Higher spins in the symmetric orbifold of K3

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Abstract: The symmetric orbifold of K3 is believed to be the CFT dual of string theory on AdS$_3$ × S$^3$ × K3 at the tensionless point. For the case when the K3 is described by the orbifold $\mathbb{T}^4/\mathbb{Z}_2$, we identify a subsector of the symmetric orbifold theory that is dual to a higher spin theory on AdS$_3$. We analyse how the BPS spectrum of string theory can be described from the higher spin perspective, and determine which single-particle BPS states are accounted for by the perturbative higher spin theory.
1. Introduction

One of the long standing puzzles in theoretical physics is the determination of the symmetry algebra of string theory. Hints of an enlarged symmetry in string theory have been found by studying the high energy limit \([1, 2, 3]\), where a large set of higher spin symmetries emerge. In asymptotically AdS backgrounds, this limit can be studied in quite some detail thanks to the AdS/CFT correspondence \([4]\), as it corresponds to free gauge theories on the boundary \([\text {4, 5, 6}]\). The extended higher-spin gauge symmetries of the bulk are then dual to the large number of conserved currents that emerge in the free limit of the dual theory.

This idea has sparked considerable interest in studying this higher-spin subsector in isolation from the rest of the dynamics, which has led to a number of weak-weak
'vector-like' dualities between Vasiliev [8] higher-spin theories on AdS and CFTs with extended symmetries on the boundary [3, 10].

In the AdS$_3$/CFT$_2$ context, the relevant higher-spin bulk theories have been conjectured to be dual to minimal model CFTs [11]. In this paper, we are particularly interested in theories with $\mathcal{N} = 4$ supersymmetry that arise as the near-horizon limit of D1-D5 systems; in this case the bulk geometry is AdS$_3 \times S^3 \times \mathcal{M}_4$, where $\mathcal{M}_4$ is either $T^4$ or $K3$. In the tensionless limit of string theory, these backgrounds are believed to be dual to the free symmetric orbifold $\text{Sym}_N \mathcal{M}_4$, where $N = Q_1 Q_5$ is the product of the D1 and D5 brane charges. On the other hand, their higher-spin subsector should be dual to minimal models with $\mathcal{N} = 4$ symmetry.

It is then natural to try and understand the relation between the minimal model CFTs dual to the higher-spin subsector of such theories, and the CFTs (in the present case, the symmetric orbifold theory) describing the full string theory spectrum. This has been answered in [12] for the case of AdS$_3 \times S^3 \times T^4$ in the limit where the volume of the torus is very large. In this limit, the perturbative part of the Vasiliev AdS$_3$ theory is captured by a subsector of the large level limit of particular Wolf space cosets [13, 14, 15, 16, 17]. Indeed, it was shown in [12] that the relevant subsector is described by the $U(N-1)$ invariant states of the $4N$ free bosons and fermions that make up the $T^{4N}$ theory of the symmetric orbifold; this is naturally a subsector of the $S_N$ invariant states of the symmetric orbifold since $S_N \subset U(N-1)$. The symmetric orbifold can then be regarded as another modular invariant of the coset theory. All its states can be organised in terms of the $\mathcal{W}$-algebra associated to the coset CFT, but it does not possess any light states.

In this paper we want to find the analogous relation for the case where $\mathcal{M}_4 = K3$. We shall concentrate on a specific K3 sigma-model, namely the one where K3 can be described by the orbifold $K3 = T^4/Z_2$. The situation is then a little different from what was considered in [12] since the symmetric orbifold of K3 does not actually contain a contraction of the Wolf-space large $\mathcal{N} = 4$ $\mathcal{W}_\infty[0]$ algebras as a symmetry algebra — this follows, for example, from the fact that the elliptic genus of the symmetric orbifold of K3 does not vanish, while that of the large $\mathcal{N} = 4$ theories is always trivial. The higher spin – CFT duality that is relevant for the K3 case is therefore slightly different from what was considered before in [21]: at $\lambda = 0$ both the higher spin algebra and the Wolf space cosets contain a subtheory, and it is the duality between these subtheories that is of relevance here. In particular, we show that the full spectrum of the symmetric K3-orbifold can be organised in terms of representations of an appropriate subalgebra $W^s_\infty$ of $\mathcal{W}_\infty[0]$.

One added benefit of repeating the analysis for K3 (rather than $T^4$) is that the elliptic genus of the K3 symmetric orbifold is non-trivial. Thus it is natural to look

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1This subsector is not modular invariant by itself and needs to be completed with additional states. The usual diagonal modular invariant leads to a plethora of light states, see e.g. [18], whose precise interpretation from a bulk viewpoint has not been understood in detail, see however [3, 20].
at the BPS spectrum of the theory, and we analyse the BPS spectrum of supergravity or string theory on $\text{AdS}_3 \times S^3 \times K3$ in terms of the representation theory of $\mathcal{W}_{\infty}^\kappa$. This allows us to determine the part of the BPS spectrum that is captured by the perturbative higher spin theory. As it turns out, only a tiny fraction of the single-particle BPS states of string theory are actually contained in the perturbative higher spin theory; for example, out of the 21 chiral primaries whose descendant are the exactly marginal operators of the K3 symmetric orbifold, only 2 are part of the perturbative higher spin theory.

The paper is organised as follows. In Section 2, we introduce the modified higher spin – CFT duality that is relevant in our case, and we explain how it is related to the symmetric orbifold of K3. In Section 3 we show that the chiral algebra of the K3 symmetric orbifold can indeed be described in terms of representations of the relevant $\mathcal{W}_{\infty}^\kappa$ algebra. In Section 4 we study the chiral primaries of the symmetric orbifold from the higher spin perspective; we identify all low-lying states of the symmetric orbifold in terms of $\mathcal{W}_{\infty}^\kappa$ representations, and then specifically identify the BPS states among them. We also enumerate the BPS states of the perturbative higher spin theory and show that there are only two single-particle states among them. Finally, Section 5 contains our conclusions.

2. The small $\mathcal{N} = 4$ higher-spin duality

Let us begin by reviewing the minimal model holography with large $\mathcal{N} = 4$ superconformal symmetry. The relevant higher spin theory is based on the Lie superalgebra $\text{shs}_2[\lambda]$, and was argued to be dual to the Wolf space coset CFTs $[21]$

\[
\frac{\mathfrak{su}(N+2)^{(1)}_{k+N+2}}{\mathfrak{su}(N)^{(1)}_{k+N+2} \oplus \mathfrak{u}(1)^{(1)}_\kappa},
\]

where $\kappa = 2N(N+2)(N+k+2)$. These cosets contain the large $\mathcal{N} = 4$ superconformal algebra $A_\lambda$ with $\lambda = \frac{N+1}{N+k+2}$ (the reader is encouraged to consult $[21]$ for more details); further evidence for this duality was given in $[22, 23, 24, 25, 26]$.

