On Anomalies in Orbifold Theories

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Abstract: We study the issue of gauge invariance in five-dimensional theories compactified on an orbifold $S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ in the presence of an external $U(1)$ gauge field. From the four-dimensional point of view the theory contains a tower of Kaluza-Klein Dirac fermions with chiral couplings and it looks anomalous at the quantum level. We show that this “anomaly” is cancelled by a topological Chern-Simons term which is generated in the effective action when the gauge theory is regularized introducing a spontaneous breakdown of the parity symmetries. In the presence of a classical background gauge field, the fermionic current acquires a parity-violating vacuum expectation value, thus generating the suitable Chern-Simons term and a gauge invariant theory. Our results reflect the profound relation between anomalies in orbifold quantum field theories with odd dimensions and parity-conservation.

Keywords: Anomalies, Field Theories in Higher Dimensions, Orbifold Theories.
1. Introduction

The presence of extra-dimensions is a crucial ingredient in theories explaining the uni-
fication of gravity and gauge forces. A typical example is string theory where more than 
three spatial dimensions are necessary for the consistency of the theory. It has recently 
become clear that extra-dimensions may be very large and could be even testable in accel-
erator experiments [1]. Of special interest are the theories on orbifold spaces [2] which are 
obtained compactifying the extra-dimensions and imposing a discrete symmetry acting on 
the higher dimensional coordinates. The four-dimensional (4D) low energy effective field 
theory coming from an orbifold compactification contains a tower of Kaluza-Klein (KK) 
states and may be chiral. The corresponding 4D orbifold gauge theory may therefore be 
anomalous [3] – signaling the breaking of gauge invariance – unless anomaly cancellation 
takes place through some non-trivial mechanism such as the Green-Schwarz [4] or the bulk 
inflow mechanisms [5].

The gauge anomaly in five-dimensional (5D) theories compactified on an $S^1/\mathbb{Z}_2$ orbifold 
with chiral boundary conditions for a single bulk fermion with unit charge under an abelian 
gauge group $U(1)$ was first discussed in Ref. [6]. The anomaly – defined as the five dimensional 
divergence of the current – lives entirely on the orbifold fixed planes

$$\partial_M J^M = \frac{1}{2} \left[ \delta(y) + \delta(y - \pi R) \right] Q(x, y),$$

where $J^M$ is the 5D fermionic current and

$$Q(x, y) = \frac{g_5^2}{32\pi^2} F_{\mu\nu} (x, y) \tilde{F}_{\mu\nu} (x, y),$$

is the 4D chiral anomaly in the external gauge potential $A_M(x, y)$\footnote{In our notation: $M = \left[ (\mu = 0, 1, 2, 3), 5 \right]$ and $y = x^5$ is the fifth coordinate compactified on a circle with radius $R$; $g_5$ is the 5D gauge coupling constant.}. Therefore the long distance four dimensional anomaly cancellation ensures the consistency of the higher dimensional orbifold theory. However, it was recently claimed that this phenomenon does not persist [7] in a five-dimensional field theory with a $U(1)$ gauge field and a charged fermion, compactified on the orbifold $S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2')$. Despite the fact that the orbifold projections remove both fermionic zero modes, gauge anomalies localized at the fixed points were found

$$\partial_M J^M = \frac{1}{4} \left[ \delta(y) - \delta(y - \pi R/2) + \delta(y - \pi R) - \delta(y - 3\pi R/2) \right] Q(x, y).$$

The 4D effective theory is anomaly-free because anomalies cancel after integration over 
the fifth dimension, but gauge invariance is broken, spoiling the consistency of the 5D theory. This result would be important for phenomenologically interesting models as the one discussed in Ref. [8] whose light spectrum contains just the zero modes of the Standard Model fields with an anomaly-free fermion content.
The goal of this paper is to show that gauge invariance in models with five-dimensions compactified on an orbifold $S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ can be maintained if the theory is regularized introducing a spontaneous breakdown of the parity symmetries. This procedure leads to the appearance in the effective action of a Chern-Simons (CS) topological term whose gauge variation exactly cancels the anomalous term. The “anomaly” now appears as a parity-violating topological term in the ground-state current rather than as a topological term in $\partial_M \langle J^M \rangle$.

