SEMICLASSICAL STRINGS AND ADS/CFT

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Abstract

We discuss AdS/CFT duality in the sector of “semiclassical” string states with large quantum numbers. We review the coherent-state effective action approach, in which similar 2d sigma model actions appear from the $AdS_5 \times S^5$ string action and from the integrable spin chain Hamiltonian representing the $N=4$ super Yang-Mills dilatation operator. We consider mostly the leading-order terms in the energies/anomalous dimensions which match but comment also on higher-order corrections.

1. INTRODUCTION

The $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group is a family of conformal theories parametrized by the two numbers – $N$ and $g_{YM}$. Four-dimensional conformal theories have apparently much less symmetry than their two-dimensional cousins and thus should be much harder to solve (i.e. to determine their spectrum of dimensions of conformal primary operators and their correlation functions). There are strong indications that this problem may simplify in the planar ($N \to \infty$, $\lambda \equiv g_{YM}^2 N=$fixed) limit. In the large $N$ limit there are no formal objections to integrability of a 4-d quantum field theory, and, indeed, the AdS/CFT duality [1, 2, 3] implies the existence of a hidden integrable 2-d structure corresponding to that of $AdS_5 \times S^5$ string sigma model (on a sphere or a cylinder). A major first step towards the solution of the SYM theory would be to determine the spectrum of anomalous dimensions $\Delta(\lambda)$ of the primary operators built out of products of local gauge-covariant fields.

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The AdS/CFT duality implies the equality between the $AdS_5$ energies $E$ of quantum closed string states (as functions of the effective string tension $T = \sqrt{\frac{\lambda}{2\pi}} = \frac{\lambda}{2\pi}$ and quantum numbers like $S^5$ angular momenta $J_i$) and the dimensions $\Delta$ of the corresponding local SYM operators (see, e.g., [4]). To check the duality one would like also to understand how strings “emerge” from the field theory, in particular, which (local, single-trace) gauge theory operators [5] correspond to which “excited” string states and how one may verify the matching of their dimensions/energies beyond the well-understood BPS/supergravity sector. One would then like to use the duality as a guide to deeper level of the structure of quantum SYM theory. For example, the results motivated by comparison to string theory may allow one to “guess” the general structure of the SYM anomalous dimension matrix and may also suggest new methods of computing anomalous dimensions in less supersymmetric gauge theories.

Below we shall review some recent progress in checking AdS/CFT correspondence in a subsector of non-BPS string/SYM states with large quantum numbers.

1.1 GENERALITIES

Let us start with brief remarks on the SYM and the string sides of the duality. The SYM theory contains a gauge field, 6 scalars $\phi_m$ and 4 Weyl fermions, all in adjoint representation of $SU(N)$. It has global conformal and R-symmetry, i.e. is invariant under $SO(2,4) \times SO(6)$. To determine (in the planar limit) scaling the dimensions of local gauge-invariant operators one, in general, needs to find the anomalous dimension matrix to all orders in $\lambda$ and then to diagonalize it. The special case is that of chiral primary or BPS operators (and their descendants) $\text{tr}(\phi_{(m_1} \cdots \phi_{m_k)})$ whose dimensions are protected, i.e. do not depend on $\lambda$. The problem of finding dimensions appears to simplify also in the case of “long” operators containing large number of fields under the trace. One example is provided by “near-BPS” operators [6] like $\text{tr}(\Phi_1^J \Phi_2^J \cdots) + \ldots$, where $J \gg n$, and $\Phi_k = \phi_k + i\phi_{k+3}$, $k = 1, 2, 3$. Below we shall consider “far-from-BPS” operators like $\text{tr}(\Phi_1^{J_1} \Phi_2^{J_2} \cdots) + \ldots$, with $J_1 \sim J_2 \gg 1$.

The type IIB string action in $AdS_5 \times S^5$ space has the following structure

$$I = -\frac{1}{2} T \int d\tau \int_0^{2\pi} d\sigma \left( \partial^p Y^\mu \partial_\mu Y^\nu \eta_{\mu\nu} + \partial^p X^m \partial_n X^n \delta_{mn} + \ldots \right), \quad (1.1)$$

where $Y^\mu Y^\nu \eta_{\mu\nu} = -1$, $X^m X^n \delta_{mn} = 1$, $\eta_{\mu\nu} = (- + + + +)$, $T = \sqrt{\frac{\lambda}{2\pi}}$ and dots stand for the fermionic terms [7] that ensure that this model
defines a 2-d conformal field theory. The closed string states can be classified by the values of the Cartan charges of the obvious symmetry group $SO(2,4) \times SO(6) - E, S_1, S_2, J_1, J_2, J_3$, i.e. by the $AdS_5$ energy, two spins in $AdS_5$ and 3 spins in $S^5$. The mass shell (conformal gauge constraint) condition then gives a relation $E = E(Q,T)$. Here $T$ is the string tension and $Q = (S_1, S_2, J_1, J_2, J_3; n_k)$ where $n_k$ stand for higher conserved charges (analogs of oscillation numbers in flat space). The BPS (chiral primary) string states are point-like (supergravity modes), near-BPS (BMN) states are nearly pointlike, while generic semiclassical far-from-BPS states are represented by extended closed string configurations.

According to the AdS/CFT duality quantum closed string states in $AdS_5 \times S^5$ should be dual to quantum SYM states at the boundary, i.e. in $R \times S^3$ or, via radial quantization, to local single-trace operators at the origin of $R^4$. Such operators have the following structure $\text{tr}(D_1 S_1 + D_2 S_2 \Phi_1 \Phi_2 \Phi_3 ...)+...$ (where scalars and covariant derivatives may be also replaced by gauge field strength factors and fermions). The energy of a string state should then be equal to the dimension of the corresponding SYM operator, $E(Q,T) = \Delta(Q,\lambda)$, where on the SYM side the charges $Q$ should characterise the eigen-operator of the anomalous dimension matrix. By analogy with the flat space case and ignoring $\alpha'$ corrections (i.e. assuming $R \to \infty$ or $\alpha' \to 0$) the excited string states are expected to have energies $E \sim m \sim \frac{1}{\sqrt{\alpha'}} \sim \lambda^{1/4}$ [2]. This represents a non-trivial prediction for strong-coupling asymptotics of SYM dimensions. The asymptotics may, however, be different in the limits where the charges $Q$ are also large, e.g., of order $\lambda^{1/2}$ as in the semiclassical limit [8].

In general, the natural (inverse-tension) perturbative expansion on the string side will be given by $E = \sum_{n=-1}^{\infty} \frac{2n}{(\sqrt{\lambda})^n}$, while on the SYM side the usual planar perturbation theory will give the eigenvalues of the anomalous dimension matrix as $\Delta = \sum_{n=0}^{\infty} a_n \lambda^n$. The AdS/CFT duality implies that the two expansions should be the strong-coupling and the weak-coupling asymptotics of the same function. To check the relation $E = \Delta$ is then a non-trivial problem, except in the case of 1/2 BPS (single-trace chiral primary) operators which are dual to the supergravity states when the energies/dimensions are protected from corrections [4] and thus can be matched on the symmetry grounds.
1.2 SEMICLASSICAL STRING STATES: BMN AND BEYOND

For generic non-BPS states the situation with checking the duality looked hopeless until the remarkable suggestion of [6] and then of [8] that a progress can be made by (i) concentrating on a subsector of states with large or “semiclassical” values of quantum numbers, $Q \sim T \sim \sqrt{\lambda}$ and (ii) considering a new limit

$$Q \to \infty , \quad \tilde{\lambda} \equiv \frac{\lambda}{Q^2} \text{ fixed} . \quad (1.2)$$

On the string side $\frac{Q}{\sqrt{\lambda}} = \frac{1}{\sqrt{\tilde{\lambda}}}$ plays the role of a semiclassical parameter (like rotation frequency) which can then be taken to be large. The energy of such states is $E = Q + f(Q, \lambda)$, where $f \to 0$ in the $\lambda \to 0$ limit. The duality implies that such semiclassical string states (as well as states represented by small fluctuations near them) should be dual to “long” SYM operators with a large canonical dimension, i.e. containing large number of fields or derivatives under the trace. In this case the duality map becomes more explicit.

The simplest possibility is to start with a BPS state that carries a large quantum number and consider small fluctuations near it, i.e. a set of near-BPS states characterised by a large parameter [6]. The only non-trivial example of such a BPS state is represented by a point-like string moving along a geodesic in $S^5$ with a large angular momentum $Q = J$. Then $E = J$ and the dual operator is $\text{tr} \Phi^J$, $\Phi = \phi_1 + i\phi_2$. The small

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The history of the BMN limit is somewhat non-trivial. It started with the important observation that the Penrose limit of the $AdS_5 \times S^5$ space [9] leads to a maximally supersymmetric plane wave geometry supported by the Ramond-Ramond 5-form flux. Remarkably, the $AdS_5 \times S^5$ string theory action [7] in this limit (i.e. Green-Schwarz string action in plane-wave background) becomes essentially quadratic and thus its spectrum can be found explicitly [10]. Motivated by that, ref. [6] gave a dual SYM interpretation of the corresponding string states, suggested that string energies can be compared to perturbative SYM dimensions computed in the same limit, and indeed directly checked this to the first non-trivial order. Ref. [8] then explained that, on the string-theory side, the BMN limit is nothing but a semiclassical expansion near a particular (point-like) string solution. This interpretation was further clarified and extended in [11, 12], suggesting, in particular, how one can in principle compute corrections to the BMN limit (which was later done in full detail in [13] and especially in [14]). This made it clear that the plane-wave connection is not fundamental but rather is a special feature of semiclassical expansion near a certain string configuration (represented by massless $AdS_5 \times S^5$ geodesic wrapping $S^5$). Expanding near other string solutions leads to other special “Penrose-type” limits of string geometry (geometry “seen” by a fundamental string probe) which are described by the corresponding quadratic fluctuation actions. Expansions near a class of “fast” string configurations (discussed below) for which the world sheet becomes null in the effective zero-tension limit [15, 16, 17] may be interpreted as a stringy generalization of the Penrose limit (keeping in mind, of course, that this analogy applies only in the strict zero-tension limit).
closed strings representing near-by fluctuations are ultrarelativistic, i.e.
their kinetic energy is much larger than their mass. They are dual to SYM operators of the form \( \text{tr}(\Phi^J \ldots) \) where dots stand for a small number of other fields and/or covariant derivatives (one needs to sum over different orders of the factors to find an eigenstate of the anomalous dimension matrix). The energies of the small fluctuations happen to have the structure \([10, 6]\)

\[
E = J + \sqrt{1 + n^2 \tilde{\lambda}^2} K + O(\frac{1}{J}) = J + K + k_1 \tilde{\lambda} + k_2 \tilde{\lambda}^2 + \ldots . \tag{1.3}
\]

One can argue in general \([11, 12]\) and check explicitly \([13, 14]\) that higher-order quantum string sigma model corrections are indeed suppressed in the limit (1.2), i.e. in the large \( J \), fixed \( \tilde{\lambda} \equiv \frac{\lambda}{J^2} = \lambda' \) limit. A remarkable feature of this expression is that \( E \) is analytic in \( \tilde{\lambda} \), suggesting direct comparison with perturbative SYM expansion in \( \lambda \).

