On conditional scalar increment and joint velocity–scalar increment statistics

Hengbin Zhang, Danhong Wang and Chenning Tong

Department of Mechanical Engineering, Clemson University, Clemson, SC 29634-0921, USA
E-mail: ctong@ces.clemson.edu

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Abstract. Conditional velocity and scalar increment statistics are usually studied in the context of Kolmogorov’s refined similarity hypotheses and are considered universal (quasi-Gaussian) for inertial-range separations. In such analyses the locally averaged energy and scalar dissipation rates are used as conditioning variables. Recent studies have shown that certain local turbulence structures can be captured when the local scalar variance $\langle \phi''^2 \rangle_r$ and the local kinetic energy $k_r$ are used as the conditioning variables. We study the conditional increments using these conditioning variables, which also provide the local turbulence scales. Experimental data obtained in the fully developed region of an axisymmetric turbulent jet are used to compute the statistics. The conditional scalar increment probability density function (PDF) conditional on $\langle \phi''^2 \rangle_r$ is found to be close to Gaussian for $\langle \phi''^2 \rangle_r$ small compared with its mean and is sub-Gaussian and bimodal for large $\langle \phi''^2 \rangle_r$, and therefore is not universal. We find that the different shapes of the conditional PDFs are related to the instantaneous degree of non-equilibrium (production larger than dissipation) of the local scalar. There is further evidence of this from the conditional PDF conditional on both $\langle \phi''^2 \rangle_r$ and $\chi_r$, which is largely a function of $\langle \phi''^2 \rangle_r/\chi_r$, a measure of the degree of non-equilibrium. The velocity–scalar increment joint PDF is close to joint Gaussian and quad-modal for equilibrium and non-equilibrium local velocity and scalar, respectively. The latter shape is associated with a combination of the ramp–cliff and plane strain structures. Kolmogorov’s refined similarity hypotheses also predict a dependence of the conditional PDF on the degree of non-equilibrium. Therefore, the quasi-Gaussian (joint) PDF, previously observed in the context of Kolmogorov’s refined similarity hypotheses, is only one of the conditional PDF shapes of inertial range turbulence. The present study suggests that such analyses provide a connection between statistical and structural descriptions of inertial-range turbulence.

1 Author to whom any correspondence should be addressed.
1. Introduction

Velocity and scalar increments have been used extensively to study the fine-scale structure of turbulent velocity and scalar fields [1]–[9]. The increments are often analysed in the context of Kolmogorov’s refined similarity hypotheses (hereafter referred to as K62) [10]. According to the hypotheses the velocity and scalar increments for an inertial-range separation \( r \), \( \Delta u(r) = u(x + r) - u(x) \), and \( \Delta \phi(r) = \phi(x + r) - \phi(x) \) can be written as [3, 11]

\[
\Delta u(r) = V\epsilon_r^{1/3}r^{1/3}, \quad \Delta \phi(r) = V_\phi\chi_r^{1/2}\epsilon_r^{-1/6}r^{1/3},
\]

(1)

where \( \epsilon_r \) and \( \chi_r \) are the kinetic energy dissipation and scalar dissipation averaged over a sphere of radius \( r \), respectively, and \( V \) and \( V_\phi \) are universal stochastic variables independent of \( \epsilon_r \) and \( \chi_r \). The energy dissipation rate is defined as \( \epsilon = 2\nu s_{ij}s_{ij} \), where \( \nu \) and \( s_{ij} \) are the kinematic viscosity and the strain rate tensor, respectively. The scalar dissipation rate is defined as \( \chi = D(\partial \phi/\partial x_j)(\partial \phi/\partial x_j) \), where \( D \) is the molecular diffusivity. According to equation (1), the normalized conditional probability density function (PDF) of \( \Delta u(r) \) conditional on \( \epsilon_r \) and that of \( \Delta \phi(r) \) conditional on \( \epsilon_r \) and \( \chi_r \) (specifically \( \epsilon_r^{1/3} \) and \( \chi_r\epsilon_r^{-1/6} \), which provide velocity and scalar scales at scale \( r \)) are universal. Previous experimental and numerical results support these predictions and show that the conditional PDFs are usually quasi-Gaussian [3, 11, 12], i.e. with kurtosis close to three but with non-zero skewness (only when the separation is comparable to the dissipation scales do the PDFs deviate significantly from Gaussian and become bimodal [3, 11, 12]). Therefore, for an inertial-range separation, conditional velocity and scalar increments, statistics conditional on their respective scales are generally considered to be universal.

However, non-universal inertial-range increment conditional PDFs ranging from near Gaussian to bimodal with kurtosis much smaller than three have recently been observed [9]. The conditional velocity increment PDF conditioned on the local kinetic energy

\[
k_r = \frac{1}{2} \int |u(x') - \langle u \rangle_r(x)|^2 G(x' - x) \, dx',
\]

(2)

is found to be not far from Gaussian when the local energy is comparable to or smaller than its mean value but becomes bimodal when the local energy is large [9]. Here \( \langle u \rangle_r \) is the locally averaged velocity and \( G(x) \) is a box filter function.
Other non-universal conditional inertial-range statistics have also been observed. An investigation of the velocity filtered joint density function [13] (FJDF, which is essentially the subgrid-scale joint PDF used in large eddy simulation) shows that the FJDF conditional on the local turbulent kinetic energy \( k_r \) has an approximately joint-normal shape when \( k_r \) is small compared with its mean and that the conditional local velocity field is in quasi-equilibrium, i.e., the production of \( k_r \) is less than or equal to the locally averaged energy dissipation. Here we refer to both cases as quasi-equilibrium because the conditional local velocity has very similar characteristics. When \( k_r \) is large, the conditional FJDF is sub-Gaussian and the conditional local velocity is in non-equilibrium. The study also suggests that when the local energy is large, there exist linear-flow structures, such as plane strain, axisymmetric contraction and expansion in the local velocity fields.

