Energy Level Quasi-Crossings: Accidental Degeneracies or Signature of Quantum Chaos?

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Abstract

In the field of quantum chaos, the study of energy levels plays an important role. The aim of this review paper is to critically discuss some of the main contributions regarding the connection between classical dynamics, semi-classical quantization and spectral statistics of energy levels. In particular, we analyze in detail degeneracies and quasi-crossings in the eigenvalues of quantum Hamiltonians which are classically non-integrable.

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1 Introduction

In the last few years, many attempts have been made by authors working in different fields to point out some characteristic properties of quantal systems whose classical analogs are chaotic [1-4].

In this review paper we shall focus our attention mainly on the connection between quasi-degeneracies of the energy levels of a given quantum Hamiltonian $\hat{H}$ and the chaotic behavior of its classical limit.

As stressed by Berry [5], if a system is without symmetries then degeneracies of the energy levels are considered to be accidental. In fact, for a typical Hamiltonian $\hat{H}$ representing a bound quantal system, no two of the energy levels $E$ will coincide. Nevertheless, if one embeds $\hat{H}$ in a population smoothly parameterised by variables $\vec{R} = (X, Y, Z, ...)$ then degeneracies of energy levels can rise from negligible to inevitable. On the basis of Von Neumann and Wigner’s famous theorem [6] for typical families $\hat{H}(\vec{R})$ of real Hamiltonians, two parameters are necessary to produce a degeneracy, while if $\hat{H}(\vec{R})$ is Hermitian (and not real) three parameters are necessary.

A very different way of embedding a given Hamiltonian in a two-parameter family is to consider it as a member of a family labelled by a single complex parameter $C$ [7]. It is possible for levels to degenerate at isolated points in $C$-plane, even for one-dimensional problems [8]. But, as discussed by Berry [7], with a complex $C$ the operators are not Hermitians, the eigenvalues are not real, and at degeneracies their surfaces have a branch-point, rather than conical structure.
2 Quasi-Crossing and Chaos

As previously discussed, a family of generic hamiltonians $\hat{H}(X)$ (real or Hermitian) depending on a single real parameter $X$ will not exhibit degeneracies: in the plane $E, X$ the eigenvalue curves $E_n(X)$ will not cross [6]. Instead, the curves will display quasi-crossings (QC) whose local form (which arises from a close approach to a degeneracy in an extended parameter space) is that of a pair of hyperbolas (slices of a cone or hypercone) [5]. Such QC are now familiar in calculations of the quantal spectra of families of classical non-integrable systems.

If the family consists of Hamiltonians $\hat{H}(X)$ which are not generic, but are special in some way, then degeneracies can be expected to occur. An important case is families of separable systems. For example, in two dimensions a particle of unitary mass in a rectangular box with side ratio $X$ and characteristic length $L$ has levels labelled by quantum numbers $m, n$ with energies

$$E_{m,n} = \frac{\hbar^2 \pi^2}{2L^2} (m^2 + X^2 n^2) ,$$

which can degenerate when $X^2$ is rational. Berry and Tabor [9] showed that for quantum systems whose classical analogs are integrable, the distribution $P(s)$ of nearest-neighbor spacings $s_i = (\tilde{E}_{i+1} - \tilde{E}_i)$ of the unfolded levels $\tilde{E}_i$ follows the Poisson distribution

$$P(s) = \exp(-s) .$$

It means that quasi-degeneracies of energy levels have the maximal probability to occur. Instead, for multidimensional systems whose classical motion is
quasi-integrable, i.e. perturbation of integrable systems, Berry [5] suggested that multiple QC in the quantal spectrum are associated with classical chaos. In this case, the probability of quasi-degeneracies is zero, thus there is level repulsion, and $P(s)$ is quite well reproduced by

$$P(s) = \frac{\pi}{2} s \exp \left( -\frac{\pi}{4} s^2 \right),$$  

which is the so-called Wigner distribution [10].

Note that, the early idea of separating the spectrum of energy levels in regular and chaotic components was proposed by Percival [11]. Moreover, in general, multiple QC are not always associated with classical chaos. In fact, in pseudo-integrable systems (rational billiards), all trajectories are confined to $N$-dimensional invariant surfaces (multiple-handled spheres rather than tori) and there is no chaos in the sense of exponential separation. Nevertheless, Richens and Berry [12] found multiple QC in the spectrum of a family of such systems. Isolated QC can occur even in integrable systems, for example in the double-well potential.

The research field on quasi-crossings and their connection with quantum chaos is very wide. For reasons of space, in the next sections we focus our attention only on a few selected examples taken from molecular, nuclear and particle physics.
3 Molecular Spectroscopy

In the field of molecular spectroscopy, Marcus [13-15] studied the semiclassical eigenvalues of the Hénon-Heiles Hamiltonian [16]

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 + w_x^2 x^2 + w_y^2 y^2 \right) + \lambda x (y^2 + \eta x^2) , \]  

(4)

with coordinates \( x \) and \( y \), momenta \( p_x \) and \( p_y \), and \( w_x \) and \( w_y \) incommensurable frequencies. \( \lambda \) and \( \eta \) are real parameters. A typical classical trajectory is given in Fig. 1. On the ellipse, the potential energy equals the total energy. As shown by Marcus [13-15], within the region ABCD, the corresponding quantum mechanical wave function is large and oscillatory, while, outside that region, it dies away exponentially with distance. By calculating \( \oint \vec{p} \cdot d\vec{r} \) along the curves AB and BC, setting them equal to \((n_x + 1/2)\hbar\) and \((n_y + 1/2)\hbar\) respectively, semiclassical eigenvalues were calculated [13-15]. The agreement with the quantum mechanical eigenvalues is excellent.

