., 2001). If this measured spectral function were used to predict the structure of $N(k, E)$, then the quasiparticle interference features would be weak (Capriotti et al., 2003) (in proportion to $Z^2$), which is something of a surprise given the apparent robustness of the experimental features. One might also expect the interference features to be broadened significantly by the short quasiparticle lifetimes at the probed energies. Perhaps the most important consequence is that the intensities of the structures should have a dramatic temperature dependence, going to zero

![FIG. 10 Dispersion of the peaks (colored symbols) in $N(k, E)$ (from STM measurements on near optimally doped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+x}$) at wavevectors a) $k_1$ (along the (1, 0) direction) and b) $k_2$ (along the (1, 1) direction) from Fig. 4 of Hoffman et al. (2002b). The solid lines are from a global fit to the ARPES spectrum, as described in the text.](image-url)
as $Z$ goes to zero at $T_c$. This provides a crucial test to which we shall return shortly.

In the second place, $\Delta_q$ determined from the dispersion of the quasiparticle-like peak in ARPES differs significantly, especially at low energies, from that obtained by Hoffman et al. (2002b) and McElroy et al. (2003) in STM. This is illustrated in Fig. 10, as well. The solid lines in the figure were computed (using the method of Wang and Lee (2003)) from a global fit to the measured ARPES spectrum, as follows: 1) The location of the Fermi surface, $\epsilon(q) = 0$ was computed from the phenomenological “band structure” that has been obtained (Damascelli et al., 2002; Norman et al., 1994) from a fit to ARPES experiments. 2) The gap function along the Fermi surface was assumed to be of the simple $d$-wave form, $\Delta_q = \Delta_0 \cos(q_a a) - \cos(q_y a)$, a form that has been widely found to fit the measured ARPES spectrum in optimally and overdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, although significant deviations from this form are often seen in underdoped samples (Mesot et al., 1999). The gap observed in ARPES experiments corresponds to $\Delta_0 \sim 35-45$ meV (Ding et al., 1995; Fedorov et al., 1999; Loeser et al., 1997) near optimal doping - for purposes of the figure, we have adopted $\Delta_0 = 40$ meV. The solid black circle in the figure was obtained from the location of the nodal point (which determines the zero energy limit of all dispersion curves) as determined from ARPES experiments; this is a feature of the ARPES spectrum that has been looked at in great detail by several groups (Ding et al., 1997; Valla et al., 1999). There is agreement$^{40}$ with 1% accuracy that the nodal point, where $E(q) = 0$, is $q = (2\pi/a)(0.195, 0.195)$. (Note that the phenomenological band structure was determined from a fit to older, less precise ARPES data, which is why the line in Fig. 10 does not quite approach the correct zero energy limit; manifestly, correcting this discrepancy would only exacerbate the disagreement between STM and ARPES.)

It is clear from Fig. 10, as originally emphasized by Howald et al. (2002), that there are significant discrepancies between the ARPES and STM results, especially below 20 meV. In particular, in the region of $k$ space near the expected stripe-ordering wave vector, $k = Q_{ch} \sim (2\pi/a)(\frac{1}{4}, 0)$, the STM spectrum is considerably less dispersive than would be expected on the basis of the ARPES data; it appears that the peak in $N(k, E)$ largely stops dispersing when it reaches this magic wave vector (see for instance the discussion surrounding Eq.(4.7)).$^{41}$

It is not clear a priori how much significance one should attach to the discrepancies between ARPES data and the quasiparticle interpretation of the STM. Both are highly surface sensitive probes, so there is always the issue of whether either is telling us anything about the bulk properties of the materials, but this worry does not affect the comparison between the two sets of measurements.

It seems significant to us that the most serious quantitative differences between the ARPES dispersion and those inferred from a quasiparticle interpretation of the STM data occur at low energies, and are particularly pronounced for $k$’s near $Q_{ch}$, where effects of incipient stripe order are largest. In this context, it is worth noting that in the STM studies of Howald et al. (2003) shown in Fig. 9, a peak at $k = Q_{ch}$ is seen in all the curves with $E \leq 15$ meV all the way to $E = 0$! (Hoffman et al. (2002b) and McElroy et al. (2003) do not report spectra below about 6 meV. It is possible that, as suggested by Howald et al. (2003), that this difference is a consequence of the stripe signal being washed out due to the existence of many domains in the larger field of view used in the experiment of Hoffman et al. (2002b).)

Moreover, in their STM study of the magnetic-field-induced structure in vortex-core halos, Hoffman et al. (2002a) observed oscillations at precisely these same wave vectors, $k = Q_{ch}$, in the energy-integrated LDOS (in a 12 meV window about 0 voltage). The amplitude of these latter oscillations is very large compared with that of the disorder-induced signal, and the dominant contribution comes from energies around 7 meV.

We believe that the issues of interpretation can be definitively resolved by studies of the temperature and impurity concentration dependence of the signal. Since the contribution to $N(k, E)$ from quasiparticle interference is proportional to $Z^2$, and since $Z$ is observed in ARPES experiments to have a strong temperature dependence and vanish above $T_c$, it should be very easy to quench the interference signal by heating; what is left at temperatures approaching $T_c$ cannot be due to quasiparticle interference. And, of course, above $T_c$ in overdoped samples, and above a suitable pseudogap temperature in underdoped samples, the superconducting gap vanishes, so only the more usual rings can possibly arise from quasiparticle interference. Conversely, as discussed previously, light Zn doping is known to pin stripes in $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$, so it is reasonable to expect the same effect in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$. Thus, stripe-related signals should be strengthened, and made less sensitive to thermal depinning by small concentrations of Zn impurities in the Cu-O planes.

STM studies at elevated temperatures are technically demanding, but recently Vershinin et al. (2003) have achieved atomic scale resolution at temperatures well than a peak! The same is apparent in Figs. 2 and 3 of Wang and Lee (2003).

$^{40}$ We thank P. Johnson for pointing this out to us.

$^{41}$ The same discrepancy between ARPES and STM at low energies can be seen directly from full microscopic calculations of $N(k, E)$ for non-interacting quasiparticles. For instance, $N(k, E)$ was recently computed by Polkovnikov et al. (2002a) for non-interacting quasiparticles scattering from a point impurity; it is apparent from their Fig. 4 that, at energies of 20 meV and below, there is a local minimum in the neighborhood of $k = Q_{ch}$, rather than a peak! The same is apparent in Figs. 2 and 3 of Wang and Lee (2003).
above $T_c$ on a Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ surface. They report that for $T \sim T_c/2$ many of the peaks in $N(k,E)$ have, indeed, been extinguished. However, the low energy portion of the peak at $k_{12}$ (in Fig. 10) remains unattenuated, and even survives, well above $T_c$. This effect clearly cannot be explained by the quasiparticle interference mechanism. An explanation in terms of pinned stripes seems much more plausible (although the persistence of a stripe-induced signal at such an elevated temperature requires further investigation).

One further feature of all the data that has been reported to date is worth commenting on is the large (facets of 2 or 3) differences that are observed (Hoffman et al., 2002b) in the peak intensities at $Q_{\alpha \alpha}$ and $R(Q_{\beta \beta})$. Systematic experiments have not, yet, been conducted to verify whether this effect is real, or an experimental artifact. If it is not an artifact, then this observation is among the first microscopic pieces of evidence of a strong local tendency to stripe orientational (i.e., nematic) order. In addition, such large anisotropies are something that cannot be accounted for in any simple way by quasiparticle interference (nor local checkerboard order). It is important to bear in mind, however, that even if the observed anisotropy reflects nematic order, it is expected to decrease in magnitude (Howald et al., 2002) in direct proportion to $A^{-1/2}$, where $A$ is the area of the field of view, due to the unavoidable domain structure produced by quenched disorder.

C. Detecting nematic order

While stripe order necessarily implies nematic order, the converse is not true. Although nematic order involves the spontaneous breaking of a spatial (point-group) symmetry, when there is no accompanying breaking of translation symmetry, even the identification of the ordered state is somewhat subtle, and this holds doubly for fluctuation effects. Moreover, since quenched disorder is always relevant and results in domain structure, true macroscopic measurements of spontaneous nematic symmetry breaking are not possible. The difficulty of detecting the rotational-symmetry breaking associated with ordered stripes is illustrated by the case of La$_{2-x}$Sr$_x$NiO$_{4+\delta}$, a non-superconducting structural analogue of La$_{2-x}$Sr$_x$CuO$_4$. Even transmission electron microscopy, which is capable of measuring over a fairly small sample area, tends to yield superlattice diffraction peaks for stripe order that reflect the 4-fold symmetry of the NiO$_2$ planes in the tetragonal crystal structure (Chen et al., 1993). Only recently have electron diffraction patterns consistent with the 2-fold symmetry of an ordered stripe domain been reported (Li et al., 2003). Thus, almost all tests of nematic order in solids necessarily involve the observation of an unreasonably large, and strongly temperature dependent anisotropy in the electronic response to a small symmetry breaking field which favors the $x$ direction over the $y$ direction.

1. Transport anisotropies

An example of a system in which this approach has been successfully applied is provided by the 2DEG in quantum Hall devices. While the fractional quantum Hall effect dominates the physics at very high magnetic fields, at intermediate magnetic fields (so that more than one Landau level is occupied), it has recently been discovered (Du et al., 1999; Lilly et al., 1999a) that there are a set of anisotropic states which have been identified (Cooper et al., 2002; Fradkin and Kivelson, 1999; Fradkin et al., 2000) as being quantum Hall nematics, and which can be thought of as melted versions of the long predicted (Fukuyama et al., 1979; Kouklakov et al., 1996; Moeessner and Chalker, 1996) quantum Hall smectic (or stripe) phases. These phases are characterized by a large resistance anisotropy, $\rho_{xx} \gg \rho_{yy}$, which sets very strongly below a characteristic temperature, $T_n \sim 100$ mK. The precise origin of the symmetry breaking field which aligns the nematic domains in these experiments has not been unambiguously determined (Cooper et al., 2001; Zhu et al., 2002). However, by applying (Cooper et al., 2002; Lilly et al., 1999b; Pan et al., 2000c) an in-plane magnetic field, the magnitude of the symmetry breaking field can be varied, and the transition can be significantly rounded (giving evidence (Cooper et al., 2002) that local stripe order persists up to temperatures well in excess of $T_n$), and even the orientation of the nematic order can be switched, resulting in a state with $\rho_{xx} \ll \rho_{yy}$.

Experiments that involve such fine control of an external symmetry-breaking field are considerably harder to carry out in the context of the cuprates. Some of the relevant materials, such as Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and La$_{2-x}$Sr$_x$CuO$_4$ with $x > 0.05$, have an orthorhombic axis at $45^\circ$ to the expected stripe directions (i.e., a nematic phase can be defined in terms of spontaneous breaking of the mirror plane which lies along the orthorhombic a-axis.) In these materials, if one wishes to align the nematic domains, one must apply a suitable external symmetry-breaking field such as a uniaxial strain or in-plane magnetic field; however, such experiments tend to be challenging.

When the principle axes of an orthorhombic phase lie parallel to the expected stripe directions, the orthorhombicity (typically $< 2\%$ in cuprates) plays the role of a small, external symmetry-breaking field. Examples where this occurs are superconducting YBa$_2$Cu$_3$O$_{6+y}$ with $0.35 \leq y \leq 1$ and non-superconducting La$_{2-x}$Sr$_x$CuO$_4$ with $0.02 \leq x < 0.05$. In both of these cases, resistivity anisotropies as large as a factor of 2 have been observed (Ando et al., 2002b) in detwinned single crystals. Moreover, as in the quantum Hall case, this anisotropy is strongly temperature dependent: in La$_{2-x}$Sr$_x$CuO$_4$, $|\rho_{xx}/\rho_{yy} - 1| < 10\%$ for temperatures in excess of $T_n \sim 150$ K. Polarization dependent measurements of the infra-red conductivity on a detwinned $x = 0.03$ crystal of La$_{2-x}$Sr$_x$CuO$_4$ reveal that the frequency
dependence of the conductivity anisotropy has a scale comparable to \( kT \) (Basov, 2002). Large anisotropies in the frequency-dependent conductivity have also been observed (Basov et al., 1995) in \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) and in \( \text{YBa}_2\text{Cu}_4\text{O}_8 \), although in those cases, some part of the conductivity anisotropy must be due directly to the Cu-O chains. Taken together, these various observations are circumstantial evidence of a fair degree of local stripe order, and of the presence of a nematic phase; the trouble being that, without a somewhat quantitative theory (which does not exist), it is hard to say how large an observed resistance anisotropy must be to be accepted as being an “unreasonably large” response to the orthorhombicity, that is, one that can only be understood in terms of the alignment of nematic domains.

