Can a periodically driven particle resist laser cooling and noise?

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Studying a particle trapped by a time-dependent periodic anharmonic potential, we identify large amplitude trajectories stable for millions of oscillation periods in the presence of stochastic laser cooling. The competition between energy gain from the potential’s time-dependence and the damping leads to the stabilization of such stochastic limit cycles, in the regions of phase-space where the frequency of the particle oscillation is close to a multiple of the periodic drive. As a result, the steady-state phase-space distribution develops multiple peaks instead of converging to the region where the trap forces vanish. Our results suggest that such distinct nonequilibrium behaviour can be observed in realistic radio-frequency traps with laser-cooled ions, suggesting that Paul traps offer a well-controlled test-bed for studying transport and dynamics of microscopically driven systems.

An atomic ion trapped in near-vacuum is a highly isolated system whose quantum motion can be controlled exquisitely \cite{レファレンス}. Notwithstanding this, it is a system where chaos and randomness at the microscopic level give rise to intriguing classical states of motion.

Paul traps are based on radio-frequency (10–200 MHz) time-periodic potentials \cite{レファレンス}. The time-dependent drive affects the dynamics qualitatively and the trapping is based on dynamical stabilization, akin to a stabilization of an inverted pendulum \cite{レファレンス}. Even for motion of one particle in one spatial dimension, the time-dependent drive renders the Hamiltonian phase-space effectively three-dimensional (3D), counting the time (which can be treated as periodic), the space coordinate and the momentum. For a general anharmonic potential, this results in a complex phase-space structure (see below). With dynamics that can be tuned through different regimes of integrability and chaos, Paul traps provide a good setup for studying microscopic Hamiltonian transport \cite{レファレンス}.

Laser cooling is widely used in ion trapping \cite{レファレンス}. With the cooling beam turned on, the ion may be expected to be damped to the minimum of the effective potential or, in some cases \cite{レファレンス}, heat up or diffuse to a larger amplitude where it may escape from the trap. However, in nonequilibrium scenarios, the peaks of the spatial probability distribution may not necessarily coincide with the minimum of the potential. Rather, the distribution may develop new maxima and, even more strikingly, a particle may be captured in complex stochastic limit cycles or show hysteretic behaviour \cite{レファレンス}.

In this Letter, we show that the anharmonicity and periodic drive in Paul traps can capture an ion at a large amplitude motion, corresponding to a stable limit cycle even in the presence of damping by laser cooling and the associated randomness. The time-dependence of the potential is a critical ingredient for such stochastic limit cycles since, in contrast to a time-independent setup, it prevents the damping from erasing the signatures of the Hamiltonian phase-space in the long-time limit. Instead, the stochastic dynamics reflects the complex structure of the underlying Hamiltonian phase-space, with multiple peaks of the probability distribution of the ion emerging away from the effective potential minimum. Hence, although the realization that we focus on is that of a trapped ion, the basic required ingredients (time-dependence, anharmonicity, and weak damping), are relevant to many dynamical systems, as we briefly discuss in the conclusions.

Model. We consider dynamics in a time-periodic potential in the presence of weak damping and first examine the stability of limit cycles in a model that can be expanded analytically in one spatial dimension. In the second step we use this expansion to demonstrate that this mechanism remains robust under the complex stochastic process of laser cooling in a realistic trap potential.

We first consider a time-dependent anharmonic potential,

\begin{equation}
V(z, t) = \frac{1}{2}a_z(z - z_s)^2 + qV_2(z)\cos 2t,
\end{equation}

where the time \(t\) is in units of \(2/\Omega\) with \(\Omega\) being the angular frequency of the trap rf-drive and the coordinate is in units of a length-scale \(d\), defined below. The charge and mass of the ion and the trap geometry parameters and voltages are absorbed into the nondimensional parameters \(a_z\) and \(q\). Setting \(z_s\) to be a point where \(V_2(z)\) vanishes and choosing the parameters appropriately, makes \(z_s\) a stationary point with stable motion in the phase-space around it. Since the periodic drive’s frequency has been rescaled to 2, the trap potential is \(\pi\)-periodic.

