Estimates of Hilbert modular cusp forms of half-integral and integral weight

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Abstract
Let \( \Gamma \) be a cocompact, discrete, and irreducible subgroup of \( \text{PSL}_2(\mathbb{R})^n \). Let \( \nu \) be a unitary character of \( \Gamma \). For \( k \in 1/2 \mathbb{Z} \), let \( S^k_\nu(\Gamma) \) denote the complex vector space of cusp forms of weight-\( \tilde{k} = (k, \ldots, k) \) and nebentypus \( \nu^{2k} \) with respect to \( \Gamma \). We assume that \( \omega_{X, \nu} \), the line bundle of cusp forms of weight-\( 1/2 := (1/2, \ldots, 1/2) \) with nebentypus \( \nu \) over \( X \) exists. Let \( \{f_1, \ldots, f_j(\tilde{k})\} \) denote an orthonormal basis of \( S^\tilde{k}_\nu(\Gamma) \).

In this article, we show that as \( k \to \infty \), the sum \( \sum_{i=1}^j y^k |f_i(z)|^2 \) is bounded by \( O(k^n) \), where the implied constant is independent of \( \Gamma \). Furthermore, we extend these results to the case when \( k \in 2 \mathbb{Z} \), and to the case when \( \Gamma \) is commensurable with the Hilbert modular group \( \Gamma_K := \text{PSL}_2(O_K) \), where \( K \) is a totally real number field of degree \( n \geq 2 \), and \( O_K \) is the ring of integers of \( K \).

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1 Introduction
The main aim of this article is to extend the heat kernel approach from [2] to study Hilbert modular cusp forms. J. Jorgenson and J. Kramer have successfully applied heat kernel analysis to study automorphic forms, and derived optimal estimates for various arithmetic invariants. In [2], using a different approach from geometric analysis, we derived similar estimates for integral weight cusp forms as in [8], and extended these estimates to half-integral weight cusp forms.

In the current article we extend the methods from [2] to study estimates of half-integral and integral weight Hilbert modular forms. The method can be extended to study automorphic forms of arbitrary Shimura varieties.

Notation
Let \( \mathbb{H} \) denote the hyperbolic upper half-plane. Let \( \mu_\mathbb{H} \) denote the \((1,1)\)-form on \( \mathbb{H} \) corresponding to the hyperbolic metric. For \( z = x + iy \in \mathbb{H} \), it is given by the following formula

\[
\mu_\mathbb{H}(z) := \frac{i}{2} \cdot \frac{dz \wedge d\overline{z}}{y^2}.
\]

For \( (z_1, \ldots, z_n) \in \mathbb{H}^n \), let \( \mu_{\mathbb{H}^n} \) denote the \((1,1)\)-form on \( \mathbb{H}^n \), given by the following formula

\[
\mu_{\mathbb{H}^n}(z) := \sum_{i=1}^n \mu_\mathbb{H}(z_i).
\]

Let \( \mu_{\mathbb{H}^n}^{\text{vol}} \) denote volume form corresponding to \( \mu_{\mathbb{H}^n} \).

Let \( \Gamma \subset \text{PSL}_2(\mathbb{R})^n \) be a cocompact, discrete, and irreducible (as in the sense of [9], and described in section [3]) subgroup acting on \( \mathbb{H}^n \), via fractional linear transformations. We
assume that $\Gamma$ admits no elliptic elements. So, the quotient space $X := \Gamma \backslash \mathbb{H}$ admits the structure of a compact complex manifold of complex dimension $n$. We further assume that $\mu_{\mathbb{H}^n}$ defines a Kähler metric on $X$, which is compatible with the complex structure of $X$.

Let $\nu$ denote a unitary character associated to $\Gamma$. For $k \in \mathbb{R}$, let $S^{\tilde{\nu}}_{k}(\Gamma)$ denote the complex vector space of cusp forms of weight-$\tilde{\nu} = (k, \ldots, k)$ and nebentypus $\nu^{2k}$ with respect to $\Gamma$. Similarly, for $k \in \mathbb{Z}$, let $S^{\nu}_{k}(\Gamma)$ denote the complex vector space of cusp forms of weight-$\nu^{k} = (k, \ldots, k)$ and nebentypus $\nu^{k/2}$ with respect to $\Gamma$.

