First integrals for charged perfect fluid distributions

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Abstract

We study the evolution of shear-free spherically symmetric charged fluids in general relativity. We find a new parametric class of solutions to the Einstein-Maxwell system of field equations. Our charged results are a generalisation of earlier treatments for neutral relativistic fluids. We regain the first integrals found previously for uncharged matter as a special case. In addition an explicit first integral is found which is necessarily charged.

Key words: Einstein-Maxwell equations; exact solutions; charged fields

1 Introduction

Solutions of the Einstein-Maxwell system of equations are important in relativistic astrophysics as they may be used to describe charged compact objects with strong gravitational fields such as dense neutron stars. Several recent treatments, including the works of Ivanov [1] and Sharma et al [2], demonstrate that the presence of the electromagnetic field affects the values of redshifts, luminosities and maximum mass of a compact relativistic star. The electromagnetic field cannot be ignored when considering the gravitational evolution of stars composed of quark matter as pointed out by Mak and Harko [3] and Komathiraj and Maharaj [4]. Therefore exact models describing the formation and evolution of charged stellar objects, within the context of full general

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relativity, are necessary. Electromagnetic fields play a role in gravitational collapse, the formation of naked singularities, and the collapse of charged shells of matter onto existing black holes (as indicated by Lasky and Lun [5,6]). Significant electric fields are also present in phases of intense dynamical activity, in collapsing configurations, with time scales of the order of the hydrostatic time scale for which the usual stable equilibrium configurations assumptions are not reliable (as shown in the treatments of Di Prisco et al [7] and Herrera et al [8]). It is interesting to note that Maxwell’s equations play a role in several other scenarios, including the evolution of cosmological models in higher dimensions. De Felice and Ringeval [9] considered braneworld models, exhibiting Poincare symmetry in extra-dimensions, which admit wormhole configurations.

Spherical symmetry and a shear-free matter distribution are simplifying assumptions usually made when seeking exact solutions to the Einstein field equations with neutral matter. The field equations may then be reduced to a single partial differential equation. What is interesting about this equation is that it can be treated as an ordinary differential equation. A general class of solutions was first found by Kustaanheimo and Qvist [10]. Comprehensive treatments of the uncharged case are provided by Srivastava [11] and Sussman [12]. The generalisation to include the electromagnetic field is easily performed and is described by the Einstein-Maxwell system. The field equations are again reducible to a single partial differential equation, now containing a term corresponding to charge. A review of known charged solutions, admitting a Friedmann limit, is contained in the treatment of Krasinski [13]. A detailed investigation of the mathematical and physical features of the Einstein-Maxwell system has been performed by Srivastava [14] and Sussman [15,16] respectively.

The objective of this paper is to investigate the integrability properties of the governing partial differential equation that contains a term corresponding to charge, for shear-free fluids. This investigation is performed using an elementary approach suggested by Srivastava [11]. In Section 2, we reduce the Einstein-Maxwell field equations, generalising the transformation due to Faulkes [17], to a single nonlinear second order partial differential equation that governs the behaviour of charged fluids. As in the uncharged case, this equation can be treated as an ordinary differential equation. In Section 3, we derive a first integral of the governing equation by generalising the technique of Srivastava [11] first used for uncharged fluids. This first integral is subject to two integrability conditions expressed as nonlinear integral equations. We transform the integrability conditions, in Section 4, into a new system of differential equations which can be integrated in terms of quadratures. In Section 5 we comprehensively investigate the nature of the factors of the quartic arising in the quadrature. Finally, in Section 6, we discuss the results obtained and comment on some of the physical aspects. The approach of Khalique et al [18] using a group classification may be helpful in providing further insights.
in future work.

