Massive Supersymmetric Quantum Gauge Theory

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Abstract

We continue the study of the supersymmetric vector multiplet in a purely quantum framework. We obtain some new results which make the connection with the standard literature. First we construct the one-dimensional physical Hilbert space taking into account the (quantum) gauge structure of the model. Then we impose the condition of positivity for the scalar product only on the physical Hilbert space. Finally we obtain a full supersymmetric coupling which is gauge invariant in the supersymmetric sense in the first order of perturbation theory. By integrating out the Grassmann variables we get an interacting Lagrangian for a massive Yang-Mills theory related to ordinary gauge theory; however the number of ghost fields is doubled so we do not obtain the same ghost couplings as in the standard model Lagrangian.

PACS: 11.10.-z, 11.30.Pb

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1 Introduction

The supersymmetric gauge theories are constructed using the vector supersymmetric multiplet (see also [23, 22, 21, 9, 1, 6, 14, 20, 15, 16, 17, etc.) This is a superfield for a \( N = 1 \) supersymmetric theory which describes only particles of spin \( s \leq 1 \) and it contains in particular exactly one particle of spin 1 in the one-particle Hilbert space. In the standard literature quoted above this model is considered mainly as a classical field theory and the quantization is performed by path integral method. We have considered in detail the vector multiplet in a previous paper [10] in a pure quantum framework. We have analyzed there the question of positivity for the whole multiplet and have obtained some restrictions on the free parameters of the model (the mass and some free parameters appearing in the causal commutator). Then we have analyzed the gauge structure associated to the vector multiplet (constructing some natural ghost fields which are chiral fields) and tried to construct an interaction Lagrangian built only from superfields and gauge invariant in a supersymmetric sense. One can obtain non-trivial solutions for this problem but it is not so easy to find a solution which gives, after integrating the Grassmann variables, the usual interaction for a system of massive Yang-Mills fields. In fact, our ansatz for the coupling in [10] was not general enough for this goal; this shortcoming will be removed here.

In this paper we consider only vector multiplets of non-null mass and we use the gauge structure already determined in [10]. Using the explicit expression of the gauge charge \( Q \) (we remind that \( Q \) is an operator acting in a Hilbert space \( H \) with a non-degenerate sesqui-linear form \( < \cdot , \cdot > \) verifying \( Q^2 = 0 \) such that the physical Hilbert space is \( H_{phys} \equiv Ker(Q)/Im(Q) \)) we are able to determine the structure of the one-dimensional physical Hilbert space \( Ker(Q)/Im(Q) \cap H^{(1)} \); then we impose the condition of positivity of the scalar product only on this sub-space. We obtain weaker conditions on the parameters of the model than in our previous work [10]. This is done in Section 2. Then we make a more general ansatz for the supersymmetric interaction Lagrangian and we succeed in finding a solution having all the required properties. We require that the corresponding interaction Lagrangian \( t(x) \) should be renormalizable (so it must have canonical dimension \( \leq 4 \)) and it should follow by integrating out the Grassmann variables

\[
t(x) \equiv \int d\theta^2 d\bar{\theta}^2 T(x, \theta, \bar{\theta});
\]

moreover we require that \( T(x, \theta, \bar{\theta}) \) is gauge invariant in the supersymmetric sense:

\[
[Q, T(x, \theta, \bar{\theta})] = \mathcal{D}^a T_a(x, \theta, \bar{\theta}) + \mathcal{D}_a \bar{T}^a(x, \theta, \bar{\theta})
\]

where

\[
\mathcal{D}_a \equiv \frac{\partial}{\partial \theta^a} - i\sigma_{ab}^\mu \bar{\theta}^b \partial_\mu \quad \mathcal{D}_a \equiv -\frac{\partial}{\partial \bar{\theta}^a} + i\sigma_{ba}^\mu \theta^b \partial_\mu;
\]

we call SUSY-divergences the expressions appearing in the right hand side of formula (1.2). A priori one should include in the right hand side of (1.2) space-time total divergence \( \partial_\mu T^\mu \) also. But if we use the identity

\[
\{\mathcal{D}_a, \mathcal{D}_b\} = -2i\sigma_{ab}^\mu \partial_\mu
\]
we can eliminate the space-time divergence by redefining the expressions $T_a$ and $\bar{T}^a$.

We get a possible interaction for a system of massive Yang-Mills fields verifying (1.1) and (1.2). Because the super-ghost fields are chiral we obtain a model with twice as many ghost fields as the usual standard model. As a consequence the resulting coupling is different from that in the standard model. Whether this has implications for phenomenology remains to be investigated.

