On variational formulation of current drive problem in uniformly magnetized relativistic plasma

Y.M. Hu and Y.J. Hu

Institute of Plasma Physics, Chinese Academy of Sciences, Hefei, Anhui 230029, People’s Republic of China
Center for Magnetic Fusion Theory, Chinese Academy of Sciences, Hefei, Anhui 230029, People’s Republic of China

E-mail: yeminhu@ipp.ac.cn

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Abstract
A fully relativistic extension of the variational principle with the modified test function for the Spitzer function with momentum conservation in the electron–electron collision is investigated in uniformly magnetized plasma. The term of the momentum conserving constraint in Hirshman’s variational calculation is studied. The model developed is extended for arbitrary temperatures and covers exactly the asymptotic for $u \gg 1$ when $Z_{\text{eff}} \gg 1$, and the results obtained are suited to facilitate the development of a rigorous variational formulation of current drive efficiency in tokamak plasma.

Keywords: variational principle, current drive efficiency, relativistic plasma

(Some figures may appear in colour only in the online journal)

1. Introduction

Current drive, either inductive or non-inductive, is a key component of tokamak operation. Accurate and computationally efficient predictions of current drive efficiency, especially for future tokamak fusion devices, are highly desirable. The best method for calculating current drive efficiency known up to date is to use the Green’s-function techniques, which were pioneered by Hirshman [1] in his classical collision theory of beam-driven plasma currents, and simultaneously introduced for electron current drive techniques by Fisch and Boozer, [2] and also shown to be equivalent to adjoint techniques using the Spitzer function by Antonsen and Chu [3]. In the Green’s-function formulation the current drive efficiency is expressed in terms of an integral in velocity space of the product of the current drive response function (the Green’s function) and a source function, which represents physical effects of a current drive scheme. At the same time, the response function is nothing but the Spitzer function, the electron response to a parallel electric field applied to the plasma.

As for ITER plasmas and those of future fusion devices, relativistic effects cannot be neglected in the current drive calculations in an electron-based current drive scheme, due to high electron temperature. In particular, from theoretical studies on electron cyclotron current drive for ITER plasmas in recent years [4, 5] it becomes clear that the full Linearized Coulomb collision integral including the field-particle contributions, which preserves momentum conservation in the e–e collisions, will be needed to give accurate predictions of the current drive efficiency in the high temperature plasmas. In this case, the exact calculation of the current drive response function is very involved and computationally expansive. A variational formulation of the problem should be useful.

In this work, we extend Hirshman’s original calculation of the current drive response function in uniformly magnetized plasma to the relativistic case. His calculation is variational and based on a generalized variational principle derived from the classical transport theory [6]. We will use
the fully relativistic Coulomb collision operators, developed by Beliaev and Budker [7] as well as Braams and Karney, [8] to describe the electron–electron and electron–ion collisions in relativistic plasmas. The work will be focused on the issue of construction of appropriate variational basis functions. The response functions obtained variationally are to be compared with the numerical solutions to the exact differential-integral equation (the relativistic Spitzer–Harm equation), which have been previously obtained by Braams and Karney. Moreover, we will also explore the difference between the response functions obtained from Hirshman’s variational principle and those from the standard variational principle of the classical transport theory for a wide range of electron temperature, $T_e$, and effective ion charge, $Z_{\text{eff}}$. Thus, the results obtained here could facilitate the development of a rigorous variational formulation of current drive efficiency in toroidal geometry.

The rest of this paper is organized as follows. First, in section 2, the green’s function formulation and Hirshman’s variational principle for the current drive response function are investigated and also the momentum conservation constraint in approximate variational solution proposed by Hirshman is discussed. Second, in section 3, we extend Hirshman’s variational calculation of the current drive response function to the relativistic case and make a choice of trial function to carry out the variational calculation of the current drive response function, and also the lorentz limit and the finite effective ion charge case are investigated. In section 4, we study the variational calculation without the momentum conserving constraint and show how the corresponding solution would behave in comparison with the one with the constraint.

2. The green’s function formulation and hirshman’s variational principle for the current drive response function

In a uniformly magnetized plasma, the plasma response to the external perturbation due to an electron-based current drive scheme is usually described by the linearized Fokker–Planck equation:

$$C_e(f_{ei}) \equiv C_{ee}(f_{ei}f_{em}) + C_{ee}(f_{em}f_{ei}) + C_{ei}(f_{ei}) = \frac{\partial f_{ei}}{\partial t}_{\text{CD}}.$$  

(1)

Here, $f_{ei} = f_{em} + f_{ei} + \cdots$, $f_{em}$ is the electron Maxwellian distribution and $f_{ei}$ is the first order perturbed distribution function; $C_{ee}$ and $C_{ei}$ are, respectively, the Coulomb collision operators for electron–electron and electron–ion collisions; $(\frac{\partial f_{ei}}{\partial t})_{\text{CD}}$ is the source function representing the effects on electrons due to the current drive perturbation.

For the relativistic plasma, we choose $\bar{u} \equiv \beta \frac{m_e}{c} = \gamma \hat{v}$ with $\gamma = \sqrt{1 + u_{\parallel}^2}$, momentum per unit mass, as the independent variable in the phase space and the electron distribution is normalized so that $\int d^3\bar{u} f_{ei}(\bar{u}) = n_e$, where $n_e$ is the electron density. In this case, the electron Maxwellian distribution is given as

$$f_{em}(u)d^3u = \frac{n_e m_e}{4\pi c T_e^{\frac{3}{2}} K_0(1/\theta)} \exp\left[\left(-\frac{1}{2}\frac{m_e c^2}{T_e}\right)\right] d^3u,$$

(2)

where $\Theta \equiv T_e m_e^{-2} = T_e/511 \text{ keV}$ and $K_0(x)$ is the nth-order of the modified Bessel function of the second kind. We will be only interested in the strongly magnetized limit in which $f_{ei}$ can be assumed to have azimuthal symmetry about the external magnetic field, $\hat{B}$, i.e.

$$f_{ei}(\bar{u})d^3u = f_{ei}(u, \theta)(2\pi u_d^2 \sin \theta d\theta du),$$

(3)

where $u_\parallel = u \cos \theta = \hat{u} \cdot \hat{B}/B$ and $u_e = u \sin \theta$.

