Fundamentals of continuum mechanics – classical approaches and new trends

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Abstract. Continuum mechanics is a branch of mechanics that deals with the analysis of the mechanical behavior of materials modeled as a continuous manifold. Continuum mechanics models begin mostly by introducing of three-dimensional Euclidean space. The points within this region are defined as material points with prescribed properties. Each material point is characterized by a position vector which is continuous in time. Thus, the body changes in a way which is realistic, globally invertible at all times and orientation-preserving, so that the body cannot intersect itself and as transformations which produce mirror reflections are not possible in nature. For the mathematical formulation of the model it is also assumed to be twice continuously differentiable, so that differential equations describing the motion may be formulated. Finally, the kinematical relations, the balance equations, the constitutive and evolution equations and the boundary and/or initial conditions should be defined. If the physical fields are non-smooth jump conditions must be taken into account.

The basic equations of continuum mechanics are presented following a short introduction. Additionally, some examples of solid deformable continua will be discussed within the presentation. Finally, advanced models of continuum mechanics will be introduced.

1. Introduction
The basic ideas with respect to continuum mechanics are given in a German textbook [1] which is written in the style of famous Russian school of profs’ A. I. Lurie (1901–1980), V. A. Pal’mov and P. A. Zhilin (1942–2005). These professors presenting from the 60th of the last century up to the beginning of 2000s various courses at the Leningrad Polytechnic Institute/Peter the Great State Polytechnical University containing various elements of Continuum Mechanics. Lurie was more focused on elasticity problems. Pal’mov’s idea was to combine continuum mechanics with rheological modeling. Zhilin presented, together with the three-dimensional approach, special courses devoted to one- and two-dimensional continua. Unfortunately, a part of monographs, related to various courses, have been published only in Russian [2–5] in which the following topics were typically discussed. The remaining monograph were translated [6–8].

The common idea of their courses was

- the use of the direct tensor notation,
- the clear split of material-independent and material-dependent equations,
- the introduction of five balance equations only,

1 The paper is dedicated to Alexander Manzhurov’s 60th birthday.
• the formulation of boundary conditions, and
• the formulation of initial conditions.

It should be noted that the courses were strongly based on mathematics. But the starting point was always mechanics (or physics) and only after that mathematics. This is partly in contrast to Clifford Truesdell (1919–2000) and his followers.

These items are discussed briefly as follows. At first, the elements are presented for the case of classical continua, sometimes named Cauchy (after Augustin-Louis Cauchy, 1789–1857) or Boltzmann (after Ludwig Boltzmann, 1844–1906) continua, and then some advanced models are introduced. The latter step is not final since many new models were introduced in recent years.

Continuum mechanics is a branch of mechanics that deals with the analysis of the mechanical behavior of materials and structures modeled as a continuous mass rather than as discrete particles. Maybe the French mathematician Augustin-Louis Cauchy was the first to formulate such models in the 19th century. Up to now, one has to understand that the starting point of continuum mechanics is not only the axioms of Isaac Newton (1643–1727), but also the ideas of Leonhard Euler (1707–1783). With the help of Newton’s axiom, we will get only the simplest theory based on force interactions. Euler mentioned that there is more: for the creation of the beam theory it was necessary to assume the independence of forces and moments. Nowadays, research on developing new theories, based on Newton’s axioms, is being carried out in several directions.

Some elements of continuum mechanics should be introduced from the very beginning. Firstly, one has to assume a space: let us consider here the Euclidean three-dimensional space — in special cases two-dimensional or one-dimensional spaces will be used. The four-dimensional space, discussed, for example, by Minkowski (1864–1909) and mentioned in [9, 10], is not applied here. Next step is the introduction of the (pseudo) time $t \geq 0$. After that, one has to define the body, with/without mass, which has a volume and a surface with an outward-pointing normal. Last but not least, homogeneity and isotropy should be assumed among other things. A continuum theory must be valid for solids and fluids. In addition, the theory should be scale-independent. Namely, not only the theory is applicable the the macro-scale but also it should provide proper results in the case of meso, micro or nano.

The details of the direct tensor notation will be not given here. However, a very good presentation of the tensor calculus is given in the appendices of [6, 7] and in the recent published textbook [11]. All these references followed the ideas of Josiah Willard Gibbs (1839–1903) and Max Lagally (1881–1945), see for example [12, 13].

2. Material-independent equations
In this section, the material-independent equations will be introduced, i.e., the equations are independent from the individual material response. The kinematical relations are based only on geometrical considerations and contain the changes of the geometry. Forces and stresses are the basic reasons for the geometrical changes in the classical continuum mechanics. The balances are physical principles formulated here in the general sense for the continuum volume. We have to distinguish between balance equation and inequality. There are four balance equations, the fifth one is an inequality necessary for the process direction estimation. Boundary and initial conditions allowing to solve initial-boundary problems. Jump conditions are necessary if the physical fields are non-smooth.