2 Field equations

We consider the shear-free motion of a spherically symmetric perfect fluid in the presence of the electromagnetic field. We choose a coordinate system \( x^i = (t, r, \theta, \phi) \) which is both comoving and isotropic. In this coordinate system the metric can be written as

\[
 ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\lambda(t,r)} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]
\]  

(1)

where \( \nu \) and \( \lambda \) are the gravitational potentials. We are investigating the general case of a self-gravitating fluid in the presence of the electromagnetic field without placing arbitrary restrictions on the potentials. For this model the Einstein equations are supplemented with Maxwell equations. The Einstein field equations for a charged perfect fluid can be written as the system

\[
 \rho = \frac{3}{e^{2\nu}} \frac{\lambda_t^2}{4} - \frac{1}{e^{2\lambda}} \left( 2\nu_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r} \right) - \frac{E^2}{r^4 e^{4\lambda}} \]  

(2a)

\[
p = \frac{1}{e^{2\nu}} \left( -3\lambda_t^2 - 2\lambda_t \nu + 2\nu \lambda_t \right) + \frac{1}{e^{2\lambda}} \left( \lambda_r^2 + 2\nu \lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r} \right) + \frac{E^2}{r^4 e^{4\lambda}} \]  

(2b)

\[
p = \frac{1}{e^{2\nu}} \left( -3\lambda_t^2 - 2\lambda_t \nu + 2\nu \lambda_t \right) + \frac{1}{e^{2\lambda}} \left( \nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} + \frac{\lambda_r}{r} + \lambda_{rr} \right) - \frac{E^2}{r^4 e^{4\lambda}} \]  

(2c)

\[
0 = \nu_r \lambda_r - \lambda_{tr} \]  

(2d)

Maxwell’s equations yield

\[
 E = r^2 e^{\lambda - \nu} \Phi_r, \quad E_r = \sigma r^2 e^{3\lambda} \]  

(3)

In the above \( \rho \) is the energy density and \( p \) is the isotropic pressure which are measured relative to the four-velocity \( u^a = (e^{-\nu}, 0, 0, 0) \). Subscripts refer to partial derivatives with respect to that variable. The quantity \( E = E(r) \) is an arbitrary constant of integration and \( \sigma \) is the proper charge density of the fluid. We interpret \( E \) as the total charge contained within the sphere of radius \( r \) centred around the origin of the coordinate system. Note that \( \Phi_r = F_{10} \) is the only nonzero component of the electromagnetic field tensor \( F_{ab} = \phi_{b,a} - \phi_{a,b} \).
where \( \phi_a = (\Phi(t, r), 0, 0, 0) \). The Einstein-Maxwell system (2)-(3) is a coupled system of equations in the variables \( \rho, p, E, \sigma, \nu \) and \( \lambda \).

The system of partial differential equations (2) can be simplified to produce an underlying nonlinear second order equation. Equation (2d) can be written as

\[
\nu_r = (\ln \lambda) r
\]

Then (2b) and (2c) imply

\[
\left[ e^\lambda \left( \lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r} \right) + \frac{2E^2 e^{-\lambda}}{r^4} \right]_r = 0
\]

and the potential \( \nu \) has been eliminated. The Einstein field equations (2) can therefore be written in the equivalent form

\[
\begin{align*}
\rho &= 3e^{2h} - e^{-2\lambda} \left( 2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r} \right) - \frac{E^2}{r^4 e^\lambda} \quad (4a) \\
p &= \frac{1}{\lambda} e^{3\lambda} \left[ e^\lambda \left( \lambda_r^2 + \frac{2\lambda_r}{r} \right) - e^{3\lambda + 2h} - \frac{E^2}{r^4 e^\lambda} \right]_r \quad (4b) \\
e^\nu &= \lambda t e^{-h} \quad (4c) \\
e^\lambda \left( \lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r} \right) &= -\tilde{F} - \frac{2E^2}{r^4 e^\lambda} \quad (4d)
\end{align*}
\]

for a charged relativistic fluid. In the above \( h = h(t) \) and \( \tilde{F} = \tilde{F}(r) \) are arbitrary constants of integration. Equation (4d) is the condition of pressure isotropy generalised to include the electric field. To find an exact solution of the field equations, we need to specify the functions \( h, \tilde{F} \) and \( E \) and solve equation (4d) for \( \lambda \). We can then compute the quantities \( \rho \) and \( p \) from (4a) and (4b), and \( \sigma \) follows from (3).

