Metaphors in textbooks of differential equations

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Abstract. This research is an approach to the study of metaphors in the teaching of differential equations, present in textbooks. The objective is to analyse the cognitive metaphors that facilitate learning in the search for the solution of homogeneous linear differential equations. The methodology used was content analysis applied to a subjective sample of four university texts. The analysis allowed to find that the texts that use linear algebra make it easier the teaching of differential equations, and it is concluded that the metaphors from linear algebra are fundamentals in the teaching of linear differential equations, the basic concept facilitates the learning for the construction of the general solution of linear differential equations.

1. Introduction
The idea of developing this work arises from educational practice and researches on the metaphors in mathematics derived from linear algebra, centered on some concepts exposed by differential equation books, studied in the light of the didactic transposition, under the assumption that in the teaching of Mathematics, the presentation of mathematical contents in textbooks corresponds to modified versions with respect to the original or wise version. In order to achieve a better understanding of mathematical concepts, authors of books and researchers make use of a repertoire of strategies of the same mathematics and language, to facilitate the learning of the exposed topics; also the teachers in their explanations use different techniques according to the context, interests, needs, prior knowledge and motivation that students have regarding a particular topic.

According to Lizcano [1], the discourse in general is full of metaphors, these sometimes go unnoticed both for those who expose the speech and for those who listen or read it, scientific concepts are metaphorical, mathematics because of its abstract nature is clearly metaphorical, its objects of study are also abstract [2], hence geometric figures, graphs and concepts are metaphorical expressions of school mathematical discourse.

The current textbooks of mathematics resort to metaphors in the presentation of curricular contents, the metaphors used in mathematics are not fictitious or arbitrary because this science is not a language in the strict sense, language allows expressions to give a universe of meanings [3], mathematics has the virtue of not leading to any interpretation, there is no possibility of disparate interpretations [4], mathematical concepts are precise, they are in practice absolute decisions taken in a given theoretical framework.

Taking into account the ideas presented, this work is important because It points to the ideal path to follow in the teaching of differential equations, and because it shows another facet of the absent metaphor in research on the subject.
2. General theory of metaphor

The origin of the metaphors goes back to the oral traditions to the XIII century before our era, the written metaphors appear approximately twenty-eight centuries ago, some cases are evidenced in the Homeric Iliad, however, its precise definition is given by Aristotle who expresses it in his poetic and rhetoric works. For Aristotle metaphor is "translation of someone else's name, either from the genus to the species, or from the species to the genus, or from one species to another, or by analogy, “the metaphors implicitly allude to an enigma, so that it is evident that are well transported” [4].

Recognizing the metaphor as a substitution or analogy referred to more similarity or approximation because there is no comparative link in it, this belongs to the class of tropes for being a rhetorical figure expressed in a figurative sense, translates the meaning of a word, phrase or expression of the language, to constitute a relation of identification where one term replaces the other, metaphors “are ways of saying one thing comparing it with another” [5], metaphor is a conceptual projection in which inferences are transferred from one domain to another [2], is a cognitive instrument to establish implications between subjects belonging to two different domains [6], it is the understanding of one domain in terms of another [7].

In the words of Aristotle, the metaphor is the substitution of one word for another, or of one expression or sentence for another in the same context or in a different context; a name or a descriptive term is transferred to some other object, but analogous to it, insofar as it is proportional to what is correctly applicable [6]; the metaphor manifests by itself the essential capacity of the mind to express relationships that transcend the immediate or customary meaning; the metaphorical statement is the result of saying something that is in the literal normal sense, in another expression given in figurative language in a special or general way. The use of metaphors supposes that the original or main expression is in a context attributable to a certain linguistic category, be it a science, a topic or any other area, this context is called source or starting domain; the translation produces a subsidiary term or proposition, which occurs in another context called destination or destination domain. Using a metaphor to explain the metaphor, there is a mapping or application $f$ of a source domain $A$ in a destination domain $B$, $f: A \rightarrow B$ such that $f$ (main expression) = subsidiary expression [6], that is, the metaphor found an implication between themes belonging to two different domains producing a change of meaning whose foundation is not simple, obeys a moment of creation, a key that allows understanding the meaning of what is expressed, consequently, a metaphor is good or appropriate when it is achieved get that moment that facilitates the understanding of the concept, reorganizing the known in a new format [7].

The metaphor, in addition to fulfilling the functions of communication or establishing meanings, is presented as a linguistic device for the oblivious organization of a reality, building new concepts from others already established, acting as a “mechanism through which we construct new concepts based on in the existing ones; we build on the unknown from the known” [8]; the metaphor is the connection between the source domain and the destination domain, it is the effect that builds bridges between the two domains, it is a double conceptual ambush that generates the path to understanding. Starting from modern theories, the metaphor replaces one denomination with another, it is a supplanting of a common word or idea with a new one, it is substitution of a common use name for a new one.

