Classical and quantum dynamics of confined test particles in brane gravity

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Abstract

A model is constructed for the confinement of test particles moving on a brane. Within the classical framework of this theory, confining a test particle to the brane eliminates the effects of extra dimensions, rendering them undetectable. However, in the quantized version of the theory, the effects of the gauge fields and extrinsic curvature are pronounced and this might provide a hint for detecting them. As a consequence of confinement, the mass of the test particle is shown to be quantized. The condition of stability against small perturbations along extra dimensions is also studied and its relation to dark matter is discussed.

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1. Introduction

The first exploration of the idea of extra dimensions was made by Kaluza and Klien (KK). In this theory the gravitational and electromagnetic interactions have a common origin. This unification and its generalization to Yang–Mills interactions can only be done in the presence of extra dimensions. Another reason to study extra coordinates comes from string theory which is a candidate theory for quantum gravity, being formulated consistently in spaces with extra dimensions. The third reason to study extra dimensions arises from the cosmological constant problem (CCP). This is one of the most severe problems facing gravitational and particle physics. The CCP can be put in two categories. The first is the question of why the observed value of the vacuum energy density is so small that the ratio of its experimental value to the theoretical value is of the order of $-120$. The other is to understand why the observed vacuum energy density is not only small, but also, as current Type Ia supernova observations seem to indicate, of the same order of magnitude as the present mass density of the universe [1].
If the extra dimensions exist, then the natural question that one might ask is: why can we not travel through them and by which mechanism are they detectable? In KK theory the extra dimensions have compact topology with a certain compactification scale \( l \) so that at the scales much larger than \( l \), the extra dimensions should not be observable. They only become visible when one probes at very short distances of order \( l \). Another way to display the invisible nature of extra dimensions in low-energy scales is to assume that the standard matter is confined to a four-dimensional submanifold (brane) embedded in a higher dimensional manifold (bulk), while the extra dimensions are probed by gravitons. An example comes from Horava-Witten’s M-theory [2], where the standard model of interactions contained in the \( E_8 \times E_8 \) heterotic string theory is also confined to a 3-brane, but gravitons propagate in the 11-dimensional bulk.

In this paper, we have studied the dynamics of test particles confined to a brane at classical and quantum levels. In doing so, we have assumed an \( m \)-dimensional bulk space through which a 4D brane can move. The existence of more than one extra dimension and certain Killing vector fields suggest that the twisting vector fields, as in classical Kaluza–Klein theory, play the role of the gauge fields. We then move on to study the classical dynamics of a confined test particle. This in turn requires an algebraic constraint to be imposed on the extrinsic curvature. An interesting question would be to investigate the effects of small perturbations along the extra dimensions. It turns out that within the classical limits, the particle remains stable under small perturbations and the effects of extra dimensions are not felt by the test particle, hence making them undetectable in this way. At the quantum level, however, since the uncertainty principle prevents the wavefunction of a test particle from being exactly localized on the brane, the gauge fields and extrinsic curvature effects become pronounced in the Klein–Gordon equation induced on the brane. Ultimately, the quantum fluctuations of the brane cause the mass of a test particle to become quantized. The cosmological constant problem is also addressed within the context of this approach. We show that the difference between the values of the cosmological constant in particle physics and cosmology stems from our measurements in two different scales, small and large.

2. Geometrical preliminary of the model

Consider the background manifold \( \overline{V}_4 \) isometrically embedded in a pseudo-Riemannian manifold \( V_m \) by the map \( \mathcal{Y} : \overline{V}_4 \to V_m \) such that

\[
\mathcal{G}_{AB} \mathcal{Y}_A^{,\mu} \mathcal{Y}_B^{,\nu} = \overline{g}_{\mu\nu}, \quad \mathcal{G}_{AB} \mathcal{Y}_A^{,\mu} \mathcal{N}_B^{,\mu} = 0, \quad \mathcal{G}_{AB} \mathcal{N}_A^{,\mu} \mathcal{N}_B^{,\mu} = g_{ab} \equiv \epsilon_a; \quad \epsilon_a = \pm 1
\]

(1)

where \( \mathcal{G}_{AB} (\mathcal{g}_{\mu\nu}) \) is the metric of the bulk (brane) space \( V_m (\overline{V}_4) \) in arbitrary coordinates, \( \{ \mathcal{Y}^A \} \) (\( \{ x^\mu \} \)) is the basis of the bulk (brane) and \( \mathcal{N}_a^A \) are \( (m - 4) \) normal unite vectors, orthogonal to the brane. The perturbation of \( \mathcal{V}_4 \) in a sufficiently small neighbourhood of the brane along an arbitrary transverse direction \( \xi \) is given by

\[
\mathcal{Z}^A (x^\mu, \xi^a) = \mathcal{Y}^A + (\xi^a) \mathcal{Y}^A
\]

(2)

where \( \xi \) represents the Lie derivative. By choosing \( \xi^a \) orthogonal to the brane, we ensure gauge independency [3] and have perturbations of the embedding along a single orthogonal extra direction \( \mathcal{N}_a \) giving local coordinates of the perturbed brane as

\[
\mathcal{Z}^A_{,\mu} (x^\mu, \xi^a) = \mathcal{Y}^A_{,\mu} + \xi^a \mathcal{N}_a^A_{,\mu} (x^\nu),
\]

(3)

where \( \xi^a \) (\( a = 0, 1, \ldots, m - 5 \)) is a small parameter along \( \mathcal{N}_a^A \) that parametrizes the extra non-compact dimensions. Also one can see from equation (2) that since the vectors \( \mathcal{N}_a^A \) depend only on the local coordinates \( x^\mu \), they do not propagate along the extra dimensions

\[
\mathcal{N}_a^A (x^\nu) = \mathcal{N}_a^A + \xi^b \{ \mathcal{N}_b, \mathcal{N}_a \}^A = \mathcal{N}_a^A.
\]