For $k \in \mathbb{R}$ (or $k \in \mathbb{Z}$), let $j_{\tilde{k}}$ denote the dimension of $S^{\tilde{\nu}}_{k}(\Gamma)$, and let $\{f_{1}, \ldots, f_{j_{\tilde{k}}}\}$ denote an orthonormal basis of $S^{\tilde{\nu}}_{k}(\Gamma)$ with respect to the Petersson inner-product.

For $k \in \mathbb{R}$ (or $k \in \mathbb{Z}$), let $\mathcal{B}_{X,\nu}^{\tilde{k}}(z) := \sum_{i=1}^{j_{\tilde{k}}} \left( \prod_{j=1}^{n} \eta^{k}_{j} |f_{i}(z)|^{2} \right)$. When the nebentypus $\nu$ is trivial, we denote $\mathcal{B}_{X,\nu}^{\tilde{k}}(z)$ by $\mathcal{B}_{X}^{\tilde{k}}(z)$.

We further assume that $\omega_{X,\nu}$ and $\Omega_{X,\nu}$, the line bundles of cusp forms of weight-$\tilde{\nu} := (2, \ldots, 2)$ and weight-$1/2 := (1/2, \ldots, 1/2)$, respectively, with nebentypus $\nu$, over $X$ exist.

With notation as above, for $k \in \mathbb{R}$ (or $k \in \mathbb{Z}$), we prove the following estimate

$$\lim_{k \to \infty} \sup_{z \in X} \frac{1}{k^{n}} \mathcal{B}_{X,\nu}^{\tilde{k}}(z) = O(1),$$

where the implied constant is independent of $\Gamma$.

Let $K$ be a totally real number field of degree $n \geq 2$, and let $\mathcal{O}_{K}$ denote its ring of integers. Let $\Gamma_{K} := \text{PSL}_{2}(\mathcal{O}_{K})$ denote the Hilbert modular group, and let $\Gamma$ now be commensurable with $\Gamma_{K}$. We further assume that $\Gamma$ does not admit elliptic elements, which implies that $X = \Gamma \backslash \mathbb{H}^{n}$ admits the structure of a noncompact complex manifold of complex dimension $n$ with Kähler metric $\mu_{\mathbb{H}^{n}}$. Then, for any $z \in X$ and $k \in \mathbb{R}$, we prove the following estimate

$$\lim_{k \to \infty} \sup_{z \in X} \frac{1}{k^{n}} \mathcal{B}_{X,\nu}^{\tilde{k}}(z) \leq \frac{1}{8\pi};$$

and for $k \in \mathbb{Z}$, we prove that

$$\lim_{k \to \infty} \sup_{z \in X} \frac{1}{k^{n}} \mathcal{B}_{X,\nu}^{\tilde{k}}(z) \leq \frac{1}{2\pi}.$$

**Applications and existing results** When $X$ is compact, in proving estimate (1), for any $z \in X$ and $k \in \mathbb{R}$, we first arrive at the following estimate

$$\lim_{k \to \infty} \frac{1}{k^{n}} \mathcal{B}_{X,\nu}^{\tilde{k}}(z) = O(1),$$


where the implied constant is independent of $\Gamma$. Furthermore, for $k \gg 0$, adapting the Selberg trace formula method from [9] (Theorem 1.6 on p. 79) to compute $j_k$ the dimension of $S^k_\nu(\Gamma)$, one can show that $j_k = O(k^n)$. Using which, we derive that

$$\lim_{k \to \infty} \frac{1}{j_k} B_{X,\nu}^k(z) = O(1).$$

So let

$$\lim_{k \to \infty} \frac{1}{j_k} B_{X,\nu}^k(z) = C.$$

Then, for any $z \in X$ and $k \in 1/2 \mathbb{Z}$, we observe that

$$\lim_{k \to \infty} \frac{1}{Cj_k} B_{X,\nu}^k(z) \mu_{\text{shyp}}(z) = \mu_{\text{shyp}}(z),$$

which proves an equidistribution result for a set of orthonormal basis of Hilbert modular cusp forms.

