ANOMALIES AND INVERTIBLE FIELD THEORIES

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ABSTRACT. We give a modern geometric viewpoint on anomalies in quantum field theory and illustrate it in a 1-dimensional theory: supersymmetric quantum mechanics. This is background for the resolution of worldsheet anomalies in orientifold superstring theory.

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1. Introduction

The subject of anomalies in quantum field theories is an old one, and it is well-trodden. There is a huge physics literature on this topic of anomalies, for which one entree is [Be]. Important work in the early 1980s [AS1, AgW, AgG, ASZ] tied the study of local anomalies to the Atiyah-Singer topological index theorem, and extensions to global anomalies [W1, W2] were not far behind. These ideas were quickly fit in to geometric invariants in index theory, such as the determinant line bundle and the $\eta$-invariant. Indeed, many developments in geometric index theory at that time were directly motivated by the physics. A geometric picture of anomalies emerged from this interaction [F1, §1].

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One impetus to reconsider the settled canon on anomalies is a rather sticky enigma: world-sheet anomalies in Type II superstring orientifolds. That was the subject of my lecture at String-Math 2013, and it will be elaborated elsewhere. Here we take the opportunity to introduce a modern geometric viewpoint on anomalies (§2), to illustrate it in a simpler theory (§3), and to introduce some topology which is crucial in resolving worldsheet orientifold anomalies (§4).

The modern point of view rests on the observation that the anomaly itself is a quantum field theory. It should be expected that anomalies, which are computed as part of a quantum field theory, obey the locality principles of quantum field theory. The anomaly is a very special type of theory: it is invertible. If in addition an invertible theory is topological, then it reduces to a map of spectra in the sense of stable algebraic topology. This presents us with the opportunity to employ more sophisticated topological arguments. We remark that an anomalous quantum field theory is a relative quantum field theory [FT], related to the anomaly.

The simpler theory we revisit here is supersymmetric quantum mechanics (QM) with a single supersymmetry. It was used in the 1980s to give a physics derivation of the Atiyah-Singer index theorem. This physical system describes a particle moving in a Riemannian manifold $X$. The quantum operator which represents the single supersymmetry is the basic Dirac operator on $X$, whose definition requires a spin structure. In the physics a spin structure is required to cancel an anomaly in the quantization of the fermionic field. This is technically much simpler if we assume that $X$ is even-dimensional and oriented, which we do in §3. In §5 we analyze the anomaly without that simplifying assumption. One consequence is that if $X$ is odd-dimensional, it is most natural to consider the Hilbert space of the theory to be a module over a complex Clifford algebra with an odd number of generators. This is well-known in differential geometry in the Clifford linear Dirac operator construction [LM], and it seems natural for the physics as well. (See Remark 5.10.)

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2. Anomalies

The reader may wish to consult previous expositions of anomalies in [F1], [F2], and [FM].

2.1. Fields and field theories: formal view

An $n$-dimensional\textsuperscript{1} quantum field theory

\begin{equation}
Z: \text{Bord}_{(n-1,n)}(\mathcal{F}) \to \text{Vect}_{\text{top}}
\end{equation}

is, formally, a functor from a geometric bordism category of $(n-1)$- and $n$-dimensional manifolds with fields $\mathcal{F}$ to the category of complex topological vector spaces. Unraveling this definition we find that to a closed $n$-manifold—that is, a compact manifold without boundary—the theory assigns a

\textsuperscript{1}Here $n$ is the spacetime dimension. For supersymmetric quantum mechanics we have $n = 1$. 
number $Z(M) \in \mathbb{C}$, the partition function. To a closed $(n-1)$-manifold $N$ is attached a topological vector space $Z(N)$, often called the quantum Hilbert space. The Hilbert space inner product exists if $Z$ is unitary. Compact $n$-dimensional bordisms map under $Z$ to continuous linear maps. For example, if $M$ is a closed $n$-manifold, and $B_1 \cup \cdots \cup B_r \subset M$ a disjoint union of open $n$-balls, then

\[ Z(M \setminus B_1 \cup \cdots \cup B_r): Z(S_1) \otimes \cdots \otimes Z(S_r) \longrightarrow \mathbb{C} \]

encodes correlation functions of local operators, where $S_j = \partial B_j$ and all boundaries are incoming in the bordism. (In a general quantum field theory we take a limit as the radii of the balls shrink to zero.) See [Se1] for a recent exposition of this geometric definition of quantum field theory, due to Segal.

The fields $\mathcal{F}$ in (2.1) are, from the point of view of the theory $Z$, background fields; any fluctuating fields have already been integrated out. Formally, fields are a simplicial sheaf $\mathcal{F}$ on the category of $n$-manifolds and local diffeomorphisms. Fix a closed $n$-manifold $M$. Then the fields $\mathcal{F}(M)$ on $M$ form an iterated fiber bundle. There are topological fields (orientations, spin structures, framings, etc.) and geometric fields (scalar fields, metrics, connections, spinor fields, etc.) The definition of some fields depends on other fields (e.g., a spinor field depends on a metric and spin structure), which is why $\mathcal{F}(M)$ is an iterated fibration and not a Cartesian product. Some fields have internal symmetries, and so $\mathcal{F}(M)$ is typically an infinite dimensional higher stack. Examples of fields with internal symmetries include spin structures, connections (gauge fields), and higher gauge fields such as the $B$-field in string theory. The sheaf condition encodes the locality of fields and allows the construction of a bordism category with an arbitrary collection of fields. The manifolds $M, N, B_j, S_j$ in the previous paragraph and going forward are assumed endowed with fields, though the fields are not always made explicit in the notation.

A field theory $\alpha$ is invertible if for every closed $(n-1)$-manifold $N$ with fields the vector space $\alpha(N)$ is a line and if for each $n$-dimensional bordism $M: N_0 \rightarrow N_1$ with fields the linear map $\alpha(M): \alpha(N_0) \rightarrow \alpha(N_1)$ is invertible. In particular, the partition function $\alpha(M) \in \mathbb{C}$ of a closed $n$-manifold is nonzero. The natural algebraic operation on field theories is multiplication—tensor product of the quantum vector spaces and numerical product of the partition functions—and ‘invertibility’ refers to that operation. For example, the vector space $\mathbb{C}$ is the identity under tensor product of vector spaces, and a vector space $V$ has an inverse $V'$—i.e., there exists an isomorphism $V \otimes V' \cong \mathbb{C}$—if and only if dim $V = 1$.

A lagrangian theory is specified by a collection of fields $\mathcal{F}$—both background and fluctuating—and, for each $n$-manifold $M$, a function

\[ A = A(M): \mathcal{F}(M) \longrightarrow \mathbb{C} \]

called the exponentiated action. Note that despite the name, there is not necessarily a well-defined action which would be its logarithm.

**Example 2.4.** In supersymmetric QM with values in a fixed Riemannian manifold $X$, the manifold $M$ is 1-dimensional and $\mathcal{F}(M)$ consists of 4 fields: a metric on $M$, a spin structure on $M$, and if we assume a symmetric formal $n$-dimensional tubular neighborhood of $N$ is given.
a smooth map $\phi: M \to X$, and a spinor field $\psi$ on $M$ with values in $\phi^*TX$. The metric, spin structure, and $\phi$ are independent of each other, but we need all three to define the space of spinor fields $\psi$. Also, the fermionic field $\psi$ is odd in the sense of supermanifolds [DM], so the exponentiated action (2.3) is not really a complex-valued function on fields, but as we only consider bosonic fields in the sequel we do not dwell on this.

If the fields $\mathcal{F}$ include fermionic fields, as in supersymmetric QM, then there is an odd vector bundle $\mathcal{F} \to \mathcal{F}'$ with fibers the fermionic fields and base the bosonic fields. The fermionic fields can be integrated out to give a theory with only bosonic fields $\mathcal{F}'$. Each fermionic path integral contributes the pfaffian of a Dirac operator to the effective exponentiated action $A_{\text{eff}} = A_{\text{eff}}(M)$ on $\mathcal{F}'(M)$. The pfaffian may vanish, so $A_{\text{eff}}$ is not necessarily an invertible theory. The Feynman procedure next calls for integration of $A_{\text{eff}}$ over the bosonic fields $\mathcal{F}'(M)$, and this brings in all the analytic interest of quantum field theory: one needs to construct a well-defined measure on $\mathcal{F}'(M)$ to define the integral.

