On the connectivity of Milnor fiber for mixed functions

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Dedicated to Professor Lê Dũng Tráng for his 70’s birthday

Abstract. In this note, we prove the connectivity of the Milnor fiber for a mixed polynomial \( f(z, \bar{z}) \), assuming the existence of a sequence of smooth points of \( f^{-1}(0) \) converging to the origin. This result gives also another proof for the connectivity of the Milnor fiber of a non-reduced complex analytic function which is proved by A. Dimca

1. Introduction

Let \( f(z, \bar{z}) = \sum_{\nu, \mu} c_{\nu, \mu} z^\nu \bar{z}^\mu \) be a mixed polynomial of \( n \)-variables \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) which has a singularity at the origin. We say that \( f \) satisfies Hamm-Lê condition at the origin if there exists a positive number \( r_0 \) such that for any \( 0 < r_1 \leq r_0 \), there exists a positive number \( \delta(r_1) \) such that the hypersurface \( f^{-1}(\eta) \cap B_r^{2n} \) is non-empty, non-singular and it intersects transversely with the sphere \( S_{r}^{2n-1} \) for any \( r, \eta \) with \( r_1 \leq r \leq r_0 \) and \( 0 \neq |\eta| \leq \delta(r_1) \). Here \( B_r^{2n} \) is the ball \( \{ z \in \mathbb{C}^n \mid ||z|| \leq r \} \) of radius \( r \) and \( S_{r}^{2n-1} \) is the boundary sphere of \( B_r^{2n} \). We call such a positive number \( r_0 \) a stable Milnor radius. If \( f \) is a holomorphic function, Hamm-Lê condition is always satisfied (Hamm-Lê [H-L]). For a strongly non-degenerate mixed function, this condition is also satisfied provided \( f(z, \bar{z}) \) is either convenient ([O2]) or locally uniformly tame along vanishing coordinate sub-spaces ([O3]). The following assertion is immediate from Hamm-Lê condition and Ehresman’s fibration theorem ([W]).

Lemma 1.1 ([H-L, O3]). Assume that \( f(z, \bar{z}) \) be a mixed function of \( n \)-variables with \( n \geq 2 \) which satisfies the Hamm-Lê condition. Choose \( r_0 > 0 \) as above and take arbitrary positive numbers \( r, r_1 \), with \( 0 < r_1 \leq r \leq r_0 \) and take a positive number \( \delta \) with \( \delta \leq \delta(r_1) \). Consider the following tubular set and its boundary

\[
E(r, \delta) := \{ z \in B_r^{2n} \mid |f(z)| \leq \delta \}
\]

\[
\partial E(r, \delta) := \{ z \in B_r^{2n} \mid |f(z, \bar{z})| = \delta \}.
\]
Then the mappings
\[ f : E(r, \delta) \setminus f^{-1}(0) \to D_\delta \setminus \{0\} \text{ and} \]
\[ f : \partial E(r, \delta) \to S^1_\delta \]
are locally trivial fibrations where \( D_\delta := \{ \zeta \mid |\zeta| \leq \delta \} \) and \( S^1_\delta = \partial D_\delta \) (the boundary circle). They are homotopically equivalent and their isomorphism classes do not depend on the choice of \( r \) and \( \delta \).

This is called the tubular Milnor fibration of \( f \) at the origin and the fiber \( f^{-1}(\delta) \cap B_r \) is called the Milnor fiber at the origin. It is known that the tubular Milnor fibration is equivalent to the spherical Milnor fibration (see [M, O2])
\[ \varphi : S_r \setminus K \to S^1, \quad \varphi(z) = f(z, \bar{z})/|f(z, \bar{z})| \]

2. Statement of the result

2.1. Connectivity of Milnor fibers of holomorphic functions. We first recall basic facts about the connectivity of the Milnor fibers of holomorphic functions. Assume that \( f : (U, 0) \to (\mathbb{C}, 0) \) be a holomorphic function with a singularity at the origin where \( U \) is an open neighborhood of the origin. A fundamental result for the connectivity is

**Theorem 2.1 (Milnor [M]).** The Milnor fiber \( F \) has a homotopy type of an \((n - 1)\)-dimensional CW-complex. If further the origin is an isolated singularity, \( F \) is \((n - 2)\)-connected.

