Phase transition of a scalar field theory at high temperatures

Hidenori SONODA
Physics Department, Kobe University, Kobe 657-8501, Japan
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Abstract

At high temperatures a four dimensional field theory is reduced to a three dimensional field theory. In this letter we consider the $\phi^4$ theory whose parameters are chosen so that a thermal phase transition occurs at a high temperature. Using the known properties of the three dimensional theory, we derive a non-trivial correction to the critical temperature.

At very high temperatures the thermal properties of a 3+1 dimensional field theory are given by an effective three dimensional field theory. In this letter we use an effective field theory to discuss the thermal phase transition of the $\phi^4$ theory at a high temperature. The idea is simple: we choose the parameters of the $\phi^4$ theory in such a way that the phase transition occurs at a high temperature for which the three dimensional reduction is a good approximation. We can then use the known properties of the three dimensional $\phi^4$ theory to discuss the physics near and at the transition temperature. In ref. [1] the main interest was in computing the free energy density of the massless theory at high temperatures, and the thermal phase transition of a massive theory was not discussed. The method of effective field theory has been used extensively to discuss the thermal phase transitions of the electroweak theory. [2] Applied to the simpler $\phi^4$ theory, the method can give non-trivial results more easily because the three dimensional $\phi^4$ theory is much better understood than the gauge theories.

The four dimensional theory is defined by the lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 + \text{counterterms} , \quad (1)$$

1E-mail: sonoda@phys.sci.kobe-u.ac.jp
where the counterterms are chosen in the $\overline{\mathrm{MS}}$ scheme with $\bar{\mu} = 1$. We choose $m^2$ to be negative so that the $Z_2$ symmetry is broken spontaneously at zero temperature. Then a continuous transition occurs approximately at temperature $T_c \simeq \sqrt{-\frac{24m^2}{\lambda}}$. The $Z_2$ symmetry is restored at temperatures $T > T_c$. Observe that we can choose $-m^2$ to be of order $T_c$ and $\lambda$ of order $\frac{1}{T_c}$. If we take $T_c$ large, i.e., the coupling $\lambda$ small, so that the physical mass $m_{ph}$ is much smaller than $T_c$, we can reduce the four dimensional theory to an effective three dimensional theory, and we can use the well-known results on the phase transition of the three dimensional $\phi^4$ theory. Near the transition, $m_{ph}$ is small, and the condition $m_{ph} \ll T_c$ is bound to be satisfied.

The three-dimensional $\phi^4$ theory is defined by the lagrangian

$$L_{eff} = \frac{1}{2} \partial_\mu \Phi \partial_\mu \Phi + \frac{m_3^2}{2} \Phi^2 + \frac{\lambda_3}{4!} \Phi^4 + \frac{\delta m_3^2}{2} \Phi^2,$$

where the spatial dimension is $3 - \epsilon$, and the renormalization is done in the $\overline{\mathrm{MS}}$ scheme with

$$\delta m_3^2 = \frac{1}{2\epsilon} C \lambda_3^2,$$

and

$$C = -\frac{1}{6(4\pi)^2}.$$

The parameters $m_3^2$ and $\lambda_3$ are related to the temperature $T$ and the parameters of the four dimensional theory as follows:

$$\lambda_3 = T \left( \lambda + O(\lambda^2) \right)$$

$$m_3^2 = m^2 \left( 1 + \frac{\lambda}{(4\pi)^2} (\ln T + k) + O(\lambda^2) \right)$$

$$+ \frac{\lambda T^2}{24} \left( 1 + \frac{\lambda}{(4\pi)^2} (-\ln T + k - 2\Delta) + O(\lambda^2) \right),$$

where the constants are given by

$$k = \ln 4\pi - \gamma , \quad \Delta = \ln 4\pi - 1 - \frac{\zeta'(-1)}{\zeta(-1)} \simeq -0.454 .$$

($\gamma \simeq 0.577$ is Euler’s constant, and $\zeta(z)$ is Riemann’s zeta function.) Note that with the choice

$$m^2 = O(T) , \quad \lambda = O \left( T^{-1} \right) ,$$

2The calculations in ref. [1] largely depend on the earlier calculations in ref. [4].
the above approximations (tree for \( \lambda_3 \), one-loop for the \( m^2 \) term in \( m_3^2 \), and two-loop for the \( T^2 \) term) give all the contributions which survive the limit \( T \to \infty \): the higher order corrections vanish as \( T \to \infty \).

The parameters \( \lambda_3 \) and \( m_3^2 \) of the three dimensional theory satisfy the following renormalization group (RG) equations:

\[
\frac{d}{dt} \lambda_3 = \lambda_3, \quad \frac{d}{dt} m_3^2 = 2m_3^2 + C\lambda_3^2.
\]  

(9)

Note that these are consistent with eqns. (5, 6) and the one-loop RG equations of the four-dimensional theory:

\[
\frac{d}{dt} \lambda \simeq -\frac{3\lambda^2}{(4\pi)^2}, \quad \frac{d}{dt} m^2 \simeq \left(2 - \frac{\lambda}{(4\pi)^2}\right)m^2.
\]  

(10)

\((\frac{d}{dt} \ T = T \ by \ convention.)\)