2.1. Kinematics
The starting point for any kinematical relation is the introduction of configurations. Generally, a minimum of two configurations is employed: the reference configuration at the initial moment $t_0$ and the actual configuration at the moment $t$. It is obvious that instead of time we have a
(pseudo)time. It is our decision to fix the moment $t_0$ (the real moment of the beginning is unknown). In addition, it should be underlined that in many textbooks the initial configuration is related to the Lagrangian (named after Joseph-Louis Lagrange, 1736–1813) description and the reference to the Eulerian one. However historically, this is not correct: the Lagrangian description was introduced by Leonhard Euler (1707–1783) in 1762 and the Eulerian by Jean-Baptiste le Rond d’Alembert (1717–1783) in 1752 \cite{14, 15}. It should be noted that many other configurations may also exist: stress-free configuration, damaged configuration, etc. For solids, the Lagrangian description is preferred whereas the Eulerian description is used for fluids. But this restrictive split is under question if we deal, for example, with plasticity problems.

Any configuration can be introduced with the help of a reference point and a position vector. A coordinate system at the beginning is not necessary, but may be helpful in solving problems. At the same time, one has to select either a fixed- or a moving reference point. The latter allows to increase the number of distinct configurations.

In mechanics, geometrical changes are related to the motion and to the strains. At this stage, it is necessary to make a distinction by introducing a property of the medium: rigidity or deformability. In the first case, all material points of the body have the same motion (translations and rotations), while the distance between each two arbitrary points remains the same at all times. In the second case, we disregard the rigid motions and focus on the geometrical changes in the infinitesimal surrounding of the material points. It can be shown that these changes are related to extension/shortening and angle changes. In the case of extension/shortening, we look at the length of infinitesimal small radius-vector whereas in the second case, the focus is on the angle between two infinitesimal small radius-vectors. Note that in some applications, both approaches should be adopted concurrently.

The radius-vector approach for description of the material point positions results in two strain measures in the literature. After introducing the radius-vector in both configurations that means $\mathbf{R}$ for the reference configuration and $\mathbf{r}$ for the actual one, the displacement vector can be expressed as:

$$\mathbf{u}(\mathbf{R}, t) = \mathbf{r}(\mathbf{R}, t) - \mathbf{R}$$

with $\mathbf{r}(t = t_0) = \mathbf{R}$. This approach is the common way to present the Cauchy’s strains in the geometrically-linear theory. The second approach is related to the deformation gradient — better gradient of the radius-vector, see \cite{7}:

$$\mathbf{F} = (\nabla \mathbf{r})^T$$

This property is more appropriate for the case of large strains since after some manipulation a more clear representation of the strains and rigid body motions is obtained. Note that linearizing each one of the relations coincides with the other one. More details about different strain measures are presented, for example, in \cite{2, 7}. Concerning the linearization procedure, a excellent method is given in \cite{16, 17}.

With the strain measures and strain tensors one has only one part of the kinematics. In addition, the velocity vector, the acceleration vector, the strain rate tensor, and the strain gradient rate should be introduced. The calculation of time derivatives can be readily done in the case of geometrically-linear theory while it is more complicated in geometrically non-linear problems. One problem arises from the objectivity of these derivatives for instance. Different approaches are presented in the literature, see \cite{5} and \cite{6} among others. Some actual statements are presented in \cite{18, 19}.

2.2. **Forces and stresses**

The following classification of the external loading can be given:

(i) **Natural loading models**
Figure 1. Force stress vector.

- body / mass / volume loading (forces, moments), and
- surface / contact loading (forces, moments).

(ii) Artificial loading models
- line loading (forces, moments), and
- single point loading (forces, moments).

Note that as in the Course of General Mechanics forces and moments are introduced. In addition, it should be mentioned that a moment is not only a moment of the force (torque) or a pair of forces — it can defined as an independent quantity. Dimensional analysis provides the following results for the single loads:

- dimension of force $[F] = N$ and
- dimension of moment $[M] = Nm$. 

Hence, one can conclude that:

- body loads are expressed per unit volume,
- surface loads are expressed per unit area, and
- line loads are expressed per unit length.

Let us introduce body loadings:

- body force $\rho(r,t)k(r,t) = k\dot{V}(r,t)$, and by analogues
- $\rho(r,t)l(r,t) = l\dot{V}(r,t)$ the body moment.

