THE DIOPHANTINE PROBLEM FOR RINGS OF EXPONENTIAL POLYNOMIALS

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Abstract. One of the main open problems regarding decidability of the existential theory of rings is the analogue of Hilbert’s Tenth Problem (HTP) for the ring of entire holomorphic functions in one variable. In the direction of a negative solution, we prove unsolvability of HTP for rings of exponential polynomials. This provides the first known case of HTP for a ring of entire holomorphic functions in one variable strictly containing the polynomials. The technique of proof consists of an interaction between Arithmetic, Analysis, Logic, and Functional Transcendence.

1. Introduction

By the work of Davis-Putnam-Robinson [DPR61] and Matijasevich [M70] we know that Hilbert’s tenth problem is unsolvable. More generally, let $A$ be a ring and let $A_0$ be a recursive subring of $A$. The analogue of Hilbert’s tenth problem for $A$ with coefficients in $A_0$ asks for an algorithm which decides the solvability in $A$ of polynomial equations with coefficients in $A_0$. See the surveys [Ph94], [PhZ00], [Po08], and [Koe14] for a presentation of results and open problems in the context of extensions of Hilbert’s tenth problem.

Let $R$ be the ring of complex entire functions in one variable $z$ of the form $\sum_{j=1}^{n} p_j \exp(q_j)$, where $p_j, q_j \in \mathbb{C}[z]$ for each $j$. Holomorphic functions of this type are usually called exponential polynomials (of finite order) and there is considerable interest in their value distribution properties going back at least to [Ri29]. See [HITW18] and [GSW20] for some recent results and an overview of this topic.

The analogue of HTP for $R$ with coefficients in $\mathbb{Z}$ is decidable by Tarski’s Theorem, because $\mathbb{C} \subseteq R$, and any solution in $R$ to a system of polynomial equations over the integers can be evaluated at $z = 0$ to get a complex solution. Our main result is a negative solution to Hilbert’s tenth problem on $R$ with coefficients in $\mathbb{Z}[z]$.

Theorem 1.1. Let $R'$ be a subring of $R$ containing the variable $z$. The ring $\mathbb{Z}$ is positive existentially interpretable in the ring $R'$ over the language $L_z = \{0, 1, z, +, \times, =\}$. In particular, the analogue of Hilbert’s tenth problem for $R'$ with coefficients in $\mathbb{Z}[z]$ has a negative answer.

In favorable circumstances, we also have a strengthening of the previous result.

Theorem 1.2. Let $R'$ be a subring of $R$ containing the variable $z$. If the ring of constants $R'_\text{cst} = \mathbb{C} \cap R'$ is positive existentially definable in $R'$ over $L_z$, then $\mathbb{Z}$ is positive existentially definable in $R'$ over $L_z$. Furthermore, this is the case if $R'_\text{cst} = \mathbb{C}$. In particular, $\mathbb{Z}$ is positive existentially definable in $R$ over the language $L_z$.

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Theorems 1.1 and 1.2 lie in the context of the study of model-theoretic aspects of rings which may be constructed by arithmetic operations and composition from the usual functions one encounters in elementary algebra and calculus; this topic dates back at least to Tarski’s high school algebra problem, see [Wi00]. An important case is the ring \( \mathbb{C}[z]^{E} \) obtained from \( \mathcal{R} \) by closing under composition of functions; see for instance [Dr84], [HR84], and [HRS89]. Elements of \( \mathbb{C}[z]^{E} \) are a more general kind of exponential polynomials, and \( \mathcal{R} \) is precisely the subring of \( \mathbb{C}[z]^{E} \) consisting of those holomorphic functions \( f \in \mathbb{C}[z]^{E} \) of finite order (in the sense of growth in complex analysis).

One of the major open problems in the area is the analogue of Hilbert’s Tenth Problem for the ring \( \mathcal{R} \) of entire holomorphic functions in one variable \( z \) with coefficients in \( \mathbb{Z}[z] \). Equivalently, the problem is whether the positive existential theory of the ring \( \mathcal{R} \) over the language \( \mathcal{L}_{z} \) is decidable. In more geometric terms, the question is equivalent to asking for an algorithm that takes as input algebraic varieties fibred over the affine line \( \pi: X \to \mathbb{A}^{1} \) (all defined over \( \mathbb{Q} \)) and decides whether there is a complex holomorphic section of \( \pi \).

Before discussing how our results fit into the context of Hilbert’s tenth problem for \( \mathcal{R} \), let us briefly recall some related results. The first order theory of \( \mathcal{R} \) over \( \mathcal{L}_{z} \) is undecidable [Ro51]. If instead of \( \mathcal{R} \) one considers the ring of rigid analytic functions in one variable \( z \) over a non-archimedean field \( k \), then undecidability of the positive existential theory over \( \mathcal{L}_{z} \) is proved in [LP95] when \( k \) has characteristic 0, and in [GP15] when \( k \) has positive characteristic. A negative solution to the analogue for Hilbert’s tenth problem for rings of complex holomorphic functions in at least two variables is proved in [PhV18] over a language including the variables and a predicate for evaluation. Regarding (possibly transcendental) meromorphic functions, much less is known and we refer the reader to [V03, P17, PhV18]. See also [PhZ08] for connections between these problems and questions in number theory.

