Initial energy density and gluon distribution from the Glasma in heavy-ion collisions

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We estimate the energy density and the gluon distribution associated with the classical fields describing the early-time dynamics of the heavy-ion collisions. We first decompose the energy density into the momentum components exactly in the McLerran-Venugopalan model, with the use of the Wilson line correlators. Then we evolve the energy density with the free-field equation, which is justified by the dominance of the ultraviolet modes near the collision point. We also discuss the improvement with inclusion of nonlinear terms into the time evolution. Our numerical results at RHIC energy are fairly consistent with the empirical values.

I. INTRODUCTION

Color Glass Condensate (CGC) provides us with a theoretical foundation for the weak-coupling description of soft partons associated with highly energetic hadrons [1, 2]. The number of the soft partons grows with increasing energy due to quantum branching, which eventually leads to merging and saturation of the partons in the projectile at a certain energy characterized by the scale $Q_s$. Such soft (wee) partons are relevant not only in the diffractive process in deep inelastic scatterings but also for thermalization in relativistic heavy-ion collisions [3]. In the latter case, the dense transient system at mid-rapidity is initially created through the interactions between the soft partons from the incident nuclei.

In the Au-Au collisions at the top energy $\sqrt{s_{NN}} = 200$ GeV of Relativistic Heavy Ion Collider (RHIC), the relevant scale of Bjorken’s $x$ is roughly estimated as $p_T/\sqrt{s_{NN}} \sim 10^{-2}$ with presuming that the bulk of the initial medium is composed of the gluons of momenta $p_T \lesssim 1$ GeV. Then, the phenomenological Golec-Biernat–Wüsthoff fit [4, 5], multiplied by a nuclear enhancement factor $A^{1/3} = 5.8$ (see Refs. [4, 5]) gives rise to $Q_s^2 \simeq 2$ GeV$^2$. This implies that the bunch of the incoming gluons with $p_T \lesssim Q_s$ in the incident nuclei are in the saturation regime. These gluons are to be described as the classical Weiszäcker-Williams fields [6, 7] in the first approximation. Such a classical field picture is expected to be more reliable at the energy of Large Hadron Collider (LHC), $\sqrt{s_{NN}} = 5500$ GeV. At this energy the saturation scale for the Pb-Pb collisions will be around $Q_s^2 \simeq 5.2$ GeV$^2$, i.e., $(1.6)^2$ times larger than that of RHIC. Since it will be shown that the energy density and multiplicity will scale with $Q_s$ and $Q_s^2$, respectively, the former will become $1.6^3 = 4.1$ times larger and the latter $1.6^2 = 2.6$ times larger at LHC than at RHIC. At the same time, the typical time scale at LHC will be $1.6^{-1} = 0.63$ times shorter than at RHIC [10, 11]. In this paper we will address the initial stage of the nuclear collisions at RHIC energy taking the saturation scale as $Q_s^2 = 1$–2 GeV$^2$. [This uncertainty comes from the choice of relevant $x$.]

There are a number of efforts to solve the classical Yang-Mills equations of motion in order to determine the early-time evolution in the heavy-ion collisions [8, 9]. The numerical studies have been performed quite successfully so far [12, 13, 14, 15, 16, 17, 18, 19] and also the analytical techniques are developing recently [17, 18, 19] based on the McLerran-Venugopalan (MV) model [2] in the CGC picture.

This paper is a continued attempt from Ref. [18] to give an analytical estimate for the energy deposit and the gluon production from the colliding coherent fields. It is well known that the initial gauge configuration contains only the longitudinal fields [8, 9, 12, 13]. These longitudinal fields rapidly decay and the nonzero transverse fields are generated in the early-time evolution. However, if we expand the evolution of the energy density in a power series of $\tau$ from the collision time $\tau = 0$, we find that an ultraviolet (UV) divergence in each term, especially in the zeroth order term, i.e. initial energy density. This divergence originates from the perturbative tail of the gluon distribution in the MV model.

In Ref. [18] an ansatz of the logarithmic form was proposed by one of the authors to resum this UV singularity. Here we take another way. Because the origin of the divergence is in the UV regime where couplings between the classical fields are unimportant, one should be able to tame the divergent terms within the perturbative framework. This is indeed the case, as argued in Ref. [21] in a different context and emphasized also in the numerical simulation [16]. In this paper we shall resum all the most UV-singular terms, which results in solving the free-field equation. The solution is the Bessel function in the boost-invariant expanding geometry. At finite proper time $\tau$ we can send the UV cutoff to infinity. Moreover we find that the energy density behaves as $1/\tau$ at large $\tau$, consistently with the free-streaming behavior. In fact,
the limit $\tau \to 0$ reproduces the original UV divergence of the energy density as it should.

Our present approach consists of the exact initial condition of the MV model and the following perturbative time evolution. The saturation effect is fully taken into account in the initial condition in this sense. We first neglect any nonlinear effect in the time dependence, and therefore the validity of this first treatment is limited to the very early time when the UV modes dominate the dynamics. This approach recovers the same analytical result as Ref. [21] in a simple manner. Although the free-streaming region is no longer in the reliable range of the approximation, it is worth estimating the energy and the gluon multiplicity there and comparing them with the empirical values. Next we will examine the size of the nonlinear effect in the time evolution by including a mean-field type resummation in the equation.

Our analyses should be useful to grasp deeper understanding of the qualitative feature inherent to the Glasma [15], a transient stage from the coherent CGC state decaying to the thermalized plasma state. From our study two remarks are here in order. 1) Although the initial energy at the collision point in the MV model contains the UV divergence [16, 21], the momentum spectrum of the energy content and the gluon distribution is well-defined as we will explicitly compute. The spectrum is quite informative on its own. 2) Another potential application is the analytical approach toward the Glasma instability [18, 19, 22] found in the numerical simulation [23]. Because the instability presumably resides in the very early time where our description of the time evolution works, it is a feasible strategy to examine the stability of field fluctuations on top of the perturbative time evolution. We will list more future outlooks in the final section.

This paper is organized as follows: In the next section we introduce the MV model. Then, in Sec. III we calculate the correlation functions of the gluon fields and fix the MV model parameter corresponding to $Q_s$ for a given infrared regulator. Section IV is devoted to a guide for our calculation procedures which are divided into the following four sections; the proper time expansion is discussed in Sec. V, the energy density and the gluon distribution at $\tau = 0$ are evaluated in Sec. VI, the time evolution is convoluted in Sec. VII, and the time evolution is augmented with nonlinear terms in Sec. VIII. Our discussions and outlooks are in Sec. IX.

II. MODEL

For the purpose of describing a longitudinally expanding system it is convenient to adopt the Bjorken coordinates of the proper time $\tau$ and the space-time rapidity $\eta$, defined respectively by

$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \frac{1}{2} \ln \left[ \frac{t + z}{t - z} \right]. \quad (1)$$

It should be noted that $\eta$ above is different from the pseudo-rapidity in momentum space which is often denoted by $\eta$ in literature. The metric tensor associated with the Bjorken coordinates is $g_{\tau \tau} = 1$, $g_{\eta \eta} = -\tau^2$, $g_{\tau z} = g_{\eta z} = -1$, and zero otherwise.

We will work in the radial gauge, $A_r = 0$, throughout this paper. The canonical momenta (chromo-electric fields) in this gauge read from the Lagrangian as [12, 13, 20, 23]

$$E^i = \frac{\delta (\tau L)}{\delta (\partial_\tau A_i)} = \tau \partial_\tau A_i, \quad (2)$$

$$E^\eta = \frac{\delta (\tau L)}{\delta (\partial_\tau A_\eta)} = \frac{1}{\tau} \partial_\tau A_\eta. \quad (3)$$

Here we include $\tau$ in front of $L$ in Eqs. (2) and (3) originating from $\sqrt{|g|}$ in the integral measure. The following equations of motion are derived from Hamilton’s equations:

$$\partial_\tau E^i = -\frac{\delta (\tau H)}{\delta A^i} = \frac{1}{\tau} D_\eta F_{\eta i} + \tau D_j F_{ji}, \quad (4)$$

$$\partial_\tau E^\eta = -\frac{\delta (\tau H)}{\delta A_\eta} = \frac{1}{\tau} D_j F_{j\eta} \quad (5)$$

with the Hamiltonian,

$$H = \text{tr} \left[ \frac{1}{\tau^2} E^i E^i + E^\eta E^\eta + \frac{1}{\tau^2} B^i B^i + B^\eta B^\eta \right]. \quad (6)$$

These four equations (2–5) are the basic ingredients for the classical description valid in the early stage when small-$x$ partons are abundant and quantum corrections are still negligible; in momentum rapidity $Y (= \ln(1/x))$ space the validity region is bounded as $\ln(1/\alpha_s) \ll Y \ll 1/\alpha_s$ specifically.

