Fuzzy Instantons

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Abstract

We present a series of instanton-like solutions to a matrix model which satisfy a self-duality condition and possess an action whose value is, to within a fixed constant factor, an integer $l^2$. For small values of the dimension $n^2$ of the matrix algebra the integer resembles the result of a quantization condition but as $n \to \infty$ the ratio $l/n$ can tend to an arbitrary real number between zero and one.
1 Introduction and motivation

The ideas behind the construction of solitons and instantons can be to a certain extent extended to noncommutative geometries, at least to some of them. Within the present context the word soliton refers to a finite-energy smooth solution with no reference to conserved quantities; an instanton is a smooth finite-action solution with euclidean signature, generally involving Yang-Mills fields whose field equations reduce to self-duality conditions. In ordinary geometry they are both stable because of a topological obstruction to their decay. To a certain extent topology now becomes ill-defined and propagators regular. Soliton solutions thus lose somewhat their specificity. It is nevertheless interesting to study their structure on a fuzzy space. It was sometime after Dirac presented his magnetic monopole \[1\] solution that it was shown \[2\] to be a singular limit of a soliton solution. Since the monopole charge is the first Chern number of a line bundle over the 2-sphere, to obtain a regular solution this bundle has to be combined with another of equal and opposite monopole charge. The gauge group has to be enlarged to \(SU_2\) and a Higgs field added so that the topological twist can be encoded as the winding number of a map of the sphere at infinity in configuration space into the unit sphere of the \(SU_2\) Lie algebra. The integer in turn can be identified as an element of the second homotopy of the homogeneous space \(SU_2/U_1\). We recall this history because the monopole suggests one path from simple electromagnetism to the Yang-Mills-Higgs-Kibble action with an \(SU_2\) gauge group and a Higgs field with values in the adjoint representation.

Another way \[3\] is the study of electromagnetism on a noncommutative Kaluza-Klein-type extension of space-time with \[4\] the matrix algebra \(M_2\) as additional internal structure. If we consider Maxwell theory on this algebra then the extra matrix factor transforms the Maxwell potential into a \(U_2\)-gauge multiplet and a triplet of Higgs scalars. The ’t Hooft-Polyakov solution can be considered then as a ‘Maxwell’ solution on the extended algebra \(\mathcal{A}\) and in this sense one can say that with the extra noncommutative factor the core region of the Dirac monopole has been regularized. To define a smooth structure we must introduce a calculus or algebra of forms, a graded differential algebra \(\Omega^* (\mathcal{A})\) over \(\mathcal{A}\).

The theory which we call Maxwell’s on space-time can acquire quite a different aspect when applied \[3\] to more exotic spaces. We have just seen that the inclusion of a simple matrix factor in the algebra can radically change the singularity structure of the original theory. There are further examples of this, all of which are constructed by a procedure which resembles the blowing up of singularities in algebraic geometry except that in the present case a singular point is regularized not by expanding it into a higher dimensional variety but rather by surrounding it with a finite ‘fuzzy’ structure. An enlightening comparison is with the description \[4\] of the core regions of planar nematic liquid crystals. The order-parameter of such a region can ‘escape into the third dimension’ thereby resolving itself into a smooth configuration or it can remain, ill-defined, in the plane.

By definition the local ‘electromagnetic’ gauge group of a geometry with structure algebra \(\mathcal{A}\) is the set of unitary elements of the algebra. If \(\mathcal{A} = M_n\) then this yields \(U_n\) as ‘local’ gauge group. A simple example is afforded by the quantum approximations to a classical spin for large quantum numbers. If \(s\) is the spin then the spin operators lie in \(M_n\) with \(n = 2s + 1\). In a way which can be made precise the algebra \(M_n\) tends to the algebra of smooth functions on the sphere and the group \(U_n\) tends to the group \(U_l\) of local \(U_1\) transformations. We shall refer to a manifold possessing a filtering of its calculus by a sequence of differential algebras...
over algebras of matrices as being fuzzy. As a fuzzy version of the space $\mathbb{R}^3$ in which a monopole lives one can choose the simplest: the direct sum of all matrix algebras.

In Section 2 we shall very briefly compare electromagnetism on ordinary flat $\mathbb{R}^3$ with the same theory on the slightly more structured ‘space’ which is a Kaluza-Klein extension thereof by the algebra $M_2$ of $2 \times 2$ matrices over the complex numbers. Although it is necessary to introduce the complex numbers to describe it, the geometry of this latter ‘2-point’ space is real. We examine also the analogous case of a time-dependent fuzzy-sphere [4, 7]. The ‘abelian’ gauge theory on the fuzzy sphere has several ground states which have been called [4] ‘instantons’ but which in the present context should rather be dubbed ‘solitons’. In Section 3 we find instanton-like solutions which tunnel between two of these solitons. The formalism of the following section could be said to describe ‘fuzzy’ spherical $D2$-branes [8], in which case the instanton would be said to tunnel between two such configurations, one in which the branes coincide and the other in which they are completely separated. In the spherical case the separation is quantized and the instanton can have a topological interpretation. An summary of some results on projectors used is given in the Appendix.