Here $\rho = \rho(r,t)$ is the density. Examples of body forces are the weight force, the inertia force and potential force. The body moments in the classical theory will be ignored.

After introduction of forces and moments, we can define various stress vectors

- the force stress vector
  
  \[ t = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}, \]

- and the couple stress vector
  
  \[ m = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A}. \]

As usual, the second one is ignored in the textbooks of strength of materials. There are several arguments why the couple stresses can be ignored among which one is presented in [20].

Let us introduce an infinitesimal small surface element $dA$ with the unit normal $n$ (figure 1). The acting force stress vector does not have the same value over the surface but the difference of any two neighbouring magnitudes is negligible. From the definition of moment, it follows that the moment of the couple is infinitesimal since a small difference is multiplied by a small distance. Therefore, the couple stress vector can be ignored in this case.

A counterexample is the case of stress vectors on the crack tip: even over an infinitesimal small distance, the change in the force stress vector can be huge. Such cases initiate the arguments over
introducing couple stresses. Maybe this was the reason for extending the appendix in [20] and to publish another book on elasticity taking into account asymmetric stress tensors in addition to the symmetric ones [21].

**Note** Similar derivations can be established for both the actual and the reference configuration. Below we are performing the procedures for the actual configuration. That means writing $\nabla$ implies $\nabla_R$, also $t$ and $T$ are the Cauchy stress vector and tensor in the actual configuration, respectively.

The components of the stress vector can be presented in Cartesian coordinates as

$$\mathbf{t} = t_n \mathbf{n} + t_t \mathbf{e}_t = t_n \mathbf{n} + t_{t_1} \mathbf{e}_{t_1} + t_{t_2} \mathbf{e}_{t_2},$$

where $\mathbf{e}_{t_1}$ and $\mathbf{e}_{t_2}$ are two arbitrary tangential directions on the surface $dA$. Since we have a Cartesian coordinate system $\mathbf{n} \perp \mathbf{e}_{t_1}$, $\mathbf{n} \perp \mathbf{e}_{t_2}$, and $\mathbf{e}_{t_1} \perp \mathbf{e}_{t_2}$, so $\mathbf{n}$, $\mathbf{e}_{t_1}$, and $\mathbf{e}_{t_2}$ form an arbitrary orthonormal unit base.

Following Cauchy, a lemma can be written down

$$\mathbf{t}(\mathbf{r}, \mathbf{n}, t) = \mathbf{n} \cdot \mathbf{T}(\mathbf{r}, t)$$

connecting the stress vector with the stress tensor. Cauchy got his lemma from equilibrium considerations [22]. Note that in many textbooks [23, for example] the lemma is presented in a different way

$$\mathbf{t}(\mathbf{r}, \mathbf{n}, t) = \mathbf{T}(\mathbf{r}, t) \cdot \mathbf{n}.$$ 

This is equivalent to the linear mapping of two vector spaces with the help of a second rank tensor. Both formulations coincide if the stress tensor is symmetrical.

As in General Mechanics now the resultant force

$$\mathbf{F}^R = \int_V \rho \mathbf{k} \, dV + \int_A \mathbf{t} \, dA$$

and the resultant moment

$$\mathbf{M}^R = \int_V (\mathbf{r} \times \rho \mathbf{k}) \, dV + \int_A (\mathbf{r} \times \mathbf{t} + \mathbf{m}) \, dA$$

can be introduced. Taking into account only force actions the static equilibrium can be formulated for the continuum with $\mathbf{F}^R = 0$ and $\mathbf{M}^R = 0$

$$\int_V \rho \mathbf{k} \, dV + \int_A \mathbf{t} \, dA = 0, \quad \int_V (\mathbf{r} \times \rho \mathbf{k}) \, dV + \int_A (\mathbf{r} \times \mathbf{t}) \, dA = 0.$$

With the divergence theorem (formulated independently by Carl Friedrich Gauß, 1777–1855, and Michail Wassiljewitsch Ostrogradski, 1801–1862) for the first equilibrium equation

$$\int_A \mathbf{t} \, dA = \int_A \mathbf{n} \cdot \mathbf{T} \, dA = \int_V \nabla \cdot \mathbf{T} \, dA$$

one gets

$$\int_V (\rho \mathbf{k} + \nabla \cdot \mathbf{T}) \, dV = 0$$

and the local form can be obtained (if all fields are smooth)

$$\nabla \cdot \mathbf{T} + \rho \mathbf{k} = 0$$
Applying the divergence theorem to the second equilibrium equation the symmetry condition for the stress tensor \((T = T^T)\) can be established. Finally, with the d’Alambert’s Principle one gets the dynamic equations (Euler’s equation of motion)
\[
\int_V \rho k \, dV + \int_A t \, dA - \int_V \rho \ddot{r} \, dV = 0.
\]
Applying again the divergence theorem we obtain
\[
\int_V (\rho k + \nabla \cdot T - \rho \ddot{R}) \, dV = 0
\]
and finally, if all fields are smooth, the local form is obtained
\[
\nabla \cdot T + \rho k = \rho \ddot{r}.
\]

