The low temperature behavior the Casimir-Polder energy for conductive plane

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The low temperature expansion of the free energy of atom/plane system is considered for general symmetric form of tensor conductivity of the plane. It is shown that the first correction is proportional to second order of the temperature \(\sim T^2\) and comes from TM mode. The agreement of the expansion and exact expressions for different models of conductivity is numerically demonstrated.

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I. INTRODUCTION

Van der Waals dispersion forces play an important role in different physical, biological as well as chemical phenomena [1]. In the case of interaction between particle and plate it is commonly referred to as the Casimir-Polder force [2]. The van der Waals force is very important for interaction of graphene with microparticles [3], where finite conductivity of graphene plays essential role [4]. At a short range the energy rises as the third power of inverse distance between the microparticle and the plate. The retardation of the interaction should be taken into account at large distances and the interaction energy falls down as the fourth power of distance. At separations larger than a few micrometers, thermal effects become dominant.

Thermal corrections for van der Waals energy of the system atom/slab and atom/graphene were considered in Ref. [5]. It was shown that the correction to the Casimir-Polder free energy is proportional to forth degree of temperature \(\sim T^4\) in the case atom and ideal plane. Different models were considered to describe graphene namely, i) hydrodynamical model [6], ii) density-density correlation function [7] and iii) the Dirac model [8]. In framework of Dirac model was found tensor of conductivity of graphene with and temporal and spatial dispersions and dependence of temperature and chemical potential [9].

In the present paper we consider low temperature expansion of the free Casimir-Polder energy for atom/plane system taking into account general symmetric form of the plane’s conductivity tensor. We found that the first correction is quadratic over temperature \(\sim T^2\). Numerically we justify low temperature expansion for three different models of conductivity – constant conductivity model, the Drude-Lorentz model and conductivity calculated in the context of polarization tensor approach.

The paper is organized as follows. In Sec. II, we briefly consider general structure of the conductivity tensor. Section III presents different representations of the Casimir-Polder energy. In Sec. IV we derive the main expressions for low temperature expansion and in Sec. V we numerically compare exact expressions and low temperature approximations. In Sec. VI we discuss the obtained results. Appendix A devotes for different models of graphene’s conductivity and in Appendix B we obtain low temperature expansion of the conductivity.

II. THE STRUCTURE OF THE TENSOR CONDUCTIVITY

Let us consider conductive 2D infinitely thin layer positioned perpendicular to axes \(z\). We suppose that the anisotropic Ohm low, \(j_s = \sigma E\), is satisfied on the plane, where \(j_s\) is surface current and \(\sigma\) is conductivity tensor. The latter, in general, depends on the frequency \(\omega\), wave vector \(k\), velocity \(v\) and other scalar parameters such as temperature \(T\) and chemical potential \(\mu\). It has the following structure [10] (\(i, j = x, y\))

\[
\sigma_{ij} = A\delta_{ij} + Bk_ik_j + Ck_iv_j + Dv_i v_j + E\varepsilon_{ij},
\]

where the constant \(E\) describes parity-odd part of conductivity [11] and \(\varepsilon_{ij}\) is complete antisymmetric tensor. We consider here parity-even part of conductivity without velocity \(v = 0\):

\[
\sigma_{ij} = A\delta_{ij} + Bk_ik_j.
\]

The eigenvalues of this tensor,

\[
\sigma^{te} = A, \quad \sigma^{tm} = A + k^2B,
\]

are the conductivities of TE and TM modes. Indeed, boundary conditions for TE \((E_z = 0)\) and TM \((H_z = 0)\) modes have the following form

\[
\text{TE} : [H_z] = 0, \quad [H_z'] = +4\pi i\omega \sigma^{te} H_z,
\]

\[
\text{TM} : [E_z'] = 0, \quad [E_z] = -\frac{4\pi i}{\omega} \sigma^{tm} E_z',
\]

where \([f] = f_2 - f_1\) means jump of function at the layer. Therefore, we observe that eigenvalues \(\sigma^{te}\) and \(\sigma^{tm}\) play the role of conductivity for TE and TM modes, respectively.
Using boundary conditions (4) for scattering process we obtain the transmission and reflection coefficients
\[ r_{\text{te}}(\omega, k_z) = -\frac{\eta_{\text{te}}}{\eta_{\text{te}} + \frac{k_z}{\omega}}, \quad t_{\text{te}} = 1 + r_{\text{te}}, \]
\[ r_{\text{tm}}(\omega, k_z) = -\frac{\eta_{\text{tm}}}{\eta_{\text{tm}} + \frac{k_z}{\omega}}, \quad t_{\text{tm}} = 1 - r_{\text{tm}}, \] (5)
where \( k_z = \sqrt{\omega^2 - k_\perp^2} \) and \( \eta_{\text{te,tm}} = 2\pi\sigma_{\text{te,tm}}^\alpha. \)

