New aspects of one-dimensional $\mathcal{N} = 4$-extended supersymmetry

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Abstract. In this talk I will present very recent results concerning the $\mathcal{N} = 4$-extended supersymmetry in one-dimension. The new features consist in the construction of the non-minimal linear multiplets realized by 8 bosonic and 8 fermionic components, the construction of non-linear realizations induced by the supersymmetrization of the second Hopf fibration and, finally, the construction of a new superalgebra with $\mathcal{N} = 4$ fermionic generators (two of them nilpotent) obtained by the dimensional reduction of the four-dimensional twisted super-Yang-Mills theory.

1. Introduction
In this talk I will summarize the main results, concerning the $\mathcal{N} = 4$ extended one-dimensional supersymmetry, obtained in the last year in three separate papers.[1, 2, 3]

These new results are part of a systematic investigation of the properties of the Extended one-dimensional supersymmetry, started approximately ten years ago, focused on investigating the properties of the supersymmetry representations and of their associated invariant actions. One of the aim of the investigation consists in providing constraints on the construction of higher-dimensional supersymmetric theories based on the information encoded in the rich one-dimensional structural data, which allows the use of powerful mathematical tools. This line of research has been worked out and discussed in several works, see. [4]–[17].

The three main results I will discuss in this talk concern:

i) the construction of the non-minimal linear representations of the $\mathcal{N} = 4$-extended supersymmetry realized on a multiplet of 8 bosonic and 8 fermionic fields. Since this multiplet can (minimally) carry at most $\mathcal{N} = 8$ supersymmetries, these results finds application in the context of partial supersymmetry breaking (we recall that a $D = 4$ $\mathcal{N} = 2$ supersymmetric theory admits $\mathcal{N} = 8$ supercharges, while a $D = 4$ $\mathcal{N} = 1$ supersymmetric theory admits $\mathcal{N} = 4$ supercharges);

ii) the geometrical construction of non-linear realization of the $\mathcal{N} = 4$ supersymmetry obtained by the supersymmetrization of the 2nd Hopf fibration;

iii) a non-trivial realization of a new superalgebra with 4 supercharges, obtained by dimensional reduction of the twisted super-Yang-Mills theory. The superalgebra under consideration, contrary to the dimensional reduction of the ordinary supersymmetry, possesses generators which are nilpotent (therefore behaving like Grassmann parameters or BRST charges).
For completeness in an appendix the main definitions and properties of the representations of the one-dimensional supersymmetry are reported.

2. $\mathcal{N} = 4$ Non-minimal linear supermultiplets

The 1D $\mathcal{N}$-Extended Superalgebra, with $\mathcal{N}$ odd generators $Q_I$ ($I = 1, 2, \ldots, \mathcal{N}$) and a single even generator $H$ satisfying the (anti)-commutation relations

$$\{Q_I, Q_J\} = \delta_{IJ}H, \quad [H, Q_I] = 0,$$

is the superalgebra underlying the Supersymmetric Quantum Mechanics [18]. In recent years the structure of its linear representations has been unveiled by a series of works (upon which the present investigation is based) [4]–[15].

The linear representations under considerations (supermultiplets) contain a finite, equal number of bosonic and fermionic fields depending on a single coordinate (the time). The operators $Q_I$ and $H$ act as differential operators. The linear representations are characterized by a series of properties which, for sake of consistency, are reviewed in the appendix.

The minimal linear representations (also called irreducible supermultiplets) are given by the minimal number $n_{\text{min}}$ of bosonic (fermionic) fields for a given value of $\mathcal{N}$. The value $n_{\text{min}}$ is given [4] by the formula

$$\mathcal{N} = 8l + m, \quad n_{\text{min}} = 2^l G(m),$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

$G(m)$ appearing in (2) is the Radon-Hurwitz function

| $m$  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| $G(m)$ | 1 | 2 | 4 | 4 | 8 | 8 | 8 | 8 |

Non-minimal linear representations have been discussed in [6, 14, 15, 1]. The maximal finite number $n_{\text{max}}$ of bosonic (fermionic) fields entering a non-minimal representation is given by [14, 15]

$$n_{\text{max}} = 2^{\mathcal{N}-1}.$$

An important subclass of non-minimal representations is given by the reducible but indecomposable supermultiplets (see [6]). For this subclass the associated graph is connected (there is a path connecting any two given vertices).

