Charge qubits in the ultrastrong coupling regime

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We study the feasibility of reaching the ultrastrong (USC) and deep-strong coupling (DSC) regimes of light-matter interaction, in particular at resonance condition, with a superconducting charge qubit, also known as Cooper-Pair box (CPB). We show that by shunting the charge qubit with a high-impedance LC-circuit, one can maximally reach both USC and DSC regimes exceeding the classical upper bound $|g| \leq \sqrt{\omega q/2}$ between two harmonic systems with frequencies $\omega_q$ and $\omega_r$. In our case, the fundamental model corresponds to an enhanced quantum Rabi model, which contains a displacement field operator that breaks its internal parity symmetry. Furthermore, we consider a multipartite device consisting of two CPBs ultrastrongly coupled to an oscillator as a mediator and study a quantum state transfer protocol between a pair of transmon qubits, all of them subjected to local incoherent noise channels with realistic parameters. This work opens the door for studying light-matter interactions beyond the quantum Rabi model at extreme coupling strengths, providing a new building block for applications within quantum computation and quantum information processing.

I. INTRODUCTION

Macroscopic devices exhibiting quantum behavior have attracted much attention in the last three decades. The idea of having access to a system with tailored and tunable parameters paves the way for developing protocols to solve particular computational problems. Among the available quantum architectures, outstanding results [1] have been achieved with superconducting circuits [2–4], where one can study and engineer light-matter interactions in what is known as the field of circuit quantum electrodynamics (cQED) [5–9], in opposition to cavity quantum electrodynamics (CQED) [10–12]. In this platform, one can design circuits containing Josephson junctions [13, 14], lumped elements that exhibit a wide range of anharmonicity, to mimic atomic energy spectra. These artificial atoms have been classified into three different types depending on the main degree of freedom [15]: charge qubits [16–20], phase qubits [21–23], and flux qubits [24–26]. On the other hand, LC oscillators and transmission line resonators encode (approximate) single, and multimode light degrees of freedom [27–31], respectively.

The fast development of cQED technology was triggered by the success of a system based on the charge qubit presenting coherent quantum interaction with a superconducting resonator [5, 6]. This anharmonic system was made of a Josephson junction, with energy ($E_J$), in parallel with a capacitor, with dominating kinetic energy ($E_C \geq E_J$). However, this system proved to be very sensitive to charge fluctuations, and consequently, has shown very low coherence times. Later on, better performances were achieved with transmon qubits [18, 19], a more harmonic version of the CPB where the potential energy dominates the kinetic energy ($E_J \gg E_C$). Likewise, modified versions [32] of the original flux qubit, i.e. shunting the junctions with big capacitors in parallel, led to more harmonic and reproducible designs and are competing with transmon-like technology. Another leap in decay time performance was achieved with the fluxonium qubit [33], where their original lack of good coherence properties has been recently tackled [34].

These flexible technological accomplishments from the community have pushed forward for achieving regimes previously unobserved in natural systems such as the ultrastrong (USC) [35], and deep-strong (DSC) coupling regime [36]. In the minimal USC regime scheme, the coupling strength between a qubit (matter) and a harmonic oscillator mode (light) is comparable to or larger than the subsystem energies of the two coupled systems. Here, the complete quantum Rabi model (QRM) [37–39] is necessary to unveil the correct non-classical physics [40–42]. This is due to the fact that the rotating wave approximation (RWA), leading to the Jaynes-Cummings model is no longer valid [36, 39, 43]. Distinct spectroscopic features of this system have been experimentally probed with superconducting flux qubits embedded in cQED setups [41, 44–46], and transmon qubits [47–49] as well as quantum dots [50] coupled to high impedance resonators. More recently, these features of the USC regime have been observed in a decelerated frame [51]. The QRM presents a discrete parity symmetry and an anharmonic energy spectrum that provide a set of resources for quantum information and quantum simulations [52–62].

All different proposals for building superconducting qubits coupled to harmonic oscillators (light photons) suggest the existence of a natural trade-off between anharmonicity on the qubit, the strength that the coupling between them could have, and the noise experienced in the device. In charge qubits, considerable anharmonicity increases the sensitivity to the voltage (charge) fluctuations that controls it [16, 17]. For flux qubits, the anharmonic energy spectrum produces high sensitivity to external magnetic flux [24, 26]. In these devices, attenuating the anharmonicity may lead to the enhancement of their coherence times. Nevertheless, the coupling strength that they can achieve with harmonic degrees of freedom de-
increases [63, 64, 66]. In fact, it is already known that the coupling $g$ between two physical oscillators with frequencies $\omega_q$ and $\omega_r$ is bounded by $g \leq \sqrt{\frac{e^2}{\hbar c_0^2}} / \alpha$ [35, 47, 67]. However, such restriction does not seem to exist when we consider a harmonic oscillator coupled to an anharmonic system, e.g., a qubit. Moreover, it has been shown that the topology of the connection between the defined anharmonic and harmonic degrees of freedom plays a crucial role in this bound [64].

In this article, we investigate theoretically the spectroscopic and dynamical properties of a charge superconducting qubit ($\omega_q$) coupled to an LC oscillator ($\omega_r$) in its most general (mixed series/parallel) configuration beyond the USC regime. Here, we aim to find experimentally feasible sets of circuit parameters such that we have both resonance and USC/DSC regime, readily $g \sim \{\omega_r, \omega_q\}$ and $\omega_r \simeq \omega_q$, thus beating the previously mentioned limit. In this regime, we find that the standard quantum Rabi model is no longer valid for describing the light-matter interaction in this circuit. Furthermore, we prove that an extension of the proposed USC qubit-oscillator system, now with two qubits and an oscillator, can be used as a new quantum information building block, the mediator of a quantum state transfer (QST) protocol, considering archetypical noise sources.

The remainder of the paper is structured as follows. In Sec. II we derive the Hamiltonian of a single and two charge qubits coupled to the LC oscillator. Then we study numerically and analytically the coupling limit of the model we proposed, and the conditions needed for the different parties to be both on resonance, and in the USC/DSC coupling regime. Furthermore, we provide feasible circuit parameters for the experimental implementation of both the single and the two-CPB models to achieve the resonant USC/DSC regime. In Sec. III we study the properties of the two-qubit model, and use it as the incoherent mediator for performing QST. Finally, we summarize the main conclusions of this work in Sec. IV.

II. COUPLING RATIO OF THE LIGHT-MATTER INTERACTION

In recent decades, there has been an increasing interest to explore and push the light-matter coupling to its limits. Constraint by the fine-structure constant $\alpha \approx 1/137$, coupling parameters in CQED are bound. On the other hand, engineered systems in circuit QED have not been found to present such limitation [63], and experimental and theoretical efforts to find the true coupling limit between subsystems have been prolific [35, 41, 44–46, 48, 64].

In this section, we review and extend the works of [35, 64, 67], and study the coupling limit of the light-matter interaction with a charge qubit coupled to an LC oscillator on and off resonance. We show that the existing bound on the coupling versus local frequencies ratio of two harmonic systems is no longer valid here, i.e., there is no fundamental limit for an anharmonic system, like the charge qubit, that prevents it from reaching the non-perturbative USC regime [43] on resonance. Finally, we extend this result to a two-qubit and oscillator circuit, which we later use in the QST protocol.

A. Hamiltonian description

Let us study the circuit depicted in Fig. 1(a), consisting of a Cooper-pair box (CPB) generically coupled to an LC oscillator. The CPB is formed by a Josephson junction with Josephson energy $E_J$ and capacitance $C_J$. This junction couples to a superconducting island providing Cooper-pairs from a voltage source $V_g$ through a gate capacitance $C_g$. The oscillator is formed by a lumped inductor $L_r$ and capacitor $C_r$. There are ground connections from the oscillator nodes through capacitors $C_c$ and $C_p$. Notice that this circuit recovers the purely series and parallel coupling configurations in the limit of $C_p \to \infty$, and $C_c \to 0$, respectively [64].

