Inflationary cosmology proposes that the early Universe undergoes accelerated expansion, driven, in simple scenarios, by a single scalar field, or inflaton. The form of the inflaton potential determines the initial spectra of density perturbations and gravitational waves. We show that constraints on the duration of inflation together with the BICEP3/Keck bounds on the gravitational wave background imply that higher derivatives of the potential are nontrivial with a confidence of 99%. Such terms contribute to the scale-dependence, or running, of the density perturbation spectrum. We clarify the “universality classes” of inflation in this limit and models with a very small gravitational wave background typically have a larger running. If a gravitational wave background is not observed by pending experiments, the running may be at the threshold of detectability. Correlated expectations for the running and gravitational wave background are thus an avenue via which future observations may yield insight into possible inflationary mechanisms.

Now forty years old, inflation [1] is the de facto description of the very early Universe. The clear consequences of generic inflationary models are well-verified: the Universe is spatially flat, almost homogeneous and isotropic with Gaussian, adiabatic perturbations [2, 3] that induce large scale correlations in the polarization and temperature of the microwave background [4]. The one ambiguous observable is the primordial gravitational wave background. Constraints have progressively tightened [5–8] to the point that the latest BICEP3/Keck data permits an amplitude of at most 4% that of the density perturbations [9]. A gravitational wave background is often viewed as the “smoking gun” of inflation. However, while alternatives to inflation do not generate detectable gravitational waves [10, 11] this is also true of many viable inflationary models. Moreover, algebraically simple slow roll scenarios with large gravitational-wave signals need to be “protected” by near-symmetries [12] and while many such models have been proposed (e.g. [13]) nature need not make use of them.

Physically, the amplitudes of the density and gravitational wave perturbations (expressed via their ratio, r), depend on the height V and slope V’ of the potential. The spectral index of the density perturbations ns further involves the second derivative, V''. Given a single field slow-roll prior, constraints on ns, and r are inputs for the inflationary inverse problem: the reconstruction of the potential from observational data [14].

The fundamental dynamics of this system are not mysterious and the correlation between the running and the duration of inflation is well-known [15–19]. However, we show that with the latest BICEP3/Keck release all viable implementations of slow-roll inflation with only V, V’ and V” as free parameters imply that the Universe grows more than $e^{65}$ times larger after astrophysically relevant perturbations leave the horizon, with 99% confidence. Without exotic matter in the post-inflationary universe, this is inconsistent with long-standing constraints on the duration of inflation [19–22] and the accelerated phase can only terminate appropriately if higher derivatives are nontrivial or the potential is discontinuous.

A nontrivial $V'''$ modifies the duration of inflation relative to that predicted using only $V'$ and $V''$. For any $n_s$ and $r$ one can fix $V'''$ to yield a specified duration of inflation. However, this induces scale dependence in $V''$, contributing to the running of the spectral index, $\alpha_s = dn_s/d\ln k$ where k is the comoving wavenumber. Experiments now under development are sensitive to $r \gtrsim 10^{-4}$ [23, 24]. If these give a null result we show that $\alpha_s < -10^{-3}$ if the duration of the inflationary phase is realistic, given three nontrivial slow-roll parameters. This is several times larger than $\alpha_s$ in simple models [19] and at the threshold of detection by upcoming experiments.

Given the many inflationary mechanisms and the possibility of further higher-order terms, a large running is not a necessary consequence of inflation if $r$ is very small. However, these correlated expectations for $r$ and $\alpha_s$ are highly informative and will be tested by future astrophysical measurements.

