A $p$-ARTON MODEL FOR MODULAR CUSP FORMS

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To a modular form, we propose to associate (an infinite number of) complex-valued functions on $p$-adic numbers $\mathbb{Q}_p$ for each prime $p$. We elaborate on the correspondence and study its consequences in terms of the Mellin transform and the $L$-function related to the form. Further, we discuss the case of products of Dirichlet $L$-functions and their Mellin duals, which are convolution products of $\vartheta$-series. The latter are intriguingly similar to nonholomorphic Maass forms of weight zero as suggested by their Fourier coefficients.

Keywords: modular cusp forms, $p$-adic wavelets, theta functions, $L$-functions

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1. Introduction

The modular groups $SL(2, \mathbb{Z})$ and its subgroups, while of interest to mathematicians for their myriad manifestations, also play an important role in models of statistical mechanics, where they are related to the partition function viewed as a function of the complexified temperature and other external parameters. Holomorphic modular forms, with definite transformation properties under a subgroup of the modular group, are essential ingredients in conformally invariant quantum field theories in two dimensions, and more specifically in string theory and black hole physics, where both holomorphic and nonholomorphic modular forms arise (see e.g., [1], [2] and the references therein). The usual holomorphic modular forms are defined as $q$-series expansions in which the coefficients satisfy certain multiplicative properties. The coefficients of a modular cusp form define an associated $L$-series, the two being related by a Mellin transform. Thanks to the multiplicative properties of its coefficients, an $L$-series admits an Euler product representation over prime numbers.

Another class of $L$-series is the family of Dirichlet $L$-series, of which the Riemann zeta function is the best known member [3], associated to the multiplicative Dirichlet characters. Each of these also admits an Euler product representation in terms of prime numbers. The product form allows one to connect the local factors to the $p$-adic number fields $\mathbb{Q}_p$. We have recently constructed [4] pseudodifferential operators (as generalizations of the notion of the Vladimirov derivative in [5]) that incorporate the Dirichlet characters.

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and shown that the \( L \)-series can be expressed as the trace of an operator in an appropriate vector space. These operators act on the Bruhat–Schwarz class of locally constant complex-valued square-integrable functions on \( \mathbb{Q}_p \). The traces of these operators are well defined, being restricted to operators that act on the subspace \( \mathcal{H}_- \subset L^2(p^{-1}\mathbb{Z}_p) \) of functions spanned by the Kozyrev wavelets [6] that are locally constant functions with compact support in \( p^{-1}\mathbb{Z}_p \subset \mathbb{Q}_p \).

In a larger collaboration [7], we have also initiated a program to construct a random ensemble of unitary matrix models (UMM) for the prime factors of the Riemann zeta function, and then combine them to obtain a UMM for the latter. We find that the partition function of the UMM can be written as a trace of the Vladimirov derivative restricted to \( \mathcal{H}_- \). In this approach, the Riemann zeta function is essentially a partition function in the sense of statistical mechanics (see also [8]–[12] for “physical models” of \( L \)-functions), a key difference being that we use the orthonormal basis in \( \mathbb{Q}_p \) provided by Kozyrev wavelets. In [4], we also constructed pseudodifferential operators corresponding to \( L \)-series of modular cusp forms. We showed that a family of Kozyrev wavelets, known to be the eigenfunctions of the Vladimirov derivative, is the set of common eigenfunctions of all these pseudodifferential operators.

The purpose of this article is to propose an association between a modular cusp form and complex-valued functions in \( L^2(\mathbb{Q}_p) \), one function for each prime \( p \). More precisely, the correspondence is between the Fourier expansion of the cusp forms and the functions on \( \mathbb{Q}_p \). In other words, we argue that a cusp form is equivalent to a vector in \( \otimes_p L^2(\mathbb{Q}_p) \). This decomposition reminds us of the parton model of hadrons (which, like primes, begins fortuitously with \( p \)). A \( p \)-adic Mellin transform of these vectors, when combined for all primes, is shown to be related to the \( L \)-series corresponding to the cusp form. We also discuss Hecke operators in terms of the raising and lowering operators on the wavelet basis. This requires the definition of an appropriate inner product at the level of the \( p \)-artons. We examine two possibilities and study their properties. In an attempt to elaborate on these, we propose to define a class of toy \( L \)-functions (by taking a product of two Dirichlet \( L \)-functions) that mimic the Euler product form of \( L \)-functions associated with modular forms. As a pleasant surprise, we find that the objects associated with these bear an intriguing resemblance to nonanalytic Maass forms [13]. It is likely that these are indeed toy examples of Maass-like forms, however, we have only been able to verify their behavior under the \( \text{Im} \, z \to -1/\text{Im} \, z \) transformation of the modular group.

In what follows, we review a few relevant facts concerning holomorphic cusp forms of the modular groups related to \( SL(2, \mathbb{Z}) \) (Sec. 2) and wavelets on the \( p \)-adic numbers (Sec. 3) that also help us establish the notation used. In Sec. 4, we propose to associate complex-valued functions, one for each prime \( p \), which we call the \( p \)-artons, with a cusp form, and discuss their Mellin transforms. Realization of the Hecke operators in this description is addressed in Sec. 5, where we propose possible inner products on the spaces of \( p \)-artons and their Mellin duals. Finally, in Sec. 6, we study the modular objects associated to products of two Dirichlet \( L \)-functions and point out their relation to nonanalytic Maass forms. We conclude in Sec. 7 with a brief summary and a comment on the holographic nature of the correspondence proposed in this paper.

2. Modular forms and associated \( L \)-functions

The discrete subgroup\(^1\)

\[
SL(2, \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \, ad - bc = 1 \right\}
\]

\(^1\)More precisely, the relevant groups are the projective special linear groups \( PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\} \) and \( PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm 1\} \).
of the special linear group $SL(2, \mathbb{R})$ is the symmetry group of lattices $\Lambda$ in the complex plane $\mathbb{C}$ [14]–[17]. It is sometimes called the full modular group, and has the following congruence subgroups of finite indices:

\[
\begin{align*}
\Gamma_0(N) &= \{ \gamma \in SL(2, \mathbb{Z}) \mid c \equiv 0 \mod N \}, \\
\Gamma_1(N) &= \{ \gamma \in SL(2, \mathbb{Z}) \mid a, d \equiv 1 \text{ and } c \equiv 0 \mod N \}, \\
\Gamma(N) &= \{ \gamma \in SL(2, \mathbb{Z}) \mid a, d \equiv 1 \text{ and } b, c \equiv 0 \mod N \}
\end{align*}
\]

(since the conditions are empty for $N = 1$, $\Gamma(1)$ is the full modular group $SL(2, \mathbb{Z})$.) They are ordered as $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma(1)$. Of these, $\Gamma(N)$, called the principal congruence subgroup, is the kernel of the homomorphism $SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/N\mathbb{Z})$.

If $\Gamma$ is a discrete subgroup such that $\Gamma(N) \subset \Gamma \subset \Gamma(1)$, where $N$ is the smallest such integer, it is referred to as a congruence subgroup of level $N$. The natural action of the group $SL(2, \mathbb{R})$ on the upper half-plane $\mathbb{H} = \{ z : \text{Im } z > 0 \}$ restricts to that of $\Gamma$, which partitions it into equivalence classes. A fundamental domain $F$ is a subset of $\mathbb{H}$ representing a $\Gamma$-equivalence class. For example, the fundamental domain of the full modular group $\Gamma(1)$ is $\{ z \in \mathbb{H} \mid -1/2 \leq \text{Re } z \leq 1/2, |z| \geq 1 \}$ (ignoring some double counting of points on the boundaries).

A modular form [15] $f : \mathbb{H} \to \mathbb{C}$ of weight $k \in \mathbb{N}$ and level $N \in \mathbb{Z}$ associated to a Dirichlet character $\chi_N$ modulo $N$, is a holomorphic form on the upper half-plane $\mathbb{H}$ that transforms, under the action of the discrete subgroup $\Gamma(N) \subset \Gamma \subset \Gamma(1)$, as

\[
f(\gamma z) \equiv f \left( \frac{az + b}{cz + d} \right) = \chi_N(d)(cz + d)^k f(z), \quad \gamma \in \Gamma \subset SL(2, \mathbb{Z}).
\]

The modular form vanishes identically unless $k$ is an even integer.

Using $z \to z + 1$ in (2), we see that a modular form of the full modular group $\Gamma(1)$ (i.e., of level 1) is a periodic function in $z$, and therefore it has the following Fourier expansion ($q$-expansion) in $q = e^{2\pi iz}$:

\[
f(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi inz} = \sum_{n=0}^{\infty} a(n)q^n.
\]

A cusp form is a modular form that vanishes as $\text{Im } z \to i\infty$, or equivalently, at $q = 0$. Thus, $a(0) = 0$ for a cusp form. It is conventional, and often convenient, to normalize the first coefficient of a cusp form to $a(1) = 1$, which is what we assume in what follows.

