THREE-PHASE FREAK WAVES

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Abstract. Three-phase finite-gap solutions of the focusing non-linear Schrödinger, Kadomtsev-Petviashvili and Hirota equations are constructed. These solutions have a behavior of almost-periodic “freak waves”.

Introduction

This study was motivated by the intention to demonstrate the behavior of three-phase extreme waves. In the last time it was realized that the simplest and most universal model for such waves is the focusing nonlinear Schrödinger equation (NLS)

\[ ip_t + p_{xx} + 2|p|^2p = 0, \quad i^2 = -1, \]

Equation (1) is used when describing distribution on the surface of the ocean weakly nonlinear quasi-monochromatic wave packets with a relatively steep fronts since 1968 [49]. An application of this equation to the problems of nonlinear optics was known earlier [7]. Since the equation (1) is a model of first approximation, it will appear in simulations of many weakly nonlinear phenomena. Fields of application of this equation are from plasma physics [32] to financial markets [48].

One of equation (1) property is a modulation instability, leading to the appearance of so-called “freak waves” (in hydrodynamics known as “rogue waves”) [2]. These waves are localized in space and time amplitude’s peaks. In the last 20 years, first in hydrodynamics, and then in nonlinear optics, these waves have been the object of numerous theoretical and experimental studies [2]. Such attention to the problem of “freak waves” is due, in particular, losses from destruction “rogue waves” oil platforms, tankers, container ships and other large vessels.

There are many more precise and more complex models, which give a more fine description of “freak waves” [2]. These models can be divided into two classes. For some models, like to equation (1) can be applied analytical methods. Other models are non-integrable and can be solved by numerical methods only. Analytical methods include:

- inverse scattering transform method;
- finite-gap integration method;
- Bäcklund transform method;
- Darboux transform method;
- Hirota method.

In the present work we use a finite-gap integration method. This method was created in the works of Dubrovin, Novikov, Marchenko, Lax, McKean, van Moerbeke, Matveev, Its, Krichever [11, 15, 24, 25, 30, 33, 35, 37, 39] (see also review article [36]). It should be mentioned that another

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method of constructing finite-gap solutions of integrable nonlinear equations exists \[27, 28, 38, 40\]. Let us remark that first method is based on Baker-Akhiezer function but second method is based on some Fays identities \[16\]. In our paper we use first method and Its’ and Kotlyarov’s classic formulas \[23, 26\] (see also \[6\]).

Our aim here is to show a behavior of three-phase algebro-geometric solutions of NLS, KP-I and Hirota equations. The first section of this paper contains the basic notations and classic formulas for algebro-geometric solutions of considered integrable non-linear equations. The second section of the paper is devoted to the periodicity of three-phase solutions of NLS, KP-I and Hirota equations. In the third section we consider an example of three-phase algebro-geometric solution of KP-I and Hirota equations for different values of parameters.

1. **Finite-gap multi-phase solutions of the NLS equation**

The nonlinear differential equations that are integrated by methods of the algebraic geometry, can be obtained as a compatibility condition of system of the ordinary linear differential equations with spectral parameter \[6, 19, 20\]. In particular, let us consider the following equations \[19, 22, 42\]

\[
\begin{align*}
Y_x &= iY, \\
Y_z &= 2iY, \\
Y_t &= 2iY,
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{U} &= -\lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} 0 & i\psi \\ -i\phi & 0 \end{pmatrix}, \\
\mathcal{V} &= 2\lambda\mathcal{U} + \mathcal{V}_0, \\
\mathcal{W} &= 4\lambda^2\mathcal{U} + 2\lambda\mathcal{V}_0 + \mathcal{W}_1,
\end{align*}
\]

\(\lambda\) is a spectral parameter. Using these equations and additional relations

\[
(Y_x)_z = (Y_z)_x, \quad (Y_x)_t = (Y_t)_x
\]

one can easy obtain so-called equations of zero curvature

\[
\begin{align*}
\mathcal{U}_z - \mathcal{V}_x + \mathcal{V}\mathcal{U} - \mathcal{U}\mathcal{V} &= 0, \\
\mathcal{U}_t - \mathcal{W}_x + \mathcal{W}\mathcal{U} - \mathcal{U}\mathcal{W} &= 0,
\end{align*}
\]

which should be valid for all values of spectral parameter \(\lambda\). Respectively, it follows from eqs. \(3\) that matrices \(\mathcal{V}_0, \mathcal{W}_0, \mathcal{W}_1\) have to look like

\[
\begin{align*}
\mathcal{V}_0 &= \begin{pmatrix} -i\phi & -\psi_x \\ -\phi_x & i\psi \end{pmatrix}, \\
\mathcal{W}_1 &= \begin{pmatrix} \psi_x\phi - \psi\phi_x & 2i\psi^2\phi - i\psi_x \\ -2i\psi\phi_x + i\phi_{xx} & \psi\phi_x - \psi_x\phi \end{pmatrix},
\end{align*}
\]

