Rough path continuity equations with discontinuous coefficients - regularization by fractional Brownian motion

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Abstract

We consider the stochastic continuity equation perturbed by a fractional Brownian motion and the drift is allowed to be discontinuous. We show that for almost all paths of the fractional Brownian motion there exists a solution to the equation.

1 Introduction

We study the stochastic continuity equation

$$\partial_t \mu_t + \text{div}(b \mu_t) + \text{div}(\mu_t \dot{B}^H_t) = 0,$$

where $\mu_0$ is a given measure, $\dot{B}^H_t$ is fractional noise. We allow for low regularity on the drift, i.e. we assume $b \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}^d))$.

We are looking for a spatially measure-valued solution, meaning that for a test function $\eta \in C_c^\infty(\mathbb{R}^d)$, we should have

$$\partial_t \mu_t(\eta) = \mu_t((b(t, \cdot), \nabla \eta)) + \langle \mu_t(\nabla \eta), \dot{B}^H_t \rangle$$

where $\mu_t(\eta) := \int_{\mathbb{R}^d} \eta(x) d\mu_t(x)$, $\mu_t(\nabla \eta) = (\mu_t(\partial_{x_1} \eta), \ldots, \mu_t(\partial_{x_d} \eta))$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^d$. We will choose the noise to be a fractional Brownian motion with $H < 1/2$. The noise is not differentiable so one should integrate the equation in time and recast the last term as $\int_0^t \langle \mu_s(\nabla \eta), dB^H_s \rangle$. But even at this level it is not a priori clear how to make sense of this term. In fact, since $B^H_t$ is not a semi-martingale there is no Itô theory to define this integral. Since $B^H_t$ is Hölder continuous of order strictly smaller than $H < 1/2$ the integration theory of Young is also out of reach.

Instead we shall interpret the integral in the rough path setting, meaning we shall use iterated integrals of $B^H_t$ and the theory of controlled paths to define the integral.

We will show that for every initial condition $\mu_0$ and $P$-a.e. path of the fractional Brownian motion there exists a solution to the continuity equation.

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More specifically, we will show that the solution is on the form $\mu_t = (\phi_t)\mu_0$, i.e. the push forward of $\mu_0$ by the flow map

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + B_t^H.$$ (1)

To see this, heuristically, take $\eta \in C_c^\infty(\mathbb{R}^d)$ and suppose we have some kind of Itô-Stratonovich-formula for the fractional Brownian motion in the rough path setting. We should have

$$\eta(\phi_t(x)) = \eta(x) + \int_0^t \langle \nabla \eta(\phi_r(x)), b(r, \phi_r(x)) \rangle dr + \int_0^t \langle \nabla \eta(\phi_r(x)), dB_r^H \rangle.$$ Integrating w.r.t. $\mu_0$ produces the desired formula.

This means that if we can prove existence of a solution to (1) and an Itô-formula compatible with the rough path setting, we get a solution to the continuity equation.

Very recently, the authors in [1] show existence of a unique solution to (1). The result will be included in Section 4.

1.1 Notation

For Banach spaces $V, W$ we denote $\mathcal{L}(V; W)$ the set of all continuous linear mappings from $V$ to $W$. For simplicity we denote $\mathcal{L}(V) := \mathcal{L}(V; \mathbb{R})$. If the spaces $V$ and $W$ are finite dimensional, and we can identify $\mathcal{L}(V \otimes W)$ with $\mathcal{L}(V; \mathcal{L}(W))$. In particular, for a sufficiently smooth function $f : V \to W$ the $k$th derivative is considered as a map $D^k f : V \to \mathcal{L}(V^{\otimes k}; W)$.

For the entire paper we fix a $T > 0$ and define the simplex $\Delta^{(p)}(s, t) := \{(r_1, \ldots, r_p) \in [0, T]^p : s < r_1 < \cdots < r_p < t\}$. For $\gamma > 0$ denote by $C_2^\gamma([0, T]; V)$ the space of all functions $f : \Delta^{(2)}(0, T) \to V$ such that $\|f\|_\gamma := \sup_{s < t} \frac{\|f(s, t)\|}{|t - s|^\gamma} < \infty$. Given a function $X : [0, T] \to V$ its increment is denoted $X_{s,t} := X_t - X_s$.

For an integer $p$ the $p$-step truncated tensor algebra

$$T^{(p)}(\mathbb{R}^d) := \bigoplus_{n=0}^p (\mathbb{R}^d)^{\otimes n}$$

is equipped with the product $(a \otimes b)^{(n)} = \sum_{k=0}^n a^{(n-k)} \otimes b^{(k)}$.

We recall the following Taylor formula for a function $f : V \to W$ that is $m + 1$ times differentiable

$$f(x) - f(y) = \sum_{k=1}^m \frac{D^k f(y)}{k!} (x - y)^{\otimes k} + R^f_m(x, y)$$ (2)

where $|R^f_m(x, y)| \lesssim |x - y|^{m+1}$. More specifically, we shall use the explicit formula

$$R^f_m(x, y) = \frac{1}{m!} \int_0^1 D^{m+1} f(y + u(x - y))(1 - u)^m du (x - y)^{\otimes (m+1)}. \quad (3)$$
2 Elements of Controlled Rough Paths

The theory of rough paths was first introduced by Terry Lyons in the late 90’s, see [9]. His insight was that even though solutions to ODE’s driven by rough signals are typically not continuous as a function of the signals themselves, adding extra information, namely the iterated integrals of the driving signals, one obtains a topology for which there is continuity of the solutions. The theory was further developed by Gubinelli, [6] and [7], who introduced the notion of controlled paths which defines spaces that are well suited for constructing solutions of the rough ODE’s.

In the present paper we shall use controlled paths as one of our main tools. See [5] for an introduction.

Throughout this section we fix some \( \gamma \in (0, \frac{1}{2}) \) and let \( p \) be the integer part of \( \frac{1}{\gamma} \). A \( \gamma \)-rough path is a mapping \( X : \Delta^{(2)}(0, T) \to T^{(p)}(\mathbb{R}^d) \)

\[
(s, t) \mapsto (1, X_{st}^{(1)}, \ldots, X_{st}^{(p)})
\]

that satisfies an algebraic (Chen’s) relation

\[
X_{st} = X_{su} \otimes X_{ut},
\]

and an analytic relation

\[
|X_{st}^{(n)}| \lesssim |t - s|^{n\gamma} \quad n = 1, \ldots, p.
\]

We denote by \( \mathcal{C}^\gamma \) the set of all rough paths equipped with the metric

\[
\varrho_{\gamma}(X, \tilde{X}) := \sum_{n=1}^{p} \sup_{t \neq s} \frac{|X_{st}^{(n)} - \tilde{X}_{st}^{(n)}|}{|t - s|^{n\gamma}}
\]

Given a function \( X \in C^1([0, T]; \mathbb{R}^d) \) we can consider its canonical lift to a rough path

\[
X_{st} := (1, X_{st}, \int_s^t X_{sr} \otimes \dot{X}_{r} \, dr, \ldots, \int_{\Delta^{(p)}(s,t)} \dot{X}_{r_1} \otimes \cdots \otimes \dot{X}_{r_p} \, dr_1 \cdots dr_p).
\]

