Optimal regularity of SPDEs with additive noise

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Abstract
The sample-function regularity of the random-field solution to a stochastic partial differential equation (SPDE) depends naturally on the roughness of the external noise, as well as on the properties of the underlying integro-differential operator that is used to define the equation. In this paper, we consider parabolic and hyperbolic SPDEs on \((0, \infty) \times \mathbb{R}^d\) of the form
\[
\partial_t u = \mathcal{L} u + g(u) + \dot{F} \quad \text{and} \quad \partial_t^2 u = \mathcal{L} u + c + \dot{F},
\]
with suitable initial data, forced with a space-time homogeneous Gaussian noise \(\dot{F}\) that is white in its time variable and correlated in its space variable, and driven by the generator \(\mathcal{L}\) of a genuinely \(d\)-dimensional Lévy process \(X\). We find optimal Hölder conditions for the respective random-field solutions to these SPDEs. Our conditions are stated in terms of indices that describe thresholds on the integrability of some functionals of the characteristic exponent of the process \(X\) with respect to the spectral measure of the spatial covariance of \(\dot{F}\). Those indices are suggested by references [45, 46] on the particular case that \(\mathcal{L}\) is the Laplace operator on \(\mathbb{R}^d\).

Key words and phrases. Stochastic partial differential equation, Gaussian noise, Lévy process, characteristic exponent, optimal Hölder regularity.

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1 Introduction

We consider a semilinear stochastic partial differential equation (SPDE) of the following parabolic type,
\[
\begin{cases}
\partial_t u = \mathcal{L} u + g(u) + \dot{F} & \text{on } (0, \infty) \times \mathbb{R}^d, \\
\text{subject to } u(0) = u_0 & \text{on } \mathbb{R}^d,
\end{cases}
\]
(1.1)
as well as a hyperbolic stochastic PDE of the form,
\[
\begin{cases}
\partial_t^2 u = \mathcal{L} u + c + \dot{F} & \text{on } (0, \infty) \times \mathbb{R}^d, \\
\text{subject to } u(0) = u_0 \text{ and } \partial_t u(t)|_{t=0} = v_0 & \text{on } \mathbb{R}^d,
\end{cases}
\]
(1.2)
where \(\mathcal{L}\) denotes the generator of a genuinely \(d\)-dimensional Lévy process (see (2.4)), \(g : \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}, \) and \(u_0 : \mathbb{R}^d \to \mathbb{R}\) and \(v_0 : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}\) are deterministic functions. The quantity \(\dot{F}\) is a centered space-time Gaussian noise whose covariance is, somewhat informally, defined as
\[
\text{Cov}[\dot{F}(t_1, x), \dot{F}(t_2, y)] = \delta_0(t_1 - t_2) \Gamma(x - y) \quad \text{for every } t_1, t_2 \geq 0 \text{ and } x, y \in \mathbb{R}^d,
\]
(1.3)

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where \( \Gamma \) is a tempered, nonnegative-definite Borel measure on \( \mathbb{R}^d \). We can understand \( \hat{F} \) more carefully via its action on a rapidly decreasing test function \( \phi \in \mathcal{S}(\mathbb{R}^{1+d}) \) as follows: Let
\[
\hat{F}(t, \phi) = \int_{(0,t) \times \mathbb{R}^d} \phi(s, x) F(ds, dx) \quad [t > 0],
\]
where the stochastic integral is a Wiener integral normalized to ensure that
\[
\text{Cov}[\hat{F}(t_1, \phi_1), \hat{F}(t_2, \phi_2)] = \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}^d} \Gamma(dx) \left( \phi_1(s) * \tilde{\phi}_2(s) \right) \quad \text{for every } \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).
\]
In this formula, the symbol “\( * \)” denotes the convolution operator in the spatial variable, and \( \tilde{\phi}(t, x) = \phi(t, -x) \) defines the reflection of \( \phi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R} \) in its space variable.

Let \( \mu \) denote the Fourier transform on \( \mathbb{R}^d \), normalized so that
\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(x) dx \quad \text{for all } f \in L^1(\mathbb{R}^d) \text{ and } \xi \in \mathbb{R}^d.
\]
We can then let \( \mu = \hat{\Gamma} \) and \( \psi = -\mathcal{L} \) respectively denote the spectral measure for the spatial aspect of the noise and the characteristic exponent of the underlying Lévy process (see §2). Assume that \( g : \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous. An extension of the theory of Dalang [16] (see also Brzézniak and van Neerven [9] and Khoshnevisan and Kim [33]) ensures that for every \( d \geq 1 \) (respectively, \( d \in \{1, 2, 3\} \)) the SPDE (1.1) (respectively, (1.2)) has a random-field solution if
\[
\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + \text{Re}(\psi(\xi))} < \infty.
\]
Moreover, this condition is optimal in the sense that it is necessary as well as sufficient when \( g \) is a constant function. Therefore, throughout this paper we assume that Dalang’s condition (1.5) holds in order to ensure that (1.1) and (1.2) are well posed.

The principal aim of this paper is to establish optimal Hölder regularity conditions for the random-field solution to (1.1) and (1.2), respectively. For the stochastic heat equation (1.1), we restrict attention to random initial data \( u_0 \) that is independent of the noise \( \hat{F} \) and satisfies moment-type conditions; see Proposition 2.6 for details. In the case of the wave equation (1.2), we will consider null initial data, although in spatial dimension \( d \in \{1, 2, 3\} \) it would be possible to consider non-null smooth initial data as well [20, 40]. The next two theorems contain the main findings of this paper.

**Theorem 1.1.** Let \( \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\} \) denote the random-field solution to the parabolic SPDE (1.1).

(a) For every \( x \in \mathbb{R}^d \), the random function \( t \mapsto u(t, x) \) is a.s. locally Hölder continuous provided that
\[
\int_{\mathbb{R}^d} |\psi(\xi)|^a \mu(d\xi) < \infty \quad \text{for some } a \in (0, 1).
\]

(b) If \( g \) is a constant and (1.6) fails, then \( t \mapsto u(t, x) \) is a.s. not locally Hölder continuous for every \( x \in \mathbb{R}^d \).

(c) The function \( (t, x) \mapsto u(t, x) \) is a.s. locally Hölder continuous provided that
\[
\int_{\mathbb{R}^d} ||\xi||^{2b} \mu(d\xi) < \infty \quad \text{for some } b \in (0, 1).
\]

(d) If \( g \) is a constant and (1.7) fails, then \( x \mapsto u(t, x) \) is a.s. not locally Hölder continuous for every \( t > 0 \).

When \( g \) is constant and the initial condition \( u_0 \) vanishes, assertions (a) and (c) of Theorem 1.1 follow respectively from (4.3) and (4.4) in Theorem 4.1. In the constant-g case, parts (b) and (d) of Theorem 1.1 follow from assertion 2. of Theorem 4.1. Theorem 4.6 below yields (a) and (c) in the more general case that \( g \) is non constant and the initial condition \( u_0 \) satisfies the hypotheses of Proposition 2.6. We mention also that Theorem 4.1 and Theorem 4.6 provide bounds on the optimal Hölder indices of the processes in question.

The preceding is a presentation of optimal Hölder regularity for parabolic SPDEs. The following is a hyperbolic counterpart of these results.
Theorem 1.2. Let \( \{u(t, x); \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\} \) denote the random-field solution to (1.2), and assume that \( g \) is a constant function and the initial functions \( u(0) \) and \( \partial_t u(0+, \cdot) \) both vanish. Then,

(i) If (1.6) holds then for, every \( x \in \mathbb{R}^d \), the random function \( t \mapsto u(t, x) \) is a.s. locally Hölder continuous.

(ii) If condition (1.6) fails then for every \( x \in \mathbb{R}^d \), \( t \mapsto u(t, x) \) is a.s. not locally Hölder continuous.

(iii) The function \( (t, x) \mapsto u(t, x) \) is a.s. locally Hölder continuous provided that (1.7) holds.

(iv) If condition (1.7) fails, then for every \( t > 0 \), \( x \mapsto u(t, x) \) is a.s. not locally Hölder continuous.

These results are proved in Theorem 5.2, where optimal Hölder indices are also given. In particular, parts (i) and (iii) follow from (5.6), (5.7), respectively, while the assertions (ii) and (iv) follow from part 2 of Theorem 5.2. In the last part of Section 6, we discuss briefly why an extension of Theorem 1.2 to a non constant \( g \) and (or) non vanishing initial conditions seems to be at present out of reach.

Remark 1.3. 1. Theorems 1.1 and 1.2 sometimes have relations to SPDEs driven by fractional powers of \( \mathcal{L} \); see Section 7.

2. One can read from Theorems 1.1 and 1.2 that condition (1.7) always implies (1.6). This implication can be more directly read off from the following classical estimate, which is itself a ready consequence of the Lévy-Khintchine formula (2.2) below:

\[
\sup_{\xi \in \mathbb{R}^d} \frac{|\psi(\xi)|}{1 + \|\xi\|^2} < \infty. \tag{1.8}
\]

There can be no converse implication. Indeed, Proposition 2.1 yields a non-trivial example where (1.6) holds but (1.7) does not; see also Example 2.3. In this way, we can see from Theorems 1.1 and 1.2 that optimal Hölder regularity in the space variable implies optimal Hölder regularity in the time variable, but not conversely.

3. It should be possible to combine the methods of this paper and those of Jacob [30] and Schilling [49] to extend our results to the case where \( \mathcal{L} \) is the generator of a nice, Lévy like, Markov process. Though we have not considered such a generalization, it would be a worthwhile endeavor since a celebrated theorem of Çinlar, Jacod, Protter, and Sharpe [14] asserts that every nice (quasi continuous, to be sure) Markov process that is a semimartingale is a time-change of a locally Lévy-like process.

Parts (b) and (d) of Theorem 1.1, and (ii) and (iv) of Theorem 1.2 are the main contributions of those results. Consider the case that \( \mathcal{L} = \Delta \), the Laplace operator. Then \( \psi(\xi) = \|\xi\|^2 \), and Conditions (1.6) and (1.7) coincide and are both equivalent to

\[
\int_{\mathbb{R}^d} \frac{\mu(dx)}{(1 + \|x\|^2)^{1+\eta}} < \infty, \quad \text{for some } \eta \in (0, 1). \tag{1.9}
\]

The work of Sanz-Solé and Sarra [45, 46] implies the sufficiency of this condition for the local Hölder continuity of the solution to the heat equation with nonlinear noise and nonlinear drift terms. In that case, Theorem 1.1 completes the circle by providing the necessity of (1.6) in the case that \( g \) is constant and the noise term is additive. Concerning the wave equation (1.2) in spatial dimension \( d \in \{1, 2, 3\} \) (with \( \mathcal{L} = \Delta \)), we see from [45, Section 4] that condition (1.9) implies the Hölder continuity of the sample paths of the random-field solution. When \( \Gamma \) is a Riesz kernel, the optimality of the Hölder index is proven in [20, Chapter 5]. Thanks to the results of Sections 3 and 5 we see that, in the case of a constant function \( g \), and for any dimension \( d \geq 1 \), Condition (1.9) is necessary and sufficient for the local Hölder continuity of the sample paths of the solution.

Next we say a few things about notation. Throughout, we write \( \|x\| \) for the Euclidean norm of any \( x \in \mathbb{R}^d \), and \( \|V\|_{k} = \{\mathbb{E}(|V|^k)\}^{1/k} \) for every \( k \geq 1 \) and all random variables \( V \). Moreover, we write \( f \lesssim g \) when the functions \( f = f(x) \) and \( g = g(x) \) are nonnegative and there exists a number \( c > 0 \) such that \( f \leq cg \) pointwise. And naturally, \( g \gtrsim f \) is another way to say \( f \lesssim g \). If \( f \lesssim g \) and \( g \lesssim f \), we write \( f \asymp g \). In similar fashion, we might say that \( f \leq g \) uniformly for all \( x \) in some set \( S \) when the respective restrictions \( f_S \) and \( g_S \) of \( f \) and \( g \) to \( S \) satisfy the uniformly bound \( f_S \leq cg_S \) for some \( c > 0 \). All vectors in \( \mathbb{R}^d \) are consistently treated as column vectors. As usual, \( 1_F \) denotes the indicator function of \( F \subset \mathbb{R}^d \).

We conclude the Introduction with a few remarks about our methods and the existing literature. As has been observed on multiple occasions in the literature, the only general known method for establishing Hölder
regularity of a stochastic process is to appeal to a suitable form of the Kolmogorov continuity theorem, which has to do with the moments of the increments of that process. In fact, there are beautiful formulations of this assertion that are rigorous theorems in their own right; see Hahn and Klass [24], Ibragimov [29], and Konô [35]. Therefore, it does not come as a surprise to know that the sufficiency half of our work also uses Kolmogorov’s continuity theorem. Our present challenge stems from the fact that a direct estimation/computation of the moments of the increments of the processes here do not naturally lend themselves to a form that can be used in conjunction with the Kolmogorov continuity theorem. In the case that \( \mathcal{L} \) is the Laplace operator, Sanz-Solé and Sarrà [45, 46] overcome this challenge by appealing to Hölder’s inequality, after they apply a stochastic version of the factorization method in semigroup theory; see also Dalang and Sanz-Solé [19] and Li [34]. Here, we use a simple but powerful idea from fractional calculus (Lemma A.3) that does not rely much on the particular form of \( \mathcal{L} \) and yields optimal results. There is a large literature that considers Hölder regularity of stochastic PDEs; see for example Balan and Chen [1], Balan, Jolis, and Quer-Sardanyons [2], Balan, Quer-Sardanyons, and Song [3], Chen and Dalang [11], Dalang and Sanz-Solé [19, 20], Hu and Le [25], Li [34], and Liu and Du [37]. There is also a large literature in which Hölder regularity is a key step in further analysis of the solution. For a small representative sampler of that literature, see Bezdek [7], Boulanba, Eddhabi, and Mellouk [8], Chen and Kim [12, 13], Dalang, Khoshnevisan, and Nualart [17], Dalang and Sanz-Solé [21], Dalang and Pu [18], Faugeras and Inglis [23], Hu, Nualart, and Song [26], Hu, Huang, Nualart, and Sun [27], Huang, Nualart, Viitasaari, and Zhang [28], Lun and Warren [38], Misiats, Stanzhytskyi, and Yip [41], Nualart [42], Rippl and Sturm [44], and Sanz-Solé and Šišek [47]. Our methods can yield a different understanding of some of these undertakings, and offer potential for extensions.

2 Lévy processes

We begin our discussion by making a few brief remarks and observations about Lévy processes. More details, and further information, on the general theory of Lévy processes can be found in the monographs of Bertoin [6], Kyprianou [36], and Sato [48]. Jacob [30] and Schilling [49] include the theory of Lévy (and Lévy like) processes from the point of view of harmonic analysis, particularly useful for the discussions that follow.

