Hadamard Renormalization of a 2-Dimensional Dirac Field

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The Hadamard renormalization procedure is applied to a free, massive Dirac field ψ on a 2-dimensional Lorentzian spacetime. This yields the state-independent divergent terms in the Hadamard bispinor \( G^{(1)}(x, x') = \frac{1}{2} \left\langle \psi^*(x'), \psi(x) \right\rangle \) as \( x \) and \( x' \) are brought together along the unique geodesic connecting them. Subtracting these divergent terms within the limit assigns \( G^{(1)}(x, x') \), and thus any operator expressed in terms of it, a finite value at the coincident point \( x' = x \). In this limit, one obtains a quadratic operator instead of a bispinor. The procedure is thus used to assign finite values to various quadratic operators, including the stress-energy tensor. Results are presented covariantly, in a conformally-flat coordinate chart at purely spatial separations, and in the Minkowski metric. These terms can be directly subtracted from combinations of \( G^{(1)}(x, x') \) - themselves obtained, for example, from a numerical simulation - to obtain finite expectation values defined in the continuum.

I. MOTIVATION

Considering that one’s surroundings tend to at least seem four-dimensional, two-dimensional models play a surprisingly central role in quantum field theory. This is largely a matter of convenience: they tend to be more readily soluble by nonperturbative methods than their higher-dimensional counterparts. The Thirring model \( ^1 \) of self-interacting 2D fermions, for example, can be solved nonperturbatively in the massless \( ^2 \) and massive cases \( ^3 \). Enought results have by now been gathered about two dimensional conformal field theories \( ^5 \) to form practically a new discipline.

Such nonperturbative studies often (e.g. \( ^6 \)) exploit an identification between two-component Dirac spinors and continuum limits of spin chains. Here, Dirac spinors in the continuum are mapped, for example by “staggering” \( ^7, ^8 \), to continuum limits of fermionic lattice operators. These, in turn, can be exactly related to Pauli operators by a so-called Jordan-Wigner transformation \( ^10 \).

This strategy has enjoyed something of a renaissance of late, because the numerical simulation of spin-chain Hamiltonians using “matrix product state” \( ^11-18 \) algorithms has grown sufficiently mature for application to lattice field theory \( ^19-33 \). Such studies encounter a recurrent difficulty during computations of certain “quadratic” expectation values. These diverge in the continuum limit, and some sort of counter-term must be subtracted to get finite results. Most of the standard renormalization techniques used in perturbation theory are not available in a lattice field theoretic setting because the “bare” results are found directly in coordinate space.

A similar problem occurs during studies of quantum field theory in curved spacetime \( ^34-35 \). On general manifolds, one has neither a preferred vacuum state nor spatial homogeneity, which again requires one to regulate and renormalize directly in coordinate space. Among the various techniques for doing this, covariant point-splitting followed by Hadamard renormalization \( ^36-42 \) is particularly interesting for application to numerics, since it works by subtracting pre-computed terms from externally-supplied two-point functions, which themselves are straightforward to numerically compute.

The Hadamard procedure is especially useful if the relevant quantum field is in curved spacetime. The Einstein tensor vanishes identically in two spacetime dimensions, but curved two-dimensional spacetimes are still interesting. Most of the familiar QFT-incurved-spacetime effects such as Hawking radiation \( ^43 \) occur in both 2 \( ^44 \) and 4 dimensions, but the former are much simpler technically. They also arise theoretically from “dilaton” theories \( ^45-47 \), obtained, for example, by restricting Einstein gravity to spherical symmetry and then compactifying. Even in flat spacetime, Hadamard renormalization is of some use. Though the actual divergences in that case can be calculated more straightforwardly, the Hadamard procedure establishes their independence of the particular quantum state.

Once the terms have been computed, renormalization is as simple as subtracting them from correlation functions, but unfortunately the initial computation is a bit involved. Early studies found results in 4 dimensions, first for scalar, and then for vector and spinor fields. More recent work has concerned a scalar field in dimensions from 2 up to 6 \( ^41 \), and a spinor field in AdS spacetime \( ^42 \). The immediately relevant case of a spinor field in two dimensions, though, seems to be missing. This is the most useful case for application to matrix product state simulations, whose expense scales with the spatial dimension and with the local Hilbert space of the lattice model. I develop the missing results here.

