KERNELS IN TROPICAL GEOMETRY AND A JORDAN-HÖLDER THEOREM

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ABSTRACT. A correspondence exists between affine tropical varieties and algebraic objects, following the classical Zariski correspondence between irreducible affine varieties and the prime spectrum of the coordinate algebra in affine algebraic geometry. In this context the natural analog of the polynomial ring over a field is the polynomial semiring over a semifield, but one obtains homomorphic images of coordinate algebras via congruences rather than ideals, which complicates the algebraic theory considerably.

In this paper, we pass to the semifield \( F(\lambda_1, \ldots, \lambda_n) \) of fractions of the polynomial semiring, for which there already exists a well developed theory of kernels; this approach enables us to switch the structural roles of addition and multiplication and makes available much of the extensive theory of chains of homomorphisms of groups. The parallel of the zero set now is the 1-set. (Idempotent semifields correspond to lattice ordered groups, and the kernels to normal convex \( l \)-subgroups.)

These notions are refined in the language of supertropical algebra to \( \nu \)-kernels and 1\(^\nu\)-sets, lending more precision to tropical varieties when viewed as sets of common roots of polynomials. The \( \nu \)-kernels corresponding to (supertropical) hypersurfaces are the 1\(^\nu\)-sets of corner internal rational functions. The kernels corresponding to “usual” tropical geometry are the regular, corner-internal kernels.

In analogy to Hilbert’s celebrated Nullstellensatz which provides a correspondence between radical ideals and zero sets, we develop a correspondence between 1\(^\nu\)-sets and a well-studied class of \( \nu \)-kernels of the rational semifield called polars, originating from the theory of lattice-ordered groups. This correspondence becomes simpler and more applicable when restricted to a special kind of kernel, called principal, intersected with the kernel generated by \( F \). We utilize this theory to study tropical roots in tropical geometry.

As an application, we develop algebraic notions such as composition series and convexity degree, along with notions having a geometric interpretation, like reducibility and hyperdimension, leading to a tropical version of the Jordan-Hölder theorem.

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be the diagonal ray emanating from \((2,2)\). One way out is to introduce the “balancing condition,” given in rather general terms in Yu [34], but this is tricky in higher dimensions and does not relate so well to the algebraic structure.

A more promising direction is to develop an analog of the celebrated Zariski correspondence, in order to understand tropical geometry in terms of the algebraic structure of various semirings† refining the polynomial algebra over the max-plus algebra. Such a correspondence is one of the traditional ways of treating algebraic geometry in terms of universal algebra, as described in [4, Theorem 1.1] and [26], and was adapted to tropical algebra by means of corner roots, as described in [20]; a more structural description of roots was given in [13].

Let \(K = \mathbb{C}\{\{t\}\}\) be the field of Puiseux series on the variable \(t\), which is the set of formal series of the form \(f = \sum_{k=k_0}^{\infty} c_k t^{k/N}\) where \(N \in \mathbb{N}\), \(k_0 \in \mathbb{Z}\), and \(c_k \in \mathbb{C}\). Then \(K\) is an algebraically closed field equipped with a valuation map \(\text{val} : K^* = \mathbb{C}\{\{t\}\} \setminus \{0\} \rightarrow \mathbb{Q} \subset \mathbb{R}\) defined by

\[
\text{val}(f) = - \min_{c_k \neq 0} \{k/N\}.
\]

The \textit{tropicalization} of a Laurent polynomial \(f = \sum c_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\) is defined to be \(\text{trop}(f) : \mathbb{R}^n \rightarrow \mathbb{R}\) given by

\[
\text{trop}(f)(\omega) = \max(\text{val}(c_u) + \omega \cdot u)
\]

where \(u \in \mathbb{N}\) is the power vector and \(\cdot\) is the scalar product.

Given a Laurent polynomial \(f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\), one defines its \textit{tropical hypersurface} as

\[
T(\text{trop}(f)) = \{\omega \in \mathbb{R}^n : \text{the maximum in } \text{trop}(f) \text{ is achieved at least twice}\}
\]

To develop tropical algebraic geometry further, one has the choice either of working directly at the level of polynomials over Puiseux series, and then tropicalizing, or first tropicalizing and then utilizing a Zariski correspondence at the tropical level. The latter is attractive from the point of view of being able to work directly with a simpler concept, and one is led to continue the algebraic study of tropical geometry by means of corner roots of polynomials, developing algebraic geometry over the ordered group \((\mathbb{Q}, +)\); some relevant references are [3, 6, 11, 22].

There has also been recent interest in algebraic geometry over monoids, and much of the basic algebraic material can be found in [5]. But many problems arise in formulating algebraic geometry directly over the max-plus algebra, not the least of which is the failure of the max-plus algebra to reflect the uncertainty involved in taking the value of the sum of two polynomials. This was dealt with in [13, 9], by refining the max-plus algebra to a “supertropical semifield†” \(F\) and, even more generally, to a “layered” semifield†. 6 also treats tropical algebraic geometry in terms of valuations.

When developing tropical affine geometry, it is natural to try to obtain the Zariski-type correspondence between varieties and coordinate semirings. This approach is used for our proposed definition of tropical (affine) variety, in parallel to classical algebraic geometry. Then the usual constructions of schemes and sheaves can be obtained for “semiringed spaces” by noting that negation is not needed in the proofs; we do not go into the details here, for lack of space, but this is to be treated in [17].

Our approach to affine tropical geometry also focuses on a Zariski correspondence parallel to the Zariski correspondence in classical algebraic geometry, which is obtained from the passage between ordered groups and semifields†. Since tropical geometry often is understood in terms of corner roots of polynomials over the target of the order valuation, one should investigate the semiring† \(R = F[\lambda_1, \ldots, \lambda_n]\) of polynomials over a semifield† \(F\), as well as its homomorphic images.

Unfortunately, in general semiring† theory, such homomorphisms are not attained by means of (prime) ideals, but rather via congruences, which are defined as subsets of \(R \times R\) rather than of \(R\). Since congruences replace ideals in the study of homomorphic images, one is led to the study of congruences of semifields†. Several authors, [6, 21, 22] have explored “bend congruences.” But the theory of congruences is much more complicated than the study of ideals of rings, since it involves substructures of \(R \times R\) rather than \(R\). Furthermore, the polynomial algebra over a semifield† need not be Noetherian, cf. [15].

The main innovation of this paper is to bypass these difficulties by switching the roles of multiplication and addition (which is natural enough, since we started with the max-plus algebra). One can view \(F[\lambda_1, \ldots, \lambda_n]\) as a lattice-ordered (multiplicative) monoid and pass to its group of fractions, focusing on the group structure. This point of view leads us to lattice-ordered groups, and their corresponding idempotent semifields†. Put another way, \(F[\lambda_1, \ldots, \lambda_n]\) is not a semifield†, but rather a cancellative semiring†, so we need to pass to its semifield† of fractions \(F(\lambda_1, \ldots, \lambda_n)\), which is called the semifield† of \textit{rational functions} over \(F\). Homomorphisms of idempotent semifields† have been studied long ago in the literature [8, 52, 31], where the homomorphic images are described in terms of what they call (semifield†) \textit{kernels}, which as noted in [8] are just the convex
(normal) $l$-subgroups of the corresponding lattice-ordered groups. Since semifield\textsuperscript{1} kernels are subgroups which can be described algebraically, cf. \cite[Theorem 3.2]{8}, they are more amenable to the classical structure theorems of group theory (Noether isomorphism theorems, Jordan-Hölder theorem, etc.) than congruences.

The parallel notion to a zero set in algebraic geometry now is the set of points which when substituted into a function give the value 1 instead of 0, so we call these sets 1\textsuperscript{ν}-sets. Our ultimate goal is a 1:1 correspondence between sets of corner roots (which in our language are “ghost roots”) of polynomials and 1\textsuperscript{ν}-sets of rational functions.

A kernel is principal if it is generated by one element. One big advantage of the use of kernels is that the product of finitely many principal kernels is a principal kernel, and thus finite intersections of hypersurfaces still can be described in terms of principal kernels.

We come upon various correspondences, but for some mysterious reason, we need to intersect down to the kernel generated by the constant functions, which we call $\langle F \rangle$, in order to obtain the correspondence given in Corollary [8.0.13]\textsuperscript{†}.

The lattice of (tangible) finitely generated corner loci corresponds to the lattice of principal (tangible) kernels of $\langle F \rangle$.

This result, a consequence of Theorem [8.0.13]\textsuperscript{†} to be described soon, enables one to transfer the geometric theory to kernels of semifields\textsuperscript{1}. The preparation for its proof requires a careful description of different kinds of kernels and takes the bulk of this paper.

The application of kernels to tropical geometry was developed by T. Perri in his doctoral dissertation (2013) which essentially is given in \cite[23]{23}. But it turns out that in many cases this notion is equivalent to the bend congruences of \cite[6]{6} mentioned earlier. This will be explained in Remark [0.1.15]\textsuperscript{†}.

To obtain our results algebraically, we need a slight modification of the results about semifields\textsuperscript{†} in the literature. As explained in \cite[12]{12} and elaborated in \cite[11]{11}, there are other semirings\textsuperscript{†} $R$ covering the usual idempotent max-plus algebra, which provide better tools for examining the algebraic structure arising from valuations on Puiseux series, and these semirings\textsuperscript{†} are no longer embeddable into semifields\textsuperscript{1}. We list the relevant structures in increasing level of refinement:

- The max-plus algebra
- Supertropical algebra
- Layered tropical algebras
- Exploded layered algebras

In each of the latter three cases, although the multiplicative monoid is not ordered, there is an underlying partially ordered monoid, so we can modify the results about (semifield) kernels in the literature to suit our purposes. Furthermore, in the supertropical and layered situations, there is a “ghost map” $\nu$ from $R$ to the set of “ghost elements” $G$, which enables us to compare elements in $R$ via their images in the ordered monoid $G$. Our emphasis is in the supertropical setting, since we feel that the theory can be described most concisely in that context, even though much of \cite[23]{23} is formulated in terms of the max-plus algebra.

Let us proceed with a more detailed overview of this paper. Since both semirings\textsuperscript{†} and their kernels may be unfamiliar to many researchers, we review ordered monoids and their semirings\textsuperscript{†} in \cite[2]{2} After an introduction to the supertropical theory in \cite[3]{3} the basic notions of semifield\textsuperscript{1} kernels (and their supertropical analogs) are presented in \cite[4]{4}.

The pertinence to tropical geometry is given in \cite[5]{5}. Ghost roots are replaced by kernel roots, those points whose value at each function in the kernel is 1; these are called 1\textsuperscript{ν}-sets, which are the kernel-theoretic analogues of corner loci. The initial correspondence between certain kernels and 1\textsuperscript{ν}-sets is given in Theorem [5.3.6]\textsuperscript{†}.

The interplay of corner roots of polynomials and kernel roots comes in Section [6] In applying the theory to sets of polynomials defining tropical varieties, one quickly sees that there still are too many kernels; i.e., some of the varieties defined by kernels do not satisfy basic tropical conditions such as the balancing condition. Although, in our opinion all kernels define interesting tropical varieties, there is no way to obtain such basic notions as dimension without specifying certain kinds of kernels to obtain the tropical varieties arising from hypersurfaces. These kernels, called corner internal kernels cf. Definition [7.1.1]\textsuperscript{†} lie at the heart of our theory:

A kernel is corner internal if it is generated by a rational function $f = \frac{g}{h}$ such that every ghost root of $g + h$ is a kernel root of $f$.

There are two main ways of passing from hypersurfaces, one given in the basic “hat construction” of \cite[6.1]{6} leading to Theorem [6.1.7]\textsuperscript{†} stating that the correspondences $f \mapsto \hat{f}$ and $h \mapsto \hat{h}$ induce 1:1 correspondences between hypersurfaces and 1\textsuperscript{ν}-sets of corner internal rational functions.

\textsuperscript{†} 

\textsuperscript{1}
Another, subtler way, of describing corner internal rational functions is given in Definition 7.3.6 and Theorem 7.3.9. In §6.4 we specify the regular rational functions, which locally are not the identity, and thus have tropical significance, distinguishing between two general types of nontrivial principal $1^e$-sets in $F^{(n)}$:

- $1^e$-sets not containing a region of dimension $n$;
- $1^e$-sets that do contain a region of dimension $n$.

The kernels defining “traditional” tropical hypersurfaces are the regular corner internal kernels.

We further narrow down the kernels of interest, finally showing how one obtains a strong Zariski correspondence using regular principal kernels. The corresponding hypersurfaces are of the expected codimension 1. We would rather deal with principal kernels since they correspond to hypersurfaces.

There is a basic question as to over which semifields $\mathbb{F}$ we should define our kernels. Although most of the results are for kernels of $F(\lambda_1, \ldots, \lambda_n)$, the rational fractions of the polynomial semiring $\mathbb{R}$ over an arbitrary supertropical semifield $F$, in §8.2 we add the condition that the underlying semifield $\mathbb{F}$ be complete. At times we need to assume that $F$ is complete, archimedean, and divisible, in order to obtain principal kernels.

Most of §8 involves intersecting down to $\langle F \rangle$, defined as the kernel generated by $F$. In Theorem 8.0.13 we formulate a major result:

All (regular) principal kernels of $\langle F \rangle$ are generated by (regular) corner internal principal kernels.

See Example 8.6.2 as to how this result applies to the familiar tropical line. Its proof requires some more machinery developed later on, and requires us to take $F$ to be complete under the order topology, say $F = \mathbb{R}$, the supertropical version of the real numbers (viewed as a max-plus structure).

This decomposition also gives rise to basic geometric tropical concepts such as dimension, which is seen to match the intuitive geometric notion. Towards this end, we define a hyperplane kernel, or HP-kernel, a hyperspace-fraction kernel, or HS-kernel, a hyperspace-fraction kernel, or HS-kernel, is a principal kernel of $F(\lambda_1, \ldots, \lambda_n)$ generated by a single Laurent monomial. More generally, a hyperspace-fraction kernel, or HS-kernel, is a principal kernel of $F(\lambda_1, \ldots, \lambda_n)$ generated by a rational function $f \sim (F) \sum_{i=1}^n |f_i|$ where the $f_i$ are non-proportional non-constant Laurent monomials, and corresponds to a hyperplane in polyhedral geometry. On the other hand, we also investigate order kernels, which correspond to a half planes. Together they are the building blocks of polyhedral geometry, and can be used to construct any polyhedron.

The height of an HS-kernel is given in Definition 11.1.9 and the convex dimension then is given in Definition 11.3.10.

This dimension is catenary, in the sense of Theorem 11.4.7.

If $R$ is a region kernel and $L$ is an HS-kernel of $F(\Lambda)$, then

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(F(\Lambda)) - d_{\text{conv}}(L).$$

In particular, $d_{\text{conv}}(F(\Lambda)/LR) = n - d_{\text{conv}}(L)$, leading us to the Jordan-Hölder type Theorem 12.0.5.

If a kernel $L$ is irreducible in the lattice of kernels finitely generated by HP-kernels of $F(\Lambda)$, then $\text{ht}(L)$ is its convex dimension. Moreover, every factor of a descending chain of HS-kernels of maximal length is an HP-kernel.

Let us describe some more ingredients in the proof of Theorem 8.0.13. In order to move back and forth between geometry and algebra, we want functions with the same $1^e$-set to generate the same kernel. This fails in general, and forces us to restrict to bounded kernels in the $\nu$-semifield $\langle F \rangle$ of rational functions, cf. §8.2.2. The interplay of principal bounded and unbounded kernels is given in Proposition 8.3.1.

Although different principal kernels can have the same $1^e$-sets, we remedy this by intersecting with $\langle F \rangle$, cf. Proposition 8.4.2. We get the desired 1:1 correspondence between principal $1^e$-sets and principal sub-kernels of $\langle F \rangle$ in Theorem 8.4.3. The wedge decomposition is introduced in Definition 8.5.2 and its existence is proved in Theorem 8.5.3.

But we still need more preparation to prove Theorem 8.0.13. Our classes of kernels are further subdivided in §9.4 into “HO-kernels” and another kind of kernel which can be bypassed when restricting from $F(\Lambda)$ to $\langle F \rangle$. HO-kernels are decomposed as products of “HS-kernels” and “region kernels” in Theorem 9.4.8.

Every principal kernel $\langle f \rangle$ of $F(\lambda_1, \ldots, \lambda_n)$ can be written as the intersection of finitely many principal kernels

$$\{K_i : i = 1, \ldots, q\} \text{ and } \{N_j : j = 1, \ldots, m\},$$

whereas each $K_i$ is the product of an HS-kernel and a region kernel, whereas each $N_j$ is a product of “bounded from below” kernels and (complementary) region kernels.

This enables us to reduce to finite decompositions and finally prove Theorem 8.0.13.
The lattice of principal subkernels of the kernel \( \langle F \rangle \) (viewed as a semifield\(^1\)) is generated by the principal corner internal kernels, and the sublattice of regular principal kernels is generated by the regular, principal corner internal kernels.

Just as the Hilbert Nullstellensatz shows that affine varieties correspond to radical ideals of the polynomial algebra, here affine tropical varieties correspond to certain kernels called polars, defined intrinsically in terms of the natural orthogonality relation that \( f \perp g \) for \( f, g \in F(X) \) if their minimum always takes on the value 1.

Polars are best understood as kernels in the completion of the natural image of \( F(\lambda_1, \ldots, \lambda_n) \) inside the semiring\(^1\) of functions from some subset \( S \subseteq F^{(n)} \) to \( F \), but we can take the restriction back to \( F(\lambda_1, \ldots, \lambda_n) \). Unfortunately, finitely generated polars need not be finitely generated as kernels, which is another reason for us to restrict to bounded kernels.

We discuss some of their basic properties in \([11]\) leading up to the basic correspondence of Theorem \([10.3.9]\).

There is a 1 : 1 correspondence from the polars of \( \mathcal{R}(\Lambda) \) to the \( 1^\nu \)-sets in \( \mathcal{R}^n \), given by \( B \mapsto 1_{\log}(B) \) and \( Z \mapsto Kern(Z) \), which restricts to a correspondence between the principal polars (as polars) of \( \mathcal{R}(\Lambda) \) and the principal \( 1^\nu \)-sets in \( \mathcal{R}^n \).

In this way we reduce kernels to finitely generated kernels, which then are principal, and we obtain the appropriate Zariski correspondence in its entirety.

## 2. Algebraic Background

We start by reviewing some familiar notions that are needed extensively in our exposition.

### 2.1. Semi-lattice ordered monoids, groups, and semirings.

The passage to the max-plus algebra in tropical mathematics can be viewed via ordered groups and, more generally, ordered monoids and semirings, so we start with them, drawing on the review given in \([12]\). This background material is well known, and largely can be found in \([29]\). Recall that a monoid \( (M, \cdot, 1) \) is just a semigroup with an identity, i.e., unit element, denoted as 1. We work here with Abelian monoids, in which the operation is commutative. Our semigroups without identity are written in additive notation, and monoids are written in multiplicative notation.

**Definition 2.1.1.** A (set-theoretic) semi-lattice \( L \) is a partially ordered set with an (associative) binary “sup” operation \( \lor \), which means:

\[
(2.1) \quad a, b \leq a \lor b, \quad \text{and if } a, b \leq c \text{ then } a \lor b \leq c.
\]

**Definition 2.1.2.** An element \( a \) of a semigroup \( (S,+) \) is **idempotent** if \( a + a = a \). A **band** is a semigroup in which every element is idempotent. A band \( (S,+) \) is **bipotent** if \( a + b \in \{a, b\} \) for each pair of elements.

**Lemma 2.1.3.** Any semi-lattice can be viewed as a band, where we define \( a + b = a \lor b \).

*Proof.* To check associativity of addition we have \((a + b) + c = \text{sup}\{a, b, c\} = a + (b + c)\). Furthermore \(a + a = a \lor a = a\). \(\square\)

**Definition 2.1.4.** A **semi-lattice ordered monoid** is a monoid \( M \) that is also a semi-lattice with respect to the operation \( \lor \) and satisfies the following property:

\[
(2.2) \quad g(a \lor b) = ga \lor gb
\]

for all elements \( a, b, g \in M \).

A **partially ordered monoid** is a monoid \( M \) with a partial order satisfying

\[
(2.3) \quad a \leq b \quad \text{implies} \quad ga \leq gb,
\]

for all elements \( a, b, g \in M \).

It follows readily that if \( a \leq b \) and \( g \leq h \), then \( ga \leq gb \leq hb \).

This definition is subtle. The group \( (\mathbb{R}, +) \) is ordered, under this definition, but \( (\mathbb{R}, \cdot) \) is not; for example, \( a > b \) in \( (\mathbb{R}, \cdot) \) does not imply \( -a > -b \), but rather \( -a < -b \).

Clearly every semi-lattice ordered monoid is partially ordered with respect to the partial order \( a \leq b \) iff \( a \lor b = b \).
Definition 2.1.5. An ordered monoid is a partially ordered monoid for which the given partial order is a total order.

A lattice ordered monoid is a monoid $M$ that is also a lattice with respect to the operations $\lor$ and $\land$ satisfying $2.2$ for both $\lor$ and $\land$.

In case a semi-lattice ordered monoid $M$ is a group $G$, one defines the dual semi-lattice $(G, \land)$ via

$$a \land b = (a^{-1} \lor b^{-1})^{-1}.$$  

Now $G$ becomes a lattice, and we call $(G, \lor, \land)$ a lattice ordered group if $G$ is a lattice ordered monoid.

Lemma 2.1.6. Suppose $(G, \lor)$ is a semi-lattice ordered Abelian group with respect to a given partial order $\leq$. Then $(G, \land)$ is indeed a semi-lattice, with respect to which $G$ also becomes a lattice ordered group satisfying $(a \lor b)^{-1} = a^{-1} \land b^{-1}$.

Proof. $a \leq b$ iff $b^{-1} = (ab)^{-1}a \leq (ab)^{-1}b = a^{-1}$, so we get the dual partial order $\geq$. Hence $(G, \land)$ is a semi-lattice with respect to $\geq$, since

$$a \land b = (a^{-1} \lor b^{-1})^{-1} = \inf \{a^{-1}, b^{-1}\} = \inf \{(a^{-1})^{-1}, (b^{-1})^{-1}\} = \inf \{a, b\}.$$  

Then $g(a \land b) = g(a^{-1} \lor b^{-1})^{-1} = (g^{-1}(a^{-1} \lor b^{-1})^{-1})^{-1} = (g^{-1}a^{-1} \lor g^{-1}b^{-1})^{-1} = ga \land gb$, as desired.

The last assertion follows from (2.4) taking $a^{-1}, b^{-1}$ instead of $a, b$.  

The duality in Lemma 2.1.6 shows that it suffices to consider only $\lor$ (or only $\land$). Every partially ordered group that is a semi-lattice is a lattice ordered group, seen by using $g^{-1}$ instead of $g$ in (2.2).

In any ordered group we define the open interval of distance $b$ around $a$ given by $\{g : ab^{-1} < g < ab\}$, giving rise to the order topology.

Although idempotence pervades the theory, it turns out that what is really crucial for many applications is the following well-known fact:

Lemma 2.1.7. In any band, if $a + b + c = a$, then $a + c = a$.

Proof. $a = a + b + c = (a + b + c) + c = a + c$.  

Let us call a semigroup proper if it satisfies the condition of Lemma 2.1.7.

Remark 2.1.8 (31 Theorem 4.3]). A proper semigroup cannot have additively invertible elements other than 0, since if $a + c = 0$, then $a = a + 0 = a + a + c$, implying $a = a + c = 0$.

A submonoid $T$ of an Abelian monoid $M$ is called cancellative if $ab = ac$ for $a \in T$ and $b, c \in M$ implies $b = c$. In this case, when $T = M$, we say that the Abelian monoid $M$ is cancellative.

A monoid $M$ is power-cancellative (called torsion-free in [3]) if $a^n = b^n$ for some $n \in \mathbb{N}$ implies $a = b$. Any ordered, cancellative monoid is power-cancellative and infinite.

2.1.1. Divisibility.

We say that a monoid $M$ is $\mathbb{N}$-divisible (also called radicalizable in the tropical literature, but that terminology conflicts with [33] and radical theory) if for each $a \in M$ and $m \in \mathbb{N}$ there is $b \in M$ such that $b^m = a$. For example, $(\mathbb{Q}, +)$ is $\mathbb{N}$-divisible. There is a standard construction of the divisible hull of a cancellative monoid $M$, given in [29], which is semi-lattice ordered when $M$ is semi-lattice ordered; namely,

$$\forall a \lor \forall b = \sup\{a^n \lor b^m : a, b, n, m \in \mathbb{N}\}.$$  

This also was discussed in [31] Remark 2.3.

2.2. Semirings without zero.

Definition 2.2.1. A semiring (semiring without zero, at times called a hemiring) is a set $R := (R, +, \cdot, 1)$ equipped with binary operations $+$ and $\cdot$ and distinguished element $1_R$ such that:

(i) $(R, +)$ is an Abelian semigroup;
(ii) $(R, \cdot, 1_R)$ is a monoid with identity element $1_R$;
(iii) Multiplication distributes over addition.
(iv) $R$ contains elements $r_0$ and $r_1$ with $r_0 + r_1 = 1_R$.

Note that (iv) is automatic if $(R, +)$ is a band, since then $1_R + 1_R = 1_R$. For the purposes of this paper, a domain is a commutative semiring whose multiplicative monoid is cancellative.

Definition 2.2.2. A semifield is a domain in which every element is (multiplicatively) invertible.
(In other words, its multiplicative monoid is an Abelian group. In \[31\] commutativity is not assumed, but we make this assumption to avoid distraction from our applications.) In particular, the max-plus algebras \((\mathbb{N}, +), (\mathbb{Q}, +),\) and \((\mathbb{R}, +)\) are semifields\(^\dagger\), whose multiplication now is given by \(+\). We also have the trivial semifield\(^\dagger\) \((1)\). In the literature it is customary to write the operations as \(\oplus\) and \(\odot\), but we use the usual algebraic notation of \(+\) and \(\cdot\) for addition and multiplication respectively, to emphasize the structural aspects of the theory.

**Digression:** The customary definition of semiring \([7]\) also requires the existence of a zero element:

A *semiring* is a semiring\(^\dagger\) \(R\) with a zero element \(0_R\) satisfying

\[
a + 0_R = a, \quad a \cdot 0_R = 0_R = 0_R \cdot a, \quad \forall a \in R.
\]

(Note that in the definition of semiring\(^\dagger\) one could then take \(r_0 = 0_R\) and \(r_1 = 1_R\).) We use semifields\(^\dagger\) instead of semifields since the zero element usually can be adjoined formally, and often is irrelevant, and the concepts are easier to define when we do not need to treat the zero element separately. Also, the language of semifields\(^\dagger\) is more appropriate to geometry over tori, which are direct products of groups.

Not every semiring \(R\) can be obtained by adjoining \(0\) to a semiring\(^\dagger\), since \(R \setminus 0\) need not be closed under multiplication. However, this is the case for domains, by definition.

A semiring\(^\dagger\) \(R\) is *idempotent* if the semigroup \((R, +)\) is a band. A semiring\(^\dagger\) is *bipotent* if the semigroup \((R, +)\) is bipotent. Thus, the max-plus algebra, viewed as a semiring\(^\dagger\), is bipotent.

Our basic structures are idempotent semifields\(^\dagger\), denoted as \(F\) or \(\mathbb{S}\) throughout. (Usually \(F\) is the underlying semifield\(^\dagger\), contained in \(\mathbb{S}\).) Let us recall an idea of Green from the theory of idempotent semigroups.

**Proposition 2.2.3** \(([32], \S 4)\). *Any semi-lattice ordered Abelian monoid \(M\) becomes an idempotent (commutative) semiring\(^\dagger\), which we denote as \(R\), via the usual max-plus procedure; we define the new multiplication on \(R\) to be the operation given originally on \(M\), and addition on \(R\) is defined as in Lemma 2.1.3:*

\[
a + b := a \vee b
\]

(viewed in \(M\).) Conversely, *any idempotent commutative semiring\(^\dagger\) \(R\) becomes a semi-lattice ordered Abelian monoid by reading (2.6) backwards.*

This provides a category isomorphism from the category of semi-lattice ordered Abelian monoids, whose morphisms are monoid homomorphisms which also are semi-lattice homomorphisms (i.e., \(\varphi(a \vee b) = \varphi(a) \vee \varphi(b)\)), to the category of idempotent commutative semiring\(^\dagger\).

**Proof.** Distributivity follows from (2.2). Furthermore \(a + a = a \vee a = a\).

Conversely, if \(a \leq b\) and \(b \leq a\) we have \(a = a + b = b\), so \(\leq\) is antisymmetric; we need to show that \(a + b\) is the sup of \(a\) and \(b\). In other words, if \(a \leq c\) and \(b \leq c\) then \(a + b \leq c\). But we are given \(a + c = c\) and \(b + c = c\), so

\[
a + b + c = (a + c) + (b + c) = c + c = c,
\]

as desired. \(\square\)

When \(M\) is cancellative, then \(R\) is a domain\(^\dagger\). When \(M\) is a group, then \(R\) is a semifield\(^\dagger\).

Any semifield\(^\dagger\) without negatives is proper, by \([8]\) Proposition 20.37].

**Lemma 2.2.4** \(([30], \text{Property 2.6})\). *Every domain\(^\dagger\) is torsion-free as a monoid.*

**Proof.** If \(a^n = 1\), then \(a(a^{n-1} + a^{n-2} + \cdots + 1) = (1 + a^{n-1} + a^{n-2} + \cdots + a)\), implying \(a = 1\). \(\square\)

**Definition 2.2.5.** A semiring\(^\dagger\) \(R\) is *ordered* if \((R, +)\) and \((R, -)\) are both ordered monoids.

**Remark 2.2.6.** The multiplicative group of every ordered semifield\(^\dagger\) is a torsion-free group, i.e., all of its elements not equal to 1 have infinite order.

Here is another important property:

**Definition 2.2.7.** A semiring\(^\dagger\) \(R\) is *Frobenius* if it satisfies the identity

\[
(a + b)^n = a^n + b^n, \quad \forall a, b \in R.
\]

It is clear that any bipotent semiring\(^\dagger\) is Frobenius, although the converse may fail (in view of Proposition 3.5.2]. Also, any homomorphic image of a Frobenius semiring\(^\dagger\) is Frobenius.
**Definition 2.2.8.** A lattice $P$ is said to be **complete** if all of its bounded subsets have both a supremum and an infimum.

A sublattice of $P$ is said to be **completely closed** if all of its subsets have both a supremum and an infimum in $P$.

An idempotent semiring† is said to be **complete** if its underlying lattice (with $\lor$ as $+$ and $\land$ as $\cdot$) is complete.

### 2.3. Localization.

Since we are mainly interested in (proper) idempotent semifields†, we need a method of passing from semirings† to semifields†.

**Remark 2.3.1.** There is a well-known localization procedure with respect to multiplicative subsets of Abelian monoids, described in detail in [2]. Namely, for any submonoid $S$ of a monoid $M$, we define an equivalence on $S \times M$ by saying $(s_1, a_1) \sim (s_2, a_2)$ iff there is $s \in S$ such that $ss_2a_1 = ss_1a_2$; then the localization $S^{-1}M$ is the set of equivalence classes $\{[(s, a)]: s \in S, a \in M\}$, written as $\frac{a}{s}$. $S^{-1}M$ is a monoid via the operation

$$
\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.
$$

There is a homomorphism $M \rightarrow S^{-1}M$ given by $a \mapsto \frac{a}{1}$. This map is $1:1$ precisely when $a \neq b$ implies $\frac{a}{1} \neq \frac{b}{1}$; in other words, when the submonoid $S$ of $M$ is cancellative.

If the monoid $M$ itself is cancellative, then localizing with respect to all of $M$ yields its **group of fractions**. In this case $\frac{a}{s} = \frac{a'}{s'}$ iff $ss_2a_1 = ss_1a_2$.

If the monoid $M$ is ordered, then $S^{-1}M$ is also ordered, by putting $\frac{a}{s} \leq \frac{a'}{s'}$ iff $as' \leq a's$ for some $s \in S$.

When $R$ is a commutative semiring† and $S$ is a submonoid of $(R, \cdot)$, we endow $S^{-1}R$ with addition given by

$$
\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.
$$

When $(R, \cdot)$ is ordered, this is compatible with the monoid order on $S^{-1}R$, in the sense of Proposition 2.2.3.

Clearly, the localization of an idempotent (resp. proper, resp. Frobenius) semiring† is idempotent (resp. proper, resp. Frobenius).

Furthermore, if $R$ is a domain†, and $S = R$, then we call $S^{-1}R$ the **semifield† of fractions** of $R$.

**Remark 2.3.2.** These considerations are compatible with localization and the divisible hull. For example, one can define $\lor$ on the monoid $S^{-1}M$ via

$$
\frac{a}{s} \lor \frac{b}{s} = \frac{a \lor b}{s}.
$$

For the divisible hull, we would use the analog of Equation 2.5.

**Proposition 2.3.3.** Suppose $R, R'$ are domains†, with a semiring† homomorphism $\varphi : R \rightarrow R'$, and suppose $S$ is a submonoid of $R$. Then there is a homomorphism $\bar{\varphi} : S^{-1}R \rightarrow \bar{\varphi}(S)^{-1}R'$, given by $\bar{\varphi}(\frac{a}{s}) = \frac{\varphi(a)}{\varphi(s)}$.

**Proof.** This is standard. First we need to show that $\bar{\varphi}$ is well-defined: If $\frac{a_1}{s_1} = \frac{a_2}{s_2}$ then $ss_2a_1 = ss_1a_2$ so $\varphi(s)\varphi(s_2)\varphi(a_1) = \varphi(s)\varphi(s_1)\varphi(a_2)$, implying $\varphi(s_2)\varphi(a_1) = \varphi(s_1)\varphi(a_2)$, and thus $\bar{\varphi}(\frac{a_1}{s_1}) = \bar{\varphi}(\frac{a_2}{s_2})$.

Now $\bar{\varphi}\left(\frac{a_1}{s_1} + \frac{a_2}{s_2}\right) = \bar{\varphi}\left(\frac{a_1a_2}{s_1s_2}\right) = \frac{\varphi(a_1a_2)}{\varphi(s_1s_2)} = \bar{\varphi}\left(\frac{a_1}{s_1}\right) + \bar{\varphi}\left(\frac{a_2}{s_2}\right)$. \(\square\)

### 2.4. Congruences.

A **congruence** $\Omega$ of an algebraic structure $A$ is an equivalence relation $\equiv$ that preserves all the relevant operations and relations; we call $\equiv$ the **underlying equivalence** of $\Omega$. One can view a congruence more formally as a subalgebra of $A \times A$. For ease of notation we write $a \equiv b$ when $(a, b) \in \Omega$. In the case of semirings†, $\Omega$ being a congruence means that $ai \equiv bi$ for $i = 1, 2$ implies $a_1 + a_2 \equiv b_1 + b_2$ and $a_1a_2 \equiv b_1b_2$.

**Remark 2.4.1.** We recall some key results of [18 §2]:

(i) Given a congruence $\Omega$ of an algebraic structure $R$, one can endow the set

$$
A/\Omega := \{[a]: a \in A\}
$$

of equivalence classes with the same (well-defined) algebraic structure, and the map $a \mapsto [a]$ defines an onto homomorphism $A \rightarrow A/\Omega$. 

(ii) In the opposite direction, for any homomorphism $\varphi : A \to A'$, one can define a congruence $\Omega_{\varphi}$ on $A$ by saying that

$$a \equiv_{\varphi} b \quad \text{iff} \quad \varphi(a) = \varphi(b).$$

We call $\Omega_{\varphi}$ the congruence of $\varphi$. Then $\varphi$ induces a $1:1$ homomorphism $\bar{\varphi} : A/\Omega_{\varphi} \to A'$, via $\bar{\varphi}(\bar{a}) = \varphi(a)$, for which $\varphi$ factors through

$$A \to A/\Omega_{\varphi} \to A',$$

as indicated in [18, p. 62]. Thus the homomorphic images of $A$ correspond to the congruences defined on $A$.

**Definition 2.4.2.** A congruence $\Omega$ on an Abelian monoid $M$ is cancellative when the monoid $M/\Omega$ is cancellative. The congruence $\Omega$ on an Abelian monoid $M$ is power-cancellative when the monoid $M/\Omega$ is power-cancellative, i.e., if $a_k^i \equiv a_k^j$ for some $k \geq 1$ implies $a_1 \equiv a_2$.

For the moment, let $\mathcal{C}(M)$ denote a given set of cancellative and power-cancellative congruences on $M$. A congruence $\Omega \in \mathcal{C}(M)$ is irreducible if it is not the proper intersection of two congruences in $\mathcal{C}(M)$. This seems to be the most general useful tropical venue for defining a spectrum on $M$, when we also take the given order into account. It is not exactly the same as $\text{Spec}(M)$ defined in [5] (for there, $A/I$ denotes the Rees quotient with respect to the monoid ideal $I$), but the underlying idea is analogous, and much of [5] can be lifted with the same proofs.

**Lemma 2.4.3.** Any congruence $\Omega$ on a semiring $^1 R$ extends to its localization $S^{-1}R$ by a cancellative sub-monoid $S$, by putting $\frac{a}{s} \equiv \frac{a'}{s'}$ when $as' \equiv a's$.

**Proof.** It is standard and easy that $\equiv$ extends to the given equivalence on $S^{-1}R$, so we need check merely that the given operations are preserved: If $\frac{a_1}{s_1} \equiv \frac{a_1'}{s_1'}$ and $\frac{a_2}{s_2} \equiv \frac{a_2'}{s_2'}$, then $a_1s_1'a_2s_2' \equiv a_1's_1a_2's_2$ and thus

$$\frac{a_1a_2}{s_1s_2} = \frac{a_1a_2'}{s_1s_2'} \equiv \frac{a_1'a_2'}{s_1's_2'}.$$

Likewise,

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1s_2 + a_2s_1}{s_1s_2} \equiv \frac{a_1's_2' + a_2's_1'}{s_1's_2'} \equiv \frac{a_1'}{s_1'} + \frac{a_2'}{s_2'}.$$

2.5. The function monoid and semiring $^1$.

The supertropical structure will permit us to detect corner roots of tropical polynomials in terms of the algebraic structure, by means of ghosts. To see this most clearly, we introduce another structure in universal algebra.

**Definition 2.5.1.** The function algebra $\text{Fun}(S, A)$ is the set of functions from a set $S$ to an algebra $A$.

This can be viewed as the direct product of copies of $A$, indexed by $S$, so has the parallel algebraic structure as $A$, seen by evaluating the relevant operations at each element of $S$. As special cases, we have:

**Definition 2.5.2.** The function monoid $\text{Fun}(S, M)$ is the set of functions from a set $S$ to a monoid $M$.

The function semiring $^1 \text{Fun}(S, R)$ is the set of functions from a set $S$ to a semiring $^1 R$.

Thus, when $M$ is a monoid, $\text{Fun}(S, M)$ becomes a monoid under the componentwise operation, $fg(a) = f(a)g(a)$, and is cancellative when $M$ is cancellative. If $M$ is semi-lattice ordered, then so is $\text{Fun}(S, M)$, where we define $(f \lor g)(a) = f(a) \lor g(a) = f(a) + g(a)$.

When $R$ is a semiring $^1$, $\text{Fun}(S, R)$ becomes a semiring $^1$ under componentwise operations, and is idempotent (resp. proper) when $R$ is idempotent (resp. proper).

When $R$ is a semifield $^1$, $\text{Fun}(S, R)$ becomes a semifield $^1$.

Note that the semiring $^1 \text{Fun}(S, R)$ need not be idempotent even if $R$ is idempotent, since some of the evaluations of $f + g$ might come from $f$ and others from $g$. As in Proposition 2.2.3 we identify the semiring $^1$ structure of $\text{Fun}(S, R)$ with the semilattice operation $f \lor g = f + g$. When $R$ is a semifield $^1 F$, we can also define

$$(2.7) \quad f \land g = (f^{-1} + g^{-1})^{-1}.$$

**Remark 2.5.3.** Viewing $F$ as an ordered group, one sees easily that $(f \land g)(a) = \min\{f(a), g(a)\}$, and this is how we think of it, although $(2.7)$ is easier to handle formally.
Often $S$ is taken to be $F^{(n)}$, the Cartesian product of $n$ copies of $F$. But using $S \subseteq F^{(n)}$ enables one to lay the foundations of algebraic geometry.

For any subset $S' \subseteq S$, there is a natural homomorphism

$$\text{Fun}(S, F) \to \text{Fun}(S', F),$$

given by restriction of functions.