2 The Gauge Structure of the Vector Multiplet

We use the notations and definitions from [10]. The vector multiplet is by definition given by the expression

$$V(x, \theta, \bar{\theta}) = C(x) + \theta \chi(x) + \bar{\theta} \bar{\chi}(x) + \theta^2 \phi(x) + \bar{\theta}^2 \phi^\dagger(x) + \theta \sigma^\mu \bar{\theta} v_\mu(x) + \bar{\theta} \bar{\sigma}^\mu \theta \bar{v}_\mu(x) + \theta^2 \theta \lambda(x) + \bar{\theta} \bar{\theta} \lambda(x) + \theta \bar{\theta} \bar{\theta} \bar{\theta} d(x)$$

(2.1)

with the reality condition:

$$V^\dagger = V;$$

(2.2)

here $C$ and $d$ are real scalar fields, $\phi$ is a complex scalar field, $v_\mu$ is a hermitian vector field, $\chi_a$ and $\lambda_a$ are Dirac spinor fields and $\theta_a$ are Grassmann parameters. The major difference with respect to [10] is that we do not suppose that the Hilbert space in which these fields live has a positive defined scalar product; we suppose that we have only a indefinite sesqui-linear form $<\cdot, \cdot>$. The construction of the fields is done in the same way as in the usual case by applying these fields on the vacuum state $\Omega$, because Borchers algebra can be constructed regardless of the positive-defined character of the sesqui-linear form. We also suppose that all these fields are of mass $m$:

$$(\partial^2 + m^2)V = 0$$

(2.3)

and we take $m$ to be strictly positive.

We remind from [10] the decomposition of $V$ which will be very useful in the following:

**Proposition 2.1** The vector superfield $V$ can be written as follows

$$V = \sum_{j=0}^{2} P_j V$$

(2.4)

where the expressions $P_j, \ j = 0, 1, 2$ are given by

$$P_0 \equiv -\frac{1}{8m^2} D^a \bar{D}^2 D_a, \quad P_1 \equiv \frac{1}{16m^2} D^2 \bar{D}^2, \quad P_2 \equiv \frac{1}{16m^2} \bar{D}^2 D^2.$$ 

(2.5)

The expressions $P_j, \ j = 0, 1, 2$ are projectors on the mass shell i.e. they verify

$$P_j P_k = 0, \ \forall j \neq k,$$

$$P_j^2 V = P_j V, \ \forall j.$$ 

(2.6)
The components \( V_j \equiv P_j V, \quad j = 1, 2 \) of \( V \) verify
\[
\mathcal{D}_a V_1 = 0, \quad \mathcal{D}_a V_2 = 0.
\] (2.7)

The three components are called \textit{chiral}, \textit{anti-chiral} and \textit{transversal} components of \( V \), respectively. The appellative transversal is justified by the explicit formula:
\[
V_0 = -\frac{2}{m^2} D'
\]
\[
D' = d' - \frac{i}{2} \theta \sigma^\mu \partial_\mu \lambda' + \frac{i}{2} \partial_\mu \lambda' \sigma^\mu \bar{\theta} - \frac{1}{2} \left( \theta \sigma^\mu \bar{\theta} \right) \left( m^2 g_{\mu \rho} + \partial_\mu \partial_\rho \right) v^\rho
\]
\[
- \frac{m^2}{4} \left( \bar{\theta}^2 \lambda' + \bar{\theta} \theta \lambda' \right) - \frac{m^2}{4} \theta^2 \bar{\theta}^2 \ d'
\] (2.8)

where
\[
\lambda'_a \equiv \lambda_a + \frac{i}{2} \sigma^\mu b_\mu \lambda^b
\]
\[
d' \equiv d - \frac{m^2}{4} C;
\] (2.9)
in particular the superfield \( V_0 \) contains only one Majorana spinor \( \lambda' \), one scalar field \( d' \) and a real vector field
\[
v'_\mu \equiv \left( g_{\mu \rho} + \frac{1}{m^2} \partial_\mu \partial_\rho \right) v^\rho
\] (2.10)
verifying the transversality condition
\[
\partial^\mu v'_\mu = 0.
\] (2.11)

Next we remind the generic form of the causal commutator of the vector superfield. Let us consider the causal commutator
\[
[V(X_1), V(X_2)] = -i D(X_1; X_2) \ 1;
\] (2.12)
then we have according to theorem 3.7 of [10]
\[
D(X_1; X_2) = \sum_{j=1}^{4} c_j \ D_j(X_1; X_2)
\] (2.13)
where
\[
D_1(X_1; X_2) = \exp[i \ (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \ \partial_\mu] \ D_m(x_1 - x_2)
\]
\[
D_2(X_1; X_2) = (\theta_1 - \theta_2)^2 (\bar{\theta}_1 - \bar{\theta}_2)^2 \ D_m(x_1 - x_2)
\]
\[
D_3(X_1; X_2) = \exp[i \ (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \ \partial_\mu][((\theta_1 - \theta_2)^2 + (\bar{\theta}_1 - \bar{\theta}_2)^2] \ D_m(x_1 - x_2)
\]
\[
D_4(X_1; X_2) = i \ \exp[i \ (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \ \partial_\mu][((\theta_1 - \theta_2)^2 - (\bar{\theta}_1 - \bar{\theta}_2)^2] \ D_m(x_1 - x_2).
\] (2.14)
It is useful to write down the corresponding causal (anti)commutation relations of the component fields. A straightforward computation leads to:

