Morita Equivalence of $W^*$-Correspondences and Their Hardy Algebras

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Abstract
Muhly and Solel developed a notion of Morita equivalence for $C^*$-correspondences, which they used to show that if two $C^*$-correspondences $E$ and $F$ are Morita equivalent then their tensor algebras $T_+(E)$ and $T_+(F)$ are (strongly) Morita equivalent operator algebras. We give the weak* version of this result by considering (weak) Morita equivalence of $W^*$-correspondences and employing Blecher and Kashyap’s notion of Morita equivalence for dual operator algebras. More precisely, we show that weak Morita equivalence of $W^*$-correspondences $E$ and $F$ implies weak Morita equivalence of their Hardy algebras $H^\infty(E)$ and $H^\infty(F)$. We give special attention to $W^*$-graph correspondences and show a number of results related to their Morita equivalence.

Keywords  Hardy algebras · Morita equivalence · $W^*$-correspondence · Graph correspondence

Mathematics Subject Classification Primary 16D90 · 30H10 · 47L80 · 47L45

1 Introduction
Given a von Neumann algebra $A$ and a $W^*$-correspondence $E$ over $A$, Muhly and Solel constructed an algebra $H^\infty(E)$ which they called the Hardy algebra of $E$ [12]. This algebra is a noncommutative generalization of the classic Hardy algebra $H^\infty(\mathbb{T})$ of bounded analytic functions on the open unit disc. More precisely, when $E = A = \mathbb{C}$, $H^\infty(E)$ is the classical Hardy space $H^\infty(\mathbb{T})$. When $A = \mathbb{C}$ and $E = \mathbb{C}^n$, $H^\infty(E)$ is the free semigroup algebra $\mathcal{L}_n$ studied by Popescu [16], Davidson and Pitts [5] and others. This Hardy algebra is a dual operator subalgebra of $\mathcal{L}(\mathcal{F}(E))$, the adjointable
operators of the Fock space of $E$, generated by diagonal and creation operators. When $E$ is a correspondence derived from a directed graph $G$, $H^\infty(E)$ is a dual operator algebra version of what algebraists call the path algebra of $G$.

Kiiti Morita’s 1958 groundbreaking paper [9] contains the main ideas of what later became known as Morita equivalence, an extremely important concept in the study of the algebraic structure of rings. Following the dissemination of Morita’s ideas, mainly by H. Bass and P. Gabriel in the early 1960s, many other notions of Morita equivalence have been developed, including notions of Morita equivalence for selfadjoint algebras, operator algebras, groupoids, group $*$-algebras, finite groups, Poisson manifolds, non commutative smooth tori, tensor categories, semigroups and star products. In [11], Muhly and Solel introduced a notion of (strong) Morita equivalence for dual operator algebras. Such notions were developed to show that if two dual operator algebras are (weakly) Morita equivalent then their Hardy algebras and their representations. We show that if $(E, A)$ is a $W^*$-graph correspondence, which they used to show that if two $C^*$-correspondences $E$ and $F$ are (strongly) Morita equivalent then their tensor algebras $T_+(E)$ and $T_+(F)$ are (strongly) Morita equivalent operator algebras. At that time however, there was no clear notion of Morita equivalence for dual operator algebras. Such notions were developed ten years later in the work of Blecher, Kashyap, Eleftherakis and Paulsen ([1,7,8]). Motivated by Muhly and Solel’s work, we consider (weak) Morita equivalence of $W^*$-correspondences, and use Blecher and Kashiap’s notion of Morita equivalence for dual operator algebras to show that if two $W^*$-correspondences $E$ and $F$ are (weakly) Morita equivalent then their Hardy algebras $H^\infty(E)$ and $H^\infty(F)$ are (weakly) Morita equivalent dual operator algebras.

In the last section, we concentrate on Morita equivalence of $W^*$-graph correspondences, their Hardy algebras and their representations. We show that if $(E, A)$ is a $W^*$-graph correspondence then any two faithful normal representations $\sigma$ and $\tau$ of $A$ give rise to Morita equivalent dual correspondences $(E^\sigma, \sigma(A)')$ and $(E^\tau, \tau(A)')$. Then we consider the induced representations $\sigma^{\mathcal{F}(E)}$ and $\tau^{\mathcal{F}(E)}$ of the Hardy algebra $H^\infty(E)$ and show that the commutants of $\sigma^{\mathcal{F}(E)}(H^\infty(E))$ and $\tau^{\mathcal{F}(E)}(H^\infty(E))$ are (weakly) Morita equivalent dual operator algebras. We also study equivalence bimodules and the relation between graphs and the Morita equivalence of their $W^*$-correspondences.

2 Preliminaries

A right $C^*$-module $E$ over a $C^*$-algebra $A$ is said to be selfdual if every continuous $A$-module map $f : E \to A$ is of the form $f(\cdot) = \langle y, \cdot \rangle$, for some $y \in E$. We say that $E$ is a right $W^*$-module if $E$ is a selfdual right $C^*$-module over a $W^*$-algebra. We write $\mathcal{L}_A(E)$ (or simply $\mathcal{L}(E)$) for the space of adjointable $A$-module maps on $E$. An $A \to B$ $W^*$-correspondence is a right $W^*$-module $E$ over $B$ for which there exists a unital normal $*$-homomorphism $\varphi : A \to \mathcal{L}_B(E)$. We then say that $E$ is a $W^*$-correspondence from $A$ to $B$, and we denote it by $A \! \otimes B$. If $A = B$ then we say that $E$ is a $W^*$-correspondence over $A$. In this case, we might also denote the correspondence by $(E, A)$. The center of a $W^*$-correspondence $(E, A)$ is the set $\mathcal{Z}(E) = \{ x \in E : a \cdot x = x \cdot a \text{ for all } a \in A \}$. We will sometimes abbreviate “weak*” to “$w^*$.”
The $W^*$-module tensor product $\overline{\otimes}_A$ (sometimes written as the composition tensor product $X\overline{\otimes}_\sigma Y$) is defined to be the self-dual completion (the weak*-completion) of the $C^*$-module interior tensor product $X \otimes \sigma Y$. When there is no risk of confusion, we will simply write $X\overline{\otimes}Y$. The $W^*$-module tensor product is functorial and associative. If $E$ is a $W^*$-correspondence from $A$ to $B$ and $F$ is a $W^*$-correspondence from $B$ to $C$ then $E\overline{\otimes}AF$ is a $W^*$-correspondence from $A$ to $C$ with inner product given by $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{E\overline{\otimes}F} = \langle y_1, \varphi((x_1, x_2)_E) y_2 \rangle_F$ and left/right actions given by $a \cdot (x \otimes y) \cdot c = (a \cdot x) \otimes (y \cdot c) = (\varphi_E(a)x) \otimes (y \cdot c)$. In particular, given a $W^*$-correspondence $E$ over $A$ and a Hilbert space $H$ equipped with a normal representation $\sigma$ of $A$, we can form the Hilbert space $E\overline{\otimes}_\sigma H$, where we have $\langle x_1 \otimes h_1, x_2 \otimes h_2 \rangle = \langle h_1, \sigma((x_1, x_2)_E) h_2 \rangle$.

A $W^*$-correspondence isomorphism between two $W^*$-correspondences $(E_1, A_1)$ and $(E_2, A_2)$ is a pair $(\sigma, \psi)$ where $\sigma : A_1 \to A_2$ is an isomorphism of $W^*$-algebras and $\psi : E_1 \to E_2$ is a vector space isomorphism, such that for $e, f \in E_1$ and $a, b \in A_1$, we have $\psi(a \cdot e \cdot b) = \sigma(a) \cdot \psi(e) \cdot \sigma(b)$ and $\langle \psi(e), \psi(f) \rangle = \langle \sigma(e), f \rangle$. Such $\psi$ must be weak*-homeomorphic because the predual of a $W^*$-module is unique.

If $A$ and $B$ are $W^*$-algebras, then an $A$-$B$ $W^*$-equivalence bimodule is an $A$-$B$ $W^*$-bimodule $X$ which is a $w^*$-full right $W^*$-module over $B$ and a $w^*$-full left $W^*$-module over $A$, such that the two (left and right) inner products of $X$ are compatible in the sense that $A(x, y) \cdot z = x \cdot (y, z) B$ for all $x, y, z \in X$. If $AX_B$ and $C Y_D$ are $W^*$-equivalence bimodules then a $W^*$-equivalence bimodule isomorphism (as defined in [6, Definition 1.16 and Remark 1.19]) is a triple $(\sigma, \phi, \pi)$, where $\sigma : A \to C$ and $\pi : B \to D$ are $W^*$-algebra isomorphisms and $\phi : X \to Y$ is a vector space isomorphism such that $\phi(a \cdot e \cdot b) = \sigma(a) \cdot \phi(e) \cdot \pi(b)$, $\langle \phi(e), \phi(f) \rangle_D = \pi(\langle e, f \rangle_B)$ and $C \langle \phi(e), \phi(f) \rangle = \sigma(\langle e, f \rangle)$.

Given a representation $\sigma$ of $A$, an operator $T \in \mathcal{L}(E)$ and an operator $S \in \sigma(A)'$, the map $x \otimes h \to Tx \otimes Sh$ defines a bounded operator on $E\overline{\otimes}_\sigma H$ denoted by $T \otimes S$. In particular, the representation of $\mathcal{L}(E)$ resulting from letting $S = I$, is Rieffel’s induced representation of $\mathcal{L}(E)$ induced by $\sigma$. This representation is denoted by $\sigma^E$. That is, $\sigma^E(T) = T \otimes I$. Likewise, we say that the composition $\sigma^E \circ \varphi$ is the representation of $A$ on $E\overline{\otimes}_\sigma H$ induced by $E$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $E$ is a $W^*$-correspondence over a $W^*$-algebra $A$ then we can form the tensor powers $E\overline{\otimes}^n$, $n \geq 0$, where $E\overline{\otimes}^0 = A$. For each $n$, $E\overline{\otimes}^n$ is a $W^*$-correspondence over $A$ with the inner product defined inductively. The ultraweak direct sum $\mathcal{F}(E) := \bigoplus_{n \in \mathbb{N}_0} E\overline{\otimes}^n$ is a $W^*$-correspondence over $A$ called the Fock space over $E$. The left action of $A$ on $\mathcal{F}(E)$ is given by the map $\varphi_\infty$ defined by $\varphi_\infty(a) = \text{diag}(a, \varphi(a), \varphi(2)(a), \varphi(3)(a), \ldots)$ where $\varphi(n)(a)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (\varphi(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n \in E\overline{\otimes}^n$. Given $x \in E$, the creation operator $T_x \in \mathcal{L}(\mathcal{F}(E))$ is defined by $T_x(\eta) = x \otimes \eta$, $\eta \in \mathcal{F}(E))$. That is,

$$
\varphi_\infty(a) = \begin{pmatrix}
\varphi(0)(a) & 0 & \cdots \\
0 & \varphi(1)(a) & \cdots \\
0 & 0 & \cdots
\end{pmatrix}
$$

and

$$
T_x = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
T_x(1) & 0 & \cdots & 0 \\
0 & T_x(2) & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
$$
The tensor algebra over $E$, denoted $T_+(E)$ is defined to be the norm closed subalgebra of $\mathcal{L} (\mathcal{F}(E))$ generated by $\varphi_\infty(A)$ and $\{ T_x : x \in E\}$. The ultraweak closure of $T_+(E)$ in $\mathcal{L} (\mathcal{F}(E))$ is called the Hardy Algebra of $E$, and is denoted by $H^\infty(E)$.

As shown in [12], the completely contractive representations of $H^\infty(E)$ are determined by pairs $(T, \sigma)$ where $\sigma : A \to B(H)$ is a normal $*$-representation of $A$ and $T : E \to B(H)$ is a linear, completely contractive $w^*$-continuous representation of $E$ satisfying $T(abx) = \sigma(a)T(x)\sigma(b)$ for all $x \in E$ and $a, b \in A$. The linear map $\tilde{T}$ defined on the algebraic tensor product $E \otimes H$ by $\tilde{T}(x \otimes h) = T(x)h$ extends to an operator of norm at most 1 on the completion $E \overline{\otimes}_\sigma H$. The pairs $(T, \sigma)$ are called the completely contractive covariant representations of $E$. The bimodule property of $T$ is equivalent to the equation $\tilde{T} (\sigma(\varphi(a)) = \tilde{\varphi}(\varphi(a) \otimes I) = \sigma(a)\tilde{T}$ for all $a \in A$, which means that $\tilde{T}$ intertwines the representations $\sigma$ and $\sigma^E \circ \varphi$ of $A$ on $H$ and $E \otimes H$ respectively. The space composed of all these intertwiners is called the intertwinning space, and it is usually denoted as $\mathcal{I}(\sigma^E \circ \varphi, \sigma) \sigma^\ast$. The space $\sigma^\ast$ is itself a $W^\ast$-correspondence over $\sigma(A)'$ with the actions given by $a \cdot \eta = (I_E \otimes a)\eta$ and $\eta \cdot a = \eta a$ for $\eta \in E$ and $a \in A$. The $\sigma(A)'$-valued inner product is given by $\langle \eta, \xi \rangle = \eta^\ast \xi$.

A dual operator algebra is an operator algebra $A$ which is also a dual operator space. Any weak$^*$-closed subalgebra of $B(H)$ is a dual operator algebra and conversely, for any dual operator algebra $A$, there is a Hilbert space $H$ and a $w^*$-continuous completely isometric homomorphism $\varphi : A \to B(H)$. By the Krein-Smulian theorem, $\varphi(A)$ is a weak$^*$-closed subalgebra of $B(H)$, so we can identify $A$ with $\varphi(A)$ as dual operator algebras. A normal representation of a dual operator algebra is a completely contractive, $w^*$-continuous homomorphism $\varphi : A \to B(H)$. The category of normal representations of $A$ is denoted by $A\mathcal{M}$. The objects of $A\mathcal{M}$ are pairs $(H, \varphi)$ where $H$ is a Hilbert space and $\varphi : A \to B(H)$ is a unital completely contractive, $w^*$-continuous homomorphism. If $(H_i, \varphi_i)$, $i = 1, 2$, are objects in $A\mathcal{M}$, the morphisms are given by $\text{Hom}_A (H_1, H_2) = \{ T \in B(H_1, H_2) : T \varphi_1(a) = \varphi_2(a)T, a \in A\}$.