Also that, accordingly, \(\mathcal{W} = 2\lambda\mathcal{V} + \mathcal{W}_1\). Conditions \(3\) lead also to additional system of equations (parities). First system is the coupled nonlinear Schrödinger equation

\[
\begin{align*}
\begin{cases}
  i\psi_x + \psi_{xx} - 2\psi^2\phi = 0, \\
  i\phi_x - \phi_{xx} + 2\psi^2\phi = 0,
\end{cases}
\end{align*}
\]

and the second system is the coupled modified Korteweg-de Vries equation

\[
\begin{align*}
\begin{cases}
  \psi_t + \psi_{xxx} - 6\psi\psi_x = 0, \\
  \phi_t + \phi_{xxx} - 6\psi\phi_x = 0.
\end{cases}
\end{align*}
\]
These two systems of the nonlinear differential equations are closely connected with two other ones. Namely, by differentiating eqs. (5) on \(x\) and substituting them in (6), one obtains the coupled modified two-dimensional nonlinear Schrödinger equation in a cone coordinates \[31\]

\[
\begin{align*}
\psi_t + \psi_{xx} + 2i(\psi\phi_x - \phi\psi_x) = 0, \\
\phi_t - \phi_{xx} + 2i(\phi\psi_x - \psi\phi_x) = 0,
\end{align*}
\]

Also the functions \(\psi(x,t,-\alpha t)\) and \(\phi(x,t,-\alpha t)\) are solutions of the coupled integrable Hirota equation (\(\alpha \in \mathbb{R}\)) \[8\]

\[
\begin{align*}
\psi_t + \psi_{xx} - 2|\psi|^2\psi - i\alpha(\psi_{xxx} - 6|\psi|^2\psi_x) = 0, \\
\phi_t - \phi_{xx} + 2\psi^2\phi - i\alpha(\phi_{xxx} - 6\psi\phi\phi_x) = 0,
\end{align*}
\]

if \(\psi(x,z,t)\) and \(\phi(x,z,t)\) are solutions of (5) and (6).

Systems of the nonlinear differential equations (5), (6) are the first two integrable systems from the AKNS hierarchy \[19\]. One of features of finite-gap multi-phase solutions of the integrable nonlinear equations is that fact that in some sense they are the solutions of all hierarchy. Particulary, our solutions can be used for constructing solutions of generalized nonlinear Schrödinger equation \[47\]. Substitutions \(\phi = \pm \psi\) into eq. (5) give us a standard form of the nonlinear Schrödinger equation. Particularly, for \(\phi = -\psi\) equations (5) transform to (1) \[13, 22, 26\] and equations (8) transform to the integrable Hirota equation \[4, 8, 21, 34\] \[9\]

\[
\begin{align*}
\psi_t + \psi_{xx} + 2|\psi|^2\psi - i\alpha(\psi_{xxx} - 6|\psi|^2\psi_x) = 0.
\end{align*}
\]

It is also easy to check that for any \(\psi\) and \(\phi\), that satisfy both (5) and (6) simultaneously, the function \(u(x,z,t) = -2\psi\phi\) is a solution of the Kadomtsev-Petviashvili-I equation (KP-I) \[10\]  

\[
3u_{zzx} = (4u_t + u_{xxx} + 6|\psi|^2\psi_x)_x.
\]

In the case \(\phi = \pm \overline{\psi}\) this solution is a real function.

Finite-gap solutions of systems (5), (6) are parameterized by the hyperelliptic curve \(\Gamma = \{(\chi, \lambda)\}\) of the genus \(g\) \[19, 42\]:

\[
\chi^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j),
\]

The branch points \((\lambda = \lambda_j, j = 1, \ldots, 2g+2)\) of this curve are the endpoints of the spectral arcs of continuous spectrum of Dirac operator (2a). Infinitely far point of the spectrum corresponds two different points \(\mathcal{P}_\pm\) on the curve \(\Gamma\). In the case \(\phi = -\overline{\psi}\) the curve \(\Gamma\) has the form

\[
\Gamma : \chi^2 = \prod_{j=1}^{g+1} (\lambda - \lambda_j)(\lambda - \overline{\lambda}_j) = \lambda^{2g+2} + \sum_{j=1}^{2g+2} \lambda_j \overline{\lambda}_j^{2g+2-j}, \quad \exists \chi_j = 0, \quad \exists (\lambda_j) \neq 0.
\]

