WEAKLY INTEGRABLE CAMASSA-HOLM-TYPE EQUATIONS

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Abstract. Series of deformed Camassa-Holm-Type equations are constructed using the Lagrangian deformation and Loop algebra splittings. They are weakly integrable in the sense of modified Lax pairs.

Keywords: Loop algebra splitting, Lagrangian deformation, Camassa-Holm equation

1. Introduction

The Camassa-Holm (CH) equation is an important type of shallow water wave equation in fluid dynamics and is known to be integrable [1]. It can be written in the following form,

\[ u_t + 2\kappa u_x - uu_{xt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.1) \]

where \( u \) represents the fluid velocity for the \( x \) direction and \( \kappa \) is the wavenumber. There are many reduction forms of this equation, such as the Benjamin-Bona-Mahony (BBM) equation [2]:

\[ u_t + u_x - uu_{xt} + uu_x = 0 \quad (1.2) \]

and the Korteweg-de Vries (KdV) equation:

\[ u_t = uu_{xxx} + 6uu_x. \quad (1.3) \]

Mathematically, (1.1) can be derived from an asymptotic expansion in the Hamiltonian for Euler’s equation, or by the tri-Hamiltonian duality from the KdV equation [3].

The CH equation is completely integrable in the sense that it admits a Lax pair, a bi-Hamiltonian structure and an infinite sequence of conservation laws (cf. [4,5]). Besides standard traveling wave and (multi) peakon solutions ( [7,8]), the CH equation has solutions with the presence of breaking waves [9], that is, the wave profile remains bounded while the slope of the wave blows up. This non-local dispersion gains a lot of interests in the research of integrable systems (cf. [10,11]). Recently, some modifications of the CH equation (CH-type equations) are studied by both mathematicians and physicists. For

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example, we mention here the modified CH equation with cubic nonlinearity \[13\] and the \(\mu\)-CH equation (\[14,15\]).

Arnaudon \[16\] developed a theory of Lagrangian reduction by using the Sobolev norm \(H^1\) in the Lagrangian instead of the standard \(L^2\) norm, from this, deformed equations corresponding to some classical soliton equations can be derived. These deformed equations are called weakly integrable in the sense of the deformed Lax pair.

Usually, it is not sure whether the deformed equations are completely integrable. But in some special cases, integrable equations can be obtained. For example, the Camassa-Holm equation can be considered as the deformed equation from the KdV equation. This makes the Lagrangian reduction method highly nontrivial.

Another example is the Camassa-Holm-nonlinear Schrödinger (CH-NLS) equation:

\[
\hat{q}_t = i q_{xx} \pm 2 i \hat{q}(|q|^2 - \alpha^2 |q_x|^2), \quad \hat{q} = q - \alpha^2 q_{xx}, \alpha \in \mathbb{R}. \tag{1.4}
\]

Although its integrability is still unknown, its solitary wave solutions preserve some typical soliton properties. Such behavior is studied in Ref. \[17\]. This work motivates us to study the deformed type equations.

One efficient scheme to construct integrable systems in the literature is the loop algebra splitting method; series of soliton equations can be derived from this way (cf. \[18–21\]). For example, the KdV equation, the nonlinear Schrödinger equation, the derivative nonlinear Schrödinger equation (\[22\]) etc. It is well known that when a soliton hierarchy comes from splitting, it is highly possible that we can find a series of Bäcklund or Darboux transformations to construct explicit soliton and rational solutions (cf. \[23\]).

Next we will give a brief introduction of these equations that we will study. All of them describe completely integrable systems. The NLS equation is the loop algebra splitting method; series of soliton equations can be derived from this way (cf. \[18–21\]). For example, the KdV equation, the nonlinear Schrödinger equation, the derivative nonlinear Schrödinger equation (\[22\]) etc. It is well known that when a soliton hierarchy comes from splitting, it is highly possible that we can find a series of Bäcklund or Darboux transformations to construct explicit soliton and rational solutions (cf. \[23\]).
rogue waves \cite{44-50} in the self-focusing case, and single dark solitons, multi-
dark soliton solutions in the self-defocusing case \cite{51}.

