Exact Analytical Solution of Ivancevic Options Pricing Model (IOPM) or Schrödinger’s Equation via ADM and SBA Methods

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Authors’ contributions
This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information
DOI: 10.9734/ARJOM/2022/v18i330366

Received: 20 January 2022
Accepted: 24 March 2022

Review Article
Published: 08 April 2022

Abstract
This paper is devoted to the study of the general equation of the Ivancevic option pricing model (IOPM) or Schrödinger’s equation and to determine its analytical solution via the methods of numerical analysis ADM and SBA. The Ivancevic option pricing model is an adaptive wave model that is a nonlinear wave alternative to the standard Black-Scholes option pricing model, it is also a model that links quantum mechanics and financial mathematics.

Keywords: Ivancevic or Shrodinger model; Adomian Method (ADM); SBA method; Successive approximations.

2010 Mathematics Subject Classification: 35A25; 9324; 97I50; 44Axx.

1 Introduction
The classical Black-Scholes model (BSM) is an important financial model for option pricing an valuation. In this paper, we are interested in the determination of the analytical solution of the
general equation of the Ivancevic [1] or Schrödinger [2] model in quantum mechanics. It is about
the equations:

\[(E) : \begin{cases} \\
\frac{i}{\tau} \frac{\partial w(t, x)}{\partial t} + \varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} + q |w(t, x)|^{2p} w(t, x) = 0; p \in \mathbb{N}^* \\
w(0, x) = \beta e^{iax} 
\end{cases}\]
and

\[(F) : \begin{cases} \\
\frac{i}{\tau} \frac{\partial w(t, x)}{\partial t} + \varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} + \mu w(t, x) + q |w(t, x)|^{2p} w(t, x) = 0; p \in \mathbb{N}^* \\
w(0, x) = \beta e^{iax} 
\end{cases}\]

where \(\varepsilon > 0, \mu > 0\) and \(q > 0\).

2 Description of Numerical Method ADM and SBA

2.1 Numerical method ADM

Consider the functional equation below:

\[Fw = f \quad (2.1)\]

where \(F\) is an operator defined in the Hilbert space \(H\) in \(H\), \(f\) is a given function in \(H\) and \(w\) is
the unknown function. Let us decompose as follows

\[F = L - R - N \quad (2.2)\]

Where \(L\) is the linear part of inverse \(L^{-1}\), \(R\) the linear remainder and \(N\) the nonlinear part, (2.1)
becomes:

\[Lw - Rw - Nw = f \quad (2.3)\]

Applying \(L^{-1}\) to (2.3), we get the Adomian canonical form [3]:

\[w = \theta + L^{-1} f + L^{-1} Rw + L^{-1} Nw \quad (2.4)\]

where

\[L\theta = 0.\]

Let us determine the solution of (2.1) in the form of a convergent series [4]

\[w = \sum_{n=0}^{+\infty} w_n \]

and

\[Nw = \sum_{n=0}^{+\infty} A_n < +\infty \]

where the

\[A_n = A_n (w_0, w_1, ..., w_n)\]

are Adomian polynomials [5]. We get the following Adomian algorithm [6]:

\[\begin{cases} w_0 = \theta + L^{-1} f \\
w_{n+1} = L^{-1} Rw_n + L^{-1} A_n; n \geq 0. \end{cases}\]
2.2 The Adomian polynomials

**Definition 2.1.** The Adomian polynomials are defined by:
\[
\begin{align*}
A_0 & = N(w_0) \\
A_n & = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{k=0}^{\infty} \lambda^k w_k \right) \right]_{\lambda=0} : n \geq 1
\end{align*}
\]

**Theorem 2.1.** The Adomian polynomials are calculated using the formula:
\[
\left[ \frac{d^n}{d\lambda^n} \sum_{k=0}^{n} \lambda^k A_k \right]_{\lambda=0} = \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{k=0}^{n} \lambda^k w_k \right) \right]_{\lambda=0}
\]