The positive existential theory of the ring of complex polynomials \( \mathbb{C}[z] \) over \( \mathcal{L}_{z} \) is undecidable [De78], and there is abundant literature on analogues of Hilbert’s tenth problem for \( \mathcal{H} \) of entire holomorphic functions in one variable \( z \) and polynomial solutions are used in [De78] and elsewhere. Elements of \( \mathbb{C}[z] \) by closing under composition of \( \mathcal{L}_{z} \), for a subring containing the exponential function. It is worth pointing out that \( \mathcal{R} \) is natural subring of \( \mathcal{H} \) not just from the point of view of logic, but also from the point of view of value distribution; for instance, in [GSW20] it is shown that \( \mathcal{R} \) is radically closed in \( \mathcal{H} \) by means of Nevanlinna theory.

The proofs of Theorems 1.1 and 1.2 involve an interaction between Arithmetic, Analysis, Logic, and Functional Transcendence. Let us briefly outline the structure of the argument.

First, the logical side builds on work of [De78] which concerns rings of polynomials and uses Pell equations; see [PhZ00] and the references therein for other cases where these ideas are used. However, in our case additional technical difficulties arise, essentially because \( f(1) = 0 \) is not the same as \( (z-1)|f \) in \( \mathcal{R} \).

The functional Pell equation that we study is

\[ x^2 - (z^2 - 1)y^2 = 1 \]

(in the unknowns \( x \) and \( y \)) over \( \mathcal{R} \). The following theorem is our key technical result.

**Theorem 1.3.** Equation \( (1.1) \) has the same solutions over \( \mathcal{R} \) and over \( \mathbb{Z}[z] \).

Functional Pell equations with polynomial coefficients are those of the form \( X^2 - DY^2 = 1 \), where \( D \in \mathbb{C}[z] \) is a polynomial without multiple zeros. They are some of the oldest studied functional polynomial equations since Abell, see [Z13], [Z14], and [Ko11] for some recent developments.

Equation \( (1.1) \) has an infinite number of polynomial solutions, which can be given the structure of an abelian group isomorphic to \( \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \). These polynomial solutions are used in [De78] and elsewhere to approach Hilbert’s tenth problem over various rings of functions. Roughly speaking, Theorem 1.3 allows us to approach Hilbert’s tenth problem for \( \mathcal{R} \) using the Pell equation method, but there are some serious technical complications (as mentioned before). We explain the details in Section 5.
The proof of Theorem 1.3 is given in Section 4 and it has three main ingredients. First, we prove a theorem that gives a complete description of all solutions of Equation (1.1) in $\mathcal{H}_C$ in terms of functions in a suitable quadratic extension of $\mathcal{H}_C$ (Theorem 2.3). Secondly, we use an extension of the Borel-Caratheodory theorem on an auxiliary Riemann surface which, together with Theorem 2.4, allows us to severely restrict the kind of functions in $\mathcal{R}$ that can appear as solutions of Equation (1.1)—see Theorem 3.4. Finally, we use results from functional transcendence (a consequence of the Ax-Schanuel theorem) to show that the solutions of Equation (1.1) in $\mathcal{R}$ after the restrictions imposed by Theorem 3.4 have a trivial transcendental part (Proposition 4.2).

We finish this introduction by mentioning that there are other natural extensions of the ring language where one may consider a similar problem. For instance, one may include in the language a predicate symbol for the non-constant functions, i.e. one may want to ask whether a given polynomial equation has or does not have solutions in $\mathcal{H}$.

Notation and basic facts.

- $B$ is the Riemann surface associated to the curve $w^2 = z^2 - 1$. Points in $B$ will be written in coordinates $(z, w)$, and $\pi$ will denote the projection $\pi(z, w) = z$. So, $B$ is a connected Riemann surface and $\pi: B \to C$ is a surjective, proper holomorphic map of degree 2.
- $\mathcal{M}_C$ is the quotient field of $\mathcal{H}_C$, namely, the field of meromorphic functions on $\mathbb{C}$.
- $\mathcal{M}_B$ is the field of complex of meromorphic functions on $B$. It is a quadratic extension of $\mathcal{M}_C$ by means of the inclusion $\mathcal{M}_C \to \mathcal{M}_B$ defined by pull-back. Indeed, we have $\mathcal{M}_B = \mathcal{M}_C(w) \simeq \mathcal{M}_C(\sqrt{z^2 - 1})$ where $w$ is the holomorphic function on $B$ defined by the $w$-coordinate, as this function satisfies $w^2 = z^2 - 1$. The extension $\mathcal{M}_B/\mathcal{M}_C$ is Galois with non-trivial automorphism determined by $w \mapsto -w$.
- For $x, y \in \mathcal{M}_C$, we write $N_r(x + yw) = (x + yw)(x - yw)$ for the norm of the quadratic extension $\mathcal{M}_B/\mathcal{M}_C$.
- $\mathcal{H}_B$ is the subring of $\mathcal{M}_B$ of holomorphic functions on $B$. Every $h \in \mathcal{H}_B$ can be written in a unique way as $f + gw$, with $f, g \in \mathcal{H}_C$. Thus, $\mathcal{H}_B = \mathcal{H}_C[w]$.

