NON-HERMITEAN RANDOM MATRIX THEORY: method of hermitean reduction

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Abstract

We consider random non-hermitean matrices in the large $N$ limit. The power of analytic function theory cannot be brought to bear directly to analyze non-hermitean random matrices, in contrast to hermitean random matrices. To overcome this difficulty, we show that associated to each ensemble of non-hermitean matrices there is an auxiliary ensemble of random hermitean matrices which can be analyzed by the usual methods. We then extract the Green’s function and the density of eigenvalues of the non-hermitean ensemble from those of the auxiliary ensemble. We apply this “method of hermitization” to several examples, and discuss a number of related issues.
“I found it difficult,....., to keep my mind from wandering into the magical world of random matrices.”
F. J. Dyson

1 Introduction

When Wigner [1] first proposed treating the Hamiltonians of complex systems as random matrices, he quite naturally took the matrices to be hermitean (or real symmetric for time reversal invariant systems) in accordance with the fundamental postulates of quantum mechanics. In the last few years, there has been increasing interest in considering non-hermitean random matrices in the context of various physical problems. (We will use the term “non-hermitean” to include “real but not symmetric.”) We will mention a few examples: QCD at finite chemical potential [2], nuclear decay [3], dissipative systems [4, 5], and neural networks [6]. In all these examples, the random non-hermitean matrices were taken from a Gaussian distribution. Recently, Hatano and Nelson [7] have used non-hermitean random matrices to study the pinning of magnetic flux lines in high temperature superconductors.

In this paper, we discuss some aspects of random non-hermitean matrix theory, hoping to clarify some of the existing (and growing, see e.g. [8] for some recent examples) literature, which in the authors’s opinion, contains a number of confusing and confused statements. We develop a method to deal with non-hermitean matrices, by reducing the non-hermitean problem to an auxiliary hermitean problem. Unlike many examples that appear in the literature, our formalism is generic and is not limited to Gaussian ensembles. We then present some specific results. Some of our results may already be known in the literature. In particular, as we were completing this work, a series of very interesting papers by Zahed and collaborators [9] studying non-hermitean matrices appeared.

A basic tool in studying hermitean random matrices $\phi$ (henceforth all matrices

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1F.J. Dyson, “Selected Papers of Freeman Dyson with Commentary” (American Mathematical Society, International Press, 1996), p. 39.
will be taken to be \( N \times N \) with \( N \) tending to infinity unless otherwise specified) is the Green’s function defined by

\[
G(z) = \frac{1}{N} \text{tr} \left( \frac{1}{z - \phi} \right)
\]

where \( < \cdots > \) denotes averaging over the random distribution from which the matrices \( \phi \) are drawn. Diagonalizing \( \phi \) by a unitary transformation we have

\[
G(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{z - \lambda_k}
\]

where the \( N \) real numbers \( \{\lambda_k\} \) are the eigenvalues of \( \phi \). Thus, \( G(z) \) is a meromorphic function with poles at the eigenvalues of \( \phi \). In the large \( N \) limit, the poles merge into a cut (or several cuts) on the real axis. Powerful theorems from the theory of analytic functions can then be brought to bear to the problem of determining \( G(z) \) [10]. All of this is well-known. In contrast, for a non-hermitean matrix, the eigenvalues invade the complex plane. For example, for non-hermitean matrices \( \phi \) generated with the probability \( P(\phi) = \frac{1}{Z} e^{-N \text{tr} \phi^\dagger \phi} \), Ginibre determined long ago [11] that the eigenvalues are uniformly distributed over a disk of radius unity in the complex plane. Another simple example is presented in the Appendix. Thus, we lose the powerful aid of analytic function theory. To get oriented, consider the simple example of the function

\[
g(z) = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{z - e^{i\theta}} = \oint \frac{dw}{2\pi i} \frac{1}{w(z-w)}
\]

which appears to depend only on \( z \). In fact, \( g(z, z^*) = \theta (|z|^2 - 1) / z \) is clearly non-analytic, and depends on both \( z \) and \( z^* \). Speaking mathematically, one would say that this integral defines two functions, one for \( |z| > 1 \) and one for \( |z| < 1 \).

In the theory of hermitean random matrices, a powerful method consists of the Feynman diagrammatic expansion[12] in which one expands \( G(z) = \sum_k \langle \text{tr} \phi^k \rangle / z^{k+1} \) as a series in \( 1/z \), the “bare quark propagator.” This method is clearly no longer available in studying non-hermitean random matrices. The knowledge of \( G(z) \) as a series in \( 1/z \) can no longer tell us anything about the behavior \( G(z) \) for small \( z \), as
the simple example above makes clear. The eigenvalues fill a two-dimensional region rather than a one-dimensional region, as is the case for hermitean matrices.

A renormalization group inspired method used in [13, 14, 15, 16] implicitly involves an expansion in $1/z$ and is thus also not available for dealing with non-hermitean matrices without suitable further developments.

This paper is organized as follows. In Section 2 we review some of the basic formalism. We also make a tentative connection with the theory of quaternions. Section 3 is the heart of this paper which consists of showing that the Green’s functions and density of eigenvalues of the random non-hermitean matrix may be determined from those of an auxiliary hermitean matrix. The hermitean auxiliary ensemble may be analyzed by the usual methods. We refer to this process as the “Method of Hermitization”. In Section 4 we show how to determine the boundary of the eigenvalue distribution of the complex matrix without ever determining its density of eigenvalues $\rho(x, y)$. In Section 5 we present some simple applications of the hermitization method. In Section 6 we discuss some aspects of “energy-level” dynamics of non-hermitean hamiltonians. In particular, we show how the celebrated phenomenon of level repulsion in the case hermitean hamiltonians changes when one discusses non-hermitean hamiltonians. We present a simple (yet instructive) example in the Appendix.
2 Some Basic Formalism

In this section, we will derive the central formula for studying the eigenvalue distribution of random non-hermitean matrices. While this formula is already known in the literature, we hope to clarify a number of misleading statements made in its derivation. In order to be absolutely clear, we will begin with some almost laughably elementary facts.

For \( z = x + iy \) we define

\[
\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)
\]

so that \( \partial z = 1 \). Clearly, \( \partial z^* = 0 \). (We denote complex conjugation by *.) It follows that \( \partial z^{*n} = 0 \quad (n \geq 0) \), and \( \partial f(z^*) = 0 \) in a region in which \( f \) has a series expansion in powers of \( z^* \). As is well-known, this fails if \( f \) does not have a series expansion. In the simplest example, we have

\[
\partial \frac{1}{z^*} = \pi \delta(x) \delta(y),
\]

which we can easily prove by integrating the left and right hand sides over a small square centered at the origin. Similarly, we define

\[
\partial^* \equiv \frac{\partial}{\partial z^*} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),
\]

so that \( \partial^* z^* = 1 \). We have, obviously, \( \partial^* (1/z) = \pi \delta(x) \delta(y) \).

Let us now calculate \( \partial \partial^* \log (zz^*) = \partial \partial^* (\log z + \log z^*) \). At this point it is important to keep in mind that \( \partial \log z \) is not \( \frac{1}{z} \) in the cut plane! (\( \partial \log z \) is \( \frac{1}{z} \) on the multi-sheeted Riemann surface of course.) Instead, the correct formula is

\[
\partial \log z = \frac{1}{z} + \pi \theta(-x) \delta(y)
\]

with the standard choice of cutting \( \log z \) along the negative real axis. Thus,

\[
\partial \partial^* (\log z + \log z^*) = \partial^* \left( \frac{1}{z} + \pi \theta(-x) \delta(y) \right) + \text{c.c.}
\]

\[
= \pi \delta(x) \delta(y) + \pi \delta(x) \delta(y) - \pi \delta(x) \delta(y),
\]

(2.5)
where the last term comes from \((\partial + \partial^*) \pi \theta(-x) \delta(y) = -\pi \delta(x) \delta(y)\). We finally obtain
\[
\partial \partial^* \log (zz^*) = \pi \delta(x) \delta(y) . \tag{2.6}
\]

Notice that if we had glibly used \(\partial \log z = \frac{1}{z}\), as was sometimes done in the literature, we would have missed the last term in (2.5) and obtained an erroneous formula off by a factor of 2 from the correct formula, thus leading to an endless stream of apparent paradoxes.

With these basic facts established we are now ready to deal with the average density of eigenvalues of a non-hermitean matrix \(\phi\). We diagonalize the matrix by a similarity transformation
\[
\phi = S^{-1} \Lambda S \tag{2.7}
\]
where \(\Lambda\) denotes a diagonal matrix with elements \(\lambda_i, i = 1, ..., N\). Taking the hermitean conjugate we have \(\phi^\dagger = S^\dagger \Lambda^* S^{-1\dagger}\). By definition, the density of eigenvalues is
\[
\rho(x, y) = \langle \frac{1}{N} \sum_i \delta(x - \text{Re} \lambda_i) \delta(y - \text{Im} \lambda_i) \rangle. \tag{2.8}
\]

Given the preceding discussion, we obtain from (2.6) that
\[
\partial \partial^* \text{tr } (z - \Lambda)^{-1} = \frac{\pi}{N} \sum_i \delta(x - \text{Re} \lambda_i) \delta(y - \text{Im} \lambda_i). \tag{2.9}
\]

This formula has been known in the literature (see e.g., [3]), but there a seemingly unnatural small regulator had been introduced to make the argument of the logarithm in (2.9) strictly positive to avoid the branch point at 0. Our derivation of (2.6) suggests that that regulator is not really necessary.

Our expression (2.9) is manifestly symmetric in \(z\) and \(z^*\). It arises quite naturally in calculations involving Grassmann variables[2, 3] where \(\det (z - \phi)(z^* - \phi^\dagger)\) is the
“fermion determinant”. However, it involves the logarithm inside the average, as well as two derivatives. Thus, unless one has a simple way of calculating the average in (2.9) (e.g., the replica method in the case of Gaussian ensembles [3]), (2.9) is not a practical expression for $\rho(x, y)$. In most cases it is easier to calculate the Green’s function

$$G(z, z^*) = \langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \rangle = \langle \frac{1}{N} \sum_i \frac{1}{(x - x_i) + i(y - y_i)} \rangle,$$

(2.10)

(with $\lambda_i = x_i + iy_i$) than to do the average $\langle \frac{1}{N} \text{tr} \log (z - \phi)(z^* - \phi^*) \rangle$. Then, from (2.8) and from the complex-conjugate of (2.2), we have

$$\rho(x, y) = \frac{1}{\pi} \partial^* G(z, z^*),$$

(2.11)

which is a simpler representation of $\rho(x, y)$ than (2.9).