### 2.6. Archimedean monoids.

**Definition 2.6.1.** A partially ordered monoid $(G, \cdot)$ is called archimedean if $a^\mathbb{Z} \leq b$ implies that $a = 1$.

A semiring† $(R, +, \cdot)$ is archimedean if it is archimedean as a partially ordered monoid.

This reduces to the usual definition when $G$ is totally ordered, but we need this more general version, because of the next example.

**Example 2.6.2.** The semifield $\mathbb{N}$ is ordered and archimedean, as are $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$.

**Remark 2.6.3.** If $R$ is archimedean then $\text{Fun}(S, R)$ also is archimedean.

### 3. The tropical environment

We would like to see how this all fits into tropical mathematics. Tropical mathematics involves using the Puiseux valuation to pass from the field of Puiseux series to its value group $(v)_{\mathbb{R}}$.

Although this looks rather trivial, it becomes useful for us to have two copies of the next example.

**Example 2.6.2.** The semifield $\mathbb{N}$ is ordered and archimedean, as are $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$.

**Remark 2.6.3.** If $R$ is archimedean then $\text{Fun}(S, R)$ also is archimedean.

### 3.1. The translation to monoids.

To view tropicalization at the tropical level, we bring in a version of valuations reminiscent of [10, Definition 2.1]:

**Definition 3.1.1.** An m-valuation of a monoid $M$ is a monoid homomorphism $v : M \to G$ to a semi-lattice ordered monoid $G$.

We can arrange for $v$ to be onto, by replacing $G$ by $v(R)$.

**Remark 3.1.2.** One reason that we introduce this definition is to enable us to replace $M$ by $M/\Omega$, where $\Omega$ is defined via $a \equiv b$ if $v(a) = v(b)$. By Noether’s isomorphism theorem, $v$ induces an isomorphism $M/\Omega \to G$. Although this looks rather trivial, it becomes useful for us to have two copies of $G$ at our disposal.

**Remark 3.1.3.** For any semilattice-ordered Abelian group $G$, and any submonoid $S \subset M$, any m-valuation $v : M \to G$ extends to a m-valuation $S^{-1}M \to G$ given by $v(\frac{a}{s}) = v(r) - v(s)$.

### 3.2. The standard supertropical semifield†.

We are ready to bring in the algebraic structure that reflects the properties of the Puiseux valuation.

**Definition 3.2.1.** A $\nu$-domain † is a quadruple $(R, \mathcal{T}, \nu, \mathcal{G})$ where $R$ is a semiring† and $\mathcal{T} \subset R$ is a cancellative multiplicative submonoid and $G \triangleleft R$ is endowed with a partial order, together with an idempotent homomorphism $\nu : R \to G$, with $\nu|_{\mathcal{T}}$ onto, satisfying the conditions:

$$a + b = a \quad \text{whenever} \quad \nu(a) > \nu(b).$$

$$a + b = \nu(a) \quad \text{whenever} \quad \nu(a) = \nu(b).$$

$\mathcal{T}$ is called the tangible submonoid of $R$. $\mathcal{G}$ is called the ghost ideal.

(In practice, $\nu$ is induced from an m-valuation of a field, for example the Puiseux valuation.)

(The last condition is introduced to enable us to move back and forth between tangible and ghost elements.)

**Remark 3.2.2.** This definition is quite close to that of [3], where $\mathcal{T}$ would be the tangible elements and $\mathcal{G}$ the ghost ideal $(1+1)R$. The main differences are as follows:

- Here the target $G$ need not be ordered, so that we can handle semirings of polynomials directly.
- We do not require a zero element.

We write $a^\nu$ for $\nu(a)$, for $a \in R$. We write $a \cong^\nu b$ if $a^\nu = b^\nu$, and say that $a$ and $b$ are $\nu$-equivalent. Likewise we write $a \geq^\nu b$ (resp. $a >^\nu b$) if $a^\nu \geq b^\nu$ (resp. $a^\nu > b^\nu$).
Remark 3.2.3. Any semiring\( ^\dagger \) homomorphism \( \varphi : R \to R' = (R', T', G', \nu') \) of \( \nu \)-domains\( ^\dagger \) satisfies
\[
\varphi(a') = \varphi(a + a) = \varphi(a) + \varphi(a) = \varphi(a')
\]
for all \( a \) in \( R \).

Strictly speaking, the supertropical \( \nu \)-domains\( ^\dagger \) can never be semifields\( ^\dagger \) since the ghost elements are not invertible. Accordingly, we frame the next definition:

Definition 3.2.4. A \( \nu \)-domain\( ^\dagger \) \( R \) is \nu-bipotent if \( a + b \in \{ a, b, a' \} \) for all \( a, b \in R \). \( \nu \)-semifield\( ^\dagger \) \( S \) is a \( \nu \)-domain\( ^\dagger \) for which the tangible submonoid \( T \) is an Abelian group. A supertropical semifield\( ^\dagger \) is a \( \nu \)-semifield\( ^\dagger \) \( F = (F, T, \nu, G) \) for which \( F = T \cup G \) and \( G \) is totally ordered, a special case of [13].

Example 3.2.5. In the case where \( F = T = G, \) with \( \nu = 1_F \), we are back to the semifield\( ^\dagger \) theory. We call this the degenerate case. Although we are not interested in this case, it shows us how to generalize to the \( \nu \)-theory. Namely, we somehow must “lift” the usual theory from \( G \) to \( F \).

Remark 3.2.6. Any supertropical semifield\( ^\dagger \) is \( \nu \)-bipotent.

Example 3.2.7. Any supertropical semifield\( ^\dagger \) in the sense of [13] is a \( \nu \)-semifield\( ^\dagger \), in the sense of this paper.

The most important example, which we denote as \( A \), is defined by taking the tangible submonoid to be \((\mathbb{R}, +)\), together with another ghost copy.

Note that \( \nu\|T \) is an \( m \)-valuation. (This is a major reason why we are interested in \( m \)-valuations.) In this spirit, here is our main example, but we will need our more general definition when we consider functions, and in particular, polynomials. We follow Remark 3.1.2.

Example 3.2.8. Given a monoid \( M \) and an ordered group \( G \) with an isomorphism \( \nu : M \to G \), we write \( a' \) for \( \nu(a) \). The standard supertropical monoid \( R \) is the disjoint union \( T \cup G \) where \( T \) is taken to be \( M \), made into a group by starting with the given products on \( M \) and \( G \), and defining \( ab' \) and \( a'b \) to be \( (ab)' \) for \( a, b \in M \).

We extend \( \nu \) to the ghost map \( \nu : R \to G \) by taking \( \nu|_M = \nu \) and \( \nu_G \) to be the identity on \( G \).

We make \( R \) into a semiring\( ^\dagger \), called the standard supertropical semifield\( ^\dagger \), by defining

\[
a + b = \begin{cases} 
  a & \text{for } a >_\nu b; \\
  b & \text{for } a <_\nu b; \\
  a' & \text{for } a \equiv_\nu b.
\end{cases}
\]

\( R \) is never additively cancellative, since
\[
a + a' = a' = a' + a'.
\]

In order for Condition (iv) of Definition 2.2.1 to hold, we need the valuation \( \nu \) to be nontrivial, in the sense that there exists \( a \in R \) such that \( a <_\nu 1 \).

If needed, we could formally adjoin a zero element in a new component \( R_0 \).

The standard supertropical monoid \( R \) is a cover of the max-plus algebra of \( G \), in which we can “resolve” additive idempotents, in the sense that \( a + a = a' \) instead of \( a + a = a \).

The point here is that although we are interested in \( M = T \), we have our semi-lattice ordered monoid structure in \( G \), which itself can be viewed as an idempotent semiring\( ^\dagger \) with unit element \( 1_R \) (instead of \( 1_R \)). Thus, we obtain \( \nu \)-versions of the previous notions by lifting from \( G \).

The ghost ideal \( G \) is to be treated much the same way that one treats the 0 element in commutative algebra. The standard supertropical semifield\( ^\dagger \) works well with linear algebra, as seen for example in [14], providing many of the analogs to the classical Cayley-Hamilton-Frobenius theory, but our interest here will be its geometric significance.

3.2.1. \( \nu \)-Localization.

If \( R := (R, T, G, \nu) \) is a \( \nu \)-domain\( ^\dagger \), then we call \( F := T^{-1}R \) the \( \nu \)-semifield\( ^\dagger \) of fractions \( \text{Frac}_\nu R \) of \( R \).

Lemma 3.2.9. \( \text{Frac}_\nu R \) is a \( \nu \)-semifield\( ^\dagger \) in the obvious way.

Proof. Define \( \nu(a + b') = \frac{a + b'}{1} \). \( T^{-1}T \) is the group of tangible elements. \( \square \)
3.3. The \((\ast)\)-operation.

Strictly speaking, since a \(\nu\)-semifield\(^{\dagger}\) is not a semifield\(^{\dagger}\), we need to modify our definition of inverse. Considering \(\mathcal{G}\) as a group in its own right, with unit element \(1^{\nu}\), we define \(a^\ast\) to be the inverse of \(a\) in the appropriate group to which it belongs \((\mathcal{T} \setminus \mathcal{G})\). Specifically, we have:

**Remark 3.3.1.** Any supertropical semifield\(^{\dagger}\) \((F, \mathcal{T}, \mathcal{G}, \nu)\) has a monoid automorphism \((\ast)\) of order 2 given by

\[
a^\ast = a^{-1}, \quad (a^\nu)^\ast = (a^{-1})^\nu, \quad a \in \mathcal{T}.
\]

**Remark 3.3.2.** The map \((\ast)\) is a monoid automorphism of order 2, in the sense that \((a^\ast)^\ast = a\), and \((ab)^\ast = a^\ast b^\ast\) for all \(a, b \in F\). But \((\ast)\) reverses the partial order induced by \(\nu\), and thus does not preserve addition.

**Remark 3.3.3.** There is ambiguity in defining \(a \land b\) in a supertropical semifield\(^{\dagger}\), since we can only compare \(\nu\)-values. So we define

\[
a \land b = (a^\ast + b^\ast)^\ast.
\]

But now \(a \land a = a^\nu\).

3.3.1. The \(\nu\)-norm.

Since the element \(1_F\) plays an important role, the following notion will be useful.

**Definition 3.3.4.** Suppose \(F = (F, \mathcal{T}, \mathcal{G}, \nu)\) is a supertropical semifield\(^{\dagger}\). The \(\nu\)-norm \(|a|\) of an element \(a \in F\) is \(a + a^\ast\).

In the degenerate case of Example 3.2.5, \(|a| = a + a^{-1}\). In the max-plus semifield\(^{\dagger}\) this is the parallel of the usual absolute value of \(a\).

**Remark 3.3.5.** (i) \(|a|^2 = 1^\nu + |a^2| \geq \nu 1\).

(ii) \(|a| \equiv \nu 1\) iff \(a \equiv \nu 1\).

3.4. The function \(\nu\)-domain\(^{\dagger}\).

This supertropical structure also permits us to detect corner roots of tropical polynomials in terms of the algebraic structure by means of ghosts. To see this most clearly, we provide a \(\nu\)-structure for the function semiring\(^{\dagger}\) of Definition 2.5.2.

**Example 3.4.1.** If \((R, \mathcal{T}, \mathcal{G}, \nu)\) is a \(\nu\)-domain\(^{\dagger}\), then \(\text{Fun}(S, R)\) is a \(\nu\)-domain\(^{\dagger}\) with respect to the map \(\nu : \text{Fun}(S, R) \to \text{Fun}(S, \mathcal{G})\) given by

\[
\nu(f)(a) = \nu(f(a)).
\]

Here the ghosts are \(\text{Fun}(S, \mathcal{G})\). One might expect the tangibles to be \(\text{Fun}(S, \mathcal{T})\), but this would exclude polynomials with tangible coefficients, which are our main interest. We come back to this issue shortly.

When \(F\) is a \(\nu\)-semifield\(^{\dagger}\), then \(\text{Fun}(S, F)\) also is a \(\nu\)-semifield\(^{\dagger}\), seen by pointwise verification.

We extend the \((\ast)\)-map to \(\text{Fun}(S, F)\).

**Definition 3.4.2.** For \(F\) a supertropical semifield\(^{\dagger}\) and \(f \in \text{Fun}(S, F)\), define \(f^\ast\) via \(f^\ast(a) = f(a)^\ast\) for all \(a \in F\).

By Remark 3.3.2 the map \(f \mapsto f^\ast\) defines a monoid automorphism of \(\text{Fun}(S, F)\) of order 2. In this way, we have the following lattice operations on \(\text{Fun}(S, F)\):

\[
f \lor g = f + g; \quad f \land g = (f^\ast + g^\ast)^\ast.
\]

**Definition 3.4.3.** For any semitropical semifield\(^{\dagger}\) \(F = (F, \mathcal{T}, \mathcal{G}, \nu)\) and any subset \(A \subseteq \text{Fun}(S, F)\), \(A^+\) denotes \(\{f \in A : f >_\nu 1\}\).

**Remark 3.4.4.** If \(R\) is a \(\nu\)-domain\(^{\dagger}\), then so is \(R^+\).

3.4.1. Polynomials and Laurent polynomials.

\(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\) always denotes a finite commuting set of indeterminates; often \(n = 1\), and we have a single indeterminate \(\lambda\).

Given a semiring\(^{\dagger}\) \(R\), we have the polynomial semiring\(^{\dagger}\) \(R[\Lambda]\). Just as in [13], we view polynomials in \(R[\Lambda]\) as functions. More precisely, for any subset \(S \subseteq R^{(n)}\), there is a natural semiring\(^{\dagger}\) homomorphism

\[
\psi : R[\Lambda] \to \text{Fun}(S, R),
\]

obtained by viewing a polynomial as a function on \(S\).
Remark 3.4.5. If \( R = (R, \mathcal{T}, G, \nu) \) is a \( \nu \)-domain\(^\dagger\), then so is \((R[\Lambda], \mathcal{T}[\Lambda], G[\Lambda], \nu)\) (where \( \lambda^\nu = 1^\nu_{R\Lambda} \)).

This definition is concise, but we should note the difficulty that two polynomials, one tangible, and one non-tangible, can describe the same function, for example \( x^2 + 6 \) and \( (\lambda + 3)^2 = x^2 + 3\lambda^\nu + 6 \). One can remedy this formally by identifying polynomials agreeing on a dense subset of a variety.

The \( \nu \)-domain\(^\dagger\)
\[ (F(\Lambda)^+, T^+(\lambda_1, \ldots, \lambda_n), \nu, G^+(\lambda_1, \ldots, \lambda_n), \nu) \]
is of special significance.

Any valuation \( \nu : R \to G \) extends to a map \( \tilde{\nu} : R[\Lambda] \to G[\Lambda] \) given by
\[ \tilde{\nu} \left( \sum a_i \lambda_i^{i_1} \cdots \lambda_i^{i_n} \right) = \sum \nu(a_i) \lambda_i^{i_1} \cdots \lambda_i^{i_n}. \]

When \( R \) is a semifield\(^\dagger\), the same analysis is applicable to the domain\(^\dagger\) of Laurent polynomials \( R[\Lambda, \Lambda^{-1}] \), since the homomorphism \( \lambda_i \mapsto a_i \) then sends \( \lambda_i^{-1} \mapsto a_i^{-1} \). But the algebraic structure of choice in this paper is the following (and its \( \nu \)-analogue described below):

Definition 3.4.6. When \((R, \mathcal{T}, G, \nu)\) is a \( \nu \)-domain\(^\dagger\), \( R(\Lambda) \) denotes \( \text{Frac}_\nu R[\Lambda] \), the \( \nu \)-semifield\(^\dagger\) of fractions of \( R[\Lambda] \) obtained by localizing at \( \mathcal{T}[\Lambda] \).

The instance of importance for us is when \( R = F \) is a supertropical semifield\(^\dagger\), and we call \( F(\Lambda) \) the \( \nu \)-semifield\(^\dagger\) of rational functions (over \( F \)); this is our main subject of investigation.

Viewed in \( \text{Fun}(F^{(n)}, F) \), \( F(\Lambda) \) is contained in \( \{ fg^* : f, g \in F[\Lambda] \} \), and they behave similarly. We use the former \((F(\Lambda))\) since it is intuitively clearer, although probably the latter may be technically superior.

(Likewise, we could also define \( R[\Lambda]_{\text{rat}} \) where the exponents of the indeterminates \( \lambda_i \) are taken to be arbitrary rational numbers; we could define substitution homomorphisms when \((R, \mathcal{T}, G, \nu)\) is power-cancellative and divisible.)

These can all be viewed as elementary sentences in the appropriate language, so model theory is applicable to polynomials and their (tropical) roots, to be considered shortly.

Remark 3.4.7. Here are the semilattice operations on \( F(\Lambda) \):

Suppose \( f = \frac{h}{g} \) and \( f' = \frac{h'}{g'} \). Then
\[
\begin{align*}
  f \vee f' &= f + f' = \frac{h + gh'}{gg'}; \\
  f \wedge f' &= f f'(f + f')^* = \frac{hh'}{gg'}(hh' + gh')^* = hh'(hh' + gh')^*.
\end{align*}
\]

When \( hh' + gh' \) is tangible, we have
\[
  f \wedge f' = \frac{hh'}{hh' + gh'}. \tag{3.5}
\]

3.5. Frobenius and archimedean properties.

Definition 3.5.1. A \( \nu \)-domain\(^\dagger\) \( R \) is \( \nu \)-Frobenius if it satisfies the identity
\[
(a + b)^n \cong_{\nu} a^n + b^n, \quad \forall a, b \in R.
\]

\( R \) is \( \nu \)-archimedean if \( a_\nu^\nu \leq b \) implies that \( a \cong_{\nu} 1 \).

Proposition 3.5.2. If \( R \) is a \( \nu \)-Frobenius \( \nu \)-domain\(^\dagger\), then so is \( \text{Fun}(S, R) \).

Proof. Pointwise verification. \( \square \)

Proposition 3.5.3. If the idempotent semifield\(^\dagger\) \( F \) is \( \nu \)-archimedean, then \( \text{Fun}(F^{(n)}, F) \) is \( \nu \)-archimedean.

Proof. Let \( f, g \in \text{Fun}(F^{(n)}, G) \) such that \( f^Z \leq_{\nu} g \). If \( a \in F^{(n)} \) then by our assumption \( f(a)^k \leq_{\nu} g(a) \) for all \( k \in \mathbb{Z} \). Since \( F \) is \( \nu \)-archimedean and \( f(a), g(a) \in F \), we have that \( f(a) = 1 \). Since this holds for any \( a \in F^{(n)} \) we have \( f(F^{(n)}) \cong_{\nu} 1 \), implying \( f \cong_{\nu} 1 \). \( \square \)

Lemma 3.5.4. If a supertropical semifield \( F \) is \( \nu \)-Frobenius, then \( k \)-th roots in \( \text{Fun}(F^{(n)}, F) \) are unique.

Proof. It is enough to check this pointwise in \( F \). Suppose \( a^k = b^k \). Then \( (a + b)^k \cong_{\nu} a^k + b^k \in G \), implying \( a + b \in G \). If moreover, \( a^k \) is tangible then so is \( a \) and \( b \), so \( a = b \). But if \( a^k \in G \), then \( a \) and \( b \) are both ghost, so \( a = b \). \( \square \)
Remark 3.5.5. Let \((\mathcal{S}, \cdot, +)\) be a divisible semifield\(^1\). By Lemma 3.5.4 we can uniquely define any rational power of the elements of \(\mathcal{S}\). In this way, \(\mathcal{S}\) becomes a vector space over \(\mathbb{Q}\), rewriting the multiplicative operation \(\cdot\) on \(\mathcal{S}\) as addition and defining
\[
(m/n) \cdot a = \alpha^{m/n}.
\]

Furthermore, when \(\mathcal{S}\) is complete, we can define real powers as limits of rational powers, and \(\mathcal{S}\) becomes a vector space over \(\mathbb{R}\). In this way we can apply linear algebra techniques to \((\mathcal{S}, \cdot)\).

3.6. Appendix to \[3\] The layered structure.

Although the standard supertropical semifield\(^1\) is adequate for many situations, it fails to detect the multiplicity of a root of a tropical polynomial. Furthermore, serious difficulties are encountered in attempting to establish a useful intrinsic differential calculus on the supertropical structure. Also, some basic supertropical verifications require ad hoc arguments.

These drawbacks are resolved by refining the ghost ideal into different “layers,” following a construction originating in \[32\] (given for the case that \(L\) is a group), and then given in \[19\] Proposition 5.1, as explained in \[9\] Construction 3.2.2.

Construction 3.6.1. Suppose we are given a cancellative ordered monoid \(M\), viewed as a \(\nu\)-domain\(^1\) as above. For any positively ordered semiring\(^1\) \(L\) we define the semiring\(^1\) \(\mathcal{A}(L, M)\) to be set-theoretically \(L \times M\), where we rewrite the element \((\ell, a)\) as \([\ell]a\) and for \(k, \ell \in L, a, b \in M\), we define multiplication componentwise, i.e.,
\[
[k]a [\ell]b = [k\ell](ab),
\]
and addition from the rules:
\[
[k]a + [\ell]b = \begin{cases} [k]a & \text{if } a > b, \\ [\ell]b & \text{if } a < b, \\ [k+\ell]a & \text{if } a = b. \end{cases}
\]

\(R := \mathcal{A}(L, \mathcal{G})\) is indeed a semiring\(^1\). We identify \(a \in M\) with \([1]a \in R_1\). There is a sort map \(s : R \to L\) given by \(s([\ell]a) = \ell\). In most of our applications, the smallest nonzero element of the sorting set \(L\) is 1, so the tangible part of \(R\) is just \(R_1 = \{a \in R : s(a) = 1\}\).

There are two ways of utilizing this construction.

3.6.1. Layering the non-tangible elements.

Here, rather than a single ghost level, we take the indexing set \(L\), which itself is an ordered semiring\(^1\). The ghosts are \(\{a \in R : s(a) > 1\}\), and correspond to the ghosts in the standard supertropical theory. Often \(L = \mathbb{N}\) under classical addition and multiplication.

But strictly speaking, this assignment is not supertropical since we do not have a well-defined \(\nu\)-map. This can be attained by taking the direct limit of the \(\{L_k : k > 1\}\), but we do not concern ourselves with that here. A slicker solution is to adjoin 0 formally to \(L\) and attach a new ghost layer \(R_0\) to form the ghost ideal; then \(\nu\) is given by \([1]a \mapsto [0]a\).

Definition 3.6.2. We call \(\mathcal{A}(L, M)\) a layered domain\(^1\); when \(M\) moreover is a group, we call \(\mathcal{A}(L, M)\) a layered 1-semifield\(^1\).

Note that when \(L\) is multiplicatively cancellative (such as \(L = \mathbb{N}\)), so is Construction 3.6.1 although the standard supertropical construction is not (since \(a^\nu b = a^\nu b^\nu\)).

We see that the theory rests on the monoid \(M\) together with the semiring \(L\) which can be viewed as the sorting semiring\(^1\).

The familiar max-plus algebra is recovered by taking \(L = \{1\}\), whereas the standard supertropical semifield\(^1\) is obtained when \(L = \{1, \infty\}\), where \(R_1\) and \(R_\infty\) are two copies of \(\mathcal{G}\), with \(R_1\) the tangible submonoid of \(\mathcal{G}\) and \(R_\infty\) being the ghost copy. Other useful choices of \(L\) include \(\{1, 2, \infty\}\), and \(\mathbb{N}\) or \(\mathbb{N} \cup \{\infty\}\).

In order to deal with tropical integration as anti-differentiation, one should consider sorting sets \(\mathbb{Q}_{>0}\) and \(\mathbb{R}_{>0}\), but this lies outside our present scope.

If \(R\) is a layered domain\(^1\) (with respect to \(L\)) with \(T = R_1\), then we call \(F := T^{-1} R\) the 1-semifield\(^1\) of fractions \(\text{Frac}_L R\) of \(R\).

Proposition 3.6.3. \(F = \text{Frac}_L R\) is indeed a layered 1-semifield\(^1\), where we extend the sort map \(s\) to \(s : F \to L\) by defining \(s([a]_\mathcal{A}) = s(a)\).

Proof. This is almost immediate from the definition together with Remark 2.3.1. \(\square\)
By convention, \([\ell]_{\lambda}\) denotes \([\ell]_{1_R}\lambda\). Thus, any monomial can be written in the form \(h = \alpha \lambda_1^{i_1} \cdots \lambda_n^{i_n}\) for \(i = (i_1, \ldots, i_n)\), where \(s(h) = s(\alpha)\).

**Remark 3.6.4.** One could also go in the other direction. Suppose \((R, \nu, G)\) is a supertropical \(\nu\)-domain. \(L = \{a \in R : a \geq 1\}\), is called the layering sub-semiring of \(R\). \(L\) can also be considered the “characteristic” sub-\(\nu\)-domain of \(R\). This is like \(F_1\) in the literature.

Note that \(L\) is a sub-semiring of \(R\) of characteristic 1.

**Proposition 3.6.5.** Suppose \(R = (R, T, G, \nu)\) is a \(\nu\)-domain. Then \(LT \cong L \times T\) as a monoid, and with addition as in Equation (3.8), is isomorphic to \(\mathcal{A}(L, T)\) as a semiring.

**Proof.** It is easy to check that \(L\) is a semiring, with \(\nu(L) = 1_G\). If \(k, \ell \in L\), with \(ka = \ell b\) for \(a, b \in T\), then \(a^\nu = b^\nu\), so \(a = b\) and thus \(k = \ell\) by cancelation, proving \(LT \cong L \times T\) as a monoid.

We write \([\ell]_a\) for \((\ell, a)\). If \(a < \nu\), we have \([k]_a < [\ell]_b\), so \([k]_a + [\ell]_b = [\ell]_b\), and by symmetry we need only check for \(a \equiv \nu b\). But then \([k]_a + [\ell]_a = [k+\ell]_a\).

\(\square\)

### 3.7. The layered (\(\ast\))-operation.

There also is a layered version of \((\ast)\).

**Definition 3.7.1.** For any layered 1-semifield \([\ell]\) and \(a = [\ell]_\alpha \in F\), we define \(a^\ast = [\ell]_{\alpha^{-1}}\).

In other words, we take the inverse, but with the same layer.

**Remark 3.7.2.** The map \((\ast)\) preserves layers, by definition.

Since tangible polynomials are not tangible pointwise, in the sense that any root takes on a ghost value, we extend Definition 3.7.1 to Fun\((S, F)\), just as in Definition 3.7.2.

Since we are interested in polynomials, we introduce the layered polynomial domain \(F[\Lambda]_L\), and \(F(\Lambda)_L := \{fg^\ast : f, g \in F[\Lambda]_L, g\) tangible\}.

### 4. Kernels and \(\nu\)-kernels

Although the theory of congruences applies generally in universal algebra and in particular for a semiring \(\uparrow R\), congruences are difficult to work with since they involve subsets of \(R \times R\) rather than \(R\) itself. When \(R\) is a semifield \(\uparrow S\), one can get around this by switching the roles of addition and multiplication. Although we are interested in the \(\nu\)-structure, we present the definitions without \(\nu\), in order to get to the underlying ideas more quickly.

#### 4.1. Kernels of Semiring \(\uparrow S\)

**Definition 4.1.1.** A kernel of a semiring \(\uparrow S\) is a subgroup \(K\) which is convex in the sense that if \(a, b \in K\) and \(\alpha, \beta \in S\) with \(\alpha + \beta = 1_S\), then \(\alpha a + \beta b \in K\).

**Note 4.1.2.** Usually one takes \(S\) to be a semifield \(\uparrow S\), which is our assumption through the remainder of this subsection, but we frame the definition a bit more generally in order to be able later to handle \(\nu\)-semifields, which technically are not semifields. In some of the literature semifields are not required to be commutative, and then \(K\) is required to be a normal subgroup; we forego this generality.

Following [30] Proposition (1.1)], we have the following key correspondence.

**Proposition 4.1.3.** If \(\Omega\) is a congruence on a semifield \(\uparrow S\), then \(K_{\Omega} = \{a \in S : a \equiv 1\}\) is a kernel. Conversely, any kernel \(K\) of \(S\) defines a congruence according to [8] Definition 3.1, i.e., \(a \equiv b\) iff \(\frac{a}{b} \equiv 1\). If \(S\) is the semifield of the lattice-ordered group \(G\), then the semifield \(S/\Omega\) of the lattice-ordered group \(G/K\).

A quick proof that \(\Omega\) is indeed a congruence is given in [31] Proposition 1.1].

**Remark 4.1.4.** We recall some basic facts about kernels.

(i) \([31]\) Corollary 1.1], [30] Property 2.4 Any kernel \(K\) is convex with respect to the order of Proposition 2.2.3 in the sense that if \(a \leq b \leq c\) with \(a, c \in K\), then \(b \in K\). This is seen by passing to the factor semiring \(S/\Omega\) of (i) and applying Lemma 2.1.7.

(ii) \([31]\) Proposition 2.3. If \(|a| \not\in K\), a kernel, then \(a \not\in K\).

(iii) \([31]\) The product \(K_1K_2 = \{ab : a \in K_1, b \in K_2\}\) of two kernels is a kernel, in fact the smallest kernel containing \(K_1 \cup K_2\). (This follows at once from (i).)
(iv) The intersection of kernels is a kernel. Thus, for any set \( S \subset \mathcal{S} \) we can define the kernel \( \langle S \rangle \) generated by \( S \) to be the intersection of all kernels containing \( S \).

(v) \([31]\) Theorem 3.5. Any kernel generated by a finite set \( \{s_1, \ldots, s_m\} \) is in fact generated by the single element \( \sum_{i=1}^{m} (s_i + s_i^{-1}) \). (Follows easily from (ii).)

(vi) The kernel generated by \( a \in S \) is just the set of finite sums \( \{\sum_i b_i a^i : b_i \in S, \sum b_i = 1\} \).

(vii) \([3]\) Theorem 3.8]. If \( K \) is a kernel of a semifield \( S \) and the semifield \( S/K \) is idempotent, then \( K \) is a sub-semifield of \( S \). (This is because for \( a, b \in K \) the image of \( a + b \) is \( 1K + 1K = 1K \).)

(viii) Let \( K \) be a kernel of a semifield \( S \). For every \( a \in S \), if \( a^n \in K \) for some \( n \in \mathbb{N} \) then \( a \in K \). (Indeed, passing to \( S/K \), which is torsion free by Lemma \([22,4]\) it suffices to note that if \( a^n = 1 \) then \( a = 1 \).)

(ix) The kernel of a kernel is a kernel.

We also need the following generalization of (vi):

**Proposition 4.1.5.** \([3]\) Proposition (3.13)] Let \( S \) be a semifield and let \( N \) be a (normal) subgroup of \( (S, \cdot) \). Then the smallest kernel containing \( N \) is

\[
\left\{ \sum_{i=1}^{n} s_i h_i : n \in \mathbb{N}, h_i \in N, s_i \in S \text{ such that } \sum_{i=1}^{n} s_i = 1 \right\}.
\]

**Proposition 4.1.6.** \([29]\) Theorem (2.2.5]) The lattice of kernels of an idempotent semifield \( S \) is a complete distributive lattice and satisfies the infinite distributive law.

**Theorem 4.1.7.** \([3]\) Theorems (3.4) and (3.5)] Let \( \phi : S_1 \to S_2 \) be a semifield homomorphism. Then the following hold:

1. For any kernel \( L \) of \( S_1 \), \( \phi(L) \) is a kernel of \( \phi(S_1) \).
2. For a kernel \( K \) of \( \phi(S_1) \), \( \phi^{-1}(K) \) is a kernel of \( S_1 \). In particular, for any kernel \( L \) of \( S_1 \) we have that \( \phi^{-1}(\phi(L)) = KL \) is a kernel of \( S_1 \).

In particular, \( \phi^{-1}(1) \) is a kernel.

**Corollary 4.1.8.** There is an injection \( S/(K_1 \cap K_2 \cap \cdots \cap K_t) \to \prod_{i=1}^{t} S/K_i \), for any kernels \( K_i \) of a semifield \( S \), induced by the map \( f \mapsto (fK_i) \).

We have the three fundamental isomorphism theorems.

**Theorem 4.1.9.** \([33]\) Let \( S \) be an idempotent semifield \( S^\# \) and \( K, L \) be kernels of \( S \).

1. If \( U \) is a sub-semifield of \( S \), then \( U \cap K \) is a kernel of \( U \), and \( K \) a kernel of the sub-semifield \( UK = \{u \cdot k : u \in U, k \in K\} \) of \( S \), and one has the isomorphism \( U/(U \cap K) \cong UK/K \).

2. \( L \cap K \) is a kernel of \( L \) and \( K \) a kernel of \( LK \), and the group isomorphism \( L/(L \cap K) \cong LK/K \)

is a semifield isomorphism.

3. If \( L \subseteq K \), then \( K/L \) is a kernel of \( S/L \) and one has the semifield isomorphism \( S/K \cong (S/L)/(K/L) \).

**Theorem 4.1.10.** Let \( L \) be a kernel of a semifield \( S \). Every kernel of \( S/L \) has the form \( K/L \) for some uniquely determined kernel \( K \supseteq L \), yielding a lattice isomorphism

\[ \{\text{Kernels of } S/L\} \to \{\text{Kernels of } S \text{ containing } L\} \]

given by \( K/L \mapsto K \).

**Proof.** From the theory of groups, there is such a bijection for normal subgroups. We note by Theorem 4.1.7 that the homomorphic image and pre-image of a kernel are kernels. \( \square \)

**Definition 4.1.11.** A semifield \( S^\# \) which contains no kernels but the trivial ones, \( \{1\} \) and itself, is called simple.

**Remark 4.1.12.** A kernel \( K \) of a semifield \( S \) is a maximal kernel if and only if the semifield \( S/K \) is simple.
4.1.1. Principal kernels.

**Definition 4.1.13.** For a subset $S$ of a semifield $S$, denote by $\langle S \rangle$ the smallest kernel in $S$ containing $S$, i.e., the intersection of all kernels in $S$ containing $S$. A kernel $K$ is said to be **finitely generated** if $K = \langle S \rangle$ where $S$ is a finite set. If $K = \langle a \rangle$ for some $a \in S$, then $K$ is called a **principal kernel**.

$\mathcal{P}(K)$ denotes the set of principal subkernels of a kernel $K$. $\mathcal{P}(K)$ turns out to be a lattice, and one of the keys of our study.

**Lemma 4.1.14.** [30 Property 2.3] Let $K$ be a kernel of an idempotent semifield $S$. Then for $a, b \in S$,

\[(a) = \{x \in S : \exists n \in \mathbb{N} \text{ such that } a^{-n} \leq x \leq a^n \}.\]

**Proposition 4.1.15.** [30 Proposition (3.1)]

\[(a) = \{x \in S : \exists n \in \mathbb{N} \text{ such that } |a|^{-n} \leq x \leq |a|^n \}.\]

**Corollary 4.1.16.** [30] $\langle a \rangle = \langle a^k \rangle$, for any element $a$ of $S$ and any $0 \neq k \in \mathbb{Z}$.

**Corollary 4.1.17.** [30] $\langle a \rangle = \langle |a| \rangle$, for any element $a$ of $S$.

A direct consequence of Proposition 4.1.15 and Corollary 4.1.17 is

**Corollary 4.1.18.** For any $a \in S$,

\[(a) = \{x \in S : \exists n \in \mathbb{N} \text{ such that } |a|^{-n} \leq x \leq |a|^n \}.\]

**Definition 4.1.19.** A semifield $\langle \rangle$ is said to be **finitely generated** if it is finitely generated as a kernel. If $S = \langle a \rangle$ for some $a \in S$, then $a$ is called a **generator** of $S$ and $S$ is said to be a **principal semifield**.

**Corollary 4.1.20.** Every nontrivial principal semifield $S$ has a generator $a > 1$.

**Proof.** Let $u \in S \setminus \{1\}$ be a generator of $S$. By Lemma 4.1.14 the element $|u|$ is also a generator of $S$ which yields that the element $a = |u|^2 = u^2 + u^{-2} + 1 \geq 1$ is a generator of $S$ too, by Proposition 4.1.15. But $a \neq 1$, so $a > 1$.

**Theorem 4.1.21.** If an idempotent semifield $S$ has a finite number of generators $a_1, \ldots, a_n$, then $S$ is a principal semifield, generated by $a = |a_1| + \cdots + |a_n|$.

**Proof.** By Lemma 4.1.14 $a_1, \ldots, a_n$ are contained in the kernel $\langle a \rangle \subseteq S$; hence, $S = \langle a \rangle$ as desired.

**Proposition 4.1.22.** Every idempotent archimedean semifield $S$ is simple.

**Proof.** We may assume that $S \neq \{1\}$. Take $a \in S$ such that $a > 1$. Since $S$ is archimedean, for every $b \in S$ there exists $m \in \mathbb{N}$ such that $a^{-m} \leq b \leq a^m$, so $b \in \langle a \rangle$ by Proposition 4.1.15. Thus $\langle a \rangle = S$ and our claim is proved.

**Notation 4.1.23.** By Proposition 4.1.22, $S = \langle a \rangle$ for each $a \neq \{1\}$.

The kernel $\langle S \rangle$ in $S(\Lambda)$ is somewhat bigger, containing all functions $af + bg$ where $f + g$ is the constant function $1$. This has a significant role, discussed in \[8.2.2\] below, where we describe $S(\Lambda)$ more explicitly.

**Proposition 4.1.24.** For any $\gamma_1, \ldots, \gamma_n \in S$ the kernel $\langle \frac{\alpha_{\gamma_1}}{\gamma_1}, \ldots, \frac{\alpha_{\gamma_n}}{\gamma_n} \rangle$ is a maximal kernel of $S(\Lambda)$.

**Proof.** The quotient is isomorphic to $S$, which is simple.

Here are more properties of kernels of an idempotent semifield and their generators.

**Proposition 4.1.25.** [29 Theorem 2.2.4(d)] For any $V, W \subseteq S$, any kernel $K$ of $S$ and $f, g \in S$ the following statements hold:

\[
\begin{align*}
(i) \quad & \langle V \rangle \langle W \rangle = \langle V \cup W \rangle = \langle \{ |x| + |y| : x \in V, y \in W \} \rangle, \\
(ii) \quad & \langle V \rangle \cap \langle W \rangle = \langle \{ |x| \wedge |y| : x \in V, y \in W \} \rangle, \\
(iii) \quad & \langle (K, f) \rangle \cap \langle (K, g) \rangle = K \langle |f| \wedge |g| \rangle.
\end{align*}
\]

**Lemma 4.1.26.**

\[(4.4) \quad |g| |h| \in K \iff |g| + |h| \in K.
\]

for any kernel $K$ of an idempotent semifield $S$. 


Proof. \(|g| \leq |g||h|\) and \(|h| \leq |g||h|\) since \(|g|, |h| \geq 1\), and thus

\[|g| + |h| = \sup(|g|, |h|) \leq |g||h|.
\]

On the other hand,

\[(|g| + |h|)^2 = |g|^2 + |g||h| + |h|^2 \geq |g||h|,
\]

so, by Corollary 4.1.18 \((|g||h|) = (|g| + |h|)\) and (4.4) follows. \(\square\)

Corollary 4.1.27. Let \(S\) be an idempotent semifield\(^1\). Then the intersection and product of two principal kernels are principal kernels. Namely, for every \(f, g \in S\)

\[
(4.5) \quad (f) \cap (g) = (|f| \cap |g|) : (f)(g) = (|f||g|).
\]

Proof. Taking \(V = \{f\}\) and \(W = \{g\}\) in Proposition 4.1.25 yields the first equality, whereas for the second equality, Lemma 4.1.28 yields

\[
(f)(g) = (|f| + |g|),
\]

and again we apply the proposition. \(\square\)

Corollary 4.1.28. The set of principal kernels of an idempotent semifield\(^1\) forms a sublattice of the lattice of kernels (i.e., a lattice with respect to intersection and multiplication).

Corollary 4.1.29. For any generator \(a\) of a semifield\(^1\) \(F\), \(F(\Lambda) = \langle a \rangle \prod_{i=1}^{n} (\Lambda_i)\), and \(F(\Lambda)\) is a principal semifield\(^1\) with generator \(\sum_{i=1}^{n} |\Lambda_i| + |a|\).

Proof. By Corollary 4.1.28 it is enough to prove that any monomial \(f\) in \(F[\Lambda]\) belongs to \((\sum_{i=1}^{n} |\Lambda_i| + |a|)\), which follows from Theorem 4.1.21. \(\square\)

4.2. \(\nu\)-Congruences and \(\nu\)-kernels.

