Notes on a Theorem of Benci-Gluck-Ziller-Hayashi

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Abstract

We use constrained variational minimizing methods to study the existence of periodic solutions with a prescribed energy for a class of second order Hamiltonian systems with a \( C^2 \) potential function which may have an unbounded potential well. Our result can be regarded as complementary to the well-known theorem of Benci-Gluck-Ziller and Hayashi.

Key Words: \( C^2 \) second order Hamiltonian systems, periodic solutions, constrained variational minimizing methods.

2000 Mathematical Subject Classification: 34C15, 34C25, 58F.

1. Introduction

Based on the earlier works of Seifert([20]) in 1948 and Rabinowitz([18,19]) in 1978 and 1979, Benci ([4]), and Gluck-Ziller ([11]), and Hayashi([13]) published work examining the periodic solutions for second order Hamiltonian systems

\[
\ddot{q} + V'(q) = 0 \quad (1.1)
\]

\[
\frac{1}{2}|\dot{q}|^2 + V(q) = h \quad (1.2)
\]

with a fixed energy. Utilizing the Jacobi metric and very complicated geodesic methods with algebraic topology, they proved the following general theorem:

**Theorem 1.1** Suppose \( V \in C^2(R^n, R) \). If the potential well

\[
\{ x \in R^n | V(x) \leq h \}
\]
is bounded and non-empty, then the system (1.1)-(1.2) has a periodic solution with energy $h$.

Furthermore, if $V'(x) \neq 0$, $\forall x \in \{x \in \mathbb{R}^n | V(x) = h\}$, then the system (1.1)-(1.2) has a nonconstant periodic solution with energy $h$.

For the existence of multiple periodic solutions for (1.1)-(1.2) with compact energy surfaces, we can refer to Groessen([12]) and Long[14] and the references therein.

In 1987, Ambrosetti-Coti Zelati[2] successfully used Clark-Ekeland’s dual action principle and Ambrosetti-Rabinowitz’s Mountain Pass theorem to study the existence of $T$-periodic solutions of the second-order equation

$$-\ddot{x} = \nabla U(x),$$

where

$$U = V \in C^2(\Omega; \mathbb{R})$$

such that

$$U(x) \to \infty, x \to \Gamma = \partial \Omega;$$

with $\Omega \subset \mathbb{R}^n$ a bounded convex domain. Their principle result is the following:

**Theorem 1.2** Suppose

1. $U(0) = 0 = \min U$
2. $U(x) \leq \theta(x, \nabla U(x))$ for some $\theta \in (0, \frac{1}{2})$ and for all $x$ near $\Gamma$ (superquadraticity near $\Gamma$)
3. $(U''(x)y, y) \geq k|y|^2$ for some $k > 0$ and for all $(x, y) \in \Omega \times \mathbb{R}^N$.

Let $\omega_N$ be the greatest eigenvalue of $U''(0)$ and $T_0 = (2/\omega_N)^{1/2}$. Then $-\ddot{x} = \nabla U(x)$ has for each $T \in (0, T_0)$ a periodic solution with minimal period $T$.

The dual variational principle and Mountain Pass Lemma again proved the essential ingredients for the following theorem of Coti Zelati-Ekeland-Lions [8] concerning Hamiltonian systems in convex potential wells.

**Theorem 1.3** Let $\Omega$ be a convex open subset of $\mathbb{R}^n$ containing the origin $O$. Let $V \in C^2(\Omega, \mathbb{R})$ be such that

(V1). $V(q) \geq V(O) = 0, \forall q \in \Omega$
(V2). $\forall q \neq O, V''(q) > 0$
(V3). $\exists \omega > 0$, such that

$$V(q) \leq \frac{\omega}{2} ||q||^2, \forall ||q|| < \epsilon$$

and
(V4). \( V''(q)^{-1} \to 0, \|q\| \to 0 \) or, \( (V4)'\). \( V''(q)^{-1} \to 0, q \to \partial \Omega \).