It has been well established that the initial condition is uniquely determined by boundary matching at the singularities of the color source in the limit of vanishing longitudinal extent due to the Lorentz contraction [8, 9, 20]. The initial fields are thus known as

$$A_i(0) = \alpha_i^{(1)}(0) + \alpha_i^{(2)}(0), \quad A_\eta(0) = 0, \quad E_i'(0) = 0, \quad E_\eta'(0) = ig \left[ \alpha_i^{(1)}(0), \alpha_i^{(2)}(0) \right], \quad (7)$$

where $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ are the pure gauge fields in the space-like region extending from the right-moving nucleus traveling on the $x^+$ axis and the left-moving nucleus traveling on the $x^-$ axis, respectively. They are the gauge transformation from the light-cone solution [24];

$$\alpha_i^{(1)}(x_{\perp}) = -\frac{1}{ig} V_{\infty}(x_{\perp}) \partial_\tau V_{\infty}^\dagger(x_{\perp}),$$

$$\alpha_i^{(2)}(x_{\perp}) = -\frac{1}{ig} W_{\infty}(x_{\perp}) \partial_\tau W_{\infty}^\dagger(x_{\perp}) \quad (8)$$

with the Wilson lines, which are color SU($N_c$) matrices.
in the fundamental representation, defined by

\[
V_{x-}^{1}(x_{\perp}) = \mathcal{P}_{-} \exp \left[ -ig \int_{-\infty}^{\infty} dz_{-} \frac{\rho^{(1)}(x_{\perp}, z_{-})}{\partial^{2}} \right],
\]

\[
W_{x+}^{1}(x_{\perp}) = \mathcal{P}_{+} \exp \left[ -ig \int_{-\infty}^{\infty} dz_{+} \frac{\rho^{(2)}(x_{\perp}, z_{+})}{\partial^{2}} \right].
\]

Here \( \mathcal{P}_{\pm} \) represents the path ordering with respect to \( x^{\pm} \). We remark that \( \rho^{(1)} \) is a static color source in \( x^{+} \) (i.e. \( x^{+}\)-independent) and \( \rho^{(2)} \) is static in \( x^{-} \) due to the Lorentz time dilatation. Although we assume \( \rho^{(1)}(x_{\perp}, x^{-}) \propto \delta(x^{-}) \) and \( \rho^{(2)}(x_{\perp}, x^{+}) \propto \delta(x^{+}) \) in the end, we have to keep both the \( z \)-integral and the path ordering, which encompasses the random color distribution along the longitudinal extent \[1, 23, 27, 28, 29, 30\].

Now that we have fixed the initial condition for the equations of motion, we should have a unique solution. In principle the solution is given in terms of \( V \) and \( W \) through the initial condition, and any physical observable \( \mathcal{O} \) at later time could be determined once \( \rho^{(1)} \) and \( \rho^{(2)} \) are given. Since the color sources \( \rho^{(1)} \) and \( \rho^{(2)} \) will fluctuate randomly in event by event, observed \( \mathcal{O} \) should be an ensemble average over the color source distribution \( \mathcal{W}[\rho^{(1)}, \rho^{(2)}] \), that is,

\[
\langle \mathcal{O} \rangle = \int \mathcal{D}\rho^{(1)} \mathcal{D}\rho^{(2)} \mathcal{W}[\rho^{(1)}, \rho^{(2)}] \mathcal{O}[V, W].
\]

This is a general form for the observable expectation value in the CGC framework.

The MV model is founded on the Gaussian approximation to \( \mathcal{W}[\rho^{(1)}, \rho^{(2)}] \) as formulated in Ref. \[22\], where any averaged quantity is expressed in terms of the two-point function,

\[
\langle \rho^{(m)}(x_{\perp}, z) \rho^{(n)}(y_{\perp}, z') \rangle = g^{2} \mu^{2}(z) \delta^{mn} \delta(z - z') \delta^{(2)}(x_{\perp} - y_{\perp}).
\]

Here \( m \) and \( n \) distinguish the right-moving (1) and left-moving (2) nucleus, \( a \) and \( b \) are the color indices, and \( \mu^{2}(z) \) is the only dimension parameter in the MV model. All the physical quantities are given in terms of (integrated) \( \mu^{2} \) and importantly \( \mu^{2} \) has a tight connection to the parton saturation scale \( Q_{s}^{2} \).

With these preliminaries, we are ready to compute physical quantities of our interest in the MV model, once the model scale \( \mu^{2} \) and the coupling constant \( g \) are fixed. Before proceeding to the next section, let us elucidate a simple expression for \( \alpha^{(n)} \) for later use. Here we only deal with \( \alpha^{(1)} \) of the right-moving nucleus and suppress the superscript in the rest of this section and in the next section. Similar formulae obviously hold for \( \alpha^{(2)} \), too.

Substituting Eq. (9) into Eq. (8), we can explicitly write down the expression,

\[
\alpha_{i} = -\frac{1}{2g} \int_{-\infty}^{\infty} dx_{-} \left[ \rho(x_{-}) \right] V_{x-}^{1} = \int_{-\infty}^{\infty} dx_{-} \left[ -i g \frac{\partial \rho(x_{-})}{\partial^{2}} \right] V_{x-}^{1} = \int_{-\infty}^{\infty} dx_{-} \frac{\partial \rho^{(1)}(x_{-})}{\partial^{2}} V_{x-}^{1} - \int_{-\infty}^{\infty} dx_{-} \frac{\partial \rho^{(2)}(x_{-})}{\partial^{2}} V_{x-}^{1}.
\]

where \( T_{F}^{\mu} \)'s are the SU(\( N_{c} \)) algebra in the fundamental representation and \( V_{x} \) denotes the Wilson line in the adjoint representation whose components are given by \( V_{x}^{ab} = 2\text{tr}[T_{F}^{a}V_{F}^{2}T_{F}^{b}V_{F}^{1}] \) in terms of the fundamental Wilson lines.

### III. GAUSSIAN AVERAGE

In this section we list the useful formulae for the Gaussian average over the color source distribution. Using Eq. (12) we can explicitly write the gauge field correlator in a form of

\[
\langle \alpha^{a}_{i}(x_{\perp}) \alpha^{b}_{j}(y_{\perp}) \rangle = \int_{-\infty}^{\infty} dx_{-} dy_{-} \left( \frac{\partial \rho^{(1)}(x_{-}, x_{-})}{\partial^{2}} \right) \left( \frac{\partial^{2}\rho^{(1)}(y_{-}, y_{-})}{\partial^{2}} \right) \left( \frac{\partial^{2}\rho^{(2)}(y_{-}, y_{-})}{\partial^{2}} \right) \left( \frac{\partial^{2}\rho^{(2)}(x_{-}, x_{-})}{\partial^{2}} \right) \left( \frac{\partial^{2}\rho^{(2)}(x_{-}, x_{-})}{\partial^{2}} \right)
\]

\[
= \delta^{ab} g^{2} \int_{-\infty}^{\infty} dx_{-} \frac{\partial^{2}\rho^{(a)}(x_{-}, x_{-})}{\partial^{2}} \frac{\partial^{2}\rho^{(b)}(y_{-}, y_{-})}{\partial^{2}} L(x_{\perp} - y_{\perp}) \mu^{2}(x_{-}) \frac{\partial^{2}\rho^{(2)}(y_{-}, y_{-})}{\partial^{2}} \frac{\partial^{2}\rho^{(2)}(x_{-}, x_{-})}{\partial^{2}} L(x_{\perp} - y_{\perp})
\]

\[
= \delta^{ab} g^{2} \mu^{2}(x_{-}) \delta(x_{-} - y_{-}) L(x_{\perp} - y_{\perp}) \frac{\partial^{2}\rho^{(a)}(x_{-}, x_{-})}{\partial^{2}} \frac{\partial^{2}\rho^{(b)}(y_{-}, y_{-})}{\partial^{2}} L(x_{\perp} - y_{\perp})
\]

\[
= \delta^{ab} g^{2} \mu^{2}(x_{-}) \delta(x_{-} - y_{-}) L(x_{\perp} - y_{\perp}) \frac{\partial^{2}\rho^{(a)}(x_{-}, x_{-})}{\partial^{2}} \frac{\partial^{2}\rho^{(b)}(y_{-}, y_{-})}{\partial^{2}} L(x_{\perp} - y_{\perp})
\]

and

\[
L(x_{\perp}) = \frac{1}{(2\pi)^{2}} \int d^{2}k_{\perp} e^{ik_{\perp} \cdot x_{\perp}} \left( \frac{2}{k_{\perp}^{2} + m^{2}} \right)^{2},
\]

The momentum integration in Eq. (13) has infrared (IR) divergence. We regularize this by the gluon mass \( m \neq 0 \) and regard \( m \) as a model parameter controlling the IR behavior. Then the integration can be performed analytically. For convenience we introduce the notation \[25, 28\],

\[
\Gamma(x_{\perp}) = \frac{2}{2\pi m^{2}} \left[ 1 - m|x_{\perp}| K_{1}(m|x_{\perp}|) \right] \simeq \frac{1}{8\pi} \frac{|x_{\perp}|^{2} \ln(m^{2}|x_{\perp}|^{2} \lambda^{2})}{1 - m|x_{\perp}| K_{1}(m|x_{\perp}|)}.
\]
where $K_1(x)$ is the modified Bessel function, and $\lambda = e^{\gamma_e - 1/2}/2 \simeq 0.54$. The second line is an approximate expression for small $m|x_\perp|$.