### III. The Casimir-Polder Free Energy

The system under consideration consists of atom and conductive plane with distance \( d \) between atom and plane. Using the rarefied procedure of Lifshitz [12] the Casimir-Polder (CP) energy can be given as a sum of TM and TE contributions [4],
\[ \mathcal{E}_{\text{tm}} = \int d^2k_\perp \int_0^{\infty} \frac{d\lambda}{\kappa} \alpha(\lambda) \left( \lambda^2 - 2\kappa^2 \right) t_{\text{tm}}(\lambda, \kappa)e^{-2\alpha\kappa}, \]
\[ \mathcal{E}_{\text{te}} = \int d^2k_\perp \int_0^{\infty} \frac{d\lambda}{\kappa} \alpha(\lambda) \lambda^2 r_{\text{te}}(\lambda, \kappa)e^{-2\alpha\kappa}, \] (6)
where \( \kappa = \sqrt{\lambda^2 + \kappa^2} \) and \( \alpha \) is polarizability of atom.

To take into account temperature we have to change \( \int_0^{\infty} \lambda \to 2\pi T \int_{n=0}^\infty ' \) and \( \lambda \to \xi_n (\kappa \to \kappa_n = \sqrt{\lambda^2 + \xi_n^2} \), where \( \xi_n = 2\pi n T \) being the Matsubara frequencies. We obtain the following expressions for free energy [5]
\[ \mathcal{F}_{\text{tm}} = \frac{T}{2\pi} \sum_{n=0}^{\infty} \int d^2k_\perp \alpha_n (\xi_n^2 - 2\kappa_n^2) t_{\text{tm}}(\xi_n, \kappa_n)e^{-2\alpha\kappa_n}, \]
\[ \mathcal{F}_{\text{te}} = \frac{T}{2\pi} \sum_{n=0}^{\infty} \int d^2k_\perp \alpha_n \xi_n^2 r_{\text{te}}(\xi_n, \kappa_n)e^{-2\alpha\kappa_n}, \] (7)
where \( \alpha_n = \alpha(\xi_n) \). The ideal case appears by formal limit \( \eta_{\text{te,tm}} \to \infty \) (\( r_{\text{te,tm}} \to -1 \), \( t_{\text{te,tm}} \to 1 \)).

Taking into account the Poisson summation formula (see, for example, Ref. [1]) we obtain following expression for free energy
\[ \mathcal{F}_{\text{tm}} = -3\alpha(0)/8\pi a^4 - \text{Casimir-Polder energy for ideal plane/atom.} \]

### IV. The Low Temperature Expansions

To analyze \( \Delta \mathcal{F} \) we use Erdélyi’s lemmas for asymptotic expansion integrals [13]. For completeness we reproduce them below.

**Lemma 1**
\[ \int_0^a x^{\alpha-1}f(x)e^{i\lambda x}dx = \sum_{n=0}^\infty a_n\Lambda^{-(n+\beta)}, \]
where \( a_n = f^{(n)}(0)\frac{\Gamma(n+\beta)}{n!}e^{i\pi(n+\beta)}. \)

**Lemma 2**
\[ \int_0^a x^{\alpha-1}f(x)\ln x e^{i\lambda x}dx = \sum_{n=0}^\infty b_n(\Lambda)\Lambda^{-(n+\beta)}, \]
where \( b_n = f^{(n)}(0)\frac{\Gamma(n+\beta)}{n!}e^{i\pi(n+\beta)} \times \left[ -\ln \Lambda + \psi(n+\beta) + \frac{i\pi}{2} \right], \)
and \( \psi(x) = \Gamma'(x)/\Gamma(x) \). Both Lemmas are valid as \( \Lambda \to \infty \), \( f^{(n)}(a) = 0 \) and \( \beta > 0 \).