Then, for every \( T < \frac{2\pi}{\sqrt{\omega}} \), the system (1.1) has a solution with minimal period \( T \).

In Theorems 1.2 and 1.3, the authors assumed the convex conditions for potentials and potential wells in order to apply Clark-Ekeland’s dual variational principle. We observe that Theorems 1.1-1.3 essentially make the assumption

\[
V(x) \to \infty, x \to \Gamma = \partial \Omega
\]

so that all potential wells are bounded. We wish to generalize Theorems 1.1-1.3 from two directions: (1) We dispense with the convex assumption on potential functions, (2) \( V(x) \) can be uniformly bounded, and the potential well can be unbounded.

In 1987, D.Offin ([16]) generalized Theorem 1.1 to some non-compact cases for \( V \in C^3(R^n, R) \) under complicated geometric assumptions on the potential wells; however, these geometric conditions appear difficult to verify for concrete potentials. In 2009, Berg-Pasquotto-Vandervorst ([5]) studied the closed orbits on non-compact manifolds with some complex topological assumptions.

Using simpler constrained variational minimizing method, we obtain the following result:

**Theorem 1.4** Suppose \( V \in C^2(R^n, R) \), \( h \in R \) satisfies

1. \( V(-q) = V(q) \)
2. \( V'(q)q > 0, \forall q \neq 0 \)
3. \( 3V'(q)q + (V''(q)q, q) \neq 0, \forall q \neq 0 \)
4. \( \exists \mu_1 > 0, \mu_2 \geq 0 \), such that
   \[
   V'(q) \cdot q \geq \mu_1 V(q) - \mu_2
   \]
5. \( \lim_{|q| \to \infty} \text{sup} |V(q) + \frac{1}{2}V''(q)q| \leq A \)

Then the system (1.1) – (1.2) has at least one non-constant periodic solution with the given energy \( h \).

**Corollary 1.5** Suppose \( V(q) = a|q|^{2n}, a > 0 \), then the system \( \forall h > 0, (1.1) - (1.2) \) has at least one non-constant periodic solution with the given energy \( h \).

**Remark 1** Suppose \( V(x) \) is the following well-known \( C^\infty \) function:

\[
V(x) = e^{-\frac{1}{|x|}}, \forall x \neq 0; \\
V(0) = 0.
\]

Then \( V(x) \) satisfies \( (V_1) - (V_5) \) if we take \( \mu_1 = \mu_2 > 0 \) and \( A = 1 \) in Theorem 1.4, but \( (V_6) \) does not hold.

**Proof** In fact, it’s easy to check \( (V_1) - (V_5) \):
1. It’s obvious for \( (V_1) \).
(2). For \((V_2)\) and \((V_3)\), we notice that
\[
V'(x)x = \frac{1}{|x|} e^{\frac{1}{|x|}} > 0, \forall x \neq 0,
\]
\[
(V''(x)x, x) = e^{\frac{1}{|x|}} \left( -\frac{2}{|x|} + \frac{1}{|x|^2} \right)
\]
\[
3V'(x)x + (V''(x)x, x) = e^{\frac{1}{|x|}} \left( \frac{1}{|x|} + \frac{1}{|x|^2} \right) > 0, \forall x \neq 0.
\]

(3). For \((V_4)\), we set
\[
w(x) = \left( \frac{1}{|x|} - \mu_1 \right) e^{\frac{1}{|x|}}; \quad x \neq 0, w(0) = 0.
\]
We will prove \(w(x) > -\mu_1\); in fact,
\[
w'(x) = \left[ \frac{1}{|x|} - (1 + \mu_1) \right] \frac{x}{|x|^2} e^{\frac{1}{|x|}}; x \neq 0, w'(0) = 0.
\]
From \(w'(x) = 0\), we have \(x = -\frac{1}{1+\mu_1}\) or 0 or \(\frac{1}{1+\mu_1}\).