Let $k \in 2\mathbb{Z}$ and $\Gamma$ be commensurable with the Hilbert modular group $\text{PSL}_2(\mathcal{O}_K)$. Furthermore, let the nebentypus $\nu$ be trivial. Then, using an infinite series representation for $B_{X}^k(z)$, Liu has proved the above equidistribution result in [11]. Using Liu’s technique, in [5], Codgell and Luo extended the equidistribution result to Siegel modular cusp forms.

Sup-norm bounds for automorphic forms, especially, the ones associated to Fuchsian subgroups of $\text{PSL}_2(\mathbb{R})$ is a widely studied topic in number theory. However, sup-norm bounds for Hilbert modular cusp forms are yet to be explored. But in principle, most of the methods used to study sup-norm bounds for automorphic forms associated to Fuchsian subgroups should extend to higher dimensions.

We now state relevant results associated to Fuchsian subgroups of $\text{PSL}_2(\mathbb{R})$. For the rest of this section, let $N \in \mathbb{N}$ with $N$ square-free.

Let $f$ any Hecke-normalized newform of $\Gamma_0(N)$ with trivial nebentypus and of weight 2. Then, in [1], Abbes and Ullmo proved the following estimate

$$\sup_{z \in \mathbb{H}} \left| y f(z) \right| = O_{\varepsilon}(N^{\frac{1}{2} + \varepsilon}),$$

for any $\varepsilon > 0$, and the implied constant depends on $\varepsilon$.

Let $f$ any Hecke-Maass cuspidal newform of $\Gamma_0(N)$ with trivial nebentypus and Laplacian eigenvalue $\lambda > 0$. Then, in [6], Harcos and Templier proved the following estimate

$$\sup_{z \in \mathbb{H}} \left| f(z) \right| = O_{\lambda,\varepsilon}(N^{-\frac{1}{6} + \varepsilon}),$$

for any $\varepsilon > 0$, and the implied constant depends on the eigenvalue $\lambda$ and $\varepsilon$.

For $k \in 1/2 \mathbb{Z}$, let $f$ be any cusp form of $\Gamma_0(4N)$ with nebentypus $\nu$ and of weight-$k$. Furthermore, let $f$ be normalized with respect to the Petersson inner-product. Then, in [7], Kiral has derived the following estimate

$$\sup_{z \in \mathbb{H}} \left| y^k f(z) \right|^2 = O_{k,\varepsilon}(N^{\frac{k}{2} - \frac{1}{18} + \varepsilon}),$$

for any $\varepsilon > 0$, and the implied constant depends on the weight $k$ and $\varepsilon$. 

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Using heat kernel techniques, Jorgenson, and Kramer have re-proved the result of Abbes and Ullmo (estimate (4) in [10]). In [8], Friedman, Jorgenson and Kramer extended their method from [10], to derive sup-norm bounds for $B^\ell_X(z)$. When $X$ is a compact hyperbolic Riemann surface, they proved the following estimate

$$\sup_{z \in X} B^\ell_X(z) = O(k),$$

where the implied constant is independent of $X$. When $X$ is a noncompact hyperbolic Riemann surface of finite volume, they proved the following estimate

$$\sup_{z \in X} B^\ell_X(z) = O(k^{3/2}),$$

(5)

where the implied constant is independent of $X$. The estimates of Jorgenson and Kramer are optimal, as shown in [8].

For $k \in \mathbb{R}_{>0}$ with $k > 2$, $B^\ell_X(z,\nu)$ can be represented by an infinite series, which is uniformly convergent in $z \in X$. Using which, Steiner has extended the bounds of Jorgenson and Kramer to real weights and nontrivial nebentypus.

Let $\Gamma$ be any subgroup of finite index in $SL_2(\mathbb{Z})$, and let $A$ be a compact subset of $X$. Then, in [12], for $k \in \mathbb{R}_{>0}$ with $k \gg 1$, Steiner has derived the following estimates

$$\sup_{z \in A} B^\ell_X(z,\nu) = O_A(k),$$

(6)

where the implied constant depends on the compact subset $A$; and

$$\sup_{z \in X} B^\ell_X(z,\nu) = O_X(k^{3/2}),$$

where the implied constant depends on $X$.

