Non-symmetric Macdonald’s polynomials

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Recently Eric Opdam [O2] in the differential case and then Ian Macdonald [M3] in the difference $q,t$-setting introduced remarkable orthogonal polynomials. In contrast to major known families they linearly generate the space of all (non-symmetric) polynomials. Their meaning still needs to be clarified. At the present time we believe in their importance mainly because they are eigenfunctions of the differential [C4,C5] and difference [C1,C2] Dunkl operators. The latter operators play a preponderant role in the representation theory of (affine and) double affine Hecke algebras and related harmonic analysis. Anyway the Hecke algebra technique works better for non-symmetric polynomials than for their celebrated symmetric counterparts defined in [M1,M2].

Paper [O2] is mostly about analytic theory of graded affine Hecke algebras defined by Lusztig. Its first algebraic part is a generalization (and a simplification) of [O1] where the Macdonald conjectures were proved in the differential case. In particular, it contains the presentation of the symmetric (Jacobi) polynomials in terms of non-symmetric ones, the formula for the norms of the latter, and the interpretation of the shift operator [O1,He] via the anti-symmetric polynomials. Macdonald announced in [M3] the difference analogues of these results (when $t = q^k$, $k \in \mathbb{Z}$).

In the present paper, we prove Macdonald’s statements for arbitrary $q,t$, also establishing the duality-evaluation theorem, the recurrence theorem, and the basic facts on non-symmetric polynomials at roots of unity, including a description of the projective action of $SL_2(\mathbb{Z})$ on them (generalizing the representations from [Ki]). We follow [C3] devoted to the same questions in the symmetric case. The main point is the definition of the difference spherical Fourier transform based on the double affine Hecke algebras (generalizing the Hankel transform from [D,J]). For instance, it readily results in the norm-formulas conjectured by Macdonald and proved in [C2] and clarifies why they are so surprisingly simple. Hopefully, this version of the Macdonald theory can be extended to the elliptic case (see [C6]) without serious difficulties.

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Once the Fourier transform appeared, we cannot restrict ourselves to symmetric functions anymore. Even the classical multi-component Fourier transform requires at least the coordinate functions and the corresponding differentiations. It reveals itself at many levels.

First, it is easier to operate with the double affine Hecke algebra than with its (very complicated) subalgebra of symmetric operators. Second, promising applications are expected in arithmetic, where the symmetric elliptic functions have no particular importance. Although much was done by means of the characters of Kac-Moody algebras (see [K]), certainly they and their \( q, t \)-analogues are not enough. Then, non-symmetric polynomials seem more relevant to incorporate the Ramanujan \( 1 \Psi_1 \)-summation and its generalizations into the Macdonald theory. As to physics, they can be transformed into eigenfunctions of the so-called spin-Calogero-Sutherland hamiltonians [C5] and its difference counterparts. We also mention [O2,HO], which contain a lot of analytic evidence on the same point.

In spite of all these, there should exist deeper relations to the representation theory and the combinatorics. Till now, there hasn’t been any interpretation of the non-symmetric Opdam-Macdonald polynomials as characters or generalized characters [EK] (the equivalence of the spin-CS model and the affine Knizhnik-Zamolodchikov equations from [C5] indicates that it could exist). Our technical achievements are far ahead of the understanding of their true place.

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1. Affine root systems

Let \( R = \{ \alpha \} \subset \mathbb{R}^n \) be a root system of type \( A, B, ..., F, G \) with respect to a euclidean form \( (z, z') \) on \( \mathbb{R}^n \ni z, z' \), \( W \) the Weyl group generated by the reflections \( s_{\alpha} \). We assume that \( (\alpha, \alpha) = 2 \) for long \( \alpha \). Let us fix the set \( R_+ \) of positive roots (\( R_- = -R_+ \)), the corresponding simple roots \( \alpha_1, ..., \alpha_n \), and their dual counterparts \( a_1, ..., a_n, a_i = \alpha_i^\vee \), where \( \alpha_i^\vee = 2\alpha_i / (\alpha_i, \alpha_i) \). The dual fundamental weights \( b_1, ..., b_n \) are determined from the relations \( (b_i, \alpha_j) = \delta_i^j \) for the Kronecker delta. We will also introduce the dual root system \( R^\vee = \{ \alpha^\vee, \alpha \in R \} \), \( R_i^\vee \), and the lattices

\[
A = \oplus_{i=1}^n \mathbb{Z}a_i \subset B = \oplus_{i=1}^n \mathbb{Z}b_i,
\]

\( A_\pm, B_\pm \) for \( Z_\pm = \{ m \in \mathbb{Z}, \pm m \geq 0 \} \) instead of \( \mathbb{Z} \). (In the standard notations, \( A = Q^\vee \), \( B = P^\vee \) - see [B].) Later on,

\[
\nu_\alpha = \nu_{\alpha^\vee} = (\alpha, \alpha), \quad \nu_i = \nu_{\alpha_i}, \quad \nu_R = \{ \nu_\alpha, \alpha \in R \},
\]
\[
\rho_\nu = (1/2) \sum_{\nu_0 = \nu} \alpha = (\nu/2) \sum_{\nu_1 = \nu} b_i, \quad \text{for } \alpha \in \mathbb{R}_+,
\]

(1.1)

\[
r_\nu = \rho_\nu^\vee = (2/\nu)\rho_\nu = \sum_{\nu_1 = \nu} b_i, \quad 2/\nu = 1, 2, 3.
\]

The vectors \( \tilde{\alpha} = [\alpha, k] \in \mathbb{R}^n \times \mathbb{Z} \subset \mathbb{R}^{n+1} \) for \( \alpha \in \mathbb{R}, k \in \mathbb{Z} \) form the affine root system \( \mathbb{R}^a \supset \mathbb{R} (z \in \mathbb{R}^n \) are identified with \([z, 0])\). We add \( \alpha_0 \overset{\text{def}}{=} [-\theta, 1] \) to the simple roots for the maximal root \( \theta \in \mathbb{R} \). The corresponding set \( \mathbb{R}_+ \) of positive roots coincides with \( \mathbb{R}_+ \cup \{[\alpha, k], \alpha \in \mathbb{R}, k > 0\} \).

We denote the Dynkin diagram and its affine completion with \( \{\alpha_j, 0 \leq j \leq n\} \) as the vertices by \( \Gamma \) and \( \Gamma^a \). Let \( m_{ij} = 2, 3, 4, 6 \) if \( \alpha_i \) and \( \alpha_j \) are joined by 0, 1, 2, 3 laces respectively. The set of the indices of the images of \( \alpha_0 \) by all the automorphisms of \( \Gamma^a \) will be denoted by \( O (O = \{0\} \) for \( E_8, F_4, G_2 \)). Let \( O^* = r \in O, r \neq 0 \). The elements \( b_r \) for \( r \in O^* \) are the so-called minuscule weights \((b_r, \alpha) \leq 1 \) for \( \alpha \in \mathbb{R}_+ \).

Given \( \tilde{\alpha} = [\alpha, k] \in \mathbb{R}^a, b \in B, \) let

\[
s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)]
\]

for \( \tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1} \).

The affine Weyl group \( W^a \) is generated by all \( s_{\tilde{\alpha}} \) (we write \( W^a = < s_{\tilde{\alpha}}, \tilde{\alpha} \in \mathbb{R}_+ > \)). One can take the simple reflections \( s_j = s_{\alpha_j}, 0 \leq j \leq n \) as its generators and introduce the corresponding notion of the length. This group is the semi-direct product \( W \ltimes A' \) of its subgroups \( W = < s_\alpha, \alpha \in \mathbb{R}_+ > \) and \( A' = \{a', a \in A\} \), where

\[
a' = s_{\alpha}s_{[\alpha, 1]} = s_{[-\alpha, 1]}s_\alpha \quad \text{for } a = \alpha^\vee, \alpha \in R.
\]

The extended Weyl group \( W^b \) generated by \( W \) and \( B' \) (instead of \( A' \)) is isomorphic to \( W \ltimes B' \):

\[
(wb')(z, \zeta) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W, b \in B.
\]

Given \( b_+ \in B_+ \), let

\[
\omega_{b_+} = w_0w_{0}^+ \in W, \quad \pi_{b_+} = b_+^\prime(\omega_{b_+})^{-1} \in W^b, \quad \omega_i = \omega_{b_+}, \pi_i = \pi_{b_+},
\]

where \( w_0 \) (respectively, \( w_{0}^+ \)) is the longest element in \( W \) (respectively, in \( W_{b_+} \)) generated by \( s_i \) preserving \( b_+ \) relative to the set of generators \( \{s_i\} \) for \( i > 0 \).

To describe \( W^b \) as an extension of \( W^a \) we need the elements \( \pi_r = \pi_{b_+}, r \in O \). They leave \( \Gamma^a \) invariant and form a group denoted by \( \Pi \), which is isomorphic to \( B/A \) by the natural projection \( \{b_r \rightarrow \pi_r\} \). As to \( \{\omega_r\} \), they preserve the set \( \{-\theta, \alpha_i, i > 0\} \). The relations \( \pi_r(\alpha_0) = \alpha_r = (\omega_r)^{-1}(-\theta) \) distinguish the indices \( r \in O^* \). Moreover (see e.g. \( [C2]\)):

\[
W^b = \Pi \ltimes W^a, \quad \text{where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j, 0 \leq j \leq n.
\]

(1.6)
We extend the notion of the length to $W^b$. Given $\nu \in \nu_R$, $r \in O^*$, $\tilde{w} \in W^a$, and a reduced decomposition $\tilde{w} = s_{j_1} \ldots s_{j_k} s_{j_1}$ with respect to $\{s_j, 0 \leq j \leq n\}$, we call $l = l(\tilde{w})$ the length of $\tilde{w} = \pi_r \tilde{w} \in W^b$. Setting

$$\lambda(\tilde{w}) = \{\tilde{a}^1 = \alpha_{j_1}, \tilde{a}^2 = s_{j_1}(\alpha_{j_2}), \tilde{a}^3 = s_{j_1} s_{j_2}(\alpha_{j_3}), \ldots \},$$

$$\ldots, \tilde{a}^l = \tilde{w}^{-1}s_{j_l}(\alpha_{j_l})\},$$

one can represent

$$l = |\lambda(\tilde{w})| = \sum_{\nu} l_\nu, \quad \text{for} \quad l_\nu = l_\nu(\tilde{w}) = |\lambda_\nu(\tilde{w})|,$$

(1.7)

$$(\nu = \nu([\alpha, k]) \overset{\text{def}}{=} \nu_\alpha.$$  

To interpret the length geometrically, let us introduce the following (affine) action of $W^b$ on $z \in \mathbb{R}^n$:

$$\langle wb \rangle(z) = w(b + z), \quad w \in W, b \in B,$$

(1.8)

$$s_\tilde{a} (z) = z - ((z, \alpha) + k)\alpha, \quad \tilde{a} = [\alpha, k] \in \mathbb{R}^a,$$

and the affine Weyl chamber:

$$C^n = \bigcap_{j=0}^{n} L_{\alpha_j}, \quad L_{\tilde{a}} = \{z \in \mathbb{R}^n, (z, \alpha) + k > 0\}.$$

Then (see e.g. [C2]):

$$\lambda_\nu(\tilde{w}) = \{\tilde{a} \in \mathbb{R}^n, \langle C^n \rangle \not\subset \tilde{w}(L_{\tilde{a}}), \nu(\tilde{a}) = \nu\}$$

$$\lambda_\nu(\tilde{w}) = \{\tilde{a} \in \mathbb{R}^n, l_\nu(\tilde{w}s_{\tilde{a}}) < l_\nu(\tilde{w})\}.$$  

(1.9)

It coincides with (1.8) due to the relations

$$\lambda_\nu(\tilde{w} \tilde{u}) = \lambda_\nu(\tilde{u}) \cup \tilde{u}^{-1}(\lambda_\nu(\tilde{w})), \quad \lambda_\nu(\tilde{w}^{-1}) = -\tilde{w}(\lambda_\nu(\tilde{w}))$$

(1.10)

$$\text{if} \quad l_\nu(\tilde{w} \tilde{u}) = l_\nu(\tilde{w}) + l_\nu(\tilde{u}).$$

Let us generalize the definition of $\omega_{b_+}, \pi_{b_+}$. See [C2], Definition 1.1, Proposition 1.3, and Theorem 1.4.