Our results for orbifolds theories are reminiscent of the well-known phenomenon present in theories with an odd number of non-compact dimensions where the parity-violating part of the vacuum current induced by the classical background gauge field implies the presence in the effective action of a CS topological invariant which is odd under parity transformations \[9\]. This demonstrates the profound interrelation between anomalies in orbifold quantum field theories with odd dimensions and parity-conservation.

The paper is organized as follows. In section 2 we summarize the computation leading to the anomaly in the theory compactified on $S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and with a KK tower of 4D chiral fermions with opposite parities under the two $\mathbb{Z}_2$’s. In section 3 we discuss the CS counterterm suitable to cancel the anomaly and in section 4 we present the explicit derivation of the CS counterterm coming from a regularization method which spontaneously breaks the parity symmetries. Finally, section 5 contains our conclusions.

2. The 5D vector current

Consider a 5D fermion (Dirac) living in $\mathbb{R}^4 \times \mathcal{P}$ coupled with a $U(1)$ external gauge field. The compact component of the space is a circle $S^1$ of radius $R$ modulo some discrete group $\mathbb{Z}_2 \times \mathbb{Z}_2'$ (or $\mathbb{Z}_2$) which in general has fixed points. As a result, the resulting space in an orbifold. The action we consider is

\[ S = \int d^5x \left[ i\bar{\Psi} \Gamma^M \partial_M \Psi - g_5 \bar{\Psi} \Gamma^M \Psi A_M \right] \]  \hspace{1cm} (2.1)

where

\[ \Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \Gamma^4 = -i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -i \gamma^5, \quad \sigma^\mu = (1, \bar{\sigma}), \quad \bar{\sigma}^\mu = (1, -\bar{\sigma}). \]  \hspace{1cm} (2.2)

The two orbifold projections act on the spacetime points as

\[ \begin{align*}
\mathbb{Z}_2 : \quad (x^\mu, y) &\rightarrow (x^\mu, y') = (x^\mu, 2\pi R - y) \\
\mathbb{Z}_2' : \quad (x^\mu, y) &\rightarrow (x^\mu, y') = (x^\mu, \pi R - y) \quad y \in [0, 2\pi R].
\end{align*} \]  \hspace{1cm} (2.3)

and $y$ is identified with $y'$. On the 5D spinor the following condition is imposed

\[ \Psi(x, y') = \epsilon \gamma^5 \Psi(x, y), \quad \epsilon = \pm 1. \]  \hspace{1cm} (2.4)

We shall denote by $\epsilon$ and $\epsilon'$ the parity of the field. Notice that a 5D mass term $M$ is forbidden unless it has a non-trivial profile in the bulk with parities $(-, -)$. In \[7\] was claimed that the 5D current

\[ J^M = \bar{\Psi} \Gamma^M \Psi \]  \hspace{1cm} (2.5)
though is classically conserved, has an anomalous divergence at the quantum level. The argument is based on the result for the covariant anomaly for chiral fermions in four dimensions. Indeed one can rewrite the action (2.1) as a collection of 4D massive Dirac fermions by expanding $\Psi$ in terms of the complete set formed by the solutions of free Dirac equation in 5D

$$
\Psi(x, y) = \sum_n [\psi_{nR}(x) \phi_n^R(y) + \psi_{nL}(x) \phi_n^L(y)] ;
$$

(2.6)

where $\psi_n$ is a 4D Dirac fermion of mass $M_n$. The KK modes $\phi_{nL/R}$ satisfy

$$
\phi_n^R M_n - \frac{d}{dy} \phi_n^L = 0 ;
$$

(2.7)

$$
\phi_n^L M_n + \frac{d}{dy} \phi_n^R = 0 .
$$

(2.8)

From Eq. (2.4) we get the following transformation rules under the orbifold projections

$$
\phi_{nL/R}(-y) = \pm \epsilon \phi_{nL/R}(y) \quad \phi_{nL/R}(\pi R - y) = \pm \epsilon' \phi_{nL/R}(y) .
$$

(2.9)

The action (2.1) can be written, using the orthogonality of the KK modes, as

$$
S = \int d^4x \left[ \sum_n \left( \bar{\psi}_n i \gamma^\mu \partial_\mu \psi_n - M_n \bar{\psi}_n \psi_n \right) - g_5 \sum_{n,m} \left( j_{Lmn}^{\mu} A_{\mu mn} + j_{Rmn}^{\mu} A_{\mu mn}^R - i j_{5mn} A_{5mn} \right) \right] ;
$$