It is difficult to directly extend the methods of Ullmo and Abbes, Harcos and Templier, and that of Kiral to Hilbert modular cusp forms. It is possible to extend Steiner’s method, but the results could be restricted to groups commensurable with the Hilbert modular group.

The method of Jorgenson and Kramer relies on the analysis of the infinite series representation of the heat kernel associated to the Riemann surface. It is possible to extend their heat kernel analysis to higher dimensions, namely to study Hilbert modular cusp forms and Siegel modular cusp forms of integral weight. However, one has to address certain nontrivial convergence issues, while doing so. Their method cannot be extended to cusp forms with nontrivial nebentypus, nor to the case of cusp forms of half-integral weight.

## 2 Heat kernels on compact complex manifolds

In this section, we recall the main results from [4] and [3], which we use in the next section.

Let $(M, \omega)$ be a compact complex manifold of dimension $n$ with natural Hermitian metric $\omega$. Let $L$ be a positive Hermitian holomorphic line bundle on $M$ with the Hermitian metric given by $\|s(z)\|_L^2 := e^{-\phi(z)}|s(z)|^2$, where $s \in L$ is any section, and $\phi(z)$ is a real-valued function defined on $M$.

For any $k \in \mathbb{N}$, let $\square_k := (\overline{\partial} + \partial)^2$ denote the $\overline{\partial}$-Laplacian acting on smooth sections of the line bundle $L^{\otimes k}$. Let $K^k_{M,L}(t; z, w)$ denote the smooth kernel of the operator $e^{-\frac{2\pi t}{k}}\square_k$. 
We refer the reader to p. 2 in [4], for the details regarding the properties which uniquely characterize the heat kernel $K_{M,L}^k(t; z, w)$. When $z = w \in M$, the heat kernel $K_{M,L}^k(t; z, z)$ admits the following spectral expansion

$$K_{M,L}^k(t; z, z) = \sum_{n \geq 0} e^{-\frac{2t}{k} \lambda_n^k \|\varphi_n(z)\|_{L^2}}.$$  \hspace{1cm} (7)

where $\{\lambda_n^k\}_{n \geq 0}$ denotes the set of eigenvalues of $\Box_k$ (counted with multiplicities), and $\{\varphi_n\}_{n \geq 0}$ denotes a set of associated orthonormal eigenfunctions.

Let $H^0(M, L^\otimes k)$ denote the vector space of global holomorphic sections of the line bundle $L^\otimes k$, and let $\{s_i\}$ denote an orthonormal basis of $H^0(M, L^\otimes k)$. For any $z \in M$, the following function is called the Bergman kernel associated to the line bundle $L^\otimes k$

$$B_{M,L}^k(z) := \sum_i \|s_i(z)\|^2_{L^2}.$$ \hspace{1cm} (8)

The above definition is independent of the choice of orthonormal basis of $H^0(M, L^\otimes k)$.

For any $z \in M$ and $t \in \mathbb{R}_{>0}$, from the spectral expansion of the heat kernel $K_{M,L}^k(t; z, w)$ described in equation (7), it is easy to see that

$$B_{M,L}^k(z) \leq K_{M,L}^k(t; z, z) \quad \text{and} \quad \lim_{t \to \infty} K_{M,L}^k(t; z, z) = B_{M,L}^k(z).$$ \hspace{1cm} (9)

Let

$$c_1(L)(z) := \frac{i}{2\pi} \partial \bar{\partial} \phi(z)$$ \hspace{1cm} (10)

denote the curvature form of the line bundle $L$ at the point $z \in M$. Let $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of $\partial \bar{\partial} \phi(z)$ at the point $z \in M$. Then, with notation as above, from Theorem 1.1 in [4], for any $z \in M$ and $t \in (0, k^2 \varepsilon)$, for a given $\varepsilon > 0$ not depending on $k$, we have

$$\lim_{k \to \infty} \frac{1}{k^n} K_{M,L}^k(t; z, z) = \prod_{j=1}^n \frac{\alpha_j}{(4\pi)^n \sinh(\alpha_j t)},$$ \hspace{1cm} (11)

and the convergence of the above limit is uniform in $z$.