We will denote by \( \mathcal{C}^\gamma_0 \) the closure of the canonical lift of \( C^1([0, T]; \mathbb{R}^d) \) in the rough path topology \( \mathcal{C}^\gamma \). An element \( X \in \mathcal{C}^\gamma_0 \) will be referred to as a geometric rough path and it satisfies the identity

\[
\text{sym}(X_{st}^{(n)}) = \frac{1}{n!} \left( X_{st}^{(1)} \right)^{\otimes n}.
\]

Given a rough path \( X \in \mathcal{C}^\gamma \), we shall say that a mapping

\[
Y : [0, T] \longrightarrow \bigoplus_{n=1}^{p} \mathcal{L}((\mathbb{R}^d)^{\otimes n})
\]

\[
t \longmapsto (Y_{t}^{(1)}, \ldots, Y_{t}^{(p)})
\]

is sometimes written \( \mathcal{C}^\gamma_0 \) in the litterature, whereas \( \mathcal{C}^\gamma \) is reserved for paths satisfying \( \mathcal{C}^\gamma_0 \). While \( \mathcal{C}^\gamma_0 \) is strictly included in \( \mathcal{C}^\gamma \) one can use “geodesic approximations” and interpolation to show \( \mathcal{C}^\gamma \subset \mathcal{C}^{\gamma'} \subset \mathcal{C}^\gamma_0 \) for \( \gamma' < \gamma \), so that one can still approximate elements satisfying \( \mathcal{C}^\gamma_0 \) at the expense of choosing a smaller \( \gamma \).
is a controlled (by \(X\)) path if the implicitly defined functions

\[
Y_{st}^{(k)} := Y_t^{(k)} - \sum_{n=k}^{p} Y_s^{(n)} X_{st}^{(n-k)} \quad k = 1, \ldots, p.
\]

is such that \(Y^{(k)}_{st} \in C_2((p+1-k)\gamma([0, T]; L((\mathbb{R}^d)^{\otimes k})), \) i.e.

\[
\|Y_{st}^{(k)}\| \lesssim |t-s|^{(p+1-k)\gamma}.
\]  

(8)

We denote by \(\mathcal{D}_X^{\gamma}\) the set of all paths controlled by \(X\), and we equip this linear space with the semi-norm

\[
\|Y\|_X = \sum_{k=1}^{p} \|Y^{(k)}\|_{(p+1-k)\gamma}.
\]

Conditioned on \((Y_0^{(1)}, \ldots, Y_0^{(p)})\) we get a norm which controls the \(\| \cdot \|_\infty\)-norm of \(Y\) in the following way. We have \(Y_t^{(k)} = Y_0^{(k)} + \sum_{n=k}^{p} Y_0^{(n)} X_{nt}^{(n-k)}\) so that

\[
\|Y^{(k)}\|_\infty \leq T^{(p+1-k)\gamma} \|Y^{(k)}\|_{(p+1-k)\gamma} + \sum_{n=k}^{p} |Y_0^{(n)}| \|X\|^{(n-k)\gamma} T^{(n-k)\gamma}
\]

\[
\lesssim \|Y\|_X + \varrho_\gamma(0, X)|Y_0|.
\]

If we consider two paths \(Y\) and \(\tilde{Y}\), controlled by \(X\) and \(\tilde{X}\) respectively, we introduce the “distance”

\[
\|Y; \tilde{Y}\|_{X, \tilde{X}} := \sum_{k=1}^{p} \|Y^{(k)} - \tilde{Y}^{(k)}\|_{(p+1-k)\gamma}.
\]

Similar as above we can find the following estimate

\[
\max_{n=1; \ldots, p} \|Y^{(n)} - \tilde{Y}^{(n)}\|_\infty \leq \|Y; \tilde{Y}\|_{X, \tilde{X}} + \varrho_\gamma(X, 0)|Y_0 - \tilde{Y}_0|
\]  

(9)

We define the total space

\[
\mathcal{C}^{\gamma} \times \mathcal{D}^{\gamma} := \bigsqcup_{X \in \mathcal{C}^{\gamma}} \{X\} \times \mathcal{D}^{\gamma}_X
\]

equipped with its natural topology, i.e. the weakest topology such that

\[
\mathcal{C}^{\gamma} \times \mathcal{D}^{\gamma} \rightarrow \mathcal{C}^{\gamma} \times \bigoplus_{k=1}^{p} C_2((p+1-k)\gamma([0, T]; L((\mathbb{R}^d)^{\otimes k})))
\]

\[
(X, Y) \mapsto \left( X, \oplus_{k=1}^{p} Y^{(k)}_{st} \right)
\]

is continuous.
2.1 Integration of Controlled Rough Paths

Following [5] we denote by $C^{\alpha,\beta}_{2}([0,T])$ the space of functions $\Xi : \Delta^{(2)}([0,T]) \to \mathbb{R}$ such that

$$\|\Xi\|_{\alpha} := \sup_{s<t} \frac{|\Xi_{st}|}{|t-s|^\alpha} < \infty \quad \text{and} \quad \|\delta\Xi\|_{\beta} := \sup_{s<u<t} \frac{|\delta\Xi_{sut}|}{|t-s|^\beta} < \infty$$

where $\delta\Xi_{sut} := \Xi_{st} - \Xi_{su} - \Xi_{ut}$. We equip the space with the norm $\|\Xi\|_{\alpha,\beta} := \|\Xi\|_{\alpha} + \|\delta\Xi\|_{\beta}$. The following result is sometimes referred to as the “sewing lemma”:

**Lemma 2.1.** Assume $0 < \alpha \leq 1 < \beta$. Then there exists a unique continuous linear map $I : C^{\alpha,\beta}_{2}([0,T]) \to C^{\alpha}([0,T])$ such that $(I\Xi)_{0} = 0$ and

$$|(I\Xi)_{st} - \Xi_{st}| \lesssim |t-s|^\beta.$$

More specifically,

$$I(\Xi)_{st} = \lim_{|P| \to 0} \sum_{[u,v] \in P} \Xi_{uv}$$

(10)

where $P$ denotes a partition of $[s,t]$ and $|P|$ its mesh. The limit can be taken along any sequence of partitions and is independent of this choice.

For a proof, see [5]. It is clear from (10) that $C^{\theta}_{2}([0,T]) \subset \ker(I)$ for $\theta > 1$.

We are ready to define the integral of a controlled rough path. For $X \in \mathcal{C}^{\gamma}$ and $Y \in \mathcal{D}^{p\gamma}X$ let

$$\Xi_{st} := \sum_{n=1}^{p} Y_{s}^{(n)} X_{s}^{(n)}.$$

Chen’s relation (4) gives $X_{s}^{(n)} = \sum_{k=0}^{n} X_{s}^{(n-k)} \otimes X_{s}^{(k)}$, so that

$$\delta\Xi_{sut} = \sum_{n=1}^{p} Y_{s}^{(n)} (X_{s}^{(n)} - X_{s}^{(n)}) = \sum_{n=1}^{p} Y_{u}^{(n)} X_{ut}^{(n)}$$

$$= \sum_{n=1}^{p} Y_{s}^{(n)} \sum_{k=1}^{n} X_{s}^{(n-k)} \otimes X_{s}^{(k)} = \sum_{n=1}^{p} Y_{u}^{(n)} X_{ut}^{(n)}$$