(4)
The above assumptions lead to the embedding equations of the perturbed geometry
\[ g_{\mu\nu} = G_{AB} \delta^A_{\mu} \delta^B_{\nu}, \quad g_{\mu a} = G_{AB} \delta^A_{\mu} \epsilon^B_a, \quad G_{AB} N^A_a N^B_b = g_{ab}. \] (5)
If we set \( N^A_a = \delta^A_a \), the metric of the bulk space can be written in the matrix form (Gaussian frame)
\[ G_{AB} = \begin{pmatrix} g_{\mu\nu} + A_{\mu c} A^c_{\nu} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}, \] (6)
where
\[ g_{\mu\nu} = \bar{g}_{\mu\nu} - 2\xi^a \bar{K}_{\mu a} + \xi^a \xi^b g^{ab} \bar{K}_{\mu a \nu}, \] (7)
is the metric of the perturbed brane, so that
\[ K_{\mu a \nu} = -G_{AB} \delta^A_{\mu} \delta^B_{a \nu}, \] (8)
represents the extrinsic curvature of the original brane (second fundamental form). Also, we use the notation \( A_{\mu c} = \xi^d A_{\mu cd} \)
\[ A_{\mu cd} = G_{AB} N^A_{\mu} \delta^B_{c}, \] (9)
represents the twisting vector field (normal fundamental form). Any fixed \( \xi^a \) shows a new brane, enabling us to define an extrinsic curvature similar to the original one by
\[ K_{\mu a \nu} = -G_{AB} \delta^A_{\mu} \delta^B_{a \nu} = \bar{K}_{\mu a \nu} - \xi^b \left( \bar{K}_{\mu a \nu} \bar{K}_{\nu b} + A_{\mu c} \epsilon^c_{b} \right). \] (10)
Note that definitions (7) and (10) require
\[ K_{\mu a \nu} = -\frac{1}{2} \frac{\partial G_{\mu \nu}}{\partial \xi^a}, \] (11)
which is the generalized York’s relation and shows how the extrinsic curvature propagates as a result of the propagation of metric in the direction of extra dimensions. In general, the new submanifold is an embedding in such a way that the geometry and topology of the bulk space do not become fixed [3]. We now show that if the bulk space has certain Killing vector fields, then \( A_{\mu ab} \) transform as the components of a gauge vector field under the group of isometries of the bulk space. Under a local infinitesimal coordinate transformation for extra dimensions we have
\[ \bar{\xi}^a = \xi^a + \eta^a. \] (12)
Assuming the coordinates of the brane are fixed \( x'^{\mu} = x^{\mu} \) and defining
\[ \eta^a = M^a b \xi^b, \] (13)
then in the Gaussian coordinates of the bulk space (6) we have
\[ g'_{\mu a} = g_{\mu a} + g_{\mu b} \eta^b_a + g_{ba} \eta^b_{\mu} + \eta^b g_{\mu a b} + O(\xi^2), \] (14)
hence the transformation of \( A_{\mu ab} \) becomes
\[ A'_{\mu ab} = \frac{\partial g'_{\mu a}}{\partial \xi^b} = \frac{\partial g'_{\mu a}}{\partial \xi^b} - \eta^b \frac{\partial g'_{\mu a}}{\partial x^b}. \] (15)
Now, using \( \eta^a_{\mu} = M^a b (x^{\mu}) \) and \( \eta^a_{\mu} = M^a b_{\mu} \xi^b \) we obtain
\[ A'_{\mu ab} = A_{\mu ab} - 2A_{\mu c} \epsilon^c_{b} M_{b_{\mu}} b_{\mu} \] (16)
This is exactly the gauge transformation of a Yang–Mills gauge potential. In our model, the gauge potential can only be present if the dimension of the bulk space is equal to or greater than six \((m \geq 6)\), because the gauge fields \( A_{\mu ab} \) are antisymmetric under the exchange of
extra coordinate indices \(a\) and \(b\). For example, let the bulk space have an isometry group \(SO(p−1, q−3)\). Then if \(L^{ab}\) is the Lie algebra generators of this group, we have
\[
[L^{ab}, L^{cd}] = C^{abcd}_{pq} L^{pq},
\]
where \(C^{abcd}_{pq}\) are the Lie algebra structure constants defined by
\[
C^{abcd}_{pq} = 2 g^{[b}_{p} g^{a][c} \delta^{d]}_{q}. \tag{17}
\]
On the other hand, if \(F^{\mu\nu} = F^{\mu\nu ab} L^{ab}\) is to be the curvature associated with the vector potential \(A^{\mu} = A^{\mu ab} L^{ab}\), we have
\[
F^{\mu\nu} = A^{\nu, \mu} - A^{\mu, \nu} + \frac{1}{2} [A^{\mu}, A^{\nu}], \tag{18}
\]
or in component form
\[
F^{\mu\nu ab} = A^{\nu, \mu ab} - A^{\mu, \nu ab} + \frac{1}{2} \epsilon^{\alpha \mu \nu \rho \lambda} A^{\alpha \mu \lambda} A^{\nu \rho}. \tag{19}
\]