2.3. Balances
2.3.1. General balance equation. Let us introduce some quantity named balance variable of the continuum \(\Psi(r, t)\) and defined in the actual configuration. The integration over all body points results in \(Y(t)\)
\[
Y(t) = \int_V \Psi(r, t) \, dV.
\]
With \(dV = \det F \, dV_0\) the recalculation of the properties for the reference configuration is possible.

The rate of \(Y(t)\) is influenced by actions onto the volume and through the surface (flux)
\[
\frac{D}{Dt} Y(t) = \frac{D}{Dt} \int_V \Psi(r, t) \, dV = \int_A \Phi(r, t) \, dA + \int_V \Xi(r, t) \, dV,
\]
where \(D/Dt\) is the total time derivative, \(\Phi\) and \(\Xi\) are named flux and supply, respectively. The visualization of the general balance equation is given in figure 2.

It should be underlined that the general balance equations can be introduced in a different form with three terms on the right-hand side [23]. Instead of one surface and one volume integral for the flux and the supply one surface and two volume integrals for the flux, the supply and the production are introduced. But there are only few applications for production. If Cauchy’s lemma is valid for the general form after application of the divergence theorem one gets again the local form.

Note If the right-hand side of the general balance is equal to zero we have a conservation law. In other words, the first integral is a constant.
2.3.2. Balance of mass. In solid mechanics applications, the formulation of the balance equation for mass starts with

\[ m = \int_V \rho(r, t) \, dV = \text{const}, \]

where \( m \) is the mass. Then the integral form of the balance of mass is

\[ \frac{Dm}{Dt} = \frac{D}{Dt} \int_V \rho(r, t) \, dV = 0 \]

and the local form is

\[ \frac{D}{Dt} (dm) = \frac{D}{Dt} (\rho \, dV) = 0. \]

From this equation the continuity equation can be deduced

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \]

with the velocity \( \mathbf{v} = \dot{r} \). If \( \rho = \text{const} \) with respect to the time the incompressibility condition is following as

\[ \nabla \cdot \mathbf{v} = 0. \]

2.3.3. Balance of momentum. By introducing the linear momentum of the continuum, the integral form can be written down as

\[ \frac{D}{Dt} \int_V \rho(r, t) \mathbf{v}(r, t) \, dV = \int_A \mathbf{t}(r, t) \, dA + \int_V \rho(r, t) \mathbf{k}(r, t) \, dV. \]

The local form is

\[ \frac{D}{Dt} \rho(r, t) \mathbf{v}(r, t) = \nabla \cdot \mathbf{T}(r, t) + \rho(r, t) \mathbf{k}(r, t). \]

2.3.4. Balance of moment of momentum. By introducing the angular momentum of the continuum, the integral form can be written down as

\[ \frac{D}{Dt} \int_V \mathbf{r} \times \rho(r, t) \mathbf{v}(r, t) \, dV = \int_A \mathbf{t} \times \mathbf{r}(r, t) \, dA + \int_V \mathbf{r} \times \rho(r, t) \mathbf{k}(r, t) \, dV. \]

If the balance of momentum is valid as given in the previous section, the local form results again in the symmetry condition for the stress tensor.

2.3.5. Balance of energy. Let us assume at the beginning that we have only mechanical actions. In this case, the balance of energy, which is equivalent to the first law of thermodynamics, can be presented as

\[ \frac{D}{Dt} \int_V \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + u \right) \rho \, dV = \int_A \mathbf{t} \cdot \mathbf{v} \, dA + \int_V \mathbf{k} \cdot \mathbf{v} \rho \, dV. \]

The kinetic and potential energy is influenced only by surface and volume forces. If the balance of moment The local form can be expressed as

\[ \rho \dot{u} = \mathbf{T} \cdot (\nabla \mathbf{v})^T = \mathbf{T} \cdot \mathbf{D}. \]

The general first law of thermodynamics states that:
The changes in time of the total energy $W$ within the volume is equal to the heat flux $Q$ and the power of all external loadings $P_{\text{ext}}$

$$\frac{D}{Dt} W = P_{\text{ext}} + Q,$$

where $W = U + K$ is the sum of the inner energy $U$ and the kinetic energy $K$.