Inspired by the importance of these interesting developments about the anal-
ysis of deformed NLS-type equations, we will construct the deformed equations
of the derivative NLS (DNLS) equation. The DNLS equation is one of these
nonlinear evolution equations, which is of much importance in mathematics
and physics. Under special reduction condition, the rogue waves, bright and
dark solitons etc., can be obtained by different seed solution \cite{52}. Similarly,
the \((2 + 1)\)-dimensional DNLS equation was also investigated \cite{53}. We men-
tion here the Hirota equation, which is a combination of the NLS equation and
the modified KdV equation. It is well known that if an integrable equation is
a combination of two known equations that are also integrable, the low-order
solutions are based on solutions of the two equations \cite{54}. The same applies
to the discrete Hirota equation \cite{55}. And it is decisive to obtain the most general
solutions of the Hirota equation, which provide a lot of information about its
intrinsic structure, so different types of exact solutions were given in Ref. \cite{56}.

The main goal of this paper is to combine the Lagrangian deformation un-
der the loop algebra splitting scheme to get a systematic way to construct
weakly integrable equations. To illustrate our method, we will show some de-
formed equations such as the CH-NLS, CH-DNLS, CH-\(G\)NLS, and CH-Hirota
equations.

We arrange this paper as follows. In Sec. 2, we introduce the Lie algebra
splitting and Lagrangian deformation. In Sec. 3, we use our method to derive
series of deformed soliton equations. We will end in the last Section with some
discussion of the obtained results and future extension of this work.

## 2. Lie algebra splitting and Lagrangian deformation

In this Section, we give a brief review of the loop algebra splitting and
Lagrangian deformation.

**Loop algebra splitting.**

Let \(G\) be a compact Lie subgroup of \(GL(n, \mathbb{C})\), with \(\mathcal{G}\) its Lie algebra. Denote
\(L(G)\) by the group of smooth loops from \(S^1\) to \(G\), and \(\mathcal{L}(\mathcal{G})\) its Lie algebra. That is,

\[
\mathcal{L}(\mathcal{G}) = \left\{ \sum_{i \leq n_0} A_i \lambda^i \mid A_i \in \mathcal{G}, n_0 \in \mathbb{Z} \right\},
\]  

(2.1)

Let \(L(G)_+\) and \(L(G)_-\) be subgroups of \(L(G)\), and \(L(G)_+ \cap L(G)_- = \{e\}\), where
\(e\) is the identity element. Then we have \(\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{G})_+ \oplus \mathcal{L}(\mathcal{G})_-\), which is a
direct sum of linear subspaces of \(\mathcal{L}(\mathcal{G})\). A **vacuum sequence** \(\mathcal{J} = \{J_1, J_2, \cdots \}\)
is a sequence of commuting elements in \(\mathcal{L}(\mathcal{G})_+\). Let \(\pi_+\) be a projection of \(\mathcal{L}(\mathcal{G})\)
onto $\mathcal{L}(\mathcal{G})_\pm$ along $\mathcal{L}(\mathcal{G})_\pm$. The phase space of evolution is defined as:

$$\mathcal{M} = \pi_+(g_-J_1g_-^{-1}), \quad g_- \in L(\mathcal{G})_-. \quad (2.2)$$

Usually, $\mathcal{M} = J_1 + u$ for some $u \in \mathcal{G}$. And this is an affine space.