Besides IR singularity, UV divergence also exists in $\partial^2 L(x_\perp), (\partial^2)^2 L(x_\perp)$ and so on, due to the Delta function correlation $\delta(2)(x_\perp)$ in Eq. (11), as explicitly shown below in this section.

In Eq. (13), we have also defined the Wilson line correlator by

$$\langle V_{A_x}^{\{a}(x_\perp)V_{A_x}^{\{b}(y_\perp)\rangle = \delta^{ab}\text{adj}(x^-; x_\perp - y_\perp).$$

(17)

We note that Eq. (14) is the building block and its iterative use in the left-hand side of Eq. (17) leads to $C_{\text{adj}}(x^-; x_\perp)$. The answer is already given in literature \cite{27, 28, 30} as

$$C_{\text{adj}}(x^-; x_\perp) = \exp\left[-\frac{N_c}{2}g^4 \int_{-\infty}^{\infty} dz \mu^2(z) \Gamma(x_\perp)\right].$$

(18)

This factor encodes the effect of the multiple interactions with the color source in the nucleus. We see that $C_{\text{adj}}(x^-; x_\perp \to 0_\perp) \to 1$ as it should be due to color transparency.

To simplify the discussion we assume here $\mu^2(x^-) = \theta(x^-)\theta(\epsilon - x^-)\mu_A^2/\epsilon$ with $\epsilon \to 0^+$ (this can be relaxed in more careful calculations) so that

$$\int_{-\infty}^{\infty} dx^- \mu^2(x^-) = \mu_A^2.$$  

(19)

Then the integration over $x^-$ results in

$$\int_{-\infty}^{\infty} dx^- \mu^2(x^-) C_{\text{adj}}(x^-; x_\perp) = \mu_A^2 \bar{C}_{\text{adj}}(x_\perp)$$

(20)

with

$$\bar{C}_{\text{adj}}(x_\perp) = \frac{1}{am^2\Gamma(x_\perp)} \left(1 - \exp\left[-am^2\Gamma(x_\perp)\right]\right).$$

(21)

We have used a dimensionless parameter:

$$a \equiv \frac{N_c(g^2\mu_A)^2}{2m^2},$$

(22)

in accord with Refs. \cite{27, 28, 29}. Here in this paper, however, we regard the parameter $a$ as the IR parameter controlling how much of the non-perturbative effect in the initial fields contributes to the momentum integration once the saturation scale is fixed [see discussions below]. Finally we find from Eq. (13),

$$\langle \alpha^a_i(x_\perp) \alpha^b_j(y_\perp) \rangle = \delta^{ab}\delta^2 \bar{C}_{\text{adj}}(x_\perp - y_\perp) \partial_i^x \partial_j^y L(x_\perp - y_\perp).$$

(23)

Let us discuss a rough estimate for the parameter $a$ here. We will check at the end of this section that $g^4\mu_A$ is nearly identified with the saturation scale $Q_s$ (which also depends on $a$). The IR cutoff is difficult to fix uniquely but a natural expectation is $m \simeq g\mu_A$, from the comparison between our cutoff scheme and the color neutrality as a result of the quantum evolution \cite{26, 31} with an uncertain prefactor. The strong coupling constant at the scale $Q_s$ at RHIC energy is $\alpha_s \simeq 0.3$ (or $g \simeq 2$). This implies $a \simeq 3g^2/2 = 6$. The physical meaning of this $m$ value is that the classical description breaks down there. On the other hand, if we take $m$ of order of the confinement scale $\sim 1$ fm, i.e., $m \simeq 200$ MeV, we would have $a \simeq (3 \cdot 1 \sim 2$ GeV$^2)/(2 \cdot 0.2^2$ GeV$^2) = 37 \sim 75$ for $Q_s^2 = 1 \sim 2$ GeV$^2$. In this estimate $m$ is the scale where the CGC picture in the perturbative regime breaks down. From these considerations the reasonable range for $a$ would be 10–100 at the RHIC energy. It should be noted that $a$ used in the numerical simulation may be much larger; there $m$ is provided by the system size of order 10 fm and $a$ may be then hundreds times larger. Nevertheless, because the IR property in the numerical implementation of the MV model is totally different from the analytical formulation \cite{24}, we cannot make a direct comparison to the numerical results. In this paper we will specifically choose four cases: $a = 10, 25, 100, and 500$ for comparison, keeping in mind that the physical value is around $a = 10 \sim 40$.

It is important to note that the above $\bar{C}_{\text{adj}}(x_\perp)$ exactly corresponds to the gluon propagator \cite{24} and should be distinguished from

$$C_A(x_\perp) = \exp\left[-am^2\Gamma(x_\perp)\right],$$

(24)

which appears in the gluon distribution \cite{27, 28, 32}. In the case of dense-dilute collisions it is possible to convert $\bar{C}_{\text{adj}}(x_\perp)$ into $C_A(x_\perp)$ by gauge rotation but this is not the case if the dense-dense collisions are concerned in a symmetric way.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{$C_{\text{adj}}(x_\perp)$ as a function of dimensionless $m\sqrt{a|x_\perp|} = \sqrt{N_c/2}Q_s^2\mu_A|x_\perp|$ in the case that $a = N_c(g^2\mu_A)^2/(2m^2)$ is 25 which is the same choice as Ref. \cite{28}. The solid curve is the result from the numerical integration and the dashed curve represents the approximation by Eq. (18) which eventually blows up in the IR region.
} \end{figure}
The part involving $\tilde{C}_{adj}(x_\perp)$ describes the transverse color correlation in the random distribution. It shows moderate damping in a distant region as plotted by the (red) solid curve in Fig. 1 for $a = 25$. We show the approximation using Eq. (16) by the (green) dashed curve, and $C_A(x_\perp)$ by the (blue) dotted curve for reference. From the comparison between the solid and dotted curves we immediately realize that $\tilde{C}_{adj}(x_\perp)$ retains a long-range tail than $C_A(x_\perp)$. Indeed, we can readily notice that $\Gamma(x_\perp) \to (2\pi m^2)^{-1}$ for $m|x_\perp| \gg 1$ and thus

$$\tilde{C}_{adj}(x_\perp) \to \frac{2\pi}{a} \left(1 - e^{-a/2\pi}\right),$$

which is nearly 0.5, 0.25, 0.063, and 0.013 for $a = 10, 25, 100,$ and 500 respectively; the asymptotic value is fixed by the IR cutoff $a$.

Equation (13) is a useful expression. Here we enumerate quantities necessary for the evaluation of the initial energy; the equal-point gauge field correlator is

$$\langle \alpha_i^a(x_\perp) \alpha_j^b(x_\perp) \rangle = -\frac{1}{2} \delta^{ab} \delta_{ij} g^2 \mu_A^2 (\partial^2 L(0_\perp)), \quad (26)$$

and the derivative correlations read

$$\langle \left(\partial_k \alpha_i^a(x_\perp)\right) \left(\partial_l \alpha_j^b(x_\perp)\right) \rangle = \frac{1}{8} \delta^{ab} g^2 \mu_A^2 \left(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}\right) (\partial^2 L(0_\perp))$$

$$- N_c g^4 \mu_A^2 \delta_{ij} \delta_{kl} (\partial^2 L(0_\perp))^2, \quad (27)$$

and also

$$\langle \alpha_i^a(x_\perp) \partial^2 \alpha_j^b(x_\perp) \rangle = -\frac{1}{4} \delta^{ab} g^2 \mu_A^2 \left[2 \delta_{ij} (\partial^2 L(0_\perp)) - N_c g^4 \mu_A^2 \delta_{ij} (\partial^2 L(0_\perp))^2\right], \quad (28)$$

where we can explicitly estimate the integrals as

$$\partial^2 L(0_\perp) = -\int_\Lambda \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + m^2} = -\frac{1}{2\pi} \ln \frac{\Lambda}{m}, \quad (29)$$

and

$$(\partial^2)^2 L(0_\perp) = \int_\Lambda \frac{d^2k_\perp}{(2\pi)^2} = \frac{1}{4\pi} \Lambda^2$$

for $\Lambda \gg m$. We see that the derivatives of the correlation $L(x_\perp)$ at the origin contain the UV divergence.

Finally in this section, let us make the connection between the scale parameter, $g^2 \mu_A$, and the saturation scale $Q_s$. The definition of the saturation scale is not unique. We apply the same scheme as adopted in Ref. [32] using the Fourier transform of $C_A(x_\perp)$, which we denote as $C_A(k_\perp)$. Since $k_\perp^2 C_A(k_\perp)$ is interpreted as a gluon distribution function, we define the saturation scale as the peak position of this function. Bear in mind that we have to keep the same definition for $Q_s$ in order to make a consistent comparison with other empirical analyses. 