The collision term $C_{ei}(f_{ei}, f_{em})$ is for electrons colliding off the Maxwellian background. It can be expressed as the divergence of a momentum-space flux which consists of a diffusion term and a friction term involving $f_{ei}$. The collision term $C_{ei}(f_{ei}, f_{em})$ is an integral operator acting on $f_{ei}$. It is sometimes referred to as the field-particle contributions to the collision integral. In dealing with electron–electron collision processes, the presence of both collision terms is necessary to ensure momentum conservation. A general method for calculating these collision terms for relativistic plasma has been developed by Braams and Karney. The explicit analytic expression for $C_{ei}(f_{ei}, f_{em})$ and that for $C_{ei}(f_{em}, f_{ei})$ in the case of $f_{ei}(u, \theta) = f_{ei}^{(1)}(u) \cos \theta$ have been given in [8]. A brief summary of their results needed in the present calculations is presented in the appendix. For electron–ion collisions, we take the limit of infinite massive ion, $m_i m_e \to 0$, and neglect small parallel streaming velocity of ions. Thus, the electron–ion collision term for the relativistic plasma is given as

$$C_{ei}(f_{ei}) = n_e Z_{\text{eff}}^2 \left( \frac{u_e}{u} \right)^2 \frac{\partial}{\partial \theta} \frac{\sin \theta}{\partial \theta} f_{ei} = n_e Z_{\text{eff}}^2 \frac{\partial}{\partial \theta} f_{ei},$$

(4)

where $\mathcal{L} = \frac{1}{2 \sin \theta} \frac{\partial}{\partial \theta} \frac{\sin \theta}{\partial \theta} \frac{\partial}{\partial \theta}$ is the pitch-angle scattering operator, $x \equiv u/u_e$ with $u_e = \sqrt{2 T_e/m_e}$, $Z_{\text{eff}}$ is the effective ion charge, and $\nu_{ei} = \frac{n_e e^2 \lambda_{\text{De}}}{4 \pi \sin \theta u_e}$ is the electron–electron collision frequency. Note that in the non-relativistic limit, $\gamma \to 1$ and $u \to v$, we recover the well-known result that the electron–ion pitch angle scattering rate is inversely proportional to $v^3$.

The straightforward approach for determining the driven current in a current drive problem is to find the appropriate $(\partial f_{ei}/\partial t)_{\text{CD}}$ for the problem, solve equation (1) to determine the electron distribution function $f_{ei}$, and then find the current by evaluating the integral

$$j_\parallel = -e \int d^3u f_{ei}(u) \frac{\gamma}{\gamma},$$

(5)

For the relativistic Spitzer–Harm problem,

$$\left( \frac{\partial f_{ei}}{\partial t} \right)_{\text{CD}} = -\frac{e E_\parallel}{m_e} \frac{\partial f_{ei}}{\partial \theta} = \frac{e E_\parallel \nu_{ei}}{T_e} f_{em}.$$  

(6)

where $E_\parallel$ is the parallel inductive electric field.
The Green’s-function formulation gives a way of determining the driven current without having to find \( f_0 \) first. The key concept is to introduce the current drive response function which satisfies the ‘adjoint’ equation:

\[
C_\ell'(g) = \nu_0 \left( \frac{v_h}{u_e} \right) f_{em} \equiv \nu_0 \left( \frac{\lambda \cos \theta}{\gamma} \right) f_{em}. 
\]  

(7)

By using the self-adjointness of the linearized collision operator,

\[
\int d^3 u f_{em}^{-1} C_\ell'(f) = \int d^3 u f_{em}^{-1} C_\ell(g),
\]

(8)

where \( f_{em}^{-1} \) serves as the weight for the inner product of two distribution functions in momentum space, together with equations (5) and (7), it is straightforward to show

\[
j_h = -\frac{\nu_h}{\nu_0} \int d^3 u f_{em}^{-1} \left( \frac{\partial f_\ell}{\partial t} \right)_{CD}.
\]

(9)

Note that the response \( g \) so defined has the dimensions of the electron distribution function \( f_0 \) and they share the same boundary conditions in momentum space. Moreover, by comparing the source term in equation (7) with equation (6), it is easily recognized that \( g \) is the solution to the Spitzer-Härn equation with an equivalent electric field of \( E_{eq}^* = \frac{\nu_0 f_{em}}{e^*} \).

Hereafter equation (7) will be referred to as the normalized Spitzer-Härn equation. Once it is solved, any other current drive problem is reduced to performing the integral given in equation (8). This is the main advantage of the Green’s-function formulation.

We now turn to Hirshman’s variational principle for the response function \( g \). To solve equation (7) variationally, consider the variational functional

\[
S_3 = \int d^3 u f_{em}^{-1} g C_\ell(g) - 2 \nu_0 \int d^3 u \left( \frac{v_h}{u_e} \right) g f_{em},
\]

(10)

which is just the entropy production rate in the classical transport theory for the equivalent Spitzer-Härn problem. By considering variation of \( S_3 \) with respect to \( g \) and using equation (8), it is easy to see that \( \delta S_3 = 0 \) yields equation (7). In practical applications of the variational principle, one often does not obtain the exact solution to equation (7). To guarantee momentum conservation in approximate variational solution, Hirshman proposed an additional constraint be imposed on trial functions: \([1]\)

\[
\int d^3 u u_h \left[ C_\ell'(g) - \nu_0 \left( \frac{v_h}{u_e} \right) f_{em} \right] = 0,
\]

(11)

which is the first momentum moment of the normalized Spitzer-Härn equation. The generalized variational principle can be stated as follows: \( \delta S_3 = 0 \) with

\[
S_\ell = S_3 - 2 \frac{\lambda}{u_e} \int d^3 u u_h \left[ C_\ell'(g) - \nu_0 \left( \frac{v_h}{u_e} \right) f_{em} \right],
\]

(12)

where \( \lambda \) is the Lagrangian multiplier to be determined by the constraint, equation (11). We note in passing that the equation (10) can be simplified to

\[
\int d^3 u \left[ \nu \left( \frac{v_h}{u_e} \right) f_{em} \right] = \nu_0 \int d^3 u \left[ \frac{\lambda \cos \theta}{\gamma} f_{em} \right]
\]

(13)

by noticing the momentum conserving property of the linearized e-e collision operator, \( C_{ee}(ufem, fem) + C_{ee}(fem, uefem) = 0 \). For a given, \( \delta S_\ell = 0 \) leads to

\[
C_\ell'(g) = \nu_0 \frac{v_h}{u_e} f_{em} + \lambda Z_{eff} \left( \frac{\nu_h \gamma}{\lambda u_e} \right) f_{em}.
\]

(14)

Comparison of this equation with equation (7) shows that \( \lambda \) must vanish in the case of the exact solution. Therefore, the smallness of \( \lambda \) will be a necessary condition for a good approximate variational calculation.