It is possible to write (4d) in a simpler form by eliminating the exponential factor \( e^\lambda \). We use the transformation, first introduced by Faulkes [17] for neutral fluids, which has the adapted form

\[
x = r^2, \quad y = e^{-\lambda}, \quad f(x) = \frac{\tilde{F}}{4r^2}, \quad g(x) = \frac{E^2}{2r^6}
\]

Then (4d) becomes

\[
y_{xx} = f(x)y^2 + g(x)y^3 \quad (5)
\]

which is the fundamental equation governing the behaviour of a shear-free charged fluid. Observe that (5) is a nonlinear partial differential equation since \( y = y(t, x) \). When \( g = 0 \) then \( y_{xx} = f(x)y^2 \) for a neutral fluid which has been studied by Maharaj et al [19] and others.
3 A charged integral

It would appear that we need to specify the functions $f(x)$ and $g(x)$ to integrate (5). However it is possible ab inito to generate a first integral without choosing $f(x)$ and $g(x)$ if we generalise a technique first suggested by Srivastava [11], and extended by Maharaj et al [19]. The first integral generated is subject to a system of integral equations in $f(x)$ and $g(x)$ which can be rewritten as differential equations.

Rather than choose $f(x)$ and $g(x)$ we seek general conditions that reduce the order of (5) to produce a first order differential equation. We can formally integrate (5) to obtain

$$y_x = \int f(x)y^2dx + \int g(x)y^3dx$$

$$= f_I y^2 + g_I y^3 - 2 \int f_I yy_x dx - 3 \int g_I y^2 y_x dx$$

Equation (6)

For convenience we have used the notation

$$\int f(x)dx = f_I, \quad \int g(x)dx = g_I$$

Integrating $f_I yy_x$ by parts and utilising (5) gives the result

$$\int f_I yy_x dx = f_{I1}y y_x - \int f_{I1}y^2 dx - \int f f_{I1}y^3 dx - \int g f_{I1}y^4 dx$$

Equation (7)

We substitute (7) in (6) to obtain

$$y_x = f_I y^2 + g_I y^3 - 2 f_{I1} y y_x + 2 \int f_{I1} y^2 dx$$

$$+ 2 \int f f_{I1} y^3 dx + 2 \int g f_{I1} y^4 dx - 3 \int g_I y^2 y_x dx.$$  

Equation (8)

We continue this process and evaluate the integrals of $f_{I1}y^2, f f_{I1}y^3$ and $g f_{I1}y^4$ in (8) using integration by parts. Eventually we arrive at the expression

$$y_x = f_I y^2 + g_I y^3 - 2 f_{I1} yy_x + 2 f_{II1} y^2 + 2 (f f_{II}) y^3 + 2 (g f_{II}) y^4$$

$$- \frac{2}{3} \int \left\{ 2 f f_{III} + 3 (f f_{II}) + \frac{3}{2} g_I \right\} \left( \frac{dy^3}{dx} \right) dx$$

$$- \int \left\{ g f_{III} + 2 (g f_{II}) \right\} \left( \frac{dy^4}{dx} \right) dx.$$  

Equation (9)
For a meaningful result the integrals on the right hand side of (9) must be eliminated.

We note that these integrals can be determined if \(2f_{f_{III}} + 3(f_{f_{II}})_I + \frac{3}{2}g_I\) and \(g_{f_{III}} + 2(g_{f_{II}})_I\) are constants. This observation yields the following result

\[
\tau_0(t) = -y_x + f_I y^2 + g_I y^3 - 2f_{f_{II}} y y_x + 2f_{f_{III}} y_x^2 + 2[(f_{f_{II}})_I - \frac{1}{3}K_0] y^3 \\
+ [2(g_{f_{II}})_I - K_1] y^4
\]  

subject to the integrability conditions

\[
K_0 = 2f_{f_{III}} + 3(f_{f_{II}})_I + \frac{3}{2}g_I \quad (11a) \\
K_1 = g_{f_{III}} + 2(g_{f_{II}})_I \quad (11b)
\]

where \(K_0\) and \(K_1\) are constants, and the quantity \(\tau_0(t)\) is an arbitrary function of integration. We have therefore established that a first integral of the field equation (5) is given by (10) subject to conditions (11) which are integral equations. On the surface it appears that the functions \(f(x)\) and \(g(x)\) are free. However equations (11a) and (11b) effectively determine the forms of the functions \(f(x)\) and \(g(x)\); they are constrained by the integrability conditions (11).

4 Integrability conditions

It is not easy to solve the nonlinear integral equations (11). However we can transform these equations into an equivalent system comprising a first order and a fourth order ordinary differential equation which are more convenient to work with.