We stress here the role of instantons as mediators between different stable vacuum sectors of a matrix geometry. Several other aspects of instantons have also been carried over [9, 10, 11, 12, 13] into various noncommutative geometries including the fuzzy sphere [14, 15, 16]. Similar calculations have been carried out [17] on the torus.

2 Abelian gauge theory

In this section we recall very briefly some formulae which will be useful in studying product structures. Let $\mathbb{P}^1(\mathbb{C}) = S^2$ be smoothly embedded in $\mathbb{R}^3$. A Dirac monopole will refer to a principle $U_1$ bundle over $S^2$ or to an associated complex line bundle $L$ over $\mathbb{P}^1(\mathbb{C})$ or to an extension of the former to a $U_1$ bundle over $\mathbb{R}^3$, either of which can be completely classified by a single integer $k$. Expressed algebraically a Dirac monopole is an irreducible module over the algebra $\mathcal{C}(S^2) \subset \mathcal{C}(\mathbb{R}^3)$; a ‘t Hooft-Polyakov monopole or soliton is exactly the same object over the algebra $\mathcal{A} = \mathcal{C}(S^2) \otimes M_2$. More precisely, if one consider ‘Maxwell’s’ theory on the two algebras and look for a solution which tends in the asymptotic region to the monopole, then if $\mathcal{A} = \mathcal{C}(\mathbb{R}^3)$ one obtains the Dirac monopole and if $\mathcal{A} = \mathcal{C}(\mathbb{R}^3) \otimes M_2$ the ‘t Hooft-Polyakov soliton. This is an obvious remark which follows directly from the definition of a differential calculus over a product structure which is given in the Appendix. It is perhaps the simplest example of the regularizing capabilities of noncommutative extensions of manifolds with singularities.

There are many noncommutative versions of $\mathbb{R}^3$, all of which necessarily break the inhomogeneous invariance group $ISO_3$ of $\mathbb{R}^3$. We mention in particular the $q$-deformed version of Faddeev et al. [18] which has an interesting quantum group as invariance ‘group’. Otherwise one can for example split the space into the product $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ of the complex line times the real line [11], a procedure which respects the subgroup $ISO_2 \times \mathbb{R} \subset ISO_3$. We shall choose to split space into the product $\mathbb{R}^3 = S^2 \times \mathbb{R}_+$ which respects the subgroup $SO_3 \subset ISO_3$. Although we consider here exclusively the case of the canonical flat $\mathbb{R}$ the same reasoning will apply to a curved manifold provided the soliton in question is of dimension $r$ and placed
in a region of typical Gaussian curvature $K$ such that $r^2 K \ll 1$. We replace the algebra $\mathcal{C}(\mathbb{R}^3)$ of smooth functions of compact support by the noncommutative algebra

$$\mathcal{A} = \bigoplus_{1}^{\infty} M_n.$$  

Each $M_n$ describes a fuzzy 2-sphere $[19]$ of radius $r_n$ determined by the possible values of the Casimir operator. If we introduce a length scale $\sqrt{\bar{k}}$ then the possible values of $r$ are given, for large $n$, by the sequence of solutions to the equation

$$n = \frac{4\pi r^2}{2\pi \bar{k}}. \quad (2.1)$$

This is to be interpreted as meaning that the ‘2-sphere’ of radius $r$ describes a ‘space’ consisting of $n$ cells of area $2\pi \bar{k}$. The interpretation is taken from quantum mechanics. It parallels also that of a spin or of the Landau-orbit structure of an electron gas in a constant normal magnetic field. The $n$ ‘points’ are the ground states of the Landau orbits; the remaining $n(n-1)$ degrees of freedom of $M_n$ become in the limit delocalized states. [19]

Space acquires an onion-like structure with an infinite sequence of concentric fuzzy spheres at the radii given by the Casimir relation (2.1), which can also be written

$$\text{Area}[S^2] = 2\pi kn$$

as a quantization condition on the area. A large number of differential calculi can be put on a matrix algebra, a subset $\Omega_m(M_n)$ of which can be described [19] in terms of the action of a group $SU_m$ with $m$ belonging to a subset of the integers less than $n$. We shall restrict our considerations here to the simplest situation with $m = 2 \ll n$.

The differential calculus is a product calculus; on the first factor it is that of the fuzzy sphere and along the generator $r$ the ordinary one in the finite-difference approximation. In this example the electromagnetic potential can be thought of as the Higgs field. Its vacuum value is the ‘Dirac operator’ $\theta$ and it tends at infinity to the Dirac-monopole solution of change $-1$. There should be no obstruction to considering higher monopole charge [20]. A sequence of projective modules describing in the limit the higher monopole charge configurations has been constructed[21]. The Chern numbers which arise there are noninteger and converge in the commutative limit to the integer Chern numbers of the complex line bundles over the two-sphere. To describe the Schwarzschild instanton in terms of matrices we must replace the factor $M_2$ in $\mathcal{A}$ by $M_l$ with $l \gg 2$. That is, we must consider the extensions

$$\mathcal{A} \mapsto \mathcal{A}' \otimes \mathcal{A}'' = \bigoplus_{l, n = 1}^{\infty} (M_l \otimes M_n)$$

to the tensor product of two copies of $\mathcal{A}$. The monopoles become the first two elements in either one of the factor series.