3. Choice of trial function and variational calculation of the current drive response function

In this section, we extend Hirshman’s variational calculation of the current drive response function to the relativistic case. To construct the variational trial function, consider first the Lorentz-gas limit in which \( Z_{eff} \gg 1 \) and e-e collisions are ignorable. This model has the exact solution

\[
g_{LG} = -\frac{1}{Z_{eff}} \frac{1}{\gamma^2} \int d^3 u \frac{v_h}{u_e} f_{em} \cos \theta = -\frac{1}{Z_{eff}} \frac{x^4}{\gamma^2} f_{em} \cos \theta, 
\]

(15)

where \( x = u/u_e \). It suggests that an appropriate trial function for the variational calculation could have the form of \( g \propto x(d_\alpha x + d_\alpha^2 x + \cdots) / \gamma^2 \). In the following, we choose to work on the two sets of trial functions:

\[
g \equiv -f_{em} \lambda \cos \theta = -f_{em} \left( \frac{x}{\gamma} \right) (d_\alpha x + d_\alpha^2 x + d_\alpha^3 x + d_\alpha^4 x) \cos \theta
\]

(16)

with \( \alpha = 1, 2 \). The two functions have different high energy asymptotic behaviors, but in the non-relativistic limit they approach to the same trial function used in [8]. The motivation to consider the case of \( \alpha = 1 \) is that there is no a priori reason that the choice of \( \alpha = 2 \) would be better than that of \( \alpha = 1 \) for finite \( Z_{eff} \). Also, the trial functions with the same high energy asymptotic behavior of \( \alpha = 1 \) have been used recently by Marushchenko et al. [4] in a study of electron cyclotron current drive for weakly relativistic tokamak plasma. It would be of some interest to compare the response functions derived from different values of \( \alpha \) for a wide range of \( T_e \) and \( Z_{eff} \) to ascertain the regime of their applicability.

Define the dimensionless matrix elements of the collision operator for the basis functions, \( \phi_i(x, \theta) \equiv \phi_i \cos \theta, \phi_i = x^{i+1} f_{\gamma^i} \):

\[
M_{ij} \equiv -\frac{2}{ne^\gamma \gamma} \int d^3 u \phi_i \phi_j \equiv M_{ij} \quad \text{for} \quad i, j = 1, 2, 3, 4.
\]

(17)

Also,

\[
M_{ij} \equiv -\frac{2}{ne^\gamma \gamma} \int d^3 u (x \cos \theta) \phi_i \phi_j \quad \text{for} \quad j = 1, 2, 3, 4
\]

(18)
where \( \tau_{ee}^{-1} = \frac{4\epsilon e^2}{3\gamma}\). Notice that there is no \( \gamma \) factor associated with the function \( x \cos \theta \) in the definition of \( M_{0j} \) which are to be used in the momentum conserving constraint. The matrix element \( M_{ij} \) consists of two parts:

\[
M_{ij} = \bar{m}_{ij}^e(\Theta) + Z_{\text{eff}}\bar{m}_{ij}^e(\Theta),
\]

(19)

where \( \bar{m}_{ij}^e \) and \( \bar{m}_{ij}^e \) are, respectively, contributions from electron–electron and electron–ion collisions and are functions of \( \Theta = T_e/m_e e^2 \); \( \bar{m}_{ij}^e(0) \) and \( \bar{m}_{ij}^e(0) \) have been exactly evaluated in [8]. For finite \( \Theta \), the fully relativistic e–e collision operator is quite complicated. We did not find useful analytic expressions for \( \bar{m}_{ij}^e \) as a function of \( \Theta \) and choose to calculate them directly using numerical integration. The e–i collision terms \( \bar{m}_{ij}^i \) are readily evaluated using the Lorentz collision operator. So are \( M_{0j} \) (see equation (18)). They can be expressed as

\[
\bar{m}_{ij}^i = \frac{\sqrt{\pi}}{2} \left( \frac{x^{i+j-1}}{\gamma^{2i-1}} \right),
\]

(20)

\[
M_{0j} = Z_{\text{eff}}\frac{\sqrt{\pi}}{2} \left( \frac{x^{i+j-1}}{\gamma^{a-1}} \right)
\]

(21)

with

\[
\left( \ldots \right) \equiv \int d^4u f_{\text{em}} \ldots \int d^4f_{\text{em}}.
\]

(22)

The computation of these matrix elements is discussed in the appendix.

To carry out the variational calculation using the trial functions specified in equation (16), we substitute it into equation (12) and obtain

\[
\delta_{\lambda} = n_{e e}^{-1} \left\{ -\frac{1}{2} \sum_{i,j=1}^4 M_{ij}d_id_j + \sum_{j=1}^4 s_jd_j - \lambda \left( \sum_{j=1}^4 M_{0j}d_j - \eta \right) \right\}
\]

(23)

with

\[
s_j = \frac{\sqrt{\pi}}{2} \left( \frac{x^{i+j}}{\gamma^{a+2}} \right)
\]

(24)

\[
\eta = \frac{\sqrt{\pi}}{2} \left( \frac{x^2}{\gamma} \right) = 3\sqrt{\pi} \frac{4}{4}
\]

(25)

The variation of \( S_{\lambda} \) in equation (20) with respect to \( d_i \) yields

\[
\sum_{j=1}^4 M_{0j}d_j + \lambda M_{0i} = s_i \quad \text{for } i = 1, 2, 3, 4.
\]

(26)

We might solve this system of \( 4 \times 4 \) linear equations for \( d_i \) by regarding the Lagrangian multiplier \( \lambda \) as a constant, and then determine \( \lambda \) using the momentum conserving constraint

\[
\sum_{j=1}^4 M_{0j}d_j = \eta
\]

(27)

or consider \( \lambda \) as the 5th unknown and solve equations (24) and (25) simultaneously for the five unknowns \( (d_1, d_2, d_3, d_4, \lambda) \). The system of \( 5 \times 5 \) linear equations can be written in a matrix form:

\[
\begin{pmatrix}
M_{ij} & M_{0j}^T & d_i \\
M_{0i} & 0 & \lambda
\end{pmatrix}
\]

(28)

where \( (M)_{ij} = M_{ij}(M_{0j}) \equiv (M_{0j})_i = M_{0j}(d)_j = d_i, \text{ and } (s)_i = s_i \).

