Restrictions and stability of time-delayed dynamical networks

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Abstract
In this paper we present a criteria for the global stability of general time-delayed dynamical networks. We show that under our criteria a network’s stability is invariant with respect to the removal of time delays and the addition of single type time delays. As modifying a network’s delays can have a destabilizing effect on the system’s dynamics, this introduces a new and stronger form of global stability, which we call intrinsic stability. To carry out this analysis we introduce a family of graph transformations that can be used to maintain or modify the spectral radius of a graph. We then introduce the notion of an implicit delay and show that by removing a network’s implicit delays the result is a lower dimensional system, which we term a network restriction. We demonstrate that such restrictions can be used to obtain improved estimates of a network’s global stability. The effectiveness of our approach is illustrated by applications to various classes of Cohen–Grossberg neural networks.

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1. Introduction
The study of networks in nature, science, and technology is an extremely active area of research. As a natural first step, much of this work has focused on understanding the graph-theoretic structure (topology) of these systems. This has lead to numerous new concepts and techniques designed to analyse the static graph structure of a network [2, 8, 9, 12, 14, 15, 19].

Besides having an underlying graph structure most real networks are also dynamic. That is, each network element has a state that is time-dependent. Additionally, because of finite processing speeds and the need to transmit signals over distances most real networks are also
time-delayed. In such complex systems these time delays are often a source of instability and the cause of poor performance.

To investigate the dynamical properties of networks, a general approach was introduced in [1]. The major idea is that a network’s dynamics can be analysed in terms of three key features; (i) the internal local dynamics of the network elements, (ii) the interactions between the network elements and (iii) the topology or structure of the network’s graph of interactions.

In this paper we extend this formalism to investigate the dynamics of time-delayed networks by considering dynamical networks which have time-delayed interactions. The type of dynamics we consider in this paper is whether a time-delayed network has an evolution that is stable, i.e. whether a network has a globally attracting fixed point. We develop a stability criteria that allows us, under mild assumptions, to analyse the stability of any time-delayed network and as a special case any systems of simultaneous equations involving delays (see corollary 1). This stability criteria deals with symmetric as well as nonsymmetric delays, which is useful in analysing real networks which typically do not have a delay distribution that is symmetric.

We also differentiate between two types of delays; those that are single and those that are of a multiple delay type. A network has single delays if the delays between any two network elements happen on a single time scale. Otherwise, the network is said to have a multiple type delay. We show that if a network is stable, using our criteria, then this stability is invariant with respect to the removal of delays and the addition of single type delays (see theorems 4.2 and 4.4). As the addition or removal of delays can destabilize a system our stability criteria introduces a new and stronger form of global stability, which we call intrinsic stability.

Because the removal and addition of delays has a transformatve effect on the structure of a network we introduce the concept of a bounded radial transformation and isoradial expansion for graphs (see definitions 5.2 and 5.8). Both procedures allow one to transform a network’s graph of interactions while either maintaining or modifying the graph’s spectral radius in a specific way. This theory of radial graph transformations is used to prove our main results regarding the addition and removal of time delays.

In the latter half of the paper we observe that in an undelayed network there is a time-delayed interaction between two elements if after some period of time the one influences the other. This we call an implicit time-delay. By choosing a particular subset of network elements we show that it is possible to associate to a network a time-delayed network in which the implicit delays between these elements are made explicit. By removing these time delays (or equivalently the network’s implicit time delays) the result is a lower dimensional network, which we call a network restriction.

We show that a network restriction can be used to gain improved estimates of the original network’s stability (see theorem 6.5). The reason for this improvement is that network restrictions concentrate the information stored in the network to a smaller set of interactions and elements in the ‘restricted’ network. This consolidation of information leads to improved local estimates and ultimately to an improved estimate of the network’s stability.

At each step in this paper these concepts and results are applied to a class of dynamical networks known as Cohen–Grossberg neural networks [6] whose stability has received a considerable amount of attention [7, 10, 13, 16–18]. By applying our theory to such systems we are not only able to illustrate the use of our theory but able to derive new criteria for the stability of both the delayed and undelayed versions of this class of networks.

The results and topics in this paper are organized as follows. Section 2 introduces the concept of a dynamical network and gives a sufficient condition under which such systems have a globally attracting fixed point. Section 3 then defines a time-delayed dynamical network and extends this stability criteria to networks with time delays. The notion of single and
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multiple time delays are introduced in section 4 in which we investigate the effect of adding and removing time delays from a dynamical network. This allows us to formally define the concept of intrinsic stability.

In section 5 we introduce our theory of radial graph transformations, which includes bounded radial transformations and isoradial expansions. These are then used to prove the theorems given in section 4. In the final section of this paper we introduce the notion of implicit time delays and use this to define the concept of a network restriction. Here, by use of radial transformations, we demonstrate that a network restriction can be used to obtain improved stability estimates of a dynamical network.

Lastly, we note that although our procedure deals with the spectra of graphs (equivalently matrices), our techniques are not limited to the investigation of linear systems and are applicable without any modifications to general nonlinear dynamical systems.

2. Dynamical networks and global stability

Dynamical networks, or networks of interacting dynamical systems, are composed of (i) local dynamical systems (ii) interactions between these local systems and (iii) the network’s graph of interactions. The network’s local systems describe the dynamics of the network elements in isolation. The interactions and graph of interactions describe the dynamics and structure (topology) of the interaction between these elements respectively.

2.1. Dynamical networks

Following the approach in [1, 3] we let \( \varphi_i : X_i \to X_i \) be maps on the complete metric space \((X_i, d)\) for \( i \in \mathcal{I} = \{1, \ldots, n\} \) where

\[
L_i = \sup_{x_i \neq y_i \in X_i} \frac{d(\varphi_i(x_i), \varphi_i(y_i))}{d(x_i, y_i)} < \infty.
\]

(1)

From the individual spaces \((X_i, d)\) we define \((X, d_{\text{max}})\) to be the metric space on \( X = \bigoplus_{i=1}^{n} X_i \) with distance given by

\[
d_{\text{max}}(x, y) = \max_{i \in \mathcal{I}} \{d(x_i, y_i)\}, \quad x, y \in X.
\]

We let \((\varphi, X)\) be the direct product of the systems \((\varphi_i, X_i)\) over \( i \in \mathcal{I} \), which we refer to \((\varphi, X)\) as a collection of local systems.

**Definition 2.1.** A map \( F : X \to X \) is called an interaction if for every \( j \in \mathcal{I} \) there exists a nonempty collection of indices \( \mathcal{I}_j \subseteq \mathcal{I} \) and a continuous function

\[
F_j : \bigoplus_{i \in \mathcal{I}_j} X_i \to X_j,
\]

where \( F_j(x) = F_j(x|_{\mathcal{I}_j}) \) for \( j \in \mathcal{I} \) and \( x \in X \). The superposition of the interaction and local systems

\[
\mathcal{F}(x) = (F \circ \varphi)(x) \quad \text{for } x \in X
\]

generates the dynamical system \((\mathcal{F}, X)\), which is a dynamical network.

Suppose \( F \) satisfies the following Lipschitz condition for finite constants \( \Lambda_{ij} \geq 0 \):

\[
d(F_j(x), F_j(y)) \leq \sum_{i \in \mathcal{I}_j} \Lambda_{ij} d(x_i, y_i)
\]

(2)
for all \( x, y \in X \) and \( i, j \in I \). The Lipschitz constants \( \Lambda_{ij} \) in equation (2) form the nonnegative matrix \( \Lambda \in \mathbb{R}^{n \times n} \) where \( \Lambda_{ij} = 0 \) if \( i \notin I_j \). We call the matrix

\[
\Lambda^T \cdot \text{diag}[L_1, \ldots, L_n] = 
\begin{pmatrix}
\Lambda_{11} L_1 & \cdots & \Lambda_{1n} L_n \\
\vdots & \ddots & \vdots \\
\Lambda_{n1} L_1 & \cdots & \Lambda_{nn} L_n
\end{pmatrix}
\]

a stability matrix of \((F, X)\).

Recall that a dynamical system \( f : M \to M \) on a metric space \( M \) has a globally attracting fixed point \( \tilde{x} \in M \) if for any \( x \in M \), \( \lim_{k \to \infty} f^k(x) = \tilde{x} \). If a system has a globally attracting fixed point we call it globally stable or simply stable.

Specifically, for dynamical networks, we consider the following question. Given a collection of local systems \((\phi, X)\) does a particular interaction \(F : X \to X\) lead to a stable dynamical network? To address this question we let \( \rho(A) \) denote the spectral radius of a matrix \( A \in \mathbb{C}^{n \times n} \), i.e. if \( \sigma(A) \) are the eigenvalues of \( A \) then

\[
\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.
\]

The following sufficient condition for the stability of \((F, X)\) can be found in [3] (see theorem 2.5).

**Theorem 2.2.** Suppose \( A \) is a stability matrix the dynamical network \((F, X)\). If \( \rho(A) < 1 \) then the dynamical network \((F, X)\) has a globally attracting fixed point.

Roughly speaking, theorem 2.2 states that a network has a global attractor if the instability of either its local systems or interaction is dominated by the stability of the other.

**Definition 2.3.** If \( \rho(A) < 1 \) where \( A \) is a stability matrix of \((F, X)\) then we say that this network is intrinsically stable.

For now the term intrinsically stable means that a network is globally stable. In section 4 we show that if a network is intrinsically stable then it remains stable even if single type delays are added to its interaction (see theorem 4.4). As the addition of single type delays can destabilize a network in general then intrinsic stability is a stronger form of global stability.

Suppose the maps \( \psi : X \to X \) and \( F : X \to X \) are continuously differentiable and each \( X_i \subseteq \mathbb{R} \) is a closed but possibly unbounded interval. If the constants

\[
L_i = \max_{x \in X} |\psi'(x_i)| < \infty; \quad \text{and}\quad \Lambda_{ij} = \max_{x \in X} |(DF)_{ij}(x)| < \infty
\]

where \( DF \) is the matrix of first partial derivatives of \( F \) then we write \( F \in C_1^\infty(X) \). For \( F \in C_1^\infty(X) \) the matrix

\[
A_F = \Lambda^T \cdot \text{diag}[L_1, \ldots, L_n]
\]

given by (3) and (4) can be shown to be a stability matrix of \((F, X)\). If \( F \in C_1^\infty(X) \) then we let \( \rho(F) = \rho(A_F) \) denote the spectral radius of \((F, X)\). The matrix \( A_F \) is optimal in the following sense.

For matrices \( A, B \in \mathbb{R}^{n \times n} \) we write \( A \leq B \) if \( A_{ij} \leq B_{ij} \) for each \( 1 \leq i, j \leq n \). If it is known that \( 0 \leq A \leq B \), where \( 0 \) is the zero matrix, then \( \rho(A) \leq \rho(B) \) (see [11], for instance). This can be used to show the following.