**Theorem 2.1.** ([21]) Given $\xi : \mathbb{R} \to \mathcal{M}$, there exists unique $Q_j(\xi) \in \mathcal{L}(\mathcal{G})$ such that:

$$\begin{cases}
[\partial x + \xi, Q_j(\xi)] = 0,
Q_j(J_1) = J_j, \quad Q_j(\xi) \text{ is conjugate to } J_j.
\end{cases} \quad (2.3)$$

The $j$-th flow in the $G$-hierarchy is of the form:

$$\xi_{t_j} = [\partial x + \xi, (Q_j(\xi))_+]. \quad (2.4)$$

Let $\tau$ be an involution of $G$, such that $d\tau$ is conjugate linear in $\mathcal{G}$. Then the fixed point set $\mathcal{U}$ of $\tau$ in $\mathcal{G}$ is a real form. Let $\sigma$ be another complex linear involution on $\mathcal{U}$ and $K(\mathcal{P})$ the eigenspace of $\sigma$ of eigenvalue $1$. Hence $\mathcal{U} = K + \mathcal{P}$ is a Cartan decomposition. For $\xi \in \mathcal{G}$, we use $\xi_G(\mathcal{P})$ to denote the $K(\mathcal{P})$ component of $\xi$. Let

$$\mathcal{L}_r(\mathcal{G}) = \left\{ \sum A(\lambda) \in \mathcal{L}(\mathcal{G}) \mid \tau(A(\bar{\lambda})) = A(\lambda) \right\}.$$ Consider the following splitting of $\mathcal{L}_r(\mathcal{G})$:

$$\mathcal{L}_r(\mathcal{G})_+ = \left\{ \sum A_i \lambda_i \in \mathcal{L}(\mathcal{U}) \right\}, \quad \mathcal{L}_r(\mathcal{G})_- = \left\{ \sum A_i \lambda_i \in \mathcal{L}(\mathcal{U}) \right\}. \quad (2.5)$$

Let $a \in K$, such that $ad(a^2) = -id_\mathcal{P}$. Then given $u \in C^\infty(R, \mathcal{P})$, according to Theorem 2.1, there exists unique $Q(u, \lambda) = a\lambda + Q_0(u) + Q_{-1}(u)\lambda^{-1} + \cdots \in \mathcal{L}(\mathcal{U})$ such that $Q(u, \lambda)$ is conjugate to $a\lambda$, and

$$[\partial x + a\lambda + u, Q(u, \lambda)] = 0, \quad (2.6)$$

The $j$-th flow in the $U$-hierarchy is

$$[\partial x + a\lambda + u, \partial t + (Q(u, \lambda)\lambda^{(j-1)})_+] = 0, \quad (2.7)$$

and the Lax pair is the following flat $U$-value connection one form:

$$(a\lambda + u)d_x + (a\lambda^j + u\lambda^{j-1} + \cdots Q_{1-j}(u))dt.$$

**Example 2.2** ($U(n)$NLS hierarchy).

Let $\mathcal{L}(u(n)) = \{ A(\lambda) = \sum A_i \lambda^i \mid A_i \in su(n), A(\lambda) = -\bar{A(\lambda)} \}$, consider the following splitting of $\mathcal{L}(su(n))$:

$$\begin{cases}
\mathcal{L}_+(u(n)) = \{ \sum_{i \geq 0} A_i \lambda^i \mid A_i \in u(n) \}, \\
\mathcal{L}_-(u(n)) = \{ \sum_{i \leq 0} A_i \lambda^i \mid A_i \in u(n) \}.
\end{cases}$$
Let \( a = \frac{i}{2}I_{k,n-k} \), and \( J_1 = a\lambda \), then \( \mathcal{J} = \{ a\lambda^i \mid i \geq 1 \} \) is a vacuum sequence.

