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1. Introduction

Disorder effects in quantum electronic systems have led to a variety of novel phases. Fermionic systems have played a special role in our understanding of such effects. Indeed for fermions, the Pauli principle prevent the fermions to be trapped macroscopically in the minima of the random potential, making the non interacting case worthwhile to study. Disorder then leads to the rich physics of Anderson localization. Using both scaling theories and sophisticated field theoretical techniques, it is now known that electrons are localized by disorder in one and two dimensions, whereas a mobility edge exists in three dimensions. [7, 3, 1, 35, 12, 11]

The situation becomes much more complicated when one wants to take into account the electron-electron interaction. Such a question was crucial for the understanding of doped semiconductors [33]. In addition recent experiments in two dimensional electron gas systems have prompted the question of whether a metal-insulator transition could exist in interacting systems (see [2] and references therein), stimulating further interest in this problem.

On the theoretical side the question is extremely complicated. Most of the theoretical approaches used for free electrons either fail or become much more complicated when interactions are included which makes it more difficult to obtain unambiguous answers. Perturbative calculations or renormalization group calculations can be made for weak interactions. Unfortunately they scale to strong coupling, which leaves the question of the large scale/low energy physics still open [4, 14, 27].

In fact, even the pure system is interesting. Indeed for weak interactions it is reasonable to expect to have a Fermi liquid behavior, at least for three dimensional systems. Upon increasing the interaction is was predicted long ago by Wigner that
the electrons would crystallize [36]. Such crystallization can be induced by other means, decreasing the density of particles or quenching their kinetic energy by applying a magnetic field.

This suggests another line of attack for the fermionic disordered problem: start from the Wigner crystal phase and study the effect of disorder on such a phase. In the crystalline phase one can expect the statistics to be less important and thus the problem to be more tractable. Such a problem falls in the more general category of disordered elastic systems, which exhibit competition between elastic forces that like some ordered structure (perfect crystal for a periodic structure, flat structures for manifolds) and disorder. Physical systems entering in this category range from manifold (magnetic domain walls, wetting interfaces etc.), periodic classical systems (vortex lattice, colloids, magnetic bubbles, charges spheres, etc.), quantum systems (Luttinger liquids, stripe phases, charge and spin density waves, Wigner crystal). There has been an immense activity in these various domains and many recent progress and it is out of question in these few pages to cover such a vast topic. I will thus avoid completely here the topic of classical disordered systems. The reader who want to know more on that subject and see the links between the quantum systems discussed here and the classical problems is referred to the review [16], where this topic is discussed. This review provides references to other relevant review papers on this vast subject.

For the quantum problems I will confine the discussion to the two dimensional Wigner crystal. Rather than to try to embark on a review on the subject, I will focuss on some specific points of this problem. I will mostly insist on the basic concepts and present some little additional complements on the question of the compressibility in such systems. These notes are thus not self contained and rely heavily on other published material, both for contents and references. These few pages can thus be viewed as an “appendix” of the following papers:

- The basic concepts of quantum disordered elastic systems are summarized in the review paper [22]. The reader is thus referred to this paper for the basic physical ideas in this field and for the technology needed to treat such problems. In addition this paper contains of course references to further material. This paper also treats in detail the interesting case of the one dimensional disordered electron gas. This is a situation relevant for systems such as quantum wire or nanotubes. In that case electrons are known to lead to a non fermi liquid state, the Luttinger liquid, and the disorder effects are particularly drastic.

- The Wigner crystal is examined in details in [9, 10]. These papers contain a detailed discussion of the physical issues, problems and comparison with the experiments. They describes all the technical details of the approach that we have used and that I will briefly discuss in the paper, and give references to other relevant papers published on the subject.
The plan of these notes is the following: In Sec. 2 I will recall the experimental questions as well as the minimal ingredients underlying the description of a disordered Wigner crystal. Sec. 3 discusses the standard approach used to treat such problems in the past and its comparison with the recent experiments. Sec. 4 will very briefly present the results of [9, 10] with an emphasis on the differences between the approach used in Sec. 3. Sec. 5 discusses in details the question of the compressibility and capacitance measurements. Some aspects of the dynamics are presented in Section 6. Conclusions and perspectives are presented in Section 7.