Marcus [13] also discussed in great detail the difference between a statistical and non statistical wave function: “A statistical (stochastic) wavefunction is described as one that yields averages for dynamical quantities that are approximately equal to their microcanonical averages at the energy: when the states are sufficiently dense, a microcanonical average can be computed by averaging over all quantum states in a small interval \((E, E + \delta E)\). For large enough quantum numbers, the classical microcanonical average could also be used for comparison. A nonstatistical wavefunction corresponds semiclassically to an invariant torus when the latter exists, and would yield, instead, an average approximately equal to the classical average over that torus.”
A statistical wavefunction could arise in the following way. One assumes that, in the absence of QC, the principal effect of the perturbation $\lambda$ is to distort the shape and extent of the region largely occupied by the wavefunction, but not to otherwise change it drastically. In that case, if one plots the eigenvalues versus some perturbation parameter (e.g. $\lambda$ in equation (4)) and there is no QC, then each state would have a nonstatistical eigenfunction. If, however, two such eigenvalues approach each other and then repel as the perturbation parameter is increased, one has a QC. Such QC of two energy levels does not yet convey a statistical character on each of the two wavefunctions. Rather, in the vicinity of the QC, each wavefunction has some of the characteristics of the two wavefunctions that would have arisen had the QC not occurred. A QC (see Fig. 2) is an analogue of an isolated classical resonance: a vibrational frequency, which corresponds semiclassically to a difference of eigenvalues, becomes nearly zero in the vicinity of the quasi crossing, as it does in the classical case.

As pointed out by Marcus [13], one way of obtaining a statistical wavefunction is to have many overlapping QC, an analogue of overlapping resonances in the classical case. An example of multiple QC is given in Fig. 3.

In classical mechanics, a generic Hamiltonian written in action-angle variable $(\vec{J}, \vec{\theta})$ is given by

$$H(\vec{J}, \vec{\theta}) = H_0(\vec{J}) + \lambda V(\vec{J}, \vec{\theta}),$$

where $H_0$ is the integrable part and $\lambda V(\vec{J}, \vec{\theta})$ a perturbation. A resonance
occurs when there is some value \( \vec{J}^* \) of \( \vec{J} \), such that one finds

\[
\vec{m} \cdot \vec{\omega}(\vec{J}^*) = 0 ,
\]

(6)

where \( \vec{\omega} = \partial H_0 / \partial \vec{J} \). The resonance is an isolated resonance when one needs to consider only one set of integers \( m_i \) for which the previous condition holds. In this case, one also recalls that the Hamiltonian is integrable (even when multiples of \( m \) are included); a canonical transformation to new variables \( (\vec{I}, \vec{\phi}) \) can be introduced such that the Hamiltonian is a function only of the new action variables \( \vec{I} \). For small \( \lambda \), one can make a 2nd order Taylor expansion about the resonant action variable to obtain an approximate Hamiltonian. This Hamiltonian has the form of the pendulum Hamiltonian, for which one finds librations and rotations, divided by the separatrix. Following Chirikov [17], the onset of widespread chaos begins when two or more resonance regions overlap, thus when two or more separatrices cross. By performing the semiclassical quantization (Bohr-Sommerfeld) of the classical Hamiltonian, one can determine librational and rotational energy levels. The semiclassical effect of the crossing of separatrices is the crossing of semiclassical energy levels that are close to the separatix. In actual fact, quantum mechanically one finds only quasi-crossing because of the previously discussed theorem of Von Neumann and Wigner [6] on quantum degeneracies.

