On the particularities of Bose-Einstein condensation of quasiparticles

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Abstract

An attempt is made to determine the difference between Bose-Einstein condensation of particles and quasiparticles. An equation is obtained for the number of particles in a Bose-Einstein condensate as a function of the total number of particles in the system. This equation is also written for quasiparticles taking account of their creation by external pumping and the presence of equilibrium thermal excitations in the system. Analyzing both equations, the chemical potential of the pumped quasiparticles and their number in the condensate are found as a function of the pumping intensity. A condition under which the Bose-Einstein condensation of low-energy quasiparticle excitations starts and occurs at any, including quite high, temperatures is found.

1. It is well known that the Bose-Einstein condensation (BEC) is one of the remarkable macroscopic quantum phenomena which can occur in one or another collective of particles or quasiparticles (QP) with integer spin. There are now a number of examples of the more or less direct experimental observation of BEC, among which we call attention to BEC in rarified gases of Bose atoms (Refs. 1 and 2; see also the review Ref. 3). Its critical temperature is extremely low ($\sim 10^{-6}$ K) because of the low density of the particles and the relatively large mass of even the lightest atoms. In this sense, quasiparticles are more interesting. Quite high quasiparticle densities are experimentally fully achievable and the effective masses, as a rule, are of the order the electronic masses (to say nothing of the existence of massless quasiparticles\textsuperscript{4}). Possibly, for this reason the question of the BEC of QP for the example of large-radius excitons (and biexcitons) was raised about 50 years ago,\textsuperscript{5} and the investigation of their condensation became very popular especially for indirect excitons or in heterostructures.\textsuperscript{6} The recent experiments on BEC of excitonic polaritons in microcavities also merit equal attention\textsuperscript{7,8}. Here the condensation temperature, though it increases by an order of magnitude, is still quite low—about several degrees or fractions of a degree.

As concerns the short-radius dipole-active excitons, their condensation has turned out to be practically unobservable because of their short lifetime relative to radiative decay and therefore the impossibility of attaining the densities required for BEC. The high-density limitation substantially weakens for magnons, which also can be regarded as a type of Frenkel excitons (or, which is the same thing, short-radius excitons), which possess a long lifetime. Apparently, it is this fact that helps to achieve such high magnon densities, so that their
condensation, as asserted in Ref. 10, is observed in perfect yttrium-iron garnet films with temperatures which are anomalously high and compared with room (!) temperature. The works mentioned above and’ the works of Demokritov’s group11,12 devoted to BEC magnons make it necessary to analyze whether or not the conditions at which BEC of QP becomes possible at truly high temperatures are indeed satisfied, since nothing like this can be achieved for particles and excitons.

2. First, we note a remarkable feature of BEC, which, being essentially a phase transition, can occur in an ideal gas, which makes it possible to calculate many physical characteristics and observable quantities of this process to a high degree of accuracy. In addition, although objects which do not interact with one another at all are unlikely to exist, the concept of ideality could be even more characteristic for QP, which interact quite weakly even at relatively high densities.

Nonetheless, it is not ideality (or weak ideality) that lies at the basis of the main difference between particles and QP of the Bose type. The fundamental difference between them is that if at a given density the particles are characterized by a finite value of the chemical potential $\mu$, then $\mu = 0$ for equilibrium QP, since their density itself is determined by only the temperature. This means that the realization and observation of BEC of QP requires converting the system (for example, by special nonthermal action) into an excited (in other words, a nonequilibrium) state, which remains in existence for a sufficiently long time (in any case, appreciably greater than the equilibration time and the QP lifetime). Then the (quasi)equilibrium distribution of QP which is established and which corresponds to BEC becomes accessible for experimental study. But, on the other hand, it also follows from this that one must take great care when describing the BEC of QP because of the problematic nature of using the thermodynamic approach and the nonstrictness of the application of various limits.