In studies of scalar filtered density function (FDF) [14, 15], it was found that the FDF conditional on the local scalar variance (or the subgrid-scale scalar variance)

\[
\langle \phi'^2 \rangle_r = \frac{1}{2} \int |\phi(x') - \langle \phi \rangle_r(x)|^2 G(x' - x) \, dx' \tag{3}
\]

is close to Gaussian and bimodal when the local scalar variance is small and large compared with its mean, respectively. There is evidence that the local scalar field is in quasi-equilibrium and non-equilibrium, respectively. These studies also show that the bimodal FDF for large local variance is due to the ramp–cliff (or diffusion layer) structure in the local SGS scalar. Such a structure has also been found to be the cause of the small-scale anisotropy observed in scalar fields [16, 17].

Since universality (at least in the context of K62) is considered a key concept in understanding the inertial-range turbulence, the observed non-universal conditional PDFs require further attention. In the present work we investigate the conditional scalar increment PDF and the velocity–scalar increments JPDF conditional on the local scalar variance and the local kinetic energy. We also examine the influence of the locally averaged scalar and energy dissipation rates on the conditional PDF and JPDF. Because \( \langle \phi'^2 \rangle_r \) and \( k_r \) are inertial-range variables and provide local scalar and velocity scales, conditional statistics conditional on them are important for understanding the properties of the inertial-range turbulence. Such conditioning provides an alternative probabilistic decomposition of velocity and scalar increments to that of K62 and can reveal the new structure of the local turbulence. In addition, the impact of the local velocity and scalar structures on the conditional increment statistics are important for studying the scaling of inertial-range turbulence as the structures may lead to saturation of the inertial-range scaling components of structure functions [18].

We use experimental data obtained in the fully developed region of an axisymmetric turbulent jet with passive scalar fluctuations to study velocity–scalar increment PDFs. The rest of the paper is organized as follows. In section 2 we outline the experimental apparatus and flow conditions. The experimental results are discussed in section 3 followed by further discussions and the conclusions (in section 4).

2. Experimental facilities and flow conditions

The jet facility was housed in a large, air-conditioned room. The jet was produced with an assembly of a nozzle and a plenum chamber (figure 1), which contains a section of flow-straightening honeycomb and three stages of damping screens. The assembly was mounted...
Figure 1. A schematic of the experimental set-up including a magnified view of the X-wire array. The array is used to perform averaging in the transverse direction in addition to the streamwise averaging by invoking Taylor’s hypothesis.

vertically on a 5 × 5 ft² grill portion of the floor to allow the flow of entrainment air (figure 1). The flow downstream of the nozzle was surrounded by a circular screen (1/16” mesh size and 6 ft in diameter) to reduce the disturbances in the room. A collection hood was installed at a downstream distance of 260 nozzle diameters (3.9 m) to minimize the effects of the ceiling on the jet. The hood was connected to an exhaust fan with the flow rate adjusted by a throttle. Jet air supply was heated with a pipe heater before entering the plenum chamber, producing an excess temperature (above the ambient) of 20°C at the nozzle exit. The jet nozzle had a fifth-order polynomial profile with a large contraction ratio (≈100), producing a nearly top-hat velocity profile at the nozzle exit.

All measurements were made for a jet exit velocity $U_j$ of 40 m s$^{-1}$. The jet Reynolds number $Re_j$ based on the nozzle diameter $D_j$ and the jet exit velocity $(U_j D_j/\nu)$ was 40 000, where $\nu$ is the kinematic viscosity. The corresponding Taylor microscale Reynolds number $R_\lambda = \langle u^2 \rangle^{1/2} / \lambda/\nu$ was approximately 260, where $\langle u^2 \rangle^{1/2}$ is the rms streamwise velocity fluctuation and $\lambda$ is the Taylor microscale. Data were collected at $x/D_j = 80$, well into the self-similar (fully developed) region of the jet. The flow conditions are presented in table 1. We limit our measurement location to the jet centreline to avoid flow reversal and to minimize errors arising from the use of Taylor’s hypothesis. The mean axial velocity on the jet centreline $U_c$ at this downstream location was 3.07 m s$^{-1}$ and the resulting $U_j/U_c$ value was comparable with previous results [19]–[21]. The excess mean temperature was approximately 1.25°C and the normalized temperature was
similar to those in previous studies [22]. The Kolmogorov scale \( \eta = (\nu^3/\epsilon)^{1/4} \) and the scalar dissipation scale \( \eta_\phi = (\gamma^3/\epsilon)^{1/4} \) were 0.16 and 0.22 mm respectively. Here \( \gamma \) and \( \epsilon \) are the thermal diffusivity and the energy dissipation rate respectively. Under these flow conditions the Kolmogorov frequency of the signals \( (U_c/2\pi \eta = 2.5 \text{ kHz}) \) can be fully resolved by the sensors.

The integral scale of the velocity field \( \ell \) is estimated to be 75 mm using \( \langle u_1^2 \rangle^{3/2}/\langle \epsilon \rangle \). This value is close to the value 80 mm estimated using the empirical formula given by Tennekes and Lumley [23]. The non-buoyant (momentum dominated) region was determined to be \( x/D_j \leq 120 \) by a criterion based on a jet Froude number (the ratio of the square of the Reynolds number to the Grashof number) given by Chen and Rodi [24]. The effects of the initial jet-to-air density ratio (\( \approx 0.88 \)) on the properties of the jet, such as the spreading rate and the rms fluctuations of velocity and temperature, were small [25, 26]. Thus, our measurement locations were well within the region in which the buoyancy effects were negligible and the temperature fluctuations were dynamically passive.