An equivalent description of modular forms is in terms of scaling functions on $\mathbb{C}/\Lambda$, where $\Lambda$ is a lattice left invariant by the action of a subgroup of the modular group.

Modular forms of weight $k$ form a finite-dimensional complex vector space $M_k(\Gamma(1))$, and the subset of cusp forms is a subspace $S_k(\Gamma(1))$. Similar notions exist for modular forms of the congruence subgroups $\Gamma \subset \Gamma(1)$ of level $N$. However, a cusp form of a congruence subgroup $\Gamma$ is required to vanish as $z$ approaches certain rational points on $\mathbb{R} = \partial \mathbb{H}$ (equivalently, at certain points on the unit circle $|q| = 1$), in addition to $z \to i\infty$ ($q = 0$). These additional points in the fundamental domain are images of $\text{Im } z \to i\infty$.

The Dirichlet series of a cusp form $f = \sum_n a(n)/q^n$ is defined by the coefficients in its $q$-expansion,

\[
L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = 1 + \frac{a(2)}{2^s} + \frac{a(3)}{3^s} + \frac{a(4)}{4^s} + \cdots,
\]

\[A \text{ Dirichlet character (modulo } N) \text{ is a group homomorphism } \chi_N \in Hom(G(N), \mathbb{C}^*) \text{ from the multiplicative group } G(N) = (\mathbb{Z}/N\mathbb{Z})^* \text{ of invertible elements of } \mathbb{Z}/N\mathbb{Z} \text{ to } \mathbb{C}^*. \text{ It is a multiplicative character. It is customarily extended to all integers by setting } \chi_N(m) = 0 \text{ for all } m \text{ that share common factors with } N [14].} \]
where we have used the normalization \( a(1) = 1 \). This series converges uniformly to a holomorphic function of \( s \) in the half-plane to the right of \( \text{Re} \ s = \sigma + 1 \) as long as the coefficients \( |a(n)| \) are bounded by some power \( n^\sigma \). The corresponding \( L \)-function associated with the cusp form \( f \) is then defined by an analytic continuation to the complex \( s \)-plane. For a cusp form of weight \( k \), the series above converges in \( \text{Re} \ s > (k + 1)/2 \). The series is also related to the cusp form as

\[
L(s, f) = M[f](s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy y^{s-1} f(iy),
\]

i.e., it is the Mellin transform of \( f(iy) \).

The discriminant function

\[
\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi iz}.
\]

is an example of a cusp form of weight 12 (and level 1) of the full modular group. Ramanujan noticed that the coefficients in its \( q \)-expansion

\[
\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n
\]

have the following properties:

- \( \tau(mn) = \tau(m)\tau(n) \) if \( \gcd (m, n) = 1 \);
- \( \tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-1}) \) for \( p \) a prime and \( m > 0 \);
- \( |\tau(p)| \leq 2p^{11/2} \).

The function \( \tau: \mathbb{N} \rightarrow \mathbb{Z} \) is known as the Ramanujan \( \tau \)-function. (The first two properties were proved by Mordell, while the proof for the bound was provided by Deligne.) More generally, the coefficients \( a(n) \) of a modular form of weight \( k \) and level \( N \) satisfy

\[
a(mn) = a(m)a(n), \text{ if } \gcd (m, n) = 1;
\]

\[
a(p^{m+1}) = a(p)a(p^m) - \chi(p)p^{k-1}a(p^{m-1}) \text{ for } p \text{ a prime and } m > 0.
\]

The coefficient function \( a: \mathbb{N} \rightarrow \mathbb{C} \) is said to define a multiplicative character [14]. The convergence of the series also puts a bound on the growth of the coefficients, which for a cusp form is \( |a(p)| \leq Cp^{(k-1)/2} \) for \( C \) of order 1. Due to the above properties of its coefficients, the \( L \)-function of a cusp form \( f \) in (4) also admits the Euler product form

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - a(p)p^{-s} + \chi(p)p^{k-1}p^{-2s}}.
\]

This is analogous to the Dirichlet \( L \)-functions in Eq. (48) below, but in contrast to that case, each local factor

\[
L_p(s, f) = (1 - a(p)p^{-s} + \chi(p)p^{k-1}p^{-2s})^{-1}
\]

in \( L(s, f) \) is quadratic in \( p^{-s} \).
3. A wavelet basis for complex-valued functions on $\mathbb{Q}_p$

The prime factors in the Euler product of the Dirichlet $L$-functions, Eq. (48), as well as the $L$-functions associated with cusp forms, can be related to complex-valued functions on the $p$-adic space $\mathbb{Q}_p$. Using the ultrametric $p$-adic norm, which itself is an example of a complex-valued function $|\cdot|_p : \mathbb{Q}_p \to \mathbb{C}$, we can write the prime factor $(1 - p^{-s})^{-1}$ in the Riemann zeta function as

$$
\frac{1}{1 - p^{-s}} \equiv \zeta_p(s) = \frac{p}{p - 1} \int_{\mathbb{Z}_p} dx \frac{x^{-s}}{|x|_p^{s}}, \quad \text{Re } s > 0, \tag{10}
$$

where $dx$ is the (complex-valued) translation-invariant Haar measure on $\mathbb{Q}_p$. The integral can also be thought of as

$$
\frac{p}{p - 1} \int_{\mathbb{Z}_p} d^s x \frac{x^s}{|x|_p^s}, \quad \text{Re } s > 0, \tag{11}
$$

where the integral is over the nonzero elements in $\mathbb{Z}_p$ with respect to the multiplicative (scale invariant) measure $d^s x = dx/|x|_p$ of $\mathbb{Q}_p^\times$ viewed as a multiplicative group. The integrand $|x|_p^s$, a multiplicative character on $\mathbb{Q}_p$, is another complex-valued function.

We introduce two other functions

$$
\chi_p(x) = e^{2\pi ix}, \quad \Omega_p(x, x_0) = \begin{cases} 1 & \text{if } |x - x_0|_p \leq 1, \\ 0 & \text{otherwise}, \end{cases} \quad x, x_0 \in \mathbb{Q}_p, \tag{12}
$$

where the first one is an additive character and the second is an indicator function for a unit ball centered at $x_0$ [5], [18]. This additive character defines the Fourier transform $\mathcal{F}[f]$ of a function $f : \mathbb{Q}_p \to \mathbb{C}$ as

$$
\mathcal{F}[f](k) = \int_{\mathbb{Q}_p} dx \chi_p(kx)f(x),
$$

and its inverse $\mathcal{F}^{-1}$. The indicator function $\Omega_p(x)$ retains its form after the Fourier transform, i.e., $\mathcal{F}[\Omega_p](k) = \Omega_p(k)$. In this sense, it is an analogue of the Gaussian function $e^{-\pi x^2}$ on $\mathbb{R}$. These are the ingredients of a family of orthonormal functions [6] on $\mathbb{Q}_p$

$$
\psi^{(p)}_{n,m,j}(x) = p^{-n/2}e^{\frac{2\pi i}{p} p^n x} \Omega_p(p^n x - m), \quad \int_{\mathbb{Q}_p} dx \psi^{(p)}_{n,m,j}(x)\psi^{(p)}_{n',m',j'}(x) = \delta_{nn'}\delta_{mm'}\delta_{jj'}, \tag{13}
$$

where $n \in \mathbb{Z}$, $m \in \mathbb{Q}_p/\mathbb{Z}_p$, $j = 1, \ldots, p - 1$. These functions, which we refer to as the Kozyrev wavelets, provide an orthonormal basis for the set $L^2(\mathbb{Q}_p)$ of square-integrable Bruhat–Schwarz (locally constant) functions on $\mathbb{Q}_p$. These wavelets can be considered to be analogous to the (generalized) Haar wavelets on $\mathbb{R}$. All these functions can be obtained from the mother wavelets $\psi_{0,0,j}(x)$ by the action of the affine group “$ax + b$” with suitable choices for $a$ and $b$.

The usual definition of the derivative of a function does not work due to the totally disconnected topology of $\mathbb{Q}_p$. However, a pseudodifferential operator, called the generalized Vladimirov derivative [5], [19], is defined by the integral kernel

$$
D_{(p)}^\alpha f(x) = \frac{1}{\Gamma_{(p)}(-\alpha)} \int_{\mathbb{Q}_p} dx' \frac{f(x') - f(x)}{|x' - x|_p^{\alpha + 1}}, \quad \Gamma_{(p)}(-\alpha) = \int_{\mathbb{Q}_p} \frac{dx}{|x|_p} e^{2\pi i x} |x|_p^{-\alpha}, \tag{14}
$$

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for $\alpha$ in a suitable half-plane of $\mathbb{C}$ and elsewhere by analytic continuation when appropriate. The operator $\log_p D_\alpha$ can be defined in the limit $\alpha \to 0$ as

$$
\log_p D_\alpha = \lim_{\alpha \to 0} \frac{D_\alpha^\alpha - 1}{\alpha \ln p}
$$

(15)
such that $\log_p D_\alpha \psi^{(p)}_{n,m,j}(x) = (1 - n)\psi^{(p)}_{n,m,j}(x)$. The Kozyrev wavelets are eigenfunctions of the Vladimirov derivatives

$$
D_\alpha^{\alpha} \psi^{(p)}_{n,m,j}(x) = p^{\alpha(1-n)}\psi^{(p)}_{n,m,j}(x)
$$

(16)
corresponding to the eigenvalues $p^{\alpha(1-n)}$. These wavelets are also common eigenfunctions of a more general class of Vladimirov derivatives twisted by a multiplicative character [4].