It’s easy to see that \(w(x)\) is strictly increasing on \((-\infty, -\frac{1}{1+\mu_1}]\) and \([0, \frac{1}{1+\mu_1}]\) but strictly decreasing on \([\frac{1}{1+\mu_1}, 0]\) and \([\frac{1}{1+\mu_1}, +\infty)\). We notice that
\[
\lim_{|x| \to +\infty} w(x) = -\mu_1,
\]
and
\[
w(0) = 0.
\]
So
\[
w(x) > -\mu_1.
\]
When we take \(\mu_2 = \mu_1 > 0\), \((V_4)\) holds.

(4). For \((V_5)\), we have
\[
V(x) + \frac{1}{2} V'(x)x = e^{\frac{1}{|x|}} (1 + \frac{1}{2} \frac{1}{|x|}) < 1, \forall x \neq 0;
\]
\[
V(0) + \frac{1}{2} V'(0)0 = 0.
\]

Corollary 1.6 Given any \(a > 0\), \(n \in \mathbb{N}\), suppose \(V(x) = a|x|^{2n} + e^{\frac{1}{|x|}}, x \neq 0, V(0) = 0\). Then \(\forall h > 1\), the system \((1.1) - (1.2)\) has at least one non-constant periodic solution with the given energy \(h\).

Remark 2 The potential \(V(x)\) in Remark 1 is noteworthy since the potential function is non-convex and bounded which satisfies neither of the conditions of Theorems 1.1-1.3, Offin’s geometrical conditions, nor Berg-Pasquotto-Vandervorst’s complex topological assumptions. Notice the special properties for our potential well. It is a
bounded set if $h < 1$, but for $h \geq 1$ it is $\mathbb{R}^n$ - an unbounded set. We also notice that the symmetrical condition on the potential simplified our Theorem 1.4 and it's proof; it seems interesting to observe to obtain non-constant periodic solutions if the symmetrical condition is deleted.

2 A Few Lemmas

Let

$$H^1 = W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) = \{ u : R \rightarrow \mathbb{R}^n, u \in L^2, \dot{u} \in L^2, u(t + 1) = u(t) \}$$

Then the standard $H^1$ norm is equivalent to

$$\| u \| = \| u \|_{H^1} = \left( \int_0^1 |\dot{u}|^2 dt \right)^{1/2} + | \int_0^1 u(t) dt |.$$

**Lemma 2.1** ([1]) Let

$$M = \{ u \in H^1 | \int_0^1 (V(u) + \frac{1}{2} V''(u) u) dt = h \}.$$  

If $(V_3)$ holds, then $M$ is a $C^1$ manifold with codimension 1 in $H^1$.

Let

$$f(u) = \frac{1}{4} \int_0^1 |\dot{u}|^2 dt \int_0^1 V'(u)u dt$$

and $\tilde{u} \in M$ be such that $f'(\tilde{u}) = 0$ and $f(\tilde{u}) > 0$. Set

$$\frac{1}{T^2} = \frac{\int_0^1 V'(\tilde{u})\tilde{u} dt}{\int_0^1 |\dot{\tilde{u}}|^2 dt}.$$  

If $(V_2)$ holds, then $\tilde{q}(t) = \tilde{u}(t/T)$ is a non-constant $T$-periodic solution for (1.1)-(1.2).

When the potential is even, then by Palais's symmetrical principle ([17]) and Lemma 2.1, we have

**Lemma 2.2** ([1]) Let

$$F = \{ u \in M | u(t + 1/2) = -u(t) \}$$

and suppose $(V_1) - (V_3)$ holds. If $\tilde{u} \in F$ be such that $f'(\tilde{u}) = 0$ and $f(\tilde{u}) > 0$,then $\tilde{q}(t) = \tilde{u}(t/T)$ is a non-constant $T$-periodic solution for (1.1)-(1.2). In addition, we have

$$\forall u \in F, \int_0^1 u(t) dt = 0.$$  

Recall the following two classic results.
Lemma 2.3 (Sobolev-Rellich-Kondrachov\cite{15},\cite{22})
\[ W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \]
and the imbedding is compact.

