Microreversibility, fluctuation relations, and response properties in 1D Kitaev Chain

Fan Zhang and Jiayin Gu
School of Physics, Peking University.

H. T. Quan
School of Physics, Peking University, Beijing, 100871, China
Collaborative Innovation Center of Quantum Matter, Beijing 100871, China and
Frontiers Science Center for Nano-optoelectronics, Peking University, Beijing, 100871, China
(Dated: July 26, 2022)

We analytically calculate the cumulant generating function of energy and particle transport in an open 1D Kitaev chain by utilizing the Keldysh technique. The joint distribution of particle and energy currents obeys different fluctuation relations in different regions of the parameter space as a result of $U(1)$ symmetry breaking and energy conservation. We discuss the thermoelectricity of the Kitaev chain as a three terminal system and derive an analytical expression of the maximum work power. The response theory up to the second order is explicitly checked, and the result is consistent with the relations derived from the fluctuation relation.

I. INTRODUCTION

Microreversibility, which is a fundamental symmetry of the physical laws, imposes remarkable constraints on the nonequilibrium dynamics of a system. The most famous example is the celebrated Onsager-Casimir reciprocal relation, which states that the matrix of linear kinetic coefficient is symmetric $[1–3]$. This relation greatly reduces the number of response coefficients in a transport process, thus finding wide applications in transport experiments. Another example is the fluctuation-dissipation relation (FDR), which relates the dissipation or response in a nonequilibrium process to the properties in equilibrium $[4]$. Recently, a new family of nonequilibrium relations, called fluctuation relations (FR) have been discovered $[5–15]$. The derivation of the fluctuation relations only relies on the microreversibility of the system, and does not depend on the microscopic details. These fluctuation relations generalize the above two relations from the linear-response regime to regimes arbitrarily far away from equilibrium. From these relations, one can not only easily reproduce the results in linear-response theory such as the Green-Kubo formula, but also obtain relations of higher-order response coefficients $[16–23]$. The most general form of a fluctuation relation about the entropy production in an open system can be written as

$$P(\Delta \Omega) = P(-\Delta \Omega) = e^{\Delta \Omega},$$

where $P(\Delta \Omega)$ is the probability distribution of entropy production $\Delta \Omega$. According to principles of thermodynamics, one can relate $\Delta \Omega$ to various physical observables, such as particle number $N$, exchanged heat $Q_h$, and applied work $W$. In an open system without driving, there is no work done on the system. The entropy production is associated with the exchange of particles and energy; the corresponding FR is termed as exchange FR $[20]$. In the derivation of exchange FR, two ingredients are used. One is the microreversibility of the equation of motion and the other is the particle and energy conservation. Whereas the former is well recognized, the latter is implicit and taken for granted since the conservation law is a result of $U(1)$ symmetry and time-translation symmetry. However, in condensed matter physics, the $U(1)$ symmetry can be explicitly broken in some systems described by low-energy effective Hamiltonian, such as the BCS Hamiltonian of the superconductor. The breaking of $U(1)$ symmetry implies that the particle number is not conserved in the transport process, and it can lead to new forms of exchange FR. In a previous work $[27]$, we show that the exchange FR of particle current in the 1D Kitaev chain in the steady state takes various forms for different parameters. It is due to the presence of a paring term such as $\Delta c_{j+1}^\dagger c_j^\dagger$ which explicitly breaks $U(1)$ symmetry. The competition between the paring potential, the hopping amplitude, and the chemical potential gives rise to different microscopic transport processes, namely, the normal transport (NT), the local Andreev reflection (LAR) and the crossed Andreev reflection (CAR). Each particle current component satisfies a steady-state FR. In this article, we go one step further and study the joint probability distribution of energy and particle transport. We use the Keldysh technique to analytically calculate the full counting statistics of the particle and energy currents. We will show that the joint distribution of particle and energy currents obeys different exchange FRs due to $U(1)$ symmetry breaking and energy conservation. We will study its response, linear and nonlinear, and calculate the response coefficients. These response coefficients are used to demonstrate a family of response relations derived from the FR. In the linear response regime, we
also discuss the thermoelectricity of the Kitaev chain as a three terminal system.

Our paper is structured as follows. We introduce the open 1D Kitaev chain model and analytically calculate its full counting statistics of energy and particle in Sec. II. We discuss the exchange FR in Sec. III. In Sec. IV, we study the response properties of the Kitaev chain. In Sec. V, we discuss our results and make a summary.

II. MODEL AND FULL COUNTING STATISTICS

We consider a Kitaev chain connected to two reservoirs. The set up is shown in Fig. 1. A nanowire is put above an s-wave superconductor (SC) and couples to two reservoirs. Due to the proximity effect, the Cooper pairs can leak into the nanowire, and turns the low-energy effective Hamiltonian of the nanowire into a 1D Kitaev chain [28–30]. The whole Hamiltonian is

\[ H = H_K + \sum_{\alpha = 1,2} H_{\alpha} + \hat{H}_1, \]

where the Hamiltonian of the Kitaev chain is

\[ H_K = -\mu \sum_{j=1}^{N} \left( \hat{c}_j^\dagger \hat{c}_j - \frac{1}{2} \right) \]  
+ \sum_{j=1}^{N-1} \left( -\hbar \hat{c}_j^\dagger \hat{c}_{j+1} + \Delta \hat{c}_j^\dagger \hat{c}_{j+1} + h.c. \right) \]

with \( \mu, \hbar, \Delta \) the chemical potential, the hopping amplitude and the superconducting gap. \( \hat{c}_j^\dagger \) and \( \hat{c}_j \) are the creation and annihilation operators of electrons on site \( j \); \( N \) is the site number of the Kitaev chain. The reservoirs are described by the free fermion Hamiltonian

\[ \hat{H}_\alpha = \sum_j \left( \hbar \omega_{\alpha j} - \mu_\alpha \right) \hat{c}_j^\dagger \hat{c}_j, \quad \alpha = 1,2. \]

Here, \( \omega_{\alpha j} \) denotes the energy of the \( j \)th state of reservoir \( \alpha \) whose chemical potential is \( \mu_\alpha \). We assume linear couplings between the Kitaev chain and the reservoirs

\[ \hat{H}_1 = \sum_j \lambda_{1j} \left( \hat{c}_j^\dagger \hat{c}_{j1} + \hat{c}_j^\dagger \hat{c}_{j1}^\dagger \right) + \sum_j \lambda_{2j} \left( \hat{c}_{2j}^\dagger \hat{c}_N + \hat{c}_{2j}^\dagger \hat{c}_{2j}^\dagger \right) \]

with \( \lambda_{\alpha j} \) the coupling strength. We adopt the two-point measurement scheme. We assume that the initial state is a product state, and every part is prepared in its thermal equilibrium state \( \hat{\rho}_0 = e^{-\beta_0 \hat{H}_K - \sum_{\alpha=1,2} \beta_\alpha \hat{H}_\alpha} / \text{Tr} \left[ e^{-\beta_0 \hat{H}_K - \sum_{\alpha=1,2} \beta_\alpha \hat{H}_\alpha} \right] \), where \( \beta_0 \) and \( \beta_\alpha \) are the initial temperatures of the Kitaev chain and reservoir \( \alpha \), respectively. We measure the particle number \( N_\alpha = \sum_j \hat{c}_j^\dagger \hat{c}_j \) and the energy \( \hat{H}_\alpha \) of reservoir \( \alpha \) simultaneously at the initial time \( t = 0 \) and a latter time \( t = \tau \). The particle (energy) exchanged between the reservoir and the chain during a time interval \([0, \tau]\) is defined to be the difference between the two outcomes which is denoted as follows

\[ \Delta X = (\Delta N_1, \Delta N_2, \Delta E_1, \Delta E_2) = (\Delta N, \Delta E). \]

We also define the operator \( \hat{X} = (\hat{N}_1, \hat{N}_2, \hat{H}_1, \hat{H}_2) \). The moment generating function (MGF) which is the Fourier transform of the probability distribution \( P(\Delta X) \) is defined as

\[ Z(\lambda) = \int d\Delta X \: P(\Delta X) e^{i \Delta N \xi \cdot \Delta E \eta} = \text{Tr} \left[ \hat{\rho}_0 U^\dagger (\tau,0) e^{i \hat{X} \lambda} U (\tau,0) e^{-i \hat{X} \lambda} \right] \]

with the counting fields \( \lambda = (\xi, \eta) = (\xi_1, \xi_2, \eta_1, \eta_2) \) and \( U(\tau,0) \) the unitary evolution operator of the total Hamiltonian.

