Hilbert-Polya conjecture and Generalized Riemann Hypothesis

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Abstract

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Extending a classical integral representation of Dirichlet L-functions associated to a non trivial primitive character $\chi$ we define associated functions $B^L_{s}(y, z)$ which are eigenfunction of the Hermitian operator:

$$H = \frac{1}{2} \left( \frac{\partial^2}{\partial^2 z} + z^2 \right) + i \left( y \frac{\partial}{\partial y} + \frac{1}{2} \right)$$

These eigenfunctions have $i(s - \frac{1}{2})$ as eigenvalues.

We prove that if $s$ is a non trivial zero of such a Dirichlet L-functions with $Re(s) < \frac{1}{2}$, then:

• the associated eigenfunction $B^L_{s}(z, y)$ is square integrable.

• $H$ "is Hermitian" for this function: $<HB^L_{s}, B^L_{s}> = <B^L_{s}, HB^L_{s}>$.

We deduce from this (using the idea of Hilbert-Polya and finding a contradiction) the Generalized Riemann Hypothesis:

the non trivial zero of a Dirichlet L-functions lie on the critical line $Re(s) = \frac{1}{2}$.

This results correspond to a weak form of the Hilbert-Polya conjecture (as for $Re(s) = \frac{1}{2}$ the eigen-functions presented here are not square integrable).

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1 Introduction

Hilbert and Polya had the intuition that a Hermitian operator was "hidden" behind the Zeta function non trivial zeros (zero located in the critical strip\(^1\)): leading to the conjecture that the imaginary parts of these zeros correspond to a Hermitian operator eigenvalues. In this way the Hilbert-Polya conjecture is closely linked to the Riemann Hypothesis\(^2\).

In this article we show that considering the imaginary parts of $L(s, \chi)$ zeros as eigenvalues of the hermitian operator $H$ leads to the demonstration of the Generalized Riemann Hypothesis.

In the Section 2 we provide an explanation on our motivations. Showing the basic ideas leading to the study of presented two dimensional operator and eigenfunctions. (Reader only interested in the result should jump directly to Section 3)

In the section 3 we provide the main properties of $B^L_{s}$ functions as well as the main theorem of this article with the required explanations so that the reader can understand clearly "how it works".

Complements on demonstrations are given in Appendix.

In this article, for $\chi$ a given character, we note $L(s, \chi)$ (or simply $L_s$) the associated Dirichlet L-functions. Except if explicitly mentioned, the Dirichlet characters $\chi$ considered all along this article, are non trivial

\(^1\)We define the "critical strip" in this article as the domain of the complex plane defined by $0 < Re(z) < 1$.

\(^2\)Riemann Hypothesis: all non trivial zero of Zeta function lie on the critical line ($Re(z) = \frac{1}{2}$). Refer to [12], [5] and [7] for an overview of the Zeta function theory and description of Riemann Hypothesis.
and primitive, so that we have: $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{nx^{s}}$ for $\text{Re}(s) > 0$. We note $\zeta_s$ the Zeta function ($\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\text{Re}(s) > 1$). When we refer to zeros of $\zeta_s$ or $L_s$ we always refer to zeros located (strictly) in the critical strip.

We note also: $\alpha = e^{-\frac{i\pi}{2}}$ (which has following interesting property: $i\alpha = \pi$)

2 Motivations

This paragraph presents some of the ideas leading to the main result of this article\(^3\). Proposed approach is very different from the one in \([11]\) as we do not use any trace formula as a basis, and we do not think that the presented operator (as such) has a trace formula, this is why presented result correspond to what we call a "weak" Hilbert-Polya conjecture.

As noted in some articles (For example in \([3, 10]\) or \([11]\)) the one dimensional Anti-Hermitian\(^4\) operator $D = x \frac{d}{dx} + \frac{1}{2}$ seems to be in a "certain way" linked to the Zeta function. (In its original form or as "transformed" to become the Inverted Harmonic Oscillator)

Noting $f_s(x) = x^{-s}$ we see that $f_s$ is eigenfunction of $D$:

$$Df_s(x) = (\frac{1}{2} - s)f_s(x)$$

If we imagine that this function $f_s$ is square integrable (and null at infinity) when $s$ is zero of the Zeta function then, as $D$ is anti Hermitian we would have:

$$(\frac{1}{2} - s) < f_s(x), f_s(x) >= - < Df_s(x), f_s(x) >$$

$$= - < f_s(x), Df_s(x) >$$

$$= - (\frac{1}{2} - s) < f_s(x), f_s(x) >$$

And we would simply found $s + \pi = 1$ meaning: $\text{Re}(s) = \frac{1}{2}$.

Unfortunately $f_s$ is not square integrable for $s$ zero of Zeta function ($f_s$ is never square integrable). And "nothing is happening" to the function $f_s$ when $s$ is zero of Zeta.

We can remark that if $s$ is on the critical line (so if $s = \frac{1}{2} + i\lambda$ with $\lambda$ real) then: $f_s \overline{f_s} = \frac{1}{\pi}$, meaning that in this case $f_s$ is making "its maximum effort" to be square integrable: in the sense that if $\text{Re}(s)$ is higher or lower than $\frac{1}{2}$ we have one side of the integral $\int_{0}^{\infty} x^{-2\text{Re}(s)} dx$ convergent and the other one divergent. The problem is that this "maximum effort" ends up with the worse situation: $\int_{0}^{\infty} x^{-1} dx$ is divergent at zero and at infinity.

We will see in this article that finally the function we use are the $x^{-s}$ but they are correctly "enveloped" so that when $s$ is zero of $L_s$ or Zeta function they are helped to be square integrable\(^5\). (See the term $x^{-s}$ and $x^{s-1}$ in the definition of $B^L_s$).

On the other hand we can remark (with $\chi$ a non trivial primitive character) that a function such as $G_s(x) = \sum_{n=1}^{\infty} \chi(n) g_s(nx)$

(with $g_s(x)$ admitting following asymptotic: $g_s(x) \sim x^{-s} + O(x^{-s-1})$) could be an interesting candidate to have "something happening" when $s$ is zero of the $L(s, \chi)$ function.

In this case, the asymptotic of $G(x)$ at infinity is:

$$G(x) \sim \sum_{n=1}^{\infty} \left( \chi(n) (nx)^{-s} + \chi(n) O((nx)^{-s-1}) \right)$$

$$\sim L(s, \chi) x^{-s} + O(x^{-s-1})$$

\(^3\)Reader interested in more details could refer to our previous article \([2]\)

\(^4\)This operator is anti-Hermitian for the scalar product $< f, g >= \int_{-\infty}^{\infty} f \overline{g} \, dx$

\(^5\)This term is nearly square integrable in the sense that: adding an infinitely small epsilon to $\frac{1}{2}$ will make it square integrable in infinity and dually: subtracting an infinitely small epsilon to $\frac{1}{2}$ will make it square integrable in $0$.
So when \( s \) is zero of \( L(s, \chi) \) we have simply: \( G(x) \sim O(x^{-s-1}) \)
and we see that \( G_s(x) \) (which is originally not integrable at infinity) becomes integrable when \( s \) is zero of
\( L(s, \chi) \). This simple example is of course just theoretical, but it illustrates the sort of attended "cancellation
effect" that we would like to have so that our function becomes square integrable when \( s \) is a zero.