Except for very small values of $N$ and $k$, the chiral algebra contains higher spin currents in addition to the ones associated to the superconformal symmetry. These can be organised into multiplets $R^{(s)}$ of the superconformal algebra, where $s = 1, 2, \ldots$ labels the spin of the highest-weight state,

\[
\begin{align*}
    s & : \quad (1, 1) \\
    s + \frac{1}{2} & : \quad (2, 2) \\
    R^{(s)} : \quad s + 1 & : (3, 1) \oplus (1, 3) \\
    s + \frac{3}{2} & : \quad (2, 2) \\
    s + 2 & : \quad (1, 1).
\end{align*}
\]
The quantum numbers shown in the column on the right refer to the two \( \mathfrak{su}(2) \) algebras of \( A_\lambda \). Together with the superconformal algebra generators, they form the large \( \mathcal{N} = 4 \ W_\infty[\lambda] \) algebra, which is studied in detail in \([24, 25]\).

In order to obtain a higher-spin algebra with small \( \mathcal{N} = 4 \) symmetry, we now take the limit \( k \to \infty \), for which \( \lambda \to 0 \). In this limit, the large \( \mathcal{N} = 4 \) algebra contracts to the small \( \mathcal{N} = 4 \) algebra together with 4 free bosons and 4 free fermions. In terms of the coset description, taking the limit \( k \to \infty \) leads to the continuous orbifold of the form \([12]\) (see also \([27, 28]\))

\[
(T^4)^{N+1} / U(N) .
\] (2.3)

There are also matter fields corresponding to the \( (0; f) \otimes (0; f^*) \) and \( (0; f^*) \otimes (0; f) \) degrees of freedom of the CFT, and the perturbative part of the bulk higher spin theory is described by the CFT subsector

\[
\mathcal{H}^{(\text{pert})} = \bigoplus_\Lambda (0; \Lambda) \otimes (0; \Lambda^*) .
\] (2.4)

The authors of \([12]\) found an interesting non-diagonal modular completion of the above Hilbert space, which corresponds to the degrees of freedom of string theory on \( \text{AdS}_3 \times S^3 \times T^4 \) in the tensionless limit — the dual CFT can then be described in terms of the symmetric orbifold \( \text{Sym}_{N+1}(T^4) \), all of its states can be organised into representations of the \( W_\infty[0] \) algebra.

It is natural to expect that a similar analysis could be carried out for string theory on \( \text{AdS}_3 \times S^3 \times K3 \), since the dual theory also enjoys small \( \mathcal{N} = 4 \) supersymmetry. However, we find a small difficulty in this: the chiral algebra of \( \text{Sym}_{N}(K3) \) does not contain the \( W_\infty[0] \) algebra described above as a subalgebra. This is due to the fact that the supersymmetry algebra of \( K3 \) is not a contracted version of the large \( \mathcal{N} = 4 \) algebra, as can be easily seen for example by noticing that the elliptic genus of \( K3 \) does not vanish. However, the chiral algebra at the symmetric orbifold point does contain the \( W_\infty[0] \) algebra, which is obtained from \( W_\infty[0] \) by removing the 4 free bosons and fermions, and for \( \lambda = 0 \), the resulting coset theory can be described as (see Section 2.1 below)

\[
(T^4)^{N} / U(N) .
\] (2.5)

Note that now the superconformal algebra contains only one \( \mathfrak{su}(2) \) algebra, which is nothing else but the R-symmetry algebra of the small \( \mathcal{N} = 4 \) superconformal algebra.

Analogously, the bulk theory can be based on the higher-spin algebra \( \text{shs}_2[0] \), which is obtained from \( \text{shs}_2[0] \) upon removing the generators

\[
(1 + k) \otimes E_{\alpha\beta} ,
\] (2.6)

which can never appear in commutators, see \([21]\) for our conventions.\(^2\) Since the representation theory of the \( W_\infty[0] \) algebra is largely identical to that of \( W_\infty[0] \) — in

\(^2\)At \( \lambda = 0, \nu = -1 \), and hence the commutator \( [\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}(1 - k) \) is proportional to \( (1 - k) \).
particular, representations of $W_\infty$ can also be labeled by pairs $(\Lambda_+; \Lambda_-)$, and the wedge characters agree — it is immediate that we can deduce a higher spin/CFT correspondence of the form

$$\text{higher spin theory based on shs}_2^s \leftrightarrow (\mathbb{T}^4)^N/U(N) .$$

(2.7)

We want to show in this paper that string theory on $\text{AdS}_3 \times S^3 \times K3$ can be interpreted within this framework.

### 2.1 The continuous orbifold and the symmetric orbifold

It was shown in [12], see also [28], that the large $k$ limit of the cosets (2.1) can be described by the continuous orbifold (2.3). In this limit, removing the 4 free bosons and fermions corresponds to removing the `center-of-mass' $\mathbb{T}^4$ in (2.3), and the remaining fields transform as

$$\begin{align*}
\text{bosons:} & \quad 2 \cdot (N,1) \oplus 2 \cdot (\overline{N},1) \\
\text{fermions:} & \quad (N,2) \oplus (\overline{N},2),
\end{align*}$$

(2.8)

where the labels refer to the representations of the $U(N)$ and $SU(2)$ R-symmetry, respectively. The complex boson field of the higher spin theory corresponds to the representations $(0;f) \otimes (0;f^*)$ and $(0;f^*) \otimes (0;f)$ of the dual CFT. As in the duality considered in [12], the perturbative part of the higher spin theory based on shs$_2^s$ is thus captured by the states of the form $(0; \Lambda) \otimes (0; \Lambda^*)$, where $\Lambda$ runs over the $U(N)$ representations with finitely many boxes and anti-boxes, i.e., by eq. (2.4).