$$= \sum_{k=1}^{p} \sum_{n=k}^{n} Y_{s}^{(n)} X_{s}^{(n-k)} \otimes X_{s}^{(k)} = \sum_{k=1}^{p} Y_{u}^{(n)} X_{ut}^{(k)}$$

$$= \sum_{k=1}^{p} \left( \sum_{n=k}^{n} Y_{s}^{(n)} X_{s}^{(n-k)} - Y_{u}^{(k)} \right) X_{ut}^{(k)}$$

$$= - \sum_{k=1}^{p} Y_{su}^{(k)} X_{ut}^{(k)}.$$

From (5) and (8) each term can be bounded by $C|t-s|^{(p+1)\gamma}$ for an appropriate constant $C$. Consequently $|\delta\Xi_{sut}| \lesssim |t-s|^{(p+1)\gamma}$. Since $(p+1)\gamma > 1$ we arrive at the following definition:
Definition 2.2. Let $X \in \mathcal{C}^\gamma$ and let $Y \in \mathcal{D}_X^{p\gamma}$. We define the rough path integral of $Y$ w.r.t. $X$ as
\[
\int_s^t Y_r dX_r := (I\Xi)_{st} \tag{11}
\]
with $I$ and $\Xi$ as above.

Remark 2.3. For a smooth path $X$ with its geometric lift (6) the rough path integral and the usual calculus coincide, i.e.
\[
\int_s^t Y_r dX_r = \int_s^t Y_r \dot{X}_r dr,
\]
for all $Y \in C^\gamma([0,T]; \mathcal{L}(\mathbb{R}^d))$. Indeed, we may define $Y(n) = 0$ for $n = 2, \ldots, p$.

Even though in general (8) is not satisfied for $k = 1$, if we define
\[
\Xi_{st} := Y_{st},
\]
we get $\delta\Xi_{sut} = -Y_{su}X_{ut}$ so that $\Xi \in C^{1,1+\gamma}_2([0,T])$.

The rest of this section is devoted to obtaining a “local Lipschitz”-type estimate when we regard the above as a mapping
\[\mathcal{C}^\gamma \times \mathcal{D}^{p\gamma} \to C_2^{(p+1)\gamma}([0,T]).\]
Indeed, let $X, \tilde{X} \in \mathcal{C}^\gamma$ and let $Y$ and $\tilde{Y}$ be controlled by $X$ and $\tilde{X}$ respectively. Define $\tilde{\Xi}$ as before and
\[
\tilde{\Xi}_{st} := \sum_{n=1}^p Y_{s}^{(n)} \tilde{X}_{st}^{(n)}.
\]

Lemma 2.4. Assume $\varrho_\gamma(0,X), \|Y\|_{X,0} \leq M$ for some constant $M$, and similarly for $\tilde{X}$ and $\tilde{Y}$. Then there exists a constant $C_M$ such that
\[
\|\Xi - \tilde{\Xi}\|_{\gamma,(p+1)\gamma} \leq C_M (\|Y_0 - \tilde{Y}_0\| + \|Y; \tilde{Y}\|_{X,\tilde{X}} + \varrho_\gamma(X,\tilde{X})).
\]

Proof. We begin by decomposing
\[
\Xi_{st} - \tilde{\Xi}_{st} = \sum_{n=1}^p Y_s^{(n)} \tilde{X}^{(n)}_{st} - \sum_{n=1}^p Y_s^{(n)} \tilde{X}_{st}^{(n)} = \sum_{n=1}^p Y_s^{(n)} (\tilde{X}^{(n)}_{st} - \tilde{X}_{st}^{(n)}) + \sum_{n=1}^p (Y_s^{(n)} - \tilde{Y}_s^{(n)}) \tilde{X}_{st}^{(n)}
\]
so that
\[
|\Xi_{st} - \tilde{\Xi}_{st}| \leq \sum_{n=1}^p \|Y_s^{(n)}\|_\infty \|\tilde{X}_{st}^{(n)} - \tilde{X}^{(n)}_{st}\|_{n\gamma} |t-s|^{n\gamma}
\]
\[+ \sum_{n=1}^p \|Y_s^{(n)} - \tilde{Y}_s^{(n)}\|_\infty \|\tilde{X}^{(n)}_{st}\|_{n\gamma} |t-s|^{n\gamma}
\]
\[\leq |t-s|^\gamma \max_{n=1,\ldots,p} \|Y_s^{(n)}\|_\infty \varrho_\gamma(X,\tilde{X})
\]
\[+ |t-s|^\gamma \varrho_\gamma(0,\tilde{X}) \max_{n=1,\ldots,p} \|Y_s^{(n)} - \tilde{Y}_s^{(n)}\|_\infty.
\]
Using (9) we can find a constant \( \tilde{C}_M \) such that
\[
\|\Xi - \bar{\Xi}\|_{\gamma} \leq \tilde{C}_M (\|Y; \bar{Y}\|_{X; \bar{X}} + |Y_0 - \bar{Y}_0| + \rho_\gamma(X; \bar{X})).
\]

Similarly,
\[
\delta\Xi_{sut} - \delta\bar{\Xi}_{sut} = -\sum_{n=1}^{p} \frac{Y^{(n)}(n)}{(p+1-\gamma)\gamma} X^{(n)}_{sut} + \sum_{n=1}^{p} \frac{Y^{(n)}(n)}{(p+1-\gamma)\gamma} \bar{X}^{(n)}_{sut}
\]
\[
= -\sum_{n=1}^{p} \frac{Y^{(n)}(n)}{(p+1-\gamma)\gamma} (X^{(n)}_{sut} - \bar{X}^{(n)}_{sut}) + \sum_{n=1}^{p} \frac{Y^{(n)}(n)}{(p+1-\gamma)\gamma} (Y^{(n)}(n) - \bar{Y}^{(n)}(n)) \bar{X}^{(n)}_{sut}
\]
so that
\[
\|\delta(\Xi - \bar{\Xi})\|_{(p+1)\gamma} \leq \sum_{n=1}^{p} \frac{Y^{(n)}(n)}{(p+1-\gamma)\gamma} X^{(n)}_{n\gamma} + \sum_{n=1}^{p} \frac{Y^{(n)}(n)}{(p+1-\gamma)\gamma} \bar{X}^{(n)}_{n\gamma}
\]
\[
\leq M(\rho_\gamma(X; \bar{X}) + \|Y; \bar{Y}\|_{X; \bar{X}}).
\]

\[\Box\]

### 2.2 Controlling solutions of ODE's

In this section we will show how to control solutions to ODE’s perturbed by a rough path \( X \in \mathcal{C}_\gamma \). Fix a function \( b \in C^b_b([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \) and denote by \( \phi_t(x) \) the solution of the perturbed ODE
\[
\phi_t(x) = x + \int_0^t b(r, \phi_s(x))dr + X_t.
\tag{12}
\]

When there is no chance of confusion we shall denote the solution to \tag{12} by \( \phi_t \) for notational convenience. Notice that we shall later on be interested in \( \phi_t \) as a function of \( x \), but for this section we leave it fixed.

We have
\[
\phi_{st} = \int_s^t b(r, \phi_r)dr + X_{st} =: R^{\phi}_{st} + X_{st}
\]
where \( |R^{\phi}_{st}| \leq |t - s| \) by the boundedness of \( b \). Let \( f \in C^b_b(\mathbb{R}^d; \mathbb{R}^d) \), so that we can view \( D^k f : \mathbb{R}^d \to \mathcal{C}((\mathbb{R}^d)^{\otimes (k+1)}) \). We shall lift the composition \( f(\phi) \) to a controlled path in \( \mathcal{C}^b_{X} \).