If $A$ and $B$ are dual operator algebras, a dual operator $A-B$-bimodule is a non-degenerate operator $A-B$-bimodule $X$, which is also a dual operator space, such that the module actions are separately weak$^*$-continuous. If $X$ and $Y$ are right operator modules over $B$, then we write $CB_B(X, Y)$ for the set of completely bounded right $B$-module maps from $X$ to $Y$. If $X$ and $Y$ are left operator modules over $A$, then we write $AC_B(X, Y)$ for the set of completely bounded left $A$-module maps from $X$ to $Y$. Similarly, we write $w^*CB_B(X, Y)$ for the set of $w^*$-continuous completely bounded right $B$-module maps from $X$ to $Y$.

### 3 Morita Equivalence of $W^*$-Correspondences and Hardy Algebras

In 2000, Muhly and Solel introduced a notion of (strong) Morita equivalence for $C^*$-correspondences [11, Definition 2.1]. This notion can be extended to $W^*$-correspondences in the following way: $W^*$-correspondences $A_E A$ and $B_FB_B$ are called (weakly) Morita equivalent if the $W^*$-algebras $A$ and $B$ are weakly Morita equivalent via a $W^*$-equivalence bimodule $X$ for which there is an $A-B$ $W^*$-correspondence iso-
morphism $W$ from $X \otimes_B F$ onto $E \otimes_A X$. In this case, we will write $E \xrightarrow{WME} X F$. Recall that $X \otimes_B F$ and $E \otimes_A X$ are the self dual completions of the balanced $C^*$-module interior tensor products and that the $C^*$-module interior tensor product coincides with the module Haagerup tensor product. Throughout this section, $A E_A$ and $B F_B$ are $W^*$-correspondences over the $W^*$-algebras $A$ and $B$.

In [4], Blecher, Muhly and Paulsen generalized Rieffel’s strong Morita equivalence of $C^*$-algebras ([18]) to general operator algebras. Their generalization is a natural variation of the theory of Morita equivalence that one finds in pure algebra, where the description of Morita equivalence is given in terms of Morita contexts (these contexts are also found in the pure algebra literature under the name: sets of pre-equivalence data). Their definition is the following: Let $A$ and $B$ be unital or approximately unital operator algebras. Let $X$ be an $A$-$B$ operator bimodule, and let $Y$ a $B$-$A$ operator bimodule. Let $(\cdot, \cdot)$ be a completely bounded bilinear map from $X \times Y$ to $A$, balanced over $B$. Let $[\cdot, \cdot]$ be a completely bounded bilinear map from $Y \times X$ to $B$, balanced over $A$. The 6-tuple $(A, B, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ is called a (strong) Morita context for $A$ and $B$ if the module actions are completely contractive and:

- $(x_1, y) \cdot x_2 = x_1 \cdot (y, x_2)$, $x_1, x_2 \in X$, $y \in Y$.
- $[y_1, x] \cdot y_2 = y_1 \cdot (x, y_2)y_1$, $y_2 \in Y, x \in X$.
- The linear map from $X \otimes_Y Y$ to $A$ determined by $(\cdot, \cdot)$ is a complete quotient map onto $A$.
- The linear map from $Y \otimes_X X$ to $B$ determined by $[\cdot, \cdot]$ is a complete quotient map onto $B$.

As shown in [4], a (strong) Morita context determines a pair of equivalence functors between the categories of operator modules of both operator algebras in the context. It also determines an equivalence between the categories of Hilbert modules of both operator algebras. Furthermore, the Morita context gives rise to an isomorphism between the lattices of ideals of both operator algebras in the context. One important shortcoming of this notion of strong Morita equivalence is that if the two operator algebras $A$ and $B$ that we are comparing, are dual operator algebras then the strong Morita context does not capture this duality. More precisely, the two functors that are derived from the context, do not give an equivalence between the categories of normal representations of $A$ and $B$.

In [7], Eleftherakis formulated a version of Morita theory for dual operator algebras using ternary rings of operators (TROs) and a relation called $\Delta$-equivalence. In [8], Eleftherakis and Paulsen showed that this notion of $\Delta$-equivalence is equivalent to the notion of weak* stable isomorphism of dual operator algebras. In [1], Blecher and Kashyap introduced a new notion of weak Morita equivalence of dual operator algebras which includes most of the examples of Morita-like equivalence (in the dual setting) found in the literature. Also, this approach contains the notion of stable isomorphism given by Eleftherakis and Paulsen. That is, if two unital dual operator algebras are weak* stably isomorphic then they are weakly Morita equivalent in the sense of [1].

In the following definition [1, Definition 3.2], $A$ and $B$ are dual unital operator algebras, $X$ and $Y$ are dual operator bimodules, $X$ is an $A$-$B$ bimodule and $Y$ is an $B$-$A$-bimodule. Furthermore, assume that $(\cdot, \cdot) : X \times Y \to A$ and $[\cdot, \cdot] : Y \times X \to B$ are separately weak*-continuous completely contractive bilinear maps. We will use the 6-
We say that $A$ is weakly Morita equivalent to $B$ if there exists a 6-tuple as above, there exist $w^*$-dense operator algebras $A'$ and $B'$ in $A$ and $B$ respectively, there exists a $w^*$-dense operator $A'-B'$-submodule $X'$ in $X$, and a $w^*$-dense operator $B'-A'$-submodule $Y'$ in $Y$, such that the subcontext $(A', B', X', Y')$, together with restrictions of the maps $(\cdot, \cdot)$ and $[\cdot, \cdot]$, is a strong Morita context in the sense of [4, Definition 3.1]. The 6-tuple $(A, B, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ is called a weak Morita Context.

Our goal in this section is to show that if two $W^*$-correspondences $(E, A)$ and $(F, B)$ are (weakly) Morita equivalent then their Hardy algebras $H^\infty(E)$ and $H^\infty(F)$ are weakly Morita equivalent and weak* Morita equivalent.

**Lemma 3.1** Let $X$ be an $A$-$B$ $W^*$-equivalence bimodule. Then the pairs $(I_A, m_A)$ and $(I_B, m_B)$, where $m_A : X \otimes_B \tilde{X} \rightarrow A$ and $m_B : \tilde{X} \otimes_A X \rightarrow B$ are defined by $m_A(x \otimes \tilde{y}) = A \langle x, y \rangle$ and $m_B(\tilde{x} \otimes y) = (x, y)_B$, are $W^*$-correspondence isomorphisms.

**Proof** The two identities are obviously $W^*$-algebra isomorphisms.

\[
m_A(a \cdot (x \otimes \tilde{y}) \cdot b) = m_A((a \cdot x) \otimes (\tilde{y} \cdot b)) = m_A((a \cdot x) \otimes (b^* \cdot \tilde{y}))
\]

\[
=m_A \langle a \cdot x, b^* \cdot \tilde{y} \rangle = a_A \langle x, y \rangle b = am_A(x \otimes y)b
\]

\[
\langle m_A(x \otimes \tilde{y}), m_A(z \otimes \tilde{w}) \rangle_A = \langle A \langle x, y \rangle, A \langle z, w \rangle \rangle_A = \langle A \langle x, y \rangle \cdot A \langle z, w \rangle \rangle_A
\]

\[
= \langle A \langle x, y \rangle, A \langle z, w \rangle \rangle_A = \langle \tilde{y}, w \cdot \langle x, z \rangle_B \rangle_A = \langle \tilde{y}, w \rangle \cdot \langle x, z \rangle_B^* \rangle_A
\]

\[
= \langle (x \otimes \tilde{y}) \cdot (z \otimes \tilde{w}) \rangle_A
\]

That is, $m_A$ preserves both left and right inner products. So it is isometric, hence injective with closed range. Since $m_A$ is defined in terms of the left inner product of $X$, and by definition, $X$ is a $w^*$-full left $W^*$-module over $A$, $m_A$ has $w^*$-dense range in $A$. So $m_A$ is surjective, thus $(I_A, m_A)$ is a $W^*$-correspondence isomorphism. Similarly, $(I_B, m_B)$ is also a $W^*$-correspondence isomorphism. \qed

**Theorem 3.2** If two $W^*$-correspondences $(E, A)$ and $(F, B)$ are (weakly) Morita equivalent then their Hardy algebras $H^\infty(E)$ and $H^\infty(F)$ are weakly Morita equivalent and weak* Morita equivalent (as dual operator algebras) in the sense of [1].

**Proof** We model our proof on the proof given in [11] for the $C^*$ case. And $A$ and $B$ are weakly Morita equivalent $W^*$-algebras via a $W^*$-equivalence bimodule $A X B$ and there is a $W^*$-correspondence isomorphism $W : X \otimes_B F \rightarrow E \otimes_A X$. Let $\mathcal{I}$ (and $\mathcal{I}^w$) denote the norm closure in $A$ (and the $w^*$-closure in $A$) of the range of the $A$-valued inner product on $X$. Form the linking $W^*$-algebra $L$ for $\tilde{X}$. That is, let

\[
L = \left( \begin{array}{c}
B \\
X \\
A
\end{array} \right).
\]
Let \( Y_1 \) be the first column, \( \begin{pmatrix} B \\ X \end{pmatrix} \). This is a \( W^\ast \)-module over \( B \) since it is the column sum of \( B \) and \( X \). Likewise, the second column, \( Y_2 = \begin{pmatrix} \tilde{X} \\ A \end{pmatrix} \), is a \( W^\ast \)-module over \( A \).

\( CB_B(X) = B_B(X) = L_B(X) = M(K_B(X)) \cong M(\mathcal{T}) = T^w = A \) and \( CB_B(B) = B \). Since \( X \) is a selfdual space, \( CB_B(B, X) \cong X \) and \( CB_B(X, B) \cong \tilde{X} \). So we have that \( L \cong CB_B(Y_1) \). Similarly, \( CB_A(\tilde{X}) \cong B, CB_A(A, \tilde{X}) \cong \tilde{X}, CB_A(\tilde{X}, A) \cong X \) and \( CB_A(A) \cong A \). So \( L \cong CB_A(Y_2) \). That is, \( CB_B(Y_1) \) and \( CB_A(Y_2) \) are \( W^\ast \)-algebras. For more information on the previous identifications, see [2, 8.5.5, 8.1.15, 8.5.3, 8.5.1, 8.5.13, 8.5.5 (1), 2.6.1, 3.5.4 (2)] for example.

Now we have that \( CB_B(Y_1, F) \) is a module over the \( W^\ast \)-algebra \( CB_B(Y_1) = L \), where right multiplication is given by composition and the inner product is given by \( \langle T, S \rangle = T^*S \in CB_B(Y_1) = L \). By [3, equation († †)], \( CB_B(Y_1, F) = (Y_1 \hat{\otimes}_BF_\ast)^\ast \), where \( \hat{\otimes}_B \) is the module operator space projective tensor product. So \( CB_B(Y_1, F) \) is a \( W^\ast \)-module over \( CB_B(Y_1) = L \) (by [2, corollary 8.5.7]). Similarly, \( CB_A(Y_2, E) \) is a \( W^\ast \)-module over \( CB_A(Y_2) = L \). So their sum \( Z = CB_B(Y_1, F) \oplus CB_A(Y_2, E) \) is a \( W^\ast \)-module over \( L \). Since \( CB_B(Y_1, F) \) can be written as \( (CB_B(B, F), CB_B(X, F)) \) and \( CB_A(Y_2, E) \) is \( (CB_A(\tilde{X}, E), CB_A(A, E)) \), we have

\[
Z = \begin{pmatrix} CB_B(B, F) & CB_B(X, F) \\ CB_A(\tilde{X}, E) & CB_A(A, E) \end{pmatrix}
\]

The right action of \( L \) on \( Z \) is realized as the usual matrix multiplication, and the inner product is given by

\[
\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} (T_{11})^* & T_{12}^* \\ T_{21}^* & T_{22}^* \end{pmatrix}, \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in L
\]

Since \( CB_B(B, F) \cong F, CB_B(X, F) \cong F \hat{\otimes} \tilde{X}, CB_A(\tilde{X}, E) \cong E \hat{\otimes} X \) and \( ACB(A, E) \cong E \), we have

\[
Z = \begin{pmatrix} F & F \hat{\otimes}_B \tilde{X} \\ E \hat{\otimes}_A X & E \end{pmatrix}
\]

The right action then becomes

\[
\begin{pmatrix} f_1 & f_2 \hat{\otimes} \tilde{x} \\ e_1 \hat{\otimes} y & e_2 \end{pmatrix} \begin{pmatrix} b \\ z \end{pmatrix} = \begin{pmatrix} f_1b + IF \otimes m_B(f_2 \hat{\otimes} \tilde{x} \otimes u) & f_1 \hat{\otimes} \tilde{z} + f_2 \hat{\otimes} \tilde{x}a \\ e_1 \otimes yb + e_2 \otimes u & IE \otimes m_A(e_1 \otimes y \otimes \tilde{z}) + e_2a \end{pmatrix}
\]

and the inner product is
Let $\varphi_Z : L \to \mathcal{L}(Z)$ be defined by

$$
\varphi_Z \left( \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \right) \left( \begin{pmatrix} f_1 & f_2 \otimes \tilde{x} \\ e_1 \otimes v & e_2 \end{pmatrix} \right) = \left( \begin{pmatrix} \varphi_F(b) f_1 & \varphi_F(b) f_2 \otimes \tilde{z} \\ \varphi_E(a) e_1 \otimes v & \varphi_E(a) e_2 \end{pmatrix} \right),
$$

where $\tilde{\mathcal{W}} : \tilde{X} \otimes_A E \to F \otimes_B \tilde{X}$ is the isomorphism defined in [11, Section 2, pg 116]. By [11, Proposition 2.6], $\varphi_Z$ is a *-homomorphism, and since $W$ is $w^*$-continuous (being a $W^*$-correspondence isomorphism), $\varphi_Z$ is normal. Thus $Z$ is a $W^*$-correspondence over $L$ (where the left action of $L$ on $Z$ is given by $\varphi_Z$).