2.3 Numerical method SBA

Consider the functional equation below:
\[
F w = f
\]
where \( F \) is an operator defined in the Hilbert space \( H \) in \( H \), \( f \) is a given function in \( H \) and \( w \) is the unknown function. Let us decompose as follows
\[
F = L - R - N
\]

Where \( L \) is the linear part of inverse \( L^{-1} \), \( R \) the linear remainder and \( N \) the nonlinear part, (2.1) becomes:
\[
Lw - Rw - Nw = f
\]

Applying \( L^{-1} \) to (2.3), we get the Adomian canonical form:
\[
w = \theta + L^{-1}f + L^{-1}Rw + L^{-1}Nw
\]

where
\[
L \theta = 0.
\]
Equation (2.5) is the Adomian canonical form [7]. Using the successive approximations [1], we get:
\[
w^k = \theta + L^{-1}(f) + L^{-1}(R(w^k)) + L^{-1}(N(w^{k-1})); k \geq 1
\]

This let’s to the following Adomian algorithm:
\[
\begin{align*}
\{ w_0^k & = \theta + L^{-1}(f) + L^{-1}(N(w^{k-1})); k \geq 1 \\
w_n^k & = L^{-1}(R(w_{n-1}^k)), n \geq 1 \}
\end{align*}
\]

(2.10)

The Picard principle is then applied to equation (2.7): let \( w^0 \) be such that \( N(w^0) = 0 \), for \( k = 1 \), we get:
\[
\begin{align*}
\{ w_0^1 & = \theta + L^{-1}(f) + L^{-1}(N(w^0)) \\
w_n^1 & = L^{-1}(R(w_{n-1}^1)), n \geq 1 \}
\end{align*}
\]

(2.11)

If the series \( \sum_{n=0}^{\infty} w_n^1 \) converges, then \( w^1 = \left( \sum_{n=1}^{\infty} w_n^1 \right) \)

For \( k = 2 \), we get:
\[
\begin{align*}
\{ w_0^2 & = \theta + L^{-1}(f) + L^{-1}(N(w^1)) \\
w_n^2 & = L^{-1}(R(w_{n-1}^2)), n \geq 1 \}
\end{align*}
\]

(2.12)
Consider the following Cauchy problem:

$w(x, t) = \varepsilon \Delta w(x, t) + \mu w(x, t) + Nw(t, x), (t, x) \in \Omega$

$w(0, x) = h(x)$

Associated to the problem $(p)$, the SBA algorithm is given as:

$$(pSBA) : \begin{cases} w_0^n(t, x) = h(x) + L_t^{-1} \left[ N(w^{k-1}(t, x) \right]; k \geq 1 \\ w_{n+1}^1(t, x) = L_t^{-1} \left[ \varepsilon \Delta w_n^1(t, x) + \mu w_n^1(t, x) \right]; n \geq 0 \end{cases}$$

$(H_1) :$ There is $w^0(t, x)$ at the step $k = 1$, such as $Nw^0(t, x) = 0$.

$(H_2) :$ At the step $k = 1$, $w^1(t, x)$ is the solution of:

$$\begin{cases} w_0^1(t, x) = h(x) \\ w_{n+1}^1(t, x) = L_t^{-1} \left[ \varepsilon \Delta w_n^1(t, x) + \mu w_n^1(t, x) \right]; n \geq 0 \end{cases}$$

$(H_3) :$ At the step $k = 2$, $Nw^1(t, x) = 0$. So, the algorithm:

$$(pSBA) : \begin{cases} w_0^2(t, x) = h(x) + L_t^{-1} \left[ N(w^{k-1}(t, x) \right]; k \geq 1 \\ w_{n+1}^2(t, x) = L_t^{-1} \left[ \varepsilon \Delta w_n^2(t, x) + \mu w_n^2(t, x) \right]; n \geq 0 \end{cases}$$

is convergent for $k \geq 2$ and we obtain: $w^1(t, x) = w^2(t, x) = \ldots = w^k(t, x)$.

From which the unique solution of the problem $w(t, x) = \lim_{k \to +\infty} w^k(t, x)$.