3. Classical dynamics of test particles and confinement

In this section, we apply the above formalism to derive the 4D geodesic equation for a particle confined to a brane and the conditions for the confinement. The geodesic equation for a test particle travelling in the neighbourhood of the brane world in \(V_m\) are taken to be
\[
\frac{dU^A}{d\lambda} + \Gamma^A_{\beta\gamma} U^\beta U^\gamma = \frac{\mathcal{F}^A}{M^2}, \tag{20}
\]
where \(U^A = \frac{dx^A}{d\lambda}\), \(G_{AB} U^A U^B = \kappa, M\) is the mass of a test particle in the bulk space, \(\lambda\) is an affine parameter and \(\mathcal{F}^A\) is some non-gravitational force that is responsible for the confinement of test particles and defined by the potential \(V\) such that \(\mathcal{F}^a = -\nabla^a V\). We require \(\mathcal{F}^A\) to satisfy three general conditions: firstly, its defining potential has a deep minimum on the non-perturbed brane; secondly, it depends only on extra coordinates; and thirdly, the gauge group representing the subgroup of the isometry group is preserved by the potential. Here, \(\kappa = -1, 0, 1\) to allow for massive, null and tachyonic particles, respectively. One can decompose these equations by using the Gaussian form of the bulk space metric. In this frame, the Christoffel symbols of \(V_m\) can be written as
\[
\Gamma^\mu_{ab} = \Gamma^\mu_{a\beta} + \frac{1}{2} \{A_{a\beta} F^\mu_{\beta\gamma} + A_{b\gamma} F^\mu_{\gamma\beta} \} - K_{a\beta\gamma} A^{\mu\gamma},
\]
\[
\tilde{\Gamma}^\mu_{aa} = -K^\mu_{aa} - \frac{1}{2} F^\mu_{aa},
\]
\[
\tilde{\Gamma}^\mu_{ab} = \Gamma^\mu_{bc}, \tag{21}
\]
\[
\tilde{\Gamma}^\mu_{ba} = -\frac{1}{2} A^{\mu\beta} F^\beta_{a\beta} + A^\mu_{ba},
\]
where \(\Gamma^\mu_{ab}\) is the Christoffel symbol induced on the perturbed brane. Note that for obtaining these relations we have used the inverse of \(G_{AB}\)
\[
G^{AB} = \begin{pmatrix}
\delta^{\mu\nu} & -A^{\mu\alpha}
-\frac{A^{\mu\alpha}}{A^{ab} + A^\alpha_{a} A^{\beta b}}
\end{pmatrix}, \tag{22}
\]
hence the geodesic equations (20) in the Gaussian frame split into the following equations:
\[
\frac{du^\mu}{d\lambda} + (\Gamma^\mu_{ab} + A_{a\beta} F^\mu_{\beta\gamma} - K_{a\beta\gamma} A^{\mu\gamma}) u^\beta u^\gamma - (2 K^\mu_{aa} + F^\mu_{aa}) u^\alpha u^\alpha = \frac{\mathcal{F}^\mu}{M^2}, \tag{23}
\]
\[
\frac{d\bar{u}}{d\lambda} + \left( \nabla_{\alpha} A^\alpha_{\beta} + A^a_{\mu} A^\alpha_{\beta} F_{\beta \mu \alpha} + K^a_{\alpha \beta} + A^a_{\mu \lambda} A^{\mu \alpha \beta} \right) \bar{u}^a \bar{u}^\beta
- \left( 2A^a_{\alpha \beta} + A^a_{\mu \lambda} F_{\mu \beta} \right) \bar{u}^a \bar{u}^\beta = \frac{\mathcal{F}^a}{M^2},
\]
and the normalization condition \( g_{\alpha \beta} \bar{u}^\alpha \bar{u}^\beta = \kappa \) becomes
\[
g_{\alpha \beta} \bar{u}^\alpha \bar{u}^\beta + g_{ab} (\bar{u}^a + A^a_{\mu} \bar{u}^\mu) (\bar{u}^b + A^b_{\nu} \bar{u}^\nu) = \kappa,
\]
where \( u^\mu = \frac{dx^\mu}{d\lambda} \) and \( \bar{u}^a = \frac{d\bar{x}^a}{d\lambda} \). Now, if a test particle is confined to the original non-perturbed brane through the action of the force \( \mathcal{F} \) and if we impose the first of the constraining condition which implies that \( \mathcal{F}^a \) can be expanded as a power series in \( \xi^a \) about the minimum of the corresponding potential, that is
\[
\mathcal{F}^a = -\omega^a \omega_b \xi^b + \mathcal{O}(\xi^2),
\]
then the components of the force vanish on the brane with \( u^a = 0 = \bar{\xi}^a \). This is so because according to our assumptions the potential is not a function of the brane coordinates \( x^\mu \) and therefore \( \mathcal{F}^\mu = 0 \). Also \( \mathcal{F}^a = 0 \) since the potential has a minimum on the brane. Equations (23), (24) and (25) thus reduce to
\[
\frac{d\bar{u}}{ds} + \Gamma^\alpha_{\alpha \beta} \bar{u}^\alpha \bar{u}^\beta = 0,
\]
\[
\frac{d\bar{u}}{ds} + \bar{K}^a_{\alpha \beta} \bar{u}^\alpha \bar{u}^\beta = 0,
\]
\[
\bar{u}^\mu \bar{u}_\mu = \kappa,
\]
where \( \bar{u}^a = \frac{dx^a}{ds}, \bar{u}^\alpha = \frac{dx^\alpha}{ds} \) and \( s \) is an affine parameter on the brane. Here, in the spirit of our third assumption about the general behaviour of the potential we have assumed that \( \mathcal{V} \) preserves the symmetries of the gauge fields. However, for later convenience we assume that our gauge symmetry is that of the group \( SO(p-1,q-3) \) and hence we may write \( \mathcal{V} \) in a symmetric form given by \( \mathcal{V} = \frac{1}{2} \omega^a g_{ab} \xi^b \). Since \( \mathcal{V} \) has a deep minimum, we can neglect \( \xi^2 \) and its higher orders when expanding the force. To cancel these terms, we consider \( \omega \) to be much larger than the inverse of the scale of curvatures \( \rho^{-1} \) on \( V_4 \), or more specifically \( \omega \gg \rho^{-2} \). Following [4], we adsorb the scale of \( \omega \) into a small dimensionless parameter \( \varepsilon \), that is \( \omega = \varepsilon \omega_0 \), so that \( \omega \) becomes of the same order as \( \rho^{-2} \). The smaller the ‘squeezing’ parameter \( \varepsilon \) the deeper is the minimum of \( \mathcal{V} \) and the system is more squeezed on the original brane. Thus, \( \varepsilon \) plays the role of a natural perturbation parameter. For a confined particle, equation (28) requires that acceleration along the extra coordinate must vanish, i.e. \( \frac{d\bar{u}}{ds} = 0 \). This will occur if
\[
\bar{K}^a_{\alpha \beta} \bar{u}^\alpha \bar{u}^\beta = 0.
\]
This is an algebraic constraint equation that must hold for the confinement to happen, in agreement with the results obtained in 5D in [5]. As a special case, equation (30) is satisfied when \( \bar{K}^a_{\alpha \beta} = 0 \). In this case, the brane is a totally geodesic submanifold, that is, any geodesic of \( V_4 \) is also a geodesic of \( V_m \). A totally geodesic submanifold is a multidimensional analogue of a geodesic line. A Riemannian manifold containing a totally geodesic submanifold cannot be arbitrary. Ricci [6] has given a system of differential equations that a Riemannian submanifold has to satisfy in order to admit totally geodesic submanifolds.