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2. Analytic solutions of Pell’s Equation

Let $t$ be the standard variable on the Riemann surface $\mathbb{C}^\times$ and let $\mathcal{H}_{C^\times}$ be the ring of complex holomorphic functions on $\mathbb{C}^\times$.

Lemma 2.1. We have $\mathcal{H}_{C^\times} = \{t^n \cdot \exp(h) : n \in \mathbb{Z} \text{ and } h \in \mathcal{H}_{C^\times}\}$.

Proof. The exponential exact sequence of sheaves on $\mathbb{C}^\times$ gives the exact sequence in cohomology

$$(0) \to \mathbb{Z} \to \mathcal{H}_{C^\times} \xrightarrow{\exp} \mathcal{H}_{C^\times}^\times \xrightarrow{\gamma} H^1(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z},$$

where $\varphi(f) = (2\pi i)^{-1} \oint f'/f$ and $\gamma$ is the circle of radius 1 around the origin positively oriented. Since $\varphi(t) = 1$, we conclude that the class of $t$ in the multiplicative group $\mathcal{H}_{C^\times}^\times / \exp(\mathcal{H}_{C^\times}) \simeq \mathbb{Z}$ is a generator.

Lemma 2.2. We have $\mathcal{H}_{B^\times} = \{(z + w)^n \cdot \exp(h) : n \in \mathbb{Z} \text{ and } h \in \mathcal{H}_B\}$.
Proof. The map \( \psi: B \to \mathbb{C}^\times \) given by \( t = \psi(z, w) = z + w \) is an isomorphism of Riemann surfaces. So we can conclude with Lemma 2.1.

For \( f, g \in \mathcal{H}_\mathbb{C} \), the norm map \( \text{Nr}: \mathcal{H}_B^\times \to \mathcal{H}_C^\times \) satisfies:

\[
\text{Nr} \left( (z + w)^n \exp(f + gw) \right) = (z + w)^n \exp(f + gw) \cdot (z - w)^n \exp(f - gw) = \exp(2f).
\]

Lemma 2.3. We have \( \ker(\text{Nr}: \mathcal{H}_B^\times \to \mathcal{H}_C^\times) = \{ \pm (z + w)^n \exp(gw) : n \in \mathbb{Z} \text{ and } g \in \mathcal{H}_\mathbb{C} \} \).

Proof. For \( f \in \mathcal{H}_\mathbb{C} \), we have \( \exp(2f) = 1 \) if and only if \( f \in \pi i \mathbb{Z} \). We conclude by Lemma 2.2.

Writing the elements of \( \mathcal{H}_B \) as \( x + yw \) for \( x, y \in \mathcal{H}_\mathbb{C} \), we have \( x + yw \in \ker(\text{Nr}: \mathcal{H}_B^\times \to \mathcal{H}_C^\times) \) if and only if \( x^2 - (z^2 - 1)y^2 = \text{Nr}(x + yw) = 1 \). Using Lemma 2.3 this proves the following result.

Theorem 2.4. The solutions \((x, y)\) of Equation (1.1) over \( \mathcal{H}_\mathbb{C} \) are of the form

\[
x + yw = \pm (z + w)^n \exp(hw)
\]

where \( n \) is a rational integer and \( h \in \mathcal{H}_\mathbb{C} \).

3. Growth

Given an open, connected Riemann surface \( S \), a proper degree \( n \) holomorphic map \( p: S \to \mathbb{C} \), a point \( z_0 \in \mathbb{C} \), and a holomorphic function \( h: S \to \mathbb{C} \), we define the following functions for \( r \geq 0 \):

\[
M_{S, z_0}(h, r) = \max_{|p(s) - z_0| \leq r} |h(s)|, \quad A_{S, z_0}(h, r) = \max_{|p(s) - z_0| \leq r} \Re(h(s))
\]

where \( \Re \) denotes the real part. In the classical case when \( p: S \to \mathbb{C} \) is taken as \( \text{Id}_\mathbb{C}: \mathbb{C} \to \mathbb{C} \) and \( z_0 = 0 \), we denote these maps simply by \( M_C(h, r) \) and \( A_C(h, r) \).