**Proposition 1.1.** Given $b \in B$, the decomposition $b = \pi_b \omega_b, \omega_b \in W$ can be uniquely determined from the following equivalent conditions

i) $\omega_b(b) = b_- \in B_-$ and $(\alpha, b) \neq 0$ if $(-\alpha) \in R_+ \ni \omega_b(\alpha),$

ii) $\omega_b(b) \in B_-$ and $l(\omega_b)$ is the smallest possible,

iii) $l(\pi_b) + l(\omega_b) = l(b)$ and $l(\omega_b)$ is the biggest possible,

iv) $\lambda(\omega_b) = \{\alpha \in R_+, (\alpha, b) > 0\}$. 

□
It results from i) and iii), that \( \omega_b \pi_b = b \). As to (1.5), \( \omega_{b_+} \) really sends \( b_+ \) to \( B_+ \) and is the smallest with this property. We will also apply the formulas (see ibid.):

\[(1.12) \quad l_\nu(b) = \sum_\alpha |(b, \alpha)|, \quad \alpha \in R_+, \nu_\alpha = \nu \in \nu_R,\]

\[(1.13) \quad \lambda(b) = \{\tilde{\alpha}, \alpha \in R_+, (b, \alpha) > k \geq 0\} \cup \{\tilde{\alpha}, \alpha \in R_-, (b, \alpha) \geq k > 0\},\]

\[(1.14) \quad \lambda(\pi_b) = \{\tilde{\alpha}, \alpha \in R_-, (b, \alpha) > k > 0 \text{ if } (\alpha, b) < 0, (b, \alpha) \geq k > 0 \text{ if } (\alpha, b) > 0\},\]

where \( \tilde{\alpha} = [\alpha, k] \in R_+^+, b \in B, | \cdot | \) is the absolute value.

\textbf{Convexity.} Let us introduce two orderings on \( B \). Here and further \( b_\pm \) are the unique elements from \( B_\pm \) which belong to the orbit \( W(b) \). Namely, \( b_- = \omega_b \pi_b = \omega_b(b) \), \( b_+ = w_0(b_-) = \omega_b(b) \). So the equality \( c_- = b_- \) (or \( c_+ = b_+ \)) means that \( b, c \) belong to the same orbit. Set

\[(1.15) \quad b \leq c, c \geq b \text{ for } b, c \in B \text{ if } c - b \in A_+,\]

\[(1.16) \quad b \leq c, c \geq b \text{ if } b_- \leq c_- \text{ or } b_- = c_- \text{ and } b \leq c.\]

We use \( <, >, \prec, \succ \) respectively if \( b \neq c \). For instance,

\[c \succ b_+ \iff b_+ > c > b_-, \quad c \succ b_- \iff c \in W(b_-) \text{ or } c > b_+ .\]

The following sets

\[(1.17) \quad \sigma(b) \overset{\text{def}}{=} \{c \in B, c \geq b\}, \quad \sigma_+(b) \overset{\text{def}}{=} \{c \in B, c \succ b\},\]

\[\sigma_+(b) \overset{\text{def}}{=} \{c \in B, c_- > b_-\} = \sigma_+(b_+).\]

are convex. Moreover \( \sigma_+ \) is \( W \)-invariant. By convex, we mean that if \( c, d = c + r \alpha^\vee \in \sigma \) for \( \alpha \in R_+, r \in \mathbb{Z}_+ \), then

\[\{c, c + \alpha^\vee, ..., c + (r - 1)\alpha^\vee, d\} \subset \sigma.\]

Actually all the elements from \( \sigma(b) \) strictly between \( c \) and \( d \) (i.e. \( c + q\alpha, 0 < q < r \)) belong to \( \sigma_+(b) \). Let us adapt Proposition 1.5 from [C2] to the setup of the present paper:

\textbf{Proposition 1.2.} \( i) \) Given \( b \in B, \tilde{\alpha} = [\alpha, k] \in \lambda(b) \), let \( b = \hat{w}(0), \hat{w} = bs_{\tilde{\alpha}}, c = \hat{w}(0) \). Then \( \hat{w} = cw \) for \( w \in W \) and \( c \in \sigma(b) \). More exactly,

\[(1.18) \quad \{\alpha > 0, k > 0\} \Rightarrow b > c > s_\alpha(b) \Rightarrow c \in \sigma_+(b) \]

\[(1.19) \quad \{\alpha < 0, k < (\alpha, b)\} \Rightarrow s_\alpha(b) > c > b \Rightarrow c \in \sigma_+(b) \]

\[\{\alpha > 0, k = 0\} \Rightarrow c = b, \{\alpha < 0, k = (\alpha, b)\} \Rightarrow c = s_\alpha(b) > b.\]

\( ii) \) Let \( \hat{w} = bs_{\tilde{\alpha}_1} ... s_{\tilde{\alpha}_m} \), where we take \( \tilde{\alpha}_i \) from (1.7) for any sequence

\[1 \leq i_1 < i_2 < ... < i_m \leq l = l(b). \text{ Then } c = \hat{w}(0) \in \sigma(b). \text{ Moreover, } c \in \sigma_+(b)\]
iff at least one $\bar{\alpha}^q = [\alpha, k]$ has $k > 0$, and $c \in \sigma_+(b)$ iff at least one of them has $0 < k < (b, a)$.

The ordering $\succ$ on $b \in B$ and the proposition (in a bit different but equivalent form) were applied in [C2] to describe the structure of the operators $Y_b$ (see below). On the other hand, this ordering appeared in [O2] (and then recently in [M3]) to introduce the non-symmetric orthogonal polynomials. The coincidence of these orderings is not by chance. It results from the duality and the Recurrence Theorem below.

2. Double affine Hecke algebras

We put $m = 2$ for $D_{2k}$ and $C_{2k+1}$, $m = 1$ for $C_{2k}, B_k$, otherwise $m = |\Pi|$. Let us set

$$t_\tilde{\alpha} = t_{\nu(\tilde{\alpha})}, \quad t_j = t_{\alpha_j}, \quad \text{where} \quad \tilde{\alpha} \in \mathbb{R}^a, 0 \leq j \leq n,$$

$$X_\tilde{b} = \prod_{i=1}^{n} X_i^{k_i} q^{k_i} \text{ if } \tilde{b} = [b, k],$$

$$\text{for } b = \sum_{i=1}^{n} k_i b_i \in B, \quad k \in \frac{1}{m} \mathbb{Z}.$$ 

Here $q, \{t_\nu, \nu \in \nu_R\}, X_1, \ldots, X_n$ are considered as independent variables.

Later on $C_q$ is the field of rational functions in $q^{1/m}$, $C_q[X] = C_q[X_b]$ means the algebra of polynomials in terms of $X_i^{\pm 1}$ with the coefficients depending on $q^{1/m}$ rationally. We replace $C_q$ by $C_{q,t}$ if the functions (coefficients) also depend rationally on $\{t_{\nu^{1/2}}\}$.

Let $([a, k], [b, l]) = (a, b)$ for $a, b \in B$, $[\alpha, k]^{\nu} = 2[\alpha, k]/(\alpha, \alpha)$, $a_0 = \alpha_0$, $\nu_{\alpha^\nu} = \nu_\alpha$. We also introduce the map $O^* \ni r \rightarrow r^*$, $\alpha_r^* = \pi_r^{-1}(\alpha_0)$.

**Definition 2.1.** The double affine Hecke algebra $\mathcal{H}$ (see [C1,C2]) is generated over the field $C_{q,t}$ by the elements $\{T_j, 0 \leq j \leq n\}$, pairwise commutative $\{X_b, b \in B\}$ satisfying (2.1), and the group $\Pi$ where the following relations are imposed:

1. $(T_j - t_j^{1/2})(T_j + t_j^{-1/2}) = 0, 0 \leq j \leq n$;
2. $T_i T_j T_i \ldots = T_j T_i T_j \ldots \text{, } m_{ij} \text{ factors on each side}$;
3. $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$;
4. $T_i X_b T_i = X_{\theta(b)} X_{\theta(b)}^{-1} \text{ if } (b, \alpha_i) = 1, 1 \leq i \leq n$;
5. $T_0 X_b T_0 = X_{\theta(b)} X_{\theta(b)}^{-1} \text{ if } (b, \theta) = -1$;
6. $T_i X_b = X_b T_i \text{ if } (b, \alpha_i) = 0 \text{ for } 0 \leq i \leq n$;
7. $\pi_r X_b \pi_r^{-1} = X_{\omega_r^{-1}(b)} q^{(b_r, \nu)}$, $r \in O^*$. 