We recall that a differential calculus can be completely defined in terms of the left and right module structure of the $\mathcal{A}$-module of 1-forms $\Omega^1(\mathcal{A})$. We shall restrict our attention to the case where this module is a submodule of a module of rank $n$ which is free as a left or
right module and which possesses a special basis $\theta^a$, $1 \leq a \leq n$, which commutes with the elements $f$ of the algebra.

$$[f, \theta^a] = 0.$$  \hspace{1cm} (2.2)

The differential $d$ is a projection onto $\Omega^1(A)$ of the image of a map $d$ given by the expression

$$df = e_a f \theta^a = [\lambda_a, f] \theta^a.$$  \hspace{1cm} (2.3)

One can rewrite this equation as

$$df = -[\theta, f],$$  \hspace{1cm} (2.4)

if one introduces \cite{22} the ‘Dirac operator’

$$\theta = -\lambda_a \theta^a.$$  \hspace{1cm} (2.5)

The differential is then of the form

$$df = -[\theta, f] = -e_a f \theta^a.$$  \hspace{1cm} (2.6)

The differential of the relations $R$ which define the algebra must vanish and the largest module of 1-forms is obtained by supposing that these are the only relations in $\Omega^1(A)$.

In the present case the $\lambda_a$ satisfy the commutation relations $[\lambda_a, \lambda_b] = C^{c}_{ab} \lambda_c$ with $C_{abc} = r^{-1} \epsilon_{abc}$ and they have units of inverse length. The forms $\theta^a$ anti-commute. The form $\theta$ can be considered as a flat connection:

$$d\theta + \theta^2 = 0.$$  

The fuzzy 2-solitons are in 1-1 correspondence with the projectors and their number is given by the (Hardy-Ramanujan) partition function $p(n)$.

The calculus we use is a 3-dimensional calculus on the Hopf fibration over the 2-sphere. This is necessary because the 2-sphere is not parallelizable. We would like therefore to have an expression for the integral of a ‘function’ on the fuzzy 2-sphere as integral of a 3-form. If $\alpha$ is a $d$-form over a $d$-dimensional manifold then under Hodge duality it corresponds to an element $a$,

$$\alpha = a \theta^1 \cdots \theta^d \longrightarrow \text{Vol}(V)a$$

of the algebra which has scale dimension $L^{-d}$. It seems quite natural to define the integral of $a$ so that

$$\int \alpha = \text{Vol}(V)n^{-1}\text{Tr}(a).$$

With a second length scaling becomes rather ambiguous. The identity (2.1) can be considered as a relation between $\bar{k}$ and $n$ with $r$ fixed or as a relation between $r$ and $n$ with $\bar{k}$ fixed. The limits when $n \rightarrow \infty$ are respectively the sphere of radius $r$ and the noncommutative plane. In the case at hand the volume $V$ is a circle bundle over one of these objects. We would like to include the case where the circle and the sphere scale differently. We introduce therefore a constant $\epsilon$ which is the ratio of the corresponding radii and scale $r \rightarrow n^{\gamma}r$. The integral is given in terms of the trace as

$$\int \alpha = \pi^2 r \epsilon n^{3\gamma} k \text{Tr}(a).$$  \hspace{1cm} (2.7)
When $\gamma = 0$ the radius $r$ remains fixed and when $\gamma = 1/2$ the length scale $k$ remains fixed. If we require that the radius of the circle remain fixed under a change of $r$ we must choose $\epsilon = n^{-\gamma}$.

In the abstract theory of integration on noncommutative algebras [22] there is but one dimension in which the integral makes sense and all matrix algebras are considered to be of dimension zero. We shall enlarge the algebra by adding a time variable and look for time-dependent configurations which tunnel between two static configurations. These are referred to as (fuzzy) instantons.

### 3 Fuzzy Instantons

Let $\theta^a$ be a real frame [23, 24, 25] for the differential calculus over the matrix factor and let $\theta^0 = dt$ be the standard de Rham differential along the real line. The differential $df$ of $f$ can be written as

$$df = e_a f \theta^a + \dot{f} \theta^0, \quad \dot{f} = \partial_t f$$

Over the algebra $\mathcal{A}$ we introduce the product calculus described in Section 3. An arbitrary 1-form $\omega$ can be expanded in the basis $\theta^\alpha = (\theta^0, \theta^a)$. Using the notation of the Appendix we can write

$$\omega = \omega^\alpha \theta^\alpha$$

and therefore in terms of $\phi = \omega - \theta$ the curvature

$$\Omega = d\omega + \omega^2 = \frac{1}{2} \Omega_{\alpha\beta} \theta^\alpha \theta^\beta$$

can be written as

$$\Omega_{0a} = \dot{\phi}_a + e_a \phi_0 + [\phi_0, \phi_a], \quad \Omega_{ab} = [\phi_a, \phi_b] - C^c_{ab} \phi_c.$$  