2.1 The Lévy-Khintchine formula

Let \( X = \{X(t)\}_{t \geq 0} \) be a Lévy process on \( \mathbb{R}^d \) that starts at the origin. Recall that this means that \( X(0) = 0 \), \( t \mapsto X(t) \) is càdlàg, and \( X(t+s) - X(s) \) is independent of \( \{X(r); 0 \leq r \leq s\} \) and distributed as \( X(t) \) for all \( s,t \geq 0 \). According to the Lévy-Khintchine formula [6, 30, 36, 48, 49], we can characterize the law of the entire \( X \) via the distribution of \( X(1) \) using the formula,

\[
E e^{i\xi \cdot X(t)} = e^{-t\psi(\xi)} \quad [t \geq 0, \xi \in \mathbb{R}^d].
\]

In the preceding, \( \psi : \mathbb{R}^d \to \mathbb{C} \) is called the characteristic exponent of \( X \), and has the form,

\[
\psi(\xi) = -i\mathbf{a} \cdot \xi + \frac{i}{2} \xi \cdot A \xi + \int_{\mathbb{R}^d} \left[ 1 - e^{iy \cdot \xi} + iy \cdot \xi \mathbb{1}_{B(0,1)}(y) \right] \nu(dy),
\]

where \( B(0,1) = \{ y \in \mathbb{R}^d : ||y|| \leq 1 \} \), \( \mathbf{a} \in \mathbb{R}^d \) is a constant,

\[
A = \mathbb{Q}' \mathbb{Q},
\]

for a \( d \times d \) matrix \( \mathbb{Q} \), and \( \nu \) is the Lévy measure of \( X \); that is, \( \nu \) is a \( \sigma \)-finite Borel measure \( \mathbb{R}^d \) that satisfies \( \nu(\{0\}) = 0 \) and \( \int_{\mathbb{R}^d} (1 + ||x||)^2 \nu(dx) < \infty \). It is apparent from (2.2) that \( \psi \) is continuous, and \( \text{Re}\psi \) is a real-valued, in fact non negative, function. We tacitly use these facts throughout.

It is well known that \( X \) is a Feller process whose generator \( \mathcal{L} \) is a pseudo-differential operator with constant symbols and Fourier multiplier \( \mathcal{L} = -\psi \); that is, \( \int_{\mathbb{R}^d} (\mathcal{L}f)(x)g(x) \, dx = -(2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{g}(\xi)\psi(\xi) \, d\xi \) for every \( f, g \in \mathcal{S}(\mathbb{R}^d) \). In particular, test functions of rapid decrease form a core for the domain of the definition of the generator \( \mathcal{L} \) of \( X \), and the action of \( \mathcal{L} \) on \( f \in \mathcal{S}(\mathbb{R}^d) \) is described as follows:

\[
\mathcal{L}f = \mathbf{a} \cdot \nabla f + \frac{i}{2} \text{Tr} \left( \mathbb{Q}' \nabla^2 f \mathbb{Q} \right) + \int_{\mathbb{R}^d} \left[ f(\cdot + y) - f(\cdot) - (y \cdot \nabla f)\mathbb{1}_{B(0,1)}(y) \right] \nu(dy),
\]
where $\nabla^2 = \nabla \nabla'$ denotes the Hessian matrix and $Q$ was defined in (2.3).

Formulas (2.2) and (2.4) are analytic ways of saying that we can decompose $X$ as

$$X(t) = a t + Q B(t) + Y(t) \quad [t \geq 0], \quad (2.5)$$

where $B$ is a $d$-dimensional, standard Brownian motion and $Y$ is an independent, pure-jump Lévy process. Note that the so-called $A$-Brownian motion $QB$ is a mean-zero Gaussian process with $\text{Cov}(QB(s), QB(t)) = \min(s, t) A$ for all $s, t \geq 0$, whence comes the name. The vector $a$ is sometimes referred to as the drift. Finally, we point out that the structure theory of Lévy process implies that – and hinges on – the fact that $\nu(A)$ is precisely equal to the expected number of jumps of $X$ (equiv. $Y$) in a Borel set $A \subset \mathbb{R}^d$.

In the Introduction, we mentioned that a standing assumption of this paper is that the underlying Lévy process is “genuinely $d$-dimensional.” By this we mean that throughout we assume the following:

$$\psi^{-1}(0) = 0, \quad (2.6)$$

In order to see what this condition says, suppose to the contrary that there exists $\xi_0 \neq 0 \in \mathbb{R}^d$ such that $\psi(\xi_0) = 0$. According to the Lévy-Khintchine formula (2.1),

$$0 = \Re \psi(\xi_0) = \frac{1}{2} \|Q \xi_0\|^2 + \int_{\mathbb{R}^d} [1 - \cos(\xi_0 \cdot y)] \nu(dy).$$

Therefore, $Q \xi_0 = 0$ and $\cos(\xi_0 \cdot y) = 1$ for $\nu$-almost every $y \in \mathbb{R}^d$. The former condition asserts that the rows of $Q$ are orthogonal to $\xi_0$, and the latter condition is another way to say that $\nu$ is concentrated on the 0-dimensional set $Z = \{y \in \mathbb{R}^d : y \cdot \xi_0 \in (2\pi \mathbb{Z})^d\}$. A final look at (2.2) now shows that $a : \xi_0 = 0$ also. Because the line of reasoning can be reversed, the description of this paragraph characterizes (2.6) and explains the reason for using it to describe “non degeneracy”: (2.6) says that either $Q \nabla^2 Q$ is strongly elliptic, or $\nu$ does not concentrate on $Z$, or both. The class of all (possibly degenerate) Lévy processes on $\mathbb{R}^d$ includes and extends the class of all continuous-time random walks, also known as compound Poisson processes. Thus, the non concentration of $\nu$ on $Z$ is an extension of the notion of genuinely $d$-dimensional continuous-time random walks, as can be found for example in Spitzer [51]. Condition (2.6) is natural enough that some authors assume it tacitly.

### 2.2 Fractal indices and examples

It is convenient to introduce two “fractal indices” that play a fundamental role in the formulation of the results of the paper. These indices show a trade-off between the singularity of the pseudo-differential operator $\mathcal{L}$ and the spectral spatial covariance of the noise $F$.

Our first regularity fractal index is

$$\text{IND} = \sup \left\{ a \in (0, 1) : \int_{\mathbb{R}^d} |\psi(\xi)|^a \mu(d\xi) < \infty \right\}, \quad (2.7)$$

and our second fractal index is

$$\text{IND} = \sup \left\{ b \in (0, 1) : \int_{\mathbb{R}^d} \|\xi\|^{2b} \mu(d\xi) < \infty \right\}. \quad (2.8)$$

By convention, $\sup \emptyset = 0$. Observe that (1.6) holds iff $\text{IND} > 0$, whereas $\text{IND} > 0$ iff (1.7) holds. Furthermore, (1.8) readily yields

$$\text{IND} \leq \text{IND}. \quad (2.9)$$

Let us briefly discuss some examples. The first was announced earlier in the Introduction.

**Proposition 2.1.** One can construct a pair $(\psi, \Gamma)$ that satisfies (1.5), (2.6), and $0 = \text{IND} < \text{IND} = 1$.

**Proof.** We consider spatial dimension one, that is $d = 1$, and first construct $\Gamma$ via $\Gamma(0) = 0$ and

$$\frac{d\Gamma(x)}{dx} = \frac{1}{\pi} \int_0^\infty s \exp(-s) \frac{ds}{s^2 + x^2} \text{ for all } x \in \mathbb{R} \setminus \{0\}.$$ 

This defines a positive and positive definite and symmetric probability measure whose Fourier transform is given by $d\mu(\xi)/d\xi = (1 + |\xi|)^{-1}$ for $\xi \in \mathbb{R}$. We plan to construct a symmetric Lévy process whose
characteristic exponent satisfies (2.6) and \( \psi(\xi) = \text{Re}\psi(\xi) \asymp (\log \xi)^2 \) uniformly for all \( \xi \geq \epsilon^2 \). Because \( \psi \) is an even function, one can directly check, using (2.7) and (2.8), that this construction yields \( 0 = \text{IND} < \text{IND} = 1 \) and completes the proof. Therefore, it remains to construct such a characteristic exponent \( \psi \).

Consider the particular case of the Lévy-Khintchine formula (2.2) with \( \alpha = 0 \), \( A = 0 \) and \( d\nu(x)/dx = -|x|^{-1} \log |x| \) if \( 0 < |x| < 1/e \) and \( d\nu(x)/dx = 0 \) otherwise; \( \nu \) is a bona fide Lévy measure since \( \int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2) \nu(dx) < \infty \). Also,

\[
\psi(\xi) = 2 \int_0^{1/e} (1 - \cos(|x|\xi)) \log(1/|x|) \frac{dx}{x} = 2 \int_0^{1/e} (1 - \cos y) \log \left( \frac{|\xi|}{y} \right) \frac{dy}{y} \geq \log \xi \int_1^{\sqrt{\xi}} \left( 1 - \cos \frac{y}{\xi} \right) \frac{dy}{y},
\]

for \( \xi \geq \epsilon^2 \). Moreover, the first identity holds for all \( \xi \in \mathbb{R} \). This readily shows that \( \psi(\xi) > 0 \) if \( \xi \neq 0 \). Thus (2.6) follows. Also, since \( \cos x \leq 0 \) on every interval \((\frac{2\pi}{2n}, \frac{2\pi}{2n} + \pi)\), \( n \) ranges over \( \mathbb{N} \), we may write

\[
\psi(\xi) \geq \log \xi \sum_{n \in \mathbb{N}} \int_{(n\pi/2) + \pi < y \leq \sqrt{\xi}} \frac{2}{n + 2},
\]

uniformly for all \( \xi \geq \epsilon^2 \). This immediately yields \( \psi(\xi) \geq (\log \xi)^2 \) uniformly for all \( \xi \geq \epsilon^2 \). Conversely,

\[
\psi(\xi) \leq 2 \int_0^{1/e} \left( 1 - \frac{\cos y}{y} \right) \log(\xi/y) \frac{dy}{y} + 4 \log \xi \int_1^{\sqrt{\xi}} \frac{dy}{y} \lesssim \int_0^{1} y \log(\xi/y) \frac{dy}{y} + (\log \xi)^2 \lesssim (\log \xi)^2,
\]

valid uniformly for all \( \xi \geq \epsilon^2 \). This completes the proof.

These examples show, among other things, that there is no general relationship between \( \text{IND} \) and \( \text{IND} \).

**Example 2.2.** First consider a Lévy process which satisfies (2.2) for a non-singular matrix \( A = Q \cdot Q \) in terms of the representation (2.5), this means that \( X \) has a non-degenerate Brownian component. In that case, \( \text{Re}\psi(\xi) \geq \frac{1}{2} \xi \cdot A \xi \gtrsim \| \xi \|^2 \), uniformly for all \( \xi \in \mathbb{R}^d \), and the nondegeneracy condition (2.6) holds. In light of (1.8), we can see that Dalang’s condition (1.5) holds iff \( \int_{\mathbb{R}^d} (1 + \| \xi \|)^{-2} \mu(d\xi) < \infty \). Moreover,

\[
\text{IND} = \text{IND} = 1 - \inf \left\{ c \in (0, 1) : \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + \| \xi \|^2c^2} < \infty \right\}
\]

and (1.7) and (1.6) are equivalent to one another, as well as to the condition \( 0 < \text{IND} \leq 1 \), which is the condition of Sanz-Solé and Sarrà [45, 46].

The above example covers all Lévy processes that have a non-degenerate Gaussian component. In particular the case \( \psi(\xi) = \frac{1}{2} \xi \cdot A \xi \), which corresponds to the Brownian motion. In the remainder of our examples we consider Lévy processes with a degenerate Gaussian component. For simplicity of exposition, for the remainder of this section we consider in fact only \( A = 0 \).

**Example 2.3** (Compound Poisson processes). The process \( X \) is compound Poisson (or a continuous-time random walk) iff \( \nu \) is a finite measure. Let us recall that (2.6) holds iff \( \nu \) does not concentrate on the countable set \( \{ y \in \mathbb{R}^d : y \cdot \xi_0 \in (2\pi Z)^d \} \) for any \( \xi_0 \in \mathbb{R}^d \setminus \{0\} \). This rules out examples such as the standard 1-dimensional Poisson process. According to (2.2), \( \psi(\xi) = -i\hat{a} \cdot \xi + \int_{\mathbb{R}^d} (1 - \exp(iy \cdot \xi)) \nu(dy) \), where \( \hat{a} = a + \int_{B_{(0, 1)}} y \nu(dy) \). Since \( 0 \leq \text{Re}\psi(\xi) \leq 2\nu(\mathbb{R}^d) \), we can see that Dalang’s condition (1.5) holds iff the spectral measure \( \mu \) is finite. This implies that \( \Lambda(dx) = f(x) \, dx \), for a uniformly continuous, bounded, and positive-definite function \( f : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) [22, p. 145]. Suppose then that the preceding conditions are met. Then,

\[
\text{IND} = \sup \left\{ b \in (0, 1) : \int_{\mathbb{R}^d} \| \xi \|^{2b} \mu(d\xi) < \infty \right\} \quad \text{and} \quad \text{IND} = \sup \left\{ c \in (0, 1) : \int_{\mathbb{R}^d} |\hat{a} \cdot \xi|^c \mu(d\xi) < \infty \right\},
\]

where sup \( \emptyset = 0 \), as before. Because \( |\hat{a} \cdot \xi| \leq \| \hat{a} \| \| \xi \| \), we have \( \frac{1}{2} \text{IND} \leq \text{IND} \) and the inequality can be strict because \( \text{IND} = 1 \) generically when \( \hat{a} = 0 \).
Example 2.4 (Radially symmetric stable processes). Choose and fix some $\alpha \in (0, 2)$ and let $dv(x)/dx = C\|x\|^{-d-\alpha}$, where $C > 0$ is chosen to ensure that $\psi(\xi) = \|\xi\|^\alpha$ for all $\xi \in \mathbb{R}^d$. The resulting process $X$ is called the radially symmetric $\alpha$-stable process. Clearly, (2.6) holds, and the generator is given by the following principle-value integral,

$$(\mathcal{L}f)(x) = C \int_{\mathbb{R}^d} \frac{f(x + y) - f(y)}{\|y\|^{d+\alpha}} \, dy \text{ for all } x \in \mathbb{R}^d \text{ and } f \in \mathcal{S}(\mathbb{R}^d);$$

see (2.4). That is, $\mathcal{L}$ is a fractional Laplace operator $\mathcal{L} = (-\Delta)^{\alpha/2}$. In this case, Dalang’s condition holds iff $\int_{\mathbb{R}^d}(1 + \|\xi\|)^{-\alpha} \mu(dx) < \infty$. And if this holds then

\[ \text{IND} = \sup \left\{ a \in (0, 1) : \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + \|\xi\|^{\alpha(1-a)}} < \infty \right\}, \quad \text{IND} = \sup \left\{ b \in (0, 2/\alpha) : \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + \|\xi\|^{\alpha(1-b)}} < \infty \right\}. \]