4 Nuclear Models
4.1 ZVW Model

Various properties of nuclear spectra can be described by an ensemble of random matrices, for example the Gaussian Orthogonal Ensemble [18]. However, realistic Hamiltonians of atomic nuclei contain a large regular part, that takes into account shell structure. Zirnbauer, Verbaaschot and Weidenmuller [19], using the Hamiltonian
\[ \hat{H} (\lambda) = \hat{H}_0 + \lambda \hat{V}_{GOE}, \] (7)
studied the competition between order as induced by \( \hat{H}_0 \) and chaos characteristic of situations where the random part of the interaction \( \hat{V}_{GOE} \) dominates. In practice, they took a diagonal \( \hat{H}_0 \) with equally spaced eigenvalues, but they stress that their conclusions are valid irrespective of the special choice for \( \hat{H}_0 \). In particular, they found complete mixing of eigenstates whenever the norm of \( \lambda \hat{V}_{GOE} \) is comparable or greater than the norm of \( \hat{H}_0 \). Moreover, they studied the distribution of "branch points", defined as the complex values of \( \lambda \) for which two eigenvalues of the Hamiltonian (7) coincide. The branch points \( \lambda \) of \( \hat{H}_0 \) are the solutions of the system
\[ \text{det}(E - \hat{H}(\lambda)) = 0, \]
\[ \frac{\partial}{\partial E} \text{det}(E - \hat{H}(\lambda)) = 0. \] (8)
On the real \( \lambda \)-axis, the branch points are observable as quasi-crossings of two eigenvalues of \( \hat{H}(\lambda) \). (As discussed in the introduction, according to the theorem of ref. [6], no branch points can occur for real values of \( \lambda \).) If \( \lambda \) is large enough, one passes a large number of QC and, as a result, the wave functions become more and more mixed. The main conclusion of the author
of [19] is that the distribution of branch points, namely the distribution of QC for real values of \( \lambda \), is strongly peaked at \( \lambda \sim 1 \): for this value of \( \lambda \) the GOE term begins to dominate over the regular term \( \hat{H}_0 \). For larger values of \( \lambda \) the distribution of branch points decreases and eventually goes to zero.

### 4.2 LMG Model

Another example, taken from nuclear physics, is the three-level Lipkin-Mashow-Glick (LMG) model [20,21], whose quantum Hamiltonian is

\[
\hat{H} = \sum_{k=0}^{2} \epsilon_k \hat{G}_{kk} - \frac{V}{2} \sum_{kl=0}^{2} \hat{G}_{kl}^2,
\]

where

\[
\hat{G}_{kl} = \sum_{m=1}^{M} \hat{a}_{km}^+ \hat{a}_{lm}
\]

are the generators of the SU(3) group. This model describes \( M \) identical particles in three, \( M \)-fold degenerate, single-particle levels \( \epsilon_i \). Like the authors of refs. [20,21], we assume \( \epsilon_2 = -\epsilon_0 = \epsilon = 1 \), \( \epsilon_1 = 0 \), a vanishing interaction for particles in the same level and an equal interaction for particles in different levels. For the SU(3) model a semi-quantal Hamiltonian can be defined as

\[
H(p_1, p_2, q_1, q_2; M) = \langle q_1p_1, q_2p_2; M | \frac{\hat{H}}{M} | q_1p_1, q_2p_2; M \rangle,
\]

where \(| q_1p_1, q_2p_2; M \rangle\) is the coherent state, given by:

\[
| q_1p_1, q_2p_2; M \rangle = \exp [z_1 G_{01} + z_2 G_{02}] |00 \rangle,
\]

with:

\[
\frac{1}{\sqrt{2M}} (q_k + i p_k) = \frac{z_k}{\sqrt{1 + z_1^* z_1 + z_2^* z_2}}, \quad k = 1, 2
\]
and $|0> = \Pi_{k=1}^M a_0^+ k \ldots 0>$ is the ground state. Here $1/M$ plays the role of the Planck constant $\hbar$.

As discussed in great detail in [20-25], the classical Hamiltonian can be obtained in the thermodynamic limit

$$H_{cl} = \lim_{M \to \infty} H(p_1, p_2, q_1, q_2; M) =$$

$$-1 + \frac{1}{2}(p_1^2 + q_1^2) + (p_2^2 + q_2^2) + \frac{1}{4} \chi [(q_1^2 + q_2^2)^2 - (p_1^2 + p_2^2)^2]$$

$$- (q_1^2 - p_1^2)(q_2^2 - p_2^2) - 4q_1q_2p_1p_2 - 2(q_1^2 + q_2^2 - p_1^2 - p_2^2) \ldots 14$$

with $\chi = MV/\epsilon$. The phase space has been scaled to give $(q_1^2 + q_2^2 + p_1^2 + p_2^2) \leq 2$.

In order to analyze the stability of the system, we calculated the periodic orbits of this model using Hamilton’s equation of the classical Hamiltonian [22]. Such equations can be written as

$$\dot{\vec{x}} = J \nabla H_{cl}(\vec{x}, \chi), \quad 15$$

where

$$\vec{x} = (x_1, x_2, x_3, x_4) = (q_1, q_2, p_1, p_2), \quad \nabla = \left( \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right) \quad 16$$

and $J$ is the $4 \times 4$ symplectic matrix

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad 17$$

where $I_2$ is the $2 \times 2$ identity matrix. In terms of the stability matrix $S(t)$, defined in the usual way

$$S_{ij}(t) = \frac{\partial x_i(t)}{\partial x_j(0)}, \quad 18$$
the largest Lyapunov exponent can be written as

\[
\lambda(\vec{x}) = \lim_{t \to \infty} \frac{1}{t} \log |S(t)| ,
\]

(19)

where \(|S(t)|\) is the norm of the matrix \(S(t)\). This matrix can be calculated by solving its equations of motion:

\[
\dot{S}(t) = J \frac{\partial^2 H(\vec{x})}{\partial x^2} S(t) ,
\]

(20)

with the initial conditions

\[
S(0) = I ,
\]