**Proof.** At step $k = 1$, we have the following algorithm:

$$(p_1) : \begin{cases} w_0^1(t, x) = h(x) \\ w_{n+1}^1(t, x) = L_t^{-1} \left[ \varepsilon \Delta w_n^1(t, x) + \mu w_n^1(t, x) \right]; n \geq 0 \end{cases}$$

According to hypothesis $(H_1)$ and $(H_2)$, the solution of $(p_1)$ is $w^1(t, x) = \sum_{n=0}^{+\infty} w_n^1(t, x)$. According to the hypothesis $(H_3)$, at step $k = 2$, $Nw^1(t, x) = 0$.

We get the following algorithm:

$$(p_2) : \begin{cases} w_0^2(t, x) = h(x) \\ w_{n+1}^2(t, x) = L_t^{-1} \left[ \varepsilon \Delta w_n^2(t, x) + \mu w_n^2(t, x) \right]; n \geq 0 \end{cases}$$

Thus, we obtain the same algorithm as in step $k = 1$, then $w^2(t, x) = w^1(t, x)$. Thus, in a recursive way it will be for each step $k \geq 2$, $w^1(t, x) = w^2(t, x) = w^3(t, x) = \ldots$

Then the solution of the problem $(p)$ is $w(t, x) = \lim_{k \to +\infty} w^k(t, x)$.
Suppose that the problem \((p)\) has two distinct solutions \(w(t, x) \neq v(t, x)\), and consider their difference \(\varphi(t, x) = w(t, x) - v(t, x)\).

For each solution, we have:

\[
\begin{align*}
\left\{ 
 w_k^0(t, x) &= h(x) + L^{-1}_t \left[ N(w_k^{k-1}(t, x)) \right] ; k \geq 1 \\
 w_{n+1}^k(t, x) &= L^{-1}_t \left[ \varepsilon \Delta w_n^k(t, x) + \mu w_n^k(t, x) \right] ; n \geq 0
\end{align*}
\]

and

\[
\begin{align*}
\left\{ 
 v_k^0(t, x) &= h(x) + L^{-1}_t \left[ N(v_k^{k-1}(t, x)) \right] ; k \geq 1 \\
v_{n+1}^k(t, x) &= L^{-1}_t \left[ \varepsilon \Delta v_n^k(t, x) + \mu v_n^k(t, x) \right] ; n \geq 0
\end{align*}
\]

\(\forall k \geq 1, N(w_k^{k-1}(t, x)) = 0\) and \(N(v_k^{k-1}(t, x)) = 0\), so we obtain:

\[
\begin{align*}
\left\{ 
 w_k^0(t, x) &= h(x) ; k \geq 1 \\
 w_{n+1}^k(t, x) &= L^{-1}_t \left[ \varepsilon \Delta w_n^k(t, x) + \mu w_n^k(t, x) \right] ; n \geq 0
\end{align*}
\]

and

\[
\begin{align*}
\left\{ 
 v_k^0(t, x) &= h(x) ; k \geq 1 \\
v_{n+1}^k(t, x) &= L^{-1}_t \left[ \varepsilon \Delta v_n^k(t, x) + \mu v_n^k(t, x) \right] ; n \geq 0
\end{align*}
\]

For the difference we get:

\[
\begin{align*}
\left\{ 
 \varphi_k^0(t, x) &= 0 ; k \geq 1 \\
 \varphi_{n+1}^k(t, x) &= L^{-1}_t \left[ \varepsilon \Delta \varphi_n^k(t, x) + \mu \varphi_n^k(t, x) \right] ; n \geq 0
\end{align*}
\]

from which

\[
\begin{align*}
\varphi_0^k(t, x) &= 0 \\
\varphi_1^k(t, x) &= 0 \\
\vdots \\
\varphi_n^k(t, x) &= 0, \forall n \geq 0
\end{align*}
\]

Thus \(\varphi_n^k(t, x) = \sum_{n=0}^{+\infty} \varphi_n^k(t, x) = 0\) and \(w(t, x) = v(t, x)\) which contradicts our hypothesis. Therefore the problem \((p)\) has a unique solution \(w(t, x)\).