We note that there have been more than a few attempts (see, for example, Refs. 13-15) at describing the kinetic behavior of the nonequilibrium QP collective (specifically, ferromagnons) under one or another form of pumping—pulsed, noise, and so forth, which can then condense. In such strict approaches, additional information must be taken into account: on the interaction of QP with one another, their interaction with other objects — phonons or defects, the character of the pumping, the temporal evolution, which makes the model and the calculations quite complex. All this, being undoubtedly important, nonetheless does not appear to be determining for BEC itself. In complete correspondence with Bose and Einstein, we understand it as a phenomenon of temperature redistribution of a prescribed number of particles over the states and the accumulation of a substantial fraction of them (particles) in the lowest of them (states). Even if we confine ourselves to the assumption that pumping has created only QP and that fast relaxation made them essentially equilibrium states, we can raise many questions which follow from experiments.10–12 Neither is our objective to examine the coherent manifestations of the QP collective accumulated in this manner in the lowest (excited) state, which, certainly, is an interesting and topical problem. We shall study the particularities which make it possible for QP to condense at such high temperatures, including, in this case, room temperatures; this problem is limited and has not yet been thoroughly analyzed.

3. It is well known that the long-wavelength elementary quasiparticle excitations determine the low-energy part of the spectrum of any particular system. On the other hand, their interaction with one another is the weakest interaction. As mentioned above, this makes it possible to consider, to a first approximation, the subsystem of QP to be an ideal gas. Let
ε(k) = ε₀ + [ε(k) − ε₀] ≡ ε₀ + ε_{kin}(k) be the QP energy, and let ε₀ = ε(k₀) correspond to the lowest excited state. Then, according to the assumption that the number of QP is conserved as a result of their external interaction or, in other words, the fact that they have a finite chemical potential, we write the number of QP in the kth quantum state at fixed temperature as

\[ n(k, \mu) = \frac{1}{\exp\left\{\frac{\epsilon(k) - \mu}{k_B T}\right\} - 1}. \]  

(1)

Instead of the quantity μ, we introduce, following Ref. 16, a different thermodynamic variable

\[ n_0 = \frac{1}{\exp(\epsilon_0 - \mu)/k_B T} - 1, \]  

(2)

whence \( \mu = \epsilon_0 - k_B T \ln[(n_0 + 1)/n_0] \). Using the expression (2), the occupation numbers (1) can be easily rewritten in the form

\[ n(k, \mu) \rightarrow n(k, n_0) = \frac{1}{(1 + 1/n_0) \exp(\epsilon_{kin}(k)/k_B T) - 1} = \frac{n_k(T)}{1 + [n_k(T) + 1]/n_0}, \]  

(3)

where we have introduced the following notation for the number of particles

\[ n_k(T) = \frac{1}{\exp[\epsilon_{kin}(k)/k_B T] - 1}, \]  

(4)

which, as one can see from its definition, does not depend on the gap in the QP spectrum. The replacement (2), eliminating the QP chemical potential from the analysis, facilitates the study of the most interesting and important situation where the nonphysical asymptotic limit \( n_0 \to \infty \) corresponds to the generally accepted condition for BEC \( \mu \to \epsilon_0 \).\(^1\)

For sufficiently large values of \( n_0 (\gg n_k(T)) \), which, generally speaking, should correspond to the BEC regime, the following expansion becomes valid:

\[ n_k(T, n_0) = n_k(T) \left\{ 1 - \left(\frac{n_k(T) + 1}{n_0}\right) + \left(\frac{n_k(T) + 1}{n_0}\right)^2 - \ldots\right\}, \]  

(5)

which is a direct indication that the quantity \( n_k(T) \) introduced above is simply the maximum possible (corresponding to the formal condition \( n_0 \to \infty \)) occupation number of the corresponding state for fixed \( T \).

