Entropy Corrections for a Charged Black Hole of String Theory

Alexis Larrañaga

January 4, 2011

National University of Colombia, National Astronomical Observatory.
Bogota, Colombia.

ealarranaga@unal.edu.co

Abstract

We study the entropy of the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) charged black hole, originated from the effective action that emerges in the low-energy of string theory, beyond semiclassical approximations. Applying the properties of exact differentials for three variables to the first law thermodynamics we derive the quantum corrections to the entropy of the black hole. The leading (logarithmic) and non leading corrections to the area law are obtained.

PACS: 04.70.Dy, 04.70.Bw, 11.25.-w
KeyWords: quantum aspects of black holes, thermodynamics, strings and branes

1 Introduction

When studying black hole evaporation by Hawking radiation using the quantum tunneling approach, a semiclassical treatment is used to study changes in thermodynamical quantities. The quantum corrections to the Hawking temperature and the Bekenstein-Hawking area law have been studied for the Schwarzschild, Kerr and Kerr-Newman black holes \cite{1,2} as well as BTZ black holes \cite{3}.

It has been realized that the low-energy effective field theory describing string theory contains black hole solutions which can have properties which are qualitatively different from those that appear in ordinary Einstein gravity. Here we will analyse the quantum corrections to the entropy of the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) charged black hole \cite{4}, which is an exact classical solution of the low-energy effective heterotic string theory with a finite amount of charge. To obtain the quantum corrections we use the
criterion for exactness of differential of black hole entropy from the first law of thermodynamics for two parameters. We find that the leading correction term is logarithmic, while the other terms involve ascending powers of inverse of the area.

In the quantum tunneling approach, when a particle with positive energy crosses the horizon and tunnels out, it escapes to infinity and appear as Hawking radiation. Meanwhile, when a particle with negative energy tunnels inwards it is absorbed by the black hole and as a result the mass of the black hole decreases. Therefore, the essence of the quantum tunneling argument for Hawking radiation is the calculation of the imaginary part of the action. If we consider the action \( I(r, t) \) and make an expansion in powers of \( \hbar \) we obtain

\[
I(r, t) = I_0(r, t) + \hbar I_1(r, t) + \hbar^2 I_2(r, t) + \ldots \tag{1}
\]

\[
= I_0(r, t) + \sum_i \hbar^i I_i(r, t), \tag{2}
\]

where \( I_0 \) gives the semiclassical value and the terms from \( O(\hbar) \) onwards are treated as quantum corrections. The work of Banerjee and Majhi [6] shown that the correction terms \( I_i \) are proportional to the semiclassical contribution \( I_0 \). Since \( I_0 \) has the dimension of \( \hbar \), the proportionality constants should have the dimension of inverse of \( \hbar \). In natural units (\( G = c = k_B = 1 \)), the Planck constant is of the order of square of the Planck Mass. Therefore, from dimensional analysis the proportionality constants have the dimension of \( M^{-2i} \), where \( M \) is the mass of black hole, and the series expansion becomes

\[
I(r, t) = I_0(r, t) + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} I_0(r, t) \tag{3}
\]

\[
I_0(r, t) \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right), \tag{4}
\]

where \( \beta_i \)'s are dimensionless constant parameters. If the black hole has other macroscopic parameters such as angular momentum or electric charge, one can express this expansion in terms of those parameters, as done in [1] and [3]. In this work, the dimensional analysis suggest that the constants of proportionality for charged rotating black holes have the dimensions of \( (r_H^2 - 2Q^2e^{-2\phi_0})^{-1} \), so we will use an expansion in terms of the horizon radius and the electric charge as

\[
I(r, t) = I_0(r, t) \left( 1 + \sum_i \beta_i \frac{\hbar^i}{(r_H^2 - 2Q^2e^{-2\phi_0})^{2i}} \right). \tag{5}
\]

Using this expansion we will calculate the quantum corrections to the entropy of the GMGHS black hole.
2 Entropy as an Exact Differential

In order to perform the quantum corrections to the entropy of the black hole we will follow the analysis of [1, 3]. The first law of thermodynamics for charged black holes is

\[ dM = TdS + \Phi dQ, \]

where the parameters \( M \) and \( Q \) are the mass and charge of the black hole, while \( T, S \) and \( \Phi \) are the temperature, entropy and electrostatic potential, respectively. This equation can be re-written as

\[ dS (M, Q) = \frac{1}{T} dM - \Phi \frac{T}{T} dQ, \]

from which one can infer that in order for \( dS \) to be an exact differential, the thermodynamical quantities must satisfy

\[ \frac{\partial}{\partial Q} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Phi}{T} \right). \]

