Marginals, Measurable Modifications of Stochastic Processes, and the Product Lifting Problem

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Abstract. This paper deals with the problem of measurable lifting modification for stochastic processes in its most general form and with the 'product lifting problem'. Solutions to the positive are reduced to the existence of marginals with respect to product probability spaces between the ordinary product and the product whose probability measure is the restriction of the skew product of the factor probabilities to the σ-algebra obtained by adjoining either the right or left nil-null sets to the ordinary product algebra. We discuss the problem of the existence of (strong) marginals.

Key Words: lifting, product strong lifting, marginals, nil null sets, skew product measures, measurable stochastic processes, lifting topologies, theta operator.

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Introduction.

This paper is concerned with two topics from probability theory, i.e. that of the measurable lifting modification for stochastic processes and that of the 'product lifting problem'. It turns out, that the notion of the marginal from the pioneering paper [25] lies at the basis of both problems.

Previous papers on measurable lifting modification imposed additional assumptions, such as separability by D.L. Cohn [4] and M. Talagrand [32], stability in [32] and [24], again separability in [31] though the probability measure could be quite far from the ordinary product measure but at the expense of the existence of a product regular conditional probability. D.L. Cohn’s paper allows processes taking their values in compact metrizable spaces. We extend D.L. Cohn’s work by replacing the latter class by the more general class of all strongly lifting compact spaces from [1] and [29] which, in the general case, seems to be the most general class where measurable lifting modification makes sense in view of Definition 4.4(b) and Remark [1]. The product of complete probability spaces for two basic probability spaces involved in both of the above problems up to now was usually taken as the 'ordinary product' or its completion, in case of topological probability spaces with τ-additive probabilities as their τ-additive product, available since Ressel [26]. In case of Radon factors the latter is just the
Radon product. It is crucial for this paper to allow more generally products with \( \sigma \)-algebra between the ordinary product and the \( \sigma \)-algebra obtained from the ordinary product \( \sigma \)-algebra by adjoining either the right or left nil-sets (see Lemma 1.1), taking as probability the restriction of the right or left skew product of the factor probabilities, compare Definition 1.4. All these probability spaces produce the same measure algebra, consequently the same hyperstonean space, see Definition 1.4. In this case our main result Theorem 4.8 tells us, that the existence of measurable lifting modifications for stochastic processes with range in a strongly lifting compact space is equivalent to the existence of a marginal, in the topological case (leaving unchanged continuous random variables) to the existence of a strong marginal, see Corollary 4.9. This transfers the existence of measurable lifting modifications to the problem of the existence of (strong) marginals. For the completion of the ordinary product, marginals exist as admissible densities or admissibly generated liftings as defined in [21]; their definition by transfinite recursion is very involved but, e.g. in case of separable \( \sigma \)-algebras (in particular for topological probability spaces over Polish spaces) ordinary induction will do. Unfortunately there exist even Radon probability factor spaces without strong admissible densities nor strong admissibly generated liftings, e.g. for the hyperstonean space of the Lebesgue probability space over \([0,1]\) in both factors, see Example 3.6 and there exist marginals being not admissible, see Remarks 5.8. If we adjoin either all left or all right nil-null sets to the ordinary product \( \sigma \)-algebra, every (strong) lifting or density (even weaker types) become a marginal by Proposition 3.8. But it may happen for certain liftings that this is to the best and this occurs in the paper of D.L. Cohn [4] (using R.M. Dudley [6], [7]) for a strong lifting due to Fremlin/Mokobodzki [10], which does not become a marginal if we adjoin to the ordinary product all two sided nil-null sets, see Remark 4.11.

In its most general setting (compare Lemma 1.10 (iv)) the 'product lifting problem' is that of permanence for the existence of strong liftings under the \( \tau \)-additive product for \( \tau \)-additive probabilities of full support in the factors. It is even unsolved in the particular case, that both factor spaces are hyperstonian spaces. The interest in this problem traces back to a result of A. and C. Ionescu Tulcea [18], concerning the equivalence of the existence of strict disintegrations with the existence of strong liftings. It is already known from [20], that the completed ordinary product is too 'small' for a solution to the positive, since in general this product does not contain all open sets of the product topology (for an elementary example see S. Gryllakis and S. Grekas [15]), thus prompting the question, whether we can achieve a solution to the positive, just by enlarging the ordinary product 'properly'? By Proposition 5.1 for a solution to the positive the existence of strong liftings in the factors is a necessary condition. If one of these is a marginal, in addition, and if the \( \sigma \)-algebra in the product contains the \( \tau \)-additive product \( \sigma \)-algebra of the lifting topologies for the strong liftings in the factors, the existence of strong lifting in the product follows, see Corollary 5.5. Using
a permanence result for marginals under inverse measure preserving maps from Proposition 3.10, we derive conditions implying for the canonical strong lifting of a hyperstonian space to be a marginal in case of the Radon product of this space with itself. As a special instance it follows, that the Radon product of a hyperstonian space associated to a Polish space with itself admits a product strong lifting, see Corollary 5.7.

We need section 2 about the $\tau$-additive product as a preparation for Theorem 5.4 and section 3 for generalizing the results of [25] about marginals from the ordinary product to our more general situation, where Proposition 3.10 is completely new.

1. Preliminaries.

$\mathbb{N}$ and $\mathbb{R}$ stand for the natural and the real numbers, respectively. By $\mathcal{P}(X)$ we denote the set of all subsets of the set $X$ and we write $M^c := X \setminus M$ for $M \in \mathcal{P}(X)$. $Y^X$ is the space of all functions from $X$ into $Y$. For a given probability space $(X, \Sigma, \mu)$ we denote by $(\Sigma, \mu)_0$, or by $\Sigma_0$ for simplicity, the $\sigma$-ideal of all $\mu$-null sets in $\Sigma$. If $A \in \Sigma$ we write $\Sigma \cap A$ for the $\Sigma$-algebra of all $E \cap A$ with $E \in \Sigma$. For $A, B \in \mathcal{P}(X)$ we write $A \subseteq \mu B$, if $A \setminus B \in \Sigma_0$ and $A =_\mu B$, if $A \subseteq \mu B$ and $B \subseteq \mu A$ and $f =_\mu g$ means $\{f \neq g\} \in \Sigma_0$ for $f, g \in \mathbb{R}^X$.

We denote by $(\Sigma/\mu, \tilde{\mu})$ the measure algebra of $(X, \Sigma, \mu)$, where $\Sigma/\mu$ is the space of all equivalence classes $A^* := A \setminus \mu$ for $A \in \Sigma$ modulo the $\sigma$-ideal $\Sigma_0$ and $\tilde{\mu}(A^*) := \mu(A)$ for $A \in \Sigma$. The completion of $(X, \Sigma, \mu)$ will be written $(X, \hat{\Sigma}, \hat{\mu})$. For topological spaces $(X, \Sigma)$ we write $\mathcal{B}(X)$ or $\mathcal{B}(\Sigma)$ (if we have to distinguish different topologies over $X$) for its Borel $\sigma$-field over $X$.

**Lemma 1.1.** Let be given a $\sigma$-ideal $\mathcal{I}$ in $\mathcal{P}(X)$.

(i) $\sigma(\Sigma \cup \mathcal{I}) := \Sigma_\mathcal{I} = \{E \Delta P : E \in \Sigma \wedge P \in \mathcal{I}\}$ for every measurable space $(X, \Sigma)$.

(ii) Given a probability space $(X, \Sigma, \mu)$, there exists a probability measure $M$ on $\Sigma_\mathcal{I}$ extending $\mu$ such that $\mathcal{I} = (\Sigma_\mathcal{I}, \mu_\mathcal{I})_0$ if and only if $\mathcal{I} \cap \Sigma = (\Sigma, \mu)_0$. In this case $(X, \Sigma_\mathcal{I}, \mu_\mathcal{I})$ is complete, and the measure algebras $\Sigma/\mu$ and $\Sigma_\mathcal{I}/\mu_\mathcal{I}$ are Boolean isomorphic.

For a proof see [16], Ia, and II and [8], II, Aufgabe 6.2.

If $(X, \Sigma, \mu)$ is a probability space and $\mathcal{G}$ a topology over $X$, the quadruple $(X, \mathcal{G}, \Sigma, \mu)$ is a topological probability space provided $\mathcal{G} \subseteq \Sigma$. We denote by $\mathcal{B}(\mathcal{G})$ the completion of $\mathcal{B}(\mathcal{G})$ under the measure $\mu|\mathcal{B}(\mathcal{G})$, and we apply the notions of $\tau$-additive probability measure $\mu$, support of $\mu$ (written $\text{supp}_\tau(\mu)$ or simply $\text{supp}(\mu)$ if the topology is obvious from the context), and Radon probability space in the sense of [14]. We call a topological probability space complete if its underlying probability space $(X, \Sigma, \mu)$ is complete.

**Definitions 1.2.** For measurable space $(X, \Sigma)$ we consider for maps $\delta \in \mathcal{P}(X)^\Sigma$ satisfying for $A, B, A_n \in \Sigma$ and $n \in \mathbb{N}$ the following:
\((N)\) \(\delta(\emptyset) = \emptyset\), \((E)\) \(\delta(X) = X\), \((M)\) \(A \subseteq B\) implies \(\delta(A) \subseteq \xi(B)\),
\((\vartheta)\) \(\delta(A) \cap \delta(B) = \delta(A \cap B)\), \((C)\) \(\delta(A^c) = [\delta(A)]^c\),
\((F)\) \(\delta(A) \cap \delta(B) \subseteq \delta(A \cap B)\), \((O)\) \(\delta(A) \cap \delta(B) = \emptyset\) if \(A \cap B = \emptyset\),
\((V)\) \(\delta(A^c) \cap \delta(A) = \emptyset\), \((\Pi)\) \(\bigcap_{1 \leq i \leq n} \delta(A_i) = \emptyset\) if \(\bigcap_{1 \leq i \leq n} A_i = \emptyset\).

For a given measure space \((X, \Sigma, \mu)\) we consider the following conditions.
\((L0)\) \(\delta \in \Sigma^\Sigma\) and \(\delta(A) \subseteq \mu A\). \((L1)\) \(\delta(A) = \mu A\).
\((L2)\) \(\delta(A) = \delta(B)\) if \(A = \mu B\).

We call a map \(\delta \in \Sigma^\Sigma\) satisfying \((L1), (L2), (N)\) and \((E)\) a primitive lifting for \(\mu\) and denote by \(P(\mu)\) the class of all primitive liftings and write \(Z(\mu)\) for the set of all \(\delta \in P(\mu)\) satisfying \((Z)\) for \(Z = M, C, V, O, \Pi, F, \vartheta\) and \(\Lambda(\mu) := \vartheta(\mu) \cap C(\mu)\). \(\delta \in \vartheta(\mu)\) if \(\mu, \Pi, M(\mu), O(\mu)\) and \(\Lambda(\mu)\) is usually called a (lower) density, monotone lifting, orthogonal lifting and (Boolean) lifting with respect to \(\mu\), respectively. \(M^*(\mu)\) and \(\vartheta^*(\mu)\) is the set of all \(\delta \in \Sigma^\Sigma\) satisfying \((L0), (L2), (N), (E)\), as well as \((M)\) and \((\vartheta)\), respectively. The elements of \(M^*(\mu)\) and \(\vartheta^*(\mu)\) are called monotone semi-liftings and semi-densities, respectively.

Given a measurable space \((X, \Sigma)\) and a topology \(\mathfrak{S}\) over \(X\) any \(\delta \in P(X)^\Sigma\) is called \((\mathfrak{S}, )\) strong if \(G \subseteq \delta(G)\) for all \(G \in \mathfrak{S}\). We denote by \(Z_{\mathfrak{S}}(\mu)\) and \(Z_{\mathfrak{S}}(\mu)\) the set of all strong \(\delta \in Z(\mu)\) for \(Z = M, C, V, O, \Pi, F, \vartheta, \Lambda\).

If \((X, \Sigma)\) is a measurable space, \(\delta \in P(X)^\Sigma\) we define its adjoint \(\delta^c \in P(X)^\Sigma\) by means of \(\delta^c(A) := [\delta(A^c)]^c\) and its upper hull \(\delta^m \in P(X)^\Sigma\) by means of \(\delta^m(A) := \bigcup_{A \supseteq B \in \Sigma} \delta(B)\) for \(A \in \Sigma\) with basic properties
\(\delta \leq \delta^c\) if and only if \(\delta\) satisfies \((V)\) and
\(\delta^m = \min\{\xi \in P(X)^\Sigma : \delta \leq \xi\} \) and \(\xi\) satisfies \((M)\},\) respectively.

For measurable space \((X, \Sigma, \mu)\) and \((X, Y, \Upsilon)\) with \(\Sigma \otimes T \subseteq Y\) if \(\gamma \in P(X, Y)^T\), \(\varphi \in \gamma \otimes \delta\), we call \(\varphi\) a product of \(\gamma\) and \(\delta\), written \(\varphi \in \gamma \otimes \delta\), \(\varphi(A \times B) = \delta(A) \times \varphi(B)\) for all \(A \in \Sigma\) and \(B \in T\).

Given a complete probability space \((X, \Sigma, \mu)\), one can associate with every \(\delta \in \vartheta^*(\mu)\) two topologies \(t_\delta\) and \(\tau_\delta\), where \(t_\delta\) is the topology with basis \(\{\delta(A) : A \in \Sigma\}\) and \(\tau_\delta := \{A \in \Sigma : A \subseteq \delta(A)\}\). \(\tau_\delta \subseteq \Sigma\) by Proposition 3.2 and \(t_\delta \subseteq \tau_\delta\) if and only if \(\delta \leq \delta \circ \delta\) (in particular, if \(\delta \in \vartheta(\mu)\)). \(t_\delta\) and \(\tau_\delta\) are Hausdorff, if \(\delta(\Sigma)\) separates the points of \(X\). \((X, \tau_\delta, \Sigma, \mu)\) is a topological probability space with \(\tau\)-additive measure and \(\text{supp}(\mu) = X\), \(\mathfrak{B}(t_\delta) = \Sigma = \mathfrak{B}(\tau_\delta)\), and \(\delta \in \vartheta(\mu)\) for given \(\delta \in \vartheta(\mu)\) (compare \([30], Theorem 4.1)\).

For an arbitrary probability space \(X, \Sigma, \mu\) we call the compact Radon probability space \((Y, \mathcal{F}, T, \nu)\) its associated hyperstonian space, where \((Y, \mathcal{F})\) is the Stone space of its measure algebra \((\Sigma/\mu, \mathfrak{P})\) in the sense of \([13], 321K\). For every \(A \in T\) there exists a unique open-closed set \(\sigma(A) \in A^\ast\) and \(\Lambda_{\mathcal{F}}(\nu) = \{\sigma\} ; \sigma\) is called the canonical strong lifting of the hyperstonian space. According to \([13], 311I, \mathfrak{F}\) is a Hausdorff zero-dimensional topology (cf. e.g \([13], 3A3Ad, for the definition). Since \(\sigma[T]\) is a basis of the topology \(\mathfrak{F}\) we get \(\mathfrak{F} = t_\sigma\).
Definition 1.3. For $\mathcal{X} \subseteq \mathcal{P}(X)$, $\mathcal{Y} \subseteq \mathcal{P}(Y)$, and $\mathcal{H} \subseteq \mathbb{R}^Y$ we define the skew products

$$\mathcal{X} \times \mathcal{Y} := \{ E \subseteq X \times Y : \exists N_E \in \mathcal{X} \forall x \notin N_E (E_x \in \mathcal{Y}) \},$$

$$\mathcal{X} \times \mathcal{Y} := \{ E \subseteq X \times Y : \exists M_E \in \mathcal{Y} \forall y \notin M_E (E^y \in \mathcal{X}) \},$$

and

$$\mathcal{X} \times \mathcal{H} := \{ f \subseteq \mathbb{R}^{X \times Y} : \exists N_f \in \mathcal{X} \forall x \notin N_f (f_x \in \mathcal{H}) \},$$

where $E_x := \{ y \in Y : (x, y) \in E \}$ and $E^y := \{ x \in X : (x, y) \in E \}$ for $E \subseteq X \times Y$, $f_x(y) := f(x, y)$ and $f^y(x) := f(x, y)$ for $f \in \mathbb{R}^{X \times Y}$, $x \in X$, and $y \in Y$.

Definitions and notations 1.4. Let be given complete probability spaces $(X, \Sigma, \mu)$ and $(Y, \nu, \mu)$. 

(a) We define the $\sigma$-ideal $\mathfrak{N} := \Sigma_0 \times T_0$ of all right nil null sets, the $\sigma$-ideal $\mathfrak{N} := \Sigma_0 \times T_0$ of all left nil null sets in $\mathcal{P}(X \times Y)$, and $\mathfrak{N}_2 := \mathfrak{N} \cap \mathfrak{N} \cap \mathfrak{N}$ of all (two-sided) nil null sets in $\mathcal{P}(X \times Y)$. Note

$$\mathfrak{N} = \{ E \subseteq X \times Y : \{ x : E_x \notin T_0 \} \in \Sigma_0 \} \text{ for complete } (X, \Sigma, \mu)$$

and

$$\mathfrak{N} = \{ E \subseteq X \times Y : \{ y : E^y \notin \Sigma_0 \} \in T_0 \} \text{ for complete } (Y, \nu, \mu).$$

(b) If $\mu \times T = \{ E \subseteq X \times Y : \exists N_E \in \Sigma_0 \forall x \notin N_E (E_x \in T) \}$ and $[x \in N_E \mapsto \nu(E_x)] \in L^\infty(\Sigma \cap N_E) \} \} \} \}$ define the right skew product of $\mu$ and $\nu$ by $(\mu \times \nu)(E) := \int_{N_E} \nu(E_x)\mu(dx)$ for every $E \subseteq \mu \times T$. We get a Dynkin class $\mu \times T$ (compare [8] Kapitel I, 6.4 for definition), and $\mathfrak{N} = \{ E \subseteq \mu \times T : (\mu \times \nu)(E) = 0 \}$. There is a similar definition for a left Dynkin class $\Sigma \times \nu$, left skew product of $\mu \times \nu$, and $\mathfrak{N} = \{ E \subseteq \Sigma \times \nu : (\mu \times \nu)(E) = 0 \}$. (c) We write $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$ for the 'ordinary' (also 'usual') product probability space of the probability spaces $(X, \Sigma, \mu)$ and $(Y, \nu, \nu)$ and $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$ for its completion and for $\mathcal{I} = \mathfrak{N}_2, \mathfrak{N}$ we get $\mathcal{I} \cap \mathfrak{N} \cup \mathfrak{N}$ and Lemma [1] implies that the probability measure $\mu \otimes \nu := (\mu \otimes \nu)_{\mathcal{I}}$ is the unique extension of $\mu \otimes \nu$ onto $\Sigma \otimes T := (\Sigma \otimes T)_{\mathcal{I}}$, where

$$\Sigma \otimes T \subseteq \Sigma \otimes T \subseteq \Sigma \otimes \mathfrak{N}_2 \subseteq \Sigma \otimes \mathfrak{N}_2 \subseteq \mu \times T,$$

and

$$(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$$

is a complete probability space for $\mathcal{I} = \mathfrak{N}_2, \mathfrak{N}$, and the measure algebras

$$\Sigma \otimes \mathfrak{N}_2 \otimes \mu \otimes \nu, \Sigma \otimes \mathfrak{N}_2 \otimes \mu \otimes \nu, \Sigma \otimes T / \mu \otimes \nu$$

are Boolean isomorphic. It follows, that the hyperstonian spaces of the probability spaces $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$, $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$, and $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$ for $\mathcal{I} = \mathfrak{N}_2, \mathfrak{N}$ coincide, since the hyperstonian space of a probability space $(X, \Sigma_2, \mu)$ depends only on its measure algebra $(\Sigma_2 / \mu_2, \mu_2)$.