These two representations for $\rho(x, y)$ and the relation between them can be interpreted by recognizing (2.9) as a two dimensional Poisson equation for the electrostatic potential $-\frac{1}{2} \langle \frac{1}{N} \text{tr} \log (z - \phi)(z^* - \phi^*) \rangle$ created by the charge density $\rho(x, y)$. This connection between eigenvalue distributions of complex matrices and two-dimensional electrostatics has been long known in the literature [3, 5]. Continuing along this line, consider the Green’s function $G(z, z^*)$ in (2.11). If we define the electric field $\vec{E} = (\text{Re} G, -\text{Im} G)$ then from the definition of $G(z, z^*)$ we have

$$\vec{E}(x) = \int d^2 y \rho(\vec{y}) \frac{x - \vec{y}}{|x - \vec{y}|^2},$$

(2.12)

and thus (2.11) is simply the statement of Gauss’ law for this electrostatic problem.

We end this section with a speculative remark. As we pointed out earlier, in the theory of random hermitean matrices it is well known that it is much easier to work with $G(z) = \langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \rangle$ rather than with the density of eigenvalues $\rho(\mu)$ directly, since the power of analytic function can be brought to bear on $G(z)$. Of course, one can go from $\rho(x)$ to $G(z)$ with the identity

$$\frac{1}{\pi} \text{Im} \frac{1}{x - i\epsilon} = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \to \delta(x).$$

(2.13)

Here, for random non-hermitean matrices our formula (2.9) gives the density of eigenvalues $\rho(x, y)$ directly, but this formula is awkward to work with because of the
logarithm. Its direct analog in the theory of random hermitean matrices would be
\( \rho(x) = \frac{\partial}{\partial x} \langle \frac{1}{N} \text{tr} \theta(x - \phi) \rangle \), which would also be extremely awkward to work with. It would thus be desirable to define a quantity analogous to \( G(z) \) and develop a method for calculating it. Reasoning along these lines we are led to an attempt to write a function of a quarternionic variable, in the same way that going from \( \rho(x) \) to \( G(z) \) we went from a function of a real variable to a function of a complex variable. Indeed, the obvious analog of (2.13) is easy to find, namely,

\[
\frac{1}{2j} \left[ \frac{1}{(z - j\epsilon)|z - j\epsilon|} + i \frac{1}{(z - j\epsilon)|z - j\epsilon|} \right]
\]

\[= \frac{\epsilon}{|z - j\epsilon|^3} = \frac{\epsilon}{(x^2 + y^2 + \epsilon^2)^{3/2}} \rightarrow 2\pi \delta(x) \delta(y) . \tag{2.14} \]

Here \( z = x + iy \) is a complex number, \( \epsilon \) is a small positive real number and \( \{1, i, j, k\} \) is the standard basis of the quaternion algebra. (The absolute value of a quaternion is defined by \( |a + ib + jc + kd| \equiv \sqrt{a^2 + b^2 + c^2 + d^2} \).)

Unfortunately, (2.14) as it stands, is less useful than (2.9), because it leads to the quaternionic Green’s function \( G(q) \) (\( q \) being a quateronionic variable)

\[
G(q) = \langle \frac{1}{N} \sum_i \frac{1}{(q - \lambda_i)|q - \lambda_i|} \rangle \tag{2.15}
\]

which involves the absolute value operation explicitly, and thus cannot be written as a simple trace like (1.1). Note, however, that if we split \( q \) quite generally into \( q = z + jw \) (\( z, w \) being ordinary complex variables), then \( G(z + jw) \) is manifestly non-analytic in \( z \) as \( w \rightarrow 0 \), even before taking the average.

We pose this as an interesting problem, namely to find a useful quaternionic generalization of (1.1) for random non-hermitean matrices, and to develop the analogue of the work of Brézin et al. [10] associated with it.

\[\text{2 One may of course write an equivalent formula with } j \text{ replaced everywhere by the third quaternion basis element } k. \text{ Also, } \epsilon \text{ may be taken complex.}\]
3 The Hermitization Method: Reduction to Random Hermitean Matrices

We have emphasized that a straightforward diagrammatic method is not allowed for non-hermitean matrices. Somewhat remarkably, we can arrive at a diagrammatic method indirectly. We will show that the problem of determining the eigenvalue density of random non-hermitean matrices can be reduced to the problem of determining the eigenvalue density of random hermitean matrices, for which the diagrammatic method may be applied.

We start with the representation
\[ \rho(x,y) = \frac{1}{\pi} \partial^* \left\langle \frac{1}{N} \text{tr} \log (z - \phi)(z^* - \phi^\dagger) \right\rangle \] (namely, Eq. (2.9)). By standard manipulations we observe that
\[ \left\langle \text{tr} \left( N \log (z - \phi)(z^* - \phi^\dagger) \right) \right\rangle = \langle \text{tr}_{(2N)} \log H \rangle - i\pi N^2 , \]
where \( H \) is the hermitean \( 2N \times 2N \) matrix Hamiltonian
\[ H = \begin{pmatrix} 0 & \phi - z \\ \phi^\dagger - z^* & 0 \end{pmatrix}. \] (3.1)

Thus, if we can determine
\[ F(\eta; z, z^*) = \frac{1}{2N} \langle \text{tr}_{(2N)} \log (\eta - H) \rangle \] (3.2)
we can determine \( \rho(x,y) \).

Consider now the propagator associated with \( H \), namely,
\[ G_\mu(\eta; z, z^*) = \left\langle \left( \frac{1}{\eta - H} \right)^\mu \right\rangle \] (3.3)
where \( \eta \) is a complex variable and the indices \( \mu \) and \( \nu \) run over all possible \( 2N \) values. Here we followed a common practice and borrowed some terminology from gauge field theory: we may consider \( \phi, \phi^\dagger \) as “gluons” (in zero space-time dimensions), which interact with a \( 2N \) dimensional multiplet of “quarks” \( \psi^\mu \), with a complex mass matrix (the “inverse propagator”)
\[ G^{-1}_0 = \begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix} \] (3.4)
(expressed in terms of its $N \times N$ blocks.) The crucial point is that since $H$ is hermitian, $G^\mu_\nu$ can be determined by the usual methods of hermitian random matrix theory. In particular, as we already mentioned, the diagrammatic evaluation of (3.3) is essentially the expansion of $G^\mu_\nu$ in powers of $1/\eta$, with interaction vertices $H$. This is a well defined procedure for large $\eta$, and it converges to a function which is analytic in the complex $\eta$ plane, except for the cut (or cuts) along the real axis which contain the eigenvalues of $H$. After summing this series (and thus determining $G^\mu_\nu(\eta; z, z^*)$ in closed form), we are allowed to set $\eta \to 0$ in (3.3). Speaking colloquially, we may say that the crucial maneuver here is that while we cannot expand in powers of $z$, we can arrange for $z$ to “hitch a ride” with $\eta$ and at the end of the ride, throw $\eta$ away.

We can now calculate $\rho(x, y)$ by two different methods, each with its advantages. The first is to simply observe that

$$-\frac{1}{H} = \begin{pmatrix} 0 & \frac{1}{z^* - \phi^\dagger} \\ \frac{1}{z - \phi} & 0 \end{pmatrix}.$$ 

Thus, the quantity $\frac{1}{z - \phi}$ is simply the lower left block of $G^\mu_\nu(\eta = 0; z, z^*)$. In other words, once we have $G^\mu_\nu(\eta; z, z^*)$ we can set $\eta$ to zero and use (2.10) and (2.11) to write

$$\rho(x, y) = \frac{1}{N \pi} \partial^\nu \text{tr}_{(2N)} \left[ \begin{pmatrix} 0 & \frac{1}{N} \\ 0 & 0 \end{pmatrix} G(0; z, z^*) \right].$$

(3.5) This observation is the basis for many of the calculations in [9]. We will illustrate this discussion with a simple example below.

An alternative is to take the trace of $G^\nu_\mu$:

$$G^\mu_\nu(\eta; z, z^*) = \frac{1}{2N} \left( \frac{\text{tr}_{(2N)} 1}{\eta - H} \right) = \frac{\eta}{N} \left( \text{tr}_{(N)} \frac{1}{\eta^2 - (z^* - \phi^\dagger)(z - \phi)} \right),$$

(3.6) from which one can determine (3.2).

We refer to this procedure of calculating $G(z, z^*) = \frac{1}{N} \left( \text{tr} \frac{1}{z - \phi} \right)$ and $\rho(x, y)$ by manipulating the hermitian matrix $H$ as the “Method of Hermitization”.