The above discussion means that at the classical level a test particle does not feel the effects of extra dimensions at low energy. Later, we show that at the quantum level, the effects of extra dimensions would become manifest as the gauge fields and a potential characterized by the extrinsic curvature.
4. Stabilization

At this point it would be interesting to consider what would happen if the position of the particle was perturbed along a direction normal to the brane. In other words, how stable the particle is confined to the brane. However, before doing so, it would be necessary to make some of the concepts to be used in what follows more transparent and clear. Let us then start by taking a quick look at the five-dimensional brane world scenario according to the formulation presented in [7]. This would help us grasp the salient points of our discussion more easily.

In the SMS formalism, Einstein equations in the bulk space can be written in the form

\[(b)G_{AB} = \kappa_5^2 T_{AB},\]  

(31)

where \((b)G_{AB}\) is the Einstein tensor in the bulk space, \(\kappa(5) = 1/M_*^3\), \(M_*\) being the bulk energy scale and

\[T_{AB} = -\Lambda(5)G_{AB} + \delta(\xi)S_{AB}.\]  

(32)

Here, \(S_{AB}\) is the energy–momentum tensor on the brane with \(S_{AB}N^A = 0\). This tensor consists of two parts, that is

\[S_{\mu\nu} = -\sigma g_{\mu\nu} + T_{\mu\nu},\]  

(33)

where \(\sigma\) is the tension of the brane in 5D and \(T_{\mu\nu}\) is the energy–momentum tensor of ordinary matter on the brane. We note that the existence of the \(\delta\) function in the energy–momentum tensor (32) leads to the usual Israel–Lanczos junction conditions

\[\begin{align*}
[g_{\mu\nu}] &= 0 \quad \text{and} \quad [K_{\mu\nu}] = -\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu}S\right),
\end{align*}\]  

(34)

where \([X] = \lim_{\xi \to 0^+} X - \lim_{\xi \to 0^-} X\). Now, imposing \(Z_2\) symmetry on the bulk space and considering the brane as fixed, this symmetry determines the extrinsic curvature of the brane in terms of the energy–momentum tensor

\[K_{\mu\nu}^+ = \frac{1}{2} \kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu}S\right).\]  

(35)

Now, to move any further we have to write the Einstein field equations. There are two ways in which the brane world Einstein field equations differ from what is customary. Firstly, they do not constitute a closed system in that they contain an unspecified electric part of the Weyl tensor, where it can only be specified in terms of the bulk properties. Secondly, they contain a quadratic term in the energy–momentum tensor of the brane \(S_{\mu\nu}\), which is important for the evolution of baby universe models [8]. However, this is not the end of the story. According to [9], the splitting of the right-hand side of the brane-Einstein equations into a term characterizing the bulk \((\mathcal{E}_{\mu\nu})\) and a term on the brane containing linear and quadratic terms of the energy–momentum tensor is highly non-unique. Since in the present model we do not restrict ourselves to one extra dimension, there will be no suitable Israel–Lanczos junction conditions and so the usual handling of the brane-Einstein equations would break down. The reason for this is that if the number of extra dimensions exceeds 1, the brane cannot be considered as a boundary between two regions.