(d) Given probability spaces $(X, \Sigma, \mu)$, $(Y, \nu, \nu)$, and $(X \times Y, \nu, \nu)$ the following conditions will be of constant use.
Examples 1.6. In general map $f$ the continuum hypothesis R.M. Dudley exhibits a non-in $\mathbb{R}$ by Haupt and Pauc. Therefore, in [6] and [7] under assumption of Bledsoe Morse nilsets coincide with the Nilmengen Satz 1, due to the equivalence of (1) and (3). In probability spaces these nilsets in the sense of Bledsoe Morse coincides with to the notion of integral over probability spaces as usual today, the system of nilsets of nilsets in the sense of Bledsoe Morse coincides with the notion of integral in [17] and its relation implying Bledsoe and Morse define nilsets $E$ one $\text{supp } E$. By e.g. [8], Chapter V, Section 2, Beispiel 2.3 (b) (tracing back to (a) and consider $E$ integrals $b$)

Remark 252K. For another example showing that for Radon probability spaces $(X, \mathcal{G}, \mathcal{B}(\mathcal{G}), \mu)$ and $(Y, \mathcal{I}, \mathcal{B}(\mathcal{I}), \nu)$ the completion $(X \times Y, \mathcal{G} \times \mathcal{I}, \mathcal{B}((\mathcal{G} \times \mathcal{I})), \beta(\mathcal{G} \times \mathcal{I}))$ is their Radon product with Radon product measure $\mu \otimes_R \nu := \beta(\mathcal{G} \times \mathcal{I})$, see [26], Theorem 1.

Definition 1.8. By [26], Section 1, Theorem 1 for $\tau$-additive topological probability spaces $(X, \mathcal{G}, \mathcal{B}(\mathcal{G}), \mu)$ and $(Y, \mathcal{I}, \mathcal{B}(\mathcal{I}), \nu)$ there exists exactly one $\tau$-additive extension $\beta(\mathcal{G} \times \mathcal{I})$ of the product measure $\mu \otimes \nu$ to the Borel $\Sigma$-algebra $\mathcal{B}(\mathcal{G} \times \mathcal{I})$ called the $\tau$-additive product measure on $\mathcal{B}(\mathcal{G} \times \mathcal{I})$, such that $(X \times Y, \mathcal{G} \times \mathcal{I}, \beta(\mathcal{G} \times \mathcal{I}))$ satisfies $[C]$ and $[\overline{C}]$, where $\mathcal{B}(\mathcal{G}) \otimes \mathcal{B}(\mathcal{I}) \subseteq \mathcal{B}(\mathcal{G} \times \mathcal{I}) \subseteq \mu \times \mathcal{B}(\mathcal{I}) \cap \mathcal{B}(\mathcal{G}) \times \nu$.

Remark 1.7. $[P_0]$ and $[C]$ implies $[P]$. $[P]$ implies $(\tau_\gamma \times \tau_\delta) \cap \mathcal{Y}_0 = \{\emptyset\}$, i.e. $\text{supp}_{\tau_\gamma \times \tau_\delta}(\nu) = X \times Y$ for complete probability spaces $(X, \mathcal{G}, \mu)$ and $(Y, T, \nu)$ with $\nu \in (\mu)$ and $\delta \in (\nu)$.

Definition 1.8. By [26], Section 1, Theorem 1 for $\tau$-additive topological probability spaces $(X, \mathcal{G}, \mathcal{B}(\mathcal{G}), \mu)$ and $(Y, \mathcal{I}, \mathcal{B}(\mathcal{I}), \nu)$ there exists exactly one $\tau$-additive extension $\beta(\mathcal{G} \times \mathcal{I})$ of the product measure $\mu \otimes \nu$ to the Borel $\Sigma$-algebra $\mathcal{B}(\mathcal{G} \times \mathcal{I})$ called the $\tau$-additive product measure on $\mathcal{B}(\mathcal{G} \times \mathcal{I})$, such that $(X \times Y, \mathcal{G} \times \mathcal{I}, \beta(\mathcal{G} \times \mathcal{I}))$ satisfies $[C]$ and $[\overline{C}]$, where $\mathcal{B}(\mathcal{G}) \otimes \mathcal{B}(\mathcal{I}) \subseteq \mathcal{B}(\mathcal{G} \times \mathcal{I}) \subseteq \mu \times \mathcal{B}(\mathcal{I}) \cap \mathcal{B}(\mathcal{G}) \times \nu$.

For Radon probability spaces $(X, \mathcal{G}, \mathcal{B}(\mathcal{G}), \mu)$ and $(Y, \mathcal{I}, \mathcal{B}(\mathcal{I}), \nu)$ the completion $(X \times Y, \mathcal{G} \times \mathcal{I}, \mathcal{B}(\mathcal{G} \times \mathcal{I}), \beta(\mathcal{G} \times \mathcal{I}))$ is their Radon product with Radon product measure $\mu \otimes_R \nu := \beta(\mathcal{G} \times \mathcal{I})$, see [26], Theorem 1.
Definition 1.9. For given topological probability space \((X, \mathcal{S}, \Sigma, \mu)\) by
\[
\theta_{\mathcal{S}, \mu}(A) := \bigcup \{ G \in \mathcal{S} : G \subseteq \mu A \} \text{ for all } A \in \Sigma
\]
we define a map \(\theta := \theta_{\mathcal{S}, \mu} \in \mathcal{P}(X)^\Sigma\) called the \textbf{theta operator}.

Lemma 1.10. Let \(b\) be a basis for the topology.
(i) \(\theta(A) = \bigcup \{ G \in b : G \subseteq \mu A \} \text{ for all } A \in \Sigma\).
(ii) \(\theta\) satisfies \((E)\) and \((\vartheta)\), \(\theta\) satisfies \((N)\) if and only if \(\Sigma_0 \cap \mathcal{S} = \{\emptyset\}\), i.e. \(\text{supp}(\mu) = X\), and
\[
\theta = \min \{ \zeta \in \mathcal{P}(X)^\Sigma : \zeta \text{ satisfies (L2)}, (M), \text{ and it is strong} \}
\]
\[
= \min \{ \zeta \in \mathcal{P}(X)^\Sigma : \zeta \text{ satisfies (L2)}, (\vartheta), \text{ and it is strong} \}.
\]
(iii) For complete \((X, \Sigma, \mu)\) with \(\text{supp}(\mu) = X\) the following statements are all equivalent.
(a) \(\theta\) satisfies \((L0)\),
(b) \(\theta \in \vartheta^*_\mathcal{S}(\mu)\),
(c) \(M^*_\Sigma \neq \emptyset\),
(d) \(\vartheta^*_\mathcal{S}(\mu) \neq \emptyset\),
(e) \(\mu\) is \(\tau\)-additive.
In this case \(\theta_{\mathcal{S}, \mu} = \min M^*_\mathcal{S}(\mu) = \min \vartheta^*_\mathcal{S}(\mu)\).
(iv) Given a topological probability space \((X, \mathcal{S}, \Sigma, \mu)\), \(\text{supp}(\mu) = X\) and \(\mu\) \(\tau\)-additive are necessary conditions for the existence of a strong monotone lifting for \(\mu\).

Statements (i) and (ii) are immediate by the definition of \(\theta\) and we get (iii) and (iv) by [22], Theorem 4.4 and Proposition 3.11, respectively.

2. \(\tau\)-Products of Probability Spaces Under Density Topologies

Throughout what follows let be given complete probability spaces \((X, \Sigma, \mu)\) and \((Y, T, \nu)\) together with \(\gamma \in \vartheta(\mu)\) and \(\delta \in \vartheta(\nu)\) and write for short \(\mathcal{B}_\gamma := \mathcal{B}(\tau_\gamma) = \Sigma, \mathcal{B}_\delta := \mathcal{B}(\tau_\delta) = T, \mathcal{B}_{\gamma, \delta} := \mathcal{B}(\tau_\gamma \times \tau_\delta), \beta_{\gamma, \delta} := \beta(\tau_\gamma \times \tau_\delta), \) where the equalities \(\mathcal{B}(\tau_\gamma) = \Sigma\) and \(\mathcal{B}(\tau_\delta) = T\) follow by [30], Theorem 4.1, (ix), and \(\mathcal{B}_{\gamma, \delta}\) for the completion of \(\mathcal{B}_{\gamma, \delta}\) under \(\beta_{\gamma, \delta}\).

Lemma 2.1.
(i) \(\mathcal{B}(\tau_\gamma) \otimes \mathcal{B}(\tau_\delta) \subseteq \Sigma \otimes T = \mathcal{B}_\gamma \otimes \mathcal{B}_\delta \subseteq \mathcal{B}_{\gamma, \delta} \subseteq \Sigma \otimes \mathcal{S}_\Sigma T = (\mathcal{B}_\gamma)_{\mathcal{S}_\Sigma} \subseteq \Sigma \otimes \mathcal{S}_\Sigma T = (\mathcal{B}_\gamma)_{\mathcal{S}_\Sigma} \subseteq \mathcal{B}_{\gamma, \delta} \subseteq \Sigma \otimes \mathcal{T}_\Sigma T = (\mathcal{B}_\gamma)_{\mathcal{T}_\Sigma} \subseteq \Sigma \otimes \mathcal{T}_\Sigma T = (\mathcal{B}_\gamma)_{\mathcal{T}_\Sigma}\), and \((\mathcal{B}_\gamma)_{\mathcal{T}_\Sigma} = \beta_{\gamma, \delta}\) for \(\mathcal{I} = \mathcal{N}_2, \mathcal{N}_\mathcal{R}\).
(ii) \((X \times Y, \mathcal{B}_{\gamma, \delta}, \beta_{\gamma, \delta})\) and \((X \times Y, \mathcal{B}_{\gamma, \delta}, \beta_{\gamma, \delta})\) are probability spaces satisfying \([P_0]\), \([P]\), \([C]\), \([\overline{C}]\), and \([S]\).
(iii) \(\theta_{\gamma, \times \times, \delta, \beta_{\gamma, \delta}} \in \vartheta_{\gamma, \times \times, \delta, \beta_{\gamma, \delta}}\).
(iv) If \((X \times Y, \mathcal{V}, \nu)\) is a complete probability space satisfying \([P_0]\) and there exists a \(\zeta \in \vartheta^*(\nu) \cap \gamma \otimes \delta\), it follows \(\mathcal{B}_{\gamma, \delta} \subseteq \mathcal{V}\) (equivalently \(\tau_\gamma \times \tau_\delta \subseteq \mathcal{V}\)). If, in addition, \([C]\) holds for \((X \times Y, \mathcal{V}, \nu)\), it follows \(\nu|_{\mathcal{B}_{\gamma, \delta}} = \beta_{\gamma, \delta}\).
(v) For topological probability spaces \((X, \mathcal{S}, \Sigma, \mu)\) \((Y, \mathcal{I}, T, \nu)\) with \(\tau\)-additive measures \(\mu\) and \(\nu\), \(\text{supp}(\mu) = X\) and \(\text{supp}(\nu) = Y\), \(\gamma \in \vartheta_{\mathcal{S}}(\mu)\), and \(\delta \in \vartheta_{\mathcal{T}}(\nu)\), it follows \(\mathcal{S} \subseteq \tau_\gamma, \mathcal{I} \subseteq \tau_\delta, \) and \(\mathcal{B}(\mathcal{S}) \otimes
\[ \mathcal{B}(\tau) \subset \mathcal{B}(G \times \tau) \subset \mathcal{B}_{\gamma,\delta} \subset \mathcal{B}_{\gamma,\delta} \subset \Sigma \otimes \mathfrak{M}_2 \subset \Sigma \otimes \mathfrak{M}_1 \] for \( \mu \otimes \tau \nu \| \mathcal{B}(G \times \tau) = \beta(G \times \tau) \) for \( \tau = \mathfrak{M}_2, \mathfrak{M}_1 \).

(vi) \( \mathcal{B}(t_\gamma \times t_\delta) = \mathcal{B}_{\gamma,\delta} \) (completion with respect to the probability measure \( \beta_{\gamma,\delta} \)).

(vii) \( \langle Y_1 \times Y_2, t_{\sigma_1} \times t_{\sigma_2}, \mathcal{B}(t_{\sigma_1} \times t_{\sigma_2}), \tilde{\beta}_{\tau_{\sigma_1},\tau_{\sigma_2}} \rangle \) is the Radon product of the hyperstonian spaces \( \langle Y_j, \Sigma_j, T_j, \nu_j \rangle \) with canonically strong lifting \( \sigma_j \), \( j = 1, 2 \), satisfying \([P_0], [C], [S]\) and \( \mathcal{B}_{\sigma_1,\sigma_2} \subset \mathcal{B}(t_{\sigma_1} \times t_{\sigma_2}) \).

**Proof.** Ad (i) : The first equality in (i) and the first three inclusions are clear. For the fourth \( \tau_\gamma \times \tau_\delta \subset \Sigma \otimes \mathfrak{M}_1 \) will be sufficient. For this reason, write \( G := \bigcup_{i \in I} A_i \times B_i \) for \( A_i \subset \tau_\gamma \) and \( B_i \subset \tau_\delta \) if \( i \in I \) for \( G \in \tau_\gamma \times \tau_\delta \). If \( I \) is the set of all finite subsets of \( I \) it follows

\[ G = \bigcup_{F \in [I]} G_F \quad \text{for} \quad G_F := \bigcup_{i \in F} A_i \times B_i \in \Sigma \otimes T \quad \text{for} \quad F \in [I] \]

with directed upwards family \( \langle G_F \rangle_{F \in [I]} \). Since \( \beta := \beta_{\gamma,\delta} \) is \( \tau \)-additive, it follows \( \beta(G) = \sup_{F \in [I]} \beta(G_F) \) and we can choose \( F_n \in [I] \) with \( F_n \supseteq F_{n+1} \) for \( n \in \mathbb{N} \) and \( \beta(G) = \sup_{n \in \mathbb{N}} \beta(G_{F_n}) \), implying \( G = \bigcup_{n \in \mathbb{N}} G_{F_n} =: H \in \Sigma \otimes T \) and with \( N := G \setminus H \in \mathcal{B}_{\gamma,\delta} \) it follows

\[ 0 = \beta(N) = \int_{N_G} \nu(N_x) \mu(dx) = \int_{M_G} \mu(N_y) \nu(dy), \]

implying \( N \in \mathfrak{M}_2 \). For every \( G \in \tau_\gamma \times \tau_\delta \), there exists a \( H \in \Sigma \otimes T \) with \( G \Delta H \in \mathfrak{M}_2 \). This implies \( \tau_\gamma \times \tau_\delta \subset \Sigma \otimes \mathfrak{M}_2 \), consequently \( \mathcal{B}_{\gamma,\delta} \subset \Sigma \otimes \mathfrak{M}_2 \).

Q \( \subset N \in \mathcal{B}_{\gamma,\delta} \) and \( \beta(N) = 0 \) implies \( Q \in \mathfrak{M}_2 \) and we get \( \mathcal{B}_{\gamma,\delta} \subset \Sigma \otimes \mathfrak{M}_2 \).

For \( I = \mathfrak{M}_2, \mathfrak{M}_1 \) we get from \( \mathcal{B}_{\gamma,\delta} \subset \Sigma \otimes \mathfrak{M}_1 \) first \( \langle \mathcal{B}_{\gamma,\delta} \rangle_I \subset \Sigma \otimes \mathfrak{M}_1 \) and from \( \Sigma \otimes \mathfrak{M}_1 \subset \mathcal{B}_{\gamma,\delta} \) the converse inclusion \( \Sigma \otimes \mathfrak{M}_1 \subset \langle \mathcal{B}_{\gamma,\delta} \rangle_I \).

(ii) is obvious and (iii) follows by Lemma 3.6. It is sufficient to show \( \tau_\gamma \times \tau_\delta \subset \Sigma \).

Ad (iv) : First note, that \( \tau_\gamma \) is a topology with \( \tau_\gamma \in \Sigma \) by [22], Proposition 3.6. It is sufficient to show \( \tau_\gamma \times \tau_\delta \subset \Sigma \).

Ad (v) : Clearly \( \mathcal{B}(t_\gamma \times t_\delta) \subset \mathcal{B}_{\gamma,\delta} \). For the converse inclusion note, that \( G \in \tau_\gamma \times \tau_\delta \) can be written \( G = \bigcup_{i \in I} A_i \times B_i \) with \( A_i \subset \tau_\gamma \), \( B_i \subset \tau_\delta \) for every \( i \in I \). This implies \( \tau_\gamma \times \tau_\delta \subset \Sigma \otimes \mathfrak{M}_2 \), consequently \( \mathcal{B}_{\gamma,\delta} \subset \Sigma \otimes \mathfrak{M}_2 \).

\[ G := \bigcup_{i \in I} (G_i \times H_i) \in \tau_\gamma \times \tau_\delta, \quad \text{i.e.} \quad G 

\text{for} \quad F \text{ is the system of all finite subsets of} \ I \ \text{we get} \ G := \bigcup_{E \in F} G_E \text{ for} \ G_E := \bigcup_{i \in E} G_i \times H_i, \ E \in F \text{ and by the} \ \tau \text{-additivity of} \ \beta := \beta_{\gamma,\delta} \ \text{it follows} \ \beta(G) = \sup_{E \in F} \beta(G_E) = \sup_{n \in \mathbb{N}} \beta(G_{E_n}) \text{ for some} \ E_n \subset F \ \text{with} \ E_n \subseteq E_{n+1}, \ n \in \mathbb{N}. \ \text{For the countable subset} \ P \ \text{of} \ I \ \text{we get} \ G = G := \bigcup_{i \in P} (G_i \times H_i) \in \tau_\gamma \times \tau_\delta, \ \text{i.e.} \ G \in \mathcal{B}(t_\gamma \times t_\delta), \ \text{and so} \ \mathcal{B}_{\gamma,\delta} \subset \mathcal{B}(t_\gamma \times t_\delta), \ \text{implying} \ \mathcal{B}_{\gamma,\delta} \subset \mathcal{B}(t_\gamma \times t_\delta). \]
Ad (vii): Since $\mathfrak{T}_1 = t_{\sigma_1}$, we get $T_j = \mathfrak{B}_c(t_{\sigma_j}) = \widehat{\mathfrak{B}}(t_{\sigma_j}) = \mathfrak{B}(\tau_{\sigma_j})$ for $j = 1, 2$. It follows that

$$(Y_1 \times Y_2, \mathfrak{T}_1 \times \mathfrak{T}_2, T_1 \otimes_R T_2, \nu_1 \otimes_R \nu_2) = (Y_1 \times Y_2, t_{\sigma_1} \times t_{\sigma_2}, \widehat{\mathfrak{B}}(t_{\sigma_1} \times t_{\sigma_2}), \widehat{\mathfrak{B}}_c(t_{\sigma_1} \times t_{\sigma_2}))$$

is the Radon product of the hyperstonian spaces $(Y_j, \mathfrak{T}_j, T_j, \nu_j)$, $j = 1, 2$, satisfying $[P_0], [C]$, and $[S]$ and $\mathfrak{B}_{\sigma_1, \sigma_2} \subseteq \widehat{\mathfrak{B}}(t_{\sigma_1} \times t_{\sigma_2})$ since $\widehat{\mathfrak{B}}(t_{\sigma_1} \times t_{\sigma_2}) = \mathfrak{B}_{\sigma_1, \sigma_2}$ by (vi).

\begin{proof}

By Remark 3.1 and Proposition 3.1 from [25].
\end{proof}

**Example 2.2.** [9] exhibits a hyperstonian space $(X, \mathfrak{A}, \Sigma, \mu)$ with canonical strong lifting $\sigma$ derived from a diffuse probability space such that $\mathfrak{A} \times \mathfrak{A} \not\subseteq \Sigma \otimes \Sigma$. Since $\mathfrak{A} = t_\sigma$, the topology with basis $\{\sigma(A) : A \in \Sigma\}$ and $t_\sigma \subseteq \tau_\sigma$, this implies $\tau_\sigma \times \tau_\sigma \not\subseteq \Sigma \otimes \Sigma$ and with [30], Theorem 4.1 together with Lemma 3.1, we get

$$\mathfrak{B}(t_\sigma) \otimes \mathfrak{B}(t_\sigma) = \mathfrak{B}(\tau_\sigma) \otimes \mathfrak{B}(\tau_\sigma) = \Sigma \otimes \Sigma \subseteq \mathfrak{B}_{\tau_\sigma, \tau_\sigma} \subseteq \Sigma \otimes_{\mathfrak{g}_2} \Sigma,$$

implying $\mathfrak{A} \times \mathfrak{A} \subseteq \Sigma \otimes_{\mathfrak{g}_2} \Sigma$.