Measurements of the local scalar variance, locally averaged turbulent kinetic energy and locally averaged energy and scalar dissipation rates require spatial averaging of turbulent fields. Here we use one-dimensional averaging in the streamwise direction. A two-dimensional averaging technique using a sensor array developed by Tong et al [27] has been used previously to study velocity increments [9]. In this averaging technique, the streamwise averaging was performed by invoking Taylor’s hypothesis and the cross-stream averaging was realized with a number of sensors (three in [9]) aligned in the cross-stream direction (figure 1). While the conditional velocity increment statistics for one- and two-dimensional averaging are similar [9], the results for conditional scalar increment differ when there is a ramp–cliff structure in the conditional local scalar, due to the discrete nature of the averaging in the cross-stream direction (discussed further in section 4). Therefore, in the present study we use one-dimensional averaging in the streamwise direction to study the increment (the centre probe), and only use two-dimensional averaging to obtain locally averaged strain rate (section 3.2). Three separations, 10, 20 and 40 mm, were used. These correspond to \( r/\ell = 0.13, 0.27 \) and 0.53, and \( r/\eta = 63, 125 \) and 250, respectively with the largest being close to the integral length scale \( \ell \).

To minimize the error associated with invoking Taylor’s hypothesis, instantaneous convection velocity obtained by low-pass filtering of the streamwise velocity component was used. The filter width for this purpose is twice the largest averaging domain size for obtaining the local variances and the locally averaged dissipation rates so that the convection velocity is relatively uniform within an averaging domain.

Temperature fluctuations were measured with platinum resistance wires of 0.625 \( \mu \text{m} \) in diameter, capable of a maximum frequency response of approximately 5 kHz. This was determined by analysing the heat transfer to (or from) the wire. Antonia et al [28] performed analyses and tests for platinum (90%)–rhodium (10%) wires using a pulse input and obtained similar results. (The wire response was affected by \( \rho c \) of the wire material, which is approximately the same for both platinum and the platinum–rhodium alloy. Here \( \rho \) and \( c \) are density and specific heat respectively.) The wire length \( l_w \) was approximately 0.4 mm (etched portion), giving a wire length/diameter ratio of approximately 600–700. This wire length is

| \( \langle U \rangle \) | \( \langle u_1^2 \rangle^{1/2} \) | \( \langle u_2^2 \rangle^{1/2} \) | \( R_s \) | \( \langle \epsilon \rangle \) | \( \eta \) | \( \langle \chi \rangle/\langle \phi^2 \rangle \) | \( \eta_\phi \) | \( \ell \) |
|-------------|-------------|-------------|---------|-------------|---------|-----------------|---------|-----|
| 3.07 m s\(^{-1}\) | 0.72 m s\(^{-1}\) | 0.61 m s\(^{-1}\) | 229 | 5.25 m\(^2\) s\(^{-3}\) | 0.16 mm | 5.52 s\(^{-1}\) | 0.22 mm | 75 mm |
chosen to achieve a sufficiently high spatial resolution \((l_w/\eta = 2.3, \text{ see } [29])\), while minimizing errors due to heat conduction to the prongs. The wires were connected to DC bridges with ultra-low-noise amplifiers. The probe current was set at 100 \(\mu\)A so that the velocity contamination of the temperature signal was negligible.

Simultaneous velocity measurements were made with three X-wire probes operated by TSI IFA 100 hot-wire anemometers with an overheat ratio of 1.8. The X-wire probes are placed approximately 0.75 mm from their pairing cold wires. The probes were calibrated using a modification of a method of Browne et al [30]. In this method a velocity–yaw angle relation at zero yaw angle for each wire was obtained. A yaw-angle–effective-wire-angle relation was obtained with yaw-angle calibration at a fixed velocity (3 m s\(^{-1}\) in the present study). The effective angles were then used as ‘geometric’ wire angles in computing the two velocity components. Due to the high signal-to-noise ratio of the resistance-wire temperature device, a very low excess temperature (1.25°C at the measurement location) can be used, rendering the temperature contamination of hot wires negligible. For the statistics considered, the differences between the corrected and uncorrected results are within 2%. The velocity and temperature signals were low-pass filtered at 5 kHz and amplified by Krohn-Hite 3364 filters. The signals were digitized at 5000 samples s\(^{-1}\) by a 12-bit National Instrument A/D converter (PCI-6071E) which has a maximum sampling rate of 1.25 \(\times\) \(10^9\) samples s\(^{-1}\) so that the inter-channel delay is much shorter than the sample interval. Approximately \(2 \times 10^8\) samples were used in the data analyses.

3. Results and discussions

In this section the measured conditional PDFs of scalar and velocity–scalar increments for three separations \(r = 250\eta, 125\eta\) and 63\(\eta\) are presented. The smallest of the separations is approaching the dissipation scales whereas the largest is close to the velocity integral length scale. We focus on the results for \(r = 125\eta\), which are most representative of inertial-range turbulence in the present study.