In the spirit of the above discussions and because of the fact that we are using a confining potential approach rather than matter localization by a delta function in \(T_{AB}\), the extrinsic curvature would be independent of the matter content of the brane, in contrast to using the junction conditions. Thus, the arguments presented in [9] regarding the appearance or vanishing of the quadratic terms of the energy–momentum tensor in the Einstein field
equations on the brane world would be rendered unnecessary. Let us then start by contracting
the Gauss–Codazzi relations \[13\]
\[ R_{\alpha \beta \gamma \delta} = 2 g^{\mu \nu} K_{\alpha \beta \gamma b} = g_{\mu \nu} A_{\gamma b} + R_{\alpha \beta \gamma \delta} Z^B A_{\gamma b} Z^C Z^D , \]
where \( R_{\alpha \beta \gamma \delta} \) and \( R_{\alpha \beta \gamma b} \) are the Riemann tensors for the bulk and the brane, respectively,
one obtains
\( (b)G_{AB} Z_{\alpha} Z_{\beta} = G_{\mu \nu} - Q_{\mu \nu} - g^{ab} R_{AB} N^A_a K^b_b g_{\mu \nu} + g^{ab} R_{AB} N^A_a Z^b_{\mu} Z^C Z^D , \)
where \( G_{\mu \nu} \) is the Einstein tensor of the brane and
\( Q_{\mu \nu} = g^{ab} (K_{\mu a} K_{\nu b} - K_{\mu b} K_{\nu a}) - \frac{1}{2} (K_{a a} K^a_b - K_a K^b_a) g_{\mu \nu} \),
with \( K_a = K_{\mu a} \). Note that directly from the definition of \( Q_{\mu \nu} \), it follows that it is
independently a conserved quantity, that is \( Q_{\mu \nu} ; \mu = 0 \).
Now, in order to substitute for the terms proportional to the bulk space in equation (37),
we use Einstein equation
\( (b)G_{AB} + (b) \Lambda G_{AB} = \alpha^* S_{AB} \),
where \( \alpha^* = 1/M^m e^{-2} \). Also, \( \Lambda \) is the cosmological constant of the bulk space with \( S_{AB} \)
consisting of two parts
\( S_{AB} = T_{AB} + \frac{1}{2} \nu G_{AB} \),
where \( T_{AB} \) is the energy–momentum tensor of the matter confined to the brane through
the action of the confining potential \( \nu \). The contracted Bianchi identities \( (b)G_{AB} ; A = 0 \) imply
\( (T^{AB} + \frac{1}{2} \nu G^{AB}) ; A = 0 \).
If we take \( T^{AB} = T_{AB} \) as the energy–momentum tensor of the test particle, the equations of motion
(20) are obtained. Also, since there is no singular behaviour in \( T^{AB} \), we obtain the following
junction condition:
\[ [K_{\mu \nu}] = 0. \]
The application of \( Z_2 \) symmetry now causes the brane to become simply totally geodesic, that is
\( K^b_b = 0 \). It should be mentioned at this point that we do not apply such a symmetry to our
model since doing so would impose restrictions on the type of bulk space one may wish to
consider.
Now, by decomposing the Riemann tensor into the Weyl and Ricci tensors and Ricci
scalar, respectively, we finally obtain
\[ G_{\mu \nu} = \frac{1}{\alpha} \tau_{\mu \nu} - \frac{m - 4}{(m - 1)(m - 2)} \alpha \tau g_{\mu \nu} \]
\[ = \left[ \frac{(m - 1)(m - 2) + 2(m - 4) + (m - 1)(m - 2)^2}{(m - 1)(m - 2)^2} \right] (b) \Lambda g_{\mu \nu} + Q_{\mu \nu} + \epsilon_{\mu \nu} , \]
where \( \epsilon_{\mu \nu} = g^{ab} C_{ABCD} N^A_a N^B_b N^C c D \) and following \( [3] \) we find \( \alpha^* \sim (\alpha + 1/M^2) V \). Here,
\[ \alpha = 1/M^2 \pi, \quad 1/M^2 = \int (K^\mu \nu K_{\mu \nu} + K^\mu K_\mu) \sqrt{g} d^4x \]
and \( V \) is the volume of the extra space. As was mentioned before, \( Q_{\mu \nu} \) is a conserved quantity which according to \([10]\) may be considered as an energy–momentum tensor of a dark
energy fluid representing the \( x \)-matter, the more common phrase being ‘X-Cold-Dark Matter’
(XCDM). This matter has the most general form of the equation of state which is characterized by the following conditions \[11\]: first it violates the strong energy condition at the present epoch for \(\omega_x < -1/3\) where \(p_x = \omega_x \rho_x\); second it is locally stable, i.e., \(c_s^2 = \delta p_x / \delta \rho_x \geq 0\); and third, the causality is granted, that is \(c_s \leq 1\). Ultimately, we have three different types of ‘matter’, on the right-hand side of equation (43) namely, ordinary confined conserved matter (see equation (41)) represented by \(\tau_{\mu\nu}\), the matter represented by \(Q_{\mu\nu}\) which is independently conserved and will be discussed in what follows and finally, the Weyl matter represented by \(E_{\mu\nu}\). It follows that \(E_{\mu\nu} \sim t\cdot e\).

It is now appropriate to go back to the task at hand and study the stabilization of the confinement of our test particle. Since \(\xi^a\) is ‘small’ in our approximation, equation (24) up to \(O(\xi^2)\) becomes, calculating all the quantities on the non-perturbed brane and disregarding the bar from hereon,

\[
\frac{d^2 \xi^a}{d\lambda^2} + \left( A^a_{\beta;\alpha} + K^a_{\alpha\beta} \right) u^\alpha u^\beta = -\frac{\omega^2}{M^2 \varepsilon^2} \xi^a. \tag{44}
\]

Now, using equation (10) and condition (30) the above equation simplifies to

\[
\frac{d^2 \xi^a}{d\lambda^2} + \left[ \left( A^a_{\beta;\alpha} - K^a_{\alpha\beta} K^\gamma_{\rho;\beta} - A^a_{\alpha\beta} A^\gamma_{\beta\gamma} \right) u^\alpha u^\beta + \frac{\alpha^2}{M^2 \varepsilon^2} \delta^a \right] \xi^b = 0. \tag{45}
\]

For stabilizing the particle on the brane it would be necessary for the term in the square brackets in equation (31) to be positive. This suggests that we should write it in terms of the quantities defined above such that our brane scenario acquires meaningful interpretations. In an arbitrary \(m\)-dimensional bulk space the Ricci equations are given by \[13\]

\[
F_{\gamma\delta ab} = -2g_{\mu\nu} K_{\gamma\mu a} K_{\delta\nu b} - R_{ABCD} N^A_a N^B_b Y^C_{\gamma\delta}, \tag{46}
\]

Now, using the decomposition of the Riemann tensor in terms of the Weyl tensor, one can show that

\[
R_{ABCD} N^A_a N^B_b Y^C_{\gamma}, Y^D_{\delta} = 0, \tag{47}
\]

and hence the Ricci and Codazzi relations lead to

\[
F_{\mu\nu ab} = Q_{\mu\nu ab} - Q_{\mu\nu ab}, \tag{48}
\]

where

\[
Q_{\mu\nu ab} = K_{\mu\gamma a} K_{\nu\delta b} - h_a K_{\mu\nu b} - \frac{1}{2} g_{\mu\nu} \left( K_{a;\delta}^a K_{\alpha;\beta} - h_a h_b \right). \tag{49}
\]

The above considerations lead us to obtain the generalized XCDM corresponding to an extrinsic quantity \(Q_{\mu\nu ab}\) given by

\[
Q_{\mu\nu ab} = -8\pi G \left[ \left( p_x + \rho_x \right) u^\mu u^\nu + p_x g_{\mu\nu} \right] g_{ab} + A_{\mu\nu ab} - A_{\mu\nu ab}, \tag{50}
\]

Multiplication of equations (38) and (50) by \(u^\mu u^\nu\) results in

\[
\left( K_{\mu\gamma a} K_{\nu\delta b} - A_{\mu\gamma a} A_{\nu\delta b} \right) u^\mu u^\nu = -4\pi G \left( 3p_x + \rho_x \right) g_{ab} + 4D_{\mu} A_{\mu ab}, \tag{51}
\]

where \(D_{\mu} A_{\muab} = A_{\muab} - A_{\muab} \). Hence if we assume that, as a generalized Coulomb gauge, \(D_{\mu} A_{\muab} = 0\), then from equations (45) and (51) we obtain

\[
\frac{\omega^2}{M^2 \varepsilon^2} + 4\pi G (3\omega_x + 1) p_x > 0. \tag{52}
\]

Since we know from the accelerated expanding universe that \(\omega_x < -1/3\), the second term in equation (52) is negative. However, since \(\varepsilon \to 0\) the first term is large and we should not worry about the particle being confined to the brane.
5. Quantization

To describe the quantum dynamics of a test particle confined to a semi-Riemannian submanifold embedded in a generic semi-Riemannian manifold that satisfies the Einstein field equations (39), three different approaches may be adapted. One is the confining approach in which, as was discussed before, a confining potential forces a particle to stay on the brane by a non-gravitational force. Both of the other approaches follow Dirac’s general prescription to treat a constrained system, with two different types of constraints, namely the usual and the conservative constraints.