Replacing the $C^*$-module interior tensor product (which is the same as the module Haagerup tensor product) with the $W^*$-module tensor product, and replacing the direct sum with the ultraweak direct sum in [11, Lemmas 2.7, 2.8 and 2.10], we have that the Fock space $\mathcal{F}(Z)$ can be written in the form

$$
\mathcal{F}(Z) \cong \left( \begin{array}{cc} \mathcal{F}(F) & \mathcal{F}(F) \otimes_B \tilde{X} \\ \mathcal{F}(E) \otimes_A X & \mathcal{F}(E) \end{array} \right).
$$

Form the operator algebras $\mathcal{T}(Z)$, $\mathcal{T}_+(Z)$ and $H^\infty(Z)$ associated with $Z$. Consider the submodule

$$
\mathcal{F}'(Z) = \left( \begin{array}{cc} \mathcal{F}(F) & 0 \\ \mathcal{F}(E) \otimes_A X & 0 \end{array} \right) \subset \mathcal{F}(Z).
$$

By the definition of the map $\varphi_Z$, $\mathcal{F}'(Z)$ is invariant for the diagonal operators in $\mathcal{L}(\mathcal{F}(Z))$. By [11, Lemma 2.9], $\mathcal{F}'(Z)$ is also invariant for the creation operators in $\mathcal{L}(\mathcal{F}(Z))$. Thus, $\mathcal{F}'(Z)$ is invariant for $H^\infty(Z)$. Furthermore, by [11, Lemma 3.1], the representation of $\mathcal{T}(Z)$ obtained by restricting the action of $\mathcal{T}(Z)$ to $\mathcal{F}'(Z)$ is faithful. That is, we can study the action of $\mathcal{T}(Z)$ on $\mathcal{F}(Z)$ by just studying the action of $\mathcal{T}(Z)$ on $\mathcal{F}'(Z)$. 
Write $p$ for the projection in $\mathcal{L}(\mathcal{F}'(Z))$ onto \[
\begin{pmatrix}
\mathcal{F}(F) \\
0
\end{pmatrix}
\] and $q$ for the projection onto \[
\begin{pmatrix}
0 \\
\mathcal{F}(E) \otimes A X
\end{pmatrix}.
\]
Next, we show that $pH^\infty(Z) \cong H^\infty(F)$ and $qH^\infty(Z) \cong H^\infty(E)$. Let
\[
\xi = \begin{pmatrix}
h_1 \\
h_2 \otimes \tilde{w} \\
k_1 \otimes v \\
k_2
\end{pmatrix} \in \mathbb{Z}^m
\]
and let $f \in F^{\otimes l}$. We can view $f$ as the element \[
\begin{pmatrix}
f_0 \\
0
\end{pmatrix} \in \begin{pmatrix}
F^{\otimes l} \\
0
\end{pmatrix} \subset \mathcal{F}'(Z).
\]
By \cite{11, Lemma 2.9},
\[
T_\xi p \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = T_\xi \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = \xi \otimes \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = \begin{pmatrix}
h_1 \otimes f \\
k_1 \otimes W_l(v \otimes f)
\end{pmatrix},
\]
where $W_k = (I_E \otimes W_{k-1})(W_1 \otimes I_{F^{\otimes (k-1)}})$. Hence
\[
pT_\xi p \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = \begin{pmatrix}
h_1 \otimes f \\
0
\end{pmatrix},
\]
which we write as $pT_\xi p = T_h_1$. For
\[
\lambda = \begin{pmatrix}
b \\
x \\
a
\end{pmatrix} \in \mathcal{L},
\]
we have
\[
\varphi_\infty(\lambda) p \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = \varphi_\infty(\lambda) \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = \begin{pmatrix}
\varphi_{F^{\otimes l}}(b) f \\
W_l(x \otimes f)
\end{pmatrix}.
\]
Hence
\[
p\varphi_\infty(\lambda) p \begin{pmatrix}
f_0 \\
0
\end{pmatrix} = \begin{pmatrix}
\varphi_{F^{\otimes l}}(b) f \\
0
\end{pmatrix},
\]
So $p\varphi_\infty(\lambda) p = \varphi_\infty(b)$. That is, the generators of the algebra $p\mathcal{T}_+(Z) p$ are identified with the generators of $\mathcal{T}_+(F)$. Thus $p\mathcal{T}_+(Z) p \cong \mathcal{T}_+(F)$ and $pH^\infty(Z) p \cong H^\infty(F)$.

Similarly, we can view the element $e \otimes u \in E^{\otimes l} \otimes X$ as the element \[
\begin{pmatrix}
0 \\
e \otimes u \\
0
\end{pmatrix} \in \mathcal{F}'(Z).
\]
So by \cite{11, Lemma 2.9},
\[
T_\xi q \begin{pmatrix}
0 \\
e \otimes u \\
0
\end{pmatrix} = T_\xi \begin{pmatrix}
0 \\
e \otimes u \\
0
\end{pmatrix} = \xi \otimes \begin{pmatrix}
0 \\
e \otimes u \\
0
\end{pmatrix} = \begin{pmatrix}
c(h_2 \otimes \tilde{w}, W_l^{-1}(e \otimes u)) \\
k_2 \otimes e \otimes u
\end{pmatrix}.
\]
where \( c : F^\otimes m \otimes \tilde{X} \times X \otimes F^\otimes l \to F^\otimes m \otimes F^\otimes l \) is a bilinear map which is not relevant for our purposes. So

\[
q T_\xi q \left( \begin{array}{ccc} 0 & 0 & 0 \\ e \otimes u & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} k_2 & 0 & 0 \\ e \otimes u & 0 & 0 \end{array} \right).
\]

which we write as \( q T_\xi q = T_{k_2} \otimes I_x \). For

\[
\lambda = \left( \begin{array}{c} b \\ x \\ a \end{array} \right) \in L,
\]

we have

\[
\varphi_\infty(\lambda) \left( \begin{array}{ccc} 0 & 0 & 0 \\ e \otimes u & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} (m_B \otimes I_{F^\otimes l}(I_{\tilde{X}} \otimes W^{-1}(\tilde{y} \otimes e \otimes u)) \varphi_{E^\otimes l}(a) \otimes e \otimes u & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right).
\]

Hence

\[
q \varphi_\infty(\lambda) q \left( \begin{array}{ccc} 0 & 0 & 0 \\ e \otimes u & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ \varphi_{E^\otimes l}(a) \otimes e \otimes u & 0 \\ 0 & 0 \end{array} \right).
\]

So \( q \varphi_\infty(\lambda) q = \varphi_\infty(a) \otimes I_X \). Since the map from \( \mathcal{L}(\mathcal{F}(E) \otimes_A X) \) to \( \mathcal{L}(\mathcal{F}(E)) \) taking \( T \otimes I_X \) to \( T \) is an isomorphism ([11, Lemma 2.12]), we have that \( qT^+_+(Z)q \cong T^+_+(E) \) and \( qH^\infty(Z)q \cong H^\infty(E) \).

Next, we show that \( H^\infty(E) \) and \( H^\infty(F) \) are weakly Morita equivalent (as dual operator algebras) in the sense of [1]. First note that \( pH^\infty(Z)p \) and \( qH^\infty(Z)q \) are unital dual operator algebras with identities \( p\varphi_\infty(1_L)p = \varphi_\infty(1_B) \) and \( q\varphi_\infty(1_L)q = \varphi_\infty(1_A) \) respectively.

Let \( (\cdot, \cdot) : pH^\infty(Z)q \times qH^\infty(Z)p \to pH^\infty(Z)p \) and \( [\cdot, \cdot] : qH^\infty(Z)p \times pH^\infty(Z)q \to qH^\infty(Z)q \) be given by:

\[
(p\alpha q, q\beta p) \to p\alpha q \beta p
\]

\[
[q\alpha p, p\beta q] \to q\alpha p \beta q
\]

respectively, and let \( (\cdot, \cdot)_t : pT^+_+(Z)q \times qT^+_+(Z)p \to pT^+_+(Z)p \) and \( [\cdot, \cdot]_t : qT^+_+(Z)p \times pT^+_+(Z)q \to qT^+_+(Z)q \) be the respective restrictions of \( (\cdot, \cdot) \) and \( [\cdot, \cdot] \).

As shown in [11, Theorem 3.2 (3)],

\[
(pT^+_+(Z)p, qT^+_+(Z)q, pT^+_+(Z)q, qT^+_+(Z)p, (\cdot, \cdot)_t, [\cdot, \cdot]_t)
\]

is a strong Morita context in the sense of [4, Definition 3.1]. In particular, the multiplication maps \( (\cdot, \cdot) \) and \( [\cdot, \cdot] \) are completely contractive bilinear maps. Since for any Hilbert space \( H \), the product in \( B(H) \) is separately weak*-continuous, we have that \( (\cdot, \cdot) \) and \( [\cdot, \cdot] \) are separately weak* continuous (here we are using the identification of \( H^\infty(Z) \), an abstract dual operator algebra, with a subalgebra of \( B(H) \) via a complete isometric homomorphism which is a \( w^* \)-homeomorphism).
Then, since $pT_+(Z)p \overset{w^*}{\sim} = pH^\infty(Z)p$, $qT_+(Z)q \overset{w^*}{\sim} = qH^\infty(Z)q$, $pT_+(Z)p \overset{w^*}{\sim} = pH^\infty(Z)q$ and $qT_+(Z)q \overset{w^*}{\sim} = qH^\infty(Z)p$, we have that

$$(pH^\infty(Z)p, qH^\infty q, pH^\infty(Z)q, qH^\infty(Z)p, \langle \cdot, \cdot \rangle, \langle [], [] \rangle)$$

is a weak Morita context. Thus $H^\infty(E) \cong qH^\infty(Z)q$ and $H^\infty(F) \cong pH^\infty(Z)p$ are weakly Morita equivalent in the sense of [1, Definition 3.2].

In the same way that a strong Morita context gives rise to a linking algebra $\mathcal{L}$ (see [4, Chapter 3]), a weak Morita context gives rise to a weak linking algebra $\mathcal{L}^\omega$, which is a dual operator algebra. The construction of $\mathcal{L}^\omega$ is given in [1, Section 4]. In our case, the linking operator algebra of the strong Morita context $(T_+(F), T_+(E), pT_+(Z)q, qT_+(Z)p)$ is

$$\mathcal{L} = \left( \begin{array}{cc} T_+(F) & pT_+(Z)q \\ qT_+(Z)p & T_+(E) \end{array} \right)$$

and the weak linking algebra of the weak Morita context $(H^\infty(F), H^\infty(E), pH^\infty(Z)q, qH^\infty(Z)p)$ is

$$\mathcal{L}^\omega = \left( \begin{array}{cc} H^\infty(F) & pH^\infty(Z)q \\ qH^\infty(Z)p & H^\infty(E) \end{array} \right).$$

$\mathcal{L}$ can be identified completely isometrically with a weak*-dense subalgebra of $\mathcal{L}^\omega$. Adapting [1, 4] to our algebras, we have that $(H^\infty(F), \mathcal{L}^\omega, H^\infty(F) \ominus^f pH^\infty(Z)q, H^\infty(F) \ominus^c qH^\infty(Z)p)$ and $(H^\infty(E), \mathcal{L}^\omega, H^\infty(E) \ominus^f qH^\infty(Z)p, H^\infty(E) \ominus^c pH^\infty(Z)q)$ are weak*-Morita contexts. The next corollary follows.

**Corollary 3.3** If two $W^*$-correspondences $(E, A)$ and $(F, B)$ are weakly Morita equivalent, then $H^\infty(F)$ and $H^\infty(E)$ are weakly Morita equivalent to $\mathcal{L}^\omega$.

Furthermore, applying the map $L_N$ in [1, Theorem 3.6] to the weak linking algebras of the weak* Morita contexts $(H^\infty(F), \mathcal{L}^\omega, H^\infty(F) \ominus^f pH^\infty(Z)q, H^\infty(F) \ominus^c qH^\infty(Z)p)$ and $(H^\infty(E), \mathcal{L}^\omega, H^\infty(E) \ominus^f qH^\infty(Z)p, H^\infty(E) \ominus^c pH^\infty(Z)q)$, we have:

**Corollary 3.4** If two $W^*$-correspondences $(E, A)$ and $(F, B)$ are weakly Morita equivalent, then $w^*CB_{H^\infty(F)}(H^\infty(F) \ominus^c qH^\infty(Z)p) \cong \mathcal{L}^\omega \cong w^*CB_{H^\infty(E)}(H^\infty(E) \ominus^c pH^\infty(Z)q)$ completely isometrically and $w^*$-isomorphically.

## 4 $W^*$-Graph Correspondences

A directed graph $G = (G^0, G^1, r, s)$ consists of two countable sets $G^0, G^1$ and functions $r, s : G^0 \to G^1$ identifying the range and source of each edge. The $W^*$-correspondence $(E, A)$ associated to $G = (G^0, G^1, r, s)$ is given by:
A = ℓ_{|G^0|}^∞ \quad E = \left\{ x : G^1 \rightarrow \mathbb{C} \mid \sup_{v \in G^0} \left\{ \sum_{x(e)=v} |x(e)|^2 \right\} < \infty \right\}

The left and right actions are given by \((a \cdot x \cdot b)(e) = a(r(e))x(e)b(s(e))\), where \(a, b \in A\) and \(x \in E\). The inner product is given by \((x, y)_A(v) = \sum_{x(e)=v} x(e)y(e)\).

This definition of a W*-graph correspondence is equivalent to the one given by Solel in [20, pg 3], where the W*-correspondence is defined in terms of matrices. Let \(I, J\) be indexing sets with \(|I| = |G^0|\) and \(|J| = |G^1|\). We will write the elements of \(A\) as \(a = (a_i)\) or as \(a = \sum_{i \in I} a_i \delta_{v_i}\), and the elements of \(E\) as \((z_j)\) or as \(x = \sum_{j \in J} z_j \delta_{e_j}\), \((\delta_{v_i})\) and \((\delta_{e_j})\) denote the point masses of a vertex and an edge respectively. Note that \(A\) is the \(\mathcal{W}^{*}\)-closure of \(c_0(G^0)\) (by \(\ell^\infty = c_0^{**}\) and Goldstine’s theorem). The norm of \(A\) is given by \(||a|| = \sup_{i \in I} |a_i|\). The norm of \(E\) is given by \(||x|| = \|(x, x)_A\|_2^\frac{1}{2} = (\sup_{v \in G^0} \sum_{x(e)=v} |x(e)|^2)^\frac{1}{2}\). Note that \(E\) is a subspace of \(\ell^\infty_{|G^1|}\), which may also be viewed as a disjoint union \(\bigsqcup_{v \in G^0} \ell^2_{|s^{-1}(v)|}\).

For each vertex \(v_i \in G^0\), we have \(\delta_{v_i}^2 = \delta_{v_i} = \delta_{v_i}^*\). That is, for each \(i \in I\), \(\delta_{v_i}\) is a projection. If \(\sigma\) is a faithful normal representation of \(A\) on a Hilbert space \(H\), then \(\sigma(\delta_{v_i})\) is also a projection. So for any \(a = (a_i) \in A\), we have \(\sigma(a) = \bigoplus_{i \in I} a_i I_{H_i}\), a direct sum of uniformly bounded operators on \(H = \bigoplus_{i \in I} H_i\), the Hilbert space direct sum. Then \(\sigma(A) = \bigoplus_{i} \mathbb{C}_i I_{H_i}\), where \(\bigoplus_{i} \mathbb{C}_i\) denotes the \(\infty\)-direct sum (for more information on this sum, see for example [2, 1.2.17]). Since \(\sigma\) is faithful, it is isometric (\(||\sigma(a)|| = \sup_{i} a_i I_{H_i} = \sup_{i} a_i = ||a||\)), and so are its amplifications \(\sigma_n\). So \(\sigma\) is completely isometric. The dimension of each Hilbert space \(H_i\) is the multiplicity \(m_i\) of the one-dimensional representation \(\sigma(\sum_{i=1}^{G^0} a_i \delta_{v_i}) = a_i\). So \(\sigma\) is completely determined up to unitarily equivalence by the sequence \((m_1, m_2, \ldots)\) of these multiplicities. Since \(\sigma\) is faithful, \(0 < m_i \leq \infty\). Thus \(H\) can be written as \(H = \bigoplus_{i \in I} \mathbb{C}^{m_i}\), where \(\mathbb{C}^{\infty}\) is interpreted as \(\ell^2\).