Figure 2 is a plot for $k_\perp^2 C_A(k_\perp)$ as a function of the dimensionless variable $k_\perp/(m \sqrt{a})$ where $m \sqrt{a} = \sqrt{N_c/2} g^2 \mu_A$. The peak position defines the saturation scale. From Fig. 2 it reads $Q_s/(m \sqrt{a}) = 0.497, 0.558, 0.720, and 0.903 for $a = 10, 25, 100,$ and 500, which correspond to $m = 0.64 Q_s, 0.36 Q_s, 0.14 Q_s, and 0.050 Q_s$, respectively. We summarize these relations in Table I.

| $a$   | $m$  | $g^2 \mu_A$ |
|------|------|-------------|
| 10   | 0.64 | 1.65 $Q_s$  |
| 25   | 0.36 | 1.46 $Q_s$  |
| 100  | 0.14 | 1.13 $Q_s$  |
| 500  | 0.050| 0.90 $Q_s$  |

TABLE I: IR cutoff $m$ and the MV model parameter $g^2 \mu_A$ determined from $k_\perp^2 C_A(k_\perp)$ for various $a$.

IV. CALCULATION STEPS

In the subsequent sections we will elucidate the time evolution of the initial energy density and the gluon distribution. Since the computation process is involved, we outline the derivation in advance here.

Step I) We will carry out the proper time expansion; the equations of motion are solved in a power series of $\tau$. We will then find that each term contains the UV singularity. We decompose the divergent initial energy density into the Fourier modes which give the gluon distribution if divided by the gluon energy. We take full account of the saturation effect.

Step III) We will figure out the time evolution for the UV modes from the equations of motion. We confirm
that the UV singularity is regularized at finite \( \tau \), and fix the initial condition for the time evolution by matching it to Step II).

**Step IV** We will improve the time evolution by inclusion of the nonlinear terms in the Gaussian approximation. We finally read the initial energy density and the gluon number at the formation time when the time dependence shows free-streaming behavior.

V. PROPER TIME EXPANSION – STEP I)

We shall begin with a naive expansion for the classical fields in terms of the proper time \( \tau \) as in Ref. [17], and in the next section we resum the leading order terms in the UV singularity coming from the Gaussian average. It should be stressed that the gauge fields for a given color source are free from the singularity and have no difficulty in the \( \tau \)-expansion.

The first order terms are the contributions at \( \tau = 0 \), that is, the initial fields as given in Eq. (7). Let us now evaluate the associated chromo-electric and chromo-magnetic fields. It is trivial to read the initial chromo-electric fields from the initial condition as

\[
E_i^{(0)} = 0, \\
E_i^{(1)} = ig\left[\alpha_i^{(1)}, \alpha_i^{(2)}\right],
\]

(31)

and noting \( \alpha_i^{(n)} \)'s are pure gauge solutions, we find the chromo-magnetic fields being

\[
B_i^{(0)} = 0, \\
B_i^{(1)} = F_{12(0)},
\]

(32)

where

\[
F_{ij(0)} = -ig\left[\alpha_i^{(1)}, \alpha_j^{(2)}\right] + \left[\alpha_i^{(2)}, \alpha_j^{(1)}\right].
\]

(33)

These results are recognized as the fundamental property of the Glasma initial state; the longitudinal fields between two color sheets are predominant in the initial state of matter.

To proceed further away from \( \tau = 0 \), we apply the expansion, i.e.,

\[
O(\tau) = \sum_{n=0}^{\infty} O(n)\tau^n,
\]

(34)

for arbitrary fields \( O \) given in terms of the classical gauge fields. It is easy to confirm that the terms for odd \( n \) are all vanishing due to time reversal symmetry. The non-trivial contribution to the gauge field starts at the second-order terms which turn out to be

\[
A_i^{(2)} = \frac{1}{2} E_i^{(2)} = \frac{1}{2} D_j(0) F_{ji(0)}, \\
A_\eta^{(2)} = \frac{1}{2} E_\eta^{(2)}.
\]

(35)

Then we can explicitly write down the second-order contributions to the initial chromo-electric and chromo-magnetic fields:

\[
E_i^{(2)} = \frac{1}{2} D_j(0) F_{ji(0)} = -\epsilon^{ij} \frac{1}{2} D_j(0) B_\eta^{(0)},
\]

(36)

\[
E_\eta^{(2)} = \frac{1}{2} D_j(0) F_{\eta j(2)} = \frac{1}{2} D_j(0) D_j(0) E_\eta^{(0)},
\]

and

\[
B_i^{(2)} = \epsilon^{ij} F_{ij(2)} = \epsilon^{ij} \frac{1}{2} D_j(0) E_\eta^{(0)},
\]

(37)

\[
B_\eta^{(2)} = F_{12(2)} = \frac{1}{2} D_j(0) D_j(0) B_\eta^{(0)}.
\]

Here we have defined the anti-symmetric tensor \( \epsilon^{12} = -\epsilon^{21} = 1 \) in the transverse coordinates. The duality relation between the electric and magnetic fields is manifest itself in Eqs. (36) and (37).

VI. INITIAL ENERGY DENSITY AND GLUON DISTRIBUTION – STEP II)

The energy density is an ensemble average of the Hamiltonian \( \mathcal{H} \), i.e. \( \varepsilon = \langle \mathcal{H} \rangle \). In this section we compute \( \varepsilon \) and decompose it into the Fourier components. The merit of the Fourier decomposition is that each Fourier mode is free from UV singularity and we realize how the \( k_\perp \)-integration of the spectrum diverges. Besides, we can define the gluon distribution associated with the field intensity at each momentum.

A. Zeroth Order

Right after the collision at \( \tau = 0 \) only the longitudinal fields along the \( \eta \) direction have nonzero values, and therefore we calculate the following energy density:

\[
\varepsilon^{(0)} = \frac{1}{3} \left\langle \text{tr} \left[ E_\eta^{(0)} E_\eta^{(0)} + B_\eta^{(0)} B_\eta^{(0)} \right] \right\rangle.
\]

(38)

Noting that \( \Gamma(x_\perp - y_\perp) \rightarrow 0 \) in the limit of \( y_\perp \rightarrow x_\perp \), we can drop \( \hat{C}_{\text{adj}}(x_\perp - y_\perp) \) in Eq. (25). With a shorthand notation as \( \langle \alpha_i^{(m)} a_j^{(n)} \rangle = \delta_{mn} \delta^{\alpha \beta} \delta_{ij} \langle \alpha \alpha \rangle \), we can express the energy density in a form of

\[
\varepsilon^{(0)} = 2N_c(N_c^2 - 1)g^2\langle \alpha \alpha \rangle^2 = g^6\mu_A^4 \cdot \frac{3}{\pi^2} \left[ \ln \frac{\Lambda}{m} \right]^2,
\]

(39)

from Eq. (29) for \( N_c = 3 \).

It is interesting and practically useful to look into the energy content in momentum space, which is also necessary when we consider the gluon distribution. To this end, let us calculate the following quantity:

\[
\varepsilon^{(0)}(k_\perp) = \frac{1}{V} \left\langle \text{tr} \left[ E_\eta(0)(-k_\perp) E_\eta(0)(k_\perp) + B_\eta(0)(-k_\perp) B_\eta(0)(k_\perp) \right] \right\rangle,
\]

(40)
so that the $k_\perp$-integration of $\varepsilon_0(k_\perp)$ recovers the energy density $\varepsilon_0$. Then we have to evaluate the spatially separated correlation function of the chromo-electric and chromo-magnetic fields:

$$
\langle \text{tr} E_0^\eta(\mathbf{x}_\perp) E_0^\eta(\mathbf{y}_\perp) \rangle = \frac{1}{2} N_c (N_c^2 - 1) g^2 \times \left( \langle \alpha_1(\mathbf{x}_\perp) \alpha_1(\mathbf{y}_\perp) \rangle^2 + \langle \alpha_1(\mathbf{x}_\perp) \alpha_2(\mathbf{y}_\perp) \rangle^2 + \langle \alpha_2(\mathbf{x}_\perp) \alpha_1(\mathbf{y}_\perp) \rangle^2 + \langle \alpha_2(\mathbf{x}_\perp) \alpha_2(\mathbf{y}_\perp) \rangle^2 \right)
$$

$$
= \frac{1}{2} N_c (N_c^2 - 1) g^6 \mu_A^4 \left[ \tilde{C}_{\text{adj}}(\mathbf{x}_\perp - \mathbf{y}_\perp) \right]^2 \times \left[ \left[ \partial^2_1 L(x^\perp - y^\perp) \right]^2 + 2(\partial_1 \partial_2 L(x^\perp - y^\perp))^2 \right] .
$$