We let

\[ f_{III} = \mathcal{F} \]

so that \(f_{II} = \mathcal{F}_x, f_I = \mathcal{F}_{xx}\) and \(f = \mathcal{F}_{xxx}\). Then it is possible to rewrite (11b) as

\[
(g\mathcal{F})_x + 2g\mathcal{F}_x = 0 \quad (12)
\]

Note that the integral equation (11b) has been transformed to a first order differential equation in \(\mathcal{F}\). Equation (12) is integrable and we obtain

\[
g = \mathcal{K}_0\mathcal{F}^{-3} \quad (13)
\]

where \(\mathcal{F} = \mathcal{F}(x)\) and \(\mathcal{K}_0\) is an arbitrary constant.
Similarly we can eliminate \( g \) in (11a), with the help of (13), to get the result

\[
\mathcal{F}\mathcal{F}_{xxx} + \frac{5}{2}\mathcal{F}_x\mathcal{F}_{xxx} = -\frac{3}{4}\mathcal{K}_0\mathcal{F}^{-3}
\] (14)

Therefore the integral equation (11a) has been transformed to a fourth order differential equation in \( \mathcal{F} \). Equation (14) can be integrated repeatedly to yield

\[
\mathcal{F}^{-1} = \mathcal{K}_4 + \mathcal{K}_3 \int \mathcal{F}^{-3/2} dx + \mathcal{K}_2 \left( \int \mathcal{F}^{-3/2} dx \right)^2 - \frac{1}{6}\mathcal{K}_1 \left( \int \mathcal{F}^{-3/2} dx \right)^3
+ \frac{1}{32}\mathcal{K}_0 \left( \int \mathcal{F}^{-3/2} dx \right)^4
\] (15)

where the \( \mathcal{K}_i \) are arbitrary constants.

We can rewrite (15) in a simpler form if we let

\[
u = \int \mathcal{F}^{-3/2} dx
\] (16)

so that

\[u_x = \left( \mathcal{F}^{-1} \right)^{3/2}\] (17)

which is a first order equation in \( u \). The equivalent integral representation is

\[x - x_0 = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4)^{3/2}}
\] (18)

where \( x_0 \) is a constant. The quadrature (18) can be evaluated in terms of elliptic integrals. We can summarise our result as follows: the first integral (11), with \( g = \mathcal{K}_0\mathcal{F}^{-3} \), \( f = \mathcal{F}_{xxx} \) and \( \mathcal{F} \) given by (18) via (16), represents a particular class of solutions of (5).

To obtain solutions in closed form, satisfying the integrability conditions (11), we need to evaluate the integral (18). Particular solutions in terms of elementary functions are admitted. In general the solution will be given in terms of special functions. We can express the solutions to (11) in the parametric form as follows

\[
f(x) = \mathcal{F}_{xxx} \] (19a)

\[g(x) = \mathcal{K}_0\mathcal{F}^{-3} \] (19b)

\[u_x = \mathcal{F}^{-3/2} = [G'(u)]^{-1} \] (19c)

\[x - x_0 = G(u) \] (19d)
where we have set

\[ G(u) = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4)^{3/2}} \]  

(20)

If we set \( g = 0 \) (which forces \( \mathcal{K}_0 = 0 \)) then the charge vanishes and the system (19) becomes

\[ f(x) = F_{xxx} \]  

(21a)

\[ u_x = F^{-3/2} = [G'(u)]^{-1} \]  

(21b)

\[ x - x_0 = G(u) \]  

(21c)

where

\[ G(u) = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4)^{3/2}}, \]  

(22)

This corresponds to the results found by Maharaj et al. [19] for a neutral shear-free gravitating fluid. Thus their first integral is contained in our class of charged models (19)-(20).

5 Particular solutions

Nine cases arise from the solution (19)–(20) depending on the nature of the factors of the polynomial \( \mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 \).