Let \( \varphi : S_r^{2n-1} \setminus K \to S^1 \) be the spherical Milnor fibration. The proof depends on Morse theory. Milnor proved that the fiber has a homotopy type of an \( n - 1 \) dimensional CW-complex, by showing that the index of a suitable Morse function is less than or equal to \( n - 1 \). Secondly, if the origin is an isolated singularity at the origin, \( F \) is a manifold with boundary and the inclusion \( F \subset \bar{F} \) is a homotopy equivalence in the sphere \( S_r^{2n-1} \) and the vanishing \( H_j(F) = 0 \) for \( 0 < j < n - 1 \) follows from the Alexander duality and the homotopy equivalence \( F_s \subset S_r^{2n-1} \setminus \bar{F} \) where \( F_s = \varphi^{-1}(-1) \). The simply connectedness of \( F \) for \( n \geq 3 \) follows from handle body argument.

Kato and Matsumoto further generalized this assertion as follows.

**Theorem 2.2 (Kato and Matsumoto [K-M]).** Assume that \( s \) is the dimension of the critical points locus at the origin. Then \( F \) is \((n - s - 2)\)-connected.

Their proof depends on the above result of Milnor and an inductive argument using Whitney stratification.

2.2. Mixed functions. For mixed functions, no similar connectivity statement is known. A main reason is that the tangent space of a mixed hypersurface has no complex structure and Morse function argument does not help so much as in the holomorphic case.

The following is our result which is a first step for the connectivity of the Milnor fiber of a mixed function. For the proof, we use an elementary but completely different viewpoint which is nothing to do with Morse functions. We use a one-parameter group of diffeomorphisms which contains the monodromy map.
2.3. Proof of Theorem 2.3 Let $r_0$ be a positive number which satisfies the Hamm-Lê condition. Take a mixed smooth point $w \in f^{-1}(0)$ with $\|w\| < r_0$. Put $r := \|w\|$. Fix positive numbers $r_1$ with $r_1 < r$ and $\delta \leq \delta(r_1)$ and we consider the Milnor fibration

\[ (*) \quad f : \partial E(r_0, \delta) \to S^1_\delta. \]

Let $F_\theta := f^{-1}(\delta e^{i\theta}) \cap B_{r_1}$ for $\theta \in \mathbb{R}$ be the Milnor fiber over $\delta e^{i\theta}$. We assume also the sphere with radius $r$, $S^2_{r_1}$ intersects transversely with the hypersurface $f^{-1}(0)$ at the smooth point $w$. For the proof, we use a certain one-parameter family of diffeomorphisms associated with the tubular Milnor fibration $(\ast)$. Recall that a one-parameter family of the characteristic diffeomorphisms $h_{\theta}$, $\theta \in \mathbb{R}$ are constructed by integrating a given horizontal vector field $V$. Here a vector field $V$ is called a horizontal vector field on $\partial E(r_0, \delta)$ if it satisfies the following property:

\[ T f_p(V(p)) = \frac{\partial}{\partial \theta} (f(p)), \quad p \in \partial E(r_0, \delta) \]