![FIG. 1. (a) Illustration for a Cooper-pair box capacitively coupled to an LC oscillator. The CPB corresponds to a Josephson junction with Josephson energy $E_J$, and capacitance $C_J$. The LC oscillator corresponds to a capacitance $C_r$, and $C_p$, in series with an inductor $L_r$. We describe the system with two independent flux node variables, $\Phi_J$ and $\Phi_r$. (b) Extended symmetrical circuit with two Josephson junctions and an LC oscillator. Both $\Phi_0$ and $\Phi_1$ node variables can be eliminated in the analysis.](image)

We describe the electrical circuit in terms of the flux node variables related to the voltage drops as $\Phi = \int_{-\infty}^{t} V(t')dt'$, leading to the following Lagrangian

$$L_1 = \frac{1}{2} \dot{\Phi}_T C \ddot{\Phi} - \tilde{V}_g \Phi^T C v - \frac{\tilde{\Phi}^2}{2L_r} + E_J \cos(\varphi_J), \quad (1)$$

here, $\tilde{\Phi} = \{\Phi_J, \Phi_r\} \leftrightarrow$ correspond to the flux vector describing the CPB and the LC oscillator, and $\varphi_J = \Phi_J / \phi_0$ is the superconducting flux quantum, and $2e$ is the charge of a Cooper-pair. Notice that we have eliminated the flux coordinate $\Phi_1$ since it corresponds to a passive node (for more details see Appendix A). $C$ is the capacitance matrix of the circuit, and $C_c$ is the gate capacitance vector. We obtain the circuit Hamiltonian by com-
puting the canonical conjugate momenta \( Q_\ell = \partial L_1 / \partial \dot{\Phi}_\ell \), and applying the Legendre transformation leading to

\[
H_1 = \frac{1}{2C_J}(Q_J - 2en_g)^2 - E_J \cos(\varphi_J) + \frac{Q_r^2}{2C_r} + \frac{\Phi_r^2}{2L_r} - \frac{Q_rQ_J}{C_{Jr}},
\]

where \( Q_J \) is the CPB charge, and \( Q_r \) is the oscillator charge. Furthermore, \( \tilde{C}_J, \tilde{C}_r, \) and \( C_{Jr} \) correspond to the effective Josephson, oscillator, and coupling capacitances, respectively. Finally, \( n_g \) is the effective gate-charge number, which depends on the collective voltage bias \( \tilde{V}_g \), and we set it throughout the whole article to the sweet spot of the CPB, i.e. \( n_g = 0.5 \). The quantization of the Hamiltonian in Eq. (2) is done by promoting the annihilation and creation operators \( \hat{a}, \hat{a}^\dagger \) satisfying the commutation relation \( [\hat{a}, \hat{a}^\dagger] = \hbar/2 \), and \( \tilde{Q}_r = \sqrt{\tilde{C}_r} / 2(a - a^\dagger) \), quantifies the excess of Cooper-pairs on the CPB, \( e^2 / 2\tilde{C}_J \) is the charge energy of the CPB, and

\[
g = \omega_r \gamma \sqrt{\frac{Z_r}{2R_Q}},
\]

is the coupling strength between the CPB and the LC oscillator, where \( Z_r = \sqrt{L_r / \tilde{C}_r} \) is the effective impedance of the oscillator, \( R_Q = \hbar / (2e^2) \) is the reduced quantum resistance, and \( \gamma = \tilde{C}_r / C_{Jr} \) is the ratio between the dressed resonator capacitance and the coupling capacitance.

The same analysis can be extended to a system composed of two CPBs coupled to the same LC oscillator as shown in Fig. 1(b). Following the same procedure, we obtain

\[
H_2 = \sum_{\ell=1}^{2} H^{\text{cpb}}_\ell + \hbar \omega_r a^\dagger a + \hbar \sum_{\ell=1}^{2} g_r \hat{n}_{\ell,1} (a + a^\dagger) + g_{12} \hat{n}_{1,1} \hat{n}_{2,1},
\]

where \( H^{\text{cpb}}_\ell = 4E_C (\hat{n}_J - n_g)^2 - E_J \cos(\hat{\varphi}_J) \) is the Hamiltonian of the \( \ell \)-th CPB, and again, we assume that both CPBs work on the sweet spot i.e. \( n_g = 0.5 \). Here, \( g_r \) is the coupling strength between the \( \ell \)-th CPB and LC oscillator, and \( g_{12} \) stands for the direct coupling parameter between the two CPBs, defined as

\[
g_\ell = \omega_r \gamma \sqrt{\frac{Z_r}{2R_Q}}, \quad g_{12} = -\frac{1}{R_Q C_{12}},
\]

with \( \gamma_\ell = \tilde{C}_r / C_{\ell r} \), and \( \ell \in \{1, 2\} \). Moreover, \( \tilde{C}_r \) is the dressed resonator capacitance, \( C_{\ell r} \) is the coupling capacitance between the \( \ell \)-th CPB and the oscillator, \( C_{12} \) is the dressed coupling capacitance between the CPBs. Notice that even though both CPBs have no direct coupling between them, an effective capacitive coupling arises, through the capacitance matrix inversion (see Appendix C for more details). Furthermore, we have discarded the constant term \( O(n_g^2) \) which does not modify the equations of motion.

### B. Limit of the light-matter coupling ratio

Now that we have obtained the exact Hamiltonians for the circuits above, let us study the limit of the light-matter coupling ratio. As previously mentioned, we consider the CPB working on the sweet spot, where its low-lying energy spectrum can in certain conditions be truncated to an effective two-level system, with transition frequency \( \hbar \omega_q = E_J \) and charge operator \( \hat{n}_J = (\mathbb{1} + \sigma^z) / 2(a - a^\dagger) \), satisfying the commutation relation \([a, a^\dagger] = 1\), and the effective oscillator frequency \( \omega_r = 1 / \sqrt{\tilde{C}_r L_r} \). Finally, we obtain the quantum Hamiltonian as follows

\[
H_1 = 4E_C (\hat{n}_J - n_g)^2 - E_J \cos(\hat{\varphi}_J) + \hbar \omega_r a^\dagger a + \hbar g_r \hat{n}_J (a + a^\dagger),
\]

where \( \tilde{g} = g / 2 \) corresponds to the normalized coupling strength. Notice that for small \( \tilde{g} \), the coupling term \( [a + a^\dagger] \) can be adiabatically eliminated. However, as \( \tilde{g} \) increases, this contribution is not negligible anymore, and Eq. (7) describes the light-matter interaction between a CPB and a single electromagnetic mode rather differently from the QRM

\[
H_{\text{QRM}} = \frac{\hbar \omega_q}{2} \sigma^z + \hbar \omega_r a^\dagger a + \hbar \tilde{g} \sigma^y (a + a^\dagger).
\]

Furthermore, this coupling term \( [a + a^\dagger] \) induces the breaking of the parity symmetry \( \mathbb{Z}_2 \). Alternatively, by displacing the annihilation and creation operators with a constant, we are left with the QRM plus a \( \sigma_z \) additional term in the qubit Hamiltonian. Consequently, it is possible to access any state of the system because of the lack of internal selection rules [65]. Fig. 2 shows the energy spectrum of both systems i.e. \( H_1 \) and \( H_{\text{QRM}} \) as a function of the ratio \( \tilde{g} / \omega_r \). For \( \tilde{g} / \omega_r \in [0, 0.3] \), i.e. within the perturbative USC [43], the energy spectrum of both models coincide, confirming the approximation routinely used in the literature [5, 6]. Out of the perturbative USC/DSC regimes, i.e. \( \tilde{g} / \omega_r \in [0.3, 1] \), the two models do not describe the same physical phenomena.

In the past decades, there has been a great interest to know the limits of the fundamental coupling between an anharmonic system (matter) and a single-mode resonator (light) [35, 63, 64, 67]. In the context of the qubit approximation the following ratio is a compact figure of merit for the strength of the coupling relative to the characteristic energies of the qubit and the harmonic part

\[
R = \frac{g}{\sqrt{\omega_r \omega_q}} = \frac{1}{C_{\ell r}} \sqrt{\frac{\tilde{C}_r}{2R_Q E_J}}.
\]
We put it forward as a figure of merit also beyond the qubit approximation. It can be readily appreciated that the coupling ratio is inversely proportional to the Josephson energy $\sqrt{E_J}$ of the CPB, and can, in principle, be arbitrarily increased. This formula generalizes those defined in [35, 64], which are proposed for resonance, i.e. $\omega_r = \omega_q$,

$$R_{\text{res}} \equiv \frac{2\tilde{g}}{\omega_r} = \gamma x,$$

where $x = \sqrt{Z_r/(2R_Q)}$ is a dimensionless parameter, and the capacitance ratio is

$$\gamma = \frac{C_g C_p}{C_g (C_c + C_p) + C_j (C_c + C_p + C_j)}.$$

Observe that by considering $C_g, C_p \gg C_c, C_j$, the ratio $R_{\text{res}} \approx x$, in other words, it increases with the rise of the resonator impedance [64]. Thus, there is no fundamental restraint preventing us from reaching the USC/DSC regime when the anharmonic subsystem works as a CPB. Conversely, we can recover the upper bound [35, 47, 67] taking the harmonic (transmon) limit of the Josephson variable, i.e. replacing the pure Josephson term with a linear inductor ($L_J$) in Fig. 1(a), leading to

$$g_o = \frac{\omega_j \omega_r C_j \tilde{C}_r}{4C_j^2} \leq \frac{\sqrt{\omega_r \omega_q}}{2}.$$

Here, $g_o$ is the effective coupling strength between the harmonic systems, and $\omega_q = 1/\sqrt{C_j L_J}$ is the frequency of the transmon-oscillator, see Appendix B for more details.