$\square$
Given \( \tilde{w} \in W^a, r \in O \), the product

\[
T_{\pi, \tilde{w}} \stackrel{\text{def}}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \quad \text{where} \quad \tilde{w} = \prod_{k=1}^{l} s_{i_k}, \ l = l(\tilde{w}),
\]

does not depend on the choice of the reduced decomposition (because \( \{ T \} \) satisfy the same "braid" relations as \( \{ s \} \) do). Moreover,

\[
T_{\tilde{v}}T_{\hat{w}} = T_{\tilde{v}\hat{w}} \quad \text{whenever} \quad l(\tilde{v}\hat{w}) = l(\tilde{v}) + l(\hat{w}) \quad \text{for} \quad \tilde{v}, \hat{w} \in W^b.
\]

In particular, we arrive at the pairwise commutative elements

\[
Y_b = \prod_{i=1}^{n} Y_i^{k_i} \quad \text{if} \quad b = \sum_{i=1}^{n} k_i b_i \in B, \quad \text{where} \quad Y_i \stackrel{\text{def}}{=} T_{b_i},
\]
satisfying the relations

\[
T_i^{-1}Y_bT_i^{-1} = Y_bY_{a_i}^{-1} \quad \text{if} \quad (b, a_i) = 1,
\]

\[
T_iY_b = Y_bT_i \quad \text{if} \quad (b, a_i) = 0, \ 1 \leq i \leq n.
\]

Let us introduce the following elements from \( C_i^n \):

\[
t^{\pm \rho} \stackrel{\text{def}}{=} (l_t(b_1)^{\pm 1}, \ldots, l_t(b_n)^{\pm 1}), \quad \text{where}
\]

\[
l_t(\tilde{w}) \stackrel{\text{def}}{=} \prod_{\nu \in \nu_R} l_{\nu}^{(\tilde{w})/2}, \ \tilde{w} \in W^b,
\]

and the corresponding evaluation maps:

\[
X_i(t^{\pm \rho}) = l_t(b_i)^{\pm 1} = Y_i(t^{\pm \rho}), \ 1 \leq i \leq n.
\]

For instance, \( X_{a_i}(t^{\rho}) = l_t(a_i) = t_i \) (see (1.12)).

We will establish the duality of non-symmetric polynomials applying the following theorem ([C2],[C3]).

**Theorem 2.2.** i) The elements \( H \in \mathfrak{S} \) have the unique decompositions

\[
H = \sum_{w \in W} g_wT_wf_w, \ g_w \in C_{q,t}[X], \ f_w \in C_{q,t}[Y].
\]

ii) The map

\[
\varphi : X_i \rightarrow Y_i^{-1}, \ Y_i \rightarrow X_i^{-1}, \ T_i \rightarrow T_i
\]

\[
t_{\nu} \rightarrow t_{\nu}, \ q \rightarrow q, \ \nu \in \nu_R, \ 1 \leq i \leq n.
\]

can be extended to an anti-involution \((\varphi(AB) = \varphi(B)\varphi(A))\) of \( \mathfrak{S} \).

iii) The linear functional on \( \mathfrak{S} \)

\[
[\sum_{w \in W} g_wT_wf_w] = \sum_{w \in W} g_w(t^{\rho})l_t(w)f_w(t^{\rho})
\]
is invariant with respect to $\varphi$. The bilinear form

$$[G, H] \overset{\text{def}}{=} [\varphi(G)H], \quad G, H \in \mathfrak{H},$$

is symmetric ($[G, H] = [H, G]$) and non-degenerate.

The map $\varphi$ is the composition of the involution (see [C1])

$$\varepsilon : X_i \to Y_i, \quad Y_i \to X_i, \quad T_i \to T_i^{-1},$$

$$\tau_i \to \tau_i^{-1}, \quad q \to q^{-1}, \quad 1 \leq i \leq n,$$

and the main anti-involution from [C2]

$$X_i^* = X_i^{-1}, \quad Y_i^* = Y_i^{-1}, \quad T_i^* = T_i^{-1},$$

$$\tau_i \to \tau_i^{-1}, \quad q \to q^{-1}, \quad 0 \leq i \leq n.$$

Let us give the explicit formulas for the action of $\varphi, \varepsilon$ on $T_0$:

$$\varphi(T_0) = Y_{\theta}^{-1}T_0X_{\theta}^{-1} = T_{s\theta}^{-1}X_{\theta}^{-1},$$

$$\varepsilon(T_0) = X_{\theta}T_0^{-1}Y_{\theta} = X_{\theta}T_{s\theta}.$$

The next theorem from [C3] will be used to obtain a projective action of $GL_2(\mathbb{Z})$ on the restricted non-symmetric polynomials when $q, t$ are roots of unity.

**Theorem 2.3.** i) Adding $q^{1/2m}$, the following maps can be uniquely extended to automorphisms of $\mathfrak{H}$, preserving each of $T_1, \ldots, T_n, t$ and $q$:

$$\tau_+ : X_0 \to X_b, \quad Y_r \to X_rY_0q^{-(b_r, b_r)/2}, \quad Y_\theta \to X_0^{-1}T_0^{-2}Y_\theta,$$

$$\tau_- : Y_b \to Y_b, \quad X_r \to Y_rX_rq^{(b_r, b_r)/2}, \quad X_\theta \to T_0X_0Y_\theta^{-1}T_0,$$

$$\omega : Y_b \to X_b^{-1}, \quad X_r \to X_\theta^{-1}Y_rX_rq^{(b_r, b_r)}, \quad X_\theta \to T_0^{-1}Y_\theta^{-1}T_0,$$

where $b \in B$, $r \in O^*$, $X_0 = qX_\theta^{-1}$.

Moreover,

$$\tau_- = \varepsilon\tau_+\varepsilon = \varphi\tau_+\varphi, \quad \omega = \tau_+^{-1}\tau_-^{-1} = \tau_-\tau_+^{-1}\tau_-. $$

ii) The above maps give automorphisms of $\mathfrak{H}$ and the (elliptic braid) group $\mathfrak{B}$ generated by the elements $\{X_b, Y_b, T_i, \tau_i, q^{1/2m}\}$, satisfying the relations (i)-(vi) from Definition 2.1 and (2.4). Let $\mathfrak{A}_o$ be the group of its automorphisms modulo the conjugations by the elements from the center $Z(\mathfrak{B})$ of the group $\mathfrak{B}$ generated by $\{T_1, \ldots, T_n\}$. Considering the images of $\varepsilon$ (see (2.12)), $\tau_-, \omega$ in $\mathfrak{A}_o$ we obtain the homomorphism $GL_2(\mathbb{Z}) \to \mathfrak{A}_o$:

$$
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix} \to \varepsilon, \quad 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \to \tau_+, \\
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \to \omega, \quad 
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} \to \tau_-.
$$
3. Basic representation

Setting

\begin{equation}
\tilde{b} = \prod_{i=1}^{n} x_i^{k_i} q^{k_i} \text{ if } \tilde{b} = [b, k], b = \sum_{i=1}^{n} k_i b_i \in B, \ k \in \frac{1}{m} \mathbb{Z},
\end{equation}

for independent \( x_1, \ldots, x_n \), we consider \( \{X\} \) as operators acting in \( C_{q,t}[x] = C_{q,t}[x_1^\pm 1, \ldots, x_n^\pm 1] \):

\begin{equation}
X_b(p(x)) = x_b p(x), \quad p(x) \in C_{q,t}[x].
\end{equation}

The elements \( \hat{w} \in W_b \) act in \( C_q[x] \) by the formulas:

\begin{equation}
\hat{w}(x_b) = x_{\hat{w}(b)}. \quad \text{In particular:}
\end{equation}

\begin{equation}
\pi_r(x_b) = x_{\omega_r^{-1}(b)} q^{(b_r^*, b)} \text{ for } \alpha_r = \pi_r^{-1}(\alpha_0), \ r \in O^*.
\end{equation}

The Demazure-Lusztig operators (see [KL1, KK, C1], and [C2] for more detail)

\begin{equation}
\hat{T}_j = t_j^{1/2} s_j + (t_j^{1/2} - t_j^{-1/2})(X_{a_j} - 1)^{-1}(s_j - 1), \quad 0 \leq j \leq n.
\end{equation}

act in \( C_{q,t}[x] \) naturally. We note that only \( \hat{T}_0 \) depends on \( q \):

\begin{equation}
\hat{T}_0 = t_0^{1/2} s_0 + (t_0^{1/2} - t_0^{-1/2})(qX_0^{-1} - 1)^{-1}(s_0 - 1),
\end{equation}

where \( s_0(X_i) = X_i X_0^{(b_i, \theta)} q^{(b_i, \theta)}. \)

**Theorem 3.1.** The map \( T_j \rightarrow \hat{T}_j, \ X_b \rightarrow x_b \) (see (2.1, 3.2)), \( \pi_r \rightarrow \pi_r \) (see (3.4)) induces a \( C_{q,t} \)-linear homomorphism from \( \mathfrak{H} \) to the algebra of linear endomorphisms of \( C_{q,t}[x] \). This representation is faithful and remains faithful when \( q, t \) take any non-zero values assuming that \( q \) is not a root of unity (see [C2]). The image \( \hat{H} \) is uniquely determined from the following condition:

\begin{equation}
\hat{H}(f(x)) = g(x) \text{ for } H \in \mathfrak{H} \text{, if } H f(X) - g(X) \in \left\{ \sum_{i=0}^{n} H_i(T_i - t_i) + \sum_{r \in O^*} H_r(\pi_r - 1), \text{ where } H_i, H_r \in \mathfrak{H} \right\}.
\end{equation}

\( \square \)

We will also reformulate Lemmas 2.4 and 3.2 from [C2] in the next proposition. They result directly from Proposition 1.2.
\textbf{Proposition 3.2.} \ i) Given $b \in B$,
\begin{equation}
Y_b = \pi_b \gamma^b_{\omega_b} T_{\omega_b} + \sum_{c \succ b, w \in W} (cw) g^w_b, \quad \text{where } g^w_b \in C_{q,t}(X),
\end{equation}
\begin{equation}
\gamma^b_{\omega_b} = \prod_{\alpha \in \lambda(\pi_b)} \frac{t_{\alpha - 1}^{1/2} X_{\alpha} - t^{-1/2}_{\alpha}}{X_{\alpha} - 1} = \prod_{a \in R^+} \frac{t^{1/2}_{\alpha} (q_{\alpha}) - t^{-1/2}_{\alpha}}{(q_{\alpha}) - 1} \quad \text{for}
\end{equation}
\[(a^\vee, b) > 0 \Rightarrow (a^\vee, b_-) < k < 0, \ (a^\vee, b) < 0 \Rightarrow (a^\vee, b_-) \leq k < 0.\]

\text{ii) Given } b \in B,
\begin{equation}
\dot{Y}_b = T_{\omega_b} \dot{\phi}^b_{\omega_b} \pi^b_{\omega_b} - 1 + \sum_{c \succ b, w \in W} f^w_b (cw)^{-1} \quad \text{for } f^w_b \in C_{q,t}(X),
\end{equation}
\begin{equation}
\dot{\phi}^b_{\omega_b}(X; t) = \gamma^b_{\omega_b}(X; t^*) = \gamma^b_{\omega_b}(X^{-1}; t), \ t_{\nu}^* = t_{\nu}.\]