The problem of determining the current drive response function is now reduced to solve this matrix equation, equation (28).

We proceed to determine variationally the response function using the formulation given in equations (16)–(28) for electron temperatures in the range of \( \Theta = 0.0–0.2 \) and at various values of \( Z_{\text{eff}} \). The variational determined response functions are to be compared with the numerical (or exact) solutions of the normalized Spitzer–Harm equation (equation (7)). For the purpose of presenting the computational results, it is convenient to define a dimensionless Maxwellian distribution:

\[
f_0(x) = \frac{\sqrt{2\Theta}}{K_0(\Theta^{-1})} \exp(-\gamma/\Theta)
\]

(29)

which is derived from \( n_{ef}(x)(d^i e^2/2\pi) = f_{\text{em}}(u)d^4u \). To give a quantitative measure of the difference between the approximate variational solution, \( g_\lambda \), and the numerical (or exact) solution, \( g_N \), we define

\[
\Delta \equiv \sqrt{\frac{\int d^4u f_{\text{em}}^{-1}(g_\lambda - g_N)^2}{\int d^4u f_{\text{em}}^{-1}g_N^2}}.
\]

(30)

This measures the average deviation of two functions in the functional space. Using equations (6), (8) and (16), the conductivity may be evaluated as

\[
\sigma = \frac{2n_e e^2}{3m_e e^2 \gamma} \left( \frac{x}{\gamma} \right).
\]

(31)

For the trial functions given in equation (16), the variationally determined conductivity is given as

\[
\sigma_\lambda = \frac{2n_e e^2}{3m_e e^2} \sum_{i=1}^4 d_i \left( \frac{x^{i+j}}{\gamma^{a+1}} \right).
\]

(32)

Since Braams and Karney have published an extensive table of conductivity for a relativistic plasma, we present our results for conductivities using their normalization:

\[
\sigma(Z_{\text{eff}}, \Theta) \equiv \sigma/\sigma_{\text{BK}}.
\]

(33)

where \( \sigma_{\text{BK}} \equiv \frac{1}{2\pi} \frac{n_e e^2}{Z_{\text{eff}} m_e e^2} \).

3.1. The lorentz limit

Consider the results for the Lorentz-gas model first. In this limit,

\[
M_{ij} = \bar{m}_{ij}^e + Z_{\text{eff}}\bar{m}_{ij}^e \approx Z_{\text{eff}}\bar{m}_{ij}^e.
\]
In the case of $\alpha = 2$, from equations (20), (21), (24) and (25), we have
\[ M_{ij} = Z_{\text{eff}} \frac{\sqrt{\pi}}{2} \left( \frac{x_i^{i+2}}{\gamma^3} \right) = Z_{\text{eff}} \delta_i \quad \text{for } i = 1, 2, 3, 4, \]  
(34)
and
\[ M_{03} = Z_{\text{eff}} \frac{\sqrt{\pi}}{2} \left( \frac{x^2}{\gamma} \right) = Z_{\text{eff}} \eta \]  
(35)
According to equation (28), these imply that $d_i = \delta_{ij} Z_{\text{eff}}$ and $\lambda = 0$, which recover the exact results of the Lorentz limit. For the case of $\alpha = 1$, we can only obtain approximate variational solutions. We have solved equation (28) numerically for various electron temperatures. At the highest temperature considered, $\Theta = 0.2$.
\[ Z_{\text{eff}}(x) = \gamma^{-1} x^2 ( -0.324 + 0.911 x + 0.321 x^2 - 0.0396 x^3 ) \]
\[ Z_{\text{eff}} = -0.00233, \]
(36)
The functions $Z_{\text{eff}}(x) f_i(x)$ and $Z_{\text{eff}}(x)$ are, respectively, compared with the exact results in figure 1. It is observed that except at the low energy limit ($x \lesssim 0.3$), where the function becomes negative, and in the high energy regime ($x > 4.5$), there is point-wise agreement shown in both figures. Also, the average deviation $\Delta_{ij}$ is computed to be less than 1.5%. At lower temperatures, we observe even better agreement between the two functions. The conductivities of the relativistic Lorentz gas and those determined from the variational calculation at various temperatures and the results are listed in table 1. Since conductivity is the variational quantity, excellent agreement between the variational and exact results is expected and is indeed observed. Also listed in table 1 are the average deviation $\bar{\lambda}$ and the normalized $\bar{\alpha} \equiv \lambda / \sqrt{\sum d_i^2}$ as function of $\Theta$. It shows that both $\bar{\lambda}$ and $\bar{\alpha}$ are much less than 1, as expected from a good variational calculation. It is interesting to note that $\lambda$ vanishes and change sign at $\Theta \approx 0.03$.
This indicates that $|\lambda| \ll 1$ can only be a necessary condition, but not sufficient condition for a good variational calculation. Nevertheless, $|\lambda|$ and $\Delta$ are somehow correlated as shown in figure 2. We conclude this part of discussions by noting that despite they don’t have the correct high-energy asymptotic behavior of $\chi(x)$, the trial function with $\alpha = 1$ still performs very well in the model calculations.