Following a standard procedure of constructing a finite-gap solutions \[6, 13, 42\], let us to choose on \(\Gamma\) a canonical basis of cycles \(\gamma^j = (a_1, \ldots, a_g, b_1, \ldots, b_g)\) with matrix of intersection indices

\[
C_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

From the condition \(\phi = -\overline{\psi}\) it following that this basis of cycles satisfies the transformation relations \[6, 13\]

\[
\tilde{\gamma}_1 a = -a, \quad \tilde{\gamma}_1 b = b + K a.
\]
where $\tau_1$ is anti-holomorphic involution, $\tau_1: (\chi, \lambda) \to (\overline{\chi}, \overline{\lambda})$.

Let us also consider normalized holomorphic differentials $dU_j$:

$$\oint_{a_k} dU_j = \delta_{kj}, \quad k, j = 1, \ldots, g,$$

and a matrix of periods $B$ of the curve $\Gamma$:

$$B_{kj} = \oint_{b_k} dU_j, \quad k, j = 1, \ldots, g.$$

It is well known (see, for example [5, 13]) that the matrix $B$ is a symmetric matrix with positively defined imaginary part.

Let us introduce in consideration $g$-dimensional Riemann theta function with characteristics $\eta, \zeta \in \mathbb{R}^g$ [5, 13, 16]:

$$\Theta[\eta'; \zeta'; (p)|B] = \sum_{m \in \mathbb{Z}^g} \exp\{\pi i (m + \eta)' B (m + \eta) + 2\pi i (m + \eta)' (p + \zeta)\},$$

where $B$ is a matrix of periods, $p \in \mathbb{C}^g$ and summation passes over an integer $g$-dimensional lattice.

Let us also define normalized Abelian integrals on the $\Gamma$: of the second kind, $\Omega_j(P)$, $j = 1, 2, 3$ and the third kind, $\omega_0(P)$, with the following asymptotic at infinitely far points $\mathcal{P}_{\infty}^\pm$:

$$\oint_{a_k} d\Omega_1 = \oint_{a_k} d\Omega_2 = \oint_{a_k} d\Omega_3 = \oint_{a_k} d\omega_0 = 0, \quad k = 1, \ldots, g,$$

$$\Omega_1(P) = \mp i \left(\lambda - K_1 + O(\lambda^{-1})\right), \quad P \to \mathcal{P}_{\infty}^\pm,$$

$$\Omega_2(P) = \mp i \left(2\lambda^2 - K_2 + O(\lambda^{-1})\right), \quad P \to \mathcal{P}_{\infty}^\pm,$$

$$\Omega_3(P) = \mp i \left(4\lambda^3 - K_3 + O(\lambda^{-1})\right), \quad P \to \mathcal{P}_{\infty}^\pm,$$

$$\omega_0(P) = \mp (\ln \lambda - \ln K_0 + O(\lambda^{-1})) \times \mathcal{P} \to \mathcal{P}_{\infty}^\pm.$$

Let us denote by $2\pi i U, 2\pi i V, 2\pi i W$ the vectors of $b$-periods of Abelian integrals of the second kind $\Omega_1(P), \Omega_2(P), \Omega_3(P)$, respectively.

**Theorem 1** ([5, 12]). Function

$$Y(P, x, z, t) = \begin{pmatrix} y_1(P, x, z, t) & y_1(\tau_0 P, x, z, t) \\ y_2(P, x, z, t) & y_2(\tau_0 P, x, z, t) \end{pmatrix},$$

where $\tau_0$ is hyperelliptic involution, $\tau_0: (\chi, \lambda) \to (-\chi, \lambda)$,

$$y_1(P, x, z, t) = \frac{\Theta(U(P) + Ux + Vz + Wt - X)\Theta(Z)}{\Theta(U(P) - X)\Theta(Ux + Vz + Wt + Z)} \times \exp\{\Omega_1(P)x + \Omega_2(P)z + \Omega_3(P)t + i\Phi(x, z, t)\},$$

$$y_2(P, x, z, t) = \rho \frac{\Theta(U(P) + Ux + Vz + Wt + \Delta - X)\Theta(Z - \Delta)}{\Theta(U(P) - X)\Theta(Ux + Vz + Wt + Z)} \times \exp\{\Omega_1(P)x + \Omega_2(P)z + \Omega_3(P)t - i\Phi(x, z, t) + \omega_0(P)\},$$
is the eigenfunction of the Dirac operator \((2a)\) with functions

\[
\psi(x, z, t) = \frac{2K_0 \Theta(Z)\Theta(Ux + Vz + Wt + Z - \Delta)}{\Theta(Z - \Delta)\Theta(Ux + Vz + Wt + Z)} \exp\{2i\Phi(x, z, t)\},
\]