We would like to relate this continuous orbifold to the symmetric orbifold describing string theory on $\text{AdS}_3 \times S^3 \times K3$ in the tensionless limit, namely

$$\text{Sym}_N(K3) \equiv (K3)^N/S_N .$$

(2.9)

Furthermore, it is easiest to establish the relation between the two theories at a particular point of the moduli space of $K3$, namely the orbifold point

$$K3 = \mathbb{T}^4/\mathbb{Z}_2 .$$

(2.10)

For then, the symmetric orbifold in (2.9) can be written as

$$\text{Sym}_N(K3) = (\mathbb{T}^4)^N/(S_N \ltimes \mathbb{Z}_2^N) ,$$

(2.11)

where the group $S_N \ltimes \mathbb{Z}_2^N$ is the semidirect product of the symmetric group $S_N$ and $N$ copies of $\mathbb{Z}_2$, and $S_N$ acts on $\mathbb{Z}_2^N$ by permuting the factors in the obvious way. The untwisted sector of this theory consists of $4N$ free bosons and fermions that transform as

$$\begin{align*}
\text{bosons:} & \quad 4 \cdot (N,1) \\
\text{fermions:} & \quad (N,2) \oplus (\overline{N},2),
\end{align*}$$

where the labels refer to the representations of the $U(N)$ and $SU(2)$ R-symmetry, respectively. The complex boson field of the higher spin theory corresponds to the representations $(0;f) \otimes (0;f^*)$ and $(0;f^*) \otimes (0;f)$ of the dual CFT. As in the duality considered in [12], the perturbative part of the higher spin theory based on shs$_2^s$ is thus captured by the states of the form $(0; \Lambda) \otimes (0; \Lambda^*)$, where $\Lambda$ runs over the $U(N)$ representations with finitely many boxes and anti-boxes, i.e., by eq. (2.4).

We would like to relate this continuous orbifold to the symmetric orbifold describing string theory on $\text{AdS}_3 \times S^3 \times K3$ in the tensionless limit, namely
with respect to $S_N \ltimes \mathbb{Z}_2^N \times SU(2)$, where $SU(2)$ is again the R-symmetry of the small $\mathcal{N} = 4$ algebra. Here $N$ denotes the $N$-dimensional representation of $S_N \ltimes \mathbb{Z}_2^N$, where the permutation group acts in the usual way (by matrices with one 1 in each row and column), while $\mathbb{Z}_2^N$ is described by the diagonal matrices with $\pm 1$ along the diagonal. We should stress that this representation is an irreducible representation of $S_N \ltimes \mathbb{Z}_2^N$. One way to see this is to note that, as a representation of $S_N$, it decomposes as $N \cong 1 + (N - 1)$, where the 1 is generated by the sum of all $N$ basis vectors. However, since this vector is not invariant under the action of $\mathbb{Z}_2^N$, there is no non-trivial invariant subspace, and the representation is irreducible.

For the following it will be important that we have the obvious embedding

$$S_N \ltimes \mathbb{Z}_2^N \hookrightarrow U(N). \quad (2.13)$$

Under this embedding, both the $\mathbf{N}$ and the $\overline{\mathbf{N}}$ representations of $U(N)$ branch into the $N$ of $S_N \ltimes \mathbb{Z}_2^N$. From this, and comparing (2.8) with (2.12), we conclude that the untwisted sector of (2.5) is a subsector of the untwisted sector of (2.11). As in the case of $\mathbb{T}^4$, the partition function of the symmetric orbifold provides then a non-diagonal modular invariant completion for (2.4).

3. Chiral algebra

In this section, we decompose the chiral algebra of the symmetric orbifold of K3 in terms of representations of the small $\mathcal{W}_\infty^s$ algebra described in the previous section.

3.1 The chiral algebra of $\text{Sym}_N(K3)$

We want to show that, for sufficiently large $N$, the vacuum character $Z_{\text{vac}}(q,y)$ of $\text{Sym}_N(K3)$ can be decomposed as

$$Z_{\text{vac}}(q,y) = \sum_{\Lambda} n(\Lambda) \chi_{(0;\Lambda)}(q,y), \quad (3.1)$$

where $\chi_{(0;\Lambda)}$ are the $\mathcal{W}_\infty^s$ characters of the representations $(0;\Lambda)$, which in turn belong to the untwisted sector of the continuous orbifold (2.5). The non-negative integers $n(\Lambda)$ denote the multiplicity of the trivial representation of $S_N \ltimes \mathbb{Z}_2^N$ in the branching of the $U(N)$ representation $\Lambda$ under (2.13); they can be computed using standard character techniques.

The vacuum character of $\text{Sym}_N(K3)$ can be easily deduced from the vacuum character of K3 by using the DMVV formula [29]. In fact, if we write

$$Z_{\text{chiral}}^R(K3) = \text{Tr}_R(-1)^F q^{L_0} y^{\phi_0} = \sum_{\Delta,\ell} c(\Delta, \ell) q^{\Delta} y^{\ell}, \quad (3.2)$$
the corresponding chiral character of the symmetric orbifold is given by

$$
\sum_{k=0}^{\infty} Z_{R}^\text{chiral} (\text{Sym}_N(K3)) = \prod_{\Delta, \ell} \frac{1}{(1 - pq^{\Delta / \ell})^{c(\Delta, \ell)}.}
$$

(3.3)

Here the subscript R refers to the fact that we are working in the Ramond sector. It is then easy to recover the chiral character in the NS sector by spectral flow. For sufficiently large N, the result is (we suppress the overall factor of $q^{-N/4}$)

$$
Z_{\text{vac}} = 1 + q \left( y^2 + \frac{1}{y^2} + 4 \right) + q^{3/2} \left( 8y + \frac{8}{y} \right) + q^2 \left( y^4 + \frac{1}{y^4} + 8y^2 + 8 + \frac{8}{y^2} + 30 \right) + q^{5/2} \left( 8y^3 + \frac{8}{y^3} + 64y + \frac{64}{y} \right) + q^3 \left( y^6 + \frac{1}{y^6} + 8y^4 + \frac{8}{y^4} + 93y^2 + \frac{93}{y^2} + 248 \right) + O(q^{7/2}).
$$

(3.4)

In analogy with [12], we can decompose this character into coset representations as

$$
Z_{\text{vac}} = \chi(0,0,0,0) + \chi(0,2,0,0) + \chi(0,0,0,2) + 2\chi(0,4,0,0) + 2\chi(0,0,0,4) + 2\chi(0,0,0,0) + \chi(0,0,0,2) + \chi(0,0,0,1,1) + \chi(0,1,0,0) + 2\chi(0,0,0,2) + 3\chi(0,0,0,0) + 3\chi(0,0,0,0,0) + 3\chi(0,0,0,1,1) + \chi(0,0,0,1,2) + 2\chi(0,0,0,1,2) + 4\chi(0,0,0,0,0) + 4\chi(0,0,0,0,2) + 4\chi(0,0,0,0,4) + 4\chi(0,0,0,0,2) + 4\chi(0,0,0,0,1,1) + 4\chi(0,0,0,0,0,0,1,1) + 4\chi(0,0,0,0,0,0,0,0,1,1) + \chi(0,0,0,0,0,0,0,0,0) + O(q^{4/2}).
$$

(3.5)

and the multiplicities agree with the group theoretic prediction for $n(\Lambda)$ from above. Here the wedge characters of the representations $(0; \Lambda)$ agree with those given in [12], and the only difference concerns the contribution of the modes outside the wedge — these differ from [12] by the absence of the 4 free bosons and fermions, i.e., the free fermion and free boson contributions in eq. (B.5) of [12].