To gain further insight into these ratios, we aim to find the optimal set of circuital parameters that allow us to exceed the classical upper bound set by the ratio $R$. For this purpose, we generate an objective function that maximizes the coupling strength $\tilde{g}$ constrained to $\omega_q = \omega_r$ (resonance condition). A candidate (but not unique) objective function reads

$$F_1 = 1 - \frac{g |\langle e| \hat{n}_J |g\rangle|}{\Delta_+} \cdot \left[1 - \frac{2|\Delta_-|}{\Delta_+}\right],$$

where $\omega_q$ is the CPB frequency, and $|\langle e| \hat{n}_J |g\rangle|$ corresponds to the matrix element of the charge operator. We compute both quantities considering the CPB truncated up to its five lower energy states. Also, $\Delta_\pm = \omega_q \pm \omega_r$ corresponds to the detuning between the CPB and the resonator frequencies. We minimise over the space parameter in Table I for the one CPB case, and Table II for two CPBs case (with an accordingly modified objective function), respectively, where the circuital parameters were explored in the value ranges $C = (0.11, 550)$ fF for the capacitances, $L = (100, 600)$ nH for the inductances, and $E_J = 2\pi(8, 11)$ GHz for the Josephson energies [9, 68]. For the single CPB case, we assume no constraints between parameters, whereas for the two CPBs we consider that the grounding capacitors $C_{ce}$ are identical during
the optimization subroutine. See below in the following subsection further details about the physical realization of circuits with those parameters.

In Fig. 3(a), we plot $R$ as a function of the ratio $E_J/E_C$, color-classified by a resonance ratio $\beta = \omega_q/\omega_r$, fixing all circuit parameters except the Josephson energy. Naturally, on approaching the CPB regime ($E_J \sim E_C$), we can reach values of $R > 1$ (see Fig. 3(a)), beating the oscillator (transmon) limit. Moreover, in Fig. 3(b), $R$ is plotted as a function of $E_J$ for different values of the resonator impedance $Z_r$, keeping fixed the frequency $\omega_r$. For instance, by considering $C_r \rightarrow C_r/\mu$, and $L_r \rightarrow \mu L_r$, we obtain a new impedance smaller than the previous one $Z_r \rightarrow \mu Z_r$, while the frequency remains the same. Points (squares, bullets, and crosses) in Fig. 3(b) correspond to the values obtained through optimization, with the color-classifying their resonance condition $\beta$ as in the previous figure, while solid lines are computed from Eq. (9) and a data-set point to guide the eye. This figure highlights the known fact [35, 63, 64] that the larger the resonator’s impedance, the larger the coupling ratio $R$ is, consolidating the result obtained in Eq. (10) i.e., $R \propto \sqrt{Z_r/2RQ}$.

### C. Choice of Parameters

In addition, by minimizing $F_1$ in Eq. (13), we find circuit parameters for experimental implementation of the devices depicted in Fig. 1 such that their frequencies and coupling strengths beat the upper classical limit $2\tilde{g} \leq \sqrt{\omega_q \omega_r}$, while imposing the resonance condition. We summarize these values and their corresponding Hamiltonian parameters/ratios in Table I for the single CPB coupled to the LC oscillator. We find that the explored parameters are consistent with previous cQED experiments [68, 69], i.e., within the 5-11 GHz frequency range. Notice that, we must have a high-impedance oscillator in order to obtain a large coupling strength, which can be achieved either by building an array of Josephson junctions having a large effective inductance (values of $L \leq 2.5 \mu$H have already been engineered [69–71]), or by increasing the kinetic inductance of a superconducting film [72].

| Parameters | Results |
|-----------|---------|
| $C_g$ [fF] | $9.67 \omega_q / 2\pi$ (GHz) | $6.63$ |
| $C_J$ [fF] | $3.96 \omega_q / 2\pi$ (GHz) | $6.63$ |
| $C_r$ [fF] | $0.14 E_J/E_C$ | $1.66$ |
| $C_p$ [fF] | $1.07 Z_r (\kappa \Omega)$ | $3.69$ |
| $L_r$ [nH] | $79.53 \tilde{g}/2\pi$ (GHz) | $3.87$ |
| $E_J/h$ (GHz) | $160 \tilde{R}$ | $1.22$ |

**Table I.** Optimal values for the circuit parameters and the corresponding Hamiltonian parameters/ratios of the one-CPB model depicted in Fig. 1(a), where the circuit values are obtained by minimizing the objective function $F_1$ in Eq. (13).

We extend our study for two-CPBs coupled to the LC oscillator (depicted in Fig. 1(b)). Here, it is convenient to define

$$R = \frac{2\tilde{g}_{\ell}}{\omega_q \omega_r},$$

the analogous coupling ratio to Eq. (9) for the $\ell$th CPB with frequency $\omega_{q,\ell}$ as follows (see Appendix C for details)

$$R_{\ell} = \frac{2\tilde{g}_{\ell}}{\omega_q \omega_{r,\ell}} = \frac{1}{C_{r,\ell}} \sqrt{\frac{C_r}{2RQE_{J,\ell}}}. \quad (14)$$

As we are interested in comparing the coupling strength between the qubits with the LC oscillator, we constrain the optimization problem to suppress the direct qubit-qubit interaction i.e., $g_{12} \ll \tilde{g}_{\ell}$, with the new objective function

$$F_2 = 1 - f_{12} \sum_{\ell=1}^{N_{\ell}} \left( \frac{g_{\ell}}{\Delta_{\ell}^+} \right)^2 \left[ 1 - \frac{2 |\Delta_{\ell}^-|}{\Delta_{\ell}^+} \right], \quad (15)$$

here, $f_{12} = 1 - g_{12}/(\omega_{q,1} + \omega_{q,2})$ constrains the qubit-qubit coupling, and $\Delta_{\ell}^+ = \omega_{q,\ell} \pm \omega_r$. Finally, $(g_{\ell})$ and $(\Delta_{\ell}^-)$ are the ground and first excited states of the $\ell$-th CPB, respectively. In Table II, we summarize the optimal circuit values and their corresponding Hamiltonian parameters for the aforementioned situation. The optimization routine gives us a slight asymmetry in the total capacitance in parallel with the junction, irrelevant for the qubit frequencies, which are well approximated by the Josephson energies in the CPB regime, but naturally providing different coupling strengths $\tilde{g}_{\ell}$.

| Parameters | Results |
|-----------|---------|
| $C_g$ [fF] | $7.15 \omega_q / 2\pi$ (GHz) | $8.76$ |
| $C_J$ [fF] | $0.48 \omega_{q,1} / 2\pi$ (GHz) | $8.76$ |
| $C_g$ [fF] | $7.47 \omega_{q,2} / 2\pi$ (GHz) | $8.76$ |
| $C_J$ [fF] | $0.23 E_{J1}/E_{C1}$ | $1.26$ |
| $C_r$ [fF] | $3.20 E_{J2}/E_{C2}$ | $1.17$ |
| $C_r$ [fF] | $0.22 Z_r (\kappa \Omega)$ | $9.13$ |
| $L_r$ [nH] | $170 \tilde{g}/2\pi$ (GHz) | $8.34$ |
| $E_{J1}/h$ (GHz) | $8.78 \tilde{g}_{12}/2\pi$ (GHz) | $9.20$ |
| $E_{J2}/h$ (GHz) | $8.78 \tilde{g}_{12}/2\pi$ (GHz) | $0.56$ |
| $R_1$ | $1.91$ |
| $R_2$ | $2.11$ |

**Table II.** Optimal values for the circuit parameters and the corresponding Hamiltonian parameters/ratios of the two-CPB model depicted in Fig. 1(b), where the circuit values are obtained by minimizing the objective function $F_2$ in Eq. (15). We have considered two asymmetric CPBs and equal grounding capacitors $C_{g0} = C_{g1} = C_g$ in the optimization subroutine.

In summary, we have numerically shown that a (or more) nonlinear system(s) coupled to a high-impedance LC circuit can beat the pure oscillators’ (classical) limit $2\tilde{g} \leq \sqrt{\omega_q \omega_r}$ by engineering suitable parameters. Having obtained and analyzed the Hamiltonian for the two CPBs USC-coupled to an oscillator, let us study the feasibility of using this compound unit as an incoherent mediator [73] to perform a quantum state transfer protocol.
III. INCOHERENT MEDIATOR FOR QUANTUM STATE TRANSFER

Among the several different tasks required to perform quantum processing, the coherent and fast controllable interaction between two subsystems is crucial [74, 75]. However, a successful information exchange often relies on having a highly controllable and coherent third-party quantum system that acts as a mediator/coupler. A radically different approach is to design a protocol which is insensitive to the state of the mediator [73], with the advantage that the coupler does not require preparation or postselection [75].

