Fermionic Operators from Bosonic Fields in 3+1 Dimensions.

A. Kovner*

Theory Division, T-8, Los Alamos National Laboratory, MS B-285
Los Alamos, NM 87545

B. Rosenstein**

Institute of Physics, Academia Sinica
Taipei, 11529
Taiwan, R.O.C.

Abstract

We present a construction of fermionic operators in 3+1 dimensions in terms of bosonic fields in the framework of $QED_4$. The basic bosonic variables are the electric fields $E_i$ and their conjugate momenta $A_i$. Our construction generalizes the analogous construction of fermionic operators in 2+1 dimensions. Loosely speaking, a fermionic operator is represented as a product of an operator that creates a pointlike charge and an operator that creates an infinitesimal t’Hooft loop of half integer strength. We also show how the axial $U(1)$ transformations are realized in this construction.

*KOVNER@PION.LANL.GOV
**BARUCH@PHYS.SINICA.EDU.TW
The problem of bosonization in dimension higher than two is a long standing one. Many attempts have been made to generalize the 1+1 dimensional Mandelstam construction to higher dimensions \[1\]. However, they all have an unattractive feature that the bosonized theories turn out to be nonlocal. Recently, using certain dynamical assumptions, one of us and P. Kurzepa succeeded to perform bosonization of QED in 2+1 dimensions \[4\]. The resulting bosonic theory was found to be local and strongly interacting. As expected, in 3+1 dimensions the corresponding construction is significantly more complicated. In this letter we perform a first step of the analogous program in 3+1 dimension: the construction of fermionic operators in terms of bosonic fields.

We consider quantum electrodynamics with one Dirac fermion in 3+1 dimensions. Our basic philosophy and motivation to consider this particular theory are basically same as in 2+1 dimensions. They are described in detail in ref.\[4\] and need not be repeated here. Apart of obvious specific 3+1 dimensional features, like a nonabelian representation of the rotation group and different discrete symmetries, note that now the axial anomaly is present. This is different from 2+1 but reminiscent of 1+1 QED. We elaborate later on its implementation. We will work in the Hamiltonian formalism, and find the temporal gauge (\(A_0 = 0\)) to be most convenient to our purposes. The Hamiltonian is

\[
H = \frac{1}{2}E^2 + \frac{1}{2}B^2 + \bar{\psi} \gamma^i (i\partial_i + eA_i)\psi + m\bar{\psi}\psi
\] (1)

It is supplemented by Gauss’s constraint,

\[
\partial_i E_i = e\psi^\dagger \psi
\] (2)

We take the same approach as in ref.\[4\]. We would like to construct gauge invariant fermionic fields in terms of bosonic fields \(E_i\) and their conjugate momenta, which would solve the constraint eq. \[4\]. In principle, as a next step we would like to calculate fermionic bilinears, including the components of the energy momentum tensor, in terms of these bosonic fields, and thereby bosonise the theory. In this paper we will only perform the first step of this program, namely construct the fermionic operators.

The fields \(\psi_\alpha\) that appear in eq.\[1\] are not gauge invariant. We therefore define

\[
\psi^{CG}_\alpha(x) = \psi_\alpha(x)\Phi(x)
\] (3)

where,

\[
\Phi(x) = e^{ie}\int d^2y e_i(y-x)A_i(y)
\] (4)

and

\[
e_i(x) = -\frac{1}{4\pi} \frac{x_i}{|x|^3}
\] (5)

\[1\] The term "bosonization" is sometimes understood as obtaining an effective low energy description of fermionic theories in terms of bosonic fields. This in many cases is a straightforward exercise \[3\]. This is not what we mean by bosonization in this paper. Here we have in mind an exact mapping between the fermionic and bosonic Hilbert spaces, which should give an exact bosonic description of a fermionic theory at all energy scales.
is the electric field of a point like charge. The operator defined this way is gauge invariant, and coincides with the original fermionic field in the Coulomb gauge, in which the phase factor $\Phi(x)$ in eq.3 becomes unity.