**Lemma 2.5.** Assume \( X \) is a geometric rough path. Then the mapping \( s \mapsto (f(\phi_s), \ldots, D^{p-1} f(\phi_s)) \) belongs to \( \mathcal{C}^b_{X} \), i.e. if we introduce the ad-hoc notation
\[
f(\phi)^{(k)}_{st} := D^k f(\phi_t) - \sum_{n=k}^{p-1} D^n f(\phi_s) X_{st}^{(n-k)} \quad k = 0, 1, \ldots, p - 1
\]
we have \( f(\phi)^{(k)}_{ST} \in C^b_{2}([0, T]; \mathcal{C}((\mathbb{R}^d)^{\otimes (k+1)})) \).

\[7\]
Proof. Begin by writing
\[ \phi_{st}^\otimes n = (R_{st}^\phi + X_{st})^\otimes n = \sum_{q=0}^{n} \binom{n}{q} \text{sym}((R_{st}^\phi)^{(n-q)} \otimes X_{st}^\otimes q) \]

For a sufficiently smooth function \( g : \mathbb{R}^d \to \mathcal{L}(V) \) where \( V \) is a finite-dimensional Banach space, we have from Taylor’s formula
\[ g(\phi_t) - g(\phi_s) = \sum_{n=1}^m \frac{D^n g(\phi_s)}{n!} (\phi_{st})^\otimes n + R_m(\phi_s, \phi_t) \]
\[ = \sum_{n=1}^m D^n g(\phi_s) X_{st}^{(n)} + R_m(\phi_s, \phi_t) \]
\[ + \sum_{n=1}^m \sum_{k=1}^n \binom{n}{q} \frac{D^n g(\phi_s)}{n!} ((R_{st}^\phi)^{(n-q)} \otimes X_{st}^\otimes q). \]

In the above we have used that \( X \) satisfies (11) so that \( D^n g(\phi_s) X_{st}^{(n)} = D^n g(\phi_s) X_{st}^{(n)} \) since \( D^n g \) only acts on symmetric tensors. Furthermore, the second term \( \lesssim |\phi_{st}|^{m+1} \lesssim |t-s|^{(m+1)\gamma} \), and the third term \( \lesssim |t-s| \). With \( g = D^k f \) and \( m = p-k-1 \) it follows that \( f(\phi)^{(k')} \in C_2^{(p-k)\gamma}([0,T]; \mathcal{L}(\mathbb{R}^d)^{(k+1)}) \), thus proving the lemma.

\[ \square \]

Corollary 2.6. For \( f \in C_0^p(\mathbb{R}^d; \mathbb{R}^d) \) we may define \( \int f(\phi_r)dX_r \) as the rough path integral of \( f(\phi) \) w.r.t. \( X \) as in (11).

2.3 Stability w.r.t. the driving path

The purpose of this section is to prove local Lipschitz continuity of the mapping
\[ \mathcal{C}^\gamma \to \mathcal{C}^\gamma \otimes \mathcal{D}^{p\gamma} \]
\[ \mathcal{X} \mapsto (\mathcal{X}, f(\phi)) \]

where \( \phi \) is the solution to (12), \( f \in C_0^p(\mathbb{R}^d; \mathbb{R}^d) \) and \( f(\phi) \) denotes the lift as described in the previous section. We begin with some trivial bounds, namely let \( \mathcal{X} \in \mathcal{C}^\gamma \) and denote by \( \tilde{\phi} \) the solution to (12) when we replace \( \mathcal{X} \) by \( \mathcal{X} \), i.e.
\[ \tilde{\phi}_{st} = \int_s^t b(r, \tilde{\phi}_r)dr + \tilde{X}_{st} \]
\[ =: R_{st}^\phi + \tilde{X}_{st}. \]

One can check that (see [3], Lemma A.7)
\[ \|\phi - \tilde{\phi}\|_\gamma \leq C(T, b') \|X - \tilde{X}\|_\gamma. \]

Clearly this implies \( \|\phi - \tilde{\phi}\|_\gamma \lesssim \vartheta_\gamma(\mathcal{X}, \mathcal{X}) \) and also \( \|R^\phi - R^\tilde{\phi}\|_\gamma \lesssim \vartheta_\gamma(\mathcal{X}, \mathcal{X}). \)

It follows that \( \|\phi^{\otimes n} - \tilde{\phi}^{\otimes n}\|_{n\gamma} \lesssim \vartheta_\gamma(\mathcal{X}, \mathcal{X}) \) by induction: assume this holds for \( n-1 \). Then
\[ |\phi_{st}^{\otimes n} - \tilde{\phi}_{st}^{\otimes n}| \leq |\phi_{st}^{(n-1)}| |\phi_{st} - \tilde{\phi}_{st}| + |\phi_{st}^{(n-1)} - \tilde{\phi}_{st}^{(n-1)}| |\tilde{\phi}_{st}| \]
\[ \leq 2|t-s|^{n\gamma} \vartheta_\gamma(\mathcal{X}, \mathcal{X}) \]
\[ \leq 2|t-s|^{n\gamma} \vartheta_\gamma(\mathcal{X}, \mathcal{X}). \]
by the induction hypothesis combined with \(\mathbf{13}\).

The main result of this section is the following.

**Lemma 2.7.** Assume \(\varphi_s(X, 0), \varphi_s(\tilde{X}, 0) \leq M\) and \(f \in C^k_δ(\mathbb{R}^d; \mathbb{R}^d)\). Then there exists a constant \(C_M\) such that

\[
\|f(\phi); f(\tilde{\phi})\|_{X, \tilde{X}} \leq C_M \varphi_s(X, \tilde{X}).
\]

Proof. We shall use the formula \(\mathbf{13}\) to show that \(\|f(\phi)^{(k)} - f(\tilde{\phi})^{(k)}\|_{(p-k)\gamma} \leq C_M \varphi_s(X, \tilde{X})\), which will prove the claim. To this end for a function \(g\) smooth enough, we have that the remainder term of the Taylor expansion satisfies

\[
R_{m,n}^a(\phi_s, \phi_t) - R_{m,n}^a(\tilde{\phi}_s, \tilde{\phi}_t) = \int_0^1 \frac{(1-r)^{m+1}}{m!} D^{m+1} g(\phi_s + r\phi_{st}) dr \bigg( \phi_{st}^{(m+1)} - \tilde{\phi}_{st}^{(m+1)} \bigg)
\]

\[
+ \int_0^1 \frac{(1-r)^{m+1}}{m!} \left( D^{m+1} g(\phi_s + r\phi_{st}) - D^{m+1} g(\tilde{\phi}_s + r\tilde{\phi}_{st}) \right) dr \left( \tilde{\phi}_{st}^{(m+1)} \right).
\]

For the first term above we have \(\lesssim |t-s|^{(m+1)\gamma} \|D^{m+1} g\|_\infty \varphi_s(X, \tilde{X})\). For the second term we use, uniformly in \(r \in [0, 1]\)

\[
|D^{m+1} g(\phi_s + r\phi_{st}) - D^{m+1} g(\tilde{\phi}_s + r\tilde{\phi}_{st})| \leq \|D^{m+1} g\|_\infty (|\phi_s - \tilde{\phi}_s| + r|\phi_{st} - \tilde{\phi}_{st}|) \lesssim \|D^{m+1} g\|_\infty \varphi_s(X, \tilde{X}).
\]