More subtle investigations of orientational symmetry breaking can be undertaken by studying the transport in the presence of a magnetic field. A very clever approach along these lines was introduced by Noda et al. (1999). They applied a voltage along one axis of the \( \text{CuO}_2 \) planes and a magnetic field perpendicular to the planes in order to break the 4-fold symmetry of the crystal structure and to obtain evidence of one-dimensional charge transport. In a stripe-ordered state, the geometry used should be sensitive primarily to those domains in which the stripes are aligned parallel to the direction of the applied voltage. For \( x \leq \frac{1}{2} \), they found that, within the stripe-ordered phase, the transverse conductivity tends to zero at low temperature while the longitudinal conductivity remains finite. (This effect has been explained in terms of the electron-hole symmetry of a \( \frac{1}{2} \)-filled charge stripe (Emery et al., 2000; Prelovšek et al., 2001).)

(Ando et al., 1999) have observed a remarkable anisotropy of the resistivity tensor induced by an in-plane magnetic field in non-superconducting \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) with \( y = 0.32 \) and \( y = 0.3 \). (Presumably, these samples are antiferromagnetic.) The results were interpreted as evidence for nematic stripe order. Alternative explanations in terms of anisotropy associated with spin-orbit coupling in the antiferromagnetic phase have also been proposed (Jánossy et al., 2000; Moskvin and Panov, 2002). This example illustrates the difficulty in making a unique association between a bulk anisotropy and microscopic stripe order.

Finally, we note that there is a very direct way to detect nematic order using light scattering. This approach is well known from studies of classical nematic liquid crystals (de Gennes, 1974). Recently Rühhausen and coworkers (Rühhausen et al., 2000; Yoon et al., 2000) have used light scattering techniques to study the behavior of the low-frequency dielectric tensor of the manganite \( \text{Bi}_{1-x}\text{Ca}_x\text{MnO}_3 \) near and below the charge-ordering transition at \( T_{\text{co}} \sim 160K \). In these experiments, a pronounced anisotropy of the dielectric tensor was found with a sharp temperature-dependence near \( T_{\text{co}} \). Raman scattering studies by the same group show that long range positional stripe order sets in only at much lower temperatures. These experiments suggest that this managanite is in a nematic state below the transition at \( T_{\text{co}} \), and long range charge-stripe order occurs at much lower temperatures.

2. Anisotropic diffraction patterns

A more microscopic approach is to directly measure the nematic order parameter, \( Q_\perp \) of Eq. (2.1), by diffraction. This was done (more or less) by Mook et al. (2000), in neutron scattering studies of the magnetic dynamic structure factor of a partially (2 to 1) detwinned sample of \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) with \( y = 0.6 \) and \( T_c \approx 60 K \). No elastic scattering corresponding to actual stripe order was detected; however, well developed structure was observed in the inelastic spectrum at the two-dimensional wave vectors \( Q \approx (0.5 \pm 0.1, 0.5) \) and \( Q \approx (0.5, 0.5 \pm 0.1) \) in units of \( 2\pi/a \). Remarkably, the intensities of the peaks at \( Q \) were found to be about a factor of 2 larger than those of the \( Q \) peaks, consistent with the supposition that, in a single twin domain, the incommensurate inelastic structure is entirely associated with ordering vectors perpendicular to the chain direction, i.e., \( Q \parallel 1 \).

More recently, inelastic neutron scattering studies of Stock et al. (2003) on a nearly single-domain sample of the Ortho-II phase of \( \text{YBa}_2\text{Cu}_3\text{O}_{6.5} \) (Stock et al., 2002) have revealed substantial structure down to the lowest energies in the magnetic structure factor. These spectra are highly anisotropic about the Néel ordering vector: For a scan perpendicular to the chain direction, there is a broad flat-topped peak (reminiscent of earlier results on underdoped YBCO (Sternlieb et al., 1994)) which is strongly suggestive of two barely resolved incommensurate peaks at the expected stripe ordering wavevector. However, a scan along the chain direction reveals a single, sharp peak at the commensurate wave vector \( \pi/a \). Moreover, in the normal state, this structure is observed at all energies below the “resonant peak” energy, \( \omega \approx 25 \text{ meV} \), down to the lowest energies probed.

Although the presence of chains in this material certainly means that there is no symmetry operation that interchanges the \( a \) and \( b \) axes, the copper-oxide planes are nearly tetragonal. Thus, it seems to us that the extreme anisotropy of the inelastic scatterings very strong evidence of a nematic liquid phase in this material. As pointed out by Mook et al. (2000), this conclusion also offers a potential explanation for the observed (Basov et al., 1995) large but nearly temperature independent superfluid anisotropy in the \( a-b \) plane.

Interesting anisotropies have also been observed in optical phonon branches of \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) with \( y = 0.6 \). The identification by Mook et al. (2000) of an anomalous broadening of a bond-bending mode at a wave vector expected for charge-stripe order is potentially the most direct evidence of nematic order. However, conflicting results have been reported by Pintschovius et al. (2002), who have instead observed a zone-boundary softening of the bond-stretching mode propagating along \( b \) but not
for that along a. This latter anisotropy is unlikely to be directly associated with a stripe modulation wave vector since the anomaly occurs along the direction parallel to the Cu-O chains. Of course, this does not necessarily rule out a connection with stripes, as the softening might be associated with anisotropic screening due to charge fluctuations along the stripes.

In an attempt to understand the effects of stripe order on phonons, a neutron scattering study (Tranquada et al., 2002) was recently performed on La$_{1.69}$Sr$_{0.31}$NiO$_4$. Although the charge-ordering wave vector did not play an obvious role, the high-energy bond-stretching mode propagating parallel (and perpendicular) to the stripe modulation exhibited an energy splitting toward the zone boundary, while along the Ni-O bond direction (at 45° to the stripes) a softening from zone center to zone boundary was observed with a magnitude similar to that in the cuprates. A better understanding of the nature of the relevant electron-phonon coupling processes is required to make progress here.

3. STM imaging of nematic order

Because it is a local but spatially resolved probe, STM is actually the optimal probe of nematic order. One way it can be used, which is illustrated in Fig. 11, has been explored by Howald et al. (2003, 2002). What is shown here is a filtered version of \( N(r, E) \) measured on a patch of surface of a very slightly overdoped crystal of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ \( (T_c = 86 \text{ K}) \). Specifically, Howald et al. defined a filtered image

\[
N_f(r, E) = \int d\mathbf{r}' f(\mathbf{r} - \mathbf{r}') N(\mathbf{r}', E) \tag{5.1}
\]

where in the present case, the filter function, \( f \), has been defined so as to accentuate the portions of the signal associated with stripe order

\[
f(\mathbf{r}) \propto \Lambda^2 e^{-r^2/4\Lambda^2} [\cos(\pi x/2a) + \cos(\pi y/2a)]. \tag{5.2}
\]

Clearly, \( f(\mathbf{r}) \rightarrow \delta(\mathbf{r}) \) when \( \Lambda \rightarrow \infty \), while \( N_f = N(Q_{\text{ch}}, E) + N(R[Q_{\text{ch}}], E) + \text{c.c.} \) in the limit \( \Lambda \rightarrow 0 \). For intermediate values of \( \Lambda \), the filtered image shows only that portion of the signal we have associated with pinned stripes. The roughly periodic structure in the image has period \( 4a \). We know independently from the analysis in Section V.B that there is prominent structure in the raw data with this period, but even were there not, the filtering would build in such structure. However, what is clear from the image is that there is a characteristic domain structure, within which the stripes appear to dominantly lie in one direction or the other. The domain size is seen to be on the order of 100 Å \( \sim 25a \), which is large compared to \( \Lambda^{-1} \) and, more importantly, roughly independent (Howald et al., 2003) of the precise value of \( \Lambda \). This domain size is a characteristic correlation length of the pinned nematic order.

FIG. 11 A filtered version of the local density of states map, \( N_f(r, E) \), of the local density of states on a surface patch of a Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ crystal. Here, \( E = 15 \text{ meV} \) and the distances on the x and y axes are measured in angstroms. The filter is defined in Eq. (5.1) with \( \Lambda = (2\pi/15a) \). The arrows point along the directions of the Cu-O bonds.

This particular method of analysis builds directly on the realization of the nematic as a melted stripe-ordered state. Indeed, looking at the figure, one can clearly identify dislocations and disclinations in what looks like a fairly well developed locally ordered stripe array.

More generally, STM could be used to directly measure the two independent components of a suitably defined traceless symmetric tensorial density. The simplest such quantities are

\[
Q_{xx}(r, E) = [\partial_x^2 - \partial_y^2]N(r, E) \tag{5.3}
\]

\[
Q_{xy}(r, E) = 2\partial_x\partial_y N(r, E)
\]

Of course, these quantities, like the local density of states itself, will typically have features which reflect the interference between elementary excitations, and other extraneous information. To obtain a better view of the long wave-length nematic correlations, we should again integrate these densities over a suitable energy interval, \( \Omega \), and filter out the short wavelength components:

\[
\hat{Q}_f(r) = \int_0^\Omega dE \frac{\Omega}{\delta} \int d\mathbf{r}' f(\mathbf{r} - \mathbf{r}') Q(\mathbf{r}', E) \tag{5.4}
\]

where \( f \) might be a Gaussian filter, as in Eq. (5.2), but without the cosine factors. A map of \( \hat{Q}_f \) should produce a domain structure, similar to that shown in Fig. 11, but without all the short wavelength detail. Where the domain size, \( L_N \), is large compared to \( a \), the resulting
picture should look qualitatively the same independent of
the range over which the signal is coarse-grained, so long
as \( L_\text{N} \gg \Lambda^{-1} \gg a \). The above procedure should work
well if there are no other long-wavelength features in the
data. Unfortunately, for BSCCO, the inhomogeneities in
the gap structure (Howald et al., 2002; Lang et al., 2000)
harmer such a procedure.

D. “1/8” anomaly

Many members of the lanthanum cuprate family of
high-temperature superconductors exhibit strong singular-
larities in the doping dependence of various interesting
low-temperature properties at \( x = 1/8 \); together, these
phenomena are referred to as the “1/8 anomaly.” For
instance, in La\(_{2-x}\)Ba\(_x\)CuO\(_4\) and La\(_{1.6-x}\)Nd\(_{0.4}\)Sr\(_x\)CuO\(_4\),
there is (Crawford et al., 1991; Moodenbaugh et al.,
1988) a deep minimum in \( T_c(x) \); in La\(_{2-x}\)Ba\(_x\)CuO\(_4\) and
La\(_{2-x}\)Sr\(_x\)CuO\(_4\) there is a pronounced (Crawford et al.,
1990; Zhao et al., 1997) maximum in \( \alpha(x) \), the isotope
exponent \( \alpha \equiv \frac{d \log(T_c)}{d \log(M)} \); and in La\(_{2-x}\)Sr\(_x\)CuO\(_4\)
there is a pronounced (Panagopoulos et al., 2002) mini-
num in the superfluid density, \( n_s(x) \). One of the central
inferences drawn by Tranquada et al. (1995b) following
the discovery of stripe order in La\(_{1.6-x}\)Nd\(_{0.4}\)Sr\(_x\)CuO\(_4\) is
that this 1/8 anomaly is associated with a commensurate
lock-in of the stripe structure. At \( x = 1/8 \) the preferred
spacing between charge stripes is 4 lattice constants, so
there is an additional commensuration energy which sta-
bilizes stripe order at this particular hole density.

While it is possible to imagine other forms of CDW
order which would similarly be stabilized at \( x = 1/8 \),
there can be no doubt that in this family of materials
the 1/8 effect is associated with stripe order. This has
been confirmed, for example, in La\(_{2-x}\)Sr\(_x\)CuO\(_4\) where
quasi-elastic magnetic scattering from spin-stripe order
has been detected (Kimura et al., 1999; Suzuki et al.,
1998) for \( x \) in the neighborhood of 1/8; in low-energy in-
elastic measurements, (Yamada et al., 1998) they have
shown that the magnetic peak width is most narrow at \( x = 1/8 \).
Correspondingly, slow (probably glassy) spin fluctuations
have been detected by \( \mu \text{SR} \) in the same material with a
somewhat arbitrarily defined onset temperature, \( T_g(x) \),
which has (Panagopoulos et al., 2002) a pronounced peak
at \( x = 1/8 \). The fact that quasi-elastic magnetic
order as detected by neutron scattering onsets at a con-
siderably higher temperature than that detected by \( \mu \text{SR} \)
is clearly a consequence of the inevitable glassiness of a
density-wave transition in the presence of quenched
order; it reflects the differences in the time scales of
the two probes, not the presence of two distinct or-
dering phenomena. More recently, new experiments on
La\(_{2-x}\)Sr\(_x\)CuO\(_4\) as a function of \( x \) show pronounced sin-
ularities in the \( x \) dependence of the \( c \)-axis Josephson
plasma edge (Basov, 2002) and in the low-temperature
thermal conductivity (Takeya et al., 2002); these effects
can be interpreted straightforwardly in terms of a peak in
the stability of the charge-stripe order at \( x = 1/8 \). Fur-
thermore, charge-stripe order has now been detected di-
rectly by neutron diffraction in La\(_{1.875}\)Ba\(_{0.125-x}\)Sr\(_x\)CuO\(_4\)
(Fujita et al., 2002).