The phase space defined by (2.2) is of the form:

\[
\xi = J_1 + u = a\lambda + \begin{pmatrix} 0 & q \\ -q^t & 0 \end{pmatrix}, \quad q \in C^\infty(\mathbb{R}, \mathbb{C}^{k \times (n-k)}).
\]

Solve \( Q(u, \lambda) = a\lambda + Q_0(u) + Q_{-1}(u)\lambda^{-1} + \cdots \in \mathcal{L}(u(n)) \) from (2.3), and get

\[
Q_0(u) = u, \quad Q_{-1}(u) = i\begin{pmatrix} -q\bar{q}^t & q_x \\ \bar{q}_x^t & q^t q \end{pmatrix},
\]

\[
Q_{-2}(u) = \begin{pmatrix} q_x\bar{q}^t - q\bar{q}_x^t & -q_{xx} - 2q\bar{q}^t q \\ \bar{q}_x^t q - \bar{q}^t q_x \end{pmatrix}.
\]

The second flow \( u_t = [\partial_x + u, Q_{-1}] = [Q_{-2}, a] \), which is the matrix NLS equation (or \( U(n)\) NLS):

\[
q_t = i(q_{xx} + 2q\bar{q}^t q). \tag{2.8}
\]

Lagrangian deformation

**Definition 2.3.** (16) Let \( \mathcal{L}(U) = \sum_{i \leq m_0} \xi_i\lambda^i, \xi_i \in U \). Define a projection \( P_k \) on \( \mathcal{L}(U) \) as follows:

\[
P_k(Z) = \sum_{k<i \leq m_0} Z_i\lambda_i + (Z_{k-1})\lambda\lambda_{k-1} + \sum_{i<k-1} Z_i\lambda_i, \quad Z = \sum_{i \leq m_0} Z_i\lambda_i. \tag{2.9}
\]

By using the \( H^1 \)-norm instead of \( L^2 \)-norm, the generated operator becomes \( \hat{\mathcal{L}} = a\lambda + \hat{u} \), where \( \hat{u} = u - \alpha^2u_{xx}, \alpha \in \mathbb{R} \). Set

\[
\hat{Q}(u) = a\lambda + u + \hat{Q}_{-1}(u)\lambda^{-1} + \hat{Q}_{-2}(u)\lambda^{-2} + \cdots.
\]

Instead of using (2.3), we can use the following equation to solve first several coefficients of \( \hat{Q}(u) \), although the process is not purely algebraic.

\[
P_j([\partial_x + a\lambda + \hat{u}, \partial_t + (\hat{Q}(u)\lambda^{j-1})_+]) = 0. \tag{2.10}
\]

The \( j \)-th deformation flow is

\[
(\hat{u})_t = ((\hat{Q}_{j-1})_x + [\hat{u}, \hat{Q}_{j-1}])_+ = ((\hat{Q}_{j-1})_+)_x + [\hat{u}, (\hat{Q}_{j-1})_x]. \tag{2.11}
\]

Equations constructed through this process are called weak complete integrable, and (2.10) is called the weak Lax pair.

From Lagrangian deformation and the Lax pair of the KdV equation, the CH equation can be derived. In Ref. [16], several deformed equations are derived corresponding to the NLS equation, modified KdV equations and flows constructed from Lie algebra \( so(3) \). In the rest of this paper, we will derive some other CH-type equations by the Lie algebra splitting and the weak Lax pairs. For the standard ones (without Lagrangian deformation), more details can be found in Ref. [57].
In this Section, we will use the modified loop algebra splitting method with Lagrangian deformation to construct series of new weak completely integrable equations. To explain the algorithm, we carry out in detail the computation of Camassa-Holm-nonlinear Schrödinger (CH-NLS) equation as a first example.