The structure constants $C^c_{ab}$ are those of $SU_2$ and for convenience they contain an extra factor $r^{-1}$: $C_{abc} = r^{-1} \epsilon_{abc}$. The covariant derivative $D_a$ is defined by

$$D_a \phi_b = [\phi_a, \phi_b] - \frac{1}{2} C^c_{ab} \phi_c.$$  

The equations we use are the analogs

$$D_a \Omega^{\alpha\beta} = 0,$$

of the usual Maxwell equations. We recall that we have changed the ‘space’ rather than the theory; the local gauge group remains the set of unitary elements of the algebra. The equations become

$$D_a \Omega^{a0} = 0, \quad D_0 \Omega^{0b} + D_a \Omega^{ab} = 0.$$  

The first equation yields

$$D_a \Omega^{a0} = [\phi_a, \Omega^{a0}] = -[\phi^a, \dot{\phi}_a] + [\phi^a, e_a \phi_0] + [\phi^a, [\phi_a, \phi_0]].$$

If we look for solutions with $\phi_0 = 0$ then we see that $[\phi^a, \dot{\phi}_a] = 0$ which implies that $\dot{\phi} \propto \phi$. If we choose the Coulombe gauge there is one particularly simple Ansatz given by $\phi_a = e(t) \lambda_a$. This can be also written as

$$\phi = -e(t) \theta, \quad \omega = (1 - e(t)) \theta.$$
The form of the Ansatz is not gauge-invariant. The $e(t)$ is a priori an arbitrary element of the algebra.

The field strength is

$$\Omega = e(\theta e - \theta)\theta + \dot{e}\theta\theta^0.$$ (3.1)

Since $\theta$ is the ‘Dirac’ operator this is the same as the usual expression

$$\Omega = edede$$

when $e$ is a projector, that is in the present context, when $t \rightarrow \pm \infty$. We shall compute only the simple case with $e(t)$ in the center. Using the relation (3.1) we find that the (euclidean) field equations reduce to

$$r^2 \ddot{f} - f(f - 1)(2f - 1) = 0.$$ (3.2)

The roots $f = 0, 1$ correspond to the two stable ground states of the system and the root $f = 1/2$ to the unstable solution [4]. The instanton solution we give below interpolates between the first two, passing through the third.

An instanton is described by two partitions of an arbitrary integer $n$, one which characterizes the soliton at $t \rightarrow -\infty$ and the other at $t \rightarrow +\infty$. In the case with $e$ in the center the condition (3.5) simplifies to the differential equation

$$r^2 \dot{f}^2 = f^2(f - 1)^2,$$ (3.2)

a nonlinear first-order equation, which is also a first integral of the field equations. The general solution

$$f = \frac{1}{2}(1 + \tanh(\frac{t}{2r} + b)), \quad b \in \mathbb{R}$$ (3.3)

interpolates between $f(-\infty) = 0$ and $f(+\infty) = 1$. This has the same form as the classical double-well instanton [26].

We define the integral as usual in terms of the trace and use the definitions of Sections 2 to define the action as an integral of a 4-form. We set, that is

$$S = \frac{1}{2} \int_{S^3 \times \mathbb{R}} \Omega \ast \Omega.$$ (3.4)

The product is in the algebra of forms; the star refers to duality. As in the commutative case we use the duality condition (3.5) to write the action (formally) proportional to the 2nd-Chern number:

$$S = \pm \frac{1}{2} \int_{S^3 \times \mathbb{R}} \Omega^2.$$ (3.5)

To calculate $S$ however it is more convenient to use the fact that the 4-form $\text{Tr}(\Omega^2)$ is the exterior derivative of a 3-form,

$$\int_{S^3 \times \mathbb{R}} \Omega^2 = \int_{S^3 \times \mathbb{R}} dK, \quad K = \omega d\omega + \frac{2}{3} \omega^3.$$
We can express then the action

\[ S = \frac{1}{2} \int_{S^3} K(+\infty) - \frac{1}{2} \int_{S^3} K(-\infty) \]

as an ordinary volume integral. We apply Stokes' theorem only along the (ordinary) time axes; in general it makes less sense as a relation amongst traces of operators. Our solution tunnels between \( f = 0 \) at \( t = -\infty \) to \( f = 1 \) at \( t = \infty \). That is \( K(+\infty) = 0 \) and

\[ K(-\infty) = -\frac{1}{3} \int \theta^3. \]

But

\[ \int \theta^3 = -\int \lambda_a\lambda_b\lambda_c\theta^a\theta^b\theta^c = \frac{1}{4\pi^2}\epsilon n^{2+3\gamma}. \]

We find then that the action of the instanton solution (3.3) in the case of most interest, with \( \epsilon = 1 \) and \( \gamma = 0 \), is

\[ S[\gamma = 0] = \frac{1}{3!} \int_{S^3} \theta^3 = \frac{1}{4!}\pi^2 n^2. \]

It is singular in the limit when \( n \to \infty \). It would remain finite if we let \( \epsilon \) scale as

\[ \epsilon = n^{-2-3\gamma}. \]

In the case of the instanton \( T_l^n \) which tunnels

\[ [l, 1, \cdots 1] \to [1, \cdots 1] \]

the action (when \( \gamma = 0 \)) is given by

\[ S[T_l^n] = \frac{1}{3!} \int_{S^3} \theta^3 = \frac{1}{4!}\pi^2 l^2. \quad (3.4) \]

This is the basic relation which we find. It involves a new integer \( l \) which in the commutative limit can combine with \( n \) to form an irrational number, the limit of the sequence \( l/n \).