Indeed, it can be shown that the first three \( \tilde{\lambda}, \tilde{\lambda}^2 \) and \( \tilde{\lambda}^3 \) terms in the expansion of the square root agree precisely with the one \([6]\), two \([19]\) and three \([20, 21, 23]\) loop terms in the anomalous dimensions of the corresponding operators. There is also (for a 2-impurity \( K = 2 \) case) an argument \([24]\) suggesting how the full \( \sqrt{1 + n^2 \tilde{\lambda}^2} \) expression may appear on the perturbative SYM side (for reviews of various aspects of the BMN limit see also \([25]\)). However, the general proof of the consistency of the BMN limit in the SYM theory (i.e. that the usual perturbative expansion can be rewritten as an expansion in \( \tilde{\lambda} \) and \( \frac{1}{J} \)) remains to be given. Also, to explain why the string and the SYM expressions match one should show that the string limit (first \( J \rightarrow \infty \), then \( \tilde{\lambda} = \frac{\lambda}{J^2} \rightarrow 0 \)) and the SYM limit (first \( \lambda \rightarrow 0 \), then \( J \rightarrow \infty \)) produce exactly the same expressions for the energies/dimensions, even though, in general, the two limits may not commute, cf. \([26, 27, 28]\).

If one moves away from the near-BPS limit and considers, e.g., a non-supersymmetric state with a large angular momentum \( Q = S \) in \( AdS_5 \) \([8]\), a similar direct quantitative check of the duality is no longer possible: here the classical energy is not analytic in \( \lambda \) and quantum corrections are no longer suppressed by powers of \( \frac{1}{Q} \) (but as usual are suppressed by powers of \( \frac{1}{\sqrt{\lambda}} \)). However, it is still possible to demonstrate a remarkable qualitative agreement between \( S \)-dependence of the string energy and of the SYM anomalous dimension. The energy of a folded closed string rotating at the center of \( AdS_5 \) which is dual \([8]\) to twist 2 operators on the SYM side \( (\text{tr}(\Phi_k^* D^S \Phi_k), \ D = D_1 + iD_2, \text{ and similar operators with spinors and gauge bosons that mix at higher loops} [32, 33]) \) has the
following form when expanded at large $S$:

$$ E = S + f(\lambda) \ln S + O(S^0) \quad (1.4) $$

On the string side

$$ f(\lambda)_{\lambda > 1} = c_0 \sqrt{\lambda} + c_1 + \frac{c_2}{\sqrt{\lambda}} + ... , \quad (1.5) $$

where $c_0 = \frac{1}{\pi}$ is the classical [8] and $c_1 = -\frac{3}{\pi} \ln 2$ is the 1-loop [11] coefficient. On the gauge theory side one finds the same $S$-dependence of the anomalous dimension with the perturbative expansion of the $\ln S$ coefficient being

$$ f(\lambda)_{\lambda > 1} = a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3 + ... \quad (1.6) $$

where $a_1 = \frac{1}{2\pi^2}$ [31], $a_2 = -\frac{1}{96\pi^2}$ [32], and $a_3 = \frac{11}{360\times 64\pi^2}$ [33]. Like in the case of the SYM entropy [34], here one expects the existence of a smooth interpolating function $f(\lambda)$ that connects the two perturbative expansions. In fact, observing that the factor $\frac{1}{\pi}$ in (1.5) and factor $\frac{1}{\lambda}$ in (1.6) seem to factorize, one can suggest a simple square root type interpolating formula for $f(\lambda)$ that seem indeed to give a good fit [32, 33] (cf. also the discussion end of section 4).

### 1.3 MULTISPIN STRING STATES

One may wonder still if examples of quantitative agreement between string energies and SYM dimensions found for the near-BPS (BMN) states exist also for more general non-BPS string states. Indeed, it was noticed already in [11] that a string state that carries large spin in $AdS_5$ as well as large spin $J$ in $S^5$ has, in contrast to the above $J = 0$ case, an analytic expansion of its energy in $\tilde{\lambda} = \frac{\lambda}{J^2}$, just as in the BMN case with a large oscillation number $K \sim S$. It was observed in [35] that semiclassical string states carrying several large spins (with at least one of them being in $S^5$) have a regular expansion of their energy $E$ in powers of $\lambda$ and it was then suggested, by analogy with the near-BPS case, that the expansion of $E$ in small effective tension or $\tilde{\lambda}$ may be possible to match with the perturbative expansion of the SYM dimensions.

For a classical rotating closed string solution in $S^5$ one has $E = \sqrt{\lambda} \mathcal{E}(w_i), J_i = \sqrt{\lambda} w_i$ so that $E = E(J_i, \lambda)$. The required key property is that (in contrast to the case of a single spin in $AdS_5$) there should be no $\sqrt{\lambda}$ factors in the expansion of the classical energy $E$ in small $\lambda$

$$ E = J + c_1 \frac{\lambda}{J} + c_2 \frac{\lambda^2}{J^3} + ... = J \left[ 1 + c_1 \tilde{\lambda} + c_2 \tilde{\lambda}^2 + ... \right] . \quad (1.7) $$
Here $J = \sum_{i=1}^{3} J_i$, $\tilde{\lambda} \equiv J / \sqrt{\lambda}$ and $c_n = c_n \left( \frac{J}{\sqrt{\lambda}} \right)$ are functions of ratios of the spins which are finite in the limit $J_i \gg 1$, $\tilde{\lambda} =$fixed.

The simplest example of such solution is provided by a circular string rotating in two orthogonal planes in $S^3$ part of $S^5$ with the two angular momenta being equal, $J_1 = J_2$ [35]:

$X_1 \equiv X_1 + iX_2 = \cos(n\sigma) \ e^{i w \tau}$, \quad $X_2 \equiv X_3 + iX_4 = \sin(n\sigma) \ e^{i w \tau}$,

and with the global $AdS_5$ time being $t = \kappa \tau$. The conformal gauge constraint implies $\kappa^2 = w^2 + n^2$ and thus

$$E = \sqrt{J^2 + n^2 \lambda} = J \left[ 1 + \frac{1}{2} n^2 \tilde{\lambda} - \frac{1}{8} n^4 \tilde{\lambda}^2 + ... \right], \quad (1.8)$$

where $J = J_1 + J_2 = 2J_1$. For fixed $J$ the energy thus has a regular expansion in string tension (in contrast to what happens in flat space where $E = \sqrt{\frac{2}{\alpha' \lambda}}$).

Similar expressions (1.7) are found also for more general rigid multi-spin closed strings [35, 36, 37, 38, 39, 40]. In particular, for a folded string rotating in one plane of $S^5$ and with its center of mass orbiting along big circle in another plane the coefficients $c_n$ are transcendental functions (expressed in terms of elliptic integrals) [37]. More generally, the 3-spin solutions are described by an integrable Neumann model [38, 39] and the coefficients $c_n$ in the energy are expressed in terms of genus two hyper-elliptic functions. The reason why choosing a particular string ansatz one gets an integrable effective 1-d model lies in the integrability of the original $S^5 = SO(6)/SO(5)$ classical sigma model [30] (see also [44]).

To be able to hope to compare the classical energy to the SYM dimension one should be sure that higher string $\alpha'$ corrections are suppressed in the limit $J \to \infty$, $\tilde{\lambda} =$fixed. Formally, this is of course the case since $\alpha' \sim \frac{1}{\sqrt{\lambda}} \sim \frac{1}{J \sqrt{\lambda}}$; what is more important, the $\frac{1}{J}$ corrections are again analytic in $\tilde{\lambda}$ [36], i.e., as in the BMN case, the expansion in large $J$ and small $\tilde{\lambda}$ is well-defined on the string side,

$$E = J \left[ 1 + \tilde{\lambda}(c_1 + \frac{d_1}{J} + ...) + \tilde{\lambda}^2(c_2 + \frac{d_2}{J} + ...) + ... \right], \quad (1.9)$$

with the classical energy (1.7) being the $J \to \infty$ limit of the exact expression.

The reason for this particular form of the energy (1.9) can be explained as follows [11, 12, 40]. We are computing the $AdS_5 \times S^5$ super-string sigma model loop corrections to the mass of a stationary solitonic solution on a 2-d cylinder (no IR divergences). This theory is conformal (due to the crucial presence of the fermionic fluctuations) and thus
it does not depend on a UV cutoff. The relevant 2d fluctuations are massive and their masses scale as \( w \sim \sqrt{\lambda} \). As a result, the inverse mass expansion is well-defined and the quantum corrections should be proportional to positive powers of \( \tilde{\lambda} \). This was explicitly demonstrated by a 1-loop computation in [36, 66].

Similar expressions are found for the energies of small fluctuations near a given classical solution: as in the BMN case, the fluctuation energies are suppressed by extra factor of \( J \), i.e.

\[
\delta E = \tilde{\lambda}(k_1 + \frac{m_1}{J} + ...) + \tilde{\lambda}^2(k_2 + \frac{m_2}{J} + ...) + ... .
\]  

### 1.4 ADS/CFT DUALITY: NON-BPS STATES

Assuming that the same large \( J \) limit is well-defined also on the SYM side, one should then be able to compare the coefficients in (1.9) to the coefficients in the anomalous dimensions of the corresponding SYM operators \( \text{tr}(\Phi_J^1 \Phi_J^2 \Phi_J^3) + ... \) (and also do similar matching for near-by fluctuation modes) [35]. In practice, what is known (at least in principle) is how to compute the dimensions in the different limit: by first expanding in \( \lambda \) and then expanding in \( \frac{1}{J} \). One may expect that this expansion of anomalous dimensions may take the form equivalent to (1.9), i.e.

\[
\Delta = J + \lambda(a_1 \frac{1}{J} + b_1 + ...) + \lambda^2(a_2 \frac{1}{J^2} + b_2 + ...) + ... ,
\]  

and that the respective coefficients in \( E \) and \( \Delta \) may agree with each other. The subsequent work [41, 42, 43, 44, 45, 46, 27, 47, 48, 49] did verify this structure of \( \Delta \) and, moreover, established the general agreement between the two leading coefficients \( c_1, c_2 \) in \( E \) (1.9) and the one-loop and two-loop coefficients \( a_1, a_2 \) in \( \Delta \) (1.11) (as usual, by “\( n \)-loop” term in \( \Delta \) we mean the term multiplied by \( \lambda^n \)).