Lemma 2.4 (Eberlein-Smulian \cite{21}) A Banach space \( X \) is reflexive if and only if any bounded sequence in \( X \) has a weakly convergent subsequence.

Definition 2.1 (Tonelli,\cite{15}) Let \( X \) be a Banach space and \( M \subset X \). If it the case that for any sequence \( \{x_n\} \subset M \) strongly convergent to \( x_0 \) \((x_n \to x_0)\), we have \( x_0 \in M \), then we call \( M \) a strongly closed (closed) subset of \( X \); if for any \( \{x_n\} \subset M \) weakly convergent to \( x_0 \) \((x_n \rightharpoonup x_0)\), we have \( x_0 \in M \), then we call \( M \) a weakly closed subset of \( X \).

Let \( f : M \to \mathbb{R} \).
(i). If for any \( \{x_n\} \subset M \) strongly convergent to \( x_0 \), we have
\[ \liminf f(x_n) \geq f(x_0), \]
then we say \( f(x) \) is lower semi-continuous at \( x_0 \).
(ii). If for any \( \{x_n\} \subset M \) weakly convergent to \( x_0 \), we have
\[ \liminf f(x_n) \geq f(x_0), \]
then we say \( f(x) \) is weakly lower semi-continuous at \( x_0 \).

Using his variational principle, Ekeland proved

Lemma 2.5 (Ekeland\cite{9}) Let \( X \) be a Banach space and \( F \subset X \) a closed (weakly closed) subset. Suppose that \( \Phi \) defined on \( X \) is Gateaux-differentiable and lower semi-continuous (or weakly lower semi-continuous) and that \( \Phi|_F \) restricted on \( F \) is bounded from below. Then there is a sequence \( x_n \subset F \) such that
\[ \Phi(x_n) \to \inf_F \Phi \quad \text{and} \quad \|\Phi|_F'(x_n)\| \to 0. \]

Definition 2.2 (\cite{9,10}) Let \( X \) be a Banach space and \( F \subset X \) a closed (weakly closed) subset. Suppose that \( \Phi \) defined on \( X \) is Gateaux-differentiable. If it is true that whenever \( \{x_n\} \subset F \) such that
\[ \Phi(x_n) \to c \quad \text{and} \quad \|\Phi|_F'(x_n)\| \to 0, \]
then \( \{x_n\} \) has a strongly convergent (weakly convergent) subsequence, we say \( \Phi \) satisfies the \((PS)_{c,F} \) \(((WPS)_{c,F}) \) condition at the level \( c \) for the closed subset \( F \subset X \).

Using Lemma 2.5, it is easy to prove the following lemma.

Lemma 2.6 Let \( X \) be a Banach space,
(i). Let \( F \subset X \) be a closed subset. Suppose that \( \Phi \) defined on \( X \) is Gateaux-differentiable and lower semi-continuous and bounded from below on \( F \). If \( \Phi \) satisfies \((PS)_{\inf \Phi, F}\) condition, then \( \Phi \) attains its infimum on \( F \).

(ii). Let \( F \subset X \) be a weakly closed subset. Suppose that \( \Phi \) defined on \( F \) is Gateaux-differentiable and weakly lower semi-continuous and bounded from below on \( F \). If \( \Phi \) satisfies \((WPS)_{\inf \Phi, F}\) condition, then \( \Phi \) attains its infimum on \( F \).

3 The Proof of Theorem 1.4

We prove the Theorem as a sequence of claims.

Claim 3.1 If \((V_1) - (V_6)\) hold, then for any given \( c > 0 \), \( f(u) \) satisfies the \((PS)_{c,F}\) condition; that is, if \( \{u_n\} \subset F \) satisfies

\[
f(u_n) \to c > 0 \quad \text{and} \quad f|_F'(u_n) \to 0, \tag{3.1}\]

then \( \{u_n\} \) has a strongly convergent subsequence.