Another remark concerns the fact that in one dimension the operator \( D \) does not offer a lot of flexibility: only the \( x^{-s} \) are eigenfunctions of this operator, whereas if we consider the same operator extended in
dimension two:

\[
P = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}
\]
we have more possibilities for eigenfunctions.

For example functions like: \( x^n y^k g(xy) \) are eigenfunctions of this operator (independently of the choice of
\( g(t) \)).

If we consider here a function \( g(t) \) asymptotic to a constant: \( g(t) \sim K + O(\frac{1}{t}) \) and define

\[
h_s(x, y) = x^{-s} y^{-\frac{3}{2}} \sum_{n=1}^{\infty} \chi(n) \frac{g(nxy)}{n^s}
\]
we have (at least formally):

\[
P h_s = (\frac{1}{2} - s) h_s
\]

And we see on this simple example how the additional dimension allows the introduction of what we call
the "cancellation effect" into a function.

\[
h_s(x, y) \sim x^{-s} y^{-\frac{3}{2}} \sum_{n=1}^{\infty} \chi(n) \frac{K}{n^s} + O(x^{-s-1} y^{-\frac{3}{2}})
\]

This makes \( h_s \) square integrable at infinity.

In one dimension we were condemned to have \( \sum_{n=1}^{\infty} \chi(n) \frac{x^n}{n^s} = x^s L(s, \chi) \) (The \( L(s, \chi) \) function "escapes"
from the sum), whereas here, in two dimensions on this example we see that values on the domain edge (at
infinity) depends on \( s \) (zero or not zero).

Another problem is that the operator \( x \frac{d}{dx} + \frac{1}{2} \) does not have square integrable eigenfunctions
but this can be changed by transforming \( D \) into the well known Harmonic Oscillator operator: \( H_0 = \frac{d^2}{dx^2} - x^2 \).

We provide below the main result of this transform allowing to pass from \( D \) to \( H_0 \).

Consider \( T \) the transform on real functions defined by \( T : f \rightarrow e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{(x-iy)^2} f(t) dt \)

If \( \phi(x) = i(x \frac{d}{dx} + \frac{1}{2}) f(x) \) Then \( T(\phi)(x) = -\frac{i}{2} (\frac{d^2}{dx^2} - x^2) T(f)(x) \)

And to have the operator Hermitian (and to have eigenfunctions not increasing exponentially) we need to
"twist" the operator: posing \( \alpha = e^{-\frac{ix}{2}} \) we have:

\[
T(\phi)(\alpha x) = \frac{1}{2} (\frac{d^2}{dx^2} + x^2) T(f)(\alpha x)
\]

The new operator is now Hermitian.

So if we consider the initially mentioned two dimensional operator, by adding artificially \( \frac{1}{2} \) we obtain:

\[
i \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) = i \left( x \frac{\partial}{\partial x} + \frac{1}{2} - y \frac{\partial}{\partial y} - \frac{1}{2} \right)
\]

Using the transform \( T \) presented above transforming \( x \) to \( z \), then changing \( z \) by \( iz \) and multiplying by \( -1 \) we obtain the following operator:

\[
H = \frac{1}{2} \left( \frac{d^2}{dz^2} + z^2 \right) + i \left( y \frac{d}{dy} + \frac{1}{2} \right)
\]

\[6\]And this is one of the main problem of Riemann Hypothesis: \( \int x^{-s} dx \) cannot converge at same time at infinity and 0.

[7]Nearly a Weierstrass transform
This is the operator we present in this article: an operator "naturally" associated to L-functions (an association of Inverted Harmonic Oscillator and its "original" form: $D$).

The idea is that the operator in variable $y$ is identical to the original one (the variable $y$ provides the link with $L(s, \chi)$ function: square integrability in $y$ at infinity will be ensured only when $s$ is zero of $L(s, \chi)$) whereas the part in $z$ allows to see the phenomenon and provide the integral which allows to have a representation of $L(s, \chi)$ function. (This explanation is only qualitative as the eigenfunctions we define, are not a "pure product" of a function in $y$ and a function in $z$, the eigenfunctions we will considered are interlaced in $y$ and $z$.)

Some eigenfunctions of the operator $H$ are:

$$e^{iz^2/2}y^{-1/2} \int_0^\infty e^{-(\alpha z + \sqrt{\pi} x)^2} g(xy) \, x^{-s} \, dx$$

And taking functions $g(\frac{t}{n})$ (we can take the function $g$ we want), multiplying by $n^{s-1}$ we have:

$$e^{iz^2/2}y^{-1/2} \int_0^\infty e^{-(\alpha z + \sqrt{\pi} x)^2} g(\frac{xy}{n}) n^{s-1} \, x^{-s} \, dx$$

Then changing $x$ by $nx$ and summing on $n$ with weighting by $\chi(n)$:

$$e^{iz^2/2}y^{-1/2} \int_0^\infty \sum_{n=1}^\infty \chi(n) e^{-(\alpha z + \sqrt{\pi} nx)^2} g(xy) \, x^{-s} \, dx$$

We remark that this type of function is linked to the $L_s$ function: if $g$ is a constant function then this eigenfunction is proportional to the $L_s$ function. Moreover we recognize, for $z = 0$ and $g(xy) = 1$, the classical integral used to demonstrate the $L(s, \chi)$ functional equation (integral which is proportional to the $L(s, \chi)$ function):

$$\int_0^\infty \sum_{n=1}^\infty \chi(n) e^{-\pi n^2 x^2} \, x^{-s} \, dx$$

All these remarks are the basis of this article and the proof of Generalized Riemann Hypothesis presented here\(^8\).

### 3 The main Theorem

#### 3.1 On the Generalized Riemann Hypothesis

In this paragraph we explicit the main steps and ideas leading to the main result of this article. The complement of demonstrations are given in Appendix.

We fix: $\chi$ an even, not trivial, primitive character of modulus $q$, $s$ a complex strictly in the critical strip, and $g(t)$ a bounded function in $C^2 \mathbb{R}^+$. We define then the function $B_s^L$ associated to the $L$-function $L(s, \chi)$ on the domain $D = \{(y, z) \in \mathbb{R}^+ \times \mathbb{R}^+\}$

$$B_s^L(y, z) = e^{iz^2/2}y^{-1/2} \int_0^\infty \left( \sum_{n=1}^\infty \chi(n) e^{-(\alpha z + \sqrt{\pi} nx)^2} \right) g(xy) \, x^{-s} \, dx$$

$$- A_\chi e^{iz^2/2}y^{-1/2} \int_0^\infty \left( \sum_{n=1}^\infty \chi(n) e^{-(\alpha z + \sqrt{\pi} nx)^2} \right) g(\frac{y}{qx}) \, x^{s-1} \, dx$$

\(^8\)Some articles can be considered as linked to the present work\(^8\)\(^9\) and especially \(^11\) where the proposed operator is not so different from the one presented here.
With \( \alpha = e^{-i\frac{\pi}{4}} \)

Notice that \(-(\alpha z + \sqrt{\pi n} x)^2\) and \(-\overline{\alpha z + \sqrt{\pi n} x}^2\) have always negative real part for \(x, z\) and \(n\) positive. We need first to ensure that this function is well defined for \(y\) and \(z\) fixed in the domain of definition.

The first integral converges in \(\infty\): for \(s\) in the critical strip the function \(\sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z + \sqrt{\pi n} x)^2}\) approaches 0 for \(x \to \infty\) faster than any inverse power of \(x\). For the convergence in zero a Poisson formula helps to see that convergence is also ensured (See in Appendix). Situation is identical for second integral.