We will need the following version of the Borel-Carathéodory theorem, which follows from Corollaire 7 in [Ch04] under the usual convention that a holomorphic map \( h: S \to \mathbb{C} \) (with the previous notation) can be seen as a holomorphic multivalued function on \( \mathbb{C} \) via the rule \( z \mapsto \{ h(s) : s \in p^{-1}(z) \} \) (multivalued functions of this type are classically known as algebroid maps.)

Lemma 3.1 (Borel-Carathéodory for algebroid functions). Let \( S \) be an open, connected Riemann surface having a proper, degree \( n \) holomorphic map \( p: S \to \mathbb{C} \). Let \( h: S \to \mathbb{C} \) be a holomorphic function and let \( z_0 \in \mathbb{C} \) be such that \( \{ h(s) : s \in p^{-1}(z_0) \} = \{ 0 \} \). For all \( 0 < r < R \) we have

\[
M_{S, z_0}(h, r) \leq C_n(r, R) \cdot A_{S, z_0}(h, R),
\]

where \( C_n(r, R) = 2 \left( (R/r)^{1/n} - 1 \right)^{-1} \).

We prove the following consequence:

Lemma 3.2. Let \( f, g, h \in \mathcal{H}_\mathbb{C} \) be such that \( \exp(hw) = f + gw \) as holomorphic functions on \( B \). Then for all \( r \geq 74 \) we have

\[
M_C(h, r) \leq \frac{6}{r} \log \max \{ M_C(f, 2r), M_C(g, 2r) \} + \frac{12 \log r}{r}.
\]

Proof. We will be using the previous definitions with \( p: S \to \mathbb{C} \) taken as \( \text{Id}_\mathbb{C}: \mathbb{C} \to \mathbb{C} \) and \( \pi: B \to \mathbb{C} \).

First of all, we note that for any \( \rho > 0 \) we have

\[
M_{B, 0}(w, \rho) = \max_{|\pi(b)| = \rho} |w(b)| = \max_{|z| = \rho} \sqrt{|z^2 - 1|} = \sqrt{\rho^2 + 1}.
\]

For each \( b \in B \) we have

\[
|f(\pi(b)) + g(\pi(b))w(b)| = |\exp(h(\pi(b)))w(b)| = \exp(\Re(h(\pi(b)))w(b))
\]

from which we deduce

\[
\exp(A_{B, 0}(hw, \rho)) = M_{B, 0}(f + gw, \rho)
\]

\[
\leq M_{B, 0}(f, \rho) + M_{B, 0}(g, \rho) \cdot M_{B, 0}(w, \rho)
\]

\[
= M_C(f, \rho) + M_C(g, \rho) \sqrt{\rho^2 + 1}.
\]

Hence,

\[
A_{B, 0}(hw, \rho) \leq \log \max \{ M_C(f, \rho), M_C(g, \rho) \} + \log(2\sqrt{\rho^2 + 1}).
\]
Let $r \geq 74$. Note that $hw$ satisfies $\{(hw)(b) : \pi(b) = 1\} = \{0\}$. Lemma 3.1 applied to $hw : B \to \mathbb{C}$ with $n = 2$ and $z_0 = 1$ gives

$$M_{B,1}(hw, r + 1) \leq C_2(r + 1, 2r - 1) \cdot A_{B,1}(hw, 2r - 1) \leq 5 \cdot A_{B,1}(hw, 2r - 1)$$

where we used the bound $C_2(r + 1, 2r - 1) \leq 5$, which is valid because we chose $r \geq 74$.

Writing $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ we observe that $D(0, r) \subseteq D(1, r + 1)$ and $D(1, 2r - 1) \subseteq D(0, 2r)$. These inclusions together with the fact that holomorphic functions and their real parts are harmonic, give the bounds

$$M_{B,0}(hw, r) \leq M_{B,1}(hw, r + 1) \leq 5 \cdot A_{B,1}(hw, 2r - 1) \leq 5 \cdot A_{B,0}(hw, 2r)$$

and from the bound (3.1) with $\rho = 2r$ we conclude

$$M_{B,0}(hw, r) \leq 5 \left( \log \max \{M_C(f, 2r), M_C(g, 2r)\} + \log(2\sqrt{4r^2 + 1}) \right).$$

Notice that we have

$$M_{B,0}(hw, r) \geq M_{B,0}(h, r) \cdot \inf_{|w(b)| = r} |w(b)| = M_C(h, r) \cdot \sqrt{r^2 - 1}$$

and

$$\frac{5}{\sqrt{r^2 - 1}} < \frac{6}{r} \quad \text{and} \quad 2\sqrt{4r^2 + 1} < r^2.$$

Finally, we get

$$M_C(h, r) \leq \frac{5}{\sqrt{r^2 - 1}} \log \max \{M_C(f, 2r), M_C(g, 2r)\} + \frac{5 \cdot \log(2\sqrt{4r^2 + 1})}{\sqrt{r^2 - 1}}$$

$$\leq \frac{6}{r} \log \max \{M_C(f, 2r), M_C(g, 2r)\} + \frac{12 \log r}{r}.$$

\[\square\]

**Lemma 3.3.** Let $h \in \mathbb{H}_\mathbb{C}$. Let $\alpha \geq 0$ be a real number. If $M_C(h, r) = O(r^\alpha)$, then $h$ is a polynomial of degree at most $\lfloor \alpha \rfloor$.