In numerical simulations we take \(V_2(z)\) to be the potential of a model five-wire surface trap \cite{レファレンス} along an axis perpendicular to the electrodes (with \(d\) the width of each of the electrodes carrying the rf-modulated voltage), and \(V_2 = \frac{d}{\pi} [\arctan \left(\frac{z}{d}\right) - \arctan \left(\frac{-z}{d}\right)]\) which van-
of chaotic motion, in regions where the anharmonic oscillation frequency of the ion is at a rational ratio with the drive frequency. Figure 1(b) shows a close-up on the trajectories around one island in a chain which has \( s = 11 \) islands.

Limit cycles. A trajectory starting at the center of one island of an island chain constitutes a periodic orbit repeating itself after \( t = s \tau \), with the ion moving between the \( s \) island centres in a fixed order. When adding dissipation, the ion is attracted from (almost) the entire island area towards the periodic orbit, a phenomenon previously studied mostly in terms of chaotic maps\([22,24]\). To model the mechanism of trapping in the island chain we start by adding a friction coefficient \( \gamma > 0 \) to the ion’s equation of motion:

\[
\ddot{z} = F(z,t) - \gamma \dot{z}, \quad F(z,t) = -\partial V(z,t)/\partial z. \tag{2}
\]

A numerical simulation of the time evolution using two (different) initial conditions is shown in Fig. 1(c)-(d). In (c), the ion is damped toward \( z_s \), while in (d) it gets trapped in the chain of \( s = 11 \) islands.

The limit cycle is a generic feature that results from the interplay of the nonlinearity of the Hamiltonian forces acting on the ion in the vicinity of island chains, where the time-dependent drive counteracts the damping. Assume that \( \bar{z}(t) \) is an \( s \pi \)-periodic and time-reversal invariant orbit that connects the island centres for \( \gamma = 0 \), i.e. \( \bar{z}(t) = F(\bar{z}(t),t) \). We write

\[
z(t) = \bar{z}(t) + u(t), \quad \bar{z} = \sum_n B_{2n}e^{i2nt/s}, \tag{3}
\]

where \( n \in \mathbb{Z} \) and \( B_{2n} = B_{-2n} \), and substitute Eq. (3) into Eq. (2). After linearizing the motion around \( \bar{z}(t) \), we get

\[
\ddot{u} + f(t)u = -\gamma (\dot{\bar{z}}(t) + \dot{u}), \quad f(t) = -\frac{\partial F}{\partial z} (\bar{z}(t),t). \tag{4}
\]

Substituting \( u(t) \) by the ansatz \( u(t) = w(t)e^{-\gamma t/2} \) gives

\[
\ddot{w} + \left[ f(t) - \frac{\gamma^2}{4} \right] w = g(t), \quad g(t) \equiv -\gamma \dot{\bar{z}}(t)e^{\gamma t/2}. \tag{5}
\]

The general solution for \( w \) is composed of the sum of a particular solution growing exponentially as \( e^{\gamma t/2} \), and the two linearly independent solutions of the homogeneous equation [with \( g(t) = 0 \), see below]. Since \( \bar{z} \) is \( s \pi \)-periodic and \( F \) is \( \pi \)-periodic, the function \( f(t) \) in Eq. (4) can be Fourier expanded in the form \( f = \sum F_{2n}e^{i2nt/s} \).

A particular solution of Eq. (4) can be obtained by substituting \( \bar{z}_0(t) = e^{\gamma t/2} \sum W_{2n}e^{i2nt/s} \), which gives an inhomogeneous system of recursion relations for \( W_{2n} \), with a unique solution under general conditions\([24]\). The homogeneous equation in \( w \) [Eq. (5)] with \( g(t) = 0 \), whose
coefficients are periodic in time, is a Hill equation [23].