Using that \(L(\chi, s)\) has a functional equation it is possible to fix \(A_0^\chi\) so that \(B_s^L(0, 0) = 0\). We detail below how.

Considering a primitive, non trivial, even character of modulus \(q\), its \(L_q\)-function functional equation can be deduced from following relation (which is a Poisson summation formula, posing \(\tau(\chi)\) the Gauss sum):

\[
\sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 x^2} = \frac{\tau(\chi)}{q} x \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 x^2/\pi^2}
\]

Using this and definition of \(B_s^L\) we have:

\[
y^s B_s^L(y, 0) = \int_0^{\infty} (\sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2 x^2}g(x y) x^{-s} dx - A_s^\chi \int_0^{\infty} (\sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2 x^2})g(\frac{y}{q} x) x^{s-1} dx
\]

So using to transform first integral and then making the change of variable \(x \to \frac{y}{q x}\) we find:

\[
y^s B_s^L(y, 0) = (\frac{\tau(\chi)}{q^{1-s}} - A_s^\chi) \int_0^{\infty} (\sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2 x^2})g(\frac{y}{q} x) x^{s-1} dx
\]

Hence posing \(A_s^\chi = \frac{\tau(\chi)}{q^{1-s}}\) ensures that for all \(y \in \mathbb{R}^+\): \(B_s^L(y, 0) = 0\)

In other terms: the functional equation ensures the annihilation of \(B_s^L\) at its edge on line \(z = 0\).

An other property of \(B_s^L\) is to be eigenfunction of the Hermitian operator \(\mathbf{H}\):

\[
\mathbf{H} = \frac{1}{2} \left( \frac{\partial^2}{\partial z^2} + z^2 \right) + i \left( y \frac{\partial}{\partial y} + \frac{1}{2} \right)
\]

And on the domain \(\mathcal{D}\) we have:

\[
\mathbf{H} B_s^L = i(s - \frac{1}{2}) B_s^L
\]

This result (which is simply the result of a calculus: both terms defining \(B_s^L\) are eigenfunctions of \(\mathbf{H}\) with same eigenvalues) is linked to the differential equation satisfied by Parabolic Cylinder functions.

We summarize this first set of properties for \(B_s^L\) (which are valid for all \(s\) in the critical strip).

**Proposition 3.1** The function \(B_s^L\) defined by \(\mathbb{2}\), with \(\chi\) an even, non trivial, primitive character, are well defined on \(\mathcal{D} = \{(y, z) \in \mathbb{R}^+ \times \mathbb{R}^+\}\) for \(s\) in the critical strip and verifies on this domain:

- \(\mathbf{H} B_s^L = i(s - \frac{1}{2}) B_s^L\)
- \(B_s^L(y, 0) = 0\)

We need to fix a specific \(g\) function to go further. From now on we fix \(g(t) = e^{-t}\) so that we have:

\[
B_s^L(y, z) = e^{\frac{2\pi i}{q} y^{-\frac{1}{2}}} \int_0^{\infty} \left( \sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z + \sqrt{\pi n} x)^2} \right) e^{-\pi y x^2} x^{-s} dx
\]

\[
- \frac{\tau(\chi)}{q^{1-s}} e^{i\frac{\pi^2}{4} y^{-\frac{1}{2}}} \int_0^{\infty} \left( \sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z + \sqrt{\pi n} x)^2} \right) e^{-\frac{\pi x^2}{y}} x^{s-1} dx
\]

\(^9\mathbf{H}\) is Hermitian for scalar product \(\langle f, g \rangle = \iint f \overline{g} dy dz\) on the space of function square integrable with correct boundary properties on the domain of integration considered.
We can now present the last three remaining properties of $B^L_s$, appearing when $s$ is a non trivial zero of $L(s, \chi)$, and leading to the Generalized Riemann Hypothesis.

**The first property** is what we call the "cancellation effect": when $s$ is zero of $L(s, \chi)$, if $g(t)$ is chosen to be constant, then $B^L_s = 0$. This is consequence of annihilation of following integrals in this case:\(^{10}\)

\[
\int_0^\infty \sum_{n=1}^\infty \chi(n)e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2} x^{-s} dx = 0
\]

\[
\int_0^\infty \sum_{n=1}^\infty \overline{\chi(n)}e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2} x^{s-1} dx = 0
\]

This is obvious by a change of variable taking $n$ out of the integral and considering that when $s$ is zero of $L(s, \chi)$ then $(1-s)$ is also zero of $L(s, \overline{\chi})$ (due to the classical $L(s, \chi)$ functional equation).

**The second property** of $B^L_s$ is that if $s$ is zero of $L(s, \chi)$ such that $Re(s) < \frac{1}{2}$ then with our choice of $g(t) = e^{-t}$, then $B^L_s$ is square integrable on the domain $D$.

This property has two root causes:

- Firstly the "cancellation effect" which gives following equality for first integral:

\[
\int_0^\infty \left( \sum_{n=1}^\infty \chi(n)e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2}\right) e^{-xy} x^{-s} dx = \int_0^\infty \left( \sum_{n=1}^\infty \chi(n)e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2}\right)(e^{-xy} - 1) x^{-s} dx
\]

and we have same type of equality for second integral:

\[
\int_0^\infty \left( \sum_{n=1}^\infty \chi(n)e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2}\right)e^{-\frac{y}{x^2}} x^{s-1} dx = \int_0^\infty \left( \sum_{n=1}^\infty \chi(n)e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2}\right)(e^{-\frac{y}{x^2}} - 1) x^{s-1} dx
\]

So we see that one form will be used to prove that integral approaches zero for $y \rightarrow \infty$ whereas the second one will be used to prove that integral approaches zero for $y \rightarrow 0$.

- Secondly the good behavior (on $x$ variable) of $\sum_{n=1}^\infty \chi(n)e^{-\left(\alpha z + \sqrt{\pi n}x\right)^2}$ in zero (due to existence of Poisson summation formula for Dirichlet character series) and infinity (due to $-n^2 x^2$ in the exponential).

**The third property**: when $s$ is zero of $L(s, \chi)$ the operator $H$ is Hermitian for the function $B^L_s$:

\[
<HB^L_s, B^L_s> = <B^L_s, HB^L_s>
\]

In order to "move" $H$ from one side (of scalar product) to other side, several integration by parts are required, and remaining "brackets terms" need to disappear, this is the case (Details are given in Appendix):

- For $y$ variable it is due to $yB^L_s \overline{B^L_s}$ approaching zero for $y \rightarrow 0$ and for $y \rightarrow \infty$ (because $s$ is zero of $L(s, \chi)$).

- For $z$ variable it is due to $\frac{\partial B^L_s}{\partial z} \overline{B^L_s}$ approaching zero at infinity (because $s$ is zero of $L(s, \chi)$: so in this case $B^L_s$ is a square integrable function and therefore approaches zero for $z \rightarrow \infty$) and being null in $z = 0$ (due to $B^L_s(y, 0) = 0$).

We summarize the three properties in proposition below.