(17)

\[
\phi(x, z, t) = 2\rho K_0 \frac{\Theta(Z - \Delta)\Theta(Ux + Vz + Wt + Z + \Delta)}{\Theta(Z)\Theta(Ux + Vz + Wt + Z)} \exp\{-2i\Phi(x, z, t)\},
\]

for any \(z, t\) and \(\rho \neq 0\). The functions (17) satisfy the equations (5) and (6). Here \(\Delta\) is vector of holomorphic Abelian integrals, calculated along a path, connecting \(\mathcal{P}_\infty^-\) and \(\mathcal{P}_\infty^+\), without crossing any of basic’s cycles,

\[
\Delta = \mathcal{U}(\mathcal{P}_\infty^+) - \mathcal{U}(\mathcal{P}_\infty^-), \quad \Phi(x, z, t) = K_1x + K_2z + K_3t,
\]

\[
\mathbf{X} = \mathcal{K} + \sum_{j=1}^g \mathcal{U}(\mathcal{P}_j), \quad \mathbf{Z} = \mathcal{U}(\mathcal{P}_\infty^+) - \mathbf{X},
\]

\(\mathcal{K}\) is a vector of Riemann constants \([5][13][16][25]\), \(\mathcal{P}_j, j = 1, \ldots, g\) is a non-special divisor. If the spectral curve \(\Gamma\) satisfies the condition \([11]\) then the following equalities hold

\[
|\psi|^2 = -4K_0^2 \frac{\Theta(Ux + Vz + Wt + Z - \Delta)\Theta(Ux + Vz + Wt + Z + \Delta)}{\Theta^2(Ux + Vz + Wt + Z)},
\]

\[
\Im \mathbf{U} = \Im \mathbf{V} = \Im \mathbf{W} = \Im \mathbf{Z} = 0, \quad K_0^2 < 0.
\]

It is easy to see that corresponding solution of KP-I equation (10) has the form

\[
u(x, z, t) = -8K_0 \frac{\Theta(Ux + Vz + Wt + Z - \Delta)\Theta(Ux + Vz + Wt + Z + \Delta)}{\Theta^2(Ux + Vz + Wt + Z)},
\]

and that the square of amplitude of solution of Hirota equation (9) equals

\[
|\psi_H|^2(x, t) = -4K_0^2 \frac{\Theta(Ux + (V - \alpha W)t + Z - \Delta)\Theta(Ux + (V - \alpha W)t + Z + \Delta)}{\Theta^2(Ux + (V - \alpha W)t + Z)}.
\]

2. Features of three-phase solutions

In a case \(g = 3\) basis of normalized holomorphic differentials is defined by the formula \([6][13]\):

\[
d\mathcal{U}_k = (c_{k1}\lambda^2 + c_{k2}\lambda + c_{k3}) \frac{d\lambda}{\lambda},
\]

where

\[
\mathbf{C} = (\mathbf{A}^t)^{-1}, \quad A_{jm} = \oint_{\mathcal{A}_j} \lambda^{3-m} \frac{d\lambda}{\lambda}.
\]

It follows from equation (\(\ell\) is an arbitrary path on \(\Gamma\))

\[
\int_{\mathcal{A}_\ell} d\omega = \int_{\ell} \tau^* d\omega,
\]

that

\[
\mathcal{A}_{jm} = \oint_{\mathcal{A}_j} \lambda^{3-m} d\lambda = \oint_{\mathcal{A}_j} \tau^1 (\lambda^{3-m} \frac{d\lambda}{\lambda}) = \oint_{\mathcal{A}_j} \lambda^{3-m} d\lambda,
\]

\[
\mathcal{A}_{jm} = \oint_{\mathcal{A}_j} \lambda^{3-m} \frac{d\lambda}{\lambda} = - \oint_{\mathcal{A}_j} \lambda^{3-m} \frac{d\lambda}{\lambda} = -A_{jm}.
\]
Therefore $\mathcal{A} = -A$ and $\mathcal{C} = -C$. Dealing similarly with integrals on $b$-cycles, we obtain
\begin{equation}
\mathcal{B} = -B - K \quad \text{or} \quad \Re B = -\frac{1}{2} K.
\end{equation}