Our ansatz for the bosonic form of the fermionic operator is inspired by the results of ref.[4]. In 2+1 dimensions the fermionic operators had the following basic structure:

$$\psi_{2+1}(x) = \lim_{|\epsilon| \to 0} \int d\hat{\epsilon} \Gamma(\hat{\epsilon}) V^{1/2}(x + \epsilon) \Phi(x) V^{*1/2}(x - \epsilon)$$

(6)

The different factors in this expression have a clear physical meaning. The operator $\Phi$ is the exact 2 + 1 dimensional analog of the operator defined in eq.4. It creates the electric field of a point electric charge at the point $x$. The role of this factor is to ensure the correct transformation properties of $\psi$ under the global electric charge transformation. The operator $V(x)$ ($V^*(x)$) is the operator that creates a magnetic vortex with the elementary flux $2\pi/e$ ($2\pi/e$). This is a local scalar field in the framework of QED$_{2+1}$. Its appearance in eq.6 is natural, since bosonization should be akin to duality transformation, and $V(x)$ is in fact the dual field in the theory.[5]. The fact that it is the square root of $V$ that appears in the equation, makes the operator $\psi$ double valued. This is a necessary property, since otherwise $\psi(x)$ could not have the correct spinorial rotational properties. It is also in agreement with general arguments of ref. [7].$V^*$ also necessarily appears in the same expression, since $\psi$ should not carry magnetic flux, and therefore the flux that is created by $V$ is cancelled by $V^*$. This particular combination also insures anticommutativity of $\psi(x)$. To make the product of local operators appearing in the definition of $\psi$ onto the spin 1/2 representation of the rotation group.

To generalize this construction we should first understand what is the generalization of the vortex operator $V$ to 3+1 dimensions. The operator $V$ now should create an infinitesimally thin line of magnetic flux. It will depend therefore on the curve. The explicit construction has been given in refs.[8], [9].

$$V(C) = \text{exp} \left\{ \frac{i}{e} \int d^2y [a_i(y - x_c)E_i(y) + b(x_c - y)\partial_1E_i(y)] \right\}$$

(7)

Here $a_i(x)$ is a vector potential of an infinitesimally thin magnetic vortex along the curve $C$: $\epsilon_{ijk}\partial_j a_k(x) = 2\pi \tau_i(x, C) \delta^{\perp}_{\text{perp}}(x)$, where $\tau_i(x, C)$ is the unit tangent vector to the curve $C$ at point $x$, and $\delta^{\perp}_{\text{perp}}(x)$ is the two dimensional delta function in the plane normal to $\tau_i$. The function $b(x)$ satisfies $\partial_i [b(x)]_{\text{mod}2\pi} = a_i(x)$. Since $a_i(x)$ has a nonvanishing curl, the function $b(x)$ must have a surface of singularities ending at the curve $C$. For example, for a straight line $C$ running along the $x_3$ axis one has

$$a_i(x) = \epsilon_{ij} \frac{x_i}{x_1^2 + x_2^2}, \quad i = 1, 2; \quad a_3(x) = 0$$

$$b(x) = \Theta(x)$$

(8)

with $\Theta$ the polar angle in the $x_1 - x_2$ plane (Fig.1.). An alternative representation of $V$...
Figure 1: The function $\Theta(x)$.

Figure 2: The standard set of curves $C$.

is obtained by integrating by parts the $\partial_i E_i$ term in the exponential:

$$V(C) = \exp \left\{ \frac{2\pi i}{e} \int_S d^2 s \, i E_i + \frac{i}{|y| \to \infty} \int_{\partial C} d^2 \ell (b(x_c - y) E_i) \right\}$$

(9)

where the integration is over the surface $S$ of discontinuities of the function $b$, which has the curve $C$ as its boundary, and the second term in the exponential is the integral over the surface at spatial infinity. Note, that even though the surface $S$ enters formally the definition of $V$, the operator actually does not depend on it, but depends rather only on the boundary $C$. This is the consequence of the fact, that changing the surface changes the phase by $2\pi/e \oint ds E_i$, which is an integer multiple of $2\pi$ due to quantization of electric charge $[8]$.

Let us now define a standard set of operators $V_\eta(x, \hat{n})$, which are associated with the set of curves in Fig. 2. Those consist of half a circle with the center at the point $x$ and radius $\eta$ in the plane specified by the unit normal vector $\hat{n}$, and two semiinfinite segments of a straight line in the direction of the third axis. Note, that $V_\eta^*(x, -\hat{n})$ is obtained from $V_\eta(x, \hat{n})$ by a three dimensional parity transformation. Note, also that the product $V_\eta(x, \hat{n})V_\eta^*(x, -\hat{n})$, is an operator that creates a unit strength closed t’Hooft loop, in the plane normal to $\hat{n}$, centered at $x$ with radius $\eta$.