\textbf{Proof.} We will remind that the main step is the presentation:
\begin{equation}
\dot{Y}_b = b G_{\alpha_1^*} \cdot \cdots \cdot G_{\alpha_l^*}, \ \dot{a}^1 = \alpha_1^\vee, \ \dot{a}^2 = s_{j_1} (\alpha_{j_2}^\vee), \ \dot{a}^3 = s_{j_1} s_{j_2} (\alpha_{j_3}^\vee), \ldots,
\end{equation}
where $b = \pi_r s_{j_1} \cdots s_{j_l}$, $l = l(b), r \in O$, $\dot{a} = \tilde{\alpha}^\vee = \hat{\alpha} / \nu_{\alpha},$
\begin{equation}
G_{\alpha_1^*} = G_{\tilde{a}} = t^{-1/2}_{\alpha} + (t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}) (X_{\alpha} - 1)^{-1} (1 - s_{\alpha}),
\end{equation}
\begin{equation}
G_{\alpha_1^*} = G_{\tilde{a}} \text{ if } \alpha > 0, \ G_{\alpha^*} = G_{\tilde{a}^*} = G_{-\tilde{a}} = G_{-\alpha^*} \text{ otherwise }.
\end{equation}

We note that (3.10) also provides that the coefficients of the operator $\dot{Y}_b$ of any $\dot{w} \neq b$ are zero at the point $\Diamond = (X_1 = \ldots = X_n = 0)$. Here the order of the coefficients (from $C_{q,t}(X)$) and $\dot{w}$ does not matter since $\Diamond$ is $W^b$-invariant. Thus
\begin{equation}
\dot{Y}_b(\Diamond) = \prod_{\nu} t_{\nu}^{(b, \nu)} b.
\end{equation}

Next, we need an extended version of Proposition 3.6 from [C2].

\textbf{Proposition 3.3.} The operators $\{Y_i, 1 \leq i \leq n\}$ preserve $\Sigma(b) \overset{\text{def}}{=} \bigoplus_{c \in \sigma(b)} C_{q,t} x_c$ and the $\Sigma_c(b)$ (defined for $\sigma_c(b)$) for arbitrary $b \in B$. The operators $\{T_j, 0 \leq j \leq n\}$ preserve $\Sigma_+(b) = \Sigma_c(b_+)$:
\begin{equation}
\dot{T}_j(x_b) \mod \Sigma_+(b)
\end{equation}
\begin{equation}
= t_{j}^{1/2} s_{j} x_b + (t_{j}^{1/2} - t_{j}^{-1/2}) x_b \quad \text{if } (b, \alpha_j) < 0,
\end{equation}
\begin{equation}
= t_{j}^{-1/2} s_{j} x_b \quad \text{if } (b, \alpha_j) > 0, = t_{j}^{1/2} x_b \quad \text{if } (b, \alpha_j) = 0.
\end{equation}
Proof. Formulas (3.13) are verified directly. Similarly, assuming that \( \alpha > 0 \) in \( \tilde{\alpha} = [\alpha, k] \),
\[
G_{\tilde{\alpha}^\vee}(x_b) \mod \Sigma_s(b) = t_{\hat{\alpha}}^{1/2}x_b + (t_{\hat{\alpha}}^{1/2} - t_{\hat{\alpha}}^{-1/2})s_{\tilde{\alpha}}(x_b) \quad \text{if} \quad (b, \alpha) \leq 0,
\]
\[
= t_{\hat{\alpha}}^{-1/2}x_b \quad \text{if} \quad (b, \alpha) > 0.
\]
Replacing \( \tilde{\alpha} \), \( t_{\alpha} \) by \( -\tilde{\alpha} \), \( t_{\alpha}^{-1} \) we can use the same formulas for \( G_{\tilde{\alpha}^\vee} \) when \( \alpha < 0 \).

Relations (3.13) and the formulas \( \pi_r(x_b) = x_{\pi_r(b)} \) for \( r \in O \) induce an action of the \textit{affine Hecke algebra} \( \mathcal{H}_Y \) generated by \( T_j(0 \leq j \leq n) \) and the group \( \Pi \) in the space

\[
V(b_-) \overset{\text{def}}{=} \Sigma(b_-)/\Sigma_+(b_-) = C_{q,t}[W(b)] = C_{q,t} \otimes C[W(b)].
\]

The \( \mathcal{H}_Y \)-module \( V(b_-) \) is irreducible (for generic \( q \)). It can be described as the induced representation generated by \( x_+ (= x_{b_+}) \) and satisfying the following defining conditions:
\[
T_j(x_+) = t_j^{1/2} \quad \text{if} \quad (b, \alpha_j) = 0, \quad 1 \leq j \leq n,
\]
\[
Y_a(x_+) = q^{(\alpha, b_+)} \prod_{\nu} t_{\nu}^{(\omega_{b_+}(a), \rho_\nu)} x_+, \quad a \in B.
\]

We also note that (3.13) can be rewritten in \( V(b_-) \) as follows:
\[
s_j(x_b) = (t_j^{1/2}T_j(x_b))^{-1} \quad \text{if} \quad (b, \alpha_j) < 0,
\]
\[
= t_j^{1/2}T_j(x_b) \quad \text{if} \quad (b, \alpha_j) > 0.
\]

Hence this representation corresponds to the natural action of \( W^b \) on the indices of \( T_{\tilde{w}} \) after proper normalization.

4. Orthogonal polynomials

The coefficient of \( x^0 = 1 \) (the \textit{constant term}) of a polynomial \( f \in C_{q,t}[x] \) will be denoted by \( \langle f \rangle \). Let
\[
\mu = \prod_{a \in R^+} \prod_{i=0}^{\infty} \frac{(1 - x_a q_a^i)(1 - x_a^{-1} q_a^{i+1})}{(1 - x_a t_a q_a^i)(1 - x_a^{-1} t_a q_a^{i+1})},
\]
where \( q_a = q^2/\nu \) for \( \nu = \nu_a \).

The coefficients of \( \mu_1 \overset{\text{def}}{=} \mu/\langle \mu \rangle \) are from \( C(q, t) \), where the formula for the constant term of \( \mu \) is as follows (see [C2]):
\[
\langle \mu \rangle = \prod_{a \in R^+} \prod_{i=1}^{\infty} \frac{(1 - x_a (t^\nu q_a^i)^2)}{(1 - x_a (t^\nu t_a q_a^i)(1 - x_a (t^\nu t_a^{-1} q_a^i))}.
\]
Here $x_b(t^{\pm \rho} q^{\rho}) = q^{(b,c)} \prod_{\nu} t_{\nu}^{\pm (b, \rho_{\nu})}.$

We note that $\mu_1^* = \mu_1$ with respect to the involution

$$x_b^* = x_{-b}, \quad t^* = t^{-1}, \quad q^* = q^{-1}.$$ 

If $t_{\nu} = q_{\nu}^{k_{\nu}}$ for $k_{\nu} \in \mathbb{Z}_+$ then $\mu \in \mathbb{C}(q, t)$ (see (6.12)).

Setting

$$\langle f, g \rangle = \langle \mu_1 f, g^* \rangle = \langle g, f^* \rangle \quad \text{for} \quad f, g \in \mathbb{C}(q, t)[x],$$

we introduce the non-symmetric Macdonald polynomials $e_b(x), b \in B_{-}$, by means of the conditions

$$\langle f, x_c \rangle = 0 \quad \text{for} \quad c \in \sigma_* = \{ c \in B, c \succ b \}$$

in the setup of Section 1. They can be determined by the Gram - Schmidt process because the pairing is non-degenerate and form a basis in $\mathbb{C}(q, t)[x]$. We also note that $w_0(e_b(x^{-1})) = e_{w_0(-b)}$ since $-w_0$ does not change the ordering $\succ$.

This definition is due to Macdonald [M3] who generalized Opdam’s non-symmetric polynomials introduced in the degenerate (differential) case in [O2]. He also established the connection with the $Y$-operators introduced in the degenerate (differential) case in [C5] play the role of Dunkl operators from [C5] next. In Opdam’s paper, the trigonometric Dunkl operators from [C5] play the role of $\{Y_b\}$.

The notations are from Proposition 1.1 and (1.1). We identify the operators $H \in \mathfrak{h}$ with their images $H$ and use the involution $\bar{x}_a = x^{-1}_a, \quad \bar{q} = q, \quad \bar{t} = t, \quad a \in B$.

**Theorem 4.1.** The polynomials $\{e_b, b \in B\}$ are eigenvectors of the operators $\{L_f \defeq f(Y_1, \cdots, Y_n), f \in \mathbb{C}_q,t[x]\}$:

$$\begin{align*}
L_f(e_b) &= f(\# b)e_b, \quad \text{where} \quad \# b \defeq \pi_b = b\omega_b^{-1}, \\
L_f(e_b) &= f(\# b)e_b, \quad \pi_b = b\omega_b^{-1}, \\
x_a(bw) &\defeq x_a(q^t - w(\rho)) = q^{(a,b)} \prod_{\nu} t_{\nu}^{-w(\rho_{\nu}, a)}, \quad w \in W.
\end{align*}$$

**Proof.** Due to [C2], $\langle Hf, g \rangle = \langle f, H^* g \rangle$ for any $H \in \mathfrak{h}$ for the anti-involution $^*$ from (2.13). Hence the operators $\{Y_b\}$ are unitary relative to $\langle \ , \ \rangle$. Since they leave all $\Sigma(a), \Sigma_+(a)$ invariant (Proposition 3.3), their eigenvectors in $\mathbb{C}_q,t[x]$ are exactly $\{e\}$. The eigenvalues are readily calculated by means of formula (3.12).

It is worth mentioning that

$$x_a(\# b) = q^{(a,b)} \prod_{\nu} t_{\nu}^{-(\rho_{\nu}, a)},$$

where

$$\rho_{\nu}(b) = \omega_b^{-1}(\rho_{\nu}) = \rho_{\nu} - \sum_{(b, \alpha) \leq 0} \alpha, \quad \alpha \in R_+, \quad \nu = \nu_{\alpha}.$$
The theorem results immediately in the orthogonality of \( \{e_b\} \) for pairwise distinct \( b \). Macdonald also gives the formula for the squares of \( e_b \) (for \( t_{\nu} = q^k, \ k \in \mathbb{Z}_+ \)) and writes that he deduced it from the corresponding formula in the \( W \)-symmetric case ([C2]). A direct simple proof (based on the duality) will be a subject of the next section. Now we come to the connection between \( e_b \) for \( b \) from the same \( W \)-orbit.