**Proof.** This is immediate from Cauchy’s estimate for Taylor coefficients. \[\square\]

We recall that a holomorphic function $f \in \mathbb{H}_\mathbb{C}$ is said to have finite order if for some real number $\alpha > 0$ one has $M_C(f, r) = O(\exp(r^\alpha))$. The infimum of all such numbers $\alpha$ is the order of $f$. From the previous two lemmas we deduce:

**Theorem 3.4.** Let $f, g, h \in \mathbb{H}_\mathbb{C}$ be such that $\exp(hw) = f + gw$ as holomorphic functions on $B$. If $f$ and $g$ have order at most $\beta \geq 1$, then $h \in \mathbb{C}[z]$ is a polynomial of degree at most $\lfloor \beta \rfloor - 1$. In particular, if $f$ and $g$ are of finite order, then $h \in \mathbb{C}[z]$.

4. **Functional transcendence and proof of Theorem 1.3**

In this section, we prove Theorem 1.3.

Let $K$ be a field containing $\mathbb{C}$. We say that $a_1, \ldots, a_n \in K$ are linearly independent over $\mathbb{Q}$ modulo constants if there is no non-trivial relation

$$\lambda_1 a_1 + \cdots + \lambda_n a_n = c$$

with $c \in \mathbb{C}$ and $\lambda_j \in \mathbb{Q}$ for each $j$.

The following result is a well known consequence of Ax’s theorem [AV1] concerning Schanuel’s conjecture on power series —see [Pi13] or [Pi15].

**Theorem 4.1** (Ax-Lindemann-Weierstrass theorem). Let $W$ be an algebraic variety over $\mathbb{C}$, with function field $\mathbb{C}(W)$. If $a_1, \ldots, a_n \in \mathbb{C}(W)$ are linearly independent over $\mathbb{Q}$ modulo constants, then $\exp(a_1), \ldots, \exp(a_n)$, as functions on $W$, are algebraically independent over $\mathbb{C}(W)$.

**Proposition 4.2.** Let $a \in \mathbb{C}[z]$ and $f, g \in \mathbb{R}$. If $\exp(aw) = f + gw$ as holomorphic functions on $B$, then $a = 0$. 

Lemma 5.2. The set \(aw\) relation between \(\exp(R)\) hence, \(P_{f,g} \in \mathbb{C}[z]\) (for \(k = 1, ..., N\)). Applying Theorem 1.1 with \(W = B\), we obtain that \(aw, b_1, ..., b_N\) are linearly dependent over \(\mathbb{Q}\) modulo constants. Since \(w\) is quadratic over \(\mathbb{C}(z)\), we deduce that \(b_1, ..., b_N\) are linearly dependent over \(\mathbb{Q}\) modulo constants, so that we have \(\sum_{k=1}^{N} \lambda_k b_k = c_0\) for some \(\lambda_k \in \mathbb{Q}\) and \(c_0 \in \mathbb{C}\). Modulo relabelling, we can assume \(\lambda_N \neq 0\). Let \(P_k \in \mathbb{C}[z]\) be such \(b_N = c + \sum_{k=1}^{N-1} \alpha_k P_k\), where \(c \in \mathbb{C}\) and \(\alpha_k \in \mathbb{Z}\) (so \(b_k\) is an integer times \(P_k\)). Replacing \(b_1, ..., b_N\) in Eq. (4.1) by their expression in terms of the \(P_k\), we obtain an algebraic relation between \(\exp(aw)\), \(\exp(P_1)\), ..., \(\exp(P_{N-1})\). By Theorem 4.1, we deduce that \(P_1, ..., P_{N-1}\) are linearly dependent modulo constants. Repeating this process, we get an algebraic relation between \(\exp(aw)\) and \(\exp(Q)\), hence \(Q\) is constant, which is absurd.

Proof of Theorem 5.3. Every solution of Equation (1.1) in \(\mathbb{Z}[z]\) is in \(\mathbb{R}\). Conversely, let \(x, y \in \mathbb{R}\) be a solution of Equation (1.1). By Theorem 2.2, \(x\) and \(y\) satisfy \(x + y u^w = \pm (z + w)^n\). \(\exp(hw)\) for some \(n \in \mathbb{Z}\) and \(h \in \mathbb{H}\). Let \(f, g \in \mathbb{H}\) be such \(f + g w = (x + y w) \cdot (z + w)^{-n}\). We notice that \(f, g\) lie in \(\mathbb{R}\) (expanding the product) and \(f + g w = \pm \exp(hw)\). Hence, by Theorem 5.3 we have that \(h \in \mathbb{C}[z]\), and by Proposition 1.2 we have \(h = 0\). Hence, \(x + y w = \pm (z + w)^n\) and we get \(x, y \in \mathbb{Z}[z]\) by expanding this expression and using \(w^2 = z^2 - 1\).