Because $\alpha < 2$, two different regimes emerge: On one hand, if $\mu(\mathbb{R}^d) = \infty$, then $\text{IND} = (\alpha/2)\text{IND}$ and the latter is of course given by the previous display. [Send $\alpha \uparrow 2$ in order to somewhat informally return to Example 2.2.] On the other hand, if $\mu(\mathbb{R}^d) < \infty$, then it can happen that $\text{IND} > (\alpha/2)\text{IND}$; in fact, in that case, careful examination of the preceding display yields the following:

\[ \text{IND} = 1 \quad \text{and} \quad \text{IND} = \frac{\alpha}{2} + \frac{1}{2} \sup \left\{ c \in (0, 2 - \alpha) : \int_{\mathbb{R}^d} \|\xi\|^c \mu(d\xi) < \infty \right\}. \]

Example 2.5 (Asymmetric Cauchy processes on $\mathbb{R}$). Set $d = 1$ and $a = 0$, and select $dv(x)/dx = c_1 x^{-2} dx$ if $x > 0$ and $dv(x)/dx = c_2 x^{-2} dx$ if $x < 0$, where $c_1, c_2 > 0$ are fixed numbers. The resulting process is the so-called Cauchy process on $\mathbb{R}$, and is described by $\psi(\xi) = |\xi| + i\xi \log |\xi|$ for $\xi \neq 0$, and $\psi(0) = 0$. The asymmetry parameter $h$ depends on $c_1$ and $c_2$ and can be made to be any number in $[-2/\pi, 2/\pi]$ by suitably adjusting $c_1, c_2$. The symmetric case $h = 0$ is subsumed by Example 2.4 [$d = \alpha = 1$]. We now look at the asymmetric case, $h \neq 0$. Dalang’s condition (1.5) holds iff $\int_{\mathbb{R}}(1 + |\xi|)^{-1} \mu(d\xi) < \infty$, the same as it does in the symmetric case. Moreover, in the asymmetric case, $|\psi(\xi)| \sim |h\xi \log |\xi||$ as $\xi \to \pm \infty$. Since $\log |\xi| = |\xi|^{\alpha(1)}$ as $|\xi| \to \infty$, it follows that, regardless of the strength of the asymmetry parameter $h$,

\[ \text{IND} = \frac{1}{2} \text{IND} \quad \text{and} \quad \text{IND} = \sup \left\{ a \in (0, 1) : \int_{\mathbb{R}} \frac{\mu(d\xi)}{1 + |\xi|^{1-a}} < \infty \right\}. \]

2.3 The Kolmogorov-Fokker-Planck equation

The fundamental solution to the corresponding heat operator $\partial_t - \mathcal{L}$ is the solution to the integro-differential equation $\partial_t \varphi = \mathcal{L}\varphi$ on $(0, \infty) \times \mathbb{R}^d$, subject to $\varphi(0) = \delta_0$ on $\mathbb{R}^d$. Since $\mathcal{L} = -\psi$, we can apply the Fourier transform in order to see that the fundamental solution to $\partial_t - \mathcal{L}$ is the following probability-measure-valued function of $t$:

\[ p(t, F) = \mathbb{P}\{X(t) \in F\} \quad \text{for all } t \geq 0 \text{ and Borel sets } F \subset \mathbb{R}^d. \]

These are the so-called transition functions of the Lévy process $X$.

The preceding discussion tells us that if $u_0$ is a tempered, nonnegative-definite Borel measure on $\mathbb{R}^d$ then the associated Kolmogorov-Fokker-Planck equation, $\partial_t f = \mathcal{L}f$ on $(0, \infty) \times \mathbb{R}^d$, subject to $f(0) = u_0$ on $\mathbb{R}^d$, is well posed, and has a unique solution that is a function from $\mathbb{R}_+$ to the space of tempered distributions (Borel probability measures, in fact), and that unique solution is in fact described by $f(t, \cdot) = p(t) * u_0$ at time $t \geq 0$. The following addresses the local Hölder regularity of that solution when $u_0$ is a random field satisfying some moment-type constraints.

Proposition 2.6. Let $u_0 = \{u_0(x) : x \in \mathbb{R}^d\}$ be a random field. Assume that there exist real numbers $k \geq 1$ and $\eta \in (0, 1)$, and a constant $C := C(k) > 0$ such that

\[ \sup_{x \in \mathbb{R}^d} E \left( |u_0(z)|^k \right) < \infty \quad \text{and} \quad E \left( |u_0(x) - u_0(y)|^k \right) \leq C \|x - y\|^\eta \quad \text{uniformly for all } x, y \in \mathbb{R}^d. \]

Then, there exists $c = c(k) > 0$ such that

\[ E \left( \left| (p(t + \varepsilon) * u_0)(-x) - (p(t) * u_0)(-y) \right|^k \right) \leq c \left( \varepsilon^{1/2} + \|x - y\| \right)^\eta, \]
uniformly for all \( t \geq 0, \varepsilon \in [0,1], \) and \( x,y \in \mathbb{R}^d \).

If (2.11) holds for every \( k \geq 1, \) then for every \( a \in (0, \eta), \) \((t,x) \mapsto (p(t) * u_0)(-x)\) a.s. belongs to \( C^{a/2,a}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d) \).

The proposition rests on the following result, which is a quantitative version of the assertion that every Lévy process on \( \mathbb{R}^d \) is locally subdiffusive.

**Lemma 2.7.** \( E(\|X(t)\|^2 \wedge 1) \leq t \) uniformly for all \( t \in [0,1] \).

**Proof.** Let \( X_t \) denote the \( i \)th coordinate process of \( X \) and observe that \( E(\|X(t)\|^2 \wedge 1) \lesssim \sum_{i=1}^{d} E(|X_i(t)|^2 \wedge 1) \), uniformly for all \( t \geq 0 \). Since every \( X_i \) is a one-dimensional Lévy process, the above reduces the problem to a one-dimensional problem. Therefore, we may and will assume without loss in generality that \( d = 1 \). In this case, (2.5) reduces to the existence of real numbers \( a,q \geq 0 \) such that \( X(t) = at + qB(t) + Y(t) \) for all \( t \geq 0 \), where \( B \) is a standard linear Brownian motion and \( Y \) is an independent one-dimensional, pure-jump Lévy process. It follows from this that
\[
E(|X(t)|^2 \wedge 1) \lesssim t^2 + E(|B(t)|^2 \wedge 1) + E(|Y(t)|^2 \wedge 1) \lesssim t + E(|Y(t)|^2 \wedge 1),
\]uniformly for all \( t \in [0,1] \). Thanks to the structure theory of Lévy processes, we may write \( Y(t) = S(t) + L(t) \) for all \( t \geq 0 \) where \( S \) and \( L \) are independent one-dimensional Lévy processes, and
\[
L(t) = \sum_{s \in [0,t]} (\Delta Y)(s)\mathbf{1}_{\{|\Delta Y(s)| > 1\}}
\]for all \( t \geq 0 \), where \( (\Delta Y)(s) = Y(s) - Y(s-) \). The letters “\( L \)” and “\( S \)” respectively refer to the “large jumps” and the “small jumps” of \( Y \). The collection of all of the jumps of \( Y \) that are \( >1 \) in magnitude has a Poisson distribution with parameter \( tv(\mathbb{R} \setminus [-1,1]) \). Therefore, for every \( t \geq 0 \), \( P\{L(s) \neq 0 \text{ for some } s \in [0,t]\} = 1 - \exp\{-tv(\mathbb{R} \setminus [-1,1])\} \leq t \), and hence
\[
E(|Y(t)|^2 \wedge 1) \lesssim E(|S(t)|^2 \wedge 1) + P\{L(t) \neq 0\} \lesssim E(|S(t)|^2) + t,
\]uniformly for all \( t \geq 0 \). The process \( S \) is Lévy with zero drift and zero diffusion, and its Lévy measure is the restriction of \( \nu \) to \([-1,1]\). Therefore, the structure theory of Lévy processes implies that the stochastic process \( \{|S(t)|^2 - t \int_{-1}^{1} y^2 \nu(dy); t \geq 0\} \) is a mean-zero martingale, whence \( E(|S(t)|^2) = t \int_{-1}^{1} y^2 \nu(dy) \) for every \( t \geq 0 \). Combine with (2.14) and (2.13) to complete the proof.

**Proof of Proposition 2.6.** The moment bound (2.12) and Kolmogorov’s continuity theorem together imply the Hölder regularity statement. Therefore, it suffices to establish (2.12).

Without any loss in generality, we can and will assume that \( u_0 \) and \( X \) are independent. With this in mind, let us note that \( (p(t) * u_0)(-x) = E[u_0(x + X(t))] \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \). Therefore, if we write \( E(\cdots) = E(\cdots | u_0) \), and denote by \( L^k(u_0) \) the space \( L^k(\Omega) \) endowed with a regular version of the conditional probability \( P(\cdots | u_0) \), we can deduce from disintegration, (2.11), and the triangle inequality that
\[
\| (p(t) * u_0)(-x) - (p(t) * u_0)(-z) \|_{L^k(u_0)} \lesssim E \left( \| u_0(x + X(t)) - u_0(z + X(t)) \|_{L^k(u_0)} \right) \lesssim C^{1/k} \| x - z \|^\eta,
\]valid uniformly for all \( k \geq 1, x, z \in \mathbb{R}^d, \) and \( t \geq 0 \). On the other hand, set \( D = \sup_{x \in \mathbb{R}^d} \| u_0(z) \|_{k} \) to infer from (2.11) that, for all \( k \geq 1, \) the following holds uniformly for all \( x \in \mathbb{R}^d, \) and \( t, \varepsilon \geq 0 \):
\[
\| (p(t + \varepsilon) * u_0)(-x) - (p(t) * u_0)(-x) \|_{L^k(u_0)} \lesssim E \left( \| u_0(x + X(t + \varepsilon)) - u_0(x + X(t)) \|_{L^k(u_0)} \right) \lesssim E \left( \| X(t + \varepsilon) - X(t) \|^\eta \wedge 2D \right) \lesssim E \left( \| X(t) \|^\eta \wedge 1 \right) \lesssim \{ E(\| X(t) \|^2 \wedge 1) \}^{\eta/2},
\]thanks to Jensen’s inequality. Lemma 2.7 then implies that for every \( k \geq 1, \)
\[
E \left( \| (p(t + \varepsilon) * u_0)(-x) - (p(t) * u_0)(-x) \|^k \right) \lesssim \varepsilon^{k/2},
\]uniformly for all \( x \in \mathbb{R}^d, \) and \( t, \varepsilon \geq 0 \). Combine this with (2.15) to conclude (2.12) whence the lemma.
3 The stochastic convolution

In this section we study the sample-function regularity of the Fourier-Laplace type stochastic convolution,

\[ \int_{(0,t) \times \mathbb{R}^d} \varphi(t-s, x-y) \hat{F}(s, y) \, ds \, dy := \int_{(0,t) \times \mathbb{R}^d} \varphi(t-s, x-y) F(ds, dy) := (\varphi * F)(t, x), \]

(3.1)
of a deterministic function \( \varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) and the noise \( \hat{F} \), using ideas of Wiener on stochastic integration.

Fix \( \varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) deterministic, and \( t_1, t_2 \geq 0, x_1, x_2 \in \mathbb{R}^d \). If we formally, and somewhat liberally, exchange expectations with integrals, and if we treat \( \delta_0 \) and \( \Gamma \) in (1.3) as nice functions, then we see from (3.3) and (3.1) that

\[
\text{Cov}[(\varphi * F)(t_1, x_1), (\varphi * F)(t_2, x_2)] = \int_{(0, t_1) \times (0, t_2)} ds_1 \, ds_2 \int_{\mathbb{R}^d \times \mathbb{R}^d} dy_1 \, dy_2 \, \varphi(t_1 - s_1, x_1 - y_1) \varphi(t_2 - s_2, x_2 - y_2) E[\hat{F}(s_1, y_1) \hat{F}(s_2, y_2)]
\]

(3.2)

The first two identities are not rigorous and contain ill-defined integrals. However, the final expression makes perfectly good sense as a Lebesgue integral when \( \varphi \) is a nice space-time function. Under appropriate conditions on \( \varphi \) one can in fact apply inverse Fourier transforms to see that

\[
\text{Cov}[(\varphi * F)(t_1, x_1), (\varphi * F)(t_2, x_2)] = 2\pi^{-d} \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}^d} \mu(d\xi) \hat{\varphi}(t_1 - s, \xi) \hat{\varphi}(t_2 - s, \xi) e^{i(x_1 - x_2) \cdot \xi},
\]

(3.3)

where \( \hat{\varphi}(t, \xi) = \int_{\mathbb{R}^d} \exp(i \xi \cdot x) \varphi(x) \, dx \), whenever the preceding integral makes sense as a Lebesgue integral, and we recall that \( \mu = \hat{\Gamma} \) is a tempered Borel measure on \( \mathbb{R}^d \) since \( \Gamma \) is a tempered, nonnegative-definite Borel measure on \( \mathbb{R}^d \).

In order to rigorously construct the stochastic convolution, one merely reverse-engineers the preceding “calculation” and starts with (3.2) as a definition. We outline the details next.

**Definition 3.1.** Let \( \mathcal{K} \) denote the class of all functions \( \varphi : \mathbb{R}_+ \to S'((\mathbb{R}^d)) \) such that:

1. \( \hat{\varphi}(t) \in L^2(\mu) \) for every \( t \geq 0 \); and
2. There is a version \( \hat{\varphi} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C} \) that is jointly measurable that satisfies

\[
0 < \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\varphi(s, \xi)|^2 < \infty \quad \text{for every } t > 0.
\]

(3.3)

We refer to elements of \( \mathcal{K} \) as kernels.

The fundamental solutions to some PDEs provide typical examples of kernels. The strict positivity condition in (3.3) is there to avoid degenerate cases.

Now, choose and fix a kernel \( \varphi \in \mathcal{K} \), and denote the quantity on the right-hand side of (3.2) by \( \langle (t_1, x_1), (t_2, x_2) \rangle \). We may view the latter as a pairing between \( (t_1, x_1) \) and \( (t_2, x_2) \) for all \( (t_1, x_1), (t_2, x_2) \in \mathbb{R}_+ \times \mathbb{R}^d \). One can readily see that this pairing defines a nonnegative-definite Hermitian product on \( (\mathbb{R}_+ \times \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathbb{R}^d) \). The general theory of Gaussian processes then ensures that there exists a \((d+1)\)-parameter Gaussian process \( \varphi * F = \{ (\varphi * F)(t, x) ; t \geq 0, x \in \mathbb{R}^d \} \) whose covariance is given by the above pairing; that is, \( \varphi * F \) satisfies (3.2). Observe that \( \varphi * F \) is stationary in space. One can in fact construct \( \varphi * F \) for a slightly larger family of non-random integrands by replacing Item (2) above by the condition that the function \( t \mapsto \|\hat{\varphi}(t)\|_{L^2(\mu)} \) is in \( L^2_{\text{loc}}((\mathbb{R}_+)) \). But the presently assumed joint measurability of \( \hat{\varphi} \) is convenient.