For $\mathcal{N} = 4$ we have that $n_{\text{min}} = 4$ and $n_{\text{max}} = 8$. As a consequence, there are only two subclasses of non-minimal $\mathcal{N} = 4$ representations. Besides the irreducible but indecomposable subclass, we have the subclass of fully reducible representations given by the direct sum of two minimal $\mathcal{N} = 4$ representations (the associated graph is disconnected and given by two separate minimal $\mathcal{N} = 4$ graphs. We present here the complete list of inequivalent $\mathcal{N} = 4$ non-minimal supermultiplets associated to connected graphs (therefore providing reducible but indecomposable representations).

Up to equivalence, there exists a unique length-2 $\mathcal{N} = 4$ root supermultiplet, based on a connected graph, of field content $(8, 8)$.

The inequivalent non-minimal supermultiplets of length-3 are given by the table below. One should note that, for field content $(k, 8, 8 - k)$ with $k = 2, 3, 4, 5, 6$, inequivalent supermultiplets
are discriminated by their respective connectivity symbol (see the appendix) and are named according to the given label. The table further reports the dually related supermultiplet (see the appendix). We have

| field content: | label: | connectivity symbol: | dual supermultiplet: |
|----------------|--------|----------------------|----------------------|
| (1, 8, 7)      |        | 4_4 + 4_3            | (7, 8, 1)            |
| (2, 8, 6)      | A      | 2_4 + 4_3 + 2_2     | (6, 8, 2)_A         |
|                | B      | 8_3                  | (6, 8, 2)_B         |
| (3, 8, 5)      | A      | 1_4 + 3_3 + 3_2 + 1_1 | (5, 8, 3)_A            |
|                | B      | 4_3 + 4_2            | (5, 8, 3)_B         |
| (4, 8, 4)      | A      | 1_4 + 6_2 + 1_0     | self-dual           |
|                | B      | 4_3 + 4_1            | self-dual           |
|                | C      | 2_3 + 4_2 + 2_1     | self-dual           |
|                | D      | 8_2                  | self-dual           |
| (5, 8, 3)      | A      | 1_3 + 3_2 + 3_1 + 1_0 | (3, 8, 5)_A            |
|                | B      | 4_2 + 4_1            | (3, 8, 5)_B         |
| (6, 8, 2)      | A      | 2_2 + 4_1 + 2_0     | (2, 8, 6)_A         |
|                | B      | 8_1                  | (2, 8, 6)_B         |
| (7, 8, 1)      |        | 4_1 + 4_0            | (1, 8, 7)            |

(5)

3. \( \mathcal{N} = 4 \) Non-linear realizations from the 2nd Hopf fibration

The geometrical construction of the \( \mathcal{N} = 4 \) non-linear realization acting on the multiplets of field-content \((3, 4, 1)\) and \((2, 4, 2)\) (see the appendix) is obtained by the supersymmetrization of the second Hopf fibration.\(^2\)

We present at first the four (bosonic) Hopf maps (for \( k = 1, 2, 4, 8 \)), associated to the four division algebras, that can be illustrated by the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}^{2k} & \xrightarrow{p} & \mathbb{R}^{k+1} \\
\rho \downarrow & & \downarrow \rho' \\
S^{2k-1} & \xrightarrow{h} & S^k
\end{array}
\]

which connects four spaces (two Euclidean spaces and two spheres) which, for later convenience, can be identified as \( I, II, III, IV \) according to

\[
\begin{array}{ccc}
I & \xrightarrow{p} & II \\
\rho \downarrow & & \downarrow \rho' \\
III & \xrightarrow{h} & IV
\end{array}
\]

The four arrows correspond to the following maps:
- the bilinear map \( p : I \rightarrow II \), sending coordinates \( \vec{u} \in \mathbb{R}^{2k} \) into coordinates \( \vec{x} \in \mathbb{R}^{k+1} \) according to

\[
p : \quad \vec{u} \mapsto x_i = u^T \gamma_i u,
\]