3.1. Camassa Holm-nonlinear Schrödinger-type equations.
In this case, \( G = GL(2, \mathbb{C}) \) and \( \tau(g) = (g)^{-1} \). Hence the fixed point set of \( \tau \) is \( \mathcal{U} = u(2) \). Let \( \sigma(g) = I_{1,1}gI_{1,1} \), where \( I_{1,1} = \text{diag}(1, -1) \). Then

\[
\mathcal{K} = \mathbb{R}iI_{1,-1}, \quad \mathcal{P} = \begin{pmatrix} 0 & r \\ -\overline{r} & 0 \end{pmatrix}. \tag{3.1}
\]

Let \( a = \frac{1}{2}\text{diag}(i, -i) \). Then the phase space \( \xi = a\lambda + u = a\lambda + \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \) from the standard loop algebra splitting. And the second flow in the \( u(2) \)-hierarchy is the NLS equation,

\[
q_t = i(q_{xx} + 2|q|^2 q). \tag{3.2}
\]

By Lagrangian deformation, we have \( \hat{u} = \begin{pmatrix} 0 & \hat{q} \\ -\bar{\hat{q}} & 0 \end{pmatrix} \), where \( \hat{q} = q - \alpha^2 q_{xx} \).

Apply (2.10) for \( j = 2 \) and compute the coefficient of \( \lambda^i \) in

\[
\begin{align*}
\lambda: & \quad u_x + [\hat{u}, u] + [a, \hat{\mathcal{Q}}^{-1}(u)]], \\
\lambda^0: & \quad (\hat{u}_t = (\hat{\mathcal{Q}}^{-1}(u)) = [\hat{u}, \hat{\mathcal{Q}}^{-1}(u)].
\end{align*}
\]

By equating \( (\hat{u}_t = (\hat{\mathcal{Q}}^{-1}(u))_x) \), we get

\[
(\hat{\mathcal{Q}}^{-1}(u))_x + [\hat{u}, (\hat{\mathcal{Q}}^{-1}(u))] = 0.
\]

Hence \( (\hat{\mathcal{Q}}^{-1}(u))_x = i(\alpha^2 |q_x|^2 - |q|^2)I_{1,1} \).

Then the \( \mathcal{P} \)-component gives the CH-NLS equation \((1.4)\):

\[
\hat{q}_t = iq_{xx} - 2i\bar{q}(\alpha^2 |q_x|^2 - |q|^2) = iq_{xx} - 2i(q - \alpha^2 q_{xx})(\alpha^2 |q_x|^2 - |q|^2).
\]

Remark 3.1. In the process of solving \((\hat{\mathcal{Q}}^{-1}(u))_x \), the equation we have is for \(((\hat{\mathcal{Q}}^{-1}(u)))_x \). Therefore, \((\hat{\mathcal{Q}}^{-1}(u))_x = i(\alpha^2 |q_x|^2 - |q|^2)I_{1,1} \) up to a constant in
$x$. This is different to the standard Lie algebra splitting theory, where this step can be solved algebraically, and it can be proved that this constant is zero.

**Remark 3.2.** From a direct computation, we can see that $h = |\hat{q}|^2$ is conserved under the flow (1.4).

Next we consider the weak Lax pair for $j = 3$. Following a similar computation, we can get

$$
\hat{Q}^{-1}(u) = i \left( \alpha^2 |q_x|^2 - |q|^2 \frac{q_x}{\bar{q}_x} |q|^2 - \alpha^2 |q_x|^2 \right),
$$

$$
\hat{Q}^{-2}(u) = \left( \frac{q_x \bar{q} - q \bar{q}_x}{\bar{q}_x x - 2(\alpha^2 |q_x|^2 - |q|^2)\bar{q}} - q_{xx} + 2(\alpha^2 |q_x|^2 - |q|^2)\bar{q} \right).
$$

Therefore, the third CH-NLS flow (the CH-mKdV equation) is:

$$
\hat{q}_t = -q_{xxx} + 2((\alpha^2 |q_x|^2 - |q|^2)\bar{q})_x + 2(q\bar{q}_x - q_x \bar{q})\hat{q}. \quad (3.3)
$$

Now we can generalize the argument to $n$-dimensions and derive the CH-$U(n)$NLS equation. Since the process is similar, we only list the result below.