We are now going to generalize the $2+1$ dimensional construction of the Fermi operators, by basically changing the product of a vortex and antivortex into an infinitesimally small t’Hooft loop of half integer strength. Consider therefore the following operators

$$\chi(x) = k \Lambda^{3/2} e^{-\frac{3\pi i}{e} \int d^3 y \partial_i E_i V_\eta^{1/2}(x, \hat{n}) \Phi(x) V_\eta^{* 1/2}(x, -\hat{n})}$$

(10)

The first exponential factor here is the same as in $2+1$ dimensions $[3]$. We have also introduced the ultraviolet cutoff $\Lambda$ so that $\chi$ has a dimension of the canonical fermi field. The constant $k$ in eq. (10) is should be chosen so as to normalize correctly the fermionic bilinears (see $[3]$). The operator $\Phi(x)$ is defined in eq. (4) in terms of an operator $A_i$. This operator $A_i$ should be understood as a variable conjugate to $E_i$

$$[E_i(x), A_j(y)] = i\delta^3(x - y)\delta_{ij}$$

(11)

We also require, that $\epsilon_{ijk} \partial_j A_k = B_i$. In these respects $A_i(x)$ is similar to the original vector potential. One should remember, however, that since we are working on the gauge invariant subspace of QED$_4$, $A_i$ is a gauge invariant operator. In this sense, it can be thought of as a vector potential in a particular gauge. Up to now, we did not need to specify this gauge condition. We will return to this point later.

\[\text{We pick the third direction arbitrarily, for definiteness. As will become clear later, none of our calculations depend on it.}\]
The operator $V^{1/2}$ ($V^{*1/2}$) obviously has different locality properties than the operator $V$. Whereas $V_C$ has support only on the curve $C$, and the discontinuity surface of the function $b(x)$ is unphysical, this is not quite true for $V^{1/2}$ ($V^{*1/2}$). The latter operator is double valued. By moving the surface of discontinuities from $S_1$ to $S_2$, we multiply $V^{1/2}$ by a phase factor $\exp\left\{\frac{i\pi}{\eta} \int_{S_1} - \int_{S_2} ds^i E_i \right\}$, which due to quantization of electric charge can be either 1 or $-1$. Fixing the surface of discontinuities, is therefore equivalent to specifying what branch of the square root we choose in the definition of $V^{1/2}$.

Although we do not have to specify this choice, and can just think about $V^{1/2}$ as a single valued operator defined on a double cover, it is sometimes convenient to visualize it as having a definite surface of discontinuities. We will always think about this surface as being parallel to the $[1,3]$ plane. Note, that it is still true that $V^{*1/2}(x, -\hat{n})$ the and $V^{1/2}(x, \hat{n})$ are related by a parity transformation, and therefore the surface of discontinuities associated with the former operator, is also the parity transform of the surface of discontinuities associated with the latter. Therefore, the surface of discontinuities associated with the closed loop, $V^{1/2}_\eta(x, \hat{n})V^{*1/2}_\eta(x, -\hat{n})$ in the limit of small $\eta$ is just the whole plane $[1,3]$. This doublevaluedness, or effective nonlocality in the definition of $V^{1/2}$ is directly responsible for the anticommutativity of the operators defined in eq. 10. Using the fact that for the function $\Theta(x)$ defined in eq. 8 satisfies $\Theta(x - y) - \Theta(y - x) = \pm \pi$ for any two points $x$ and $y$, we find:

$$\chi_{\hat{n}}(x)\chi_{\hat{n}'}(y)|_{x - y|\gg \eta} = \chi_{\hat{n}'}(y)\chi_{\hat{n}}(x) \exp\{i(\Theta(x - y) - \Theta(y - x))\} = -\chi_{\hat{n}'}(y)\chi_{\hat{n}}(x)$$  (12)

The anticommutation relations between $\chi(x)$ and $\chi^\dagger(y)$ also follow immediatelly for $x \neq y$. In fact, the canonical anticommutator at coincident points is obtained, quite generally, only in the limit where the regulator in the definition of fermionic operators is removed $\eta \to 0$. This is also true in other dimensionalities ($[1]$, $[4]$). In this limit the operator $\chi$ does not depend on the direction $\hat{n}$ and one obtains

$$\chi(x)\chi^\dagger(x) + \chi^\dagger(x)\chi(x) = 2k^2\Lambda^3$$  (13)

which for an appropriate choice of $k$ becomes a $\delta$ - function in the continuum limit.

