Percolation games, probabilistic cellular automata,
and the hard-core model

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Abstract

Let $p \in (0, 1)$, and let each site of $\mathbb{Z}^2$ be closed with probability $p$ and open with probability $1 - p$, independently for different sites. Consider the following two-player game: a token starts at the origin, and a move consists of moving the token from its current site $x$ to an open site in $\{x + (0,1), x + (1,0)\}$; if both these sites are closed, then the player to move loses the game. Is there positive probability that the game is drawn with best play – i.e. that neither player can force a win? This is equivalent to the question of ergodicity of a certain one-dimensional probabilistic cellular automaton (PCA), which has already been studied from several perspectives, for example in the enumeration of directed animals in combinatorics, in relation to the golden-mean subshift in symbolic dynamics, and in the context of the hard-core model in statistical physics. The ergodicity of the PCA has been given as an open problem by several authors. Our main result is that the PCA is ergodic for all $p$, and that no draws occur for the game on $\mathbb{Z}^2$. A related game, in which the winner is the first player to reach a closed site, is shown to correspond to another PCA. Here we also show that the PCA is ergodic and the game has no draws. There are several natural extensions to dimension $d \geq 3$, and we conjecture that these games have positive probability of a draw for small $p$; we prove this in a variety of cases using a connection to phase transitions for the hard-core model on appropriately defined $(d - 1)$-dimensional lattices.

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1 Introduction

Let each site of $\mathbb{Z}^2$ be closed with probability $p$ and open with probability $1 - p$, independently for different sites. Consider the following two-player game: a token starts at the origin. The players move alternately; if the token is currently at $x$, a move consists of moving it to $x + (0,1)$ or to $x + (1,0)$. The token is only allowed to move to an open site; if both the sites $x + (0,1)$ and $x + (1,0)$ are closed,
then the player whose turn it is to move loses the game. The entire configuration of open and closed sites is known to both players at all times. We call this the percolation game on \( \mathbb{Z}^2 \).

If \( p \) exceeds one minus the critical probability for directed site percolation, then, with probability 1, only finitely many sites can be reached from the origin along directed paths of open sites, and so the game must end in finite time. In particular, one or other player must have a winning strategy. Suppose on the other hand that \( p \) is less than one minus the critical probability; is there now a positive probability that neither player has a winning strategy? In that case we say that the game is a draw, with the interpretation that it continues for ever with best play. (When \( p = 0 \) the game is clearly always a draw).

Related questions are considered in [HM] and [BHMW], in which the underlying graph is respectively a Galton-Watson tree, and a random subset of the grid with undirected moves.

In our case of a random subset of the grid with directed moves, the outcome of the game started from each site (winning, losing, draw) can be interpreted in terms of the evolution of a certain one-dimensional discrete-time probabilistic cellular automaton (PCA); the state of the PCA at a given time relates to the outcomes of the sites on a given NW-SE diagonal of \( \mathbb{Z}^2 \).

The PCA has alphabet \( \{0, 1\} \) and universe \( \mathbb{Z} \), so that a configuration at a given time is an element of \( \{0, 1\}^\mathbb{Z} \). (The three game outcomes will correspond to the two states of the PCA via a coupling of two copies of the PCA). The evolution of the PCA is as follows. Given a configuration \( \eta \) at some time \( t \), the configuration \( \eta_{t+1} \) at time \( t+1 \) is obtained by updating each site \( n \in \mathbb{Z} \) simultaneously and independently, according to the following rule.

- If \( \eta_t(n−1) = \eta_t(n) = 0 \), then \( \eta_{t+1}(n) = 0 \) with probability \( p \) and 1 with probability \( 1−p \).
- Otherwise (i.e. if at least one of \( \eta_t(n−1) \) and \( \eta_t(n) \) is 1), \( \eta_{t+1}(n) = 0 \) set to 0 with probability 1.

This PCA has already been studied from a number of different perspectives. It is closely related to the enumeration of directed lattice animals, which are classical objects in combinatorics. The link was originally made by Dhar [Dha83], and subsequent work includes [BM98, LBM07] – see also Section 4.2 of the survey of Mairese and Marcovici [MM14a] for a short introduction. It also has strong connections to the hard-core lattice gas model in statistical physics (which also has applications for example to the modelling of communications networks) – see Section 3 of this paper. The case \( p = 1/2 \) in particular relates to the question of the measure of maximal entropy of the golden mean subshift – see [Elo96] and also [Mar13, Chapter 8].

It is straightforward to show that for \( p \) sufficiently large, the PCA is ergodic (that is, it has a unique stationary distribution, and the iterates of any initial distribution converge to that stationary distribution). The question of whether it is ergodic for all \( p \in (0, 1) \) has been mentioned as an open problem by several authors, for example in [TVS90, LBM07, MM14a]. A link between the PCA and the percolation game was already pointed out by [LBM07], and it is relatively easy to show that the percolation game has positive probability of a draw if and only if the PCA is non-ergodic.

Our main result is that the PCA is ergodic for all \( p \in (0, 1) \). We prove this by showing that, with probability 1, no draws occur for the percolation game.

PCA that are defined on \( \mathbb{Z} \) and whose alphabet and neighbourhood are both of size 2 are sometimes called elementary PCA. A variety of tools have been developed to study their ergodicity. Under an additional assumption of left-right symmetry of the update rule, these PCA are defined by only three parameters: the probabilities to update a cell to state 1 if its neighbourhood is in state 00, 11, or 01 (which is the same as for 10). Existing methods can be used to handle more than 90% of the volume of the cube defined by this parameter space, but the PCA we study belongs to an open domain of the cube where none of the previously known criteria hold ([TVS90, Chapter 7]).

We also give analogous results for a certain misère form of the game which we call the target game, in which the first player to move to a closed site wins. We show that this game corresponds to another elementary PCA, which we prove to be ergodic also. Therefore the probability of a drawn position in the target game is 0. (For other possible misère variants of the game, the question of existence of draws remains open.)

The link between the games and the two PCA is explained in Section 2, where the main result (see Theorem 1) is stated. The main ergodicity results are proved in Section 4.

In Section 3 we describe a connection between the PCA corresponding to the percolation game and a hard-core model. We then use this idea to study generalisations of the percolation game to dimensions \( d \geq 3 \). There are many natural such generalisations of the two-dimensional game, corresponding to different choices of the set of possible moves in the lattice \( \mathbb{Z}^d \). For several such choices, we prove that the probability of a draw is now positive for \( p \) sufficiently small (see Theorem 2). For example, we
prove this when \( d \geq 3 \), and the allowed moves are from \( x \in \mathbb{Z}^d \) to \( x \pm e_i + e_d \) for any \( 1 \leq i \leq d - 1 \). (Here \( e_i \) is the \( i \)th coordinate vector). However, we do not know whether the existence of draws is monotone in \( p \).

For several other natural choices of allowed moves in \( d \geq 3 \), we conjecture that the probability of a draw is again positive for small \( p \). In some cases this would follow from the existence of multiple Gibbs measures for a hard-core model on a suitable graph in one dimension lower, which could likely be proved by standard methods. In other cases there is a more fundamental obstacle, in that our method of reducing to the hard-core model breaks down. This applies in particular to the canonical case in which moves are allowed from \( x \) to any \( x + e_i \). We conjecture that this game also has positive probability of a draw for \( p \) sufficiently small when \( d \geq 3 \).