We insert fermionic coherent states and write it in the form of contour functional integral; by utilizing the Keldysh technique [27–32], we obtain the MGF of the open Kitaev chain in the long time limit \( \tau \to \infty \).

\[ Z(\lambda) = \prod_{\omega} \sqrt{Z(\lambda; \omega)}, \]

where the MGF of every mode is composed of three components: the normal transport, the crossed Andreev reflection, and the local Andreev reflection

\[ Z(\lambda; \omega) = \frac{Z_{NT}(\xi_1 - \xi_2) + Z_{CAR}(\xi_1 + \xi_2) + Z_{LAR}(\xi_1, \xi_2)}{Z_{NT}(0) + Z_{CAR}(0) + Z_{LAR}(0)}. \]
The three components are given by
\[ Z_N (\xi_1, \xi_2) = \left \{ C_1 + T_1 \left[ n_{1e} \bar{n}_{2h} (e^{i(\xi_1 - \xi_2)} e^{i\omega \eta_1} - 1) + \bar{n}_{1e} n_{2h} (e^{-i(\xi_1 - \xi_2)} e^{-i\omega \eta_1} - 1) \right] \right \} \]
\[ \times \left \{ \bar{C}_1 + \bar{T}_1 \left[ n_{1h} \bar{n}_{2h} (e^{-i(\xi_1 - \xi_2)} e^{i\omega \eta_1} - 1) + n_{1h} n_{2h} (e^{i(\xi_1 - \xi_2)} e^{-i\omega \eta_1} - 1) \right] \right \}, \]
\[ Z_{CAR} (\xi_1, \xi_2) = \left \{ C_2 + T_2 \left[ n_{1e} \bar{n}_{2h} (e^{i(\xi_1 + \xi_2)} e^{i\omega \eta_1} - 1) + \bar{n}_{1e} n_{2h} (e^{-i(\xi_1 + \xi_2)} e^{-i\omega \eta_1} - 1) \right] \right \} \]
\[ \times \left \{ \bar{C}_2 + \bar{T}_2 \left[ n_{1h} \bar{n}_{2h} (e^{-i(\xi_1 + \xi_2)} e^{i\omega \eta_1} - 1) + n_{1h} n_{2h} (e^{i(\xi_1 + \xi_2)} e^{-i\omega \eta_1} - 1) \right] \right \}, \]
\[ Z_{LAR} (\xi_1, \xi_2) = \left \{ C_3 + T_3 \left[ n_{1e} \bar{n}_{1h} (e^{i2\xi_1} - 1) + \bar{n}_{1e} n_{1h} (e^{-i2\xi_1} - 1) \right] \right \} \]
\[ \times \left \{ C_4 + T_4 \left[ n_{2e} \bar{n}_{2h} (e^{i2\xi_2} - 1) + n_{2e} n_{2h} (e^{-i2\xi_2} - 1) \right] \right \} . \]

Here for simplicity, we eliminate the redundancy of the counting fields of energy by introducing a new counting field \( \eta_a = \eta_1 - \eta_2 \). The fermionic occupation numbers of electrons and holes in reservoirs are denoted by \( n_{ae}(\omega) = f_a(\omega - \mu_a) \) and \( n_{ah}(\omega) = f_a(\omega + \mu_a) \) respectively, where \( f_a(\omega) = 1/[e^{\beta \omega} + 1] \). \( n_{ae}(\omega) \equiv 1-n_{ah}(\omega) \) and \( n_{ah}(\omega) \equiv 1-n_{ae}(\omega) \). \( T_j(\omega) \) and \( C_j(\omega), j = 1, 2, 3, 4 \) are the transmission and reflection amplitudes of mode \( \omega \); \( T_j(\omega) = T_j(-\omega) \) and \( C_j(\omega) = C_j(-\omega) \). We emphasize that this form of MGF is valid for arbitrary number of sites and arbitrary parameters. Different numbers of sites correspond to different amplitudes \( C_j \) and \( T_j \) but do not affect the remaining expressions of the MGF. In the following, we will only use \( \eta_a \) as the counting field for energy flow, i.e., \( \eta = \eta_a \).

The reduction of the number of counting fields of energy is a consequence of energy conservation. Note that the information of the initial state of the Kitaev chain is lost in the long-time limit.

### III. FLUCTUATION RELATION

From the explicit form of MGF, we observe that the transport process is composed of independent bidirectional processes of mode \( \omega \). Every process consists of three subprocesses. The three subprocesses are the NT, the CAR, and the LAR. The NT corresponds to transferring one electron (hole) from the left reservoir to the right reservoir, while the CAR corresponds to a process in which an incoming electron from the left reservoir is turned into an outgoing hole in the right reservoir [33, 34]. As a result, one electron from each reservoir is injected into the SC to form a Cooper pair. The LAR corresponds to the process in which an incident electron from one reservoir is converted into a backscattered hole. The CAR and LAR break the particle conservation in two reservoirs, which is due to the presence of a non-zero pairing potential \( \Delta \). If we take \( \Delta = 0 \), \( T_2 \), \( T_3 \) and \( T_4 \) vanish and only \( T_1 \) is nonzero. In this case, the number of conservation law recovers to two and only two counting fields \( \bar{\xi}_a \equiv \xi_1 - \xi_2, \eta_a \) are needed to generate the cumulants of the currents. On the other hand, the particle conservation is also recovered if we take into account the third reservoir, the superconductor, which does not appear explicitly in the Hamiltonian of the open Kitaev chain.

The MGF satisfies a symmetry relation
\[ Z(\xi, \eta_a) = Z(-\xi + iA_N, -\eta_a + iA_E), \quad \text{(9)} \]
where \( A_N = (\beta_1 \mu_1, \beta_2 \mu_2) \) and \( A_E = \beta_2 - \beta_1 \) are the affinities. The symmetry of the MGF implies an exchange of the joint probability distribution [12, 13, 18]
\[ P(\Delta N_1, \Delta N_2, \Delta E_1) = e^{\Delta N_1 \mu_1 \beta_1 + \Delta N_2 \mu_2 \beta_2} e^{\Delta E_1 (\beta_2 - \beta_1)}. \quad \text{(10)} \]