Together with the bound \(\|\tilde{\phi}_{st}^{(m+1)}\| \lesssim |t-s|^{(m+1)\gamma}\) we see that

\[
\|R_{m,n}^a(\phi, \phi) - R_{m,n}^a(\tilde{\phi}, \tilde{\phi})\|_{(m+1)\gamma} \lesssim \varphi_s(X, \tilde{X}).
\]

Fix integers \(q \geq 1\) and \(n \geq 0\). Using the estimate \(|a \otimes b - a' \otimes b'| \leq |a - a'||b| + |a'|(b - b')|\) repeatedly, it is easy to check that

\[
|Dg(\phi_s)(X_{st}^{\otimes n} \otimes (R_{st}^{\otimes q})) - Dg(\tilde{\phi}_s)(\tilde{X}_{st}^{\otimes n} \otimes (R_{st}^{\otimes q}))| \lesssim |t-s| \varphi_s(X, \tilde{X}).
\]

These facts combined with \(\mathbf{13}\) gives

\[
\|f(\phi)^{(k)} - f(\tilde{\phi})^{(k)}\|_{(p-k)\gamma} \lesssim \varphi_s(X, \tilde{X})
\]

which ends the proof of the lemma.

\[\square\]

Combining the above Lemma, Lemma 2.4 and Remark 2.3 we get

**Corollary 2.8.** Let \(X \in \mathcal{C}^\gamma_\delta\). Then there exists a family of smooth paths \(X^\epsilon\)

such that

\[
\int_0^1 f(\phi^\epsilon_r) \dot{X}^\epsilon_r dr \to \int_0^1 f(\phi_r) dX_r \quad \text{in } C^\gamma([0, T]),
\]

as \(\epsilon \to 0\).
2.4 Stability w.r.t. the drift

Let us fix $X \in C^\gamma$ and we consider the ODE (12). Assume we have a sequence of functions $b_\epsilon$ such that there exists a solution of for every $\epsilon > 0$ to

$$\phi_\epsilon^t = x + \int_0^t b_\epsilon(r, \phi_\epsilon^r)dr + X_t.$$  

We will show stability in the sense of controlled rough paths when we assume that $\phi_\epsilon$ converges in an appropriate topology. This convergence will be shown to hold in Proposition 4.2 for our particular case.

**Lemma 2.9.** Assume $\phi_\epsilon$ converges in $C^\gamma$ to the solution of (12). Then for any $f \in C^{p \gamma}(\mathbb{R}_d; \mathbb{R}_d)$ we have that the lift of $f(\phi_\epsilon)$ converges in $D^{p \gamma}$ to $f(\phi)$, and as $\epsilon \to 0$

$$\int_0^t f(\phi_\epsilon^r)dX_r \to \int_0^t f(\phi_r)dX_r.$$

where the above convergence is in $C^\gamma$.

**Proof.** Note that the second claim follows from the first in connection with Remark 2.1.

To see the first claim, one has to show

$$\lim_{\epsilon \to 0} \|f(\phi) - f(\phi_\epsilon)\|_{(p-k)\gamma} = 0$$

for all $k = 0, 1, \ldots, p-1$. The proof follows the same lines as the proof of Lemma 2.7 with minor modifications, noting that $X = \tilde{X}$.

2.5 An Itô-Stratonovich formula

For the sake of being self-contained, we include a change-of-variable formula for our particular case. Let $\eta \in C^\infty_c(\mathbb{R}^d)$ and assume $\phi$ solves (12). If $X$ is a smooth path usual calculus yields,

$$\frac{d}{dt} \eta(\phi_t) = \langle D\eta(\phi_t), b(t, \phi_t) \rangle + \langle D\eta(\phi_t), \dot{X}_t \rangle.$$

We can generalize this to geometric rough paths.

**Lemma 2.10.** Suppose $\eta \in C^\infty_c(\mathbb{R}^d)$ and $X$ is a rough path above $X$. Then we have

$$\eta(\phi_t) = \eta(x) + \int_0^t \langle D\eta(\phi_r), b(r, \phi_r) \rangle dr + \int_0^t D\eta(\phi_r)dX_r.$$

where the last term is the rough path integral.

**Proof.** Let $0 \leq u \leq v \leq t$ and use Taylor’s formula to write, as in

$$\eta(\phi)_{uv} = \sum_{n=1}^p \frac{D^n \eta(\phi_u)}{n!} (\phi_{uv})^n + R^\eta_p(\phi_u, \phi_v)$$

$$= D\eta(\phi_u)R^\phi_{uv} + \sum_{n=1}^p D^n \eta(\phi_u) X_{uv}^{(n)} + \Xi_{uv}.$$
where

$$\Xi_{uv} := R^p_{uv}(\phi_u, \phi_v) + \sum_{n=2}^{p} \sum_{q=1}^{n-1} \frac{D^n \eta(\phi_u)}{n!} ((P^0_{uv})^{(n-q)} \otimes X^{q}_{uv})$$

and notice that $\Xi \in C_{2+\gamma}^1([0, T]) \subset \ker(I)$. We have

$$\lim_{|P| \to 0} \sum_{[u,v]} \langle D \eta(\phi_u), b(r, \phi_r)dr \rangle = \lim_{|P| \to 0} \int_0^t (\sum_{[u,v]} D \eta(\phi_u)1_{[u,v]}(r), b(r, \phi_r))dr$$

$$= \int_0^t \langle D \eta(\phi_r), b(r, \phi_r) \rangle dr$$

where we used continuity of $D \eta$ and dominated convergence in the last step to take in the limit. Note that the above reasoning does not use any regularity requirements on $b$.

Finally, we have

$$\eta(\phi_t) - \eta(x) = I(\eta(\phi))_{0,t} = \lim_{|P| \to 0} \sum_{[u,v]} \eta(\phi)_{uv}$$

$$= \lim_{|P| \to 0} \sum_{[u,v]} \left( D \eta(\phi_u) R^p_{uv} + \sum_{n=1}^{p} D^n \eta(\phi_u) X^{(n)}_{uv} + \Xi_{uv} \right)$$

$$= I(D \eta(\phi) R^p) + I(\sum_{n=1}^{p} D^n \eta(\phi) X^{(n)}) + I(\Xi)$$

$$= \int_0^t \langle D \eta(\phi_r), b(r, \phi_r) \rangle dr + \int_0^t D \eta(\phi_r) dX_t$$

by definition of the rough path integral.

\[ \square \]

### 2.6 Integrated ODE’s

To emphasize that the solution of (12) depends on the initial value $x$, we denote its solution by $\phi(x)$, i.e.

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x))dr + X_t.$$  

Let $\nu$ be a finite signed measure on $\mathbb{R}^d$, and $f = (f^{(1)}, \ldots, f^{(d)}) \in C^p_b(\mathbb{R}; \mathbb{R}^d)$. In later chapters we shall be interested in expressions on the form

$$\nu(f(\phi)) := \left( \int_{\mathbb{R}} f^{(1)}(\phi(x)) d\nu(x), \ldots, f^{(d)}(\phi(x)) d\nu(x) \right) \in L_{\mathbb{R}^d}$$

as a controlled path in order to define $\int_0^t \nu(f(\phi_r)) dX_r$ in the rough path sense. Similar results as the previous chapters holds, summarized below.