In Section 5 we conclude with several further open problems.

2 Percolation games and probabilistic cellular automata

2.1 The PCA for the percolation game

Consider the percolation game on \( \mathbb{Z}^2 \) as defined in the introduction. Given a starting vertex and a configuration of open and closed sites, a strategy for one or the other player is a map that assigns a legal move (if one exists) to each vertex. A winning strategy is one that results in a win for that player, whatever strategy the other player uses. If one player has a winning strategy then we say that the game is a win for that player, and a loss for the other player. If neither player has a winning strategy then the game is said to be a draw (with the interpretation that the game will continue forever with best play).

Suppose \( x \) is an open site of \( \mathbb{Z}^2 \). Let \( \eta(x) \) be \( W \), \( L \) or \( D \) according to whether the game started with the token at \( x \) is win for the first player, a loss for the first player, or a draw, respectively. If \( x \) is a closed site, it is convenient to set \( \eta(x) = W \). (We can imagine that a player is allowed to move the token to \( x \), but with the effect that the game is then immediately won by the opponent).

Let \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) be the standard basis vectors of \( \mathbb{Z}^2 \). For \( x \in \mathbb{Z}^2 \), define the set \( N(x) = \{ x + e_1, x + e_2 \} \), the sites to which the token can move from \( x \). By considering the first move, we have the following recursion for the status of the sites:

\[
\eta(x) = \begin{cases} 
  L & \text{if } \eta(y) = W \text{ for all } y \in N(x) \\
  W & \text{if } \eta(y) = L \text{ for some } y \in N(x) \\
  D & \text{otherwise}
\end{cases}
\]  

For \( k \in \mathbb{Z} \), let \( S_k \) be the set \( \{ x = (x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 = k \} \), a NW-SE diagonal of \( \mathbb{Z}^2 \). The recursion (1) gives us the values \( \eta(x), x \in S_k \) in terms of the values \( \eta(x), x \in S_{k+1} \) together with the information about which sites in \( S_k \) are closed.

In this way we can regard the configurations on successive diagonals \( S_k \), as \( k \) decreases, as successive states of a one-dimensional PCA. Let us introduce the following recoding: \( W = 0, L = 1, D = ? \). The PCA evolves as follows: given the values for sites in \( S_{k+1} \), each value \( \eta(x), x \in S_k \) is derived independently using the values \( \eta(x + e_1) \) and \( \eta(x + e_2) \), according to the scheme given in Figure 1 (where a * represents any symbol from \( \{0, ?, 1\} \)).

![Figure 1: The probabilistic cellular automaton (PCA) \( F_p \)](image)

We denote the corresponding PCA \( F_p \). Although we have defined it as a process in the plane, we
can also regard it as a PCA on \( \mathbb{Z} \) with a configuration in \( \{0,?,1\}^\mathbb{Z} \) evolving in time by setting

\[
\eta_t(n) = \eta_t((-t-n,n)).
\] (2)

Formally, we take \( F_p \) to be the operator on the set of distributions on \( \{0,?,1\}^\mathbb{Z} \) representing the action of the PCA: if \( \mu \) is the distribution of a configuration in \( \{0,?,1\}^\mathbb{Z} \), then \( F_p \mu \) is the distribution of the configuration obtained by performing one update step of the PCA. A stationary distribution \( F_p \) is a distribution \( \mu \) such that \( F_p \mu = \mu \). More generally, \( \mu \) is \( k \)-periodic if \( F_p^k \mu = \mu \), and periodic if it is \( k \)-periodic for some \( k \geq 1 \).

In the setting of the percolation game, translation invariance of the whole process on \( \mathbb{Z}^2 \) implies that the distribution of the configuration on the diagonal \( S_k \) does not depend on \( k \); that is, the distribution of \( \eta((k-n,n)), n \in \mathbb{Z} \) does not depend on \( k \) and is a stationary distribution of \( F_p \). In addition, this distribution is invariant under the action of translations of \( \mathbb{Z} \).

We next note two useful monotonicity properties for the PCA \( F_p \).

**Lemma 1.** Let \( \mu \) and \( \nu \) be probability distributions on \( \{0,?,1\}^\mathbb{Z} \).

(i) If \( \mu \leq \nu \), where \( \leq \) denotes stochastic domination with respect to the coordinatewise partial order induced by \( 0 < ? < 1 \), then \( F_p \mu \geq F_p \nu \). (Note the reversal of the inequality).

(ii) If \( \mu \not\leq \nu \), where \( \not\leq \) denotes stochastic domination with respect to the coordinatewise partial order induced by \( 0 < ? < 1 \), then \( F_p \mu \not\leq F_p \nu \).

**Proof.** We can use the recursion (1) to give a coupling of a single step of the PCA \( F_p \) started from two different configurations. Suppose we fix values \( \eta(x), x \in S_{k+1} \) and \( \eta'(x), x \in S_{k+1} \), in such a way that \( \eta(x) \leq \eta'(x) \) for all \( x \in S_{k+1} \) (where \( \leq \) is the coordinatewise order on configurations induced by \( 0 < ? < 1 \)). Now use (1) to obtain values \( \eta(x) \) and \( \eta'(x) \) for \( x \in S_k \), using the same realisation of closed and open sites in \( S_k \) in each case. It is straightforward to check that in that case \( \eta(x) \geq \eta'(x) \) for each \( x \in S_k \). Hence the operator \( F_p \) is decreasing in the desired sense.

Similarly, if \( \eta(x) \not\leq \eta'(x) \) for all \( x \in S_{k+1} \), then we obtain \( \eta(x) \not\leq \eta'(x) \) also for each \( x \in S_k \). So in this case the operator \( F_p \) is increasing as desired. \( \square \)

The restriction of the PCA \( F_p \) to configurations that do not contain the symbol \( ? \) is a well-defined PCA, which we denote by \( A_p \). The update rule of the PCA imposes a kind of hard-core constraint: a 1 is allowed only if both predecessors are 0; see Figure 2.

![Figure 2: PCA \( A_p \)](image)

In the terminology of Bušić et al. [BMM11, BMM13], the PCA \( F_p \) is the envelope PCA of \( A_p \). A copy of the PCA \( F_p \) can be used to represent a coupling of two or more copies of the PCA \( A_p \), started from different initial conditions. The symbol \( ? \) represents a site whose value is not known, i.e. one which may differ between the different copies.

In particular, consider starting copies of the PCA \( A_p \) from several different initial conditions, represented by configurations set on the diagonal \( S_k \) for some fixed \( k > 0 \). As in the proof of Lemma 1(i) a natural coupling is provided by the recursion (1), using the same realisation of the closed and open sites on \((S_r : r < k)\). If we start three copies \( \eta, \eta' \) and \( \eta'' \) from some given initial conditions on \( S_k \) with \( \eta''(x) = ? \) for all \( x \in S_r \) (so that \( \eta'' \) is maximal for the ordering \( \leq \) in Lemma 1(ii)), then we have that \( \eta(x) \leq \eta''(x) \) and \( \eta'(x) \leq \eta''(x) \) for all \( x \in S_r \) with \( r < k \). This implies that if \( \eta(x) \neq \eta''(x) \), then \( \eta''(x) = ? \).