**Proposition 3.2** If $s$ is zero of $L(s, \chi)$, with $Re(s) < \frac{1}{2}$, and $B^L_s$ defined as in \(^{12}\) then we have following three properties:

\(^{10}\)This is the "cancellation effect" we were speaking about in the "motivation" paragraph
• \[ \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n)e^{-\alpha z + \sqrt{n}z^2} x^{-s} dx = \int_{0}^{\infty} \sum_{n=1}^{\infty} \overline{\chi(n)}e^{-\overline{\alpha} z + \sqrt{n}z^2} x^{s-1} dx = 0 , \text{ for all fixed } z. \]

• \( B_s^L \) is square integrable\(^{11} \) on domain \( D \)

• \( B_s^L \) admits \( H \) as a Hermitian operator: \( \langle HB_s^L, B_s^L \rangle = \langle B_s^L, HB_s^L \rangle \)

We will see in demonstration of main theorem that the previous properties are sufficient to conclude that L-functions associated to even, non trivial, characters have their zeros on the critical line.

For odd characters we have similar results but the function to be considered is different (due to a different functional equation and integral representation). For non trivial odd characters we define\(^{12} \) on \( D \):

\[
B_s^L(y,z) = e^{-i \frac{z^2}{4} y^2} \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n)n x e^{-(\alpha z + \sqrt{n}z^2)} e^{-xy} x^{-s} dx \tag{8}
\]

\[+ iq^2 \tau(\chi) e^{i \frac{z^2}{4} y^2} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)n x e^{-(\overline{\alpha} z + \sqrt{n}z^2)}}{\sqrt{\pi n}} x^{-s-1} dx \]

And it is not difficult to check (adapting step by step demonstration made in the case of an even character) that in this case the properties associated to \( B_s^L \) are nearly the same:

• \( B_s^L(y,0) = 0 \)

• \( \mathbf{H}B_s^L = i(s-\frac{1}{2})B_s^L \)

"cancellation effect" is there: \[ \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n)n e^{-(\alpha z + \sqrt{n}z^2)} x^{1-s} dx = 0 = \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)n e^{-(\overline{\alpha} z + \sqrt{n}z^2)}}{\sqrt{\pi n}} x^{s} dx \]

• \( B_s^L \) is square integrable for \( s \) zero of \( L(s, \chi) \) such that \( \text{Re}(s) < \frac{1}{2} \)

• \( \langle HB_s^L, B_s^L \rangle = \langle B_s^L, HB_s^L \rangle \) (Again for \( \text{Re}(s) < \frac{1}{2} \))

Using previous properties and the classical property stating that Hermitian operators on Hilbert spaces have real eigenvalue\(^{13} \), we can conclude.

**Theorem 3.3** :

Considering \( \chi \), a Dirichlet character, the non trivial zero of the \( L(s, \chi) \) function are located on the critical line.

We have seen that for \( \chi \) a primitive character (not trivial), the function \( B_s^L \) associated to a non trivial zero\(^{14} \) of \( L(s, \chi) \) with \( \text{Re}(s) < \frac{1}{2} \) is square integrable and included in the Hilbert space for which \( H \) is Hermitian, so in this case the following equality holds:

\[ i(s-\frac{1}{2}) < B_s^L, B_s^L > = \langle HB_s^L, B_s^L \rangle = \langle B_s, HB_s^L \rangle = i(s-\frac{1}{2}) < B_s^L, B_s^L > \]

We then deduce that: \( i(s-\frac{1}{2}) = i(s-\frac{1}{2}) \) and: \( \text{Re}(s) = \frac{1}{2}, \) which is in contradiction with our hypothesis that \( \text{Re}(s) < \frac{1}{2} \).

\(^{11}\)For the classical scalar product \(< f, g > = \int f \overline{g} dydz \)

\(^{12}\)Note the \( y^2 \) instead of \( y^{-\frac{1}{2}} \) which is compensated by the \( x \) added under the integrand: these modification compensate each other in the asymptotic of \( B_s \) (Example for \( y \to \infty \) the added \( x \) adds a factor \( y^{-1} \) to the asymptotic \( y^{-\frac{1}{2}} \) becomes again \( y^{-\frac{1}{2}} \))

\(^{13}\)This is exactly the property at the origin of Hilbert-Polya conjecture

\(^{14}\)We call non trivial zeros the zeros located strictly in the critical strip.
So the zeros of the \( L(s, \chi) \) functions cannot have their real part strictly lower than \( \frac{1}{2} \), and using the functional equation which states that if \( s \) is a zero then \( (1-s) \) is also a zero, we conclude that all zeros of \( L(s, \chi) \) have their real part on the critical line.

We can now deduce that the Zeta function as also its non trivial zeros on the critical line using the fact that Zeta function has the same zero in the critical strip as the Dirichlet Eta function, defined by

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.
\]

For trivial characters it is the same: their \( L(s, \chi) \) functions are proportional to the Zeta function and their non trivial zeros are on the critical line. For non primitive characters: their zeros are also zeros of primitive characters so they lie on the critical line.

Finally we conclude: non trivial zeros of \( L(s, \chi) \) functions (for all Dirichlet characters) have their zeros on the critical line \( \square \)

All \( L(s, \chi) \) functions (for different non trivial primitive characters) are associated to \( B^L_s \) functions which are the eigenvalues of the same operator. We can say that the zeros are forced to lie on the "same line" by the operator \( H \).

This Hermitian operator explains that the zeros are on the critical line but it does not explain their position on the line (it does not provide the quantification explanation of the zeros, does not explain their distribution on the line), moreover we do not have square integrable eigenfunctions associated to these zeros. In this sense we only prove what we call a "weak version" of Hilbert-Polya conjecture.

4 Conclusion

The demonstration given here of the Generalized Riemann Hypothesis let open many questions and will lead naturally to further developments (the reader already guessed the main one: Extended Riemann Hypothesis...) but we prefer to list these questions and developments in a separate article.

Hence as a conclusion we will just summarize our result.

To a Dirichlet L-function (associated to a non trivial Dirichlet character) we have associated a Hermitian operator \( H \) and a family of eigenfunctions \( B^L_s \) such that (posing \( s = \frac{1}{2} + i \lambda \)):

\[
HB^L_s = -\lambda B^L_s
\]

in the case \( s \) is a zero of \( L_s \) with \( \Re(s) < \frac{1}{2} \), then we deduce \( \lambda \) is real from the relation:

\[
< HB^L_s, B^L_s > = < B^L_s, HB^L_s >
\]

This construction shows that we obtain a contradiction by assuming that a zero of a Dirichlet L-functions with primitive character does not lie on the critical strip. Therefore existing zeros in the critical strip lie on the critical line.

This method can be applied to the study of other L-functions which, providing certain conditions\(^{15}\) will have their associated B-functions verifying the same conditions as above and therefore have their zeros on the critical line defined by their functional equation.

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5 Appendix - Complements on the main theorem proof

In this paragraph we provide details on the demonstration of properties of \( B^L_s \) with \( \chi \) a non trivial even primitive Dirichlet character. To simplify notation we define:

\[
G_{\chi}(x, \alpha z) = \sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z + \sqrt{\pi n}x)^2}
\]

\(^{15}\)Conditions are: Functional equation and having their function \( G(x, \alpha z) \) (associated to L-function in same way as in his article - with correct adaptation) with good property when \( x \to 0 \).
5.1 The function \( B_L^I(y, z) \) is well defined

We need to demonstrate that the integrals defining \( B_L^I \) are well defined. This is done below for the first integral by analyzing the behavior of \( G_\chi(x, \alpha z) \) for \( x \) approaching zero (case of other integral is exactly similar).