The free Casimir-Polder energy \( \Delta \mathcal{F} \) maybe represented in the following form
\[ \frac{\Delta \mathcal{F}}{\mathcal{E}_{\text{CP}}} = \frac{8}{3} \sum_{l=1}^\infty \int_0^{\infty} dz e^{i\Lambda z} \left( Y_{\text{tm}} + Y_{\text{te}} \right), \]
where \( \Lambda = \frac{l}{aT} \) and
\[ Y_{\text{tm}}(z) = \frac{\alpha(\lambda)}{\alpha(0)} \int_z^\infty \frac{e^{-2s} \left( 2s^2 - z^2 \right)}{s + z/\eta_{\text{tm}}} ds, \]
\[ Y_{\text{te}}(z) = \frac{\alpha(\lambda)}{\alpha(0)} \int_z^\infty \frac{e^{-2s} z^3}{s + z/\eta_{\text{te}}} ds. \] (13)
Here \( \lambda = z/a \) and \( k = \sqrt{z^2 - z^2/a} \) are used for frequency and wave-vector, correspondingly.

First of all, let us consider the case without spatial dispersion, \( \eta = \eta(\lambda) \). Straightforward integration in Eq. (13) gives
\[ Y_{\text{tm}} = \frac{\alpha e^{-2z}}{2\alpha(0)\eta_{\text{tm}}}(\eta_{\text{tm}} (2z^2 - z(1+2z)\eta_{\text{tm}} + (1+z)^2 \eta_{\text{tm}}^2)) \]
where $\Gamma(a, b)$ is incomplete gamma function.

Expansion at point $z = 0$ contents logarithmic contribution

$$Y_{tm, te} = \sum_{m=0}^{\infty} A_{tm}^{\text{tm, te}} z^m + \ln z \sum_{m=3}^{\infty} B_{tm}^{\text{tm, te}} z^m. \quad (15)$$

Taking into account this expansion and Lemmas we obtain expansion up to 4th power of $T$ for the energy at low temperature

$$\frac{\Delta F_{tm, te}}{E_{\text{CP}}} = -\frac{\chi^2}{9} A_1^{tm} + \frac{\chi^4}{90} \left\{ A_3^{tm, te} + B_3^{tm, te} \left( \ln \frac{\chi}{2\pi} - \gamma_E + \frac{90\zeta'_R(4)}{\pi^4} + \frac{11}{6} \right) \right\}, \quad (16)$$

where $\chi = 2\pi aT$ and $\zeta_R(s)$ is Riemann zeta-function.

From Eqs. (14) we obtain in manifest form

$$A_1^{tm} = -\frac{1}{2\eta_{tm}}, A_1^{te} = 0,$$

$$A_3^{tm} = -\frac{1}{4d^2\eta_{tm}} \left( \alpha'' - \frac{\eta''}{\eta_{tm}} + \frac{2\eta''}{\eta_{tm}^2} - \frac{\eta''}{\eta_{tm}^2} \right) - \frac{2}{\eta_{tm}} + \frac{1}{\eta_{tm}^2} + \frac{1}{\eta_{tm}} + \left( \gamma_E + \ln \left\{ \frac{2(1 + \eta_{tm})}{\eta_{tm}} \right\} \right) B_3^{tm},$$

$$A_3^{te} = \left( \gamma_E + \ln \left\{ 2(1 + \eta_{te}) \right\} \right) B_3^{te},$$

$$B_3^{tm} = \frac{2}{\eta_{tm} - 1}, B_3^{te} = -\eta_{te}. \quad (17)$$

Taking into account these expressions we obtain asymptotic expansion of free CP energy

$$\frac{\Delta F_{tm}}{E_{\text{CP}}} = -\frac{\chi^2}{18\eta_{tm}} + \frac{\chi^4}{270} \left\{ 1 - \frac{6}{\eta_{tm}} + \frac{\eta''}{\eta_{tm} \alpha''} \right\} + \frac{\eta''}{\eta_{tm}^2} + \frac{2}{\eta_{tm}^2} + \frac{6}{\eta_{tm}} + \ln \left\{ \frac{1 + \eta_{tm}}{\pi \eta_{tm}} \right\} \right\},$$

$$\Delta F_{te} \frac{E_{\text{CP}}}{E_{\text{CP}}} = -\frac{\chi^4\eta_{te}}{90} + \ln \left\{ \frac{1 + \eta_{te}}{\pi} + \frac{90\zeta'_R(4)}{\pi^4} \right\}. (18)$$

where functions $\alpha$ and $\eta$ and their derivatives are considered at zero argument.