In terms of the game, we have the following interpretation: if the origin \( O \) is an open site, and \( \eta''(O) = 0 \) (respectively \( \eta''(O) = 1 \)) then the first (respectively second) player can force a win within at most \( k \) moves of the game.
Definition 1. A PCA is said to be ergodic if it has a unique stationary distribution and if from any initial distribution, the iterates of the PCA converge to that stationary distribution (in the sense of convergence in distribution with respect to the product topology).

The ergodicity of an envelope PCA implies the ergodicity of the original PCA, but the converse is not true in general. In our case, however, we can use the monotonicity property in Lemma [1](i) to show that the two are equivalent.

**Proposition 1.** The PCA $F_p$ is ergodic if and only if $A_p$ is ergodic.

**Proof.** It is clear from the definitions that if $F_p$ is ergodic, then $A_p$ is also ergodic. Conversely, suppose that $A_p$ is ergodic. Let $\mu$ be a distribution on $\{0,?,1\}^Z$, and let $\delta_0$ and $\delta_1$ the distributions concentrated on the configurations “all 0’s” and “all 1’s”. Then $\delta_0 \leq \mu \leq \delta_1$, so by Lemma [1](i), for $k \geq 0$ we have either $F_p^k \delta_0 \leq F_p^k \mu \leq F_p^k \delta_1$ or $F_p^k \delta_0 \geq F_p^k \mu \geq F_p^k \delta_1$, according to whether $k$ is even or odd. But $F_p^k \delta_0 = A_p^k \delta_0$ and $F_p^k \delta_1 = A_p^k \delta_1$, and by ergodicity of $A_p$, the latter two sequences converge as $k \to \infty$ to the same measure $\pi$, so $F_p^k \mu$ also converges to $\pi$. Thus $F_p$ is also ergodic. \(\square\)

**Proposition 2.** For each $p \in [0,1]$, the percolation game has probability 0 of a draw if and only if $A_p$ is ergodic.

**Proof.** If $A_p$ is ergodic then so is $F_p$, and so the unique invariant distribution of $F_p$ has no ? symbols. But we know that the distribution of the configuration along a diagonal $S_k$ is invariant for $F_p$. Hence with probability 1, there are no sites from which the game is drawn.

For the converse, let $\omega$ be a random configuration of open and closed sites on $Z^2$ chosen according to the percolation measure. Consider any site $x \in S_0$. If the game started from $x$ is not a draw, then (since at each turn the player to move has only finitely many options) one player has a strategy that guarantees a win in fewer than $N$ moves, where $N \in \mathbb{N}$ is a finite random variable that depends on $\omega$. Consequently, if we assign any configuration of states $0,?,1$ to $S_N$ and compute the resulting states on $\{S_n : 0 \leq n < N\}$ using the recursion (1) and the configuration $\omega$ of open and closed sites, the resulting state at $x$ is the same as its state for the percolation game on $Z^2$ with configuration $\omega$.

Let $\gamma$ be the random configuration of game outcomes on $S_0$ arising from $\omega$. Also, fix a distribution $\nu$ on $\{0,1\}^Z$, and let $\gamma_n$ be the configuration on $S_0$ that results from assigning a configuration with law $\nu$ to $S_n$, independent of $\omega$, and applying (1) as described above. By the argument in the previous paragraph, if the probability of a draw is 0, then $\gamma_n$ converges almost surely to $\gamma$ (in the product topology). Hence also the distribution of $\gamma_n$ converges to that of $\gamma$. But $\gamma_n$ has distribution $A_p^\nu \nu$, so $A_p^\nu \nu$ converges as $n \to \infty$ to the distribution of $\gamma$, which does not depend on $\nu$. Hence $A_p$ is ergodic. \(\square\)

### 2.2 The target game and the corresponding PCA

Recall that in the percolation game described above, moves to closed sites are forbidden, and a player who cannot move loses. Clearly, an equivalent formulation is that any move is allowed, but a player who moves to a closed site is deemed to lose. We can reverse this rule, so that instead a player moving to a closed site wins; this can be seen as a misère version of the percolation game. We call this new game the target game.

As before, we can introduce a PCA to describe the status of the positions. The recursion (1) still applies when $x$ is open, but now if $x$ is closed we set instead $\eta(x) = L$. The equivalents of $F_p$ and $A_p$ for the target game are the PCA $G_p$ and $B_p$ defined as shown in Figures 3 and 4.

![Figure 3: PCA $G_p$](image-url)
We may see the PCA $B_p$ as a composition of Stavskaya’s PCA and the flip operator. Stavskaya’s PCA (see for example [TVS+90]) is given by the local rule which sets $\eta_t(n) = 0$ with probability $p$ and otherwise $\eta_t(n) = \max\{\eta_{t-1}(n), \eta_{t-1}(n-1)\}$, and provides a non-trivial example of a one-dimensional PCA where ergodicity fails. However, the ergodicity of the PCA $B_p$ had not been studied, and, as in the case of $A_p$, it does not satisfy any of the previously known criteria for ergodicity.

Using exactly the same arguments as for Proposition 1 and Proposition 2, we have the following results.

**Proposition 3.** The PCA $G_p$ is ergodic if and only if $B_p$ is ergodic.

**Proposition 4.** The target game has probability 0 of a draw if and only if $B_p$ is ergodic.

In Section 4 we will show that, for any $p > 0$, any stationary distribution for $F_p$ or for $G_p$ is concentrated on $\{0, 1\}^2$; the probability of a ? symbol is 0. From this we obtain our main result:

**Theorem 1.** For any $p \in (0, 1)$, the PCAs $A_p$ and $B_p$ are both ergodic, and the probability of a draw is 0 for both the percolation game and the target game on $\mathbb{Z}^2$.

## 3 Reversible invariant measures and the hard-core model

### 3.1 The two-dimensional percolation game

In this section we discuss the relationship between the PCA for the percolation game and the hard-core model. We start in the two-dimensional case where the setting is simplest to understand, but our main application of the ideas will be in Section 3.2, when we use them to show that certain higher-dimensional games have positive probability of a draw when $p$ is small.

Consider the PCA $A_p$. This PCA is known to belong to the family of one-dimensional PCA having a Markovian stationary distribution [BCM69] TVS+90 MM14a. This Markovian measure is described by the following transition matrix on the set $\{0, 1\}$:

$$
P = \begin{pmatrix}
  p_{0,0} & p_{0,1} & 1 - p \\
  p_{1,0} & p_{1,1} & 0
\end{pmatrix} = \begin{pmatrix}
  \frac{2 - p - \sqrt{p(4 - 3p)}}{2(1-p)} & \frac{2p^2 - 3p + \sqrt{p(4 - 3p)}}{2(1-p)} \\
  \frac{-p + \sqrt{p(4 - 3p)}}{2(1-p)} & \frac{2 - p - \sqrt{p(4 - 3p)}}{2(1-p)}
\end{pmatrix},
$$

(see Section 4.2 of MM14a – note that $p$ there corresponds to our $1 - p$). In fact, the evolution of the PCA in this equilibrium is reversible; the distribution of the two-dimensional space-time diagram obtained (via the correspondence at (2)) is invariant under reflection in the line $x_1 + x_2 = k$ for any $k$.