(2.10)

where

$$
A_{\mu L/R}^{mn} = \int_0^{2\pi R} dy \ \phi_n^{L/R}(y) \phi_m^{L/R}(y) A_\mu(x, y) ;
$$

(2.11)

$$
A_{5mn} = \int_0^{2\pi R} dy \ \phi_n^{L}(y) \phi_m^{R}(y) A_5(x, y) ;
$$

and

$$
j_{Lmn}^{\mu} = \bar{\psi}_m \gamma^\mu P_{L/R} \psi_n ;
$$

$$
j_{5mn} = \bar{\psi}_L \psi_n - \bar{\psi}_R \psi_L .
$$

(2.12)

Let us now reproduce the results in [6] and [7]. After setting $A_5 = 0$ by a gauge choice\(^2\), from the classical equations of motion and the well known result for the anomalous divergence of chiral current in 4D one has

$$
\partial_\mu j_{Lmn}^{\mu} = i \left[ \bar{\psi}_m M_m P_L \psi_n - \bar{\psi}_m P_L M_n \psi_n \right] - \frac{g_5^2}{32\pi^2} \left( F_{\mu\nu}^{L} \tilde{F}_{\mu\nu}^{L} \right)_{mn} ;
$$

(2.13)

$$
\partial_\mu j_{Rmn}^{\mu} = i \left[ \bar{\psi}_m M_m P_R \psi_n - \bar{\psi}_m P_R M_n \psi_n \right] + \frac{g_5^2}{32\pi^2} \left( F_{\mu\nu}^{R} \tilde{F}_{\mu\nu}^{R} \right)_{mn} .
$$

(2.14)

\(^2\)Strictly speaking this is not allowed: one is using gauge invariance before showing that it is still a good symmetry at the quantum level.
On the other hand, the 5D current can be written in terms of the 4D currents
\[ J^\mu(x, y) = \sum_{mn} \left[ \phi_R^m(y) \phi_R^n(y) j^\mu_{rmn}(x) + \phi_L^m(y) \phi_L^n(y) j^\mu_{lmn}(x) \right] . \]
\[ J^5 = -i \sum_{mn} \phi_L^m(y) \phi_R^n(y) j_{5mn}(x) . \] (2.15)

The fifth-dimensional structure of the divergences is recovered noticing that at the classical level
\[ \partial_y J^5 = \sum_{mn} i \left[ M_n (\phi_R^m \phi_R^n - \phi_L^m \phi_L^n) \tilde{\psi}_m (P_L - P_R) \psi_n \right] . \] (2.16)

Combining Eqs. (2.13), (2.14), (2.15) and (2.16), one finally gets
\[ \partial_M J^M = \frac{g_5^2}{32\pi^2} f(y) F_{\mu\nu} \tilde{F}^{\mu\nu} , \quad f(y) = \sum_m (\phi_R^m \phi_R^m - \phi_L^m \phi_L^m) . \] (2.17)

It should be stressed that in deriving (2.17) one is tacitly supposing that (2.16) is still valid at the quantum level and all the quantum effects are encoded in Eqs. (2.13-2.14); we will show that actually this is not the case.

The sum over the KK modes in (2.17) can be computed and it reads in the case \( \mathcal{P} = \mathbb{Z}_2 \times \mathbb{Z}'_2 \) for the various choices of parity for the fermion
\[ f^{(++)}(y) = - f^{(-)}(y) = \frac{1}{4R} \sum_{n=-\infty}^{+\infty} \delta(y/R - n\pi/2) ; \]
\[ f^{(+-)}(y) = - f^{(-+)}(y) = \frac{1}{4R} \sum_{n=-\infty}^{-1} (-1)^n \delta(y/R - n\pi/2) . \] (2.18) (2.19)

In particular, if the fermions have opposite parities (+, −) and (−, +), one recovers Eq. (1.3). For the case \( \mathcal{P} = \mathbb{Z}_2 \) one finds
\[ f^{(+)}(y) = - f^{(-)}(y) = \frac{1}{2R} \sum_{n=-\infty}^{+\infty} \delta(y/R - \pi n) , \] (2.20)
which reproduces Eq. (1.1).

3. Deformed Chern-Simons counterterm

In general, the manifestation of an anomaly at the quantum level reflects the failure of removing the ultraviolet divergences and – at the same time – preserving all the classical symmetries of the theory.