**Proposition 2.11.** Retain the hypotheses and notations respectively from Corollary 2.6, Corollary 2.8 and Lemma 2.9. The following holds.
1. The rough path integral \( \int_0^t \nu(f(\phi_\epsilon))dX_\epsilon \) is well defined.

2. Let \( X \in C^\gamma \). Then there exists a family of smooth paths \( X^\epsilon \) such that
   \[
   \int_0^t \nu(f(\phi_\epsilon))dX_\epsilon \to \int_0^t \nu(f(\phi_\epsilon))dX_\epsilon \quad \text{in } C^\gamma([0,T]),
   \]
   as \( \epsilon \to 0 \).

3. If \( \nu(f(\phi_\epsilon)) \to \nu(f(\phi)) \) in \( C^\gamma \) we have
   \[
   \int_0^t \nu(f(\phi_\epsilon))dX_\epsilon \to \int_0^t \nu(f(\phi))dX_\epsilon \quad \text{in } C^\gamma([0,T]),
   \]
   as \( \epsilon \to 0 \).

Proof. Begin with the first assertion. Integrating (13) w.r.t. \( \nu \) gives
   \[
   \int_{\mathbb{R}^d} f(\phi(x))_{st}^{(k)} d\nu(x) = \sum_{n=1}^{p-k-1} \sum_{q=1}^n \int_{\mathbb{R}^d} \frac{Df^{(k+n)}(\phi_s(x))}{q!} (R_{st}^{\phi(x)} \otimes q \otimes X_{st}^{(n-q)} d\nu(x)
   \]
   \[
   + \int_{\mathbb{R}^d} R_{p-k-1}^{\phi(x)}(\phi_s(x), \phi_t(x)) d\nu(x).
   \]
   Since \( \nu \) is finite and \( b \) is bounded we get for each \( k, n \) and \( q \)
   \[
   |\int_{\mathbb{R}^d} \frac{Df^{(k+n)}(\phi_s(x))}{q!} (R_{st}^{\phi(x)} \otimes q \otimes X_{st}^{(n-q)} d\nu(x)| \lesssim |t-s|.
   \]
   Furthermore
   \[
   |\int_{\mathbb{R}^d} R_{p-k-1}^{\phi(x)}(\phi_s(x), \phi_t(x)) d\nu(x)| \lesssim \int_{\mathbb{R}^d} |\phi_s(x)|^{p-k} d\nu(x) \lesssim |t-s|^{(p-k)\gamma},
   \]
   so that \( \int_{\mathbb{R}^d} f(\phi(x)) d\nu(x) \) is a controlled path and
   \[
   \left( \int_{\mathbb{R}^d} f(\phi(x)) d\nu(x) \right)_{st}^{(k)} = \int_{\mathbb{R}^d} f(\phi(x))_{st}^{(k)} d\nu(x).
   \]
   Using linearity, boundedness of \( b \) and dominated convergence the reader is invited to complete the remaining proofs.

3 Fractional Brownian motion and Girsanov’s theorem

Let \( B = \{ B_t, t \in [0,T] \} \) be a 1-dimensional fractional Brownian motion (fBm) with Hurst parameter \( H \in (0, \frac{1}{2}) \), i.e. a centered Gaussian process with covariance
   \[
   R_H(t,s) := E[B_tB_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
   \]
   Observe that \( B \) has stationary increments and Hölder continuous trajectories of index \( H - \epsilon \) for all \( \epsilon > 0 \).
Denote by \( \mathcal{E} \) the set of step functions on \([0, T]\) and denote by \( \mathcal{H} \) the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the inner product

\[
(1_{[0,t]}, 1_{[0,s]})_{\mathcal{H}} = R_H(t, s).
\]

The mapping \( 1_{[0,t]} \mapsto B_t \) can be extended to an isometry between \( \mathcal{H} \) and a Gaussian subspace of \( L^2(\Omega) \).

For a function \( f \in L^2([a, b]) \), we define the left fractional Riemann-Liouville integral by

\[
I^\alpha_0 f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - y)^{\alpha - 1} f(y) dy
\]

for \( \alpha > 0 \). Denote by \( I^\alpha_0 \) \((L^2([a, b]))\) the image of \( L^2([a, b]) \) under \( I^\alpha_0 \) and by \( D^\alpha_0 \) its inverse.

We define \( K_H(t,s) \) as

\[
K_H(t,s) = c_H \left( H + \frac{1}{2} \right) s^{H-\frac{1}{2}} \left( D^{\frac{H-\frac{1}{2}}{2}}_{t-} u^{H-\frac{1}{2}} \right)(s),
\]

and write \( K_H \) for the operator from \( L^2([0, T]) \) onto \( I^H_0 \) \((L^2)\) associated with the kernel \( K_H(t,s) \).

It follows that

\[
R_H(t,s) = \int_0^t K_H(t,u)K_H(s,u) du.
\]

Moreover, if \( W = \{W_t : t \in [0, T]\} \) is a standard Brownian motion \( B \) can be represented as

\[
B_t = \int_0^t K_H(t,s) dW_s. \tag{15}
\]

A \( d \)-dimensional fractional Brownian motion is a \( d \)-dimensional process where the components are independent \( 1 \)-dimensional fractional Brownian motions.

**Theorem 3.1** (Girsanov’s theorem for fBm). Let \( u = \{u_t, t \in [0, T]\} \) be an \( \mathbb{R}^d \)-valued, \( \{\mathcal{F}_t\}_{t \in [0, T]} \)-adapted process with integrable trajectories and set \( \tilde{B}_t = B_t + \int_0^t u_s ds, \ t \in [0, T]. \) Assume that

(i) \( \int_0^T u_s ds \in (I^H_0 + \frac{1}{2}(L^2([0, T])))^d, \ P\text{-a.s.} \)

(ii) \( E[\xi_T] = 1 \) where

\[
\xi_T := \exp \left\{ - \int_0^T \left( K^{-1}_H \left( \int_0^r u_s ds \right) (s), dW_s \right) - \frac{1}{2} \int_0^T \left| K^{-1}_H \left( \int_0^r u_s dr \right) (s) \right|^2 ds \right\}.
\]

Then the shifted process \( \tilde{B} \) is an \( \{\mathcal{F}_t\}_{t \in [0, T]} \)-fractional Brownian motion with Hurst parameter \( H \) under the new probability \( \tilde{P} \) defined by \( \frac{d\tilde{P}}{dP} = \xi_T \).

Moreover, for every \( p > 1 \) we have \( E[\|\xi_T\|^p] \leq C_p(\|b\|_\infty) \), where \( C_p(\cdot) \) is an increasing function.

For a proof we refer to [11]. In particular, the moment-estimate if found in the proof of Theorem 3, [11].