(In addition, the stationary distribution is itself reversible as a Markov chain on $\mathbb{Z}$.

One way to understand the presence of this Markovian reversible stationary distribution is to consider the **doubling graph** of the PCA, corresponding to two consecutive times of its evolution [Vas78 KV80 TVS+90]. This is an undirected bipartite graph, connecting sites between which there is an influence induced by the rules of the PCA.

As in Section 2.1 we can think of a configuration of the PCA as indexed by a diagonal $S_k = \{(x_1, x_2) : x_1 + x_2 = k\}$ of $\mathbb{Z}^2$. A time-step of the PCA then corresponds to moving from a configuration on $S_{k+1}$ to a configuration on $S_k$.

As before, let $N(x) = \{x + e_1, x + e_2\}$ for $x \in S_k$. The elements of $N(x)$ are in $S_{k+1}$ and are the sites to which the token may move from sites $x$; they are the sites whose values appear on the right-hand side of the recurrence in (1) for the value $\eta(x)$. Then the bijection

$$\phi(x) = x + e_1 + e_2$$

(3)
from $S_k$ to $S_{k+2}$ has the property that
\[ y \in N(x) \text{ iff } \phi(x) \in N(y). \] (4)

Let $G_k$ be the undirected bipartite graph with vertex set $S_k \cup S_{k+1}$, and an edge joining $x \in S_k$ and $y \in S_{k+1}$ if $y \in N(x)$.

The graphs $G_k$ are isomorphic to each other for $k \in \mathbb{Z}$. The doubling graph is a graph $G$ which is isomorphic to each $G_k$. We can also interpret the doubling graph as the image of $\mathbb{Z}^2$ under the equivalence relation $x \equiv \phi(x)$. We can take $G$ to be $\mathbb{Z}$, with nearest-neighbour edges, as shown in Figure 5. Consider the map
\[ v((x_1, x_2)) = x_1 - x_2 \] (5)
from $\mathbb{Z}^2$ to $\mathbb{Z}$; restricted to the set $S_k \cup S_{k+1}$, this gives an isomorphism between $G_k$ and $G$, for any $k$.

![Figure 5: The doubling graph $G$ isomorphic to $\mathbb{Z}$, shown on the left in correspondence with two successive diagonals $S_k$, $S_{k+1}$ of $\mathbb{Z}^2$.](image)

We now introduce the hard-core model on a graph with vertex set $V$. A Gibbs measure for the hard-core model with activity $\lambda > 0$ is a distribution on configurations $\eta \in \{0, 1\}^V$ such that
\[
P(\eta(x) = 1 \mid \{\eta(y), y \neq x\}) = \begin{cases} \frac{\lambda}{1 + \lambda} & \text{if } \eta(y) = 0 \text{ for all neighbours } y \text{ of } x \\ 0 & \text{otherwise} \end{cases}
\]

We will consider the hard-core model on the doubling graph $G$ with vertex set $V = \mathbb{Z}$. This is a bipartite graph, with bipartition $V = V_0 \cup V_1$ where $V_0$ is the set of even vertices and $V_1$ is the set of odd vertices. We consider the following two update procedures for configurations on $\{0, 1\}^V$. For an “odd” update, for each vertex $x \in V_1$ independently, resample $\eta(x)$ according to the values at its two neighbours, setting $\eta(x) = 0$ with probability 1 if either of the neighbours takes value 1, and otherwise setting $\eta(x) = 1$ with probability $1 - p$. For an “even” update, do the same for vertices in $V_0$. Set $\lambda = \frac{1}{p} - 1$, so that $1 - p = \frac{\lambda}{1 + \lambda}$. If a distribution on $\{0, 1\}^V$ is a Gibbs measure for the hard-core model with activity $\lambda$, then it is invariant under both of these update procedures (this is a version of Glauber dynamics for the hard-core model).

Take some even $k \in \mathbb{Z}$. Suppose we start from a configuration on $\{0, 1\}^V$ which, via the isomorphism between $G$ and $G_k$ under which $V_0$ maps to $S_k$ and $V_1$ to $S_{k+1}$, corresponds to a configuration in $\{0, 1\}^{S_k \cup S_{k+1}}$. We perform an odd update, resampling the sites of $V_1$, leading to a new configuration on $\{0, 1\}^V$. Considering now $\phi$ as an isomorphism between $G_k$ and $G_{k-1}$, which maps $V_0$ to $S_k$ and $V_1$ to $S_{k+1}$, the updated configuration on $\{0, 1\}^V$ corresponds to a configuration in $\{0, 1\}^{S_{k-1} \cup S_k}$, whose values at the sites in $S_k$ are left unchanged. We can interpret the update as generating a configuration on $S_{k-1}$ from a configuration on $S_k$, and the procedure is identical to that which occurs in one iteration of the PCA $A_p$.

If we then perform an even update, resampling the sites of $V_0$, we can pass in the same way to a configuration on the sites of $S_{k-2} \cup S_{k-1}$, which corresponds to the next step of the PCA.

Continuing to perform odd and even updates alternately, we reproduce the evolution of the PCA. A Gibbs measure on $G$ is characterised by its restriction to the vertices of one half of the bipartition,
say $V_0$. Since the measure is preserved by the updates, the distribution on $\{0, 1\}^{V_0}$ gives a 2-periodic distribution for the PCA.

In fact, for any $\lambda$ there is a unique Gibbs measure for the hard-core model on $\mathbb{Z}$, which is a Markov chain, with transition matrix $Q$, say; projecting this distribution onto either the even or the odd sites gives the same distribution $\mu_p$ which is the Markov chain with transition matrix $P = Q^2$. This $\mu_p$ is a stationary distribution for the PCA $A_p$. See Figure 6 for an illustration.

Figure 6: The Markovian measure corresponding to a Gibbs measure for the hard-core model on the doubling graph $V$ yields a Markovian measure $\mu_p$ on each of the two vertex classes $V_0$ and $V_1$. Since the Gibbs measure is invariant under the update procedures, the measure $\mu_p$ is invariant for the PCA.

The reversibility property for the stationary distribution $\mu_p$ follows from the reversibility of the process of configurations on $G$ under the update procedure (this is essentially the standard reversibility property for Glauber dynamics and is easy to verify by checking the detailed balance equations). In fact, from the uniqueness of the Gibbs measure for the hard-core model on $G$, one can deduce quite easily that there is only one reversible stationary distribution for the PCA $A_p$. However, this argument does not preclude the existence of other non-reversible stationary distributions. By the claimed ergodicity (Theorem 1), such distributions in fact do not exist, but this will be proved using a different argument, in Section 3.2.

In contrast, in the next section we will use the implication in the other direction; in situations where there exist multiple Gibbs measures for the hard-core model, we can conclude that there are multiple periodic distributions for the corresponding PCA; then the PCA is non-ergodic, and draws occur with positive probability in the corresponding game.