In this section, we follow the latter approach by Cardenas et al. and consider a theoretical mediator described by Hamiltonian in Eq. (5), approximating the CPBs by two-level systems $(H_\text{USC} \approx H_2)$ capacitively coupled to two transmons with coupling strength $\lambda_1$ and $\lambda_2$. Since the CPB and the transmon have positive and negative anharmonicity, respectively, there is a choice of parameters for which the two transmons only couple to the oscillator in the one-excitation manifold [76], thus obtaining the Hamiltonian of the QST protocol

$$H \approx H_\text{USC} + \sum_{m=1}^{2} \sum_{k=0}^{2} \omega_{k,m} \sigma_{k,m} + \sum_{m=1}^{2} \lambda_m \hat{n}_m(a + a^\dagger),$$

(16)

where $\omega_{k,m}$ corresponds to the frequency of the $k$-th energy level for the $m$-th transmon, $\sigma_{k,m} = |\tilde{k}_m\rangle \langle \tilde{k}_m|$ stands for the $k$-th projector of the $m$-th transmon, and $\hat{n}_m$ stands for the charge operator of the $m$-th transmon. Due to the low

relative anharmonicity of the transmons, $\alpha_{r,m} = (\omega_{12,m} - \omega_{01,m})/\omega_{01,m} = -0.096$ [18], where $\omega_{a,b,m} = \omega_{b,m} - \omega_{a,m}$, we first regard them here as three-level systems.

Now we explore the performance of the QST mediator operating in the perturbative USC regime and beyond [43]. The protocol starts by tuning the first energy transition of the transmons on resonance with a forbidden energy transition concerning the coupling operator $(a + a^\dagger)$ of the mediator. For instance, for $\tilde{\omega}_f/\omega_r = 0.3$, and $\tilde{g}_f/\omega_r = 0.5$, we have prohibited transitions $|\psi_1\rangle \leftrightarrow |\psi_3\rangle (\Delta_{13})$ and $|\psi_1\rangle \leftrightarrow |\psi_4\rangle (\Delta_{14})$, respectively, as shown in Fig. 5, where $|\psi_q\rangle$ is the $q$-th eigenstate of the USC Hamiltonian $(H_\text{USC} = \sum \varepsilon_q |\psi_q\rangle \langle \psi_q|)$, with transition energy $\Delta_{jk} = \varepsilon_k - \varepsilon_j$.

Afterwards, we prepare the system in the state $|0\rangle |\psi_0\rangle |1\rangle \in \mathcal{H}_{\text{transmon1}} \otimes \mathcal{H}_{\text{USC}} \otimes \mathcal{H}_{\text{transmon2}}$. Figs. 6(a) and (b) show the population inversion between the states $|0\rangle |\psi_0\rangle |1\rangle$ and $|1\rangle |\psi_0\rangle |0\rangle$, for the system evolving under the Hamiltonian in Eq. (16), using the aforementioned forbidden transitions. The population inversions occur at time scales $t = 13.41$ ns for $\tilde{g}_f/\omega_r = 0.3$, and $t = 169.01$ ns for $\tilde{g}_f/\omega_r = 0.5$, with $\omega_r = 2\pi \times 8.13$ GHz. Notice that, as most of the population is in the two lowest states of the transmons, the state $|2\rangle$ does not participate in the system dynamics. Consequently, we truncate both transmons to two-level systems and write the Hamiltonian in Eq. (16) as

$$H = H_\text{USC} + \sum_{m=1}^{2} \frac{\omega_{01,m}}{2} \tau_m^x + \sum_{m=1}^{2} \lambda_m \tau_m^x (a + a^\dagger),$$

(17)

where $\tau_m^x$ and $\tau_m^x$ are the first and third Pauli matrices describing the $m$-th two-level transmon.

Let us study now the robustness of the QST protocol under loss mechanisms, including energy relaxation and depolarization channels. In the USC/DSC regime, the weak-coupling master equation formalism used in earlier quantum optics is not valid, because the dissipative dynamics strongly depends on the interaction between the subsystems. The corrected method [42] requires to write the system-environment Hamiltonian in a diagonal basis for the USC/DSC system, leading to two principal physical consequences. Firstly, the decay

FIG. 5. The value of matrix element $\langle \psi_j | a + a^\dagger | \psi_k \rangle$ for different coupling ratios (a) $\tilde{g}_{1(2)}/\omega_r = 0.3$, and (b) $\tilde{g}_{1(2)}/\omega_r = 0.5$, where $\langle \psi_{j(k)} \rangle$ is the corresponding eigenstate of the mediator Hamiltonian in Eq. (16). The numerical calculations have been performed with $\omega_{q,1(2)} = \omega_r$. 

FIG. 4. Illustration of the QST protocol. An LC oscillator (green box) capacitively coupled to a pair of CPBs (blue boxes), and connected with grounded capacitors $C_0$ and $C_1$. This subsystem constitutes our mediator. We weakly couple two additional transmons (red boxes) to the LC oscillator, through capacitors $C_5$ and $C_2$. The illustration in the upper left corner is the schematic QST diagram.
 FIG. 6. Population inversion between the states $|1\rangle \langle \psi_0 | 0 \rangle$ (blue), and the state $|0\rangle \langle \psi_0 | 1 \rangle$ (orange), evolving under the Hamiltonian in Eq. (16), with three-level transmons, and coupling parameter (a) $\tilde{g} = 0.3\omega_r$, and (b) $\tilde{g} = 0.5\omega_r$. Population inversion between the states $|0\rangle \langle \rho Th \otimes | 0 \rangle | 0 \rangle$ (blue), and the state $|1\rangle \langle \rho Th \otimes | 1 \rangle | 1 \rangle$ (orange), for a thermal state at $T = 100 \text{ mK}$, calculated from the master equation in Eq. (18), with two-level transmons, and coupling parameter (c) $\tilde{g} = 0.3\omega_r$, and (d) $\tilde{g} = 0.5\omega_r$. The above calculations have been performed using parameters $\omega_q(1,2) = \omega_r = 2\pi \times 8.13(\text{GHz})$, and $\lambda_1(2) = 0.02\omega_r$.

Rates depend on the energy transition of each level. Secondly, the coupling operator $\sigma_z$ is not diagonal in the dressed basis. Therefore, the depolarizing channel has an additional component producing energy relaxation on the system. We encompass all these effects into the following master equation in the Lindblad form [42, 77, 78] (see Appendix D for more details)

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_{m=1}^2 [\gamma_m D[\tau_m^z] \rho(t) + \gamma_{\phi m} D[\tau_m^z] \rho(t)] + \sum_{j,k \neq j} \Gamma_{jk}^{\rho} \rho(t) + \Gamma_{\phi j}^{\rho} \rho(t)$$

where the Hamiltonian $H$ is given by Eq. (17), $\sigma_{jk} = |\psi_j\rangle \langle \psi_k |$ corresponds to an operator acting on the USC system, $D[O] \rho = 1/2(2O^\dagger \rho O - \rho O^\dagger O - O^\dagger O \rho)$ is the dissipator, and $\tau_m^z, \tau_m^z$ correspond to the noise channels of the $m$-th transmon, which dissipates with ratio $\gamma_m$, and loses coherence at rate $\gamma_{\phi m}$, respectively. Moreover, $\Gamma_{jk}^{\rho}$ and $\Gamma_{\phi j}^{\rho}$ correspond to the dressed heating and decay rates in the mediator, given by

$$\Gamma_{jk}^{\rho} = \Gamma_{jk}^{\rho} + \sum_{k} \frac{\langle \Delta_{kj}^\rho \rangle}{(\Delta_{kj}^\rho + 1)}$$

where $\Delta_{kj}^\rho = 1/(\exp(h\Delta_{kj}/k_B T) - 1)$ is the number of thermal photons of energy $h\Delta_{kj}$ at temperature $T$ [77], with $k_B$ the Boltzmann constant. $\Gamma_{\phi j}^{\rho}$ is the relaxation rate provided by the resonator, and $\Gamma_{jk}^{\phi}$ is the relaxation rate of the $\ell$-th charge qubit due to the coupling with its thermal bath. Finally, $\Gamma_{\phi j}^{\phi}$ stands for the effective relaxation rate due to the fact that $\sigma_z$ is not diagonal in the dressed basis. These rates are defined as

$$\Gamma_{jk}^{\rho} = \frac{\kappa \Delta_{kj}}{\omega_r} \langle \sigma_j^+ \sigma_k^- | \rho \rangle, \langle \psi_j^+ \sigma_k^- | \psi_k \rangle, \Gamma_{\phi j}^{\phi} = \frac{\gamma_{\phi j}}{\omega_{\phi j}^2} \langle \psi_j^+ \sigma_j^x | \psi_k \rangle, \langle \psi_j^+ \sigma_j^x | \psi_k \rangle.$$
FIG. 7. Fidelities of the QST process between the state \( \rho_2 \) and the state \( \rho(t) \), which is the solution of the master equation in Eq. (18), at \( T = 50 \) mK for (a) \( \tilde{g}/\omega_r = 0.3 \), and (b) \( \tilde{g}/\omega_r = 0.5 \).

