ON STRUCTURE OF $L^2$-HARMONIC FUNCTIONS FOR ONE-DIMENSIONAL DIFFUSIONS

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ABSTRACT. In this note we analyse the harmonic functions in $L^2$-sense for an irreducible diffusion on an interval.

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1. DIRICHLET FORMS ASSOCIATED WITH ONE-DIMENSIONAL DIFFUSIONS

Let $I := [l, r)$ be an interval where $l$ or $r$ may or may not be in $I$ and $m$ be a positive Radon measure on $I$ with full topological support. The notation $m(l+) < \infty$ (resp. $m(r-) < \infty$) means that for some $\varepsilon > 0$, $m((l, l + \varepsilon)) < \infty$ (resp. $m((r - \varepsilon, r)) < \infty$). Otherwise write $m(l+) = \infty$ (resp. $m(r-) = \infty$). When $l = -\infty$ (resp. $r = \infty$), $l + \varepsilon$ (resp. $r - \varepsilon$) in this meaning is replaced by some constant in $(l, r)$. Clearly, if $l \in I$ (resp. $r \in I$), then $m(l+) < \infty$ (resp. $m(r-) < \infty$). Fix a point $e \in \bar{I} := (l, r]$ and without loss of generality assume that $m(\{e\}) = 0$. Denote by

$$S(I) := \{s : I \to \mathbb{R} : s \text{ is continuous and strictly increasing, } s(e) = 0\}$$

the family of all scale functions on $I$. Define

$$s(l) := \lim_{x \downarrow l} s(x) \geq -\infty, \quad s(r) := \lim_{x \uparrow r} s(x) \leq \infty.$$ 

Set $\bar{I} := [l, r]$ to be the interval containing the boundary points even if $l$ or $r$ is infinite. If a function $f$ is undefined in $l$ or $r$, we understand $f(l)$ or $f(r)$ the limit $\lim_{x \to l \text{ or } r} f(x)$. Take $s \in S(\bar{I})$ and set for $x \in \bar{I}$,

$$\sigma(x) := \int_e^x \int_{e}^{\xi} m(\eta)ds(\xi), \quad \mu(x) := \int_e^x \int_{e}^{\xi} ds(\eta)m(d\xi).$$

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Remark scale function on \( J \).

Definition 1.1. Given \( s \in \mathcal{S}(\bar{I}) \), \( l \) (resp. \( r \)) is called (with respect to \( (s, m) \))

1. regular, if \( \sigma(l), \mu(l) < \infty \) (resp. \( \sigma(r), \mu(r) < \infty \));

2. exit, if \( \sigma(l) < \infty, \mu(l) = \infty \) (resp. \( \sigma(r) < \infty, \mu(r) = \infty \));

3. entrance, if \( \sigma(l) = \infty, \mu(l) < \infty \) (resp. \( \sigma(r) = \infty, \mu(r) < \infty \));

4. natural, if \( \sigma(l) = \mu(l) = \infty \) (resp. \( \sigma(r) = \mu(r) = \infty \)).

In addition, \( l \) (resp. \( r \)) is called absorbing, if \( l \) (resp. \( r \)) is regular and \( l \notin J \) (resp. \( r \notin J \)). It is called reflecting, if it is regular and contained in \( J \).

Remark 1.2. Note that \( r \) is regular, if and only if \( s(r) < \infty \) and \( m(r) < \infty \). If \( r \) is exit, then \( s(r) < \infty \) and \( m(r) = \infty \). If \( r \) is entrance, then \( s(r) = \infty \) and \( m(r) < \infty \). If \( r \) is natural, then at least one of \( m(r) \) and \( s(r) \) must be \( \infty \).

The endpoint \( l \) or \( r \) is called approachable if \( s(l) > -\infty \) or \( s(r) < \infty \). Given a function \( f \) on \( I \), \( f \ll ds \) means that \( f \) is absolutely continuous with respect to \( ds \), i.e. there exists an absolutely continuous function \( g \) on \( s(I) = \{ s(x) : x \in I \} \) such that \( f = g \circ s \). Meanwhile \( df/ds := g' \circ s \). Note that if \( l \) or \( r \) is approachable, then any function \( f \) with \( f \ll ds \) and \( df/ds \in L^2(I, ds) \) admits the finite limit \( f(l) := \lim_{x \downarrow l} f(x) \) or \( f(r) := \lim_{x \uparrow r} f(x) \). Particularly \( f \in C([l, r]) \) or \( f \in C((l, r)) \); see [1] §2.2.3.

What we are concerned with is a regular and irreducible Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(I, m) \). To state its representation in the following lemma, denote by \( \mathcal{S}(\bar{I}) \) the family of all scale functions \( s \in \mathcal{S}(\bar{I}) \) satisfying the condition: For \( j = l \) or \( r \), if \( j \in I \) and \( m(\{j\}) > 0 \), then \( |s(j)| < \infty \).