### 3.2 Higher-dimensional percolation games

We will consider various generalisations of the percolation game to $\mathbb{Z}^d$ for $d \geq 3$. In some cases (but not all) a symmetry condition analogous to (4) holds, and we can obtain, via a generalisation of the idea of doubling graph, a correspondence to a hard-core model on a suitable lattice, now in two or more dimensions. For some such lattices, it is known that there are multiple Gibbs measures for the hard-core model with sufficiently high $\lambda$. This is enough to imply the existence of draws (and non-ergodicity of the corresponding higher-dimensional PCAs) in these cases, when $p$ is sufficiently low.

For each of the games below, let each site $x$ of $\mathbb{Z}^d$ be closed with probability $p$ and open with probability $1 - p$, independently. Let $N(x)$ be the set of sites to which a player can move the token from site $x$ (provided the destination site is open). As before, if the token is at $x$ and all the sites in $N(x)$ are closed, then the player to move loses the game. In each case we will have the recursion (1) for the status of the sites.

**Game 1.** Let $N(x) = \{x + e_i, 1 \leq i \leq d\}$. Then $|N(x)| = d$.

This is perhaps the most natural extension to higher dimensions, but it appears that an argument using the hard-core model does not apply here, due to the lack of a symmetry property analogous to (4).

**Game 2.** $N(x) = \{x \pm e_i + e_d, 1 \leq i \leq d - 1\}$. Here $|N(x)| = 2(d - 1)$. Since any step preserves parity, it is natural to restrict to the set of even sites $(\mathbb{Z}^d)_{\text{even}} = \{x \in \mathbb{Z}^d : \sum x_i \text{ is even}\}$.

In two dimensions, the game is isomorphic to the original game on $\mathbb{Z}^2$. For general $d$, the symmetry condition (4) holds if we set $\phi(x) = x + 2e_d$. In the same way as we used the diagonals $S_k$ before, states of the relevant PCA are now identified with configurations on the cross-sections $T_k = \{x \in (\mathbb{Z}^d)_{\text{even}} : x_d = k\}$. A single time-step of the PCA corresponds to deriving a configuration on $T_k$ from one on $T_{k+1}$; specifically, for each site $x \in T_k$ independently, set $\eta(x) = 0$ if $\eta(y) = 1$ for some $y \in N(x)$, and otherwise set $\eta(x) = 1$ with probability $1 - p$ and $\eta(x) = 0$ with probability $p$. Just as in Section 2.1 this PCA extends to its envelope PCA with alphabet $\{0, 1\}$, whose iteration is given by (1).
To obtain the doubling graph, we take the graph $G_k$ with vertex set $T_k \cup T_{k+1}$, with an edge between $x \in T_k$ and $y \in T_{k+1}$ when $y \in N(x)$. Equivalently, this is the projection of $(\mathbb{Z}^d)_{\text{even}}$, with edges given by all pairs $(x, y)$ such that $y \in N(x)$, under the equivalence relation given by $x \equiv x + 2e_d$ for all $x$.

Let the doubling graph $G$ be the standard cubic lattice $\mathbb{Z}^{d-1}$ with nearest neighbour edges; then $G$ is isomorphic to $G_k$ for all $k$. Indeed, the map

$$v((x_1, \ldots, x_d)) = (x_1, \ldots, x_{d-1})$$

from $(\mathbb{Z}^d)_{\text{even}}$ to $\mathbb{Z}^{d-1}$ gives an isomorphism from $G_k$ to $G$ when restricted to the set $T_k \cup T_{k+1}$.

The doubling graph is bipartite, and as before, an update consists of resampling the configuration at all the sites of one of the classes of the bipartition, keeping the value at a site as 0 if any neighbour has value 1, and otherwise setting the value to 1 with probability $p = \frac{1}{1+\lambda}$ and 0 with probability $1-p$.

As in Section 3.1, the alternation of even and odd updates corresponds, via the isomorphism between $G$ and $G_k$ for each $k$, to the evolution of the PCA. Any Gibbs measure for the hard-core model with parameter $\lambda$ is invariant under both of the updates, and hence gives a measure for the PCA which is $2$-periodic. Since any Gibbs measure is preserved by either of the updates, it is characterised by its restriction to either half of the bipartition; hence if there are multiple Gibbs measures for the hard-core model, then there are multiple $2$-periodic measures for the PCA. Then the PCA is non-ergodic, and (using equivalent arguments to those in Proposition 1 and Proposition 2), we will argue that the game is drawn with positive probability.

**Game 3.** Let $N(x) = \{x \pm e_1 \pm e_2 \pm \cdots \pm e_{d-1} + e_d\}$, so that $|N(x)| = 2^{d-1}$. Each step changes the parity of every coordinate, so we restrict to the set $\mathbb{Z}_{\text{bcc}}^d = \{x \in \mathbb{Z}^d : x_i \equiv x_j \mod 2 \text{ for all } i, j\}$. Putting an edge between $x$ and $y$ whenever $y \in N(x)$, we obtain the body-centred cubic lattice in $d$ dimensions. This consists of two copies of $(2\mathbb{Z})_d^4$, each offset from the other by $(1, 1, \ldots, 1)$, so that each point of one lies at the centre of a unit cube of the other; the edges are given by joining each point to the $2^d$ corners of the surrounding unit cube.

States of the relevant PCA are identified with configurations on the cross-sections $T_k = \{x : x_d = k\}$. Again we set $\phi(x) = x + 2e_d$ and the symmetry condition (4) holds.

Let $G_k$ be the graph with vertex set $T_k \cup T_{k+1}$ and edges between $x$ and $y$ when $y \in N(x)$. The doubling graph $G$ isomorphic to $G_k$ for each $k$ is now the body-centred cubic lattice in $d-1$ dimensions. The map $v(x) = (x_1, x_2, \ldots, x_{d-1})$ from $\mathbb{Z}_{\text{bcc}}^d$ to $\mathbb{Z}_{\text{bcc}}^{d-1}$ restricts to an isomorphism between $G_k$ and $G$ for each $k$.

When $d = 2$ or $d = 3$ the game is equivalent to Game 2 above, but for $d \geq 4$ the games are different. In the same way as before, non-uniqueness for the hard-core model on the relevant $(d-1)$-dimensional body-centred cubic lattice will imply the existence of draws for the game.

**Figure 7:** A section of the three-dimensional body-centred cubic lattice (which is the doubling graph for the PCA associated to Game 3 when $d = 4$). The two underlying copies of $\mathbb{Z}^3$ are shown with red and with blue vertices. The black lines are the edges of the body-centred cubic lattice, and the dotted red lines show the nearest-neighbour edges in the red copy of $\mathbb{Z}^3$.

**Game 4.** Let $N(x) = \{x + \sum_{i \in S} e_i\}$, for any $S \subset \{1, \ldots, d\}$ with $1 \leq |S| \leq d-1$. So a move of the game corresponds to incrementing at least one, but not all, of the coordinates by one. Now $|N(x)| = 2^d - 2$. 