\( (\gamma_i \) are the Euclidean gamma matrices of \( \mathbb{R}^{k+1} \));
- the restrictions \( \rho, \rho' \) on spheres, where \( \rho : I \rightarrow III \) and \( \rho' : II \rightarrow IV \);
- the hopf map \( h : II \rightarrow IV \), admitting \( S^{k-1} \) as a fiber (for \( k = 8 \), \( S^7 \) is a parallelizable manifold but not, properly speaking, a group-manifold due to the nonassociativity of the octonions; for \( k = 1 \), \( S^0 \equiv \mathbb{Z}_2 \)).
For \( k = 1, 2, 4, 8 \) the map (8) preserves the norm, allowing to induce the map \( h \) from \( p: \)

\[
u^T u = R \mapsto x^T x = r, \quad \text{with} \quad r = R^2.
\]

By setting \( k = 2^l \), the four Hopf maps \( h \) will be referred to (for \( l = 0, 1, 2, 3 \) respectively) as the 0\(^{th}\), 1\(^{st}\), 2\(^{nd}\) and 3\(^{rd}\) Hopf map.

In the following we will give a detailed description of the supersymmetric extension of the first Hopf map \( (k = 2) \), corresponding to the diagram

\[
\begin{array}{ccc}
\mathbb{R}^4 & \xrightarrow{p} & \mathbb{R}^3 \\
\downarrow \rho & & \downarrow \rho' \\
\mathbb{S}^3 & \xrightarrow{h} & \mathbb{S}^2
\end{array}
\]

In the supersymmetric extension \( \mathbb{R}^4 \) is replaced by the \( \mathcal{N} = 4 \) root supermultiplet \( (4, 4) \) whose four bosonic (target) coordinates correspond to the coordinates of \( \mathbb{R}^4 \). The off-shell supermultiplets extending \( II, III \) and \( IV \) are induced by applying, respectively, the map \( p \) and the restriction \( \rho \) to \( (4, 4) \), as well as the restriction \( \rho' \) on the induced supermultiplet generalizing \( III \).

For our purposes it will be convenient to define the target coordinates of the supermultiplets extending \( III \) (\( IV \)) in terms of the stereographic projection of the \( I (II) \) target coordinates.

For \( k = 2 \) we can express the three Euclidean gamma matrices \( \gamma_i \) as

\[
\gamma_1 = \tau_1 \otimes \mathbf{1}_2, \quad \gamma_2 = \tau_A \otimes \tau_A, \quad \gamma_3 = \tau_2 \otimes \mathbf{1}_2,
\]

where

\[
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

With this convention the bilinear map \( p \) is explicitly presented as

\[
\begin{align*}
x_1 &= 2(u_1u_3 + u_2u_4), \\
x_2 &= 2(u_1u_4 - u_2u_3), \\
x_3 &= u_1^2 + u_2^2 - u_3^2 - u_4^2.
\end{align*}
\]

It is invariant under the \( \sigma \) transformation \( (\sigma^2 = -1) \), given by

\[
\sigma: \quad u_1 \mapsto u_2, \quad u_2 \mapsto -u_1, \quad u_3 \mapsto u_4, \quad u_4 \mapsto -u_3.
\]

Induced by the \( \mathcal{N} = 4 \) \( (4, 4) \) root supermultiplet, three more (inequivalent) \( \mathcal{N} = 4 \) off-shell supermultiplets are obtained. They correspond to the supersymmetric extensions of \( II, III \) and \( IV \). The supertransformations extending \( II \) are all linear. This supermultiplet has field content \( (3, 4, 1) \) and will therefore be denoted as \( (3, 4, 1)_{\text{lin}} \). The supermultiplets extending \( III \) and \( IV \) possess supertransformations which are differential polynomials in their component fields. Since their field content is, respectively, \( (3, 4, 1) \) and \( (2, 4, 2) \), the supermultiplets will be denoted as \( (3, 4, 1)_{\text{nl}} \) and \( (2, 4, 2)_{\text{nl}} \), respectively.