(21)

where \(I\) is the \(4 \times 4\) identity matrix. The calculation of the full set of Lyapunov exponents is related to that of the eigenvalues \(\sigma_i\) of the matrix \(S(T)\), with \(T\) the period of the periodic orbit:

\[
\lambda_i(\vec{x}) = \lim_{t \to \infty} \frac{1}{T} \log \sigma_i .
\]

(22)

Using the unitarity nature of \(S\), a periodic orbit is unstable if

\[
Tr(S) > 4 \quad \text{or} \quad Tr(S) < 0 ,
\]

(23)

and stable if

\[
0 < Tr(S) < 4 .
\]

(24)

It is interesting to study the change in stability of periodic trajectories as a function of the coupling constant \(\chi\). In Fig. 4 the ratio between the number of stable periodic orbits and the number of total orbits with period \(T < 30\) is plotted versus \(\chi\). For the coupling constant \(\chi \in (0, 3]\), \(T_{\min} \simeq 3\), as shown in Ref. [22]. As can be seen, the sensitivity of the orbits to a small change
of $\chi$ is quite different for different values of $\chi$, reflecting the transition order-chaos as the coupling constant increases. For the sake of completeness, in Fig. 5 the density $\rho(\omega)$ of the periodic orbits is shown as a function of the frequency $\omega = 2\pi/T$. Like the power spectrum, $\rho(\omega)$ presents few peaks in the integrable or quasi-integrable region and a quasi-continuum spectrum in the chaotic region.

In order to apply the quantal criteria to our system, the eigenvalues of Hamiltonian (9) must be calculated. A natural basis can be written $|bc\rangle$, meaning $b$ particles in the second single-particle level, $c$ in the third and, of course, $M - b - c$ in the first level. In this way $|00\rangle$ is the ground state with all the particles in the lowest level. We can write the basis states as

$$|bc\rangle = \left(\frac{1}{b!c!}\right)^{1/2} \hat{G}^b_{21} \hat{G}^c_{31} |00\rangle ,$$

where $(1/b!c!)^{1/2}$ is the normalizing constant.

From the commutation relation of $G_{kl}$ we can calculate expectation values of $\hat{H}/M$ and thus, eigenvalues and eigenstates of $\hat{H}/M$. In this way the energy spectrum range is independent of the number of particles:

$$\langle b'c'|\hat{H}/M|bc\rangle = \frac{1}{M}(-M + b + 2c)\delta_{b'b'}\delta_{cc'} - \frac{\chi^2}{2M^2} Q_{b'c',bc} ,$$

where

$$Q_{bc,b'c'} = \sqrt{b(b-1)(M-b-c+1)(M-b-c+2)}\delta_{b-2,b'}\delta_{cc'} + \sqrt{(b+1)(b+2)(M-b-c)(M-b-c-1)}\delta_{b+2,b'}\delta_{cc'} + \sqrt{c(c-1)(M-b-c+1)(M-b-c+2)}\delta_{b,b'}\delta_{c-2,c'}$$

(26)

(27)
The expectation values \( \langle \hat{H}/M \rangle \) are real and symmetric. For any given number of particles \( M \), we can set up the complete basis state, calculate the matrix elements of \( \langle \hat{H}/M \rangle \), and then diagonalize \( \langle \hat{H}/M \rangle \) to find its eigenvalues. \( \langle \hat{H}/M \rangle \) connects states with \( \Delta b = -2, 0, 2 \) and \( \Delta c = -2, 0, 2 \) only, which simplifies matters. States with \( b, c \) even; \( b, c \) odd; \( b \) even and \( c \) odd; \( b \) odd and \( c \) even are grouped together. Thus \( \langle \hat{H}/M \rangle \) becomes block diagonal, containing four blocks that can be diagonalized separately; these matrices are referred to as \( ee, oo, oe \) and \( eo \). When the parameter \( \chi = 0 \), the Hamiltonian consists of two oscillators and there are many degeneracies, but for \( \chi \neq 0 \) these degeneracies are obviously broken (see Fig. 6). For a large number of particles (semiclassical limit), we calculated the density of quasicrossings outside the degeneracy region as a function of the parameter \( \chi \) (see Fig. 7):

\[
\rho(\chi) = \frac{\Delta N}{\Delta \chi},
\]

where \( \Delta N \) is the number of quasicrossing in the parameter range \( \Delta \chi = 0.01 \). To obtain \( \Delta N \) we fixed three values \( \chi - \Delta \chi, \chi \) and \( \chi + \Delta \chi \) and imposed that

\[
s_i(\chi - \Delta \chi) > s_i(\chi), \quad s_i(\chi + \Delta \chi) > s_i(\chi),
\]

where \( s_i(\chi) = E_{i+1}(\chi) - E_i(\chi) \). The results (Fig. 8) show a maximum of quasicrossings for \( \chi = 2 \) for all classes, in agreement with the transition to chaos of Fig. 9.
In order to study the sensitivity of energy levels to small changes of the parameter \( \chi \) we used the statistics \( \Delta^2(E) \) defined in the usual way

\[
\Delta^2(E) = \left| E_i(\chi + \Delta \chi) + E_i(\chi - \Delta \chi) - 2E_i(\chi) \right| ,
\]

which measures the curvature of \( E_i \) in a small range \( \Delta \chi \). To remove the secular variation of the level density, each spectrum was mapped into one which has a constant level density by a numerical procedure described in Ref. [26]. Fig. 8 shows \( \Delta^2(E) \) for different values of \( \chi \); we note that the maximum value of \( \Delta^2(E) \) corresponds to the \( \chi = 2 \) value.