As stated in Sect. 2, attached to each faithful normal representation \(\sigma\) of \(A\), there is a dual correspondence \(E^{\sigma}\), which is a \(W^{*}\)-correspondence over \(\sigma(A)^{\prime}\). In the following theorem, we identify the elements of \((E^{\sigma})^*\) when \((E, A)\) is a \(W^{*}\)-graph correspondence.

**Theorem 4.1** If \((E, A)\) is a \(W^{*}\)-graph correspondence and \(\sigma : A \rightarrow B(H)\) is a faithful normal representation of \(A\), then the elements of \((E^{\sigma})^*\) are block matrices \((T_{ij})\) where \(T_{ij} \in B(H_{s(e_j)}, H_{v_i})\) and \(T_{ij} = 0\) if \(r(e_j) \neq v_i\).

**Proof** Let \(x \in E, a \in A\) and \(h \in H\). \(\sigma(a) = \sigma(\sum_{i=1}^{G^0} a_i \delta_{v_i}) = \sum_{i=1}^{G^0} a_i \sigma(\delta_{v_i})\). So \(\sigma(A) = \bigoplus_{i} \mathbb{C}_i a_i I_{H_i}\) and \(H = \bigoplus_{i=1}^{G^0} H_i\). Sometimes, for clarity, we might also write \(\sigma(A) = \bigoplus_{i} a_i \delta_{v_i} I_{H_{v_i}}\) and \(H = \bigoplus_{i=1}^{G^0} H_{v_i}\). Let \(x \otimes h \in E \otimes_{\sigma} H\).

\[
x \otimes h = \left( \sum_{j=1}^{G^1} z_j \delta_{e_j} \right) \otimes h = \sum_{j=1}^{G^1} z_j \delta_{e_j} \cdot \delta_{s(e_j)} \otimes h
\]
\[
It \text{ follows from } [13, \text{Lemma 4.12 (2)}] \text{ and the definition of left and right actions on } W_t, \text{ the block on row } e_j.
\]

Our next goal is to show that if \((E, A)\) is a \(W^*\)-correspondence derived from a directed graph \(G\) and \(\sigma, \tau\) are two faithful normal representations of \(A\), then \((\sigma \mathcal{F}(E)(H^\infty(E)))'\) is weakly Morita equivalent to \((\tau \mathcal{F}(E)(H^\infty(E)))'\).

\textbf{Corollary 4.2} If \((E, A)\) is a \(W^*\)-graph correspondence and \(\sigma : A \to B(H)\) is a faithful normal representation of \(A\), then the elements of \(\mathcal{F}(E)\) are block matrices \((T_{ij})\) such that \(T_{ij} = \begin{cases} z_j H_{s(e_j)} & \text{if } e_j \text{ is a loop} \\ 0 & \text{otherwise} \end{cases}\).

\textbf{Proof} It follows from [13, Lemma 4.12 (2)] and the definition of left and right actions in a \(W^*\)-graph correspondence. \qed

\subsection{4.1 Commutants of Induced Representations of The Hardy Algebra}

Our next goal is to show that if \((E, A)\) is a \(W^*\)-correspondence derived from a directed graph \(G\) and \(\sigma, \tau\) are two faithful normal representations of \(A\), then \((\sigma \mathcal{F}(E)(H^\infty(E)))'\) is weakly Morita equivalent to \((\tau \mathcal{F}(E)(H^\infty(E)))'\).
Lemma 4.3 The $W^*$-algebras $\sigma(A)'$ and $\tau(A)'$ are weakly Morita equivalent.

Proof If $\sigma$ and $\tau$ are faithful normal representations of $A$ on $H$ and $K$ respectively, then

$$H = \bigoplus_{q \in I} H_q \quad \text{and} \quad K = \bigoplus_{q \in I} K_q,$$

$$\sigma(A) = \bigoplus_{q} \mathbb{C}_q I_{H_q} \quad \text{and} \quad \tau(A) = \bigoplus_{q} \mathbb{C}_q I_{K_q},$$

$$\sigma(A)' = \bigoplus_{q} B(H_q) \quad \text{and} \quad \tau(A)' = \bigoplus_{q} B(K_q).$$

Let $X = \bigoplus_{q} B(K_q, H_q)$. Let $\bigoplus_{q} T_q, \bigoplus_{q} S_q \in X$. We show that $X$ with inner products

$$\langle \bigoplus_{q} T_q, \bigoplus_{q} S_q \rangle_{\tau(A)'} = \bigoplus_{q} T_q^* S_q$$

$$\sigma(A)'/\langle \bigoplus_{q} T_q, \bigoplus_{q} S_q \rangle = \bigoplus_{q} T_q S_q^*$$

is a $\sigma(A)' - \tau(A)'$ equivalence bimodule, where the left and right actions are given by regular matrix multiplication. First, we check that the two previous equations do define inner products on $X$. Let $x = \bigoplus_{q} T_q, y = \bigoplus_{q} S_q, z = \bigoplus_{q} U_q \in X, a = \bigoplus_{q} R_q \in \tau(A)'$ and $\lambda, \mu \in \mathbb{C}$. Then

$$\langle x, \lambda y + \mu z \rangle_{\tau(A)'} = \left\langle \bigoplus_{q} T_q, \lambda \bigoplus_{q} S_q + \mu \bigoplus_{q} U_q \right\rangle_{\tau(A)'}$$

$$= \left\langle \bigoplus_{q} T_q, \bigoplus_{q} (\lambda S_q + \mu U_q) \right\rangle_{\tau(A)'} = \bigoplus_{q} T_q^* (\lambda S_q + \mu U_q)$$

$$= \lambda \bigoplus_{q} T_q^* S_q + \mu \bigoplus_{q} T_q^* U_q$$

$$= \lambda \left\langle \bigoplus_{q} T_q, \bigoplus_{q} S_q \right\rangle_{\tau(A)'} + \mu \left\langle \bigoplus_{q} T_q, \bigoplus_{q} U_q \right\rangle_{\tau(A)'}$$

$$= \lambda \langle x, y \rangle_{\tau(A)'} + \mu \langle x, z \rangle_{\tau(A)'}$$

$$\langle x, y \cdot a \rangle_{\tau(A)'} = \left\langle \bigoplus_{q} T_q, \bigoplus_{q} S_q \cdot \left( \bigoplus_{q} R_q \right) \right\rangle_{\tau(A)'} = \left\langle \bigoplus_{q} T_q, \bigoplus_{q} S_q R_q \right\rangle_{\tau(A)'}$$
Since each diagonal entry \((T_q^\ast T_q)_{jj}\) of \(T_q^\ast T_q\) is the sum \(\sum_j (T_{ij})^2\) of the squares of the entries on column \(j\) of \(T_q\), we have that \(T_q^\ast T_q = 0\) implies \(T_q = 0\). So \(\langle x, x \rangle_{\tau(A)'} = (\bigoplus_{q \in I} T_q^\ast T_q)_{\tau(A)'} = \bigoplus_{q \in I} T_q^\ast T_q = 0\) implies \(T_q = 0\) for all \(q \in I\). So \(x = \bigoplus_{q \in I} T_q = 0\). Thus \(\langle \cdot, \cdot \rangle_{\tau(A)'}\) is a right inner product on \(X\). Similarly, \(\sigma(A)'\langle \cdot, \cdot \rangle\) is a left inner product on \(X\).

Now we show that \(X\) is a \(w^*\)-full left Hilbert \(\sigma(A)\)'-module and a \(w^*\)-full right Hilbert \(\tau(A)\)'-module. Let \(M \in \tau(A)\). So \(M = \bigoplus_{q \in I} M_q\) (where \(M_q \in B(K_q)\)). Assume for the moment, each Hilbert space \(K_q\) has finite dimension \(n_q\). For each \(q \in I\), \(M_q = \sum_{i,j=1}^{n_q} m_{ij}^{(q)} E_{ij}^{(q)}\), where \(m_{ij}^{(q)} \in \mathbb{C}\) and \(\{E_{ij}^{(q)}\}_{i,j}\) is the usual (matrix unit) basis for \(B(K_q)\). That is, \(E_{ij}^{(q)}\) is the \(n_q \times n_q\) matrix with 1 in the \(i, j\) entry and zeros everywhere else. Let \(\langle T_{ij}^{(q)} \rangle_{i,j}\) be a matrix basis for \(B(K_q, H_q)\). Then \(E_{ij}^{(q)} = T_{ij}^{(q)\ast} T_{ij}^{(q)} = \langle T_{ij}^{(q)} \rangle^* \langle T_{ij}^{(q)} \rangle\). So \(M_q = \sum_{i,j=1}^{n_q} m_{ij}^{(q)} E_{ij}^{(q)} = \sum_{i,j=1}^{n_q} m_{ij}^{(q)} \langle T_{ij}^{(q)} \rangle^* \langle T_{ij}^{(q)} \rangle\). Thus \(M = \bigoplus_{q \in I} M_q = \bigoplus_{q \in I} \sum_{i,j=1}^{n_q} m_{ij}^{(q)} E_{ij}^{(q)} = \bigoplus_{q \in I} \sum_{i,j=1}^{n_q} m_{ij}^{(q)} \langle T_{ij}^{(q)} \rangle^* \langle T_{ij}^{(q)} \rangle\). Likewise, since \(E_{ij}^{(q)} = T_{ij}^{(q)\ast} T_{ij}^{(q)} = \langle T_{ij}^{(q)} \rangle^* \langle T_{ij}^{(q)} \rangle\), we have \(M = \bigoplus_{q \in I} M_q = \bigoplus_{q \in I} \sum_{i,j=1}^{n_q} m_{ij}^{(q)} E_{ij}^{(q)} = \bigoplus_{q \in I} \sum_{i,j=1}^{n_q} m_{ij}^{(q)} \langle T_{ij}^{(q)} \rangle^* \langle T_{ij}^{(q)} \rangle\).

Then, since for each \(q\), \(\mathbb{K}(K_q)\) is \(w^*\) dense in \(B(K_q)\) (by Goldstine’s theorem), we have that \(X = \bigoplus_{q \in I} B(K_q, H_q)\) is a \(w^*\)-full left Hilbert \(\sigma(A)\)'-module and a \(w^*\)-full right Hilbert \(\tau(A)\)'-module.

Next, we show that \(\sigma(A)'\) acts as adjointable operators on \(X_{\tau(A)'}\) and \(\tau(A)'\) acts as adjointable operators on \(\sigma(A)'X\). Let \(x, y \in X, a \in \sigma(A)'\) and \(b \in \tau(A)'\). Then

\[
\langle a \cdot x, y \rangle_{\tau(A)'} = \left( \left( \bigoplus_{q \in I} N_q \right) \cdot \bigoplus_{q \in I} T_q \cdot \bigoplus_{q \in I} S_q \right)_{\tau(A)'} = \left( \bigoplus_{q \in I} N_q T_q \cdot \bigoplus_{q \in I} S_q \right)_{\tau(A)'}
\]
\[= \bigoplus_{q \in I} T_q^* N_q^* S_q = \left\langle \bigoplus_{q \in I} T_q, \bigoplus_{q \in I} N_q^* S_q \right\rangle_{\tau(A)'} \]

\[= \left\langle \bigoplus_{q \in I} T_q, \left( \bigoplus_{q \in I} N_q \right)^* \cdot \bigoplus_{q \in I} S_q \right\rangle_{\tau(A)'} = \langle x^*, y \rangle_{\tau(A)'} \]

and

\[\sigma(A)' \langle x \cdot b, y \rangle = \left\langle \left( \bigoplus_{q \in I} T_q \right) \cdot \left( \bigoplus_{q \in I} M_q \right) \cdot \bigoplus_{q \in I} S_q \right\rangle = \left\langle \bigoplus_{q \in I} T_q M_q, \bigoplus_{q \in I} S_q \right\rangle \]

\[= \left\langle \bigoplus_{q \in I} T_q, \left( \bigoplus_{q \in I} N_q \right)^* \cdot \bigoplus_{q \in I} S_q \right\rangle = \langle x, y \cdot b \rangle \]

Next, we show that the two inner products are compatible.

For all \( x = \bigoplus_{q \in I} T_q, y = \bigoplus_{q \in I} S_q, z = \bigoplus_{q \in I} U_q \in X \), we have

\[\sigma(A)' \langle x \cdot b, y \rangle \cdot z = \left\langle \left( \bigoplus_{q \in I} T_q \right) \cdot \left( \bigoplus_{q \in I} M_q \right) \cdot \bigoplus_{q \in I} U_q \right\rangle = \left\langle \bigoplus_{q \in I} T_q M_q, \bigoplus_{q \in I} U_q \right\rangle \]

\[= \left\langle \bigoplus_{q \in I} T_q, \left( \bigoplus_{q \in I} S_q \right)^* \cdot \bigoplus_{q \in I} U_q \right\rangle = \langle x \cdot y \cdot b \rangle \]

Finally, each \( B(K_q, H_q) \) is the dual of the space \( S^1((H_q, K_q)) \) of trace class operators. So \( X = \bigoplus_{q=1}^{\infty} B(K_q, H_q) = (\bigoplus_{q \in I} S^1(K_q, H_q))^* \) is a dual space. Here, \( \bigoplus_{q \in I} \) denotes the 1-direct sum. For more information on this direct sum and its \( \ell^1 \)-norm, see [2, 1.4.13] for example. Thus \( X \) is a \( \sigma(A)' - \tau(A)' \) W*-equivalence bimodule. That is, \( \sigma(A)' \) and \( \tau(A)' \) are weakly Morita equivalent.

\[\sum_{q=1}^{\infty} B(K_q, H_q) \]

**Theorem 4.4** If \((E, A)\) is a \( W^* \)-graph correspondence and \( \sigma : A \to B(H), \tau : A \to B(K) \) are faithful normal representations of \( A \), then \((E^\sigma, \sigma(A)') \) WME \( \sim \) \((E^\tau, \tau(A)') \).