Section II establishes notation, while Section III re-
views the well-understood theory of two-point functions on curved spacetime. Section IV details the replacement of quadratic expectation values with divergent limits of correlation functions. Section V then shows how to compute the divergent terms in the correlation functions, and Section VI how to relate the latter to those regulating the quadratic operators. Section VII then specializes these results to a general conformally flat metric at zero coordinate time separation, and to Minkowski spacetime.

As an accessory to the main text, Mathematica notebooks that adapt the package xTensor to the manipulation of bispinors and to their coin-

II. THE FREE DIRAC FIELD

Consider a globally hyperbolic, 2 dimensional spacetime $\mathcal{M}$ with metric $g_{\mu\nu}(x)$ and Lorentzian signature. At each point $x \in \mathcal{M}$, install a two-component spinor $\psi(x)$, forming there a spinor representation of the Lorentz group. To clarify notation the Gothic spinor indices will usually be suppressed, $\psi(x) \rightarrow \psi$. The Lorentz representations at each point bear no connection with one another $\textit{a priori}$, but can be related by introducing a set of frame fields

\[ e^a_\mu(x) e^b_\nu(x) g_{ab} = g_{\mu\nu}(x), \]  
\[ e^a_\mu(x) e^a_\nu(x) = \delta^a_\nu, \]  
\[ e^a_\mu(x) e^b_\mu(x) = \delta^a_b, \] 

where $\delta^a_b$ is the Kronecker delta. Latin “Lorentz” indices are raised and lowered by the Minkowski metric $g_{ab} = \text{diag}(-1, 1)$. We can use the frame fields to define a “spin” connection in terms of the Christoffel connection, and thus to define covariant differentiation of spinors.

Products of spinors get mapped into scalars by the curved-spacetime gamma matrices $\Gamma_{\mu}(x)$. These relate to their flat-spacetime counterparts $\gamma_{a}$ by the anticommutation relations

\[ \{\gamma^a, \gamma^b\} = 2\eta^{ab}, \]  
\[ \{\Gamma^\mu(x), \Gamma^\nu(x)\} = 2g^{\mu\nu}(x). \] 

The former imply the latter, given

\[ \Gamma^\mu(x) = e^a_\mu(x) \gamma^a. \] 

We will denote covariant derivatives interchangeably with semicolons and nablas (e.g. $\nabla_\mu T^\nu = T^\nu_{\nu,\mu}$), and partial derivatives interchangeably with colons and dels (e.g. $\partial_\mu T^\nu = T^\nu_{\nu,\mu}$). The covariant derivative of a spinor $\psi_{\mu}$ is given by

\[ \psi;\mu = \psi_\mu + \zeta_\mu \psi, \]  
\[ \bar{\psi};\mu = \bar{\psi}_\mu - \bar{\psi} \zeta_\mu, \] 

where the spin connection $\zeta_\mu$ stands in for

\[ \zeta_\mu = \frac{1}{2} \omega_{\mu ab} \Sigma^{ab}, \]  
\[ \Sigma_{ab} \equiv \frac{1}{4} [\gamma^a, \gamma^b]. \] 

Some references call $\omega_{\mu ab}$ the spin connection. Whatever its name, it relates to the Christoffel connection by

\[ \omega^a_{\mu b} = -e_b^\nu (\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu \nu} e^a_\lambda). \] 

In the torsion-free geometries we consider, we have

\[ \omega_{\mu ab} = -\omega_{\mu ba}. \] 

The failure of the spin covariant derivative to commute is measured by

\[ [\nabla_\mu, \nabla_\nu] \psi = \frac{1}{2} \Sigma_{ab} R^{\mu ab}_{\mu \nu} \psi, \] 

where $R^{\mu \nu \sigma \tau}$ is the Riemann curvature tensor. Partial derivatives commute, of course. We furthermore have

\[ \gamma^\mu_\mu = 0, \]  
\[ e^a_\mu = 0. \] 