Let us stress again that this protocol strongly depends on the level structure of the mediator, and its coupling operators to the adjacent information-exchanging partners. In this case, we have used the particular light-mode forbidden transitions of the USC-unit, but any other analogous scheme could be potentially implemented. In that sense, using the same proposed circuit but in the strong coupling regime (where the RWA is applicable) will not produced any similar results.

IV. CONCLUSIONS

We have demonstrated that by coupling a quantum system with strong anharmonicity, e.g., a charge qubit, to a microwave resonator, one can beat the coupling limit between transitions of two harmonic subsystems. In particular, we have shown that it is possible to enhance the coupling and achieve the resonant USC and DSC coupling regimes by increasing the impedance of the oscillator, requiring the use of either high-kinetic inductors or a chain of Josephson junctions. In this coupled system, an extension of the quantum Rabi model with additional field operator terms, constitutes the fundamental model describing the light-matter interaction. This model does not preserve the parity symmetry intrinsic to the archetypical quantum Rabi model.

In addition, we have proposed a compound CPB-based USC unit as an incoherent (thermally populated) mediator to perform a fast quantum state transfer protocol improving a previous proposal with flux qubits. We remark that this effect crucially relies on the level structure of the mediator being in the USC regime, dynamics that would otherwise be forbidden had the rotating-wave approximation been allowed. Our theoretical results encourage further experimental exploration of the physics of the light-matter interaction with large couplings and multi-mode resonators, and its possible applications for quantum simulation and quantum information processing.

ACKNOWLEDGEMENTS

We thank J. Braumüller, P. Forn-Díaz and I. L. Egusquiza for enlightening discussions and comments on the manuscript. The authors acknowledge support from Spanish MCIU/AEI/FEDER (PGC2018-095113-B-I00), Basque Government IT986-16, projects QMICS (820505) and OpenSuperQ (820363) of EU Flagship on Quantum Technologies, EU FET Open Grants Quromorphic and EPIQUS, Shanghai STCSM (Grant No. 2019SHZDZX01-ZX04). A. P.-R. thanks Basque Government Ph.D. grant PRE-2016-1-0284.

Appendix A: One CPB coupled to an LC oscillator

For completeness, we derive the Hamiltonian of the one-qubit system depicted in Fig. 1(a) by first computing the Lagrangian with node-flux variables

\[
L_1 = \frac{C_x}{2} \dot{\Phi}_1^2 + \frac{C_x}{2} \dot{\Phi}_r^2 + \frac{C_c}{2} (\Phi_r + \dot{\Phi}_1)^2 + \frac{C_p}{2} \dot{\Phi}_p^2 \\
+ \frac{C_g}{2} (\Phi_J + \dot{\Phi}_r + \dot{\Phi}_1 - V_g)^2 - \frac{\Phi_1^2}{2L_r} + E_J \cos(\varphi_J),
\]

where \( \Phi_1, \Phi_r, \) and \( \Phi_J \) are the fluxes describing the capacitance \( C_p \), the LC oscillator, and the CPB, respectively. The flux velocity \( \dot{\Phi}_1 \) corresponds to a passive node (does not contain its respective generalise coordinate) which can be eliminated noticing that the Euler-Lagrange equation reads \( \partial_t(\partial L_1/\partial[\dot{\Phi}_1]) = 0 \). This relation permits to write \( \Phi_1 \) in terms of the remaining fluxes

\[
\dot{\Phi}_1 = \frac{C_g}{C} (V_g - \dot{\Phi}_J) - (C_c + C_g) \dot{\Phi}_r + V_1(0),
\]

where we have defined \( \tilde{C} = C_c + C_g + C_p \), and \( V_1(0) = \dot{\Phi}_1(0) \) is an initial voltage bias (integration constant). By replacing this expression in Eq. (A1), we can obtain the Lagrangian only in terms of \( \Phi_r \) and \( \Phi_J \)

\[
L_1 = \frac{1}{2} \ddot{\Phi}^T C \ddot{\Phi} - V_g \ddot{\Phi}^T C_v - \frac{\Phi_1^2}{2L_r} + E_J \cos(\varphi_J),
\]

where we have defined the coordinates’ vector as \( \tilde{\Phi} = (\Phi_J, \Phi_r)^T \), we have collected the two voltage biases in one term \( V_g = V_g + V_1(0) \tilde{C}/C_g \), and we discarded the term proportional to \( V_g^2 \), which does not contribute to the dynamics. \( C \) is the capacitance matrix, and \( C_v \) is the gate capacitance vector, and are defined as

\[
C = \frac{1}{C} \left( \begin{array}{ccc}
C_J & C_g(C_c + C_p) & C_gC_p \\
C_cC_p & C_r & C_p(C_c + C_g)
\end{array} \right),
\]

\[
C_v = \frac{1}{C} \left( \begin{array}{c}
C_g(C_c + C_p) \\
C_gC_p
\end{array} \right).
\]
We compute the canonical conjugate momenta to the flux variables, readily, \( \tilde{Q} = \frac{\partial L}{\partial \dot{\Phi}} = C \dot{\Phi} - \dot{V}_g C_o \), and apply the Legendre transformation to get the Hamiltonian
\[
H_1 = \frac{[C^{-1}]_{1,1}}{2} Q_r^2 - E_J \cos(\varphi_J) + \frac{[C^{-1}]_{1,2}}{2} Q_o^2 + \frac{\Phi_o^2}{2L_r} + [C^{-1}]_{1,2} Q_r Q_o + \gamma_{r,g} Q_r \dot{V}_g + \gamma_{r,g} Q_o \dot{V}_g. \tag{A5}
\]
Here, \([C^{-1}]_{j,k}\) is the \(j,k\)-th element of the inverse capacitance matrix. The coefficients of \(Q_o \dot{V}_g\) and \(Q_r \dot{V}_g\) are defined as
\[
\gamma_{r,g} = \sum_{j=1}^2 [C^{-1}]_{1,j} C_{o,j}, \quad \gamma_{r,g} = \sum_{j=1}^2 [C^{-1}]_{2,j} C_{r,j}, \tag{A6}
\]
where \([C_{o,j}]\) is the \(j\)-th element of the gate capacitance vector \(C_o\). Notice that by applying the transformation \(Q_r \to Q_r - \gamma_{r,g} \dot{V}_g / [C^{-1}]_{1,2}\) in Eq. (A5), we eliminate the term proportional to \(Q_r \dot{V}_g\), while introducing an additional gate charge term to the CPB. After algebraic manipulations, we can write the interaction between \(Q_o\) and the gate voltage in a compact form leading to the following Hamiltonian
\[
H_1 = \frac{1}{2C_J} (Q_J - 2en_g)^2 - E_J \cos (\varphi_J) - \frac{Q_r Q_J}{C_J},
\]
\[
+ \frac{Q_o^2}{2C_r} + \frac{\Phi_o^2}{2L_r}, \tag{A7}
\]
where \(n_g = -\gamma_{r,g} \dot{V}_g \hat{C}_{r,J} / 2e - \gamma_{r,g} \hat{C}_{o,J} \hat{C}_{r,J} \hat{V}_g / 2e C_J\) is the effective gate-charge number, which is tunable by controlling the voltage \(V_g\). In this article, we always assume that the CPBs work on their sweet spot i.e. \(n_g = 0.5\). The effective Josephson, resonator, and coupling capacitances are defined as follows
\[
\hat{C}_J = \frac{1}{[C^{-1}]_{1,1}}, \quad \hat{C}_r = \frac{1}{[C^{-1}]_{2,2}}, \quad C_{J,r} = \frac{1}{[C^{-1}]_{1,2}}. \tag{A8}
\]
We can canonically quantize the circuit Hamiltonian in Eq. (A7) by promoting the conjugated variables to quantum operators, \(\hat{\phi}_r = i\sqrt{\hbar \omega_r L_r / 2 (a - a)}\), \(\hat{Q}_r = \sqrt{\hbar \omega_r \hat{C}_r / 2 (a + a)}\), \(\hat{Q}_o = 2e \tilde{n}_o\), and \(e^{i \tilde{\phi}_r} = \sum_{n_J} |n_J + 1\rangle \langle n_J|\), where \(\omega_r = 1 / \sqrt{\hat{C}_r L_r}\) is the frequency of the resonator, and \(a\) (\(a^\dagger\)) are the annihilation (creation) operators of the quantum harmonic oscillator satisfying the commutation relation \([a, a^\dagger] = 1\), such that \([\hat{Q}_r, \hat{Q}_o] = i\hbar\). The Cooper-pair number \(\tilde{n}_o\) and phase \(\tilde{\phi}_r\) operators satisfy the commutation relation \([e^{i \tilde{\phi}_r}, \tilde{n}_o] = e^{i \tilde{\phi}_r}\). After the above transformations, we arrive at the Hamiltonian in Eq. (3) of the main text
\[
H_1 = 4E_C (\tilde{n}_o - n_g)^2 - E_J \cos (\tilde{\phi}_r) + \hbar \omega_r a^\dagger a
\]
\[
+ \hbar \tilde{g}_o (a + a^\dagger),
\]
where \(E_C = e^2 / 2 \hat{C}_J\) is the charge energy, and the coupling strength between the CPB and the resonator reads
\[
g = \omega_r \gamma \sqrt{\frac{Z_r}{2R_Q}}, \quad \gamma = \frac{\hat{C}_r}{C_{J,r}}
\]
where \(Z_r = \sqrt{L_r / \hat{C}_r}\) is the impedance of the effective resonator mode and \(R_Q = \hbar / (2e)^2\) is the reduced quantum resistance.