Since our interest is in the eigenvalues, which depend only on the quantum number $n$ related to scaling (and not $m$ and $j$ related to translation and phase), we restrict our attention to the set of eigenfunctions

$$
\psi^{(p)}_{n,0,1}(x) = p^{-n/2} \chi_p(p^{n-1}x) \Omega_p(p^n x) \quad \leftrightarrow \quad |1 - n\rangle_p,
$$

(17)
where we use the alternative ket-vector notation labeled by the eigenvalue of the wavelet.

We also define the lowering and raising operators $a_+^{(p)}$ and $a_-^{(p)}$ on the wavelets,

$$
a_+^{(p)} \psi^{(p)}_{n,0,1}(x) = \psi^{(p)}_{n+1,0,1}(x) \quad \leftrightarrow \quad a_-^{(p)} |n\rangle_p = |n+1\rangle_p
$$

(18)
which change the scaling quantum number by one. With these operators $a_+^{(p)}$, together with $\log_p D_\alpha^{(p)}$, we can define $J_\pm^{(p)} = a_\pm^{(p)} \log_p D_\alpha^{(p)}$ and $J_3^{(p)} = \log_p D_\alpha^{(p)}$, whose commutator algebra generates the sl(2, $\mathbb{R}$) symmetry of the wavelets [20].

We are mainly interested in the subset spanned by the wavelets

$$
\{|m\rangle_p \mid m = 0, 1, 2, \ldots \} \quad \leftrightarrow \quad \{\psi^{(p)}_{n,0,1}(x) \mid n = 1, 0, -1, -2, \ldots \}
$$

(19)
to define the subspace $\mathcal{H}^{(p)} \subset L^2(p^{-1} \mathbb{Z}_p)$. It consists of wavelets supported on the compact subset $p^{-1} \mathbb{Z}_p$ and the corresponding eigenvalues of the operator $D_\alpha^{(p)}$ on these basis functions are $\{1, p, p^2, \ldots \}$, i.e., positive integer powers of $p$. When restricted to this subspace, we demand that the lowering operator $a_+^{(p)}$ annihilate the mother wavelet corresponding to the lowest eigenvalue of the Vladimirov derivative$^3$

$$
a_+^{(p)} |0\rangle_p = 0 \quad \leftrightarrow \quad a_+^{(p)} \psi_{1,0,1}(x) = 0.
$$

(20)
Thus, the wavelet $\psi^{(p)}_{1,0,1}(x)$ is like a “ground state” or “lowest-weight state” in this subspace, and the other wavelets arise from repeated applications of the raising operator $a_+^{(p)}$ to it: $|n\rangle = (a_+^{(p)})^n |0\rangle_p$.

4. Vectors for modular (cusp) forms

We are now ready to propose an association between a modular cusp form and functions on $\otimes_p \mathbb{Q}_p$, more precisely, on $\otimes \mathcal{H}^{(p)}$, i.e., those spanned by the wavelets in subspace (19). For definiteness, we discuss the association for cusp forms of the principal congruent subgroup $\Gamma(N)$ of the modular group.

First, we use prime factorization of a natural number $n \in \mathbb{N}$ to relate it to a wavelet in $\otimes \mathcal{H}^{(p)} \subset \otimes_p \mathbb{Q}_p$ as

$$
n = \prod_{p \text{ prime}} p^{n_p} \quad \leftrightarrow \quad \bigotimes_p |n_p\rangle_p = |n_2\rangle \otimes |n_3\rangle \otimes |n_5\rangle \otimes \cdots.
$$

(21)

$^3$By an abuse of notation, we continue to use the same symbol for the restrictions of the raising and lowering operators to $\mathcal{H}^{(p)}$. Because it is the subspace that is of primary interest to us, this should hopefully not be a cause of confusion.
Since all but a finite number of $n_p$'s are zero, the associated wavelets are the lowest-weight or ground states $\langle 0 \rangle_{(p)}$ corresponding to the wavelets $\psi_{0,0,1}^{(p)}(x)$ for the prime $p$. Only a finite number of wavelets are therefore nontrivial in this correspondence. A similar association between natural numbers as vectors in the Hilbert space of fictitious quantum systems (dubbed arithmetic gas or primon gas) using the prime factorization has been made in Refs. [8]–[11]. The present proposal relates to a mathematically well-defined space of space of fictitious quantum systems (dubbed arithmetic gas or primon gas) using the prime factorization

\[ \sum_{n=1}^{\infty} a(n)q^n = \sum_{n_p=0}^{\infty} \left( \prod_{p} a(p^{n_p}) \right) q^{\prod p^{n_p}} \quad \rightarrow \quad |f \rangle = \sum_{n_p=0}^{\infty} \bigotimes_{p} a(p^{n_p}) |n_p \rangle_{(p)} = \]

where we use the multiplicative property of the coefficients in Eq. (8) for arguments that are coprime to each other. It is understood that for $n_p = 0$, the coefficients $a(1) = 1$ for all $p$. We note that in the foregoing, most of the terms are also “trivial,” i.e., most vectors correspond to the ground state $\langle 0 \rangle_{(p)} = |n_p = 0 \rangle \in \mathcal{H}_{(p)}$. In the last line of (22), we use the fact that ket vectors of a higher occupation number $n_p \geq 1$ can be obtained from the ground state by the action of raising operator (18):

\[ |n_p \rangle = a_{n_p}^{-1}|0 \rangle_{(p)} \]

The vector $|f \rangle \in \otimes_p \mathcal{H}_{(p)}$ appears to be a complicated one, and is in fact entangled. However, thanks to the multiplicative property of coefficients (8), it is simplified to a product form

\[ |f \rangle = \sum_{n_p=0}^{\infty} \bigotimes_{p} a(p^{n_p}) a_{n_p}^{-1} |0 \rangle_{(p)} = (1 - a(p) a_{-} + p^{k-1} \chi(p) a_{-}^{2})^{-1} |0 \rangle_{(p)} \equiv \bigotimes_{p} |f_{(p)} \rangle \]

as can be seen by expanding the right-hand side and comparing it with the left-hand side. We would like to think of the vector $|f_{(p)} \rangle \in \mathcal{H}_{(p)}$ as the $p$th $p$-arton, i.e., the “part” of the cusp form $f$ at the prime $p$. It is interesting to note that the operator that acts on the “ground state” to generate the $p$-arton resembles the form of the local $L$-function at a prime $p$ in (9). Indeed, that was a clue to find the factorization above.

All this can be equivalently expressed in terms of the complex-valued wavelet functions if we introduce the coordinate basis, familiar in quantum mechanics, consisting of generalized kets $\{|x_{(p)} \rangle, x_{(p)} \in \mathbb{Q}_p \}$ that satisfy the orthonormality condition

\[ \langle x_{(p)} | x'_{(p')} \rangle = \delta_{pp'} \delta(x_{(p)} - x'_{(p)}) \]

with $\delta(x_{(p)} - x'_{(p)})$ being the Dirac $\delta$-functions. In this basis, the components $\langle x_{(p)} | f_{(p)} \rangle$ (also called the wave function) of $|f_{(p)} \rangle$ are

\[ f_{(p)}(x_{(p)}) \equiv \langle x_{(p)} | f_{(p)} \rangle = \sum_{n_p=0}^{\infty} a(p^{n_p}) |x_{(p)} \rangle |n_p \rangle = \sum_{n_p=0}^{\infty} a(p^{n_p}) \psi_{0,0,1}^{(p)}(x_{(p)}) \]

i.e., a linear combination of wavelets compactly supported on $p^{-1}\mathbb{Z}_p \subset \mathbb{Q}_p$. 

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Summary. According to this correspondence, a cusp form $f : \mathbb{H} \to \mathbb{C}$ is equivalent to the infinite set of functions $f(p) : \mathbb{Q}_p \to \mathbb{C}$, one function for each prime $p$. The two are equivalent in the sense that from $f$ we can obtain $(f(2), f(3), f(5), \ldots)$ and vice versa.