From (11) and (12) one can derive the square of the Dirac operator

\[ (\gamma^\mu \nabla_\mu)^2 = \Box + \frac{1}{4} R, \] 

where $\Box \equiv \nabla^\mu \nabla_\mu$ is the covariant wave operator. The field’s dynamics will be set by the free Dirac action,

\[ S[\psi, \bar{\psi}] = - \int d^2x \sqrt{-g} \bar{\psi} \left( \frac{1}{2} \gamma^\mu \nabla_\mu - m \right) \psi. \] 

where $\bar{\psi} \equiv i \gamma^0 \psi^\dagger$ is called the Dirac adjoint and left-right arrows over a derivative operator indicate, for example,

\[ \bar{\psi} \overset{\leftrightarrow}{\nabla}_\mu \psi = \bar{\psi} \psi_\mu - \bar{\psi}_\mu \psi. \] 

Variation of (15) yields the Dirac equation and its adjoint

\[ \psi_\mu - m \psi = 0 \]  
\[ \bar{\psi}_\mu + m \psi = 0 \]
The theory can now be quantized by various equivalent means. After canonical quantization, for example, one has

\[ \{ \bar{\psi}(t, x^a), \psi(t, y^a) \} = \delta(x - y), \]

where \( \delta(x - y) \) is the Dirac delta distribution. This immediately implies that “quadratic” operators involving products of fields, being formally proportional \( \delta^2(0) \), have ill-defined expectation values. Hadamard renormalization is one of several means of defining them.

The procedure runs as follows: first, a quadratic operator is regularized by covariant point-splitting \[37, 39, 40\]. Thus, it is substituted with the zero-separation limit of the product of linear operators at different points, as they are parallel-dragged towards one another along the unique geodesic connecting them. It is assumed that said linear operators yield well-defined correlation functions with respect to the given quantum state. This assumption can be viewed as a restriction of the quantum state to a special class called “Hadamard states”. The divergences in the coincidence limit are then computed by appeal to a construction due to Hadamard \[36\]. Finally, the coincidence limit is renormalized, by subtracting the computed divergences before it is taken. Since only finite terms remain, the renormalized expectation value obtained in this way is also finite.

III. TWO-POINT FUNCTIONS ON MANIFOLDS

Before outlining the Hadamard procedure in detail, it is helpful to review some of the well-understood machinery of two-point functions on manifolds. As is standard, we label indices transforming at \( x' \) with a prime. Thus \( T_{\mu\nu}(x, x') \) transforms separately as a vector with respect to coordinate transformations at either the “base” point \( x \) or the “field” point \( x' \). We denote a biterm or bispinor’s “coincidence” limit from \( x' \) to \( x \) along the unique geodesic connecting those points with square brackets. For example

\[ [B(x, x')] = \lim_{x' \to x} B(x, x'), \]

where it is understood that the limit is to be taken along the coincident geodesic.

One is typically interested in using coincidence limits to construct “covariant expansions” of tensors (spinors) in terms of bitensors (bispinors). These are morally similar to Taylor series, with the separation measured by Synge’s world function \( \sigma(x, x') \), a biscalar numerically equal to one-half the squared geodesic proper interval between \( x' \) and \( x \). We will adopt the idiosyncratic, though standard, practice of denoting covariant derivatives of \( \sigma(x, x') \) without a semicolon: \( \sigma^\mu(x, x') \equiv \sigma_{,\mu}(x, x') \), for example. The most important facts about \( \sigma(x, x') \) are

\[ [\sigma(x, x')] = [\sigma^\mu(x, x')] = 0, \]

\[ \sigma^\mu(x, x') \sigma_\mu(x, x') = 2 \sigma(x, x'). \]

The latter identity, in particular, implies that expressions involving \( \sigma^\mu(x, x') \) scale numerically like \( O(\sigma^{1/2}(x, x')) \) during the coincidence limit. Outside of Minkowski space, primed indices will generically suffer a nontrivial parallel transport as the coincidence limit is taken. We express this using the spinor parallel propagator, defined by

\[ \mathcal{J}^A_{\bar{B}}(x, x') \equiv \mathcal{J}, \]

\[ \mathcal{J}^B_{\bar{A}}(x, x') \equiv \mathcal{J}^{-1} \]

The identities \[22a\] and \[23a\] are, respectively, the parallel transport equations for spinors and vectors. Thus, contraction with a parallel propagator implements parallel transport along the coincident geodesic.