The natural question is therefore whether there exists a local counterterm \( S_{ct} \) such that the new action \( S' = S_{5D} + S_{ct} \) leads to a conserved 5D vector current in the orbifold theory \( S^1 / (\mathbb{Z}_2 \times \mathbb{Z}'_2) \) maintaining the symmetries of classical action.
Consider the vacuum functional $Z[A]$

$$Z[A] = e^{iW[A]} = \int D[\Psi] e^{iS[A,\Psi]} ;$$  \hspace{1cm} (3.1)

a would-be anomaly $G$ shows up in a non-zero variation of the connected generating functional $W$ under a gauge transformation $\delta A_M = -\partial_M \lambda$ of the the vector potential

$$\delta_\lambda W[A] = i \int d^5x \lambda G[A] .$$  \hspace{1cm} (3.2)

As a result, one has for the divergence of the current

$$\partial_M \langle J^M \rangle_{\text{conn.}} = G[A] , \quad \frac{\delta W[A]}{\delta A_M} = \langle J^M \rangle_{\text{conn.}} .$$  \hspace{1cm} (3.3)

$G$ is defined modulo a local functional of $A$, and the presence of a true anomaly is related to the impossibility of finding a suitable $S_{\text{ct}}$ such that $G$ is zero. The natural candidate for $S_{\text{ct}}$ in our case is the following deformed Chern-Simons term

$$S_{\text{CS}} = \int u(y) A \wedge F \wedge F = \int d^5x \epsilon^{MNPQ} A_M F_{NP} F_{PQ} u(y) .$$  \hspace{1cm} (3.4)

The counter term $S_{\text{CS}}$ will be consistent with the orbifold projections if $u$ has parity $(-,-)$. The gauge variation of the new connected generating functional is given by

$$\delta_\lambda W' = i \int d^5x \lambda \left[ G[A] + F_{\mu\nu} \tilde{F}^{\mu\nu} \partial_y u(y) \right] ,$$  \hspace{1cm} (3.5)

and the condition of vanishing divergence for the current gives the following constraint on the function $u(y)$

$$\partial_y u(y) + \frac{g_5^2}{32\pi^2} f(y) = 0 .$$  \hspace{1cm} (3.6)

In the case $(+, -)$ and $(+,-)$ a solution is easily found

$$u^{(+)}(y) = -u^{(-)}(y) = -\frac{g_5^2}{64\pi^2} \text{sgn}_\pi(\pi - y/R) ;$$  \hspace{1cm} (3.7)

where $\text{sgn}_\pi(x)$ is the sign function periodically extended with period $\pi$. The other cases are more subtle; if we suppose that $u(y) = u(y + 2\pi R)$ then Eq. (3.6) can be solved only if

$$\int_0^{2\pi R} f(y) dy = 0 .$$  \hspace{1cm} (3.8)

As a result, there is no periodic $u$ which solves Eq. (3.6) in the cases of fermions with parities $(+, +)$ $(-, -)$ where a chiral zero mode is present in the spectrum, the same result holds for the case of a single $Z_2$.

Since the CS term (3.4) one needs to add in order to preserve the gauge invariance of the theory depends on the odd function $u(y)$, it becomes clear that gauge invariance may be maintained if one is ready to give up parity-conservation. In other words, there are two
ways to regulated ultraviolet divergences in the calculation of the effective theory. The first way maintains parity as a good symmetry, but does not maintain gauge invariance. The second way introduces a parity non-conserving regulator, but maintains gauge invariance. If the second option is adopted, the “anomaly” appears as a parity-violating topological term in the ground-state current \( \langle J^M \rangle \) rather than as a topological term (\( \sim \hat{F} F \)) in \( \partial_M \langle J^M \rangle \). The “anomaly” causes the physical ground-state current to violate the parity symmetries of the original action.

4. The parity violating vacuum current

We now derive an explicit expression for the induced vacuum current in the orbifold theory \( S^1/(\mathbb{Z}_2 \times \mathbb{Z}'_2) \) for a 5D fermion with parity \((+\,+)\) and \((-\,+\) and for simplicity with vanishing bulk mass. We consider a gauge field background which is uniform in the bulk. The generic result can then be derived by general covariance arguments.