3. Marginals

The following definition extends Definition 3.1 from [25].

**Definition 3.1.** Let be given a measurable space $(X \times Y, \Upsilon)$.

(a) Given a measurable space $(Y, T)$ for $\delta \in \mathcal{P}(Y)^T$ define $\delta_\bullet = \delta_\bullet(T, \Upsilon)$ by

$$\delta_\bullet(E) := \{(x, y) \in X \times Y : E_x \in T \land y \in \delta(E_x)\} \text{ for every } E \in \Upsilon.$$

(b) Given a measurable space $(X, \Sigma)$ for $\gamma \in \mathcal{P}(X)^\Sigma$ define

$$\gamma^\bullet(E) := \{(x, y) \in X \times Y : E^y \in \Sigma \land x \in \gamma(E^y)\} \text{ for every } E \in \Upsilon.$$

Below we will discuss only $\delta_\bullet$ and $\xi_\bullet$, since the corresponding results for $\gamma^\bullet$ and $\eta^\bullet$ are easily derived from those of $\delta_\bullet$ and $\xi_\bullet$ by interchanging the roles of the factor spaces, respectively. **Without special comment we will mark a statement for $\gamma^\bullet$ by putting the symbol $\perp$ after the number of the corresponding result for $\delta_\bullet$.** The next two results extent Remark 3.1 and Proposition 3.1 from [25].

**Remark 3.2.** Let be given measurable spaces $(Y, T)$ and $(X \times Y, \Upsilon)$, and $\delta \in T^T$.

(a) $E_x \in T$ implies $[\delta_\bullet(E)]_x = \delta(E_x) \text{ otherwise } [\delta_\bullet(E)]_x = \emptyset$ for every $E \in \Upsilon$ and $x \in X$.

(b) $\delta_\bullet$ satisfies the condition (Z), provided $\delta$ does for $Z = N, E, V, O, \Pi, F$. If $\delta$ satisfies (F), then $(\delta_\bullet)^m$ satisfies (O). $\delta \in T^T$ implies $[\delta_\bullet(E)]_x \in T$ for every $E \in \Upsilon$ and $x \in X$.

(c) If $A \times B \in \Upsilon$ for $A \in \Sigma$ and $B \in T$ we get $\delta_\bullet(A \times B) = A \times \delta(B)$.

(d) Suppose that $(X \times Y, \Upsilon)$ has the property $[S_0]$. Then $\delta_\bullet$ satisfies (M), (O), and (C), provided $\delta$ has these properties, respectively.

(e) If $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(X \times Y, \Upsilon, \nu)$ are probability spaces such that $(X, \Sigma, \mu)$ is complete and $[C]$ is satisfied, then for $E, F \in \Upsilon$ with $E =_\nu F$ it follows for all $y \in Y$ the equality $[\delta_\bullet(E)]^y =_\mu [\delta_\bullet(F)]^y$. 
(f) If $\mathcal{G}$ is a topology over $X$ and $\mathcal{T} \subseteq T$ a topology over $Y$ such that $\mathcal{G} \times \mathcal{T} \subseteq \mathcal{Y}$, $\mathcal{T} \subseteq T$ and $\delta$ is $\mathcal{T}$-strong and satisfies the conditions $(N)$ and $(M)$, it follows that $\delta_*$ is $\mathcal{G} \times \mathcal{T}$-strong.

**Proof.** $E \in \mathcal{G} \times \mathcal{T}$ can be written as $E = \bigcup_{i \in I} G_i \times H_i$ with $G_i \in \mathcal{G}$, $H_i \in \mathcal{T}$ for $i \in I$. For every $x \in X$ we get $E_x = \bigcup \{G_i : i \in I, x \in H_i\} \in \mathcal{T} \subseteq T$, implying $[\delta_*(E)]_x = \delta(E_x) \supseteq \delta(G_i) \supseteq G_i$ for $i \in I$ and $x \in H_i$, and by the latter $[\delta_*(E)]_x \supseteq E_x$ for every $x \in X$, i.e. $\delta_*$ is $\mathcal{G} \times \mathcal{T}$-strong. \hfill $\Box$

The next Lemma extends [25] Proposition 3.1 to our more general situation.

**Lemma 3.3.** For probability spaces $(X, \Sigma, \mu)$, $(Y, \mathcal{T}, \nu)$, and complete probability space $(X \times Y, \mathcal{Y}, \mu \otimes \nu)$ such that $[\mathcal{C}]$ is satisfied, we consider the following statements for $\delta_* := \delta_*(T, \mathcal{Y})$, where $\delta \in T^T$ satisfies $(L1)$, $(L2)$ and $(N)$.

1. $\delta_*(E) \in \mathcal{Y}$ for every $E \in \mathcal{Y}$;
2. $\delta_*(E)_x = E_x$ for every $E \in \mathcal{Y}$.
3. There exists a $\varphi \in \mathcal{Y}^T$ satisfying (L1), such that $\delta([\varphi(E)]_x) = [\varphi(E)]_x \in T$ for all $E \in \mathcal{Y}$ and $x \in X$.
4. There exists a $\varphi \in \mathcal{Y}^T$ satisfying (L1), such that for all $E \in \mathcal{Y}$ there exists a set $N_E \in \Sigma_0$ with $\delta([\varphi(E)]_x) = [\varphi(E)]_x \in T$ for all $x \notin N_E$.
5. There exists a $\psi \in \vartheta(\nu)$, such that $\delta([\psi(E)]_x) = [\psi(E)]_x \in T$ for every $E \in \mathcal{Y}$ and $x \in X$.
6. There exists a $\zeta$ satisfying (L1), (N), (E), (M), and (O) such that $\delta([\zeta(E)]_x) = [\zeta(E)]_x \in T$ for every $E \in \mathcal{Y}$ and $x \in X$.

Then (i) to (iv) are all equivalent and for complete $(Y, T, \nu)$, and $\delta \in \vartheta(\nu)$ they are all equivalent to (v).

If $(Y, T, \nu)$ is complete and $\delta \in M(\nu) \cap O(\nu)$, in addition, it follows that (i) to (iv) are all equivalent to (vi). The $\varphi$ appearing in (iii) can be chosen $\varphi = \delta_*$ and satisfies $(Z)$, if every $\delta$ does for $Z = N, E, V, O, \Pi, F$.

**Proof.** Note, that the implications (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (iv) trivially hold true. For (i) $\Rightarrow$ (ii) replace in the corresponding proof of [25] Proposition 3.1 ‘$\mathcal{G} \otimes T$’ by ‘$\mathcal{Y}$’ and ‘$\mu \otimes \nu$’ by ‘$\nu$’. Get (ii) $\Rightarrow$ (iii) by defining $\varphi := \delta_*$. 

Ad (iv) $\Rightarrow$ (i) : Let be given a $\varphi \in \mathcal{Y}^T$ such that for all $E \in \Theta$ there exists a set $N_E \in \Sigma_0$ with $\delta([\varphi(E)]_x) = [\varphi(E)]_x$ for all $x \notin N_E$. Since $\varphi(E) = E$ for $E \in \mathcal{Y}$, condition [C] implies the existence of a $M_E \in \Sigma_0$ with $[\varphi(E)]_x \in T$ and $[\varphi(E)]_x = E_x$ for every $x \notin M_E$. Put $P_E := N_E \cup M_E$.

Then $P_E \in \mathcal{G}_0$ and $[\varphi(E)]_x = \delta([\varphi(E)]_x) = \delta(E_x) = [\delta_*(E)]_x$ for every $x \notin P_E$, so $\varphi(E) \cap (P^c_E \times \mathcal{Y}) = \delta_*(E) \cap (P^c_E \times \mathcal{Y})$, implying $\delta_*(E)_x = E_x$ for any $E \in \mathcal{Y}$. By the completeness of $(X \times Y, \mathcal{Y}, \nu)$, the latter implies $\delta_*(E) \in \mathcal{Y}$ for any $E \in \mathcal{Y}$.

Ad (ii) $\Rightarrow$ (v) : Now let $\delta \in \vartheta(\nu)$ and put $\xi := (\delta_*)^m$. Then $\xi$ satisfies $(\vartheta)$ by Remark [23] (b), and since $\delta_*$ satisfies (N) by the same Remark, $\xi$ also satisfies (N), consequently $\xi$ satisfies (V). Therefore,

\[ E =_v \xi_*(E) \subseteq \xi(E) \subseteq \xi^c(E) \subseteq (\delta_*)^c(E) =_v E \]
by assumption \((ii)\), implying \(\xi(E) = _v E\) for every \(E \in \mathcal{Y}\) and with the completeness of \((X \times Y, \mathcal{Y}, \nu)\) also \(\xi(E) \in \mathcal{Y}\). We can choose an \(\eta \in \vartheta (v)\) to define \(\tilde{\xi}(E) := \xi(\eta(E))\) for every \(E \in \mathcal{Y}\). Clearly \(\tilde{\xi}\) satisfies \((L1), (L2), (N), (E),\) and \((\vartheta)\). By \(\xi(E) = _v \eta(E) = _v E, \tilde{\xi}\) satisfies \((L1),\) i.e. \(\tilde{\xi} \in \vartheta (v)\).

For \(E \in \mathcal{Y}\) and \(x \in X\) we get

\[
[\tilde{\xi}(E)]_x = [\xi(\eta(E))]_x = ([\delta_\bullet^m(\eta(E))]_x = \bigcup \{\delta(A)_x : A \in \mathcal{Y}, A \subseteq \eta(E)\}
\]

where \(\bigcup \{\delta(A)_x : A \in \mathcal{Y}, A \subseteq \eta(E), A_x \in T\} \in T\) follows by e.g. [30], Theorem 4.1, and at this point completeness of \((Y, T, \nu)\) is required, (note that \([\delta_\bullet(A)]_x = \delta(A_x)\) if \(A_x \in T\) and \(= \emptyset\) otherwise) and \([\tilde{\xi}(E)]_x \subseteq \delta(\tilde{\xi}(E)_x)\), compare [30] Theorem 3.9.

Define \(\psi\) by \([\psi(E)]_x := \delta(\tilde{\xi}(E)_x)\) for every \(E \in \mathcal{Y}\) and \(x \in X\) satisfying clearly \((N), (E), (\vartheta)\) and \((L2), (\vartheta)\) for \(\psi\) implies \(V\) for \(\psi\), i.e. \(\psi \leq \psi^c\) and we get for \(E \in \mathcal{Y}\) that \(E = _v \xi(E) \subseteq \psi(E) \subseteq \psi^c(E) \subseteq \xi^c(E) = _v E\). By the completeness of \((X \times Y, \mathcal{Y}, \nu)\) the latter implies \(\psi(E) \in \mathcal{Y}\) and \((L1)\) \(\psi(E) = _v E\) for every \(E \in \mathcal{Y}\), i.e. \(\psi \in \vartheta (v)\), satisfying \(\delta([\psi(E)]_x) = [\psi(E)]_x\) for every \(E \in \mathcal{Y}\) and \(x \in X\).

Ad \((vi) \implies (iii)\) is obvious. For the converse implication let \(\delta \in M(\mu) \cap O(\mu)\) in addition and by \((iii)\) choose \(\varphi = \delta_\bullet \in \mathcal{Y}\) satisfying \((L1), (N), (E)\) such that \(\delta ([\varphi(E)]_x) = [\varphi(E)]_x\) for all \(E \in \mathcal{Y}\) and \(x \in X\) for \(\varphi\) satisfying \((O)\). Then \(\varphi^m\) satisfies \((N), (E),\) and \((M)\). For \(E, F, G, H \in \mathcal{Y}\) with \(E \cap F = \emptyset, G \subseteq E,\) and \(H \subseteq F\) we get \(\varphi(G) \cap \varphi(H) = \emptyset\), implying \(\varphi^m(E) \cap \varphi^m(F) = \emptyset\), i.e. \(\varphi^m\) satisfies \((O)\). By the latter get \(E = _v \varphi(E) \subseteq \varphi^m(E) \subseteq (\varphi^m)^c(E) \subseteq \varphi^c(E) = _v E\) and by completeness of \((X \times Y, \mathcal{Y}, \nu)\) this implies \(\varphi^m(E) \in \mathcal{Y}\) and \((L1)\) for \(\varphi^m\). Again by e.g. [30] Theorem 4.1 we get \([\varphi^m(E)]_x \in T\) for every \(E \in \mathcal{Y}\) and \(x \in X\) (and at this point completeness of \((Y, T, \nu)\) is required). This implies \([\varphi(E)]_x = \delta([\varphi^m(E)]_x) \subseteq \delta(\tilde{\xi}(E)_x)\). If we define \(\zeta\) by \([\zeta(E)]_x := \delta([\varphi^m(E)]_x)\) for \(E \in \mathcal{Y}\) and \(x \in X\), then \(\zeta\) satisfies \([\zeta(E)]_x \in T\) for \(x \in X, (N), (E), (M), (O)\), and \([\varphi(E)]_x = \delta([\varphi^m(E)]_x) \subseteq [\zeta(E)]_x \subseteq [\varphi^c(E)]_x \subseteq [\varphi^m(E)]_x\) for every \(x \in X\), i.e. \(E = _v \varphi(E) \subseteq \zeta(E) \subseteq \varphi^c(E) = _v E\) implying \(\zeta(E) \in \mathcal{Y}\) for \(E \in \mathcal{Y}\) and \((L1)\) for \(\zeta\) as well as \(\delta([\zeta(E)]_x) = \delta(\tilde{\xi}(E)_x)\).

**Definition 3.4.** For probability spaces \((X, \Sigma, \mu), (Y, T, \nu),\) and \((X \times Y, \mathcal{Y}, \nu)\) satisfying \([C]\), we call \(\delta \in T^Y\) satisfying \((L1), (L2)\) and \((N)\) a \(Y\)-marginal with respect to \(\nu\), if condition \((iv)\) of Lemma 3.3 is fulfilled.

**Lemma 3.5.** Let be given probability spaces \((X, \Sigma, \mu)\) and \((Y, T, \nu)\) with \((Y, T, \nu)\) complete, and ideals \(I, J\) in \(P(X \times Y)\) with \((\Sigma \otimes T)_{0} \subseteq I \subseteq J \subseteq \mathfrak{A}\), such that \((X \times Y, \Sigma \otimes Z T, \mu \otimes Z \nu)\) satisfies \([P_0]\) and \([C]\) for \(Z = I, J\). If \(\delta \in M(\nu) \cap O(\nu)\) is a \(Y\)-marginal with respect to \(\mu \otimes T \nu\), it is also \(Y\)-marginal with respect to \(\mu \otimes J \nu\). The same holds true, if we replace \(\mathfrak{A}\) by \(\mathfrak{A}\).
Proof. Note, that \((X \times Y, \Sigma \otimes T, \mu \otimes \nu)\) is complete for \(Z = \mathcal{I}, \mathcal{J}\), \([P]\) and \([C]\) imply \([P]\) by Remark \([1.7]\), consequently \(\mathcal{I} \cap \Sigma \otimes T = \mathcal{J} \cap \Sigma \otimes T = (\Sigma \otimes T)_0\).

By Lemma \([3.3]\), \((vi)\), there exists a \(\zeta : \Sigma \otimes T \mapsto \Sigma \otimes T\) satisfying \((L1), (N), (E), (M)\) and \((O)\) such that for every \(E \in \Sigma \otimes T\) we have \(\delta([\zeta(E)]_x) = [\zeta(E)]_x\) for \(x \in X\).

We can choose a density \(\eta \in \vartheta(\nu)\) to define \(\psi(E) := \zeta(\eta(E))\) for every \(E \in \Sigma \otimes T\). Clearly, \(\psi\) satisfies \((L1), (L2), (N), (E), (M)\) and \((O)\). For \(E \in \Sigma \otimes T\) and \(x \in X\) we get

\[
[\psi(E)]_x = [\zeta(\eta(E))]_x = \delta([\zeta(\eta(E))]_x) = \delta([\psi(E)]_x).
\]

For every \(E \in \Sigma \otimes T\) there exists a \(F \in \Sigma \otimes T\) with \(E = F\) a.e.\((\mu \otimes \nu)\), where \(F = F_1\) a.e.\((\mu \otimes \nu)\) for another \(F_1 \in \Sigma \otimes T\) by implies \(F = F_1\) a.e.\((\mu \otimes \nu)\) and for this reason by \((L2)\) for \(\psi\) we can unambiguously define \(\varphi : \Sigma \otimes T \mapsto \Sigma \otimes T \subseteq \Sigma \otimes \mathcal{J}\) by \(\varphi(E) := \psi(F) = \psi(F_1)\) for \(E, F, F_1\) as above, the latter equation by \((L2)\) for \(\psi\). Then get

\[
\delta(\varphi(E)]_x = \delta(\psi(F)]_x) \tag{4}
\]

and \(\varphi\) satisfies \((L1)\) since \(\varphi(E) = \psi(F) = \mathcal{I} F = \mathcal{J} E\). Now apply Lemma \([3.3]\) \((iii)\). \(\square\)

Example 3.6. Given complete probability spaces \((X, \Sigma, \mu)\) and \((Y, T, \nu)\), for the involved definition of the non-empty classes \(\vartheta(\nu)\) of all admissible densities and the class \(AGA(\nu)\) of all admissibly generated liftings we refer to \([30]\) pp.1138 and p.1139, respectively. Every \(\delta \in \vartheta(\nu) \cup AGA(\nu)\) is a \(Y\)-marginal with respect to \(\mu \otimes \nu\) in the sense of Definition \([3.3]\), see \([30]\) Theorem 6.21 and Theorem 6.22, for proofs see \([24]\), Theorems 2.9 and 2.13. But for complete topological probability spaces \((X, \mathcal{S}, \Sigma, \mu)\) and \((Y, \mathcal{T}, T, \nu)\) the classes \(\vartheta(\nu)\) and \(AGA(\nu)\) of all strong elements in \(\vartheta(\nu)\) and \(AGA(\nu)\), respectively, might be empty, e.g. for the hypersonian space (a Radon probability space) of the Lebesgue probability space over \([0,1]\), compare e.g. \([30]\), p.1162ff.