Suppose that $f(Y) = 0$ defines our brane. The usual approach employs this equation as the constraint condition and corresponds to a dynamical system defined by the d’Alembert–Lagrange principle [5]. However, one may use another approach in which the constraint is given by $\frac{df}{ds} = 0$ where $s$ is the affine parameter on the brane [12]. This means that no dynamical motion normal to the brane is allowed. The confining approach is a rather straightforward one whereas there is much to be said about the approach that follows Dirac’s prescription. First, there is the well-known problem of the ordering of operators. Second, as was mentioned above, the equations expressing the constraints are not unique. At the classical level the above three approaches lead to the same classical equations of motion. However, at the quantum level, they generally lead to results which do not agree with each other. In the rest of this section we concentrate on the confining approach since realistically, in any real system, the transverse direction contains a number of atoms so any layer cannot become smaller than $\hbar$ which is of order of the magnitude of atomic dimensions, in units in which time and mass are of order one. As regards to our brane, the above discussion would motivate us to use the confining approach since we assume that our brane has a finite small thickness. It also suggests to link the squeezing parameter, determined by the constraining potential, to the quantum scale given by $\hbar$, that is

$$\varepsilon = a\hbar^b,$$

where $a$ is a dimensional constant so as to making $\varepsilon$ dimensionless.

We now focus attention on the quantum aspects of the problem. The Hamiltonian of the system in the coordinates of the bulk space is

$$H = \frac{1}{2} P_A P^A + V - \frac{1}{2} M^2 = 0.$$  

(54)

Here, we have added to the Hamiltonian the confinement potential $V$, since by taking the covariant derivative of $H$ we expect to obtain equation (20). Before considering the constraint, the dynamics of the quantum particle is described by the Klein–Gordon (KG) equation in the bulk space given by

$$\left(-\frac{1}{2}G^{AB}\nabla_A \nabla_B - \frac{1}{2}M^2 + V\right)\psi = 0,$$

(55)

with the normalization condition for the wavefunction given by

$$\int |\psi|^2 d^m Z = 1.$$  

(56)

By changing the coordinates to $\{x^\mu, \xi^a\}$, the Dalambertian in the KG equation becomes

$$\Delta = -\frac{1}{2G^{\mu\nu}} \frac{\partial}{\partial \xi^a} G^{AB} |G|^{1/2} \frac{\partial}{\partial \xi^B}$$

(57)

and the normalization condition (56) takes the form

$$\int |\psi|^2 |G|^{1/2} dx^4 d\xi^{m-4} = 1.$$  

(58)
Since our goal is to have an effective dynamics on the brane, we rescale the wavefunction in such a way as to make it normalized in $L^2(\mathcal{V}_4)$ instead of $L^2(\mathcal{V}_m)$. This aim is achieved by the following rescalings:

$$\Phi = \frac{|\bar{g}|^{1/4}}{|g|^{1/4}} \Phi_1, \quad \Box = \frac{|\bar{g}|^{1/4}}{|g|^{1/4}} \Delta \frac{|g|^{1/4}}{|\bar{g}|^{1/4}},$$

(59)

where $\bar{g}$ is the determinant of the non-perturbed metric $\bar{g}_{\mu\nu}$. The rescaled wavefunction then satisfies the normalization condition

$$\int d^4x d\xi \Phi^2 |\bar{g}|^{1/2} = 1.$$  

(60)

Now, using the explicit form of the Gaussian metric we obtain

$$\Box = -\frac{1}{2|g|^{1/4}} \partial_a |g|^{1/2} \partial^a \frac{1}{|g|^{1/4}} \partial_a + \partial_\mu g^{\mu\nu} \xi^a |g|^{1/2} \partial_\nu \frac{1}{|g|^{1/4}} \partial_\nu + \partial_\mu g^{\mu\nu} \xi^a A^a_{\nu a} |g|^{1/2} \partial_\nu \frac{1}{|g|^{1/4}} \partial_\nu .$$

(61)

Defining $\mathcal{D}_\mu = \partial_\mu - \frac{i}{2} A_\mu$, where $A_\mu = i A^{ab}_{\mu} L_{ab}$ and $L_{ab}$ are the Lie algebra operators, the Dalambertian (62) can be rewritten as

$$\Box = -\frac{1}{2|g|^{1/4}} \partial_a |g|^{1/2} \partial^a \frac{1}{|g|^{1/4}} \partial_a + \partial_\mu g^{\mu\nu} |g|^{1/4} \partial_\nu \partial_\nu |g|^{1/4} \partial_\nu + \frac{1}{8} g^{\mu\nu} g^{\rho\sigma} (g_{ab} \tilde{K}_{a\mu}^b \tilde{K}_{b\nu}^a - 2 g_{ab} \tilde{K}_{a\mu}^a \tilde{K}_{b\nu}^b).$$

(62)

In the previous section, we saw that the parameter $\epsilon$ can be used as a perturbation parameter. By changing the extra coordinates as $\xi^a \rightarrow \epsilon \xi^a$, the Dalambertian (62) and confinement potential can be expanded in powers of $\epsilon$, leading to

$$\epsilon \Box = \Box^{(0)} + \epsilon \Box^{(1)} + \cdots ,$$

(63)

and

$$\mathcal{V}(\xi) = \frac{1}{2\epsilon^2} \omega^2 g_{ab} \xi^a \xi^b + \mathcal{O}(\epsilon^3),$$