**TWO PARAMETER SLOW-ROLL MODELS**

Single-field inflationary scenarios are governed by the Einstein-Klein-Gordon equations,

$$H^2 = \frac{1}{3M_p^2} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right),$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0.$$

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where the symbols have their usual meanings and we use the reduced Planck mass, $M_P$. During the accelerated phase $\phi$ evolves monotonically and is thus a “clock”. Eq. (2) can be rearranged to show that $dH/d\phi$ is proportional to $-d\phi/dt$, so

$$V(\phi) = \frac{3M_P^2}{2}H(\phi)^2 - M_P^4H'(\phi)^2.$$  \hspace{1cm} (4)

For the purposes of parameter counting, we assume that the Potential and Hubble Slow Roll formulations are interchangeable. The Hubble Slow Roll hierarchy [25] provides a more succinct account of the dynamics,

$$\epsilon(\phi) \equiv 2M_P^2 \left[\frac{H'(\phi)}{H(\phi)}\right]^2,$$  \hspace{1cm} (5)

$$\ell\lambda_H \equiv (2M_P)^\ell \frac{(H')^{\ell-1} d^{\ell+1}H}{d\phi^{\ell+1}}, \quad \ell \geq 1.$$  \hspace{1cm} (6)

with the convention that $\eta = 1/\lambda_H$ and $\xi = 2\lambda_H$. The number of e-folds that will elapse before inflation ends is $N = -\ln(a/a_{\text{end}})$, where $a_{\text{end}}$ is the scale factor as inflation completes. Noting $H = \dot{a}/a$, the “flow equations” are

$$\frac{d\epsilon}{dN} = 2\epsilon(\eta - \epsilon),$$  \hspace{1cm} (8)

$$\frac{d\eta}{dN} = -\epsilon\eta + \xi,$$  \hspace{1cm} (9)

$$\frac{d^{\ell+1}\lambda_H}{dN} = [\ell(\eta - \epsilon) \times (\ell+1)\lambda_H + \ell\lambda_H + \lambda_H],$$  \hspace{1cm} (10)

where $N$ is now the independent variable. Accelerated expansion occurs when $\dot{a} > 0$ or equivalently $\epsilon < 1$. If $\ell\lambda_H = 0$ for all $\ell \geq M$ at some $\phi_0$ the system remains closed as it evolves [25][27], with $M$ nontrivial slow-roll parameters. The amplitude of the potential is a further free parameter but scales out of the dynamics.

The foregoing treatment is exact but key observables are expressed in the slow roll approximation, or

$$n_s = 1 + 2\eta - 4\epsilon - 2(1 + C)\epsilon^2 - \frac{1}{2}(3 - C)\xi,$$  \hspace{1cm} (11)

$$r = 16\epsilon[1 + 2C(\epsilon - \eta)],$$  \hspace{1cm} (12)

$$\alpha_s = -\frac{1}{1 - \epsilon} \frac{d\phi}{dN}$$  \hspace{1cm} (13)

where $C = -2 + \ln 2 + \gamma$, $C = 4(\gamma + \ln 2) - 5$, and $\gamma$ is the Euler-Mascheroni constant. Finally, $dN/d\ln k = -1/(1 - \epsilon)$ is the rate at which modes leave the horizon and $\epsilon \to 0$ in the de Sitter limit where $H$ is constant.

A two-term hierarchy maps $n_s$ and $r$ to an inflationary trajectory. Fig. 2 shows the constraints on $n_s$ and $\ln r$ derived from the BK15 [7] and BK18 [9] datasets (whose analyses were published in 2018 and 2021, respectively), together with Planck and Baryon Acoustic Oscillation data overlaid with the duration of inflation computed with two slow-roll terms. Fig. 2 shows the marginalised distributions for $N$; BK18 yields $P(N < 65) \approx 0.0024$. Provided the post-inflationary universe is not dominated by matter whose stiffness exceeds that of radiation, $N < 65$ is a generic bound on the amount of inflation after the pivot leaves the horizon [21]. Subject to this proviso on the post-inflationary equation of state, the full cosmological dataset now excludes all inflationary models described by the first two slow-roll parameters.