2. Basic Questions

2.1. EXPERIMENTS

Although the theoretical concept of a Wigner crystal is easy to grasp it is much more difficult to check experimentally that such a phase is realized in nature. Indeed for other crystal states such as the vortex lattice or the charge density waves, imaging techniques (decoration, neutrons, X-rays) allow to directly see the crystal structure. In the case of the Wigner crystal no direct imaging has been possible so far, although it might become feasible in a near future. One is thus forced to see whether indirect measurements (mostly transport, sound absorption) can be consistently interpreted by assuming a Wigner crystal. The first evidences for a Wigner crystal are shown in Fig. 1. The optical conductivity, an example of which is shown in Fig. 2, provides detailed information as we will discuss it in more details in Sec. 4.

In all these experiments there is no direct evidence of the crystal structure. In order to know whether the transport experiments can be considered as a proof or not of the existence of the Wigner crystal is it thus specially important to have a reliable theory that allows to compute the transport properties. Such a task is far from being trivial given the complexity of the problem.

2.2. ELASTIC DESCRIPTION

Starting from the full electronic Hamiltonian (fermions with interactions and disorder) is a near impossible task. In the crystal phase some simplifications can be made since the particles are now discernable by their position. This allows for a minimal phenomenological model to describe such a crystal: one assumes that the particle are characterized by an equilibrium position $R^0_i$ and a displacement $u_i$ relative to this equilibrium position. In order to define uniquely the displacement one should not have topological defects such as dislocations in the crystal. I will come back to this point in Sec. 3. From the original quantum problem one has to define the “particles” of the crystal. If the wavefunction is localized enough then one can indeed ignore the exchange between the various sites and thus define “particles” that have a size given by the extension of the wavefunction as indicated
in Fig. 3. Of course the density fixes the lattice spacing $a$. These two lengthscales (size of particle, lattice spacing) are independent and should be kept. Then one retains for the energy the phonon modes of the crystal. This leads to the action (see [10] for more details):

\[
S = \frac{1}{2} \int \frac{1}{\beta} \sum_{\omega_n} u_L(q, \omega_n)(\rho_m \omega_n^2 + c_L(q))u_L(-q, -\omega_n) \quad (1)
\]

\[
+ u_T(q, \omega_n)(\rho_m \omega_n^2 + c_T(q))u_T(-q, -\omega_n)
\]

\[
+ \rho_m \omega_n [u_L(q, \omega_n)u_T(-q, -\omega_n) - u_T(q, \omega_n)u_L(-q, -\omega_n)]
\]
Figure 2. Optical conductivity for various densities for a 2DEG under strong magnetic field. The peak at a characteristic frequency (pinning frequency) is again an expected characteristic of a pinned crystalline structure (see Sec. 3.2) (from [29]).

\[ + \int d^2r \int_0^\beta d\tau V(r)\rho(r, \tau) \]

where we have used the decomposition of the displacements in longitudinal and transverse modes \( \bar{u} = \frac{2}{q} u_L(q) + (\frac{2}{q} \wedge z) u_T(q) \). \( \int_q \) denote the integration over the Brillouin zone \( \int_{BZ} \frac{d^2q}{(2\pi)^2} \), and the \( \omega_n \) are the standard Matsubara frequencies. The third term in (1) comes from the Lorentz force. \( \rho_m \approx \frac{m}{\pi a^2} \) is the mass density and \( \omega_c = eB/m \) the cyclotron frequency.

\( C_{L,T}(q) \) are the elastic coefficients for the longitudinal and transverse modes respectively. These coefficients can be obtained from an expansion of the coulomb correlation energy of the WC in terms of the displacements [8, 30]. Since the longitudinal mode describes compressional modes, it is drastically affected by the coulomb repulsion thus \( c_L(q) \propto q \), whereas the transverse mode describes shear and thus \( c_T(q) \propto q^2 \) as in elastic media with only short range interactions.