Let us fix a set \( \{\varepsilon = \varepsilon_{\nu} \in \{\pm 1\}\} \) ensuring the condition \( \varepsilon_{\nu} = 1 \) if \( \nu = \nu_j \) for at least one index \( j \) such that \( s_j(b_\ldots) = b_\ldots \). We keep the same notations \( \varepsilon_a, \varepsilon_j \) as for \( t \). We introduce "the \( \varepsilon \)-intertwiners" (see e.g. [C2,C4]) as follows:

\[
\Phi^{(\varepsilon)} = (T_j + (1/2 - t_j^{-1/2})(Y_{a_j}^{-1} - 1)^{-1})^{-1}, \quad \phi_j^{(\varepsilon)} = \varepsilon_j t_j^{\varepsilon_j/2} + (1/2 - t_j^{-1/2})(Y_{a_j}^{-1} - 1)^{-1} = \varepsilon_j(t_j^{1/2} + (1/2 - t_j^{-1/2})(Y_{a_j}^{-\varepsilon_j} - 1)^{-1})
\]

for \( 1 \leq j \leq n \). They belong to the proper localization of the affine Hecke algebra \( \mathcal{H}_Y \) and satisfy the same relations as \( \{s_j\} \) do. Hence

\[
\Phi^{(\varepsilon)}(w) = \Phi_{j_1}^{(\varepsilon)} \ldots \Phi_{j_n}^{(\varepsilon)} \quad \text{for} \quad w = s_{j_1} \ldots s_{j_n} \in W
\]

are well defined and \( \Phi \) is a homomorphism of \( W \).

**Proposition 4.2.** Let \( b \in B, \ w = \omega_b: b \rightarrow b_\ldots \) (Proposition 1.1). Then (see (4.5)):

\[
e^{b_\ldots} = \prod_{(a^\gamma, b) > 0} \frac{\varepsilon_a t_a - x_a^{-\varepsilon_a}(\#b)}{1 - x_a^{-\varepsilon_a}(\#b)} \Phi^{(\varepsilon)}(e_b), \quad a \in R^+_+.
\]

**Proof.** The intertwiners permute \( Y: \Phi^{(\varepsilon)}(Y_{a}) = Y_{w(a)}\Phi^{(\varepsilon)}(w) \). Hence the r.h.s of (4.9) is proportional to \( e^{b_\ldots} \). We put the denominators on the right using \( G \) from (3.11) and formula (4.8):

\[
\Phi^{(\varepsilon)}(e_b) = w \left( G_{a^1} + s_{a^1}(t_{a^1}^{1/2} - t_{a^1}^{-1/2})(x_{a^1}^{(\#b)} - 1)^{-1} \right) \ldots \left( G_{a^n} + s_{a^n}(t_{a^n}^{1/2} - t_{a^n}^{-1/2})(x_{a^n}^{(\#b)} - 1)^{-1} \right)
\]

\[
\prod_{a^\gamma \in (\lambda(w))^\gamma} \varepsilon_a t_a^{-1/2} - t_a^{1/2} x_a^{-\varepsilon_a}(\#b) \varepsilon_a x_a^{-\varepsilon_a}(\#b)
\]

where

\[
\{a^1 = a_{1j_1}, a^2 = s_{a_1}(a_{2j_2}), a^3 = s_{a_2}s_{a_{2j_3}}, \ldots, \} = (\lambda(w))^\gamma.
\]

Due to Proposition 1.1 and formula (3.14), the conditions \( a^i > 0, (b, a^i) > 0 \) give that each quantity \( G_{a^i} + \ldots \) acts as the corresponding \( t_{a^i}^{-1/2} \) on the leading \( x \)-term. So the coefficient of \( x^{b_\ldots} \) in (4.10) equals

\[
\prod_{a^\gamma \in (\lambda(w))^\gamma} \varepsilon_a (1 - x_a^{-\varepsilon_a}(\#b))/(1 - t_a x_a^{-\varepsilon_a}(\#b)).
\]
Symmetric polynomials. The above formula results in the following explicit expressions for the \( \varepsilon, t \)-symmetrizations of \( e_{b_-} \). To introduce them we need

\[
P_t^\varepsilon = \sum_{w \in W} \prod_{\nu} (\varepsilon_\nu t_\nu^{1/2})^{\varepsilon_\nu} l_\nu(w) T_w,
\]

\[
F_\varepsilon = \sum_{w \in W} \Phi(\varepsilon) w.
\]

Let us check (see (3.13) and [C2], Proposition 4.6) that \( F_\varepsilon \) is divisible on the left by \( P_t^\varepsilon \) (i.e. \( F_\varepsilon = P_t^\varepsilon( ) \)) and, moreover, \( P_t^+ \equiv P_t^{t=1} \) (cf. Corrolary 4.7, ibidem). We def note the constant set \( \{ \varepsilon = \pm \} \) by \( \pm \).

Using \( \varphi \) from Section 2 (there is also a straightforward way via the induced representations of \( \mathcal{H}_Y \)),

\[
\varphi(\Phi^{\varepsilon}) = s_j \text{ for } \varepsilon_j = +1,
\]

\[
= - \frac{t_j^{1/2} X_{a_j} - t_j^{-1/2}}{t_j^{-1/2} X_{a_j} - t_j^{1/2}} s_j \text{ for } \varepsilon_j = -1
\]

in the basic representation. Hence \( \varphi(\Phi^{\varepsilon}) + 1 \) is divisible on the right by \( T_j + \varepsilon_j t_j^{-\varepsilon_j}/2 \). Applying \( \varphi \) one more time we get the required.

**Proposition 4.3.** The polynomial

\[
p^{(\varepsilon)}_{b_-} \equiv \sum_{b \in W(b_-)} \Phi^{\varepsilon} \omega_b(e_{b_-}) = \sum_{b \in W(b_-)} \prod_{(a', b) > 0, a \in R^+_b} \varepsilon_a t_a - x_a^{\varepsilon_a}(\#b) e_b
\]

is \( \varepsilon, t \)-symmetric, i.e. \( P_t^\varepsilon(p) \) is proportional to \( p \). Moreover, it is \( W \)-invariant when \( \varepsilon = + \).

**Proof.** Because \( \varepsilon_\nu = 1 \) for \( \nu = \nu_j \) if \( s_j(b_-) = b_- \) for some \( j \), \( F_\varepsilon(e_{b_-}) \) is proportional to \( p^{(\varepsilon)}_{b_-} \). Then it is necessary just to use (4.9) divided by \( \Phi_w \).  

We note that the representation of \( \mathcal{H}_Y \) in the Macdonald polynomials for \( b \in W(b_-) \) is isomorphic to \( V(b_-) = \Sigma(b_-)/\Sigma_+(b_-) \) (see (3.15). The image of \( p^{(\varepsilon)}_{b_-} \) is \( m_{b_-} \) (which provides another way to fix it uniquely).

The \( + \)-symmetrizations \( p^{(\varepsilon)}_{b_-} \) are the Macdonald polynomials [M1,M2]. More precisely, he defined a basis \( \{ p_b, b \in B_- \} \) in the space \( C_{q,t}[x]^W \) of all \( W \)-invariant polynomials by the conditions

\[
p_b - m_b \in \Sigma_+(b), \quad \langle p_b, m_c \rangle = 0, \quad \text{when } c \succ b_+.
\]
for the monomial symmetric functions $m_b = \sum_{c \in W(b)} x_c$. One can also introduce $\{p\}$ as eigenvectors for the ($W$-invariant) operators $L_f$, $f \in \mathbb{C}_{q,t}[x]^W$:

(4.15) \[ L_f(p_b) = f(q^bt^{-\rho})p_b, \ b \in B_. \]

Applying any elements from $\mathcal{H}_Y$ to $e_c (c \in W(b_-))$ we get solutions of (4.15), because symmetric $Y$-polynomials are central in $\mathcal{H}_Y$ (due to I. Bernstein). Since $p_b^{(\varepsilon)}$ are of this kind, Proposition 4.3 readily gives the coincidence $p_b = p_b^{(+)} (b = b_-)$ and the coefficients of the decomposition of $p_b$ in terms of $e_c, c \in W(b)$. The formula for these coefficients was announced in [M3] (where $t_\nu = q^k, k \in \mathbb{Z}_+$). In the differential case, the coefficients (for arbitrary $\varepsilon$) were calculated in [O2].

The elements $p_b^{(-)}, b \in B_-$, are also quite remarkable. The map

\[
p_b \rightarrow p_b' = p_b^{(-)}/\det_t \text{ for } \det_t = \prod_{a \in R^+_t} \left( (a x_a)^{1/2} - (a x_a)^{-1/2} \right)
\]

is exactly the action of the shift operator from [C2]. Namely, $p_b'$ is proportional to the Macdonald symmetric polynomial for $t'_\nu = t_\nu q^k, b' = b + \rho$. In the differential case this observation is due to Opdam (ibid.). It is closely connected with the main theorem from [FV]. Macdonald also uses this approach to the shift operators in [M3] (the difference case).

5. Duality, applications

First of all we will use Theorem 2.2 to define the Fourier pairing. In the classical theory the latter is the inner product of a function and the Fourier transform of another function. In this and the next sections we will continue to identify the elements $H \in \mathfrak{H}$ with their images $\hat{H}$. The following pairing on $f, g \in \mathbb{C}_{q,t}[x]$ is symmetric and non-degenerate:

\[
[f, g] = [f(X), g(X)] = [\varphi(f(X))g(X)] = [\bar{f}(X)g(X)] = \{L_\bar{f}(g(x))\}(t^{-\rho})
\]

(5.1) \[ \bar{x}_b = x_{-b} = x_b^{-1}, \bar{q} = q, \bar{t} = t, \]

where $L_\bar{f}$ is from Theorem 4.1, and we used the main defining property (3.7) of the representation from Theorem 3.1.

The Fourier adjoint $\varphi(L)$ of any $\mathbb{C}_{q,t}$-linear operator $L$ acting in $\mathbb{C}_{q,t}[x]$ is defined from the relations:

(5.2) \[ [L(f), g] = [f, \varphi(L)(g)], \ f, g \in \mathbb{C}_{q,t}[x]. \]

This anti-involution ($\varphi^2 = \text{id}$) extends $\varphi$ from (2.9) by construction. If $f \in C_{q,t}[x]$, then $\varphi(L_f) = \bar{f}(X)$. We arrive at the following theorem:
Duality Theorem 5.1. Given $b, c \in B$ and the corresponding Macdonald’s polynomials $e_b, e_c$,

\begin{align}
& e_b(\#) e_c(\#) = e_b(t^{-\rho(c)}) e_c(t^{-\rho}) = [e_b, e_c] = e_c(\#) e_b(\#) = e_c(t^{-\rho(b)}) e_b(t^{-\rho}) = [e_c, e_b]
\end{align}

in the notations from (4.6).

To complete the theorem we will calculate $e_b(t^{-\rho})$ together with the norms $\langle e_b, e_b \rangle$ of the renormalized Macdonald polynomials $e_b \overset{def}{=} e_b/e_b(t^{-\rho})$ by means of the Recurrence Theorem.