Using equations (10) and (11), in Theorem 2.1 in [4], Bouche derived the following estimate

$$\lim_{k \to \infty} \frac{1}{k^n} B_{M,L}^k(z) = O(\det(\omega(c_1(L)(z)))),$$ \hspace{1cm} (12)

and the convergence of the above limit is uniform in $z \in M$.

When $M$ is a noncompact complex manifold, using micro-local analysis of the Bergman kernel, in [3], Berman derived the following estimate

$$\limsup_{k \to \infty} \frac{1}{k^n} B_{M,L}^k(z) \leq \det(\omega(c_1(L)(z))).$$ \hspace{1cm} (13)

Unlike Bouche, Berman worked directly with the Bergman kernel $B_{M,L}^k(z)$. His proof relies on comparison of $B_{M,L}^k$ with the Bergman kernel of the trivial line bundle with constant metric defined over $\mathbb{C}^n$. He then used certain strong inequalities from the theory of elliptic partial differential equations to derive estimate (13).
3 Estimates of cusp forms

As in section 1, let $K$ be a totally real number field of degree $n \geq 2$, and let $\mathcal{O}_K$ be its ring of integers. Let $\sigma_1, \ldots, \sigma_n$ be the $n$-real embeddings of the number field $K$. Then, the the Hilbert modular group $\Gamma_K := \text{PSL}_2(\mathcal{O}_K)$ can be seen as a discrete subgroup of $\text{PSL}_2(\mathbb{R})^n$, via the following map

$$i : \Gamma_K \hookrightarrow \text{PSL}_2(\mathbb{R})^n \quad (a_{ij})_{1 \leq i,j \leq n} \mapsto \left( (\sigma_1(a_{ij}))_{1 \leq i,j \leq n}, \ldots, (\sigma_n(a_{ij}))_{1 \leq i,j \leq n} \right).$$

We say that $\Gamma$ is commensurable with $\Gamma_K$, if there exists a $g \in \text{PSL}_2(\mathbb{R})^n$ such that $g \Gamma g^{-1}$ is a finite index subgroup of $i(\Gamma_K)$.

Any $\gamma := (\gamma_1, \ldots, \gamma_n) \in \text{PSL}_2(\mathbb{R})^n$, where

$$\gamma_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$$

defines the following linear transformation on $\mathbb{H}^n$

$$\gamma : \mathbb{H}^n \longrightarrow \mathbb{H}^n \quad (z_1, \ldots, z_n) \mapsto (\gamma_1(z_1), \ldots, \gamma_n(z_n)),$$

where $\gamma_i(z_i) := \frac{a_i z_i + b_i}{c_i z_i + d_i}$.

A subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})^n$ is irreducible, if the restriction of each of the $n$ projections

$$\rho_j : \text{PSL}_2(\mathbb{R})^n \longrightarrow \text{PSL}_2(\mathbb{R}) \quad (1 \leq j \leq n)$$

to $\Gamma$ is injective.

Let $\Gamma$ be a discrete, irreducible subgroup of $\text{PSL}_2(\mathbb{R})^n$, with no elliptic elements. Let $X := \Gamma \backslash \mathbb{H}^n$ denote the quotient space. For the rest of this article, we assume that $\Gamma$ is one of the following two types:

Type(1): The group $\Gamma$ is cocompact, which implies that $X$ admits the structure of a compact complex manifold of complex dimension $n$. We assume that $\mu_{\mathbb{H}^n}$, the natural metric on $\mathbb{H}^n$ defines a Kähler metric on $X$.

Type(2): The group $\Gamma$ is commensurable with the Hilbert modular group $\Gamma_K$, which implies that $X$ admits the structure of a noncompact complex manifold of complex dimension $n$ with cusps. The number of cusps is equal to the class number of $K$. The hyperbolic metric $\mu_{\mathbb{H}^n}$, which is the natural metric on $\mathbb{H}^n$ defines a Kähler metric on $X$.

For the convenience of the reader, in this section, we adopt the following notation. We denote half-integers by $\kappa$, and even integers by $k$.