Consider for \( z \) fixed \( f_z(x) = e^{-(\alpha z + \sqrt{n}x)^2} \), we can apply the modified Poisson summation formula to this function (as this function decreases exponentially fast in \( \infty \) and \(-\infty\)), reminding we have here an even non trivial primitive character \( \chi \), we obtain:

\[
G_\chi(x, \alpha z) = \sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z + \sqrt{n}x)^2}dx = \frac{\tau(\chi)}{x} \sum_{n=1}^{\infty} \chi(n)\hat{f}(\frac{n}{x})
\]  

(9)

With \( \hat{f}(y) = \int_{-\infty}^{\infty} e^{-(\alpha z + \sqrt{n}x)^2} e^{i xy}dx = 2 \int_0^{\infty} e^{-(\alpha z + \sqrt{n}x)^2} \cos(xy)dx \)

And in [1] page 15 we find:

\[
\hat{f}(y) = 2 \frac{e^{iz^2}}{4} \left( e^{\frac{(2\alpha z - iy)^2}{4\pi}} \text{Erfc}(\frac{2\alpha \sqrt{\pi} z - iy}{2 \sqrt{\pi}}) + e^{\frac{(2\alpha z + iy)^2}{4\pi}} \text{Erfc}(\frac{2\alpha \sqrt{\pi} z + iy}{2 \sqrt{\pi}}) \right)
\]

(10)

With following asymptotic property:

\[
\hat{f}(y) \sim \frac{e^{iz^2}}{y^2 - 4i \pi z^2} \quad \text{Hence we also have:} \quad \hat{f}(\frac{n}{x}) \sim \frac{e^{iz^2}}{n^2 - 4i \pi z^2 x^2}
\]

(11)

And finally:

\[
G_\chi(x, \alpha z) \xrightarrow{z \to 0} e^{iz^2} \frac{2\alpha \sqrt{\pi} z}{n^2 - 4i \pi z^2 x^2}
\]

(12)

From this asymptotic we deduce that \( G_\chi(x, \alpha z) \) is asymptotic to zero for \( x \) approaching zero: the integral \( \int_0^{\infty} G_\chi(x, \alpha z)g(xy) x^{-s}dx \) converges in zero for \( s \) in the critical strip. (Identically the second integral is well defined for \( s \) in the critical strip). (In the case of odd characters the \( G_\chi \) function to consider is the following: \( G_\chi(x, \alpha z) = \sum_{n=1}^{\infty} \chi(n)x e^{-(\alpha z + \sqrt{n}x)^2} \) and Fourier transform is found in [1] page 16, leading also to an equivalent of type \( G_\chi \sim k x \) for \( x \to 0 \))

5.2 The differential equation verified by \( B_L^I(y, z) \)

We do not present full calculations (which are fastidious but not difficult). The main basic first step is to show that:

\[
T_{s,a,\alpha}(y, z) = e^{-iz^2} \int_0^{\infty} e^{-(\alpha z + \sqrt{n}x)^2} g(xy^n) x^{-s}dx \]

verifies (calculate derivatives by \( z \) then integrate by parts in \( x \)):

\[
\frac{1}{2} \left( \frac{\partial^2}{\partial z^2} + z^2 \right) T_{s,a,\alpha} + \frac{i}{a} \left( y \frac{\partial}{\partial y} \right) T_{s,a,\alpha} = i(s - \frac{1}{2}) T_{s,a,\alpha}
\]

(13)

---

16 For \( x \) approaching infinity there is no problem due to exponential in \( x^2 \)

17 For a continuous function decreasing sufficiently fast at \( +/- \infty \) \( (x > 0) \): \( \sum_{m=-\infty}^{\infty} \chi(m) f \left( \frac{mx}{q} \right) = \frac{\tau(\chi)}{x} \sum_{n=-\infty}^{\infty} \chi(-n) f \left( \frac{n}{x} \right) \)
then to deduce that we have also the relation:
\[-\frac{1}{2} \left( \frac{\partial^2}{\partial^2 z} + z^2 \right) T_{1-s, -a, \overline{\pi}} - \frac{i}{\alpha} \left( y \frac{\partial}{\partial y} \right) T_{1-s, -a, \overline{\pi}} = i \left( \frac{1}{\alpha} - s \right) T_{1-s, -a, \overline{\pi}} \tag{14}\]

This gives the result: both terms entering into definition of $B^L_s$ are eigenfunction of $H$ with same eigenvalue.

It has to be highlighted that this result is linked to the property of Cylindric Parabolic functions (showing the close link between these functions and the Zeta function). In [6] p687 - 19.5.3, we find their integral
\[\int_{-\infty}^{\infty} e^{-zx - \frac{i}{2} x^2} e^{-\frac{1}{2} z} dx\]

5.3 The square integrability of $B_s(y, z)$

We want to show the square integrability of $B_s(y, z)$ on its domain of definition $D$ when $s$ is zero of $L(s, \chi)$:

\[\langle B^L_s, B^L_s \rangle = \int_{0}^{\infty} \int_{0}^{\infty} B^L_s(y, z) B^L_s(z, y) dy \, dz \tag{16}\]

Before we remark that $G_s(x, \alpha z)$ is bounded for $(x, z) \in D$

For $x$ and $z$ positive and not null we have:

\[|G_s(x, \alpha z)| = \left| \sum_{n=1}^{\infty} \chi(n) e^{-(\alpha z + \sqrt{\pi} nx)^2} \right| \leq \sum_{n=1}^{\infty} e^{-\sqrt{\pi} n x z - \pi n^2 x^2}\]

showing $|G_s(x, \alpha z)|$ is bounded on all domain of the form $(x, z) \in [a, \infty] \times [0, \infty]$ (with $a > 0$).

And using asymptotic of $G_s(x, \alpha z)$ for $x$ approaching zero (use (8) and correct behavior in $z$ of further terms of $G_s$ asymptotic), we see that $|G_s(x, \alpha z)|$ is also bounded for $(x, z) \in [0, a] \times [0, \infty]$.

We conclude $\exists M'/|G_s(x, \alpha z)| < M'$ on $R^+2$ (identically $\exists M'/|G_s(x, \overline{\pi} z)| < M'$ on $R^+2$).

For the demonstration of square integrability of $B_s$ on $D$, we note $I_1$ (resp. $I_2$) the first (resp. second) integral in the definition of $B^L_s$ in [2]:

\[I_1(y, z) = \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n) e^{-(\alpha z + \sqrt{\pi} nx)^2} e^{-xy} x^{-s} dx \tag{17}\]
\[I_2(y, z) = \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n) e^{-(\pi z + \sqrt{\pi} nx)^2} e^{-\frac{y}{x}} x^{s-1} dx \tag{18}\]

For the demonstration we will split $D$ in different parts, so that on each we can use the expression of $I_1$ which is convenient. Note that the "cancellation effect" (see [5]) ensures that $I_1(0, z) = I_2(0, z) = 0$ (so $I_1$ and $I_2$ are well defined and continuous on $R^+2$) and allows us to remove 1 to the exponential in the integrand depending on domain considered (see [6] and [7]).