We want to show that the association proposed in (22)–(24) has other ramifications. For this, we consider a Mellin transformation of the wavelets. There are different proposals for $p$-adic Mellin transforms [18], [21], but all involve integrands with multiplicative characters on $\mathbb{Q}_p$. We use the following definition,\(^4\) which is similar to the one in Ref. [21]:

$$\tilde{g}_\omega(s) = \mathcal{M}_{(p, \omega)}[g](s) = \int_{\mathbb{Q}_p^\times} d^\times x \, e^{\frac{2\pi i}{p} x |x|_p} |x|_p^s g(x), \quad s \in \mathbb{C}, \quad \ell = 0, 1, \ldots, p - 1,$$

(25)

where $d^\times x = dx/|x|_p$ is the scale-invariant measure and the kernel contains the unitary character $\omega(x) = e^{\frac{2\pi i}{p} x |x|_p}$. The character, which determines a phase depending on the leading $p$-adic “digit” of $x$, has been mentioned in [5]. The transformation satisfies the scaling property

$$\mathcal{M}_{(p, \omega)}[g(\alpha x)](s) = |\alpha|_p^{-s} \mathcal{M}_{(p, \omega)}[g(x)](s)$$

similar to the usual Mellin transforms, with $\omega_{\ell} \to \omega_{\ell, \alpha|_p}$, which is also a primitive $p$th root of unity if $\omega_{\ell}$ is. In particular, for $\alpha = p^n$, the unitary character does not change, and hence

$$\mathcal{M}_{(p, \omega)}[g(p^n x)](s) = p^{-n s} \mathcal{M}_{(p, \omega)}[g(x)](s),$$

can be used to relate the Mellin transform of the wavelets. The inverse Mellin transform

$$\mathcal{M}^{-1}_{(p, \omega)}[\tilde{g}_\omega](x) = \sum_{\ell=0}^{p-1} e^{-\frac{2\pi i}{p} x |x|_p} \int_0^{2\pi} dt |x|_p^{-\ell} \tilde{g}_\omega(it)$$

(26)

involves a discrete Fourier transform of the character $\omega$ and the path of integration is from 0 to $2\pi i/\ln p$ along $t$, the imaginary axis in the complex $s$-plane.

With this definition, the Mellin transform of the Kozyrev wavelet $\psi_{n,0,1}(x)$ is

$$\mathcal{M}_{(p, \omega)}[\psi_{n,1,0}](s) = -\left(\frac{1}{p(1-p^{-s})} - \frac{1}{p^s - 1} \delta_{\ell,0} - \delta_{\ell,p-1}\right)_p p^{n(s-1/2)}.$$  

(27)

It can be verified by an explicit calculation that the inverse transform of the above is the Kozyrev wavelet. We note that even though the contribution of the last term to the inverse transform adds to zero, while the contributions from the first two terms reproduce the wavelet function, the discrete Fourier transform involving the character $\omega$ is absolutely crucial for the inverse transform to work. As an aside, it is interesting to note that the Mellin transform of a Kozyrev wavelet is related to the eigenvalue of the generalized Vladimirov derivative $D^{-s}$.

From (24) and (27), we obtain

$$\mathcal{M}_{(p, \omega)}[f(p)(x(p))](s) = c_p(\ell, s) \sum_{n_p=0}^{\infty} a(p^{n_p}) p^{(1-n_p)(s-1/2)},$$

(28)

\(^4\)In [18], the multiplicative character in the kernel is taken to be unitary, which is satisfied if $s$ in (25) is purely imaginary. The definition in [21] (see definitions 2.8.4 and 2.8.5) uses a normalized unitary character $\omega$, which has a conductor $p^N$, which in this case is $N = 1$.  

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where
\[ c_p(\ell, s) = -\frac{1}{p(1 - p^{-s})} + \frac{\delta_{\ell,0}}{p^s - 1} + \delta_{\ell,p-1} = \begin{cases} 
- p^{-s} \Gamma_p(s), & \ell = 0, \\
p^{-1} \zeta_p(s), & \ell = 1, \ldots, p - 2, \\
p^{-1} \zeta_p(s - 1), & \ell = p - 1, 
\end{cases} \] (29)
is the \( n_p \)-independent factor in the parenthesis in (27).

We can now combine the results for all primes: the Mellin transform of the “wave function” of (24) is
\[ M(p, \omega) \left[ \left( \xi(2), \xi(3), \xi(5), \ldots \right) \right](s) = \prod_p M(p, \omega) \left[ f(p) \xi(p) \right](s) = \prod_p c_p(\ell, s) p^{s-1/2} \cdot L \left( s - \frac{1}{2}, f \right). \] (30)
Curiously, the \( L \)-function obtained this way has its argument shifted from \( s \) to \( s - 1/2 \). The infinite product in the prefactor also depends on \( \ell \), which in turn depends on \( p \).

5. Raising, lowering, and Hecke operators

It would be instructive to understand the proposed decomposition of a modular form in terms of \( p \)-adic wavelets in the context of the profound Hecke theory of modular forms. However, this is outside the scope of our present understanding. We need to know how to extend the definition of the Kozyrev wavelets, which are defined on \( \mathbb{Q}_p \), to the projective space \( \mathbb{P}(\mathbb{Q}_p) \sim \mathbb{Q}_p \cup \{\infty\} \) such that their transformation under the full group \( GL(2, \mathbb{Q}_p) \) can be addressed. For now, we set a more modest goal of an operational understanding of the Hecke operators \( T(m), m \in \mathbb{N} \) and their algebraic properties.

We recall that the Hecke operators \( T(m), m \in \mathbb{N} \) are a set of commuting operators whose action on the modular form \( f(z) = \sum a(n)e^{2\pi inz} \) is to return the coefficients in the \( q \)-expansion as eigenvalues [14]–[17], [21]:
\[ T(m)f(z) = a(m)f(z). \] (31)
In other words, a modular form is an eigenvector of the Hecke operators with the eigenvalues given by the coefficients in its \( q \)-expansion. They satisfy
\[ T(m)T(n) = T(mn) \quad \text{for} \quad m \nmid n, \]
\[ T(p)T(p^{\ell}) = T(p^{\ell+1}) + \chi(p)p^{k-1}T(p^{\ell-1}). \] (32)
Alternatively, the action of the Hecke operator \( T(m) \) on the underlying lattice can be understood as a sum of sublattices of index \( m \):
\[ T(m)\Lambda = \sum_{[\Lambda',\Lambda] = m} \Lambda'. \]

The Hecke operator \( T(m) \) can be written as a sum of two operators [14], [15], the first of which, \( V(m) \), gives a new series by replacing each \( q^n \) in \( f \) with \( q^{mn} \),
\[ (V(m)f)(z) = \sum_{n=1}^{\infty} a(n)q^{mn} = \sum_{n=1}^{\infty} a(n)e^{2\pi imnz} = f(mz) \] (33)
while the action of the second, \( U(m) \), is to define a new series by keeping only those terms \( q^n \) that are divisible by \( m \):

\[
(U(m)f)(z) = \sum_{n=1}^{\infty} a(n) q^{n/m} = \frac{1}{m} \sum_{j=0}^{m-1} f\left(\frac{z + j}{m}\right).
\]

We note that \( U(m)V(m) = 1 \), but since \( V(m)U(m) \) deletes terms in (3) not divisible by \( m \), \( V(m)U(m) \neq 1 \).

A Hecke operator for a prime argument can be written as

\[
T(p) = U(p) + \chi(p)p^{k-1}V(p).
\]

Thus, if we set \( (T(p)f)(z) = \sum b(n) q^n \), then \( b(n) = a(pm) + \chi(p)p^{k-1}a(n/p) \).

The actions of the \( U \) and \( V \) operators remind us of the raising and lowering operators (18) on the wavelets. Consequently,

\[
\sum_{n=0}^{\infty} a(p^n)|n\rangle \xrightarrow{a_{+}} \sum_{n=1}^{\infty} a(p^n)|n+1\rangle = \sum_{n=1}^{\infty} a(p^{n-1})|n\rangle,
\]

\[
\sum_{n=0}^{\infty} a(p^n)|n\rangle \xrightarrow{a_{-}} \sum_{n=1}^{\infty} a(p^n)|n-1\rangle = \sum_{n=0}^{\infty} a(p^{n+1})|n\rangle =
\]

\[
= a(p) \sum_{n=0}^{\infty} a(p^n)|n\rangle - \chi(p)p^{k-1} \sum_{n=1}^{\infty} a(p^{n-1})|n\rangle.
\]

Hence, we propose to define the Hecke operator for a prime argument

\[
T(p) = a_{+} + \chi(p)p^{k-1}a_{-}
\]

to obtain

\[
T(p)|f(p)\rangle = a(p)|f(p)\rangle.
\]

It is easy to see that this leads to the correct eigenvalues (31).