Today there are several studies on the metaphor with a variety of approaches in different areas of knowledge, including artificial intelligence, theory of science and cognitive psychology. From the cognitive point of view, metaphors lead students to make inferences, interpret representations, establish connections between concepts, generate conceptual models and infer general principles, this implies a separation of the metaphor from its linguistic aspect to take an interpretative path and make possible the understanding of a phenomenon or a concept, leading to take a look at the situation in another direction [8].

3. Classification of metaphors

Cognitive science classifies metaphors into three classes: metaphorical expressions, conceptual or cognitive metaphors and image metaphors. The first two were explained in a previous paragraph,
however it is emphasized that the conceptual metaphor is an intellectual procedure, to achieve the learning of an original concept corresponding to the domain of wise knowledge; it is a supplement of our intellectual arm, and represents logically the fishing rod or the net said Ortega y Gasset; it is an “indispensable mental instrument for the construction of representations of reality, not only aesthetic but also those structured in the scientific record” [9]; it allows us to apprehend what is beyond our understanding and to understand what is remote or elusive [8].

The image metaphors, called in some cases visual metaphors, correspond to the representation of an idea or concept through a scheme, a graphic or any other pictorial expression; in this way graphic representations of the atom, the solar system, the earthly paradise, etc., correspond to this type of metaphor. In other words, image metaphors are imaginary schemes or abstract structures that arise from human experience, are kinesthetic in nature and refer to aspects of human activity, reflecting in some cases experiences of living or forms of conceptualization [10].

Taking into account that metaphors are translations, and regardless of the type to which they belong, they are in essence translation problems from one domain to another [11] consider classifying them according to the following possibilities of domains that conform them, which correspond to the following structures: use the same source and destination domain, M→M; metaphors whose domains are similar, M1→M2; and metaphors with different source and destination domains, M→P.

This last classification is applicable to all metaphors, therefore it is adjustable to the research that is presented, without neglecting that the metaphors in mathematics are of a cognitive nature, it is pertinent to look at the metaphors in mathematics, as in the processes of teaching and learning, the conceptual or cognitive metaphor is of connoted importance for being a tool or sapient framework, very related to the knowledge.

4. Metaphors and mathematics
Since the appearance of mathematics in texts the metaphor was a powerful element to facilitate its understanding. Starting from the Pythagorean school, the metaphor is presented as a representation or image, used in some demonstrations, for example, the sum of the first n odd natural numbers is n²; in Euclid’s elements is defined, “number is a multitude composed of units”, we present a cognitive metaphor with M→M structure. In mathematics there is a wide variety of metaphors ranging from the initial to the university level, defining numbers as collections of objects, to say that adding is joining objects, multiplication is an abbreviated sum, function is a rule of assignment of elements of a set to those of another set, a rational is the quotient of two integers, etc.; Graphic representations, symbolic images, actions, gestures and descriptions that use vocabulary that does not belong to the domain of mathematics are also metaphors [12].

Whenever metaphors are used, there is something that does not fit into the formal mathematical context, however, they allow the concept to be represented in the minds of the students, since the experience of didactic work has shown that a certain level of representation is acquired. assimilate the concept, attenuating the uncertain relationship between the mind and the external universe called the uncertainty principle of knowledge, ideas and definitions emerge from human experience, emerge from cognitive and bodily mechanisms of human beings, this cognitive mechanism that allows to conceptualize the abstract in terms of the concrete is the conceptual metaphor [13].

Much of the reasoning in argumentation or deductive processes are metaphorical, most abstract concepts are defined in terms of metaphors and only through metaphor it is possible to theorize about abstract concepts whose understanding is not immediate [5]. In the classroom it is very common to use metaphors, some of them are: when algorithmic procedures are used, for example, when working with equalities it is said “what is adding on one side happens to subtract the other side, what is multiplying go to the other side dividing”; “The function y = x² is a parabola”; “A function is an assignment rule”; “The inverse derivative operator is the integral”; “At each point on the line there is a real number”; and many others. In short, the importance of conceptual metaphors in mathematics lies in that these are constituted in a cognitive mechanism incorporated and used by the mind to conceive mathematical objects, then beyond the linguistic aspect metaphors are the product of thought [14].
5. Methodology
The research is of qualitative and descriptive nature, framed in the qualitative paradigm and a broad referential framework that includes content analysis, because it describes and analyzes textbooks. It focuses on the topic of linear equations of the course known as Calculus of differential equations or differential equations, taught at the university level; an intentional sample of four textbooks was taken, which by tradition were or are the most used in the teaching of the course, exploring the bibliography recorded in curricular programs or syllabus of an accredited university that has science and engineering programs, and they were also used in a large number of Colombian universities. The selected books were Lambe and Tranter [15], Derrick and Grossman [16], Boyce and Diprima [17] and Zill [18]. Other criteria for the choice of the texts was to look for a difference of approximate years between the publication dates in different decades, thus allowing some contrasts in the presentation of the contents.