Throughout, \(R = (R, T, G, \nu)\) is a \(\nu\)-domain\(^1\). Any congruence of \(G\) also can be viewed as a congruence of \(R\), but we also want to bring \(T\) into play. The next observation is due to Izhakian.

Remark 4.2.1. Any congruence \(\Omega\) satisfies the property that if \((a, b) \in \Omega\) then \((a', b') = (1^\nu, 1^\nu)(a, b) \in \Omega\).

Definition 4.2.2. A \(\nu\)-congruence \(\Omega\) is a congruence of \(\Omega\) for which \((a, b) \in \Omega\) iff \((a', b') \in \Omega\).

A \(\nu\)-kernel of a \(\nu\)-semifield\(^1\) \(F\) is a subgroup \(K\) which is \(\nu\)-convex in the sense that if \(a, b \in K\) and \(\alpha, \beta \in F\)

\[
\alpha + \beta \equiv_\nu 1_F, \text{ then } a \alpha + \beta b \in K.
\]

We write \(a_1 \equiv_\nu a_2\) when \(a_1^\nu \equiv_\nu a_2^\nu\).

Remark 4.2.3. For any congruence \(\Omega\) of \(G\), \(\nu^{-1}(\Omega) = \{(a, b) : a \equiv_\nu b\}\) is a \(\nu\)-congruence of \(R\).

There is a natural correspondence between \(\nu\)-congruences (resp. \(\nu\)-kernels) of \(R\) and congruences (resp. kernels) of \(G\), given as follows:

Any \(\nu\)-congruence \(\Omega = \{(a, b) : a, b \in R\}\) of \(R\) defines a congruence \(\Omega' = \{(a', b') : (a, b) \in \Omega\}\) of \(G\).

Conversely, if \(\Omega'\) is a congruence of \(G\), then \(\nu^{-1}(\Omega')\) is a \(\nu\)-congruence of \(R\).

Any \(\nu\)-kernel \(K\) of a \(\nu\)-semifield\(^1\) \(F\) defines a kernel \(K'\) of \(G\). Conversely, if \(K\) is a kernel of \(G\), then \(\nu^{-1}(K)\) is a \(\nu\)-kernel of \(F\).

If \(K = K_\Omega\), then \(\nu^{-1}(K) = K_{\nu^{-1}(\Omega)}\).

Theorem 4.2.4. Given a \(\nu\)-congruence \(\Omega\) on a \(\nu\)-semifield\(^1\) \(R\), corresponding to the \(\nu\)-kernel \(K\) of \(\text{Frac}_R\), the factor semiring \(\bar{R} = R/\Omega\) is a \(\nu\)-semifield\(^1\) with respect to the idempotent map \(\bar{\nu} : \bar{R} \to \bar{G} = G/\nu(K)\) defined by \(\bar{\nu}(Ka) = Ka^\nu\).

Proof. If \(a \equiv_\nu b\) then \(ab^* \in K\), implying \(Ka = Kb\). This implies \(\bar{\nu}\) is well-defined, and its target is the semifield\(^1\) \(G\).

Writing \(\bar{a}\) for the image of \(a\) in \(\bar{R}\), we write \(a \equiv_\nu b\) to denote that \(\bar{a} \equiv_\nu \bar{b}\), or equivalently \(\bar{a}^\nu \equiv_\nu \bar{b}^\nu\).

Remark 4.2.5. We translate the facts from Remark 4.1.4 to \(\nu\)-kernels:

(i) Given a \(\nu\)-congruence \(\Omega\) on a \(\nu\)-semifield\(^1\) \(F\), we define \(K_\Omega = \{a \in F : a \equiv_\nu 1\}\). Conversely, given a \(\nu\)-kernel \(K\) of \(F\), we define the \(\nu\)-congruence \(\Omega\) on \(F\) by \(a \equiv b\) iff \(ab^\nu \equiv_\nu 1_F\).

(ii) Any \(\nu\)-kernel \(K\) is \(\nu\)-convex, in the sense that if \(a \leq_\nu b \leq_\nu c\) with \(a, c \in K\), then \(b \in K\).

(iii) If \(a \in K\), a \(\nu\)-kernel, then \(a \in K\).

(iv) The product of two \(\nu\)-kernels is a \(\nu\)-kernel.

(v) The intersection of \(\nu\)-kernels is a \(\nu\)-kernel. Thus, for any set \(S \subset F\) we can define the \(\nu\)-kernel \(\langle S \rangle\) generated by \(S\) to be the intersection of all \(\nu\)-kernels containing \(S\).

(vi) Any \(\nu\)-kernel generated by a finite set \(\{s_1, \ldots, s_m\}\) is generated by the single element \(\sum_{i=1}^{m} (|s_i|)\).
(vii) The ν-kernel generated by \( a \in F \) is just the set of finite sums \( \{ \sum b_i a^i : b_i \in F, \sum b_i \equiv_\nu 1 \} \).

(viii) If \( K \) is a ν-kernel of a ν-semifield \( F \) and the ν-semifield \( F/K \) is ν-idempotent, then \( K \) is a sub-ν-semifield \( F \). (This is because for \( a,b \in K \) the image of \( a + b \) is ν-equivalent to \( 1K + 1K = 1K \).)

4.3. Digression: Fractional kernels.

Having a way of transferring information to and from semifields \( \dagger \), let us introduce fractional ν-kernels, to be able to pass to ν-domains \( \dagger \). This should be a useful tool in the further study of kernels in tropical mathematics.

**Definition 4.3.1.** A fractional ν-kernel \( K \) on a ν-domain \( R \) is a ν-kernel on the ν-semifield \( \text{Frac}_\nu R \).

**Remark 4.3.2.** Extending Remark 4.2.1, given a ν-congruence \( \Omega \) on a ν-domain \( R \) with ν-semifield \( \dagger \) of fractions \( F \), we can define the fractional kernel \( K_\Omega = \{ ab^\nu \in F : a,b \in R \} \equiv_\nu b \} \). (Or, in other words, extending \( \Omega \) to \( F \) as in Lemma 2.4.3, we require that \( ab^\nu \equiv_\nu 1_F \).)

A ν-congruence \( \Omega \) on \( R \) is called ν-cancellative if \( a_1 b \equiv_\nu a_2 b \) for some \( b \in R \) implies \( a_1 \equiv_\nu a_2 \).

**Theorem 4.3.3.** There is a 1:1 correspondence between ν-cancellative congruences of a ν-domain \( R \) and fractional ν-kernels, given by \( \nu \rightarrow K_\Omega \). Any homomorphism of ν-domains \( R \rightarrow R' \) (for \( R' \) an arbitrary ν-domain) gives rise to a fractional ν-kernel, and \( \nu \) is the congruence corresponding to \( R \rightarrow R/K \).

**Proof.** We extend \( \Omega \) to \( F \) as in Lemma 2.4.3. As noted in [8, Theorem 3.2], \( K_\Omega \) is clearly a congruence, since \( a + b \equiv_\nu a + b \equiv_1 F \) for all \( a,b \in K_\Omega \).

In the other direction, we restrict this to \( R \).

Given a homomorphism \( R \rightarrow R' \), we compose it with the injection of \( R' \) into its semifield \( \dagger \) of fractions \( F' \), and then extend this naturally to a homomorphism of semifields \( \dagger \) \( F \rightarrow F' \), thereby obtaining a kernel. \( \square \)

5. The supertropical connection: Corner loci and K-varieties

In this section we apply our theory to tropical geometry. Let us recall the basic notions concerning supertropical varieties. Throughout, \( F \) denotes a ν-semifield \( \dagger \).

5.1. Corner loci.

In [13, Section (5.2)] Izhakian and Rowen have generalized the notion of (tangible) corner root to \( F[\Lambda] \) (over a supertropical semifield \( \dagger \) \( F \)) as follows:

**Definition 5.1.1.** Suppose \( f \in F[\Lambda] \) be a supertropical polynomial, written \( f = \sum_{i=1}^k f_i \) where each \( f_i \) is a monomial. A point \( a \in F^{(n)} \) is said to be a ghost root of \( f \) if \( f(a) \in G \), i.e., if \( f \) obtains a ghost value at \( a \).

This happens in one of the following cases:

1. There are two distinct monomials \( f_i \) and \( f_s \) of \( f \) such that \( f(a) = f_s(a) = f_i(a) \).

2. There exists a ghost monomial \( f_i \) of \( f \) such that \( f(a) = f_i(a) \).

**Remark 5.1.2.**

1. If \( a,b \in F^n \) are ghost roots of \( f,g \in F[\Lambda] \) respectively, then both \( a \) and \( b \) are ghost roots of the product \( f g \).

2. In view of Proposition 3.5.2 any ghost root \( a \) of \( f^k \) for \( k \geq 1 \) is a ghost root of \( f \).

**Definition 5.1.3.** A set \( A \subseteq T^{(n)} \) is said to be a corner locus if \( A \) is a set of the form

\[
A = \{ x \in F^n : \forall f \in S, x \text{ is a ghost root of } f \}
\]

for some \( S \subseteq F[\Lambda] \). We write \( C_{\text{loc}}(S) \) to denote the corner locus defined by \( S \).

**Lemma 5.1.4.** Suppose \( f = f' + f'' \in F(\Lambda) \). Then \( a \in C_{\text{loc}}(f) \) iff one of the following hold:

1. \( a \in C_{\text{loc}}(f') \) with \( f'(a) \geq f''(a) \);
2. \( a \in C_{\text{loc}}(f'') \) with \( f''(a) \geq f'(a) \);
3. \( f'(a) \equiv_\nu f''(a) \).

**Proof.** These are the only ways to get a ghost value. \( \square \)

This concise formulation enables us to treat tropical varieties algebraically as simultaneous roots of sets of polynomials. In particular, a hypersurface now is viewed as the corner locus of a single polynomial. When the polynomials are tangible, the hypersurfaces are those from usual tropical geometry.

We write \( 2^S \) for the power set of a set \( S \).

In view of Definition 5.1.3 we define an operator \( C_{\text{loc}} : 2^{F[\Lambda]} \rightarrow 2^{F^{(n)}} \) given by

\[
C_{\text{loc}} : S \subseteq F[\Lambda] \mapsto C_{\text{loc}}(S).
\]

We now proceed to study the behavior of the \( C_{\text{loc}} \) operator.
Remark 5.1.5. If \( S_A, S_B \subseteq F[\Lambda] \) then
\[
(5.3) \quad S_A \subseteq S_B \Rightarrow C_{\text{loc}}(S_B) \subseteq C_{\text{loc}}(S_A).
\]

Let \( \{S_i\}_{i \in I} \) be a family of subsets of \( F[\Lambda] \) for some index set \( I \). Then \( \bigcap_{i \in I} C_{\text{loc}}(S_i) \) is a corner locus and
\[
(5.4) \quad \bigcap_{i \in I} C_{\text{loc}}(S_i) = C_{\text{loc}} \left( \bigcup_{i \in I} S_i \right); \quad \bigcup_{i \in I} C_{\text{loc}}(S_i) \subseteq C_{\text{loc}} \left( \bigcap_{i \in I} S_i \right).
\]

In particular, \( C_{\text{loc}}(S) = \bigcap_{f \in S} C_{\text{loc}}(f) \).

Proof. First, equality \((5.3)\) is a direct set-theoretical consequence of the definition of corner loci. In turn this implies that \( C_{\text{loc}}(\bigcup_{i \in I} S_i) \subseteq C_{\text{loc}}(S_i) \) for each \( i \in I \) and thus \( C_{\text{loc}}(\bigcup_{i \in I} S_i) \subseteq \bigcap_{i \in I} C_{\text{loc}}(S_i) \). Conversely, if \( x \in F(n) \) is in \( \bigcap_{i \in I} C_{\text{loc}}(S_i) \) then \( a \in C_{\text{loc}}(S_i) \) for every \( i \in I \), which means that \( a \) is a common ghost root of \( \{f : f \in S_i\} \). Thus \( a \) is a common ghost root of \( \{f : f \in \bigcup_{i \in I} S_i\} \) which yields that \( a \in C_{\text{loc}}(\bigcup_{i \in I} S_i) \). For the second equation (inclusion) in \((5.4)\), \( \bigcap_{i \in I} S_i \subseteq S_j \) for each \( j \in I \). Thus, by \((5.3)\), \( C_{\text{loc}}(S_j) \subseteq C_{\text{loc}}(\bigcap_{i \in I} S_i) \), and so, \( \bigcup_{i \in I} C_{\text{loc}}(S_i) \subseteq C_{\text{loc}}(\bigcap_{i \in I} S_i) \).

The next lemma deals with the case where \( A \) and \( B \) are finitely generated.

Lemma 5.1.6. If \( A = C_{\text{loc}}(f_1, \ldots, f_s) \) and \( B = C_{\text{loc}}(g_1, \ldots, g_t) \), then \( A \cap B = C_{\text{loc}}(f_1, \ldots, f_s, g_1, \ldots, g_t) \), and \( A \cup B = C_{\text{loc}}(\{f_ig_j\}_{i=1,j=1}^{s,t}) \).

Proof. In view of Definition 5.1.3 and Remarks 5.1.2 and 5.1.5 we only need to prove that \( A \cup B = C_{\text{loc}}(\{f_ig_j\}_{i=1,j=1}^{s,t}) \). Indeed, for each \( 1 \leq i \leq s \) and each \( 1 \leq j \leq t \), \( A \subseteq C_{\text{loc}}(f_i) \) and \( B \subseteq C_{\text{loc}}(g_j) \). Thus \( A \cup B \subseteq C_{\text{loc}}(f_i) \cup C_{\text{loc}}(g_j) = C_{\text{loc}}(\{f_ig_j\}_{i=1,j=1}^{s,t}) \).

On the other hand, if \( a \notin A \cup B \) then there exist some \( i_0 \) and \( j_0 \) such that \( a \notin C_{\text{loc}}(f_{i_0}) \) and \( a \notin C_{\text{loc}}(g_{j_0}) \). Thus \( a \notin C_{\text{loc}}(f_{i_0}) \cup C_{\text{loc}}(g_{j_0}) = C_{\text{loc}}(f_{i_0}g_{j_0}) \). So \( a \notin C_{\text{loc}}(\{f_ig_j\}_{i=1,j=1}^{s,t}) \), proving the opposite inclusion. \( \square \)

In [16] the following density condition was introduced to pick out those varieties defined by several polynomials, which are more in line with the usual tropical viewpoint.

Definition 5.1.7. A corner locus \( Z \) is admissible if it satisfies the following property:
If \( f \) and \( g \) agree on a \( \nu \)-dense subset of \( Z \) then \( f = g \).

A variety is an admissible corner locus that is not the union of two admissible corner loci.

Likewise, when \( R \) is a layered 1-semifield\(^1\), we say that a tangible element \( a \in F^{(n)}_1 \) is a layered root of \( f(\lambda_1, \ldots, \lambda_n) \) if \( s(f(a)) > 1 \), and build the parallel theory.

5.2. 1'-sets.

We present a method for describing a ‘corner locus’ by the analogous concept for kernels, which we call a 1'-set. This in turn sets the stage for applying the theory of kernels to tropical geometry, focusing on the correspondence between kernels and 1'-sets. We introduce the geometric notion of ‘1'-set’ and the algebraic notion of ‘kernel of a 1'-set’, the respective analogs of affine varieties and their ideals. To obtain a Zariski-type correspondence, we define a pair of operators \( I_{\text{loc}} \) and \( \text{Kern} \) where \( I_{\text{loc}} \) maps a kernel to its 1'-set and \( \text{Kern} \) maps a 1'-set to its corresponding kernel.

Throughout this section, we take \( F \) to be a \( \nu \)-semifield\(^1\). \( F(\Lambda) \) was defined in Definition 3.4.6

Definition 5.2.1. A kernel root of \( f \in \text{Fun}(F^{(n)}, F) \) is an element \( a \in F^{(n)}_1 \) such that \( f(a) \cong_\nu 1_F \).

For \( S \subseteq F(\Lambda) \), define the subset \( I_{\text{loc}}(S) \) of \( F^{(n)} \) as
\[
(5.5) \quad I_{\text{loc}}(S) = \{a \in F^{(n)} : f(a) \cong_\nu 1, \forall f \in S\}.
\]

We write \( I_{\text{loc}}(f) \) for \( I_{\text{loc}}(\{f\}) \).

Lemma 5.2.2. The following hold for \( f, g \in F(\Lambda) \):

(i) \( I_{\text{loc}}(fg) = I_{\text{loc}}(f) \cap I_{\text{loc}}(g) \) for all \( f, g \geq_\nu 1 \).
(ii) \( I_{\text{loc}}(f + g) = I_{\text{loc}}(f) \cup I_{\text{loc}}(g) \) for all \( f, g \leq_\nu 1 \).
Proof. (i) \((f + g)(\mathbf{a}) = f(\mathbf{a}) + g(\mathbf{a}) \equiv_\nu 1\) if and only if \(f(\mathbf{a}) \equiv_\nu 1\) and \(g(\mathbf{a}) \equiv_\nu 1\) (since both \(f, g \geq_\nu 1\)). Analogously the same holds for \(f(\mathbf{a})g(\mathbf{a}) \equiv_\nu 1\).

(ii) \((f \wedge g)(\mathbf{a}) = f(\mathbf{a}) \wedge g(\mathbf{a}) \equiv_\nu 1\) if and only if \(f(\mathbf{a}) \equiv_\nu 1\) or \(g(\mathbf{a}) \equiv_\nu 1\). \(\square\)

**Lemma 5.2.3.** \(1_{\text{loc}}(f) = 1_{\text{loc}}(f^{-1}) = 1_{\text{loc}}(|f|) = 1_{\text{loc}}(f \wedge f^{-1})\).

*Proof.* Clearly \(f(\mathbf{a}) \equiv_\nu 1\) iff \(f^{-1}(\mathbf{a}) \equiv_\nu 1\), in which case \(|f| = f^{-1}(\mathbf{a}) \equiv_\nu 1\). Conversely, if \(|f| \equiv_\nu 1\) then both \(f(\mathbf{a}), f^{-1}(\mathbf{a}) \leq_\nu 1\), implying \(f(\mathbf{a}), f^{-1}(\mathbf{a}) \equiv_\nu 1\). The last assertion is by Lemma 5.2.2(ii). \(\square\)

**Definition 5.2.4.** A subset \(Z \subset F(n)\) is said to be a **1'-set** if there exists a subset \(S \subset F(\Lambda)\) such that \(Z = 1_{\text{loc}}(S)\).

**Proposition 5.2.5.** For \(S_1 \subset F(\Lambda)\) the following statements hold:

\(1\) \(S_1 \subseteq S_2 \Rightarrow 1_{\text{loc}}(S_2) \subseteq 1_{\text{loc}}(S_1)\).

\(2\) \(\bigcap_{i \in I} 1_{\text{loc}}(S_i) = 1_{\text{loc}}\left(\bigcup_{i \in I} S_i\right)\) for any index set \(I\) and in particular, \(1_{\text{loc}}(S) = \bigcap_{f \in S} 1_{\text{loc}}(f)\).

*Proof.* The assertions are formal. \(\square\)

**Lemma 5.2.6.** \(1_{\text{loc}}(S_1) = 1_{\text{loc}}(\langle S_1 \rangle)\).

*Proof.* For any \(f, g\) such that \(f(\mathbf{a}) \equiv_\nu g(\mathbf{a}) \equiv_\nu 1\), \((fg)(\mathbf{a}) = f(\mathbf{a})g(\mathbf{a}) \equiv_\nu 1 \cdot 1 = 1\), \(f^*(\mathbf{a}) \equiv_\nu f(\mathbf{a})^* \equiv_\nu 1^{-1} = 1\), and \((f + g)(\mathbf{a}) = f(\mathbf{a}) + g(\mathbf{a}) \equiv_\nu 1 + 1 \equiv_\nu 1\). To show that convexity is preserved for \(f_1, ..., f_t \in F(\Lambda)\) with \(\sum_{i=1}^t f_i \equiv_\nu 1\) and for any \(g_1, ..., g_t\) in \(1_{\text{loc}}(S_1)\), we note that

\[
\left(\sum_{i=1}^t f_i g_i\right)(\mathbf{a}) = \sum_{i=1}^t f_i(\mathbf{a}) g_i(\mathbf{a}) \equiv_\nu \sum_{i=1}^t f_i(\mathbf{a}) \cdot 1 \equiv_\nu \sum_{i=1}^t f_i(\mathbf{a}) \equiv_\nu 1.
\]

\(\square\)

**Proposition 5.2.7.** \([23]\) Proposition (4.2.6)] Let \(K_i\) be kernels of \(F(\Lambda)\), and let \(Z_i = 1_{\text{loc}}(K_i)\) be their corresponding 1'-sets. Then the following statements hold:

\(5.6\) \(1_{\text{loc}}(K_1 K_2) = Z_1 \cap Z_2;\)

\(5.7\) \(1_{\text{loc}}(\cap K_i) = \cup Z_i.\)

Note 5.2.8. The proposition fails miserably when we take \(\wedge\) of an infinite set. For example, we could take \(f_m\) to be a series of constants approaching 1. Then \(\inf\{f_m\} = 1\), whose 1'-set is \(F(n)\), although \(1_{\text{loc}}(f_m) = \emptyset\) for each \(m\).

**Definition 5.2.9.** A 1'-set \(Z\) is said to be **principal** if there exists \(f \in F(\Lambda)\) such that \(Z = 1_{\text{loc}}(f)\).

**Definition 5.2.10.** Denote the collection of 1'-sets in \(F(n)\) by 1'-set\((F(n))\) and the collection of principal 1'-sets in \(F(n)\) by 1-Set. ((\(F(n)\)) will be understood in the context.)

Although principal 1'-sets are analogous to hypersurfaces, they are more pervasive because of Lemma 5.2.2.

**Proposition 5.2.11.** P1-Set is closed under finite unions and intersections.

*Proof.* By the lemma. \(\square\)

As a special case of Proposition 5.2.11, we have

**Corollary 5.2.12.** For \(f, g \in F(\Lambda)\),

\(5.8\) \(1_{\text{loc}}(\langle f \rangle g) = 1_{\text{loc}}(f) \cap 1_{\text{loc}}(g);\)

\(5.9\) \(1_{\text{loc}}(\langle f \rangle \cap g) = 1_{\text{loc}}(f) \cup 1_{\text{loc}}(g).\)

**Proposition 5.2.13.** If \(\langle f \rangle\) is a principal kernel generated by \(f \in F(\Lambda)\), then \(1_{\text{loc}}(f) = 1_{\text{loc}}(\langle f \rangle)\). Consequently, \(1_{\text{loc}}(f) = 1_{\text{loc}}(f')\) for any generator \(f'\) of \(f\).

*Proof.* By Corollary 4.1.13, \(g \in F(\Lambda)\) is in \(\langle f \rangle\) if and only if there exists some \(n \in \mathbb{N}\) such that \(|f|^{-n} \leq_\nu g \leq_\nu |f|^n\). Hence, for every \(g \in \langle f \rangle\), \(1 = 1^{-n} \leq_\nu g(\mathbf{a}) \leq_\nu 1^n = 1\) for each \(\mathbf{a} \in 1_{\text{loc}}(f)\), implying \(g(\mathbf{a}) \equiv_\nu 1\).

In particular, \(1_{\text{loc}}(f) = 1_{\text{loc}}(\langle f \rangle) = 1_{\text{loc}}(\langle f' \rangle) = 1_{\text{loc}}(f').\) \(\square\)
5.3. Kernels of $1^\nu$-sets.

In view of Lemma 5.2.6, the operator $1_{\text{loc}}$ can be restricted to kernels. In the other direction, we now construct an operator that associates a kernel of the semifield of fractions $F(\Lambda)$ to any subset of $F^{(n)}$. Then we note that the operator $\text{Kern}$ and the operator $1_{\text{loc}}$ defined in the previous subsection are inverses of each other.

**Definition 5.3.1.** Given a subset $Z$ of $F^{(n)}$, we define

\begin{equation}
\text{Kern}(Z) = \{ f \in F(\Lambda) : f(a) \equiv_\nu 1, \forall a \in Z \}.
\end{equation}

$\text{Kern}(Z)$ is a $\nu$-kernel of $F(\Lambda)$, by the same argument as given in Lemma 5.2.6

**Remark 5.3.2.** The following statements hold for $Z, Z_i \subset F^{(n)}$:

1. If $Z_1 \subseteq Z_2$, then $\text{Kern}(Z_2) \subseteq \text{Kern}(Z_1)$.
2. $\text{Kern}(\bigcup_{i \in I} Z_i) = \bigcap_{i \in I} \text{Kern}(Z_i)$.
3. $K \subseteq \text{Kern}(1_{\text{loc}}(K))$ for any kernel $K$ of $F(\Lambda)$.
4. $Z \subseteq 1_{\text{loc}}(\text{Kern}(Z))$.

**Lemma 5.3.3.** If $Z \subset F^{(n)}$, then $\text{Kern}(Z)$ is a $\nu$-kernel.

**Proof.** It is closed under multiplication, and if $g_1 + g_2 \equiv_\nu 1$ then $(g_1f + g_2f)(a) = g_1(a) + g_2(a) \equiv_\nu a$. □

**Definition 5.3.4.** A $K$-kernel of $F(\Lambda)$ is a kernel of the form $\text{Kern}(Z)$, where $Z$ is a $1^\nu$-set.

**Lemma 5.3.5.**

1. $1_{\text{loc}}(\text{Kern}(Z)) = Z$ for any $1^\nu$-set $Z$.
2. $\text{Kern}(1_{\text{loc}}(K)) = K$, for any $K$-kernel $K$.

**Proof.** This is a standard argument, applying Remark 5.3.2 to the reverse inclusion of Proposition 5.2.6. □

By Lemma 5.3.5 we have

**Theorem 5.3.6.** There is a $1 : 1$, order-reversing correspondence

\begin{equation}
\{1^\nu\text{-sets of } F^{(n)}\} \to \{K\text{-kernels of } F(\Lambda)\},
\end{equation}

given by $Z \mapsto \text{Kern}(Z)$; the reverse map is given by $K \mapsto 1_{\text{loc}}(K)$.

One of the main goals in this paper is to find an intrinsic characterization of $K$-kernels, especially the principal $K$-kernels; this is only done in Corollary 8.0.14.

**Definition 5.3.7.** The $K$-variety of a set $S \subseteq \text{Fun}(F^{(n)},F)$, denoted $1_{\text{loc}}(S)$, is $\bigcap\{1_{\text{loc}}(f) : f \in S\}$. (Usually $S$ is taken to be finite.)

We need to cut down the class of $K$-varieties for tropical applications, in view of the following examples.

**Example 5.3.8.**

1. Let $f = \lambda_1 + \lambda_2$ and $g = 1_F$.
2. Let $f = \lambda_1 + 1_F$ and $g = \lambda_1^3 + \lambda_1^2 + 1_F$.
3. Let $f = \lambda_1 + \lambda_2 + 1_F$ and $g = \lambda_2^3 + \lambda_2^2 + 1_F$.

   In each of (1)–(3), $1_{\text{loc}}(\frac{f}{g})_T = \{ (\alpha,1_F) : \alpha \leq_\nu 1_F \} \cup \{ (1_F,\alpha) : \alpha \leq_\nu 1_F \}$ is not a variety.

4. Let $f = \lambda_1 + \lambda_2 + 1_F$ and $g = \lambda_1 + \lambda_2$. Then $1_{\text{loc}}(\frac{f}{g})_T = \{ (\alpha,\alpha) : \alpha \geq_\nu 1_F \}$

5.4. The coordinate $\nu$-semifield$^1$ of a $1^\nu$-set.

**Definition 5.4.1.** For $X \subset F^{(n)}$, The coordinate $\nu$-semifield$^1$ $F(X)$ of a $1^\nu$-set $X$ is the set of restriction of the rational functions $F(\Lambda)$ to $X$.

\[ \phi_X : F(\Lambda) \to F(X) \]

denotes the restriction map $h \mapsto h|_X$.

**Proposition 5.4.2.** $\phi_X$ is an onto semifield$^1$ homomorphism.

**Proof.** Straightforward verification. □

**Proposition 5.4.3.** $F(X)$ is a $\nu$-domain$^1$, isomorphic to $F(\Lambda)/\text{Kern}(X)$. 
Proof. The restriction map has kernel equal to those functions \( f \) which restrict to 1, which is \( \text{Kern}(X) \). □

When \( X' \supseteq X \) is another \( l^* \)-set, further restriction gives us a semiring homomorphism \( F[X] \to F[X'] \), and chains of these homomorphisms give us an algebraic view of dimension, which is studied at the end of \[25\].

6. The transition between tropical varieties and K-varieties

In view of Lemma 5.3.3 we would like to pass back and forth from tropical varieties to K-varieties. This is one of our main themes.

6.1. The hat construction.

We start by passing to the corner kernel locus from the corner locus obtained from (super)tropical polynomials. Towards this end, we formulate the following notion. We say a function \( f \) dominates \( g \) at \( a \) if \( f(a) \geq g(a) \); \( f \) dominates \( g \) if \( f \) dominates \( g \) at each point \( a \).

Definition 6.1.1. A molecule is a rational function \( hg^* \) where \( h \) is a monomial and \( g \) is a polynomial. The molecule is tangible if \( h \) and \( g \) are tangible. (In this case, \( hg^* = \frac{h}{g} \).

Definition 6.1.2. Given a polynomial \( f \in F[\Lambda] = \sum_{i=1}^{k} f_i \), written as a sum of monomials, define the molecules of \( f \) to be the molecules

\[
\hat{f}_i = f_i \left( \sum_{j \neq i} f_j \right)^* \in F(\Lambda),
\]

and

\[
\hat{f} = \sum_{i=1}^{k} \hat{f}_i.
\]

Usually \( f \) is tangible, in which case we have

\[
\hat{f}_i = \frac{f_i}{\sum_{j \neq i} f_j} \in F(\Lambda),
\]

Definition 6.1.3. Given \( f = \sum h \) written as a sum of monomials, and selecting one of these monomials \( h = h_i \), define \( f_{(h)} \) to be \( \sum_{h \neq h_j} h_j \). Then

\[
\hat{f} = \sum_{h} hf_{(h)}^*.
\]

Lemma 6.1.4. Given \( f = \sum h \) written as a sum of monomials, then \( \hat{f} \in T^+(\lambda_1, \ldots, \lambda_n)_\nu \).

Proof. Write \( f = \sum h \) and \( \hat{f} = \sum_{h} hf_{(h)}^* \). If some monomial \( h \) dominates at \( a \), then

\[
\hat{f}(a) = hf_{(h)}^*(a) \geq 1_F.
\]

Thus, we may assume that two monomials \( g, h \) dominate at \( a \), and then

\[
\hat{f}(a) = g f_{(g)}^*(a) + hf_{(h)}^*(a) = 1_F'.
\]

□

Lemma 6.1.5. The following conditions are equivalent:

(i) \( \hat{f}(a) = 1_F' \);
(ii) \( \hat{f}(a) \equiv \nu 1_F \);
(iii) \( a \in C_{\text{loc}}(f) \).

Proof. (i) Each side says that the numerator of \( \hat{f}_i \) dominates the denominator.
(ii) Clearly \( \hat{f}_i(a) \equiv \nu 1_F \) iff, for some \( j \neq i \), \( f_i(a) = f_j(a) \) which dominates each other monomial. This means \( f_i(a) + f_j(a) = f(a)^\nu \), implying \( f(a) = f(a)^\nu \in G \). □
Proof. (i) $\rightarrow$ (ii). Obvious.

(ii) $\rightarrow$ (iii). By Lemma 6.1.4 there are two monomials $f_i$ and $f_j$ that dominate at $a$, and so

$$1'_{F} = \hat{f}_i(a) + \hat{f}_j(a),$$

which then is $\hat{f}(a)$.

(iii) $\rightarrow$ (i). $f_i(a) = \hat{f}_j(a) \cong \mu 1'_{F}$, so

$$1'_{F} = \hat{f}_i + (a) = \hat{f}_j(a) = \hat{f}(a).$$

$\square$

Corollary 6.1.6. $1_{\text{loc}}(\hat{f}) = C_{\text{loc}}(f)$.

A principal corner locus is a set of ghost roots of a supertropical polynomial.

Corollary 6.1.7. Any principal corner locus is a principal $1'$-set.

The layered structure gives us more information:

Corollary 6.1.8. Over a layered semifield $F$, the number of monomials dominating $f$ at a point $a$ is precisely $s(\hat{f}(a))$.

Remark 6.1.9. For $S \subset \mathcal{F}(F[\Lambda])$, let $\hat{S} = \{ \hat{f} : f \in S \} \subseteq \mathcal{F}(\Lambda)$. Then by Remark 5.1.10 and Proposition 5.2.10,

$$C_{\text{loc}}(S) = \bigcap_{g \in \hat{S}} 1_{\text{loc}}(g) = 1_{\text{loc}}(\hat{S}).$$

Thus, the map $C_{\text{loc}}(f) \mapsto 1_{\text{loc}}(\hat{f})$ extends to a map

$$\Phi : C_{\text{loc}}(F^{(n)}) \rightarrow 1^{\prime} \text{-Set}(F^{(n)}),$$

where $\Phi : C_{\text{loc}}(f) \mapsto 1^{\prime} \text{-Set}(\hat{f})$. In particular, taking only finitely generated corner loci, and recalling that finite intersections and unions of principal $1'$-sets are principal $1'$-sets, $\Phi$ sends every finitely generated corner locus to a principal $1'$-set.

Lemma 6.1.10. [23 Lemma (10.2.5)] Let $f = \sum_{i=1}^{k} f_i \in F[\Lambda]$. Then for $1 \leq i, j \leq k$ such that $i \neq j$ and $f_i(a) \cong \mu f_j(a)$.

$$\text{(6.4) Either } \hat{f}_i(a) \cong \mu \hat{f}_j(a) \cong \mu 1 \text{ or } \hat{f}_i(a) \cong \mu \hat{f}_j(a) \text{ are dominated by } \hat{f} \text{ at } a.$$

Proof. Take the two cases, where either $f_i$ and $f_j$ both dominate, or neither dominates. $\square$

Lemma 6.1.11. For $f, g \in F[\Lambda]$ with $g$ tangible, over a layered 1-semifield $F$ (where $\Lambda = \{ \lambda_1, \ldots, \lambda_n \}$), the union of 1-loc($\frac{f}{g}$) with the corner loci of $f$ and $g$, is the corner locus of $f + g$.

Proof. $\frac{f}{g}(a) \cong \mu 1_{F}$ iff $f(a) \cong \mu g(a)$, which happens when $f(a) + g(a) \in \mathcal{G}$. $\square$

Proposition 6.1.12. An element $a \in F^{(n)}$ is a kernel root of $\hat{f}$ iff $a$ is a ghost root of $f$.

Proof. Apply Lemma 6.1.11 taking $g = f(h_k)$. $\square$

We can state this more explicitly.

Proposition 6.1.13. Suppose $f = \sum_{i=1}^{t} h_i \in F[\lambda_1, \ldots, \lambda_n]$, written as a sum of monomials. Then

$$\hat{f} = f^t \left( \prod_{i=1}^{t} f_{h_i} \right)^{\ast}.$$

Proof. (i) Since $\hat{f} = \sum_{i} h_i f_{h_i}^\ast = (\sum_{i} h_i \prod_{j \neq i} f_{h_j})(\prod_{j} f_{h_j})^\ast$, it suffices to prove that $\sum_{i} h_i \prod_{j \neq i} f_{h_j} = f^t$. But each $h_i$ appears in both sides, and every other product of length $t$ of the $h_j$ appears as a ghost in both sides. $\square$

We get Proposition 6.1.12 as a consequence, since the kernel roots must be precisely those $a$ for which each $\prod_{h_i(a)} \cong \mu f(a)$, which are the ghost roots.

Remark 6.1.14. Applying Proposition 6.1.13 to $\hat{f}(h_i)$ and observing that $f$ is dominated at each ghost root in two monomials, by definition, we see that the $1'$-set of $\hat{f}(h_i)$ is precisely the $1'$-set of $f_{(h_i)}(\prod_{j \neq i} f_{h_j})^\ast$. From this point of view, we can cut down one summand when passing to $1'$-sets.
Remark 6.1.15. The hat construction in Perri’s dissertation [23] is mathematically equivalent to the bend congruences of [6, 22, 21] (up to $\nu$-equivalence). Namely, take a polynomial $f = \sum f_i = h + f(h)$ where $h = f_j$ is one of the $f_i$ dominating at $a$ and $f(h) := \sum_{i \neq j} f_i$; the bend congruence is defined to be the congruence in $F[\Lambda]$ generated by $(f, f(h))$. But this could be matched with the principal kernel defined by the rational function

$$\frac{f}{f(h)} = \sum_i \frac{f_i}{f(h)} \cong \frac{h}{f(h)} \cong \frac{h_j}{f_j},$$

and we can reverse directions.

6.2. Examples of $1'$-sets.

Example 6.2.1. In these examples, we write $a = (a_1, \ldots, a_n)$.

(i) Take the tropical line $f = \lambda_1 + \lambda_2 + 1$. Its corresponding $1'$-set is defined by the rational function

$$(6.5) \quad \hat{f} = \frac{\lambda_1}{\lambda_2 + 1} + \frac{\lambda_2}{\lambda_1 + 1} + \frac{1}{\lambda_1 + \lambda_2} = \frac{(\lambda_1 + \lambda_2 + 1)^3}{(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2)}.$$ 

Moreover, in Remark 6.1.14 it is shown that any of the three terms above can be omitted, and for example we could use instead

$$(6.6) \quad \frac{\lambda_1}{\lambda_2 + 1} + \frac{\lambda_2}{\lambda_1 + 1} = \frac{\lambda_1^2 + \lambda_2^2 + \lambda_1 + \lambda_2}{(\lambda_1 + 1)(\lambda_2 + 1)}.$$ 

Although these two functions (6.5) and 6.6 define the same $1'$-set, they define different kernels, since $\lambda_1 + \lambda_2$ is not in the kernel generated by $\frac{\lambda_1}{\lambda_2 + 1} + \frac{\lambda_2}{\lambda_1 + 1}$. Indeed, if we take the point $a = (\alpha, \beta)$ for $\alpha < \beta < 1$, we get $\frac{1}{\beta}$ in (6.6) but $\beta$ in (6.5), and the their ratio can be whatever we want, and so the condition of Proposition 6.1.15 fails. This ambiguity motivates much of the later theory of this paper.

(ii) The ghost roots of $a_2 \leq \nu = a_1$ and $a_1 \leq a_2 \equiv \nu$ 1 provide two rays, but we are missing the ray $1 \equiv \nu = a_1 \equiv a_2$.

(iii) The $1'$-set of the rational function $\frac{\lambda_1}{\lambda_2 + 1} + \frac{\lambda_2}{\lambda_1 + 1}$ is comprised only of the ray $1 \equiv a_1 \equiv a_2$.

(iv) Next, we take the tropical line $g = \lambda_1 + \lambda_2 + 2$. Its corresponding $1'$-set is defined by the rational function $\hat{g} = \frac{\lambda_1}{\lambda_2 + 1} + \frac{\lambda_2}{\lambda_1 + 2}$. Its corresponding $1'$-set is defined by the rational function $\hat{g}$.

(v) We take the corner locus defined by the common ghost roots of $f$ and $g$, which correspond to the $1'$-set defined by the rational function $\hat{f} + \hat{g}$. Taking the three dominant monomials yields $\frac{\lambda_1}{\lambda_2 + 1} + \frac{\lambda_2}{\lambda_1 + 1} + \frac{2}{\lambda_1 + \lambda_2}$, which again cannot come from a single polynomial.

6.3. From the kernel locus to hypersurfaces.

The other direction is more straightforward.

Definition 6.3.1. Given $f = \frac{h}{g} \in F(\Lambda)$ for $g, h \in F[\lambda_1, \ldots, \lambda_n]$, define $\widehat{f} = g + h$.

Remark 6.3.2. Conceivably, one could have $f = \frac{h}{g} = \frac{h'}{g'} \in T^+(\lambda_1, \ldots, \lambda_n)$, so the definition technically depends on how we define the representation. But in this case, $h'g' = gh$. Thus, any ghost root $a$ of $h$ is a ghost root of $g$ or $h'$, so we get the same ghost roots except when both the numerator and denominator have a common ghost root.