Finally, the last term describes the coupling to disorder, modelled here by a random potential \( V \). The density of particles

\[ \rho(r) = \sum_i \delta(r - R_i - u_i) \]  

where \( \delta \) is a \( \delta \)-like function of range \( l_c \) (see Figure 3) and \( u_i \equiv u(R_i) \). Since the disorder can vary at a lengthscale \( r_f \) \textit{a priori} shorter or comparable to the lattice spacing \( a \), the continuum limit \( u_i \to u(r) \), valid in the elastic limit \( |u_i - u_{i+1}| \ll a \) should be taken with care in the disorder term [17, 18]. This can be done using
Figure 3. The three length characterizing the Wigner crystal. The size $l_c$ of the "particles" in the crystal (at low temperature it is essentially given by the extension of the wavefunction around the equilibrium position, at large temperatures it is controlled by the thermal fluctuations and is the Lindemann length), $a$ the lattice spacing is controlled by the density of particles, and the disorder is correlated over a length $r_f$. The inset shows the triangular structure of the Wigner crystal. Particles are labeled by an equilibrium position $R_i$ and a displacement $u_i$. (From [10])

The decomposition of the density in terms of its Fourier components

$$\rho(r) \simeq \rho_0 - \rho_0 \nabla \cdot u + \rho_0 \sum_{K \neq 0} e^{iK \cdot (r - u(r))}$$  \hspace{1cm} (3)

where $\rho_0$ is the average density and $K$ are the reciprocal lattice vectors of the perfect crystal. The finite range of $\delta$ is recovered [18] by restricting the sum over $K$ to momentum of order $K_{\text{max}} \sim \pi/l_c$. The disorder is often assumed gaussian, a limit valid when there are many weak pins

$$V(r)V(r') = \Delta_{r_f}(r - r')$$  \hspace{1cm} (4)

$\Delta_{r_f}$ is a delta-like function of range $r_f$ which is the characteristic correlation length of the disorder potential (see Figure 3).

These characteristics lengthscales and Hamiltonian define the minimal model needed to describe a pinned Wigner crystal.
TABLE I. $q$ dependence of the eigenmodes in a Wigner crystal in the absence of magnetic field or for a very strong field.

| Mode   | zero field       | High magnetic field |
|--------|------------------|---------------------|
| $\omega_-(q)$ | $\propto q^2$ (trans.) | $\propto \frac{q^{3/2}}{\omega_c}$ |
| $\omega_+(q)$ | $\propto q$ (long.) | $\sim \omega_c$ |

2.3. CONSEQUENCES FOR PURE SYSTEM

For a pure system the consequences of the quadratic Hamiltonian (1) are easy to carry out. The eigenmodes of the system are easy to compute. In the absence of field the longitudinal one is plasmon like, whereas the transverse one is phonon-like. In the presence of a large magnetic field the two modes are mixed, giving the eigenmodes of Table I. These are the modes that were probed in the sound absorption experiment shown in Fig. 1.

For the crystal, the current is simply given by $J = e\rho_0 \partial_t u$, making thus the conductivity very simple to compute since it is essentially the correlator of the displacements (up to a factor $\omega$). In the absence of magnetic field the optical conductivity is a simple $\delta$ function at zero frequency, which traduces the fact that the crystal slides when submitted to an external force. In the presence of a finite magnetic field the electrons describes cyclotron orbits and the peak in conductivity is pushed to the cyclotron frequency $\omega_c$.

Of course these results are for the pure system only and the crucial question is to determine how the disorder changes the above results, in order to make contact with the experiments.

3. Conventional wisdom

3.1. BASIC IDEAS AND CONVENTIONAL WISDOM

In order to know how disorder can modify the above results and lead to pinning, it is necessary to solve the full problem (1), an herculean task. People have thus resorted to approximations. Various such approximation are presented in the review [22]. Based on the various approximate solutions a conventional wisdom on how a disordered elastic system should behave has emerged, as shown in Fig. 4. It was believed that because of disorder the crystal is “broken” into crystallites whose characteristic size is the pinning length. Topological defects (dislocations etc.) in the crystals would be generated at about the same lengthscale. All positional order in the crystal is thus lost beyond the “crystallites”. Each crystallite can thus be seen as pinned practically individually.
Figure 4. The traditional image of a disordered elastic system. The system is “broken” into crystallites of size \( R_a \). The size corresponds to relative displacements of the order of the lattice spacing between edges of the “crystallite”. Topological defects (e.g. dislocations) are argued to be induced by the disorder at the same characteristic lengthscale \( R_a \). All positional order is thus lost beyond \( R_a \). As we now know (see Sec. 4) this physical image is incorrect.