Proof First, we prove the constrained set \( F \neq \emptyset \) under our assumptions. Using the notation of [1], for \( a > 0 \) let

\[
g_u(a) = g(au) = \int_0^1 [V(au) + \frac{1}{2} V'(au) au] dt. \tag{3.2}\]

By the assumption \((V_3)\), we have

\[
\frac{d}{da} g_u(a) \neq 0 \tag{3.3}\]

and so \( g_u \) is strictly monotone. By \((V_5)\), we have

\[
\lim_{a \to +\infty} g_u(a) \leq A \tag{3.4}\]

By \((V_4)\), we notice that

\[
g_u(0) = V(O) \leq \frac{\mu_2}{\mu_1}. \tag{3.5}\]

So for \( V(O) < h < A \), the equation \( g_u(a) = h \) has a unique solution \( a(u) \) with \( a(u)u \in M \).

By \( f(u_n) \to c \), we have

\[
\frac{1}{4} \int_0^1 |\dot{u}_n(t)|^2 dt \cdot \int_0^1 V'(u_n) u_n dt \to c, \tag{3.6}\]

and by \((V_4)\) we have
$$h = \int_0^1 (V(u_n) + \frac{1}{2} < V'(u_n), u_n >) dt \leq \left( \frac{1}{\mu_1} + \frac{1}{2} \right) \int_0^1 V'(u_n) u_n dt + \frac{\mu_2}{\mu_1}. \quad (3.7)$$

By (3.6) and (3.7) we have

$$\int_0^1 V'(u_n) u_n dt \geq \frac{h - \frac{\mu_2}{\mu_1}}{\frac{1}{2} + \frac{1}{\mu_1}}. \quad (3.8)$$

Condition (V_6) provides \( h > \frac{\mu_2}{\mu_1} \). Then (3.6) and (3.8) imply \( \int_0^1 |\dot{u}_n(t)|^2 dt \) is bounded and \( \|u_n\| = \|\dot{u}_n\|_{L^2} \) is bounded.

We know that \( H^1 \) is a reflexive Banach space, so by the embedding theorem, \( \{u_n\} \) has a weakly convergent subsequence which uniformly strongly converges to \( u \in H^1 \). The argument to show \( \{u_n\} \) has a strongly convergent subsequence is standard, and we can refer to Lemma 3.5 of Ambrosetti-Coti Zelati [1].

Claim 3.2 \( f(u) \) is weakly lower semi-continuous on \( F \).

Proof For any \( u_n \subset F \) with \( u_n \rightharpoonup u \), by Sobolev’s embedding Theorem we have the uniform convergence:

$$|u_n(t) - u(t)|_\infty \to 0.$$ 

Since \( V \in C^1(R^n, R) \), we have

$$|V(u_n(t)) - V(u(t))|_\infty \to 0.$$ 

By the weakly lower semi-continuity of norm, we have

$$\liminf \left( \int_0^1 |\dot{u}_n|^2 dt \right)^{\frac{1}{2}} \geq \left( \int_0^1 |\dot{u}|^2 dt \right)^{\frac{1}{2}}.$$ 

Calculating we see

$$\liminf \left( \int_0^1 |\dot{u}_n|^2 dt \right) \geq \int_0^1 |\dot{u}|^2 dt,$$ 

and

$$\liminf f(u_n) = \liminf \frac{1}{4} \int_0^1 |\dot{u}_n|^2 dt \int_0^1 V'(u_n) u_n dt \geq \frac{1}{4} \int_0^1 |\dot{u}|^2 dt \int_0^1 V''(u) u dt = f(u).$$

Claim 3.3 \( F \) is a weakly closed subset in \( H^1 \).

Proof This follows easily from Sobolev’s embedding Theorem and \( V \in C^1(R^n, R) \).