Lemma 1.3. Let \( I, m \) be given as above. Then \((\mathcal{E}, \mathcal{F})\) is a regular, irreducible and strongly local Dirichlet form on \( L^2(I, m) \), if and only if there exists a unique scale function \( s \in \mathcal{S}(\bar{I}) \) such that

\[
\mathcal{F} = \{ f \in L^2(I, m) : f \ll ds, df/ds \in L^2(I, ds), f(j) = 0 \text{ if } j \text{ is absorbing for } j = l \text{ or } r \},
\]

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_I \frac{df}{ds} \frac{dg}{ds} ds, \quad f, g \in \mathcal{F}.
\]

Proof. Necessity. Let \((\mathcal{E}, \mathcal{F})\) be such a Dirichlet form. Applying [4] Theorem 2.1 and the irreducibility, we can obtain a unique interval \( J \subset I \), called effective interval, and a unique adapted scale function on \( J \) in the sense of [4] (2.2) representing \((\mathcal{E}, \mathcal{F})\). Note that \( J \) must be ended by \( l \) and \( r \), because otherwise \( I \setminus J \) would become a non-trivial \( m \)-invariant set (with respect to \((\mathcal{E}, \mathcal{F})\)) as violates the irreducibility. Denote by the adapted scale function on \( J \) by \( s \). Then \( s \in \mathcal{S}(\bar{I}) \). To show \( s \in \mathcal{S}(\bar{I}) \), argue by contradiction and suppose that \( r \in I, m(\{r\}) > 0 \) and \( s(r) = \infty \). The adaptedness of \( s \) indicates that \( r \notin J \). Hence \( \{r\} \) is a non-trivial \( m \)-invariant set, as leads to a contradiction. Therefore \( s \in \mathcal{S}(\bar{I}) \) and the expression of \((\mathcal{E}, \mathcal{F})\) can be obtained by [4] Theorem 2.1.

Sufficiency. Let \( J := (l, r) \), where \( j \in J \) if and only if \( j \) is reflecting with respect to \((s, m)\) for \( j = l \) or \( r \). Then \( J \subset I \) and \( s \) is adapted to \( J \). By [4] Theorem 2.1, \((\mathcal{E}, \mathcal{F})\) gives a regular and strongly local Dirichlet form on \( L^2(I, m) \) with the effective interval \( J \) and adapted scale function \( s \). The condition \( s \in \mathcal{S}(\bar{I}) \) implies that \( m(\bar{J}) = 0 \) and hence \((\mathcal{E}, \mathcal{F})\) is irreducible. That completes the proof. □
Remark 1.4. The associated Markov process $X = (X_t)_{t \geq 0}$ of $(\mathscr{E}, \mathscr{F})$ is a diffusion process on $I$ with no killing inside whose scale function is $s$ and speed measure is $m$; see, e.g., [6 V§7].

From now on we denote by $I_e$ the effective interval of $(\mathscr{E}, \mathscr{F})$ as described in [4, §2.3]. More precisely, $I \subset I_e \subset I$, and $j \in I_e$ if and only if $j$ is reflecting for $j = t$ or $r$. In addition, $j \in I \setminus I_e$ implies that $|s(j)| = \infty$ and $m(\{j\}) = 0$. Any singleton contained in $I_e$ is of positive capacity and $I \setminus I_e$ is $\mathscr{E}$-polar. Particularly, $\mathscr{F} \subset C(I_e)$. The family
\[ \mathcal{C}_I := \{ \varphi \circ s : \varphi \in C^\infty_c(s(I_e)) \} \tag{1.2} \]
is a special standard core of $(\mathscr{E}, \mathscr{F})$.

Notations. Given an interval $J$, $C(J)$, $pC(J)$ and $C^\infty_c(J)$ stand for the families of all continuous functions, all non-negative continuous functions and all smooth functions with compact support on $J$ respectively.

2. Solutions of harmonic equation

Given a constant $\alpha > 0$, consider the following equation
\[ \frac{1}{2} \frac{d}{ds} \frac{du}{ds}(x) = \alpha u(x), \quad x \in \tilde{I}. \tag{2.1} \]
A solution of (2.1) means $u \in L^1_{loc}(\tilde{I}, m)$ such that $u \ll ds$ and a $ds$-a.e. version $v$ of $du/ds$ satisfies
\[ 2\alpha \int_{(x, y]} u(\xi)m(d\xi) = v(y) - v(x), \quad \forall x, y \in \tilde{I}, x \neq y. \tag{2.2} \]
Note that (2.2) indicates that $v$ is right continuous and $v(r) := \lim_{x \uparrow r} v(x)$ (resp. $v(l) := \lim_{x \downarrow l} v(x)$) is well defined if $u|_{[x, r]} \in L^1([x, r], m)$ (resp. $u|_{[l, x]} \in L^1([l, x], m)$).

This section is devoted to presenting two particular solutions of (2.1) as obtained in [5 Chapter II]; see also [3 §5.12]. Set, for $n = 0, 1, 2, \cdots, x \in \tilde{I},$
\[ u^0(x) \equiv 1, \quad u^1(x) = \sigma(x), \quad u^{n+1}(x) = \int_x^y u^n(\xi)m(d\xi)ds(y). \]
Denote
\[ u(x) = \sum_{n=0}^{\infty} \alpha^n u^n(x), \quad x \in \tilde{I}, \]
and introduce the functions
\[ u_+(x) = u(x) \int_x^r u(y)^{-2}ds(y), \quad x \in \tilde{I} \]
\[ u_-(x) = u(x) \int_l^x u(y)^{-2}ds(y), \quad x \in \tilde{I}. \]
The following lemma due to [5] is crucial to our treatment.