**Proposition 1.** Suppose \( F \in C_1^\infty(X) \). If \( B \) is any stability matrix of \((F, X)\) then \( \rho(F) \leq \rho(B) \).
Proof. Suppose \( F \in C^1_\infty(X) \) and that \( B = \tilde{\Lambda}^T \cdot \text{diag}[\tilde{L}_1, \ldots, \tilde{L}_n] \) is a stability matrix of \((F, X)\). For \( e_i \) the \( i \)th standard basis vector of \( \mathbb{R}^n \) and \( x \in X \) let \( h \neq 0 \) such that \( y = x + he_i \in X \). Then by (2),

\[
d(F_j(x), F_j(y)) \leq \tilde{\Lambda}_{ij}d(x_i, x_i + h) = \tilde{\Lambda}_{ij}|h|.
\]

Hence, \( |F(x) - F(y)|/|h| \leq \tilde{\Lambda}_{ij} \). Taking the limit as \( h \to 0 \) we have

\[
|DF_{ji}(x)| \leq \tilde{\Lambda}_{ij} \quad \text{for all} \quad x \in X.
\]

By setting \( \Lambda_{ij} = \max_{x \in X} |DF_{ji}(x)| \) it follows that \( 0 \leq \Lambda \leq \tilde{\Lambda} \).

Similarly, one can show that \( L_i = \max_{x \in X} |\phi_i'(x)| \leq \tilde{L}_i \). Hence, the matrix \( A_F \leq B \) implying \( \rho(F) \leq \rho(B) \).

For \( F \in C^1_\infty(X) \), proposition 1 implies that \( \rho(F) \) is optimal for directly determining via theorem 2.2 whether \((F, X)\) is intrinsically stable.

Remark 1. In what follows we formally consider those dynamical networks \((F, X)\) for which \( F \in C^1_\infty(X) \). However, the results in this paper can be extended to networks satisfying (1) and (2) by slight modifications of our arguments. We consider the class \( C^1_\infty(X) \) to simplify the exposition.

2.2. Cohen–Grossberg neural networks

Let \((\phi_i, \mathbb{R})\) be the local systems

\[
\phi_i(x_i) = (1 - \epsilon)x_i + c_i
\]

where \( c_i, \epsilon \in \mathbb{R} \) and \( 1 \leq i \leq n \). We note that these systems are stable if and only if \(|1 - \epsilon| < 1\).

The question we consider is what kind of interaction on these local systems will lead to a stable dynamical network.

For the local systems (5) we are specifically interested in interactions of the form

\[
C_j(x) = x_j + \sum_{i=1}^{n} W_{ij} \phi_i(x_i - c_i) \quad \text{for all} \quad x \in X.
\]

where \( \epsilon \neq 1 \), \( W \in \mathbb{R}^{n \times n} \) and \( \phi_i : \mathbb{R} \to \mathbb{R} \) is any smooth sigmoidal function with Lipschitz constant \( L \geq 0 \). The reason we consider this particular interaction is that the dynamical network \((C, \mathbb{R}^n)\) with \( C = C \circ \phi \) is then given by

\[
C_j(x) = (1 - \epsilon)x_j + \sum_{i=1}^{n} W_{ij} \phi(x_i) + c_j.
\]

which is a special case of a Cohen–Grossberg neural network in discrete time [6].

For such neural networks the variable \( x_i \) represents the activation of the \( i \)th neuron population. The function \( \phi_i \) describes the \( i \)th neuron population’s response to inputs. The matrix \( W \) gives the interaction strengths between each of the \( i \)th and \( j \)th neuron populations which describes how the neurons are connected within the network. The constants \( c_i \) indicate constant inputs from outside the system.

There has been considerable interest in determining stability conditions for such networks, see [7, 10, 13]. To apply our theory to this problem we denote by \(|W|\) the matrix with entries \(|W|_{ij} = |W_{ij}|\). The following result gives a general stability condition for this class of Cohen–Grossberg neural networks.
Theorem 2.4 (Stability of Cohen–Grossberg neural networks). Let \((C, \mathbb{R}^n)\) be the Cohen–Grossberg network given by (7) where \(\phi_i\) has Lipschitz constant \(L\). If \(|1 - \epsilon| + L\rho(|W|) < 1\) then \((C, \mathbb{R}^n)\) has a globally attracting fixed point.

Proof. The claim is that, for the local systems and interaction given by (5) and (6), the matrix 
\[ A = \tilde{\Lambda}^T \cdot \text{diag}[L_1, \ldots, L_n] \]
with 
\[ \max_{x \in \mathbb{R}^n} |\phi_i'(x_i)| = L_i = |1 - \epsilon|, \quad \text{and} \]
\[ \max_{x \in \mathbb{R}^n} |(DC)_{ji}(x)| \leq \tilde{\Lambda}_{ij} = \begin{cases} 
\frac{1 + W_{ji}L}{1 - \epsilon} & \text{for } i = j, \\
\frac{W_{ji}L}{1 - \epsilon} & \text{for } i \neq j 
\end{cases} \]
is a stability matrix of \((C, \mathbb{R}^n)\). To see this note that the constants 
\[ \Lambda_{ij} = \max_{x \in \mathbb{R}^n} |(DC)_{ji}(x)| \]
satisfy (2). Since \(\Lambda_{ij} \leq \tilde{\Lambda}_{ij}\) then the constants \(\tilde{\Lambda}_{ij}\) must also satisfy (2) verifying the claim. As the matrix \(A\) has the form 
\[ A = \begin{bmatrix}
|1 - \epsilon| + |W_{11}L| & |W_{12}L| & \ldots & |W_{1n}L| \\
|W_{21}L| & |1 - \epsilon| + |W_{22}L| & \ldots & |W_{2n}L| \\
\vdots & \vdots & \ddots & \vdots \\
|W_{n1}L| & |W_{n2}L| & \ldots & |1 - \epsilon| + |W_{nn}L| 
\end{bmatrix} \]
the spectral radius of the matrix \(A\) is then 
\[ \rho(A) = \rho \left(|1 - \epsilon| + L \cdot \rho(|W|)\right) = |1 - \epsilon| + L \cdot \rho(|W|). \]
Hence, theorem 2.2 implies that \((C, \mathbb{R}^n)\) is stable if \(|1 - \epsilon| + L\rho(|W|) < 1\). \(\square\)

To the best of our knowledge theorem 2.4 is a new stability criteria for this class of neural networks. However, the major goal of this paper is to extend such results to the case in which the network’s interactions include time delays. Before doing so we first consider the case in which a dynamical network has trivial local dynamics.

2.3. Dynamical networks with no local dynamics

For a given dynamical network it is always possible to absorb the dynamics of the local systems into the interaction by considering the dynamical network \((\mathcal{F}, X)\) to have the interaction \(F \circ \phi\) and no local dynamics. In this sense the theory we present here can be used to deal with general dynamical systems, which typically do not have local dynamics, i.e. the dynamics of each component in the absence of the others cannot be explicitly defined.

If for a set of local systems \((\phi, X)\) we have a given interaction \((F, X)\) then the network \((\mathcal{F}, X)\) can always be considered as a network with no local dynamics. If only the local systems \((\phi, X)\) are given the question becomes, what kind of interaction induces a specific type of network dynamics? For instance, what type of interaction induces stability.

The difference between these points of view is the first focuses on the network interactions while the second emphasizes local dynamics. Both are relevant, but for the moment we consider the case in which we have a fixed interaction as this will allows us to analyse time-delayed networks in the following sections.

By absorbing the network’s local dynamics into its interaction we may consider \((\mathcal{F}, X)\) to be the dynamical network with interaction \(F \circ \phi\) and no local dynamics, i.e. the local dynamics
is given by the identity map \( I_d : X \rightarrow X \). We let \((\mathcal{F}, X)\) denote the network \((\mathcal{F}, X)\) considered as a network without local dynamics.

**Proposition 2.** Assuming \( \mathcal{F} \in C^1_{\infty}(X) \) then \( \rho(\mathcal{F}) \leq \rho(\mathcal{F}) \) where \( A_\Xi \) is given by

\[
(A_\Xi)_{ij} = \max_{x \in X} |D\mathcal{F}^j_i(x)|. \tag{8}
\]

**Proof.** As \((\mathcal{F}, X)\) has no local dynamics and the continuously differentiable interaction \( \mathcal{F} = F \circ \varphi \) then \( L_i = 1 \) satisfies (3) and \( \max_{x \in X} |D\mathcal{F}^j_i(x)| \) satisfies (4). Since \( \text{diag}[L_1, \ldots, L_n] = I \), the \( n \times n \) identity matrix, then equation (8) holds.

To show \( \rho(\mathcal{F}) \leq \rho(\mathcal{F}) \) note that

\[
(A_\Xi)_{ij} = \max_{x \in X} |D\mathcal{F}^j_i(x)| = \max_{x \in X} \left| \frac{\partial F_j^i(x)}{\partial x_i} \right| \leq \max_{x \in X} \left| \frac{\partial F_j^i(x)}{\partial x_i} \right| \max_{x \in X} |\varphi'(x_i)| = \Lambda_{ij} L_j.
\]

Hence, \( A_\Xi \leq \Lambda^T \cdot \text{diag}[L_1, \ldots, L_n] \) implying \( \rho(A_\Xi) \leq \rho(\Lambda^T \cdot \text{diag}[L_1, \ldots, L_n]) \) and the result follows.

Proposition 2 has the following consequence. As \( \rho(\mathcal{F}) \leq \rho(\mathcal{F}) \) then by considering \((\mathcal{F}, X)\) rather than \((\mathcal{F}, X)\) we obtain an improved estimate of the network’s stability since these systems have equivalent dynamics. The reason for this improved estimate is that in \((\mathcal{F}, X)\) we have forced more information about the network’s dynamics together, i.e. the local dynamics and interaction are combined.

This technique of compressing network information is the basic idea used later in section 6 to get improved stability estimates of \((\mathcal{F}, X)\) by removing its implicit time delays. However, before considering such improvements we first turn our attention to analysing the stability of general time-delayed networks.

### 3. Global stability of time-delayed dynamical networks

Our first task in this section is to extend our mathematical framework to dynamical networks with time delays. This requires that we allow the state \( \mathcal{F}^{k+1}(x) \) of the dynamical network \((\mathcal{F}, X)\) to depend not only on \( \mathcal{F}^k(x) \) but on some subset of the previous \( T \) states \( \mathcal{F}^k(x), \mathcal{F}^{k-1}(x), \ldots, \mathcal{F}^{k-(T-1)}(x) \) of the system.

#### 3.1. Time-delayed dynamical networks

Given the fixed integer \( T \geq 1 \) let \( T = \{0, -1, \ldots, -T + 1\} \). We define the product space

\[
X^T = \bigoplus_{t \in T} X
\]

where a point \( x = (x^0, \ldots, x^{-T+1}) \in X^T \) has coordinates \( (x)^T = x^T \in X \) and \( (x)^t = x^t \in X_i \) for \( (i, t) \in I \times T \). We say \( x^T \) is \( x \) at time \( \tau \) and \( x^t \) is the \( i \)th component of \( x^T \).