\[
[C(x), C(y)] = -i \, c_1 \, D_m(x - y) \\
[C(x), d(y)] = -i \, c_2 \, D_m(x - y) \\
[C(x), \phi(y)] = -i \, (c_4 - ic_3) \, D_m(x - y) \\
[\phi(x), \phi^\dagger(y)] = -i \left( \frac{m^2}{4} c_1 + c_2 \right) D_m(x - y) \\
[\phi(x), d(y)] = \frac{m^2}{4} \, (c_4 - ic_3) \, D_m(x - y) \\
[\phi(x), v_\mu(y)] = (c_3 + ic_4) \, \partial_\mu D_m(x - y) \\
[d(x), d(y)] = \frac{im^4}{16} \, c_1 \, D_m(x - y) \\
\]

\[
[v_\mu(x), v_\rho(y)] = i \, c_1 \, \partial_\mu \partial_\rho \, D_m(x - y) + i \left( \frac{m^2}{2} c_1 - 2c_2 \right) g_{\mu\rho} D_m(x - y) \\
{\{\chi_a(x), \chi_b(y)\} = 2(c_4 - ic_3) \, \epsilon_{ab} \, D_m(x - y),} \\
{\{\chi_a(x), \bar{\chi}_b(y)\} = c_1 \, \sigma_{\mu\nu} \, \partial_\mu D_m(x - y),} \\
{\{\lambda_a(x), \lambda_b(y)\} = -\frac{m^2}{2} (c_4 - ic_3) \, \epsilon_{ab} \, D_m(x - y),} \\
{\{\lambda_a(x), \bar{\lambda}_b(y)\} = \frac{m^2}{4} c_1 \, \sigma_{\mu\nu} \, \partial_\mu D_m(x - y),} \\
{\{\chi_a(x), \bar{\lambda}_b(y)\} = -2i \, c_2 \, \epsilon_{ab} \, D_m(x - y),} \\
{\{\chi_a(x), \bar{\chi}_b(y)\} = -i \, (c_4 + ic_3) \, \sigma_{\mu\nu} \, \partial_\mu D_m(x - y) \quad (2.15)}
\]

and all other (anti)commutators are zero.

Finally we give the gauge structure of the vector multiplet. We need three ghost superfields \(U, \bar{U}, H\) (the first two fermionic and the last one bosonic) which are analogues of the usual ghost fields of the standard model [18]. We take these fields to be chiral

\[
\mathcal{D}_a \, U = 0 \quad \mathcal{D}_a \, \bar{U} = 0 \quad \mathcal{D}_a \, H = 0. \quad (2.16)
\]

In components we have

\[
H(x, \theta, \bar{\theta}) = h(x) + 2 \, \bar{\theta} \bar{\psi}(x) + i \, (\theta \sigma^\mu \bar{\theta}) \, \partial_\mu h(x) + \bar{\theta}^2 \, f(x) - i \, \bar{\theta}^2 \, \theta \sigma^\mu \partial_\mu \bar{\psi}(x) \\
+ \frac{m^2}{4} \, \bar{\theta}^2 \bar{\theta}^2 \, h(x) \quad (2.17)
\]

where \(h, f\) and are scalar and \(\psi\) is a Dirac field. Analogously we have:

\[
U(x, \theta, \bar{\theta}) = u(x) + 2i \, \bar{\theta} \bar{\zeta}(x) + i \, (\theta \sigma^\mu \bar{\theta}) \, \partial_\mu u(x) + \bar{\theta}^2 \, g(x) + \bar{\theta}^2 \, \theta \sigma^\mu \partial_\mu \bar{\zeta}(x) + \frac{m^2}{4} \, \bar{\theta}^2 \bar{\theta}^2 \, u(x) \quad (2.18)
\]

and, respectively

\[
\bar{U}(x, \theta, \bar{\theta}) = \bar{u}(x) - 2i \, \theta \bar{\zeta}(x) + i \, (\theta \sigma^\mu \bar{\theta}) \, \partial_\mu \bar{u} + \bar{\theta}^2 \, \bar{g}(x) - \bar{\theta}^2 \, \theta \sigma^\mu \partial_\mu \bar{\zeta}(x) + \frac{m^2}{4} \, \theta^2 \bar{\theta}^2 \, \bar{u}(x) \quad (2.19)
\]
where \( u, g \) (resp. \( \bar{u}, \bar{g} \)) are scalar fields of Fermi statistics and \( \zeta \) (resp. \( \bar{\zeta} \)) are Dirac fields of Bose statistics. The superfields \( U, \bar{U}, H \) are of the same mass \( m \) as the vector multiplet.

Now we give the definition of the gauge charge. In ordinary quantum gauge theory, one gauges away the unphysical degrees of freedom of a vector field \( v_\mu \) using ghost fields. Suppose that the vector field is of positive mass \( m \); then one enlarges the Hilbert space with three ghost fields \( v, \bar{v}, \phi \) such that: (a) All three are scalar fields; (b) All them have the same mass \( m \) as the vector field; (c) The Hermiticity properties with respect to the indefinite sesqui-linear form are:

\[
\phi^\dagger = \phi, \quad v^\dagger = v, \quad \bar{v}^\dagger = -\bar{v};
\]

(2.20)

(d) The last two ones \( v, \bar{v} \) are fermionic and \( \phi \) is bosonic; (e) The commutation relations for the ghost fields are:

\[
[\phi(x), \phi(y)] = -i \, D_m(x - y), \quad \{v(x), \bar{v}(y)\} = -i \, D_m(x - y)
\]