\section{Resolution via Numerical Method ADM}

Consider the following equation

\[
\begin{align*}
(i \partial w(t, x) + \varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} + q |w(t, x)|^{2p} w(t, x) = 0
\end{align*}
\]

Let us determine the canonical form of Adomian, the equation

\[
\begin{align*}
(i \partial w(t, x) + \varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} + q |w(t, x)|^{2p} w(t, x) = 0; p \in \mathbb{N}^+
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\partial w(t, x) + i\varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} +iq |w(t, x)|^{2p} w(t, x)
\end{align*}
\]
from which we obtain the canonical form:

\[ w(t,x) = w(0,x) + \varepsilon \int_0^t \frac{\partial^2 w(z,x)}{\partial x^2} \, dz + i\varepsilon \int_0^t |w(z,x)|^{2p} \, w(z,x) \, dz. \]

Thus, we obtain the Adomian algorithm:

\[
\begin{align*}
&\left \{ \\
&w_0(t,x) = w(0,x) \\
&w_n+1(t,x) = i\varepsilon \int_0^t \frac{\partial^2 w_n(z,x)}{\partial x^2} \, dz + i\varepsilon \int_0^t A_n(z,x) \, dz; n \geq 0
\end{align*}
\]

Let’s calculate the polynomials:

\[
\begin{align*}
&\mathcal{A}_0 = [\beta^{2p} w_0] \\
&\mathcal{A}_1 = w_1(0,x) + \mu w_0 + \varepsilon w_0 \mathcal{W}_0 + \varepsilon w_0 \mathcal{A}_0 \mathcal{W}_0 \\
&\mathcal{A}_2 = 2(aw_0)p w_2 + 2p(aw_0 + bw_0)p^{-1}(w_1) + p(p - 1)(2(aw_0 + bw_0 + cw_0))p^{-2}(w_0).
\end{align*}
\]

Let’s calculate the terms: \( w_0(t,x), w_1(t,x), \ldots \)

we obtain thus:

\[
\begin{align*}
&w_0(t,x) = \beta e^{i\alpha x} \\
&w_1(t,x) = \beta i(t - \beta^2 + q |\beta|^{2p}) e^{i\alpha x} \\
&w_2(t,x) = \beta \left[ i(t - \beta^2 + q |\beta|^{2p}) \right]^2 e^{i\alpha x} \\
&w_3(t,x) = \beta \left[ i(t - \beta^2 + q |\beta|^{2p}) \right]^3 e^{i\alpha x} \\
&\ldots \\
&w_n(t,x) = \beta \left[ i(t - \beta^2 + q |\beta|^{2p}) \right]^n e^{i\alpha x}
\end{align*}
\]

Therefore, the solution of problem (E) obtained by the ADM method is:

\[ w(t,x) = \sum_{n=0}^{+\infty} n! w_n(t,x) = \beta \exp \left[ i \left( -\alpha^2 + q |\beta|^{2p} \right) t + ax \right]. \]

Consider the following equation

\[ (E) : \int \frac{i}{\partial t} \frac{\partial w(t,x)}{\partial t} + \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + \mu w(t,x) + q |w(t,x)|^{2p} w(t,x) = 0; p \in \mathbb{N}^* \]

Let us determine the canonical form of Adomian, the equation

\[ \int \frac{i}{\partial t} \frac{\partial w(t,x)}{\partial t} + \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + \mu w(t,x) + q |w(t,x)|^{2p} w(t,x) = 0; p \in \mathbb{N}^* \]

is equivalent to

\[ \frac{\partial w(t,x)}{\partial t} + i\mu w(t,x) + q |w(t,x)|^{2p} w(t,x) \]

from which we obtain the canonical form:

\[ w(t,x) = w(0,x) + i\varepsilon \int_0^t \frac{\partial^2 w(z,x)}{\partial x^2} \, dz + i\mu \int_0^t w(z,x) \, dz + iq \int_0^t |w(z,x)|^{2p} \, w(z,x) \, dz. \]