From the form of the CS term, one expects that the vacuum expectation value of the vector current acquires a non-vanishing component along the fifth direction

\[
\langle J^5(x,y) \rangle = -\frac{i}{Z[A]} \frac{\delta Z[A]}{\delta A_5(x,y)}. \tag{4.1}
\]

From this term one can then deduce the CS term in the effective action. We first introduce a new fermion with five-dimensional mass term \( M(y) \bar{\Psi} \Psi \) which at the end of the computation we decouple from the theory by taking the mass to infinity (Pauli-Villars regularization). Since the fermionic bilinear \( \bar{\Psi} \Psi \) is odd under both the \( \mathbb{Z}_2 \) and the \( \mathbb{Z}'_2 \) parities, the function \( M(y) \) has to be odd under both reflection symmetries. This amounts to saying the we admit in the action a spontaneous breakdown of the discrete symmetries \( \mathbb{Z}_2 \) and the \( \mathbb{Z}'_2 \).

For instance, we can choose \( M(y) = \kappa \text{sgn}_\pi(\pi/2 - y/R) \), where \([\kappa] = 1\). Eventually we will let \( \kappa \to \infty \). With a mass term the equations for the KK modes read

\[
\phi^R_n M_n - M(y) \phi^L_n - \frac{d}{dy} \phi^L_n = 0; \tag{4.2}
\]

\[
\phi^L_n M_n - M(y) \phi^R_n + \frac{d}{dy} \phi^R_n = 0. \tag{4.3}
\]

In the limit of large \( \kappa \) one can infer from Eqs. (4.2) and (4.3) that the mass eigenvalues \( M_n \) satisfy the relation

\[
\tan \left( \frac{\pi}{2} \alpha_n R \right) = -\frac{\alpha_n}{\kappa}, \tag{4.4}
\]

where

\[
M^2_n = \alpha^2_n + \kappa^2. \tag{4.5}
\]

The mass eigenvalues reduce to \( M^2_n = (2n-1)^2/R^2 + \kappa^2 \) and the corresponding eigenfunctions are given by

\[
\phi^R_n = \frac{1}{M_n \sqrt{\pi R}} \sin \left( \frac{(2n-1)}{R} y \right),
\]

\[
\phi^L_n = \frac{1}{M_n \sqrt{\pi R}} \left\{ M(y) \sin \left( \frac{(2n-1)}{R} y \right) + \frac{(2n-1)}{R} \cos \left( \frac{(2n-1)}{R} y \right) \right\}. \tag{4.6}
\]
The crucial point is that the eigenfunction $\phi_n^R$ picks up a piece which depends upon the parity-violating mass term $M(y)$.

Our goal is to show that the decoupling procedure leaves a finite CS term in the regularized action $S_{\text{reg}} = S_{5D}[A] - S_{5D}[A,M]$ when the five-dimensional mass is taken to infinity.

Consider the one-loop diagram in Fig. 1 where the fermions $\psi_{n L/R}$ run in the loop. The relevant contribution to $\langle \tilde{J}^5(q_1 + q_2, y) \rangle$ reads

$$
\frac{g_5^2}{\pi R} \sum_{n>0} \int \frac{d^4 p}{(2\pi)^4} A_\mu(q_1) A_\sigma(q_2) \frac{q_1 \nu q_2 \rho M_n \phi_n^R(y) \phi_n^L(y)}{(p^2 + M_n^2)(p + q_1)^2 + M_n^2} \text{Tr} \left( \gamma^5 \Gamma^\mu \Gamma^\nu \Gamma^\sigma \Gamma^\rho \right). \tag{4.7}
$$

The fact that the eigenfunction $\phi_n^R$ has a piece proportional to $M(y)$ gives rise to a finite non-vanishing contribution left-over in the large $\kappa$ limit. Indeed, performing the integral over the four momenta $p$ for small external momenta and using Eq. $\left[ 4.6 \right]$, we find that the vacuum expectation value of the current $\langle \tilde{J}^5(q_1 + q_2, y) \rangle$ gets a finite contribution proportional to

$$
\sum_{n>0} M_n^{-1} \phi_n^L(y) \phi_n^R(y) \frac{M(y)}{\pi R} - \frac{1}{2\pi i} \int_C dp_5 \frac{\left( \sin^2 p_5 y \right)}{p_5^2 + \kappa^2} \mathcal{P}(p_5), \tag{4.8}
$$