The large drop in \( T_c \) at \( x = 1/8 \) is not found in
La\(_{2-x}\)Sr\(_x\)CuO\(_4\) but is not observed in La\(_{2-x}\)Sr\(_x\)CuO\(_4\); how-
ever, such a dip in the doping dependence of \( T_c \) can be
induced in La\(_{2-x}\)Sr\(_x\)CuO\(_4\) (centered at \( x = 0.15 \)) by
substitution of 1% Zn for Cu, as shown some time ago by
Koike et al. (1992) Zn substitution enhances local mag-
netic order at low temperature near the dip minimum,
as detected by \( \mu \text{SR} \) (Panagopoulos et al., 2002; Watan-
abe et al., 2002). Given the clear association between
the 1/8 anomaly and stripe order in La\(_{2-x}\)Sr\(_x\)CuO\(_4\), an
indirect method to look for the presence of local stripe
order in other families of cuprate superconductors is to
search for whether a 1/8 anomaly can be induced by Zn
substitution. In the case of YBa\(_2\)Cu\(_3\)O\(_6+y\), already exists
in the literature strong evidence from the work of Tallon
and collaborators (Tallon et al., 1995) that the “60-K plate-
au” in the \( y \) dependence of \( T_c \) is not, primarily, a
reflection of some form of oxygen ordering in the chain
layers, as is commonly assumed, but is rather a barely
resolved 1/8 anomaly: this conclusion is also supported
by a Sr-substitution study (Akoshima and Koike, 1998)
and by a recent study of transport properties (Ando and
Segawa, 2002). To further test this idea, Koike and col-
leagues (Akoshima and Koike, 1998; Akoshima et al.,
2000; Watanabe et al., 2000) studied the doping depen-
dence of \( T_c \), and of \( T_g \) measured by \( \mu \text{SR} \), in lightly Zn
doped YBa\(_2\)Cu\(_3\)O\(_6+y\) and Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_8+\delta\). They have
found that there is some tendency for light Zn dopping
to produce a peak in \( T_c(x) \) and a more impressive
peak in \( T_g(x) \) at \( x \approx 1/8 \) in both YBa\(_2\)Cu\(_3\)O\(_6+y\) and
Bi\(_2\)Sr\(_2\)CaCu\(_2\)O\(_8+\delta\). Related results have been reported
(Ando et al., 2002a; Yang et al., 2000) for the single-layer
system Bi\(_2\)Sr\(_2\)La\(_x\)CuO\(_8+\delta\). While it is clear that more
work is needed to test the connection between the “1/8
anomaly” and stripe pinning in these various systems, we
consider this a promising approach to the problem, as it
permits evidence of local order to be obtained using a
variety of probes which can be applied in materials for
which large crystals, and/or easily cleaved surfaces are
not easily obtained.

E. Other probes

It is clear that the most direct evidence for stripes
comes from techniques that can provide images of
charge, spin, and/or lattice modulations in real space
(STM) or in reciprocal space (diffraction techniques).
We have already discussed neutron and X-ray diffrac-
tion, but explicit mention should also be made of
transmission electron microscopy (TEM). The charge
stripes in La\(_{2-x}\)Sr\(_x\)NiO\(_4+\delta\) were first detected by TEM
(Chen et al., 1993), and a recent study has provided
high-resolution TEM images of local stripe order in

\[
\begin{align*}
\frac{\mu \text{SR}(x)}{\mu \text{SR}(x) = 1} \approx \frac{x}{x} & \quad (\text{Panagopoulos et al.}) \\
\text{La}_{1.875} \text{Ba}_{0.125-x} \text{Sr}_x \text{CuO}_4 & \quad (\text{Fujita et al.}) \\
\text{La}_{2-x} \text{Sr}_x \text{CuO}_4 & \quad \text{not observed} \quad (\text{Lu et al.}) \\
\text{La}_{2-x} \text{Sr}_x \text{CuO}_4 & \quad \text{induced} \quad (\text{Koike et al.}) \\
\text{YBa}_2 \text{Cu}_3 \text{O}_6+y & \quad \text{Sr-substitution} \quad (\text{Akoshima and Koike}) \\
\text{Bi}_2 \text{Sr}_2 \text{CaCu}_2 \text{O}_8+\delta & \quad \text{Zn substitution} \quad (\text{Ando and Segawa})
\end{align*}
\]
La_{1.725}S\textsubscript{0.275}NiO_{4} (Li et al., 2003). So far, TEM studies have not provided positive evidence for stripes in any cuprates, but, though challenging, it should be possible to do so.

Less direct but extremely valuable information comes from techniques that are sensitive to local order. We have already mentioned evidence for local, static hyperfine fields and slowly fluctuating hyperfine fields obtained by \(\mu\)SR. Besides providing a practical measure of local magnetic order as a function of doping (Klauss et al., 2000; Niedermayer et al., 1998), \(\mu\)SR can detect the distribution of local hyperfine fields in a sample with relatively uniform order (Nachumi et al., 1998), as well as being able to detect inhomogeneous magnetic or superconducting order (Savici et al., 2002). Related information can be obtained by NMR and NQR techniques, where the latter also provides information on the distribution of local electric-field gradients. Some of the first evidence for spatial inhomogeneity of magnetism in lightly-doped \(La_{2-x}Sr_xCuO_4\), consistent with stripe-like behavior, was obtained in a La NMR experiment by Cho et al. (1992). Later experiments (Goto et al., 1997, 1994; Oh sugi et al., 1994; Tou et al., 1993) (including La and Cu NMR and NQR) provided evidence for local magnetic order near \(x = 1/8\) in \(La_{2-x}Sr_xCuO_4\) and \(La_{2-x}Ba_xCuO_4\).

The work by Imai’s group (Hunt et al., 1999) suggesting that local stripe order in variants of \(La_{2-x}Sr_xCuO_4\) could be detected through the “wipeout effect” of Cu NQR has motivated a considerable number of further studies using Cu and La NMR and NQR (Curro et al., 2000a; Hunt et al., 2001; Julien et al., 1999, 2001; Simović et al., 2002; Singer et al., 1999; Suh et al., 1999, 2000; Teitelbaum et al., 2001, 2000). Although there has been some controversy over details of the interpretation (and in certain cases, there may be differences between samples), it is now generally agreed that NQR and NMR are sensitive to the onset of slow spin fluctuations whose appearance tends to correlate with the onset of local charge stripe pinning as determined by diffraction. In particular, the glassy nature of the ordering has been investigated. In the case of Eu-doped \(La_{2-x}Sr_xCuO_4\), related information has been obtained by electron-spin resonance detected from a very low density of Gd impurities (Kataev et al., 1997, 1998). A recent Cu NQR and Y NMR study (Singer and Imai, 2002) has shown that similar signatures are observed in Ca-doped \(YBa_2Cu_3O_6\), consistent with the \(\mu\)SR study of (Niedermayer et al., 1998), and suggestive of the presence of pinned stripes in that system at low temperatures. Direct evidence for local spatial inhomogeneities in \(La_{2-x}Sr_xCuO_4\) has been obtained from studies of NMR line broadening (Haase et al., 2000) and frequency-dependent NQR relaxation rates (Singer et al., 2002).

Very recently, a fascinating study (Haase and Slichter, 2003) of the NMR/NQR spectra in \(YBa_2Cu_3O_{6+y}\) has found evidence of two distinct planar O environments in the unit cell, but only one Cu environment. This finding is not consistent with any form of translation-symmetry breaking. It is, however, suggestive of substantial nematic order, in which the O sites midway between two copper sites in one direction (for instance, the chain direction) has substantially larger hole-density than the O’s on the bonds in the perpendicular direction. Of course, because the crystal is orthorhombic, this does not truly imply any symmetry breaking, but it should be possible, in principle, to establish whether the magnitude of the effect is out of proportion with the small distortions produced by the orthorhombicity.

Several recent NMR studies (Curro et al., 2000b; Kakuyanagi et al., 2002a; Mitrović et al., 2001, 2002) have exploited the magnetic-field dependence of the technique to probe the spatial variation of nuclear spin-lattice relaxation rates in the vortex lattice state of cuprate superconductors. Vortex cores are regions of suppressed superconductivity, so if there is a competing spin-stripe ordered phase, it will be enhanced and pinned (Demler et al., 2001; Kivelson et al., 2002; Zhang et al., 2002) in the neighborhood of the vortex core. Indeed, \(^{17}\)O NMR measurements on \(YBa_2Cu_{3}O_{6+y}\) and \(YBa_2Cu_{3}O_8\) indicate antiferromagnetic-like spin fluctuations and a reduced density of states associated with the vortex cores (Kakuyanagi et al., 2002a; Mitrović et al., 2002). Similar results, along with evidence of static spin ordering in the vortex cores at low temperatures, have been reported (Kakuyanagi et al. 2002b) in \(T_{2}Ba_{2}Cu_{3}O_{6+\delta}\). A \(\mu\)SR study of the details of the magnetic-field distribution in \(YBa_2Cu_3O_{6.5}\) suggests the presence of a small local hyperfine field near vortex cores (Miller et al., 2002). For completeness, we note that while one neutron scattering study has indicated very weak, field-induced, elastic antiferromagnetic scattering in superconducting \(YBa_2Cu_3O_{6+y}\) (Vaknin et al., 2000), two other studies, focusing on the effect of a magnetic field on inelastic scattering, found no low frequency enhancement (Bourges et al., 1997a; Dai et al., 2000).

It has been demonstrated that stripe order has an impact on phonon heat transport (Hess et al., 1999). The thermal conductivity increases slightly on cooling through the charge-ordering transition, exhibiting a normal peak at \(\sim 25\) K. The suppression of the latter peak in superconducting \(La_{2-x}Sr_xCuO_4\) has been attributed to the scattering of phonons by fluctuating stripes (Baberski et al., 1998). The doping dependence of the thermal conductivity measured in \(YBa_2Cu_3O_{6+y}\) and in mercury-cuprates has been interpreted as evidence for a 1/8 anomaly in those materials (Cohn et al., 1999). An enhancement of the thermal conductivity in \(La_{1.88}Ba_{0.12}CuO_4\) was originally noted by Sera et al. (1990)

Indirect evidence of some form of local charge order can also be gleaned from the response of a system to electromagnetic radiation. For instance, Raman scattering

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42 For a discussion of the electromagnetic signatures of CDW order,
(Blumberg et al., 2002) was recently used to reveal charge ordering, and more importantly (for present purposes) to detect the effect of collective CDW motion at higher temperatures in \( \text{Sr}_2\text{Cu}_2\text{O}_3 \), a ladder system with a local electronic structure very similar to that of the cuprate superconductors. Careful measurements of the optical response of optimally doped \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) have revealed (Bernhard et al., 2001; Homes et al., 2000) infrared active phonons with oscillator strength comparable to those in the undoped insulating compound. This has plausibly interpreted as giving evidence of “fluctuating charge inhomogeneities” in the copper-oxide planes - the point being that for the phonons to be unscreened at that frequency, their local environment on some scale must be insulating.