3.2. The general collision case
Now let’s split $\bar{m}_{ij}$ into two terms:
\[ \bar{m}_{ij} = \bar{m}_{ij}^{\text{eff}} + \bar{m}_{ij}^{\text{eff}} \]  
(37)
with
\[ \bar{m}_{ij}^{\text{eff}} \equiv \frac{2}{n_c e} \int d^4 u \phi_i(\theta) C_{\text{eff}}(f_{\text{rem}} f_{\text{em}}) \cos \theta, \]  
(38)
Making use of equations (A.2) and (A.6),
\[ m_{ij}^{\text{eff}} = \frac{8 \pi}{3 n_c \gamma} \int d\phi \partial \phi_i(\theta) \partial \phi_j(\theta) + 2 D_{uu} \frac{\partial \phi_i(\theta)}{\partial u} \frac{\partial \phi_j(\theta)}{\partial u} \]  
(40)
\[ = \frac{\sqrt{\pi}}{2} \int d\phi \partial \phi_i(\theta) \partial \phi_j(\theta) + 2 D_{uu} \frac{\partial \phi_i(\theta)}{\partial u} \frac{\partial \phi_j(\theta)}{\partial u} \]  
(41)
\[ = -\frac{\sqrt{\pi}}{2} \int d\phi \partial \phi_i(\theta) \partial \phi_j(\theta) + 2 D_{uu} \frac{\partial \phi_i(\theta)}{\partial u} \frac{\partial \phi_j(\theta)}{\partial u} \]  
(42)
\[ K_{ij} = c \int_0^u K(u, u') \frac{\partial^2 f_{\text{em}}(u')}{\partial u' \partial u} du' \]  
(44)
\[ K_{ij} = c \int_0^\infty K(u, u') \frac{\partial^2 f_{\text{em}}(u')}{\partial u' \partial u} du' \]  
(45)
Here $D_{uu} \equiv \frac{4 \pi D_{uu}}{n_c u}$, $D_{uu} \equiv \frac{4 \pi D_{uu}}{n_c u}$, $\phi' = \phi(x')$, $K_i = \frac{16 \pi^2 n_c^2}{u^2} K_{ii}$, $K_{ij} = \frac{16 \pi^2 n_c^2}{u^3} K_{ij}$, and in equation (40) we have made use of the facts:
\[ D_{uu} \frac{\partial \phi_i(\theta)}{\partial u} \frac{\partial \phi_j(\theta)}{\partial u} = -D_{uu} m_{ij} \frac{f_{\text{em}}(u)}{T_e} \frac{\partial \phi_i(\theta)}{\partial u} = F_u \frac{\partial \phi_i(\theta)}{\partial u}. \]
We have used Matlab to carry out the calculation of $\bar{m}_{ij}^{\text{eff}}$ and $\bar{m}_{ij}^{\text{eff}}$ using the above formulas. Tables of $\bar{m}_{ij}^{\text{eff}}$ and $\bar{m}_{ij}^{\text{eff}}$ for $\alpha = 1, 2$ were made and listed in table 1. From the above tables it can be found that the matrix elements with $\alpha = 2$ are in good agreement with those with $\alpha = 1$ when $\Theta < 0.02$, but when $\Theta > 0.02$ there will be a sensible difference between the two cases. Figure 3 show the functions $\chi(x)$ and $\chi(x) f_i(x)$ obtained by a variational calculation with $\alpha = 1, 2$ are, respectively, compared with the numerical results solved by Karney’s method [10] in the case of $Z_{\text{eff}} = 1$. It is clearly observed that $\alpha = 2$ is better than $\alpha = 1$ when $x \gg 1$, which is important for the driving noninductive current by LHW or ECW which rely on superthermal electrons (i.e. $x \gg 1$).

4. Variational calculation without the momentum conserving constraint
In all the calculations reported in section 3, we have found that $|\lambda|, |\bar{\alpha}| \ll 1$. Consequently, this leads us to consider the variational calculation without the momentum conserving constraint and study how the corresponding solution would behave in comparison with the one with the constraint. For
the conductivity, since it is the variational quantity, we would expect the calculation without the constraint to give equally good or better results than the one with the constraint if a good set of basis functions is used. The crucial issue for the variational solution without the constraint is whether it could have a good momentum conserving property. This will be the focus of the following discussions.

To perform the variational calculation without the constraint, we solve the reduced $4 \times 4$ matrix equation:

$$M_d s = (46)$$

and denote the solution of $d$ by $d_f$, that of $g$ by $g_f$, of $\chi$ by $\chi_f$, and the normalized conductivity by $\bar{\sigma}_f$ (see equation (31)). To examine the momentum conserving property of $g_f$, we define the quality factor $Q_f$ for momentum conservation:

$$Q_f = \frac{\int d^3 w_H C_{gs}(g_f)}{\int d^3 w_{eff} \bar{n}_f^4 (\gamma u_e)} - 1 = \frac{4}{3\sqrt{\pi}} \sum_{j=1}^{4} M_0(d_f)_j - 1. \quad (47)$$

It is the fractional difference between the momentum loss due to electron–ion collisions and the momentum input from the source (the equivalent ‘parallel electric field’ given in the normalized Spitzer–Harm equation). When the momentum is perfectly conserved, $Q_f = 0$.

Consider first the non-relativistic limit ($\gamma \rightarrow 1, u \rightarrow v, \Theta \rightarrow 0$). In this limit, the exact analytic expressions for all the matrix elements have been given in [8]. We make use of them to solve equation (46) and obtain

$$d_f = (3.125 41 - 1.617 17 Z_{eff} - 0.199 117 Z_{eff}^2) G_f(Z_{eff}), \quad (48a)$$

$$d_{f2} = (-3.240 52 + 5.546 32 Z_{eff} + 0.548 559 Z_{eff}^2) G_f(Z_{eff}), \quad (48b)$$

$$d_{f3} = (1.398 31 - 2.838 12 Z_{eff} + 0.071 1957 Z_{eff}^2 + 0.031 8837 Z_{eff}^4) G_f(Z_{eff}), \quad (48c)$$

$$d_{f4} = (-0.210 823 + 0.490 98 Z_{eff} - 0.006 901 51 Z_{eff}^2) G_f(Z_{eff}), \quad (48d)$$

where

$$G_f(Z_{eff}) = (0.007 380 12 + 0.700 442 Z_{eff} + Z_{eff}^2 + 0.341 69 Z_{eff}^3 + 0.031 8837 Z_{eff}^4). \quad (48e)$$

To facilitate the comparison, we reproduce the solution of the constrained calculation originally given in [8] (see equation (23) of [8]), and that for $\lambda$:
Table 2. $\bar{m}_{ij}^{\alpha T}$ with $\alpha = 2$.

| $\Theta \setminus ij$ | 11  | 12  | 13  | 14  | 22  | 23  | 24  | 33  | 34  | 44  |
|-----------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                     | 1.41421 | 2.33664 | 4.06586 | 7.47364 | 4.37522 | 8.42040 | 16.8490 | 17.6335 | 37.9633 | 87.1979 |
| 0.01                  | 1.31080 | 2.16143 | 3.75956 | 6.92134 | 4.06240 | 7.85548 | 15.8195 | 16.6015 | 36.1190 | 84.1565 |
| 0.02                  | 1.21980 | 2.00806 | 3.49281 | 6.44283 | 3.78948 | 7.36516 | 14.9332 | 15.7132 | 34.5600 | 81.7113 |
| 0.05                  | 1.00235 | 1.64439 | 2.86488 | 5.32415 | 3.14476 | 6.21485 | 12.8803 | 13.6567 | 31.0738 | 76.8477 |
| 0.1                   | 0.75823 | 1.24038 | 2.17348 | 4.10198 | 2.42961 | 4.94837 | 10.6645 | 11.4391 | 27.5864 | 73.5852 |
| 0.2                   | 0.48609 | 0.79445 | 1.41482 | 2.76494 | 1.63349 | 3.53920 | 8.25928 | 9.03361 | 24.3909 | 74.9196 |