Main Theorem. Given $b \in B$, let $b_-, b_+ = w_0(b_-)$ be the corresponding elements from $W(b) \cap B_\pm$, $b^0 = -w_0(b)$. Then

\begin{align}
& e_b(t^{-\rho}) = x_{b_-}(t^\rho) \prod_{a \in R^+_+} \left( \frac{1 - q_a^j t a x_a(t^\rho)}{1 - q_a t a x_a(t^\rho)} \right), \\
& \langle e_b, e_b \rangle = \prod_{a \in R^+_+} \left( \frac{t_a^{1/2} - q_a^j t a^{-1/2} x_a(t^\rho)}{t_a^{-1/2} - q_a^{j_1} t a^{1/2} x_a(t^\rho)} \right),
\end{align}

where the products are over the same set $J_b = \{ j \}$:

\begin{align}
& 0 < j < (a^\vee, b_+) \text{ if } (a, b^0) > 0, \quad 0 < j \leq (a^\vee, b_+) \text{ if } (a, b^0) < 0.
\end{align}

We mention that there is a straightforward passage to non-reduced root systems and to $\mu$ introduced for $\alpha \in R_+$ instead of $a \in R^+_+$. As to the latter case, it is necessary just to replace the indices $a$ by $\alpha$ ($q_a \to q$, $\rho \to r$) in the formulas for $\{ e_a \}$. In the $W$-symmetric case these statements (the Macdonald conjectures) are from [C2,C3]. In [M3], Macdonald gives a formula for the norms of $e_b$. Hopefully it coincides with (5.5) after the multiplication by $e_b(t^{-\rho}) e_b(t^{-\rho})^\ast$ and then by $\langle \mu \rangle$. In his paper, $t_\nu = q^k, k \in \mathbb{Z}$. The differential case is due to Opdam.

Discretization. Let us establish the recurrence relations for the Macdonald polynomials generalizing the three-term relation for the $q$-ultraspherical polynomials (Askey, Ismail) and the Pieri rules. We follow [C3] where the symmetric case was considered. We need to go to the lattice version of the functions and operators. The discretization of functions $g(x)$ in $x \in \mathbb{C}^n$ and
the operators acting on such functions is defined as in Theorem 4.1:
\[ \delta x_a(bw) = x_a(q^{b_t - w(\rho)}) = q^{(a,b)} \left( t_{\nu}^{-w(\rho),a} \right), \]
(5.7)
\[ (\delta \hat{u}(\delta g))(bw) = \hat{u}(\delta g^{-1}bw), \quad \hat{u} \in W^b, \]
\[ (\delta X_a(\delta g))(bw) = x_a(bw) \delta g(bw). \]

It is a homomorphism. The image is the space \( \text{Funct}(W^b, \mathbb{C}_{q,t}) \) of functions on \( W^b \) and operators acting on such functions. We will sometimes omit \( \delta \) and put \( g(\hat{w}) \) instead of \( \delta g(\hat{w}) \) etc.

Given an arbitrary linear combination of functions \( \{\phi_{\hat{w}}( ), \hat{w} \in W^b \} \), we can also apply the above operators to the sufficies:
\[ \delta(g(x)\hat{u}) \left( \sum_{\hat{w} \in W^b} c_{\hat{w}}\phi_{\hat{w}}( ) \right) = \sum_{\hat{w} \in W^b} c_{\hat{w}}g(\hat{w})\phi_{\hat{w}^{-1}}( ), \quad c_{\hat{w}} \in \mathbb{C}. \]
(5.8)

It is an anti-homomorphism, i.e.
\[ \delta(GH) = \delta H \delta G \quad \text{for operators } G, H. \]

We will mostly use the discretizations \( \epsilon_b(\hat{w}) = e_b(\hat{w})/e_b(0) \) of the renormalized Macdonald polynomials \( \epsilon_b(x) = e_b(x)/e_b(t^{-\rho}) \), and especially \( \epsilon_b(\#b) = \epsilon_c(\#b) \), where \( \#c = \pi_c = \xi_c^{-1} \). See (4.5),(4.6), and Theorem 5.1. Sometimes we drop \( \# \) and write \( \epsilon_b(c) \) instead of \( \epsilon_b(\#c) \). For example, \( \epsilon(0) \) always means \( \epsilon(\#) \). Vice versa, we will consider the sufficies \( b \) as elements from \( W^b \) via the same map \( b \rightarrow \#b \).

Given a polynomial \( f \in \mathbb{C}[x] \), we construct the operator \( L_f = f(Y) \), go to its discretization \( \delta L_f \), and finally introduce the recurrence operator \( \Lambda_f = \delta L_f \) acting on the sufficies \( \hat{w} \in B \) of any \( \mathbb{C} \)-valued functions \( \phi_{\hat{w}}( ) \). We write \( \Lambda_a \) when \( f = x_a, a \in B \).

**Recurrence Theorem 5.2.** For arbitrary \( a, b \in B, f \in \mathbb{C}[x] \),
\[ \Lambda_f(\epsilon_b(x)) = \bar{f}(x)e_b(x), \quad \Lambda_a(\epsilon_b(x)) = x_a^{-1}e_b(x), \]
(5.9)
where \( \bar{f}(x) = f(x^{-1}) \). The operators \( \Lambda \) (acting on \( \#b \)) are well defined. It means that they do not produce the indices which do not belong to \( \#B = \{\pi_b, b \in B\} \).

**Proof.** We can rewrite (4.15) as follows:
\[ \delta L_f(\delta e_b) = \bar{f}(\#b) \delta e_b. \]
(5.10)
Replacing \( e \) by \( \epsilon \) and using the duality we yield:
\[ \delta L_f(\epsilon_b(c)) = \bar{f}(\#) \delta e_b(c), \]
\[ \Lambda_f(\epsilon_b(c)) = \delta \hat{f}(c) \delta e_b(c), \]
(5.11)
and (5.9) if we can ensure that $\Lambda_f$ does not create polynomials $\epsilon_c$ with the indices apart from $\# B$. The latter will be checked in the next section.

The theorem has many applications. For instance, we can prove the Main Theorem. To demonstrate this let (see Proposition 3.2)

$$
Y_a = \sum_{b \geq a, w \in W} (bw)g^b_a, \quad \Lambda_a = \sum_{b \geq a, w \in W} \delta(g^b_a)g^{bw},
$$

$$
Y_{-a} = \sum_{b \geq a, w \in W} f^{bw}_a, \quad \Lambda_{-a} = \sum_{b \geq a, w \in W} \delta(bw)^{-1}\delta(f^{bw}_a).
$$

Thanks to the theorem:

$$
\Lambda_a \epsilon_b = x_a^{-1} \epsilon_b = \sum_{\tilde{w}} g^\tilde{w}_a(\tilde{w}^{-1} \# b) \epsilon_{\tilde{w}^{-1} \# b},
$$

$$
\Lambda_{-a} \epsilon_b = x_a \epsilon_b = \sum_{\tilde{w}} f^{\tilde{w}}_a(\# b) \epsilon_{\tilde{w} \# b}.
$$

For example,

$$
g^0_a(\#) = \langle \mu_1 x_a^{-1} \rangle = \langle \mu_1 x_a \rangle^* = f^0_a(\#)^* \\
g^0_b(\#) = \langle \mu_1 x_a^{-1} \rangle = \langle \mu_1 x_a \rangle^* = f^0_b(\#)^*.
$$

are the coefficients of $\mu_1$ (we remind that $g(\#) = g(t^{-\rho})$). Their description is one of the main open problems in the Macdonald theory (we hope to consider it in the next papers).

**Proposition 5.3.** The presentation $x_a = \sum_{b \geq a} f^{\# b}(\#) \epsilon_b$ leads to:

$$
f^{\# b}(\#)^* \epsilon_b, \epsilon_b = \langle \epsilon_b, x_a \rangle = g^{\# b}_a(\#),
$$

$$
\epsilon_b(\#) = f^{\# b}_b(\#), \quad \langle \epsilon_b, \epsilon_b \rangle = g^{\# b}_b(\#)(f^{\# b}_b(\#)^*)^{-1}.
$$

**Proof.** The first relation results from (5.13) for $b = 0$. It gives readily the formula for $e_a(\#)$. Then

$$
\langle \epsilon_b, x_a \rangle = \langle \epsilon_b, \sum_b f^{\# b}_a(\#) \epsilon_b \rangle = (f^{\# b}_a(\#)^*)^* \langle \epsilon_b, \epsilon_b \rangle.
$$

On the other hand,

$$
\langle \epsilon_b, x_a \rangle = \langle \mu_1 \epsilon_b x_a^{-1} \rangle = g^{\# b}_a(\#),
$$

which is (5.14). Letting $a = b$, we come to the last formula for the norm.

**Proof of the Main Theorem.** In fact, the coefficients $f^{\# b}_b(\#), g^{\# b}_b(\#)$ were calculated in Proposition 3.2. We need only to substitute the evaluation of $T_{\omega_b}$ at $\#$, that is $l_t(\omega_b) = \prod_{\nu} l^{I_{\nu}(w)/2}$.

6. Roots of unity
Let us assume that $q$ is a primitive $N$-th root of unity for $N \in \mathbb{N}$ and first consider $t$ as an indeterminate parameter. More precisely, we will operate over the field $\mathbb{Q}_t^0 \overset{\text{def}}{=} \mathbb{Q}(q_0, t)$ where we fix $q_0$ such that $q_0^{2m} = q$ ($q_0$ belongs to a proper extension of $\mathbb{Q}$). Actually all formulas will hold even over the localization of $\mathbb{Z}[q_0, t]$ by $t^{r_1}q^{s_1}(1 - t^{r_2}q^{s_2}) \not= 0$, $r_1, s_i \in \mathbb{Z}$.

The pairing
\begin{equation}
B \times B \ni a \times b \to q^{(a,b)} \overset{\text{def}}{=} q_0^{2m(a,b)}
\end{equation}
acts through $B_N \times B_N$, where $B_N \overset{\text{def}}{=} B/K_N$, $K_N$ is its radical.

Following the previous section we restrict the functions $\{x_b\}$ and the operators $\{Y_b\}$ to the $W^b$ using the pairing (6.1) and the formulas $x_a(bw) = x_a(q^{b}t^{-w(p)}, b \in B, w \in W$. Given $w \not= u$ and any $b, c$, there exists $a \in B$ such that $x_a(bw) \not= x_a(cu)$ ($t$ is generic). Hence the discretization maps via $W_N^b \overset{\text{def}}{=} B_N \times W$.

The $T, Y$-operators are well defined over $\mathbb{Q}_0^0$ since their denominators are products of the binomials $(x_aq^k - 1)$ for $a \in R^i, k \in \mathbb{Z}$. The latter remain non-zero when evaluated at $q^{b}t^{-w(p)}$ since $(a, \rho)$ never equals 0 ($x_a(t^{-w(p)})$ always contains $t$). Hence the discretizations of these operators exist too. More exact information about the properties of these coefficients can be extracted from Proposition 5.3.