3.1. Conditional scalar increment PDF

For all the scales considered, the PDF of the logarithm of the local scalar variance is close to Gaussian [14] indicating approximately log-normal PDF for \(\langle \phi''^2 \rangle_r\). The conditional PDF of the scalar increments is generally not far from Gaussian for \(\langle \phi''^2 \rangle_r < \langle \phi''^2 \rangle \) (figure 2). This is consistent with the near-Gaussian conditional scalar PDF which suggests a well-mixed conditional inertial-range scalar. Throughout the paper the conditional PDFs are normalized using the conditional means and the local variances of the variables concerned, unless otherwise noted. For large \(\langle \phi''^2 \rangle_r\), the conditional PDF becomes bimodal and the peaks become sharper as \(\langle \phi''^2 \rangle_r\) increases, again consistent with the bimodal conditional scalar PDF observed previously [14]. Such a conditional increment PDF shape has been observed previously for dissipation-scale separations [11], but not for an inertial-range separation. It may be closely related to the presence of ramp–cliff (diffusion-layer) structure in the scalar field previously observed (see e.g., [16, 31]). In such a structure two portions of well-mixed scalar are separated by a sharp interface (cliff); therefore the scalar increment has a fixed magnitude while its sign depends on the orientation of the interface. In the jet, the scalar values are on average higher on the upstream side; consequently, \(\Delta\phi\) across a cliff has a higher probability to take negative values, resulting in a
Figure 2. Conditional PDFs of the scalar increment conditional on the local scalar variance for three streamwise separations: (a) $r = 63\eta$; (b) $125\eta$; and (c) $250\eta$. The values of the normalized local scalar variance $\langle \phi'^2 \rangle_r/\langle \phi'^2 \rangle$ are given in the legend.
positive skewness of the increment PDF. The results also show that as the separation approaches the dissipation scales, the conditional PDF becomes more bimodal. This is because, in the dissipation scales, the scalar field essentially has a diffusion layer profile and the scalar increment will be bimodal. For such a separation, $\langle \phi'^2 \rangle_r$ is proportional to $\chi_r$ and therefore the conditional-increment PDFs conditional on the two variables respectively are similar.

Because a well-mixed and a poorly mixed local scalar generally have close to Gaussian and bimodal PDFs respectively [14, 15], the results in figure 2 suggest that $\langle \phi'^2 \rangle_r/\langle \phi'^2 \rangle_r$ can be used as an indicator of the mixedness of the local scalar. The change from Gaussian to bimodal conditional PDF shows that, when conditioned on the scalar scale given by the local scalar variance ($\langle \phi'^2 \rangle_r^{1/2}$), the scalar increments have non-universal PDFs. This is in contrast with the universal, quasi-Gaussian conditional-increment PDFs conditional on $\chi_r \epsilon_r^{-1/3}$ previously observed [3, 9] as hypothesized in K62. Note that, in K62, the parameters are $\chi_r$ and $\epsilon_r$; therefore the mean scalar and energy dissipation rates do not influence the conditional local turbulence. On the other hand, previous studies [13, 14] and the present work show that the mean local scalar variance and the mean local kinetic energy are important parameters determining the conditional statistics of the local turbulence. Therefore, the local turbulence depends on both the external parameters (which determine the average local variance) as well as the internal parameters (the instantaneous local variance).

Previous studies [9, 13] have shown that the non-universal conditional distributions of the inertial-range velocity are due to its degree of non-equilibrium under different conditions. There is also evidence suggesting that the non-universal conditional inertial-range scalar distributions are related to the degree of non-equilibrium of the local scalar fields. To examine the possible causes for the qualitatively different shapes of conditional PDFs when conditioned on $\chi_r$ and $\langle \phi'^2 \rangle_r$, we compute the conditional production rate of $\langle \phi'^2 \rangle_r$, $P_{\phi r} = -\langle u_j \phi \rangle_r - \langle u_j \rangle_r \langle \phi \rangle_r [\partial \langle \phi \rangle_r / \partial x_j]$, conditional on $\langle \phi'^2 \rangle_r$ and $\chi_r$, respectively (figure 3). The averaging domain size $r$ is $125 \eta$ which is used for the rest of the paper. When conditioned on $\langle \phi'^2 \rangle_r$, the conditional mean of $P_{\phi r}$ normalized

**Figure 3.** Conditional mean of the local scalar variance production $P_{\phi r}$, conditional on $\langle \phi'^2 \rangle_r$ and $\chi_r$, respectively. The averaging domain size $r = 125 \eta$ is used for the rest of the paper.
by \( \langle \chi_r \mid \langle \phi''^2 \rangle_r \rangle \) is small for \( \langle \phi''^2 \rangle_r / \langle \phi''^2 \rangle_r < 1 \) and increases with \( \langle \phi''^2 \rangle_r \) \( \sim \langle \phi''^2 \rangle_r^{0.8} \), indicating that the local scalar field changes from quasi-equilibrium to non-equilibrium as \( \langle \phi''^2 \rangle_r \) increases. (A decaying scalar field has similar characteristics as a quasi-equilibrium scalar field.) On the other hand, when conditioned on the scalar dissipation, \( \mathcal{P}_{\phi r} / \chi_r \) increases by only a factor of two over two orders of magnitude increase in \( \chi_r \). The results suggest that when conditioned on \( \chi_r \), the conditional inertial-range scalar is on average in quasi-equilibrium whereas when conditioned on \( \langle \phi''^2 \rangle_r \), the degree of non-equilibrium of the conditional inertial-range scalar increases with \( \langle \phi''^2 \rangle_r \). This provides further evidence that the close-to-Gaussian conditional increment PDFs are a result of quasi-equilibrium scalar as implied in Kolmogorov’s hypothesis, and suggests that the bimodal conditional PDFs are closely related to the non-equilibrium of the conditional inertial-range scalar.