5.1 Case I: One order-four linear factor

If \( \mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 \) has one repeated linear factor then we have

\[ \mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4 = (a + bu)^4, b \neq 0 \]

We evaluate the integral in (20) to obtain

\[ G(u) = -\frac{1}{5b}(a + bu)^{-5} \]  

(23a)

\[ f(x) = \frac{24}{75}(5b)^{4/5}(x - x_0)^{-11/5} \]  

(23b)

\[ g(x) = \mathcal{K}_0(5b)^{-12/5}(x - x_0)^{-12/5} \]  

(23c)

In this case it is possible to invert the integral (18) and then write \( u = u(x) \). The first integral (10) has the form
\[ \tau_0(t) = -y_x - \frac{4}{15}(5b)^{4/5}(x - x_0)^{-6/5}y^2 - \frac{5}{7}K_0(5b)^{-12/5}(x - x_0)^{-7/5}y^3 \]
\[ -\frac{8}{3}(5b)^{4/5}(x - x_0)^{-1/5}yy_x + \frac{10}{3}(5b)^{4/5}(x - x_0)^{4/5}y_x^2 \]
\[ -2 \left[ \frac{3856}{10815}(5b)^{8/5}(x - x_0)^{-7/5} - \frac{15}{14}K_0(5b)(x - x_0)^{-7/5} \right]y^3 \]
\[ -\frac{5}{3}K_0(5b)^{-8/5}(x - x_0)^{-8/5}y^4 \]  

(24)

where we have used the functional forms in (23). The first integral (24) corresponds to a shear-free spherically symmetric charged fluid which does not have an uncharged limit since \( K_0 \neq 0 \). If \( K_0 = 0 \) then the polynomial becomes cubic which is a contradiction. The charged integral (24) \( (E \neq 0, K_0 \neq 0, b \neq 0) \) is a new solution to the Einstein-Maxwell field equations.

5.2 Case II: One order-three linear factor

If \( \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 \) has two linear factors, one of which is not repeated, then we have

\[ \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 = (a + bu) (u + c)^3 \]

We use the computer package Mathematica [20] to determine the integral in (20) to obtain

\[ G(u) = \frac{2\sqrt{(a + bu)(u + c)}}{35(a - bc)^5} \left[ \frac{35b^4}{a + bu} + \frac{93b^3}{u + c} - \frac{29b^2(a - bc)}{(u + c)^2} \right. \]
\[ + \frac{13b(a - bc)^2}{(u + c)^3} - \frac{5(a - bc)^3}{(u + c)^4} \]  

(25)

expressed completely in terms of elementary functions. In this case, if \( g = 0, \mathcal{K}_0 = 0 \) and \( b = 0 \), then (25) becomes

\[ G(u) = a^{-3/2} \left( -\frac{2}{7} \right) (u + c)^{-7/2} \]  

(26)

and hence using (21) we find

\[ f(x) = a^{2/7} \left( \frac{48}{343} \right) \left( -\frac{7}{2} \right)^{6/7} (x - x_0)^{-15/7} \]  

(27)

Note that (27) is related to the result obtained by Maharaj et al [19].
Again setting $g = 0$, $K_1 = 0$, in (10) we get

$$\psi_0(t) = -y_x + f_1y^2 + g_1y^3 - 2f_{11}yy_x + 2f_{111}y_x^2 + 2[(f_{111})_1 - \frac{1}{3}K_0]y^3$$

which was the first integral for uncharged matter found by Maharaj et al [19].

Also observe that if $g = 0$, $K_1 = 0$, $f(x) = (ax + b)^{-15/7}$ then (10) yields

$$\phi_0(t) = -6y_x - \frac{21}{4a}(ax + b)^{-8/7}y^2 - \frac{3}{2} \left( \frac{7}{a} \right)^2 (ax + b)^{-1/7}yy_x$$

$$+ \frac{1}{4} \left( \frac{7}{a} \right)^3 (ax + b)^{6/7}y_x^2 - \frac{1}{6} \left( \frac{7}{a} \right)^3 (ax + b)^{-9/7}y^3$$

(28)

which was found by Srivastava [11]. Also with $g = 0$, $K_1 = 0$, $f(x) = x^{-15/7}$ in (10) (or if we set $a = 1, b = 0$ in (28)) we have

$$\varphi_0(t) = -6y_x - \frac{21}{4}x^{-8/7}y^2 - \frac{3}{2} \cdot \frac{7}{a}x^{-1/7}yy_x + \frac{1}{4} \cdot \frac{7}{a}x^{6/7}y_x^2$$

$$- \frac{1}{6} \cdot \frac{7}{a}x^{-9/7}y^3$$

which was established by Stephani [21]. Therefore the first integral (10) is a charged generalisation of the particular Maharaj et al [19], Srivastava [11] and Stephani [21] neutral models.