Thus, we obtain the Adomian algorithm:

\[
\begin{align*}
&w_0(t,x) = w(0,x) \\
&w_n+1(t,x) = i\varepsilon \int_0^t \frac{\partial^2 w_n(z,x)}{\partial x^2} \, dz + i\mu \int_0^t w_n(z,x) \, dz + iq \int_0^t A_n(z,x) \, dz; n \geq 0
\end{align*}
\]
Let’s calculate the polynomials: $A_0, A_1, A_2, ...$

\[
\begin{align*}
A_0 &= |\beta|^{2p} w_0 \\
A_1 &= w_1 (w_0 w_{w_0})^p + p w_0 (w_1 w_{w_0} + w_{w_0})^{p-1} \\
A_2 &= 2 (w_{w_0} w_0)^p w_2 + 2p (w_{w_0} w_1 + w_{w_0} w_{w_0})^{p-1} (w_1) + p (p - 1) (2 (w_{w_0} w_2 + w_1 w_{w_0} + w_{w_0} w_0))^{p-2} (w_0) \\
&\vdots
\end{align*}
\]

Let’s calculate the terms: $w_0(t,x), w_1(t,x), w_2(t,x), ...$

we thus obtain:

\[
\begin{align*}
w_0(t,x) &= \beta e^{i\alpha x} \\
w_1(t,x) &= \beta t (\mu - \varepsilon \alpha^2 + q |\beta|^{2p}) e^{i\alpha x} \\
w_2(t,x) &= \beta \frac{[i t (\mu - \varepsilon \alpha^2 + q |\beta|^{2p})]}{2!} e^{i\alpha x} \\
w_3(t,x) &= \beta \frac{[i t (\mu - \varepsilon \alpha^2 + q |\beta|^{2p})]}{3!} e^{i\alpha x} \\
&\vdots
w_n(t,x) &= \beta \frac{[i t (\mu - \varepsilon \alpha^2 + q |\beta|^{2p})]}{n!} e^{i\alpha x}
\end{align*}
\]

Therefore, the solution of problem $(E)$ obtained by the ADM method is:

\[
w(t,x) = \sum_{n=0}^{+\infty} w_n(t,x) = \beta \exp \left[ i \left( (\mu - \varepsilon \alpha^2 + q |\beta|^{2p}) t + a x \right) \right].
\]

\section{Resolution via Numerical Method SBA}

Consider the following equation

\[
(E) : \begin{cases}
\frac{i}{\partial t} w(t,x) + \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + q |w(t,x)|^{2p} w(t,x) = 0 \\
w(0,x) = \beta e^{i\alpha x}
\end{cases}
\]

Let us determine the canonical form of Adomian, the equation

\[
\frac{i}{\partial t} w(t,x) + \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + q |w(t,x)|^{2p} w(t,x) = 0
\]

is equivalent to

\[
\frac{\partial w(t,x)}{\partial t} = i \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + i q |w(t,x)|^{2p} w(t,x)
\]

(4.2)

By putting

\[
N w(t,x) = i q |w(t,x)|^{2p} w(t,x)
\]

from which we obtain the Adomian [7] canonical form:

\[
w(t,x) = w(0,x) + i \varepsilon \int_0^t \frac{\partial^2 w(z,x)}{\partial x^2} dz + \int_0^t N w(z,x) dz.
\]

(4.3)

Applying to (4.3) the method of successive approximations [8], we obtain:

\[
w^k(t,x) = w^k(0,x) + i \varepsilon \int_0^t \frac{\partial^2 w^k(z,x)}{\partial x^2} dz + \int_0^t N w^{k-1}(z,x) dz, k \geq 1
\]

(4.4)
We thus obtain the SBA algorithm [9]:

\[
\begin{cases}
    w_0^k(t, x) = u^k(0, x) + \int_0^t Nw^{k-1}(z, x) \, dz \quad k \geq 1 \\
    w_{n+1}^k(t, x) = i\varepsilon \int_0^t \frac{\partial^2 w_n^k(z, x)}{\partial x^2} \, dz \quad n \geq 0
\end{cases}
\tag{4.5}
\]