In the inertial range, the local scalar variance production \( \mathcal{P}_{\phi r} \) should scale as \( \langle \phi''^2 \rangle_r / \tau_{\phi r} \), where \( \tau_{\phi r} \) is the local scalar time scale. Therefore, to understand the variations of the degree of non-equilibrium, it is useful to examine the relationship between \( \langle \phi''^2 \rangle_r \) and \( \chi_r \). In figure 4 we plot two conditional means, \( \langle \langle \phi''^2 \rangle_r \mid \chi_r \rangle \) and \( \langle \chi_r \mid \langle \phi''^2 \rangle_r \rangle \). The former increases as \( \chi_r \) (the results are similar when \( \chi_r \epsilon_r^{-1/3} \) is used as the conditioning variable), consistent with the K62 prediction

\[
\langle \phi''^2 \rangle_r = V_{\phi 2} \chi_r \epsilon_r^{-1/3} r^{2/3},
\]

where \( V_{\phi 2} \) is also a universal stochastic variable independent of both \( \chi_r \) and \( \epsilon_r \). However, the latter (as well as \( \langle \chi_r \epsilon_r^{-1/3} \mid \langle \phi''^2 \rangle_r \rangle \), not shown) scales approximately as \( \langle \phi''^2 \rangle_r^{0.7} \); therefore it is at variance with the K62 prediction of \( \sim \langle \phi''^2 \rangle_r \). A possible reason for the deviation of \( \langle \chi_r \mid \langle \phi''^2 \rangle_r \rangle \) from the K62 scaling is that the conditional inertial-range scalar is in quasi-equilibrium when conditioned on \( \chi_r \), whereas when conditioned on \( \langle \phi''^2 \rangle_r \), it can be in non-equilibrium: for a given value of \( \chi_r \), there is generally a corresponding amount of spectral transfer (scales as \( \langle \phi''^2 \rangle_r / \tau_{\phi r} \)), therefore the scalar is in quasi-equilibrium. On the other hand, when \( \langle \phi''^2 \rangle_r \), (therefore the spectral transfer) is increased, \( \chi_r \) generally lags behind since it takes time for the increased spectral transfer.
rate to reach the dissipation scales, resulting in non-equilibrium local scalar and a smaller scaling exponent for \( \langle \chi_r | \langle \phi'^2 \rangle_r \rangle \).

To further understand the effects of the non-equilibrium spectral transfer on the conditional increment PDFs, it is useful to vary the degree of non-equilibrium independent of the local scalar variance when computing the conditional PDFs. One way to do this is to use \( \chi_r \) as an additional conditioning variable. Then the balance between the production and dissipation can be altered by changing the values of \( \chi_r \), with \( \langle \phi'^2 \rangle_r \) fixed. To verify this, we plot in figure 5 the isocontours of the conditional production, conditional on both \( \langle \phi'^2 \rangle_r \) and \( \chi_r \). In general, the conditional production increases with \( \langle \phi'^2 \rangle_r \), and the dependence on \( \chi_r \) is much weaker, especially for large \( \langle \phi'^2 \rangle_r \) and \( \chi_r \), indicating that the degree of non-equilibrium indeed can be changed by varying \( \chi_r \) with \( \langle \phi'^2 \rangle_r \) fixed. Figure 6 gives the conditional increment PDF, conditional on both \( \langle \phi'^2 \rangle_r \) and \( \chi_r \). For small \( \langle \phi'^2 \rangle_r \) (\( = 0.3 \langle \phi'^2 \rangle \)) and \( \chi_r \geq 0.3 \langle \chi \rangle \) (figure 6(a)) \( P_{\Delta \phi | \langle \phi'^2 \rangle_r, \chi_r} \) is close to Gaussian. Under these conditions \( \chi_r \) is close to or larger than the conditional mean \( \langle \chi_r | \langle \phi'^2 \rangle_r \rangle \) and \( P_{\phi} \) is smaller than \( \chi_r \) (see figure 7), i.e. the inertial-range scalar is in quasi-equilibrium. As \( \chi_r \) decreases, i.e., as the degree of non-equilibrium increases, \( P_{\Delta \phi | \langle \phi'^2 \rangle_r, \chi_r} \) becomes bimodal. Similarly, for large \( \langle \phi'^2 \rangle_r \) (\( = 3.0 \langle \phi'^2 \rangle \)) (figure 6(b)), \( P_{\Delta \phi | \langle \phi'^2 \rangle_r, \chi_r} \) is not far from Gaussian for \( \chi_r \geq 3 \langle \chi \rangle \). Therefore, the shape of the conditional PDF is strongly dependent on the degree of non-equilibrium of the inertial-range scalar.

The dependence of \( P_{\Delta \phi | \langle \phi'^2 \rangle_r, \chi_r} \) on \( \langle \phi'^2 \rangle_r \) and \( \chi_r \) observed above suggests that a single variable characterizing the degree of non-equilibrium may be used to parametrize the conditional PDF. In figure 7 the ratio of the conditional production to \( \chi_r \) is given as isocontours (figure 5 plotted in a different way). Along each isocontour, the degree of non-equilibrium is constant. The isocontours for large \( P_{\phi} / \chi_r \) values (>3) have approximately unit slope, indicating that, along them, \( \langle \phi'^2 \rangle_r / \chi_r \) is constant. This suggests that \( \langle \phi'^2 \rangle_r / \chi_r \) can be used as a parameter to quantify the degree of non-equilibrium and its influence on the conditional PDF.

To quantify the dependence of the conditional PDFs on \( \langle \phi'^2 \rangle_r / \chi_r \), we compute a bimodality parameter \([32]\) \( B = K - S^2 \) as a function of \( \langle \phi'^2 \rangle_r / \chi_r \) for three \( \langle \phi'^2 \rangle_r \) values (figure 8), where

\[
\langle \phi'^2 \rangle_r = \langle \phi'^2 \rangle_r = 3 \langle \chi \rangle \quad \text{for} \quad \chi_r \geq 3 \langle \chi \rangle
\]

\[
\langle \phi'^2 \rangle_r = 10 \langle \chi \rangle \quad \text{for} \quad \chi_r \geq 10 \langle \chi \rangle
\]

\[
\langle \phi'^2 \rangle_r = \langle \phi'^2 \rangle_r = 100 \langle \chi \rangle \quad \text{for} \quad \chi_r \geq 100 \langle \chi \rangle
\]
Figure 6. Conditional PDFs of the scalar increment conditional on the local scalar variance and the locally averaged scalar dissipation rate. The values of $\langle \phi''^2 \rangle_r / \langle \phi''^2 \rangle$ are (a) 0.3 and (b) 3.0. The values of $\chi_r / \langle \chi \rangle$ are given in the legend.