The homogeneous solutions $u(t)$ determine the stability of $u(t)$ [since the exponential growth of the inhomogeneous $w_0$ is cancelled when going back to $u$]. In fact, Eq. (5) with $\gamma = g(t) = 0$ determines the linear stability of the periodic orbit in the Hamiltonian case. If the area of the islands about the periodic orbit is not too small, perturbations about the periodic orbit are stable for a range of amplitudes of the motion, which implies that the motion would be linearly stable also for a small non-vanishing value of $\gamma$ in Eq. (5). The friction coefficient taken here is $\gamma = 1 \times 10^{-5}$ which guarantees that the damping is a slow process compared to the ion dynamics [26]. We note in particular that to leading order the effect of the damping is eliminated from the dynamics, which explains the numerical observation that although the ion is gradually damped towards the island chain centres [Fig. 1(d)], it is being damped much more slowly than outside the island chain [Fig. 1(c)].

**Laser-cooling.** To treat laser cooling more realistically, we assume a beam that is uniform over all positions in $z$ and apply a recently developed semiclassical theory of laser-cooling that is valid for motion within the time-dependent potential of Paul traps [18]. We approximate the ion as a two-level system, whose excited level has a dependent potential of Paul traps [18]. We approximate the ion as a two-level system, whose excited level has a dependent potential of Paul traps [18].

We find that relatively large values of the laser detuning $\Delta$ are required to stabilize the periodic orbit. For detunings within the range $-24 \Gamma \lesssim \Delta \lesssim -27 \Gamma$, we find a stationary distribution in the action within the island chain with a strong (exponential) suppression of the probability of $I$ values away from the island centres. The lifetime within the island can be estimated (by calculating the mean first passage time at the island boundary [20]) to exceed tens of seconds around $\Delta = -26.8 \Gamma$. In addition we find other ranges of the detuning with lifetimes smaller by an order of magnitude (for example around $\Delta = -20 \Gamma$). Figure 2(a) shows the action drift coefficient for a few representative values of the detuning, and Fig. 2(b) shows the corresponding steady-state distributions. For $\Delta = -27.5 \Gamma$ we see that the ion is canonical time-dependent transformation to the Hamiltonian action-angle coordinates $(I, \theta)$ describing the linearized motion about the periodic orbit. The value of $I$ is 0 for the periodic orbit $\bar{z}(t)$ since the area bounded by the corresponding phase-space torus is 0. By the linearity of the expanded motion, the transformation can be obtained in analytic closed form, with the coefficients of the Fourier expansion calculated numerically [24 27 28].

Averaging over the angle $\theta$, we obtain an effective Fokker-Planck equation for the probability distribution $P(I, t)$, which is a probability density function that depends on time and action only [18],

$$\partial_t P(I, t) = -\partial_I S(I, t) \equiv -\partial_I \left[ \Pi_I P \right] + \frac{1}{2} \partial_{II} \left[ \Pi_{II} P \right],$$

with $S(I, t)$ a probability flux, $\Pi_I(I)$ an action drift coefficient and $\Pi_{II}(I)$ a diffusion coefficient. If we find a region of action where the ion remains bounded for a very long time (as determined by the Fokker-Planck dynamics), we can assume an approximately stationary probability distribution in the action of the form

$$P(I) \propto \left[ \Pi_{II}(I) \right]^{-1} \exp \left\{ 2 \int dI' \Pi_I(I')/\Pi_{II}(I') \right\}.$$
The stabilization of a particle in a large amplitude periodic motion in a certain phase-space region, due to a combination of time-dependence, damping and noise is reminiscent of a nonequilibrium phase transition \cite{38}, and relates to the dynamics of active inertial systems \cite{39,40} and Hamiltonian and Brownian ratchets \cite{41}, which are a basic model of transport \cite{42,43}. Transport in a mixed phase-space is especially rich \cite{44,50} and the ability to control and accurately measure motion in complex time-dependent potentials make ion-trap experiments suitable for quantitative tests of such ideas \cite{51,52}. The possibility of exploring quantum effects in phase-space \cite{49,53,54} with a single trapped ion could constitute a promising future direction.