Let’s apply Picard’s principle [10], at step \( k = 1 \), \( Nw^0(t, x) = 0 \), if \( w^0(t, x) = 0 \), hence

\[
\begin{cases}
    w_0^1(t, x) = \beta e^{i\alpha x} + \int_0^t Nw^0(z, x) \, dz \quad k \geq 1 \\
    w_{n+1}^1(t, x) = i\varepsilon \int_0^t \frac{\partial^2 w_n^1(z, x)}{\partial x^2} \, dz \quad n \geq 0
\end{cases}
\]

Therefore we have

\[
\begin{align*}
    w_1^1(t, x) &= \beta e^{i\alpha x} \\
    w_2^1(t, x) &= -i\alpha^2 \beta e^{i\alpha x} \\
    w_3^1(t, x) &= -\frac{1}{2} i\alpha^2 \beta^2 e^{i\alpha x} \\
    w_4^1(t, x) &= \frac{1}{6} i\alpha^2 \beta^3 e^{i\alpha x} \\
    &\vdots \\
    w_n^1(t, x) &= \beta \left( -\varepsilon i\alpha^2 t \right)^n e^{i\alpha x}, n \geq 0
\end{align*}
\]

from which at step \( k = 1 \), we obtain:

\[
w^1(t, x) = \lim_{p \to +\infty} \beta e^{i\alpha x} \sum_{p=0}^{n} \left( -\varepsilon i\alpha^2 t \right)^p = \beta \exp \left[i (\alpha x - \varepsilon a^2 t)\right].
\]

Then let’s calculate \( Nw^1(t, x) \)

\[
Nw^1(t, x) = i\varepsilon \left| w^1(t, x) \right|^{2p} w^1(t, x) - iq^2 \beta w^1(t, x) \neq 0
\]

therefore, we modify problem (E) into an equivalent problem:

\[
(E) : \begin{cases}
    i\frac{\partial w(t, x)}{\partial t} + i\varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} + q \beta |\beta|^{2p} w(t, x) + \tilde{N}w(t, x) = 0
\end{cases}
\]

where

\[
\tilde{N}w(t, x) = q |w(t, x)|^{2p} w(t, x) - q |\beta|^{2p} w(t, x)
\]

Therefore, we obtain:

\[
\frac{\partial w(t, x)}{\partial t} = \varepsilon i\frac{\partial^2 w(t, x)}{\partial x^2} + q \beta |\beta|^{2p} w(t, x) + i\tilde{N}w(t, x)
\]

then the canonical form of Adomian [10]

\[
w(t, x) = \beta e^{i\alpha x} + \varepsilon i \int_0^t \frac{\partial^2 w(z, x)}{\partial x^2} \, dz + q |\beta|^{2p} \int_0^t w(z, x) \, dz + i \int_0^t \tilde{N}w(z, x) \, dz
\]

The new algorithm is then:

\[
\begin{cases}
    w_0^k(t, x) = \beta e^{i\alpha x} + \int_0^t \tilde{N}w^{k-1}(z, x) \, dz ; k \geq 1 \\
    w_{n+1}^k(t, x) = \varepsilon i \int_0^t \frac{\partial^2 w_n^k(z, x)}{\partial x^2} \, dz + q |\beta|^{2p} i \int_0^t w_n^k(z, x) \, dz; n \geq 0
\end{cases}
\]
Let’s determine $w^1(t, x)$

\[
\begin{cases}
  w_0^1(t, x) = \beta e^{iax} \\
  w_1^1(t, x) = \beta i (q |\beta|^{2p} - a^2 x) t e^{iax} \\
  w_2^1(t, x) = \beta \frac{i}{2} (q |\beta|^{2p} - a^2 x)^2 t^2 e^{iax} \\
  w_3^1(t, x) = \beta \frac{i}{3!} (q |\beta|^{2p} - a^2 x)^3 t^3 e^{iax} \\
  \vdots \\
  w_n^1(t, x) = \beta \frac{i}{n!} (q |\beta|^{2p} - a^2 x)^n t^n e^{iax}
\end{cases}
\]

hence

\[
w^1(t, x) = \beta e^{iax} \sum_{n=0}^{\infty} \frac{(i(q |\beta|^{2p} - a^2 x)t)^n}{n!} = \beta \exp \left[ i \left( (q |\beta|^{2p} - a^2 x) t + ax \right) \right]
\]

We thus obtain $\tilde{N}w^1(t, x) = 0$.