### 3.2 The generating fields of the chiral algebra

In order to compare the extended chiral algebra we just found to the coset $\mathcal{W}$-algebra, it is useful to identify its independent generators. The small $\mathcal{N} = 4$ $\mathcal{W}_\infty$ algebra is generated by 8 fields for each (half-integer) spin greater than 1, as well as 4 fields at $s = 1$. This information can be conveniently encoded in the generating function $J(q,y)$, defined by

$$
J(q,y) = q \left( y^2 + \frac{1}{y^2} + 2 \right) + q^{3/2} \left( 4y + \frac{4}{y} \right) + q^2 \left( y^2 + \frac{1}{y^2} + 6 \right).
$$

(3.6)
where the power of $y$ keeps track of the U(1) charge of the corresponding generator. The analogous generating function for the extended chiral algebra of the K3 symmetric orbifold, which counts the independent fields at each dimension, can be calculated by removing all the contributions from descendants and products of lower level states from (3.4), and we get

$$J_{K3}(q,y) = q \left( y^2 + \frac{1}{y^2} + 4 \right) + q^{3/2} \left( 8y + \frac{8}{y} \right) + q^2 \left( 3y^2 + \frac{3}{y^2} + 15 \right) + q^{5/2} \left( 16y + \frac{16}{y} \right) + q^3 \left( 15y^2 + \frac{15}{y^2} + 57 \right) + \mathcal{O}(q^{7/2}). \quad (3.7)$$

As in the case of [30], see eq. (2.9), this generating function is related to the character of a single copy of K3 as

$$J_{K3} = (1 - q) \left( Z_{\text{chiral}}^{\text{NS}}(K3) - 1 \right), \quad (3.8)$$

where $Z_{\text{chiral}}^{\text{NS}}(K3)$ is the untwisted chiral partition function of one copy of K3 in the NS sector. Following the same logic as in [30], we have checked that we can organise the generating function (3.7) in terms of coset representations as

$$J_{K3}(q,y) = J(q,y) + (1 - q) \sum_{m,n \geq 0} \chi(0;[m,0,0,n])(q,y), \quad (3.9)$$

where the prime on the summation symbol means that we sum over $m$ and $n$ for which $m + n$ is even, and we exclude the cases $(m,n) = (0,0)$ and $(1,1)$. This describes rather succinctly the additional generating fields that need to be added to the small $\mathcal{N} = 4 \mathcal{W}_\infty$-algebra in order to obtain the extended chiral algebra of (2.9).

4. Chiral primaries

In this section we shall study more general states of the symmetric orbifold that do not belong to the chiral algebra. We shall focus on the chiral primaries, and specifically on those of dimension $(\tilde{h}, \tilde{h}) = (1/2, 1/2)$ since their descendants contain singlet states of the R-symmetry with $(h, \tilde{h}) = (1, 1)$ that describe exactly marginal deformations preserving the superconformal algebra. We will see that states with $h = \tilde{h} = 1/2$ arise from the untwisted as well as the (12)-twisted sectors of the symmetric orbifold, and we will identify their $\mathcal{W}$-algebra representations. We will also study how many of the BPS states (and in particular of the 21 chiral primaries associated to the exactly marginal deformations of the K3 symmetric orbifold) are accounted for in the perturbative Vasiliev theory.

4.1 Chiral primaries in $\mathcal{N} = (4, 4)$ theories

Let us begin by reviewing the BPS spectrum of $\mathcal{N} = (4, 4)$ theories. The representations of the small $\mathcal{N} = (4, 4)$ algebra are characterised by the conformal dimension
\((h, \bar{h})\) as well as the R-symmetry quantum numbers \((\bar{j}, \bar{j})\) under the left and right \(\text{SU}(2)_R\). In our conventions, the unitarity bound is

\[
h \geq \frac{j}{2}, \quad \bar{h} \geq \frac{j}{2}.
\] (4.1)

Representations that saturate any of the two bounds are shorter than generic representations, those that saturate both bounds are called \textit{chiral primaries}. As a consequence, these representations are characterised by the two \(\text{SU}(2)\) quantum numbers as \((j, \bar{j})_S\), where \(S\) stands for ‘short’. A particularly interesting set of chiral states are those of the form \((1, 1)_S\). These short multiplets contain four (descendant) states with \((h, \bar{h}) = (1, 1)\) and \((j, \bar{j}) = (0, 0)\), that describe exactly marginal deformations preserving the small \(\mathcal{N} = 4\) superconformal algebra.

A very useful property of \(\mathcal{N} = (4, 4)\) (as well as \(\mathcal{N} = (2, 2)\)) theories in 2d is that the spectrum of chiral primaries is bounded from above. More precisely we have \([31]\)

\[
h \leq \frac{c}{6}.
\] (4.2)

The spectrum of these theories can then be encoded in the so-called generalised Poincaré polynomial, defined as

\[
P_{t, \bar{t}} = \text{Tr} \, t^{J_0} \bar{t}^{\bar{J}_0},
\] (4.3)

where the trace is taken over the chiral primaries only. For supersymmetric sigma models with target space \(\mathcal{M}\), the Poincaré polynomial is given by \([32]\)

\[
P_{t, \bar{t}} = \sum_{p,q} h^{p,q} t^p \bar{t}^q,
\] (4.4)

where the \(h^{p,q}\) are the Betti numbers of \(\mathcal{M}\). For symmetric orbifolds \(\mathcal{M}^N/S_N\), the Poincaré polynomial can be computed from the formula \([33, 34]\)

\[
\sum_N Q^N P_{t, \bar{t}}(\mathcal{M}^N/S_N) = \prod_{m=1}^{\infty} \prod_{p,q} (1 + (-1)^{p+q+1} Q^{m+p+\frac{d}{2}(m-1)} t^{\frac{d}{2}(m-1)} \bar{t}^{\frac{d}{2}(m-1)} (-1)^{p+q+1} h^{p,q}),
\] (4.5)

where \(d\) is the complex dimension of \(\mathcal{M}\).