To compute \( \Delta \) one is to diagonalize the anomalous dimension matrix defined on a set of “long” scalar operators, and this is obviously a non-trivial problem. The important step to this goal was made in [41] where it was observed that the one-loop planar dilatation operator in the scalar sector can be interpreted as a Hamiltonian of an integrable \( SO(6) \) spin chain and thus can be diagonalized even for large length \( L = J \) by the Bethe ansatz method.\(^2\) In the simplest case of the “\( SU(2) \)” sector of operators \( \text{tr}(\Phi_J^1 \Phi_J^2) + ... \) built out of two chiral scalars, the dilatation

\(^2\) Relations between 1-loop anomalous dimension matrix for a certain class of composite operators and integrable spin chain Hamiltonians were observed previously in the large \( N \) QCD context [63] (for a review and connections to AdS/CFT see [64]).
operator can be interpreted as “spin up” and “spin down” states of periodic XXX\textsubscript{1/2} spin chain with length \( L = J = J_1 + J_2 \). Then the 1-loop dilatation operator becomes equivalent to the Hamiltonian of the ferromagnetic Heisenberg model

\[ D_1 = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{J} (I - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}). \quad (1.12) \]

By considering the thermodynamic limit \((J \to \infty)\) of the corresponding Bethe ansatz equations, the proposal of [35] was confirmed at the leading order of expansion in \( \lambda \) [42, 43]: for eigen-operators with \( J_1 \sim J_2 \gg 1 \) it was shown (i) that \( \Delta - J = \lambda a_1 + ... \), and (ii) a remarkable agreement was found between \( a_1 = a_1(J_1(J_2)) \) and the coefficient \( c_1 \) in the energies (1.9) of various 2-spin string solutions. It was also possible also to match (as in the BMN case) the energies of fluctuations near the circular \( J_1 = J_2 \) solution with the corresponding eigenvalues of (1.12) [42].

Similar leading-order agreement between string energies and SYM dimensions was observed also in other sectors of states with large quantum numbers:

1. in the \( SU(3) \) sector: for specific solutions [35, 38, 39] with 3 spins in \( S^5 \) which are dual to the operators \( \text{tr}(\Phi_{1}^{J_1} \Phi_{2}^{J_2} \Phi_{3}^{J_3}) + ... \) [45, 50];

2. in the \( SL(2) \) [51] sector: for a folded string state [11] with one spin in \( AdS_5 \) and one spin in \( S^5 \) (with \( E = J + S + \lambda c_1(S) + ... \) [11, 35]) which is dual to the operators \( \text{tr}(D^5 \Phi^J) + ... \) [43];

3. in a “subsector” of \( SO(6) \) states containing pulsating (and rotating) solutions which again have regular [52] expansion of the energy in the limit of large oscillation number \( L \), i.e. \( E = L + c_1 \lambda + ... \) [42, 45].

This agreement between the leading-order terms in the expansion of energies of certain semiclassical string states and dimensions of the corresponding “long” SYM operators leaves, however, many questions, in particular:

(i) How to understand this agreement beyond specific examples, i.e. in a more universal way?

(ii) Which is the precise relation between profiles of string solutions and the structure of the dual SYM operators?

(iii) How to characterise the set of semiclassical string states and dual SYM operators to which this direct relation should apply?

(iv) Why the agreement holds at all, i.e. why the two limits (first \( J \to \infty \), and then \( \lambda \to 0 \), or vice versa) taken on the string and the SYM sides give equivalent results to the first two orders in expansion in \( \lambda \)? Why/when it does not work to all orders in expansion in \( \lambda \) (and \( \frac{1}{J} \))?
The questions (i),(ii) were addressed in [46, 48, 55, 56, 57] using the low-energy effective action approach for coherent states; an alternative approach based on matching the general solution (and the integrable structure) of the string sigma model with that of the thermodynamic limit of the Bethe ansatz in the $SU(2)$ sector was developed in [47]. The question (iii) was addressed in [15, 16, 17, 57], and the question (iv) – in [27, 28, 29]. Still, our understanding of why there is a direct agreement with gauge theory at the first two $\tilde{\lambda}$ and $\tilde{\lambda}^2$ orders of expansion and why it does not [43, 28] continue to the $\tilde{\lambda}^3$ order is still rather rudimentary.\footnote{Similar (dis)agreements were found for the $1/J$ corrections to the BMN states [14].}

Below we shall review the effective action approach as developed in [46, 48, 56, 57], concentrating mostly on the leading (“1-loop”) order in expansion in $\tilde{\lambda}$.

2. EFFECTIVE ACTIONS FOR COHERENT STATES

The suggestion of how to understand the agreement between leading-order terms in the multispin string energies and the corresponding one-loop anomalous dimensions in a universal way was made in [46] and was clarified and elaborated further in [48, 57]. The key idea was that instead of comparing particular solutions one should try to match effective sigma models which appear on the string side and the SYM side. Another related idea of [46, 48, 57] was that since the “semiclassical” string states carrying large quantum numbers are represented in the quantum theory by coherent states, one should be comparing coherent string states to coherent SYM states (and thus to coherent states of the spin chain). In view of the ferromagnetic nature of the dilatation operator (1.12), in the thermodynamic limit $J = J_1 + J_2 \to \infty$ with fixed large number of impurities (i.e. with fixed $J_2$) it is favorable to form large clusters of spins. Then a “low-energy” approximation and continuum limit should apply, leading to an effective “non-relativistic” sigma model for a coherent-state expectation value of the spin operator.

At the same time, on the string side, taking the “large space-time energy” (or large $J$) limit directly in the classical string action produces a reduced “non-relativistic” sigma model that describes in a universal way the leading-order $O(\tilde{\lambda})$ corrections to energies of all string solutions in the two-spin sector. The resulting sigma model action turns out to agree exactly [46] with the semiclassical coherent state action found from the $SU(2)$ sector of the spin chain in the $J \to \infty$, $\tilde{\lambda} =$ fixed limit. This demonstrates how a string action can directly “emerge” from a gauge
theory in the large-$N$ limit and provides a direct map between the “coherent” SYM states (or the corresponding operators built out of two holomorphic scalars) and all two-spin classical string states. Furthermore, the correspondence established at the level of the action implies also (i) the matching of the integrable structures and (ii) the matching of the fluctuations around particular solutions and thus it goes beyond the special examples of rigidly rotating strings.

2.1 COHERENT STATES

Let us briefly review the definition of coherent states (see, e.g., [53]). For a harmonic oscillator ($[a,a^\dagger] = 1$) one can define the coherent state $|u\rangle$ as $a|u\rangle = u|u\rangle$, where $u$ is a complex number. Equivalently, $|u\rangle = R(u)|0\rangle$, where $R = e^{ua^\dagger - u^*a}$ so that acting on the vacuum $|0\rangle$ the operator $R$ is simply proportional to $e^{ua^\dagger}$. Note that $|u\rangle$ can be written as a superposition of the eigenstates $|n\rangle$ of the harmonic oscillator Hamiltonian, $|u\rangle \sim \sum_{n=0}^{\infty} \frac{u^n}{\sqrt{n!}} |n\rangle$. An alternative definition of a coherent state is that it is a state with minimal uncertainty for both the coordinate $\hat{q} = \frac{1}{\sqrt{2}} (a + a^\dagger)$ and the momentum $\hat{p} = -\frac{i}{\sqrt{2}} (a - a^\dagger)$ operators, $\Delta \hat{q}^2 = \Delta \hat{p}^2 = \frac{1}{2}$, $\Delta \hat{p}^2 = \langle \hat{q}^2 \rangle - (\langle \hat{q}\hat{p} \rangle)^2$. For that reason it is the “best” approximation to a classical state. If one defines a time-dependent state $|u(t)\rangle = e^{-iHt}|u\rangle$ then the expectation values of $\hat{q}$ and $\hat{p}$, i.e. $\langle u|\hat{q}|u\rangle = \frac{1}{\sqrt{2}} (u + u^*)$, $\langle u|\hat{p}|u\rangle = -\frac{i}{\sqrt{2}} (u - u^*)$ follow the classical trajectories.

Starting instead of the Heisenberg algebra with the $SU(2)$ algebra $[S_3, S_{\pm}] = \pm S_{\pm}$, $[S_+, S_-] = 2S_3$ and considering the $s = 1/2$ representation where $S = \frac{1}{2} \hat{\sigma}$ one can define a spin coherent state as a linear superposition of spin up and spin down states: $|u\rangle = R(u)|0\rangle$. Here $R = e^{\alpha S_+ - \alpha^* S_-}$, $|0\rangle = |1/2, 1/2\rangle$ and $u$ is a complex number. An equivalent way to label the coherent state is by a unit 3-vector $\vec{n}$ defining a point of $S^2$. Then $|\vec{n}\rangle = R(\vec{n})|0\rangle$ where $|0\rangle$ corresponds to a 3-vector $(0, 0, 1)$ along the 3rd axis. One can write $\vec{n} = U^\dagger \hat{\sigma} U$, $U = (u_1, u_2)$, and then $R(\vec{n})$ is an $SO(3)$ rotation from a north pole to a generic point of $S^2$ defined by $\vec{n}$. The key property of the coherent state is that $\vec{n}$ determines the coherent state expectation value of the spin operator:

$$\langle \vec{n}|\vec{S}|\vec{n}\rangle = \frac{1}{2} \vec{n} \cdot \vec{n}.$$  

(2.1)

Similar definition of coherent states can be given in the case when $SU(2)$ is replaced by a generic group $G$. Given a semisimple group $G$ with the Cartan basis of its algebra $(H_i, E_\alpha, E_{-\alpha})$ $[[H_i, H_j]] = 0$, $[H_i, E_\alpha] = \alpha_i E_\alpha$, $[E_\alpha, E_{-\alpha}] = \alpha^i H_i$, $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$ whose interpretation will
be a symmetry group of a quantum Hamiltonian (acting in a unitary irreducible representation Λ on the Hilbert space $V_Λ$) one may define a set of coherent states by choosing a particular state $|0⟩$ (with $⟨0|0⟩ = 1$) in $V_Λ$ and acting on it by the elements of $G$. A subgroup $H$ of $G$ that leaves $|0⟩$ invariant up to a phase $(Λ(h)|0⟩ = e^{iφ(h)}|0⟩)$ is called maximum stability subgroup. One may then define the coset space $G/H$, the elements of which $(g = ωh, h ∈ H, ω ∈ G/H, Λ(g) = Λ(ω)Λ(h))$ will parametrize the coherent states, $|ω, Λ⟩ = Λ(ω)|0⟩$.

