Symmetries and Order in Cluster Nuclei

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Abstract. It is shown that the rotational band structure of the cluster states in \(^{12}\)C and \(^{16}\)O can be understood in terms of the underlying discrete symmetry that characterizes the geometrical configuration of the \(\alpha\)-particles, i.e. an equilateral triangle for \(^{12}\)C, and a regular tetrahedron for \(^{16}\)O. The structure of rotational bands provides a fingerprint of the underlying geometrical configuration of \(\alpha\)-particles. Finally, some first results are presented for odd-cluster nuclei.

INTRODUCTION

The concept of symmetries has played an important role in nuclear structure physics, both continuous and discrete symmetries. Examples of continuous symmetries are isospin symmetry [1], Wigner’s combined spin-isospin symmetry [2], the (generalized) seniority scheme [3, 4], the Elliott model [5] and the interacting boson model [6]. Discrete symmetries have been used in the context of collective models to characterize the intrinsic shape of the nucleus, such as axial symmetry for quadrupole deformations [7], octupole [8], tetrahedral [9, 10] and octahedral [10, 11] symmetries for deformations of higher multipoles.

A different application is found in \(\alpha\)-particle clustering in light nuclei to describe the geometrical configuration of the \(\alpha\) particles. Early work on \(\alpha\)-cluster models goes back to the 1930’s with studies by Wheeler [12], and Hafstad and Teller [13], followed by later work by Brink [14, 15] and Robson [16]. Recently, there has been a lot of renewed interest in the structure of \(\alpha\)-cluster nuclei, especially for the nucleus \(^{12}\)C [17]. The measurement of new rotational excitations of the ground state [18, 19, 20] and the Hoyle state [21, 22, 23, 24] has stimulated a large theoretical effort to understand the structure of \(^{12}\)C ranging from studies based on the semi-microscopic algebraic cluster model [25], antisymmetrized molecular dynamics [26], fermionic molecular dynamics [27], BEC-like cluster model [28], (no-core) shell models [29, 30], \textit{ab initio} calculations based on lattice effective field theory [31, 32], and the algebraic cluster model [20, 33, 34].

In this contribution, I discuss some properties of the \(\alpha\)-cluster nuclei \(^{12}\)C and \(^{16}\)O in the algebraic cluster model, and present some first results for odd-cluster nuclei in the framework of the cluster shell model.

ALGEBRAIC CLUSTER MODEL

The Algebraic Cluster Model (ACM) is an interacting boson model (IBM) to describe the relative motion of \(n\)-body clusters in terms of a spectrum generating algebra (SGA) of \(U(\nu + 1)\) where \(\nu = 3(n - 1)\) represents the number of relative spatial degrees of freedom. In the IBM, \(\nu\) denotes the five quadrupole degrees of freedom, leading to a SGA of \(U(6)\). For the two-body problem the ACM reduces to the \(U(4)\) vibron model [35], for three-body clusters to the \(U(7)\) model [33, 36] and for four-body clusters to the \(U(10)\) model [34, 37]. The ACM has a very rich symmetry structure. In addition to continuous symmetries like the angular momentum, in case of \(\alpha\)-cluster nuclei the Hamiltonian has to be invariant under the permutation of the \(n\) identical \(\alpha\) particles. Since one does not consider the excitations of the \(\alpha\) particles themselves, the allowed cluster states have to be symmetric under the permutation group \(S_n\).

The potential energy surface corresponding to the \(S_n\) invariant ACM Hamiltonian gives rise to several possible equilibrium shapes. In addition to the harmonic oscillator (or \(U(3n-3)\) limit) and the deformed oscillator (or \(SO(3n-2)\) limit), there are other solutions which are of special interest for the applications to \(\alpha\)-cluster nuclei. These cases
correspond to a geometrical configuration of $\alpha$ particles located at the vertices of an equilateral triangle for $^{12}$C and of a regular tetrahedron for $^{16}$O. Even though they do not correspond to dynamical symmetries of the ACM Hamiltonian, one can still obtain approximate solutions for the rotation-vibration spectrum

$$E = \begin{cases} \omega_1(v_1 + \frac{1}{2}) + \omega_2(v_2 + 1) + \kappa L(L + 1) & \text{for } n = 3 \\ \omega_1(v_1 + \frac{1}{2}) + \omega_2(v_2 + 1) + \omega_3(v_3 + \frac{1}{2}) + \kappa L(L + 1) & \text{for } n = 4 \end{cases}$$