The next step is to simplify the sum over the Kaluza-Klein states using the contour trick from finite temperature field theory which allows one to identify the different possible divergences. We can rewrite the sum as

$$
\sum_{n>0} \frac{\sin^2 [y(2n-1)/R]}{[(2n-1)^2/R^2 + \kappa^2]} = \frac{1}{2\pi i} \int_C dp_5 \frac{\sin^2 p_5 y}{p_5^2 + \kappa^2} \mathcal{P}(p_5), \tag{4.9}
$$

where the contour $C$ is the line from the left to the right below the real axis and another line from the right to the left above this axis and enclosing the poles at $p_5 = (2n - 1)/R$ of the function $\mathcal{P} = -1/(\pi R / \cot \frac{1}{2} \pi R p_5)$. The constant parts $\pm \frac{1}{2} \pi R$ of the pole functions
\( \mathcal{P}(p_5) \) correspond to the pure 5D divergences in the propagators and produce the finite part we are looking for. Putting all the pieces together we find

\[
\langle J^5(x^\mu, y) \rangle = -\frac{g_5^2}{64\pi^2} \frac{M(y)}{4\kappa} F_{\mu\nu}(x, y) \tilde{F}^{\mu\nu}(x, y). \tag{4.10}
\]

Thus, the physical ground-state current violates the parity symmetries of the original action and we discover that a finite CS counterterm is produced

\[
\mathcal{L}_{\text{CS}} = -\frac{g_5^2}{64\pi^2} \frac{M(y)}{4\kappa} A_5(x, y) F_{\mu\nu}(x, y) \tilde{F}^{\mu\nu}(x, y). \tag{4.11}
\]

The ratio \( M(y)/\kappa \) does not depend upon \( \kappa \) and changes sign around the fixed points. We can safely take the limit \( \kappa \to \infty \) and find

\[
\mathcal{L}_{\text{CS}} = -\frac{g_5^2}{128\pi^2} \frac{1}{2} \text{sgn}_\pi(\pi/2 - y/R) A_5(x, y) F_{\mu\nu}(x, y) \tilde{F}^{\mu\nu}(x, y) \tag{4.12}
\]

which is the precisely the form needed in the expression (3.4) to cancel the anomaly. As a result, the cancellation of the divergence in Eq. (2.17) may be thought as coming from \( \langle \partial_y J^5 \rangle \), which can be non-vanishing only if the vacuum is not invariant the projections \( \mathbb{Z}_2 \) and \( \mathbb{Z}'_2 \) thus spontaneously breaking the parity. Consequently, \( \langle \partial_y J^5 \rangle \) gets an extra contribution due to the nontrivial shape of the function \( \partial_y M(y) \).

Let us close with some comments about theories with a single \( \mathbb{Z}_2 \). The reader might expect that the relation between spontaneous breaking of the reflection symmetry and gauge symmetry leading to the cancellation of the anomaly described in this paper might work in that case too. However, this is not the case. The basic difference is that in 5D theories compactified on \( S^1/\mathbb{Z}_2 \) the odd function that one should introduce in the CS term to reproduce the singular structure in Eq. (1.1) is monotone in the bulk space with jumps every \( \pi R \). The problem is that such a function is not periodic. Notice also that, with a single orbifold projection, when we take the limit \( \kappa \to \infty \), the fermion does not decouple due to the chiral zero mode whose bulk wave-function becomes peaked at one of the fixed points.

5. Conclusions

In this paper we have studied the issue of gauge invariance in models with five-dimensions compactified on an orbifold \( S^1/(\mathbb{Z}_2 \times \mathbb{Z}'_2) \). If, upon reduction to four-dimensions, the theory contains a tower of KK chiral fermions with opposite parities under the two \( \mathbb{Z}_2 \)'s, the theory looks anomalous at the quantum level. This would be anomaly can be cancelled by a suitable topological CS term which is made even under the two reflection symmetries by the presence of an odd function which breaks spontaneously the parity symmetries. We have shown that such a CS term is generated if the gauge theory is regularized introducing a spontaneous breakdown of the parity symmetries. This occurs because the fermionic current acquires a vacuum expectation value induced by the classical background gauge field. This vacuum expectation value breaks spontaneously the \( \mathbb{Z}_2 \) parities. This phenomenon is analogous to what happens in theories with an odd number of non-compact dimensions.
where the parity-violating part of the vacuum current induced by a classical background
gauge field implies the presence in the effective action of a CS topological invariant which
is odd under parity transformations.

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