The careful reading of these sources was based on the didactic micro-structural analysis because it deals with a specific topic, this includes content analysis, cognitive analysis, analysis of instruction and performance analysis, all of them necessary for analyze the structure of concepts, their relationship with other concepts and the procedures involved in learning these, especially to establish those terms in which the metaphor has intervened in its formation; however, reference is made only to content analysis and cognitive analysis.

In general terms, content analysis as a technique present throughout the work, allows inferring objective conclusions from the information examined, which in some way reflects the manifest meaning in a tacit or hidden form that the author of each text intends to communicate [twenty-one]; in this case, the latent or indirect are the metaphors that appear in the exhibition of contents whether textual, images or representations within mathematics, for this reason the analysis of performance is not taken.

6. Results and discussion
The texts examined initiate the presentation of the theme introducing the concept of differential equation and the classification of these in ordinary and partial, then expose the differential equations of first order, then the second order and higher orders by studying the homogeneous linear differential equations and not homogeneous with constant coefficients, they continue with some linear differential equations whose coefficients are not constant and can be solved by power series, then they go to the Laplace transform to solve equations that involve continuous functions by sections; finally, in a generalized way, they deal with specific topics about systems of differential equations, numerical methods, differential equations in partial derivatives and theory of stability present in the Derrick-Grossman and Boyce-Diprima texts. All the texts include applications in each chapter, except Zill's book that devotes special chapters to modeling with differential equations.

The content analysis applied to the linear differential equations of second order and other higher orders, to the researched texts, indicates that the order of the topics follows the direction indicated above, presenting the texts by Derrick and Grossman, and Lambe and Tranter in a single chapter differential equations of second order and above including applications; the text of Boyce and Diprima studies in separate chapters those of second order and higher order, including applications only in those of second order; Zill's text includes in a single chapter the equations of all orders, but exposes the applications in a separate chapter. None of the texts includes applications that involve differential equations beyond those of second order, except the structural analysis examples presented in Zill and Lambe and Tranter, corresponding to bending deflection and flexural stiffness and critical rotation speed of a column, where a fourth-order differential equation result.

The cognitive analysis applied to the contents is related to the metaphors present in the theoretical exposition. Referring to the homogeneous differential equations of the second order \( ay'' + by' + cy = 0 \), the search for solutions is based on the displacement that is made of the solution space of the second order differential operators \( L = aD^2 + bD + c; \ a, b, c \in \mathbb{R} \), up to the vector space of the second degree polynomials \( ar^2 + br + c; \ a, b, c \in \mathbb{R} \), where the solutions are related with the roots of the associated
polynomial equation \(ar^2 + br + c = 0\), metaphorically called in the texts characteristic equation or auxiliary equation in Lambe-Tranter. In this case a metaphor of the type \(M_1 \to M_2\) is given, then making use of the inverse mapping the roots obtained from \((r - \alpha)(r - \beta) = 0\) are expressed in terms of operators such as \((D - \alpha)(D - \beta)y = 0\), where the solutions are in the kernel or kernel of the operator; then, according to the basic theory of linear algebra, the core is a vector space, a base for this specific operator contains two functions that are linearly independent, the base of this vector space is metaphorically called the fundamental set of solutions and the functions are called linearly independent solutions. It is known that if a set is a base of a vector space, any element of space is obtained by a linear combination, in the same way the general solution of the differential equation is formed, applying the so-called superposition principle, which is the expression metaphorical of the property that has the base of a vector space as generator set.

In the solution of the linear differential equations of higher order with constant coefficients, it is pointed out that the differential operators with constant coefficients operate as polynomials; thus, metaphorically, multiplying two operators is to make the composition of operators, multiplication obeys the commutative law [18]; factoring the characteristic or auxiliary polynomial, means expressing the operator as the composition of first and second order operators; to look for the general solution is to find the roots of the characteristic or auxiliary equation, and then find the functions that are annihilated or annulled by each operator, that is, those that are in the nucleus of each operator, and then assemble the linear combination of all the which are linearly independent, which must be equal in number to the order \(n\) of the operator, because the dimension of the vector space formed by the operator's nucleus is precisely \(n\).