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For \( d = 2 \) the game is the same as ever. For \( d \geq 3 \) we have some new features. Consider the configuration on the diagonals \( S_k = \{ x : \sum x_i = k \} \). In order to generate the configuration on \( S_k \), we need to know the configuration not just on \( S_{k+1} \), but now in fact on \( S_{k+1}, \ldots, S_{k+d-1} \), so a notion of a “state” of the PCA needs to be generalised. If we consider the evolution of a process whose states correspond to configurations on \( S_k \), as \( k \) decreases, the process is no longer Markovian but now has a memory of length \( d - 1 \); instead we can consider a Markovian system in which a state corresponds to a configuration on \( d - 1 \) successive layers \( S_k, S_{k+1}, S_{k+d-2} \).

Rather than a doubling graph, we now need what could be called a “\( d \)-tupling” graph. The symmetry condition \( \{4\} \) holds with \( \phi(x) = x + \sum_{i=1}^{d} e_i \). If we take the quotient of \( \mathbb{Z}^d \), with an edge \((x, y)\) whenever \( y \in N(x) \), under the equivalence relation \( x \equiv \phi(x) \) for all \( x \), then we obtain a graph which is \((d - 1)\)-dimensional and \( d \)-partite. For \( d = 3 \), this corresponds to the triangular lattice, as shown on the left of Figure \( \PageIndex{8} \) for example via the map

\[
v(x) = x_1(1, 0) + x_2\left(-\frac{1}{2}, \sqrt{3} \right) + x_3\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).
\]

Instead of alternating odd and even updates as before, now we repeat a cyclic sequence of updates for each of the \( d \) classes of the graph. Again, any hard-core Gibbs measure is invariant under all of these updates, and existence of multiple Gibbs measures implies the existence of multiple \( d \)-periodic distributions for the PCA, and hence of existence of draws for the game.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{trian_hex.png}
\caption{Triangular lattice and hexagonal lattice.}
\end{figure}

**Game 5.** Fix \( r \) with \( 1 \leq r \leq d \), and now restrict to sites \( x \in \mathbb{Z}^d \) such that \( \sum x_i \equiv 0 \) or \( r \) mod \( d \). For \( x \equiv 0 \) mod \( d \), let \( N(x) = \{ x + \sum_{i \in S} e_i \} \), for any \( S \subset \{1, \ldots, d\} \) with \( |S| = r \). Meanwhile, for \( x \equiv r \) mod \( d \), let \( N(x) = \{ x + \sum_{i \in S} e_i \} \), for any \( S \subset \{1, \ldots, d\} \) with \( |S| = d - r \).

Now \( |N(x)| = \binom{d}{r} \) for all \( x \). Replacing \( r \) by \( d - r \) gives an equivalent game so we may assume \( 1 \leq r \leq d/2 \). The symmetry condition \( \{4\} \) holds with \( \phi = x + \sum_{i=1}^{d} e_i \). For \( d = 2 \) (and hence \( r = 1 \)) the game is the familiar two-dimensional game. For \( d = 3 \) and \( r = 1 \), we get \( |N(x)| = 3 \) and the doubling graph is the two-dimensional hexagonal lattice (shown on the right of Figure \( \PageIndex{8} \)); this is the image of \( \{ x \in \mathbb{Z}^2 : \sum x_i \equiv 0 \text{ or } 1 \text{ mod } 3 \} \), with edges between \( x \) and \( y \) where \( y \in N(x) \), under the map \( \{5\} \).

For \( d = 4 \) and \( r = 2 \), the game is isomorphic to the \( d = 4 \) case of Game 2 above, and so the doubling graph is the standard cubic lattice \( \mathbb{Z}^3 \).

For \( d = 4 \) and \( r = 1 \), we have \( |N(x)| = 4 \), and the doubling graph is the so-called diamond cubic graph (see for example Section 6.4 of [CS99]). This graph may, for example, be represented as

\[\{(y_1, y_2, y_3) \in \mathbb{Z}^3 : y_1 \equiv y_2 \equiv y_3 \text{ mod } 2 \text{ and } y_1 + y_2 + y_3 \equiv 0 \text{ or } 1 \text{ mod } 4\},\]
with edges between nearest neighbours (which are at distance $\sqrt{3}/4$). This is the image of $\{x \in \mathbb{Z}^d : \sum x_i \equiv 0 \text{ or } 1 \mod 3\}$, with edges between $x$ and $y$ where $y \in N(x)$, under the map

$$
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} =
\begin{pmatrix}
x_1 - x_2 - x_3 + x_4 \\
-x_1 + x_2 - x_3 + x_4 \\
x_1 + x_2 - x_3 - x_4
\end{pmatrix}
$$

(see [NS08]). See the right of Figure 9 for an illustration.

Figure 9: The cubic lattice and the diamond cubic graph in three dimensions.

We note that several of the lattices appearing above have appeared in a closely related context in [Dha83], where the link is made between the enumeration of directed lattice animals in $d$ dimensions and the hard-core model on appropriately defined $(d-1)$-dimensional lattices.

**Theorem 2.** For each of the following games, there is positive probability of a draw for sufficiently small $p$: Game 2 for all $d \geq 3$; Game 3 for $d = 3$ and $d = 4$; Game 4 for $d = 3$; and Game 5 for $d = 3$ (with $r = 1$) and $d = 4$ (either with $r = 1$ or with $r = 2$).

**Proof.** The cases listed correspond to those where the existence of multiple Gibbs measures for the hard-core model on the relevant doubling (or $d$-tupling) graph has been shown, when the activity parameter $\lambda$ is sufficiently high. For the standard cubic lattice in any dimension greater than 1, the result goes back to Dobrushin [Dob65]. Other models in two and three dimensions were covered by Heilmann [Hei74] and Runnels [Run75], including the triangular and hexagonal lattices in two dimensions and the body-centered cubic and diamond lattices in three dimensions.

In each case, the existence of multiple Gibbs measures yields the existence of multiple periodic measures for the PCA associated to the game ($d$-periodic in the case of Game 4, and 2-periodic in the other cases). Hence the PCA is not ergodic, and the same argument used in the proof of Proposition 2 shows that this non-ergodicity implies that the probability of a draw for the percolation game is positive. Note that this direction of the argument uses only the monotonicity of the type given in Lemma 1(ii). The monotonicity of the type given in Lemma 1(i) is not required (and it does not hold in the case of Game 4, where the underlying graph is no longer bipartite; from a given starting site, there are sites which can be reached later in the game with either one of the two players to move). Monotonicity of the type given in Lemma 1(ii) can be shown for all the games in this section via exactly the same coupling procedure as in Section 2.1 based on the recursion (1).

We expect that in fact the hard-core model has a phase transition in all the cases considered above (for $d \geq 3$, when the lattices described have dimensions 2 and above). For example, a version of the Peierls argument should apply; but this is less easy than, for example, for the Ising model, since the definition of a contour for the hard-core model is less straightforward and depends on the details of the graph.