where $T f_p : T_p \partial E(r_0, \delta) \to T f_p(S^1_\delta)$ is the tangential map and furthermore $V$ is tangent to $S^2_{r_1} \cap \partial E(r_0, \delta)$ on the boundary and $\frac{\partial}{\partial \theta}$ is the unit angular vector field along $S^1_\delta$. In other word, $\frac{\partial}{\partial \theta}$ is the tangent vector of the curve $t \mapsto \delta e^{it}$. As $\partial E(r_0, \delta)$ is a compact manifold with boundary, there is an integral $\varphi : \partial E(r_0, \delta) \times \mathbb{R} \to \partial E(r_0, \delta)$ such that $\varphi(p, 0) = p$ and $\varphi(p, t) = \varphi(p, 0) + \delta e^{it}$ for $-\infty < t < \infty$ is the integral of $V$ starting at $p$ for $t = 0$. Let $h_\theta : \partial E(r_0, \delta) \to \partial E(r_0, \delta)$, $\theta \in \mathbb{R}$ be the corresponding one parameter family of characteristic diffeomorphisms, which are defined by $h_\theta (p) = \varphi(p, \theta)$. Note that $\{h_\theta\}$ satisfy the property

\[ h_\theta(F_\eta) = F_{\eta + \theta}, \quad \eta, \theta \in \mathbb{R}. \]

They also satisfy the equalities

\[ h_0 = \text{id}, \quad h_\theta \circ h_\xi = h_{\theta + \xi}, \quad \theta, \xi \in \mathbb{R}. \]

Using one-parameter family $\{h_\theta, \theta \in \mathbb{R}\}$, the monodromy map $h : F \to F$ is given by the restriction $h_{2\pi} : F$ with $F = F_0$.

Now we construct $\mathcal{V}$ more carefully. We take a local real-analytic chart $U$ with local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ centered at the given smooth point $w \in f^{-1}(0)$ such that $f(x_1, y_1, \ldots, x_n, y_n) = x_n + iy_n$. We take $\delta$ sufficiently small and we take a normal disc $D$ of $f^{-1}(0)$ centered at $w$ so that $|f(z)| = \delta$ for any $z \in \partial D$ and thus $\partial D \subset \partial E(r_0, \delta)$. In the above coordinates, we can assume $D = \{(0, \ldots, 0, x_n, y_n) | x_n^2 + y_n^2 \leq \delta^2\}$. By the transversality assumption, we may assume that $V$ is tangent to $\partial D$ which implies that $h_{2\pi}(z) = z$ for $z \in \partial D$. Note that the family of tubular neighborhoods $\{E(r, \delta) \mid \delta < \delta(r)\}$ and the family of disk neighborhoods $\{B^2_{s}, s > 0\}$ are cofinal neighborhood systems of the origin. Thus $E(r, \delta)$ is contractible and $E \setminus f^{-1}(0)$ is connected by the assumption
coding $f^{-1}(0) = 2$, $\partial E(r, \delta)$ is also connected, as $E(r_0, \delta) \setminus f^{-1}(0)$ and $\partial E(r_0, \delta)$ are homotopy equivalent.

Now we are ready to prove the connectivity of the Milnor fiber $F_0$. Fix a point $p \in \partial D \cap F_0$ and take an arbitrary point $q \in F_0$. As $\partial E(r, \delta)$ is connected, we can find a path $\sigma : [0, 1] \to \partial E(r_0, \delta)$ such that $\sigma(0) = q$ and $\sigma(1) = p$. We will show that $q \in F_0$ can be joined to $p$ by a path in the fiber $F_0$, modifying the path $\sigma$. Consider first the closed loop $f \circ \sigma : (I, \{0, 1\}) \to (S^1, \delta)$ and let $m$ be the rotation number, that is

$$m = \frac{1}{2\pi} \int_0^1 d\arg f(\sigma(\theta)).$$

Let $\omega : I \to \partial D$ be the clockwise rotation $m$-times along the boundary of $D$ starting at $p$. Consider the path $\sigma'$ which is given as the composition of paths $\sigma' := \sigma \cdot \omega : I \to \partial E(r, \delta)$ and define a function $\psi : [0, 1] \to \mathbb{R}$ by

$$\psi(t) = \int_0^t d\arg f(\sigma'(t))dt.$$

Note that $\psi(t) \equiv \arg f(\sigma'(t))$ modulo $2\pi$ and $\psi(0) = \psi(1) = 0$. This follows from the observation that the rotation number of $f \circ \sigma'$ is zero by the definition of $\omega$.