IV. COVARIANT POINT-SPLITTING

To illustrate the Hadamard procedure, consider the following operators: the condensate \( C_I(x) \) and \( \mu \)-currents \( C^\mu(x) \), given formally by

\[ C_I(x) = m \langle \bar{\psi}(x) \psi(x) \rangle, \]

\[ C^\mu(x) = \langle \bar{\psi}(x) \Gamma_\mu \psi(x) \rangle \]

and the stress-energy tensor \( T_{\mu\nu}(x) \), given formally by

\[ T_{\mu\nu}(x) \equiv \frac{-2}{\sqrt{-g(x)}} \frac{\delta S(x)}{\delta g_{\mu\nu}(x)} \]

\[ = \frac{1}{4} \bar{\psi}(x) \left( \Gamma_\mu(x) \nabla_\nu + \Gamma_\nu(x) \nabla_\mu \right) \psi(x). \]

Note we use \( T_{\mu\nu} \) to refer to the expectation value of the stress-energy tensor, not the latter as an operator. In the limit of flat spacetime, the condensate provides a measure of chiral symmetry breaking, while
the currents form a conserved quantity. The stress-energy tensor serves as the source of the semiclassical gravitational field.

To point-split these expressions, we parallel-drag the adjoint spinor $\bar{\psi}(x)$ to the field point $x'$. To make eventual contact with the Dirac equation, it is convenient to express the point-split expressions in terms of the “Hadamard” bispinor \[ G(1)_{A}^{B'}(x, x') = \frac{1}{2} \left\{ \left[ \psi_{A}(x), \bar{\psi}^{B'}(x') \right] \right\}. \quad (27) \]

The angle brackets here instruct us to take the expectation value of the enclosed expression with regard to the given quantum state, upon which $G(1)(x, x')$ also implicitly depends. Due to the comma, the square bracket in (27) is the commutator, not the coincidence limit. The point-split expressions \[ C_{I}(x) \rightarrow [C_{I}(x, x')] = -m \lim_{x' \rightarrow x} \text{Tr} \mathcal{J} G(1)(x, x'), \]

\[ C_{\mu}(x) \rightarrow [C_{\mu}(x, x')] = - \lim_{x' \rightarrow x} \text{Tr} \mathcal{J} \mathcal{\Gamma}_{\mu} G(1)(x, x'), \]

\[ T_{\mu\nu}(x) \rightarrow [T_{\mu}(x, x')] \]

\[ = \frac{1}{8} \lim_{x' \rightarrow x} \text{Tr} \mathcal{J} \mathcal{\Gamma}_{\mu} \left( G(1)_{\nu'}(x, x') - g_{\nu'}^{\prime} G(1)_{\nu'}(x, x') \right) \quad (28c) \]

The traces here are over the suppressed spinor indices. The arrows remind us that the point-splitting is still a formal manipulation, since the limits are ill-defined.

We will find that, having restricted attention to a so-called “Hadamard” quantum state, all the divergences in $G(1)(x, x')$, and thus in each of (28), are determined only by the Lagrangian and by the geometry local to $x$ and $x'$. Thus, subtracting from $G(1)(x, x')$ all its locally-determined terms $G(1)(x, x')$ yields a bispinor $G^{\text{ren.}}(x, x')$ with a well-defined coincidence limit. In turn, we obtain the locally-determined terms in each of (28) by making the substitution $G(1)(x, x') \rightarrow ˜G(1)(x, x')$. Renormalized expressions are then obtained by subtraction. Specifically,

\[ C_{I}^{\text{ren.}}(x) = \left[ C_{I}(x, x') - ˜C_{I}(x, x') \right], \]

\[ C_{\mu}^{\text{ren.}}(x) = \left[ C_{\mu}(x, x') - ˜C_{\mu}(x, x') \right], \]

\[ T_{\mu\nu}^{\text{ren.}}(x) = \left[ T_{\mu}(x, x') - ˜T_{\mu}(x, x') \right]. \]

These renormalized expressions have well-defined coincidence limits.

\section{V. COMPUTATION OF LOCALLY-DETERMINED TERMS}

To employ the definitions \[ G(1)(x, x') \] we must compute the locally-determined terms $G(1)(x, x')$, $C_{I}(x, x')$, $C_{\mu}(x, x')$, and $T_{\mu\nu}(x, x')$. This is outlined in the following subsections.