This definition depends on a choice of group $G$, its representation $Λ$ and the vector $|0⟩$. It is natural to assume also that $|0⟩$ is an eigenstate of the Hamiltonian $H$, e.g., a ground state. For a unitary representation $Λ$ we may choose $H^†_i = H_i, E^†_α = E_−α$ and select $|0⟩$ to be the highest-weight vector of the representation $Λ$, i.e. demand that it is annihilated by “raising” generators and is an eigen-state of the Cartan generators: (i) $E_α|0⟩ = 0$ for all positive roots $α$; (ii) $H_i|0⟩ = h_i|0⟩$. In addition, we may demand that $|0⟩$ is annihilated also by some “lowering” generators, i.e. (iii) $E_−β|0⟩ = 0$ for some negative roots $β$; the remaining negative roots will be denoted by $γ$. Then the coherent states are given by

$$|ω, Λ⟩ = \exp[∑_γ (w_γ E_−γ − w^*_γ E_γ)] |0⟩ ,$$

where $γ$ are the negative roots for which $E_γ|0⟩ \neq 0$. $w_γ$ may be interpreted as coordinates on $G/H$ where $H$ is generated by $(H_1, E_α, E_−α)$.

For example, in the case of $G = SU(3)$ with the Cartan basis $(H_1, H_2, E_α, E_β, E_α+β, E_−α, E_−β, E_−α−β)$ and with $|0⟩$ being the highest-weight of the fundamental representation, i.e. $E_−β|0⟩ = 0, E_−α|0⟩ \neq 0, E_−α−β|0⟩ \neq 0$, the subgroup $H$ is generated by $(H_1, H_2, E_β, E_−β)$, i.e. is $SU(2) × U(1)$ and $G/H = SU(3)/(SU(2) × U(1)) = CP^2$ (see also [56]). We shall apply this general definition of coherent states in section 3.

2.2 LANDAU-LIFSHITZ MODEL FROM SPIN CHAIN

In general, one can rewrite the usual phase space path integral as an integral over the overcomplete set of coherent states (for the harmonic oscillator this is simply a change of variables from $q, p$ to $u = \frac{1}{\sqrt{2}}(q + ip)$):

$$Z = \int [du] \ e^{iS[u]} , \quad S = \int dt \left( ⟨u|i\frac{d}{dt}|u⟩ - ⟨u|H|u⟩ \right).$$

The first (“Wess-Zumino” or “Berry phase”) term in the action $\sim iu^α \frac{d}{dt}u$ is the analog of the usual $pq$ term in the phase-space action. Applying this to the case of the Heisenberg spin chain Hamiltonian (1.12) one ends
up with with the following action for the coherent state variables \( \vec{n}_l(t) \) at sites \( l = 1, \ldots, J \) (see, e.g., [54]):

\[
S = \int dt \sum_{l=1}^J \left[ \vec{C}(n_l) \cdot \vec{n}_l - \frac{\lambda}{2(4\pi)^2} (\vec{n}_{l+1} - \vec{n}_l)^2 \right].
\]  

(2.4)

Here \( dC = \epsilon^{ijk} dn_j \wedge dn_k \) (i.e. \( \vec{C} \) is a monopole potential on \( S^2 \)). In local coordinates (at each site \( l \)) one has \( \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), \( \vec{C} \cdot d\vec{n} = \frac{1}{2} \cos \theta d\phi \). In the limit \( J \to \infty \) with fixed \( \tilde{\lambda} = \frac{\lambda}{J^2} \) (which we are interested in) we can take a continuum limit by introducing the 2-d field \( \vec{n}(t,\sigma) = \{\vec{n}(t,\frac{2\pi}{J} l), l = 1, \ldots, J \}. \) Then

\[
S = J \int dt \int_0^{2\pi} d\sigma \left[ \vec{C} \cdot \partial_t \vec{n} - \frac{1}{8} \tilde{\lambda}(\partial_\sigma \vec{n})^2 + \ldots \right],
\]  

(2.5)

where dots stand for higher derivative terms suppressed by \( \frac{1}{J} \). Since \( J \) appears as a factor in front of the action, in the limit \( J \to \infty \) all quantum corrections should be also suppressed by \( \frac{1}{J} \), and thus the above action can be treated classically. The corresponding equations of motion

\[
\partial_t n_i = \frac{1}{2} \tilde{\lambda} \epsilon_{ijk} n_j \partial^2 n_k
\]  

(2.6)

are the Landau-Lifshitz equations for a classical ferromagnet. An alternative derivation of them is based on first writing down the Heisenberg equation for the time evolution of the spin operator directly from the spin chain Hamiltonian, then considering the coherent state expectation value and finally taking the continuum limit.

2.3 LANDAU-LIFSHITZ MODEL FROM STRING ACTION

The action (2.5) should be describing the coherent states of the Heisenberg spin chain in the above thermodynamic limit. One may wonder how a similar “non-relativistic” action may appear on the string side where one starts with the usual “relativistic” sigma model (1.1). To obtain such an effective action one is to perform the following sequence of steps [46, 48, 57]:

(i) isolate a “fast” coordinate \( \alpha \) whose momentum \( p_\alpha \) is large for a class of string configurations we consider;
(ii) gauge-fix \( t \sim \tau \) and \( p_\alpha \sim J \) (or \( \tilde{\alpha} \sim \sigma \) where \( \tilde{\alpha} \) is “T-dual” to \( \alpha \));
(iii) expand the action in derivatives of “slow” or “transverse” coordinates (to be identified with \( \vec{n} \)).

To illustrate this procedure let us consider the \( SU(2) \) sector of string states carrying two large spins in \( S^5 \), with string motions restricted
to $S^3$ part of $S^5$. The relevant part of the $AdS_5 \times S^5$ metric is then $ds^2 = -dt^2 + dX_idX_i^*$, with $X_iX_i^* = 1$. Let us set

$$X_1 = X_1 + iX_2 = u_1e^{i\alpha}, \quad X_2 = X_3 + iX_4 = u_2e^{i\alpha}, \quad u_iu_i^* = 1.$$ 

Here $\alpha$ will be a coordinate associated to the total spin in the two planes (which in general will be the sum of orbital and internal spin). $u_i$ (defined modulo $U(1)$ gauge transformations) will be the “slow” coordinates determining the “transverse” string profile. Then

$$dX_idX_i^* = (d\alpha + C)^2 + Du_iDu_i^*, \quad C = -iu_i^*du_i, \quad Du_i = du_i - iCu_i,$$

where the second $|Du_i|^2$ term represent the metric of $CP^1$ (this parametrisation corresponds to the Hopf fibration $S^3 \sim S^1 \times S^2$). Introducing $\vec{n} = U^\dagger\vec{\sigma}U$, $U = (u_1, u_2)$ we get

$$dX_idX_i^* = (D\alpha)^2 + \frac{1}{4}(d\vec{n})^2, \quad D\alpha = d\alpha + C(n). \quad (2.7)$$

Writing the resulting sigma model action in phase space form and imposing the (non-conformal) gauge $t = \tau$, $p_\alpha = \text{const} = J$, one gets [48] the same action $(2.5)$ with the WZ term $\vec{C} \cdot \partial_t \vec{n}$ originating from the $p_\alpha D\alpha$ term in the phase-space Lagrangian (cf. its origin on the spin chain side as an analog of the $pq$ in the coherent state path integral action).

An equivalent approach [57] leading to the same action $(2.5)$ is based on first applying a 2-d duality (or “T-duality”) transformation $\alpha \to \tilde{\alpha}$ and then choosing the “static” gauge $t = \tau$, $\tilde{\alpha} = \frac{1}{\sqrt{\lambda}}\sigma$ with $\sqrt{\lambda} \equiv J/\sqrt{\lambda}$. Indeed, starting with

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{pq}(-\partial_p\partial_qt + D_p\alpha D_q\alpha + D_pu_i^*D_qu_i) \quad (2.8)$$

and applying the 2-d duality in $\alpha$ we get

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{pq}(-\partial_p\partial_qt + \partial_p\tilde{\alpha}\partial_q\tilde{\alpha} + D_pu_i^*D_qu_i) + \epsilon^{pq}C_p\partial_q\tilde{\alpha}. \quad (2.9)$$

Thus the “T-dual” background has no off-diagonal metric component but has a non-trivial NS-NS 2-form coupling in the $(\tilde{\alpha}, u_i)$ sector. It is useful not to use conformal gauge here. Eliminating the 2-d metric $g^{pq}$ we then get the Nambu-type action

$$\mathcal{L} = \epsilon^{pq}C_p\partial_q\tilde{\alpha} - \sqrt{h}, \quad (2.10)$$

where $h = |\text{det} h_{pq}|$ and $h_{pq} = -\partial_p\partial_qt + \partial_p\tilde{\alpha}\partial_q\tilde{\alpha} + D_pu_i^*D_qu_i$. If we now fix the static gauge, $t = \tau$, $\tilde{\alpha} = \frac{1}{\sqrt{\lambda}}\sigma$, we finish with the action
\[ I = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{L}, \text{ where} \]
\[ \mathcal{L} = C_t - \sqrt{(1 + \lambda|D_\sigma u_i|^2)(1 - |D_t u_i|^2) + \frac{1}{4}\lambda(D_t u_i^* D_\sigma u_i + \text{c.c.})^2}. \]  
(2.11)

Making the key assumption that the evolution of \( u_i \) in \( t \) is slow, i.e. the time derivatives are suppressed (which can be implemented by rescaling \( t \) by \( \tilde{\lambda} \) and expanding in powers of \( \tilde{\lambda} \)), we find, to the leading order in \( \tilde{\lambda} \),
\[ \mathcal{L} = -i u_i^* \partial_t u_i - \frac{1}{2}\lambda|D_\sigma u_i|^2. \]  
(2.12)

This is the same as the \( CP^1 \) Landau-Lifshitz action (2.5) when written in terms of \( \bar{n} \). Thus the string-theory counterpart of the WZ term in the spin-chain coherent state effective action originates from the 2-d NS-NS WZ term in the action for the “T-dual” coordinate \( \tilde{\alpha} \) upon the static gauge fixing of the latter [57].