### 2.2. Anomalies: traditional view

The anomaly is a geometric, rather than analytic, obstruction to integrating $A_{\text{eff}}$ over $\mathcal{F}'(M)$. Namely, it may happen that rather than a global function, the effective exponentiated action $A_{\text{eff}}$ is a section of a complex line bundle

$$\alpha(M) \to \mathcal{F}'(M).$$

Furthermore, in a unitary theory $\alpha(M)$ carries a hermitian metric and compatible covariant derivative. Typically $\alpha(M)$ is a tensor product of more primitively defined line bundles. For example, if $A_{\text{eff}}$ is obtained by integrating out fermionic fields, then some factors of $\alpha(M)$ are Pfaffian line bundles of families of Dirac operators parametrized by $\mathcal{F}'(M)$. To obtain a function to formally integrate over $\mathcal{F}'(M)$ we require a setting of the quantum integrand, a section $\mathbf{1}$ of (2.5) which we demand be flat and have unit norm. Then the desired quantum integrand is the ratio $A_{\text{eff}}/\mathbf{1}$.

From this lagrangian point of view, the anomaly is the obstruction to the existence of $\mathbf{1}$. The local anomaly is the curvature of (2.5); if the curvature vanishes, the global anomaly is the holonomy. If all holonomies are trivial, then the local and global anomalies vanish. Vanishing holonomy implies the existence of $\mathbf{1}$, though $\mathbf{1}$ is unique only up to a phase on each component of $\mathcal{F}'(M)$. Said differently, the set of trivializations on each component is a torsor over the circle group of unit norm complex numbers.

There is also a hamiltonian point of view on anomalies [Sc2], [Fa], [NAg]. To an $n$-dimensional manifold $N$ a non-anomalous field theory assigns a fiber bundle over $\mathcal{F}'(N)$ whose fibers are complex topological vector spaces. In an anomalous theory $F$ the fibers of the bundle

$$F(N) \to \mathcal{F}'(N)$$

are rather complex projective spaces. This is in line with expectations in quantum mechanics: the space of pure states in a quantum system is a complex projective space. “Integrating” over

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$^3$An object of $\text{Bord}_{(n-1,n)}(\mathcal{F}')$ is really the germ of an $n$-manifold neighborhood of $N$ and the fields are defined on that neighborhood.
bosonic fields, or canonical quantization, involves taking $L^2$ sections of a vector bundle.\textsuperscript{4} Again there is an analytic difficulty—construct a measure on the space of bosonic fields—and a geometric difficulty—lift the projective bundle to a vector bundle. The obstruction

\begin{equation}
\alpha(N) \longrightarrow \mathcal{F}'(N)
\end{equation}

to the existence of a lift is the anomaly. Topologically, this obstruction is a twisting of complex $K$-theory, or a gerbe (see [FHT, ASe], for example). It describes a twisted notion of ‘complex vector bundle’, exactly as a complex line bundle describes a twisted notion of ‘complex-valued function’. In a unitary theory there is also differential geometry—the obstruction is a “differential twisting” of complex $K$-theory—just as in a unitary theory the line bundle (2.5) carries a metric and connection. For example, the local hamiltonian anomaly is measured by a 3-form on $\mathcal{F}'(N)$.

A **hamiltonian setting** is a trivialization of the anomaly (2.7). If the anomaly vanishes, then on each component of $\mathcal{F}'(N)$ the trivializations form a torsor over the Picard groupoid of flat hermitian line bundles.

### 2.3. Anomalies: modern view

As quantum field theory is local on spacetime, we require that the bundles $\alpha(M)$ and $\alpha(N)$ be *local* functions of $M$ and $N$. The same is required for trivializations of anomalies. Locality is encoded by demanding that the anomalies (2.5) and (2.7) fit together as parts of an *invertible extended* $(n + 1)$-dimensional field theory\textsuperscript{5}

\begin{equation}
\alpha : \text{Bord}_{(n-1,n,n+1)}(\mathcal{F}') \longrightarrow \Sigma^{n+1}I_{\mathbb{R}/\mathbb{Z}}.
\end{equation}

‘Extended’ means that $\alpha$ has values on manifolds with corners of dimensions $n + 1$, $n$, and $n - 1$. We remark that the numerical invariants of closed $(n + 1)$-manifolds include the holonomies of the anomaly line bundle (2.5). There is flexibility in choosing the codomain in (2.8). Here we take a universal choice, the Pontrjagin or Brown-Comenetz dual $I_{\mathbb{R}/\mathbb{Z}}$ of the sphere spectrum [HS, Appendix B], shifted up in degree. In §3.4 and §5 we make more economical choices. After exponentiation: $\alpha(W)$ is a complex number of unit norm for a closed $(n + 1)$-manifold $W$; $\alpha(M)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded complex line for a closed $n$-manifold $M$; and $\alpha(N)$ is a gerbe with various $\mathbb{Z}/2\mathbb{Z}$-gradings for a closed $(n - 1)$-manifold $N$. Since the theory is invertible, (2.8) factors though the quotient of the bordism 2-category obtained by inverting all morphisms. As the bordism category is symmetric monoidal what is obtained is a *spectrum* in the sense of algebraic topology; see [L, §2.5]. A theorem of Galatius-Madsen-Tillmann-Weiss [GMTW] identifies it as an unstable approximation to a Thom spectrum. For the anomaly of supersymmetric quantum mechanics, there are non-topological fields—the metric and the map to the target—so it is not automatic that the anomaly

\textsuperscript{4}In fact, one takes sections over the space of classical solutions which are flat along some polarization, but here we only focus on the formal geometric difficulty to do with projectivity of the fibers, not the polarization.

\textsuperscript{5}If $\alpha$ is unitary and not topological, then we promote $\alpha$ to a differential field theory in the sense that the line bundles and gerbes are smooth over smooth parameter spaces and carry metrics and connections. In supersymmetric QM the anomaly is topological, so we will not pursue this here and tacitly assume that $\alpha$ is topological.
is topological. Nonetheless, it is. In particular, the factorization of (2.8) is a map of spectra, so is amenable to analysis via techniques of homotopy theory.

As stated earlier, an anomalous theory \( F \) is an example of a relative quantum field theory [FT]. Thus it is a map

\[
F: 1 \longrightarrow \tau_{\leq n} \alpha
\]

of \( n \)-dimensional field theories from the trivial theory to the \( n \)-dimensional truncation of \( \alpha \). To a closed \( n \)-manifold \( M \) with fields it attaches an element \( F(M) \) of the complex line \( \alpha(M) \), and to a closed \( (n-1) \)-manifold \( N \) with fields it attaches a complex vector space \( F(N) \) twisted by the gerbe \( \alpha(N) \).

The anomaly is trivializable if \( \alpha \) is isomorphic to the trivial theory, and a trivialization of the anomaly, or setting, is a choice of isomorphism

\[
\mathbb{1}: \alpha \xrightarrow{\sim} 1
\]

as field theories. This general formulation encodes the locality of the setting of the quantum integrand as well as the locality of the anomaly itself.

### 3. Supersymmetric quantum mechanics

Supersymmetric quantum mechanics (QM) with minimal supersymmetry was used in [Ag, FWi] to give a physics derivation of the Atiyah-Singer index theorem for a single Dirac operator. An account geared to mathematicians appears in [W3], and a mathematically precise take on the argument was given in [Bi], inspired by [At1]. We restrict our attention here to the anomaly and its trivialization, which is a prerequisite to having a well-defined quantum mechanical theory.

Supersymmetric QM is a 1-dimensional theory of a particle moving in a Riemannian manifold \( X \). The theory is defined on 1-manifolds \( M \) equipped with a background metric and spin structure. There are two fluctuating fields on \( M \) which are integrated out in the quantum theory. First, a map \( \phi: M \rightarrow X \) which represents the trajectory of a particle. Then there is an odd field \( \psi \) which is a spinor field on \( M \) with coefficients in the pullback tangent bundle \( \phi^*TX \rightarrow M \). The lagrangian density [Detal, pp. 647–656] has kinetic terms for these fields:

\[
\mathcal{L} = \frac{1}{2} \left\{ \langle \frac{d\phi}{dt}, \frac{d\phi}{dt} \rangle + \langle \psi, D\psi \rangle \right\} |dt|,
\]

where \( t \) is a local coordinate on \( M \) with \( d/dt \) of unit length and \( D \) is the Dirac operator on \( M \), coupled to the pullback bundle \( \phi^*TX \rightarrow M \). A spin structure on \( M \) can be identified as a real line bundle \( L \rightarrow M \) equipped with an isomorphism \( L^\otimes 2 \xrightarrow{\sim} T^*M \). Multiplication and integration over \( M \), assuming \( M \) is closed, gives a self-dual pairing on spinor fields with respect to which the Dirac operator \( D \) is formally skew-adjoint. The spinor fields, which are sections of \( L \rightarrow M \), are real, as is the skew-adjoint Dirac operator. We do not dwell on the precise meaning of the kinetic term for fermions.
3.1. Lagrangian anomaly

Integrate out the fermionic field $\psi$, assuming that the 1-manifold $M$ is closed. In the notation of §2 this is fermionic integration over the fibers of $\mathcal{F}(M) \to \mathcal{F}'(M)$. The result is standard: ignoring the kinetic term for $\phi$, which plays no role in anomaly analysis, we obtain the pfaffian of the Dirac operator, which is a section pfaff $D$ of the Pfaffian line bundle $[F3, §3]$

\[
\begin{align*}
\text{Pfaff } D &\longrightarrow \mathcal{F}'(M) .
\end{align*}
\]

Furthermore, this real line bundle carries a metric and compatible covariant derivative. Thus locally there are two unit norm sections $\mathbb{1}$; an orientation of $\text{Pfaff } D \to \mathcal{F}'(M)$—which is a topological trivialization and may not exist—picks out a global section.