$K$ and $S$ are the kurtosis and skewness of the conditional scalar increments. This parameter takes the value of unity for any double-delta-function PDF regardless of its symmetry, and therefore is a better measure of the bimodality than the kurtosis, which increases with the asymmetry of the PDF. Figure 8 shows that $B$ is approximately independent of $\langle \phi''^2 \rangle_r$, and is largely a function of $\langle \phi''^2 \rangle_r / \chi_r$ alone. Therefore, the conditional PDF is largely determined by the degree of non-equilibrium. For large $\langle \phi''^2 \rangle_r / \chi_r$ values, $B$ appears to decrease monotonically towards unity (the slight increase of $B$ for very large $\langle \phi''^2 \rangle_r / \chi_r$ is due to insufficient statistical convergence there), indicating that the conditional increment PDF approaches a double-delta function for strong non-equilibrium local scalar.

We note that, although K62 assumes quasi-equilibrium spectral transfer, it actually allows the dependence of the conditional PDF on $\langle \phi''^2 \rangle_r / \chi_r \epsilon_r^{-1/3}$, which can also be interpreted as a measure of the degree of non-equilibrium of the local scalar. Using equations (1) and (4), the
Figure 7. Isocontours of the conditional local scalar variance production \( \langle P_{\phi r} | \langle \phi''^2 \rangle_r, \chi_r \rangle / \chi_r \).

Figure 8. Dependence of the bimodality parameter on \( \langle \phi''^2 \rangle_r / \chi_r \). The values of \( \langle \phi''^2 \rangle_r / \langle \phi''^2 \rangle \) are given in the legend. \( B \) is essentially determined by \( \langle \phi''^2 \rangle_r / \chi_r \), independent of \( \langle \phi''^2 \rangle_r \).

The conditional increment can be expressed as

\[
\{ \Delta \phi | \langle \phi''^2 \rangle_r, \chi_r^{1/2} \epsilon_r^{-1/6} \epsilon_r^{1/3} \} = \{ V_{\phi} | V_{\phi^2}, \chi_r^{1/2} \epsilon_r^{-1/6} \epsilon_r^{1/3} \} \chi_r^{1/2} \epsilon_r^{-1/6} \epsilon_r^{1/3}
\]

\[
= \{ V_{\phi} | V_{\phi^2} \} \chi_r^{1/2} \epsilon_r^{-1/6} \epsilon_r^{1/3}.
\]

(5)

Therefore, K62 predicts that the conditional increment PDF depends only on \( V_{\phi^2} \). This variable can also be written as the ratio of \( \langle \phi''^2 \rangle_r / \epsilon_r^{-1/3} r^{2/3} \) to \( \chi_r \). The former can be considered as a spectral
transfer rate across the scalar $r$; hence, $V_{\phi}$ is a measure of the degree of non-equilibrium. Therefore, equation (5) shows that K62 predicts that the conditional increment PDF is determined by the degree non-equilibrium. This predicted variable characterizing the degree non-equilibrium is different from $\langle \phi'^2 \rangle_r/\chi_r$ used in figure 8. However, since the spectral transfer depends largely on $\langle \phi'^2 \rangle_r$, $\langle \phi'^2 \rangle_r/\chi_r$ should be very similar $\langle \phi'^2 \rangle_r/\chi_r \epsilon_r^{-1/3}$ (at least when $\epsilon_r$ is not used as a separate conditioning variable), although the latter variable contains the time scale given by a dissipation scale variable rather than an inertial-range one. We examine the differences between these two variables computing the conditional local variance production and the conditional PDFs and the results (not shown) are very similar to those given in figures 6 and 8. Therefore the results in these figures are consistent with the K62 prediction. However, this situation is in contrast with the previous results for the velocity increments. Zhang and Tong [9] showed that the conditional velocity increment PDF, conditional on $k_r/\epsilon_r$ and $\epsilon_r$, depends largely on $k_r/\epsilon_r$ whereas K62 predicts that the PDF depends on $k_r/\epsilon_r^{2/3}$. This difference between the conditional velocity increments and conditional scalar increments with respect to the K62 predictions requires further investigation.

The results provide further evidence that the universal conditional PDFs predicted by K62 are due to the quasi-equilibrium local scalar. This study also suggests the possibility that not only equilibrium inertial-range turbulence has universal statistics, as predicted by K62, but for a given degree of non-equilibrium, non-equilibrium inertial-range turbulence also has universal statistics, independent of the scale and flow type.

Conditional sampling and averaging, like those discussed above, essentially can be viewed as a statistical decomposition of inertial-range turbulence. The decomposition based on the K62 predictions in equation (1) (with the locally averaged energy and scalar dissipation rates as conditioning variables) results in conditional turbulence that is on average in quasi-equilibrium. Consequently, it does not capture the underlying turbulence structures such as ramp–cliff and plane strain. On the other hand, the conditioning based on the degree of non-equilibrium (with the local variances or the local variances and the locally averaged dissipation rates as conditioning variables) can capture these structures and potentially lead to a natural statistical description of them. Therefore, such conditioning may reconcile statistical and structural descriptions of inertial-range turbulence and provide a connection between them. It can also provide information for structure-based models of small-scale turbulence such as those of Lundgren [33], She and Leveque [34] and Saffman and Pullin [35]. In contrast, the locally averaged dissipation rates, when used as conditioning variables alone, result in quasi-equilibrium inertial-range turbulence containing no structure at the scale concerned, and therefore may be incompatible with structure-based models.