It follows from bilinear relations of Riemann (see, for example, [5, 6, 13]) that coordinates of vectors $U, V, W$ can be written as
\begin{align*}
U_m &= -i \left( \frac{dU_m}{d\xi_-} \bigg|_{\xi_- = 0} - \frac{dU_m}{d\xi_+} \bigg|_{\xi_+ = 0} \right), \\
V_m &= -2i \left( \frac{d^2U_m}{d\xi_-^2} \bigg|_{\xi_- = 0} - \frac{d^2U_m}{d\xi_+^2} \bigg|_{\xi_+ = 0} \right), \\
W_m &= -2i \left( \frac{d^3U_m}{d\xi_-^3} \bigg|_{\xi_- = 0} - \frac{d^3U_m}{d\xi_+^3} \bigg|_{\xi_+ = 0} \right),
\end{align*}
where $\xi_{\pm} = 1/\lambda$ are local parameters in the neighborhood of infinitely far points $P_{\pm\infty}$. Calculating derivatives, we obtain relations
\begin{align*}
U_m &= -2i\chi c_m, \quad V_m = 2i\chi_1 c_m - 4ic_m, \\
W_m &= i(4\chi_2 - 3\chi_1^2) c_m + 4i\chi_1 c_m - 8ic_m,
\end{align*}
or
\begin{equation}
(U, V, W) = iC \begin{pmatrix}
-2 & 2\chi_1 & 4\chi_2 - 3\chi_1^2 \\
0 & -4 & 4\chi_1 \\
0 & 0 & -8
\end{pmatrix}.
\end{equation}

It follows from (23) that the vectors $U, V, W$ are real and linearly independent. Therefore $U, V, W$ are basis vectors in $\mathbb{R}^3$. Hence any vector from $\mathbb{R}^3$ can be presented in the form of the linear combinations of these vectors. In particular, for the vectors of the periods of the three-dimensional theta-functions $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ we can write the following relations
\begin{equation*}
e_k = \lambda_k U + \mathcal{Z}_k V + \mathcal{T}_k W.
\end{equation*}
Therefore three-phase solutions [19] of equation KP-I [10] is a periodic function in a three-dimensional space
\begin{equation*}
u(x + \lambda_k, z + \mathcal{Z}_k, t + \mathcal{T}_k) = u(x, z, t).
\end{equation*}
If a three-phase solution of [10] has a form of freak waves then maxima of its amplitude are in nodes of a three-dimensional lattice with edges $(\lambda_k, \mathcal{Z}_k, \mathcal{T}_k)$. These edges can be found by an inversion of the matrix $(U, V, W)$:
\begin{equation*}
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mathcal{Z}_1 & \mathcal{Z}_2 & \mathcal{Z}_3 \\
\mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3
\end{pmatrix} = (U, V, W)^{-1} = i \begin{pmatrix}
1/2 & \chi_1/4 & \chi_2/4 - \chi_1^2/16 \\
0 & 1/4 & \chi_1/8 \\
0 & 0 & 1/8
\end{pmatrix} A^t.
\end{equation*}
Therefore for three-phases solutions of equation KP-I [10] it is possible to describe their behaviour as following: after a time interval $\Delta t = \mathcal{T}_k$ a surface of solution $u(x, z)$ reproduces itself with a shift on plane XOZ on a vector $(\lambda_k, \mathcal{Z}_k)$.

As the three-phase solution of the equations [1] depends on two coordinates, $x$ and $z$, and the third coordinate $t$ is considered as parameter, the value of amplitude of this solution depends on the distance between the nodes of the given three-dimensional lattice and a plane $t = t_0$. Hence,
in the difference of a case of the two-phase solution [43,45,46], where change of initial phase $Z$ led to trivial shift of the solution on plane $XOZ$, the amplitude of the three-phase solution (17) of equations (1) depends on a choice of initial phase $W_{t_0} + Z$ a little bit more complicated.

3. AN EXAMPLE OF THREE-PHASE SOLUTION

Let us consider a spectral curve $\Gamma_3 = \{\chi, \lambda\}$ of genus $g = 3$:

\begin{equation}
\Gamma_3 : \chi^2 = ((\lambda - \lambda_0)^4 - 2a^2(\lambda - \lambda_0)^2 \cos 2\varphi + a^4)((\lambda - \lambda_0)^4 - 2b^2(\lambda - \lambda_0)^2 \cos 2\varphi + b^4),
\end{equation}

where $0 < a < b$, $\pi/4 < \varphi < \pi/2$.

Let us choose the basis of cycles on $\Gamma_3$ as it is shown on fig. 1.