In this section we make the following hypothesis, which we relax in §5.

**Assumption 3.3.** The target manifold $X$ is even-dimensional and oriented.

**Theorem 3.4.** Given Assumption 3.3, the topological equivalence class in $H^1(\mathcal{F}'(M); \mathbb{Z}/2\mathbb{Z})$ of the lagrangian anomaly $\text{Pfaff } D \to \mathcal{F}'(M)$ is the transgression of $w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$.

Because $\mathcal{F}'(M)$ includes the field $\phi: M \to X$, there is an evaluation map which is the top arrow in the diagram

\[
\begin{diagram}
\mathcal{F}'(M) \times M & \xrightarrow{e} & X \\
\pi_1 \downarrow & & \\
\mathcal{F}'(M) & & 
\end{diagram}
\]

(3.5)

The vertical map is projection onto the first factor. Transgression is the composition $(\pi_1)_* \circ e^*$ on mod 2 cohomology. The pushforward

\[
(\pi_1)_* : H^2(\mathcal{F}'(M) \times M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(\mathcal{F}'(M); \mathbb{Z}/2\mathbb{Z})
\]

in mod 2 cohomology is defined without any orientation data on the fibers of $\pi_1$. Notice that the anomaly is purely topological; it is independent of the background metric on $M$. It also turns out to be independent of the background spin structure on $M$, as is clear from the formula in the theorem. Theorem 3.4 is well-known. The proof we sketch here, which is based on the topological Atiyah-Singer index theorem, appears in [FW, (5.22)].

**Proof.** The manifold $M$ is a finite union of circles, and since under disjoint union $\text{Pfaff } D$ is multiplicative and the transgression of $w_2(X)$ is multiplicative, it suffices to consider $M = S^1$. Also, the class

\[
[\text{Pfaff } D] \in H^1(\mathcal{F}'(M); \mathbb{Z}/2\mathbb{Z})
\]

(3.7)
is determined by its pairing with the fundamental class of smooth loops \( S^1 \to \mathcal{F}'(M) \). Pull back (3.5) over a single loop to obtain a family

\[
S^1 \times M \xrightarrow{\epsilon} X \\
\pi_1 \downarrow \\
S^1
\]

(3.8)

of circles parametrized by the circle. The Atiyah-Singer theorem [AS2] computes the value of (3.7) on the base circle as a pushforward in \( KO \)-theory, where the base circle has the bounding spin structure:

\[
\langle [\text{Pfaff } D], [S^1] \rangle = \pi_*^{S^1 \times M} (e^*TX) \in KO^{-2}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}.
\]

(3.9)

No matter what the spin structure on the circle \( M \), the torus \( S^1 \times M \) has the bounding spin structure, whence (3.9) is independent of the spin structure on \( M \). For the bounding spin structure the \( KO \)-pushforward \( \pi_*^{S^1 \times M} \) of the trivial bundle vanishes, so we can replace \( e^*TX \) by the reduced virtual bundle, and now by excision we replace the torus \( S^1 \times M \) with the 2-sphere. Then the \( KO \)-pushforward becomes the suspension isomorphism, and since \( KO^0(S^2) \cong \mathbb{Z}/2\mathbb{Z} \) via the second Stiefel-Whitney class, it follows that

\[
\langle [\text{Pfaff } D], [S^1] \rangle = \langle \phi^*w_2(X), [S^1 \times M] \rangle,
\]

(3.10)

as desired. \( \square \)

Remark 3.11. The Atiyah-Singer index theorem computes the Pfaffian line bundle as a transgression in \( KO \)-theory. Since \( M \) has very small dimension, and because we make the simplifying Assumption 3.3, a very simple truncation of \( KO \)-theory suffices, namely mod 2 cohomology. When we drop Assumption 3.3 in §5 the Pfaffian will be computed by a somewhat larger truncation of \( KO \)-theory.

Remark 3.12. The lagrangian anomaly is a complex line bundle, the complexification of (3.2), so its equivalence class in \( H^2(\mathcal{F}'(M); \mathbb{Z}) \) is the integral Bockstein of the equivalence class of the real bundle (3.2). Since integral Bockstein \( \beta_{\mathbb{Z}} \) commutes with transgression, that equivalence class is the transgression of

\[
W_3(X) = \beta_{\mathbb{Z}}w_2(X) \in H^3(X; \mathbb{Z}).
\]

(3.13)

But since supersymmetric QM is unitary, the anomaly bundle carries a metric and connection. In this case the connection is flat of order two—all holonomies are \( \pm 1 \)—and is encoded precisely by the real structure, i.e., by the real Pfaffian line bundle (3.2).
3.2. Hamiltonian anomaly

For more details on parts of this subsection, see [Detal, pp. 372–373] and [Detal, pp. 679–681].

It suffices to consider a connected 0-manifold, so a point $N = \text{pt}$. Technically, we should embed $N$ in a germ of a Riemannian 1-manifold, but that plays no role since ultimately the anomaly is topological. We also have a spin structure on the augmented tangent bundle, augmented in the sense that we add a trivial bundle to make it 1-dimensional. Up to isomorphism this is determined by a sign, comparing the orientation underlying the spin structure to the standard orientation on the real line $\mathbb{R}$. We take the sign to be $+$. The space of classical solutions to the Euler-Lagrange equations derived from the lagrangian (3.1) is a symplectic supermanifold, and for the partial quantization which integrates out the fermionic field $\psi$ we work with a fixed $\phi$. In canonical quantization we only consider $\phi, \psi$ which satisfy the classical equations of motion, a second order ODE for $\phi$ and a first order ODE for $\psi$. The space of classical solutions $\phi, \psi$ on $\mathbb{R} \times N$ (time cross space) may, after choosing an initial time, be identified with the supersymplectic manifold

\begin{equation}
\pi^*\Pi TX \longrightarrow TX,
\end{equation}

where $\pi: TX \rightarrow X$ is the tangent bundle with its symplectic structure derived from the Riemannian metric, via the induced isomorphism $TX \xrightarrow{\cong} T^*X$ and the standard symplectic structure on the cotangent bundle. The fibers of (3.14) are the parity-reversed tangent spaces, which have an odd symplectic structure given by the Riemannian metric. The quantization problem for the constant $\phi \equiv x$ is to quantize the odd symplectic vector space $\Pi T_x X$. Assumption 3.3 that $X$ is even dimensional ensures the existence of a complex polarization, which is the parity reversal of a half-dimensional isotropic subspace $W \subset T_x X \otimes \mathbb{C}$ of the complexified tangent space. This induces a complex structure on $T_x X$, and we demand that the induced orientation agree with the orientation given in Assumption 3.3. Write the polarization as a decomposition

\begin{equation}
T_x X \otimes \mathbb{C} \cong W \oplus \overline{W}.
\end{equation}

The quantum Hilbert space is then the space of functions on $\Pi W$, which we identify with the $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra $\mathcal{H} = \bigwedge^* W \cong \bigwedge^* \overline{W}$. Complex linear functions on $\Pi T_x X$ act as operators on $\mathcal{H}$: elements of $(\Pi W)^* \cong \Pi \overline{W}$ act by exterior multiplication and elements of $(\Pi \overline{W})^* \cong \Pi W$ by contraction. These are the standard creation and annihilation operators, and they generate the action of the Clifford algebra built on $T_x X^* \otimes \mathbb{C}$.