Moreover, it is proven in \([24]\), Theorem 4.1, that given complete topological probability spaces \((X, \mathcal{S}, \Sigma, \mu)\), \((Y, \mathcal{T}, T, \nu)\) with \(\Sigma := \hat{\mathcal{B}}(\mathcal{S})\) and \(T := \hat{\mathcal{B}}(\mathcal{T})\), and a strong lifting \(\rho \in L_{\mathcal{S}}(\mu)\), every \(\sigma \in AGA(\nu)\) is a \(X\)-marginal with respect to \(\mu \otimes \nu\) and \(\Sigma \otimes T = \hat{\mathcal{B}}(\mathcal{S} \times \mathcal{T})\); hence every \(\sigma \in AGA(\nu)\) is a \(X\)-marginal with respect to \(\mu \otimes \nu\).

Note that, if \((Y, T, \nu)\) is a complete non-atomic probability space then, according to \([24]\), Theorem 4.3, an admissibly generated lifting \(\delta \in \vartheta(\nu)\) cannot be an admissible density. Consequently, we get that an admissibly generated lifting \(\delta \in \vartheta(\nu)\) is always a \(Y\)-marginal density with respect to \(\mu \otimes \nu\) but it cannot be an admissible density. Therefore, the class of all admissible densities is a proper subclass of those of all \(Y\)-marginal densities with respect to \(\mu \otimes \nu\).
According to [23] Theorem 2.1 for every non-atomic topological probability spaces \((Y, \mathfrak{F}, T, \nu)\) with supp\((\nu) = Y\), second countable \((Y, \mathfrak{F})\) having a countable basis \(\{B_n : n \in \mathbb{N}\}\) such that \(\mu(\partial B_n) = 0\) for every \(n \in \mathbb{N}\) (if \((Y, \mathfrak{F})\) is regular, then it is metrizable and this condition is satisfied, in particular for Polish spaces), then the collection \(A\theta_\delta(\nu)\) of all strong \(\delta \in A\theta(\nu)\) is non-empty and if \((X, T, \nu)\) is complete, then \(AGA_\delta(\nu) \neq \emptyset\).

**Lemma 3.7.** Let be given a probability space \((X, \Sigma, \mu)\) and complete probability spaces \((Y, \mathfrak{F}, T, \nu)\) such that \([C]\) is satisfied. For every \(Y\)-marginal \(\xi \in \theta(\nu)\) there exists a \(Y\)-marginal \(\eta \in \Lambda(\nu)\) with respect to \(\nu\) with \(\xi \leq \eta \leq \xi^c\) and a \(\rho \in \Lambda(\nu)\) such that \(\eta([\rho(E)]_x) = [\rho(E)]_x\) for every \(E \in \mathfrak{F}\) and \(x \in X\).

**Proof.** By Lemma 3.3 we can choose a \(\psi \in \theta(\nu)\) such that \(\xi([\psi(E)]_x) = [\psi(E)]_x\) for every \(E \in \mathfrak{F}\) and \(x \in X\).

By [22] Corollary 2.5, there exist \(\eta \in \Lambda(\nu)\) and \(\phi \in \Lambda(\nu)\) with \(\xi \leq \eta \leq \xi^c\) and \(\psi \leq \phi \leq \psi^c\). For \(E \in \mathfrak{F}\) and \(x \in X\), it follows \([\psi(E)]_x = \xi([\psi(E)]_x) \leq \eta([\phi(E)]_x)\), in particular \([\psi(E^c)]_x \leq \eta([\phi(E^c)]_x) = [\eta([\phi(E)]_x)]^c\), i.e. \([\eta([\phi(E^c)]_x) \leq [\psi(E^c)]_x\). Defining \(\rho\) by means of \([\rho(E)]_x := \eta([\phi(E)]_x)\) for every \(E \in \mathfrak{F}\) and \(x \in X\), we get \(E =_\nu \psi(E) \subseteq \rho(E) \leq \psi(E) =_\nu E\), implying \(\rho(E) =_\nu E\) for every \(E \in \mathfrak{F}\), i.e. \(\rho\) satisfies (L1), but all other properties for \(\rho \in \Lambda(\nu)\) are immediate and \(\eta([\rho(E)]_x) = \eta([\phi(E)]_x) = [\rho(E)]_x\) for every \(E \in \mathfrak{F}\) and \(x \in X\) and again by Lemma 3.3 \(\eta\) is a \(Y\)-marginal with respect to \(\nu\).

**Proposition 3.8.** Let be given probability spaces \((X, \Sigma, \mu)\), \((Y, T, \nu)\), and \((X \times Y, \mathfrak{F}, \mu \times \nu)\) satisfying \([C]\). If \((Y, T, \nu)\) is complete every \(\delta \in T^\mathfrak{F}\) satisfying (L1), (L2) and (N) is a \(Y\)-marginal with respect to \(\nu^\mathfrak{F}\).

**Proof.** Let \(\delta^*_\mathfrak{F} := \delta^*\mathfrak{F}\). For \(E \in \mathfrak{F}\) we can choose a \(F \in \mathfrak{F}\), such that \(E\Delta F \in \mathfrak{F}\), i.e. there exists a \(N_E \in \mathfrak{F}_0\), such that \(E_x =_\nu F_x\) for every \(x \notin N_E\). By \([C]\) we can choose a \(M_F \in \mathfrak{F}_0\), such that \(E_x \in T\) for every \(x \in N_E\). It follows \(Q := N_E \cup M_F \in \mathfrak{F}_0\) and \(E_x \in T\) for every \(x \notin Q\) by completeness of \((Y, T, \nu)\), implying \(\delta^*_\mathfrak{F}(E)_x = \delta(E_x) =_\nu E_x \in T\) for every \(x \notin Q\), i.e. \(\delta^*_\mathfrak{F}(E) =_\nu E\). Since by Lemma 3.1 \(\mathfrak{F} = (\mathfrak{F}_\mathfrak{F}, \nu^\mathfrak{F})\) and \((X \times Y, \mathfrak{F}, \mu \times \nu)\) is complete and satisfies \([C]\) too, we have \(\delta^*_\mathfrak{F}(E) =_\nu^\mathfrak{F} E\) for every \(E \in \mathfrak{F}\), i.e. by Lemma 3.3 \(\delta\) is a \(Y\)-marginal with respect to \(\nu^\mathfrak{F}\).

**Remark 3.9.** Given probability spaces \((X, \Sigma, \mu)\) and \((Y, T, \nu)\) we call a \(\Sigma\)-\(T\)-measurable map \(f : X \to Y\) inverse measure preserving for \(\mu\) and \(\nu\), if \(\nu(B) := \mu(f^{-1}(B))\) for \(B \in T\), written \(\mu \circ f^{-1} = \nu\). Put \(f^{-1}(T) := \{f^{-1}(B) : B \in T\}\). In this situation we get the following.

(a) Let \(\Sigma = f^{-1}(T)\) in addition.

(i) \((Y, T, \nu)\) is complete, if \((X, \Sigma, \mu)\) is complete and \(f\) is surjective.

(ii) For every \(\zeta : T \to \mathcal{P}(Y)\) satisfying (L2) we define uniquely \(f^{-1}([\zeta]) := \delta : \Sigma \to \mathcal{P}(X)\) satisfying (L2) by means of \(\delta(f^{-1}(B)) = f^{-1}([\zeta(B)])\)
for every $B \in T$. $f^{-1}[\zeta]$ satisfies (Z) if $\zeta$ does for $Z = N, E, V, O, M, \Pi, F, \vartheta, C, (L_j)$ and $j=0,1$. $f^{-1}[\zeta] \in \vartheta^*(\mu)$ if $\zeta \in \vartheta^*(\nu)$. $f^{-1}[\zeta] \in Z(\mu)$ if $\zeta \in Z(\nu)$ for $Z = \vartheta, \Lambda$.

(b) Let $\Sigma \neq f^{-1}[T]$ and $Z = \vartheta, \Lambda$. Define the $\sigma$-subalgebra $\eta := \sigma(\Sigma \cup f^{-1}[T]) = \{A_{\Delta B} : A \in \Sigma_0, B \in f^{-1}[T]\}$ and for $\zeta \in Z(\mu)\nu$ define $\delta := f^{-1}[\zeta] \in Z(\mu|f^{-1}[T])$ according to (a), (ii). Unambiguously define $\delta \in Z(\mu|\eta)$ by means of

\[
(2) \quad \tilde{\delta}(A_{\Delta f^{-1}[B]}) := \delta(f^{-1}[B]) \quad \text{for} \quad A \in \Sigma_0 \quad \text{and} \quad B \in T.
\]

$\tilde{\delta}$ has an in general not uniquely defined extension $\delta \in Z(\mu)$ by Theorem 2.5 and Theorem 2.6 satisfying $\delta(f^{-1}[B]) = f^{-1}[\zeta(B)]$ for every $B \in T$ and we write $\delta \in f^{-1}[\zeta]$.

**Proposition 3.10.** Let be given complete probability spaces $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(\bar{Y}, \bar{T}, \bar{\nu})$ together with an inverse measure preserving $(T, \bar{T}$-measurable) surjection $f : Y \rightarrow \bar{Y}$. For a $\sigma$-subalgebra $\bar{Y}$ of $\mu \times \bar{T}$ and $\bar{\nu} = \mu \times \nu|\bar{Y}$, such that $(X \times \bar{Y}, \bar{\nu})$ is complete and $N \times \bar{Y} \in \bar{Y}$ for every $N \in \Sigma$ put $Y := \{(id_X \times f)^{-1}[E] : E \in \bar{Y}\}$, $\upsilon := \mu \times \nu|Y$. Given $\delta \in \nu(\bar{\nu})$ choose $\delta \in f^{-1}[\delta]$ with $\delta \in \nu(\bar{\nu})$ according to Remark 3.9. (b).

(i) $\bar{Y} \subseteq \mu \times T$ and $\bar{\nu} = \upsilon \circ (id_X \times f)^{-1}$. $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(X \times Y, \bar{Y}, \upsilon)$ as well as $(X, \Sigma, \mu)$, $(\bar{Y}, \bar{T}, \bar{\nu})$, and $(X \times \bar{Y}, \bar{\nu})$ satisfy $[C]$.

(ii) $\delta \in Z(\nu)$ if $\delta \in Z(\bar{\nu})$ for $Z = \vartheta, \Lambda$. $\delta : \bar{Y} \rightarrow \mathcal{P}(X \times Y)$ satisfies

\[
(Z) \quad \text{for} \quad Z = N, E, F.
\]

(iii) If $\delta$ is a $\bar{Y}$-marginal with respect to $\bar{\nu}$, it follows that $\delta$ is a $Y$-marginal with respect to $\upsilon$.

**Proof.** $N \times \bar{Y} \in \bar{Y}$ for $N \in \Sigma_0$ implies $N \times Y = N \times f^{-1}[\bar{Y}] = (id_X \times f)^{-1}[N \times \bar{Y}] \in \bar{Y}$ and $\upsilon(N \times Y) = (\mu \times \nu)(N \times Y) = 0$, i.e. $N \times Y \in \bar{Y}$ for every $N \in \Sigma_0$.

Note

\[
(3) \quad \left[ (id_X \times f)^{-1}[E] \right]_x = f^{-1}[E_x] \quad \text{for every} \quad E \subseteq X \times Y \quad \text{and} \quad x \in X.
\]

Ad (i) : For $E \in \bar{Y} \subseteq \mu \times \bar{T}$ there exists a $N_E \in \Sigma_0$ with $E_x \in \bar{T}$ for every $x \notin N_E$ and this implies by $[3]$ that $\left[ (id_X \times f)^{-1}[E] \right]_x = f^{-1}[E_x] \in T$ for every $x \notin N_E$ and the map

\[
X \in N_E \mapsto \nu\left( \left[ (id_X \times f)^{-1}[E] \right]_x \right) = \nu(f^{-1}[E_x]) = \upsilon(E_x)
\]

is in $L^\infty(\Sigma \cap N_E^c)$, i.e. $\bar{Y} \subseteq \mu \times T$. By the latter $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(X \times Y, \bar{Y}, \upsilon)$ as well as $(X, \Sigma, \mu)$, $(\bar{Y}, \bar{T}, \bar{\nu})$, and $(X \times \bar{Y}, \bar{\nu})$ satisfy $[C]$ and

\[
\bar{\nu}(E) = \int_{N_E} \nu(f^{-1}[E_x])\mu(dx) = \int_{N_E} \nu\left( \left[ (id_X \times f)^{-1}[E] \right]_x \right)\mu(dx) = (\mu \times \nu)\left( (id_X \times f)^{-1}[E] \right) = \upsilon((id_X \times f)^{-1}[E]),
\]

\[
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\]
i.e. $\pi = v \circ (\text{id}_X \times f)^{-1}$.

Ad (ii) : $\delta \in Z(\nu)$ for $Z = \vartheta, \Lambda$ by Remark 3.9(b). $\delta_x$ satisfies $(Z)$ if $\delta$ does for $Z = N, E, F$ by Remark 3.2(b) and Remark 3.9(b).

Ad (iii) : If $\delta$ is $Y$-marginal with respect to $\pi$, it follows by Lemma 3.3, that there exists a $\zeta \in \vartheta(\pi)$ such that

$\delta((\zeta(E))_x) = [\zeta(E)]_x \in T$ for every $E \in \mathcal{T}$ and $x \in X$.

Since $\zeta \in \vartheta(\pi)$, defining $\zeta := (\text{id}_X \times f)^{-1}[\zeta]$ we get $\zeta \in \vartheta(\nu)$. According to Remark 3.9(b), get $\zeta \in \vartheta(\nu)$ by means of

$\zeta(N \Delta (\text{id}_X \times f)^{-1}[E]) := \zeta((\text{id}_X \times f)^{-1}[E])$ for $N \in (\mathcal{T})_0$ and $E \in \mathcal{T}$.

For fixed $x \in X$, $N \in (\mathcal{T})_0$, and $E := (\text{id}_X \times f)^{-1}[E]$, if $E \in \mathcal{T}$ get

$[\zeta(N \Delta E)]_x = [\zeta((\text{id}_X \times f)^{-1}[E])]_x = [\delta((\text{id}_X \times f)^{-1}[E])]_x$

$= f^{-1}([\zeta(E)]_x) = f^{-1}([\delta(E)]_x) = \delta(f^{-1}([\zeta(E)]_x)

= \delta([\zeta((\text{id}_X \times f)^{-1}[E])]_x) = \delta([\zeta(N \Delta E)]_x)

i.e. condition $[\zeta(N \Delta E)]_x = \delta([\zeta(N \Delta E)]_x)$ holds true, implying together with Lemma 3.3 that $\delta$ is a $Y$-marginal with respect to $\hat{\vartheta}$.

4. MEASURABLE MODIFICATION OF STOCHASTIC PROCESSES WITH VALUES IN STRONGLY LIFTING COMPACT SPACES

In this section, we examine the problem of the existence of a measurable lifting modification of a measurable process. For this reason we need liftings for functions instead of sets. Below we report on the (purely technical) standard procedure for passing from sets to functions used also in [10].

In Theorem 4.8 we get a characterization of liftings converting measurable processes into their measurable modifications, extending in this way Theorem 5.2L from [21].

Given a measurable space $(X_\Sigma)$ and a topological space $K$ we write $\mathcal{M}(\Sigma, K)$ for the space of all $\Sigma$-$\mathcal{B}(K)$-measurable maps from $X$ into $K$, put $L^0(\Sigma) := \mathcal{M}(\Sigma, \mathbb{R})$. $L^\infty(\Sigma)$ is the space of all (strictly) bounded (i.e. $\|f\|_\infty := \sup_{x \in X} |f(x)| < \infty$) $f \in L^0(\Sigma)$ and $l_\infty(X) := L^\infty(\mathcal{P}(X))$.

As usual, for given probability space $(X, \Sigma, \mu)$ we write also $\mathcal{M}(\mu, K)$, $L^0(\mu)$, and $L^\infty(\mu)$ instead of $\mathcal{M}(\Sigma, K)$, $L^0(\Sigma)$, and $L^\infty(\Sigma)$, respectively.

For a topological spaces $(X, \Sigma)$ we denote by $C(X, Y)$ the space of all continuous functions from $X$ into $Y$, $C(X, \mathbb{R})$ and $C_b(X) := C(X) \cap C_b(X)$, $C_l(X)$ and $C_u(X)$ is the space of all lower and upper semi-continuous functions from $X$ into $\mathbb{R}$, respectively.

Remark 4.1. Let be given a probability space $(X, \Sigma, \mu)$.

(a) For $\gamma \in \mathcal{P}(X)^\Sigma$ we define $\gamma^0, \gamma_0 : L^0(\mu) \to L^0(\mu)$ by $\gamma^0(f)(x) := \inf\{r \in \mathbb{Q} : x \in \gamma\{f < r\}\}$ and $\gamma_0(f)(x) := \sup\{r \in \mathbb{Q} : x \in \rho\{f > r\}\}$ for $f \in L^0(\mu)$ and $x \in X$ (inf $\emptyset := \infty$, sup $\emptyset := -\infty$). $\gamma_\infty = \gamma_0|L^\infty(\mu)$ and $\gamma^\infty = \gamma^0|L^\infty(\mu)$.
(b) For \( p = 0, \infty \) and \( \rho \in \Lambda(\mu) \), it follows that \( \rho^p = \rho\_p \) is uniquely determined by \( \rho \) via \( \rho_p(x_A) = \chi_{\rho(A)} \) for every \( A \in \Sigma \) and for \( f, g \in \mathcal{L}^p(\mu) \) and \( a \in \mathbb{R} \) get

(i) \( \rho_p(f) = \mu(f) \), (ii) \( \rho_p(f) = \rho_p(g) \) if \( f = \mu g \), and (e) \( \rho_p(a) = a \).

For every \( a \in \mathbb{R} \) and \( f, g \in \mathcal{L}^\infty(\mu) \) get

(i) \( \rho_\infty(f \wedge g) = \rho_\infty(f) \wedge \rho_\infty(g) \), \( \rho_\infty(f \vee g) = \rho_\infty(f) \vee \rho_\infty(g) \),

(ii) \( \rho_\infty(fg) = \rho_\infty(f) + \rho_\infty(g) \),

(iii) \( \rho_\infty(\rho(f)g) = \rho_\infty(\rho(f) \rho_\infty(g)) \) for every \( a \in \mathbb{R} \) and \( f, g \in \mathcal{L}^\infty(\mu) \).