5. Interpretablity and definability

For \(n \in \mathbb{Z}\), we let \(x_n, y_n \in \mathbb{C}[z]\) be defined by \(x_n + y_n \sqrt{z^2 - 1} = (z + \sqrt{z^2 - 1})^n\). Let us recall from [De78] that

(D1) \(x_n, y_n \in \mathbb{Z}[z]\) for each \(n \in \mathbb{Z}\).

(D2) The value of \(y_n\) at \(z = 1\) is equal to \(n\).

(D3) The pairs \((\pm x_n, y_n)\) for \(n \in \mathbb{Z}\) are precisely all the solutions of Equation (1.1) in \(\mathbb{C}[z]\).

In this section we let \(\mathbb{R}' \subseteq \mathbb{R}\) be a subring containing the variable \(z\). We remark that \(\mathbb{Z}[z] \subseteq \mathbb{R}' \subseteq \mathbb{R}\).

As in the introduction, we consider the language \(\mathcal{L}_z = \{0, 1, z, +, \times, =\}\) and we view \(\mathbb{R}'\) as an \(\mathcal{L}_z\)-structure in the obvious way.

Let us make a slight abuse of notation to simplify formulas: When we write a positive existential \(\mathcal{L}_z\)-formula to define a set or relation on \(\mathbb{R}'\), we can use symbols that stand for functions or relations that are already known to be positive existentially definable on \(\mathbb{R}'\) over \(\mathcal{L}_z\).

Let us define the set \(\mathbb{R}'_2 = \{f \in \mathbb{R}' : \text{there exists } p \in \mathbb{Z}[z] \text{ and } u \in \mathbb{R}' \text{ such that } f - p = (z - 1)u\}\). The main reason to work with this slightly technical definition is that it is not clear whether the binary relation \(f(1) = g(1)\) is positive existentially definable in \(\mathbb{R}'\) over \(\mathcal{L}_z\), but in \(\mathbb{R}'_2\) such difficulties are manageable.

Note that the previous definition of \(\mathbb{R}'_2\) is not a first-order definition, because we do not know whether \(\mathbb{Z}[z]\) is first-order definable in \(\mathbb{R}'\). Nevertheless, we will be able to show that \(\mathbb{R}'_2\) is first-order positive existentially definable in \(\mathbb{R}'\).

Lemma 5.1. We have that \(\mathbb{R}'_2\) is a ring and \(\mathbb{Z}[z] \subseteq \mathbb{R}'_2 \subseteq \{f \in \mathbb{R}' : f(1) \in \mathbb{Z}\}\).

Proof. The inclusions are clear. Also \(\mathbb{R}'_2\) is closed under addition. Regarding multiplication, let \(f, g \in \mathbb{R}'_2\). Take \(p, q \in \mathbb{Z}[z]\) and \(u, v \in \mathbb{R}'\) such that \(f - p = (z - 1)u\) and \(g - q = (z - 1)v\). We have \((z - 1)ug = fg - pq = fg - (q + (z - 1)v)p = fg - pq - (z - 1)pv\) hence, \(fg - pq = (z - 1)(ug + pv)\) where \(pq \in \mathbb{Z}[z]\) and \(ug + pv \in \mathbb{R}'\).

Lemma 5.2. The set \(\mathbb{R}'_2\) and the binary relation \(\text{Val}\) defined by

\[\text{Val}(f, g) : \quad f, g \in \mathbb{R}'_2 \text{ and } f(1) = g(1)\]

are positive existentially definable in \(\mathbb{R}'\) over \(\mathcal{L}_z\).
Proof. Consider the following positive existential $\mathcal{L}_2$-formula on the free variable $T$:

$$\phi(T) : \exists h \exists g, ((h^2 - (z^2 - 1)g^2 = 1) \land (\exists u, T - g = (z - 1)u)).$$

Let $f \in \mathcal{H}$. We claim that $f \in \mathcal{H}_u$ if and only if $\mathcal{H}$ satisfies $\phi(f)$.