Recursively we have:

\[
w^1(z, x) = w^2(z, x) = \ldots = w^k(z, x)
\]

so the solution of problem (E) is:

\[
w(t, x) = \beta \exp \left[ i \left( (q |\beta|^{2p} - a^2 x) t + ax \right) \right].
\]

Consider the following problem:

\[(F) : \begin{cases}
  i \frac{\partial w(t, x)}{\partial t} + \varepsilon \frac{\partial^2 w(t, x)}{\partial x^2} + \mu w(t, x) + q |w(t, x)|^{2p} w(t, x) = 0
\end{cases}
\]

We obtain the following Adomian algorithm:

\[
w(t, x) = w(0, x) + i\varepsilon \int_0^t \frac{\partial^2 w(z, x)}{\partial x^2} dz + i\mu \int_0^t w(z, x) dz + iq \int_0^t Nw(z, x) dz \quad (4.6)
\]

where

\[
Nw(t, x) = |w(t, x)|^{2p} w(t, x)
\]

Let us apply the method of successive approximations to (4.6),

\[
\begin{cases}
  w^k(t, x) = w^k(0, x) + i\varepsilon \int_0^t \frac{\partial^2 w^k(z, x)}{\partial x^2} dz + i\mu \int_0^t w^k(z, x) dz + iq \int_0^t Nw^{k-1}(z, x) dz, k \geq 1
\end{cases} \quad (4.7)
\]

We are looking for the solution of (F) in the form of a series [11]

\[
w^k(t, x) = \sum_{n=0}^{+\infty} w_n^k(t, x)
\]

At each step $k \geq 1$, we have the following algorithm [12]

\[
\begin{cases}
  w_0^k(t, x) = w^k(0, x) + iq \int_0^t Nw^{k-1}(z, x) dz \\
  w_{n+1}^k(t, x) = i\varepsilon \int_0^t \frac{\partial^2 w_n^k(z, x)}{\partial x^2} dz + i\mu \int_0^t w_n^k(z, x) dz; n \geq 0
\end{cases}
\]
Let’s calculate the terms of the series

\[ w^k(t, x) = \sum_{n=0}^{\infty} w_n^k(t, x) \]

At step \( k = 1 \), for \( w^0(t, x) = 0 \), we have \( Nw^0(t, x) = 0 \) and we obtain:

\[
\begin{align*}
  w_0^1(t, x) &= \beta e^{iax} \\
  w_1^1(t, x) &= i(\mu - a^2) e^{iax} \\
  w_2^1(t, x) &= \frac{(i((\mu - a^2) t)^2 e^{iax}}{2!} \\
  w_3^1(t, x) &= \frac{(i((\mu - a^2) t)^3 e^{iax}}{3!} \\
  \vdots \\
  w_n^1(t, x) &= \frac{(i((\mu - a^2) t)^n e^{iax}}{n!}
\end{align*}
\]

therefore

\[
w^1(t, x) = \beta e^{iax} \sum_{n=0}^{\infty} \frac{(i((\mu - a^2) t)^n}{n!} = \beta \exp \left[ i((\mu - a^2) t + ax) \right]
\]

Calculate \( N^2(t, x) \), we have:

\[
Nw^1(t, x) = q |w^1(t, x)|^{2p} w^1(t, x) = q |\beta|^{2p} w^1(t, x) \neq 0
\]