The goal of this section is to find close-to-optimal local Hölder regularity results for stochastic convolutions of the above type. For the remainder of this section we choose and fix a nonrandom number \( T > 0 \) and a kernel \( \varphi \in \mathcal{K} \), and define four indices that play a key role in the regularity of the sample functions of \( \varphi * F \). In
the particular examples of the stochastic heat and wave equations, we will prove in Lemma 4.2 and Lemma 5.3, respectively, their relationship with the indices \( \overline{\text{ND}} \) and \( \text{ND} \) (defined in (2.7) and (2.8), respectively).

The first index is defined as follows:

\[
\mathcal{I}_R(T) = \sup \left\{ \eta \in (0,1) : \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \|\xi\|^\eta \|\hat{\varphi}(s,\xi)\|^2 < \infty \right\},
\]

where \( \sup \emptyset = 0 \).

**Hölder regularity in space**

**Proposition 3.2.** If \( \mathcal{I}_R(T) > 0 \) then, for all \( a \in (0, \mathcal{I}_R(T)) \)

\[
\| (\varphi * F)(t,h) - (\varphi * F)(t,0) \|_2 \lesssim \|h\|^{a/2},
\]

uniformly for all \( t \in [0,T] \) and \( h \in \mathbb{R}^d \) that satisfy \( \|h\| \leq 1 \).

On the other hand, if \( \mathcal{I}_R(T) = 0 \), then

\[
\limsup_{h \to 0} \|h\|^{-a} \| (\varphi * F)(t,h) - (\varphi * F)(t,0) \|_2 = \infty \quad \text{for every } a > 0 \text{ and } t \geq T.
\]

Furthermore, for any \( a \in (\mathcal{I}_R(T), \infty) \),

\[
\limsup_{h \to 0} \|h\|^{-a} \| (\varphi * F)(t,h) - (\varphi * F)(t,0) \|_2 = \infty \quad \text{for every } t \geq T.
\]

**Proof.** First, let us suppose that \( \mathcal{I}_R(T) > 0 \) for a fixed \( T > 0 \). Choose and fix some \( a \in (0, \mathcal{I}_R(T)) \). The definitions of \( \mathcal{K} \) and \( \mathcal{I}_R(T) \) together imply that

\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \left( 1 + \|\xi\|^a \right) \|\hat{\varphi}(s,\xi)\|^2 < \infty.
\]

(3.5)

According to (3.2), for any \( h \in \mathbb{R}^d \),

\[
E \left( \| (\varphi * F)(t,h) - (\varphi * F)(t,0) \|^2 \right) = \frac{2}{(2\pi)^d} \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \|\hat{\varphi}(s,\xi)\|^2 \left[ 1 - \cos(h \cdot \xi) \right]
\]

\[
\leq \frac{2}{(2\pi)^d} \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \|\hat{\varphi}(s,\xi)\|^2 \left( \|\xi\|^2 \|h\|^2 + 1 \right),
\]

(3.6)

uniformly for all \( t \in [0,T] \). Define

\[
g(\varepsilon) = \sup_{t \in [0,T]} \sup_{h \in \mathbb{R}^d, \|h\| \leq \varepsilon} E \left( \| (\varphi * F)(t,h) - (\varphi * F)(t,0) \|^2 \right) \quad \text{for all } \varepsilon > 0.
\]

As part of the definition of \( \mathcal{I}_R(T) \), we are assured that \( a \in (0,1) \); see (3.4). Therefore, (3.6) and Lemma A.1 of the appendix (with \( b = 2 \) and \( c := 2a < 2 \) together show that \( \int_0^1 \varepsilon^{1-2a} g(\varepsilon) \, d\varepsilon < \infty \). Since \( g \) is non decreasing and measurable, Lemma A.3 of the appendix ensures that \( g(\varepsilon) \lesssim \varepsilon^{2a} \) uniformly for all \( \varepsilon \in (0,1) \). The first assertion of the proposition follows.

For the second part, we suppose \( \mathcal{I}_R(T) = 0 \) for a fixed \( T > 0 \). Since \( \varphi \in \mathcal{K} \), (3.4) assures us that

\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \|\hat{\varphi}(s,\xi)\|^2 \|\xi\|^a = \infty \quad \text{for all } t \geq T \text{ and } a \in (0,1).
\]

Choose and fix \( a \in (0,1) \) and \( t \geq T \). In the following we will use the fact that

\[
\int_{\mathbb{R}^d} \left( \frac{1 - \cos(\xi \cdot h)}{\|h\|^d + a} \right) \, dh = c \|\xi\|^a \quad \text{for all } \xi \in \mathbb{R}^d,
\]

(3.7)
where $c = c(d, a) > 0$ is a real number. This holds because the left-hand side of (3.7) converges whenever $a \in (0, 1)$ and defines a radial function of $\xi$.

Define
\[ f(h) = E \left( |(\varphi * F)(t, h) - (\varphi * F)(t, 0)|^2 \right) \quad \text{for every } h \in \mathbb{R}^d. \]

We may combine the first line of (3.6) with (3.7) to see that, for all $\xi \in \mathbb{R}^d$ and $t \geq T$,
\[
\int_{\mathbb{R}^d} f(h) \frac{dh}{\|h\|^{d+a}} = \frac{2c}{(2\pi)^d} \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \ |\hat{\varphi}(s, \xi)|^2 \|\xi\|^a = \infty \tag{3.8}
\]

Define $f_n = \sup_{\|x\| \leq \exp(-n)} f(x)$ for every $n \in \mathbb{Z}_+$ and observe that
\[
\sum_{n=0}^{\infty} e^{na} f_n \geq \sum_{n=1}^{\infty} f_{n-1} \int_{e^{-n} \leq \|h\| \leq e^{-n+1}} \frac{dh}{\|h\|^{d+a}} \geq \int_{\|h\| \leq 1/e} f(h) \frac{dh}{\|h\|^{d+a}}.
\]

Because of (3.6), $f$ is bounded uniformly from above and therefore, $\int_{\|h\| \leq 1/e} f(h)\|h\|^{-d-a} dh = \infty$. Hence by (3.8) $\int_{\|h\| \leq 1/e} f(h)\|h\|^{-d-a} dh = \infty$. Consequently, $\sum_{n=0}^{\infty} e^{na} f_n = \infty$. This proves that $\ell = \sup_{n \geq 1} e^{nb} f_n = \infty$ whenever $b > a$ for $\sum_{n=0}^{\infty} e^{na} f_n \leq \ell \sum_{n=0}^{\infty} e^{-n(b-a)} < \infty$ otherwise. Because $\ell = \infty$, a monotonicity argument implies that $\limsup_{h \to 0} \|h\|^{-b} f(h) = \infty$ for every $b > a$. Since $a > 0$ is arbitrary, this completes the proof of the proposition.

For the proof of the last statement, we observe that, if $a \in (I_R(T), \infty)$ then
\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\hat{\varphi}(s, \xi)|^2 \|\xi\|^a = \infty
\]

Defining $f_T(h) = E \left( |(\varphi * F)(T, h) - (\varphi * F)(T, 0)|^2 \right)$, with the same arguments used to obtain (3.8), we have $\int_{\mathbb{R}^d} f_T(h) \frac{dh}{\|h\|^{d+a}} = \infty$. This yields the assertion. \hfill \square

In the particular instances of the heat and wave equation, it turns out that $I_R(T)$ is independent of $T$: see Lemmas 4.2 and 5.3, respectively.

**Hölder regularity in time**

We introduce the remaining relevant indices of this section:

\[
I_H = \sup \left\{ b \in (0, 1) : \int_0^1 \frac{ds}{s^{\alpha}} \int_{\mathbb{R}^d} \mu(d\xi) |\hat{\varphi}(s, \xi)|^2 < \infty \right\},
\]

\[
I_H(T) = \sup \left\{ b \in (0, 1) : \int_0^1 \frac{ds}{s^{1+b}} \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\hat{\varphi}(s + \varepsilon, \xi) - \hat{\varphi}(s, \xi)|^2 < \infty \right\}, \tag{3.9}
\]

\[
\mathcal{I}_H(T) = \sup \left\{ b \in (0, 1) : \int_0^1 \frac{ds}{s^{1+b}} \sup_{r \in [0, \varepsilon]} \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\hat{\varphi}(s + r, \xi) - \hat{\varphi}(s, \xi)|^2 < \infty \right\},
\]

where $T > 0$. Clearly, $0 \leq I_H(T) \leq I_H(T) \leq 1$.

**Proposition 3.3.** Assume $I_H \wedge I_H(T) > 0$. Then, for any $b \in (0, I_H \wedge I_H(T))$,
\[
\|(\varphi * F)(t + \varepsilon, 0) - (\varphi * F)(t, 0)\|_2 \lesssim \varepsilon^{b/2},
\]

uniformly for all $t \in [0, T]$ and $\varepsilon \in (0, 1)$.

On the other hand, for all $t > T$ and $b > I_H \wedge I_H(T)$,
\[
\limsup_{\varepsilon \to 0^+} \varepsilon^{-b/2} \|(\varphi * F)(t + \varepsilon, 0) - (\varphi * F)(t, 0)\|_2 = \infty.
\]
Proof. Choose and fix \( t \in [0, T] \). Owing to (3.2), we may write for every \( \varepsilon > 0 \),

\[
\mathbb{E} \left( \left| (\varphi * F)(t + \varepsilon, 0) - (\varphi * F)(t, 0) \right|^2 \right) = \frac{I_1(\varepsilon) + I_2(\varepsilon)}{(2\pi)^d},
\]

where

\[
I_1(\varepsilon) = \int_t^{t+\varepsilon} ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(t + \varepsilon - s, \xi) \right|^2 = \int_{0}^{\varepsilon} ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(s, \xi) \right|^2,
\]

\[
I_2(\varepsilon, t) = \int_{0}^{t} ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(s + \varepsilon, \xi) - \hat{\varphi}(s, \xi) \right|^2.
\]

Tonelli’s theorem implies that, for every \( a > 0 \),

\[
\int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} I_1(\varepsilon) = \int_{0}^{1} ds \int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(s, \xi) \right|^2 = \frac{1}{a} \int_{0}^{1} \left( \frac{1}{s^a} - 1 \right) ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(s, \xi) \right|^2.
\]

Assume that \( a \in (0, I_H) \). Because \( \varphi \in K \), it follows that

\[
\int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} I_1(\varepsilon) < \infty.
\]

Since \( I_1 \) is monotone, we can apply the integral test from calculus to obtain \( \sum_{n=0}^{\infty} c_{an} I_1(e^{-n}) < \infty \). Consequently, if \( a \in (0, I_H) \) then \( I_1(e^{-n}) = o(e^{an}) \) as \( n \to \infty \) along integers; see also Lemma A.3 of the appendix. We deduce from this the following:

For every \( a \in (0, I_H) \) there exists a number \( C > 0 \) such that \( I_1(\varepsilon) \leq C e^a \) for all \( \varepsilon \in (0, 1) \).

We also learn from Lemma A.3 that for every \( a > I_H \),

\[
\limsup_{\varepsilon \to 0^+} \varepsilon^{-a} I_1(\varepsilon) = \infty.
\]

For all \( a > 0 \), define

\[
\mathcal{A}(a) = \int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} I_2(\varepsilon) = \int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} \int_{0}^{t} ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(s + \varepsilon, \xi) - \hat{\varphi}(s, \xi) \right|^2,
\]

\[
\mathcal{B}(a) = \int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} \sup_{r \in [0, c]} I_2(r) = \int_{0}^{1} \frac{d\varepsilon}{\varepsilon^{1+a}} \sup_{r \in [0, c]} \int_{0}^{T} ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \hat{\varphi}(s + r, \xi) - \hat{\varphi}(s, \xi) \right|^2.
\]

The justifications of (3.10) and (3.11) can now be repurposed in order to show the following:

1. If \( a \in (0, \overline{I}_H(T)) \), then \( \mathcal{B}(a) < \infty \) and therefore \( I_2(\varepsilon, t) \leq \varepsilon^b \) uniformly for all \( t \in [0, T], \varepsilon \in (0, 1) \) and \( b \in (0, a) \). Along with (3.10), this yields the first statement of the proposition;
2. \( \mathcal{A}(a) = \infty \) for every \( a > \overline{I}_H(T) \). It follows that \( \limsup_{\varepsilon \to 0^+} \varepsilon^{-a} I_2(\varepsilon, T) = \infty \). Together with (3.11), this implies the second statement of the proposition.

\[\Box\]

4 The linear heat equation

In this section we return to SPDEs, and study the following linearization of the heat equation (1.1):

\[
\begin{aligned}
\left[ \partial_t H = \mathcal{L} H + \tilde{F} \right. & \quad \text{on } (0, \infty) \times \mathbb{R}^d, \\
\left. \text{subject to } H(0) = 0 \right. & \quad \text{on } \mathbb{R}^d.
\end{aligned}
\]

The general theory of Dalang [16] ensures that (4.1) has a random-field solution if and only if (1.5) holds. Let us briefly review why this is the case in the present, simple context, of the heat equation (4.1).
Because the fundamental solution to the operator $\partial_t - \mathcal{L}$ is $p(t, \cdot) = \mathbb{P}\{X(t) \in \cdot\}$ [see for example the discussion around (2.10)] the stochastic heat equation (4.1) has a random-field solution if and only if the stochastic convolution $p*F$ is a well-defined Gaussian random field. This means that we have to have $p \in \mathcal{K}$. Since $p(t)$ is a Borel probability measure for every $t \geq 0$, (2.6), and (2.1) together imply that (4.1) has a random-field solution if and only if

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \left| e^{-s\psi(\xi)} \right|^2 = T \mu\{0\} + \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} \left( \frac{1 - e^{-2TRe\psi(\xi)}}{Re\psi(\xi)} \right) \mu(d\xi) < \infty \quad \text{for all } T > 0.$$

Because $Re\psi \geq 0$ and $u^{-1}(1 - e^{-u}) \asymp (1 + u)^{-1}$ uniformly for all $u > 0$, this shows that (1.5) is necessary and sufficient for (4.1) to have a random-field solution.