Let $B(N) \subset B$ be a fundamental domain of the group $K_N$. It means that the map $B(N) \ni b \to \# b \in W_N^b$ is an isomorphism. Further we identify these two sets, putting
\begin{equation}
B(N) = \{\beta_1, \ldots, \beta_d\} = W_N^b, \quad \text{where } d = |W_N^b|, \quad \beta_1 = 0.
\end{equation}
The images of $\hat{w} \in W^b$ in $B(N)$ will be denoted by $\hat{w}'$ (i.e. $\hat{w}' = \beta_0$ if $\hat{w} = \# \beta_0 \mod K_N$). One may assume that $-w_0(B(N)) = B(N)$ for the longest element $w_0$.

Abusing the notations, we write $g(b)$ where $b \in B$ instead of $\delta g(\# b)$. Correspondingly, by $b'$ we mean the image of $\# b$ in $B(N)$, that is the image of $b$ in $B_N$.

Let us consider (temporarily) the case when $N$ is coprime with the order $|B/A| = |O|$ taking $q_0^2 = q^{1/m}$ in the $N$-th roots of unity. Then $K_N = NP \cap B$ for the weight lattice $P = \oplus_{i=1}^{n} \mathbb{Z}\omega_i$ generated by the $\omega_i$ (dual to $a_i$). We can take the following fundamental domain:
\begin{equation}
B(N) \subset \{c \in W(b_-), \quad \text{where } b_- = - \sum_{i=1}^{n} k_i b_i \in B_-, \}
\end{equation}
$0 \leq k_i \leq N$ if $(2/\nu_i, N) = 1$, $0 \leq k_i \leq \nu_i N/2$ otherwise, removing $b_-$ such that $k_i = 0, k_j = N/(2/\nu_i, N)$ for at least one pair of the indices. Moreover, if $c_- \in a_- + NP_+$ then we do not take $a \in W(a_-)$ when
\(\omega_a \in \{\omega_c, c \in W(e_-)\}\). Recall that \(2/\nu_i = 2/(\alpha_i, \alpha_i) = 1, 2, 3, \#b = b\omega_b^{-1}\) (see Section 1).

Let us demonstrate that the Macdonald polynomials \(e_b\) are well defined for \(b \in B(N)\) (later we will see that they always exist). We introduce them directly from (4.15), using that the \(Y\)-operators preserve any subspaces

\[
\Sigma^0_a(b) = \oplus_{c>b} Q_a^0 x_c, \quad \text{and} \quad \Sigma^0(b) = \Sigma^0_a(b) \oplus Q_a^0 x_b, \quad b, c \in B.
\]

It is necessary to check that given \(B \ni c > b\), there exists at least one \(a \in B\) such that \(x_a(q^{b}t^{-\rho(b)}) \neq x_a(q^{c}t^{-\rho(c)})\) for

\[
\rho(b) = \omega_b^{-1} = \rho - \sum \alpha, \quad \alpha \in R_+, \quad (b, \alpha) \leq 0
\]

(see (4.6)). Then the eigenvalues of \(Y\) will separate \(e_b\) from the elements from \(\Sigma^0_a(b)\) and we can argue by induction.

It is obvious if \(\omega_b \neq \omega_c\). Otherwise \(k_i < N(2/\nu_i, N)^{-1}\) and \(c_- > b_-\). Then we repeat the corresponding reasoning from [C3], Section 5.

The discretizations \(e_b(\hat{\omega})\) are well defined too and depend only on the images \(\hat{\omega}'\) because \(\epsilon_{\omega}'\) are linear combinations of \(x_a, a \in B\). We see that \(\{\epsilon_{\beta}(\hat{\omega}')\}\) form a basis in the space \(V_N \overset{\text{def}}{=} \text{Funct}(B(N), Q^0_b)\) of all \(Q^0_b\)-valued functions on \(B(N)\). Indeed, they are non-zero and the action of the \(\{\delta_{Y_a}\}\) ensures that they are linearly independent.

**The end of the proof of Theorem 5.2.** First of all, let us rewrite formally relation (5.9) for \(f = x_a\) as follows:

\[
(6.4) \quad x_a^{-1} \epsilon_b(x) = \Lambda^\#_a(\epsilon_b(x)) + \sum_{\hat{\omega} \notin \#B} M^\hat{\omega}_{ab} \epsilon_{\hat{\omega}}(x).
\]

Here \(\hat{\omega}\) form a finite set \(E = E(a, b)\) \((E \cap \#B = \emptyset)\), \(M^\hat{\omega}_{ab}\) are rational functions of \(q, t\). The truncation \(\Lambda^\#_a\) of \(\Lambda_a\) is uniquely determined by the condition that it does not contain \(\hat{w}\) moving \(\#b\) to elements apart from \(\#B\). Assuming that \(N\) is sufficiently big the discretization gives the relation (see (5.11)):

\[
(6.5) \quad x_a^{-1}(c) \epsilon_b(c) = \Lambda^\#_a(\epsilon_b(c)) + \sum_{\hat{\omega} \notin \#B} M^\hat{\omega}_{ab} \epsilon_{\hat{\omega}'}(c), \quad c \in B(N),
\]

for \(x_a(c) = \delta x_a(c)\). Here \(\epsilon_{\hat{\omega}'} = \epsilon_{\hat{\omega}}\) for \(\hat{\omega} \in E\). This substitution was impossible before the discretization. We remind that the formula with \(\epsilon_{c}(\hat{\omega})\) in place of \(\epsilon_{\hat{\omega}}(c)\) is always true. Because \(c\) is taken from \(B(N)\) the discretization of \(\epsilon_c\) exists. Therefore we can replace the argument \(\hat{\omega}\) by \(\hat{\omega}' \in B(N) \subset B\), and then go from \(\epsilon_c(\hat{\omega}')\) to \(\epsilon_{\hat{\omega}}(c)\) thanks to the duality.

As to \(M^\hat{\omega}_{ab}\), they are the values of the coefficients of \(\delta Y_a\) and are well defined when \(q^N = 1\) (enlarging \(N\) we can get rid of singularities in \(q\) even if \(M\) are arbitrary rational).
On the other hand:

\[(6.6)\]

\[
x^{-1}_a \epsilon_b(x) = \sum_{h \in B} K^h_{ab} \epsilon_h(x),
\]

where the coefficients \(K^h_{ab}\) are rational functions of \(q, t, \{h\}\) form a finite set \(H = H(a, b) \subset B\). The discretization gives that

\[(6.7)\]

\[
x^{-1}_a(c) \epsilon_b(c) = \sum_{h \in B} K^h_{ab} \epsilon_h(c), \quad c \in B(N).
\]

We pick \(N\) to avoid possible singularities.

Since \(N\) is sufficiently big, the eigenvalues of the \(Y\)-operators distinguish all \(\epsilon_d(c)\) for \(d \in (E)' \cup H\). It holds only for generic \(t\) (say, when \(t = 1\) it is wrong). Comparing (6.5) and (6.7) we conclude that \(M^\hat{w}_{ab} = 0\) for all \(\hat{w} \in E\), when \(q^N = 1\). Using again that \(N\) is arbitrary (big enough, coprime with \(|O|\)) we get that the actions of \(\Lambda\#\) and \(\Lambda_a\) coincide on \(\epsilon_b\), i.e. the latter operator does not create the indices not from \(#B\).

Let us go back to the general case (we drop the condition \((N, |O|) = 1\)). Once the Recurrence Theorem has been established we can use Proposition 5.3 without any reservation. It readily gives that the Macdonald polynomials \(\epsilon_b, \epsilon_b, \delta\) are well defined for arbitrary \(b \in B\) because \(f^b_{\#}(t^{-\gamma}) \neq 0\) (formulas (5.4) and (5.15)). Moreover, \(\delta \epsilon_b = \delta \epsilon_b\) if and only if \(b' = c'\). Hence the restricted Macdonald polynomials \(\epsilon_{\beta_i}(\hat{w}), 1 \leq i \leq d\) (see (6.2)) form a basis in \(V_N = \text{Funct}(B(N), Q_0^t)\). Indeed, \(\delta \epsilon_{\beta_i}(c')\) are eigenvectors of the \(\delta L\)-operators separated by the eigenvalues. They are always non-zero since \(\epsilon_0(0) = 1\). Hence they are linearly independent over \(Q_0^t\) and form a basis in \(V_N\). Every \(\delta \epsilon_b\) is an \(L\)-eigenvector and coincides with one of them (when \(\beta_i = b\)).

Similarly, \(\delta L_{\epsilon_b} = \delta L_{\epsilon_c}\) if and only if \(b' = c'\) because the latter condition is necessary and sufficient to ensure the coincidence of the sets of eigenvalues. We will also use the basis of the delta-functions \(\delta_{\beta_i}(\beta^j) \overset{\text{def}}{=} \delta_{ij}\) separated by the action of \(\{\delta x_a\}\).

**Proposition 6.1.** The discretization map supplies \(V_N\) with the structure of an \(\hat{S}\)-module which is irreducible. The Fourier pairing is well defined on \(V_N\) and induces the anti-involution \(\varphi\).

**Proof.** The radical of the Fourier pairing (5.1) contains the kernel of the discretization map \(Q_0^t[x] \to V_N\). Its restriction to \(V_N\) is non-degenerate since

\[(6.8)\]

\[
\Pi = (\epsilon_{ij}), \quad \text{where} \quad \epsilon_{ij} = \langle \epsilon_{\beta_i}, \epsilon_{\beta_j} \rangle = \epsilon_{\beta_i} (\delta^j),
\]

is the matrix connecting the bases \(\{\epsilon\}\) and \(\{\delta\}\). The corresponding anti-involution coincides with \(\varphi\). Thus \(V_N\) is semi-simple.
If $V_N$ is reducible then $\epsilon_{\beta_1} = 1$ generates a proper $H$-submodule ($\neq V_N$). But it takes non-zero values at any points of $B(N)$. Hence its $H$-span must contain all $\delta_{\beta_1}$. We come to a contradiction.

When $t$ are roots of unity. Till the end of the paper $t_\nu = q^{k_\nu}_{\nu}$ for $k_\nu \in \mathbb{Z}_+$, $\nu \in \nu_R$. The $L$-operators act in $Q^0[x]$ for $Q^0 \overset{df}{=} Q(q)$. Let $J \subset Q^0[x]$ be the radical of the pairing $[,]$. It is an ideal and an $H$-submodule. The quotient (a ring and an $H$-module) $\mathcal{V} = Q^0[x]/J$ is finite dimensional over $Q^0$. It results from Proposition 6.1 as generic $t$ approaches $q^{k_\nu}$.