5.3.1 $I_1$ on the domain $(y, z) \in [0, A] \times [B, \infty[\]

As $s$ is zero of $L(s, \chi)$ we use cancellation effect to write:

\[I_1 = \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n) e^{-(\alpha z + \sqrt{\pi} nx)^2} (e^{-xy} - 1) x^{-s} dx\]

And by change of variable:

\[I_1 = e^{iz^2} \int_{0}^{\infty} e^{-\sqrt{\pi} \alpha z x} e^{-\pi x^2} \sum_{n=1}^{\infty} \chi(n) (e^{-\frac{x}{n}} - 1) \left( \frac{x}{n} \right)^{-s} \frac{dx}{n}\]
Moreover we have following asymptotic for $y$ fixed and $x \to 0$:

\[
e^{-\pi x^2} \sum_{n=1}^{\infty} \chi(n)(e^{-\frac{x}{\pi}} - 1) \left(\frac{x}{n}\right)^{-s} \frac{1}{n} \sim -\sum_{n=1}^{\infty} \chi(n) \frac{n}{n^2-y} x^{1-s}
\]

We apply the Watson lemma (See [13] page 20) to find asymptotic of $I_1$ for $z \to \infty$ ($c$ is a constant):

\[I_1 \sim c e^{iz^2} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2-y} (\sqrt{2\pi n}z)^s
\]

And we conclude that on the domain considered (as $y \in [0, A]$) there exists a constant $K$ such that:

\[|I_1| < Ky^2 z^{Re(s)-2}
\]

(19)

5.3.2 $I_1$ on the domain $(y, z) \in [0, A] \times [0, B]$

$I_1(y, z)$ being continuous on this domain the only possible problem for $B_s$ integrability could be for $y$ tending to zero (as we need to ensure square integrability of $y^{-1}|I_1|^2$ we need to estimate asymptotic of $I_1$ when $y \to 0$).

We start by bounding $|I_1|:

\[|I_1| \leq \int_{0}^{\infty} \sum_{n=1}^{\infty} e^{-\sqrt{2\pi n}y} e^{-\pi^2 x^2} (1 - e^{-xy}) x^{-Re(s)} dx \leq \int_{0}^{\infty} \sum_{n=1}^{\infty} e^{-\sqrt{2\pi n}y} (1 - e^{-xy}) x^{-Re(s)} dx
\]

Then by change of variable ($x \to \frac{x}{ny}$):

\[|I_1| \leq (zy)^{-Re(s)-1} \int_{0}^{\infty} e^{-\sqrt{2\pi y} x} \sum_{n=1}^{\infty} (1 - e^{-\frac{x}{n}}) \left(\frac{x}{n}\right)^{-Re(s)} \frac{dx}{n}
\]

And using the following asymptotic for $x \to 0$: \(\sum_{n=1}^{\infty} (1 - e^{-\frac{x}{n}}) \left(\frac{x}{n}\right)^{-Re(s)} \sim \frac{1}{n} \sim \frac{1}{n} \sum_{n=1}^{\infty} \frac{1}{n^{Re(s)-2}}\)

we apply Watson lemma (for $\frac{\sqrt{2\pi y}}{y} \to \infty$) and find following bound on this domain, for $z$ fixed non null, there exists a constant $K$ such that:

\[|I_1| \leq K(zy)^{-Re(s)-1} \frac{1}{z} y^{2-Re(s)} = Kz^{Re(s)-2} y
\]

(20)

So for all fixed (not null) $I_1$ is bounded by a term in $O(y)$ (this will compensate the $y^{-\frac{1}{2}}$ in $B_s$ definition and ensure convergence in 0 of integral on variable $y$)

5.3.3 $I_1$ on the domain $(y, z) \in [A, \infty] \times [0, B]$

Here we need the behavior of $I_1$ for $y \in [0, B]$ and $y \to \infty$:

\[I_1 = \int_{0}^{\infty} e^{-xy} \sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z+\sqrt{\pi n}x)^2} x^{-s} dx
\]

Using (12) we have $G(x, z) = \sum_{n=1}^{\infty} \chi(n)e^{-(\alpha z+\sqrt{\pi n}x)^2}$ asymptotic to $K_z x + o(x)$ for $x \to 0$ ($K_z$ a constant depending on $z$ and bounded on $[0, B]$).

So applying Watson lemma (for $y \to \infty$) and as $z \in [0, B]$, we deduce there exists a constant $K'$ such that on this domain:

\[|I_1| \leq K'y^{Re(s)-2}
\]

(21)

5.3.4 $I_1$ on the domain $(y, z) \in [A, \infty] \times [B, \infty]$

\[I_1 = e^{iz^2} \int_{0}^{\infty} \sum_{n=1}^{\infty} \chi(n)e^{-\sqrt{2\pi n}x} e^{-\pi^2 x^2} e^{-xy} x^{-s} dx = e^{iz^2} \sum_{n=1}^{\infty} \chi(n) \int_{0}^{\infty} e^{-x(\sqrt{2\pi n}z+y)} e^{-\pi^2 x^2} x^{-s} dx
\]

And for $y \to \infty$ and $z \to \infty$ we can use Watson lemma term by term to find ($c$ is a constant):
\[ I_1 \sim c e^{iz^2} \sum_{n=1}^{\infty} \frac{\chi(n)(\sqrt{2\pi\alpha zn} + y)^{s-1}}{n^{2-s} (\sqrt{2\pi\alpha z})^{2-s}} \]

Now we need to consider two cases: \( y < z \) and \( y \geq z \):

**For** \( y \geq z \) we keep the asymptotic found:

\[ I_1 \sim c e^{iz^2} \sum_{n=1}^{\infty} \chi(n)(\sqrt{2\pi\alpha zn} + y)^{s-1} \]  

(22)

For \( y < z \) we need to use that \( s \) is zero of \( L(s, \chi) \) in order to see that the asymptotic can be modified:

\[ \sum_{n=1}^{\infty} \chi(n)(\sqrt{2\pi\alpha zn} + y)^{s-1} = \sum_{n=1}^{\infty} \chi(n)(\sqrt{2\pi\alpha zn})^{s-1} \left( 1 + \frac{y}{\sqrt{2\pi\alpha zn}} \right) \]

And we have following asymptotic in this case:

\[ \sum_{n=1}^{\infty} \chi(n)(\sqrt{2\pi\alpha zn} + y)^{s-1} \sim \sum_{n=1}^{\infty} \chi(n)(\sqrt{2\pi\alpha zn})^{s-1} \left( 1 + (s-1)\frac{y}{\sqrt{2\pi\alpha zn}} \right) \]

\[ L(s, \chi) = 0 \text{ hence: } \sum_{n=1}^{\infty} \chi(n)(\sqrt{2\pi\alpha zn})^{s-1} = 0 \]

And we conclude that on the half of the domain considered (asymptotic valid for \( y < z \), with \( c' \) a constant):

\[ I_1 \sim c' e^{iz^2} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{2-s} (\sqrt{2\pi\alpha z})^{2-s}} \]

(23)

We perform now the same analysis of \( I_2 \) behavior with the same split of \( D \) domain:

5.3.5 \( I_2 \) on the domain \((y, z) \in [0, A] \times [0, B]\)

We need to assess behavior of \( I_2 \) for \( y \rightarrow 0 \) and \( z \) fixed. As \( G_\chi \) is bounded by \( M \):

\[ |I_2| = \left| \int_0^\infty \left( \sum_{n=1}^{\infty} \chi(n)e^{-(\pi z + \sqrt{\pi zn})^2} \right) (e^{-x\frac{y}{n^2}} - 1) x^{s-1} dx \right| \leq \int_0^\infty M(1 - e^{-x\frac{y}{n^2}}) x^{Re(s)-1} dx \]