Under certain conditions, one of the three current components dominates the transport process and Eq. (10) is reduced to a simpler FR. We consider three different cases in the following. The first case is when the pairing potential \( \Delta = 0 \), i.e., the Kitaev chain is a conventional conductor. The transmission amplitudes of CAR \( T_2 \) and LAR \( T_3, T_4 \) vanish. The gain of particles in one reservoir is equal to the loss of particles in the other reservoir, i.e., \( \Delta N_1 = -\Delta N_2 \). The FR reads
\[ \frac{P(\Delta N_1, \Delta E_1)}{P(-\Delta N_1, -\Delta E_1)} = e^{\Delta N_1 (\mu_1 \beta_1 - \mu_2 \beta_2)} e^{\Delta E_1 (\beta_2 - \beta_1)}. \quad \text{(11)} \]

which is the conventional FR of two terminal systems. If we introduce a nonzero pairing potential \( \Delta \neq 0 \) and turn off the hopping term \( h = 0 \), only CAR will occur [35]. In this case, \( \Delta N_1 = \Delta N_2 \), and the FR becomes
\[ \frac{P(\Delta N_1, \Delta E_1)}{P(-\Delta N_1, -\Delta E_1)} = e^{\Delta N_1 (\mu_1 \beta_1 + \mu_2 \beta_2)} e^{\Delta E_1 (\beta_2 - \beta_1)}. \quad \text{(12)} \]

Two points are worth emphasizing. The first one is that when we apply symmetric bias, i.e., \( \mu_1 \beta_1 = -\mu_2 \beta_2 \), the probability distribution \( P(\Delta N_1) \) is symmetric about \( \Delta N_1 = 0 \), and gives zero mean particle current but nonzero energy current. The second one is that when we apply equal bias, i.e., \( \mu_1 \beta_1 = \mu_2 \beta_2 \), the nonzero particle current signatures the presence of a nonzero pairing potential. The third case is the Majorana case \( \Delta = h, \mu = 0 \). Two Majorana zero modes will emerge and localize at the ends of the Kitaev chain. The NT and CAR are
The regime for TSC phase is reduced to about 20 sites. The boundary of a Kitaev chain in the thermodynamic limit is \( |\mu/h| < 2 \) for a 3-site chain and about \( |\mu/h| < 1.7 \) for 20-site system; both are denoted as dashed vertical lines. The phase boundary of a Kitaev chain in the thermodynamic limit is \( |\mu/h| < 2 \). The shrink of the TSC phase is a manifestation of the finite-size effect.

In the third case, there is no energy transport, since LAR effectively transports two electrons with opposite energy to the chain. The net exchange of particle is two, while the net exchange of energy is zero. It is worth mentioning that the expression of transmission coefficient of LAR \( T_3 \) in the third case is independent of the site number \( 36 \). It can be proven that the Kitaev chain in the third case is equivalent to a three-level system. It is also worth mentioning that the above discussion of the MZM case applies when the localized MZM has no overlap, i.e., the system is in the topological superconductor (TSC) phase. For a infinite-long chain, the TSC phase appears when \( |\mu/h| < 2 \) and \( \Delta \neq 0 \) \( 27 \). For a short chain, the TSC phase will shrink in the phase diagram due to the finite-size effect (see Fig. 2). In summary, all three subprocesses contribute to the particle current, but only NT and CAR contribute to the energy flow.

### IV. RESPONSE THEORY

In this section, we first review the response theory which is derived from the exchange FR \( 16, 18, 19 \), including the well-known results in linear response theory, such as the Onsager reciprocal relation and FDR. Then we obtain the exact expression of linear and nonlinear response coefficients of the Kitaev chain and discuss thermoelectricity in this model.

#### A. Response theory from FR

The cumulants can be generated from the cumulant generating function (CGF) which is defined as

\[
\mathcal{F}(\lambda, A) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \ln Z(\lambda, A),
\]

where we update the definition of counting fields \( \lambda = \{\lambda_1, \lambda_2, \lambda_3\} \equiv \{\xi_1, \xi_2, \eta_0\} \). The affinities are denoted as \( A = \{A_N, A_E\} \). The CGF inherits the symmetry in the MGF [Eq. (9)], namely,

\[
\mathcal{F}(\lambda, A) = \mathcal{F}(-\lambda + iA, A).
\]

All the cumulants of the energy and particle currents can be obtained by successive derivatives with respect to the counting fields, and then setting all these fields equal to zero. For example, the mean value (first cumulant), the diffusivities (second cumulant), and higher cumulants are given by

\[
J_j(A) \equiv \frac{\partial \mathcal{F}(\lambda, A)}{\partial (i\lambda_j)} \bigg|_{\lambda = 0},
\]

\[
D_{jk}(A) \equiv \frac{1}{2} \frac{\partial^2 \mathcal{F}(\lambda, A)}{\partial (i\lambda_j) \partial (i\lambda_k)} \bigg|_{\lambda = 0},
\]

\[
C_{jkl}(A) \equiv \frac{\partial^3 \mathcal{F}(\lambda, A)}{\partial (i\lambda_j) \partial (i\lambda_k) \partial (i\lambda_l)} \bigg|_{\lambda = 0}.
\]

As Refs. \( 16, 18, 19 \) point out, the third cumulants \( C_{ijk} \) characterizes the magnetic-field asymmetry of the fluctuations. When there is no term breaking time-reversal symmetry, that is, no magnetic field, \( C_{jkl} = 0 \).

At equilibrium \( (A = 0) \), the mean current vanishes. We notice that the mean current can be expanded in powers of the affinities close to equilibrium

\[
J_j = \sum_k L_{j,k} A_k + \frac{1}{2} \sum_{k,l} M_{j,k,l} A_k A_l + \ldots .
\]

This expansion implies a definition of the response coefficients

\[
L_{j,k} = \left. \frac{\partial^2 \mathcal{F}(\lambda, A)}{\partial (i\lambda_j) \partial A_k} \right|_{\lambda = 0},
\]

\[
M_{j,k,l} = \left. \frac{\partial^3 \mathcal{F}(\lambda, A)}{\partial (i\lambda_j) \partial A_k \partial A_l} \right|_{\lambda = 0},
\]

The response coefficients and the cumulants satisfy a family of universal relations, which can be derived from the exchange FR \( 16, 18, 19 \). The first-order response relations are nothing but the FDR

\[
L_{j,k} = D_{jk}(A = 0),
\]
and the Onsager reciprocal relation

$$L_{kJ} = L_{jk},$$

where the second relation is from the symmetry of $D_{jk} = D_{kj}$. Thus, we see that the two main cornerstones of linear response theory are encoded in the exchange FR.

As for nonlinear response at equilibrium, we have similar relations \[\tag{19}\]

$$M_{ijk} = \left( \frac{\partial^2 D_{ij}}{\partial A_k \partial A_l} + \frac{\partial^2 D_{ik}}{\partial A_j \partial A_l} + \frac{\partial^2 D_{il}}{\partial A_j \partial A_k} + \frac{\partial C_{ijkl}}{2} \right) \bigg|_{A=0},$$

$$N_{ijk} = \left( \frac{\partial D_{ij}}{\partial A_k} + \frac{\partial D_{ik}}{\partial A_j} + \frac{\partial D_{il}}{\partial A_k} \right) \bigg|_{A=0},$$

where $N_{ijk}$ is the third-order response coefficient

$$N_{ijk} = \frac{\partial^3 J_i(A)}{\partial A_j \partial A_k \partial A_l} \bigg|_{A=0}.$$

In the following, we will consider response properties up to the second order.