ARPES experiments can also be interpreted as giving indirect evidence of local stripe order. At an empirical level, this can be done by looking (Zhou et al., 1999, 2001) at the evolution of the ARPES spectra in a sequence of materials in the \( \text{La}_2\text{CuO}_4 \) family in which various types of stripe order have been detected; this provides a basis for identifying similar features in the ARPES spectrum of materials in which stripe order has not been established. For instance, stripe long-range order produces a tendency for the Fermi surface in the antinodal region of the Brillouin zone to be flat (one-dimensional), and to suppress spectral weight in the nodal region. Some of these same features, especially the differing impact on the nodal and antinodal regions, are induced by Zn doping of \( \text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+y} \), as well, plausibly reflecting (White et al., 1999) the tendency of Zn to pin local stripe order. More broadly, many of the most striking features of the observed (Campuzano et al., 2003; Damascelli et al., 2002; Fedorov et al., 1999; Ino et al., 1999; Valla et al., 1999) ARPES spectra can be naturally interpreted in terms of an underlying quasi one-dimensional electronic structure (Carlson et al., 2003). Examples of this are the disappearance of the the coherent quasiparticle peak (Carlson et al., 2001) upon heating above \( T_c \), the dichotomy between the widths of the peaks as a function of momentum and energy in the normal state spectrum (Gweon et al., 2001; Orgad et al., 2001), and the general structure of the spectra and the evolution of the Fermi surface with doping (Eroles et al., 2001; Granath et al., 2001, 2002; Ichioka and Machida, 1999; Salkola et al., 1996; Zachar et al., 2000, 2001).

Finally, several probes have been used to search for the inhomogeneous distribution of bond-lengths expected from the lattice response to local charge stripe order. Pair-distribution-function (PDF) analysis of scattering data and extended X-ray absorption fine structure (EXAFS) spectroscopy can detect instantaneous distributions of nearest-neighbor bond lengths. Both techniques have the potential to provide indirect evidence of dynamical, as well as static, stripes; however, whether they have sufficient sensitivity to detect the very small bond-length variations associated with charge modulation in the cuprates remains a topic of some contention. (Božin et al., 2000) reported a doping-dependent broadening of the in-plane Cu-O bond-length distribution in \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) detected by PDF analysis of neutron powder diffraction data. While the reported doping dependence is intriguing, the maximum enhancement of the bond-length spread is much too large to be compatible with the Debye-Waller factors measured in a single-crystal neutron diffraction experiment on \( \text{La}_{1.85}\text{Sr}_{0.15}\text{CuO}_4 \) by Braden et al. (2001) A much more extreme discrepancy occurs with the EXAFS studies of \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) (and other cuprates) by Bianconi’s group (Bianconi et al., 1996; Lanzara et al., 1996; Saini et al., 1997). They report a low-temperature splitting of the in-plane Cu-O bond lengths of 0.08 Å, which is not only incompatible with a large number of diffraction studies, but is also inconsistent with EXAFS analysis performed by other groups (Haskel et al., 2000; Niemöller et al., 1998). Large atomic mean-square displacements detected in \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) by ion channeling (Sharma et al., 2000) have been attributed to transitions associated with dynamic stripes, but the magnitude of the displacements and the temperature dependence are difficult to reconcile with other experiments.

In short, these various local measures of the distribution of lattice displacements are, in principle, a very good way to look for local charge-ordering tendencies, but it is important to reconcile the results obtained with different techniques. Where apparent inconsistencies exist, they must be resolved before unambiguous inferences can be made.

VI. WEAK AND STRONG COUPLING PERSPECTIVES

Order in simple metals is typically thought of as arising from a Fermi surface instability caused by the weak residual interactions between the lowest energy quasiparticles. Specifically, density wave order occurs when the band structure is such that there are nearly nested segments of the Fermi surface. Thus, even at temperatures above the ordering temperature, or if the interactions are slightly weaker than the critical strength needed for ordering, there will often be structure in the appropriate dynamical structure factors due to this near nesting which in some sense reflect the fact that the metal is close to an ordered state. In this section, we will analyze the connection between this weak-coupling Fermi-liquid theoretical, as well as static, stripes; however, whether they have sufficient sensitivity to detect the very small bond-length variations associated with charge modulation in the cuprates remains a topic of some contention. (Božin et al., 2000) reported a doping-dependent broadening of the in-plane Cu-O bond-length distribution in \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) detected by PDF analysis of neutron powder diffraction data. While the reported doping dependence is intriguing, the maximum enhancement of the bond-length spread is much too large to be compatible with the Debye-Waller factors measured in a single-crystal neutron diffraction experiment on \( \text{La}_{1.85}\text{Sr}_{0.15}\text{CuO}_4 \) by Braden et al. (2001) A much more extreme discrepancy occurs with the EXAFS studies of \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) (and other cuprates) by Bianconi’s group (Bianconi et al., 1996; Lanzara et al., 1996; Saini et al., 1997). They report a low-temperature splitting of the in-plane Cu-O bond lengths of 0.08 Å, which is not only incompatible with a large number of diffraction studies, but is also inconsistent with EXAFS analysis performed by other groups (Haskel et al., 2000; Niemöller et al., 1998). Large atomic mean-square displacements detected in \( \text{YBa}_2\text{Cu}_3\text{O}_{6+y} \) by ion channeling (Sharma et al., 2000) have been attributed to transitions associated with dynamic stripes, but the magnitude of the displacements and the temperature dependence are difficult to reconcile with other experiments.

In short, these various local measures of the distribution of lattice displacements are, in principle, a very good way to look for local charge-ordering tendencies, but it is important to reconcile the results obtained with different techniques. Where apparent inconsistencies exist, they must be resolved before unambiguous inferences can be made.
complete understanding of the effects of weak residual interactions on the properties of a well formed Fermi liquid in a simple metal. However, only in special circumstances is a comparably sound and complete understanding available in a strongly interacting system. For instance, in the weak coupling limit, considerable useful information about the quasiparticle spectrum can be deduced directly from transport data, because Boltzmann transport theory relates the two properties in a precise manner. The theory of transport in a strongly interacting system is, in our opinion, not established. The upshot of this is that it is much easier to point to experiments that show that simple, weak coupling ideas cannot be safely applied in many highly correlated materials, than to point to experiments that show that strong coupling notions can be.

A. Distinctions in principle

1. Thermodynamic distinctions

As mentioned in the introduction, stripe-ordered states are ultimately defined by a specific set of broken symmetries. From this viewpoint, there is no distinction between the strong coupling limit, in which spin stripes can be viewed as micro phase separation of charges into an array of rivers with strips of nearly insulating antiferromagnet between them, and the weak-coupling Hartree-Fock description of a SDW phase which opens small gaps on various nearly nested segments of the Fermi surface. Indeed, in many circumstances, the two limits are adiabatically connected as a function of the interaction strength. However, this is not guaranteed. For instance, a stripe-ordered state can be either metallic or insulating. At \( T = 0 \), metallic and insulating stripes are thermodynamically distinct states of matter.

There is a more subtle possible distinction possible between metallic spin-stripe states in the weak and strong coupling limits. Certainly, in the weak-coupling limit, the ungapped portions of the Fermi surface support well-defined quasiparticles, so the system remains a Fermi liquid unless, at still lower temperatures, it suffers an additional ordering transition, say to a superconducting state. On the other hand, in the strong coupling limit, the metallic behavior in the stripe ordered phase may or may not be well characterized by Fermi liquid power laws. It is now established that non Fermi liquid power laws can, in principle, exist in more than one dimension.\(^{43}\)

The character of charge-stripe order tends to more easily distinguish between strong and weak coupling limits. For repulsive interactions, weak coupling Hartree-Fock theories always produce charge order which is parasitic on the fundamental spin order. In Landau theory, there is a cubic coupling between CDW and SDW order which, whenever the primary order parameter is the SDW order, leads to CDW order which onsets at the same \( T_c \), but with a strength proportional to the square of the SDW order: \( \langle \rho_2 \rangle \propto |S_Q|^2 \sim (T_c - T)^{2\beta} \). By contrast, in a strong coupling picture of micro-phase separation, if the spin-ordering is only triggered when the holes agglomerate into stripes, one would typically expect the stripe ordering transition to be first order, or for the charge ordering to precede spin ordering. Thus, the measured sequence of transitions provide sharp criteria that can be used to discriminate between weak and strong coupling.

In the thermal or quantum disordered phase proximate to a stripe ordered state, sharp thermodynamic distinctions between the strong and weak coupling limits are harder to draw. The only possible thermodynamic distinction is, again, that in the weak-coupling limit the disordered state is necessarily a Fermi liquid, while in strong coupling it may either be a Fermi liquid or a non Fermi liquid.

2. Quantitative distinctions

Even where no sharp thermodynamic distinctions exist, there are very many clear physical differences between the weak and strong coupling limits. Of these, the most obvious is the relative importance of electron quasiparticles with a sharply defined Fermi surface. In the weak coupling limit, the quasiparticle is the essential building block in terms of which all other properties are derived. In the strong coupling limit, by contrast, even where it might be the case that well-defined quasiparticles (with very small weight, \( Z \)) are recovered in the strict \( T \to 0 \) limit, such coherence may require such low temperatures as to be only of academic interest. Thus, on a practical level, “effective non-Fermi liquids,” in which the majority of the electronic excitations in the experimentally relevant ranges of temperature, energy, and wave-number are not well defined quasiparticles, cannot be sensibly treated in terms of a weak coupling approach. Conversely, where well defined quasiparticles are everywhere manifest, there is an a priori reason to prefer a weak coupling viewpoint.

Another set of important of quantitative distinctions concern the \( \mathbf{k} \) and \( \omega \) dependences of the magnetic dynamical structure factor, \( S(\mathbf{k}, \omega) \). In a one band-model of non-interacting electrons, the integral of \( S \) over all \( \mathbf{k} \) and \( \omega \) is \((1/8)(1-x^2)\) while in the large interaction (local moment) limit it is \((1/4)(1-x)\), which is not dramatically different. However, in weak coupling, the integrated intensity comes from a very broad range of energies, of order \( E_F \), and momenta, of order \( 2k_F \), while for an antiferromagnet the integral is dominated by energies less than the exchange energy, \( J \), and is highly peaked near the magnetic ordering vector, whether or not the system is actually ordered. Put another way, in weak coupling the structure factor is dominated by the particle hole con-

\(^{43}\)In particular the existence of a non Fermi liquid “sliding” phase in a stripe ordered system has been shown (Emery et al., 2000; Vishwanath and Carpentier, 2001) for a class of interacting Luttinger liquids.
the "distance" of the system from the nearest quantum
titude of the spin, but more physically should be viewed as
scales for magnetic excitations.

A wave can be studied to get an idea of the relevant energy
disordered state, so long as \( T \) is small. For momenta \( q > 2k_F \) it is

\[
S(q, \omega) = \frac{k_F^2}{2\pi^2 v_F q} \sqrt{1 - \left( \frac{q}{2k_F} - \frac{\omega}{v_F q} \right)^2}
\]

when \( v_F q \left( \frac{2\pi}{2k_F} - 1 \right) < \omega < v_F q \left( \frac{2\pi}{2k_F} + 1 \right) \) and zero otherwise. (The expression for \( q < 2k_F \) is somewhat more complicated.) Clearly, the spectral weight is distributed smoothly over the entire support of this particle-hole continuum, \( \ldots \), over a range of momenta of order \( 2k_F \) and of energies of order the band width. From the weak coupling perspective the chief qualitative effect of interactions is to produce (quasi)bound states that manifest themselves as peaks at or below threshold of the continuum. In the broken-symmetry phase these bound states are Goldstone modes, which descend down to zero energy and lose weight to the new Bragg peaks. What is important to note here is that even in the broken-symmetry phase, most of the spectral weight remains in the incoherent background spread more or less uniformly over energies of order band width.

In contrast, consider the zero temperature structure factor of an insulating antiferromagnet computed to first order in a \( 1/S \) expansion, where \( S \) is formally the magnitude of the spin, but more physically should be viewed as the “distance” of the system from the nearest quantum disordered phase. If, on a square lattice

\[
S_{ij}(q, \omega) = S(S - A) \langle \hat{m}_i \hat{m}_j \rangle \delta(q - Q_{AF})
\]

where \( \omega_k = 2SJ \sqrt{4 - (\cos k_x + \cos k_y)^2} \) is the spin wave dispersion (which depends on the lattice and the details of the exchange interactions), \( \hat{m} \) is a unit vector parallel to the magnetization, and \( A = 0.3932 + \ldots \). (Higher order terms in \( 1/S \) typically produce finite lifetimes for all but the lowest energy spin waves, and also give rise to a multi spin wave continuum.) As promised, the basic energy scale is set by \( J \), and a large portion of the total spectral weight is found in a small region of \( \mathbf{k} \) space around \( k = \pi \).

Note that the energy scale \( J \) can be deduced from the value of the spin wave energy near the zone boundary, \( \omega_{k_0} = 4SJ = 2J \), for \( k_0 = (\pi, 0) \). Even in the thermally disordered state, so long as \( T \ll J \), the zone edge spin wave can be studied to get an idea of the relevant energy scales for magnetic excitations.