(c) Let \( \rho \in \Lambda(\mu) \). For \( f, g \in \mathcal{L}^0(\mu) \) we confine pointwise arithmetics to the following rules in \( \mathbb{R} \) which seem to be accepted generally, i.e. we avoid the 'doubtful sums' \( \infty - \infty \) and \( -\infty + \infty \), defined differently throughout literature. For \( a \in \mathbb{R} \) put

\[
a + \infty = \infty + a = \infty \quad \text{for} \quad a \in (-\infty, \infty] \quad \text{and} \quad a - \infty = -\infty + a = -\infty \quad \text{for} \quad a \in (-\infty, \infty), \quad a(\pm \infty) = (\pm \infty)a = \mp \infty \quad \text{for} \quad a \in (-\infty, 0), \quad \text{and} \quad 0(\pm \infty) = (\pm \infty)0 = 0,
\]

Then get for every \( f, g \in \mathcal{L}^0(\mu) \)

(i) \( \rho_0(fg) = \rho_0(f) \cdot \rho_0(g) \), in particular \( \rho_0(a \cdot f) = a \cdot \rho_0(f) \) for every \( a \in \mathbb{R} \),

(ii) \( \rho_0(f + g) = \rho_0(f) + \rho_0(g) \), if \( a \leq f, g \in \mathcal{L}^0(\mu) \) for some \( a \in \mathbb{R} \),

(iii) \( \rho_0(f^\pm) = (\rho_0(f))^\pm \), \( \rho_0(0) = \rho_0(f^+) - \rho_0(f^-) \), and \( \rho_0(|f|) = \rho_0(f) \),

(iv) \( |\rho_0(f)| \wedge |\rho_0(g)| = 0 \) and \( |\rho_0(f + g)| \leq |\rho_0(f) + \rho_0(g)| \), if \( |f| \wedge |g| = 0 \).

Remark 4.2. \((X, \mathcal{S}, \Sigma, \mu)\) a topological probability space with \( \gamma : \Sigma \rightarrow \mathcal{P}(X) \)

(a) \( f \leq \gamma_0(f) \) for every lower semi continuous and \( f \geq \gamma^0(f) \) for every upper semi continuous function \( f : X \rightarrow \mathbb{R} \).

(b) If \( \gamma \) satisfies the condition \( (O) \), it follows \( \gamma_0(f) = \gamma^0(f) = f \) for every continuous function \( f : X \rightarrow \mathbb{R} \).

The following result extends Lemma 3.2 from [21]. Its proof runs in a similar way.

**Lemma 4.3.** Let \( p = 0, \infty \), let be given probability spaces \((X, \Sigma, \mu), (Y, T, \nu)\), and \((X \times Y, \mathcal{Y}, \nu)\), and let be given \( \delta \in \mathcal{P}(Y)^T \) and \( \varphi \in \mathcal{P}(X \times Y)^T \) both satisfying (M). If for every \( E \in \mathcal{Y} \) there exists a \( N_E \in \Sigma_0 \) such that \( \delta^E = \delta(x | x) \) for every \( x \in N_E \), it follows that for every \( f \in \mathcal{L}^p(v) \) there exists a \( N_f \in \Sigma_0 \) such that \( \delta|_x = \delta|_x \) for every \( x \notin N_f \); \( N_f = \emptyset \) for every \( f \in \mathcal{L}^p(v) \), if \( N_E = \emptyset \) for every \( E \in \mathcal{Y} \).

For given completely regular Hausdorff topological space \( K \), \( C_b(K) \) denotes its space of all bounded continuous real valued functions, \( \mathcal{B}_0(K) \) its **Baire \( \sigma \)-field**, i.e. the \( \sigma \)-field generated by all \( h \in C_b(K) \), and write \( \mathcal{M}_0(\mu, K) := \mathcal{M}(\Sigma, \mathcal{B}_0(K)) \) for the space of all **Baire measurable functions** for given probability space \((X, \Sigma, \mu)\). Note \( \mathcal{M}(\mu, K) = \mathcal{M}_0(\mu, K) \) for metrizable \( K \).

Let be given a complete probability space \((X, \Sigma, \mu)\) and a completely regular Hausdorff topological space \( K \). Write \( i_K : K \rightarrow \beta K \) for the canonical
injection of $K$ into $\beta K$, the Stone-Cech compactification of $K$ and $\overline{h} \in C_b(\beta K) = C(\beta K)$ for the unique extension of $h \in C_b(K)$ onto $\beta K$ satisfying $\overline{h} \circ i_K = h$ and defining a bijection $h \in C_b(K) \rightarrow \overline{h} \in C(\beta K)$.

If $\rho \in \Lambda(\mu)$, then every $F \in \mathcal{M}_0(\mu, K)$ induces a map $\rho'(F) \in \mathcal{M}(\mu, \beta K)$, for $\beta K$ the Stone-Cech compactification of $K$, defined by

$$\overline{h} \circ \rho'(F) = \rho_\infty(h \circ F) = \rho_\infty(h \circ i_k \circ F) = \overline{h} \circ \rho'(i_K \circ F)$$

for $h \in C_b(K)$. This defining equation for $\rho'(F)$ is very often shortened to $h \circ \rho'(F) = \rho_\infty(h \circ F)$ for $h \in C_b(K)$, e.g. in [1], with the understanding, that $F$ is considered as a 'map of $X$ into $\beta K$' and $h$ is identified with its extension to $C(\beta K)$, for simplicity.

**Definitions 4.4.** Let be given a complete probability space $(X, \Sigma, \mu)$ and a completely regular Hausdorff topological space $K$.

(a) Any $F \in \mathcal{M}_0(\mu, K)$ is called **lifting compact**, if for every $\rho \in \Lambda(\mu)$ there exists a $N \in \Sigma_0$, such that $\rho'(F)(x) \in K$ for every $x \notin N$, see [2].

(b) Any $F \in \mathcal{M}_0(\mu, K)$ is called **strongly lifting compact**, if $F$ is lifting compact and $\rho'(F) = \mu F$, see [1] p.213.

(c) A completely regular Hausdorff topological space $T$ is called **(strongly) lifting compact**, if for every complete probability space $(X, \Sigma, \mu)$, every Baire measurable map $F : X \rightarrow T$ is (strongly) lifting compact.

**Remark 4.5.** Let be given a complete probability space $(X, \Sigma, \mu)$, a completely regular Hausdorff topological space $T$, and $\rho \in \Lambda(\mu)$.

(a) For $F, G \in \mathcal{M}_0(\mu, K)$ it follows $\rho'(F) = \rho'(G)$ from $F =_{\mu} G$.

(b) For $G \in \mathcal{M}_0(\mu, K)$ put $G^* := \{F \in \mathcal{M}_0(\mu, K) : G =_{\mu} F\}$ and get $N^*_G := \{x \in X : \rho'(G)(x) \notin K\} \in \Sigma_0$ such that $\rho'(G)(x) \in K$ for every $x \in N^*_G$.

We may choose $x_0 \in N^*_G$. If $F \in G^*$ define $\overline{F}(x) := \rho'(F)(x)$ for every $x \in N^*_G$ and $\overline{F}(x_0) := \rho'(F)(x_0)$ for every $x \in N^*_G$.

It follows $\overline{F}(G)[X] \subseteq K$ for every $G \in \mathcal{M}_0(\mu, K)$ and by (a) we get $\overline{F}(F) = \overline{F}(G)$, i.e. $\overline{F}(F) = \overline{F}(G)$ for (strongly) lifting compact $F, G \in \mathcal{M}_0(\mu, K)$ with $F =_{\mu} G$. Clearly $\overline{F}(F) =_{\mu} F$ for strongly lifting compact $F \in \mathcal{M}_0(\mu, K)$ and $\overline{F}(F) \in \mathcal{M}_0(\mu, K)$, the latter since $\overline{F}(F) =_{\mu} \rho'(F)$ and since $(X, \Sigma, \mu)$ is complete.

Note, that $\rho'(F)(x) \in K$ implies $\overline{F}(F)(x) = \rho'(F)(x)$, if $F \in \mathcal{M}_0(\mu, K)$ and $x \in X$.

(c) For compact $K$ it follows $\rho'(G)[X] \subseteq K$ for every $G \in \mathcal{M}_0(\mu, K)$, and $\overline{F} = \rho'$.

(d) For metrizable $K$ it follows $\mathcal{M}_0(\mu, K) = \mathcal{M}(\mu, K)$.

(e) Every subspace of a compact metrizable space $K$ is strongly lifting compact and in this case Definition 4.4 (c) coincides with that of D.L. Cohn [H]. For more information about strongly lifting compact spaces, compare [1] and [29].

(f) The compact metrizable $K := \mathbb{R}$ is strongly lifting compact, $\mathcal{M}_0(\mu, \mathbb{R}) = \mathcal{L}_0(\mu)$, and $\overline{F} = \rho' = \rho_0$.  

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Proof. Since $\mathbb{R}$ is compact $\overline{\rho} = \rho'$ follows by (c). Ad $\rho' = \rho_0$ consider the strictly increasing homomorphism $H : \mathbb{R} \to K := [-1, 1]$ defined by

$$H(x) := \pm 1 \quad \text{for} \quad x = \pm \infty \quad \text{and} \quad H(x) := x/(1 + |x|) \quad \text{for} \quad x \in \mathbb{R}.$$ 

First get

$$H \circ \rho_0(F) = \rho_\infty(H \circ F) \quad \text{for} \quad F \in \mathcal{L}^0(\mu). \quad (5)$$

We have to distinguish $\rho'$ (defined by $h \circ \rho'(F) = \rho_\infty(h \circ F)$ for $h \in C_0(\mathbb{R})$ if $F \in \mathcal{L}^0(\mu)$) from $\rho'_K$, defined by $g \circ \rho'_K(G) = \rho_\infty(g \circ G)$ for $g \in C_0(K)$ if $G \in \mathcal{M}^0(\mu, K)$. For $\text{id}_K \in C_0(K)$ the latter implies

$$\rho'_K(G) = \rho_\infty(G) \quad \text{for} \quad G \in \mathcal{M}^0(\mu, K). \quad (6)$$

Since $\rho'(F) = h^{-1} \circ \rho_\infty(h \circ F)$, with (5) and (6) it follows $\rho'(F) = \rho_0(F)$ for $F \in \mathcal{L}^0(\mu)$.

(g) If, in addition, there is given a topology $\mathcal{G}$ over $X$ making $(X, \mathcal{G}, \Sigma, \mu)$ a topological probability space and $\rho \in \Lambda_\mathcal{G}(\mu)$, it follows $\overline{\rho}(F) = \rho'(F) = F$ for every continuous $F : X \to K$ with $K$ compact.

(h) Given for another complete probability space $(Y, T, \nu)$ an inverse measure-preserving map $f : X \to Y$, a $\zeta \in \Lambda(\nu)$, and $\rho \in f^{-1}[\zeta]$, for every $F \in \mathcal{M}_0(\nu, K)$ it follows $F \circ f \in \mathcal{M}_0(\mu, K)$ and $\rho'(F \circ f) = \zeta'(F) \circ f$ and if $\overline{\zeta}(F)(y) = \zeta'(F)(y) \in K$ for $y \notin N_F \in T_0$ and $\overline{\zeta}(F)(y) = \zeta'(F)(y_0) \in K$ for some $y_0 \notin N_F$ define $\overline{\rho}$ for $M_F := f^{-1}[N_F] \subset \Sigma_0$ and some $x_0 \in M_F$ getting $\overline{\rho}(F \circ f) = \overline{\zeta}(F) \circ f$. In particular, for $F \in \mathcal{L}^p(\nu)$ get $F \circ f \in \mathcal{L}^p(\mu)$ and $\rho_p(F \circ f) = \overline{\rho}_p(F) \circ f$, if $p = 0, \infty$.

Lemma 4.6. Let be given a strongly lifting compact space $K$ and complete probability spaces $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(X \times Y, \mathcal{Y}, \upsilon)$ satisfying $[C]$. If $\rho \in C(\nu)$ is a $Y$-marginal with respect to $\upsilon$, there exists a $\zeta \in C(\upsilon)$ such that $\rho'(\zeta'(F)) = [\zeta'(F)]_x$ for every $F \in \mathcal{M}_0(\upsilon, K)$ and $x \in X$.

Proof. First note, that $[h \circ f]_x = h \circ (f_x)$ for every $h \in C_b(K)$, for every $f \in \mathcal{M}_\infty(\upsilon, K)$, and $x \in X$.

By Lemmas 3.3 and 3.7 there exists a $\zeta \in C(\upsilon)$ such that $\rho(\zeta(E)) = [\zeta(E)]_x$ for every $E \in \mathcal{Y}$ and $x \in X$. By Lemma 4.3 this implies $[\zeta(\upsilon)]_x \in \mathcal{L}^\infty(\upsilon)$ and $\rho_\infty([\zeta(\upsilon)]_x) = [\zeta(\upsilon)]_x$ for every $f \in \mathcal{L}^\infty(\upsilon)$ and $x \in X$.

In what follows let $h \in C_b(K)$, $F \in \mathcal{M}_0(\upsilon, K)$ and $x \in X$, then get $h \circ [\zeta'(F)]_x = [h \circ \zeta'(F)]_x = [\zeta(\upsilon) \circ F]_x \in \mathcal{L}^\infty(\upsilon)$, implying $[\zeta'(F)]_x \in \mathcal{M}_0(\upsilon, K)$ for every $x \in X$ (since $\mathcal{B}_0(K)$ is generated by the $h \in C_b(K)$) and

$$h \circ [\zeta'(F)]_x = [h \circ \zeta'(F)]_x = [\zeta(\upsilon) \circ F]_x = \rho_\infty([\zeta(\upsilon) \circ F]_x) = \rho_\infty(h \circ [\zeta'(F)]_x) = h \circ \rho'(\zeta'(F))_x$$

for every $h \in C_b(K)$, implying $\rho'(\zeta'(F))_x = [\zeta'(F)]_x$. \(\square\)

Definition 4.7. Let be given a completely regular Hausdorff space $K$ and complete probability spaces $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(X \times Y, \mathcal{Y}, \upsilon)$ satisfying $[C]$.

For every family $\{Q_x\}_{x \in X}$ in $K^Y$, we write $\overline{Q}$ for the function in $K^{X \times Y}$
uniquely defined by } Q_x := Q(x, y), \text{ for } x \in X \text{ and } y \in Y. \text{(a)} \text{ A family } \{Q_x\}_{x \in X} \text{ is called a stochastic process over } (Y, T, \nu) \text{ with values in } K, \text{ if } Q_x \in \mathcal{M}_0(\nu, K) \text{ for every } x \in X.

(b) \text{ A family } \{Q_x\}_{x \in X} \text{ is called a bounded stochastic process over } (Y, T, \nu), \text{ if } \{Q_x\}_{x \in X} \text{ is called a stochastic process over } (Y, T, \nu) \text{ with values in } \mathbb{R}, \text{ such that } Q_x \in L^\infty(\nu) \text{ for every } x \in X.

(c) \text{ A stochastic process } \{Q_x\}_{x \in X} \text{ over } (Y, T, \nu) \text{ with values in } K \text{ is } \Upsilon\text{-measurable for a probability space } (X \times Y, \Upsilon, \nu), \text{ if } \mathcal{Q} \in \mathcal{M}_0(\nu, K).

(d) \text{ Stochastic processes } \{Q_x\}_{x \in X} \text{ and } \{R_x\}_{x \in X} \text{ over } (Y, T, \nu) \text{ with values in } K \text{ are equivalent, if } Q_x =_\nu R_x \text{ for every } x \in X. \text{ } \{Q_x\}_{x \in X} \text{ is then called a modification of } \{R_x\}_{x \in X}, \text{ and vice versa.}

Theorem 4.8. For complete probability spaces } (X, \Sigma, \mu), \text{ } (Y, T, \nu), \text{ and } (X \times Y, \Upsilon, \nu) \text{ satisfying } [C], \text{ the following statements are all equivalent for } \rho \in \Lambda(\nu).

(i) \rho \text{ is a } \Upsilon\text{-marginal with respect to } \nu.

(ii) } \langle \rho_\infty(Q_x) \rangle \in X \text{ is } \Upsilon\text{-measurable for every } \Upsilon\text{-measurable bounded stochastic process } \{Q_x\}_{x \in X} \text{ over } (Y, T, \nu).

(iii) } \langle \rho_0(Q_x) \rangle \in X \text{ is } \Upsilon\text{-measurable for every } \Upsilon\text{-measurable stochastic process } \{Q_x\}_{x \in X} \text{ over } (Y, T, \nu) \text{ with values in } \mathbb{R}.

(iv) } \langle \mathcal{P}(Q_x) \rangle \in X \text{ is } \Upsilon\text{-measurable for every } \Upsilon\text{-measurable stochastic process } \{Q_x\}_{x \in X} \text{ over } (Y, T, \nu) \text{ with values in a strongly lifting compact space } K.

In (iv) there exists a } N \in \Sigma_0, \text{ such that } \mathcal{P}(Q_x) = \rho'(Q_x) \text{ for every } x \notin N. \langle \rho_\infty(Q_x) \rangle \in X, \langle \rho_0(Q_x) \rangle \in X, \text{ and } \langle \mathcal{P}(Q_x) \rangle \in X \text{ are equivalent to } \langle Q_x \rangle \in X \text{ in (ii), (iii), and (iv), respectively.}

Proof. } \text{ Ad (i) } \Longrightarrow \text{ (iv): Let } \langle Q_x \rangle \in X \text{ be a measurable stochastic process over } (Y, T, \nu) \text{ with values in } K. \text{ By Lemma 4.6 there exists a } \zeta \in \Lambda(\nu) \text{ such that } \rho'(\zeta(Q')) = [\zeta(Q')]_x \text{ for every } F \in \mathcal{M}_0(\nu, K) \text{ and } x \in X. \text{ By } \zeta(Q) = \zeta \nu, \text{ i.e. } N := \{\zeta(Q) \neq Q\} \in \Upsilon_0, \text{ by } [C] \text{ there exists a } M \in \Sigma_0 \text{ such that } N_x \in T_0 \text{ for every } x \notin M. \text{ Let } x \notin M \text{ and for } y \notin N_x \text{ and } h \in C_0(K) \text{ get } (h \circ \zeta(Q'))(x) = [h \circ \zeta(Q')]_x(y) = [h \circ \zeta(Q)]_x(y) = (h \circ Q)(x), \text{ i.e. } h \circ \zeta(Q) \notin h \circ Q_x, \text{ implying } h \circ \rho'(\zeta(Q'))_x = \rho_\infty(h \circ \zeta(Q'))_x = \rho_\infty(h \circ Q_x) \text{ for } h \in C_0(K) \text{ and together with Lemma 4.6 the latter implies } \rho'(\zeta(Q'))_x = \rho'(\zeta(Q'))_x = \rho'(Q_x) \text{ for every } x \notin M. \text{ Writing } P_x := \rho'(Q_x) = \rho'(Q_x) \text{ for } x \in X, \text{ it follows } \{P \neq \zeta(Q')\} \subseteq M \times Y \in \Upsilon_0, \text{ i.e. } P =_\nu Q_x = \mathcal{M}_0(\nu, K), \text{ applying completeness of } (X \times Y, \Upsilon, \nu). \text{ By } P_x = [\zeta(Q')]_x \in \mathcal{M}_0(\nu, K), \text{ the latter again by Lemma 4.6 we have } P_x \in \mathcal{M}_0(\nu, K) \text{ for every } x \notin M.