Table 3. $\bar{m}_{ij}^{\alpha F}$ with $\alpha = 1$.

| $\Theta \setminus ij$ | 11  | 12  | 13  | 14  | 22  | 23  | 24  | 33  | 34  | 44  |
|-----------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                     | -1.22934 | -1.85234 | -3.00517 | -5.21189 | -2.96101 | -5.03527 | -9.07085 | -8.90116 | -16.5628 | -31.6900 |
| 0.01                  | -1.21097 | -1.84097 | -3.02133 | -5.31462 | -2.96876 | -5.10606 | -9.32707 | -9.12845 | -17.2215 | -33.4061 |
| 0.02                  | -1.19411 | -1.83189 | -3.04180 | -5.42799 | -2.98053 | -5.18558 | -9.60643 | -9.37658 | -17.9375 | -35.2791 |
| 0.05                  | -1.15086 | -1.81552 | -3.12452 | -5.82514 | -3.03559 | -5.46969 | -10.5750 | -10.2371 | -20.4258 | -41.8806 |
| 0.1                   | -1.09598 | -1.81310 | -3.31458 | -6.64780 | -3.17458 | -6.06581 | -12.5912 | -12.0223 | -25.7168 | -56.4590 |
| 0.2                   | -1.02130 | -1.85414 | -3.80496 | -8.75548 | -3.54989 | -7.58480 | -17.9847 | -16.7508 | -40.7855 | -101.588 |

Table 4. $\bar{m}_{ij}^{\alpha F}$ with $\alpha = 2$.

| $\Theta \setminus ij$ | 11  | 12  | 13  | 14  | 22  | 23  | 24  | 33  | 34  | 44  |
|-----------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                     | -1.22934 | -1.85234 | -3.00517 | -5.21189 | -2.96101 | -5.03527 | -9.07085 | -8.90116 | -16.5628 | -31.6900 |
| 0.01                  | -1.15416 | -1.74358 | -2.84398 | -4.97275 | -2.79538 | -4.78030 | -8.68260 | -8.49975 | -15.9487 | -30.7760 |
| 0.02                  | -1.08637 | -1.64610 | -2.70053 | -4.76258 | -2.64784 | -4.55507 | -8.34409 | -8.14920 | -15.4245 | -30.0279 |
| 0.05                  | -0.91757 | -1.40543 | -2.35020 | -4.26036 | -2.29705 | -4.01191 | -7.55379 | -7.32142 | -14.2441 | -28.5071 |
| 0.1                   | -0.71476 | -1.11898 | -1.93865 | -3.69089 | -1.86289 | -3.38639 | -6.69220 | -6.40489 | -13.0755 | -27.4518 |
| 0.2                   | -0.46798 | -0.77004 | -1.43764 | -3.02219 | -1.34865 | -2.64051 | -5.75060 | -5.37460 | -12.0737 | -27.8495 |

Figure 3. Comparisons of the results of the variational calculation of $\chi(x)$ (left) and $\chi(x)_{\alpha}(x)$ (right) for $\alpha = 1, 2$ with the corresponding exact results.

$$d_1 = (4.39723 - 2.31986 Z_{\text{eff}} - 0.283476 Z_{\text{eff}}^2)/G(Z_{\text{eff}}), \quad (49a)$$

$$d_2 = (-4.62688 + 8.05286 Z_{\text{eff}} + 0.792705Z_{\text{eff}}^2)/G(Z_{\text{eff}}), \quad (49b)$$

$$d_3 = (2.00597 - 4.13612 Z_{\text{eff}} + 0.108513Z_{\text{eff}}^2$$

$$+ 0.046714Z_{\text{eff}}^3)/G(Z_{\text{eff}}), \quad (49c)$$

$$d_4 = (-0.303672 + 0.716192Z_{\text{eff}} - 0.0110833Z_{\text{eff}}^2)/G(Z_{\text{eff}}), \quad (49d)$$

$$\lambda = (\alpha = 0.0146711 + 0.0522889 Z_{\text{eff}} - 0.0105684 Z_{\text{eff}}^2$$

$$- 0.00192144Z_{\text{eff}}^3)(Z_{\text{eff}} G(Z_{\text{eff}})), \quad (49e)$$

where

$$G(Z_{\text{eff}}) = Z_{\text{eff}}(1 + 0.292901Z_{\text{eff}})$$

$$+ 1.15971Z_{\text{eff}} + 0.159488Z_{\text{eff}}^3). \quad (49f)$$

Using these results, the normalized conductivities are evaluated to be
\[
\bar{\sigma}_f(\text{Z}_{\text{eff}}, 0) = Z_{\text{eff}}(3.01937 + 8.25957Z_{\text{eff}} + 3.76967Z_{\text{eff}}^2 + 0.407032Z_{\text{eff}}^3)/G_f(\text{Z}_{\text{eff}}).
\]

Also, the quality factor for momentum conservation is given as
\[
Q_f(\text{Z}_{\text{eff}}, 0) = 5.552 \times 10^{-3}(-1.32934 + 4.79226Z_{\text{eff}} - 0.9576Z_{\text{eff}}^2 - 0.174101Z_{\text{eff}}^3)/G_f(\text{Z}_{\text{eff}}).
\]

Although \(d_i\) and \(d_f\) have difference dependences on \(\text{Z}_{\text{eff}}\), their values are quite close to each other for \(1 \leq \text{Z}_{\text{eff}} < \infty\). The largest discrepancies, which are at most a few percent, occur at \(\text{Z}_{\text{eff}} = 1\):

\[
\chi_c(x) = x^2(0.598267 + 1.40693x - 0.658637x^2 + 0.133879x^3),
\]

\[
\chi_f(x) = x^2(0.628967 + 1.37137x - 0.642229x^2 + 0.131285x^3).
\]