Lemma 2.1. \hspace{1cm} \begin{enumerate}
\item $1 + \sigma(x)\alpha \leq u(x) \leq \exp(\alpha \sigma(x))$. For $x > e$ (resp. $x < e$),
\[ \int_x^r u(y)^{-2}ds(y) \leq \frac{1}{\alpha(1 + \alpha \sigma(x))m((e, x])}, \]
\[ \left( \text{resp.} \int_l^x u(y)^{-2}ds(y) \leq \frac{1}{\alpha(1 + \alpha \sigma(x))m((x, e])} \right). \]
\item $u, u_\pm \in pC(\tilde{I})$ are solutions of (2.1). \end{enumerate}
Theorem 3.1. \( u_+ \) is decreasing. If \( r \) is not entrance, then \( u_+(r) = 0 \). \( du_+/ds \) is increasing and if \( r \) is entrance or natural, then \( du_+/ds(r) = 0 \).

(4) \( u_- \) is increasing in \( x \). If \( l \) is not entrance, then \( u_-(l) = 0 \). \( du_-/ds \) is increasing in \( x \) and if \( l \) is entrance or natural, then \( du_-/ds(l) = 0 \).

\[ \square \]

3. Analytic treatment of harmonic functions

Let \( (\mathcal{E}^0, \mathcal{F}^0) \) be the part Dirichlet form of \((\mathcal{E}, \mathcal{F})\) on \( \hat{I} \), i.e.
\[
\mathcal{F}^0 := \{ f \in \mathcal{F} : f = 0 \text{ q.e. on } I \setminus \hat{I} \},
\]
\[
\mathcal{E}^0(f, g) := \mathcal{E}(f, g), \quad f, g \in \mathcal{F}^0.
\]

It is a regular Dirichlet form on \( L^2(\hat{I}, m) \) having a special standard core
\[
\mathcal{E}^0 := \{ \varphi \circ s : \varphi \in C^\infty_c(s(\hat{I})) \}.
\]

For any \( \alpha > 0 \), \( \mathcal{F}^0 \) is a closed subspace of \( \mathcal{F} \) under the norm \( \| \cdot \|_{\mathcal{E}_\alpha} \) and the following direct product decomposition holds true:
\[
\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{H}_\alpha,
\]
where \( \mathcal{H}_\alpha := \{ f \in \mathcal{F} : \mathcal{E}_\alpha(f, g) = 0, \forall g \in \mathcal{F}^0 \} \). The function in \( \mathcal{H}_\alpha \) is called \( \alpha \)-harmonic.

We note that \( \mathcal{F}^0 = \mathcal{F} \) if and only if \( I \setminus \hat{I} \) is \( \mathcal{E} \)-polar, i.e. neither \( l \) nor \( r \) is reflecting, or equivalently, \( I_e = \hat{I} \). In this case \( \mathcal{H}_\alpha = \{ 0 \} \) for all \( \alpha > 0 \). When \( r \) is reflecting (resp. \( l \) is reflecting), Lemma 2.1(1) yields that \( u_-(r) \in (0, \infty) \) (resp. \( u_+(l) \in (0, \infty) \)). Meanwhile set
\[
u^\alpha_l(x) := \frac{u_-(x)}{u_-(r)}, \quad x \in (l, r];
\]
\[\text{resp. } \nu^\alpha_r(x) := \frac{u_+(x)}{u_+(l)}, \quad x \in [l, r)\] (3.1)

It is worth pointing out that \( u_+ \) and \( u_- \) are linear independent; see [5, II, §3#5]. The result below characterizes the \( \alpha \)-harmonic functions.

**Theorem 3.1.** For \( \alpha > 0 \), the following holds:

\[
\mathcal{H}_\alpha = \begin{cases} 
\{0\}, & \text{when } I_e = (l, r); \\
\text{span}\{u^\alpha_l\}, & \text{when } I_e = (l, r]; \\
\text{span}\{u^\alpha_r\}, & \text{when } I_e = [l, r); \\
\text{span}\{u^\alpha_l, u^\alpha_r\}, & \text{when } I_e = [l, r].
\end{cases}
\]

**Proof.** Only the case \( I_e = (l, r] \) will be treated, and the other cases can be proved by a similar way.

We first show that \( u^\alpha_l \in \mathcal{F} \). To do this, take \( \varepsilon > 0 \). On account of Lemma 2.1(2), \( u^\alpha_l \) is a solution of (2.1). Denote the \( ds \)-a.e. version of \( du^\alpha_l/\text{d}s \) satisfying (2.2) still by \( du^\alpha_l/\text{d}s \).