Schematically, we have

\[
\begin{array}{ccc}
(4, 4) & \rightarrow & (3, 4, 1)_{\text{lin}} \\
\downarrow & & \downarrow \\
(3, 4, 1)_{\text{nl}} & \rightarrow & (2, 4, 2)_{\text{nl}}
\end{array}
\]
Their component fields are parametrized according to

\[(u_1, u_2, u_3, u_4; ψ_1, ψ_2, ψ_3, ψ_4) \rightarrow (x_1, x_2, x_3; µ_1, µ_2, µ_3, µ_4; f)\]

\[(w_1, w_2, w_3; ξ_1, ξ_2, ξ_3, ξ_4; g) \rightarrow (z_1, z_2; η_1, η_2, η_3, η_4; h_1, h_2)\]

(16)

The greek letters have been employed to denote the fermionic fields; \(\vec{u}, \vec{x}, \vec{w}, \vec{z}\) denote the bosonic target coordinates of the respective supermultiplets, while \(f, g, h_{1,2}\) denote the auxiliary fields.

We present at first the supersymmetry transformations of the four supermultiplets above (the presentation of the transformations explicitly connecting their component fields, namely the “arrows” in (16), will be given in the next Section). In the following tables the entries give the supertransformations of the component fields under the action of the \(Q_I\) supersymmetry operator.

The \(N = 4\) (linear) \((4,4)\) root supermultiplet can be explicitly presented as

| \(Q_1\) | \(Q_2\) | \(Q_3\) | \(Q_4\) |
|---|---|---|---|
| \(u_1\) | \(ψ_1\) | \(ψ_2\) | \(ψ_3\) | \(ψ_4\) |
| \(u_2\) | \(ψ_2\) | \(-ψ_1\) | \(ψ_4\) | \(-ψ_3\) |
| \(u_3\) | \(ψ_3\) | \(-ψ_4\) | \(-ψ_1\) | \(ψ_2\) |
| \(u_4\) | \(ψ_4\) | \(ψ_3\) | \(-ψ_2\) | \(-ψ_1\) |

(17)

The \((3,4,1)_{lin}\) supermultiplet is given by

| \(Q_1\) | \(Q_2\) | \(Q_3\) | \(Q_4\) |
|---|---|---|---|
| \(x_1\) | \(µ_1\) | \(-µ_2\) | \(-µ_3\) | \(µ_4\) |
| \(x_2\) | \(µ_2\) | \(µ_1\) | \(-µ_4\) | \(-µ_3\) |
| \(x_3\) | \(µ_3\) | \(µ_4\) | \(µ_1\) | \(µ_2\) |
| \(µ_1\) | \(x_1\) | \(x_2\) | \(x_3\) | \(-f\) |
| \(µ_2\) | \(x_2\) | \(-x_1\) | \(f\) | \(x_3\) |
| \(µ_3\) | \(x_3\) | \(-f\) | \(-x_1\) | \(-x_2\) |
| \(µ_4\) | \(f\) | \(x_3\) | \(-x_2\) | \(x_1\) |

(18)

The \((3,4,1)_{nl}\) supermultiplet is given by

| \(Q_1\) | \(Q_2\) | \(Q_3\) | \(Q_4\) |
|---|---|---|---|
| \(w_1\) | \(ξ_1 + \frac{1}{4}(w_1ξ_4)\) | \(ξ_2 + \frac{1}{4}(w_1ξ_3)\) | \(ξ_2 - \frac{1}{4}(w_1ξ_2)\) | \(ξ_4 - \frac{1}{4}(w_1ξ_1)\) |
| \(w_2\) | \(ξ_2 + \frac{1}{4}(w_2ξ_4)\) | \(-ξ_1 + \frac{1}{4}(w_2ξ_3)\) | \(ξ_4 - \frac{1}{4}(w_2ξ_2)\) | \(-ξ_3 - \frac{1}{4}(w_2ξ_1)\) |
| \(w_3\) | \(ξ_3 + \frac{1}{4}(w_3ξ_4)\) | \(-ξ_4 + \frac{1}{4}(w_3ξ_3)\) | \(-ξ_1 - \frac{1}{4}(w_3ξ_2)\) | \(ξ_2 - \frac{1}{4}(w_3ξ_1)\) |
| \(ξ_1\) | \(w_1 - \frac{1}{2}(w_1g + ξ_1ξ_4)\) | \(-w_2 + \frac{1}{2}(w_2g - ξ_1ξ_3)\) | \(-w_3 + \frac{1}{2}(w_3g + ξ_1ξ_2)\) | \(-w_4 + \frac{1}{2}(w_4g - ξ_1ξ_1)\) |
| \(ξ_2\) | \(w_2 - \frac{1}{2}(w_2g + ξ_2ξ_4)\) | \(w_1 - \frac{1}{2}(w_1g + ξ_2ξ_3)\) | \(-g\) | \(w_3 - \frac{1}{2}(w_3g + ξ_2ξ_2)\) |
| \(ξ_3\) | \(w_3 - \frac{1}{2}(w_3g + ξ_3ξ_4)\) | \(g\) | \(w_1 - \frac{1}{2}(w_1g + ξ_3ξ_1)\) | \(-w_2 + \frac{1}{2}(w_2g - ξ_3ξ_3)\) |
| \(ξ_4\) | \(g\) | \(-w_3 + \frac{1}{2}(w_3g + ξ_4ξ_4)\) | \(-w_2 + \frac{1}{2}(w_2g + ξ_4ξ_3)\) | \(w_1 - \frac{1}{2}(w_1g + ξ_4ξ_2)\) |
| \(g\) | \(ξ_4\) | \(-ξ_3\) | \(-ξ_2\) | \(-ξ_1\) |