We use in an essential way the fact that the differential calculus over the matrices is based on a module of 1-forms which is free of rank 3. With the addition of an euclidean time variable this yields a differential calculus of rank 4. We can thus introduce a duality

\[ *(\theta^a\theta^0) = \frac{1}{2} g^{ad} \epsilon_{bed} \theta^b \theta^c, \quad *(\theta^a\theta^b) = g^{ac} g^{bd} \epsilon_{cde} \theta^d \theta^0 \]

and look for the special set of solutions for \( \epsilon(t) \) which satisfy the self-duality conditions

\[ \Omega = \pm * \Omega. \quad (3.5) \]

This clarifies the duality in the fuzzy spheres found previously [27] and could also shed light on the appearance of Chern-Simons terms in the work or Alekseev et al. [28].
In terms of components the self-duality condition can be written
\[ \Omega_{ab} = g^{cd} \epsilon_{abcd} \Omega_{cd} \]
and therefore the self-dual and anti-self-dual parts of the curvature 2-form can be written
\[ \Omega^\pm = \frac{1}{2} \Omega_{ab} \theta^a \theta^b, \quad \Omega^\pm_{ab} = \frac{1}{2} (\Omega_{ab} \pm g^{cd} \epsilon_{abcd} \Omega_{cd}) . \]

In the projector case these become
\[ \Omega^\pm_{ab} = [\epsilon \lambda_a, \epsilon \lambda_b] - C_{ab}^c \epsilon \lambda_c \pm r C_{ab}^c \dot{\epsilon} \lambda_c . \]

If we raise the indices of the components and introduce the 3-dual
\[ \Omega^c = \frac{1}{2} g^{cd} \epsilon_{abcd} \Omega_{cd} \]
then the duality condition can be written
\[ \dot{\epsilon} \lambda^c = \Omega^c [\epsilon \lambda_a] . \quad (3.6) \]

As usual, from the inequality
\[ \int (\Omega \pm * \Omega) * (\Omega \pm * \Omega) \geq 0 \]
we can conclude that the action \( S[\omega] \) of any configuration \( \omega \) ‘in the same sector’ is bounded below by \( S[T^n_n] \):
\[ |S[\omega]| \geq S[T^n_n] . \]

The integer \( n \) measures the ‘amplitude’ of the instanton and \( l \) measures the extension. In a conformal-invariant theory the value of the action would not depend on these two parameters.

4 Fuzzy Solitons

Because the radius of the fuzzy sphere is related to a Casimir operator and lies in the center of the algebra one can consider field configurations which are concentrated entirely on one sphere. These we refer to as (fuzzy) 2-solitons, with the alias ‘fuzzy \( D2 \)-branes’. Implicit in the expression for the energy density
\[ \epsilon(r) = \frac{1}{2} \Omega_{\alpha\beta} \Omega^{\alpha\beta}(r) \]
is a choice of differential calculus along the radial direction \( r = x^4 \). The Greek indices take the values 1...4. If we embed the solution in the continuous line and use the radial parameter \( r \) instead of euclidean time in the solution we gave in the previous section we find a (fuzzy) 3-soliton, with the alias ‘fuzzy \( D3 \)-brane’. As we shall see in the next section the equilibrium configuration is a distribution which satisfies one of the ‘duality’ conditions \( (3.6) \). It is given, for large \( n \), by the function
\[ f(l) = \frac{1}{2} (1 - \tanh(2\pi k^{-1} r^2)) = \frac{1}{2} (1 - \tanh(\pi l)), \quad l \leq n , \]
the same as (3.3) below but with the replacement

\[ -t \mapsto \pi k^{-1} r^3 \]

and with \( b = 0 \). The solution has a maximum at the origin and drops off exponentially toward infinity.

To write an expression for the energy of the \( l \)th sphere we use the definition of integral given above. For large \( n \) then and for all \( l \lesssim n \) we set

\[ E^n_l = \int_{r_l}^{r_{l+1}} \epsilon(r). \]

If \( \gamma = 0 \) we see that \( E^n_l = E_l \) is independent of \( n \). The sum

\[ E^n = \sum_{0}^{n} E^n_l \]

is the total energy.