(2.21)

and the rest of the (anti)commutators are zero. Then one introduces the gauge charge \( Q \) according to:

\[
Q \Omega = 0 \quad Q^\dagger = Q
\]

\[
[Q, v_\mu] = i \partial_\mu v, \quad [Q, \phi] = i \, m \, v
\]

\[
\{Q, v\} = 0, \quad \{Q, \bar{v}\} = -i \, (\partial^\mu v_\mu + m \, \phi).
\]

(2.22)

It can be proved that this gauge charge is well defined by these relations i.e. it is compatible with the (anti)commutation relations. Moreover one has \( Q^2 = 0 \) so the factor space \( \text{Ker}(Q)/\text{Im}(Q) \) makes sense; it can be proved that this is the physical space of an ensemble of identical particles of spin 1. For details see [18], [8].

In [10] we have generalized this structure for the vector multiplet:

\[
Q \Omega = 0 \quad Q^\dagger = Q
\]

(2.23)

and

\[
[Q, V] = U - U^\dagger
\]

\[
\{Q, U\} = 0.
\]

\[
\{Q, \bar{U}\} = -\frac{1}{16} \, D^2 \, \bar{D}^2 V - i \, m \, H
\]

\[
[Q, H] = i \, m \, U.
\]

(2.24)

Let us note in particular

\[
[Q, D'] = 0.
\]

(2.25)

These commutators and anticommutators with \( Q \) define the gauge variation \( d_Q \) of the operators as the graded bracket with \( Q \). It is interesting to translate (2.24) in terms of the component fields of the multiplet. One notices in this way that the fields

\[
v_\mu, u + u^\dagger, \bar{u} - \bar{u}^\dagger, h + h^\dagger.
\]

(2.26)
have the same gauge structure as (2.22). However, let us emphasize that our model will be
different from the usual standard model because the number of ghost fields is doubled.

Now we consider a generic form for the one-particle states from the Hilbert space. It can
be written in a supersymmetric way as follows:

$$\Psi = \int f_1(X)V(X) + \int f_2(X)U(X) + \int f_3(X)U(X)^\dagger + \int f_4(X)\bar{U}(X)$$
$$\quad + \int f_5(X)\bar{U}(X)^\dagger + \int f_6(X)H(X) + \int f_7(X)H(X)^\dagger$$ (2.27)

with $f_j, j = 1, \ldots, 7$ supersymmetric test functions. The integration is over $d^4x d^2\theta d^2\bar{\theta}$. Let us
note that the writing (2.27) is unique iff we require

$$P_2 f_j = 0, \quad j = 2, 4, 6 \quad P_1 f_j = 0, \quad j = 3, 5, 7$$ (2.28)
because of the chirality condition. Indeed we have for instance

$$\int f_2(X)U(X) = \int (P_2 f_2)(X)U(X)$$ (2.29)
and similar relations for the other chiral fields. Also because of (2.3) we can suppose that all
these test functions verify Klein-Gordon equation.

If we impose the condition $\Psi \in \text{Ker}(Q)$ i.e. $Q\Psi = 0$ then we immediately find

$$P_1 f_1 = P_2 f_1 = 0$$ (3.30)
$$f_4 = 0, \quad f_5 = 0.$$ (3.31)

Now the generic expression of an element from $\text{Im}(Q)$ is

$$\Psi' = \int (g_1 + img_6)(X)U(X) + \int (-g_1 + img_7)(X)U(X)^\dagger$$
$$\quad - \frac{1}{16} \int \bar{D}^2 D^2 g_4 + \bar{D}^2 \bar{D}^2 g_5)(X)V(X) - im \int g_4(X)H(X) + im \int g_5(X)H(X)^\dagger.$$ (3.32)

If we take the test functions $g_j, j = 1, \ldots, 7$ convenient, then we can arrange such that

$$\Psi - \Psi' = \int f_1(X)V(X)$$ (3.33)
where $f_1$ is restricted only by (2.30). Moreover one can see that in every equivalence classes
from $\text{Ker}(Q)/\text{Im}(Q) \cap \mathcal{H}^{(1)}$ there exists one and only one element of transverse form; so the
equivalence classes are indexed by supersymmetric transversal functions. Equivalently, only
the transversal part of the vector superfield is producing physical states. This statement is the
rigorous form of the so-called Wess-Zumino gauge.