We now derive a dispersive representation of $F(\eta; z, z^*)$ in terms of $G(\eta; z, z^*)$. Recall that the eigenvalue density of $H$ is given by the discontinuity of $G$ across the
cut, namely,
\[ \omega(\mu; z, z^*) \equiv \frac{1}{2N} \langle \text{tr}_{(2N)} \delta(\mu - H) \rangle = \frac{1}{\pi} \text{Im} \mathcal{G}(\mu - i\epsilon)_{|\epsilon \to 0^+}. \]  

(3.7)

From (2.4) we then find
\[ \frac{\partial}{\partial \eta} F(\eta; z, z^*) = \mathcal{G}(\eta; z, z^*) + \pi \left[ 1 - \Omega(\text{Re} \eta) \right] \delta(\text{Im} \eta), \]  

(3.8)

where \( \Omega(\text{Re} \eta; z, z^*) \) is the integrated density of eigenvalues of \( H \),
\[ \Omega(\mu; z, z^*) = \frac{1}{2N} \langle \text{tr}_{(2N)} \theta(\mu - H) \rangle. \]  

(3.9)

As a result of the “chiral” block structure of \( H \), its spectrum consists of pairs of eigenvalues with equal magnitudes and opposite signs. Namely, if \( \xi \) is an eigenvalue of \( H \), so is \( -\xi \). This symmetry of the spectrum of \( H \) around \( \mu = \text{Re} \eta = 0 \) implies \( \Omega(0) = \frac{1}{2} \). Thus, setting \( \eta = is \) pure imaginary we obtain
\[ \frac{\partial}{\partial s} F(is; z, z^*) = i\mathcal{G}(is; z, z^*) + \frac{i\pi}{2} \delta(s). \]  

(3.10)

Integrating over \( s \) we have
\[ F(i0+; z, z^*) = F(is; z, z^*) - i \int_{0^+}^{\infty} ds \mathcal{G}(is; z, z^*) \]  

(3.11)

where the lower integration limit avoids the \( \delta(s) \) piece in (3.10). From the definition of \( F \), we have that \( F(\eta; z, z^*) \to \log \eta \), independent of \( z \), as \( \eta \) tends to infinity. Thus, applying the operator \( \frac{2i}{\pi} \partial \partial^* \) to both sides of (3.11), we finally obtain
\[ \rho(x, y) = -\frac{2i}{\pi} \int_{0^+}^{\infty} ds \partial \partial^* \mathcal{G}(is; z, z^*). \]  

(3.12)

Thus, the density of eigenvalues of the non-hermitean random matrices \( \phi \) is determined by an integral over \( \mathcal{G}(\eta; z, z^*) \). The determination of \( \mathcal{G}(\eta; z, z^*) \) involves only the well-known and more elementary problem of determining the density of eigenvalues of the hermitean random matrices \( H \), which is the essence of our method of hermitization.
In fact, $H$ in (3.1) consists of the sum of the deterministic matrix

$$H_0 = \begin{pmatrix} 0 & -z \\ -z^* & 0 \end{pmatrix} \quad (3.13)$$

and the random matrix

$$V = \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix}. \quad (3.14)$$

Note that the deterministic piece has $N$ eigenvalues $+r = |z|$ and $N$ eigenvalues $-r$. This problem of adding a deterministic matrix and a random matrix has been extensively discussed in the literature [12, 17, 18, 19, 20, 21]. The Green’s function (3.6) and the density of eigenvalues (3.7) are related by the standard expression

$$G(\eta; z, z^*) = \int_{-\infty}^{\infty} d\mu \frac{\omega(\mu; z, z^*)}{\eta - \mu} = \int_{0}^{\infty} d\mu \omega(\mu; z, z^*) \frac{2\eta}{\eta^2 - \mu^2}. \quad (3.15)$$

Here we used the fact that $\omega(\mu; z, z^*) = \omega(-\mu; z, z^*)$, namely, the symmetry of the spectrum of $H$ mentioned above.

Using the “dispersive representation” (3.15) we can derive a compact formal expression for the density of eigenvalues of the non-hermitean random matrix $\phi$. Inserting (3.15) into (3.12) we obtain

$$\rho(x, y) = -\frac{4}{\pi} \int_{0}^{\infty} d\mu \partial^\ast \Omega(\mu; z, z^*) \int_{0}^{\infty} ds \frac{s}{s^2 + \mu^2}. \quad (3.16)$$

Note that the “+” can now be safely removed from the lower limit of the $s$ integral. Also, note that while the $s$ integral appears to be logarithmically divergent, it is in fact convergent thanks to the “sum rule” $\int_{-\infty}^{\infty} d\mu \omega(\mu; z, z^*) = 1$. To make this last observation explicit, let us recall that $\omega(\mu; z, z^*) = \frac{\partial \Omega(\mu; z, z^*)}{\partial \mu}$, where $\Omega(\mu, z, z^*)$ is the integrated density of eigenvalues defined in (3.9). We can thus integrate over $\mu$ by parts in (3.16), thereby bringing in $\Omega(\mu; z, z^*)$. We obtain finally the compact expression

$$\rho(x, y) = -\frac{4}{\pi} \int_{0}^{\infty} d\mu \partial^\ast \frac{\Omega(\mu; z, z^*)}{\mu}. \quad (3.17)$$
To summarize, we have obtained a formalism (the “Method of Hermitization”) for reducing the problem of dealing with random non-hermitean matrices to the well-studied problem of dealing with random hermitean matrices. Given the non-hermitean matrix $\phi$, we study the hermitean matrix $H$ instead. By whatever method one prefers, once one has determined the quark propagator $G^{\mu}_{\nu}$ (or its trace, the Green’s function $G(\eta; z, z^*)$), one can in principle obtain $\rho(x, y)$ using (3.5) or (3.17). Whether that can be done in practice is of course another story.

Note that a “shadow” of the variable $\eta$ appears in Section 3 of the first paper in[9], as a small real regulator $\lambda$. This regulator explicitly “breaks” what these authors refer to as the “holomorphic symmetry”, by coupling “quarks” and “conjugate quarks”, in an analogous manner to the way a small explicit quark mass breaks the chiral symmetry in QCD. Then $N$ tends to infinity first, followed by $\lambda \to 0$. After these limits were taken, in that particular order, one observes that $\langle \text{tr}(z - \phi)^{-1} \rangle$ is not a holomorphic function. Since by now $\lambda = 0$, these authors refer to this phenomenon as “spontaneous breakdown of holomorphic symmetry”. From the point of the hermitization method, we see that it is rather misleading to treat $\eta \equiv \lambda$ as a small regulator. On the contrary - one expands in powers of $1/\eta$ for $\eta$ large, and only after summing this perturbative series for the hermitean matrix $H$, one is allowed to set $\eta = 0$. From this point of view, as well as from the simple example given in the Introduction, the non-holomorphy of $\langle \text{tr}(z - \phi)^{-1} \rangle$ is not more mysterious than the fact that $\phi$ has complex eigenvalues.

It is worth emphasizing that the formalism developed here has nothing to do with large $N$ as such. The formalism is of course indifferent to the method one may choose to use to determine $G^{\mu}_{\nu}(\eta; z, z^*)$. The problem of determining the Green’s function of chiral matrices such as $H$ has been discussed by numerous authors [10, 20, 22, 23]. In some cases, results can be obtained for finite $N$ [22].

It is also clear that the formalism here is totally independent of the form of the probability distribution $P(\phi, \phi^\dagger)$. In special cases one can say more. For example, suppose that $P(\phi, \phi^\dagger)$ is invariant under

$$
\phi \to e^{i\alpha} \phi, \quad \phi^\dagger \to e^{-i\alpha} \phi^\dagger.
$$

(3.18)
For instance, this holds if $P$ is constructed out of $\text{tr} \ (\phi^\dagger \phi)^m$, $(\text{tr} \ \phi^\dagger \phi)^n$, etc. Then, clearly, $\langle \frac{1}{N} \text{tr} \ \log \ (z^* - \phi^\dagger)(z - \phi) \rangle$ is a function of $|z|^2 = r^2$ rather than of $z$ and $z^*$ separately. The formalism just given simplifies somewhat: the propagator $G_\mu^\nu(\eta; z, z^*)$ and the functions $\mathcal{G}(\eta; z, z^*)$, $\omega(\mu; z, z^*)$, and $\Omega(\mu; z, z^*)$ do not depend on $z$ and $z^*$ separately, but only on $r = |z|$. Acting on such functions, the operator $\partial \partial^*$ in (3.17) reduces to

$$\partial \partial^* = \frac{d}{dr^2} r^2 \frac{d}{dr^2} = \frac{1}{4r} \frac{d}{dr} r \frac{d}{dr}$$

(3.19)

The density of eigenvalues $\rho(x, y) = \rho(r)$ is obviously circularly symmetric. In particular, the Gaussian problem considered by Ginibre is of this class.

In some problems, one might be interested in the density of zero eigenvalues. Expanding $\Omega(\mu; r) = \Omega_0(\mu) + r^2 \Omega_1(\mu) + \cdots$ as a series in $r^2$, and inserting into (3.17) we obtain formally a compact expression for the density of zero eigenvalues $\rho(0) = \int \frac{d\mu}{\mu} \Omega_1(\mu)$

It is worth emphasizing again the point we made in the Introduction, namely that we can no longer expand in powers of $1/z$ and use the diagrammatic expansion directly. Consider in particular the class of distribution we just mentioned. Suppose we are allowed to expand in powers of $\frac{1}{z}$, then we would compute

$$\rho(x, y) = \frac{1}{\pi} \partial \partial^* \langle \frac{1}{N} \text{tr} \ \log \ (z - \phi) \rangle + c.c. \]

$$= \delta(x) \delta(y) + \frac{1}{\pi} \partial \partial^* \frac{1}{N} \sum_k \langle \text{tr} \ \phi^k \rangle \frac{z^k}{z^k} + c.c. = \delta(x) \delta(y),$$

since in each term $\langle \text{tr} \ \phi^k \rangle$ vanishes (for $k > 0$) by the symmetry of the distribution $P(\phi, \phi^\dagger)$. We would have concluded erroneously that $\rho(x, y)$ is concentrated in a $\delta$-function spike at the origin.

In contrast, it is of course perfectly legitimate to expand in powers of $1/\eta$ and use the diagrammatic method to compute $G_\mu^\nu(\eta; z, z^*)$ associated with the hermitean matrix problem.

A similar point worth re-emphasizing is that the formula (2.11) for $\rho(x, y)$ is also not very useful as it stands unless we have a way of computing $G(z, z^*)$. What we have to do is to compute $G_\mu^\nu(\eta; z, z^*)$ by a diagrammatic method and then use (3.3).
Let us illustrate some of this discussion with a simple example, the Gaussian problem first worked out by Ginibre[11], namely the density of eigenvalues of non-hermitean matrices randomly generated according to the normalized probability distribution

\[ P(φ) = \frac{1}{Z} e^{-\text{tr} φ^† φ}. \]  

(3.20)

According to our method of hermitization, we can reduce this problem to that of obtaining the density of eigenvalues of the hermitean Hamiltonian \( H \) in (3.1), consisting of a deterministic matrix and a hermitean Gaussian random matrix of a chiral form. As remarked earlier, this belongs to a class of problems long discussed in the literature[12, 17, 18, 19, 20, 21].