**B. Linear response in Kitaev chain**

From Eq. (19), we obtain the expression of the particle current $J_i^N$ from the left reservoir and $J_i^N$ from the right reservoir, as well as the energy current $J_i^E$ from the left reservoir

$$J_i^N = \int \frac{d\omega}{2\pi} \left[ \tilde{T}_1(p_{e\rightarrow c} - p_{c\rightarrow e}) + \tilde{T}_2(p_{e\rightarrow h} - p_{c\rightarrow h}) + \tilde{T}_3(n_{1e} \bar{n}_{1h} - \bar{n}_{1e} n_{1h}) \right],$$

$$J_i^N = \int \frac{d\omega}{2\pi} \left[ \tilde{T}_1(p_{c\rightarrow e} - p_{e\rightarrow c}) + \tilde{T}_2(p_{e\rightarrow h} - p_{c\rightarrow h}) + \tilde{T}_4(n_{2e} \bar{n}_{2h} - \bar{n}_{2e} n_{2h}) \right],$$

$$J_i^E = \int \frac{d\omega}{2\pi} \left[ \tilde{T}_1(p_{e\rightarrow c} - p_{e\rightarrow c}) + \tilde{T}_2(p_{c\rightarrow h} - p_{c\rightarrow h}) \right],$$

where $p_{e\rightarrow c} = n_{1e} \bar{n}_{2e}$, $p_{e\rightarrow h} = \bar{n}_{1e} n_{2h}$, $p_{c\rightarrow h} = n_{1e} \bar{n}_{2h}$, and $p_{c\rightarrow e} = \bar{n}_{1e} n_{2e}$. The transmission coefficients are given by $\tilde{T}_{1,2} = C_{1,2} \mathcal{T}_{1,2}/\mathcal{C} \mathcal{C}$ with $\mathcal{C} = C_1 \mathcal{C}_1 + C_2 \mathcal{C}_2 + C_3 \mathcal{C}_3$. The energy current of the right reservoir $J_i^E$ is equal to the opposite of $J_i^F$. Three independent currents are consistent with three affinities. As mentioned before, the energy current is carried by the particles participating in the NT and the CAR processes, while all the three transport processes contribute to the particle flow. From Eq. (19), we obtain the linear response coefficients $L_{\alpha,\beta}$ relevant to $J_1$ as (For simplicity, we label the currents $(J_1^N, J_2^N, J_3^F)$ as $(J_1, J_2, J_3)$)

$$L_{1,1} = \int \frac{d\omega}{2\pi} \left[ \tilde{T}_1(\omega) + \tilde{T}_2(\omega) + 2 \tilde{T}_3(\omega) \right] n_{1e} \bar{n}_{1e},$$

$$L_{1,2} = \int \frac{d\omega}{2\pi} \left[ -\tilde{T}_1(\omega)n_{2e} \bar{n}_{2e} + \tilde{T}_2(\omega) n_{2h} \bar{n}_{2h} \right],$$

$$L_{1,3} = \int \frac{d\omega}{2\pi} \left[ \tilde{T}_1(\omega) + \tilde{T}_2(\omega) + 2 \tilde{T}_3(\omega) \right] n_{1e} \bar{n}_{1e}.$$

The linear response coefficients $L_{\alpha,\beta}$ relevant to $J_2$ are

$$L_{2,1} = \int \frac{d\omega}{2\pi} \left[ -\tilde{T}_1(\omega) + \tilde{T}_2(\omega) \right] n_{1e} \bar{n}_{1e},$$

$$L_{2,2} = \int \frac{d\omega}{2\pi} \left[ \tilde{T}_1(\omega) + \tilde{T}_2(\omega) + 2 \tilde{T}_4(\omega) \right] n_{2e} \bar{n}_{2e},$$

$$L_{2,3} = \int \frac{d\omega}{2\pi} \left[ -\tilde{T}_1(\omega) + \tilde{T}_2(\omega) \right] n_{1e} \bar{n}_{1e}.$$

From Eq. (28) and Eq. (32), it seems that Onsager reciprocal relation is apparently violated due to the presence of LAR. But actually, we have $\tilde{T}_3(\omega) = \tilde{T}_3(-\omega)$ as a consequence of the particle-hole symmetry, so the term containing $\tilde{T}_3$ in Eq. (28) is an odd function of $\omega$ at zero affinity and vanishes after the integration. Hence, the Onsager reciprocal relation remains valid in our model.

In experiment, the more familiar linear response coefficients are the electrical conductance $G^c$, thermal conductance $K$, and Seebeck coefficient $S$. In previous studies on the thermoelectricity of 1D Kitaev chain or two Majorana zero modes (MZMs), the system has been treated as a two-terminal system \[\tag{38}\] \[\tag{39}\] Landauer-Büttiker formula is invoked to obtain the currents, such as Eqs. (34) in Ref. (38) (in our notation)

$$J_i^N = \int \frac{d\omega}{2\pi} \tilde{T}(\omega)[n_{1e}(\omega) - n_{2e}(\omega)],$$

$$J_i^E = \int \frac{d\omega}{2\pi} \omega \tilde{T}(\omega)[n_{1e}(\omega) - n_{2e}(\omega)].$$

The above expression of particle current neglects the fact that there are in total three current components. In Appendix A we show that Eq. (35) differs from Eq. (34) obtained in the framework of FCS, and thus gives half the quantized electrical conductance $e^2/h$ (the correct one is $2e^2/h$). The expression of energy current Eq. (36) is also incorrect in the transmission term $\tilde{T}(\omega)$, due to the fact that $\tilde{T}(\omega)$ incorrectly includes contributions from LAR which is not involved in the energy transport. The correct expression of currents and a detailed discussion of
two MZMs coupled to two reservoirs as a three-terminal system are given in Appendix A.

In fact, the presence of three independent currents (and three affinities) in this model implies that it is a genuine three-terminal system, where the third terminal is the grounded SC. It can be compared to the phonon-thermoelectric systems, e.g., a double quantum dots (QDs) in contact with two metals and a phonon substrate [10, 11]. The phonon bath absorbs or releases heat but does not exchange particles with the two QDs, while the grounded SC in our system exchanges Cooper pairs with the nanowire but does not exchange energy.

In a three-terminal system, we write the relation between the currents and affinities in the linear response regime as

\[
\begin{pmatrix}
  J^N_{1}
  J^Q_{1}
  J^N_{2}
  J^Q_{2}
\end{pmatrix} =
\begin{pmatrix}
  L_{1,1} & L_{1,2} & L_{1,3} \\
  L_{2,1} & L_{2,2} & L_{2,3} \\
  L_{3,1} & L_{3,2} & L_{3,3}
\end{pmatrix}
\begin{pmatrix}
  \delta \mu_1 / T \\
  \delta \mu_2 / T \\
  \delta T / T^2
\end{pmatrix},
\]

where we take the temperature of the right reservoir as the reference temperature \( T = T_2 \) and \( \delta T = T_1 - T_2 \). The chemical potential of the grounded SC is \( \mu_{SC} = 0 \) and \( \delta \mu_{SC} = \mu_a - \mu_{SC} = \mu_a \). The heat current from the left reservoir is defined as \( J^Q_i = J^F_i - \mu_i J^N_i \) as a result of thermal laws [14]. Following Ref. [12], the electrical conductance is obtained under the isothermal condition, i.e.,

\[
G_{ij} = \left( \frac{\delta J^N_j}{\delta \mu_j} \right)_{\delta T = 0, \delta \mu_k = 0} = \frac{e^2}{T} \frac{L_{1,1}}{L_{2,1} L_{2,2}}.
\]

(37)

Here, \( G_{11} \) and \( G_{22} \) are the local electrical conductances and \( G_{12} = G_{21} \) is the non-local electrical conductance. The Seebeck coefficients are obtained as the ratio of voltage difference and temperature difference when there are no electrical currents, i.e.,

\[
S_{j3} = -\left( \frac{\delta \mu_j}{e \delta T} \right)_{J^N_k = 0}.
\]

We find

\[
S_{13} = \frac{1}{e T} \frac{L_{1,3} L_{2,2} - L_{1,2} L_{2,3}}{L_{1,1} L_{2,2} - L_{1,2} L_{2,1}},
\]

\[
S_{23} = \frac{1}{e T} \frac{L_{1,1} L_{2,3} - L_{1,3} L_{2,1}}{L_{1,1} L_{2,2} - L_{1,2} L_{2,1}}.
\]

(38)

The Peltier coefficient is related to Seebeck coefficient by \( \Pi_{3j} = T S_{j3} \). Thermal conductance is defined as the ratio of heat current and temperature difference when the particle current is zero, i.e.

\[
K = \left( \frac{J^Q_j}{\delta T} \right)_{J^N_k = 0} = \frac{1}{T^2} (L_{3,3} - L_{3,1} S_{11} - L_{3,2} S_{21}).
\]

(39)

From Eqs. (37, 39), we recognize that the electrical conductance between the left reservoir and the chain is still given by \( G^{-1}_{11} = L_{1,1} / T \), which is identical to the two-terminal case. Nevertheless, the expressions for Seebeck coefficients \( S_{j3} \) and thermal conductance \( K \) are different from the two-terminal case [15].