\subsection{A. The Hadamard Form}

One first assumes that $G(1)(x, x')$ is a homogeneous solution to the Dirac equation

\[ (\Gamma^{\mu}\nabla_{\mu} - m)G(1)(x, x') = 0, \quad (30) \]

a first order hyperbolic PDE. We would like to appeal to a theorem concerning solutions to second order hyperbolic PDEs. To this end, define the auxiliary propagator $\mathcal{G}(x, x')$ implicitly by

\[ (\Gamma^{\mu}\nabla_{\mu} + m)\mathcal{G}(x, x') = G(1)(x, x'). \quad (31) \]

Inserting (31) into (30) and applying (14) reveals that $\mathcal{G}(x, x')$ obeys the Klein-Gordon-like equation

\[ \left( \nabla^{\mu}\nabla_{\mu} + \frac{1}{4} R - m \right) \mathcal{G}(x, x') = 0, \quad (32) \]

a second-order hyperbolic PDE. A theorem due to Hadamard \[ \text{[36]} \] now guarantees that under certain smoothness assumptions, $\mathcal{G}(x, x')$ must take the “Hadamard” form

\[ \mathcal{G}(x, x') = \frac{1}{4\pi} \left( V(x, x') \ln \mu\sigma + W(x, x') \right) \quad (33) \]

where $V(x, x')$ and $W(x, x')$ are analytic bispinors. The dimensionful “renormalization parameter” $\mu$ is undetermined by the procedure, but if desired it can be fixed by requiring one of (29) to take a particular value in some circumstance. For $\mathcal{G}(x, x')$ of the Hadamard form \[ \text{[33]}, \] we have

\[ \tilde{\mathcal{G}}(x, x') = \frac{1}{4\pi} V(x, x') \ln \mu. \sigma. \quad (34) \]

As we will soon discover, the requirement that $\tilde{\mathcal{G}}(x, x')$ be of the form \[ \text{[33]} \] places a restriction upon the quantum state. States meeting this requirement are said to be “Hadamard”. Heuristically, such states locally resemble the Minkowski vacuum, and the Hadamard condition is taken \[ \text{[55][57]} \] as a prerequisite for physical reasonableness.
B. Expansion in $\sigma(x, x')$.

Via (34), we can determine $\tilde{G}(x, x')$ by computing $V(x, x')$. To do so, insert the Ansatz expansions

$$
V(x, x') = \sum_{i=0}^{\infty} V_i(x, x') \sigma^{(i)}(x, x'),
$$

(35a)

$$
W(x, x') = \sum_{i=0}^{\infty} W_i(x, x') \sigma^{(i)}(x, x').
$$

(35b)

into (33) and then (32). Having done so, the demand that each power of $\sigma$ separately vanish yields a set of recurrence relations for $V_i(x, x')$ and $W_i(x, x')$, along with a boundary condition for $V_0(x, x')$. They are

$$
2(n+1)^2 V_{n+1} + 2(n+1) V_{n+1; \mu} \sigma^{\mu} - 2(n+1) V_{n+1} \Delta^{-1/2} \Delta_{\mu}^{\nu} \sigma^{\nu}
$$

$$
+ (\square_x - m^2 + \frac{1}{4} R) V_n = 0,
$$

(36a)

$$
2(n+1)^2 W_{n+1} + 2(n+1) W_{n+1; \mu} \sigma^{\mu} - 2(n+1) W_{n+1} \Delta^{-1/2} \Delta_{\mu}^{\nu} \sigma^{\nu} + 4(n+1) V_{n+1}
$$

$$
+ 2 V_{n+1; \mu} \sigma^{\mu} - V_{n+1} \Delta^{-1/2} \Delta_{\mu}^{\nu} \sigma^{\nu}
$$

$$
+ (\square_x - m^2 + \frac{1}{4} R) W_n = 0,
$$

(36b)

$$
V_0; \mu \sigma^{\mu} - V_0 \Delta^{-1/2} \Delta_{\mu}^{\nu} \sigma^{\nu} = 0.
$$

(36c)

The biscalar $\Delta^{1/2}$, called the van Vleck-Morette determinant, appears here via the identity

$$
\square_x \sigma = (d-1) - 2 \Delta^{-1/2} \Delta^{1/2} \sigma^{\mu},
$$

(37)

where $d=1$ is the spatial dimension. Note that (36) are formally identical to those obtained in [41] for a scalar field, apart from the differing connection in the covariant derivatives.