3.2. COMPARISON WITH EXPERIMENTS

This physical image inspired from pioneering theories used for charge density waves or for the pinning of vortex lattices (see [22] for more details and references) allows to compute, using some approximations, the optical conductivity. Essentially the crystallites will be held by a pinning potential and respond at a given frequency, the pinning frequency, related to the pinning length, as shown in Fig. 5. The coupling between the various crystallites lead to a broadening of the peak, very often assumed to be lorentzian.

When one compares the experiments with these theoretical predictions the agreement is qualitatively wrong. Let us show for example the density dependence of the pinning peak (see Fig. 5). The predicted density dependence of the pinning frequency would be from the above mentioned approximations \( \omega_p \propto n^{1/2} \). This result would be totally opposite to the data which shows a decrease of the pinning frequency with the density. Such important problems when one tries to compare with the data could cast serious doubts on the interpretation of the insulating phase in terms of a Wigner crystal and quite naturally other interpretations for this phase have been proposed [39].

Moreover note that in fact the conventional description shown in Fig. 4 would in fact invalidate the very use of an elastic approximation such as (1) to compute the peak in the optical conductivity. Indeed the peak in such approach would be controlled by the pinning length \( R_a \) of the crystallites. But at that lengthscale topological defects are argued to occur. Such defects are not taken into account in the elastic description and could in principle modify the results for the peak. Strictly speaking the elastic theory could thus only be applied for frequencies \( \omega \gg \omega_p \) (see Fig. 6).
Variation of the pinning frequency with the density for a 2DEG under a strong magnetic field. The pinning frequency decreases with the density in all systems (different symbols), in contradiction with the naive calculations based on the physical image shown in Fig. 4. Such calculations would lead to $\omega_p \propto n^{1/2}$. (From [28])

4. Bragg glass and disordered Wigner crystal

The situation is in fact much better than one could think based on the naive approach exposed in Sec. 3. In fact the discrepancy lies in the fact that the theory used to connect the elastic Hamiltonian (1) to the transport properties and based on the physical ideas shown in Fig. 4, is in fact incorrect. Fortunately, it has been possible with recent “theoretical technology” to obtain a quite complete solution of (1). I will not review here the method or solution but refer the reader to [19, 22] for the general technology and to [9, 10] for the Wigner crystal solution. The agreement with experiments is now quite good (see [10] for a full discussion). In particular one finds a decrease of the pinning frequency with the density as $\omega_p \propto n^{-3/2}$, as well as a good magnetic field dependence of the pinning frequency. This very good agreement gives a good confirmation that the insulating phase in the 2DEG under strong magnetic field is indeed a Wigner crystal collectively pinned by impurities.

Of course despite this good agreement some points still remain open. Among them the question of the low frequency behavior and the magnetic field dependence of the width of the peak. I will not discuss these questions further and refer the reader to [15, 10] for further discussions of these issues.

I want to insist here on two important physical features which are apparent in the solution [9, 10] and whose physics deserves to be explained in detail.
The first important point is that the two characteristic lengthscales (size of particle and lattice spacing) define two characteristic lengthscales via the displacement field. The first one $R_c$, known as the Larkin length, correspond to the distance for which relative displacements are of the order of the size of the particle $u(R_c) - u(0) \sim l_c$. The second $R_a$ is the one for which the relative displacements are of the order of the lattice spacing $u(R_a) - u(0) \sim a$. Since in the Wigner crystal $l_c$ and $a$ are quite different $R_c$ and $R_a$ corresponds to quite different lengthscales and have in general quite different dependence in the various parameters. This is to be contrasted with charge density waves for which $l_c \sim a$ due to the nearly sinusoidal density modulation and thus $R_c \sim R_a$. Thus borrowing directly approximate solutions that have been developed for this case is dangerous and gives part of the physics incorrectly. As can be checked from the solution [9, 10] the pinning frequency $\omega_p$ is controlled by the lengthscale $R_c$ and not the lengthscale $R_a$. This is physically reassuring since one knows [24] for classical systems that $R_c$ is the length that controls the threshold force, and thus is naturally associated with pinning. The distinction is important since $R_c$ depends on the size of the particle $l_c$. This gives, for example for the case of strong magnetic field for which $l_c$ is just the cyclotron orbit, additional magnetic field dependence to the pinning frequency.