\section*{Appendix B: Two-oscillator model}
In this section, we show that there exists a fundamental limit for the coupling strength between two harmonic systems. For doing so, we consider the circuit depicted in Fig. 1(a), and we change the Josephson junction of the CPB by a linear inductor \(L_J\), and remove the gate voltage \(V_g\). The Hamiltonian for this system reads
\[
H_o = \frac{Q_J^2}{2C_J} + \frac{\Phi_J^2}{2L_J} + \frac{Q_o^2}{2C_o} + \frac{\Phi_o^2}{2L_o} - \frac{Q_r Q_J}{C_J},
\]
which has the similar expression as the Hamiltonian in Eq. (A7) except that we replace the Josephson energy term with the potential energy \(\Phi_J^2 / 2L_J\), and neglect the terms proportional to \(V_g\). The dressed capacitances \(C_J, C_o, C_J, C_o\) are identical to those defined in Eq. (A8). Next, we quantize the Hamiltonian \(H_o\) by promoting the classical conjugated variables to quantum operators, \(\hat{Q}_o = \sqrt{\hbar \omega_q \hat{C}_o / 2 (b + b)}\), \(\hat{\phi}_J = i \sqrt{\hbar \omega_q L_J / 2 (b - b)}\), which satisfy the commutation relation \([\hat{\phi}_J, \hat{Q}_J] = i\hbar\), and \(\omega_q = 1 / \sqrt{\hat{C}_J L_J}\) is the frequency of the oscillator, where \(b (b)\) is the corresponding annihilation (creation) operator satisfying the bosonic commutation relation \([b, b] = 1\). For the resonator, we use the procedure as in the previous Appendix. The quantized Hamiltonian reads
\[
\hat{H}_o = \hbar \omega_q b^\dagger b + \hbar \omega_r a^\dagger a - \hbar \bar{g}_o (a + a^\dagger)(b + b),
\]
where the coupling strength between the harmonic system is defined as
\[
g_o = \frac{\gamma_o}{2} \sqrt{\omega_r \omega_q},
\]
with the dimensionless ratio
\[
\gamma_o = \sqrt{\frac{\hat{C}_J \hat{C}_o}{C_J^2}} \sqrt{\frac{1}{(1 + \frac{C_p C_o + C_J C_r}{C_o C_p})(1 + \frac{C_p C_J + C_o C_c}{C_o C_p})}}
\]
written in terms of the bare system capacitances. Notice that in the limit \(C_p, C_c \gg C_o, C_J, \gamma_o \to 1\). For generic values of the capacitors, we see that \(2g_o / \sqrt{\omega_r \omega_q} \leq 1\), i.e. there exists a fundamental limit for the value of the coupling strength between the subsystems.

\section*{Appendix C: Two CPBs coupled to an LC oscillator}
We derive here the Hamiltonian of the two-qubit system, as shown in Fig. 1(b) considering identical grounded capacitances i.e. \(C_{c0} = C_{c1} = C_c\). We first calculate the Lagrangian of the circuit following the same procedure as in Appendix A,
i.e. writing the passive node \( \Phi_0 \) in terms of \( \Phi_{J1}, \Phi_{J2}, \) and \( \Phi_r \)
to obtain
\[
L_2 = \frac{1}{2} \tilde{\Phi}^T \mathbf{C} \tilde{\Phi} - \tilde{V}_{g1} \tilde{\Phi}^T \mathbf{C} \mathbf{v}_1 - \tilde{V}_{g2} \tilde{\Phi}^T \mathbf{C} \mathbf{v}_2 - \frac{\Phi^2}{2L_r} \\
+ E_{J1} \cos (\varphi_{J1}) + E_{J2} \cos (\varphi_{J2}), \quad (C1)
\]
respectively. We again apply the Legendre transformation, and we eliminate the interaction between the gates voltages and the resonator analogously to Appendix A obtaining
\[
H_2 = \sum_{\ell=1}^{2} \left[ \frac{1}{2C_{J\ell}} (Q_{J\ell} - 2en_{g\ell})^2 - E_{J\ell} \cos (\varphi_{J\ell}) \right] \\
+ \frac{Q^2}{2C_r} + \frac{\Phi^2}{2L_r} - \frac{Q_{J1} Q_{J2}}{C_{1r}} - \frac{Q_{J1} Q_{J2}}{C_{2r}} - \frac{Q_{J1} Q_{J2}}{C_{12}} \quad (C3)
\]
where \( n_{g\ell} \) is the effective gate-charge number of the \( \ell \)-th CPB, which is tunable through the gate voltages and again assumed to be at the sweet spot, i.e. \( n_{g\ell} = 0.5 \). The effective Josephson, resonator, and coupling capacitances are defined as
\[
\hat{C}_{J\ell} = \frac{1}{[\mathbf{C}^{-1}]_{\ell,\ell}}, \quad \hat{C}_{r} = \frac{1}{[\mathbf{C}^{-1}]_{2,2}}, \\
\hat{C}_{lr} = \frac{1}{[\mathbf{C}^{-1}]_{\ell,3}}, \quad \hat{C}_{12} = \frac{1}{[\mathbf{C}^{-1}]_{1,2}}
\]
respectively, where \( [\mathbf{C}^{-1}]_{j,k} \) is the \( jk \)-th element of the inverse capacitance matrix. After the quantization procedure and a unitary transformation we arrive to Hamiltonian
\[
H_2 = \sum_{\ell=1}^{2} \hat{H}_{\ell}^{\text{pb}} + \omega_r a^\dagger a + h \sum_{\ell=1}^{2} g_{\ell} \hat{n}_{J\ell} (a + a^\dagger) \\
+ \hat{n}_{12} \hat{n}_{J1} \hat{n}_{J2},
\]
where \( \hat{H}_{\ell}^{\text{pb}} = 4E_{C\ell} (\hat{n}_{J\ell} - n_{g\ell})^2 - E_{J\ell} \cos (\varphi_{J\ell}) \) corresponds to the Hamiltonian of the \( \ell \)-th CPB, with the charge energy \( E_{C\ell} = e^2 / 2C_{J\ell} \) and the coupling strengths
\[
g_{\ell} = \omega_r \gamma_{\ell} \sqrt{\frac{Z_r}{2R_Q}}, \quad g_{12} = - \frac{1}{R_Q \hat{C}_{12}},
\]
which correspond to the interaction between the \( \ell \)-th CPB and the LC resonator, and the two CPBs, respectively. Moreover, the ratio \( \gamma_{\ell} = \hat{C}_{r} / \hat{C}_{lr} \).

### Appendix D: Derivation of the Master Equation

Here we derive the master equation (ME) to describe the interaction between the transmons-USC unit and the environment, which is modeled as an infinite set of quantum harmonic oscillators [42]. We include the effects of energy loss/gain in the transmons, CPBs, and LC oscillator, and the depolarizing channel for each CPB independently. In ultrastrongly coupled systems, the standard (perturbative) master equation description leads to unphysical results [77]. To avoid this, we express the USC operators in the dressed basis of the whole system.