The rotational structure of the ground-state band depends on the point group symmetry of the geometrical configuration of the $\alpha$ particles and is summarized in Table 1.

| ACM | $U(7)$ | $U(10)$ |
|-----|-------|--------|
| Point group | $D_{3h}$ | $T_d$ |
| Geometry | Triangle | Tetrahedron |
| G.s. band | $0^+$ | $0^+$ |
| $2^+$ | $3^-$ | $3^-$ |
| $4^+$ | $4^+$ | $4^+$ |
| $5^-$ | $6^{++}$ | $6^+$ |

The triangular configuration with three $\alpha$ particles has point group symmetry $D_{3h}$ [33]. Since $D_{3h} \sim D_3 \times P$, the transformation properties under $D_{3h}$ are labeled by parity $P$ and the representations of $D_3$ which is isomorphic to the permutation group $S_3$. The corresponding rotation-vibration spectrum is that of an oblate top: $v_1$ represents the vibrational quantum number for a symmetric stretching $A$ vibration, $v_2$ denotes a doubly degenerate $E$ vibration. The rotational band structure of $^{12}$C is shown in the left panel of Fig. 1.

**FIGURE 1.** (Color online) Rotational band structure of the ground-state band, the Hoyle band (or $A$ vibration) and the bending vibration (or $E$ vibration) in $^{12}$C (left) [20], and the ground-state band (closed circles), the $A$ vibration (closed squares), the $E$ vibration (open circles) and the $F$ vibration (open triangles) in $^{16}$O (right) [34].
The tetrahedral group $T_d$ is isomorphic to the permutation group $S_4$. In this case, there are three fundamental vibrations: $v_1$ represents the vibrational quantum number for a symmetric stretching $A$ vibration, $v_2$ denotes a doubly degenerate $E$ vibration, and $v_3$ a three-fold degenerate $F$ vibration. The right panel of Fig. 1 shows the rotational band structure of $^{16}$O, where the bands are labeled by $(v_1, v_2, v_3)$.

**ELECTROMAGNETIC TRANSITIONS**

For transitions along the ground state band the transition form factors are given in terms of a product of a spherical Bessel function and an exponential factor arising from a Gaussian distribution of the electric charges [33]

$$F(0^+ \rightarrow L^P; q) = c_L j_L(q\beta) e^{-q^2/4\alpha}.$$  \hfill (1)

The transition form factors depend on the parameters $\alpha$ and $\beta$. The value of $\beta$ is determined from the first minimum in the elastic form factor giving $\beta = 1.74$ (fm) [33] for $^{12}$C and $\beta = 2.07$ (fm) [34] for $^{16}$O. The value of $\alpha$ is determined by the size of the $\alpha$-particle to be $\alpha = 0.56$ (fm)$^{-2}$ [38].

The transition probabilities $B(EL)$ along the ground state band can be extracted from the form factors in the long wavelength limit

$$B(EL; 0^+ \rightarrow L^P) = \frac{(Ze)^2}{4\pi} c_L^2 \beta^{2L},$$

with

$$c_L^2 = \begin{cases} \frac{2L+1}{4L} \left[ 1 + 2P_L(-\frac{1}{\beta}) \right] & \text{for } n = 3 \\ \frac{2L+1}{4L} \left[ 1 + 3P_L(-\frac{1}{\beta}) \right] & \text{for } n = 4 \end{cases}$$

This means that the transition probabilities $B(EL)$ for different multiplicities $L$ are related by the product of a numerical factor and a power of $\beta$.

$$\begin{align*}
B(E3; 3^+_1 \rightarrow 0^+_1) &= \frac{5}{2} \beta^2, \\
B(E2; 2^+_1 \rightarrow 0^+_1) &= \frac{9}{16} \beta^4,
\end{align*}$$

for $^{12}$C, and

$$\begin{align*}
B(E4; 4^+_1 \rightarrow 0^+_1) &= \frac{7}{15} \beta^2, \\
B(E6; 6^+_1 \rightarrow 0^+_1) &= \frac{32}{45} \beta^6,
\end{align*}$$

for $^{16}$O. The good agreement for the $B(EL)$ values for the ground band in Table 2 shows that both in $^{12}$C and in $^{16}$O the positive and negative parity states indeed merge into a single rotational band. The large values of $B(EL; L^P \rightarrow 0^+_1)$ indicate a collectivity which is not predicted for simple shell model states. The relation between octupole and quadrupole transitions in $^{12}$C, and that between octupole and hexadecupole transitions in $^{16}$O are beautifully explained by Eqs. (2) and (3). The charge radii can be obtained from the slope of the elastic form factors in the origin, and are found to be in good agreement with the experimental value.