The Clifford module $\mathcal{H}$ depends on the choice of polarization (3.15). The underlying projective space $\mathbb{P}\mathcal{H}$ is independent of the polarization. Thus, without any choice of polarization, partial hamiltonian quantization along the fibers of (3.14) produces a bundle $\pi^*\mathcal{P} \rightarrow TX$ of complex projective spaces, where

\begin{equation}
\mathcal{P} \longrightarrow X
\end{equation}

is the bundle of projective complex spin representations. In other words, if $SO(X) \rightarrow X$ is the oriented orthonormal frame bundle with structure group $SO_{2n}$, then (3.16) is the bundle associated
to the projective spin representation \( SO_{2m} \to \text{Aut}(\mathbb{P}) \). The projective bundle (3.16), pulled back to \( TX \), is one model for the hamiltonian anomaly (2.7). Another model is the pullback of the bundle of complex Clifford algebras

\[
\text{Cliff}^C(TX) \to X,
\]

formed as the associated bundle to the conjugation action \( SO_{2m} \to \text{Aut}(\text{Cliff}^C_{2m}) \) on the standard complex Clifford algebra.

The bundles (3.16) and (3.17) are both standard models for the gerbe represented by the integral Bockstein (3.13) of the second Stiefel-Whitney class of \( X \). As in Remark 3.12 the hamiltonian gerbe carries flat differential geometric data which amount to the real gerbe represented by the bundle of real Clifford algebras

\[
\text{Cliff}(TX) \to X.
\]

Its equivalence class is precisely the second Stiefel-Whitney class \( w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \). From the field theory point of view, a fiber of (3.17) is the operator algebra generated by \( \psi \).

### 3.3. Trivializing the lagrangian and hamiltonian anomalies

We show that a spin structure on \( X \) induces a canonical trivialization of the lagrangian anomaly (3.2) and the hamiltonian anomaly (3.18), where for both we incorporate the real structures.

The statement for the lagrangian anomaly follows from a “categorification” of Theorem 3.4. Namely, Theorem 3.4 is a topological formula for the equivalence class of the lagrangian anomaly; it is a topological index theorem. What we construct now is an isomorphism of the Pfaffian line bundle with a real line bundle which represents the transgression of \( w_2(X) \). A spin structure on \( X \) induces a trivialization of this line bundle and so, via this isomorphism, a trivialization of the lagrangian anomaly. The argument appears in [DFM, §5.2] for the nonbounding spin structure on the circle; here we give a few more details and treat the bounding spin structure as well.

The Pfaffian line bundle (3.2) carries a Quillen metric. The points of unit norm in each fiber form a \( \mathbb{Z}/2\mathbb{Z} \)-torsor, and from the torsor we can canonically reconstruct the fiber as a real line with metric. The torsor is canonically equivalent to \( \pi_0(\text{Pfaff} D_\phi \setminus \{0\}) \), where \( \text{Pfaff} D_\phi \) is the fiber over \( \phi \). As in §3.1 it suffices to take \( M = S^1 \). Fix \( \phi : S^1 \to X \) and let \( E = \phi^*TX \to S^1 \) be the pullback tangent bundle, which is an oriented real vector bundle with metric and covariant derivative. Let \( SO(E) \to S^1 \) be its bundle of oriented orthonormal frames. Let \( \Gamma_\phi \) be the space of sections of \( SO(E) \to S^1 \), which is nonempty. Since the group of homotopy classes of maps \( S^1 \to SO_{2m} \) is cyclic of order two, \( \pi_0(\Gamma_\phi) \) is a \( \mathbb{Z}/2\mathbb{Z} \)-torsor.

**Theorem 3.19.** After a universal choice, there is a canonical isomorphism

\[
\pi_0(\text{Pfaff} D_\phi \setminus \{0\}) \xrightarrow{\cong} \pi_0(\Gamma_\phi).
\]

The universal choice is a path in the special orthogonal group from 1 to \(-1\); cf., Remark 3.24.
Corollary 3.21. A spin structure on $X$ determines a trivialization $\mathbb{1}$ of Pfaff $D_\phi$.

The fact that the trivialization is canonical, given the spin structure on $X$, means that the trivializations of the lines Pfaff $D_\phi$ patch to a smooth trivialization of the lagrangian anomaly (3.2).

Proof of Corollary 3.21. A spin structure on $X$ induces a spin structure on $E \to S^1$. Let $\text{Spin}(E) \to SO(E) \to S^1$ be the corresponding $\text{Spin}_{2m}$-bundle of frames. The space of sections of $\text{Spin}(E) \to S^1$ is connected, so maps into a single component of $\Gamma_\phi$. □

Proof of Theorem 3.19. For convenience let the metric on the circle $S^1$ have total length 1. Choose a periodic coordinate $t$ so that $\xi = d/dt$ has unit length and is properly oriented. Let $L \to S^1$ denote the spin structure, which is a real line bundle with metric equipped with an isomorphism $L \otimes^2 \cong \mathbb{R}$ of its square with the trivial bundle of rank one. Either $L \to S^1$ is the trivial bundle (nonbounding spin structure) or the M"obius bundle (bounding spin structure). The real skew-adjoint Dirac operator $D_\phi$ may be identified with the covariant derivative operator $\nabla_\xi$ on sections of $L \otimes E \to S^1$.

Suppose first that $L \to S^1$ is the trivial bundle. If $e \in \Gamma_\phi$ is a pointwise oriented orthonormal basis of sections of $E \to S^1$, then $\nabla_\xi(e) = A(e) \cdot e$ for some function $A(e) : S^1 \to \mathfrak{so}_{2m}$. Up to a constant element of $SO_{2m}$ we can choose $e$ so that $A(e)$ is a constant skew-symmetric matrix $A \in \mathfrak{so}_{2m}$, whose eigenvalues $a\sqrt{-1}$ satisfy $-\pi < a \leq \pi$. Then the holonomy of $E$ around $S^1$ is $\exp(A) \in SO_{2m}$. Let $\mathcal{H}$ be the real Hilbert space of $L^2$ sections of $E$ and $W \subset \mathcal{H}$ the subspace spanned by the $2m$ sections which comprise the framing $e$. The algebraic direct sum $\bigoplus_{k \in \mathbb{Z}} e^{2\pi ikt}W$ is dense in $\mathcal{H}$. Furthermore, the absolute value of the eigenvalues of $\nabla_\xi$ on $e^{2\pi ikt}W$ is bounded below by $(2|k| - 1)\pi$, whence $\nabla_\xi$ is invertible on the orthogonal complement to $W$. It follows directly from the construction [F3, §3] of the Pfaffian line that Pfaff $D_\phi = \text{Pfaff} \nabla_\xi$ is canonically isomorphic to $\text{Det} W^\ast$. There is an induced isomorphism

$$\pi_0(\text{Pfaff} D_\phi \setminus \{0\}) \xrightarrow{\cong} \pi_0(\text{Det} W^\ast \setminus \{0\}) \quad (3.22)$$

of $\mathbb{Z}/2\mathbb{Z}$-torsors. The latter is the $\mathbb{Z}/2\mathbb{Z}$-torsor of orientations of $W$. Now an ordered basis of $W$ is a sequence of $2m$ sections of $E \to S^1$ which are linearly independent at each point, so after applying Gram-Schmidt determines an element of $\Gamma_\phi$. This induces an isomorphism

$$\pi_0(\text{Det} W^\ast \setminus \{0\}) \xrightarrow{\cong} \pi_0(\Gamma_\phi) \quad (3.23)$$

and the isomorphism (3.20) is the composition of (3.22) and (3.23).

If $L \to S^1$ is the M"obius bundle, then the preceding argument gives an isomorphism of the Pfaffian line with the components of the space $\Gamma_L$ of sections of $SO(E \otimes L) \to S^1$, where $SO(E \otimes L)$ is the oriented orthonormal frame bundle of $E \otimes L$. Fix a path $g(t)$, $0 \leq t \leq 1$, in $SO_{2m}$ with $g(0) = 1$ and $g(1) = -1$. Then if $e \in \Gamma_L$ is a section of $SO(E) \to S^1$, the product $e \cdot g$ is a section of $SO(E \otimes L)$. There is an induced isomorphism of $\mathbb{Z}/2\mathbb{Z}$-torsors $\pi_0(\Gamma_L) \xrightarrow{\cong} \pi_0(\Gamma_\phi)$. □

Remark 3.24. The isomorphism $\Gamma_L \to \Gamma_\phi$ depends on the choice of path $g$ and the induced isomorphism $\pi_0(\Gamma_L) \to \pi_0(\Gamma_\phi)$ depends on the homotopy class of $g$ rel boundary. There are two such homotopy classes. Therefore, the isomorphism of Theorem 3.19, and so the trivialization of Corollary 3.21, depends on this universal choice.
**Remark 3.25.** Theorem 3.19 is an example of a “categorified index theorem”. We expect in general that isomorphisms in theorems of this type depend on universal choices.