In conclusion, in the study of the transition from order to chaos, there is, in agreement with the authors of [11] and [20], a good correspondence between the classical approach, based on the stability matrix and Lyapunov exponents, and the quantal one, based on the quasicrossing distribution and the \( \Delta^2(E) \) statistics.

For the sake of completeness in Fig. 9 the distribution \( P(s) \) of spacings \( s \) between adjacent levels for the eo class (nearest-neighbor spacing distribution) has been calculated and compared to the Brody distribution [27]:

\[
P(s) = \alpha (q + 1) s^q \exp (-\alpha s^{q+1}) ,
\]

with

\[
\alpha = \left( \Gamma \left( \frac{q + 2}{q + 1} \right) \right)^{q+1} , \quad 0 \leq q \leq 1 ,
\]

where \( \Gamma(x) \) is the factorial function. This distribution interpolates between the Poisson distribution \( (q = 0) \) and the Wigner distribution \( (q = 1) \). As can be seen from Fig. 11, this statistic also confirms the smooth transition from the regular to the chaotic regime discussed in the papers [20,21].
5 Particle Physics and Field Theory

In this section we apply a quantal analog [28] of the Chirikov resonance criterion [17] to study the suppression of classical chaos in the spatially homogeneous SU(2) Yang-Mills-Higgs (YMH) system [29-33]. The quantal Chirikov criterion means that one applies the semiclassical quantization to calculate the critical value of the parameters corresponding to the intersection of two neighboring quantal separatrices.

The SU(2) YMH system describes a Higgs scalar field $\phi$, coupled to the Yang-Mills vector fields $A^a_\beta, a = 1, 2, 3$. The Yang-Mills fields are the gauge fields of the SU(2) group [34]. The Higgs field has the familiar sombrero-shaped potential

$$V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4.$$  \hspace{1cm} (33)

The SU(2) YMH system is a simplified version of the scalar QCD, which requires the SU(3) group and 8 Yang-Mills vector fields [34]. Nevertheless, the equations of motion of the SU(2) YMH system are complex and strongly nonlinear. Some simplifications can be made by working in the (2+1)-dimensional Minkowski space ($\beta = 0, 1, 2$) and choosing spatially homogenous Yang-Mills and the Higgs fields. Thus, one considers the system in the region where space fluctuations of fields are negligible compared to their time fluctuations [29-33]. In the gauge $A^a_0 = 0$ and using the real triplet representation for the Higgs field, the Hamiltonian of the system becomes

$$H = \frac{1}{2}(\dot{A}^2_1 + \dot{A}^2_2) + \dot{\phi}^2 + g^2[\frac{1}{2} \vec{A}^2_1 - \frac{1}{2}(\vec{A}_1 \cdot \vec{A}_2)^2 +$$

$$+ (\vec{A}^2_1 + \vec{A}^2_2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] + V(\vec{\phi}),$$  \hspace{1cm} (34)
where $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$, $\vec{A}_1 = (A^1_1, A^2_1, A^3_1)$, and $\vec{A}_2 = (A^1_2, A^2_2, A^3_2)$.

When $\mu^2 > 0$ the potential $V$ has a minimum in $|\vec{\phi}| = 0$, but for $\mu^2 < 0$ the minimum is:

$$|\vec{\phi}_0| = \left(\frac{-\mu^2}{4\lambda}\right)^{\frac{1}{2}} = v$$

which is the non-zero Higgs vacuum. This vacuum is degenerate and after spontaneous symmetry breaking, the physical vacuum can be chosen $\vec{\phi}_0 = (0,0,v)$. If $A^1_1 = q_1$, $A^2_2 = q_2$, and the other components of the Yang-Mills fields are zero, in the Higgs vacuum the Hamiltonian of the system is:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + g^2 v^2 (q_1^2 + q_2^2) + \frac{1}{2}g^2 q_1^2 q_2^2,$$

(35)

where $p_1 = \dot{q}_1$ and $p_2 = \dot{q}_2$. Obviously $w^2 = 2g^2v^2$ is the mass term of the Yang-Mills fields.

Classical chaos was demonstrated in a pure Yang-Mills system [29], i.e. in a zero Higgs vacuum. The effect of a non-zero Higgs vacuum can be analyzed by using the quantal analog of the Chirikov criterion [30]. We introduce the action-angle variables by the canonical transformation

$$q_i = (\frac{2I}{\omega})^{\frac{1}{2}} \cos \theta_i, \quad p_i = (2I \omega)^{\frac{1}{2}} \sin \theta_i, \quad i = 1, 2.$$

(36)

The Hamiltonian becomes

$$H = (I_1 + I_2)\omega + \frac{1}{v^2}I_1 I_2 \cos^2 \theta_1 \cos^2 \theta_2.$$

(37)

By the new canonical transformation into slow and fast variables:

$$A_1 = I_1 + I_2, \quad A_2 = I_1 - I_2,$$

$$\theta_1 = \chi_1 + \chi_2, \quad \theta_2 = \chi_1 - \chi_2,$$

(38)
\( H \) can be written:

\[
H = A_1 \omega + \frac{1}{4v^2}(A_1^2 - A_2^2) \cos^2(\chi_1 + \chi_2) \cos^2(\chi_1 - \chi_2). \tag{39}
\]

We now eliminate the dependence on the angles to the order \( 1/v^4 \) by resonant canonical perturbation theory [34]. First we average on the fast variable \( \chi_1 \).