Let us introduce the four terms

- $K = \frac{1}{2} \int_V \mathbf{v} \cdot \mathbf{v} \rho \, dV$,
- $U = \int_m u \, dm = \int_V \rho u \, dV$,
- $P_{\text{ext}} = \int_A \mathbf{t} \cdot \mathbf{v} \, dA + \int_V \mathbf{k} \cdot \mathbf{v} \rho \, dV$,
- $Q = \int_V \rho r \, dV - \int_A \mathbf{n} \cdot \mathbf{h} \, dA$.

Finally, we have

$$\frac{D}{Dt} \int_V \left( u + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) \rho \, dV = \int_A \mathbf{t} \cdot \mathbf{v} \, dA + \int_V \mathbf{k} \cdot \mathbf{v} \rho \, dV - \int_A \mathbf{n} \cdot \mathbf{h} \, dA + \int_V \rho r \, dV.$$

With the help of the Cauchy lemma and after some manipulations the local form can be deduced (it is assumed that the balance of momentum is valid):

$$\frac{D u}{Dt} + T \cdot D - \nabla \cdot \mathbf{h} + \rho r = 0.$$

2.3.6. Balance of entropy. The last balance cannot be formulated in the standard form since now we have two formulations: as an equation for non-dissipative processes or as an inequality for dissipative processes. The decision on the type of process can only be made after introducing statements concerning the material behavior.

Let us introduce the second law of thermodynamics in the integral form

$$\frac{D}{Dt} \int_V \rho s \, dV \geq \int_V \frac{r}{\Theta} \rho \, dV - \int_A \frac{\mathbf{n} \cdot \mathbf{h}}{\Theta} \, dA,$$

where $s$ is the specific entropy, $r$ is an energy source, $\Theta$ denotes the absolute temperature, and $\mathbf{h}$ is the heat flux vector. The changes in time of the entropy within the volume under consideration is not smaller than the rate of the outer entropy flux. Again after some manipulations

$$\int_A \frac{\mathbf{n} \cdot \mathbf{h}}{\Theta} \, dA = \int_V \nabla \cdot \frac{\mathbf{h}}{\Theta} \, dV = \int_V \left( \nabla \cdot \frac{\mathbf{h}}{\Theta} - \frac{\mathbf{h} \cdot \nabla \Theta}{\Theta^2} \right) \, dV$$

and taking into account

$$\frac{1}{\Theta} \mathbf{h} \cdot \nabla \Theta = \mathbf{h} \cdot \nabla \ln \Theta$$

we get the local form

$$\rho \theta \frac{D s}{Dt} - \rho r - \nabla \cdot \mathbf{h} - \mathbf{h} \cdot \nabla \ln \theta \geq 0.$$
Figure 3. Temperature field $\Theta_1 < \Theta_2 < \Theta_3$.

The underlined terms are also in the first law of thermodynamics. Combining both equations we get

$$\rho \frac{D\theta}{Dt} - \frac{Du}{Dt} + T \cdot D - h \cdot \nabla \ln \theta \geq 0.$$  

With the Helmholtz free energy

$$u - \Theta s = f$$

we can define the dissipation function

$$T \cdot D - \rho (\dot{f} + s \dot{\theta}) = \Phi \geq 0.$$  

Finally, we obtain

$$\Phi - h \cdot \nabla \ln \theta \geq 0.$$  

Since $\Phi$ is positive, the remaining part can be written

$$h \cdot \nabla \ln \theta \leq 0.$$  

The limiting cases are defined by the following two processes:

- $h = 0$ adiabatic process and
- $\nabla \Theta = 0$ isothermal process.

Assuming a non-dissipative process $\Phi = 0$ with temperature field (figure 3).

The following solution is valid only for

$$\angle (h, \nabla \Theta) > \frac{\pi}{2}.$$  

Note that the perpendicular case is expected.

Going back to the first law

$$\rho \Theta \frac{D\theta}{Dt} = T \cdot D - \rho \left( \frac{Df}{Dt} + s \frac{D\Theta}{Dt} \right) + \rho r - \nabla \cdot h = \Phi + \rho r - \nabla \cdot h$$

the following situations can be considered:

- non-dissipative process: $\Phi = 0$
  $$\rho \Theta \frac{D\theta}{Dt} = \rho r - \nabla \cdot h,$$
  which is a heat transfer equation,

- isothermal process: no heat transfer, mechanical and thermal processes are decoupled, and
- adiabatic process with $h = 0$
  $$\rho \Theta \frac{D\theta}{Dt} = \rho r.$$
2.4. Boundary and initial conditions

The balance equations in the local form are a system of coupled differential equations in terms of the position-vector and time. That means for solving the system one should formulate boundary and initial equations.