Now we can deform $\sigma'$ using characteristic diffeomorphisms $h_t$, $t \in \mathbb{R}$ into a path in the fiber $F_0$ as follows. Define a modified path

$$\hat{\sigma}(t) := h_{-\psi(t)}(\sigma'(t)), \quad 0 \leq t \leq 1.$$

Then we have the equality

$$\arg f(\hat{\sigma}(t)) = -\psi(t) + \arg f(\sigma'(t)) \equiv 0, \quad 0 \leq t \leq 1.$$

Thus the deformed path $\hat{\sigma}$ is entirely included in $F_0$. By the construction, $\hat{\sigma}(0) = q$ and $\hat{\sigma}(1) = p$ and $\hat{\sigma}$ is the path which connect $q$ and $p$ in $F_0$.

\begin{theorem}
2.4. Generalization of Main Theorem for holomorphic functions.\end{theorem}

The following is known by A. Dimca [11]. We give another direct proof of this assertion, using a similar argument as in the proof of Theorem 2.3.

THEOREM 2.5. (Proposition 2.3, [11]) Assume that $f$ is a germ of holomorphic function at the origin of $\mathbb{C}^n$, $n \geq 2$ with $f(0) = 0$ and assume that $f$ is factored as $f_1^{n_1} \cdots f_r^{n_r}$ where $f_1, \ldots, f_r$ are irreducible in $\mathcal{O}_n$ and mutually coprime. Put $n_0 = \gcd(n_1, \ldots, n_r)$. Then Milnor fiber of $f$ at the origin is connected if and only if $n_0 = 1$.

Here $\mathcal{O}_n$ is the ring of germs of holomorphic functions at the origin.

PROOF. Let $F$ be the Milnor fiber of $f$. We can write $f = g^{n_0}$ with $g = f_1^{n_1/n_0} \cdots f_r^{n_r/n_0}$ and $F$ is diffeomorphic to disjoint sum of $n_0$ copies of the Milnor fibers of $g$. Thus $F$ is not connected if $n_0 \geq 2$. Assume $n_0 = 1$. Take a stable radius $r_0$ which satisfies Harn-Lê condition. For each $1 \leq i \leq r$, we consider the reduced irreducible component $V_i = \{f_i = 0\}$ and take a non-singular point $p_i \in V_i \setminus \bigcup_{j \neq i} V_j$ with $\|p_i\| < r_0$. Choose a positive number $r < r_0$ with $r \leq \|p_i\|$, $i = 1, \ldots, r$ and we consider Milnor fibration

$$f : \partial E(r_0, \delta) \to S^1, \quad \delta \leq \delta(r)$$

We assume that in a sufficiently small neighborhood $U_i$ of $p_i$, any analytic branch of $f^{1/n_i}$ is a well-defined single-valued function. Taking a one branch $\bar{f}_i$ and we
take \( \tilde{f}_i \) as the last coordinate function of a complex analytic coordinate system \( z_i = (z_{i1}, \ldots, z_{in}) \) in \( U_i \), i.e., \( \tilde{f}_i(z_i) = z_{in} \). Consider a normal disk at \( p_i \) which is defined by \( D \delta = \{(0, \ldots, 0, z_{in}) | |z_{in}| \leq \delta^{1/n_i}\} \) where we assume that \( \delta \) is sufficiently small so that \( D \delta \subset U_i \). Note that \( \tilde{f}_i \) is locally written as \( \tilde{f}_i = f_i \cdot u_i \) where \( u_i \) is a unit in \( U_i \) and \( f(\partial D \delta) = S_{\delta}^i \) and \( \partial D \delta \subset \partial E(r, \delta) \). Construct a one parameter family of characteristic diffeomorphisms \( h_t, t \in \mathbb{R} \) as before such that \( \partial D \delta \) is stable by \( h_t \) and \( h_t|\partial D \delta \) is the rotation of angle \( t/n_i \), under the identification \( f_t : \partial D \delta \rightarrow S_{\delta}^i/n_i \). (Note that \( f : \partial D \delta \rightarrow S_{\delta}^i/n_i \) is the \( n_i \)-th rotation.) \( F \cap \partial D \delta \) can be identified with \( n_i \)-th roots of \( \delta \) and we put them as \( F \cap \partial D \delta = \{p_{i,0}, \ldots, p_{i,n_i-1}\} \) so that the monodromy \( h := h_{2\pi} \) acts simply as a cyclic permutation \( p_{i,j} \mapsto p_{i,j+1} \). Using a similar discussion as in the proof of Theorem 2.3 for a given \( q \in F \), we can connect \( q \) to some point in \( F \cap D_j \) for any \( j \). To see this, we first take a path \( \sigma \) so that \( \sigma(0) = q \) and \( \sigma(1) = p_{j,0} \), then in the argument of the proof of Theorem 2.3 we simply replace \( \omega \) by \( m/n_j \) rotation in the clockwise direction along \( \partial D_j \) and do the same argument, where \( m \) is the rotation number of \( f \circ \sigma \). Note that \( \omega \) need not be a closed loop but the image of \( \omega \) by \( f \) is a closed loop and it gives \( -m \) rotation.