Computing the chromo-magnetic part in the same way, we obtain almost the same expression as Eq. (41) but with the opposite sign in front of the $[\partial_1 \partial_2 L(x^\perp - y^\perp)]^2$ term. This term cancels out in the sum of the chromo-electric and chromo-magnetic parts. The Fourier transformation of the sum gives rise to

$$
\varepsilon_0(\mathbf{k}_\perp) = \frac{1}{V} \int d^2 x^\perp d^2 y^\perp e^{-i \mathbf{k} \cdot (x^\perp - y^\perp)} \times \left( \langle \text{tr} E_0^\eta(\mathbf{x}_\perp) E_0^\eta(\mathbf{y}_\perp) + B_0^\eta(\mathbf{x}_\perp) B_0^\eta(\mathbf{y}_\perp) \rangle \right) = \frac{1}{2} N_c (N_c^2 - 1) g^6 \mu_A^4
$$

$$
\times \int \frac{d^2 q_1\perp}{(2\pi)^2} \frac{d^2 q_2\perp}{(2\pi)^2} \frac{d^2 q_3\perp}{(2\pi)^2} \left( \tilde{C}_{\text{adj}}(q_1\perp) \tilde{C}_{\text{adj}}(q_2\perp) / (q_3\perp + m^2)[(k'\perp - q_3\perp)^2 + m^2] \right) .
$$

We denote $k'_\perp = k_\perp - q_1\perp - q_2\perp$, and $\tilde{C}_{\text{adj}}(q_\perp)$ is the Fourier transform of $C_{\text{adj}}(x_\perp)$. We can further perform the $q_{3\perp}$-integration to reach the following:

$$
\varepsilon_0(\mathbf{k}_\perp) = \frac{1}{4\pi} N_c (N_c^2 - 1) g^6 \mu_A^4 \int \frac{d^2 q_1\perp}{(2\pi)^2} \frac{d^2 q_2\perp}{(2\pi)^2} \frac{\tilde{C}_{\text{adj}}(q_1\perp) \tilde{C}_{\text{adj}}(q_2\perp)}{k'_\perp \sqrt{k'_\perp + 4m^2} + k'_\perp} \ln \left[ \frac{\sqrt{k'_\perp + 4m^2} + k'_\perp}{\sqrt{k'_\perp + 4m^2} - k'_\perp} \right] .
$$

At this point it is easy to confirm that Eq. (43) reproduces Eq. (39) when integrated over $k_\perp$. In the presence of the $k_\perp$-integration, we can change the variable from $k_\perp$ to $k'_\perp = k_\perp - q_1\perp - q_2\perp$, and then we can perform the $q_{1,2\perp}$-integrations independently of the $k_\perp$-integration. Using the sum rule,

$$
\int \frac{d^2 q_{1,2\perp}}{(2\pi)^2} \tilde{C}_{\text{adj}}(q_{1,2\perp}) = \tilde{C}_{\text{adj}}(x_\perp = 0_\perp) = 1 ,
$$

we have

$$
\varepsilon_0 = \frac{1}{4\pi} N_c (N_c^2 - 1) g^6 \mu_A^4 \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{m^2} \tilde{T}_{\text{pert}}(k_\perp/m) = \frac{1}{8\pi^2} N_c (N_c^2 - 1) g^6 \mu_A^4 \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{m^2} \tilde{T}_{\text{pert}}(k_\perp/m) .
$$

Figure 3 shows $\sqrt{a} k_\perp C_{\text{adj}}(k_\perp)$ as a function of the dimensionless variable $k_\perp/(m\sqrt{a})$ for $a = 10, 25, 100, 500$. where we have used $\Lambda \gg m$ and this is exactly identical to Eq. (39). Here, we have defined the perturbative contribution by a function,

$$
\tilde{T}_{\text{pert}}(\xi) = \frac{1}{\xi} \ln \left[ \frac{\sqrt{\xi^2 + 4} + \xi}{\sqrt{\xi^2 + 4} - \xi} \right] .
$$

It is worth mentioning that $\tilde{T}_{\text{pert}}(\xi \to 0) \to 0.5$ which is finite owing to IR regularization.

We need to evaluate $\tilde{C}_{\text{adj}}(k_\perp)$ appearing in the integral (43). In Fig. 3, we plot $\sqrt{a} k_\perp C_{\text{adj}}(k_\perp)$ as a function of $k_\perp/(m\sqrt{a})$. As is already mentioned previously, $\tilde{C}_{\text{adj}}(x_\perp)$ has a constant tail $(2\pi/a)(1 - \exp[-a/(2\pi)])$ at large distances due to the IR cutoff, which gives rise to a Dirac delta function $\delta(2\perp)$ in momentum space. $\tilde{C}_{\text{adj}}(k_\perp)$ represents the effect of the multiple scattering. For smaller $a$ with fixed $g^2 \mu_A$, larger part of IR region is cutoff and therefore $\tilde{C}_{\text{adj}}(k_\perp)$ spreads over smaller $k_\perp$ region and has larger weight of $\delta(2\perp)$. For larger $a$, more multiple interactions are effective and $\tilde{C}_{\text{adj}}(k_\perp)$ has the larger spread and smaller weight of $\delta(2\perp)$. The perturbative limit corresponds to no multiple scattering.

Using the numerical results shown in Fig. 3 and the Dirac delta function with the weight $(2\pi/a)(1 - \exp[-a/(2\pi)])$, we can explicitly evaluate $\varepsilon_0(k_\perp)$ performing the numerical integration of $q_{1\perp}$ and $q_{2\perp}$. In the perturbative limit where $\tilde{C}_{\text{adj}}(q_{1,2\perp})$ factor is replaced with $\delta(2\perp)$ in Eq. (43), the $k_\perp$-spectrum of the energy content is simply fixed by $\tilde{T}_{\text{pert}}(k_\perp/m)$. We show this perturbative limit of $\tilde{T}_{\text{pert}}(k_\perp/m)$ by the (black) dotted curve in Fig. 4.

The saturation effect changes the functional form of $\tilde{T}_{\text{pert}}(\xi)$ into the one depending on $a$ which we denote by $T(a; \xi)$, i.e.,

$$
\varepsilon_0(k_\perp) = \frac{1}{4\pi a^2 m^2} N_c (N_c^2 - 1) g^6 \mu_A^4 T(a; k_\perp/m) .
$$

Figure 4 shows $T(10; \xi)$ by the (red) solid curve, $T(25; \xi)$ by the (green) dashed curve, $T(100; \xi)$ by the (blue)
FIG. 4: \( T(\alpha; \xi) \) in the initial energy density as a function of the dimensionless variable \( \xi (= k_\perp/m) \); the (black) dotted curve represents the perturbative limit \( T_{\text{pert}}(\xi) \) (where \( C_{\text{adj}}(q_{1,2,1}) \) is replaced with \( (2\pi)^2 \delta^{(2)}(q_{1,2,1}) \)), the (red) solid curve for \( a = 10 \), the (green) dashed curve for \( a = 25 \), the (blue) dotted curve for \( a = 100 \), and the (purple) dot-dashed curve for \( a = 500 \).

dotted curve, and \( T(500; \xi) \) by the (purple) dot-dashed curve. We notice at a glance at Fig. 4 that the saturation effect is larger for larger \( a \) in the low \( k_\perp \) region. Although the functional form of \( C_{\text{adj}}(k_\perp) \) has a large suppression for small \( a \) in view of Fig. 4, the resulting spectrum in Fig. 4 turns out to be closer to the perturbative one when \( a \) is small. This is because \( C_{\text{adj}}(q_{1,2,1}) \) contains the Dirac delta function \( \delta^{(2)}(q_{1,2,1}) \) with a larger weight for smaller \( a \).

If we integrate the above \( \varepsilon_{(0)}(k_\perp) \) over \( k_\perp \) the initial energy density becomes \( a \)-independent and should be reduced to Eq. (15). As we will discuss later, the spectrum should follow the perturbative time evolution, and then we will have the energy density as a function of time that is \( a \)-dependent and UV finite (but \( m \) dependent).

B. Second Order

We are now going into the contributions of \( \mathcal{O}(\tau^2) \). There arise two kinds of terms with and without spatial derivatives. The derivative terms bring about the UV singularity from the Gaussian average in the MV model. The non-derivative terms have more gauge fields in place of derivatives. We will discuss each contribution in order.

1. Longitudinal fields

We calculate first the longitudinal field energy of \( \mathcal{O}(\tau^2) \). Because the longitudinal field has an \( \mathcal{O}(\tau^0) \) contribution, the second-order terms in the energy come from the cross terms between the zeroth and the second order in the fields. That is,

\[
\varepsilon_{(2)}^L = \frac{1}{2} \left\langle \mathcal{T} \right. \left[ E_{(0)}^{\mu} \partial^2 E_{(0)}^{\mu} + B_{(0)}^{\nu} \partial^2 B_{(0)}^{\nu} \right] \right. \\
= \frac{1}{2} \left\langle \mathcal{T} \right. \left[ E_{(0)}^{\mu} D_{(0)}^{\mu} E_{(0)}^{\mu} + B_{(0)}^{\nu} D_{(0)}^{\nu} B_{(0)}^{\nu} \right] \right. . \\
(48)
\]

The derivative part yields

\[
\frac{1}{2} \left\langle \mathcal{T} \right. \left[ E_{(0)}^{\mu} \partial^2 E_{(0)}^{\mu} + B_{(0)}^{\nu} \partial^2 B_{(0)}^{\nu} \right] \right. \\
= 2g^2 N_c (N_c^2 - 1) \langle \alpha \rangle \langle \alpha \partial^2 \alpha \rangle \\
= 6g^2 \mu_A^2 \partial^2 L(0_{\perp}) \left[ (\partial^2)^2 L(0_{\perp}) - 3g^4 \mu_A^2 (\partial^2 L(0_{\perp}))^2 \right] \\
= -g^6 \mu_A^4 \cdot \frac{3}{2\pi^2} \Lambda^2 \ln \frac{\Lambda}{m} + g^{10} \mu_A^6 \cdot 9 \frac{4}{8\pi^3} \left[ \ln \frac{\Lambda}{m} \right]^3 . \\
(49)
\]