We fix the hopping amplitude \( h = 1 \), and plot electrical conductance \( G \) for different \( \mu \) at different temperatures for a three-site model in Fig. 3. We adopt the asymmetric effective coupling strength \( \Gamma_1 = 0.5, \Gamma_2 = 0.1 \). We find the local electrical conductance [Fig. 3(a, b)] is nearly unity for low \( T \) in the region \( |\mu| < h \). It is consistent with the spectrum of a three-site model in Fig. 2 which shows that the nanowire hosts Majorana modes at two ends of the wire when \( |\mu| < h \). In contrast, the nonlocal electrical conductance [Fig. 3(b)] is small at low temperature and is nonzero near the gap-opening region. The low-temperature feature of the electrical conductances of the left and right reservoirs is quite similar even though the coupling strength are asymmetric. As the temperature increases, high-energy modes begin to get involved, and the asymmetry in the coupling strength affect the conductance dramatically. For example, at \( T = 0.2 \), the electrical conductance of the left reservoir \( G_{11} \) is nearly three times of the right one \( G_{22} \) [see Fig. 3(d)].

In Fig. 4, we show the thermal conductance \( K / T \) as a function of \( T \) and \( \mu / h \). We see that the \( K / T \) is nearly zero in the region \( |\mu| < h \) and increases dramatically near the gap opening region \( |\mu| / h \sim 1 \), which is different from the behavior of local electrical conductance. The zero value of thermal conductance is due to the fact that LAR does not transfer energy. It can also be seen in Fig. 3(b). When \( \mu = 0 \), the system is in the exactly solvable case. Two Majorana modes are perfectly localized at two ends of the chain. NT and CAR vanish and only LAR is present. There is no energy transport, so the heat conductance is always zero regardless of the temperature. Another interesting feature is that the peak at low temperature \( T = 0.02 \) is close to half of a thermal conductance quantum \( (1/2) \pi^2 k_B^2 / 3h \). Similar feature is also seen in the two MZMs case (see Appendix A). In the two MZMs case, we prove that the half quantization is exact when \( \epsilon_M / \Gamma = 2 \) for large gap \( \epsilon_M \) and large coupling \( \Gamma \) (see Appendix A). Here in the Kitaev chain, it is still an open question if the half quantization is exact (in the thermodynamic limit) or rather accidental, and if it is a feature of Majorana physics or anything else.

As a thermoelectric device, we can discuss its power and efficiency. According to the first law of thermodynamics, the work power \( \dot{W} \) is defined as (positive work means that the system outputs power)

\[
\dot{W} + \sum_{\alpha=1,2} J^Q_{\alpha} = 0
\]

\[
\Rightarrow \dot{W} = \sum_{\alpha=1,2} J^Q_{\alpha} = - \sum_{\alpha=1,2} \mu_{\alpha} J^N_{\alpha},
\]

(40)

where \( J^Q_2 = J^F_2 - \mu_2 J^N_2 \) and we use the energy conser-
We assume that three special cases, in which only one current component is present. We assume that \( \mu_1 > \mu_2 \) and \( T_1 < T_2 \) without loss of generality. In the first (NT) case, the two particle currents are opposite to each other. \( J^N_1 = -J^N_2 \). The work power is \( \dot{W} = -(\mu_1 - \mu_2)J^N_1 = -\delta \mu J^N_1 \) with \( \delta \mu = \mu_1 - \mu_2 \).

In order to generate a positive work, the signs of \( J^N_1 \) and \( \delta \mu_1 \) must be opposite to each other. From the FR \[11\], a negative \( J^N_1 \) requires \( \beta_1 \mu_1 < \beta_2 \mu_2 \) which implies that \( \mu_2 > 0 \) and \( \beta_2 > \beta_1 \mu_1 / \mu_2 \). In the second (CAR) case, \( J^N_1 = J^N_2 \). The work power is \( \dot{W} = -(\mu_1 + \mu_2)J^N_1 \). From the FR \[12\], a negative \( J^N_1 \) requires \( \beta_1 \mu_1 + \beta_2 \mu_2 < 0 \), which implies \( \mu_2 < 0 \) and \( \beta_2 > \beta_1 \mu_1 / |\mu_2| \) if we let \( \mu_2 > 0 \). In the third (LAR) case, \( J^E_1 = J^E_2 = 0 \). The work is \( \dot{W} = -\mu_1 J^E_1 - \mu_2 J^E_2 \). From FR \[13\], \( \beta_\alpha \mu_\alpha \) has the same sign as current \( J^E_\alpha \). Hence, the work power is always negative and the Kitaev chain can’t serve as a useful heat engine. In Fig. 5 we fix \( \beta_2 = 1 \), \( \mu_1 = 2 \), and \( \mu_2 = \pm 1 \) [plus for Fig. 5(a); minus for Fig. 5(b)], then vary \( \beta_2 / \beta_1 = T_1 / T_2 \) and \( \mu \). We see that the efficiency \( \eta \) is highly asymmetric about \( \mu \) in the NT case [Fig. 5(a)], while it is symmetric about \( \mu \) in the CAR case. We further note that the maximum of \( \eta \) locates at \( \mu = 0 \) in the CAR case.

In the linear response regime, the power can be expressed as

\[
\dot{W} = -\sum_\alpha \mu_\alpha J^E_\alpha = -\sum_{i,j=1,2} \mu_i \frac{L_{i,j}}{T} \mu_j - \mu_i L_{i,3} \frac{\delta T}{T^2}.
\]

Optimizing \( \dot{W} \) at a fixed temperature gradient, we find the maximum work power as

\[
W_{\text{max}} = \frac{1}{4} \delta T^2 (G_{11}S^2_{11} + G_{22}S^2_{21} + 2G_{12}S_{11}S_{21}),
\]

with the condition

\[
(\mu_1, \mu_2) = eT (S_{11}, S_{21}).
\]

Eq. (41) can be seen as a generalization of the maximum power \( W_{\text{max}}' = G S^2 2T^2 / 4 \) in two terminal case [44].
Figure 5. Efficiency of a four-site Kitaev chain as a heat engine or refrigerator. The left reservoir is always hotter than the right reservoir. The system operates as a heat engine when $J_1^R > 0$ and $W > 0$ and as a refrigerator when $J_2^R > 0$ and $W < 0$. (a) Only NT is present ($h = 1$, $\Delta = 0$, $\mu_2 = 1$). (b) Only CAR is present ($h = 0$, $\Delta = 1$, $\mu_2 = -1$). The efficiency is symmetric about $\mu = 0$ in (b). The other parameters are chosen to be $\Gamma_1 = \Gamma_2 = 0$, $T_2 = 1$, $\mu_1 = 2$. All efficiencies are measured in the corresponding Carnot efficiencies, i.e., $\eta_{\text{C,HE}} = 1 - T_2/T_1$ and $\eta_{\text{C,Ref}} = 1/(T_1/T_2 - 1)$.