Let us first apply the by-now well-known diagrammatic method to calculate the Green’s function \( G(η; z, z^*) \equiv G(η; r) \) in the large \( N \) limit. Recall that the quark self-energy matrix \( Σ^μ_ν \) and the quark propagator \( G^μ_ν \) in (3.3) are related by

\[ G^μ_ν = \left( \frac{1}{G^{-1} - Σ} \right)^μ_ν. \]  

(3.21)

Since \( Σ \) is the sum over all one-quark irreducible graphs we have the Dyson-Schwinger identity

\[ Σ^μ_ν = \langle VGV \rangle^μ_ν, \]  

(3.22)

where \( V \) was defined in (3.14). After a simple calculation we find from (3.20) and (3.22) that

\[ Σ^μ_ν = G(η; r)δ^μ_ν. \]  

(3.23)

Thus, from (3.21) and (3.23) we observe that the diagonal blocks of \( G^μ_ν \) are equal and proportional to the \( N \) dimensional unit matrix. By “index democracy” this proportionality constant is nothing but the Green’s function \( G(η; r) \equiv G \). We thus have

\[ G^μ_ν = \frac{1}{r^2 - (η - G)^2} \begin{pmatrix} G - η & z \\ z^* & G - η \end{pmatrix}. \]  

(3.24)

Tracing both sides of (3.24) we obtain the “gap equation”

\[ G^3 - 2ηG^2 + (1 + η^2 - r^2)G - η = 0. \]  

(3.25)
The relevant solution can be immediately written down using the classic Cardano formula. (Of the three roots, the relevant root is the one which goes as \(1/\eta\) as \(\eta\) tends to infinity, as is evident from the definition of \(G(\eta; r)\).) We will not bother to write the expression explicitly. Then, following (3.3) we find

\[
\rho(x, y) = \frac{1}{\pi} \partial^* \frac{z^*}{r^2 - G^2(0; r)}. \tag{3.26}
\]

Solving (3.23) at \(\eta = 0\) we find that either \(G(0; r) = 0\) or \(r^2 - G^2(0; r) = 1\). Plugging \(G = 0\) into (3.23) we obtain an expression proportional to \(\delta(x)\delta(y)\) which is nothing but the zero density outside the eigenvalue distribution. In the other case, substituting \(r^2 - G^2(0; r) = 1\) in (3.26) we obtain \(\pi \rho = \partial^* z^* = 1\), which corresponds to a uniform \(\rho\) inside the disk of eigenvalues. These two possibilities match at \(r = 1\) which must therefore correspond to the boundary of the eigenvalue distribution. Thus, finally

\[
\pi \rho_{\text{Ginibre}}(x, y) = \begin{cases} 
1, & r < 1 \\
0, & r > 1 
\end{cases} \tag{3.27}
\]

in agreement with [11], but the derivation here is considerably simpler. It is worthwhile mentioning at this point that the matching of these two solutions at \(r = 1\) arises naturally in a replica saddle-point calculation of \(\langle \frac{1}{N} \text{tr} \log (z^* - \phi^\dagger)(z - \phi) \rangle \sim F(0; r)\) along the lines of [3], as we now describe briefly. Before the replica limit \(n \to 0\) is taken, one deals with \(\langle \det^n (z^* - \phi^\dagger)(z - \phi) \rangle\). This expression may be considered as the averaged fermion determinant in the fluctuating gluon field \(\phi\) when there are \(n\) fermion flavors degenerate in mass. The role of fermion mass is played by \(r = |z|\). For \(r < 1\) there are two large \(N\) saddle points, call them A and B, that govern the integral over certain replica matrix fields in the expression for the averaged determinant. These saddle points are obviously in \(1 - 1\) correspondence with the two solutions of (3.25) at \(\eta = 0\) that we mentioned before. One of these saddle points, say A, dominates the other (i.e.; its integrand is larger.) However, when \(r > 1\), the previously dominating saddle-point A disappears, and the determinant is given entirely by saddle-point B. The two expressions for this averaged determinant as well as
their first derivatives with respect to $r$ match continuously at $r = 1$. In other words, the fermion “free energy” $\log \langle \det^n (z^* - \phi^\dagger)(z - \phi) \rangle$ undergoes a second order phase transition at $r = 1$ as the fermion mass is varied.
4 Boundaries of the Eigenvalue Distribution

In some applications of random non-hermitean matrices, one is primarily interested in the domains over which the density of eigenvalues is non-zero. In this section, we will develop a formalism for determining the boundary of these domains.

First, let us get oriented. We have already remarked that the deterministic piece in $H$ has $N$ eigenvalues $+r = |z|$ and $N$ eigenvalues $-r$. For small $r = |z|$ we would expect the random piece in $H$ to smear these two adjacent delta function spikes in $\omega(\mu; z, z^*)$ into one continuous lump. In other words, the density of eigenvalues $\omega$ as a function of $\mu$ is non-zero along a single segment. For large $r$, the deterministic part in $H$ will dominate, and we expect $\omega$ as a function of $\mu$ to separate into two lumps centered at $\pm r$. On the other hand, in the original non-hermitean matrix problem, as $|z|$ increases, we expect to arrive at the boundary of the domain outside of which $\rho(x, y)$ vanishes.

It is thus natural and intuitive to expect that the criterion for determining the boundary in question is to locate the points in the complex $z$ plane at which $\omega(\mu; z, z^*)$ splits into two lumps. Note that since $\omega$ is an even function of $\mu$, as mentioned earlier, the split first occurs at $\mu = 0$. We can readily formalize this expectation. Consider the positive semi-definite hermitean matrix

$$h^2 = (z^* - \phi^\dagger)(z - \phi)$$

whose eigenvalues we denote by $h^2_i$. Then from (3.7) we have

$$\omega(\mu; z, z^*) = \frac{1}{2N} \left( \sum_{i=1}^{N} [\delta(\mu - h_i) + \delta(\mu + h_i)] \right).$$

Thus, if $\omega(0; z, z^*)$ vanishes, that implies $h$ and hence $z - \phi$ (considered averaged over the randomness of course) do not have any zero eigenvalues, or equivalently, $\phi$ does not have any eigenvalue equal to $z$. This proves our expectation.

How do we determine the boundary of the domains over which $\rho(x, y)$, the density of eigenvalues of the non-hermitean matrices $\phi$, is non-zero? To proceed, we will now broaden our horizons and discuss the problem for an entire class of which the Ginibre
problem that concluded the previous section is an example, namely the class in which
the Green’s function $G(\eta; \{r\})$ of the associated hermitean problem is determined by
the solution of a polynomial equation $F(G) = 0$. In general, as we will see, there can
be a whole set of parameters $\{r\}$; in Ginibre’s problem there is only one real variable
$r$.

Let us decompose

$$G = u + iv$$

(4.3)

into its real and imaginary parts and imagine moving along just below the real axis
(that is, we will take $\eta = \mu - i\epsilon$ as in the previous section.) Note that for $\eta$ real, the
coefficients in the polynomial $F(G)$ are real as well. There will be segments over which
$v$ is non-zero; these segments are where the density of eigenvalues of the hermitean
matrix $H$ is non-zero. (Up to an irrelevant factor, $v > 0$ is just a convenient notation
for what we called $\omega$.) Let us now imagine moving along such a segment. We would
like to determine the value of $\mu$ (call it $\mu_c$) at which $v$ vanishes, in other words, the
endpoints of the segment over which $v$ is non-zero.

Let us decompose the equation $F(G) = 0$ into its real and imaginary parts and
expand for $v$ small:

$$F(u) + h(u)v^2 + O(v^4) = 0 \quad \text{and}$$

$$v((F'(u) + O(v^2)) = 0. \quad (4.4)$$

Here $h(u)$ is a function whose detailed form does not concern us, and $F'$ denotes the
first derivative of $F$ with respect to its argument. All these functions $F$, $F'$, and $h$
are of course also functions of $\mu$ (and of $\{r\}$ which we think of as fixed.)

At $\mu = \mu_c$, $v$ vanishes by definition and hence according to the first equation in
(4.3) $F(u)$ also vanishes, generically as $F(u) \sim (\mu - \mu_c)$. Thus, we conclude that
$v \sim |\mu - \mu_c|^2$. (This type of argument is of course reminiscent of the arguments
used in Landau in his theory of phase transition. As a matter of fact, it explains
the robustness of the square root singularity in $\omega(\mu)$ near the edges of the eigenvalue
distribution for many ensembles of hermitean matrices.

For \( \mu \) away from \( \mu_c \), \( v \) is non-zero, and thus we may take out the factor of \( v \) from the second equation in (4.3) and write \( F'(u) + O(v^2) = 0 \). Letting \( \mu \) go to \( \mu_c \), we argue by continuity that \( F'(u) \) also vanishes at \( \eta_c \).

Should the reader feel somewhat uneasy over the continuity argument, we can obtain the same conclusion by differentiating the second equation in (4.4) with respect to \( \mu \). Using \( v \sim |\mu - \mu_c|^2 \), we have that as \( \mu \) approaches \( \mu_c \), \( \frac{dv}{d\mu} \) tends to infinity while \( \frac{dv^3}{d\mu} \) tends to zero. We reach the same conclusion above.

An alternative argument is to write the polynomial \( F(G) = \prod_i (G - G_i) \) in terms of its roots \( G_i \). Since all coefficients in \( F(G) \) are real (we assume that \( \eta = \mu - i\epsilon \)), the \( G_i \) are real or appear as complex-conjugate pairs. In particular, the “physical” root \( G(\mu; \{ r \}) \) is generically complex, and thus appears with its complex-conjugate partner. As \( \mu \to \mu_c \) (from within the segment along which \( v > 0 \)) \( v \to 0 \) and thus the two roots \( G(\mu; \{ r \}) \) and \( G^*(\mu; \{ r \}) \) collide, that is to say, become equal (and real, so that \( v(\mu_c) = 0 \)). Thus, \( F(G) \) has generically a double zero at \( \mu = \mu_c \), and so both \( F \) and its first derivative \( F' \) vanish at that point.

To summarize, to determine the boundary or edge of the density of eigenvalues of the hermitean random matrix \( H \) we solve the two equations \( F(u) = 0 \) and \( F'(u) = 0 \) to obtain \( \mu_c \) as a function of the parameters \( \{ r \} \). In general, there will a whole set of \( \mu_c \)'s.

This method gives us more than what we need to know in the context of the non-hermitean problem. To determine the boundary of the density of eigenvalues of the non-hermitean random matrix \( \phi \), we are interested in the point, as we vary \( \{ r \} \), when one of the \( \mu_c \) appears at the origin. Thus, we finally obtain the equations to determine the boundary in question, namely

\[
F(u)|_{\mu_c=0} = 0 \quad \text{and} \quad F'(u)|_{\mu_c=0} = 0. \quad (4.5)
\]
To see how simply this procedure works apply it to Ginibre’s problem in which 
\[ F(u) = u^3 - 2\eta u^2 + (1 + \eta^2 - r^2)u - \eta. \]
We obtain almost instantly \( r = 1. \) Indeed, it is known from Ginibre’s work, as well as from our derivation in Section 3, that the eigenvalues are distributed over a disk of radius unity.