Theorem 6.3.3. Suppose $f \in F[\lambda_1, \ldots, \lambda_n]$. Then

$$C_{\text{loc}}(\hat{f}) = 1_{\text{loc}}(\hat{f}) = C_{\text{loc}}(f).$$

Proof. Using Proposition 6.1.13 and its proof, we can compute $\hat{f}$ explicitly, as $f^t + \prod f_i = f^t$ (since each $f_i \leq \nu \hat{f}$), and $\prod f_i$ does not contain any $h_i$, which are the only tangible monomials of $f^t$.

In this way, we can pass from the corner locus of a single polynomial to a 1-set, and back. This procedure breaks down for tropical varieties defined by more than one polynomial, as seen in Example 6.2.1(v).

Going the other way is trickier. Suppose $f = \frac{h}{g}$. Then $\hat{f} = g + h$.

Example 6.3.4. Take $f = \frac{h}{g} \in F(\Lambda)$, for $g = \lambda_1$ and $h = 2\lambda_1 + 4$. Then $\hat{f} = h$ has the ghost root 2, which is not in the 1-set of $f S$. 
7. Distinguishing the canonical tropical varieties

By “canonical” tropical variety we mean the kind of variety that arises in the usual study of tropical geometry, cf. [20], although there really is no truly “canonical” definition. One possibility is the definition of variety given above. One major objective is to indicate how to obtain these via kernels. In this section, following Perri’s dissertation, [23], we present a method for describing a canonical tropical variety in terms of a single rational function, thereby enabling us to pass from corner loci to $1^\nu$-sets which often are principal. This sets the path for applying the study conducted in the previous sections to tropical geometry. We begin our discussion by introducing the notion of “corner internal” $1^\nu$-sets, which characterize the $1^\nu$-sets and kernels arising from tropical varieties. Then we bring in “regular” rational functions, which avoid degeneracy in the variety.

7.1. Corner internal functions.

First we want to know which kernels come from corner loci. Example 6.3.4 shows that we can have $f = \frac{h}{g}$, where $a$ is a ghost root of $h$ and thus of $f = h + g$, but having $h(a) >_\nu g(a)$, and thus $a$ is not in the $1$-set of $f$. Excluding this possibility leads to our next definition.

**Definition 7.1.1.** A rational function $f \in F(\Lambda)$ is corner internal if we can write $f = \frac{h}{g}$ for polynomials $g, h$, with $g \in T[\Lambda]$, such that every ghost root of $g + h$ is a kernel root of $f$.

In other words, if $g(a) \leq_\nu h(a) \in G$ or $h(a) \leq_\nu g(a) \in G$, then $g(a) \equiv_\nu h(a)$.

**Remark 7.1.2.** Suppose $f = hg^\ast$. Any ghost root of $g + h$ is either a ghost root of $h$, or of $g$, or else satisfies $g(a) \equiv_\nu h(a)$ in which case it is automatically in $1_{loc}(f)$. Thus, corner internality is equivalent to showing that $C_{loc}(g) \cup C_{loc}(h) \subseteq 1_{loc}(f)$.

**Remark 7.1.3.** When considering whether $f \in F(\Lambda)$ is corner internal, the choice of $g$ and $h$ is important, which we call the canonical way of writing $f$. For example, $\lambda + 1 \in F(\Lambda)$ is trivial corner internal, although $\frac{(\lambda + 1)(\lambda + 1)}{\lambda + \alpha}$ for $\alpha >_\nu 1$ is not - since substituting $\alpha$ for $\lambda$ we get $\frac{\alpha^2 + \alpha^2}{\alpha + \alpha} = \frac{(\alpha^2)\alpha^2}{\alpha^2}$. Thus $\alpha$ is a ghost root of the numerator, which surpasses the denominator since $(\alpha^2)^\nu >_\nu \alpha^\nu$.

Also note that $g + h$ must be tangible, since otherwise some open set around $a$ is in $C_{loc}(g + h)$ but not in $1_{loc}(f)$. Thus, we write $f = \frac{h}{g}$ instead of $f = h g^\ast$. In particular, $f$ must be reduced as a fraction.

Even though the canonical way of writing $f = \frac{h}{g}$ may not be unique, the following observation is enough for our purposes.

**Lemma 7.1.4.** Suppose $f = \frac{h}{g} = \frac{h'}{g'}$ are two canonical ways of writing a corner internal rational function $f$. Then $C_{loc}(h + g) = C_{loc}(h' + g')$.

**Proof.** If $a \in C_{loc}(h + g)$, then either $g(a) \equiv_\nu h(a)$ and we are done, or say $a \in C_{loc}(g)$. But then $h(a) \leq_\nu g(a)$, implying $h'(a) \leq_\nu g'(a)$, and also $a \in C_{loc}(h'g')$ since $h'g' = h'g$. We are done if $a \in C_{loc}(g')$, so we may assume that $a \in C_{loc}(h)$. Hence $h(a) \leq_\nu g(a) \in G$, implying $g(a) \equiv_\nu h(a)$, and again we are done.

Thus, $C_{loc}(f)$ does not depend on the canonical way we write the corner internal rational function $f$.

**Proposition 7.1.5.** A rational function $f$ is corner internal iff $C_{loc}(f) = 1_{loc}(f)$, for any canonical way of writing $f = \frac{h}{g}$.

**Proof.** $(\Leftarrow)$ is by definition.

$(\Rightarrow)$ Any kernel root $a$ must satisfy $g(a) \equiv_\nu h(a)$, implying $a$ is a ghost root of $g + h$.

**Lemma 7.1.6.** For any polynomial $f \in F[\Lambda]$, the function $\hat{f}$ is corner internal.

**Proof.** $C_{loc}(\hat{f}) = 1_{loc}(\hat{f})$ in view of Theorem 6.3.3.

**Theorem 7.1.7.** The correspondences $f \mapsto \hat{f}$ and $h \mapsto h$ induce a 1:1 correspondence between hypersurfaces and $1^\nu$-sets of corner internal rational functions.

**Proof.** Combine Proposition 7.1.5 and Lemma 7.1.6.

Having established the importance of being corner internal, let us delve deeper into the elementary properties.

By symmetry, a rational function $f \in F(\Lambda)$ is corner internal if and only if $f^{-1}$ is corner internal.
Remark 7.1.8. If \( f = \frac{h}{g}, \ f' = \frac{h'}{g'} \in F(\Lambda) \), then
\[
f + f' = \frac{hg' + gh'}{gg'},
\]
and
\[
|f| = f + f^{-1} = (h^2 + g^2)(gh)^*.
\]
By Proposition 3.5.2, any ghost root of \( h^2 + g^2 \) or of \( h^2 + g^2 + gh \cong (h + g)^2 \) is a ghost root of \( g + h \).

**Proposition 7.1.9.** A rational function \( f \in F(\Lambda) \) is corner internal if and only if \( |f| \) is corner internal.

**Proof.** \(|f|\) and \( f \) have the same 1'-sets.

\((\Rightarrow)\) Write \( f = \frac{h}{g} \) canonically. Then \(|f| = hg^* + gh^* = (h^2 + g^2)(gh)^*\). By Remark 7.1.8, any ghost root \( a \) of \( g^2 + h^2 \) is a ghost root of \( g + h \) and thus a kernel root of \( f \), by definition, or else \( g(a) = h(a) \), yielding a kernel root of \( f \). But then \( a \in 1_{loc}(|f|) \).

\((\Leftarrow)\) If \( a \) is a ghost root of \( g + h \), then, again by Remark 7.1.8 \( a \) is a ghost root of \( g^2 + h^2 + gh \) and thus by hypothesis is in \( 1_{loc}(|f|) = 1_{loc}(f) \).

**Lemma 7.1.10.** If \( f \in F(\Lambda) \) is corner internal, then \( f^k \), for any \( k \in \mathbb{N} \) and \( \sum_{i=1}^{m} f^{d(i)} \) with \( d(1) < \ldots < d(m) \) in \( \mathbb{N} \), are corner internal.

**Proof.** Everything follows easily from Remark 7.1.8.

We recall \( f \wedge g \) from (3.2) after Definition 3.5.2.

**Remark 7.1.11.** For intuition, in view of Remark 3.5.3, \( (f \wedge g)(a) \cong \min\{f(a)^\nu, g(a)^\nu\} \).

**Proposition 7.1.12.** If \( f, f' \in F(\Lambda) \) are corner internal, then \( |f| \wedge |f'| \) is corner internal.

**Proof.** Since \( f \) and \( f' \) are corner internal, \( |f| \) and \( |f'| \) also are corner internal. Write \( |f| = \frac{h}{g} \) and \( |f'| = \frac{h'}{g'} \) in the canonical ways. Now,
\[
1 \leq \nu |f| \wedge |f'| = (gh^* + g'h^*)^* = hh'(gh' + g'h)^*,
\]
so
\[
h(a)h'(a) \geq \nu (gh' + g'h)(a).
\]
We need to show that any ghost root \( a \) of \( hh' + gh' + g'h \) satisfies
\[
(hh' + gh' + g'h)(a) \cong \nu h(a)h'(a).
\]
\leq \nu \) follows from (7.2), so we need to show \( \geq \nu \). Clearly \( a \) is a ghost root of \( hh' + gh' \), \( hh' + g'h \), or \( gh' + g'h \). In the latter case, \( h(a)h'(a) \leq_a (gh' + g'h)(a) \), and we are done. Thus by symmetry we may assume that \( a \in C_{loc}(hh' + gh') = C_{loc}((g + h)h') \). If \( a \in C_{loc}(g + h) \) then \( a \in 1_{loc}(|f|) \) and thus in \( 1_{loc}(|f| \wedge |f'|) \). So we are done unless \( a \in C_{loc}(h') \). If \( h'(a) \geq \nu g'(a) \), then \( a \in C_{loc}(g' + h') = 1_{loc}(f) \) and we are done. Thus we may assume that \( h'(a) \leq \nu g'(a) \), implying \( g'(a)h(a) \geq \nu h'(a)h(a) \), and again we are done.

**Corollary 7.1.13.** Let \( f \in F(\Lambda) \). Then \( f \) is corner internal if and only if \( |f| \wedge |\alpha| \) is corner internal for any \( \alpha \neq 1 \) in \( F \).

**Proof.** Write \( f = \frac{h}{g} \). Since \( |\alpha| > 1 \), the ghost roots of \( |\alpha|h + |\alpha|g + h \) and of \( |\alpha|(g + h) \) are the same, and thus the kernel roots of \( |f| \) and of \( |f| \wedge |\alpha| \) are the same.

**Proposition 7.1.14.** There is a \( 1:1 \) correspondence between principal corner internal 1'-sets and principal corner-loci.

**Proof.** Take \( f, f' \in F(\Lambda) \) corner internal.

If \( 1_{loc}(f') = 1_{loc}(f) \), then
\[
1_{loc}(f') = 1_{loc}(f) = C_{loc}(f) = 1_{loc}(\tilde{f}).
\]
Conversely, if \( C_{loc}(f') = C_{loc}(f) \), then
\[
C_{loc}(f') = C_{loc}(f) = 1_{loc}(\tilde{f}) = C_{loc}(\tilde{f}),
\]
by Theorem 6.3.3.
7.2. Corner internal kernels.

**Definition 7.2.1.** A principal kernel $K$ of $F(\Lambda)$ is said to be **corner internal** if it has a corner internal generator. In this case, the $1^\nu$-set $1_{\text{loc}}(K)$ corresponding to $K$ is said to be a **corner internal** $1^\nu$-set.

**Corollary 7.2.2.** Any finite intersection of principal corner internal kernels is a principal corner internal kernel.

**Proof.** By induction, it is enough to show that if $K$ and $K'$ are principal corner internal kernels, then so is $K \cap K'$. Write $K = \langle f \rangle$ and $K' = \langle f' \rangle$. By Proposition 7.1.12 we may assume that $f, f' \geq 1$. By Proposition 7.1.12 $f \wedge f'$ is corner internal, which generates $K \cap K'$.

**Remark 7.2.3.** If $f, g \in F(\Lambda)$ are corner internal then $|f| + |g|$ need not be corner internal. Thus the collection of corner internal kernels is not a lattice. In our study we thus take the lattice generated by principal corner internal kernels. These elements will be shown to correspond to finitely generated corner loci that are not necessarily principal.

7.3. The hat-construction for corner internal kernels.

Although the $f \mapsto \hat{f}$ correspondence given above yields a fast and effective correspondence from corner loci of polynomials to $1^\nu$-sets, it does not work so well on arbitrary rational functions $f \in F(\Lambda)$, so we turn to a subtler but more thorough correspondence, which “explains” what makes a kernel corner internal. We start with the special case of a polynomial $f = \sum_i f_i \in F[\Lambda]$, written as a sum of monomials $f_i$.

**Definition 7.3.1.** Given $f = \sum_i f_i \in F[\Lambda]$, define the rational function $\hat{f} = \bigwedge_{i=1}^k |\hat{f}_i|$.

**Proposition 7.3.2.** $C_{\text{loc}}(f) = 1_{\text{loc}}(\hat{f})$.

**Proof.** We have seen in Theorem 6.3.3 that $a \in C_{\text{loc}}(f)$ iff $|\hat{f}_i|(a) = 1$ for some $i$. But each $|\hat{f}_i|(a) \geq 1$, so we conclude with Lemma 5.2.3.

Thus, we have an alternative approach to that of Theorem 6.3.3 motivating some of the intricate computations we are about to make.

**Theorem 7.3.3.** Suppose $f = hg^* \in T(\Lambda)$ is a rational function, where $h = \sum_i h_i$ and $g = \sum_j g_j$ written as sums of monomials. Then $f$ is corner internal if and only if the following conditions hold:

$$
(7.3) \quad \left( \bigcup_{i=1}^k 1_{\text{loc}}(\hat{h}_i) \right) \cap 1_{\text{loc}}(f^* + 1) \subseteq 1_{\text{loc}}(f)
$$

$$
(7.4) \quad \left( \bigcup_{j=1}^m 1_{\text{loc}}(\hat{g}_j) \right) \cap 1_{\text{loc}}(f + 1) \subseteq 1_{\text{loc}}(f).
$$

**Proof.** Note that $f + 1 = (g + h)g^*$, whereas $f^* + 1 = (g + h)h^*$. In view of Remark 7.1.2 we need to show that $C_{\text{loc}}(h) \subseteq 1_{\text{loc}}(f)$ is equivalent to (7.3), since the other condition is symmetric (with respect to exchanging $g$ and $h$). By definition of corner internality, any ghost root $a$ of $h$ must satisfy $h(a) \leq \nu g(a)$, implying $a \in 1_{\text{loc}}(f + 1)$. Thus, checking this at each ghost root, we have $1_{\text{loc}}(\hat{h}_i) \subseteq 1_{\text{loc}}(f + 1)$ for each $i$, or equivalently,

$$
(7.5) \quad \langle f + 1 \rangle \subseteq \langle \hat{h}_i \rangle.
$$

The steps are reversible.

Now, intersecting both sides with $1_{\text{loc}}(f^* + 1)$ yields

$$
1_{\text{loc}}(\hat{h}_i) \cap 1_{\text{loc}}(f^* + 1) \subseteq 1_{\text{loc}}(f + 1) \cap 1_{\text{loc}}(f^* + 1).
$$

Note that this step also is reversible since $1_{\text{loc}}(f^* + 1) \cup 1_{\text{loc}}(f + 1) = F^{(n)}$.

Passing again from $K$-varieties to kernels yields

$$
\langle f \rangle \subseteq \langle \hat{h}_i \rangle \langle f^* + 1 \rangle.
$$

Moreover, since the above inclusion holds for every $i \in \{1, \ldots, k\}$ we conclude that

$$
\left( \bigcup_{i=1}^k 1_{\text{loc}}(\hat{h}_i) \right) \cap 1_{\text{loc}}(f^* + 1) \subseteq 1_{\text{loc}}(f),
$$
as desired. 

**Remark 7.3.4.** Note that
\[
\bigcup_{i=1}^{k} 1_{\text{loc}}(\hat{h}_i) = 1_{\text{loc}} \left( \bigwedge_{i=1}^{k} |\hat{h}_i| \right) = 1_{\text{loc}}(\hat{h})
\]
and similarly
\[
\bigcup_{j=1}^{m} 1_{\text{loc}}(\hat{g}_j) = 1_{\text{loc}}(\hat{g}).
\]
Thus we can rewrite (7.3) and (7.4) as
\[
1_{\text{loc}}(\hat{h}) \cap 1_{\text{loc}}(f^* + 1) \subseteq 1_{\text{loc}}(f) \quad \text{and} \quad 1_{\text{loc}}(\hat{g}) \cap 1_{\text{loc}}(f + 1) \subseteq 1_{\text{loc}}(f),
\]
or as
\[
1_{\text{loc}}(\hat{h}) \cap 1_{\text{loc}}(f^* + 1) \subseteq 1_{\text{loc}}(f) \quad \text{and} \quad 1_{\text{loc}}(\hat{g}) \cap 1_{\text{loc}}(f + 1) \subseteq 1_{\text{loc}}(f).
\]
We conclude that \(\hat{f}\) is corner internal for any \(f \in F(\Lambda)\).

**Remark 7.3.5.** In view of Theorem 7.3.3, given \(f = \frac{h}{g} \in F(\Lambda)\), in order to obtain a corner internal rational function whose K-hypersurface contains \(1_{\text{loc}}(f)\) one must adjoin both
\[
\left( \bigcup_{i=1}^{k} 1_{\text{loc}}(\hat{h}_i) \right) \cap 1_{\text{loc}}(f^* + 1)
\]
and
\[
\left( \bigcup_{j=1}^{m} 1_{\text{loc}}(\hat{g}_j) \right) \cap 1_{\text{loc}}(f + 1)
\]
to the K-hypersurface of \(f\).

**Definition 7.3.6.** Define the map \(\Phi_{CI}: F(\Lambda) \rightarrow F(\Lambda)\) by
\[
\Phi_{CI}(f) = |f| \wedge \left( |f^* + 1| + \hat{h} \right) \wedge \left( |f + 1| + \hat{g} \right),
\]
where \(f = \frac{h}{g}\).

**Remark 7.3.7.** \(\Phi_{CI}(f) = \left( |f^* + 1| + \left| |f| \wedge \hat{h} \right| \right) \wedge \left( |f + 1| + \left| |f| \wedge \hat{g} \right| \right),\) since \(|f^* + 1|, |f + 1| \leq |f|\). This is the fraction whose K-hypersurface is formed by adjoining all the necessary points to \(1_{\text{loc}}(f)\) to obtain corner internality. Hence,
\[
\langle \Phi_{CI}(f) \rangle = \left( (f^* + 1) \left( (f) \cap \bigcap_{i=1}^{k} \langle \hat{h}_i \rangle \right) \right) \cap \left( (f + 1) \left( (f) \cap \bigcap_{j=1}^{m} \langle \hat{g}_j \rangle \right) \right).
\]

**Corollary 7.3.8.** Let \(f = \frac{h}{g} \in F(\Lambda)\) be a rational function, where \(h = \sum_{i=1}^{k} h_i\) and \(g = \sum_{j=1}^{m} g_j\) are written as sums of monomials in \(F[x_1, \ldots, x_n]\). Then
\[
1_{\text{loc}}(\hat{h} + g) = 1_{\text{loc}}(f) \cup \left( \left( \bigcup_{i=1}^{k} 1_{\text{loc}}(\hat{h}_i) \right) \cap 1_{\text{loc}}(f^* + 1) \right) \cup \left( \left( \bigcup_{j=1}^{m} 1_{\text{loc}}(\hat{g}_j) \right) \cap 1_{\text{loc}}(f + 1) \right).
\]
Thus
\[
1_{\text{loc}}(\hat{h} + g) = 1_{\text{loc}}(\Phi_{CI}(f))
\]

**Proof.** Apply Lemma 7.3.2 and Remark 7.3.4 to the theorem, to get the three parts. \(\square\)

**Theorem 7.3.9.** If \(f \in F(\Lambda)\), then \(\langle \Phi_{CI}(f) \rangle\) is corner internal, and \(1_{\text{loc}}(\Phi_{CI}(f)) \supseteq 1_{\text{loc}}(f)\). Furthermore, \(1_{\text{loc}}(\Phi_{CI}(f)) = 1_{\text{loc}}(f)\) if and only if \(f\) is corner internal.
Proof. The first claim follows from Corollary 7.3.8 where \( \hat{f} \) is corner internal by Theorem 6.3.3. The second claim is straightforward from Remark 7.3.7 since

\[
1_{\text{loc}} \left( |f| \wedge (|f^* + 1| + \hat{h}) \right) = 1_{\text{loc}}(f) \cup 1_{\text{loc}} \left( |f^* + 1| + \hat{h} \right) = 1_{\text{loc}}(f + 1 + \hat{g}).
\]

The last statement follows Theorem 7.3.3 \( \square \)

**Proposition 7.3.10.** For any \( f = \frac{h}{g} \in F(\Lambda) \),

\[
\langle \Phi_{CI} \left( \sum_{i=1}^{k} f^{d(i)} \right) \rangle = \langle \Phi_{CI}(f) \rangle
\]

whenever \( 0 < d(1) < \cdots < d(k) \), and

\[
\langle \Phi_{CI}(f^k) \rangle = \langle \Phi_{CI}(f) \rangle
\]

for any \( k \in \mathbb{Z} \setminus \{0\} \).

**Proof.** By the Frobenius property, \( \sum_{i=1}^{k} f^{d(i)} = \sum_{i=1}^{k} \frac{h^{s+t} + g^{s+t}}{h^{g^t} g^t} \) where \( t = |d(k)| \) and \( s = |d(1)| \). So

\[
\Phi_{CI} \left( \sum_{i=1}^{k} f^{d(i)} \right) = \Phi_{CI} \left( \frac{h^{s+t} + g^{s+t}}{h^{g^t} g^t} \right) = h^{s+t} + g^{s+t} + h^s g^t = (\hat{h} + g)^{s+t}.
\]

(The hats are over the entire expressions.) But \( \langle (\hat{h} + g)^{s+t} \rangle = (\hat{h} + g) = \langle \Phi_{CI}(f) \rangle \), since \( \Phi_{CI}(\frac{k}{g}) = h^k + g^k = (h + g)^k \). Thus, \( \langle (\hat{h} + g)^k \rangle = (\hat{h} + g) = \langle \Phi_{CI}(f) \rangle \). \( \square \)

**Corollary 7.3.11.** Let \( f \in F(\Lambda) \) be such that \( f = u_1 \wedge \cdots \wedge u_k \) where \( u_1, \ldots, u_k \in F(\Lambda) \) are corner internal. Then \( f \) is corner internal and

\[
1_{\text{loc}}(\Phi_{CI}(f)) = 1_{\text{loc}}(\Phi_{CI}(u_1) \wedge \cdots \wedge \Phi_{CI}(u_k)).
\]

**Proof.** \( f \) is corner internal, by Corollary 7.2.2. Replacing \( f \) by \(|f| \), we may assume that \( f \geq 1 \) (and thus each \( u_i \geq 1 \)). Since \( u_i \) is corner internal, \( 1_{\text{loc}}(\Phi_{CI}(u_i)) = 1_{\text{loc}}(u_i) \) for \( i = 1, \ldots, k \), and thus

\[
1_{\text{loc}}(\Phi_{CI}(u_1) \wedge \cdots \wedge \Phi_{CI}(u_k)) = 1_{\text{loc}}(u_1 \wedge \cdots \wedge u_k) = 1_{\text{loc}}(f).
\]

(7.9) holds since \( f \) is corner internal. \( \square \)

### 7.4. Regularity.

In the standard tropical theory, varieties have the property that their complement is dense, whereas in the supertropical theory the root set say of \( \lambda^2 + 2^0 \lambda + 3 \) contains the closed interval \([1, 2]\). We would like to handle this issue through kernels and their \( 1^\nu \)-sets.

The relation \( f(a) = \sum_{g} h(g) \wedge a \); \( \leq_{\nu} 1 \) is studied locally in the following sense: For any \( a \in F(n) \), there is at least one monomial \( h_{i0} \) of the numerator and at least one monomial \( g_{j0} \) of the denominator which are dominant at \( a \). If more than one monomial at \( a \) is dominant, say \( \{h_{ik}\}_{k=1} \) and \( \{g_{jm}\}_{m=1} \), then we have additional relations of the form \( h_{i0} = h_{ik} \) and \( g_{j0} = g_{jm} \).

Our motivating example: The relation \( 1_F + \lambda = 1_F \) holds for all \( a \leq_{\nu} 1_F \). This will be called an order relation.

In general, the dominant monomials of both numerator and denominator at some point define relations \( \{h_{ik} = g_{jm} : 0 \leq k \leq s, 0 \leq m \leq t\} \) on regions of \( F(n) \). Every such relation can be converted by multiplying by inverses of monomials to obtain a relation of the form \( 1 = \phi(\Lambda) \) with \( \phi \in F(\Lambda) \setminus F \) a Laurent monomial, and thus reduces the dimension. Note that in the case in which \( h_{i0} \) and \( g_{j0} \) singly dominate and are the same monomial, no extra relation is imposed on the region described above, so we are left only with the order relations defining the region.

In this way, two types of principal \( 1^\nu \)-sets emerge from two distinct types of kernels, characterized by their generators, distinguished via the following definition.

**Definition 7.4.1.** A rational function \( f = \frac{h}{g} \) is regular at a point \( a \) in \( 1_{\text{loc}}(f) \) if each \( \nu \)-neighborhood of \( a \) contains a point on which \( h \) and \( g \) do not agree. Otherwise \( f \) is irregular at \( a \). \( f \) is regular at a set \( S \) if it is regular at every point \( a \) in \( S \). \( f \) is regular if it is regular at \( 1_{\text{loc}}(f) \). \( \text{Reg}(F(\Lambda)) \) denotes the set of regular rational functions.
Note that any rational function which is regular at a point \( a \) is also regular at the region containing \( a \). Another way of stating this condition, writing \( f = \frac{h}{g} \) is to define a leading Laurent monomial of \( f \) to be of the form \( \frac{h_i^{\lambda}}{g_i^{\nu}} \) where \( h_i \) is a dominant monomial of \( h \) and \( g_i \) is a dominant monomial of \( g \). Of course \( f \) will have several leading Laurent monomials at \( a \) if \( a \) is a ghost root of \( h \) or \( g \). The regularity condition is that \( f \) possesses some leading Laurent monomial \( \neq 1_{F} \). For example, \( f = \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_3} \) is regular at \((1,1,1) \) but not at \((2,1,1) \).

**Lemma 7.4.2.** If \( f \) is regular, then any other generator \( f' \) of \( \langle f \rangle \) is regular.

**Proof.** If \( f' \) were not regular at \( a \), then its sole leading Laurent monomial at \( a \) would be \( 1_{F} \), implying that some \( \nu \)-neighborhood of \( a \) is in \( 1_{loc}(f') \) and thus of \( 1_{loc}(f) \), contrary to \( f \) possessing some leading Laurent monomial \( \neq 1_{F} \).

**Lemma 7.4.3.** If \( f, g \in \text{Reg}(F(\Lambda)) \) such that \( f \neq 1_{F} \) and \( g \neq 1_{F} \), then the following elements are also in \( \text{Reg}(F(\Lambda)) \):

\[
f^*, \ f^k \quad \text{with} \quad k \in \mathbb{Z}, \ f + g, \ |f|, \ f \wedge g.
\]

**Proof.** Follows at once from Lemma 7.4.2 and Proposition 4.1.17. A more direct argument for \( |f| \) is that writing \( f = \frac{h}{g} \), we have

\[
|f| = \frac{h^2 + g^2}{gh},
\]

but the leading Laurent monomial is either that of \( h^2 (gh)^* = f \) or that of \( g^2 (gh)^* = f^* \).

**Definition 7.4.4.** The relation \( f(\mathbf{a}) \cong_\nu 1_{F} \) is regular at \( \mathbf{a} \) if \( f \) is regular at \( \mathbf{a} \). Otherwise \( f(\mathbf{a}) \cong_\nu 1_{F} \) is an order relation.

A 1*-set is regular if it can be written as \( 1_{loc}(S) \) where each \( f \in S \) is regular.

**Lemma 7.4.5.** If \( A = 1_{loc}(f_1, ..., f_s) \) and \( B = 1_{loc}(g_1, ..., g_t) \) are regular then \( A \cap B \) and \( A \cup B \) are regular.

**Proof.** Immediate from Lemma 5.1.6.

Considering only dominant monomials at the neighborhood of \( \mathbf{a} \), our local relations fall into two distinct cases:

- An order relation of the form \( 1_{F} + g \cong_\nu 1_{F} \) with \( g \in F(\Lambda) \), i.e., \( g(\mathbf{a}) \leq_\nu 1_{F} \). The resulting quotient \( \nu \)-semifield\(^1\) \( F(\Lambda) / \langle 1 + g \rangle \) does not reduce the dimension at such a point \( \mathbf{a} \), but only imposes new order relations on the variables.
- A regular relation; this reduces the dimensionality of the image of \( F(\Lambda) \) in the quotient semifield\(^1\).

We aim to characterize those elements of \( F(\Lambda) \) that do not translate (locally) to order relations but only to regular relations (locally). This will allow us to characterize those relations which correspond to corner loci (tropical varieties in tropical geometry). The kernels corresponding to these relations will be shown to form a sublattice of the lattice of principal kernels (which is itself a sublattice of the lattice of kernels).

**Example 7.4.6.** If \( f \in F(\Lambda) \) such that \( f \neq f + 1 \) (i.e., 1 is essential in \( f + 1 \)), then \( f + 1 = \frac{f + 1}{1} \) is not regular since 1 being essential in the numerator coincides with the denominator over some nonempty region.

**Example 7.4.7.** Consider the map \( \phi : F(\Lambda) \rightarrow F(\Lambda) / (\lambda + 1) \) given by \( \lambda \mapsto 1 \equiv_\nu 1 \). This map imposes the relation \( \lambda + 1 \equiv_\nu 1 \) on \( F(\Lambda) \), which is just the order relation \( \lambda \equiv_\nu 1 \). Under the map \( \phi \), \( \lambda \) is sent to \( \lambda = \lambda (\lambda + 1) \), where now, in \( \text{Im} (\phi) = F(\Lambda) \), \( \lambda \) and 1 are comparable, as opposed to the situation in \( F(\Lambda) \), where \( \lambda \) and 1 are not comparable. If instead of \( \lambda + 1 \), we consider \( |\lambda| + 1 \equiv_\nu \lambda + \lambda^{-1} + 1 \), then as \( |\lambda| \geq_\nu 1 \) the relation \( |\lambda| + 1 \equiv_\nu 1 \) means \( |\lambda| \equiv_\nu 1 \), as yielded by the substitution map sending \( \lambda \) to 1. Note that \( |\lambda| \) and 1 are comparable in \( F(\Lambda) \), since, as mentioned above, \( |\lambda| \geq_\nu 1 \), which is equivalent to the relation imposed by the equality \( |\lambda| + 1 \equiv_\nu |\lambda| \) or equivalently by \( |\lambda|^{-1} + 1 \equiv_\nu 1 \). The kernel \( \langle |\lambda|^{-1} + 1 \rangle \), since \( |\lambda|^{-1} + 1 \equiv_\nu 1 \), is just the trivial kernel \( \langle 1 \rangle = \{1\} \).

**Note 7.4.8.** As seen in Proposition 4.1.15 and Corollary 4.1.18, order relations affect the structures of kernels in a semifield\(^1\). For instance, consider the principal kernel in a semifield\(^1\) \( F \) generated by an element \( a \in F \). Then \( b \not\in (a) \) for any element \( b \in F \) such that \( b \) not comparable to \( a \).
8. The role of the kernel \(\langle F \rangle\) of \(F(\Lambda)\) in the lattice of regular principal kernels

Our overall objective in this section and the next is to describe the theory in terms of principal kernels.

Definition 8.0.9. Denote the sublattice of principal kernels of a \(\nu\)-semifield \(S\) by \(\mathcal{P}(S)\).

Definition 8.0.10. A principal kernel \(K = \langle f \rangle\) of \(F(\Lambda)\) is regular if \(f \in F(\Lambda)\) is regular. In this case the 1-set \(1_{\text{loc}}(f)\) corresponding to \(K\) also is called regular.

Proposition 8.0.11. The set of regular principal kernels forms a sublattice of \(\mathcal{P}(F(\Lambda))\).

Proof. This follows directly from Corollary 4.1.27 and Lemma 7.8.3. \(\square\)

8.0.1. Corner loci and principal \(1^\nu\)-sets.

Suppose \(X \subseteq F(\nu)\), and consider rational functions restricted to \(X\). By Remark 5.1.9 and Lemma 5.1.6, the collection of corner loci is closed under intersections, while the collection of finitely generated principal corner loci is also closed under finite unions. Also, \(C_{\text{loc}}(\emptyset) = X\) and \(C_{\text{loc}}(\alpha) = \emptyset\) for \(\alpha \neq 1\).

By Lemma 7.4.6, the regular finitely generated principal corner loci comprise a sublattice, which we want to investigate.

Remark 8.0.12. By Remark 6.1.9 and Proposition 8.0.11 applied to the correspondence of Theorem 7.4.7, the lattice generated by principal corner internal kernels with respect to (finite) products and intersections corresponds to the lattice of finitely generated corner loci. Note that \(1_{\text{loc}}(1) = X\) and \(1_{\text{loc}}(\alpha) = \emptyset\) for \(\alpha \neq 1\).

To delve deeper, we need to turn to the kernel \(\langle F \rangle\) of \(F(\Lambda)\).

Theorem 8.0.13. [23, Theorem (13.5.2)] The lattice \(\mathcal{P}(\langle F \rangle)\) is generated by the principal corner internal kernels, and the sublattice of regular principal kernels is generated by the regular, principal corner internal kernels.

The proof is rather long, requiring the concept of bounded kernels, so we defer it until §9.

Corollary 8.0.14. The lattice of (tangible) finitely generated corner loci corresponds to the lattice of principal (tangible) kernels of \(\langle F \rangle\). Intersections of (supertropical) hypersurfaces correspond to principal \(1^\nu\)-sets and kernels, whereas intersections of tangible hypersurfaces correspond to tangible principal \(1^\nu\)-sets and kernels.

Proof. We use the correspondence of Remark 8.0.12 between principal (tangible) corner-loci and principal (tangible) corner internal kernels of \(\langle F \rangle\). \(\square\)

Thus, supertropical varieties correspond to principal \(1^\nu\)-sets and kernels, while tropical varieties correspond to regular principal \(1^\nu\)-sets and kernels.

In order to bypass the ambiguity between kernels and \(1^\nu\)-sets encountered in Example 6.2.1(i), we introduce one particular sub-semifield \(\Lambda\) of \(F(\Lambda)\) of considerable interest.

8.1. The kernel \(\langle F \rangle\).

Assume that \(F := (F, T, \nu, G)\) is an archimedean \(\nu\)-bipotent \(\nu\)-semifield \(\Lambda\).

Remark 8.1.1. \(\langle \alpha \rangle = \langle \beta \rangle\) for any \(\alpha, \beta \neq 1\) in \(F\), in view of Proposition 8.1.5.

Definition 8.1.2. \(\langle F \rangle\) denotes the kernel given in Remark 8.1.1.

The kernel \(\langle F \rangle\) is preserved under any homomorphism \(\phi\) for which \(\phi(F) \neq 1\). In this subsection we show that \(\langle F \rangle\) retains all the information in \(F(\Lambda)\) needed for the important family of principal \(1^\nu\)-sets and provides a 1:1 correspondence between kernels and \(1^\nu\)-sets.

The kernel \(\langle F \rangle\) is much more sophisticated than what one might think at first blush, because of the convexity condition. Namely, any function lying between two constants is in the kernel. For example, \(3 \wedge |\lambda|\) is constant except in the interval between \(\frac{1}{3}\) and 3, where it descends to 1 and then increases back to 3.

We begin by introducing the motivating example for this section.

Example 8.1.3. The \(1^\nu\)-set corresponding to the principal kernel \(\langle \lambda \rangle\) of \(F(\Lambda)\) is the set of \(a \equiv_\nu 1\). For any \(\alpha \neq 1\) in \(F\), we also have the principal kernel \(\langle \lambda \rangle \cap \langle \alpha \rangle = \langle |\lambda| \wedge |\alpha| \rangle = \langle \frac{|\lambda| |\alpha|}{|\lambda| + |\alpha|} \rangle\). As \(\lambda \not\in \langle \lambda \rangle \cap \langle \alpha \rangle\), we conclude that \(\langle \lambda \rangle \supset \langle \lambda \rangle \cap \langle \alpha \rangle\).

But \(\langle |\lambda| \wedge |\alpha| \rangle \rangle = \langle \lambda \rangle \rangle 1\) if \(a \equiv_\nu 1\). It follows at once that \(1_{\text{loc}}(\langle |\lambda| \wedge |\alpha| \rangle) = 1_{\text{loc}}(\lambda)\).

This ambiguity can be bypassed by intersecting all kernels with \(\langle F \rangle\).
Proposition 8.1.4. If $K$ is a $K$-kernel of $F(\Lambda)$, then $K' := K \cap \langle F \rangle$ is a $K$-kernel of $\langle F \rangle$ satisfying $1_{\text{loc}}(K) = 1_{\text{loc}}(K')$.

Proof. $K = \text{Kern}(1_{\text{loc}}(K))$ since $K$ is a $K$-kernel. Fix $\alpha \in F \setminus \{1\}$. By Proposition 5.2.7, $\langle f \rangle \cap \{1\} = \{1\}$ for all $f \in K$ and for any $f \in K$, $f(\alpha) \cong 1$ for some $\alpha \in F(\Lambda)$ if and only if $f(\alpha) \cong 1$ so $1_{\text{loc}}(K') = 1_{\text{loc}}(K)$. Thus $K' = K \cap \langle F \rangle = \text{Kern}(1_{\text{loc}}(K')) \cap \langle F \rangle = \text{Kern}_{\langle F \rangle}(1_{\text{loc}}(K'))$, and so $K'$ is a $K$-kernel of $\langle F \rangle$. \hfill \Box

We want $K'$ to be unique with this property, but for this we need to assume that $F$ is complete and $\nu$-archimedean, cf. Proposition 8.4.2 below.

Thus, we can pass down to $\langle F \rangle$, leading us to the next definition.

Definition 8.1.5. Define the equivalence relation

\begin{equation}
\begin{aligned}
\phi(f) &\sim_{\langle F \rangle} \phi(f') \iff \langle f \rangle \cap \langle F \rangle = \langle f' \rangle \cap \langle F \rangle.
\end{aligned}
\end{equation}

as kernels of $F(\Lambda)$. The equivalence classes are $\phi(f) = \{f' : f'$ is a generator of $\langle f \rangle \cap \langle F \rangle\}$.

Proposition 8.1.6. For any kernel $K$ of $F(\Lambda)$, if $K \cap \langle F \rangle = \{1\}$, then $K = \{1\}$.

Proof. For any $\alpha \in F(\Lambda)$, $|f(\alpha)| \cap |\alpha| \cong 1$ if and only if $|f(\alpha)| \cap |\alpha| \cong 1$. \hfill \Box

Lemma 8.1.7. $\mathcal{P}(\langle F \rangle) = \{\langle f \rangle \cap \langle F \rangle : \langle f \rangle \in \mathcal{P}(F(\Lambda))\}$.

Proof. $\langle f \rangle \cap \langle F \rangle = \langle |f| \cap |\alpha| \rangle \in \mathcal{P}(\langle F \rangle)$ for any $\langle f \rangle \in \mathcal{P}(F(\Lambda))$. \hfill \Box

Definition 8.1.8. Fixing $\alpha > 1$, define the map $\omega : F(\Lambda)^+ \to \langle F \rangle^+$ by

$$
\omega(f) = |f| \cap |\alpha|.
$$

Lemma 8.1.9. $\omega$ is a lattice homomorphism.

Proof. Indeed, $\omega(|f| + |g|) = (|f| + |g|) \cap |\alpha| = (|f| \cap |\alpha|) + (|g| \cap |\alpha|) = \omega(|f|) + \omega(|g|)$ and $\omega(|f| \cdot |g|) = (|f| \cdot |g|) \cap |\alpha| = (|f| \cap |\alpha|) \cdot (|g| \cap |\alpha|) = \omega(|f|) \cdot \omega(|g|)$. \hfill \Box

$\omega$ induces a lattice map

$$
\Omega : \mathcal{P}(F(\Lambda)) \to \mathcal{P}(F(\Lambda)) \cap \langle F \rangle
$$

such that $\Omega(\langle f \rangle) = \langle \omega(|f|) \rangle = \langle |f| \cap |\alpha| \rangle = \langle f \rangle \cap \langle F \rangle$.

Lemma 8.1.10. (i) $\Omega$ preserves both intersections and products of kernels.