Now we discuss the diffusivity. Here we only consider the diffusivity of energy and particle transport of the left reservoir. The diffusivity is symmetric to its index and has six independent components. Near equilibrium, the diffusivities are

$$D_{jk}(0) = \int \frac{d\omega}{2\pi} f(\omega)[1 - f(\omega)] \left( \begin{array}{ccc} \tilde{T}_1 + \tilde{T}_2 + 2\tilde{T}_3 & -\tilde{T}_1 + \tilde{T}_2 & \omega(\tilde{T}_1 + \tilde{T}_2) \\ -\tilde{T}_1 + \tilde{T}_2 & \tilde{T}_1 + \tilde{T}_2 + 2\tilde{T}_4 & \omega(-\tilde{T}_1 + \tilde{T}_2) \\ \omega(\tilde{T}_1 + \tilde{T}_2) & \omega(-\tilde{T}_1 + \tilde{T}_2) & \omega^2(\tilde{T}_1 + \tilde{T}_2) \end{array} \right).$$

They are equal to the linear response coefficients $L_{jk}$ [Eq. (26-34)]. Thus, we explicitly verify the fluctuation-dissipation relation in our model. In relevance to the experiment, the diffusivity is related to the zero-frequency noise power by

$$D(A) = \frac{1}{2} S(0, V),$$

where the zero-frequency noise power is defined as the Fourier transform of the symmetric current correlation

$$S_{ij}(\omega, V) = \int dt e^{i\omega t} \left\langle (\hat{J}_i(t) - J_i)(\hat{J}_j(0) - J_j) + (\hat{J}_j(0) - J_j)(\hat{J}_i(t) - J_i) \right\rangle.$$

Here $\hat{J}_j = \partial_t \hat{X}_j$ is the current operator. A quantized electrical conductance $G = 2e^2/h = 1/\pi$ (in natural unit) implies a quantized $S(0, V)/T = 4/h = 2/\pi$ by the fluctuation-dissipation relation.

C. Nonlinear response in Kitaev chain

In this section, we go beyond the linear response regime, and check the nonlinear response relation Eq. (22) explicitly.
The second-order response coefficients \( M_{i,jk} \) \((i,j,k = 1, 2, 3)\) at zero affinity are given by

\[
M_{1,ij} = \int \frac{d\omega}{2\pi} p_0(1 - p_0)(1 - 2p_0) \begin{pmatrix}
\tilde{T}_1 + \tilde{T}_2 & 0 & \omega(\tilde{T}_1 + \tilde{T}_3 + 2\tilde{T}_3) \\
0 & \tilde{T}_1 - \tilde{T}_2 & 0 \\
\omega(\tilde{T}_1 + \tilde{T}_2 + 2\tilde{T}_3) & 0 & \omega^2(\tilde{T}_1 + \tilde{T}_2)
\end{pmatrix},
\]

\[
M_{2,ij} = \int \frac{d\omega}{2\pi} p_0(1 - p_0)(1 - 2p_0) \begin{pmatrix}
-\tilde{T}_1 + \tilde{T}_2 & 0 & 0 \\
-\tilde{T}_1 - \tilde{T}_2 & \tilde{T}_1 - \tilde{T}_2 & \omega(-\tilde{T}_1 + \tilde{T}_2) \\
0 & \omega(-\tilde{T}_1 + \tilde{T}_2) & \omega^2(-\tilde{T}_1 + \tilde{T}_2)
\end{pmatrix},
\]

\[
M_{3,ij} = \int \frac{d\omega}{2\pi} p_0(1 - p_0)(1 - 2p_0) \begin{pmatrix}
\omega(\tilde{T}_1 + \tilde{T}_2) & 0 & \omega^2(\tilde{T}_1 + \tilde{T}_2) \\
0 & -\omega(\tilde{T}_1 + \tilde{T}_2) & 0 \\
\omega^2(\tilde{T}_1 + \tilde{T}_2) & 0 & \omega^3(\tilde{T}_1 + \tilde{T}_2)
\end{pmatrix}.
\]

Accordingly, the derivative of diffusivities \( D_{ij} \) are given by

\[
\frac{\partial D_{1i}}{\partial A_j} = \frac{1}{2} \int \frac{d\omega}{2\pi} p_0(1 - p_0)(1 - 2p_0) \begin{pmatrix}
\tilde{T}_1 + \tilde{T}_2 & \tilde{T}_1 - \tilde{T}_2 & \omega(\tilde{T}_1 + \tilde{T}_3 + 4\tilde{T}_3) \\
-\tilde{T}_1 + \tilde{T}_2 & -\tilde{T}_1 - \tilde{T}_2 & \omega(-\tilde{T}_1 + \tilde{T}_2) \\
\omega(\tilde{T}_1 + \tilde{T}_2) & \omega(\tilde{T}_1 - \tilde{T}_2) & \omega^2(\tilde{T}_1 + \tilde{T}_2)
\end{pmatrix},
\]

\[
\frac{\partial D_{2i}}{\partial A_j} = \frac{1}{2} \int \frac{d\omega}{2\pi} p_0(1 - p_0)(1 - 2p_0) \begin{pmatrix}
-\tilde{T}_1 + \tilde{T}_2 & -\tilde{T}_1 - \tilde{T}_2 & \omega(-\tilde{T}_1 + \tilde{T}_2) \\
\tilde{T}_1 + \tilde{T}_2 & \tilde{T}_1 - \tilde{T}_2 & \omega(\tilde{T}_1 + \tilde{T}_2 + 4\tilde{T}_4) \\
\omega(-\tilde{T}_1 + \tilde{T}_2) & -\omega(\tilde{T}_1 + \tilde{T}_2) & -\omega^2(\tilde{T}_1 - \tilde{T}_2)
\end{pmatrix},
\]

\[
\frac{\partial D_{3i}}{\partial A_j} = \frac{1}{2} \int \frac{d\omega}{2\pi} p_0(1 - p_0)(1 - 2p_0) \begin{pmatrix}
\omega(\tilde{T}_1 + \tilde{T}_2) & \omega(\tilde{T}_1 - \tilde{T}_2) & \omega^2(\tilde{T}_1 + \tilde{T}_2) \\
-\omega(-\tilde{T}_1 + \tilde{T}_2) & \omega(\tilde{T}_1 + \tilde{T}_2) & \omega^2(\tilde{T}_1 - \tilde{T}_2) \\
\omega^2(\tilde{T}_1 + \tilde{T}_2) & \omega^2(\tilde{T}_1 - \tilde{T}_2) & \omega^3(\tilde{T}_1 + \tilde{T}_2)
\end{pmatrix}.
\]

It is easy to see that the nonlinear response relation Eq. (22) is satisfied, namely

\[
M_{i,jk} = \frac{\partial D_{ij}}{\partial A_k} + \frac{\partial D_{ik}}{\partial A_j}.
\]

Higher-order response relations can be checked similarly.

V. SUMMARY

In this article, we analyze transport in 1D open Kitaev chain. We obtain a general form of MGF of energy and particle transport at finite temperature. The explicit expression of MGF allows us to extract the fluctuation relations in a straightforward manner. The energy current is carried by the particles involved in the NT process and the CAR process, while the particle current is also carried by the LAR process in addition to the above two processes. We find that the joint distribution of particle and energy currents obeys different fluctuation relations in different regions of the parameter space as a result of \(U(1)\) symmetry breaking and energy conservation. Moreover, we study the response properties of the Kitaev chain. Explicitly, we calculate the response coefficients, and find that they are consistent with the relations derived from the fluctuation relation. In addition, in the linear response regime, we treat the open Kitaev chain as a three-terminal system instead of two-terminal system and discuss its thermoelectrical properties. The electrical conductance is quantized when the Kitaev chain hosts two Majorana modes at two ends as expected. The thermal conductance, however, exhibits a peak (up to half thermal conductance quantum) around the region of the gap opening. The work power and the operation of thermoelectric device (based on the Kitaev chain) is also discussed. We find that a Kitaev chain in the topological superconductor phase always consumes energy, but it can operate as a heat engine or refrigerator otherwise.