A spin structure on $X$ leads more directly to a trivialization of the Hamiltonian anomaly (3.17). Recalling the discussion in §3.2 we solve the quantization problem by the $\mathbb{Z}/2\mathbb{Z}$-graded bundle of complex spinors, which is a vector space lift of (3.16). In terms of the bundle of algebras (3.17), let $\text{Spin}(X) \to X$ denote the spin structure, a principal $\text{Spin}_{2m}$-bundle. Left multiplication by $\text{Spin}_{2m} \subset \text{Cliff}_{2m}$ on $\text{Cliff}_{2m}$ induces a real vector bundle over $X$ which is a bundle of invertible bimodules between (3.18) and the constant bundle of algebras with fiber $\text{Cliff}_{2m}$. (See §4 for a discussion of the 2-category of algebras; invertible bimodules are isomorphisms, also known as Morita equivalences.) Upon complexification the latter bundle is Morita isomorphic to the trivial bundle of algebras, since $\text{Cliff}_{2m}^C$ is Morita trivial. This Morita viewpoint on spin structures is emphasized in [ST].

### 3.4. The anomaly as an invertible field theory

The modern view in §2.3 is that the anomaly in supersymmetric QM is a 2-dimensional invertible extended field theory $\alpha_{\text{analytic}}$. We do not give a direct analytic construction of the entire field theory from Dirac operators—we have pieces of it in previous subsections—though that would be an interesting general undertaking in geometric index theory. Rather, we use the index theory carried out in the previous subsections to motivate a direct topological definition of a field theory $\alpha = \alpha_{\text{topological}}$, which should be isomorphic to $\alpha_{\text{analytic}}$.

Recall that the fields $\mathcal{F}$ of supersymmetric QM consist of a metric, spin structure, map $\phi$, and fermionic field $\psi$. The anomaly in question occurs after integrating out $\psi$, so naively we expect it to depend on the three background fields. However, as is clear from Theorem 3.4 and the discussion in §3.2, it is independent of the metric and spin structure.\(^\text{6}\) Furthermore, up to isomorphism it only depends on the homotopy class of $\phi$, since the anomaly is flat: a flat line bundle for a family of 1-manifolds and a flat gerbe for a family of 0-manifolds. Therefore, the anomaly has a purely topological description.

Let $\text{Bord}_2(X)$ denote the bordism 2-category of 0-, 1-, and 2-manifolds equipped with a map to $X$. (See [L] for an exposition of bordism multicategories and [Ay] for bordism categories of manifolds with general geometric structures.) As the anomaly theory is invertible, it factors through the geometric realization of $\text{Bord}_2(X)$, which inverts all the morphisms. According to a theorem of Galatius-Madsen-Tillmann-Weiss [GMTW], the result is the 0-space of the smash product

\[
\Sigma^2 MTO_2 \wedge X_+.
\]

Here $MTO_2$ is the Thom spectrum of the virtual vector bundle $-V \to BO_2$, the negative of the canonical 2-plane bundle over the classifying space of $O_2$. The ‘+’ denotes a disjoint basepoint. An invertible topological field theory is a spectrum map out of (3.26); we take the codomain to be a shift of the Eilenberg-MacLane spectrum $H\mathbb{Z}/2\mathbb{Z}$ for mod 2 cohomology. (In §2.3 we discussed

\(^6\) after some universal choice; see Remark 3.24.
a universal choice, the Pontrjagin dual of the sphere, but for this example the simpler Eilenberg-MacLane spectrum suffices and captures the theory more precisely.) That map is the composition

\[(3.27) \quad \alpha : \Sigma^2 MTO \wedge X \xrightarrow{\text{id} \wedge w_2} \Sigma^2 MTO \wedge K(\mathbb{Z}/2\mathbb{Z}, 2) \xrightarrow{\text{Thom}} \Sigma^2 H\mathbb{Z}/2\mathbb{Z} \]

To construct the first map we represent the second Stiefel-Whitney class of the tangent bundle by a map \( X \xrightarrow{w_2} K(\mathbb{Z}/2\mathbb{Z}, 2) \) into the appropriate Eilenberg-MacLane space. The second spectrum is the Thom spectrum of \( \mathbb{R}^2 - V \to BO_2 \times K(\mathbb{Z}/2\mathbb{Z}, 2) \), where \( \mathbb{R}^2 \to BO_2 \times K(\mathbb{Z}/2\mathbb{Z}, 2) \) is the vector bundle with constant fiber \( \mathbb{R}^2 \). The Thom isomorphism identifies the second cohomology of the Thom spectrum with the second cohomology of the (suspension spectrum of the) base, and the map labeled ‘Thom’ is the composition of the Thom isomorphism and projection onto the second factor. Intuitively, if \( S \) is a space, then for \( m = 0, 1, 2 \) a map of \( S \times S^m \) into \((3.26)\) is a parametrized family over \( S \) of closed \( m \)-manifolds equipped with a map to \( X \):

\[(3.28) \quad M \xrightarrow{\phi} X \xrightarrow{\pi} S \]

The value of \( \alpha \), computed as composition with \((3.27)\), is a map \( S \to K(\mathbb{Z}/2\mathbb{Z}, 2 - m) \) whose homotopy class is the transgression \( \pi_* \phi^* w_2(X) \).

A spin structure on \( X \) can be identified with a null homotopy of the map \( X \to K(\mathbb{Z}/2\mathbb{Z}, 2) \) representing \( w_2(X) \), which induces a null homotopy of the first map in \((3.27)\) and then, by composition, of \((3.27)\) as well. This is a trivialization \((2.10)\) of the anomaly theory \( \alpha \).

### 4. Central simple algebras and topology

Real vector spaces are a model for real \( K \)-theory in a precise sense, and in this section we describe models for various truncations of and modules over real \( K \)-theory. We do not give proofs of the statements made here; we hope to provide them elsewhere.

Traditionally \([At2]\) real \( K \)-theory is defined on a compact space \( S \) as the universal abelian group constructed from the monoid of equivalence classes of real vector bundles on \( S \), with the monoid operation being direct sum. Tensor product of vector bundles makes this \( K \)-theory group into a ring. Combining with the suspension construction in topology one obtains \emph{connective ko-theory}, which only has nonzero cohomology in nonpositive degrees. Equivalently, the homotopy groups of the spectrum \( ko \) are only nonzero in nonnegative degrees, hence the adjective ‘connective’.\(^7\) In somewhat different terms \([Se3]\): the 0-space of the connective spectrum \( ko \) is the classifying space of the symmetric monoidal topological category of real vector spaces and isomorphisms. In this section we introduce other symmetric monoidal topological (multi-)categories and their classifying connective spectra.

\(^7\)Periodic \( KO \)-theory is constructed from connective \( ko \)-theory by inverting Bott periodicity. We remark that in the topological index theory argument of \(\S 3.1\) we could have used \( ko \) in place of \( KO \).
4.1. Some $ko$-modules

The nonzero homotopy groups of $ko$ form the Bott song:

\[(4.1) \quad \pi_{0,1,2,\ldots}(ko) = \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots \}.\]

Just as with spaces, spectra have Postnikov towers and Postnikov truncations. For example, the Eilenberg-MacLane spectrum $\Sigma^2 H\mathbb{Z}/2\mathbb{Z}$ in (3.27) is the truncation of $ko$ which just keeps $\pi_2$. We introduce a richer truncation which keeps the first several homotopy groups

\[(4.2) \quad R := \pi_{\leq 4}ko = ko\langle 0 \cdots 4 \rangle.
\]

and kills all higher homotopy groups. This can be done \[B\] so that $R$ is a ring spectrum. Downshifts of $R$ have negative homotopy groups, which we truncate by taking connective covers. For example, we denote the connective cover of $\Sigma^{-1}R$ as $R^{-1}$. Just as we can consider ordinary cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$, there is a spectrum $R_{\mathbb{R}/\mathbb{Z}}$ which represents $R$-cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$. The Postnikov truncation $R$, its shifts, and their connective covers are all module spectra over the ring spectrum $ko$. We also introduce another module spectrum $E$, which we define below. For reference we record here the nonzero homotopy groups of these spectra:

\[
\begin{array}{cccccccc}
\pi_4 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 \\
\pi_3 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 \\
\pi_2 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{R}/\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} & 0 & 0 \\
\pi_1 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{R}/\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
\pi_0 & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
\end{array}
\]

This chart also displays the cohomology groups of a point: for any spectrum $h$ and $q \in \mathbb{Z}$ we have $h^{-q}(pt) \cong \pi_q h$.