(ii) If $\langle f \rangle$ is a kernel that is bounded from below, then $\langle f \rangle \cap \langle F \rangle = \langle F \rangle$. In fact, every principal kernel whose $1^\nu$-set is the empty set is mapped to $\langle F \rangle$.

(iii) $1_{\text{loc}}(\Omega(\langle f \rangle)) = 1_{\text{loc}}(\langle f \rangle \cap \langle H \rangle) = 1_{\text{loc}}(\langle f \rangle) \cup \emptyset = 1_{\text{loc}}(\langle f \rangle)$.

(iv) Any subkernel $K \subset (\alpha)$ must satisfy $K \cap \langle F \rangle = \{1\}$.

Proof. (i) By Proposition 8.1.4

(ii) By Remark 8.2.11 and Lemma 8.1.7

(iii) $1_{\text{loc}}(\langle \alpha \rangle) = \emptyset$ and $1_{\text{loc}}(\langle f \rangle \cap \langle g \rangle) = 1_{\text{loc}}(\langle f \rangle) \cup 1_{\text{loc}}(\langle g \rangle)$, for any principal kernel $\langle f \rangle$.

(iv) Any $\alpha \neq 1$ generates $\langle F \rangle$. \hfill \Box

We now show that restricting $1_{\text{loc}}$ and $\text{Kern}$ to $\langle F \rangle$ does not affect the $1^\nu$-sets and that each $K$-kernel of $F(\Lambda)$ restricts to a $K$-kernel in $\langle F \rangle$.

In Proposition 5.1.23, we have shown using the substitution homomorphism $\psi$ that any point $a = (\alpha_1, \ldots, \alpha_n) \in F(\Lambda)$ corresponds to the maximal kernel

$$
\left\langle \frac{\lambda_1}{\alpha_1}, \ldots, \frac{\lambda_n}{\alpha_n} \right\rangle = \left\langle \frac{\lambda_1}{\alpha_1}, \ldots, \frac{\lambda_n}{\alpha_n} \right\rangle = \left\langle \frac{\lambda_1}{\alpha_1}, \ldots, \frac{\lambda_n}{\alpha_n} \right\rangle.
$$

For example, consider the homomorphism $\psi : F(\Lambda) \to F$ given by $\lambda_1 \mapsto 1$. By Theorem 5.1.17, the kernel of its restriction $\psi_{\langle F \rangle} : \langle F \rangle \to \psi(\langle F \rangle) = F$ is $\text{Kern}(\psi_{\langle F \rangle}) = \text{Kern}(\psi) \cap \langle F \rangle = \langle \lambda_1 \rangle \cap \langle F \rangle$. Thus, the proposition applies to $\langle F \rangle$ providing the maximal kernel $\langle x \rangle \cap \langle F \rangle$. We will now show that any maximal kernel of $\langle F \rangle$ has that form.
Proposition 8.1.11. If $K$ is a maximal kernel of $\langle F \rangle$, then

$$K = \Omega \left( \left\langle \frac{\lambda_1}{\alpha_1}, \ldots, \frac{\lambda_n}{\alpha_n} \right\rangle \right)$$

for suitable $\alpha_1, \ldots, \alpha_n \in F$.

Proof. Denote $L_a = (|\frac{\lambda_1}{\alpha_1}| + \ldots + |\frac{\lambda_n}{\alpha_n}|) \wedge |\alpha|$ with $\alpha \neq 1$, for $a = (\alpha_1, \ldots, \alpha_n)$. By Remark 8.1.10 we may assume that $1_{\text{loc}}(K) \neq \emptyset$, since the only kernel corresponding to the empty set is $\langle F \rangle$ itself. If $a \in 1_{\text{loc}}(K)$, then as $1_{\text{loc}}(L_a) = \{a\} \subseteq 1_{\text{loc}}(K)$, we have that $\langle L_a \rangle \supseteq K$. Thus, the maximality of $K$ implies that $K = \langle L_a \rangle$. \hfill $\Box$

8.2. Completions of idempotent $\nu$-semifields$^\dagger$.

We recall Definition 2.2.8 of “complete,” viewing a $\nu$-semifield$^\dagger$ as a lattice. In essence it is enough to consider $\mathbb{R}$ from Example 3.2.7.

8.2.1. Kernels of $\nu$-archimedean idempotent $\nu$-semifields$^\dagger$.

Theorem 8.2.1. [29] Theorem 2.3.10 (Hölder)] The following statements are equivalent for an idempotent semifield$^\dagger$ $\mathcal{S}$.

1. $\mathcal{S}$ is simple.
2. $\mathcal{S}$ is totally ordered and archimedean.
3. $\mathcal{S}$ can be embedded into the max-plus algebra of $(\mathbb{R}, +)$.

The supertropical version: Any $\nu$-archimedean supertropical semifield$^\dagger$ $\mathcal{S} = (\mathcal{S}, T, \nu, G)$ can be embedded into $\mathcal{R}$.

Corollary 8.2.2. [29] Any complete divisible totally ordered $\nu$-archimedean supertropical semifield$^\dagger$ is isomorphic to $\mathcal{R}$.

To avoid duplication for kernels having the same $1^\nu$-set, our next step is to work with $\mathcal{R}$ and restrict to sub-kernels of $\langle \mathcal{R} \rangle$ in $\mathcal{R}(\Lambda)$.

Definition 8.2.3. A subset $A$ of the poset $P$ is called co-initial in $P$ if for every $x \in P$ there exists some $f \in A$ such that $f \leq x$. $A$ is co-final in $P$ if for every $x \in P$ there exists some $f \in A$ such that $x \leq f$.

Definition 8.2.4. The subset $A$ of the idempotent semifield$^\dagger$ $\mathcal{S}$ is called left dense in $\mathcal{S}$ if $A^+$ (cf. Definition 8.4.3) is co-initial in $\mathcal{S}^+$, and $A$ is called right dense in $\mathcal{H}$ if $A^+$ is co-final in $\mathcal{S}^+$.

Definition 8.2.5. A completion of the idempotent semifield$^\dagger$ $\mathcal{S}$ is a pair $(H, \theta)$ where $H$ is a complete idempotent $\nu$-semifield$^\dagger$ and $\theta : \mathcal{S} \rightarrow H$ is a monomorphism whose image is dense (left and right) in $H$.

Remark 8.2.6. Each $\nu$-archimedean idempotent $\nu$-semifield$^\dagger$ has a unique completion up to isomorphism. The proof follows the standard lines, with details given in [29] Theorem (2.3.4)].

Remark 8.2.7. When the supertropical semifield$^\dagger$ $F$ is $\nu$-archimedean, any kernel $K$ of $F(\Lambda)$ also has a completion, which we denote as $\hat{K} \subseteq F(\Lambda)$, which by Corollary 8.2.9 is a kernel of $F(\Lambda)$.

Theorem 8.2.8. [29] Theorem 2.3.6] Let $\mathcal{S}$ be an $\nu$-archimedean sub-$\nu$-semifield$^\dagger$ of a complete idempotent $\nu$-semifield$^\dagger$ $\hat{\mathcal{S}}$. Then the following are equivalent:

1. $\hat{\mathcal{S}}$ is the completion of $\mathcal{S}$.
2. $\mathcal{S}$ is left dense in $\hat{\mathcal{S}}$, and if $A$ is an idempotent subsemifield of $\hat{\mathcal{S}}$ that is complete and contains $\mathcal{S}$, then $A = \hat{\mathcal{S}}$.

Corollary 8.2.9. [29] Suppose $\mathcal{S}$ is a left dense $\nu$-archimedean idempotent sub-$\nu$-semifield$^\dagger$ of the complete idempotent $\nu$-semifield$^\dagger$ $\hat{\mathcal{S}}$. Then the kernel of $\hat{\mathcal{S}}$ generated by a kernel $K$ of $\mathcal{S}$, is the completion of $K$ in $\hat{\mathcal{S}}$.

By Remark 8.2.10, $\mathcal{R}(\Lambda)$ has a unique completion to a complete $\nu$-archimedean idempotent semifield$^\dagger$ $\mathcal{R}(\Lambda)$ in $\text{Fun}(\mathcal{R}(\Lambda), \mathcal{R})$. By Theorem 8.2.8 $\mathcal{R}(\Lambda)$ is dense in $\mathcal{R}(\Lambda)$. 

[29] Makespan for specific references.
8.2.2. Bounded rational functions.

$F$ is assumed to be an archimedean supertropical semifield$^1$.

Definition 8.2.10. A rational function $f \in F(\Lambda)$ is said to be bounded from below if there exists some $\alpha \geq \nu$ in $F$ such that $|f| \geq \alpha$. $f$ is said to be bounded from above (or bounded, for short) if there exists some $\alpha \leq \nu$ in $F$ such that $|f| \leq \alpha$.

Remark 8.2.11. [23 Remarks (5.1.3-4), (5.1.10-12)] Let $\langle f \rangle$ be a principal kernel of $F(\Lambda)$. Then

(i) $f$ is bounded from below if and only if $\langle f \rangle \supseteq \langle \alpha \rangle$ for some $\alpha > 1$ in $F$. Moreover, any generator $g \in \langle f \rangle$ is bounded from below.

(ii) $f$ is bounded if and only if $\langle f \rangle \subseteq \langle \alpha \rangle$ for some $\alpha \in F$. Moreover, any generator $g \in \langle f \rangle$ is bounded.

Definition 8.2.12. A principal kernel $\langle f \rangle$ of $F(\Lambda)$ is said to be bounded from below, if it is generated by a rational function bounded from below. $\langle f \rangle$ is said to be bounded if it is generated by a bounded rational function.

Proposition 8.2.13. [23 Corollary (5.1.8)] $1_{\text{loc}}(\langle f \rangle) = \emptyset$ if and only if $f$ is bounded from below.

Lemma 8.2.14. $\langle f \rangle = \{ f \in F(\Lambda) : f$ is bounded $\}$.

Proof. The assertion follows from Remark 8.2.11(ii), since $f \in \langle f \rangle$ if and only if $\langle f \rangle \subseteq \langle FW \rangle$.

Lemma 8.2.15. If $F$ is complete, then $\text{Fun}(X,F)$ is bounded from below, say by $h \in \text{Fun}(X,F)$. Then for any $a \in X$ the set $W := \{ f(a) : f \in X \}$ is bounded from below by $h(a)$ and thus has an infimum $\bigwedge_{f \in W} f(a)$. It is readily seen that the function $g \in \text{Fun}(F^{(n)}, F)$ defined by $g(a) = \bigwedge_{f \in W} f(a)$ is an infimum for $X$, i.e., $g = \bigwedge_{f \in W} f$. Analogously, if $W$ is bounded then $\langle \bigwedge_{f \in W} f(a) \rangle = \bigwedge_{f \in W} f(a)$ is the supremum of $W$.

8.3. The kernel $\langle \mathcal{R} \rangle$ of $\mathcal{R}(\Lambda)$ as bounded functions.

Proposition 8.3.1. [23 Proposition (5.2.1)] For any principal bounded kernel $\langle f \rangle \in \mathcal{P}(\langle \mathcal{R} \rangle)$, there exists an unbounded kernel $\langle f' \rangle \in \mathcal{P}(\mathcal{R}(\Lambda))$, such that

$\langle f \rangle = \langle f' \rangle \cap \langle \mathcal{R} \rangle$.

In particular, $\langle f' \rangle \supset \langle f \rangle$ and $1_{\text{loc}}(f') = 1_{\text{loc}}(f)$.

Proof. Suppose $f \in \langle \mathcal{R} \rangle$ is bounded. Then there exists some $\beta_1 \in \mathcal{R}$ such that $|f(a)| \leq \nu \alpha_1 \in \mathcal{R}$. Similarly for each $2 \leq i \leq n$ there exists some $\beta_i \in \mathcal{R}$ such that $|f(a)| \leq \nu \alpha_i$ whenever $\alpha_i \geq \nu \beta_i$. As $|f|$ is continuous we may assume that $\alpha_i = \alpha$ are all the same. Now define the function

$f' = (|\beta^{-1}_1 | \lambda_1 | + \ldots + |\beta^{-1}_n | \lambda_n | + 1) | f |$.

where $\beta = \sum_{i=1}^n |\beta_i|$. Write $g(\lambda_1, \ldots, \lambda_n) = |\beta^{-1}_1 | \lambda_1 | + \ldots + |\beta^{-1}_n | \lambda_n | + 1$. Let

$S = \{ a = (a_1, \ldots, a_n) \in F^{(n)} : |a_i| > \beta \forall i \}$.

Then for every $a \in S$, $f'(a) = g(a) + |\alpha|$. Moreover, for every $b = (b_1, \ldots, b_n) \notin S$ there exists some $j$ such that $|b_j| \leq \beta$ thus we have that $(\beta^{-1}_j | b_1 | + \ldots + \beta^{-1}_j | b_n |) \leq 1$ and so $g(b) = 1$. By construction $1_{\text{loc}}(f) \subseteq 1_{\text{loc}}(g)$, so $1_{\text{loc}}(f') = 1_{\text{loc}}(|g| + |f|) = 1_{\text{loc}}(g) \cap 1_{\text{loc}}(f) = 1_{\text{loc}}(f)$. Finally $f'$ is not bounded since $|g|$ is not bounded and $|f'| = |g| + |f|$ we have that $|f| \leq |f'|$, so $f \in \langle f' \rangle$. On the other hand, since $f'$ is not bounded, clearly $f' \notin \langle f \rangle$. Finally, $g(\alpha) \geq 1$ for any $a \in S$. Thus, $f'(a) \wedge |\alpha| \equiv_{\nu} (g(a) + |\alpha|) \wedge |\alpha| \equiv_{\nu} |\alpha|$, while for $a \notin S$ $f'(a) \wedge |\alpha| \equiv_{\nu} (g(a) + |f(a)|) \wedge |\alpha| \equiv_{\nu} (1 + |f(a)|) \wedge |\alpha| \equiv_{\nu} |f(a)| \wedge |\alpha| \equiv_{\nu} |f(a)|$, since $|f| \leq |\alpha|$. So we get that $|f'| \wedge |\alpha| \equiv_{\nu} |f|$ which means that $\langle f \rangle = \langle f' \rangle \cap \langle \mathcal{R} \rangle$. (Note that $f' = |f'|$ by definition, since $f' \geq 1$.)

8.4. Principal $1^\flat$-sets and bounded kernels.

It turns out that $\langle \mathcal{R} \rangle$ possesses enough distinct bounded copies of the principal kernels of $\mathcal{R}(\Lambda)$ to represent faithfully the principal $1^\flat$-sets.

Remark 8.4.1. The restriction of the image of the operator $\text{Kern} : 2^{2^{\mathcal{R}(\Lambda)}} \to 2^{2^{\mathcal{R}(\Lambda)}}$ is

(8.2) $\text{Kern}_{\langle \mathcal{R} \rangle}(Z) = \{ f \in \langle \mathcal{R} \rangle : f(a_1, \ldots, a_n) = 1, \forall (a_1, \ldots, a_n) \in Z \} = \text{Kern}(Z) \cap \langle \mathcal{R} \rangle$.

Furthermore, the assertions of this subsection apply to $\text{Kern}_{\langle \mathcal{R} \rangle}$ and the restriction $1_{\text{loc}}|_{\langle \mathcal{R} \rangle} : 2^{\langle \mathcal{R} \rangle} \to 2^{\mathcal{R}(\Lambda)}$ of $1_{\text{loc}}$ to $2^{\langle \mathcal{R} \rangle}$. 
When there is no ambiguity, we denote $Kern_{\langle \mathcal{A} \rangle}$ and $1_{\text{loc}} |_{\mathcal{A}}$ respectively as $Kern$ and $1_{\text{loc}}$.

Applying Lemma 8.1.10 to the semifield $^\dagger \mathcal{A}$, we see that

$$\Omega : \mathcal{P}(\mathcal{A}(\Lambda)) \to \mathcal{P}(\langle \mathcal{A} \rangle)$$

is a lattice homomorphism of the lattice $(\mathcal{P}(\mathcal{A}(\Lambda)) , \cdot \cap)$ onto $(\mathcal{P}(\langle \mathcal{A} \rangle), \cdot \cap)$, such that $1_{\text{loc}}(\langle f \rangle) = 1_{\text{loc}}(\Omega(\langle f \rangle))$.

Let $f \in \mathcal{P}(\langle \mathcal{A} \rangle)$ and let $A = \{ g \in \mathcal{A}(\Lambda) : \Omega(\langle g \rangle) = \langle f \rangle \}$. Define the kernel $K = \langle A \rangle$ of $\mathcal{A}(\Lambda)$. Then by Remark 8.1.10 if $g \in K$ then $1_{\text{loc}}(g) = 1_{\text{loc}}(f)$.

We aim for the $1:1$ correspondence

$$(f) \in \mathcal{P}(\langle \mathcal{A} \rangle) \leftrightarrow 1_{\text{loc}}(f)$$

between the principal $1^\nu$-sets in $\mathcal{A}(\Lambda)^{(n)}$ and the kernels in $\mathcal{P}(\langle \mathcal{A} \rangle)$. If $1_{\text{loc}}(g) = 1_{\text{loc}}(f)$, then $1_{\text{loc}}(\Omega(g)) = 1_{\text{loc}}(f)$ since $1_{\text{loc}}(g) = 1_{\text{loc}}(\Omega(g))$, and $\Omega(g), f \in \mathcal{P}(\langle \mathcal{A} \rangle)$. Thus in view of the above $\Omega(\langle g \rangle) = \langle f \rangle$. Consequently $1_{\text{loc}}(K) = 1_{\text{loc}}(f)$, and $K$ is the maximal kernel of $\mathcal{A}(\Lambda)$ having this property. Our next result justifies the use of $\Omega$.

**Proposition 8.4.2.** If $\langle f \rangle \subseteq \langle \mathcal{A} \rangle$ and $h \in \langle f \rangle$ is such that $1_{\text{loc}}(h) = 1_{\text{loc}}(f)$, then $h$ is a generator of $\langle f \rangle$.

**Proof.** The assertion is obvious in the special case for which $\langle f \rangle \cong_{\nu} \{1\}$. So, as $1_{\text{loc}}(h) = 1_{\text{loc}}(f)$ we may assume that $f$ and $h$ are not $\nu$-equivalent to $1$. If $\langle f \rangle = \langle \alpha \rangle$, then $1_{\text{loc}}(h) = 1_{\text{loc}}(f) = 0$ implies by Lemma 8.1.10 (3) that $\langle h \rangle = \langle \mathcal{A} \rangle = \langle f \rangle$.

Suppose there is a rational function $h \in \langle f \rangle$ which is not a generator of $\langle f \rangle$ but satisfying $1_{\text{loc}}(h) = 1_{\text{loc}}(f)$. By Corollary 8.1.13 for each $k \in \mathbb{N}$ there exists some $a \in \mathcal{A}^{(n)}$ for which $|f(a)| >_{\nu} |h(a)|^k$. For any $k \in \mathbb{N}$, define the set $U_k = \{ a : |f(a)| >_{\nu} |h(a)|^k \}$. As $\mathcal{A}$ is dense, for any $a \in U_k$ there exists a neighborhood $B_a \subset U_k$ containing $a$ such that for all $a' \in B_a$, $|f(a')| > |h(a')|^k$. Now, since $h$ and $f$ are bounded, the $U_k$ are bounded regions inside $\mathcal{A}^{(n)}$. Taking the closures, we may assume that the $U_k$ are closed (nonempty).

Since $1_{\text{loc}}(h) = 1_{\text{loc}}(f)$, $|f(a)| >_{\nu} |h(a)|^k$ implies that $|h(a)|, |f(a)| >_{\nu} 1$, so, by the definition of $U_k$ we get the sequence of strict inclusions $U_1 \supset U_2 \supset \cdots \supset U_k \supset \cdots$. Thus, since $\mathcal{A}$ is complete, there exists an element $b \in B = \bigcap_k B_k$. Now, for $a \notin 1_{\text{loc}}(h)$, $|h(a)| > 1$, and thus there exists some $r = r(a) \in \mathbb{N}$ such that $|h(a)|^r >_{\nu} |f(a)|$ and thus $a \notin B$ and $b \notin \mathcal{A}^{(n)} \setminus 1_{\text{loc}}(h)$. On the other hand, if $b \in 1_{\text{loc}}(h)$ then $b \in 1_{\text{loc}}(f)$ so $1 \cong_{\nu} |f(b)| \leq_{\nu} |h(b)| \equiv_{\nu} 1$. Thus $\bigcap_k B_k = \emptyset$, contradiction.

Using Theorem 8.3.1 and Proposition 8.1.3 we conclude:

**Theorem 8.4.3.** There is a $1:1$, order-reversing correspondence

$$\{\text{principal } 1^\nu\text{-sets of } \mathcal{A}^{(n)}\} \leftrightarrow \{K - \text{principal kernels of } \langle \mathcal{A} \rangle\},$$

given by $Z \mapsto Kern_{\langle \mathcal{A} \rangle}(Z)$; the reverse map is given by $K \mapsto 1_{\text{loc}}(K)$.

**Proof.** Every principal kernel gives rise to a principal $1^\nu$-set by the definition of $1_{\text{loc}}$. The reverse direction follows from Proposition 8.4.2 as every principal kernel which produces a principal $1^\nu$-set via $1_{\text{loc}}$ is in fact a $K$-kernel.

**Proposition 8.4.4.** Let $\langle f \rangle$ be a principal kernel in $\mathcal{P}(\langle \mathcal{A} \rangle)$. Then $\langle f \rangle$ is a $K$-kernel.

**Proof.** We need to show that $Kern(1_{\text{loc}}(f)) \subseteq \langle f \rangle$.

Let $h \in \langle \mathcal{A} \rangle$ such that $h \in Kern(1_{\text{loc}}(f))$. Then $h(x) = 1$ for every $x \in 1_{\text{loc}}(f)$ and so $1_{\text{loc}}(f) \subseteq 1_{\text{loc}}(h)$. If $|h| \leq |f|^k$ for some $k \in \mathbb{N}$ then $h \in \langle f \rangle$. Thus in particular we may assume that $h \neq 1$. Now, by Corollary 8.2.12 $1_{\text{loc}}(\langle f \rangle \cap \langle h \rangle) = 1_{\text{loc}}(f) \cup 1_{\text{loc}}(h) = 1_{\text{loc}}(h)$. Since $h \neq 1$, $1_{\text{loc}}(h) \notin \mathcal{A}(\Lambda)$ and thus $\langle f \rangle \cap \langle h \rangle \neq \{1\}$. Again, $1_{\text{loc}}(\langle f \rangle \cap \langle h \rangle) = 1_{\text{loc}}(f) \cup 1_{\text{loc}}(h) = 1_{\text{loc}}(f)$ by Corollary 8.2.12.

Thus $\langle f, h \rangle = (\langle f \rangle \cap \langle h \rangle) \neq \langle \mathcal{A} \rangle$ for otherwise $1_{\text{loc}}(f) = \emptyset$. Consequently for $g = |f| \wedge |h|$, the kernel $K = \langle g \rangle = \langle f \rangle \cap \langle h \rangle$ satisfies $\{1\} \neq K \subseteq \langle f \rangle$, implying $g \in \langle f \rangle$ and $1_{\text{loc}}(g) = 1_{\text{loc}}(h)$. Thus by Proposition 8.4.2 $g$ is a generator of $\langle h \rangle$, so $\langle h \rangle = K \subseteq \langle f \rangle$ as desired.

**8.5. The wedge decomposition in $\langle f \rangle$.**

**Lemma 8.5.1.** Suppose $f = \sum_{i=1}^{k} f_i \in F[x_1, ..., x_n]$ is a supertropical polynomial written as the sum of its monomials, then $\check{f} := \bigwedge_{i=1}^{k} \check{f}_i$ is corner internal.
Proof. \( \tilde{f} \geq 1 \). For any given \( i \in \{1, \ldots, k\} \), then Remark \[ 7.3.10 \] yields \( 1_{\text{loc}}(\Phi_{CI}(\tilde{f}_i)) = 1_{\text{loc}}(\Phi_{CI}(\check{f}_i)) = 1_{\text{loc}}(\check{f}_i) \). Then

\[
1_{\text{loc}}(\Phi_{CI}(\tilde{f})) = 1_{\text{loc}}(\bigwedge_{i=1}^{k} \Phi_{CI}(\tilde{f}_i)) = 1_{\text{loc}}(\bigwedge_{i=1}^{k} \check{f}_i)
\]

so \( \Phi_{CI}(\tilde{f}) = \check{f} = \bigwedge_{i=1}^{k} \check{f}_i \), which by Theorem \[ 7.3.9 \] is corner internal. \( \square \)

Definition 8.5.2. A wedge decomposition of a rational function \( f \in F(x_1, \ldots, x_n) \) is an expression

\[
|f| = \bigwedge_{i} |u_i|
\]

for \( u_i \in F(x_1, \ldots, x_n) \). This wedge decomposition is associated to a kernel intersection \( \langle f \rangle = \bigcap K_i \) if each \( K_i = \langle u_i \rangle \).

Note by Lemmas \[ 5.2.2 \) ii) and \[ 5.2.3 \) that \( 1_{\text{loc}}(f) = \bigcup_{i} 1_{\text{loc}}(u_i) \).

Theorem 8.5.3. If \( \langle f \rangle = \bigcap K_i \) is an intersection of (principal) subkernels of \( \langle \mathcal{A} \rangle \), this then kernel intersection has an associated wedge decomposition.

Proof. It is enough to prove the assertion when \( \langle f \rangle = \langle u \rangle \cap \langle v \rangle \) for \( u, v \in \langle \mathcal{A} \rangle \), and then to conclude by induction. Let \( \tilde{f} = |u| \cap |v| \), \( \langle f \rangle = \langle f \rangle \), so there exist some \( q_1, \ldots, q_k \in \mathcal{A}(\Lambda) \) such that \( \sum_{i=1}^{k} q_i = 1 \) and \( |f| = \sum_{i=1}^{k} q_i \tilde{f}^{d(i)} \) with \( d(i) \in \mathbb{Z}_{\geq 0} \) \( d(i) \geq 0 \) since \( |f| \geq 1 \). Thus

\[
f = \sum_{i=1}^{k} q_i (|u| \cap |v|)^{d(i)} = \left( \sum_{i=1}^{k} q_i |u|^{d(i)} \right) \wedge \left( \sum_{i=1}^{k} q_i |v|^{d(i)} \right) = |g| \wedge |h|
\]

where \( g = |g| = \sum_{i=1}^{k} q_i |u|^{d(i)} \) and \( h = |h| = \sum_{i=1}^{k} q_i |v|^{d(i)} \). Thus

\[ 1_{\text{loc}}(f) \geq 1_{\text{loc}}(g) \geq 1_{\text{loc}}(u); \quad 1_{\text{loc}}(f) \geq 1_{\text{loc}}(h) \geq 1_{\text{loc}}(v). \]

We claim that \( |g| \) and \( |h| \) generate \( \langle |u| \rangle \) and \( \langle |v| \rangle \), respectively. Since \( \langle f \rangle = \langle f \rangle \), we see that \( \tilde{f}(a) = 1 \Leftrightarrow |f|(a) = 1 \) for any \( a \in \mathcal{A}^{(n)} \). Let \( q_j \tilde{f}^{d(j)} \) be a dominant term of \( |f| \) at \( a \), i.e.,

\[ |f|(a) \geq_{\nu} \sum_{j}^{k} q_j(a)(\tilde{f}(a))^{d(j)} \geq_{\nu} q_j(a)(\tilde{f}(a))^{d(j)}. \]

Then \( \tilde{f}(a) \geq_{\nu} 1 \Leftrightarrow q_j(a)(\tilde{f}(a))^{d(j)} \geq_{\nu} 1 \). Hence, \( \tilde{f}(a) \geq_{\nu} 1 \Leftrightarrow q_j(a) = 1 \).

Now, consider \( a \in 1_{\text{loc}}(g) \). Then \( g(a) = \sum_{i=1}^{k} q_i |u|^{d(i)} \geq_{\nu} 1 \). Let \( q_i |u|^{d(i)} \) be a dominant term of \( g \) at \( a \). If \( q_i(a) \geq_{\nu} 1 \) then \( |u|^{d(i)}(a) \geq_{\nu} 1 \) and thus \( a \in 1_{\text{loc}}(u) \).

So we may assume that \( q_i(a) <_{\nu} 1 \) (since \( \sum_{i=1}^{k} q_i \geq_{\nu} 1 \)) and so, as above, \( q_i \tilde{f}^{d(i)} \) is not a dominant term of \( |f| \) at \( a \). Hence,

\[ |u(a)|^{d(i)} = q_j(a)|u(a)|^{d(i)} <_{\nu} q_j(a)|u(a)|^{d(i)} \geq_{\nu} g(a) \geq_{\nu} 1, \]

for any dominant term of \( |f| \) at \( a \). Thus

\[ q_j(a)(\tilde{f}(a))^{d(j)} = q_j(a)(|u(a)| \wedge |v(a)|)^{d(j)} \leq q_j(a)|u(a)|^{d(j)} <_{\nu} 1. \]

On the other hand, \( \tilde{f}(a) \geq_{\nu} 1 \) since \( 1_{\text{loc}}(f) \geq 1_{\text{loc}}(g) \), and thus \( q_j(a)(\tilde{f}(a))^{d(j)} \geq_{\nu} 1 \), contradicting \( 1_{\text{loc}}(g) \leq 1_{\text{loc}}(u) \), so, by the above, \( 1_{\text{loc}}(g) = 1_{\text{loc}}(u) \), which in turn yields that \( g \) is a generator of \( \langle |u| \rangle = \langle u \rangle \), by Proposition \[ 5.4.2 \). The proof for \( h \) and \( |v| \) is analogous. Consequently, we have that \( 1_{\text{loc}}(g) = 1_{\text{loc}}(|g|) = 1_{\text{loc}}(|u|) \) and \( 1_{\text{loc}}(h) = 1_{\text{loc}}(|v|) \), as desired. \( \square \)

8.6. Example: The tropical line revisited.

For ease of notation, we write \( x \) for \( \lambda_1 \) and \( y \) for \( \lambda_2 \).

Note 8.6.1. In the following example we consider the rational function

\[ \hat{f} = \left| \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{x+y} \right| \wedge |a| \in \langle \mathcal{A} \rangle \]

for any \( a \in \mathcal{A} \setminus \{1\} \).
Example 8.6.2. Let \( f = x+y+1 \) be the tropical line, considered already in Example 6.2.1(i). Its corresponding \( 1^- \)-set is defined by the rational function \( \hat{f} = \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{x+y} \), and so, its corresponding kernel in \( \mathcal{R}(x, y) \) is \( \langle \hat{f} \rangle \) and the bounded copy in \( \langle \mathcal{R} \rangle \) is \( \langle \hat{f} \rangle \cap \langle \mathcal{R} \rangle \). As mentioned above \( \hat{f} \sim \langle \mathcal{R} \rangle \) \( \left( \frac{y}{y+1} \right) \cap \left( \frac{x}{x+1} \right) \cap \langle x+y \rangle \),
and any of the three above terms can be omitted. Thus we have that

\[
\langle \hat{f} \rangle \cap \langle \mathcal{R} \rangle = \left( \frac{x}{y+1} \right) \cap \left( \frac{y}{x+1} \right) \cap \langle x+y \rangle \cap \langle \mathcal{R} \rangle,
\]
where each of the kernels comprising the intersection (excluding \( \langle \mathcal{R} \rangle \)) is contained in each of the other kernels. (In the third kernel from the left we choose to take \( x+y \) as a generator instead of its inverse.) Now, taking logarithms, it can be seen that \( 1_{\text{loc}}((x+y)) \) is precisely the union of the bounding rays of the third quadrant.

Furthermore,

\[
\langle x+y \rangle = (|x+y| + |x|) \cap (|x+y| + |y|),
\]
since

\[
x+y \sim \langle \mathcal{R} \rangle |x+y| = |x+y| + (|x| \wedge |y|) = (|x+y| + |x|) \wedge (|x+y| + |y|).
\]

This wedge decomposition of \( x+y \) is quite natural. The geometric locus of the equation \( |x| \wedge |y| = 1 \) in logarithmic scaling is precisely the union of the \( x \)-axis corresponding to \( |x| \cong \nu 1 \) and the \( y \)-axis corresponding to \( |y| \cong \nu 1 \).

Using the wedge decomposition, we can define any segment and ray in \( \mathcal{R}^2 \) by means of principal kernels, so the only irreducible \( 1^- \)-sets turn out to be the points in the plane. This does not hamper us in developing geometry, since we may also restrict our attention to sublattices of the lattice of principal kernels.

9. Hyperspace-kernels and region kernels

In order to prove Theorem 8.0.13 we separate principal kernels into two classes of kernels, HO-kernels and bounded from below kernels. The latter kernels are eliminated by passing to the kernel \( \langle F \rangle \). Using these classes of kernels we introduce the hyperspace-region decomposition given in Theorem 9.4.3.

9.1. Hyperspace-kernels.

Remark 9.1.1. Though we consider the semifield* of rational functions \( F(\lambda_1, \ldots, \lambda_n) \), most of the results introduced in this section are applicable to any finitely generated sub-semifield† of \( F(\Lambda) \) over \( F \). In particular, \( \langle F \rangle \subset F(\lambda_1, \ldots, \lambda_n) \) is just another case of a finitely generated semifield† over \( F \), taking the generators \( \lambda_1 \wedge |\alpha| \) for \( 1 \leq i \leq n \), and \( \alpha \in F \setminus \{1\} \). In this case, the generators are bounded, and we specifically designate the results that are true only for unbounded generators.

Definition 9.1.2. An \( \mathcal{L} \)-monomial is a non-constant Laurent monomial \( f \in F(\lambda_1, \ldots, \lambda_n) \); i.e., \( f = \frac{h}{g} \) with \( h, g \in F[\lambda_1, \ldots, \lambda_n] \) non-proportional monomials.
Remark 9.1.3. Whenever $F$ is $\nu$-archimedean, $\mathcal{L}$-monomials in $F(\lambda_1, \ldots, \lambda_n)$ are not bounded; i.e., for any $\mathcal{L}$-monomial $f$ there does not exist $\alpha \in F$ for which $|f| \leq |\alpha|$.

Definition 9.1.4. A rational function $f \in F(\lambda_1, \ldots, \lambda_n)$ is a hyperspace-fraction, or HS-fraction, if $f \sim (F)$ $\sum_{i=1}^{t} |f_i|$ where the $f_i$ are non-proportional $\mathcal{L}$-monomials.

Remark 9.1.5. HS-fractions in $F(\lambda_1, \ldots, \lambda_n)$ are not bounded, since $|f_j| \leq \sum_{i=1}^{t} |f_i| = f$.

Definition 9.1.6. A hyperplane kernel, or HP-kernel, for short, is is a principal kernel of $F(\lambda_1, \ldots, \lambda_n)$ generated by an $\mathcal{L}$-monomial.

A hyperspace-fraction kernel, or HS-kernel, for short, is is a principal kernel of $F(\lambda_1, \ldots, \lambda_n)$ generated by a hyperspace fraction.

By definition, any HP-kernel is regular. Also, a fortiori, every HP-kernel is an HS-kernel.

Definition 9.1.7. $\Omega(F(\Lambda))$ is the lattice of kernels finitely generated by HP-kernels of $F(\Lambda)$, i.e., every element $(f) \in \Omega(F(\Lambda))$ is obtained via finite intersections and products of HS-kernels.

Proposition 9.1.8. Any principal HS-kernel is a product of distinct HP-kernels, and thus is in $\Omega(F(\Lambda))$.

Proof. If $(f)$ is an HS-kernel, then $(f) = \langle \sum_{i=1}^{t} f_i \rangle = \prod_{i=1}^{t} (f_i)$ with $f_i \in F[\lambda_1, \ldots, \lambda_n]$ non-proportional $\mathcal{L}$-monomials.

Corollary 9.1.9. Any HS-Kernel is regular.

Proof. Apply Proposition [8.0.11]

Definition 9.1.10. A $1^{\nu}$-set in $F^n$ is a hyperplane $1^{\nu}$-set (HP-1$^{\nu}$-set for short) if it is defined by an $\mathcal{L}$-monomial. A $1^{\nu}$-set in $F^n$ is a hyperspace-fraction $1^{\nu}$-set (HS-1$^{\nu}$-set for short) if it is defined by an HS-fraction.

Corollary 9.1.11. A $1^{\nu}$-set is an HS-1$^{\nu}$-set if and only if it is an intersection of HP-1$^{\nu}$-sets.

Proof. As $1_{\text{loc}}((f)\langle g\rangle) = 1_{\text{loc}}((f)) \cap 1_{\text{loc}}((g))$, the assertion follows directly from Proposition [9.1.8]

9.2. Region kernels.

Our last kind of kernel marks out a tropical region, and thus is called a “region kernel.”

Lemma 9.2.1. Let $(f)$ be an HP-kernel, with $F$ divisible. If $w \in (f)$ is an $\mathcal{L}$-monomial, then $w^s = f^k$ for some $s, k \in \mathbb{Z} \setminus \{0\}$.

Proof. By assumption $\langle w \rangle \subseteq (f)$, implying $1_{\text{loc}}(w) \supseteq 1_{\text{loc}}(f)$. Assume that $w^s \neq f^k$ for any $s, k \in \mathbb{Z} \setminus \{0\}$. We will show that there exists some $b \in F^n$ such that $b \in 1_{\text{loc}}(f) \setminus 1_{\text{loc}}(g)$ for some $g \in (f)$, which clearly is impossible.

Let $(p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \mathbb{Z}^n$ be the vectors of degrees of $\lambda_1, \ldots, \lambda_n$ in the Laurent monomials $f$ and $w$. Since $w$ and $f$ are nonconstant, $(p_1, \ldots, p_n), (q_1, \ldots, q_n) \neq (0)$. Since $F$ is divisible, we may take appropriate roots and assume that $\gcd(p_1, \ldots, p_n) = \gcd(q_1, \ldots, q_n) = 1$. Since $w \in (f)$ we have $|w| \leq |f|^m$ for some $m \in \mathbb{N}$, so if $\lambda_i$ occurs in $w$ it must also occur in $f$. Finally, since $w^s \neq f^k$ for any $k \in \mathbb{Z} \setminus \{0\}$ we may also assume that $(p_1, \ldots, p_n) \neq (q_1, \ldots, q_n)$, for otherwise $w = \alpha f$ for some $\alpha \neq 1$ and thus $1_{\text{loc}}(f) \subseteq 1_{\text{loc}}(f) \cap 1_{\text{loc}}(w) = 1_{\text{loc}}(f) \cap 1_{\text{loc}}(\alpha f) = \emptyset$.

a contradiction. Suppose that $\lambda_i^j$ occurs in $w$ for some $l \in \mathbb{Z} \setminus \{0\}$ such that $\lambda_i$ is not identically 1 on $1_{\text{loc}}(w)$ (and thus also on $1_{\text{loc}}(f)$). Thus there exists some $k \in \mathbb{Z} \setminus \{0\}$ such that $\lambda_i^k$ occurs in $f$. Then $\lambda_i$ does not occur in the Laurent monomial $g := w^{-k} f^j \in (f)$. Without loss of generality, assume that $j = 1$. If $a = (a_1, \ldots, a_n) \in 1_{\text{loc}}(f)$, then $g(a) = w(a)^{-1} f(a) = 1$. By assumption that $w^s \neq f^k$, there exists $\lambda_i$ occurring in $f$ and not in $w$. Take $b = (1, \alpha_2, \ldots, \beta_i, \ldots, \alpha_n)$ with $b \in F$ occurring in the $t$-th component, such that $\beta_i \alpha_i = \alpha_i^s \alpha_i^t$. Then as $\lambda_i$ is not identically 1 over $1_{\text{loc}}(f)$, we can choose $b \in 1_{\text{loc}}(w)$ such that $f(b) = 1$ but $g(b) \neq 1$.

Proposition 9.2.2. Let $f$ be an $\mathcal{L}$-monomial. Then $w \in (f)$ is an $\mathcal{L}$-monomial if and only if $w$ is a generator of $(f)$.

Proof. The claim follows from Lemma [9.2.1] and the property that $(g^k) = (g)$.
Lemma 9.2.8. The \( \mathcal{L} \)-binomial \( o \) defined by an \( \mathcal{L} \)-monomial \( f \) is the rational function \( 1 + f \).

The complementary \( \mathcal{L} \)-binomial \( o^c \) of \( o \) is \( 1 + f^{-1} \). By definition \( (O^c)^c = O \).