### 3.2. Conditional velocity–scalar increment JPDF

Previous studies have shown that the conditional velocity-increment PDF for an inertial range separation is close to Gaussian when conditioned on the locally averaged energy dissipation rate [3] but when conditioned on the local kinetic energy it can be close to Gaussian or bimodal [9]. Here we examine the conditional JPDF of velocity and scalar increments. We show the results for $\Delta v$ and $\Delta \phi$ because they are representative of the JPDF in isotropic turbulence (see the discussion in the next paragraph). When conditioned on $\epsilon_r$ and $\chi_r$, $P_{\Delta v, \Delta \phi | \epsilon_r, \chi_r}$ are close to joint Gaussian for a range of $\epsilon_r$ and $\chi_r$ values (figure 9), consistent with K62. When conditioned
on the local kinetic energy and the local scalar variance, the $P_{\Delta v, \Delta \phi \mid \epsilon_r, \chi_r}$, has different shapes depending on the values of the conditioning variables. For small $\langle \phi''^2 \rangle_r$ and $k_r$, it is close to joint normal (figure 10(a)). For small $\langle \phi''^2 \rangle_r$ but large $k_r$, it is bimodal in $\Delta v$, indicating a well-mixed local scalar but non-equilibrium local velocity. Previous results [13] have shown that when $k_r$ is large the local turbulence is under rapid distortion and the local velocity field contains plain strain. Figure 10(b) suggests that the plane strain does not seem to alter the distribution of a well-mixed scalar. For large $\langle \phi''^2 \rangle_r$ but small $k_r$, $P_{\Delta v, \Delta \phi \mid \epsilon_r, \chi_r}$ is bimodal in $\Delta \phi$, indicating a non-equilibrium local scalar but equilibrium local velocity. Although the velocity is in quasi-equilibrium at scale $r$, the bimodal JPDF suggests that there may be non-equilibrium velocity at smaller scales. This suggests that the diffusion layers under such conditions are highly wrinkled. When both $\langle \phi''^2 \rangle_r$ and $k_r$ are large, $P_{\Delta v, \Delta \phi \mid \epsilon_r, \chi_r}$, is quad-modal with one peak in each quadrant of the $\Delta v-\Delta \phi$ plane. This JPDF shape has not been previously observed for inertial-range turbulence and is a result of the combination of a local plane strain and ramp–cliff (diffusion layer) structure.
We note that previous results [9] have suggested that the conditional PDF of longitudinal velocity increment $\Delta u_1$ measured in a jet is less bimodal than the lateral increment $\Delta v_1$. This is probably a result of the anisotropy of the velocity field at large $k_r$ because for isotropic turbulence they should have the same PDF shape [9]. In the fully developed region of a jet the ramp–cliff structure is preferentially oriented [31], suggesting the same for the velocity field (plane strain). We obtain the orientation of the velocity field from the eigenvectors of the locally averaged strain rate. The eigenvectors are computed using a modified strain rate

$$S'_{ij} = S_{ij} - \frac{1}{2}(S_{11} + S_{22}) \delta_{ij}, \quad i, j = 1, 2,$$

where $S_{ij}(x) = \int S_{ij}(x') G(x' - x) \, dx'$ is the locally averaged strain rate, and $S_{ij}$ and $G(x)$ are the strain rate and a box-averaging function of size $r$ in both the $x$ and $y$ directions, approximated by the sensor array. Subtracting the trace $\frac{1}{2}(S_{11} + S_{22})$ of the strain rate in the $x$–$y$ plane removes the influence of the out of plane flows (axisymmetric contraction or expansion due to $S_{33}$), therefore $S'_{ij}$ represents the strain rate of a two-dimensional flow in the $x$–$y$ plane. In figure 11 the conditional JPDF of the compressive eigenvector $\lambda I_1$ (multiplied by the}

![Figure 10. Conditional JPDF of $\Delta v$ and $\Delta \phi$ conditional on $k_r$ and $\langle \phi''^{(2)} \rangle_r$. The values of $k_r/\langle k_r \rangle$ and $\langle \phi''^{(2)} \rangle_r/\langle \phi''^{(2)} \rangle$ are (a) 0.06, 0.3; (b) 6.0, 0.3; (c) 0.06, 8.0; and (d) 6.0, 8.0.](http://www.njp.org/)

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corresponding eigenvalue) conditional on $k_r$ is used to characterize the strain field. The vector is arranged to point in the upstream direction.

For small $k_r$, the eigenvector $\lambda I_1$ is largely isotropically distributed with slightly higher probabilities to be in the cross-stream ($y$) direction. However, for large $k_r$ the JPDF has two branches with peaks near $(1.0, \pm 1.0)$, indicating that $\lambda I_1$ is preferentially oriented at $\pm 45^\circ$ to the jet axis. Therefore, for a streamwise separation, $\Delta u$ produced by plane strain tends to be negative whereas $\Delta v$ will have equal probabilities to take positive and negative values, producing unimodal and bimodal conditional PDFs for $\Delta u$ and $\Delta v$, respectively. Consequently, the conditional JPDF of $\Delta u$ and $\Delta \phi$ for large $k_r$ and $\langle \phi''^2 \rangle$, remains bimodal. Therefore, the results for $\Delta v$ are qualitatively the same as those in isotropic turbulence.