There are different types of boundary conditions:

- A boundary condition which specifies the value of the function itself is a Dirichlet boundary condition (Peter Gustav Lejeune Dirichlet, 1805–1859).
- A boundary condition which specifies the value of the normal derivative of the function is a Neumann boundary condition (Carl Gottfried Neumann, 1832–1925).
- A boundary condition which specifies the sum of the value of the function itself multiplied by a constant and the value of the normal derivative of the function multiplied by another constant is a Robin boundary condition (Victor Gustave Robin, 1855–1897).
- If the boundary has the form of a curve or surface that gives a value to the normal derivative and the variable itself then it is a Cauchy boundary condition.
- Last but not least there are mixed boundary conditions that on one part of the boundary is given one condition, on another part — another condition.

If we have a pure mechanical problem, the simplest boundary conditions are related to the displacements or the stress vector on the boundary. In more complicated problems, the conditions should also be formulated for the temperature or the heat flux vector, etc.

With respect to the time, we have differential equations of second order in the case of a pure mechanical problem. That means one can formulate initial conditions for the function itself or the first time derivative. Examples are a prescribed displacement vector and/or velocity velocity at the beginning \( t_0 \). If both values are zero, we get the static solution only (no motion or constant velocity for all points of the continuum). In more complex problems, we have more initial conditions. The temperature problem, for example, is a first order differential equation with respect to time. That means the initial temperature can be prescribed.

2.5. Jump conditions

Up to now, it was assumed that the physical fields are continuously differentiable within the volume in both configurations. This assumption was also introduced for the Reynolds transport theorem (Osborne Reynolds, 1842–1912) necessary for the recalculation of the actual balances into balances in the reference configuration and the divergence theorem. If a surface within a material body exists with a discontinuous physical quantity, it is called a singular surface [23]. Examples are shock front in supersonic flows, surface between two different bodies (ski-snow, atmosphere-ocean) or the front between two phases. In all these cases, the balance can be formulated in the integral form but they should incorporate additional terms expressing the jump conditions. Details concerning the jump conditions are presented in [23–27] among others.

3. Material-dependent equations

Now our focus is on the specific (individual) response of the given material under an arbitrary load. We should include the information on material behavior since the number of governing equation is not equal to the number of unknowns in the set of governing equations. For example, consider a homogeneous three-dimensional solid which is only mechanically loaded. In this case, we have the following governing equations:

- balance of mass — one scalar equation,
- balance of momentum — one vectorial or three scalar equation,
- balance of moment of momentum — one vectorial or three scalar equation,
• balance of energy — one scalar equation, and
• balance of entropy — one scalar expression.

The unknown are:
• the density (one scalar),
• the displacement vector (one vector or three scalars),
• the stress tensor (one second rank tensor or 9 scalars), and
• the inner energy (one scalar).

It is obvious that we have only 8 governing equations, but 14 unknowns. To close the gap, we need more equations which can be established only with the help of information on the behavior of the materials. The material behavior itself is specific to each material, and thus a universal constitutive equation cannot be established.

3.1. Constitutive equations

The formulation of constitutive equations cannot be based on some physical principles (exception is that the requirements of the second of thermodynamics/balance of entropy should be fulfilled). The following modeling principles are established in the literature:
• inductive approach that is moving from the simplest to more complex models,
• deductive approach that is moving from a general frame to more specific cases, and
• rheological modeling which is a combination of both approaches.

The constitutive equations contain parameter, parameter functions, etc. which should be identified with the help of
• experimental observations (real or virtual),
• mathematical analysis, and
• theory of symmetry [28].

The last one should be used together with the Curie-Neumann’s principle (Pierre Curie, 1859–1906, Fritz Ernst Neumann, 1798–1895). In addition, sometimes the models are restricted by some constraints (for example, the incompressibility condition). There are a lot of references in this matter among which is [17]. Herein, the deductive approach and the problems of identification are discussed. In addition, a classification of the material behavior can be done as follows:
• spontaneous material behavior with the subclasses of elastic and plastic behavior, and
• time-dependent material behavior with the subclasses of visco-elastic and visco-plastic behavior.

Formulation of suitable constitutive and evolution equations finally should be proved by
• checking the correctness of the formulation and the adequateness in terms of thermodynamic considerations and
• experimental validation or falsification, etc.