Given the local systems \((\varphi, X)\) we define the function \( \bar{\varphi} : X^T \rightarrow X^T \) as follows. For \( x = (x^0, \ldots, x^{-T+1}) \in X^T \) let

\[
\bar{\varphi}(x^0, x^{-1}, \ldots, x^{-T+1}) = (\varphi(x^0), x^{-1}, \ldots, x^{-T+1})
\]

so that only the present \((\tau = 0)\) time step of \( x \) is effected by \( \varphi \).
Definition 3.1. A map $H : X^T \to X$ is called a \textit{time-delayed interaction} if for all $j \in I$ there is a nonempty set $\mathcal{I}_j \subseteq I \times T$ and a function

$$H_j : \bigsqcup_{(i, \tau) \in \mathcal{I}_j} X^T_i \to X_j.$$  

The map $H$ is defined by $H_j(x) = H_j(x|_{\mathcal{I}_j})$ for $j \in I$ and $x \in X^T$. The superposition of the local systems and time-delayed interaction

$$\mathcal{H}(x) = (H \circ \bar{\phi})(x) \quad \text{for} \ x \in X^T$$

generates the \textit{time-delayed dynamical network} $(\mathcal{H}, X^T)$. The \textit{orbit} of the point $(x^0, x^{-1}, \ldots, x^{-T+1}) \in X^T$ under $\mathcal{H}$ is the sequence $\{x^k\}_{k=-T}^{\infty}$ where

$$x^{k+1} = \mathcal{H}(x^k, x^{k-1}, \ldots, x^{k-T+1}).$$

Here, we assume that both $\phi$ and $H$ are continuously differentiable so that $\mathcal{H} \in C^1_{\infty}(X^T)$. As in section 2 this assumption is for convenience. We note that the results in this section can be extended to those time-delayed dynamical networks that are Lipshitz continuous on a complete metric space (see remark 1).

Remark 2. For more generality we could assume that the local systems $(\varphi, X^T)$ are also time delayed. However, if such is the case then we can absorb these local time delays into the delayed interaction. In this way the mathematical framework we present here can be used to investigate the more general case of interacting systems with time delays in both the local systems and interaction.

Again we are concerned with finding sufficient conditions under which a time-delayed dynamical network has a globally attracting fixed point. A \textit{fixed point} of $(\mathcal{H}, X^T)$ is an $\tilde{x} \in X$ such that $\tilde{x} = \mathcal{H}(\tilde{x}, \ldots, \tilde{x})$. The fixed point $\tilde{x} \in X$ is a \textit{global attractor} of $(\mathcal{H}, X^T)$ if for any initial condition $(x^0, x^{-1}, \ldots, x^{-T+1}) \in X^T$ the limit

$$\lim_{k \to \infty} x^k = \tilde{x}.$$  

As before we call the time-delayed dynamical network $(\mathcal{H}, X^T)$ \textit{stable} if it has a global attractor.

As time-delayed networks are similar in many ways to undelayed networks it is tempting to use the results given in section 2 to determine whether $(\mathcal{H}, X^T)$ is stable. However, in order to use this theory we must first modify the function $\mathcal{H}$.

Let $\tilde{\mathcal{H}} : X^T \to X^T$ be the map defined by

$$\tilde{\mathcal{H}}(x) = (\mathcal{H}(x), x^0, x^{-1}, \ldots, x^{-T+2})$$

for $x = (x^0, x^{-1}, \ldots, x^{-T+1}) \in X^T$. Writing $\tilde{\mathcal{H}}(x) = (\tilde{\mathcal{H}} \circ I d)(x)$, where $I d$ is the identity map, allows us to consider $(\tilde{\mathcal{H}}, X^T)$ as the dynamical network with interaction $\tilde{\mathcal{H}}$ and no local dynamics. We call $(\tilde{\mathcal{H}}, X^T)$ the dynamical network associated with $(\mathcal{H}, X^T)$, which has components

$$\tilde{\mathcal{H}}_i^j (x) = \begin{cases} H_i(x) & \tau = 0, \\ x^i_j & -T < \tau < 0 \end{cases} \quad \text{for} \ i \in I. \quad (9)$$

Remark 3. If the system $(\varphi, X)$ is invertible it is possible to define $(\tilde{\mathcal{H}}, X^T)$ as a dynamical network with the nontrivial local systems $(\tilde{\varphi}, X^T)$ by letting

$$\tilde{H}(x) = (H(x), \varphi^{-1}(x^0), x^{-1}, \ldots, x^{-T+2})$$

be the system’s interaction. We define $(\tilde{\mathcal{H}}, X^T)$ to have the interaction $\tilde{\mathcal{H}}$ and no local dynamics for the sake of generality.
Using (8) the stability matrix $A_{\tilde{H}}$ has entries given by
\[
(A_{\tilde{H}})_{ij} = \max_{x \in X^T} \{|D\tilde{H}(x)|\} \quad \text{for } i, j \in I \times T.
\] (10)

Additionally, as $(\tilde{T}^k(x))^0 = H(x^0, x^{k-1}, \ldots, x^{-T+1})$ for any initial condition $x = (x^0, x^{-1}, \ldots, x^{-T+1}) \in X^T$ and $k \geq 0$ then the following holds.

Lemma 3.2. The time-delayed network $(\mathcal{H}, X^T)$ is stable if and only if its associated dynamical network $(\tilde{\mathcal{H}}, X^T)$ is stable.

In light of lemma 3.2 we let $\rho(\mathcal{H}) = \rho(\tilde{\mathcal{H}})$ denote the spectral radius of $(\mathcal{H}, X^T)$ and say that $(\mathcal{H}, X^T)$ is intrinsically stable if $\rho(\mathcal{H}) < 1$. By combining this result with theorem 2.2 it is possible to investigate the dynamic stability of $(\mathcal{H}, X^T)$ via the dynamical network $(\tilde{\mathcal{H}}, X^T)$.

Corollary 1. If $\rho(\mathcal{H}) < 1$ then the time-delayed dynamical network $(\tilde{\mathcal{H}}, X^T)$ is globally stable.

By associating a time-delayed network with a network that formally does not have delays it is possible to use the theory developed in [3] to study the stability of time-delayed dynamical networks. This approach is illustrated in the following example.

Example 1. Consider the time-delayed dynamical network $(\mathcal{H}, X^T)$ given by
\[
\mathcal{H}(x^k, x^{k-1}, x^{k-2}, x^{k-3}) = \begin{bmatrix}
(1 - \epsilon)x_1^{k-1} + 2a \tanh(bx_2^{k-1} - c_1) \\
(1 - \epsilon)x_2^{k-1} + 2a \tanh(bx_1^{k-1} - c_2)
\end{bmatrix}
\] (11)

with local systems $\phi_i(x_i) = (1 - \epsilon)x_i + c_i$ for $i = 1, 2$ and interaction
\[
H(x^k, x^{k-1}, x^{k-2}, x^{k-3}) = \begin{bmatrix}
x_1^{k-1} + 2a \tanh(bx_2^{k-1} - c_1) \\
2x_2^{k-1} + 2a \tanh(bx_1^{k-1} - c_2)
\end{bmatrix}
\] (12)

where $a, b, c_i \in \mathbb{R}$, $X = \mathbb{R}^2$, $T = 4$, and $\epsilon \neq 1$. Note that this system has the form of a Cohen–Grossberg network considered in section 2 but with time delays. The function $\phi_i(x_i) = \tanh(\mathcal{L}x_i)$ is a standard function considered in the theory of Cohen–Grossberg neural networks (e.g. see [18]).

The dynamical network $(\tilde{\mathcal{H}}, X^T)$ associated with this system has components

\[
\tilde{\mathcal{H}}(x) = \begin{bmatrix}
\mathcal{H}_0(x_1^{-1}, x_2^{-3}) \\
\mathcal{H}_1(x_0) \\
\mathcal{H}_2^{-1}(x_1^{-1}) \\
\mathcal{H}_2^{-3}(x_1^{-2}) \\
\mathcal{H}_3^{-1}(x_1^{-1}, x_2^{-1}) \\
\mathcal{H}_3^{-2}(x_2^{-3})
\end{bmatrix}
= \begin{bmatrix}
(1 - \epsilon)x_1^{-1} + 2a \tanh(bx_2^{-3} + c_1) \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

Using equation (10) we compute the stability matrix $A_{\tilde{H}}$ to be

\[
A_{\tilde{H}} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Taking the spectral radius of this network we find that
\[ \rho(\mathcal{H}) = \sqrt{|1 - \epsilon| + \sqrt{|1 - \epsilon|^2 + 8|ab|}}. \] (13)

The spectral radius \( \rho(\mathcal{H}) < 1 \) or \((\mathcal{H}, X^T)\) is intrinsically stable if and only if \(|1 - \epsilon| + 2|ab| < 1\). By corollary 1 the time-delayed network \((\mathcal{H}, X^T)\) is stable if this condition holds.

We now turn our attention to determining how modifications of a network’s time-delays effect the network’s stability.

4. Manipulating time delays

In this section we consider modifications of a networks’s time delays. Our first result is that, if a time-delayed dynamical network is known to be intrinsically stable then removing these time delays will not effect its stability. This result is novel in the sense that a time-delayed network (or general delayed dynamical system) may become unstable if its delays are removed (see example 2). Hence, an intrinsically stable network with time delays has a stronger form of stability than just globally stability as it remains stable even when its delays are removed.

We also show that it is possible to destabilize a dynamical network \((\mathcal{F}, X)\) by adding delays to the system even if \(\rho(\mathcal{F}) < 1\). As such delays have a specific form this will enable us to classify network delays into two categories. The first class, which we call a multiple type delay, can have a destabilizing effect on the network. The second class of delays, called single type delays, do not have this effect. We show that if single type delays are added to the network \((\mathcal{F}, X)\) where \(\rho(\mathcal{F}) < 1\) the resulting network is necessarily stable.

4.1. Undelayed dynamical networks

Here we formalize the notion of modifying a system’s time delays. Our first objective is to describe the removal of delays from a time-delayed dynamical network.

**Definition 4.1.** Let \((\mathcal{H}, X^T)\) be a time-delayed dynamical network. The map \(\mathcal{U} : X \to X\) given by
\[ \mathcal{U}(x) = \mathcal{H}(x, \ldots, x) \quad \text{for } x \in X \]
generates the undelayed dynamical network \((\mathcal{U}, X)\) with local systems \((Id, X)\) and interaction \(\mathcal{U} : X \to X\).

**Remark 4.** If the local systems \((\varphi, X)\) are invertible we can define \((\mathcal{U}, X)\) to be the dynamical network with local systems \((\varphi, X)\) and interaction
\[ U(x) = H(x, \varphi^{-1}(x), \varphi^{-1}(x), \ldots, \varphi^{-1}(x)) \]
for \(x \in X\). For the sake of generality though we define \((\mathcal{U}, X)\) to have the interaction \(\mathcal{U} : X \to X\) and no local dynamics similar to our definition of \((\mathcal{H}, X^T)\).