In addition, Dalang’s theory ensures that when the solution exists as a random field, it automatically is continuous in $L^2(\Omega)$ in the following rather strong sense:

$$\lim_{\varepsilon \to 0^+} \sup_{s,t \in [0,T]} \sup_{x,y \in \mathbb{R}^d} \mathbb{E} \left( |H(t,x) - H(s,y)|^2 \right) = 0 \quad \text{for all } T > 0.$$

In particular, Doob’s separability theory ensures that under condition (1.5), the random field $H(t,x)$ is Lebesgue measurable [more precisely, has a Lebesgue-measurable modification]. In this section we identify when exactly $H$ has locally Hölder-continuous sample functions. Before we discuss those details, let us point out that, as a consequence of the mentioned separability, Kallianpur’s zero-one law for Gaussian processes (Kallianpur [32]; see also Cambanis and Rajput [10]) tells us that for all $s > 0$ and $y \in \mathbb{R}^d$:

$$\mathbb{P}\{H(\cdot,y) \text{ is locally Hölder continuous over } \mathbb{R}_+\} = 0 \text{ or } 1,$$

$$\mathbb{P}\{H(s,\cdot) \text{ is locally Hölder continuous over } \mathbb{R}^d\} = 0 \text{ or } 1,$$

$$\mathbb{P}\{H \text{ is locally Hölder continuous over } \mathbb{R}_+ \times \mathbb{R}^d\} = 0 \text{ or } 1.$$

Therefore, we sometimes simply say that $H$ has [or does not have, depending on the case] locally Hölder continuous samples when we really mean to say that $H$ a.s. has [or a.s. does not have, depending on the case] a modification that is locally Hölder continuous.

The goal of this section is to prove the following result. Throughout we assume (1.5). It might help to also remind that if $\text{IND} > 0$ then $\text{IND} > 0$, but the converse need not hold; see (2.9) and Proposition 2.1.

**Theorem 4.1.** 1. If $\text{IND} > 0$, then

$$\mathbb{P}\left\{ H(\cdot,x) \in C^\alpha_{\text{loc}}(\mathbb{R}^d) \right\} = 1 \quad \text{for every } x \in \mathbb{R}^d \text{ and } 0 < \alpha < \frac{1}{2} \text{IND},$$

and if $\text{IND} > 0$, then

$$\mathbb{P}\left\{ H \in C^{\alpha,\beta}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d) \right\} = 1 \quad \text{for every } 0 < \alpha < \frac{1}{2} \text{IND} \text{ and } 0 < \beta < \text{IND}.$$  

2. If $\text{IND} = 0$, then $t \mapsto H(t,0)$ is not Hölder continuous; and if $\text{IND} = 0$, then $x \mapsto H(t,x)$ is not Hölder continuous for every $t > 0$.

Note that Theorem 1.1 reduces to a weaker version of Theorem 4.1 when $g \equiv u_0 \equiv 0$.

As the first step of the proof of Theorem 4.1 we evaluate the indices $I_H$, $\mathfrak{I}_H$, and $\mathfrak{T}_H$ in terms of the indices $\text{IND}$, $\text{IND}$ and $I_R(T)$ defined in (2.7), (2.8), (3.4), respectively.

**Lemma 4.2.** $I_R(T) = 2 \text{IND}$ for all $T > 0$, and $I_H = \mathfrak{I}_H(T) = \mathfrak{T}_H(T) = 2 \text{IND}$.

**Proof.** Recall the transition functions of the process $X$ from (2.10) and appeal to the non-degeneracy condition (2.6) and the Lévy-Khintchine formula (2.2) to see that for every $T > 0$ and $\eta > 0$ fixed,

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) ||\xi||^\eta |\hat{p}(s,\xi)|^2 = \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \frac{\hat{p}(s,\xi)}{1 + Re\psi(\xi)} \right|^\eta e^{-2sRe\psi(\xi)} \asymp \int_{\mathbb{R}^d} ||\xi||^\eta \mu(d\xi).$$

This proves that $I_R(T) = 2 \text{IND}$ for all $T \geq 0$. 

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Next, we observe that if \( b \in (0, 1) \), then
\[
\int_0^1 \frac{ds}{s^b} \int_{\mathbb{R}^d} \mu(d\xi) \ |\hat{\psi}(s, \xi)|^2 = \int_{\mathbb{R}^d} \mu(d\xi) \int_0^1 \frac{ds}{s^b} e^{-2s\text{Re}\psi(\xi)} \asymp \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{[1 + \text{Re}\psi(\xi)]^{1-b}},
\]
thanks to (2.6), the Tonelli theorem, and Lemma A.2 of the Appendix. This proves that \( \mathcal{I}_H = \overline{\mathbb{N}}. \)

For the next stage of the proof, we observe that for every \( t > 0 \),
\[
\mathcal{F}(t) := \int_0^t \int_{\mathbb{R}^d} \mu(d\xi) \ |\hat{\psi}(s + r, \xi) - \hat{\psi}(s, \xi)|^2 = \int_0^t \int_{\mathbb{R}^d} \mu(d\xi) \ |e^{-s\psi(\xi)} - e^{-(s+r)\psi(\xi)}|^2 \asymp \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)}.
\]
Therefore, uniformly for all \( a > 0 \),
\[
\sup_{r \in [0, \varepsilon]} \mathcal{F}(r) \lesssim \sup_{r \in [0, \varepsilon]} \int_{\mathbb{R}^d} |1 \wedge r\psi(\xi)|^a \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)} = \int_{\mathbb{R}^d} |1 \wedge \varepsilon\psi(\xi)|^a \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)}.
\]
This readily implies the following for every \( a > 0 \) fixed:
\[
\int_0^1 \frac{d\varepsilon}{\varepsilon^{1+a}} \sup_{r \in [0, \varepsilon]} \mathcal{F}(r) \lesssim \int_{\mathbb{R}^d} |\psi(\xi)|^a \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)}.
\]
Because of Dalang’s condition (1.5), the preceding implies that \( \mathcal{I}_H(T) \geq \overline{\mathbb{N}} \); see (3.9) and (2.7).

Next we prove that for every \( a > 0 \) fixed,
\[
\int_0^1 \frac{d\varepsilon}{\varepsilon^{1+a}} \mathcal{F}(\varepsilon) \gtrsim \int_{|\psi| \geq 2} |\psi(\xi)|^a \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)}.
\]
(4.5)
Because Dalang’s condition holds, this will show that \( \mathcal{I}_H(T) \leq \overline{\mathbb{N}} \), and completes the proof of the lemma.

First, we note that for all \( \xi \in \mathbb{R}^d \) and \( r \geq 0 \),
\[
|1 - e^{-r\psi(\xi)}| = (1 - e^{-r\text{Re}\psi(\xi)})^2 + 2(1 - \cos[|\xi|\text{Im}\psi(\xi)]) e^{-r\text{Re}\psi(\xi)} \gtrsim (1 \wedge r^2(|\text{Re}\psi(\xi)|^2) + (1 - \cos[|\xi|\text{Im}\psi(\xi)]) e^{-r\text{Re}\psi(\xi)}.
\]
It follows from a few lines of computation that
\[
\int_0^1 \frac{d\varepsilon}{\varepsilon^{1+a}} \mathcal{F}(\varepsilon) \gtrsim \int_{\mathbb{R}^d} \frac{|\text{Re}\psi(\xi)|^a \mu(d\xi)}{1 + \text{Re}\psi(\xi)} + \mathcal{B},
\]
where
\[
\mathcal{B} = \int_0^1 \frac{d\varepsilon}{\varepsilon^{1+a}} \int_{\mathbb{R}^d} (1 - \cos[\varepsilon|\text{Im}\psi(\xi)|]) e^{-\varepsilon\text{Re}\psi(\xi)} \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)}.
\]
Since the integrand of the above integral is nonnegative,
\[
\mathcal{B} \gtrsim \int_{|\text{Im}\psi(\xi)| \geq \text{Re}\psi(\xi) \geq 1} \frac{\mu(d\xi)}{1 + \text{Re}\psi(\xi)} \int_0^1 \frac{d\varepsilon}{\varepsilon^{1+a}} (1 - \cos[\varepsilon|\text{Im}\psi(\xi)|]) e^{-\varepsilon\text{Re}\psi(\xi)}
\]
\[
\gtrsim \int_{|\text{Im}\psi(\xi)| \geq \text{Re}\psi(\xi) \geq 1} \frac{|\text{Im}\psi(\xi)|^a \mu(d\xi)}{1 + \text{Re}\psi(\xi)} \int_0^{|\text{Im}\psi(\xi)|} \frac{ds}{s^{1+a}} (1 - \cos s) e^{-s},
\]
after a change of variables \( s = \varepsilon|\text{Im}\psi(\xi)| \). Therefore, we replace \( \int_0^{|\text{Im}\psi(\xi)|} \) (\( \cdots \)) with \( \int_0^1 (\cdots) \) to find that
\[
\mathcal{B} \gtrsim \int_{|\text{Im}\psi(\xi)| \geq \text{Re}\psi(\xi) \geq 1} \frac{|\psi(\xi)|^a \mu(d\xi)}{1 + \text{Re}\psi(\xi)}.
\]
and hence
\[
\int_0^1 \frac{d\varepsilon}{\varepsilon^{1+a}} \mathcal{F}(\varepsilon) \gtrsim \int_{\mathbb{R}^d} \frac{|\text{Re}\psi(\xi)|^a \mu(d\xi)}{1 + \text{Re} \psi(\xi)} + \int_{|\text{Im}\psi(\xi)| \geq \text{Re} \psi(\xi) \geq 1} \frac{|\psi(\xi)|^a \mu(d\xi)}{1 + \text{Re} \psi(\xi)}
\gtrsim \int_{|\psi(\xi)| \geq 2} \frac{|\psi(\xi)|^a \mu(d\xi)}{1 + \text{Re} \psi(\xi)}.
\]

Since \(\psi\) is continuous it is locally bounded. Because \(\mu\) is locally finite, this proves that \(\mathcal{I}_H(T) \leq \text{IND}\) and completes the proof (see (3.9) and (2.7)).

We will also have need for the following classical fact from real analysis.

**Lemma 4.3** (The Paley-Zygmund inequality \(\{43, \text{Lemma } \gamma\}\)). If \(V\) is a non-negative, mean-one random variable in \(L^2(\Omega)\), then \(\mathbb{P}\{V \geq \lambda\} \geq (1 - \lambda)^2 / \mathbb{E}(V^2)\) for every \(\lambda \in (0, 1)\).

We are ready for the following.

**Proof of Theorem 4.1.** First, suppose that \(\text{IND} > 0\). Choose and fix an arbitrary number \(a \in (0, \text{IND})\) and appeal to Lemma 4.2 and Proposition 3.3 in order to see that
\[
\mathbb{E}\left(|H(t + \varepsilon, x) - H(t, x)|^2\right) \lesssim \varepsilon^a,
\]
uniformly for all \(t \in [0, T], \varepsilon \in (0, 1),\) and \(x \in \mathbb{R}^d\). [It might help to also recall that the law of \(t \mapsto H(t, x)\) does not depend on \(x\).] Because
\[
\mathbb{E}(W^n) = (n - 1)!![\text{Var}(W)]^{n/2}
\]
for every even integer \(n \geq 1\), (4.6) and for every centered normally distributed random variable \(W\) on \(\mathbb{R}\), it follows that for every integer \(n\) fixed,
\[
\sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left(|H(t + \varepsilon, x) - H(t, x)|^n\right) \lesssim \varepsilon^{an/2},
\]
uniformly for every \(\varepsilon \in (0, 1)\). Therefore, the Kolmogorov continuity theorem implies that \(\mathbb{P}\{H(\cdot, x) \in C^\alpha_{\text{loc}}(\mathbb{R}^d)\} = 1\) for every \(x \in \mathbb{R}^d\) and \(\alpha \in (0, a/2)\). Since \(a \in (0, \text{IND})\) is otherwise arbitrary, this proves (4.3).

We verify (4.4) next. With this in mind, let us assume that \(\text{IND} > 0\), equivalently \(\mathcal{I}_R(T) > 0\) (by Lemma 4.2). Choose and fix an arbitrary number \(b \in (0, \mathcal{I}_R(T))\) and appeal to Proposition 3.2 in order to see that
\[
\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left(|H(t, x + h) - H(t, x)|^2\right) \lesssim \|h\|^b,
\]
uniformly for every \(h \in \mathbb{R}^d\) that satisfies \(\|h\| \leq 1\). Therefore, (4.6) implies that for every integer \(n \geq 1\),
\[
\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left(|H(t, x + h) - H(t, x)|^n\right) \lesssim \|h\|^{nb/2}
\]
uniformly for every \(h \in \mathbb{R}^d\) that satisfies \(\|h\| \leq 1\). Our assumption that \(\text{IND} > 0\) implies also that \(\text{IND} > 0\); see (2.9). Therefore, (4.7) also holds. Consequently, for every integer \(n \in \mathbb{N}\) and all \(a \in (0, \text{IND})\) and \(b \in (0, \text{IND})\), all fixed,
\[
\mathbb{E}\left(|H(t, x) - H(s, y)|^n\right) \lesssim |t - s|^an/2 + \|x - y\|^{bn},
\]
uniformly for all tuples \((t, x)\) and \((s, y)\) on a (given and fixed) bounded rectangle. Because \(a \in (0, \text{IND})\) and \(b \in (0, \text{IND})\) are otherwise arbitrary, Kolmogorov’s continuity theorem implies (4.4).

Next, we suppose that \(\mathbb{P}\{H(\cdot, 0) \in C^\gamma_{\text{loc}}(\mathbb{R}_+)\} = 1\) for some \(\gamma > 0\) and aim to prove that \(\text{IND} > 0\). Note that
\[
\mathbb{P}\left\{\lim_{\varepsilon \to 0^+} \sup_{s \in (0, 1)} \frac{|H(s + \varepsilon, 0) - H(s, 0)|}{\varepsilon^\theta} = 0\right\} = 1 \quad \text{for every } \theta \in (0, \gamma).
\]
Among other things, this implies the existence of two numbers \( \varepsilon_0 \in (0, 1) \) and \( K > 1 \) such that
\[
\inf_{\varepsilon \in (0, \varepsilon_0)} \inf_{s \in (0, 1)} P \left\{ \left| H(s + \varepsilon, 0) - H(s, 0) \right|^2 < K\varepsilon^{2\theta} \right\} > 1 - \frac{1}{2\pi}. \tag{4.11}
\]

Apply Lemma 4.3 to the random variable \( V := |H(s + \varepsilon, 0) - H(s, 0)|/\|H(s + \varepsilon, 0) - H(s, 0)\|_1 \) to see that
\[
P \left\{ \left| H(s + \varepsilon, 0) - H(s, 0) \right| \geq \frac{1}{2}\|H(s + \varepsilon, 0) - H(s, 0)\|_1 \right\} \geq \frac{\|H(s + \varepsilon, 0) - H(s, 0)\|_2^2}{4\|H(s + \varepsilon, 0) - H(s, 0)\|_2^2} = \frac{1}{2\pi},
\]
thanks to the elementary fact that \( E|W| = \sqrt{2/\pi} E(|W|^2) \) for every centered normal random variable on \( \mathbb{R} \).