Defining } R_x := P_x \text{ for } x \notin M \text{ and } R_x := \mathcal{P}(Q_x) \text{ for } x \in M \text{ again } \{P \neq \mathcal{P}\} \subseteq M \times Y \in \Upsilon_0 \text{ implies } R \in \mathcal{M}_0(\nu, K) \text{ and } R_x = \rho'(Q_x) \text{ for every } x \notin M. \text{ For } x \in M \text{ clearly } R_x = \mathcal{P}(Q_x) \in \mathcal{M}_0(\nu, K) \text{ by definition and also } R_x =_\nu Q_x \text{ and for } x \notin M \text{ since } K \text{ is strongly lifting compact also } R_x =_\nu Q_x, \text{ i.e. } \mathcal{P}(R_x) \in X \text{ and } (\mathcal{P}(Q_x))_{x \in X} \text{ are equivalent.}
Ad $(iv) \Rightarrow (iii)$ note, that the compact metrizable $\mathbb{R}$ is strongly lifting compact and that $\overline{\rho} = \rho_0$ by Remark 4.5 ($f$). For the implication $(iii) \Rightarrow (ii)$ note that $\rho_0|\mathcal{L}^\infty(\nu) = \rho_\infty$.

Ad $(ii) \implies (i)$: For $f \in \mathcal{L}^\infty(\nu)$ there exists a $N_f \in \Sigma_0$ such that $f_X \in \mathcal{L}^\infty(\nu)$ for every $x \notin N_f$. Put $Q_x := f_X$ for $x \notin N_f$ and $Q_x := 0$ otherwise. It follows $\overline{Q} = _v f$. This implies that $\langle Q_x \rangle_{x \in X}$ is a $\mathcal{Y}$-measurable real-valued process in $\mathcal{L}^\infty(\nu)$ and by assumption $\langle \rho_\infty(Q_x) \rangle_{x \in X}$ is a $\mathcal{Y}$ measurable process, i.e. $\langle \rho_\infty \rangle_{\bullet}(\overline{Q}) \in \mathcal{L}^\infty(\nu)$, since $R_x := \rho_\infty(Q_x) = \rho_\infty(\overline{Q}) = [\langle \rho_\infty \rangle_{\bullet}(\overline{Q})]_x$ for $x \in X$, i.e. $\overline{R} = \langle \rho_\infty \rangle_{\bullet}(\overline{Q})$.

By $\rho_\infty(Q_x) = \rho_\infty(f_X)$ for $x \notin N_f$ we get $\langle \rho_\infty \rangle_{\bullet}(\overline{Q}) = \rho_\infty(f_X) = \rho_\infty(Q_x) = \rho_\infty(Q_x)$, for $x \notin N_f$ implying $\rho_\infty(f_x) = v \langle \rho_\infty \rangle_{\bullet}(\overline{Q}) = \overline{R} \in \mathcal{L}^\infty(\nu)$. By completeness of $(X \times \mathcal{Y}, \upsilon, v)$ this implies $\langle \rho_\infty \rangle_{\bullet}(f) \in \mathcal{L}^\infty(\nu)$ for every $f \in \mathcal{L}^\infty(\nu)$.

But the latter for $f = \chi_E$ with $E \in \mathcal{Y}$ yields $\langle \rho_\infty \rangle_{\bullet}(\chi_E) = \chi_{\rho_{\bullet}(E)} \in \mathcal{L}^\infty(\nu)$ or $\rho_{\bullet}(E) \in \mathcal{Y}$ for any $E \in \mathcal{Y}$; hence applying Lemma 3.3 we get that $\rho$ is a $\mathcal{Y}$-marginal with respect to $\upsilon$.

\[
\square
\]

**Corollary 4.9.** Let $(X, \mathcal{S}, \Sigma, \mu)$, $(Y, \mathcal{T}, T, \nu)$ be complete topological probability spaces with $\text{supp}(\nu) = Y$, and let $(X \times Y, \mathcal{S} \times \mathcal{T}, \mathcal{Y}, \upsilon)$ be a complete topological probability space satisfying $[P_0]$, $[C]$, and $[S]$. For $\rho \in \Lambda(\nu)$ the following statements are all equivalent.

(i) $\rho \in \Lambda(\nu)$ is a strong $\mathcal{Y}$-marginal with respect to $\upsilon$.

(ii) $\langle \rho_\infty(Q_x) \rangle_{x \in X}$ is $\mathcal{Y}$-measurable for every $\mathcal{Y}$-measurable bounded stochastic process $\langle Q_x \rangle_{x \in X}$ over $(Y, T, \nu)$ and $Q_x \in C_l(Y) \cap \mathcal{L}_\infty(X)$ for (some) $x \in X$ implies $Q_x \leq \rho_\infty(Q_x)$.

(iii) $\langle \rho_0(Q_x) \rangle_{x \in X}$ is $\mathcal{Y}$-measurable for every $\mathcal{Y}$-measurable stochastic process $\langle Q_x \rangle_{x \in X}$ over $(Y, T, \nu)$ with values in $\mathbb{R}$ and $Q_x \in C_l(Y)$ for (some) $x \in X$ implies $Q_x \leq \rho_0(Q_x)$.

(iv) $\langle \rho_\infty(Q_x) \rangle_{x \in X}$ is $\mathcal{Y}$-measurable for every $\mathcal{Y}$-measurable bounded stochastic process $\langle Q_x \rangle_{x \in X}$ over $(Y, T, \nu)$ and $Q_x \in C_b(Y)$ for (some) $x \in X$ implies $\rho_\infty(Q_x) = Q_x$.

(v) $\langle \rho_0(Q_x) \rangle_{x \in X}$ is $\mathcal{Y}$-measurable for every $\mathcal{Y}$-measurable stochastic process $\langle Q_x \rangle_{x \in X}$ over $(Y, T, \nu)$ with values in $\mathbb{R}$ and $Q_x \in C(Y)$ for (some) $x \in X$ implies $\rho_0(Q_x) = Q_x$.

(vi) $\overline{\rho}(Q_x) \in \mathcal{Y}$-measurable for every $\mathcal{Y}$-measurable stochastic process $\langle Q_x \rangle_{x \in X}$ over $(Y, T, \nu)$ with values in a strongly lifting compact space $K$ equivalent to $\langle Q_x \rangle_{x \in X}$ and $Q_x$ continuous implies $\overline{\rho}(Q_x) = \rho'(Q_x) = Q_x$.

Statement (i) implies all other ones and (i) is equivalent to each of (ii), (iii), (j), and (jjj), if (jj) and (jjj) result from (ii) and (iii) by replacing $\mathcal{C}_l(Y)$ by $\mathcal{C}_b(Y)$ and $Q_x \leq \rho_\infty(Q_x)$ by $Q_x \geq \rho_\infty(Q_x)$' in (ii) and $Q_x \leq \rho_0(Q_x)$ by $Q_x \geq \rho_0(Q_x)$' in (iii). For uniformizable $Y$ all above statements are equivalent.
Proof. (i) implies all other statements by Theorem 4.8 Remark 4.2 and Remark 4.5 (g).

For the converse implications, note that clearly (vi) \(\Rightarrow\) (v) \(\Rightarrow\) (iv) and (iii) \(\Rightarrow\) (ii). So we left to show the implications (iii) \(\Rightarrow\) (i) and (iv) \(\Rightarrow\) (i). By Theorem 4.8 each of the statements (iii) \(\Rightarrow\) (i) and (iv) \(\Rightarrow\) (i) implies (i) except of \(\rho\) being \(G\)-strong. For this reason we left to show (i) if (iii) or (iv) holds true. For this reason for \(H \in \mathfrak{S}\) we consider the (constant) stochastic process \(\langle Q_x \rangle_{x \in X}\) defined by \(Q_x := [\chi_{x \times H}]_x = \chi_H\) for \(x \in X\). By (iii) we get \(\chi_H = Q_x \leq \rho_\infty(Q_x) = \rho_\infty(\chi_H) = \chi_{\rho(H)}\), implying \(H \subseteq \rho(H)\) for every \(H \in \mathfrak{S}\), i.e. \(\rho\) is strong.

If (iv) holds and \(Y\) is uniformizable (i.e. it is Hausdorff and \(3\frac{1}{2}\), see [27] 6.2.1), note \(\chi_H = \sup \mathcal{H}\) for \(\mathcal{H} := \{h : 0 \leq h \in C_0(Y)\}\) by [27] 6.3.4. For \(h \in \mathcal{H}\) we consider the (constant) stochastic process \(\langle Q_x \rangle_{x \in X}\) defined by \(Q_x := h\). By (iv) we get \(h = \rho_\infty(h) \leq \rho_\infty(\chi_H)\) for every \(h \in \mathcal{H}\), consequently \(\chi_H \leq \rho_\infty(\chi_H) = \chi_{\rho(H)}\), implying again \(H \subseteq \rho(H)\), i.e. \(\rho\) is strong. \(\square\)

Corollary 4.10. For given complete probability spaces \((X, \Sigma, \mu)\) and \((Y, T, \nu)\) we get for every \(\rho \in A(\nu)\) the following statements.

(i) \(\langle \rho_\infty(Q_x) \rangle_{x \in X}\) is \(\Sigma \otimes \Theta T\)-measurable for every \(\Sigma \otimes \Theta T\)-measurable bounded stochastic process \(\langle Q_x \rangle_{x \in X}\) over \((Y, T, \nu)\) and it is equivalent to \(\langle Q_x \rangle_{x \in X}\).

(ii) \(\langle \rho(Q_x) \rangle_{x \in X}\) is \(\Sigma \otimes \Theta T\)-measurable for every \(\Sigma \otimes \Theta T\)-measurable stochastic process \(\langle Q_x \rangle_{x \in X}\) over \((Y, T, \nu)\) with values in a strongly lifting compact space \(K\) and it is equivalent with \(\langle Q_x \rangle_{x \in X}\).

If we choose \(\rho \in A\Lambda(\nu) \neq \emptyset\) (see Example 3.6 in (i) and (ii) we can replace above \(\Sigma \otimes \Theta T\) by \(\Sigma \otimes T\) throughout.

Proof. We apply Theorem 4.8 for the complete probability space \((X \times Y, \mathcal{Y}, \mathcal{T}, \mu \otimes \nu)\) satisfying [C]. By Proposition 3.8 \(\rho\) is a \(Y\)-marginal with respect to \(\mu \otimes \nu\).

Every \(\rho \in A\Lambda(\nu)\) is a \(Y\)-marginal with respect to \(\Sigma \otimes T\) by definition, see Example 3.6 and again apply Theorem 4.8 \(\square\)

Remark 4.11. Under [CH] D.L. Cohn [4] considers topological probability spaces \((X, \mathfrak{S}, \Sigma, \mu)\) and \((Y, \mathfrak{T}, \mathcal{T}, \nu)\) with \((Y, T, \nu)\) complete, \(X\) a Borel subset of \(\mathbb{R}\), \(\mu\) a regular Borel measure, \(\nu\) the completion of a regular Borel measure. It can easily be seen, that every regular Borel measure \(\mu\) on a Hausdorff topological space (see D.L. Cohn [5] p.189/190 for definition) is \(\tau\)-additive and also its completed measure.

In this situation D.L. Cohn exhibits a \(\Sigma \otimes T\)-measurable stochastic process \(\langle Q_x \rangle_{x \in X}\) with values in the compact metrizable space \(K := [-\infty, \infty]\) (i.e. \(Q_x \in L^0(\nu)\)) and a \(\delta \in A\mathcal{T}(\nu)\) such that \(\langle \delta_0(Q_x) \rangle_{x \in X}\) is not \(\Sigma \otimes T\)-measurable.

D.L. Cohn’s process is on the basis of an example of R.M. Dudley [5] in such a manner that the function \(\overline{Q}\) defined by \(\overline{Q}(x, y) := \delta_0(Q_x)(y)\) for \((x, y) \in X \times Y\) is just the non \((\Sigma \otimes T)_{\mathfrak{M}}\)-measurable function (consequently not \((\Sigma \otimes T)_{\mathfrak{M}}\)-measurable function), where \(\mathfrak{M}\) denotes the system of all Bledsoe Morse
nilsets (not only $\Sigma \otimes T$-measurable function) constructed by R.M. Dudley, see [6, 5813]. Since $M = \mathfrak{M}$ by Remark 1.5, it follows that $\langle \delta_0(Q_x) \rangle_{x \in X}$ is not even $\Sigma \otimes \mathfrak{N}_2$ $T$-measurable, but according to Corollary 4.10 it can be chosen $\Sigma \otimes \mathfrak{N} T$-measurable, i.e. the worst case happens here.

Note to the contrary, that there always exists a $\delta \in A\theta(\nu) \cup A\Theta(\nu)$ (see Example 3.6) being a $Y$-marginal with respect to $\mu \otimes \nu$ and for these we can apply Corollary 4.10 too. Clearly these $\delta$ are $Y$-marginals with respect to $\mu \otimes \mathfrak{N}_2 \nu$.

In particular this witnesses, that the $\rho \in \Lambda(\mu)$ applied by D.L. Cohn (and due to Fremlin/Mokobodzki [10]) is not in $A\Theta(\nu)$ nor in $A\theta(\nu)$.

Concerning Corollary 4.9 let us mention first, that there are compact Radon probability spaces without strong lifting (see [13], 439S) and even when for such spaces $(X, S, \Sigma, \mu)$ and $(Y, T, T, \nu)$ there exist strong liftings it may happen, that for their ordinary product $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$ there exists no strong $Y$-marginal with respect to $\mu \otimes \nu$, e.g. we can take $(X, S, \Sigma, \mu) = (Y, T, T, \nu)$ the hyperstonian space of the Lebesgue probability space over $[0, 1]$ by [25] Theorem 5.1, where its unique canonical strong lifting $\sigma$ is no $Y$-marginal with respect to $\nu \otimes \nu$. Note that $Z_T(\nu) = \{\sigma\}$ for $Z = C, O, \Pi, F, \vartheta, \Lambda$, which means that in classes $Z_T(\nu)$ for $Z = C, O, \Pi, F, \vartheta$ providing less structure we don’t find strong $Y$-marginals with respect to $\nu \otimes \nu$.

The above raises also the question, whether in this case $\sigma$ is a $Y$-marginal with respect to $\nu \otimes \mathfrak{N}_2 \nu$ ? Fortunately we know, that the canonical strong lifting $\sigma$ of the above hyperstonian space is a $Y$-marginal with respect to $\nu \otimes \nu$ by Proposition 3.8. This means in this situation, that in Corollary 4.9 we can take $\Upsilon = \Sigma \otimes \mathfrak{N} T$, if we want regularization respecting (semi) continuous functions.

Remark 4.12. The above results apply Boolean liftings which are constructed by an application of the axiom of choice. We may therefore ask for results not using this axiom. Along these lines, it is well known, that densities can be obtained as Possel derivatives using Vitali derivation bases without the axiom of choice, see e.g. [30], p.1146. Let us state the following result sufficient for stochastic processes mostly used in applications, i.e. for extended real-valued stochastic processes generalizing Theorem 5.2 from [24].

(a) For complete probability spaces $(X, \Sigma, \mu)$, $(Y, T, \nu)$, and $(X \times Y, \Upsilon, \upsilon)$ satisfying $[C]$ and $\delta \in \vartheta(\nu)$ (the proof works even for $\delta \in M(\nu) \cap O(\nu)$) the following statements are all equivalent.

(i) $\delta$ is a $Y$-marginal with respect to $\upsilon$.

(ii) $\langle \delta_\infty(Q_x) \rangle_{x \in X}$ is $\Upsilon$-measurable for every $\Upsilon$-measurable bounded stochastic process $\{Q_x\}_{x \in X}$ over $(Y, T, \upsilon)$.

(iii) $\langle \delta_0(Q_x) \rangle_{x \in X}$ is $\Upsilon$-measurable for every $\Upsilon$-measurable stochastic process $\{Q_x\}_{x \in X}$ over $(Y, T, \upsilon)$ with values in $\overline{\mathbb{R}}$.

In (ii) we may replace $\delta_\infty$ by $\delta^\infty$ and in (iii) $\delta_0$ by $\delta^0$.  

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The proof of this result is by obvious changes in the proof of Theorem 4.8. 

Let \( p = 0, \infty \). As to be expected, for \( \delta \in \vartheta(\nu) \) the arithmetical rules for \( \delta_p \) and \( \delta^p \) are less effective as for \( \rho \in \Lambda(\nu) \) above, i.e. we have only \[ \delta_p \leq \delta^p, \delta(\infty, f), \delta(\infty, f) \in L^\infty(\mu) \text{ and } \| \delta_0(f) \|_\infty = \| \delta^0(f) \|_\infty \leq \| f \|_\infty := \sup_{x \in X} | f(x) | \] for every \( f \in L^\infty(\mu) \).

\( \delta_p \) and \( \delta^p \) satisfy (11), (12), (e).

For \( f, g \in L^p(\mu) \) get \( \delta_p(a f) = a \delta_p(f) \) and \( \delta^p(a f) = a \delta^p(f) \) for \( 0 \leq a \in \mathbb{R} \), \( \delta_p(f) \land \delta_p(g) = \delta_p(f \land g) \), and \( \delta^p(f) \lor \delta^p(g) = \delta^p(f \lor g) \) for \( f, g \in L^p(\mu) \).

For \( -\infty < a \leq f, g \in L^0(\mu) \), \( a \in \mathbb{R} \) get \( \delta_p(f) \delta_p(g) \leq \delta_p(f g) \leq \delta^p(f g) \). 

(c) If, in addition, there is given a topology \( \mathcal{F} \) over \( Y \) such that \( (Y, \mathcal{F}, T, \nu) \) is a topological probability space with \( \text{supp}(\nu) = X \) and a strong \( \delta \), it follows \( Q_x \leq \delta_p(Q_x), \delta^p(Q_x) \leq Q_x \), and \( \delta_p(Q_x) = \delta^p(Q_x) = Q_x \) for lower semi-continuous, upper semi-continuous, and continuous \( Q_x, x \in X \), for \( p = \infty \) in (ii) and \( p = 0 \) in (iii). Concerning the existence of strong marginals with respect to densities we have the same situation as for liftings discussed in Remark 4.11.

Remark 4.13. (Compare [24] Theorem 5.5 and Corollary 5.6) Given complete probability spaces \( (X, \Sigma, \mu) \) and \( (Y, T, \nu) \), call \( f : X \times Y \to \mathbb{R} \) separately measurable, if \( f_x \in L^0(\nu) \) for \( \mu \)-a.e. \( x \in X \) and \( f^y \in L^0(\mu) \) for \( \nu \)-a.e. \( y \in Y \). For the definition of stable sets of measurable functions compare [24] 9-1-1. Call a stochastic process \( \langle Q_x \rangle_{x \in X} \) separately measurable, if \( \mathcal{F} \) is. Call it stable, if \( \langle Q_x \rangle_{x \in X} \) is.

It follows by Theorem 4.8 and [24] Proposition 5.4, that the following statements are all equivalent for \( \rho \in \Lambda(\nu) \).

(i) \( \rho \) is a \( Y \)-marginal with respect to \( \mu \mathcal{F} \).

(ii) \( \langle \rho_0(Q_x) \rangle_{x \in X} \) is \( \Sigma \mathcal{F} T \)-measurable for every separately measurable and stable bounded stochastic process \( \langle Q_x \rangle_{x \in X} \) over \( (Y, T, \nu) \).

(iii) \( \langle \rho_0(Q_x) \rangle_{x \in X} \) is \( \mathcal{Y} \)-measurable for every separately measurable and stable stochastic process \( \langle Q_x \rangle_{x \in X} \) over \( (Y, T, \nu) \) with values in \( \mathbb{R} \).

(iv) \( \langle \rho_\infty(f_x) \rangle_{x \in X} \) is \( \Sigma \mathcal{F} \mathcal{T} \)-measurable for every separately measurable function \( f : X \times Y \to \mathbb{R} \) with \( \{ f_x : x \in X \} \subseteq L^\infty(\nu) \) stable.