One observes from (36) that $V(x, x')$ is completely determined by the mass and the locally-available spacetime geometry. On the other hand, $W(x, x')$ additionally depends upon the biscalar $W_0(x, x')$, which is not constrained by any boundary condition. Thus, $W_0(x, x')$ must contain any additional information distinguishing different two-point functions from one another. This notably includes the quantum state: all Hadamard states with the same mass and on the same background have the same $\tilde{G}(x, x')$.

C. Solving for $V(x, x')$

Each time we take a derivative of $V(x, x')$, its scaling with $\sigma$ during the coincidence limit will be reduced by a factor of $\sigma^{1/2}$. Since $\mathcal{T}_{\mu\nu}(x, x')$ depends on second derivatives of $\mathcal{Q}(x, x')$ in order to compute $\mathcal{T}_{\mu\nu}$, we need $V(x, x')$ up to $O(\sigma)$. In light of (33), we in turn need $V_0(x, x')$ up to $O(\sigma)$, and $V_1(x, x')$ up to $O(1)$. To find $V_0(x, x')$ we must solve the boundary equation (36c). To do so, we make the Ansatz $V_0(x, x') = a S(x, x') S(x, x')$, where $a$ is a constant, $S(x, x')$ is a biciscalar, and $S(x, x')$ is a bispinor. Comparing with (36c), we find

$$
S_{\mu}(x, x') \sigma^{\mu} = 0,
$$

(38a)

$$
[S(x, x')] = -a \mathbb{1}.
$$

(38b)

The choice $a = -1$ yields agreement with standard flat-spacetime QFT [58]. In that case (38) is just the definition of the spin parallel propagator [22], so that $S(x, x') = J(x, x')$. We thus have exactly

$$
V_0(x, x') = -\Delta^{1/2} J.
$$

(39)

It is not possible to make a covariant expansion of $J$. A covariant expansion of $\Delta^{1/2}$, on the other hand, can be found in [40], for example. Up to $O(\sigma)$ it is

$$
\Delta^{1/2} = 1 + \frac{1}{12} R_{\mu\nu} \sigma^{\mu} \sigma^{\nu} + O(\sigma^{3/2}),
$$

(40)

and thus

$$
V_0(x, x') = -J \left( 1 + \frac{1}{12} R \sigma \right) + O(\sigma^{3/2}),
$$

(41)

where the well-known relations

$$
R_{\mu\nu\tau} = \frac{1}{2} R (g_{\mu\nu} g_{\tau} - g_{\mu\tau} g_{\nu}),
$$

(43a)

$$
R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu},
$$

(43b)

which are specific to 2D, were used.

To get $V_1(x, x')$ up to $O(1)$, we follow [40] and first write $\mathcal{J}^{-1} V_1(x, x')$ as a Taylor-like “covariant” expansion [59],

$$
\mathcal{J}^{-1} V_1(x, x') = v_0(x) + v_1(x) \sigma^{\mu} + \ldots
$$

(44)

where the coefficients $v_i(x)$ depend only on $x$. Taking the coincidence limit of both sides we have

$$
v_0(x) = [V_1(x, x')] .
$$

(45)

Now set $n = 0$ in (36a), insert (42), and take the coincidence limit to find

$$
[V_1(x, x')] = \frac{1}{24} (-12 m^2 + 5 R) \mathbb{1},
$$

(46)

$$
V_1(x, x') = \mathcal{J} \frac{1}{24} (-12 m^2 + 5 R) = O(\sigma^{1/2}).
$$

(47)

Combining (47) and (42) with (35), we find

$$
\mathcal{J}^{-1} V(x, x') = -\mathbb{1} - (1/2) (m^2 + (1/4) R) \sigma \mathbb{1}.
$$

(48)

Except for the presence of the spinor identity, this is the same result as reported for the scalar field in [41].
VI. DIVERGENCES OF POINT-SPLIT OPERATORS

We now must insert (48) into (54) and then (51) to find \( \tilde{G}^{(1)}(x, x') \), and then the latter into each of (28) to find \( \tilde{C}_i(x, x') \), \( \tilde{C}_\mu(x, x') \), and \( \tilde{T}_{\mu\nu}(x, x') \).