Once again we apply Lemma 4.3, this time with \( V := |H(s + \varepsilon, 0) - H(s, 0)|^2/\|H(s + \varepsilon, 0) - H(s, 0)\|_2^2 \) and then we appeal to (4.6) with \( n = 4 \), in order to find that
\[
P \left\{ \left| H(s + \varepsilon, 0) - H(s, 0) \right|^2 \geq \frac{\|H(s + \varepsilon, 0) - H(s, 0)\|_2^2}{2\pi} \right\} \geq \frac{1}{2\pi}. \tag{4.12}
\]

If \( E_1 \) and \( E_2 \) are two events on the same probability space and satisfy \( P(E_1) + P(E_2) > 1 \), then \( P(E_1 \cap E_2) \geq P(E_1) + P(E_2) - 1 > 0 \), thanks to Bonferroni’s inequality. We apply this fact with \( E_1 \) and \( E_2 \) denoting respectively the event in (4.11) and (4.12) in order to see that
\[
\sup_{\varepsilon \in (0, 1)} E \left( |H(s + \varepsilon, 0) - H(s, 0)|^2 \right) \leq 2\pi K\varepsilon^{2\theta}, \tag{4.13}
\]
uniformly for every \( \varepsilon \in (0, \varepsilon_0) \). Because this is true for every \( \theta \in (0, \gamma) \), the second half of Proposition 3.3 implies that \( \mathcal{I}_H \land \mathcal{I}_H(T) > 0 \). Equivalently, \( \text{IND} > 0 \), because of Lemma 4.2.

Similarly, we may use the second half of Proposition 3.2 together with the Paley-Zygmund inequality (Lemma 4.3) to prove that if \( x \mapsto H(t, x) \) is in \( C^\beta(\mathbb{R}^d) \) for some \( \beta > 0 \) and \( t > 0 \), then \( \mathcal{I}_H(T) > 0 \). By Lemma 4.2 this is equivalent to \( \text{IND} > 0 \). This concludes the proof of the theorem. \( \square \)

5 The linear wave equation

In this section we study the regularity and well-posedness of the stochastic wave equation:
\[
\begin{align*}
\frac{\partial_t^2 W}{\partial t} &= LW + \mathbb{F} \quad \text{on} \quad (0, \infty) \times \mathbb{R}^d, \\
\text{subject to} \quad W(0) &= \partial_t W(t) = 0 \quad \text{on} \quad \mathbb{R}^d.
\end{align*} \tag{5.1}
\]
In order to be able to accomplish this goal we will restrict attention to the symmetric case; that is, we assume that the underlying Lévy process \( X \) has the same law as the Lévy process \( -X \). This symmetry condition is equivalent to the following Fourier-analytic property:
\[
\psi(\xi) = \psi(-\xi) = \text{Re}\psi(\xi) \geq 0 \quad \text{for all} \quad \xi \in \mathbb{R}^d. \tag{5.2}
\]

The restriction to \( \mathcal{S}(\mathbb{R}^d) \) of the generator \( \mathcal{L} \) of a symmetric Lévy process is self-adjoint on \( L^2(\mathbb{R}^d) \). Furthermore, in this case the characteristic exponent of \( X \) reduces to
\[
\psi(\xi) = \frac{1}{2} \xi \cdot A\xi + \int_{\mathbb{R}^d} \left[ 1 - \cos(y \cdot \xi) \right] \nu(dy) \quad \text{for all} \quad \xi \in \mathbb{R}^d \tag{5.3}
\]
(see (2.2)).

Throughout this section we also assume that
\[
\lim_{\|\xi\| \to \infty} \psi(\xi) = \infty, \tag{5.4}
\]
which turns out to be a non-degeneracy condition for the law of the random vector $X(1)$. In Remark 5.1 below we discuss further this hypothesis.

For every $\xi \in \mathbb{R}^d$, define

$$\hat{\varphi}(t, \xi) = t \text{sinc} \left( t \sqrt{\psi(\xi)} \right), \quad [t \geq 0],$$

where $\text{sinc}(a) = \sin(a)/a$ for $a \neq 0$ and $\text{sinc}(0) = 1$.\footnote{Some authors use instead the slightly different scaling $\sin(\pi a)/(\pi a)$ for the sinc function; see for example Baumann and Stenger [4].} Thanks to (5.4) and the continuity of $\psi$—ensured by (5.3)—one can check that, for any $t > 0$, the mapping $\xi \mapsto \hat{\varphi}(t, \xi)$ is a tempered distribution; in fact it is a pseudo measure [5, 31].

Notice that for any fixed $\xi \in \mathbb{R}^d$, the function $t \mapsto \hat{\varphi}(t, \xi)$ satisfies the second order linear differential equation

$$\frac{d^2}{dt^2} \hat{\varphi}(t, \xi) + \psi(\xi) \hat{\varphi}(t, \xi) = 0, \quad t \geq 0,$$

subject to initial conditions $\varphi(0, \xi) = 0, \frac{d}{dt} \varphi(0, \xi) = 1$. Thus, we set $\varphi(t, \cdot) := \mathcal{F}^{-1} \hat{\varphi}(t, \cdot)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform operator (acting on the spatial variable of $\varphi$), in order to see that $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, and $\varphi$ is the fundamental solution of the linear wave operator $\partial_t^2 - \mathcal{L}$.

Referring to Section 3, the random-field solution to the wave equation (5.1) is the Gaussian process defined by the stochastic convolution $\varphi \ast F = \{\varphi \ast F(t, \cdot) : t \geq 0, x \in \mathbb{R}^d\}$. This process is well defined if and only if $\varphi \in \mathcal{K}$. Since (5.4) implies that $\{\psi < 1\}$ is bounded, in the context of this section, it follows that

$$\varphi \in \mathcal{K} \quad \text{if and only if} \quad \int_0^T \mu(d\xi) \left[ \text{sinc} \left( t \sqrt{\psi(\xi)} \right) \right]^2 < \infty \quad \text{for all } T > 0.$$  

From the identity $\int_0^c \sin^2(x) \, dx = \frac{1}{2} [1 - \frac{1}{2} \sin(2c)]$, valid for every $c > 0$, we deduce

$$\int_0^T t^2 |\text{sinc}(tK)|^2 \, dt = \frac{T}{2K^2} [1 - \sin(2KT)] \sim \frac{T}{1 + K^2}, \quad (5.5)$$

uniformly for all $K, T \geq 0$ such that $KT \geq 1$. Consequently, $\varphi \in \mathcal{K}$ if and only if Dalang’s condition (1.5) holds; because of (5.2) it reads

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + \psi(\xi)} < \infty.$$  

Remember that we are assuming (1.5) throughout the paper.

Condition (5.4) has relations to other parts of mathematics, and is worthy of inspection in its own right. Therefore, we pause to include some remarks about (5.4). Throughout the following remarks we make extensive references to (2.10). Also, we recall that a finite Borel measure on $\mathbb{R}^d$ is Rajchman if its Fourier transform vanishes at infinity [39]. Thanks to the Lévy-Khintchine formula (5.3), we can interpret (5.4) as saying that $p(t)$ is a Rajchman measure for some $t > 0$.

**Remark 5.1.**

1. If $p(t)$ is absolutely continuous for some $t > 0$ then (5.3) implies (5.4) thanks to the Riemann-Lebesgue lemma. In particular, if $\exp\{-t\psi\} \in L^1(\mathbb{R}^d)$ for some $t > 0$, then (5.4) holds.

2. In the case that $p(t)$ is radially symmetric for all $t > 0$ and $d \geq 2$, Zabczyk [53, p. 245] has proven that (5.4) holds if every excessive function of $X$ is lower semicontinuous. Moreover [53, Example (4.6)], either $X$ is Poisson, or $p(t, dx) \ll dx$, in which case (5.4) holds automatically.

3. If $A$ is non singular, then $\psi(\xi) \gtrless \|\xi\|^2$ uniformly for all $\xi \in \mathbb{R}^d$, whence (5.4) holds.

4. Since $\psi \leq 2\nu(A(R^d))$ on the null space of $A$ [see (5.3)] if $A$ is singular and $\nu(A(R^d)) < \infty$, then (5.4) fails.

5. Suppose the absolutely continuous part of $\nu$ is infinite; that is, $h = dn/dx \notin L^1(\mathbb{R}^d)$. We can deduce from (5.3) that $\psi(\xi) \gtrsim \int_{S(\delta)} [1 - \cos(y \cdot \xi)] h(y) \, dy$ for every $\xi \in \mathbb{R}^d$ and $\delta > 0$, where $S(\delta) = \{y \in \mathbb{R}^d : \|y\| > \delta\}$. Thus, $\lim_{\|\xi\| \to \infty} \psi(\xi) \gtrsim \lim_{\delta \to 0} \int_{S(\delta)} h = \infty$ by the Riemann-Lebesgue lemma, and hence (5.4) fails.

We are ready to study the regularity properties of the solution, assuming (1.5). It might help to first recall (2.9).
Theorem 5.2. 1. If \( \text{IND} > 0 \), then
\[
P \{ W(\cdot, x) \in C^\alpha_{\text{loc}}(\mathbb{R}^d) \} = 1 \text{ for every } x \in \mathbb{R}^d \text{ and } 0 < \alpha < \frac{1}{2} \wedge \text{IND};
\]
and if \( \text{IND} > 0 \), then
\[
P \{ W \in C^{\alpha, \beta}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d) \} = 1 \text{ for every } 0 < \alpha < \frac{1}{2} \wedge \text{IND} \text{ and } 0 < \beta < \text{IND}.
\]

2. If \( \text{IND} = 0 \), then \( t \mapsto W(t, 0) \) is not Hölder continuous; and if \( \text{IND} = 0 \), then \( t \mapsto W(t, x) \) is not Hölder continuous for every \( t > 0 \).

Theorem 5.2 is proved using the same argument as Theorem 4.1 was, except the following index computation replaces the role of Lemma 4.2.

Lemma 5.3. \( I_R(T) = 2\text{IND} \) for all \( T > 0 \), \( I_H = 1 \), and \( I_H(T) = T_H(T) = 2\text{IND} \).

Proof: We proceed in stages.
Stage 1. Because \( |\text{sinc}(x)| \leq 1 \wedge |x|^{-1} \) for all \( x \in \mathbb{R} \setminus \{0\} \) and \( 1 \wedge z^{-1} \approx (1 + z)^{-1} \) uniformly for all \( z \in \mathbb{R}_+ \),
\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\varphi(s, \xi)|^2 \|\xi\|^{2a} \leq \int_0^T s^2 ds \int_{\mathbb{R}^d} \mu(d\xi) \left( 1 \wedge \frac{1}{\psi(\xi)} \right) \|\xi\|^{2a} = \int_{\mathbb{R}^d} \|\xi\|^{2a} \mu(d\xi) 1 + \psi(\xi).
\]
This proves that \( I_R(T) = 2\text{IND} \) for all \( T > 0 \).

The first inequality in (5.8) can be reversed. Indeed, by Tonelli’s theorem and (5.5),
\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\varphi(s, \xi)|^2 \|\xi\|^{2a} \geq \int_{\psi \leq 1/T^2} \|\xi\|^{2a} \mu(d\xi) 1 + \psi(\xi)
\]
uniformly for all \( T > 0 \).

The condition (5.4) ensures that \( \{ \xi \in \mathbb{R}^d : \psi(\xi) \leq C \} \) is compact for every choice of \( C > 0 \). Hence, \( \int_{\psi \leq 1/T^2} \|\xi\|^{2a} (1 + \psi(\xi))^{-1} \mu(d\xi) \) is finite for every \( a > 0 \). Therefore, the preceding two displays show that, for any \( a, T > 0 \),
\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\varphi(s, \xi)|^2 \|\xi\|^{2a} < \infty \text{ if and only if } \int_{\mathbb{R}^d} \|\xi\|^{2a} \mu(d\xi) 1 + \psi(\xi) < \infty.
\]
This is more than enough to show that \( I_R(T) \leq 2\text{IND} \) for all \( T > 0 \), which in turn proves that \( I_R(T) = 2\text{IND} \) for all \( T > 0 \), thanks to the first portion of the proof.

Stage 2. Similarly as in (5.8), for every \( b \in (0, 3) \),
\[
\int_0^1 ds \int_{\mathbb{R}^d} \mu(d\xi) |\varphi(s, \xi)|^2 \leq \int_0^1 ds s^{2-b} \int_{\mathbb{R}^d} \mu(d\xi) \left( 1 \wedge \frac{1}{\psi(\xi)} \right) \geq \int_{\mathbb{R}^d} \mu(d\xi) 1 + \psi(\xi).
\]
We are assuming that \( \int_{\mathbb{R}^d} (1 + \psi(\xi))^{-2} \mu(d\xi) < \infty \). Thus, we see from the definition of \( I_H \) (see (3.9)) that \( b \leq I_H \leq 1 \) for every \( b \in (0, 1) \). It follows that \( I_H = 1 \).

Stage 3. Next we observe that \( |\sin x - \sin y| \leq 2 \wedge |y - x| \) for all \( x, y \in \mathbb{R} \), and hence
\[
\int_0^1 \frac{dz}{z + b} \sup_{r \in [0, e]} \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\varphi(s + r, \xi) - \varphi(s, \xi)|^2
\]
\[
\approx \int_{\mathbb{R}^d} \frac{d\xi}{\psi(\xi)} \int_0^1 \frac{dz}{z + b} (1 + e^2 \psi(\xi))
\]
\[
= \frac{\mu(\psi \leq 1)}{2 - b} + \left( \frac{1}{2 - b} + \frac{1}{b} \right) \int_{\psi > 1} \frac{\mu(d\xi)}{\psi(\xi)^{1 - (b/2)}} - \frac{1}{b} \int_{\psi > 1} \frac{\mu(d\xi)}{\psi(\xi)}
\]
\[
\approx \int_{\mathbb{R}^d} |\psi(\xi)|^{b/2} \mu(d\xi) 1 + \psi(\xi) \text{ for every fixed } b \in (0, 2).
\]
This proves that \( I_H(T) \geq 2\text{IND} \).
For the complementary bound we notice that for every $b, t > 0$,

$$
\mathcal{C} := \int_0^1 \frac{d\epsilon}{\epsilon+\beta} \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \left| \dot{\phi}(s+\epsilon, \xi) - \dot{\phi}(s, \xi) \right|^2
$$

$$
\geq \int_{\psi \geq 16K(t)} \frac{\mu(d\xi)}{1 + \psi(\xi)} \int_0^1 \frac{d\epsilon}{\epsilon+\beta} \sin \left( (s+\epsilon)\sqrt{\psi(\xi)} \right) - \sin \left( s\sqrt{\psi(\xi)} \right) \right|^2
$$

$$
\geq \int_{\psi \geq 16K(t)} \frac{[\psi(\xi)]^{(b-1)/2} \mu(d\xi)}{1 + \psi(\xi)} \int_0^t \frac{d\psi(\xi)}{\psi(\xi)} \int_0^{\psi(\xi)} \frac{d\psi(\xi)}{v^{1+b}} |\sin(u+v) - \sin(u)|^2,
$$

where $K(t) := (\pi/t)^2 \vee 1$.