And we conclude (by change of variable \( x \rightarrow xy \)) that on this domain, there exists a constant \( N \) such that:

\[ |I_2| < Ny^{Re(s)} \]  

(24)

5.3.6 \( I_2 \) on the domain \((y, z) \in [0, A] \times [B, \infty] :\)

\[ I_2 = \int_0^\infty \left( \sum_{n=1}^{\infty} \chi(n)e^{-(\pi z + \sqrt{\pi zn})^2} \right) e^{-\frac{y}{n^2}} x^{s-1} dx \]

\[ I_2 = e^{-iz^2} \int_0^\infty e^{-\sqrt{2\pi\alpha zn}x} e^{-\pi x^2} \sum_{n=1}^{\infty} \chi(n) e^{-\frac{y}{n^2}} \left( \frac{x}{n} \right)^{s-1} \frac{dx}{n} \]

Using here that for any \( k > 1 \) in 0: \( e^{-\pi x^2} \sum_{n=1}^{\infty} \chi(n) e^{-\frac{y}{n^2}} \left( \frac{x}{n} \right)^{s-1} \frac{1}{n} = o(x^k) \)

A variant of Watson lemma shows that in this case \( I_2(z) \) approaches 0 at infinity faster than any inverse power of \( z \). Moreover the bound (23) made in previous paragraph (domain \([0, A] \times [0, B]\)) is also valid on the present domain.

We deduce the following bound on the domain:

\[ |I_2| < N'z^{-k} \text{ and: } |I_2| < Ny^{Re(s)} \]  

(25)

(for all \( k > 1 \) chosen, there exists such a constant \( N' \))
5.3.7 \( I_2 \) on the domain \((y, z) \in [A, \infty] \times [0, B]\)

We need to study the behavior of \( I_2 \) for \( y \to \infty \).

\[
I_2 = \int_0^\infty \left( \sum_{n=1}^\infty \chi(n) e^{-\left(\pi z + \sqrt{\pi n} x\right)^2} e^{-\frac{x}{y}} \right) x^{s-1} dx
\]

By change of variable \((x \text{ changed to } \frac{x}{y})\) we have:

\[
I_2 = \int_0^\infty e^{-\frac{x}{y}} \left( \sum_{n=1}^\infty \chi(n) e^{-\left(\pi z + \sqrt{\pi n} \frac{x}{y}\right)^2} \right) x^{s-1} dx
\]

And as the function: \( \left( \sum_{n=1}^\infty \chi(n) e^{-\left(\pi z + \sqrt{\pi n} \frac{x}{y}\right)^2} \right) x^{-s-1} \) approaches zero (for \(x \to 0\)) faster than any power of \(x\), we deduce by Watson lemma, that for \(k > 1\), there exists \(N'\) such that on this domain:

\[
|I_2| < N'y^{-k}
\]

5.3.8 \( I_2 \) on the domain \((y, z) \in [A, \infty] \times [B, \infty]\)

\[
I_2 = \int_0^\infty \left( \sum_{n=1}^\infty \chi(n) e^{-\left(\pi z + \sqrt{\pi n} x\right)^2} \right) e^{-\frac{x}{y}} x^{s-1} dx
\]

\[
I_2 = e^{-iz^2} \int_0^\infty e^{-\sqrt{\pi} \alpha x^2} e^{-\pi x^2} \left( \sum_{n=1}^\infty \chi(n) e^{-\frac{n}{\pi} \alpha} \right) \left( \frac{x}{n} \right)^{-s-1} dx
\]

Using here that for any \(k > 1\), for \(\frac{x}{y} \to 0\):

\[
|\sum_{n=1}^\infty \chi(n) e^{-\frac{n}{\pi} \alpha} (\frac{x}{n})^{-s-1}| \frac{1}{n} = o\left(\frac{x}{y}\right)^k
\]

By a variant of Watson lemma we have that for \(k > 1\), there exists a constant \(N\) such that on this domain:

\[
|I_2| < Ny^{-k}z^{-k-1}
\]

We can now conclude on the square integrability of \(B^L_s\) on each part of the domain of integration \(D\) using previous results on each domain and:

\[
|B^L_s(y, z)|^2 \leq \frac{1}{y} (|I_1|^2 + |I_2|^2)
\]

- **On domain** \((y, z) \in [0, A] \times [B, \infty]\):
  It is not difficult to see that \(|B^L_s|^2\) is integrable on this domain using our results for \(I_1\) and \(I_2\): (19), (20) and (21).
  - **On domain** \((y, z) \in [0, A] \times [0, B]\):
    Only possible problem is in \(y\) variable near zero, but using (21), (24) we have: \(\exists N', |B^L_s|^2 < N'y^{2Re(s)-1}\), so as \(s\) is strictly in the critical strip \((Re(s) > 0)\) we can integrate \(|B^L_s|^2\) in \(y\) on \([0, A]\). We conclude on the square integrability of \(B^L_s\) on this domain.
  - **On domain** \((y, z) \in [A, \infty] \times [0, B]\):
    We use here (21), (28) and (25) to see that: \(\exists K, \forall z \in [0, B]\) such that we have \(|B^L_s|^2 < Ky^{-2Re(s)}\) on this domain. Hence as \(B^L_s\) function is continuous in \(z\) we conclude on the square integrability of \(B^L_s\) on this domain.
  - **On domain** \((y, z) \in [A, \infty] \times [B, \infty]\):
    It is immediate that \(\frac{1}{y}(|I_1|^2 + |I_2|^2)\) is square integrable using (21), (23) and (24).

For integrability of \(\frac{1}{y} |I_1|^2\) we first consider \(y \geq z\):

On this sub domain, using (23), we see that we need to show that:
\[
J = \int_A^\infty \int_B^\infty \frac{1}{y} \sum_{n=1}^\infty \chi(n) \sqrt{2\pi\alpha zn + y} s^{-1} \sum_{n=1}^\infty \chi(p) \sqrt{2\pi\alpha zp + y} s^{-1} dz \ dy \text{ converges.}
\]

A direct calculation gives:
\[
\int (\sqrt{2\pi\alpha zn + y}) s^{-1} (\sqrt{2\pi\alpha zp + y}) s^{-1} dz = \frac{(\sqrt{2\pi\alpha zp + y})^s}{\sqrt{2\pi\alpha (pz + y)}} \frac{\Gamma(\frac{1}{2} + s, y) \Gamma(\frac{1}{2} + s, y)}{\Gamma(\frac{1}{2} + s)} 2F_1 (\frac{1}{2} + s - \frac{1}{2}, \frac{1}{2} + s + 1; \frac{\sqrt{2\pi\alpha zp + y}}{\alpha y \sqrt{2\pi\alpha (pz + y)}})
\]

and this integral taken in \( z = y \) is asymptotic for \( y \to \infty \) to a term of the form \( Ky^{s-1} \), then multiplying by \( \frac{1}{y} \) we obtain: \( Ky^{s-2} \) which shows that \( J \) is converging for \( \text{Re}(s) < \frac{1}{2} \) (Integral taken in a constant \( B \) and multiplied by \( \frac{1}{y} \) is also convergent). We conclude that \( J \) is well defined in this case and \( B_s \) square integrable on this domain.