**Proof** Let \( X = \bigoplus_{i=1}^{G^0} B(K_i, H_i) \) be the \( \sigma(A)' - \tau(A)' \) equivalence bimodule given in Lemma 4.3. We show that the map
gives a $W^*$-correspondence isomorphism, where $x, y \in X$ and $(I_E \otimes x^*) : E \otimes (A') H \rightarrow E \otimes (A') K$ is defined by $(I_E \otimes x^*)(\xi \otimes h) = \xi \otimes x^* h$.

Let $x, y \in \sigma(A') X\tau(A')$, $\eta \in E^\sigma$, $a, b \in \tau(A')$. Note that $\varphi$ is well defined, since for any $c, d \in \sigma(A')$, we have $\varphi((x \cdot c \otimes \eta \cdot d \otimes y) = \varphi(c^* x \otimes \eta d \otimes y) = (I_E \otimes x^*) \eta dy = (I_E \otimes x^*) (c \cdot \eta)(d \cdot y) = \varphi(\tilde{x} \otimes c \cdot \eta \otimes d \cdot y)$.

$$\varphi(a \cdot (\tilde{x} \otimes \eta \otimes y) \cdot b) = \varphi(((x \cdot a^* \otimes \eta \otimes (y \cdot b)) = \varphi((\tilde{x} \cdot a^* \otimes \eta \otimes (yb)) = (I_E \otimes a x^*) \eta y b = (I_E \otimes a)(I_E \otimes x^*) \eta y b = a \cdot ((I_E \otimes x^*) \eta y) \cdot b = a \cdot \varphi(\tilde{x} \otimes \eta \otimes y) \cdot b$$

Let $\tilde{x}_1 \otimes \eta_1 \otimes y_1, \tilde{x}_2 \otimes \eta_2 \otimes y_2 \in \widetilde{X} \otimes E^\sigma \otimes X$. Then

$$\langle \varphi(\tilde{x}_1 \otimes \eta_1 \otimes y_1), \varphi(\tilde{x}_2 \otimes \eta_2 \otimes y_2) \rangle_{\tau(A')} = \langle (I_E \otimes x_1^*) \eta_1 y_1, (I_E \otimes x_2^*) \eta_2 y_2 \rangle_{\tau(A')} = y_1^* \eta_1^* (I_E \otimes x_1)(I_E \otimes x_2^*) \eta_2 y_2$$

That is, $\varphi$ preserves the inner product. So it is isometric, hence injective with closed range. Now we show $\varphi$ is surjective. Each element $S \in E^\tau$ has $|L|$ nonzero blocks $S_q$, where $L$ is a set with $|G^0| \leq |L| \leq |G^1|$. Each $S_q \in B(K_{s(eq)}R_{r(eq)})$. For $Q \in B(K_{s(eq)}R_{r(eq)})$, $P \in B(H_{s(eq)}R_{r(eq)})$. $R \in B(H_{\varphi}R_{\varphi})$, $R P Q \in B(K_{s(eq)}R_{r(eq)})$. Let $M_Q \in X$ be an element with all zero blocks except for $Q$, $M_R \in E^\sigma$ be an element with all zero blocks except for $P$, and $M_R \in (I_E \otimes \tilde{X})$ be an element with all zero blocks except for $R$. Assume for the moment that all Hilbert spaces $H_j$ and $K_j$ are finite dimensional. That is, the multiplicity of the representation of $\delta_v$ is finite for all $v \in G^0$. $S_q = \bigoplus_{i,j} s_{ij} E_{ij}$, where $s_{ij} \in C \otimes \{E_{ij}\}_{i,j}$ is a matrix basis for $B(K_{s(eq)}R_{r(eq)})$. Let $\{T_{ij}\}_{i,j}$ be a matrix basis for $B(K_{s(eq)}R_{r(eq)})$, $\{Y_{ij}\}_{i,j}$ be a matrix basis for $B(H_{s(eq)}R_{r(eq)})$, and $\{Z_{ij}\}_{i,j}$ be a matrix basis for $B(H_{\varphi}R_{\varphi})$. Then $E_{ij} = Z_{ij} Y_{ij} T_{ij}$. So for $x, y \in X, \eta \in E^\sigma$ and $H, K$ finite dimensional, the products $(I_E \otimes \tilde{x}) \eta y$ span $E^\tau$. If the representation of $\delta_{s(eq)}$ or $\delta_{r(eq)}$ on $K$ is not finite dimensional then since $B(K_{s(eq)}R_{r(eq)}) = \mathbb{K}(K_{s(eq)}R_{r(eq)})^{**}$, the span of the finite dimensional products $E_{ij} = Z_{ij} Y_{ij} T_{ij}$ is $w^*$-dense in $B(K_{s(eq)}R_{r(eq)})$ by Goldstine’s theorem. Summing over all $q \in L$, we have that the span of these finite
dimensional products is $w^*$-dense in $E^\tau$. So $\varphi$ is surjective, thus a $W^*$-correspondence isomorphism.

Since by Lemma 3.1, $X\frown_{\tau(A)}X \cong \sigma(A)'$ and $E^\sigma \cong \sigma(A)'\overline{\otimes}_{\sigma(A)'}E^\sigma$ as $W^*$-correspondences, we have:

$$E^\sigma \overline{\otimes}_{\sigma(A)'}X \cong \sigma(A)'\overline{\otimes}_{\sigma(A)'}E^\sigma \overline{\otimes}_{\sigma(A)'}X \cong X\overline{\otimes}_{\tau(A)}X \overline{\otimes}_{\sigma(A)'}E^\sigma \overline{\otimes}_{\sigma(A)'}X$$

as $W^*$-correspondences. So $(E^\sigma, \sigma(A)')_{W^*E} \cong (E^\tau, \tau(A)')$. \hfill \qed

In [10], Muhly and Solel defined the induced representations $\rho$ of $H^\infty(E)$, which play a central role in the study of Hardy algebras. Indeed, these induced representations (in the sense of Rieffel [17]) appear in most of the work related to Hardy algebras. In [14], Muhly and Solel showed how the commutant of $\rho(H^\infty(E))$ can be expressed in terms of induced representations of $H^\infty(E^\tau)$. More precisely, let $\sigma : A \to B(H)$ be a normal representation of $A$ on a Hilbert space $H$ and form the Hilbert space $\mathcal{F}(E) \otimes_{\sigma} H$. The induced covariant representation of $E$ determined by $\sigma$ is the representation $(T, \varphi_{\infty} \otimes I_H)$ where $T : E \to B(\mathcal{F}(E) \otimes_{\sigma} H)$ is defined by $T(\xi)(\eta \otimes h) = (\xi \otimes \eta) \otimes h$ for $\xi \in E$ and $\eta \otimes h \in \mathcal{F}(E) \otimes_{\sigma} H$. The representation of $H^\infty(E)$, induced by $\sigma$, denoted by $\sigma_{\mathcal{F}(E)}$, is the integrated form of $(T, \varphi_{\infty} \otimes I_H)$. For $X \in H^\infty(E)$, $\sigma_{\mathcal{F}(E)}(X)$ is also written as $X \otimes I_H$. Define a map $U : \mathcal{F}(E^\sigma) \otimes_I H \to \mathcal{F}(E) \otimes_I H$ (where $I$ denotes the identity representation of $\sigma(A)'$ in $B(H)$) by

$$U(\eta_1 \otimes \eta_2 \otimes \cdots \eta_n \otimes h) = (I_{E^\sigma \otimes_I (n-1)} \otimes \eta_1)(I_{E^\sigma \otimes_I (n-2)} \otimes \eta_2) \cdots (I_E \otimes \eta_{n-1})\eta_n h$$

By [12, Lemma 3.8], $U$ is a Hilbert space isometric isomorphism and by [12, Theorem 3.9], the representation $\rho$ of $H^\infty(E^\sigma)$ on $\mathcal{F}(E) \otimes_{\sigma} H$, defined by the formula

$$\psi(X) = U \iota_{\mathcal{F}(E^\sigma)}(X)U^*$$

is an ultraweakly homeomorphic, completely isometric isomorphism from $H^\infty(E^\sigma)$ onto $\sigma_{\mathcal{F}(E)}(H^\infty(E))'$. Likewise, the map $\nu$, defined by

$$\nu(X) = U^* \sigma_{\mathcal{F}(E)}(X)U$$

is an ultraweakly homeomorphic, completely isometric isomorphism from $H^\infty(E)$ onto $\iota_{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma))'$.\hfill \qed

Theorem 4.5 If $(E, A)$ is a $W^*$-graph correspondence and $\sigma : A \to B(H)$, $\tau : A \to B(K)$ are faithful normal representations of $A$, then $(\sigma_{\mathcal{F}(E)}(H^\infty(E)))'_{W^*E} \cong (\tau_{\mathcal{F}(E)}(H^\infty(E)))'_{W^*E}$.\hfill \qed

Proof By Theorem 4.4, $(E^\sigma, \sigma(A)')_{W^*E} \cong (E^\tau, \tau(A)')$. Then by Theorem 3.2, $H^\infty(E^\sigma)_{W^*E} \cong H^\infty(E^\tau)$. So by the isomorphism $\psi$ above, we have
\[(\sigma \mathcal{F}(E)(H^\infty(E)))' \cong H^\infty(E^\sigma) \cong (\tau \mathcal{F}(E)(H^\infty(E)))'\]

Note also that if \((E, A)\) is a graph correspondence and \(\sigma : A \to B(H)\), \(\tau : A \to B(K)\) are faithful normal representations of \(A\), then the map \(\nu\) above, gives us

\[(\nu \mathcal{F}(E)(H^\infty(E)))' \cong (\nu \mathcal{F}(E^\sigma)(H^\infty(E^\tau)))'.\]

### 4.2 Morita Equivalence of \(W^*\)-Graph Correspondences

Let \(X\) be a countable set, \(A = C(X)\) (with the sup norm) and let \(C(X) \mathcal{X}_{C(X)} = A \mathcal{X}_A\) be a \(W^*\)-equivalence bimodule. By [15, Theorem 3.11] and Zorn’s lemma, \(\mathcal{X}\) has an orthonormal basis \(\mathcal{A}\) consisting of mutually orthogonal non zero partial isometries. That is, for each \(e_i \in \mathcal{A}\), \((e_i, e_i)\) is a nonzero orthogonal projection in \(A\), and for each \(g \in \mathcal{X}\), \(g = \sum_i e_i(e_i, g)\). In particular, \(\sum_i \Theta_{e_i, e_i} = I\mathcal{X}\) where \(\Theta_{e_i, e_i}\) is the rank-one operator defined by \(\Theta_{e_i, e_i}(z) = e_i \cdot (e_i, z)_A\). The elements of \(A\) are linearly independent, otherwise there would be \(e_j \in \mathcal{A}\) such that \(e_j = \sum_{i \neq j} z_i e_i (z_i \in \mathbb{C})\). But then we would have \(0 < \langle e_j, e_j \rangle_A < \langle e_j, \sum_{i \neq j} z_i e_i \rangle_A = \sum_{i \neq j} z_i < \langle e_j, e_i \rangle_A = \sum_{i \neq j} z_i(0) = 0\).

Since \(\mathcal{X} = \ell^\infty\) can be identified with \(C(\beta \mathbb{N})\), where \(\beta \mathbb{N}\) denotes the Stone Cech compactification of \(\mathbb{N}\), the maximal ideals of \(C(X)\) are \(\{I_x\}_{x \in X}\) where \(I_x = \{\sum_{y \in X} a_y \delta_y : a_y \in \mathbb{C}, \sup|a_y| < \infty\text{ and }y \neq x\}\). The maximal \(C(X)\)-\(C(X)\)-submodules of \(\mathcal{X}\) are \(\{\mathcal{X}_j\}_{j \in \{1, \ldots, n\}}\) where \(\mathcal{X}_j = \{\sum_i z_i e_i : z_i \in \mathbb{C}, \sup|z_i| < \infty\text{ and }i \neq j\}\). Since the Rieffel correspondence of \(\mathcal{A} \mathcal{X}_A\) pairs maximal ideals of \(\mathcal{A} = C(X)\) with maximal submodules of \(\mathcal{X}\), we have that \(\dim(\mathcal{X}) = |\mathcal{A}| = \dim(C(X)) = |X|\).

If the corresponding submodule (under the Rieffel correspondence) for the maximal ideal \(I_x\) is the maximal submodule \(\mathcal{X}_j\), then \(\mathcal{X}_j = \mathcal{X} \cdot I_x\) [19, Lemma 3.23]. So \(e_i \cdot \delta_y \neq e_j\) for all \(e_i \in \mathcal{A}\). But by Cohen’s factorization theorem, \(e_j = e \cdot a\) for some \(e \in \mathcal{X}, a \in C(X)\). So we must have \(e_i \cdot \delta_x = e_j\) for some \(e_i \in \mathcal{A}\). Then we have \(e_i = e_j\) (otherwise we would have \(e_i \cdot \delta_x = e_j\) for \(i \neq j\) and \(0 < \langle e_j, e_j \rangle_{C(X)} = \langle e_j, e_i \rangle_{C(X)} \cdot \delta_x = \langle e_j, e_i \rangle_{C(X)} \cdot 0 = 0\)). Thus the element \(x \in X\) (and therefore \(\delta_x \in C(X)\)) gets uniquely paired up with the element \(e_j \in \mathcal{A}\). Likewise, each basis element \(\delta_x \in C(X)\) gets uniquely paired up with a basis element \(e_i \in \mathcal{X}\) by the right action relation \(e_i \cdot \delta_y = e_j\). So we have a bijection \(R\) between the basis elements \(\{e_i\}\) in \(\mathcal{X}\) and the basis elements \(\{\delta_y\}\) in \(C(X)\). Applying the same analysis to the Rieffel correspondence between \(C(X)\) and \(\mathcal{X}\), but now with \(C(X)\) giving the left action on \(\mathcal{X}\), we have another bijection \(L : \{\delta_y\}_{y \in X} \to \mathcal{A}\). Thus \(\sigma = R \circ L\) is a permutation of \(\{\delta_x : x \in X\}\), or equivalently, \(\sigma\) is a permutation of \(X\) given by the Rieffel correspondence of \(C(X)\mathcal{X}_{C(X)}\). Note that \(\delta_y \cdot L(\delta_x) \cdot \delta_w = L(\delta_x)\) if \(y = x\) and \(w = \sigma(x)\). If we let \(A_A\mathcal{A}_{\sigma}\) denote the algebra \(A\) (viewed as a bimodule over itself) with a modified right action and right inner product (given by \(\sigma\), we obtain the following result:
Lemma 4.6 $C(X)\mathcal{X}_C(X) = \mathcal{X}_A$ is a $W^*$-equivalence bimodule if and only if $\mathcal{X}_A$ is of the form $A\mathcal{X}_A$, where $\sigma = R \circ L$ is the permutation given by the Rieffel correspondence of $\mathcal{X}_A$.