With the exception of the spectrum $R$, the 0-space of each spectrum in (4.3) can be realized as the classifying space of a symmetric monoidal topological category. (Perhaps that is true for $R$ also, but we do not know any such model.) The objects in these categories are either lines or algebras which are $\mathbb{Z}/2\mathbb{Z}$-graded. As per custom we use ‘super’ as a synonym for ‘$\mathbb{Z}/2\mathbb{Z}$-graded’. For example, real metrized superlines—that is, real inner product spaces of dimension 1 with a $\mathbb{Z}/2\mathbb{Z}$ grading—are the objects of a symmetric monoidal category. Morphisms are degree-preserving isometries. The monoidal structure is tensor product, and the symmetry encodes the Koszul sign rule. Every object is invertible under tensor product—invertible vector spaces are lines—and every morphism is also invertible, thus superlines form a Picard groupoid. It is easy to see that there are two equivalence classes of objects—the even and odd line—and that the automorphism group of any object is cyclic of order 2. The classifying spectrum of this Picard groupoid appears in (4.3): it is $E^{-1}$. To
prove that statement requires\textsuperscript{8} checking the $k$-invariant between $\pi_0$ and $\pi_1$. Complex hermitian superlines also form a Picard groupoid. The group of equivalence classes of objects is $\pi_0 \cong \mathbb{Z}/2\mathbb{Z}$ and the automorphism group of any object is $\pi_1 \cong \mathbb{R}/\mathbb{Z}$, the latter realized via exponentiation as the group $\mathbb{T}$ of unit norm complex numbers. Now there are two choices: we can use the continuous topology or the discrete topology for the morphisms. The classifying spectrum with the continuous topology is $R^{-2}$ and with the discrete topology it is $R^{-3}_{\mathbb{R}/\mathbb{Z}}$. We term the latter ‘flat’ since over a space $S$ the abelian group $R^{-3}_{\mathbb{R}/\mathbb{Z}}(S)$ classifies flat super hermitian complex line bundles.

Fix a field $k$. There is a 2-category $\mathcal{C}$ whose objects are $k$-algebras, whose 1-morphisms are bimodules, and whose 2-morphisms are intertwiners. Invertible 1-morphisms are Morita equivalences, so $\mathcal{C}$ is sometimes called the Morita 2-category. We obtain a Picard 2-groupoid by keeping only invertible objects and morphisms. A basic theorem asserts that the invertible algebras are precisely the central simple algebras, and their equivalence classes make up the Brauer group of $k$. The $\mathbb{Z}/2\mathbb{Z}$-graded version was proved by Wall \cite{Wa}; see also \cite{De}. Now assume $k = \mathbb{R}$ or $k = \mathbb{C}$. Just as one can introduce metrics on lines, so too we can introduce “metrics” on invertible superalgebras and invertible supermodules, and they are used implicitly in the sequel to cut down groups of 2-automorphisms from $\mathbb{C}^\times$ to $\mathbb{T}$. We remark that every central simple superalgebra over $\mathbb{R}$ or $\mathbb{C}$ is Morita equivalent to a Clifford algebra.

The following table summarizes the Picard groupoids of superlines and Picard 2-groupoids of invertible superalgebras and their classifying spectra:

| spectrum        | Picard (2-)groupoid                  |
|-----------------|--------------------------------------|
| $R^{-1}$        | complex central simple superalgebras |
| $R^{-2}_{\mathbb{R}/\mathbb{Z}}$ | flat complex central simple superalgebras |
| $E$             | real central simple superalgebras    |
| $R^{-2}$        | complex superlines                  |
| $R^{-3}_{\mathbb{R}/\mathbb{Z}}$ | flat complex superlines             |
| $E^{-1}$        | real superlines                     |

The third line can be taken as a definition of $E$, but the other lines require proofs, which are fairly routine checks of homotopy groups and $k$-invariants.

For the spectra which appear in (4.4) the generalized cohomology groups of a space $S$ are equivalence classes of bundles of superlines or invertible superalgebras. Thus, for example, $E^{-1}(S)$ is the abelian group of real superline bundles up to equivalence. Bundles of superalgebras, however, do not suffice to realize all classes in $R^{-1}(S)$, for example. We should allow replacement of $S$ by a locally equivalent groupoid \cite[Appendix A]{FIIT} and take fiber bundles of invertible superalgebras, glued together using fiber bundles of invertible supermodules and invertible intertwiners. In this paper we only encounter global bundles of Clifford algebras, so do not need groupoid replacements.

\textsuperscript{8}given that $E$ has already been defined!
4.2. Some maps between $ko$-modules

Define $\eta$ as the nonzero element

\[(4.5) \quad \eta \in R^{-1}(pt) \cong ko^{-1}(pt)\]

and $\theta$ as a generator

\[(4.6) \quad \theta \in E^0(pt)\]

of the cyclic group $E^0(pt)$ of order 8. They can be represented by Clifford algebras on a 1-dimensional vector space. We use the same symbols $\eta, \theta$ for multiplication by these elements.

Let

\[(4.7) \quad \beta_{\mathbb{Z}}: R^{-q+1}_{\mathbb{R}/\mathbb{Z}} \to R^{-q}, \quad q \in \mathbb{Z},\]

be the connecting homomorphism derived from the fiber sequence $R \to R_\mathbb{R} \to R_{\mathbb{R}/\mathbb{Z}}$ of spectra. Finally, there is a complexification map

\[(4.8) \quad \gamma: E^{-q} \to R^{-2(q+2)}_{\mathbb{R}/\mathbb{Z}}, \quad q = 0, 1.\]

We interpret various multiplication and coboundary maps as geometric realizations of functors between the Picard groupoids (4.4) and also the symmetric monoidal groupoid of real vector spaces, whose classifying spectrum is $ko$.

**Proposition 4.9.**

(i) The assignment of the real Clifford algebra $\text{Cliff}(V)$ to a real vector space $V$ induces the spectrum map $\theta: ko \to E$, multiplication by (4.6).

(ii) The assignment of the complex Clifford algebra $\text{Cliff}^C(V)$ to a real vector space $V$ induces the spectrum map $\eta: ko \to R$, multiplication by (4.5).

(iii) The assignment of the complexification $A_C$ to a real central superalgebra $A$ induces the spectrum map $\gamma: E \to R^{-2}_{\mathbb{R}/\mathbb{Z}}$ in (4.8).

(iv) The assignment of the complexification $L_C$ to a real superline $L$ induces the spectrum map $\gamma: E^{-1} \to R^{-3}_{\mathbb{R}/\mathbb{Z}}$ in (4.8).

(v) The spectrum map $\beta_{\mathbb{Z}}: R^{-2}_{\mathbb{R}/\mathbb{Z}} \to R^{-1}$ forgets the flat structure.

(vi) The spectrum map $\beta_{\mathbb{Z}}: R^{-2}_{\mathbb{R}/\mathbb{Z}} \to R^{-2}$ forgets the flat structure.

(vii) The Postnikov truncation $ko^{-1} \to \pi_{\leq 1}ko^{-1} \cong E^{-1}$ is multiplication by $\theta$ in (4.6).

For example, if $S$ is a space and $L \to S$ a real line bundle over $S$ with equivalence class $[L] \in E^{-1}(S)$, then (iv) asserts that $\gamma[L] \in R^{-3}_{\mathbb{R}/\mathbb{Z}}(S)$ is the equivalence class of the complexification $L_C \to S$, which carries a natural flat structure, and (vi) asserts that $\beta_{\mathbb{Z}}\gamma[L] \in R^{-2}(S)$ is the equivalence class of the complex line bundle $L_C \to S$ if we disregard the flat structure. We remark that statement (iii) can be used as the definition of the map $\gamma$. 
Remark 4.10. There is one more theorem of this kind which is relevant here. According to \cite{ABS} elements of $ko^{-1}(pt)$ are represented by supermodules over the Clifford algebra $\text{Cliff}(\mathbb{R})$ with a single generator of square $-1$. To such a module $W^0 \oplus W^1$ we assign the real superline $\text{Det}(W^0)^*$, which is even or odd according to $\dim W^0 \pmod{2}$. This induces a map $ko^{-1} \to E^{-1}$. The theorem is that this map is the one in (vii). This construction relates (vii) to the Pfaffian superline bundle of a family of Dirac operators on 1-manifolds, if we use the Clifford linear Dirac operator \cite{LM}.