It is convenient to use the expansion (5) when writing thermodynamically equilibrium quantities. Specifically, the total (average) number of particles becomes

\[ N = \sum_k g_k n(k, n_0) = n_0 + \sum_{k \neq k_0} g_k n(k, n_0) \equiv n_0 + N_{\text{exc}}(T, n_0), \]  

(6)

where \( g_k \) is the degeneracy of the state with wave vector \( k \), and

\[ N_{\text{exc}}(T, n_0) = \sum_{k \neq k_0} g_k n_k(T) \left\{ 1 - \left(\frac{n_k(T) + 1}{n_0}\right) + \left(\frac{n_k(T) + 1}{n_0}\right)^2 - \ldots\right\} \approx N_{\text{exc}}(T) - \frac{\delta N_{\text{exc}}(T)}{n_0}, \]  

(7)
corresponds to the total number of QP in the excited states. In the latter equality, the
following number is singled out:

\[ N_{\text{exc}}(T) = \sum_{k \neq k_0} g_k n_k(T), \]  

(8)

which gives the maximum number (achieved under the same condition (see above) \( n_0 \to \infty \)) of all thermal excitations, and the coefficient

\[ \delta N_{\text{exc}}(T) = \sum_{k \neq k_0} g_k n_k(T)[n_k(T) + 1], \]  

(9)

is determined, to first order in \( 1/n_0 \), by the fluctuations of the quantity \( (7) \). It is evident that irrespective of the specific form of the QP dispersion law \( \varepsilon_{\text{kin}}(k) \) the quantities \( N_{\text{exc}}(T) \) and \( \delta N_{\text{exc}}(T) \) are monotonically increasing functions of the temperature, which is easily seen from the definitions, since as \( T \) increases, all occupation numbers increase.

Substituting (7) into (6), taking account of (7) and (8) and assuming \( N \gg 1 \) (and, correspondingly, \( n_0 \gg 1 \)), we easily obtain the equation

\[ N \simeq n_0 + N_{\text{exc}}(T) - \frac{\delta N_{\text{exc}}(T)}{n_0} \]  

(10)

for finding the number \( n_0 \) of Bose-condensed QP. We find from this equation that the behavior of the desired number

\[ n_0 \equiv n_0(T) = \frac{1}{2} \left\{ N - N_{\text{exc}}(T) + \sqrt{\left[ N - N_{\text{exc}}(T) \right]^2 + 4\delta N_{\text{exc}}(T)} \right\} \]  

(11)

for large \( N \) and small \( \delta N_{\text{exc}}(T)/N^2 \) changes quite sharply when the temperature crosses a certain value \( T_c \) which is the solution of the equation

\[ N_{\text{exc}}(T_c) = N, \]  

(12)

The Eq. (11) shows directly that the critical temperature introduced in this manner depends on the total number of particles in the system and the specific form of the function \( N_{\text{exc}}(T) \). The explicit expression for the latter (as also for \( T_c \)) is determined by the spectrum, the dimensionality, and even the form of the system as well as by the conditions at the boundaries of the system.

If we now introduce the dimensionless density \( n_{\text{BEC}}(T) \equiv n_0(T)/N \) of the Bose condensate, then in the limit \( \delta N_{\text{exc}}(T)/N^2 \to 0 \) it becomes a nonanalytic function of the temperature:

\[ n_{\text{BEC}}(T) = \left[ 1 - \frac{N_{\text{exc}}(T)}{N} \right] \theta \left[ 1 - \frac{N_{\text{exc}}(T)}{N} \right], \]  

(13)

where \( \theta(x) \) is a step function. Such behavior of the density (12) makes it possible to interpret it as the order parameter of the BEC process in a given collective of particles or QP. We shall not dwell on this, but we shall focus our attention on the question, raised above, concerning the particularities which distinguish QP from particles.

4. Indeed, thus far the analysis could refer to both of these objects, since the chemical potential was excluded from it. However, it is well known and has already been mentioned
that for QP $\mu = 0$, to that any system at finite $T$ will contain one or another — thermally excited — quantity of them (compare Eq. (11)):

$$n^{(th)}_k(T) = \frac{1}{\exp[\varepsilon_k/k_BT] - 1}.$$  \hspace{1cm} (14)

In complete analogy with Eq. (2), it is also possible to single out the equilibrium thermal occupation of the lowest state:

$$n^{(th)}_0(T) \equiv n^{(th)}_0 = \frac{1}{\exp(\varepsilon_0/k_BT) - 1},$$  \hspace{1cm} (15)

which is a given (increasing) function of the temperature.