Doing this, inserting the expansions found in [33, 39, 40], and dropping any terms that vanish at coincidence, one obtains expressions in terms of geometric tensors, \( \sigma \), and \( \sigma_\mu \) only. This is conceptually straightforward, but a bit tiresome in practice, due to the length of the intermediate expressions involved. I have written some Mathematica notebooks, based on the package xTensor [41–52], to assist with such spinor and bitensor manipulations. They can be found at [53].

Following this procedure yields our central results,

\[
\tilde{C}_i(x, x') = -\frac{m^2}{2\pi} \ln \mu \sigma \quad (49a)
\]

\[
\tilde{C}_\mu(x, x') = -\frac{\sigma_\mu}{2\pi \sigma} \quad (49b)
\]

\[
\tilde{T}_{\mu\nu}(x, x') = \frac{1}{4\pi} \left[ \frac{g_{\mu\nu} - \sigma_\mu \sigma_\nu}{\sigma^2} + \frac{R}{6} \left( \frac{\sigma_\mu \sigma_\nu}{\sigma} - \frac{5}{4} g_{\mu\nu} \right) \right] + \frac{m^2}{8\pi} \left[ \frac{\sigma_\mu \sigma_\nu}{\sigma} + g_{\mu\nu}(1 + \ln \mu \sigma) \right]. \quad (49c)
\]

Note that the trace of (49c) is

\[
\tilde{T}^\mu_{\mu}(x, x') = \frac{R}{48\pi} + \frac{m^2}{2\pi} \left( 1 + \frac{1}{2} \ln \mu \sigma \right), \quad (50)
\]

which, when \( m = 0 \), differs from the standard CFT result \( \tilde{\Gamma} \) for free Dirac fermions by a factor of \( 1/2 \). If desired, this can be corrected using a procedure due to Moretti [60]. In this procedure, the stress-energy tensor is redefined to include a factor proportional to the Lagrangian, which vanishes classically. Thus, define

\[
\Theta_{\mu\nu}(x, x') \equiv g_{\mu\nu}\tilde{J}^{-1}(\Gamma^\rho\nabla_\rho - m)G^{(1)}(x, x'). \quad (51)
\]

The correction \( \Theta_{\mu\nu}(x, x') \) vanishes for the “classical” \( G^{(1)}(x, x) \) which solves the Dirac equation. However, it does not vanish after the replacement \( G^{(1)}(x, x') \rightarrow \tilde{G}^{(1)}(x, x') \). Instead,

\[
\Theta_{\mu\nu}(x, x') = \frac{g_{\mu\nu}}{\pi} \left( \frac{R}{24\pi} - m^2 g_{\mu\nu} \right). \quad (52)
\]

Now define

\[
\tilde{T}^\mu_{\mu}(x, x') \equiv \tilde{T}_{\mu\nu}(x, x') + a\tilde{\Theta}_{\mu\nu}(x, x'). \quad (53)
\]

Then the demand

\[
\tau^{\text{new,}\mu}_{\mu} = -\frac{R}{24\pi} \quad (54)
\]

implies \( a = -\frac{1}{4} \), and thus

\[
\tau^{\text{new,}\mu}_{\mu}(x, x') = \frac{1}{4\pi} \left[ \frac{g_{\mu\nu}}{\sigma} - \frac{\sigma_\mu \sigma_\nu}{\sigma^2} + \frac{R}{2} \left( \frac{\sigma_\mu \sigma_\nu}{3\sigma} - \frac{1}{2} g_{\mu\nu} \right) \right] + \frac{m^2}{8\pi} \left[ \frac{\sigma_\mu \sigma_\nu}{\sigma} + g_{\mu\nu}(3 + \ln \mu \sigma) \right]. \quad (55)
\]