We refer to \((\mathcal{U}, X)\) as the undelayed version of \((\mathcal{H}, X^T)\). The following result relates the stability of the undelayed network \((\mathcal{U}, X)\) to the stability of \((\mathcal{H}, X^T)\).

**Theorem 4.2.** If \(\rho(\mathcal{H}) < 1\) then the undelayed version of this network \((\mathcal{U}, X)\) is intrinsically stable.
An important consequence of theorem 4.2 is that if a time-delayed network is intrinsically stable then the network’s stability is invariant under the removal of its time delays. We save the proof of theorem 4.2 for section 5 but illustrate its use with the following example.

**Example 2.** Consider the time-delayed dynamical network \((H, X^T)\) given by

\[ H(x^k, x^{k-1}) = (\alpha + \gamma)x^k + \alpha x^{k-1} \]

with the local system \(\varphi(x) = (\alpha + \gamma)x\) and interaction \(H(x^k, x^{k-1}) = x^k + \alpha x^{k-1}\) where \(X^T = \mathbb{R}^2\) and \(\alpha, \gamma \in \mathbb{R}\). This system has the associated dynamical network

\[ \tilde{H}(x) = \begin{bmatrix} \alpha + \gamma & \alpha \\ x^0 & x^{-1} \end{bmatrix}, \]

which is linear and therefore stable if the matrix of first partial derivatives

\[ D\tilde{H}(0) = \begin{bmatrix} \alpha + \gamma & \alpha \\ 0 & 0 \end{bmatrix} \]

has a spectral radius strictly within the unit circle. In contrast, the stability matrix of \((H, X^T)\) is given by

\[ A_{H} = \begin{bmatrix} \alpha + \gamma & |\alpha| \\ 0 & 1 \end{bmatrix} \cdot \]

For \(\Omega_1 = \{ (\alpha, \gamma) \in \mathbb{R}^2 : \rho(D\tilde{H}(0)) < 1 \}\) and \(\Omega_2 = \{ (\alpha, \gamma) \in \mathbb{R}^2 : \rho(A_{H}) < 1 \}\), which are shown in figure 1 (left), we have the strict inclusion \(\Omega_2 \subset \Omega_1\). That is, there are parameter values for which \((H, X^T)\) is stable but \(\rho(H) \geq 1\).

Recall that theorem 2.2 gives a sufficient condition stating that if \(\rho(M_H) < 1\) then \((H, X^T)\) is stable. However, theorem 4.2 additionally guarantees that if this condition holds then the undelayed version of this network is also stable. In this example the undelayed system \((U, X)\) is given by

\[ U(x) = (2\alpha + \gamma)x \quad \text{for } x \in \mathbb{R}, \]

which has a globally attracting fixed point if and only if \(|2\alpha + \gamma| < 1\).

Letting \(\Omega_3 = \{ (\alpha, \gamma) \in \mathbb{R}^2 : |2\alpha + \gamma| < 1 \}\), figure 1 (right) shows the inclusion \(\Omega_2 \subset (\Omega_1 \cap \Omega_3)\). Hence, for \((\alpha, \gamma) \in \Omega_2\) the time-delayed network \((H, X^T)\) and its undelayed version \((U, X)\) are stable as guaranteed by theorem 4.2 and corollary 1.
Example 2 also points out the following. Since $\Omega_1 \not\subseteq \Omega_3$ it is possible to destabilize a network, or more generally a dynamical system, by removing its time delays. Conversely, given that $\Omega_3 \not\subseteq \Omega_1$ it is also possible to destabilize a system by introducing time delays. Therefore, the converse of theorem 4.2 does not hold in general. However, this theorem has a partial converse if we restrict ourselves to specific types of time-delayed interactions. Recall from definition 3.1 that a time-delayed interaction is a function $H : X^T \rightarrow X$ given by

$$H_j : \bigoplus_{(i, \tau) \in I_j} X^\tau_i \rightarrow X_j$$

where the set $I_j \subseteq I \times T$.

**Definition 4.3.** The time-delayed network $(H, X^T)$ is said to have single delays if for all $j \in I$ the set $I_j \subseteq I \times T$ has the property

$$(i, \mu) \in I_j \Rightarrow (i, \nu) \not\in I_j \quad \text{for all} \quad \nu \neq \mu.$$ 

If a $(H, X^T)$ does not have this property we say it has a multiple type delay.

**Theorem 4.4.** Suppose $(H, X^T)$ has single type delays. Then $\rho(H) < 1$ if and only if $\rho(U) < 1$.

As with the other theorems in this section we prove theorem 4.4 in section 5. Combining theorems 2.2 and 4.4 allows for the following corollary.

**Corollary 2.** Suppose $(H, X^T)$ has single type delays. If either $\rho(H) < 1$ or $\rho(U) < 1$ then both $(H, X^T)$ and $(U, X)$ are globally stable.

Phrased another way, corollary 2 states that if a network is intrinsically stable then this stability is invariant under the removal or addition of single type delays.

**Example 3.** Consider the time-delayed dynamical network $(H, X^T)$ given by (11) in example 1. The undelayed version of $(H, X^T)$ is the network $(U, X)$ given by

$$U(x) = \begin{bmatrix}
(1 - \epsilon)x_1 + 2a \tanh(bx_2) + c_1 \\
(1 - \epsilon)x_2 + 2a \tanh(bx_1) + c_2
\end{bmatrix}$$

for $a, b, c_i \in \mathbb{R}$, $X = \mathbb{R}^2$, and $\epsilon \neq 1$. Computing the network’s stability matrix we find

$$A_U = \begin{bmatrix}
1 - \epsilon & 2|ab| \\
2|ab| & |1 - \epsilon|
\end{bmatrix}$$

from which it follows that

$$\rho(U) = |1 - \epsilon| + 2|ab|. \quad (14)$$

As can be seen from (12) the network $(H, X^T)$ has single delays. In particular, corollary 2 implies that $(H, X^T)$ is stable if $|1 - \epsilon| + 2|ab| < 1$.

Observe that the same condition for the stability of $(H, X^T)$ was formulated in example 1. However, this equivalence of conditions does not imply that the number of computations required to compute $\rho(H)$ versus $\rho(U)$ are equal, as can be seen by comparing equations (13) and (14). In fact, from a computational point of view, it is always easier to analyse the stability of an undelayed dynamical network $(U, X)$ than the associated time-delayed network $(H, X^T)$.

If a time-delayed dynamical network $(H, X^T)$ has a multiple type delay then the conclusions of theorem 4.4 do not necessarily hold. For example, the dynamical network $(U, X)$ in example 2 has spectral radius $\rho(U) = 0 < 1$ for $(\alpha, \gamma) = (1, -2)$ although the associated time-delayed network does not have a globally attracting fixed point at these...
parameter values. The reason is that the time-delayed network \((H, X^T)\) in example 2 has a multiple type delay.

Combining theorems 4.2 and 4.4, a network that is intrinsically stable remains stable under the removal of time delays and the introduction of single type delays. This is potentially significant for the design of real networks since these systems are inherently time-delayed. This is, if the goal is to build a stable network one could design it to be not only stable but intrinsically stable in which case the network’s stability would be more robust to changes in its delays.

To apply the results of section 4.1 to the class of Cohen–Grossberg networks considered in this paper suppose \((C, X^T)\) is the time-delayed dynamical network given by

\[
C_j(x^k, \ldots, x^{k-(T-1)}) = (1 - \epsilon)x_j^{k-\tau_{jj}} + \sum_{i=1}^{n} W_{ij} \phi(x_j^{k-\tau_{ij}}) + c_j \tag{15}
\]

with local systems \(\psi_i(x_i) = (1 - \epsilon)x_i + c_i\) and time-delayed interaction

\[
C(x^k, \ldots, x^{k-(T-1)}) = x_j^{k-\tau_{jj}} + \sum_{i=1}^{n} W_{ij} \phi \left( \frac{x_i^{k-\tau_{ij}} - c_i}{1 - \epsilon} \right). \tag{16}
\]

Here, \(1 \leq i, j \leq n\), \(0 \leq \tau_{ij} \leq T - 1\), and \(\epsilon \neq 1\). As before \(W \in \mathbb{R}^{n \times n}\), \(X = \mathbb{R}^n\), and \(\phi : \mathbb{R} \to \mathbb{R}\) is a smooth function with the Lipschitz constant \(L\).

Equations (15) and (16) describe a class of time-delayed Cohen–Grossberg networks. We note that the stability analysis of time-delayed Cohen–Grossberg networks has attracted much interest \([16–18]\). Since \((C, \mathbb{R}^n T)\) has single time delays then the following result is a consequence of the proof of theorems 2.4 and 4.4.

**Theorem 4.5 (Stability of time-delayed Cohen–Grossberg networks).** Let \((C, \mathbb{R}^n T)\) be the time-delayed Cohen–Grossberg network given by (15) where \(\phi_i\) has the Lipschitz constant \(L\). If \(|1 - \epsilon| + L\rho(|W|) < 1\) then \((C, \mathbb{R}^n T)\) has a globally attracting fixed point.

5. **Isoradial graph and matrix transformations**

To better understand how delays effect the stability of a dynamical network we first consider how modifications in a network’s delays effect the network’s graph structure. This leads us to introduce the notion of an isoradial graph (matrix) transformation, which is a transformations that preserves the spectral radius of a graph (matrix). We then use this concept to prove theorems 4.2 and 4.4.

5.1. **Graph structure of a dynamical network**

To each dynamical network \((F, X)\) there is an associated weighted directed graph called the network’s graph of interactions. A weighted directed graph \(G = (V, E, \omega)\) is a graph with vertex set \(V\) and edges \(E\) where the function \(\omega\) gives the weight of each edge. For \(V = \{v_1, \ldots, v_n\}\) we denote the directed edge from \(v_i\) to \(v_j\) by \(e_{ij}\) having weight \(\omega(e_{ij})\).

**Definition 5.1.** The graph \(\Gamma_F = (V, E, \omega)\) with vertex set \(V = \{v_1, \ldots, v_n\}\), edges \(E = \{e_{ij} : i \in I_j, j \in I_i\}\), and edge weights \(\omega(e_{ij}) = (A_F)_{ij}\) is called the graph of interactions of \(F(X)\).

The vertex \(v_i \in V\) of \(\Gamma_F\) corresponds to the \(i\)th element of the dynamical network \((F, X)\). Also, the edge \(e_{ij} \in E\) if and only if the \(j\)th component of the interaction \(F(x)\) depends on
the $i$th coordinate of $x$. The weight $\omega(e_{ij})$ measures the strength of the interaction between the $i$th and $j$th network elements.