The second important point concern the possibility to use the elastic theory. The elastic theory is in fact much more stable to the presence of topological defects than initially anticipated. In $d = 3$ it is now known that below a certain threshold of disorder no topological defects can be induced by the disorder. The disordered elastic system is in a Bragg glass state [18] with a quasi long range positional order, much more ordered than the image of Fig. 4 suggests (see e.g. [16] for a discussion and references on this point). In $d = 2$ the situation is marginal, and defects appear in the ground state, but at distance $R_d$ much larger (for weak disorder) than the lengthscale $R_a$ and not at that lengthscale [18, 26]. Dislocations will thus spoil the results of the elastic theory for the optical conductivity only well below the peak, as shown on Fig. 6. This implies that the theory is a reliable tool to compute the characteristics of the peak and above, and thus most of the a.c. transport.

5. Compressibility

5.1. COMPRESSIBILITY IN CHARGED SYSTEMS

Naively one relates the compressibility to the density-density correlation function by

$$\kappa = \lim_{q \to 0} \langle \rho(q, \omega_n = 0) \rho(-q, \omega_n = 0) \rangle$$

(5)

The case of the disordered system will be discussed in Sec. 5.4, but let us look first at the pure system. The compressibility is simply (only the longitudinal mode
Figure 6. (a) If dislocations occurred at scale $R_a$ and the pinning frequency was controlled by $R_a$, as was naively believed, the elastic theory is incapable of giving any reliable information on the pinning peak. It would be necessary to include dislocations from the start. (b) As was discussed in the text, dislocations occur in fact at $R_D \gg R_a$ and the pinning peak depends on $R_c \ll R_a$. Thus the pinning peak is given quantitatively by a purely elastic theory. It is necessary to take into account topological defects such as dislocations only at much lower frequencies, and in particular if one wants to obtain reliable results for the d.c. transport. (from [10])

\[ \kappa(q) = \lim_{q \to 0} \frac{q^2}{c_L(q)} \]  

(6)

If only short range interactions are present in the system the longitudinal mode is a phonon-like mode $c_L(q) \propto q^2$ and the one recovers a finite compressibility. On the other hand if one has long range Coulomb interactions $c_L(q) \propto |q|$ and the compressibility becomes zero. This is simply due to the fact that (5) measures the density response to a change of chemical potential while keeping the neutralizing background unchanged. A charged systems thus does not remain neutral, hence the infinite compressibility.

One has thus to define the compressibility more precisely. Based on the correlations a standard substraction procedure consists in keeping only the “irreducible” part of the density-density correlation function, i.e. define the compressibility as

\[ \kappa_{\text{irr}} = \kappa - V \kappa \]  

(7)

where $V$ is the long range Coulomb potential. However it is unclear how this procedure is related to the standard way of measuring the compressibility, i.e. the capacitance measurements (see below). Many derivations of the compressibility use instead directly a derivative of the free energy with respect to the number of particles. The free energy can be computed for a neutral system for an arbitrary number of particles which solves the above-mentioned problem. Unfortunately very often the calculation is only possible in some sort of approximate way such as an Hartree-Fock approximation. Here again the link with the direct measurements of the compressibility is not clear. Using such procedures, so called “negative”
compressibilities are found for some range of the interactions, for interacting electrons. Similarly, experiments measure such negative “compressibilities” [13, 23] (see [2] for further references and discussion on this question).

5.2. CAPACITANCE MEASUREMENTS

In order to make the physics of such negative compressibility more transparent, I will discuss now a very simple way to compute them. This way is hopefully more physically transparent than the standard derivations, and has the advantage to be easily extensible to the Wigner crystal in presence of disorder. It is in fact a direct calculation of the quantity that is actually measured to determine the “compressibility”, i.e. the capacitance of a system made by the 2DEG [13]. For simplicity I take here a capacitor formed of two identical systems, as shown in Fig. 7. Taking one system and one metallic plate would not change the results in an essential way. The Hamiltonian of the system is thus

$$H = H_1^0 + H_2^0 + \sum_{\alpha,\beta=1,2} \int_{r,r'} \frac{1}{2} V(r-r') [\rho_\alpha(r) - \rho_0] [\rho_\beta(r') - \rho_0] + \frac{\mu}{2} \int_r [\rho_1(r) - \rho_2(r)]$$

(8)

If one assumes that the system is neutral in the absence of $\mu$, then the charge on one plate when a potential $\mu$ is applied is

$$\langle \rho_1 \rangle = \frac{\mu}{2} [\langle \rho_1 \rho_1 \rangle - \langle \rho_1 \rho_2 \rangle]$$

(9)

in linear response. (9) give directly the capacitance $\langle \rho_1 \rangle / \mu$.