The order kernel of the semifield\(^1 \) \( F(\lambda_1, ..., \lambda_n) \) defined by \( f \) is the principal kernel \( O = \langle o \rangle \) for the \( \mathcal{L} \)-binomial \( o = 1 + f \).

The complementary order kernel \( O^c \) of \( O \) is \( \langle o^c \rangle \).

Since \( f \) is an \( \mathcal{L} \)-monomial, so is \( f^{-1} \) and thus \( O^c \) is an order kernel.

Lemma 9.2.4. Let \( O = (1 + f) \) be an order kernel of \( F(\lambda_1, ..., \lambda_n) \). Then
\[
O \cap O^c = (1) \quad \text{and} \quad OO^c = (f).
\]

Proof. \( O \cap O^c = \langle |1 + f| \rangle \) and \( OO^c = \langle |1 + f| + |1 + f^{-1}| \rangle \) is \( (1 + f + f^{-1}) = (1 + |f|) = \langle |f| \rangle = \langle f \rangle \)
(notice that \( (1 + f), (1 + f^{-1}) \geq_\nu 1 \) implies \(|1 + f| \cong_\nu 1 + f \) and \(|1 + f^{-1}| \cong_\nu 1 + f^{-1} \)). \( \square \)

Definition 9.2.5. A rational function \( f \in F(\lambda_1, ..., \lambda_n) \) is said to be a region fraction if \( \text{1}_{\text{loc}}(f) \) contains some nonempty open interval.

Lemma 9.2.6. \( f \sim \langle F \rangle \sum_{i=1}^t |a_i| \) is a region fraction iff, writing \( a_i = 1 + f_i \) for \( \mathcal{L} \)-monomials \( f_i \), we have \( \sum_{i=1}^t f_i \geq_\nu \sum_{i=1}^t f_i^{-1} \) for every \( i \neq j \).

Proof. Since \( 1 + f_i \geq_\nu 1 \) for every \( i \), we have that
\[
\sum_{i=1}^t |a_i| = \sum_{i=1}^t |1 + f_i| = \sum_{i=1}^t (1 + f_i) = 1 + \sum_{i=1}^t f_i.
\]
Thus a region fraction \( r \) can be defined as \( r \sim (F) 1 + \sum_{i=1}^t f_i \), so the last condition of the definition can be stated as \( f_j \neq f_i^{-1} \) for any \( 1 \leq i, j \leq t \). If there exist \( k \) and \( \ell \) for which \( f_{\ell} \cong_\nu f_k^{-1} \), then
\[
\sum_{i=1}^t |a_i| \cong_\nu |f_k| + (1 + \sum_{i \neq k, \ell} f_i) \cong_\nu |f_k| + \left| \sum_{i \neq k, \ell} f_i \right|,
\]
and thus \( \text{1}_{\text{loc}}(r) = \text{1}_{\text{loc}}(\sum_{i=1}^t |a_i|) = \text{1}_{\text{loc}}(f_k) \cap \text{1}_{\text{loc}}(1 + \sum_{i \neq k, \ell} f_i) \subseteq \text{1}_{\text{loc}}(f_k) \). \( \square \)

Definition 9.2.7. A region kernel is a principal kernel generated by a region fraction.

Lemma 9.2.8. A principal kernel \( K \) is a region kernel if and only if it has the form
\[
K = \prod_{i=1}^v O_i
\]
for order kernels \( O_1, ..., O_v \in \mathcal{P}(F(\lambda_1, ..., \lambda_n)) \).

Proof. If \( f = \sum_{i=1}^t |a_i| \) is a generating region fraction of \( K \), then
\[
K = \langle F \rangle = \left( \sum_{i=1}^t |a_i| \right) = \prod_{j=1}^v \langle a_i \rangle.
\]

Conversely, if \( K = \prod_{i=1}^v O_i \) then writing \( O_i = \langle a_i \rangle \), we see that \( \sum_{i=1}^t |a_i| \) is a region fraction generating \( K \), since \( \left( \sum_{i=1}^t |a_i| \right) = \prod_{i=1}^v O_i = K \). \( \square \)

Lemma 9.2.9. Any order kernel \( f = 1 + \frac{h}{g} \) (for monomials \( g \) and \( h \)) is corner-internal.

Proof. \( f = 2 + \frac{h}{g} \). Any ghost root \( a \) of \( f = g^\nu + h \) is either a ghost root of \( g \) dominating \( h \), or a ghost root of \( g + h \); in either case it is a kernel root of \( f \). \( \square \)

Lemma 9.2.10. If \( \langle f \rangle \neq (1) \) is a regular kernel and \( \langle o \rangle \) is an order kernel, then \( \langle f \rangle \langle o \rangle \) is regular.

Proof. Write \( f = \frac{h}{g} \) and \( o = 1 + \frac{h'}{g'} \) with \( h' \) and \( g' \) monomials in \( F[\lambda_1, ..., \lambda_n] \). Since regularity does not depend on the choice of generator of the kernel and since
\[
\langle f \rangle \langle o \rangle = \langle |f| + |o| \rangle = \langle |f| + o \rangle,
\]
we check the condition on \( |f| + o \). Then \( |f| + o = \left( \frac{h}{g} \right) + 1 + \frac{h'}{g'} \). Since \( \left( \frac{h}{g} \right) \geq_\nu 1 \) we have
\[
|f| + o = \left( \frac{h}{g} \right) + 1 + \frac{h'}{g'} \cong_\nu \frac{(h^2 + g^2)g' + ghh'}{ghg'}.
\]
Since \( g \neq h \), we are done unless \( g' >_\nu h' \), and Frobenius enables us to reduce to \( \frac{k^2 + g^2}{g^n} = |f| \). But then we are done since \(|f|\) is regular, by Lemma 7.4.3.

**Proposition 9.2.11.** \( \langle f \rangle \langle \alpha_1 \rangle \cdots \langle \alpha_k \rangle \) is regular, for any HP-kernel \( \langle f \rangle \neq 1 \) and order kernels \( \langle \alpha_1 \rangle, \ldots, \langle \alpha_k \rangle \).

**Proof.** Iterate Lemma 9.2.10 noting that every HP-kernel is regular.

9.3. Geometric interpretation of HS-kernels and region kernels.

**Remark 9.3.1.** In view of Theorem 8.2.1, for the case that \( F \) is a divisible, \( \nu \)-archimedean supertropical semifield\(^{\dagger} \), we may take \( F = \mathcal{R} \).

By Remark 8.3.4, any \( L \)-monomial \( f \) may be considered as a linear functional over \( \mathbb{Q} \) and thus the HP-kernel given by the equation \( f = 1 \) over \( \mathcal{R} \) translates to \( f = 0 \) over \( (\mathcal{R}, +)^{(n)} \) (in logarithmic notation), where \( f \) is the linear functional obtained from \( f \) by applying (8.3).

**Lemma 9.3.2.** If \( f \) is an \( L \)-monomial in \( F(\lambda_1, \ldots, \lambda_n) \), then \( f \) is completely determined by the set of vectors \( \{w_0, \ldots, w_n\} \) for any \( w_i = (\alpha_{i,1}, \ldots, \alpha_{i,n}, f(\alpha_{i,1}, \ldots, \alpha_{i,n})) \in F^{(n+1)} \) where \( \alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,n}) \subset F^{(n)} \) such that \( w_0, \ldots, w_n \) are in general position (are not contained in an \( (n-1) \)-dimensional affine subspace of \( F^{(n+1)} \)).

**Proof.** Writing \( f(\lambda_1, \ldots, \lambda_n) = \alpha \Pi_{i=1}^n \lambda_i^{k_i} \) with \( k_i \in \mathbb{Z} \), we have \( \alpha = f(a_0) \Pi_{i=1}^n a_0^{k_i} \). After \( \alpha \) is determined, since \( w_0, \ldots, w_n \) are in general position the set \( \{a_1, \ldots, a_n, b = (f(a_1), \ldots, f(a_n))\} \subset F^{(n)} \) define a linearly independent set of \( n \) linear equations in the variables \( k_i \), and thus determine them uniquely.

Consider the HS-kernel of \( F(\lambda_1, \ldots, \lambda_n) \) defined by the HS-fraction \( f = \sum_{i=1}^t |f_i| \) where \( f_1, \ldots, f_t \) are \( L \)-monomials. Then \( f(a) \cong \nu \) if and only if \( f_i(a) \cong \nu \) for each \( i = 1, \ldots, t \), which translates to a homogenous system of linear equations (over \( \mathbb{Q} \)) of the form \( f_i = 0 \) where \( f_i \) is the logarithmic form of \( f_i \). This way \( 1_{\text{loc}}(f) \subset F^{(n)} \cong (F^+)^n \) is identified with an affine subspace of \( F^{(n)} \) which is just the intersection of the \( t \) affine hyperplanes defined by \( f_i = 0, 1 \leq i \leq t \). Analogously, the \( L \)-binomial \( o = 1 + g \) has \( \nu \)-value \( 1 \) if and only if \( g \not\cong \nu \), giving rise to the half space of \( \mathbb{R} \) defined by the weak inequality \( g \leq 0 \). Thus, the region kernel defined by \( r = \sum_{i=1}^t |o_i| = 1 + \sum_{i=1}^t g_i \) where \( o_i = 1 + g_i \) are \( L \)-binomials, yields the nondegenerate polyhedron formed as the intersection of the affine halfspaces each defined using the \( g_i \) corresponding to \( o_i \).

9.4. The HO-decomposition.

The following example motivates our subsequent discussion. Any point \( a = (\alpha_1, \ldots, \alpha_n) \) in \( F^{(n)} \) is just \( 1_{\text{loc}}(f_a) \), where

\[
|f_a(\lambda_1, \ldots, \lambda_n)| = \left| \frac{\lambda_1}{\alpha_1} \right| + \cdots + \left| \frac{\lambda_n}{\alpha_n} \right| \in F(\lambda_1, \ldots, \lambda_n).
\]

We would like \( \langle f_a \rangle \) to encode the reduction of dimension from \( F^{(n)} \) to the point \( \{a\} \).

For each \( 1 \leq k \leq n \) define \( f_{k,a} = \left| \frac{\lambda_1}{\alpha_1} \right| + \cdots + \left| \frac{\lambda_k}{\alpha_k} \right| \) and \( f_0 = 1 \), and consider the chain of principal HS-kernels

\[
\langle f_a \rangle = \langle f_{n,a} \rangle \supset \langle f_{n-1,a} \rangle \supset \cdots \supset \langle f_1,a \rangle \supset \{1\}.
\]

The factors \( \langle f_{k,a} \rangle/\langle f_{k-1,a} \rangle \) are the quotient semifields\(^{\dagger} \)

\[
\prod_{j=1}^k \left\langle \frac{\lambda_j}{\alpha_j} \right\rangle / \prod_{j=1}^{k-1} \left\langle \frac{\lambda_j}{\alpha_j} \right\rangle \cong \left\langle \frac{\lambda_k}{\alpha_k} \right\rangle / \left( \prod_{j=1}^{k-1} \left\langle \frac{\lambda_j}{\alpha_j} \right\rangle \right) \cap \left\langle \frac{\lambda_k}{\alpha_k} \right\rangle,
\]
a nontrivial homomorphic image of an HP-kernel of the quotient semifield
\[ F(\lambda_1, \ldots, \lambda_n) \big/ \left( \left( \prod_{j=1}^{k-1} \langle \frac{1}{\lambda_j} \rangle \right) \cap \langle \lambda_k \rangle \right). \]

We claim that this chain of HS-kernels can be refined to a longer descending chain of principal kernels descending from \( \langle f_a \rangle \). Indeed, the kernels \( \langle \lambda_1 + |\lambda_2| \rangle \) and \( \langle \lambda_1 \rangle \) both are semifields, and \( \langle \lambda_1 \rangle \) is a subkernel of \( \langle \lambda_1 + |\lambda_2| \rangle \). Consider the substitution map \( \phi \) sending \( \lambda_1 \) to 1. Then \( \text{Im} (\phi) = \langle 1 + |\lambda_2| \rangle \). The kernel \( \langle |\lambda_2| \rangle \) is not simple as a principal kernel of the semifield \( \langle |\lambda_2| \rangle F(\lambda_2) \), for the chain \( \langle |\lambda_2| \rangle \supset \langle 1 + |\lambda_2| \rangle \supset \langle 1 \rangle \) is the image of the refinement
\[ \langle \lambda_1 + |\lambda_2| \rangle \supset \langle |\lambda_1 + \lambda_2| + |\lambda_1| \rangle \supset \langle |\lambda_1| \rangle \]
(since \( \phi(|\lambda_1 + \lambda_2| + |\lambda_1|) = \phi(|\lambda_1| + |\lambda_2|) = |1 + |\lambda_2| + |1| = |1 + \lambda_2| \).)

On the other hand, \( 1 + \lambda_2 \) is an order kernel which induces the order relation \( \lambda_2 \leq 1 \) on the \( \nu \)-semifield \( \langle |\lambda_2| \rangle \).

In view of these considerations, we would like to exclude order kernels and ask:
- Can \( \mathcal{E} \) be refined to a longer descending chain of HS-kernels descending from \( \langle f_a \rangle \)?
- Are the lengths of descending chains of HS-kernels beginning at \( \langle f_a \rangle \) bounded?
- Can any chain of HS-kernels be refined to such a chain of maximal length?

We provide answers to these three questions, for which the chain \( \mathcal{E} \) is of maximal unique length common to all chains of HS-kernels descending from \( \langle f_a \rangle \). Our method is to provide an explicit decomposition of a principal kernel \( f \) as an intersection of kernels of two types: The first, called an \( \text{HO-kernel} \), is a product of an HS-kernel and a region kernel. The second is a product of a region kernel and a bounded from below kernel. Whereas the first type defines the \( 1^\nu \)-set of \( f \), the second type corresponds to the empty set and thus has no effect on the geometry. This latter type is the source of ambiguity in relating a \( 1^\nu \)-set to a kernel, preventing the kernel of the \( 1^\nu \)-set from being principal.

When intersected with \( F \), the factors in the decomposition coming from kernels of the second type are degenerate. Restriction to \( F \) thus removes our ambiguity, and each HO-kernel (intersected with \( F \)) is in \( 1 : 1 \) correspondence with its \( 1^\nu \)-set (the segment in the \( 1^\nu \)-set defined by \( f \)). Then the ‘HO-part’ is unique and independent of the choice of the kernel generating the \( 1^\nu \)-set.

Geometrically, the decomposition to be described below is just the decomposition of a principal \( 1^\nu \)-set defined by \( f \) to its “linear” components. Each component is obtained by bounding an affine subspace of \( F(n) \) defined by an appropriate HS-fraction (which in turn generates an HS-kernel) using a region fraction (generating a region kernel). Although the HS-fraction and region fraction defining each segment may vary when we pass from one generator of a principal kernel to the other, the HS-kernels and region kernels are independent of the choice of generator, for they correspond to the components of the \( 1^\nu \)-set of \( f \).

**Construction 9.4.1.** Take a rational function \( f \in F(\lambda_1, \ldots, \lambda_n) \) for which \( \lambda_{\text{loc}}(f) \neq \emptyset \). Replacing \( f \) by \( |f| \), we may assume that \( f \geq 1 \). Write \( f = \sum h_i g_j \) where \( h_i \) and \( g_j \) are monomials in \( F[\lambda_1, \ldots, \lambda_n] \). For each \( a \in \lambda_{\text{loc}}(f) \), let
\[ H_a \subseteq H = \{ h_i : 1 \leq i \leq k \}; \quad G_a \subseteq G = \{ g_j : 1 \leq j \leq m \} \]
be the sets of dominant monomials at \( a \); thus, \( h_i(a) = g_j(a) \) for any \( h_i \in H_a \) and \( g_j \in G_a \). Let \( H_a^c = H \setminus H_a \) and \( G_a^c = G \setminus G_a \). Then, for any \( h' \in H_a^c \) and \( h'' \in H_a^c, h'(a) + h''(a) = h'(a), \) or, equivalently, \( 1 + \frac{h''(a)}{h'(a)} = 1 \). Similarly, for any \( g' \in G_a^c \) and \( g'' \in G_a^c, g'(a) + g''(a) = g'(a) \) or, equivalently, \( 1 + \frac{g''(a)}{g'(a)} = 1 \).

Thus for any such \( a \) we obtain the relations
\begin{align*}
(9.2) & \quad \frac{h'}{g'} = 1, \quad \forall h' \in H_a, g' \in G_a, \\
(9.3) & \quad 1 + \frac{h''}{h'} = 1; \quad 1 + \frac{g''}{g'} = 1, \quad \forall h' \in H_a, h'' \in H_a^c, g' \in G_a, g'' \in G_a^c.
\end{align*}

As \( a \) runs over \( \lambda_{\text{loc}}(f) \), there are only finitely many possibilities for \( H_a \) and \( G_a \) and thus for the relations in (9.2) and (9.3); we denote these as \( \theta_1(i), \theta_2(i) \), \( i = 1, \ldots, q \).

In other words, for any \( 1 \leq i \leq q, \) the pair \( \theta_1(i), \theta_2(i) \) corresponds to a kernel \( K_i \) generated by the corresponding elements
\[ \frac{h'}{g'}, \left( 1 + \frac{h''}{h'} \right), \text{ and } \left( 1 + \frac{g''}{g'} \right), \]
where \( \left\{ \frac{h'}{g'} = 1 \right\} \in \theta_1 \) and \( \{ 1 + \frac{h''}{h'} = 1 \}, \{ 1 + \frac{g''}{g'} = 1 \} \in \theta_2 \).

Reversing the argument, every point satisfying one of these \( q \) sets of relations is in \( \lambda_{\text{loc}}(f) \). Hence,
\begin{align*}
(9.4) & \quad \lambda_{\text{loc}} \left( (f) \cap (F) \right) = \lambda_{\text{loc}}(f) = \bigcup_{i=1}^{q} \lambda_{\text{loc}}(K_i) = \bigcup_{i=1}^{q} \lambda_{\text{loc}}(K_i \cap (F)) \\
& \quad = \lambda_{\text{loc}} \left( \bigcap_{i=1}^{q} (K_i \cap (F)) \right),
\end{align*}

Hence \( (f) \cap (F) = \bigcap_{i=1}^{q} K_i \cap (F) \), since \( (f) \cap (F), \bigcap_{i=1}^{q} K_i \cap (F) \in \mathcal{P}((F)) \). \( \bigcap_{i=1}^{q} K_i \) provides a local description of \( f \) in a neighborhood of its \( 1^\nu \)-set.
Let us view this construction globally. We used the 1rst-set of $f$ to construct $\bigcap_{i=1}^{q} K_i$. Adjoining various points $a$ in $F^{(a)}$ might add some regions, complementary to the regions defined by (9.3) in $\theta_q(i)$ for $i = 1, \ldots, q$, over which $\frac{\partial}{\partial t} \neq 1$, $v_i' \in H_a, v_i' \in G_a$ for each $a$, i.e., regions over which the dominating monomials never agree. Continuing the construction above using $a \in F^{(a)} \setminus \cap_{j=1}^{q} N_j$ similarly produces a finite collection of, say $t \in Z_{\geq 0}$, kernels generated by elements from (9.3) and their complementary order fractions and by elements of the form (9.2) (where now $\frac{\partial}{\partial t} \neq 1$ over the region considered). Any principal kernel $N_j = (\gamma_j), 1 \leq j \leq t$, of this complementary set of kernels has the property that $1_{\text{loc}}(N_j) = 0$, and thus by Corollary 8.2.13, $N_j$ is bounded from below. As there are finitely many such kernels there exists small enough $\gamma > 1$ in $T$ for which $|q_j| \wedge \gamma = \gamma$ for $j = 1, \ldots, t$. Thus $\bigcap_{j=1}^{t} N_j$ is bounded from below and thus $\bigcap_{j=1}^{t} N_j \subseteq \langle F \rangle$ by Remark 8.2.13.

Piecing this together with (9.4) yields $f$ over all of $F^{(a)}$, so we have

\begin{equation}
\langle f \rangle = \bigcap_{i=1}^{q} K_i \cap \bigcap_{j=1}^{t} N_j.
\end{equation}

So, $\langle f \rangle \cap \langle F \rangle = \bigcap_{i=1}^{q} K_i \cap \bigcap_{j=1}^{t} N_j \cap \langle F \rangle = \bigcap_{i=1}^{q} K_i \cap \langle F \rangle$.

In this way, we see that intersecting a principal kernel $\langle f \rangle$ with $\langle F \rangle$ ‘chops off’ all of the bounded from below kernels in (9.3) (the $N_j$’s given above). This eliminates ambiguity in the kernel corresponding to $1_{\text{loc}}(f)$. Finally we note that if $1_{\text{loc}}(f) = 0$, then $\langle f \rangle = \bigcap_{j=1}^{t} N_j$ for appropriate kernels $N_j$ and $\langle f \rangle \cap \langle F \rangle = \langle F \rangle$.

Remark 9.4.2.

(i) If $K_1$ and $K_2$ are such that $K_1 \cap K_2 \cap F = \{1\}$ (i.e., $1_{\text{loc}}(K_1) \cap 1_{\text{loc}}(K_2) \neq 0$), then the sets of $L$-monomials $\theta_1$ of $K_1$ and of $K_2$ are not the same (although one may contain the other), for otherwise together they would yield a single kernel via Construction 9.4.1.

(ii) The kernels $K_i$, being finitely generated, are in fact principal, so we can write $K_i = (\gamma_i)$ for rational functions $k_1, \ldots, k_q$. Let $(f) \cap \langle F \rangle = \bigcap_{i=1}^{q} (K_i \cap \langle F \rangle) = \bigcap_{i=1}^{q} (|k_i| \wedge |\alpha|) = \bigcap_{i=1}^{q} (|k_i| \wedge |\alpha|)$ with $\alpha \in F \setminus \{1\}$. By Theorem 8.3.3 for any generator $f'$ of $\langle f \rangle \cap \langle F \rangle$ we have that $|f'| = \bigcap_{i=1}^{q} |k_i|$ with $k_i \sim (F) |k_i| \wedge |\alpha|$ for every $i = 1, \ldots, q$. In particular, $1_{\text{loc}}(k_i) = 1_{\text{loc}}(|k_i| \wedge |\alpha|) = 1_{\text{loc}}(k_i)$. Thus the kernels $K_i$ are independent of the choice of generator $f$, being defined by the components $1_{\text{loc}}(k_i)$ of $1_{\text{loc}}(f)$.

We now provide two instances of Construction 9.4.1. In what follows, we always make use of the above notation for the different types of kernels involved in the construction. When denoting kernels, $R$ stands for “region”, $N$ for “null” (which are bounded from below), and $L$ for “linear” (representing HS-kernels, which are unbounded).

Example 9.4.3. $F = (F, T, v, \gamma)$. Let $f = |\lambda_1| \wedge |\alpha| = \frac{\alpha |\lambda_1|}{\alpha + |\lambda_1|} \in F(\lambda_1, \lambda_2)$, where $\alpha > 1$ in $T$. The order relation $\alpha \leq \nu |\lambda_1|$ translates to the relation $\alpha + |\lambda_1| \geq \nu |\lambda_1|$ or equivalently to $\alpha|\lambda_1|^{-1} + 1 \geq \nu$. Over the region defined by this relation we have $f = \frac{\alpha}{\alpha \pm \lambda_1} = |\lambda_1|$. Similarly, its complementary order relation $\alpha \geq \nu |\lambda_1|$ translates to $\alpha^{-1}|\lambda_1| + 1 = 1$ (via $|\lambda_1| + \alpha \geq \nu$) over whose region $\frac{\alpha}{\alpha \pm \lambda_1} = |\lambda_1|$. So

\begin{equation}
\langle f \rangle = \bigcap_{i=1}^{q} K_i \cap \bigcap_{j=1}^{t} N_j \cap \langle F \rangle = \bigcap_{i=1}^{q} K_i \cap \langle F \rangle.
\end{equation}

where $R_{1,1} = \langle \alpha^{-1}|\lambda_1| + 1 \rangle$, $L_{1,1} = \langle |\lambda_1| \rangle$, $R_{2,1} = \langle |\lambda_1|^{-1} + 1 \rangle$, and $N_{2,1} = \langle \alpha \rangle$. Geometrically $R_{1,1}$ is a strip containing the axis $\lambda_1 = 1$, and $R_{2,1}$ is the complementary region. The restriction of $f$ to $R_{1,1}$ is $|\lambda_1|$ while $f$ restricted to $R_{2,1}$ is $\alpha$. Deleting $N_{2,1}$ we still have $1_{\text{loc}}(f) = 1_{\text{loc}}(R_{1,1}L_{1,1})$, although $R_{1,1}L_{1,1}$ properly contains $\langle F \rangle$.

Example 9.4.4. Let $f = |\lambda_1 + 1| \wedge \alpha \in F(\lambda_1, \lambda_2)$ for some $\alpha > 1$ in $T$. First note that $|\lambda_1 + 1| = |\lambda_1 + 1| \wedge \alpha > 1$ since $|\lambda_1 + 1| \geq |\lambda_1 + 1|$, allowing us to rewrite $f$ as $(\lambda_1 + 1) \wedge \alpha$. Then $f = \frac{\alpha |\lambda_1 + 1|}{(\alpha + |\lambda_1 + 1|)} = \frac{\alpha \lambda_1 + \lambda_2}{\alpha + \lambda_2}$. The order relation $\alpha \leq \nu |\lambda_1 + 1$ translates to the relation $\alpha + \lambda_1 \geq \nu |\lambda_1 + 1|$, or equivalently to $\alpha |\lambda_1 + 1|^{-1} \geq \nu$, over whose region $\frac{\alpha |\lambda_1 + 1|}{\alpha + |\lambda_1 + 1|} = \frac{\alpha \lambda_1 + \lambda_2}{\alpha + \lambda_2} = \alpha + \frac{\lambda_1}{\lambda_2}$. Similarly, the complementary order relation $\alpha \geq \nu |\lambda_1 + 1$ translates to $\alpha^{-1}|\lambda_1 + 1| + 1 \geq \nu$, over whose region $\frac{\alpha}{\alpha \pm \lambda_1 + 1} = \frac{\alpha \lambda_1 + \lambda_2}{\alpha + \lambda_2} = \alpha + \frac{\lambda_1}{\lambda_2}$. So

\begin{equation}
\langle f \rangle \cap \langle F \rangle = \bigcap_{i=1}^{q} K_i \cap \bigcap_{j=1}^{t} N_j \cap \langle F \rangle = \bigcap_{i=1}^{q} K_i \cap \langle F \rangle.
\end{equation}

where $R_{1,1} = \langle \alpha^{-1}|\lambda_1 + 1 \rangle$, $\Omega_{1,2} = \langle |\lambda_1 + 1| \rangle$, $R_{2,1} = \langle \alpha \lambda_1^{-1} + 1 \rangle$ and $N_{2,1} = \langle \alpha \rangle$. But $1_{\text{loc}}(R_{2,1}L_{2,1}) \subseteq 1_{\text{loc}}(N_{2,1}) = 0$. So $L_{1,1} = \bigcup_{i=1}^{q} (\Omega_{1,2}) \cap (R_{2,1}N_{2,1}) = 1_{\text{loc}}(\lambda_1 + 1) \wedge \nu = 1_{\text{loc}}(\lambda_1 + 1)$.

Now suppose $L_i$ are HP-kernels and $\Omega_i$ are order kernels, and let $L = \prod L_i$ and $R = \prod \Omega_i$. As can be seen easily from examples 9.4.3 and 9.4.4 by substituting any $L$-monomial for $\lambda_1$ and any order fraction for $\lambda_1 + 1$, $\langle L \cap R \rangle = \prod L_i \cap (\Omega_i \cap \langle F \rangle) = \bigcap \langle \bigcap (\Omega_i \cap (N_i, R_i^\gamma)) \bigcap \langle (\Omega_i R_i^\gamma, \cap M_j R_j^\gamma) \bigcap \langle F \rangle \rangle \bigcap \langle F \rangle = \bigcap \langle \bigcap \Omega_i \cap N_i \cap \langle F \rangle \rangle$.
where the $R_i$ and $R_i'$ are region kernels and $N = \left(\prod_i N_i\right)\left(\prod_j M_j\right)$. The kernels $N_i, M_j$, and thus $N$, are bounded from below; $\langle F \rangle \subseteq N$, and the right side is $(LR) \cap (F)$ where $R'$ is a region kernel.

Note that the $O_j$s involve the $R_i$s, the $R_i'$s and the $O_j$s. Also note that intersecting with $\langle F \rangle$ keeps the HS-kernel unchanged in the new decomposition.

As the $N_j$s in (9.5), being bounded from below, do not affect $1_{\text{loc}}(f)$, we put them aside for the time being and proceed to study the structure of the kernels $K_i$ and their corresponding $1^\nu$-sets.

Take one of these kernels $K_i$. Recall that $K_i$ is generated by a set comprised of $\mathcal{Z}$-monomials and order elements. Let $L_i, j, 1 \leq j \leq u$ and $O_i, k, 1 \leq k \leq v$, be the HP-kernels and the order kernels generated respectively by these $\mathcal{Z}$-monomials and order elements. Then we can write

\[ K_i = L_i R_i = \left(\prod_{j=1}^u L_i, j\right) \left(\prod_{k=1}^v O_i, k\right), \]

where $L_i = \prod_{j=1}^u L_i, j$ is an HS-kernel and $R_i = \prod_{k=1}^v O_i, k$ is a region kernel. By assumption, $1_{\text{loc}}(K_i) \neq \emptyset$ since at least one point of the $1^\nu$-set was used in its construction. Moreover, one cannot write $L = M_1 \cap M_2$ for distinct HS-kernels $M_1$ and $M_2$, for otherwise the construction would have produced two distinct kernels, one with $M_1$ as its HS-kernel and the other with $M_2$ as its HS-kernel, rather than $K_i$.

Let us formalize this situation.

**Definition 9.4.5.** A rational function $f \in F(\lambda_1, ..., \lambda_n)$ is an **HO-fraction** if it is the sum of an HS-fraction $f'$ and a region fraction $\alpha f$.

**Definition 9.4.6.** A principal kernel $K \in \mathcal{P}(F(\lambda_1, ..., \lambda_n))$ is said to be an **HO-kernel** if it is generated by an HO-fraction.

Note that any HS-kernel or any region kernel is an HO-kernel.

**Lemma 9.4.7.** A principal kernel $K$ is an HO-kernel if and only if $K = LR$ where $L$ is an HS-kernel and $R$ is a region kernel.

**Proof.** ($\Rightarrow$) Write $K = \langle f \rangle$, where $f = f' + \alpha f$ is an HO-fraction. Thus $K = \langle f' + \alpha f \rangle = \langle f' \rangle \langle \alpha f \rangle$ where $\langle f' \rangle$ is an HS-kernel and $\langle \alpha f \rangle$ is a region kernel.

($\Leftarrow$) Write the HO-fraction $f = f' + \alpha f$ where $f'$ is an HS-fraction generating $L$ and $\alpha$ is a region fraction generating $R$; then $\langle f \rangle = \langle f' + \alpha f \rangle = \langle f' \rangle \langle \alpha f \rangle = LR = K$.

**Theorem 9.4.8.** Every principal kernel $\langle f \rangle$ of $F(\lambda_1, ..., \lambda_n)$ can be written as the intersection of finitely many principal kernels

\[ \{K_i : i = 1, ..., q\} \text{ and } \{N_j : j = 1, ..., m\}, \]

whereas each $K_i$ is the product of an HS-kernel and a region kernel

\[ K_i = L_i R_i = \prod_{j=1}^{t_i} L_{i, j} \prod_{k=1}^{k_i} O_{i, k}, \]

while each $N_j$ is a product of bounded from below kernels and (complementary) region kernels. For $\langle f \rangle \in \mathcal{P}(\langle F \rangle)$, the $N_j$ can be replaced by $\langle F \rangle$ without affecting the resulting kernel.

**Proof.** Let $K = LR$ be an HO-kernel with $R$ a region kernel and $L$ an HS-kernel. By Proposition 9.1.8 and Lemma 9.2.8 we have that $L = \prod_{i=1}^u L_i$ for some HP-kernels $L_1, ..., L_u$ and $R = \prod_{j=1}^v O_j$ for some order kernels $O_1, ..., O_v$. Thus

\[ K = LR = \left(\prod_{i=1}^u L_i\right) \left(\prod_{j=1}^v O_j\right), \]

where $u \in \mathbb{Z}_{\geq 0}, \ v \in \mathbb{N}, \ L_1, ..., L_u$ are HP-kernels and $O_1, ..., O_v$ are order kernels.

Let $K_1$ and $K_2$ be region kernels (respectively HS-kernels) such that $(K_1 K_2) \cap F = \{1\}$. Then $K_1 K_2$ is a region kernel (respectively HS-kernel). Consequently, if $K_1$ and $K_2$ are HO-kernels such that $(K_1 K_2) \cap F = \{1\}$, then $K_1 K_2$ is an HO-kernel. Indeed, the assertions follow from the decomposition $K_i = L_i O_i = (\prod_{j=1}^{u_i} L_{i, j})(\prod_{k=1}^{v_i} O_{i, k})$ for $i = 1, 2$, so that

\[ K_1 K_2 = (L_1 L_2)(O_1 O_2) = \left(\prod_{i=1}^{u_1} L_{1, i}\right) \left(\prod_{j=1}^{v_1} O_{1, j}\right) \left(\prod_{i=1}^{u_2} L_{2, i}\right) \left(\prod_{j=1}^{v_2} O_{2, j}\right) = LO, \]

with the appropriate $u_i, v_i$ taken for $i = 1, 2$.

**Proposition 9.4.9.**

- If $\langle f \rangle$ is an HS-kernel, then the decomposition degenerates to $\langle f \rangle = K_1$ with $K_1 = L_1 = \langle f \rangle$.
- If $\langle f \rangle$ is a region kernel, then $\langle f \rangle = K_1$ with $K_1 = R_1 = \langle f \rangle$.
- $\langle f \rangle$ is an irregular kernel if and only if there exists some $t_0 \in \{1, ..., q\}$ such that $K_{t_0} = R_{t_0} = \prod_{k=1}^{v_{t_0}} O_{k, t_0}$.
- $\langle f \rangle$ is a regular kernel if and only if $K_i$ is comprised of at least one HP-kernel, for every $i = 1, ..., q$. 


Lemma 9.6.1. Let \( \langle h \rangle \) be a principal kernel. Then its decomposition given in Corollary 9.4.11 is given by

\[
1_{\text{loc}}\langle h \rangle = 1_{\text{loc}}\left( \left\langle \frac{\lambda_1}{\lambda_2} \right\rangle \cap (1 + \lambda_2^{-1}) \right) \cup 1_{\text{loc}}\left( \left\langle \lambda_1 \right\rangle \cap (1 + \lambda_2) \right) \cup 1_{\text{loc}}\left( \left\langle |\lambda_1| + |\lambda_2| \right\rangle \right)
\]

\[\xymatrix{\lambda_2 \\
1_{\text{loc}}\langle \frac{\lambda_1}{\lambda_2} \rangle \\
1_{\text{loc}}\left( \left\langle \frac{\lambda_1}{\lambda_2} \right\rangle \cap (1 + \lambda_2^{-1}) \right) \\
1_{\text{loc}}\langle \lambda_1 \rangle \\
1_{\text{loc}}\left( \left\langle \lambda_1 \right\rangle \cap (1 + \lambda_2) \right) \\
1_{\text{loc}}\left( \left\langle |\lambda_1| + |\lambda_2| \right\rangle \right) \\
}\]

**Figure 3.** $1_{\text{loc}}\langle \frac{\lambda_1}{\lambda_2} \rangle = 1_{\text{loc}}\left( \left\langle \frac{\lambda_1}{\lambda_2} \right\rangle \cap (1 + \lambda_2^{-1}) \right) \cup 1_{\text{loc}}\langle \lambda_1 \rangle \cup 1_{\text{loc}}\left( \left\langle |\lambda_1| + |\lambda_2| \right\rangle \right)$

**Proof.** (i) By Proposition 9.4.8, \( \langle f \rangle = \prod_{j=1}^{t} L_j \) where \( L_j \) is an HP-kernel for each \( j \). Write \( f = \sum_{i} c_i h_i g_j \) for monomials \( h_i, g_j \in F[\Lambda] \). If \( \langle f \rangle \) is irregular, then at some neighborhood of a point \( a \in F^\wedge \) we have some \( i_0 \) and \( j_0 \) for which \( h_{i_0} \approx g_{j_0} \) where \( \langle g_{j_0}(a) \rangle \approx h_{i_0}(a) \) for every \( i \neq i_0 \) and \( j \neq j_0 \). The kernel corresponding to (the closure) of this region has its relations \((\ref{eq:decomposition})\) degenerating to 1 = 1 as \( \sum_{i} c_i h_i g_j \) over the region, thus is given only by its order relations of \((\ref{eq:decomposition})\).

The last three assertions are direct consequences of \((\ref{eq:decomposition})\); namely, if \( \langle f \rangle \) is either an HS-kernel or a region kernel, \( \langle f \rangle \) already takes on the form of its decomposition. The fourth is equivalent to the third.

\[\Box\]

9.5. Conclusion of the proof of Theorem 8.0.13

**Definition 9.5.1.** The HO-decomposition of a principal kernel \( \langle f \rangle \) is its decomposition given in Theorem 9.4.8. In the special case where \( \langle f \rangle \in \mathcal{P}(\langle F \rangle) \), all bounded from below terms of the intersection are equal to \( \langle f \rangle \).

**Definition 9.5.2.** For a subset \( S \subseteq F(\lambda_1, ..., \lambda_n) \), denote by HO\((S)\) the family of HO-fractions in \( S \), by HS\((S)\) the family of HS-fractions in \( S \), and by HP\((S)\) the family of \( \mathcal{L} \)-monomials in \( S \).

**Remark 9.5.3.** HP\((S)\) \( \subseteq \) HS\((S)\) \( \subseteq \) HO\((S)\) for any \( S \subseteq F(\lambda_1, ..., \lambda_n) \), since every \( \mathcal{L} \)-monomial is an HS-fraction and every HS-fraction is an HO-fraction.

**Example 9.5.4.** Consider the kernel \( \langle f \rangle \) where \( f = \sum_{i} c_i h_i g_j \in F(\lambda_1, ..., \lambda_n) \). The points on the \( 1^\wedge \)-set of \( f \) define three different HS-kernels: \( \langle \frac{\lambda_1}{\lambda_2} \rangle \) (corresponding to \( \lambda_1 = \lambda_2 \)) over the region \( \{ \lambda_2 \geq 1 \} \) defined by the region kernel \( (1 + \lambda_2^{-1}) \), \( \langle \lambda_1 \rangle \) (corresponding to \( \lambda_1 = 1 \)) over the region \( \{ \lambda_2 \leq 1 \} \) defined by the region kernel \( (1 + \lambda_2) \), and \( \langle |\lambda_1| + |\lambda_2| \rangle \) (corresponding to the point defined by \( \lambda_1 = \lambda_2 = 1 \)). Thus by Construction 9.4.11

\[
\langle f \rangle = \left( \left\langle \frac{\lambda_1}{\lambda_2} \right\rangle \cap (1 + \lambda_2^{-1}) \right) \cap (1 + \lambda_2) \cap \left\langle |\lambda_1| + |\lambda_2| \right\rangle \cap \langle f \rangle
\]

and

\[
1_{\text{loc}}\langle f \rangle = \left( 1_{\text{loc}}\left( \left\langle \frac{\lambda_1}{\lambda_2} \right\rangle \cap (1 + \lambda_2^{-1}) \right) \right) \cup \left( 1_{\text{loc}}\langle \lambda_1 \rangle \cap (1 + \lambda_2) \right) \cup \left( 1_{\text{loc}}\left( |\lambda_1| + |\lambda_2| \right) \right) \cap \langle f \rangle
\]

The third component of the decomposition (i.e., the HS-kernel \( \langle |\lambda_1| + |\lambda_2| \rangle \)) could be omitted without effecting \( 1_{\text{loc}}\langle f \rangle \).

The decomposition is shown (in logarithmic scale) in Figure 3 where the first two components are the rays emanating from the origin and the third component is the origin itself.

9.6. The lattice generated by regular corner-internal principal kernels.

Recall from Proposition 9.2.14 that the principal kernel \( \langle f \rangle \langle a_1 \rangle \cdots \langle a_k \rangle \) is regular, for any HP-kernel \( \langle f \rangle \neq 1 \) and order kernels \( \langle a_1 \rangle, ..., \langle a_k \rangle \).