Complete analogy with $2 + 1$ dimensions would require at this point to project the operators $\chi$ onto appropriate representation of the rotation group. However, before doing that we have to address one more question, which was not present in $2 + 1$ d. This is the action on the fermionic fields of the axial $U_A(1)$ transformation. To implement the correct action of $U_A(1)$, we need to know the form of the axial charge in terms of bosonic variables. We infer this from the axial anomaly equation of QED$_4$.

$$\partial_\mu J^\mu_5 = -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma}$$  (14)

We take therefore the bosonic representation of the axial current as:

$$J^\mu_5 = -\frac{e^2}{8\pi^2} \epsilon^{\mu\nu\lambda\sigma} A_\mu F_{\lambda\sigma}$$  (15)
and in particular
\[ J_5^0 = -\frac{e^2}{4\pi^2}A_i B_i \] (16)

This is completely analogous to $1+1$ dimensional case. We want to stress, that eq. (16) should not be thought of as a guess, but rather as the equation defining what we mean by the operator $A_i$. Up to now, we have only required that $A_i$ is a canonical conjugate of the electric field $E_i$, and that $\epsilon_{ijk} \partial_j A_k = B_i$. Eq. (16) defines also the longitudinal part of $A_i$ in terms of the gauge invariant quantity $J_5^0$, in a way consistent with the anomaly equation. This equation therefore can be understood as the gauge condition referred to earlier.

Let us see what are the transformation properties of the operators $\chi$ under the axial transformation. The operator $\Phi(x)$, clearly commutes with $J_5^0$, since it only involves the momentum $A$. The relevant commutator is, therefore

\[ e^{i\alpha Q^5}V^{1/2}_\eta(y, \hat{n})V^{1/2}_\eta(x, -\hat{n})e^{-i\alpha Q^5} = V^{1/2}_\eta(x, \hat{n})V^{1/2}_\eta(x, -\hat{n})e^{i\alpha \chi(x)} \] (17)

Here $\phi_\eta(x)$ is the magnetic flux through the surface bounded by the circle of radius $\eta$ with the center in $x$ lying in the plane with normal $\hat{n}$. In other words it is magnetic flux through the area bounded by the closed t'Hooft loop created by the operator $V^{1/2}_\eta(x, \hat{n})V^{1/2}_\eta(x, -\hat{n})$.

We can now define eigenoperators of the axial charge with eigenvalues $\pm 1$

\[
\begin{align*}
\chi_\eta(x)^+ &= \int d\alpha e^{i\alpha Q^5} \chi_\eta(x)e^{-i\alpha Q^5}e^{i\alpha} = \int d\alpha \chi_\eta(x)e^{i\alpha (\frac{\delta}{2\pi} \phi_\eta(x) - 1)} \\
\chi_\eta(x)^- &= \int d\alpha e^{i\alpha Q^5} \chi_\eta(x)e^{-i\alpha Q^5}e^{i\alpha} = \int d\alpha \chi_\eta(x)e^{i\alpha (\frac{\delta}{2\pi} \phi_\eta(x) + 1)}
\end{align*}
\] (18)

The integral over the transformation parameter $\alpha$ can be now easily performed, and we find

\[
\begin{align*}
\chi_\eta(x)^+ &= \chi_\eta(x)\delta(\phi_\eta(x) - 1) \\
\chi_\eta(x)^- &= \chi_\eta(x)\delta(\phi_\eta(x) + 1)
\end{align*}
\] (19)