1 This is of course a personal and probably biased opinion!
As a warmup let us shown that in the RPA approximation (9) leads back to the standard results for the compressibility. With (8) it is easy to check that the susceptibilities \( \chi_{\alpha\beta} = \langle \rho_\alpha \rho_\beta \rangle \) are given, in RPA, by
\[
\begin{pmatrix}
\chi_{11} \\
\chi_{12}
\end{pmatrix} = -\begin{pmatrix}
\chi^0 V_{11} & \chi^0 V_{12} \\
\chi^0 V_{12} & \chi^0 V_{11}
\end{pmatrix} \begin{pmatrix}
\chi_{11} \\
\chi_{12}
\end{pmatrix} + \begin{pmatrix}
\chi^0 \\
0
\end{pmatrix}
\]
(10)
where \( \chi^0 \) is the bare (i.e. for \( H^0 \) only) density-density correlation function in one of the systems. It is easy to solve (10) to obtain for the \( (q \text{ dependent}) \) capacitance
\[
\chi_{11} - \chi_{12} = \frac{\chi^0}{1 + \chi^0(V_{11} - V_{12})}
\]
(11)
The Fourier transform of the Coulomb potentials are given by
\[
V_{11} - V_{12} = \int d^2 r e^{iqr} \left[ \frac{1}{r} - \frac{1}{\sqrt{r^2 + d^2}} \right] = (2\pi)(1 - e^{-qd})
\]
(12)
since the two plates are at a distance \( d \). The true capacitance is the limit \( q \to 0 \) which leads to
\[
C = \frac{1}{(2\chi^0(q = 0))^{-1} + 4\pi d}
\]
(13)
On thus recovers that the capacitance is the sum of a geometrical one \( C_{\text{geom}} \) and one due to the electron gas inside the plates \( C_{\text{el}} \)
\[
\frac{1}{C} = \frac{1}{C_{\text{geom}}} + \frac{1}{C_{\text{el}}}
\]
(14)
The geometrical one is the standard \( 1/(4\pi d) \) result. For a simple electron gas \( \chi^0(q = 0)^{-1} \) is simply the screening length \( \lambda \) of the electron gas. One thus recovers that the geometrical distance \( d \) between the plates is increased by the screening length \( \lambda \) on each side.

5.3. WIGNER CRYSTAL

One can use the general formula (9) to compute the capacitance for the Wigner crystal. One substitutes in (1) the density decomposition (3). The \( \nabla u \) terms give directly the contribution of the long range part of the Coulomb interaction
\[
H_{\text{long-range}} = \frac{1}{2} \rho^2 \sum_q \sum_{\alpha\beta=1,2} [V_{\alpha\beta}(q) u^q_L(q) u_{\beta}(-q)]
\]
(15)
Since \( V_{11}(q) \sim 1/q \), (15) gives obviously the part proportional to \( q \) in the elastic coefficients for an isolated plane. The higher harmonics give the regular part (i.e.
the part proportional to \( q^2 \) in the elastic coefficients. Such a way to determine the coefficient is equivalent the calculation of the coefficients in \([8]\). Taking a pure system the Hamiltonian becomes (only the \( \omega_n = 0 \) term of the longitudinal part needs to be computed to have the compressibility)

\[
H = \left( \begin{array}{cc}
  u_1^L(q) & \rho_0^2 q^2 V_{11}(q) \\
  u_2^L(q) & \rho_0^2 q^2 V_{12}(q)
\end{array} \right)
\left( \begin{array}{c}
  c_{SR}^L(q) + \rho_0^2 q^2 V_{11}(q) \\
  c_{SR}^L(q) + \rho_0^2 q^2 V_{12}(q)
\end{array} \right)^T
\left( \begin{array}{c}
  u_1^L(-q) \\
  u_2^L(-q)
\end{array} \right)
\]

(16)

where \( c_{SR}^L(q) \) is the “short range” part of the elastic coefficients. Using \([9]\) and the expression of the density for small \( q \) from \((3)\) \( \rho_L(q) = -\rho_0 q u_L(q) \) one gets for the capacitance an equation like (14), where now

\[
\frac{1}{C_{el}} = \lim_{q \to 0} \frac{2c_{SR}^L(q)}{\rho_0^2 q^2}
\]