ACKNOWLEDGMENTS

We acknowledge support from the National Science Foundation of China under grants 11775001, 11825501, and 12147162.

Appendix A: Two-Terminal Majorana Junction

In this appendix, we study the transport of two Majorana modes localized at two ends of a nanowire. This model has been extensively studied in the literature, since it is simple enough but still captures the main features of the Majorana physics. The Hamiltonian of the whole sys-
The energy currents from the left and the right reservoirs are
\[ J_N^1 = \int \frac{d\omega}{2\pi} 4\Gamma^2 \left[ (n_{1e} n_{1h} - n_{1h} n_{1e}) (4\Gamma^2 + \epsilon_M^2 + \omega^2) + \frac{d\omega}{2\pi} \right] T_N(\omega) (n_{1e} - n_{1h}), \]
\[ J_N^2 = \int \frac{d\omega}{2\pi} 4\Gamma^2 \left[ (n_{2e} n_{2h} - n_{2h} n_{2e}) (4\Gamma^2 + \epsilon_M^2 + \omega^2) + \frac{d\omega}{2\pi} \right] T_N(\omega) (n_{2e} - n_{2h}), \]
where the transmission coefficient is
\[ T_N(\omega) = \frac{4\Gamma^2}{(4\Gamma^2 + \omega^2)^2 + (8\Gamma^2 - 2\omega^2) \epsilon_M^2 + \epsilon_M^4}. \]

The energy currents from the left and the right reservoirs are
\[ J_E^1 = \int \frac{d\omega}{2\pi} \omega \left[ \frac{4\Gamma^2 \epsilon_M^2 (n_{1e} + n_{1h} - n_{2e} - n_{2h})}{(4\Gamma^2 + \omega^2)^2 + (8\Gamma^2 - 2\omega^2) \epsilon_M^2 + \epsilon_M^4} + \frac{d\omega}{2\pi} \right] T_E(\omega) (n_{1e} - n_{2e}), \]
\[ J_E^2 = \int \frac{d\omega}{2\pi} \omega \left[ \frac{4\Gamma^2 \epsilon_M^2 (n_{1e} + n_{1h} - n_{2e} + n_{2h})}{(4\Gamma^2 + \omega^2)^2 + (8\Gamma^2 - 2\omega^2) \epsilon_M^2 + \epsilon_M^4} + \frac{d\omega}{2\pi} \right] T_E(\omega) (n_{2e} - n_{1e}), \]
with
\[ T_E(\omega) = \frac{4\Gamma^2 \epsilon_M^2}{(4\Gamma^2 + \omega^2)^2 + (8\Gamma^2 - 2\omega^2) \epsilon_M^2 + \epsilon_M^4}. \]

The net effect of the NT and the CAR in the particle current \( J_N^N \) is to convert an electron in the left reservoir to an hole in the same reservoir. It can be seen by considering current \( j(\omega) \) through a single channel \( \omega \). From the MGF Eqs. (A1, A2), the current components for a single channel \( \omega \) from NT and CAR are
\[ j_{NT}(\omega) = T_1(n_{1e} - n_{2e}) - T_1(n_{1h} - n_{2h}), \quad j_{CAR}(\omega) = T_2(n_{1e} - n_{2h}) - T_2(n_{1h} - n_{2e}). \]
Since $T_1 = T_2 = \bar{T}_1 = \bar{T}_2$, we have $j_{NT} + j_{CAR} = 2\bar{T}_1(n_{1e} - n_{1h})$ which indicates the whole process is equivalent to a LAR. Again, we see that $J_1^N \neq J_2^N$ generally and $J_F^L = -J_F^R$. Previously, some studies, e.g., Refs. [38, 39] treat the system as a two-terminal system and use the Landauer-Büttiker formula, which is incorrect (in fact $G$ will be half of the correct value) according to the full-counting statistics.

1. Linear response regime

In the framework of three-terminal system, the linear response matrix for two MZMs reads

$$ L = \begin{pmatrix} L_{1,1} & 0 & 0 \\ 0 & L_{2,2} & 0 \\ 0 & 0 & L_{3,3} \end{pmatrix} $$

with

$$ L_{1,1} = L_{2,2} = \int \frac{d\omega}{2\pi} \frac{1}{2 \cosh^2 \frac{\beta \omega}{2}} T_N(\omega) \frac{1}{\omega}, $$

$$ L_{3,3} = \int \frac{d\omega}{2\pi} \frac{1}{\omega^2} 2\omega T_E(\omega) \frac{1}{4 \cosh^2 \frac{\beta \omega}{2}}. $$

(A6)

The electrical conductance is

$$ G = \frac{e^2}{T} \begin{pmatrix} L_{1,1} & 0 \\ 0 & L_{2,2} \end{pmatrix}, $$

which shows that there is no non-local conductance. The Seebeck coefficients are all zero. Thermal conductance is given by

$$ K = \frac{1}{T^2} L_{3,3}. $$

In Fig. 6 we show $K(\epsilon_M, T)$ and $G(\epsilon_M, T)$. The behavior of $G$ is consistent with previous studies: it is quantized at $2e^2/h$ at zero gap $\epsilon_M = 0$ at low temperature. The thermal conductance differs substantially. It vanishes at zero gap regardless of the temperature, and increases to the maximum at finite gap. Interestingly, the maximum of $K/T$ is about half thermal conductance quantum $(1/2)\pi^2 k_B^2/3h$. As the temperature increases, the quantization is smeared out gradually. In the following, we demonstrate that the quantization of $K/T$ is in fact exact. We measure the energy in the unit of $T$, i.e., we scale $\omega \rightarrow \beta \omega$, $\Gamma \rightarrow \beta \Gamma$, $\epsilon_M \rightarrow \beta \epsilon_M$. Then $K$ can be written as

$$ K = T \int \frac{d\omega}{2\pi} \frac{\omega^2 T_E(\omega)}{2 \cosh^2 \frac{\beta \omega}{2}} \frac{1}{4 \cosh^2 \frac{\beta \omega}{2}}. $$

The integral reaches its maximum in the limit $\Gamma \rightarrow \infty$ and $\epsilon_M \rightarrow \infty$. In this limit, we can approximate the integrand $T_E(\omega) \approx T_E(0)$ and carry out the integral

$$ K/T = T \int \frac{d\omega}{2\pi} \frac{\omega^2}{2 \cosh^2 \frac{\beta \omega}{2}} \frac{1}{2 \cosh^2 \frac{\beta \omega}{2}} = \frac{\pi}{3} T_E(0) $$

$$ = \frac{\pi}{3} \frac{4 \Gamma^2 e_M^2}{(4 \Gamma^2 + e_M^2)^2} \leq \frac{\pi}{12} = \frac{e^2 k_B^2}{3h}, $$

where the equality is obtained at $\epsilon_M/T = 2$ and we resort to SI unit in the last equality. We show $K/T$ as a function of $\Gamma/T$ and $\epsilon_M/T$ in Fig. 7.