Classically all the vacuum modes have energy equal to zero; including the zero-point fluctuations this will no longer be the case. For example the completely reducible and the irreducible configuration will have vacuum energy given respectively by

\[ E_1 \simeq \frac{1}{2} \hbar \omega \cdot n^2, \quad E_n \simeq \frac{1}{2} \hbar \omega \cdot \sum 2l(2l+1), \quad \omega = \sqrt{2}/r. \]

In the first there are \( n^2 \) modes of equal frequency \( \omega \); in the second, for each \( l \lesssim n \) there are \( 2(2l+1) \) modes of frequency \( l\omega \). The remaining vacuum energies lie between these two values. To discuss this it is convenient to introduce a Fock space.

## 5 Fock space

We have found an instanton solution which tunnels between \( f = 0 \) and \( f = 1 \). If we return to the definition of the curvature as a functional of the fields \( \phi \) we see that this corresponds to a transition from the irreducible representation of dimension \( n \) to the completely reducible representation of the same dimension. The former corresponds to the partition \([n]\) of \( n \); the latter to the partition \([1, \ldots, 1]\). Let \([n_1, \ldots, n_k]\) be an arbitrary partition of \( n \), a set of non-increasing integers whose sum is equal to \( n \). To it corresponds a representation of \( SO_3 \) which is the sum of irreducible representations of dimension \( n_i \). By the same construction there is, for each index \( i \), an instanton which tunnels between \([n_i]\) and \([1, \ldots, 1]\). There are perhaps other instantons, those which correspond to transitions between non-trivial projectors. If \([n_1, \ldots, n_k]\) is a partition of \( n \) into \( k \) integers then the corresponding representation is block diagonal and reducible to \( k \) representations of dimensions \( n_i^2 - 1 \) for \( 1 \leq i \leq k \). Let \( e_i \) be the projector onto the \( i \)th sector. An arbitrary projector \( e \) can be written in the form

\[ e = \sum_i \epsilon_i e_i, \quad \epsilon_i = 0, 1. \]
The corresponding expression (3.1) for the field strength is given by
\[ \Omega = \sum_i \Omega_i \]
with each \( \Omega_i \) of the form (3.1). Each corresponding \( f_i \) evolves independently according to its field equation. The instantons tunnel therefore between different partitions. Since we have found no others we shall suppose that (3.3) is the only type of instanton. It is the only \( SO_3 \)-invariant type.

If one quantizes the system one obtains a bosonic Fock space of ordinary ‘vacuum modes’. Each of the \( p(n) \) minima of the potential gives rise to a tower of states in general different from each other. Besides these modes there is also a Fock space \( F \) of ‘tunneling modes’, each with only one quantum number, an integer \( l \). We set
\[ |\cdots n_{i_1}, \cdots n_{i_k}, \cdots \rangle = [l_1, \cdots l_k]. \]

The integer \( n_{i_j} \) in the \( l_j \)th position indicates the presence of \( n_{i_j} \) tunneling modes with quantum number \( l_j \), each of which is an irreducible representation of rank \( l_j \). The tunneling modes interact with all the vacuum modes and change their energy eigenvalues in a rather complicated way. Without the tunneling modes the vacuum modes do not interact so we consider the tunneling modes as responsible for the dynamics. They have a single quantum number, the integer \( l \). An instanton gas is an ensemble of tunneling modes, considered as a Bose gas. The integer \( n \) cannot be considered as conserved and we can suppose then that the chemical potential vanishes. The best description of the dynamics is through some examples.

1. In Fock-space notation the basic transition is of the general form
\[ |0, \cdots n_{i_1}, \cdots 0, \cdots \rangle \leftrightarrow |l_i, \underbrace{0, \cdots 0}_l \cdots \rangle. \]

2. More generally we have
\[ |0, \cdots n_{i_1} - 1, \cdots n_{i_1}, \cdots n_{i_k}, \cdots \rangle \leftrightarrow |l_i, \underbrace{0, \cdots 0}_l n_{i_k}, \cdots \rangle. \]

One of the \( n_{i_1} \) irreducible components of dimension \( l_j \) in the given representation ‘decays’ into \( l_j \) representations of dimension zero, or the inverse. The probability of this transition is proportional to the barrier penetration rate
\[ p[T_{l_i}^n] = A[T_{l_j}^n]e^{-S[T_{l_j}^n]} \]

The \( A[T_{l_i}^n] \) is a WKB amplitude difficult to calculate in general.

3. The transition whereby \( n \) ‘decays’ into \( n - 1 \) and 1 can be written in the Fock-space notation
\[ |0, \cdots 0, \cdots 1_n \rangle \leftrightarrow |1, \cdots 0, \cdots 1_{n-1}, 0 \rangle. \]

By our assumption that the only transition is that between the irreducible and completely reducible representation the transition can only proceed via the intermediate state
\[ |0, \cdots 0, \cdots 1_n \rangle \leftrightarrow |n, \cdots 0, \cdots 0 \rangle \leftrightarrow |1, \cdots 0, \cdots 1_{n-1}, 0 \rangle. \quad (5.1) \]
4. All the different partitions are ground states since the action of each vanishes at least in the classical approximation. The tunneling phenomena would lift this degeneracy and make some partitions more favorable. Consider the case \( n = 2 \) with its two partitions \([2]\) and \([1, 1]\). These are not large values of \( n \) and the ‘exact’ formula must be used, obtained by replacing \( n^2 \) by \( n^2 - 1 \). The two degenerate levels split by an amount proportional to the transition probability \( p = Ae^{-3\pi^2/4} \).