Both \( u^\alpha_l \) and \( du^\alpha_l/\text{d}s \) are of bounded variation on \((l + \varepsilon, r - \varepsilon)\), and \( u^\alpha_r \) is continuous. Then it follows from (2.2) that
\[
\int_{l+\varepsilon}^{r-\varepsilon} \left( \frac{du^\alpha_r}{ds} \right)^2 \text{d}s = \int_{l+\varepsilon}^{r-\varepsilon} \frac{du^\alpha_l}{ds} \text{d}u^\alpha_r
\]
\[
= \frac{du^\alpha_r}{ds} \cdot u^\alpha_r \bigg|_{l+\varepsilon}^{r-\varepsilon} - 2\alpha \int_{(l+\varepsilon, r-\varepsilon)} u^\alpha_r(\xi)^2 m(d\xi)
\]
This yields that
\[ \int_{l+\varepsilon}^{r-\varepsilon} \left(\frac{du}{ds}\right)^2 ds + 2\alpha \int_{(l+\varepsilon,r-\varepsilon]} u_\alpha^\sigma(\xi)^2 m(d\xi) = \left. \frac{du}{ds} \cdot u_\alpha^\sigma \right|_{l+\varepsilon}^{r-\varepsilon}. \] (3.2)

Since \( u_\alpha^\sigma \) is positive and increasing and \( u_\alpha^\sigma(r) = 1 \), it follows that \( u_\alpha^\sigma(l) := \lim_{x \downarrow l} u_\alpha^\sigma(x) \) is finite. Note that \( du_\alpha^\sigma/ds \geq 0 \) since \( u_\alpha^\sigma \) is increasing. In addition, (2.2) yields that \( du_\alpha^\sigma/ds \) is increasing and \( du_\alpha^\sigma/ds(r) := \lim_{y \to r} du_\alpha^\sigma/ds(y) \) is finite. Letting \( \varepsilon \downarrow 0 \) in (3.2) and noticing that \( u_\alpha^\sigma(l) = 0 \) if \( l \) is absorbing in view of Lemma 2.1 (4), we get that \( u_\alpha^\sigma \in \mathcal{F} \).

Next we assert that \( u_\alpha^\sigma \in \mathcal{H}_\alpha \). Let \( f \in \mathcal{C}_f = \{ \varphi \circ s : \varphi \in C_0^\infty(s(I)) \} \). Mimicking the derivation of (3.2), one can obtain that
\[
\int_{l+\varepsilon}^{r-\varepsilon} \frac{du_\alpha^\sigma}{ds} f \cdot ds + 2\alpha \int_{(l+\varepsilon,r-\varepsilon]} u_\alpha^\sigma(\xi) f(\xi) m(d\xi) = \left. \frac{du}{ds} \cdot f \right|_{l+\varepsilon}^{r-\varepsilon}.
\]

Note that \( \lim_{x \downarrow l} f(r - \varepsilon) = \lim_{x \uparrow l} f(l + \varepsilon) = 0 \). Hence \( \mathcal{E}_\alpha(u_\alpha^\sigma, f) = 0 \). Since \( \mathcal{C}_f \) is a special standard core of \( (\mathcal{E}^0, \mathcal{F}^0) \), we get \( u_\alpha^\sigma \in \mathcal{H}_\alpha \).

Thirdly we prove that \( \mathcal{H}_\alpha \subset \text{span}\{u_+, u_-\} \). Let \( h \in \mathcal{H}_\alpha \). Take an arbitrary interval \( J := (a, b) \subset [a, b] \subset I_e \) with \( e \in J \). For any \( f \in \mathcal{F} \) with \( \text{supp}[f] \subset J \), it holds \( \mathcal{E}_\alpha(h, f) = 0 \). Set
\[ F_h(x) := \int_x^h \varphi(m)d\xi. \]

Then \( F_h \) is of bounded variation on \([a, b]\) and \( f \in C([a, b]) \) is also of bounded variation. Hence
\[ \int_J f(x) h(x) m(dx) = \int_{[a, b]} f(x)dF_h(x) = -\int_{[a, b]} F_h(x) df(x). \]

It follows from \( \mathcal{E}_\alpha(h, f) = 0 \) that
\[ \int_{[a, b]} \left( \frac{dh}{ds}(x) - 2\alpha F_h(x) \right) df(x) = 0. \]

Note that \( \mathcal{C}_J := \{ \varphi \circ s : \varphi \in C_0^\infty(s(J)) \} \subset \mathcal{F} \) and \( \text{supp}[f] \subset J \) for any \( f \in \mathcal{C}_J \). This yields that \( dh/ds - 2\alpha F_h \) is constant on \( J \). Particularly, \( h|_{[a, b]} \in C([a, b]) \) is a solution of the equation (2.1) restricted to \( J \). In view of [5, II, §4], one can conclude that \( h \in \text{span}\{u_+, u_-\} \).

Finally it suffices to prove \( u_+ \notin \mathcal{F} \). In fact, it is straightforward to verify that \( du_+ / ds(r) \cdot u_+(r) \) is finite. Since \( u_+ \geq 0 \) satisfies (2.2) and \( du_+ / ds(e) = 0 \), it follows that \( du_+ / ds \) is increasing and
\[ \frac{du_+}{ds}(l) \leq \frac{du_+}{ds}(e) = -\frac{1}{u(e)} = -1. \]

If \( \sigma(l) = \infty \), then Lemma 2.1 (1) yields that \( u_+(l) = \infty \). An analagous derivation of (3.2) leads to \( u_+ \notin \mathcal{F} \). If \( \sigma(l) < \infty \), then Lemma 2.1 (1) implies that
\[ u_+(l) \geq \int_l^r u(y)^{-2} ds(y) > 0. \]