Let $G = GL(n, \mathbb{C})$ with $\mathcal{G} = gl(n, \mathbb{C})$ its Lie algebra. Then $U = u(n)$ is the real form under the involution $\tau(g) = \bar{g}^t$. Let $\sigma$ be the involution of $G$ defined by $\sigma(g) = I_{k,n-k}g I_{k,n-k}$, where $I_{k,n-k} = \text{diag}(I_k, -I_{n-k})$. Then

$$
K = u(k) \times u(n-k), \quad \mathcal{P} = \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix}, X \in \mathbb{C}^{k \times (n-k)}.
$$

Let $a = \frac{i}{2} I_{k,n-k}$, then $u = \begin{pmatrix} 0 & q \\ -q^t & 0 \end{pmatrix}$, where $q \in \mathbb{C}^{k \times (n-k)}$. Under Lagrangian decomposition, we get $\hat{u} = \begin{pmatrix} 0 & \hat{q} \\ -\hat{q}^t & 0 \end{pmatrix}$, with $\hat{q} = q - \alpha^2 q_{xx}$. From the weak Lax pair (2.11) for $j = 2$, we get

$$
\hat{Q}^{-1}(u) = i \left( \alpha^2 q_x \hat{q}_x - q \hat{q}_x + q_x \hat{q} - \alpha^2 \hat{q}_x q_x \right).
$$

Therefore, the CH-$U(n)$NLS equation is:

$$
\hat{q}_t = i q_{xx} + i(\hat{q} \hat{q} - \alpha^2 \hat{q}_x q_x) + (qq^t - \alpha^2 q_x q_x^t)\hat{q}. \quad (3.4)
$$

In particular, when $n = 2$, this equation becomes (1.4).

### 3.2. The CH-DNLS equation.

In this Section, we start with the Lax pair of the DNLS equation. Then we consider the constraint under the Lagrangian deformation. The construction of the DNLS equation can be found in Ref. [57].
Let $G = SU(2)$, and $L(SU(2))$ be the group of smooth loops from $S^1$ to $SU(2)$, and $\mathcal{L}(su(2))$ its Lie algebra. Define an involution $\sigma$ on $su(2)$ as following:

$$\sigma(A) = I_{1,1} A I_{1,1}^{-1}, \quad I_{1,1} = \text{diag}(1, -1).$$

(3.5)

Let $K$ and $P$ denote the $1$ and $-1$ eigenspaces of $\sigma$, respectively, then

$$K = \mathbb{R} i I_{1,1}, \quad P = \begin{pmatrix} 0 & r \\ -\bar{r} & 0 \end{pmatrix}.$$ 

Furthermore, $\sigma$ induces an involution on $\mathcal{L}(sl(2, \mathbb{C}))$ such that

$$\sigma(A(\lambda)) = I_{1,1} A(-\lambda) I_{1,1}^{-1}.$$ 

Let $L_\sigma(su(2))$ be the subalgebra of $\mathcal{L}(su(2))$ consisting of fixed points of $\sigma$, and consider the following splitting of $\mathcal{L}(su(2))$:

$$\begin{cases} 
\mathcal{L}_+^\sigma(su(2)) = \{ \sum_{i \geq 1} A_i \lambda^i \in \mathcal{L}(su(2)) \}, \\
\mathcal{L}_-^\sigma(su(2)) = \{ \sum_{i \leq 0} A_i \lambda^i \in \mathcal{L}(su(2)) \}.
\end{cases}$$

Given $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, write

$$Q(u, \lambda) = a \lambda^2 + Q_1(u) \lambda + Q_0(u) + Q_{-1}(u) \lambda^{-1} + \ldots \in L_\sigma(su(2)).$$

Such $Q(u, \lambda)$ can be solved uniquely by the following system:

$$\begin{cases} 
[\partial_x + a \lambda^2 + u \lambda, Q(u, \lambda)] = 0, \\
Q(u, \lambda)^2 = -\lambda^4 I_2.
\end{cases}$$

Then the DNLS equation can be written as

$$q_t = iq_{xx} - (|q|^2 q)_x.$$ 

(3.6)

Now we construct the deformed equation for $\dot{u} = \begin{pmatrix} 0 & \dot{q} \\ -\bar{q} & 0 \end{pmatrix}$ as follows.