![Figure 1. Canonical basis of cycles on $\Gamma_3$](image)

It is easy to check that the anti-holomorphic involution $\tau_1$ transforms the canonical basis of cycles accordingly to relations (12) with the matrix

\[
K = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 
\end{pmatrix}.
\]

Also there are three holomorphic involutions on $\Gamma_3$:

\[
\tau_0 : (\chi, \lambda) \rightarrow (-\chi, \lambda),
\]
\[
\tau_2 : (\chi, \lambda) \rightarrow (\chi, 2\lambda_0 - \lambda),
\]
\[
\tau_3; (\chi, \lambda) \rightarrow (a^2b^2(\lambda - \lambda_0)^{-4}\chi, \lambda_0 + ab(\lambda - \lambda_0)^{-1}).
\]

As a corollary the curve $\Gamma_3$ covers two following curves:

(1) $\Gamma_1 = \Gamma_3/\tau_2$ of genus $g = 1$

\[
\Gamma_1 : \chi_+^2 = (t^2 - 2a^2t \cos 2\varphi + a^4)(t^2 - 2b^2t \cos 2\varphi + b^4),
\]

(2) $\Gamma_2 = \Gamma_3/(\tau_0\tau_2)$ of genus $g = 2$

\[
\Gamma_2 : \chi_-^2 = t(t^2 - 2a^2t \cos 2\varphi + a^4)(t^2 - 2b^2t \cos 2\varphi + b^4),
\]
where \( t = (\lambda - \lambda_0)^2, \chi_+ = \chi, \chi_- = (\lambda - \lambda_0)\chi, \) and
\[
\frac{dt}{\chi^+} = 2(\lambda - \lambda_0)d\lambda, \quad \frac{d\tau}{\chi} = 2(\lambda - \lambda_0)^2d\lambda, \quad \frac{d\tau}{\chi^-} = 2d\lambda.
\]
The curves \( \Gamma_1 \) and \( \Gamma_2 \) are shown on fig. 2 and 3 where \( t_1 = b^2e^{2i\varphi}, t_2 = a^2e^{2i\varphi}. \)

**Figure 2.** The curve \( \Gamma_1 \)

The coverings generate the following transformations of cycles:
\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\rightarrow
S
\begin{pmatrix}
a_1^2 \\
a_1^2 \\
a_2^2
\end{pmatrix}
+ P
\begin{pmatrix}
b_1^2 \\
b_2 \\
b_3
\end{pmatrix},
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\rightarrow
Q
\begin{pmatrix}
a_1^2 \\
a_1^2 \\
a_2^2
\end{pmatrix}
+ R
\begin{pmatrix}
b_1^2 \\
b_2 \\
b_3
\end{pmatrix},
\]
where
\[
S = \begin{pmatrix}
-1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & -1
\end{pmatrix},
\quad P = \begin{pmatrix}
0 & -2 & 0 \\
0 & 0 & 2 \\
1 & 1 & 1
\end{pmatrix},
\quad Q = \begin{pmatrix}
-1 & 1 & 0 \\
0 & 0 & -1 \\
1 & 1 & -1
\end{pmatrix},
\quad R = \begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & -1
\end{pmatrix}.
\]

Let us remind that these matrices should satisfy to relations
\( S^tQ = Q^tS, \quad R^tP = P^tR, \quad S^tR - Q^tP = nI, \)
where \( I \) is identity matrix, \( n = 2 \) is number of covering sheets.

Due to involution \( \tau_3, \) the curve \( \Gamma_2 \) covers two elliptic curves \( \Gamma_\pm \) (fig. 4 and 5):
\[
\Gamma_\pm: \quad \nu_\pm = (s \pm 2ab)(s^2 - 2(a^2 + b^2)s \cos 2\varphi + a^4 + b^4 + 2a^2b^2 \cos 4\varphi),
\]
where
\[
s = t + \frac{a^2b^2}{t}, \quad \nu_\pm = \frac{t \pm ab}{t^2 - \chi^-}, \quad \frac{ds}{\nu_\pm} = \frac{(t \mp ab)dt}{\chi^-}.
\]

The coverings of \( \Gamma_2 \) on \( \Gamma_\pm \) generate the following mappings of cycles
\[
\begin{pmatrix}
a_1^2 \\
a_2^2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
a_+ \\
a_-
\end{pmatrix},
\begin{pmatrix}
b_1^2 \\
b_2^2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
b_+ \\
b_-
\end{pmatrix},
\]
As a result we have
\[
\begin{pmatrix}
a_1 \\ a_2 \\ a_3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
a^+_1 \\ a^+_2 \\ a^+_3
\end{pmatrix}
+ \begin{pmatrix}
0 & -2 & -2 \\ 0 & 2 & -2 \\ 0 & -2 & 2
\end{pmatrix}
\begin{pmatrix}
b_1 \\ b_2 \\ b_3
\end{pmatrix},
\]
\[
\begin{pmatrix}
b_1 \\ b_2 \\ b_3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
a^+_1 \\ a^+_2 \\ a^+_3
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
b_1 \\ b_2 \\ b_3
\end{pmatrix}.
\]