Like the bimodal velocity and scalar PDFs, the quad-modal JPDF may also be associated with non-equilibrium local velocity and scalar. In figure 12 we plot the conditional JPDF,
Figure 12. Conditional JPDF of $\Delta v$ and $\Delta \phi$ conditional on $k_r$, $\langle \phi''^2 \rangle_r$, $\epsilon_r$ and $\chi_r$. The normalized values of these variables are 1.4, 0.8, 0.1 and 0.1, respectively.

Conditional on four variables $\langle \phi''^2 \rangle_r$, $k_r$, $\epsilon_r$, and $\chi_r$. The values of $\langle \phi''^2 \rangle_r$ and $k_r$ are comparable to their respective mean values, for which the JPDF conditioning on them alone is still close to a joint-Gaussian. However, $\epsilon_r$ and $\chi_r$ are much smaller than their respective mean values, thereby resulting in non-equilibrium local velocity and scalar. The resulting conditional JPDF is quad-modal, similar to that in figure 10(d), strongly suggesting that the quad-modal JPDF is a result of non-equilibrium local velocity and scalar.

4. Further discussions and conclusions

Conditional PDFs of scalar and conditional JPDFs of velocity–scalar increments are studied using data obtained in the fully developed region of an axisymmetric turbulent jet with passive temperature fluctuations. For inertial-range separations, the conditional scalar increment PDF conditional on the local scalar variance $\langle \phi''^2 \rangle_r$ is found to be close to Gaussian for $\langle \phi''^2 \rangle_r$ small compared with its mean value and becomes sub-Gaussian and bimodal for large $\langle \phi''^2 \rangle_r$. The bimodal conditional PDF is associated with the ramp–cliff structure in the scalar field. This is in contrast with the universal, quasi-Gaussian conditional PDF conditioning on $\chi_r \epsilon_r^{-1/3}$ previously observed in the context of K62. For separations close to the dissipation scales, the conditional PDF is almost always sub-Gaussian, similar to the PDF conditional on the locally averaged dissipation rate previously observed.

Further analyses show that the different shapes of the conditional PDF are largely a result of the changes in the degree of non-equilibrium (production larger than dissipation) of the local scalar. We find that the conditional local scalar variance production increases faster than $\chi_r$ as $\langle \phi''^2 \rangle_r$ increases; therefore the degree of non-equilibrium of the local scalar increases with $\langle \phi''^2 \rangle_r$. We also explicitly change the degree of non-equilibrium by varying $\chi_r$ with $\langle \phi''^2 \rangle_r$ fixed since the production depends only weakly on $\chi_r$. The conditional PDF conditional on both $\langle \phi''^2 \rangle_r$ and $\chi_r$ is largely a function of $\langle \phi''^2 \rangle_r / \chi_r$, which is a measure of the degree of non-equilibrium, and is independent of $\langle \phi''^2 \rangle_r$. For large $\langle \phi''^2 \rangle_r / \chi_r$, the bimodality parameter appears to approach the unity limit, the value for a double-delta-function PDF.
Although K62 assumes quasi-equilibrium transfer, it allows a dependence of the conditional PDF on \( \langle \phi'^2 \rangle_r / \chi_r \epsilon_r^{-1/3} \), which can also be viewed as a measure of the degree of non-equilibrium. We find that results obtained using this variable are essentially the same as those using \( \langle \phi'^2 \rangle_r / \chi_r \). This suggests that K62 also allows non-universal conditional increment PDFs.

The JPDF of \( \Delta v \) and \( \Delta \phi \) conditional on \( k_r \) and \( \langle \phi'^2 \rangle_r \) is close to joint normal when both variables are small and is bimodal when one of them is large. When both variables are large, the conditional JPDF is quad-modal with one peak in each quadrant of the \( \Delta v - \Delta \phi \) plane. Such a conditional increment JPDF has not been observed previously and is consistent with the presence of a combination of local plane-strain and ramp–cliff structure in the local turbulence. The shape of the conditional JPDF is largely determined by \( k_r / \epsilon_r \) and \( \langle \phi'^2 \rangle_r / \chi_r \), i.e. the degrees of non-equilibrium of the local velocity and scalar.

The results in the present study are obtained with one-dimensional averaging and sampling. Although two-dimensional averaging using the sensor array yields results for conditional velocity increments similar to those obtained by using one-dimensional averaging [9], some results for scalar increments are different for the two types of averaging due to the spacing between the sensors in the cross-stream direction. The most significant difference occurs when \( \langle \phi'^2 \rangle_r / \chi_r \) is large. Under such conditions, there is generally a ramp–cliff structure in the local scalar which, however, can lie between two adjacent probes in the array, contributing little to the measured \( \chi_r \). Therefore, such a local scalar will be sampled falsely as one with a much smaller \( \chi_r \) value, producing a spurious conditional PDF. Consequently, when both \( \langle \phi'^2 \rangle_r \) and \( \chi_r \) are used as conditioning variables, one-dimensional averaging and sampling is more accurate than the array sampling. This is in contrast with the situation of the velocity field, in which a plane strain can be captured by the array regardless of its orientation; therefore the one-dimensional and array results are similar. We note that this conditional sampling problem with the array does not affect the measurements of scalar FDF because an FDF does not depend on the orientation of the scalar structure; therefore the spurious samples are averaged out.

The present study suggests that the quasi-Gaussian conditional increment PDF is only one of the PDF shapes of the inertial-range turbulence largely determined by the degree of non-equilibrium, which are allowed by K62. It also raises the possibility that for a given degree of non-equilibrium, the conditional inertial-range increments turbulence has universal statistics independent of the separation and the large scales. In addition, the conditional statistics studied provides a connection to statistical and structural descriptions of the inertial-range turbulence and are potentially useful for developing structure-based models for small-scale turbulence.

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