According to Eq. (10), the work power is

$$ W = J_1^Q + J_2^Q = -\mu_1 J_1^N - \mu_2 J_2^N = -G(\mu_1^2 + \mu_2^2), $$

which is always negative. The heat current is

$$ J_1^Q = -L_{3,3} \delta T, \quad J_2^Q = \frac{L_{3,3}}{T^2} \delta T. $$

Assume $\delta T > 0$, then $J_1^Q > 0$ and $J_2^Q < 0$. It indicates that the two MZMs as a thermoelectric device, always consumes energy and cannot operate as a heat engine.
Figure 7. (a) The thermal conductance \( K/T \) as a function of \( \Gamma/T \) and \( \epsilon_M/T \). The solid line is \( 2\Gamma = \epsilon_M \). (b) The thermal conductance \( K/T \) as a function of \( \epsilon_M/\Gamma \) at \( T = 0.02 \). The maximum of \( K/T \) is half of a thermal conductance quantum \( \pi^2k_B^2/6\hbar \), which is obtained in the limit \( \epsilon_M \to \infty \), \( \Gamma \to \infty \) and \( \epsilon_M/\Gamma = 2 \). The points deviating from the red line correspond to small \( \epsilon_M \) and small \( \Gamma \).

2. Nonlinear transport

The explicit expression of the currents has a consequence on the response coefficients. The occupation number

\[
n_{1e}(\omega) - n_{1h}(\omega) = \frac{\sinh A_1}{\cosh [(\beta_2 - A_3)\omega] + \cosh A_1}
\]

is an odd function of \( A_1 \), thus all response coefficients corresponding to even power term of \( A_1 \) vanishes. Eq. (A7) is also an even function of \( \omega \). Differentiating with respect to \( A_3 \) won’t change the parity of Eq. (A7). So the response coefficients only correspond to odd powers of \( A_1 \) are nonzero. Similar consideration applies to \( J_2^N \). Since \( J_2^N \) does not depend on \( A_3 \), the response coefficients of \( J_2^N \) are diagonal. From the expression of currents Eqs. (A4,A5), we find the second-order response coefficients

\[
M_{1,13} = M_{1,31} = \int \frac{d\omega}{2\pi} 2\omega T_N p_0 (1 - p_0)(1 - 2p_0),
\]

\[
M_{3,33} = \int \frac{d\omega}{2\pi} 2\omega^3 T_N p_0 (1 - p_0)(1 - 2p_0),
\]

\[
M_{3,22} = \int \frac{d\omega}{2\pi} (-2\omega) T_N p_0 (1 - p_0)(1 - 2p_0),
\]

\[
M_{3,11} = \int \frac{d\omega}{2\pi} 2\omega T_E p_0 (1 - p_0)(1 - 2p_0),
\]

where \( p_0 = 1/(e^{\beta_2\omega} + 1) \). All other second-order coefficients vanish.

The diffusivities are (due to the symmetry of the index, only six of them are independent)

\[
D_{11}(A) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ T_1(-n_{1e}\bar{n}_{2e} + \bar{n}_{1e}n_{2e} + n_{1e}\bar{n}_{2h} + \bar{n}_{1e}n_{2h}) + 2C_4T_3(n_{1e}\bar{n}_{1h} + \bar{n}_{1e}n_{1h}) - (j_1^N)^2 \right],
\]

\[
D_{22}(A) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ T_1(n_{1e}\bar{n}_{2e} + \bar{n}_{1e}n_{2e} + n_{1e}\bar{n}_{2h} + \bar{n}_{1e}n_{2h}) + 2C_3T_4(n_{2e}\bar{n}_{2h} + \bar{n}_{2e}n_{2h}) - (j_2^N)^2 \right],
\]

\[
D_{33}(A) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ \omega^2T_1(n_{1e}\bar{n}_{2e} + \bar{n}_{1e}n_{2e} + n_{1e}\bar{n}_{2h} + \bar{n}_{1e}n_{2h}) - (j_1^E)^2 \right],
\]

\[
D_{12}(A) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ T_1(-n_{1e}\bar{n}_{2e} - \bar{n}_{1e}n_{2e} + n_{1e}\bar{n}_{2h} + \bar{n}_{1e}n_{2h}) + 2T_3T_4(n_{1e} - n_{1h})(n_{2e} - n_{2h}) - j_1^N j_2^N \right],
\]

\[
D_{13}(A) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ T_1(n_{1e}\bar{n}_{2e} + \bar{n}_{1e}n_{2e} + n_{1e}\bar{n}_{2h} + \bar{n}_{1e}n_{2h}) - j_1^N j_1^E \right],
\]

\[
D_{23}(A) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ T_1(-n_{1e}\bar{n}_{2e} - \bar{n}_{1e}n_{2e} + n_{1e}\bar{n}_{2h} + \bar{n}_{1e}n_{2h}) - j_2^N j_1^E \right].
\]

At zero affinity, they reduce to

\[
D_{11}(A = 0) = \int \frac{d\omega}{2\pi} 2T_N p_0 (1 - p_0) = L_{1,1},
\]

\[
D_{22}(A = 0) = \int \frac{d\omega}{2\pi} 2T_N p_0 (1 - p_0) = L_{2,2},
\]

\[
D_{33}(A = 0) = \int \frac{d\omega}{2\pi} 2\omega T_N p_0 (1 - p_0) = L_{3,3},
\]

\[
D_{13}(A = 0) = \int \frac{d\omega}{2\pi} 2T_E p_0 (1 - p_0) = L_{1,3},
\]

\[
D_{12}(A = 0) = D_{23}(A = 0) = 0
\]
as expected. We find that although the mean currents of the left and the right reservoirs are decoupled, the diffusivity encodes the information of the two LARs. The derivatives of diffusivity at zero affinities are

\[
\begin{align*}
\frac{\partial D_{1i}}{\partial A_j} &= \int \frac{d\omega}{2\pi} p_0(1-p_0)(1-2p_0) \begin{pmatrix}
0 & 0 & \omega(T_1 + 2C_4T_3) \\
0 & 0 & 0 \\
\omega T_1 & 0 & 0
\end{pmatrix}, \\
\frac{\partial D_{2i}}{\partial A_j} &= \int \frac{d\omega}{2\pi} p_0(1-p_0)(1-2p_0) \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \omega T_1 \\
0 & -\omega T_1 & 0
\end{pmatrix}, \\
\frac{\partial D_{3i}}{\partial A_j} &= \int \frac{d\omega}{2\pi} p_0(1-p_0)(1-2p_0) \begin{pmatrix}
\omega T_1 & 0 & 0 \\
0 & -\omega T_1 & 0 \\
0 & 0 & \omega^3 T_1
\end{pmatrix}.
\end{align*}
\]

The symmetric sum of \( D_{ij,k} \) is

\[
\begin{align*}
\frac{\partial D_{1i}}{\partial A_j} + \frac{\partial D_{1j}}{\partial A_i} &= \begin{pmatrix}
0 & 0 & D_{11,3} + D_{13,3} \\
0 & 0 & 0 \\
D_{13,1} + D_{11,3} & 0 & 0
\end{pmatrix}, \\
\frac{\partial D_{2i}}{\partial A_j} + \frac{\partial D_{2j}}{\partial A_i} &= 0, \\
\frac{\partial D_{3i}}{\partial A_j} + \frac{\partial D_{3j}}{\partial A_i} &= \begin{pmatrix}
2D_{13,1} & 0 & 0 \\
0 & 2D_{33,2} & 0 \\
0 & 0 & 2D_{33,3}
\end{pmatrix}.
\end{align*}
\]

Compare to the expression of \( M_{i,j,k} \) we verify Eq. (22)

\[
M_{i,j,k} = \left( \frac{\partial D_{ij}}{\partial A_k} + \frac{\partial D_{ik}}{\partial A_j} \right) \bigg|_{A=0}.
\]

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In fact, the CAR occurs when the site number is even. If the site number is odd, NT rather than CAR will occur.

In fact, the number of sites should be larger than 3.

The relation between $\Gamma_\alpha$ and $\lambda_{\alpha j}$ can be found in our previous paper [27].