(19)
Finally, the \((2,4,2)_{nl}\) supermultiplet is given by

\[
\begin{array}{|c|c|c|c|c|}
\hline
Q_1 & Q_2 & Q_3 & Q_4 \\
\hline
z_1 & \eta_1 + \frac{i}{2}(z_1\eta_3) & -\eta_2 + \frac{i}{2}(z_1\eta_4) & -\eta_3 + \frac{i}{2}(z_1\eta_1) & \eta_4 + \frac{i}{2}(z_1\eta_2) \\
z_2 & \eta_2 + \frac{i}{2}(z_2\eta_3) & \eta_1 + \frac{i}{2}(z_2\eta_4) & -\eta_4 + \frac{i}{2}(z_2\eta_1) & -\eta_3 + \frac{i}{2}(z_2\eta_2) \\
\hline
\eta_1 & \frac{1}{r}(z_1h_1 + \eta_1\eta_3) & \frac{1}{r}(z_2h_1 + \eta_1\eta_4) & h_1 & -h_2 - \frac{1}{r}(\eta_1\eta_2) \\
\eta_2 & \frac{1}{r}(z_1h_1 + \eta_1\eta_4) & \frac{1}{r}(z_2h_1 + \eta_2\eta_4) & h_2 - \frac{1}{r}(\eta_1\eta_2) & -\eta_1 - \frac{1}{r}(z_1h_1 + \eta_1\eta_3) \\
\eta_3 & h_1 & -h_2 - \frac{1}{r}(\eta_3\eta_4) & -\eta_1 - \frac{1}{r}(z_1h_1 + \eta_2\eta_3) & \eta_2 + \frac{i}{2}(z_2h_1 + \eta_2\eta_4) \\
\eta_4 & h_2 - \frac{1}{r}(\eta_3\eta_4) & \eta_1 & \eta_4 + \frac{i}{2}(z_2h_1 + \eta_3\eta_4) & -h_1 \\
\hline
h_1 & \eta_1 & \eta_1 & \eta_1 & \eta_1 \\
\eta_2 & \frac{1}{r}(h_1\eta_4 - h_2\eta_2) & -\eta_3 + \frac{1}{r}(h_1\eta_3 + h_2\eta_4) & \eta_4 + \frac{i}{2}(h_1\eta_2 + h_2\eta_1) & -\eta_1 + \frac{i}{2}(h_1\eta_1 + h_2\eta_2) \\
\hline
\end{array}
\]

(20)

The constant parameters \(R\) (entering \((3,4,1)_{nl}\)) and \(r\) (entering \((2,4,2)_{nl}\)) can be reabsorbed (set equal to 1) through a suitable rescaling of the component fields. It is however convenient to present them explicitly to show that in the contraction limit (for \(R,r \rightarrow \infty\)) the linear supermultiplets \((3,4,1)_{lin}\) and, respectively, \((2,4,2)_{lin}\) are recovered.