Using the relation (14), the numbers (13) can be easily put into a form analogous to (3):

$$n^{(th)}_k(T) \rightarrow n^{(th)}_k(T, n^{(th)}_0) \equiv n_k(T) \frac{1}{1 + n_k(T) + 1 n^{(th)}_0},$$  \hspace{1cm} (16)

where, remarkably, the notation $n_k(T)$ is identical to the occupation number (4) given above. Formally, the expression (15) is exact, but the number of thermal excitations, generally speaking, does not have the quantitative advantages over the expression used in the expansion (7) (at least, the leading terms) over the occupation numbers $n_k(T)$ (or, especially, the numbers $n^{(th)}_k(T)$) as $n_0$ over $n_k(T)$ in the BEC regime. However, it can be supposed that because the occupation numbers of the excited states decrease exponentially, the error of the series expansion in powers of $1/n^{(th)}_0$ will not be substantial.\textsuperscript{20} Of course, for small $n^{(th)}_0$ the difference between the particles and QP becomes less noticeable. However, if one stays within the assumption that the initial temperatures are sufficiently high (or energy of the elementary excitations is relatively low), then the expansion can be written approximately as (compared Eq. (7))

$$n^{(th)}_k(T, n^{(th)}_0) = n_k(T) \left\{ 1 - \left( \frac{n_k(T) + 1}{n^{(th)}_0} \right) + \left( \frac{n_k(T) + 1}{n^{(th)}_0} \right)^2 \cdots \right\},$$  \hspace{1cm} (17)

and Eq. (17) which follows immediately

$$N_{th} \simeq n^{(th)}_0 + N_{exc}(T) - \frac{\delta N_{exc}(T)}{n^{(th)}_0},$$  \hspace{1cm} (18)

where, as mentioned, the quantities $N_{exc}(T)$ and $\delta N_{exc}(T)$ do not change, even if nonthermal (specifically, pumped) QP are present in the system.

We now introduce their number according to the obvious expression

$$n_k(T) = n^{(th)}_k(T) + n^{(pump)}_k(T)$$  \hspace{1cm} (19)

for each $k$. Then, according to Eqs. (7) and (17) it is easily shown that for all $k \neq k_0$

$$n^{(pump)}_k(T) = n_k(T) [n_k(T) + 1] \left[ \frac{1}{n^{(th)}_0(T)} - \frac{1}{n_0(T)} \right] \left[ 1 + \frac{n_k(T) + 1}{n_0(T)} \right]^{-1} \left[ 1 + \frac{n_k(T) + 1}{n^{(th)}_0(T)} \right]^{-1}.\hspace{1cm} (20)$$
The latter expression simplifies substantially provided that the proposed (and actually experimentally realizable\cite{10-12}) ratio of the parameters of the system obtains: \( k_B T \gg \varepsilon_0, \mu \). Indeed, we find from Eq. (20), taking account of the definition (2), that in this case the increment to the number of excitations created in all states, with the exception of the lowest, by pumping can be represented in the quite simple form

\[
n_k^{(\text{pump})}(T) \approx \frac{\mu}{k_B T} n_k(T)[n_k(T) + 1]. \tag{21}
\]

In other words pumping increases the initial occupation numbers somewhat, but very little because of the “temperature suppression factor” that appears \( \mu/k_B T \ll 1 \).

At the same time the density of the pumped QP in a Bose condensate always behaves completely differently (see Eqs. (2) and (15):

\[
n_0^{(\text{pump})}(T) = n_0(T) - n_0^{(\text{th})}(T) \approx \frac{k_B T}{\varepsilon_0} \frac{\mu}{(\varepsilon_0 - \mu)}, \tag{22}
\]

or can be arbitrarily large because of the possibility \( \mu \rightarrow \varepsilon_0 \) (except for the exact equality \( \mu = \varepsilon_0 \)). It is remarkable that, conversely, in contrast to all \( n_k(T) \) with \( k \neq k_0 \) the occupation numbers \( n_0^{(\text{pump})}(T) \) increase additionally on account of (compare Eq. (21)) the “temperature intensification factor” \( k_B T/\varepsilon_0 \gg 1 \).