By Taylor’s theorem, $\sin(u+v) - \sin(u) \geq v/\sqrt{2}$ uniformly for all

$$
v \in \left[0, \frac{\pi}{8}\right] \quad \text{and} \quad u \in \bigcup_{n=0}^{\infty} J_n \text{ where } J_n = \left[2n\pi, 2n\pi + \frac{\pi}{8}\right].
$$

Therefore, for every $b, t > 0$,

$$
\mathcal{C} \geq \int_{\psi \geq 16K(t)} \frac{[\psi(\xi)]^{(b-1)/2} \mu(d\xi)}{1 + \psi(\xi)} \sum_{n \in \mathbb{Z}_+} \int_{J_n} \int_0^{\pi/8} \int_0^{\psi(\xi)} \frac{d\psi(\xi)}{\psi(\xi)} \int_0^{\psi(\xi)} \frac{d\psi(\xi)}{\psi(\xi)} |\sin(u+v) - \sin(u)|^2
$$

$$
\geq \int_{\psi \geq 16K(t)} \frac{[\psi(\xi)]^{(b-1)/2} \mu(d\xi)}{1 + \psi(\xi)} \max \left\{ n \in \mathbb{Z}_+ : 2\pi n + \frac{\pi}{8} \leq t\sqrt{\psi(\xi)} \right\} \mu(d\xi).
$$

Because of the choice of $K(t)$, we have $t\sqrt{\psi(\xi)} \geq 4t\sqrt{K(t)} \geq 4\pi$ on the set $\{\xi \in \mathbb{R}^d : \psi(\xi) \geq 16K(t)\}$; therefore,

$$
\max \left\{ n \in \mathbb{Z}_+ : 2\pi n + \frac{\pi}{8} \leq t\sqrt{\psi(\xi)} \right\} \geq \max \left\{ n \in \mathbb{Z}_+ : n \leq \frac{15t}{32\pi} \sqrt{\psi(\xi)} \right\}
$$

$$
\geq \frac{15t}{32\pi} \sqrt{\psi(\xi)} - 1 \geq \frac{7t}{32\pi} \sqrt{\psi(\xi)}.
$$

Thus, for every fixed $b, t > 0$,

$$
\mathcal{C} \geq \int_{\psi \geq 16K(t)} [\psi(\xi)]^{b/2} [1 + \psi(\xi)]^{-1} \mu(d\xi).
$$

Because (5.4) implies that $\int_{\psi \leq 16K(t)} [\psi(\xi)]^{b/2} [1 + \psi(\xi)]^{-1} \mu(d\xi) < \infty$, it follows that

$$
\mathcal{C} < \infty \quad \implies \quad \int_{\mathbb{R}^d} [\psi(\xi)]^{b/2} [1 + \psi(\xi)]^{-1} \mu(d\xi) < \infty.
$$

This shows that $\mathcal{I}_H(T) \leq 2\mathbb{N} \mathbb{D}$ (see (2.7) and (3.9)), and completes the proof. \qed

### 6 Nonlinear SPDEs

This section is about extending the results of Section 4 to the stochastic heat equation with a nonlinear drift term. In the last part, a brief discussion on the difficulties for a similar extension concerning the wave equation is included.

Consider the SPDE (1.1). We suppose that the function $g$, defined on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \times \Omega$ with values in $\mathbb{R}$, is $\mathcal{B}_{\mathbb{R}_+ \times \mathbb{R}^d} \otimes \mathcal{F}$-measurable and adapted to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ associated to the noise $\tilde{\eta}$. That is, for every number fixed $t \geq 0$, $(x, z, \omega) \mapsto g(t, x, z; \omega)$ is $\mathcal{B}_{\mathbb{R}^d \times \mathbb{R}} \otimes \mathcal{F}_t$-measurable. Furthermore, we fix $T > 0$ and assume the following:
Global Lipschitz continuity. There exists a constant \( c_1(T) > 0 \) such that for all \( (t, x; \omega) \in [0, T] \times \mathbb{R}^d \times \Omega \) and \( z_1, z_2 \in \mathbb{R} \),
\[
|g(t, x, z_1; \omega) - g(t, x, z_2; \omega)| \leq c_1(T)|z_1 - z_2|.
\]

Uniform linear growth. There exists a constant \( c_2(T) > 0 \) such that for all \( (t, x; \omega) \in [0, T] \times \mathbb{R}^d \times \Omega \) and \( z \in \mathbb{R} \),
\[
|g(t, x, z; \omega)| \leq c_2(T)(1 + |z|).
\]

In the sequel, we write \( c_T = c_1(T) \lor c_2(T) \).

As regards the initial condition, we assume that \( u_0 = \{u_0(x); x \in \mathbb{R}^d\} \) is a random field that is independent of the noise \( F \), and satisfies some assumptions made explicit in the specific statements.

As per the theory of Walsh \([52]\), the random-field solution to \((1.1)\) is defined as the unique solution to the following integral equation: For any \( t > 0 \) and \( x \in \mathbb{R}^d \),
\[
u(t, x) = (p(t) \ast u_0)(-x) + \int_0^t ds \int_{\mathbb{R}^d} p(t-s, dy) g(s, y, u(s, y-x)) + H(t, x),
\]
a.s., where \( p(t, \cdot) = \mathbb{P}\{X(t) \in \cdot\} \) – see \((2.10)\) – and \( H(t, x) = (p \ast F)(t, x) \) is the random-field solution to \((4.1)\). It might help to also recall that the first term on the right-hand side – that is \((p(t) \ast u_0)(-x) = \mathbb{E}(u_0(x) + X(t))\) – defines the semigroup of \( X \).

If the condition \( \sup_{x \in \mathbb{R}^d} \mathbb{E}(|u_0(x)|^2) < \infty \) holds, then we may apply a well-known fixed point argument to prove the existence of a jointly measurable and adapted process \( \{u(t, x); t \in [0, T] \times \mathbb{R}^d\} \) that satisfies \((6.1)\) for every \( (t, x) \in [0, T] \times \mathbb{R}^d \) a.s., and
\[
\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|u(t, x)\|_2 < \infty.
\]
Moreover, if \( \sup_{x \in \mathbb{R}^d} \mathbb{E}(|u_0(x)|^k) < \infty \) for some \( k \geq 2 \), then \((6.2)\) holds with the norm \( \| \cdot \|_2 \) there replaced by \( \| \cdot \|_k \) (see e.g. \([16]\) for an approach to the proof).

Suppose first that there exists a non-random constant \( a_0 \in \mathbb{R} \) such that \( g(z) = a_0 \) for all \( z \in \mathbb{R} \). Then, \((6.1)\) yields \( u(t, x) = (p(t) \ast u_0)(-x) + a_0 t + H(t, x) \). It now follows from Proposition \(2.6\) that \( u \) is locally Hölder continuous iff \( H \) is. Consequently, Theorem \(4.1\) tells us that \( u \) is locally Hölder continuous iff \( H \) is constant. This proves half of Theorem \(1.1\). Thanks to the Kolmogorov continuity theorem, the following two propositions (Propositions \(6.2\) and \(6.3\)) together imply the sufficiency of the condition \( H \) > \( 0 \) in the general case, and complete the proof of Theorem \(1.1\).

**Lemma 6.1.** Let \( T > 0 \) and \( k \geq 2 \), and assume that
\[
K(T) := \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|u(t, x)\|_k < \infty.
\]
Set
\[
I(t, x) = \int_0^t ds \int_{\mathbb{R}^d} p(t-s, dy) g(s, y, u(s, y-x)) \quad \text{for every } t \geq 0 \text{ and } x \in \mathbb{R}^d.
\]
Then, there exists a constant \( C_T > 0 \) such that for every \( t \in (0, T] \) and \( \varepsilon \geq 0 \),
\[
\sup_{x \in \mathbb{R}^d} \|I(t + \varepsilon, x) - I(t, x)\|_k \leq C_T \left( \varepsilon + \int_0^t \sup_{y \in \mathbb{R}^d} \|u(s + \varepsilon, y) - u(s, y)\|_k \, ds \right) .
\]
**Proof.** Write \( I(t + \varepsilon, x) - I(t, x) = I_1(t, x; \varepsilon) + I_2(t, x; \varepsilon) \), where
\[
I_1(t, x; \varepsilon) = \int_t^{t+\varepsilon} ds \int_{\mathbb{R}^d} dy \, p(t + \varepsilon - s, dy) g(s, y, u(s, y-x)),
\]
\[
I_2(t, x; \varepsilon) = \int_0^t ds \int_{\mathbb{R}^d} \{p(t + \varepsilon - s, dy) - p(t - s, dy)\} g(s, y, u(s, y-x)).
\]
To estimate $I_1(t, x; \varepsilon)$, we apply Minkowski’s inequality and use the uniform linear growth of the function $g$. In this way we find that
\[
\|I_1(t, x; \varepsilon)\|_k \leq c_T \int_t^{t+\varepsilon} ds \int_{\mathbb{R}^d} p(t + \varepsilon - s, dy) (1 + \|u(s, y + x)\|_k) \leq c_T K(T) \varepsilon, \tag{6.5}
\]
uniformly for all $t \in (0, T]$ and $x \in \mathbb{R}^d$.

Next, we estimate $I_2(t, x; \varepsilon)$ by writing $I_2(t, x; \varepsilon) = I_{2,1}(t, x; \varepsilon) + I_{2,2}(t, x; \varepsilon) + I_{2,3}(t, x; \varepsilon)$, where
\[
\begin{align*}
I_{2,1}(t, x; \varepsilon) &= \int_0^\varepsilon ds \int_{\mathbb{R}^d} p(t + \varepsilon - s, dy) g(s, y, u(s, y - x)), \\
I_{2,2}(t, x; \varepsilon) &= \int_{t-\varepsilon}^t ds \int_{\mathbb{R}^d} p(t - s, dy) g(s, y, u(s, y - x)), \\
I_{2,3}(t, x; \varepsilon) &= \int_0^{t-\varepsilon} ds \int_{\mathbb{R}^d} p(t - s, dy) [g(s, y, u(s + \varepsilon, y - x)) - g(s, y, u(s, y - x))].
\end{align*}
\]
The same arguments that were used to estimate the term $I_1(t, x; \varepsilon)$ yield
\[
\|I_{2,1}(t, x; \varepsilon)\|_k + \|I_{2,2}(t, x; \varepsilon)\|_k \leq c_T K(T) \varepsilon, \tag{6.6}
\]
valid uniformly for all $t \in (0, T]$ and $x \in \mathbb{R}^d$. Finally, because of the Lipschitz continuity property of $g$,
\[
\|I_{2,3}(t, x; \varepsilon)\|_k \leq c_T \int_0^t \sup_{y \in \mathbb{R}^d} \|u(s + \varepsilon, y) - u(s, y)\|_k \, ds,
\]
uniformly for all $t \in (0, T]$ and $x \in \mathbb{R}^d$. Combine this with (6.5), (6.6) in order to deduce (6.4).

**Proposition 6.2.** We assume that $u_0$ satisfies the hypotheses of Proposition 2.6. Then,
\[
\sup_{t \in (0, T]} \sup_{x \in \mathbb{R}^d} \|u(t + \varepsilon, x) - u(t, x)\|_k \lesssim \varepsilon^{\frac{1}{2}(\eta \wedge \text{IND}) + o(1)} \text{ as } \varepsilon \downarrow 0,
\]
where $k \geq 2$ and $\eta \in (0, 1]$ are given in (2.11).

**Proof.** Fix $\varepsilon \in (0, 1)$. Thanks to (6.1) we may write, for all $x \in \mathbb{R}^d$, $t \in (0, T]$ and $\varepsilon > 0$, the decomposition:
\[
u(t + \varepsilon, x) - u(t, x) = J(t, x; \varepsilon) + [I(t + \varepsilon, x) - I(t, x)] + [H(t + \varepsilon, x) - H(t, x)], \tag{6.7}
\]
where $J(t, x; \varepsilon) = (p(t + \varepsilon) * u_0)(x) - (p(t) * u_0)(x)$.

Proposition 2.6 ensures that
\[
\|J(t, x; \varepsilon)\|_k \leq c_1^1 \varepsilon^{n/2}, \tag{6.8}
\]
uniformly for every $t > 0, x \in \mathbb{R}^d$. Furthermore, we recall that
\[
\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \mathbb{E} (\|H(t + \varepsilon, x) - H(t, x)\|_k) \lesssim \varepsilon^{n/2}, \tag{6.9}
\]
for any $a \in (0, \text{IND})$, uniformly in $\varepsilon \in (0, 1)$ (see (4.7)).

Set
\[
f_\varepsilon(t) := \sup_{x \in \mathbb{R}^d} \|u(t + \varepsilon, x) - u(t, x)\|_k, \quad 0 < t \leq T.
\]

As was mentioned before, the assumptions of the proposition imply the validity of (6.3). Then from the estimates (6.8), (6.9) together with Lemma 6.1, we see that for any $\varepsilon \in (0, 1)$ and for any $\delta \in (0, \frac{1}{2}(\eta \wedge \text{IND}))$,
\[
f_\varepsilon(t) \lesssim \varepsilon^\delta + \int_0^t f_\varepsilon(s) \, ds,
\]
uniformly in $\varepsilon \in (0, 1)$. The proposition follows from Gronwall’s lemma since $\delta \in (0, \frac{1}{2}(\eta \wedge \text{IND}))$ can be otherwise arbitrary. \qed
Proposition 6.3. The initial condition $u_0$ is as in Proposition 2.6. Then,
\[\sup_{t \in (0, T]} \sup_{x \in \mathbb{R}^d} \|u(t, x + h) - u(t, x)\|_k \leq \|h\|^\eta + o(1)\]
where $k \geq 2$ and $\eta \in (0, 1]$ are given in (2.11).