VII. SPECIALIZATION TO CONFORMALLY FLAT COORDINATES

Our original motivation in computing (49a), (49b), and (49c) was to regularize numerically-generated data. These will typically be in some specific coordinate chart, localized to equal-time hypersurfaces with \( t = t' \). Thus, we outline here how to specialize the given example to a coordinate system, using conformally flat coordinates

\[
g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu} \quad (56)
\]

and a purely spatial coordinate separation as a prototype. Such coordinates are always available in 2D. The connections and the curvature scalar satisfy the component equations

\[
\Gamma^0_{\ 00} = \Gamma^0_{\ 11} = \Gamma^1_{\ 01} = \frac{\Omega_0}{\Omega}, \quad (57a)
\]

\[
\Gamma^0_{\ 01} = \Gamma^1_{\ 00} = \Gamma^1_{\ 11} = \frac{\Omega_1}{\Omega}, \quad (57b)
\]

\[
\zeta_0 = \frac{i}{2} \Omega_0 \gamma^5, \quad (57c)
\]

\[
\zeta_1 = \frac{i}{2} \Omega_1 \gamma^5, \quad (57d)
\]

\[
R = 2 \left( \frac{\Omega_2}{\Omega^4} - \frac{\Omega_0^2}{\Omega^4} - \frac{\Omega_1^2}{\Omega^4} - \frac{\Omega_{00}}{\Omega^3} \right), \quad (57e)
\]

while the gamma matrices satisfy

\[
\Gamma^\mu = \Omega^{-1} \gamma^\mu, \quad (58a)
\]

\[
\Gamma_\mu = \Omega \gamma_\mu. \quad (58b)
\]

Coordinate expansions of \( \sigma_\mu \) can be found in [61]. Specialized to \( 56 \) with \( t' = t \), they yield

\[
\sigma_0 = \frac{1}{2} \Omega_0 \eta_{\mu\mu} + O(r^3) \quad (59a)
\]

\[
\sigma_1 = -\Omega^2 r - \frac{1}{2} \Omega_1 \eta_{\mu\mu} + O(r^3) \quad (59b)
\]

where \( r \equiv x' - x \). We can find expansions of \( \sigma \) by inserting these expansions into the identity \( \sigma = \frac{1}{2}\sigma^\mu \sigma_\mu \).
and then the result into a Laurent expansion of $\frac{1}{\sigma}$ about $\sigma = 0$. The results are

\[ \tilde{\mathcal{C}}_I = -\frac{m^2}{2\pi} \ln \frac{1}{2} \mu^2 \Omega^2, \]  

\[ \tilde{\mathcal{C}}_0 = \frac{\Omega_0}{2\pi \Omega}, \]  

\[ \tilde{\mathcal{C}}_1 = -\frac{1}{\pi r} + \frac{\Omega_1}{2\pi \Omega}. \]  

Further specializing to Minkowski space, $\Omega = 1$, we have

\[ \tilde{T}_{00} = \frac{1}{2\pi r^2} + \frac{1}{2\pi r} + \frac{m^2}{8\pi} \left( 1 + \ln \frac{1}{2} \mu^2 \Omega^2 \right) - \frac{5}{96\pi} R \Omega^2 + \frac{1}{24\pi} \frac{\Omega_1^2}{\Omega^2} - \frac{1}{6\pi} \frac{\Omega_{11}}{\Omega} + \frac{5}{24\pi} \frac{\Omega_0^2}{\Omega^2}, \]  

\[ \tilde{T}_{01} = \frac{1}{2\pi} \left( \frac{\Omega_0 \frac{\Omega_1}{r} + \frac{\Omega_1}{2\pi} - \frac{\Omega_{01}}{3\Omega} \right), \]  

\[ \tilde{T}_{11} = \frac{1}{2\pi r^2} + \frac{1}{2\pi r} - \frac{m^2}{8\pi} \left( 3 + \ln \frac{1}{2} \mu^2 \Omega^2 \right) + \frac{1}{32\pi} R \Omega^2 + \frac{1}{24\pi} \frac{\Omega_1^2}{\Omega^2} - \frac{1}{6\pi} \frac{\Omega_{11}}{\Omega} + \frac{5}{24\pi} \frac{\Omega_0^2}{\Omega^2}, \]  

\[ \Theta_{00} = \frac{a}{\pi} \Omega^2 \left( m^2 + \frac{R}{24} \right), \]  

\[ \Theta_{01} = 0, \]  

\[ \Theta_{11} = -\Theta_{00}. \]

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