We now specialise the discussion above to the case at hand where \(\mathcal{M} = \text{K3}\). The Betti numbers can be read off from the Hodge diamond

\[
\begin{array}{ccc}
1 & & \\
0 & 0 & \\
1 & 20 & 1 \\
0 & 0 & \\
1
\end{array}
\] (4.6)

– 9 –
It is then easy to show that for sufficiently large $N$, we have
\[
P_{t,t}(K3^N/S_N) = 1 + (t^2 + 21 t \bar{t} + \bar{t}^2) + (t^4 + 22 t^3 \bar{t} + 254 t^2 \bar{t}^2 + 22 t \bar{t}^3 + \bar{t}^4) + \cdots . \tag{4.7}
\]
In particular, there are 21 representations of the form $(1,1)$. In the following sections, we will compute the $W_\infty$ representations associated to these 21 chiral states, and identify which of them belong to the perturbative Vasiliev sector. We shall start out, more generally, by determining the coset representations of all states with $\tilde{h} = \frac{1}{2}$.

### 4.2 The untwisted sector

Let us begin by analysing the $\tilde{h} = \frac{1}{2}$ states coming form the untwisted sector of the orbifold $(2.11)$. They arise from pairing left- and right-moving states that are separately not invariant under $S_N \ltimes \mathbb{Z}_2^N$, but which form a singlet of $S_N \ltimes \mathbb{Z}_2^N$ when paired together; for example, the simplest such states are of the form
\[
\sum_{i=1}^N \psi^{i(\alpha)} \tilde{\psi}^{*i(\beta)} , \tag{4.8}
\]
where $\psi^{i(\alpha)}$ and $\tilde{\psi}^{*i(\beta)}$ represent any of the left- or right-moving fermions, respectively.

In order to count the relevant states we look at the terms proportional to $q^{1/2}$ in the untwisted sector of the symmetric orbifold of K3 that do not come from the vacuum sector (which describes states that are separately invariant).\(^3\) From the DMVV formula [29], we get
\[
\mathcal{Z}_{1/2}^U = \sqrt{q} \left( 2y + \frac{2}{y} \right) + 4q + q^{3/2} \left( 2y^3 + \frac{2}{y^3} + 14y + \frac{14}{y} \right) + q^2 \left( 24y^2 + \frac{24}{y^2} + 68 \right) \\
+ q^{5/2} \left( 2y^5 + \frac{2}{y^5} + 24y^3 + \frac{24}{y^3} + 160y + \frac{160}{y} \right) \\
+ q^3 \left( 24y^4 + \frac{24}{y^4} + 264y^2 + \frac{264}{y^2} + 604 \right) + \mathcal{O}(q^{7/2}) , \tag{4.9}
\]
which can be decomposed into coset representations as
\[
\mathcal{Z}_{1/2}^U = \chi_{(0,1,0,0,0,0)} + \chi_{(0,0,0,1,0,0,0)} + 2\chi_{(0,3,0,0,0,0,0)} + 2\chi_{(0,0,0,0,0,0,0)} \\
+ \chi_{(1,0,0,0,0,0,0)} + \chi_{(0,0,0,0,0,1,0,0,0)} + 2\chi_{(0,2,0,0,0,0,0,0,0)} + 2\chi_{(1,0,0,0,0,0,0,0,0)} \\
+ 4\chi_{(0,5,0,0,0,0,0,0,0,0,0)} + 4\chi_{(0,0,0,0,0,0,0,0,0,0,0,0)} + 3\chi_{(3,1,0,0,0,0,0,0,0,0,0,0)} + 3\chi_{(0,0,0,0,0,0,0,0,0,0,0,0,0)} \\
+ 2\chi_{(0,1,2,0,0,0,0,0,0,0,0,0)} + 2\chi_{(0,0,0,0,0,2,0,0,0,0,0,0,0,0,0)} + \chi_{(0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0)} + \chi_{(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} \\
+ 5\chi_{(0,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} + 5\chi_{(0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} + 2\chi_{(0,2,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} + 2\chi_{(0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} \\
+ 2\chi_{(0,0,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} + 2\chi_{(0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} + 6\chi_{(0,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)} + 6\chi_{(0,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)}.
\]

\(^3\)We should stress that both the twisted and untwisted sector of the K3 partition function will contribute to this $q^{1/2}$ order. Here we describe the states that arise in the untwisted sector — the twisted sector contributions will be described in the following subsections.
\[ \chi(0, [0, 0, 0, 0, 0, 0]) + \chi(0, [0, 1, 0, 0, 0, 0]), \chi(0, [0, 0, 0, 0, 0, 0]) + 3 \chi(0, [0, 0, 0, 0, 1, 0]) \] + O(q^{7/2}). \]

(4.10)

As before, the multiplicities are determined by the embedding \( S_N \ltimes \mathbb{Z}_2^N \hookrightarrow U(N) \), but now we consider the multiplicity of the \( N \) representation of \( S_N \ltimes \mathbb{Z}_2^N \) (which is the representation in which the \( \hat{\psi}^{i\beta} \) transform), instead of the trivial representation.

There are 4 BPS states of the form \((1, 1)\) that arise in this sector, and that come from the leading terms in \(|Z_{1/2}^U|^2\); they are

\[ (0, f) \otimes (0, f^*), \quad (0, f^*) \otimes (0, f), \quad (0, f^*) \otimes (0, f^*) \] (4.11)

where \( f = [1, 0, \ldots, 0, 0] \) and \( f^* = [0, 0, \ldots, 0, 1] \) are the fundamental and anti-fundamental representations of \( U(N) \), respectively. Note that only two of them, namely

\[ (0, f) \otimes (0, f^*) \quad \text{and} \quad (0, f^*) \otimes (0, f) \] (4.12)

belong to the perturbative Vasiliev sector, see eq. (2.4).

4.3 The twisted sectors of \((\mathbb{T}^4)^N / (S_N \ltimes \mathbb{Z}_2^N)\)

We now discuss the low-lying states arising from the twisted sectors of (2.11). Since we need to use various elements and subgroups of \( S_N \ltimes \mathbb{Z}_2^N \), it is convenient to fix some notation.

The elements of \( S_N \ltimes \mathbb{Z}_2^N \) can be uniquely characterised by a collection of signs, \( s = (s_1, s_2, \ldots, s_N) \), \( s_i = \pm \), followed by an arbitrary permutation \( \pi \). Consequently, we will use the notation \( \pi_s \) for the elements of this group. To avoid clutter, we will omit from the notation the last consecutive + signs, e.g., \((-, +, +, \ldots, +) = (-)\). The trivial permutation will be denoted by 1.

The twisted sectors of (2.11) are labeled by conjugacy classes of \( S_N \ltimes \mathbb{Z}_2^N \). We are particularly interested in sectors that contain states with \( \bar{h} = 1/2 \). We will see in the following that the relevant conjugacy classes are

\[ [1_{(-)}], \quad [(12)], \quad [(12)(-)]. \quad (4.13) \]

We will examine each of them in turn.