The non-derivative part, on the other hand, gives

\[
- \frac{1}{2} g^2 \left\langle \mathcal{T} \right. \left[ E_{(0)}^{\mu} [A_{(0)}], [A_{(0)}], E_{(0)}^{\mu} \right] + B_{(0)}^{\nu} [A_{(0)}], [A_{(0)}], B_{(0)}^{\nu} \right] \right. \\
= -7g^4 N_c^2 (N_c^2 - 1) \langle \alpha \rangle^3 = -g^{10} \mu_A^6 \frac{8}{8\pi^3} \left[ \ln \frac{\Lambda}{m} \right]^3 . \\
(50)
\]

The sum of these gives rise to the second-order term of the longitudinal field energy:

\[
\varepsilon_{(2)}^L = -g^6 \mu_A^4 \frac{3}{2\pi^2} \Lambda^2 \ln \frac{\Lambda}{m} - g^{10} \mu_A^6 \frac{45}{8\pi^3} \left[ \ln \frac{\Lambda}{m} \right]^3 . \\
(51)
\]

2. Transverse fields

The computation for the transverse fields is parallel to the previous case. Since the Hamiltonian \( (\mathcal{O}(\tau^2) \) possesses \( 1/\tau^2 \) in front of the transverse fields, the square of the \( \mathcal{O}(\tau^2) \) fields contributes to the energy at \( \mathcal{O}(\tau^2) \). Note that there is no \( \mathcal{O}(\tau^0) \) transverse field. We thus evaluate,

\[
\varepsilon_{(2)}^T = \frac{1}{4} \left\langle \mathcal{T} \right. \left[ E_{(2)}^{\mu} E_{(2)}^{\mu} + B_{(2)}^{\nu} B_{(2)}^{\nu} \right] \right. \\
= \frac{1}{4} \left\langle \mathcal{T} \right. \left[ D_{(0)}^{\mu} E_{(0)}^{\mu} D_{(0)}^{\mu} E_{(0)}^{\mu} + D_{(0)}^{\nu} B_{(0)}^{\nu} D_{(0)}^{\nu} B_{(0)}^{\nu} \right] \right. . \\
(52)
\]

The derivative terms give

\[
\frac{1}{4} \left\langle \mathcal{T} \right. \left[ \partial_\mu E_{(0)}^{\nu} \partial_\mu E_{(0)}^{\nu} + \partial_\mu B_{(0)}^{\nu} \partial_\mu B_{(0)}^{\nu} \right] \right. \\
= g^2 N_c (N_c^2 - 1) \langle \alpha \rangle \langle \partial_\mu \alpha \partial_\mu \alpha \rangle \\
= -3g^6 \mu_A^4 \partial^2 L(0_{\perp}) \left[ (\partial^2)^2 L(0_{\perp}) - 3g^4 \mu_A^2 (\partial^2 L(0_{\perp}))^2 \right] \\
= g^6 \mu_A^4 \cdot \frac{3}{4\pi^2} \Lambda^2 \ln \frac{\Lambda}{m} - g^{10} \mu_A^6 \cdot 9 \frac{4}{8\pi^3} \left[ \ln \frac{\Lambda}{m} \right]^3 , \\
(53)
\]

which is nothing but a negative half of the corresponding derivative part in the longitudinal fields. We can also
check that the non-derivative terms are a negative half of the longitudinal counterparts as well. Hence we find
\[ \varepsilon_{(2)}^T = -\frac{1}{2}\varepsilon_{(2)} \] (54)

VII. TIME EVOLUTION – STEP III)

We must take the limit of \( \Lambda \to \infty \) in the end because the UV singularity is not physical in contrast to the IR property which is physical. In the situation with large \( \Lambda \) only the derivative terms \( \propto \Lambda^2 \) are predominant in the \( \mathcal{O}(\tau^2) \) corrections. We shall therefore drop the non-derivative contributions for the moment. In fact we could take account of those effects and find them negligible in the early-time region.

A. Energy Density

In Ref. [18] one of the authors proposed the following logarithmic ansatz to resum the UV diverging terms:
\[ \varepsilon(\tau) \simeq \varepsilon(0) + \varepsilon(2) \tau^2 \]
\[ \simeq g^6 \mu_A^4 \cdot \frac{3}{4\pi^2} \ln \frac{\Lambda^2}{m^2} \left( \frac{\ln \frac{\Lambda^2}{m^2} - \frac{1}{2} \Lambda^2 \tau^2}{m^2 + \frac{1}{4} m^2 \Lambda^2 \tau^2} \right) \]
\[ \to g^6 \mu_A^4 \cdot \frac{3}{4\pi^2} \left( \ln \frac{\Lambda^2}{m^2 + \frac{1}{4} m^2 \Lambda^2 \tau^2} \right)^2 . \] (55)

A nice feature of this ansatz is that the \( \Lambda \to \infty \) limit is well defined, i.e.,
\[ \varepsilon(\tau) = \frac{1}{g^2} (g^2 \mu_A)^4 \frac{3}{\pi^2} \left[ \ln \frac{2}{m\tau} \right]^2 . \] (56)

This is a concise pocket formula. We will later confirm that Eq. (55) works well to give a nice approximation to the perturbative time evolution of the Bessel function.

In the rest of this section let us elaborate the correct resummation of the the highest-order divergent terms at each order of \( \tau \). The idea is simple: We have understood that the UV singularity is attributed to the spatial derivative. Then, it is natural to anticipate that the solution of the equations of motion with the most singular spatial derivatives retained resums the UV singularity. This is indeed the case, as is explicitly done here.

The highest-derivative contribution is the solution of the equations of motion:
\[ \partial_\tau E^i = \tau \partial_\tau \left( \partial_\tau A_i - \partial_i A_\tau \right) = \tau \partial^2 P^T_{ij} A_j , \] (57)
\[ \partial_\tau E^\eta = \frac{1}{\tau} \partial^2 A_\eta . \] (58)

From the above differential equations for \( A_i \) and \( A_\eta \), the time dependence is deduced as
\[ A_i(\tau, k_\perp) = A_{i(0)}(k_\perp) J_0(k_\perp \tau) , \]
\[ A_\eta(\tau, k_\perp) = A_{\eta(2)}(k_\perp) \frac{2\tau}{k_\perp} J_1(k_\perp \tau) . \] (59)

We have dropped the transverse projection \( P^T_{ij} \) because the initial condition \( E_{i(2)}^{(2)} \) given as Eq. (59) is transverse in the leading order of the spatial derivative. We can find \( E^i \) and \( E^\eta \) with the use of the definition of the canonical momenta \( \mathcal{E}_i \) and \( \mathcal{E}_\eta \). In the leading order of the spatial derivative we have \( B^\eta = \partial_1 A_2 - \partial_2 A_1 \) and \( B^i = \epsilon^i j \partial_j A_\eta \), so that \( B^\eta \) and \( B^i \) have the time-dependence same as \( \mathcal{E}_i \) and \( \mathcal{E}_\eta \), respectively. In summary, we have
\[ E^\eta(\tau, k_\perp) = E_{\eta(0)}(k_\perp) J_0(k \tau) , \]
\[ E^i(\tau, k_\perp) = E_{i(2)}(k_\perp) \frac{2\tau}{k_\perp} J_1(k \tau) , \]
\[ B^n(\tau, k_\perp) = B^n_{(0)}(k_\perp) J_0(k \tau) , \]
\[ B^i(\tau, k_\perp) = B^i_{(2)}(k_\perp) \frac{2\tau}{k_\perp} J_1(k \tau) . \] (60)

It follows that the longitudinal field energy component can be summed up to be
\[ \varepsilon^T(\tau, k_\perp) = \frac{6}{\pi m^2} g^6 \mu_A^4 T(\tau, k_\perp/m) \left[ J_0(k \tau) \right]^2 \] (61)
with \( N_c = 3 \) substituted. It is easy to see that the \( \mathcal{O}(\tau^2) \) term exactly corresponds to the \( \Lambda^2 \) terms in \( \varepsilon^T_{(2)} \) if expanded. Likewise, the transverse field energy becomes
\[ \varepsilon^T(\tau, k_\perp) = \frac{6}{\pi m^2} g^6 \mu_A^4 T(\tau, k_\perp/m) \left[ J_1(k \tau) \right]^2 , \] (62)
whose coefficient is fixed so as to reproduce Eq. (54) in the \( \tau \)-expansion. The total energy density at each moment thus evolves as
\[ \varepsilon(\tau, k_\perp) = \varepsilon^{(0)}(k_\perp) \left( [J_0(k \tau)]^2 + [J_1(k \tau)]^2 \right) ; \] (63)
and the integrated one is
\[ \varepsilon(\tau) = \frac{1}{2\pi} \int d\xi \xi T(\tau, \xi) \left\{ [J_0(\xi m \tau)]^2 + [J_1(\xi m \tau)]^2 \right\} , \] (64)
where we have defined a dimensionless function,
\[ I_E \left( a; m \tau \right) = \int_0^\infty d\xi \xi T(\tau, \xi) \left\{ [J_0(\xi m \tau)]^2 + [J_1(\xi m \tau)]^2 \right\} . \] (65)

Although our derivation is rather straightforward, this time dependence according to the Bessel function is exactly the same as the one given in Ref. [21]. It should be noted here that this Bessel function form has been discussed in the earliest work in Ref. [8] and confirmed in the numerical simulation in Ref. [12] in the weak field limit. However, we do not assume the weak field limit at all in our argument but the UV dominance at \( \tau \ll 1/Q_s \) is sufficient to justify this time dependence. The same evolution is also found in the Abelian model in Ref. [19].