(v) \( \langle \rho_0(f_x) \rangle_{x \in X} \) is \( \Sigma \mathcal{F} \mathcal{T} \)-measurable for every separately measurable function \( f : X \times Y \to \mathbb{R} \) with \( \{ f_x : x \in X \} \) stable.

5. Strong marginals and products

Given sets \( X, Y \) for \( \mathcal{A} \subseteq \mathcal{P}(X) \) put \( 3(\mathcal{A}) := \{ A \times Y : A \in \mathcal{A} \} \) and for \( \mathcal{Y} \subseteq \mathcal{P}(Y) \) write \( 3(\mathcal{Y}) := \{ X \times B : B \in \mathcal{Y} \} \).

Proposition 5.1. Let be given complete probability spaces \( (X, \Sigma, \mu) \), \( (Y, T, \nu) \), and \( (X \times Y, \mathcal{Y}, \nu) \).

(a) If \( 3(\mathcal{T}) \subseteq \mathcal{Y} \) for given \( \zeta \in \mathcal{P}(X \times Y) \) for \( B \in \mathcal{T} \) define \( \delta(B) := \{ y \in Y : [\zeta(X \times B)]^y = \mu X \} = \{ y \in Y : \mu([\zeta(X \times B)]^y) = 1 \} \).
(i) For $Y_0 \subseteq \Sigma_0 \times T_0$ it follows condition (L1) for $\delta$ if $\zeta(E) = E$ for every $E \in 3(T)$.
(ii) For $\mathcal{F}(T_0) \subseteq Y_0$ it follows condition (L2) for $\delta$ from (L2) for $\zeta$.
(iii) $\delta$ satisfies condition (Z), if $\zeta$ does for $Z = M, V, O, \Pi, F, C, \vartheta, \Lambda$.
(iv) Let be given topologies $\mathcal{G} \subseteq \Sigma$ and $\mathcal{T} \subseteq T$ over $X$ and $Y$, respectively, such that $\mathcal{G} \times \mathcal{T} \subseteq Y$. Then $\delta$ is $\mathcal{G} \times \mathcal{T}$-strong if $\zeta$ is $\mathcal{G} \times \mathcal{T}$-strong.

(v) If $\mathcal{F}(T_0) \subseteq Y_0 \subseteq \Sigma_0 \times T_0$ and there are given topologies $\mathcal{G}$, $\mathcal{T}$, and $\mathcal{G} \times \mathcal{T}$ over $X$, $Y$, and $X \times Y$ contained in $\Sigma$, $T$, and $Y$, respectively, then $Z_{\mathcal{G} \times \mathcal{T}}(v) \neq \emptyset$ implies $Z_{\mathcal{T}}(v) \neq \emptyset$ for $Z = P, M, V, O, \Pi, F, C, \vartheta, \Lambda$.

(b) If $(X \times Y, Y)$ satisfies [P] and there are given topologies $\mathcal{G}$, $\mathcal{T}$, and $\mathcal{G} \times \mathcal{T}$ over $X$, $Y$, and $X \times Y$ contained in $\Sigma$, $T$, and $Y$, respectively, then $Z_{\mathcal{G} \times \mathcal{T}}(v) \neq \emptyset$ implies $Z_{\mathcal{G}}(\mu), Z_{\mathcal{T}}(v) \neq \emptyset$ for $Z = P, M, V, O, \Pi, F, C, \vartheta, \Lambda$.

**Proof.** Ad (i): If $\zeta \in \mathcal{P}(X \times Y)^T$ satisfies (L1) we have $\zeta \in Y^T$ by completeness of $(X \times Y, Y, v)$. By $\zeta(X \times B) = v, X \times B$ for $B \in T$ we get with $Y_0 \subseteq \Sigma_0 \times T_0$ a $M_B \in T_0$ such that $[\zeta(X \times B)]^v = [X \times B]^v$ for every $y \notin M_B$, implying $[\zeta(X \times B)]^v \Delta B \subseteq M_B$ and by completeness of $(Y, T, v)$ it follows $\delta(B) = v, B$ and by $B \in T$ also $\delta(B) \in T$, i.e. $\delta$ satisfies (L1).

It is now routine to verify (ii) to (v).

(b) holds since [P] implies $\mathcal{F}(\Sigma), \mathcal{F}(T) \subseteq \Sigma \otimes T$ and $Y_0 \subseteq \Sigma_0 \times T_0 \cap \Sigma_0 \times T_0$ because $v(E) = \int v(E_\mu)(d\mu) = \int(\mu(\nu)(dy))$ for $E \in \Sigma \otimes T$.

**Example 5.2.** Let $(X, \mathcal{G}, \Sigma, \mu)$ be a compact Radon probability space with $\text{supp}(\mu) = X$ and $\Lambda_{\mathcal{G}}(\mu) = \emptyset$ (see [14] 439S) and let $(Y, \mathcal{T}, v)$ be the Lebesgue probability space over $[0, 1]$ satisfying $AG\Lambda_{\mathcal{G}}(\mu) = \emptyset$ by Example 3.6, consequently there exists a strong $Y$-marginal with respect to $\tilde{\beta}$ by Example 3.6 if we take their Radon product $(X \times Y, \mathcal{G} \times \mathcal{T}, \mathcal{B}(\mathcal{G} \times \mathcal{T}), \tilde{\beta} := \tilde{\beta}(\mathcal{G} \times \mathcal{T}))$.

Assume, if possible $\Lambda_{\mathcal{G} \times \mathcal{T}}(\tilde{\beta}) = \emptyset$. This implies $\Lambda_{\mathcal{G}}(\mu) = \emptyset$, a contradiction, i.e. $\Lambda_{\mathcal{G} \times \mathcal{T}}(\tilde{\beta}) = \emptyset$.

**Lemma 5.3.** Given complete probability spaces $(X, \Sigma, \mu)$ and $(Y, T, v)$ together with $\gamma \in \vartheta(\mu)$ and $\delta \in \vartheta(v)$ and a probability space $(X \times Y, \mathcal{Y}, v)$ satisfying [P] and $\tau_\gamma \times \tau_\delta \subseteq \mathcal{Y}$, put $\theta := \theta_{\tau_\gamma \times \tau_\delta}$ in $(\tau_\gamma \times \tau_\delta)^\mathcal{T}$.

(i) $[\theta(E)]_x \subseteq \delta([\theta(E)]_x)$ and $[\theta(E)]_y \subseteq \gamma([\theta(E)]_y)$ for every $E \in \mathcal{Y}$, $x \in X$, and $y \in Y$.
(ii) $\theta_{\tau_\gamma \times \tau_\delta} = \theta$ and $M_{\tau_\gamma \times \tau_\delta}(v) = M_{\tau_\gamma \times \tau_\delta}(v)$.
(iii) If $\gamma \in \Lambda(\mu), \delta \in \Lambda(v)$, and $\zeta \in \mathcal{P}(X \times Y)^T$ satisfies (\theta) and $\theta \leq \zeta$, it follows $\zeta \in \gamma \otimes \delta$.

**Proof.** Assertion (i) is immediate by definition of $\theta$.

Ad (ii): $\tau_\gamma \times \tau_\delta \subseteq \tau_\gamma \times \tau_\delta$ implies $\theta_{\tau_\gamma \times \tau_\delta} \leq \theta$. For $E \in \mathcal{Y}$, $A \in \tau_\gamma$, and $B \in \tau_\delta$ with $A \times B \subseteq_{\mathcal{Y}} E$ follows $A \times B \subseteq_{\mathcal{Y}} \gamma(A) \times \delta(B)$ and by [P]
\[ A \times B = \gamma(A) \times \delta(B), \text{ implying } \gamma(A) \times \delta(B) \subseteq E \text{ and then } \gamma(A) \times \delta(B) \subseteq \theta_{t_\gamma \times t_\delta}(E). \text{ This implies } \theta(E) \subseteq \theta_{t_\gamma \times t_\delta}(E). \]

Note that for \( \psi \in M^*(v) \) we have \( \psi \in M^*_\theta(A \times \delta) \) if and only if \( \psi \geq \theta_{t_\gamma \times t_\delta}(v) \) and \( \psi \in M^*_{t_\gamma \times t_\delta}(v) \) if and only if \( \psi \geq \theta. \) Now apply condition \( \theta = \theta_{t_\gamma \times t_\delta}(v). \)

Ad (iii) : By (ii) and [P] we get for \( A \in \Sigma \) and \( B \in T \) that
\[ \gamma(A) \times \delta(B) \subseteq \theta_{t_\gamma \times t_\delta}(\gamma(A) \times \delta(B)) = \theta_{t_\gamma \times t_\delta}(A \times B) = \theta(A \times B) \subseteq \zeta(A \times B) \]
and this implies \( (\gamma(A) \times Y)^c \subseteq \zeta(A^c \times Y) \subseteq [\zeta(A \times Y)]^c, \) i.e. \( \zeta(A \times Y) \subseteq \gamma(A) \times Y \) and in the same way \( \zeta(X \times B) \subseteq X \times \delta(B) \) and with (\text{\textit{iv}}) for \( \zeta \) we find
\[ \zeta(A \times B) = \zeta((A \times Y) \cap (X \times B)) = \zeta(A \times Y) \cap \zeta(X \times B) \leq (\gamma(A) \times Y) \cap (X \times \delta(B)) = \gamma(A) \times \delta(B), \]
therefore \( \zeta(A \times B) = \gamma(A) \times \delta(B). \)

By Proposition 5.1 (b), only the existence of strong liftings in both coordinates is necessary for the existence of strong lifting in the product, if the probability measure in the product space extends the 'ordinary' product probability measure. According to Example 5.2 the existence of a strong marginal in only one coordinate will not suffice for the conclusion of a strong lifting in the product. Crucially for the proof of the next Theorem is the application of the theta-operation \( \theta \) in order to start a transfinite induction (carried through in the form of Zorn’s Lemma). For this reason such a proof was impossible by the methods developed in [25].

**Theorem 5.4.** Let \((X, \Sigma, \mu), (Y, T, \nu)\) be complete probability spaces, \( \gamma \in \Lambda(\mu), \delta \in \Lambda(\nu), \) and let \((X \times Y, \Upsilon, \upsilon)\) be a complete probability space satisfying [P\text{\textsubscript{0}}], [C], [S], \( \mathcal{B}_{\gamma, \delta} \subseteq \Upsilon, \) and \( \upsilon | \mathcal{B}_{\gamma, \delta} = \beta_{\gamma, \delta}. \) If \( \delta \) is a \( Y \)-marginal with respect to \( \upsilon, \) there exists a \( \zeta_0 \in \Lambda(\nu) \) such that

(i) \( [\zeta_0(E)]_x = \delta([\zeta_0(E)]_x) \) for all \( E \in \Upsilon \) and \( x \in X, \)
(ii) \( \theta := \theta_{t_\gamma \times t_\delta}(v) \leq \zeta_0, \)
(iii) \( \zeta_0 \in \gamma \otimes \delta, \)
(iv) \( \zeta_0 \) is \( \tau_\gamma \times \tau_\delta \)-strong.

**Proof.** By Remark 5.7 Conditions [P\text{\textsubscript{0}}] and [S] imply [P] for \((X \times Y, \Upsilon, \upsilon).\)

Assume first that the probability space \((X, \Sigma, \mu)\) is purely atomic. We then may assume that \( X = \bigcup_{n \in \mathbb{N}} \{x_n\} \) with each point \( x_n \) being of positive \( \mu \)-measure. Define \( \zeta_0 \in \Lambda(\nu) \) by means of \( \zeta_0(E) := \bigcup_{n \in \mathbb{N}} \{x_n\} \times \delta(E_{x_n}) \) for all \( E \in \Upsilon. \) The lifting \( \gamma \in \Lambda(\mu) \) with \( \gamma(\{x_n\}) := \{x_n\} \) for all \( n \in \mathbb{N} \) is the unique lifting on the family of all subsets of \( X, \) and \( \tau_\gamma = \mathcal{P}(X). \) Clearly \( \zeta_0 \in \Lambda(\upsilon) \cap (\gamma \otimes \delta) \) satisfies all the above conditions (i) to (iv).

For this reason, we may and do assume for the rest of the proof that the space \((X, \Sigma, \mu)\) is non-atomic.

We define the set \( \mathcal{H} \) of all \( \zeta \in \mathcal{P}(X \times Y)^T \) such that

(a) \( \zeta \) satisfies (L2), (E), (N), (\text{\textit{iv}});
(b) \( \forall E \in \Upsilon \ (\zeta(E) \subseteq E); \)

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(c) \( \forall x \in X \forall E \in \Upsilon (|\zeta(E)|_x \in T \text{ and } |\zeta(E)|_x \subseteq \delta(|\zeta(E)|_x)); \\
(d) \theta \leq \zeta.

First let us note, that Lemmas 1.10 and 5.3 yield \( \theta \in \mathcal{H}, \) consequently \( \mathcal{H} \neq \emptyset. \) We consider \( \mathcal{H} \) under the partial order \( \preceq, \) defined for \( \zeta, \xi \in \mathcal{H} \) by \( \zeta \preceq \xi, \) if \( \zeta(E) \subseteq \xi(E) \) for each \( E \in \Upsilon. \)

Claim 1. There exists a maximal element in \( \mathcal{H}. \)

Proof. In view of Zorn’s Lemma it will suffice to show, that each chain \( \langle \zeta_{\alpha} \rangle_{\alpha \in A} \subseteq \mathcal{H} \) has a dominating element in \( \mathcal{H}. \) Such an element \( \zeta \) is given for each \( E \in \Upsilon \) by \( \zeta(E) := \bigcup_{\alpha \in A} \zeta_{\alpha}(E). \) Clearly \( \zeta \) satisfies (a) and (d).

Since \( |\zeta(E)|_x = \bigcup_{\alpha \in A} |\zeta_{\alpha}(E)|_x \text{ and } |\zeta_{\alpha}(E)|_x \subseteq \delta(|\zeta_{\alpha}(E)|_x) \) for all \( \alpha \in A, \) all \( x \in X, \) and all \( E \in \Upsilon, \) it follows from [30], Theorem 3.9, that \( |\zeta(E)|_x \in T \) and \( |\zeta(E)|_x \subseteq \delta(|\zeta(E)|_x), \) consequently
\[
|\zeta(E)|_x = \bigcup_{\alpha \in A} |\zeta_{\alpha}(E)|_x \subseteq \bigcup_{\alpha \in A} \delta(|\zeta_{\alpha}(E)|_x) \subseteq \delta(|\zeta(E)|_x),
\]
implies (c) for \( \zeta. \)

For proving (b), by [S] we may choose for every \( F \in \Upsilon \) a set \( E \in \Upsilon \) with \( E =_v F \) and \( E \in T \) for every \( x \in X. \) Since \( \zeta \) satisfies (L2), it is sufficient to prove (b) for \( E \) instead of \( F. \) For each \( \alpha \in A \) put \( N_{E,\alpha} := \zeta_{\alpha}(E) \setminus E \in \Upsilon_0. \) We then get
\[
(7) \quad |\zeta_{\alpha}(E)|_x \setminus E_x = |N_{E,\alpha}|_x \in T \quad \text{for all } \alpha \in A \text{ and all } x \in X,
\]
while by [C] for each \( \alpha \in A \) there exists a null set \( M_{E,\alpha} \in \Sigma_0 \) such that \( |N_{E,\alpha}|_x \in T_0 \) for all \( x \notin M_{E,\alpha}. \)

By condition (7) we obtain for all \( x \in X \) and all \( \alpha \in A \) that
\[
\delta(|\zeta_{\alpha}(E)|_x \setminus E_x) = \delta(|N_{E,\alpha}|_x) \in T,
\]
implying, - in virtue of [30], Theorem 3.9 -, that
\[
\bigcup_{\alpha \in A} \delta(|\zeta_{\alpha}(E)|_x \setminus E_x) = \bigcup_{\alpha \in A} \delta(|N_{E,\alpha}|_x) \in T
\]
and
\[
\nu\left(\bigcup_{\alpha \in A} \delta(|\zeta_{\alpha}(E)|_x \setminus E_x)\right) = \lim_{\alpha \in A} \nu\left(\delta(|N_{E,\alpha}|_x)\right),
\]
the latter since \( \langle \delta(|N_{E,\alpha}|_x)_{\alpha \in A} \rangle \) is an increasing family of open sets in \( \tau_3 \) and \( \nu \) is \( \tau \)-additive with respect to the topology \( \tau_3 \) (see [22], Proposition 3.11). Consequently, there exists a sequence \( \langle \alpha_n \rangle_{n \in \mathbb{N}} \) in \( A \) such that
\[
\bigcup_{\alpha \in A} \delta(|N_{E,\alpha}|_x) = \bigcup_{n \in \mathbb{N}} \delta(|N_{E,\alpha_n}|_x) \text{ a.e.}(\nu).
\]
This implies the existence of a set \( M_E := \bigcup_{n \in \mathbb{N}} M_{E,\alpha_n} \in \Sigma_0 \) such that \( \bigcup_{\alpha \in A} \delta(|N_{E,\alpha}|_x) \in T_0 \) for all \( x \notin M_E. \)
From the latter together with condition (7) we obtain together with the completeness of \((Y,T,\nu)\) that

$$\bigcup_{\alpha \in A} \delta \left( [\zeta_{\alpha}(E)]_x \setminus E_x \right) \in T_0 \quad \text{for all} \quad x \notin M_E.$$  

But then \(\bigcup_{\alpha \in A} \delta([\zeta_{\alpha}(E)]_x) \setminus E_x \in T_0\) for all \(x \notin M_E\), since for all \(x \notin M_E\) we get

$$\bigcup_{\alpha \in A} \delta \left( [\zeta_{\alpha}(E)]_x \setminus E_x \right) = \bigcup_{\alpha \in A} \delta([\zeta_{\alpha}(E)]_x) \cap \delta(E_x^c) =_\nu \bigcup_{\alpha \in A} \delta([\zeta_{\alpha}(E)]_x) \setminus E_x.$$  

So taking into account the inclusion \(\bigcup_{\alpha \in A} [\zeta_{\alpha}(E)]_x \subseteq \bigcup_{\alpha \in A} \delta([\zeta_{\alpha}(E)]_x)\) we infer that \([\zeta(E)]_x \setminus E_x \in T_0\) for all \(x \notin M_E\). Therefore we obtain (remember \([\zeta(E)]_x \in T\) by \([30], \text{Theorem} \ 3.9\))

$$\zeta(E) \cap (M_E^c \times Y) = \{(x,y) \in M_E^c \times Y : y \in [\zeta(E)]_x\}$$

$$\subseteq \{(x,y) \in M_E^c \times Y : y \in \delta([\zeta(E)]_x)\}$$

$$\subseteq \{(x,y) \in M_E^c \times Y : y \in \delta(E_x)\}$$

$$= \delta_*(E) \cap (M_E^c \times Y),$$

implying \(\zeta(E) \subseteq_\nu \delta_*(E) =_\nu E\), since \(\delta\) is \(Y\)-marginal with respect to \(\nu\) (see Definition \([3.4]\) and Lemma \([3.3]\)). But the latter yields \(\zeta(E) \setminus E \subseteq_\nu \delta_*(E) \setminus E \in \Upsilon_0\) (the latter again by Lemma \([3.3]\)). Therefore, completeness of \((X \times Y, \Upsilon, \nu)\) implies \(\zeta(E) \setminus E \in \Upsilon_0\), i.e. \((b)\) holds true.  