In ideal case [5] the free energy for low temperatures has first correction $\sim \chi^4$

$$\frac{\mathcal{F}}{E_{\text{CP}}} \bigg|_{T \to 0} = 1 - \frac{\chi^4}{135}. \quad (19)$$

One comment is in order. Above expansions are valid if arguments of incomplete gamma functions in Eqs. (14) are small, that is for $\chi \ll \frac{\eta_{tm}}{1 + \eta_{tm}}$ and $\chi \ll \frac{1}{1 + \eta_{te}}$. Therefore, we may take limit to ideal case only for TM mode in Eq. (18). To consider ideal case for TE mode we have to take limit $\eta_{te} \to \infty$ first of all in Eq. (14) and then make expansion over $z$.

The main term of expansion $\chi^2/18\eta_{tm}$ is the same for $k \neq 0$. Indeed, to obtain $A_1^{tm}$ we may take derivative of Eq. (13) with respect of $z$ and then take limit $z \to 0$. By proceed that way we obtain the same form of main term where $\eta_{tm}$ is calculated for $\lambda = k = 0$.

Therefore, we observe that for all models of conductivities the main term of low temperature expansion proportional to $\chi^2$.

Let us consider now zero term in Poisson representation

$$\frac{F_0^{\text{tm}}}{E_{\text{CP}}} = -\frac{4}{3} \int_0^{\infty} z^3 dz \int_0^1 dx \frac{\alpha(\lambda)}{\alpha(0)} (2x - 2 \eta_{te}) r^{tm}(x, 1) e^{-2z},$$

$$\frac{F_0^{\text{tm}}}{E_{\text{CP}}} = -\frac{4}{3} \int_0^{\infty} z^3 dz \int_0^1 dx \frac{\alpha(\lambda)}{\alpha(0)} x^2 r^{te}(x, 1) e^{-2z}. \quad (20)$$

As noted above it coincides exactly with that obtained for zero temperature in Ref. [4], but with additional dependence on the temperature and chemical potential through dependence of conductivity on these parameters (see Sec. A 3). These expressions tend to $1/2$ for ideal ($\eta \to \infty$) case and for $a \to \infty$.

V. NUMERICAL ANALYSIS

Let us compare numerically the formulas obtained (18) with exact numerical calculations for different models of conductivity. Let us denote for simplicity $\delta F_n = \Delta F/E_{\text{CP}}$ where $n = 0$ corresponds to exact expression calculated numerically and $n = 1, 2$ corresponds to first ($\sim \chi^2$) and second ($\sim \chi^4$) approximations in (18). Numerically we use (7) and subtract (20). To estimate error we plot relative error function $E_n = (\delta F_n - \delta F_0)/\delta F_0 \cdot 100\%$.

For definiteness we consider Hydrogen atom in framework of one-oscillator model (see Ref. [4]) and distance $a = 10nm$ between atom and plane of graphene. Then the interval of temperatures $T \in [0, 100^\circ K]$ corresponds to interval of parameter $\chi \in [0, 2.7 \cdot 10^{-3}]$. Different models of graphene’s conductivity briefly discussed in Appendix A.

A. Constant conductivity model

First model is constant conductivity model. For graphene we use universal conductivity, $e^{\sigma} = e^2/4\hbar$ and then $\eta_{tm} = \eta_{te} = \eta_{gr} = 0.0114$. Fig. 1 illustrates numerical evaluation exact expression and approximations obtained and relative error. We observe that relative error for second approximation is not more than $1\%$ up to $T = 100K$. 

\[ \text{Insert Fig. 1 here} \]
B. Drude-Lorentz model

In the case of Drude-Lorentz 7-oscillator model of conductivity agreement is not so good. The point is that even the conductivity at zero frequency equals to graphene universal conductivity, but derivatives of first and second order are very huge. The contributions to $\chi^4$ contain terms

$$\eta = 0.0114, \quad \frac{\eta^4}{a\eta} = 8.76 \cdot 10^3, \quad \frac{\eta^4}{a^2\eta} = -1.85 \cdot 10^7,$$  \quad (21)

and the ratio of second term $\sim \chi^4$ and first $\sim \chi^2$ is 0.67 for $T = 10^8K$ and 67 for $T = 10^9K$. For this reason the low temperature expansion is valid for very low temperature (see Fig. 2).