Proof. The proof is similar to that of Proposition 6.2, but simpler. Write
\[u(t, x + h) - u(t, x) = J_1(t, x; h) + J_2(t, x; h) + [H(t, x + h) - H(t, x)],\]
where
\[J_1(t, x; h) = (p(t) * u_0)(x + h) - (p(t) * u_0)(x),\]
\[J_2(t, x; h) = \int_0^t ds \int_{\mathbb{R}^d} p(t, dy) [g(s, y, u(s, x + h - y)) - g(s, y, u(s, x - y))].\]
Proposition 2.6 ensures that $\|J_1(t, x; h)\|_k \leq \|h\|^\eta$, and since $g$ is uniformly global Lipschitz continuous, applying Minkowski’s inequality we deduce
\[\|J_2(t, x; h)\|_k \leq \int_0^t \sup_{w \in \mathbb{R}^d} \|u(s, w + h) - u(s, w)\|_k ds,\]
all valid uniformly for all $x, h \in \mathbb{R}^d$ and $t \in (0, T]$. Set
\[\tilde{f}_h(t) = \sup_{x \in \mathbb{R}^d} \|u(t, x + h) - u(t, x)\|_k \quad 0 < t \leq T.\]
From the above discussion we see that
\[\tilde{f}_h(t) \leq c h^\eta + c \int_0^t \tilde{f}_h(s) ds + \sup_{s \in (0, T]} \|H(s, x + h) - H(s, x)\|_k \quad (0 < t \leq T). \tag{6.10}\]
Because of the estimate (4.9) and the identity $I_R(T) = 2^{\text{IND}}$ (see Lemma 4.2), for every $\gamma \in (0, \text{IND})$,
\[\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|H(s, x + h) - H(s, x)\|_k \lesssim |h|^{\gamma},\]
uniformly for all $h \in \mathbb{R}^d$ that satisfy $\|h\| \leq 1$. Consequently, (6.10) yields
\[\tilde{f}_h(t) \leq c_1 h^{\eta + \text{IND}} + c_2 \int_0^t \tilde{f}_h(s) ds \quad \text{for every} \quad t \in (0, T].\]
The proposition follows from Gronwall’s lemma. \[\square\]

Assume that the hypotheses of Proposition 2.6 on moments of $u_0$ hold for any $k \geq 2$. Then Propositions 6.2 and 6.3 and Kolmogorov’s continuity theorem imply the following.

Theorem 6.4. 1. If $\text{IND} > 0$ then for every $x \in \mathbb{R}^d$, $u(\cdot, x)$ is a.s. in $C^\alpha_{\text{loc}}(\mathbb{R}_+)$ for any $\alpha \in (0, \frac{1}{2}(\eta \wedge \text{IND})).$

2. If $\text{IND} > 0$ then with probability one $u \in C^\alpha_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ for every $\alpha \in (0, \frac{1}{2}(\eta \wedge \text{IND}))$ and $\beta \in (0, \eta \wedge \text{IND})$. When $u_0 \equiv g \equiv 0$, this becomes assertions (4.3) and (4.4) of Theorem 4.1, respectively.

It would be interesting to have a counterpart of Theorem 6.4 for the stochastic wave equation. Unfortunately, this question seems at the moment out of reach. The last part of the section is devoted to describe some of the problems to be solved to progress in that direction.

Consider the stochastic nonlinear wave equation:
\[\begin{align*}
\partial_t^2 u &= Lu + g(u) + \dot{F} \quad \text{on} \ (0, \infty) \times \mathbb{R}^d, \\
\text{subject to} \quad u(0) &= \partial_t u(0+), \quad \text{on} \ \mathbb{R}^d, \tag{6.11}
\end{align*}\]
where \( g : \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous function, and we assume that the characteristic exponent \( \psi \) satisfies (5.4) and \( \psi(\xi) = \psi(-\xi) \). Similarly to the case of the heat equation, a stochastic process \( \{ u(t, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \} \) is called a random-field solution to (6.11) if it satisfies some appropriate measurability conditions and for every \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \),

\[
 u(t, x) = \int_0^t ds \int_{\mathbb{R}^d} dy \ G(t - s, x - y) \ g(u(s, y)) + W(t, x),
\]

a.s., where \( G(t, \cdot) \) denotes here the fundamental solution of the wave operator \( \partial_t^2 - \mathcal{L} \), and \( W \) is the random-field solution to (5.1). Remember that \( W \) is defined by the stochastic convolution \( G \ast F \), and also that

\[
 \hat{G}(t, \xi) = t \text{sinc} \left( t \sqrt{\psi(\xi)} \right) \quad \text{for } t \geq 0, \ \xi \in \mathbb{R}^d.
\]

When \( \psi(\xi) = \| \xi \|^2 \), and for any spatial dimension \( d \geq 1 \), the pathwise integral \( \int_0^t ds \int_{\mathbb{R}^d} dy \ G(t - s, x - y) \ g(u(s, y)) \) in (6.12) is rigorously defined in [15, Proposition 3.4]. A further analysis shows that the validity of this proposition can be extended to symmetric characteristic exponents \( \psi \) (as described above) if the fundamental solution \( G \) satisfies the following:

(a) \( G(t, \cdot) \in \mathcal{C}_c^\infty \) (the space of Schwartz distributions of rapid decrease);

(b) the function \( (s, \xi) \mapsto \hat{G}(s, \xi) \) is measurable;

(c) \( \int_0^T \sup_{\xi \in \mathbb{R}^d} |\hat{G}(s, \xi)|^2 \, ds < \infty \);

(d) \( \lim_{h \downarrow 0} \int_0^T \sup_{\xi \in \mathbb{R}^d} \sup_{s < r < s + h} |\hat{G}(r, \xi) - \hat{G}(s, \xi)|^2 \, ds = 0 \).

This is indeed the case, as we now argue.

According to [50, Théorème IX, p.244], condition (a) is equivalent to saying that \( G(t, \cdot) \ast \varphi \in \mathcal{S}(\mathbb{R}^d) \) for every \( \varphi \in \mathcal{D}(\mathbb{R}^d) \). Since \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), this follows from the lower bound \( t \text{sinc}(t \sqrt{\psi(\xi)}) \gtrsim [1 + \psi(\xi)]^{-1} \) along with the assumption (5.4).

Assertion (b) is a consequence of the continuity of \( \hat{\psi} \). Property (c) follows from the upper bound \( t \text{sinc} \left( t \sqrt{\psi(\xi)} \right) \lesssim [1 + \psi(\xi)]^{-1} \). As regards (d), using (6.13), we have

\[
|\hat{G}(r, \xi) - \hat{G}(s, \xi)| \leq \frac{2 \wedge |r - s| \sqrt{\psi(\xi)}}{\sqrt{\psi(\xi)}} \leq |r - s|
\]

(see stage 3 in the proof of Lemma 5.3). This yields \( \sup_{\xi \in \mathbb{R}^d} \sup_{s < r < s + h} |\hat{G}(r, \xi) - \hat{G}(s, \xi)| \leq h \) and therefore, (d) holds.

We deduce that the hypotheses required in the application of [15, Theorem 4.2] to the particular case when the coefficient \( a \) there vanishes and \( \beta = g \), are satisfied. This yields at once two facts:

1) The expression (6.12) is well-defined, that means, the integral \( \int_0^t ds \int_{\mathbb{R}^d} dy \ G(t - s, x - dy) \ g(u(s, y)) \) is a \( L^2 \)-random variable. In fact, setting \( Z(s, x) := g(u(s, y)) \), one has

\[
\left\| \int_0^t ds \int_{\mathbb{R}^d} G(t - s, x - dy) \ g(u(s, y)) \right\|^2_2 \leq \left( \sup_{0 \leq s \leq t} \sup_{x \in \mathbb{R}^d} \| Z(s, x) \|^2 \right) \int_0^t ds \sup_{\xi \in \mathbb{R}^d} |\hat{G}(s, \xi)|^2.
\]

2) There exists a stochastic process \( \{ u(t, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \} \) that satisfies (6.12) and

\[
\sup_{0 \leq s \leq t} \sup_{x \in \mathbb{R}^d} \| u(s, x) \|_2 < \infty.
\]

However, even in the case \( \psi(\xi) = \| \xi \|^2 \), the extension of (6.14) to \( L^p \) moments, with \( p > 2 \), is an open question. Recall that, in Proposition 6.3 (relative to the stochastic heat equation), the existence of \( L^p \) moments of any order \( p > 2 \), is fundamental for the proof.

Comparing with the nonlinear heat equation (6.1), we see that the drift \( g \) in (6.12) is more particular and moreover, the initial conditions are null. This is to ensure a stationary-type property which is crucial in the theory developed in [15], and as a consequence, to give a rigorous meaning to (6.11). It is an open and rather speculative project to build an integration theory à la Conus-Dalang without stationary constraints on the integrands.
7 A comment on fractional powers of \( L \)

Suppose \( X \) is symmetric and that there exists \( q > 0 \) such that

\[
\psi(|\xi|) \geq \|\xi\|^q
\]

uniformly for all \( \xi \in \mathbb{R}^d \). \hfill (7.1)

We can deduce from (1.8) that \( q \leq 2 \). Property (7.1) says that, in some sense, the law of \( X(1) \) is smoother than the law of a radially symmetric \( q \)-stable process. Under these conditions,

\[
\int_{\mathbb{R}^d} \frac{\|\xi\|^{2b} \mu(d\xi)}{1 + \psi(|\xi|)} \leq \int_{\mathbb{R}^d} \frac{|\psi(|\xi|)|^{2b/q} \mu(d\xi)}{1 + |\psi(|\xi|)|},
\]

which, together with (2.9), yields

\[
\text{IND} > 0 \iff \text{IND} > 0 \iff \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\psi(|\xi|)|} < \infty \text{ for some } \tau \in (0, 1). \hfill (7.2)
\]

Whenever \( \tau \in (0, 1) \), we can introduce an independent \( \tau \)-stable subordinator \( S \) (see Bertoin \[6\]); that is \( S \) is a Lévy process with \( S(0) = 0 \) and nondecreasing sample functions, normalized such that \( \text{E} \exp(-\lambda S(1)) = \exp(-\lambda^\tau) \) for every \( \lambda \geq 0 \). Define \( X'(t) = X(S(t)) \) to be the process \( X \), subordinated by \( S \). Then, disintegration shows that \( X' \) is a symmetric Lévy process with \( \text{E} \exp(i \xi \cdot X'(1)) = \exp(-|\psi(|\xi|)|^\tau) \) for all \( \xi \in \mathbb{R}^d \). Moreover, the generator of \( X' \) is the \( \tau \)-th power of \( L \) — denoted by \( (-L)^\tau \) — whose multiplier is defined via symbolic calculus as \( -\psi^\tau \). In the present setting, conditions (1.6) and (1.7) are equivalent to one another, as well as the final condition in (7.2). Now we observe that the final condition in (7.2) is simply Dalang's condition (1.5) for the Lévy process \( X' \). In this way we can see from Dalang’s theory \[16\] and Theorems 1.1 and 4.1 that, in the present setting where \( L \) is self-adjoint and satisfies (7.1), the optimal condition for the Hölder regularity condition of the solution to (1.1) is that the following SPDE has a random field solution for some \( \tau \in (0, 1) \):

\[
\begin{align*}
\partial_t v &= (-L)^\tau v + \hat{F} \quad \text{on } (0, \infty) \times \mathbb{R}^d, \\
\text{subject to } v(0) &= u_0 \quad \text{on } \mathbb{R}^d,
\end{align*}
\]

Analogously, the optimal condition for Hölder regularity of the solution to (1.2) is equivalent to the existence of random-field solution of the following initial-value problem for some \( \tau \in (0, 1) \):

\[
\begin{align*}
\partial_t^2 v &= (-L)^\tau v + \hat{F} \quad \text{on } (0, \infty) \times \mathbb{R}^d, \\
\text{subject to } v(0) &= \partial_t(v(0)) = 0 \quad \text{on } \mathbb{R}^d.
\end{align*}
\]

A Appendix

Lemma A.1. If \( 0 < c < b \), then uniformly for all \( a > 0 \),

\[
\int_0^1 \left( (a \varepsilon)^b \wedge 1 \right) \frac{d\varepsilon}{\varepsilon^{1+c}} \asymp \begin{cases} a^b \wedge a^c & \text{if } 0 < c < b, \\ \infty & \text{if } c \geq b. \end{cases}
\]

Lemma A.2. If \( T > 0 \) is fixed, then \( \int_0^T s^{-a} \exp(-sb) \, ds \asymp (1 + b)^{-1+a} \), uniformly for every \( a \in (0, 1) \) and \( b \geq 0 \).

Lemma A.3. If \( f : (0, 1) \to \mathbb{R}_+ \) is measurable, then

\[
\sup \left\{ b > 0 : \int_0^1 f(t) \frac{dt}{t^{1+b}} < \infty \right\} \supseteq \sup \left\{ a > 0 : \limsup_{t \to 0^+} \frac{f(t)}{t^a} < \infty \right\}.
\]

where \( \sup \emptyset = 0 \). If \( f \) is non decreasing and measurable, then we can replace “\( \sup \)” with an identity.
Proof. First, consider the case that \( f \) is non decreasing and measurable. Let \( I \) denote the quantity on the left-hand side of the identity of the lemma, and \( J \) the quantity on the right. The integral test from calculus tells us that the integral in the definition of \( I \) is finite if \( I = \sum_{n=1}^{\infty} e^{-bn} f(e^{-n}) \) is finite. If \( c \in (0, I) \), then \( S_n < \infty \) and hence \( e^{-cn} f(e^{-n}) \to 0 \) as \( n \) tends to infinity along integers. Apply monotonicity to see that 
\[
\lim_{t \to 0^+} t^{-c} f(t) = 0.
\]
This shows that every \( c < I \) satisfies \( c \leq J \), which is to say that \( I \leq J \). It remains to prove that \( J \leq I \).

On one hand, if \( J = 0 \) then the above proves that \( I = J = 0 \). On the other hand, if \( J > 0 \) and \( c \in (0, J) \), then \( \limsup_{t \to 0^+} t^{-c} f(t) < \infty \). This is equivalent to saying that \( c = \sup_{n \in \mathbb{N}} \left| e^{bn} f(e^{-n}) \right| < \infty \), which in turn implies that \( S_b \leq c \sum_{n=1}^{\infty} e^{-n(b-c)} \) \( < \infty \) whenever \( b > c \). This proves that every \( c < J \) satisfies \( c \leq I \), whence \( J \leq I \), and completes the proof of the identity \( I = J \).

Finally, if \( f \) is not necessarily monotone, then we observe that \( f \leq \bar{f} \) where \( \bar{f}(t) = \sup_{s \in [0,t]} f(s) \). If \( b > I \), then \( \int_0^1 f(t) t^{-1-b} dt = \infty \) and hence \( \int_0^1 \bar{f}(t) t^{-1-b} dt = \infty \). We may apply the already-proved portion of the lemma to conclude that \( \limsup_{t \to 0^+} t^{-b} \bar{f}(t) = \infty \), which is another way to say that \( \lim_{t \to 0^+} t^{-b} f(t) = \infty \), and hence \( b \geq J \). Since this is true for every \( b > I \), it follows that \( I \geq J \).

\[
\square
\]

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