From (25), (26) and from the relations
\[
\frac{d\lambda}{\chi} = \frac{1}{4ab \nu_-} ds - \frac{1}{4ab \nu_+} ds,
\]
\[
\frac{\lambda d\lambda}{\chi} = \frac{1}{2} \frac{dt}{\chi_+} + \frac{\lambda_0}{4ab \nu_-} ds - \frac{\lambda_0}{4ab \nu_+} ds,
\]
\[
\frac{\lambda^2 d\lambda}{\chi} = \frac{\lambda_0}{\chi_+} dt + \frac{\lambda_0^2 + ab}{4ab \nu_-} ds - \frac{\lambda_0^2 - ab}{4ab \nu_+} ds,
\]
it follows that the matrices \(C\) and \(B\) equal
\[
C = \begin{pmatrix}
\mathcal{C}_1 + \mathcal{C}_3 & -2\lambda_0 (\mathcal{C}_1 + \mathcal{C}_3) & (\lambda_0^2 - ab) \mathcal{C}_1 + (\lambda_0^2 + ab) \mathcal{C}_3 \\
\mathcal{C}_1 & \mathcal{C}_2 - 2\lambda_0 \mathcal{C}_1 & (\lambda_0^2 - ab) \mathcal{C}_1 - \lambda_0 \mathcal{C}_2 \\
\mathcal{C}_3 & \mathcal{C}_2 - 2\lambda_0 \mathcal{C}_3 & (\lambda_0^2 + ab) \mathcal{C}_3 - \lambda_0 \mathcal{C}_2
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
ib_1 + ib_3 & ib_1 - 1/2 & ib_3 - 1/2 \\
ib_1 - 1/2 & ib_1 + ib_2 & ib_2 - 1/2 \\
ib_3 - 1/2 & ib_2 - 1/2 & ib(\mathcal{B}_2 + \mathcal{B}_3)
\end{pmatrix}.
\]
where
\[
\begin{align*}
c_1 &= \frac{1}{2(\alpha_1 - 2\beta_1)}, \quad c_2 = \frac{1}{2\alpha_2}, \quad c_3 = \frac{1}{2(\alpha_3 - 2\beta_3)}, \\
i_b &= \frac{\alpha_1}{2(\alpha_1 - 2\beta_1)}, \quad i_b^2 = \frac{\beta_2}{2\alpha_2}, \quad i_b^3 = \frac{\alpha_3}{2(\alpha_3 - 2\beta_3)}, \\
\alpha_1 &= \frac{1}{2} \int_{a_+} ds, \quad \alpha_2 = \frac{1}{2} \int_{a_1} dt, \quad \alpha_3 = \frac{1}{2} \int_{a_+} ds, \\
\beta_1 &= \frac{1}{2} \int_{b_+} ds, \quad \beta_2 = \frac{1}{2} \int_{b_1} dt, \quad \beta_3 = \frac{1}{2} \int_{b_+} ds.
\end{align*}
\]

From the structure of $B$-matrix and from matrix version of Appel’s theorem \[41\] it follows that the Riemann theta function of curve $\Gamma_3$ equals:
\[
(27) \quad \Theta(p|B) = f(\bar{p}_1, \bar{p}_2, \bar{p}_3) =
\begin{align*}
\vartheta_3(\bar{p}_1 | h_1)\vartheta_3(\bar{p}_2 | h_2)\vartheta_3(\bar{p}_3 | h_3) + \vartheta_4(\bar{p}_1 | h_1)\vartheta_4(\bar{p}_2 | h_2)\vartheta_4(\bar{p}_3 | h_3) + \\
+ \vartheta_1(\bar{p}_1 | h_1)\vartheta_2(\bar{p}_2 | h_2)\vartheta_1(\bar{p}_3 | h_3) + \vartheta_1(\bar{p}_1 | h_1)\vartheta_4(\bar{p}_2 | h_2)\vartheta_2(\bar{p}_3 | h_3),
\end{align*}
\]
where $\bar{p}_j = p_j + p_{j+1} - p_{j+2}, p_{j+3} = p_j, h_j = \exp(-4b_j)$.