1 These and other plots in this paper were constructed by resampling chains made available by the BICEP/Keck collaboration.
We now extend the Hubble Slow Roll expansion to third order, so that \( \xi \) is non-zero. This can increase the scale-dependence of \( \eta / N \), as \( \alpha_s \approx -2 \xi \) when \( \epsilon \) is small. Fig. 3 overplots the \( n_s \) and \( r \) constraints with contours showing the running resulting from choosing \( \xi \) such that \( N = 55 \) when the pivot leaves the horizon. The running is generically larger than in “standard” inflationary models [19] but still well inside recent constraints; e.g. \( d \ln \eta / d \ln k = -0.006 \pm 0.013 \) [6].

This adds nuance to statements that \( n_s - 1 \sim -1/N \) and \( \alpha_s \sim 1/N^2 \) which hold empirically for many simple models [19], These expectations have been formalised in the Potential Slow Roll expansion [23][29][31], leading to what are sometimes referred to as “universality classes” [30]. In this framework \( \epsilon_V = M_p^2 (V'/V)^2 / 2 \), \( \eta_V = M_p^2 V''/V \) and \( \xi_V = M_p^2 V'''/V^2 \) and

\[
\frac{d \epsilon_V}{dN} \approx M_p^2 \left( \frac{V'}{V} \right)^2 \left[ \frac{V''}{V} - \left( \frac{V'}{V} \right)^2 \right].
\]

We write \( n_s - 1 = -\alpha / N \), where \( \alpha \) is a constant a little larger than unity. Dropping higher order terms and accounting for the difference between \( \eta \) and \( \eta_V \) we can set this equal to Eq. [11] or \( n_s \sim 1 - 6 \epsilon_V + 2 \eta_V \) to find a differential equation for \( \epsilon_V(N) \) (e.g. [31]). In the low \( r \) limit the solution has the form \( \epsilon_V \sim 1/(AN^2) \) where \( A \) is a large constant. Physically, this ensures that \( \eta_V \) and \( \epsilon_V \) are tightly correlated even when \( \epsilon_V \ll \eta_V \). However if \( r \lesssim |n_s - 1|^2 \) it would seem that \( \xi_V \) cannot be self-consistently ignored, since it contributes to the scale dependence of \( \eta_V \) via

\[
\frac{d \eta_V}{dN} \approx M_p^2 \left[ \frac{V'}{V} \right] \left[ \frac{V''}{V} - \left( \frac{V'}{V} \right)^2 \right].
\]

and the second term can be far smaller than the first.

This regime corresponds to the Low-\( \epsilon \) limit of the Hubble Slow Roll hierarchy, and with three terms

\[
\frac{d \eta}{dN} \approx \xi, \quad \frac{d \xi}{dN} \approx \xi \eta.
\]

These equations can be solved [18], showing

\[
\xi(N) = \frac{\eta(N)^2 - \eta_0^2}{2} + \xi_s
\]

where the star subscript denotes a value at the pivot. To a good approximation \( \eta(N) = \eta_s - \xi_s \Delta N \) for astrophysically relevant modes, where \( \Delta N \) is the number of e-folds after the pivot leaves the horizon. In particular, the relationship \( \xi \sim |n_s - 1|^2 \sim 1/N^2 \) is supplemented by an additive constant in the Low-\( \epsilon \) limit. Physically, this yields a near-inflection point in the potential, where both \( \epsilon \) and \( n_s - 1 \) are necessarily very small.

**Running and the End of Inflation**

Recalling that \( r \sim (V'/V)^2 \), we can identify three regimes; \( V' > V'' \), \( V' \sim V'' \) and \( V' \ll V'' \) (with \( M_p = 1 \)). The first requires \( r \gtrsim 0.01 \) and is close to being ruled out, the second is eliminated if \( r \lesssim 10^{-4} \), a threshold which will be within reach by 2030 [23][24].