Claim 3.4 The functional \( f(u) \) has positive lower bound on \( F \)
**Proof** By the definitions of \( f(u) \) and \( F \) and the assumption \((V_2)\), we have

\[
    f(u) = \frac{1}{4} \int_0^1 |\dot{u}|^2 dt \int_0^1 (V'(u)u) dt \geq 0, \forall u \in F.
\]

Furthermore, we claim that

\[
    \inf f(u) > 0;
\]

otherwise, \( u(t) = \text{const} \), and by the symmetrical property \( u(t + 1/2) = -u(t) \) we have \( u(t) = 0, \forall t \in R \). But by assumptions \((V_4)\) and \((V_6)\) we have

\[
    V(0) \leq \frac{\mu_2}{\mu_1} < h,
\]

which contradicts the definition of \( F \) since \( V(0) = h \) if we have \( 0 \in F \). Now by Lemmas 3.1-3.4 and Lemma 2.6, we see that \( f(u) \) attains the infimum on \( F \), and we know that the minimizer is nonconstant.

**Acknowledgements**

The authors sincerely thank Professor P. Rabinowitz who brought the paper of D. Offin ([16]) to our attention.

**References**

[1] A. Ambrosetti, V. Coti Zelati, Closed orbits of fixed energy for singular Hamiltonian systems, Arch. Rat. Mech. Anal. 112(1990), 339-362.

[2] A. Ambrosetti, V. Coti Zelati, Solutions with minimal period for Hamiltonian systems in a potential well, Ann. Inst. H. Poincare, Analyse Non Lineare 4(1987), 235-242.

[3] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. of Functional Analysis, 14(1973), 349-381.

[4] V. Benci, Closed geodesics for the Jacobi metric and periodic solutions of prescribed energy of natural Hamiltonian systems, Ann. Inst. Henri Poincare Anal. Non Lineaire 1(1984), 401-412.

[5] J. Berg, F. Pasquotto, R. Vandervorst, Closed characteristics on non-compact hypersurfaces in \( R^{2n} \), Math. Ann. 343(2009), 247-284.

[6] K. C. Chang, Infinite dimensional Morse theory and multiplicity problems, Birkhauser, 1993.

[7] G. Cerami, Un criterio di esistenza per i punti critici su varietà illimitate, Rend. dell. accademia di sc. lombardo 112(1978), 332-336.

[8] V. Coti Zelati, I. Ekeland and P. L. Lions, Index estimates and critical points of functionals not satisfying Palais-Smale, Ann. Scuola Norm Sup. Pisa 17(1990), 569-581.
[9] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer, 1990.

[10] N. Ghoussoub, D. Preiss, A general mountain pass principle for locating and classifying critical points, Ann. Inst. Henri Poincare Anal. NonLineaire 6 (1984), 321-330.

[11] H. Gluck and W. Ziller, Existence of periodic motions of conservative systems, in Seminar on minimal submanifolds, E. Bombieri Ed., Princeton Univ. Press, 1983.

[12] E. W. C. Van Groesen, Analytical mini-max methods for Hamiltonian break orbits with a prescribed energy, JMAA 132 (1988), 1-12.

[13] K. Hayashi, Periodic solutions of classical Hamiltonian systems, Tokyo J. Math., 1983.

[14] Y. Long, Index Theory for Symplectic Paths with Applications, Basel: Birkhauser, 2002.

[15] J. Mawhin, M. Willem, Critical Point Theory and Applications, Springer, 1989.

[16] D. Offin, A class of periodic orbits in classical mechanics, JDE, 66 (1987), 90-117.

[17] Palais R., The principle of symmetric criticality, CMP 69 (1979), 19-30.

[18] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math. 31 (1978), 157-184.

[19] P. H. Rabinowitz, Periodic solutions of a Hamiltonian systems on a prescribed energy surface, JDE 33 (1979), 336-352.

[20] H. Seifert, Periodischer bewegungen mechanischer system, Math. Zeit 51 (1948), 197-216.

[21] K. Yosida, Functional Analysis, Springer, Berlin, 1978.

[22] W. P. Ziemer, Weakly differentiable functions, Springer, 1989.