Meanwhile \( m(l+) = \infty \) leads to \( u_+ \notin L^2(I, m) \) and \( m(l+) < \infty \) corresponds to an absorbing endpoint \( l \). In the latter case \( u_+ \notin \mathcal{F} \) because \( u_+(l) \neq 0 \). That completes the proof. \( \square \)

Remark 3.2. The third and final steps in this proof would be simplified if we apply the probabilistic representation of harmonic functions (4.1). In the case \( I_e = (l, r) \), (4.1)

\[
\int_{l+\varepsilon}^{r-\varepsilon} \left(\frac{du}{ds}\right)^2 ds + 2\alpha \int_{(l+\varepsilon,r-\varepsilon]} u_\alpha^\sigma(\xi)^2 m(d\xi) = \left. \frac{du}{ds} \cdot u_\alpha^\sigma \right|_{l+\varepsilon}^{r-\varepsilon}. \] (3.2)
The main result characterizing harmonic functions is as follows. The following hold:

Lemma 3.3. \( H \) before moving on, we prepare a lemma to give the expression of \( \mathcal{F}_e \). Proof. See \([2, \text{Theorem 2.2}]\).

When \(|s(l)| + |s(r)| < \infty\), set
\[
 u^0_l(x) := \frac{s(r) - s(x)}{s(r) - s(l)}, \quad u^0_r(x) := \frac{s(x) - s(l)}{s(r) - s(l)}, \quad x \in \hat{I}.
\]

The main result characterizing harmonic functions is as follows.

Theorem 3.4. The following hold:

1. When \( I_e = (l, r) \), \( \mathcal{H} = \{0\} \) if \( s(l) > -\infty \) or \( s(r) < \infty \); otherwise \( \mathcal{H} = \text{span}\{1\} \).
2. When \( I_e = [l, r) \), \( \mathcal{H} = \text{span}\{u^0_l\} \) if \( s(r) < \infty \); otherwise \( \mathcal{H} = \text{span}\{1\} \).
3. When \( I_e = (l, r) \), \( \mathcal{H} = \text{span}\{u^0_l, u^0_r\} \).
4. When \( I_e = [l, r) \), \( \mathcal{H} = \text{span}\{1\} \).

Proof. (1) In this case \( \mathcal{F}_e^0 = \mathcal{F}_e \). Note that \((\mathcal{E}, \mathcal{F})\) is transient, if and only if \( s(l) > -\infty \) or \( s(r) < \infty \); see, e.g., \([1, \text{Theorem 2.2.11 and Example 3.5.7}]\). Under the transience, \( \mathcal{E}(f, f) = 0 \) for \( f \in \mathcal{F}_e \) implies \( f = 0 \). Hence \( \mathcal{H} = \{0\} \). When \((\mathcal{E}, \mathcal{F})\) is recurrent, it follows from \([1, \text{Theorem 5.2.16}]\) that \( f \) is constant for any \( f \in \mathcal{F}_e \) with \( \mathcal{E}(f, f) = 0 \). To the contrary, the recurrence of \((\mathcal{E}, \mathcal{F})\) implies that \( 1 \in \mathcal{F}_e \) and \( \mathcal{E}(1, f) = 0 \) for any \( f \in \mathcal{F}_e \). These yield \( \mathcal{H} = \text{span}\{1\} \).

(2) It is straightforward to verify that \( u^0_l \in \mathcal{F}_e, \mathcal{E}(u^0_l, f) = 0 \) for any \( f \in \mathcal{F}_e \) if \( s(r) < \infty \), and \( 1 \in \mathcal{F}_e, \mathcal{E}(1, f) = 0 \) for any \( f \in \mathcal{F}_e \) if \( s(r) = \infty \). Hence \( \text{span}\{u^0_l\} \subset \mathcal{H} \) if \( s(r) < \infty \) and \( \text{span}\{1\} \subset \mathcal{H} \) if \( s(r) = \infty \). To the contrary, take \( h \in \mathcal{H} \). For any \( f \in \mathcal{F}_e \), \( \mathcal{E}(h, f) = 0 \) implies that
\[
\int_{l}^{r} \frac{dh}{ds} df = 0. \tag{3.3}
\]
Since \( \mathcal{E}_f = \{\varphi \circ s : \varphi \in C_{\infty}^{c}(s(\hat{I}))\} \subset \mathcal{F}_e^0 \), it follows from \(3.3\) that \( dh/ds \) is constant, i.e., \( h = c_1 \cdot s + c_2 \) for two constants \( c_1 \) and \( c_2 \). Consequently Lemma \(3.3\) yields that \( h = c \cdot u^0_l \) for some constant \( c \) if \( s(r) < \infty \) and \( h \) is constant if \( s(r) = \infty \).

(3) This case can be treated by an analagous way to (2).

(4) It is easy to verify that \( u^0_l, u^0_r \in \mathcal{F}_e \) and \( \mathcal{E}(u^0_l, f) = \mathcal{E}(u^0_r, f) = 0 \) for any \( f \in \mathcal{F}_e^0 \). Thus \( \text{span}\{u^0_l, u^0_r\} \subset \mathcal{H} \). To the contrary, take \( h \in \mathcal{H} \) and \(3.3\) indicates that \( h = c_1 s + c_2 \) for some constants \( c_1, c_2 \). Then there exist another two constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \) such that \( h = \tilde{c}_1 u^0_l + \tilde{c}_2 u^0_r \). Therefore \( \mathcal{H} \subset \text{span}\{u^0_l, u^0_r\} \).