**Lemma 9.6.1.** Let \( K \in \mathcal{P}(\langle F \rangle) \) and let

\[
K = \langle LR \rangle \cap \langle F \rangle = \left( \prod_{j=1}^{u} L_j \right) \left( \prod_{k=1}^{v} O_k \right) \cap \langle F \rangle
\]
be the decomposition of \( K \), as given in (9.6), where \( L_1, \ldots, L_n \) are HP-kernels and \( O_1, \ldots, O_r \) are order kernels. If \( u \neq 0 \), i.e., \( L \neq 1 \), then \( K \) is regular.

Proof. Indeed, \( K = \langle \bigcap_{i=1}^{n} L_i \rangle (L_1 \prod_{j=1}^{r} O_j) \cap \langle F \rangle \). By Proposition 9.2.11 \( (L_1 \prod_{j=1}^{r} O_j) \) is regular since \( L_1 \) is an HP-kernel. Thus \( K \) is a regular kernel, since a product of regular kernels is regular and since intersection with \( \langle F \rangle \) does not affect regularity.

We are finally ready to prove Theorem 8.0.13.

Proof. Let \( \langle f \rangle \) be a principal kernel and let

\[
\langle f \rangle = \left( \bigcap_{i=1}^{q} K_i \right) \cap \langle F \rangle, \quad K_i = \prod_{j=1}^{k} L_{i,j} \prod_{k=1}^{r} O_{i,k}
\]

be its HO-decomposition. By Lemma 9.2.9 each HP-kernel \( L_{i,j} \) and each order kernel \( O_{i,k} \) are corner internal. Thus \( \langle f \rangle \) as a finite product of principal corner internal kernels is in the lattice generated by principal corner-internal kernels.

For the second assertion, if \( \langle f \rangle \) is regular, then by Theorem 9.4.8 for every \( 1 \leq i \leq q \), we have that \( L_{i,1} \neq 1 \). Thus by Lemma 9.6.1 each \( K_i \) is a product of principal regular corner-internal kernels. Thus \( \langle f \rangle \) is in the lattice generated by principal regular corner-internal kernels. We conclude with Proposition 8.0.11.

10. Polars: an intrinsic description of \( K \)-kernels

To characterize \( K \)-kernels intrinsically, we need a kind of orthogonality relationship, and introduce a new kind of kernel, called polar, borrowed from the theory of lattice ordered groups (29 section (2.2)). We fix \( X \subseteq F(\Omega) \). \( S \) is always assumed to be an idempotent 1-semifield, often \( F(X) \) or even Fun(\( F(\Omega), F \)). In this section, we lay out the general basics of the theory, although its full strength is only obtained in the following sections when the underlying semifield \( F \) is taken to be \( \nu \)-divisible, \( \nu \)-archimedean and complete, i.e., \( F = \mathbb{R} \).

10.1. Basic properties of polars.

Definition 10.1.1. We write \( f \perp g \) for \( f, g \in F(X) \) if \( |f(a)| \wedge |g(a)| \leq 1 \) for all \( a \in X \), i.e., \( \rho_{\text{loc}}(f) \cup \rho_{\text{loc}}(g) = X \).

For subsets \( K, L \) of \( F(X) \) we write \( K \perp L \) if \( f \perp g \) for all \( f \in K \) and \( g \in L \). This is not necessarily implied by \( (\wedge K) \perp (\vee L) \), as evidenced by Note 5.2.8. For \( V \subseteq S \), we define

\[
V^\perp = \{ g \in S : g \perp V \}
\]

Such a set \( V^\perp \) is called a polar in the literature.

For \( f \in S \) we write \( f^\perp \) for \( \{ f \}^\perp \). Thus, \( f^\perp = \langle f \rangle^\perp \). The set of all polars in \( S \) is denoted as \( \mathcal{P}(S) \).

Remark 10.1.2. \( f \perp g \) iff \( f \perp \langle g \rangle \). \( f \perp g \) iff \( \langle f \rangle \perp \langle g \rangle \).

Remark 10.1.3 (29). If \( K \subseteq S \), then \( K^\perp = \langle K \rangle^\perp \). Consequently,

\[
\langle K \rangle^\perp \perp \langle K \rangle = K^\perp \perp K = K
\]

Thus, \( \perp \perp K \) iff \( K \subseteq L^\perp \), for any kernels \( K, L \).

Although this kind of property cannot arise in classical algebraic geometry since a variety cannot be the proper union of two algebraic sets, it is quite common in the tropical setting. For example \( (1 + f) \perp (1 + f^{-1}) \). The usual properties of orthogonality go through here.

Lemma 10.1.4. The following statements are immediate consequences of Definition 10.1.1. For any \( K, L \subseteq S \),

\[
\begin{align*}
(i) \quad & K \subseteq L \Rightarrow K^\perp \supseteq L^\perp. \\
(ii) \quad & K \subseteq K^{\perp \perp}. \\
(iii) \quad & K^\perp = K^{\perp \perp \perp}. \\
(iv) \quad & K \text{ is a polar iff } K^{\perp \perp} = K.
\end{align*}
\]

Proof. The first two assertions are obvious, and the third follows from using \( K^\perp \) in (ii), applying (i). Finally, if \( K \) is a polar, then \( K = V^\perp \) for some \( V \subseteq S \), and \( K^{\perp \perp} = (V^\perp)^\perp = V^\perp = K \).

The following facts, taken from [29], can be checked pointwise.

Theorem 10.1.5. For any subset \( V \subseteq S \), \( V^\perp \) is a \( K \)-kernel of \( S \).

Proof. As noted in [29] Theorem 2.2.4(e), \( K := V^\perp \) is a convex (abelian) subgroup, and thus a kernel. It remains to show that \( K = \ker(1_{\text{loc}}(K)) \). Clearly \( K \subseteq 1_{\text{loc}}(K) \), so we need to show that any \( g \in \ker(1_{\text{loc}}(K)) \) belongs to \( K \). We are given \( 1_{\text{loc}}(g) \geq 1_{\text{loc}}(K) \) and \( K \perp v \) for each \( v \in V \), so \( 1_{\text{loc}}(g) \geq 1_{\text{loc}}(K) \) implies that \( 1_{\text{loc}}(g) \cup 1_{\text{loc}}(v) \geq 1_{\text{loc}}(K) \cup 1_{\text{loc}}(v) = X \), i.e., \( g \perp V \), as desired.

Proposition 10.1.6. \( (\mathcal{P}(S), \cdot, \cap, \perp, \{1\}, S) \) is a complete Boolean algebra.
Proof. [20 Theorem 2.2.5]; If \( \{K_i : i \in I\} \) are subsets of \( S \), then
\[
\left( \bigcup_{i \in I} K_i \right) ^\perp = \bigcap_{i \in I} K_i ^\perp.
\]

Closure under complements is a consequence of (ii).

**Proposition 10.1.7.** For any subset \( V \subseteq S \), \( V ^\perp \) is the minimal polar containing \( V \).

**Proof.** By definition, \( V ^\perp \) is a polar containing \( S \). Let \( P \supseteq S \) be a polar. Then \( S ^\perp = (S ^\perp ) ^\perp \subseteq (P ^\perp ) ^\perp = P \).

**Definition 10.1.8.** Let \( S \subseteq S \). We say that a polar \( P \) is generated by \( S \) if \( P = S ^\perp \). If \( S = \{f\} \) then we also write \( f ^\perp \) for the polar generated by \( \{f\} \).

**Definition 10.1.9.** A polar \( P \) of \( S \) is principal if there exists some \( f \in S \) such that \( P = f ^\perp (= (f) ^\perp ) \).

**Lemma 10.1.10.** The collection of principal polars is a sublattice of \( (\mathcal{P}l(S), \cdot, \cap) \).

**Proof.** For any \( f, g \in S \),
\[
(f \cap g) ^\perp = (|f| \cap |g|) ^\perp = (|f| \cap |g|) ^\perp, \quad \text{and}
\]
and
\[
(f + g) ^\perp = (|f| + |g|) ^\perp = (|f| + |g|) ^\perp.
\]

by Corollary [3.5](#).

**Remark 10.1.11.** Any kernel \( K \) of an idempotent semifield \( S \) cannot be orthogonal to a nontrivial subkernel of \( K ^\perp \).

**Proof.** Any kernel \( L \) of \( K ^\perp \) is a kernel of \( S \). If \( K \perp L \), then \( L \subseteq K ^\perp \cap K ^\perp = \{1\} \), yielding \( L = \{1\} \).

10.2. Polar over complete Archimedean semifields.

In Theorem [3.1.5](#) we showed that \( \mathcal{K} \)-kernels of \( F(\Lambda) \) correspond to \( 1^\perp \)-sets in \( F^\perp (\Lambda) \). Our aim here is to characterize these special kind of kernels as polars, as a converse to Theorem 10.1.5. In analogy to algebraic geometry, polars play the role of ‘radical ideals,’ corresponding to root sets by an analog to the celebrated Nullstellensatz theorem, but for this we need to make extra assumptions on the underlying semifield.

10.3. The polar-\( 1^\perp \)-set correspondence.

Recall that \( \mathcal{R} \) is presumed to be a \( \nu \)-bipotent, divisible, archimedean and complete \( \nu \)-semifield. In this subsection we concentrate on \( \mathcal{R}(\Lambda) \). Doing so, we consider the natural extensions to \( \mathcal{R}(\Lambda) \) of the operators \( 1_{\text{loc}} \) and \( \mathcal{K} \) defined in [3.2](#) with respect to \( \mathcal{R}(\Lambda) \). We write \( \mathcal{K}(\mathcal{R}(\Lambda)) \) to denote the restriction of \( \mathcal{K} \) to \( \mathcal{R}(\Lambda) \).

Recall “completely closed” from Definition [2.4.3](#).

**Lemma 10.3.1.** All kernels of \( \mathcal{R}(\Lambda) \) that are completely closed are completions of kernels of \( \mathcal{R}(\Lambda) \).

**Proof.** By Theorem [3.1.1](#), the kernels of \( \mathcal{R}(\Lambda) \) are precisely \( \{K \cap \mathcal{R}(\Lambda) : K \text{ is a kernel of } \mathcal{R}(\Lambda)\} \). Since \( \mathcal{R}(\Lambda) \) is dense in \( \mathcal{R}(\Lambda) \) for every kernel \( K \) of \( \mathcal{R}(\Lambda) \), the kernel \( L = K \cap \mathcal{R}(\Lambda) \) of \( \mathcal{R}(\Lambda) \) is dense in \( K \). Since \( L \subseteq K \), we conclude that \( L = K_i \), i.e., \( L = K \) if and only if \( K \) is completely closed.

**Proposition 10.3.2.** Each completely closed kernel \( K \) of \( \mathcal{R}(\Lambda) \) defines a unique kernel of \( \mathcal{R}(\Lambda) \) given by \( L = K \cap \mathcal{R}(\Lambda) \) for which \( L = K \).

**Example 10.3.3.** Consider the kernel \( K = (|\lambda_1| \wedge |\alpha|) \in \mathcal{P}(\mathcal{R}(\Lambda)) \) and its subset \( X = \{|\lambda_1| \wedge |\alpha| : n \in \mathbb{N}\} \). Then \( |\lambda_1| = \bigvee_{f \in A} f \in K \). Hence \( \lambda_1 \subseteq K \) and \( f \lambda_1 f \subseteq |\lambda_1| \).

**Remark 10.3.4.** \( 1_{\text{loc}}(\mathcal{K}) \) is the kernel of every completely closed kernel \( K \) of \( \mathcal{R}(\Lambda) \).

**Proof.** First note that \( 1_{\text{loc}}(\mathcal{K}) \subseteq 1_{\text{loc}}(K) \) since \( K \subseteq \mathcal{K} \) in \( \mathcal{R}(\Lambda) \). Now, let \( X = 1_{\text{loc}}(K) \). For any nonempty subset \( A \) of \( K \), if \( \bigvee_{f \in A} f \) is in \( \mathcal{R}(\Lambda) \) then for any \( a \in X \),
\[
\left( \bigvee_{f \in A} f \right) (a) = \bigvee_{f \in A} f(a) = \bigvee_{f \in A} 1 \equiv_{\nu} 1,
\]
yielding \( 1_{\text{loc}}(\bigvee_{f \in A} f) \supseteq X \) (cf. Lemma [2.3.10](#)). Similarly, if \( \bigwedge_{f \in A} f \) is in \( \mathcal{R}(\Lambda) \) then for any \( a \in X \) we have \( (\bigwedge_{f \in A} f)(a) = \bigwedge_{f \in A} f(a) = \bigwedge_{f \in A} 1 \equiv_{\nu} 1 \), yielding \( 1_{\text{loc}}(\bigwedge_{f \in A} f) \supseteq X \). We conclude that \( 1_{\text{loc}}(\mathcal{K}) = 1_{\text{loc}}(K) \).

**Proposition 10.3.5.** Every \( \mathcal{K} \)-kernel of \( \mathcal{R}(\Lambda) \) is completely closed.
Proof. By Lemma 5.3.5 if \( Z = 1_{\text{loc}}(K) \) then \( 1_{\text{loc}}(\text{Kern}(Z)) = Z \). Let \( Z = 1_{\text{loc}}(K) \subseteq \mathfrak{R}^{(n)} \) where \( K = \text{Kern}(Z) \), a kernel of \( \mathfrak{R}(\Lambda) \). \( 1_{\text{loc}}(f) \supseteq Z \) for any \( f \in K \). Now, if \( \bigvee_{f \in A} f \in \mathfrak{R}(\Lambda) \) and \( \bigwedge_{f \in A} f \in \mathfrak{R}(\Lambda) \), then by Lemma 5.2.15 for any \( a \in Z \)

\[
\left( \bigvee_{f \in A} f \right)(a) = \bigvee_{f \in A} f(a) = 1 \cong \nu 1 \quad \text{and} \quad \left( \bigwedge_{f \in A} f \right)(a) = \bigwedge_{f \in A} f(a) = 1 \cong \nu 1.
\]

Thus

\[
1_{\text{loc}} \left( \bigvee_{f \in A} f \right), 1_{\text{loc}} \left( \bigwedge_{f \in A} f \right) \supseteq Z.
\]

So \( \bigvee_{f \in A} f, \bigwedge_{f \in A} f \in K \), and \( K \) is completely closed in \( \mathfrak{R}(\Lambda) \).

Remark 10.3.6. Proposition 10.3.5 is not true when taking \( K \)-kernels of \( \mathfrak{R}(\Lambda) \) instead of \( \mathfrak{R}(\Lambda) \). Let \( \alpha \in \mathfrak{R} \) such that \( \alpha > 1 \). Consider the subset \( X = \{|\lambda| \mid \alpha : n \in \mathbb{N} \} \). Then \( X \subset (\lambda_1), \bigvee_{f \in X} f = \alpha \) (the constant function) and \( 1_{\text{loc}}(f) = \{1\} \subseteq \mathfrak{R} \) for every \( f \in X \). Thus \( \alpha = (\alpha)(1) = (\bigvee_{f \in X} f)(1) = 1 \) and \( \alpha \) is not in the preimage of \( 1_{\text{loc}}(\lambda_1) \). So we deduce that \( K \)-kernels of \( \mathfrak{R}(\Lambda) \) need not be completely closed in \( \mathfrak{R}(\Lambda) \), and thus not polars since every polar is completely closed by Proposition 10.3.5. Also note that \( \alpha \in \lambda_1^{\perp}, \text{yielding } \lambda_1^{\perp} \cdot \lambda_1 = (\lambda_1^{\perp})^* = \{1\} \).

Theorem 10.3.7. The following properties of a kernel \( K \) of \( \mathfrak{R}(\Lambda) \) are equivalent:

(i) \( K \) is a \( K \)-kernel.
(ii) \( K \) is a polar.
(iii) \( K \) is completely closed.

Proof. (i) \( \Rightarrow \) (iii) By Proposition 10.3.5 \( K \)-kernel of \( \mathfrak{R}(\Lambda) \) is completely closed.

(iii) \( \Rightarrow \) (ii) See [29, Theorem 2.3.7]. One checks the condition of Lemma 10.1.13.

(ii) \( \Rightarrow \) (i) By Theorem 10.1.13.

Proposition 10.3.8. For any \( f \in \mathfrak{R}(\Lambda) \),

\[
1_{\text{loc}}(f^{\perp}) = 1_{\text{loc}}(f^{\perp}) \quad \text{and} \quad f^{\perp} = \text{Kern}(1_{\text{loc}}(f)).
\]

Proof. \( 1_{\text{loc}}(f^{\perp}) \subseteq 1_{\text{loc}}(f) \) since \( f^{\perp} \supseteq (f) \). Now, let \( K \) be the \( K \)-kernel containing \( f \) such that \( 1_{\text{loc}}(K) = 1_{\text{loc}}(f) \). \( K \) is a polar by Lemma 10.1.14(iv). By Proposition 10.1.17 \( K \supseteq f^{\perp} \), and so

\[
1_{\text{loc}}(f) = 1_{\text{loc}}(K) \subseteq 1_{\text{loc}}(f^{\perp}) \subseteq 1_{\text{loc}}(f).
\]

But \( f^{\perp} \) is a polar and thus a \( K \)-kernel, implying \( f^{\perp} = \text{Kern}(1_{\text{loc}}(f^{\perp})) = \text{Kern}(1_{\text{loc}}(f)). \)

Theorem 10.3.9. There is a \( 1:1 \) correspondence

\[
\mathcal{P}lr(\mathfrak{R}(\Lambda)) \leftrightarrow 1'-\text{Set}(\mathfrak{R}^{(n)})
\]

between the polars of \( \mathfrak{R}(\Lambda) \) and the \( 1' \)-sets in \( \mathfrak{R}^{(n)} \), given by \( B \mapsto 1_{\text{loc}}(B) \) and \( Z \mapsto \text{Kern}(Z) \).

This correspondence restricts to a correspondence between the principal polars of \( \mathfrak{R}(\Lambda) \) and the principal \( 1' \)-sets in \( \mathfrak{R}^{(n)} \).

Proof. By Lemma 10.1.14(iv), \( B \) is a polar if \( B \) is a \( K \)-kernel; thus, \( \text{Kern}(1_{\text{loc}}(B)) = B \). If \( B = f^{\perp} \) for some \( f \in \mathfrak{R}(\Lambda) \), then \( 1_{\text{loc}}(B) = 1_{\text{loc}}(f) \) by Proposition 10.3.8, yielding \( \text{Kern}(1_{\text{loc}}(f^{\perp})) = \text{Kern}(1_{\text{loc}}(f)) = f^{\perp} \). The restriction to principal polars follows from Lemma 10.1.10.

Corollary 10.3.10. Let

\[
B = \{ B \cap \mathfrak{R}(\Lambda) : B \in \mathcal{P}lr(\mathfrak{R}(\Lambda)) \}.
\]

There is a \( 1:1 \) correspondence

\[
B \leftrightarrow 1'-\text{Set}(\mathfrak{R}^{(n)})
\]

given by \( K \mapsto 1_{\text{loc}}(K) \) and \( Z \mapsto \text{Kern}(Z) \cap \mathfrak{R}(\Lambda) = \text{Kern}(\mathfrak{R}(\Lambda))(Z) \), which restricts to a correspondence

\[
\text{Principal kernels of } \mathcal{B} \leftrightarrow \text{P1-Set}.
\]

Furthermore, \( \text{Kern}(\mathfrak{R}(\Lambda))(1_{\text{loc}}(K)) = \overline{K} \cap \mathfrak{R}(\Lambda) \) for any kernel \( K \) of \( \mathfrak{R}(\Lambda) \).

Proof. Since \( (\mathcal{P}lr(\mathfrak{R}(\Lambda)), \cap) \) is a lattice, the first assertion follows by applying Proposition 10.3.2 to the correspondence in Theorem 10.7. The second assertion is by Theorem 10.3.9 and 10.3, and the last by Proposition 10.3.7 and Remark 5.2.7.

Here is an analog to Proposition 5.4.8: For \( X \subseteq \mathfrak{R}^{(n)} \) we define the restriction map \( \phi_X : \mathfrak{R}(\Lambda) \to \mathfrak{R}(X) \) by \( f \mapsto f|_X \).
Proposition 10.3.11. For any 1*-set $X = 1_{\text{loc}}(S) \subseteq \mathcal{R}^n$ with $S \subseteq \mathcal{R}(\Lambda)$, $\phi_X$ is a homomorphism and

$$10.5 \quad \mathcal{R}(\Lambda)/K_S \cong \mathcal{R}[X].$$

where $K_S = \mathcal{S}^{\perp \perp} \cap \mathcal{R}(\Lambda)$ with $\mathcal{S}^{\perp \perp}$ is taken in the completion $\bar{\mathcal{R}}(\Lambda)$ of $\mathcal{R}(\Lambda)$ in $\text{Fun}(\mathcal{R}^n, \mathcal{R})$.

Proof. Corollary [10.3.10] implies that

$$\text{Kern}(\phi_X) = \{ g \in \mathcal{R}(\Lambda) : g|_X = 1 \} = \{ g \in \mathcal{R}(\Lambda) : g \in \mathcal{S}^{\perp \perp} \} = K_S.$$

We conclude with the isomorphism theorems [11.1.9].

10.3.1. The restriction of the coordinate $\nu$-semifield.

Corollary 10.3.12. For any (principal) 1*-set $X = 1_{\text{loc}}(f) \subseteq \mathcal{R}^n$, the restriction map of $\phi_X$ to $\mathcal{R}$ satisfies

$$10.6 \quad \langle (\mathcal{S})/((\mathcal{S}) \cap \langle \mathcal{R} \rangle) \rangle \cong \mathcal{R}[X].$$

Proof. By Proposition 10.4.3, $\text{Kern}(\phi_X) = \{ g \in \langle \mathcal{S} \rangle : g|_X = 1 \} = \{ g \in \langle \mathcal{S} \rangle : g \in \langle f \rangle \} = \langle f \rangle \cap \langle \mathcal{R} \rangle$, so we conclude with the isomorphism theorems [11.1.9].

11. Dimension

Throughout, $F$ denotes a divisible, $\nu$-archimedean $\nu$-semifield$^\dagger$. Having the HS kernels at our disposal, our next major goal is to use them to define dimension, which will be seen to be uniquely defined. We return to the notation of Construction [9.7.1] as indicated after Remark [9.7.2].

Let $(f) \subseteq \langle F \rangle$ be a principal kernel and let $(f) = \bigcap_{i=1}^s K_i$, where

$$K_i = (L_i \cdot R_i) \cap \langle f \rangle = (L_i \cap \langle f \rangle) \cdot (R_i \cap \langle f \rangle) = L_i' \cdot R_i'$$

is its (full) HO-decomposition; i.e., for each $1 \leq i \leq s$, $R_i \in \mathcal{P}(F)$ is a region kernel and $L_i \in \mathcal{P}(F)$ is either an HS-kernel or bounded from below (in which case $L_i' = \langle f \rangle$). Then by Corollary 10.4.8, we have the subdirect decomposition

$$\langle f \rangle/\langle f \rangle \hookrightarrow \prod_{i=1}^t \langle f \rangle/K_i = \prod_{i=1}^t \langle f \rangle/L_i' \cdot R_i'$$

where $t \leq s$ is the number of kernels $K_i$ for which $L_i' \neq \langle f \rangle$ (for otherwise $\langle f \rangle/K_i = \{ 1 \}$ and can be omitted from the subdirect product).

Example 11.0.13. Consider the principal kernel $\langle \lambda_1 \rangle \in \mathcal{P}(F(x, \lambda_2))$. For $\alpha \in F$ such that $\alpha > 1$, we have the following infinite strictly descending chain of principal kernels

$$\langle \lambda_1 \rangle \supset \langle \lambda_1 + |\lambda_2 + 1| \rangle \supset \langle \lambda_1 + |\alpha^{-2} \lambda_2 + 1| \rangle \supset \langle \lambda_1 + |\alpha^{-3} \lambda_2 + 1| \rangle \supset \ldots$$

and the strictly ascending chain of 1*-sets corresponding to it.

1-loc($\lambda_1\rangle \subset 1-loc(\lambda_1 + |\lambda_2 + 1|) \subset \cdots \subset 1-loc(\lambda_1 + |\alpha^{-k} \lambda_2 + 1|) \subset \cdots$ =

1-loc($\lambda_1\rangle \subset 1-loc(\lambda_1) \cap 1-loc(\lambda_2 + 1) \subset \cdots \subset 1-loc(\lambda_1) \cap 1-loc(\alpha^{-k} \lambda_2 + 1) \subset \cdots$.

Example 11.0.14. Again, consider the principal kernel $\langle x \rangle \in \mathcal{P}(F(x, y))$. Then

$$\langle x \rangle = \langle \|x| + (|y| + 1) \cdot \|\frac{1}{y} \rangle + 1) \rangle = \langle \|x| + (|y| + 1) \cdot \|\frac{1}{y} \rangle + 1) \rangle.$$

So, we have the nontrivial decomposition of 1-loc($x$) as 1-loc($\|x| + |y| + 1) \cdot |1-loc(\|\frac{1}{y} \rangle + 1) \rangle$ (note that 1-loc($\|x| + |y| + 1) = 1-loc(x) \cap 1-loc(y+1)$). Furthermore, 1-loc($\|x| + |\frac{1}{y} \rangle + 1 = 1-loc(x) \cap 1-loc(\|\frac{1}{y} \rangle + 1) \rangle$). In a similar way, using complementary order kernels, one can show that every principal kernel can be nontrivially decomposed to a pair of principal kernels.

11.1. Irreducible kernels.

Examples [11.0.13] and [11.0.14] demonstrate that the lattice of principal kernels $\mathcal{P}(F(\Lambda))$ (resp. $\mathcal{P}(\mathcal{F}(\Lambda))$) is too rich to define reducibility or finite dimension. (See [1] for a discussion of infinite dimension.) Moreover, these examples suggest that this richness is caused by order kernels. This motivates us to consider $\Theta$-reducibility for a suitable sublattice of kernels $\Theta \subset \mathcal{P}(F(\Lambda))$ (resp. $\Theta \subset \mathcal{P}(\mathcal{F}(\Lambda))$).

There are various families of kernels that could be utilized to define the notions of reducibility, dimensionality, and so forth. We take $\Theta$ to be the sublattice generated by HP-kernels, because of its connection to the (local) dimension of the linear spaces (in logarithmic scale) defined by the 1*-set corresponding to a kernel. Namely, HP-kernels, and more generally HS-kernels, define affine subspaces of $F^n$ (see [21 §9.2]). We work with Definition 9.1.7.

Definition 11.1.1. A kernel $(f) \in \Omega(F(\Lambda))$ is reducible if there are $(g), (h) \in \Omega(F(\Lambda))$ for which $(g), (h) \not\subseteq (f)$ but $(g) \cap (h) \subseteq (f)$.

Lemma 11.1.2. $(f)$ is reducible ifff $(f) = (g) \cap (h)$ where $(g) \neq (f)$ and $(f) \neq (h)$. 
Proof. Assume \((f)\) admits the stated condition. If \((f) \supseteq (g) \cap (h)\), then \((f) = (f) \cdot (g) = ((g) \cdot (f)) \cap ((h) \cdot (f))\). Thus \((f) = (g) \cdot (f)\) or \((f) = (h) \cdot (f)\), implying \((f) \supseteq (g)\) or \((f) \supseteq (h)\). The converse is obvious. \(\Box\)

Lemma 11.1.3. Let \((f)\) be an HP-kernel. Then for any HP-kernels \((g)\) and \((h)\) such that \((f) = (g) \cap (h)\) either \((f) = (g)\) or \((f) = (h)\) in \(\mathcal{L}\). In other words, every HP-kernel is irreducible.

Proof. If \((f) = (g) \cap (h)\) then \((f) \subseteq (g)\) and \((f) \subseteq (h)\). As both \(f\) and \(g\) are \(\mathcal{L}\)-monomials (up to equivalence), Lemma 11.2 yields \((f) = (g)\), which in turn, by Lemma 11.2 implies that \((f)\) is irreducible. \(\Box\)

Corollary 11.1.4. Any HS-kernel \((f)\) is irreducible.

Proof. If \((f) = (g) \cap (h)\) for HP-kernels \((g)\) and \((h)\), then \((f) \subseteq (g)\) and \((f) \subseteq (h)\). But \((f)\) is a product \((f_1) \cdots (f_t)\) of finitely many HP-kernels. For each \(1 \leq j \leq t\), \((f_j) \subseteq (g)\) yielding \((g) = (f_j)\) by Lemma 11.2 and so \((f) = (g)\).

Corollary 11.1.5. If \((f) = (g) \cap (h)\) for HS-kernels \((g)\) and \((h)\), then either \((f) = (g)\) or \((f) = (h)\).

Proof. Otherwise, since \((g)\) and \((h)\) are finite products of HP-kernels, there are HP-kernel \((g') \subseteq (g)\) and \((h') \subseteq (h)\) such that \((g') \not\subseteq (f)\) and \((h') \not\subseteq (f)\). But \((g') \cap (h') \subseteq (g) \cap (h) = (f)\), implying \((f)\) is reducible, contradicting Corollary 11.1.4.

Proposition 11.1.6. The irreducible kernels in the lattice generated by HP-kernels are precisely the HS-kernels.

Proof. This follows from Corollary 11.1.4 since all proper intersections in the lattice generated by HP-kernels are reducible. (Note that HP-kernels are also HS-kernels.) \(\Box\)

Definition 11.1.7. The hyperspace spectrum of \(F(\Lambda)\), denoted \(HSpec(F(\Lambda))\), is the family of irreducible kernels in \(\Omega(F(\Lambda))\).

Corollary 11.1.9. \(HSpec(F(\Lambda))\) is the family of HS-kernels in \(\Omega(F(\Lambda))\), which is precisely the family of HS-fractions of \(F(\Lambda)\).

Definition 11.1.9. A chain \(P_0 \subset P_1 \subset \cdots \subset P_t\) in \(HSpec(F(\Lambda))\) of HS-kernels of \(F(\Lambda)\) is said to have length \(t\). An HS-kernel \(P\) has height \(t\) (denoted \(\text{hgt}(P) = t\)) if there is a chain of length \(t\) in \(HSpec(F(\Lambda))\) terminating at \(P\), but no chain of length \(t + 1\) terminates at \(P\).

Remark 11.1.10. Let \(L\) be a kernel in \(\mathcal{P}(F(\Lambda))\). Consider the canonical homomorphism \(\phi_a : F(\Lambda) \to F(\Lambda)/L\). Since the image of a principal kernel is generated by the image of any of its generators, \(\phi_a(a) = (\phi_a(f))\) for any HP-kernel \((f)\). Choosing \(f\) to be an \(\mathcal{L}\)-monomial, \((\phi_a(f))\) is a nontrivial HP-kernel in \(F(\Lambda)/L\) if and only if \(\phi_a(f) \not\in L\). Thus, the set of HP-kernels of \(F(\Lambda)\) mapped to HP-kernels of \(F(\Lambda)/L\) is \(\{\langle g\rangle : \langle g\rangle \cdot \langle F\rangle \supseteq \phi_a^{-1}(L) \cdot \langle F\rangle\}\). As \(\phi_a\) is an \(F\)-homomorphism, it respects \(\vee, \wedge\) and \(|\cdot|\), and thus \(\phi_a((\Omega(F(\Lambda)) \cap, \cdot)) = (\Omega(F(\Lambda)/L), \cap, \cdot)\). In fact Theorem 11.1.7 yields a correspondence identifying \(HSpec(F(\Lambda)/L)\) with the subset of \(HSpec(F(\Lambda))\) which consists of all HS-kernels \(P\) of \(F(\Lambda)\) such that \(P \cdot \langle F\rangle \supseteq L \cdot \langle F\rangle\).

Lemma 11.1.11. The above correspondence extends to a correspondence identifying \(\Omega(F(\Lambda)/L)\) with the subset \(\Omega(F(\Lambda))\). Under this correspondence, the maximal HS-kernels of \(F(\Lambda)/L\) correspond to maximal HS-kernels of \(F(\Lambda)\), and reducible kernels of \(F(\Lambda)/L\) correspond to reducible kernels of \(F(\Lambda)\).

Proof. The latter assertion is obvious since \(\wedge\) is preserved under homomorphisms. For the first assertion, \((F(\Lambda)/L)/(P/L) \cong F(\Lambda)/P\) by Theorem 11.1.9 so simplicity of the quotients is preserved. Hence, so is maximality of \(P/L\) and \(P\). \(\Box\)

Definition 11.1.12. The Hyperdimension of \(F(\Lambda)\), written \(\text{Hd}(F(\Lambda))\) (if it exists), is the maximal height of the HS-kernels in \(F(\Lambda)\).

11.2. Decompositions.

Let us garner some information about reducible kernels, from rational functions. Suppose \(f \in F(\Lambda)\). We write \(f = \sum_{i=1}^k f_i\) where each \(f_i\) is of the form \(g_i h_i^*\) with \(g_i, h_i \in F[\Lambda]\) and \(g_i\) a monomial. (Thus \(f_i(a) = \frac{g_i(a)}{h_i(a)}\) when \(h_i(a)\) is tangible.) We also assume that this sum is irredundant in the sense that we cannot remove any of the summands and still get \(f\). If each time the value \(1\) is attained by one of the terms \(f_i\) in this expansion and all other terms attain values \(\leq 1\), then \(\hat{f} = \bigwedge_{i=1}^k [f_i]\) defines the same \(1^*\)-set as \(f\). Moreover, if \(f \in (F)\) then \(\hat{f} \wedge |\alpha| \in (F)\), for \(\alpha \in F \setminus \{1\}\) is also a generator of \((f)\). The reason we take \(\hat{f} \wedge |\alpha|\) is that we have no guarantee that each of the \(f_i\)'s in the above expansion is bounded.

We can generalize this idea as follows:

We call \(f \in F(\Lambda)\) reducible if we can write \(f = \sum_{i=1}^k f_i\) as above, such that for every \(1 \leq i \leq k\) the following condition holds:

\[ f_i(a) \cong a, \quad f_j(a) \leq a, \quad \forall j \neq i. \]
Definition 11.2.1. Let $f \in F(\Lambda)$. A $\Theta$-decomposition of $f$ is an expression of the form

$$|f| = |u| \wedge |v|$$

with $u, v$ $\Theta$-elements in $F(\Lambda)$.

The decomposition is said to be trivial if $f \sim_{(F)} u$ or $f \sim_{(F)} v$ (equivalently $|f| \sim_{(F)} |u|$ or $|f| \sim_{(F)} |v|$). Otherwise, the decomposition is said to be nontrivial.

Lemma 11.2.2. Suppose $f \in F(\Lambda)$ is a $\Theta$-element. Then $\langle f \rangle$ is reducible if and only if there exists some generator $f'$ of $f$ that has a nontrivial $\Theta$-decomposition.

Proof. If $\langle f \rangle$ is reducible, then there exist kernels $\langle u \rangle$ and $\langle v \rangle$ in $\Theta$ such that $\langle f \rangle = \langle u \rangle \cap \langle v \rangle$ where $\langle f \rangle \neq \langle u \rangle$ and $\langle f \rangle \neq \langle v \rangle$. Since $\langle u \rangle \cap \langle v \rangle = |u| \wedge |v|$ we have the nontrivial $\Theta$-decomposition $f' = |u| \wedge |v|$ (which is a generator of $\langle f \rangle$).

Conversely, assume that $f' = |u| \wedge |v|$ is a nontrivial $\Theta$-decomposition for some $f' \sim_{(F)} f$. Then $\langle f \rangle = \langle f' \rangle = \langle u \rangle \cap \langle v \rangle$. Since the decomposition $f' = |u| \wedge |v|$ is nontrivial, we have that $u \not\sim_{(F)} f'$ and $v \not\sim_{(F)} f'$, and thus $\langle u \rangle = \langle f' \rangle = \langle f \rangle$. Similarly, $\langle v \rangle \neq \langle f \rangle$. Thus, by definition, $\langle f \rangle$ is reducible.

We can equivalently rephrase Lemma 11.2.2 as follows:

Remark 11.2.3. $f$ is reducible if and only if some $f' \sim_{(F)} f$ has a nontrivial $\Theta$-decomposition.

A question immediately arising from Definition 11.2.1 and Lemma 11.2.2 is:

If $f \in F(\Lambda)$ has a nontrivial $\Theta$-decomposition and $g \not\sim_{(F)} f$, does $g$ also have a nontrivial $\Theta$-decomposition? If so, how is this pair of decompositions related?

In the next few paragraphs we provide an answer to both of these questions, for $\Theta = P(\mathcal{A})$.

Remark 11.2.4. \(\sum_{i=1}^{k} s_i(a_i \wedge b_i) d(i) = \left(\sum_{i=1}^{k} s_i a_i^{d(i)}\right) \wedge \left(\sum_{i=1}^{k} s_i b_i^{d(i)}\right), \forall s_1, \ldots, s_k, a_1, \ldots, a_k, b_1, \ldots, b_k \in F(\Lambda), \) and $d(i) \in \mathbb{N}_{\geq 0}$.

Remark 11.2.5. If $h_1, \ldots, h_k \in F(\Lambda)$ such that each $h_i \geq 1$, then $\sum_{i=1}^{k} s_i h_i \geq 1$ for every $s_1, \ldots, s_k \in F(\Lambda)$ such that $\sum_{i=1}^{k} s_i \geq 1$.

Theorem 11.2.6. (For $\Theta = P(\mathcal{A})$.) If $\langle f \rangle$ is a (principal) reducible kernel, then there exist $\Theta$-elements $g, h \in F(\Lambda)$ such that $|f| = |g| \wedge |h|$ and $|f| \not\sim_{(F)} |g|, |h|$.

Proof. If $\langle f \rangle$ is a principal reducible kernel, then there exists $f' \sim_{(F)} f$ such that $f' = |u| \wedge |v| = \min(|u|, |v|)$ for $\Theta$-elements $u, v \in \mathcal{A}$ with $f' \not\sim_{(F)} |u|, |v|$. Then $\langle f \rangle \in \langle f' \rangle$ since $f'$ is a generator of $f$, so there exist $s_1, \ldots, s_k \in F(\Lambda)$ such that $\sum_{i=1}^{k} s_i = 1$ and $|f| = \sum_{i=1}^{k} s_i f'i^{d(i)}$ with $d(i) \in \mathbb{N}_{\geq 0}$.

Thus

$$f = \sum_{i=1}^{k} s_i (|u| \wedge |v|) d(i) = \sum_{i=1}^{k} s_i (|u|) d(i) = \min\left(\sum_{i=1}^{k} s_i |u|^{d(i)}, \sum_{i=1}^{k} s_i |v|^{d(i)}\right) = |g| \wedge |h|$$

where $g = |g| = \sum_{i=1}^{k} s_i |u|^{d(i)}$, $h = |h| = \sum_{i=1}^{k} s_i |v|^{d(i)}$.

Now $\langle |f| \rangle \subseteq \langle |g| \rangle \subseteq \langle |u| \rangle$ and $\langle |f| \rangle \subseteq \langle |h| \rangle \subseteq \langle |v| \rangle$; implying $1_{\text{loc}}(f) \supseteq 1_{\text{loc}}(g) \supseteq 1_{\text{loc}}(u)$ and $1_{\text{loc}}(f) \supseteq 1_{\text{loc}}(h) \supseteq 1_{\text{loc}}(v)$.

We claim that $|g|$ and $|h|$ generate $\langle |u| \rangle$ and $\langle |v| \rangle$, respectively. Indeed, $1_{\text{loc}}(f') = 1_{\text{loc}}(|f|)$, since $f' \sim_{(F)} |f|$ and thus for any $a \in F(\Lambda)$, $f'(a) = 1 \Leftrightarrow |f|(a) = 1$. Let $s_j f'(a)^{d(i)}$ be a dominant term of $|f|$ at $a$, i.e.,

$$|f| \cong_{\nu} \sum_{k=1}^{\nu} s_j(a)(f'(a)^{d(i)} \cong_{\nu} s_j(a)(f'(a)^{d(i)}.$$}

Then $f(a) \cong_{\nu} 1 \Leftrightarrow s_j(a)(f'(a)^{d(i)} \cong_{\nu} 1$. If $f'(a) \cong_{\nu} 1$, then $(f'(a)^{d(i)} \cong_{\nu} 1$, so $s_j(a) \cong_{\nu} 1$. Now, for $a \in 1_{\text{loc}}(g)$.