(17)

(again the factor of 2 comes from the fact that here I took two identical plates). The electronic one corresponds to the propagator where only the short range part of the elastic coefficients is kept. Using \([8]\) \( c_{SR}^L(q) = -\omega_0^2 (0.18..) (aq)^2 \), where \( \omega_0 = \frac{4\pi e^2}{\sqrt{3}ma^3} \), one finds for the Wigner crystal a “negative” compressibility. The fact that a system of discrete charges can lead to such effects has been noted before for classical Wigner crystals (see e.g. \([31]\) and references therein). The method presented here allows to easily determine this “compressibility” and is trivially applicable to the disordered case.

One should note that if the distance \( d \) between the plates is large compared to the lattice spacing of the Wigner crystal it is safe to throw away the higher harmonics in the coupling term \( \rho_1 \rho_2 \) since they behave as \( e^{-Kd} \) where \( K \) are the vectors of the reciprocal lattice. The capacitance is then indeed only given by the properties of a single system as in (17). This is not true if \( d \sim a \) (which can be the case experimentally) in that case the higher harmonics of the coupling will also contribute to the capacitance, which then cannot be trivially related to the intrinsic properties of a single Wigner crystal. These questions will be addressed in more details elsewhere.

5.4. VARIATIONAL COMPRESSIBILITY

Finally let us consider the effects of disorder. The method discussed in Sec. 4 gives straightforwardly the density-density correlation function which is simply related to the displacement-displacement correlation. Since it is a Gaussian approximation one can easily use the capacitance method shown above. The capacitance is thus given by (17) but where one should use the propagator in the presence of disorder. It is given by (within the variational approximation used \([9, 10]\)) by:

\[
\langle uu \rangle = \frac{1}{\rho_m \omega_n^2 + c_L(q) + \Sigma(1 - \delta_{n,0}) + I(\omega_n)}
\]

(18)
where \( \Sigma \) and \( I(\omega_n) \) are respectively a constant and a function related to the disorder verifying \( I(0) = 0 \).

At \( \omega_n = 0 \) (18) leads to a compressibility in identical to the one of the pure system, and thus also “negative”. This is the thermodynamics results, but the disorder can in principle lead to an additional twist. The thermodynamic compressibility is obviously given by the correlation function in imaginary time at \( \omega_n = 0 \). For the Wigner crystal this leads back to the compressibility of the pure system. If however one makes the analytic continuation \( i\omega_n \rightarrow \omega + i\epsilon \), one gets a quite different result. (18) becomes

\[
\langle uu \rangle = \frac{1}{-\rho_m \omega^2 + c_L(q) + \Sigma + I(\omega)}
\]

Taking then the limit \( \omega \rightarrow 0 \) first as should be done to get the compressibility (and not the transport which corresponds to the opposite limit \( q \rightarrow 0 \) first \( \omega \rightarrow 0 \) after) one has

\[
\langle uu \rangle = \frac{1}{c_L(q) + \Sigma}
\]

which leads to a zero compressibility because of the presence of the mass \( \Sigma \) due to the disorder. One thus may expect in an experiment for a disordered Wigner crystal a drastic change in the behavior of the “compressibility” (i.e. the measured capacitance) of the system as a function of the frequency. At extremely low frequency, one measures the thermodynamic compressibility (negative correction for the Wigner crystal as discussed above). When the frequency increases one measures (20), i.e. a zero compressibility. What is the characteristic frequency separating these two behavior is still unclear the moment, although it is obviously related to the coupling to the external environment. This phenomenon seems to be the equivalent for the Wigner crystal of the Coulomb gap, in usual interacting electronic systems, where a gap linked to the Fermi level appears.

6. Dynamics

We have seen in Sec. 4 that the a.c. transport is a very efficient way to probe the crystalline nature of the electronic system. Getting the d.c. transport is a much more complicated task. For the full quantum system this problem is still a tough cookie, although it is clear that the same methods than used for classical system (functional renormalization group for example) should give good results. One can however gain considerable intuition on what to expect by looking at the classical equivalent.