To summarize:

(i) \((t, \tilde{\alpha})\) are the “longitudinal” coordinates that are gauge-fixed (with \( \tilde{\alpha} \) playing the role of the string direction or the spin chain direction on the SYM side);

(ii) \(U = (u_1, u_2)\) or \(\bar{n} = U^\dagger \bar{\sigma} U\) are the “transverse” coordinates that determine the semiclassical string profile and also the structure of the coherent operator on the SYM side, \(\text{tr}(\Pi_\sigma u_i(\sigma)\Phi_i)\) (see [46, 57] and below).

The agreement between the low-energy effective actions on the spin chain and the string side explains not only the matching of energies of coherent states representing configurations with two large spins (and also the matching of near-by fluctuations) but also the equivalence of the integrable structures (which was observed on specific examples in [44, 45]).

### 2.4 HIGHER ORDERS IN \( \lambda \)

The above leading-order agreement in \( SU(2) \) sector has several generalizations.

First, we may include higher-order terms on the string side. Expanding (2.11) in \( \tilde{\lambda} \) and eliminating higher powers of time derivatives by field redefinitions (which can be done since the leading-order equation of motion is first order in time derivative) we end up with [48] \((n' \equiv \partial_\sigma n)\)

\[
\mathcal{L} = \bar{C} \cdot \tilde{n} - \frac{\lambda^2}{8} \tilde{n}^2 + \frac{\lambda^2}{32} (\tilde{n}^2 - \frac{3}{4} \tilde{n}^4) \\
- \frac{\tilde{\lambda}^3}{64} |\tilde{n}|^2 \tilde{n}^2 \tilde{n}^2 \tilde{n}^2 - \frac{7}{4} \tilde{n}^2 \tilde{n}^2 \tilde{n}^2 \tilde{n}^2 - \frac{25}{2} (\tilde{n}^2 \tilde{n}^2 \tilde{n}^2 \tilde{n}^2)^2 + \frac{13}{16} \tilde{n}^6 \tilde{n}^6 \tilde{n}^6 \tilde{n}^6 \tilde{n}^6 + O(\tilde{\lambda}^4). \]  
(2.13)
The same $\tilde{\lambda}^2$ term is obtained [48] in the coherent state action on the spin chain side by starting with the sum of the 1-loop dilatation operator (1.12) and the 2-loop term found in [20]

$$D_2 = \frac{2\lambda^2}{(4\pi)^2} \sum_{l=1}^{J}(Q_{l,l+2} - 4Q_{l,l+1}) , \quad Q_{k,l} \equiv \frac{1}{2}(I - \vec{\sigma}_k \cdot \vec{\sigma}_l) . \quad (2.14)$$

The $\tilde{\eta}^4$ term originates from a non-trivial quantum correction on the spin chain side. This explains the matching of energies and dimensions to the first two orders in $\tilde{\lambda}$, observed on specific examples (using the generalized Bethe ansatz on the spin chain side) in [27]. The equivalent general conclusion about 2-loop matching was obtained in the integrability-based approach in [47].

The order-by-order agreement seems to break down at $\tilde{\lambda}^3$ term. A natural explanation [27, 28] is that the string limit (first $J \to \infty$, then $\tilde{\lambda} \to 0$) and the SYM limit (first $\lambda \to 0$, then $J \to \infty$) need not produce the same result when applied to the 2-parameter functions $E = \Delta(\lambda, J)$. A proposal of how to “complete” the perturbative gauge-theory expression to make the agreement with string theory manifest appeared in [28]; ref. [29] also suggested a possible form of generalized Bethe ansatz on the string side that would naturally interpolate between the string and gauge theory results (see also [62]). These proposals also suggest a resolution of the order $\tilde{\lambda}^3$ disagreement [14] between the string theory and gauge theory predictions for $\frac{1}{J}$ corrections to the BMN spectrum.

A possible explanation of why we found the agreement of $\tilde{\lambda}$ and $\tilde{\lambda}^2$ terms is that the structure of the dilatation operator at one and two loop orders is, in a sense, fixed by the BMN limit, which thus essentially determines the low energy effective action in a unique way. This is no longer so starting with the 3-loop order, where the dilatation operator already contains [20] the 4-spin $QQ$ interactions (cf. (2.14)) which do not contribute to anomalous dimensions in the strict BMN limit.

By analogy with non-renormalization (due to underlying supersymmetry) of few leading terms in low-energy effective actions, one may suggest that here the 3-loop problem may be related to the appearance of non-trivial interpolating functions as coefficients of the $Q^n$ ($n \geq 2$) terms in the dilatation operator. This would, in particular, explain why different structures of the $QQ$ terms are found in the “gauge theory” [20] and “string theory” [62] limits.

As was suggested in [27, 28], the disagreement between the string and gauge theory results at 3-loop order in $\lambda$ and the leading order in $J$ can be repaired by adding “wrapping” contributions to the dilatation operator (and thus to the Bethe ansatz relations) on the gauge theory side. To
illustrate this possibility, let us consider the circular solution case \([35]\), and use the function \(\frac{\lambda^J}{(1+\lambda^J)}\) \([28]\), which is equal to 1 in the string theory limit \((J \to \infty \text{ with fixed } \frac{\lambda}{J^2} \equiv \tilde{\lambda})\) but zero in the perturbative gauge theory limit, in order to interpolate between the different \(\frac{\lambda^3}{J^5}\) coefficients as follows:

\[
\Delta = J + \frac{\lambda}{2J} - \frac{\lambda^2}{8J^3} + \frac{\lambda^3}{16J^5} \frac{\lambda J^{-3}}{(1+\lambda) J^{-3}} + \ldots .
\]  

(2.15)

This expression agrees with both the string result \((E = \sqrt{J^2 + \lambda}\) in (1.8) with \(n = 1)\) and the perturbative gauge theory result \((\Delta_{\text{pert}} = J + \frac{\lambda}{2J} - \frac{\lambda^2}{8J^3} + 0 \times \lambda^3 + \ldots \) \([27]\)).

Let us add few details about the coherent-state expectation value of higher-loop SYM dilatation operator. This expectation value appears in the action in the coherent state path integral (2.3) of the quantum spin chain. Written in terms of independent permutations or \(Q_{l,k} \equiv I - P_{l,k} = \frac{1}{2}(1 - \sigma_l \cdot \sigma_k)\) the “\(r\)-loop” term in the planar dilatation operator is expected \([20, 61]\) to contain \(Q\) in maximal power \(\left[\frac{r+1}{2}\right]\), i.e. \(D_1, D_2 \sim \sum Q, \ D_3, D_4 \sim \sum Q + \sum QQ, \) etc. Explicitly, \(D = \sum_{r=0}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} D_r\), \(D_r = \sum_{l=1}^{J} D_r(l)\),

(2.16)

where as in (1.12) \([41]\) and (2.14) \([20]\)

\[D_0 = I, \quad D_1 = 2Q_{l,l+1}, \quad D_2 = -2(4Q_{l,l+1} - Q_{l,l+2}) . \]

(2.17)

The 3-loop term \([20, 61]\) is a special case of a 2-parameter family \([21, 62]\)

\[D_3(\alpha, c) = 4(15Q_{l,l+1} - 6Q_{l,l+2} + Q_{l,l+3}) + b_1Q_{l,l+2}Q_{l+1,l+3} + b_2Q_{l,l+3}Q_{l+1,l+2} + b_3Q_{l,l+1}Q_{l+2,l+3} \]

(2.18)

\[b_1 = 4 + 2c - \alpha , \quad b_2 = -4 - 4c + \alpha , \quad b_3 = -2c + 5\alpha . \]

(2.19)

The choice of \(c = 0, \ \alpha = 0\) corresponds to the 3-loop gauge theory operator of \([20]\) whose form is fixed \([61]\) by the superconformal algebra and the structure of Feynman diagrams (and the BMN scaling). This choice is also consistent with integrability of the spin chain. To preserve integrability \([21]\) one should set \(\alpha = 0\) (this is the parameter \(\alpha_2\) of \([21]\)), while to have consistency with the gauge-theory perturbative expansion \([62]\) one should set \(c = 0\) \((c \equiv c_4\) in \([62]\)). The case of \(\alpha = 2, \ c = 0\) corresponds to the operator mentioned in \([21]\) which seemed to agree...
with some string-theory results. The case of $\alpha = 0$, $c = 1$ is the “string" operator [62] which should correspond to the “string" modification of the Bethe ansatz equations in [29].

Starting with an $SU(2)$ coherent state satisfying (2.1) for which
\[
\langle n | Q_{l,k} | n \rangle = \frac{1}{2} (1 - n_l \cdot n_k) = \frac{1}{4} (n_l - n_k)^2 ,
\]
computing $\langle n | D | n \rangle$ and then taking the continuum limit (by introducing a spatial coordinate $0 < \sigma \leq 2\pi$, and $n(\sigma) = \frac{2\pi l}{J} \partial_{\sigma} n + \ldots$, etc.) we find, using Taylor expansion and dropping a total derivative over $\sigma$, see [48],
\[
\lambda (4\pi)^2 \langle n | D_1 | n \rangle \rightarrow \frac{\lambda}{8} \left[ \vec{n}''^2 + O(\frac{1}{J^2} \partial^4 \vec{n}) \right] , \quad \tilde{\lambda} = \frac{\lambda}{J^2} ,
\]
\[
\lambda^2 (4\pi)^4 \langle n | D_2 | n \rangle \rightarrow -\frac{\tilde{\lambda}^2}{32} \left[ \vec{n}''''^2 + O(\frac{1}{J^2} \partial^6 \vec{n}) \right] ,
\]
\[
\lambda^3 (4\pi)^6 \langle n | D_3 | n \rangle \rightarrow \frac{\tilde{\lambda}^3}{64} \left[ k_1 \frac{J^2}{(2\pi)^2} \vec{n}^4 + k_2 \vec{n}''^2 \vec{n}'''^2 + k_3 (\vec{n}''''^2)^2 + O(\frac{1}{J^2} \partial^8 \vec{n}) \right] .
\]

Here
\[
k_1 = \frac{1}{16} (16b_1 + 9b_2 + b_3) = \frac{1}{8} (14 - 3c - \alpha)
\]
\[
k_2 = -\frac{1}{96} (64b_1 + 45b_2 + b_3) = -\frac{1}{48} (38 - 27c - 7\alpha) ,
\]
\[
k_3 = -\frac{5}{48} (32b_1 + 9b_2 + 5b_3) = -\frac{5}{24} (46 + 9c + \alpha) .
\]

Note a relation: $k_1 + k_2 + \frac{1}{10} k_3 = 0$. The problematic scaling-violating term $J^2 \partial^4 (\vec{n}^2)^4$ in (2.23) does not cancel automatically; it should cancel after one takes into account quantum corrections (which survive the continuum limit beyond the order $\tilde{\lambda}$ approximation [48]). Quantum corrections are expected also to be important in order to demonstrate the equivalence of the spin-chain result (for the “string" choice of $c = 1$, $\alpha = 0$) with the string-theory result in (2.13). Verifying this equivalence beyond the quadratic $\vec{n}^2$-terms (which obviously agree) remains an open problem.