As $(\mathcal{H}, X^T)$ is a dynamical network it also has a graph of interactions $\Gamma_{\mathcal{H}}$. For convenience, we let this be the graph of interactions of the time-delayed dynamical network $(\mathcal{H}, X^T)$ so that $\Gamma_{\mathcal{H}} = \Gamma_{\mathcal{H}}^t$. The vertex $v^t_i$ of $\Gamma_{\mathcal{H}}$ corresponds to the component $\mathcal{H}^t_i$ of $\mathcal{H}$ and represents the $i$th network element but $t$ time steps in the past. Each edge from the vertex $v^t_i$ to the vertex $v^t_j$ has unit weight since $\mathcal{H}^t_i \mathcal{H}^t_j = \mathcal{H}^{t+1}$. This corresponds to the fact that these edges simply pass along information until this information is used to update the state of some network element. The graph of interactions of the time-delayed network $(\mathcal{H}, X^T)$ from example 1 and its undelayed version $(\mathcal{U}, X)$ considered in example 3 are shown in figure 2.

Single and multiple network delays of $(\mathcal{H}, X^T)$ can be characterized in terms of the path and cycle structure of $\Gamma_{\mathcal{H}}$. To make this precise, a path $P$ in $G = (V, E, \omega)$ is an ordered sequence of distinct vertices $v_1, \ldots, v_m \in V$ such that $e_{i,i+1} \in E$ for $1 \leq i \leq m - 1$. The vertices $v_1, \ldots, v_{m-1}$ are the interior vertices of $P$. In the case that the interior vertices are distinct, but $v_1 = v_m$, then $P$ is a cycle. A cycle $v_1, \ldots, v_m$ is called a loop if $m = 1$.

A time-delayed network $(\mathcal{H}, X^T)$ has single type delays if there is at most one path of the form $v^0_i, v^{-1}_i, \ldots, v^{-|ab|}_i, v^0_j$ from $v^0_i$ to $v^0_j$ in $\Gamma_{\mathcal{H}}$ for all $i, j \in \mathbb{Z}$. If there are two or more such paths then the network has a multiple type delay. For instance, the graph $\Gamma_{\mathcal{H}}$ in figure 2 (left) indicates that the network $(\mathcal{H}, X^T)$ has single type delays.

### 5.2. Isoradial and bounded radial transformations

In this section we introduce a transformation that allows us to modify the structure of a graph while either preserving or modifying its spectral radius in a specific way. More precisely, we let the weighted adjacency matrix of a graph $G = (V, E, \omega)$ be the matrix $M(G)$ with entries $M(G)_{ij} = \omega(e_{ij})$ where $M(G)_{ij} = 0$ if $e_{ij} \notin E$. We let $\sigma(G) = \sigma(M(G))$ and $\rho(G) = \rho(M(G))$ denote the eigenvalues and spectral radius of $G$. We say $G$ is a nonnegative graph if its adjacency matrix $M(G)$ is nonnegative.

**Definition 5.2.** For the nonnegative graph $G = (V, E, \omega)$ let $G_{\theta}$ be the graph $G$ in which $e_{ij} \in E$ is replaced as in figure 3 where $\omega_1, \omega_2 \geq 0, \omega_1 + \omega_2 = \omega(e_{ij})$, and $\theta > 0$. We call $G_{\theta}$ a bounded radial transformation of $G$.

Since for every graph $G$ there is a corresponding matrix $M(G)$ and every matrix is the adjacency matrix of some graph then the transformations described in definition 5.2 can
equivalently be viewed as a matrix transformation. Before considering such transformations in general we first consider the following special case.

**Lemma 5.3.** Let $G_\theta$ be a bounded radial transformation of $G$. If $\theta = \rho(G)$ then $\rho(G_\theta) = \rho(G)$.

Lemma 5.3 states that $G_\theta$ is an isoradial transform of $G$ if $\theta = \rho(G)$, i.e. a transformation that preserves the spectral radius of $G$. In order to prove lemma 5.3 we use the theorem of Perron and Frobenius and the following standard terminology.

A directed graph $G$ is called strongly connected if there is a path from each vertex in the graph to every other vertex or if $G$ consists of a single vertex. The strongly connected components of $G$ are its maximal strongly connected subgraphs. If $M \in \mathbb{R}^{n \times n}$ then $M$ is said to be irreducible if the graph $G$ with adjacency matrix $M$ is strongly connected.

**Theorem 5.4 (Perron–Frobenius).** Let $M \in \mathbb{R}^{n \times n}$ and suppose that $M$ is irreducible and nonnegative. Then

(a) $\rho(M) > 0$;
(b) $\rho(M)$ is an eigenvalue of $M$;
(c) $\rho(M)$ is an algebraically simple eigenvalue of $M$; and
(d) the left and right eigenvectors $x$ and $y$ associated with $\rho(M)$ have strictly positive entries.

If a graph $G$ is not strongly connected then it has strongly connected components $S(G)_1, \ldots, S(G)_N$. Let $M_j$ be the adjacency matrix of the graph $S_j(G)$. Then the eigenvalues of $M$ are

$$\sigma(M) = \bigcup_{j=1}^{N} \sigma(M_j)$$

from which it follows that

$$\rho(M) = \max_{1 \leq j \leq N} \rho(M_j).$$

We call a strongly connected component $S_j(G)$ trivial if it consists of a single vertex without loop in which case $\sigma(S_j(G)) = \{0\}$. We now give a proof of lemma 5.3.

**Proof.** Let $G = G_\theta$ be a bounded radial transformation of $G = (V, E, \omega)$ where $\theta = \rho(G)$, $V = \{v_2, \ldots, v_n\}$, and the edge $e_{23} \in E$ is replaced as in figure 3 in which $v_1 = v_k$. Let $\theta I$ have the block form

$$M(G) - \theta I = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A$ is a $1 \times 1$ matrix. Assuming $G$ is strongly connected then $A = [\theta]$ is invertible by part (a) of the Perron–Frobenius theorem. Using the identity

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \cdot \det(D - CA^{-1}B)$$
we have \( \det(M(G) - \theta I) = \theta \det(D - \theta^{-1}CB) \) where
\[
(D - \theta^{-1}CB)_{ij} = \begin{cases} 
\omega_1 + \omega_2 & \text{for } i = 2, j = 3, \\
D_{ij} & \text{otherwise}.
\end{cases}
\]
As \( \omega_1 + \omega_2 = \omega(e_{23}) \) then \( D - \theta^{-1}CB = M(G) - \theta I \) so that
\[
\det(M(G) - \theta I) = \theta \det(M(G) - \theta I) = 0
\]
implying \( \theta \in \sigma(G) \).

Writing \( G \) as a function of \( \omega_1 \) the claim is that \( \rho(G(\omega_1)) = \theta \) for \( \omega_1 \in [0, \omega(e_{ij})] \). To verify this we note that \( G(0) \) has the strongly connected components \( G \) and \( v_i \) where \( v_i \) is trivial. Hence, equation (18) implies \( \rho(G(0)) = \theta \). As \( \theta \in \sigma(G(\omega_1)) \) and \( \rho(G(\omega_1)) \) is continuous with respect to \( \omega_1 \) for \( \omega_1 \in [0, \omega(e_{ij})] \) then parts (b) and (c) of the Perron–Frobenius theorem imply \( \rho(G(\omega_1)) = \theta \). This verifies the claim completing the proof. \( \square \)

The reason we call the graph \( G_\theta \) a bounded radial transform is found in the following lemma.

**Lemma 5.5.** If \( G \) is nonnegative then \( \rho(G) < \rho(G_\theta) < \theta \) if and only if \( \theta > \rho(G) \).

That is, the spectral radius of the graph \( G_\theta \) is bounded by \( \theta \) so long as \( \theta > \rho(G) \). To prove lemma 5.5 we require the following result, which can be found in [11].

**Proposition 3.** Let \( A, B \in \mathbb{R}^{n \times n} \) be nonnegative and suppose that \( A \) is irreducible. Then \( \rho(A + B) > \rho(A) \) if \( B \neq 0 \).

Hence, the spectral radius of a nonnegative irreducible matrix is strictly monotone in each of its entries. This allows us to give the following proof of lemma 5.5.

**Proof.** Let \( M_\theta = M(G_\theta) \) and \( M = M(G) \). Supposing \( G \) is nonnegative and strongly connected then \( M_\theta(M) \) is nonnegative and irreducible. Moreover, \( M_\theta(M) \) is the adjacency matrix of the isoradial transformation \( G \) so lemma 5.3 implies that \( \rho(M_\theta(M)) = \rho(M) \). This together with proposition 3 implies that for \( \gamma > 1 \),

\[
\rho(M_\theta(M)) < \rho(M_\gamma(M)) \leq \rho(M_\gamma(M)) \).
\]

Since \( \rho(M_\gamma(M)) = \gamma \rho(M) \) then letting \( \theta = \gamma \rho(M) \) we have
\[
\rho(M) < \rho(M_\theta(M)) < \theta \quad \text{if and only if } \theta > \rho(M)
\]
as \( \theta > 0 \). This implies the result in the case that \( G \) is strongly connected.

Suppose \( G \) is nonnegative but not strongly connected. In this case \( G \) can be decomposed into its strongly connected components
\[
S(G)_1, \ldots, S(G)_N.
\]
Let \( M_m \) be the adjacency matrix of the strongly connected component \( S(G)_m \) for \( 1 \leq m \leq N \). As \( G \) is nonnegative each \( M_m \) is nonnegative.

If \( e_{ij} \) is an edge of \( S(G)_m \) then we consider two cases. First, suppose \( S(G)_m \) is nontrivial. Then the graph \( G_\theta \) with adjacency matrix \( M_\theta(G) \) has the strongly connected components
\[
S(G)_1, \ldots, S_\theta(G)_m, \ldots, S(G)_N
\]
where \( (M_\theta)_m \) is the nonnegative adjacency matrix of \( S_\theta(G)_m \). As the adjacency matrix of a nontrivial strongly connected component is irreducible then \( \rho((M_\theta)_m) < \theta \) if and only if \( \rho(M) < \theta \). From equation (18),
\[
\rho(M_\theta) = \max(\rho(M_1), \ldots, \rho((M_\theta)_m), \ldots, \rho(M_N)) = \max(\rho(M), \rho((M_\theta)_m)).
\]
Hence, if $\rho(M) < \theta$ then $\rho\left((M_0)_{\ell}\right) < \theta$ implying $\rho(M_0) < \theta$. Conversely, if $\rho(M_0) < \theta$ then $\rho(M) < \theta$ and the result holds in this case.

If $\mathcal{S}(G)_\ell$ is trivial then $\omega_1, \omega_2 = 0$ and $M_0$ has the strongly connected components

$\mathcal{S}(G)_1, \ldots, \mathcal{S}(G)_N, \mathcal{S}(G)_{N+1}$

where $\mathcal{S}(G)_{N+1}$ is trivial with single vertex $v_k$. Hence, $\sigma(M_{N+1}) = [0]$ implying $\rho(M_0) = \rho(M)$ by equation (17). Hence, the result follows in the case where the edge $e_{ij}$ is an edge of the component $\mathcal{S}(G)_\ell$.