Assume $f \in \mathcal{H}_u$ and let $p \in \mathbb{Z}[z]$, $v \in \mathcal{H}$ with $f - p = (z - 1)v$. Then $n = f(1) = p(1) \in \mathbb{Z}$ and we can take $h = x_a$ and $g = y_a$, which belong to $\mathcal{H}$ by (D1). With this choice $h^2 - (z^2 - 1)g^2 = 1$ holds by (D3), and $g(1) = n$ by (D2). Furthermore, $p - q \in \mathbb{Z}[z]$ satisfies $(p - g)(1) = 0$, so there is $q \in \mathbb{Z}[z]$ with $p - g = (z - 1)q$. Since $q \in \mathbb{Z}[z] \subseteq \mathcal{H}_Z$ we have $v + q \in \mathcal{H}$ and $f - g = (z - 1)(v + q)$. Choosing $u = v + q$, we see that $\mathcal{H}$ satisfies $\phi(f)$.

Conversely, assume that $\mathcal{H}$ satisfies $\phi(f)$ and choose $h, g, u \in \mathcal{H}$ with $h^2 - (z^2 - 1)g^2 = 1$ and $f - g = (z - 1)u$. By Theorem 1.2 we have $h, g \in \mathbb{Z}$. Since $f - g = (z - 1)u$, we get $f \in \mathcal{H}_u$.

Finally, we claim that $\mathcal{H}$ satisfies $\phi(f)$ and choose $h, g, u \in \mathcal{H}$ with $h^2 - (z^2 - 1)g^2 = 1$ and $f - g = (z - 1)u$. By Theorem 1.2 we have $h, g \in \mathbb{Z}$. Since $f - g = (z - 1)u$, we get $f \in \mathcal{H}_u$.

Let us recall that an interpretation of a structure $(M_1, \mathcal{L}_1)$ in a structure $(M_2, \mathcal{L}_2)$ is a function $\theta : A \to M_1$ where $A \subseteq M_2^r$ for certain $r \geq 1$, such that

- (Int1) $\theta$ is surjective onto $M_1$
- (Int2) $A$ is $\mathcal{L}_2$-definable in $M_2$, and
- (Int3) for each symbol $s \in \mathcal{L}_1$ with realization $s^{M_1} \subseteq M_2^k$, the pre-image of $s^{M_1}$ under $k$-th cartesian power of $\theta$ is $\mathcal{L}_2$-definable in $M_2$.

In other words, $M_1$ is the quotient of a definable subset of $M_2$, namely $A$, by a definable equivalence relation (induced by the preimages).

It is a standard fact that if there is an interpretation of $(M_1, \mathcal{L}_1)$ in $(M_2, \mathcal{L}_2)$ and if the first order theory of $(M_1, \mathcal{L}_1)$ is undecidable, then so is the first order theory of $(M_2, \mathcal{L}_2)$.

The interpretation is said to be positive existential if the $\mathcal{L}_2$-formulas used in (Int2) and (Int3) are positive existential. In this case, if the positive existential theory of $(M_1, \mathcal{L}_1)$ is undecidable, then so is the positive existential theory of $(M_2, \mathcal{L}_2)$.

For short, we write “p.e.” instead of “positive existential”.

Proof of Theorem 1.3. Let us check that the function $\theta : \mathcal{H}_Z \to \mathbb{Z}$ given by $\theta(f) = f(1)$ gives a p.e. interpretation of $(\mathbb{Z}, 0, 1, +, \times, =)$ in the $\mathcal{L}_2$-structure $\mathcal{H}$.

Since $\mathbb{Z} \subseteq \mathcal{H}_Z$, (Int1) holds. As $\mathcal{H}_Z$ is p.e. $\mathcal{L}_2$-definable in $\mathcal{H}$ (by Lemma 5.2) we get (Int2) with a p.e. $\mathcal{L}_2$-formula. The pre-images of $0^2$ and $1^2$ are defined by $\text{Val}(f, 0)$ and $\text{Val}(f, 1)$ respectively. The pre-image of $\mathbb{Z}^2$ is defined by $\text{Val}(f, g)$. The pre-image of $\mathbb{Z}^2$ is defined by $(f, g, h \in \mathcal{H}_Z) \land \text{Val}(f + g, h)$, and the case of $\times^2$ is similar since $\mathcal{H}_Z$ is a ring. By Lemma 5.2 we obtain (Int3) with p.e. $\mathcal{L}_2$-formulas.

Finally, undecidability of the p.e. $\mathcal{L}_2$-theory of $\mathcal{H}$ follows from the fact that the positive existential theory of $(\mathbb{Z}, 0, 1, +, \times, =)$ is undecidable.

Proof of Theorem 1.4. Assume first that $\mathcal{H}_\text{cst}$ is p.e. $\mathcal{L}_2$-definable in $\mathcal{H}$. We note that $\mathbb{Z} = \mathcal{H}_\text{cst} \cap \mathcal{H}_Z$, so $\mathbb{Z}$ is p.e. $\mathcal{L}_2$-definable in $\mathcal{H}$ by Lemma 5.2.