4.3.1 \( \mathbb{Z}_2 \) torus twist

Let us consider first the conjugacy class containing the \( \mathbb{Z}_2 \) element \( 1_{(-)} \), i.e., the generator of the \( \mathbb{Z}_2 \) inversion of the underlying \( \mathbb{T}^4 \). It is easy to see that the centraliser subgroup is

\[ C_{1_{(-)}} = \mathbb{Z}_2 \times (S_{N-1} \ltimes \mathbb{Z}_2^{N-1}) \] (4.14)
and that the representation $N$ decomposes as
\[ N \cong 1 \oplus (N - 1) \,, \] (4.15)
where the 1 is odd under the $\mathbb{Z}_2$. Thus the situation is very similar to what was considered in section 7.2 of [12].

Let us concentrate on the states that transform trivially (separately for left- and right-movers) with respect to the $(S_{N-1} \ltimes \mathbb{Z}_2^{N-1})$ factor of the centraliser (4.14), but are either even or odd under the first $\mathbb{Z}_2$. Using the explicit transformation properties of the bosonic and fermionic modes under the centraliser, we can compute the first few terms of the corresponding characters, and we find
\[ Z^T_+ = q^{1/2} \left( (y + y^{-1}) + 8q^{1/2} + (y^3 + 19y + 19y^{-1} + y^{-3})q \right. \]
\[ + (24y^2 + 112 + 24y^{-2})q^{3/2} + \cdots \right) , \] (4.16)
\[ Z^T_- = q^{1/2} \left( 2 + 4(y + y^{-1})q^{1/2} + (4y^2 + 32 + 4y^{-2})q \right. \]
\[ + (4y^3 + 76y + 76y^{-1} + 4y^{-3})q^{3/2} + \cdots \right) , \] (4.17)
where the sign in $Z^T_+$ refers to the eigenvalue under the first $\mathbb{Z}_2$ in (4.14). Using similar techniques as in [12], these characters can be decomposed as
\[ Z^T_+ = \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}-1,0,\ldots,0]) + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+1,0,\ldots,0]) \]
\[ + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}-2,0,\ldots,0]) + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+2,0,\ldots,0]) + \mathcal{O}(q^2) , \] (4.18)
\[ Z^T_- = \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2},0,\ldots,0]) + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+3,0,\ldots,0]) \]
\[ + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+1,0,\ldots,0]) + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+0,0,\ldots,0]) \]
\[ + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+4,0,\ldots,0]) + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+2,0,\ldots,0]) + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+0,0,\ldots,0]) \]
\[ + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+2,0,\ldots,0]) \] \[ + \chi([\frac{k}{2},0,\ldots,0],[\frac{k}{2}+2,0,\ldots,0]) + \mathcal{O}(q^2) . \] (4.19)

To understand the multiplicities, we can follow the same logic as in [12]. First we note that the ground state is degenerate due to the fermionic zero modes — acting with them changes only the first entry of the Dynkin label $\Lambda_+ = \{k/2 + l_0, \Lambda'\}$. The even/odd separation is then determined by the index $P = (l_0 + \sum_i \Lambda'_i) \mod 2$, while the multiplicity with which each representation appears in the decomposition equals the number of $(S_{N-1} \ltimes \mathbb{Z}_2^{N-1})$ singlets contained in the corresponding $U(N-1)$ representation. This then reduces to the same computation as in the untwisted sector.

It is clear from the structure of $Z^T_+$ that the ground states of $|Z^T_+|^2$ describe a BPS state of the form $(1, 1)_S$. Since there are 16 $\mathbb{Z}_2$-twisted sectors in $K3 = T^4/\mathbb{Z}_2$, there are a total of 16 such states that sit in the $W^\Sigma_\infty$ representations
\[ 16 \cdot ([k/2,0,\ldots,0],[k/2-1,0,\ldots,0]) \otimes ([k/2,0,\ldots,0],[k/2-1,0,\ldots,0]) \right) . \] (4.20)

None of these states belongs to the perturbative Vasiliev sector.
4.3.2 $S_2$-twisted sector of $K3^N/S_N$

Next we consider the conjugacy class $[(12)]$ corresponding to the permutation $(12)$ of the $N$ copies. The centraliser subgroup is in this case

$$C_{(12)} = (S_2 \times \mathbb{Z}_2^D) \times (S_{N-2} \rtimes \mathbb{Z}_2^{N-2}) ,$$  \hspace{1cm} (4.21)

where $\mathbb{Z}_2^D$ is the diagonal subgroup of $\mathbb{Z}_2^2$, and the product $S_2 \times \mathbb{Z}_2^D$ is not semidirect. The standard representation now breaks into

$$N \cong 1^{(+,-)} \oplus 1^{(-,-)} \oplus (N - 2) ,$$  \hspace{1cm} (4.22)

where $s_1$ and $s_2$ in $1^{(s_1,s_2)}$ indicate the charges with respect to $S_2$ and $\mathbb{Z}_2^D$, respectively.

As before we shall only consider the states that are separately singlets with respect to the $(S_{N-2} \rtimes \mathbb{Z}_2^{N-2})$ factor of the centraliser. Then there are four such sectors, depending on the eigenvalues $(s_1, s_2)$ under the $(S_2 \times \mathbb{Z}_2^D)$ group. Since the fermionic zero-modes have charge $(-,-)$ with respect to both $\mathbb{Z}_2$’s, there will be two $(+,-)$ ground states of charge $y$ and $y^{-1}$ respectively, and two $(-,-)$ ground states. By acting with the oscillators on the various groundstates, we get

$$Z_{(+,+)}^{T} = q^{1/2} \left( (y + y^{-1}) + 8q^{1/2} + (2y^3 + 24y + 24y^{-1} + 2y^{-3})q ight)$$

$$+ (40y^2 + 160 + 40y^{-2})q^{3/2} + \cdots ,$$  \hspace{1cm} (4.23)

$$Z_{(-,-)}^{T} = q^{1/2} \left( 2 + 4(y + y^{-1})q^{1/2} + (6y^2 + 40 + 6y^{-2})q ight)$$

$$+ (8y^3 + 112y + 112y^{-1} + 8y^{-3})q^{3/2} + \cdots ,$$  \hspace{1cm} (4.24)

$$Z_{(+,-)}^{T} = (2y^2 + 4 + 2y^{-2})q + \cdots ,$$  \hspace{1cm} (4.25)

$$Z_{(-,+)}^{T} = (4y + 4y^{-1})q + \cdots .$$  \hspace{1cm} (4.26)