If $v_i \in \mathcal{S}(G)_\ell$ and $v_j \in \mathcal{S}(G)_m$ for $\ell \neq m$ then the graphs associated with $M$ and $M_0$ have the same nontrivial strongly connected components. Equation (18) then implies that $\rho(M_0) = \rho(M)$ and the result follows for the case where $e_{ij}$ does not belong to a strongly connected component. As this exhausts all cases this completes the proof. \hfill \Box

**Example 4.** Consider the graph of interactions $\Gamma_\mathcal{U}$ shown in figure 4 (left) of the undelayed network $(\mathcal{U}, X)$ from example 3. By a series of bounded radial transformations it is possible to transform $\Gamma_\mathcal{U}$ into the graph $\Gamma_1$ shown on the right. This is done at each step by letting $\theta = 1$ and $\omega_2 = 0$ so that the edge being modified is bisected with one edge retaining the original edge weight and the other receiving the weight 1. Importantly, we note that lemma 5.5 implies $\rho(\mathcal{U}) < 1$ if and only if $\rho(\Gamma_1) < 1$.

Although the graph $\Gamma_1$ in figure 4 is not the graph of interactions of the network $(\mathcal{H}, X^T)$ shown in figure 1 the two have the a similar structure. To quantify in what sense two graphs are similar we introduce the following.

**Definition 5.6.** For the graph $G = (V, E, \omega)$, let $S \subseteq V$ such that each cycle of $G$ contains a vertex in $S$. If each vertex of $V$ belongs to a path or cycle that begins and ends with a vertex of $S$ then $S$ is called a complete structural set of $G$.

For the graph $G$ let $s_{00}(G)$ be the set of all complete structural sets of $G$. For $S = \{v_1, \ldots, v_m\} \in s_{00}(G)$ let $\mathcal{I}_S = \{1, \ldots, m\}$ denote the index set of $S$.

**Definition 5.7.** For $S \in s_{00}(G)$ and $i, j \in \mathcal{I}_S$ let $\mathcal{B}_{ij}(G; S)$ be the set of paths and cycles of $G$ from $v_i$ to $v_j$ that have no interior vertices in $S$. The set

$$\mathcal{B}_S(G) = \bigcup_{i, j \in \mathcal{I}_S} \mathcal{B}_{ij}(G; S)$$

is called the branch set of $G$ with respect to $S$.  

![Figure 4. A transform $\Gamma_1$ of the graph $\Gamma_\mathcal{U}$ from example 3, via a sequence of bounded radial transformations.](image-url)
Suppose $G = (V, E, \omega)$ and $S \in st_0(G)$. If the branch $\beta = v_1, \ldots, v_m \in B_S(G)$ let $\Omega_G(\beta)$ be the ordered sequence

$$\Omega_G(\beta) = \omega(e_1), \omega(e_2), \ldots, \omega(e_{m-1,m})$$

for $m > 1$ and $\omega(e_1)$ if $m = 1$. Suppose $S = \{v_1, \ldots, v_m\}$ is a complete structural set of the graphs $G$ and $K$. The branch set $B_{ij}(G; S)$ is isomorphic to $B_{ij}(K; S)$ if there is a bijection $b : B_{ij}(G; S) \rightarrow B_{ij}(K; S)$ such that $\Omega_G(\beta) = \Omega_K(b(\beta))$ for each $\beta \in B_{ij}(G; S)$. If such a map exists we write $B_{ij}(G; S) \simeq B_{ij}(K; S)$. If

$$B_{ij}(G; S) \simeq B_{ij}(K; S) \quad \text{for each } 1 \leq i, j \leq m$$

we say $B_S(G)$ is isomorphic to $B_S(K)$ and write $B_S(G) \simeq B_S(K)$.

For the graph $G$ and $S \in st_0(G)$ suppose $\alpha = v_1, \ldots, v_m$ and $\beta = u_1, \ldots, u_n$ are branches in $B_S(G)$. These branches are said to be independent if

$$\{v_2, \ldots, v_{m-1}\} \cap \{u_2, \ldots, u_{n-1}\} = \emptyset.$$

That is, $\alpha$ and $\beta$ are independent if they share no interior vertices.

**Definition 5.8.** For the graphs $G$ and $K$ let $S \in st_0(G), st_0(K)$. Suppose

(i) $B_S(G) \simeq B_S(K)$;
(ii) the branches of $B_S(K)$ are independent; and
(iii) each vertex of $G$ and $K$ belongs to a branch of $B_S(G)$ and $B_S(K)$ respectively. Then we call $K$ an isoradial expansion of $G$ with respect to $S$ and write $K = \mathcal{X}_S(G)$.

As a corollary to theorem 4.2 in [3] we have the following result.

**Theorem 5.9.** Let $G = (V, E, \omega)$ and $S \in st_0(G)$. Then $\rho(\mathcal{X}_S(G)) = \rho(G)$.

**Example 5.** Consider the graph $\Gamma_{\mathcal{N}}$ shown in figure 5 (right), which is the graph of interactions of the time-delayed network $(\mathcal{H}, X^T)$ from example 1. As each cycle of $\Gamma_{\mathcal{N}}$ passes through a vertex of $S = \{v_1^0, v_2^0\}$ and each vertex of this graph belongs to a cycle then $S \in st_0(\Gamma_{\mathcal{N}})$.

The isoradial expansion $\mathcal{X}_S(\Gamma_{\mathcal{N}})$ is shown in figure 5 (left), which has spectral radius $\rho(\mathcal{X}_S(\Gamma_{\mathcal{N}})) = \rho(\Gamma_{\mathcal{N}})$.

Note that the graph $\mathcal{X}_S(\Gamma_{\mathcal{N}})$ is identical to the graph $\Gamma_1$ from example 4, which was shown to have spectral radius $\rho(\Gamma_1) < 1$ if and only if $\rho(\mathcal{U}) < 1$. Hence, $\rho(\mathcal{U}) < 1$ if and only if $\rho(\mathcal{H}) < 1$ where $(\mathcal{U}, X)$ is the undelayed version of the time delayed network $(\mathcal{H}, X^T)$. That
is, using isoradial expansions together with bounded radial transformations it is therefore possible to show the stability of a delayed network implies the stability of the network without delays and vice versa.

Our goal in the remainder of this section is to use isoradial expansions and bounded radial transformations to prove theorems 4.4 and 4.4. A proof of theorem 4.4 is the following.

**Proof.** Let \( (\mathcal{H}, X^T) \) be a time-delayed network. The claim is that \( S = \{v^0_i : i \in I\} \) is a complete structural set of \( \Gamma_H \). To see this note that if \( v_1, \ldots, v_m \) is a path or cycle in \( \Gamma_H \) containing no vertices of \( S \) then (9) implies there is some \( i \in I \) such that \( v_\ell = v_\ell^{-} \) for \( 1 \leq \ell \leq m \). As each \( v_x \) is distinct then in order to form a cycle of \( \Gamma_H \) we need an element of \( S \). Moreover, each vertex of \( \Gamma_H \) belongs to a path (cycle) of the form \( v_\ell, v_\ell^{-1}, \ldots, v_\ell^{-r}, v_\ell^0 \) (where \( i = j \)). Hence, \( S \subseteq st_0(\Gamma_H) \).

Suppose \( (\mathcal{H}, X^T) \) has single type delays. Then for \( i, j \in I \) there is at most one branch in \( B_S(\Gamma_H) \) from \( v^0_i \) to \( v^0_j \) implying the same is true of the isoradial expansion \( \bar{X}_S(\Gamma_H) \). If \( \alpha = v^0_i, \bar{v}^{-1}_i, \ldots, \bar{v}^{-p}_i, v^0_j \) is a branch in \( B_S(\bar{X}_S(\Gamma_H)) \) it has the form shown in figure 6 (left) where \( i = (i, -p) \) and \( j = (j, 0) \) are in \( I \times I \).

Via a sequence of bounded radial transformations with \( \omega_1 = (A_{\bar{H}})_{ij}, \omega_2 = 0, \) and \( \theta = 1 \) the single edge of \( \Gamma_u \) shown in figure 6 (right) can be replaced by the branch \( \alpha \). Lemma 5.5 then implies that \( \rho(\Gamma_u) < 1 \) if and only if \( \rho(\bar{X}_S(\Gamma_H)) < 1 \). For the same reason, replacing each \( \beta = v^0_{\ell}, \bar{v}^{-1}_i, \ldots, \bar{v}^{-q}_i, v^0_m \in B_S(\bar{X}_S(\Gamma_H)) \) by a single edge from \( v^0_{\ell} \) to \( v^0_m \) with weight \( (A_{\bar{H}})_{m\ell} \) results in the graph \( \Gamma \) with the property

\[
\rho(\Gamma) < 1 \quad \text{if and only if} \quad \rho(\bar{X}_S(\Gamma_H)) < 1 \tag{19}
\]

For \( i, j \in I \) the adjacency matrix \( M(\Gamma) \) of \( \Gamma \) has entries

\[
M(\Gamma)_{ij} = (A_H)_{ij} = \max_{x \in X} |(D\bar{H})_{ij}(x, \ldots, x)| = \max_{x \in X} |(DU)_{ij}(x)| = (A_{\bar{H}})_{ij} \tag{20}
\]

where the second to last inequality follows from the fact that \( (H, X^T) \) has only single type delays. Hence, \( \rho(\Gamma) = \rho(\bar{U}) \). The reason that the entries \( (A_{\bar{H}})_{ij} \) and \( (DU)_{ij}(x) \) are assumed to have no local dynamics. The fact that \( \rho(\bar{H}) = \rho(\bar{X}_S(\Gamma_H)) \), via theorem 5.9, together with (19) implies that

\[
\rho(\bar{U}) < 1 \quad \text{if and only if} \quad \rho(\bar{H}) < 1.
\]

This completes the proof. \( \square \)

We now give a proof of theorem 4.2.