Let us also mention that one can sum up all terms in the string effective Hamiltonian that are of second order in $\vec{n}$ but to arbitrary order in
σ-derivatives \([48]\)

\[
\mathcal{L} = \bar{C} \cdot \dot{n} - \frac{1}{4} \bar{n} \left( \sqrt{1 - \lambda \partial^2} - 1 \right) \bar{n} + O(\bar{n}^3) .
\]  

(2.25)

This expression is in agreement with the leading-order results (2.13) and with the exact BMN spectrum (see also \([60]\)). The coherent analogs of BMN states correspond to small fluctuations near the vacuum state \(\bar{n}_0 = (0, 0, 1)\). On the spin chain side these correspond (in discrete version) to the microscopic spin wave excitations or magnons.

2.5 OTHER SECTORS

It is possible to extend the approach of \([46, 48]\) to other sectors of rotating string states \([55, 56, 57]\). First, one is to identify subsectors of operators of the SYM theory which are closed under renormalization at least to one-loop. Such bosonic subsectors are:

(i) the three-spin “SU(3)” sector of string configurations with all three \(S^5\) angular momenta \((J_1, J_2, J_3)\) being non-zero. They are dual to chiral operators \(\text{tr}(\Phi^{J_1}_1 \Phi^{J_2}_2 \Phi^{J_3}_3) + ...\). These form a set closed only under one-loop renormalization \([20]\), but in the limit when \(L = J_1 + J_2 + J_3\) is large one can treat this sector as closed even beyond one loop (mixings with fermionic operators are suppressed by \(1/L\) \([59]\)).

(ii) the “SO(6)” sector of generic (pulsating and rotating) string motions in \(S^5\) which are dual to operators built out of 6 real scalars. Examples of pulsating string states were considered in \([52, 45]\) and more generally in \([57]\); this sector will be discussed in the next section. As was pointed out in \([59]\), one can also consider \(SO(3)\) and \(SO(4)\) subsectors of the \(SO(6)\) sector which are again closed modulo \(1/L\) corrections.

(iii) the two-spin “SL(2)” sector of string configurations with one \(AdS_5\) spin \(S = S_1\) and one \(S^5\) angular momentum \(J = J_3\), which are dual to operators \(\text{tr}(D^{S_1}_{J_1+J_2} \Phi^J) + ...\) (forming a set closed under renormalization to all orders \([51, 64]\)).

(iv) the three-spin “SU(1, 2)” sector of string configurations with two \(AdS_5\) angular momenta \(S_1, S_2\) and one \(S^5\) spin \(J = J_3\), dual to operators \(\text{tr}(D^{S_1}_{J_1+J_2} D^{S_2}_{J_3+J_4} \Phi^J) + ...\) which form a set closed under one-loop renormalization.

Operators carrying more general combinations of non-zero spins from the list \((S_1, S_2, J_1, J_2, J_3)\) mix with fermionic and field-strength operators already at one loop and would require to consider the full superspin chains \([51, 65]\); on the string side one would then need to include fermions of the GS action \([18]\). It may happen that one can again isolate some more general subsectors closed modulo \(1/L\) corrections, but it appears that in order to apply a derivation of the reduced sigma model action
on the string side one would need to impose additional constraints on the form of string configurations [56].

It is indeed straightforward to generalize [55, 56] the leading-order agreement observed in the $SU(2)$ sector to the $SU(3)$ sector of states with three large $S^5$ spins $J_i$, $i = 1, 2, 3$, finding the $CP^2$ analog of the $CP^1$ Landau-Lifshitz Lagrangian in (2.5),(2.12) [56] $\mathcal{L} = -iu_i^* \partial_0 u_i - \frac{1}{2} \bar{\lambda} |D_1 u_i|^2$ on both the string and the spin chain sides.

Similar conclusion is reached [56] (see also [69]) in the $SL(2)$ sector of $(S,J)$ states dual to operators like $\text{Tr}(D^S \Phi J) + \ldots$. Like in the $SU(2)$ sector here the one-loop dilatation operator $D_1$ may be interpreted as the $XXX_{-1/2}$ Heisenberg spin chain Hamiltonian [51]. The corresponding coherent states (related to the $SU(2)$ ones by an “analytic continuation” from the 2-sphere to the 2-hyperboloid, $\vec{n} \rightarrow \vec{\ell}$, $\eta^{ij}\ell_i\ell_j = -\ell_1^2 + \ell_2^2 + \ell_3^2 = -1$) are defined so that for the $SL(2)$ generators $S_i$ one has $\langle \ell | S_i | \ell \rangle = -\frac{1}{2}\ell_i$ and then [56]

$$\langle \ell | (D_1)_{SL(2)} | \ell \rangle \rightarrow \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{J} \ln \left[ \frac{1}{2} (1 - \eta_{ij} \ell_i^l \ell_j^{l+1}) \right]$$

$$= \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{J} \ln \left[ 1 + \frac{1}{4} \eta_{ij} (\ell_i^l - \ell_i^{l+1})(\ell_j^l - \ell_j^{l+1}) \right].$$

(2.26)

Note that this is the direct $(- + +)$ signature analog on the classically integrable lattice Hamiltonian for the Heisenberg magnetic [58]. Since we are interested in comparing to the semiclassical string case, $S$ as well as $J$ should be large, and, in view of the ferromagnetic nature of the spin chain, this effectively amounts to a low-energy semiclassical limit of the chain. Considering the limit $J \rightarrow \infty$ with fixed $\bar{\lambda} = \frac{\lambda}{J^2}$ we get, as in the $SU(2)$ case, $\ell(\sigma_l) = \ell(\frac{2\pi l}{J}) \equiv \ell_l$ and $\ell_{l+1} - \ell_l = \frac{2\pi}{J^2} \partial_\sigma \ell + \mathcal{O}(\frac{1}{J^4} \partial^4 \ell)$. To the one loop order, i.e. with only one power of $\lambda$ in (2.26), in expanding the logarithm we need to keep only the order $\frac{1}{J^2}$ term, i.e. the term quadratic in first derivatives. This leads to

$$\langle \ell | (D_1)_{SL(2)} | \ell \rangle \rightarrow J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ \frac{\bar{\lambda}}{8} \eta_{ij} \ell_i^l \ell_j^l + \mathcal{O}(\frac{1}{J^2} \partial^4 \ell) \right].$$

(2.27)

As in the $SU(2)$ case, the same term in the action (containing also the WZ term) is found on the string theory side. This implies the general agreement between the string and SYM theory predictions for the energies/dimensions at leading order in $\lambda$ in the $SL(2)$ sector and thus generalizes the previous results [43] found for particular solutions.
3. GENERAL FAST MOTION IN $S^5$ AND $SO(6)$ SCALAR OPERATORS

One would like to try to understand the general conditions on string states and SYM operators for which the above correspondence works, and incorporate also states with large oscillation numbers. Here we will follow [56, 57] (a closely related approach was developed in [16, 17, 18]). For strings moving in $S^5$ with large oscillation number the energy is $E = L + c_1 \tfrac{\lambda}{L} + \ldots$, i.e. it is again regular in the limit $L \to \infty$, $\tilde{\lambda} = \tfrac{\lambda}{L^2} \to 0$ [52]. The leading-order duality relation between string energies and anomalous dimensions in this case was checked in [42, 45, 59]. The general condition on string solutions for which $E/L = f(\tilde{\lambda})$ has a regular expansion in $\tilde{\lambda}$ appears to be that the world sheet metric should degenerate [16] in the $\tilde{\lambda} \to 0$ limit, i.e. the string motion should be ultra-relativistic in the small effective string tension limit (in the strict tensionless limit the corresponding states become BPS [15]).

For example, in the conformal gauge the 2-d induced metric in general scales as $g_{00} = -\kappa^2 + \ldots$ (assuming $t = \kappa \tau$, etc.), or, after a rescaling of the 2-d time coordinate, $g_{00} = -1 + O(\tilde{\lambda}) + \ldots$, where we used that $\kappa = \frac{1}{\sqrt{\tilde{\lambda}}}$. For the fast-moving strings, the leading $O(1)$ term in the metric gets cancelled out, and thus the metric degenerates in the $\tilde{\lambda} \to 0$ limit.

In the strict tensionless $\tilde{\lambda} \to 0$ limit each string piece is following a geodesic (big circle) in $S^5$, while switching on tension leads to a slight deviation from geodesic flow, i.e. to a nearly-null world surface [16]. The dual coherent SYM operators are then “locally BPS”, i.e. each string bit corresponds to a BPS linear combination of 6 scalars (see below). In general, the scalar operators can be written as

$$\mathcal{O} = C_{m_1\ldots m_L} \text{tr}(\phi_{m_1} \ldots \phi_{m_L}) \,. \quad (3.1)$$

The planar 1-loop dilatation operator $D_1$ acting on $C_{m_1\ldots m_L}$ was found in [41] to be equivalent to an integrable $SO(6)$ spin chain Hamiltonian

$$H_{m_1\ldots m_L}^{n_1\ldots n_L} = \frac{\lambda}{(4\pi)^2} \sum_{i=1}^L \left( \delta_{m_i m_{i+1}} \delta^{n_i n_{i+1}} + 4 \delta_{[m_i}^{n_i} \delta_{m_{i+1}]^{n_{i+1}}} \right) \,. \quad (3.2)$$

To find the analog of the coherent-state action (2.12) we choose a natural set of coherent states $\Pi_i |v_i\rangle$, where at each site $|v\rangle = R(v)|0\rangle$. Here $R$ is an $G = SO(6)$ rotation and $|0\rangle$ is the BPS vacuum state corresponding to $\text{tr}(\phi_1 + i\phi_2)L$ or $v_{(0)} = (1, i, 0, 0, 0, 0)$, which is invariant under the subgroup $H = SO(2) \times SO(4)$. Then the rotation $R(v)$ and thus the
coherent state is parametrized by a point in

\[ G/H = SO(6)/[SO(4) \times SO(2)] , \]

i.e. \( v \) belongs to the Grassmanian \( G_{2,6} \) \[ [56] \]. \( G_{2,6} \) is thus the coherent state target space for the spin chain sigma model since it parametrizes the orbits of the half-BPS operator \( \phi_1 + i \phi_2 \) under the \( SO(6) \) rotations. This is the space of 2-planes passing through zero in \( R^6 \), or the space of big circles in \( S^5 \), i.e. the moduli space of geodesics in \( S^5 \) \[ [17] \]. It can be represented also as an 8-dimensional quadric in \( CP^5 \): a complex 6-vector \( v_m \) should be subject, in addition to \( v_m v_m^* = 1 \) (and gauging away the common phase) also to \( v_m v_m = 0 \) condition. Taking the limit \( L \to \infty \) with fixed \( \tilde{\lambda} = \lambda L^2 \) and the continuum limit \( v_m(t) \to v_m(t, \sigma) \) we then get the \( G_{2,6} \) analog of the \( CP^2 \) action \( (2.5),(2.12) \)

\[ S = L \int dt \int_0^{2\pi} d\sigma \left( -iv_m^* \partial_t v_m - \frac{1}{2} \tilde{\lambda} |D\sigma v_m|^2 \right) , \quad (3.3) \]

\[ v_m v_m^* = 1 , \quad v_m v_m = 0 , \quad D\sigma v_m = \partial_\sigma v_m - (v^* \partial_\sigma v) v_m . \]

This is also a generalization of the \( CP^2 \) action found in the \( SU(3) \) sector \[ [56] \].