(64)

where

$$\Box^{(0)} = -\frac{1}{2g} \partial_\mu g^{\nu\rho} \partial_\nu \partial_\rho,$$

(65)

and

$$\Box^{(1)} = -\frac{1}{2|g|^{1/4}} \mathcal{D}_\mu \mathcal{D}_\nu |g|^{1/2} \mathcal{D}_\nu + \frac{1}{8} g^{\mu\nu} g^{\rho\sigma} (g_{ab} \tilde{K}_{a\mu}^b \tilde{K}_{b\nu}^a - 2 g_{ab} \tilde{K}_{a\mu}^a \tilde{K}_{b\nu}^b).$$

(66)

Thus, the $\epsilon \rightarrow 0$ limit can be unambiguously achieved by considering $\epsilon \mathcal{H}$, so that

$$\epsilon \mathcal{H} = H_0 + \epsilon H,$$

(67)

where

$$H_0 = \Box^{(0)} + \frac{1}{2\epsilon^2} \omega^2 g_{ab} \xi^a \xi^b,$$

(68)

and

$$H = \Box^{(1)} - \frac{1}{2} M^2 .$$

(69)

In the last step for obtaining an effective KG equation on the brane, we need to ‘freeze’ the extra degrees of freedom and thus assume

$$\Phi(x, \xi) = \Sigma_\rho \phi_\rho(x) \theta_\rho(\xi),$$

(70)
so that the index $\beta$ runs over any degeneracy that exists in the spectrum of the normal degrees of freedom. The KG equation associated with the normal degrees of freedom describes \((m - 4)^2\) uncoupled harmonic oscillators

\[
\frac{1}{\varepsilon} \left( \Box^{(0)} + \frac{1}{2} \omega^2 g_{ab} \xi^a \xi^b \right) \theta_\beta = E_0 \theta_\beta.
\]  

(71)

At this point, it would be appropriate to justify the use of the simple harmonic oscillator eigenvalues to describe the system when the classical potential is only taken to be parabolic near the brane. The curvature radii of the background \(V_4\) are the \(4 \times m\) values \(l_a^{\mu}\) of \(\xi^a\), satisfying the homogeneous equation [13]

\[
(g_{\mu\nu} - \xi^a \tilde{K}_{\mu\nu a}) \, dx^\mu = 0,
\]

(72)

where \(a\) is fixed. The single scale of curvature \(\rho\) is the smallest of these solutions, corresponding to the direction in which the brane deviates more sharply from the tangent plane. Considering all contributions of \(l_a^{\mu}\) in such a way that the smaller solution of (72) prevails, the scale of curvature may be expressed as

\[
\frac{1}{\rho} = \sqrt{\tilde{g}^{\mu\nu} \frac{1}{l_a^{\mu} l_b^{\nu}}}. \tag{73}
\]

Since equation (7) can also be written as

\[
g_{\mu\nu} = \tilde{g}^{ab}(\tilde{g}_{\mu a} - \xi^a \tilde{K}_{\mu a}) (\tilde{g}_{\nu b} - \xi^b \tilde{K}_{\nu b}),
\]

(74)

it follows that the above equation becomes singular at the solutions of equation (72). Therefore, \(g_{\mu\nu}\) and consequently the metric of the bulk, described by equation (6), also becomes singular at the points determined by those solutions. Of course this is not a real singularity but a property of the Gaussian system. However, this singularity breaks the continuity and regularity of the Gauss–Codazzi–Ricci equations which are constructed with this system. Therefore, it also represents a singularity for the wave equation of the graviton written in the Gaussian system of the bulk. Hence, the scale of curvature \(\rho\) sets a local limit for the region in the bulk accessed by gravitons associated with those high-frequency waves. Since we have assumed \(\omega/\varepsilon \gg \rho^{-2}\), the above approximation regarding the confining potential is well justified all the way before the singularity is reached by the extra dimensions. Note that changing the extra coordinates makes the divergence in the harmonic potential in equation (71) disappear and we can study the \(\varepsilon \to 0\) limit unambiguously by considering \(\varepsilon H\). Equation (71) has the largest contribution of the order \(O(1/\varepsilon)\) to the 4D mass of the particle. Now by projecting the KG equation onto the space of degenerate states \(|\theta_\beta\rangle\) the effective KG equation in 4D becomes

\[
\left[ -\frac{1}{\varepsilon} \left( \partial_\mu I - i A_\mu \right) \tilde{g}^{\mu\nu} \left( \partial_\nu I - i A_\nu \right) + Q - m^2 \right] \phi(x) = 0, \tag{75}
\]

where

\[
A_\mu = \frac{i}{2} A_{ab} (i \mathcal{L}_{ab}), \quad Q = \frac{1}{4} \left( 2 g_{ab} \tilde{K}_a \tilde{K}_b - 2 \tilde{K}_{a\alpha a} \tilde{K}_{\alpha b} \right) I,
\]

\[
m^2 = (M^2 + E_0) I, \tag{76}
\]

with \(\langle \mathcal{L}_{ab}\rangle\) being the matrix obtained by bracketing \(\mathcal{L}_{ab}\) between the eigenstates corresponding to \(E_0\) and resulting in components different from zero if the normal wavefunction lies in a degenerate, non-trivial representation of \(SO(p - 1, q - 3)\) and \(I\) is the \(n \times n\) identity matrix \((n\) is the order of degeneracy in equation (71)). In obtaining equation (71) we have used the inverse of \(g_{\mu\nu}\). However, since the inverse of \(g_{\mu\nu}\) cannot be obtained in an exact form, one should resort to an expansion of its terms obtaining

\[
g^{\mu\nu} = \tilde{g}^{\mu\nu} + 2 \sqrt{\varepsilon} \xi^a \tilde{K}_{a\mu} \tilde{K}_{\nu a} + 3 \varepsilon \xi^a \xi^b \tilde{K}_{a\alpha a} \tilde{K}_{b\beta b} + O(\varepsilon^{3/2}). \tag{77}
\]
Equation (75) shows that the extrinsic contributions stem from the momentum-independent potential \( Q \) and the minimally coupled gauge field \( A_\mu \). In the classical equation of motion these quantities do not exist and are therefore purely quantum mechanical effects arising from higher dimensions. On the other hand, the equation of 4D mass, the last equation in (76), shows that the observable mass is a quantized quantity. If the mass of a particle in the bulk space is taken to be zero according to induced matter theory \([14]\), then

\[
m^2 = \frac{\omega \sum_a \epsilon_a}{\varepsilon} \left( n_a + \frac{1}{2} \right).
\]  