#### 1. Energy relaxation

Let us begin by obtaining a ME for the energy relaxation process, following the procedure of Beaudoin et al. [42], where we assume that the USC system is weakly coupled to the environment through the \( X \) quadrature of the LC oscillator and \( \sigma^x \) for the \( \ell \)-th CPB. We describe this situation with the following Hamiltonian (\( \hbar = 1 \))
\[
H_{SE} = H_{USC} + H_B + H_{SB} \quad (D1)
\]
\[
= \sum_j \varepsilon_j \sigma_{jj} + \sum_l n_l b_l^\dagger b_l + \sum_l (c + c^\dagger)(h_l b_l + h_l^\dagger b_l^\dagger).
\]
Here, the first term corresponds to the USC Hamiltonian expressed in its diagonal basis, with the projector \( \sigma_{nm} = |\psi_n\rangle \langle \psi_m| \), the second term to the Hamiltonian describing the infinite quantum harmonic oscillator with frequency \( \nu_l \), and \( b_l (b_l^\dagger) \) is the ladder operator for the \( l \)-th bath mode, and the last terms to the system-environment interaction. Depending on whether we are calculating relaxation rates for the resonator or the CPBs, the operator \( c + c^\dagger \) will correspond to the field quadrature \( X \) for the resonator, or to the 2x2 Pauli matrix for the \( \ell \)-th CPB, expressed in the dressed basis as \( c + c^\dagger = \sum_{jk} C_{jk} \sigma_{jk} \), with the real and symmetric matrix element \( C_{jk} = C_{jk}^* \). In the interaction
picture, it takes the form
\[
\hat{H}_{SB}(t) = \sum_{j,k} C_{jk} \sigma_{jk} (h_i b_i e^{-i\epsilon_j t} + h_i^* b_i^* e^{i\epsilon_j t}) e^{i\Delta_{jk} t}, \tag{D2}
\]
where \(\Delta_{jk} = \epsilon_j - \epsilon_k\) is the energy transition between the jth and kth energy level of the USC system. Now, we split the interaction Hamiltonian in Eq. (D2) into two parts
\[
\hat{H}_{SB}(t) = \sum_{j,k} C_{jj} \sigma_{jj} (h_i b_i e^{-i\epsilon_j t} + h_i^* b_i^* e^{i\epsilon_j t})
\]
\[
+ \sum_{l} \sum_{j,k>j} C_{jk} \sigma_{jk} \left[ (h_i b_i e^{-i(\epsilon_j + \Delta_{jk}) t}) + h_i^* b_i^* e^{i(\epsilon_j + \Delta_{jk}) t}) \right]. \tag{D3}
\]

We can simplify the interaction Hamiltonian defining the operators
\[
\hat{S}_1 = \sum_{j,k>j} S_1 e^{-i\Delta_{jk} t}, \quad \hat{S}_2 = \sum_{j,k>j} S_2 e^{i\Delta_{jk} t}, \tag{D4}
\]
where \(S_1 = C_{jk} \sigma_{jk}\), \(S_2 = C_{jk} \sigma_{kj}\), and \(S_3 = \sum_j C_{jj} \sigma_{jj}\), and similarly for bath operators
\[
\hat{B}_1(t) = \sum_l h_i^* b_l e^{-i\epsilon_k t}, \quad \hat{B}_2(t) = \sum_l h_i b_l e^{-i\epsilon_k t}. \tag{D5}
\]

By replacing Eq. (D4), and Eq. (D5) in Eq. (D3), we obtain
\[
\hat{H}_{SB}(t) = (\hat{S}_1 + \hat{S}_2 + \hat{S}_3) (\hat{B}_1(t) + \hat{B}_2(t)).
\]

In the interaction picture, the dynamics of the system-bath Hamiltonian is described by the von Neumann equation
\[
\frac{d\rho_{SB}(t)}{dt} = -i[H_{SB}(t), \rho_{SB}(t)].
\]

The formal solution of this equation reads
\[
\dot{\rho}_{SB}(t + \Delta t) = \rho_{SB}(t) - i \int_t^{t+\Delta t} dt_1 [\hat{H}_{SB}(t_1), \rho_{SB}(t_1)].
\]

We replace this solution on the von Neumann equation, and we trace over the degrees of freedom of the environment for approximately describing the dynamics of the subsystem, obtaining
\[
\Delta \dot{\rho}(t) = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt_1 \text{Tr}_B [\hat{H}_{SB}(t_1), \rho(t) \otimes \rho_B] - \int_t^{t+\Delta t} dt_1 \text{Tr}_1 \text{Tr}_2 \left[ \hat{H}_{SB}(t_1), \dot{\rho}(t) \otimes \rho_B \right], \tag{D6}
\]
where we have retained the terms in the interaction Hamiltonian up to second order. The standard way to proceed here, is to introduce two approximations to simplify this equation. Firstly, we assume that the state of the system-environment is separable \(\forall t\), also known as the Born approximation [86], i.e. \(\rho_{SB}(t) = \rho(t) \otimes \rho_B\), where \(\rho(t)\) and \(\rho_B\) are the density matrices of the USC and bath subsystems, respectively. Secondly, we apply the Markov approximation, which presumes different time-scales for the system and environment dynamics.

Let us perform the change of variable \(\tau = t_1 - t_2\), leading to \(\int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 = \int_0^{\Delta t} d\tau \int_t^{t+\Delta t} dt_1\), where \(\Delta t\) and \(\tau\) are the time-scales describing the dynamics of the system and the environment, respectively. The Markov approximation states that \(\tau \ll \Delta t\) meaning that the correlation function’s time is shorter than the system ones. This allow us to change the limit of integration of both integrals as follows [78]

\[
\frac{\Delta \dot{\rho}(t)}{\Delta t} = -i \sum_{\Delta_{jk}, \Delta'_{jk'}} \frac{1}{\Delta t} \left\{ \int_0^{\Delta t} d\tau e^{i\Delta_{jk} \tau} \tilde{G}_{11}(\tau) \int_t^{t+\Delta t} dt_1 e^{i(\Delta_{jk} - \Delta'_{jk'}) \tau} \left[ S_1 \rho(t) S_1^\dagger - S_1^\dagger S_1 \rho(t) \right] \right. \\
+ \left. \int_0^{\Delta t} d\tau e^{-i\Delta_{jk} \tau} \tilde{G}_{22}(\tau) \int_t^{t+\Delta t} dt_1 e^{i(\Delta_{jk} - \Delta'_{jk'}) \tau} \left[ S_2 \rho(t) S_2^\dagger - S_2^\dagger S_2 \rho(t) \right] \right\} + \text{H.c.,}
\]

where we have performed the secular approximation neglecting the fast oscillating terms proportional to \(e\{\pm i\Delta_{jk} \tau\} + e\{\pm i\Delta'_{jk} \tau\} \}

\[
\text{Tr}[\tilde{G}_{11}(\tau) B_1 \rho_B] \]
reads
\[
\tilde{G}_{11}(\tau) = \sum_l e^{-i\epsilon_l \tau} |h_l|^2 \left\{ \tilde{n}(\epsilon_l, T) + 1 \right\}, \tag{D7}
\]
\[
\tilde{G}_{22}(\tau) = \sum_l e^{-i\epsilon_l \tau} |h_l|^2 \tilde{n}(\epsilon_l, T),
\]

Finally, the correlations functions of the bath \(G_{\alpha\beta}(\tau) = \text{Tr}[\tilde{G}_{\alpha\beta}(\tau) B_1 \rho_B] \) reads
\[
\tilde{G}_{11}(\tau) = \sum_l e^{-i\epsilon_l \tau} |h_l|^2 \left\{ \tilde{n}(\epsilon_l, T) + 1 \right\}, \tag{D7}
\]
\[
\tilde{G}_{22}(\tau) = \sum_l e^{-i\epsilon_l \tau} |h_l|^2 \tilde{n}(\epsilon_l, T),
\]
where $\tilde{n}(\nu_i, T) = [e^{\nu_i/k_BT} - 1]^{-1}$ is the average of thermal photons of the $i$th mode [77]. In the Schrödinger picture, we see that the only terms contributing correspond to $k = k'$, and $j = j'$ [78] (second secular approximation) leading to

$$
\frac{d\rho(t)}{dt} = -i[H_{\text{USC}} + H_{LS}^1, \rho(t)] \\
+ \sum_{\Delta k_j} W_{11}(\Delta k_j) \left[ S_1 \rho(t) S_1^\dagger - S_1^\dagger S_1 \rho(t) \right] \\
+ \sum_{\Delta k_j} W_{22}(\Delta k_j) \left[ S_2 \rho(t) S_2^\dagger - S_2^\dagger S_2 \rho(t) \right] + \text{H.c.,}
$$

with $W_{11}(\Delta k_j) = \int_0^\infty dt e^{i\Delta k_j \tau} \bar{G}_{11}(\tau)$, and $W_{22}(\Delta k_j) = \int_0^\infty dt e^{-i\Delta k_j \tau} \bar{G}_{22}(\tau)$ corresponding to the Fourier transform of the correlation function of the bath. Now we write the ME in its standard form as follows

$$
\frac{d\rho(t)}{dt} = -i[H_{\text{USC}} + H_{LS}^1, \rho(t)] + \sum_{\Delta k_j} \mathcal{J}(\Delta k_j) \\
= \left[ \tilde{n}(\Delta k_j, T) + 1 \right] D[S_1] \rho(t) + \tilde{n}(\Delta k_j, T) D[S_2] \rho(t),
$$