We have also simulated the Hamiltonian motion within the full time-dependent potential of the five-wire trap in three spatial dimensions for the parameters analyzed above, verifying that this mechanism is robust up to large amplitudes of motion in the transverse coordinates. Island chains frequently develop in surface traps with voltages in the upper range of experimentally relevant values, and they can have very large relative sizes \cite{6}. This is true also for other trap types, since the potential often attains some anharmonic contributions which become relevant above some energy or spatial scale. Hence we expect that such limit cycles can be observed in many existing traps. The resulting motion bares similarities to frequency-locked trajectories predicted in the relative coordinate of two interacting particles in a cylindrical Paul trap, and observed with microspheres \cite{31,32}.

To conclude, we have analyzed the dynamics of a particle driven by periodic trap potential and the laser. The balance between the time-dependence of the trap and the damping by laser-cooling results in the emergence of new peaks of the probability distribution within the islands of the mixed phase-space. Beyond a deeper understanding of the motion of laser-cooled ions in a Paul trap, our results are also interesting purely from the point of view of study of driven stochastic systems. By employing a time-dependent transformation to the action-angle coordinates, the truly stochastic, noncoherent part of the motion can be separated from the Hamiltonian motion through the center of the islands. As can be derived from Eq. \ref{eq:7}, the distribution of Eq. \ref{eq:7} (effectively stationary for a long time on the scale of the dynamics) is a zero-current steady state \cite{15,29}. This provides an example of a system driven far-out-of-equilibrium which is nevertheless described by effective dynamics that obey detailed-balance (since the probability flux is zero). Therefore, beyond the specific physical system we study, our calculation suggests a new method for obtaining the steady-state distributions for nonequilibrium systems through a mapping of the angle-averaged dynamics to that of an equilibrium system \cite{33,37}.

Heated away from the center of the island and the distribution develops a peak at an intermediate action region. For $\Delta = -35\Gamma$ the drift coefficient oscillates and multiple peaks emerge within each island. In this case the exponential suppression of $I$ away from the island centers is not large enough, implying that the lifetime of the limit cycle would be possibly too short to be observed. However, larger islands or a larger ratio of the laser wavelength to the natural lengthscale $d$, would make them long-lived. Figure \ref{fig:2} also shows that the action drift coefficient is negative throughout most of the island chain, indicating that the ion will drift towards the maximum of the quasi-stationary action distribution from any point within. Due to the stochastic nature of laser cooling, an ion starting from outside the island chain has a finite probability to diffuse or drift onto one of the islands, however elucidating this mechanism is beyond the scope of the current work. For the photon scattering rate with the most stable limit cycle we obtain $1.5 \times 10^{6} \text{s}^{-1} \times s_{L}$, with $s_{L} \ll 1$ the saturation parameter. This rate can be detected and distinguished from the much higher rate at the potential minimum.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{(a) The action drift coefficient (in nondimensional units, see the text for the parameters) as a function of the action $I$ of linearized motion expanded about the periodic orbit going through the island centres of the chain with $s = 11$ islands in Fig. \ref{fig:1}(b). A few values of the detuning are shown [see legend in panel (b)], with the $I$ axis extending roughly up to the size of (each) island. At action values where the drift coefficient is positive the ion is effectively heated by scattering laser photons, while where it is negative the ion is being cooled. (b) The resulting approximately stationary probability distribution [Eq. \ref{eq:7}]. We find a few ranges of the detuning for which the ion is well cooled within the island chain, with an exponential decay away from the center guaranteeing a lifetime distribution $[8]$. We find a few ranges of the detuning where the drift coefficient is positive the ion is effectively heated by scattering laser photons, while where it is negative the ion is being cooled. (b) The resulting approximately stationary probability distribution [Eq. \ref{eq:7}]. We find a few ranges of the detuning for which the ion is well cooled within the island chain, with an exponential decay away from the center guaranteeing a very long lifetime estimated to be $(\Delta = -26.8\Gamma)$ of order tens of seconds. For higher detunings a local peak emerges and even multiple peaks can be found within each island, although for these parameters they approach the island edges, implying that the ion will escape the island from these regions.}
\end{figure}
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