Finally, distinctions can be drawn based on the sensitivity of the various signatures of order or proximate order to various weak perturbations: weak disorder, as we have discussed extensively above, can serve to pin fluctuating order, and hence enhance the small \( \omega \) portion of \( S(q, \omega) \). However, in weak coupling, the principal effect of weak disorder is to scatter the quasiparticles, and thus broaden any Fermi surface features (\( \ldots \), e.g. nested segments) which might give rise to peaks at particular \( q \)'s in \( S(q, \omega) \). Thus, in weak coupling, we expect weak quenched disorder to suppress, rather than enhance low frequency signatures of stripe order or incipient stripe order. Distinct responses to an applied magnetic field can also be expected to differentiate the weak and strong coupling limits. In particular, where the zero-field ground state is superconducting, induced order in vortex cores occurs under broad circumstances in the strong-coupling limit where there are competing orders; this sort of phenomena is generally much less prominent in the weak coupling limit. These issues are discussed in a number of good recent reviews - see footnotes 23 and 24.

3. Intermediate coupling

It is worth remembering that in many materials, the interaction strength is comparable to the Fermi energy, \( \ldots \), stability of matter (or some memory of the Virial theorem) conspires to place systems in the awkward regime of intermediate coupling. Here, neither the weak nor strong coupling approaches are well justified, and one is typically forced to extrapolate results beyond their regime of validity. In this case, it is generally sensible to study the problem from both the strong and weak coupling perspectives. Some features (such as those that are most sensitive to the presence of a particle-hole continuum) may be best viewed from the weak coupling perspective, while others (such as the role of collective modes) may be better viewed from strong coupling.

One traditional way to understanding this regime is to include the effects of interactions in the context of the random phase approximation (RPA). For the complex spin susceptibility \( \chi \), one writes

\[
\chi(k, \omega) = \frac{\chi_0(k, \omega)}{1 - Uk\chi_0(k, \omega)},
\]

where \( \chi_0 \) is the susceptibility for noninteracting electrons, and \( U \) characterizes the interaction. In the case of dominantly forward-scattering interactions (such as the long-range piece of the Coulomb interaction), the RPA summation of bubble diagrams is justified even when the interactions are relatively strong. For the case of short-range interactions, no such systematic justification exists to the best of our knowledge. However, the results obtained from RPA are explicit and often intuitively appealing. It is certainly reasonable, at least on a phenomenological level, to compare the results of an RPA treatment with experiment to determine whether the physics
is simply connected to the weakly-interacting limit, or on the other hand whether the interactions produce qualitatively new phenomena.

One of the first RPA calculations was performed by Bulut et al. (1990) to model the antiferromagnetic spin fluctuations detected by NMR; for the interaction, they selected $U_{\mathbf{k}} = U$, the on-site Coulomb repulsion. Littlewood et al. (1993) found significant structure in the bare $\chi_0(k, \omega)$ from nearly-nested features of the Fermi surface. By suitable choice of the chemical potential, and inclusion of band narrowing by a factor of 2–4 compared to band structure calculations, they were able to find reasonable agreement with the early inelastic neutron scattering measurements of low-energy magnetic excitations in La$_{2-x}$Sr$_x$CuO$_4$ (Cheong et al., 1991). Calculations for YBa$_2$Cu$_3$O$_{6+y}$ (Monthoux and Pines, 1993; Si et al., 1993) have typically found it necessary to employ the RPA expression with $U_{\mathbf{k}} = -J [\cos(k_x a) + \cos(k_y a)]$ (or a similar form) in order to obtain reasonable consistency with experiment.44 Several recent calculations (Brinckmann and Lee, 1999; Norman, 2000; Onufrieva and Pfleuty, 2002) have addressed the resonance peak in YBa$_2$Cu$_3$O$_{6+y}$ and the downward-dispersing excitations that appear at slightly lower energies (Arai et al., 1999; Bourges et al., 2000; Ito et al., 2002).

In addition, we note that the lack of dispersion of the spin gap that develops below $T_c$ in La$_{2-x}$Sr$_x$CuO$_4$ (Lake et al., 1999), together with the enhancement of intensity above the gap (Mason et al., 1996), have been interpreted in terms of the spin response of the electrons that participate in the superconducting state (Lake et al., 1999; Mason et al., 1996; Morr and Pines, 2000).

### B. Experimental evidence supporting a strong-coupling perspective in the cuprates

From a purely theoretical perspective, it may not be possible to determine a priori whether there is an advantage to either a weak- or strong-coupling approach; however, we believe that numerous experimental results point decisively towards the strong-coupling picture. Many of the relevant results involve measurements of magnetic correlations, and we have collected selected experimental quantities in Table I. (Some explanations of how the parameter values are taken from experiment are given in App. B.) Below we highlight some of the important observations, and explain their significance.

- Magnetism in undoped cuprates is quantitatively consistent with super-exchange between local magnetic moments on copper ions.

The undoped parent compounds, such as La$_2$CuO$_4$ and YBa$_2$Cu$_3$O$_6$ are antiferromagnetic insulators with a charge-transfer gap of ~2 eV (Kastner et al., 1998). The ordered magnetic moments are consistent with one unpaired spin per planar Cu ion, after zero-point spin fluctuations, given accurately by spin-wave theory (Igarashi, 1992), are taken into account. The effective superexchange energy, $J_{\text{eff}}$ ~ 0.1 eV, determined from measurements of the spin-wave dispersion, has been calculated from ab initio cluster models (Muñoz et al., 2000; Van Oosten et al., 1996), consistent with the local superexchange mechanism.

If the antiferromagnetic order corresponded to a spin-density wave due to Fermi-surface nesting, one would expect the magnetic correlations and the optical gap to disappear at the Néel temperature. Neither of these things happens; these materials remain correlated insulators in the disordered state, and the spin correlations are well described by a nonlinear sigma model (Chakravarty et al., 1989) with parameters derived from the ordered state (Kastner et al., 1998).45

- With light doping, local magnetic moments change little, even though long-range Néel order is destroyed.

In a μSR study of lightly-doped La$_{2-x}$Sr$_x$CuO$_4$ and Y$_{1-x}$Ca$_x$Ba$_2$Cu$_3$O$_6$, Niedermayer et al. (1998) showed that there is essentially no change in the low-temperature ordered moment per planar Cu as the long-range antiferromagnet order is destroyed by increasing the hole concentration. This behavior has also been detected in NMR studies (Chou et al., 1993). In the “spin-glass” regime that occurs in the doping range between the antiferromagnetic and superconducting phases, the ordered moments decrease gradually (Niedermayer et al., 1998). We now know that for La$_{2-x}$Sr$_x$CuO$_4$ with 0.02 $\lesssim x < 0.06$ the low-temperature phase involves ordering of magnetic moments in a diagonal spin-stripe structure (Fujita et al., 2002; Wakimoto et al., 1999). That identical behavior occurs in Y$_{1-x}$Ca$_x$Ba$_2$Cu$_3$O$_6$ is supported by an NQR/NMR study (Singer and Imai, 2002).

- The energy scale of magnetic excitations and the strength of the dynamic structure factor at frequencies less than $J$ change only modestly with doping.

One can see from Table I that $J_{\text{eff}}$ decreases a relatively small amount with doping. There is also evidence in some underdoped YBa$_2$Cu$_3$O$_{6+y}$ samples for effects reminiscent of spin-wave-like dispersion at energies above 50 meV (Bourges et al., 1997b; Fong et al., 2000). The

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44 Anderson (1997) has made a strong and compelling argument that antiferromagnetic exchange cannot be properly treated in RPA, and that the approach is internally inconsistent.

45 The recent observation by Coldea et al. (2001) that the spin-wave dispersion at large energies is better fit if a four-spin exchange interaction is included in the microscopic Heisenberg model is still consistent with a system of localized spins.
TABLE I Various quantities characterizing the strength of antiferromagnetic correlations in doped and undoped cuprates and nickelates. All are determined at low temperature. The quantity \( p \) is the nominal hole concentration per planar Cu; for YBa\(_2\)Cu\(_{6+x}\)O\(_y\) and YBa\(_2\)Cu\(_4\)O\(_8\) this was estimated from the “universal” formula relating \( T_c/T_c^{\text{max}} \) and \( p \) from Presland et al. (1991). The acronyms used under “phase” are: AFI = antiferromagnetic insulator, CSO = charge-stripe order, SSO = spin stripe order, SC = superconducting order. \( m \) is the magnetic moment per planar Cu, and \( m/m_0 \) is the moment relative to the AFI parent. \( \int S(k, \omega) \) is the dynamical structure factor, integrated over a Brillouin zone and over energy up to 100 meV, and normalized to the AFI parent. \( J_{\text{eff}} \) is the effective super-exchange energy characterizing the maximum energy of the magnetic excitations (2\( J_{\text{eff}} \)) measured by neutron scattering, or obtained from Raman scattering assuming that the two-magnon peak is at 2.7\( J_{\text{eff}} \). Further discussion is given in Appendix B.

| material                        | \( p \) | phase   | \( m \) \((\mu B)\) | \( m/m_0 \) | \( \int S(k, \omega) \) | \( J_{\text{eff}} \) \((\text{meV})\) | \( T_{\text{crit}} \) | \( T_N \) \( \approx \) |
|---------------------------------|--------|---------|----------------|------------|----------------|----------------|----------------|--------|
| La\(_2\)CuO\(_4\)              | 0      | AFI     | 0.60(5)\(^a\)  | 1          | 1                | 152(6)\(^p\)   | \( T_N \approx 300 \) K |
| La\(_2\)Cu\(_{6+y}\)(Zn,Mg)\(_0\)\(_2\)CuO\(_4\) | 0      | AFI     | 0.72(5)\(^c\)  | 1          | 1                | \( T_N \approx 100 \) K |
| La\(_{1.84}\)Ba\(_{0.12}\)CuO\(_4\) | 0.12   | CSO, SSO| 0.60(2)\(^d\)  | 1          | 1                | \( T_{\text{iso}} \approx 35 \) K |
| La\(_{1.46}\)Nd\(_0\)\(_{0.5}\)Sr\(_{0.12}\)CuO\(_4\) | 0.12   | CSO, SSO| 0.60(2)\(^d\)  | 1          | 1                | \( T_{\text{iso}} \approx 50 \) K |
| La\(_2\)Cu\(_{4.11}\)          | 0.16   | SSO, SC | 0.62(2)\(^f\)  | 1          | 1                | \( T_{\text{iso}} = T_c = 42 \) K |
| La\(_{1.84}\)Sr\(_{0.16}\)CuO\(_4\) | 0.14   | SC      | 0                | 0          | 0.4\(^g\)        | 140(10)\(^q\)   | \( T_c = 35 \) K |
| YBa\(_2\)Cu\(_3\)O\(_6\)\(_1\) | 0      | AFI     | 0.55(3)\(^h\)  | 1          | 1                | 125(8)\(^i\)   | \( T_N \approx 400 \) K |
| YBa\(_2\)Cu\(_3\)O\(_6\)\(_5\) | 0.09   | SC      | 0                | 0          | 0.5\(^i\)        | \( T_c \approx 52 \) K |
| YBa\(_2\)Cu\(_3\)O\(_6\)\(_6\) | 0.10   | SC      | 0                | 0          | 0.6\(^k\)        | 110(15)\(^k\)   | \( T_c = 63 \) K |
| YBa\(_2\)Cu\(_3\)O\(_6\)\(_7\) | 0.11   | SC      | 0                | 0          | 0.3\(^l\)        | 125(5)\(^m\)   | \( T_c \approx 70 \) K |
| YBa\(_2\)Cu\(_4\)O\(_8\)      | 0.13   | SC      | 0                | 0          | 0.4\(^m\)        | 125(5)\(^m\)   | \( T_c = 80 \) K |
| La\(_2\)NiO\(_4\)             | 0      | AFI     | 1.0(1)\(^n\)    | 1          | 1                | 31(1)\(^o\)     | \( T_N \approx 330 \) K |
| La\(_2\)NiO\(_4\)\(_133\)    | 0.27   | CO      | 0.8(1)\(^p\)    | 0.80(5)    | 1                | \( T_{\text{iso}} = 110 \) K |
| La\(_{2-x}\)Sr\(_{x}\)NiO\(_4\) | \( \sim 0.33 \) | CO       | 1.11(5)\(^q\)   | 20(1)\(^q\) | \( T_{\text{iso}} = 200 \) K |

\(^a\)From neutron diffraction (Yamada et al., 1987).
\(^b\)From inelastic neutron scattering (Coldea et al., 2001).
\(^c\)From neutron diffraction (Vajk et al., 2002).
\(^d\)Local moment relative to La\(_2\)CuO\(_4\) from \( \mu\)SR (Nachumi et al., 1998).
\(^e\)From two-magnon Raman scattering, assuming that the observed peak is at 2.7\( J \) (Gozar et al., 2003; Nachumi et al., 2002).
\(^f\)Local moment relative to La\(_2\)CuO\(_4\) from \( \mu\)SR (Savici et al., 2002).
\(^g\)From inelastic neutron scattering (Hayden et al., 1996a).
\(^h\)From neutron diffraction (Vajk et al., 1994).
\(^i\)From inelastic neutron scattering (Hayden et al., 1996b) and two-magnon Raman scattering (Blumberg et al., 1996, 1994).
\(^j\)Based on inelastic neutron scattering results (Fong et al., 2000; Hayden et al., 1996b).
\(^k\)Based on inelastic neutron scattering results (Daï et al., 1999; Hayden et al., 1996b).
\(^l\)Based on inelastic neutron scattering results (Fong et al., 2000; Hayden et al., 1996b).
\(^m\)From two-magnon Raman scattering (Blumberg et al., 1996, 1994).
\(^n\)From neutron diffraction (Wang et al., 1992).
\(^o\)From inelastic neutron scattering (Yamada et al., 1991).
\(^p\)From neutron diffraction (Tranquada et al., 1995a).
\(^q\)Local moment relative to La\(_2\)NiO\(_4\) from \( \mu\)SR (Jestad et al., 1999).
\(^r\)From inelastic neutron scattering (Boothroyd et al., 2003; Bourges et al., 2003).

partially-integrated dynamical structure factor decreases with doping, but remains substantial.