Thus the proof is reduced to show that the points \( p_{1,0}, \ldots, p_{1,n_1-1} \) are in the same connected component of \( F \).

In particular taking \( q = p_{1,0} \), we can find a path \( \ell_j \) in \( F \) which connects \( p_{1,0} \) to some \( p_{j,a} \) for any \( j, a = 2, \ldots, r \). Let \( \ell_j^a = h^a(\ell_j) \) the image of \( \ell_j \) by \( h^a \). Then it connects \( p_{1,a} \) to \( p_{j,a} \) in \( F \). Now we consider the image of \( \ell_j^a \), \( a = 0, \ldots, n_1 - 1 \) under \( h^{n_j} \) which fixes \( p_{j,a} \), but the other end \( p_{1,a} \) of \( \ell_j^a \) goes to \( p_{1,n_1+a} \). As \( F \) is stable by the monodromy map \( h = h_{2\pi} \), this image is also a path in \( F \). Thus \( p_{1,a} \) and \( p_{1,n_1+a} = h^{n_j}(p_{1,a}) \) are connected by the path \( \ell_j^a \cdot (h^{n_j} \circ (\ell_j^a)^{-1}) \) in \( F \) for any \( a \). See Figure 1 for \( a = 0 \). As we have assumed \( \gcd(n_1, \ldots, n_r) = 1 \), there exist integers \( a_1, \ldots, a_r \) so that we can write \( 1 = \sum a_i n_i \). Then

\[
p_{1,1} = h(p_{1,0}) = h^{\sum a_i n_i}(p_{1,0}) = h^{a_1 n_1}(\cdots (h^{a_r n_r}(p_{1,0}) \cdots)).
\]

Put \( p_{1,a_1} := h^{a_1 n_1}(p_{1,0}) \) and \( p_{1,a_{j+1}} := h^{a_j n_1 + a_{j+1}}(p_{1,a_j}) \) for \( j = 2, \ldots, r - 1 \). Then points \( p_{1,a_1}, \ldots, p_{1,a_r} = p_{1,1} \) are in the same connected component of \( p_{1,0} \) in \( F \). Thus \( p_{1,1} \) and \( p_{1,0} \) are in the same component. Repeating the same argument, we conclude that \( p_{1,0}, \ldots, p_{1,n_1-1} \) are all in the same connected component of \( F \). Combining the above observation, this proves that \( F \) is connected.

\[\square\]