The functions $\vartheta_j(p|h)$ are Jacobi elliptic theta functions \[1\]:
\[
\begin{align*}
\vartheta_1(p|h) &= 2 \sum_{m=1}^{\infty} (-1)^{m-1} h^{(m-1/2)^2} \sin[(2m - 1)\pi p], \\
\vartheta_2(p|h) &= 2 \sum_{m=1}^{\infty} h^{(m-1/2)^2} \cos[(2m - 1)\pi p], \\
\vartheta_3(p|h) &= 1 + 2 \sum_{m=1}^{\infty} h^{m^2} \cos(2m\pi p), \\
\vartheta_4(p|h) &= 1 + 2 \sum_{m=1}^{\infty} (-1)^{m} h^{m^2} \cos(2m\pi p).
\end{align*}
\]

Using the reduced form of theta-function \[27\] and values for vectors of periods, one obtains the following formula for the square of absolute value of the three-phase solution \[18\] of the focusing NLS equation \[1\]
\[
(28) \quad |\psi|^2 = -4K_0^2 f(k_1 x + \kappa_1 t + \delta_1, k_2 z + \delta_2, k_3 x + \kappa_3 t + \delta_3) \times \\
\times f(k_1 x + \kappa_1 t - \delta_1, k_2 z - \delta_2, k_3 x + \kappa_3 t - \delta_3) \times \{f(k_1 x + \kappa_1 t, k_2 z, k_3 x + \kappa_3 t)\}^{-2},
\]
where the function $f(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ is defined by equation \[27\], and
\[
\begin{align*}
k_1 &= -4i\xi_1, \quad k_2 = -8i\xi_2, \quad k_3 = -4i\xi_3, \\
\kappa_1 &= 4k_1(3\lambda_0^2 - ab + (a^2 + b^2) \cos(2\varphi)), \\
\kappa_3 &= 4k_3(3\lambda_0^2 + ab + (a^2 + b^2) \cos(2\varphi)).
\end{align*}
\]

It follows from \[27\], \[28\] that for $\lambda_0 = 0$ the amplitude of the constructed solution of NLS equation \[1\] is a periodic function in $z$, and for $\lambda_0 = 0, \varphi = \frac{1}{2} \arccos \left( \frac{\pm ab}{a^2 + b^2} \right)$ it is a periodic function in $z$ and in $t$. 
Let us recall that the three-phase solution \( u(x, z, t) \) of the KP-I equation (10) and the square of amplitude \( |\psi_H(x, t)|^2 \) of three-phase solution of Hirota equation (9) can be constructed from (28) by relations \( u(x, z, t) = 2 |\psi(x, z, t)|^2 \) and \( |\psi_H(x, t)|^2 = |\psi(x, t, -\alpha t)|^2 \).

The three-phase solution of KP-I equation for \( ab = 1, \sqrt{b/a} = 1.3, \varphi = 0.3\pi \) at the different moment of time \( t \) and for \( \lambda_0 = 0 \) is presented on the fig. 6-9. The same solution for \( \lambda_0 = k_2/(4k_1) \) one can see on the fig. 10-13. It is easy to see all three phase of solution on figures 6-13. Two phase are shortwave and third phase is a long-wave envelope. One can see also on fig. 6-9 that the solution for \( \lambda_0 = 0 \) is periodic in \( z \), and that the long-wave envelope move to the right side.

**Figure 6.** Three-phase solution of KP-I equation for \( \lambda_0 = 0, t = 0 \)

**Figure 7.** Three-phase solution of KP-I equation for \( \lambda_0 = 0, t = 0.1 \)

**Figure 8.** Three-phase solution of KP-I equation for \( \lambda_0 = 0, t = 0.2 \)

**Figure 9.** Three-phase solution of KP-I equation for \( \lambda_0 = 0, t = 0.3 \)
The three-phase solution of Hirota equation for $ab = 1$, $\sqrt{b/a} = 1.3$, $\varphi = 0.3\pi$, $\alpha = 0.1$ and for different values of $\lambda_0$ is presented on fig. 14-17. It is easy to see all three phase of solution only on fig. 15.

**Concluding remarks**

Let us remark that the considered solution can not be used for a degeneration to quasi-rational one because odd number of phases. I.e. very popular solutions can not be obtained as a limit of three-phase solution. Analyzing a family of quasi-rational solutions from [9][10][17][18] we make the following conjecture.
(1) Quasi-rational rank $n$ solution of NLS equation can be obtained as a limit of $2n$-phase elliptic solution.

(2) Spectral curve of corresponding elliptic solution should be a 2-sheet unramified covering over spectral curve of $n$-phase elliptic solution of KdV equation.

The rank 1 quasi-rational solution (Peregrine soliton) was obtained as limit of two-phase elliptic one in [44], and for obtaining rank 2 quasi-rational solution we should use a spectral curve of genus $g = 4$.

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