This yields

\[
\frac{1}{2\pi} \int_0^{2\pi} d\chi_1 \cos^2(\chi_1 + \chi_2) \cos^2(\chi_1 - \chi_2) = \frac{1}{8}(2 + \cos 4\chi_2), \tag{40}
\]

and

\[
\bar{H}_{cl} = A_1 \omega + \frac{1}{32v^2}(A_1^2 - A_2^2)(2 + \cos 4\chi_2). \tag{41}
\]

The dependence on \( \chi_2 \) is now eliminated by a second canonical transformation. The Hamilton-Jacobi equation for the perturbation part is

\[
[A_1^2 - (\frac{\partial S}{\partial \chi_2})^2](2 + \cos 4\chi_2) = K, \tag{42}
\]

\[
\frac{\partial S}{\partial \chi_2} = \pm \sqrt{\frac{A_1^2(2 + \cos 4\chi_2) - K}{2 + \cos 4\chi_2}}. \tag{43}
\]

and thus the Hamiltonian becomes

\[
\bar{H} = B_1 + \frac{1}{32v^2} K(B_1, B_2), \tag{44}
\]

where

\[
B_1 = A_1, \quad B_2 = \frac{1}{2\pi} \oint d\chi_2 \frac{\partial S}{\partial \chi_2}. \tag{45}
\]

It appears from the structure of this equation that the motion of our system is similar to that of a simple pendulum: for \( 0 < K < B_1^2 \) rotational motion,
for $K = B_1^2$ separatrix, and for $B_1^2 < K < 3B_1^2$ librational motion. On the separatrix we have $B_1^2(2 + \cos 4\chi_2) = K$, and:

$$B_2 = \pm \frac{2}{\pi} \int_a^b dx \sqrt{\frac{B_1^2(2 + \cos 4x) - K}{2 + \cos 4x}},$$

(46)

where $a = -\frac{\pi}{4}$, $b = \frac{\pi}{4}$ for rotational motion, and $a = \phi_-(K, B_1)$, $b = \phi_+(K, B_1)$ for librational motion, with:

$$\phi_\pm(K, B_1) = \pm \frac{1}{4} \arccos\left(\frac{K}{B_1^2} - 2\right).$$

(47)

The appearance of a separatrix (which is not immediately obvious in the $(p, q)$ coordinates) accounts, as is well known, for the stochastic layers originating near it [35]. This corresponds to local irregular behavior of the quantum spectrum; one of its manifestations is the local shrinking of the level spacing and the tendency to avoided crossings [28]. Note that the resonant perturbation theory of Hamiltonian (35) can be quite easily extended also to the second order (see [36]).

The approximate Hamiltonian depends only on the actions so that a semiclassical quantization formula can be obtained by the Bohr-Sommerfeld quantization rules [34]. Set $B_1 = m_1\hbar$ and $B_2 = m_2\hbar$, then, up to terms of order $\hbar$, the quantum spectrum is

$$E_{m_1, m_2} = m_1\hbar\omega + \frac{1}{32v^2}K(m_1\hbar, m_2\hbar),$$

(48)

where $K$ is implicitly defined by the relation

$$m_2\hbar = \pm \frac{2}{\pi} \int_a^b dx \sqrt{\frac{(m_1\hbar)^2(2 + \cos 4x) - K}{2 + \cos 4x}},$$

(49)

with $a = -\frac{\pi}{4}$, $b = \frac{\pi}{4}$ for $0 < K < (m_1\hbar)^2$, and $a = \phi_-(K, B_1)$, $b = \phi_+(K, B_1)$ for $(m_1\hbar)^2 < K < 3(m_1\hbar)^2$. 18
On the separatrix, where \( K = (m_1\hbar)^2 \), \( m_2 = \pm \alpha m_1 \), with:

\[
\alpha = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dx \sqrt{\frac{1 + \cos 4x}{2 + \cos 4x}}.
\]  

(50)

It is clear that for \( m_1 \) fixed, the function \( K \), and hence the semiclassical energy \( E_{m_1, m_2} \), is a decreasing function of the secondary quantum number \( m_2 \), and we have a quantum multiplet [34]. We can calculate the value of the coupling constant \( 1/v^2 \) corresponding to the intersection of the separatrices of two neighboring quantum multiplets

\[
(m_1 + 1)\hbar \omega + \frac{1}{32v^2} K[(m_1 + 1)\hbar, \alpha(m_1 + 1)\hbar] = m_1\hbar \omega + \frac{1}{32v^2} K(m_1\hbar, \alpha m_1\hbar),
\]  