Let $\nu$ be a unitary character of the group $\Gamma$. We assume that $\omega_X, \nu$, the line bundle of cusp forms of weight-$1/2 = (1/2, \ldots, 1/2)$ with nebentypus $\nu$ over $X$ exists. Then, for any $\kappa \in 1/2 \mathbb{Z}$, cusp forms of weight-$\kappa = (\kappa, \ldots, \kappa)$ and nebentypus $\nu^{2\kappa}$ with respect to $\Gamma$ are global sections of the line bundle $\omega_X^{2\kappa}$.

Furthermore, for any $f \in \omega_X, \nu$, i.e., $f$ is a cusp form of weight-$1/2$ and nebentypus $\nu$ with respect to $\Gamma$ and $z = (z_1 = x_1 + y_1, \ldots, z_n = x_n + y_n)$, the Petersson metric on the line
bundle \( \omega_{X, \nu} \) is given by

\[
\|f(z)\|_{\omega_{X, \nu}}^2 := \left( \prod_{i=1}^{n} y_i^{1/2} \right) |f(z)|^2.
\]  

(14)

Similarly, we assume that \( \Omega_{X, \nu} \), the line bundle of cusp forms of weight-\( \tilde{k} \) with nebentypus \( \nu \) over \( X \) exists. Then, for any \( k \in 2\mathbb{Z} \), cusp forms of weight-\( \tilde{k} \) and nebentypus \( \nu^{k/2} \) with respect to \( \Gamma \) are global sections of the line bundle \( \Omega_{X, \nu}^{\otimes k/2} \).

Furthermore, for any \( f \in \Omega_{X, \nu} \), i.e., \( f \) is a cusp form of weight-\( \tilde{k} \) and nebentypus \( \nu \) with respect to \( \Gamma \) and \( z = (z_1 = x_1 + y_1, \ldots, z_n = x_n + y_n) \), the Petersson metric on the line bundle \( \Omega_{X, \nu} \) is given by

\[
\|f(z)\|_{\Omega_{X, \nu}}^2 := \left( \prod_{i=1}^{n} y_i^{2} \right) |f(z)|^2.
\]

Remark 1. For any \( z \in X \) and \( \kappa \in 1/2\mathbb{Z} \), from the definition of the Bergman kernel \( B^\kappa_{X,\Omega_{X, \nu}}(z) \) for the line bundle \( \omega_{X, \nu} \) from equation (8), we have

\[
B^\kappa_{X,\omega_{X, \nu}}(z) = B^\kappa_{X, \nu}(z).
\]

(15)

Similarly, for any \( z \in X \) and \( k \in 2\mathbb{Z} \), from the definition of the Bergman kernel \( B^{k/2}_{X,\Omega_{X, \nu}}(z) \) for the line bundle \( \Omega_{X, \nu} \) from equation (8), we have

\[
B^{k/2}_{X,\Omega_{X, \nu}}(z) = B^{k}_{X, \nu}(z).
\]

Theorem 2. Let notation be as above, and we assume that \( \Gamma \) is of Type (1), i.e., \( X \) is compact. Then, for \( \kappa \in 1/2\mathbb{Z} \), we have

\[
\lim_{\kappa \to \infty} \sup_{z \in X} \frac{\kappa^{-n}}{\kappa^n} B^\kappa_{X, \nu}(z) = O(1);
\]

and for \( k \in 2\mathbb{Z} \), we have

\[
\lim_{k \to \infty} \sup_{z \in X} \frac{k^{-n}}{k^n} B^{k}_{X, \nu}(z) = O(1),
\]

where the implied constants in both the above equations are independent of \( \Gamma \).