The quadrupole moment for $^{12}$C can be calculated as

$$Q_{21} = \frac{2}{7} Ze \beta^2,$$

i.e. a positive value, as is to be expected for an oblate deformation, and is found to be in good agreement with a recent measurement [42].
The spin-orbit interaction is taken as 

\[ V_{so} (\vec{r}) = V_{0,so} \frac{1}{2} \left[ \frac{1}{r} \frac{\partial V (\vec{r})}{\partial r} \left( \hat{\vec{s}} \cdot \hat{l} \right) + \left( \hat{\vec{s}} \cdot \hat{l} \right) \frac{1}{r} \frac{\partial V (\vec{r})}{\partial r} \right] \]  

Finally, for protons one has to include the Coulomb potential which is obtained by convoluting the charge density with the Green’s function as 

\[ V_C (\vec{r}) = \frac{Ze^2}{n} \int \rho (\vec{r'}) \frac{1}{|\vec{r} - \vec{r'}|} d\vec{r'} \]  

We then obtain the single-particle energy levels and intrinsic states as a function of \( \beta \) for each configuration \((n = 2, 3, 4)\) by solving the single-particle Schrödinger equation 

\[ H = \frac{\vec{p}^2}{2m} + V (\vec{r'}) + V_{so} (\vec{r'}) + V_C (\vec{r'}) \]  

In the case of neutrons \( V_C (\vec{r}) = 0 \).
FIGURE 2. Single-particle energies in a cluster potential with $D'_{3h}$ triangular symmetry (left) and with $T'_{d}$ tetrahedral symmetry (right). In the left panel, the single-particle levels are labeled by $E_{1/2}$ (black), $E_{5/2}$ (red) and $E_{3/2}$ (blue), and in the right panel by $E_{1/2}$ (black), $E_{3/2}$ (red) and $G_{3/2}$ (blue). For $\beta = 0$ the ordering of the single-particle orbits is $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$ and (almost degenerate) $1d_{5/2}$, $2s_{1/2}$.

Fig. 2 shows how the single-particle orbits are split as a function of the deformation parameter $\beta$, the relative distance of the $\alpha$ particles with respect to the center of mass. The single-particle levels can be classified according to the irreducible representations (irreps) of the corresponding double point group, $E_{1/2}$, $E_{5/2}$ and $E_{3/2}$ for the case of triangular symmetry (point group $D'_{3h}$) and $E_{1/2}$, $E_{3/2}$ and $G_{3/2}$ for tetrahedral symmetry (point group $T'_{d}$), in the notation of Herzberg [46]. The $E$ states are double degenerate and the $G$ states fourfold degenerate. In neither case, angular momentum and parity are good quantum numbers (with the exception for $\beta = 0$).

The construction of representations of the double groups $D'_{3h}$ and $T'_{d}$ was carried out for applications to molecular physics by Herzberg [46] and to crystal physics by Koster et al. [47]. Here I discuss an application in nuclear physics [48]. Table 3 shows the decomposition of single-particle orbits into irreps double point groups $D'_{3h}$ and $T'_{d}$. For example, both the $p_{3/2}$ and $d_{3/2}$ orbits transform as $G_{3/2}$ under $T'_{d}$, whereas under $D'_{3h}$ the $p_{3/2}$ orbit splits into $E_{5/2}$ and $E_{3/2}$, and the $d_{3/2}$ into $E_{1/2}$ and $E_{3/2}$.

Fig. 3 shows the rotational bands for the case of a single-particle coupled to a triangular configuration of $\alpha$ particles. The left panel shows the result for the ground-state band of even-cluster nuclei, as has been observed in $^{12}\text{C}$. The ground-state band consists of a series of $K$-bands with $K = 3k$ ($k = 0, 1, 2, \ldots$) and angular momenta $L = 0, 2, 4, \ldots$, for $K = 0$ and $L = K, K + 1, K + 2, \ldots$, for $K \neq 0$. The parity is given by $P = (-1)^K = (-1)^{k}$.