C. Polarization tensor approach

The last model is polarization tensor model of conductivity developed in Refs. [9]. In this case the conductivity depends on frequency, wave-vector, temperature and chemical potential $\eta_{\text{hm,te}}(\lambda, k, T, \mu)$. In the case under consideration, $\lambda = 0$,

$$\frac{\eta_{\text{hm,te}}}{\eta_{\text{gr}}} = \frac{4m}{\pi \lambda} \left\{ 1 + \left( \frac{\lambda}{2m} \right)^2 - \frac{1}{(2m)} \operatorname{arctan} \left( \frac{\lambda}{2m} \right) \right\} + \frac{16}{\pi \lambda} \int_0^\infty \frac{dz}{4z^2 + m^2} \left\{ \frac{1}{e^{\frac{z}{\mu \lambda}} + 1} + \frac{1}{e^{\frac{\mu \lambda}{z}} + 1} \right\}. \quad (22)$$

In the static limit, $\lambda \to 0$, the conductivity is divergent (but transmission and reflection coefficients (5) tend to that for ideal case). To make it finite we cut $\lambda$ on minimal value $\gamma$. The threshold parameter $\gamma$ appears in natural way in framework of Kubo approach calculation of conductivity [14] as a scattering rate.

Numerical evaluations (see Figs. 3, 4) demonstrate good agreement expansion (18) with exact expression (7).

Let us consider now zero temperature term $F_0$ given by Eqs. (9), (20). In the framework of the model under consideration it depends on the temperature and chemical potential. Expansion of conductivity over temperature is given in Appendix B.

For $m = \mu = 0$ the first correction $\sim T^3$ (see Eq. (B12)) and reads

$$\eta_{\text{te}} = \eta_{\text{hm}} = \eta_{\text{gr}} + \frac{48}{\pi} \zeta R(3) \frac{T^3}{\lambda^3}. \quad (23)$$
Therefore,

\[
\frac{F_0}{E_{CP}} = \frac{F_0^{T=0}}{E_{CP}} + T^3 \beta \tag{24}
\]

where

\[
\beta = \frac{64 \zeta R(3) \eta_{gr}}{\pi} \int_0^{\infty} \frac{x^3 dz}{\alpha(0)} x e^{-2z} \left\{ \frac{2 - x^2}{(x + \eta_{gr})^2} + \frac{x^2}{(1 + x \eta_{gr})^2} \right\}, \tag{25}
\]

For \( a = 10 nm \) and \( \gamma = 0.1 eV \) we have \( \beta = 2.33 \cdot 10^{-12} \) where \( T(K) \).

For \( m = 0.1 eV \) and \( \mu = 0.05 eV \) the first correction is exponentially small \( \sim e^{-\frac{m}{\mu}} \) (see Eq. (B8)). Fig. 5 illustrates temperature correction of zero temperature term due to conductivity dependence of temperature and chemical potential.

\[ \text{FIG. 5. Polarization tensor approach for conductivity. Zero term } \frac{(F_0 - F_0^{T=0})}{E_{CP}}. \text{ Left panel: } m = \mu = 0 \text{ and } \gamma = 0.1 eV. \text{ Black curve is exact expression (20) and blue is first approximation (24). Right panel: } m = 0.1 eV, \mu = 0.05 eV \text{ and } \gamma = 0.1 eV. \text{ Temperature contribution is exponentially small up } 100^5 K. \]

VI. CONCLUSION

We have obtained the analytic expression of low temperature expansion of the Casimir-Polder (van der Waals) energy for a system which contains an atom and conductive plane. The conductivity is characterized by symmetric 2D conductivity tensor. The eigenvalues of this tensor are conductivity of TE and TM modes. The main term of expansion \( \sim T^2 \) comes from TM mode, the next terms \( \sim T^4 \) and \( \sim T^4 \ln T \). Numerical analysis shows good agreement expansion obtained with exact expressions.

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Appendix A: Models of conductivity

In this appendix we consider different models of graphene conductivity which were used for numerical analysis in Sec. V. We normalize conductivity to the universal conductivity of graphene \( \sigma_{gr} = e^2 / 4 \hbar \) and mark these quantities by overline.

1. Constant conductivity

It is well-known that the graphene conductivity is a constant \( (\sigma_{gr} = e^2 / 4 \hbar) \) over a relatively large frequency range, near infrared to optical [15]. For this reason we consider the model in which the conductivity equals to this value for whole frequencies. In this case

\[ \overline{\sigma} = I, \tag{A1} \]

and \( \overline{\sigma}_{te} = \overline{\sigma}_{tm} = 1 \). This approximation of conductivity was used intensively in Ref. [4].