Then we consider the sub domain defined by \( y < z \):

Using (28), on this sub domain there exists a constant \( K \) such that
\[
\left| \frac{I_1}{y} \right| < Ky^{2\text{Re}(s) - 4},
\]

And as \( \int_B^\infty \int_A^y yz^{2\text{Re}(s) - 4} dz \ dy \) converges, for \( \text{Re}(s) < \frac{1}{2} \), we conclude on the square integrability of \( B_s \) on this sub domain also.

Note: \( B_s \) is "nearly" square integrable for \( \text{Re}(s) = \frac{1}{2} \) on the full domain, the only reason it is not is due to \( I_1 \) on the domain \( (y, z) \in [A, \infty] \times [B, \infty] \). This forces us to conclude to the Generalized Riemann Hypothesis in this article using a contradiction (not a direct proof as it would be the case if \( B_s \) was also square integrable for \( s = \frac{1}{2} \)). This raises instantly the following question: is there another another \( B_s \) function such that for \( s \) zero of \( L_s \) located on the critical line we have \( B_s \) square integrable?

5.4 \( H \) is a Hermitian operator for \( B_s \).

We want to show that (we continue to consider here \( s \) zero of \( L(s, \chi) \) with \( \text{Re}(s) < \frac{1}{2} \)):
\[
\int_0^\infty \int_0^\infty H B_s(z, y) B_s(z, y) dz \ dy = \int_0^\infty \int_0^\infty B_s(z, y) \overline{H B_s(z, y)} dz \ dy \tag{29}
\]

On variable \( y \) there is not problem to have the operator \( iy(\frac{\partial}{\partial y} + \frac{1}{2}) \) moving from left to right using integration by parts:
\[
\int_0^\infty iy \left( \frac{\partial B_s}{\partial y} + \frac{B_s}{2} \right) dz \ dy = \left[ iy B_s \overline{B_s} \right]_0^\infty + \int_0^\infty B_s \left( iy \frac{\partial B_s}{\partial y} + \frac{B_s}{2} \right) dz \ dy \tag{30}
\]

So we need to take care of the term in brackets. Using our asymptotic value of \( B_s \) for \( z \) fixed and for \( y \) "sufficiently near" to zero, (13), (20), (24), (25) we deduce that for \( y \) sufficiently near zero, there exists a constant \( C \) such that (equation (24) being the lowest bound with \( y^{\text{Re}(s)} \), we multiply by \( y^{-\frac{1}{2}} \) to obtain the bound for \( B_s \); then square it and multiply by \( y \) to obtain the bound below):
\[
|y B_s \overline{B_s}| < C y y^{\text{Re}(s) - \frac{1}{2}} y^{\text{Re}(s) - \frac{1}{2}} = C y^{2\text{Re}(s)} \tag{31}
\]

So \( iy B_s \overline{B_s} \) taken in zero is zero.

Moreover, for \( z \) fixed and \( y \) approaching infinity we have (using that bounds of (21), (26) can be done with \( B \) as big as we want): \(|y B_s \overline{B_s}| < Ky^{\text{Re}(s) - \frac{1}{2}} y^{\text{Re}(s) - \frac{1}{2}} = Ky^{2\text{Re}(s) - 5}\)

As \( s \) is strictly in the critical strip we conclude that \(|y B_s \overline{B_s}|_0^\infty = 0 \) and on variable \( y \) the operator is Hermitian for \( B_s \).

On variable \( z \) we have to be careful as our analysis of different domains for square integrability shows that we did not proved that:
\[
\int_0^\infty z^2 B_s \overline{B_s} dz \text{ converges (due to the operator adding a } z^2 \text{ in the integrand and our bound of } |B_s|^2 \text{ being only of type } \frac{1}{z^2} \text{ for } z \text{ approaching infinity).}
\]
On the other hand we know that: \( \int_0^\infty \left( \frac{\partial^2 B_s}{\partial^2 z} + z^2 B_s \right) B_s \, dz \) converges.

So we avoid the difficulty by integrating up to \( V \), making the integration by parts to move \( H \) on the right and then make \( V \) tending to infinity.\(^{18}\) We integrate by parts the second derivative of \( B_s \):

\[
\int_0^V \left( \frac{\partial^2 B_s}{\partial^2 z} + z^2 B_s \right) B_s \, dz = \left[ \frac{\partial B_s}{\partial z} B_s \right]^V_0 - \int_0^V \frac{\partial B_s}{\partial z} \frac{\partial B_s}{\partial z} \, dz + \int_0^V z^2 B_s B_s \, dz \tag{32}
\]

Identically we have also:

\[
\int_0^V B_s \left( \frac{\partial^2 B_s}{\partial^2 z} + z^2 B_s \right) \, dz = \left[ \frac{\partial B_s}{\partial z} B_s \right]^V_0 - \int_0^V \frac{\partial B_s}{\partial z} \frac{\partial B_s}{\partial z} \, dz + \int_0^V z^2 B_s B_s \, dz \tag{33}
\]

Hence the difference between these integrals is:

\[
\left[ \frac{\partial B_s}{\partial z} B_s \right]^V_0 - \left[ \frac{\partial B_s}{\partial z} B_s \right]^V_0 \tag{34}
\]

The terms of these brackets taken in zero (as \( B_s(y, 0) = 0 \)) is null, so we need just to prove that

\[
\lim_{V \to \infty} \frac{\partial B_s}{\partial z} B_s(V) = 0,
\]

and considering that for \( y \) fixed we have: \( \lim_{z \to \infty} B_s(y, z) = 0 \), it is sufficient to show that for \( y \) fixed, \( |\frac{\partial B_s}{\partial z}| \) is bounded when \( z \to \infty \).

Calculating \( \frac{\partial B_s}{\partial z} \) we see that the only terms that could be problematic is (for other terms appearing in the derivative of \( B_s \) we use directly results (19) and (25) which are valid for all fixed \( y \):

\[
y^{-\frac{1}{2}} \int_0^\infty \left( \sum_{n=1}^{\infty} \chi(n)2\alpha \sqrt{\pi} nxe^{-\alpha z + \sqrt{\pi} nz} \right) e^{-xy} x^{-s} \, dx
\]

\[
y^{-\frac{1}{2}} \int_0^\infty \left( \sum_{n=1}^{\infty} \chi(n)2\alpha \sqrt{\pi} nxe^{-\alpha z + \sqrt{\pi} nz} \right) e^{-\frac{\alpha z}{\sqrt{\pi}} x^{s-1}} \, dx
\]

and using similar method (cancellation effect, then Watson lemma) as for demonstration of (19) and (25) it is not difficult to see that these terms are bounded for \( z \to \infty \).

So we have: \( \lim_{z \to \infty} \frac{\partial B_s}{\partial z} B_s = 0 \) and we conclude that (34) tends to zero for \( V \) tending to infinity, meaning that following equality holds:

\[
\int_0^\infty \left( \frac{\partial^2 B_s}{\partial^2 z} + z^2 B_s \right) B_s \, dz = \int_0^\infty B_s \left( \frac{\partial^2 B_s}{\partial^2 z} + z^2 B_s \right) \, dz \tag{35}
\]

There is now no difficulty to conclude that for \( s \) zero of \( L(s, \chi) \):

\[
< H B_s, B_s > = < B_s, H B_s > \tag{36}
\]

This is ending the demonstration of property 2 \( \Box \)

\(^{18}\)This small complication should not occult the fact that what is here important to notice is the way the brackets terms of integration by part disappear due to symmetry of \( B_s \).
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