If the magnetic excitations corresponded to electron-hole excitations across the Fermi surface, one would expect that the magnetic energy scale should increase to a value comparable to the Fermi energy. For that scale to be the same as \( J_{\text{eff}} \) of the undoped phase would seem to be an incredible coincidence. With a substantial increase in energy scale, one would expect a corresponding decrease in the integrated dynamical structure factor.
Similar trends in terms of a modest reduction of $J_{\text{eff}}$ with doping and the survival of spin-wave-like excitations are found in stripe-ordered La$_{2-x}$Sr$_x$NiO$_4$ (Boothroyd et al., 2003; Bourges et al., 2003). This system is definitely in the strong-coupling limit, as it remains semiconducting above the charge-ordering temperature (Kat-suufuji et al., 1996).

- The $k$ dependence of the magnetic neutron scattering measured from superconducting samples of YBa$_2$Cu$_3$O$_{6+y}$ is consistent with spin density on planar Cu ions.

The magnetic scattering cross section is proportional to the square of the magnetic form factor (Fourier transform of the magnetization density); this is true for both elastic and inelastic scattering. An early analysis of inelastic magnetic scattering at different $k$ points in superconducting YBa$_2$Cu$_3$O$_{6+y}$ samples revealed an anisotropy (Rossat-Mignod et al., 1992) that was later shown to be consistent with the anisotropy due to the magnetic form factor for a single spin in a 3$d_{x^2−y^2}$ orbital (Shamoto et al., 1993).

The coherence of spin excitations within the CuO$_2$ bilayers of YBa$_2$Cu$_3$O$_{6+y}$ also results in a bilayer structure factor. The structure factor is a sinusoidal function of momentum transfer perpendicular to the bilayers, with a period that is proportional to the spacing between the layers of the centers of spin density. The observed period is quantitatively consistent with the intra bilayer spacing of the Cu atoms, but is incompatible with the spacing of the oxygen atoms, which is significantly smaller (Tranquada et al., 1992).

For electronic states at the Fermi level, there is a substantial amount of weight from planar O 2$p$ states (Takahashi et al., 1988). As a result, one would expect that magnetic scattering due to electron-hole excitations should, in real space, have significant weight associated with the oxygen sites. The measured modulations (Bourges et al., 1996; Fong et al., 1996; Tranquada et al., 1992) are not compatible with a significant component corresponding to the oxygen spacing.

- When stripe order is observed in more heavily doped samples, the magnetic moments are large.

As indicated in the table, the local magnetic moments detected in stripe-ordered phases by $\mu$SR are comparable to the moment found in undoped La$_2$CuO$_4$. The large moments imply a strong modulation of hole density. The magnetic form factor determined by neutron diffraction is consistent with that expected for spin moments (Tranquada et al., 1996).

- When stripe order is observed, charge orders before the spins, and the ordering wave vector grows with doping.

When charge and spin-stripe order are observed, as in La$_{1.6−x}$Nd$_{0.4}$Sr$_x$CuO$_4$, the charge order appears at a higher temperature than the stripe order (Ichikawa et al., 2000; Tranquada et al., 1996). (In La$_{1.875}$ (Ba,Sr)$_{0.125}$CuO$_4$, the ordering temperatures for charge and spin are very close (Fujita et al., 2002); nevertheless, the charge order parameter grows more rapidly than the spin order parameter.) It follows (Zachar et al., 1998) that the charge order is not driven by the spin ordering.

This result is natural for a strong-coupling picture of stripe correlations, but it poses a considerable challenge for the weak-coupling approach. In the latter, if one attributes the spin-stripe order to a nesting instability of the Fermi surface, then one must find a distinct nesting feature, with a spanning wave vector corresponding to $Q_{\text{ch}}$, to explain the charge-stripe order. Of course it is also necessary to explain the doping dependence of the ordering wave vectors. According to ARPES studies, the spanning wave vector near $k = (2\pi/a)(\frac{1}{2}, 0)$ is close to $Q_{\text{ch}}$. However, its variation with doping is opposite to that of $Q_{\text{ch}}$ (Ding et al., 1997; Ino et al., 2002); with increasing $x$, the hole-like Fermi-surface approaches closer to the $(\pi, 0)$ point, so the spanning wave vector decreases, while $Q_{\text{ch}}$ increases. We are not aware of a plausible explanation for the combined charge and spin modulations from the weak-coupling perspective.

- Substituting Zn impurities into the CuO$_2$ planes pins stripe order in La$_{2−x}$Sr$_x$CuO$_4$.

The substitution of Zn into the CuO$_2$ planes causes a significant increase in normal-state resistivity without modifying the carrier density (Fukuzumi et al., 1996). It also wipes out the dispersive feature near the nodal point observed by ARPES in the normal state (White et al., 1999). We have already discussed the fact that Zn doping can induce static stripe order in La$_{2−x}$Sr$_x$CuO$_4$ (see Fig. 8). These results are incompatible with a mechanism for stripe order based on Fermi-surface nesting. The Zn impurities break translational symmetry and cause considerable scattering of the charge carriers. Any sharp features at the Fermi surface are smeared out in the presence of the Zn. Thus the Zn should destroy, not induce, a nesting instability.

- There is no evidence of well-defined quasiparticles in the normal state of under- and optimally-doped cuprates.

There are many features of the doped system which suggest that, for the most part, there are no well defined quasiparticles. ARPES spectra in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and La$_{2−x}$Sr$_x$CuO$_4$ in some cases do exhibit features with a well defined dispersion, and by looking at the spectral intensity integrated over a small energy window about the Fermi energy, a Fermi surface of sorts is observed (Dessau et al., 1993; Ding et al., 1996; Ino et al., 2002). However, in no case has a peak been observed with a width small compared to its mean, which we take to be the definition of a well defined quasiparticle. In some ranges of temperature and $k$, there is no well defined quasiparticle peak, at
all. Near the nodal points, there are marginally defined quasiparticle, in the sense that there is certainly a clear peak in the spectral function, but its width is approximately twice its mean energy (Valla et al., 1999). There is corroborating evidence that any quasiparticles are at best “marginal” which comes from the T-linear dependence of the normal state resistivity, and various other indirect (but bulk) measurements (Anderson, 1987, 1992; Lee, 1999; Varma et al., 1989).

In addition, there is considerable indirect evidence that the familiar Fermi liquid power laws are strikingly absent. NMR experiments reveal significant temperature dependence to the nuclear 1/T1/T and dramatic violations of the Korringa relation, not only in the pseudogap regime, but even in the high temperature regime at optimal doping (Takigawa et al., 1991). The optical conductivity has a clearly non-Drude form (Collins et al., 1989), above and beyond the peculiar temperature dependence of the DC conductivity. The Hall number, as well, is anomalously temperature dependent (Wang et al., 1987).

All together, these results strongly imply that a picture of weakly interacting quasiparticles has limited validity in the cuprates. The one exception to this concerns the low-energy behavior (E ≪ Δ0) of the cleanest superconducting materials, deep in the superconducting state. Here, considerable indirect evidence exists that there are remarkably long lived nodal quasiparticles which dominate the physics (Sutherland et al., 2003). However, direct evidence of these sharply defined excitations has yet to emerge in any single-particle experiment such as ARPES or STM.

VII. CONCLUSIONS

There are many reasons why identifying and studying fluctuations associated with the presence of “nearby” ordered states has become one of the main thrusts in the study of cuprate superconductors and related materials. To some extent, these states are interesting just because they occur somewhere in the (multidimensional) phase diagram of these intriguing materials. Quantum critical points associated with some of these orders have been proposed to be the explanation of the notoriously peculiar high-temperature (“normal state”) behavior observed in many experiments. The possibility that certain local orders are inextricably linked with the phenomenon of high-temperature superconductivity is the most enticing reason of all. However, before we can determine whether a particular form of local order, such as local stripe order, could possibly be central to the problem of high temperature superconductivity, we need to determine whether or not it is ubiquitous in the families of materials which exhibit high-temperature superconductivity. Regardless of one’s motivation, it is clear that the ability to identify the proximity to an ordered phase by the signatures of “fluctuating order” is very useful. This paper has focused on practical considerations pertinent to this task.

A combination of rather general scaling considerations concerning quantum critical phenomena, and specific insights gleaned from the solvable models studied, has lead us to articulate a number of “lessons” concerning the optimal way of obtaining information about nearby ordered states from experiment. 1) The information is best obtained from the low-frequency part of the dynamic structure factor, preferably integrated over a small, but nonzero range of frequencies, with the scale of frequencies set by the characteristic frequency of quantum fluctuations, $E_G/h$. 2) Weak disorder can make it easier, especially for static probes, to image the local order, as it can pin the fluctuations without greatly disrupting the intrinsic correlations. 3) Experiments which reveal strongly dispersing features generally give information about the elementary excitations of the system; however, distinguishing dispersing features that arise from well defined quasiparticles from the multiparticle continua characteristic of quantum critical points can be exceedingly subtle. Specifically, even where one set of experiments can be sensibly interpreted in terms of band structure effects, care must be taken in interpreting this as evidence of well defined electron-like quasiparticles. 4) Several aspects of the E and k dependence of the LDOS (measurable by STIM) in the presence of weak disorder allow one, in principle, to distinguish interference effects due to the scattering of the elementary excitations from impurities, from the effects of pinned incipient order. Interference effects in 2D produce peaks along curves in k space which disperse as a function of energy in a manner which is directly related to the quasiparticle dispersion relations, such as could be measured in ARPES, and they may (or may not) have a strongly energy dependent phase. Pinning of incipient order produces peaks at well defined points in k space which depend only weakly on energy, and generally have an energy independent phase. 5) It will often be true that interference effects and collective pinning will jointly produce complicated k and E dependent properties in the local density of states that arise from a combination of both effects, especially in energy ranges in which the interference and pinning features lie at nearby values of k.

In Section IV of this paper, we applied these ideas to an analysis of the evidence of local stripe order in a number of neutron scattering and STM measurements on various cuprate superconductors. The evidence of both spin- and charge-stripe order is unambiguous in the La2CuO4 family of high-temperature superconductors. However, there is increasingly strong evidence of substantial local charge-stripe order, and probably nematic order as well, in Bi2Sr2CaCu2O8±δ and YBa2Cu3O6+y. Conversely, no evidence of incipient stripe order has been reported to date in the electron-doped cuprates, Ne2−xCe4CuO4±δ and Pr2−xCe4CuO4±δ.

Note Added: After this work was completed, we received an advanced copy of a review article by (Sachdev,
which has overlapping material with the present paper, and which reaches similar conclusions concerning the effects of incipient stripe order on the STM spectra.

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APPENDIX A: Luttinger liquids as one-dimensional quantum critical charge ordered states

In Section III we discussed how a single impurity induces charge order in the simplest and best understood quantum critical system, the Tomonaga-Luttinger Model of a one-dimensional electron gas. In this Appendix we give a summary of the physics and of essential technical aspects of the discussion of Section III. In this Appendix we will only present the aspects of the theory relevant to the derivation of the expressions used in Section III. There are a number of excellent reviews which cover the theory of the 1DEG at great depth and we refer the interested reader to that literature (Emery, 1979; Fradkin, 1991; Gogolin et al., 1998; Stone, 1994).