Let us thus consider the case of a two dimensional classical crystal, submitted to an external force (here due to the electric field). As was shown in [20], periodic moving periodic systems have a quite specific dynamics. Indeed due to the existence of the periodicity in the direction transverse to the direction of motion, the
Figure 8. (Left) for a pure crystal the particles move in straight lines. When the particles move an applied transverse force tilts the trajectories leading to linear response. (Right) In a disordered system the moving system is still in a glassy state (see text). The particle move along rough channels that are the best compromise between the elasticity and the disorder. Although the particles themselves move the channels themselves are pinned, and a transverse pinning force $F_{cy}$ exists. Thus if a transverse force is applied there is no response (at $T = 0$) in the transverse direction until $F_y > F_{cy}$. The transverse pinning force decreases with the longitudinal velocity as shown in the figure above.

motion cannot average completely over the disorder. The moving system is thus submitted to a random potential, which leads to a channel like motion as shown in the right hand side of Fig. 8.

The channels are the best compromise between the elastic energy and the remaining disorder. Their very existence have an important consequence if one tries to make the system move in the transverse direction. Indeed although the particles do move along the channels, the channels themselves are pinned. This means that even above the longitudinal threshold field, if one tries to apply a force in the transverse direction a transverse critical force still exists, as shown on Fig. 8. The value of this transverse critical force can be computed by simple scaling arguments [20] or by more sophisticated renormalization group techniques [21, 25, 6].

For the two dimensional electronic system, putting a magnetic field is a simple way of applying a transverse force. If the lattice is sliding at velocity $v$, it is submitted to a Lorentz transverse force $F_L = evB$. The existence of the transverse critical force $F_{tr}$ thus implies that the channel structure should not slide as long as $F_L < F_{tr}$. There will thus be no hall voltage generated. On the other hand
when $F_L > F_{tr}$ the channel structure should slide and a Hall voltage exists. The *periodicity* of the crystalline structure thus implies that one needs a *finite* longitudinal current before a Hall voltage exists.

Such experiment has been performed in the systems under strong magnetic field [32], and the results are shown in Fig. 9. A finite longitudinal current is clearly needed to develop a Hall voltage, in good agreement with the existence of a transverse threshold. The existence of such an effect is a direct probe of the crystalline (existence of a transverse periodicity) nature of the phase.

7. Conclusions and perspectives

In these short notes I have discussed how the concepts developed to deal with disordered elastic systems can be fruitfully applied to interacting electrons. They allow to investigate the effect of the disorder on a Wigner crystal.

Both the a.c. transport and thermodynamic quantities such as the compressibility can be reliably computed. There is good agreement with the predicted d.c. transport and the observed behavior of a 2DEG under a strong magnetic field, making a strong case for a Wigner crystal phase in such systems. The compressibility is found to be negative both for a pure Wigner crystal and in the presence of disorder, and detailed comparison with experiments on that point would be clearly fruitful. Even if computing the full d.c. transport is beyond reach at the present, some properties can be obtained. In particular the periodicity of the crystal should lead to the existence of a transverse pinning which should entail a shift in the Hall response, as observed experimentally.

We see that there are thus efficient, if not easy, ways via transport to check for the presence of a crystalline phase. Most of the experiments discussed in the preceding sections were for a 2DEG under a strong magnetic field. It would of course be extremely interesting to use the same techniques to probe the 2DEG...
in the absence of the magnetic field, and analyze the experiments in the line of what was discussed above to check for the existence of a Wigner crystal in these systems. Among the interesting possible experiments one can note:

− Measurement of the optical conductivity. In particular the density dependence of the pinning peak can be directly checked against the theoretical predictions of the pinned Wigner crystal.
− If the optical measurements exist, a comparison between the threshold field in the d.c. transport and the pinning frequency.
− The Hall tension vs the longitudinal current (i.e. the measure of the transverse pinning force)
− Although not discussed in these notes, noise measurements are also a good way to probe the periodic nature of the systems (see e.g. [34] and references therein).

8. Acknowledgements

The bulk of the work discussed in these notes results from a fruitful and enjoyable collaboration with R. Chitra and P. Le Doussal, both of whom I would like to specially thank. I would also like to thank E. Abrahams, A. Yacoby and C.M. Varma for many interesting discussions on the compressibility in charged systems and F.I.B. Williams for many enlightening discussions on the Wigner crystal.
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