5.1. Inner product I. The vector space of modular (cusp) forms of weight \( k \) is equipped with the Petersson inner product, defined as

\[
\langle f | g \rangle = \int_{F} d^{2}z (\text{Im} \, z)^{k-2} f^{*}(z)g(z)
\]

which is invariant under the action of the modular group. The Hecke operators, in general, are not Hermitian, but satisfy

\[
T^{\dagger}(p) = \chi^{*}(p)T(p),
\]

where Hermitian conjugation is defined in terms of inner product (37). In other words, \((\chi^{*})^{1/2}T\) is a Hermitian operator.

How does \( T(p) \) in (36), acting on the space of vectors \( \{|f(p)\rangle\} \), behave under conjugation?

First, we need to define an inner product. At this point, we do not have an understanding of the transformation properties of the wavelets under the local modular group \( GL(2, \mathbb{Q}_{p}) \), and therefore we do not know how to define a \( GL(2, \mathbb{Q}_{p}) \)-invariant inner product analogous to the one in (37). We examine two choices, both seemingly natural, in turn.
The first is defined similarly to the Petersson inner product above, but using only the subgroup of scaling transformations, as
\[
\langle \hat{f}_p | g_p \rangle = \int_{\mathbb{Q}_p^\times} d^\times x |x|^{k-1} \hat{f}_p^*(x) g_p(x) dx. \tag{38}
\]

We recall that the effects of the operators \(a_{\pm} \) on the wavelets is to scale \( x \) by a factor of \( p^{\pm 1} \) (as well as change the overall normalization such that the raised/lowered wavelet functions remain orthonormal). By a change of the variable of integration in inner product (38), it is easy to verify that
\[
a_{\pm}^\dagger = p^{k-1} a_{\mp}, \quad a_{\mp}^\dagger = p^{1-k} a_{\pm}.
\]
Hence, it follows that \( T^\dagger(p) = \chi^*(p) T(p) \) and
\[
\langle a_q(p) - \chi(p) a_q^\dagger(p) | \hat{f}_p | g_p \rangle = 0.
\]
Therefore, either \( \hat{f}_p = g_p \), in which case the expression in the parentheses vanishes, thus establishing the Hermiticity of \( T(p) \), or the two vectors associated to distinct cusp forms are orthogonal, \( \langle \hat{f}_p | g_p \rangle = 0 \).

We hasten to add that the arguments (and hence the conclusions) above are formal. This is because the manipulations by scaling and redefining the variable of integration work only for functions in \( L^2(\mathbb{Q}_p) \). We do, however, need to restrict to a subspace \( L^2(p^{-1} \mathbb{Z}_p) \), in which scaling is not a symmetry because it may take us out of the subspace. Consequently, the relevant operators have to be projected back to the subspace of our interest. It is not obvious that the process of conjugation commutes with the projection. The arguments in 5.2 based on another proposal for the inner product does not rely on scaling.

We present an interesting observation, the analysis of the implication of which is left for the future. In inner product (38), we write the function \( g_p(x) \) as the inverse (generalized) Mellin transform of its (generalized) Mellin transform, as defined in [21], to obtain
\[
\langle \hat{f}_p | g_p \rangle = \int_{\mathbb{Q}_p^\times} d^\times x |x|^{(k-1)/2} \hat{f}_p^*(x) \mathcal{M}_\omega^{-1} [ \mathcal{M}_\omega |x|[(k-1)/2] g_p ](x) dx = \\
\int_{\mathbb{Q}_p^\times} d^\times x |x|^{(k-1)/2} \hat{f}_p^*(x) \sum_{\omega \in \text{mod } p^\times} \frac{\ln p}{2\pi} \int_0^{2\pi} dt |x|^{-it} \omega^*(x) \mathcal{M}_\omega |x|[(k-1)/2] g_p(it) = \\
\sum_{\omega \in \text{mod } p^\times} \frac{\ln p}{2\pi} \int_0^{2\pi} dt \left( \mathcal{M}_\omega [\hat{f}_p] \left( \frac{k-1}{2} + it \right) \right) \mathcal{M}_\omega [g_p] \left( \frac{k-1}{2} + it \right), \tag{39}
\]
which is a Parseval-type identity. It is interesting to note that the argument of the Mellin transform in the last line is \( k/2 - 1/2 + it \), which is shifted by half from the position of the conjectured zeroes of the \( L \)-function (4) on the line \( \text{Re } s = k/2 + it \). However, the poles of the local \( L \)-function in (9) are likely to be on the line \( \text{Re } s = (k - 1)/2 + it \). This can be verified for those cases where the character \( \chi \) is trivial, for example, \( L(s, \Delta) \), whose coefficients are the Ramanujan \( \tau \)-functions.

5.2. Inner product II. To discuss the other inner product and its applications, it turns out to be more convenient to modify Kozyrev wavelets (13) as \( \Psi_{n,m,j}^{(p)}(x) = |x|^{k/2} \psi_n^{(p)}(x) \) to define a set that is orthonormal with respect to the scale-invariant measure \( d^\times x \) on \( \mathbb{Q}_p^\times \). This modification is described in the Appendix. We also redefine \( p \)-artonic modular forms (24) by rescaling the coefficients with a weight-dependent factor as
\[
f(p)(x) = \sum_{n_p=0}^{\infty} \sum_{m=0}^\infty p^{-k/2 - n_p} a(p^{n_p}) \Psi_{1-n_p,0,1}^{(p)}(x).
\]
The rescaling is motivated by the bound on the growth of the coefficients of cusp forms. The inner product we define is the simple overlap integral

$$ (f_{(p)}|g_{(p)}) = \int_{\mathbb{Q}_p} d^\times x f_{(p)}^*(x)g_{(p)}(x) $$

(41)

of the modified $p$-armonic wave functions.

We use orthonormality conditions (A.2) to express the action of raising and lowering operators (18), (20) on function (40) to write

$$ a_+ f_{(p)}(x) = \sum_{n_p=1}^\infty \int_{\mathbb{Q}_p} d^\times y \Psi_{2-n_p,0,1}^{(p)}(x)\Psi_{1-n_p,0,1}^{(p)*}(y)f_{(p)}(y), $$

$$ a_- f_{(p)}(x) = \sum_{n_p=0}^\infty \int_{\mathbb{Q}_p} d^\times y \Psi_{n_p,0,1}^{(p)}(x)\Psi_{1-n_p,0,1}^{(p)*}(y)f_{(p)}(y). $$

(42)

From this, we can verify that $a_\pm^\dagger = a_\mp$ with respect to inner product (41). Moreover, using the same set of equations, we obtain

$$ a_+ f_{(p)}(x_{(p)}) = p^{\frac{k-1}{2}} a(p)\Psi_{1,0,1}^{(p)}(x_{(p)}) + p^{\frac{k-1}{2}} \sum_{n_p=1}^\infty p^{-\frac{k-1}{2} n_p} a(p^{n_p+1})\Psi_{1-n_p,0,1}^{(p)}(x_{(p)}), $$

$$ a_- f_{(p)}(x_{(p)}) = p^{\frac{k-1}{2}} \sum_{n_p=1}^\infty p^{-\frac{k-1}{2} n_p} a(p^{n_p-1})\Psi_{1-n_p,0,1}^{(p)}(x_{(p)}). $$

Hence, the combination

$$ T(p)f_{(p)} \equiv (a_+^{(p)} + \chi(p)a_-^{(p)})f_{(p)} = p^{-(k-1)/2}a(p)f_{(p)}, \quad T^\dagger(p) = \chi^*(p)T(p), $$

(43)

behaves like the Hecke operator $T(p)$ (cf. (31) and Sec. 5.1). Thanks to these equations,

$$ \int_{\mathbb{Q}_p} d^\times x g_{(p)}^*(x)(T(p) - \chi(p)T^\dagger(p))f_{(p)}(x) = 0 $$

implies that

$$ p^{-(k-1)/2}(a_f(p) - \chi(p)a_g^*(p))\int_{\mathbb{Q}_p} d^\times x g_{(p)}^*(x)f_{(p)}(x) = 0 $$

(44)

where we conclude that $f_{(p)}$ and $g_{(p)}$ are orthogonal (with respect to inner product (41)) and that $(\chi^*)^{1/2}(p)a_f(p)$ are real.