In view of Claim 1 we can choose a maximal element \(\zeta_0\) of \(\mathcal{H}\) by Zorn’s Lemma, satisfying \((a), (b), (c),\) and \((d)\). Since condition \([P]\) holds true for \((X \times Y, \Upsilon, \nu)\), the equality \(\theta = \theta_{\nu} \times_\delta \nu\) follows by Lemma \([5.4]\) (ii), implying together with \((d)\) assertion (ii). By the same Lemma \([5.3]\) (iii), assertion (iii) follows by (ii), while assertion (iv) is immediate by (ii).

Claim 2. For every \(E \in \Upsilon\) and \(x \in X\)

$$[\zeta_0(E)]_x \in T \quad \text{and} \quad \nu([\zeta_0(E)]_x \cup [\zeta_0(E^c)]_x) = 1.$$  

**Proof.** Since \(\zeta_0\) satisfies (c) we have \([\zeta_0(E)]_x \in T\) for every \(E \in \Upsilon\) and \(x \in X\). Assume that there exist \(H \in \Upsilon\) and \(x_0 \in X\) such that \([\zeta_0(H)]_{x_0} \cup [\zeta_0(H^c)]_{x_0} \neq Y\) a.e.\(\nu\). Let \(W := \delta \left( ([\zeta_0(H)]_{x_0} \cup [\zeta_0(H^c)]_{x_0})^c \right) \) and let \(\tilde{\zeta}\) be defined by means of

$$[\tilde{\zeta}(E)]_x := \begin{cases} [\zeta_0(E)]_{x_0} \cup (W \cap [\zeta_0(H \cup E)]_{x_0}) & \text{if} \quad x \neq x_0 \\ [\zeta_0(E)]_{x_0} & \text{if} \quad x = x_0 \end{cases}$$

for \(E \in \Upsilon\). It is clear, that \(\zeta_0(E) \subseteq \tilde{\zeta}(E)\) for each \(E \in \Upsilon\); hence \(\tilde{\zeta}\) is \(\tau_{\gamma} \times \tau_{\delta}\)-strong, and it is easy to check that \(\tilde{\zeta}\) possesses all the density properties except of \((L1)\). Consequently, applying Lemma \([1, 10]\) we get \(\theta \leq \tilde{\zeta}\), i.e. \(\tilde{\zeta}\) satisfies \((d)\). Clearly \([\tilde{\zeta}\) satisfies condition (c)].
To show that \( \tilde{\zeta} \in \mathcal{H} \), let us fix an arbitrary \( E \in \Upsilon \). We then infer that
\[
\tilde{\zeta}(E) \cap (\{x_0\}^c \times Y) = \bigcup_{x \in \{x_0\}^c} \left( \{x\} \times [\tilde{\zeta}(E)]_x \right) = \bigcup_{x \in \{x_0\}^c} \left( \{x\} \times [\zeta_0(E)]_x \right) 
\]
\[
= [\zeta_0(E)]_x \cap (\{x_0\}^c \times Y) \subseteq \nu \cap (\{x_0\}^c \times Y).
\]
By completeness of \((X \times Y, \Upsilon, \nu)\) the latter yields condition \( \tilde{\zeta}(E) \subseteq \nu \), i.e. condition \((b)\) holds true for \( \tilde{\zeta} \). Consequently \( \zeta \in \mathcal{H} \). Since \([\tilde{\zeta}(H)]_{x_0} = [\zeta_0(H)]_{x_0} \cup W \neq [\zeta_0(H^c)]_{x_0} \), we see that \( \zeta \) and \( \zeta_0 \) are different maps, contradicting the maximality of \( \zeta_0 \).

Ad \((i)\): Again \([\zeta_0(E)]_x \in T\) for every \( E \in \Upsilon \) and \( x \in X \) by \((c)\). Let \( E \in \Upsilon \) and \( x \in X \). Since by Claim 2 we have \([\zeta_0(E)]_x \cup [\zeta_0(E^c)]_x = \nu \), it follows \( \delta([\zeta_0(E)]_x) \cup \delta([\zeta_0(E^c)]_x) = \nu \), i.e. \( \delta([\zeta_0(E^c)]_x) = \nu \), implying \( \zeta_0 \in \mathcal{H} \) as \( \zeta \) is a \( \nu \)-marginal with respect to \( \zeta_0 \).

By completeness of \((X \times Y, \Upsilon, \nu)\) the latter yields \( \zeta_0 \in \mathcal{H} \), i.e. condition \((b)\) holds true for \( \zeta_0 \), while \((c)\) is obvious by definition for \( \zeta \), consequently \( \zeta, \zeta_0 \in \mathcal{H} \) and the maximality of \( \zeta_0 \) implies \( \zeta = \zeta_0 \), i.e. \((i)\) holds true.

By \((b)\) for \( E^c \) instead of \( E \in \Upsilon \) it follows with \((C)\) for \( \zeta_0 \), that \( E \backslash [\zeta_0(E)] = [\zeta_0(E^c)] \subseteq \Upsilon_0 \). But also \( \zeta_0 \backslash [\nu, \zeta_0] \subseteq \Upsilon_0 \), implying \( E \subseteq [\nu, \zeta_0] \subseteq \Upsilon_0 \) and by completeness of \((X \times Y, \Upsilon, \nu)\) we obtain \( \zeta_0(E) \in \Upsilon_0 \) and \( \zeta_0(E) = \nu \), i.e. \( \zeta_0 \) satisfies the condition \((L1)\), implying that \( \zeta \in \Lambda(\nu) \).

**Corollary 5.5.** Let \((X, \mathcal{G}, \Sigma, \mu), (Y, \mathcal{T}, T, \nu)\) be complete topological probability spaces with \( \tau \)-additive probability measures \( \mu, \nu \) and \( \text{supp}(\mu) = X \), \( \text{supp}(\nu) = Y \), \( \gamma \in \Lambda_0(\mu), \delta \in \Lambda_\Sigma(\nu), \) and let \((X \times Y, \Upsilon, \nu)\) be a complete probability space satisfying \([P_0]\), \([C]\), \([S]\), \( \mathcal{B}_{\gamma, \delta} \subseteq \Upsilon \), and \( \nu|_{\mathcal{B}_{\gamma, \delta}} = \beta_{\gamma, \delta} \). It follows that \((X \times Y, \mathcal{G} \times \mathcal{T}, \Upsilon, \nu)\) is a complete topological probability space with \( \tau \)-additive probability measure \( \nu \), \( \text{supp}(\nu) = X \times Y \), \( \mathcal{B} \subseteq \mathcal{B}(\mathcal{G} \times \mathcal{T}) \subseteq \mathcal{B}(X) \times \mathcal{B}(\mathcal{T}) \subseteq \mathcal{B}(X) \times \mathcal{B}(\mathcal{T}) \), \( \beta_{\gamma, \delta}|_{\mathcal{B}} = \nu|_{\mathcal{B}} = \mu \otimes \epsilon \), \( \nu \). If, in addition, \( \delta \) is a \( \nu \)-marginal with respect to \( \nu \), there exists a \( \zeta \in \Lambda_{\mathcal{G} \times \mathcal{T}}(\nu) \) such that
\[
(i) \ [\zeta(E)]_x = \delta([\zeta(E)]_x) \quad \text{for all} \ E \in \Upsilon \text{ and } x \in X,
\]
The latter implies for particular, taking $(X \times Y, \nu) := (X \times Y, \Sigma \otimes \mathcal{R} T, \mu \otimes \mathcal{R} \nu)$ the above holds true for arbitrary $\delta \in \Lambda_{\Sigma}(\nu)$, where

$$\mathcal{B}(\Sigma) \otimes \mathcal{B}(\Xi) \subseteq \mathcal{B}(\Sigma \times \Xi) \subseteq \mathcal{B}(\Sigma) \subseteq \Sigma \otimes \mathcal{R} T.$$ 

**Proof.** $\gamma \in \Lambda_{\Sigma}(\mu)$ and $\delta \in \Lambda_{\Sigma}(\nu)$ implies $\Sigma \subseteq \tau_\gamma$ and $\Xi \subseteq \tau_\delta$, $\mathcal{B}(\Sigma) \otimes \mathcal{B}(\Xi) \subseteq \mathcal{B}(\Sigma \times \Xi) \subseteq \mathcal{B}_{\gamma,\delta} \subseteq \mathcal{Y}$, and $\nu|\mathcal{B}(\Sigma \times \Xi) = \beta(\Sigma \times \Xi)$ by [C]. By Definition 1.5 $\beta := \beta(\Sigma \times \Xi)$ is $\tau$-additive and $\text{supp}(\beta) = X \times Y$ and this implies, that $\nu$ is $\tau$-additive and $\text{supp}(\nu) = X \times Y$.

By Theorem 5.4 there exists a $\zeta \in \Lambda(\nu) \cap \gamma \otimes \delta$ with $[\zeta(E)]_x = \delta ([\zeta(E)]_x)$ for all $E \in \mathcal{Y}$ and $x \in X$, and $\theta_{\tau_\gamma \times \tau_\delta, \nu} \leq \zeta$. $\Sigma \times \Xi \subseteq \tau_\gamma \times \tau_\delta$ implies $\theta_{\Sigma \times \Xi, \nu} \leq \theta_{\tau_\gamma \times \tau_\delta, \nu} \leq \zeta$. The latter implies that $\zeta$ is $\Sigma \times \Xi$- and $\tau_\gamma \times \tau_\delta$-strong.

In case $(X \times Y, \Sigma \times \Xi, \nu) := (X \times Y, \Sigma \times \Xi, \Sigma \otimes \mathcal{R} \mu, \mu \otimes \mathcal{R} \nu)$ note that by Proposition 3.3 every $\delta \in \Lambda_{\Sigma}(\nu)$ is a $Y$-marginal with respect to $\mu \otimes \mathcal{R} \nu$. □

**Lemma 5.6.** For a compact Radon probability space $(X, \Sigma, \mu)$ with $\Sigma := \hat{\mathcal{B}}(\Sigma)$ and $\Lambda_{\Sigma}(\mu) \neq \emptyset$, and its hyperstonian space $(Y, \Xi, \nu)$ there exists a continuous inverse measure preserving surjection $g : Y \to X$. If $\mathcal{I} := (\mathcal{B}(\Xi \times \mathcal{I}))_0$ and $J := (\mathcal{B}(\Xi \times \mathcal{G}))_0$ we consider $\mathcal{Y} := \mathcal{B}(\Xi \times \mathcal{G})$, $\mathcal{V} := \nu \otimes \mathcal{R} \mu$, $\mathcal{Y} := (\nu \otimes \mathcal{R} \nu)$. 

(i) $\mathcal{B}(\Sigma \times \Xi) = (id_\mathcal{Y} \times g)^{-1}[\mathcal{B}(\Sigma \times \Xi)]$, $\mathcal{V} = \nu \circ (id_\mathcal{Y} \times g)^{-1}$, $(id_\mathcal{Y} \times g)^{-1}[\mathcal{Y}] \subseteq \mathcal{Y}$ and $\mathcal{Y} = \sigma(\mathcal{Y} \cup \mathcal{I})$.

(ii) The canonical strong lifting $\sigma$ is a $Y$-marginal with respect to $\nu \otimes \mathcal{R} \nu$, if there exists a $X$-marginal $\rho \in \Lambda_{\Sigma}(\mu)$ with respect to $\nu \otimes \mathcal{R} \mu$.

**Proof.** By 4.11(c) the map $g$, defined as in 1.17, is continuous and inverse measure preserving and there exists an inverse measure preserving map $h : X \to Y$ such that $g(h(x)) = x$ for every $x \in X$. The latter equation implies the surjectivity of $g$.

Ad (i): By 4.11(c) for the canonical homomorphism $\pi : \Sigma \to \Sigma/\mu$ and the closed-and-open set $s(\pi(K))$ in $Y$ corresponding to $\pi(K) \in \Sigma/\mu$ under the Stone isomorphism the map $g$ is defined by saying that $K \subseteq X$ is compact (i.e. closed) and $z \in s(\pi(K))$ then $g(z) \in K$. This implies $g^{-1}[K] = s(\pi(K))$. Since the closed-and-open subsets of $Y$ form a basis for the topology $\Xi$, for closed subset $A$ in $Y$ there exist compact $K_i$ in $X$ for $i \in I$, such that $A = \bigcap_{i \in I} s(\pi(K_i)) = \bigcap_{i \in I} g^{-1}[K_i] = g^{-1}\big[\bigcap_{i \in I} K_i\big]$. For closed set $\bigcap_{i \in I} K_i$ in $X$, proving

$$\mathcal{I} = g^{-1}[\mathcal{G}].$$

The latter implies

$$\mathcal{B}(\mathcal{I} \times \mathcal{I}) = (id_\mathcal{Y} \times g)^{-1}[\mathcal{B}(\mathcal{I} \times \mathcal{I})].$$
For $E \subseteq Y \times X$ we have $|(\text{id}_Y \times g)^{-1}[E]|_y = g^{-1}[E_y]$ for every $y \in Y$. For $E \in T \otimes \Sigma$ it follows
\[
(\nu \otimes \nu)((\text{id}_Y \times g)^{-1}[E]) = \int \nu(g^{-1}[E_y])\nu(dy) = \int \mu(E_y)\nu(dy) = (\nu \otimes \mu)(E),
\]
i.e.
\[
(11) \quad \nu \otimes \mu = (\nu \otimes \nu) \circ (\text{id}_Y \times g)^{-1}.
\]
For $E \in \mathcal{Y}$ there exists a set $F \in \mathcal{B}(\mathcal{X} \times \mathcal{S})$ such that $E = F \nu \otimes_R \mu$-a.e.. Consequently, applying (11) we get
\[
(\nu \otimes_R \mu)(E) = (\nu \otimes \mu)(F) = (\nu \otimes \nu)((\text{id}_Y \times g)^{-1}[F]) = (\nu \otimes_R \nu)((\text{id}_Y \times g)^{-1}[E]);
\]
and by (9) this implies that $\nu \otimes\nu$ is $\delta$-marginal with respect to $\nu \otimes \sigma$, i.e.

\[
\delta(g^{-1}[G]) = g^{-1}[\rho(G)] \supseteq g^{-1}[G]
\]
and by (9) this implies that $\delta$ is $\mathcal{X}$-strong, i.e. $\delta = \sigma$. By Proposition 3.10 (iii), this implies, that $\sigma$ is a $Y$-marginal with respect to $\nu \otimes_R \nu$. \hfill \box

**Corollary 5.7.** Let be given a compact Radon probability space $(X, \mathcal{S}, \Sigma, \mu)$ with $\Sigma := \mathcal{B}(\mathcal{S})$.

(i) $AG \Lambda_\mu \neq \emptyset$ implies that the canonical strong lifting $\sigma$ of its hyperstonian space is a $Y$-marginal with respect to $\nu \otimes \mathcal{Z}$ and $\Lambda_\Sigma(\nu \otimes \mathcal{Z}) \cap (\sigma \otimes \sigma) = \emptyset$. For $\mathcal{Z} = \mathcal{N}_2, R$.

(ii) The conclusion of (i) holds true for every non-atomic compact Radon probability space $(X, \mathcal{S}, \Sigma, \mu)$ with second countable $(X, \mathcal{S})$ having a countable basis $\{B_n : n \in \mathbb{N}\}$ such that $\mu(\partial B_n) = 0$ for every $n \in \mathbb{N}$ (if $X, \mathcal{S}$ is regular, then it is metrizable and this condition is satisfied, in particular for Polish spaces).

**Proof.** Ad (i): By Example 3.6 the set of all strong admissibly generated (Boolean) liftings $AG \Lambda_\mu$ for $\mu$ is non-empty and every $\rho \in AG \Lambda_\mu$ is a $Y$-marginal with respect to $\nu \otimes_R \mu$. By Lemma 5.6 (ii), this implies, that the canonical strong lifting $\sigma$ of its hyperstonian space $(Y, \mathcal{X}, T, \nu)$ is a $Y$-marginal with respect to $\nu \otimes_R \nu$. Since the complete topological probability
space \((Y \times Y, \mathcal{I} \otimes \mathcal{I}, \nu \otimes_R \nu)\) satisfies \([P_0], [C], [S]\), and \(\mathcal{B}_{\sigma,\sigma} \subseteq \mathcal{B}(\mathcal{I} \otimes \mathcal{I})\) by Lemma 2.1 (i), and \(\nu \otimes_R \nu|\mathcal{B}_{\sigma,\sigma} = \beta_{\sigma,\sigma}\) it follows by Corollary 5.5 that \(\Lambda_{\mathcal{I} \otimes \mathcal{I}}(\nu \otimes_R \nu) \cap (\sigma \otimes \sigma) \neq \emptyset\).

Since \(I \subseteq \mathcal{N}_2 \subseteq \mathcal{N}\), the probability space \((Y \times Y, T \otimes \mathcal{N}_2, T, \nu \otimes \mathcal{N}_2)\) satisfies \([P_0]\) and \([C]\), and \(\sigma\) is \(Y\)-marginal with respect to \(\nu \otimes \mathcal{N}_2\), it follows by Lemma 5.5 that \(\Lambda_{\mathcal{I} \otimes \mathcal{I}}(\nu \otimes \mathcal{N}_2) \cap (\sigma \otimes \sigma) \neq \emptyset\).

(ii) is immediate from (i) by Example 3.6.

Remarks 5.8. (a) In [20] it was seen, that under the continuum hypothesis for the Radon product of the hyperstonian space \((Y, \mathcal{I}, T, \nu)\) with its self of a non-atomic topological probability space with topology of weight \(\leq \mathfrak{c}\) (the cardinality of \(\mathbb{R}\)), it follows \(\Lambda_{\mathcal{I} \otimes \mathcal{I}}(\nu \otimes_R \nu) \cap (\sigma \otimes \sigma) \neq \emptyset\) by applying the result of Mokobodzki/Fremlin [10] already applied by Cohn in [4] in a situation where it turned out that such a lifting was not a \(Y\)-marginal with respect to a probability measure generated by adjoining the two-sided nil sets to the 'usual product probability', see Remark 4.11.

(b) In [25], Theorem 5.1, it has been proven that the canonical strong \(\sigma\) lifting of the hyperstonian space \((Y, \mathcal{I}, \mathcal{B}_c(Y), \nu)\) of the Lebesgue probability space on \([0, 1]\) is not a \(Y\)-marginal with respect to \(\nu \otimes \mathcal{N}_2\). In the same paper [25], Question 5.1, it was raised the question what is the situation in case of Radon product of hyperstonian spaces. Corollary 5.7 answers to the positive the above question in the case of hyperstonian spaces of non-atomic compact Radon probability spaces \((X, \mathcal{G}, \Sigma, \mu)\) with second countable topological space \((X, \mathcal{G})\) having a countable basis \(\{B_n : n \in \mathbb{N}\}\) such that \(\mu(\partial B_n) = 0\) for every \(n \in \mathbb{N}\) (in particular this covers the case of Polish probability spaces). But note, that \(\sigma\) is neither an admissible density nor an admissibly generated lifting by Example 3.6. The same Corollary 5.7 answers to the positive the part of Problem 457Z (a) from [14], concerning the strong lifting problem for products of hyperstonian spaces at least in case of hyperstonian spaces of Polish probability spaces.

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