**Remark 2.6.** The special case \( f = z_1^{a_1} \cdots z_k^{a_k} \) has been considered in Example (3.7), [O1]. For mixed function case, the above argument does not work. There does not exist any correspondence between irreducible components of \( f^{-1}(0) \) as real algebraic sets and the irreducible factors of \( f(z, \bar{z}) \) as a polynomial in \( \mathbb{C}[z, \bar{z}] \) or a germ of an analytic function of \( 2n \) variables. For example, let \( f(z, w, \bar{z}, \bar{w}) \) be the strongly homogeneous mixed polynomial of two variables which come from the Rhie’s Lens equation \( \varphi_n = 0 \),

\[
\varphi_n(z) := \bar{z} - \frac{z^{n-2}}{z^{n-1} - a^{n-1}} - \frac{\varepsilon}{z}, \quad n \geq 2
\]

where \( \varepsilon, a \) are sufficiently small positive numbers \( 0 < \varepsilon < a < 1 \). \( f \) is obtained as the homogenization of the numerator of \( \varphi_n \). Then \( f \) is irreducible but \( f^{-1}(0) \) has \( 5n - 5 \) line components in \( \mathbb{C}^2(\mathbb{O}4, \mathbb{R}) \).
3. Application

3.1. Convenient strongly non-degenerate mixed functions. Let \( f(z, \bar{z}) \) be a strongly non-degenerate convenient mixed polynomial. Then the Hamm-Lê condition is satisfied (Lemma 28, [O2]). Furthermore the mixed hypersurface \( V = f^{-1}(0) \) has an isolated singularity at the origin (Corollary 20, [O2]). Thus the assumption of Theorem 2.3 is satisfied and we have

**Theorem 3.1.** Let \( f(z, \bar{z}) \) be a strongly non-degenerate convenient mixed function and \( n \geq 2 \). Then the Milnor fiber is connected.

3.2. Non-convenient mixed polynomials. Let \( f(z, \bar{z}) \) be a strongly non-degenerate mixed polynomial which is not convenient. In general, such a mixed hypersurface defined by \( f \) has a non-isolated singular locus. Let \( I \) be a non-empty subset of \( \{1, \ldots, n\} \). We use the following notations.

\[
\mathbb{C}^I = \{ z | z_j = 0, \forall j \notin I \},
\]

\[
\mathbb{C}^{*I} = \{ z | z_j = 0 \iff j \notin I \}.
\]

We say \( \mathbb{C}^I \) is a vanishing coordinate subspace for \( f \) if the restriction of \( f \) to the coordinate subspace \( \mathbb{C}^I \), denoted by \( f^I \), is identically zero. Otherwise, we say \( \mathbb{C}^I \) a non-vanishing coordinate subspace. Put \( V^I \) be the union of \( V \cap \mathbb{C}^{*I} \) for all \( I \) such that \( \mathbb{C}^I \) is a non-vanishing coordinate subspace. We know that \( V^I \) is mixed non-singular as a germ at the origin (Theorem 19, [O2]). We say that \( f \) is uniformly locally tame on the vanishing coordinate subspace \( \mathbb{C}^I \) if there exists a positive number \( \varepsilon > 0 \) and for any non-negative weight vector \( P = (p_1, \ldots, p_n) \in \mathbb{Z}^n_{\geq 0} \) with \( I(P) = I \) where \( I(P) := \{ i | p_i = 0 \} \), the face function \( f_P \) is strongly non-degenerate as a function of \( \{ z_j | j \notin I \} \) for any fixed \( z_I = (z_i)_{i \in I} \in \mathbb{C}^{*I} \) with \( \sum_{i \in I} |z_i|^2 \leq \varepsilon \). A strongly non-degenerate mixed function which is uniformly locally tame along any vanishing
coordinate subspaces satisfies Hamm-Lê condition (Proposition 11, [O3]). Thus we have

**Theorem 3.2.** Assume that \( f(z, \bar{z}) \) is a strongly non-degenerate mixed function which is uniformly locally tame along vanishing coordinate subspaces and \( n \geq 2 \). We assume also that \( V^\sharp \) is a non-empty germ of mixed variety. Then the Milnor fiber is connected.

A similar but weaker result is proved for a strongly mixed homogeneous polynomial ([O5]). As a next working problem, we also propose a conjecture about the fundamental groups of the Milnor fiber.

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