That completes the proof. \(\square\)
4. Probabilistic counterparts of harmonic functions

Recall that \( X \) is the diffusion process associated with \((\mathcal{E}, \mathcal{F})\). Denote by \( \zeta \) the lifetime of \( X \) and set

\[ \tau := \inf \{ t > 0 : X_t \notin (l, r) \}. \]

Then \( X_\tau \in \{ l, r \} \) for \( \tau < \zeta \). For convenience, write \( \mathcal{H}_0 := \mathcal{H} \) and set \( u^0_{\alpha} := 1 \) (resp. \( u^0_{\alpha} := 1 \)) if \( I_e = [l, r] \) and \( s(r) = \infty \) (resp. \( I_e = (l, r] \) and \( s(l) = -\infty \)). The harmonic functions admit the following probabilistic representation.

**Theorem 4.1.** Assume that \( I_e \neq \emptyset \). Then for any \( \alpha \geq 0 \) and \( x \in I_e \),

\[
\begin{align*}
    u^\alpha_{l}(x) &= \mathbb{E}_x \left[ e^{-\alpha \tau} : X_\tau = l, \tau < \zeta \right], & \text{if } l \in I_e; \\
    u^\alpha_{r}(x) &= \mathbb{E}_x \left[ e^{-\alpha \tau} : X_\tau = r, \tau < \zeta \right], & \text{if } r \in I_e.
\end{align*}
\]

**Proof.** Set \( \varphi^\alpha_{j}(\cdot) := \mathbb{E}_\cdot \left[ e^{-\alpha \tau} : X_\tau = j, \tau < \zeta \right] \) for \( j = l \) or \( r \). Suppose first that \( \alpha > 0 \), or that \( \alpha = 0 \) and \((\mathcal{E}, \mathcal{F})\) is transient. Then

\[ \mathcal{H}_\alpha = \{ \mathbb{E}_\cdot \left[ e^{-\alpha \tau} f(X_\tau) : \tau < \zeta \right] : f \in \mathcal{F} \text{ for } \alpha > 0 \text{ and } f \in \mathcal{F}_c \text{ for } \alpha = 0 \}; \quad (4.1) \]

see [1] Theorems 3.2.2 and 3.4.2. When \( I_e = [l, r] \), \( X_\tau = l \) for \( \tau < \zeta \). Hence

\[ \mathcal{H}_\alpha = \text{span}\{ \varphi^\alpha_{l} \}. \]

It follows from Theorems 5.1 and 5.4 that \( \varphi^\alpha_{l} = cu^0_{\alpha} \) for some constant \( c \). Note that \( \varphi^\alpha_{l}(l) = 1 \) because \( \mathbb{P}_l(\tau = 0) = 1 \), and \( u^0_{\alpha}(l) = 1 \). Therefore \( c = 1 \). Another case \( I_e = (l, r] \) can be treated analogically and we can obtain that \( u^\alpha_{r} = \varphi^\alpha_{r} \). When \( I_e = [l, r] \), \( \zeta = \infty \) and \( X_\tau \in \{ l, r \} \). Then \( \mathcal{H}_\alpha = \text{span}\{ \varphi^\alpha_{l}, \varphi^\alpha_{r} \} \). Meanwhile \( \varphi^\alpha_{l}(l) = \varphi^\alpha_{r}(r) = 1 \) and \( \varphi^\alpha_{l}(r) = \varphi^\alpha_{r}(l) = 0 \). Therefore Theorem 5.1 yields \( u^\alpha_{l} = \varphi^\alpha_{l} \) and \( u^\alpha_{r} = \varphi^\alpha_{r} \).

Finally consider \( \alpha = 0 \) and \((\mathcal{E}, \mathcal{F})\) is recurrent. Particularly \( \zeta = \infty \). When \( I_e = [l, r] \) or \( I_e = (l, r] \), it follows from [1] Theorem 3.5.6 (2) that \( \varphi^0_{j}(x) = \mathbb{P}_x(\sigma_j < \infty) = 1 \) for \( j = l \) or \( r \) and \( x \in I_e \), where \( \sigma_j := \inf \{ t > 0 : X_t = j \} \). Hence \( \varphi^0_{l} = u^0_{j} \). When \( I_e = [l, r] \), [1] Theorem 3.4.8 implies that \( \varphi^0_{l} \), \( \varphi^0_{r} \in \mathcal{H} \). Since \( \varphi^0_{l}(l) = \varphi^0_{r}(r) = 1 \) and \( \varphi^0_{l}(r) = \varphi^0_{r}(l) = 0 \), applying Theorem 3.4.8 we can conclude that \( u^0_{l} = \varphi^0_{l} \), \( u^0_{r} = \varphi^0_{r} \). That completes the proof. \( \square \)

The special case \( \alpha = 0 \) leads to the characterization of the first hitting times of the endpoints \( l \) and \( r \).