4. New 1D superalgebras from dimensional reduction of twisted SYM

On a Euclidean 4-manifold with \(SU(2)\) holonomy the \(N = 1\) superYang-Mills theory can be expressed in twisted form, both for the vector and the scalar multiplets \((3,4,1)\) and \((2,4,2)\). [19, 20, 21, 22, 23, 24]. In the twisted formulation the spinors are described as holomorphic and antiholomorphic forms, in such a way that both multiplets are decomposed as follows

\[
\begin{align*}
(3, 4, 1) & : A_m, \bar{A}_{\bar{m}}, \psi_m, \chi_{\bar{m}n}, \chi, h, \\
(2, 4, 2) & : \Phi, \bar{\Phi}, \psi_m, \chi_{mn}, \bar{\chi}; B_{\bar{m}n}, T_{mn}.
\end{align*}
\]

(21)

Here we use the complex space coordinates \(z^m, \bar{z}^\bar{m}\), where \(m = 1, 2\) are \(SU(2)\) indices and \(\bar{m} = 1, 2\) are the complex conjugate ones. One has a complex structure \(J^j_i\), with \(J^2 = -1\), \(J_{m\bar{n}} = -J_{\bar{n}m}\), \(J_{mn} = 0 = J_{\bar{m}\bar{n}} = 0\), which can be used as a metric, according to \(X_m Y_{\bar{n}} = X_{\bar{m}} Y_m Y_{\bar{n}}\).

The counting of the degrees of freedom is as follows. For the first multiplet, \((3,4,1)\), we have \(4 = 2 + 2\) gauge components \(A_m, \bar{A}_{\bar{m}}\) with one gauge degree of freedom (leading to \(4 - 1 = 3\) physical degrees of freedom), \(4\) fermions \((\psi_m, \chi_{\bar{m}n}, \chi)\) and 1 bosonic auxiliary field \((h)\). For the \((2,4,2)\) multiplet we have 2 propagating bosons \((\Phi, \bar{\Phi})\), 4 fermions \((\psi_m, \chi_{mn}, \bar{\chi})\) and 2 bosonic auxiliary fields \((B_{\bar{m}n}, T_{mn})\).

The link between analytically continued Majorana spinors \(\lambda^\alpha, \bar{\lambda}_{\dot{\alpha}}\) and their holomorphic-antiholomorphic decompositions as in Eq. (21) is given by the following formula

\[
\psi_m = \lambda^\alpha \sigma_\mu \alpha e_\mu^m, \\
\chi_{\bar{m}n} = \bar{\lambda}_{\dot{\alpha}} \bar{\sigma}_\mu \dot{\alpha} e_\mu^{\bar{m}} e_n, \\
\chi = \delta^\alpha_{\dot{\alpha}} \lambda_{\dot{\alpha}}.
\]

(22)

The twist formula (22) can be considered as a mere (complex) change of variables. The Dirac Lagrangian is expressed as

\[
\tilde{\lambda} \gamma^\mu D_\mu \lambda = \chi_{\bar{m}n} D_{[m} \psi_{n]} + \chi D_{\bar{m}} \psi_m.
\]

(23)

The four (ordinary) Poincaré supercharges are reexpressed, in the twisted supersymmetry formulation, as the operators \(s, s_\mu, s_m\) which close, up gauge transformations of the r.h.s.,
the superalgebra

\[
\{ s, s_m \} = \partial_m + \delta \text{gau.}(A_m), \\
\{ s_p, s_{mn} \} = J_{\bar{n}}(\partial_n + \delta \text{gau.}(A_n))
\] (24)

(the remaining anticommutators are all vanishing).

The balance between bosonic and fermionic degrees of freedom can be reached through a gauge-fixing, which can be imposed by setting, e.g., \( A_2 = 0 \). On the other hand, further constraints are necessary in order to maintain the compatibility of the supersymmetry algebra with the gauge-fixing. The net result is that the dependence on the coordinates \( m = 1, 2, \bar{m} = 1 \) must be frozen, leaving a one-dimensional theory which depends on the single variable \( \bar{2} \). The resulting 4-generator dimensionally reduced superalgebra is expressed (for the twisted \((3, 4, 1)\) multiplet) by