5. The role of pumping is obvious — to create a definite number of QP. But how is it related with the chemical potential? To answer this question, we take account of the fact that (see Eq. (6)) the total number \( N \) of QP in the system can also be divided into two contributions: the temperature-determined equilibrium part \( N_{\text{th}} \) and the pumping-determined number \( N_{\text{pump}}: N = N_{\text{th}} + N_{\text{pump}} \). As is well-known,\cite{18} under electromagnetic pumping with intensity \( I_{\text{pump}} \) this number changes in accordance with the simplest balance equation

\[
dN/dt = I_{\text{pump}} - (N - N_{\text{th}})/\tau_{\text{rel}} \]

where \( \tau_{\text{rel}} \) is an effective equilibration time after pumping is switched off (a more accurate analysis is given in the Appendix). It follows immediately from this equation that because of the obvious equality\cite{3} \( dN_{\text{th}}/dt = 0 \) actually becomes \( dN_{\text{pump}}/dt = I_{\text{pump}} - N_{\text{pump}}/\tau_{\text{rel}} \) and that the stationary number (for times \( t \gg \tau_{\text{rel}} \), when \( dN/dt = 0 \)) is the total number of pumped particles, equal to \( N_{\text{pump}}^{\text{st}} = I_{\text{pump}}\tau_{\text{rel}} \). Assuming that the main number \( (22) \) of the latter accumulates precisely in the lowest state, or that \( n_0^{(\text{pump})}(T) \approx N_{\text{pump}}^{\text{st}} \), we easily arrive at

\[
I_{\text{pump}}\tau_{\text{rel}} = \frac{k_B T}{\varepsilon_0} \frac{\mu}{(\varepsilon_0 - \mu)}, \tag{23}
\]

which enables us to write the chemical potential of the pumped (and only pumped) QP:

\[
\mu = \varepsilon_0 - \frac{I_{\text{pump}}\tau_{\text{rel}}}{(k_B T/\varepsilon_0 + I_{\text{pump}}\tau_{\text{rel}})}, \tag{24}
\]

This expression shows that for relatively weak pumping, when \( I_{\text{pump}}\tau_{\text{rel}} \ll k_B T/\varepsilon_0 \), the chemical potential of the QP is \( \mu \approx \varepsilon_0^2 I_{\text{pump}}\tau_{\text{rel}}/k_B T \), and for strong pumping when \( I_{\text{pump}}\tau_{\text{rel}} \gg k_B T/\varepsilon_0 (\gg 1) \) its value goes (see Eq. (24)) to its limit \( \varepsilon_0 \), always satisfying the physically necessary condition \( \mu < \varepsilon_0 \). It is easy to show that the difference

\[
\varepsilon_0 - \mu \approx \frac{k_B T}{N_{\text{pump}}^{\text{st}}} \rightarrow \frac{k_B T}{n_0^{(\text{pump})}(T)} \approx \frac{k_B T}{I_{\text{pump}}\tau_{\text{rel}}}, \tag{25}
\]
and, in consequence, as follows from Eqs. (23) and (25), the thermodynamic properties of the collective of QP condensed as a result of BEC should depend only on the ratio $I_{\text{pump}}\tau_{\text{rel}}/k_B T$. It appears that such scaling behavior can be checked experimentally.