2. Drude-Lorentz model

We use conductivity of graphite alongside to planes which is approximated with high precision by Drude-Lorentz model consisting of a Drude term and seven Lorentz oscillators according to [16]:

\[ \sigma(\omega) = \frac{f_0 \omega_p^2}{\gamma_0 - i \omega} + \sum_{j=1}^{7} \frac{i \omega f_j \omega_p^2}{\omega^2 - \omega_j^2 + i \omega \gamma_j}. \tag{A2} \]

We multiply it on the length scale \( d = 0.2245 nm \) which is close to interplane distance of graphite \( d_{gr} = 0.3345 nm \) and obtain \( \overline{\sigma}_{te} = \overline{\sigma}_{tm} = \overline{\sigma} \) and

\[ \overline{\sigma} = \overline{\sigma} I, \tag{A3} \]

where

\[ \overline{\sigma}(\lambda) = \frac{\sigma_0 \gamma_0}{\gamma_0 + \lambda} + \sum_{j=1}^{7} \frac{\lambda \sigma_{j} \gamma_{j}}{\lambda^2 + \lambda_j^2 + \lambda \gamma_j}. \tag{A4} \]

With scale \( d = 0.2245 nm \) we obtain right limit for small frequencies \( \overline{\sigma}(0) = 1 \). Here, \( \gamma_j \) is the relaxation time and \( \omega_j \) is the characteristic frequency for the \( j \)-th term. All parameters of this model maybe found in Ref. [4].

3. Polarization tensor approach

In Ref. [9] was used relation between \((2 + 1)D\) polarization tensor, \( \Pi_{\mu \nu} \) and tensor of conductivity, \( \sigma_{ij} \),

\[ \sigma_{\mu \nu} = \frac{\Pi_{\mu \nu}}{i \omega}, \tag{A5} \]
and obtained polarization tensor in general form, which depends on \( \lambda, k, T, \mu \). Due to gauge invariance the polarization tensor has only two independent components, for example, \( \Pi_{00} \) and \( \Pi_{tr} = \Pi^{\mu\nu}g_{\mu\nu} = \Pi_{00} - \Pi_{11} - \Pi_{22} \).

In Ref. [9] has obtained elegant representation of polarization tensor. Taking into account these expressions and expressions (3) for TE and TM conductivities we have

\[
\sigma_{te} = \frac{4}{e^2\lambda} \left( \Pi_{tr} - \frac{\lambda^2 + k^2}{k^2}\Pi_{00} \right), \quad \sigma_{tm} = \frac{4\lambda}{e^2k^2}\Pi_{00}. \quad (A6)
\]

According with Ref. [9] we divide conductivity into

\[
\Delta \sigma_{te} = \frac{8}{\pi \lambda} \Re \int_{m}^{\infty} dz \frac{\left( \frac{q^2}{r} \right) - \left( \frac{q^2}{r} \right)^2}{r(q^2v_F^2 + 4m^2k^2v_F^2 + q\lambda r)} \Theta, \quad \Delta \sigma_{tm} = \frac{8}{\pi \lambda} \Re \int_{m}^{\infty} dz \frac{\left( \frac{q^2}{r} \right) - \left( \frac{q^2}{r} \right)^2}{r(q^2v_F^2 + 4m^2k^2v_F^2 + q\lambda r)} \Theta,
\]

where \( \Theta = (e^{\frac{\mu}{\lambda}} + 1)^{-1} + (e^{\frac{-\mu}{\lambda}} + 1)^{-1}, \quad r = \sqrt{k^2(q^2v_F^2 + 4m^2k^2v_F^2)} \) and \( q = \lambda - 2iz \).

In the gapless case, \( m = 0 \), we obtain a little bit simple expressions

\[
\Delta \sigma_{te} = \frac{8}{\pi \lambda} \Re \int_{0}^{\infty} dz \frac{\left( \frac{q^2}{r} \right) - \left( \frac{q^2}{r} \right)^2}{r(q^2v_F^2 + 4m^2k^2v_F^2 + q\lambda r)} \Theta, \quad \Delta \sigma_{tm} = \frac{8}{\pi \lambda} \Re \int_{0}^{\infty} dz \frac{\left( \frac{q^2}{r} \right) - \left( \frac{q^2}{r} \right)^2}{r(q^2v_F^2 + 4m^2k^2v_F^2 + q\lambda r)} \Theta, \quad (A9)
\]

with \( r_0 = \sqrt{q^2 + k^2v_F^2} \) and \( \sigma_{te}^0 = k_F/\lambda, \quad \sigma_{tm}^0 = \lambda/k_F \).