**Remark 2.2** The preceding argument depends essentially on the positivity of the mass: for a
null mass we do not have anymore the projection operators $P_j$. Also one should generalize this
argument to multi-particle states. This is a difficult technical problem.
From the preceding analysis it follows that we have to impose the positivity of the scalar product only in the sector generated by the fields appearing in the superfield $D' = -\frac{m^2}{2} V_0$ i.e. $d', \lambda'$ and the transversal part of $v_{\mu}$. If we do this by going through the proof of theorem 3.8 of [10], we get only the restriction

$$c_2 \leq 0.$$  

One can see this directly from

$$[d'(x), d'(y)] = -\frac{im^4}{8} \left( c_1 - 2 \frac{c_2}{m^2} \right) D_m(x - y)$$

$$[v'_\mu(x), v'_\nu(y)] = i \left( c_1 - 2 \frac{c_2}{m^2} \right) (\partial_\mu \partial_\nu + m^2 g_{\mu\nu}) D_m(x - y)$$

$$\{\lambda'_a(x), \lambda'_b(y)\} = -\frac{m^2}{2} \left( c_4 - ic_3 \right) \epsilon_{ab} D_m(x - y),$$

and

$$\{\lambda'_a(x), \bar{\lambda}'_b(y)\} = -2 c_2 \sigma_{ab} \partial_\mu D_m(x - y).$$  (2.35)

If we want the field $v_{\mu}$ to give a renormalizable theory then we should also put

$$c_1 = 0$$  (2.36)

in such a way that the causal commutator of $v_{\mu}$ does not have derivatives in the right hand side $[2.16]$. In particular, the solution $D(X_1; X_2) = -D_2(X_1; X_2)$ suggested by the supersymmetry literature is acceptable from the point of view of positivity.

We also mention that the preceding analysis is compatible with the following SUSY-scalar product

$$< f_1, f_2 > = -\int f_1(X_1) \bar{D}_2(X_1; X_2) f_2(X_2)$$  (2.37)

like is suggested in [3].

3 A Massive Supersymmetric Gauge Invariant Coupling

We consider the following interaction Lagrangian:

$$T = \sum_{j=1}^{7} T^{(j)}$$  (3.1)

with

$$T^{(1)} = f^{(1)}_{jkl} \left[ V_j (\mathcal{D}^a V_k) (\mathcal{D}^2 \mathcal{D}_a V_k) : -H.c. \right]$$

$$T^{(2)} = f^{(2)}_{jkl} : V_j (U_k + U_k^\dagger)(\bar{U}_l + \bar{U}_l^\dagger) :$$

$$T^{(3)} = f^{(3)}_{jkl} : (H_j + H_j^\dagger) (U_k - U_k^\dagger)(\bar{U}_l + \bar{U}_l^\dagger) :$$

$$T^{(4)} = f^{(4)}_{jkl} : (H_j + H_j^\dagger) (H_k - H_k^\dagger) V_l :$$

$$T^{(5)} = f^{(5)}_{jkl} : (H_j + H_j^\dagger) V_k D_l' :$$

$$T^{(6)} = f^{(6)}_{jkl} : (H_j + H_j^\dagger) V_k V_l :$$

$$T^{(7)} = f^{(7)}_{jkl} : (H_j + H_j^\dagger) (H_k - H_k^\dagger) D_l' :$$  (3.2)
where \( f_{jkl}^{(j)} \), \( j = 1, \ldots, 7 \) are some constants to be determined later. The last term is new by comparison to [10]. Because of the identity

\[
: (H_j + H_j^\dagger) (H_k - H_k^\dagger) D'_i : + : (H_j - H_j^\dagger) (H_k + H_k^\dagger) D'_i : := \text{SUSY - divergence} \quad (3.3)
\]

we can suppose that

\[
f_{jkl}^{(7)} = -f_{klj}^{(7)}. \quad (3.4)
\]

We can also take

\[
f_{jkl}^{(6)} = f_{jlk}^{(6)}. \quad (3.5)
\]

As discussed in [10], the list (3.2) is suggested by the fact that after integrating out the Grassmann variables one gets coupling terms of the same form as in ordinary gauge theory.

We impose the supersymmetric gauge invariance condition (1.2) from the Introduction. Noticing that \( d_Q D' = 0 \) we have

\[
d_Q T^{(1)} = -32 f_{jkl}^{(1)} : (U_j + U_j^\dagger) V_k D'_l + \text{SUSY - div}
\]

\[
d_Q T^{(2)} = f_{jkl}^{(2)} : (U_j - U_j^\dagger) (U_k + U_k^\dagger) (\bar{U}_l + \bar{U}_l^\dagger) : \]

\[
\quad + 2 f_{jkl}^{(2)} : V_j (U_k + U_k^\dagger) D'_l + f_{jkl}^{(2)} m^2_l : V_j (U_k + U_k^\dagger) V_l
\]

\[
+ i f_{jkl}^{(2)} m_l : V_j (U_k + U_k^\dagger) (H_l - H_l^\dagger) : \]

\[
d_Q T^{(3)} = i f_{jkl}^{(3)} m_j : (U_j + U_j^\dagger) (U_k - U_k^\dagger) (\bar{U}_l + \bar{U}_l^\dagger) : \]

\[
\quad + 2 f_{jkl}^{(3)} : (H_j + H_j^\dagger) (U_k - U_k^\dagger) D'_l + f_{jkl}^{(3)} m^2_l : (H_j + H_j^\dagger) (U_k - U_k^\dagger) V_l
\]

\[
+ i f_{jkl}^{(3)} m_l : (H_j + H_j^\dagger) (U_k - U_k^\dagger) (H_l - H_l^\dagger) : \]

\[
d_Q T^{(4)} = i f_{jkl}^{(4)} m_j : (U_j + U_j^\dagger) (H_k - H_k^\dagger) V_l : \]