If \( dS \) is an exact differential, we can write the entropy \( S(M, J, Q) \) in the integral form

\[ S (M, Q) = \int \frac{1}{T} dM - \int \Phi \frac{T}{dQ} dQ - \int \left( \frac{\partial}{\partial Q} \left( \int \frac{1}{T} dM \right) \right) dQ. \]

3 Standard Entropy of the GMGHS Black Hole

The low energy effective action of the heterotic string theory in four dimensions is given by

\[ A = \int d^4x \sqrt{-g} e^{-\phi} \left( -R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{8} F_{\mu\nu} F^{\mu\nu} \right), \]

where \( R \) is the Ricci scalar, \( G_{\mu\nu} \) is the metric that arises naturally in the \( \sigma \) model,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

is the Maxwell field associated with a \( U(1) \) subgroup of \( E_8 \times E_8 \), \( \phi \) is the dilaton field and

\[ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} - [\Omega_3 (A)]_{\mu\nu\rho}, \]

where \( B_{\mu\nu} \) is the antisymmetric tensor gauge field and
\[
\Omega_3 (A)_{\mu\nu\rho} = \frac{1}{4} (A_\mu F_{\nu\rho} + A_\nu F_{\rho\mu} + A_\rho F_{\mu\nu})
\]

is the gauge Chern-Simons term. Considering \( H_{\mu\nu\rho} = 0 \) and working in the conformal Einstein frame, the action becomes

\[
A = \int d^4x \sqrt{-g} \left( -R + 2 (\nabla \phi)^2 + e^{-2\phi} F^2 \right),
\]

where the Einstein frame metric \( g_{\mu\nu} \) is related to \( G_{\mu\nu} \) through the dilaton,

\[
g_{\mu\nu} = e^{-\phi} G_{\mu\nu}.
\]

The charged black hole solution, known as the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) solution, is given by [4, 5]

\[
ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 \left( 1 - \frac{Q^2 e^{-2\phi_0}}{Mr} \right) d\Omega^2
\]

where

\[
e^{-2\phi} = e^{-2\phi_0} \left( 1 - \frac{Q^2 e^{-2\phi_0}}{Mr} \right)
\]

\[
F = Q \sin \theta d\theta \wedge d\phi
\]

and \( \phi_0 \) is the asymptotic constant value of \( \phi \) at \( r \to \infty \). Note that this metric become Schwarzschild’s solution if \( Q = 0 \). The GMGHS solution has a spherical event horizon at

\[
r_H = 2M
\]

and its area is given by

\[
A = \int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = 4\pi \left( r_H^2 - 2Q^2 e^{-2\phi_0} \right).
\]

Equation (20) tell us that the area of the horizon goes to zero if

\[
r_H^2 = 2Q^2 e^{-2\phi_0},
\]

i.e. the GMGHS solution becomes a naked singularity if

\[
M^2 \leq \frac{1}{2} Q^2 e^{-2\phi_0}.
\]

The Hawking temperature is

\[
T_H = \frac{\kappa}{2\pi} = \frac{\hbar}{8\pi M}.
\]
which is independent of charge. Finally, the electric potential computed on the horizon of the black hole is

$$\Phi = \frac{Q}{r_H} e^{-2\phi_0}. \quad (24)$$

One can easily check that thermodynamical quantities for the GMGHS black hole satisfy condition (8), making $dS$ an exact differential. Thus, the integral form of the entropy (9) gives

$$S_0 (M, Q) = \frac{1}{T_H} \left[ M^2 - \frac{e^{-2\phi_0}}{2} Q^2 \right] \quad (25)$$

that corresponds to the standard black hole entropy

$$S_0 (M, Q) = \frac{A}{4\hbar} = \pi \left( \frac{r_H^2 - 2Q^2 e^{-2\phi_0}}{\hbar} \right). \quad (26)$$

**4 Quantum Correction of the Entropy**

When considering the expansion for the action (5), it affects the Hawking temperature by introducing some correction terms [1, 3, 6]. Therefore the temperature is now given by

$$T = T_H \left( 1 + \sum_i \beta_i \frac{\hbar^i}{(r_H^2 - 2Q^2 e^{-2\phi_0})^{i}} \right)^{-1}, \quad (28)$$

where $T_H$ is the standard Hawking temperature and the terms with $\beta_i$ are quantum corrections to the temperature. It is not difficult to verify that the conditions to make $dS$ an exact differential are satisfied when considering this new form for the temperature. Therefore, the entropy with correction terms is given by

$$S (M, Q) = \int \frac{1}{T} dM = \int \frac{1}{T_H} \left( 1 + \sum_i \beta_i \frac{\hbar^i}{(r_H^2 - 2Q^2 e^{-2\phi_0})^{i}} \right) dM \quad (29)$$

or

$$S (M, Q) = \int \frac{1}{T_H} dM + \int \frac{\beta_1}{T_H (r_H^2 - 2Q^2 e^{-2\phi_0})} \frac{\hbar}{dM} + \int \frac{\beta_2}{T_H (r_H^2 - 2Q^2 e^{-2\phi_0})^2} \frac{\hbar^2}{dM} + \ldots \quad (30)$$