Write $\dot{Q}(u, \lambda) = a \lambda^2 + u \lambda + \dot{Q}_0(u) + \dot{Q}_{-1}(u) \lambda^{-1} + \ldots$, with

$$\dot{Q}_0(u) = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad \dot{Q}_{-1}(u) = \begin{pmatrix} 0 & B \\ -\bar{B} & 0 \end{pmatrix}.$$ 

Consider the following weak Lax pair:

$$P_5([\partial_x + a \lambda^2 + \dot{u} \lambda, \partial_t + (\dot{Q}(u) \lambda^2)_+]) = 0,$$ 

(3.7)

and compare the coefficient of $\lambda^2$ of (3.7). The first non-trivial equation comes from the coefficient of $\lambda^3$:

$$u_x + [a, \dot{Q}_{-1}(u)] + [\dot{u}, Q_0(u)] = 0.$$
Therefore,
\[
q_x + i\beta - 2A\hat{q} = 0. \tag{3.8}
\]
Compute the coefficient of \(\lambda^2\) in (3.7) to get:
\[
(\hat{Q}_0(u)_x + [\hat{u}, \hat{Q}_{-1}(u)] = 0.
\]
Hence \(A_x = \hat{q}\beta - B\hat{q}\). Together with (3.8) and note that \(\bar{A} = -A\), we can solve \(A = i(\alpha^2|q_x|^2 - |q|^2)\) (up to a constant in \(x\)).

Therefore, the CH-DNLS equation is
\[
\hat{q}_t = iq_{xx} + 2((\alpha^2|q_x|^2 - |q|^2)\hat{q})_x, \quad \hat{q} = q - \alpha^2q_{xx}. \tag{3.9}
\]

Remark 3.3. In Ref. \([57]\), the authors used the loop algebra splitting method to get hierarchies of the generalized DNLS equations, for example,
\[
q_t = i\frac{q_{xxx} - (2\theta + 1)|q|^2q_x - (2\theta - 1)q^2\bar{q}_x + (\frac{1}{2}\theta + 2\theta^2)i|q|^4q}{\theta \in \mathbb{R}}. \tag{3.10}
\]
By choosing different values of \(\theta\), the DNLSI, DNLSII, and DNLSIII are derived.

But the deformed DNLSII and DNLSIII equations do not come from the weak Lax pair. That is, equation in the form of (3.7) is not solvable. This means we may need to find more reductions in order to make it well-posed.

3.3. CH-Hirota equation.

The Hirota equation is a combination of the NLS equation and the mKdV equation \([58]\):
\[
q_t = \beta i(q_{xx} + 2q|q|^2) - \gamma(q_{xxx} + 6|q|^2q_x), \quad \beta, \gamma \in \mathbb{R}. \tag{3.11}
\]
It can be derived from the following Lax pair:
\[
[\partial_x + a\lambda + u, \partial_t + \gamma a\lambda^3 + (\beta a + \gamma u)\lambda^2 + Q_{-1}(u)\lambda + Q_{-2}(u)] = 0,
\]
where
\[
a = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad Q_{-1}(u) = \begin{pmatrix} -\gamma|q|^2 & \gamma iq_x + \beta q \\ \gamma i\bar{q}_x - \beta\bar{q} & \gamma i|q|^2 \end{pmatrix},
\]
\[
Q_{-2}(u) = \begin{pmatrix} -\gamma(q\bar{q}_x - q_x\bar{q}) - \beta i|q|^2 & \beta iq_x - \gamma q_{xx} - 2\gamma|q|^2q_x \\ \beta i\bar{q}_x + \gamma\bar{q}_{xx} + 2\gamma\bar{q}|q|^2 & \gamma(q\bar{q}_x - q_x\bar{q}) + \beta i|q|^2 \end{pmatrix}.
\]
Under the Lagrangian reduction, where \(G = SU(2)\), \(K, P\) are as in (3.1), and \(\hat{u} = \begin{pmatrix} 0 & \hat{q} \\ -\hat{q} & 0 \end{pmatrix}\), \(\hat{q} = q - \alpha^2q_{xx}\).