Finally, using Eqs. (7) and (18), and subtracting the latter from the former, gives

$$n_0 - n_0^{(th)} + \frac{n_0 - n_0^{(th)}}{n_0 n_0^{(th)}} \delta N_{\text{exc}}(T) = I_{\text{pump}}\tau_{\text{rel}},$$

where the unknown is the number $n_0(T)$ of QP in the condensate (or, which is the same thing, the number $n_0^{(pump)}(T)$. Evidently, $n_0(T) = n_0^{(th)}(T)$ in the absence of pumping. In addition, we find from Eq. (26) that, specifically, for weak pumping, when $n_0^{(pump)}(T) \ll n_0^{(th)}(T)$, the number of QP in the lowest level increases as

$$n_0(T) \approx n_0^{(th)}(T) + \left(1 - \frac{\varepsilon_0^2}{(k_B T)^2} \delta N_{\text{exc}}(T)\right) I_{\text{pump}}\tau_{\text{rel}}.$$

For strong pumping and when nonequilibrium has been reached $n_0^{(pump)}(T) \gg n_0^{(th)}(T)$, the number of QP in the lowest level is given by the relation

$$n_0(T) \approx n_0^{(pump)}(T) = I_{\text{pump}}\tau_{\text{rel}} - \frac{\varepsilon_0}{k_B T} \delta N_{\text{exc}}(T).$$

Although it is obvious that the latter number also increases as the pump intensity, a “transition” occurs from one straight line onto another displaced below the origin of the coordinates, which also can be checked experimentally.

However, if the small corrections to the linear behavior of the number of QP in the Bose condensate as a function of the pumping intensity are neglected, which is precisely what was assumed in the derivation of the chemical potential, then we arrive at a seemingly paradoxical result: the pumped low-frequency QP are always in the BEC regime. However, one must remember that the so-called temperatures must satisfy the inequality $k_B T \gg \varepsilon_0$.

This can be stated differently: the computed change of the occupation numbers of the quasiparticle states (in the experiments Refs. 10-12 this is a ferromagnetic plate exposed to external pumping) shows that the initial—thermally equilibrium—QP “prevent” the accumulation of new QP in all states except the lowest one (see Eqs. (21) and (22)). For this reason, it seems that pumping-created QP have nothing to do but to collect exclusively in it. As follows from our analysis, this will occur at all times (of the order of the action of the pumping pulse) irrespective of the temperature as soon as, once again, $k_B T \gg \varepsilon_0$. Thus, the satisfaction of this inequality makes the temperature a factor which not only does not impede (neglecting relaxation processes, which decrease $\tau_{\text{rel}}$) but rather even promotes BEC to some extent. Of course, as mentioned above, the latter assertion relies on the assumption that the temperature remains constant. In addition, temperature stability in a real experiment requires special attention. However, it is worth repeating that when high values of the temperature are reached, its relative increase as a result of irradiation processes should not be too large.

6. The results obtained above show that certain types of QP, specifically, QP with low activation energy (or at temperatures not exceeding them) under external pumping can (more accurately, must) accumulate only in the lowest possible state, which corresponds completely with the BEC phenomenon. In addition, such accumulation is accompanied by relaxation
from the strongly nonequilibrium state into (thermal) a (quasi)equilibrium state which is found to be indistinguishable from a Bose-condensed state. It appears that experimental verification (including by measuring the Mandel'shtam-Brillouin scattering\textsuperscript{10−12,19}) of the computational results obtained in the present work could show that there is a large difference between BEC particles (for example, Bose atoms of light metals) and low-frequency QP. This difference could be manifested, first and foremost, even in its (BEC) most important characteristic—the critical temperature. Other features (specifically, coherent and fluctuation properties) of BEC for QP of different nature, with different dispersion laws, and in systems with different dimensionality will be analyzed elsewhere.