In terms of the coset representations, the first two characters then decompose as

$$Z_{(+,+)}^{T} = \chi((\frac{1}{2},0,...,0,\frac{1}{2},0,...,0)) + \chi((\frac{3}{2},0,...,0,\frac{1}{2},0,...,0))$$

$$+ 2\chi((\frac{1}{2},0,...,0,\frac{3}{2},1,0,...,0)) + 2\chi((\frac{3}{2},0,...,0,\frac{1}{2},1,0,...,0))$$

$$+ \chi((\frac{1}{2},0,...,0,\frac{1}{2},1,0,...,0)) + \chi((\frac{3}{2},0,...,0,\frac{3}{2},1,0,...,0)) + O(q^2) ,$$  \hspace{1cm} (4.27)

$$Z_{(-,-)}^{T} = \chi((\frac{1}{2},0,...,0,\frac{1}{2},0,...,0)) + \chi((\frac{1}{2},0,...,0,\frac{1}{2},1,0,...,0))$$

$$+ \chi((\frac{1}{2},0,...,0,\frac{1}{2},1,0,...,0)) + 2\chi((\frac{3}{2},0,...,0,\frac{3}{2},1,0,...,0)) + 2\chi((\frac{1}{2},0,...,0,\frac{3}{2},1,0,...,0))$$

$$+ 2\chi((\frac{1}{2},0,...,0,\frac{3}{2},1,0,...,0)) + 2\chi((\frac{1}{2},0,...,0,\frac{1}{2},1,0,...,0)) + 2\chi((\frac{3}{2},0,...,0,\frac{3}{2},1,0,...,0))$$

$$+ \chi((\frac{1}{2},0,...,0,\frac{1}{2},1,0,...,0)) + O(q^{5/2}) .$$  \hspace{1cm} (4.28)

The multiplicities can be determined exactly as in the previous subsection, however now the relevant embedding is $S_{N-2} \rtimes \mathbb{Z}_2^{N-2} \hookrightarrow U(N - 1)$. It is then convenient to
consider the chain of embeddings $S_{N-2} \ltimes \mathbb{Z}_2^{N-2} \hookrightarrow U(N - 2) \subset SU(N - 1)$, where the $N - 1$ of $U(N - 1)$ decomposes into $(N - 2) \oplus 1$ of $U(N - 2)$. As a consequence, besides the $S_{N-2} \ltimes \mathbb{Z}_2^{N-2}$ invariant excitations from the $N - 2$ part, there will be additional contributions from the $U(N - 2)$ singlet. However, the $\mathbb{Z}_2^D$ projection implies that only the states built from an even number of such singlets will survive.

Now, only the sector $|Z_{(+,+)}|^2$ gives rise to a chiral primary state of the form $(1, 1)_S$, whose $\mathcal{W}_S^\infty$ representation is

$$([k/2, 0, \ldots, 0], [k/2 - 1, 0, \ldots, 0]) \otimes ([k/2, 0, \ldots, 0], [k/2 - 1, 0, \ldots, 0]) .$$

This is of the same form as the states in (4.20). In particular, this BPS state does not appear in the perturbative part of the Vasiliev theory.

### 4.3.3 $S_2/\mathbb{Z}_2$ twisted sector

Finally, we examine the case where the twist is in the conjugacy class $[(12)_{(-)}]$. The centraliser is then

$$C_{(12)_{(-)}} = \mathbb{Z}_4 \times (S_{N-2} \ltimes \mathbb{Z}_2^{N-2}) ,$$

where the group $\mathbb{Z}_4$ is generated by the element $(12)_{(-)}$. The standard representation breaks now into

$$N \cong 1^{(i)} \oplus 1^{(-i)} \oplus (N - 2) ,$$

where $1^{(i)}$ and $1^{(-i)}$ have eigenvalue $i$ and $-i$ under the element $(12)_{(-)}$ respectively. The corresponding fields will then have moding $\frac{1}{4} + n$ and $-\frac{1}{4} + n = \frac{3}{4} + n'$ in this twisted sector, respectively. In particular, the ground state is non-degenerate, and its ground-state energy equals

$$h = |\frac{1}{4}| + | - \frac{1}{4}| = \frac{1}{2} ,$$

thus potentially allowing for a BPS state $(1, 1)_S$. We will only consider states that are separately (for left- and right-movers) a singlet under the $(S_{N-2} \ltimes \mathbb{Z}_2^{N-2})$ factor of the centraliser; there are then four sectors $Z_P$, $P = 0, 1, 2, 3$, where $Z_P$ has eigenvalue $i^P$ under the $\mathbb{Z}_4$ generator of $(12)_{(-)}$. Expanding out the relevant characters we get

$$Z_0^T = q^{1/2} \left( 1 + (8y + 8y^{-1})q^{1/2} + (15y^2 + 104 + 15y^{-2})q \right.$$  
$$\left. + (16y^3 + 392y + 392y^{-1} + 16y^{-3})q^{3/2} + \cdots \right) ,$$

$$Z_1^T = (2y + 2y^{-1})q^{3/4} + (4y^2 + 40 + 4y^{-2})q^{5/4}$$  
$$\left. + (4y^3 + 16y + 16y^{-1} + 4y^{-3})q^{7/4} + \cdots \right) ,$$

$$Z_2^T = (y^2 + 14 + y^{-2})q + (64y + 64y^{-1})q^{3/2}$$

(4.33)  
(4.34)  
(4.35)
\[ (y^4 + 118y^2 + 562 + 118y^{-2} + y^{-4})q^2 + \cdots , \]
\[ \mathcal{Z}_3^T = 4q^{3/4} + (24y + 24y^{-1})q^{5/4} + (44y^2 + 248 + 44y^{-2})q^{7/4} + \cdots . \]  
(4.36)

We can also write these characters in terms of coset representations, and the explicit formulae are given in the appendix. Note that none of these sectors contains a \((1, 1)_{S}\) BPS state — indeed, only \(Z_0\) has \(h = \frac{1}{2}\), but the corresponding ground state is a singlet, and does not transform in the \(j = \frac{1}{2}\) of the R-symmetry SU(2).