In the special case when $\mathbb{C} \subseteq \mathcal{H}$ we note that for $v \in \mathcal{H}$, we have $v \in \mathbb{C}$ if and only if $\mathcal{H}$ satisfies the formula $\exists c, (v^2 = f^2 + 1)$. This is because the curve $y^2 = x^5 + 1$ has geometric genus 2, so it admits no non-constant meromorphic parametrizations by Picard's theorem.

References

[A71] J. Ax, *On Schanuel’s Conjectures*, Ann. of Math. (2) 93 (1971), 252–268.

[Ch04] Y. Chen, *Inégalité de Borel-Carathéodory et lemme de Schwarz pour les multifonctions analytiques (French)* [Borel-Carathéodory inequality and Schwarz lemma for analytic multifunctions] Complex Var. Theory Appl. 49 (2004), no. 10, 747–757.

[DPR61] M. Davis, H. Putnam, J. Robinson, *The decision problem for exponential diophantine equations*, Ann. of Math. (2) 74 (1961), 425–436.

[De78] J. Denef, *The diophantine problem for polynomial rings and fields of rational functions*, Trans. Amer. Math. Soc. 242 (1978), 391–399.
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J. Guo, C. Sun, J. T.-Y. Wang, On the dth Roots of Exponential Polynomials and Related Problems Arising from the Green-Griffiths-Lang Conjecture. J. Geom. Anal. 31 (2021), no. 5, 5201–5218.

J. Heittokangas, K. Ishizaki, K. Tohge, Z.-T. Wen, Zero distribution and division results for exponential polynomials. Israel J. Math. 227 (2018), no. 1, 397–421.

C. W. Henson, L. Rubel, Some applications of Nevanlinna theory to mathematical logic: identities of exponential functions. Trans. Amer. Math. Soc. 282 (1984), no. 1, 1–32.

T. Pheidas, Extensions of Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), J. Denef, L. Lipshitz, T. Pheidas and J. V. Geel (eds.), Contemp. Math., 270 (1999), no. 4, 1301–1309.

M. Asai, The Diophantineness of enumerable sets, (Russian) Dokl. Akad. Nauk SSSR 191 (1970), 279–282.

H. Pasten, Definability of Frobenius orbits and a result on rational distance sets, Monatsh. Math. 182 (2017), no. 1, 99–126.

T. Pheidas, Extensions of Hilbert's tenth problem, J. Symb. Log. 59 (1994), no. 2, 372–397.

T. Pheidas and K. Zahidi, Undecidability of existential theories of rings and fields: A survey, In: “Hilbert’s tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999)”, J. Denef, L. Lipshitz, T. Pheidas and J. V. Geel (eds.), Contemp. Math., 270 (1999), Providence, RI, 2000, 49–105.

T. Pheidas and K. Zahidi, Decision problems in algebra and analogues of Hilbert’s tenth problem, In: “Model theory with applications to algebra and analysis.”, Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie (eds.), London Math. Soc. Lecture Note Ser. 350, Cambridge Univ. Press, Cambridge, 2008, Vol. 2, 207–235.

T. Pheidas, X. Vidaux Hilbert’s tenth problem for complex meromorphic functions in several variables with a place, to appear in Int. Math. Res. Not. IMRN arXiv:1711.09412 [math.LO].

J. Pila, Modular Az-Lindemann-Weierstrass with derivatives. Notre Dame J. Form. Log. 54 (2013), no. 3-4, 553–565.

J. Pila, Functional transcendence via o-minimality. In: “O-minimality and diophantine geometry”, G. O. Jones and A. J. Wilkie (Eds), London Math. Soc. Lecture Note Ser. 421, Cambridge Univ. Press, Cambridge, 2015, 66–99.

B. Poonen, Undecidability in Number Theory, Notices Amer. Math. Soc. 55 (2008), no. 3, 344–350.

J. Ritt, On the zeros of exponential polynomials, Trans. Amer. Math. Soc. 31 (1929), no. 4, 680–686.

R. Robinson, Undecidable rings., Trans. Amer. Math. Soc. 70 (1951), 137–159.

X. Vidaux, An analogue of Hilbert’s 10th problem for fields of meromorphic functions over non-Archimedean valued fields, J. Number Theory 101 (2003), no. 1, 48–73.

A. Wilkie, On exponentiation—a solution to Tarski’s high school algebra problem. Connections between model theory and algebraic and analytic geometry, Quad. Mat., 6, Dept. Math., Seconda Univ. Napoli, Caserta, (2000), 107–129.

U. Zannier, Elementary integration of differentials in families and conjectures of Pink, In: “Proceedings of the International Congress of Mathematicians — Seoul 2014, Vol. II”, S. Y. Jang, Y. R. Kim, D.-W. Lee and I. Yie (eds.), Kyung Moon Sa, Seoul, 2014, 531–556.

U. Zannier, unlikely intersections and Pell’s equations in polynomials, In: “Trends in contemporary mathematics”, V. Ancona and E. Strickland (eds.), Springer INdAM Ser., 8, Springer, Cham, 2014, 151–169.

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