In all the examples discussed here and below, \( F(G) \) is a polynomial of odd degree in \( G. \) In Ginibre’s problem \( F \) is a cubic, while in the examples below, \( F \) is a quintic. Furthermore, from (3.13) and the chiral structure of \( H, \) we know that \( \text{Re} \, G(\eta) \) is an odd function of \( \eta. \) Thus, at \( \eta = \mu = 0 \) the real part of \( G \) vanishes. In general, the imaginary part does not vanish of course, but precisely when the density of eigenvalues \( \omega(\mu) \) splits into two lumps, the imaginary part of \( G \) also vanishes at \( \mu = 0. \) It is clear from (3.13) and the symmetry of \( \omega(\mu) \) that in the cut complex plane, away from the cut, \( G(\eta) \) is an odd function of its argument. Thus, if we write \( F(G) = \sum_k c_k(\eta)G^k, \) the coefficient functions \( c_j(\eta) \) must be even in \( \eta \) for odd \( j, \) and odd for even \( j, \) namely, obey
\[
c_j(-\eta) = (-1)^{j+1}c_j(\eta). \tag{4.6}\]
One can check by inspection of (3.25) that this holds in particular for Ginibre’s polynomial.

This suggests a computationally simpler procedure. Given the polynomial equation \( F(G) = 0, \) first set \( \eta \) to zero. Now from (4.6) \( c_0(\eta = 0) = 0, \) and so we should divide \( F(G) \) by \( G \) and set \( G \) to zero (because we wish to identify the points \( \{r\} \) where \( \mu_c = 0. \) At the risk of being repetitive but for the sake of clarity, let us apply this to Ginibre’s polynomial \( G^3 - 2\eta G^2 + (1 + \eta^2 - r^2)G - \eta = 0. \) After the first step, we have \( G^3 + (1 - r^2)G = 0. \) After the second step, we have \( r = 1 \) as desired.

The advantage of this procedure is that in the computation of \( G \) from the equations analogous to (3.21) and (3.23), we can from the beginning set \( \eta \) to zero and then compute various quantities to either zeroth order or first order in \( G. \) This results in enormous simplifications.
5 Further Examples of the Application of the Method of Hermitization

The development of the method of hermitization in the previous two sections was formal and generic. In the last part of Section 3 we demonstrated the method of hermitization by applying it to Ginibre's problem. In this Section we provide further examples to the application of this method.

5.1 Deterministic plus random, both non-hermitean

Consider a situation in which the fluctuating non-hermitean matrix $\phi$ is shifted by a non-hermitean deterministic piece $h_0$. For simplicity, we take $h_0$ to be diagonal, and also assume that $\phi$ is drawn from the Gaussian distribution (3.20). Then our discussion of Ginibre's problem (Eq. (3.21) et. seq.) follows through with the obvious shifts $z \rightarrow z - h_0$ and $z^* \rightarrow z^* - h_0^\dagger$. In particular, (3.24) now reads

$$G_{\mu \nu} = \frac{1}{R^2 - (\eta - G)^2} \begin{pmatrix} G - \eta & z - h_0 \\ z^* - h_0^\dagger & G - \eta \end{pmatrix},$$

(5.7)

where $R^2$ is the $N \times N$ diagonal matrix $(z - h_0)^\dagger(z - h_0)$. Setting $\eta = 0$ and tracing both sides of (5.7), the “gap-equation” (3.25) changes into

$$G = G \frac{1}{N \text{tr}} \frac{1}{R^2 - G^2}.$$

(5.8)

Eq. (5.8) has a trivial solution $G = 0$ and a non-trivial solution which satisfies $(1/N)\text{tr}[1/(R^2 - G^2)] = 1$. Note that according to the analysis in Section 4, the boundary of the eigenvalue distribution occurs when the non-trivial solution of (5.8) vanishes as well, and is thus given by

$$\frac{1}{N \text{tr}} \frac{1}{(z - h_0)^\dagger(z - h_0)} = 1.$$

(5.9)

Extracting the lower left block of $G_{\mu \nu}$ and tracing, we find from (5.5) that

$$\rho(x, y) = \frac{1}{N \pi} \frac{\partial^\dagger \text{tr}(z - h_0)^\dagger}{R^2 - G^2}.$$

(5.10)
Substituting $G = 0$ into (5.10) we find $\rho(x, y) = (1/N) \, \text{tr} \, \delta^{(2)}(z - h_0)$, which is simply the spectrum of eigenvalues of $h_0$, and it obviously vanishes outside the domain of the eigenvalue distribution of the random matrix. We thus associate the solution $G = 0$ (5.8) with the region of zero eigenvalue density outside the domain of eigenvalue distribution, as we already observed in our analysis of Ginibre’s problem at the end of Section 3. The eigenvalue distribution is bounded inside the curve (5.9), and the density of eigenvalues is given by (5.10) with $G$ being the non-trivial solution of (5.8).

It is instructive to work out some simple examples of this case.

### 5.1.1 Circular deterministic spectrum

Assume that the eigenvalues of $h_0$ lie equally spaced along a circle of radius $a$. In the limit $N \to \infty$ they form a uniform continuum. Defining

$$ I = \oint \frac{dw}{2\pi i} \, \frac{1}{w^2 - \frac{r^2 + a^2 - G^2}{ar} w + 1}, $$

where $w = \exp i\theta$ runs along the unit circle, we have for the non-trivial factor of (5.8)

$$ I = -ar. \tag{5.12} $$

Let us denote the roots of $w^2 - [(r^2 + a^2 - G^2)/ar]w + 1 = 0$ by $w_1$ and $w_2$. Note that the roots of the quadratic equation satisfy $w_1 w_2 = 1$. Thus, one of the roots, call it $w_{in}$, always lies inside the unit circle, and the other, call it $w_{out}$, outside. We obtain

$$ I = \oint \frac{dw}{2\pi i} \, \frac{1}{w - w_1} \frac{1}{w - w_2} = \frac{1}{w_{in} - w_{out}}. $$

Then (5.12) reduces to $(a^2 + r^2 - G^2)^2 - 4a^2 r^2 = 1$. There are obviously two roots for $G^2$, but one of them is positive for all values of $r$, and thus cannot correspond to the eigenvalue distribution. The remaining (physical) root is

$$ G^2 = r^2 + a^2 - \sqrt{1 + 4a^2 r^2}. \tag{5.13} $$

The boundary of the eigenvalue distribution occurs at $G = 0$, namely at

$$ r^2 = a^2 \pm 1. \tag{5.14} $$

Thus, if $a > 1$, the original circle of eigenvalues swells into the annulus $\sqrt{a^2 - 1} < r < \sqrt{a^2 + 1}$, while for $a < 1$, the eigenvalues fill a disk of radius $\sqrt{a^2 + 1}$. Finally,
from (5.10) we find the density inside the annulus (or the disk) as

$$\rho(x, y) = \frac{1}{\pi} \left(1 - \frac{a^2}{\sqrt{1 + 4a^2r^2}}\right).$$  \hspace{1cm} (5.15)

Note that there are more eigenvalues near the outer edge. For the annulus case \(a > 1\) the densities at the outer and inner edges are given by \(\pi \rho(\text{outer}) = 1 - a^2/(2a^2 + 1)\) and \(\pi \rho(\text{inner}) = 1 - a^2/(2a^2 - 1)\). For the disk case \(a < 1\) the density at the origin is given by \(\pi \rho(0) = 1 - a^2\) and is thus reduced from the density in Ginibre’s case by a factor of \(1 - a^2\).

In this example the eigenvalues of the deterministic piece \(h_0\) were located along a circle. It would be interesting to generalize this example to situations where the eigenvalues of \(h_0\) are located along an arbitrary curve in the complex plane.

### 5.1.2 Two point deterministic spectrum

In our next example we consider an \(h_0\) with eigenvalues \(\pm \epsilon\), equally degenerated. With no loss of generality we may always take \(\epsilon\) real and positive. The non-trivial factor of (5.8) is now

$$\frac{1}{|z - \epsilon|^2 - G^2} + \frac{1}{|z + \epsilon|^2 - G^2} = 2.$$  \hspace{1cm} (5.16)

Setting \(G = 0\) in (5.16) we conclude that the boundary is determined by

$$y^4 + (2x^2 + 2\epsilon^2 - 1)y^2 + x^4 - (2\epsilon^2 + 1)x^2 + \epsilon^4 - \epsilon^2 = 0.$$  \hspace{1cm} (5.17)

Clearly, the eigenvalue distribution is an even function in both \(x\) and \(y\). For \(\epsilon = 0\) this is simply Ginibre’s problem and the eigenvalues are distributed uniformly on a disk of unit radius centered around the origin. As we increase \(\epsilon\) from zero, the disk is distorted: it gets squeezed along the \(y\) axis and stretched along the \(x\) axis (in both directions). We find that for \(\epsilon < 1\) the eigenvalues are still concentrated in a single blob centered at the origin and confined within \(x^2 < (2\epsilon^2 + 1 + \sqrt{1 + 8\epsilon^2})/2\). However, at \(\epsilon = 1\), what used to be the disk is now completely pinched and separates into two identical lobes that touch only at the origin. As we increase \(\epsilon\) above 1, these two lobes...
separate and their extent along the $x$ axis is determined by $(2\epsilon^2 + 1 - \sqrt{1 + 8\epsilon^2})/2 < x^2 < (2\epsilon^2 + 1 + \sqrt{1 + 8\epsilon^2})/2$. The expression for $\rho(x, y)$, (Eq. (5.10)) is not particularly illuminating and we do not bother to write it here. More generally, if $h_0$ has $K$ different eigenvalues $\epsilon_i$, Ginibre’s disk would get torn into $K$ different blobs as we scale $h_0$ up, obviously.