If a primordial gravitational wave background is not detected in the coming decade, any viable single-field model will satisfy \( V' \ll V'' \) and is thus squarely inside the Low-\( \epsilon \) regime. Fig. 4 shows the likely values of \( \alpha_s \) on the \( n_s - \ln r \) plane for three different choices of the total number of e-foldings. If \( r \lesssim 10^{-4} \) then \( \alpha_s < -10^{-3} \) for any self-consistent three parameter scenario.

Fig. 5 shows the individual and combined limits on \( |\alpha_s| \) expected from CHIME [33] and SPHEREx [34], together with CMB-S4 [23]. Each experiment measures \( \alpha_s \) with

\[
^2 \text{The solution for } \eta(N) \text{ is given in full in Ref. [18], where it is referred to as a “2-parameter, Low-\( \epsilon \) model.}
\]
FIG. 4. The running $\alpha_s \times 10^3$ is plotted in the $n_s - \ln r$ plane for $N = 45, 55$ and $60$, assuming a three-parameter slow-roll hierarchy. When $r \lesssim 10^{-4}$ we see that $\alpha_s < -10^{-3}$ for all values of $n_s$ consistent with presently available data.

FIG. 5. Forecast $n_s$-$\alpha_s$ constraints with CMB-S4, SPHEREx [S], CHIME [C]. The best combination promises to measure $\alpha_s$ to about $2.2 \times 10^{-3}$ at 95% confidence [22].

an accuracy of, at best, $5 \times 10^{-3}$ but their combined sensitivity is similar to the expected running if $r \lesssim 10^{-4}$. All these experiments aim to provide results by 2030. Consequently, if the early Universe passes through an accelerated phase it is reasonable to hope to have evidence that either $r$ or $\alpha_s$ is non-zero in a decade from now.

DISCUSSION

We have updated the priors for single-field, slow-roll inflation in the light of the latest cosmological data. We find that inflationary models specified by only the values of $V'$ and $V''$ at the pivot scale do not lead to self-consistent predictions for the duration of inflation, with a confidence of greater than 99%.

This analysis sheds light on inflationary universality classes for very small $r$. Universality classes arise from viewing expressions for $n_s$ as differential relationships. Conversely, the Hubble Slow Roll parameters are akin to coefficients in a Taylor expansion but the "flow equations" describe their running [25]. The first two terms set $\sqrt{r}$ and $|n_s - 1|$ but near an extremum of $V(\phi)$ (or $H(\phi)$, since $V' = 0$ implies $H' = 0$) $r$ is very small and $V''$ is the next-to-leading order term.

The scenarios with $r \lesssim 10^{-4}$ examined here correspond to potentials with a near-stationary inflexion point. Our conclusions could change if $3\lambda_H$ and above are nontrivial, allowing $\xi$ and thus the running to be small at the pivot. That said, it is natural to retain only as many terms in an expansion as needed. Moreover, potentials of the form $V \sim V_0 - V_2 \phi^2 - V_4 \phi^4$ struggle to generate low values of $r$ while matching constraints on $n_s$ [35]. This overall discussion could be further sharpened by adopting a Bayesian model comparison framework [6, 33, 37]. Note too, if the inflationary patch of the potential is small it is more likely that models will be sensitive to the initial spatial configuration of the inflaton [38-40].

Given the vast range of inflationary mechanisms and the possibility of higher-order slow roll terms, the inverse relationship between $r$ and $\alpha_s$ explored here creates a correlated expectation for these observables, rather than a firm prediction for all inflationary models. However, for fixed $n_s$ there is also an inverse correlation between $\alpha_s$ and $N$, which depends on both the overall inflationary scale [17, 21] and the possibly complicated and nonlinear physics of the post-inflationary universe [41-47].

The current roadmap for observational cosmology will provide the capability to detect a gravitational wave signal with $10^{-4} \lesssim r \lesssim 10^{-2}$ and $|\alpha_s| \gtrsim 10^{-3}$ in the coming decade. Without a detection of gravitational waves, this will real pressure on the relationship between $\alpha_s$, $r$ and $N$ explored here. Either outcome would represent major progress for early universe cosmology.
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