(51)

and so

\[
\frac{1}{v^2} = \frac{-32\hbar \omega}{K[(m_1 + 1)\hbar, \alpha(m_1 + 1)\hbar] - K(m_1\hbar, \alpha m_1\hbar)}.
\]  

(52)

In this way, we have the quantal counterpart [28] of the method of overlapping resonances developed by Chirikov [17]. The denominator can be evaluated by the Taylor expansion and finally

\[
\frac{1}{v^2} = \left[ -8\omega \frac{\partial K}{\partial B_1} - \alpha \frac{\partial K}{\partial B_2} \right]_{B_1 = m_1\hbar, B_2 = \alpha m_2\hbar}.
\]  

(53)

K is implicitly defined by the relation

\[
F[B_1, B_2, K(B_1, B_2)] = B_2 - \frac{\pi}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dx \sqrt{\frac{B_1^2(2 + \cos 4x) - K}{2 + \cos 4x}} = 0,
\]  

(54)

or

\[
F(B_1, B_2, K) = B_2 - \Phi(B_1, K) = 0.
\]  

(55)

As a function of \( \Phi \), \( 1/v^2 \) can be written:

\[
\frac{1}{v^2} = \lim_{K \to B_1^2} \left[ \frac{8\omega \frac{\partial \Phi}{\partial K}}{\alpha - \frac{\partial \Phi}{\partial B_1}} \right]_{B_1 = m_1\hbar},
\]  

(56)
where
\[ \frac{\partial \Phi}{\partial K} = -\frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dx \frac{1}{\sqrt{(2 + \cos 4x)[B_1^2(2 + \cos 4x) - K]}} \]
and
\[ \frac{\partial \Phi}{\partial B_1} = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dx \sqrt{\frac{B_1^2(2 + \cos 4x)}{B_1^2(2 + \cos 4x) - K}}. \]  

(57)

The result is
\[ \frac{1}{v^2} = \frac{16\omega}{m_1 \hbar}, \]  

(58)

where \( m_1 \hbar \simeq E \) (the energy of the system) and \( \omega = (2v^2g^2)^{\frac{1}{2}} \). Therefore the chaos-order transition depends on the parameter \( \lambda = v^3 g / E \): if \( 0 < \lambda < \sqrt{2}/32 \), a relevant region of the phase-space is chaotic, but if \( \lambda > \sqrt{2}/32 \) the system becomes regular. This result shows that the value of the Higgs field in the vacuum \( v \) plays an important role. The system is regular for large values of \( v \) in agreement with numerical calculations [30] and the Toda Criterion of negative Gaussian curvature [32,33]. The Yang-Mills coupling constant \( g \) has the same role. Instead, if \( v \) and \( g \) are fixed there is an order-chaos transition increasing the energy \( E \).

In conclusion, we have shown that, for the spatially homogenous SU(2) YMH system, the quantum resonance criterion, which describes the onset of widespread chaos associated to semiclassical crossing between separatrices of different quantum multiplets, gives an analytical estimation of the classical chaos-order transition as a function of the Higgs vacuum, the Yang-Mills coupling constant and the energy of the system.
6 Conclusions

It is important to stress that multiple quasi-crossings are \textit{not} associated with classical chaos, but simply with the \textit{onset} of chaos. Although, the results of Marcus are not conclusive, those of Weidenmuller and our own suggest this conclusion. Note that the Chirikov criterion gives a qualitative picture of the transition to chaos, as well as its quantal analog. The semiclassical quantization of fully chaotic systems must be performed by using a path integral technique. Instead, the results we have presented are based on the torus quantization of approximate classical Hamiltonians, which are obtained by means of a perturbative approach.
References

1. M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Berlin, 1990).

2. A.M. Ozorio de Almeida, *Hamiltonian Systems: Chaos and Quantization* (Cambridge University Press, Cambridge, 1990).

3. M.T. Lopez-Arias, V.R. Manfredi and L. Salasnich, Rivista del Nuovo Cimento, vol. **17**, n. 5 (1994).

4. V.R. Manfredi and L. Salasnich, Int. J. Mod. Phys. B **13**, 2343 (1999).

5. M.V. Berry, in *Chaotic Behavior in Quantum Systems*, Ed. G. Casati, NATO ASI Series B: Physics 120 (Plenum Press, New York, 1985).