This equation can be written as
\[ S(M, Q) = S_0 + S_1 + S_2 + ..., \]  

where \( S_0 \) is the standard entropy given by equation (27) and \( S_1, S_2, ... \) are quantum corrections. The first of these terms is

\[ S_1 = \beta_1 \hbar \int \frac{1}{T_H (r_H^2 - 2Q^2 e^{-2\phi_0})} dM. \]  

Solving the integral, we obtain

\[ S_1 = \pi \beta_1 \ln \left| r_H^2 - 2Q^2 e^{-2\phi_0} \right|. \]  

The following terms can be written, in general, as

\[ S_j = \beta_j \hbar^j \int \frac{1}{T_H (r_H^2 - 2Q^2 e^{-2\phi_0})^j} dM \]  

By calculating the integral, we obtain

\[ S_j = \pi \beta_j \hbar^{j-1} \frac{1}{1-j} (r_H^2 - 2Q^2 e^{-2\phi_0})^{1-j} \]  

for \( j > 1 \). Therefore, the entropy with quantum corrections is given by

\[ S(M, Q) = \frac{\pi (r_H^2 - 2Q^2 e^{-2\phi_0})}{\hbar} + \pi \beta_1 \ln |r_H^2 - 2Q^2 e^{-2\phi_0}| + \sum_{j>1} \frac{\pi \beta_j \hbar^{j-1}}{1-j} (r_H^2 - 2Q^2 e^{-2\phi_0})^{1-j}. \]  

Using equation (20), and doing a re-definition of the \( \beta_i \), we can write the entropy in terms of the area of the horizon as

\[ S(M, Q) = \frac{A}{4\hbar} + \pi \beta_1 \ln |A| + \sum_{j>1} \frac{\pi \beta_j \hbar^{j-1}}{1-j} \left( \frac{A}{4\pi} \right)^{1-j}. \]  

The first term in this expansion is the usual semiclassical entropy while the second term is the logarithmic correction found earlier for some geometries and using different methods \[7, 8\]. The value of the coefficients \( \beta_i \) can be evaluated using other approaches, such as the entanglement entropy calculation. Finally note that the third term in the expansion is an inverse of area term similar to the one obtained by S. K. Modak \[7\] for the rotating BTZ black hole, for the charged BTZ black hole \[8\] and also in the general cases studied in \[1\] and \[3\].
5 Conclusion

As is well known, the Hawking evaporation process can be understood as a consequence of quantum tunneling in which some particles cross the event horizon. The positive energy particles tunnel out of the event horizon, whereas, the negative energy particles tunnel in, resulting in black hole evaporation. Using this analysis we have studied the quantum corrections to the entropy of the GMGHS black hole of heterotic string theory. With the help of the conditions for exactness of differential of entropy we obtain a power series for entropy. The first term is the semiclassical value, while the leading correction term is logarithmic as has been found using other methods\cite{7,8}. The other terms involve ascending powers of the inverse of the area. This analysis shows that the quantum corrections to entropy obtained in the literature \cite{1,3}, also hold for the black hole of string theory studied here.

Acknowledgements. This work was supported by the Universidad Nacional de Colombia. Project Code 2010100.

References

[1] R. Banerjee and S.K. Modak, JHEP 05, 063 (2009)
[2] R.B. Mann and S.N. Solodukhin. Phys. Rev. D54, 3932 (1996)
[3] M. Akbar, K. Saifullah [arXiv:1002.3581]; [arXiv:1002.3901]
[4] G.W. Gibbons, Nucl. Phys. B207, 337 (1982); G.W. Gibbons and K. Maeda, ibid. B298, 741 (1988); D. Garfinkle, G.T. Horowitz, and A. Strominger, Phys. Rev. D43, 3140 (1991); 45, 3888(E) (1992)
[5] Shao-Wen Wei, Yu-Xiao Liu, Ke Yang, Yuan Zhong. [arXiv:1002.1553]
[6] R. Banerjee and B.R. Majhi. JHEP 06 (2008) 095 [arXiv:0805.2220]
[7] S. K. Modak. Phys. Lett. B671 (2009) 167-173
[8] A. Larrañaga. [arXiv:1002.3410]