5.3 Case III: One order-two linear factor; one order-one quadratic factor

If $\mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4$ has two factors, one linear and repeated and the other is irreducible to linear factors, then we have

$$\mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4$$

$$= (a + bu + cu^2)(u + d)^2, b^2 - 4ac < 0.$$
\[ G(u) = \]
\[- \frac{1}{(a - bd + cd^2) u^2} + \frac{5(b - 2cd)}{2(a - bd + cd^2)u} \]
\[- \frac{15(b - 2cd)^4 - 62c(b - 2cd)^2(a - bd + cd^2) + 24c^2(a - bd + cd^2)^2}{2(a - bd + cd^2)[4c(a - bd + cd^2) - (b - 2cd)^2]} \]
\[- \frac{c(b - 2cd)[15(b - 2cd)^2 - 52c(a - bd + cd^2)]u}{2(a - bd + cd^2)\Delta} \times \]
\[ \frac{2\sqrt{(a - bd + cd^2) + (b - 2cd)u + cu^2}}{8(a - bd + cd^2)^3} \]
\[ \int \frac{du}{u\sqrt{a - bd + cd^2 + (b - 2cd)u + cu^2}} \quad (29) \]

where \( \Delta = 4(a - bd + cd^2)c - (b - 2cd)^2 \) and the integral on the right hand side can be expressed in terms of elementary functions. The exact form of the integral depends on the signs of \( a - bd + cd^2 \) and \( \Delta \) (see Gradshteyn and Ryzhik [22], equations 2.266 and 2.269.6).

5.4 Case IV: One order-two linear factor; two order-one linear factors

With one repeated and two non-repeated linear factors we have

\[ \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 = (a + bu)(cu + d)(u + e)^2 \]

In this case the expression for the integral in (20) can be evaluated with the help of the computer package Mathematica [20]. The resulting expression is expressible in terms of only elementary functions. This expression is very lengthy and not illuminating, and is therefore not included in this work.

5.5 Case V: Two order-two linear factors

If \( \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 \) has two linear factors each of which is repeated, then we have

\[ \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 = (a + bu)^2(u + c)^2. \]

The integral in (20) may be easily determined so that
\[ G(u) = \left[ 6b^2 \ln \frac{u + c}{a + bu} + 3b^2(a - bc) \right. \]
\[ - \frac{(a - bc)^2}{2(u + c)^2} \left. + \frac{b^2(a - bc)^2}{2(a + bu)^2} + \frac{3b(a - bc)}{u + c} \right] \]
\[ \frac{1}{(a - bc)^3} \]

Thus for the case of two order-two linear factors the integral can be expressed completely in terms of elementary functions.

### 5.6 Case VI: No repeated linear factors

If \( \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 \) has no repeated linear factors, then we have

\[ \mathcal{K}_4 + \mathcal{K}_3u + \mathcal{K}_2u^2 - (1/6)\mathcal{K}_1u^3 + (1/32)\mathcal{K}_0u^4 = e(a + u)(b + u)(c + u)(d + u), \quad e \neq 0 \]

In this case we obtain, in terms of elementary functions and elliptic integrals, the result [23]

\[ G(u) = \]
\[ \frac{2e^{-3/2}}{(a - b)\sqrt{(a + u)(b + u)(c + u)(d + u)}} \left[ \frac{(a + u)(b + u)}{(b - c)(a - d)} \right. \]
\[ - \frac{2}{(b - d)^2} \left. + \frac{1}{(b - d)(c - d)} + \frac{1}{(a - c)(c - d) +} \right] + \frac{b + u}{a - c} \left[ \frac{2(d + u)}{(a - b)(a - d)^2} \right. \]
\[ - \frac{1}{(b - c)(b - d)} - \frac{1}{(a - d)(b - d)} \left. - \frac{1}{(a - d)(c - d)} \right] \]
\[ - \frac{4e^{-3/2}}{(a - b)\sqrt{b - d}} \left[ \frac{1}{(a - d)^2(c - d)\sqrt{a - c}} + \frac{\sqrt{a - c}}{(a - b)(b - c)^2(b - d)} \right. \]
\[ + \frac{a - b - c + d}{(c - d)^2(a - c)^{3/2}(b - c)} \left. E(\alpha, p) \right] \]
\[ + \frac{2e^{-3/2}}{(a - c)^{3/2}(b - d)^{3/2}(b - c)(a - d)} \times \]
\[ \left[ \frac{2(a + b - c - d)^2}{(b - c)(a - d)} + \frac{(a - b - c + d)^2}{(a - b)(c - d)} \right] \]
\[ F(\alpha, p), \quad (0 < d < c < b < a) \quad (31) \]