6. J. Von Neumann and E. Wigner, Phys. Z. **30**, 467 (1929).

7. M.V. Berry and M. Wilkinson, Proc. R. Soc. Lond. A **392**, 15 (1984).

8. C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Mc Grow-Hill, New York, 1978).

9. M.V. Berry and M. Tabor, J. Phys. A **10**, 371 (1977).

10. O. Bohigas, M.J. Gannoni and C. Schmit, Phys. Rev. Lett. **52**, 1 (1884).

11. I.C. Percival, Adv. Chem. Phys. **36**, 1 (1977).

12. P.J. Richens and M. Berry, Physica D **1**, 495 (1981).
13. R.A. Marcus, Ann. N.Y. Acad. Sci. **357**, 169 (1980).

14. D.W. Noid, M.L. Koszykowski, M. Tabor and R.A. Marcus, J. Chem. Phys. **72**, 6169 (1980).

15. R. Ramaswamy and R.A. Marcus, J. Chem. Phys. **74**, 1385 (1981).

16. M. Hénon and C. Heiles, Astr. J. **69**, 73 (1964).

17. B.V. Chirikov, Phys. Rep. **52**, 265 (1979).

18. M.L. Mehta, *Random Matrices and the Statistical Theory of Energy levels* (Academic Press, New York, 1967).

19. H.R. Zirnbauer, J.J.M. Verbaarschot and H.A. Weidenmuller, Nucl. Phys. A **411**, 161 (1983).

20. R. Williams and S. Koonin, Nucl. Phys. A **391**, 72 (1982).

21. D. Meredith, S. Koonin and M. Zirnbauer, Phys. Rev. A **37**, 3499 (1988).

22. V.R. Manfredi and L. Salasnich, Z. Phys. A **343**, 1 (1992).

23. L. Demattè, V.R. Manfredi and L. Salasnich, in *From Classical to Quantum Chaos*, Ed. G.F. Dell’Antonio, S. Fantoni and V.R. Manfredi, SIF Conference Proceedings **41**, pp. 111 (Editrice Compositori, Bologna, 1993).

24. V.R. Manfredi, L. Salasnich and L. Dematte, Phys. Rev. E **47**, 4556 (1993).
25. V.R. Manfredi and L. Salasnich, Int. J. Mod. Phys. B 9, 3219 (1995).

26. V.R. Manfredi, Lett. Nuovo Cimento 40, 135 (1984).

27. T.A. Brody, Lett. Nuovo Cimento 7, 482 (1973).

28. S. Graffi, T. Paul, H.J. Silverstone, Phys. Rev. Lett. 59, 255 (1987).

29. G. K. Savvidy: Phys. Lett. B 130, 303 (1983).

30. L. Salasnich, Phys. Rev. D 52, 6189 (1995).

31. L. Salasnich, Mod. Phys. Lett. A 12, 1473 (1997).

32. L. Salasnich, Physics of Atomic Nuclei, 61, 1878 (1998).

33. L. Salasnich, J. Math. Phys. 40, 4429 (1999).

34. S. Graffi, V. R. Manfredi, L. Salasnich: Mod. Phys. Lett. B 7, 747 (1995).

35. A. J. Lichtenberg, M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, Berlin, 1983).

36. L. Salasnich, Meccanica 33, 397 (1998).
Figure Captions

Figure 1: A trajectory of Hamiltonian (4) with $w_x$ and $w_y$ incommensurable (adapted from Ref. 13).

Figure 2: An example of a QC of two pairs of quantum states of the same symmetry, in a plot of eigenvalue $E$ versus changes in the parameter $\lambda$ for the Henon-Heiles system. Also given is an actual crossing of states $(12, \pm 12)$ (split) with $(13, \pm 5)$ – an allowed crossing, since these two pairs of states are of different symmetry (adapted from Ref. 13).

Figure 3: An example of overlapping QC. Plot of eigenvalues vs parameter $k_{122}$ in the Hamiltonian $H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + w_x x^2 + w_y y^2 + w_z z^2) + k_{122} x y^2 + k_{133} x z^2 + k_{233} y z^2 + k_{111} x^3 + k_{222} y^3 + k_{1122} x^2 y^2 + k_{2233} y^2 z^2$, with $w_x : w_y : w_z = 2 : 1 : 1$. Only eigenvalues for eigenfunctions even in $y$ are plotted. The absence of hidden symmetries (producing actual crossings) was assumed in joining the points (adapted from Ref. 13).

Figure 4: Ratio between the number of stable periodic orbits and the number of total periodic orbits vs $\chi$, with $T < 30$; $T_{\text{min}} \simeq 3$ (adapted from Ref. 22).

Figure 5: The density of the periodic orbits in different regions: (a) $\chi = 2$ and $E < -1$; (b) $\chi = 2$ and $E > -1$; (c) $\chi = 100$ and $E < -28$; (d) $\chi = 100$ and $-28 < E < 25$ (adapted from Ref. 23).

Figure 6: A group of 40 energy levels as a function of the parameter $\chi$ for the eo class (adapted from Ref. 23).

Figure 7: Density of quasicrossings vs $\chi$ for all classes (adapted from Ref. 24).
Figure 8: $\Delta^2(E)$ vs $E$ for different values of $\chi$ for the eo class; $M = 102$; (a) $\chi = 0.5$, (b) $\chi = 2$; (c) $\chi = 3$, (d) $\chi = 5$ (adapted from Ref. 23).

Figure 9: $P(s)$ vs $s$ for different values of $\chi$ for the eo class; $M = 102$; (a) $\chi = 0.5$, (b) $\chi = 2$; (c) $\chi = 3$ (adapted from Ref. 24).
\[ \frac{\Delta N_{st}}{\Delta N_{tot}} \]
Energy (a.u.)

$\Delta^2(E)$

(d)

(c)

(b)

(a)