Proof. We prove the theorem for \( \kappa \in 1/2\mathbb{Z} \), and the case for \( k \in 2\mathbb{Z} \) follows automatically with notational changes. For any \( z \in X \), from the definition of the curvature form from equation (10), and from the definition of the Petersson inner-product for the line bundle \( \omega_{X, \nu} \) from equation (14), we have

\[
c_1(\omega_{X, \nu})(z) = -\frac{i}{2\pi} \partial \bar{\partial} \log \left( \prod_{i=1}^{n} y_i^2 \right) = -\frac{i}{2\pi} \sum_{i=1}^{n} \partial \bar{\partial} \log (y_i^2).
\]

(16)

For any \( 1 \leq i \leq n \), we compute

\[
\frac{i}{2\pi} \partial \bar{\partial} \log (y_i^{1/2}) = \frac{i}{4\pi} \partial \bar{\partial} \log \left( \frac{z_i - \bar{z}_i}{2i} \right) = -\frac{i}{4\pi} \partial \left( \frac{\partial \bar{z}_i}{z_i - \bar{z}_i} \right) = \frac{i}{4\pi} \frac{dz_i \wedge d\bar{z}_i}{(z_i - \bar{z}_i)^2} = -\frac{i}{16\pi} \cdot \frac{dz_i \wedge d\bar{z}_i}{y_i^2} = -\frac{1}{8\pi} \mu_{\mathbb{H}}(z_i).
\]

(17)
Combining equations (16) and (17), we arrive at
\[ c_1(\omega_X, \nu)(z) = \frac{1}{8\pi} \sum_{i=1}^{n} \mu_H(z_i) = \frac{1}{8\pi} \mu\mathbb{H}^{n}(z), \]
which shows that the line bundle \( \omega_X, \nu \) is positive, and
\[ \det_{\mu\mathbb{H}^{n}} \left( c_1(\omega_X, \nu)(z) \right) = (1/8\pi)^{n}. \]  
(18)

So applying estimate (12) to the complex manifold \( X \) with its natural Hermitian metric \( \mu\mathbb{H}^{n} \) and the line bundle \( \omega_X \otimes 2\kappa \), and using equations (15) and (18), we find that
\[ \lim_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) = \lim_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \omega_X, \nu}^{2\kappa}(z) = O \left( 2\det_{\mu\mathbb{H}^{n}} \left( c_1(\omega_X, \nu)(z) \right) \right) = O(1). \]
As the above limit convergences uniformly in \( z \in X \), and as \( X \) is compact, we have
\[ \sup_{z \in X} \lim_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) = \lim_{\kappa \to \infty} \sup_{z \in X} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) = O(1), \]
which completes the proof of the theorem.

**Corollary 3.** Let notation be as above, and we assume that \( \Gamma \) is of Type (2), i.e., \( X \) is a noncompact complex manifold with cusps. Then, for \( z \in X \) and \( \kappa \in 1/2\mathbb{Z} \), we have
\[ \limsup_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) \leq \frac{1}{(8\pi)^n}, \]
and for \( k \in 2\mathbb{Z} \), we have
\[ \limsup_{k \to \infty} \frac{1}{k^n} B_{X, \nu}^k(z) \leq \frac{1}{(2\pi)^n}. \]

*Proof.* The proof of the theorem follows from estimate (13), and from similar arguments as in Theorem 2.

**Corollary 4.** Let notation be as above, and we assume that \( \Gamma \) is of Type (2), i.e., \( X \) is a noncompact complex manifold with cusps. Let \( A \) be a compact subset of \( X \). Then, for any \( z \in A \) and \( \kappa \in 1/2\mathbb{Z} \), we have the following estimate
\[ \lim_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) = O_A(1); \]
and for \( k \in 2\mathbb{Z} \), we have
\[ \lim_{k \to \infty} \frac{1}{k^n} B_{X, \nu}^k(z) = O_A(1); \]
and the implied constant depends on \( A \).

*Proof.* We prove the theorem for \( \kappa \in 1/2\mathbb{Z} \), and the case for \( k \in 2\mathbb{Z} \) follows automatically with notational changes. From Corollary 3 for any \( z \in A \) and \( \kappa \in 1/2\mathbb{Z} \), we have
\[ \limsup_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) \leq \frac{1}{(8\pi)^n}. \]
As \( A \subset X \) is compact, we can find a constant \( C \) (which depends on \( A \)) such that
\[ \lim_{\kappa \to \infty} \frac{1}{\kappa^n} B_{X, \nu}^\kappa(z) \leq C, \]
which completes the proof of the corollary.

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