\[
\quad + i f_{jkl}^{(4)} m_k : (H_j + H_j^\dagger) (U_k - U_k^\dagger) V_l + f_{jkl}^{(4)} : (H_j + H_j^\dagger) (H_k - H_k^\dagger) (U_l - U_l^\dagger) \]

\[
d_Q T^{(5)} = i f_{jkl}^{(5)} m_j : (U_j + U_j^\dagger) V_k D'_l : \]

\[
\quad + f_{jkl}^{(5)} : (H_j + H_j^\dagger) (U_k - U_k^\dagger) D'_l : \]

\[
d_Q T^{(6)} = i f_{jkl}^{(6)} m_j : (U_j + U_j^\dagger) V_k V_l : \]

\[
\quad + 2 f_{jkl}^{(6)} : (H_j + H_j^\dagger) (U_k - U_k^\dagger) V_l : \]

\[
d_Q T^{(7)} = 2i f_{jkl}^{(7)} m_k : (H_j + H_j^\dagger) (U_k - U_k^\dagger) D'_l : + \text{SUSY - div} \quad (3.6)
\]

To obtain the last expression one must use the identity

\[
: (U_j + U_j^\dagger) (H_k - H_k^\dagger) D'_l : + : (U_j - U_j^\dagger) (H_k + H_k^\dagger) D'_l : := \text{SUSY - divergence}. \quad (3.7)
\]
The gauge invariance condition now leads to the following system:

\[
\begin{align*}
    f_{jkl}^{(2)} m_l^2 + i f_{jkl}^{(6)} m_j &= -(k \leftrightarrow l) \\
    -32 f_{jkl}^{(1)} + 2 f_{jkl}^{(2)} + i f_{jkl}^{(5)} m_j &= 0 \\
    f_{jkl}^{(2)} - i f_{jkl}^{(3)} m_k &= 0 \\
    f_{jkl}^{(2)} m_l + f_{jlk}^{(4)} m_j &= 0 \\
    2 f_{jkl}^{(3)} + f_{jkl}^{(5)} + 2 i f_{jkl}^{(7)} m_k &= 0 \\
    f_{jkl}^{(3)} m_l^2 + i f_{jkl}^{(4)} m_k + 2 f_{jkl}^{(6)} &= 0 \\
    i f_{jkl}^{(3)} m_l + f_{jlk}^{(4)} &= 0
\end{align*}
\]  

(3.8)

The solution of this system is determined by the expressions \( f_{jkl}^{(1)} \) and \( f_{jkl}^{(2)} \); explicitly

\[
\begin{align*}
    f_{jkl}^{(3)} &= -\frac{i}{m_j} f_{jkl}^{(2)} \\
    f_{jkl}^{(4)} &= -\frac{m_k}{m_j} f_{jlk}^{(2)} \\
    f_{jkl}^{(5)} &= \frac{2i}{m_j} (f_{jkl}^{(2)} - 16 f_{jkl}^{(1)}) \\
    f_{jkl}^{(6)} &= \frac{i}{2m_j} (m_k^2 f_{jlk}^{(2)} + m_l^2 f_{jkl}^{(2)}) \\
    f_{jkl}^{(7)} &= \frac{16}{m_j m_k} f_{jkl}^{(1)}
\end{align*}
\]  

(3.9)

where we also have

\[ f_{jkl}^{(1)} = -f_{jkl}^{(1)}. \]  

(3.10)

If one computes the corresponding expression \( t \) (see the Introduction) by integrating out the Grassmann variables one gets, up to finite renormalizations, the following expressions:

\[
\begin{align*}
    \int d\theta^2 d\bar{\theta}^2 T^{(1)} &= 4i f_{jkl}^{(1)} : v^\mu_j v^\nu_k (\partial_\mu v_\nu - \partial_\nu v_\mu) : + \cdots \\
    \int d\theta^2 d\bar{\theta}^2 T^{(2)} &= \frac{i}{2} f_{jkl}^{(2)} : v^\mu_j [(u_k + u_k^\dagger) \partial_\mu (\bar{u}_l - \bar{u}_l^\dagger) + \partial_\mu (u_k - u_k^\dagger) (\bar{u}_l + \bar{u}_l^\dagger)] : + \cdots \\
    \int d\theta^2 d\bar{\theta}^2 T^{(3)} &= \frac{1}{4} f_{jkl}^{(3)} \left\{ (m_j^2 + m_k^2 + m_l^2) : (h_j + h_j^\dagger) (u_k - u_k^\dagger) (\bar{u}_l - \bar{u}_l^\dagger) + (m_j^2 + m_k^2 - m_l^2) : (h_j - h_j^\dagger) (u_k + u_k^\dagger) (\bar{u}_l + \bar{u}_l^\dagger) : - (m_j^2 + m_k^2 + m_l^2) : (h_j + h_j^\dagger) (u_k + u_k^\dagger) (\bar{u}_l - \bar{u}_l^\dagger) : - (m_j^2 + m_k^2 - m_l^2) : (h_j - h_j^\dagger) (u_k - u_k^\dagger) (\bar{u}_l + \bar{u}_l^\dagger) : \right\} + \cdots \\
    \int d\theta^2 d\bar{\theta}^2 T^{(4)} &= \frac{i}{2} f_{jkl}^{(4)} : [(h_j + h_j^\dagger) \partial_\mu (h_k + h_k^\dagger) + (h_j - h_j^\dagger) \partial_\mu (h_k - h_k^\dagger)] v^\mu_l : + \cdots \\
    \int d\theta^2 d\bar{\theta}^2 T^{(5)} &= -\frac{1}{4} f_{jkl}^{(5)} m_l^2 : (h_j + h_j^\dagger) v^\mu_k v^\nu_l : + \cdots 
\end{align*}
\]
\[
\int d\theta^2 d\bar{\theta}^2 T^{(6)} = \frac{1}{2} \ f^{(6)}_{jkl} : (h_j + h_j^\dagger) \ v_k^\mu \ v_l^\mu : + \cdots \\
\int d\theta^2 d\bar{\theta}^2 T^{(7)} = \frac{i}{2} \ f^{(7)}_{jkl} (m_j^2 - m_k^2 + m_l^2) \\
x : [(h_j + h_j^\dagger) \ \partial_\mu (h_k + h_k^\dagger) + (h_j - h_j^\dagger) \ \partial_\mu (h_k - h_k^\dagger)] \ v^\mu_l : + \cdots \quad (3.11)
\]