We keep the same abbreviations:

\begin{equation}
\delta g(bw) = g(q^{b-w(\rho_k)}) \quad \text{however} \
\delta g(b) = g(\#b).
\end{equation}

By the construction, all $\{Y_a\}$-eigenvectors in $\mathcal{V}$ have pairwise distinct eigenvalues (the difference of any two of them with the same sets of eigenvalues and coinciding evaluations at 0 belongs to $J$). Applying this to $\epsilon_0 = 1$ generating $\mathcal{V}$ as an $H$-module we establish the irreducibility of $\mathcal{V}$ (use the pairing $[,]$ and follow Proposition 6.1).

Next, we will introduce the restricted Macdonald pairing (cf. (4.1),(4.3)):

\begin{equation}
\langle f(x), g(x) \rangle' \overset{df}{=} \sum_{c \in B_N} \mu(c) f(c) g(c) \quad \text{for} \quad f, g \in Q^0[x]^W,
\end{equation}

\begin{equation}
\mu = \prod_{a \in \mathbb{R}_+^\vee} (1 - x_a q_a^{-k_a} - 1 \cdot x_a \cdots (1 - x_a^{-1} q_a^{-k_a}) (1 - x_a^{-1} q_a^{k_a}).
\end{equation}

Here $\mu(c) = \delta \mu(c) = \mu(\#c)$.

The same verification as in [C2], Proposition 4.2 gives that

\begin{equation}
\langle Af, g \rangle' = \langle f, A^* g \rangle'
\end{equation}

for the anti-involution $*$ from (2.13) considered on the operators from $H$ acting on polynomials.

Let us assume that:

\begin{equation}
q_a^{(\rho_k_a)+i} \neq 1 \text{ for all } a \in \mathbb{R}_+^\vee, \; i = -k_a + 1, \ldots, k_a,
\end{equation}
In the simply laced case \((A, D, E)\), it is equivalent to the condition \(N > k((\rho, \theta) + 1)\).

**Lemma 6.2.** The natural map \(\mathcal{V} \to \mathcal{\hat{V}} \overset{\text{def}}{=} \text{Funct}(\hat{\mathcal{B}}, \mathbb{Q}^0)\) is an isomorphism which supplies \(\mathcal{\hat{V}}\) with the structure of a non-zero irreducible \(\mathcal{H}\)-module. Both pairings \([\ , \ ], \langle \ , \ \rangle\) are well defined and non-degenerate on \(\mathcal{\hat{V}}\).

**Proof.** The radical \(J'\) of the pairing \(\langle \ , \ \rangle'\) in \(\mathbb{Q}^0[x]^W\) is an \(\mathcal{H}\)-submodule. It equals the space of all functions \(f(x)\) such that \(\delta f(c) = 0\) for \(c\) from the subset \(B' \subset B_N\) where \(\delta\mu\) is non-zero. The set \(B'\) contains 0, since \(\mu'(t^\rho) \neq 0\) because of condition (6.14). Hence the linear span \(J + J'\) (that is an \(\mathcal{H}\)-submodule) does not coincide with the entire \(\mathbb{Q}^0[x]\), and the irreducibility of \(\mathcal{V}\) results in \(J + J' = J\).

Introducing now the delta-functions \(\hat{\delta}_i = \delta_{\beta_i}\), we can define the \(\epsilon\)-functions \(\{\hat{\epsilon}_i\}\) from the orthogonality and evaluation conditions

\[
(6.15) \quad [\hat{\epsilon}_i, \hat{\delta}_j] = C_i \delta_{ij} \quad \text{and} \quad \hat{\epsilon}_i(0) = 1, , 1 \leq i, j \leq \partial.
\]

They are eigenvectors of the \(Y_{\alpha}\)-operators with the eigenvalues \(x_{\alpha}^{-1}(\beta_i^*)\) and linearly generate \(\mathcal{\hat{V}}\). The sets of eigenvalues are pairwise distinct and \(\langle \hat{\pi}_i, \hat{\pi}_i\rangle' \neq 0\).

Presumably the \(\epsilon\)-functions are the discretizations of certain restricted Macdonald’s polynomials and the above scalar products can be calculated explicitly but we will not discuss this here.

We will use that \((K_N, \tau^\nu) \in N\mathbb{Z}\), where \(\tau^\nu = (2/\nu)\rho^\nu \in B\). Let us impose one more restriction:

\[
(6.16) \quad q^{(a,a)/2} = q_0^{m(a,a)} = 1 \quad \text{for} \quad a \in K_N, \nu \in \nu_R.
\]

If \(q_0\) is a primitive root of degree \(2mN\) then \(K_N = N\mathbb{Z} \cap B\) for the root lattice \(Q = \oplus_{i=1}^{n} \mathbb{Z}\alpha_i\) (see (6.1)). This condition obviously holds true for even \(N\) (all roots systems). For odd \(N\), it is necessary to exclude \(B_{n}, C_{4l+2}\). In the latter case, \(B \subset Q, m = 1\) and we can pick \(q_0\) in the roots of unity of degree \(N\).

**Theorem 6.3.** Introducing \(\hat{\Pi} = (\hat{\epsilon}_i(\hat{\beta}_j^*))\) (see (6.8,6.10)) for \(\hat{\epsilon}_i = \epsilon_{\beta_i}\), let

\[
(6.17) \quad \mathcal{T}_+ = \text{Diag}(q^{(\hat{\beta}_i^*, \hat{\beta}_j^*)/2} x_{\beta_i}(t^\rho)), \quad \mathcal{T}_- = \Pi \mathcal{T}_-^{-1}\Pi^{-1}, \quad \Omega = \mathcal{T}_+^{-1}\mathcal{T}_-\mathcal{T}_+^{-1}.
\]

The conjugations be these elements induce the automorphisms \(\tau_\pm, \omega\) of \(\mathcal{H}\) acting in \(\mathcal{\hat{V}}\). Let us decompose \(\mathcal{\hat{V}} = \oplus\mathcal{\hat{V}}_{\chi}\) relative to the central characters \(\chi\) of \(\mathcal{H} = \mathbb{Q}^0[T_1, \ldots, T_n]\). The following map gives a projective action of the group \(\mathcal{SL}_2(\mathbb{Z})\) in every \(\mathcal{\hat{V}}_{\chi}\):

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \mathcal{T}_+, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \to \mathcal{T}_-, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to \Omega.
\]
Proof. Setting \( x_b = q^{a_b}, z_{a+b} = z_a + z_b, z_i = z_{b_i}, a(z_b) = z_b - (a, b), a, b \in \mathbb{R}^n \), we introduce formally the Gaussian \( \gamma = q^{\sum_{i=1}^n z_i(z_i^0 - a_i^0)} \), which satisfies the following (defining) difference relations:

\[
\begin{align*}
    b_j(\gamma) &= q^{(1/2)\sum_{i=1}^n(z_i - (b_j, b_i))(z_i - a_i^0)} = \\
    \gamma q^{-z_i + (b_j, b_i)/2} &= x_j^{-1}\gamma q^{(b_j, b_i)/2} \quad \text{for} \quad 1 \leq j \leq n.
\end{align*}
\]

(6.18)

The Gaussian commutes with \( T_j \) for \( 1 \leq j \leq n \) because it is \( W \)-invariant. When \( b_r \) are minuscule \( (r \in O^* \) ), we use directly formulas (2.4, 3.5) to check that

\[
\gamma(X)Y_\tau \gamma(X)^{-1} = X_\tau q^{-(b_r, b_r)/2}Y_\tau = \tau_+(Y_\tau).
\]

A straightforward calculation gives that

\[
\begin{align*}
    \gamma(X)T_0 \gamma(X)^{-1} &= \tau_+(T_0) = X_0^{-1}T_0^{-1}, \\
    \tau_-(T_0) &= T_0, \quad \omega(T_0) = X_\theta^{-1}Y_\theta^{-1}T_0.
\end{align*}
\]

Hence the conjugation by \( \gamma \) induces \( \tau_+ \).

The formula for \( \tau_+ \) describes multiplication by \( \gamma \) in \( \tilde{V} \) (up to a constant factor) in the basis of delta-functions. Really,

\[
\begin{align*}
    \gamma(c) &= \delta \gamma(c) = q^{\sum_{i=1}^n \zeta_i c_i / 2} \quad \text{for} \\
    \zeta_i &= \log_q(x_i(q^{-r}t^r)) = (b_i, c) + \log_q(\prod_{i} t_\nu^{-rhythms}(\nu, b_i)).
\end{align*}
\]

(6.20)

Hence \( \gamma(c) = gq^{(c, c)/2x_c(t^{-r})} \) for \( g = q^{(\nu_r, \rho_s)} \). Since the matrix \( \tau_+ \) is important up to proportionality one can drop the constant \( g \). We see that changing \( c \) by any elements from \( K_n \) does not influence \( \gamma(c) \) because of the condition (6.16), which makes the multiplication by \( \gamma \) well defined.

Next, the automorphism \( \tau_- = \varphi \tau_+ \varphi \) corresponds to \( \tilde{\Pi} T_+^{-1} \tilde{\Pi}^{-1} \), and the matrix \( \Omega \) from (6.17) induces \( \omega = \tau_+^{-1}\varphi \tau_+ \varphi \tau_+^{-1} \) in the same delta-basis. Indeed, \( \tau_- \) is the application of \( \gamma(Y) \). It multiplies \( \pi_b \) by \( \gamma(t^\rho q^{-b}) \) whereas \( \gamma(X) \) multiplies \( \delta_b \) by \( \gamma(t^{-r}q^b) \) (so we need to inverse \( T_+ \)). More formally, one can use the equation \([T_+^{-1}f, g] = [f, T_-g]\).

Finally, any relations from \( SL_2(\mathbb{Z}) \) hold for these matrices up to proper central elements of \( H \) (Theorem 2.3). Thus the last statement results directly from the irreducibility of \( \tilde{V} \). \hfill \Box

The theorem is a non-symmetric version of the last theorem from [C3]. The latter in its turn generalizes the construction due to Kirillov [Ki] (in the case of \( A_n \)) and is directly related to Theorem 13.8 from [K] when \( t = q \). Following [C3] one can extend the above map to the projective \( GL_2(\mathbb{Z}) \). The biggest projective representations of \( SL_2(\mathbb{Z}) \) or \( GL_2(\mathbb{Z}) \) can be obtained from the eigenvalues of the element \( T_+^2 \) in \( \tilde{V} \). These groups act projectively in the corresponding spaces of eigenvectors.
Presumably the results from the last section have counterparts for generic $q, t$ in the analytic setting. They are connected with the monodromy representation of the double affine Knizhnik-Zamolodchikov equation and the main theorem from [KL2]. They also might help to renew elliptic functions towards the Ramanujan theories.

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