where $D[O] \rho(t) = 1/2[\mathcal{O} \rho(t) \mathcal{O}^\dagger - \mathcal{O}^\dagger \mathcal{O}, \rho(t)]_+$ is the Lindbladian, $\mathcal{J}(\omega) = \sum_i 2\pi |h_i|^2 \delta(\omega - \nu_i)$ is the spectral density of the bath, and $H_{LS}^1$ is the Lamb-shift Hamiltonian

$$
H_{LS}^1 = \sum_{\Delta k_j} \delta_{11}(\Delta k_j) S_1^\dagger S_1 + \delta_{22}(\Delta k_j) S_2^\dagger S_2,
$$

with $\delta_{aa}(\omega) = -i[W_{aa}(\omega) - W_{a*}(\omega)]/2$. We recall that in the weak-coupling limit, and upon the use of a system-environment coupling model with a natural but far-enough cutoff [31], the Lamb-shift terms commute with the system Hamiltonian [86] just modifying the compound eigenfrequencies (of the USC unit here). Due to that, we assume their presence to barely alter the results in the QST protocol and thus we neglect them. Sharper predictions will require the thorough study of the different phenomena imposing cutoffs in the system. As previously advanced, we have considered ohmic environments characterized by the spectral density

$$
\mathcal{J}(\Delta k_j) = \frac{\zeta \Delta k_j}{\omega},
$$

where $\zeta$ and $\omega$ correspond to $\gamma_\ell$ (bare relaxation rate) and $\omega_{q, \ell}$ (frequency) for the $\ell$-th CPB, or to $\kappa$ and $\omega$, for the LC oscillator, respectively. For the sake of simplicity we assume in the following that $\gamma_\ell = \gamma$. Finally, we obtain the ME for the relaxation of both qubits and oscillator

$$
\frac{d\rho(t)}{dt} = -i[H_{\text{USC}}, \rho_S(t)] + \sum_{j,k>j} (\Gamma^\ell_{jk} + 2 \Gamma^\ell_{j\ell}) \\
\left[ \tilde{n}(\Delta k_j, T) + 1 \right] D[\sigma_{jk}] \rho(t) + \tilde{n}(\Delta k_j, T) D[\sigma_{kj}] \rho(t),
$$

where

$$
\Gamma^\ell_{jk} = \frac{\kappa \Delta k_j}{\omega_{q, \ell}} |(X)_{jk}|^2, \quad \text{and} \quad \Gamma^\ell_{j\ell} = \frac{\gamma \Delta k_j}{\omega_{q, \ell}} |(\sigma^\ell_{j})_{jk}|^2
$$

correspond to the dressed photon leakage rate for the resonator, and to the dressed qubit relaxation rate of $\ell$th CPB respectively, with $(\sigma^\ell_{j})_{jk} = \langle \psi_j | \sigma^\ell_{j} | \psi_k \rangle$, and $(X)_{jk} = \langle \psi_j | a + a^\dagger | \psi_k \rangle$.

2. Depolarizing channels and energy relaxation due to $\sigma^z$

In this subsection, we derive a master equation considering the same free Hamiltonian for the system and environment, but changing the interaction term used in previous subsection for

$$
H_{SB} = \sum_{\ell} \sigma^\ell_{xx}(h_{b\ell} + h^\dagger_{b\ell}),
$$

where the operator $\sigma^\ell_{xx}$ corresponds to the $z$-Pauli matrix of the $\ell$-th CPB. This will provide us with a phenomenological description of long-time charge noise for the CPBs [18, 82]. Following the previous procedure, we write $H_{SB}$ in the interaction picture

$$
\bar{H}_{SB}(t) = \sum_{jkl} e^{i\Delta k_j t} Z_{jk} \sigma_{jk} \left( h_{b\ell} e^{-i\omega t} + h^\dagger_{b\ell} e^{i\omega t} \right),
$$

where $\sigma^\ell_{xx} = \sum_{jkl} Z_{jk} \sigma_{jk}$, with $Z_{jk} = Z_{jk} = \langle \psi_j | \sigma^\ell_{xx} | \psi_k \rangle$. As in the previous section, the system operators are renamed as follow

$$
\bar{S}_1 = \sum_{j,k>j} S_1 e^{-i\Delta k_j t}, \quad \bar{S}_2 = \sum_{j,k>j} S_2 e^{i\Delta k_j t}, \quad \bar{S}_3 = \sum_{j,j>j} S_3,
$$

where $S_1 = Z_{jk} \sigma_{jk}$, $S_2 = Z_{jk} \sigma_{kj}$, and $S_3 = \sum_j Z_{jj} \sigma_{jj}$. Similarly for the bath, where its operators $\bar{B}_1$ and $\bar{B}_2$ were defined in Eq. (D5). After performing the Born-Markov approximation, with the secular one, we obtain
where $\tilde{G}_{\alpha \alpha}(\tau)$ ($\alpha = 1, 2$) is the correlation function defined in Eq. (D7), $G_{33}^r(\tau) = \lim_{\nu \to 0} \sum_k e^{i\nu t} |h_k|^2 \bar{n}(\nu, T) + e^{-i\nu t} |h_k|^2 \bar{n}(\nu, T) + 1$, and $S_T^\prime = Z_j^{\gamma_1} \sigma_j^{\gamma_1} K_j^{\gamma_1}$. In this case, we cannot neglect the terms proportional to $S_T$ for the depolarizing channel spectral density. Therefore, this interaction will be reflected in the total master equation with both a pure depolarizing and a relaxation channels [42].

In the Schrödinger picture, and after applying the secular approximations, we obtain

$$\frac{d\rho(t)}{dt} = -i[H_{\text{USC}}, \rho(t)] + \sum_{\ell = 1}^{2} \sum_{j, k > j} \Gamma_{\phi, \ell}^{jk} \frac{\tilde{n}(\Delta_{jk}, T)}{2} \rho(t)$$

$$+ \sum_{\ell = 1}^{2} \sum_{j} \Gamma_{\phi, \ell}^{j} \frac{\tilde{n}(\Delta_{jk}, T)}{} D[\sigma_{jk}](\rho(t)) + 2 \sum_{\ell = 1}^{2} \sum_{j} \Gamma_{\phi, \ell}^{j} \frac{\tilde{n}(\Delta_{jk}, T)}{} D[\sigma_{jj}](\rho(t))$$

where

$$\Gamma_{\phi, \ell}^{jk} = \frac{\gamma_{CPB}^{\ell}}{\omega_q^{\ell}} \omega_T |Z_{jk}|^2, \quad \Gamma_{\phi, \ell}^{j} = \frac{\gamma_{CPB}^{\ell}}{\omega_q^{\ell}} \omega_T |Z_{jj}|^2$$

are the dressed off-diagonal depolarizing channel rate producing energy relaxation, and dressed pure depolarizing channel rate of the $\ell$-th CPB. Here also, $\gamma_{CPB}$ is the bare depolarizing channel rate of both CPBs, and the thermal frequency $\omega_T = k_B T$.

3. Complete master equation

Finally, we describe the dissipative dynamics for the external transmons with standard weak-coupling limit master equation terms [18, 78, 86], obtaining for the complete model

$$\frac{dp(t)}{dt} = -i[H, \rho(t)] + \sum_{m=1}^{2} \left[ \gamma_m D(\tau_{m}) \rho(t) + \gamma_{\phi, m} D(\tau_{m}) \rho(t) \right] + \sum_{j, k > j} \left[ \Gamma_{\phi, \ell}^{jk} D[\sigma_{jk}](\rho(t)) + \Gamma_{\phi, \ell}^{jk} D[\sigma_{jk}](\rho(t)) \right]$$

$$+ \sum_{\ell = 1}^{2} \sum_{j} \Gamma_{\phi, \ell}^{j} D[\sigma_{jj}](\rho(t))$$

where the Hamiltonian $H$ is given by Eq. (17), $\gamma_m$ and $\gamma_{\phi, m}$ are the relaxation and depolarizing channel rates of the $m$-th external transmon, $\Gamma_{\phi, \ell}^{jk}$ and $\Gamma_{\phi, \ell}^{jk}$ correspond to the effective decay rates of the system defined as

$$\Gamma_{\phi, \ell}^{jk} = \left[ \Gamma_{\phi, \ell}^{jk} + \sum_{\ell = 1}^{2} \left( \Gamma_{\phi, \ell}^{jk} + \Gamma_{\phi, \ell}^{jk} \right) \right] \frac{\tilde{n}(\Delta_{jk}, T)}{} + 1,$$

$$\Gamma_{\phi, \ell}^{jk} = \left[ \Gamma_{\phi, \ell}^{jk} + \sum_{\ell = 1}^{2} \left( \Gamma_{\phi, \ell}^{jk} + \Gamma_{\phi, \ell}^{jk} \right) \right] \frac{\tilde{n}(\Delta_{jk}, T)}{}$$

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