\[
\begin{array}{|c|c|c|c|}
\hline
s & s_T & s_\bar{T} & s_{12} \\
\hline
A_1 & \psi_1 & 0 & -a\chi & 0 \\
A_2 & \psi_2 & a\chi & 0 & 0 \\
A_\bar{T} & 0 & 0 & -\chi_{\bar{T}} & a\psi_1 \\
\psi_1 & 0 & 0 & A_1 + ah & 0 \\
\psi_2 & 0 & -ah & A_2 & 0 \\
\chi_{\bar{T}} & -\bar{A}_T & 0 & 0 & a(\bar{A}_1 + ah) \\
\chi & h & 0 & 0 & 0 \\
h & 0 & 0 & \tilde{\chi} & 0 \\
\hline
\end{array}
\] (25)

The mixed transformations can be eliminated by field redefinitions

\[
\begin{align*}
z_1 &= A_1, \\
z_2 &= \bar{A}_1 - aA_2, \\
z_3 &= A_\bar{T} + aA_2, \\
g &= \bar{A}_1 + 2ah,
\end{align*}
\] (26)

By introducing the basis of four operators

\[
s_\pm = s \pm s_T, \quad N_\pm = \frac{1}{\alpha}(s_{12} \pm s_T),
\] (27)

one ends up with a non-trivial realization of a generalized supersymmetry which consists of ordinary supersymmetry operators and nilpotent operators. The four operators \( s_\pm \) and \( N_\pm \) are indeed mutually anticommuting and satisfy

\[
s_\pm^2 = \pm \partial_t, \quad N_\pm^2 = 0.
\] (28)

The \((3, 4, 1)\) twisted multiplet transforms as

\[
\begin{array}{|c|c|c|c|}
\hline
s_+ & s_- & N_+ & N_- \\
\hline
z_1 & \xi_1 & \xi_2 & 0 & 0 \\
z_2 & -\xi_3 & \xi_4 & \xi_1 & \xi_2 \\
z_3 & -\xi_4 & \xi_3 & -\xi_2 & \xi_1 \\
\xi_1 & \bar{z}_1 & -g & 0 & 0 \\
\xi_2 & g & -\bar{z}_1 & 0 & 0 \\
\xi_3 & -\bar{z}_2 & -\bar{z}_3 & \bar{z}_1 & g \\
\xi_4 & -\bar{z}_3 & -\bar{z}_2 & g & \bar{z}_1 \\
g & \bar{\xi}_2 & \bar{\xi}_1 & 0 & 0 \\
\hline
\end{array}
\] (29)
This is non-trivial realization (a similar results holds for the twisted (2, 4, 2) multiplet as well),
of the $\mathcal{N} \equiv [n_+ = 1, n_- = 1, n_0 = 2]$ generalized supersymmetry
\begin{align}
\{Q_I, Q_J\} &= \eta_{IJ} H, \\
[H, Q_I] &= 0,
\end{align}
where $\eta_{IJ}$ is a diagonal metric with $n_\pm$ diagonal entries $\pm 1$ e $n_0$ diagonal entries 0.

The presence of a nilpotent operator implies that the twisted theory is a topological theory
defined in terms of a BRST cohomology. On the other hand the transformations (22) allows us
to relate the topological theory to the ordinary supersymmetric theory.

5. Conclusions
The new $\mathcal{N} = 4$ one-dimensional supermultiplets which have been presented in this talk, namely
the non-minimal linear supermultiplets, the non-linear supermultiplets induced by the second
Hopf fibration and the twisted supermultiplets induced by the dimensional reduction of the
twisted superYang-Mills theory, all admit invariant actions which define their associated $\mathcal{N} = 4$
sigma-models. The actions are not reported here for lack of space.

The supermultiplets here presented can be regarded as building blocks to construct different
realizations of supersymmetries (non-minimal, non-linear, twisted) for larger values of $\mathcal{N} > 4$.

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5.2. Appendix
For completeness we report the definitions, applied to the cases used in the text, of the properties
characterizing the linear representations of the one-dimensional $\mathcal{N}$-Extended Superalgebra. In
particular the notions of mass-dimension, field content, dressing transformation, connectivity
symbol, dual supermultiplet and so on, as well as the association of linear supersymmetry
transformations with graphs, will be reviewed following [4, 6, 5, 11, 12, 1]. The Reader can
consult these papers for broader definitions and more detailed discussions.