Proof Let $A\mathcal{X}_A$ denote the algebra $A$ (viewed as a $W^*$-bimodule over itself) with the right and left actions given by:

$$\delta_x \cdot \delta_y = \begin{cases} \delta_x & \text{if } y = \sigma(x) \\ 0 & \text{otherwise} \end{cases} \quad \delta_y \cdot \delta_x = \begin{cases} \delta_x & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

and the right and left inner products given by:

$$\langle \delta_x, \delta_y \rangle_A = \begin{cases} \delta_{\sigma(x)} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad A \langle \delta_x, \delta_y \rangle = \begin{cases} \delta_x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

So that if $\sum_{x \in X} a_x \delta_x$, $\sum_{x \in X} c_x \delta_x \in A_\sigma$ and $\sum_{y \in X} b_y \delta_y \in A$, then the right and left actions and inner products are:

$$\sum_{x \in X} a_x \delta_x \cdot \sum_{y \in X} b_y \delta_y = \sum_{x \in X} a_x b_{\sigma(x)} \delta_x$$
$$\sum_{y \in X} b_y \delta_y \cdot \sum_{x \in X} a_x \delta_x = \sum_{x \in X} a_x b_x \delta_x$$
$$\langle \sum_{x \in X} a_x \delta_x, \sum_{y \in X} c_y \delta_y \rangle_A = \sum_{x \in X} a_x c_x \delta_{\sigma(x)} = \sum_{x \in X} a_{\sigma^{-1}(x)} c_{\sigma^{-1}(x)} \delta_x$$
$$A \langle \sum_{x \in X} a_x \delta_x, \sum_{y \in X} c_y \delta_y \rangle = \sum_{x \in X} a_x c_x \delta_x$$

First we check that $A\mathcal{X}_A$ is a $W^*$-equivalence bimodule:

Since $\{\delta_x : x \in X\}$ spans $C(X) = A$ and $\{\delta_{\sigma^{-1}(x)}, \delta_{\sigma^{-1}(x)}\}C(X) = \delta_x$ and $C(X)\delta_x = \delta_x$, we have that $A_\sigma$ is a $w^*$-full left Hilbert $A$-module and a $w^*$-full right Hilbert $A$-module.

Let $s, t \in A_\sigma, a \in A$.

$$\langle a \cdot s, t \rangle_A = \left\langle \left( \sum_{y \in X} a_y \delta_y \right), \sum_{w \in X} t_w \delta_w \right\rangle_A = \left\langle \sum_{x \in X} a_x z_x \delta_x, \sum_{w \in X} t_w \delta_w \right\rangle_A = \sum_{x \in X} \overline{a_x} z_x t_x \delta_{\sigma(x)} = \left\langle \sum_{x \in X} z_x \delta_x, \sum_{w \in X} a_w t_w \delta_w \right\rangle_A = \langle s, a^* \cdot t \rangle_A$$
\[A(s \cdot a, t) = A\left( \sum_{x \in X} z_x \delta_x \cdot \left( \sum_{y \in X} a_y \delta_y \right), \sum_{w \in X} t_w \delta_w \right)\]
\[= A\left( \sum_{x \in X} a_{\sigma(x)} z_x \delta_x, \sum_{w \in X} t_w \delta_w \right) = \sum_{x \in X} a_{\sigma(x)} z_x t_x \delta_x\]
\[= A\left( \sum_{x \in X} z_x \delta_x, \sum_{w \in X} a_{\sigma(w)} t_w \delta_w \right)\]
\[= A\left( \sum_{x \in X} z_x \delta_x, \sum_{w \in X} t_w \delta_w \right) \cdot \left( \sum_{y \in X} \overline{a_y} \delta_y \right) = A\langle s, t \cdot a^* \rangle\]

Let \(r, s, t \in A_\sigma\).

\[A(r, s) \cdot t = A\left( \sum_{y \in X} r_y \delta_y, \sum_{x \in X} z_x \delta_x \right) \cdot \sum_{w \in X} t_w \delta_w = \left( \sum_{x \in X} r_x \overline{z_x} \delta_x \right) \cdot \sum_{w \in X} t_w \delta_w\]
\[= \sum_{x \in X} r_x \overline{z_x} t_x \delta_x = \sum_{y \in X} r_y \delta_y \cdot \left( \sum_{x \in X} z_x t_x \delta_{\sigma(x)} \right)\]
\[= \left( \sum_{y \in X} r_y \delta_y \right) \cdot \left( \sum_{x \in X} z_x \delta_x, \sum_{w \in X} t_w \delta_w \right) = r \cdot \langle s, t \rangle_A\]

Since \(A\) has an operator space predual (being a \(W^*\)-algebra), it is a selfdual \(C^*\)-module over itself. Thus \(A_{A,A}\) is a \(W^*\)-equivalence bimodule.

Let \(\psi : A_{A,A} \to A\hat{x}_A\) be the linear extension of the bijection \(L : \{\delta_y\}_{y \in X} \to A\) that we encountered above when we studied the Rieffel correspondence of \(A\hat{x}_A\). For any element \(e = \sum_{x \in X} z_x \delta_x \in A_\sigma\), we have \(\psi(e) = \psi(\sum_{x \in X} z_x \delta_x) = \sum_{x \in X} z_x L(\delta_x)\). We show now that \(\psi\) is a \(W^*\)-equivalence bimodule isomorphism. Recall that \(\delta_y \cdot L(\delta_x) \cdot \delta_w = L(\delta_x)\) in \(A\hat{x}_A\) if \(y = x\) and \(w = \sigma(x)\). Let \(a, b \in A\) and \(e \in A_\sigma\). Then

\[
\psi(a \cdot e \cdot b) = \psi \left( \sum_{y \in X} a_y \delta_y \cdot \sum_{x \in X} z_x \delta_x \cdot \sum_{w \in X} b_w \delta_w \right) = \psi \left( \sum_{x \in X} a_x z_x b_{\sigma(x)} \delta_x \right)\]
\[= \sum_{x \in X} \psi(a_x z_x b_{\sigma(x)} \delta_x) = \sum_{x \in X} a_x z_x b_{\sigma(x)} L(\delta_x)\]
\[= \sum_{y \in X} a_y \delta_y \cdot \sum_{x \in X} z_x L(\delta_x) \cdot \sum_{w \in X} b_w \delta_w\]
\[= \left( \sum_{y \in X} a_y \delta_y \right) \cdot \psi \left( \sum_{x \in X} z_x \delta_x \right) \cdot \sum_{w \in X} b_w \delta_w = a \cdot \psi(e) \cdot b\]
So $\psi$ is a bimode map. Note that if the Rieffel correspondence pairs up $e_j \in \mathcal{X}$ and $\delta_x \in C(X)$ by $e_j : \delta_x = e_j$, then since $1 = ||\langle e_j, e_j \rangle_{C(X)}|| = ||\langle e_j : \delta_x, e_j : \delta_x \rangle_{C(X)}|| = ||\delta_x \langle e_j, e_j \rangle_{C(X)} \delta_x || = ||\delta_x \langle e_j, e_j \rangle_{C(X)} \delta_x ||$ and $\delta_x \delta_y = 0$ for all $x \neq y$, we must have $\langle e_j, e_j \rangle_{C(X)} = \delta_x = R(e_j)$. Likewise, for any $e_i \in \mathcal{X}$ and $\delta_x \in C(X)$ paired up by $\delta_x : e_j = e_j$, we must have $C(X) \langle e_j, e_j \rangle = \delta_x$. Thus

$$\langle L(\delta_x), L(\delta_x) \rangle_{C(X)} = R(L(\delta_x)) = \sigma(\delta_x) = \delta_{\sigma(x)} \quad \text{and}$$

$$C(X) \langle L(\delta_x), L(\delta_x) \rangle = L^{-1}(L(\delta_x)) = \delta_x$$

So if $e, f \in A_{\sigma}$, we have

$$\langle e, f \rangle_A = \left\langle \sum_{x \in X} z_x \delta_x, \sum_{y \in X} z_y \delta_y \right\rangle_A = \sum_{x \in X} \overline{z_x} z_x \langle L(\delta_x), L(\delta_x) \rangle_A = \sum_{x \in X} \overline{z_x} \langle L(\delta_x), L(\delta_x) \rangle_A$$

$$= \left\langle \sum_{x \in X} z_x L(\delta_x), \sum_{y \in X} z_y L(\delta_y) \right\rangle_A = \left\langle \psi \left( \sum_{x \in X} z_x \delta_x \right), \psi \left( \sum_{y \in X} z_y \delta_y \right) \right\rangle_A$$

$$= \langle \psi(e), \psi(f) \rangle_A$$

$$A \langle e, f \rangle_A = \left\langle \sum_{x \in X} z_x \delta_x, \sum_{y \in X} z_y \delta_y \right\rangle_A = \sum_{x \in X} \overline{z_x} \langle L(\delta_x), L(\delta_x) \rangle_A = \left\langle \sum_{x \in X} z_x L(\delta_x), \sum_{y \in X} z_y L(\delta_y) \right\rangle_A$$

$$= \left\langle \psi \left( \sum_{x \in X} z_x \delta_x \right), \psi \left( \sum_{y \in X} z_y \delta_y \right) \right\rangle_A = \langle \psi(e), \psi(f) \rangle_A.$$ 

Thus $\psi$ preserves both inner products (so it is injective). Since $A$ spans $\mathcal{X}$, and $\psi(L^{-1}(e_i)) = L((L^{-1}(e_i)) = e_i$, $\psi$ is surjective. Thus an isomorphism. $\square$

We can also view $A A_{\sigma} A$ as a graph correspondence. More precisely, let $G_{\sigma} = (G_0^0, G_1^1, r, s)$ be the directed graph given by $G_0^0 = X, G_1^1 = \{e_x\}_{x \in X}, r, s : G_1^1 \rightarrow G_0^0$ given by $r(e_x) = x$ and $s(e_x) = \sigma(x)$. Then the graph correspondence $C(G_0^0) C(G_1^1) C(G_0^0)$ associated to $G_{\sigma}$ with the usual actions and inner products:

$$\sum_{x \in X} a_x \delta_{e_x} \cdot \sum_{y \in X} b_y \delta_{e_y} = \sum_{x \in X} a_x b_{\sigma(x)} \delta_{e_x}$$

$$\sum_{y \in X} b_y \delta_{e_y} \cdot \sum_{x \in X} a_x \delta_{e_x} = \sum_{x \in X} a_x b_x \delta_{e_x}$$

$$C(X) \left\langle \sum_{x \in X} a_x \delta_{e_x}, \sum_{y \in X} c_y \delta_{e_y} \right\rangle_{C(X)} = \sum_{x \in X} a_x c_{\sigma(x)} \delta_{\sigma(x)} = \sum_{x \in X} a_{\sigma^{-1}(x)} c_{\sigma^{-1}(x)} \delta_x$$

$$\sum_{x \in X} a_x \delta_{e_x} \cdot \sum_{y \in X} c_y \delta_{e_y} = \sum_{x \in X} a_x c_x \delta_x$$
is isomorphic to $A A \sigma A \cong A X_A$ via the map $\omega : C(G^1_\sigma) \rightarrow A_\sigma$ given by $\omega(\delta_\epsilon) = \delta_\chi$. Note that $|G^1_\sigma| = |G^0_\sigma|$ and $r, s$ are bijections. So if the graph $G_\sigma$ is finite then $G_\sigma$ is either a cycle or a disconnected union of cycles (given by the cycle decomposition of $\sigma$). Note also that each permutation $\sigma$ of $X$ gives an equivalence bimodule $A C(G^1_\sigma)_A \cong A A \sigma A$.

If $[X]$ denotes the isomorphism class of $A X_A$, then we have:

**Lemma 4.7** $P = \{[X] : A \xrightarrow{WME} X A\}$ is a group with the operation given by $[X] * [Y] = [X \overline{\otimes}_A Y]$.

**Proof** First we show that if $X$ and $Y$ are $A \cdot A$ $W^*$-equivalence bimodules, then so is $X \overline{\otimes}_A Y$. If $X$ is a $w^*$-full right $A$-module, and by Cohen’s factorization theorem, $Y = A \cdot Y$. Thus $(X, X)_A \cdot Y$ is $w^*$-dense in $Y$. So $(X \overline{\otimes}_A Y, X \overline{\otimes}_A Y)_A = (Y, (X, X)_A \cdot Y)_A$ is $w^*$-dense in $(Y, (X, X)_A \cdot Y)_A$, which is $w^*$-dense in $A$, since $Y$ is a $w^*$-full right $A$-module. So $(X \overline{\otimes}_A Y, X \overline{\otimes}_A Y)_A$ is $w^*$-dense in $A$. Thus $X \overline{\otimes}_A Y)_A$ is a full right Hilbert $A$-module. Likewise, $X \overline{\otimes}_A Y)_A$ is a full left Hilbert $A$-module.