The texts of Derrick-Grossman, Zill and Boyce-Diprima use the metaphorical figure of the annihilating or annihilating operator; that of Lambe and Tranter makes mention of the expression complementary function, exposed in a tax manner based on the metaphorical concept of independent solutions that can be translated as linear independence; however, for the operator \(D^n\) consider the division by this interpreted as integration, however, the text of Boyce - Diprima states in one of its paragraphs that the fundamental joint expression is part of the vector space \(V\) of the functions that satisfy the differential equation \(a\gamma'' + b\gamma' + cy = 0\) as follows:

Since every member of \(V\) can be expressed as a linear combination of two linearly independent members \(y_1\) and \(y_2\), this pair is said to form a basis for \(V\). This concludes that \(V\) is two-dimensional: therefore, it is analogous in many respects to the space of geometric vectors in a plane. Then it will be seen that the set of solutions of a homogeneous linear differential equation of \(n\)th order forms a vector space of dimension \(n\) and that any set of \(n\) linearly independent solutions of the differential equation forms a basis for space.

The Derrick and Grossman text also points to the concept of base in the following terms: Let \(y_1, y_2\) be two linearly independent solutions of the equation \(y''(x) + a(x)y'(x) + b(x)y(x) = 0\). We will show that any other solution can be written as a linear combination of \(y_1\) and \(y_2\). This notorious fact means that once we have found two linearly independent solutions of the differential equation, we have found, essentially, all the solutions.

In this case, when referring to linear combination the metaphor is given by analogy; and by substitution, by replacing the condition of being the base a generator set by the circumstance of obtaining essentially all the solutions. Under these assumptions, the reader must cognitively infer that a general solution must contain functions that are linearly independent and that generate at the same time any other solution; that is, the metaphor proposes to leave implicit in the mind the concept of base from the practical, making a replacement of the original concept in the same domain if it is equated to the concept of vector space, or in similar domains if we consider each particular space.

Zill's text proposes to find the general solution of a homogeneous linear differential equation of order \(n\) by stating the general solution using the fundamental set of solutions of the homogeneous linear differential equation of \(n\)th order. Metaphorically replaces the fact that the subset \(S = \{y_1, y_2, ..., y_n\}\) of the vector space of all the solutions, is a system generating this space, which corresponds to the linear envelope generated by \(S\), in this case it is exactly the total space of solutions. The concept of
linear envelope is an abstract concept, replaced by the general solution concept corresponding to a structural metaphor; in the style of Lakoff and Johnson [5], one thing is put in place of another in figurative language using a new name.

There are other metaphors disseminated throughout the exposition of techniques to find the general solution of the homogeneous linear differential equation of second order with constant coefficients; Thus for example, repeated roots instead of multiplicity of the root, polynomial operator or operator polynomial instead of linear differential operator are metaphors of displacement type \( M \rightarrow P \), the graphical representation of the general solution of the differential equation is a metaphor of image; so is the representation of the solution of the initial value problem.

The analysis of instruction applied to the four texts allows to affirm the existence of a mapping between the linear differential operators of the second order and the polynomials in a variable of degree two, in such a way that the search for the roots of the polynomial leads to assembling the solution general of the differential equation, giving the cognitive step of the vector space of said polynomials to the core vector space of the operator; later, the mapping technique is extended to the solution of homogeneous linear differential equations with constant coefficients of order \( n \).

7. Conclusions

It is proved that the correct path to follow to determine the general solution of a linear differential equation with constant coefficients is that of using metaphors of the linear apex related to the basic concept of a vector space, hence it is important to choose a text in that direction.

The books analyzed show a thin line, which separates the theory on the basis of a vector space from the so-called fundamental set of solutions, used as a cognitive metaphor to learn to construct the general solution of a homogeneous linear differential equation of order \( n \); in effect, it is the same but in terms of another language because the source and destination domains belong both to mathematics and share the common expressions, linear combination and linearly independent set, where the same meaning is subtly imposed.

The use of metaphors from linear algebra is crucial in the teaching of differential equations to understand why the solution of a homogeneous differential equation, the sample of similarities that occurs in differential operators and polynomials, is fundamental in the explanation of the events that involve obtaining the general solution of a homogenous linear differential equation and its construction, using the factorization to find the basis of a kernel, the metaphors used reveal analogies that can only be seen through them; therefore, without this theoretical body, the teaching of homogeneous linear differential equations would translate into a recipe for bland techniques and procedures without any justification.

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