where we have let

\[ \alpha = \arcsin \frac{(a - c)(d + u)}{(a - d)(c + u)}, \quad p = \frac{(b - c)(a - d)}{(a - c)(b - d)} \]

In [31], \( F(\alpha, p) \) is the elliptic integral of the first kind and \( E(\alpha, p) \) is the elliptic integral of the second kind. This result is similar to one of the results obtained
by Maharaj et al [19]. However their uncharged model is not regainable from the expression above as the polynomial here is necessarily quartic.

5.7 Case VII: One order-two quadratic factor

If $K_4 + K_3u + K_2u^2 - (1/6)K_1u^3 + (1/32)K_0u^4$ has one repeated quadratic irreducible factor, then we have

$$K_4 + K_3u + K_2u^2 - (1/6)K_1u^3 + (1/32)K_0u^4 = (a + bu + cu^2)^2$$

In this case we obtain

$$G(u) = \frac{b + 2cu}{4ac - b^2} \left[ \frac{1}{2(a + bu + cu^2)^2} + \frac{3c}{(4ac - b^2)(a + bu + cu^2)} \right] + \frac{6c^2}{(4ac - b^2)^2} \int \frac{du}{a + bu + cu^2}$$

which can be expressed in terms of only elementary functions. The exact form of the integral depends on the sign of $4ac - b^2$ (see Gradshteyn and Ryzhik [22], equations 2.172 and 2.173.2).

5.8 Case VIII: Two order-one quadratic factors

With two non-repeated quadratic factors we have

$$K_4 + K_3u + K_2u^2 - (1/6)K_1u^3 + (1/32)K_0u^4 = (a + bu + cu^2)(d + eu + u^2)$$

In this case the expression for the integral in (20), using the computer package Mathematica [20], is obtainable but is not included in this work as it is very lengthy. It may be expressed in terms of elementary functions and elliptic integrals.

5.9 Case IX: One order-one cubic factor

If $K_4 + K_3u + K_2u^2 - (1/6)K_1u^3 + (1/32)K_0u^4$ has one irreducible cubic factor, then we have

$$K_4 + K_3u + K_2u^2 - (1/6)K_1u^3 + (1/32)K_0u^4 = (a + bu + cu^2 + du^3)(e + u)$$

The integral in (20) can again be found with the help of the computer package Mathematica [20]. It is given in terms of elementary functions, elliptic integrals.
and special functions. However it is so lengthy that it is also not included in this work.

6 Discussion

In this paper we have modelled the behaviour of shear-free charged fluids, and reduced the solution of the Einstein-Maxwell system of field equations to a single nonlinear partial differential equation. By treating this equation as an ordinary differential equation, a first integral was found using elementary methods. It is remarkable to note that the first integral is obtainable without specifying the arbitrary functions contained in the governing equation. The first integral is subject to a system of two integral equations which were replaced by a system of two differential equations which can be integrated up to a quadrature. Consequently we have found a new class of parametric solutions to the Einstein-Maxwell system for a charged gravitating shear-free fluid. The new solution is given by the parametric equations (19)-(20).

A detailed analysis of the factors of the quartic arising in the quadrature was performed. Two cases of interest arise. Firstly, we are in a position to explicitly invert the quadrature when there is one repeated linear factor and explicitly write the first integral. Then the model has to be necessarily charged. We believe that this is a new result. Secondly, we can explicitly invert the quadrature when there is one order-three linear factor. This case contains that of vanishing charge and we regain the results of Maharaj et al [19], Srivastava [11] and Stephani [21]. In the remaining cases the functions $G(u)$ is a complicated combination of elementary functions or/and special functions. In these remaining cases it is not possible, except maybe for special parameter values, to invert the integral and write expressions for $f(x)$ and $g(x)$ explicitly.