Let us consider some special limits.

\[
\Delta \sigma_{te} = \frac{8}{\pi \lambda} \Re \int_{m}^{\mu} dz \frac{\left( \frac{q^2}{r} \right) - \left( \frac{q^2}{r} \right)^2}{r(q^2v_F^2 + 4m^2k^2v_F^2 + q\lambda r)} \Theta(\mu - m), \quad \Delta \sigma_{tm} = \frac{8}{\pi} \Re \int_{m}^{\mu} dz \frac{\left( \frac{q^2}{r} \right) - \left( \frac{q^2}{r} \right)^2}{r(q^2v_F^2 + 4m^2k^2v_F^2 + q\lambda r)} \Theta(\mu - m), \quad (A12)
\]

where \( \Theta(x) \) is step function. The zero terms have the same form (A8). Therefore, we have additional contribution due to chemical potential. If \( \mu \leq m \) the conductivity is defined by zero temperature contribution (A8).
In the case of zero mass gap, $m = 0$, the conductivity is zero if $\mu \leq 0$ and it reads
\[
\Delta \sigma_{te} = \frac{8}{\pi \lambda} \Re \int_0^\mu dz \frac{q (q^2 - \lambda^2)}{r_0(qk_F + \lambda r_0)} \theta(\mu),
\]
\[
\Delta \sigma_{tm} = \frac{8}{\pi k_F} \Re \int_0^\mu dz \frac{q r_0}{\lambda k_F r_0 + q \lambda} \theta(\mu).
\]
(A13)

The expansion over $T$ up to $T^4$ obtained in Appendix B.

4. Kubo approach

The tensor of conductivity $\sigma_{ij}(\lambda, k, T)$ was obtained in Ref. [14] in framework of Kubo approach. The eigenvalues in this case have the following form
\[
\bar{\sigma}_{tm} = \int_0^\infty dx \int_0^{2\pi} d\varphi \left\{ K_- \sin^2 \varphi + K_+ \cos^2 \varphi \right\},
\]
(A14)

where
\[
K_\mp = \frac{4}{\pi^2} \tanh(\nu_+ \mp \nu_-) \mp \tanh(\nu_+ \mp \nu_-) \left( 1 + (\nu_+ \mp \nu_-)^2 \right),
\]
\[
\nu_\pm = \sqrt{x^2 + \frac{b^2}{4}} \pm bx \cos \varphi,
\]
and $\gamma$ is scattering rate. Three parameters $\lambda, k, T$ are combined in two dimensionless parameters
\[
p = \frac{\lambda + \gamma}{2T}, \quad b = \frac{k\nu_F}{2T}.
\]
(A15)

Let us consider different limits.

a. $k \to 0$

We obtain that
\[
\bar{\sigma}_{te} = \bar{\sigma}_{tm} = \frac{4 \ln 2}{\pi p} + \frac{2}{\pi} \int_0^{\infty} \frac{\tanh(\frac{b^2}{4})}{x^2 + 1} dx.
\]
(A16)

The same expression maybe obtained from that in framework of polarization tensor approach (A10), (A11) taking into account scattering rate $\gamma$.

The conductivity without spatial dispersion was also obtained in Ref. [15].

b. $T \to 0$

The conductivities read
\[
\bar{\sigma}_{te} = -\frac{4}{\pi} \frac{2 + b^2}{b^2} \arctan \frac{b^2}{2 + \frac{4}{\pi b}} - \frac{4 + 3b^2}{b^2 \sqrt{1 + b^2}},
\]
\[
\bar{\sigma}_{tm} = -\bar{\sigma}_{te} - b + \frac{2 + b^2}{\sqrt{1 + b^2}}.
\]
(A17)

In the limit $b \to 0$ ($k \to 0$ or $\lambda \to \infty$) we obtain $\bar{\sigma}_{te} = \bar{\sigma}_{tm} = 1$, that is $\bar{\sigma} = I$.