Let $x, y \in A X \overline{\otimes}_A Y$ and $a, b \in A$. Then $(a \cdot x, y)_A = \langle a \cdot (x_1 \otimes y_1), x_2 \otimes y_2 \rangle = \langle a \cdot x_1 \otimes (y_1 \otimes y_2), y_2 \rangle = \langle (y_1, (a \cdot x_2 \otimes y_2)_A = \langle (x, a^* \cdot y)_A \rangle A \langle x \cdot b, y \rangle = \langle (x_1 \otimes y_1), a^* \cdot (x_2 \otimes y_2)_A = \langle a^* \otimes (y_1 \otimes y_2), y_2 \rangle = \langle (x_1 \otimes y_1 \cdot b \cdot x_2 \otimes y_2), y_2 \rangle = \langle (x, x_2 \cdot A \otimes y_2 \cdot b^*_A = \langle (x_1 \otimes y_1, x_2 \otimes y_2 \cdot b^*_A = \langle (x, y \cdot b^*_A = \langle a \cdot x, y \rangle_A = \langle a \cdot x_1 \otimes y_1, x_2 \otimes y_2 \cdot b \cdot (y_1 \otimes y_2 \cdot b^*_A = \langle a \cdot x_1 \otimes y_1, x_2 \otimes y_2 \cdot b^*_A = \langle (x, y \cdot z)_A$. Thus $X \overline{\otimes}_A Y_A$ is a $W^*$-equivalence bimodule. Since $X \overline{\otimes}_A Y_A = A E \cong A E \overline{\otimes}_A A_A$, the identity of $P$ is $[A]$. By Lemma 3.1, $[X]^{-1} = [X]$. Thus $P$ is a group. □

**Lemma 4.8** If $\sigma, \tau \in S_X$ then $A A_\sigma A \cong A A_\tau A$ as $W^*$-equivalence bimodules.

**Proof** If $a = \sum_{x \in X} a_x \delta_x \in A$, denote $\sum_{x \in X} a_{\sigma(x)} \delta_x$ by $a_\sigma$. Consider the triple $(\iota, \iota, \iota) : A A_\sigma A \rightarrow A A_\sigma A$ where $\iota$ is the identity map on $A$ and $\iota : A \rightarrow A$ is given by $a_\tau^{-1}(a) = a_{\tau^{-1}}$. That is, $\iota(a) = a_{\tau^{-1}}$. Then

$$\iota(a \cdot b) = \sum_{x \in X} a_x \delta_x \cdot \sum_{x \in X} b_x \delta_x = \iota \left( \sum_{x \in X} a_x \delta_x \cdot \sum_{x \in X} b_x \delta_x \right) = \sum_{x \in X} a_x b_{\sigma(x)} \delta_x = \iota \left( \sum_{x \in X} a_x \delta_x \cdot \sum_{x \in X} b_{\sigma(x)} \delta_x \right) = \iota \left( \sum_{x \in X} a_x \delta_x \right) \cdot \tau \left( \sum_{x \in X} b_{\sigma(x)} \delta_x \right) \cdot \iota \left( \sum_{x \in X} b_x \delta_x \right) = \iota(a) \cdot \iota(e) \cdot \iota(b).$$

So $(\iota, \iota, \iota)$ is a bimodule homomorphism.

$$\iota \left( \sum_{x \in X} z_x \delta_x \right) \cdot \tau \left( \sum_{x \in X} w_x \delta_x \right) = \left( \sum_{x \in X} z_x \delta_x, \sum_{x \in X} w_x \delta_x \right)_A.$$
\[ \sum_{x \in X} z_{\tau^{-1}(x)} w_{\tau^{-1}(x)} \delta_x = \pi \left( \sum_{x \in X} z_{\sigma^{-1}(x)} w_{\sigma^{-1}(x)} \delta_x \right) = \pi \left( \sum_{x \in X} z_{x} \delta_x, \sum_{x \in X} w_{x} \delta_x \right)_A \]

and \[ A \left( \sum_{x \in X} z_{x} \delta_x, \sum_{x \in X} w_{x} \delta_x \right) = \sum_{x \in X} z_{x} \delta_x \sum_{x \in X} w_{x} \delta_x \]

So \((\iota, \iota, \pi)\) preserves inner products. Thus \((\iota, \iota, \pi)\) is a \(W^*\)-equivalence bimodule isomorphism. \(\square\)

By Lemmas 4.6 and 4.8, \(P = \{ [X] : A \overset{WME}{\sim} X A \} \) consists of only one element:

**Theorem 4.9** If \(A = C(X)\) for some set \(X\), then \(P = \{ [X] : A \overset{WME}{\sim} X A \} = \{ [A] \} \).

Now consider the \(W^*\)-equivalence bimodule \(A_\sigma \otimes_\sigma A_{\tau A}\). Since this bimodule is balanced over \(A\), we have that \(\delta_x \cdot \delta_y = \delta_x \cdot \delta_y\) if and only if \(\sigma(x) = z = y\). Thus the non zero elements of \(A_\sigma \otimes_\sigma A_{\tau A}\) are of the form \(\sum (z_x \delta_x \otimes w_{\sigma(x)} \delta_{\sigma(x)})\), where \(z_x, w_{\sigma(x)} \in \mathbb{C}\). Note that if \(\sigma, \tau \in S_X\), then Lemmas 4.7 and 4.8 say that \(A_\sigma \otimes_\sigma A_{\tau A}\) is isomorphic to \(A A_A\). Here we give an explicit \(W^*\)-isomorphism between these two \(W^*\)-equivalence bimodules. Consider the triple

\[ (\omega, \psi, \pi) : A_\sigma \otimes_\sigma A_{\tau A} \rightarrow A A_A \]

where \(\omega : A \rightarrow A\) is given by \(\omega(a) = a_{\sigma^{-1}}\) (that is, \(\pi(\sum_{x \in X} a_x \delta_x) = \sum_{x \in X} a_{\sigma^{-1}(x)} \delta_x\)), \(\psi : A_\sigma \otimes_\sigma A_{\tau A} \rightarrow A\) is given by \(\psi(\sum_{x \in X} z_x \delta_x \otimes A \sum_{x \in X} w_x \delta_x) = \psi(\sum_{x \in X} z_x \delta_x \otimes w_{\sigma(x)} \delta_{\sigma(x)}) = \sum_{x \in X} z_x w_{\sigma(x)} \delta_{\sigma(x)}\) and \(\pi : A \rightarrow A\) is given by \(\pi(a) = a_{\tau}\). That is, \(\pi(\sum_{x \in X} a_x \delta_x) = \sum_{x \in X} a_{\tau(x)} \delta_x\). Then

\[
\begin{align*}
\psi \left( \sum_{x \in X} a_x \delta_x \cdot \left( \sum_{x \in X} z_x \delta_x \otimes A \sum_{x \in X} w_x \delta_x \right) \cdot \sum_{x \in X} b_x \delta_x \right) &= \psi \left( \sum_{x \in X} a_x z_x \delta_x \otimes w_{\sigma(x)} b_{\tau(\sigma(x))} \delta_{\sigma(x)} \right) \\
&= \psi \left( \sum_{x \in X} a_x z_x \delta_x \otimes w_{\sigma(x)} b_{\tau(\sigma(x))} \delta_{\sigma(x)} \right) = \sum_{x \in X} a_x z_x w_{\sigma(x)} b_{\tau(\sigma(x))} \delta_{\sigma(x)} \\
&= \sum_{x \in X} a_x \delta_{\sigma(x)} \cdot \sum_{x \in X} z_x w_{\sigma(x)} \delta_{\sigma(x)} \cdot \sum_{x \in X} b_{\tau(\sigma(x))} \delta_{\sigma(x)}
\end{align*}
\]
\[ = \sum_{x \in X} a_{\sigma^{-1}(x)} \delta_x \cdot \sum_{x \in X} z_x w_{\sigma(x)} \delta_{\sigma(x)} \cdot \sum_{x \in X} b_{\tau(x)} \delta_x \]
\[ = \omega \left( \sum_{x \in X} a_x \delta_x \right) \cdot \psi \left( \sum_{x \in X} z_x \delta_x \otimes w_{\sigma(x)} \delta_{\sigma(x)} \right) \cdot \pi \left( \sum_{x \in X} b_x \delta_x \right) \]

So \((\omega, \psi, \pi)\) is a bimodule homomorphism.

\[
\left\langle \psi \left( \sum_{x \in X} z_x \delta_x \otimes_A \sum_{x \in X} w_x \delta_x \right), \psi \left( \sum_{x \in X} u_x \delta_x \otimes_A \sum_{x \in X} v_x \delta_x \right) \right\rangle_A
\]
\[
= \left\langle \psi \left( \sum_{x \in X} z_x \delta_x \otimes w_{\sigma(x)} \delta_{\sigma(x)} \right), \psi \left( \sum_{x \in X} u_x \delta_x \otimes v_{\sigma(x)} \delta_{\sigma(x)} \right) \right\rangle_A
\]
\[
= \left\langle \sum_{x \in X} z_x w_{\sigma(x)} \delta_{\sigma(x)}, \sum_{x \in X} u_x v_{\sigma(x)} \delta_{\sigma(x)} \right\rangle_A = \sum_{x \in X} z_x w_{\sigma(x)} u_x v_{\sigma(x)} \delta_{\sigma(x)}
\]
\[
= \pi \left( \sum_{x \in X} \langle w_{\sigma(x)} \delta_{\sigma(x)}, z_x \delta_x, u_x \delta_x \rangle_A \cdot v_{\sigma(x)} \delta_{\sigma(x)} \right)
\]
\[
= \pi \left( \sum_{x \in X} \langle w_{\sigma(x)} \delta_{\sigma(x)}, z_x \delta_x, u_x \delta_x \rangle A \cdot v_{\sigma(x)} \delta_{\sigma(x)} \right)
\]
\[
= \pi \left( \sum_{x \in X} \langle z_x \delta_x \otimes w_{\sigma(x)} \delta_{\sigma(x)}, \sum_{x \in X} u_x \delta_x \otimes v_{\sigma(x)} \delta_{\sigma(x)} \rangle_A \right)
\]
\[
= \pi \left( \sum_{x \in X} z_x \delta_x \otimes \sum_{x \in X} w_x \delta_x, \sum_{x \in X} u_x \delta_x \otimes \sum_{x \in X} v_x \delta_x \right) \right\rangle_A
\]

and

\[
\left\langle \psi \left( \sum_{x \in X} z_x \delta_x \otimes A \sum_{x \in X} w_x \delta_x \right), \psi \left( \sum_{x \in X} u_x \delta_x \otimes A \sum_{x \in X} v_x \delta_x \right) \right\rangle
\]
\[
= A \left\langle \psi \left( \sum_{x \in X} z_x \delta_x \otimes A \sum_{x \in X} w_x \delta_x \right), \psi \left( \sum_{x \in X} u_x \delta_x \otimes A \sum_{x \in X} v_x \delta_x \right) \right\rangle
\]
\[
= A \left\langle \sum_{x \in X} z_x w_{\sigma(x)} \delta_{\sigma(x)}, \sum_{x \in X} u_x v_{\sigma(x)} \delta_{\sigma(x)} \right\rangle = \sum_{x \in X} z_x w_{\sigma(x)} u_x v_{\sigma(x)} \delta_{\sigma(x)}
\]
\[
= \omega \left( \sum_{x \in X} z_x w_{\sigma(x)} u_x v_{\sigma(x)} \delta_x \right) = \omega \left( \sum_{x \in X} A \langle z_x \delta_x, u_x v_{\sigma(x)} \delta_{\sigma(x)} \rangle \right)
\[\omega \left( \sum_{x \in X} A \langle z_x \delta_x, u_x \delta_x \cdot v_{\sigma(x)}w_{\sigma(1)}\delta_{\sigma(x)} \rangle \right)\]

\[= \omega \left( \sum_{x \in X} A \langle z_x \delta_x, u_x \delta_x \cdot (v_{\sigma(x)}\delta_{\sigma(x)}, w_{\sigma(1)}\delta_{\sigma(x)}) \rangle \right)\]

\[= \omega \left( A \left( \sum_{x \in X} z_x \delta_x \otimes w_{\sigma(x)}\delta_{\sigma(x)}, \sum_{x \in X} u_x \delta_x \otimes v_{\sigma(x)}\delta_{\sigma(x)} \right) \right)\]

\[= \omega \left( A \left( \sum_{x \in X} z_x \delta_x \otimes w_{\sigma(x)}\delta_{\sigma(x)}, \sum_{x \in X} u_x \delta_x \otimes v_{\sigma(x)}\delta_{\sigma(x)} \right) \right)\]

So \((\omega, \psi, \pi) : A A \sigma \otimes A A \tau A \rightarrow A A \) preserves inner products. Thus, it is injective. For each \(\sum_{x \in X} ax \delta x \in A A\), \(\psi \left( \sum_{x \in X} ax \delta_{\sigma^{-1}(x)} \otimes \sum_{x \in X} \delta x \right) = \psi \left( \sum_{x \in X} ax \delta_{\sigma^{-1}(x)} \otimes \delta x \right) = \sum_{x \in X} ax \delta x\). So \((\omega, \psi, \pi)\) is surjective. Thus a \(W^*\)-equivalence bimodule isomorphism.

**Lemma 4.10** Let \(A E A\) and \(B F B\) be two \(W^*\)-correspondences. If \(A E A \cong B F B\) then \(A E A \overset{WME}{\sim} B F B\).

**Proof** If \(A E A \cong B F B\) then there is a \(W^*\)-correspondence isomorphism \((\pi, \phi) : A E A \rightarrow B F B\), where \(\phi\) is a vector space isomorphism and \(\pi : A \rightarrow B\) is a \(W^*\)-algebra isomorphism. Then \(B\) is an \(A-B\) \(W^*\)-equivalence bimodule with the left action given by \(a \cdot b = \pi(a)b\), right action given by multiplication in \(B\) and inner products given by \(\langle b_1, b_2 \rangle = b_1^* b_2\) and \(A \langle b_1, b_2 \rangle = \pi^{-1}(b_1 b_2^*)\).

We show that \((\iota, \varphi) : B \bar{E} \bar{A} \overset{\sim}{\rightarrow} B F B\) is a \(W^*\)-correspondence isomorphism, where \(\iota\) is the identity map and \(\varphi : \bar{B} \bar{E} \bar{A} \overset{\sim}{\rightarrow} F B\) is defined by \(\varphi(b \otimes e \otimes c) = b^* \cdot \phi(e) \cdot c\). Let \(e, g \in E\) and \(a, b, c, d, \alpha, \beta \in B\). Then

\[
\varphi(\alpha \cdot (b \otimes e \otimes c) \cdot \beta) = \varphi(b \cdot \alpha^* \otimes e \otimes c \cdot \beta) = \varphi(b \alpha^* \otimes e \otimes c \beta) = \alpha \cdot (b^* \cdot \phi(e) \cdot c) \cdot \beta = \alpha \cdot \varphi(b \otimes e \otimes c) \cdot \beta
\]

So \((\iota, \varphi, \iota)\) is a correspondence homomorphism.
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\[ \langle \varphi(\tilde{b} \otimes e \otimes c), \varphi(\tilde{a} \otimes g \otimes d) \rangle_B \]

\[ = \langle b^* \cdot \phi(e) \cdot c \cdot a^* \cdot \phi(g) \cdot d \rangle_B = c^* \langle b^* \cdot \phi(e), a^* \cdot \phi(g) \rangle_Bd \]

\[ = c^* \langle \phi(e), b \cdot a^* \cdot \phi(g) \rangle_Bd = c^* \langle \phi(e), ba^* \cdot \phi(g) \rangle_Bd \]

\[ = c^* \langle \phi(e), \pi(\pi^{-1}(ba^*) \cdot g) \rangle_Bd = c^* \langle \phi(e), \phi(\pi^{-1}(ba^*) \cdot g) \rangle_Bd \]

\[ = c^* \pi((e, \pi^{-1}(ba^*), g)A)d = \langle c, \pi((e, \pi^{-1}(ba^*) \cdot g)A) \rangle_Bd \]

\[ = \langle c, \langle e, \pi^{-1}(ba^*) \rangle_Bd \rangle_B = \langle c, \langle b \rangle_B, a \rangle_Bd \]

\[ = \langle \tilde{b} \otimes e \otimes c, \tilde{a} \otimes g \otimes d \rangle_B \]

So $(\iota, \varphi, \iota)$ preserves the inner product. Thus it is injective. Since for each $f \in F$, there is $e \in E$ such that $\phi(e) = f$, we have $\varphi(1 \otimes e \otimes 1) = f$. So $\varphi$ is surjective, thus a $W^*$-correspondence isomorphism.