The results for the case of the coupling of a single-particle level with $E_{5/2}$, $E_{1/2}$ and $E_{3/2}$ symmetry to the ground-state of a triangular configuration with $A'_{1}$ symmetry is presented in the 2nd, 3rd and 4th panels of Fig. 3. The symmetry character under $D'_{3h}$ is given by the products $A'_{1} \otimes E_{5/2} = E_{5/2}$, $A'_{1} \otimes E_{1/2} = E_{1/2}$, and $A'_{1} \otimes E_{3/2} = E_{3/2}$.
TABLE 3. Splitting of single-particle orbits into representations of the double point groups $D'_{3h}$ (left) and $T'_d$ (right).

|       | $D'_{3h}$ |       | $T'_d$ |
|-------|-----------|-------|--------|
|       | $E_{1/2}$ | $E_{3/2}$ | $E_{1/2}$ | $E_{3/2}$ | $G_{3/2}$ |
| $s_{1/2}$ | 1 0 0 | 1 0 0 |       |       |       |
| $p_{1/2}$ | 0 1 0 | 0 1 0 |       |       |       |
| $s_{3/2}$ | 0 1 1 | 0 0 1 |       |       |       |
| $d_{3/2}$ | 1 0 1 | 0 0 1 |       |       |       |
| $d_{5/2}$ | 1 1 1 | 0 1 1 |       |       |       |
| $f_{5/2}$ | 1 1 1 | 1 0 1 |       |       |       |
| $f_{7/2}$ | 2 1 1 | 1 1 1 |       |       |       |

respectively. Just as for even-cluster nuclei, each representation for odd-cluster nuclei consists of a series of $K$-bands given by [47, 48]

\[
\begin{align*}
E_{1/2} : & \quad K^p = 1/2^+, 5/2^-, 7/2^-, \ldots, \\
E_{5/2} : & \quad K^p = 1/2^-, 5/2^+, 7/2^+, \ldots, \\
E_{3/2} : & \quad K^p = 3/2^-, 9/2^+, \ldots,
\end{align*}
\] (10)

with angular momenta $J = K, K + 1, K + 2, \ldots$.

An application of this scheme to the rotational bands in $^{13}$C is in progress [48].

FIGURE 3. Structure of rotational bands for a triangular configuration of $\alpha$ particles in even-cluster nuclei (first panel) and odd-cluster nuclei with $E_{3/2}, E_{1/2}$ and $E_{3/2}$ symmetry (second, third and fourth panel). Each rotational band is labeled by the quantum numbers $K^p$. 

| $D_{3h} : A'_1$ | $D'_{3h} : E_{5/2}$ | $D'_{3h} : E_{1/2}$ | $D'_{3h} : E_{3/2}$ |
|----------------|---------------------|---------------------|---------------------|
| $4^-$ | $9^+$ | $9^+$ | $9^+$ |
| $3^-$ | $7^-$ | $7^+$ | $7^+$ |
| $2^+$ | $5^-$ | $5^+$ | $5^+$ |
| $0^+$ | $1$ | $1$ | $1$ |
| $3^-$ | $1$ | $5^+$ | $7^+$ |
| $1$ | $5^+$ | $7^+$ | $9^+$ |
SUMMARY AND CONCLUSIONS

In this contribution, I reviewed a study of the cluster states in $^{12}$C and $^{16}$O in the framework of the ACM in which they were interpreted as arising from the rotations and vibrations of a triangular and tetrahedral configuration of $\alpha$ particles, respectively. An analysis of both the rotation-vibration spectra and electromagnetic transition rates shows strong evidence for the occurrence of $D_{3h}$ and $T_d$ symmetry in $^{12}$C and $^{16}$O, respectively. In both cases, the ground state band consist of positive and negative parity states which merge to form a single rotational band. This interpretation is validated by the observance of strong $B(E2)$ values. The rotational sequences can be considered as the fingerprints of the underlying geometric configuration (or point-group symmetry) of $\alpha$ particles.

In the second part, I discussed the Cluster Shell Model which is very similar in spirit to the Nilsson model, in which the single-particle levels are split in the deformed field of the core nucleus. In the CSM, the levels are split in the deformed field generated by the cluster potential with triangular $D_{3h}$ or tetrahedral $T_d$ symmetry. As a first result, I presented the quantum numbers of the deformed single-particle levels and, for the case of $^{12}$C plus a nucleon, the angular momentum content of the rotational bands. The study of odd-cluster nuclei is currently in progress. The first results for $^9$Be and $^9$B have been published [49], and the systems with three- and four- $\alpha$ particles plus a nucleon (particle or hole) are underway.

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