The average deviation of the two functions is estimated to be \(\Delta \chi \approx 0.3\%\) using a formula similar to equation (30). The normalized conductivities obtained from the constrained and unconstrained calculations \((\bar{\sigma}_c\) and \(\bar{\sigma}_f\)) along with the numerical results of Braams and Karney \((\sigma_N)\) are compared in the left table of table 5 for \(\Theta = 0\). As one can see that there is excellent agreement between these results. In comparison with \(\bar{\sigma}_c\) and \(\bar{\sigma}_f\), the difference is quite insignificant. Also listed in the left table of table 5 are the normalized \(\bar{\lambda}\) for the constrained calculation and the quality factor for momentum conservation \(Q_f\) of the unconstrained calculation. For \(1 \leq \text{Z}_{\text{eff}} < \infty\), both \(|\bar{\lambda}|\) and \(|Q_f|\) are much less than 1, but when \(\text{Z}_{\text{eff}} \ll 1\) both \(|\bar{\lambda}|\) and \(|Q_f|\) would become large and so the momentum constraint also become important. The right of table 5 show the same case for \(\Theta = 0.2\), it can find that there are similar conclusions and change trends of \(|\bar{\lambda}|\) and \(|Q_f|\) changing with \(\text{Z}_{\text{eff}}\). Figure 4 shows \(|\bar{\lambda}|\) and \(|Q_f|\) as functions of \(\text{Z}_{\text{eff}}\) for \(\Theta = 0\) and \(\Theta = 0.2\), it is found that the two quantities are always strongly correlated.

### 5. Summary

A variational calculation of the current drive response function with different trial function is described in this paper. The trial function with the cases of \(\alpha = 1, 2\) (see equation (16)) is considered, and the results obtained shows that the choice of \(\alpha = 2\) is better than that of \(\alpha = 1\) (which has been used by Marushchenko et al) for finite \(\text{Z}_{\text{eff}}\), which can be explained simply by that for the Lorentz limit the exact analytic solution of the response is in proportion to \(1/\gamma^2\). Hirshman’s variational calculation with and without the momentum conserving constraint are discussed. It is found that Lagrangian multiplier \(\lambda\) and the quality factor for momentum conservation \(Q_f\) are positive correlated and also they are as functions of the effective ion charge \(\text{Z}_{\text{eff}}\). We also found that the additional momentum constraint term would be significant for the current trial function expansion only when \(\text{Z}_{\text{eff}} \ll 1\), and the constraint term would be unnecessary for a general cases \(\text{Z}_{\text{eff}} \geq 1\). In addition, the results of variational calculation of the matrix elements \(M_{ij}\) for a wide range of electron temperature, \(T_e\) and effective ion charge, \(\text{Z}_{\text{eff}}\) may be used as data inputs, then the matrix elements functions of \(T_e\) and \(\text{Z}_{\text{eff}}\) can be obtained by interpolation method, which is readily to apply to develop a code for current drive efficiency in toroidal geometry, in fact, our present model of the variational formulation of current drive has been extended to the tokamak plasma and also applied to the improvement of the code TORAY-GA successfully, the corresponding work may be published in the near future.

### Table 5. Left: \(\Theta = 0\), Right: \(\Theta = 0.2\).

| \(\text{Z}_{\text{eff}}\) | \(\bar{\sigma}_N\) | \(\bar{\sigma}_v\) | \(\bar{\sigma}_f\) | \(\bar{\lambda}\) | \(Q_f\) |
|---|---|---|---|---|---|
| 0.01 | 3.85501 | 4.14797 | 2.14176 | -0.214 | -0.491 |
| 0.1 | 4.52631 | 4.64483 | 4.42418 | -1.62e-2 | -5.44e-2 |
| 1.0 | 7.42898 | 7.39947 | 7.42388 | 5.14e-3 | 6.22e-3 |
| 2.0 | 8.75460 | 8.74570 | 8.75326 | 1.06e-3 | 1.95e-3 |
| 5.0 | 10.3912 | 10.3962 | 10.3908 | -8.74e-4 | -1.40e-3 |
| 10.0 | 11.3301 | 11.3357 | 11.3294 | -1.51e-3 | -1.61e-3 |
| \(\infty\) | 12.7662 | 12.7662 | 12.7662 | 0 | 0 |
| 0.01 | 3.62277 | 3.84796 | 1.97087 | 6.63e-1 | -4.92e-1 |
| 0.1 | 3.97973 | 4.08707 | 3.85327 | 2.40e-2 | -6.15e-2 |
| 1.0 | 5.43667 | 5.41344 | 5.43312 | 6.34e-3 | 5.30e-3 |
| 2.0 | 6.06243 | 6.04899 | 6.06037 | 3.12e-3 | 3.09e-3 |
| 5.0 | 6.80431 | 6.80251 | 6.80404 | 4.36e-4 | 4.19e-4 |
| 10.0 | 7.21564 | 7.21623 | 7.21557 | 2.67e-4 | -2.10e-4 |
| \(\infty\) | 7.82693 | 7.82693 | 7.82693 | 0 | 0 |
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Appendix

In this appendix we will discuss the computation of these matrix elements in equations (17)–(21) as well as asymptotic analysis. The electron–electron collision operator is

\[ C_{ee}(f_{em} \chi \cos \theta, f_{em}) = C_{ee}(f_{em} \chi \cos \theta, f_{em}) + C_{ee}(f_{em} \chi \cos \theta) \]  

(A.1)

Here, \( C_{ee}(f_{em} \chi \cos \theta, f_{em}) \) is the collision-off-the Maxwellian-background term which can be expressed as

\[ C_{ee}(f_{em} \chi \cos \theta, f_{em}) = f_{em} \cos \theta \left[ \frac{1}{u^2} \frac{\partial}{\partial u} \left( u^2 D_{uu} \frac{\partial \chi}{\partial u} \right) - \frac{m_e u}{\gamma T_e} \left( \frac{\partial \chi}{\partial u} - 2 D_{00} \frac{\partial \chi}{u^2} \right) \right] \]

(A.2)

with

\[ D_{uu} = \frac{4 \pi \nu_{ee} \rho_e}{n_e} \left[ \int_0^\infty \left( \frac{2 e^2 \gamma^2}{u^2} \right) f_{0(2)} - \frac{8 e^2}{u^2} f_{0(3)} \right] \frac{u^2}{\gamma^2} f_{em}(u') du' \]

\[ + \int_0^\infty \left( \frac{2 e^2 \gamma^2}{u^2} \right) f_{0(2)} \frac{8 e^2}{u^2} f_{0(3)} \frac{u^2}{\gamma^2} f_{em}(u') du' \]  

(A.3)

\[ F_u = - \frac{m_e u}{\gamma T_e} D_{uu} \]  