Our expression for the initial energy density \( \varepsilon(0) \) is a function of the proper time \( \tau \), the QCD coupling \( g \), the
Interestingly enough, the sum of two is a smooth function of \( x \) to \( 2/(\pi x) \) at large \( x \). The dotted curve starting from 1 represents \([J_0(x)]^2\) and the other one starting from 0 represents \([J_1(x)]^2\).

MV model scale \( g^2 \mu_A \), and the IR cutoff \( m \) or its ratio \( a \) to \( g^2 \mu_A \). The Bessel function \([J_0(k_{\perp} \tau)]^2 + [J_1(k_{\perp} \tau)]^2\) encompasses the time dependence which we plot in Fig. 4. Interestingly enough, the sum of two is a smooth function, though each of \([J_0(k_{\perp} \tau)]^2\) and \([J_1(k_{\perp} \tau)]^2\) is an oscillatory function of \( k_{\perp} \tau \). This is reminiscent of the trigonometric function. Moreover, we can prove that

\[
[J_0(k_{\perp} \tau)]^2 + [J_1(k_{\perp} \tau)]^2 \to \frac{2}{\pi k_{\perp} \tau},
\]

asymptotically in the \( k_{\perp} \tau \to \infty \) limit. This analytical feature results in the behavior of \( \varepsilon(\tau \to \infty) \propto 1/\tau \) that is the same as the free-streaming expansion [21].

B. Gluon Distribution

As for the gluon distribution, we shall adopt here a working definition in accord with the preceding works [8, 12, 13, 14, 33]. That is, we assume a harmonic oscillator and count the number of quanta. The concrete procedure is that we divide the field energy with momentum \( k_{\perp} \) by the (free) gluon energy quanta \( k_{\perp} \) to infer the gluon distribution contained in the fields, which is consistent with the multiplicity computation by the reduction formula [34]. We define in this way the gluon distribution as

\[
n(\tau, k_{\perp}) = \frac{1}{\sqrt{k_{\perp}^2 + m^2}} \varepsilon(\tau, k_{\perp}),
\]

where we put the gluon mass \( m \). From this we readily reach the expression for the gluon distribution as

\[
n(\tau) = \frac{1}{2\pi} \int dk_{\perp} n(\tau, k_{\perp}) = \frac{3}{\pi^2 m} \cdot \frac{1}{g^2} (g^2 \mu_A)^4 I_N(a; m\tau),
\]

We are now ready to give a numerical estimate for the initial energy and the multiplicity from our formulation. The key equations are Eqs. (64), (65), (68), and (69). From these expressions we can write the total energy and the multiplicity per rapidity as

\[
\frac{dE(\tau)}{d\eta} = \pi R_A^2 \tau(\tau)
\]

\[
= \frac{3\pi R_A^2}{\pi^2 m} \cdot \frac{1}{g^2} (g^2 \mu_A)^4 (m\tau) I_E(a; m\tau),
\]

\[
\frac{dN(\tau)}{d\eta} = \pi R_A^2 \tau(\tau)
\]

\[
= \frac{3\pi R_A^2}{\pi^2 m^2} \cdot \frac{1}{g^2} (g^2 \mu_A)^4 (m\tau) I_N(a; m\tau).
\]

We plot \((m\tau) I_E(a; m\tau)\) and \((m\tau) I_N(a; m\tau)\) appearing in the above expressions in Fig. 6 for various \(a\) using \( T(a; k_{\perp}/m) \) shown in Fig. 9. It is clear that both \((m\tau) I_E(m\tau)\) and \((m\tau) I_N(m\tau)\) behave approaching constant for large \(m\tau\) as we have mentioned before.

The validity of the CGC (classical field) approximation breaks down when the system becomes dilute near \( g^2 \mu_A \tau \sim m\sqrt{\alpha} \tau \sim 1 \). In the very early stage of the evolution, on the other hand, the UV modes dominate in

FIG. 5: \([J_0(x)]^2 + [J_1(x)]^2\) as a function of \( x \) which goes to zero as \( 2/(\pi x) \) at large \( x \). The dotted curve starting from 1 represents \([J_0(x)]^2\) and the other one starting from 0 represents \([J_1(x)]^2\).

FIG. 6: \((m\tau) I_E(m\tau)\) and \((m\tau) I_N(m\tau)\) used in Eqs. (70) and (71), respectively, as a function of \( m\sqrt{\alpha} \tau \) for various \(a\).
the physical quantities and the logarithmic ansatz \[ (\ln[\Lambda^2/m^2])^2 \] works quite well. We see this fact from the comparison in Fig. 7 which is the same plot as Fig. 6 without the factor \( m \tau \). This is quite a non-trivial statement. We started from \( \tau = 0 \) with the exact initial condition provided by the MV model. Then we found that, as long as \( m\sqrt{a} \tau \) is much smaller than unity (or simply \( m \tau < 1 \)), only the UV modes control the time evolution of the energy and there is only little effect from the IR sector which changes with \( a \). Hence, the remaining \( a \) dependence is only the trivial scaling in terms of \( m \) with \( m \sqrt{a} \tau \) fixed as long as \( \tau \) is small.

From Fig. 6 we can give physical numbers for the energy and the multiplicity deduced in the 1/\( \tau \)-scaling region. We first set \( \pi R_A^2 \approx 150 \text{ fm}^2 \) and \( g \approx 2 \) which are less ambiguous.

In the case of \( a = 10 \), we have obtained \( m \approx 0.64Q_s \) and \( (g^2\mu_A)^2 \approx 2.70Q_s^2 \). The asymptotic values of \((m\tau)I_E(a,m\tau)\) and \((m\tau)I_N(a,m\tau)\) at \( \tau \to \infty \) are respectively 1.43 and 0.47. Thus, we have \( dE/d\eta = 1.4 \times 10^4 \text{ GeV} \) and \( dN/d\eta = 5.0 \times 10^3 \) in the case of \( Q_s^2 = 2 \text{ GeV}^2 \). If \( Q_s^2 = 1 \text{ GeV}^2 \), \( dE/d\eta \) and \( dN/d\eta \) get smaller by factors 2\( 3/2 \) and 2, respectively. For other values of \( a \), we can get the estimates in the same way. We summarize our results in Table II. We note that \( c \) listed in Table II is the gluon liberation factor defined by \[ \frac{1}{\pi R_A^2} \frac{dN}{d\eta} = c \frac{N_c^2 - 1}{\pi g^2 N_c} Q_s^2. \] (72)

It is interesting to note that the energy is less sensitive to \( a \) in the range of our interest while the multiplicity rises significantly with increasing \( a \). This is because \( a \) controls the IR scale \( m \) to which the average gluon energy \( (dE/d\eta)/(dN/d\eta) \) is proportional. In other words, with increasing \( a \) we include more IR modes, which contribute to \( dN/d\eta \) more than \( dE/d\eta \).

![Figure 7: Comparison between the log-ansatz \((\ln[2/m\tau])^2\) (green dashed curve) and the numerically evaluated \(I_E(m\tau)\) (red solid curves) for \( a = 10, 25, 100, \) and 500 from the top to the bottom.](image)

| \( a \) | \( dE/d\eta \) (GeV) | \( dN/d\eta \) | \( c \) |
|---|---|---|---|
| 10 | \((4.9 - 14) \times 10^3\) | \((2.5 - 5.0) \times 10^3\) | 3.04 |
| 25 | \((4.9 - 14) \times 10^3\) | \((4.0 - 7.9) \times 10^3\) | 4.86 |
| 100 | \((3.5 - 10) \times 10^3\) | \((5.6 - 11) \times 10^3\) | 6.88 |
| 500 | \((2.6 - 7.3) \times 10^3\) | \((7.7 - 16) \times 10^3\) | 9.47 |

TABLE II: Total energy and gluon distribution estimated in our formulation for various \( a \) with the choice of \( Q_s^2 = 1 - 2 \text{ GeV}^2 \). [Lower and upper values correspond to \( Q_s^2 = 1 \text{ GeV}^2 \) and \( Q_s^2 = 2 \text{ GeV}^2 \), respectively.] We also list the gluon liberation factor \( c \).