Various further extensions can be made while still preserving the correspondence to the hard-core model. For example, in any of the cases above, we can augment the set of allowable moves from site $x$ to be $N(x) = N(x) \cup \{\phi(x)\}$. The change to the update rule is that if the value at a site of the doubling
graph is 1 before an update, it must switch to 0. Again one can show that hard-core Gibbs measures are stationarity under such updates, but now with the activity parameter $\lambda$ equal to $1 - p$ rather than $1/p - 1$ as before. (To verify the stationarity, one can start by checking the detailed balance condition for an update at a single site; then if the measure is stationary for the update at any single site, it is also invariant under simultaneous updates at any set of non-neighbouring sites.) Note that now as $p \to 0$, we have $\lambda \to 1$ rather than $\lambda \to \infty$. Hence to show existence of draws, and non-ergodicity of the corresponding PCA, we would need multiplicity of Gibbs measures for some $\lambda < 1$. For the case of the standard cubic lattice, Galvin and Kahn [GK04] show that this holds for sufficiently high dimension, so that we can deduce the existence of draws for this variant of Game 2 when $d$ is sufficiently large.

4 Ergodicity of the probabilistic cellular automata on $\mathbb{Z}$

4.1 Description of the PCA

We are interested in the following two PCA, denoted by $A_p$ and $B_p$. We will prove that both of them are ergodic for any parameter $p \in (0, 1)$.

$$
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{array}
$$

Figure 10: PCA $A_p$ (percolation game)

$$
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
$$

Figure 11: PCA $B_p$ (target game)

We recall that the envelope PCA of $A_p$ and $B_p$, with alphabet $\{0, ?, 1\}$, are denoted by $F_p$ and $G_p$ and defined as represented on Figures 12 and 13.

$$
\begin{array}{ccc}
0 & 0 & * \\
1 & 1 & * \\
\end{array}
\quad
\begin{array}{ccc}
? & 0 & ? \\
0 & ? & ? \\
? & ? & ? \\
\end{array}
$$

Figure 12: PCA $F_p$, the envelope of $A_p$

Let us introduce the deterministic cellular automaton defined as shown on Figure 14. (Note that $D = F_0 = G_0$).

We also introduce the random operator $R_p^0$ on $\{0, ?, 1\}^\mathbb{Z}$ that changes each symbol into a 0 with probability $p$ (making no change if it was already a 0), independently for different sites, and similarly the random operator $R_p^1$ that changes each symbol into a 1 with probability $p$, independently for different sites.
Then we have $F = R^0_p \circ D$ and $G = R^1_p \circ D$.

### 4.2 Introduction of a weight on configurations

Next we establish a property of the deterministic operator $D$.

In a given configuration of $\{0, ?, 1\}^\mathbb{Z}$, let us weight the occurrences of the symbol $?$ as follows:

- if a symbol $?$ is followed by a 0 and then by a 1 (pattern $?01$), then it counts for 3,
- if a symbol $?$ is followed by a 0 and then by something other than a 1, it counts for 2,
- otherwise, a symbol $?$ counts for 1.

The distribution of a configuration $(\eta_n, n \in \mathbb{Z})$ is shift-invariant if $(\eta_n, n \in \mathbb{Z})$ and $(\eta_{n+k}, n \in \mathbb{Z})$ have the same distribution for each $k \in \mathbb{Z}$, and reflection-invariant if $(\eta_n, n \in \mathbb{Z})$ and $(\eta_{-n}, n \in \mathbb{Z})$ have the same distribution.

If $\mu$ is a distribution, and $x \in \{0, ?, 1\}^n$ is a finite word, we write $\mu(x) := \mu(\{\eta : (\eta_1, \ldots, \eta_n) = x\})$ for the corresponding cylinder probability. For a shift-invariant distribution $\mu$ on $\{0, ?, 1\}^\mathbb{Z}$, we introduce the quantity:

$$\mu(?01) + \mu(?0) + \mu(?).$$

which is the expected weight per site under $\mu$.

**Lemma 2.** If $\mu$ is a shift-invariant and reflection-invariant distribution on $\{0, ?, 1\}^\mathbb{Z}$, then

$$D\mu(?01) + D\mu(?0) + D\mu(?) \leq \mu(?01) + \mu(?0) + \mu(?)$$

**Proof.** By looking at the possible preimages of each pattern, we obtain the following three equalities:

$$D\mu(?) = \mu(??) + \mu(0?) + \mu(?0),$$

$$D\mu(?0) = \mu(??1) + \mu(0?1) + \mu(?01),$$

$$D\mu(?01) = 0.$$  

Summing, and using reflection invariance to deduce $\mu(0?) = \mu(?0)$, we obtain

$$D\mu(?01) + D\mu(?0) + D\mu(?) - \mu(?01) - \mu(?0) - \mu(?01) = \mu(??) + \mu(0?) + \mu(?01) + \mu(??1) + \mu(0?1)$$

$$\leq \mu(??) + \mu(0?) + \mu(?1)$$

$$= \mu(??) + \mu(?0) + \mu(?1)$$

$$= \mu(?)$$.
Here is an informal way to explain the above result. Let us consider a symmetric version of the weight system that we have introduced: for each symbol ?, we add its right-weight, as introduced above, to its left-weight, which is equal to 3 if it the previous letter is a 0 and if there is a 1 before it (pattern 10?), to 2 if the previous letter is a 0 and if there is something else than a 1 before, and to 1 otherwise. (Since in Lemma 2 we consider only reflection-invariant measures, working with the symmetric weight is equivalent to working with the original weight).

Thus, the total weight of the symbol ? in the pattern 1?1 is equal to 1 + 1 = 2, while in the pattern 10??1, the weight of the first ? symbol is 3 (left) + 1 (right) = 4, and the weight of the second one is equal to 1 + 1 = 2.

Figure 15 shows an example of evolution of the deterministic CA $D$ from an initial configuration represented at the top (with time going down the page). The weights of the symbols ? appearing in the space-time diagram are shown in red. As illustrated in the figure, from a pattern 1?1, the symbol ? disappears and the weight thus decreases, but in other cases the total weight is locally preserved.

4.3 Proof of ergodicity

**Proposition 5.** For $0 < p < 1$, the PCA $F_p$ has no stationary measure in which the symbol ? appears with positive probability.

**Proof.** It suffices to show that there are no such shift-invariant and reflection-symmetric invariant measures in which the symbol ? appears with positive probability. For consider iterating the PCA starting from the measure $\delta_i$ concentrated on the configuration with ? at all sites. By Lemma 1(ii), the probability $F^n_p\delta_i(\cdot)$ is non-increasing, and if there is any stationary distribution $\mu$ with positive probability of ?, then $F^n_p\delta_i(\cdot)$ is bounded below by $\mu(\cdot)$ for all $n$, and so does not converge to 0. Then any limit point of the sequence of Césaro sums of $F^n_p\delta_i$ is an stationary distribution which has positive probability of ?, and which is also shift-invariant and reflection-symmetric.