5. Supersymmetric QM with a general target

We revisit anomalies in supersymmetric QM, only now we drop Assumption 3.3. Thus the target $X$ is an arbitrary Riemannian manifold. Supersymmetric QM is still defined; the fields and lagrangian (3.1) are unchanged. It still makes sense to integrate out the fermionic field $\psi$ to obtain a relative theory. In this section we identify the anomaly theory $\alpha$, which is an invertible extended 2-dimensional topological field theory.

It is easiest to begin with the hamiltonian anomaly, which is the value of $\alpha$ on a point. The discussion in §3.2 carries over: the space of classical solutions is still the supersymplectic manifold (3.14). However, if $X$ is odd dimensional there is no polarization and if $X$ is not oriented there is no oriented polarization. Instead we consider quantization from the operator algebra viewpoint. Namely, at each point of $X$ the operator algebra in the fermionic system with field $\psi$ is the complex Clifford algebra $\text{Cliff}(T_x X)$. In the family of fermionic system parametrized by constant maps $\phi$ into $X$, the family of operator algebras is the bundle (3.17) of complex Clifford algebras. A quantization is a complex vector bundle $E \to X$ and an isomorphism of $\text{Cliff}(T X) \to X$ with the bundle of endomorphisms $\text{End}(E) \to X$ Furthermore, the Riemannian metric on $X$ induces a metric structure\footnote{We alluded to this type of metric structure before (4.4), but did not define it. It is something we expect in a unitary quantum field theory.} which in this case is flat and in fact is induced from the bundle (3.18) of real Clifford algebras. Applying (4.4) and Proposition 4.9 we conclude that the equivalence class of the flat bundle (3.17) of complex central simple superalgebras is

\[ [\text{Cliff}(TX)] = \gamma \theta[TX] \in R_{\mathbb{Z}/2}^2(X). \]

This is hamiltonian anomaly: the obstruction to finding the vector bundle $E \to X$.

We can also analyze the lagrangian anomaly. The real Pfaffian line bundle (3.2) is defined as in §3.1, but now it is $\mathbb{Z}/2\mathbb{Z}$-graded by the mod 2 index. (Under Assumption 3.3 the Pfaffian line bundle is even, so we did not encounter the $\mathbb{Z}/2\mathbb{Z}$-grading previously.) The following result is expressed in terms of transgression using (3.5). Recall that we are studying a family of real skew-adjoint Dirac operators on a spin 1-manifold $M$.

**Theorem 5.2.** The topological equivalence class in $E^{-1}(\mathfrak{F}'(M))$ of the lagrangian anomaly Pfaff $D \to \mathfrak{F}'(M)$ is $\theta(\pi_1), e^*\theta[TX]$.
Here \([TX] \in ko^0(X)\) is the \(ko\)-theory class of the tangent bundle of \(X\), and the pushforward \((\pi_1)_*\) in \(ko\)-theory uses the spin structure on \(M\).

**Proof.** The Atiyah-Singer topological index theorem \([\text{AS2}]\) identifies \((\pi_1)_*e^*[TX] \in ko^{-1}(\mathcal{F}(M))\) as the index of the family of Dirac operators. The Pfaffian line bundle is computed by the low-
est 2-stage Postnikov truncation of \(ko\), and Proposition 4.9(iv) implies that it is computed as multiplication by \(\theta\).

See Remark 4.10 for a more direct relationship between the Pfaffian line bundle and the \(ko\)-index.

**Remark 5.3.** By Proposition 4.9(iv) the equivalence class of the flat complex superline bundle obtained from the real Pfaffian superline bundle is

\[\gamma \theta (\pi_1)_*e^*[TX] \in R_{\mathbb{R}/\mathbb{Z}}^{-3}(\mathcal{F}(M)).\]

Motivated by (5.1) and (5.4) we posit a direct definition of the anomaly field theory \(\alpha\), as in §3.4. In this general case we have already seen in Theorem 5.2 that the spin structure on “spacetime” \(M\) enters, so we expect a theory on the bordism 2-category \(\text{SpinBord}_2\) of 0-, 1-, and 2-dimensional spin manifolds equipped with a map to \(X\). The Madsen-Tillmann spectrum \(\Sigma^2 \text{MT Spin}_2 \wedge X_+\) is its geometric realization, which now replaces (3.26), and we let \(\alpha\) take values in the spectrum \(\Sigma^{-2} R_{\mathbb{R}/\mathbb{Z}}\) whose 0-space classifies flat complex central simple superalgebras. Analogous to (3.27), we define \(\alpha\) as the composition

\[\alpha: \Sigma^2 \text{MT Spin}_2 \wedge X_+ \xrightarrow{\text{id} \wedge [TX]} \Sigma^2 \text{MT Spin}_2 \wedge \langle ko_0 \rangle_+ \xrightarrow{\theta \circ \text{Thom}} E \xrightarrow{\gamma} \Sigma^{-2} R_{\mathbb{R}/\mathbb{Z}} \]

We have chosen a map \(X \to ko_0\) into the 0-space of the \(K\)-theory spectrum which represents \([TX] \in ko^0(X)\). The second map in (5.5) is the composition of the Thom isomorphism in \(ko\)-theory for spin bundles \([\text{ABS}]\), a projection map, and multiplication by \(\theta\). Since \(\theta\) commutes with transgression—it is pulled back from a point—we can rewrite (5.5) by first acting by \(\gamma \theta\) and then applying the Thom isomorphism for the theory \(R_{\mathbb{R}/\mathbb{Z}}\):

\[\alpha: \Sigma^2 \text{MT Spin}_2 \wedge X_+ \xrightarrow{\text{id} \wedge [\gamma [TX]]} \Sigma^2 \text{MT Spin}_2 \wedge \langle (R_{\mathbb{R}/\mathbb{Z}})^{-2} \rangle_+ \xrightarrow{\text{Thom}} \Sigma^{-2} R_{\mathbb{R}/\mathbb{Z}} \]

Suppose \(W\) is a closed 2-manifold with spin structure \(\sigma\) and a smooth map \(\phi: W \to X\). We compute \(\alpha(W, \sigma, \phi) \in \mathbb{R}/\mathbb{Z}\). The map \(\gamma\) in (5.5) simply includes \(E^{-2}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \cong R_{\mathbb{R}/\mathbb{Z}}^{-4}(\text{pt})\) when evaluated on a 2-manifold. Furthermore, by the looping of Proposition 4.9(vii) the map \(\theta\) does nothing in this case. Hence we identify

\[\alpha(W, \sigma, \phi) = \pi^W_* \phi^*[TX] \in ko^{-2}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z},\]

where \(\pi^W: W \to \text{pt}\) and we pushforward in \(ko\)-theory using the spin structure on \(W\). The Atiyah-Singer index theorem identifies this as the mod 2 index of the Dirac operator on \(W\) coupled to \(\phi^*TX\).

Denote \(\text{Arf}_W(\sigma) = \pi^W_* (1) \in \mathbb{Z}/2\mathbb{Z}\), where \(1 \in ko^0(W)\) is the unit. \(\text{Arf}_W\) is a quadratic function on spin structures \([\text{At3}]\). Set \(w_q = w_q(X), q = 1, 2.\)
Proposition 5.8. $\alpha(W, \sigma, \phi) = (\dim X) \text{Arf}_W(\sigma) + \text{Arf}_W(\sigma + \phi^*w_1) + \text{Arf}_W(\sigma) + \langle \phi^*w_2, [W] \rangle$.

Proof. Write $[TX] = \dim X + ([TX] - \dim X)$ to pick off the first term and reduce to evaluating the $ko$-pushforward on a virtual bundle of rank zero. That bundle can be written as $[L_w] - 1$ plus a class $z \in ko^0(W)$ of rank zero with vanishing first Stiefel-Whitney class, where $L_w \to W$ is the real line bundle with Stiefel-Whitney class $w = \phi^*(w_1)$. The pushforward of $[L_w]$ in spin structure $\sigma$ equals the pushforward of 1 in spin structure $\sigma + w$. This explains the middle two terms in the formula. Finally, the class $z$ can be represented by a map from $W$ into the 2-skeleton of $ko_0$ which is trivial on the first Postnikov section, so a map into $K(\mathbb{Z}/2\mathbb{Z}, 2)$. That map can be taken to be trivial off of a ball in $W$, and since the $ko$-pushforward of that class is easily seen to be independent of spin structure, by the bordism invariance of the pushforward we can replace $W$ by a 2-sphere. Now the pushforward is the suspension isomorphism, and the last term in the formula results. □

Remark 5.9. The cobordism hypothesis [L] asserts that the extended topological field theory $\alpha$ is determined by its value (5.1) on a point. But the cobordism hypothesis is overkill for an invertible topological theory as we can define it directly by specifying the map (5.6) (or equivalently (5.5)) of spectra.