Consider a one-dimensional system of interacting spin one-half fermions (electrons). We will denote by \( \Psi_\sigma(x) \) the Fermi field for an electron with spin \( \sigma \) varying and hence are well described by an effective continuum Hamiltonian density \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \) where

\[
\mathcal{H}_0 = -i v_F \sum_{\sigma=\uparrow, \downarrow} \left( \psi_{+., \sigma}^\dagger \partial_x \psi_{+., \sigma} - \psi_{-., \sigma}^\dagger \partial_x \psi_{-., \sigma} \right)
\]  

(A2)

is the Hamiltonian density for non-interacting electrons in the low-energy regime where the dispersion is linearized; here \( v_F \) is the Fermi velocity. As usual, irrelevant operators which account for corrections to the linear dispersion are not included (see however below for caveats). The effects of interactions are included in \( \mathcal{H}_{\text{int}} \).

The best way to describe the physics of the 1DEG is by means of bosonization methods. Bosonization is the statement that the low-energy spectrum of the 1DEG is exhausted by a long lived bosonic excitation described by a field \( \phi_\sigma(x) \) which represents particle-hole fluctuations of spin \( \sigma \) near the Fermi points \( \pm k_F \). It turns out that the field \( \phi_\sigma \) also describes the phase fluctuations of a 2D CDW. Consequently the electron density operator \( \rho_\sigma(x) \) is decomposed into a long wavelength piece \( j_0^\sigma(x) \) and a \( 2k_F \) piece related to the density wave order parameter. The long wavelength electron density electron density for spin projection \( \sigma \), \( j_0^\sigma(x) \), is:

\[
j_0^\sigma(x) = \psi_{+., \sigma}^\dagger \psi_{+., \sigma} + \psi_{-., \sigma}^\dagger \psi_{-., \sigma}
\]  

(A3)

and the long wavelength current density with spin projection \( \sigma \), \( j_1^\sigma(x) \), is:

\[
j_1^\sigma(x) = \psi_{+., \sigma}^\dagger \psi_{-., \sigma} - \psi_{-., \sigma}^\dagger \psi_{+., \sigma}
\]  

(A4)

Here \( j_0^\sigma \) and \( j_1^\sigma \) are operators normal-ordered with respect to the filled Fermi sea.

The density and current density operators obey the equal-time bosonic commutation relations

\[
[j_0^\sigma(x), j_1^\nu(y)] = \frac{i}{\pi} \partial_x \delta(x-y) \delta_{\sigma, \nu}
\]  

(A5)

where \( \sigma, \nu = \uparrow, \downarrow \). One can identify \( j_0^\sigma(x) \) and \( j_1^\sigma(x) \) with

\[
j_0^\sigma(x) = \frac{1}{\sqrt{\pi}} \partial_x \phi_\sigma(x)
\]

\[
j_1^\sigma(x) = \frac{1}{\sqrt{\pi}} \Pi_\sigma(x)
\]  

(A6)

where \( \phi_\sigma(x) \) is a bose field and \( \Pi_\sigma(x) \) is its canonically conjugate momenta; they obey the canonical equal-time commutation relations

\[
[\phi_\sigma(x), \Pi_\nu(y)] = i \delta_{\sigma, \nu} \delta(x-y).
\]  

(A7)

The (slowly varying) Fermi fields \( \psi_{\pm, \sigma}(x) \) can themselves be reconstructed from the bose field \( \phi_\sigma(x) \):

\[
\psi_{\pm, \sigma}(x) = \frac{1}{\sqrt{2\pi\alpha}} N_\sigma \ e^{i \sqrt{\pi} (\pm \phi_\sigma(x) - \theta_\sigma(x))}
\]  

(A8)

(here \( \alpha \sim v_F / D \) is a short-distance cutoff and \( D \) is the fermion band width) where we have introduced the dual field \( \theta_\sigma(x) \), defined by the identity

\[
\partial_x \theta_\sigma(x) = \Pi_\sigma(x)
\]  

(A9)
and $N_s$, the Klein factor, is an operator which guarantees that Fermi fields with different spin labels anti-commute with each other.

It is convenient to introduce the charge and spin boson fields $\phi_c$ and $\phi_s$,

$$\phi_c = \frac{1}{\sqrt{2}} (\phi^c + \phi^c_\uparrow)$$
$$\phi_s = \frac{1}{\sqrt{2}} (\phi^s + \phi^s_\downarrow).$$

(A10)

The Hamiltonian density of the interacting system can be written a sum of operators which are marginal (with scaling dimension equal to 2), relevant (with scaling dimension smaller than 2) and irrelevant (with scaling dimension larger than 2). For an incommensurate 1DEG the effective low-energy Hamiltonian, which contains only marginal operators, has the spin-charge separated form

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s$$

(A11)

where the charge Hamiltonian density $\mathcal{H}_c$ in bosonized form has the universal (Tomonaga-Luttinger) form

$$\mathcal{H}_c = \frac{v_c}{2} K_c \Pi_c^2 + \frac{v_c}{2K_c} (\partial_x \phi_c)^2$$

(A12)

where $v_c$ is the charge velocity and $K_c$ is the charge Luttinger parameter. T For repulsive interactions, $\phi_c$ has the spin-charge separated form

$$\mathcal{H}_s = \frac{v_s}{2} K_s \Pi_s^2 + \frac{v_s}{2K_s} (\partial_x \phi_s)^2$$

(A13)

The non-universal charge and spin velocities and the Luttinger parameters encode the dependence of this low energy theory on the microscopic parameters of the system. Quite generally, for a system with repulsive interactions the charge Luttinger parameter obeys the inequality $K_c < 1$, and for spin rotationally invariant interactions the spin Luttinger parameter satisfies the equality $K_s = 1$. Typically the charge and spin velocities satisfy the inequality $v_c > v_s$. In the weak coupling limit, and neglecting all irrelevant operators, simple expressions for $K_c$, $v_c$ and $v_s$ in terms of the backscattering and forward scattering amplitudes can be written down. 46

Physical observables of the 1DEG have simple expressions in the bosonized theory. Since the Tomonaga-Luttinger model is strictly quadratic in boson fields, it allows for a straightforward computation of the correlation functions of all observables of interest, both at zero and finite temperature, as well as for different types of boundary conditions. For the purposes of this review it will be sufficient to note that the electron density has the decomposition

$$\rho(x) = \frac{2k_F}{\pi} j_0(x) + e^{i2k_F x} \mathcal{O}_{CDW}(x) + e^{-i2k_F x} \mathcal{O}_{CDW}(x)^\dagger$$

(A14)

$$\mathcal{O}_{CDW} = \sum_\sigma \psi^\dagger_+, \sigma \psi_-, \sigma$$

(A16)

is the CDW order parameter (i.e., the $2k_F$ amplitude of the charge density), and $\mathcal{O}_{k_F}$ is the $4k_F$ CDW order parameter (which we will not discuss here).

In particular this implies that the charge dynamical structure factor for wave vectors close to $2k_F$, discussed in Section III, is the (retarded) correlation function of the CDW order parameter:

$$S_{CDW}(x,t) = \langle \mathcal{O}_{CDW}(x,t) \mathcal{O}_{CDW}(0,0) \rangle$$

(A17)

whereas for small wave vectors it is given instead by the (retarded) density correlator

$$S_0(x,t) = \langle j_0(x,t) j_0(0,0) \rangle$$

(A18)

In particular, the spectral function for $S_{CDW}(x,t)$, which we will denote by $S_{CDW}(k,\omega)$, has the scaling form

$$\tilde{S}_{CDW}(k,\omega) = \frac{v_c}{16\pi^4} \left( \frac{\pi T \alpha}{v_c} \right)^{K_c-1} \Phi_{CDW} \left( \frac{v_c k}{\pi T}, \frac{\omega}{\pi T} \right)$$

(A19)

where $\Phi_{CDW}(x,y)$ is the (dimensionless) scaling function (Orgad et al., 2001)

$$\Phi_{CDW}(x,y) \equiv \int_{-\infty}^\infty du \int_{-\infty}^\infty dv \ h_{K_c} \left( \frac{u + v}{2} \right) \times h_{K_s} \left( \frac{\sqrt{r u + v}}{2} \right) h_{K_s} \left( \frac{r u - v}{2} \right)$$

(A20)

Here $r = v_s/v_c < 1$ and $h_{K_s}(z)$ is given by (Orgad et al., 2001):

$$h_{K_s}(z) = \text{Re} \left[ (2i)^{\nu} B \left( \frac{\nu - iz}{2}, 1 - \nu \right) \right]$$

(A21)

where $B(x,y)$ is the Euler Beta function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

(A22)
and \( \Gamma(z) \) is the Gamma function.

We now turn to the calculation of the tunneling density of states discussed in Section III. There we presented the behavior of the \( 2k_F \) component of the tunneling density of states \( N(2k_F + q, E) \), the Fourier Transform in \( x \) and \( t \), of the \( 2k_F \) component of the electron spectral function.

At zero temperature and for a semi-infinite system with \( 0 \leq x < L \) (with \( L \to \infty \)), the \( 2k_F \) component of the fermion Green’s function, \( g_{2k_F}(x,t) \), is given by (Eggert, 2000; Eggert et al., 1996; Mattsson et al., 1997)

\[
g_{2k_F}(x,t) \equiv \langle \psi_{\uparrow,\sigma}(x,t)\psi_{\downarrow,-\sigma}(x,0) \rangle \sim 1 \frac{\alpha^a+b-1/2}{2\pi (v_s - 2x)^{1/2}} \left( v_c \tau + 2x \right)^b
\]

As with the structure factor, \( N(q + 2k_F, E) \) can also be expressed in terms of a scaling function \( \Phi \) as

\[
N(q + 2k_F, E) = \int_{-\infty}^{\infty} dt e^{-i(qx - Et)} g_{2k_F}(x,t) \]

\[
= \frac{B}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} \Phi \left( \frac{2E}{v_c}, \frac{E}{K_B T} \right)
\]

where \( \tau = t + i0^+ \). Notice that in Eq. (A24) we have kept the dependence on the short distance cutoff \( \alpha \); consequently, \( g_{2k_F}(x,t) \) naively has units of \( \alpha^{-1} \). However, in a Luttinger liquid the fermion operator has an anomalous scaling dimension. In this semi-infinite geometry the scaling dimension is governed by the exponents \( a, b \) and \( c \), which are given by

\[
a = (K_e + 1)^2, \quad b = (K_e - 1)^2, \quad c = \frac{1}{4} \left( \frac{1}{K_e} - K_e \right)
\]

At finite temperature \( T > 0 \), \( g_{2k_F}(x,t;T) \) becomes

\[
|g_{2k_F}(x,t;T)| \sim \frac{\pi T \alpha}{(v_s - 2x)^{1/2}} \left( v_c \tau + 2x \right)^b
\]

where \( K_e \) is the dimensionless quantity \( v_c/v_s \). \( B \) is the dimensionless quantity

\[
B = \frac{e^{-i\pi(c+1)/2}}{\Gamma(a + b + c - \frac{3}{2})}
\]

The behavior of \( N(2k_F + q, E; T) \) for \( T > 0 \) is shown in Fig. 3 and Fig. 4 in Section III. As \( T = 0 \) the scaling function \( \Phi(2E/v_c q) \) is given by

\[
\Phi(u) = \sqrt{\frac{v_c}{v_s} u^{c+1}} \int_0^1 dt (1 - t)^{a+b-\frac{3}{2}} (1 - ut)^{-a} (1 + u^*t)^{-b} \left( 1 - \frac{v_c}{v_s} ut \right)^{-\frac{b}{2}}
\]

Here \( u = 2|E|/qv_c + i0^+ \). We have obtained analytic expressions for the singular pieces of \( N(k,E) \) at \( T = 0 \), for a general value of the charge Luttinger parameter in the range of interest \( 0 < K_e < 1 \), and general \( v_c/v_s \). \( N(k,E) \) has power law singularities as \( 2E/v_c q \to \pm 1 \), \( 2E/v_c q \to 0 \), and \( 2E/v_c q \to \infty \) and \( 2E/v_q q \to v_s/v_c \).

Physically the non-analyticities at \( 2E/v_c q = \pm 1 \) and \( 2E/v_q q = 1 \) represent the threshold of the continuum of propagating excitations of appropriate type moving to the right and to the left respectively. The singularity at \( 2E/v_c q \to \infty \) corresponds to the \( 2k_F \) CDW stabilized (pinned) by the boundary.