We make certain points, related to the mathematics, to clarify the approach followed in this paper. Firstly, the Einstein-Maxwell system has been studied extensively in the past with the objective of finding exact solutions. We have considered earlier treatments and, in particular the comprehensive analyses of Krasinski [13] and Stephani [21], and have not found any reference to the first integrals established in this paper. Secondly, we have generated the first integral (10) mathematically following the approach of Maharaj et al [19] for uncharged fluids. We believe that this is an elegant approach and may be used in other investigations for the gravitational field or other physical systems. Our new class of solutions may be useful in this context, and could provide a deeper insight into the behaviour of the gravitational field. A comprehensive mathematical analysis of the integrability properties of (5) using the symmetry properties of the equation may provide further solutions and insights. For example the treatment of Halburd [24], for the uncharged shear-free case,
established an equivalence with the generalised Chazy equation and provided a new class of integrable equations.

We now make certain comments, related to the physics, related to our results. Firstly, a natural question is whether the models generated here are physically meaningful. It is difficult to perform a general qualitative study of the physical features of the models because of the complexity of the functions involved. However we have found explicit charged first integrals and earlier uncharged first integrals are regained. Other parameter values may also lead to acceptable models. These factors point to physical reasonability. Secondly, the line element (1) is written in terms of isotropic and comoving coordinates; the spacetime is shear-free. The introduction of charge, and assumptions made in the integration, do not affect (1) so that the spacetime remains shear-free. Thirdly, charged shear-free models in the presence of heat flow are of crucial importance in relativistic astrophysics [7,8] and influence the range of temperature profiles of the models discussed here. In [25] it was shown that, for the most general spherically symmetric line element with acceleration, the causal transport equation reduces to

\[ \beta(qB)_t T^{-\sigma} + A(qB) = -\alpha \frac{T^{3-\sigma}(AT)_r}{B} \]  

(32)

where

\[ \kappa = \gamma T^3 \tau_c, \tau = \left( \frac{\beta \gamma}{\alpha} \right) \tau_c = \beta T^{-\sigma} \]  

(33)

with \( \gamma \geq 0, \alpha \geq 0, \beta \geq 0 \) and \( \sigma \geq 0 \). Then it is possible to integrate this equation for general metric functions \( A \) and \( B \). When \( \beta = 0 \), all noncausal solutions of (32) are given by

\[ (AT)^{4-\sigma} = \frac{\sigma - 4}{\alpha} \int A^{4-\sigma} qB^2 \, dr + F(t) \quad \sigma \neq 4 \]  

(34)

\[ \ln(AT) = -\frac{1}{\alpha} \int qB^2 \, dr + F(t) \quad \sigma = 4 \]  

(35)

where \( F(t) \) is an arbitrary function of integration which is fixed by the expression for the temperature of the star at its surface \( \Sigma \). For causal solutions (\( \beta \neq 0 \)) two solutions are provided. In the case of constant mean collision time, i.e. \( \sigma = 0 \), (32) is simply integrated to yield

\[ (AT)^4 = -\frac{4}{\alpha} \left[ \beta \int A^3 B(qB)_t \, dr + \int A^4 qB^2 \, dr \right] + F(t) \]  

(36)

The only nonconstant mean collision time solution is given for \( \sigma = 4 \):
Our results can be incorporated into the above framework with $A = e^\nu$ and $B = e^\lambda$ in the astrophysical context.

The main objective of this paper was to show that an earlier method to obtain uncharged first integrals is extendible to the Einstein-Maxwell system of equations. We have shown that this is possible and particular charged first integrals have been explicitly obtained. It would be beneficial to study the models generated in terms of the original metric, at least in particular cases for simple forms of $f(x)$ and $g(x)$, and to evaluate the evolution of the charged relativistic fluid in terms of the fluid and electromagnetic variables. Other quantities requiring attention are the spacetime structure, energy conditions, Riemann invariants and causality. This is outside the scope of our present treatment and will be considered in future work.

Acknowledgements
MCK and KSG thank the National Research Foundation and the University of KwaZulu-Natal for financial support. SDM acknowledges that this work is based on research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation.

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