### 5.2 Two point deterministic spectrum plus hermitean random

As a complication of the previous subsection, we will now discuss the problem in which $\phi = h_0 + M$ consists of the sum of a deterministic piece $h_0$ and an hermitean random matrix $M$ which we will take from the Gaussian distribution $P(M) = \frac{1}{Z} e^{-\frac{1}{2} \text{tr} M^2}$. This case is more complicated than the previous cases because the hermiticity of $M$ amounts to a constraint on the “gluon propagator”. Our interest in this problem is partly motivated by the flux line pinning problem to be discussed in Section 6. In that problem the eigenvalues of $h_0$ trace out an ellipse. Here we content ourselves with the much simpler case of $h_0$ being a diagonal matrix, with half its eigenvalues equal to $i\epsilon$ and the other half equal to $-i\epsilon$ (with $\epsilon$ real and positive.) The qualitative features of the eigenvalue distribution is clear. For $\epsilon = 0$, the eigenvalues are restricted to the real axis and follow Wigner’s semi-circle distribution. As $\epsilon$ increases, the eigenvalues invade the complex plane, and eventually, as $\epsilon$ increases past some critical value $\epsilon_c$ we expect the domain over which the density of eigenvalues is non-zero to break up into two “blobs” centered around $\pm i\epsilon$. Note the obvious symmetries of the shape of the eigenvalue distribution: if $(x, y)$ is a point on the boundary, so are $(\pm x, \pm y)$. Here we shall content ourselves with calculating the boundary of the eigenvalue distribution only. Due to the specific structure of $h_0$ we break $M$ into $(N/2) \times (N/2)$ blocks

$$M = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

in terms of which the Gaussian distribution reads

$$P(M) = \frac{1}{Z} e^{-\frac{1}{2} \text{tr} (A^2 + B^2) - N \text{tr} C^T C},$$

(5.19)
(In general, of course, we can consider taking the Gaussian widths of $A$, $B$, and $C$ to be different, but we will not entertain this possibility here.) With $\phi$ and $M$ given above, the deterministic piece of the Hamiltonian (3.1) in terms of its $(N/2) \times (N/2)$ blocks is

$$H_0 \equiv -G^{-1}_0(\eta = 0) = \begin{pmatrix}
0 & 0 & i\epsilon - z & 0 \\
0 & 0 & 0 & -i\epsilon - z \\
-i\epsilon - z^* & 0 & 0 & 0 \\
0 & i\epsilon - z^* & 0 & 0
\end{pmatrix},$$

(5.20)

and the random part is

$$V = \begin{pmatrix}
0 & 0 & A & C \\
0 & 0 & C^\dagger & B \\
A & C & 0 & 0 \\
C^\dagger & B & 0 & 0
\end{pmatrix}.$$  

(5.21)

Speaking picturesquely, and adding to the “quark-gluon” language mentioned earlier, we can refer to the (block) four dimensional basis as “flavor”, with flavor assignments

$$\psi = \begin{pmatrix}
u \\
c \\
d \\
s
\end{pmatrix},$$

(5.22)

containing two “generations”, $(u,d)$ and $(c,s)$. With this flavor assignment, we see from the interaction $\psi^\dagger V \psi$ that the gluons corresponding to $A$, $B$ and $C$ all act like weak interactions gauge bosons. $A$ and $B$ act within their appropriate generation, while $C$ and $C^\dagger$ communicate between the two generations\footnote{To see this more clearly, and also for computational simplicity, it is convenient to change the basis by interchanging rows as well as columns 2 and 3. In the new basis the $A$'s in $V$ will migrate to the upper left corner, the $B$'s to the lower right corner, the $C$'s to the upper right block, and the $C^\dagger$'s to the lower left block, such that in the new basis $V$ will become simply $M \otimes \sigma_1$. However, we do not use the transformed basis in the text explicitly, to avoid unnecessary repetitions of formulas.} Since we are only interested in determining the boundary of the eigenvalue distribution, from now on, we
set \( \eta = 0 \), so that \( \mathcal{G}_\mu^\nu(\eta = 0) \) is hermitean. From (3.22) and (5.19) we then find that the self-energy matrix has the form

\[
\Sigma = \begin{pmatrix}
g_{33} + g_{44} & 0 & g_{31} + g_{42} & 0 \\
0 & g_{33} + g_{44} & 0 & g_{31} + g_{42} \\
g_{31}^* + g_{42}^* & 0 & g_{11} + g_{22} & 0 \\
0 & g_{31}^* + g_{42}^* & 0 & g_{11} + g_{22}
\end{pmatrix},
\]

(5.23)

where the non-zero blocks are proportional to \( \frac{1}{N} \) with proportionality coefficients that are sums of

\[
g_{ab} = g_{ba}^* = \frac{1}{N} \text{tr}(\mathcal{G})_{ab \text{ block}}, \quad a, b = 1, \ldots, 4,
\]

(5.24)

where the trace is carried over the \((N/2) \times (N/2)\) blocks. By “index democracy” and the texture of \( H \), we obviously have \( g_{11} = g_{22} \) and \( g_{33} = g_{44} \). Then clearly \( \mathcal{G} = (g_{11} + g_{33})/2 \). Substituting (5.23) into (3.21) we obtain a system of coupled “gap equations” for the coefficients \( g_{ab} \). We are interested only in the \( g_{ab} \) that appear explicitly in \( \Sigma \).\footnote{Note also that the trace of the lower left block of \( \mathcal{G}_\mu^\nu \) is \( g_{11} + g_{42} \). Thus the \( g_{ab} \) that do not appear in (5.23) are also irrelevant for the purpose of calculating \( \rho(x, y) \) from (2.11).}

Introducing the quantity \( D \equiv (g_{11} - g_{33})/2 \) we obtain the equations for these elements as

\[
D = \mathcal{G}(|g_{13}|^2 - |g_{24}|^2 - 4\mathcal{G}D) \\
\mathcal{G} = \mathcal{G}(|g_{13}|^2 + |g_{24}|^2 - 2(\mathcal{G}^2 + D^2)) \\
g_{13} = (|g_{13}|^2 - \mathcal{G}^2)(z - i\epsilon - g_{13}^* - g_{24}^*) \\
g_{24} = (|g_{24}|^2 - \mathcal{G}^2)(z + i\epsilon - g_{13}^* - g_{24}^*).
\]

(5.25)

These are 6 real equations in 2 complex plus 2 real unknowns. Being interested only in determining the boundary of the eigenvalue distribution, we only need the non-trivial factor of the gap equations to first order in \( \mathcal{G}(0) \). From the first equation in (5.25) we find that \( D = 0 \) when \( \mathcal{G} = 0 \) (for the trivial, as well as for the non-trivial
factor of the gap equation), thus eliminating one real unknown and one real equation. Then, dividing the second equation in (5.25) by $G$ and setting $G = 0$ we obtain

\[
|g_{13}|^2 + |g_{24}|^2 = 1 \\
(z^* + i\epsilon - g_{13} - g_{24})g_{13} = 1 \\
(z^* - i\epsilon - g_{13} - g_{24})g_{24} = 1 ,
\]

which constitute 5 real equations in 2 complex unknowns, thus producing a single real constraint, namely, the desired equation for the boundary. A direct elimination of one of the $g'$s in (5.26) produces a cubic equation in the other $g$, which has to be solved explicitly. Then, the correct root of that cubic has to be fed into the constraint to obtain the boundary. We will content ourselves with studying perturbatively the two limits $\epsilon << 1$ and $\epsilon >> 1$.

We first examine the small $\epsilon$ limit. It is useful to introduce the combinations

\[
g_{13} + g_{24} = g\sqrt{2}, \quad g_{13} - g_{24} = f\sqrt{2} \quad (5.27)
\]

and the parametrizations

\[
z^* = 2\sqrt{2} \sin\beta, \quad i\epsilon = \lambda\sqrt{2} . \quad (5.28)
\]

Then the appropriate linear combinations of the equations (5.26) become

\[
g^2 - 2g \sin \beta - \lambda f + 1 = 0 \\
gf - 2f \sin \beta - \lambda g = 0 \\
|f|^2 + |g|^2 = 1 . \quad (5.29)
\]

At $\epsilon = 0$ we should recover the ordinary Wigner semi-circular behavior associated with (5.19)

\[
g_{13}^{(0)} = g_{24}^{(0)} = \frac{z^* - \sqrt{z^*}^2 - 8}{4} \quad (5.30)
\]

namely,

\[
g^{(0)} = \frac{z^* - \sqrt{z^*}^2 - 8}{2\sqrt{2}} = -ie^{i\beta}, \quad f^{(0)} = 0 . \quad (5.31)
\]
Thus, in a systematic expansion of \((5.29)\) in powers of \(\lambda\) around \(\lambda = 0\), \(f\) is always a small quantity, which is the reason for introducing the combinations \((5.27)\). The eigenvalues lie on the real axis in the segment \(|x| \leq 2\sqrt{2}\) which is the (degenerate) boundary of the eigenvalue distribution. In particular, along this “boundary”, the angle \(\beta = \beta^{(0)}\) \((0 \leq \beta^{(0)} < 2\pi)\) is real, of course, in accordance with the third equation in \((5.29)\). Let us now solve the first two equations in \((5.29)\) to leading order in \(\lambda\). We find

\[
\begin{align*}
g &= i e^{i\beta} \left( -1 + \frac{\lambda^2 e^{i\beta}}{2 \cos \beta} \right), \\
f &= \lambda e^{2i\beta}.
\end{align*}
\]

(5.32)

Substituting these quantities into the third equation in \((5.29)\) we find that to this order

\[
\beta = \beta^{(0)} - i\lambda^2,
\]

(5.33)

and thus, we find from \((5.28)\) that to this order the boundary is

\[
x^2 + \frac{4y^2}{\epsilon^4} = 8,
\]

(5.34)

which shows that the original segment containing the eigenvalues of \(M\) swells into an ellipse, with a semi-minor axis equal to \(2\sqrt{2}\lambda^2 = \sqrt{2}\epsilon^2\).