**Proof.** For the time-delayed network \( (\mathcal{H}, X^T) \) the proof of theorem 4.4 implies that \( S = \{v^0_i : i \in I\} \) is a complete structural set of \( \Gamma_H \). Suppose then that there is a single branch \( \alpha = v^0_i, \bar{v}^{-1}_i, \ldots, \bar{v}^{-p}_i, v^0_j \in B_S(\bar{X}_S(\Gamma_H)) \) from \( v^0_i \) to \( v^0_j \) as shown in figure 6 (left) with
$\hat{i} = (i, -p)$ and $\hat{j} = (0, j)$ in $\mathcal{I} \times \mathcal{T}$. Following the proof of theorem 4.4 the branch $\alpha$ can be replaced with a single edge as in figure 6 (right) such that
\[ \rho(\Gamma_\alpha) < 1 \quad \text{if and only if} \quad \rho(\chi_\lambda(\Gamma_H)) < 1. \]

Suppose then that there are two branches
\[ \alpha = v_i^0, v_i^{-1}, \ldots, v_i^{-p}, v_j^0, \quad \beta = v_j^0, v_j^{-1}, \ldots, v_j^{-q}, v_j^0 \]
in $\mathcal{B}_S(\chi_\lambda(\Gamma_H))$ from $v_i^0$ to $v_j^0$ as shown in figure 7 (left) where $\hat{i} = (i, -p)$ and $\hat{j} = (i, -q)$ are in $\mathcal{I} \times \mathcal{T}$. By repeated use of lemma 5.5 with $\omega_2 = 0$, and $\theta = 1$ the branch $\alpha$ can be replaced with a single edge and $\beta$ by two edges as in the intermediate graph $\Gamma_{\alpha/\beta}$ in figure 7 (center). By another use of lemma 5.5 with $\omega_1 = (\alpha_\beta; j), \omega_2 = (\alpha_\beta; i)$; and $\theta = 1$ it follows that the graph $\Gamma_{\alpha/\beta}$ shown in figure 6 (right) has the property
\[ \rho(\Gamma_{\alpha/\beta}) < 1 \quad \text{if and only if} \quad \rho(\chi_\lambda(\Gamma_H)) < 1. \]

Continuing in this manner suppose there are branches $\alpha_1, \ldots, \alpha_m \in \mathcal{B}_S(\chi_\lambda(\Gamma_H))$ from $v_i^0$ to $v_j^0$. Through a sequence of bounded radial transformations these branches can be replaced by a single edge with weight
\[ \omega_{ij} = \sum_{\ell=1}^{m} (A_{\tilde{R}})_{i,\ell} \quad \text{where} \quad i_{\ell} = (i, |\alpha_{\ell}| + 1) \in \mathcal{I} \times \mathcal{T}. \]

The claim is that $(A_{\bar{U}})_{ij} < \omega_{ij}$ where $(\bar{U}, X)$ is the undelayed version of $(\mathcal{H}, X^T)$. Indeed, as $\alpha_1, \ldots, \alpha_m \in \mathcal{B}_S(\chi_\lambda(\Gamma_H))$ are the branches of $\chi_\lambda(\Gamma_H)$ from $v_i^0$ to $v_j^0$ then the function $\mathcal{H}_j : X^T \rightarrow X$ depends on the variables indexed $i$ given by
\[ x_{i_{\ell}} = |\alpha_{\ell}| + 1 \quad \text{for} \quad 1 \leq \ell \leq m. \]

Therefore, $(A_{\bar{U}})_{ij} \leq \omega_{ij}$ as
\[
\max_{x \in X} |(D\bar{U})_{ji}(x)| = \max_{x \in X} |(D\mathcal{H})_{ji}(x, \ldots, x)| \leq \max_{x \in X^T} \sum_{\ell=1}^{m} |(D\mathcal{H})_{i_{\ell},i}(x)| = \omega_{ij}.
\]

Let $\Gamma$ be the graph in which each set of branches in $\chi_\lambda(\Gamma_H)$ from $v_i^0$ to $v_j^0$ is replaced by the edge with weight $\omega_{ij}$ for all $i, j \in \mathcal{I}$. Then equation (21) implies that $\rho(\bar{U}) \leq \rho(\Gamma)$. Since $\Gamma$ can be transformed via a sequence of bounded radial transformations with $\theta = 1$ into $\chi_\lambda(\Gamma_H)$ then lemma 5.5 implies $\rho(\Gamma) < 1$ if and only if $\rho(\chi_\lambda(\Gamma_H)) < 1$. The fact that $\rho(\chi_\lambda(\Gamma_H)) = \rho(\Gamma_H)$ implies that if $\rho(\mathcal{H}) < 1$ then $\rho(\bar{U}) < 1$ completing the proof. \[\square\]
6. Restrictions of dynamical networks

One of the major obstacles in understanding the dynamics of a network (or high dimensional system) is that the information needed to do so is spread throughout the various network elements and interactions (or system components). In the remainder of this paper we consider whether it is possible to consolidate this information to gain improved estimates of a network’s stability.

6.1. Implicit delays

Suppose we consider two different elements of a dynamical network. Even if there is no direct interaction between the two the one may influence the dynamics of the other after some number of time steps $\tau > 1$. This amounts to an implicit time-delayed interaction between these elements.

In terms of the graph structure of $(F, X)$ there is an implicit time-delayed interaction from the $i$th to the $j$th network element if there is a path in $\Gamma_F$ from $v_i$ to $v_j$. By choosing a particular subset of vertices, i.e. network elements, we show here that it is possible to associate a time-delayed network with $(F, X)$ whose delays represent the implicit delays between these network elements.

To formally define this time-delayed network let $x^{k+1} = F(x^k)$ and suppose $S \in st_0(\Gamma_F)$. For $j \in I_S$ let $F_{j,0}(x^k) = F_j(x^k)$. For $\ell \geq 1$ we recursively define $F_{j,\ell}(x^k, \ldots, x^{k-\ell})$ to be the function

$$F_{j,\ell-1}(x^k, \ldots, x^{k-\ell+1})$$

in which the variable $x_i^{k-\ell+1}$ is replaced by the function $F_i(x^{k-\ell})$ for all $i \notin I_S$. If for some $m > 0$ each variable of $F_{j,m-1}(x^k, \ldots, x^{k-m})$ is indexed by an element of $I_S$ then we let

$$H_S F_j(x^k_S, \ldots, x^{k-m}_S) = F_{j,m-1}(x^k_S, \ldots, x^{k-m}_S), \quad (22)$$

where each $x^{k-m}_S \in X_S = X|_{I_S}$.

By choosing $S$ to be a complete structural set we are guaranteed that the function $H_S F_j$ exists, i.e. there is some $m < \infty$ such that each variable of $F_{j,m-1}$ is indexed by an element of $I_S$. The reason for this is that otherwise there would be a cycle of $\Gamma_F$ which contained no elements of the structural set $S$.

**Definition 6.1.** For $S \in st_0(\Gamma_F)$ let $(H_S F, X_S^T)$ be the dynamical network with components defined by equation (22) for $j \in I_S$. This network is called the time-delayed version of $(F, X)$ with respect to $S$.

The time-delayed network $(H_S F, X_S^T)$ is consider as a network with no local dynamics since the compositions needed to construct this system do not in general allow for a decomposition into an interaction and local systems.

**Example 6.** Consider the dynamical network $(F, X)$ given by

$$F(x) = \begin{bmatrix} \tanh(x_6) + c \\ \tanh(x_1) + c \\ \tanh(x_2) + \tanh(x_5) + c \\ \tanh(x_3) + c \\ \tanh(x_4) + c \\ \tanh(x_2) + \tanh(x_5) + c \end{bmatrix}, \quad x \in \mathbb{R}^6$$
with local systems $\varphi_i(x_i) = \tanh(x_i)$ and interaction

$$F(x) = \begin{cases} x_{i-1} + c & \text{for } i = 1, 2, 4, 5 \\ x_2 + x_5 + c & \text{for } i = 3, 6 \end{cases}$$

where the indices are taken mod 6, $X = \mathbb{R}^6$ and $c \in \mathbb{R}$. The network $(\mathcal{F}, X)$ can be thought of as an undelayed Cohen–Grossberg network where we allow $\epsilon = 1$.

Formally, $(\mathcal{F}, X)$ can be written as the time-delayed network $x^{k+1} = \mathcal{F}(x^k)$ with

$$x^{k+1} = \begin{bmatrix} \tanh(x_1^k) + c \\ \tanh(x_2^k) + c \\ \tanh(x_3^k) + \tanh(x_4^k) + c \\ \tanh(x_5^k) + c \\ \tanh(x_6^k) + \tanh(x_5^k) + c \end{bmatrix}, \quad x^k \in \mathbb{R}^6.$$

As can be seen from figure 8, the vertex set $S = \{v_1, v_2, v_4, v_5\}$ is a complete structural set of $\Gamma_\mathcal{F}$. To construct $(\mathcal{H}_S \mathcal{F}, X^T_S)$ we let $\mathcal{F}_{j,0}(x^k) = \mathcal{F}_j(x^k)$ for each $j \in I_S$ and note that $\mathcal{H}_S \mathcal{F}_j(x^k) = \mathcal{F}_{j,0}(x^k)$ for $j = 2, 5$ since each variable of these two functions is indexed by an element of $I_S$. Replacing the variables $x_3^k$ and $x_5^k$ by $\mathcal{F}_3(x^{k-1})$ and $\mathcal{F}_5(x^{k-1})$ respectively in $\mathcal{F}_{j,0}(x^k)$ for $j = 1, 4$ yields the function

$$\mathcal{F}_{j,1}(x^k, x^{k-1}) = \tanh(\tanh(x_2^{k-1}) + \tanh(x_4^{k-1}) + c) + c.$$ 

As each of this functions variables is indexed by an element of $I_S$ then the function $\mathcal{H}_S \mathcal{F}_j(x^k, x^{k-1}) = \mathcal{F}_{j,1}(x^k, x^{k-1})$ for $j = 1, 4$.

The time-delayed network $(\mathcal{H}_S \mathcal{F}, X^T_S)$ associated with $(\mathcal{F}, X)$ is then given by

$$x^{k+1} = \begin{bmatrix} \mathcal{H}_1(x_2^{k-1}, x_4^{k-1}) \\ \mathcal{H}_2(x_1^k) \\ \mathcal{H}_3(x_2^{k-1}, x_4^{k-1}) \\ \mathcal{H}_4(x_5^{k-1}, x_6^{k-1}) \\ \mathcal{H}_5(x_3^k) \end{bmatrix} = \begin{bmatrix} \tanh(\tanh(x_2^{k-1}) + \tanh(x_4^{k-1}) + c) + c \\ \tanh(x_1^k) + c \\ \tanh(\tanh(x_2^{k-1}) + \tanh(x_4^{k-1}) + c) + c \\ \tanh(x_5^{k-1}) + \tanh(x_6^{k-1}) + c \\ \tanh(x_3^k) + c \end{bmatrix}$$

where $T = 2$. The system $(\mathcal{H}_S \mathcal{F}, X^T_S)$ is the network $(\mathcal{F}, X)$ in which the implicit delays between the elements indexed by $S$ are made explicit.

By writing $(\mathcal{F}, X)$ in the form of a delayed network $x^{k+1} = \mathcal{F}(x^k)$ by convention the initial condition $x^0 \in X$ has the orbit $\{x^k\}_{k \geq 0}$ under the action of $\mathcal{F}$. For $x^0 \in X$ it follows from our construction that

$$\mathcal{F}_j(x^k) = \mathcal{H}_S \mathcal{F}_j(x^k, x^{k-T+1})$$
for each \( j \in I_S \) and \( k \geq T - 1 \). That is, \((H_S, X^1_T)\) and \((F, X)\) have the same dynamics when restricted to \( S \). Since every vertex of \( \Gamma_F \) belongs to a branch of \( B_0(\Gamma_F) \) this implies the following result.

**Lemma 6.2.** For \( S \in st_0(\Gamma_F) \) the time-delayed network \((H_S, X^1_S)\) is stable if and only if \((F, X)\) is stable.