We can now better formulate the problem of the classical limit. We recall that this limit is singular from several points of view. The volume form, for example, is closed but not exact in the limit but it is exact for all finite \( n \). We found that the action of the transition between the two extreme partitions of \( n \) diverges with \( n \) but we claim that the limiting bundle has a well-defined Chern number. In the analogy with statistical physics the euclidean-time world sheet of each fuzzy surface appears as a probability distribution. The action is the energy of the configuration considered in one dimension higher. Since we are working in one dimension lower than the ‘physical one’ [sic.] the instantons resemble in all respects solitons. From the ‘gas’ point of view the penetration probability is to be thought of as a creation probability from the vacuum, that is, a distribution probability for the existence of a configuration. By standard WKB-approximation calculations we have an estimate of the barrier-penetration probability \( p_l \). In terms of an amplitude \( A[T^n] \) it is given by \[ p_l = A[T^n] e^{-\pi^2 l^2/4}, \quad l \lesssim n, \quad l \gg 1. \]

We shall use the language of statistical physics because of the interpretation of multiple paths as an instanton gas. As a first approximation we shall suppose that the amplitude is independent of \( l \) and we shall write

\[ p_l = e^{\beta F - \pi^2 l^2/4}, \]

in terms of \( \beta F = \log A \). This is the probability distribution of a classical Bose-Einstein gas with energy spectrum

\[ E_l = \frac{1}{2} \pi^2 l^2. \]

As in ordinary physics a temperature at which the entropy term and the energy term just cancel could be indicative of a phase transition. One interesting temperature which must appear also is a maximal or Hagedorn temperature \( T_H \). Noncommutative geometry certainly requires that \( T_H \) exist and further that

\[ T_H^2 k \lesssim 1. \]

In the recent literature there are calculations of such temperatures within the context of matrix models. To a certain extent the expression \( \beta = \sqrt{k} \) could be said to play this role. From the expression of the statistical distribution it appears that a configuration with more isolated \( D \)-branes is more probable. We can conclude that they repulse due to the instanton exchange. This would seem to contradict the results of Gao & Yang.

6 Speculative conclusion

We have calculated transition instantons between various vacua of ‘Maxwell’s theory’ on different matrix geometries. There are two ‘mode gases’ which one can introduce. The
modes around the classical vacua, the ‘momentum’ modes, interact with a gas of tunneling modes, the instantons; otherwise the gas is free. The Hopf fibration has Chern number \( c_1 = -1 \) and for \( n \) large the fuzzy sphere is a good approximation. One can say then that the instanton tunnels between a good and very bad approximation to the sphere but not between two distinct topological sectors. If one take the time variable together with the circle of the fibration then by compactifying one obtains a second sphere. The space becomes the euclidean Schwarzschild solution, whose instanton number is the product of an electric monopole charge and a magnetic monopole charge equal to \(-1\). The electric charge must be identified as a topological invariant of the matrix factor. This is strictu sensu neither possible nor desirable. But the infinite potential barrier associated with a topological obstruction can be replaced by a very large barrier which would tend to infinity with the integer \( n \). We imagine the action of the partition \([n]\) as a quantum of energy for a state of the bosonic Fock space described above and that the action of the element of the instanton which tunnels to the classical manifold appears as the associated free energy. We refrained from speculating about possible relations between the limit of the tunneling modes and winding modes.

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7 Appendix: projectors

If we consider a manifold \( V \) of dimension \( d \) as a subset of some higher-dimensional euclidean space \( \mathbb{R}^n \) then the algebra of functions \( \mathcal{A} = C(V) \) can be defined as a formal algebra in terms of generators and relations. The coordinates \( x^i \) of \( \mathbb{R}^n \) are the generators of an algebra \( \hat{\mathcal{A}} \) which satisfy only the commutation relations \([x^i, x^j] = 0\). Within \( \hat{\mathcal{A}} \) there is an ideal defined by constraint relations \( R(a)(x^i) \) which define \( V \) as a submanifold. If \( \hat{\mathcal{A}} \) is noncommutative then we suppose it to be still characterized by commutation relations but we shall no longer suppose that the constraint relations necessarily generate a non-trivial 2-sided ideal. The projection of \( \hat{\mathcal{A}} \) onto \( \mathcal{A} \) which describes the embedding in the commutative limit will be a vector space map which to a certain approximation only can be considered an algebra epimorphism. If one introduce the moving frame \( dx^i \) on \( \mathbb{R}^n \) then the latter acquires the structure of a flat differential manifold. This defines by the embedding, that is, by the relations, a moving frame \( \theta^a \) on \( V \) and a splitting of the module of sections \( \mathcal{A}^a \) of \( T^* V \) into a direct sum

\[
\mathcal{A}^a = \Omega^1(V) \oplus \mathcal{N}
\]

of the module of 1-forms and a complement. The metric and the frame on \( V \) are determined by the embedding relations. The construction is most elegantly described in terms of projec-
tors. Similar constructions on ordinary and graded spheres have been reviewed in the recent literature [34, 35, 36]. The noncommutative generalization seems to be folklore.