where by \( \cdots \) we mean terms containing the superpartners from the corresponding multiplets.

It seems encouraging that the first term is in agreement with the usual gauge coupling of the vector fields in Yang-Mills theory. After Grassmann integration one finds out that \( T^{(5)} \) is producing a non-renormalizable expression. So in the following we consider the case

\[
f^{(5)}_{jkl} = 0. \quad (3.12)
\]

In this case one can re-express everything in terms of

\[
f_{jkl} \equiv f_{jkl}^{(1)} \quad (3.13)
\]

which verifies

\[
f_{jkl} = - f_{kjl} \quad (3.14)
\]
as follows:

\[
\begin{align*}
f^{(2)}_{jkl} &= 16 \ f_{jk}^l, \\
f^{(3)}_{jkl} &= \frac{-16i}{m_j} f_{jkl}, \\
f^{(4)}_{jkl} &= - \frac{16m_k}{m_j} f_{jlk} \\
f^{(6)}_{jkl} &= \frac{8i}{m_j} (m_k^2 f_{jlk} + m_l^2 f_{jkl}) , \\
f^{(7)}_{jkl} &= \frac{16}{m_j m_k} f_{jkl} .
\end{align*} \quad (3.15)
\]

This is our massive supersymmetric quantum gauge model. Let us also remark that the doubling of the number of ghost fields cannot be avoided. The only way of halving this number would be to consider that the super-ghost multiplets are of Wess-Zumino type. However, this assumption together with (2.24) leads to a contradiction.

A strange solution corresponding to the case \( f^{(5)}_{jkl} \neq 0 \) was already discovered in [10].

4 Conclusions

Our main result is the existence of a physically interesting massive super-gauge theory satisfying the strong supersymmetric gauge condition (1.2). This condition is the true supersymmetric extension of first order perturbative gauge invariance [15], [8]. It expresses gauge invariance in terms of the chronological products using the gauge structure of the free asymptotic fields only, whereas other formulations of gauge invariance (classical or quantum) deal with interacting fields. The usefulness of (1.2) is evident from the fact that it determines the theory essentially unique under the renormalizable theories of the form (3.2).
In ordinary gauge invariance we know that \( f_{jkl} \) has to be totally antisymmetric. Assuming this we find the coupling

\[
T = f_{jkl} \left[ : V_j (\mathcal{D}^a V_k) (\mathcal{D}^2 \mathcal{D}_a V_k) : - H.c. \right.
\]

\[
+16 : V_j (U_k + U_k^\dagger) (\tilde{U}_l + \tilde{U}_l^\dagger) : - \frac{16i}{m_j} : (H_j + H_j^\dagger) (U_k - U_k^\dagger) (\tilde{U}_l + \tilde{U}_l^\dagger) : 
\]

\[
+ \frac{16m_k}{m_j} : (H_j + H_j^\dagger) (H_k - H_k^\dagger) V_l : + \frac{8i}{m_j} (m_l^2 - m_k^2) : (H_j + H_j^\dagger) V_k : 
\]

\[
- \frac{16}{m_j m_k} : (H_j + H_j^\dagger) (H_k - H_k^\dagger) V_l : \right]
\]