In what follows we investigate the undelayed version of the system \((H_S, X^1_S)\), which we call a restriction of the dynamical network \((F, X)\).

### 6.2. Restricting dynamical networks

The time-delayed dynamical network \((H_S, X^1_S)\) allows one to view \((F, X)\) as a lower dimensional time-delayed network. The reason \((H_S, X^1_S)\) is dimensionally smaller is that in its construction we absorb the dynamics of the elements of \((F, X)\), which are not indexed by \( I_S \), into the time-delayed interaction of \((H_S, X^1_S)\). In this section we show that this type of network compression can be used to gain improved stability estimates of \((F, X)\).

**Definition 6.3.** For \( S \in st_0(\Gamma_F) \) let \((R_S, X_S)\) be the undelayed version of the time-delayed network \((H_S, X^1_S)\). We call this network the restriction of \((F, X)\) to \( S \).

Although lemma 6.2 states that the stability of \((F, X)\) and \((H_S, X_S)\) are equivalent there is no guarantee that the stability of \((R_S, X_S)\) implies the stability of \((H_S, X^1_S)\). To determine when the stability of a restriction implies the stability of the original network we define the following.

**Definition 6.4.** Let \( S \) be a complete structural set of the graph \( G \). Then \( S \) is called a basic structural set of \( \Gamma_F \) if

\[
|B_I(\Gamma_F; S)| \leq 1 \quad \text{for all } i, j \in I_N.
\]

If \( S \) is a basic structural set of \( G \) then we write \( S \in st_B(G) \). The notion of a basic structural set allows for the following theorem.

**Theorem 6.5.** For the network \((F, X)\) suppose \( S \in st_B(\Gamma_F) \). If \( \rho(R_S) < 1 \) then \((F, X)\) is stable.

**Proof.** Given the dynamical network \((F, X)\) suppose that \( S \in st_B(\Gamma_F) \). Under this assumption the claim is that the time-delayed system \((H_S, X^1_S)\) has single type delays. To see this suppose \( v_1, v_2, \ldots, v_l \) is a branch of \( B_0(\Gamma_F) \). Then, in particular, the function \( F_{j,0}(x^k) = F_j(x^k) \) depends on the variable \( x^{k-1}_j \) and \( F_{j-1}(x^k) \) on the variable \( x^{k-2}_j \) implying \( F_{j,1}(x^k, x^{k-1}) \) is a function of \( x^{k-1}_{j-2} \).

Continuing in this manner it follows that

\[
F_{j,j-2} = F_{j,j-3}(x^k, \ldots, x^{k-1+j-1})
\]

is a function of \( x^{k-j+1} \). As the index \( 1 \in I_S \) then the component \( H_S(x_j) \) must depend on the variable \( x^{k-j+1} \). Therefore, \( H_S(x_j) \) depends on the variable \( x^{k-j+2} \) if there is a branch \( v_i, \ldots, v_j \in B_S(\Gamma_F) \). Conversely, if \( H_S(x_j) \) depends on the variable \( x^{k-j+2} \) then there must be a branch in \( B_0(\Gamma_F) \) from \( v_i \) to \( v_j \). As \( S \) is a basic structural set then the time-delayed network \((H_S, X^1_S)\) has single type delays verifying the claim.

Assuming \( \rho(R_S) < 1 \) then theorem 4.4 implies \( \rho(H_S) < 1 \). Hence, \((H_S, X^1_S)\) is stable by theorem 2.2 and lemma 6.2 can be used to show that \((F, X)\) is stable. \( \square \)
Phrased another way, if the restriction \((R_S,F, X_S)\) is intrinsically stable and \(S \in st_B(\Gamma_F)\) then theorem 6.5 states that the unrestricted network \((F, X)\) must be globally stable. The proof of theorem 6.5 also implies that the set \(S \in st_B(\Gamma_F)\) if and only if the time-delayed network \((H_S,F, X_S^T)\) has single type delays. By removing this type of implicit delays from a network the resulting restriction can be used to determine the stability of the original network. This procedure is illustrated in the following example.

**Example 7.** Let \((F, X)\) be the dynamical network considered in example 6, which has the stability matrix \(A_F\) given by

\[
A_F = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

As one can compute \(\rho(F) = 2^{1/3}\) implying \(\rho(F) > 1\). Hence, theorem 2.2 cannot be used to determine if the dynamical network \((F, X)\) is stable.

By removing the delays from \((H_S,F, X_S^T)\) found in (23) the restriction

\[
R_S(F)(x) = \begin{bmatrix}
R_SF_1(x_2, x_3) \\
R_SF_2(x_1) \\
R_SF_4(x_2, x_3) \\
R_SF_5(x_4)
\end{bmatrix} = \begin{bmatrix}
\tanh(\tanh(x_2) + \tanh(x_3) + c) + c \\
\tanh(x_1) + c \\
\tanh(\tanh(x_2) + \tanh(x_3) + c) + c \\
\tanh(x_4) + c
\end{bmatrix}.
\]

Since the branches of \(B_S(\Gamma_F)\) are given by

- \(B_{12}(\Gamma_F; S) = v_1, v_2;\)
- \(B_{21}(\Gamma_F; S) = v_2, v_1;\)
- \(B_{24}(\Gamma_F; S) = v_2, v_3, v_4\)
- \(B_{45}(\Gamma_F; S) = v_4, v_5;\)
- \(B_{54}(\Gamma_F; S) = v_5, v_4;\)
- \(B_{51}(\Gamma_F; S) = v_5, v_6, v_1\)

then \(S \in st_B(\Gamma_F)\). To compute a stability matrix of the restriction \((R_S,F, X_S)\) we note that

\[
(D_{R_S,F})_{ij} = \text{sech}^2(x_i)\text{sech}[\tanh(x_2) + \tanh(x_3) + c] \\
\leq \text{sech}[\tanh(x_2) + \tanh(x_3) + c] \leq \text{sech}^2(c - 2)
\]

for \(i = 1, 4, j = 2, 5, \text{ and } c \geq 2\). Similarly, \((D_{R_S,F})_{ij} \leq \text{sech}^2(c + 2)\) for \(c \leq -2\).

Considering the case \(c \geq 2\) we can compute

\[
A_{R_S,F} \leq \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
\text{sech}^2(c - 2) & 0 & \text{sech}^2(c - 2) & 0 \\
0 & 0 & 0 & 1 \\
\text{sech}^2(c - 2) & 0 & \text{sech}^2(c - 2) & 0
\end{bmatrix}
\]

from which it follows that

\[
\rho(R_S,F) \leq \frac{2\sqrt{2}e^2}{e^{c-2} + e^c}.
\]

Therefore, if \(c > 2 + \ln(1 + \sqrt{2}) \approx 2.88\) then \(\rho(R_S,F) < 1\). Similarly, using \((D_{R_S,F})_{ij} \leq \text{sech}^2(c + 2)\) for \(c \leq -2\) we have \(\rho(R_S,F) < 1\) if \(c < -2 - \ln(1 + \sqrt{2})\). For \(|c| > 2.88\) theorem 6.5 together with 2.2 imply that \((F, X)\) is stable. By slight abuse of notation we let the graph \(G_{R_S,F}\) in figure 8 (right) have the weights given by the matrix in (24). We stress the fact that only by use of a network reduction were we able to deduce the stability of \((F, X)\).
Remark 5. We note that the restriction \((\mathcal{R}_S \mathcal{F}, X_S)\) can be constructed without first constructing \((\mathcal{H}_S \mathcal{F}, X_S)\). This can be done by letting \(\mathcal{F}_{j, \ell}(x)\) be the function \(\mathcal{F}_{j,\ell-1}(x)\) in which each variable \(x_i \in X\) is replaced by \(\mathcal{F}_i(x)\) if \(i \notin I_S\). If each variable of \(\mathcal{F}_{j,m-1}(x)\) is indexed by an element of \(I_S\) for some \(m > 0\) then the function \(\mathcal{F}_{j,m-1}(x_S) = \mathcal{R}_S \mathcal{F}_j(x_S)\) for \(j \in I_S\).

In certain cases it may be possible to sequentially restrict a dynamical network and thereby sequentially improve ones estimates of the original unrestricted network’s stability. For instance, the network \((\mathcal{R}_S \mathcal{F}, X_S)\) in example 7 can be restricted to the set \(\tilde{S} = \{v_1, v_4\} \in \text{stB}(\Gamma_{\mathcal{R}_S \mathcal{F}})\) (see figure 8). Moreover, it is possible to use restrictions to get improved stability estimates of time-delayed network as the following corollary to theorems 4.4 and 6.5 indicates.

Corollary 3. Suppose \((\mathcal{H}, X^T)\) has single type delays. If \((\mathcal{U}, X)\) has the restriction \((\mathcal{R}_S \mathcal{U}, X|_S)\) with \(S \in \text{stB}(\Gamma_\mathcal{U})\) and spectral radius \(\rho(\mathcal{R}_S \mathcal{U}) < 1\) then \((\mathcal{H}, X^T)\) is stable.

We note that the network restrictions introduced here are similar to the procedure of expanding dynamical networks found in [3] as both allow for improved estimates of a network’s stability. However, the computational effort involved in finding a network restriction is far less when compared with the procedure of constructing and analysing a network expansion.

7. Concluding remarks

This paper analyses the global stability of a general class of time-delayed dynamical networks. To carry out this analysis we continue the investigation of useful network transformations initiated in [3]. The transformations we consider here modify the spectral radius of a graph in a prescribed way, which allow us to investigate the stability of networks in which time delays have either been added or removed.

This analysis leads us to differentiate between two types of network delays. The first and simpler type are single delays between network elements while the second more complicated type are multiple delays or delays that happen on multiple time scales between the same network elements. The stability condition we introduce in this paper is novel in that a network that is known to be stable via this criteria is also stable if its time delays are removed or single time delays are added. As this does not hold for general stability the stability we introduce here, which we call intrinsic stability, is a new and stronger form of global stability.

This form of stability is potentially important in the construction of real networks as such networks are inherently time-delayed. Specifically, if a network is designed to be intrinsically stable then its stability is much more robust to changes effecting its delays then if it were only designed to be stable.

A notion of an implicit delay is also introduced in this paper. By removing these implicit delays we demonstrate that the resulting lower dimensional system or network restriction can be used to gain improved estimates of the network’s stability. A key ingredient in this procedure is the new notion of a basic structural set of the network’s graph of interactions. These sets form a subclass of structural sets introduced in [3].

The theory developed in this paper is illustrated by various examples of Cohen–Grossberg neural networks. Importantly, this approach to analysing the global stability of networks is not limited in any way to this class of networks and can be applied to any time-delayed or undelayed dynamical system with only slight restrictions. We fully expect that this theory of network transformations will continue to prove useful in the analysis of other network features, structures, and types of dynamics as it has already been used to obtain improved estimates of matrix spectra [4] and escape rates in open dynamical systems [5].
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