Following the steps leading to the equality in (39), we find the corresponding Parseval-type identity for this inner product:

$$ (f_{(p)}|g_{(p)}) = \int_{\mathbb{Q}_p} d^\times x f_{(p)}^*(x)\mathcal{M}_{\omega}^{-1}[\mathcal{M}_\omega[g_{(p)}]](x) = $$

$$ = \sum_{\omega \pmod{p^\infty}} \frac{\ln p}{2\pi} \int_0^{2\pi} dt \mathcal{M}_\omega^*[f_{(p)}](it)\mathcal{M}_\omega[g_{(p)}](it) = $$

$$ = \frac{\ln p}{2\pi} \int_0^{2\pi} dt L_{\ell(p)}^*(\frac{k-1}{2} + it)L_{\ell(p)}^*(\frac{k-1}{2} + it). $$

(45)
In arriving at the last step above, we used the following. In the Mellin transform

\[ \mathcal{M}_\omega[f_\ell](it) = c_\ell(it)p^{it}L_{\ell}(p^{\frac{k-1}{2} + it}), \]

only the prefactor depends on \( \omega(\ell) \), and therefore the sum in the discrete Fourier transform yields 1 for the argument \( s = it \) (see the Appendix), and the \( t \)-integral leads to a Kronecker delta, with which one of the sums is evaluated trivially. Thus, both the left- and right-hand sides of (45) reduce to

\[ \sum_{np} p^{-(k-1)n_p} a_\ell^p(p^n) a_\mu(p^n). \]

The left-hand side vanishes if \( f \) and \( g \) arise from distinct modular forms. Then the above is an orthogonality condition for the corresponding modular \( L \)-functions. It should be instructive to verify this by a direct computation, which we have unfortunately not been able to do. Instead, we offer an indirect argument.

We parameterize the “roots” of the denominator of the local function \( L_p(s,f) \) (9), which is quadratic in \( p^{-s} \), as \( a_1(p) = p^{(k-1)/2} e^{i\alpha_1(p)} \) and \( a_2(p) = p^{(k-1)/2} e^{-i\alpha_2(p)} \), whence

\[ a(p) = p^{(k-1)/2}(e^{i\alpha_1(p)} + e^{i\alpha_2(p)}) = 2 \cos \frac{1}{2}(\alpha_1 + \alpha_2)p^{(k-1)/2} e^{\frac{i}{2}(\alpha_1 - \alpha_2)}, \]

\[ \chi(p) = e^{i(\alpha_1(p) - \alpha_2(p))}. \]

We note that this is consistent with the condition on the growth of the coefficients \( a(p) \). The function \( L_p(s,f) \) is the generating function

\[ \frac{1}{1 - 2t \cos \theta + t^2} = \sum_{n=0}^{\infty} U_n(\cos \theta)t^n \quad \text{(46)} \]

of the family of orthogonal Chebyshev polynomials of type II, denoted by \( U_n(x) \), with \( \theta = (\alpha_1 + \alpha_2)/2 \) and \( t = p^{(k-1)/2} - s e^{i(\alpha_1 - \alpha_2)/2} \). The Chebyshev polynomials of type II satisfy the three-term recursion relation

\[ U_{n+1}(\xi) = 2\xi U_n(\xi) - U_{n-1}(\xi), \quad U_0(\xi) = 1, \quad U_1(\xi) = 2\xi. \quad \text{(47)} \]

In terms of trigonometric functions, \( U_n(\cos \theta) = \sin((n+1)\theta)/\sin \theta \). Furthermore, thanks to these recursion relations, the polynomials satisfy multiplicative property (8). The orthogonality of the local functions \( L_p(s,f) \), and hence of the product functions \( L(s,f) \) follows as a consequence of the orthogonality of the Chebyshev polynomials. The appearance of the Chebyshev polynomials of type II in this context has been noticed\(^5\) in Refs. [22], [23]. In the next section, we study a simpler class of functions, for which these properties are manifest by construction.

**6. Products of Dirichlet L-functions and Maass-like forms**

Since the modular forms and the associated \( L \)-functions are rather complicated objects, we instead study a family of functions related to the Dirichlet \( L \)-function (corresponding to the Dirichlet character \( \nu \))

\[ L(s,\nu) = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \nu(p)p^{-s}} \quad \text{(48)} \]

\(^5\)We became aware of these results after submitting a version of this paper to the arXiv.
which is simpler, as we see below. We cannot use these \( L \)-functions (48) directly because, unlike the \( L \)-functions associated with a modular cusp form (9), where the local factors at prime \( p \), \((1 - a(p)p^{-s} + \chi(p)p^{k-1}p^{-2s})^{-1}\), are quadratic functions of \( p^{-s} \), the local factors \( L_p(s, \nu) = (1 - \nu(p)p^{-s})^{-1} \) above are linear. Therefore, in order to mimic the properties of a modular \( L \)-function, we consider a product of two Dirichlet \( L \)-functions to define the function \( 2L(s, \nu) \) as

\[
2L(s, \nu) = L(s, \nu)L(s, \nu^*) = \prod_p \frac{1}{(1 - \nu(p)p^{-s})(1 - \nu^*(p)p^{-s})} = \prod_p \frac{1}{1 - 2 \cos(\arg \nu_p)p^{-s} + p^{-2s}} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]

where \( \nu^* \) is the complex conjugate of the character \( \nu \) (it is further assumed that \( \nu \) is not a principal character) and

\[
a(n) = \sum_{d|n} \nu(d)\nu^*(n/d)
\]

is the convolution of characters [24]. We note that the function \( 2L(s, \nu) \) is still meromorphic as a function of \( s \). Formally, it has the same form as (9) with

\[
a(p) = \nu(p) + \nu^*(p) = 2 \cos(\arg \nu_p),
\]

\( k = 1 \) and \( \chi = 1 \) (the trivial character).

The local factor \( 2L_p(s, \nu) \) at a prime \( p \) can be recognized as generating function (46), whence

\[
2L_p(s, \nu) = \frac{1}{1 - 2 \cos(\arg \nu_p)p^{-s} + p^{-2s}} = \sum_{\eta_p=0}^{\infty} U_{\eta_p}(\cos \xi)p^{-s\eta_p},
\]

where \( U_{\eta_p}(\xi) \) are the Chebyshev polynomials of type II of degree \( n_p \) in \( \xi = \arg \nu_p \). The property (the second condition in (8)) of the coefficients in \( L \)-series (49) is a recursion relation (47) satisfied by the Chebyshev polynomials. Finally, from the two equations above, we find that the coefficients are

\[
a(n) = \sum_{d|n} \nu(d)\nu^*(n/d) = \prod_{p: n=1^n}^p U_{\eta_p}(\cos(\arg \nu_p)).
\]

Before we proceed to discuss \( 2L(s, \nu) \) further, we note that we can define a slightly more general product

\[
2L(s, \nu_1, \nu_2^*) = L(s, \nu_1)L(s, \nu_2^*) = \prod_p \left(1 - 2e^{2i\pi(\alpha_1 - \alpha_2)} \cos \frac{\alpha_1 + \alpha_2}{2} p^{-s} + \chi(p)p^{-2s}\right)^{-1},
\]

where \( \alpha_1 = \arg \nu_1(p), \alpha_2 = \arg \nu_2(p), \chi(p) = \nu_1(p)\nu_2^*(p) = e^{i(\alpha_1 - \alpha_2)} \) and the coefficients

\[
a(p) = e^{2i\pi(\alpha_1 - \alpha_2)} U_1 \left( \cos \frac{\alpha_1 + \alpha_2}{2} \right)
\]

satisfy the multiplicative property \( a(p)a(p^r) = a(p^{r+1}) + \chi(p)p^{k-1}a(p^{r-1}) \) with \( k = 1 \). This is a special case of the parameterization discussed at the end of Sec. 5.2.

The Dirichlet \( L \)-function in (48) is related to the \( \vartheta \)-series

\[
\vartheta(z, \nu) = \sum_{n \in \mathbb{Z}} \nu(n)n^z e^{i\pi n^2 z/N},
\]

\( 1416 \)
where $\nu$ is a primitive Dirichlet character modulo $N$ and $\epsilon = (1 - \nu(-1))/2$ takes the value 0 or 1 depending on whether $\nu$ is even or odd, respectively. The series above defines a modular form of weight $1/2 + \epsilon$, level $4N^2$, and character $\nu(d)(\frac{-1}{d})^\epsilon$, where the Legendre symbol $\left(\frac{-1}{d}\right)$ gives a phase. The Mellin transform of the above is $L$-function (48):

$$L(s, \nu) = \frac{(\pi/N)^{(s+\epsilon)/2}}{2\Gamma(\frac{s+\epsilon}{2})} \int_0^\infty \frac{dy}{y} y^{(s+\epsilon)/2} \vartheta(iy, \nu).$$

(53)

We can now use this in (49) to write

$$2L(s, \nu) = \frac{(\pi/N)^{s+\epsilon}}{4\Gamma^2(\frac{s+\epsilon}{2})} \int_0^\infty dy_1 \int_0^\infty dy_2 (y_1y_2)^{\frac{s+\epsilon}{2}-1} \vartheta(iy_1, \nu)\vartheta(iy_2, \nu^*) =$$

$$= \frac{2(\pi/N)^{s+\epsilon}}{\Gamma(\frac{s+\epsilon}{2})} \sum_{n_1=1}^{\infty} \nu(n_1)n_1^{' \frac{s+\epsilon}{2}} \sum_{n_2=1}^{\infty} \nu^*(n_2)n_2^{' \frac{s+\epsilon}{2}} \int_0^\infty \frac{dy}{y} y^{s+\epsilon} \int_0^\infty \frac{dy'}{y'} e^{-\frac{\pi n_1^{' 2}y''}{N^2} - \frac{\pi n_2^{' 2}y''}{N^2}}$$

$$= \frac{4(\pi/N)^{s+\epsilon}}{\Gamma^2(\frac{s+\epsilon}{2})} \sum_{n=1}^{\infty} a(n)n^\epsilon \int_0^\infty \frac{dy}{y} y^{s+\epsilon} K_0\left(\frac{2\pi n}{N}y\right).$$

(54)

The integrals from the $L$-functions have been reorganized to be a convolution of the integrals of the two $\vartheta$-series. We have redefined the dummy integration and summation variables to $y^2 = y_1y_2$, $y'^2 = y_1/y_2$, $n = n_1n_2$, $d = n_1$ and used (51) for the coefficients. The integral over $y'$ can be evaluated from standard tables (see, e.g., [25]), or identified as the integral representation of the modified Bessel function $K_0$. Finally, the $y$-integral in (54), being a Mellin transform of the modified Bessel function, is known analytically [25]. Performing the integration, we of course reproduce (49), as expected.