Finally, we discuss trivializations of the anomaly theory $\alpha$. As described at the end of §3.3 a spin structure on $X$ trivializes the hamiltonian anomaly as long as $X$ is even-dimensional. That still applies to (5.1). If $X$ is odd-dimensional and spin, then the spin structure induces a Morita isomorphism of the flat bundle (3.17) of complex Clifford algebras with the constant bundle whose fiber is the complex Clifford algebra $\text{Cliff}_C^C = \text{Cliff}^C(\mathbb{R})$ on a single generator. We can interpret this as saying that the bundle of Hilbert spaces over $X$ obtained by quantizing $\psi$ is naturally a bundle of $\text{Cliff}_1^C$-modules. Furthermore, quantizing $\phi$ we see that the Hilbert space of supersymmetric QM is also naturally a $\text{Cliff}_1^C$-module. Should we say that the theory is anomalous, or allow that the Hilbert space of a quantum theory can be a $\text{Cliff}_1^C$-module? I opt for the latter.

Remark 5.10. This fixes a well-known problem about fermions on an odd-dimensional manifold. For example, path integral arguments [Detal, p. 682] suggest that the dimension of the Hilbert space is an integer multiple of $\sqrt{2}$ if $\dim X$ is odd. We see here that the Hilbert space is naturally a $\text{Cliff}_1^C$-module, which resolves this puzzle with the path integral.

We can see directly from (5.6) the effect of a spin structure on $X$ on the entire anomaly theory $\alpha$. A spin structure from this point of view is a homotopy of the map $X \to E_0$ representing $\theta[TX]$ to a constant map into some component of the 0-space of $E_0$. Running this homotopy through the composition (5.6) we obtain a homotopy from $\alpha$ to either (i) the trivial theory if $\dim X$ is even, or (ii) a particularly simple 2-dimensional invertible extended topological theory $\alpha'$ if $\dim X$ is odd. The theory $\alpha'$ assigns the Clifford algebra $\text{Cliff}_C^C$ to a point; the even or odd line to a spin circle, depending on whether the spin structure bounds or not; and the Arf invariant $\text{Arf}_W(\sigma)$ to a closed spin 2-manifold $W$ with spin structure $\sigma$. (See [G] for more on $\alpha'$ together with an interesting geometric application.)
References

[ABS] M. F. Atiyah, R. Bott, and A. A. Shapiro, Clifford modules, Topology 3 (1964), 3–38.

[Ag] Luis Alvarez-Gaumé, Supersymmetry and the Atiyah-Singer index theorem, Commun. Math. Phys. 90 (1983), 161.

[AgG] L. Alvarez-Gaumé and P. Ginsparg, The topological meaning of nonabelian anomalies, Nuclear Phys. B 243 (1984), no. 3, 449–474.

[AgW] L. Alvarez-Gaumé and E. Witten, Gravitational anomalies, Nucl. Phys. B234 (1983), 269.

[AS1] M. F. Atiyah and I. M. Singer, Dirac operators coupled to vector potentials, Proc. Nat. Acad. Sci. U.S.A. 81 (1984), no. 8, Phys. Sci., 2597–2600.

[AS2] [AgW], The index of elliptic operators. IV. Ann. of Math. (2) 93 (1971), 119–138.

[ASe] M. F. Atiyah and G. B. Segal, Twisted K-theory, Ukr. Mat. Visn. 1 (2004), no. 3, 287–330.

[ASZ] O. Alvarez, I. M. Singer, and B. Zumino, Gravitational anomalies and the family’s index theorem, Communications in Mathematical Physics 96 (1984), no. 3, 409–417.

[At1] M. F. Atiyah, Circular symmetry and stationary-phase approximation, Astérisque (1985), no. 131, 43–59. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983).

[At2] [At1], K-Theory, second ed., Advanced Book Classics, Addison-Wesley, Redwood City, CA, 1989.

[At3] [At2], Riemann surfaces and spin structures, Ann. Sci. École Norm. Sup. (4) 4 (1971), 47–62.

[Ay] David Ayala, Geometric cobordism categories, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Stanford University.

[B] M. Basterra, André-Quillen cohomology of commutative S-algebras, J. Pure Appl. Algebra 144 (1999), no. 2, 111–143.

[Be] Reinhold A Bertlmann, Anomalies in quantum field theory, vol. 91, Oxford University Press on Demand, 2000.

[Bi] Jean-Michel Bismut, The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem, J. Funct. Anal. 57 (1984), no. 1, 56–99.

[De] Pierre Deligne, Notes on spinors, Quantum Fields and Strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 99–135.

[Detal] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, Jacques Distler, Daniel S. Freed, and Gregory W. Moore, Irving M. Singer, and Edward Witten, Quantum fields and strings: a course for mathematicians. Vol. 1, 2, American Mathematical Society, Providence, RI, 1999. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997.

[DFM] Jacques Distler, Daniel S. Freed, and Gregory W. Moore, Spin structures and superstrings, Surveys in differential geometry. Volume XV. Perspectives in mathematics and physics, Surv. Differ. Geom., vol. 15, International Press, Somerville, MA, 2011, pp. 99–130. arXiv:0906.0795.

[DM] Pierre Deligne and John W. Morgan, Notes on supersymmetry (following Joseph Bernstein), Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 41–97.

[F1] Daniel S. Freed, Determinants, torsion, and strings, Comm. Math. Phys. 107 (1986), no. 3, 483–513.

[F2] [F1], K-theory in quantum field theory, Current developments in mathematics, 2001, Int. Press, Somerville, MA, 2002, pp. 41–87. math-ph/0206031.

[F3] [F2], On determinant line bundles, Mathematical Aspects of String Theory (S. T. Yau, ed.), Advanced Series in Mathematical Physics, vol. 1, 1986, pp. 189–238.

[Fa] L. D. Faddeev, Hamiltonian approach to the theory of anomalies, Recent Developments in Mathematical Physics (H. Mitter and L. Pittner, eds.), Internationale Universitatswoche fur Kernphysik, Schladming, Austria, vol. 26, 1987, pp. 137–159.

[FHT] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, Loop groups and twisted K-theory I, J. Topol. 4 (2011), no. 4, 737–798, arXiv:0711.1906.

[FM] Daniel S. Freed and Gregory W. Moore, Setting the quantum integrand of M-theory, Comm. Math. Phys. 263 (2006), no. 1, 89–132, arXiv:hep-th/0409135.

[FT] Daniel S. Freed and Constantin Teleman, Relative quantum field theory, Comm. Math. Phys. 326 (2014), no. 2, 459–476, arXiv:1212.1692.

[FW] Daniel S. Freed and Edward Witten, Anomalies in string theory with D-branes, Asian J. Math. 3 (1999), no. 4, 819–851, hep-th/9907189.

[FWi] D. Friedan and Paul Windey, Supersymmetric derivation of the Atiyah-Singer index and the chiral anomaly, Nucl. Phys. B235 (1984), 395.

[G] Sam Gunningham, Spin Hurwitz numbers and topological quantum field theory, arXiv:1201.1273.
[GMTW] Søren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss, The homotopy type of the cobordism category, Acta Math. 202 (2009), no. 2, 195–239, arXiv:math/0605249.

[HS] M. J. Hopkins and I. M. Singer, Quadratic functions in geometry, topology, and M-theory, J. Diff. Geom. 70 (2005), 329–452, arXiv:math/0211216.

[L] Jacob Lurie, On the classification of topological field theories, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. arXiv:0905.0465.

[LM] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, Spin geometry, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.

[NAg] Philip Nelson and Luis Alvarez-Gaumé, Hamiltonian interpretation of anomalies, Comm. Math. Phys. 99 (1985), no. 1, 103–114.

[Se1] G. B. Segal, Felix Klein Lectures 2011. http://www.mpim-bonn.mpg.de/node/3372/abstracts.

[Se2] ———, Faddeev’s anomaly in Gauss’s law. preprint.

[Se3] ———, Categories and cohomology theories, Topology 13 (1974), 293–312.

[ST] Stephan Stolz and Peter Teichner, What is an elliptic object?, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 247–343.

[W1] Edward Witten, Global anomalies, Supersymmetry and its applications: superstrings, anomalies and supergravity (Cambridge, 1985), Cambridge Univ. Press, Cambridge, 1986, pp. 21–27.

[W2] ———, Global anomalies in string theory, Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), World Sci. Publishing, Singapore, 1985, pp. 61–99.

[W3] ———, Index of Dirac operators, Quantum Fields and Strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 475–512.

[Wa] C. T. C. Wall, Graded Brauer groups, J. Reine Angew. Math. 213 (1963/1964), 187–199.

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