Let us summarize what is known about the phase transition of the three dimensional \( \phi^4 \) theory. (See [5], especially chapter 8.) Given \( \lambda_3 \), the \( Z_2 \) symmetry is exact for \( m_3^2 > m_3^2_c \), and it is spontaneously broken for \( m_3^2 < m_3^2_c \). Whether the symmetry is broken or not must be determined by a RG invariant criterion. Using \( \lambda_3 \) and \( m_3^2 \), we can construct only one independent RG invariant which can be chosen as

\[
R(m_3^2, \lambda_3) \equiv \frac{m_3^2 - C\lambda_3^2 \ln \lambda_3}{\lambda_3^2},
\]  

(11)

where the constant \( C \) is given by eqn. (3). It is trivial to check the RG invariance of \( R \) using eqns. (10). Using \( R \) we can rephrase the criterion for the transition: the symmetry is exact for \( R > R_c \), and broken for \( R < R_c \), where \( R_c \) is a constant. This implies that

\[
m_3^2_c = \lambda_3^2(R_c + C \ln \lambda_3).
\]  

(12)

The constant \( R_c \) has not been calculated analytically.

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**Figure.** RG flows of the three dimensional theory

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The critical temperature \( T_c \) of the four dimensional theory can be obtained from the condition \( m_3^2 = m_{3c}^2 \). Substituting eqn. (6) into this, we obtain the following expression for the critical temperature

\[
T_c^2 = \frac{-24m^2}{\lambda} \left[ 1 + \frac{\lambda}{(4\pi)^2} \left( -\ln(-24m^2) - 3 \ln \lambda + 24(4\pi)^2 R_c + 2\Delta \right) + O(\lambda^2) \right],
\]

where \( \Delta \) is given in eqns. (10). We observe the non-trivial logarithmic dependence on \( \lambda \).

For completeness let us add a few comments about the critical behavior of the theory. (Again a good reference is ref. [5].) Clearly the critical thermal behavior of the four dimensional theory is the same as the critical behavior of the three dimensional theory. In particular, all the critical exponents are the same. For example, near \( T = T_c \) the physical mass \( m_{ph} \) of the theory behaves as

\[
m_{ph}^2 \simeq \text{const} \cdot \left( R - R_c \right)^{-\gamma/E} \simeq \left( T_c - T \right)^{-\gamma/E},
\]

where \( \gamma/E \) is the scale dimension of the relevant parameter at the non-trivial fixed point. The approximate value of \( \gamma/E \) has been calculated by various methods: for example, the one-loop Callan-Symanzik equation gives [6]

\[
\gamma/E \simeq \frac{5}{3}.
\]

Similarly, at the critical temperature, the two-point thermal correlation function (Matsubara function) of \( \phi \) behaves as

\[
\langle \phi(r)\phi(0) \rangle_{T = T_c} \simeq T_c \langle \Phi(r)\Phi(0) \rangle_{m_3^2 = m_{3c}^2} \simeq \text{const} \cdot \left( \frac{\lambda^{-\gamma} T_c^{1-\eta}}{r^{1+\eta}} \right),
\]

where the anomalous dimension \( \eta \) is about .05.

How can we improve the high temperature approximation? At higher orders we must not only calculate the next loop order terms in eqns. (5,6), but also we must introduce irrelevant terms such as \( \Phi^6 \) (dimension three) and \( \Phi^2 \partial_{\mu}\Phi\partial_{\nu}\Phi \) (dimension four) in the three dimensional lagrangian. The calculations will be significantly more complicated. A simple analysis shows that the parameter of \( \Phi^6 \) is of order \( \frac{1}{T^4} \), and that of \( \Phi^2 \partial_{\mu}\Phi\partial_{\nu}\Phi \) is of order

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3This is similar to the appearance of the logarithm of \( \lambda_3 \) in the three dimensional massless theory.

4Irrelevant with respect to the fixed point at \( \lambda_3 = m_3^2 = 0 \) as opposed to the non-trivial infrared fixed point.
Therefore, we still do not need to introduce irrelevant terms at the next order, but at the next next order we must introduce the term $\Phi_2 \partial_\mu \Phi_2 \partial_\mu \Phi_2$.

The above calculation of the critical temperature can be easily extended to the four dimensional $O(N)$ linear sigma model whose lagrangian is given by

$$L = \frac{1}{2} \sum_{I=1}^{N} \partial_\mu \phi^I \partial_\mu \phi^I + \frac{m^2}{2} \sum_{I=1}^{N} \phi^I \phi^I + \frac{\lambda}{8} \left( \sum_{I=1}^{N} \phi^I \phi^I \right)^2 + \text{counterterms} \quad (17)$$

The critical temperature is obtained as

$$T_c^2 = \frac{-24 m^2}{(N+2)\lambda} \left[ 1 + \frac{3\lambda}{(4\pi)^2} \left\{ -\ln(-24 m^2) 
- 3 \ln((N+2)\lambda) + 8(N+2)(4\pi)^2 R_{N,c} + 2\Delta \right\} + O(\lambda^2) \right], \quad (18)$$

where the constant $R_{N,c}$ is the value of the RG invariant $R_N(m_3^2, \lambda_3) \equiv \frac{m_3^2}{(N+2)^2\lambda_3^2} - C_N \ln((N+2)\lambda_3)$

at the critical point ($C_N \equiv -\frac{1}{2(N+2)(4\pi)^2}$). In the large $N$ limit we find $R_{N,c} = 1/(8\pi^2)$.

In this letter we have computed the critical temperature of the four dimensional $\phi^4$ theory to the next leading order in the small coupling constant $\lambda$ using the effective three dimensional theory.

References

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