Appendix B: Low temperature expansion of conductivity

The temperature correction for conductivity has the following form ($i = te, tm$)
\[
\Delta \sigma_i = \int_0^\infty dz f_i(z) \Theta,
\]
(B1)

where
\[
f_{te}(z) = \frac{8}{\pi \lambda} \Re \left\{ (4m^2 + q^2)(q^2 s_F^2 + 4m^2 k^2 v_F^2) - q^2 s_F^2 \lambda^2 \right\},
\]
\[
f_{tm}(z) = \frac{8}{\pi} \Re \left\{ \frac{q(q^2 + k^2 v_F^2 + 4m^2) - \lambda r}{r + q \lambda} \right\}.
\]
(B2)

We have the following expansions in two domains ($m \geq 0$) for $T \to 0$:

I. $\mu \geq m$

\[
\Delta \sigma_i = \int_m^\infty f_i(z) dz + \sum_{n=0}^{\infty} T^{n+1} f_i^{(n)}(\mu)(-1)^n
\]
\[
\times \left\{ F_n \left[ \frac{\mu + m}{T} \right] + F_n \left[ \frac{\mu - m}{T} \right] \right\} - (1 - (-1)^n) F_n[0].
\]
(B3)

II. $\mu \leq m$

\[
\Delta \sigma_i = \sum_{n=0}^{\infty} T^{n+1} f_i^{(n)}(\mu)
\]
\[
\times \left\{ (-1)^n F_n \left[ \frac{\mu + m}{T} \right] + F_n \left[ \frac{m - \mu}{T} \right] \right\}.
\]
(B4)

Here we used notation
\[
F_n[x] = \frac{1}{n!} \int_x^{\infty} \frac{z^n dz}{e^z + 1}.
\]
(B5)

This function has the following behavior at large and small argument
\[
F_n[x]_{x \to \infty} = e^{-x} \frac{x^n}{n!}, \quad F_n[0] = (1 - 2^{-n}) \zeta(n + 1).
\]
(B6)

The above general expressions maybe simplified for three different regions of $T$:

I. $\mu > m$, $T \ll \mu - m$

\[
\Delta \sigma_i = \int_m^\mu f_i(z) dz + \frac{\pi^2}{6} T^2 f_i^{(2)}(\mu)
\]
\[
+ \frac{7\pi^4}{360} T^4 f_i^{(4)}(\mu) + O(e^{-\frac{\mu - m}{T}}).
\]
(B7)

II. $\mu < m$, $T \ll m - \mu$

\[
\Delta \sigma_i = O(e^{-\frac{m - \mu}{T}}).
\]
(B8)

III. $\mu = m$, $T \ll m$

\[
\Delta \sigma_i = \int_m^\infty f_i(z) dz + \frac{\pi^2}{6} T^2 f_i^{(2)}(\mu)
\]
\[
+ \frac{7\pi^4}{360} T^4 f_i^{(4)}(\mu) + O(e^{-\frac{\mu - m}{T}}).
\]
\[ \Delta \sigma_i = T \ln 2f_\text{e}(m) + \frac{\pi^2}{12} T^2 f'_\text{e}(m) + \frac{3}{4} \zeta_R(3) T^3 f''_\text{e}(m) \]

\[ + \frac{7\pi^4}{720} T^4 f'''_\text{e}(m) + O(e^{-\pi}). \]  

(B9)

In gapless case, \( m = 0 \),

\[ f_{\text{e}}(z) = \frac{8}{\pi \lambda} \Re \frac{q (q^2 - \lambda^2)}{r_0 (q k_F + \lambda r_0)} , \]

\[ f_{\text{m}}(m) = \frac{8}{\pi k_F} \Re \frac{q^2 r_0 - \lambda k_F}{r_0 + q \lambda} . \]  

(B10)

with \( r_0 = \sqrt{q^2 + k^2 r_0^2} \), \( q = \lambda - 2i \zeta \). We have two regions

I. \( \mu > 0, \ T \ll \mu \)

\[ \Delta \sigma_i = \int_0^\mu f_i(z) dz + \frac{\pi^2}{6} T^2 f'_\text{e}(\mu) \]

\[ + \frac{7\pi^4}{360} T^4 f'''_\text{e}(\mu) + O(e^{-\pi}). \]  

(B11)

II. \( \mu = 0, \ T \to 0 \)

\[ \Delta \sigma_i = \frac{3}{2} \zeta_R(3) T^3 f''_\text{e}(0) + \frac{15}{8} \zeta_R(5) T^5 f^{(4)}_\text{e}(0) + \ldots \]  

(B12)

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