We now turn to the other extreme limit, \(\epsilon >> 1\). Here we expect of course that the eigenvalues concentrate in two semicircular distributions around \(z = \pm i\epsilon\). For concreteness, let us concentrate on the vicinity of \(z = -i\epsilon\). Shifting \(z^* = i\epsilon + \zeta\), we find that the last two equations in \((5.26)\) become

\[
\begin{align*}
g_{24} + \frac{1}{g_{24}} &= \zeta - g_{13}, \\
(2i\epsilon + \frac{1}{g_{24}})g_{13} &= 1.
\end{align*}
\]

(5.35)

We now expand \((5.35)\) in powers of \(1/\epsilon\), for \(|\zeta|/\epsilon << 1\). It is clear that here \(g_{13}\) is the small quantity. To leading order in \(1/\epsilon\) we find the semi-circular behavior

\[
\begin{align*}
g_{24}^{(0)} &= \frac{\zeta - \sqrt{\zeta^2 - 4}}{2}, \\
g_{13}^{(1)} &= \frac{1}{2i\epsilon}.
\end{align*}
\]

(5.36)

Thus, to this order in \(1/\epsilon\), the eigenvalues follow semi-circular distribution along the segments \(|x| \leq 2, y = \pm \epsilon\), which are also the (degenerate) boundaries of the
eigenvalue distribution. Note that the width squared of this semi-circle is half that of the semi-circle in (5.31), because the eigenvalue distribution is now split into two identical blobs. It is natural to parametrize

$$\zeta = 2 \sin \gamma,$$  \hspace{1cm} (5.37)

in terms of which $$g_{24}^{(0)} = -ie^{i\gamma}.$$ Since $$\zeta$$ is real along the “boundaries” to leading order in $$1/\epsilon,$$ $$\gamma = \gamma^{(0)} \quad (0 \leq \gamma^{(0)} < 2\pi)$$ is real as well, in accordance with the first equation in (5.26). Expanding $$g_{24}$$ to the next order in $$1/\epsilon$$ we find

$$g_{24} = -ie^{i\gamma} \left(1 - \frac{1}{4\epsilon \cos \gamma}\right).$$  \hspace{1cm} (5.38)

Substituting these expressions for $$g_{24}$$ and $$g_{13}$$ into the first equation in (5.26) we find to this accuracy that on the boundary

$$\gamma = \gamma^{(0)} - \frac{i}{4\epsilon \cos \gamma^{(0)}},$$  \hspace{1cm} (5.39)

and thus from (5.37) the (lower half of the) boundary is given parametrically by

$$\left\{\begin{array}{l}
x = 2 \sin \gamma^{(0)} \\
y = -\epsilon + \frac{1}{2\epsilon}
\end{array}\right.$$  \hspace{1cm} (5.40)

We see that the effect of the first non-trivial correction is simply to pull the two semi-circle distributions towards the real axis in an amount proportional to $$1/\epsilon,$$ with no distortions. We shall present a simple explanation of this phenomenon in Section 6.

We close this subsection by determining the critical value $$\epsilon_c$$ at which the connected eigenvalue distribution corresponding to small values of $$\epsilon$$ breaks into two separate lumps. From the symmetries of the eigenvalue distribution it is clear that when this happens, the boundary passes through the origin (it then has the shape of the figure 8). Thus, substituting $$z = \beta = 0$$ in (5.29) we find a non-trivial solution $$g = 0, f = 1/\lambda,$$ and thus $$\epsilon_c = \sqrt{2}.$$
5.3 QCD at finite chemical potential

As was mentioned in the Introduction, Stephanov [2] has recently studied the chiral phase transition in QCD at finite chemical potential using non-hermitean random matrices. The Euclidean Dirac operator at chemical potential $\mu$ is non-hermitean. The matrix representation of this operator (for massless quarks) is

$$
\phi = \begin{pmatrix}
0 & C + i\mu \\
C^\dagger + i\mu & 0
\end{pmatrix}.
$$

(5.41)

The chiral components of the Dirac operator are represented here by the $N \times N$ complex matrices $C$ and $C^\dagger$. The fluctuations of the gluonic background are modeled by taking $C$ to be Gaussian random, namely, with probability distribution $P(C) = \exp(-N\text{tr}C^\dagger C)/Z$. The Green’s function and density of eigenvalues of (5.41) were calculated in [2] using the replica method, and were rederived in [4] diagrammatically. Thus, as a complementary calculation, in the following we shall apply the methods of Section 4 to calculate the boundary of the eigenvalue distribution for this ensemble, without calculating the density itself. Recall that to the sole purpose of determining the boundary, we only need the the non-trivial factor of the gap equation to first order in $G$ at $\eta = 0$. Thus, as in previous sections, we set $\eta = 0$ from now on. With $\phi$ given by (5.41), the deterministic piece of the Hamiltonian (3.1) in terms of its $N \times N$ blocks is

$$
H_0 \equiv -G_0^{-1}(\eta = 0) = 
\begin{pmatrix}
0 & 0 & -z & i\mu \\
0 & 0 & i\mu & -z \\
-z^* & -i\mu & 0 & 0 \\
-i\mu & -z^* & 0 & 0
\end{pmatrix},
$$

(5.42)
and the random part is
\[ V = \begin{pmatrix} 0 & 0 & 0 & C \\ 0 & 0 & C^\dagger & 0 \\ 0 & C & 0 & 0 \\ C^\dagger & 0 & 0 & 0 \end{pmatrix} \] (5.43)

Then from (3.22) we find that the self-energy matrix has the form
\[ \Sigma = \begin{pmatrix} g_{44} & 0 & g_{42} & 0 \\ 0 & g_{33} & 0 & g_{31} \\ g_{42}^* & 0 & g_{22} & 0 \\ 0 & g_{31}^* & 0 & g_{11} \end{pmatrix}, \] (5.44)

where the non-zero blocks are proportional to \( \mathbf{1}_N \) with proportionality coefficients
\[ g_{ab} = g_{ba}^* = \frac{1}{N} \text{tr}_{(N)} (G)_{ab \text{ block}}, \quad a, b = 1, \ldots, 4. \] (5.45)

By "index democracy" all the diagonal coefficients \( g_{kk} \) are equal, and are therefore equal to \( G \). Similarly, \( g_{42} = g_{31} \equiv G \) (they are in the same block of \( \Sigma \).) Substituting (5.44) into (3.21) we obtain a system of coupled "gap equations" for the coefficients \( g_{ab} \). We are interested only in the \( g_{ab} \) that appear explicitly in \( \Sigma \). Among the gap equations there is subset of one real equation and one complex equation that involve only the real unknown \( G \) and the complex unknown \( G \). In accordance with our discussion in Section 4, the real equation contains \( G \) as an overall factor. Following the steps indicated in Section 4 we divide this factor out and set \( G = 0 \), thereby obtaining one equation for \( G \) and a constraint. Eliminating \( G \), we find the constraint
\[ \mu^2 (4\mu^2 + x^2) [(\mu^2 - y^2)^2 x^2 + 4\mu^2 y^4 + (1 + 4\mu^2 - 8\mu^4) y^2 + 4\mu^4 (\mu^2 - 1)] = 0. \] (5.46)

5See footnote 3. Here of course, we do not have the blocks \( A \) and \( B \).

6Note also that the trace of the lower left block of \( G^\mu \) is \( (g_{31} + g_{42})/2 \equiv G \). Thus the \( g_{ab} \) that do not appear in (5.44) are also irrelevant for the purpose of calculating \( \rho(x,y) \) from (2.11).
Solving (5.46) for $x^2$ we obtain

$$x^2 = \frac{1}{(\mu^2 - y^2)^2} \left[ -4\mu^2 y^4 + (8\mu^4 - 4\mu^2 - 1)y^2 - 4\mu^4(\mu^2 - 1) \right],$$

(5.47)
in accordance with [2].
6 Energy Level Dynamics of Non-Hermitean Hamiltonians

In much of this paper, we have discussed non-hermitean random matrices without any spatial structure. Physical problems typically involve spatial structures and thus random matrices with many zero entries, with non-zero entries representing nearest neighbor hopping for instance. For such problems, even in the hermitean case, there is typically no simple analytic approach available and these problems has to be studied on a case by case basis. Here we will use the problem of driven vortex lines as an illustrative context within which to make some general qualitative comments. In the simplest model for this problem, the Hamiltonian $H$ is the sum of a deterministic but non-hermitean hopping matrix $H_0$ and a random but hermitean matrix representing pinning by impurities $V$. We have in the simplest one-dimensional case with $N$ sites

$$H_{0ij} = e^h \delta_{i+1,j} + e^{-h} \delta_{i,j+1}, \quad i, j = 1, \ldots, N$$

(6.1)

describing a particle which tends to hop in one direction more than in the opposite direction. The eigenvalues of $H_0$ (assuming periodic boundary conditions) can be easily worked out by standard methods to be

$$E_n = 2 \cos \left( \frac{2\pi n}{N} - ih \right) \quad (n = 0, 1, \ldots, N - 1),$$

(6.2)

thus tracing out an ellipse. (The limits of $h \to 0$ and $h \to \infty$ in which the ellipse collapses to a line segment on the real axis and expands to a circle are both easily understood.) The precise form of $V$ reflects what one believes to be an appropriate representation of the microscopic physics, of course. Some choices that come to mind are a random nearest hopping Hamiltonian $V_{ij} = v_i \delta_{i+1,j} + v_j^* \delta_{i,j+1}$, or a diagonal Hamiltonian describing site impurities $V_{ij} = v_i \delta_{ij}$, or a purely random hermitean matrix $V$. This last case may be most amenable to analytic study, but represents the somewhat unphysical situation of being able to hop from any site to any other site (the “infinite dimension” limit).
Let us begin our discussion of random non-hermitean hopping by making some
simple observation based on symmetry. For $H_0$ real, the non-real eigenvalues come
in complex conjugate pairs. This is true in particular for $H_0$ in (6.1), namely $E_n$
and $E_{N-n}$ in (6.2) form such a pair. Furthermore, if $\psi$ is an eigenvector (say, on the
right) of $H_0$, corresponding to an eigenvalue $E$, $\psi^*$ is an eigenvector (on the right)
corresponding to an eigenvalue $E^*$. For the specific $H_0$ in (6.1), if $N$ is even, then the
eigenvalues also come in opposite pairs: $E_n = -E_{n+N/2}$. This follows for a large class
of hopping Hamiltonians with “bipartite symmetry” under an operation in which we
multiply the basis vectors associated with the odd sites by $(-1)$.

As we crank up the hermitean $V$ we expect the eigenvalues tracing out the ellipse
to slowly migrate to the real axis, and becoming real in the infinite $V$ limit. In Figure
(1) we present numerical results for the spectrum of $H_0 + V$ (with periodic boundary
conditions), where $V$ describes random site energies. These site energies are taken
from a flat distribution, symmetric about the origin, and their range is comparable
with the size of the ellipse. We do not bother to display here how the distortion of
the ellipse varies with the range of the random site energies.