An important fact we shall need is that we can lift the fractional Brownian motion to a rough path. This was first done in [13], but we shall refer to [12] for a different construction where the authors construct the iterated integrals using a Stratonovich-Volterra-type representation.
Theorem 3.2 (Theorem 1.1. in [12]). Let $B$ be a fractional Brownian motion admitting the representation (15). For $1 \leq n \leq \lfloor \frac{1}{H} \rfloor$ define

$$B^{(n)}: \Delta^{(2)}(0, T) \to (\mathbb{R}^d)^{\otimes n}$$

componentwise, i.e. for any tuple $\{i_1, \ldots, i_n\}$ in $\{1, \ldots, d\}$, as the Stratonovich iterated integral

$$\langle B^{(n)}_{st}, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle = \sum_{j=1}^{n} (-1)^{j-1} \int_{A_j^n} \prod_{l=1}^{j-1} K(s, r_l) [K(t, r_j) - K(s, r_j)] \prod_{l=j+1}^{n} K(t, r_l) \circ dW_{i_l}^{r_l} \cdots \circ dW_{i_n}^{r_n}$$

where

$$A_j^n := \{(r_1, \ldots, r_n) \in [0, t]^n : r_j = \min(r_1, \ldots, r_n), \quad r_1 > \cdots > r_j-1 \text{ and } r_{j+1} < \cdots < r_n\}.$$ 

Then there exists a set $\Omega_B$ with full measure such that

$$B_{st} := (1, B_t - B_s, B_{st}^{(2)}, \ldots, B_{st}^{(\lfloor 1/H \rfloor)})$$

satisfies (4) and (7) on $\Omega_B$. Moreover, for $\gamma < H$ we have $|B_{st}^{(n)}| \lesssim |t - s|^n$.

Assume now that $H$ is such that $\frac{1}{H}$ is not an integer. We can choose $\gamma < H$ such that $[\frac{1}{H} \gamma] = [\frac{1}{H}]$, and from the above theorem we have, $P$-a.s., $B \in C^\gamma$.

Let us remark that for $H \in (\frac{1}{4}, \frac{1}{2})$ there exists a canonical lift of $B$ to a rough path building the iterated integral from dyadic interpolation of $B$. For the method of the current paper to work we need smaller $H$, see Section 4. When $H \in (0, \frac{1}{2})$ the dyadic interpolation fails to give a converging sequence of rough paths, see [4]. Nevertheless, the construction in [12] gives a geometric rough path so that we may approximate $B$ by a sequence of lifted smooth paths, in the rough path topology.

4 Fractional Brownian motion SDE’s

For this section we shall study a SDE driven by an additive fractional Brownian motion, i.e.

$$\phi_t(x) = x + \int_0^t b(r, \phi_r(x)) dr + B_t. \quad (16)$$

Existence and uniqueness of a solution to this equation under low regularity on $b$ was recently proved in [1] as demonstrated in the next Proposition. For proofs the reader is referred to [1].

Proposition 4.1 (Theorem 4.1 and Corollary 4.8 in [1]). Assume $H < \frac{1}{2(d \delta - 1)}$. Let $\{b_n\}_{n \geq 0} \subset C^\infty_c([0, T] \times \mathbb{R}^d)$ be a sequence of functions such that

$$\sup_{n \geq 0} (\|b_n\|_{L^\infty([0,T] \times \mathbb{R}^d)} \lor \|b_n\|_{L^1([0,T] \times \mathbb{R}^d)}) < \infty.$$
Denote by \( \phi^n_t(x) \) the solution to (15) when \( b \) is replaced by \( b_n \). Then for fixed \((t, x) \in [0, T] \times \mathbb{R}^d\) the sequence is \( \phi^n_t(x) \) is relatively compact in the strong topology of \( L(\Omega) \).

Furthermore, if \( \lim_{n \to \infty} \| b - b_n \|_{L^\infty([0, T]; L^1(\mathbb{R}^d))} = 0 \) for \( b \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}^d)) \) then \( \phi^n_t(x) \) is converging for every \((t, x) \in [0, T] \times \mathbb{R}^d\) to the unique solution of (15).

The difficult part of the proof is to get compactness of the approximating solutions. The proof of this part relies on a compactness criterion from [2] based on Malliavin calculus. Without going into too much detail the criterion is satisfied if we can bound the Malliavin derivative of \( \phi^n_t(x) \) by a constant depending only on \( \| b_n \|_{L^\infty([0, T]; L^1(\mathbb{R}^d))} \lor \| b_n \|_{L^\infty([0, T] \times \mathbb{R}^d)} \).

One has strong convergence, one can use a somewhat standard trick, see e.g. [8] or [11], to show that \( \int_0^t b_n(r, \phi^n_t(x)) dr \to \int_0^t b(r, \phi_t(x)) dr \) which gives that the limit solves (16).

With the notation of Proposition 2.11 we shall need a result to ensure convergence of \( \nu(f(\phi^n)) \) is uniform on a set of full measure.

**Proposition 4.2.** Let \( \gamma \in (0, H), f \in C^1_b(\mathbb{R}^d; \mathbb{R}^d) \) and \( \nu \) be a finite signed measure on \( \mathbb{R}^d \). Then there exists a set \( \Omega_{\gamma, \nu} \) of full measure which satisfies

\[
\lim_{n \to \infty} \nu(f(\phi^n) = \nu(f(\phi))
\]

in \( C^1([0, T]; \mathbb{R}^d) \) for all \( \omega \in \Omega_{\gamma, \nu} \).

**Proof.** We begin by showing that \( \nu(f(\phi^n)) \to \nu(f(\phi)) \) in \( L^2(\Omega) \) for every \( t \). To see this, consider

\[
E[|f(\nu(\phi^n)) - f(\nu(\phi))|^2] = E[|\nu(f(\phi^n) - f(\phi))|^2] \\
\leq |\nu|_{C^1(\mathbb{R}^d)} |D f|_{\infty} \int_{\mathbb{R}^d} E[|\phi^n_t(x) - \phi_t(x)|^2] dv(x) \\
\to 0
\]

as \( n \to \infty \) by dominated convergence, which proves the first claim.

Next we find a set universal in \( t \) for which we have pointwise in \( \omega \) convergence. Denote by \( \{ q_1 \}_{j=1}^\infty \) an enumeration of \([0, T] \cap \mathbb{Q} \). We may extract a subsequence \( \{ \nu(f(\phi^0_{q_1})), \{ n_{q_1} \}_{k=1}^\infty \} \subset \{ \nu(f(\phi_{q_1})), n_{k} \}_{n \geq 1} \) such that

\[
\lim_{k \to \infty} \nu(f(\phi_{n_{q_1}}(k,1))) = \nu(f(\phi_{q_1}(\omega)))
\]

for \( \omega \in \Omega_{q_1} \) with full measure. Furthermore, we define inductively a subsequence \( \{ \nu(f(\phi^0_{n_{q_{j+1}}}(k)))) \}_{k=1}^\infty \subset \{ \nu(f(\phi_{n_{q_{j+1}}}) \}_{n \geq 1} \) such that

\[
\lim_{k \to \infty} \nu(f(\phi^0_{n_{q_{j+1}}}(k,1))) = \nu(f(\phi_{q_{j+1}}(\omega)))
\]

for \( \omega \in \Omega_{q_{j+1}} \) with full measure. Let \( \Omega_0 = \cap_{j=1}^\infty \Omega_j \), so that we have

\[
\lim_{j \to \infty} \nu(f(\phi^0_{n_{q_{j}}}(j)))) = \nu(f(\phi_{q}(\omega)))
\]

for all \( \omega \in \Omega_0 \) and \( q \) rational.
Definition 5.1. Let \( \eta \) then for any test function \( \mu \) and \( C \),

Ascoli's theorem there exists a converging subsequence \( \nu \). Note that \( \nu \) satisfies (1) and for every \( \Omega \) that the limit coincides with \( \nu \). From (17) we see that the uniform boundedness of \( \nu \) implies that \( \| \nu \|_{H^1(\Omega)} \leq \| \nu \|_{H^1(\Omega)} \| \). Since \( (C_t \Omega) \) is a Banach space this implies that the sequence converges. By intercalation of Helder spaces we see that the claim is true if we let \( \Omega := \varnothing \).