Then we have

$$g(a) \cong_{\nu} \sum_{i=1}^{k} s_i |u| d(i) \cong_{\nu} 1.$$}

Let $s_i |u| d(i)$ be a dominant term of $g$ at $a$. If $s_i(a) \cong_{\nu} 1$ then $|u| d(i) \cong_{\nu} 1$ and thus $u = 1$, and $a = 1_{\text{loc}}(u)$. Otherwise $s_i(a) \leq 1$ (since $\sum_{i=1}^{k} s_i \cong 1$) and so, by the above, $s_i(f'(a)^{d(i)}$ is not a dominant term of $|f|$ at $a$. Thus for every index $j$ of a dominant term of $|f|$ at $a$, we have $j \neq t$ and

$$|u(a)|^{d(i)} \cong_{\nu} s_j(a)|u(a)|^{d(i)} \leq s_j(a)|u(a)|^{d(i)} \leq |u(a)|^{d(i)} \leq s_j(a)|u(a)|^{d(i)}.$$}

On the other hand, $f'(a) \cong_{\nu} 1$ since $1_{\text{loc}}(f) \supseteq 1_{\text{loc}}(g)$, implying $s_j(a)(f'(a)^{d(i)} \cong_{\nu} 1$, contradicting (11.3). Hence, $1_{\text{loc}}(g) \subseteq 1_{\text{loc}}(u)$, yielding $1_{\text{loc}}(g) = 1_{\text{loc}}(u)$, which implies that $g$ is a generator of $\langle |u| \rangle = \langle u \rangle$. The proofs for $h$ and $|v|$ are analogous.

Consequently, $g \sim_{(F)} |g| \sim_{(F)} |u|$ and $h \sim_{(F)} |h| \sim_{(F)} |v|$. Since $|f| \sim_{(F)} f' \not\sim_{(F)} |u|, |v|$ we conclude that $|f| \not\sim_{(F)} |g|, |h|$. \(\square\)
Corollary 11.2.7. For \( f \in (F) \), if \( |f| = \Lambda_{i=1}^{s} |f_i| \) for \( f_i \in (F) \), then for any \( g \sim_{(F)} f \), we have \( |g| = \Lambda_{i=1}^{s} |g_i| \), with \( g_i \sim_{(F)} f_i \) for \( i = 1, \ldots, s \).

Proof. Iterate Theorem 11.2.6.

Corollary 11.2.8. If \( (f) \) is a kernel in \( \Theta \), then \( (f) \) has a nontrivial decomposition \( (f) = (g) \cap (h) \) if and only if \( |f| \) has a nontrivial decomposition \( |f| = |g'| \wedge |h'| \) with \( |g'| \sim_{(F)} g \) and \( |h'| \sim_{(F)} h \).

Proof. If \( |f| = |g'| \wedge |h'| \) then, since \( |g'| \sim_{(F)} g \) and \( |h'| \sim_{(F)} h \) we have
\[
(f) = (|f|) = (|g'| \wedge |h'|) = (|g'|) \cap (|h'|) = (g) \cap (h).
\]
The converse is seen as in the proof of Theorem 11.2.6.

Remark 11.2.9. The righthand side of the equality, e.g., \( |g'| \sim (g) \) or \( |h'| \sim (h) \) since \( |g'| \sim (g) \) and \( |h'| \sim (h) \) if \( |g'| \sim (g) \). This motivates us to restrict the convex dependence relation to the set of \( \Theta \)-elements.

Corollary 11.2.10. Let \( S \) be a semifield and let \( a, b \in S \). We say that \( a \) and \( b \) are \( (F) \)-comparable if there exist some \( a' \sim_{(F)} a \) and \( b' \sim_{(F)} b \) such that \( |a'| \leq |b'| \) or \( |b'| \leq |a'| \).

Since \( |g| \wedge |h| = \min(|g|, |h|) \) we can utilize Remark 11.2.10 to get the following observation:

Proposition 11.2.11. A \( \Theta \)-decomposition \( f \sim_{(F)} (g) \wedge (h) \in F(\Lambda) \) is nontrivial if and only if the \( \Theta \)-elements \( g \) and \( h \) are not \( (F) \)-comparable.

Proof. If \( g \) and \( h \) are \( (F) \)-comparable, then there exist some \( g' \sim_{(F)} g \) and \( h' \sim_{(F)} h \) such that \( |g'| \geq |h'| \) or \( |h'| \geq |g'| \).

Without loss of generality, assume that \( |g'| \geq |h'| \). Then \( |g| \wedge |h| = (|g|) \cap (|h|) = (|g'|) \cap (|h'|) = (|g'| \wedge |h'|) \).

Thus \( (f) = (g) \) so \( f \sim_{(F)} g \) yielding that the decomposition is trivial.

Conversely, if \( g \) and \( h \) are not \( (F) \)-comparable then we claim that \( f \not\sim_{(F)} g \) and \( f \not\sim_{(F)} h \). We must show that \( (g) \not\subseteq (f) \) and \( (h) \not\subseteq (f) \) respectively. So assume that \( (h) \supseteq (g) \). In view of Lemma 9.2.1 and [24, Proposition 4.1.13], there exists some \( f' \sim_{(F)} f \) such that \( f' = |h|^k \wedge g \). Note that \( |h|^k \geq 1 \) and \( g \geq 1 \) so \( f' = |h|^k \wedge g \geq 1 \) and thus \( |f'| = f' \).

Finally,
\[
|f'| = |h|^k \wedge g \Leftrightarrow |f'| \leq |h|^k \Leftrightarrow |f'| \in (|h|) \Leftrightarrow f' \in (h) \Leftrightarrow f \in (h).
\]

13. Convex dependence.

Definition 13.1. An HS-fraction \( f \) of \( F(\Lambda) \) is \( F \)-convexly dependent on a set \( A \) of HS-fractions if
\[
(1.4) \quad f \in \{(g : g \in A) \} \cdot (F);
\]
otherwise \( f \) is said to be \( F \)-convexly independent of \( A \). The set \( A \) is said to be \( F \)-convexly independent if \( f \) is \( F \)-convexly independent of \( A \setminus \{f\} \), for every \( f \in A \). If \( \{a_1, \ldots, a_n\} \) is \( F \)-convexly dependent, then we also say that \( a_1, \ldots, a_n \) are \( F \)-convexly dependent.

Note that under the assumption that \( g \in (F) \setminus \{1\} \) for some \( g \in A \), the condition in (1.4) simplifies to \( f \in \{(g : g \in A) \} \).

Remark 11.3.2. By definition, an HS-fraction \( f \) is \( F \)-convexly dependent on HS-fractions \( \{g_1, \ldots, g_t\} \) if and only if
\[
(1.1) \quad \langle |f| \rangle = \langle g_1, \ldots, g_t \rangle \cdot (F) = \left( \sum_{i=1}^{t} |g_i| \right) \cdot (F) = \left( \sum_{i=1}^{t} |g_i| + |\alpha| \right),
\]

for any element \( \alpha \) of \( F \) for which \( \alpha^r \neq 1^r \).

Example 11.3.3. For any \( \alpha \in F \) and any \( f \in F(\Lambda) \),
\[
|\alpha f| \leq |f|^2 + |\alpha|^2 = (|f| + |\alpha|)^2.
\]

Thus \( \alpha f \in \langle (|f| + |\alpha|)^2 \rangle = \langle |f| + |\alpha| \rangle = \langle f \rangle \cdot (F) \). In particular, if \( f \) is an HS-fraction, then \( \alpha f \) is \( F \)-convexly dependent on \( f \).

As a consequence of Lemma 9.2.1 if two \( \mathcal{Z} \)-monomials \( f, g \), satisfy \( g \in (f) \), then \( g = (f) \). In other words, either \( g = (f) \) or \( g \not\subseteq (f) \) and \( (f) \not\subseteq (g) \). This motivates us to restrict the convex dependence relation to the set of \( \mathcal{Z} \)-monomials. This will be justified later by showing that for each \( F \)-convexly independent subset of HS-fractions of order \( t \) in \( F(\Lambda) \), there exists an \( F \)-convexly independent subset of \( \mathcal{Z} \)-monomials having order \( t \) in \( F(\Lambda) \). Let us see that convex-dependence is an abstract dependence relation.
Proposition 11.3.4. Let $A, A_i \subseteq F(\Lambda)$ be sets of HS-fractions, and let $f$ be an HS-fraction.

1. If $f \in A$, then $f$ is $F$-convexly dependent on $A$.
2. If $f$ is $F$-convexly dependent on $A$ and each $a \in A$ is $F$-convexly dependent on $A_i$, then $f$ is $F$-convexly dependent on $A_i$.
3. If $f$ is $F$-convexly dependent on $A$, then $f$ is $F$-convexly dependent on $A_0$ for some finite subset $A_0 \subseteq A$.

Proof. (1) $f \in (A) \subseteq (A) \cdot (F)$.

(2) $(A) \subseteq (A_i) \cdot (F)$ since $a$ is convexly dependent on $A_i$ for each $a \in A$. If $f$ is $F$-convexly dependent on $A$, then $f \in (A) \cdot (F) \subseteq (A_i) \cdot (F)$, so $f$ is $F$-convexly dependent on $A_i$.

(3) $a \in (A) \cdot (F)$, so by Proposition 11.3.4 there exist some $s_1, \ldots, s_k \in F(\Lambda)$ and $g_1, \ldots, g_k \in G(A \cup F) \subseteq (A) \cdot (F)$, where $G(A \cup F)$ is the group generated by $A \cup F$, such that $\sum_{i=1}^{k} s_i = 1$ and $a = \sum_{i=1}^{k} s_i g_i^{d(i)}$ with $d(i) \in Z$. Thus $a \in (g_1, \ldots, g_k)$ and $A_0 = \{g_1, \ldots, g_k\}$.

From now on, we assume that the $\nu$-semifield $F$ is divisible.

Proposition 11.3.5 (Steinitz exchange axiom). Let $S = \{b_1, \ldots, b_l\} \subseteq HP(F(\Lambda))$ and let $f$ and $b$ be elements of $HP(F(\Lambda))$. If $f$ is $F$-convexly dependent on $S \cup \{b\}$ and $f$ is $F$-convexly independent of $S$, then $b$ is $F$-convexly dependent on $S \cup \{f\}$.

Proof. We may assume that $\alpha \in S$ for some $\alpha \in F$. Since $f$ is $F$-convexly independent of $S$, by definition $f \not\in (S)$ this implies that $(S) \subseteq (S) \cdot (f)$ (for otherwise $(f) \subseteq (S)$ yielding that $f$ is $F$-convexly dependent on $S$). Since $f$ is $F$-convexly dependent on $S \cup \{b\}$, we have that $f \in (S \cup \{b\}) = (S) \cdot (b)$. In particular, we get that $b \not\in (S) \cdot (F)$ for otherwise $f$ would be dependent on $S$. Consider the quotient map $\phi : F(\Lambda) \to F(\Lambda)/(S)$. Since $\phi$ is a semifield epimorphism and $f, b \not\in (S) \cdot (F) = \phi^{-1}(F)$, we have that $\phi(f)$ and $\phi(b)$ are not in $F$ thus are $\mathcal{L}$-monomials in the semifield $\text{Im}(\phi) = F(\Lambda)/(S)$. By the above, $\phi(f) \neq 1$ and $\phi(b) \neq 1$ (since $\phi(b) = \phi(b)$). Thus, $\phi(f) = \phi(b)$ by Lemma 11.3.6. So $(S) \cdot (f) = \phi^{-1}(\phi(f)) = \phi^{-1}(\phi(b)) = (S) \cdot (b)$, consequently $b \in (S) \cdot (f) = (S) \cdot (f) = (S \cup \{f\})$, i.e., $b$ is $F$-convexly dependent on $S \cup \{f\}$.

Definition 11.3.6. Let $A \subseteq HP(F(\Lambda))$. The convex span of $A$ over $F$ is the set
$$\text{Conv}(A) = \{a \in HP(F(\Lambda)) : a \text{ is } F\text{-convexly dependent on } A\}.$$ For a semifield $F \subseteq F(\Lambda)$ such that $F \subseteq K$, a set $A \subseteq HP(F(\Lambda))$ is said to convexly span $K$ over $F$ if $\text{HP}(K) = \text{Conv}_F(A)$.

Remark 11.3.7. $\text{Conv}(\{f_1, \ldots, f_m\}) = \langle f_1, \ldots, f_m \rangle \cdot (F)$.

In view of Propositions 11.3.4 and 11.3.5 convex dependence on $HP(F(\Lambda))$ is an abstract dependence relation. Then by [28] Chapter 6, we have:

Corollary 11.3.8. Let $V \subseteq HP(F(\Lambda))$. Then any set convexly spanning $V$ contains a basis $B_V \subseteq V$, which is a maximal convexly independent subset of unique cardinality such that $\text{Conv}(B_V) = \text{Conv}(V)$.

Example 11.3.9. By Lemma 9.1.4 the maximal kernels in $\mathcal{P}(F(\Lambda))$ are HS-fractions of the form $L_{(\alpha_1, \ldots, \alpha_n)} = (\alpha_1 x_1, \ldots, \alpha_n x_n)$ for any $\alpha_1, \ldots, \alpha_n \in F$. In view of Corollary 11.3.5
$$F(\Lambda) = \text{Conv}(\{\alpha_1 x_1, \ldots, \alpha_n x_n\}),$$
i.e., $\{\alpha_1 x_1, \ldots, \alpha_n x_n\}$ convexly spans $F(\Lambda)$ over $F$. Now $\alpha x_k \not\in \cup_{j \neq k} \alpha_j x_j \cdot (F)$, since there are no order relations between $\alpha x_i$ and the elements of $\{\alpha x_j : j \neq i\} \cup \{\alpha : \alpha \in F\}$. Thus, for arbitrary $\alpha_1, \ldots, \alpha_n \in F$, $\{\alpha_1 x_1, \ldots, \alpha_n x_n\}$ is $F$-convexly independent, constituting a basis for $F(\Lambda)$.

Definition 11.3.10. Let $V \subseteq HP(F(\Lambda))$ be a set of $\mathcal{L}$-monomials. We define the convex dimension of $V$, $d_{\text{conv}}(V)$, to be $|B|$ where $B$ is a basis for $V$.

Example 11.3.11. $d_{\text{conv}}(F(\Lambda)) = n$, by Example 11.3.9.

Remark 11.3.12. If $S \subseteq HP(F(\Lambda))$, then for any $f, g \in F(\Lambda)$ such that $f, g \in \text{Conv}(S)$
$$|f| + |g| \in \text{Conv}(S) \text{ and } |f| \land |g| \in \text{Conv}(S).$$

Proof. First we prove that $|f| + |g| \in \text{Conv}(S)$. Since $(f) \subseteq (S) \cdot (F)$ and $(g) \subseteq (S) \cdot (F)$, we have $(f, g) = (|f| + |g|) = (f) \cdot (g) \subseteq (S) \cdot (F)$. $|f| \land |g| \in \text{Conv}(S)$, since $(|f| \land |g|) = (f) \cap (g) \subseteq (S) \cdot (F)$.

Remark 11.3.13. If $K$ is an HS-kernel, then $K$ is generated by an HS-fraction $f \in F(\Lambda)$ of the form $f = \sum_{i=1}^{k} |f_i|$ where $f_1, \ldots, f_k$ are $\mathcal{L}$-monomials. So,
$$\text{Conv}(K) = \langle f \rangle \cdot K = \langle f \rangle \cdot (f) \cdot \left(\sum_{i=1}^{k} |f_i| \right) = \langle f \rangle \cdot \prod_{i=1}^{k} \langle f_i \rangle$$
and so, $\{f_1, \ldots, f_k\}$ convexly spans $\langle f \rangle \cdot K$. 
Lemma 11.3.15. Suppose \( \{b_1, \ldots, b_m\} \) is a set of HS-fractions, such that \( b_1 \sim (F) \sum_{j=1}^{t_1} |f_{i,j}| \), where \( f_{i,j} \) are \( \mathcal{L} \)-monomials. Then \( b_1 \) is \( F \)-convexly dependent on \( \{b_2, \ldots, b_m\} \) if and only if all of its summands \( f_{1,r} \) for \( 1 \leq r \leq t_1 \) are \( F \)-convexly dependent on \( \{b_2, \ldots, b_m\} \).

Proof. If \( b_1 \) is \( F \)-convexly dependent on \( \{b_2, \ldots, b_m\} \), then

\[
\prod_{j=1}^{t_1} (f_{1,j}) = \left\langle \sum_{j=1}^{t_1} |f_{1,j}| \right\rangle = \left\langle b_1 \right\rangle \subseteq \left\langle \{b_1, \ldots, b_m\} \right\rangle.
\]

Hence \( f_{1,r} \in \prod_{j=1}^{t_1} (f_{1,j}) \) is \( F \)-convexly dependent on \( \{b_1, \ldots, b_m\} \) and by Remark 11.3.13, \( f_{1,r} \) is \( F \)-convexly dependent on \( \{f_{i,j} : 2 \leq i \leq m; 1 \leq j \leq t_1\} \). Conversely, if each \( f_{1,r} \) is \( F \)-convexly dependent on \( \{b_2, \ldots, b_m\} \) for \( 1 \leq r \leq t_1 \), then there exist some \( k_1, \ldots, k_{t_1} \) such that \( |f_{1,r}| \leq \sum_{i=2}^{m} \sum_{j=1}^{t_1} |f_{i,j}|^{k_r} \). Hence \( b_1 \) is \( F \)-convexly dependent on \( \{b_2, \ldots, b_m\} \), by Corollary 4.1.18. \( \square \)

Lemma 11.3.16. Let \( V = \{f_1, \ldots, f_m\} \) be a \( F \)-convexly independent set of HS-fractions, with \( f_1 \sim (F) \sum_{j=1}^{t_1} |f_{i,j}| \) for \( \mathcal{L} \)-monomials \( f_{i,j} \). Then there exists an \( F \)-convexly independent subset

\[ S_0 \subseteq S = \{f_{i,j} : 1 \leq i \leq m; 1 \leq j \leq t_1\} \]

such that \( |S_0| \geq |V| \) and Conv(S) = Conv(V).

Proof. By Remark 11.3.13, \( f_1 \) is dependent on the set of \( \mathcal{L} \)-monomials \( \{f_{i,j} : 1 \leq j \leq t_1\} \subseteq S \) for each \( 1 \leq i \leq m \), implying Conv(S) = Conv(V). By Corollary 4.1.18, \( S \) contains a maximal \( F \)-convexly independent subset \( S_0 \) such that Conv(S) = Conv(S) which, by Lemma 11.3.16, we can shrink down to a base. \( \square \)

Lemma 11.3.17. The following hold for an \( \mathcal{L} \)-monomial \( f \):

1. \( (F) \not\subseteq \langle f \rangle \).
2. If \( F(\Lambda) \) is not bounded, then \( \langle f \rangle \not\subseteq \langle F \rangle \).

Proof. By definition, an \( \mathcal{L} \)-monomial is not bounded from below. Thus \( \langle f \rangle \cap F = \{1\} \), yielding \( \langle f \rangle \not\subseteq \langle F \rangle \). For the second assertion, an HP-kernel is not bounded when \( F(\Lambda) \) is not bounded, so \( \langle f \rangle \not\subseteq \langle F \rangle \). \( \square \)

A direct consequence of Lemma 11.3.17 is:

Lemma 11.3.18. If \( F(\Lambda) \) is not bounded, then any nontrivial HS-kernel (i.e., \( \not\equiv (1) \)) is \( F \)-convexly independent.

Proof. By Lemma 11.3.17, the assertion is true for HP-kernels, and thus for HS-kernels, since every HS-kernel contains some HP-kernel. \( \square \)

11.4. Computing convex dimension.

Having justified our restriction to \( \mathcal{L} \)-monomials, we move ahead with computing lengths of chains.

Remark 11.4.1. Let \( K \) be an HS-kernel of \( F(\Lambda) \). By definition there are \( \mathcal{L} \)-monomials \( f_1, \ldots, f_t \in HP(K) \) such that \( K = (\sum_{i=1}^{t} \langle f_i \rangle) \). By Remark 11.3.13, Conv(K) is convexly spanned by \( f_1, \ldots, f_t \). Now, since Conv(K) = Conv(\( f_1, \ldots, f_t \)) and \( \{f_1, \ldots, f_t\} \) is \( HP(K) \subseteq HP(F(\Lambda)) \), by Corollary 4.1.18, \( \{f_1, \ldots, f_t\} \) contains a basis \( B = \{b_1, \ldots, b_s\} \subseteq \{f_1, \ldots, f_t\} \) of \( F \)-convexly independent elements, where \( s = d_{\text{conv}}(K) \) such that Conv(B) = Conv(\( f_1, \ldots, f_t \)) = Conv(K).

Proposition 11.4.2. For any order kernel \( o \) of \( F(\Lambda) \), if \( \mathcal{L} \)-monomials \( h_1, \ldots, h_t \) are \( F \)-convexly dependent, then the images of \( h_1, \ldots, h_t \) are \( F \)-convexly dependent in the quotient semifield \( F(\Lambda)/o \).

Proof. Denote by \( \phi_o : F(\Lambda) \rightarrow F(\Lambda)/o \) the quotient \( F \)-homomorphism. Then \( \phi_o(F) = \langle \phi_o(F) \rangle = \langle F \rangle_{F(\Lambda)/o} \).

Now, if \( h_1, \ldots, h_t \) are \( F \)-convexly dependent then there exist some \( j \), say without loss of generality \( j = 1 \), such that \( h_1 \in \langle h_2, \ldots, h_t \rangle \cdot \langle F \rangle \). By assumption and Proposition 4.1.18,

\[
\phi_o(h_1) = \phi_o(h_2, \ldots, h_t, \alpha) = \langle \phi_o(h_2), \ldots, \phi_o(h_t), \phi_o(\alpha) \rangle = \langle \phi_o(h_2), \ldots, \phi_o(h_t) \rangle \cdot \langle F \rangle
\]

Thus \( \phi_o(h_1) \) is \( F \)-convexly dependent on \( \{\phi_o(h_2), \ldots, \phi_o(h_t)\} \). \( \square \)

Conversely, we have:

Lemma 11.4.3. For any order kernel \( o \) of \( F(\Lambda) \), and any set \( \{h_1, \ldots, h_t\} \) of \( \mathcal{L} \)-monomials, if \( \phi_o(h_1), \ldots, \phi_o(h_t) \) are \( F \)-convexly dependent in the quotient semifield \( F(\Lambda)/o \) and \( \sum_{i=1}^{t} \phi_o(|h_i|) \cap F = \{1\} \), then \( h_1, \ldots, h_t \) are \( F \)-convexly dependent in \( F(\Lambda) \).
Proof. Note that $\sum_{i=1}^{t} a_i (h_i) \cap F = \{1\}$ if and only if $\bigcap_{i=1}^{t} 1\text{-loc}(h_i) \cap 1\text{-loc}(\phi) \neq \emptyset$. Translating the variables by a point $a \in \bigcap_{i=1}^{t} 1\text{-loc}(h_i) \cap 1\text{-loc}(\phi)$, we may assume that the constant coefficient of each $\mathcal{L}$-monomial $h_i$ is 1. Assume that $\phi(h_1), \ldots, \phi(h_t)$ are $F$-convexly dependent. We may assume that $\phi(h_i)$ is $F$-convexly dependent on $\phi(h_2), \ldots, \phi(h_t)$. This means by Definition 11.3 that we can take $h_{t+1} \in F$ for which $\phi(h_1) \in \langle \phi(h_2), \ldots, \phi(h_t), \phi(h_{t+1}) \rangle$. Taking the pre-images of the quotient map yields

$$\langle h_1 \rangle \cdot o \subseteq \langle h_2, \ldots, h_t, h_{t+1} \rangle \cdot o.$$ 

Take an $\mathcal{L}$-monomial $g$ such that $1 + g$ generates $o$. By Corollary 11.13 there exists some $k \in \mathbb{N}$ such that

$$h_1 + 1 + g \leq \mu (h_2 + \cdots + h_{t+1} + 1 + g)^k = |h_2|^k + \cdots + |h_{t+1}|^k + 1 + g^k. \tag{11.6}$$

As $1 + g \geq 1$ we have that $|1 + g| \mu 1 + g$, and the right hand side of Equation (11.6) equals

$$|h_2|^k + \cdots + |h_{t+1}|^k + (1 + g)^k \geq |h_2|^k + \cdots + |h_{t+1}|^k + 1 + g^k \geq \mu |h_2|^k + \cdots + |h_{t+1}|^k + g^k. \tag{11.7}$$

The last equality is due to the fact that $\sum |h_i|^k \geq 1$ so that 1 is absorbed. The same argument, applied to the left hand side of Equation (11.6), yields that

$$h_1 + g \leq \mu |h_2|^k + \cdots + |h_{t+1}|^k + g^k. \tag{11.8}$$

Assume on the contrary that $h_1$ is $F$-convexly independent of $\{h_2, \ldots, h_t\}$. Then

$$\langle h_1 \rangle \not\subseteq \langle h_2, \ldots, h_{t+1} \rangle \geq \mu \left( \sum_{i=2}^{t+1} |h_i| \right).$$

Thus for any $m \in \mathbb{N}$ there exists some $a_m \in F^n(m)$ such that

$$|h_1(a_m)| > \mu \left( \sum_{i=2}^{t+1} |h_i(a_m)| \right)^m \geq \mu \left( \sum_{i=2}^{t+1} |h_i(a_m)| \right)^m.$$ 

Thus by equation (11.8) and the last observation we get that

$$\sum_{i=2}^{t} |h_i(a_m)|^m + g(a_m) < \mu |h_1(a_m)| + g(a_m) \leq \sum_{i=2}^{t} |h_i(a_m)|^k + g(a_m)^k, \tag{11.9}$$

i.e., there exists some fixed $k \in \mathbb{N}$ such that for any $m \in \mathbb{N}$,

$$\sum_{i=2}^{t} |h_i(a_m)|^m < \mu \sum_{i=2}^{t} |h_i(a_m)|^k + g(a_m)^k. \tag{11.10}$$

For $m > k$, since $|g|^k \leq \mu |g|^m$ for any $g \in F$, we get that $\sum_{i=2}^{t} |h_i(a_m)|^m \geq \mu \sum_{i=2}^{t} |h_i(a_m)|^k$. Write

$$g^k = g(1)g^k.$$ 

Since $g^k$ is an HP-kernel, $g(1)$ is the constant coefficient of $g$ and $g^k$ is a Laurent monomial with coefficient 1.

According to the way the $a_m$ were chosen, $\sum_{i=2}^{t} |h_i(a_m)| > 1$ and $\sum_{i=2}^{t} |h_i(a_m)|^m > \mu g(a_m)^k$, and thus $g(1) < \mu g'(a_m)$ for large enough $m_0$. But $g'(a_m^{-1}) \leq \mu g'(a_m)^{-1}$ so

$$g^k(a_m^{-1}) \leq \mu g(1)g'(a_m^{-1}) \leq \mu \mu g(1)g'(a_m)^{-1} < \mu 1.$$ 

Thus (11.3) yields

$$\sum_{i=2}^{t} |h_i(a_m)^{-1}|^m \leq \mu \sum_{i=2}^{t} |h_i(a_m)^{-1}|^k,$$

a contradiction. \qed

Proposition 11.4.4. Let $R$ be a region kernel of $F(\Lambda)$. Let $\{h_1, \ldots, h_t\}$ be a set of $\mathcal{L}$-monomials such that $(R \cdot \langle h_1, \ldots, h_t \rangle) \cap F = \{1\}$. Then $h_1 \in R \cdot \langle h_2, \ldots, h_t \rangle$ are $F$-convexly dependent in the quotient semifield $F(\Lambda)/R$ if and only if $h_1, \ldots, h_t$ are $F$-convexly dependent in $F(\Lambda)$.

Proof. The ‘if’ part of the assertion follows from Proposition 11.4.2. Since $R = \prod_{i=1}^{m} o_i$ for suitable order kernels $(o_i)_{i=1}^{m}$, the ‘only if’ part follows from Lemma 11.4.3 applied repeatedly to each of these $o_i$’s. \qed

Proposition 11.4.5. Let $R \in \mathcal{P}(F(\Lambda))$ be a region kernel. Then, for any set $L$ of HS-fractions, we have

$$d_{\text{conv}}(L) = d_{\text{conv}}(L \cdot R),$$

the right side taken in $F(\Lambda)/R$.

Proof. $d_{\text{conv}}(L) \leq d_{\text{conv}}(R \cdot L)$ since $L \subseteq R \cdot L$. For the reverse inequality, let $\phi_R : F(\Lambda) \to F(\Lambda)/R$ be the quotient map. Since $L$ is a sub-semifield $F(\Lambda)$, $\phi_R^{-1}(\phi_R(L)) = R \cdot L$ by Theorem 11.1.9 and $d_{\text{conv}}(L) \geq d_{\text{conv}}(\phi_R(L))$ by Proposition 11.4.2 while $d_{\text{conv}}(\phi_R(L)) \geq d_{\text{conv}}(\phi_R(\phi_R(L)))$ by Lemma 11.4.3. Thus $d_{\text{conv}}(L) \geq d_{\text{conv}}(R \cdot L)$. \qed

In this way we see that $K \to \Omega(K)$ yields a homomorphism of kernels. Hence $\Omega$ is a natural map in the sense of Definition 2.0.2 and we can apply Theorem 2.0.3.
Remark 11.4.6. Let $R$ be a region kernel and let

$$A = \{(g) : (g) \cdot \langle F \rangle \supseteq R \cdot \langle F \rangle \}.$$  

Then $d_{\text{conv}}(F(\Lambda)/R) = d_{\text{conv}}(A)$, in view of Remark 11.4.10 and Proposition 11.4.4. As $L_a \in A$ for any $a \in 1$-loc($R$) $\neq \emptyset$ and $d_{\text{conv}}(L_a) = d_{\text{conv}}(F(\Lambda))$, we conclude that $d_{\text{conv}}(A) = d_{\text{conv}}(F(\Lambda))$.

We are ready to prove catenarity for $d_{\text{conv}}$.

Theorem 11.4.7. If $R$ is a region kernel and $L$ is an HS-kernel of $F(\Lambda)$, then

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(F(\Lambda)) - d_{\text{conv}}(L).$$

In particular,

$$d_{\text{conv}}(F(\Lambda)/LR) = n - d_{\text{conv}}(L).$$

Proof. $F(\Lambda)/LR \cong (F(\Lambda)/R)/(L \cdot R/R)$, by the third isomorphism theorem. Choose a basis for $\text{HP}(F(\Lambda)/R)$ containing a basis for $\text{HP}(L \cdot R/R)$. Then $d_{\text{conv}}(L \cdot R/R) = d_{\text{conv}}(\phi_R(L))$, by Remark 11.4.6. But $L \cdot R \cap F = \{1\}$. Hence, by Proposition 11.4.4, $d_{\text{conv}}(\phi_R(L)) = d_{\text{conv}}(L)$. So

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(A) - d_{\text{conv}}(L) = d_{\text{conv}}(F(\Lambda)) - d_{\text{conv}}(L).$$

Thus,

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(A) - d_{\text{conv}}(L) = n - d_{\text{conv}}(L).$$

\[\square\]

11.5. Recapitulation for dimensions.

In conclusion, for every principal regular kernel $(f) \in \mathcal{P}(F(\Lambda))$, we have obtained explicit region kernels

$$\{R_{t_1,1}, ..., R_{t_s,s}, R_{2,1}, ..., R_{2,t}\}$$

having trivial intersection, such that

$$\langle f \rangle = \bigcap_{i=1}^s K_i \cap \bigcap_{j=1}^t N_j$$

where $K_i = L_{t_i} \cdot R_{t_i,i}$ for $i = 1, ..., s$ and appropriate HS-kernels $L_{t_i}$ and $N_j = B_{j} \cdot R_{2,j}$ for $j = 1, ..., t$ and appropriate bounded from below kernels $B_{j}$. If $(f) \in \mathcal{P}(F(\Lambda))$, then, in view of Theorem 9.4.8, we can take $B_{j} = \langle F \rangle$ for every $j = 1, ..., t$. Note that over the various regions in $F^{(n)}$ corresponding to the region kernels $R_{t_i,j}$, $f$ is locally represented by distinct HS-fractions in $\text{HSpec}(F(\Lambda))$. In fact each region is defined so that the local HS-representation of $f$ is given over the entire region. Thus the $R_{t_i,j}$’s defining the partition of the space can be obtained as a minimal set of regions over each of which $(f)$ takes the form of an HS-kernel.

For each $j = 1, ..., t$, $d_{\text{conv}}(N_j) = \text{Hdim}(N_j) = 0$, since $N_j$ contains no elements of $\text{HP}(F(\Lambda))$, implying

$$d_{\text{conv}}(F(\Lambda)/N_j) = \text{Hdim}(F(\Lambda)/N_j) = n.$$

For each $i = 1, ..., s$, $d_{\text{conv}}(K_i) = d_{\text{conv}}(L_i) = \text{Hdim}(L_i) \geq 1$, implying

$$d_{\text{conv}}(F(\Lambda)/K_i) = \text{Hdim}(F(\Lambda)/K_i) = n - \text{Hdim}(L_i) < n.$$

Remark 11.5.1. In view of the discussion in [23] §9.2, each term $F(\Lambda)/L_i$ corresponds to the linear subspace of $F^{(n)}$ (in logarithmic scale) defined by the linear constraints endowed on the quotient $F(\Lambda)/L_i$ by the HS-kernel $L_i$. One can think of these terms as an algebraic description of the affine subspaces locally comprising 1-loc($f$).

12. The Jordan-Hölder theorem

Our final goal is to obtain a Jordan-Hölder theorem for kernels. But there are too many kernels for a viable theory in general, as discussed in [1]. If we limit our set of kernels to a given sublattice of kernels, one can use the Schreier refinement theorem [27] to obtain a version of the Jordan-Hölder Theorem.

Definition 12.0.2. $\mathcal{L}(S)$ denotes the lattice of $\nu$-kernels of a $\nu$-semifield $S$.

$\Psi$ is a natural map if for each $\nu$-semifield $S$, there is a lattice homomorphism $\Psi_S : \mathcal{L}(S) \rightarrow \mathcal{L}(S)$ such that $K \mapsto \Psi_S(K)$ is a homomorphism of kernels. We write $\Psi(S)$ for $\Psi_S(\mathcal{L}(S))$, and call the kernels in $\Theta(S) \Psi$-kernels. (We delete $S$ when it is unambiguous.)

A $\Psi(S)$-simple kernel is a minimal $\Psi$-kernel $\neq \{1\}$. A $\Psi(S)$-composition series $\mathcal{L}(K, L)$ in $\Psi(S)$ from a kernel $K$ to a subkernel $L$ is a chain

$$K = K_0 \supset K_1 \supset \ldots K_t = L$$

in $\Psi$ such that each factor is $\Psi$-simple.
By Theorem 4.1.7, \( C(K, L) \) is equivalent to the \( \Psi(S/L) \)-composition series

\[ K/L \supset K_1/L \supset \cdots \supset K_t/L = 0 \]

of \( K/L \).

Given a \( \Psi \)-kernel \( K \), we define its composition length \( \ell(K) \) to be the length of a \( \Psi \)-composition series for \( K \) (presuming \( K \) has one). By definition, \( \ell \) is the only \( \Psi \)-kernel of composition length 0. A nonzero \( \Psi \)-kernel \( K \) is simple iff \( \ell(K) = 1 \). The next theorem is a standard lattice-theoretic result of Schreier and Zassenhaus, yielding the Schreier-Jordan-Hölder Theorem, cf. [28, Theorem 3.11].

**Theorem 12.0.3.** Suppose \( K \) has a composition series

\[ K = K_0 \supset K_1 \supset \cdots \supset K_t = 0, \]

which we denote as \( C \). Then:

(i) Any arbitrary finite chain of subkernels

\[ K = N_0 \supset N_1 \supset \cdots \supset N_k \supset 0 \]

(defined as \( D \)), can be refined to a composition series equivalent to \( C \). In particular, \( k \leq t \).

(ii) Any two composition series of \( K \) are equivalent.

(iii) \( \ell(K) = \ell(N) + \ell(K/N) \) for every subkernel \( N \) of \( K \). In particular, every subkernel and every homomorphic image of a kernel with composition series has a composition series.

To apply this, we need the following observation.

**Proposition 12.0.4.** Let \( L \) be an HS-kernel in \( F(\Lambda) \) with \( 1-\text{loc}(L) \neq \emptyset \). Let \( \{h_1, ..., h_t\} \) be a set of \( \mathcal{Z} \)-monomials in \( \text{HS}(F(\Lambda)) \) such that \( \text{Conv}(h_1, ..., h_t) = \text{Conv}(L) \) and let \( L_i = \langle h_i \rangle \). Then the chain

\[ L = \bigcup_{i=1}^{u} L_i \supseteq \bigcap_{i=1}^{u-1} L_i \supseteq \cdots \supseteq L_1 \supseteq \langle 1 \rangle. \]

of HS-kernels is strictly descending if and only if \( h_1, ..., h_u \) are \( F \)-convexly independent.

**Proof.** \((\Rightarrow)\) If \( h_u \) is \( F \)-convexly dependent on \( \{h_1, ..., h_{u-1}\} \), then \( L_u = \langle h_u \rangle \subseteq \bigcap_{i=1}^{u-1} L_i \cdot \langle F \rangle \). Assume that \( L_u = \langle h_u \rangle \not\subseteq \bigcap_{i=1}^{u-1} L_i \). Then \( F \subseteq \bigcap_{i=1}^{u} L_i \), implying that \( h_u \) is not an HS-kernel. Thus \( L_u = \langle h_u \rangle \subseteq \bigcap_{i=1}^{u-1} L_i \), and the chain is not strictly descending.

\((\Leftarrow)\) \( 1-\text{loc}(\bigcap_{i=1}^{u} L_i) \subseteq 1-\text{loc}(L) \neq \emptyset \) for every \( 0 \leq t \leq u \), implying that \( \bigcap_{i=1}^{u} L_i \cap F = \{1\} \) for every \( 0 \leq t \leq u \) (for otherwise \( 1-\text{loc}(\bigcap_{i=1}^{u} L_i) = \emptyset \)). If \( \{h_1, ..., h_u\} \) is \( F \)-convexly independent then \( L_u = \langle h_u \rangle \not\subseteq \bigcap_{i=1}^{u-1} L_i \cdot \langle F \rangle \). By induction, the chain \([12.1]\) is strictly descending.

**Theorem 12.0.5.** If \( L \in \text{HS}(F(\Lambda)) \), then \( \text{hgt}(L) = d_{\text{conv}}(L) \), cf. Definition 11.3. Moreover, every factor of a descending chain of HS-kernels of maximal length is an HP-kernel.

**Proof.** By Proposition 12.0.4 the maximal length of a chain of HS-kernels descending from an HS-kernel \( L \) equals the number of elements in a basis of \( \text{Conv}(L) \); thus the chain is of unique length \( d_{\text{conv}}(L) \), i.e., \( \text{hgt}(L) = d_{\text{conv}}(L) \). Moreover, by Theorem 4.1.7(2),

\[ \prod_{i=1}^{j} L_i / \prod_{i=1}^{j-1} L_i \cong L_j / \left( L_j \cap \prod_{i=1}^{j-1} L_i \right). \]

Furthermore

\[ L_j \cdot \left( L_j \cap \prod_{i=1}^{j-1} L_i \right) \cap F = \{1\}, \]

since \( L_j \cdot \left( L_j \cap \prod_{i=1}^{j-1} L_i \right) = L_j \cap \prod_{i=1}^{j} L_i \subset \bigcap_{i=1}^{j} L_i \) and \( \bigcap_{i=1}^{j} L_i \cap F = \{1\} \). So the image of the HP-kernel \( L_j \) in \( F(\Lambda)/(L_j \cap \prod_{i=1}^{j} L_i) \) is an HP-kernel. Thus, every factor of the chain is an HP-kernel.

**Corollary 12.0.6.** \( \text{Hdim}(F(\Lambda)) = d_{\text{conv}}(F(\Lambda)) = n. \)

**Remark 12.0.7.** If \( R_1 \cap R_2 \cap \cdots \cap R_t = \{1\} \), then \( F(\Lambda) \) is a subdirect product

\[ F(\Lambda) = F(\Lambda)/(R_1 \cap R_2 \cap \cdots \cap R_t) \cong \prod_{i=1}^{t} F(\Lambda)/R_i. \]

Then for any kernel \( K \) of \( F(\Lambda) \), \( R_1 \cap R_2 \cap \cdots \cap R_t \cap K = \bigcap_{i=1}^{t} (R_i \cap K) = \{1\} \) and, since \( K \) itself is an idempotent semifield\(^\dagger\),

\[ K = K / \bigcap_{i=1}^{t} (R_i \cap K) \cong \prod_{i=1}^{t} K/(R_i \cap K) \cong \prod_{i=1}^{t} R_i K/R_i. \]
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