VIII. IMPROVEMENT – STEP IV

Let us consider here a possible improvement of our estimate. The UV dominance cannot last at later time, and the nonlinear effect may become manifest in the intermediate stage of the evolution. We then need include less singular terms from the nonlinear interactions appearing in Eq. (51). In order to estimate the size of this nonlinear effect we take the following strategy: In the Gaussian approximation we adopt here, the nonlinear terms in the Yang-Mills equations of motion produce the mean-field contributions like the mass term, that would modify Eqs. (57) and (58) as

\[
\partial_\tau E^{(1)} = \tau (\partial^2 - \kappa_\tau g^2 N_c \langle \alpha \alpha \rangle) A^T_i, \quad (73)
\]

\[
\partial_\tau E^{(2)} = \frac{1}{\tau} (\partial^2 - \kappa_\eta g^2 N_c \langle \alpha \alpha \rangle) A^\eta, \quad (74)
\]

where \( \kappa_\tau \) and \( \kappa_\eta \) are the coefficients coming from contractions of the gauge fields: \( A^T_i A^T_j A^\eta \rightarrow \delta^{ij} \delta \langle \alpha \alpha \rangle A^\eta \) and \( A^T_i A^\eta A^\eta \rightarrow \delta^{ij} \delta \langle \alpha \alpha \rangle A^\eta \), etc. Noting that the initial \( A^\eta \) has no correlation with \( A^T_i \) in the leading order of the derivative, on the one hand, we find \( \kappa_\tau = \kappa_\eta = 2 \) by these replacements in the equations of motion, though this might underestimate the effect. On the other hand, if we determine \( \kappa_\tau \) and \( \kappa_\eta \) to reproduce the expanded energy in Eqs. (51) and (54) up to \( O(\tau^2) \), we conclude \( \kappa_\tau = \kappa_\eta = 7 \) which we shall adopt here.

In Eqs. (73) and (74), we have estimated \( \langle A_i A_j \rangle \) at \( \tau = 0 \) and we did not include \( \langle A_i A^\eta \rangle \) which is \( O(\tau^4) \). From Eqs. (20) and (22) we know that \( g^2 N_c \langle \alpha \alpha \rangle = (N_c/4\pi)(g^2\mu_A)^2 \ln[\Lambda/m] \) is finite \( \tau \) (see Fig. 7), and it takes the value around \( \ln 20 \approx 3 \) to \( \ln 200 \approx 5.3 \) for \( m \tau = 0.01 \sim 0.1 \) of our interest (so that \( m\sqrt{a} \tau \) is below unity). During this interval, therefore, we may replace this logarithmic function by a constant of order \( O(1) \sim O(10) \), in effect. Let us assume rather arbitrarily the value \( \tau \) for \( \ln[\Lambda/m] \), which is within the range from 3 to 5.3, so that the mean-field effect appears only through \( k^2_\perp \rightarrow k^2_\perp + 3.5m^2a \) in the argument of the Bessel function.

![Figure 8: Shows how \((m\tau)I_E(a;m\tau)\) changes due to the replacement of \( k^2_\perp \rightarrow k^2_\perp + 3.5m^2a \). We summarize our](image)
FIG. 8: Comparison between the log-ansatz \((m\tau)(\ln[2/m\tau])^2\) (green dashed curve) and the numerically evaluated \((m\tau)I_E(a;m\tau)\) with (red solid curves) and without (black dotted curves) \(k_\perp^2 \to k_\perp^2 + 3.5m^2a\) for \(a = 10, 25, 100,\) and 500 from the top to the bottom.

| \(a\) | \(dE/d\eta\) (GeV) | \(dN/d\eta\) | \(c\) |
|------|----------------|-------------|------|
| 10   | \((2.8 - 7.8) \times 10^3\) | \((0.7 - 1.4) \times 10^3\) | 0.86 |
| 25   | \((2.5 - 7.0) \times 10^3\) | \((0.9 - 1.8) \times 10^3\) | 1.10 |
| 100  | \((1.6 - 4.5) \times 10^3\) | \((1.0 - 1.9) \times 10^3\) | 1.18 |
| 500  | \((1.0 - 2.9) \times 10^3\) | \((1.0 - 1.9) \times 10^3\) | 1.17 |

TABLE III: Total energy and gluon distribution estimated in our formulation with nonlinear terms for various \(a\) with the choice of \(Q_s^2 = 1 - 2\) GeV\(^2\).

IX. DISCUSSIONS AND OUTLOOKS

We have shown by explicit calculations in the MV model that the UV components are dominant in the energy density at \(\tau = 0\) produced by the collisions. From this observation we have evolved the exact initial energy provided by the MV model via the free field equation. We did not require the assumption of small field amplitude, but the UV dominance allows us to drop nonlinear terms in the early time evolution.

To extend the validity region of the estimate, we have augmented the time evolution with the effect of nonlinear terms in the Gaussian-type approximation. Assuming that the IR cutoff \(m\) arises from the quantum evolution \(m \simeq q_{mA}\) or from non-perturbative dynamics \(m \simeq \Lambda_{QCD}\), we have the IR parameter \(a = 10 - 40\). In this range of \(a\), we have found that the nonlinear terms tend to reduce the \(a\)-dependence of the physical observables. Let us then take \(a = 25\). Regarding the relevant saturation scale, we take \(Q_s^2 = 1\) GeV\(^2\) here because the discussion in Ref. [11] suggests that \(Q_s^2\) is close to 1 GeV\(^2\) than 2 GeV\(^2\). Thus, what we get for a realistic situation at RHIC is \(dE/d\eta \simeq 2.5 \times 10^3\) GeV and \(dN/d\eta \simeq 0.9 \times 10^3\) with \(c = 1.10\). Our result favors a larger value for \(c\) than the early numerical simulations [12], but is consistent with the analytical calculation \(c \approx 2\ln 2 \approx 1.4\) [37] and agrees well with the latest numerical evaluation [11, 32].

Our calculation seems to underestimate the total multiplicity whose empirical value is \(\simeq 1150\) at RHIC. We should, however, be aware that \(dN/d\eta\) here signifies the total number of gluons produced initially. Therefore, though \(dN/d\eta\) should not be far from the observed multiplicity, one should not expect a perfect quantitative agreement. In this sense the level of the agreement we obtained is quite acceptable and we would say that our results are consistent with the empirical values.

A rough estimate of the average energy per gluon is \((dE/d\eta)/(dN/d\eta) \simeq 2.8\) GeV. This value is greater than the largest energy anticipated in Ref. [13], above which the CGC description might not make sense. This speculation is not quite correct, however. As is known and also clear in Fig. 4 the MV model is smoothly connected to the leading-order perturbative calculation with increasing \(k_\perp\), so that the spectrum has a perturbative tail in the high momentum region and the gluon energy is not bounded by the saturation scale. In contrast, the nonlinear effect of the CGC will be the most important for the gluons with momentum of order of \(Q_s\), and the average energy per gluon scales with \(Q_s\), which is indeed the case in our formulae.

It is also intriguing to mention on the “formation time”, which can roughly read from Fig. 5 for the \(a = 25\) case. We see that the curve gets flattened around \(m\tau \simeq 0.2\). Since \(m = 0.36Q_s\), the “formation time” denoted as \(\tau_D\) which controls the convergence of \(\tau(\tau)\) (see discussions in the second paper of Ref. [12]) is estimated as \(\tau_D \simeq 0.56/Q_s \simeq 0.1\) fm or \(g^2\Lambda_{QCD}\tau \simeq 0.8\). This is pretty short as compared with the expected thermalization time \(\lesssim 1\) fm. This quantitative estimate of the “formation time” might deserve further investigation in the context of the early thermalization problem.

The future applications, in addition to the early thermalization problem, cover various aspects of phenomenology as follows. 1) The rapidity dependence with changing \(Q_s(x)\) gives the whole multiplicity distribution. 2) Finite size \(Q_s^{1(2)}(x, \perp)\) in the transverse plane yields the initial eccentricity at finite impact parameter. 3) The energy-momentum tensor specifies the initial condition connected to the kinematic or hydrodynamic description. 4) The (augmented) UV time evolution of the background fields is useful to attack the rapidity dependent instability on top of them [28]. What is addressed here is the starting point for all of these issues. We note that the point 2) is especially important because the initial eccentricity obtained in the KLN model [40, 39] is prevailing as the CGC estimate at present [40], though the confirmation from the MV model is definitely neces-
Finally, we close our discussions with a conclusion that our analytical estimates turn out to be consistent with the numerical simulations in spite of the difference in their model definitions. This is an important check for the MV model foundation.

Acknowledgments

The authors thank Raju Venugopalan for discussions. The work of H. F. is supported in part by Grants-in-Aid (19540273, 19540269) of MEXT. K. F. is supported by Japanese MEXT grant No. 20740134 and also supported in part by Yukawa International Program for Quark Hadron Sciences (YIPQS). Y. H. is supported in part by the RIKEN BNL Research Center and by the U.S. Department of Energy under Cooperative Research Agreement No. DE-AC02-98CH10886.

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