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APPENDIX

To simplify the exposition the balance equation given above for determining the number of pumped QP was written in a form which gives only the proportionality between $N_{\text{pump}}^{\text{st}}$ and the absorbed power $I_{\text{pump}}\tau_{\text{rel}}$; there are no terms which give rise to a pump threshold near which $N_{\text{pump}} \sim \sqrt{I_{\text{pump}}}$.\textsuperscript{12,20}

In this connection we recall that under the conditions of parametric electromagnetic excitation the exponential growth of the number of magnons in the system occurs when the creation rate of these QP is higher than their annihilation rate, whose determining channel are pair collisions.\textsuperscript{21} Then, to take these circumstances into account we shall write the balance equation in the form (see Ref. 19)

$$
\frac{dN}{dt} = \bar{I}_{\text{pump}}(N + 1) - \frac{N - N_{\text{th}}}{\tau_{\text{rel}}^{(1)}} - \frac{(N - N_{\text{th}})^{(2)}}{\tau_{\text{rel}}^{(2)}},
$$
or, in accordance with the notation adopted,

$$
\frac{dN_{\text{pump}}}{dt} = \bar{I}_{\text{pump}} + \left( I_{\text{pump}} - \frac{1}{\tau_{\text{rel}}^{(1)}} \right) N_{\text{pump}} - \frac{N_{\text{pump}}^2}{\tau_{\text{rel}}^{(2)}}, \quad (A1)
$$

where $\bar{I}_{\text{pump}} \equiv I_{\text{pump}}(N_{\text{th}} + 1)$, and $\tau_{\text{rel}}^{(1)}$ and $\tau_{\text{rel}}^{(2)}$ are the lifetimes of the magnons with respect to one- and two-particle processes.

It is easily shown from Eq. (A1) that the threshold behavior of the number of pumped magnons is completely determined by the sign of the second term on the right-hand side of this equation, and the last term limits the growth of their number. It is precisely the equality $I_{\text{pump}} = 1/\tau_{\text{rel}}^{(1)}$ that sets the magnitude of the desired threshold, as happens in the $S$ theory of parametric resonance.\textsuperscript{21}

As for the stationary number of pumped magnons, according to the same equation (A1) $N_{\text{pump}}^{\text{st}} \sim \bar{I}_{\text{pump}}\tau_{\text{rel}}^{(1)}$ which obtained above; near threshold, when $I_{\text{pump}}\tau_{\text{rel}}^{(1)} \sim 1$, the behavior changes and $N_{\text{pump}}^{\text{st}} \sim \sqrt{\bar{I}_{\text{pump}}\tau_{\text{rel}}^{(2)}}$; finally, if $I_{\text{pump}} \gg 1/\tau_{\text{rel}}^{(1)}$, the number $N_{\text{pump}}^{\text{st}} \approx I_{\text{pump}}\tau_{\text{rel}}^{(2)}/2 + \sqrt{(I_{\text{pump}}\tau_{\text{rel}}^{(2)}/2)^2 + \bar{I}_{\text{pump}}\tau_{\text{rel}}^{(2)}}$, i.e. it is described by a dependence which is, once
again, close to linear. This is the qualitative picture observed experimentally, where the function \( N^{\text{st}}_{\text{pump}}(I_{\text{pump}}) \approx n_{0\text{pump}}^p(I_{\text{pump}}) \) changes from linear to square-root back to linear behavior.\(^{20}\) If necessary, the corresponding dependences can be easily introduced into the relations (23)-(26), whose meaning remains intact.

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1) We note that in the theoretical study in Ref. 17 of the fluctuations of a Bose-condensate of ultracold atoms the chemical potential was also eliminated from the expression for the number of particles but in a different manner so that the noted divergence of the quantity remained.

2) Here we implicitly assume that \( n_0^{\text{th}} \gg 1 \), which can hold only under the obvious condition \( k_B T \gg \varepsilon_0 \) and holds with room to spare, for example, in experiments with BEC of ferromagnets.\(^{10-12}\) At the same time \( n_0^{\text{th}} \sim \exp(-\varepsilon/k_B T) \ll 1 \) almost always holds for excitons or QP with a large gap, whence it follows that the Boltzmann distribution describes their thermal number. Here \( n_k^{\text{th}}(T) \approx n_k(T) \exp(-\varepsilon_0/k_B T) \ll n_k(T) \).

3) Of course, neglecting a possible change of temperature, whose relative magnitude near \( 10^2 \) K cannot be large.

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