**Corollary 4.2.** Let \( \sigma_j := \inf \{ t > 0 : X_t = j \} \) for \( j = l \) or \( r \). Then the following hold:

1. When \( I_e = [l, r] \), \( \mathbb{P}_x(\sigma_l < \infty) = 1 \) for any \( x \in I_e \) if \( s(r) = \infty \). If \( s(r) < \infty \), then
   \[ \mathbb{P}_x(\sigma_l < \infty) = \frac{s(r) - s(x)}{s(r) - s(l)}, \quad x \in I_e. \]

2. When \( I_e = [l, r] \), \( \mathbb{P}_x(\sigma_r < \infty) = 1 \) for any \( x \in I_e \) if \( s(l) = -\infty \). If \( s(l) > -\infty \), then
   \[ \mathbb{P}_x(\sigma_r < \infty) = \frac{s(x) - s(l)}{s(r) - s(l)}, \quad x \in I_e. \]

3. When \( I_e = [l, r] \),
   \[ \mathbb{P}_x(\sigma_l < \sigma_r) = \frac{s(r) - s(x)}{s(r) - s(l)}, \quad x \in I_e. \]
5. Harmonic functions in generator domain

Denote by $\mathcal{L}$ with domain $\mathcal{D}(\mathcal{L})$ the $L^2$-generator of $(\mathcal{E}, \mathcal{F})$. In this section we answer the question that whether an $\alpha$-harmonic function belongs to $\mathcal{D}(\mathcal{L})$ for $\alpha > 0$. Given a function $f \in \mathcal{F}$, $\frac{df}{ds}$ is defined if there exists a $\mathcal{F}$-a.e. version of $df/ds$, still denoted by $df/ds$, and $g \in L^2(\tilde{I}, m)$ such that

$$\frac{df}{ds}(x) - \frac{df}{ds}(y) = \int_{(x,y]} g(\xi)m(d\xi), \quad l < x < y < r. \quad (5.1)$$

Meanwhile $\frac{d}{dm}(df/ds) := g$. Note that if $j \in I_e$ for $j = l$ or $r$, then $(5.1)$ implies that $df/ds(j) := \lim_{x \to j} df/ds(x)$ exists and is finite. The description of $\mathcal{L}$ is given as follows. A related consideration is referred to $[2]$.

**Lemma 5.1.** The $L^2$-generator of $(\mathcal{E}, \mathcal{F})$ is

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in \mathcal{F} : \frac{df}{dm} \in L^2(\tilde{I}, m), \frac{df}{ds}(j) = 0 \text{ if } j \in I_e, \quad m(\{j\}) = 0 \text{ for } j = l \text{ or } r \right\},$$

where we make the convention $0/0 := 0$.

**Proof.** Let $\mathcal{E}_1$ be the special standard core of $(\mathcal{E}, \mathcal{F})$ defined as $(1.2)$. Denote by $\mathcal{G}$ the right term in the first identity. Clearly $\mathcal{L} f$ given by the second identity is well defined for any $f \in \mathcal{G}$. It is straightforward to verify that $-\int_{I_e} \mathcal{L} f g dm = \mathcal{E}(f,g)$ for any $f \in \mathcal{G}$ and $g \in \mathcal{E}_1$. Hence we only need to show $\mathcal{D}(\mathcal{L}) \subset \mathcal{G}$. To do this, take $f \in \mathcal{D}(\mathcal{L})$ with $h := \mathcal{L} f \in L^2(I_e, m)$. With out loss of generality assume that $df/ds(e)$ is well defined and finite. Then $\mathcal{E}(f,g) = -\int_{I_e} h g dm$ for any $g \in \mathcal{E}_1$. Define

$$F_h(x) := \int_{(e,x]} h(\xi)m(d\xi) + \frac{1}{2} \frac{df}{ds}(e), \quad x > e$$

and

$$F_h(x) := -\int_{(x,e]} h(\xi)m(d\xi) + \frac{1}{2} \frac{df}{ds}(e), \quad x < e.$$

Set $F_h(l-) := -\int_{[l,e)} h(\xi)m(d\xi) + \frac{1}{2} \frac{df}{ds}(e)$ if $l \in I_e$. For $j = l$ or $r$, if $j \notin I_e$, then $g(j) = 0$. Hence we have

$$-\int_{I_e} h g dm = -\int_{[l,r]} g df h = \int_{l}^{r} F_h(x) dg(x) + F_h(l-) g(l) - F_h(r) g(r),$$

where we impose $F_h(l-) \cdot 0 = F_h(r) \cdot 0 = 0$ even if $F_h(l-) \text{ or } F_h(r)$ is not defined. It follows from $\mathcal{E}(f,g) = -\int_{I_e} h g dm$ that

$$\frac{1}{2} \int_{l}^{r} \frac{df}{ds}(x) dg(x) = \int_{l}^{r} F_h(x) dg(x) + F_h(l-) g(l) - F_h(r) g(r), \quad \forall g \in \mathcal{E}_1. \quad (5.2)$$
Taking all \( g \in \mathcal{C}_L \) with \( \text{supp}[g] \subset (l, r) \) and noticing \( F_h(e) = \frac{1}{2} df/\text{ds}(e) \), we can obtain that
\[
F_h(x) = \frac{1}{2} \frac{df}{\text{ds}}(x), \quad x \in (l, r).
\] (5.3)