**Mass-dimension:**
A grading, the mass-dimension $d$, can be assigned to any field entering a linear representation
(the hamiltonian $H$, proportional to the time-derivative operator $\partial \equiv \frac{d}{dt}$, has a mass-dimension
1). Bosonic (fermionic) fields have integer (respectively, half-integer) mass-dimension.

**Field content:**
Each finite linear representation is characterized by its “field content”, i.e. the set of integers
$(n_1, n_2, \ldots, n_l)$ specifying the number $n_i$ of fields of mass-dimension $d_i$ ($d_i = d_1 + \frac{i-1}{2}$, with $d_1$
an arbitrary constant) entering the representation. Physically, the $n_i$ fields of highest dimension are
the auxiliary fields which transform as a time-derivative under any supersymmetry generator.
The maximal value $l$ (corresponding to the maximal dimensionality $d_l$) is defined to be the length
of the representation (a root representation has length $l = 2$). Either $n_1, n_3, \ldots$ correspond to
the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or viceversa.
In both cases the equality $n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n$ is guaranteed.

**Dressing transformation:**
Higher-length supermultiplets are obtained by applying a dressing transformation to the
length-2 root supermultiplet. The root supermultiplet is specified by the $\mathcal{N}$ supersymmetry
operators $\tilde{Q}_i$ ($i = 1, \ldots, \mathcal{N}$), expressed in matrix form as
\begin{align}
\tilde{Q}_j &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \gamma_j \\ -\gamma_j \cdot H & 0 \end{pmatrix}, \\
\tilde{Q}_N &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1_n \\ 1_n \cdot H & 0 \end{pmatrix},
\end{align}
where the $\gamma_j$ matrices ($j = 1, \ldots, N-1$) satisfy the Euclidean Clifford algebra
\[
\{ \gamma_i, \gamma_j \} = -2 \delta_{ij} 1_n.
\] (32)
The length-3 supermultiplets are specified by the $N$ operators $Q_i$, given by the dressing transformation
\[
Q_i = D \hat{Q}_i D^{-1},
\] (33)
where $D$ is a diagonal dressing matrix such that
\[
D = \begin{pmatrix} \tilde{D} & 0 \\ \mathbf{0} & 1_n \end{pmatrix},
\] (34)
with $\hat{D}$ an $n \times n$ diagonal matrix whose diagonal entries are either 1 or the derivative operator $\partial$.

Association with graphs:

The association between linear supersymmetry transformations and $N$-colored oriented graphs goes as follows. The fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $X-Y$ plane. The $Y$ coordinate can be chosen to correspond to the mass-dimension $d$ of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the $X$ axis. The higher dimensional fields have positive, integer or half-integer values of $Y$. A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the $N \ Q_i$ supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding supersymmetry transformation connecting the two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

Each irreducible supersymmetry transformation can be presented (the identification is not unique) through an oriented $N$-colored graph with $2n$ vertices. The graph is such that precisely $N$ edges, one for each color, are linked to any given vertex which represents either a 0-mass dimension or a $\frac{1}{2}$-mass dimension field. An unoriented “color-blind” graph can be associated to the initial graph by disregarding the orientation of the edges and their colors (all edges are painted in black).

Connectivity symbol:

A characterization of length $l = 3$ color-blind, unoriented graphs can be expressed through the connectivity symbol $\psi_g$, defined as follows
\[
\psi_g = (m_1)s_1 + (m_2)s_2 + \cdots + (m_Z)s_Z.
\] (35)
The $\psi_g$ symbol encodes the information on the partition of the $n \ \frac{1}{2}$-mass dimension fields (vertices) into the sets of $m_z$ vertices ($z = 1, \ldots, Z$) with $s_z$ edges connecting them to the $n - k \ 1$-mass dimension auxiliary fields. We have
\[
m_1 + m_2 + \cdots + m_Z = n,
\] (36)
while $s_z \neq s_{z'}$ for $z \neq z'$.

Dual supermultiplet:

A dual supermultiplet is obtained by mirror-reversing, upside-down, the graph associated to the original supermultiplet.
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