(78)

This equation deserves a short discussion. If the extra coordinates are taken to be timelike then the mass will become tachyonic. However, if we have two extra dimensions, one being timelike and the other spacelike, then the zero point energy vanishes and we have massless particles. This means that the fundamental mass can be zero. In the case where they are spacelike it would be impossible to obtain a massless particle. In equation (78) as \( \varepsilon \rightarrow 0 \) the mass becomes very large. On the other hand, if we change the coordinates according to \( x^\mu \rightarrow \sqrt{\varepsilon} x^\mu \), then equation (75) gives our redefined mass

\[
\tilde{m}^2 = \omega \sum_a \epsilon_a \left( n_a + \frac{1}{2} \right).
\]  

(79)

In the first section we considered \( \omega \) to be of order \( \rho^{-2} \). This implies \( \omega \sim \Lambda \) where \( \Lambda \) is the cosmological constant of the brane. Now, inserting the appropriate units into equation (79), we obtain the fundamental mass \( \tilde{m}_0 \)

\[
\tilde{m}_0 \sim \frac{\hbar}{c \Lambda^{1/2}} \sim 10^{-65} \text{ gr.}
\]  

(80)

Since our change of coordinates amounted to \( x^\mu \rightarrow \sqrt{\varepsilon} x^\mu \), we relate this mass to the micro-world. It is the mass of a quantum perturbation in a spacetime with very small curvature measured by the astrophysical value of \( \Lambda \) as opposed to the mass sometimes inferred from the zero point or vacuum fields of particle interactions.

As we noted above, the mass denoted by \( m \) is a consequence of the large-scale gravitational effects and becomes very large according to \( m = \tilde{m}/\sqrt{\varepsilon} \). One may interpret this as having a parameter relating the large scale to that of the small. To have a feeling of \( \varepsilon \), one may use the fundamental constants to construct it. In doing so, we note that according to our guess represented by the relation (53), the following combination of \( \hbar, c, G \) and \( \Lambda \) would serve our purpose and renders a dimensionless quantity, namely

\[
\varepsilon \sim \left( \frac{\hbar G \Lambda}{c^4} \right)^2 \sim 10^{-240}.
\]  

(81)

Having made an estimate for \( \varepsilon \), we can now obtain an order of magnitude for \( m_0 \)

\[
m_0 \sim \frac{c^2}{G \Lambda^{1/2}} \sim 10^{56} \text{ gr.}
\]  

(82)

This is the same as the mass of the observable part of the universe \((10^{80} \text{ baryons of } 10^{-24} \text{ gr for each}) \) \([15]\).

Another important problem that can be addressed in the context of the present discussion is that of the cosmological constant \([16]\). As was noted in section 2, the squeezing parameter \( \varepsilon \) was introduced as a consequence of requiring a test particle to be confined to our brane. This parameter, however, opens up an opportunity in the way of observing our universe from two different angles, namely, either looking at our universe in its entirety, that is, at its large-scale structure, or do the opposite, i.e. observing it from small scales. The parameter \( \varepsilon \) provides us with the tool necessary to achieve this. It suffices to define the change of variable introduced
earlier, that is \( x'^\mu = \sqrt{\varepsilon} x^\mu \). This relation connects two very different scales: small and large. Now the question of the disparity between the values of the cosmological constant in cosmology and particle physics reduces to its measurement from two different scales. If we look at it from the large-scale point of view, we measure its astrophysical value, \( \Lambda \sim 10^{-56} \) cm\(^{-2} \). On the other hand, if one looks at its value from very small scales, it turns out to be related to that of \( \Lambda \) through the relation \( \tilde{\Lambda} = \Lambda / \varepsilon \) which is 240 orders of magnitude larger than its astrophysical value. The unusually large order of magnitude should not alarm the reader for if we had used the Planck mass in equation (82) instead of \( m_0 \), we would have obtained the usual value for the order of magnitude, that is 120. The above discussion leads us to the conclusion that the vast difference between the values of the cosmological constant simply stems from our measurements at two vastly different scales.

6. Conclusions

In this paper, we have presented a new model for the confinement of test particles on the brane. In doing so, we have obtained a confinement condition which imposes an algebraic constraint on the extrinsic curvature. This condition is particularly helpful when calculating the components of the extrinsic curvature. This is so because if the metric of the bulk space is known then the calculation of the components of the extrinsic curvature is a simple matter. However, when the only known quantity is the metric of the brane, then the Codazzi equation cannot give the components of the extrinsic curvature uniquely since there is always a constant of integration. The above-mentioned condition has the advantage of being able to determine the extrinsic curvature without ambiguity.

We also showed that a classical test particle cannot feel the effects of the extra dimensions whereas when the system is quantized, the effects of these dimensions are felt by the test particle as the gauge fields and the extrinsic potential \( Q \). Also, the mass turned out to be a quantized quantity which is related to the cosmological constant.

A major ingredient of this model is a constraining force introduced in order to confine the test particle on the brane. We assumed that the potential generating this force has a deep minimum around the brane allowing an expansion of it in terms of a parameter \( \varepsilon \) which controls the size of the minimum. This parameter plays an important role in the measurement of the value of the cosmological constant. Since the depth of the confining potential is a relative quantity depending on the scale at hand, the value of the cosmological constant depends very much on the scale defining the frame from which observations are made. This seems to be the root of the huge disparity between the values of the cosmological constant as measured in cosmic scales and that which results from particle physics.

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