One may wonder how this 8-dimensional sigma model may be related to the string sigma model on \( R \times S^5 \) where the coordinate space of transverse motions is only 4-dimensional. The crucial point is that the coherent state action is defined on the phase space (cf. the harmonic oscillator case in sect. 2.1), and \( 8 = (1 + 5) \times 2 - 2 \times 2 \) is indeed the phase space dimension of a string moving in \( R_t \times S^5 \).

On the string side, the need to use the phase space description is related to the fact that to isolate a “fast” coordinate \( \alpha \) for a generic string motion we need to specify both the position and velocity of each string piece. Given \( L = - (\partial t)^2 + (\partial X_m)^2 \) in conformal gauge \( (\dot{X} \dot{X}' = 0, \ X^2 + X'^2 = \kappa^2, \ X_m^2 = 1) \) we find that the point-like trajectories (geodesics) are described by

\[ X_m = a_m \cos \alpha + b_m \sin \alpha , \quad \alpha = \kappa \tau , \quad a_m^2 = 1, \quad b_m^2 = 1, \quad a_m b_m = 0 . \]

Equivalently,

\[ X_m = \frac{1}{\sqrt{2}}(e^{i\alpha} v_m + e^{-i\alpha} v_m^*), \quad v_m = \frac{1}{\sqrt{2}}(a_m - ib_m), \quad |v|^2 = 1, \quad v^2 = 0 , \]

where the constant vector \( v_m \) thus belongs to \( G_{2,6} \). In general, for near-relativistic string motions \( v_m \) should change slowly with \( \tau \) and \( \sigma \). Then
starting with the phase space Lagrangian for \((X_m, p_m)\)
\[
\mathcal{L} = p_m \dot{X}_m - \frac{1}{2} p_m p_m - \frac{1}{2} X'_m X'_m ,
\]
we may change the variables according to [57] (cf. again the harmonic oscillator case)
\[
X_m = \frac{1}{\sqrt{2}} (e^{i\alpha} v_m + e^{-i\alpha} v_m^*), \quad p_m = \frac{i}{\sqrt{2}} p_\alpha (e^{i\alpha} v_m - e^{-i\alpha} v_m^*) ,
\]
where \(\alpha\) and \(v_m\) now depend on \(\tau\) and \(\sigma\) and \(v_m\) again belongs to \(G_{2,6}\).

There is an obvious \(U(1)\) gauge invariance, \(\alpha \to \alpha - \beta, v_m \to e^{i\beta} v_m\). Gauge-fixing the 2-d reparametrizations by \(t \sim \tau, p_\alpha \sim L\) (or, doing an approximate T-duality \(\alpha \to \tilde{\alpha}\) and setting \(\tilde{\alpha} \sim \sigma\) as in sect. 2.3) one finds, after an additional rescaling of the time coordinate, that the phase-space Lagrangian becomes [57]:
\[
\mathcal{L} = p_\alpha D_t \alpha - \frac{1}{2} \tilde{\lambda} |D_\sigma v|^2 - \frac{1}{4} \tilde{\lambda} |e^{2i\alpha} (D_\sigma v)^2 + \text{c.c.}| .
\]

The first term here produces the WZ term \(-iv_m^* \partial_t v_m\) and the last one averages to zero since \(\alpha \approx \kappa \tau + \ldots\), where \(\kappa = (\sqrt{\lambda})^{-1} \to \infty\).

Equivalently, the \(\alpha\)-dependent terms in the action (that were absent in the \(SU(2)\) or \(SU(3)\) sectors) can be eliminated by canonical transformations [57]. We then end up with the following 8-dimensional phase-space Lagrangian for the “transverse” string motions, \(\mathcal{L} = -iv_m^* \partial_t v_m - \frac{1}{4} \tilde{\lambda} |D_\sigma v_m|^2\), which is the same as found on the spin chain side (3.3). The 3-spin \(SU(3)\) case is the special case when \(v_m = (u_1, iu_1, u_2, iu_2, u_3, iu_3)\), where \(u_i\) belongs to the \(CP^2\) subspace of \(G_{2,6}\). The agreement between the spin chain and the string sides in this general \(G_{2,6} = SO(6)/[SO(4) \times SO(2)]\) case explains not only the matching for pulsating solutions [52, 45] but also for near-by fluctuations.

Let us now discuss the reason for the restriction \(v^2 = 0\) on the spin chain side and also clarify the structure of the coherent operators corresponding to semiclassical string states. Given the scalar operator \(\mathcal{O} = C_{m_1 \ldots m_L} \text{tr} (\phi_{m_1} \ldots \phi_{m_L})\) one may obtain the Schrödinger equation for the wave function \(C_{m_1 \ldots m_L}(t)\) from
\[
S = - \int dt \left( i C^*_{m_1 \ldots m_L} \partial_t C_{m_1 \ldots m_L} + C^*_{m_1 \ldots m_L} H_{m_1 \ldots m_L} C_{n_1 \ldots n_L} \right) .
\)

\(^4\)For the coherent states we consider the corresponding equation of motion may be interpreted as a (non-trivial) RG equation for the coupling constant associated to the operator \(\mathcal{O}\).
In the limit $L \to \infty$ we may consider the coherent state description and assume the factorized ansatz [57]

$$C_{m_1 \ldots m_L} = v_{m_1} \ldots v_{m_L}, \quad (3.8)$$

where each $v_l = \{v_{m_l}\}$ ($l = 1, \ldots, L$) is a complex unit-norm 6-vector. The BPS case corresponding to the totally symmetric traceless $C_{m_1 \ldots m_L}$ is represented by $v_l = v_{l(0)}$, $v_{l(0)}^2 = 0$. Using (3.2) and substituting the ansatz for $C_{m_1 \ldots m_L}$ into the above action one finds

$$S = -\int dt \sum_{l=1}^{L} \left( iv_l^* \partial_t v_l + \frac{\lambda}{(4\pi)^2} \left[ (v_l^* v_{l+1}^*)(v_l v_{l+1}) + 2 - 2(v_l^* v_{l+1})(v_l v_{l+1})^* \right] \right), \quad (3.9)$$

where we suppressed the 6-vector indices in the scalar products. As expected [41], the coherent state expectation value of the Hamiltonian (i.e. order $\lambda$ term in (3.9)) vanishes for the BPS case when $v_l$ does not depend on $l$ and $v_l^2 = 0$. More generally, if we assume that $v_l$ is changing slowly with $l$ (i.e. $v_l \simeq v_{l+1}$), then we find that (3.9) contains a potential term $(v_l^* v_{l+1}^*)(v_l v_{l+1})$ coming from the first “trace” structure in (3.2). This term will lead to large (order $\lambda L [41]$) shifts of anomalous dimensions, invalidating low-energy expansion, i.e. prohibiting one from taking the continuum limit $L \to \infty$, $\tilde{\lambda} = \frac{\lambda}{L^2}$ = fixed, and thus from establishing direct correspondence with string theory along the lines of [46, 48, 56].

To get solutions with small variations of $v_l$ from site to site we are thus to impose

$$v_l^2 = 0, \quad l = 1, \ldots, L \quad (3.10)$$

which minimizes the potential energy coming from the first term in (3.2). This condition implies that the operator at each site is invariant under half of supersymmetries: if $v_l^2 = 0$ the matrix $v_m \Gamma^m$ appearing in the variation of the operator $v_m \phi_m$, i.e. $\delta v_m \phi_m = \frac{1}{2} \epsilon (v_m \Gamma^m) \psi$, satisfies $(v_m \Gamma^m)^2 = 0$. This means that $v_m \phi_m$ is invariant under the variations associated with the null eigenvalues [57]. One may thus call $v_l^2 = 0$ a “locally BPS” condition since the preserved combinations of supercharges in general are different for each $v_l$, i.e. the operator corresponding to $C = v_1 \ldots v_L$ is not BPS. Here “local” should be understood in the sense of the spin chain, or, equivalently, the spatial world-sheet direction.\(^5\)

\(^5\)Even SU(2) sector has in general other higher-energy states with dimensions $\sim \lambda L$ (in addition to magnons with energies $\sim \frac{\lambda}{L}$ and macroscopic spin waves with $\Delta \sim \frac{\lambda}{L}$ there are also spinons with $\Delta \sim \lambda L$) but these do not correspond to fast strings – they are not captured by the low-energy continuum limit of the coherent state path integral.

\(^6\)This generalizes the argument implicit in [46]; an equivalent proposal was made in [17]. This is related to but different from the “nearly BPS” operators discussed in [15] (which, by definition, were those which become BPS in the limit $\lambda \to 0$).
In the case when $v_l$ are slowly changing we can take the continuum limit as in [46, 48, 56] by introducing the 2-d field $v_m(t, \sigma)$ with $v_{ml}(t) = v_m(t, \frac{2\pi l}{L})$. Then (3.9) reduces to (3.3) (all higher derivative terms are suppressed by powers of $\frac{1}{L}$ and the potential term is absent due to the condition $v^2 = 0$). Then (3.9) becomes equivalent to the $G_{2,6}$ Landau-Lifshitz sigma model (3.3) which was derived from the phase space action on the string side.