(A.4)

\[ D_{00} = \frac{4 \pi \nu_{ee} \rho_e^3}{n_e} \]

\[ \times \left[ \int_0^\infty \left( \frac{2 e^2 \gamma^2}{u^2} \right) f_{0(1,2)} - \left( 1 + \frac{4 \gamma^2}{u^2} \right) f_{0(2)} + \frac{4 e^2}{u^2} f_{0(3)} \right] \frac{u^2}{\gamma^2} \]

\[ \left\{ f_{em}(u') du' + \int_0^\infty \frac{1}{u} \left( \frac{2 e^2 \gamma^2}{u^2} \right) f_{0(1,2)} \frac{u^2}{\gamma^2} f_{em}(u') du' \right\} \]  

(A.5)

where \( \gamma = \sqrt{1 + (u'/c)^2}, u_e^2 = 2 T_e m_e, \nu_{ee} = n_e e^4 \ln \Lambda (4 \pi e^2 m_e c_e)^3 \), \( j_a = j_a(u) \), \( j_b = j_b(u') \) and the specific expression of \( j_a \) was the same as that in [8].

The field-particle-contribution term, \( C_{ee}(f_{em} \chi \cos \theta) \), is given as

\[ \frac{C_{ee}(f_{em} \chi \cos \theta)}{f_{em} \cos \theta} = \frac{4 \pi \nu_{ee} \rho_e^3}{n_e} \left[ \frac{1}{\gamma} \int_0^\infty K(u, u') \frac{u'^2}{\gamma^2} f_{em}(u') \chi du' \right] \]

(A.6)

where

\[ K(u, u') = \frac{2}{u^2} \left( \frac{f_{0(1,1)}}{c^2} + f_{0(1,2)} - 10 \frac{f_{0(2,2)}}{u_e^2} \right) \]

\[ + \frac{4 \gamma^2}{u^2} + \frac{2 \gamma^2}{u^2} + 6 \frac{f_{0(2,2)}}{u_e^2} + 24 \frac{c^2 f_{0(1,12)}}{u_e^2} \]

\[ + \frac{2 \gamma^2}{u^2} + 8 \gamma^2 \frac{f_{0(2,2)}}{u_e^2}. \]

Define the matrix elements

\[ \phi_i = \frac{x^{j+1}}{\gamma^j} \]  

(A.7)

\[ M_{ij} \equiv - \frac{2}{n_e c\tau_{ee}} \int d^4 u \phi_i \chi C_{ee}(f_{em} \phi_j \cos \theta) = M_{ji} \]

for \( i, j = 1, 2, 3, 4 \),

(A.8)

\[ M_{ij} \equiv - \frac{2}{n_e c\tau_{ee}} \int d^4 u (x \cos \theta) C_{ee}(f_{em} \phi_j \cos \theta) = M_{ji} \]

for \( i, j = 1, 2, 3, 4 \),

(A.9)

\[ M_{ij} \equiv \alpha_{ij} \chi + \beta_{ij} \chi \]

(A.10)

where \( \tau_{ee} = \frac{4}{3 e^4} \).

Non-relativistic limit.

\[ \lim_{\theta \to 0} D_{uu}(x, \Theta) = \frac{1}{2 x^2} [\text{erf}(x) - x \text{erf}'(x)] \]

\[ \lim_{\theta \to 0} D_{00}(x, \Theta) = \frac{1}{4 x^2} [(2x^2 - 1) \text{erf}(x) + x \text{erf}'(x)] \]

where \( x \equiv u/u_e, \Theta \equiv x \text{erf}'(x) \equiv \frac{1}{\sqrt{x}} \exp(-x^2) \).

At \( x = 0 \) for any \( \Theta \):

\[ D_{uu}(0, \Theta) = \frac{\sqrt{2} \Theta (1 + 2 \Theta + 2 \Theta^2)}{3 \exp(1/\Theta) K_0(1/\Theta)} \]

(A.11)

\[ D_{00}(0, \Theta) = \frac{\sqrt{2} \Theta (1 + 2 \Theta + 2 \Theta^2)}{3 \exp(1/\Theta) K_0(1/\Theta)} \]

(A.12)
High-energy limit.

See Braams-Karney equations (35a) and (35b) [8]

\[(\gamma - 1) \gg \Theta \text{ or } x^2 \equiv \frac{u^2}{u_e} \gg 1 + \frac{\Theta}{2},\]

\[\tilde{D}_{uu} \rightarrow \frac{1}{2} \frac{\gamma - 1}{x^2} \left(1 - \frac{K_0}{K_1} \frac{\Theta}{\gamma^2}\right),\]

\[\tilde{D}_{\theta\theta} \rightarrow \frac{1}{2x} \left[1 - \frac{K_1}{K_2} \left(\frac{1}{2x^2} + \frac{\Theta}{\gamma}\right) + \frac{K_0}{K_2} \frac{1}{2\gamma^2 x^2}\right].\]

In the limit of \(\Theta \rightarrow 0\), \(\frac{K_1}{K_2} \rightarrow 1\) and \(\frac{K_0}{K_2} \rightarrow 1\); thus \(\tilde{D}_{uu} \rightarrow \frac{1}{2x^2}\), \(\tilde{D}_{\theta\theta} \rightarrow \frac{1}{2x} \left(1 - \frac{1}{2x^2}\right)\), which recover Linliu’s results [9].


table A1.

| \(\Theta\) | 0    | 0.01 | 0.02 | 0.05 | 0.1  | 0.2  |
|-----------|------|------|------|------|------|------|
| \(D_{uu}(0)\) | 0.376 126 | 0.376 632 | 0.377 204 | 0.379 297 | 0.383 859 | 0.395 995 |
| Numerical (at \(x = 0.0001\)) | 0.376 126 | 0.376 631 | 0.377 205 | 0.379 297 | 0.383 859 | 0.395 995 |

\[\lim_{\Theta \rightarrow 0} \tilde{D}_{\theta\theta}(0, \Theta) = \frac{2}{3\sqrt{\pi}}\]


table A2.

| \(\Theta\) | 0    | 0.01 | 0.02 | 0.05 | 0.1  | 0.2  |
|-----------|------|------|------|------|------|------|
| \(D_{uu}(0)\) | 0.376 126 | 0.384 089 | 0.391 991 | 0.415 338 | 0.453 08 | 0.524 426 |
| Numerical (at \(x = 0.0001\)) | 0.376 126 | 0.384 089 | 0.391 991 | 0.415 338 | 0.453 08 | 0.524 426 |

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Table A1.

Table A2.