The expression above means that, up to a prefactor $4(\pi/N)^{s+\epsilon}/\Gamma^2(\frac{s+\epsilon}{2})$, the function $2L(s, \nu)$ is the Mellin transform of the convolution of two $\vartheta$-series

$$(\vartheta(\nu) \ast \vartheta(\nu^*)) (iy) = \int_0^\infty \frac{dy'}{y'} \vartheta(iyy', \nu)\vartheta\left(\frac{iyy', \nu^*}{y'}\right) = \int_0^\infty \frac{dy'}{y'} \vartheta\left(\frac{iyy', \nu^*}{y'}\right) \vartheta(iyy', \nu),$$

(55)

where the last equality follows from redefining the integration variable $y' \to 1/y'$. The convolution is that of the Dirichlet character and its complex conjugate (denoted by a star above), as well as a Mellin convolution in the imaginary part of their arguments, which is shown explicitly. We know that the $\vartheta$-series has the modular property, in particular, under $y \to 1/y$ (the $S$-transformation $z \to -1/z$ of the modular group restricted to the imaginary part),

$$\vartheta(iy, \nu) = \frac{y^{1/2+\epsilon}}{\sqrt{\nu}} \tau(\nu)\vartheta(iy^*, \nu^*), \quad \tau(\nu) = \sum_{m=0}^{N-1} \nu(m)e^{2\pi im/N},$$

(56)

where $\tau(\nu)$ is the Gauss sum. Hence we find that

$$\int_0^\infty \frac{dy'}{y'} \vartheta\left(\frac{i}{yy', \nu^*}\right) = y^{1+2\epsilon} \int_0^\infty \frac{dy'}{y'} \vartheta\left(\frac{iyy', \nu^*}{y'}\right) \vartheta(iyy', \nu^*),$$

(57)

where we have used $\tau(\nu)\tau(\nu^*) = \nu(-1)N$. Thus the Mellin inverse of $2L(s, \nu)$, namely, the convolution (55) of the $\vartheta$-series, is (quasi-)modular with weight $1 + 2\epsilon$ under the transformation $y \to 1/y$.

It is interesting to note that this property, as well as the explicit expression in the right-hand side of (54), are reminiscent of the harmonic Maass waveform [13] of weight $\lambda = 0$,

$$M_{\lambda, N}(x + iy) = \sum_{n=1}^{\infty} a_n(ny)^{\epsilon} \sqrt{y} K_\lambda\left(\frac{2\pi n}{N}y\right)e^{2\pi inx/N},$$

(58)
restricted to a purely imaginary argument. The Maass waveform $M_{\lambda,N}: \mathbb{H} \to \mathbb{C}$ is a nonholomorphic “modular function” on the upper half-plane that is a square-integrable eigenfunction of the hyperbolic $\Gamma(1)$-invariant Laplacian corresponding to the eigenvalue $1/4 - \lambda^2$. Since $a_0 = 0$ in the above, it is actually a Maass cusp form. It may seem that there is a puzzle because the Maass waveform $M_{0,N}$ has zero modular weight, while the $\partial$-series related to each of the $L$-functions in (53) are of weight $1/2 + \epsilon$, and hence their (convolution) product should be of weight $1 + 2\epsilon$. This is true, but it is compensated by $y^{1/2+\epsilon}$, which is a nonholomorphic form of weight $-1-2\epsilon$. The factor of $\sqrt{y}$ was introduced by shifting the argument in the Mellin transform. It may also be noted that we lose complex analyticity by performing a Mellin transform in only the imaginary part of the argument of the $\vartheta$-series.

We propose to identify the modular object related to the series $2L(s,\nu)$ to be a Maass form of the above type,

$$\sqrt{y}f(x + iy, \nu) \to M^{(\nu)}_{0,N}(x + iy) = \sum_{n=1}^{\infty} a(n)y^{\nu} \sqrt{y} K_0\left(\frac{2\pi ny}{N}\right)e^{2\pi nx/N},$$

(59)

where $a(n)$ is related to the Dirichlet character $\nu$ by (51). We must caution the reader that this identification is tentative because we have not been able to show the transformation property of (55) under the full modular group. Even though this is only a special case of weight $\lambda = 0$ and is moreover “reducible” by construction, this toy construction of (quasi-)Maass waveforms through a product of Dirichlet modular group. Even though this is only a special case of weight $1 + 2\epsilon$, this may be useful in understanding aspects of the proposed correspondence. The following analysis is in that spirit.

Henceforth, we restrict to the case of even characters $\nu(1) = \nu(-1)$ for definiteness. The case of odd characters can be studied in an analogous fashion. We now consider the $p$-artonic modular forms corresponding to (49)–(51),

$$f(p)(\nu, x) = \sum_{n_p=0}^{\infty} a(p^{n_p})\Psi^{(p)}_{1-n_p,0,1}(x) = \sum_{n_p=0}^{\infty} \sum_{n_p=0}^{\infty} U_{n_p} \cos(\arg \nu_p) \Psi^{(p)}_{1-n_p,0,1}(x).$$

(60)

While these vectors in $\mathcal{H}^{(p)}$ are well-defined, the correspondence of the tensor product $\otimes_p f(p)(\nu, x(p))$ to a modular object on the upper half-plane $\mathbb{H}$ proposed above would require further justification. This is because Maass waveform (58), not being a meromorphic function, does not admit a $q$-series expansion. The latter form is what we had used for correspondence (22) in Sec. 4. However, the dependence of both the holomorphic and nonholomorphic modular forms (3) and (59) on the variable $x$ is of the same form. Therefore, the association proposed should correctly be thought of as that between the Fourier coefficients in the Fourier series expansion (in $x$) of the corresponding modular objects. The $L$-functions related to the modular objects are also defined with the help of these Fourier coefficients. Nevertheless, it would be desirable to make a distinction between the holomorphic and nonholomorphic forms at the level of the proposed $p$-artons.

Be that as it may, the inner product\(^6\) (41) of two such functions $f(p)(\nu_{\mathfrak{q}})$ and $g(p)(\nu_{\mathfrak{g}})$ satisfies the orthogonality condition

$$\langle f(p)(\nu_{\mathfrak{q}}), g(p)(\nu_{\mathfrak{g}}) \rangle = \int_{\mathbb{Q}_p^x} d^x x f^*(p)(\nu_{\mathfrak{q}}, x) g(p)(\nu_{\mathfrak{g}}, x) = \sum_{n_p=0}^{\infty} U_{n_p}^{(\nu_{\mathfrak{q}})}(\cos(\arg \nu_{\mathfrak{q}})) U_{n_p}^{(\nu_{\mathfrak{g}})}(\cos(\arg \nu_{\mathfrak{g}})) = \delta_{\nu_{\mathfrak{q}},\nu_{\mathfrak{g}}},$$

(61)

as a consequence of the completeness of the Chebyshev polynomials. Hence, their tensor products $f(\nu_{\mathfrak{q}})$

\(^6\)For modular forms of weight $k = 1$, the two inner products (38) and (41) are the same.
and $g(\nu_g)$

$$
(f(\nu_f)|g(\nu_g)) = \prod_p (f(p)(\nu_f)|g(p)(\nu_g)) = \prod_p \sum_{n=0}^{\infty} a_f^*(p^np) a_g(p^np) = \sum_{n=1}^{\infty} a_f^*(n) a_g(n) = 0.
$$