To the initial surprise of the authors, already for a finite amount of randomness,
there are pairs of eigenvalues which have “snapped” onto the real axis. In fact, this
“snapping phenomenon” is easy to understand qualitatively.

Repulsion of nearby energy eigenvalues is of course one of the fundamental features
of quantum mechanics, and is explained in any elementary texts as a property of $2 \times 2$
hermitean matrices. If we have a pair of eigenvalues $\pm \epsilon$, and turn on an off-diagonal
perturbation $v$ between them, the eigenvalues become $\pm \sqrt{\epsilon^2 + v^2}$, thus moving apart.
The behavior of random hermitean matrices can be understood qualitatively in this
way, and is formalized in the concept of the Dyson gas with its molecular repulsion.

Now if we have a pair of complex eigenvalues separated in the imaginary direction,
that is, if we have a pair of eigenvalues $\pm i\epsilon$, we see that under the (hermitean)
perturbation $v$ they become $\pm i\sqrt{\epsilon^2 - v^2}$. Our standard intuition fails. Two nearby
imaginary eigenvalues attract! Furthermore, they do not migrate gradually to the
Figure 1: The spectrum of (6.1) with random site energies (with 100 sites). Here $h = 0.5$ and the random site energies $|v_i| \leq 1.5$ are drawn from a flat distribution, symmetric about the origin. There are 12 real eigenvalues in all.
real axis. Rather, as soon as \( v \) becomes larger than \( \epsilon \) in magnitude, the eigenvalues snap to the real axis.

(We note that in the case of an anti-hermitean perturbation (that is, for \( v \) imaginary) the opposite behavior holds: two nearby eigenvalues separated along the real direction attract, and two nearby eigenvalues separated along the imaginary direction repel.)

As in the case of hermitean matrices, we can reaffirm the mutual effects of nearby energy levels we observed in the simple \( 2 \times 2 \) matrix example by doing a second order perturbation calculation. For simplicity, we assume that \( H_0 \) is non-degenerate. Let us then add a perturbation \( V \) to \( H_0 \). We recall that a non-hermitean Hamiltonian \( H_0 \) with eigenvalues \( \{E_n^{(0)}\} \) has a set of eigenvectors on the right \( H_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle \), as well as a set of eigenvectors on the left \( \langle \phi_n^{(0)}|H_0 = \langle \phi_n^{(0)}|E_n^{(0)} \), that form a complete bi-orthonormal system\(^{[24]}\). With these facts in mind, perturbation theory for non-hermitean Hamiltonians appears formally the same as in the purely hermitean case (as long as one remembers to make the distinction between eigenvectors on the right and eigenvectors on the left.) In particular, the correction of \( E_n^{(0)} \) to first order in \( V \) is simply \( \delta^{(1)} E_n = \langle \phi_n^{(0)}|V|\psi_n^{(0)}\rangle \), and the correction to second order in \( V \) is

\[
\delta^{(2)} E_n = \sum_{k \neq n} \frac{\langle \phi_n^{(0)}|V|\psi_k^{(0)}\rangle \langle \phi_k^{(0)}|V|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}. \tag{6.3}
\]

For a diagonal \( H_0 \), the bi-orthogonal sets coincide \( \psi_n^{(0)} = \phi_n^{(0)} \), forming a single ordinary orthogonal basis, and \( (6.3) \) assumes the same form as in perturbation theory of hermitean operators. In this case we see immediately that if \( V \) is hermitean, the original eigenvalues repel in the real direction and attract in the imaginary direction, whereas if \( V \) is anti-hermitean, they do the opposite, as expected.

To get a qualitative feel, let us analyze the effect of a pair of eigenvalues on another nearby pair. We concentrate on the \( 4 \times 4 \) piece of the Hamiltonian associated with these eigenvalues, ignoring all the other eigenvalues and take \( H = H_0 + V \) with \( H_0 = \text{diag}(i\epsilon, -i\epsilon, c + ia, c - ia) \); \( \epsilon, a > 0 \). For simplicity, we take the perturbation \( V \) hermitean with zero diagonal entries, and all other entries real and equal to \( v \) (this
should be enough to capture the qualitative behavior in a typical case.) Assuming that \( \epsilon \ll |c| \sim a \), the pair of eigenvalues of \( H \) corresponding to \( \pm i\epsilon \) would snap first, and we want to study the effect of the pair at \( c \pm ia \) on it. Cranking up (6.3) we thus find that to second order in perturbation theory, the first two eigenvalues are

\[
\pm i\epsilon \left( 1 - \frac{v^2}{2\epsilon^2} \right) - v^2 \left[ \frac{1}{c + i(a - \epsilon)} + \frac{1}{c - i(a + \epsilon)} \right],
\]

which to first order in \( \epsilon/c^2 \) reads

\[
\pm i\epsilon \left( 1 - \frac{v^2}{2\epsilon^2} - 2v^2 \frac{c^2 - a^2}{(c^2 + a^2)^2} \right) - \frac{2cv^2}{c^2 + a^2} + O \left( v^2 \epsilon^2 / a^3 \right).
\] (6.4)

We readily indentify the piece \( \pm i\epsilon \left( 1 - \frac{v^2}{2\epsilon^2} \right) \) in (6.4) as the first two terms in the expansion of \( \pm i\sqrt{c^2 - v^2} \) we mentioned earlier, namely, the exact value of these eigenvalues when \( c \pm ia \to \infty \). Thus, under the influence of \( V \), \( \pm i\epsilon \) indeed attract each other. Note that to the order to which (6.4) holds, the pair \( c \pm ia \) slows or accelerates the snapping of the first pair onto the real axis depending on whether \( c^2 < a^2 \) or \( c^2 > a^2 \). The first pair is also repelled by the pair \( c \pm ia \) along the real axis. If \( c > 0 \) it moves to the left, and if \( c < 0 \) it moves to the right.

The elementary discussion just concluded explains qualitatively the distortion of the ellipse corresponding to \( H_0 \) that we see in Figure (1). The eigenvalues near the tips of the ellipse are the first to snap to the real axis, for the obvious reason that they are closer to the real axis. They are then repelled away from the ellipse by the eigenvalues that remain off the real axis. Once on the real axis, they repel the eigenvalues that snapped earlier further along the real axis.

For random hermitean hopping Hamiltonians, the lore is that the states in the middle of the band (or possibly only one state) are extended, while the states on the two “wings” of the band are localized. For the one-dimensional case, this can be shown quite explicitly by analytical methods [25], and for the two-dimensional case, this can be studied by renormalization group arguments and verified numerically. In the present one dimensional context, we might thus expect (or conjecture) that the eigenvalues that have snapped onto the real axis correspond to localized states, while
the eigenvalues that remain complex correspond to extended states \cite{6}. In Figures
(2) and (3) we present numerical studies of the participation ratio that support this
conjecture or expectation\footnote{Out of the 12 real eigenvalues we found numerically in this case, we chose to depict the most localized state. Some of the other states corresponding to real eigenvalues may break into 2 or 3 separated lumps, but their participation ratios remain of the same order of magnitude as of the state presented on Fig. (2).}. The normalized participation ratio for a state $\psi_i$ is defined as

$$ P = \frac{1}{N} \frac{\sum_{i=1}^{N} |\psi_i|^2}{\sum_{i=1}^{N} |\psi_i|^4} $$

(6.5)

and thus should tend to 1 for extended states and to 0 for localized states.

It is worth emphasizing that the method of hermitization given in Section 3 is
independent of the spatial character of the random non-hermitean Hamiltonian. Thus,
in principle, it may be used to reduce the problem of random non-hermitean hopping

Figure 2: The eigenvector $\psi_i$ associated with one of the eigenvalues (2.9) on Fig. (1)
that snapped onto the real axis. The corresponding normalized participation ratio is
0.04.
Figure 3: The eigenvector $\psi_i$ associated with one of the complex eigenvalues ($0.2 + 0.8\ i$) on Fig. (1). The corresponding normalised participation ratio is 0.8.
to the auxiliary hermitean problem. The deterministic part of the hermitean problem can be interpreted as a nearest neighbor hopping of a particle with a binary internal state (call it an “up quark” or a “down quark”), such that whenever the particle hops it flips its internal state. In any case, it still bears the same spatial structure as in the original non-hermitean problem, thus retaining all the difficulties associated with random hermitean matrices with spatial structure.
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Appendix : A $2 \times 2$ example

In this Appendix we will perform an exceedingly elementary but instructive exercise. Consider the $2 \times 2$ non-hermitean matrix

$$
\phi = \begin{pmatrix}
0 & C + i\mu \\
C^* + i\mu & 0
\end{pmatrix}
$$

where $C$ is a Gaussian random complex variable and $\mu$ is a real parameter. This is thus the $N = 1$ limit of the problem considered by Stephanov[2], which we discussed in sub-section 5.2. The eigenvalues of $\phi$ are $\lambda = \pm [(C + i\mu)(C^* + i\mu)]^{1/2}$. Writing $C = re^{i\theta}$ we have by definition the density of eigenvalues

$$
\rho(x, y) = \int \int dr^2 \frac{d\theta}{2\pi} e^{-r^2} \delta(x - \text{Re} \lambda) \delta(y - \text{Im} \lambda) \tag{A.1}
$$

The evaluation of this integral is elementary and gives

$$
\rho(x, y) = \frac{4}{\pi} \frac{(x^2 + y^2)}{\sqrt{(\mu^2 + x^2)(\mu^2 - y^2)}} e^{-(x^2 + y^2 + \mu^2)} \theta(\mu^2 - y^2). \tag{A.2}
$$

As is indicated by the $\theta$ function, the density of eigenvalues is non-zero over an infinitely long strip with width $2\mu$. It decreases exponentially as $x \to \pm \infty$ and blows up as an inverse square root as $y \to \pm \mu$. Notice that if we impose the condition that $H$ be normal (but non-hermitean), that is $[\phi, \phi^\dagger] = 0$, $C$ is required to be real, and the density of eigenvalues collapses to

$$
\rho(x, y) = \frac{1}{2} P(x) [\delta(y - \mu) + \delta(y + \mu)] \tag{A.3}
$$

where $P(C)$ denotes the (normalized) probability of distribution of $C$, whatever one might choose. The infinitely long strip collapses into two lines.
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