There are two cases of particular interest. The simplest case is when all the constraints could be referred to as commutation relations in the sense that they all vanish in the commutative limit. The dimension of $V$ is equal to $n$ and the projection of $A^n$ onto $\Omega^1(V)$ is an invertible element of $M_n(A)$, a projector of maximal rank. Let $\tilde{\theta}^i = dx^i$ be a basis of $\Omega^1(\tilde{A})$. Then $\Omega^1(A)$ is defined by a projector $e \in M_n(\tilde{A})$ of rank $n$

$$\theta^a = e^a_i dx^i$$

a set of generators of $\Omega^1(A)$. The projection $\pi$ has an extension to all of $\Omega^*(A)$ and we can identify $e$ as a left inverse of $\pi$. We shall suppose that there exist $n$ matrices $\gamma^a \in M_n(A)$ which satisfy the relations

$$\gamma^a \gamma^b = P^{ab}_{\;\;cd} \gamma^c \gamma^d + g^{ab}. \quad (7.1)$$

This is a slight generalization of a proposal by Pusz and Woronowicz [37]. The $P$ determines [24] the structure of the algebra of forms. The algebra generated by the $\gamma^a$ is the generalized Clifford algebra and it bears the same relation to the exterior algebra as its classical counterpart. The correspondence between the Clifford and exterior algebras is only transparent at the level of 1-forms. It is given by the module map generated by $\gamma^a \mapsto \theta^a$. The extension to the entire Clifford algebra is given, for example, by the map

$$\gamma^a \gamma^b = P^{ab}_{\;\;cd} \gamma^c \gamma^d + g^{ab} \mapsto P^{ab}_{\;\;cd} \theta^c \theta^d = \theta^a \theta^b$$

and consists in dropping the ‘symmetric’ part. For this to be well defined the $P$ would have to satisfy the braid equation.

To every element $\iota \in M_n(A)$ we associate an element $e$ defined as

$$e = \frac{1}{2} (1 - \iota). \quad (7.2)$$

Then $e$ is hermitian if and only if $\iota$ is hermitian and $e$ is an idempotent if and only if $\iota$ is an involution. In particular we consider

$$\iota = \rho^{-1} \lambda_a \gamma^a.$$ 

The product here is a tensor product and $\rho \in \mathbb{C}$. The $\iota$ will be hermitian if

$$(\gamma^a)^* = \gamma^a, \quad (\rho^{-1} \lambda_a)^* = \rho^{-1} \lambda_a.$$

The differential calculus is real if and only if the $\lambda_a$ are antihermitian. Using (7.1) we can write the square of $\iota$ as

$$\rho^2 \iota^2 = \lambda_a \lambda_b g^{ab} + \lambda_a \lambda_b P^{ab}_{\;\;cd} \gamma^c \gamma^d.$$ 

It is an involution if the conditions

$$g^{ab} \lambda_a \lambda_b = \rho^2, \quad P^{ab}_{\;\;cd} \lambda_a \lambda_b = 0 \quad (7.3)$$

are satisfied. This is the case [18] for the sequence of algebras $\mathbb{R}_q^n$ for example. In these cases the condition on the norm of $\lambda$ is automatically satisfied and so $e$ is a projector if and
only if the commutation relations are satisfied. This can be stated as the proposition that the element \( e \in M_n(\mathbb{R}^n_q) \) defined in (7.2) is an hermitian idempotent if and only if all the relations which define \( \mathbb{R}^n_q \) are satisfied. One notes that in these examples the ‘holonomic’ basis does not commute with the algebra but the basis \( \theta^a \) does. Some could more properly state that the holonomic basis \( R \)-commutes and the basis \( \theta^a \) does not \( R \)-commute. In the commutative limit \( e \) tends to the identity, the unit element that is of the \( SO(n) \) group. From (2.3) it follows that

\[
[e, f] = -\frac{1}{2} [i, f] = \frac{1}{2} \rho^{-1} \gamma^a e_a f.
\]

This formula is to be compared with the Formula (2.4) for the differential. The \( \gamma \)-matrices are so defined that

\[
\det_q (\gamma^a \gamma^b) = g^{ab}.
\]

One concludes then that \((\det_q i)^2 = 1\) and that one can choose the normalization so that

\[
\det_q (i) = 1.
\]

The paradigms are the ‘quantum’ deformation \( \mathbb{R}^n_q \) of euclidean space introduced [18] by Faddeev et al. The center is generated by a single element \( r \) which one can think of as a radius. The quotient \( S^q_n \) of \( \mathbb{R}^n_q \) by the ideal generated by the central element is known as a quantum sphere. An element of \( M_n(\mathbb{R}^n_q) \) has been found [38] which is an hermitian idempotent if and only if all the relations which define \( S^{n-1}_q \) are satisfied.

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