Let us introduce the following definition: Constitutive equations connect all macroscopic phenomenological variables describing the behavior of the continuum [29]. It is clear that this definition is very general and we need some restrictions with respect to the mathematical form of the constitutive equations. Such restrictions are introduced in [30] among others. One basic definition in this paper is: Simple materials of the rank 1 are materials which are described by constitutive equations connecting local variables, e.g. the local strain tensor and the local
heat flux vector with the local stress tensor and the local temperature gradient, respectively. All statements are related to the same material point and its differential neighborhood of rank 1. As a consequence only the first gradients are in the constitutive equations, higher order gradients are ignored. The next important definition is: Process is the change of the constitutive parameters with respect of time [2]. In addition, one can assume: The behavior of the continuum in each material point is given as a set of constitutive variables which are operators with respect to time [2].

If we want to distinguish solids and fluids, several definitions are presented in the literature:

- Under a given load, a solid will have non-zero components in the stress deviator, i.e., it shows resistance against any shape change.
- In contrast, under a given load, a fluid will have only zero components in the stress deviator, i.e., it does not preserve its shape.

The starting point of the deductive approach in formulation of the constitutive equations is introducing of several axioms of the material theory [17]:
- causality,
- determinism,
- equipresence,
- material objectivity,
- local action,
- memory, and
- physical consistency.

A typical example of the set of constitutive equations is

\[ T(R, \tau) = P\{R, \theta(R, \tau), \dot{\theta}(R, \tau), \nabla_\tau \theta(R, \tau), \nabla_\tau \dot{\theta}(R, \tau), \Gamma(R, \tau) \}, \]

\[ h(R, \tau) = h\{R, \theta(R, \tau), \dot{\theta}(R, \tau), \nabla_R \theta(R, \tau), \nabla_\tau \dot{\theta}(R, \tau), \Gamma(R, \tau) \}, \]

\[ f(R, \tau) = f\{R, \theta(R, \tau), \dot{\theta}(R, \tau), \nabla_R \theta(R, \tau), \nabla_\tau \dot{\theta}(R, \tau), \Gamma(R, \tau) \}, \]

\[ s(R, \tau) = s\{R, \theta(R, \tau), \dot{\theta}(R, \tau), \nabla_R \theta(R, \tau), \nabla_\tau \dot{\theta}(R, \tau), \Gamma(R, \tau) \}, \]

describing simple thermomechanical material behavior. The constitutive parameters are: the temperature, the temperature rate, the temperature gradient, the temperature rate gradient and a strain measure. Since \(0 < \tau < t\), the constitutive equations are functionals. Other examples are given, for example, in [2, 14, 15].

Several simplifications can be made such as specification of the strain measure, neglecting aging effects and viscosity which is equivalent to removing the explicit time dependency. Among such cases, the following simplified ones are worth mentioning:

- nonlinear elastic anisotropic material behavior

\[ T(F) = 2\rho F \cdot u_C(C) \cdot F^T \]

with \(C\) as the right Cauchy-Green strain tensor \((C = F^T \cdot F)\).

- nonlinear elastic isotropic material behavior

\[ T(F) = 2\rho F \cdot (\phi_0 I + \phi_1 C + \phi_2 C^2) \cdot F^T \]

with the functions \(\phi_i\) \((i = 0, 1, 2)\) depending on the first, second and third invariants of the tensor \(C\).
The inductive approach is the more engineering way of formulation constitutive equations. For example, we start with the Hooke’s law:

- tension with \( \sigma \) — normal stress, \( E \) — Young’s modulus (named after Thomas Young, 1773–1829) and \( \varepsilon \) normal strain
  \[
  \sigma = E \varepsilon
  \]
- or torsion with \( \tau \) — shear stress, \( G \) — shear modulus and \( \gamma \) — shear strain
  \[
  \tau = G \gamma
  \]

and generalize towards the following cases:

- three-dimensional isotropic case,
- three-dimensional anisotropic case,
- nonlinear behavior,
- . . .

In each case, the thermodynamical consistency, which is guaranteed in the deductive approach, should be checked separately.

The last possibility to formulate constitutive equations is the rheological modeling [2, 8, 29]. This approach is founded on

- the introduction of some basic models, for example, related to
  - the elastic behavior,
  - the plastic behavior, and
  - the viscous behavior,
- and the assumption that the connection of basic models can be realized only
  - in parallel or
  - in series.

Then any complex behavior can be represented by these connection, for example,

- visco-elastic = elastic + viscous
- visco-plastic = plastic + viscous

The rheological modeling was introduced in [31] and discussed in [2, 8] for isotropic and anisotropic materials.

### 3.2. Evolution equations

In many cases it is enough to present the material behavior by constitutive equations discussed in the previous subsection. Sometimes the changes of the material behavior are related to evolution processes. Examples are

- the establishment of plastic zones,
- the development of damage processes,
• creep processes,
• the deformation induced anisotropy,
• the hardening and softening processes, or
• phase transformation processes.