We then modify problem (F) into an equivalent problem:

\[
(F) : \begin{cases} \\
  i \frac{\partial w(t, x)}{\partial t} + \frac{\partial^2 w(t, x)}{\partial x^2} + \mu w(t, x) + q |\beta|^{2p} w(t, x) + \tilde{N} w(t, x) = 0
\end{cases} \tag{4.8}
\]

where

\[
\tilde{N} w(t, x) = q |w(t, x)|^{2p} w(t, x) - q |\beta|^{2p} w(t, x)
\]

we have the following canonical form:

\[
w(t, x) = w(0, x) + i \int_0^t \frac{\partial^2 w(z, x)}{\partial x^2} dz + i \mu \int_0^t w(z, x) dz + q i |\beta|^{2p} \int_0^t w(z, x) dz + i \int_0^t \tilde{N} w(z, x) dz
\]

Let’s apply the method of successive approximations to (4.7),

\[
w^k(t, x) = w^k(0, x) + i \varepsilon \int_0^t \frac{\partial^2 w^k(z, x)}{\partial x^2} dz + i (\mu + q |\beta|^{2p}) \int_0^t w^k(z, x) dz + i \int_0^t \tilde{N} w^{k-1}(z, x) dz, k \geq 1
\]

Thus, at each step \( k \geq 1 \), the following algorithm is obtained:

\[
\begin{align*}
  w_0^k(t, x) &= w^k(0, x) + i \int_0^t \tilde{N} w^{k-1}(z, x) dz, k \geq 1 \\
  w_n^{k+1}(t, x) &= i \varepsilon \int_0^t \frac{\partial^2 w_n^k(z, x)}{\partial x^2} dz + i (\mu + q |\beta|^{2p}) \int_0^t w_n^k(z, x) dz, n \geq 0
\end{align*}
\]

Let’s calculate \( w^1(t, x) \) at step \( k = 1 \)

\[
\begin{align*}
  w_0^1(t, x) &= w^1(0, x) + i \int_0^t \tilde{N} w^0(z, x) dz \\
  w_n^1(t, x) &= i \varepsilon \int_0^t \frac{\partial^2 w_n^0(z, x)}{\partial x^2} dz + i (\mu + q |\beta|^{2p}) \int_0^t w_n^0(z, x) dz, n \geq 0
\end{align*}
\]

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for \( w^0(t, x) = 0 \) we have: \( Nw^0(t, x) = 0 \), hence

\[
\begin{cases}
 w^1_0(t, x) = w^k(0, x) \\
 w^1_{n+1}(t, x) = i\varepsilon \int_0^t \frac{\partial^2 w^1_n(z, x)}{\partial x^2} dz + i\mu \int_0^t w^1_n(z, x) dz + q_i |\beta|^{2p} \int_0^t w^1_n(z, x) dz; n \geq 0
\end{cases}
\]

we have:

\[
\begin{align*}
 w^1_0(t, x) &= 0 \\
 w^1_1(t, x) &= \beta e^{iax} \\
 w^1_2(t, x) &= \beta \left( it \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right) e^{iax} \right) \\
 w^1_3(t, x) &= \beta \left( \frac{2!}{3!} \left( it \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right) \right)^3 e^{iax} \right) \\
 &\vdots \\
 w^1_n(t, x) &= \beta \left( \frac{it \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right)^n}{n!} e^{iax} \right)
\end{align*}
\]

the solution at step \( k = 1 \) is

\[
w^1(t, x) = \beta e^{iax} + \sum_{n=0}^{+\infty} \frac{\left( it \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right) \right)^n}{n!} \]

\[
= \beta \exp \left[ i \left( \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right) t + ax \right) \right] 
\]

We thus obtain:

\[
w^1(t, x) = w^2(t, x) = \ldots = w^k(t, x) = \beta \exp \left[ i \left( \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right) t + ax \right) \right].
\]

Thus the solution of problem \((F)\):

\[
w(t, x) = \lim_{k \to +\infty} w^k(t, x)
\]

\[
= \beta \exp \left[ i \left( \left( \mu - a^2 \varepsilon + q |\beta|^{2p} \right) t + ax \right) \right]
\]

5 Conclusions

The SBA and ADM methods have allowed us to successfully solve the Ivancevic option pricing model (IOPM) in financial mathematics, the Schrödinger model in quantum mechanics and the classical Black-Scholes equation.

Competing Interests

Authors have declared that no competing interests exist.

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