Let \( \varepsilon > 0 \) be such that \( \varphi \in C^0(\Omega, T, [0, x]) \) and choose a subsequence \( \Omega \), we have such \( \varphi \) satisfies \( \varphi \) and for every \( \varphi \in C^0(\Omega, T, [0, x]) \).
We integrate the equation w.r.t. \( \mu_t \) to see that \( \mu_t \coloneqq (\phi_t)_\sharp \mu_0 \) solves (17) if we can use integration by parts for the rough path integral, namely

\[
\int_{\mathbb{R}^d} \int_0^t D\eta(\phi_r(x)) dX_r d\mu_0(x) = \int_0^t \mu_0(D\eta(\phi_r)) dX_r.
\]

Suppose now that \( b \in C^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \) and \( X \in \mathcal{C}^\gamma \). Let \( X^\epsilon \in C^1([0, T]; \mathbb{R}^d) \) be such that \( X^\epsilon \to X \) in \( \mathcal{C}^\gamma \). Using Section 4, we get

\[
\int_0^t \mu_0(D\eta(\phi_r)) dX_r = \lim_{\epsilon \to 0} \int_0^t \mu_0(D\eta(\phi^\epsilon_r)) \dot{X}^\epsilon_r dr
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \int_0^t D\eta(\phi^\epsilon_r)(x) \dot{X}^\epsilon_r dr d\mu_0(x)
\]

\[
= \int_{\mathbb{R}^d} \int_0^t D\eta(\phi_r(x)) dX_r d\mu_0(x)
\]

We summarize the above in a lemma.

**Lemma 5.2.** Suppose \( b \in C^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \) and \( X \in \mathcal{C}^\gamma \). Then there exists a solution to (17) and the solution is given by \( \mu_t \coloneqq (\phi_t)_\sharp \mu_0 \).

Given the previous sections, the reader will not be surprised that we can extend this to when the drift is discontinuous provided we choose the rough path to be the lift of a fractional Brownian motion with low Hurst index.

**Lemma 5.3.** Assume \( H < \frac{1}{2(d+1)} \), \( b \in L^\infty([0, T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \) and \( \mu_0 \) a finite signed measure on \( \mathbb{R}^d \). There exists a subset \( \Omega^* \subset \Omega \) with full measure such that for every \( \omega \in \Omega^* \) we have

- The fractional Brownian motion lifts to a geometric rough path \( B(\omega) \in \mathcal{C}^\gamma_0 \), \( \gamma < H \).
- There exists a solution \( \mu(\omega) \) to

\[
\mu_t(\omega) + \int_0^t \text{div}(b(r, \cdot) \mu_r(\omega)) dr + \int_0^t \text{div}(\mu_r(\omega) dB_r(\omega)) = \mu_0
\]

**Proof.** Denote by \( \Omega_B \) the set of \( \omega \in \Omega \) such that \( B(\omega) \) lifts to a rough path, \( B(\omega) \in \mathcal{C}^\gamma_0 \).

Let \( \eta \in C^\infty_c(\mathbb{R}^d) \). Consider the approximation from Section 4, i.e. we have \( \Omega_{\gamma, \eta, \mu_0} \) such that \( \lim_{n \to \infty} \mu_0(D\eta(\phi^n(\omega))) = \mu_0(D\eta(\phi(\omega))) \) in \( C^\gamma([0, T]; \mathbb{R}^d) \).

From Propositions 2.11 and 1.2 we get that

\[
\int_0^t \mu_0(D\eta(\phi^n_r(\omega))) dB_r(\omega) \to \int_0^t \mu_0(D\eta(\phi_r(\omega))) dB_r(\omega)
\]

on \( \Omega_{\gamma, \eta, \mu_0} \cap \Omega_B \).

For every \( n \) we have that \( \mu^n_t \coloneqq (\phi^n_t)_\sharp \mu_0 \) satisfies

\[
\mu^n_t(\eta) = \mu_0(\eta) + \int_0^t \mu^n_r(b_n(r, \cdot) D\eta) dr + \int_0^t \mu^n_r(D\eta) dB_r
\]

\[
\frac{1}{n}
\]

We have

\[
\int_0^t \mu^n_0(D\eta(\phi^n_r(\omega))) dB_r(\omega) \to \int_0^t \mu_0(D\eta(\phi_r(\omega))) dB_r(\omega)
\]

on \( \Omega_{\gamma, \eta, \mu_0} \cap \Omega_B \).

For every \( n \) we have that \( \mu^n_t \coloneqq (\phi^n_t)_\sharp \mu_0 \) satisfies

\[
\mu^n_t(\eta) = \mu_0(\eta) + \int_0^t \mu^n_r(b_n(r, \cdot) D\eta) dr + \int_0^t \mu^n_r(D\eta) dB_r
\]

\[
\frac{1}{n}
\]

We have

\[
\int_0^t \mu^n_0(D\eta(\phi^n_r(\omega))) dB_r(\omega) \to \int_0^t \mu_0(D\eta(\phi_r(\omega))) dB_r(\omega)
\]

on \( \Omega_{\gamma, \eta, \mu_0} \cap \Omega_B \).

For every \( n \) we have that \( \mu^n_t \coloneqq (\phi^n_t)_\sharp \mu_0 \) satisfies

\[
\mu^n_t(\eta) = \mu_0(\eta) + \int_0^t \mu^n_r(b_n(r, \cdot) D\eta) dr + \int_0^t \mu^n_r(D\eta) dB_r
\]

\[
\frac{1}{n}
\]

We have

\[
\int_0^t \mu^n_0(D\eta(\phi^n_r(\omega))) dB_r(\omega) \to \int_0^t \mu_0(D\eta(\phi_r(\omega))) dB_r(\omega)
\]
on $\Omega_B$. Denote by $\Omega_{\eta,\mu_0}$ the set of $\omega \in \Omega$ such that $\mu_t^\eta(\eta) \to \mu_t(\eta)$, so that we must have that all the above terms converges on $\Omega_{\eta,\mu_0} \cap \Omega_{\gamma,\eta,\mu_0} \cap \Omega_B$, to

$$
\mu_t(\eta) = \mu_0(\eta) + \int_0^t \mu_r(b(r,\cdot)D\eta)dr + \int_0^t \mu_r(D\eta)d\mathbf{B}_r(\omega).
$$

Let now

$$
\Omega^* := \Omega_B \cap \bigcap_{k \geq 1} \Omega_{\eta^k,\mu_0} \cap \Omega_{\gamma,\eta^k,\mu_0}
$$

where $\{\eta^k\}_{k \geq 1} \subset C_\infty^c(\mathbb{R}^d)$ is dense in $C_\infty^c(\mathbb{R}^d)$ equipped with the usual test function topology. Then $\Omega^*$ is the desired set.
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