To summarize, considering ultra-relativistic strings in $S^5$ one can isolate a fast variable $\alpha$ (a “polar angle” in the string phase space) whose momentum $p_{\alpha}$ is large. One may gauge-fix $p_{\alpha}$ to be constant $\sim L$ or set $\tilde{\alpha} \sim L \sigma$, so that $\sigma$ or the “operator direction” on the SYM side gets interpretation of “T-dual to fast coordinate” direction. As a result, one finds a local phase-space action with 8-dimensional target space (where one can not eliminate 4 momenta without spoiling the locality). This action is equivalent to the Grassmanian $G_{2,6}$ Landau-Lifshitz sigma model action appearing on the spin chain side.

As a by-product, we thus get a precise mapping between string solutions and SYM operators representing coherent spin chain states [46, 57]. Explicit examples corresponding to pulsating and rotating solutions were given in [57]. In the continuum limit we may write the operator corresponding to the solution $v(t, \sigma)$ as $\mathcal{O} = \text{tr} [\prod_{\sigma} v(t, \sigma)]$, $v \equiv v_m(t, \sigma) \phi_m$. This locally BPS coherent operator is the SYM operator naturally associated to a ultra-relativistic string solution. The $t$-dependence of the string solution thus translates into the RG scale dependence of $\mathcal{O}$, while the $\sigma$-dependence describes the ordering of scalar field factors under the trace.

In general, semiclassical string states represented by classical string solutions should be dual to coherent spin chain states or coherent operators, which are different from the exact eigenstates of the dilatation operator but which should lead to the same energy or anomalous dimension expressions. At the same time, the Bethe ansatz approach [41, 42, 43, 47] is determining the exact eigenvalues of the dilatation operator. The reason why the two approaches happen to be in agreement is that in the limit we consider the problem is essentially semiclassical, and because of the integrability of the spin chain, its exact eigenvalues are not just well-approximated by the classical solutions but are actually exactly reproduced by them for $L \rightarrow \infty$, i.e. (just as in the harmonic oscillator or flat space string theory case, cf. also [67]) the semiclassical coherent state sigma model approach happens to be exact.
4. CONCLUDING REMARKS

As discussed above, there exists a remarkable generalization of the near-BPS (BMN) limit to non-BPS but “locally-BPS” sector of string/SYM states (for related reviews see [12, 40, 22]). It remains to understand better when and why this direct relation works or fails, but the hope is to use it as a guide towards finding the string/SYM spectrum exactly in $\lambda$, at least in a subsector of states. The relation between phase-space action for “slow” variables on the string side and the coherent-state action on the SYM (spin chain) side gives a very explicit picture of how string action “emerges” on the conformal gauge theory side (with the central role played by the dilatation operator). This implies not only an equivalence between string energies and SYM dimensions (established to first two orders in expansion in the effective coupling $\lambda$) but also a direct relation between the string profiles and the structure of coherent SYM operators.

One may try also to use the duality as a tool to uncover the structure of planar SYM theory to all orders in $\lambda$ by assuming the exact correspondence between particular SYM and string states. For example, demanding the consistency with the BMN scaling limit (along with superconformal algebra) determines the structure of the full 3-loop SYM dilatation operator in the $SU(2)$ sector [20, 21]. One can also use the BMN limit to fix only part of the dilatation operator but to all orders in $\lambda$ [60]. Generalizing (1.12),(2.14) and the 3- and 4-loop expressions in [20, 21] one can organize [27, 48, 60] the dilatation operator as an expansion in powers of $Q_{k,l} = \frac{1}{2}(I - \vec{\sigma}_k \cdot \vec{\sigma}_l)$ which reflect interactions between spin chain sites,

$$D = \sum Q + \sum QQ + \sum QQQ + ... .$$

Here the products $Q...Q$ are “irreducible”, i.e. index of each site appears only once. The $Q^2$-terms first appear at 3 loops, $Q^3$-terms – at 5 loops, etc. [20, 21]. Concentrating on the order-$Q$ part $D^{(1)}$ of $D$ one can write:

$$D^{(1)} = \sum_{r=0}^{\infty} \frac{\lambda^r}{(4\pi)^r} \sum_{l=1}^{L} D_r(l) , \quad D_r(l) = 2 \sum_{k=1}^{r} a_{r,k} Q_{l,l+k} , \quad (4.11)$$

or $D^{(1)} = \sum_{l=1}^{L} \sum_{k=1}^{L-1} h_k(L, \lambda) Q_{l,l+k}$. Demanding the agreement with the BMN limit one can then determine the coefficients $a_{r,k}$ and thus the function $h_k$ explicitly to all orders in $\lambda$ [60]. In particular, for large $L$, i.e. when $D$ acts on “long” operators, one finds

$$D^{(1)} = 2 \sum_{l=1}^{L} \sum_{k=1}^{\infty} f_k(\lambda) Q_{l,l+k} , \quad f_k(\lambda) = \sum_{r=k}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} a_{r,l} , \quad (4.12)$$
where the coefficients $f_k(\lambda)$ can be summed up in terms of the standard Gauss hypergeometric function \[60\]

$$f_k(\lambda) = \left( \frac{\lambda}{4\pi^2} \right)^k \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k + 1)} \frac{\Gamma(k + 1/2)}{4\sqrt{\pi} \Gamma(k + 1)} 2F_1(k - \frac{1}{2}, k + 1; 2k + 1; -\frac{\lambda}{\pi^2}) , \quad (4.13)$$

or, equivalently,

$$f_k(\lambda) = \frac{1}{4\pi(2k - 1)} \left( \frac{\lambda}{\pi^2} \right)^k \int_0^1 du \left[ \frac{u(1 - u)}{1 + \frac{\lambda}{\pi^2} u} \right]^{k - 1/2} . \quad (4.14)$$

$f_k$ goes rapidly to zero at large $k$, so we get a spin chain with short-range interactions.

One may hope that imposing additional constraints coming from correspondence with other string solutions (and using recent insights in \[28, 29\]) may help to determine the structure of the dilatation operator further.

The function $f_k(\lambda)$ in (4.13) smoothly interpolates between the usual perturbative expansion at small $\lambda$ and $f_k(\lambda) \sim \sqrt{\lambda}$ behaviour at large $\lambda$. The latter is the expected behaviour of anomalous dimensions of “long” operators dual to “semiclassical” string states.

Similar interpolating functions should appear also in anomalous dimensions of other SYM operators, though for “short” operators the strong-coupling asymptotics of the dimensions should be $\lambda^{1/4}$. Let us consider, for example, the following dimension 4 supersymmetry descendant of the Konishi operator, $\text{tr}([\Phi_1, \Phi_2]^2)$ (which belongs to the $SU(2)$ sector and should have the same anomalous dimension as the Konishi operator). The first few terms in the perturbative $\lambda$-expansion of its anomalous dimension are known to be \[68, 20, 33, 61\]

$$\Delta_{\lambda<1} = 4 + 3\lambda_{eff} - 3\lambda_{eff}^2 + \frac{21}{4}\lambda_{eff}^3 - \frac{705}{64}\lambda_{eff}^4 + O(\lambda_{eff}^5) , \quad (4.15)$$

$$\lambda_{eff} \equiv \frac{\lambda}{4\pi^2} .$$

If one would to ignore all non-linear in $Q$ terms in the dilatation operator (4.11), then the resummed expression for the anomalous dimension would be \[60\]

$$\Delta^{(1)} = 4 + \frac{3}{2}(\sqrt{1 + 4\lambda_{eff}} - 1) , \quad (4.16)$$

which does not, however, have the expected large $\lambda$ asymptotics,

$$\Delta_{\lambda>1} = 2\lambda^{1/4} + ... = 2\sqrt{2\pi}(\lambda_{eff})^{1/4} + ... . \quad (4.17)$$
Note that one cannot reproduce such asymptotics with an interpolating expression for $\Delta$ built out of rational functions of the square of the effective string $\lambda_{\text{eff}} = T^2$: while the expansion of a rational function (like $\sqrt{a + b\lambda_{\text{eff}} + d}$) at small $\lambda$ would have the same form as (4.15), the factors of $\pi$ would not match in the strong coupling limit (4.17).

The reason for the above extra factor of $\sqrt{\pi}$ in $\Delta_{\lambda > 1}$ in (4.17) expressed in terms of $\lambda_{\text{eff}}$ can be understood following ref. [2]. The Konishi operator should correspond to the lowest-level scalar string mode. The masses of the $\text{AdS}_5 \times S^5$ string modes are, in general, non-trivial functions of the string tension (the corresponding wave equations receive $\alpha'$-corrections), but, in the large-tension limit, they should simply be the same as in flat space, i.e. (in units where $R = 1$)

$$m^2 = \frac{4n}{\alpha'} = 4n\sqrt{\lambda} = 8n\pi T, \quad T = \frac{\sqrt{\lambda}}{2\pi} = \sqrt{\lambda_{\text{eff}}}.$$  \hspace{1cm} (4.18)

Then the standard $\text{AdS}_5$ formula $\Delta(\Delta - 4) = m^2$ for the dimension of a scalar field with mass $m$ (again, expected to be valid in the large tension limit) predicts that for $n = 1$

$$\Delta - 2 = \sqrt{4 + m^2} \to m + ... = 2\sqrt{2\pi \sqrt{\lambda_{\text{eff}}} + ...}.$$ \hspace{1cm} (4.19)

It thus appears that instead of being a rational function of $\lambda_{\text{eff}}$, the dimension $\Delta$ of the Konishi operator should be a transcendental function. In fact, the hypergeometric function like the one appearing in (4.13), i.e. $2F_1(a, b; c; -k\lambda_{\text{eff}})$, seems to be a natural candidate: one can choose its arguments (and an overall coefficient) so that to match the powers of $\lambda$ and $\pi$ in both the strong and the weak coupling limits (one example with required strong-coupling asymptotics has $a = -1/4$, $b = 3/4$, $c = 3/2$, $k = 4$). With $\Delta \sim 2F_1$ (or $\Delta$ being a rational function of $2F_1$) it is possible also to satisfy the string-theory requirement that the strong-coupling expansion should be organized as an expansion in powers of the inverse string tension, i.e. $\Delta = \lambda^{1/4}(c_1 + \frac{c_2}{\sqrt{\lambda}} + \frac{c_3}{(\sqrt{\lambda})^2} + ...)$ (cf. (4.19)).

At the same time, for “long” operators with large canonical dimensions (like BMN operators or non-BPS operators discussed in the previous sections) the interpolating functions appearing in $\Delta$ may be simple rational functions like square roots: here both the weak (gauge theory) and the strong (string theory) expansions are organized in terms of $\lambda_{\text{eff}} = \frac{\lambda}{4\pi^2}$ with the leading coefficients which do not contain extra powers of $\pi$ (see also the discussion below eq. (1.6)).
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