### 4.4 A consistency check

At the beginning of the section we argued that the states with \(\bar{h} = 1/2\) are accounted for by the untwisted sector, as well as the twisted sectors corresponding to the conjugacy classes (4.13). We can now check this by comparing the full partition function, as derived by the DMVV formula [29], with the ansatz

\[ Z_{\text{ansatz}} = |Z_{\text{vac}}|^2 + |Z_{1/2}^U|^2 + 16 \left( |Z_{1/2}^T|^2 + |Z_{(+,-)}^T|^2 + |Z_{(+-,+)}^T|^2 + |Z_{(-+,-)}^T|^2 + |Z_0^T|^2 \right). \]  
(4.37)

We have checked that this ansatz reproduces correctly all terms of order \(q^{1/2}\) with arbitrary powers of \(q\) (up to the order to which we have worked out these characters, i.e., up to order \(q^{3/2}\)). This confirms that we have accounted correctly for all terms with \(\bar{h} = 1/2\). In particular, we have also found all 21 \((1, 1)_S\) BPS states: 4 come from the untwisted sector, see eq. (4.14), while 16 come from the \(Z_2\) twisted sector of the torus orbifold, see eq. (4.20), and the last one arises in the \((12)\)-twisted sector, see eq. (4.29). Of these 21 BPS states, only two are present in the perturbative Vasiliev theory, namely two of the 4 states in eq. (4.14). Note that all of the 17 BPS states arising from the twisted sector sit in the same coset representation, see eqs. (4.20) and (4.29).

### 4.5 More general BPS states

In the previous subsection we identified which of the \((1, 1)_S\) BPS states of the symmetric orbifold are accounted for by the perturbative Vasiliev theory. In this section we want to analyse this question for more general BPS states. It is not difficult to show that the BPS states that appear in the perturbative Vasiliev spectrum (2.4) are of the form

\[ (0, [0^{n_1-1}, 1, 0^{N-n_1-n_2+1}, 1, 0^{n_2-1}]) \otimes (0, [0^{n_2-1}, 1, 0^{N-n_1-n_2-1}, 1, 0^{n_1-1}]) , \]  
(4.38)

where \(n_1, n_2 = 0, 1, 2, \ldots\), and not both \(n_1\) and \(n_2\) are zero. (Note that \(n_1 = 0, n_2 = 1\), for example, corresponds to the representation \((0,f^*) \otimes (0,f)^{f^*}\).) They describe BPS states associated to

\[ (n_1 + n_2, n_1 + n_2)_S . \]  
(4.39)
However, except for the cases \((n_1, n_2) = (1, 0)\) and \((n_1, n_2) = (0, 1)\), whose top components correspond to
\[
\sum_{i=1}^{N} \psi^{(1)i} \bar{\psi}^{*(2)i}, \quad \sum_{i=1}^{N} \psi^{*(2)i} \bar{\psi}^{(1)i},
\]
respectively, they are all multi-particle, i.e., they involve more than one sum over \(i\), as dictated by the fermionic statistics. Thus the only single-particle BPS states of the Vasiliev theory are two states with \((1, 1)_S\), corresponding to \(n_1 + n_2 = 1\). Thus the perturbative Vasiliev theory only captures a tiny part of the BPS spectrum of the full string background.

5. Conclusions

In this paper we have studied how a slightly modified version of the \(\mathcal{N} = 4\) higher spin – CFT duality of [21] can be naturally realised as a subsector of the symmetric orbifold of K3 = \(T^4/Z_2\), which in turn is believed to be dual to string theory on \(\text{AdS}_3 \times S^3 \times K3\) in the tensionless limit. Most of the analysis was quite parallel to what was done in [12], but there were also important differences: the relevant \(W_\infty\) algebra is not directly a Wolf space coset, but is obtained in the limit \(k \to \infty\) upon restricting to a consistent subalgebra, see eq. (2.7). As a consequence, the structure of the branching rules (that determine the multiplicities of the corresponding \(W_\infty^s\) representations) was somewhat different to what appeared in [12]. Similarly, the structure of the twisted sectors of the symmetric orbifold of K3 = \(T^4/Z_2\) is quite rich, and we have identified all of the low-lying twisted sector states in terms of \(W_\infty^s\) representations, see sections 4.2 and 4.3. In particular, this allowed us to analyse which of the chiral primaries of the symmetric orbifold are actually contained in the perturbative higher spin theory, and we found that this is only true for a tiny number of single-particle states.

It would be interesting to understand the structure of the stringy symmetry algebra for this case; since all symmetry generators come from the untwisted sector, the relevant algebra should be a subalgebra of the stringy algebra for the \(T^4\) case, whose structure was studied in [34]. It would also be interesting to study the behaviour of the symmetry currents under the perturbation that corresponds to switching on the tension; again, because of the same reason, this should be very similar to the corresponding analysis for the \(T^4\) case, see [35].

\[\text{For the } \mathcal{N} = 3 \text{ case that was proposed in [36], a similar analysis was recently performed in [37].}\]
the K3 moduli space. Finally, it would be very interesting more generally to study implications of the gigantic stringy symmetry for various aspects of string theory.

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A. The $S_2/Z_2$ twisted sector characters

In this appendix we write the characters from the $S_2/Z_2$ twisted sector, see eqs. (4.33) – (4.36), in terms of coset representations. We find

$$Z_0^T = \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4}) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 4) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 4)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 3) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 2)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 1) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4}) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 1)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 2) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 3) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 4) + O(q^2), \quad (A.1)$$

$$Z_1^T = \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 4) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 3)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 2) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 1)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 1) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 2)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 3) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 4) + O(q^{9/4}), \quad (A.2)$$

$$Z_2^T = \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 4) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 3)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 2) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} - 1)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 1) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 2)$$

$$+ \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 3) + \chi(\frac{1}{4},0,\ldots,0,\frac{1}{4},\frac{1}{4},0,\ldots,0,\frac{1}{4} + 4) + O(q^2), \quad (A.3)$$
\[ Z_3^T = \chi([\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4},0,...,0,\frac{3}{4}-1]) + \chi([\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4}-2,0,...,0,\frac{3}{4}+1]) \\
+ \chi([\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4}-1,0,...,0,\frac{3}{4}]) + \chi([\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4}+1,0,...,0,\frac{3}{4}-2]) \\
+ \chi([\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4}+1,0,...,0,\frac{3}{4}+2]) + \chi([\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4}+2,0,...,0,\frac{3}{4}+1]) + \mathcal{O}(q^{7/4}) . \tag{A.4} \]

Note that the ground state, \((\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4},0,...,0,\frac{3}{4}])\), can be determined by the procedure outlined in [12, 28], with the relevant twist being \(\alpha = (\frac{1}{4},0,...,0,-\frac{1}{4})\). The coset representations \((\frac{3}{4},0,...,0,\frac{3}{4}],[\frac{3}{4}+l_1,\Lambda',\frac{3}{4}+l_{N-1}]\) that contribute in each sector are constrained by the selection rule

\[ P = l_1 + l_{N-1} + \sum_i \Lambda'_i \pmod{4} , \tag{A.5} \]

while the multiplicities of \(\Lambda'\) are determined in the usual way via the embedding \(S_{N-2} \ltimes \mathbb{Z}_2^{N-2} \hookrightarrow U(N-2)\).

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