Close to the right-dispersing charge-related singularity

\[
N(2k_F + q, E) \sim \frac{1}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} A_+ \left[ 1 - \frac{2E}{v_c q} \right]^{b-\frac{3}{2}}
\]

where \( A_+ \) is a finite complex coefficient determined by the strength of the singularity. An important feature of this result is that as \( E \) goes through this singularity at fixed \( q \), the phase of \( N(2k_F + q, E) \) jumps by \( \pi(b-1/2) \). (See Fig. 4.) Close to the right-dispersing spin-related
singularity at \( q \to 2E/v_c \) we find

\[
N(2k_F + q, E) \sim \frac{1}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} \left( \frac{v_s}{v_c} \right)^{a+b+c-1} A' \left[ 1 - \frac{2E}{v_c q} \right]^{2b-\frac{4}{\pi}}
\]

where \( A' \) is another complex coefficient. In the non-interacting limit \( K_c = 1 \) and \( v_s = v_c \), these two singularities coalesce into a simple pole for a particle moving to the right. However, for \( K_c < 1 \), the TLL does not have quasiparticles but massless soliton states instead. As usual, the power laws reflect the multi-soliton continuum. (More generally, the limit \( v_s/v_c \to 1 \) is somewhat subtle (Bindloss et al., 2003).) Similarly, for \( q \to -2E/v_c \)

\[
N(2k_F + q, E) \sim \frac{1}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} A_- \left[ 1 + \frac{2E}{v_c q} \right]^{b}
\]

where, once again, \( A_- \) is a finite complex coefficient specific to this singularity. However, unlike the singularity at \( E = +v_c q \), the behavior near \( E = -v_c q \) although non-analytic is not divergent since \( b > 0 \) for repulsive interactions.

In addition to propagating excitations, we also find that \( N(2k_F + q, E) \) has a singularity associated with the non-propagating CDW as \( q \to 0 \);

\[
N(2k_F + q, E) \sim \frac{1}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} \left( \frac{v_s}{v_c} \right)^{1/2} \left[ \frac{2E}{v_c q} \right]^{(1-K_c)/2}
\]

which diverges as \( q \to 0 \). This singularity also exhibits a phase jump as \( q \to 0^\pm \), equal to \( \pi(1-K_c)/2 \).

At low voltages \( E \to 0 \) and at fixed \( q \), we find

\[
N(2k_F + q, E) \sim \frac{1}{2E} \left( \frac{\alpha E}{v_c} \right)^{2b} \left( \frac{v_s}{v_c} \right)^{1/2} \left[ \frac{2E}{v_c q} \right]^{c+1}.
\]

Finally, we summarize here the calculation of the induced density of states in the weak-impurity limit, \( E > T_K \). In Section II C.1 we showed that the density of states at wavevector \( k \) and voltage \( E \) induced by an impurity potential with amplitude \( V(k) \) at wave vector \( k \) defines the susceptibility \( \chi_{\text{DOS}}(k, E) \) given in Eq. (2.8). Here we will sketch the calculation of \( \chi_{\text{DOS}}(2k_F + q, E) \) (to leading order in \( V(k) \)) for a Luttinger liquid. We must first find the Green’s function

\[
G(z_1, z_2, z_3) = \langle \psi_{+, \uparrow}^\dagger(z_1) \psi_{-, \uparrow}^\dagger(z_2) \psi_{-, \uparrow}^\dagger(z_3) \psi_{+, \uparrow}^\dagger(z_3) + \leftrightarrow \rangle
\]

which can be computed readily using bosonization methods. At \( T = 0 \) and for an infinite system the time-ordered Green’s function \( G(z_1, z_2, z_3) \), in imaginary time, is given by

\[
G(z_1, z_2, z_3) = \frac{1}{2\pi^2 \alpha^2} \left| z_1 - z_2 \right|^{-1-K_c/2} \times \left( z_3 - z_1 \right)^{-K_c/4} \left( z_2 - z_3 \right)^{-K_c/4} \left( w_1 - w_3 \right)^{-\frac{1}{2}}
\]

where we have used the charge and spin complex coordinates \( z = x + iv_c \tau \) and \( w = x + iv_c \tau \). The susceptibility \( \chi_{\text{DOS}}(2k_F + q, E) \) is obtained, upon analytic continuation to real frequencies, from

\[
\chi_{\text{DOS}}(2k_F + q, E) = \text{Im} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\tau e^{-i(kx - E\tau)} \times \int d\tau' G(x, \tau; x, 0; 0, \tau')
\]

The result for the integrated density of states \( \tilde{N}(2k_F + q, D) \) presented in Section III, Eq. (3.13) follows from integrating Eq. (A36) over energies large compared with the low-energy cutoff \( T_K \).

APPENDIX B: Experimentally determined scales of magnetism in various materials

Here we explain some of the assumptions behind the parameter values specified in Table I. For antiferromagnetic phases, the meaning of the parameters is relatively straightforward, but extracting parameters from measurements on doped systems can require extrapolations from simple models. Note that we have not addressed important issues related to possible novel forms of orbital magnetism that have been proposed (Chakravarty et al., 2001a; Varma, 1997).

1. Magnitude of the ordered moments

a. Absolute moments from neutron scattering

In the case of long-range antiferromagnetic order, the absolute magnitude of the average ordered moment per planar Cu, \( m \), is obtained from an elastic neutron diffraction measurement by comparing the intensities of the magnetic Bragg superlattice peaks with those of Bragg peaks from the chemical lattice. It has been observed that \( m \) is correlated with the Néel temperature, \( T_N \), in both \( La_2CuO_4 \) (Yamada et al., 1987) and \( YBa_2Cu_3O_6+y \) (Tranquada et al., 1988). In the earliest measurements of \( La_2CuO_4 \), samples typically contained excess oxygen, resulting in values of \( m \) and \( T_N \) significantly lower than those of the stoichiometric material. Another complication is that determination of \( m \) requires knowledge of the magnetic form factor, which turns out to be considerably more anisotropic (Shimoto et al., 1993) than was appreciated initially.
For comparison, it is worth noting that for a spin \( \frac{1}{2} \) Heisenberg model on a square lattice with only nearest neighbor interactions, the magnitude of the average ordered spin per Cu, to second order in a \( 1/S \) expansion (Igarashi, 1992), is \( \langle S \rangle = 0.307 \). For a moment due to spin only, \( q = 2 \) and \( m = g \langle S \rangle \mu_B = 0.61 \mu_B \). Although it has not been possible to measure the \( g \) factor for Cu in the layered cuprates,\(^{47} \) a typical value for a \( Cu^{2+} \) ion in a distorted octahedral environment is 2.2 (Abragam and Bleaney, 1970), which implies a typical moment of 0.67 \( \mu_B \). The fact that the observed maximum moments in \( La_2CuO_4 \) and \( YBa_2Cu_3O_{6+y} \) are 10 to 20\% smaller than this could be a result of hybridization of the Cu 3d\(_{x^2−y^2}\) orbital with O 2p\(_x\) orbitals. (By symmetry, the net spin density on oxygen sites is zero in the Néel structure.)

b. Relative moments from \( \mu \)SR

In \( \mu \)SR, one measures a precession frequency, which is directly proportional to the local hyperfine field at the \( \mu^+ \) site. Determination of the absolute moment requires knowledge of the \( \mu^+ \) location (obtained by calculation rather than experiment) and a calculation of the magnetic field at that site due to the ordered Cu moments. Alternatively, under the assumption that the \( \mu^+ \) site is unchanged by doping, one can determine the local moment in a doped sample relative to that in the parent insulator. An advantage of the \( \mu \)SR technique is that its sensitivity does not depend on the existence of long-range order. A beautiful example of this is the study of Niedermayer et al. (1998) in which it was shown that the local ordered moment (at very low temperature) changes little as long-range antiferromagnetic order is destroyed by doping in \( La_{2−x}Sr_xCuO_4 \) and \( Y_{1−x}Ca_xBa_2Cu_3O_6 \). In a complementary study, Klauss et al. (2000) have shown that substantial moments associated with local antiferromagnetic order (presumably stripe order pinned by the tetragonal structure) are present in \( La_{1.8−x}Eu_{0.2}Sr_xCuO_4 \) for 0.08 \( \lesssim x \lesssim 0.18 \).

In a stripe-ordered sample, the muons sample a distribution of local hyperfine fields, resulting in a fairly rapid damping of the precession signal (Klauss et al., 2000; Nachumi et al., 1998; Savici et al., 2002). The relative moment quoted in the table corresponds to the maximum of that distribution.

2. Integrated low-energy spectral weight

Neutron scattering measures the dynamical magnetic structure factor \( S(Q, \omega) \equiv S_{xx}(Q, \omega) \). Actually, the spin structure factor is the tensorial quantity \( S_{\alpha\beta}(Q, \omega) \), which obeys the sum rule

\[
\int_0^\infty d\omega \int_{BZ} dQ \sum_\alpha S_{\alpha\alpha}(Q, \omega) = S(S + 1),
\]

where \( S \) is the spin per magnetic ion. For cuprates in general, and superconducting samples in particular, it is challenging to measure the magnetic scattering to sufficiently high energies (with sufficient signal-to-noise ratio) that the sum rule can be evaluated; however, there do exist a few studies that allow one to evaluate the energy integral up to 100 meV. While there are significant uncertainties in evaluating the integral from published data, these should be smaller than the uncertainty in the calibration of the sample volume contributing to the signal, which for superconducting samples can be on the order 30\% (Fong et al., 2000).

Since we are interested in the change of this integrated quantity with doping, we have normalized it to the value measured for the relevant parent antiferromagnet. Note that for the latter case, the integral includes the signal in the magnetic Bragg peaks as well as that from the spin waves.

Another quantity that one can evaluate is the relative weight under the resonance peak, which appears below \( T_c \) in the superconducting state. Values, normalized to the full sum rule weight, have been tabulated by Kee et al. (2002). For \( YBa_2Cu_3O_{6+y} \) samples with \( T_c \) varying from 52 to 93 K, they found the normalized weight to be about 1.5\%; for a sample of \( Bi_2Sr_2CaCu_3O_{8+\delta} \) with \( T_c = 91 \) K, the value is \( \sim 6\% \). It should be noted that integrating \( S(Q, \omega) \) up to 100 meV for the experimental measurements on antiferromagnetic \( YBa_2Cu_3O_{6.15} \) (Hayden et al., 1996b) gives just 20\% of the full sum rule weight; hence, using the same normalization as for \( \int S(Q, \omega) \) in Table I, the weight of the resonance peak in \( YBa_2Cu_3O_{6+\delta} \) is about 0.08.

3. Energy scale of spin fluctuations

a. From neutron scattering

In the spin-wave theory of a Heisenberg model on a square lattice with only nearest-neighbor coupling, the zone-edge magnon with \( q_0 = (\pi, 0) \) has energy \( \epsilon_{q_0} = 4SJ_{\text{eff}} = 2J_{\text{eff}} \), where the final equality applies for \( S = \frac{1}{2} \). Longer-range couplings have observable effects in the spin-wave dispersion curves measured for \( La_2CuO_4 \) (Coldea et al., 2001); nevertheless, it is useful to characterize \( \epsilon_{q_{0\beta}} \), which is essentially the local energy to flip a spin, in terms of \( J_{\text{eff}} \). Assuming that a local superexchange coupling is still active between local moments in doped samples, we have used the same formula to define \( J_{\text{eff}} \) in terms of the maximum spin-excitation energy observed in superconducting cuprates.

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\(^{47}\) Despite numerous attempts, it has not been possible to detect an electron-spin resonance signal from \( Cu^{2+} \) in planar cuprates (Simon et al., 1993).
b. From two-magnon Raman scattering

While Raman scattering is not sensitive to individual spin waves, it does provide a valuable probe of two-magnon correlations in 2D square-lattice antiferromagnets (Lyons et al., 1988). The dominant part of the response has $B_{1g}$ symmetry and appears at low temperature as a strong peak in intensity, with the peak occurring at an energy of $2.8J_{\text{eff}}$ for $S = \frac{3}{2}$ and $6.8J_{\text{eff}}$ for $S = 1$ (Canali and Girvin, 1992). The scattering mechanism is believed to involve relatively short-range excitations, so that the response is not strongly sensitive to the existence of long-range order. Although there is no formal justification for it, we have used the formula for peak energy in the antiferromagnetic state in order to extract a characteristic value for $J_{\text{eff}}$ from the 2-magnon signal measured from superconducting samples. Note that there are also results available on $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ (Blumberg et al., 1997; Sugai and Hosokawa, 2000) which we have not included in Table I.

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