Particularly \( \frac{1}{2} \frac{df}{\text{ds}} = h \in L^2(\mathcal{I}, m) \). In addition, (5.2) and (5.3) yield that \( F_h(l-) = 0 \) if \( l \in I_e \) and \( F_h(r) = 0 \) if \( r \in I_e \). In the former case
\[
F_h(l-) = F_h(l) - h(l)m(\{l\}) = \frac{1}{2} \frac{df}{\text{ds}}(l) - h(l)m(\{l\}).
\]

Hence \( df/\text{ds}(l) = 0 \) if \( m(\{l\}) = 0 \) and \( h(l) = \frac{1}{2} \frac{df}{\text{ds}}(l)/m(\{l\}) \). The latter case can be treated analogically. Therefore we can obtain \( f \in \mathcal{G} \). That completes the proof. \( \square \)

Set \( C := \int_l^r u(y)^{-2} \text{ds}(y) \), which is finite due to Lemma 2.1 (1). It is straightforward to calculate that if \( l \in I_e \), then
\[
\begin{align*}
\frac{du_-}{\text{ds}}(l) &= 0, \quad u_+(l) = Cu(l) \in (0, \infty), \\
\frac{du_+}{\text{ds}}(l) &= \frac{1}{u(l)} \in (0, \infty),
\end{align*}
\]

and if \( r \in I_e \), then
\[
\begin{align*}
\frac{du_-}{\text{ds}}(r) &= 0, \quad u_+(r) = Cu(r) \in (0, \infty), \\
\frac{du_+}{\text{ds}}(r) &= \frac{1}{u(r)} \in (0, \infty),
\end{align*}
\]

Put \( c_1 := -\frac{du_-}{\text{ds}}(l)/\frac{du_+}{\text{ds}}(l) \) and \( c_r := -\frac{du_-}{\text{ds}}(r)/\frac{du_+}{\text{ds}}(r) \). Clearly \( c_1, c_r \in (0, \infty) \).

**Theorem 5.2.**

1. When \( I_e = [l, r] \), \( \mathcal{H}_a \cap D(L) = \{0\} \) if \( m(\{l\}) = 0 \) and \( \mathcal{H}_a \cap \mathcal{D}(\mathcal{L}) = \mathcal{H}_a \) if \( m(\{l\}) > 0 \).
2. When \( I_e = (l, r] \), \( \mathcal{H}_a \cap D(L) = \{0\} \) if \( m(\{l\}) = 0 \) and \( \mathcal{H}_a \cap \mathcal{D}(\mathcal{L}) = \mathcal{H}_a \) if \( m(\{l\}) > 0 \).
3. When \( I_e = [l, r] \),

\[
\mathcal{H}_a \cap \mathcal{D}(\mathcal{L}) = \begin{cases} 
\mathcal{H}_a, & m(\{l\}), m(\{r\}) > 0; \\
\text{span}\{u_+ + c_r u_-\}, & m(\{l\}) > 0, m(\{r\}) = 0; \\
\text{span}\{u_+ + c_1 u_-\}, & m(\{r\}) > 0, m(\{l\}) = 0; \\
\{0\}, & m(\{l\}) = m(\{r\}) = 0.
\end{cases}
\]

**Proof.** For the first and second assertions, it suffices to note that \( du_+/\text{ds}(l) \neq 0 \) if \( l \in I_e \) and \( du_-/\text{ds}(r) \neq 0 \) if \( r \in I_e \). Now consider \( I_e = [l, r] \). When \( m(\{l\}), m(\{r\}) > 0 \), clearly \( \mathcal{H}_a \subset \mathcal{D}(\mathcal{L}) \). When \( m(\{l\}) > 0 \) and \( m(\{r\}) = 0 \), \( h = c_1 u_+ + c_2 u_- \in \mathcal{D}(\mathcal{L}) \) if and only if
\[
c_1 \frac{du_+}{\text{ds}}(r) + c_2 \frac{du_-}{\text{ds}}(r) = 0.
\]

This amounts to \( c_2/c_1 = c_r \). Hence \( \mathcal{H}_a \cap \mathcal{D}(\mathcal{L}) = \text{span}\{u_+ + c_r u_-\} \). The third case can be treated analogically. When \( m(\{l\}) = m(\{r\}) = 0 \), \( h = c_1 u_+ + c_2 u_- \in \mathcal{D}(\mathcal{L}) \) if and only if
\[
\begin{align*}
c_1 \frac{du_+}{\text{ds}}(r) + c_2 \frac{du_-}{\text{ds}}(r) &= 0, \\
c_1 \frac{du_+}{\text{ds}}(l) + c_2 \frac{du_-}{\text{ds}}(l) &= 0.
\end{align*}
\]
Note that \( \frac{du}{ds}(l) < 0, \frac{du}{ds}(r) > 0 \) and \( u(l), u(r) > 0 \). Then we have \( c_1 = c_2 = 0 \) because
\[
\frac{d u_+}{d s}(r) \frac{d u_-}{d s}(l) - \frac{d u_-}{d s}(r) \frac{d u_+}{d s}(l) = -C u_+ \frac{du}{ds}(l) \frac{du}{ds}(r) - C u(r) \frac{du}{ds}(l) + C u(l) \frac{du}{ds}(r) < 0.
\]
That completes the proof. \( \square \)

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