Solve \(\hat{Q}(u, \lambda) = \hat{Q}(u, \lambda) = 4\gamma a\lambda + (2\beta a + 4\gamma u) + \hat{Q}_{-1}\lambda^{-1} + \ldots \in \mathcal{L}(su(2))\) from
\[
P_3([\partial_x + a\lambda + \hat{u}, \partial_t + (\hat{Q}(u)\lambda^2)_+] = 0, \tag{3.12}
\]
where $P_3$ is the projection defined in order to get

$$
\dot{Q}_{-1}(u) = \begin{pmatrix}
-\gamma i(|q|^2 - \alpha^2|q_x|^2) & \beta \hat{q} + \gamma i \hat{q}_x \\
-\beta \hat{q} + \gamma i \hat{q}_x & \gamma i(|q|^2 - \alpha^2|q_x|^2)
\end{pmatrix},
$$

$$
\dot{Q}_{-2}(u) = \begin{pmatrix}
\dot{Q}_{-2,11}(u) & \dot{Q}_{-2,12}(u) \\
-\dot{Q}_{-2,12}(u) & -\dot{Q}_{-2,11}(u)
\end{pmatrix},
$$

where

$$
\dot{Q}_{-2,11}(u) = -i \beta |\hat{q}|^2 - \gamma (q \bar{q}_x - q_x \bar{q}),
$$

$$
\dot{Q}_{-2,12}(u) = \beta i \hat{q}_x - \gamma q_{xx} - 2 \gamma \hat{q}(|q|^2 - \alpha^2|q_x|^2).
$$

Then the CH-Hirota equation is

$$
\dot{q}_t = \beta i \hat{q}_{xx} - \gamma q_{xxx} - 2 \gamma (\hat{q}(|q|^2 - \alpha^2|q_x|^2))_x + 2 \hat{q}(i \beta |\hat{q}|^2 + \gamma (q \bar{q}_x - q_x \bar{q})).
$$

(3.13)

4. Conclusion

In this paper, we tried to understand the Lagrangian deformation theory in Ref. [16] in terms of loop algebra splitting. By adding the Lagrangian reduction in the hierarchies constructed by using the loop algebra splitting, such as the $U(n)$NLS hierarchy, the DNLS equation, and the Hirota equation, we get a series of deformed equations.

This gives us an algorithm to construct new weakly integrable equations. In particular, by choosing different Lie algebra $\mathcal{G}$ and real form $U$, from the Cartan decomposition and splitting of the loop algebra $L(U)$, we can get the corresponding deformed equations. As pointed out in Sec. 3, the weak Lax pair depends on the choice of the projection $P_j$ as defined in Definition 2.3. And it can be checked that it may not work for the whole series, for example, in the case of the CH-NLS equation, when we consider the deformed equations corresponding to the forth or higher flows in the hierarchy, we may not get a well-posed equation. The next issue of interest would be find a modification that works for the whole hierarchy.

Note that the “integrability” of these equations is still an open problem: so far, we only know that the CH equation is completely integrable. Therefore, it would be worthwhile to find soliton-like solutions for these equations, which may give us a better understanding of these new types of equations. Work on this direction is underway. Also, it is well known that if an equation is completely integrable, then there exists a family of infinitely many conservation laws. As we noted in Sec. 3, another future research direction is to write down the conserved quantities for the weakly integrable equations.
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