This result has a very natural interpretation. The operator $\chi_\eta(x)^+$ is nontrivial only on states which contain a unit magnetic flux linked with the t'Hooft loop in the definition of $\chi_\eta(x)^+$. Note, that the extra factor does not change the anticommutation relations of the operators. This is obvious, since this extra piece measures magnetic field at the location of the operator $\chi(x)$, whereas the operator $\chi(y)$ creates only a closed loop of flux at its location. More precisely

\[ e^{i\alpha \chi(x)}V^{1/2}_\eta(y, \hat{n}')V^{1/2}_\eta(y, -\hat{n}') = V^{1/2}_\eta(y, \hat{n}')V^{1/2}_\eta(y, -\hat{n}')e^{i\alpha \chi(x)}e^{i\alpha \chi(C_1, C_2)} \] (20)

where $l(C_1, C_2)$ is the linking number of two small circles with radius $\eta$ centered at points $x$ and $y$ with normals $\hat{n}$ and $\hat{n}'$ respectively.
We now project these operators onto spin $1/2$ representations of the rotation group. We use the chiral representation of the Dirac $\gamma$ matrices, in which each chirality component transforms under rotations independently. The complete Dirac multiplet is then constructed as follows:

\[
\begin{align*}
\psi_1(x) &= \int d\hat{n} Y_{1/2}(\hat{n}) \chi_{\hat{n}}(x)^+ \\
\psi_2(x) &= \int d\hat{n} Y_{-1/2}(\hat{n}) \chi_{\hat{n}}(x)^+ \\
\psi_3(x) &= \int d\hat{n} Y_{1/2}(\hat{n}) \chi_{\hat{n}}(x)^- \\
\psi_4(x) &= \int d\hat{n} Y_{-1/2}(\hat{n}) \chi_{\hat{n}}(x)^- 
\end{align*}
\]

(21)

Here $Y_{\pm 1/2}(\hat{n})$ are monopole harmonics

\[
Y_{\pm 1/2}(\phi, \theta) = \mp e^{\pm \frac{\phi}{2}} (1 \mp \cos \theta)^{1/2}
\]

(22)

By construction therefore $\psi_1, \psi_2$ constitute a rotational doublet of chirality +1, and $\psi_3, \psi_4$ - a rotational doublet of chirality -1.

Eq.21 is our final result. To conclude, we have constructed in the framework of QED$_4$ the four dimensional multiplet of fermionic operators entirely in terms of bosonic fields $E_i$ and their conjugate momenta $A_i$. The fermionic operators have canonical anticommutation relations in the limit of zero regulator, carry unit electric charge, can be decompose into two representations of the axial charge with axial charges $\pm 1$ and have correct transformation properties under rotations. Note also, that under the parity transformation $\phi(\hat{n}) \rightarrow -\phi(\hat{n})$, and $\chi_{\hat{n}} \rightarrow \chi_{-\hat{n}}$. Under the parity transformation therefore the rotational multiplets $\psi_1, \psi_2$ and $\psi_3, \psi_4$ are transformed into each other, which is precisely the right parity transformation properties of Dirac fermions in the so called chiral basis of $\gamma$ matrices [11].4

The bosonic form of the fermionic operators has a very clear intuitive interpretation. The operator creates the electric field of a pointlike electric charge and the magnetic field of an infinitesimally small half integer magnetic fluxon. The extra factor required to represent the axial charge, vanishes unless the operator acts on a state which has a unit magnetic fluxon with a linking number $\pm 1$ relative to the fluxon created by the fermionic field itself. The next logical step is to calculate fermionic bilinears in terms of the bosonic fields. This calculation was rather involved already in $2 + 1$ dimensions, and is even more complicated in the present case. Work along these lines is currently in progress [12].

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4We wish to make a remark on the parity transformation properties of the operators. If we were to define the action of parity on $V^{1/2}$ as $V^{1/2}(\hat{n}) \rightarrow V^{*1/2}(-\hat{n})$ it would follow that $\chi \rightarrow e^{i\frac{\pi}{2}} \int d^{d+1}x E_i(x) \chi$ which would not give the correct parity transformation properties. However, $e^{i\frac{\pi}{2}} \int d^{d+1}x E_i(x)$ is an operator with eigenvalues $\pm 1$. Since $\chi$ is a double valued operator, we are free to change our definition of parity so that $\chi \rightarrow \chi$. This does not change the parity transformation properties of all single valued operators and is therefore perfectly consistent with the standard parity transformations of $A_i$ and $E_i$. 

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