(4.1)

or, after Grassmann integration

\[
t = \int d\theta^2 d\bar{\theta}^2 T = 4i \ f_{jkl} \left\{ 2 : v^\mu_i v^\nu_k \partial_{\nu} v_{\mu} : 
\right.
\]

\[
- 2 : v^\mu_j [(u_k + u_k^\dagger) \partial_\mu (\bar{u}_l - \bar{u}_l^\dagger) + \partial_\mu (u_k - u_k^\dagger) (\bar{u}_l + \bar{u}_l^\dagger)] : 
\]

\[
- \frac{1}{m_j} \left[ (m_j^2 + m_k^2 + m_l^2) : (h_j + h_j^\dagger) (u_k - u_k^\dagger) (\bar{u}_l - \bar{u}_l^\dagger) : 
\right.
\]

\[
- (m_j^2 + m_k^2 - m_l^2) : (h_j - h_j^\dagger) (u_k + u_k^\dagger) (\bar{u}_l + \bar{u}_l^\dagger) : 
\]

\[
- (m_k^2 + m_l^2 - m_j^2) : \left( (h_j + h_j^\dagger) (u_k + u_k^\dagger) + (h_j - h_j^\dagger) (u_k - u_k^\dagger) \right) (\bar{u}_l - \bar{u}_l^\dagger) : 
\]

\[
+ \frac{1}{m_j m_k} \left[ (3m_k^2 - m_j^2 - m_l^2) : (h_j + h_j^\dagger) \partial_\mu (h_k + h_k^\dagger) 
\right.
\]

\[
+ (m_k^2 + m_l^2 - m_j^2) \partial_\mu (h_j - h_j^\dagger) (h_k - h_k^\dagger) v^\mu_l : 
\]

\[
+ \frac{1}{m_j} (m_k^2 - m_l^2) : (h_j + h_j^\dagger) v^\mu_k v_{\mu} : \right\} + \cdots 
\]

(4.2)

where by \( \cdots \) we mean terms containing the superpartners from the corresponding multiplets.

We note for instance, that the third term of the preceding expression is absent in the ordinary Yang-Mills coupling. On the other hand the theory is not yet completely specified because there are still some free parameters in the commutations rules of \( V \) and the ghost fields. Furthermore, in second order gauge invariance one expects the necessity of a Higgs superfield as in ordinary massive gauge theory. This will be investigated in a future work.

References

[1] A. Bilal, “Introduction to Supersymmetry”, Lecture notes “Gif 2000”, [hep-th/0101055]

[2] I. L. Buchbinder, S. M. Kuzenko, “Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace”, (revised edition), IOP, 1998

[3] F. Constantinescu, “Supersymmetric Positivity and Supersymmetric Hilbert Spaces”, Lett. Math. Phys. 62 (2002) 111-125
[4] P. Deligne et. al, “Quantum Fields and Strings: A Course for Mathematicians”, vol. 1 & vol. 2, AMS publ. 2000

[5] S. Ferrara, O. Piguet, “Perturbation Theory and Renormalization of Supersymmetric Yang-Mills Theories”, Nucl. Phys. B 93 (1975) 261-302

[6] J. M. Figueroa-o’Farrill, “BUSSTEPP Lectures on Supersymmetry”, hep-th/0109172

[7] D. Freed, “Five Lectures on Supersymmetry”, AMS publ. 1999

[8] D. R. Grigore “The Standard Model and its Generalisations in Epstein-Glaser Approach to Renormalisation Theory”, hep-th/9810078, Journ. Phys. A 33 (2000) 8443-8476

[9] S. J. Gates Jr., M. T. Grisaru, M. Roček, W. Siegel, “Superspace or One Thousand and One Lessons in Supersymmetry”, Cummings, 1983, hep-th/0108200 (web edition)

[10] D. R. Grigore, G. Scharf, “The Quantum Supersymmetric Vector Multiplet and Some Problems in Non-Abelian Supergauge Theory”, hep-th/0212026, Annalen der Physik (Leipzig) 12 (2003) 643-683

[11] M. T. Grisaru, M. Roček, W. Siegel, “Improved Methods for Supergraphs”, Nucl. Phys. B 159 (1979) 429-450

[12] W. Hollik, E. Kraus, M. Roth, C. Rupp, K. Sibold, D. Stöckinger, “Renormalization of the Minimal Supersymmetric Standard Model”, Nucl. Phys. B 639 (2002) 3-65

[13] J. T. Lopuszański, “An Introduction to Symmetry and Supersymmetry in Quantum Field Theory”, World Scientific, 1991

[14] J. D. Lykken, “Introduction to Supersymmetry”, FERMILAB-PUB-96/445-T, hep-th/9612114

[15] O. Piguet, “Supersymmetry, Supercurrent, and Scale Invariance”, Lecture notes, CBPF-NF-072/96, UGVA-DPT 1996/08-938, hep-th/9611003

[16] O. Piguet, “Introduction to Supersymmetric Gauge Theories”, Lecture notes, UFES-DF-OP97/1, hep-th/9710095

[17] A. Salam, “The Electroweak Force, Grand Unification and Superunification”, Physica Scripta 20 (1979) 216-226

[18] G. Scharf, “Quantum Gauge Theories: A True Ghost Story”, Wiley, 2001

[19] P. P. Srivastava, “Supersymmetry, Superfields and Supergravity: an Introduction”, IOP Publ., 1986

[20] M. Sohnius, “Introducing Supersymmetry”, Phys. Rep. 128 (1985) 39-204
[21] S. Weinberg, “The Theory of Quantum Fields, vol. 3, Supersymmetry”, Cambridge Univ. Press, 2000

[22] P. West, “Introduction to Supersymmetry and Supergravity”, (Extended Second Edition) World Scientific, 1990

[23] J. Wess, J. Bagger, “Supersymmetry and Supergravity”, (Second Edition), Princeton Univ. Press, NJ, 1992