For a shift-invariant measure $\nu$ on $\{0, ?, 1\}^\mathbb{Z}$, we have the following three equalities:

\[
R_0^p\nu(\cdot) = (1 - p)\nu(\cdot), \\
R_0^p\nu(0) = p(1 - p)\nu(\cdot) + (1 - p)^2\nu(0) \\
R_0^p\nu(01) = p(1 - p)^2\nu(\ast 1) + (1 - p)^3\nu(01).
\]

Here $\ast$ represents an unspecified symbol to be summed over, so that for example $\nu(\ast 1) = \sum_{a=0,?,1} \mu(a1)$. Adding together these three equalities, we get

\[
R_0^p\nu(\cdot) + R_0^p\nu(0) + R_0^p\nu(01) = \nu(\cdot) + \nu(0) + \nu(01) \\
+ p(1 - p)^2\nu(\ast 1) - p^2\nu(\cdot) - p(2 - p)\nu(0) - p(3 - 3p + p^2)\nu(01).
\] (7)
Let $\mu$ be a stationary measure of $F_p$ that is shift-invariant and reflection-symmetric, and set $\nu = D\mu$. Then, since $F_p = R_p \circ D$, we have $R_p^0 \nu = \mu$. Hence by Lemma 2,

$$\nu(01) + \nu(0) + \nu(?) \leq R_p^0 \nu(01) + R_p^0 \nu(0) + R_p^0 \nu(?) .$$

Then, since $p > 0$, it follows from (7) that

$$(1 - p)^2 \nu(\ast \ast) \geq p \nu(?) + (2 - p) \nu(?) + (3 - 3p + p^2) \nu(01) \geq p \nu(?) + (2 - p) \nu(0) .$$

We now proceed to obtain bounds for the left and right sides of (8) in terms of probabilities involving $\mu$. We have $\mu(?) = R_p^0 \nu(?) = (1 - p) \nu(?)$. Also $\nu(01) = D\mu(01) = \mu(01) + \mu(11) - \mu(01) - \mu(01)\mu(11) = \mu(01) + \mu(11)$, since $F_p \mu = \mu$ and the pattern $11$ has no preimage by $F_p$. We thus obtain:

$$p \nu(?) + (2 - p) \nu(0) = \frac{p}{1 - p} \mu(?) + (2 - p) (\mu(01) + \mu(11))$$

$$= p \mu(?) + (2 - p) (\mu(01) + \mu(11)) + \frac{p^2}{1 - p} \mu(?)$$

$$\geq p \mu(11) + (2 - p) (\mu(01) + \mu(11)) + \frac{p^2}{1 - p} \mu(?)$$

$$\geq 2 \mu(11) + \mu(01) + \frac{p^2}{1 - p} \mu(?) .$$

But we have

$$(1 - p)^2 \nu(\ast \ast) = R_p^0 \nu(\ast \ast)$$

$$= \mu(\ast \ast)$$

$$= \mu(01) + \mu(01) + \mu(11)$$

$$\leq \mu(01) + 2 \mu(11) .$$

Putting together (8), (9) and (10) we obtain that $\mu(?) = 0$ as required.\[\square\]

**Proposition 6.** For $0 < p < 1$, the PCA $G_p$ has no stationary distribution in which the symbol $\ast$ appears with positive probability.

**Proof.** As in the proof of Proposition 5 it suffices to consider only stationary distributions that are shift-invariant and reflection-symmetric. For a shift-invariant measure $\nu$ on $\{0, ?, 1\}^\mathbb{Z}$, we have the following equalities:

$$R_p^1 \nu(?) = (1 - p) \nu(?)$$

$$R_p^0 \nu(0) = (1 - p)^2 \nu(0)$$

$$R_p^0 \nu(01) = p(1 - p)^2 \nu(0) + (1 - p)^3 \nu(01)$$

Thus,

$$R_p^1 \nu(?) + R_p^1 \nu(0) + R_p^1 \nu(01) = \nu(?) + \nu(0) + \nu(01)$$

$$+ p(1 - p)^2 \nu(0) - \nu(?) - p(2 - p) \nu(0) - p(3 - 3p + p^2) \nu(01)$$

Let $\mu$ be a shift-invariant, reflection-invariant, invariant distribution of $G_p$, and let $\nu = D\mu$. Then, $R_p^1 \nu = \mu$. By Lemma 2,

$$\nu(01) + \nu(0) + \nu(?) \leq R_p^1 \nu(01) + R_p^1 \nu(0) + R_p^1 \nu(?) .$$

It follows that

$$(1 - p)^2 \nu(0) \geq \nu(?) + (2 - p) \nu(?) + (3 - 3p + p^2) \nu(01) \geq (2 - p) \nu(0),$$

which is possible only if $\nu(0) = 0$. We then obtain $\mu(01) = 0$, and it follows easily that $\mu(?) = 0$.\[\square\]
Now we can quickly deduce our main result.

Proof of Theorem 1. We know that the distribution of the status (win, loss, draw) of the sites along a diagonal $S_k$ in the percolation game is a stationary distribution for $F_p$. Since by Proposition 5, $F_p$ has no stationary distribution with positive probability of $\Omega$ for any $p > 0$, the probability of a draw in the percolation game must be 0. Then by Proposition 2, the PCA $A_p$ is ergodic for each $p > 0$.

An identical argument applies for the target game, and the PCA $B_p$, using Propositions 6 and 4.

5 Open questions

(1) For the games of Section 3.2 is the probability of a draw positive for small enough $p$ in all cases when $d \geq 3$? In some cases this would follow from the existence of a phase transition occurs for the corresponding hard-core model. In other cases (including Game 1, where the allowable steps are those which increase exactly one coordinate by one) our reduction to a hard-core model breaks down, and a new approach appears to be needed.

(2) Besides the target game, there are other natural misère versions of the percolation game. For example, suppose that if the token is at $x$ and both the sites $x + (0,1)$ and $x + (1,0)$ are closed, then the next player to move wins (rather than loses). Does this game have zero probability of a draw for all $p$? Unlike the original percolation game or the target game, this game does not seem to have a convenient correspondence to a PCA with alphabet \{0,1\}.

(3) Concerning PCA, a challenge is to extend the tools for proving ergodicity. In particular, it is an open question whether every elementary (i.e. 2-state and size-2 neighbourhood) PCA on $\mathbb{Z}$ with positive rates is ergodic. Using the weighting introduced in Section 4.2 we have proved ergodicity for PCAs of the form $R^0_p \circ D$ and $R^1_p \circ D$ (which do not have positive rates). Can this method be adapted to the PCA $R^0_p \circ R^1_q \circ D$ for $p, q > 0$ (which does have positive rates), or more general elementary PCA?

(4) In situations where we have a correspondence between the game (or PCA) and the hard-core model on the doubling graph, do there exist cases where the PCA has invariant measures that are not projections of a hard-core Gibbs measure? Or is it the case that the PCA has multiple invariant distributions precisely when there are multiple hard-core Gibbs measures? Note that in general it is not known that the hard-core model has a single phase transition (i.e. that the existence of multiple Gibbs measures is a monotone property in $\lambda$), and similarly it is not clear in general whether ergodicity of the PCA (or positive probability of draws) is monotone in $p$.

(5) Undirected games provide a further challenge. For example, consider the game on $\mathbb{Z}^2$ with each site closed with probability $p$ and open with probability $1 - p$ independently as before; now a player can move from $x$ to any of the four nearest neighbours of $x$ which are open, but with the added constraint that a site that has been previously visited is forbidden. A player unable to move loses the game. Does such a game have positive probability of draws for small $p$? Since the game is undirected, the PCA approach no longer applies. In [BHMW], various results are proved for the case where the probability of closed site depends whether the site is odd or even, so that one of the two players has an advantage.

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