The description of the evolution process is based on ordinary differential equations with respect to time derivatives of first order. Special initial conditions should be introduced. For example, if we have creep or plastic evolution the initial state is assumed to be elastic.

4. Advanced models
The last part of this paper presents some new directions in continuum mechanics. This overview is not comprehensive and only few comments are given. Nevertheless, the required references are given for further reading.

4.1. Lower dimensional continuum models
These models are interesting from the theoretical point of view. Commonly, structural models (beam, rod, plate, shell, etc.) are presented in the literature as sets of equations which are reduced from a general three-dimensional equation by means of hypotheses, mathematical simplification, etc. Examples of such hypotheses are:

• Euler-Bernoulli beam theory, for example, is based on the assumption that the beam cross-sections remain planar and perpendicular to the neutral axis before and after deformation, and
• Kirchhoff plate theory, for example, assumes that the line elements of the beam remain straight and perpendicular to the mid-plane before and after deformation.

Mathematical approaches are based on power series or asymptotic integration (in both case a small parameter is assumed).

A third way is more elegant and natural. At the beginning it is assumed that we have a deformable surface or line. Then, an exact continuum theory can be deduced. The disadvantage of this approach is the identification of the parameters in the constitutive equations.

Recently overviews concerning plate and shell theories and different approaches were published in [32, 33]. A micropolar plate model was presented in [34] and extended to micropolar shells in [35]. A direct theory of rods was suggested in [36, 37]. Further discussion concerning actual trends are given in [38, 39] among others.

4.2. Nanostructures
Nanostructures are a new class of structures with special properties. In many cases they are stiffer which is a result of high specific surface to volume ratio. In the case of classical structures the bulk behavior is dominant — in the case of nanostructures the stiffness properties are under the influence of the surface behavior.

A large number of publications were released in recent years which makes it almost impossible to provide a comprehensive overview. But within a more specific framework, the pioneering works of [40, 41] suggest considering the surface effects in the continuum mechanics of the bulk material — where it is relevant. Some publications in this context are [42–45]. References for further reading are given within these publications. An overview concerning surface effects is given also in [46].
4.3. Growing Solids
In June 2015, Alexander Manzhirov has organized an IUTAM Symposium on growing solids in Moscow. From the first announcement, one can comprehend this class of solids by means of a nice description: The vast majority of objects or solids which surround us arise from some growth processes. As an example, one can present such natural phenomena as growth of biological tissues, glaciers, blocks of sedimentary and volcanic rocks, as well as space objects. Similar processes determine the specific features of many technologies in industry, including well-known technologies of crystal growth, laser deposition, solidification of melts, electrolytic formation, pyrolytic deposition, polymerization and concreting. Recent research has shown that solids, which were formed due to the processes of growth, differ essentially in their properties from solids in the traditional view. Moreover, the classical approaches of solid mechanics to the modeling of growing solids behavior fail. They have to be replaced by new ideas and methods of modern mechanics, mathematics, physics, and engineering sciences. Thus, at present, a new area of solid mechanics, which deals with the construction of adequate models for solids growth processes is forming.

There are world-wide different schools investigating growing solids. One of them was founded by N. Kh. Arutyunyan (1912–1993), who published a monograph in Russian on this topic [47]. Actual problems of the mechanics are the general theory of growth processes of solids, the numerical modeling of growth, moving boundaries and interfaces, surface effect, phase transitions, dislocations and disclinations in solids, crystal growth, growth of bone and soft tissues, laser deposition, solidification of melts, and electrolytic formation. Some of them are discussed, for example, in [48–50].

4.4. Reinforced materials
The final example of advanced materials is related to reinforced materials. By combining two or more different materials, one gets a new material with new properties. Considering the specific effective properties, it is obvious that, as a result a light-weight structural material is designed. The classical case is the are uni-directional fibre-reinforced layers. Finally, the layers can be combined and a laminate can be established. The classical ideas are published in various textbooks and monographs, for example in [51]. One of the actual problems is related to particle-reinforced matrix materials while taking into account the interphases [52–54].

5. Final Remarks
There are much more advanced models. At the moment we have a renaissance of the Cosserat theory (Eugène Cosserat, 1866–1931, François Cosserat, 1852–1914) with applications to foams [34] or bones [55]. Further developments are related to micropolar continua [56]. Another direction is related to the application of higher gradients [57–60].

Summarizing up to now Continuum Mechanics is a actual branch of Mechanics with new directions. The limits are not clear since the continuum mechanics approach is applied to problems with smaller and smaller sizes. There is no need for new theories — only the classical continuum mechanics should be improved. There is only one important item: the requirements of the continuum definition should be fulfilled.

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