Since $b B \overline{\otimes} A E \otimes B B \simeq b F$, we have $A B \overline{\otimes} B F \simeq A B \overline{\otimes} B B E \otimes A B B \simeq A A \overline{\otimes} A A B \otimes B B \simeq \widehat{A} A \otimes A A B B \simeq A A \otimes A A B B$. Thus $A A E_A \simeq B F_B$. $\square$

**Theorem 4.11** If $A E_A$ and $B D_B$ are $W^*$-graph correspondences then $A E_A \simeq B D_B$ if and only if $A E_A \simeq B D_B$.

**Proof** One direction was already shown in Lemma 4.10. Now we show the converse.

If $A E_A \simeq B D_B$ then we have $A \simeq B$, and since $A$ and $B$ are commutative, we have $A \simeq B$ (recall that if two $W^*$-algebras are Morita equivalent then their centers are isomorphic). So there is a $W^*$-algebra isomorphism $\alpha : B \to A$, such that $(\alpha, \iota) : B D_B \to A D_A$ is a $W^*$-correspondence isomorphism. Then by Lemma 4.10, we have $A E_A \simeq A D_A$. By Lemma 6.4, a $W^*$-equivalence bimodule $A \overline{\otimes} A$ is isomorphic to $A A_A$, where $A A_A$ is the same as $A A_A$ but with a modified right action and right inner product determined by some permutation $\sigma$ of $S$. Then $A E_A \simeq A D_A$ implies $A \overline{\otimes} A D \simeq E \overline{\otimes} A A$. So using Lemma 3.1, we have

\[ B D_B \simeq A D_A \simeq A A \otimes A D \simeq \widehat{A} A \otimes A A \otimes A D \simeq \widehat{A} A \otimes A E \overline{\otimes} A A \sigma \]

Thus, to show that $A E_A \simeq A D_A$, all we need to show is that $\widehat{A} A \otimes A E \overline{\otimes} A \sigma \simeq A E_A$.

Consider the pair $(\pi, \phi)$ where $\phi : A A \otimes A E \overline{\otimes} A A \sigma \to A E_A$ is given by

\[ \phi(\tilde{g} \otimes x \otimes b) = a^* \cdot x \cdot b \quad \text{and} \quad \pi : A \to A \]

is given by $\pi(c) = c$. That is, $\pi(\sum x_i \delta_{v_i}) = \sum c_i \sigma(v_i)$. Clearly, $\pi$ is a $W^*$-isomorphism. Now we show that $(\pi, \phi) : A \hateq A \sigma \otimes A E \otimes A A A A A \to A E_A$ is a $W^*$-correspondence isomorphism. Let $a, b, c, d, e \in A_A, \alpha, \beta \in A$ and $x, y \in E$.

\[ \phi(\alpha \cdot (\tilde{a} \otimes x \otimes b) \cdot \beta) = \phi((\alpha \cdot \tilde{a}) \otimes x \otimes (b \cdot \beta)) = \phi((\alpha \cdot \tilde{a} \cdot \alpha^*) \otimes x \otimes (b \cdot \beta)) \]

\[ = \phi((\alpha \cdot \tilde{a} \cdot \alpha^*) \otimes x \otimes (b \cdot \beta)) = \alpha_{\sigma} a^* \cdot x \cdot b \beta_{\sigma} \]

\[ = \alpha_{\sigma} \cdot (a^* \cdot x \cdot b) \cdot \beta_{\sigma} = \pi(\alpha) \cdot \phi(\tilde{a} \otimes x \otimes b) \cdot \pi(\beta) \]
\[ \langle \phi(\tilde{a} \otimes x \otimes b), \phi(\tilde{c} \otimes y \otimes d) \rangle_A \]
\[ = \langle a^* \cdot x \cdot b, c^* \cdot y \cdot d \rangle_A = b^* \langle a^* \cdot x, c^* \cdot y \rangle_A d \]
\[ = \pi((b^*(a^* \cdot x, c^* \cdot y)A)\sigma^{-1}) = \pi((b^*(a^* \cdot x, ac^* \cdot y)A)\sigma^{-1}) \]
\[ = \pi((b^*(ac^* \cdot y, x)A)A) = \pi(((ac^* \cdot y, x)A \cdot b, d)\sigma^{-1}) \]
\[ = \pi((x, ac^* \cdot y)A \cdot b, d)\sigma^{-1} \]
\[ = \pi((b, x, (\tilde{a}, \tilde{c})A \cdot y)A \cdot d)\sigma^{-1} \]
\[ = \pi((b, (\tilde{a} \otimes x, \tilde{c} \otimes y)A \cdot d)\sigma^{-1}). \]

So \( \phi \) is isometric, thus injective. Since for each \( e \in E, \phi(1_A \otimes e \otimes 1_A) = 1 \cdot e \cdot 1 = e \), \( \phi \) is surjective. Thus \( (\pi, \phi) \) is a \( W^* \)-correspondence isomorphism. \( \square \)

Two directed graphs \( G = (G^0, G^1, s_1, r_1) \) and \( F = (F^0, F^1, s_2, r_2) \) are isomorphic if there are two bijections \( \alpha : G^1 \to F^1 \) and \( \beta : G^0 \to F^0 \) such that for each edge \( e \in G^1 \), \( s_2(\alpha(e)) = \beta(s_1(e)) \) and \( r_2(\alpha(e)) = \beta(r_1(e)) \).

Clearly, if we draw a directed graph \( G = (G^0, G^1, s_1, r_1) \) and relabel its edges and its vertices then we produce a new graph \( F = (F^0, F^1, s_2, r_2) \) whose identical drawing implies that the two relabeling bijections \( \gamma : G^1 \to F^1 \) and \( \lambda : G^0 \to F^0 \) satisfy \( s_2(\gamma(e)) = \lambda(s_1(e)) \) and \( r_2(\gamma(e)) = \lambda(r_1(e)) \). So we obtain an isomorphic graph. In particular, if \( G^1 = F^1 \) and \( G^0 = F^0 \) then \( \gamma \) and \( \lambda \) are permutations.

**Theorem 4.12** Let \( A \mathcal{E}_A \) and \( B \mathcal{D}_B \) be \( W^* \)-graph correspondences associated to the directed graphs \( G = (G^0, G^1, s_1, r_1) \) and \( F = (F^0, F^1, s_2, r_2) \) respectively. \( A \mathcal{E}_A \cong B \mathcal{D}_B \) if and only if \( G \cong F \).

**Proof** First note that \( G \cong F \) is a particular case of having three bijections \( \alpha : G^1 \to F^1 \) and \( \beta, \gamma : G^0 \to F^0 \) such that for each edge \( e \in G^1 \), \( s_2(\alpha(e)) = \gamma(s_1(e)) \) and \( r_2(\alpha(e)) = \beta(r_1(e)) \). More precisely, \( G \cong F \) is the special case when \( \beta = \gamma \).

If \( G \) and \( F \) are isomorphic graphs, then there are two bijections \( \alpha : G^1 \to F^1 \) and \( \beta : G^0 \to F^0 \) such that for each edge \( e \in G^1 \), \( s_2(\alpha(e)) = \beta(s_1(e)) \) and \( r_2(\alpha(e)) = \beta(r_1(e)) \) then let \( \phi : E \to D \) be given by \( \phi(\delta_{e_1}) = \delta_{\alpha(e_1)} \) and \( \omega : A \to B \) be given by \( \omega(\delta_{v_1}) = \delta_{\beta(v_1)} \). Then

\[ \varphi(\delta_{r_1(e_1)} \cdot \delta_{e_1} \cdot \delta_{s_1(e_1)}) = \varphi(\delta_{e_1}) = \delta_{\alpha(e_1)} = \delta_{r_1(\alpha(e_1))} \cdot \delta_{s_2(\alpha(e_1))} \cdot \delta_{\beta(\alpha(e_1))} \cdot \delta_{\beta(\alpha(e_1))} \cdot \delta_{s_1(\alpha(e_1))} \]
\[ = \omega(\delta_{r_1(e_1)}) \cdot \varphi(\delta_{e_1}) \cdot \omega(\delta_{s_1(e_1)}) \]

and

\[ \langle \phi(\delta_{e_1}), \varphi(\delta_{e_1}) \rangle_A = \langle \delta_{\alpha(e_1)}, \delta_{\alpha(e_1)} \rangle_A = \delta_{s_2(\alpha(e_1))} = \delta_{\beta(s_1(e_1))} = \omega(\delta_{s_1(e_1)}) \]
\[ = \omega(\langle \delta_{e_1}, \delta_{e_1} \rangle_A) \]
\[
\varphi(a \cdot x \cdot b) = \varphi \left( \sum_k a_k \delta v_k \cdot \sum_i z_i \delta e_i \cdot \sum_j b_j \delta v_j \right)
\]

\[
= \varphi \left( \sum_k \sum_i \sum_j a_k z_i b_j \delta v_k \cdot \delta e_i \cdot \delta v_j \right)
\]

\[
= \varphi \left( \sum_k \sum_i z_i b_j \delta e_i \right)
\]

\[
= \sum_{r_1(e_i) = v_k} \sum_{s_1(e_i) = v_j} a_k z_i b_j \delta_{\alpha(e_i)}
\]

\[
= \sum_k \sum_i \sum j a_k z_i b_j \delta_{\beta(v_k)} \cdot \delta_{\alpha(e_i)} \cdot \delta_{\beta(v_j)}
\]

\[
= \left( \sum_k a_k \delta_{\beta(v_k)} \right) \cdot \left( \sum_i z_i \delta_{\alpha(e_i)} \right) \cdot \left( \sum_j b_j \delta_{\beta(v_j)} \right)
\]

\[
= \omega(a) \cdot \varphi(x) \cdot \omega(b)
\]

and

\[
\langle \varphi(x), \varphi(y) \rangle_B = \left\langle \sum_i z_i \delta_{\alpha(e_i)}, \sum_j y_j \delta_{\alpha(e_j)} \right\rangle_B = \sum_{i,j} \bar{z}_{i} \langle \delta_{\alpha(e_i)}, \delta_{\alpha(e_j)} \rangle_B y_j
\]

\[
= \sum_{i=j} \bar{z}_{i} \delta_{s_2(\alpha(e_i))} y_j = \sum_{i=j} \bar{z}_{i} \delta_{\beta(x_1(e_i))} y_j = \sum_{i=j} \omega(\bar{z}_{i} \delta_{s_1(e_i)} y_j)
\]

\[
= \omega\left( \sum_{i=j} (\bar{z}_{i} \delta_{e_i} \cdot y_j \delta_{e_j})_A y_j \right)
\]

\[
= \omega \left( \sum_{i,j} \langle z_i \delta_{e_i}, y_j \delta_{e_j} \rangle_A \right) = \omega \left( \sum_{i,j} \langle z_i \delta_{e_i}, y_j \delta_{e_j} \rangle_A \right)
\]

\[
= \omega \left( \sum_{i} \langle z_i \delta_{e_i}, \sum_j y_j \delta_{e_j} \rangle_A \right) = \omega(\langle x, y \rangle_A)
\]

Since for each \( \delta_{e_i} \in D \), \( \varphi(\delta_{\alpha^{-1}(e_i)}) = \delta_{e_i} \), \( \varphi \) is surjective. Thus \( (\omega, \varphi) : _AE_A \rightarrow _BD_B \) is a \( W^* \)-correspondence isomorphism.

Now we show the converse. If \( _AE_A \cong _BD_B \), then there is a \( W^* \)-correspondence isomorphism \( (\omega, \varphi) : _AE_A \rightarrow _BD_B \). Since \( \omega \) and \( \varphi \) are bijections, we have \( |G^1| = |F^1| \) and \( |G^0| = |F^0| \). Since relabeling vertices and edges gives an isomorphic graph,
we may assume that $G^0 = F^0$ and $G^1 = F^1$. Since each $\delta v_i$ is a projection, $\omega(\delta v_i) = \omega(\delta v_i)^n = \omega(\delta v_i)^0$ for all positive integers $n$. So $\omega(\delta v_i)$ is of the form $\sum_j \delta v_j$, and since $\omega$ is an isometry, we have $\omega(\delta v_i) = \delta v_i$ for some vertex $v_i$.

Let $e_i \in G^1$, $\omega(\delta_{s_1(e_i)}) = \delta v_k$ and $\varphi(\delta e_i) = \sum z_j \delta e_j$. Since $|\varphi(\delta e_i)| = 1$ (being an isometry) and $\varphi(\delta e_i) = \varphi(\delta e_i \cdot \delta s(e_i)) = \varphi(\delta e_i) \cdot \omega(\delta s(e_i)) = (\sum z_j \delta e_j) \cdot (\delta v_k) = z\delta v^{-1}_k(v_k)$, we must have, $\varphi(\delta e_i) = \delta v^{-1}_k(v_k)$. Thus $\omega$ and $\varphi$ are given by permutations. Let $\beta : G^0 \to F^0$ be the permutation given by $\beta(v_j) = v_k$ if $\omega(\delta v_j) = \delta v_k$. Let $\alpha : G^1 \to F^1$ be the permutation given by $\alpha(e_i) = e_m$ if $\varphi(\delta e_i) = \delta e_m$, $\alpha$ is of the form $\varphi(\delta e_i) \cdot \omega(\delta s(e_i)) = \varphi(\delta e_i) \cdot \varphi(\delta s(e_i)) = \omega(\delta_{s_1(e_i)}) \cdot \omega(\delta_{r_1(e_i)}) = \delta e_i \cdot \delta s(e_i) \cdot \delta r_1(e_i) = \delta e_i$. So $\alpha(e_i) = s_2(\alpha(e_i))$ and $\beta(r_1(e_i)) = r_2(\alpha(e_i))$. $G$ and $F$ are isomorphic graphs.

**Corollary 4.13** Let $A E A$ and $BDB$ be $W^*$-graph correspondences associated to the directed graphs $G = (G^0, G^1, s_1, r_1)$ and $F = (F^0, F^1, s_2, r_2)$ respectively. $A E A \sim WME_B DB$ if and only if $G \cong F$.

**Proof** It follows from Theorems 4.11 and 4.12.

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