(62)

are also orthogonal. We consider the inner product (37) of two Maass forms $M_f = \sqrt{\gamma} f(x + iy, \nu_f)$ and $M_g = \sqrt{\gamma} g(x + iy, \nu_g)$:

$$
\langle f(\nu_f)|g(\nu_g) \rangle = \int_{-N/2 < x < N/2, |x+iy| > 0} \frac{dx \, dy}{y^2} \left( \sqrt{\gamma} f(x + iy, \nu_f) \right)^* \sqrt{\gamma} g(x + iy, \nu_g).
$$

Unfortunately, we were not able to perform the integrals with the restrictions imposed by the fundamental domain. However, we can compute the above in the rectangular region $\{ -N/2 < |x| < N/2, y > 0 \}$, which contains an infinite number of copies of the fundamental domain $\mathcal{F}$. By the modular properties of the integrand and the measure, all these copies are identical, and we therefore expect to obtain the original integral multiplied by an infinite factor. That is exactly what we find. The $x$-integral in

$$
\langle f(\nu_f)|g(\nu_g) \rangle = \sum_{m,n=1}^{\infty} a_f^*(m) a_g(n) \int_{-N/2}^{N/2} dx e^{2\pi i (n-m)x/N} \int_{0}^{\infty} dy \frac{1}{K_0 \left( \frac{2\pi ny}{N} \right)} K_0 \left( \frac{2\pi ny}{N} \right)
$$

vanishes unless $m = n$, giving $\delta_{mn}$, implying that two different Maass forms are orthogonal. For the $y$-integral we use the expression [25]

$$
\int_{0}^{\infty} dy y^{-\mu} K_\alpha(a y) K_\beta(b y) = \frac{a^{\mu-1} b^\beta}{2^{\mu+2} \Gamma(1-\mu)} \Gamma \left( \frac{1-\mu+\alpha+\beta}{2} \right) \Gamma \left( \frac{1-\mu-\alpha+\beta}{2} \right) \times
$$

$$
\times \Gamma \left( \frac{1-\mu+\alpha-\beta}{2} \right) \Gamma \left( \frac{1-\mu-\alpha-\beta}{2} \right) \times
$$

$$
\times \frac{2}{\Gamma(1-\mu+\alpha+\beta)} \frac{1-\mu+\alpha+\beta}{2} \frac{1-\mu-\alpha+\beta}{2}, 1-\mu; 1-\frac{b^2}{a^2},
$$

for $\Re(a+b) > 0$, $\Re \mu < 1 - |\Re \alpha| - |\Re \beta|$. The integral we need is exactly at the border of the condition on $\mu$, and indeed the arguments of all the $\Gamma$ functions are zero, giving a divergent factor. Thus, up to an infinite normalization factor arising from the divergent $\Gamma$ functions, the modular objects related to the product $L$-functions $2L(s,\nu)$ are orthogonal nonanalytic Maass waveforms of modular weight zero. This corresponds to the orthogonality (62) obtained from the wavelet expansion.

Expression (62) can be related to the inner product of the corresponding product $L$-functions (49) by using the identity

$$
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \frac{e^{it(lnm-lnn)}}{\sqrt{mn}} = \delta(ln n - ln m) \sqrt{mn} = \delta_{mn}.
$$

We note that while the above identity for $\delta_{mn}$ works for $m^\alpha n^{1-\alpha}$ for any $0 < \alpha < 1$ in the denominator, only for $\alpha = 1/2$ does it lead to the properties required of an inner product in the following. Thus,

$$
\langle f(\nu_f)|g(\nu_g) \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_f^*(n) a_g(m) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \sum_{n=1}^{\infty} a_f^*(n) m^{1/2-it} \sum_{n=1}^{\infty} a_g(n) n^{1/2+it} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt 2L_f^* \left( \frac{1}{2} - it, \nu_f \right) 2L_g^* \left( \frac{1}{2} + it, \nu_g \right),
$$

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where in the last line, we use the analytic continuation of the power series. The final expression allows an interpretation as the inner product \( \langle 2L_f|2L_g \rangle \) of the \( L \)-functions related to the Maass forms. This is a Parseval-type identity, like that in (39), (45). Hence, the family of product \( L \)-functions, defined in (50), seems to form an orthogonal set of functions (for a fixed prime value of \( N \)). Although the \( L \)-functions are known explicitly, it is not straightforward to verify their orthogonality. It may be seen (by plotting the functions in Mathematica) that the imaginary part of the integrand is odd, and therefore its integral vanishes for \( f \neq g \), however, its real part oscillates with increasing amplitude, making it difficult to perform the integral numerically.

7. Endnote

We have used the orthonormal bases provided by wavelets [6] in \( L^2(\mathbb{Q}_p) \) to associate complex-valued functions, one function for each prime number \( p \), with a cusp form of a congruence subgroup of the modular group \( SL(2, \mathbb{Z}) \). We have studied the local functions (which may appropriately be called \( p \)-artons, borrowing a term from the physics of strong interactions) obtained through this decomposition, their Mellin transforms and the related \( L \)-functions. We leave the study of the action of the group \( GL(2, \mathbb{Q}_p) \) on these \( p \)-artons to the future. By taking a product of two Dirichlet \( L \)-functions associated to a Dirichlet character and its complex conjugate, we have also defined functions that are similar to these modular \( L \)-functions. Rather surprisingly, these turn out to behave like nonanalytic Maass waveforms of weight zero. Although being reducible by construction, these may not be of intrinsic interest mathematically, but being much simpler to work with, they may be useful toy objects to further the correspondence.

We close with the following remark. The one-to-one relation between a cusp form (a complex function on the upper half of the complex plane) and a vector in (a subspace of) \( \otimes_p L^2(\mathbb{Q}_p) \) is reminiscent of the holographic correspondence, which has dominated the landscape of research in theoretical physics in the recent decades. The conformal boundary of the upper half-plane (\( \mathbb{H} = SL(2, \mathbb{R})/U(1) \)) is the real line \( \mathbb{R} \). The latter is closely related to \( \mathbb{Q}_p \), which can be thought of as the conformal boundary of the Bruhat–Tits tree, which in turn is the coset of \( GL(2, \mathbb{Q}_p) \) by its maximal compact subgroup \( GL(2, \mathbb{Z}_p) \). The association between a modular form \( f : \mathbb{H} \to \mathbb{C} \) and the products of functions \( \otimes_p f(p) \), where \( f(p) : \mathbb{Q}_p \to \mathbb{C} \) for a prime \( p \) is a complex-valued function on \( \mathbb{Q}_p \), suggestive of a bulk–boundary correspondence in holography. It is so, since the data related to a function in the bulk of the upper half of the complex plane is in the “boundary” \( \otimes_p \mathbb{Q}_p \), which in turn is related to \( \mathbb{R} = \partial \mathbb{H} \). In this sense, the proposed relation may even be called a holographic \( p \)-arton model of modular forms. It would be interesting to explore if there are conformal field theories (and their bulk duals) related to the \( p \)-artonic objects as well as the nonanalytic Maass waveforms.

Appendix: Wavelets on \( \mathbb{Q}_p^\times \)

We modify the Kozyrev wavelets to define

\[
\Psi_{n,m,j}^{(p)}(x) = |x|^{1/2} \psi_{n,m,j}^{(p)}(x),
\]

which are naturally defined on \( \mathbb{Q}_p^\times \) because they are orthonormal with respect to the scale invariant multiplicative Haar measure \( d^\times x \):

\[
\int_{\mathbb{Q}_p^\times} \frac{dx}{|x|} \Psi_{n,0,1}^{(p)}(x) \Psi_{n',0,1}^{(p)}(x) = \int_{\mathbb{Q}_p} dx \psi_{n,0,1}^{(p)}(x) \psi_{n',0,1}^{(p)}(x) = \delta_{nn'}.
\]

We note that the wavelets above, being different from Kozyrev wavelets (13) by a coordinate dependent factor, are not equal to the constant \( p^{-n/2} \) for \( |x|_p < p^n \), although they are still locally constant functions.
on $\mathbb{Q}_p$. The raising and lowering operators in (18) and (20) act as before:

$$a_\pm^{(p)} \Psi_{n,0,1}^{(p)}(x) = \Psi_{n \pm 1,0,1}^{(p)}(x), \quad a_\pm^{(p)} \Psi_{1,0,1}^{(p)}(x) = 0,$$

but we choose to use a different notation to emphasize the fact that they act on a different space of functions. Mellin transform (25) of these modified wavelets

$$\mathcal{M}_{(p,\omega)}[\Psi_{n,0,1}^{(p)}](s) = c_p(\ell,s) p^{\ell s} = -\left(\frac{1}{p(1-p^{-s-1/2})} - \frac{1}{p^{s+1/2} - 1}\delta_{\ell,0} - \delta_{\ell,p^{-1}}\right)p^{\ell s}$$

likewise differs somewhat from (27). We note that

$$\sum_\ell |c_p(\ell,s)|^2 = 1 + \frac{1 - |p|^2}{|p^{s+1/2} - 1|^2},$$

and hence, if the argument $s$ is purely imaginary, the sum is 1.

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